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ON THE EXISTENCE AND ASYMPTOTIC STABILITY OF SOLUTIONS FOR UNSTEADY MIXING-LAYER MODELS

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Abstract. We introduce in this paper some elements for the mathematical analysis of turbulence models for oceanic surface mixing layers. We consider Richardson-number based vertical eddy diffusion models. We prove the existence of unsteady solutions if the initial condition is close to an equilibrium, via the inverse function theorem in Banach spaces. We use this result to prove the non-linear asymptotic stability of equilibrium solutions.

1. Introduction.

This paper addresses the mathematical analysis of oceanic turbulent mixing-layer models in the unsteady case. In a context of global climate change, the right computation of the Sea Surface Temperature (SST) is of high interest, as it is closely related to different aspects of the oceanic biosystems, and it has a deep impact in the evolution of polar ices (*cf.* [9]). This analysis is crucial especially in tropical regions, where the high temperatures and the wind-stress generate a well-developed surface turbulent mixing layer. This layer has two parts: the upper one is the mixed layer and the lower one is the thermocline. The mixed layer is a homogeneous layer, that presents almost constant density. The bottom of the mixed layer corresponds to the top of the thermocline, a zone of high gradients of temperature. A similar structure of the mixing layer takes place when there exists a large surface flux of salinity (*cf.* [19]). The physical description of the structure of mixing layers can be found for example in [6] or [13].

Turbulent mixing-layer models are usually vertical first-order closure turbulence models, where the eddy viscosity and diffusion are parametrized by the gradient

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Richardson number, that represents the balance between stabilizing buoyancy forces and destabilizing shearing forces. Our main purpose in this paper is to analyze the existence and asymptotic stability of global solutions for classical mixing-layer models, such as those of Pacanowski-Philander [15] and Gent [8]. The existence, stability and numerical approximation of equilibria have already been studied in [1], [2] and [4].

Let $z \in [-h, 0]$ be the vertical spatial variable, where $h > 0$ denotes the thickness of the studied flow, that must contain the mixing layer, and $t \geq 0$ be the time variable. The mixing layer is assumed to be strongly dominated by vertical fluxes, so that the velocity and density of the fluid are assumed horizontally homogeneous. The flow is turbulent and well mixed, so the vertical velocity vanishes. The model affects the mean horizontal velocity (u, v) and the mean density ρ as functions of the only variables z and t. In tropical seas, the density is just a function of the temperature through a state equation. So, the density is an idealized thermodynamic variable, which is intended to represent temperature variations. The variables u, v, ρ satisfy the Reynolds-averaged equations:

$$\begin{cases} \partial_t u - \partial_z (\nu_1 \partial_z u) = -D_1, & \\ \partial_t v - \partial_z (\nu_1 \partial_z v) = -D_2, & \text{for } t > 0 \text{ and } -h < z < 0, \\ \partial_t \rho - \partial_z (\nu_2 \partial_z \rho) = 0, & \end{cases} \tag{1}$$

where (D_1, D_2) is the horizontal pressure gradient $\nabla_H p$, that we assume to be known, and $\nu_1 = a_1 + \nu_{T1}$, $\nu_2 = a_2 + \nu_{T2}$ respectively are the total viscosity and diffusion. Here, a_1, a_2 are the laminar viscosity and diffusion, and ν_{T1} and ν_{T2} are the vertical eddy viscosity and diffusivity coefficients, that depend on the gradient Richardson number R, defined as:

$$R = R(\partial_z u, \partial_z v, \partial_z \rho) = -\frac{g}{\rho_r} \frac{\partial_z \rho}{(\partial_z u)^2 + (\partial_z v)^2}, \tag{2}$$

where $\rho_r > 0$ is a reference density. When $R >> 1$, a strongly stratified layer takes place. This corresponds to a stable configuration. When $0 < R << 1$, a slightly stratified layer takes place. This corresponds to a configuration with low stability. The case $R < 0$ corresponds to a configuration statically unstable ($\partial_z \rho > 0$). The eddy coefficients ν_{T1} and ν_{T2} are modeled by:

$$\nu_1 = f_1(R), \quad \nu_2 = f_2(R),$$

where the functions f_1 and f_2 correspond either to the Pacanowski-Philander (PP) model (*cf.* [15]):

$$f_1(R) = a_1 + \frac{b_1}{(1 + \sigma R)^2}, \quad f_2(R) = a_2 + \frac{f_1(R)}{1 + \sigma R}, \quad \sigma = 5, \tag{3}$$

or to the Gent model (*cf.* [8]):

$$f_1(R) = a_1 + \frac{b_1}{(1 + \sigma R)^2}, \quad f_2(R) = a_2 + \frac{b_1}{(1 + \sigma R)^3}, \quad \sigma = 10, \tag{4}$$

or to a more recent model proposed in Bennis et al. (*cf.* [1]):

$$f_1(R) = a_1 + \frac{b_1}{(1 + \sigma R)^2}, \quad f_2(R) = a_2 + \frac{f_1(R)}{(1 + \sigma R)^2}, \quad \sigma = 5, \tag{5}$$

with coefficients:

$$a_1 = 10^{-4}, \ b_1 = 10^{-2}, \ a_2 = 10^{-5} \ (\text{units: m}^2 \cdot \text{s}^{-1}).$$

All these models apply to well-mixed layers, for which $R > 0$, although (5) yields stable equilibria for a larger range of $R < 0$ with respect to models (3), (4) (*cf.* [2]). All of them generate mixing layer profiles in characteristic time scales of the order of several days and steady profiles in characteristic time scales of the order of one year (*cf.* [17]). The original PP and Gent models did not include imposed pressure gradients. We consider them here, the mathematical difficulty for its analysis is similar but this allows to model mixing layer flows with initial conditions that are not necessarily divergence free (*cf.* [4]).

In this paper, we shall smoothly extend f_1, f_2 to $R < 0$ by positive constant values, in order to consider unstable configurations. This models a forced return to a vertical stable configuration, and it is used in practice by physical oceanographers (see [8] for instance). We complete model (1) with suitable initial and boundary conditions:

$$\begin{cases} u = u_b(t),\ v = v_b(t),\ \rho = \rho_b(t) \text{ at the depth } z = -h, \\ \nu_1 \partial_z u = \dfrac{\rho_a}{\rho_r} V_x(t),\ \nu_1 \partial_z v = \dfrac{\rho_a}{\rho_r} V_y(t),\ \nu_2 \partial_z \rho = Q(t) \text{ at the surface } z = 0, \\ u = u_0(z),\ v = v_0(z),\ \rho = \rho_0(z) \text{ at initial time } t = 0. \end{cases} \tag{6}$$

The circulation for $z < -h$, under the boundary layer, is supposed to be known, either by observations or by a deep circulation numerical model. This justifies the choice of Dirichlet boundary conditions at $z = -h$. The Neumann boundary conditions at $z = 0$ represent the fluxes at the sea-surface that model the forcing by the atmosphere. In particular: ρ_a is the air density, $V_x(t)$ and $V_y(t)$ are respectively the stress exerted by the zonal and the meridional wind-stress, and $Q(t)$ represents thermodynamic fluxes, heating or cooling, precipitations or evaporation. We have:

$$(V_x(t), V_y(t)) = C_D \, |\mathbf{U}^a(t)| \, \mathbf{U}^a(t),$$

where $\mathbf{U}^a(t) = (u_a(t), v_a(t))$ is the air velocity, and C_D $(= 1.2 \cdot 10^{-3})$ is a friction coefficient (*cf.* [10]).

In this paper, we perform a mathematical analysis of the unsteady model (1)-(6), focused on the existence and stability of solutions close to equilibrium states. This analysis faces the difficulty that the turbulent diffusions depend on the gradient of the unknowns. In general, energy methods fail to obtain estimates in norms strong enough (these should be of $W^{2,p}$ kind) to handle uniformly bounded gradients from below and from above. In this work, we will use the inverse function theorem to show that there exists a unique solution of problem (1)-(6), close to a given equilibrium, with regularity $L^2(0, T; [H^2(I)]^3)$, and $L^2(0, T; [L^2(I)]^3)$ for its time derivative, where $I = (-h, 0)$, and $T > 0$. This result of existence will be used to prove the non-linear asymptotic stability of equilibrium solutions, for small enough data of the problem.

