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LECTURES ON THE REPRESENTATION TYPE OF A PROJECTIVE VARIETY

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Abstract. In these notes, we construct families of non-isomorphic Arithmetically Cohen Macaulay (ACM for short) sheaves (i.e., sheaves without intermediate cohomology) on a projective variety X. The study of such sheaves has a long and interesting history behind. Since the seminal result by Horrocks characterizing ACM sheaves on $\mathbb{P}^n$ as those that split into a sum of line bundles, an important amount of research has been devoted to the study of ACM sheaves on a given variety.

ACM sheaves also provide a criterium to determine the complexity of the underlying variety. This complexity is studied in terms of the dimension and number of families of undecomposable ACM sheaves that it supports, namely, its *representation type*. Varieties that admit only a finite number of undecomposable ACM sheaves (up to twist and isomorphism) are called of *finite representation type*. These varieties are completely classified: They are either three or less reduced points in $\mathbb{P}^2$, $\mathbb{P}^n_k$, a smooth hyperquadric $X \subset \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4_k$, the Veronese surface in $\mathbb{P}^5_k$ or a rational normal curve.

On the other extreme of complexity we find the varieties of *wild representation type*, namely, varieties for which there exist r-dimensional families of non-isomorphic undecomposable ACM sheaves for arbitrary large r. In the case of dimension one, it is known that curves of wild representation type are exactly those of genus larger or equal than two. In dimension greater or equal than two few examples are know and in these notes, we give a brief account of the known results.

Contents

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1. Introduction

These notes grew out of a series of lectures given by the author at the Vietnam Institute for Advanced Study in Mathematics (VIASM), Hanoi, during the period February 8 - March 7, 2014. In no case do I claim it is a survey on the representation type of a projective variety. Many people have made important contributions without even being mentioned here and I apologize to those whose work I made have failed to cite properly. The author gave 3 lectures of length 120 minutes each. She attempted to cover the basic facts on the representation type of a projective variety. Given the extensiveness of the subject, it was not possible to go into great detail in every proof. Still, it was hoped that the material that she chose will be beneficial and illuminating for the participants, and for the reader.

The projective space $\mathbb{P}^n$ holds a very remarkable property: the only undecomposable vector bundle $\mathcal{E}$ without intermediate cohomology (i.e., $\mathrm{H}^i(\mathbb{P}^n, \mathcal{E}(t)) = 0$ for $t \in \mathbb{Z}$ and $1 < i < n$), up to twist, is the structural line bundle $\mathcal{O}_{\mathbb{P}^n}$. This is the famous Horrocks' Theorem, proved in [Hor]. Ever since this result was stated, the study of the category of undecomposable arithmetically Cohen-Macaulay bundles (i.e., bundles without intermediate cohomology) supported on a given projective variety X has raised a lot of interest since it is a natural way to understand the complexity of the underlying variety X. Mimicking an analogous trichotomy in Representation Theory, in [DG] it was proposed a classification of ACM projective varieties as *finite, tame or wild* (see Definition 2.11) according to the complexity of their associated category of ACM vector bundles and it was proved that this trichotomy is exhaustive for the case of ACM curves: rational curves are finite, elliptic curves are tame and curves of higher genus are wild. Unfortunately very little is known for varieties of higher dimension and in this series of lectures I will give a brief account of known results.

The result due to Horrocks (cf. [Hor]) which asserts that, up to twist, $\mathcal{O}_{\mathbb{P}^n}$ is the only one undecomposable ACM bundle on $\mathbb{P}^n$ and the result due to Knörrer (cf. [Kn]) which states that on a smooth hyperquadric X the only undecomposable ACM bundles up to twist are $\mathcal{O}_X$ and the spinor bundles S match with the general philosophy that a "simple" variety should have associated a "simple" category of ACM bundles. Following these lines, a cornerstone result was the classification of ACM varieties of *finite representation type*, i.e., varieties that support (up to twist and isomorphism) only a finite number of undecomposable ACM bundles. It turned out that they fall into a very short list: $\mathbb{P}^n$, a smooth hyperquadric $Q \subset \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$, a rational normal curve and three or less reduced points in $\mathbb{P}^2$ (cf. [BGS, Theorem C] and [EH, p. 348]).

For the rest of ACM varieties, it became an interesting problem to give a criterium to split them into a finer classification, i.e. it is a challenging problem to find out the representation type of the remaining ones. So far only few examples of varieties of wild representation type are known: curves of genus $g \geq 2$ (cf. [DG]), del Pezzo surfaces and Fano blow-ups of points in $\mathbb{P}^n$ (cf. [MP], the cases of the cubic surface and the cubic threefold have also been handled in [CH]), ACM rational surfaces on $\mathbb{P}^4$ (cf. [MPLb]), any Segre variety unless the quadric surface in $\mathbb{P}^3$ (cf. [CMP, Theorem 4.6]) and non-singular rational normal scrolls $S(a_0, \cdots, a_k) \subseteq \mathbb{P}^N$, $N = \sum_{i=0}^{k}(a_i) + k$, (unless $\mathbb{P}^{k+1} = S(0, \cdots, 0, 1)$, the rational normal curve $S(a)$ in $\mathbb{P}^a$, the quadric surface $S(1,1)$ in $\mathbb{P}^3$ and the cubic scroll $S(1,2)$ in $\mathbb{P}^4$) (cf. [MR13, Theorem 3.8]).

Among ACM vector bundles $\mathcal{E}$ on a given variety X, it is interesting to spot a very important subclass for which its associated module $\oplus_t \mathrm{H}^0(X, \mathcal{E}(t))$ has the maximal number of generators, which turns out to be $\deg(X)\operatorname{rk}(\mathcal{E})$. This property was isolated by Ulrich in [Ulr], and ever since modules with this property have been called Ulrich modules and correspondingly Ulrich bundles in the geometric case (see [EFW] for more details on Ulrich bundles). The search of Ulrich sheaves on a particular variety is a challenging problem. In fact, few examples of varieties supporting Ulrich sheaves are known, although in [EFW] it has been conjectured that any variety supports an Ulrich sheaf. Moreover, the recent interest in the existence of Ulrich sheaves relies among other things on the fact that a d-dimensional variety $X \subset \mathbb{P}^n$ supports an Ulrich sheaf (bundle) if and only if the cone of cohomology tables of coherent sheaves (resp. vector bundles) on X coincides with the cone of cohomology tables of coherent sheaves (resp. vector bundles) on $\mathbb{P}^d$ ([ES]; Theorem 4.2). It is therefore a meaningful question to find out if a given projective variety X is of wild representation type with respect to the much more restrictive category of its undecomposable Ulrich vector bundles. We will prove that all smooth del Pezzo surfaces as well as all Segre varieties unless $\mathbb{P}^1 \times \mathbb{P}^1$ are of wild representation type and wildness is witnessed by Ulrich bundles.

Next we outline the structure of these notes. In section 2 we introduce the definitions and main properties that are going to be used throughout the paper; in particular, a brief account of ACM varieties, ACM vector bundles and Ulrich bundles on projective varieties is provided.

In section 3, we determine the representation type of any smooth del Pezzo surface S. To this end, we have to construct families of undecomposable ACM bundles of arbitrary high rank and dimension. Our construction will rely on the existence of level set of points on S and the existence of level set of points on S is related to Mustaţă's conjecture for a general set of points on a projective variety. Roughly speaking, Mustaţă's conjecture predicts the graded Betti number of a set Z of general points on a fixed projective variety X. In subsection 3.1, we will address this latter conjecture and we will prove that it holds for a general set of points Z on a smooth del Pezzo surface provided the cardinality of Z falls in certain strips explicitly described. In subsection 3.2, we perform the construction of large families of simple Ulrich vector bundles on del Pezzo surfaces obtained blowing up $s \leq 8$ points in $\mathbb{P}^2$. These families are constructed as the pullback of the kernel of certain surjective morphisms

$$\mathcal{O}_{\mathbb{P}^2}(1)^b \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2)^a$$

with chosen properties. It is worthwhile to point out that in the case of del Pezzo surfaces with very ample anticanonical divisor, we can show that these families of vector bundles could also be obtained through Serre's correspondence from a suitable general set of level points on the del Pezzo surface.

In section 4, we are going to focus our attention on the case of *Segre varieties* $\Sigma_{n_1,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i+1)-1$ for $1 \le n_1,\ldots,n_s$. It is a classical result that the quadric surface $\mathbb{P}^1\times\mathbb{P}^1 \subseteq \mathbb{P}^3$ only supports three undecomposable ACM vector bundles, up to shift: $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$, $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)$. For the rest of Segre varieties we construct large families of simple (and, hence, undecomposable) Ulrich vector bundles on them and this will allow us to conclude that they are of wild representation type. Up to our knowledge, they will be the first family of examples of varieties of arbitrary dimension for which wild representation type is witnessed by means of Ulrich vector bundles. In this section, we first introduce the definition and main properties of Segre varieties needed later. Then, we pay attention to the case of Segre varieties $\Sigma_{n,m} \subseteq \mathbb{P}^N$, $N := nm+n+m$, for $2 \le n,m$ and to the case of Segre varieties of the form $\Sigma_{n_1,n_2\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_1+1)-1$, for $2 \le n_1,\cdots,n_s$. We construct families of arbitrarily large dimension of simple Ulrich vector bundles on them by pulling-back certain vector bundles on each factor. This will allow us to conclude that they are of wild representation type. Finally, we move forward to the case of Segre varieties of the form $\Sigma_{n_1,n_2\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_1+1)-1$, for either $n_1 = 1$ and $s \ge 3$ or $n_1 = 1$, $s = 2$ and $n_2 \ge 2$. In this case the families of undecomposable Ulrich vector bundles of arbitrarily high rank will be obtained as iterated extensions of lower rank vector bundles.

In section 5, we could not resist to discuss some details that perhaps only the experts will care about, but hopefully will also introduce the non-expert reader to a subtle subject. We analyze how the representation type of a projective variety change when we change the polarization. Our main goal will be to prove that for any smooth ACM projective variety $X \subset \mathbb{P}^n$ there always exists a very ample line bundle $\mathcal{L}$ on X which naturally embeds X in $\mathbb{P}^{h^0(X,\mathcal{L})-1}$ as a variety of wild representation type.

Throughout the lectures I mentioned various open problems. Some of them and further related problems are collected in the last section of these notes.

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Notation. Throughout these notes K will be an algebraically closed field of characteristic zero, $R = K[x_0,x_1,\cdots,x_n]$, $\mathfrak{m} = (x_0,\ldots,x_n)$ and $\mathbb{P}^n = \mathrm{Proj}(R)$. Given a non-singular variety X equipped with an ample line bundle $\mathcal{O}_X(1)$, the line bundle $\mathcal{O}_X(1)^{\otimes l}$ will be denoted by $\mathcal{O}_X(l)$. For any coherent sheaf $\mathcal{E}$ on X we are going to denote the twisted sheaf $\mathcal{E}\otimes\mathcal{O}_X(l)$ by $\mathcal{E}(l)$. As usual, $\mathrm{H}^i(X,\mathcal{E})$ stands for the cohomology groups, $\mathrm{h}^i(X,\mathcal{E})$ for their dimension, $\mathrm{ext}^i(\mathcal{E},\mathcal{F})$ for the dimension of $\mathrm{Ext}^i(\mathcal{E},\mathcal{F})$ and $\mathrm{H}^i_*(X,\mathcal{E}) = \oplus_{l\in\mathbb{Z}}\mathrm{H}^i(X,\mathcal{E}(l))$ (or simply $\mathrm{H}^i_*\,\mathcal{E}$).

Given closed subschemes $X \subseteq \mathbb{P}^n$, we denote by R_X the homogeneous coordinate ring of X defined as $K[x_0, \dots, x_n]/I(X)$. As usual, the Hilbert function of X (resp. the Hilbert polynomial of X) will be denoted by $H_X(t)$ (resp. $P_X(t) \in \mathbb{Q}[t]$) and the regularity of X is defined to be the regularity of $I(X)$, i.e., $\operatorname{reg}(X) \leq m$ if and only if $\mathrm{H}^i(\mathbb{P}^n, I_X(m-i)) = 0$ for $i \geq 1$. Moreover, we know that $P_X(t) = H_X(t)$ for any $t \geq \operatorname{reg} X - 1 + \delta - n$ where δ is the projective dimension of R_X. Finally, $\Delta H_X(t)$ denotes the difference function, i.e., $\Delta H_X(t) = H_X 8t) - H_X(t-1)$.

2. Preliminaries

In this section, we set up some preliminary notions mainly concerning the definitions and basic results on ACM schemes $X \subset \mathbb{P}^n$ as well as on ACM sheaves and Ulrich sheaves $\mathcal{E}$ on X needed in the sequel.

Definition 2.1. A subscheme $X \subseteq \mathbb{P}^n$ is said to be arithmetically Cohen-Macaulay (briefly, ACM) if its homogeneous coordinate ring $R_X = R/I(X)$ is a Cohen-Macaulay ring, i.e. $\operatorname{depth}(R_X) = \dim(R_X)$.

Thanks to the graded version of the Auslander-Buchsbaum formula (for any finitely generated R-module M):

$$\operatorname{pd}(M) = n + 1 - \operatorname{depth}(M),$$

we deduce that a subscheme $X \subseteq \mathbb{P}^n$ is ACM if and only if $\operatorname{pd}(R_X) = \operatorname{codim} X$. Hence, if $X \subseteq \mathbb{P}^n$ is a codimension c ACM subscheme, a graded minimal free R-resolution of $I(X)$ is of the form:

$$0 \longrightarrow F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow R_X \longrightarrow 0 \tag{2.1}$$

with $F_0 = R$ and $F_i = \oplus_j R(-i-j)^{b_{ij}(X)}$, $1 \leq i \leq c$. The integers $b_{ij}(X)$ are called the *graded Betti numbers* of X and they are defined as

$$b_{ij}(X) = dim_k Tor^i(R/I(X), K)_{i+j}.$$

We construct the *Betti diagram* of X writing in the $(i,j)-th$ position the Betti number $b_{ij}(X)$. In this setting, minimal means that $\operatorname{im} \varphi_i \subset \mathfrak{m} F_{i-1}$. Therefore, the free resolution (2.1) is minimal if, after choosing basis of F_i, the matrices representing φ_i do not have any non-zero scalar.

Remark 2.2. For non ACM schemes $X \subseteq \mathbb{P}^n$ of codimension c the graded minimal free R-resolution of R_X is of the form:

$$0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow R_X \longrightarrow 0$$

with $F_0 = R$, $F_i = \oplus_{j=1}^{\beta_i} R(-n_j^i)$, $1 \leq i \leq p$, and $c < p \leq n$.

Notice that any zero-dimensional variety is ACM. For varieties of higher dimension we have the following characterization that will be used in this paper:

Lemma 2.3. *(cf.* [MR]*, pg. 23) If* $\dim X \geq 1$*, then* $X \subseteq \mathbb{P}^n_k$ *is ACM if and only if* $\mathrm{H}^i_*(I_X) := \oplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, I_X(t)) = 0$ *for* $1 \leq i \leq \dim X$.

Example 2.4. (i) Any complete intersection variety $X \subset \mathbb{P}^n$ is ACM.

(ii) The twisted cubic $X \subset \mathbb{P}^3$ is an ACM curve.

(iii) The rational quartic $C \subset \mathbb{P}^3$ is not ACM since $H^1(\mathbb{P}^3, I_C(1)) \neq 0$.
(iv) Segre Varieties are ACM varieties.
(v) Any standard determinantal variety $X \subset \mathbb{P}^n$ defined by the maximal minors of a homogeneous matrix is ACM.

Definition 2.5. If $X \subseteq \mathbb{P}^n$ is an ACM subscheme then, the rank of the last free R-module in a minimal free R-resolution of $I(X)$ is called the *Cohen-Macaulay type* of X.

Definition 2.6. A codimension c subscheme X of $\mathbb{P}^n$ is *arithmetically Gorenstein* (briefly AG) if its homogeneous coordinate ring R_X is a Gorenstein ring or, equivalently, its saturated homogeneous ideal, $I(X)$, has a minimal free graded R-resolution of the following type:

$$0 \longrightarrow R(-t) \longrightarrow \oplus_{i=1}^{\alpha_{c-1}} R(-n_{c-1,i}) \longrightarrow \ldots\ldots \longrightarrow \oplus_{i=1}^{\alpha_1} R(-n_{1,i}) \longrightarrow I(X) \longrightarrow 0.$$

In other words, an AG scheme is an ACM scheme with Cohen-Macaulay type 1.

Definition 2.7. Let $(X, \mathcal{O}_X(1))$ be a polarized variety. A coherent sheaf $\mathcal{E}$ on X is *Arithmetically Cohen Macaulay* (ACM for short) if it is locally Cohen-Macaulay (i.e., $\operatorname{depth} \mathcal{E}_x = \dim \mathcal{O}_{X,x}$ for every point $x \in X$) and has no intermediate cohomology:

$$\mathrm{H}^i_*(X, \mathcal{E}) = 0 \qquad \text{for all } i = 1, \ldots, \dim X - 1.$$

Notice that when X is a non-singular variety, which is going to be mainly our case, any coherent ACM sheaf on X is locally free. For this reason we are going to speak often of ACM bundles (since we identify locally free sheaves with their associated vector bundle). ACM sheaves are closely related to their algebraic counterpart, the maximal Cohen-Macaulay modules:

Definition 2.8. A graded R_X-module E is a *Maximal Cohen-Macaulay* module (MCM for short) if $\operatorname{depth} E = \dim E = \dim R_X$.

Indeed, it holds:

Proposition 2.9. *Let $X \subseteq \mathbb{P}^n$ be an ACM scheme. There exists a bijection between ACM sheaves $\mathcal{E}$ on X and MCM R_X-modules E given by the functors $E \to \widetilde{E}$ and $\mathcal{E} \to \mathrm{H}^0_*(X, \mathcal{E})$.*

The study of ACM bundles has a long and interesting history behind and it is well known that ACM sheaves provide a criterium to determine the complexity of the underlying variety. Indeed, this complexity can be studied in terms of the dimension and number of families of undecomposable ACM sheaves that it supports. Let us illustrate this general philosophy with a couple of examples (the simplest examples of varieties we can deal with have associated a simple category of undecomposable vector bundles).

Example 2.10. (1) Horrocks Theorem asserts that on $\mathbb{P}^n$ a vector bundle $\mathcal{E}$ is ACM if and only if it splits into a sum of line bundles. So, up to twist, there is only one undecomposable ACM bundle on $\mathbb{P}^n$: $\mathcal{O}_{\mathbb{P}^n}$ (cf. [Hor]).

(2) Knörrer's theorem states that on a smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$ any ACM vector bundle $\mathcal{E}$ splits into a sum of line bundles and spinor bundles. So, up to twist and dualizing, there are only two undecomposable ACM bundles on Q_{2n+1} ($\mathcal{O}_{Q_{2n+1}}$ and the

spinor bundle Σ); and three undecomposable ACM bundles on Q_{2n} ($\mathcal{O}_{Q_{2n}}$ and the spinor bundles Σ_- and Σ_+)(cf. [Kn]).

Recently, inspired by an analogous classification for quivers and for K-algebras of finite type, it has been proposed the classification of any ACM variety as being of *finite, tame or wild representation type* (cf. [DG] for the case of curves and [CH11] for the higher dimensional case). Let us recall the definitions:

Definition 2.11. Let $X \subseteq \mathbb{P}^N$ be an ACM scheme of dimension n.
(i) We say that X is of *finite representation type* if it has, up to twist and isomorphism, only a finite number of undecomposable ACM sheaves.
(ii) X is of *tame representation type* if either it has, up to twist and isomorphism, an infinite discrete set of undecomposable ACM sheaves or, for each rank r, the undecomposable ACM sheaves of rank r form a finite number of families of dimension at most n.
(iii) X is of *wild representation type* if there exist l-dimensional families of non-isomorphic undecomposable ACM sheaves for arbitrary large l.

