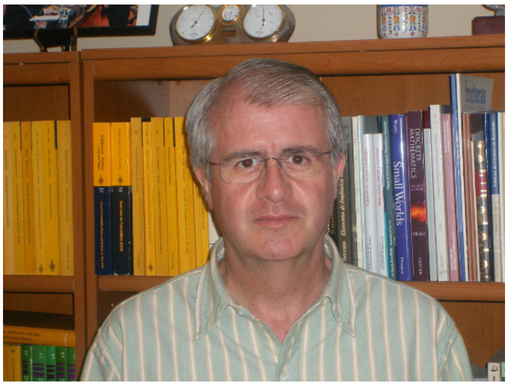
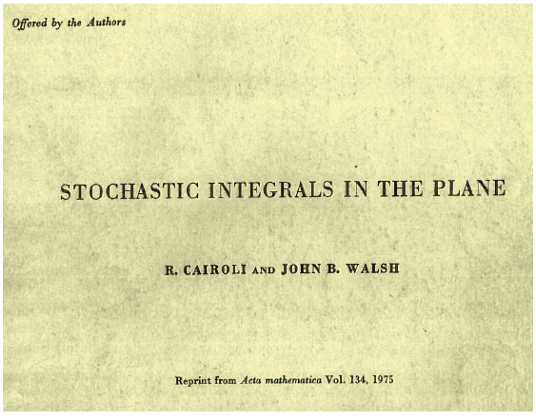


David Nualart (Barcelona, Spain, 1951) obtained his PhD in mathematics at the University of Barcelona in 1975 (advisor: Francesc Sales). He currently is Full Professor at Kansas University. His main research areas are stochastic processes and PDEs, fractional brownian motion and mathematical finance.

Stochastic integrals in the plane

DAVID NUALART*

One of the papers that has very much influenced the first steps of my mathematical career in stochastic analysis was the article on "Stochastic integrals in the plane" by Renzo Cairoli and John B. Walsh, published in *Acta Mathematica* in 1975 (see [1]). This is a very long paper, and I still keep the original reprint offered by the authors. I came across this paper during my postdoctoral stay at the "Laboratoire d'Automatique et Analyse des Systèmes", in Toulouse, in 1976, in occasion of a seminar talk given by Eugene Wong from Berkeley on multiparameter processes. At that time, being at the beginning of my career, I was interested in stochastic analysis, but I still had not found a suitable research direction. Reading this paper was a discovery for me, and I found many sources of interesting open problems and new leads to follow.

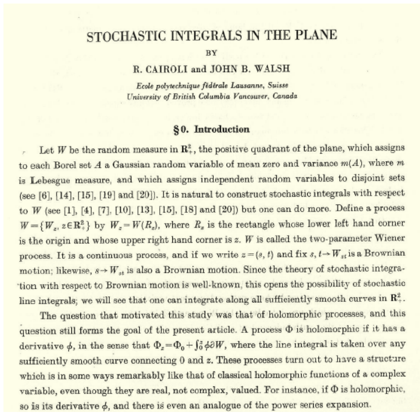


Offered by the Authors

STOCHASTIC INTEGRALS IN THE PLANE

R. CAIROLI AND JOHN B. WALSH

Reprint from *Acta mathematica* Vol. 134, 1975



STOCHASTIC INTEGRALS IN THE PLANE

BY

R. CAIROLI and JOHN B. WALSH

Ecole polytechnique fédérale Lausanne, Suisse
University of British Columbia Vancouver, Canada

§0. Introduction

Let W be the random measure in $\mathbf{R}_+^2$, the positive quadrant of the plane, which assigns to each Borel set A a Gaussian random variable of mean zero and variance $m(A)$, where m is Lebesgue measure, and which assigns independent random variables to disjoint sets (see [6], [14], [15], [19] and [20]). It is natural to construct stochastic integrals with respect to W (see [1], [4], [7], [10], [13], [15], [18] and [20]) but one can do more. Define a process $W=\{W_z, z\in\mathbf{R}_+^2\}$ by $W_z=W(R_z)$, where R_z is the rectangle whose lower left hand corner is the origin and whose upper right hand corner is z. W is called the two-parameter Wiener process. It is a continuous process, and if we write $z=(s,t)$ and fix s, $t\to W_{st}$ is a Brownian motion; likewise, $s\to W_{st}$ is also a Brownian motion. Since the theory of stochastic integration with respect to Brownian motion is well-known, this opens the possibility of stochastic line integrals; we will see that one can integrate along all sufficiently smooth curves in $\mathbf{R}_+^2$.

