Revista Matemática Iberoamericana
Vol. 1, n.° 1, 1985

The Atomic and Molecular Nature of Matter

Dedicated to Julie

Charles L. Fefferman

1. Introduction

The purpose of this article is to show that electrons and protons, interacting by Coulomb forces and governed by quantum statistical mechanics at suitable temperature and density, form a gas of Hydrogen atoms or molecules. Let us first recall elementary quantum statistical mechanics. (See [6].) We start with a box $\Omega \subset R^3$ and two parameters, $\beta > 0$ and μ real, related to temperature and density. The Hamiltonian for electrons $x_1 \dots x_N$ and protons $y_1 \dots y_{N'}$ in Ω is

$$H^{\Omega}_{N,N'} = -\varkappa_1 \Delta_x - \varkappa_2 \Delta_y + \sum_{j<k} |x_j - x_k|^{-1} + \sum_{j<k} |y_j - y_k|^{-1} - \sum_{j,k} |x_j - y_k|^{-1}. \tag{1.1}$$

Here $H^{\Omega}_{N,N'}$ acts on wave functions $\psi(x_1 \dots x_N, y_1 \dots y_{N'})$, antisymmetric in the x_j and y_k separately, and satisfying Dirichlet boundary conditions on $\Omega \times \Omega \times \dots \times \Omega$. (The coefficients $\varkappa_1$, $\varkappa_2$ are related to the electron and proton mass, and one has $\varkappa_2 \sim (\varkappa_1/2000)$. We pick units in which $\varkappa_1 + \varkappa_2 = 1$.)

Now $H^{\Omega}_{N,N'}$ has eigenfunctions $\psi_{N,N',1}, \psi_{N,N',2}, \dots$ with eigenvalues $E_{NN'1}, E_{NN'2}, \dots$. The basic idea of quantum statistical mechanics is to pick an (N, N', k) according to the probability law

$$\text{Prob. } (N, N', k) = \frac{\exp[(N+N')\mu - \beta E_{NN'k}]}{Z}, \tag{1.2}$$

where Z is a normalizing constant (the partition function) which makes $\text{Prob}(N, N', k)$ sum to 1.

Once we have picked a $\psi = \psi_{N,N',k}$ by (1.2), the probability density for finding the particles at given positions is

$$d\,\text{Prob} = |\psi(x_1 \dots x_N, y_1 \dots y_{N'})|^2 \, dx_1 \dots dx_N \, dy_1 \dots dy_{N'}. \tag{1.3}$$

In principle (1.2) and (1.3) give a complete probabilistic description of the particles in the box Ω under given β, μ. The natural mathematical problem is to describe how the particles behave for fixed β, μ as the box Ω grows large. Up to now, no rigorous results were known for this difficult problem. (See, however, important ideas in Lebowitz-Lieb [7] on the asymptotic behavior of the partition function Z, which is fundamental in thermodynamics.)

In this paper we introduce a new technique to understand the behavior of (1.2) and (1.3). For suitable β, μ we can show that quantum statistical mechanics leads to a dilute gas of isolated electrons and protons. Under an assumption to be explained in a moment, we prove that a different range of β, μ leads to a gas of isolated Hydrogen atoms, while a third range of β, μ gives a gas of diatomic Hydrogen molecules.

An estimate crucial for quantum statistical mechanics is stability of matter (Dyson-Lenard [1], Lieb-Thirring; see [8]), which we state in the form

$$H^{\Omega}_{N,N'} \geqslant -E_* \cdot (N + N' - 1) \quad \text{with } E_* \text{ independent of } N, N', \Omega. \tag{1.4}$$

The best value of the constant E_* in (1.4) profoundly influences the outcome of (1.2), (1.3). To get Hydrogen atoms, we need to assume:

$$\text{We can take} \quad E_* < \tfrac{1}{4} \quad \text{for} \quad N + N' > 2. \tag{1.5}$$

Estimate (1.5) is well established by experimental observation of Hydrogen crystals, but a rigorous mathematical proof will be hard to find. See Lieb [8] for the best results known so far. To get Hydrogen molecules requires an assumption even sharper than (1.5). For the rest of the paper, we discuss only the monatomic Hydrogen gas. The discussion for diatomic molecules is essentially the same, while the case of isolated particles is much easier.

In a later article [4], we generalize from Hydrogen to nucleii with higher charges.

2. Statement of the Theorem

Let us give a precise meaning to the idea of a gas of Hydrogen atoms. First of all, the particles must arrange themselves in electron-proton pairs. So for small $\epsilon > 0$ and large $R > 1$ we demand

All but at most ϵ percent of the particles come in pairs $\{x_k, y_k\}$ with x_k an electron, y_k a proton, and

$$|x_k - z|, |y_k - z| > R|x_k - y_k| \tag{2.1}$$

for any particle $z \neq x_k, y_k$.

Call such a pair $\{x_y, y_k\}$ an *atom*, and define its *displacement vector* to be $\xi_k = x_k - y_k$. We want the displacement vectors ξ_k to be distributed by the probability law $d\,\mathrm{Prob} = ce^{-|\xi|}\,d\xi$, as in the ground state of a single Hydrogen atom. Hence, for $E \subset R^3$ we demand

$$\left| \frac{\text{Number of atoms with } \xi_k \in E}{\text{Total number of atoms}} - c\int_E e^{-|\xi|}\,d\xi \right| < \epsilon. \tag{2.2}$$

Finally, we want the positions and displacement vectors of the different atoms to be nearly independent. To formulate this, let ρ = (Expected number of particles)/$|\Omega|$ be the density of the system, and subdivide Ω into a grid of congruent subcubes $\{Q_\alpha\}$ of volume comparable to $1/\rho$. Then subdivide each Q_α into two halves, Q'_α and Q''_α. For $E \subset R^3$, we study the events

ϵ'_α: Q'_α contains a single atom and nothing else; and the displacement vector for that atom lies in E.

ϵ''_α: Q''_α contains a single atom and nothing else; and the displacement vector for that atom lies in E.

Let

$$p' = \frac{\text{Number of } \alpha \text{ for which } \epsilon'_\alpha \text{ occurs}}{\text{Total number of } \alpha}$$
$$p'' = \frac{\text{Number of } \alpha \text{ for which } \epsilon''_\alpha \text{ occurs}}{\text{Total number of } \alpha}$$
$$p^* = \frac{\text{Number of } \alpha \text{ for which } \epsilon'_\alpha,\ \epsilon''_\alpha \text{ both occur}}{\text{Total number of } \alpha}.$$

Then the idea of independence of distinct atoms is expressed by

$$|p^* - p'p''| < \epsilon. \tag{2.3}$$

If (2.1), (2.2), (2.3) hold, then we have the right to say that our system is a gas of Hydrogen atoms. Under our assumption $E_* < \frac{1}{4}$, we shall prove the following.

Theorem. *Given $\epsilon > 0$ and $R > 1$, there exist μ, β so that on a large enough box Ω, we have* (2.1) *with probability at least* $(1 - \epsilon)$. *Moreover, for any $E \subset R^3$,* (2.2) *and* (2.3) *hold with probability at least* $(1 - \epsilon)$.

To prove the theorem, we shall study the range of μ, β given by $-1/4 + + \delta < \mu/\beta < -E_* - \delta$, $\beta \to \infty$, $\mu \to -\infty$. This corresponds to a temperature small compared to that required to ionize Hydrogen atoms ($\sim 10^5$ degrees K) and density small compared to that of a solid. These conditions are certainly reasonable for the study of a gas. In particular, we expect that if the density increases from zero and the temperature stays fixed and low, then we shall see first a gas of isolated electrons and protons, then a gas of Hydrogen atoms, next a gas of diatomic Hydrogen molecules, and ultimately, we leave the low-density regime. We make no attempt to derive practical values for β, μ.

Before passing to the proof of our theorem, we point out some respects in which it ought to be sharpened. For a fixed $\epsilon > 0$ and suitable β, μ, the probability that (2.1), (2.2) or (2.3) is violated should tend to zero as $|\Omega|$ tends to infinity. Our theorem states that these probabilities are at most ϵ. However, β, μ depend on ϵ, so that ϵ does not tend to zero for fixed β, μ as Ω grows. Thus, we know how the particles will look for suitable β, μ with probability 99 %, but we still have a 1 % chance of being utterly wrong, no matter how large the box may grow. I hope this defect may be soon remedied. In the same spirit, it would be interesting to show that no phase transitions occur in the range of β, μ under study. (In particular, the transition from atoms to diatomic molecules with increasing density occurs smoothly.)

Another point worth mentioning is that we have been speaking of scalar wave-functions ψ, i.e., spinless electrons. It is trivial to change our proofs to the case of spin-1/2 electrons and protons, but I hesitate to complicate matters further. Of course, the analogue of our theorem for H_2-molecules is stated in terms of spinning electrons. We could also have defined the events ϵ'_α, ϵ''_α using two different measurable sets E, F instead of a single E. There are also variants of (3) involving events $\epsilon'_\alpha, \epsilon''_\alpha, \epsilon'''_\alpha, \ldots, \epsilon^{(L)}_\alpha$ in place of ϵ'_α, ϵ''_α.

There is a small literature on thermodynamics, i.e., the behavior of $\lim_{|\Omega| \to \infty} \ln Z/|\Omega|$, for very low density. The reader should be warned that this literature is not entirely correct. See Hughes [5] for a correct discussion.

Here is a very crude summary of the way our proof works. Suppose first we look at statistical mechanics on a fixed large ball B of radius R. If B is held fixed while the temperature and density are taken very small depending on R, then it is easy to understand what will happen. In particular, for a suitable balance between density and temperature, B will most likely contain no particles at all; but if it contains something, then most likely it contains exactly one atom. This is where we use our assumption $E_* < \frac{1}{4}$.

Now take a huge box Ω, and cut Ω as in [7] into a huge number of balls

$\{B_{k\alpha}\}$ of various sizes R_k, and a negligible residual part. We shall compare the real system with a much simpler fictitious system in which all forces between particles in different $B_{k\alpha}$ are turned off. If the temperature and density are low enough, depending on the R_k, then each $B_{k\alpha}$ can be analyzed by the methods of the preceding paragraph. Since distinct $B_{k\alpha}$ do not interact in the fictitious system, the statistical mechanics of that system will be easy to understand. The point is to make the comparison with the real system. We succeed in doing this by showing that each observable (i.e., self-adjoint operator) A on the fictitious system induces an observable $\mathcal{A}$ on the real system, whose expected value $\langle \mathcal{A} \rangle$ can be estimated in terms of A. In particular, if A is «negative» in the sense that

$$\operatorname{Tr}\exp\{A + \mu(N+N') - \beta H_{\text{fictitious}}\} \leqslant \operatorname{Tr}\exp\{\mu(N+N') - \beta H_{\text{fictitious}}\},$$

then in the real system, $\langle \mathcal{A} \rangle$ will be negative modulo small error terms. The proof of this uses ideas from [2], [3], [7]. Once we can estimate $\langle \mathcal{A} \rangle$, the game is to pick A so that $\mathcal{A}$ expresses detailed information about the real system.

The reader should be warned that our brief summary is inaccurate and oversimplified.

Finally a fascinating problem about which almost nothing is known is to understand why matter at high density and low temperature forms a crystal, i.e., a configuration with long-range order. The frontier in our knowledge of this question involves placing points $x_1, x_2, \ldots, x_N \in R^n$ to minimize a potential $V = \sum_{j \neq k} W(x_j - x_k)$. For certain special W, both positive and negative results are available in two dimensions. Nothing is known about the three-dimensional case. See, e.g., Radin and Schulman [9]. If these matters could be settled and our present results sharpened, then maybe one could give a rigorous proof that matter undergoes phase transitions. It will take a long time to reach such deep understanding.

We now present our proof.

3. Notation

For $\Omega \subset R^3$, define $L^2_{N,N'}(\Omega)$ as the space of all square-integrable functions $\psi(x_1 \ldots x_N, y_1 \ldots y_{N'})$ on $\Omega^{N+N'}$, antisymmetric in the x's and y's separately. Define

$$H^{0,\Omega}_{N,N'} = -\varkappa_1 \sum_j \Delta_{x_j} - \varkappa_2 \sum_k \Delta_{y_k} \quad \text{acting on} \quad L^2_{N,N'}(\Omega)$$

with Dirichlet boundary conditions, and

$$H^{\Omega}_{N,N'} = H^{0,\Omega}_{N,N'} + \sum_{j<k} |x_j - x_k|^{-1} + \sum_{j<k} |y_j - y_k|^{-1} - \sum_{j,k} |x_j - y_k|^{-1}.$$

Define

$$L^2_*(\Omega) = \sum_{N,N'} \oplus L^2_{N,N'}(\Omega)$$
$$L^2_{\text{neutral}}(\Omega) = \sum_N \oplus L^2_{N,N}(\Omega)$$
$$H^\Omega_* = \sum_{N,N'} \oplus H^\Omega_{N,N'}$$
$$H^\Omega_{\text{neutral}} = H^\Omega_* \mid L^2_{\text{neutral}}(\Omega)$$
$$Z_0(\mu,\beta,\Omega,N,N') = e^{\mu(N+N')}\operatorname{Tr}\exp[-\beta H^{0,\Omega}_{N,N'}]$$
$$Z(\mu,\beta,\Omega,N,N') = e^{\mu(N+N')}\operatorname{Tr}\exp[-\beta H^{\Omega}_{N,N'}]$$
$$Z_0(\mu,\beta,\Omega) = \sum_{N,N'} Z_0(\mu,\beta,\Omega,N,N')$$
$$Z(\mu,\beta,\Omega) = \sum_{N,N'} Z(\mu,\beta,\Omega,N,N')$$
$$Z_{\text{neutral}}(\mu,\beta,\Omega) = \sum_N Z(\mu,\beta,\Omega,N,N).$$

If $x_1 \dots x_N, y_1 \dots y_{N'}$ are electrons and protons, then we sometimes write $z_1 \dots z_{N+N'}$ for a list of all the particles, with charges $\epsilon(j) = \epsilon(z_j) = 1$ if z_j is one of the y_k, -1 if z_j is one of the x_j., If $K(\cdot)$ is a kernel on R^3, and $x_1 \dots x_N, y_1 \dots y_{N'}$ are electrons and protons, then define

$$V[K] = \frac{1}{2}\sum_{j\neq k} \epsilon(j)\epsilon(k)K(z_j - z_k).$$

In particular, the Coulomb potential is $V[|x|^{-1}]$.

If A is an observable, i.e., a self-adjoint operator on $L^2_*(\Omega)$ then the expected value of A is

$$\langle A\rangle = \frac{\operatorname{Tr}(Ae^{\mu(N+N')-\beta H^\Omega_*})}{\operatorname{Tr}(e^{\mu(N+N')-\beta H^\Omega_*})}.$$

In all that follows, μ will be large negative and β will be large positive.

4. The Partition Function for a Single Ball

Fix a ball B of radius R, satisfying $e^{c_2\beta} < R < e^{c_1\beta}$, $0 < c_2 < c_1 \ll 1$. We shall estimate $\operatorname{Tr}\exp[-\beta H^B_{N,N'}]$. From (1.1), (1.4) and rescaling, we get

$$-\varkappa_1(1-\delta)\Delta_x - \varkappa_2(1-\delta)\Delta_y + V[|x|^{-1}] \geqslant -E_*(1+C\delta)\cdot(N+N'-1)$$

for $N+N' > 2$ and $0<\delta \ll 1$. Hence

$$H^B_{N,N'} \geqslant -\varkappa_1\delta\Delta_x - \varkappa_2\delta\Delta_y - E_*(1+C\delta)(N+N'-1).$$

Taking $\delta = \beta^{-1}$, we find that

$$\begin{aligned}\operatorname{Tr}\exp[-\beta H^B_{N,N'}] &\leqslant e^{(\beta E_* + C)(N+N'-1)}\operatorname{Tr}\exp(-H^{0,B}_{N,N'})\\ &\leqslant e^{(\beta E_* + C)\cdot(N+N'-1)}(\operatorname{Tr}\exp(-H^{0,B}_{1,0}))^N(\operatorname{Tr}\exp(-H^{0,B}_{0,1}))^{N'}\\ &\leqslant e^{(\beta E_* + C)\cdot(N+N'-1)}\cdot(C|B|)^{N+N'}\\ &\leqslant C'\exp[\{(E_* + 3c_1)\beta + C'\}\cdot(N+N'-1)]\\ &\qquad\qquad\text{when}\quad N+N' > 2.\end{aligned} \tag{4.1}$$

Next we look more carefully at $\operatorname{Tr}\exp[-\beta H^B_{1,1}]$. First we use the textbook separation of variables

$$H^B_{1,1} = -\varkappa_1\Delta_x - \varkappa_2\Delta_y - |x-y|^{-1} = -(\text{const})\Delta_z + (-\Delta_w - |w|^{-1}) \tag{4.2}$$

where

$$z = \text{center of mass} = \frac{\varkappa_1^{-1}x + \varkappa_2^{-1}y}{\varkappa_1^{-1} + \varkappa_2^{-1}}, \qquad w = x - y.$$

For the Hilbert space $L^2_{1,1}(B)$ we have inclusions

$$L^2_{1,1}(B) \subset L^2\{(z,w)\mid z\in B, w\in B_1\} \tag{4.3}$$

if B_1 is the ball about zero of radius $2R$ and $B_0 = B$;

$$L^2_{1,1}(B) \supset L^2\{(z,w)\mid z\in B_0, w\in B_1\} \tag{4.4}$$

if B_1 is a ball about zero and $B_0 + B_1 \subset B$. Hence we can derive upper and lower bounds for $\operatorname{Tr}\exp[-\beta H^B_{1,1}]$ by computing $I = Tr\exp[(\text{const})\beta\Delta_z - \beta(-\Delta_w - |w|^{-1})]$ on $L^2(B_0\times B_1)$. The latter breaks up as $[Tr\exp((\text{const})\beta\Delta_z)$ on $L^2(B_0)]\cdot[\operatorname{Tr}\exp(-\beta(-\Delta_w - |w|^{-1}))$ on $L^2(B_1)]$. The first factor here has the form $(c|B_0|/\beta^{3/2})(1 + 0(\beta^{-1}))$, in view of the eigenvalue asymptotics of the Laplacian on B_0 ($|B_0| > e^{3c_2\beta}$). It remains to understand $\Pi = \operatorname{Tr}\exp(-\beta(-\Delta_w - |w|^{-1}))$ on $L^2(B_1)$. Write $L^2(B_1) = C\psi_0 \oplus X$ where ψ_0 is the ground-state eigenvector of $-\Delta_w - |w|^{-1}$ with Dirichlet boundary conditions on B_1, and X is the orthocomplement of ψ_0. Now $-\Delta_w - |w|^{-1} \geqslant -c_3\cdot\Delta_w - [1/4(1-c_3)]$, $0 < c_3 < 1$ and $-\Delta_w - |w|^{-1} \geqslant (c_4 - \frac{1}{4})$ on X, with $c_4 > 0$. These estimates come from the elementary theory of the Hydrogen atom. Taking $c_3 \sim c_4/100$ and averaging, we obtain $\langle(-\Delta_w - |w|^{-1})\psi, \psi\rangle \geqslant \langle(-c_3\Delta_w + c_5 - \frac{1}{4})\psi, \psi\rangle$ for $\psi\in X$ and $c_3, c_5 > 0$. Hence by minimax,

$$\operatorname{Tr}\exp(-\beta(-\Delta_w - |w|^{-1})|_X) \leqslant \operatorname{Tr}\exp(-\beta(-c_3\Delta_w + c_5 - \tfrac{1}{4})|_{L^2(B_1)}) \leqslant e^{[(1/4)-c_6]\beta}$$

for large β, since $|B_1| < e^{3c_1}$ with $c_1 \ll 1$. So $\Pi = e^{-\beta E_0} + O(e^{[(1/4)-c_6]\beta})$, E_0 = lowest eigenvalue of $-\Delta_w - |w|^{-1}$ with Dirichlet boundary conditions on B_1. On the other hand, one sees easily that $E_0 = -\frac{1}{4} + O(e^{-c\beta})$. In fact, comparison of $-\Delta_w - |w|^{-1}$ on B_1 and on R^3 shows $E_0 \geqslant -\frac{1}{4}$; while the trial wave function $\psi(x) = \theta(x)e^{-|x|/2}$, $\theta(x) = 1$ for $x \in$ Middle half of B_1, $\theta(x) = 0$ on ∂B_1 gives $E_0 \leqslant -\frac{1}{4} + O(e^{-\text{const. radius}(B_1)})$. Therefore $\Pi = e^{\beta/4} \cdot \cdot (1 + O(e^{-c\beta}))$, and so

$$I = \frac{c|B_0|}{\beta^{3/2}} e^{\beta/4}(1 + O(\beta^{-1})), \tag{4.5}$$

provided B_0 and B_1 have radii bounded between $e^{c_2\beta}$ and $e^{c_1\beta}$, $c_1 \ll 1$.

From (4.2), (4.3), (4.5) we get $\operatorname{Tr}\exp(-\beta H^B_{1,1}) \leqslant (c|B|/\beta^{3/2})(e^{\beta/4}(1 + C\beta^{-1})$, while (4.2) (4.4), (4.5) with $B_0 = B$ dilated by $(1 - \beta^{-1})$, $B_1 = B$ dilated by a factor β^{-1} yields $\operatorname{Tr}\exp(-\beta H^B_{1,1}) \geqslant (c|B|/\beta^{3/2})(e^{\beta/4}(1 - C\beta^{-1}))$ with the same c. That is

$$\operatorname{Tr}\exp(-\beta H^B_{1,1}) = \frac{c|B|}{\beta^{3/2}} e^{\beta/4}(1 + O(\beta^{-1})). \tag{4.6}$$

Finally when $(N, N') = (1,0)$ or $(0,1)$ we have $H^B_{N,N'} = H^{0,B}_{N,N'}$ so that $\operatorname{Tr}\exp(-\beta H^B_{N,N'}) \leqslant (c|B|/\beta^{3/2})$. Analogous estimates hold for $(N, N') = (2, 0)$ or $(0, 2)$. Hence we can make a table

$$\begin{aligned}
Z(\mu, \beta, B, N, N') &= 1 \quad \text{if} \quad N = N' = 0 && (a)\\
&= (1 + O(\beta^{-1}))\frac{ce^{2\mu + \beta/4}}{\beta^{3/2}}|B| \quad \text{if} \quad N = N' = 1 && (b)\\
&\leqslant \frac{Ce^{\mu}}{\beta^{3/2}}|B| \quad \text{if} \quad (N, N') = (1, 0) \quad \text{or} \quad (0, 1) && (c)\\
&\leqslant \frac{Ce^{2\mu}}{\beta^{3/2}}|B| \quad \text{if} \quad (N, N') = (2, 0) \quad \text{or} \quad (0, 2) && (c')\\
&\leqslant C\exp\{\mu(N + N') + [(E_* + 3c_1)\beta + C'] \cdot (N + N' - 1)\} && (d)
\end{aligned}$$

if (N, N') is not one of the above. If $E_* < \frac{1}{4}$, then the quantity in braces in (d) will be less than

$$2\mu + \frac{\beta}{4} - c'(N + N' - 2) \quad \text{for } \beta \text{ large and } \quad \frac{\mu}{\beta} < -\frac{1}{4} + c''.$$

Here c', c'' are positive constants. Therefore, since $e^{c_2\beta} <$ radius $B < e^{c_1\beta}$ with $c_1 \ll 1$, we have

$$\sum_{N+N'>2} Z(\mu, \beta, B, N, N') < \frac{1}{\beta} Z(\mu, \beta, B, 1, 1)$$

for β large, $\dfrac{\mu}{\beta} < -\dfrac{1}{4} + c''$.

Bringing in $(N, N') = (0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(0,2)$ also, we can find a nonempty interval $\hat{I}$ of the form $(-\frac{1}{4}+c''', -\frac{1}{4}+c'')$ so that if β is large enough, c_1 is small enough, and $\mu/\beta \in \hat{I}$, then

$$\sum_{N+N' \neq (0,0) \text{ or } (1,1)} Z(\mu, \beta, B, N, N') < \frac{1}{\beta} Z(\mu, \beta, B, 1, 1) \tag{4.7}$$

$$Z(\mu, \beta, B, 1, 1) < \frac{1}{\beta} Z(\mu, \beta, B, 0, 0). \tag{4.8}$$

So the grand canonical ensemble on B consists most probably of a vacuum; but if it contains anything, the contents will most likely be a single Hydrogen atom.

From (a), (b), (4.7), (4.8) we get the important equation

$$Z(\mu, \beta, B) = \exp(\rho|B| \cdot (1 + O(\beta^{-1}))), \tag{4.9}$$

where

$$\rho = \frac{\text{const}\, e^{2\mu + (\beta/4)}}{\beta^{3/2}} \ll 1. \tag{4.10}$$

Evidently, we may replace Z by Z_{neutral} in (4.9).

We shall need also the following generalization of the partition function. Suppose we have balls $B_1 \ldots B_{L_0}$ with $e^{c_2\beta} <$ radius $(B_k) < e^{c_1\beta}$ as before. Fix a subset $E \subset R^3$ and a number t.

Define a Hilbert space $L^2_{1,1}(B_1 \ldots B_{L_0})$ to consist of all square integrable $\psi(x_1, y_1, x_2, y_2, \ldots, x_{L_0}, y_{L_0})$ supported in $\{x_k, y_k \in B_k (k = 1, \ldots, L_0)\}$.

On this Hilbert space, define a Hamiltonian

$$\overset{\circ}{H} = \sum_{k=1}^{L_0} (-\varkappa_1 \Delta_{x_k} - \varkappa_2 \Delta_{y_k} - |x_k - y_k|^{-1}), \text{ Dirichlet boundary conditions.}$$

Thus, each B_k contains an electron and a proton which attract each other but do not interact with the particles in the other B_k.

Next define an observable

$$G = \begin{cases} 1 & \text{if } x_k - y_k \in E \text{ for } k = 1, \ldots, s_0 \text{ but not for } k = s_0 + 1, \ldots, L_0 \\ 0 & \text{otherwise} \end{cases}$$

Then for c_1 and $|t|$ less than some small constant $c(L)$ we have

Lemma 4.1. *The trace of* $\exp(tG - \beta\overset{\circ}{H})$ *on* $L^2_{1,1}(B_1 \ldots B_{L_0})$ *is given by*

$$\prod_{k=1}^{L_0} \left(\frac{\text{const}\, e^{\beta/4} |B_k|}{\beta^{3/2}} \right) e^{tG_0} (1 + 0(\beta^{-1}) + 0(t^2))$$

with $G_0 = ((\text{const}) \int_E e^{-|x|}\, dx)^{s_0} ((\text{const}) \int_{cE} e^{-|x|}\, dx)^{L_0 - s_0}$.

Here $O(\beta^{-1})$ means less than $C(L_0)\cdot\beta^{-1}$ in absolute value, and similarly for $O(t^2)$.

We sketch the proof of Lemma 3.1. Again we can separate variables using z_k = center of mass of x_k, y_k and $w_k = x_k - y_k$. Then

$$tG - \beta\overset{\circ}{H} = +(\text{const})\,\beta\sum_k \Delta_{z_k} - \\ -\beta\left[\sum_k(-\Delta_{w_k} - |w_k|^{-1}) - \frac{t}{\beta}G(w_1\ldots w_{L_0})\right]$$

with

$$G(w_1\ldots w_{L_0}) = \begin{bmatrix} 1 & \text{if} \quad w_1\ldots w_{s_0}\in E \quad \text{but} \quad w_{s_0+1}\ldots w_{L_0}\notin E \\ 0 & \text{otherwise} \end{bmatrix}$$

As in the proof of (6), one can estimate $\operatorname{Tr}\exp(tG - \beta\overset{\circ}{H})$ above and below by

$$\prod_{k=1}^{L_0}\left(\frac{\text{const}}{\beta^{3/2}}|B_k|\right)\cdot e^{-\beta\tilde{E}}(1 + O(\beta^{-1})), \tag{4.11}$$

where $\tilde{E}$ is the lowest eigenvalue of

$$\sum_{k=1}^{L_0}(-\Delta_{w_k} - |w_k|^{-1}) - \frac{t}{\beta}G(w_1\ldots w_{L_0})$$

on a suitable product of large balls $B_1^1\times\ldots\times B_{L_0}^1$ about the origin. If $t=0$ then the w_k decouple and $\tilde{E} = -\frac{1}{4}L_0(1 + O(e^{-c\beta}))$. Perturbation theory yields

$$\tilde{E} = -\frac{L_0}{4} + O(e^{-c\beta}) - \frac{t}{\beta}G_0 + O\left[\left(\frac{t}{\beta}\right)^2\right] \quad \text{for} \quad \left|\frac{t}{\beta}\right| < c(L_0).$$

Substituting this into (4.11), we obtain the conclusion of Lemma 4.1, even with a better error term than stated there.

5. Estimates for Coulomb Systems

Take an even approximate identity $\varphi_R(x)$ of total integral one, supported in $|x| < \frac{1}{2}R$, and set $\bar{K}(x,R) = |x|^{-1} * \varphi_R * \varphi_R$. Then define

$$V_{LR}(R) = \frac{1}{2}\sum_{j,k}\epsilon(j)\epsilon(k)\bar{K}(z_j - z_k, R) = \text{“Long-Range Part of the Coulomb Potential”}. \tag{5.1}$$

Our goal in this section is to show that if $K(\cdot)$ is a kernel on R^3 which behaves roughly like $|x|^{-1}$, then $V[K] \leqslant C(H^{\Omega}_{N,N'} + CN + CN')$. Also if $K(x)$ is supported in $|x| \geqslant R$, then $V[K]$ is dominated by $V_{LR}(R) + (\text{small constant}) \cdot \cdot (H^{\Omega}_{N,N'} + CN + CN')$. The precise statements are given by Lemmas 3 and 4 below.

Now fix nucleii $y_1 \ldots y_{N'} \in R^3$, and let $\psi(x_1 \ldots x_N)$ be antisymmetric of norm 1. Let Q° be an enormous cube containing the system, take $K \gg 1$ to be picked later, and make a Calderón-Zygmund decomposition $\{Q_\nu\}$ of Q° as follows. (See [2]). We bisect Q° repeatedly, stopping at the cube Q_ν when its triple Q_ν^* contains at most K nucleii. Thus, $Q^\circ = \cup_\nu Q_\nu$, and

(*a*) Q_ν^* contains at most K nucleii.
(*b*) Q_ν^{**} contains more than K nucleii, or else the cutting process would not have reached Q_ν.
(*c*) $\bar{Q}_\mu \cap \bar{Q}_\nu \neq \phi$ implies that the side lengths δ_μ, δ_ν are comparable. Otherwise, (*b*) for the smaller cube contradicts (*a*) for the larger.
(*d*) Call Q_ν *active* if Q_ν contains at least $c \cdot K$ nucleii. Then

$$\sum_{\nu\,\text{active}} \delta_\nu^{-1} \geqslant c \sum_\nu \delta_\nu^{-1}.$$

To prove (*d*), say that Q_ν has *good geometry* if $\delta_\mu \sim \delta_\nu$ for any Q_μ intersecting Q_ν^{**}. We first check that

$$\sum_{\nu\,\text{active}} \delta_\nu^{-1} \geqslant c \sum_{\nu\,\text{good geom.}} \delta_\nu^{-1}. \tag{5.2}$$

In fact, take Q_ν with good geometry and note that only a bounded number of Q_μ can intersect Q_ν^{**}. The pigeon-hole principle therefore shows that one of these $Q\mu$ must be active, by virtue of (*b*). Hence,

$$\sum_{\nu\,\text{good geom.}} \delta_\nu^{-1} \leqslant C \sum_{\substack{\mu\,\text{active} \\ \nu\,\text{good geom.} \\ Q_\mu \cap Q_\nu^{**} \neq \phi}} \delta_\mu^{-1} \leqslant C' \sum_{\mu\,\text{active}} \delta_\mu^{-1}, \text{ which proves (5.2).}$$

Next we show

$$\sum_{\nu\,\text{good geom.}} \delta_\nu^{-1} \geqslant c \sum_\nu \delta_\nu^{-1}, \tag{5.3}$$

which together with (5.2) completes the proof of (*d*).

Observe that if Q_ν doesn't have good geometry, then some $Q_{\nu'}$ intersecting Q_ν^{**} must be much bigger or much smaller than Q_ν. If $Q_{\nu'}$ were much bigger, then $Q_\nu^{**} \subset Q_{\nu'}^*$, contradicting (*a*) and (*b*). Hence $\delta_{\nu'} < 10^{-3}\delta_\nu$, and $10^3 Q_{\nu'} \subset \subset 10^3 Q_\nu$, where $CQ = Q$ dilated about its center by a factor C. Now either $Q_{\nu'}$ has good geometry, or else we can repeat the process to find a $Q_{\nu''}$ with $\delta_{\nu''} < 10^{-3}\delta_{\nu'}$, $10^3 Q_{\nu''} \subset 10^3 Q_{\nu'}$. Continue in this way until we reach a cube with good geometry. This must happen eventually, since there are only finitely many cubes $\{Q_\nu\}$. Hence for each cube Q_ν there is a Q_μ with good geometry with $10^3Q_\mu \subset 10^3Q_\nu$. So

$$\sum_\nu \delta_\nu^{-1} \leqslant \sum_{\substack{\mu,\nu \\ \mu \text{ good geom.} \\ 10^3Q_\mu \subset 10^3Q_\nu}} \delta_\nu^{-1} \leqslant \sum_{\mu \text{ good geom.}} \left(\sum_{\substack{\text{all dyadic } Q \\ 10^3Q_\mu \subset 10^3Q}} \text{side}(Q)^{-1} \right) \leqslant$$
$$\leqslant C \sum_{\mu \text{ good geom.}} \delta_\mu^{-1},$$

which proves (5.3). Since (5.2) and (5.3) hold, we know (d).

We shall need the following estimate for functions on R^3.

Lemma 5.1. *If Q is a cube of side δ, and $\psi \in L^2(Q)$, then*

$$\int_Q \left\{ \frac{1}{40} |\nabla\psi(x)|^2 - \sum_{k=1}^K |x - y_k|^{-1} |\psi(x)|^2 \right\} dx \geqslant$$
$$\geqslant -\left(\frac{CK}{\delta} + C(K) \right) \|\psi\|^2_{L^2(Q)}. \qquad (5.4)$$

Proof. Look first at the case $\delta \leqslant \delta_0(K)$ with $\delta_0(K)$ to be picked in a moment. Set

$$V(x) = \sum_{k=1}^K |x - y_k|^{-1} \quad \text{and} \quad \psi_Q = |Q|^{-1} \int_Q \psi.$$

Then

$$\int_Q V(x)|\psi(x)|^2 dx \leqslant 2 \int_Q V(x)|\psi_Q|^2 dx + 2 \int_Q V(x)|\psi(x) - \psi_Q|^2 dx \leqslant$$
$$\leqslant \frac{CK}{\delta} \|\psi\|^2_{L^2(Q)} + 2 \int_Q V(x)|\psi(x) - \psi_Q|^2 dx, \quad \text{since} \quad |Q|^{-1} \int_Q V(x)\, dx \leqslant \frac{CK}{\delta}.$$

The last term on the right is at most

$$2\|V\|_{L^{3/2}(Q)} \cdot \|\psi(\cdot) - \psi_Q\|^2_{L^6(Q)} \leqslant CK\delta \|\nabla\psi\|^2_{L^2(Q)}$$

by Holder and Sobolev. If $\delta_0(K)$ is small enough, then $CK\delta < (1/40)$, and Lemma 5.1 follows for $\delta \leqslant \delta_0(K)$. For a cube Q of side $\delta > \delta_0(K)$, we just cut Q into subcubes $\{Q^\alpha\}$ of side $\sim\delta_0(K)$. We already know (5.4) for each of the Q^α; summing over α completes the proof of Lemma 5.1.

Antisymmetry of the wave function enters via the following observation.

Lemma 5.2. *Let L_ν denote the number of electrons x_j in Q_ν. Then*

$$\frac{1}{20} \|\nabla\psi\|^2 \geqslant c \left\langle \sum_{L_\nu \geq 2} L_\nu^{5/3} \delta_\nu^{-2} \psi, \psi \right\rangle.$$

PROOF. Suppose first that φ is antisymmetric on Q^L for a cube $Q \subset R^3$ of side δ. Then

$$\|\nabla\varphi\|^2_{L^2(Q^L)} \geqslant cL^{5/3}\delta^{-2}\|\varphi\|^2_{L^2(Q^L)} \quad \text{for} \quad L \geqslant 2.$$

This follows by expanding φ in eigenfunctions of the Neumann Laplacian on Q^L. Consequently, on $I = Q_{\nu_1} \times Q_{\nu_2} \times \ldots \times Q_{\nu_N}$ we have

$$\|\nabla\psi\|^2_{L^2(I)} \geqslant c\Big(\sum_{L_\nu \geq 2} L_\nu^{5/3}\delta_\nu^{-2}\Big)\|\psi\|^2_{L^2(I)},$$

with $L_\nu =$ (number of ν_k equal to ν) $=$ (number of electrons in Q_ν) for $(x_1 \ldots x_N) \in I$. This may be rewritten as

$$\|\nabla\psi\|^2_{L^2(I)} \geqslant c\Big\langle\Big(\sum_{L_\nu \geq 2} L_\nu^{5/3}\delta_\nu^{-2}\Big)\psi, \psi\Big\rangle_{L^2(I)}.$$

Summing over all possible choices of $\nu_1 \ldots \nu_N$, we obtain Lemma 5.2.

Next we compare the potential energy of point charges $x_1 \ldots x_N, y_1 \ldots y_{N'}$ with that of a continuous charge density $\rho(x) = \sum_k \epsilon(k)\varphi_k(x - z_k)$. Here $\varphi_k \geqslant 0$ is a spherically symmetric smooth charge density of total charge $+1$, supported in a ball of radius $(1/10)\delta(k)$, where $\delta(k) = \delta_\nu$ for the Q_ν containing z_k. Observe that

$$V(\rho) = \frac{1}{2}\int \frac{\rho(x)\rho(y)}{|x - y|}\,dx\,dy = c\int |\xi|^{-2}|\hat{\rho}(\xi)|^2\,d\xi \geqslant 0.$$

How do $V[|x|^{-1}]$ and $V(\rho)$ differ?

(α) $V(\rho)$ contains "self-energy terms" of the form

$$\frac{1}{2}\int \frac{\varphi_k(x - z_k)\varphi_k(y - z_k)}{|x - y|}\,dx\,dy$$

with no analogues in $V[|x|^{-1}]$. The self-energy terms total at most $C\sum_\nu (K + L_\nu)/\delta_\nu$.

The mean-value properties of the Coulomb potential yield

(β) $$|z_j - z_k|^{-1} \geqslant \int \frac{\varphi_j(x - z_j)\varphi_k(y - z_k)}{|x - y|}\,dx\,dy + {} + c|z_j - z_k|^{-1}\chi_{|z_j - z_k| < 10^{-2}\delta(k)}$$

which we use when $\epsilon(j) = \epsilon(k)$, $j \neq k$; and

$$(\gamma) \quad -|z_j - z_k|^{-1} \geqslant -\int \frac{\varphi_j(x - z_j)\varphi_k(y - z_k)}{|x - y|}\,dx\,dy - $$

$$- |z_j - z_k|^{-1}\chi_{|z_j - z_k| < \frac{1}{5}\delta(k)}$$

which we use when $\epsilon(j) \neq \epsilon(k)$.

From (α), (β), (γ) we obtain

$$\begin{aligned} V[|x|^{-1}] \geqslant V(\rho) - C\sum_{\nu}\frac{K + L_\nu}{\delta_\nu} + \\ + c\sum_{\nu}\sum_{x_j \in Q_\nu}\sum_{0 < |x_j - x_k| < 10^{-2}\delta_\nu} |x_j - x_k|^{-1} + \\ + c\sum_{\nu}\sum_{y_j \in Q_\nu}\sum_{0 < |y_k - y_j| < 10^{-2}\delta_\nu} |y_j - y_k|^{-1} - \\ - \sum_{\nu}\sum_{x_j \in Q_\nu}\sum_{y_k \in Q_\nu^*} |x_j - y_k|^{-1} \end{aligned} \tag{5.5}$$

If ν is active (see (d)), then by the pigeon-hole principle, some subcube of Q_ν of diameter $< 10^{-2}\delta_\nu$ will contain at least $c'K$ nucleii. Hence

$$\frac{1}{2}\sum_{y_j \in Q_\nu}\sum_{0 < |y_k - y_j| < 10^{-2}\delta_\nu} |y_j - y_k|^{-1} \geqslant \frac{c''K^2}{\delta_\nu}.$$

Similarly, if $L_\nu \geqslant K$, then

$$\frac{1}{2}\sum_{x_j \in Q_\nu}\sum_{0 < |x_k - x_j| < 10^{-2}\delta_\nu} |x_j - x_k|^{-1} \geqslant \frac{c''L_\nu^2}{\delta_\nu}.$$

Consequently, (5.5) implies

$$\begin{aligned} V[|x|^{-1}] \geqslant \Bigg[V(\rho) + c\sum_{\nu}\sum_{x_j \in Q_\nu}\sum_{0 < |x_k - x_j| < \delta_\nu \cdot 10^{-2}} |x_j - x_k|^{-1} + \\ + c\sum_{\nu}\sum_{y_j \in Q_\nu}\sum_{0 < |y_k - y_j| < 10^{-2}\delta_\nu} |y_j - y_k|^{-1}\Bigg] + \\ + \Bigg[\sum_{\nu\,\text{active}}\frac{c''K^2}{\delta_\nu} + \sum_{L_\nu \geq K}\frac{c''L_\nu^2}{\delta_\nu} - C\sum_{\nu}\frac{K + L_\nu}{\delta_\nu}\Bigg] - \\ - \sum_{\nu}\sum_{x_j \in Q_\nu}\sum_{y_k \in Q_\nu^*} |x_j - y_k|^{-1}. \end{aligned} \tag{5.6}$$

Lemma (5.1) and (a) imply

$$-\frac{1}{20}\Delta_x \geqslant 2\sum_{\nu}\sum_{x_j \in Q_\nu}\sum_{y_k \in Q_\nu^*} |x_j - y_k|^{-1} - \sum_{\nu}\left(\frac{CK}{\delta_\nu} + C(K)\right)L_\nu.$$

Adding this and the conclusion of Lemma 5.2 to (5.6), we obtain

$$-\frac{1}{10}\Delta_x + V[|x|^{-1}] \geqslant \Big[V(\rho) + + c\sum_\nu \sum_{x_j\in Q_\nu} \sum_{0<|x_k-x_j|<10^{-2}\delta_\nu} |x_j - x_k|^{-1} + + c\sum_\nu \sum_{y_j\in Q_\nu} \sum_{0<|y_k-y_j|<10^{-2}\delta_\nu} |y_j - y_k|^{-1} + + \sum_\nu \sum_{x_j\in Q_\nu} \sum_{y_k\in Q_\nu^*} |x_j - y_k|^{-1}\Big] + + \Big[\sum_\nu \frac{\tilde{c}K^2}{\delta_\nu} + \sum_{L_\nu\geq K} \frac{c''L_\nu^2}{\delta_\nu} - \sum_\nu \frac{CKL_\nu}{\delta_\nu} - - \sum_\nu C(K)L_\nu - \sum_\nu \frac{CK + CL_\nu}{\delta_\nu} + + \sum_{L_\nu\geq 2} cL_\nu^{5/3}\delta_\nu^{-2}\Big]. \tag{5.7}$$

Here we used (d) in the term with constant $\tilde{c}$. Now if K is large enough, then we have the elementary inequality

$$\left(cL_\nu^{5/3}\delta_\nu^{-2}\chi_{L_\nu\geq 2} + \frac{c''L_\nu^2\chi_{L_\nu\geq K}}{\delta_\nu} + \frac{\tilde{c}K^2}{\delta_\nu}\right) - \frac{CKL_\nu}{\delta_\nu} - \frac{CK + CL_\nu}{\delta_\nu} - - C(K)L_\nu \geqslant -E(K)L_\nu + \frac{cL_\nu}{\delta_\nu} + \frac{cK}{\delta_\nu}\cdot \tag{5.8}$$

To check (5.8), we note that $c''L_\nu^2\chi_{L_\nu\geqslant 2} + \tilde{c}K^2 - CKL_\nu - CK - CL_\nu \geqslant cL_\nu + + cK$ unless $L_\nu \sim K$. So (5.8) is obvious unless $L_\nu \sim K$. If $L_\nu \sim K$, then

$$cL_\nu^{5/3}\chi_{L_\nu\geq 2}\delta_\nu^{-2} - \frac{CKL_\nu}{\delta_\nu} - \frac{(CK + CL_\nu)}{\delta_\nu} \geqslant \frac{cL_\nu + cK}{\delta_\nu}$$

as long as $\delta_\nu < \delta_0(K)$. So (5.8) is obvious unless $L_\nu \sim K$ and $\delta_\nu \geqslant \delta_0(K)$. In this last case, (5.8) is again obvious if we just take $E(K)$ large enough. So (5.8) holds in all cases.