One of the main achievements in this field has been the classification of varieties of finite representation type (cf. [BGS], Theorem C, and [EH], pg. 348)); it turns out that they fall into a very short list: three or less reduced points on $\mathbb{P}^2$, a projective space, a non-singular quadric hypersurface $X \subseteq \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$ or a rational normal curve. As examples of a variety of tame representation type we have the elliptic curves, the Segre product of a line and a smooth conic naturally embedded in $\mathbb{P}^5$: $\varphi_{|\mathcal{O}(2,2)|} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^8$ (cf. [FM]) and the quadric cone in $\mathbb{P}^3$ (cf. [CH04], Proposition 6.1). Finally, on the other extreme of complexity lie those varieties that have very large families of ACM sheaves. So far only few examples of varieties of wild representation type are known: curves of genus $g \geq 2$ (cf. [DG]), smooth del Pezzo surfaces (see §3 of these notes) and Fano blow-ups of points in $\mathbb{P}^n$ (cf.[MP], the cases of the cubic surface and the cubic threefold have also been handled in [CH]), ACM rational surfaces on $\mathbb{P}^4$ (cf. [MPLb]), Segre varieties other than the quadric in $\mathbb{P}^3$ (see §4 of these notes or [CMP], Theorem 4.6), rational normal scrolls other than $\mathbb{P}^n$, the rational normal curve in $\mathbb{P}^n$, the quadric in $\mathbb{P}^3$ and the cubic scroll in $\mathbb{P}^4$ ([MR13], Theorem 3.8) and hypersurfaces $X \subset \mathbb{P}^n$ of degree ≥ 4 ([To], Corollary 1).

The problem of classifying ACM varieties according to the complexity of the category of ACM sheaves that they support has recently attired much attention and, in particular, the following problem is still open (for ACM varieties of dimension ≥ 2):

Problem 2.12. Is the trichotomy finite representation type, tame representation type and wild representation type exhaustive?

Very often the ACM bundles that we will construct will share another stronger property, namely they have the maximal possible number of global sections; they will be the so-called Ulrich bundles. Let us end this section recalling the definition of Ulrich sheaves and summarizing the properties that they share and that will be needed in the sequel.

Definition 2.13. Given a polarized variety $(X, \mathcal{O}X(1))$, a coherent sheaf $\mathcal{E}$ on X is said to be *initialized* if

$$\mathrm{H}^0(X, \mathcal{E}(-1)) = 0 \quad \text{but} \quad \mathrm{H}^0(X, \mathcal{E}) \neq 0.$$

Notice that when $\mathcal{E}$ is a locally Cohen-Macaulay sheaf, there always exists an integer k such that $\mathcal{E}_{init} := \mathcal{E}(k)$ is initialized.

Definition 2.14. Given a projective scheme $X \subseteq \mathbb{P}^n$ and a coherent sheaf $\mathcal{E}$ on X, we say that $\mathcal{E}$ is an *Ulrich sheaf* if $\mathcal{E}$ is an ACM sheaf and $\mathrm{h}^0(\mathcal{E}_{init}) = \deg(X)\,\mathrm{rk}(\mathcal{E})$.

The following result justifies the above definition:

Theorem 2.15. *Let $X \subseteq \mathbb{P}^n$ be an integral ACM subscheme and let $\mathcal{E}$ be an ACM sheaf on X. Then the minimal number of generators $m(\mathcal{E})$ of the associated MCM R_X-module $\mathrm{H}^0_*(\mathcal{E})$ is bounded by*

$$m(\mathcal{E}) \leq \deg(X)\,\mathrm{rk}(\mathcal{E}).$$

Therefore, since it is obvious that for an initialized sheaf $\mathcal{E}$, $\mathrm{h}^0(\mathcal{E}) \leq m(\mathcal{E})$, the minimal number of generators of Ulrich sheaves is as large as possible. MCM Modules attaining this upper bound were studied by Ulrich in [Ulr]. A complete account is provided in [EFW]. In particular we have:

Theorem 2.16. *Let $X \subseteq \mathbb{P}^N$ be an n-dimensional ACM variety and let $\mathcal{E}$ be an initialized ACM coherent sheaf on X. The following conditions are equivalent:*

(i) *$\mathcal{E}$ is Ulrich.*
(ii) *$\mathcal{E}$ admits a linear $\mathcal{O}_{\mathbb{P}^N}$-resolution of the form:*

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-N+n)^{a_{N-n}} \to \cdots \to \mathcal{O}_{\mathbb{P}^N}(-1)^{a_1} \to \mathcal{O}_{\mathbb{P}^N}^{a_0} \to \mathcal{E} \to 0.$$

(iii) *$\mathrm{H}^i(\mathcal{E}(-i)) = 0$ for $i > 0$ and $\mathrm{H}^i(\mathcal{E}(-i-1)) = 0$ for $i < n$.*
(iv) *For some (resp. all) finite linear projections $\pi : X \to \mathbb{P}^n$, the sheaf $\pi_*\mathcal{E}$ is the trivial sheaf $\mathcal{O}_{\mathbb{P}^n}^t$ for some t.*

In particular, initialized Ulrich sheaves are 0-regular and therefore they are globally generated.

Proof. See [EFW], Proposition 2.1. □

The search of Ulrich sheaves on a particular variety is a challenging problem. In fact, few examples of varieties supporting Ulrich sheaves are known, although in [EFW] has been conjectured that any variety has an Ulrich sheaf. Indeed, in [EFW], pg. 543, Eisenbud, Schreyer and Weyman leave open the following problem

Problem 2.17. (a) Is every variety (or even scheme) $X \subset \mathbb{P}^n$ the support of an Ulrich sheaf?

(b) If so, what is the smallest possible rank for such a sheaf?

Recently, after the Boij-Söderberg theory has been developed, the interest on these questions have grown up due to the fact that it has been proved ([ES2], Theorem 4.2) that the existence of an Ulrich sheaf on a smooth projective variety X of dimension n

implies that the cone of cohomology tables of vector bundles on X coincide with the cone of cohomology tables of vector bundles on $\mathbb{P}^n$.

In these series of lectures we are going to focus our attention on the existence of Ulrich bundles on smooth del Pezzo surfaces and on Segre varieties, providing the first example of wild varieties of arbitrary dimension whose wildness is witnessed by means of the existence of families of simple Ulrich vector bundles of arbitrary high rank and dimension.

3. The representation type of a del Pezzo surface

In this section, we are going to construct ACM bundles and Ulrich bundles on smooth del Pezzo surfaces, and to determine their representation type. So, let us start recalling the definition and main properties of del Pezzo surfaces.

Definition 3.1. A *del Pezzo* surface is defined to be a smooth surface X whose anti-canonical divisor $-K_X$ is ample. Its degree is defined as K_X^2. If $-K_X$ is very ample, X will be called a *strong del Pezzo surface.*

Example 3.2. AS examples of del Pezzo surfaces we have:

1. A smooth cubic surface $X \subseteq \mathbb{P}^3$.
2. A smooth quartic surface $X \subset \mathbb{P}^4$ complete intersection of two quadrics.
3. Let Y be the blow up of $\mathbb{P}^2$ at $0 \le s \le 6$ general points. Consider its embedding in $\mathbb{P}^{9-s}$ through the very ample divisor $-K_Y$ and call $X \subset \mathbb{P}^{9-s}$ its image. X is a del Pezzo surface.

The classification of del Pezzo surfaces is known and we recall it for seek of completeness.

Definition 3.3. A set of s different points $\{p_1, \dots, p_s\}$ on $\mathbb{P}^2_k$ with $s \le 8$ is in *general position* if no three of them lie on a line, no six of them lie on a conic and no eight of them lie on a cubic with a singularity at one of these points.

Theorem 3.4. *Let X be a del Pezzo surface of degree d. Then $1 \le d \le 9$ and*

(i) *If $d = 9$, then X is isomorphic to $\mathbb{P}^2$ (and $-K_{\mathbb{P}^2} = 3H_{\mathbb{P}^2}$ gives the usual Veronese embedding in $\mathbb{P}^9$).*
(ii) *If $d = 8$, then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to a blow-up of $\mathbb{P}^2$ at one point.*
(iii) *If $7 \ge d \ge 1$, then X is isomorphic to a blow-up of $\mathbb{P}^2$ at $9 - d$ closed points in general position.*

Conversely, any surface described under $(i), (ii), (iii)$ is a del Pezzo surface of the corresponding degree.

Proof. See, for instance, [Man], Chapter IV, Theorems 24.3 and 24.4, and [Dol], Proposition 8.1.9. □

Lemma 3.5. *Let X be the blow-up of $\mathbb{P}^2$ on $0 \le s \le 8$ points in general position. Let $e_0 \in Pic(X)$ be the pull-back of a line in $\mathbb{P}^2$, e_i the exceptional divisors, $i = 1, \dots, s$ and K_X be the canonical divisor. Then:*

(i) *If $s \le 6$, $-K_X = 3e_0 - \sum_{i=1}^s e_i$ is very ample and its global sections yield a closed embedding of X in a projective space of dimension*

$$\dim \mathrm{H}^0(X, \mathcal{O}_X(-K_X)) - 1 = K_X^2 = 9 - s.$$

(ii) *If* $s = 7$, $-K_X$ *is ample and generated by its global sections.*
(iii) *if* $s = 8$, $-K_X$ *is ample and* $-2K_X$ *is generated by its global sections.*

Proof. See, for instance, [Kol], Proposition 3.4. □

The construction of ACM bundles and Ulrich bundles on smooth del Pezzo surfaces is closely related (via Serre's correspondence) to the existence of level set of points.

Definition 3.6. A 0-dimensional scheme Z on a surface $X \subset \mathbb{P}^n$ is said to be *level of type* ρ if the last graded free module in its minimal graded free resolution has rank ρ and is concentrated in only one degree. Dualizing, this is equivalent to say that all minimal generators of the canonical module K_Z of Z have the same degree.

Example 3.7. Let Z be a set of 29 general points on a smooth quadric surface $Q \subset \mathbb{P}^3$. The ideal $I(Z)$ of Z has a minimal graded free resolution of the following type:

$$0 \longrightarrow R(-8)^4 \longrightarrow R(-7)^3 \oplus R(-6)^8 \longrightarrow R(-5)^7 \oplus R(-2) \longrightarrow I(Z) \longrightarrow 0.$$

Therefore, Z is level of type 4.

The existence of level set of points on a smooth del Pezzo surface is related to Mustaţă's conjecture which we will discuss in next subsection and its proof will strongly rely on the fact that we know the minimal resolution of the coordinate ring of a del Pezzo surface $X \subset \mathbb{P}^d$. Indeed, according to [H], Theorem 1, the minimal free resolution of the coordinate ring of a del Pezzo surface $X \subseteq \mathbb{P}^d$ has the form:

$$(3.1) \qquad 0 \longrightarrow R(-d) \longrightarrow R(-d+2)^{\alpha_{d-3}} \longrightarrow \dots \longrightarrow R(-2)^{\alpha_1} \longrightarrow R \longrightarrow R_X \longrightarrow 0$$

where

$$\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1} \text{ for } 1 \le i \le d-3.$$

Notice that X turns out to be AG and, in particular, $\alpha_i = \alpha_{d-2-i}$ for all $i = 1, \dots, d-2$. The Hilbert polynomial and the regularity of a del Pezzo surface X can be easily computed using the exact sequence (3.1) and we have

$$P_X(r) = \frac{d}{2}(r^2 + r) + 1 \text{ and } \operatorname{reg}(X) = 3.$$

3.1. **Mustaţă's conjecture for a set of general points on a del Pezzo surface.**

In [Mus], Mustaţă predicted the minimal free resolution of a general set of points Z in an arbitrary projective variety X; he proved that the first rows of the Betti diagram of Z coincide with the Betti diagram of X and that there are two extra nontrivial rows at the bottom. Let us recall it.

Theorem 3.8. *Let* $X \subseteq \mathbb{P}^n$ *be a projective variety with* $d = \dim(X) \ge 1$, $\operatorname{reg}(X) = m$ *and with Hilbert polynomial* P_X. *Let* s *be an integer with* $P_X(r-1) \le s < P_X(r)$ *for some* $r \ge m+1$ *and let* Z *be a set of* s *general points on* X. *Let*

$$0 \to F_n \to F_{n-1} \to \dots \to F_2 \to F_1 \to R \to R_X \to 0$$

be a minimal graded free R*-resolution of* R_X. *Then* R_Z *has a minimal free* R*-resolution of the following type*

$$0 \to F_n \oplus R(-r-n+1)^{b_{n,r-1}} \oplus R(-r-n)^{b_{n,r}} \to \cdots \to F_2 \oplus R(-r-1)^{b_{2,r-1}} \oplus R(-r-2)^{b_{2,r}}$$
$$\to F_1 \oplus R(-r)^{b_{1,r-1}} \oplus R(-r-1)^{b_{1,r}} \to R \to R_Z \to 0.$$

Moreover, if we set $Q_{i,r}(s) = b_{i+1,r-1}(Z) - b_{i,r}(Z)$,

$$Q_{i,r}(s) = \sum_{l=0}^{d-1} (-1)^l \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{n}{i}(s - P_X(r-1)).$$

Conjecture 3.9. The *minimal resolution conjecture* (MRC for short) says that

$$b_{i+1,r-1} \cdot b_{i,r} = 0 \text{ for } i = 1, \cdots, n-1.$$

Example 3.10. Let $S \subset \mathbb{P}^4$ be a smooth del Pezzo surface of degree 4. S is the complete intersection of 2 hyperquadrics in $\mathbb{P}^4$, $reg(S) = 3$ and $P_S(x) = 2x^2 + 2x + 1$. Let $Z \subset S$ be a set of 45 general points on S. We have $P_S(4) = 41 \leq 45 \leq P_S(5) = 61$.

The Betti diagram of Z looks like:

	0	1	2	3	4
0	1	-	-	-	-
1	-	2	-	-	-
2	-	-	1	-	-
3	-	-	-	-	-
4	-	16	40	28	-
5	-	-	-	-	4

The first 3 rows of the Betti diagram of Z coincide with the Betti diagram of S and there are two extra nontrivial rows without ghost terms.

Related to it there exist two weaker conjectures that deal only with a part of the minimal resolution of a general set of points:

- The *Ideal Generation Conjecture* (IGC for short) which says that the minimal number of generators of the ideal of a general set of points will be as small as possible; this conjecture can be translated in terms of the Betti numbers saying that
$$b_{1,r} b_{2,r-1} = 0.$$
- On the other extreme of the resolution the *Cohen-Macaulay type Conjecture* (CMC for short) controls the ending terms of the MFR and says that the canonical module $\mathrm{Ext}^n_R(R/I(Z), R(-n-1))$ has as few generators as possible, i.e,
$$b_{n-1,r} b_{n,r-1} = 0.$$

Remark 3.11. (1) When $X = \mathbb{P}^n$ the above conjecture coincides with the MRC for points in $\mathbb{P}^n$ stated in [Lor] which says that this resolution has no *ghost* terms, i.e, $b_{i+1,r-1} b_{i,r} = 0$ for all i. The MRC for points in $\mathbb{P}^n$ is known to hold for $n \leq 4$ (see [Gae], [BaG] and [Wal]) and for large values of s for any n (see [HiS]) but it is false in general. Eisenbud, Popescu, Schreyer and Walter showed that it fails for any $n \geq 6$, $n \neq 9$ (see [EPS]).

(2) Regarding Mustaţă conjecture, in [GMR] Giuffrida, Maggioni and Ragusa proved that it holds for any general set of points when X is a smooth quadric surface in $\mathbb{P}^3$. In [MP2], Proposition 3.10, the authors showed that it holds for any general set of $s \geq 19$ points on a smooth cubic surface in $\mathbb{P}^3$ and, in [MiP], Migliore and Patnott have been able to prove it for sets of general distinct points of any cardinality on a cubic surface $X \subseteq \mathbb{P}^3$ given that X is smooth or it has at most isolated double points.

The goal of this subsection is to prove MRC for a set Z of general points on a smooth del Pezzo surface X, when the cardinality $|Z|$ of Z falls in certain interval explicitly described later. As corollary we prove IGC and CMC for a set Z of general points on a del Pezzo surface X provided $|Z| \geq P_X(3)$.

As a main tool we use the theory of liaison. Roughly speaking, Liaison Theory is an equivalence relation among schemes of the same dimension and it involves the study of the properties shared by two schemes X_1 and X_2 whose union $X_1 \cup X_2 = X$ is either a complete intersection (CI-liaison) or an arithmetically Gorenstein scheme (G-liaison). Knowing that two sets of points are G-linked, this technique will allow us to pass from the minimal resolution of the ideal of one of them to the resolution of the other one (mapping cone process) and vice versa.

Definition 3.12. Two subschemes X_1 and X_2 of $\mathbb{P}^n$ are *directly Gorenstein linked* (*directly G-linked* for short) by an AG scheme $G \subseteq \mathbb{P}^n$ if $I(G) \subseteq I(X_1) \cap I(X_2)$, $[I(G) : I(X_1)] = I(X_2)$ and $[I(G) : I(X_2)] = I(X_1)$. We say that X_2 is *residual* to X_1 in G. When G is a complete intersection we talk about a CI-link.

When X_1 and X_2 do not share any component, being directly G-linked by an AG scheme G is equivalent to $G = X_1 \cup X_2$.

Definition 3.13. two subschemes $X_1, X_2 \subset \mathbb{P}^n$ are in the same CI-liaison class (resp. G-liaison class) if there exists $X_1 = Z_0, Z_1, ..., Z_t = X_2$ closed subschemes in $\mathbb{P}^n$ such that Z_i and Z_{i+1} are directly linked by a complete intersection (arithmetically Gorenstein) $X_i \subset \mathbb{P}^n$.

See [KMMNP] for more details on G-liaison.

Usually it is not easy to find out AG schemes to work with. The following theorem gives a useful way to construct them.

Theorem 3.14. *Let $S \subseteq \mathbb{P}^n$ be an ACM scheme satisfying condition G_1. Denote by K_S the canonical divisor and by H_S a general hyperplane section of S. Then any effective divisor in the linear system $|mH_S - K_S|$ is AG.*

Proof. See [KMMNP], Lemma 5.4. □

The main feature of G-liaison that is going to be exploited in this paper is that through the mapping cone process it is possible to pass from the free resolution of a scheme X_1 to the free resolution of its residual X_2 on an AG scheme. We have

Lemma 3.15. *Let $V_1, V_2 \subseteq \mathbb{P}^n$ be two ACM schemes of codimension c directly G-linked by an AG scheme W. Let the minimal free resolutions of $I(V_1)$ and $I(W)$ be*

$$0 \longrightarrow F_c \xrightarrow{d_c} F_{c-1} \xrightarrow{d_{c-1}} \dots F_1 \xrightarrow{d_1} I(V_1) \longrightarrow 0$$

and

$$0 \longrightarrow R(-t) \xrightarrow{e_c} G_{c-1} \xrightarrow{e_{c-1}} \dots G_1 \xrightarrow{e_1} I(W) \longrightarrow 0,$$

respectively. Then the functor $\mathrm{Hom}(-, R(-t))$ *applied to a free resolution of* $I(V_1)/I(W)$ *gives a (non necessarily minimal) resolution of* $I(V_2)$*:*

$$0 \longrightarrow F_1^\vee(-t) \longrightarrow F_2^\vee(-t) \oplus G_1^\vee(-t) \longrightarrow \dots \longrightarrow F_c^\vee(-t) \oplus G_{c-1}^\vee(-t) \longrightarrow I(V_2) \longrightarrow 0.$$

In order to achieve the main result of this subsection, we define the critical values:

$$m(r) := \frac{d}{2}r^2 + r\frac{2-d}{2}, \quad n(r) := \frac{d}{2}r^2 + r\frac{d-2}{2}.$$

Notice that

$$P_X(r-1) < m(r) < n(r) < P_X(r).$$

Our first aim is to find out the minimal graded free resolution and to prove MRC conjecture for these two specific cardinalities $m(r)$ and $n(r)$ of general set of points on a del Pezzo surface X. Since the structure of our proof requires that X contains at least a line L and moreover that the elements of the linear system $|L + rH|$ satisfy condition G_1 in order to apply the theory of generalized divisors, we need to exclude the following two particular cases: $X \cong \mathbb{P}^2$ and $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ proved in [P], Chapter II. Therefore, in this subsection $X \subseteq \mathbb{P}^d$ will stand for any del Pezzo surface except the two aforementioned sporadic cases. We also set the following notation.

(i) L is any line on X.
(ii) H denotes a general hyperplane section of X.
(iii) If C is a curve on X, H_C will be a general hyperplane section of C and K_C the canonical divisor on C.