The paper is structured as follows: In Section 2, we determine the steady states of our model, by reproducing for completeness the proof given in [4]. Section 3 concerns the existence and uniqueness of unsteady solutions for the model considered, close to a given equilibrium state. Section 4 reports the analysis of the non-linear asymptotic stability for the equilibrium states. Finally, in Section 5 we address some conclusions on this work.

2. **Equilibrium states.**

We determine in this section the steady states of system (1)-(6). We shall assume that the total viscosity and diffusion ν_1, ν_2, defined either by (3), (4) or (5), are extended in some way by positive values to $R \leq 0$. Let us consider time independent coefficients V_x^e, V_y^e, Q^e and a time-independent vector $(u_b^e, v_b^e, \rho_b^e)^T$. We assume hereafter that $Q^e < 0$. This corresponds to a negative flux of energy at the surface, that from the physical point of view should lead to stable vertical configurations. We also assume the technical hypothesis that the depth h is smaller than the values $(V_x^e \rho_a)/(\rho_r D_1)$, $(V_y^e \rho_a)/(\rho_r D_2)$, which is reasonable if D_1, D_2 are small enough. The existence of smooth equilibria for this problem is stated in [4], although we reproduce it here for the reader's convenience:

Theorem 2.1. *For any $z \in I$, assume that the implicit algebraic equation:*

$$R = G(z)\frac{[f_1(R)]^2}{f_2(R)}, \tag{7}$$

where $G(z)$ is the function defined by:

$$G(z) = -\frac{g}{\rho_r}\,\frac{Q^e}{\left(D_1 z + \dfrac{V_x^e \rho_a}{\rho_r}\right)^2 + \left(D_2 z + \dfrac{V_y^e \rho_a}{\rho_r}\right)^2}, \tag{8}$$

admits at least a solution R^e. Then, to each solution R^e there exists a unique associated smooth equilibrium solution of problem (1), given by:

$$\begin{cases} u^e(z) = u_b^e + D_1\Psi_1(z) + \dfrac{V_x^e \rho_a}{\rho_r}\Psi_2(z), \\[2ex] v^e(z) = v_b^e + D_2\Psi_1(z) + \dfrac{V_y^e \rho_a}{\rho_r}\Psi_2(z), \\[2ex] \rho^e(z) = \rho_b^e + Q^e\Psi_3(z), \end{cases} \tag{9}$$

where:

$$\Psi_1(z) = \int_{-h}^{z}\frac{s}{f_1(R^e(s))}ds,\ \Psi_2(z) = \int_{-h}^{z}\frac{1}{f_1(R^e(s))}ds,\ \Psi_3(z) = \int_{-h}^{z}\frac{1}{f_2(R^e(s))}ds,$$

and $R^e = R(\partial_z \mathbf{U}^e)$, with $\partial_z \mathbf{U}^e = (\partial_z u^e, \partial_z v^e, \partial_z \rho^e)$.

Proof. The equilibrium states of (1), if these exist, are solutions of the system:

$$\begin{cases} \partial_z\left(\nu_1 \partial_z u^e\right) = D_1, \\ \partial_z\left(\nu_1 \partial_z v^e\right) = D_2, \\ \partial_z\left(\nu_2 \partial_z \rho^e\right) = 0. \end{cases} \tag{10}$$

Integrating the three equations in (10) with respect to z, we obtain:

$$\begin{cases} \partial_z u^e = \left(D_1 z + \dfrac{V_x^e \rho_a}{\rho_r}\right)/\nu_1, \\[2ex] \partial_z v^e = \left(D_2 z + \dfrac{V_y^e \rho_a}{\rho_r}\right)/\nu_1, \\[1ex] \partial_z \rho^e = Q^e/\nu_2. \end{cases} \tag{11}$$

As by hypothesis ν_1 and ν_2 are positive, then $R(\partial_z \mathbf{U}^e) > 0$, and, as consequence, $f_1(R(\partial_z \mathbf{U}^e))$ and $f_2(R(\partial_z \mathbf{U}^e))$ are well defined. Thus, (11) is re-written as:

$$\begin{cases} \partial_z u^e = \left(D_1 z + \dfrac{V_x^e \rho_a}{\rho_r} \right) / f_1\left(R(\partial_z \mathbf{U}^e)\right), \\ \partial_z v^e = \left(D_2 z + \dfrac{V_y^e \rho_a}{\rho_r} \right) / f_1\left(R(\partial_z \mathbf{U}^e)\right), \\ \partial_z \rho^e = Q^e / f_2\left(R(\partial_z \mathbf{U}^e)\right). \end{cases} \tag{12}$$

From (2), we deduce that the equilibrium profiles $R^e = R(\partial_z \mathbf{U}^e)$ satisfy the implicit algebraic equation (7). By integrating with respect to z the three equations in (11), we deduce that to each solution R^e of (7), there corresponds a unique smooth equilibrium solutions of problem (1) given by (9). □

Observe that $\partial_z u^e \neq 0$, $\partial_z v^e \neq 0$ because we assume the depth h to be small enough. Then the associated Richardson number $R^e = R(\partial_z \mathbf{U}^e)$ is well defined.

By the expressions given by (9), we obtain that the equilibrium solutions have regularity $[C^\infty(\bar{I})]^3$. In [1] it is proved that for any flux ratio:

$$r(z) = \frac{Q^e}{\left[\left(\dfrac{D_1 \rho_r}{\rho_a} z + V_x^e \right)^2 + \left(\dfrac{D_2 \rho_r}{\rho_a} z + V_y^e \right)^2 \right]}, \tag{13}$$

there exists a unique equilibrium gradient Richardson number R^e, for each of the models (3) to (5). For zero pressure gradients ($D_1 = D_2 = 0$), R^e does not depend on z, and, as consequence, the equilibrium profiles for velocity and density are linear. The equilibrium solutions correspond to an equilibrium between destabilizing shearing forces due to the surface stress induced by the wind, and stabilizing buoyancy forces induced by the negative surface thermodynamic flux.

3. **Existence of unsteady solutions.** In this section, we prove the existence and uniqueness of solutions for the initial-boundary value problem (1)-(6), close to a given equilibrium state, by using the inverse function theorem in Banach spaces (*cf.* [5]), and a basic result of Ladyzhenskaya et al. [11] on the solvability of initial-boundary value problems for generic linear parabolic systems. Hereafter, we will use the following convention on the notation: we denote with the dot $(\cdot)$ the usual product between a matrix and a vector, to distinguish it from the component to component product between two vectors, for which we do not use any symbol. The result of Ladyzhenskaya et al. ([11]) adapted to our situation reads as:

Theorem 3.1. *Consider an initial-boundary value problem for a linear parabolic system of the form:*

$$\begin{cases} \partial_t \mathbf{W} - \partial_z \left(M \cdot \partial_z \mathbf{W} \right) = \mathbf{\Psi} \text{ for } t \in (0,T) \text{ and } z \in I = (-h, 0), \\ M \cdot \partial_z \mathbf{W} = \mathbf{\Gamma}(t) \quad \text{at } z = 0, \\ \mathbf{W} = \mathbf{W}_b(t) \quad \text{at } z = -h, \\ \mathbf{W} = \mathbf{W}_0(z) \quad \text{at } t = 0, \end{cases} \tag{14}$$

where $\mathbf{W} = (w_1, w_2, w_3)^T$, *and* M *is a* 3×3 *matrix with time-independent coefficients belonging to* $H^1(I)$, *such that all its eigenvalues have positive real part, for any* $z \in I$. *Assume that* $\mathbf{\Psi} \in L^2(0,T;[L^2(I)]^3)$, $\mathbf{\Gamma}(t) \in [L^2(0,T)]^3$, $\mathbf{W}_b(t) \in [C^0(0,T)]^3$, *and* $\mathbf{W}_0(z) \in [H^1(I)]^3$. *Then, problem (14) has a unique solution*

$\mathbf{W} \in L^2(0,T;[H^2(I)]^3)$, *with* $\partial_t \mathbf{W} \in L^2(0,T;[L^2(I)]^3)$, *and the following estimate holds:*

$$||\mathbf{W}||_{L^2(0,T;[H^2(I)]^3)} + ||\partial_t \mathbf{W}||_{L^2(0,T;[L^2(I)]^3)}$$
$$\leq C(||\mathbf{\Psi}||_{L^2(0,T;[L^2(I)]^3)} + ||\mathbf{\Gamma}||_{L^2(0,T)} + ||\mathbf{W}_b||_{L^\infty(0,T)} + ||\mathbf{W}_0||_{H^1(I)}),$$

where C *is a positive constant depending only on* I, T *and the coefficients of* M.