Substituting (5.8) into (5.7) and recalling that $\sum_\nu L_\nu = N$, we find that

$$-\frac{1}{10}\Delta_x + V[|x|^{-1}] + E(K)\cdot N \geqslant V(\rho) + \sum_\nu \frac{cL_\nu}{\delta_\nu} + \sum_\nu \frac{cK}{\delta_\nu} + c\sum_\nu \sum_{x_j\in Q_\nu} \sum_{0<|x_k-x_j|<10^{-2}\delta_\nu} |x_j - x_k|^{-1} + \sum_\nu \sum_{y_j\in Q_\nu} \sum_{0<|y_k-y_j|<10^{-2}\delta_\nu} |y_j - y_k|^{-1} + + \sum_\nu \sum_{x_j\in Q_\nu} \sum_{y_y\in Q_\nu^*} |x_j - y_k|^{-1}. \tag{5.9}$$

The terms on the right are all positive, so (5.9) implies stability of matter. Now let $\delta_+(z_j) = \min_{k \neq j} |z_j - z_k|$ = distance from z_j to its nearest neighbor. One checks easily that

$$\sum_{\substack{z_j \text{ electrons} \\ \text{in } Q_\nu}} \frac{c}{\delta_+(z_j)} \leqslant \sum_{x_j \in Q_\nu} \sum_{0 < |x_k - x_j| < 10^{-2}\delta_\nu} |x_j - x_k|^{-1} + \\ + \sum_{x_j \in Q_\nu} \sum_{y_k \in Q_\nu^*} |x_j - y_k|^{-1} + \frac{C(K + L_\nu)}{\delta_\nu}.$$

So (5.9) yields

$$c \sum_{z_j \text{ electrons}} (\delta_+(z_j))^{-1} \leqslant -\frac{1}{10}\Delta_x + V[|x|^{-1}] + E(K) \cdot N.$$

The analogous estimate for protons follows similarly (there are slight changes because of the asymmetry between electrons and protons in the last term on the right in (5.9)). Hence

$$\sum_{z_j} \delta_+^{-1}(z_j) \leqslant C\left(-\frac{1}{10}\Delta_x + V[|x|^{-1}] + C \cdot N\right). \tag{5.10}$$

From here on, we simply fix K large enough to make sure (5.9) holds, and we denote K, $E(K)$ simply by C.

Set $\delta(z_j) = \min(1, \delta_+(z_j))$. Obviously, then,

$$\sum_j \delta^{-1}(z_j) \leqslant C\left(-\frac{1}{10}\Delta_x + V[|x|^{-1}] + CN + CN'\right). \tag{5.11}$$

So far, we have regarded the nucleii $y_1 \ldots y_{N'}$ as fixed. However (5.11) for fixed nucleii implies the corresponding estimate for quantized nucleii, namely

$$\sum_j \delta^{-1}(z_j) \leqslant C(H^{\Omega}_{N,N'} + CN + CN'), \quad \text{any} \quad \Omega \subset R^3. \tag{5.12}$$

We did not even need the kinetic energy of the nucleii in (5.12). Estimate (5.12) shows that in a quantum state ψ of moderate energy, the particles are not too closely packed.

Now we are aready to estimate $V[K]$ in terms of $H^{\Omega}_{N,N'}$ for Coulomb-like potentials $K(\cdot)$. Our assumptions on K are the following rather technical estimates.

$$|\partial^\alpha K(x)| \leqslant C|x|^{-1-|\alpha|} \quad \text{for} \quad |\alpha| \leqslant 3 \tag{5.13}$$

and all x outside the annuli $\mathfrak{A}_k = \{ ||x| - R_k| < R_0 \}$, $k = 1, 2, 3, \ldots$

$$|\partial^\alpha K(x)| \leqslant CR_0|x|^{-1-|\alpha|} \quad \text{for} \quad |\alpha| \leqslant 2 \quad \text{and all } x. \tag{5.14}$$

Here $100 < R_0 < R_1 < \ldots$ are fixed radii with $R_{k+1} \geqslant 100R_k$ and $R_1 > R_0^{10}$.

Lemma 5.3. *For K satisfying* (5.13), (5.14), *we have*

$$V[K] \leqslant C(H^{\Omega}_{N,N'} + CN + CN'). \tag{5.15}$$

PROOF. We first check that $|\hat{K}(\xi)| \leqslant C|\xi|^{-2}$. In fact, write $K = K_0 + K_1$ with both terms satisfying (5.13), (5.14), $K_0(x)$ supported in $|x| < 2|\xi|^{-1}$, and $K_1(x)$ supported in $|x| > |\xi|^{-1}$. From (5.13), (5.14) we get $\|K_0\|_{L^1} \leqslant C|\xi|^{-2}$, $\|\Delta K_1(x) - \Delta K_1(x-y)\|_{L^1(dx)} \leqslant C$ for $|y| \leqslant (1/20)|\xi|^{-1}$. Hence $|\hat{K}_0(\xi)| \leqslant C|\xi|^{-2}$, $|[e^{iy\xi} - 1]\hat{K}_1(\xi)|\xi|^2| \leqslant C$. Taking $y = (1/20)\xi|\xi|^{-2}$, we get $|\hat{K}_0(\xi)|$, $|\hat{K}_1(\xi)| \leqslant C|\xi|^{-2}$, so that $|\hat{K}(\xi)| \leqslant C|\xi|^{-2}$ as claimed.

Next set $K^{\#}(x) = |x|^{-1} - cK(x)$. For $c \ll 1$ we have $\hat{K}^{\#}\{\rho\} = \frac{1}{2}\int K^{\#}(x-y)\rho(x)\rho(y)\,dx\,dy \geqslant 0$ for any charge density ρ. We shall prove

$$V[K^{\#}] \geqslant -C(H^{\Omega}_{N,N'} + CN + CN'). \tag{5.16}$$

This means $cV[K] \leqslant V[|x|^{-1}] + C(H^{\Omega}_{N,N'} + CN + CN')$. Since evidently $V[|x|^{-1}] \leqslant H^{\Omega}_{N,N'}$, (5.16) implies (5.15) and so proves Lemma 5.3.

To establish (5.16), we construct a suitable charge density ρ and compare $V[K^{\#}]$ with $K^{\#}\{\rho\} \geqslant 0$. To make ρ, first take an even, smooth function $\varphi(x)$, supported in $|x| \leqslant \frac{1}{3}$ and satisfying $\int \varphi(x)\,dx = 1$, $\int x^{\gamma}\varphi(x)\,dx = 0$ for $0 < |\gamma| < 20$. Then set $\varphi_j(x) = [\delta(z_j)]^{-3}\varphi(x/\delta(z_j))$, and define $\rho(x) = \sum_j \epsilon(z_j)\varphi_j(x - z_j)$. Comparing $V[K^{\#}]$ with $K^{\#}\{\rho\}$, we first discover self-energy terms in $K^{\#}\{\rho\}$ with no analogues in $V[K^{\#}]$. These amount to $C \cdot \sum_j \delta^{-1}(z_j)$. Next, for distinct particles z_j, z_k we compare $K^{\#}(z_j - z_k)$ with the analogous term $K^{\#} * \varphi_j * \varphi_k(z_j - z_k)$ in $K^{\#}\{\rho\}$. These differ by at most

$$\begin{aligned} |K^{\#}(z_j - z_k) - K^{\#} * \varphi_j * \varphi_k(z_j - z_k)| &\leqslant \\ \leqslant \frac{C(\delta(z_j) + \delta(z_k))^3}{|z_j - z_k|^4} &+ C(\delta(z_j) + \delta(z_k))^2 H(z_j - z_k), \end{aligned} \tag{5.17}$$

where

$$H(x) = R_0|x|^{-3} \cdot \sum_{k=1}^{\infty} \chi_{||x| - R_k| < 2R_0}.$$

(To check (5.17), just Taylor-expand $K^{\#}$ about $z_j - z_k$ to order 1 or 2, and invoke (5.13), (5.14), and the moment properties of φ.)

Consequently,

$$\begin{aligned} V[K^{\#}] \geqslant K^{\#}\{\rho\} - C\sum_j \delta^{-1}(z_j) - C\sum_{j \neq k} \frac{(\delta(z_j) + \delta(z_k))^3}{|z_j - z_k|^4} &- \\ - C\sum_{j \neq k} (\delta(z_j) + \delta(z_k))^2 H(z_j - z_k).& \end{aligned} \tag{5.18}$$

To handle the last two terms on the right, set

$$F(x) = \sum_j \delta^{-3}(z_j)\chi_{|x - z_j| < (1/3)\delta(z_j)}, \; G(x) = \sum_j \delta^{-1}(z_j)\chi_{|x - z_j| < (1/3)\delta(z_j)}.$$

Thus

$$\int F^{4/3} = \int G^4 \leqslant C\sum_j \delta^{-1}(z_j).$$

On the other hand,

$$\begin{aligned}\sum \frac{\delta^3(z_j)}{(\delta(z_j) + |z_j - z_k|)^4} &\leqslant \\ &\leqslant C\sum_j \int_{|x - z_j| < (1/3)\delta(z_j)} G(x)\left[\int \frac{\delta(z_j)}{(\delta(z_j) + |x - y|)^4} F(y)\, dy\right] dx \leqslant \\ &\leqslant C\sum_j \int_{|x - z_j| < (1/3)\delta(z_j)} G(x)F^*(x)\, dx\end{aligned}$$

(with F^* = maximal function of F; see Stein [10]) $= C\int_{R^3} GF^*\, dx \leqslant C\left(\int G^4\right)^{1/4} \left(\int F^{4/3}\right)^{3/4}$ (by the maximal theorem) $\leqslant C\sum_j \delta^{-1}(z_j)$. So

$$\sum_{j \neq k} \frac{\delta^3(z_j)}{|z_j - z_k|^4} \leqslant C' \sum_j \delta^{-1}(z_j).$$

Switching the roles of j and k, we conclude that

$$\sum_{j \neq k} \frac{(\delta(z_j) + \delta(z_k))^3}{|z_j - z_k|^4} \leqslant C'' \cdot \sum_j \delta^{-1}(z_j). \tag{5.19}$$

Similarly,

$$\sum_{j,k} \delta^2(z_j)H(z_j - z_k) \leqslant \sum_j \int_{|x - z_j| < (1/3)\delta(z_j)} G(x)\left[\int_{R^3} H^+(x - y)F(y)\, dy\right] dx$$

with $H^+(x) = \max_{|w| \leq 2} H(x + w)$. So

$$\begin{aligned}\sum_{j,k} \delta^2(z_j)H(z_j - z_k) &\leqslant \int_{R^3}\int_{R^3} G(x)H^+(x - y)F(y)\, dy\, dx \leqslant \\ &\leqslant ||H^+||_{L^1}||G||_{L^4}||F||_{L^{4/3}} \leqslant C\sum_j \delta^{-1}(z_j).\end{aligned}$$

Again switching the roles of z_j, z_k, we get

$$\sum_{j,k} (\delta(z_j) + \delta(z_k))^2 H(z_j - z_k) \leqslant C' \sum \delta^{-1}(z_j).$$

Put this and (5.19) into (5.18), and recall that $K^{\#}\{\rho\} \geqslant 0$. The result is $V[K^{\#}] \geqslant -C\sum_j \delta^{-1}(z_j)$, which implies (5.16) by virtue of (5.12).

Lemma 5.3 is proved.

We conclude this section by relating $V[K]$ to $V_{LR}(R)$ defined by (5.1), when $K(x)$ is a Coulomb-like potential supported in $|x| \geqslant R$. Note that

$$V_{LR}(R_0) = \frac{1}{2} \int \frac{\rho(x)\rho(y)}{|x-y|} \, dx \, dy \geqslant 0,$$

where $\rho(x) = \sum_j \epsilon(j)\varphi_{R_0}(x - z_j)$. As before, assume $100 < R_0 < R_1 < R_2 < < \dots$ with $R_{k+1} > 100\, R_k$ and $R_1 > R_0^{10}$. Now, however, let $K(\cdot)$ be a kernel on R^3 satisfying

$$|\partial^\alpha K(x)| \leqslant C|x|^{-1-|\alpha|} \quad \text{for} \quad |\alpha| \leqslant 2 \quad \text{and all} \quad x. \tag{5.20}$$

$$|\partial^\alpha K(x)| \leqslant C|x|^{-1-|\alpha|} \quad \text{for} \quad |\alpha| \leqslant 4 \quad \text{and all} \quad x \tag{5.21}$$

outside the annuli $\mathcal{Q}_k = \{\, ||x| - R_k| < R_0 \}$, $k = 1, 2, 3, \dots$

$$K(x) \quad \text{is supported in} \quad |x| \geqslant R_1. \tag{5.22}$$

Lemma 5.4. *If K satisfies* (5.20), (5.21) *and* (5.22), *then*

$$V[K] \leqslant C V_{LR}(R_0) + \frac{C}{R_0}(H^\Omega_{N,N'} + CN + CN'). \tag{5.23}$$

PROOF. Write $K = \tilde{K} + K'$ with $\tilde{K} = K * \varphi_{R_0} * \varphi_{R_0}$.

Then

$$V[\tilde{K}] = \frac{1}{2} \sum_{j \neq k} \tilde{K}(z_j - z_k)\epsilon(j)\epsilon(k) = \frac{1}{2} \sum_{j,k} \tilde{K}(z_j - z_k)\epsilon(j)\epsilon(k) - \frac{1}{2} \sum_j \tilde{K}(0) =$$
$$= \frac{1}{2} \int K(x-y)\rho(x)\rho(y) \, dx \, dy - \frac{1}{2} \tilde{K}(0) \cdot (N + N')$$

with $\rho(x) = \sum_j \epsilon(j)\varphi_{R_0}(x - z_j)$. Now K satisfies (5.20), (5.21), which are stronger than (5.13), (5.14). In the proof of Lemma 5.3, we saw that $|\hat{K}(\xi)| \leqslant \leqslant C|\xi|^{-2}$. Also, $\tilde{K}(0) = 0$ by (5.22). Hence,

$$V[\tilde{K}] = \frac{1}{2} \int \hat{K}(\xi)|\hat{\rho}(\xi)|^2 \, d\xi \leqslant C \int |\xi|^{-2} |\hat{\rho}(\xi)|^2 \, d\xi =$$
$$= C' \int \frac{\rho(x)\rho(y) \, dx \, dy}{|x-y|} = C'' V_{LR}(R_0). \tag{5.24}$$

On the other hand, $K^* = R_0 K'$ satisfies (5.13) and (5.14). In fact, (5.14) is immediate from (5.20), while (5.13) follows by writing $\partial^\alpha K^*(x) = = R_0 \int [\partial^\alpha K(x) - \partial^\alpha K(x-y)] \varphi_{R_0} * \varphi_{R_0}(y) \, dy$ and $|\partial^\alpha K(x) - \partial^\alpha K(x-y)| \leqslant$

$^{r}|y| \cdot \sup_{0 \leq t \leq 1} |\nabla \partial^{\alpha} K(x - ty)|$. Here we recall that $R_1 \geqslant R_0^{10}$. Applying Lemma 5.3 to K^*, we find that

$$V[K'] \leqslant \frac{C}{R_0}(H^{\Omega}_{N,N'} + CN + CN').$$

Combining this with (5.24) and recalling that $K = \tilde{K} + K'$, we obtain (5.23).

6. A Swiss Cheese

Fix radii $100 < R_1 < R_2 < \ldots < R_M$ with $R_{k+1} > 100\, R_k$, and take a cube Q^+ of diameter $\sim M^{10} R_M$. For $\overline{M}$ between $M/2$ and M, we describe how to cut Q^+ into balls of radii $R_1, R_2, \ldots, R_{\overline{M}}$ and a small left-over part.

First cut Q^+ into a grid $\{Q_\nu\}$ of cubes of side $\sim 10 R_{\overline{M}}$, and place a ball B_ν of radius $R_{\overline{M}}$ in the center of each Q_ν. Next, cut Q^+ into a grid $\{Q'_\nu\}$ of cubes of side $\sim 10 R_{\overline{M}-1}$, and place a ball B'_ν of radius $R_{\overline{M}-1}$ in the center of each Q'_ν which does not meet any of the balls already introduced. Continue in this way until we have a family $\{B_{k\alpha}\}$ of balls of radius $R_k (2 \leqslant k \leqslant \overline{M})$ in Q^+. Finally, cut Q^+ into a grid of cubes $\{Q_\nu\}$ of side $\sim R_1$, and retain those Q_α which are not contained in any of the balls $B_{k\alpha}$. In this way, we cover Q^+ by balls $B_{k\alpha}$ and cubes Q_α. Note the following properties.

$$\text{Distinct balls } B_{k\alpha}, B_{k'\alpha'} \text{ have distance } > 50 \text{ from each other.} \tag{6.1}$$

$$\sum_\alpha |B_{k\alpha}| \leqslant e^{-c(\overline{M}-k)} |Q^+| \quad \text{for} \quad 2 \leqslant k \leqslant \overline{M}. \tag{6.2}$$

$$\sum_\alpha |Q_\alpha| \leqslant e^{-c\overline{M}} |Q^+|. \tag{6.3}$$

Next suppose R^3 is cut into a grid of cubes $\{Q_\nu^+\}$, all congruent to Q^+. We can translate our covering of Q^+ to cover each of the Q_ν^+, thus obtaining a covering of all R^3 by balls $B_{k\alpha}$ of radius R_k, and cubes Q_α of side $\sim R_1$.

We introduce a partition of unity $1 = \sum_{k\alpha} \theta_{k\alpha}^2 + \sum_\alpha \theta_\alpha^2$ with the following properties.

Each $\theta_{k\alpha}(x) = \theta_k(x - x_{k\alpha})$, where $x_{k\alpha}$ is the center of $B_{k\alpha}$; and $\theta_k(x)$ is spherically symmetric, supported in $|x| \leqslant R_k$, and satisfies $|\partial^\gamma \theta_k(x)| \leqslant C_\gamma$ uniformly in k. (6.4)

Each $\theta_\alpha(x)$ is supported in $\{\operatorname{dist}(x, Q_\alpha) < 1\} = \tilde{Q}_\alpha$ and satisfies $|\partial^\gamma \theta_\alpha| \leqslant$ $\leqslant C\gamma$. (6.5)

It is easy to construct such a partition. One picks the θ_k first so that $\theta_k(x) = 1$ in $|x| < R_k - 1$, and $(1 - \theta_k^2)^{1/2} \in C^\infty$. Then the $\theta_{k\alpha}$ are defined, and $\sum_{k\alpha} \theta_{k\alpha}^2(x) + \varphi^2(x) = 1$ for some smooth function φ. Finally, one defines θ_α so that $\sum_\alpha \theta_\alpha^2 = \varphi^2$. Recall that $B_{k\alpha}$, Q_α, $\theta_{k\alpha}$, θ_α all depend on $\overline{M}((M/2) \leqslant \overline{M} \leqslant$ $\leqslant M)$. In all that follows, we will take radii $R_1 < R_2 < \ldots < R_M$ so that

$e^{c_2\beta} < R_1$ and $R_M < e^{c_1\beta}$ with $0 < c_2 < c_1 \ll 1$ to be picked later. Thus, $M \sim \beta$, and the results of Section 3 apply to all the $B_{k\alpha}$.

7. Comparison with an Exploded System

Fix $\bar{M}$ and Q^+ as in the preceding section. Thus, R^3 is covered by balls $B_{k\alpha}$ and cubes Q_α. Recall $\tilde{Q}_\alpha = \{\text{dist}(x, Q_\alpha) < 1\}$. Define

$$\mathcal{Q}^{\bar{M}} = \mathcal{Q} = \{D = B_{k\alpha} \quad \text{or} \quad \tilde{Q}_\alpha \mid D + \tau \quad \text{meets } \Omega \text{ for some} \quad \tau \in Q^+\}$$
$$\mathcal{Q}^{0\bar{M}} = \mathcal{Q}^0 = \{D \in \mathcal{Q} \mid D + \tau \subset \Omega \quad \text{for every} \quad \tau \in Q^+\}.$$

For $D \in \mathcal{Q}$, define a vector $\xi(D)$ so that the translates $\hat{D} = D + \xi(D)$ are pairwise disjoint for distinct $D \in \mathcal{Q}$. Then define the exploded set $\Omega_{ex}^{\bar{M}} = \bigcup_{D \in \mathcal{Q}} \hat{D}$. Often, we shall omit the superscript and just speak of Ω_{ex}. On Ω_{ex} we define a two-particle potential

$$K(z, z') = \begin{cases} |z - z'|^{-1} & \text{if} \quad z, z' \in \hat{D} \quad \text{with} \quad D = \text{one of the} \quad B_{k\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $K = 0$ if z, z' belong to different components of Ω_{ex}, or if both particles belong to the same $\hat{D}$ with $D = \tilde{Q}_\alpha$. For particles $z_1 \ldots z_{N+N'}$ in Ω_{ex}, with charges $\epsilon(1), \ldots, \epsilon(N + N')$, define the potential $V_{ex} = \frac{1}{2} \sum_{j \neq k} \epsilon(j)\epsilon(k)K(z_j, z_k)$. Then define the Hamiltonian

$$H_{NN'}^{ex\bar{M}} = -\varkappa_1 \Delta_x - \varkappa_2 \Delta_y + V_{ex}, \quad \text{acting on} \quad L_{N,N'}^2(\Omega_{ex}^{\bar{M}})$$

with Dirichlet boundary conditions.

For fixed $\bar{M}$ and $\tau \in Q^+$, there is a natural injection $\iota_\tau^{\bar{M}} = \iota : L_*^2(\Omega) \to L_*^2(\Omega_{ex})$, which we use to relate observables on Ω_{ex} to those on Ω. Preparing to define ι, we set $\theta(x, D) = \theta_{k\alpha}(x)$ if $D = B_{k\alpha}$, $\theta(x, D) = \theta_\alpha(x)$ if $D = \tilde{Q}_\alpha$. Thus, $\theta(x, D)$ is supported in $x \in D$, and $\sum_{D \in \mathcal{Q}} \theta^2(x - \tau, D) = 1$ for $x \in \Omega$, $\tau \in Q^+$. Now we define ι. This means that for $\psi \in L_{N,N'}^2(\Omega)$ and $\hat{x}_1 \ldots \hat{x}_N \hat{y}_1 \ldots \hat{y}_{N'} \in \Omega_{ex}$, we have to define $(\iota\psi)(\hat{x}_1 \ldots \hat{x}_N \hat{y}_1 \ldots \hat{y}_{N'})$. Each $\hat{x}_j$ belongs to a unique $\hat{D}_j$, so we can write $\hat{x}_j = x_j + \xi(D_j)$ for $x_j \in D_j$ and $D_j \in \mathcal{Q}$. Similarly, we can express $\hat{y}_k = y_k + \xi(D'_k)$ for $y_k \in D'_k$ and $D'_k \in \mathcal{Q}$. We define

$$\begin{aligned}(\iota\psi)(\hat{x}_1 \ldots \hat{x}_N, \hat{y}_1 \ldots \hat{y}_{N'}) &= \\ &= \prod_{j=1}^{N} \theta(x_j - \tau, D_j) \prod_{k=1}^{N'} \theta(y_k - \tau, D'_k) \cdot \psi(x_1 \ldots x_N, y_1 \ldots y_{N'}).\end{aligned}$$

Here, the right-hand side is interpreted as zero if any of the $x_1 \ldots x_N y_1 \ldots y_{N'}$ fail to belong to Ω. This is an isometry from $L_{N,N'}^2(\Omega)$ into $L_{N,N'}^2(\Omega_{ex})$.

It is important to compare the Hamiltonian $H^{\Omega}_{N,N'}$ with

$$\mathscr{k} = Av_{\overline{M}\tau}[(\iota_\tau^{\overline{M}})^* H^{ex\overline{M}}_{N,N'}, (\iota_\tau^{\overline{M}})].$$

Here, $Av_{\tau,\overline{M}}$ means an average over all $\tau \in Q^+$ and all $\overline{M}$ between $M/2$ and M. A computation analogous to that in [3] shows that

$$\mathscr{k} = -\varkappa_1\Delta_x - \varkappa_2\Delta_y + V[K] + G\cdot(\varkappa_1 N + \varkappa_2 N')$$

where

$$K(x) = |x|^{-1} \sum_{2\le k\le M} \lambda_k \frac{\theta_k^2 * \theta_k^2(x)}{\frac{4}{3}\pi R_k^3}$$

$$\lambda_k = Av_{\overline{M}}\Big[\sum_{\alpha\ni B_{k\alpha}\subset Q^+} |B_{k\alpha}|/|Q^+|\Big]$$

$$G = -Av_{\overline{M}}Av_\tau\Big[\sum_{k\alpha}\theta_{k\alpha}\Delta\theta_{k\alpha}(\tau) + \sum_\alpha \theta_\alpha\Delta\theta_\alpha(\tau)\Big]$$

Here, θ_k is as in (5.4). Note that unlike [3], we are able to treat electrons and protons in the same way. Therefore, our potential energy term has exactly the form $V[K]$ without the extra error terms arising in [3].

Next we use the above formulas to compare $\mathscr{k}$ with $H^{\Omega}_{N,N'}$. Recall that the radii for our Swiss cheese satisfy

$$e^{c_2\beta} < R_1 < R_2 < \ldots < R_M < e^{c_1\beta} \quad \text{with} \quad c_1 \ll 1 \text{ and } M \sim \beta.$$

We have $|G| \leqslant e^{-c\beta}$ by the geometric properties (6.1), (6.2), (6.3). Also with

$$m_k = \frac{\theta_k^2 * \theta_k^2(0)}{\frac{4}{3}\pi R_k^3},$$

we have $m_k = 1 + O(R_k^{-1})$ so that $1 \geqslant \sum_k \lambda_k m_k \geqslant 1 - e^{-c\beta}$, again by (6.1), (6.2), (6.3).

Now write

$$\mathscr{k} = H^{\Omega}_{N,N'} + G(\varkappa_1 N + \varkappa_2 N') + V[K_1 + K_2 - K_3] \tag{7.1}$$

with

$$K_1(x) = \sum_{k=1}^{M} \lambda_k\Big[|x|^{-1}\frac{\theta_k^2 * \theta_k^2(x)}{\frac{4}{3}\pi R_k^3} - m_k(|x|^{-1} - |x|^{-1} * \varphi_{R_k} * \varphi_{R_k})\Big]$$

$$K_2(x) = \Big(\sum_{k=1}^{M} \lambda_k m_k - 1\Big)|x|^{-1}$$

$$K_3(x) = \sum_{k=1}^{M} \lambda_k m_k |x|^{-1} * \varphi_{R_k} * \varphi_{R_k}.$$

Here φ_{R_k} is as in the definition (5.1) of $V_{LR}(R_k)$. As in [3], we find that MK_1 and $e^{c\beta}K_2$ satisfy (5.13), (5.14) with $R_0 = 10^3$. Therefore, Lemma 5.3 yields

$$V[K_1] \leqslant \frac{C}{M}(H^{\Omega}_{N,N'} + CN + CN') \sim C\beta^{-1}(H^{\Omega}_{N,N'} + CN + CN')$$
$$V[K_2] \leqslant Ce^{-c\beta}(H^{\Omega}_{N,N'} + CN + CN').$$

From (6.1) we get $\mathscr{h} \leqslant H - V[K_3] + C\beta^{-1}(H + CN + CN')$, so that

$$\ln \operatorname{Tr} \exp\{\mu(N + N') - \beta(H - V[K_3] + \frac{C}{\beta}(H + CN + CN'))\} \leqslant$$
$$\ln \operatorname{Tr} \exp\{\mu(N + N') - \beta\mathscr{h}\} \leqslant Av_{\overline{M}T} \ln \operatorname{Tr} \exp\{\mu(N + N') - \beta H^{ex\,\overline{M}}\}.$$

The last inequality holds because $\operatorname{Tr}\exp\{\iota^*A\iota\} \leqslant \operatorname{Tr}\exp A$ for injections ι of Hilbert spaces, and because $A \to \ln \operatorname{Tr}\exp A$ is convex; again, see [3]. We are now regarding all operators as acting on L^2_*. Our estimate may be rewritten as

$$\begin{aligned} \ln \operatorname{Tr} \exp\{(\mu - C')(N + N') - (\beta + C_1)H^{\Omega} + \beta V[K_3]\} &\leqslant \\ \leqslant Av_{\tau\overline{M}} \ln \operatorname{Tr} \exp\{\mu(N + N') - \beta H^{ex\overline{M}}\} & \\ = Av_{\overline{M}} \ln \prod_{B_{k\alpha} \in \mathfrak{Q}^{\overline{M}}} Z(\mu, \beta, B_{k\alpha}) \cdot \prod_{\tilde{Q}_\alpha \in \mathfrak{Q}^{\overline{M}}} Z_0(\mu, \beta, \tilde{Q}_\alpha), & \end{aligned} \tag{7.2}$$

since the system is made of the non-interacting subsystems $\hat{D}, D \in \mathfrak{Q}^{\overline{M}}$. Now (3.9) and its analogue for Z_{neutral} show that

$$Z(\mu, \beta, B_{k\alpha}) \leqslant Z_{\text{neutral}}(\mu, \beta + C_1, B_{k\alpha}) \cdot r(\mu, \beta, k) \quad \text{for some} \quad r(\mu, \beta, k) < 1,$$

while $Z_0(\mu, \beta, \tilde{Q}_\alpha) \leqslant \exp(Ce^{\mu}\beta^{-3/2}|\tilde{Q}_\alpha|)$. Hence (7.2) implies for some $\overline{M}$

$$\begin{aligned} &\operatorname{Tr} \exp\{(\mu - C')(N + N') - (\beta + C_1)H^{\Omega} + \beta V[K_3]\} \leqslant \\ &\leqslant \prod_{B_{k\alpha} \in \mathfrak{Q}^{\overline{M}}} (Z_{\text{neutral}}(\mu, \beta + C_1, B_{k\alpha}) \cdot r(\mu, \beta, k)) \cdot \exp(Ce^{\mu}\beta^{-3/2} \sum_{\tilde{Q}_\alpha \in \mathfrak{Q}^{\overline{M}}} |\tilde{Q}_\alpha|). \end{aligned} \tag{7.3}$$

By the method of Lebowitz-Lieb [7],

$$\prod_{B_{k\alpha} \in \mathfrak{Q}^{0\overline{M}}} Z_{\text{neutral}}(\mu, \beta + C_1, B_k\alpha) \leqslant \operatorname{Tr} \exp\{\mu(N + N') - (\beta + C_1)H^{\Omega}\}.$$

Moreover, the product

$$\prod_{B_{k\alpha} \in \mathfrak{Q}^{\overline{M}} \setminus \mathfrak{Q}^{0\overline{M}}} Z_{\text{neutral}}(\mathrm{M}, \beta + C_1, B_{k\alpha})$$

is absorbed by

$$\prod_{B_{k\alpha} \in \mathfrak{Q}^{\overline{M}}} r(\mu, \beta, k),$$

provided Ω is big enough. (To be precise, we require that $\mathrm{Vol}\{\mathrm{dist}(x, \partial\Omega) < e^{c_1\beta}\}/\mathrm{Vol}\,\Omega$ be less than a small constant depending on μ, β.) Therefore (7.3) implies

$$\begin{aligned} &\operatorname{Tr}\exp\{(\mu - C')(N + N') - (\beta + C_1)H^\Omega + \beta V[K_3]\} \leqslant \\ &\leqslant \operatorname{Tr}\exp\{\mu(N + N') - (\beta + C_1)H^\Omega\} \cdot \exp(Ce^\mu\beta^{-3/2} \sum_{\tilde{Q}_\alpha \in \bar{\mathfrak{Q}}^M} |\tilde{Q}_\alpha|). \end{aligned} \tag{7.4}$$

The last factor on the right is at most $\exp(Ce^{-\beta/4}\rho|\Omega|)$ by (4.10). Consequently, applying (7.4) to $\bar{\beta} = \beta - C_1$, we have

$$\begin{aligned} &\operatorname{Tr}\exp\{\mu(N + N') - \beta H^\Omega + ((\beta - C_1)V[K_3] - C'(N + N'))\} \leqslant \\ &\qquad\qquad \leqslant \operatorname{Tr}\exp\{\mu(N + N') - \beta H^\Omega\} \cdot \exp(Ce^{-\beta/4}\rho|\Omega|), \end{aligned}$$

which implies

$$\langle(\beta - C_1)V[K_3] - C'N - C'N'\rangle \leqslant Ce^{-\beta/4}\rho|\Omega|. \tag{7.5}$$

Recalling the definition of K_3, we see that

$$V[K_3] = \sum_k \lambda_k m_k V_{LR}(R_k) + O(R_1^{-1} \cdot (N + N')),$$

the error arising from self-energy terms in $V_{LR}(R_k)$. Since $R_1^{-1} \leqslant e^{-c_2\beta}$, estimate (7.5) yields

$$\left\langle \sum_{k=1}^M \lambda_k m_k V_{LR}(R_k) \right\rangle \leqslant \frac{C}{\beta}\langle N + N'\rangle + Ce^{-\beta/4}\rho|\Omega|. \tag{7.6}$$

Next, fix radii $R_1' < R_2' < \ldots < R_M' < R_1 < R_2 < \ldots < R_M$, so that $R_1' > e^{c_2\beta}, R_{k+1}' > 100R_k', R_1 > (R_M')^{20}, R_{k+1} > 100R_k, R_M < e^{c_1\beta}$, $M \sim \beta$, for $0 < 30c_2 < c_1 \ll 1$. Since the R_k' give rise to an ensemble of Swiss cheeses, we have the analogue of (7.6), namely

$$\left\langle \sum_{k=1}^M \lambda_k' m_k' V_{LR}(R_k') \right\rangle \leqslant \frac{C}{\beta}\langle N + N'\rangle + Ce^{-\beta/4}\rho|\Omega| \tag{7.7}$$

for Ω large enough. We use (7.7) to study the Swiss cheeses defined by $R_1 \ldots R_M$. Our starting point is (7.1). As before, we know that $V[K_2] + G(\varkappa_1 N + \varkappa_2 N') \leqslant e^{-c\beta}(H^\Omega + CN + CN'))$. Since $V_{LR}(R_k) \geqslant 0$, (7.1) implies

$$\mathscr{h} \leqslant H^\Omega + V[K_1] + e^{-c\beta}(H^\Omega + CN + CN'). \tag{7.8}$$

Now take $R_0 =$ any of the R_k', and define a cutoff function

$$\eta(x) = \begin{bmatrix} 1 & \text{for} \quad |x| > 2 \cdot R_0^{12} \\ 0 & \text{for} \quad |x| < R_0^{12} \\ & \text{smooth in between} \end{bmatrix}$$

Then Lemma 5.4 applies to $M\eta(x)K_1(x)$ if we use R_0^{12}, $R_1, R_2, \ldots$ in place of $R_1, R_2, \ldots$ in (5.21). Moreover, Lemma 5.3 applies to $e^{c\beta}(1-\eta(x))K_1(x)$. Those Lemmas yield

$$\begin{aligned} V[\eta K_1] &< \frac{C}{M} V_{LR}(R_0) + \frac{C}{MR_0}(H^\Omega + CN + CN') \\ &< \frac{C}{\beta} V_{LR}(R_0) + e^{-c\beta}(H^\Omega + CN + CN') \end{aligned}$$

and $V[(1-\eta)K_1] \leqslant e^{-c\beta}(H^\Omega + CN + CN')$. Putting these estimates into (7.8) gives

$$\mathscr{k} \leqslant H + \frac{C}{\beta} V_{LR}(R_0) + e^{-c\beta}(H^\Omega + CN + CN').$$

Recall that R_0 here can be any of the R_k', and that the coefficients $\lambda_k' m_k'$ sum approximately to 1. Therefore by taking a weighted sum of the last inequality over all the R_k', we conclude that

$$\mathscr{k} \leqslant H + \frac{C}{\beta}\left(\sum_{k=1}^{M} \lambda_k' m_k' V_{LR}(R_k')\right) + e^{-c\beta}(H^\Omega + CN + CN'). \tag{7.9}$$

8. The Expected Value of Certain Observables

In this section, we explain how to estimate the expected value of certain observables $\mathfrak{A}$ on $L^2_*(\Omega)$ in terms of information on the exploded system. Suppose for each τ, $\overline{M}$ we specify an observable $A_{ex}(\tau, \overline{M})$ on the exploded system $L^2_*(\Omega^{\overline{M}}_{ex})$, and suppose

$$\begin{aligned} &\operatorname{Tr}\exp\{A_{ex}(\tau,\overline{M}) + \mu(N+N') - \bar{\beta}H^{ex\overline{M}}\} \leqslant \\ &\qquad \leqslant e^S \operatorname{Tr}\exp\{\mu(N+N') - \bar{\beta}H^{ex\overline{M}}\} \quad \text{for each} \quad \tau, \overline{M}. \end{aligned} \tag{8.1}$$

Here S is a real number independent of τ, $\overline{M}$; and $\bar{\beta}$ very near β is to be determined. There is an induced observable

$$\mathfrak{A} = Av_{\tau,\overline{M}}[(\iota_\tau^{\overline{M}})^* A_{ex}(\tau,\overline{M})(\iota_\tau^{\overline{M}})]$$

defined on $L^2_*(\Omega)$.

Our goal here is to estimate $\langle \mathfrak{A} \rangle$. Later on, we shall pick $A_{ex}(\tau, \overline{M})$ so that our estimate on $\langle \mathfrak{A} \rangle$ gives a strong hold on what most of the particles are doing.

Start with estimate (7.9). Setting $\bar{V}_{LR} = \sum_{k=1}^{M} \lambda'_k m'_k V_{LR}(R'_k)$, we have from (7.9)

$$\begin{aligned}\ln \operatorname{Tr} \exp\{\mathfrak{A} + \mu(N+N') - \bar{\beta}H^{\Omega} - C\bar{V}_{LR} - e^{-c\bar{\beta}}(H^{\Omega} + CN + CN')\} &\leqslant \\ \leqslant \ln \operatorname{Tr} \exp\{\mathfrak{A} + \mu(N+N') - \bar{\beta}\mathfrak{k}\} &\leqslant \\ Av_{\tau\bar{M}} \ln \operatorname{Tr} \exp\{A_{ex}(\tau, \bar{M}) + \mu(N+N') - \bar{\beta}H^{ex\bar{M}}\}. \qquad (8.2)\end{aligned}$$

The last estimate holds because $\operatorname{Tr}\exp(\iota^* A \iota) \leqslant \operatorname{Tr}\exp A$ for injections ι of Hilbert spaces, and because $A \to \ln \operatorname{Tr}\exp A$ is convex.

From (8.1) and (8.2) we obtain for at least one $(\tau, \bar{M})$ that

$$\begin{aligned}\operatorname{Tr} \exp\{\mathfrak{A} + \mu(N+N') - \bar{\beta}H^{\Omega} - C\bar{V}_{LR} - e^{-c\bar{\beta}}(H^{\Omega} + CN + CN')\} &\leqslant \\ \leqslant e^{S} \operatorname{Tr} \exp\{\mu(N+N') - \bar{\beta}H^{ex\bar{M}}\} &= \\ = e^{S} \cdot \prod_{B_{k\alpha} \in \mathfrak{Q}^{\bar{M}}} Z(\mu, \bar{\beta}, B_{k\alpha}) \cdot \prod_{\tilde{Q}_{\alpha} \in \tilde{\mathfrak{Q}}^{\bar{M}}} Z_0(\mu, \bar{\beta}, \tilde{Q}_{\alpha}). \qquad (8.3)\end{aligned}$$

As in the discussion of (7.3), (7.4), (7.5), we know from (4.9), (4.10) that

$$Z(\mu, \bar{\beta}, B_{k\alpha}) \leqslant Z_{\text{neutral}}\left(\mu, \bar{\beta} + \frac{C'}{\bar{\beta}}, B_{k\alpha}\right) \cdot r(\mu, \beta, k)$$

with C' a fixed large constant and $r < 1$.