The strategy of the proof is as follows: firstly, we will establish the result for $m(2) = d+2$ points which gives the starting point for our induction process. Secondly, using G-liaison, we prove that if $m(r)$ general points on any del Pezzo surface satisfy MRC then so do $n(r)$ general points. Next we observe that if $n(r)$ general points on X have the expected minimal free resolution then $n(r)+1$ general points do as well. And, finally, we show that if $n(r) + 1$ general points on a del Pezzo surface satisfy MRC then so do $m(r + 1)$.

Since the shape of the minimal free resolution of the homogeneous ideal $I(X)$ of a del Pezzo surface of degree 3 (i.e, a cubic surface) is slightly different from that of a del Pezzo surface of degree $d \geq 4$ we need to consider apart the two cases. We only sketch the proofs in the case of degree $d \geq 4$ and we leave as exercise the case of degree 3.

Lemma 3.16. *(a) Let* $X \subseteq \mathbb{P}^d$ *be any del Pezzo surface of degree* $d \geq 4$ *and take* $C \in |(r+\epsilon)H|$, $r \geq 2, \epsilon \in \{0, 1\}$*. Then, any effective divisor* G *in the linear system* $|rH_C|$ *is AG and it has a minimal free resolution of the following form:*

$$0 \longrightarrow R(-2r-d-\epsilon) \longrightarrow R(-2r-d+2-\epsilon)^{\alpha_{d-3}} \oplus R(-r-d)^{2-\epsilon} \oplus R(-r-d-1)^{\epsilon} \longrightarrow \dots \longrightarrow M_i \longrightarrow$$

$$\longrightarrow R(-2r-\epsilon) \oplus R(-r-2)^{(2-\epsilon)\alpha_1} \oplus R(-r-3)^{\epsilon\alpha_1} \longrightarrow M_1 := R(-r)^{2-\epsilon} \oplus R(-r-1)^{\epsilon} \longrightarrow I(G|X) \longrightarrow 0$$

where $M_i := R(-2r-i+1-\epsilon)^{\alpha_{i-2}} \oplus R(-r-i)^{(2-\epsilon)\alpha_{i-1}} \oplus R(-r-i-1)^{\epsilon\alpha_{i-1}}$ *for* $i = 3, \dots, d-2$ *and* $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ *for* $1 \leq i \leq d-3$.

(b) Let $X \subseteq \mathbb{P}^3$ be a del Pezzo surface of degree 3 and take $C \in |(r+\epsilon)H|$, $r \geq 2, \epsilon \in \{0,1\}$. Then, any effective divisor G in the linear system $|rH_C|$ is AG and it has a minimal free resolution of the following form:

$$0 \longrightarrow R(-2r-3-\epsilon) \longrightarrow R(-2r-\epsilon) \oplus R(-r-3)^{2-\epsilon} \oplus R(-r-4)^{\epsilon}$$

$$\longrightarrow R(-r)^{2-\epsilon} \oplus R(-r-1)^{\epsilon} \longrightarrow I(G|X) \longrightarrow 0.$$

Proof. A curve $C \in |(r+\epsilon)H|$ has saturated ideal $I(C|X) = \mathrm{H}^0_*(\mathcal{O}_X(-r-\epsilon))$. From the exact sequence (3.1) we have:

$$0 \to \mathcal{O}_{\mathbb{P}^d}(-d) \to \mathcal{O}_{\mathbb{P}^d}(-d+2)^{\alpha_{d-3}} \to \cdots \to \mathcal{O}_{\mathbb{P}^d}(-2)^{\alpha_1} \to \mathcal{O}_{\mathbb{P}^d} \to \mathcal{O}_X \to 0 \tag{3.2}$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \leq i \leq d-3$. Twisting (3.2) with $\mathcal{O}_{\mathbb{P}^d}(-r-\epsilon)$ and taking global sections we get the minimal graded free resolution of $I(C|X)$:

$$0 \longrightarrow R(-r-d-\epsilon) \longrightarrow \ldots \longrightarrow R(-r-(i+\epsilon))^{\alpha_{i-1}} \longrightarrow$$

$$\ldots \longrightarrow R(-r-2-\epsilon)^{\alpha_1} \longrightarrow R(-r-\epsilon) \longrightarrow I(C|X) \longrightarrow 0.$$

Now we apply the horseshoe lemma to the exact sequence

$$0 \longrightarrow I(X) \longrightarrow I(C|\mathbb{P}^d) \longrightarrow I(C|X) \longrightarrow 0$$

to obtain the minimal free resolution of $I(C|\mathbb{P}^d)$:

$$0 \longrightarrow R(-r-d-\epsilon) \longrightarrow R(-r-d+2-\epsilon)^{\alpha_{d-3}} \oplus R(-d) \longrightarrow \ldots \longrightarrow$$

$$T_i := R(-r-i-\epsilon)^{\alpha_{i-1}} \oplus R(-(i+1))^{\alpha_i} \longrightarrow \ldots \longrightarrow R(-r-\epsilon) \oplus R(-2)^{\alpha_1} \longrightarrow I(C|\mathbb{P}^d) \longrightarrow 0.$$

This sequence shows that $C \subseteq \mathbb{P}^d$ is an AG variety with canonical module

$$K_C = R_C(r-1+\epsilon).$$

Therefore $I(G|C) = H^0_*(\mathcal{O}_C(-r)) = K_C(-2r+1-\epsilon)$. We apply $\mathrm{Hom}(-, R(-d-1))$ to the previous sequence and we get a minimal free resolution of K_C:

$$0 \longrightarrow R(-d-1) \longrightarrow R(r-d-1+\epsilon) \oplus R(-d+1)^{\alpha_{d-3}} \longrightarrow \ldots$$

$$\longrightarrow T'_i \longrightarrow \ldots \longrightarrow R(-1) \oplus R(r-3+\epsilon)^{\alpha_1} \longrightarrow R(r-1+\epsilon) \longrightarrow K_C \longrightarrow 0$$

where $T'_i := T^{\vee}_{d-i}(-d-1) = R(r-i-\epsilon)^{\alpha_{i-1}} \oplus R(-i)^{\alpha_{i-2}}$ for $i = 3, \ldots, d-2$. If we twist the previous sequence by $-2r+1-\epsilon$ we get the minimal resolution of $I(G|C)$:

$$0 \longrightarrow R(-2r-d-\epsilon) \longrightarrow R(-r-d) \oplus R(-2r-d+2-\epsilon)^{\alpha_{d-3}} \longrightarrow \ldots \longrightarrow T'_i(-2r+1-\epsilon) \longrightarrow \ldots$$

$$\longrightarrow R(-2r-\epsilon) \oplus R(-r-2)^{\alpha_1} \longrightarrow R(-r) \longrightarrow I(G|C) \longrightarrow 0.$$

Finally, we apply the horseshoe lemma to the short exact sequence

$$0 \longrightarrow I(C|X) \longrightarrow I(G|X) \longrightarrow I(G|C) \longrightarrow 0$$

to recover the resolution of $I(G|X)$ and we finish the proof. $\square$

Lemma 3.17. *(a) Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface and let $L \subseteq X$ be a line on it. Take $C \in |L+rH|$, $r \geq 2$, and let G be any effective divisor in the linear system $|2rH_C - K_C|$. Then, G is AG and the minimal free resolution of $I(G|C)$ has the following form:*

$$0 \longrightarrow R(-2r-d-1) \longrightarrow R(-2r-d+1)^{\alpha_1} \oplus R(-r-d)^{d-1} \longrightarrow \dots \longrightarrow$$
$$R(-2r-i)^{\alpha_{d-i}} \oplus R(-r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \longrightarrow \dots$$
$$\longrightarrow R(-2r-1) \oplus R(-r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \longrightarrow R(-r-1) \oplus R(-r-2) \longrightarrow I(G|C) \longrightarrow 0$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \leq i \leq d-3$.

(b) Let $X \subseteq \mathbb{P}^3$ be an integral cubic surface and let $L \subseteq X$ be a line on it. Take $C \in |L+rH|$, $r \geq 2$, and let G be any effective divisor in $|2rH_C - K_C|$. Then, G is AG and the minimal free resolution of $I(G|C)$ has the following form:

$$0 \longrightarrow R(-2r-4) \longrightarrow R(-2r-1) \oplus R(-r-3)^2 \longrightarrow R(-r-1) \oplus R(-r-2) \longrightarrow I(G|C) \longrightarrow 0$$

Proof. Let $L \subseteq X$ be any line. Its ideal as a subvariety of $\mathbb{P}^d$ has a resolution:

$$0 \longrightarrow R(-d+1) \longrightarrow \dots \longrightarrow R(-i)^{\binom{d-1}{i}} \longrightarrow \dots \longrightarrow R(-1)^{d-1} \longrightarrow I(L) \longrightarrow 0.$$

Applying the mapping cone process to $0 \to I(X) \to I(L) \to I(L|X) \to 0$ we get

$$0 \longrightarrow R(-d) \oplus R(-d+1) \longrightarrow \dots \longrightarrow R(-i)^{\binom{d-1}{i}+\alpha_{i-1}} \longrightarrow \dots \longrightarrow R(-1)^{d-1} \longrightarrow I(L|X) \longrightarrow 0$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \leq i \leq d-3$. Therefore, $C \in |L+rH|$ has the following minimal graded free resolution

$$0 \to R(-r-d) \oplus R(-r-d+1) \to \dots \to R(-r-i)^{\binom{d-1}{i}+\alpha_{i-1}} \to \dots \to R(-r-1)^{d-1} \to I(C|X) \to 0. \tag{3.3}$$

Now the horseshoe lemma applied to $0 \to I(X|\mathbb{P}^d) \to I(C) \to I(C|X) \to 0$ gives us

$$0 \longrightarrow R(-r-d) \oplus R(-r-d+1) \longrightarrow R(-r-d+2)^{\binom{d-1}{d-2}+\alpha_{d-3}} \oplus R(-d) \longrightarrow \dots \longrightarrow$$
$$R(-r-i)^{\binom{d-1}{i}+\alpha_{i-1}} \oplus R(-(i+1))^{\alpha_i} \longrightarrow \dots \longrightarrow R(-r-1)^{d-1} \oplus R(-2)^{\alpha_1} \longrightarrow I(C) \longrightarrow 0.$$

Since C is ACM we can apply $\mathrm{Hom}(-, R(-d-1))$ to get a resolution of K_C:

$$0 \longrightarrow R(-d-1) \longrightarrow R(-d+1)^{\alpha_1} \oplus R(r-d)^{d-1} \longrightarrow \dots \longrightarrow R(r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \oplus R(-i)^{\alpha_{d-i}}$$
$$\longrightarrow \dots \longrightarrow R(r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \oplus R(-1) \longrightarrow R(r-1) \oplus R(r-2) \longrightarrow K_C \longrightarrow 0.$$

Finally, since $G \in |2rH_C - K_C|$ we have:

$$0 \longrightarrow R(-2r-d-1) \longrightarrow R(-2r-d+1)^{\alpha_1} \oplus R(-r-d)^{d-1} \longrightarrow$$
$$\dots \longrightarrow R(-2r-i)^{\alpha_{d-i}} \oplus R(-r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \longrightarrow \dots \longrightarrow$$
$$R(-2r-1) \oplus R(-r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \longrightarrow R(-r-1) \oplus R(-r-2) \longrightarrow I(G|C) \longrightarrow 0.$$

□

Now we fix the starting point of the induction.

Lemma 3.18. *A general set Z of $m(2) = d + 2$ points on any del Pezzo surface $X \subseteq \mathbb{P}^d$ has a minimal free resolution of the following type:*

$$0 \longrightarrow R(-d-2) \longrightarrow R(-d)^{\gamma_{d-1}} \longrightarrow \dots \longrightarrow R(-3)^{\gamma_2} \longrightarrow R(-2)^{2d-1} \longrightarrow I(Z|X) \longrightarrow 0$$

with

$$\gamma_i = \sum_{l=0}^{1}(-1)^l \binom{d-l-1}{i-l} \Delta^{l+1} H_X(2+l) - \binom{d}{i}(m(2) - H_X(1)).$$

Proof. It follows from the fact that a general set Z of $d+2$ points on X is in linearly general position (i.e., any subset of Z of $d+1$ points spans $\mathbb{P}^d$). □

Fix an integer $r \geq 2$ and let $Z_{m(r)}$ and $Z_{n(r)}$ be general sets of points on X of cardinality $m(r)$ and $n(r)$, respectively. We will see that they are directly G-linked by an effective divisor G in $|rH_C|$ with C a curve in the linear system $|rH_X|$. Recall that we have:

$$P_X(r-1) < m(r) < n(r) < P_X(r).$$

Let us start with a general set $Z_{m(r)}$ of $m(r)$ points. Since $\mathrm{h}^0(\mathcal{O}_X(r)) > m(r)$ there exists a curve C in the linear system $|rH_X|$ such that $Z_{m(r)}$ lies on C. On the other hand, the inequality $n(r) > p_a(C)$ allows us to apply Riemann-Roch Theorem for curves and assure that there exists an effective divisor $Z_{n(r)}$ of degree $n(r)$ such that $Z_{m(r)} + Z_{n(r)}$ is linearly equivalent to a divisor rH_C.

Since this construction can also be performed starting from a general set $Z_{n(r)}$ of $n(r)$ points we see that a general set of $m(r)$ points is G-linked to a general set of $n(r)$ points and vice versa. Therefore as a direct application of the mapping cone process we get

Proposition 3.19. *Fix $r \geq 2$ and assume that the ideal $I(Z_{m(r)}|X)$ of $m(r)$ general points on a del Pezzo surface $X \subseteq \mathbb{P}^d$ has the minimal free resolution*

$$0 \longrightarrow R(-r-d)^{r-1} \longrightarrow R(-r-d+2)^{\gamma_{d-1,r-1}} \longrightarrow \dots$$
$$\longrightarrow R(-r-1)^{\gamma_{2,r-1}} \longrightarrow R(-r)^{(d-1)r+1} \longrightarrow I(Z_{m(r)}|X) \longrightarrow 0$$

with $\gamma_{i,r-1} = \sum_{l=0}^{1}(-1)^l \binom{d-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{d}{i}(m(r) - P_X(r-1))$. Then the ideal $I(Z_{n(r)}|X)$ of $n(r)$ general points has the minimal free resolution

$$0 \longrightarrow R(-r-d)^{(d-1)r-1} \longrightarrow R(-r-d+1)^{\beta_{d-1,r}} \longrightarrow \dots$$
$$\longrightarrow R(-r-2)^{\beta_{2,r}} \longrightarrow R(-r)^{r+1} \longrightarrow I(Z_{n(r)}|X) \longrightarrow 0$$

with $\beta_{i,r} = \sum_{l=0}^{1}(-1)^{l+1} \binom{d-l-1}{i-l} \Delta^{l+1} P_X(r+l) + \binom{d}{i}(n(r) - P_X(r-1))$.

Vice versa, if $n(r)$ general points on a del Pezzo surface $X \subseteq \mathbb{P}^d$ have the expected resolution then $m(r)$ general points do as well.

Lemma 3.20. *Let $X \subset \mathbb{P}^d$ be any del Pezzo surface. Fix $r \geq 2$ and assume that the ideal $I(Z_{n(r)}|X)$ of a set $Z_{n(r)}$ of $n(r)$ general points on $X \subseteq \mathbb{P}^d$ has the expected minimal free graded resolution. Then a set of $n(r)+1$ general points do as well.*

Proof. Since $I(Z_{n(r)}|X)$ has the expected minimal free resolution, it is generated by $r+1$ forms of degree r without linear relations. Take a general point $p \in X$ and set $Z := Z_{n(r)} \cup \{p\}$. Since $I(Z|X) \subset I(Z_{n(r)}|X)$, we can take the r generators of $I(Z|X)$ in degree r to be a subset of the generators of $I(Z_{n(r)}|X)$ in degree r; in particular, they do not have linear syzygies. We must add d generators of degree $r+1$ in order to get a minimal system of generators of $I(Z|X)$. Hence the first module in the minimal free resolution of $I(Z|X)$ is $R(-r)^r \oplus R(-r-1)^d$ which forces the remaining part of the resolution. $\square$

Proposition 3.21. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface. Fix $r \geq 2$ and assume that the ideal $I(Z_{p(r)}|X)$ of $p(r) := n(r)+1$ general points on X has the minimal free resolution*

$$0 \longrightarrow R(-r-d)^{(d-1)r} \longrightarrow R(-r-d+1)^{\delta_{d-1,r}} \longrightarrow \dots$$
$$\longrightarrow R(-r-2)^{\delta_{2,r}} \longrightarrow R(-r)^r \oplus R(-r-1)^d \longrightarrow I(Z_{p(r)}|X) \longrightarrow 0$$

with

$$\delta_{i,r} = \sum_{l=0}^{1} (-1)^{l+1} \binom{d-l-1}{i-l} \Delta^{l+1} H_X(r+l) + \binom{d}{i}(p(r) - H_X(r-1)).$$

Then the ideal $I(Z_{m(r+1)}|X)$ of $m(r+1)$ general points has the minimal free resolution

$$0 \longrightarrow R(-r-d-1)^r \longrightarrow R(-r-d+1)^{\gamma_{d-1,r}} \longrightarrow \dots$$
$$\longrightarrow R(-r-2)^{\gamma_{2,r}} \longrightarrow R(-r-1)^{(d-1)r+d} \longrightarrow I(Z_{m(r+1)}|X) \longrightarrow 0$$

with

$$\gamma_{i,r} = \sum_{l=0}^{1} (-1)^{l} \binom{d-l-1}{i-l} \Delta^{l+1} H_X(r+1+l) - \binom{d}{i}(m(r+1) - H_X(r)).$$

Proof. Let $Z_{p(r)}$ be a set of $p(r)$ general points with resolution as in the statement. Let us consider the linear system $|L+rH|$. Since, $\dim|L+rH| \geq \dim|rH| = P_X(r)-1 > p(r)$, we can find a curve $C \in |L+rH|$ passing through these $p(r)$ points. Notice that $\deg(C) = 1+rd$ and $p_a(C) = d\binom{r}{2}+r$. Since $p_a(C) < m(r+1)$ we can find an effective divisor $Z_{m(r+1)}$ of degree $m(r+1)$ such that $Z_{p(r)}$ and $Z_{m(r+1)}$ are G-linked by a divisor of degree $p(r)+m(r+1) = dr^2+dr+2 = deg(2rH_C - K_C)$. This allows us to find the resolution of $I(Z_{m(r+1)}|X)$. First we find the minimal free resolution of $I(Z_{p(r)}|C)$ using the exact sequence $0 \to I(C|X) \to I(Z_{p(r)}|X) \to I(Z_{p(r)}|C) \to 0$, the resolution of $I(C|X)$ given in (3.3) and the mapping cone process. It turns out to be:

$$0 \longrightarrow R(-r-d)^{(d-1)r+1} \longrightarrow R(-r-d+1)^{c_{d-1,r}} \longrightarrow \dots$$
$$\longrightarrow R(-r-2)^{c_{2,r}} \longrightarrow R(-r)^r \oplus R(-r-1) \longrightarrow I(Z_{p(r)}|C) \longrightarrow 0.$$

Since we know the minimal free resolution of $I(G|C)$ (see Lemma 3.17) we apply the mapping cone process to the sequence $0 \to I(G|C) \to IZ_{(p(r)}|C) \to I(Z_{p(r)}|G) \to 0$ to get

$$0 \longrightarrow R(-2r-d-1) \longrightarrow R(-r-d)^{(d-1)r+d} \oplus R(-2r-d+1)^{\alpha_1} \longrightarrow \dots \longrightarrow$$
$$R(-r-i)^{d_{i,r}} \oplus R(-2r-i+1)^{\alpha_{d-i+1}} \longrightarrow \dots \longrightarrow R(-r-2)^{d_{2,r}} \longrightarrow R(-r)^r \longrightarrow I(Z_{p(r)}|G) \longrightarrow 0.$$

($0 \to R(-2r-4) \to R(-r-3)^{2r+2} \oplus R(-2r-1) \to R(-r-2)^{d_{2,r}} \to R(-r)^r \to I(Z_{p(r)}|G) \to 0$ if $d=3$).