Actually, we will prove the existence theorem in a rather abstract framework, where all the mixing layer models considered fit as particular cases. To do that, we consider an initial-boundary value problem for the unknown vector $\mathbf{U} = (u_1, u_2, u_3)^T$ of the form:

$$\begin{cases} \partial_t \mathbf{U} - \partial_z \left(\boldsymbol{\nu}(\partial_z \mathbf{U})\partial_z \mathbf{U}\right) + \mathbf{D} = \mathbf{0}, \text{ for } t \in (0,T) \text{ and } z \in I = (-h,0), \\ \boldsymbol{\nu}(\partial_z \mathbf{U})\partial_z \mathbf{U} = \mathbf{C}(t) \quad \text{at the surface } z = 0, \\ \mathbf{U} = \mathbf{U}_b(t) \quad \text{at the depth } z = -h, \\ \mathbf{U} = \mathbf{U}_0(z) \quad \text{at initial time } t = 0, \end{cases} \tag{15}$$

and we assume the following hypotheses:

Hypothesis 1. The vector function $\boldsymbol{\nu} \in [W^{3,\infty}_{loc}(\mathbb{R}^3)]^3$, and there exists a constant $\gamma > 0$ such that its components are greater than γ on $\mathbb{R}^3$.

Hypothesis 2. Problem (15) admits an equilibrium solution:

$$\mathbf{U}^e(z) = (u^e(z), v^e(z), \rho^e(z)),$$

with at least $[H^2(I)]^3$-regularity, and the linearization of problem (15) around the equilibrium is a parabolic system of the form (14), where $M = M(\mathbf{U}^e(z))$ is such that all its eigenvalues have positive real part, for any $z \in I$.

Problem (1)-(6) in vector form is a particular case of problem (15) by considering:

- $\mathbf{U} = (u, v, \rho)^T$, $\boldsymbol{\nu} = (\nu_1, \nu_1, \nu_2)^T$, $\mathbf{D} = (D_1, D_2, 0)^T$;
- $\mathbf{C}(t) = ((\rho_a/\rho_r)V_x(t), (\rho_a/\rho_r)V_y(t), Q(t))^T$;
- $\mathbf{U}_b(t) = (u_b(t), v_b(t), \rho_b(t))^T$;
- $\mathbf{U}_0(z) = (u_0(z), v_0(z), \rho_0(z))^T$.

Note that the eddy viscosities given by models (3) to (5) are not defined for $R = -1/\sigma$ for some $\sigma > 0$, and also for $(\partial_z u, \partial_z v) = (0,0)$, as in this case R is not defined. Moreover, they generate physical instabilities for $R < -1/\sigma$. This situation is solved in practice by the modelers of mixing layers by extending the eddy viscosities to these regions with positive constant values (*cf.* [8]). Here, we adapt this technique to verify Hypothesis 1. Indeed, let us introduce the new variables:

$$\alpha = \partial_z u, \quad \beta = \partial_z v, \quad \theta = \partial_z \rho,$$

and denote $\mathbf{Z} = (\alpha, \beta, \theta)$. The gradient Richardson number, in terms of these variables, is given by:

$$R = R(\mathbf{Z}) = -\frac{g}{\rho_r}\,\frac{\theta}{(\alpha^2 + \beta^2)},$$

and, as consequence, the turbulent viscosity and diffusion previously described by models (3) to (5) are functions of $\mathbf{Z}$. For instance, model (5) reads as:

$$f_1(\mathbf{Z}) = a_1 + \frac{b_1(\alpha^2+\beta^2)^2}{(\alpha^2+\beta^2-\sigma(g/\rho_r)\theta)^2}, \quad \sigma = 5,$$

$$f_2(\mathbf{Z}) = a_2 + \frac{f_1(\mathbf{Z})(\alpha^2+\beta^2)^2}{(\alpha^2+\beta^2-\sigma(g/\rho_r)\theta)^2}, \quad \sigma = 5,$$

and similarly for model (3) and (4).

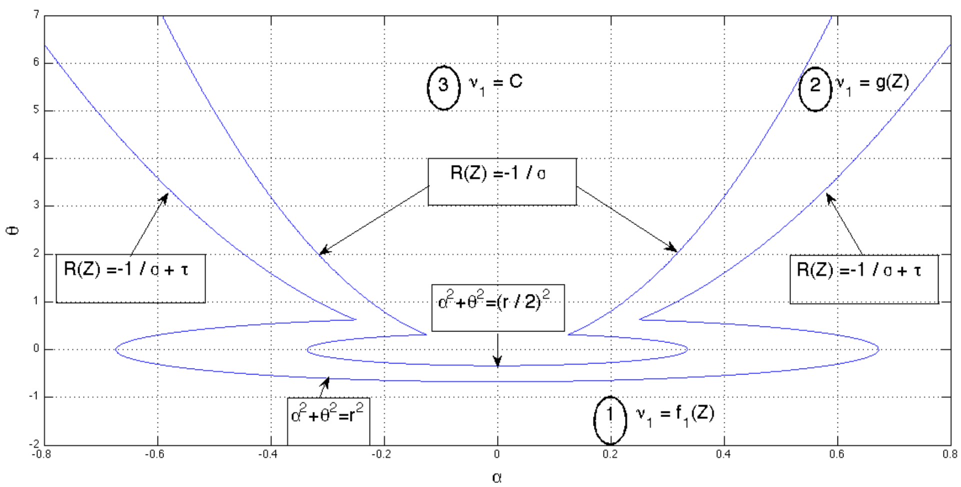

FIGURE 1. Example of a smooth extension of ν_1 in the plane (α, θ) (similarly for ν_2)

We split the $\mathbf{Z}$ space into three regions:

- A region containing the equilibria Richardson numbers, whose boundary is formed by the union of the surfaces $R(\mathbf{Z}) = -1/\sigma + \tau$ and $\alpha^2+\beta^2+\theta^2 = r^2$ for r, τ small enough (Region 1 in Figure 1, where for simplicity of presentation we have assumed $\beta = 0$).
- A region containing the physically unstable Richardson numbers, whose boundary is formed by the union of the surfaces $R(\mathbf{Z}) = -1/\sigma$ and $\alpha^2+\beta^2+\theta^2 = (r/2)^2$ (Region 3 in Figure 1).
- A buffer region whose boundary is the union of the boundaries of Regions 1 and 3 (Region 2 in Figure 1).

We then consider an extension of the total viscosity and diffusion ν_1, ν_2 such that:

(a) In Region 1: If $\mathbf{Z}$ is in Region 1, then:

$$\nu_1 = f_1(\mathbf{Z}), \quad \nu_2 = f_2(\mathbf{Z}).$$

(b) In Region 3: The functions ν_1, ν_2 take positive constant values, set by physical criteria.

(c) In Region 2: The functions ν_1, ν_2 in Region 2 are smooth extensions of their values in Regions 1 and 3, such that that $\nu_1, \nu_2 \in W^{3,\infty}_{loc}(\mathbb{R}^3)$ and there exist two positive constants γ_1, γ_2 verifying $0 < \gamma_1 \le \nu_1(\mathbf{Z}), \nu_2(\mathbf{Z}) \le \gamma_2$ for all $\mathbf{Z}$ in Region 2.