Also

$$\prod_{\tilde{Q}_{\alpha} \in \tilde{\mathfrak{Q}}^{\bar{M}}} Z_0(\mu, \bar{\beta}, \tilde{Q}_{\alpha}) \leqslant \exp(Ce^{-\beta/4}\rho|\Omega|),$$

as in Section 7. Hence, as before, an application of the Lebowitz-Lieb technique [7] shows that

$$\begin{aligned}\operatorname{Tr} \exp\{\mathfrak{A} + \mu(N+N') - \bar{\beta}H^{\Omega} - C\bar{V}_{LR} - C'\bar{\beta}^{-1}(H^{\Omega} + CN + CN')\} &\leqslant \\ \leqslant e^{S} \operatorname{Tr} \exp\{\mu(N+N') - (\bar{\beta} + C'\bar{\beta}^{-1})H^{\Omega}\} \cdot \exp(Ce^{-\bar{\beta}/4}\rho|\Omega|).&\end{aligned}$$

In other words,

$$\begin{aligned}\operatorname{Tr} \exp\{\mu(N+N') - (\bar{\beta} + C'\bar{\beta}^{-1})H^{\Omega} + [\mathfrak{A} - C\bar{V}_{LR} - C'\bar{\beta}^{-1}(CN + CN')]\} &\leqslant \\ \leqslant [\operatorname{Tr} \exp\{\mu(N+N') - (\bar{\beta} + C'\bar{\beta}^{-1})H^{\Omega}\}] \cdot \exp(S + Ce^{-\bar{\beta}/4}\rho|\Omega|). \qquad (8.4)\end{aligned}$$

Now pick $\bar{\beta}$ near β so that $\bar{\beta} + C'\bar{\beta}^{-1} = \beta$. Then (8.4) implies

$$\langle \mathfrak{A} \rangle \leqslant C\langle \bar{V}_{LR} \rangle + \frac{C''}{\beta}\langle N + N' \rangle + Ce^{-\bar{\beta}/4}\rho|\Omega| + S.$$

Recalling

$$\bar{V}_{LR} = \sum_{k=1}^{M} \lambda_k' m_k' V_{LR}(R_k')$$

and (7.7), we get from the last inequality

$$\langle \mathfrak{A} \rangle \leqslant S + \frac{C}{\beta} \langle N + N' \rangle + Ce^{-\beta/4}\rho|\Omega|. \tag{8.5}$$

In the sections to follow, we pick $A_{ex}(\tau, \overline{M})$ so that (8.1) can be verified and (8.5) gives useful information.

9. The Density of the System

For fixed $\bar{M}$ the partition function of the exploded system is

$$\begin{aligned} \operatorname{Tr} e^{\mu(N+N') - \bar{\beta}H^{ex}} &= \prod_{B_{k\alpha} \in \mathfrak{Q}} Z(\mu, \bar{\beta}, B_{k\alpha}) \cdot \prod_{\tilde{Q}_\alpha \in \mathfrak{Q}} Z_0(\mu, \bar{\beta}, \tilde{Q}_\alpha) = \\ &= \exp\left\{ \frac{(\text{const})}{\bar{\beta}^{3/2}} e^{2\mu + (1/4)\bar{\beta}} |\Omega| \cdot (1 + O(\bar{\beta}^{-1})) \right\}, \end{aligned}$$

by the results of Section 3. Hence for $0 < t < 1$, we know that

$$\begin{aligned} &\operatorname{Tr} \exp\{ \pm t(N + N' - 2\bar{\rho}|\Omega|) + \mu(N + N') - \bar{\beta}H^{ex}\} \leqslant \\ &\qquad \leqslant e^S \operatorname{Tr} \exp\{\mu(N + N') - \bar{\beta}H^{ex}\}, \end{aligned} \tag{9.1}$$

where

$$\bar{\rho} = \frac{(\text{const})e^{2\mu + (1/4)\bar{\beta}}}{\bar{\beta}^{3/2}}, \qquad S = C\bar{\rho}|\Omega|(\bar{\beta}^{-1} + t^2). \tag{9.2}$$

Writing

$$L_*^2(\Omega_{ex}) = \left[\sum_{N + N' \geq \bar{\rho}|\Omega|} \oplus L^2_{N,N'}(\Omega_{ex}) \right] \oplus \left[\sum_{N + N' < \bar{\rho}|\Omega|} \oplus L^2_{N,N'}(\Omega_{ex}) \right]$$

and applying (9.1) with the two signs $\pm$ for the two spaces in square brackets, we conclude that

$$\begin{aligned} &\operatorname{Tr} \exp\{ t|N + N' - 2\bar{\rho}|\Omega|| + \mu(N + N') - \bar{\beta}H^{ex}\} \leqslant \\ &\qquad \leqslant e^S \operatorname{Tr} \exp\{\mu(N + N') - \bar{\beta}H^{ex}\}. \end{aligned}$$

This holds for each $\bar{M}$. Moreover, the number operators N, N' commute with the injections $\iota_\tau^{\bar{M}}: L^2_*(\Omega) \to L^2_*(\Omega_{ex})$. So if we define

$$A_{ex}(\tau, \bar{M}) = t|N + N' - 2\bar{\rho}|\Omega||,$$

then $\mathcal{Q} = Av_{\tau,\bar{M}}\big(\iota_\tau^{\bar{M}}\big)^* A_{ex}(\tau, \bar{M})\big(\iota_\tau^{\bar{M}}\big)] = t|N + N' - 2\rho|\Omega||$ also. Therefore, estimate (7.5) yields

$$t\langle |N + N' - 2\bar{\rho}|\Omega||\rangle \leqslant \frac{C}{\bar{\beta}}\langle N + N'\rangle + C\bar{\rho}|\Omega|(\bar{\beta}^{-1} + t^2) \leqslant$$
$$\leqslant \frac{C}{\bar{\beta}}\langle |N + N' - 2\bar{\rho}|\Omega||\rangle + C'\bar{\rho}|\Omega|(\bar{\beta}^{-1} + t^2).$$

Picking $t = \bar{\beta}^{-1/2}$, we can absorb the first term on the right into the left-hand side, leaving us with

$$\langle |N + N' - 2\bar{\rho}|\Omega||\rangle \leqslant C\bar{\beta}^{-1/2}\bar{\rho}|\Omega|.$$

Recalling (4.10), (9.2) and $\bar{\beta} + C'\bar{\beta}^{-1} = \beta$, we can rewrite the last estimate as

$$\langle |N + N' - 2\rho|\Omega||\rangle \leqslant C\beta^{-1/2}\rho|\Omega|. \tag{9.3}$$

Hence the total number of particles clusters around the obvious guess $2\rho|\Omega|$. In view of (9.3), we can rewrite estimate (8.5) in the simpler form

$$\langle \mathcal{Q}\rangle \leqslant S + \frac{C}{\beta}\rho|\Omega|. \tag{9.4}$$

Estimate (9.4) holds when $\mathcal{Q} = Av_{\tau,\bar{M}}\big[\big(\iota_\tau^{\bar{M}}\big)^* A_{ex}(\tau, \bar{M})\big(\iota_\tau^{\bar{M}}\big)$ for $A_{ex}(\tau, \bar{M})$ satisfying (8.1), with $\bar{\beta} + C'\bar{\beta}^{-1} = \beta$ and $\bar{\beta}$ near β.

In addition to (9.3), we shall need to know later that if $F \subset \Omega$ with $F_* = \{\mathrm{dist}(x, F) < e^{c_1\beta}\}$ having volume $|F_*| < \beta^{-1}|\Omega|$, then

$$\langle \text{Number of particles in } F\rangle \leqslant C\beta^{-1}\rho|\Omega|. \tag{9.5}$$

To see this, let $A_{ex}(\tau, \bar{M}) =$ [Number of particles in those $\hat{D}$ with $(D + \tau)\cap F \neq \phi, D \in \mathcal{Q}^M$]. Since $A_{ex}(\tau, \bar{M}) = \sum_j \chi_{F_{ex}}(z_j)$ with

$$F_{ex} = \bigcup\{\hat{D} \mid (D + \tau)\cap F \neq \phi, D \in \mathcal{Q}^{\bar{M}}\},$$

one computes easily that $\big(\iota_\tau^{\bar{M}}\big)^* A_{ex}(\tau, \bar{M})\big(\iota_\tau^{\bar{M}}\big) = \sum_j V(z_j)$ with

$$V(x) = \sum_{\substack{(D+\tau)\cap F \neq \phi \\ D \in \mathcal{Q}^M}} \theta^2(x - \tau, D) \geqslant \chi_F(x).$$

Hence $\mathcal{Q} = Av_{\tau\bar{M}}\big[\big(\iota_\tau^{\bar{M}}\big)^* A_{ex}(\tau, \bar{M})\big(\iota_\tau^{\bar{M}}\big)\big] \geqslant$ (Number of particles in F).

On the other hand, for fixed τ, $\overline{M}$ we have

$$\begin{aligned}
&\operatorname{Tr}\exp\{A_{ex} + \mu(N+N') - \bar{\beta}H^{ex}\} = \\
&= \prod_{\substack{B_{k\alpha}\in\mathcal{Q} \\ (B_{k\alpha}+\tau)\cap F=\phi}} Z(\mu,\bar{\beta},B_k)\cdot \prod_{\substack{\tilde{Q}_\alpha\in\mathcal{Q} \\ (\tilde{Q}_\alpha+\tau)\cap F=\phi}} Z_0(\mu,\bar{\beta},\tilde{Q}_\alpha)\cdot \\
&\cdot \prod_{\substack{B_{k\alpha}\in\mathcal{Q} \\ (B_{k\alpha}+\tau)\cap F\neq\phi}} Z(\mu+1,\bar{\beta},B_k)\cdot \prod_{\substack{\tilde{Q}_\alpha\in\mathcal{Q} \\ (\tilde{Q}_\alpha+\tau)\cap F\neq\phi}} Z_0(\mu+1,\bar{\beta},\tilde{Q}_\alpha) \leqslant \\
&\leqslant e^S \operatorname{Tr}\exp\{\mu(N+N') - \bar{\beta}H^{ex}\} \quad \text{with} \quad S = C\bar{\rho}|F_*|,
\end{aligned}$$

by the results of Section 4. So (9.4) gives

$$\langle \text{Number of particles in } F\rangle \leqslant \langle \mathcal{A}\rangle \leqslant \frac{C}{\beta}\rho|\Omega| + C\rho|F_*|,$$

which proves (9.5).

10. Particles Form Atoms

The next application of the technique of Section 8 is to show that the vast majority of the electrons and protons pair up into "atoms". Our definition of "atom" is at first quite weak. For $0 < c' < c'' \ll 1$, we define an *atom of type* (c', c'') as an electron-proton pair $\{x_{j0}, y_{k0}\}$ with $|x_{j0} - y_{k0}| < e^{c'\beta}$ and $|x_{j0} - z_l|$, $|y_{k0} - z_l| > e^{c''\beta}$ for any particle z_l other than x_{j0}, y_{k0}.

To show that most of the particles belong to atoms, we use the observables

$$\begin{aligned}
A_{ex}(\tau,\overline{M}) &= \sum_{B_{k\alpha}\in\mathcal{Q}^{\overline{M}}} (\text{Number of particles in } \hat{B}_{k\alpha})\chi_{\hat{B}_{k\alpha}\,\text{not of ep-type}}, \\
&\equiv \sum_{B_{k\alpha}\in\mathcal{Q}^{\overline{M}}} A_{ex}(\tau,\overline{M},B_{k\alpha})
\end{aligned}$$

where $\hat{B}_{k\alpha}$ is of *ep-type* if it contains exactly one electron and one proton. For a fixed τ, $\overline{M}$ we have

$$\begin{aligned}
\operatorname{Tr}\exp\{A_{ex}(\tau,\overline{M}) + \mu(N+N') - \bar{\beta}H^{ex\overline{M}}\} &= \prod_{\tilde{Q}_\alpha\in\mathcal{Q}^{\overline{M}}} Z_0(\mu,\bar{\beta},\tilde{Q}_\alpha)\cdot \\
&\cdot \prod_{B_{k\alpha}\in\mathcal{Q}^{\overline{M}}} \operatorname{Tr}\exp\{A_{ex}(\tau,\overline{M},B_{k\alpha}) + \mu(N+N') - \bar{\beta}H^{B_{k\alpha}}\},
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Tr}\exp\{A_{ex}(\tau,\overline{M},B_{k\alpha}) + \mu(N+N') - \bar{\beta}H^{B_{k_\alpha}}\} = \\
&\qquad Z(\mu,\bar{\beta},B_{k\alpha},1,1) + \sum_{(N,N')\neq(1,1)} Z(\mu+1,\bar{\beta},B_{k\alpha},N,N').
\end{aligned}$$

Hence the results of Section 4 show that

$$\operatorname{Tr}\exp\{A_{ex}(\tau,\overline{M})+\mu(N+N')-\bar{\beta}H^{ex\overline{M}}\}\leqslant$$
$$\leqslant e^{S}\operatorname{Tr}\exp\{\mu(N+N')-\bar{\beta}H^{ex\overline{M}}\}\quad\text{with}\quad S=C\beta^{-1}\rho|\Omega|.$$

Estimate (9.4) therefore shows that

$$\langle\mathcal{A}\rangle\leqslant\frac{C}{\beta}\rho|\Omega|,\qquad \mathcal{A}=Av_{T,\overline{M}}\big[\big(\iota_\tau^{\overline{M}}\big)^*A_{ex}(\tau,\overline{M})\big(\iota_\tau^{\overline{M}}\big)\big].\tag{10.0}$$

Now we compute $\mathcal{A}$. It is convenient to work temporarily with operators which do not necessarily preserve antisymmetry of wave functions.

First let us fix τ, $\overline{M}$, $B_{k\alpha}\in\mathcal{Q}^{\overline{M}}$ and $J\subset\{1\ldots N+N'\}$. Then define an operator A_J on $L^2_*(\Omega_{ex})$ by

$$A_J\psi(\hat{z}_1\ldots\hat{z}_{N+N'})=\prod_{j\in J}\chi_{\hat{B}_{k\alpha}}(\hat{z}_j)\cdot\prod_{j\notin J}(1-\chi_{\hat{B}_{k\alpha}}(\hat{z}_j))\cdot\psi(\hat{z}_1\ldots\hat{z}_{N+N'}).$$

We have

$$(\iota\psi)(\hat{z}_1\ldots\hat{z}_{N+N})=\prod_j\theta(z_j-\tau,D_j)\psi(z_1\ldots z_{N+N'})$$

for $\psi\in L^2_*(\Omega)$, $\hat{z}_j=z_j+\xi(D_j)$ with $z_j\in D_j$, $D_j\in\mathcal{Q}^{\overline{M}}$. Hence

$$\langle A_J\iota\psi,\iota\psi\rangle=\sum_{D_1\ldots D_{N+N'}}\int\prod_{j\in J}\theta^2(z_j-\tau,B_{k\alpha})\chi_{D_j=B_{k\alpha}}\cdot$$
$$\cdot\prod_{j\notin J}\theta^2(z_j-\tau,D_J)\chi_{D_j\neq B_{k\alpha}}|\psi(z_1\ldots z_{N+N'})|^2\,dz_1\ldots dz_{N+N'}=$$
$$=\int\Big[\prod_{j\in J}\theta^2_{k\alpha}(z_j-\tau)\Big]\Big[\prod_{j\notin J}(1-\theta^2_{k\alpha})(z_j-\tau)\Big]\cdot$$
$$\cdot|\psi(z_1\ldots z_{N+N'})|^2\,dz_1\ldots dz_{N+N'}\tag{10.1}$$

Next fix τ, $\overline{M}$, $B_{k\alpha}\in\mathcal{Q}^{\overline{M}}$, and integers n, n'. We define $A^{nn'}$ on $L^2_*\big(\Omega^{\overline{M}}_{ex}\big)$ by

$$A^{nn'}\psi(\hat{z}_1\ldots\hat{z}_{N+N'})=\chi\begin{pmatrix}\text{Number of electrons in }\hat{B}_{k\alpha}=n\\ \text{Number of protons in }\hat{B}_{k\alpha}=n'\end{pmatrix}\cdot\psi(\hat{z}_1\ldots\hat{z}_{N+N'}).$$

Thus $A^{nn'}=\sum_J A_J$ over those J containing n electrons and n' protons. So (10.1) implies

$$\iota^*A^{nn'}\iota=\sum_{\substack{|J|=n\\|J'|=n'}}\prod_{j\in J}\theta^2_{k\alpha}(x_j-\tau)\cdot\prod_{j\notin J}(1-\theta^2_{k\alpha}(x_j-\tau))\cdot$$
$$\cdot\prod_{j\in J'}\theta^2_{k\alpha}(y_j-\tau)\cdot\prod_{j\notin J'}(1-\theta^2_{k\alpha}(y_j-\tau)).\tag{10.2}$$

Hence, for

$$A_{ex}(\tau,\overline{M},B_{k\alpha})=\sum_{(n,n')\neq(1,1)}(n+n')\cdot A^{nn'}$$

we can prove that

$$\iota^* A_{ex}(\tau, \overline{M}, B_{k\alpha})\iota \geqslant \begin{bmatrix} \text{Number of particles in the middle half of} \\ B_{k\alpha} + \tau, \text{ assuming at least two of those} \\ \text{particles have the same charge} \end{bmatrix} \tag{10.3}$$

and

$$\iota^* A_{ex}(\tau, \overline{M}, B_{k\alpha})\iota \geqslant \chi\begin{pmatrix} \text{There is exactly one particle in } (B_{k\alpha} + \tau), \\ \text{and it lies in the middle half of } (B_{k\alpha} + \tau) \end{pmatrix}. \tag{10.4}$$

To check (10.3) and (10.4) we write down as a consequence of (10.2)

$$\iota^* A_{ex}(\tau, \overline{M}, B_{k\alpha})\iota = \sum_{\substack{J, J' \\ (|J|, |J'|) \neq (1,1)}} (|J| + |J'|) \cdot \prod_{j \in J} \theta_{k\alpha}^2(x_j - \tau) \cdot$$

$$\cdot \prod_{j \notin J'} (1 - \theta_{k\alpha}^2(x_j - \tau)) \cdot \prod_{j \in J'} \theta_{k\alpha}^2(y_j - \tau) \cdot \prod_{j \notin J'} (1 - \theta_{k\alpha}^2(y_j - \tau)). \tag{10.5}$$

If $J_0 = \{ j \mid x_j \in \text{middle half of } B_{k\alpha} + \tau \}$, $J_0' = \{ j \mid y_j \in \text{middle half of } B_{k\alpha} + \tau \}$, then in the last equation, we restrict the sum to $J \supset J_0$ and $J' \supset J_0'$, and replace $|J| + |J'|$ by the smaller $|J_0| + |J_0'|$. If $|J_0|$ or $|J_0'| \geqslant 2$, then we never have $(|J|, |J'|) = (1,1)$. Consequently,

$$\iota^* A_{ex}(\tau, \overline{M})\iota \geqslant (|J_0| + |J_0'|) \sum_{\substack{G \subset {}^cJ_0 \\ G' \subset {}^cJ_0'}} \prod_{j \in G} \theta_{k\alpha}^2(x_j - \tau) \prod_{j \in {}^cJ_0 \setminus G} (1 - \theta_{k\alpha}^2(x_j - \tau)) \cdot$$

$$\cdot \prod_{j \in G'} \theta_{k\alpha}^2(y_j - \tau) \cdot \prod_{j \in {}^cJ'_0 \setminus G'} (1 - \theta_{k\alpha}^2(y_j - \tau)).$$

Here we use $G = J \setminus J_0$, $G' = J' \setminus J_0'$. Now the big sum on the right is simply 1, so (10.3) follows. To prove (10.4), suppose say x_{j_0} belongs to the middle half of $B_{k\alpha}$, and no other particles lie in $B_{k\alpha}$. Then we take $J = \{ j_0 \}$, $J' = \phi$ in (10.5), and we find at once that $\iota^* A_{ex}(\tau, \overline{M})\iota \geqslant 1$. The same argument works if y_{j_0} is the only particle in $B_{k\alpha}$. So (10.3) and (10.4) are proved.

For a fixed $\overline{M}$ we now sum (10.3), (10.4) over all $B_{k\alpha} \in \mathcal{Q}^{\overline{M}}$, and then average in τ. Recalling that the $B_{k\alpha}$ have radii between $e^{c_2\beta}$ and $e^{c_1\beta}$, we conclude that

$$Av_\tau\left[\left(\iota_\tau^{\overline{M}}\right)^* A_{ex}(\tau, \overline{M})\iota_\tau^{M}\right] \geqslant c. \quad \begin{pmatrix}\text{Number of particles } z_j \text{ for which at} \\ \text{least two particles } z_k, z_l \text{ of the same} \\ \text{charge lie within distance} \\ e^{(1/2)c_2\beta} \text{ of } z_j\end{pmatrix}, \tag{10.6}$$

and

$$Av_\tau\left[\left(\iota_\tau^{\overline{M}}\right)^* A_{ex}(\tau, \overline{M})\iota_\tau^{M}\right] \geqslant c. \quad \begin{pmatrix}\text{Number of particles which have} \\ \text{distance at least } e^{2c_1\beta} \text{ from all other} \\ \text{particles}\end{pmatrix}. \tag{10.7}$$

Average these estimates over $\overline{M}$, and apply (0). The conclusion is

$$\langle \text{Number of } z_j \text{ for which } B(z_j, e^{2c''\beta}) \text{ contains at least two particles of the same charge} \rangle \leqslant \frac{C}{\beta}\rho|\Omega| \tag{10.8}$$

$$\langle \text{Number of } z_j \text{ which have distance at least } e^{(1/2)c'\beta} \text{ from all other particles} \rangle \leqslant \frac{C}{\beta}\rho|\Omega| \tag{10.9}$$

provided c', c'' are small. To derive (10.8), we take a Swiss cheese with $\frac{1}{2}c_2 = 2c''$, while (10.9) requires a Swiss cheese with $\frac{1}{2}c' = 2c_1$. Now when $c' < c''$, (10.8) and (10.9) show that

$$\langle \text{Number of particles not in atoms of type } (c', c'') \rangle \leqslant \frac{C}{\beta}\rho|\Omega|. \tag{10.10}$$

Comparing this with (9.3), we see that with probability nearly 1, the great majority of particles belong to atoms of type (c', c'').

Finally, if $\{x_{j_0}, y_{k_0}\}$ form an atom of type (c', c''), then define the *displacement vector* of the atom simply as $\vec{r} = x_{j_0} - y_{k_0}$.

11. A Special Observable

In this section we compute $\mathfrak{A} = Av_{\tau,\overline{M}}[(\iota_\tau^{\overline{M}})^* A_{ex}(\tau, \overline{M})\iota_\tau^{\overline{M}}]$ for a special $A_{ex}(\tau, \overline{M})$ which is picked so that $\mathfrak{A}$ will yield strong information on the positions of the particles. Then in the next section we shall compute $\langle \mathfrak{A} \rangle$ by the method of Section 8.

To construct $A_{ex}(\tau, \overline{M})$, we begin with a few simple definitions. Recall that for fixed $\overline{M}$, a ball $\hat{B}_{k\alpha}$ is of *ep-type* if it contains exactly one electron and one proton. Given $\mathbb{S} \subset \mathfrak{Q}^{\overline{M}}$, we call $\mathbb{S}$ *monatomic* if exactly one of the $\hat{D}$, $D \in \mathbb{S}$ contains some particles, and if that $\hat{D}$ is of the form $\hat{B}_{k\alpha}$ rather than $\hat{\tilde{Q}}_\alpha$, and if finally $\hat{B}_{k\alpha}$ is of ep-type. If $\hat{B}_{k\alpha}$ is of ep-type and contains the electron x_μ and the proton y_ν, then define the *displacement vector* $\vec{r}(B_{k\alpha}) = x_\mu - y_\nu$. Similarly, if $\mathbb{S} \subset \mathfrak{Q}^{\overline{M}}$ is monatomic with $\hat{B}_{k\alpha}$ of ep-type, $B_{k\alpha} \in \mathbb{S}$, then define the displacement vector $\vec{r}(\mathbb{S}) = \vec{r}(B_{k\alpha})$.

Next, imagine Ω is partitioned into disjoint cubes $Q_1^s, Q_2^s, \ldots, Q_L^s$ of volume

$$|Q_j^s| = \frac{\lambda}{\rho}, \qquad \lambda \text{ a constant to be determined.} \tag{11.1}$$

Here L is a fixed large number to be determined, and the number of distinct s is $\sim(|\Omega|)/((\lambda/\rho)L)$, which of course grows large with Ω. A negligible part of Ω near $\partial\Omega$ may fall to be covered by the boxes Q_j^s.

Now for fixed $E \subset R^3$ and fixed $\tau, \overline{M}, s$, and for fixed sets J_1, J_2, J_3 partitioning $\{1, \ldots, L\}$, we define an event $\mathcal{E}_s = \mathcal{E}_s(\tau, \overline{M}, E, J_1, J_2, J_3)$ as follows.

$$\text{Let } \mathcal{Q}_j^s = \{D \in \mathcal{Q} \mid (D + \tau) \subset Q_j^s\}, \quad \text{and} \quad \mathcal{Q}^s = \bigcup_{j=1}^{L} \mathcal{Q}_j^s.$$

Then $\mathcal{E}_s$ means that:

$\mathcal{Q}_j^s$ is monatomic with displacement vector $\vec{r}(\mathcal{Q}_j^s) \in E$ for $j \in J_1$
$\mathcal{Q}_j^s$ is monatomic with displacement vector $\vec{r}(\mathcal{Q}_j^s) \notin E$ for $j \in J_2$
$\mathcal{Q}_j^s$ is not monatomic for $j \in J_3$.

Finally, we set $A_{ex}^s(\tau, \overline{M}) = \chi_{\mathcal{E}_s}$ and $A_{ex}(\tau, \overline{M}) = \sum_s A_{ex}^s(\tau, \overline{M})$. For fixed $\tau, \overline{M}, s$, we compute $\iota^* A_{ex}^s \iota$. To do so, look first at an arbitrary potential $\hat{V}(\hat{z}_1 \ldots \hat{z}_{N+N'})$ defined on $(\Omega_{ex})^{N+N'}$. By definition of $\iota = \iota_\tau^M$ we have

$$\langle \hat{V}\iota\psi, \iota\psi \rangle = \langle V\psi, \psi \rangle \quad \text{for} \quad \psi \in L^2_{N,N'}(\Omega), \tag{11.2}$$

with

$$V(z_1 \ldots z_{N+N'}) = \sum_{D_1 \ldots D_{N+N'} \in \mathcal{Q}} \prod_{l=1}^{N+N'} \theta^2(z_l - \tau, D_l) \cdot \\ \cdot \hat{V}(z_1 + \xi(D_1), \ldots, z_{N+N'} + \xi(D_{N+N'})). \tag{11.3}$$

Assume now $\hat{V}$ has the special form

$$\hat{V}(\hat{z}_1 \ldots \hat{z}_{N+N'}) = \prod_{j \notin \mathcal{J}} \chi_{\bigcup_{D \in \mathcal{S}} \tilde{D}}(\hat{z}_j) \cdot \prod_{j \notin \mathcal{J}} \chi_{c(\bigcup_{D \in \mathcal{S}} \tilde{D})}(\hat{z}_j) \cdot \hat{W}(\hat{z}_j, j \in \mathcal{J}) \tag{11.4}$$

for a collection $\mathcal{S} \subset \mathcal{Q}$ and $\mathcal{J} \subset \{1, 2, \ldots, N+N'\}$. Then in (11.3) we can carry out the sum over the D_l for $l \notin \mathcal{J}$, obtaining

$$V(z_1 \ldots z_N) = \left[\sum_{D_j \in \mathcal{S} \text{ for } j \in \mathcal{J}} \prod_{j \in \mathcal{J}} \theta^2(z_j - \tau, D_j) \hat{W}(z_j + \xi(D_j); j \in \mathcal{J}) \right] \cdot \\ \cdot \prod_{l \notin \mathcal{J}} \left(\sum_{D \notin \mathcal{J}} \theta^2(z_l - \tau, D) \right). \tag{11.5}$$

$$\text{Let } F = F(\tau, \overline{M}, \mathcal{S}) = \left[\bigcup_{\tilde{Q}_\alpha \in \mathcal{S}} (\tilde{Q}_\alpha + \tau) \right] \cup \left[\bigcup_{B_{k\alpha} \in \mathcal{S}} \{x \mid \operatorname{dist}(x, \partial B_{k\alpha} + \tau) < e^{\tilde{c}\beta}\} \right]$$

for $\tilde{c}$ smaller than the constants c_1, c_2 for our Swiss cheese. Later we will use the observation

$$Av_\tau \chi_{F(\tau, \overline{M}, \mathcal{S})}(x) \leqslant e^{-c\beta} \quad \text{for any } x. \tag{11.6}$$

For now, we continue to fix τ. Assume that none of the particles z_j lies in F. Then one checks easily that for $z \notin F$,

$$\sum_{D \notin \mathcal{S}} \theta^2(z - \tau, D) = 1 - \chi_{G(\mathcal{S})}(z) \quad \text{with} \quad G(\mathcal{S}) = \bigcup_{D \in \mathcal{S}} (D + \tau).$$

Moreover, if $z \notin F$ then we have $z - \tau \in D_z$ for a *unique* $D_z \in \mathcal{Q}$, and we have $\theta(z - \tau, D) = 1$ if $D = D_z$, 0 otherwise.
Setting $\gamma(z) = z + \xi(D_z)$ for $z \notin F$, and putting the above remarks into (11.5), we obtain

$$V(z_1 \dots z_{N+N'}) = \prod_{l \in \mathcal{J}} \chi_{G(\mathcal{S})}(z_l) \cdot \prod_{l \notin \mathcal{J}} (1 - \chi_{G(\mathcal{S})}(z_l)) \cdot \hat{W}(\gamma(z_j); j \in \mathcal{J}) \qquad (11.7)$$

when $z_1 \dots z_{N+N'} \notin F$.

Now specialize to the case

$$\mathcal{S} = \underline{\mathcal{Q}}^s = \bigcup_{j=1}^{L} \mathcal{Q}_j^s,$$

$\hat{W}(\hat{z}_j; j \in \mathcal{J})$ = characteristic function of the following event: After deleting all the particles $\hat{z}_l$ with $l \notin \mathcal{J}$, we find that

- (·) $\mathcal{Q}_j^s$ is monatomic with displacement vector in E for $j \in J_1$
- (·) $\mathcal{Q}_j^s$ is monatomic with displacement vector not in E for $j \in J_2$
- (·) $\mathcal{Q}_j^s$ is not monatomic for $j \in J_3$.

Thus $\hat{V}$ defined by (11.4) is the characteristic function of the event $\mathcal{E}_s \cap \{\hat{z}_j \in (\bigcup_{D \in \mathcal{S}} \hat{D})$ precisely for those $j \in \mathcal{J}\}$, while for $z_1 \dots z_{N+N'} \notin F = F(\tau, \overline{M}, \underline{\mathcal{Q}}^s)$, equation (11.7) shows that V is the characteristic function of the following event:

(a) $z_j \in G(\underline{\mathcal{Q}}^s)$ exactly for $j \in \mathcal{J}$.

(b) For $j \in J_1$, in $\mathcal{Q}_j^s$ there is a unique D with $D + \tau$ containing some particles; that D is a ball $B_{k\alpha}$, $D + \tau$ contains a single electron x_μ and a single proton y_ν; and $x_\mu - y_\nu \in E$.

(c) For $j \in J_2$, in $\mathcal{Q}_j^s$ there is a unique D with $D + \tau$ containing some particles; that D is a ball $B_{k\alpha}$, $D + \tau$ contains a single electron x_μ and a single proton y_ν; and $x_\mu - y_\nu \notin E$.

(d) For $j \in J_3$, it is *not* true that in $\mathcal{Q}_j^s$ there is a unique D with $D + \tau$ containing some particles, that D being a ball $B_{k\alpha}$ with $D + \tau$ containing a single electron and a single proton.

Sum this information over all subsets $\mathcal{J} \subset \{1, 2, \dots, N + N'\}$. Thus for fixed τ, $\overline{M}$, s we see that

$$\text{For} \qquad z_1 \dots z_{N+N'} \notin F(\tau, \overline{M}, \underline{\mathcal{Q}}^s) \qquad (11.8)$$

we have $\iota^* A^s_{ex}(\tau, \overline{M})\iota = V =$ characteristic function of the event defined by (*b*), (*c*), (*d*) above.

To clarify the meaning of this event, pick two small constants $0 < c' < c''$, and assume all the particles in $\bigcup_{j=1}^{L} Q^s_j$ belong to atoms of type (c', c'') in the sense of Section 9. We can pick c', c'' so that $0 < c' < \tilde{c} < c_2 < c_1 < c'' \ll 1$, where $\tilde{c}$ is the constant in the definition of F above, and c_1, c_2 are the constants related to the radii of the balls in the Swiss cheese ($e^{c_2\beta} <$ radius $(B_{k\alpha}) < e^{c_1\beta}$).

Assume also that there are no particles within distance $e^{2c''\beta}$ of ∂Q^s_j for $j = 1, \ldots, L$. Under our assumptions, (*b*), (*c*), (*d*) above are equivalent to

(*b*)′ For $j \in J_1$, Q^s_j contains exactly one atom, and its displacement vector lies in E.

(*c*)′ For $j \in J_2$, Q^s_j contains exactly one atom, and its displacement vector does not lie in E.

(*d*)′ For $j \in J_3$, Q^s_j fails to contain a unique atom.

Call this event $\mathcal{E}'_s$. Here, "atom" means "atom of type (c', c'')"; and (*b*), (*c*), (*d*) are equivalent to (*b*)′, (*c*)′, (*d*)′ provided:

(·) No particles lie in $F(\tau, \overline{M}, \underset{\sim}{\mathcal{Q}}^s)$.

(·) All particles in $\bigcup\limits_{j=1}^{L} Q^s_j$ belong to atoms of type (c', c'').

(·) No particles lie in $E_s = \bigcup\limits_{j=1}^{L} \{\operatorname{dist}(x, \partial Q^s_j) < e^{2c''\beta}\}$.

So for a fixed τ, $\overline{M}$, s, we know that $(\iota^{\overline{M}}_\tau)^* A^s_{ex}(\tau, \overline{M})\iota^{\overline{M}}_\tau = V$, where $V = \chi_{\mathcal{E}'_s}$ under the three assumptions just given.

Even without any assumptions, (11.2) and (11.3) show that $0 \leqslant V \leqslant 1$ always, since $A^s_{ex}(\tau, \overline{M})$ has the form $\hat{V} =$ characteristic function of an event. Consequently, we know that $|(\iota^M_\tau)^* A^s_{ex}(\tau, \overline{M})(\iota^M_\tau) - \chi_{\mathcal{E}'_s}| \leqslant \sum_{j=1}^{L}$ (Number of particles in Q^s_j not belonging to atoms of type (c', c'')) + (Number of particles belonging to $F(\tau, \overline{M}, \underset{\sim}{\mathcal{Q}}^s)$) + (Number of particles in E_s).

Average this over translates τ, and use estimate (11.6) with $\mathcal{S} = \underset{\sim}{\mathcal{Q}}^s$. The result is $|Av_\tau[(\iota^M_\tau)^* A^s_{ex}(\tau, \overline{M})(\iota^M_\tau)] - \chi_{\mathcal{E}'_s}| \leqslant \sum_{j=1}^{L}$ (Number of particles in Q^s_j not belonging to atoms of type (c', c'')) + $e^{-c\beta}$. (Number of particles in $\bigcup_{j=1}^{L} Q^s_j$) + (Number of particles in E_s). (Here we used $\chi_{F(\tau, \overline{M}, \mathcal{Q}^s)}(x) = 0$ for $x \notin \bigcup_{j=1}^{L} Q^s_j$).

Summing this over s and averaging in $\overline{M}$, we have for

$$\mathcal{A} = Av_{\tau, \overline{M}}[(\iota^{\overline{M}}_\tau)^* A_{ex}(\tau, M)(\iota^{\overline{M}}_\tau)]$$

the estimate $|\mathcal{A} - \sum_s \chi_{\mathcal{E}'_s}| \leqslant$ (Number of particles not in atoms of type (c', c'')) + (Number of particles in $\bigcup_s E_s$) + $e^{-c\beta}(N + N')$.

Since $|\bigcup_s E_s| \leqslant e^{-c\beta}|\Omega|$, estimates (9.3), (9.5), (10.10) imply

$$|\langle\mathcal{A}\rangle - \langle\sum_s \chi_{\mathcal{E}'_s}\rangle| \leqslant \frac{C}{\beta}\rho|\Omega| \quad \text{for} \quad \mathcal{A} = Av_{\tau\overline{M}}[(\iota_\tau^{\overline{M}})^* A_{ex}(\tau, \overline{M})(\iota_\tau^{\overline{M}})]. \tag{11.9}$$

Recalling that $\mathcal{E}'_s$ is the event described by $(b)'$, $(c)'$, $(d)'$ above, we see that $\langle\sum_s \chi_{\mathcal{E}'_s}\rangle$ carries a lot of information.

12. The Expected Value of the Special Observable

In this section we fix τ, $\overline{M}$ and compute $Tr\exp\{tA_{ex}(\tau, \overline{M}) + \mu(N+N') - \beta H^{ex\overline{M}}\}$ for $|t| \ll 1$ and $A_{ex}(\tau, \overline{M})$ as in Section 11.

Recall the definitions of $\mathcal{Q}_l^s$ and $\mathcal{Q}^s$, and define

$$\mathcal{Q}_{\text{extra}} = \mathcal{Q} \setminus \bigcup_{s,l} \mathcal{Q}_l^s = \{D \in \mathcal{Q} \mid D + \tau$$
$$\text{meets some} \quad \partial Q_l^s \quad \text{or lies within } \operatorname{diam}(Q_l^s) \quad \text{of} \quad \partial\Omega\}.$$

Let

$$\Omega_{ex}^s = \bigcup_{l=1}^{L}\Big(\bigcup_{D \in \mathcal{Q}_l^s} \hat{D}\Big) \equiv \bigcup_{l=1}^{L} \Omega_{ex\,l}^s.$$

We first note that H^{ex} and $A_{ex}(\tau, \overline{M})$ both break up as sums $H^{ex} = \sum_s H_{ex}^s + H_{ex}^{extra}$

$$A_{ex}(\tau, \overline{M}) = \sum_s A_{ex}^s,$$

with A_{ex}^s, H_{ex}^s acting on $L_*^2(\Omega_{ex}^s)$ and H_{ex}^{extra} acting on

$$L_*^2\Big(\bigcup_{D \in \mathcal{Q}_{extra}} \hat{D}\Big).$$

Consequently,

$$\begin{aligned} &\operatorname{Tr}\exp\{tA_{ex}(\tau, \overline{M}) + \mu(N+N') - \beta H^{ex}\} = \\ &\quad = \prod_s \operatorname{Tr}\exp\{tA_{ex}^s + \mu(N+N') - \beta H_{ex}^s \mid L_*^2(\Omega_{ex}^s)\} \cdot \\ &\quad\quad \cdot \prod_{B_{k\alpha} \in \mathcal{Q}_{extra}} Z(\mu, \beta, B_{k\alpha}) \cdot \prod_{\tilde{Q}_\alpha \in \mathcal{Q}_{extra}} Z_0(\mu, \beta, \tilde{Q}_\alpha). \end{aligned} \tag{12.1}$$

From Section 3, we know that the terms from $\mathcal{Q}_{extra}$ contribute a factor

$$\exp\Big[O\Big(\rho \sum_{D \in \mathcal{Q}_{extra}} |D|\Big)\Big] = \exp\Big[O\Big(\frac{\rho|\Omega|}{\beta}\Big)\Big]$$

for large β, Ω.

Fix s. We shall introduce some definitions to help us compute the right-hand side of (12.1).

$N(D)$, $N'(D)$ denote arbitrary assignments of a non-negative integer to each $D \in \mathfrak{Q}^s$.

$N_l(D)$, $N_l'(D)$ denote arbitrary assignments of a non-negative integer to each $D \in \mathfrak{Q}_l^s$.

Evidently, each $N(D)$, $N'(D)$ induces $N_l(D)$, $N_l'(D)$ for each l. We say that $N(D)$, $N'(D)$ is *monatomic* for $\mathfrak{Q}_l^s$, or equivalently that $N_l(D)$, $N_l'(D)$ is monatomic, if $N_l(D) = N_l'(D) = 0$ for all $D \in \mathfrak{Q}_l^s$ except for a single ball $B_{k\alpha}$ (called the *active* ball in $\mathfrak{Q}_l^s$) with $N_l(B_{k\alpha}) = N_l'(B_{k\alpha}) = 1$.

$\mathfrak{B}$ denotes an arbitrary assignment of a ball $B \in \mathfrak{Q}_l^s$ to each $l \in J_1 \bigcup J_2$. Define a set $\mathcal{E}(N(D), N'(D)) = \{(z_1 \dots z_{N+N'}) \mid \text{Each } z_j \in \Omega_{ex}^s \text{ and each } \hat{D} \text{ contains } N(D) \text{ electrons and } N'(D) \text{ protons, } D \in \mathfrak{Q}^s\}$. Given a $\mathfrak{B}$ and given $(N_l(D), N_l'(D))$ for $l \in J_3$, there is an induced

$$N(D), N'(D) = \left[\begin{array}{l} N_l(D), N_l'(D) \text{ if } D \in \mathfrak{Q}_l^s \text{ with } l \in J_3]\ 1,1 \text{ if} \\ D \in \mathfrak{Q}_l^s \text{ with } l \in J_1 \cup J_2 \text{ and } D \text{ is the} \\ \text{ball assigned to } l \text{ by } \mathfrak{B};\ 0,0 \text{ otherwise.} \end{array}\right. \tag{12.2}$$

We give the resulting set $\mathcal{E}(N(D), N'(D))$ the name $\mathcal{E}(\mathfrak{B}, (N_l(D), N_l'(D))_{l \in J_3})$.

Note that (12.2) is the most general $N(D), N'(D)$ which is monatomic for all $l \in J_1 \cup J_2$.

Now define subspaces of $L_*^2(\Omega_{ex}^s)$: $X(N(D), N'(D))$ = space of $\psi \in L_*^2(\Omega_{ex}^s)$ supported in $\mathcal{E}(N(D), N'(D))$ $X(\mathfrak{B}, (N_l(D), N_l'(D))_{l \in J_3}) = X(N(D), N'(D))$ with N, N' defined by (12.2).

We have

$$L_*^2(\Omega_{ex}^s) = \sum_{N(D), N'(D)} \oplus X(N(D), N'(D)),$$

and therefore

$$\begin{aligned} \Phi &= \operatorname{Tr} \exp\{tA_{ex}^s + \mu(N+N') - \beta H_{ex}^s \mid L_*^2(\Omega_{ex}^s)\} - \operatorname{Tr}\exp\{\mu(N+N') - \\ &\qquad - \beta H_{ex}^s \mid L_*^2(\Omega_{ex}^s)\} = \\ &= \sum_{N(D), N'(D)} [\operatorname{Tr}\exp\{tA_{ex}^s + \mu(N+N') - \beta H_{ex}^s \mid X(N(D), N'(D))\} - \\ &\qquad - \operatorname{Tr}\exp\{\mu(N+N') - \beta H_{ex}^s \mid X(N(D), N'(D))\}] = \\ &= \sum_{N(D), N'(D)} \phi(N(D), N'(D)). \end{aligned}$$

Let us recall how A_{ex}^s behaves. For $(z_1 \dots z_{N+N'}) \in \mathcal{E}(N(D), N'(D))$, we note that $\mathfrak{Q}_l^s$ is monatomic if and only if $N_l(D)$, $N_l'(D)$ is monatomic. Therefore, $A_{ex}^s = 0$ and so $\Phi(N(D), N'(D)) = 0$ unless $N_l(D), N_l'(D)$ is monatomic precisely fo $l \in J_1 \cup J_2$. Such $N(D), N'(D)$ are given by (12.2), with

$(N_l(D), N_l'(D))$ not monatomic for any $l \in J_3$. If $N(D), N'(D)$ are given by (12.2), we write $\Phi(\mathfrak{B}, (N_l(D), N_l'(D))_{l \in J_3})$ for $\Phi(N(D), N'(D))$.