Finally we obtain the minimal free resolution of $I(Z_{m(r+1)})$:

$$0 \longrightarrow R(-r-d-1)^r \longrightarrow R(-r-d+1)^{\gamma_{d-1,r}} \longrightarrow R(-r-d+2)^{\gamma_{d-2,r}} \oplus R(-d) \longrightarrow$$

$$\dots \longrightarrow R(-r-i)^{\gamma_{i,r}} \oplus R(-i)^{\alpha_i} \longrightarrow \dots \longrightarrow R(-r-1)^{(d-1)r+d} \oplus R(-2)^{\alpha_1} \longrightarrow I(Z_{m(r+1)}) \longrightarrow 0$$

$(0 \to R(-r-4)^r \to R(-r-2)^{\gamma_{2,r}} \to R(-r-1)^{2r+3} \oplus R(-3) \to I(Z_{m(r+1)}) \to 0$ if $d=3)$ from which it is straightforward to recover the predicted resolution of $I(Z_{m(r+1)}|X)$. □

We are ready to prove the MRC for $n(r)$ and $m(r)$ general points on a del Pezzo surface:

Theorem 3.22. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface. We have:*

(1) *Let $Z_{n(r)} \subseteq X$ be a general set of $n(r)$ points, $r \geq 2$. Then the minimal graded free resolution of $I(Z_{n(r)}|X)$ has the following form:*

$$0 \longrightarrow R(-r-d)^{(d-1)r-1} \longrightarrow R(-r-d+1)^{\beta_{d-1,r}} \longrightarrow R(-r-d+2)^{\beta_{d-2,r}} \longrightarrow \dots$$
$$\longrightarrow R(-r-2)^{\beta_{2,r}} \longrightarrow R(-r)^{r+1} \longrightarrow I(Z_{n(r)}|X) \longrightarrow 0.$$

where

$$\beta_{i,r} = \sum_{l=0}^{1} (-1)^{l+1} \binom{n-l-1}{i-l} \Delta^{l+1} H_X(r+l) + \binom{n}{i} (n(r) - H_X(r-1)).$$

(2) *Let $Z_{m(r)} \subseteq X$ be a general set of $m(r)$ points, $r \geq 2$. Then its minimal graded free resolution has the following form:*

$$0 \longrightarrow R(-r-d)^{r-1} \longrightarrow R(-r-d+2)^{\gamma_{d-1,r-1}} \longrightarrow \dots$$
$$\longrightarrow R(-r-1)^{\gamma_{2,r-1}} \longrightarrow R(-r)^{(d-1)r+1} \longrightarrow I(Z_{m(r)}|X) \longrightarrow 0$$

with

$$\gamma_{i,r-1} = \sum_{l=0}^{1} (-1)^{l} \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{n}{i} (m(r) - P_X(r-1)).$$

In particular, Mustaţă's conjecture works for $n(r)$ and $m(r)$, $r \geq 4$, general points on a del Pezzo surface $X \subseteq \mathbb{P}^d$.

Proof. Lemma 3.18 establishes the result for a set of $m(2)$ general points, the starting point of our induction process. Therefore, the result about the resolution of $I(Z_{n(r)}|X)$ and $I(Z_{m(r)}|X)$ follows using Lemma 3.20, Propositions 3.19 and 3.21 and applying induction. □

Next lemma controls how the bottom lines of the Betti diagram of a set of general points on a projective variety change when we add another general point.

Lemma 3.23. *Let $X \subseteq \mathbb{P}^n$ be a projective variety with $\dim(X) \geq 2$, $\operatorname{reg}(X) = m$ and with Hilbert polynomial P_X. Let s be an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$, let Z be a set of s general points on X and let $P \in X \setminus Z$ be a general point. We have*

(i) $b_{i,r-1}(Z) \geq b_{i,r-1}(Z \cup P)$ *for every i.*

(ii) $b_{i,r}(Z) \leq b_{i,r}(Z \cup P)$ *for every i.*

Proof. See [Mus], Proposition 1.7.. □

Now, we prove the main result of this subsection, namely, the MRC holds for a general set of points Z on a smooth del Pezzo surface when the cardinality of Z falls in the strips of the form $[P_X(r-1), m(r)]$ or $[n(r), P_X(r)]$, $r \geq 4$.

Theorem 3.24. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface. Let r be such that $r \geq reg(X)+1 = 4$. Then for a general set of points Z on X such that $P_X(r-1) \leq |Z| \leq m(r)$ or $n(r) \leq |Z| \leq P_X(r)$ the MRC is true.*

Proof. See [P], Chapter II, for the cases of $X \cong \mathbb{P}^2$ and $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. So let X be any other smooth del Pezzo surface. Let Z' be a general set of points of cardinality $|Z'| = n(r)$ and add general points to Z' in order to get a set of points Z of cardinality $n(r) \leq |Z| \leq P_X(r)$. By Theorem 3.22 we have that $b_{i,r-1}(Z') = 0$ for all $i = 2, \dots, d$. Therefore we can apply Lemma 3.23 to deduce that $b_{i,r-1}(Z) = 0$ for all $i = 2, \dots, d$. Thus, by semicontinuity, MRC holds for a general set of $|Z|$ points.

Now if $|Z| \leq m(r)$, we can add general points to Z in order to have a general set Z' including Z and such that $|Z'| = m(r)$. Again from the previous Theorem we have that $b_{i,r}(Z') = 0$ for all $i = 1, \dots, d-1$. So we can use again Lemma 3.23 to deduce that $b_{i,r}(Z) = 0$ for all $i = 1, \dots, d-1$ and therefore MRC holds for Z. □

As a consequence of Theorem 3.22 we prove that the number of generators of the ideal of a general set of points on a del Pezzo surface is as small as possible and so it is the number of generators of its canonical module as well. In fact, we have:

Theorem 3.25. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface. Then for a general set of points Z on X such that $|Z| \geq P_X(3)$ the Cohen-Macaulay type Conjecture and the Ideal Generation Conjecture are true.*

Proof. Let Z be a general set of points on our del Pezzo surface X. If it is the case that $n(r) \leq |Z| \leq m(r+1)$ the result has been proved on the previous theorem. So we can assume that $m(r) < |Z| < n(r)$ for some r. We know that the MRC holds for a general set $|Z'|$ of $n(r)$ points on X, $Z \subseteq Z'$ and in particular $b_{1,r}(Z') = 0$. Applying Lemma 3.23 inductively we see that $b_{1,r}(Z) = 0$. Analogously, since MRC holds for a general set Z'' of $m(r)$ points, $b_{d,r-1}(Z'') = 0$ with $Z'' \subseteq Z$. Applying once again the same Lemma we see that $b_{d,r-1}(Z) = 0$. □

In the particular case of the cubic surface, since the minimal free resolution of its points has length three, we recover one of the main results of [MP1] (see also [MP2]):

Theorem 3.26. *Let $X \subseteq \mathbb{P}^3$ be a integral cubic surface (i.e., a del Pezzo surface of degree three). Then the Minimal Resolution Conjecture holds for a general set of points on X of cardinality $\geq P_X(3) = 19$.*

3.2. **Ulrich bundles on del Pezzo surfaces.**

In this subsection, we will construct large families of ACM vector bundles on smooth del Pezzo surfaces with the maximal allowed number of global sections (the so-called Ulrich bundles) and conclude that all smooth del Pezzo surfaces are of wild representation type. This result generalizes a previous result of Pons-Llopis and Tonini [PT] (see also [CH]) which states that the cubic surface $S \subset \mathbb{P}^3$ is of wild representation type.

The proof for the degree 8 smooth del Pezzo surface $X \subset \mathbb{P}^8$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e. the Segre product of two conics naturally embedded in $\mathbb{P}^8$: $\varphi_{|\mathcal{O}(2,2)|} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^8$) is slightly different and the reader can consult [P]. So, from now on when speaking of a smooth del Pezzo surface we will understand the blow up of $\mathbb{P}^2$ at $s \leq 8$ points in general position.

Following notation from [EH], let us consider K-vector spaces A and B of respective dimension a and b. Set $V = \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$ and let $M = \mathrm{Hom}(B, A \otimes V)$ be the space of $(a \times b)$-matrices of linear forms. M is an affine space of dimension $ab(m+1)$. It is well-known that there exists a bijection between the elements $\phi \in M$ and the morphisms $\phi : B \otimes \mathcal{O}_{\mathbb{P}^m} \to A \otimes \mathcal{O}_{\mathbb{P}^m}(1)$. Taking the tensor with $\mathcal{O}_{\mathbb{P}^m}(1)$ and considering global sections, we have morphisms

$$\mathrm{H}^0(\phi(1)) : \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)^b) \longrightarrow \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(2)^a).$$

The following result tells us under which conditions the aforementioned morphisms ϕ and $\mathrm{H}^0(\phi(1))$ are surjective:

Proposition 3.27. *For $a \geq 1$, $b \geq a + m$ and $2b \geq (m+2)a$, the set of elements $\phi \in M$ such that $\phi : B \otimes \mathcal{O}_{\mathbb{P}^m} \to A \otimes \mathcal{O}_{\mathbb{P}^m}(1)$ and $\mathrm{H}^0(\phi(1)) : \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)^b) \to \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(2)^a)$ are surjective forms a non-empty open dense subset of the affine variety M that we will denote by V_m.*

Proof. See [EH], Proposition 4.1. □

Fix $m = 2$ and for a given $r \geq 2$, set $a := r$, $b := 2r$. Take an element ϕ of the non-empty subset $V_2 \subseteq M$ provided by Proposition 3.27 and consider the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{2r} \xrightarrow{\phi(1)} \mathcal{O}_{\mathbb{P}^2}(2)^r \longrightarrow 0. \tag{3.4}$$

It follows immediately that $\mathcal{F}$ is a vector bundle of rank r, being kernel of a surjective morphism of vector bundles. Let $X := Bl_Z(\mathbb{P}^2) \xrightarrow{\pi} \mathbb{P}^2$ be the low up of $\mathbb{P}^2$ at $0 \leq s \leq 8$ points in general position. Pulling-back the exact sequence (3.4) we obtain the exact sequence:

$$0 \longrightarrow \pi^*\mathcal{F} \longrightarrow \mathcal{O}_X(e_0)^b \xrightarrow{\phi(1)} \mathcal{O}_X(2e_0)^a \longrightarrow 0. \tag{3.5}$$

We can prove:

Proposition 3.28. *Let $X \xrightarrow{\pi} \mathbb{P}^2$ be the low up of $\mathbb{P}^2$ at $0 \leq s \leq 8$ points in general position and let $r \geq 2$. Let $\mathcal{F}$ be the vector bundle obtained as the kernel of a general surjective morphism between $\mathcal{O}_{\mathbb{P}^2}(1)^{2r}$ and $\mathcal{O}_{\mathbb{P}^2}(2)^r$:*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{2r} \xrightarrow{\phi(1)} \mathcal{O}_{\mathbb{P}^n}(2)^r \longrightarrow 0. \tag{3.6}$$

Then, the vector bundles $\mathcal{E}$ obtained pulling-back $\mathcal{F}$, dualizing and twisting by $H := 3e_0 - \sum_{i=1}^s e_i$

$$0 \longrightarrow \mathcal{O}_X(-2e_0 + H)^r \xrightarrow{f} \mathcal{O}_X(-e_0 + H)^{2r} \xrightarrow{g} \mathcal{E}(H) := (\pi^*\mathcal{F})^*(H) \longrightarrow 0 \tag{3.7}$$

are simple (hence, undecomposable) vector bundles of rank r on X.

Proof. See [MP], Corollary 4.5. □

The Chern classes of $\mathcal{E}(H)$ can be easily computed and we get:

$$c_1(\mathcal{E}(H)) = rH \text{ and } c_2(\mathcal{E}(H)) = \frac{H^2r^2 + (2-H^2)r}{2}.$$

Let us check that $\mathcal{E}(H)$ is an initialized Ulrich bundle. For this, we need the following computations.

Remark 3.29 (Riemann-Roch for vector bundles on a del Pezzo surface)**.** Let X be a del Pezzo surface. Since X is a rational connected surface we have $\chi(\mathcal{O}_X) = 1$. In particular, the Riemann-Roch formula for a vector bundle $\mathcal{E}$ on X of rank r has the form

$$\chi(\mathcal{E}) = \frac{c_1(\mathcal{E})(c_1(\mathcal{E}) - K_X)}{2} + r - c_2(\mathcal{E}).$$

Remark 3.30. The Euler characteristic of the involved vector bundles can be computed thanks to the Riemann-Roch formula:

$$\chi(\mathcal{O}_X(-2e_0)(lH)) = \frac{9-s}{2}l^2 - \frac{3+s}{2}l, \tag{3.8}$$

$$\chi(\mathcal{O}_X(-e_0)(lH)) = \frac{9-s}{2}l^2 + \frac{3-s}{2}l, \tag{3.9}$$

$$\chi(\mathcal{E}(lH)) = 2r\chi(\mathcal{O}_X(-e_0)(lH)) - r\chi(\mathcal{O}_X(-2e_0)(lH)) = \frac{9r-sr}{2}l^2 + \frac{9r-sr}{2}l. \tag{3.10}$$

Proposition 3.31. *Let X be a del Pezzo surface. The bundles $\mathcal{E}(H)$ given by the exact sequence (3.7) are initialized simple Ulrich bundles. Moreover, in the case of a blow-up of ≤ 7 points, they are globally generated.*

Proof. First of all, notice that $\mathrm{H}^0(\mathcal{E}^*) = \mathrm{H}^2(\mathcal{E}(-H)) = 0$. Therefore, $\mathrm{H}^2(\mathcal{E}(tH)) = 0$, for all $t \geq -1$. On the other hand, since $\mathrm{H}^2(\mathcal{O}_X(-2e_0)) = \mathrm{H}^0(\mathcal{O}_X(2e_0 - H)) = 0$ and $\mathrm{h}^1(\mathcal{O}_X(-e_0)) = -\chi(\mathcal{O}_X(-e_0)) = 0$ we obtain from the long exact sequence of cohomology associated to (3.7) that $\mathrm{H}^1(\mathcal{E}) = 0$. Since $\chi(\mathcal{E}) = 0$, we also conclude that $\mathrm{H}^0(\mathcal{E}) = 0$ and therefore $\mathrm{H}^0(\mathcal{E}(tH)) = 0$ for all $t \leq 0$. Moreover, since we also have that $\chi(\mathcal{E}(-H)) = 0$, we obtain that $\mathrm{H}^1(\mathcal{E}(-H)) = 0$.

We easily check that $\mathrm{H}^0(\mathcal{E}(H)) \neq 0$ which together with the vanishing $\mathrm{H}^0(\mathcal{E}(tH)) = 0$ for all $t \leq 0$ implies that $\mathcal{E}(H)$ is initialized.

We tensor by $\mathcal{E}$ the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

and we consider the cohomology sequence associated to it. We get

$$0 = \mathrm{H}^0(\mathcal{E}) \longrightarrow \mathrm{H}^0(\mathcal{E}_{|H}) \longrightarrow \mathrm{H}^1(\mathcal{E}(-H)) = 0.$$

This shows that $\mathrm{H}^0(\mathcal{E}_{|H}(-tH)) = 0$ for all $t \geq 0$. Then we can use this last fact together with the long exact sequence associated to

$$0 \longrightarrow \mathcal{E}(-(t+1)H) \longrightarrow \mathcal{E}(-tH) \longrightarrow \mathcal{E}_{|H}(-tH) \longrightarrow 0$$

to show inductively that $\mathrm{H}^1(\mathcal{E}(-tH)) = 0$ for all $t \geq 0$.

In order to complete the proof we need to consider two different cases:

- *X is the blow-up of $s \leq 7$ points on $\mathbb{P}^2$ in general position.* In this case, by Lemma 3.5, H is ample and generated by its global sections. Since we have just seen that $\mathcal{E}(H)$ is 0-regular with respect to H we can conclude that $\mathcal{E}(H)$ is ACM and globally generated. Moreover, $\mathrm{h}^0(\mathcal{E}(H)) = \chi(\mathcal{E}(H)) = (9-s)r = H^2 r$, i.e., $\mathcal{E}(H)$ is an Ulrich bundle.
- *X is the blow-up of 8 points on $\mathbb{P}^2$ in general position.* In this case, the argument is slightly more involved, since H is ample but not very ample. Fortunately $2H$ is ample and globally generated. First of all, since the points are in general position, $\mathrm{H}^0(\mathcal{O}_X(-e_0+H)) = 0$ and from the exact sequence (3.7) we get the following exact sequence:

$$0 \longrightarrow \mathrm{H}^0(\mathcal{E}(H)) \longrightarrow \mathrm{H}^1(\mathcal{O}_X(-2e_0+H)^r) \longrightarrow \mathrm{H}^1(\mathcal{O}_X(-e_0+H)^{2r}) \longrightarrow \mathrm{H}^1(\mathcal{E}(H)) \longrightarrow 0.$$

From this sequence and the fact that $\mathrm{h}^1(\mathcal{O}_X(-2e_0+H)) = -\chi(\mathcal{O}_X(-2e_0+H)) = 5$ and $\mathrm{h}^1(\mathcal{O}_X(-e_0+H)) = -\chi(\mathcal{O}_X(-e_0+H)) = 2$ we are forced to conclude that $\mathrm{h}^0(\mathcal{E}(H)) = r$ and $\mathrm{H}^1(\mathcal{E}(H)) = 0$. Now, from what we have gathered up to now, we can affirm that $\mathcal{E}(H)$ is 1-regular with respect to the very ample line bundle $2H$ and therefore, $\mathrm{H}^1(\mathcal{E}(H+2tH)) = 0$ for all $t \geq 0$. In order to deal with the cancelation of the remaining groups of cohomology, it will be enough to show that $\mathcal{E}(2H)$ is 1-regular with respect to $2H$, i.e., it remains to show that $\mathrm{H}^1(\mathcal{E}(2H)) = 0$. In order to do this consider the exact sequence (the cancelation of $\mathrm{H}^0(\mathcal{O}_X(-e_0+2H))$ is due to the fact that the points are in general position):

$$0 \longrightarrow \mathrm{H}^0(\mathcal{E}(2H)) \longrightarrow \mathrm{H}^1(\mathcal{O}_X(-2e_0+2H)^r) \longrightarrow \mathrm{H}^1(\mathcal{O}_X(-e_0+2H)^{2r}) \longrightarrow \mathrm{H}^1(\mathcal{E}(2H)) \longrightarrow 0.$$

Once again, we control the dimension of these vector spaces: $\mathrm{h}^1(\oplus^r \mathcal{O}_X(-2e_0+2H)) = -r\chi(\mathcal{O}_X(-2e_0+2H)) = 9r$ and $\mathrm{h}^1(\oplus^{2r}\mathcal{O}_X(-e_0+2H)) = -2r\chi(\mathcal{O}_X(-e_0+2H)) = 6r$ Therefore we are forced to have $\mathrm{h}^0(\mathcal{E}(2H)) = 3r$ and $\mathrm{H}^1(\mathcal{E}(2H)) = 0$. Notice that in this case $\mathcal{E}(3H)$ is globally generated.

□

As an immediate consequence we get:

Theorem 3.32. *Let $X \subset \mathbb{P}^d$ be a smooth del Pezzo surface of degree d. Then for any $r \geq 2$ there exists a family of dimension r^2+1 of simple initialized Ulrich bundles of rank r on X. In particular, del Pezzo surfaces are of wild representation type.*

Proof. See [MP], Theorem 4.9. □

In the last part of this subsection we consider the case of strong del Pezzo surfaces X, i.e. smooth del Pezzo surfaces with anticanonical divisor very ample. In this case, $-K_X$ provides an embedding $X \subseteq \mathbb{P}^d$, with $d = K_X^2$. Let $R := K[x_0, \ldots, x_d]$ be the graded polynomial ring associated to $\mathbb{P}^d$. Using our results on Mustaţă's conjecture explained in the previous subsection, we are going to show that the (r^2+1)-dimensional family of rank r initialized Ulrich bundles given in Theorem 3.32 could also be obtained through a version of Serre correspondence from a general set of $\frac{dr^2+(2-d)r}{2}$ points on X.