Observe that it is possible to obtain $\nu_1, \nu_2 \in W^{3,\infty}_{loc}(\mathbb{R}^3)$, because ν_1, ν_2 have C^∞-regularity in Regions 1 and 2. Also, typically ν_1, ν_2 are set to very large values in Region 2, forcing the flow to become suddenly stable. Usually, Region 2 does not explicitly appear in the computations, simply Region 1 is changed into Region 3 from a grid line to the next (*cf.* [8]).

By construction, problem (1)-(6) with smoothly extended ν_1, ν_2 verifies Hypothesis 1. In addition, it verifies Hypothesis 2. This proof is rather lengthy and we report it to Corollary 1. To prove the existence theorem for problem (15), let us define the Banach space:

$$\mathbf{X} = \left\{ L^2(0,T;[H^2(I)]^3) \text{ s.t. } \partial_t \in L^2(0,T;[L^2(I)]^3) \right\},$$

and let us consider the set $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{B}}(\mathbf{U}^e, \varepsilon) = \{\mathbf{U} \in \mathbf{X} : ||\mathbf{U} - \mathbf{U}^e||_{\mathbf{X}} < \varepsilon\}$, with ε small enough. The existence and uniqueness of solutions for problem (15) close to the equilibrium state is given by the following theorem:

Theorem 3.2. *Assume that Hypotheses 1 and 2 hold. Then, if $\mathbf{C}(t) \in [L^2(0,T)]^3$, $\mathbf{U}_b(t) \in [C^0(0,T)]^3$, $\mathbf{U}_0(z) \in [H^1(I)]^3$, and these quantities are close enough to the corresponding quantities at the equilibrium (respectively $\mathbf{C}^e$, $\mathbf{U}_b^e$ and $\mathbf{U}^e$), problem (15) admits a unique solution in an open neighborhood $\widehat{\boldsymbol{\mathcal{U}}} \subset \boldsymbol{\mathcal{U}}$, satisfying the estimate:*

$$||\mathbf{U}-\mathbf{U}^e||_{\mathbf{X}} \le C(||\mathbf{C}(t)-\mathbf{C}^e||_{L^2(0,T)}+||\mathbf{U}_b(t)-\mathbf{U}_b^e||_{L^\infty(0,T)}+||\mathbf{U}_0-\mathbf{U}^e||_{H^1(I)}), \quad (16)$$

where C is a positive constant independent of $\mathbf{U}$.

Proof. Let $\mathbf{Y}$ be the Banach space:

$$\mathbf{Y} = L^2(0,T;[L^2(I)]^3) \times [L^2(0,T)]^3 \times [C^0(0,T)]^3 \times [H^1(I)]^3,$$

and let us define the mapping:

$$\mathbf{\Phi} : \boldsymbol{\mathcal{U}} \longrightarrow \mathbf{Y},$$

$$\begin{aligned}\mathbf{\Phi}(\mathbf{U}) \quad = \quad & \{(\partial_t \mathbf{U} - \partial_z (\boldsymbol{\nu}(\partial_z \mathbf{U}) \partial_z \mathbf{U}) + \mathbf{D}), \left(\boldsymbol{\nu} \partial_z \mathbf{U}_{|_{z=0}} - \mathbf{C}^e\right), \\ & \left(\mathbf{U}_{|_{z=-h}} - \mathbf{U}_b^e\right), \left(\mathbf{U}_{|_{t=0}} - \mathbf{U}^e\right)\}.\end{aligned}$$

Observe that $\mathbf{\Phi}(\mathbf{U}^e) = \mathbf{0}$. If the hypotheses of the inverse function theorem are satisfied, we can conclude that there exists an open neighborhood $\widehat{\boldsymbol{\mathcal{U}}}$ of $\mathbf{U}^e$ in $\mathbf{X}$, $\widehat{\boldsymbol{\mathcal{U}}} \subset \boldsymbol{\mathcal{U}}$, and an open neighborhood $\boldsymbol{\mathcal{V}}$ of $\mathbf{\Phi}(\mathbf{U}^e)$ in $\mathbf{Y}$ such that $\mathbf{\Phi} : \widehat{\boldsymbol{\mathcal{U}}} \longrightarrow \boldsymbol{\mathcal{V}}$ is invertible, with continuously differentiable inverse. So, if $\mathbf{C}(t) \in [L^2(0,T)]^3$, $\mathbf{U}_b(t) \in [C^0(0,T)]^3$, $\mathbf{U}_0(z) \in [H^1(I)]^3$, and these quantities are close enough to the corresponding quantities at the equilibrium in an obvious sense, then there exists a unique solution of problem (15) in $\widehat{\boldsymbol{\mathcal{U}}}$. In addition, the inequality (16) follows because $D\mathbf{\Phi}$ is locally Lipschitz continuous (we prove it below, in **Step 1**), by using a result of Verfürth on a posteriori error estimates for nonlinear problems (*cf.* [18]). We next prove that effectively the hypotheses of the inverse function theorem are satisfied.

Step 1. $\mathbf{\Phi}$ is continuously Fréchet differentiable.

By definition of Fréchet derivative (*cf.* [5]), we have to prove that:

$$\lim_{||\mathbf{W}||_{\mathbf{X}} \to 0} \frac{||\mathbf{\Phi}(\mathbf{U}+\mathbf{W}) - \mathbf{\Phi}(\mathbf{U}) - \langle D\mathbf{\Phi}(\mathbf{U}), \mathbf{W}\rangle||_{\mathbf{Y}}}{||\mathbf{W}||_{\mathbf{X}}} = 0, \quad \forall \mathbf{U} \in \boldsymbol{\mathcal{U}}, \qquad (17)$$

where $\langle D\mathbf{\Phi}(\mathbf{U}), \mathbf{W}\rangle$ denotes the Gâteaux derivative of $\mathbf{\Phi}$ at $\mathbf{U}$. We have:

$$\begin{aligned}
&\langle D\mathbf{\Phi}(\mathbf{U}), \mathbf{W}\rangle = \frac{\mathrm{d}}{\mathrm{ds}}\mathbf{\Phi}(\mathbf{U}+\mathrm{s}\mathbf{W})_{|\mathrm{s}=0} \\
= \;& \{(\partial_t \mathbf{W} - \partial_z(\boldsymbol{\nu}(\partial_z\mathbf{U})\partial_z\mathbf{W} + \langle D\boldsymbol{\nu}(\partial_z\mathbf{U}), \mathbf{W}\rangle\partial_z\mathbf{U})), \\
& (\boldsymbol{\nu}\partial_z\mathbf{W} + \langle D\boldsymbol{\nu}(\partial_z\mathbf{U}), \mathbf{W}\rangle\partial_z\mathbf{U})_{|z=0}, (\mathbf{W})_{|z=-h}, (\mathbf{W})_{|t=0}\},
\end{aligned}$$

where:

$$\langle D\boldsymbol{\nu}(\partial_z\mathbf{U}), \mathbf{W}\rangle = \frac{\mathrm{d}}{\mathrm{ds}}\boldsymbol{\nu}(\partial_z\mathbf{U} + \mathrm{s}\,\partial_z\mathbf{W})_{|\mathrm{s}=0} = \nabla\boldsymbol{\nu}(\partial_z\mathbf{U})\cdot\partial_z\mathbf{W}.$$