Now fix $\mathfrak{B}$ and $(N_l(D), N_l'(D))_{l \in J_3}$, with none of the $(N_l(D), N_l'(D))$ monatomic. Let $B_1 \dots B_{L_0}$ be the balls assigned to $l \in J_1 \cup J_2$ by $\mathfrak{B}$, with $B_1 \dots B_{s_0}$ coming from J_1 and $B_{s_0+1} \dots B_{L_0}$ coming from J_2. We can compute $\Phi(\mathfrak{B}, (N_l(D), N_l'(D))_{l \in J_3})$ using Lemma 4.1. For, A^s_{ex} and H^s_{ex} restricted to $X(\mathfrak{B}, (N_l(D), N_l'(D))_{L \in J_3})$ are observables on a system composed of two non-interacting parts, namely $B_1 \dots B_{L_0}$ and

$$\Omega^{J_3}_{ex} = \Big(\bigcup_{l \in J_3} \bigcup_{D \in \mathfrak{Q}^s_l} \partial \hat{D}\Big).$$

The Hamiltonian breaks up as a sum of the Hamiltonian of Lemma 4.1 acting on $B_1 \dots B_{L_0}$, and an exploded Hamiltonian on $\Omega^{J_3}_{ex}$. The observable A^s_{ex} refers entirely to $B_1 \dots B_{L_0}$ and in fact agrees with G in Lemma 4.1. Therefore, we can write

$$\begin{aligned} \operatorname{Tr}\exp\{tA^s_{ex} + \mu(N+N') - \beta H^s_{ex} \,|\, X(\mathfrak{B}, (N_l(D), N_l'(D))_{l \in J_3})\} &= \\ = \operatorname{Tr}\exp\{tG + 2L_0\mu - \beta \overset{\circ}{H} \,|\, L^2_{1,1}(B_1 \dots B_{L_0})\} &\cdot \\ \cdot \prod_{l \in J_3} \Big[\prod_{B_{k\alpha} \in \mathfrak{Q}^s_l} Z(m, \beta, B_{k\alpha}, N_l(B_{k\alpha}), N_l'(B_{k\alpha})) &\cdot \\ \cdot \prod_{\tilde{Q}_\alpha \in \mathfrak{Q}^s_l} Z_0(\mu, \beta, \tilde{Q}_\alpha, N_l(\tilde{Q}_\alpha), N_l'(\tilde{Q}_\alpha))\Big]&, \end{aligned} \tag{12.4}$$

and the first term on the right can be evaluated using Lemma 4.1. In fact, we have from Lemma 4.1 that

$$\begin{aligned} &\operatorname{Tr}\exp\{tG + 2L_0\mu - \beta \overset{\circ}{H} \,|\, L_{1,1}(B_1 \dots B_{L_0})\} = \\ &\quad = \operatorname{Tr}\exp\{2L_0\mu - \beta \overset{\circ}{H} \,|\, L_{1,1}(B_1 \dots B_{L_0})\} e^{tG_0}(1 + O(t^2 + \beta^{-1})), \end{aligned}$$

with

$$G_0 = \Big(c \int_E e^{-|x|}\,dx\Big)^{s_0}\Big(c \int_{cE} e^{-|x|}\,dx\Big)^{L_0 - s_0}, \qquad s_0 = |J_1|, \qquad L_0 - s_0 = |J_2|.$$

Substituting this into (12.4), then taking $t = 0$ in (12.4) and subtracting, we obtain

$$\begin{aligned} \Phi(N(D), N'(D)) = (tG_0 + O(t^2 + \beta^{-1})) \cdot \operatorname{Tr}\{\mu(N+N') &- \\ - \beta H^s_{ex} \,|\, X(N(D), N'(D))\}& \end{aligned} \tag{12.5}$$

if $(N_l(D), N_l'(D))$ is monatomic precisely for $l \in J_1 \cup J_2$; $\Phi(N(D), N'(D)) = 0$ otherwise;

$$G_0 = \Big(\text{const} \int_E e^{-|x|}\,dx\Big)^{|J_1|}\Big(\text{const} \int_{cE} e^{-|x|}\,dx\Big)^{|J_2|}. \tag{12.6}$$

In (12.5) we wrote $tG_0 + O(t^2)$ for $(e^{tG_0} - 1)$.

Now set

$$Z^{atomic}(\mu, \beta, \mathfrak{Q}_l^s) = \sum_{N_l(D), N_l'(D) \text{ monatomic}} \prod_{B_{k\alpha} \in \mathfrak{Q}_l^s} Z(\mu, \beta, B_{k\alpha}, N_l(B_{k\alpha}), N_l'(B_{k\alpha})) \cdot$$
$$\cdot \prod_{\tilde{Q}_\alpha \in \mathfrak{Q}_l^s} Z_0(\mu, \beta, \tilde{Q}_\alpha, N_l(\tilde{Q}_\alpha), N_l'(\tilde{Q}_\alpha)) \qquad (12.7)$$

$$Z^{non-atomic}(\mu, \beta, \mathfrak{Q}_l^s) = \sum_{N_l(D), N_l'(D) \textit{ not monatomic}} (\text{same product}). \qquad (12.8)$$

Since the trace on the right in (12.5) breaks up as a product of terms corresponding to the different $\mathfrak{Q}_l^s$, we find that when (12.5) is substituted into (12.3), we get

$$\Phi = (tG_0 + O(t^2 + \beta^{-1})) \prod_{l \in J_1 \cup J_2} Z^{atomic}(\mu, \beta, \mathfrak{Q}_l^s) \cdot$$
$$\cdot \prod_{l \in J_3} Z^{non-atomic}(\mu, \beta, \mathfrak{Q}_l^s). \qquad (12.9)$$

It is easy to compute Z^{atomic} and $Z^{non-atomic}$. In fact

$$Z^{atomic}(\mu, \beta, \mathfrak{Q}_l^s) = \sum_{B_{k\alpha} \in \mathfrak{Q}_l^s} Z(\mu, \beta, B_{k\alpha}, 1, 1) = \sum_{B_{k\alpha} \in \mathfrak{Q}_l^s} \rho(1 + O(\beta^{-1}))|B_{k\alpha}|$$

(by Section 4)

$$= \rho|Q_l^s| \cdot (1 + O(\beta^{-1}))$$
$$= \lambda(1 + O(\beta^{-1}))$$

(see equation (11.1)).

On the other hand,

$$Z^{atomic}(\mu, \beta, \mathfrak{Q}_l^s) + Z^{non-atomic}(\mu, \beta, \mathfrak{Q}_l^s) =$$
$$= \prod_{B_{k\alpha} \in \mathfrak{Q}_l^s} Z(\mu, \beta, B_{k\alpha}) \cdot \prod_{\tilde{Q}_\alpha \in \mathfrak{Q}_l^s} Z_0(\mu, \beta, \tilde{Q}_\alpha)$$

(by (12.7), (12.8))

$$= \exp\{\rho|Q_l^s| \cdot (1 + O(\beta^{-1}))\} = \exp(\lambda(1 + O(\beta^{-1}))),$$

by (11.1) and Section 4 again. It follows that

$$Z^{non-atomic}(\mu, \beta, \mathfrak{Q}_l^s) = (e^\lambda - \lambda)(1 + O(\beta^{-(1/2)})),$$
$$\text{as long as } \quad \beta^{-(1/20)} < \lambda < 100. \qquad (12.10)$$

Substituting our formulas for Z^{atomic}, $Z^{non-atomic}$ into (12.9), we get

$$\Phi = (tG_0 + O(t^2 + \beta^{-1/2})) \cdot \lambda^{|J_1 \cup J_2|}(e^\lambda - \lambda)^{|J_3|}. \qquad (12.11)$$

We have also

$$\operatorname{Tr}\exp\{\mu(N+N')-\beta H^s_{ex} \mid L^2_*(\Omega^s_{ex})\} = \prod_{B_{k\alpha}\in\mathcal{Q}^s} Z(\mu,\beta,B_{k\alpha})\cdot$$

$$\cdot \prod_{\tilde{Q}_\alpha\in\mathcal{Q}^s} Z_0(\mu,\beta,\tilde{Q}_\alpha) =$$

$$= \exp\left\{\rho \sum_{B_{k\alpha}\in\mathcal{Q}^s} |B_{k\alpha}|\cdot(1+O(\beta^{-1}))\right\}\cdot\exp\left\{O\left(\beta^{-1}\rho\sum_{l=1}^{L}|Q_l^L|\right)\right\}$$

(by Section 4)

$$= \exp\left\{\rho\sum_{l=1}^{L}|Q_l^s|(1+O(\beta^{-1}))\right\} = e^{\lambda L}(1+O(\beta^{-1/2}))\Big\},$$

provided (12.10) holds and β is large enough, depending on L. Combining this and (12.11) with (12.13), we get

$$\operatorname{Tr}\exp\{tA^s_{ex}+\mu(N+N')-\beta H^s_{ex}\mid L^2_*(\Omega^s_{ex})\} =$$

$$= \left(1+tG_0\left(\frac{\lambda}{e^\lambda}\right)^{|J_1\cup J_2|}\left(\frac{e^\lambda-\lambda}{e^\lambda}\right)^{|J_3|}+O(t^2+\beta^{-1/2})\right)\cdot$$

$$\cdot\operatorname{Tr}\exp\{\mu(N+N')-\beta H^s_{ex}\mid L^2_*(\Omega^s_{ex})\}, \tag{12.12}$$

where $O(t^2+\beta^{-1/2})$ means less than $\mathrm{Const}(L)\cdot(t^2+\beta^{-1/2})$ in absolute value. Substituting (12.12) into (12.1) now gives

$$\operatorname{Tr}\exp\{tA_{ex}(\tau,\overline{M})+\mu(N+N')-\beta H^{ex}\} \leqslant e^S\operatorname{Tr}\exp\{\mu(N+N')-\beta H^{ex}\} \tag{12.13}$$

with

$$S = (\text{Number of different } s)[tG_0(\lambda e^{-\lambda})^{|J_1\cup J_2|}(1-\lambda e^{-\lambda})^{|J_3|}+O(t^2+\beta^{-1/2})].$$

Here, the number of different s is

$$\frac{|\Omega|}{\left(\frac{\lambda}{\rho}L\right)} + \text{error tending to zero as } \Omega \text{ gets big} = \frac{\rho|\Omega|}{\lambda L}(1+O(\beta^{-1})), \text{ say.}$$

Applying (8.1) and (9.4) with $A_{ex}(\tau,\overline{M})$ replaced by $tA_{ex}(\tau,\overline{M})$, we see that (12.13) yields

$$t\langle\mathfrak{A}\rangle \leqslant t(\text{Number of different } s)G_0(\lambda e^{-\lambda})^{|J_1\cup J_2|}(1-\lambda e^{-\lambda})^{|J_3|}+$$

$$+\frac{\rho|\Omega|}{\lambda L}O(t^2+\beta^{-1/2})+O(\beta^{-1}\rho|\Omega|).$$

Taking $t = \pm\beta^{-1/20}$, and comparing with (11.9), we obtain

$$\langle \textstyle\sum_s \chi_{\mathcal{E}'_s} \rangle = (\text{Number of different } s)[G_0(\lambda e^{-\lambda})^{|J_1 \cup J_2|}(1 - \lambda e^{-\lambda})^{|J_3|} + O(\beta^{-1/20})] \tag{12.14}$$

with G_0 given by (12.6), and $\mathcal{E}'_s$ defined by $(b)'$, $(c)'$, $(d)'$ in Section 11.

13. Proof of the Theorem

The idea is to use (12.14), together with a simple quantitative form of the law of large numbers, which we now set down. Suppose we have independent random variables $X_1 \ldots X_L$, with $X_j = 1$ with probability p, $X_j = 0$ with probability $1 - p$. Then $E(e^{tX_j}) = e^t p + (1 - p) = \exp(pt + O(t^2))$, uniformly in $p \in [0, 1]$. Consequently, $E(e^{t(X_1 + \cdots + X_L)}) = \exp(Lpt + O(Lt^2))$, so that

$$\text{Prob}\left\{\frac{X_1 + \ldots + X_L}{L} \geqslant p + \delta\right\} \leqslant \exp(Lpt + O(Lt^2) - tL(p + \delta)).$$

Picking $t = (\text{small const})\delta$, we obtain

$$\text{Prob}\left\{\frac{X_1 + \ldots + X_L}{L} \geqslant p + \delta\right\} \leqslant \exp(-c\delta^2 L).$$

Applying this also to $X'_j = 1 - X_j$, $p' = 1 - p$ we obtain

$$\text{Prob}\left\{\left|\frac{X_1 + \ldots + X_L}{L} - p\right| \geqslant \delta\right\} < \exp(-c\delta^2 L). \tag{13.1}$$

We apply this to a probability space defined as follows.

The points of the space are functions $f: \{1 \ldots L\} \to \{1, 2, 3\}$. Thus, each f gives rise to subsets $J_1 = \{l \mid f(l) = 1\}$, $J_2 = \{l \mid f(l) = 2\}$, $J_3 = \{l \mid f(l) = 3\}$. We fix $E \subset R^3$, and define the probability of f as $\text{Prob}(f) = \langle \sum_s \chi_{\mathcal{E}'_s} \rangle/(\text{Number of } s)$, where $\mathcal{E}'_s$ is the event defined by $(b)'$, $(c)'$, $(d)'$ in Section 10.

Formula (12.14) shows that $\text{Prob}(f)$ differs by at most $C(L)/\beta^{1/20}$ from $\text{Prob}'(f)$, defined by picking each $f(l)$ independently with probabilities

$$\text{Prob}'(f(l) = 1) = \left(\text{const} \int_E e^{-|x|}\, dx\right)(\lambda e^{-\lambda}) = p_1$$

$$\text{Prob}'(f(l) = 2) = \left(\text{const} \int_{cE} e^{-|x|}\, dx\right)(\lambda e^{-\lambda}) = p_2$$

$$\text{Prob}'(f(l) = 3) = (1 - \lambda e^{-\lambda}) = p_3.$$

Since the probability space contains only 3^L points, it follows that

$$|\text{Prob}(\mathcal{E}) - \text{Prob}'(\mathcal{E})| < \frac{C'(L)}{\beta^{1/20}} \tag{13.2}$$

for any event $\mathcal{E}$ in the probability space. We apply this to the event

$$\mathcal{E} = \left\{ \left| \left(\frac{\text{Number of } l \text{ for which } f(l) = 1}{L} \right) - p_1 \right| > \delta \right\}.$$

From (13.2), and (13.1) applied to Prob′, we conclude that

$$\text{Prob}(\mathcal{E}) \leqslant \exp(-c\delta^2 L) + \frac{C(L)}{\beta^{1/20}}.$$

Now Prob($\mathcal{E}$) has a simple interpretation.

We define

$$X_l^s = \begin{bmatrix} 1 & \text{if } Q_l^s \text{ contains an atom of type } (c', c'') \text{ and no other par-} \\ & \text{ticles, and that atom has displacement vector in } E \\ 0 & \text{otherwise} \end{bmatrix}$$

Then

$$\text{Prob}(\mathcal{E}) = \frac{\langle \text{Number of } s \text{ with } |\sum_{l=1}^{L} X_l^s - p_1 L| > \delta L \rangle}{(\text{Number of } s)}$$

Hence,

$$\left\langle \text{Number of } s \text{ with } \left| \sum_{l=1}^{L} X_l^s - p_1 L \right| > \delta L \right\rangle \leqslant \left(\exp(-c\delta_L^2) + \frac{C(L)}{\beta^{1/20}} \right) \cdot$$
$$\cdot\, (\text{Number of } s)$$

Since

$$\left\langle \left| \sum_{s,l} X_l^s - p_1 L (\text{Number of } s) \right| \right\rangle \leqslant (\delta L)(\text{Number of } s) +$$
$$+\, L \left\langle \text{Number of } s \text{ with } \left| \sum_{l} X_l^s - p_1 L \right| > \delta L \right\rangle,$$

it follows that

$$\left\langle \left| \sum_{s} X_l^s - p_1 L (\text{Number of } s) \right| \right\rangle \leqslant$$
$$\leqslant (\text{Number of } s) \left(\delta L + L \exp(-c\delta^2 L) + \frac{C(L)}{\beta^{1/20}} \right)$$

or, since $L(\text{Number of } s) = (\text{Number of boxes } Q_l^s) \equiv N_0$,

$$\left\langle \left| \sum_{l,s} X_l^s - p_1 N_0 \right| \right\rangle \leqslant N_0 (\delta + \exp(-c\delta^2 L) + C(L)\beta^{-1/20}). \tag{13.3}$$

Now take δ small first, then pick L so large that $\exp(-c\delta^2 L) < \delta$, then pick

β so large that $C(L)\beta^{-1/20} < \delta$, and finally pick Ω so large that all our estimates are valid for the β we picked. In that case, (13.3) becomes

$$\left\langle \left| \frac{\sum_{l,s} X_l^s}{(\text{Number of } l, s)} - p_1 \right| \right\rangle \leqslant 3\delta, \tag{13.4}$$

β large enough and Ω large depending on β.

Taking $E = R^3$ in (13.4), we find that the number of boxes Q_l^s containing exactly an atom of type (c', c'') is $(p_1 + O(\delta^{1/2})) \cdot N_0$ with probability $> 1 - \delta^{1/2}$. For $E = R^3$ and $\lambda \ll 1$, our defining formula for p_1 becomes $P_1 = \lambda + O(\lambda^2)$, while $N_0 =$ Number of $s, l = (|\Omega|/(\lambda/\rho)) +$ (error tending to zero with large Ω) $= (\rho|\Omega|/\lambda) \cdot (1 + O(\lambda^{10}))$ certainly. So (13.4) in this special case gives $\langle|$(Number of Q_l^s containing exactly an atom of type $(c', c'')) - \rho|\Omega|(1 + O(\lambda^2))|\rangle \leqslant (3\delta/\lambda)\rho|\Omega|$.

So if we take $\lambda < \epsilon^{10}$ and $\delta < \lambda^2$, then we find with probability at least $(1 - \epsilon)$ that the number of Q_l^s containing exactly an atom of type (c', c'') is $\rho|\Omega|(1 + O(\epsilon))$. However, we already know that with probability $> 1 - \epsilon$, all but at most $\epsilon\rho|\Omega|$ of the particles belong to atoms of type (c', c'') and the total number of particles is $2\rho|\Omega|(1 + O(\epsilon))$. So with probability $> 1 - O(\epsilon)$, we know that all but $O(\epsilon)$ fraction of the particles come from atoms of type (c', c'') which form the sole contents of one of the Q_l^s. Returning to the general case of $E \subset R^3$, we look at (13.4) and realize that with probability $> 1 - O(\epsilon)$, $\sum_{l,s} X_l^s =$ (Number of atoms with displacement vectors in E) $+ O(\epsilon \cdot$ Number of atoms), while

$$p_1 \cdot (\text{Number of } s,l) = \left(\text{const} \int_E e^{-|x|}\,dx\right)(\text{Number of atoms}) + \\ + O(\epsilon \cdot \text{Number of atoms}).$$

Therefore, (13.4) implies that with probability $> 1 - O(\epsilon)$, the fraction of atoms having displacement vectors in E is within $O(\epsilon)$ or (const $\int_E e^{-|x|}\,dx$). So we know (2.1) and (2.2).

The same technique also proves (2.3). We simply pair up the boxes Q_l^s into, say Q_{2j-1}^s, Q_{2j}^s, and define random variables

$$Y_j^s = \begin{cases} 1 & \text{if both } Q_{2j-1}^s, Q_{2j}^s \text{ contain exactly a } (c', c'')\text{-atom, and the} \\ & \text{displacement vectors of both atoms lie in } E, \\ 0 & \text{otherwise.} \end{cases}$$

Using Y_l^s in place of the X_l^s, we obtain in the notation of (2.3) that

$$p^* = \left[\left(\text{const} \int_E e^{-|x|}\,dx\right)\lambda e^{-\lambda}\right]^2 + O(\epsilon)$$

$$p', p'' = \left[\left(\text{const} \int_E e^{-|x|}\,dx\right)\lambda e^{-\lambda}\right] + O(\epsilon),$$

all with probability $> 1 - \epsilon$. This time, we need not take λ small. These last equations imply (2.3). The proof of our theorem is complete.

References

[1] F. Dyson and A. Lenard. Stability of Matter I, *J. Math. Phys.* **8** (1967), 423-434.
[2] C. Fefferman. The Uncertainty Principle, *Bull. A. M. S.,* **9** (1983) 129-206.
[3] —. The thermodynamic Limit for a Crystal, *Comm. Math. Phys.* (to appear).
[4] —. to appear.
[5] W. Hughes. Thermodynamics for Coulomb Systems: A Problem at Vanishing Density (to appear).
[6] L. D. Landau and E. M. Lifshitz. Statistical Physics, Pergamon Press (1959).
[7] J. Lebowitz and E. Lieb. The Constitution of Matter: Existence of Thermodynamics for Systems Composed of Electrons and Nucleii, *Advances in Math.* **9** (1972) 316-398.
[8] E. Lieb. Stability of Matter, *Rev. Modern Phys.* **48** (1976), 553-569.
[9] C. Radin and L. Schulman. Periodicity of Classical Ground States, *Phys. Rev. Letters* **51** No. 8 (1983) 621-622.
[10] E. M. Stein. Singular Intergrals and Differentiability Properties of Functions, Princeton Univ. Press (1970).

Charles L. Fefferman
Princeton University
Princeton, N. J. D8544
U.S.A.

Supported in part by NSF Grant N.° MCS 80-03072.

Revista Matemática Iberoamericana
Vol. 1, n.º 1, 1985

The Ternary Goldbach Problem

D. R. Heath-Brown

1. Introduction

The object of this paper is to present new proofs of the classical "ternary" theorems of additive prime number theory. Of these the best known is Vinogradov's result on the representation of odd numbers as the sums of three primes; other results will be discussed later. Earlier treatments of these problems used the Hardy-Littlewood circle method, and are highly "analytical". In contrast, the method we use here is a (technically) elementary deduction from the Siegel-Walfisz Prime Number Theorem. It uses ideas from Linnik's dispersion method, together with Vaughan's identity.

It is convenient to quote the Siegel-Walfisz Theorem here. (See Walfisz [17; Hilfssatz 3] or Davenport [6; Chapter 22] for example.)

For any constant $A > 0$ there exists $C(A) > 0$ such that

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \leq x}} \Lambda(n) = \frac{x}{\phi(k)} + O(x \exp(-C(A)(\log x)^{1/2})), \tag{1.1}$$

uniformly for $(l, k) = 1$ and $k \leqslant (\log x)^A$.

We now state our results.

Theorem 1. *For $x \geqslant 2$ define*

$$N_2(m) = \sum_{\substack{p \leq x \\ p + p' = m}} (\log p)(\log p'),$$

where p, p' run over primes. Set

$$\mathfrak{S}(m) = 2 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid m \\ p \geq 3}} \left(\frac{p-1}{p-2}\right)$$

for even m, and $\mathfrak{S}(m) = 0$ for odd m. Then for any $C > 0$ we have

$$\sum_{2x < m \leq 3x} |N_2(m) - x\mathfrak{S}(m)| \ll x^2(\log x)^{-C}.$$

Corollary 1. *For any $C > 0$ there are at most $O(x(\log x)^{-C})$ even integers $m \leqslant x$ which are not the sum of two primes.*

Corollary 2. *Every sufficiently large odd number is the sum of three primes.*

Corollary 3. *There are infinitely many sets of three distinct primes in arithmetic progression.*

Corollary 2 is the famous result of Vinogradov [15] and [16]. Proofs of Corollary 1 (via forms of Theorem 1) were given independently by van der Corput [3], Čudakov [4], [5], and Estermann [8], all using Vinogradov's method. Heilbronn [9] also discovered the result independently. It is not clear who was the first to state Corollary 3 explicitly.

Sharper versions of Corollary 1 have been obtained more recently by Vaughan [13], and by Montgomery and Vaughan [12]. In particular, the latter work proves that the exceptional set in Corollary 1 has cardinality $O(x^{1-\delta})$ for some fixed positive δ. Our results are all ineffective, since the Siegel-Walfisz Theorem (1.1) is itself ineffective. However, the estimate of Montgomery and Vaughan [12] gives an effective version of Corollary 1, and hence also of Corollaries 2 and 3.

As a by-product of our argument we shall obtain the following version of the "Barban-Davenport-Halberstam" Theorem.

Theorem 2. *For any $C > 0$ we have*

$$\sum_{k \leq x(\log x)^{-C}} \sum_{\substack{l=1 \\ (l, k)=1}}^{k} \left| \sum_{\substack{n \leq x \\ n \equiv l (\operatorname{mod} k)}} \Lambda(n) - \frac{x}{\phi(k)} \right|^2 \ll x^2(\log x)^{8-C/3}.$$

Results of this type were first obtained by Barban [1], [2], and rediscovered by Davenport and Halberstam [7]. In [2] Barban obtained the asymptotic formula

$$\sum_{k \le Q} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left| \pi(x; k, l) - \frac{\mathrm{Li}(x)}{\phi(k)} \right|^2 =$$
$$= Q\mathrm{Li}(x) + O(x^2(\log x)^{-A}) + O(Qx(\log x)^{-2} \log(x/Q))$$

for $\exp(c(\log x)^{1/2}) \leqslant Q \leqslant x$ (where A may be any positive constant, and c is an absolute constant). Moreover, when $Q = x$, he showed that the right-hand side may be replaced by

$$x\mathrm{Li}(x) + E(\mathrm{Li}(x))^2 + O(x^2(\log x)^{-A})$$

for a suitable constant E. This work anticipated some of the results of Montgomery [11; Chapter 17]. It should be noted that our proof of Theorem 2 does not use the large sieve.

The techniques used in this paper draw on ideas from Linnik's dispersion method, and from Barban [2] and Hooley [10]. Vaughan's identity [14] also plays a crucial part. In addition we shall use the function

$$\Lambda_Q(n) = \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d \mid (n,q)} d\mu(d) = \sum_{q \le Q} \frac{\mu(q)}{\phi(q)} c_q(n),$$

where $c_q(n)$ is the Ramanujan sum. The function $\Lambda_Q(n)$ is so constructed as to copy $\Lambda(n)$ in its distribution over arithmetic progressions.

We shall use the notation $L = \log x$ throughout the proof. The implied constants in the $O(.)$ and $\ll$ notations may depend on A, B and C. In general they are ineffective.

2. The distribution of $\Lambda_Q(n)$ in arithmetic progressions

In this section we investigate the properties of the function $\Lambda_Q(n)$, and show that it mimics $\Lambda(n)$. As a by-product we will establish Theorem 2.

We first note some well-known bounds that will be required from time to time. We have

$$\phi(q) \gg q(\log q)^{-1}, \qquad \sigma(q) \ll q(\log q) \tag{2.1}$$

and

$$\sum_{k \le K} (d(k))^t \ll K(\log K)^{2^t - 1}, \qquad (t = 1, 2, 3). \tag{2.2}$$

Since $d(ab) \leqslant d(a)d(b)$, we also have

$$\sum_{n \le N} (n,r)d(n) = \sum_{a|r} a \sum_{\substack{n \le N \\ (n,r)=a}} d(n) \leqslant \sum_{a|r} a \sum_{ab \le N} d(ab) \tag{2.3}$$
$$\leqslant \sum_{a|r} ad(a) \sum_{b \le N/a} d(b)$$
$$\ll \sum_{a|r} ad(a)(Na^{-1}(\log N))$$
$$\ll N(\log N) \sum_{a|r} d(a)$$
$$\ll d(r)^2 N(\log N),$$

by (2.2) with $t = 1$.

Before starting the main part of the argument we shall put (1.1) into a more convenient form, by weakening the error term to $O(xL^{1-A})$. The condition $k \leqslant L^A$ may then be dropped, since the sum on the left of (1.1) is automatically $O((1 + xk^{-1})L)$. Moreover if $(l, k) > 1$ then $p^e \equiv l (\operatorname{mod} k)$ requires $p|k$. There are then $O(\log k)$ available primes p and $O(L)$ possible exponents e. Hence

$$\sum_{\substack{n \equiv l (\operatorname{mod} k) \\ n \le x}} \Lambda(n) \ll L^3, \qquad ((l,k) > 1, k \leqslant x),$$

and clearly this is true also when $k > x$. After replacing A by $A + 1$ we can now put (1.1) into the more useful form

$$\sum_{\substack{n \equiv l (\operatorname{mod} k) \\ n \le x}} \Lambda(n) = E_{k,l} \frac{x}{\phi(k)} + O(xL^{-A}) \tag{2.4}$$

uniformly for all k, l; here we have defined

$$E_{k,l} = \begin{cases} 1, & (k,l) = 1, \\ 0, & (k,l) > 1. \end{cases}$$

We now turn to $\Lambda_Q(n)$, and start by looking at its size. Using (2.1) we have

$$|\Lambda_Q(n)| = \left| \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{\substack{d|q \\ d|n}} d\mu(d) \right|$$
$$\leqslant \sum_{d|n} d \sum_{\substack{q \le Q \\ d|q}} \frac{1}{\phi(q)}$$
$$\ll (\log Q) \sum_{d|n} d \left(\sum_{\substack{d \le Q \\ d|q}} q^{-1} \right)$$
$$\ll (\log Q) \sum_{d|n} d(d^{-1}(\log Q)).$$

Thus

$$\Lambda_Q(n) \ll d(n)(\log Q)^2. \tag{2.5}$$

Next we show that in any given arithmetic progression the functions $\Lambda_Q(n)$ and $\Lambda(n)$ behave very similarly. It is convenient to write $\Delta_Q(n) = \Lambda_Q(n) - \Lambda(n)$.

Lemma 1. *We have*

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \le x}} \Lambda_Q(n) = \frac{x}{\phi(k)} + O(QL^2) + O(xL(kQ)^{-1}d(k)), \tag{2.6}$$

for $(k, l) = 1$, *and*

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \le x}} \Delta_Q(n) \ll_A QL^2 + xL(kQ)^{-1}(k, l)d(k) + xL^{-A}, \tag{2.7}$$

for any l, *uniformly for* $1 \leqslant Q, k \leqslant x$.

By definition we have

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \le x}} \Lambda_Q(n) = \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|q} d\mu(d) \# \{n \leqslant x; d|n, n \equiv l (\mathrm{mod}\, k)\}. \tag{2.8}$$

The conditions $d|n$ and $n \equiv l(\mathrm{mod}\, k)$ are compatible only when $(d, k)|l$, in which case they define a unique residue class to modulus $kd/(d, k)$. Hence (2.8) is

$$\sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|r} d\mu(d)\{(kd)^{-1}(d, k)x + O(1)\}$$
$$= k^{-1}x \sum_{q \le Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|r} \mu(d)(d, k) + O\left(\sum_{q \le Q} \frac{\sigma(q)}{\phi(q)}\right),$$

where r is the product of those primes $p|q$ for which $(p, k)|l$. The error term of (2.9) is $O(QL^2)$ by (2.1).

Since $\mu(d)(d, k)$ is a multiplicative function of d we have, for $\mu(q) \neq 0$,

$$\sum_{d|r} \mu(d)(d, k) = \prod_{p|r} (1 - (p, k)) = \begin{cases} \mu((q, l))\phi((q, l)), & q|k, \\ 0, & q \nmid k. \end{cases}$$

We write $f(q) = \mu(q)^2\mu((q, l))\phi((q, l))/\phi(q)$, so that $f(q)$ is multiplicative.

Then

$$\sum_{\substack{q|k \\ q \le Q}} f(q) = \sum_{q|k} f(q) + O\Bigg(\sum_{\substack{q|k \\ q>Q}} |f(q)|\Bigg) = \tag{2.10}$$

$$= \sum_{q|k} f(q) + O(Q^{-1}\sum_{q|k} q|f(q)|)$$

$$= \prod_{p|k} (1 + f(p)) + O(Q^{-1} \prod_{p|k} (1 + p|f(p)|)).$$

However

$$f(p) = \begin{cases} (p-1)^{-1}, & p|k, p\nmid l, \\ -1, & p|k, p|l. \end{cases}$$

Hence (2.10) is

$$E_{k,l}\frac{k}{\phi(k)} + O\Bigg(Q^{-1}(k,l)d(k)\frac{\sigma(k)}{k}\Bigg),$$

and (2.8) becomes

$$\sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \le x}} \Lambda_Q(n) = E_{k,l}\frac{x}{\phi(k)} + O(QL^2) + O\Bigg(x(kQ)^{-1}(k,l)d(k)\frac{\sigma(k)}{k}\Bigg).$$

The estimates (2.6) and (2.7) now follow, using (2.1) and (2.4).

Our next lemma is an analogue of Theorem 2 for $\Delta_Q(n)$. For convenience we define

$$\delta_t(x,k,Q) = \sum_{l=1}^{k} \Bigg| \sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \le x}} \Delta_Q(n) \Bigg|^t, \qquad (t = 1, 2).$$

We then have:

Lemma 2. *Let $Q = L^B$ and $K \leqslant xQ^{-1}$. Then*

$$\sum_{k \le K} k^{-1}\delta_1(x,k,Q) \ll_B xQ^{-1/2}L^7 \tag{2.11}$$

for any fixed $B > 0$.

The proof falls into two parts. First we bound the sum on the left of (2.11) in terms of

$$S = \sum_{K < k \le 2K} \delta_2(x,k,Q),$$

and then we use Lemma 1 to estimate S. The second stage follows the idea used by Barban [2]. Naturally it suffices to consider the case $K = xQ^{-1}$.

For any $j \geqslant 1$ we have

$$\left|\sum_{n \equiv l(\bmod k)}\right| = \left|\sum_{m=1}^{j} \sum_{n \equiv l+mk(\bmod jk)}\right| \leqslant \sum_{m=1}^{j}\left|\sum_{n \equiv l+mk(\bmod jk)}\right|.$$

By summing over $l(\bmod k)$ we deduce that

$$\delta_1(x, k, Q) \leqslant \delta_1(x, jk, Q).$$

We proceed to average this over those j for which $jk \in (K, 2K]$. Since the number of such j is of exact order Kk^{-1} we obtain

$$Kk^{-1}\delta_1(x, k, Q) \ll \sum_{\substack{K<h\leq 2K \\ k|h}} \delta_1(x, h, Q).$$

On summing for $k \leqslant K$ this yields

$$K\sum_{k\leq K} k^{-1}\delta_1(x, k, Q) \ll \sum_{K<h\leq 2K} d(h)\delta_1(x, h, Q).$$

To obtain an estimate in terms of S we apply Cauchy's inequality, in conjunction with the case $t = 2$ of (2.2). This leads to

$$K\sum_{k\leq K} k^{-1}\delta_1(x, k, Q) \ll (K(\log K)^3)^{1/2}\Big(\sum_{K<h\leq 2K} \delta_1(x, h, Q)^2\Big)^{1/2}. \tag{2.12}$$

However, by Cauchy's inequality again, we have

$$\delta_1(x, h, Q)^2 \leqslant h\delta_2(x, h, Q) \ll K\delta_2(x, h, Q),$$

and so (2.12) yields

$$\sum_{k\leq K} k^{-1}\delta_1(x, k, Q) \ll L^{3/2}S^{1/2}. \tag{2.13}$$

We proceed to bound S. We have

$$\begin{aligned}\delta_2(x, k, Q) &= \sum_{\substack{m,n\leq x \\ k|m-n}} \Delta_Q(m)\Delta_Q(n) \\ &= \sum_{n\leq x} \Delta_Q(n)^2 + 2\sum_{\substack{m<n\leq x \\ k|m-n}} \Delta_Q(m)\Delta_Q(n).\end{aligned} \tag{2.14}$$

From (2.5) we have $\Lambda_Q(n) \ll Ld(n)$, whence $\Delta_Q(n) \ll Ld(n)$. The diagonal terms in (2.14) therefore total $O(xL^5)$, by the case $t = 2$ of (2.2). It follows that

$$S = 2\sum_{m<n\leq x} \Delta_Q(m)\Delta_Q(n) \,\#\, \{k, t; n - m = kt, K < k \leqslant 2K\} + O(xKL^5)$$

$$S = 2\sum_{1\leq t\leq xK^{-1}} \sum_{m\leq x} \Delta_Q(m) \sum_{n\equiv m(\bmod t)} \Delta_Q(n) + O(xKL^5).$$

In the innermost sum n runs over a subinterval of $(0, x]$, so that (2.7) of Lemma 1 can be applied. This yields

$$S \ll xKL^5 + \sum_t \sum_m |\Delta_Q(m)| \{xL(tQ)^{-1}(t, m)d(t)\}. \tag{2.15}$$

Note here that

$$xL(tQ)^{-1}(t, m)d(t) \gg x(tQ)^{-1} \gg x(xK^{-1}Q)^{-1} \gg xL^{-2B} \gg QL^2 + xL^{-A},$$

on taking $A = 2B$, as indeed we may. Thus the second term on the right of (2.7) is the dominant one.

We rearrange the double sum in (2.15) as

$$xLQ^{-1} \sum_t \sum_m |\Delta_Q(m)| t^{-1}(t, m)d(t) \leq xLQ^{-1} \sum_{m \leq x} |\Delta_Q(m)| \sum_{t \leq x} t^{-1}(t, m)d(t).$$

The inner sum is $O(d(m)^2L^2)$, by (2.3). Thus, since $\Delta_Q(m) \ll Ld(m)$ as before, (2.15) becomes

$$\begin{aligned} S &\ll xKL^5 + xL^3Q^{-1} \sum_{m \leq x} |\Delta_Q(m)| d(m)^2 \\ &\ll xKL^5 + xL^4Q^{-1} \sum_{m \leq x} d(m)^3 \\ &\ll xKL^5 + x^2L^{11}Q^{-1}, \end{aligned}$$

by (2.2) with $t = 3$. Lemma 2 now follows from (2.13), given our condition on K.

We can now derive Theorem 2. It follows from Lemma 2 that

$$\begin{aligned} &\sum_{k \leq K} k^{-1} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left| \sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \leq x}} \Lambda(n) - \frac{x}{\phi(k)} \right| \ll \\ &\ll \sum_{k \leq K} k^{-1} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left| \sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \leq x}} \Lambda_Q(n) - \frac{x}{\phi(k)} \right| + xQ^{-1/2}L^7. \end{aligned}$$

By (2.6) of Lemma 1 the right hand side is

$$\begin{aligned} &\ll xQ^{-1/2}L^7 + \sum_{k \leq K} \{QL^2 + xL(kQ)^{-1}d(k)\} \\ &\ll xQ^{-1/2}L^7 + KQL^2 + xQ^{-1}L^3 \\ &\ll xQ^{-1/2}L^7 + KQL^2, \end{aligned}$$

on using (2.2) with $t = 1$. However

$$\left| \sum_{\substack{n \equiv l (\mathrm{mod}\, k) \\ n \leq x}} \Lambda(n) - \frac{x}{\phi(k)} \right| \ll xk^{-1}L,$$

so that

$$\sum_{k \le K} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left| \sum_{\substack{n \equiv l (\bmod k) \\ n \le x}} \Lambda(n) - \frac{x}{\phi(k)} \right|^2$$
$$\ll xL \sum_{k \le K} k^{-1} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left| \sum_{\substack{n \equiv l (\bmod k) \\ n \le x}} \Lambda(n) - \frac{x}{\phi(k)} \right|$$
$$\ll xL(xQ^{-1/2}L^7 + KQL^2)$$
$$\ll x^2 L^{8-C/3},$$

on choosing $K = xL^{-C}$, $Q = L^B$, $B = 2C/3$. This proves Theorem 2.

3. Application of Vaughan's identity

In this section we use Vaughan's identity to estimate the sum

$$\Sigma = \sum_{2x < m \le 3x} \Big(\sum_{n \le x} \Delta_Q(n)\Lambda(m-n) \Big).$$

Here we shall take $Q = L^B$ with a large constant value for B. The identity states that for any $u, v \geqslant 1$ we have

$$\sum_{v < n \le N} f(n)\Lambda(n) = S_1 - S_2 - S_3,$$

with

$$S_1 = \sum_{c \le u} \mu(c) \sum_{r \le N/c} (\log r) f(cr),$$
$$S_2 = \sum_{k \le uv} c_k \sum_{r \le N/k} f(kr), \qquad c_k = \sum_{\substack{c \le u \\ cn = k}} \sum_{n \le v} \mu(c)\Lambda(n),$$
$$S_3 = \sum_{\substack{r > u \\ rn \le N}} \sum_{n > v} d_r \Lambda(n) f(rn), \qquad d_r = \sum_{\substack{c \mid r \\ c \le u}} \mu(c). \tag{3.1}$$

We shall take $N = 3x$, $u = Q$, $v = xQ^{-2}$ and

$$f(n) = \begin{cases} \Delta_Q(m-n), & m - x \leqslant n < m, \\ 0, & \text{otherwise.} \end{cases}$$

We proceed to estimate

$$\Sigma_i = \sum_m |S_i|,$$

for $i = 1, 2, 3$.