The question that motivated this study was that of holomorphic processes, and this question still forms the goal of the present article. A process Φ is holomorphic if it has a derivative ϕ, in the sense that $\Phi_z=\Phi_0+\int_0^z\phi\,\partial W$, where the line integral is taken over any sufficiently smooth curve connecting 0 and z. These processes turn out to have a structure which is in some ways remarkably like that of classical holomorphic functions of a complex variable, even though they are real, not complex, valued. For instance, if Φ is holomorphic, so is its derivative ϕ, and there is even an analogue of the power series expansion.

FIGURE 1: First pages of the old reprint offered by the authors with its yellowing pages that I still keep in my files.

The paper [1] is considered a fundamental work on the theory of two-parameter processes. This theory deals with stochastic processes

$$\{X_{s,t},(s,t)\in\mathbb{R}_+^2\}$$

which depend on two parameters, instead of the usual time parameter. During the 70's, and starting from the pioneering work by Cairoli and Walsh, this field was developed and got the attention of leading probabilists like Paul

*nualart@math.ku.edu.

André Meyer. An important landmark was the conference on two-parameter processes that took place in Paris in 1980, whose proceedings were published in the volume 863 of the Lecture Notes in Mathematics.

A basic ingredient in this theory is the two-parameter Brownian motion also called Brownian sheet. This is a two-parameter process

$$\{W_{s,t}, (s,t) \in \mathbb{R}_+^2\}$$

defined in a probability space $(\Omega, \mathcal{F}, P)$, which is Gaussian, with zero mean and covariance function given by

$$E(W_{s_1,t_1} W_{s_2,t_2}) = \min(s_1, s_2) \min(t_1, t_2) \, .$$

The trajectories of this process, that is, the mappings $(s,t) \mapsto W_{s,t}(\omega)$ are continuous surfaces, and for any fixed s, $t \mapsto W_{s,t}$ is a Brownian motion, and likewise, $s \mapsto W_{s,t}$ is also a Brownian motion. The purpose of the article [1] is to construct a stochastic calculus for the two-parameter Brownian motion similar to the classical Itô calculus developed by Kyoshi Itô in the 40's for the standard Brownian motion. New ingredients appear here, for instance, one can define surface integrals and also curvilinear integrals. On other hand, this calculus should be related to the theory of two-parameter martingales.

Let us introduce some basic notation of the theory of two-parameter processes. For any point $z = (s,t) \in \mathbb{R}_+^2$ we denote by R_z the rectangle $[0,s] \times [0,t]$. Also, for any $z \in \mathbb{R}_+^2$ we denote by $\mathcal{F}_z$ the σ-field generated by the random variables $\{W_\xi, \xi \in R_z\}$. We say that a process $\{\phi(z), z \in \mathbb{R}_+^2\}$ is adapted if $\phi(z)$ is $\mathcal{F}_z$-measurable for each z. The notion of predictability is stronger than adaptability and is required to define stochastic integrals. The predictable σ-field $\mathcal{P}$ of subsets of $\mathbb{R}_+^2 \times \Omega$ is generated by the sets of the form $(s,t] \times (s',t'] \times \Lambda$, where $\Lambda \in \mathcal{F}_{s,s'}$. A two-parameter process X is called predictable if the mapping $(z,\omega) \mapsto X_z(\omega)$ is measurable with respect to the predictable σ-field. These notions are similar to the one-parameter case. The main difference is the fact that the parameter space $\mathbb{R}_+^2$ is partially ordered and this creates new difficulties. In addition to the σ-fields $\mathcal{F}_z$, one can consider also the bigger σ-fields $\mathcal{F}_z^1$ and $\mathcal{F}_z^2$, generated by the random variables $\{W_{s',t}, s' \leq s\}$ and $\{W_{s,t'}, t' \leq t\}$, respectively, where $z = (s,t)$.