More precisely, as a particular case of Theorem 3.24, we have the following result:

Theorem 3.33. *Let $X \subseteq \mathbb{P}^d$ be a strong del Pezzo surface of degree d embedded in $\mathbb{P}^d$ by its very ample anticanonical divisor. Let $Z_{m(r)} \subset X$ be a general set of*

$$m(r) = \frac{1}{2}(dr^2 + (2-d)r)$$

points, $r \geq 2$. Then the minimal graded free resolution (as a R-module) of the saturated ideal of $Z_{m(r)}$ in X has the following form:

$$(3.11) \qquad 0 \longrightarrow R(-r-d)^{r-1} \longrightarrow R(-r-d+2)^{\gamma_{d-1,r-1}} \longrightarrow \dots$$
$$\longrightarrow R(-r-1)^{\gamma_{2,r-1}} \longrightarrow R(-r)^{(d-1)r+1} \longrightarrow I(Z_{m(r)}|X) \longrightarrow 0$$

with

$$\gamma_{i,r-1} = \sum_{l=0}^{1} (-1)^l \binom{d-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{d}{i}(m(r) - P_X(r-1)).$$

Theorem 3.34. *Let $X \subseteq \mathbb{P}^d$ be a strong del Pezzo surface of degree d.*

(i) *If $\mathcal{E}(H)$ is an Ulrich bundle of rank $r \geq 2$ given by the exact sequence (3.7), then there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow \mathcal{E}(H) \longrightarrow I(Z|X)(rH) \longrightarrow 0$$

where Z is a zero-dimensional scheme of degree $m(r) = c_2(\mathcal{E}(H)) = \frac{1}{2}(dr^2 + (2-d)r)$ and $\mathrm{h}^0(I(Z|X)(r-1)H) = 0$.

(ii) *Conversely, for general sets Z of $m(r) = 1/2(dr^2 + (2-d)r)$ points on X, $r \geq 2$, we recover the initialized Ulrich bundles given by the exact sequence (3.7) as an extension of $I(Z|X)(rH)$ by $\mathcal{O}_X^{r-1}$.*

Proof. (i) As $\mathcal{E}(H)$ is globally generated, $r-1$ general global sections define an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow \mathcal{E}(H) \longrightarrow I(Z|X)(D) \longrightarrow 0$$

where $D = c_1(\mathcal{E}(H)) = rH$ is a divisor on X and Z is a zero-dimensional scheme of length

$$c_2(\mathcal{E}(H)) = \frac{dr^2 + (2-d)r}{2}.$$

Moreover, since $\mathcal{E}(H)$ is initialized, $\mathrm{h}^0(I(Z|X)(r-1)H) = 0$.

(ii) Let Z be a general set of points of cardinality $m(r)$ with the minimal free resolution of (3.11). Let us denote by R_X and R_Z the homogeneous coordinate ring of X and Z. It is well-known that for ACM varieties, there exists a bijection between ACM bundles on X and Maximal Cohen Macaulay (MCM from now on) graded R_X-modules sending $\mathcal{E}$ to $\mathrm{H}^0_*(\mathcal{E})$. From the exact sequence

$$0 \longrightarrow I(Z|X) \longrightarrow R_X \longrightarrow R_Z \longrightarrow 0$$

we get $\mathrm{Ext}^1(I(Z|X), R_X(-1)) \cong \mathrm{Ext}^2(R_Z, R_X(-1)) \cong K_Z$ where K_Z denotes the canonical module of R_Z (the last isomorphism is due to the fact that $R_X(-1)$ is the canonical module of X and the codimension of Z in X is 2). Dualizing the exact sequence (3.11), we obtain a minimal resolution of K_Z:

$$\dots \longrightarrow R(r-3)^{\gamma_{d-1,r-1}} \longrightarrow R(r-1)^{r-1} \longrightarrow K_Z \longrightarrow 0.$$

This shows that K_Z is generated in degree $1-r$ by $r-1$ elements. These generators provide an extension

$$0 \longrightarrow R_X^{r-1} \longrightarrow F \longrightarrow I(Z|X)(r) \longrightarrow 0 \tag{3.12}$$

via the isomorphism $K_Z \cong \operatorname{Ext}^1(I(Z|X), R_X(-1))$. F turns out to be a MCM module because $\operatorname{Ext}^1(F, K_X) = 0$ (this last cancelation follows by applying $\operatorname{Hom}_{R_X}(-, K_X)$ to (3.12)). If we sheafiffy the exact sequence (3.12) we obtain the sequence

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow \widetilde{F} \longrightarrow I(Z|X)(r) \longrightarrow 0$$

where $\widetilde{F}$ is an ACM vector bundle on X. Using the exact sequence (3.11) we can see that $\mathrm{H}^0(I(Z|X)(r-1)H) = 0$ and $\mathrm{h}^0(I(Z|X)(rH)) = (d-1)r+1$. Therefore $\widetilde{F}$ is an initialized Ulrich bundle (i.e., $\mathrm{h}^0(\widetilde{F}) = dr$). By Theorem 2.16, $\widetilde{F}$ will be globally generated.

It only remains to show that for a generic choice of $Z_{m(r)} \subset X$, the associated bundle $\mathcal{F} := \widetilde{F}$ just constructed belongs to the family (3.7). Since $\mathcal{F}$ is an initialized Ulrich bundle of rank r with the expected Chern classes, the problem boils down to a dimension counting. We need to show that the dimension of the family of vector bundles obtained through this construction from a general set $Z_{m(r)}$ is r^2+1. Since this dimension is given by the formula $\dim Hilb^{m(r)}(X) - \dim Grass(\mathrm{h}^0(\mathcal{F}), r-1)$, an easy computation taking into account that $\dim Hilb^{m(r)}(X) = 2m(r)$ and that $\dim Grass(\mathrm{h}^0(\mathcal{F}), r-1) = (r-1)(dr-r+1)$ gives the desired result. □

As a nice application we get:

Theorem 3.35. *Let X be a smooth del Pezzo surface of degree d. Then for any $r \geq 2$ there exists a family of dimension r^2+1 of simple Ulrich bundles of rank r with Chern classes $c_1 = rH$ and $c_2 = \frac{dr^2+r(2-d)}{2}$.*

So, we conclude:

Theorem 3.36. *Smooth del Pezzo surfaces $X \subset \mathbb{P}^d$ are of wild representation type.*

4. The representation type of a Segre variety

Fix integers $1 \leq n_1, \cdots, n_s$ and set $N := \prod_{i=1}^s (n_i+1) - 1$. The goal of this section is to prove that all Segre varieties $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$ unless the quadric surface in $\mathbb{P}^3$ support families of arbitrarily large dimension and rank of simple Ulrich (and hence ACM) vector bundles. Therefore, they are all unless $\mathbb{P}^1 \times \mathbb{P}^1$ of wild representation type. To this end, we will give an effective method to construct ACM sheaves (i.e. sheaves without intermediate cohomology) with the maximal permitted number of global sections, the so-called Ulrich sheaves, on all Segre varieties $\Sigma_{n_1,\cdots,n_s}$ other than $\mathbb{P}^1 \times \mathbb{P}^1$. To our knowledge, they will be the first family of examples of varieties of arbitrary dimension for which wild representation type is witnessed by means of Ulrich bundles.

Let us start this section recalling the definition of Segre variety and the basic properties on Segre varieties needed later on. Given integers $1 \leq n_1, \cdots, n_s$, we denote by

$$\sigma_{n_1,\cdots,n_s} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \longrightarrow \mathbb{P}^N, \quad N = \prod_{i=1}^{s} (n_i+1) - 1$$

the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. The image of $\sigma_{n_1,\cdots,n_s}$ is the Segre variety $\Sigma_{n_1,\cdots,n_s} := \sigma_{n_1,\cdots,n_s}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}) \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^s (n_i+1) - 1$. Notice that in terms of very ample line bundles, this embedding is defined by means of $\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_s}}(1, \cdots, 1)$.

The equations of the Segre varieties are familiar to anyone who has studied algebraic geometry. Indeed, if we let T be the $(n_1+1) \times \cdots \times (n_s+1)$ tensor whose entries are the homogeneous coordinates in $\mathbb{P}^N$, then it is well known that the ideal of $\Sigma_{n_1,\cdots,n_s}$ is generated by the 2×2 minors of T. Moreover, we have

Proposition 4.1. *Fix integers $1 \leq n_1, \cdots, n_s$ and denote by $\Sigma_{n_1,\cdots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^s (n_i+1) - 1$, the Segre variety. It holds:*

(i) $\dim(\Sigma_{n_1,\cdots,n_s}) = \sum_{i=1}^s n_i$,
(ii) $\deg(\Sigma_{n_1,\cdots,n_s}) = \frac{(\sum_{i=1}^s n_i)!}{\prod_{i=1}^s (n_i)!}$,
(iii) $\Sigma_{n_1,\cdots,n_s}$ *is ACM, and*
(iv) $I(\Sigma_{n_1,\cdots,n_s})$ *is generated by* $\binom{N+2}{2} - \prod_{i=1}^s \binom{n_i+2}{2}$ *hyperquadrics.*

Example 4.2. (1) We consider the Segre embedding

$$\begin{gathered}\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3 \\ ((a,b),(c,d)) \mapsto (ac, ad, bc, bd).\end{gathered}$$

Set $\Sigma_{1,1} := \sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$. If we fix coordinates x, y, z, t in $\mathbb{P}^3$, we have: $I(\Sigma_{1,1}) = (xt - yz)$, $\dim(\Sigma_{1,1}) = 2$, $\deg(\Sigma_{1,1}) = 2$ and $\mathrm{Pic}(\Sigma_{1,1}) = \mathbb{Z}^2$.

(2) We consider the Segre embedding

$$\begin{gathered}\sigma_{2,3} : \mathbb{P}^2 \times \mathbb{P}^3 \longrightarrow \mathbb{P}^{11} \\ ((a,b,c),(d,e,f,g)) \mapsto (ad, ae, af, ag, \cdots, cg).\end{gathered}$$

Set $\Sigma_{2,3} := \sigma_{2,3}(\mathbb{P}^2 \times \mathbb{P}^3)$. If we fix coordinates $x_{0,0}, x_{0,1}, \cdots, x_{2,3}$ in $\mathbb{P}^{11}$, we have: $\Sigma_{2,3}$ is an ACM variety and its ideal $I(\Sigma_{2,3})$ is generated by 18 hyperquadrics. In fact, $\Sigma_{2,3}$ is a determinantal variety defined by the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix}.$$

Moreover, $\dim(\Sigma_{2,3}) = 5$, $\deg(\Sigma_{2,3}) = 10$ and $\mathrm{Pic}(\Sigma_{2,3}) = \mathbb{Z}^2$.

Let p_i denote the i-th projection of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ onto $\mathbb{P}^{n_i}$. There is a canonical isomorphism $\mathbb{Z}^s \longrightarrow \mathrm{Pic}(\Sigma_{n_1,\cdots,n_s})$, given by

$$(a_1, \cdots, a_s) \mapsto \mathcal{O}_{\Sigma_{n_1,\cdots,n_s}}(a_1, \cdots, a_s) := p_1^*(\mathcal{O}_{\mathbb{P}^{n_1}}(a_1)) \otimes \cdots \otimes p_s^*(\mathcal{O}_{\mathbb{P}^{n_s}}(a_s)).$$

For any coherent sheaves $\mathcal{E}_i$ on $\mathbb{P}^{n_i}$, we set $\mathcal{E}_1 \boxtimes \cdots \boxtimes \mathcal{E}_s := p_1^*(\mathcal{E}_1) \otimes \cdots \otimes p_s^*(\mathcal{E}_s)$. We will denote by $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \longrightarrow X_i := \mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_s}$ the natural projection and given sheaves $\mathcal{E}$ and $\mathcal{F}$ on X_i and $\mathbb{P}^{n_i}$, respectively, $\mathcal{E} \boxtimes \mathcal{F}$ stands for $\pi_i^*(\mathcal{E}) \otimes p_i^*(\mathcal{F})$. By the Künneth's formula, we have

$$\mathrm{H}^\ell(\Sigma_{n_1,\cdots,n_s}, \mathcal{E} \boxtimes \mathcal{F}) = \bigoplus_{p+q=\ell} \mathrm{H}^p(X_i, \mathcal{E}) \otimes \mathrm{H}^q(\mathbb{P}^{n_i}, \mathcal{F}).$$

While given a coherent sheaf $\mathcal{H}$ on $\Sigma_{n_1,\cdots,n_s}$, $\mathcal{H}(t)$ stands for $\mathcal{H}\otimes\mathcal{O}_{\Sigma_{n_1,\cdots,n_s}}(t,\cdots,t)$.

Let us start by determining the complete list of initialized Ulrich line bundles on Segre varieties $\Sigma_{n_1,\cdots,n_s}\subseteq\mathbb{P}^N$, $N=\prod_{i=1}^s(n_i+1)-1$. First of all, notice that it follows from Horrocks' Theorem ([Hor]) that

Lemma 4.3. *The only initialized Ulrich bundle on $\mathbb{P}^n$ is the structural sheaf $\mathcal{O}_{\mathbb{P}^n}$.*

The list of initialized Ulrich line bundles on $\Sigma_{n_1,\cdots,n_s}\subseteq\mathbb{P}^N$, $N=\prod_{i=1}^s(n_i+1)-1$, is given by

Proposition 4.4. *Let $\Sigma_{n_1,\cdots,n_s}\subseteq\mathbb{P}^N$, $N=\prod_{i=1}^s(n_i+1)-1$, be a Segre variety. Then there exist $s!$ initialized Ulrich line bundles on $\Sigma_{n_1,\cdots,n_s}$. They are of the form*

$$\mathcal{L}_{X_i}\boxtimes\mathcal{O}_{\mathbb{P}^{n_i}}(\sum_{k\neq i}n_k),$$

where $\mathcal{L}_{X_i}$ is a rank one initialized Ulrich bundle on the Segre variety $X_i:=\Sigma_{n_1,\cdots,\widehat{n_i},\cdots,n_s}\subseteq\mathbb{P}^{N'}$, $N'=\prod_{\substack{1\leq j\leq s\\ j\neq i}}(n_j+1)-1$. More explicitly, the initialized Ulrich line bundles on $\Sigma_{n_1,\cdots,n_s}$ are of the form $\mathcal{O}_{\Sigma_{n_1,\cdots,n_s}}(a_1,\dots,a_s)$ where, if we order the coefficients $0=a_{i_1}\leq\cdots\leq a_{i_k}\leq\cdots\leq a_{i_s}$ then $a_{i_k}=\sum_{1\leq j<k}n_{i_j}$.

Proof. The existence of this set of initialized Ulrich line bundles is a straightforward consequence of [EFW], Proposition 2.6. In order to see that this list is exhaustive, let us consider an initialized Ulrich line bundle $\mathcal{L}:=\mathcal{O}_{\Sigma_{n_1,\cdots,n_s}}(a_1,\dots,a_s)$ with $a_{i_1}\leq\cdots\leq a_{i_k}\leq\cdots\leq a_{i_s}$. Given that $\mathcal{L}$ is initialized, it holds that $a_{i_1}=0$. Since $\mathcal{L}$ is ACM, we have

$$\mathrm{H}^{\sum_{j=1}^k n_{i_j}}(\Sigma_{n_1,\cdots,n_s},\mathcal{L}(-\Sigma_{j=1}^k n_{i_j}-1))=0$$

for $k=1,\dots,s-1$. In particular, using Künneth's formula, it holds

$$\prod_{l=1}^k\mathrm{h}^{n_{i_l}}(\mathbb{P}^{n_{i_l}},\mathcal{O}_{\mathbb{P}^{n_{i_l}}}(a_{i_l}-\Sigma_{j=1}^k n_{i_j}-1))\cdot\prod_{l=k+1}^s\mathrm{h}^0(\mathbb{P}^{n_{i_l}},\mathcal{O}_{\mathbb{P}^{n_{i_l}}}(a_{i_l}-\Sigma_{j=1}^k n_{i_j}-1))=0,$$

from where it follows that, by induction, $a_{i_{k+1}}\leq b_{i_{k+1}}:=\Sigma_{1\leq j\leq k}n_{i_j}$ for $k=1,\dots,s-1$ (and $b_{i_1}:=0$). But, on the other hand, since an easy computation shows that

$$\mathrm{h}^0(\Sigma_{n_1,\cdots,n_s},\mathcal{O}_{\Sigma_{n_1,\cdots,n_s}}(b_1,\dots,b_s))=\frac{(\sum_{i=1}^s n_i)!}{\prod_{i=1}^s(n_i)!}=\deg(\Sigma_{n_1,\cdots,n_s})$$

we are forced to have $a_{i_j}=b_{i_j}$ for $j=1,\dots,s$. □

Corollary 4.5. *$\mathcal{O}_{\Sigma_{n,m}}(a,b)$ is an initialized Ulrich line bundle on $\Sigma_{n,m}$ if and only if $(a,b)=(0,n)$ or $(m,0)$.*

It is natural to ask if we could use these initialized Ulrich line bundles as a bricks to construct initialized Ulrich bundles of higher rank. The answer strongly depends on the values of n_i. Assume for a while that $i=2$, take $n=n_1$, $m=n_2$ and assume $n\leq m$. The main difference between the case $n=1$ and $1<n$ comes from:

$$\mathrm{Ext}^1_{\Sigma_{n,m}}(\mathcal{O}(m,0),\mathcal{O}(0,n))\neq0\Leftrightarrow n=1\text{ and }m\geq2.$$

So, if $1=n<m$, we can take a non-trivial extension $0\neq e\in\mathrm{Ext}^1_{\Sigma_{n,m}}(\mathcal{O}(m,0),\mathcal{O}(0,n))$ to construct a rank 2 undecomposable Ulrich bundle $\mathcal{E}$ on $\Sigma_{n,m}$ as an extension:

$$0 \to \mathcal{O}(0,n) \to \mathcal{E} \to \mathcal{O}(m,0) \to 0.$$

Iterating the process we will be able to construct Ulrich bundles of higher rank. If $2 \leq n \leq m$ we will need an alternative construction. So, we will distinguish to cases:

1. Case 1: $2 \leq n_1, \cdots, n_s$.
2. Case 2: $1 = n_1 \leq n_2, \cdots, n_s$.

4.1. **Representation type of $\Sigma_{n_1,\cdots,n_s}$, $2 \leq n_1, \cdots, n_s$.**

The goal of this subsection is the construction of families of arbitrarily large dimension of simple (and, hence, undecomposable) Ulrich vector bundles on Segre varieties $\Sigma_{n_1,\cdots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i+1) - 1$, for $2 \leq n_1, \cdots, n_s$.

For any $2 \leq m$ and any $1 \leq a$, we denote by $\mathcal{E}_{m,a}$ any vector bundle on $\mathbb{P}^m$ given by the exact sequence

$$0 \to \mathcal{E}_{m,a} \to \mathcal{O}_{\mathbb{P}^m}(1)^{(m+2)a} \stackrel{\phi(1)}{\to} \mathcal{O}_{\mathbb{P}^m}(2)^{2a} \to 0 \tag{4.1}$$

where $\phi \in V_m$ and V_m is the non-empty open dense subset of the affine scheme $M = \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^m}^{(m+2)a}, \mathcal{O}_{\mathbb{P}^m}(1)^{2a})$ provided by Proposition 3.27.

Note that $\mathcal{E}_{m,a}$ has rank ma and in the next Proposition we summarize the properties of these vector bundles needed later:

Proposition 4.6. *With the above notation we have:*

(i)
$$\mathrm{h}^0(\mathbb{P}^m, \mathcal{E}_{m,a}(t)) = \begin{cases} 0 & \text{for } t \leq 0, \\ a((m+2)\binom{m+t+1}{m} - 2\binom{m+t+2}{m}) & \text{for } t > 0. \end{cases}$$

(ii)
$$\mathrm{h}^1(\mathbb{P}^m, \mathcal{E}_{m,a}(t)) = \begin{cases} 0 & \text{for } t < -2 \text{ or } t \geq 0, \\ am & \text{for } t = -1 \\ 2a & \text{for } t = -2. \end{cases}$$

(iii) $\mathrm{h}^i(\mathbb{P}^m, \mathcal{E}_{m,a}(t)) = 0$ *for all* $t \in \mathbb{Z}$ *and* $2 \leq i \leq m-1$.
(iv) $\mathrm{h}^m(\mathbb{P}^m, \mathcal{E}_{m,a}(t)) = 0$ *for* $t \geq -m-1$.
(v) $\mathcal{E}_{m,a}$ *is simple.*

Proof. (i) - (iv) Since $\phi \in V_m$, by Proposition 3.27, $\mathrm{H}^0(\phi(1))$ is surjective. But, since the K-vector spaces $\mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)^{(m+2)a})$ and $\mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(2)^{2a})$ have the same dimension, $\mathrm{H}^0(\phi(1))$ is an isomorphism and therefore $\mathrm{H}^0(\mathcal{E}_{m,a}) = 0$. *A fortiori*, $\mathrm{H}^0(\mathcal{E}_{m,a}(t)) = 0$ for $t \leq 0$. On the other hand, again by the surjectivity of $\mathrm{H}^0(\phi(1))$, $\mathrm{H}^1(\mathcal{E}_{m,a}) = 0$. Since it is obvious that $\mathrm{H}^i(\mathcal{E}_{m,a}(1-i)) = 0$ for $i \geq 2$ it turns out that $\mathcal{E}_{m,a}$ is 1-regular and in particular, $\mathrm{H}^1(\mathcal{E}_{m,a}(t)) = 0$ for $t \geq 0$. The rest of cohomology groups can be easily deduced from the long exact cohomology sequence associated to the exact sequence (4.1).