To bound S_1 we use partial summation in conjunction with (2.7) of Lemma 1. This yields

$$\begin{aligned}\sum_{r \leq N/c} (\log r) f(rc) &= \sum_{(n-x)/c \leq r < m/c} (\log r) \Delta_Q(m - rc) \\ &\ll L \operatorname*{Max}_{y \leq x} \left| \sum_{\substack{s \equiv m (\bmod c) \\ s \leq y}} \Delta_Q(s) \right| \\ &\ll L(xL(cQ)^{-1}(c, m)d(c)).\end{aligned}$$

Note that, as before, the second term on the right of (2.7) dominates the other two, since A can be taken arbitrarily large. It now follows that

$$\Sigma_1 \ll \sum_{2x < m \leq 3x} \sum_{c \leq u} xL^2(cQ)^{-1}(c, m)d(c).$$

However (2.3) yields

$$\sum_{c \leq u} c^{-1}(c, m)d(c) \ll d(m)^2(\log u)^2 \ll d(m)^2 L.$$

Moreover

$$\sum_{2x < m \leq 3x} d(m)^2 \ll xL^3$$

by (2.2) with $t = 2$. Combining these estimates yields

$$\Sigma_1 \ll x^2 L^6 Q^{-1}. \tag{3.2}$$

We turn next to Σ_2. Since

$$|c_k| \leqslant \sum_{n|k} \Lambda(n) = \log k \ll L,$$

we have

$$\begin{aligned}S_2 &\ll L \sum_{k \leq uv} \left| \sum_{(m-x)/k \leq r < m/k} \Delta_Q(m - kr) \right| \\ &= L \sum_{k \leq uv} \left| \sum_{\substack{n \equiv m (\bmod k) \\ n \leq x}} \Delta_Q(n) \right|.\end{aligned}$$

As m runs over the interval $(2x, 3x]$, each congruence class (mod k) is covered $O(xk^{-1})$ times. It follows that

$$\Sigma_2 \ll Lx \sum_{k \leq uv} k^{-1} \delta_1(x, k, Q).$$

We may now apply Lemma 2 to obtain

$$\Sigma_2 \ll x^2 Q^{-1/2} L^8.$$

Lastly we examine Σ_3. We split the ranges for r and n into intervals $r \in (U, 2U]$, $n \in (V, 2V]$, where $U = u2^i$, $V = v2^j$. Since the corresponding subsum is empty unless

$$x \leqslant 4UV, \qquad UV \leqslant 3x, \tag{3.4}$$

there can be only $O(L)$ pairs of values U, V to be considered. It follows that

$$\Sigma_3 \ll L \sum_{2x < m \leq 3x} \sum_{V < n \leq 2V} \Lambda(n) \left| \sum_{U < r \leq 2U} d_r f(rn) \right|$$

for some U, V. Since $\Lambda(n) \ll L$ we can use Cauchy's inequality to obtain

$$\begin{aligned} \Sigma_3{}^2 &\ll L^2 x V L^2 \sum_{m,n} \left| \sum_{U < r \leq 2U} d_r f(rn) \right|^2 \qquad (3.5) \\ &= xVL^4 \sum_{U < r_i \leq 2U} d_{r_1} d_{r_2} \sum_{2x < m \leq 3x} \sum_{V < n \leq 2V} f(r_1 n) f(r_2 n). \end{aligned}$$

The innermost sum here is

$$S(r_1, r_2, m) = \sum_{n \in I} \Delta_Q(m - r_1 n) \Delta_Q(m - r_2 n),$$

where I is the interval

$$I = (V, 2V] \cap \left[\frac{m - x}{r_1}, \frac{m}{r_1}\right) \cap \left[\frac{m - x}{r_2}, \frac{m}{r_2}\right).$$

Let us first suppose that $r_1 = r_2$. Then, by (2.5), we have

$$S(r_1, r_1, m) \ll L \sum_{\substack{s \equiv m (\mathrm{mod}\, r_1) \\ s \leq x}} d(s)^2.$$

As before, if we sum over m, the residue classes $(\mathrm{mod}\, r_1)$ are each covered $O(xr_1^{-1}) = O(xU^{-1})$ times. Thus (2.2) with $t = 2$ yields

$$\sum_m S(r_1, r_1, m) \ll xU^{-1} L \sum_{s \leq x} d(s)^2 \ll x^2 U^{-1} L^4. \tag{3.6}$$

We now examine $S(r_1, r_2, m)$ when $r_1 < r_2$, the case $r_1 > r_2$ being essentially identical. We write $r = r_2 - r_1$ and $j = m - r_2 n$. Then

$$\begin{aligned} \sum_m S(r_1, r_2, m) &= \sum_{m,n} \Delta_Q(m - r_1 n) \Delta_Q(m - r_2 n) \qquad (3.7) \\ &= \sum_j \Delta_Q(j) \sum_n \Delta_Q(j + rn), \end{aligned}$$

where the conditions $2x < m \leqslant 3x$, $n \in I$ translate as $0 < j \leqslant x$ and

$$n \in (V, 2V] \cap (-j/r, (x - j)/r] \cap ((2x - j)/r_2, (3x - j)/r_2].$$

By (2.7) of Lemma 1 we have

$$\sum_{n} \Delta_Q(j + rn) \ll xL(rQ)^{-1}(r, j)d(r),$$

since, as before, the middle term on the right of (2.7) dominates. Now, by (2.5) and (2.3), equation (3.7) yields

$$\begin{aligned} \sum_{m} S(r_1, r_2, m) &\ll xL(rQ)^{-1}d(r) \sum_{j \le x} |\Delta_Q(j)|(r, j) \qquad (3.8)\\ &\ll xL^2(rQ)^{-1}d(r) \sum_{j \le x} d(j)(r, j)\\ &\ll x^2L^3(rQ)^{-1}d(r)^3\\ &= x^2L^3|r_1 - r_2|^{-1}Q^{-1}d(|r_1 - r_2|)^3, \qquad (r_1 \ne r_2). \end{aligned}$$

It is clear from the definition (3.1) that $|d_r| \leqslant d(r)$. Moreover, since (3.4) requires that

$$U \ll xv^{-1} = Q^2 = L^{2B},$$

we have

$$d(r) \ll r^{2/(2B)} \ll L, \qquad (r \ll U).$$

Hence, using (3.5), (3.6) and (3.8) we find

$$\begin{aligned} \Sigma_3{}^2 &\ll xVL^4\Bigg(\sum_{U < r \le 2U} |d_r|^2 x^2 U^{-1} L^4 + \\ &\qquad + \sum_{\substack{U < r_i \le 2U \\ r_1 \ne r_2}} |d_{r_1} d_{r_2}| x^2 L^3 |r_1 - r_2|^{-1} Q^{-1} d(|r_1 - r_2|)^3 \Bigg)\\ &\ll xVL^4(x^2L^6 + x^2L^8Q^{-1}\Sigma|r_1 - r_2|^{-1})\\ &\ll xVL^4(x^2L^6 + x^2L^9UQ^{-1})\\ &\ll x^4L^{10}U^{-1} + x^4L^{13}Q^{-1}\\ &\ll x^4L^{13}Q^{-1}. \end{aligned}$$

This last estimate may be combined with (3.2) and (3.3) to give

$$\Sigma \ll \Sigma_1 + \Sigma_2 + \Sigma_3 \ll x^2Q^{-1/2}L^8. \qquad (3.9)$$

4. Completion of the proof of Theorem 1 and its Corollaries

To complete the proof of Theorem 1 we need to know about

$$\sum_{n \le x} \Lambda_Q(n)\Lambda(m - n) \qquad (4.1)$$

for $2x < m \leqslant 3x$. By the definition of $\Lambda_Q(n)$ in conjunction with (2.4) this is

$$\sum_{q \leq Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|q} d\mu(d) \sum_{\substack{d|n \\ n \leq x}} \Lambda(m-n) =$$

$$= x \sum_{q \leq Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|q} \frac{d\mu(d)}{\phi(d)} E_{d,m} + O\left(xL^{-A} \sum_{q \leq Q} \frac{\mu(q)^2}{\phi(q)} \sum_{d|q} d\right).$$

Since $d\mu(d)\phi(d)^{-1}E_{d,m}$ is a multiplicative function of d, the innermost sum in the main term is

$$\prod_{\substack{p|q \\ p \nmid m}} \left(1 - \frac{p}{p-1}\right) = \frac{\mu(q)\mu((q,m))\phi((q,m))}{\phi(q)},$$

if q is square-free. Moreover the error term is

$$\ll xL^{-A} \sum_{q \leq Q} \frac{\sigma(q)}{\phi(q)} \ll xL^{2-A}Q \ll xQ^{-1},$$

by (2.1), since we may take $A = 2B + 2$. It follows that (4.1) is

$$x \sum_{q \leq Q} \frac{\mu(q)\mu((q,m))\phi((q,m))}{\phi(q)^2} + O(xQ^{-1}) =$$

$$= x \sum_{1}^{\infty} \frac{\mu(q)\mu((q,m))\phi((q,m))}{\phi(q)^2} + O\left(x \sum_{q > Q} \frac{(q,m)}{\phi(q)^2}\right) + O(xQ^{-1}).$$

The main term here is

$$x \prod_{p \nmid m} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|m} \left(1 + \frac{1}{(p-1)}\right) = x\mathfrak{S}(m)$$

and first error term is $O(xQ^{-1}Ld(m)^2)$ by (2.3), since

$$\frac{(q,m)}{\phi(q)^2} \ll \frac{(q,m)d(q)}{q^2}.$$

Now, using (3.9) together with the case $t = 2$ of (2.2), we see that

$$\sum_{2x < m \leq 3x} \left| \sum_{n \leq x} \Lambda(n)\Lambda(m-n) - x\mathfrak{S}(m) \right| \ll$$

$$\ll x^2Q^{-1/2}L^8 + x^2Q^{-1}L^4 \ll x^2Q^{-1/2}L^8.$$

Since the number of prime powers $p^e \leqslant 3x$ with $e \geqslant 2$ is $O(x^{1/2})$ we have

$$\sum_{n \leq x} \Lambda(n)\Lambda(m-n) = \sum_{\substack{p' \leq x \\ p' + p'' = m}} (\log p')(\log p'') + O(x^{1/2}L^2).$$

Thus

$$\sum_{2x < m \leq 3x} |N_2(m) - x\mathfrak{S}(m)| \ll x^2 Q^{-1/2} L^8,$$

and, as $Q = L^B$ with B arbitrary, Theorem 1 follows.

The corollaries require little comment. Since

$$\mathfrak{S}(m) \geqslant 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \gg 1$$

whenever m is even, there can be only $O(xL^{-C})$ even numbers m counted in Theorem 1 for which $N_2(m) = 0$. This gives Corollary 1. Next let n be odd, and take $x = n/3$. Then the numbers $n - p$, for odd primes $p < x$, are all even, and there are asymptotically xL^{-1} of them. Since only $O(xL^{-C})$ such numbers can have $N_2(n - p) = 0$ there must be at least one solution of $n - p = p' + p''$, if n is large enough. This proves Corollary 2. Similarly, since the number of integers $m = 2p$ in the range $2x < m \leqslant 3x$ is asymptotically $\frac{1}{2}xL^{-1}$, and only $O(xL^{-C})$ such integers can have $N_2(m) = 0$, there must be solutions of $2p = p' + p''$ with $p' \leqslant x$. Since this entails $p' \neq p(\neq p'')$, Corollary 3 is proved.

References

[1] M. B. Barban. Analogues of the divisor problem of Titchmarsh, *Vestnik Leningrad Univ. Ser. Mat. Meh. Astronom.*, **18** (1963) 4, 5-13.

[2] —. On the average error in the generalized prime number theorem, *Dokl. Akad. Nauk UzSSR*, **5** (1964), 5-7.

[3] J. G. van der Corput. Sur l'hypothèse de Goldbach pour presque tous les nombres pairs, *Acta Arith.*, **2** (1937), 266-290.

[4] N. G. Cudakov. On Goldbach's problem, *Dokl. Akad. Nauk SSSR*, **17** (1937), 331-334.

[5] —. On the density of the set of even numbers which are not representable as a sum of two odd primes, *Izv. Akad. Nauk SSSR Ser. Mat.*, **2** (1938), 25-39.

[6] H. Davenport. Multiplicative number theory, Markham, Chicago, 1967.

[7] — and H. Halberstam. Primes in arithmetic progressions, *Michigan Math. J.*, **13** (1966), 485-489.

[8] T. Estermann. On Goldbach's problem: Proof that almost all even positive integers are sums of two primes, *Proc. London Math. Soc.* (2), **44** (1938), 307-314.

[9] H. Heilbronn. *Zentralblatt*, **16** (1937), 291-292.

[10] C. Hooley. Proceedings of the International Congress of Mathematicians 1984, to appear.

[11] H. L. Montgomery. Topics in multiplicative number theory, Springer, Berlin, 1971.

[12] — and R. C. Vaughan. The exceptional set in Goldbach's problem, *Acta Arith.*, **27** (1975), 353-370.

[13] R. C. Vaughan. On Goldbach's problem, *Acta Arith.*, **22** (1972), 21-48.

[14] —. On the estimation of trigonometrical sums over primes, and related questions, Report 9, Institut Mittag-Leffler, 1977.

[15] I. M. Vinogradov, The representation of an odd number as a sum of three primes, *Dokl. Akad. Nauk SSSR*, **16** (1937), 139-142.

[16] —. Some theorems concerning the theory of primes, *Mat. Sb.* NS, **2** (1937), 179-195.

[17] A. Walfisz. Zur additiven Zahlentheorie. II, *Math. Zeit.*, **40** (1935-36), 592-607.

Roger Heath-Brown
Magdalen College
Oxford OX1 4AU
England

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 1, N.° 1, 1985

Regularity of the Boundary of a Capillary Drop on an Inhomogeneous Plane and Related Variational Problems

Luis A. Caffarelli **Avner Friedman**

1. Introduction

Consider the functional

$$J(E) = \int_{\Omega_0} |D\phi_E| + g \int_{\Omega_0} x_{n+1}\phi_E \, dx - \int_{\partial\Omega_0} \lambda(x')\phi_E \, dx' \tag{1.1}$$

where $\Omega_0 = \{x = (x', x_{n+1}), x_{n+1} > 0\}$, $x' = (x_1, \ldots, x_n)$ varies in $\mathbb{R}^n$, g is a positive constant, $\lambda(x')$ is a given function, $|\lambda(x')| < 1$, ϕ_E is the characteristic function of a set E, and E varies in the class

$$\mathcal{A} = \{E \subset \bar{\Omega}_0; E \text{ has finite perimeter } \int |D\phi_E| \}. \tag{1.2}$$

For a given positive number V set

$$\mathcal{A}_V = \{E \in \mathcal{A}, H^{n+1}(E) = V\}. \tag{1.3}$$

Problem $(\mathcal{Q}_V)$: Find E in $\mathcal{Q}_V$ such that

$$J(E) = \min_{F \in \mathcal{Q}_V} J(F).$$

For $n = 2$ this is precisely the sessile drop problem, i.e., the problem of a capillary drop occupying the set E and sitting on in inhomogeneous plane $\{x_{n+1} = 0\}$. The first term in $J(E)$ is the energy due to surface tension, the second term is the gravitational energy, and the last term is the wetting energy with contact angle $\theta(x')$ given by

$$\cos\theta(x') = \lambda(x'), \qquad 0 < \theta(x') < \pi.$$

In (the homogeneous) case $\lambda \equiv$ const., a minimizer E can be found having the form

$$E = \{x; |x'| < \rho(x_{n+1})\}$$

(see [11] [12]). In case $\lambda \not\equiv$ const., existence of a minimizer E was recently established by Caffarelli and Spruck [4] under some mild assumptions on $\lambda(x')$. For a strictly curved bottom $\partial\Omega_0$, existence was proved by Giusti [10].

We shall restrict λ to satisfy

$$0 < \lambda < 1; \tag{1.4}$$

then one can show that E is an x_{n+1}-subgraph that is

$$E = \{x; 0 \geqslant x_{n+1} < u(x'), x' \in S\} \tag{1.5}$$

for some function u with support S. In this paper we are interested in studying the boundary ∂S of S; ∂S may be conceived as the free boundary for the sessile drop problem. We prove that, for the case $n = 2$, ∂S is regular; more precisely,

$$\begin{aligned} &\text{if} \quad \lambda \in C^{m+\alpha} \quad \text{then} \quad \partial S \in C^{m+1+\alpha}; \\ &\text{if} \quad \lambda \quad \text{is analytic then} \quad \partial S \quad \text{is analytic;} \end{aligned} \tag{1.6}$$

the same holds for $3 \leqslant n \leqslant 6$ under some «flatness condition.»

Our method is based on extensions of the results of [1] and [2] to the minimal surface operator. To explain this connection, consider the functional

$$J(v) = \int_{\Omega \cap \{v > 0\}} f(x, v, \nabla v)\, dx \tag{1.7}$$

where Ω is, say, a bounded domain in $\mathbb{R}^n$ and v varies in the class of $H^{1,2}(\Omega)$ functions satisfying a boundary condition $v \doteq u^0$ on a portion $\partial_0\Omega$ of $\partial\Omega$; $u^0 \geqslant 0$. If u is a minimizer, then (see [1]) formally

$$\nabla \cdot f_p(x, u, \nabla u) - f_z(x, u, \nabla u) = 0 \quad \text{in} \quad \Omega \cap \{u > 0\} \tag{1.8}$$

where $z = u$, $p = \nabla u$, and $u = 0$,

$$f_p(x, u, \nabla u) \cdot \nabla u - f(x, u, \nabla u) = 0 \text{ on the free boundary } \Omega \cap \partial\{u > 0\}. \quad (1.9)$$

Taking in particular

$$f(x, u, \nabla u) = \sqrt{1 + |\nabla u|^2} + \frac{g}{2}u^2 - \mu u - \lambda(x) \qquad (\mu \geqslant 0), \qquad (1.10)$$

we get

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} - gu = -\mu \quad \text{in} \quad \Omega \cap \{u > 0\}, \qquad (1.11)$$

$$u = 0, \qquad \frac{1}{1 + |\nabla u|^2} = \lambda^2 \quad \text{on} \quad \Omega \cap \partial\{u > 0\}. \qquad (1.12)$$

(This incidentally shows that for a regular free boundary to exist one must assume that $|\lambda| < 1$.)

Observe now that the functional (1.1) for E a subgraph (as in (1.5)) reduces to the functional of the form (1.7) with f given by (1.10) and $\mu = 0$. The sessile drop problem, however, includes a volume constraint $H^{n+1}(E) = V$, which can actually be replaced by adding a "penalty" term $f_{\epsilon_0}(V_E)$ into the functional, where $V_E = H^{n+1}(E)$ for any $E \in \mathcal{Q}$. More precisely, Caffarelli and Spruck [4] introduce the functional

$$\tilde{J}(F) = J(F) + f_{\epsilon_0}(V_F) \qquad (F \in \mathcal{Q})$$

where $f_{\epsilon_0}(t) = (V - t)/\epsilon_0$ if $t < V$, $f_{\epsilon_0}(t) = 0$ if $t \geqslant V$ and prove that if ϵ_0 is positive and small enough then a minimizer E exists, $V_E = V$ and E is a solution of problem $(\mathcal{Q}_V)$.

The methods of the present paper apply also to the modified functional $\tilde{J}$. For the sake of clarity we shall first establish the regularity result (1.6) of the free boundary for the variational problem involving (1.7), (1.10) and then consider the sessile drop problem, indicating the minor changes in the proof.

The regularity of the free boundary for the variational problem for (1.7) was established by Alt and Caffarelli [1] in case $f(x, z, p) = |p|^2$ (corresponding to the Laplace operator), and by Alt, Caffarelli and Friedman [2] in the case of general $f(p) = F(|p|^2)$ corresponding to quasi-linear uniformly elliptic operator; the case $f = |p|^2 - Q(x)z$ with $Q > 0$ was considered by Friedman [5]. The main novelty of the present paper stems from the fact that the quasi-linear elliptic operator corresponding to (1.10) is not uniformly elliptic. Thus the crucial step is the proof that any minimizer u is Lipschitz continuous.

In §§2-4 we study the variational problem corresponding to (1.7), (1.10) and establish regularity of the minimizer and of the free boundary. In §5 we shall apply the results to the sessile drop problem as well as to other related capillary problems.

We always assume in this paper that $n \leqslant 6$; this ensures the regularity of the boundary of any perimeter minimizing set.

ADDED IN PROOF. Jean Taylor («Boundary regularity for solutions to various capillarity and free boundary problems,» *Comm. P.D.E.,* **2** (1977), 323-257) proved regularity of the free boundary surface in $\mathbb{R}^3$, using Almgren's approach.

2. The variational problem

A Borel function $v(x)$ defined in an open set $A \subset \mathbb{R}^m$ is said to be of bounded variation (BV) if

$$\int_A |Dv| \equiv \sup\Big\{\int_A v \operatorname{div} G; G = (G_1, \dots, G_m) \in C_0^1(A),$$
$$|G(x)|^2 = \sum_{i=1}^m G_i^2(x) \leqslant 1\Big\}$$

is a finite number. A Borel set $E \subset \mathbb{R}^m$ is said to have a finite perimeter in an open set $\Omega \subset \mathbb{R}^m$ if

$$\int_\Omega |D\phi_E| < \infty$$

where ϕ_E is the characteristic function of E.

We denote by E^* the one-sided Steiner symmetrization of a set E in $\mathbb{R}^m$ with respect to the plane $\Pi = \{x_m = 0\}$; more precisely, E^* lies in $\{x_m \geqslant 0\}$, $E^* \cap \{x' = x_0'\}$ consists of a single interval $0 \leqslant x_m \leqslant x_m'$ (for any $x_0' = (x_1^0, \dots, x_{m-1}^0)$), where $x = (x', x_m)$, $x' = (x_1, \dots, x_{m-1})$, and

$$H^1(E^* \cap \{x' = x_0'\}) = H^1(E \cap \{x' = x_0'\}).$$

We recall [14] that if $E \subset \{x_m \geqslant 0\}$ then

$$\int_{\{x_m > 0\}} |D\phi_{E^*}| \leqslant \int_{\{x_m > 0\}} |D\phi_E|. \tag{2.1}$$

Consider a set

$$E = \{x; x_m < u(x'), x' \in S\}$$

where $x = (x', x_m) \in \mathbb{R}^m$, $x' = (x_1, \dots, x_{m-1})$, and S is an open set in $\mathbb{R}^{m-1}$.

For any open set $A \subset S$ one defines (see [14])

$$\int_A \sqrt{1+|Du|^2}\,dx' = \sup\{\int_A (u \operatorname{div} G + G_m)\,dx';$$
$$G = (G_1, \ldots, G_{m-1}), G_i \in C_0^1(A), \sum_{i=1}^{m} G_i^2(x) \leqslant 1\}, \tag{2.2}$$

and then there holds ([14; Prop. 1.9 and (1.5)])

$$\int_{A \times \mathbb{R}} |D\phi_E| = \int_A \sqrt{1+|Du|^2}\,dx'. \tag{2.3}$$

In particular, E has a finite perimeter if and only if u is a BV function.

Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary is locally a Lipschitz graph and let u^0 be a nonnegative Lipschitz continuous function defined on $\partial\Omega$. Let

$$S_0 = \{(x, x_{n+1}); x \in \partial\Omega, 0 \leqslant x_{n+1} < u^0(x)\}$$

and denote by K_0 the class of sets in $\bar{\Omega} \times [0, \infty)$ with finite perimeter in $\Omega \times (0, \infty)$, which coincide with S_0 on $\partial\Omega \times [0, \infty)$. For $E \in K_0$, let

$$J_0(E) = \int_{\Omega \times (0,\infty)} |D\phi_E| + \int_{\Omega \times (0,\infty)} (gx_{n+1} - \mu)\phi_E - \int_{\Omega \times \{x_{n+1}=0\}} \lambda\phi_E \tag{2.4}$$

and consider the problem: Find E such that

$$E \in K_0, \qquad J_0(E) = \lim_{G \in K_0} J_0(G). \tag{2.5}$$

We assume that

$$g > 0, \qquad \mu \geqslant 0 \qquad (g, \mu \quad \text{are constants}) \tag{2.6}$$

and

$$\lambda(x) \quad \text{is Lipschitz continuous,} \qquad 0 < \lambda(x) < 1 \quad \text{in} \quad \bar{\Omega}. \tag{2.7}$$

Theorem 2.1. *There exists a solution E of problem* (2.5), *and E is a bounded set.*

PROOF. Since J_0 is clearly bounded from below, to prove existence for (2.5) it suffices to prove that J_0 is lower semicontinuous, or just that

$$\int_{\Omega \times (0,\infty)} |D\phi_E| - \int_{\Omega \times \{x_{n+1}=0\}} \lambda\phi_E$$

is lower semicontinuous; but this can be established as in [10; Th. 1.2]. The proof that a minimizer E is a bounded set is the same as in [10; Th. 2.3].

Notice that if E is in K_0 then, by (2.1), its onesided Steiner symmetrization E^* decreases the perimeter and strictly decreases the remaining part of J_0, unless $E^* = E$ a.e. Thus for a minimizer E we must have that

$$E^* = E. \tag{2.8}$$

We shall henceforth normalize E (as in [8; §3.1]) so that

$$0 < |B_\rho(X) \cap (\Omega \times (0, \infty))| < |B_\rho(X)| \qquad \forall X \in \partial E.$$

If $X^0 \in \partial E$, $X^0 \in \Omega \times (0, \infty)$ then take a small ball $B = B_r(X^0)$ contained in $\Omega \times (0, \infty)$. Clearly E is then a minimizer of

$$\tilde{J}_0(G) = \int_B |D\phi_G| + \int_B (gx_{n+1} - \mu)\phi_G$$

in the class of sets which coincide with E on ∂B. Hence, by Massari [13] (recall that $n \leqslant 6$), ∂E is in $C^{2+\alpha}$ in B and, in fact, since μ, g are constants,

$$\partial E \quad \text{is analytic in } B. \tag{2.9}$$

In view of (2.8) we can write

$$E = \{(x, x_{n+1}); 0 \leqslant x_{n+1} < u(x), x \in \Omega\} \tag{2.10}$$

for some fuction $u(x)$. In view of (2.3), $u \in BV(\Omega)$. Since E is a bounded set, we also have that

$$u(x) < C \quad \text{for all} \quad x \in \Omega \qquad (C \text{ constant}). \tag{2.11}$$

Lemma 2.2. *The function $u(x)$ is continuous in Ω.*

PROOF. Suppose $u(x^0) > 0$. If $u(x)$ is not continuous at x^0, then from (2.10) it follows that ∂E contains a vertical line segment. In view of the analyticity of ∂E, ∂E must then contain the entire interval $\{x = x^0, x_{n+1} > 0\}$, a contradiction to the boundedness of E. For the same reason, if $u(x^0) = 0$, then $u(x) \to 0$ if $x \to x^0$.

Lemma 2.3. *For any $v \in BV(\Omega) \cap C^0(\Omega)$, $v \geqslant 0$, or $v \in H^{1,2}(\Omega)$, $v \geqslant 0$,*

$$\int_\Omega \sqrt{1 + |Dv|^2}\, I_{\{v>0\}} = \int_\Omega (\sqrt{1 + |Dv|^2} - 1) + \int_\Omega I_{\{v>0\}} \tag{2.12}$$

where I_A denotes the characteristic function of a set A.

PROOF. Suppose first that $v \in BV(\Omega) \cap C^0(\Omega)$. We can approximate v by mollifiers v_m such that (see [8] [14])

$$\int_\Omega \sqrt{1 + |Dv_m|^2}\, I_A \to \int_\Omega \sqrt{1 + |Dv|^2}\, I_A$$

for $A = A_k \equiv \{ v > \frac{1}{k} \}$ $(k = 1, 2, \dots)$ and for $A = \Omega$ (notice that A_k is open). Since v_m is smooth,

$$\int_\Omega \sqrt{1 + |Dv_m|^2}\, I_{\{v_m > 0\}} = \int_\Omega (\sqrt{1 + |Dv_m|^2} - 1) + \int_\Omega I_{\{v_m > 0\}}. \tag{2.13}$$

Since v is continuous,

$$I_{\{v_m > 0\}} \geqslant I_{\{v > 1/k\}}$$

for any positive integer k, if $m \geqslant m_0(k)$. Hence

$$\int_\Omega (\sqrt{1 + |Dv_m|^2} - 1) I_{\{v_m > 0\}} \geqslant \int_\Omega (\sqrt{1 + |Dv_m|^2} - 1) I_{\{v > 1/k\}} \to$$
$$\to \int_\Omega (\sqrt{1 + |Dv|^2} - 1) I_{\{v > 1/k\}}.$$

Since k is arbitrary,

$$\int_\Omega (\sqrt{1 + |Dv|^2} - 1) I_{\{v > 0\}} \leqslant \liminf_{m \to \infty} \int_\Omega (\sqrt{1 + |Dv_m|^2} - 1) I_{\{v_m > 0\}}.$$

Using this in (2.13) we obtain

$$\int_\Omega \sqrt{1 + |Dv|^2}\, I_{\{v > 0\}} \leqslant \int_\Omega (\sqrt{1 + |Dv|^2} - 1) + \int_\Omega I_{\{v > 0\}}. \tag{2.14}$$

To prove the reverse inequality we approximate $(v - 1/m)^+$ by mollifiers $v_{mj} (j \to \infty)$ such that

$$\int_\Omega \sqrt{1 + |Dv_{mj}|^2}\, I_A \to \int_\Omega \sqrt{1 + |D(v - 1/m)^+|^2}\, I_A$$

for $A = \{ v > 0 \}$ and for $A = \Omega$. Since $I_{\{v_{mj} > 0\}} \geqslant I_{\{v > 0\}}$ if j is large enough,

$$\limsup_{j \to \infty} \int_\Omega (\sqrt{1 + |Dv_{mj}|^2} - 1) I_{\{v_{mj} > 0\}} \leqslant \limsup_{j \to \infty} \int_\Omega (\sqrt{1 + |Dv_{mj}|^2} - 1) I_{\{v > 0\}} =$$
$$= \int_\Omega (\sqrt{1 + |D(v - 1/m)^+|^2} - 1) I_{\{v > 0\}}.$$

Noting that (2.13) holds for the smooth functions v_{mj} and taking a suitable sequence $j \to \infty$, we then obtain

$$\int_\Omega \sqrt{1 + |D(v - 1/m)^+|^2} I_{\{v > 0\}} \geqslant$$
$$\geqslant \int_\Omega (\sqrt{1 + |D(v - 1/m)^+|^2} - 1) + \int_\Omega I_{\{v > 0\}}. \tag{2.15}$$

If $F = \{ (x, x_{n+1}); 0 \leqslant x_{n+1} < v(x), x \in \Omega \}$, then, by (2.3),

$$\int_\Omega \sqrt{1 + |D(v - 1/m)^+|^2} = \int_{\Omega \times (1/m, \infty)} |D\phi_F| \nearrow \int_{\Omega \times (0, \infty)} |D\phi_F|$$

as $m \to \infty$; the same holds with Ω replaced by $\Omega \cap \{ v > 0 \}$. Using these facts in (2.15) we obtain the reverse of (2.14), which completes the proof of (2.12).

If $v \in H^{1,2}(\Omega)$ then $Dv = 0$ a.e. on $\{v = 0\}$, so that

$$(\sqrt{1 + |Dv|^2} - 1) I_{\{v=0\}} = 0,$$

which immediately yields (2.12) if $v \geqslant 0$.

We introduce the functional

$$J_0(v) = \int_\Omega \sqrt{1 + |Dv|^2} I_{\{v>0\}}\, dx + \int_\Omega \left(\frac{g}{2} v^2 - \mu v\right) dx - \int_\Omega \lambda(x) I_{\{v>0\}}\, dx \tag{2.16}$$

and the admissible class

$$K = \{v \in BV(\Omega), v \geqslant 0 \quad \text{in} \quad \Omega, v = u^0 \quad \text{on} \quad \partial\Omega\}. \tag{2.17}$$

Consider the problem: Find u such that

$$u \in K \quad \text{and} \quad J_0(u) = \min_{v \in K} J_0(v). \tag{2.18}$$

We shall find it convenient to work also with the functional

$$J(v) = \int_\Omega (\sqrt{1 + |Dv|^2} - 1)\, dx + \int_\Omega \left(\frac{g}{2} v^2 - \mu v\right) dx + \\ + \int_\Omega (1 - \lambda) I_{\{v>0\}}\, dx. \tag{2.19}$$

Theorem 2.4. *Let u be defined by* (2.10) *where E is a solution of problem* (2.5). *Then u is a solution of problem* (2.18), $u \in C^0(\Omega)$ *and*

$$J(u) \leqslant J(v) \qquad \forall v \in K \cap C^0(\Omega). \tag{2.20}$$

PROOF. Since $J_0(u) = J_0(E)$ and

$$J_0(v) = J_0(G)$$

if $G = \{(x, x_{n+1}); 0 \leqslant x_{n+1} < v(x), x \in \Omega\}$, u is a solution of (2.18). The continuity of u was already established in Lemma 2.2, and (2.20) follows from the relation

$$J(v) = J_0(v) \quad \text{if} \quad v \in K \cap C^0(\Omega)$$

which is obtained using Lemma 2.3.

3. Lipschitz continuity

In this section we establish the Lipschitz continuity of the solution u asserted in Theorem 2.4.

We introduce the minimal surface operator

$$\mathcal{L}v = \operatorname{div} \frac{Dv}{\sqrt{1 + |Dv|^2}}$$

and consider the Dirichlet problem

$$\begin{aligned} \mathcal{L}w - gw &= -\mu \quad \text{in} \quad B_\epsilon(x_0), \\ w &= u \qquad \text{on} \quad \partial B_\epsilon(x_0) \end{aligned} \tag{3.1}$$

where $B_\epsilon(x_0) = \{x; |x - x_0| < \epsilon\}$ and $x_0 \in \Omega$. Set, for brevity, $B = B_\epsilon(x_0)$.

Lemma 3.1. *If ϵ is small enough then there exists a unique solution w in $C^{2+\alpha}(B) \cap C^0(\bar{B})$ of* (3.1).

By standard regularity results it follows that $w(x)$ is analytic.

PROOF. Uniqueness follows from the maximum principle. Existence is established in [7] in case $g = 0$; the proof in case $g > 0$ is similar and, for completeness, we briefly describe it. Denote the boundary values of u on ∂B by ϕ, and consider first the case where $\phi \in C^{2+\alpha}$. By comparing w (if existing) with $\pm M$ (M constant) we find that

$$\max_{\bar{B}} |w| < M \quad \text{if} \quad M > \sup |\phi|.$$

Hence $|gw| \leqslant gM$. We can now use [7; p. 285, Cor 13.5] (see (13.35), (13.36)) to deduce that

$$|Dw| \leqslant C_0 \quad \text{on} \quad \partial\Omega$$

if ϵ is small enough. By [7; p. 303, Th. 14.1] we then also get

$$|Dw| \leqslant C_0 \quad \text{in} \quad \Omega.$$

Thus $\mathcal{L}w$ is uniformly elliptic and then, by [7; p. 276, Th. 12.7],

$$[Dw]_{\alpha, \Omega} \leqslant C.$$

Having thus established an a priori $C^{1+\alpha}$ estimate on w, we can apply [7; p. 229, Th. 10.8] and deduce the existence of a $C^{2+\alpha}(\bar{B})$ solution of (3.1) in case $\phi \in C^{2+\alpha}$.

Consider next the case where ϕ is only assumed to be continuous, and approximate it uniformly by functions $\phi_m \in C^{2+\alpha}$. By the maximum principle, the corresponding solutions w_m satisfy:

$$|w_m - w_k|_{L^\infty(\Omega)} \leqslant |\phi_m - \phi_k|_{L^\infty(\partial\Omega)}.$$

Hence $w_m \to w$ uniformly in Ω.

By [7; p. 346, Cor. 15.6] (or [3]), for any $\Omega' \subset\subset \Omega$,

$$|Dw_m| \leqslant C \quad \text{in} \quad \Omega' \qquad (C = C(\Omega') \text{ constant})$$

provided $g = 0$; the proof extends with small changes to the case $g > 0$. But then also

$$[Dw_m]_{\alpha, \Omega'} \leqslant C$$

and consequently, for a subsequence, $w_m \to w$ in $C^{2+\alpha}(B) \cap C^0(\bar{B})$ where w is the desired solution.

Lemma 3.2. *The function u is an analytic solution of*

$$\mathcal{L}u - gu = -\mu \quad \text{in} \quad \{u > 0\}.$$

PROOF. Suppose $u(x_0) > 0$ and denote by w the solution of (3.1) where ϵ is sufficiently small so that $\overline{B_\epsilon(x_0)} \subset \{u > 0\}$. It suffices to show that $w \equiv u$. Consider the family of functions $w_M = w - M$ $(M > 0)$. If M is large enough then $w_M \leqslant u$ in $B = B_\epsilon(x_0)$. We decrease M until we arrive at the smallest value M_0 such that $w_{M_0} \leqslant u$. We claim that

$$M_0 = 0.$$

Indeed, if $M_0 > 0$ then there must exist a point $\bar{x} \in B$ such that $w_{M_0} = u$ at $\bar{x}$. Also,

$$\mathcal{L}w_{M_0} - gw_{m_0} = -\mu + gM_0 \quad \text{in} \quad B.$$

Recall that $(x, u(x))$ represents a smooth surface, by (2.9), and observe that the surfaces $(x, u(x))$, $(x, w_{M_0}(x))$ are tangent at $(\bar{x}, u(\bar{x}))$ and thus have a common normal $\bar{\nu}$. Using a coordinate system $(\tilde{x}, \tilde{x}_{n+1})$ in which $\bar{\nu}$ is in the direction of the $\tilde{x}_{n+1}$-axis, these surfaces can be represented in the form $\tilde{x}_{n+1} = U(\tilde{x})$ and $\tilde{x}_{n+1} = W(\tilde{x})$ respectively, and

$$\begin{aligned} \mathcal{L}U - \tilde{g}U &= -\tilde{\mu}(\tilde{x}), \\ \mathcal{L}W - \tilde{g}W &= -\tilde{\mu}(\tilde{x}) + \tilde{g}M_0 \end{aligned}$$

in a neighborhood N of $(\bar{x}, u(\bar{x}))$, where $\tilde{g}$, $\tilde{\mu}$ are the same functions in both equations and $\tilde{g} = \text{const.} > 0$. Indeed, if $\tilde{X} = (\tilde{x}, x_{n+1}) = TX$ where $X = (x, x_{n+1})$, T orthogonal matrix, $e_{n+1} = (0, 0, \ldots, 0, 1)$ and $\tilde{e} = Te_{n+1} = (b_1, \ldots, b_{n+1})$, then

$$\int gx_{n+1}\phi_G = \int gX \cdot e_{n+1}\phi_G = \int g\tilde{X} \cdot \tilde{e}\phi_G = \int gb_{n+1}\tilde{x}_{n+1}\phi_G + \int g \sum_{i=1}^{n} b_i\tilde{x}_i\phi_G$$

for any set G in K_0. Due to this change in the functional J_0 (or J) we find that

$$\tilde{g} = b_{n+1}g, \qquad \tilde{\mu}(\tilde{x}) = \mu - g \sum_{i=1}^{n} b_i\tilde{x}_i. \tag{3.2}$$

Notice that $b_{n+1} > 0$.

We now apply the maximum principle to $U - W$ and deduce that $M_0 = 0$ (and $U = M$ in N), a contradiction. We have thus proved that $M_0 = 0$ and $w \leqslant u$, and similarly $w \geqslant u$.

Later on we shall need to use radial solutions $s = s(r)$ of

$$\mathcal{L}s - \tilde{g}s = -\tilde{\mu} \quad \text{in a shell} \quad \rho < r < R, \tag{3.3}$$

$$s(R) = 0 \tag{3.4}$$

where $\tilde{g}$, $\tilde{\mu}$ are nonnegative constants and

$$\tilde{g} \ll \tilde{\mu}, \qquad \tilde{\mu} \ll 1 \tag{3.5}$$

Rewriting (3.3) in the form

$$\left(\frac{r^{n-1}s'}{\sqrt{1 + s'^2}}\right)' = (-\tilde{\mu} + \tilde{g}s)r^{n-1}$$

we find that

$$\frac{r^{n-1}s'}{\sqrt{1 + s'^2}} = \gamma - \frac{\hat{\mu}r^n}{r} \qquad (\gamma \text{ constant})$$

where $\hat{\mu} = \tilde{\mu}(1 + o(1))$ ($o(1) \to 0$ if $\tilde{g}/\tilde{\mu} \to 0$). Thus a solution is given by

$$s'(r) = -\frac{\gamma r^{1-n} - \hat{\mu}r/n}{[1 - (\gamma r^{1-n} - \hat{\mu}r/n)^2]^{1/2}}, \qquad s(R) = 0$$

provided γ is chosen so that

$$(\gamma r^{1-n} - \hat{\mu}r/n)^2 < 1.$$

Since $\hat{\mu}$ is small,

$$s'(r) \sim -\frac{\gamma r^{1-n}}{\sqrt{1-(\gamma r^{1-n})^2}} \quad \text{for} \quad \rho < r < R, \qquad s(R) = 0$$

$$\text{provided } (\gamma\rho^{1-n})^2 \leqslant 1/2. \tag{3.6}$$

We now state the main result of this section.

Theorem 3.3. $u \in C^{0,1}(\Omega)$.

PROOF. Suppose the assertion is not true. Then we can find a sequence $X^m = (x^m, y^m)$ with $y^m = u(x^m) > 0$,

$$\rho_m = \operatorname{dist}(X^m, \text{free boundary}) \to 0$$

(the free boundary is the set $\partial\{u > 0\} \times \{x_{n+1} = 0\}$), and free boundary points $\bar{X}^m = (\bar{x}^m, 0)$ such that

$$|X^m - \bar{X}^m| = \rho_m, \qquad \frac{y^m}{|x^m - \bar{x}^m|} \to 0$$

and $\operatorname{dist}(x_m, \partial\Omega) \geqslant \text{const.} > 0$.

On the line segment $\overline{x^m \bar{x}^m}$ we can clearly find a point $\tilde{x}^m$ such that

$$(\tilde{x}^m, u(\tilde{x}^m)) \in B_{\rho_m/2}(X^m) \quad \text{and} \quad |\nabla u(\tilde{x}^m)| \to 0$$

if $m \to \infty$.

The surface $y = U_m(x)$, where

$$U_m(x) = \frac{1}{\rho_m} u(x^m + \rho_m x),$$

will be denoted by S_m. By [13], $S_m \cap B_{1-\epsilon}$ are uniformly $C^{2+\alpha}$ surfaces, for any $\epsilon > 0$. Hence, for a subsequence,

$$S_m \cap B_{1-\epsilon} \to S \cap B_{1-\epsilon}$$

in $C^{2+\alpha}$ sense, for any $\epsilon > 0$. Since

$$\mathcal{L}u - gu = -\mu \quad \text{in} \quad \{u > 0\},$$

it is easily seen that

$$\mathcal{L}U_m - g_m U_m = -\mu_m \quad \text{in} \quad \tilde{S}_m$$

where $\tilde{S}_m$ is the projection of S_m on $\{x_{n+1} = 0\}$, and

$$g_m = g\rho_m^2, \qquad \mu_m = \mu\rho_m.$$

It follows that

$$S \cap B_1 \quad \text{is a minimal surface,} \tag{3.7}$$

and a graph $x_{n+1} = U(x)$.

Denote by $(z_m, U_m(z_m))$ the point corresponding to $(\tilde{x}^m, u(\tilde{x}^m))$. Then

$$|z_m| \leqslant \tfrac{1}{2}, |DU_m(z_m)| \to \infty \quad \text{if} \quad m \to \infty.$$
$$U_m(z_m) \to U(z_0).$$

We may assume that $z_m \to z_0$. Then the tangent to S at $Z^0 = (z_0, U(z_0))$ is vertical. Since S in an analytic surface, it is then given in a neighborhood W of Z^0 by

$$x_1 = w(x_2, \ldots, x_n, x_{n+1}) \tag{3.8}$$

with $\partial w/\partial x_{n+1} = 0$ at Z^0 (here we have made a suitable rotation of the axes $x_1, x_2, \ldots, x_n$). Denote by B_0 a ball such that $(x_1, w(x_2, \ldots, x_n)) \in W$ if $(x_2, \ldots, x_{n+1}) \in B_0$).