Let us set $\mathbf{f}(\mathrm{t}) = \mathbf{\Phi}(\mathbf{U}+\mathrm{t}\mathbf{W})$. Observe that $\mathbf{f}(\mathrm{t})$ belongs to $[W^{2,\infty}_{loc}(\mathbb{R})]^3$ as $\boldsymbol{\nu}$ belongs to $[W^{3,\infty}_{loc}(\mathbb{R}^3)]^3$ from Hypothesis 1. So, the numerator appearing in expression (17) can be rewritten as:

$$||\mathbf{f}(1) - \mathbf{f}(0) - \mathbf{f}'(0)||_{\mathbf{Y}} = ||\mathbf{R}_1(1)||_{\mathbf{Y}},$$

where $\mathbf{R}_1(\mathrm{t}) = \int_0^t \mathbf{f}''(\mathrm{s})(\mathrm{t}-\mathrm{s})\mathrm{ds}$ is the integral form of the reminder in the Taylor's expansion formula up to the first order. We have:

$$\mathbf{f}''(\mathrm{t}) = \left\langle D^2\mathbf{\Phi}(\mathbf{U}+\mathrm{t}\mathbf{W}), (\mathbf{W},\mathbf{W})\right\rangle = \{\mathbf{A}(\mathrm{t}), \mathbf{B}(\mathrm{t}), \mathbf{0}, \mathbf{0}\},$$

where:

$$\mathbf{A}(\mathrm{t}) = -\partial_z[2(\nabla\boldsymbol{\nu}(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z\mathbf{W})\partial_z\mathbf{W} + \langle D^2\boldsymbol{\nu}(\partial_z\widetilde{\mathbf{U}}), (\mathbf{W},\mathbf{W})\rangle\partial_z\widetilde{\mathbf{U}}],$$

$$\mathbf{B}(\mathrm{t}) = [2(\nabla\boldsymbol{\nu}(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z\mathbf{W})\partial_z\mathbf{W} + \langle D^2\boldsymbol{\nu}(\partial_z\widetilde{\mathbf{U}}), (\mathbf{W},\mathbf{W})\rangle\partial_z\widetilde{\mathbf{U}}]_{|z=0},$$

with $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}(\mathrm{t}) = \mathbf{U} + \mathrm{t}\mathbf{W}$, and:

$$\begin{aligned}
\langle D^2\boldsymbol{\nu}(\partial_z\widetilde{\mathbf{U}}), (\mathbf{W},\mathbf{W})\rangle &= \frac{\mathrm{d}}{\mathrm{ds}}\nabla\boldsymbol{\nu}(\partial_z(\widetilde{\mathbf{U}}+\mathrm{s}\mathbf{W}))_{|\mathrm{s}=0}\cdot\partial_z\mathbf{W} \\
&= \begin{pmatrix} (H\nu_1(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z\mathbf{W})^T \\ (H\nu_1(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z\mathbf{W})^T \\ (H\nu_2(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z\mathbf{W})^T \end{pmatrix}\cdot\partial_z\mathbf{W},
\end{aligned}$$

H denoting the Hessian matrix. We have to verify that:

$$\lim_{||\mathbf{W}||_{\mathbf{X}}\to 0} \frac{\left\|\int_0^1 \mathbf{A}(\mathrm{s})(1-\mathrm{s})\mathrm{ds}\right\|_{L^2(0,T;[L^2(I)]^3)}}{||\mathbf{W}||_{\mathbf{X}}} = 0, \tag{18}$$

$$\lim_{||\mathbf{W}||_{\mathbf{X}}\to 0} \frac{\left\|\int_0^1 \mathbf{B}(\mathrm{s})(1-\mathrm{s})\mathrm{ds}\right\|_{[L^2(0,T)]^3}}{||\mathbf{W}||_{\mathbf{X}}} = 0. \tag{19}$$

From Hypothesis 1, the components of $\nabla\boldsymbol{\nu}$ and $H\boldsymbol{\nu}$ belong to $W^{1,\infty}_{loc}(\mathbb{R}^3)$. Then, the following estimates hold:

$$\begin{aligned} & \left\| \int_0^1 \mathbf{A}(\mathrm{s})(1-\mathrm{s})\mathrm{ds} \right\|_{L^2(0,T;[L^2(I)]^3)} \\ \leq \;& C\left(||\mathbf{W}||^2_{L^2(0,T;[H^2(I)]^3)} + ||\mathbf{W}||^3_{L^2(0,T;[H^2(I)]^3)} \right), \\ & \left\| \int_0^1 \mathbf{B}(\mathrm{s})(1-\mathrm{s})\mathrm{ds} \right\|_{[L^2(0,T)]^3} \\ \leq \;& C\left(||\mathbf{W}||^2_{L^2(0,T;[H^2(I)]^3)} + ||\mathbf{W}||^3_{L^2(0,T;[H^2(I)]^3)} \right), \end{aligned}$$

with C a positive constant. We conclude that (18) and (19) hold, and thus $\mathbf{\Phi}$ is Fréchet differentiable.

Next, we have to prove that the Fréchet derivative $D\mathbf{\Phi}$ is continuous, i.e. we have to show, for any $\mathbf{U} \in \boldsymbol{\mathcal{U}}$, that:

$$\lim_{||\mathbf{U}-\mathbf{V}||_{\mathbf{X}}\to 0} ||D\mathbf{\Phi}(\mathbf{U}) - D\mathbf{\Phi}(\mathbf{V})||_{\mathcal{L}(\mathbf{X},\mathbf{Y})} = 0,$$

where $\mathcal{L}(\mathbf{X},\mathbf{Y})$ is the space of bounded linear maps from $\mathbf{X}$ to $\mathbf{Y}$. It is easy to check that this is reduced to prove that:

$$\begin{aligned} \lim_{||\mathbf{U}-\mathbf{V}||_{\mathbf{X}}\to 0} \left(\sup_{\mathbf{W}\in\mathbf{X},\mathbf{W}\neq\mathbf{0}} \frac{||-\partial_z[\mathbf{E}(\partial_z\mathbf{U}) - \mathbf{E}(\partial_z\mathbf{V})]||_{L^2(0,T;[L^2(I)]^3)}}{||\mathbf{W}||_{\mathbf{X}}} \right) &= 0, \\ \lim_{||\mathbf{U}-\mathbf{V}||_{\mathbf{X}}\to 0} \left(\sup_{\mathbf{W}\in\mathbf{X},\mathbf{W}\neq\mathbf{0}} \frac{||[\mathbf{E}(\partial_z\mathbf{U}) - \mathbf{E}(\partial_z\mathbf{V})]_{|z=0}||_{[L^2(0,T)]^3}}{||\mathbf{W}||_{\mathbf{X}}} \right) &= 0, \end{aligned}$$

where:

$$\begin{aligned} \mathbf{E}(\partial_z\mathbf{U}) &= \boldsymbol{\nu}(\partial_z\mathbf{U})\partial_z\mathbf{W} + \langle D\boldsymbol{\nu}(\partial_z\mathbf{U}), \mathbf{W}\rangle\partial_z\mathbf{U} \\ &= \boldsymbol{\nu}(\partial_z\mathbf{U})\partial_z\mathbf{W} + (\nabla\boldsymbol{\nu}(\partial_z\mathbf{U})\cdot\partial_z\mathbf{W})\partial_z\mathbf{U}, \end{aligned}$$

and similarly for $\mathbf{E}(\partial_z\mathbf{V})$.

By adding and subtracting the quantity $(\nabla\boldsymbol{\nu}(\partial_z\mathbf{U})\cdot\partial_z\mathbf{W})\partial_z\mathbf{V}$ to $\mathbf{E}(\partial_z\mathbf{U}) - \mathbf{E}(\partial_z\mathbf{V})$, we obtain:

$$\begin{aligned} \mathbf{E}(\partial_z\mathbf{U}) - \mathbf{E}(\partial_z\mathbf{V}) &= [\boldsymbol{\nu}(\partial_z\mathbf{U}) - \boldsymbol{\nu}(\partial_z\mathbf{V})]\partial_z\mathbf{W} + [\nabla\boldsymbol{\nu}(\partial_z\mathbf{U})\cdot\partial_z\mathbf{W}]\partial_z(\mathbf{U}-\mathbf{V}) \\ &\quad + \{[\nabla\boldsymbol{\nu}(\partial_z\mathbf{U}) - \nabla\boldsymbol{\nu}(\partial_z\mathbf{V})]\cdot\partial_z\mathbf{W}\}\partial_z\mathbf{V}. \end{aligned}$$