(v) It follows from Kac's theorem (see [Kac], Theorem 4) arguing as in [MP], Proposition 3.4 that $\mathcal{E}_{m,a}$ is simple. ∎

We are now ready to construct families of simple (hence undecomposable) Ulrich bundles on the Segre variety $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$, $2 \leq n, m$, of arbitrary high rank and dimension and to conclude that Segre varieties $\Sigma_{n,m}$ are of wild representation type. The main ingredient on the construction of simple Ulrich bundles on $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$, $2 \leq n \leq m$, will be the family of simple vector bundles $\mathcal{E}_{m,a}$ on $\mathbb{P}^m$ given by the exact sequence (4.1) as well as the vector bundles of p-holomorphic forms of $\mathbb{P}^n$, $\Omega^p_{\mathbb{P}^n} := \wedge^p \Omega^1_{\mathbb{P}^n}$, where $\Omega^1_{\mathbb{P}^n}$ is the cotangent bundle. The values of $\mathrm{h}^i(\Omega^p_{\mathbb{P}^n}(t))$ are given by the *Bott's formula* (see, for instance, [OSS], page 8).

Theorem 4.7. *Fix integers $2 \leq n \leq m$ and let $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$ be the Segre variety. For any integer $a \geq 1$ there exists a family of dimension $a^2(m^2+2m-4)+1$ of initialized simple Ulrich vector bundles $\mathcal{F} := \Omega^{n-2}_{\mathbb{P}^n}(n-1) \boxtimes \mathcal{E}_{m,a}(n-1)$ of rank $am\binom{n}{2}$.*

Proof. Let $\mathcal{F}$ be the vector bundle $\Omega^{n-2}_{\mathbb{P}^n}(n-1) \boxtimes \mathcal{E}_{m,a}(n-1)$ for $\mathcal{E}_{m,a}$ a general vector bundle obtained on $\mathbb{P}^m$ from the exact sequence (4.1). The first goal is to prove that $\mathcal{F}$ is ACM, namely, we should show that $\mathrm{H}^i(\Sigma_{n,m}, \mathcal{F} \otimes \mathcal{O}_{\Sigma_{n,m}}(t,t)) = 0$ for $1 \leq i \leq n+m-1$ and $t \in \mathbb{Z}$. By Künneth's formula

$$\mathrm{H}^i(\Sigma_{n,m}, \mathcal{F} \otimes \mathcal{O}_{\Sigma_{n,m}}(t,t)) = \bigoplus_{p+q=i} \mathrm{H}^p(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1+t)) \otimes \mathrm{H}^q(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1+t)). \tag{4.2}$$

According to Bott's formula the only non-zero cohomology groups of $\Omega^{n-2}_{\mathbb{P}^n}(n-1+t)$ are:

$$\begin{array}{ll} \mathrm{H}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1+t)) & \text{for} \quad t \geq 0 \text{ and } n \geq 3 \text{ or } t \geq -1 \text{ and } n = 2, \\ \mathrm{H}^{n-2}(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1+t)) & \text{for} \quad t = -n+1, \\ \mathrm{H}^n(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1+t)) & \text{for} \quad t \leq -n-2. \end{array}$$

On the other hand, by Lemma 4.6, the only non-zero cohomology groups of $\mathcal{E}_{m,a}(n-1+t)$ are:

$$\begin{array}{ll} \mathrm{H}^0(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1+t)) & \text{for} \quad t \geq -n+2, \\ \mathrm{H}^1(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1+t)) & \text{for} \quad -n-1 \leq t \leq -n, \\ \mathrm{H}^m(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1+t)) & \text{for} \quad t \leq -n-m-1. \end{array}$$

Therefore, using (4.2), we get

$$\mathrm{H}^i(\Sigma_{n,m}, \mathcal{F} \otimes \mathcal{O}_{\Sigma_{n,m}}(t,t)) = 0 \text{ for } 1 \leq i \leq n+m-1 \text{ and } t \in \mathbb{Z}.$$

Since for $n \geq 3$ $\mathrm{H}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-2)) = 0$ and for $n = 2$ $\mathrm{H}^0(\mathbb{P}^m, \mathcal{E}_{m,a}) = 0$ (Lemma 4.6), $\mathcal{F}$ is an initialized ACM vector bundle on $\Sigma_{n,m}$. Let us compute the number of global sections. Recall that, by Bott's formula, $\mathrm{h}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1)) = \binom{n+1}{2}$. Hence:

$$\begin{array}{rcl} \mathrm{h}^0(\mathcal{F}) = \mathrm{h}^0(\Sigma_{n,m}, \Omega^{n-2}_{\mathbb{P}^n}(n-1) \boxtimes \mathcal{E}_{m,a}(n-1)) & = & \mathrm{h}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1))\, \mathrm{h}^0(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1)) \\ & = & \binom{n+1}{2} a((m+2)\binom{m+n}{m} - 2\binom{m+n+1}{m}) \\ & = & a(\frac{(m+2)(m+n)!(n+1)!}{m!n!(n-1)!2!} - \frac{2(m+n+1)!(n+1)!}{m!(n+1)!(n-1)!2!}) \\ & = & a(\frac{n!(m+n)!}{2!(n-2)!m!n!} \cdot \frac{(n+1)(m+2)-2(m+n+1)}{n-1}) \\ & = & a\binom{n}{2}\binom{m+n}{m}\frac{m(n-1)}{n-1} \\ & = & a\binom{n}{2}\binom{m+n}{m} m \\ & = & \mathrm{rk}(\mathcal{F}) \deg(\Sigma_{n,m}) \end{array}$$

where the last equality follows from the fact that $\deg(\Sigma_{n,m}) = \binom{m+n}{m}$ and $\operatorname{rk}(\mathcal{F}) = \operatorname{rk}(\mathcal{E}_{m,a})\operatorname{rk}(\Omega^{n-2}_{\mathbb{P}^n}) = am\binom{n}{2}$. Therefore, $\mathcal{F}$ is an initialized Ulrich vector bundle on $\Sigma_{n,m}$. With respect to simplicity, we need only to observe that

$$\begin{aligned} \operatorname{Hom}(\mathcal{F},\mathcal{F}) &\cong \mathrm{H}^0(\Sigma_{n,m},\mathcal{F}^\vee\otimes\mathcal{F}) \\ &\cong \mathrm{H}^0(\mathbb{P}^n,\Omega^{n-2}_{\mathbb{P}^n}(n-1)^\vee\otimes\Omega^{n-2}_{\mathbb{P}^n}(n-1)))\otimes\mathrm{H}^0(\mathbb{P}^m,\mathcal{E}_{m,a}(n-1)^\vee\otimes\mathcal{E}_{m,a}(n-1)) \end{aligned}$$

and use the fact that $\Omega^{n-2}_{\mathbb{P}^n}$ and $\mathcal{E}_{m,a}$ are both simple.

It only remains to compute the dimension of the family of simple Ulrich bundles $\mathcal{F} := \Omega^{n-2}_{\mathbb{P}^n}(n-1)\boxtimes\mathcal{E}_{m,a}(n-1)$ on $\Sigma_{n,m}$. Since they are completely determined by a general morphism $\phi\in M := \operatorname{Hom}_{\mathbb{P}^m}(\mathcal{O}^{(m+2)a}_{\mathbb{P}^m},\mathcal{O}_{\mathbb{P}^m}(1)^{2a})$, this dimension turns out to be:

$$\dim M - \dim\operatorname{Aut}(\mathcal{O}^{(m+2)a}_{\mathbb{P}^m}) - \dim\operatorname{Aut}(\mathcal{O}_{\mathbb{P}^m}(1)^{2a}) + 1 =$$

$$= 2a^2(m+2)(m+1) - a^2(m+2)^2 - 4a^2 + 1 = a^2(m^2+2m-4)+1$$

which proves what we want. ☐

Corollary 4.8. *For any integers $2\le n,m$, the Segre variety $\Sigma_{n,m}\subseteq\mathbb{P}^{nm+n+m}$ is of wild representation type.*

Notice that in Theorem 4.7 we were able to construct simple Ulrich vector bundles on $\Sigma_{n,m}\subseteq\mathbb{P}^N$ for some scattered ranks, namely for ranks of the form $am\binom{n}{2}$, $a\ge 1$. The next goal will be to construct simple Ulrich bundles on $\Sigma_{n,m}\subseteq\mathbb{P}^{nm+n+m}$, $2\le n\le m$, of the remaining ranks $r\ge m\binom{n}{2}$.

Theorem 4.9. *Fix integers $2\le n\le m$ and let $\Sigma_{n,m}\subseteq\mathbb{P}^{nm+n+m}$ be the Segre variety. For any integer $r\ge m\binom{n}{2}$, set $r = am\binom{n}{2}+l$ with $a\ge 1$ and $0\le l\le m\binom{n}{2}-1$. Then, there exists a family of dimension $a^2(m^2+2m-4)+1+l(am\binom{n+1}{2}-l)$ of simple (hence, undecomposable) initialized Ulrich vector bundles $\mathcal{G}$ on $\Sigma_{n,m}$ of rank r.*

Proof. Note that for any $r\ge m\binom{n}{2}$, there exists $a\ge 1$ and $m\binom{n}{2}-1\ge l\ge 0$, such that $r = am\binom{n}{2}+l$. For such a, consider the family $\mathcal{P}_a$ of initialized Ulrich bundles of rank $am\binom{n}{2}$ given by Theorem 4.7. Notice that

$$\dim\mathcal{P}_a = a^2(m^2+2m-4)+1.$$

Hence it is enough to consider the case $l>0$. To this end, for any $l>0$ we construct the family $\mathcal{P}_{a,l}$ of vector bundles $\mathcal{G}$ given by a non-trivial extension

$$e: 0\to\mathcal{F}\to\mathcal{G}\to\mathcal{O}_{\Sigma_{n,m}}(0,n)^l\to 0 \tag{4.3}$$

where $\mathcal{F}\in\mathcal{P}_a$ and $e := (e_1,\dots,e_l)\in\operatorname{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0,n)^l,\mathcal{F})\cong\operatorname{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{F})^l$ with $e_1,\dots,e_l$ linearly independent.

Since

$$\begin{aligned} \operatorname{ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{F}) &= \mathrm{h}^1(\Sigma_{n,m},\Omega^{n-2}_{\mathbb{P}^n}(n-1)\boxtimes\mathcal{E}(-1)) \\ &= \mathrm{h}^0(\mathbb{P}^n,\Omega^{n-2}_{\mathbb{P}^n}(n-1))\cdot\mathrm{h}^1(\mathbb{P}^m,\mathcal{E}(-1)) \\ &= \tbinom{n+1}{2}am \\ &> m\tbinom{n}{2} \end{aligned}$$

such extension exists.

It is obvious that $\mathcal{G}$, being an extension of initialized Ulrich vector bundles, is also an initialized Ulrich vector bundle. Let us see that $\mathcal{G}$ is simple, i.e., $\operatorname{Hom}(\mathcal{G},\mathcal{G})\cong K$. If we apply the functor $\operatorname{Hom}(-,\mathcal{G})$ to the exact sequence (4.3) we obtain:

$$0\to \operatorname{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0,n)^l,\mathcal{G})\to \operatorname{Hom}(\mathcal{G},\mathcal{G})\to \operatorname{Hom}(\mathcal{F},\mathcal{G}).$$

On the other hand, if we apply $\operatorname{Hom}(\mathcal{F},-)$ to the same exact sequence we have

$$0\to K\cong \operatorname{Hom}(\mathcal{F},\mathcal{F})\to \operatorname{Hom}(\mathcal{F},\mathcal{G})\to \operatorname{Hom}(\mathcal{F},\mathcal{O}_{\Sigma_{n,m}}(0,n)^l).$$

But

$$\begin{array}{rcl}\operatorname{Hom}(\mathcal{F},\mathcal{O}_{\Sigma_{n,m}}(0,n)) & \cong & \operatorname{Ext}^{n+m}(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{F}(-n-1,-m-1))\\ & \cong & \mathrm{H}^{n+m}(\Sigma_{n,m},\mathcal{F}(-n-1,-m-n-1))\\ & = & \mathrm{H}^{n}(\mathbb{P}^n,\Omega^{n-2}_{\mathbb{P}^n}(-2))\otimes \mathrm{H}^m(\mathbb{P}^m,\mathcal{E}(-m-2))=0\end{array}\tag{4.4}$$

by Serre's duality and Bott's formula. This implies that $\operatorname{Hom}(\mathcal{F},\mathcal{G})\cong K$.

Finally, using the fact that $\operatorname{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{F})\cong \mathrm{H}^0(\mathcal{F}(0,-n))=0$ and applying the functor $\operatorname{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0,n),\cdot)$ to the short exact sequence (4.3), we obtain

$$0\to \operatorname{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{G})\to \operatorname{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{O}_{\Sigma_{n,m}}(0,n)^l)\cong K^l\xrightarrow{\phi}\operatorname{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{F}).$$

Since, by construction, the image of ϕ is the subvector space generated by $e_1,\dots,e_l$ it turns out that ϕ is injective and in particular $\operatorname{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{G})=0$. Summing up, $\operatorname{Hom}(\mathcal{G},\mathcal{G})\cong K$, i.e., $\mathcal{G}$ is simple.

It only remains to compute the dimension of $\mathcal{P}_{a,l}$. Assume that there exist vector bundles $\mathcal{F},\mathcal{F}'\in\mathcal{P}_a$ giving rise to isomorphic bundles, i.e.:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F} & \xrightarrow{j_1} & \mathcal{G} & \xrightarrow{\alpha} & \mathcal{O}_{\Sigma_{n,m}}(0,n)^l & \to & 0\\ & & & & i\|\wr & & & & \\ 0 & \to & \mathcal{F}' & \xrightarrow{j_2} & \mathcal{G}' & \xrightarrow{\beta} & \mathcal{O}_{\Sigma_{n,m}}(0,n)^l & \to & 0.\end{array}$$

Since by (4.4), $\operatorname{Hom}(\ F,\mathcal{O}_{\Sigma_{n,m}}(0,n))=0$, the isomorphism i between $\mathcal{G}$ and $\mathcal{G}'$ lifts to an automorphism f of $\mathcal{O}_{\Sigma_{n,m}}(0,n)^l$ such that $f\alpha=\beta i$ which allows us to conclude that the morphism $ij_1:\mathcal{F}\longrightarrow\mathcal{G}'$ factorizes through $\mathcal{F}'$ showing up the required isomorphism from $\mathcal{F}$ to $\mathcal{F}'$.

Therefore, since $\dim\operatorname{Hom}(\mathcal{F},\mathcal{G})=1$, we have

$$\begin{array}{rcl}\dim\mathcal{P}_{a,l} & = & \dim\mathcal{P}_a+\dim Grass(l,\operatorname{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{F}))\\ & = & \dim\mathcal{P}_a+l\dim\operatorname{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0,n),\mathcal{F})-l^2\\ & = & a^2(m^2+2m-4)+1+l(am\binom{n+1}{2}-l).\end{array}$$

□

As a by-product of the previous results we can extend the construction of simple Ulrich bundles on $\Sigma_{n,m}$, $n\geq 2$, to the case of Segre embeddings of more than two factors and get:

Theorem 4.10. *Fix integers $2\leq n_1\leq\cdots\leq n_s$ and let $\Sigma_{n_1,\dots,n_s}\subseteq\mathbb{P}^N$, $N=\prod_{i=1}^s(n_i+1)-1$ be a Segre variety. For any integer $r\geq n_2\binom{n_1}{2}$, set $r=an_2\binom{n_1}{2}+l$ with $a\geq 1$ and $0\leq l\leq n_2\binom{n_1}{2}-1$. Then there exists a family of dimension $a^2(n_2^2+2n_2-4)+1+l(an_2\binom{n_1+1}{2}-l)$*

of simple (hence, undecomposable) initialized Ulrich vector bundles on $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$ *of rank* r.

Proof. By Theorem 4.7 we can suppose that $s \geq 3$. Therefore, by [EFW], Proposition 2.6, the vector bundle of the form $\mathcal{H} := \mathcal{G} \boxtimes \mathcal{L}(n_1 + n_2)$, for $\mathcal{G}$ belonging to the family constructed in Theorem 4.9 and $\mathcal{L}$ an Ulrich line bundle on $\mathbb{P}^{n_3} \times \cdots \times \mathbb{P}^{n_s}$ as constructed in Proposition 4.4, is an initialized simple Ulrich bundle. In order to show that in this way we obtain a family of the aforementioned dimension it only remains to show that whenever $\mathcal{G} \ncong \mathcal{G}'$ then $\mathcal{H} \ncong \mathcal{H}'$, or equivalently $\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}} \ncong \mathcal{G}' \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}}$. But if there exists an isomorphism

$$\phi : \mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}} \xrightarrow{\cong} \mathcal{G}' \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}}$$

$\pi_*\phi$ would also be an isomorphism between

$$\pi_*(\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}}) \cong \mathcal{G} \text{ and } \pi_*(\mathcal{G}' \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}}) \cong \mathcal{G}'$$

in contradiction with the hypothesis. □

Corollary 4.11. *For any integers* $2 \leq n_1, \cdots, n_s$, *the Segre variety* $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^s (n_i + 1) - 1$ *is of wild representation type.*

4.2. Representation type of $\Sigma_{n_1,n_2\dots,n_s}$, $1 = n_1 \leq n_2, \cdots, n_s$.

In this subsection we are going to focus our attention on the construction of simple Ulrich bundles on Segre varieties of the form $\Sigma_{n_1,n_2\dots,n_s} \subseteq \mathbb{P}^N$ for either $n_1 = 1$ and $s \geq 3$ or $n_1 = 1$ and $n_2 \geq 2$. We are going to show that they also are of wild representation type. Opposite to the Segre varieties that we studied in the previous subsection, the Ulrich bundles on $\Sigma_{1,n_2\dots,n_s} \subseteq \mathbb{P}^N$, $N = 2\prod_{i=2}^s (n_i + 1) - 1$, will not be obtained as products of vector bundles constructed on each factor, but they will be obtained directly as iterated extensions.

Theorem 4.12. *Let* $X := \Sigma_{1,n_2\dots,n_s} \subseteq \mathbb{P}^N$ *for either* $s \geq 3$ *or* $n_2 \geq 2$. *Let* r *be an integer,* $2 \leq r \leq (\Sigma_{i=2}^s n_i - 1)\prod_{i=2}^s (n_i + 1)$. *Then:*

(i) *There exists a family* Λ_r *of rank* r *initialized simple Ulrich vector bundles* $\mathcal{E}$ *on* X *given by nontrivial extensions*

$$0 \to \mathcal{O}_X(0, 1, 1 + n_2, \dots, 1 + \Sigma_{i=2}^{s-1} n_i) \to \mathcal{E} \to \mathcal{O}_X(\Sigma_{i=2}^s n_i, 0, n_2, \dots, \Sigma_{i=2}^{s-1} n_i)^{r-1} \to 0 \tag{4.5}$$

with first Chern class $c_1(\mathcal{E}) = ((r-1)\Sigma_{i=2}^s n_i, 1, 1 + rn_2, \dots, 1 + r(\Sigma_{i=2}^{s-1} n_i))$.

(ii) *There exists a family* Γ_r *of rank* r *initialized simple Ulrich vector bundles* $\mathcal{F}$ *on* X *given by nontrivial extensions*

(4.6)
$$0 \to \mathcal{O}_X(0, 1+n_3, 1, 1+n_2+n_3, \dots, 1+\Sigma_{i=2}^{s-1} n_i) \to \mathcal{F} \to \mathcal{O}_X(\Sigma_{i=2}^s n_i, n_3, 0, n_2+n_3, \dots, \Sigma_{i=2}^{s-1} n_i)^{r-1} \to 0$$

with first Chern class $c_1(\mathcal{F}) = ((r-1)\Sigma_{i=2}^s n_i, 1 + rn_3, 1, \dots, 1 + r(\Sigma_{i=2}^{s-1} n_i))$.