Since $S \cap B_1$ is x_{n+1}-graph, it follows from the representation (3.8) that $\partial w/\partial x_{n+1} \leqslant 0$ in B_0 if, say, $\{U > 0\} \cap W$ lies to the left of S.

Differentiating the minimal surface equation $\mathfrak{L}w = 0$ with respect to x_{n+1} we find that $\partial w/\partial x_{n+1}$ satisfies a linear elliptic equation to which the strong maximum principle can be applied. Since $\partial w/\partial x_{n+1} \leqslant 0$ in B_0 whereas $\partial w/\partial x_{n+1} = 0$ at Z^0, it follows that

$$\frac{\partial w}{\partial x_{n+1}} \equiv 0 \quad \text{in} \quad B_0.$$

Consequently

$$x_1 = w(x_2, \ldots, x_n) \tag{3.9}$$

in B_0 and, by analytic continuation, the same holds throughout $S \cap B_1$. Thus $S \cap B_1$ is a cylinder whose generators are parallel to the x_{n+1}-axis. Further, since S is x_{n+1}-graph, given by $x_{n+1} = U(x)$,

$$U(x) = 0 \quad \text{in} \quad \{x_1 < w(x_2, \ldots, x_n)\}. \tag{3.10}$$

We shall now derive a contradiction to the fact that u is a minimizer. We can do it either (i) by working with u, or (ii) by working with U. It will be instructive to describe both methods.

Method (i). Since $S_m \cap B_{1-\epsilon} \to S \cap B_{1-\epsilon}$ in $C^{2+\alpha}$ (for any $\epsilon > 0$), it follows from (3.9), (3.10) that after rotating the x_{n+1}-axis by a small angle δ we have, in the new coordinate system which we again denote by $x_1, \ldots, x_n, x_{n+1}$,

$$u(x) > Md(x^m) \quad \text{in} \quad B_{\theta d(x^m)}(x^m) \qquad (0 < \theta < 1) \tag{3.11}$$

where $d(x^m) = \text{dist}(x^m, \partial\{u > 0\}), d(x^m) \to 0$ if $m \to \infty$; here $1 - \theta$ can be taken arbitrarily small and M can be taken arbitrary large provided δ is small enough and m sufficiently large.

We rescale by $d(x^m)$ so as to obtain a new function u such that

$$u(x) > M \quad \text{in} \quad B_\theta(y_0) \tag{3.12}$$

where y_0 corresponds to a particular point x^m with m large enough, $|y| = 1$, its nearest free boundary point is at the origin, and the corresponding g, μ (which are in fact $gd^2(x^m)$, $\mu d(x^m)$) satisfy (3.5), so that the radial solution s of (3.3) can be constructed as above.

Take a point z_0 in the internal $\overline{Oy_0}$ with $|z_0| \leqslant (1 - \theta)/2$ and consider the shell Σ with center z_0 and radii

$$r_1 = (1 - \theta)/2, \qquad r_2 = 1 - |z_0| + \epsilon \qquad (\epsilon > 0). \tag{3.13}$$

Introduce the function

$$w = \text{maw}\{u, s\} \tag{3.14}$$

where s is constructed as in (3.6) with $\rho = r_1$, $R = r_2$ and $r = |x - z_0|$.

In view of (3.12) and the smallness of $1 - \theta$, $1/M$, we can choose the constant γ in (3.6) such that

$$u \geqslant s \quad \text{on} \quad \{r = r_1\}$$

and

$$s'(r) \geqslant \sigma, \tag{3.15}$$

with σ large; in fact,

$$\sigma = \sigma(M, \theta) \to \infty \quad \text{if} \quad M \to \infty, \theta \to 1. \tag{3.16}$$

Notice also that

$$u \geqslant 0 = s \quad \text{on} \quad \{r = r_2\},$$

and consequently w is an admissible function. Denoting by J_Σ the part of J taken over the shell Σ, the minimality of u implies that

$$J_\Sigma(u) \leqslant J_\Sigma(w); \tag{3.17}$$

here we used Theorem 2.4 and, for simplicity, we work with the original J rather than with its scaled form.

We shall derive a contradiction to (3.17). For clarity let us first proceed in a formal way.

We have

$$J_\Sigma(u) - J_\Sigma(w) = \int_\Sigma (\sqrt{1+|Du|^2} - \sqrt{1+|Dw|^2})$$
$$+ \int_\Sigma \left[\left(\frac{g}{2}u^2 - \mu u\right) - \left(\frac{g}{2}w^2 - \mu w\right)\right] - \int_\Sigma (1-\lambda) I_{\{u=0\}}. \tag{3.18}$$

For $\nabla u = b$, $\nabla w = a$ we shall use the identity

$$\sqrt{1+|b|^2} - \sqrt{1+|a|^2} - \frac{(b-a)\cdot a}{\sqrt{1+|a|^2}} =$$
$$= \frac{(1+|b|^2)(1+|a|^2) - (1+a\cdot b)^2}{\sqrt{1+|a|^2}(\sqrt{1+|a|^2}\sqrt{1+|b|^2} + 1 + a\cdot b)}.$$

By convexity, the right hand side is always $\geqslant 0$; however on the set $\{u = 0\}$ (where, formally, $\nabla u = 0$), we have the stronger inequality

$$\geqslant \frac{|a|^2}{\sqrt{1+|a|^2}(\sqrt{1+|a|^2}+1)}.$$

Thus we obtain from (3.18)

$$J_\Sigma(u) - J_\Sigma(w) \geqslant \int_\Sigma \frac{\nabla(u-w)\cdot\nabla w}{\sqrt{1+|\nabla w|^2}}$$
$$+ \int_{\Sigma\cap\{u=0\}} \frac{s'^2}{\sqrt{1+s'^2}(\sqrt{1+s'^2}+1)}$$
$$+ \int_\Sigma \left[\left(\frac{g}{2}u^2 - \mu u\right) - \left(\frac{g}{2}w^2 - \mu w\right)\right]$$
$$- \int_\Sigma (1-\lambda) I_{\{u=0\}}. \tag{3.19}$$

Since, formally,

$$\int_\Sigma \frac{\nabla(u-w)\cdot\nabla w}{\sqrt{1+|\nabla w|^2}} = \int_\Sigma (u-w)\mathcal{L}s = \int_\Sigma (u-w)(-gs-\mu)$$
$$= \int_\Sigma (u-w)(-gw-\mu) \tag{3.20}$$

and

$$(u - w)(-gw) + \frac{g}{2}u^2 - \frac{g}{2}w^2 = \frac{g}{2}(u - w)^2 \geqslant 0,$$

we obtain

$$J_\Sigma(u) - J_\Sigma(w) \geqslant \int_{\Sigma\cap\{u=0\}} \left[\frac{s'^2}{\sqrt{1 + s'^2}(\sqrt{1 + s'^2} + 1)} - (1 - \lambda)\right]$$

$$> 0 \qquad \text{by (3.15), (3.16)}$$

(provided M is chosen large enough and $1 - \theta$ is chosen small enough, depending on λ) which is a contradiction to (3.17).

In order to carry out the preceding argument rigorously, we approximate u first by $(u - \epsilon_j)^+$ and then by mollifiers $v_{j,k} \equiv (u - \epsilon_j)^+ * \eta_k$ in a neighborhood of $\bar{\Sigma}$. Note that

$$\int_\Sigma \sqrt{1 + |D(u - \epsilon_j)^+|^2} \leqslant \int_\Sigma \sqrt{1 + |Du|^2}$$

and

$$\int_\Sigma \sqrt{1 + |Dv_{j,k}|^2} \to \int_\Sigma \sqrt{1 + |D(u - \epsilon_j)^+|^2}$$

if

$$\int_{\partial\Sigma} \sqrt{1 + |D(u - \epsilon_j)^+|^2} = 0$$

(which we may assume to be the case, by slightly changing the radii of Σ). Since also

$$I_{\{v_{j,k} > 0\}} < I_{\{u > 0\}}$$

if k is large enough (depending on ϵ_j), we may choose a sequence $u_j = v_{j,k(j)}$ such that

$$I_{\{u_j > 0\}} < I_{\{u > 0\}} \tag{3.21}$$

and

$$\int_\Sigma \sqrt{1 + |Du_j|^2} < \int_\Sigma \sqrt{1 + |Du|^2} + \eta_j, \qquad \eta_j \to 0. \tag{3.22}$$

Setting $w_j = \max(s, u_j)$ and observing that $w_j = u_j$ on ∂S, we can proceed as before (but this time rigorously) to establish that

$$J_\Sigma(u_j) - J_\Sigma(w_j) \geqslant \int_{\Sigma\cap\{u_j=0\}} \left[\frac{s'^2}{\sqrt{1 + s'^2}(\sqrt{1 + s'^2} + 1)} - (1 - \lambda)\right] \geqslant$$

$$\geqslant c\int_\Sigma I_{\{u=0\}}, \qquad c > 0. \tag{3.23}$$

On the other hand, by (3.21), (3.22),

$$J_\Sigma(u) \geqslant \limsup_{j\to\infty} J_\Sigma(u_j)$$

and by the lower semicontinuity of the perimeter

$$J_\Sigma(w) \leqslant \liminf_{j\to\infty} J_\Sigma(w_j).$$

Thus, taking $j \to \infty$ in (3.23) we find that

$$J_\Sigma(u) - J_\Sigma(w) > 0, \tag{3.24}$$

a contradiction to (3.17). This completes the proof of the Lipschitz continuity by method (i).

Method (ii). Here we work directly with the blow up limit U. First we must establish that the subgraph E_U of U is a minimizer. The proof is similar to the proof of [2; Lemma 3.3] which asserts that the blow up limit of u_m with respect to $B_{\rho_m}(x^m)$ is a minimizer. However, in that lemma it is given that $|\nabla u_m| \leqslant C$, which is not the case here. In our case one can easily show that the perimeter of the subgraphs of u_m and of $v + (1 - \eta)(u_m - u_0)$ are uniformly small outside the set $\{\eta = 1\}$ and then proceed as in [2], using the lower semicontinuity of the perimeter.

Suppose now that E_U lies in $\{x_1 < w(x_2, \ldots, x_n)\}$. Since w is analytic, the set $S \cap \{x_{n+1} \geqslant 0\}$ is regular and we can therefore repeat the proof of (3.24) working directly with the set S and with the set T_s, the subgraph of s (where s is now a radial solution of the minimal surface equation). The calculations are in fact simpler as well as rigorous (i.e., there is no need to justify the formal calculations by approximation).

The blow up limit E_U may have, however, another portion E_2 in $\{x_1 > w\}$ (we denote the portion in $\{x_1 < w\}$ by E_1). By what was said in the preceding paragraph, we have, analogously to (3.24),

$$J_\Sigma(E_1 \cup T_s) < J_\Sigma(E_1). \tag{3.25}$$

In order to derive a contradiction to the minimality of the set E_U it suffices to show that

$$J_\Sigma(E_1 \cup E_2 \cup T_s) < J_\Sigma(E_1 \cup E_2).$$

But this follows from the well known inequality

$$\mathrm{Per}(E_2 \cup T_s) \leqslant \mathrm{Per}(E_2) + \mathrm{Per}(T_s).$$

4. Regularity of the free boundary

Let Ω' be a subdomain of Ω and let M be a positive number larger than the Lipschitz coefficient of u in Ω'. Define

$$F(t) = \begin{cases} \sqrt{1+t} - 1 & \text{if} \quad 0 \leqslant t \leqslant M, \\ \sqrt{1+t} - 1 + \epsilon \dfrac{(t-M)^2}{1+(t-M)} & \text{if} \quad t > M; \end{cases}$$

if ϵ is positive then the function $f(p) = F(|p|^2)$ satisfies, for some $\beta > 0$ and all $p \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$,

$$\begin{aligned} &\beta|\xi|^2 \leqslant \sum_{i,j=1}^{n} \frac{\partial^2 f(p)}{\partial p_i \partial p_j} \leqslant \beta^{-1}|\xi|^2, \\ &\beta|p|^2 \leqslant f_p(p)p, \qquad |f_p(p)| \leqslant \beta^{-1}|p|, \\ &\beta|p|^2 \leqslant f(p) \leqslant \beta^{-1}|p|^2, \end{aligned} \tag{4.1}$$

and these are precisely the conditions which are needed for the results in [2].

Consider the functional

$$J'(v) = \int_{\Omega'} F(|\nabla v|^2)\,dx + \int_{\Omega'} \left(\frac{g}{2}v^2 - \mu v\right) dx + \int_{\Omega'} (1-\lambda) I_{\{v>0\}}\,dx$$

and the admissible class

$$K_{\Omega'} = \{v \in H^{1,2}(\Omega'), v \geqslant 0, v = u \quad \text{on} \quad \partial\Omega'\}.$$

Noting that

$$\begin{aligned} &\sqrt{1+|\nabla v|^2} \leqslant F(|\nabla v|^2) \quad \text{if} \quad v \in K_{\Omega'}, \\ &\sqrt{1+|\nabla u|^2} = F(|\nabla u|^2) \quad \text{in} \quad \Omega', \end{aligned}$$

it is clear that

$$J'(u) \leqslant J'(v) \qquad \forall v \in K_{\Omega'}. \tag{4.2}$$

Set

$$P(x, v) = \frac{g}{2}v^2 - \mu. \tag{4.3}$$

For any ball $B = B_r(x_0)$ in Ω', let v be the solution of

$$-\operatorname{div} f_P(Dv) + P_v(x, v) = 0 \quad \text{in} \quad B, \qquad v = u \quad \text{on} \quad \partial B,$$

where $f(p) = F(|p|^2)$. Since $P_v = gv$, we can apply the maximum principle to conclude that $v \geqslant 0$. It follows that v (when extended by u into $\Omega' \setminus B$) is in $K_{\Omega'}$.

Using the above remark we can now extend all the results of [2] to the present problem (4.2). (Note that in [2] f satisfies the conditions in (4.1) and $P \equiv 0$; see also [5] where $F(p) = |p|$ and $P \not\equiv 0$). In particular, the following theorems are valid:

Theorem 4.1. *If* $n = 2$ *then* (i) *if* $\lambda \in C^{k,\alpha}$ *then the free boundary is in* $C^{k+1,\alpha}$; (ii) *if* λ *is analytic then the free boundary is analytic.*

Theorem 4.2. *Let* $3 \leqslant n \leqslant 6$ *and let* D *be a domain with* $\bar{D} \subset \Omega$. *There exist positive constants* α, β, σ_0, τ, C *such that for any free boundary point* x^0 *in* D *the following is true:*

If in some coordinate system

$$u(x) = 0 \quad \text{in} \quad B_\rho(x_0) \cap \{x_n - x_n^0 > \sigma\rho\} \tag{4.4}$$

where $x^0 = (x_1^0, \ldots, x_n^0)$, $\sigma \leqslant \sigma_0$, $v \leqslant \tau_0\sigma^{2/\beta}$, *then* $\partial\{u > 0\} \cap B_{\rho/4}(x_0)$ *has the form* $x_n = g(x')$ $(x' = (x_1, \ldots, x_{n-1}))$ *with* $g \in C^{1,\alpha}$ *and*

$$|Dg(x') - Dg(\bar{x}')| \leqslant C\left|\frac{x' - \bar{x}'}{\rho}\right|^\alpha.$$

Further, $g \in C^{k+1,\gamma}$ *if* $\lambda \in C^{k,\gamma}$ *and* g *is analytic if* λ *is analytic.*

The condition (4.4) is called the flatness condition. In general, not assuming flatness, one can assert for the set S of singularities of the free boundary that $H^{n-1}(S) = 0$.

5. Applications

Consider a capillary drop on a horizontal inhomogeneous plane $\Omega_0 = \mathbb{R}^n$; the contact angle $\theta(x)$ is non-constant in general. To study this problem we introduce the functional

$$J_{\epsilon_0}(G) = \int_{Q_0} |D\phi_G| + \int_{Q_0} g x_{n+1}\phi_G - \int_{\partial Q_0} \lambda\phi_G + f_{\epsilon_0}(V_G) \tag{5.1}$$

where $Q_0 = \Omega_0 \times (0, \infty)$, $\lambda = \cos\theta$, $V_G = H^{n+1}(G)$ and

$$f_{\epsilon_0}(t) = \begin{cases} -\dfrac{1}{\epsilon_0}(t - V) & \text{if } t < V \\ 0 & \text{if } t > V \qquad (\epsilon_0 > 0) \end{cases}$$

where V is prescribed positive number (the volume of the drop). Caffarelli and Spruck [4] proved that there exists a set $E \subset \bar{Q}_0$ such that

$$J_{\epsilon_0}(E) = \min_G J_{\epsilon_0}(G), \qquad G \subset \bar{Q}_0; \tag{5.2}$$

furthermore, E is a bounded set and

$$V_E = V$$

provided ϵ_0 is small enough. (Notice that since Ω_0 is unbounded, Theorem 2.1 is not applicable to this situation.) As in §2, E is a subgraph of a function $x_{n+1} = u(x)$ with support S, say.

We may consider E as a minimizer in a smaller class K_Ω:

$$\begin{aligned} G \in K_\Omega \quad \text{if} \quad & G \subset \Omega \times [0, \infty) \quad \text{and} \\ & G \quad \text{coincides with} \quad \partial\Omega \times \{0\} \quad \text{on} \quad \partial\Omega \times [0, \infty), \end{aligned} \tag{5.3}$$

where Ω is any bounded domain which contains the set S; the integral $\int_{\partial Q_0}$ in (5.1) is replaced by $\int_{\Omega \cap \{x_{n+1}=0\}}$.

Because of the presence of the term $f_{\epsilon_0}(V_G)$, we cannot apply the results of §§2-4 directly to the present problem. However, going over the various arguments we discover that all the results remain valid with some modifications, as we shall now explain.

The fact that $\partial_0 E \equiv \partial E \cap \{x_{n+1} > 0\}$ is in $C^{2+\alpha}$ can be established by the method of Massari [13] (see also [12] for regularity of $\partial_0 E$ when the volume constraint is imposed as a side condition); the analyticity of $\partial_0 E$ follows from the existence of multipliers (see [6] [9]). We can now establish the continuity of $u(x)$ as before.

In any open set $S \subset \{u > 0\}$ there exists a point x_S such that the tangent to ∂E at $(x_S, u(x_S))$ is not vertical; thus $u(x)$ is analytic in some ball B_S with center x_S.

Take a smooth nonnegative function $u_S(x)$ with support in B_S such that $\int u_S(x)\, dx = 1$. For any $\zeta \in C_0^1(B_S)$ and for any real ϵ, $|\epsilon|$ small enough, the function $u + \epsilon\zeta - \epsilon\left(\int \zeta\right) u_S$ is an admissible function having the same volume V as u. From the inequality

$$J_{\epsilon_0}\left(u + \epsilon\zeta - \epsilon\left(\int \zeta\right) u_S\right) \geqslant J_{\epsilon_0}(u)$$

we then obtain

$$\int_{B_S} \left(\frac{\nabla u \cdot \nabla \zeta}{\sqrt{1 + |\nabla u|^2}} + g u \zeta - \mu_S \zeta \right) dx = 0 \tag{5.4}$$

where

$$\mu_S = \int_{B_S} \left(\frac{\nabla u \cdot \nabla u_S}{\sqrt{1 + |\nabla u|^2}} + g u u_S \right) dx.$$

Taking $u + \epsilon u_S - \epsilon u_{S'} - \epsilon u_{S'}$ as an admissible function with S' another open set with its corresponding $u_{S'}$ and $\mu_{S'}$, and ϵ any small real number, we find that $\mu_{S'} = \mu_S$. Further from $J_{\epsilon 0}(u + \epsilon u_S) \geqslant J_{\epsilon 0}(u)$ ($\epsilon > 0$) we find that $\mu_S \geqslant 0$. Thus

$$\mu = \mu_S \quad \text{is independent of } S, \text{ and} \quad \mu \geqslant 0. \tag{5.5}$$

From (5.4), (5.5) we deduce that

$$\mathcal{L}u - gu = -\mu \quad \text{in} \quad B_S. \tag{5.6}$$

By using local coordinates we can actually obtain a «parametric» form of (5.6) valid throughout $\partial_0 E$, whereby g, μ are to be replaced by $\tilde{g}$, $\tilde{\mu}$ (cf. (3.2)); this however will not be needed.

We shall now extend Lemma 3.2. Take any point $X^0 = (x^0, u(x^0))$ with $u(x^0) > 0$ and let $B_\delta(x^0)$ be any ball such that $u(x) > 0$ if $x \in B_\delta(x^0)$. We shall prove that u is a smooth solution of

$$\mathcal{L}u - gu = -\mu \quad \text{in} \quad B_\delta(x^0). \tag{5.7}$$

Introducing the analytic solution w of

$$\begin{aligned} \mathcal{L}w - gw = -\mu \quad &\text{in} \quad B_\delta(x^0), \\ w = u \quad &\text{on} \quad \partial B_\delta(x^0), \end{aligned} \tag{5.8}$$

it suffices to show that $w \equiv u$. Proceeding as in the proof of Lemma 3.2, we perform, at the same point in the same argument as before an orthogonal transformation $(\tilde{x}, \tilde{x}_{n+1}) = T(x, x_{n+1})$. The surfaces $x_{n+1} = w(x)$ and $x_{n+1} = u(x)$ become $\tilde{x}_{n+1} = W(\tilde{x})$ and $\tilde{x}_{n+1} = U(\tilde{x})$ respectively, ($(\tilde{x}, \tilde{x}_{n+1}) = (0, 0)$ corresponds to point $(x, x_{n+1}) = (\bar{x}, u(\bar{x}))$) and it remains to show that the analytic functions W, U in some ball $B_\rho(\tilde{O})$ with center $\tilde{O} = (0, 0)$ satisfy

$$\mathcal{L}W - \tilde{g}W = -\tilde{\mu}(\tilde{x}), \tag{5.9}$$

$$\mathcal{L}U - \tilde{g}U = -\tilde{\mu}(\tilde{x}) \tag{5.10}$$

where $\tilde{\mu}$, $\tilde{g}$ are given by (3.2).

By the manner by which the transformation T changes the functional J (see the paragraph containing (3.2)) it is clear how the corresponding Euler equation changes, namely, (5.8) changes into (5.9). Similarly by choosing S a small ball about $\tilde{x}$, (5.6) yields

$$\mathcal{L}U - \tilde{g}U = -\tilde{\mu}(\tilde{x})$$

in a small ball $B_{\tilde{\rho}}(\tilde{x}_*)$ contained in $B_\rho(\tilde{O})$; by analytic continuation it then follows that (5.10) holds throughout $B_\rho(\tilde{O})$.

Having proved (5.9), (5.10), we can now complete the proof of Lemma 3.2 as before.

We next proceed to establish the Lipschitz continuity of u, as in §2. If we use Method (i) then the proof is the same since the terms $f_{\epsilon 0}(V_E)$ cancel out when we compare $J_{\epsilon 0}(u)$ with $J_{\epsilon 0}(w)$. On the other hand, if we use Method (ii), then $f_{\epsilon 0}(t)$ must be replaced by

$$\tilde{f}_{\epsilon 0}(t) = \lim_{m\to\infty} f_{\epsilon 0}\left(\frac{t}{\rho_m^n} + V_E - V_{E\cap B_{\rho_m}(X^m)}\right).$$

Having proved Lipschitz continuity in compact subsets of Ω, we next trucate $\sqrt{1+t}$ as in §4 and consider the functional

$$J'_{\epsilon 0}(v) = \int_\Omega F(|\nabla v|^2)\,dx + \int_\Omega \frac{g}{2} v^2\,dx + \int_\Omega (1-\lambda)I_{\{v>0\}}\,dx + f_{\epsilon 0}(V_v) \quad (5.11)$$

where

$$V_v = H^{n+1}\{(x, x_{n+1}); 0 < x_{n+1} \leqslant v(x), x \in \Omega'\}.$$

and proceed as in [2].

The proof of non-degeneracy remains the same and so do all the results of [2]. However, in checking the various details one must pay attention to the term $f_{\epsilon 0}(V_v)$. If $V_v \geqslant V_u$ then $f_{\epsilon 0}(V_u) = f_{\epsilon 0}(V_v)$ and these two terms cancel out. If however $V_v < V_u$ then

$$f_{\epsilon 0}(V_v) = f_{\epsilon 0}(V_u) + 0(V_u - V_v). \quad (5.12)$$

This causes some changes in the proofs, usually trivial ones. The only slightly significant difference occurs in Theorem 4.3 of [2] where one takes $v = u - \min(u, \epsilon\zeta)$. The error in (5.12) must now be controlled by $o(r^2)$. We recall that

$$\int_{B_\rho} \zeta = O\left(\rho^2/\log\frac{\rho}{r}\right)$$

so that

$$|V_u - V_v| \leqslant C\left[\frac{\rho^2}{\log\frac{\rho}{r}} + r^2\right]\epsilon, \quad \text{and} \quad \epsilon = Cr;$$

taking $\rho = r^\theta$ with $\theta < 1$, $1 - \theta$ small, we get

$$\frac{1}{r^2}|\tilde{J}_{\epsilon 0}(V_u) - \tilde{J}_{\epsilon 0}(V_v)| \leqslant \frac{C}{r^\beta} \quad \text{for some} \quad \beta > 0.$$

We can now proceed as in [2] and obtain the extensions of Theorem 4.3, namely,

$$\frac{1}{r^2}\int_{B_r \cap \{u>0\}} ((\lambda^*)^2 - |\nabla u|^2)^+ \leqslant C/\log\frac{1}{r}$$

where

$$(\lambda^*)^2 = \frac{1}{\lambda^2} - 1.$$

The rest of the proof of theorems 4.1, 4.2 is the same as in [2]. We can therefore state, for $n = 2$:

Theorem 5.1. *The free boundary of the sessile drop problem is in $C^{k+1,\alpha}$ if $\lambda \in C^{k,\alpha}$, and it is analytic if λ is analytic.*

Remark 5.1. Consider the minimization problem for the functional ($\Omega_0 \subset \mathbb{R}^n$, $Q_0 = \Omega_0 \times (0, \infty)$)

$$\begin{aligned} \hat{J}_{\epsilon 0}(G) = \int_{Q_0} |D\phi_G| + \int_{Q_0} g x_{n+1}\phi_G - \\ - \int_{\Omega_0 \times \{0\}} \lambda\phi_G - \int_{\partial\Omega_0 \times (0,\infty)} \hat{\lambda}\phi_G + f_{\epsilon 0}(G) \qquad (5.13)\end{aligned}$$

with $G \subset \bar{Q}_0$, $\lambda = \cos\theta$, $\hat{\lambda} = \cos\hat{\theta}$. This functional is similar to (5.1); the additional term $\int \hat{\lambda}\phi_G$ represents the wetting energy on the lateral boundary of the tube Q_0. The minimization problem models capillary fluid in the tube Q_0 with a given volume V. If V is small enough then a portion of the bottom will remain dry. Theorem 5.1 immediately extends to the present case (with $n = 2$) showing that the boundary of the dry portion of the bottom is analytic.

The results of this paper also extend to functionals in which $\frac{1}{2}gv^2$ is replaced by more general functions $P(x, v)$, provided $P_v \geqslant 0$.

References

[1] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.*, **325** (1981), 105-144.

[2] H. W. Alt, L. A. Caffarelli and A. Friedman. A free boundary problem for quasi-linear elliptic equations, *Ann. Scu. Norm. Sup.*, Pisa, **11 (4)** (1984), 1-44.

[3] E. Bombieri, E. De Giorgi and M. Miranda. Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche, *Archive Rat. Mech. Anal.*, **32** (1969), 255-267.

[4] L. A. Caffarelli and J. Spruck. Existence of capillary drops on an inhomogeneous plane, to appear.

[5] A. Friedman. Axially symmetric cavities in rotational flows, *Comm. P.D.E.*, **8** (1983), 949-997.

[6] C. Gerhardt. On the capillarity problem with volume constraint, *Ann. Scu. Norm. Sup. Pisa*, **2** (**4**) (1975), 303-320.

[7] D. Gilbarg and N. S. Trudinger. Elliptic Partial Differential Equations of Second Order, Springer Verlag, vol. 224, Berlin-Heidelberg-New York, 1977.

[8] E. Giusti. Minimal Surfaces and Functions of Bounded Variation, Notes in Pure Mathematics, Australian National University, Canberra, 1977.

[9] E. Giusti. Generalized solutions for the mean curvature equation, *Pacific J. Math.*, **88** (1980), 297-321.

[10] E. Giusti. The equilibrium configuration of liquid drops, *J. Reine Angew. Math.*, **321** (1981), 53-63.

[11] E. H. A. Gonzáles. Sul problema della goccia appoggiata, *Rend. Sem. Mat. Univ. Padova*, **55** (1976), 289-302.

[12] E. Gonzáles, U. Massari and I. Tamanini. On the regularity of boundaries of sets minimizing perimeter with a volume constraint, *Indiana Univ. Math J.*, **32** (1983), 25-37.

[13] U. Massari. Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in R^n, *Archive Rat. Mech. Anal.*, **55** (1974), 357-382.

[14] M. Miranda. Superfici cartesiane generalizzate ed insiemi di perimetro localmente finito sui prodotti cartesiani, *Ann. Scu. Norm. Sup. Pisa*, **18** (1964), 515-542.

Luis A. Caffarelli
University of Chicago
Chicago, Illinois 60637
U.S.A.

Avner Friedman
Northwestern University
Evanston, Illinois 60201
U.S.A.

This work is partially supported by National Science Foundation Grants 7406375 A01 and MCS 791 5171.

Revista Matemática Iberoamericana
Vol. 1, n.º 1, 1985

Arithmetic Hilbert Modular Functions II

Walter L. Baily, Jr.

Introduction

The purpose of this paper, which is a continuation of [2, 3], is to prove further results about arithmetic modular forms and functions. In particular we shall demonstrate here a q-expansion principle which will be useful in proving a reciprocity law for special values of arithmetic Hilbert modular functions, of which the classical results on complex multiplication are a special case. The main feature of our treatment is, perhaps, its independence of the theory of abelian varieties. In that respect these developments may be considered as an extension of Hecke's thesis [13] and Habilitationsschrift [14]. We should also mention a contribution of Sugawara [34]. More recently Karel has shown how to apply such ideas to the classical case of elliptic modular functions in an adelic setting [16].

To date the furthest reaching results in this area, beyond those in the classical case, belong to a long list of distinguished contributors who have freely used the known facts about elliptic functions, elliptic curves, and abelian varieties, notably Hasse [11,12], Deuring [10], Shimura-Taniyama [32], Shimura [28, 29, 30] (and many others), Taniyama [35], Shih [33], Miyake [25], Milne-Shih [22, 23, 24], Deligne [8, 9], Borovoi [7], and Milne [21]. The last mentioned work, which uses the preceding ones together with results of Kazhdan [17, 18], contains very general results. It has recently been complemented by a work of Milne, still in unpublished form, written to put the results of [18] on a firmer basis.

However, our purpose is to develop a theory independent of abelian varieties based on the properties of the modular functions themselves and thereby also, it is hoped, to learn more about these functions and their own intrinsic arithmetic properties. The inspiration for this approach comes from the paper *Der Hilbertsche Klassenkörper eines imaginärquadratischen Zahlkörpers,* Math. Zeitschr. 64 (1956), by M. Eichler, and it is to Prof. Eichler that we wish to dedicate this article. In it we have relied most heavily on the work of Hecke, the second paper of Hasse, the papers [27, 28] of Shimura (for facts about CM-fields and reduction of algebraic varieties modulo a prime), Deligne [8] (especially for topological properties of the adelic double coset space), and Karel [16] (as will be explained later). We hope these efforts, to be continued in subsequent publications, may be of some interest to mathematicians in this field.

1. The Adelic Space

In this paper we generally follow the notation and conventions of [3]. For convenience we give some of the most frequently used notation. Let k be a totally real number field with ring of integers $\mathfrak{o}$ and $[k\colon Q] = n > 1$, let $\mathbf{A}$ resp. $\mathbf{A}(k)$ denote the adele rings of Q resp. of k, and let $\mathbf{I}(k)$ be the ideles or group of units of $\mathbf{A}(k)$, each supplied with its usual topology. The subscripts ∞ and f will denote the projections of an adelic object to its archimedean and non-archimedean components respectively and the subscript $+$ will indicate adelic objects with non-negative archimedean components. $\hat{\mathbb{Z}}$ resp. $\hat{\mathfrak{o}}$ will be the maximal compact subrings of $\mathbf{A}_f$ and of $\mathbf{A}(k)_f$ respectively. We denote by $\mathfrak{H}$ the upper half complex plane $Im z > 0$, by $\mathfrak{H}^n$ its n-th Cartesian power, and by i.e., the point $(i, \dots, i) \in \mathfrak{H}^n$.

Moreover, G' will denote the algebraic group GL_2 defined over k and G will be the group $R_{k/\mathbf{Q}}G'$ defined over $\mathbf{Q}$. There is a canonical isomorphism ϕ of $G'(\mathbf{A}(k))$ onto $G(\mathbf{A})$ and of $G'(k)$ onto $G(\mathbf{Q})$ such that if the integral structures on G' and on G are those associated to $G'(\hat{\mathfrak{o}}) = GL_2(\hat{\mathfrak{o}})$ and to $G(\hat{\mathbb{Z}})$ (with respect to suitable bases of the vector spaces on which G' and G act), then $\phi(G'(\hat{\mathfrak{o}})) = G(\hat{\mathbb{Z}})$ (cf. [5]).

Z' is the center of G' and $Z = R_{k/\mathbf{Q}}Z'$, that of G, and $G_+(\mathbb{R})/Z(\mathbb{R})$ acts effectively on $\mathfrak{H}^n$. If $\mathbb{K}_\infty$ is the isotropy group of i.e in $G_+(\mathbb{R})$ and $\mathbb{K}'_\infty = \mathbb{K}_\infty \cap G_+^{\mathrm{der}}(\mathbb{R})$, one has

$$G_+(\mathbf{A}) = G_+(\mathbb{R})G(\mathbf{A}_f) = P_+(\mathbf{A})\mathbb{K}'_\infty G(\hat{\mathbb{Z}}) \tag{1}$$

(the corrected form of §1.2(2) of [3]), where $P = R_{k/\mathbf{Q}}P'$ and P' is the group of upper triangular matrices in G'.

Let $\mathbb{K}$ be an open compact subgroup of $G(\hat{\mathbb{Z}})$ and denote by Γ or $\Gamma(\mathbb{K})$ the arithmetic subgroup $G(\mathbf{Q}) \cap G_+(\mathbb{R}) \cdot \mathbb{K}$ of $G_+(\mathbf{Q})$, whose projection into $G_+(\mathbb{R})$ we also denote by Γ or $\Gamma(\mathbb{K})$.

The space of left cosets of $\mathbb{K}\mathbb{K}_\infty$ in $G_+(\mathbf{A})$, $X_{\mathbb{K}} = G_+(\mathbf{A})/\mathbb{K}\mathbb{K}_\infty$, is the union of countably many connected components, each one of the form

$$X_\omega = \omega G_+(\mathbb{R})\mathbb{K}/\mathbb{K}\mathbb{K}_\infty, \quad \omega \in G(\mathbf{A}_f), \tag{2}$$

X_ω being complex analytically isomorphic to $\mathfrak{H}^n$. The group $G_+(\mathbf{Q})$ permutes these components under left translation in $G_+(\mathbf{A})$ and has finitely many orbits among them, the stabilizer in $G_+(\mathbf{Q})$ of X_ω being $\Gamma_\omega = \Gamma({}^\omega\mathbb{K})$. If we let $\mathfrak{C}_{\mathbb{K}}(\omega) = G_+(\mathbf{Q})\omega G_+(\mathbb{R})\mathbb{K}$, then $V_\omega = \Gamma_\omega \backslash X_\omega$ may be identified with

$$G_+(\mathbf{Q})\backslash \mathfrak{C}_{\mathbb{K}}(\omega)/\mathbb{K}\mathbb{K}_\infty,$$

and the collection of double cosets $\mathfrak{C}_{\mathbb{K}}(\omega)$ or components V_ω is in natural one-to-one correspondence with the set of elements of the group (cf. [8], Variante 2.5)

$$\mathfrak{g}_+[\mathbb{K}] = I_+(k)/k^\times k_{\infty+} \det(\mathbb{K}) \simeq I(k)_f/k_f^\times \det(\mathbb{K}),$$

where $k_\infty = \bigoplus_{v|\infty} k_v$, k_v being the completion of k at the archimedean place v, and $\det(\mathbb{K})$ is the group of $\det(\mathbf{k})$, $\mathbf{k} \in \mathbb{K}$.

Define the double coset space $V_{\mathbb{K}} = G_+(\mathbf{Q})\backslash X_{\mathbb{K}}$; this is a union

$$V_{\mathbb{K}} = \cup_{\omega \in \Omega} V_\omega, \qquad V_\omega = \Gamma_\omega \backslash X_\omega \simeq \Gamma_\omega \backslash \mathfrak{H}^n, \tag{3}$$

where Ω is a finite set of indexing representatives of the orbits of $G_+(\mathbf{Q})$ among the components X_ω. (Cf. [3].)

In this paper we consider properties of $V_{\mathbb{K}}$ in connection with arithmetic automorphic forms and functions on $G_+(\mathbf{A})$ with respect to $\mathbb{K}$ and study the arithmetic properties of such functions by means of arithmetically defined Eisenstein series on the components X_ω. We follow the ideas and program of [3] and [16], to which must be added a certain q-expansion principle and other ideas related to [13, 14] and [11, 12], as well as properties of CM-types to be found in [32, 28].

We generally adhere to the notation $\mathfrak{A}(\mathbb{K})$, $\mathfrak{A}(\mathbb{K}, w)$, etc., of §1 of [3] for the graded algebra of modular forms with respect to $\mathbb{K}$, the forms of weight w, etc.

2. Special Points and Idelic Action

We follow here the pattern of §§1.2-1.3 of [16], taking account of differences needed to accomodate the more general situation discussed in [3]. Therefore our discussion will be abbreviated by making suitable references to [3] and [16].

The group $G'_+(k)$ acts on $\mathfrak{H}^n$ in a manner analogous to that in which $GL_{2+}(\mathrm{Q})$ acts on $\mathfrak{H}$ by linear fractional transformations. If $\Sigma = (\sigma_1, \ldots, \sigma_n)$ is the set of isomorphisms of k onto subfields of $\mathbb{R}$, if $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G'_+(k)$, and $(z) = (z_1, \ldots, z_n) \in \mathfrak{H}^n$, then $S(z) = (S^{\sigma_1} \cdot z_1, \ldots, S^{\sigma_n} \cdot z_n)$, where

$$S^{\sigma_j} \cdot z_j = (\alpha^{\sigma_j} z_j + \beta^{\sigma_j})/(\gamma^{\sigma_j} z_j + \delta^{\sigma_j}).$$

We denote by K a purely imaginary quadratic extension of k. Then $S = = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G'(k)$ acts on $K - k$ by linear fractional transformations, $S \cdot \tau = = (\alpha\tau + \beta)/(\gamma\tau + \delta)$, $\tau \in K$. Consider an imbedding

$$q: K \hookrightarrow M_2(k): 2 \times 2 \quad \text{matrices over } k,$$

of K as a k-algebra such that $q(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By the Skolem-Noether theorem, the representation q of K as a k-algebra is equivalent to the regular representation of K on itself and $\det(q(x)) = x\bar{x} = N_{K/k}(x) > 0$. As a subgroup of $G'_+(k)$, $q(K^\times)$ has precisely two fixed points $\tau, \bar{\tau} \in K - k$, where $\bar{\tau}$ is the complex conjugate of τ. Conversely, if $\tau \in K - k$, then by taking τ, 1, as a k-basis of K for the regular representation, we see that each $\tau \in K - k$ defines such an imbedding $q = q_\tau$, that its complex conjugate $\bar{\tau}$ defines the conjugate imbedding, and that $q(K^\times)$ as a subgroup of $G'(k)$ has precisely the two fixed points $\tau, \bar{\tau}$. Thus we have a one-to-one correspondence between conjugate pairs of imbeddings q of K in $M_2(k)$ and complex conjugate pairs $\tau, \bar{\tau}$ of elements of $K - k$.

By a lifting of Σ, or "type" for the given CM-field K/k, we mean a set $\tilde{\Sigma} = \Phi$ of extensions $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ of Σ to a set of n imbeddings of K into $\mathbb{C}$ such that $\tilde{\sigma}_j | k = \sigma_j$, $j = 1, \ldots, n$. If $\tau \in K - k$, there is a unique lifting $\tilde{\Sigma} = \tilde{\Sigma}(\tau)$ defined by the requirement $\mathrm{Im}(\tilde{\sigma}_j(\tau)) > 0, j = 1, \ldots, n$. Conversely, given any lifting $\tilde{\Sigma}$, define the set $K_{\tilde{\Sigma}} = \{\tau \in K - k \mid \tilde{\Sigma}(\tau) = \tilde{\Sigma}\}$.

If $z = (\tau_1, \ldots, \tau_n) \in \mathfrak{H}^n$ is the fixed point of a non-central element of $G'_+(k) = G_+(\mathbf{Q})$, then [3] there is a uniquely determined purely imaginary quadratic extension $K = K_z$ of k and $\tau \in K - k$ such that if $\tilde{\Sigma}(\tau) = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_n)$, then $\tau_j = \tau^{\tilde{\sigma}_j}$, $j = 1, \ldots, n$. Moreover, from our previous discussion it follows that there is an imbedding $q = q_\tau$ of K into $M_2(k)$ such that $z = (\tau)$ is the unique fixed point of $q(K^\times) \subset G'_+(k)$ in $\mathfrak{H}^n$. In fact, the isotropy subgroup of (τ) in $G'_+(k)$ has to be a torus; since $q(K)$ is a maximal commutative subalgebra of $M_2(k)$, it follows that $q(K^\times)$ is the full isotropy group of (τ) in $G'_+(k)$.

In the future we view any point $\tau \in K - k$ as being imbedded into $\mathfrak{H}^n$ by means of $\tilde{\Sigma}(\tau)$, and write (τ) for its image there. Having fixed the CM-extension K, we refer to the set K_Σ of points (τ) for $\tau \in K - k$ as the set of special points of $\mathfrak{H}^n$ relative to the extension K/k.