Given $z = (s,t) \in \mathbb{R}_+^2$, the surface stochastic integral with respect to the Brownian sheet W on the rectangle R_z

$$\int_{R_z} \phi(\xi) dW_\xi \, ,$$

is defined for processes $\{\phi(\xi), \xi \in \mathbb{R}_+^2\}$ which are predictable and square integrable, that is,

$$E\left(\int_{R_z} \phi(\xi)^2 d\xi \right) < \infty \, ,$$

for each $z \in \mathbb{R}_+^2$. This is the counterpart of the Itô integral. In the case of a process continuous in $L^2(\Omega)$, this integral is the limit in $L^2(\Omega)$ of the Riemann sums:

$$\int_{R_z} \phi(\xi) dW_\xi = \lim_{n\to\infty} \sum_{i,j=0}^{n-1} \phi(z_{i,j}) W(\Delta_{i,j}),$$

where $z_{i,j} = (is/n, jt/n)$, $\Delta_{i,j} = (z_{i,j}, z_{i+1,j+1}]$, and $W(\Delta_{i,j})$ denotes the increment of the process W on the rectangle $\Delta_{i,j}$ defined by

$$W(\Delta_{i,j}) = W_{z_{i+1,j+1}} - W_{z_{i,j+1}} - W_{z_{i+1,j}} + W_{z_{i,j}}.$$

This integral has zero expectation and satisfies the classical Itô isometry property:

$$E\Big(\Big|\int_{R_z} \phi(\xi) dW_\xi\Big|^2\Big) = E\Big(\int_{R_z} \phi(\xi)^2 d\xi\Big).$$

This is a consequence of the fact that the process W has independent increments in disjoint rectangles, and we have considered Riemann sums based on the value of the process in the lower left corner of the rectangle.

A fundamental result in Itô calculus is the Martingale Representation Theorem that asserts that any square-integrable martingale relative to the natural fields of the Brownian motion can be written as a constant plus a stochastic integral. In order to extend this result to the framework of the two-parameter Brownian motion, we need first to introduce the notion of martingale for two-parameter processes. The simplest way to do this is to use the partial ordering on the plane: $z' = (s', t') \le z = (s, t)$ if and only if $s' \le s$ and $t' \le t$. An adapted stochastic process $M = \{M_z, z \in \mathbb{R}_+^2\}$, is called a martingale if $E(|M_z|) < \infty$ for each z, and

$$E(M_z | \mathcal{F}_{z'}) = M_{z'}$$

for each $z' \le z$. It turns out that the Martingale Representation Theorem is no longer true in the framework of the two-parameter Brownian motion. More precisely, Wong and Zakai [5] proved the following result: If $M = \{M_z, z \in \mathbb{R}_+^2\}$ is a square integrable martingale, then for each $z \in \mathbb{R}_+^2$,

$$M_z = M_0 + \int_{R_z} \phi(\xi) dW_\xi + \int_{R_z \times R_z} \psi(\xi, \xi') dW_\xi dW_{\xi'},$$

where the second integral is a double stochastic integral, and the process $\psi(\xi, \xi')$ vanishes except if $\xi = (s, t)$ and $\xi' = (s', t')$ satisfy $s < s'$ and $t > t'$, is square integrable and it satisfies a suitable predictability condition.

The stochastic integrals

$$\Big\{ \int_{R_z} \phi(\xi) dW_\xi, z \in \mathbb{R}_+^2 \Big\}$$

constitute a special type of martingales, called *strong martingales*. This means that, for any $z \le z'$, the process $M_z = \int_{R_z} \phi(\xi) dW_\xi$ satisfies

$$E(M((z,z'])|\mathcal{F}_z^1 \vee \mathcal{F}_z^2) = 0,$$

where $M((z,z'])$ denotes the rectangular increment of M, and $\mathcal{F}_z^1 \vee \mathcal{F}_z^2$ denotes the σ field generated by $\mathcal{F}_z^1$ and $\mathcal{F}_z^2$. The strong martingale property of these integrals is a consequence of the fact that the two-parameter Brownian motion W has independent increments on disjoint rectangles. Furthermore, all strong square integrable martingales vanishing on the axes are stochastic integrals of the form $M_z = \int_{R_z} \phi(\xi) dW_\xi$.