Define:

$$\mathbf{g}(\mathrm{t}) = \boldsymbol{\nu}(\partial_z(\mathbf{V} + \mathrm{t}(\mathbf{U}-\mathbf{V}))), \quad \mathbf{H}(\mathrm{t}) = \nabla\boldsymbol{\nu}(\partial_z(\mathbf{V} + \mathrm{t}(\mathbf{U}-\mathbf{V}))).$$

Then:

$$\begin{aligned} \mathbf{E}(\partial_z\mathbf{U}) - \mathbf{E}(\partial_z\mathbf{V}) &= \left(\int_0^1 \mathbf{g}'(\mathrm{s})\mathrm{ds}\right)\partial_z\mathbf{W} + [\nabla\boldsymbol{\nu}(\partial_z\mathbf{U})\cdot\partial_z\mathbf{W}]\partial_z(\mathbf{U}-\mathbf{V}) \\ &\quad + \left(\left(\int_0^1 \mathbf{H}'(\mathrm{s})\mathrm{ds}\right)\cdot\partial_z\mathbf{W}\right)\partial_z\mathbf{V}, \end{aligned}$$

where:

$$\mathbf{g}'(\mathrm{t}) = \nabla\boldsymbol{\nu}(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z(\mathbf{U}-\mathbf{V}), \quad \mathbf{H}'(\mathrm{t}) = \begin{pmatrix} \left[H\nu_1(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z(\mathbf{U}-\mathbf{V})\right]^T \\ \left[H\nu_1(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z(\mathbf{U}-\mathbf{V})\right]^T \\ \left[H\nu_2(\partial_z\widetilde{\mathbf{U}})\cdot\partial_z(\mathbf{U}-\mathbf{V})\right]^T \end{pmatrix},$$

with $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}(\mathrm{t}) = \mathbf{V} + \mathrm{t}(\mathbf{U}-\mathbf{V})$, and H denoting the Hessian matrix. As before, from Hypothesis 1, we deduce the following estimates:

$$\begin{aligned} & ||-\partial_z[\mathbf{E}(\partial_z\mathbf{U}) - \mathbf{E}(\partial_z\mathbf{V})]||_{L^2(0,T;[L^2(I)]^3)} \\ \leq\ & C\left(||\mathbf{U}-\mathbf{V}||_{L^2(0,T;[H^2(I)]^3)}||\mathbf{W}||_{L^2(0,T;[H^2(I)]^3)}\right), \end{aligned}$$

$$\begin{aligned} & ||[\mathbf{E}(\partial_z\mathbf{U}) - \mathbf{E}(\partial_z\mathbf{V})]_{|z=0}||_{[L^2(0,T)]^3} \\ \leq\ & C\left(||\mathbf{U}-\mathbf{V}||_{L^2(0,T;[H^2(I)]^3)}||\mathbf{W}||_{L^2(0,T;[H^2(I)]^3)}\right), \end{aligned}$$

with C a positive constant. We conclude that $D\boldsymbol{\Phi}$ is continuous (really, $D\boldsymbol{\Phi}$ is locally Lipschitz continuous). Thus, we have proved that $\boldsymbol{\Phi}$ is continuously Fréchet differentiable.

Step 2. The mapping $D\boldsymbol{\Phi}(\mathbf{U}^e)$ is a Banach space isomorphism from $\mathbf{X}$ onto $\mathbf{Y}$.

We need to prove the well-posedness (in the sense of Hadamard) of the linear problem:

$$\begin{cases} \partial_t\mathbf{W} - \partial_z\left[\boldsymbol{\nu}(\partial_z\mathbf{U}^e)\partial_z\mathbf{W} + \partial_z\mathbf{U}^e\left(\nabla\boldsymbol{\nu}(\partial_z\mathbf{U}^e)\cdot\partial_z\mathbf{W}\right)\right] = \boldsymbol{\Psi}, \\ \boldsymbol{\nu}(\partial_z\mathbf{U}^e)\partial_z\mathbf{W} + \partial_z\mathbf{U}^e\left(\nabla\boldsymbol{\nu}(\partial_z\mathbf{U}^e)\cdot\partial_z\mathbf{W}\right) = \boldsymbol{\Gamma}(t) \ \text{ at } z=0, \\ \mathbf{W} = \mathbf{W}_b(t) \ \text{ at } z=-h, \\ \mathbf{W} = \mathbf{W}_0(z) \ \text{ at } t=0, \end{cases} \tag{20}$$

with data $\mathbf{G} = (\boldsymbol{\Psi},\boldsymbol{\Gamma},\mathbf{W}_b,\mathbf{W}_0) \in \mathbf{Y}$. I.e., we have to prove, for any $\mathbf{G}\in\mathbf{Y}$, that this problem admits a unique solution $\mathbf{W}$ that continuously depends on $\mathbf{G}$. Note that problem (20) can be rewritten as:

$$\begin{cases} \partial_t\mathbf{W} - \partial_z\left(M^e\cdot\partial_z\mathbf{W}\right) = \boldsymbol{\Psi} \text{ for } t\in(0,T) \text{ and } z\in I, \\ M^e\cdot\partial_z\mathbf{W} = \boldsymbol{\Gamma}(t) \ \text{ at } z=0, \\ \mathbf{W} = \mathbf{W}_b(t) \ \text{ at } z=-h, \\ \mathbf{W} = \mathbf{W}_0(z) \ \text{ at } t=0, \end{cases} \tag{21}$$

where $M^e = M(\mathbf{U}^e(z))$. By Hypotheses 1 and 2, problem (20) is a well-posed coupled linear parabolic problem with $H^1(I)$ time-independent coefficients. So that, from Theorem 3.1, as $\mathbf{G} = (\boldsymbol{\Psi},\boldsymbol{\Gamma},\mathbf{W}_b,\mathbf{W}_0)\in\mathbf{Y}$, then there exists a unique $\mathbf{W}\in\mathbf{X}$, solution of problem (20), that continuously depends on the data, with the estimate $||\mathbf{W}||_{\mathbf{X}} \leq C\,||\mathbf{G}||_{\mathbf{Y}}$. □

Corollary 1. *Assume that $(V_x(t),V_y(t),Q(t))^T\in[L^2(0,T)]^3$, $(u_b(t),v_b(t),\rho_b(t))^T\in[C^0(0,T)]^3$, $(u_0(z),v_0(z),\rho_0(z))^T\in[H^1(I)]^3$, and these quantities are close enough to the corresponding quantities at the equilibrium; then, problem (1)-(6) with smoothly extended viscosities ν_1,ν_2 admits a unique solution in an open neighborhood $\widehat{\boldsymbol{\mathcal{U}}}\subset\boldsymbol{\mathcal{U}}$, satisfying the estimate (16).*

Proof. We have to prove that problem (1)-(6) with the above extension of the viscosities ν_1, ν_2 satisfies Hypotheses 1 and 2. Then, the thesis will follow by Theorem 3.2, since problem (1)-(6) will be a particular case of problem (15). Problem (15) with smoothly extended ν_1, ν_2 verifies Hypothesis 1 by construction. Moreover, it admits the equilibrium solutions (9) with $C^\infty(\bar{I})$-regularity, as shown in Theorem 2.1. The linearization of problem (1)-(6) around them is given by a system of the form (21), where M^e is the matrix:

$$M^e = \begin{pmatrix} \nu_1^e + \alpha^e \left(\dfrac{\partial \nu_1}{\partial \alpha}\right)^e & \alpha^e \left(\dfrac{\partial \nu_1}{\partial \beta}\right)^e & \alpha^e \left(\dfrac{\partial \nu_1}{\partial \theta}\right)^e \\ \beta^e \left(\dfrac{\partial \nu_1}{\partial \alpha}\right)^e & \nu_1^e + \beta^e \left(\dfrac{\partial \nu_1}{\partial \beta}\right)^e & \beta^e \left(\dfrac{\partial \nu_1}{\partial \theta}\right)^e \\ \theta^e \left(\dfrac{\partial \nu_2}{\partial \alpha}\right)^e & \theta^e \left(\dfrac{\partial \nu_2}{\partial \beta}\right)^e & \nu_2^e + \theta^e \left(\dfrac{\partial \nu_2}{\partial \theta}\right)^e \end{pmatrix}, \tag{22}$$

and, for $k = 1, 2$:

$$\nu_k^e = \nu_k(\partial_z \mathbf{U}^e) = \nu_k(\alpha^e, \beta^e, \theta^e),$$

$$\left(\frac{\partial \nu_k}{\partial \alpha}\right)^e = \left(\frac{\partial \nu_k}{\partial \alpha}\right)_{|\mathbf{Z}=\partial_z \mathbf{U}^e}, \left(\frac{\partial \nu_k}{\partial \beta}\right)^e = \left(\frac{\partial \nu_k}{\partial \beta}\right)_{|\mathbf{Z}=\partial_z \mathbf{U}^e}, \left(\frac{\partial \nu_k}{\partial \theta}\right)^e = \left(\frac{\partial \nu_k}{\partial \theta}\right)_{|\mathbf{Z}=\partial_z \mathbf{U}^e}.$$