If $\mathfrak{v}$ is a fractional ideal of k and $\tau \in K - k$, then $\mathfrak{J}_{\mathfrak{v},\tau} = \mathfrak{v}\tau + \mathfrak{o}$ is an $\mathfrak{o}$-module in K of rank two. Let $\mathfrak{R} = \mathfrak{R}_{\mathfrak{v}}(\tau)$ be its order, $\mathfrak{R} = \{x \in K \mid x \cdot \mathfrak{J}_{\mathfrak{v},\tau} \subset \subset \mathfrak{J}_{\mathfrak{v},\tau}\}$, so that $\mathfrak{J}_{\mathfrak{v},\tau}$ is a proper $\mathfrak{R}$-ideal in K. Let $\Xi_{\mathfrak{v}}(\mathfrak{R}) = \{\tau \in K - k \mid \mathfrak{R}_{\mathfrak{v}}(\tau) = = \mathfrak{R}\}$, and $\Xi_{\mathfrak{v}}(\mathfrak{R}, \tilde{\Sigma}) = \{\tau \in \Xi_{\mathfrak{v}}(\mathfrak{R}) \mid \tilde{\Sigma}(\tau) = \tilde{\Sigma}\}$.

Returning to the double coset decomposition

$$G_+(\mathbf{A}) = \bigcup_{\omega \in \Omega} G_+(\mathbf{Q})\omega G_+(\mathbf{R})\mathbb{K} \tag{4}$$

associated to the decomposition of the double coset space

$$V_{\mathbb{K}} = G_+(\mathbf{Q})\backslash G_+(\mathbf{A})/\mathbb{K}\mathbb{K}_\infty = \bigcup_{\omega \in \Omega} V_\omega \tag{5}$$

into its component varieties as in the preceding section, we recall from [3] that the double coset representatives ω may be chosen in diagonal form $\omega = \begin{pmatrix} \omega' & 0 \\ 0 & 1 \end{pmatrix}$, $\omega' \in I(k)_f$, let $\mathfrak{v} = \mathrm{id}(\omega')^{-1}$, and $\mathfrak{v}_\omega = \mathfrak{v}$, $x_\omega = x_{\mathfrak{v}}$. In particular when $\mathbb{K} = G(\hat{\mathbb{Z}})$, let $\Theta = \Omega$ and $\omega = \theta \in \Theta$, $\theta = \begin{pmatrix} \theta' & 0 \\ 0 & 1 \end{pmatrix}$, $\mathfrak{v} = \mathrm{id}(\theta')^{-1}$ and $\Theta' = = \{\theta' \mid \theta \in \Theta\}$. Then

$$G(\mathbf{A})_f = \bigcup_{\theta \in \Theta} G_+(\mathbf{Q})_f \theta G(\hat{\mathbb{Z}}). \tag{6}$$

For a fixed order $\mathfrak{R}$ in K such that $\mathfrak{R}$ contains the ring of integers of k, and for any $\theta \in \Theta$, we define the set of special points on X_θ to be the set

$$\Xi_\theta(\mathfrak{R}) = \{\theta \cdot (\tau) \in \theta G_+(\mathbb{R})G(\hat{\mathbb{Z}})/\mathbb{K}_\infty G(\hat{\mathbb{Z}}) \mid \tau \in \Xi_{\mathfrak{v}_\theta}(\mathfrak{R})\}. \tag{7}$$

In other words, identifying $\mathfrak{H}^n$ with $X_\theta = \theta \cdot \mathfrak{H}^n$, $\Xi_\theta(\mathfrak{R})$ is the set of special points of $\mathfrak{H}^n$ coming from elements $\tau \in \Xi_{\mathfrak{v}_\theta}(\mathfrak{R}) \subset K - k$. Then define $\Xi_{\theta,\infty}(\mathfrak{R})$ to be the subset of $g \in G_+(\mathbb{R})$ such that $g(\text{i.e.}) \in \Xi_\theta(\mathfrak{R})$ and define the sets

$$\Xi_A(\mathfrak{R}) = G_+(\mathbf{Q})\Xi_\infty(\mathfrak{R})G(\hat{\mathbb{Z}}), \quad \text{where} \quad \Xi_\infty(\mathfrak{R}) = \bigcup_{\theta \in \Theta} \Xi_{\theta,\infty}(\mathfrak{R}) \cdot \theta.$$

We also let

$$\Xi_\theta(\mathfrak{R}, \tilde{\Sigma}) = \{\theta \cdot (\tau) \in \Xi_\theta(\mathfrak{R}) \mid \tilde{\Sigma}(\tau) = \tilde{\Sigma}\}$$

and define $\Xi_A(\mathfrak{R}, \tilde{\Sigma})$ and $\Xi_\infty(\mathfrak{R}, \tilde{\Sigma})$ analogously. If $\mathbb{K}$ is an open compact subgroup of $G(\hat{\mathbb{Z}})$, let $\Xi_{\mathbb{K}}(\mathfrak{R})$ be the image, via the canonical projection to double cosets, of $\Xi_A(\mathfrak{R})$ in $V_{\mathbb{K}}$. In particular, if $\mathfrak{n}$ is an integral ideal in $\mathfrak{o}$ and if $\mathbb{K} = \mathbb{K}(\mathfrak{n})$ is the principal congruence subgroup of those $k \in G'(\hat{\mathfrak{o}}) = G(\hat{\mathbb{Z}}) \subset \subset M_2(\hat{\mathfrak{o}})$ such that $k - I_2 \in \mathfrak{n} \cdot M_2(\hat{\mathfrak{o}})$ (where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$), denote $V_{\mathbb{K}}$ by $V_{\mathfrak{n}}$, and $\Xi_{\mathbb{K}}(\mathfrak{R})$ by $\Xi_{\mathfrak{n}}(\mathbb{R})$. Let $\mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}})$ be the principal congruence subgroup mod $\mathfrak{n}$ of the group $\mathfrak{U}(\hat{\mathfrak{R}}) = \hat{\mathfrak{R}}^{\times}$ of unit ideles, where $\hat{\mathfrak{R}}$ is the closure of $\mathfrak{R}$ in $\mathbf{A}(K)_f$, and let $C_{\mathfrak{n}}(\mathfrak{R})$ be the group

$$C_{\mathfrak{n}}(\mathfrak{R}) = \mathbf{A}(K)^{\times}/K^{\times} \cdot \mathbf{A}(K)_{\infty}^{\times} \cdot \mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}}) \tag{8}$$

of ray classes modulo $\mathfrak{n}$ of proper $\mathfrak{R}$-ideals. If $\theta \in \Theta$ and $\tau \in \Xi_{\mathfrak{v}_\theta}(\mathfrak{R})$, and if q_τ is the imbedding associated to τ of K into $M_2(k)$, then we have

$$q_\tau(\mathfrak{U}_{\mathfrak{n}}(\hat{\mathfrak{R}}) \cap (K_f)^{\times}) \subset {}^{\theta}\mathbb{K}(\mathfrak{n}). \tag{9}$$

Let $C_{\mathfrak{R}}(\mathfrak{R})$ be the group of classes of proper fractional ideals of $\mathfrak{R}$ in K. Suppose $\gamma \in \Gamma_\theta = \Gamma({}^{\theta}G(\hat{\mathbb{Z}}))$ and $\tau \in \Xi_{\mathfrak{v}_\theta}(\mathfrak{R})$, so that $\mathfrak{A}_\tau = \mathfrak{v}\tau + \mathfrak{o}$ is a proper fractional $\mathfrak{R}$-ideal of K and so that if (as in [2]) we put $R_{\mathfrak{v}} = \left(\begin{smallmatrix} \mathfrak{o} & \mathfrak{v}^{-1} \\ \mathfrak{v} & \mathfrak{o} \end{smallmatrix}\right)$, then $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in R_{\mathfrak{v}+}^{\times}$. Then if $\tau' = \gamma \cdot \tau$, a direct calculation shows that $\mathfrak{A}_{\tau'} = \mathfrak{v}\tau' + \mathfrak{o} = (c\tau + d)^{-1}\mathfrak{A}_\tau$, therefore τ' also belongs to $\Xi_{\mathfrak{v}_\theta}(\mathfrak{R})$. Consequently the $\mathfrak{R}$-ideal class of $\mathfrak{v}\tau + \mathfrak{o}$ is constant along the orbits of Γ_θ, and similarly if $\mathfrak{A}_\tau \sim \mathfrak{A}_{\tau'}$, then $\tau' \in R_{\mathfrak{v}}^{\times} \cdot \tau$ (linear fractional operation on the right-hand side), so that Γ_θ has only finitely many orbits in $\Xi_{\mathfrak{v}_\theta}(\mathfrak{R})$.

Let $g \in G_+(\mathbf{A})$ and $\tau = g_\infty(\mathrm{i.e})$. Denote by $j(g)$ the double coset

$$G_+(\mathbf{Q})gG_+(\mathbb{R})G(\hat{\mathbb{Z}})$$

to which g belongs. Let $\mathfrak{A}$ be a proper fractional $\mathfrak{R}$-ideal of K. Since $\mathfrak{A}$ is proper and its order contains $\mathfrak{o}$, $\mathfrak{A}$ is at every finite place $\mathfrak{P}$ of K locally principal as an $\mathfrak{R}_{\mathfrak{P}}$-ideal; hence, there is a finite idele $\alpha \in I(K)_f$ such that $\mathfrak{A} = K \cap \mathbf{A}(K)_\infty \alpha \hat{\mathfrak{R}}$. If $\hat{\mathfrak{A}}$ is the $\mathfrak{o}$-ideal generated by $\mathfrak{A}$, as in [3], let $\hat{N}\mathfrak{A} = N_{K/k}\hat{\mathfrak{A}}$. Suppose $g \in \Xi_{\mathbf{A}}(\mathfrak{R})$ and that $g = \xi\theta$, $\xi \in \Xi_{\theta,\infty}(\mathfrak{R})$, $\theta \in \Theta$, and let $(\tau) = \xi(\mathrm{i.e})$ and $\mathfrak{A}_\tau = \mathfrak{v}\tau + \mathfrak{o}$ with $\mathfrak{v} = \mathrm{id}(\det(\theta))^{-1}$. Let $\gamma \in G_+(\mathbf{Q})$ and write $g' = \gamma g$. Then $\mathfrak{A}\mathfrak{A}_\tau$ is in the same proper $\mathfrak{R}$-ideal class as $\mathfrak{A}_{\tau_1} = \mathfrak{v}_1\tau_1 + \mathfrak{o}$ with $\mathfrak{v}_1 = \mathfrak{v}_{\theta_1}$ and $\tau_1 \in \Xi_{\mathfrak{v}_1}(\mathfrak{R})$ for some $\theta_1 \in \Theta$, with $\mathfrak{v}_1$ in the same narrow ideal class as $\mathfrak{v} \cdot \hat{N}\mathfrak{A}$, and $(\tau_1) = \xi_1(\mathrm{i.e})$ for some $\xi_1 \in \Xi_{\theta_1,\infty}(\mathfrak{R})$.

Lemma 1. *With the notation just introduced, we may choose* $(\tau_1) = \xi_1(\mathrm{i.e})$ *such that* $\tilde{\Sigma}(\tau_1) = \tilde{\Sigma}(\tau)$, *and then we have* $j(\theta_1\xi_1) = j(\gamma q_\tau(\alpha^{-1})\theta\xi)$.

PROOF. Of course it suffices to prove that

$$j(\theta_1\xi_1) = j(q_\tau(\alpha^{-1})\theta\xi).$$

Let $q = q_\tau$. If $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}$, let $\mathfrak{R}_{\mathfrak{p}} = \mathfrak{R} \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$ and define an $\mathfrak{R}_{\mathfrak{p}}$-module $\mathfrak{B}_{\mathfrak{p}}$ by $\mathfrak{B}_{\mathfrak{p}} = \alpha_{\mathfrak{p}}(\mathfrak{v}_{\mathfrak{p}}\tau + \mathfrak{o}_{\mathfrak{p}})$. Clearly $(\mathfrak{A}\mathfrak{A}_\tau)_{\mathfrak{p}} = \mathfrak{B}_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, thus $\mathfrak{B} = \mathfrak{A}\mathfrak{A}_\tau$ is the proper fractional $\mathfrak{R}$-ideal with localization $\mathfrak{B}_{\mathfrak{p}}$ for every $\mathfrak{p}$, and $\hat{N}\mathfrak{B}$, as defined above, is in the same narrow ideal class as $\mathfrak{v}_1$ and

$\mathfrak{v} \cdot \hat{N}\mathfrak{A}$. Then $\mathfrak{B}$ is in the same proper $\mathfrak{R}$-ideal class as $\mathfrak{A}_{\tau_1} = \mathfrak{v}_1\tau_1 + \mathfrak{o}$, with $\mathfrak{v}_1 = \mathfrak{v}_{\theta_1}$ and $\tau_1 \in \Xi_{\mathfrak{v}_1}(\mathfrak{R})$ for some $\theta_1 \in \Theta$. Then by the calculations of §3.1 of [3], we have

$$\mathfrak{v}_1 = (\Delta(\tau_1, 1)/\Delta(\tau, 1)) \cdot \hat{N}\mathfrak{A} \cdot \mathfrak{v}. \tag{10}$$

Since $\Delta(\tau, 1) = 2\,\mathrm{Im}(\tau)$, this says the principal fractional ideals $(\mathrm{Im}(\tau))$ and $(\mathrm{Im}(\tau_1))$ of k are in the same narrow ideal class, and therefore there exists a unit $\eta \in \mathfrak{o}$ such that

$$\tilde{\Sigma}(\eta\tau_1) = \tilde{\Sigma}(\tau),$$

or, if we replace τ_1 by $\eta\tau_1$, which does not change the ideal $\mathfrak{v}_1$, we may assume that $\tilde{\Sigma}(\tau_1) = \tilde{\Sigma}(\tau)$. Thus we may write

$$\mathfrak{B} = \mathfrak{v}_1 B + \mathfrak{o}B',$$

with $\tilde{\Sigma}(B/B') = \tilde{\Sigma}(\tau_1) = \tilde{\Sigma}(\tau)$. Therefore there exists $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G'_+(k)$ such that

$$P\begin{pmatrix} \tau \\ l \end{pmatrix} = \begin{pmatrix} B \\ B' \end{pmatrix}. \tag{$'$}$$

For each prime $\mathfrak{p}$ of $\mathfrak{o}$, we have

$$\mathfrak{B}_{\mathfrak{p}} = \mathfrak{v}_{1\mathfrak{p}} B + \mathfrak{o}_{\mathfrak{p}} B'$$

and also

$$\mathfrak{B}_{\mathfrak{p}} = \mathfrak{v}_{\mathfrak{p}}\alpha_{\mathfrak{p}}\tau + \mathfrak{o}_{\mathfrak{p}}\alpha_{\mathfrak{p}};$$

therefore, there exists $\omega_{\mathfrak{p}} \in \theta_{\mathfrak{p}} G'(\mathfrak{o}_{\mathfrak{p}})\theta_{1\mathfrak{p}}^{-1}$, say $\omega_{\mathfrak{p}} = \theta_{\mathfrak{p}}\gamma_{\mathfrak{p}}\theta_{1\mathfrak{p}}^{-1}$, such that

$$\omega_p\begin{pmatrix} B \\ B' \end{pmatrix} = \begin{pmatrix} \alpha_{\mathfrak{p}}\tau \\ \alpha_{\mathfrak{p}} \end{pmatrix},$$

i.e., such that

$$\omega_{\mathfrak{p}} P\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_{\mathfrak{p}}\tau \\ \alpha_{\mathfrak{p}} \end{pmatrix} \quad \text{for each prime } \mathfrak{p}.$$

Then $q(\alpha)_{\mathfrak{p}} = \omega_{\mathfrak{p}} P$ for each prime $\mathfrak{p}$. Therefore, $q(\alpha^{-1})_{\mathfrak{p}} = P^{-1}\theta_{1\mathfrak{p}}\gamma_{\mathfrak{p}}^{-1}\theta_{\mathfrak{p}}^{-1}$, or since $\alpha \in I(K)_f$,

$$q(\alpha^{-1}) = P_f^{-1}\theta_1\gamma\theta^{-1}, \qquad \gamma \in G(\hat{\mathbb{Z}}),$$

hence $q(\alpha^{-1})\theta\xi = P_f^{-1}\theta_1\gamma\xi = P^{-1}P_\infty\xi\theta_1\gamma$. Now by ('), $P_\infty\xi(\mathrm{i.e}) = \xi_1(\mathrm{i.e})$, so that

$$j(q(\alpha^{-1})\theta\xi) = j(\xi_1\theta_1) = j(\theta_1\xi_1).$$

This proves the lemma. (Cf. [16: §1.2.4]).

Now $\Xi_\infty(\mathfrak{R}) = \cup_{\theta\in\Theta}\Xi_{\theta,\infty}(\mathfrak{R})\theta = \cup_\theta \Xi_\theta(\mathfrak{R})$. Then, following Karel [16], we may introduce an action of the group $I(K)_f$ on the set $\Xi_{\mathbf{A}}(\mathbb{R})$ as follows: Define a map

$$F: I(K)_f \times G_+(\mathbf{Q}) \times \Xi_\infty(\mathfrak{R}) \times G(\hat{\mathbb{Z}}) \to G_+(\mathbf{A}) \tag{11}$$

by $F(y, \lambda, \xi\theta, \omega) = \lambda q_{\xi(\mathrm{i.e})}(y^{-1})\xi\theta\omega$, where $y \in I(K)_f$, $\lambda \in G_+(\mathbf{Q})$, $\xi \in \Xi_{\theta,\infty}(\mathfrak{R})$, and $\omega \in G(\hat{\mathbb{Z}})$. We first verify that $F(y, \lambda, \xi\theta, \omega)$ depends only on y and on the product $\lambda\xi\theta\omega \in \Xi_{\mathbf{A}}(\mathfrak{R})$. Suppose that $\lambda\xi\theta\omega = \lambda'\xi'\theta'\omega'$ with analogous meanings for the primed elements. By looking at the non-archimedean components, we see that

$$\det(\theta\theta'^{-1}) \in \det(G'(\hat{\mathfrak{o}}))\det(G'_+(k)_f),$$

so that θ and θ' represent the same double coset in (6) and therefore $\theta = \theta'$. Consequently

$$\lambda^{-1}\lambda' = \xi\xi'^{-1}\theta\omega\omega'^{-1}\theta^{-1} \in \Gamma({}^\theta G(\hat{\mathbb{Z}})),$$

the arithmetic group acting on X_θ. Put $\gamma = \lambda'^{-1}\lambda$. Then $\xi' = \gamma_\infty\xi$ so that $\theta\xi(\mathrm{i.e}) = \theta\gamma_\infty\xi'(\mathrm{i.e})$. The representation $q_\tau: K \hookrightarrow M_2(k)$ is defined by

$$q_\tau(b)\begin{pmatrix}\tau\\1\end{pmatrix} = \begin{pmatrix}b\tau\\b\end{pmatrix}, \qquad \tau \in K - k, \qquad b \in K, \tag{12}$$

and q_τ may be extended to a representation of $\mathbf{A}(K)$ into $M_2(\mathbf{A}(k))$, in particular to a representation of $\mathbb{C} = K \otimes \mathbb{R}$ into $M_2(\mathbb{R})$. Therefore, if $S = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in GL_2^+(k)$ and $S \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$, $\tau \in K - k$, we have

$$q_{S\cdot\tau} = Sq_\tau S^{-1}. \tag{13}$$

Applying this with $S = \gamma$ we obtain

$$\lambda' q_{\xi'(\mathrm{i.e})}(y^{-1})\xi'\theta'\omega' = \lambda\gamma^{-1}q_{\xi'(\mathrm{i.e})}(y^{-1})\gamma_\infty\xi\theta\omega' = \lambda\gamma_f^{-1}q_{\xi'(\mathrm{i.e})}(y^{-1})\xi\theta\omega' =$$

$$= \lambda q_{\gamma_\infty^{-1}\xi'(\mathrm{i.e})}(y^{-1})\gamma_f^{-1}\theta\xi\omega' = \lambda q_{\xi(\mathrm{i.e})}(y^{-1})\xi\gamma_f^{-1}\theta\omega' = \lambda q_{\xi(\mathrm{i.e})}(y^{-1})\xi\theta\omega,$$

since $\gamma_f^{-1} = \theta\omega\omega'^{-1}\theta^{-1}$. This says, as claimed, that $F(y, \lambda, \xi\theta, \omega)$ depends only on y and on the product $\lambda\xi\theta\omega$.

We can also see that $F(y, \lambda, \xi\theta, \omega)$ belongs to $\Xi_{\mathbf{A}}(\mathfrak{R})$. For this it suffices to prove that

$$q_{\xi(\mathrm{i.e})}(y^{-1})\theta\xi \in \Xi_{\mathbf{A}}(\mathfrak{R}) \tag{14}$$

(noting that $\theta\xi = \xi\theta$), and this is immediate from Lemma 1 since (in the notation of that lemma) $\theta_1\xi_1 \in \Xi_{\mathbf{A}}(\mathfrak{R})$.

Thus one has, in analogy with §1.2.5 of [16], a map

$$I(K)_f \times \Xi_{\mathbf{A}}(\mathfrak{R}) \to \Xi_{\mathbf{A}}(\mathfrak{R}), \tag{15}$$

written as $(y, \xi) \to y_*\xi$ for $y \in I(K)_f$, $\xi \in \Xi_{\mathbf{A}}(\mathfrak{R})$. One extends this to an action of $I(K)$ by defining the action of $I(K)_\infty$ to be trivial (which is appropriate since $q_{\xi(\text{i.e})}(\alpha).\ \xi(\text{i.e}) = \xi(\text{i.e})$ for every $\alpha \in I(K)_\infty$). In this way one obtains an action of the group $I(K)/I(K)_\infty \cdot K^\times$ of idele classes, modulo the connected component of the identity, on $G_+(\mathbf{Q})\Xi_{\mathbf{A}}(\mathfrak{R})/\mathbb{K}_\infty$ and hence on the image of $\Xi_{\mathbf{A}}(\mathfrak{R})$ in each of the double coset spaces $V_{\mathbb{K}}$ in (5). Now $V_{\mathbb{K}} = \cup_{\omega\in\Omega} V_\omega$ and it is easy to verify that

$$\det(q_{\xi(\text{i.e})}(y^{-1})) = N_{K/k}(y^{-1}), \qquad y \in I(K). \tag{16}$$

Hence, $\xi \to y * \xi$ moves a special point of the component X_ω to one of $X_{\omega'}$, where $\omega' \in \Omega$ is defined uniquely by

$$\det(\omega') \in N(y^{-1})\det(\omega)(k_+^\times)_f \det(\mathbb{K}). \tag{17}$$

Also, for $x \in \mathfrak{U}_\mathfrak{n}(\hat{\mathfrak{R}})$, $q_\tau(x)$ belongs to the group of units of $\hat{R}_{\mathfrak{d}+}$ which are $\equiv 1 \bmod \mathfrak{n}$, i.e., $q_\tau(x) \in {}^\omega\mathbb{K}(\mathfrak{n})$, where τ belongs to $\mathfrak{H}^n$ and $\mathfrak{R}$ is the order of $\tau\mathfrak{d} + \mathfrak{o}$, and $\mathfrak{d} = \text{id}.(\det(\omega)^{-1})$. Therefore if $\tau = \xi(\text{i.e})$ and $\zeta = \lambda\xi\omega\eta$ represents a point of a component V_ω of $V_{\mathbb{K}}$, where $\mathbb{K} = \mathbb{K}(\mathfrak{n})$, $\lambda \in G_+(\mathbf{Q})$, and $\eta \in \mathbb{K}(\mathfrak{n})$, then $q_\tau(x^{-1}) = {}^\omega\eta'$ for some $\eta' \in \mathbb{K}(\mathfrak{n})$ and therefore $x * \zeta = \lambda q_\tau(x^{-1})\xi\omega\eta = \lambda\xi\omega\eta'\eta$, which represents the same point of $V_{\mathbb{K}}$. This means that, as a subgroup of $I(K)$, the principal congruence subgroup $\mathfrak{U}_\mathfrak{n}(\hat{\mathfrak{R}})$ of $I(K)_f$ acts trivially on the projection $\Xi_\mathfrak{n}(\mathfrak{R})$ of $\Xi_{\mathbf{A}}(\mathfrak{R})$ into $V_\mathfrak{n}$. In other words, the ray class group

$$C_\mathfrak{n}(\mathfrak{R}) = R_\mathfrak{n}(K, \mathfrak{R}) = I(K)/I(K)_\infty \cdot K^\times \mathfrak{U}_\mathfrak{n}(\hat{\mathfrak{R}})$$

of K with respect to $\mathfrak{R}$ modulo $\mathfrak{n}$ acts on $\Xi_\mathfrak{n}(\mathfrak{R})$. It is known already from [3:§3.1] that

$$\Xi(\mathfrak{R}) = \Xi_1(\mathfrak{R}) = \cup_\mathfrak{d} \Xi_\mathfrak{d}(\mathfrak{R})/\Gamma_\mathfrak{d}^+$$

is finite, where $\mathfrak{d}$ runs over a set of representatives of narrow ideal classes of k. Since the canonical projection of $V_\mathfrak{n}$ onto $V = V_1$ has only finite fibers, it follows that $\Xi_\mathfrak{n}(\mathfrak{R})$ is a finite set in which $R_\mathfrak{n}(K, \mathfrak{R})$, of course, has only finitely many orbits.

At the same time, the relation $\tilde{\Sigma}(\tau_1) = \tilde{\Sigma}(\tau)$ from Lemma 1 implies that if $\xi \in \Xi_{\mathbf{A}}(\mathfrak{R}, \tilde{\Sigma})$ and $y \in I(K)$, then $y * \xi \in \Xi_{\mathbf{A}}(\mathfrak{R}, \tilde{\Sigma})$. Thus, the action of $I(K)$ on $\Xi_{\mathbf{A}}(\mathfrak{R})$ is also an action of $I(K)$ on $\Xi_{\mathbf{A}}(\mathfrak{R}, \tilde{\Sigma})$ for each lifting $\tilde{\Sigma}$.

The action described in [16], as well as that defined here, of the idele group on the special points of the adelic double coset space, is, of course, closely related to the action given in [31] of the idele group on the special values of arithmetic modular functions.

3. Conjugation

We now formulate a generalization of results of §1.3 of [16] to cover our present situation. The proofs, which parallel closely those of *loc. cit.*, will be mostly omitted. The purpose of these results is to effect, for each $x = \xi\theta \in \Xi_\infty(\mathfrak{R})$, an extension of the k-algebra homomorphism $q_{\xi(\mathrm{i.e})}$ to an isomorphism of a K-algebra

$$\tilde{K} = K + \iota K, \qquad \iota^2 = 1, \qquad \iota a = \bar{a}\iota \quad \text{for all} \quad a \in K,$$

with $M_2(k)$.

Let $x = \xi\theta \in \Xi_{\infty,\theta}(\mathfrak{R})$ and $z = \xi(\mathrm{i.e})$. Then $z = (\tau^{\tilde{\sigma}_1}, \ldots, \tau^{\tilde{\sigma}_n})$ for some $\tau \in K$ where $\tilde{\Sigma}(\tau) = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$. For $a \in K$, $q(a) = q_\tau(a)$ is defined by (12) of §2. We define

$$q(\iota) = \begin{pmatrix} -1 & \tau + \bar{\tau} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & x \\ 0 & 1 \end{pmatrix} \in G'_-(k),$$

where $x = 2Re\tau \in k$ and $G'_-(k) = \{M \in G'(k) \mid \det(M) \leqslant 0\}$. Then using (12) we get

$$q(\iota)q(a) = q(\bar{a})q(\iota). \tag{18}$$

Hence q, so extended to $\tilde{K}$, becomes a faithful k-algebra isomorphism of $\tilde{K}$ with $M_2(k)$. Then if $x \in M_2(k)$ satisfies $q(a)x = xq(\bar{a})$ for all $a \in K$ it follows that $xq(\iota)$ centralizes $q(K) \subset M_2(k)$ and therefore $xq(\iota) = q(b)$ for some $b \in K$, i.e., $x = q(b\iota)$.

We identify ι with the generator of $\mathrm{Gal}(K/k)$ and extend the action of $I(K)$ on

$$G_+(\mathbf{Q})\backslash G_+(\mathbf{A})/\mathbb{K}_\infty,$$

defined in §2, to an action of the semi-direct product $I(K)^\sim = I(K) \rtimes \mathrm{Gal}(K/$ $/k)$ defined by the exact sequence

$$\{1\} \to I(K) \to I(K)^\sim \to \mathrm{Gal}(K/k) \to \{1\},$$

in the following manner. If $x = \xi\theta$ as above, and if $\omega \in G(\hat{\mathbb{Z}}) = G'(\hat{\mathfrak{o}})$, let

$$\iota^*(G_+(\mathbf{Q})x\omega\mathbb{K}_\infty) = G_-(\mathbf{Q})q_\tau(\iota)_\infty x\omega\mathbb{K}_\infty,$$

where $G_-(\mathbf{Q}) = G'_-(k)$. It is easy to verify, just as in *loc. cit.*, that this definition is independent of the choice of representative $x\omega$ of the given double coset mod $G_+(\mathbf{Q})\backslash/\mathbb{K}_\infty$; that if v is such a double coset, then

$$b * \iota * v = \iota * \bar{b} * v, \qquad b \in I(K);$$

that for each prime ideal $\mathfrak{p}$ of the ring of integers $\mathfrak{o}$ of k and $\theta \in \Theta$, there exists $\delta_\mathfrak{p} \in {}^\theta G'(\mathfrak{o}_\mathfrak{p})$ such that $\delta_\mathfrak{p} q(a) \delta_\mathfrak{p}^{-1} = q(\bar{a})$ for all $a \in K_\mathfrak{p} = K \otimes_k k_\mathfrak{p}$; and then, in parallel to the proof of Lemma 1.3.2 of *loc. cit.*, that

$$\iota * (G_+(\mathbf{Q})\backslash \Xi_\mathbf{A}(\mathfrak{R})/\mathbb{K}_\infty) = G_+(\mathbf{Q})\backslash \Xi_\mathbf{A}(\mathfrak{R})/\mathbb{K}_\infty.$$

In this way, one constructs an action of $I(K)^\sim$ on the space of double cosets $G_+(\mathbf{Q})\backslash \Xi_\mathbf{A}(\mathfrak{R})/\mathbb{K}_\infty$. As in Section 2, this action commutes with right translation by elements of $G(\hat{\mathbb{Z}})$ and provides an action of

$$C_\mathfrak{n}(\mathfrak{R})^\sim = R_\mathfrak{n}(K, \mathfrak{R})^\sim = R_\mathfrak{n}(K, \mathfrak{R}) \rtimes \mathrm{Gal}(K/k)$$

on $\Xi_\mathfrak{n}(\mathfrak{R})$. (Note that complex conjugation permutes the ray classes modulo the ideal $\mathfrak{n}$ of $\mathfrak{o}$.)

At the same time, using the relations analogous to those described in the proof of Lemma 1.3.2 of [16], one sees that the action of $I(K)^\sim$ on the space of double cosets preserves each of the sets

$$G_+(\mathbf{Q})\backslash \Xi_\mathbf{A}(\mathfrak{R}, \tilde{\Sigma})/\mathbb{K}_\infty$$

and provides an action of $C_\mathfrak{n}(\mathfrak{R})^\sim$ on $\Xi_\mathfrak{n}(\mathfrak{R}, \tilde{\Sigma})$ for each lifting $\tilde{\Sigma}$.

4. Modular Forms and Eisenstein Series

4.1. If $\mathbb{K}$ is an open compact subgroup of $G(\mathbf{A}_f)$, we form the graded algebra [3: §1.2]

$$\mathfrak{A}(\mathbb{K}) = \bigoplus_{w \in \mathbb{Z}, w \geq 0} \mathfrak{A}(\mathbb{K}, w)$$

of modular forms with respect to $\mathbb{K}$ on $G_+(\mathbf{A})$. For given w, each element ϕ of $\mathfrak{A}(\mathbb{K}, w)$ induces on each component $X_\omega (\simeq \mathfrak{H}^n)$ a holomorphic modular form of weight w with respect to the arithmetic group Γ_ω and for the automorphy factor

$$j(g, z)^w = \mathrm{jac}(g, z)^w, \qquad g \in G_+(\mathbb{R}), \qquad z \in \mathfrak{H}^n,$$

where jac denotes the functional determinant. In general we adopt the notation of §1 of [3] for the graded algebras of modular forms, the graded k-subalgebras of those which are k-arithmetic, the graded algebra of homogeneous quotients of modular forms, and, in particular, for the ring of modular functions, respectively arithmetic modular functions, with respect to an open compact subgroup of $G(\mathbf{A}_f)$, or field of modular functions with respect to an arithmetic group acting on $\mathfrak{H}^n$. As remarked in *loc. cit.*, there is a standard procedure for lifting a modular form or function with respect to Γ_ω from $\mathfrak{H}^n$ $(= X_\omega)$ to one on $G_+(\mathbf{A})$ supported on the double coset $\mathfrak{C}_{\mathbb{K}}(\omega)$; we denote this lifting map by Λ_ω. There is also a standard process, given by a functor λ, for lowering a modular form ϕ on $G_+(\mathbf{A})$ of weight w with respect to $\mathbb{K}$ to a family $\{f_\omega\}_{\omega\in\Omega}$ of modular forms of weight w, where f_ω is a modular form on $\mathfrak{H}^n$ with respect to Γ_ω.

4.2. We now introduce Eisenstein series, following the constructions of [2, 3, 16], and record some basic facts about their Fourier expansions and behavior as transformed by elements of the Galois group $\mathfrak{G} = \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$.

The Eisenstein series considered in [3] are constructed as follows: Let $\mathfrak{v}$ and $\mathfrak{b}$ be fractional ideals of k representing respectively a narrow ideal class $\mathfrak{K}$ and an ideal class $\mathfrak{K}$ of k. Let $\mathfrak{n} \subset \mathfrak{o}$ be an integral non-zero ideal of $\mathfrak{o}$, and denote by ρ_1 and ρ_2 respectively elements of $\mathfrak{b}\mathfrak{v}$ and of $\mathfrak{b}$ such that

$$g.c.d.(\mathfrak{v}^{-1}\mathfrak{b}^{-1}\rho_1, \mathfrak{b}^{-1}\rho_2, \mathfrak{n}) = (1).$$

Let $\mathfrak{a}_1 = \mathfrak{v}\mathfrak{b}$, $\mathfrak{a}_2 = \mathfrak{b}$. One forms the series

$$G_w(z; \rho_1; \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = \Sigma'_{(e)} \frac{(N(\mathfrak{a}_1\mathfrak{a}_2))^w}{N(\xi_1 z + \xi_2)^{2w}}, \tag{e}$$

$$G_w^*(z; \rho_1; \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = \Sigma'_{(e)+(*)} \frac{(N(\mathfrak{a}_1\mathfrak{a}_2))^w}{N(\xi_1 z + \xi_2)^{2w}}, \tag{e*}$$

where in the first series the sum is over $\xi_i \equiv \rho_i \bmod \mathfrak{n}\mathfrak{a}_i$, $i = 1, 2$, modulo multiplication of the pair (ξ_1, ξ_2) by a totally positive unit $\eta \equiv 1 \bmod \mathfrak{n}$, and $(\xi_1, \xi_2) \neq (0, 0)$, while the conditions in the second summation are all these conditions plus the condition

$$(\mathfrak{v}^{-1}\xi_1, \xi_2) = \mathfrak{b}, \tag{*}$$

where $(\,,)$ stands for *g.c.d.* Let $q = h(\mathfrak{n})$ be the order of the group of ray classes $\bmod\, \mathfrak{n}$ in k, let $C_1, \ldots, C_q$ be the distinct ray classes $\bmod\, \mathfrak{n}$, and for each $l = 1, \ldots, q$, let $\mathfrak{r}_l$ be an integral ideal in C_l and prime to $\mathfrak{n}$, and $\varkappa_l$ be an integer of k such that $\varkappa_l \equiv 0 \pmod{\mathfrak{r}_l}$ and $\varkappa_l \equiv 1 \pmod{\mathfrak{n}}$. Then the two sets of Eisenstein series

$$\{G_w(z; \varkappa_l\rho_1, \varkappa_l\rho_2; \mathfrak{b}\mathfrak{r}_l; \mathfrak{v}; \mathfrak{n}) \mid l = 1, \ldots, q\} \tag{e_1}$$

and

$$\{ G_w^*(z; \varkappa_l \rho_1, \varkappa_l \rho_2; \mathfrak{b}\mathfrak{r}_l; \mathfrak{v}; \mathfrak{n}) \mid l = 1, \dots, q \} \tag{e_1*}$$

are linearly independent and span the same vector-space of complex-valued functions on $\mathfrak{H}^n$. Each G_w is a linear combination of the function G_w^*, and vice versa, and the coefficients in these linear combinations are given explicitly as special values of certain Dirichlet series (cf. [19], §2.2) denoted $\theta_l^{*(w)}$ and $\theta_l^{(w)}$. Moreover, if one defines

$$E_w = (-2\pi i)^{-wn} |\sqrt{d}|^{-1} G_w,$$

then E_w has a Fourier expansion of the form

$$\begin{aligned} E_w(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) &= b_0(\rho_1, \rho_2; \mathfrak{b}, \mathfrak{v}, \mathfrak{n}) + \\ &+ \frac{N\mathfrak{b}^{2w-1}}{((2w-1)!)^n N(\mathfrak{n})} \sum_{\substack{\xi_1 \equiv \rho_1 (\mathfrak{n}\mathfrak{a}_1) \\ (\xi_1)_\mathfrak{n}^+}} \sum_{\substack{\mu \in (\mathfrak{n}\mathfrak{b}\mathfrak{d}_k)^{-1} \\ \xi_1 \mu \gg 0}} sgn(N(\xi_1)) N\mu^{2w-1} e(\mu\rho_2) e(\xi_1 \mu z), \end{aligned}$$

where $(\xi_1)_\mathfrak{n}^+$ denotes that the summation over ξ_1 is, again, modulo multiplication by totally positive units $\equiv 1 \bmod \mathfrak{n}$, and $e(\,) = e^{2\pi i \cdot tr(\,)}$. All the Fourier coefficients in this expansion lie in $\mathbf{Q}_{N(\mathfrak{n})}$. Moreover, the coefficients of the linear combinations by means of wich the $E_w's$ are expressed in terms of the G_w^*'s, or vice versa, all lie in $\mathbf{Q}_{N(\mathfrak{n})}$, and if $\sigma \in \mathrm{Gal}(\mathbf{Q}_{N(\mathfrak{n})}/\mathbf{Q})$ is such that for every $N(\mathfrak{n})$-th root of unity ζ we have $\zeta^\sigma = \zeta^s$, $s \in \mathbb{Z}$, $(s, N(\mathfrak{n})) = (1)$, and if σ is applied to all the Fourier coefficients of E_w, the result is

$$E_w(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n})^\sigma = E_w(z; \rho_1, s\rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}). \tag{19}$$

By means of calculations based on Klingen's paper and in principle due to Karel, we may show that, as a consequence of (19),

$$G_w^*(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n})^\sigma = G_w^*(z; s^{-1}\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}). \tag{20}$$

This equation, which will be proved later, has a convenient formal interpretation for which we now prepare.

4.3. In [2] we calculated, in the special case of arithmetic groups commensurable with the Hilbert modular group, the explicit form of certain Eisenstein series considered in [15]. These Eisenstein series are associated to a quintuple $\mathfrak{C} = (G, P, \rho, \mathbb{K}, s)$, where G is an algebraic group, P, a parabolic subgroup, ρ, a one-dimensional character of P, all defined over $\mathbf{Q}$, $\mathbb{K}$ is an open compact subgroup of $G(\mathbf{A}_f)$, and s is a function on the double coset space

$$P_+(\mathbf{Q}) \backslash G(\mathbf{A}) / G_+(\mathbb{R}) \mathbb{K}; \tag{21}$$

and these series are constructed by means of a certain function $\phi_{\mathfrak{S}}\colon G(\mathbf{A}) \to \mathbf{C}$. In this paper we extend the calculations of [2] to a more general situation and show how, in our case, such series are related to those in section §4.2. The construction of the series we consider also depends on a certain function

$$\phi_{\mathfrak{S}}\colon G_+(\mathbf{A}) \to \mathbf{C}$$

which satisfies

$$\phi_{\mathfrak{S}}(bg) = \phi_{\mathfrak{S}}(g), \qquad g \in G_+(\mathbf{A}), \qquad b \in P_+(\mathbf{Q}),$$

where P is defined in §1, and

$$\phi_{\mathfrak{S}}(gk) = \phi_{\mathfrak{S}}(g)J(k_\infty, \text{i.e}), \qquad k_\infty \in \mathbb{K}_\infty,$$

where $J(g, z) = j(g, z)^w$ for some integer $w \geqslant 0$, and finally

$$\begin{aligned} \phi_{\mathfrak{S}}(gx) &= \phi_{\mathfrak{S}}(g), & x &\in G'(\hat{\mathfrak{o}}), \\ \phi_{\mathfrak{S}}(b) &= |\rho(b)|_{\mathbf{A}(k)}, & b &\in P_+(\mathbf{A}), \end{aligned}$$

where ρ is the rational character, defined over k, on the group P' of §1, such that $\rho\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (ad^{-1})^w$. Then one forms the series

$$\mathcal{E}_{\mathfrak{S}}(g) = \sum_{P_+(\mathbf{Q}) \setminus G_+(\mathbf{Q}) \ni \gamma} s(\gamma g)\phi_{\mathfrak{S}}(\gamma g), \tag{22}$$

where s is *a* $\mathbf{Q}$-valued function on the double coset space (21), and the series converges uniformly absolutely on compact subsets of $G_+(\mathbf{A})$ as long as $w \geqslant 2$. We have

$$G_+(\mathbf{A}) = \bigcup_{\omega \in \Omega} G_+(\mathbf{Q})\omega G_+(\mathbb{R})\mathbb{K}$$

and we may let $\Omega = \Theta \cdot H$, where H is a set of elements $\eta \in G(\mathbf{A}_f)$ such that id.$(\det(\eta))$ runs over representatives of the narrow ray classes mod $\mathfrak{n}$ contained in the principal narrow class of k, say $\eta = \begin{pmatrix} \eta' & 0 \\ 0 & 1 \end{pmatrix}$ for $\eta \in H$. (Here, for $x \in I(k)_f$, id.(x) is the ideal of k naturally associated to x.) Moreover, for each $\omega \in \Omega$, we have

$$\begin{aligned} G_+(\mathbf{Q}) &= \bigcup_{\alpha \in \mathbf{A}(\omega)} P_+(\mathbf{Q})\alpha\Gamma_\omega, \\ \Gamma_\omega &= G_+(\mathbf{Q}) \cap G_+(\mathbb{R})^\omega\mathbb{K}, \\ G_+(\mathbf{A}) &= \bigcup_{\omega \in \Omega} \bigcup_{\alpha \in \mathbf{A}(\omega)} P_+(\mathbf{Q})\alpha\omega G_+(\mathbb{R})\mathbb{K}. \end{aligned}$$

We may write

$$\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in G'_+(k) = G_+(\mathbf{Q}),$$

and let $s_{\omega,\alpha}$ be the characteristic function of the double coset

$$P_+(\mathbf{Q})\alpha\omega G_+(\mathbb{R})\mathbb{K}.$$

We assume $\mathbb{K}$ is the principal congruence subgroup $G'(\hat{\mathfrak{o}})(\mathfrak{n})$ of $G'(\hat{\mathfrak{o}})$ for some integral ideal $\mathfrak{n}$ of k, $\mathbb{K} = \mathbb{K}(\mathfrak{n})$. Then we may assume each $\theta \in \Theta$ is of the form $\left(\begin{smallmatrix}\theta' & 0\\ 0 & 1\end{smallmatrix}\right)$, where $\mathfrak{v} = \mathrm{id.}(\theta'^{-1})$ is an integral ideal prime to $\mathfrak{n}$. Moreover, a full set of representatives of the cosets C occurring in

$$I_+(k)/k_+^\times k_{\infty+}^\times \mathfrak{U}_\mathfrak{n}(\hat{\mathfrak{o}}) \qquad (\mathfrak{U}_\mathfrak{n}(\hat{\mathfrak{o}})\text{: units} \equiv 1 \bmod \mathfrak{n} \quad \text{of} \quad \hat{\mathfrak{o}})$$

for which the ideals of C belong to the narrow principal class, may be taken as units η' of the maximal compact subring $\hat{\mathfrak{o}}$ of $\mathbf{A}(k)_f$ (which are 1 at the archimedean places), and therefore each $\eta \in H$ may even be taken of the form

$$\eta = \begin{pmatrix}\eta' & 0\\ 0 & 1\end{pmatrix},$$

with $\eta' \in \hat{\mathfrak{o}}^\times$. We need the following

Lemma 2. *Let there be given two pairs (c, d) and (c', d') belonging to $(\mathfrak{v}\mathfrak{b}, \mathfrak{b})$ and such that*

$$g.c.d.(\mathfrak{v}^{-1}c, d) = g.c.d.(\mathfrak{v}^{-1}c', d') = \mathfrak{b},$$

where $\mathfrak{b}$ is an integral ideal prime to $\mathfrak{n}$. Assume $(c, d, \mathfrak{n}) = (c', d', \mathfrak{n}) = (1)$ and that $c' \equiv c$, $d' \equiv d \bmod \mathfrak{n}$. Then there exists $M'' \in R_{\mathfrak{v}+}^\times(\mathfrak{n})$ such that $\det(M'') = 1$ and $(c, d)M'' = (c', d')$ (*matrix multiplication on the right*).