Proof. To simplify we set

$$\begin{aligned}
\mathcal{A} &:= \mathcal{O}_X(0, 1, 1 + n_2, \dots, 1 + \Sigma_{i=2}^{s-1} n_i),\\
\mathcal{B} &:= \mathcal{O}_X(\Sigma_{i=2}^s n_i, 0, n_2, \dots, \Sigma_{i=2}^{s-1} n_i),\\
\mathcal{C} &:= \mathcal{O}_X(0, 1 + n_3, 1, 1 + n_2 + n_3, \dots, 1 + \Sigma_{i=2}^{s-1} n_i), \text{ and}\\
\mathcal{D} &:= \mathcal{O}_X(\Sigma_{i=2}^s n_i, n_3, 0, n_2 + n_3, \dots, \Sigma_{i=2}^{s-1} n_i).
\end{aligned}$$

We are going to give the details of the proof of statement (i) since statement (ii) is proved analogously. Recall that by Proposition 4.4, $\mathcal{A}$ and $\mathcal{B}$ are initialized Ulrich line bundles on X. On the other hand, the dimension of $\operatorname{Ext}^1(\mathcal{B},\mathcal{A})$ can be computed as:

$$\begin{aligned}\dim \operatorname{Ext}^1(\mathcal{B},\mathcal{A}) &= \mathrm{h}^1(X,\mathcal{O}_X(-\Sigma_{i=2}^s n_i,1,\dots,1))\\ &= \mathrm{h}^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-\Sigma_{i=2}^s n_i))\prod_{i=2}^s \mathrm{h}^0(\mathbb{P}^{n_i},\mathcal{O}_{\mathbb{P}^{n_i}}(1))\\ &= (\Sigma_{i=2}^s n_i-1)\prod_{i=2}^s (n_i+1).\end{aligned}$$

So, exactly as in the proof of Theorem 4.9, if we take l ($l=r-1$) linearly independent elements $e_1,\dots,e_l$ in $\operatorname{Ext}^1(\mathcal{B},\mathcal{A})$, $1\le l\le (\Sigma_{i=2}^s n_i-1)\prod_{i=2}^s(n_i+1)-1$, these elements provide with an element $e:=(e_1,\dots,e_l)$ of $\operatorname{Ext}^1(\mathcal{B}^l,\mathcal{A})\cong \operatorname{Ext}^1(\mathcal{B},\mathcal{A})^l$. Then the associated extension

$$0\longrightarrow \mathcal{A}\longrightarrow \mathcal{E}\longrightarrow \mathcal{B}^l\longrightarrow 0 \tag{4.7}$$

gives a rank $l+1$ initialized simple Ulrich vector bundle. □

Remark 4.13. (i) With the same technique, using other initialized Ulrich line bundles, it is possible to construct initialized simple Ulrich bundles of ranks covered by Theorem 4.12 with different first Chern class.

(ii) Notice that for $s=2$, we have constructed rank r simple Ulrich vector bundles on $\Sigma_{1,m}\subseteq\mathbb{P}^{2m+1}$, $r\le m^2$ as extensions of the form:

$$0\longrightarrow \mathcal{O}_{\Sigma_{1,m}}(0,1)\longrightarrow\mathcal{E}\longrightarrow\mathcal{O}_{\Sigma_{1,m}}(m,0)^{r-1}\longrightarrow 0.$$

Lemma 4.14. *Consider the Segre variety $\Sigma_{1,n_2\dots,n_s}\subseteq\mathbb{P}^N$ for either $s\ge 3$ or $n_2\ge 2$ and keep the notation introduced in Theorem 4.12. We have:*

(i) *For any two non-isomorphic rank 2 initialized Ulrich bundles $\mathcal{E}$ and $\mathcal{E}'$ from the family Λ_2 obtained from the exact sequence (4.5), it holds that* $\operatorname{Hom}(\mathcal{E},\mathcal{E}')=0$. *Moreover, the set of non-isomorphic classes of elements of Λ_2 is parameterized by*

$$\mathbb{P}(\operatorname{Ext}^1(\mathcal{B},\mathcal{A}))\cong\mathbb{P}(\mathrm{H}^1(\Sigma_{1,n_2\dots,n_s},\mathcal{O}_{\Sigma_{1,n_2\dots,n_s}}(-\sum_{i=2}^s n_i,1,\cdots,1)))$$

and, in particular, it has dimension $(\Sigma_{i=2}^s n_i-1)\prod_{i=2}^s(n_i+1)-1$.

(ii) *For any pair of bundles $\mathcal{E}\in\Lambda_2$ and $\mathcal{F}\in\Gamma_3$ obtained from the exact sequences (4.5) and (4.6), it holds that* $\operatorname{Hom}(\mathcal{E},\mathcal{F})=0$ *and* $\operatorname{Hom}(\mathcal{F},\mathcal{E})=0$.

Proof. The first statement is a direct consequence of Proposition [PT], Proposition 5.1.3. Regarding the second statement, it is a straightforward computation applying the functors $\operatorname{Hom}(\mathcal{F},-)$ and $\operatorname{Hom}(\mathcal{E},-)$ to the short exact sequences (4.5) and (4.6) respectively, and taking into account that there are no nontrivial morphisms among the vector bundles $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$. □

In the next Theorem we are going to construct families of increasing dimension of simple Ulrich bundles for arbitrary large rank on the Segre variety $\Sigma_{1,n_2\dots,n_s}$. In case $s\ge 3$ we can use the two distinct families of rank 2 and rank 3 Ulrich bundles obtained in Theorem 4.12 to cover all the possible ranks. However, when $s=2$, since there exists just a unique family, we will have to restraint ourselves to construct Ulrich bundles of arbitrary even rank. In any case, it will be enough to conclude that these Segre varieties are of wild representation type.

Theorem 4.15. *Consider the Segre variety $\Sigma_{1,n_2\dots,n_s}\subseteq\mathbb{P}^N$ for either $s\geq 3$ or $n_2\geq 2$.*

(i) *Then for any $r=2t$, $t\geq 2$, there exists a family of dimension*

$$(2t-1)(\Sigma_{i=2}^s n_i-1)\prod_{i=2}^s(n_i+1)-3(t-1)$$

of initialized simple Ulrich vector bundles of rank r.

(ii) *Let us suppose that $s\geq 3$ and $n_2=1$. Then for any $r=2t+1$, $t\geq 2$, there exists a family of dimension $\geq (t-1)((\sum_{i=2}^s n_i-1)(n_3+2)\prod_{i=4}^s(n_i+1)-1)$ of initialized simple Ulrich vector bundles of rank r.*

(iii) *Let us suppose that $s\geq 3$ and $n_2>1$. For any integer $r=an_3\binom{n_2}{2}+l\geq n_3\binom{n_2}{2}$ with $a\geq 1$ and $0\leq l\leq n_3\binom{n_2}{2}-1$, there exists a family of dimension $a^2(n_3^2+2n_3-4)+1+l(an_3\binom{n_2+1}{2}-l)$ of simple (hence, undecomposable) initialized Ulrich vector bundles of rank r.*

Proof. (i) Let $r=2t$ be an even integer and set

$$a:=\operatorname{ext}^1(\mathcal{B},\mathcal{A})=(\Sigma_{i=2}^s n_i-1)\prod_{i=2}^s(n_i+1)$$

with $\mathcal{A}$ and $\mathcal{B}$ defined as in the proof of Theorem 4.12. Denote by U the open subset of $\mathbb{P}^a\times\overset{t)}{\cdots}\times\mathbb{P}^a$, $\mathbb{P}^a\cong\mathbb{P}(\operatorname{Ext}^1(\mathcal{B},\mathcal{A}))\cong\Lambda_2$, parameterizing closed points $[\mathcal{E}_1,\cdots,\mathcal{E}_t]\in\mathbb{P}^a\times\overset{t)}{\cdots}\times\mathbb{P}^a$ such that $\mathcal{E}_i\not\cong\mathcal{E}_j$ for $i\neq j$ (i.e. U is $\mathbb{P}^a\times\overset{t)}{\cdots}\times\mathbb{P}^a$ minus the small diagonals). Given $[\mathcal{E}_1,\cdots,\mathcal{E}_t]\in U$, by Lemma 4.14, the set of vector bundles $\mathcal{E}_1,\cdots,\mathcal{E}_t$ satisfy the hypothesis of Proposition [PT], Proposition 5.1.3 and therefore, there exists a family of rank r simple Ulrich vector bundles $\mathcal{E}$ parameterized by

$$\mathbb{P}(\operatorname{Ext}^1(\mathcal{E}_t,\mathcal{E}_1))\times\cdots\times\mathbb{P}(\operatorname{Ext}^1(\mathcal{E}_t,\mathcal{E}_{t-1}))$$

and given as extensions of the form

$$0\longrightarrow\oplus_{i=1}^{t-1}\mathcal{E}_i\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}_t\longrightarrow 0.$$

Next we observe that if we consider $[\mathcal{E}_1,\cdots,\mathcal{E}_t]\neq[\mathcal{E}'_1,\cdots,\mathcal{E}'_t]\in U$ and the corresponding extensions

$$0\longrightarrow\oplus_{i=1}^{t-1}\mathcal{E}_i\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}_t\longrightarrow 0,$$

and

$$0\longrightarrow\oplus_{i=1}^{t-1}\mathcal{E}'_i\longrightarrow\mathcal{E}'\longrightarrow\mathcal{E}'_t\longrightarrow 0$$

then $\operatorname{Hom}(\mathcal{E},\mathcal{E}')=0$ and in particular $\mathcal{E}\not\cong\mathcal{E}'$. Therefore, we have a family of non-isomorphic rank r simple Ulrich vector bundles $\mathcal{E}$ on $\Sigma_{1,n_2\dots,n_s}$ parameterized by a projective bundle $\mathbb{P}$ over U of dimension

$$\dim\mathbb{P}=(t-1)\dim(\mathbb{P}(\operatorname{Ext}^1(\mathcal{E}_t,\mathcal{E}_1)))+\dim U.$$

Applying the functor $\operatorname{Hom}(-,\mathcal{E}_1)$ to the short exact sequence (4.5) we obtain:

$$0\longrightarrow\operatorname{Hom}(\mathcal{A},\mathcal{E}_1)\cong K\longrightarrow\operatorname{Ext}^1(\mathcal{B},\mathcal{E}_1)\longrightarrow\operatorname{Ext}^1(\mathcal{E}_t,\mathcal{E}_1)\longrightarrow\operatorname{Ext}^1(\mathcal{A},\mathcal{E}_1)=0.$$

On the other hand, applying $\operatorname{Hom}(\mathcal{B},-)$ to the same exact sequence we have

$$0=\operatorname{Hom}(\mathcal{B},\mathcal{E}_1)\longrightarrow\operatorname{Hom}(\mathcal{B},\mathcal{B})\cong K\longrightarrow\operatorname{Ext}^1(\mathcal{B},\mathcal{A})\cong K^a\longrightarrow\operatorname{Ext}^1(\mathcal{B},\mathcal{E}_1)\longrightarrow\operatorname{Ext}^1(\mathcal{B},\mathcal{B})=0.$$

Summing up, we obtain $\mathrm{ext}^1(\mathcal{E}_t,\mathcal{E}_1)=a-2$ and so

$$\dim \mathbb{P}=(t-1)(a-3)+ta=(2t-1)a-3(t-1).$$

(ii) Now, let us suppose that $s\geq 3$ and $n_2=1$ and take $r=2t+1$, $t\geq 2$. Let $\mathcal{E}_1,\dots,\mathcal{E}_{t-1}$ be $t-1$ non-isomorphic rank 2 Ulrich vector bundles from the exact sequence (4.5) and let $\mathcal{F}$ be a rank 3 Ulrich bundle from the exact sequence (4.6). Again, by Lemma 4.14, this set of vector bundles satisfies the hypothesis of [PT], Proposition 5.1.3 and therefore, there exists a family $\mathbb{G}$ of rank r simple Ulrich vector bundles $\mathcal{E}$ parameterized by

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_1,\mathcal{F}))\times\cdots\times\mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_{t-1},\mathcal{F}))$$

and given as extensions of the form

$$0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow\oplus_{i=1}^{t-1}\mathcal{E}_i\longrightarrow 0.$$

It only remains to compute the dimension of the family

$$\dim\mathbb{G}=(t-1)\dim(\mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_1,\mathcal{F}))).$$

Let us fix the notation

$$\begin{aligned} b:=\mathrm{ext}^1(\mathcal{B},\mathcal{C}) &= \mathrm{h}^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-\textstyle\sum_{i=2}^s n_i))\,\mathrm{h}^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1+n_3))\prod_{i=4}^s \mathrm{h}^0(\mathbb{P}^{n_i},\mathcal{O}_{\mathbb{P}^{n_i}}(1)) \\ &= (\textstyle\sum_{i=2}^s n_i-1)(n_3+2)\prod_{i=4}^s(n_i+1). \end{aligned}$$

Applying the functor $\mathrm{Hom}(-,\mathcal{F})$ to the short exact sequence (4.5) we obtain:

$$0=\mathrm{Hom}(\mathcal{A},\mathcal{F})\longrightarrow\mathrm{Ext}^1(\mathcal{B},\mathcal{F})\longrightarrow\mathrm{Ext}^1(\mathcal{E}_1,\mathcal{F})\longrightarrow\mathrm{Ext}^1(\mathcal{A},\mathcal{F}).$$

On the other hand, applying $\mathrm{Hom}(\mathcal{B},-)$ to the short exact sequence (4.6) we have

$$0=\mathrm{Hom}(\mathcal{B},\mathcal{D})\longrightarrow\mathrm{Ext}^1(\mathcal{B},\mathcal{C})\cong K^b\longrightarrow\mathrm{Ext}^1(\mathcal{B},\mathcal{F})\longrightarrow\mathrm{Ext}^1(\mathcal{B},\mathcal{D})=0.$$

Summing up, we obtain $\mathrm{ext}^1(\mathcal{E}_1,\mathcal{F})\geq b$ and therefore $\dim\mathbb{G}\geq(t-1)(b-1)$.

(iii) It follows from Theorem 4.9 and [EFW], Proposition 2.6. □

Corollary 4.16. *The Segre variety $\Sigma_{1,n_2\dots,n_s}\subseteq\mathbb{P}^N$, $N=2\prod_{i=2}^s(n_i+1)-1$, for $s\geq 3$ or $s=2$ and $n_2\geq 2$ is of wild representation type.*

Putting together Corollaries 4.8, 4.11 and 4.16, we get

Theorem 4.17. *All Segre varieties $\Sigma_{n_1,n_2\dots,n_s}\subseteq\mathbb{P}^N$, $N=\prod_{i=1}^s(n_i+1)-1$, are of wild representation type unless the quadric surface in $\mathbb{P}^3$ (which is of finite representation type).*

Slightly generalizing the arguments of this section we can extend the last Theorem and determine the representation type of any non-singular rational normal scroll. Scrolls are fascinating varieties which have been largely studied in Algebraic Geometry. Let us recall one of their possible definitions. To this end, we fix $\mathcal{E}=\oplus_{i=0}^k\mathcal{O}_{\mathbb{P}^1}(a_i)$ a rank $k+1$ vector bundle on $\mathbb{P}^1$, where $0\leq a_0\leq\ldots\leq a_k$, and $a_k>0$. Let $\mathbb{P}(\mathcal{E})=\mathbb{P}(Sym(\mathcal{E}))\xrightarrow{\pi}\mathbb{P}^1$ be the projectivized vector bundle and let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be its tautological line bundle. Then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is generated by global sections and defines a birational map $\mathbb{P}(\mathcal{E})\longrightarrow\mathbb{P}^N$, $N=\sum_{i=0}^k a_i+k$. We write $S(\mathcal{E})$ or $S(a_0,\dots,a_k)$ for the image of this map, which is a variety of dimension $k+1$ and degree $c:=\sum_{i=0}^k a_i$.

Definition 4.18. A *rational normal scroll* is one of these varieties $S(\mathcal{E})$; i.e. it is the image of the map

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^k \longrightarrow \mathbb{P}^N$$

given by

$$\sigma(x,y;t_0,t_1\cdots,t_k) := (x^{a_0}t_0, x^{a_0-1}yt_0,\cdots,y^{a_0}t_0,,\cdots,x^{a_k}t_k,x^{a_k-1}yt_k,\cdots,y^{a_k}t_k)$$

where $0 \le a_0 \le \ldots \le a_k$, and $a_k > 0$.

The most familiar examples of rational normal scrolls are $\mathbb{P}^d$, which is $S(0,\ldots,0,1)$, the rational normal curve $S(a)$ of degree a in $\mathbb{P}^a$, the quadric $S(1,1) \subset \mathbb{P}^3$ and the cubic scroll $S(1,2) \subset \mathbb{P}^4$.

There is a beautiful geometric description of rational normal scrolls. In $\mathbb{P}^N$, take $k+1$ complementary linear spaces $L_i \cong \mathbb{P}^{a_i}$ with $0 \le a_0 \le \ldots \le a_k$, and $a_k > 0$. In each L_i choose a rational normal curve C_{a_i} and an isomorphism $\phi_i : \mathbb{P}^1 \longrightarrow C_{a_i}$ (ϕ_i is constant when $a_i = 0$). Then the variety

$$S(a_0,\ldots,a_k) = \bigcup_{p\in\mathbb{P}^1} \langle \phi_0(p),\cdots,\phi_k(p)\rangle \subset \mathbb{P}^N$$

is a rational normal scroll of dimension $k+1$ and degree $c := \sum_{i=0}^k a_i$ in $\mathbb{P}^{c+k}$. Notice that rational normal scrolls are varieties of minimal degree.

This geometric description will allow us to describe the homogeneous ideal of $S(a_0,\ldots,a_k)$. Indeed, if $S(a_0,\ldots,a_k) \subset \mathbb{P}^N$, $N = \sum_{i=0}^k a_i + k$ is a rational normal scroll defined by rational normal curves $C_{a_i} \subset L_i \cong \mathbb{P}^{a_i}$, we choose coordinates $X_0^0,\cdots,X_{a_0}^0,\cdots,X_0^k,\cdots,X_{a_k}^k$ in $\mathbb{P}^N$ such that $X_0^i,\cdots,X_{a_i}^i$ are homogeneous coordinates in L_i. Then, we consider the $2\times c$ matrix with two rows and $k+1$ catalecticant blocks

$$M_{a_0,\cdots,a_k} := \begin{pmatrix} X_0^0 & \cdots & X_{a_0-1}^0 & \cdots & X_0^k & \cdots & X_{a_k-1}^k \\ X_1^0 & \cdots & X_{a_0}^0 & \cdots & X_1^k & \cdots & X_{a_k}^k \end{pmatrix}.$$

It is well known that the ideal of $S(a_0,\ldots,a_k)$ is generated by the maximal minors of $M_{a_0,\cdots,a_k}$ and we have:

Proposition 4.19. *Let $S(a_0,\ldots,a_k) \subset \mathbb{P}^N$ with $N = \sum_{i=0}^k a_i + k$, $0 \le a_0 \le \ldots \le a_k$, and $a_k > 0$ be a rational normal scroll. Set $c := \sum_{i=0}^k a_i$. It holds:*

- (i) $\dim(S(a_0,\ldots,a_k)) = k+1$ *and* $\deg(S(a_0,\ldots,a_k)) = \sum_{i=0}^k a_i$.
- (ii) *$S(a_0,\ldots,a_k)$ is ACM and $I(S(a_0,\ldots,a_k))$ is generated by $\binom{c}{2}$ hyperquadrics.*
- (iii) *$S(a_0,\ldots,a_k)$ is non-singular if and only if $a_0 > 0$ (so, $a_i > 0$ for all $0 \le i \le k$) or $S(a_0,\ldots,a_k) = S(0,\cdots,0,1) \cong \mathbb{P}^k$.*

Since we are not interested in $\mathbb{P}^k$ (according to Horrocks Theorem there is, up to twist, only one ACM bundle in $\mathbb{P}^k$, namely, $\mathcal{O}_{\mathbb{P}^k}$) and we will only deal with non-singular rational scrolls, we will assume $0 < a_i$, $0 \le i \le k$. It holds

Theorem 4.20. *All rational normal scrolls $S(a_0,\cdots,a_k) \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^s (n_i+1) - 1$, are of wild representation type unless $\mathbb{P}^{k+1} = S(0,\cdots,0,1)$, the rational normal curve $S(a)$ in $\mathbb{P}^a$, the quadric surface $S(1,1)$ in $\mathbb{P}^3$ and the cubic scroll $(S(1,2)$ in $\mathbb{P}^4$ which are of finite representation type.*

Proof. See [MR13], Theorem 3.8. □

5. Does the representation type of a projective variety depends on the polarization?

The representation type of an ACM variety $X \subset \mathbb{P}^n$ strongly depends on the chosen embedding and the goal of this section will be to prove that on an ACM projective variety $X \subset \mathbb{P}^n$ there always exists a very ample line bundle $\mathcal{L}$ on X which naturally embeds X in $\mathbb{P}^{\mathrm{h}^0(X,\mathcal{L})-1}$ as a variety of wild representation type (cf. Theorem 5.4). As immediate consequence we will have many new examples of ACM varieties of wild representation type.