We have to prove that all the eigenvalues of M^e have positive real part, for any $z \in I$. To ensure that, it is enough that three independent invariants of matrix M^e are positive, for any $z \in I$. By means of a Computer Algebra System (CAS) it is verified that, since $R^e = R(\partial_z \mathbf{U}^e) > 0$, then there exist some positive constants δ_1, δ_2, δ_3 independent of z such that:

$$\text{Trace}(M^e) = 2 f_1(R^e) + f_2(R^e) + R^e(f_2'(R^e) - 2 f_1'(R^e)) > \delta_1,$$

$$\text{Det}(M^e) = f_1(R^e)(f_1(R^e) f_2(R^e) + R^e f_1(R^e) f_2'(R^e) - 2 R^e f_2(R^e) f_1'(R^e)) > \delta_2,$$

$$\begin{aligned} \text{Trace}(\text{Adj} M^e) &= 2 f_1(R^e) f_2(R^e) + 2 f_1(R^e) R^e [f_2'(R^e) - f_1'(R^e)] \\ &- 2 f_2(R^e) R^e f_1'(R^e) + f_1(R^e)^2 > \delta_3. \end{aligned}$$

So, problem (1)-(6) with smoothly extended viscosities ν_1, ν_2 verifies Hypothesis 2. □

This result proves that $\|\partial_z \mathbf{U} - \partial_z \mathbf{U}^e\|_{L^2(0,T;[L^\infty(I)]^3)}$ can be made arbitrarily small by choosing the data for problem (15) close enough to those corresponding to the equilibrium solution $\mathbf{U}^e$. If instead we had a similar estimate in $L^\infty(0, T; [L^\infty(I)]^3)$, then Theorem 3.2 and Corollary 1 would apply to the original model with eddy viscosities given by either (3), (4) or (5). Indeed, such an estimate would imply that, for data close enough to those corresponding to the equilibrium solution, $\partial_z \mathbf{U}(x, t)$ remains in the Region 1, for all $x \in I$ and for all $t \in [0, T]$, as $\partial_z \mathbf{U}^e(x)$ lies in the interior of Region 1, for all $x \in I$. Then, $\nu_1(\partial_z \mathbf{U}) = f_1(\partial_z \mathbf{U})$ and $\nu_2(\partial_z \mathbf{U}) = f_2(\partial_z \mathbf{U})$ in $I \times (0, T)$.

Unfortunately, the existence result of Theorem (3.1) does not apply to L^∞-regularity in time (*cf.* [11]). Thus, we only may conclude a weaker result, as follows:

Theorem 3.3. *Under Hypotheses 1 and 2, the solution to problem (1)-(6) with smoothly extended viscosities ν_1, ν_2 provided by Corollary 1 is also a solution of the same problem with the eddy viscosities given by either (3), (4) or (5) in a set $I \times ((0,T) \setminus \mathcal{A})$, where $\mathcal{A}$ is a set whose Lebesgue measure tends to zero as the distance between the data:*

$$||\mathbf{C}(t) - \mathbf{C}^e||_{L^2(0,T)} + ||\mathbf{U}_b(t) - \mathbf{U}_b^e||_{L^\infty(0,T)} + ||\mathbf{U}_0 - \mathbf{U}^e||_{H^1(I)},$$

tends to zero.

Proof. As $\partial_z \mathbf{U}^e(x)$ lies in Region 1 for all $x \in I$, there exists a $\delta > 0$ such that if $|\partial_z \mathbf{U}(x,t) - \partial_z \mathbf{U}^e(x)| \le \delta$, then $\partial_z \mathbf{U}(x,t)$ lies in Region 1. Define the set:

$$\mathcal{A} = \{t \in [0,T] \text{ such that } |\partial_z \mathbf{U}(x,t) - \partial_z \mathbf{U}^e(x)| > \delta, \text{ for some } x \in I\}.$$

Then:

$$\|\partial_z \mathbf{U} - \partial_z \mathbf{U}^e\|_{L^2(0,T;[L^\infty(I)]^3)} > \delta\, |\mathcal{A}|^{1/2}.$$

As $\|\partial_z \mathbf{U} - \partial_z \mathbf{U}^e\|_{L^2(0,T;[L^\infty(I)]^3)}$ can be arbitrarily small by making small the distance between the data, then $|\mathcal{A}|$ should tend to zero as this distance tends to zero. □

Note that this result applies to all formulas for the eddy diffusions given by either (3), (4) or (5), the only need is that the coefficients a_1, b_1 and a_2 are positive. The actual values of these coefficients will affect the Lebesgue measure of the set $\mathcal{A}$, but in all cases this measure will tend to zero as the distance between the data tends to zero.

4. **Non-linear stability of the continuous equilibria.** In this section, we prove the non-linear exponential asymptotic stability of the equilibrium states, for small data of the problem (1)-(6). To do that in a more general context, we assume that Hypotheses 1 and 2 previously defined in Section 3 hold.

Let us consider an initial perturbation of a given equilibrium solution of the form:

$$\mathbf{U}_0 = \mathbf{U}^e + \mathbf{U}'_0 \in [H^1(I)]^3.$$

We consider problem (15) with the same boundary data as $\mathbf{U}^e$ and initial condition $\mathbf{U}_0$. We assume that the initial perturbation $\mathbf{U}'_0$ is small enough in $[H^1(I)]^3$-norm, in order to guarantee that the initial condition $\mathbf{U}_0$ belongs to the neighborhood of the equilibrium that ensures the existence of $\mathbf{U}$, solution of problem (15), stated in Theorem 3.2.

Theorem 4.1. *Assume that Hypotheses 1 and 2 hold. Then, for small enough data $\mathbf{C}^e$ and $\mathbf{D}$, the equilibrium solution of problem (15) is non-linearly exponentially asymptotically stable, in the sense that:*

$$||\mathbf{U}'(t)||_{L^2(I)} \le \mathrm{e}^{-\lambda t}||\mathbf{U}'_0||_{L^2(I)},$$

for some $\lambda > 0$, where $\mathbf{U}' = \mathbf{U} - \mathbf{U}^e$.