The notion of *quadratic variation* plays a basic role in Itô calculus, and it is the source of the complementary terms appearing in the classical Itô formula. The quadratic variation of a one-parameter continuous process $\{X_t, t \ge 0\}$ is defined, if it exists, as the limit in probability

$$\langle X \rangle_t = \lim_{n \to \infty} \sum_{i=0}^{n-1} (X_{(i+1)t/n} - X_{it/n})^2.$$

For example, if B_t is a Brownian motion, $\langle B \rangle_t = t$. It turns out that any continuous martingale $M = \{M_t, t \ge 0\}$ has an increasing and continuous quadratic variation $\langle M \rangle_t$. Now, the restriction of a two-parameter martingale $M = \{M_z, z \in \mathbb{R}_+^2\}$ to a continuous increasing path in the plane

$$\gamma = \{\gamma(t), 0 \le t \le 1\},$$

starting at the origin, defines a one-parameter martingale

$$M^\gamma = \{M_{\gamma(t)}, 0 \le t \le 1\}$$

and we can compute its quadratic variation $\langle M^\gamma \rangle_t$. We say that a two-parameter martingale $M = \{M_z, z \in \mathbb{R}_+^2\}$ has *path-independent variation* if

$$\langle M^\gamma \rangle_1 = \langle M^{\gamma'} \rangle_1 \,,$$

for any two paths γ and γ' such that $\gamma(1) = \gamma'(1)$. This notion was introduced by Moshe Zakai. Then, strong martingales have path-independent variation, and Cairoli and Walsh said in their paper that "We have not succeeded in proving that, in general, the converse is true, that is, that each martingale with path-independent variation is a strong martingale. However, several indications let us believe that path-independence is a second characterization of the strong martingales".

This statement le me to be interested in this challenging open problem. After working for a while, I was able to prove the surprising fact that this converse result is not true, and there are path-independent variation martingales which are not strong. The construction of such martingales is very delicate and it is obtained by an approximation procedure. I was very proud of this result which I consider my first important contribution to stochastic analysis. It was published in the proceedings of the conference in Paris devoted to two-parameter processes (see [4]).

The paper by Cairoli and Walsh [1] was actually motivated by the study of holomorphic processes in the plane. A process Φ is holomorphic if it has a derivative ϕ, in the sense that

$$\Phi_z = \Phi_0 + \int_0^z \phi \partial W \,,$$

where $\int_0^z \phi \partial W$ is a line integral taken over any sufficiently smooth curve connecting $(0,0)$ and z. These processes turn out to have a structure which is in some ways remarkably like that of classical holomorphic functions of a complex variable, even though they are real. For instance, if Φ is holomorphic, so is its derivative ϕ, and there is even an analogue of the power series expansion. Some years later, in collaboration with Ely Merzbach (see [2]) using techniques of Malliavin calculus, and more precisely, the Clark-Ocone formula to represent Φ_z as a stochastic integral, I was able to obtain a condition on the Malliavin derivative of Φ that characterizes holomorphicity. This leads to a simple proof of the power series expansion of holomorphic processes.

The line integrals together with the surface integrals allowed Cairoli and Walsh to derive in [1] a Green formula for rectangles, and, as an application, to show a two-parameter version of the classical Itô formula. An immediate application of this formula was the existence and continuity of the local time for W by means of a suitable version of Tanaka's formula.

I continued working on two-parameter processes for a while, especially on regularity properties of martingales and their two-parameter quadratic variation. For instance, in [4], I was able to prove the continuity of the quadratic variation of a square-integrable two-parameter continuous martingale, which was also an open problem. As other researchers in the field, at the beginning of the eighties I shifted my research interests to other topics like stochastic partial differential equations which are also connected with multiparametric processes.

References

[1] Cairoli, R. and Walsh, J. B.: Stochastic integrals in the plane. *Acta Math.* **134** (1975), 111–183.

[2] Merzbach, E. and Nualart, D.: Generalized holomorphic processes and differentiability. *J. Theoret. Probab.* **2** (1989), 419–432.

[3] Nualart, D.: Martingales à variation indépendante du chemin. In *Two-index random processes (Paris, 1980)*, 128–148. Lecture Notes in Math. **863**. Springer, Berlin, 1981.

[4] Nualart, D.: On the quadratic variation of two-parameter continuous martingales. *Ann. Probab.* **12** (1984), 445–457.

[5] Wong, E. and Zakai, M.: Martingales and stochastic integrals for processes with a multi-dimensional parameter. *Z. Wahrschienlichkeitstheorie und verw. Gebiete* **29** (1974), 109–122.