Let us start with a precise example to illustrate such phenomena.

Example 5.1. (1) The Segre product of two lines naturally embedded in $\mathbb{P}^3$ is an example of ACM surface of finite representation type, i.e., $\varphi_{|\mathcal{O}(1,1)|} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is a variety of finite representation type. Indeed, according to Knörrer any hyperquadric $Q_n \subset \mathbb{P}^{n+1}$ is of finite representation type ([Kn]) and, up to twist, the only undecomposable ACM bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ are: $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$, $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)$.

(2) The Segre product of two smooth conics naturally embedded in $\mathbb{P}^8$ is an example of variety of wild representation type, i.e., $\varphi_{|\mathcal{O}(2,2)|} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^8$ is an example of ACM surface of wild representation type. Indeed, any smooth del Pezzo surface is of wild representation type (see Theorem 3.36).

(3) The Segre product of a line and a smooth conic naturally embedded in $\mathbb{P}^5$ is an example of smooth ACM surface of tame representation type, i.e., $\varphi_{|\mathcal{O}(1,2)|} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ is a variety of tame representation type. Indeed, all continuous families of undecomposable ACM bundles are one-dimensional. (see Theorem [FM], Theorem 1)

This leads to the following problems

Problem 5.2. (1) Given an ACM variety $X \subset \mathbb{P}^n$, is there an integer N_X such that X can be embedded in $\mathbb{P}^{N_X}$ as a variety of wild representation type?

(2) If so, what is the smallest possible integer N_X?

We will answer affirmatively Problem 5.2 (1) and provide an upper bound for N_X. In other words, we will prove that for any smooth ACM projective variety $X \subset \mathbb{P}^n$ there is an embedding of X into a projective space $\mathbb{P}^{N_X}$ such that the corresponding homogeneous coordinate ring has arbitrary large families of non-isomorphic undecomposable graded Maximal Cohen-Macaulay modules. Actually, it is proved that such an embedding can be obtained as the composition of the "original" embedding $X \subset \mathbb{P}^n$ and the Veronese 3-uple embedding $\nu_3 : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+3}{3}-1}$. The idea will be to construct on any ACM variety $X \subset \mathbb{P}^n$ of dimension $d \geq 2$ irreducible families $\mathcal{F}$ of vector bundles $\mathcal{E}$ of arbitrarily high rank and dimension with the extra feature that any $\mathcal{E} \in \mathcal{F}$ satisfy $\mathrm{H}^i(X, \mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $2 \leq i \leq d-1$ and $\mathrm{H}^1(X, \mathcal{E}(t)) = 0$ for all $t \neq -1, -2$. Therefore, X embedded in $\mathbb{P}^{\mathrm{h}^0(\mathcal{O}_X(s))-1}$ through the very ample line bundle $\mathcal{O}_X(s)$, $s \geq 3$, is of wild representation type.

Let X will be a smooth ACM variety of dimension $d \geq 2$ in $\mathbb{P}^n$ with a minimal free R-resolution of the following type:

$$0 \longrightarrow F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow R_X \longrightarrow 0 \tag{5.1}$$

with $c = n - d$, $F_0 = R$ and $F_i = \oplus_{j=1}^{\beta_i} R(-n_j^i)$, $1 \leq i \leq c$.

For any $2 \leq n$ and any $1 \leq a$, we denote by $\mathcal{E}_{n,a}$ any vector bundle on $\mathbb{P}^n$ given by the exact sequence

$$0 \to \mathcal{E}_{n,a} \to \mathcal{O}_{\mathbb{P}^n}(1)^{(n+2)a} \stackrel{\phi(1)}{\to} \mathcal{O}_{\mathbb{P}^n}(2)^{2a} \to 0 \tag{5.2}$$

where $\phi \in V_n$ being V_n the non-empty open dense subset of the affine scheme $M = \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^n}(1)^{(n+2)a}, \mathcal{O}_{\mathbb{P}^n}(2)^{2a})$ provided by Proposition 3.27.

From now on, for any $2 \leq n$ and any $1 \leq a$, we call $\mathcal{F}_{n,a}^X$ the non-empty irreducible family of *general* rank na vector bundles $\mathcal{E}$ on $X \subset \mathbb{P}^n$ sitting in an exact sequence of the following type:

$$0 \to \mathcal{E} \to \mathcal{O}_X(1)^{(n+2)a} \stackrel{f}{\to} \mathcal{O}_X(2)^{2a} \to 0. \tag{5.3}$$

Proposition 5.3. *Let $X \subset \mathbb{P}^n$ be a smooth ACM variety of dimension $d \geq 2$. With the above notation, we have:*

(1) *A general vector bundle $\mathcal{E} \in \mathcal{F}_{n,a}^X$ satisfies*

$$\begin{array}{ll} \mathrm{H}^i_* \mathcal{E} = 0 & \textit{for } 2 \leq i \leq d-1, \\ \mathrm{H}^1(X, \mathcal{E}(t)) = 0 & \textit{for } t \neq -1, -2. \end{array}$$

(2) *A general vector bundle $\mathcal{E} \in \mathcal{F}_{n,a}^X$ is simple.*

(3) *$\mathcal{F}_{n,a}^X$ is a non-empty irreducible family of dimension $a^2(n^2+2n-4)+1$ of simple (hence undecomposable) rank an vector bundles on X.*

Proof. (1) Since $\mathrm{H}^i(X, \mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $2 \leq i \leq d-1$, and $\mathrm{H}^1(X, \mathcal{E}(t)) = 0$ for $t \neq -1, -2$ are open conditions, it is enough to exhibit a vector bundle $\mathcal{E} \in \mathcal{F}_{n,a}^X$ verifying these vanishing. Tensoring the exact sequence (5.2) with $\mathcal{O}_X$, we get

$$0 \to \mathcal{E} := \mathcal{E}_{n,a} \otimes \mathcal{O}_X \to \mathcal{O}_X(1)^{(n+2)a} \to \mathcal{O}_X(2)^{2a} \to 0. \tag{5.4}$$

Taking cohomology, we immediately obtain $\mathrm{H}^i(X, \mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $2 \leq i \leq d-1$. On the other hand, we tensor with $\mathcal{E}_{n,a}$ the exact sequence (5.1) sheafiffied

$$\begin{array}{c} 0 \longrightarrow \oplus_{j=1}^{\beta_c} \mathcal{O}_{\mathbb{P}^n}(-n_j^c) \xrightarrow{\varphi_c} \oplus_{j=1}^{\beta_{c-1}} \mathcal{O}_{\mathbb{P}^n}(-n_j^{c-1}) \xrightarrow{\varphi_{c-1}} \\ \cdots \xrightarrow{\varphi_2} \oplus_{j=1}^{\beta_1} \mathcal{O}_{\mathbb{P}^n}(-n_j^1) \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\varphi_0} \mathcal{O}_X \longrightarrow 0 \end{array}$$

and we get

$$\begin{array}{c} 0 \longrightarrow \oplus_{j=1}^{\beta_c} \mathcal{E}_{n,a}(-n_j^c) \xrightarrow{\varphi_c} \oplus_{j=1}^{\beta_{c-1}} \mathcal{E}_{n,a}(-n_j^{c-1}) \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_{i+1}} \oplus_{j=1}^{\beta_i} \mathcal{E}_{n,a}(-n_j^i) \xrightarrow{\varphi_i} \\ \cdots \xrightarrow{\varphi_2} \oplus_{j=1}^{\beta_1} \mathcal{E}_{n,a}(-n_j^1) \xrightarrow{\varphi_1} \mathcal{E}_{n,a} \xrightarrow{\varphi_0} \mathcal{E} = \mathcal{E}_{n,a} \otimes \mathcal{O}_X \longrightarrow 0. \end{array} \tag{5.5}$$

Set $\mathcal{H}_i := \ker(\varphi_i)$, $0 \leq i \leq c-2$. Cutting the exact sequence (5.5) into short exact sequences and taking cohomology, we obtain

$$\cdots \to \mathrm{H}^1(\mathbb{P}^n, \mathcal{E}_{n,a}(t)) \to \mathrm{H}^1(X, \mathcal{E}(t)) \to \mathrm{H}^2(\mathbb{P}^n, \mathcal{H}_0(t)) \to \cdots,$$

$$\cdots \to \mathrm{H}^2(\mathbb{P}^n, \oplus_{j=1}^{\beta_1}\mathcal{E}_{n,a}(-n_j^1+t)) \to \mathrm{H}^2(\mathbb{P}^n, \mathcal{H}_0(t)) \to \mathrm{H}^3(\mathbb{P}^n, \mathcal{H}_1(t)) \to \cdots ,$$
$$\cdots$$
$$\cdots \to \mathrm{H}^{c-1}(\mathbb{P}^n, \oplus_{j=1}^{\beta_{c-2}}\mathcal{E}_{n,a}(-n_j^{c-2}+t)) \to \mathrm{H}^{c-1}(\mathbb{P}^n, \mathcal{H}_{c-3}(t)) \to \mathrm{H}^c(\mathbb{P}^n, \mathcal{H}_{c-2}(t)) \to \cdots ,$$
$$\cdots \to \mathrm{H}^{c}(\mathbb{P}^n, \oplus_{j=1}^{\beta_{c-1}}\mathcal{E}_{n,a}(-n_j^{c-1}+t)) \to \mathrm{H}^{c}(\mathbb{P}^n, \mathcal{H}_{c-2}(t)) \to \mathrm{H}^{c+1}(\mathbb{P}^n, \oplus_{j=1}^{\beta_c}\mathcal{E}_{n,a}(-n_j^{c}+t)) \to \cdots ,$$

Using Lemma 4.6, we conclude that $\mathrm{H}^1(X, \mathcal{E}(t)) = 0$ for $t \neq -1, -2$.

(2) A general vector bundle $\mathcal{E} \in \mathcal{F}_{n,a}^X$ sits in an exact sequence

$$0 \to \mathcal{E} \xrightarrow{g} \mathcal{O}_X(1)^{(n+2)a} \xrightarrow{f} \mathcal{O}_X(2)^{2a} \to 0$$

and to check that $\mathcal{E}$ is simple is equivalent to check that $\mathcal{E}^\vee$ is simple. Notice that the morphism $f^\vee : \mathcal{O}_X(-2)^{2a} \longrightarrow \mathcal{O}_X(-1)^{(n+2)a}$ appearing in the exact sequence

$$0 \to \mathcal{O}_X(-2)^{2a} \xrightarrow{f^\vee} \mathcal{O}_X(-1)^{(n+2)a} \xrightarrow{g^\vee} \mathcal{E}^\vee \to 0 \qquad (5.6)$$

is a general element of the K-vector space

$$M := \mathrm{Hom}(\mathcal{O}_X(-2)^{2a}, \mathcal{O}_X(-1)^{(n+2)a}) \cong K^{n+1} \otimes K^{2a} \otimes K^{(n+2)a}$$

because $\mathrm{Hom}(\mathcal{O}_X(-2), \mathcal{O}_X(-1)) \cong \mathrm{H}^0(X, \mathcal{O}_X(1)) \cong \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \cong K^{n+1}$. Therefore, $f^\vee : \mathcal{O}_X(-2)^{2a} \longrightarrow \mathcal{O}_X(-1)^{(n+2)a}$ is represented by a $(n+2)a \times 2a$ matrix A with entries in $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Since $\mathrm{Aut}(\mathcal{O}_X(-1)^{(n+2)a}) \cong GL((n+2)a)$ and $\mathrm{Aut}(\mathcal{O}_X(-2)^{2a}) \cong GL(2a)$, the group $GL((n+2)a) \times GL(2a)$ acts naturally on M by

$$\begin{array}{ccc} GL((n+2)a) \times GL(2a) \times M & \longrightarrow & M \\ (g_1, g_2, A) & \mapsto & g_1^{-1} A g_2. \end{array}$$

For all $A \in M$ and $\lambda \in K^*$, $(\lambda Id_{(n+2)a}, \lambda Id_{2a})$ belongs to the stabilizer of A and, hence, $\dim_K Stab(A) \geq 1$. Since $(2a)^2 + (n+2)^2a^2 - 2a(n+1)(n+2)a < 0$, it follows from [Kac], Theorem 4 that $\dim_K Stab(A) = 1$. We will now check that $\mathcal{E}^\vee$ is simple. Otherwise, there exists a non-trivial morphism $\phi : \mathcal{E}^\vee \to \mathcal{E}^\vee$ and composing with $g^\vee$ we get a morphism

$$\overline{\phi} = \phi \circ g^\vee : \mathcal{O}_X(-1)^{(n+2)a} \to \mathcal{E}^\vee.$$

Applying $\mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, -)$ to the exact sequence(5.6) and taking into account that

$$\mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{O}_X(-2)^{2a}) = \mathrm{Ext}^1(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{O}_X(-2)^{2a}) = 0$$

we obtain $\mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{O}_X(-1)^{(n+2)a}) \cong \mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{E}^\vee)$. Therefore, there is a non-trivial morphism $\widetilde{\phi} \in \mathrm{Hom}(\mathcal{O}_X(-1)^{(n+2)a}, \mathcal{O}_X(-1)^{(n+2)a})$ induced by $\overline{\phi}$ and represented by a matrix $B \neq \mu Id \in \mathrm{Mat}_{(n+2)a \times (n+2)a}(K)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-2)^{2a} & \xrightarrow{f^\vee} & \mathcal{O}_X(-1)^{(n+2)a} & \xrightarrow{g^\vee} & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow C & & \downarrow B & \searrow \overline{\phi} & \downarrow \phi & & \\ 0 & \longrightarrow & \mathcal{O}_X(-2)^{2a} & \xrightarrow{f^\vee} & \mathcal{O}_X(-1)^{(n+2)a} & \xrightarrow{g^\vee} & \mathcal{E}^\vee & \longrightarrow & 0 \end{array}$$

where $C \in \mathrm{Mat}_{2a \times 2a}(K)$ is the matrix associated to $\widetilde{\phi}_{|\mathcal{O}_X(-2)^{2a}}$. Then the pair $(C, B) \neq (\mu Id, \mu Id)$ verifies $AC = BA$. Let us consider an element $\alpha \in K$ that does not belong to the set of eigenvalues of B and C. Then the pair $(B - \alpha Id, C - \alpha Id) \in GL((n+2)a) \times$

$GL(2a)$ belongs to $Stab(f)$ and therefore $\dim_K Stab(f) > 1$ which is a contradiction. Thus, $\mathcal{E}$ is simple.

(3) It only remains to compute the dimension of $\mathcal{F}^X_{n,a}$. Since the isomorphism class of a general vector bundle $\mathcal{E} \in \mathcal{F}^X_{n,a}$ associated to a morphism $\phi \in M := \mathrm{Hom}(\mathcal{O}_X^{(n+2)a}, \mathcal{O}_X(1)^{2a})$ depends only on the orbit of ϕ under the action of $GL((n+2)a) \times GL(2a)$ on M, we have:

$$\begin{aligned}\dim \mathcal{F}^X_{n,a} &= \dim M - \dim \mathrm{Aut}(\mathcal{O}_X^{(n+2)a}) - \dim \mathrm{Aut}(\mathcal{O}_X(1)^{2a}) + 1 \\ &= 2a^2(n+2)(n+1) - a^2(n+2)^2 - 4a^2 + 1 = a^2(n^2+2n-4) + 1.\end{aligned}$$

□

As an immediate consequence of the above result we can answer affirmatively Problem 5.2(1) and provide an upper bound for N_X. Indeed, we have:

Theorem 5.4. *Let $X \subset \mathbb{P}^n$ be a smooth ACM variety of dimension $d \geq 2$. The very ample line bundle $\mathcal{O}_X(s)$, $s \geq 3$, embeds X in $\mathbb{P}^{\mathrm{h}^0(\mathcal{O}_X(s))-1}$ as a variety of wild representation type.*

Proof. See [MR14], Theorem 3.4. □

Corollary 5.5. *The smallest possible integer N_X such that X embeds as a variety of wild representation type is bounded by $N_X \leq \binom{n+3}{3} - 1$.*

Proof. See [MR14], Corollary 3.5. □

6. Open problems

In this section we collect the open problems that were mentioned in the lectures, and add some more.

1. Does Mustaţă's conjecture holds for a set of general points on a smooth surface S of degree d in $\mathbb{P}^3$?

   The answer is yes if $d = 2$ (see [GMR]) or $d = 3$ (see Theorem 3.26).

   More general, does Mustaţă's conjecture holds for a set of general points on a smooth hypersurface X of degree d in $\mathbb{P}^n$?

   To my knowledge these two problems are open.

2. Fix a projective variety $X \subset \mathbb{P}^n$. As we have seen in these notes ACM bundles on X provide a criterium to determine the complexity of X. Indeed, the complexity is studied in terms of the dimension and number of families of undecomposable ACM bundles that it supports. Mimicking an analogous trichotomy in representation theory, it was proposed a classification of ACM projective varieties as finite, tame or wild representation type. We would like to know:

   Is the trichotomy finite representation type, tame representation type and wild representation type exhaustive?

   The answer is yes for smooth ACM curves. In fact, an ACM curve is of finite representation type if its genus $g(C) = 0$, of tame representation type if $g(C) = 1$, and of wild representation type if $g(C) \geq 2$. For ACM varieties of dimension ≥ 2 the answer is not known.

3. In section 5, we have seen that the representation type of an ACM projective variety strongly depends on the embedding and we have proved that given an ACM variety $X \subset \mathbb{P}^n$, there is an integer N_X such that X can be naturally embedded in $\mathbb{P}^{N_X}$ as a variety of wild representation type. So, the following question arise in a natural way:

   Given an ACM projective variety X, what is the smallest possible N_X such that X embeds in $\mathbb{P}^{N_X}$ as a variety of wild representation type?

4. In section 4, we saw that all Segre varieties $\Sigma_{n_1,\cdots,n_s} \subset \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i + 1) - 1$ are of wild representation type unless $\mathbb{P}^1 \times \mathbb{P}^1$; it follows from section 5 that the Veronese embedding $\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$, $d \geq 3$, embeds $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+d}{d}-1}$ as a variety of wild representation type. So we are led to pose the following question:

   Let $G(k, n)$ be the Grassmannian variety which parameterizes linear subspaces of $\mathbb{P}^n = \mathbb{P}(V)$ of dimension k. Embed $G(k, n)$ into $\mathbb{P}^{\binom{n+1}{k+1}-1}$ using Plücker embedding.

   Is $G(k, n) \subset \mathbb{P}^{\binom{n+1}{k+1}-1} = \mathbb{P}(\wedge^{k+1}V)$ a variety of wild representation type?

5. In section 3, we have constructed Ulrich bundles on smooth del Pezzo surfaces and, in section 4, on Segree varieties. Nevertheless few examples of varieties supporting Ulrich sheaves are known. In [EFW], pg. 43, Eisenbud, Schreyer and Weyman leave open the following interesting problems:
   (a) Is every variety (or even scheme) $X \subset \mathbb{P}^n$ the support of an Ulrich sheaf?
   (b) If so, what is the smallest possible rank for such a sheaf?

5. In subsection 3.1, we have addressed Mustaţă's conjecture for a general set of points on a del Pezzo surface. As a main tool we have used Liaison Theory and we will end these notes with a couple of open problems/questions on this fascinating Theory.
   (a) Does any zero-dimensional scheme $Z \subset \mathbb{P}^n$ belong to the G-liaison class of a complete intersection? In other words, is it glicci?
   (b) More general, is any ACM scheme $X \subset \mathbb{P}^n$ glicci?
   (c) Find new graded R-modules invariant under G-liaison.

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