Proof. In weak form, as the perturbation $\mathbf{U}'$ satisfies homogeneous boundary conditions, we have:

$$\int_{-h}^{0} (\partial_t \mathbf{U}) \cdot \mathbf{W} + \int_{-h}^{0} [\boldsymbol{\nu}(\partial_z \mathbf{U}) \partial_z \mathbf{U}] \cdot \partial_z \mathbf{W} = L(\mathbf{W}), \tag{23}$$

$$\int_{-h}^{0} (\partial_t \mathbf{U}^e) \cdot \mathbf{W} + \int_{-h}^{0} [\boldsymbol{\nu}(\partial_z \mathbf{U}^e) \partial_z \mathbf{U}^e] \cdot \partial_z \mathbf{W} = L(\mathbf{W}), \tag{24}$$

for all $\mathbf{W} \in [H^1(I)]^3$ such that $\mathbf{W}(-h) = \mathbf{0}$, where:

$$L(\mathbf{W}) = \mathbf{C}^e \cdot \mathbf{W}(0) - \int_{-h}^{0} \mathbf{D} \cdot \mathbf{W},$$

and the dot $(\cdot)$ denotes the Euclidean scalar product in $\mathbb{R}^3$. We take the difference between (23) and (24), and we add and subtract the quantity:

$$\int_{-h}^{0} [\boldsymbol{\nu}(\partial_z \mathbf{U}) \partial_z \mathbf{U}^e] \cdot \partial_z \mathbf{W}.$$

We obtain:

$$\begin{aligned} & \int_{-h}^{0} (\partial_t \mathbf{U}') \cdot \mathbf{W} + \int_{-h}^{0} [\boldsymbol{\nu}(\partial_z \mathbf{U}) \partial_z \mathbf{U}'] \cdot \partial_z \mathbf{W} \\ = & \int_{-h}^{0} \{ [\boldsymbol{\nu}(\partial_z(\mathbf{U}^e)) - \boldsymbol{\nu}(\partial_z \mathbf{U})] \, \partial_z \mathbf{U}^e \} \cdot \partial_z \mathbf{W}. \end{aligned} \tag{25}$$

Applying the same technique used in the proof of Theorem 3.2 on the right-hand side of equation (25), we have:

$$\begin{aligned} & \int_{-h}^{0} (\partial_t \mathbf{U}') \cdot \mathbf{W} + \int_{-h}^{0} [\boldsymbol{\nu}(\partial_z \mathbf{U}) \partial_z \mathbf{U}'] \cdot \partial_z \mathbf{W} \\ = & -\int_{-h}^{0} \left[\left(\int_0^1 \mathbf{g}'(\mathrm{s}) \mathrm{ds} \right) \, \partial_z \mathbf{U}^e \right] \cdot \partial_z \mathbf{W}, \end{aligned} \tag{26}$$

where $\mathbf{g}(\mathrm{t}) = \boldsymbol{\nu}(\partial_z(\mathbf{U}^e + \mathrm{t}\mathbf{U}'))$. Let us take $\mathbf{W} = \mathbf{U}'$. From (26), we deduce:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} ||\mathbf{U}'||^2_{L^2(I)} + \int_{-h}^{0} [\boldsymbol{\nu}(\partial_z \mathbf{U}) \, \partial_z \mathbf{U}'] \cdot \partial_z \mathbf{U}' \\ = & -\int_{-h}^{0} \left\{ \left[\int_0^1 \left(\nabla \boldsymbol{\nu}(\partial_z \widetilde{\mathbf{U}}) \, \partial_z \mathbf{U}' \right) \mathrm{ds} \right] \, \partial_z \mathbf{U}^e \right\} \cdot \partial_z \mathbf{U}', \end{aligned}$$

with $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}(\mathrm{s}) = \mathbf{U}^e + \mathrm{s}\mathbf{U}'$. From Hypothesis 1, we have:

$$\begin{aligned} & \frac{d}{dt} ||\mathbf{U}'||^2_{L^2(I)} + 2\gamma ||\partial_z \mathbf{U}'||^2_{L^2(I)} \\ \leq & -2 \int_{-h}^{0} \left\{ \left[\int_0^1 \left(\nabla \boldsymbol{\nu}(\partial_z \widetilde{\mathbf{U}}) \, \partial_z \mathbf{U}' \right) \mathrm{ds} \right] \, \partial_z \mathbf{U}^e \right\} \cdot \partial_z \mathbf{U}' \\ \leq & \frac{1}{\gamma} \left\| \left[\int_0^1 \left(\nabla \boldsymbol{\nu}(\partial_z \widetilde{\mathbf{U}}) \, \partial_z \mathbf{U}' \right) \mathrm{ds} \right] \, \partial_z \mathbf{U}^e \right\|^2_{L^2(I)} + \gamma ||\partial_z \mathbf{U}'||^2_{L^2(I)}, \end{aligned}$$

where we have used Young's inequality. By Hölder's inequality, we obtain:

$$\begin{aligned} & \frac{d}{dt} ||\mathbf{U}'||^2_{L^2(I)} + \gamma ||\partial_z \mathbf{U}'||^2_{L^2(I)} \\ \leq & \frac{1}{\gamma} \left\| \sup_{s \in [0,1], \, i,j=1,2,3} \left| \left[\nabla \boldsymbol{\nu}(\partial_z \widetilde{\mathbf{U}}) \right]_{ij} \right| \right\|^2_{L^\infty(I)} ||\partial_z \mathbf{U}'||^2_{L^2(I)} ||\partial_z \mathbf{U}^e||^2_{L^\infty(I)}. \end{aligned}$$

From Hypothesis 1, it follows that $||\partial_z \mathbf{U}^e||_{L^\infty(I)} \leq C/\gamma$, where C is a positive constant depending on the data, i.e. $C = C\left(||\mathbf{D}||_\infty h + ||\mathbf{C}^e||_\infty\right)$. Moreover, we have that:

$$\left\| \sup_{s \in [0,1], \, i,j=1,2,3} \left| \left[\nabla \boldsymbol{\nu}(\partial_z \widetilde{\mathbf{U}}) \right]_{ij} \right| \right\|_{L^\infty(I)} \leq \overline{C}.$$

So that, we deduce:

$$\frac{d}{dt}||\mathbf{U}'||^2_{L^2(I)} + \gamma||\partial_z \mathbf{U}'||^2_{L^2(I)} \le \Lambda||\partial_z \mathbf{U}'||^2_{L^2(I)},$$

where Λ is a positive constant depending on the data. Using Poincaré inequality, we have:

$$\frac{d}{dt}||\mathbf{U}'||^2_{L^2(I)} + \lambda||\mathbf{U}'||^2_{L^2(I)} \le 0,$$

where λ is a positive constant for small enough data. Finally, by Grönwall's lemma, we conclude:

$$||\mathbf{U}'(t)||^2_{L^2(I)} \le \mathrm{e}^{-\lambda t}\, ||\mathbf{U}'_0||^2_{L^2(I)} \overset{t\to+\infty}{\longrightarrow} 0.$$

□

Corollary 2. *Under the hypotheses of Corollary 1, the equilibrium solutions of problem (1)-(6) with smoothly extended viscosities ν_1, ν_2 are non-linearly exponentially asymptotically stable.*

Remark 1. This analysis implies a weak result on the asymptotic stability of the original problem (1)-(6) with the eddy viscosities given by either (3), (4) or (5). Indeed, from Theorem 3.3 we know that for each time interval $[0, T]$ and for each $\varepsilon > 0$ there exists a subset $\mathcal{A}_{\varepsilon,T} \subset [0, T]$ such that $\lim\limits_{\varepsilon\to 0} |\mathcal{A}_{\varepsilon,T}| = 0$, and if $||\mathbf{U}'_0||_{H^1(I)} < \varepsilon$, then $\mathbf{U}(t)$ is the solution of this original problem in $I\times((0, T)\backslash\mathcal{A}_{\varepsilon,T})$. From Theorem 4.1, this imples that:

$$||\mathbf{U}'(t)||^2_{L^2(I)} \le \mathrm{e}^{-\lambda t}\, ||\mathbf{U}'_0||^2_{L^2(I)}, \text{ if } t \notin \mathcal{A}_{\varepsilon,T}.$$

5. **Conclusions.** We have analyzed the existence of regular solutions around equilibria for oceanic turbulent mixing-layer models based on the gradient Richardson number, and we have studied the non-linear asymptotic stability of the equilibrium states. In general, it is not possible to ensure the existence of solutions for Richardson-number based turbulent models, due to the singularity presented by all the eddy coefficients defined by relations (3) to (5). In this paper, we have obtained the proof about the existence and uniqueness of solutions for the unsteady regularized version of these models around equilibria, based upon their smoothness. As consequence, it follows the asymptotic exponential stability of the equilibrium solutions, thanks to the dissipative nature of the equations of the problem. These results imply the existence of solutions of the original non-regularized models out from a small time set, as well as a weak asymptotic stability result.

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