PROOF. Let $\mathfrak{a}_1 = \mathfrak{b}\mathfrak{v}$, $\mathfrak{a}_2 = \mathfrak{b}$; these are integral ideals prime to $\mathfrak{n}$. We know ([3], §2.3) that there exist $a, a' \in \mathfrak{a}_2^{-1}$, $b, b' \in \mathfrak{a}_1^{-1}$ such that if

$$S = \begin{pmatrix}a & b\\ c & d\end{pmatrix}, \qquad S' = \begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}, \qquad \text{then } \det(S) = \det(S') = 1.$$

Clearly $M = S'^{-1}S$ belongs to $R_\mathfrak{v}^\times$. We have $S = S'M$. Since $(c, d, \mathfrak{n}) = (1)$, there is (by strong approximation) a non-singular matrix T of determinant 1 in $R_\mathfrak{v}^\times$ such that modulo $\mathfrak{n}$ we have $(c_1, d_1) = (c, d)T \equiv (c^*, 0)$, $(c^*, \mathfrak{n}) = (1)$. Since $S = S'M$, we have $(c, d) = (c', d')M$ and $(c_1, d_1) = (c, d)T = (c', d')MT = (c', d')TM' \equiv (c^*, 0) \bmod \mathfrak{n}$, where $M' = T^{-1}MT \in R_\mathfrak{v}^\times$. We have, since $c' \equiv c$, $d' \equiv d \bmod \mathfrak{n}$, $(c', d')T \equiv (c, d)T \equiv (c^*, 0) \bmod \mathfrak{n}$, hence $(c^*, 0)M' \equiv (c^*, 0) \bmod \mathfrak{n}$, and since $(c^*, \mathfrak{n}) = 1$ and $\det(M') = 1$, we have

$$M' \equiv \begin{pmatrix}1 & 0\\ \gamma' & 1\end{pmatrix} \bmod \mathfrak{n} \quad \text{for some} \quad \gamma' \in \mathfrak{v}.$$

We want $M_2 \in R_{\mathfrak{v}}$ such that $(c_1, d_1)M_2 = (c_1, d_1)$ and

$$M_2 \equiv \begin{pmatrix} 1 & 0 \\ \gamma' & 1 \end{pmatrix} \bmod \mathfrak{n}.$$

The case $d_1 = 0$ being easily handled as a special case, we assume $c_1 d_1 \neq 0$. The matrices of the form

$$\begin{pmatrix} \alpha & c_1^{-1}d_1(\alpha - 1) \\ -d_1^{-1}c_1(\alpha - 1) & 2 - \alpha \end{pmatrix}, \qquad \alpha \in k,$$

all fix (c_1, d_1), i.e., satisfy the second condition. Then the first and third conditions are expressed by

$$c_1^{-1}d_1\beta^* \in \mathfrak{v}^{-1}, d_1^{-1}c_1\beta^* \in \mathfrak{v}, \beta^* \in \mathfrak{n}, \beta^* \equiv -c_1^{-1}d_1\gamma' \bmod c_1^{-1}d_1\mathfrak{n},$$

where $\beta^* = \alpha - 1$. Since $\mathfrak{b}$ and $\mathfrak{v}$ are prime to $\mathfrak{n}$ and $\gamma' \in \mathfrak{v}$, it follows easily from the Chinese remainder theorem that such a $\beta^* \in k$ exists, hence M_2 satisfying the desired conditions also exists. Then $M'M_2^{-1} \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod \mathfrak{n}$ in $R_{\mathfrak{v}}$ and we have $(c', d')TM'M_2^{-1}T^{-1} = (c, d)TM_2^{-1}T^{-1} = (c_1, d_1)M_2^{-1}T^{-1} = = (c_1, d_1)T^{-1} = (c, d)$, while at the same time $M'M_2^{-1} \in R_{\mathfrak{v}}(\mathfrak{n})^{\times}$, therefore $M'' = TM'M_2^{-1}T^{-1} \in R_{\mathfrak{v}}(\mathfrak{n})$ and, actually, $\det(M'') = 1$. Therefore, $M'' \in \Gamma_{\mathfrak{v}}(\mathfrak{n})$. This proves the lemma.

With $\eta \in H$ of the form $\left(\begin{smallmatrix} \eta' & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\eta' \in \hat{\mathfrak{o}}^{\times}$, η normalizes $\mathbb{K}(\mathfrak{n})$. Let $\omega = \theta\eta$ and let $\tilde{\mathcal{E}}_{\omega, \alpha, w}$ be the Eisenstein series $\tilde{\mathcal{E}}_{\mathfrak{C}}$ on $G_+(\mathbf{A})$ where $\mathfrak{C} = (G, P, \rho, \mathbb{K}(\mathfrak{n}), s_{\omega, \alpha})$. Since $s_{\omega, \alpha}$ is the characteristic function of

$$P_+(\mathbf{Q})\alpha\omega G_+(\mathbb{R})\mathbb{K}(\mathfrak{n}),$$

$\tilde{\mathcal{E}}_{\omega, \alpha, w}$ is supported on the union of translates by $G_+(\mathbf{Q})$ on the left of

$$G_{\omega, \mathfrak{n}}(\mathbf{A}) = \omega G_+(\mathbb{R})\mathbb{K}(\mathfrak{n}),$$

hence the corresponding holomorphic Eisenstein series $E_{\omega, \alpha, w}$ is supported on the union of translates by $G_+(\mathbf{Q})$ on the left of

$$X_\omega = \omega G_+(\mathbb{R})\mathbb{K}(\mathfrak{n})/\mathbb{K}_\infty\mathbb{K}(\mathfrak{n}),$$

which may be identified with $\mathfrak{H}^n$ on which $\Gamma_\omega(\mathfrak{n}) = \Gamma({}^{\omega}\mathbb{K}(\mathfrak{n})) = \Gamma({}^{\theta}\mathbb{K}(\mathfrak{n}))$ acts. If $\mathfrak{v} = \mathrm{id}.(\theta'^{-1})$, $\Gamma({}^{\theta}\mathbb{K}(\mathfrak{n})) = \Gamma_{\mathfrak{v}}(\mathfrak{n})$, the principal congruence subgroup of the group $\Gamma_{\mathfrak{v}}$ defined in [2]. The calculation in [2], §5.2, applied in similar fashion to the present case shows that if

$$\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in \begin{pmatrix} \mathfrak{a}_2^{-1} & \mathfrak{a}_1^{-1} \\ \mathfrak{a}_1 & \mathfrak{a}_2 \end{pmatrix}, \qquad \mathfrak{a}_1 = \mathfrak{v}\mathfrak{b}, \qquad \mathfrak{a}_2 = \mathfrak{b},$$

and $\det(\alpha) = 1$ (which implies $(\mathfrak{a}_1^{-1}c_\alpha, \mathfrak{a}_2^{-1}d_\alpha) = (1)$), then

$$E_{\omega,\alpha,w}(z) = \sum_{(\mathfrak{v}^{-1}c,d)=\mathfrak{b},\, c\equiv c_\alpha(\mathfrak{n}\mathfrak{a}_1),\, d\equiv d_\alpha(\mathfrak{n}\mathfrak{a}_2),\,(c,d)_{\mathfrak{n}}^{+}} \frac{N(\mathfrak{a}_1\mathfrak{a}_2)^w}{N(\xi_1 z + \xi_2)^{2w}} = \tag{23}$$
$$= G_w^*(z; c_\alpha, d_\alpha; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}),$$

because by Lemma 2, every pair (c, d) satisfying the conditions of summation is obtained by applying an element $\gamma \in \Gamma_{\mathfrak{v}}(\mathfrak{n})$ to the right of (c_α, d_α): $(c, d) = (c_\alpha, d_\alpha)\gamma$. Therefore on the component $X_\omega \simeq \mathfrak{H}^n$, $\mathcal{E}_{\omega,\alpha,w}$ induces a standard congruence Eisenstein series for the congruence group $\Gamma_{\mathfrak{v}}(\mathfrak{n})$, and induces the function $\equiv 0$ on any component not in the left translates of X_ω by $G_+(\mathbf{Q})$. Hence the coefficients of the Fourier expansion of this function on each component X_ω lie in the field of $N(\mathfrak{n})$-th roots of unity. Therefore, for $\sigma \in \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ such that $\zeta_{N(\mathfrak{n})}^\sigma = \zeta_{N(\mathfrak{n})}^s$, $(s, N(\mathfrak{n})) = 1$, we need to find the result of applying σ to all the Fourier coefficients of these expansions.

5. Galois Action and Transformation Theory

5.1. The calculations in this section are based on Klingen's paper [19] and were suggested by Karel's paper [16]. We keep the notation of §4.2 and, for the greater part where there is no conflict, also that of §3 of [19].

For each $l = 1, \ldots, h(\mathfrak{n})$, let $\mathfrak{g}_l$ be an integral ideal in the ray class C_l^{-1}. In our present notation, equation (20) on p. 186 of [19] reads (where Klingen's Vorzeichencharakter may be omitted since the weight is even)

$$G_w^*(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = \sum_{l=1}^{h(\mathfrak{n})} \sum_{\mathfrak{g}\in C_l} \frac{\mu(\mathfrak{g})}{N\mathfrak{g}^{2w}} G_w(z; \rho_1, \rho_2; \mathfrak{b}\mathfrak{g}_l^{-1}; \mathfrak{v}; \mathfrak{n}),$$

and the Fourier coefficients of the Eisenstein series

$$\frac{\Delta^{1/2}}{(\pi i)^{2wn}} G_w(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = G_w^{\#}(z; \rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) \tag{24}$$

lie in $\mathbf{Q}_{N(\mathfrak{n})}$. The constant term of the series (24) is zero unless $\rho_1 \equiv 0 \bmod \mathfrak{v}\mathfrak{b}\mathfrak{n}$, but if this congruence holds, then necessarily (according to our original assumptions) we have $(\mathfrak{b}^{-1}\rho_2, \mathfrak{n}) = (1)$, and in that case, the constant term is given as

$$a_0(\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = (\pi i)^{-2wn} N(\mathfrak{a}_1\mathfrak{a}_2)^w \Delta^{1/2} \sum_{m\equiv\rho_2(\mathfrak{n}\mathfrak{b}),\,(m)_{\mathfrak{n}}^{+}} N(m)^{-2w},$$

which belongs to $\mathbf{Q}_{N(\mathfrak{n})}$ by Klingen's results. If $\sigma \in \mathfrak{G} = \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ is such that $\sigma(\zeta_{N(\mathfrak{n})}) = \zeta_{N(\mathfrak{n})}^s$, then, by §2.2 of [3],

$$a_0(\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n})^\sigma = a_0(\rho_1, s\rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}).$$

That is, for $\rho \in \mathfrak{b}$, and $(\mathfrak{b}^{-1}\rho, \mathfrak{n}) = (1)$, one has

$$((\pi i)^{-2wn} N(\mathfrak{a}_1\mathfrak{a}_2)^w \Delta^{1/2} \sum_{m \equiv \rho(\mathfrak{n}\mathfrak{b})}{}' Nm^{-2w})^\sigma = \\ = ((\pi i)^{2wn} N(\mathfrak{a}_1\mathfrak{a}_2)^w \Delta^{1/2} \sum_{\substack{m \equiv s\rho(\mathfrak{n}\mathfrak{b}) \\ (m)\mathfrak{n}^+}} Nm^{-2w}).$$

Let C be the ray class modulo $\mathfrak{n}$ to which $\mathfrak{b}^{-1}\rho$ belongs and define

$$\zeta_k(2w, C) = \sum_{\mathfrak{g} \in C} N\mathfrak{g}^{-2w}.$$

It is proved on p. 184 of [19] that the constant term may also be expressed by

$$a_0(\rho_1, \rho_2; \mathfrak{b}; \mathfrak{v}; \mathfrak{n}) = (\pi i)^{-2wn} e(\mathfrak{n}) \Delta^{1/2} N\mathfrak{v}^w \cdot \zeta_k(2w, C),$$

where, of course, a slight modification is necessary in Klingen's calculation to take account of the fractional ideal $\mathfrak{v}$, and where $e(\mathfrak{n})$ is a rational number depending on $\mathfrak{n}$ and on the structure of the units group of k. Define

$$\chi(C) = (\pi i)^{-2wn} e(\mathfrak{n}) \Delta^{1/2} N\mathfrak{v}^{2w} \cdot \zeta_k(2w, C) = \\ = (\pi i)^{-2wn} N(\mathfrak{a}_1\mathfrak{a}_2)^w \Delta^{1/2} \sum_{m \equiv \rho(\mathfrak{n}\mathfrak{b}),\ (m)\mathfrak{n}^+}{}' Nm^{-2w}.$$

Then $\chi(C)^\sigma = \chi(sC)$ and by [19] p. 186, line 3 from the bottom,

$$\sum_{t=1}^{h(\mathfrak{n})} \zeta_k(2w, CC_t^{-1}) \Big(\sum_{\mathfrak{g} \in C_t} \mu(\mathfrak{g}) N\mathfrak{g}^{-2w} \Big) = \begin{cases} 1 & \text{if } C \text{ is the principal ray mod } \mathfrak{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Here μ is the ideal-theoretic Möbius function on integral ideals of k. This can be written as

$$\sum_{t=1}^{h(\mathfrak{n})} (\pi i)^{-2wn} \Delta^{1/2} \cdot \zeta_k(2w, CC_t^{-1}) \Big(\frac{(\pi i)^{2wn}}{\Delta^{1/2}} \sum_{\mathfrak{g} \in C_t} \mu(\mathfrak{g}) N\mathfrak{g}^{-2w} \Big) = \\ = \begin{cases} 1 & \text{if } C = H(\mathfrak{n}), \text{ the principal ray mod } \mathfrak{n}, \\ 0 & \text{if otherwise.} \end{cases}$$

Use $\theta(C)$ to denote the last factor in large parentheses on the left hand side of this statement. Then $\theta(C)$ belongs to $\mathbf{Q}_{N(\mathfrak{n})}$ and the last equations read

$$\sum_{t=1}^{h(\mathfrak{n})} \chi(CC_t^{-1}) \theta(C_t) = \begin{cases} 1 & \text{if } C = H(\mathfrak{n}), \\ 0 & \text{if otherwise.} \end{cases}$$

Applying $\sigma \in \mathfrak{G}$ with s defined as above:

$$\sum_{t=1}^{h(\mathfrak{n})} \chi(sCC_t^{-1}) \theta(C_t)^\sigma = \begin{cases} 1 & \text{if } C = H(\mathfrak{n}), \\ 0 & \text{if otherwise.} \end{cases}$$

Comparing these equations and using (21), p. 187 of [19] gives

$$\theta(C_l)^\sigma = \theta(s^{-1}C_l).$$

According to Klingen

$$G_w^*(z;\rho_1,\rho_2;\mathfrak{b};\mathfrak{v};\mathfrak{n}) = \sum_{l=1}^{h(\mathfrak{n})} \theta(C_l)G_w^\#(z;\rho_1,\rho_2;C_l\mathfrak{b};\mathfrak{v};\mathfrak{n}). \tag{25}$$

Applying σ to the Fourier coefficients we get

$$G_w^*(z;\rho_1,\rho_2;\mathfrak{b};\mathfrak{v};\mathfrak{n})^\sigma = \sum_{l=1}^{h(n)} \theta(s^{-1}C_l)G_w^\#(z;\rho_1,s\rho_2;C_l\mathfrak{b};\mathfrak{v};\mathfrak{n}),$$

which, by the equation preceding (20) on p. 186 of [19], is equal to

$$\sum_{l=1}^{h(\mathfrak{n})} \theta(s^{-1}C_l)G_w^\#(z;s^{-1}\rho_1,\rho_2;s^{-1}C_l\mathfrak{b};\mathfrak{v};\mathfrak{n}) = G_w^*(z;s^{-1}\rho_1,\rho_2;\mathfrak{b};\mathfrak{v};\mathfrak{n}).$$

This proves equation (20) of §4.2.

Now by applying the above together with §2.3 of [3], it is easy to see that if we apply $\sigma \in \mathfrak{G}$ to the Fourier coefficients of the expansion of the holomorphic modular form induced on each component by $\tilde{\mathcal{E}}_{\omega,\alpha,w}$, we get the collection of holomorphic modular forms induced on each component by $\tilde{\mathcal{E}}_{\omega,{}^s\alpha,w}$, where ${}^s\alpha$ is an element of determinant one and congruent modulo $N_{k/\mathbf{Q}}(\mathfrak{n}\mathfrak{b}\mathfrak{v})^2$ to

$$\begin{pmatrix} a_\alpha & sb_\alpha \\ s^*c_\alpha & d_\alpha \end{pmatrix}, \qquad s^*s \equiv 1 \bmod N_{k/\mathbf{Q}}(\mathfrak{n}\mathfrak{b}\mathfrak{v})^2.$$

If we let $\hat{\sigma} \in \hat{\mathbb{Z}}^\times \simeq \mathfrak{G}$, and if for $\mathfrak{C} = (G,P,\rho_w,\mathbb{K},s)$ we define $\hat{\sigma}\mathfrak{C} = (G,P,\rho_w,{}^{\mu(\hat{\sigma})}\mathbb{K},\hat{\sigma}\cdot s)$, where $\hat{\sigma}\cdot s(x) = s(\mu(\hat{\sigma})^{-1}x\mu(\hat{\sigma}))$, then linearization of the result we have just obtained shows that if we apply $\hat{\sigma}$ to the Fourier coefficients of the expansions of $\lambda(\tilde{\mathcal{E}}_{\mathfrak{C}})$ on all components X_ω, the result is $\lambda(\tilde{\mathcal{E}}_{\hat{\sigma}\mathfrak{C}})$. This is analogous to Karel's result in §2.2.1 of [16].

5.2. One may then proceed precisely as in [16] to establish an operation

$$\hat{\sigma}\colon \tilde{\mathcal{E}}_{\mathfrak{C}} \to \beta(\hat{\sigma})\tilde{\mathcal{E}}_{\mathfrak{C}} = R(\mu(\hat{\sigma})^{-1})\tilde{\mathcal{E}}_{\hat{\sigma}\mathfrak{C}}$$

or $\hat{\mathbb{Z}}^\times = \mathfrak{G}$ on the Eisenstein series for $\mathbb{K}$ on $G_+(A)$. (In this equation, μ stands for a homomorphism of $\hat{\mathbb{Z}}^\times$ into the adele group as defined by

$$\mu(\hat{\sigma}) = \begin{pmatrix} \tilde{\sigma} & 0 \\ 0 & 1 \end{pmatrix}$$

in [16], and $R(x)$ is the right regular representation of a group on the functions on it.) Define $\tilde{\mathcal{E}}_{\mathfrak{C}}$ and $\lambda(\tilde{\mathcal{E}}_{\mathfrak{C}})$ to be F-rational, F being a subfield of $\mathbf{Q}_{ab}$, if $\beta(\tilde{\sigma})\tilde{\mathcal{E}}_{\mathfrak{C}} = \tilde{\mathcal{E}}_{\mathfrak{C}}$ for all $\tilde{\sigma} \in \mathrm{Gal}(\mathbf{Q}_{ab}/F)$.

With a fixed ideal $\mathfrak{n}$ and $\mathbb{K} = \mathbb{K}(\mathfrak{n})$, let $\tilde{\mathcal{E}}(\mathfrak{n})$ be the graded subalgebra of the algebra $\mathfrak{A}(\mathbb{K}(\mathfrak{n}))$ of modular forms with respect to $\mathbb{K}(\mathfrak{n})$ generated by all the Eisenstein series $\tilde{\mathcal{E}}_{\mathfrak{C}}$ of weights $w > 1$, where $\mathfrak{C}$ is defined as above. $\tilde{\mathcal{E}}_{\mathfrak{C}}$ is a linear combination with (arbitrary) rational coefficients (if s is $\mathbf{Q}$-valued) of the Eisenstein series $\tilde{\mathcal{E}}_{\alpha,\omega,w}$ discussed above. We let ϕ be an element of the ring of homogeneous quotients (with respect to its non-zero divisors) of $\tilde{\mathcal{E}}(\mathfrak{n})$, of degree zero, and form the transformation polynomial

$$T_{\phi,\mathbb{K},s_0}(g)(X) = X^N + \sum_{\nu<N} \alpha_\nu(g)X^\nu, \tag{26}$$

as in [3], §2.1(23), where $S_0 \in G(\mathbf{A}_f) \cap \hat{R}_1$, $\alpha_\nu \in \mathfrak{M}(\mathbb{K}, 0, \{\varkappa\})$. By a straightforward generalization of Karel's results ([16], §2.2.4) we have

$$\beta(\tilde{\sigma})R(M)\phi = R(M)\beta(\tilde{\sigma})\phi, \qquad \tilde{\sigma} \in \mathfrak{G}, \qquad M \in G(\mathbf{A}_f). \tag{27}$$

Since α_ν is a symmetric function of functions $R(S_j)\phi$ and the relation (27) holds with M replaced by MS_j, it follows that also

$$\beta(\tilde{\sigma})R(M)\alpha_\nu = R(M)\beta(\tilde{\sigma})\alpha_\nu. \tag{28}$$

Using this relation in conjunction with Proposition 3 and its Corollary of [3], we see that we have

Proposition 1. *Let $\mathfrak{A}(\mathbb{K}, Q_{ab})$ be the graded algebra of arithmetic modular forms on $G_+(\mathbf{A})$ for $\mathbb{K}$ and let the operators $\beta(\tilde{\sigma})$ and $R(M)$ (for $M \in G(\mathbf{A}_f)$) be defined on $\mathfrak{A}(\mathbb{K}, Q_{ab})$ by the obvious extension. Then for every $\psi \in$ $\in \mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$ one has*

$$R(M)\beta(\tilde{\sigma})\psi = \beta(\tilde{\sigma})R(M)\psi. \tag{29}$$

PROOF. For from the foregoing considerations this relation holds for a set of generators of $\mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$.

Corallary. *Relation* (29) *holds for an arbitrary element ψ of the ring of homogeneous quotients of elements of $\mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$.*

Now the results of [16], §§2.3-2.3.4 have straightforward generalizations as expressed in the following propositions.

Proposition 2. *$\mathfrak{A}(\mathbb{K}(\mathfrak{n}))$ is the integral closure of $\tilde{\mathcal{E}}(\mathfrak{n})$ in its graded ring of homogeneous quotients and we have*

$$\mathfrak{A}(\mathbb{K}(\mathfrak{n})) = \mathfrak{A}(\mathbb{K}(\mathfrak{n}), \mathbf{Q}_{N(\mathfrak{n})}) \otimes \mathbf{C}.$$

Remark. The last statement follows since each Eisenstein series $\tilde{\mathcal{E}}_{\omega,\alpha,w}$ has all Fourier coefficients in $\mathbf{Q}_{N(\mathfrak{n})}$.

If F is any subfield of $\mathbf{Q}_{ab}$, define $\psi \in \mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$ to be F-rational if $\beta(\tilde{\sigma})\psi = \psi$ for every $\tilde{\sigma} \in \mathrm{Gal}(\mathbf{Q}_{ab}/F)$. Denote the graded F-algebra of such modular forms by $\mathfrak{A}(\mathbb{K})^F$. Then we have:

Proposition 3. *Let* $\mathbb{K} = \mathbb{K}(\mathfrak{n})$. *As a graded algebra over* $\mathbf{Q}_{ab}$, $\mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$ *is generated by* $\mathfrak{A}(\mathbb{K})^{\mathbf{Q}}$.

Since for $\theta \in \mathbf{Q}_{ab}$, $\tilde{\sigma} \in \mathfrak{G}$, we have $\beta(\tilde{\sigma})(\theta\psi) = \theta^{\tilde{\sigma}}\beta(\tilde{\sigma})\psi$, this follows, as remarked in [16], from §14 of [6, AG]. Therefore

$$V_{\mathbb{K}}^* = \bigcup_{\omega \in \Omega} V_\omega^*,$$

where V_ω^* is the Satake compactification of V_ω, is an algebraic variety defined over $\mathbf{Q}$ (but not irreducible over $\mathbf{Q}_{ab}$, of course), and by the result of Borel and Narasimhan referred to in §2.5 of [3], the set of cusps $V_{\mathbb{K}}^* - V_{\mathbb{K}}$ is defined over $\mathbf{Q}$ as well, while each component V_ω^* is defined over $\mathbf{Q}_{N(\mathfrak{n})}$, and V_ω is a $\mathbf{Q}_{N(\mathfrak{n})}$— open subvariety of V_ω^*.

The conditions for $\psi \in \mathfrak{A}(\mathbb{K}, \mathbf{Q}_{ab})$ to belong to the $\mathbf{Q}$-structure are equivalent to the following: If $\omega \in \Omega$, let

$$\psi_\omega(z) = \Sigma\, a(\mu, \omega) e^{2\pi i tr(\mu z)}, \qquad \mu \in (\mathfrak{n}\mathfrak{b}\mathfrak{d}_k)^{-1},$$

be the Fourier expansion of the modular form with respect to $\Gamma_\omega = \Gamma({}^\omega\mathbb{K})$ induced by ψ on X_ω. Then for every $\tilde{\sigma} \in \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q}) = \mathfrak{G}$, one has

$$a(\mu, \omega)^{\tilde{\sigma}} = a(\mu, \mu(\tilde{\sigma})\omega).$$

6. Splitting of the Class Polynomial

6.1. The multiplier polynomial. We consider the class polynomial

$$P_{\mathfrak{v},\mathfrak{R},\phi}(X) = \prod_{\tilde{\Sigma}} P_{\mathfrak{v},\mathfrak{R},\tilde{\Sigma},\phi}(X)$$

defined for any order $\mathfrak{R}$ of K containing the maximal order of k and an arithmetic modular function ϕ for $\Gamma_{\mathfrak{v}}^+$ holomorphic on all points of the set $\Xi_{\mathfrak{v}}(\mathfrak{R})$ for given $\mathfrak{v}$, as defined in §3.3 of [3], and where

$$P_{\mathfrak{v},\mathfrak{R},\tilde{\Sigma},\phi}(X) = \prod_{j=1}^{h'} (X - \phi(o_j')),$$

$o_1', \ldots, o_{h'}'$ being a set of representatives of the orbits of $\Gamma_{\mathfrak{v}}^+$ in $\Xi_{\mathfrak{v}}(\mathfrak{R}, \tilde{\Sigma})$. In case $\mathfrak{R} = \mathfrak{O}$, the maximal order of K, it has been shown in [3: §3.3, Theorem 4] that $P_{\mathfrak{v},\mathfrak{O},\phi}(X) \in \mathbf{Q}[X]$. We shall show here, in a later section, that

$$P_{\mathfrak{v},\mathfrak{O},\tilde{\Sigma},\phi}(X) \in K^*(\tilde{\Sigma})_0[X], \tag{30}$$

where $K^*(\tilde{\Sigma})_0$ is the totally real subfield of the reflex field $K^*(\tilde{\Sigma})$ associated to the lifting $\tilde{\Sigma}$ of Σ (cf. §§2, 6.2).

If $S \in G_+(\mathbf{A}) \simeq G'_+(\mathbf{A}(k))$ and S written as a two by two matrix over $\mathbf{A}(k)$ is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, let $\nu = ad - bc \in I_+(k)$, $\mathfrak{a}$ be the $g.c.d.$ of a_f, b_f, c_f, and d_f, and $\mathfrak{U} = \mathfrak{U}(S)$ be the integral ideal $\mathfrak{a}^{-2}\nu$. If $\phi \in \mathfrak{A}(\mathbb{K}, w)$, define $\phi \mid S \in \mathfrak{A}({}^{S}\mathbb{K}, w)$ by $(\phi \mid S)(g) = \phi(gS)$ for $g \in G_+(\mathbf{A})$. Then, following the definition of §2.4(48) of [3], put (where $N = N_{k/\mathbf{Q}}$)

$$\phi \,||\, S = N(\mathfrak{U})^w \cdot \phi \mid S.$$

By Lemma 1 (*loc. cit.*), if $\phi \in \mathfrak{A}(\mathbb{K}, w, R')$, where R' is a finitely generated subring of a number field, then $\phi \,||\, S$ belongs to $\mathfrak{A}({}^{S}\mathbb{K}, w, R'')$, where R'' is integral over R' in some finite extension of that field. (N.B. The second sentence of the proof of Lemma 1 of *loc. cit.* should be corrected to read: "We may write for that $\omega \in \Omega$ such that $S \in G_\omega(\mathbf{A})$ and for each $\omega' \in \Omega$: $\omega^{-1}\omega' S = S'_{\omega' f}\omega' \mathbf{k}$ for some $S'_{\omega'} \in G_+(Q)$, $\mathbf{k} \in \mathbb{K}$." and then S' replaced by $S'_{\omega'}$ in the rest of the proof.)

Fix $S_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G'(\mathbf{A}(k)_f)$ and suppose $S_0 \in \mathfrak{S}(\omega_1)$. We write

$$C_{\mathbb{K}}(S_0) = \mathbb{K} S_0 \mathbb{K} = \bigcup_{j=1}^{N} S_j \mathbb{K}.$$

Then $\det(C_{\mathbb{K}}(S_0)) \subset \det(S_0)\det(\mathbb{K})$, so that by §1, $C_{\mathbb{K}}(S_0) \subset \mathfrak{S}_{\mathbb{K}}(\omega_1)$ and for each $\omega \in \Omega$, $\omega C_{\mathbb{K}}(S_0) \subset \mathfrak{S}_{\mathbb{K}}(\omega\omega_1)$; therefore, $\omega S_j \in S'_{j\omega f}\omega\omega_1 \mathbb{K}$ for some $S'_{j\omega} \in$ $\in G_+(Q)$, $j = 1, \ldots, N$. We introduce the so-called multiplier polynomial for any non-zero-divisor $\phi \in \mathfrak{A}(\mathbb{K}, w)$

$$M_{\phi, S_0}(g)(X) = \prod_{j=1}^{N} (X - (\phi \,||\, S_j/\phi)(g)) = X^N + \sum_{l<N} \mu_{l,\phi,S_0}(g) X^l, \tag{31}$$

where clearly μ_{l,ϕ,S_0} is a modular function for $\mathbb{K}$, and on each component X_ω, it induces a modular function $m_{l,\phi,S_0,\omega}$ for Γ_ω defined by

$$m_{l,\phi,S_0,\omega}(z) = j(g, \mathrm{i.e})^{-w(N-l)}\mu_{l,\phi,S_0}(g\omega), \qquad z \in \mathfrak{H}^n,$$

where $g \in G_+(\mathbb{R})$ is such that $z = g(\mathrm{i.e})$. Let σ_t denote de t-th elementary symmetric function in N variables and observe that for each $\omega \in \Omega$, $j = 1, \ldots, N$, and $g \in G_+(\mathbb{R})$ we have

$$(\phi \mid S_j)(g\omega) = \phi(g\omega S_j) = \phi(gS'^{-1}_{j\omega f}\omega\omega_1) = \phi(S'_{j\omega\infty}g\omega\omega_1),$$

hence, $(\phi \mid S_j)_\omega(g) = \phi_{\omega\omega_1}(S'_{j\omega\infty}g)$, and therefore for $g \in G_+(\mathbb{R})$

$$\begin{aligned}\mu_{l,\phi,S_0,\omega}(g) &= \sigma_{N-l}((\phi \,||\, S_1)_\omega(g), \dots, (\phi \,||\, S_N)_\omega(g))/\phi_\omega(g)^{N-l} \\ &= \sigma_{N-l}(\phi_{\omega\omega_1}(S'_{1\omega\infty}g), \dots, \phi_{\omega\omega_1}(S'_{N\omega\infty}g))\phi_\omega(g)^{l-N}N\mathfrak{A}^{\omega(N-l)}.\end{aligned}$$

From this and from the definitions we have for $z = g(\mathrm{i.e}) \in \mathfrak{H}^n$

$$m_{l,\phi,S_0,\omega}(z) = \sigma_{N-l}((f_{\omega\omega_1} \,||\, S'_{1\omega}/f_\omega)(z), \dots, (f_{\omega\omega_1} \,||\, S'_{N\omega}/f_\omega)(z)). \tag{31'}$$

Now for the present, we assume $S_0 \in G_+(\mathbf{Q})_f$ and $\omega_1 = 1$, and fix the order $\mathfrak{R} \supset \mathfrak{o}$ in K and assume ϕ is non-zero at all the cusps and at all the finite number of double cosets in $V_{\mathbb{K}}$ representing points of $\Xi_{\mathbf{A}}(\mathfrak{R})$. Then on each component $X_\omega = X_{\mathfrak{v}}$, $m_{l,\phi,S_0,\omega}$ is holomorphic at all the cusps and at all the points of $\Xi_{\mathfrak{v}}(\mathfrak{R})$. Let f_ω be the modular form for Γ_ω induced on X_ω by ϕ. Then as a polynomial with meromorphic coefficients on $\mathfrak{H}^n$, the multiplier polynomial takes the form

$$M_{f_\omega,S_0,\omega}(z)(X) = \prod_{j=1}^{N} (X - N(\mathfrak{U})^w(f_\omega \mid S'_{j\omega}/f_\omega)(z)).$$

We now wish to consider the roots of this polynomial for certain $z \in \Xi_{\mathfrak{v}}(\mathfrak{R})$ and for a conveniently chosen double coset $C_{\mathbb{K}}(S_0)$, for the particular case $\mathbb{K} = G(\hat{\mathbb{Z}})$.

6.2. We refer to the notation of §3.3 of [3]. Let $\tilde{\Sigma} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$ be a lifting of Σ, where each $\tilde{\sigma}_j$ is an injection of K into $\mathbf{C}$. Let $\theta \in \Theta$ be one of the double coset representatives appearing in (6), let $\omega = \theta$, $\mathbb{K} = G(\hat{\mathbb{Z}})$, and $v = \mathrm{id.}(\det(\omega)^{-1})$. Then, as in §3.3 of [3], we let $h' = h_{\mathfrak{v},\tilde{\Sigma}}$ be the number of orbits of $\Gamma_{\mathfrak{v}}^+$ in $\Xi_{\mathfrak{v}}(\mathfrak{O}, \tilde{\Sigma})$, and form the polynomial $P_{\mathfrak{v},\mathfrak{O},\tilde{\Sigma},\phi}$ where ϕ is a $\mathbf{Q}$-arithmetic modular function holomorphic on $o'_1, \dots, o'_{h'}$. Our purpose is to show that for all such ϕ this is a polynomial with coefficients in the totally real subfield $K^*(\tilde{\Sigma})_0$ of the reflex field $K^*(\tilde{\Sigma}) = \mathbf{Q}(\{\sum_{\sigma\in\Sigma}\eta^{\tilde{\sigma}} \mid \eta \in K\})$. Showing this is evidently equivalent to showing that the image $A_{\mathfrak{v}}(\mathfrak{O}, \tilde{\Sigma})$ of $\Xi_{\mathfrak{v}}(\mathfrak{O}, \tilde{\Sigma})$ in $V_{\mathfrak{v}}^*$ is a zero cycle rational over $K^*(\tilde{\Sigma})_0$. Recall that $V_{\mathfrak{v}}^*$ is itself defined over $\mathbf{Q}$.

To show this, we need to consider certain of the roots of

$$M_{\eta,S_0,\theta}(\xi)(X)$$

for $\xi \in \Xi_{\mathfrak{v}}(\mathfrak{O})$, suitable $C(S_0) = C_{G(\hat{\mathbb{Z}})}(S_0)$, and a suitably chosen modular form η with respect to $\Gamma_{\mathfrak{v}}^+$.

Let $\mathfrak{p}$ be a prime ideal of first degree in $\mathfrak{o}$ and unramified over $\mathbf{Q}$ such that $\mathfrak{p}$ splits, $\mathfrak{p} = \mathfrak{P} \cdot \bar{\mathfrak{P}}$ in the maximal order $\mathfrak{O}$ of K. Let L be the smallest Galois extension of $\mathbf{Q}$ containing K and assume that the rational prime p contained in $\mathfrak{P}$ does not ramify in L. Denote by $\mathfrak{P}$ also the extension of $\mathfrak{P}$ to an ideal in the maximal order of L, and let H be the Galois group of L over K and G be the (absolute) Galois group of L over $\mathbf{Q}$. Let $\mathfrak{P} = \mathfrak{Q}_1 \cdot \cdots \cdot \mathfrak{Q}_m$ be the factorization of $\mathfrak{P}$ into prime ideals in the maximal order of L, let f be the degree of each of these over K, and h be the order of H. Then $h = mf$, f is the order of the decomposition group of each $\mathfrak{Q}_j$ over K, and is also the order of the decomposition group of each $\mathfrak{Q}_j$ over $\mathbf{Q}$, since the absolute degree of $\mathfrak{P}$ is 1, and H permutes $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m$ transitively; therefore, H consists of precisely those $\sigma \in G$ for which σ carries any $\mathfrak{Q}_i$ into the same or some other $\mathfrak{Q}_j$, $1 \leqslant i, j \leqslant m$, so that if $\sigma \in G - H$, then the ideals $\mathfrak{P}$ and $\sigma\mathfrak{P}$ are prime to each other. Let a be a positive integer such that $\mathfrak{P}^a$ is principal, say $\mathfrak{P}^a = (\Pi)$. Then $\sigma \in G - H$ also implies that Π and Π^σ are relatively prime to each other. Thus, if $\tilde{\Sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ is any lifting of Σ and if s denotes complex conjugation, the elements

$$\Pi^{\tilde{\sigma}_1}, \ldots, \Pi^{\tilde{\sigma}_n}, \qquad s \cdot \Pi^{\tilde{\sigma}_1}, \ldots, s \cdot \Pi^{\tilde{\sigma}_n} \tag{33}$$

are the images of Π under representatives of the distinct $2n = [K : Q]$ cosets of H in G, and are therefore pairwise relatively prime. Let

$$N_{\tilde{\Sigma}}(\Pi) = \prod_{\tilde{\sigma} \in \tilde{\Sigma}} \Pi^{\tilde{\sigma}},$$

and suppose $\tilde{\Sigma}'$ is some lifting of Σ to K distinct from $\tilde{\Sigma}$ and from $s \cdot \tilde{\Sigma}$. If $\sigma \in \Sigma$, let $\tilde{\Sigma}(\sigma) = \tilde{\sigma}$ be the element of $\tilde{\Sigma}$ which extends σ to K. Let Σ' be the set of $\sigma \in \Sigma$ such that $\tilde{\Sigma}(\sigma) = \tilde{\Sigma}'(\sigma)$ and Σ'' be the set of $\sigma \in \Sigma$ such that $\tilde{\Sigma}(\sigma) = s \cdot \tilde{\Sigma}'(\sigma)$; then $\Sigma = \Sigma' \cup \Sigma''$ and Σ' and Σ'' are both non-empty. Let

$$\mathrm{M} = \mathrm{M}(\tilde{\Sigma}) = \prod_{\sigma \in \Sigma'} \Pi^{\tilde{\sigma}}, \qquad \mathrm{M}' = \mathrm{M}'(\tilde{\Sigma}) = \prod_{\sigma \in \Sigma''} \Pi^{\tilde{\sigma}}.$$

Lemma 3. $N_{\tilde{\Sigma}}(\Pi) = \mathrm{M} \cdot \mathrm{M}'$ *and, under the assumption* $\tilde{\Sigma}' \neq \tilde{\Sigma}, s \cdot \tilde{\Sigma}$, *we have for any positive integer* k

$$N_{\tilde{\Sigma}}(\Pi)^k + N_{s \cdot \tilde{\Sigma}}(\Pi)^k \neq N_{\tilde{\Sigma}'}(\Pi)^k + N_{s \cdot \tilde{\Sigma}'}(\Pi)^k.$$

PROOF. The first assertion follows from the definitions of $N_{\tilde{\Sigma}}(\Pi)$, M, and M′. Moreover we then see that $N_{s \cdot \tilde{\Sigma}}(\Pi) = s\mathrm{M} \cdot s\mathrm{M}'$, $N_{\tilde{\Sigma}'}(\Pi) = \mathrm{M} \cdot s\mathrm{M}'$, $N_{s \cdot \tilde{\Sigma}'}(\Pi) = s\mathrm{M} \cdot \mathrm{M}'$ and therefore.

$$N_{\tilde{\Sigma}}(\Pi)^k + N_{s \cdot \tilde{\Sigma}}(\Pi)^k - N_{\tilde{\Sigma}'}(\Pi)^k - N_{s \cdot \tilde{\Sigma}'}(\Pi) = (\mathrm{M}^k - s\mathrm{M}^k)(\mathrm{M}'^k - s \cdot \mathrm{M}'^k).$$

For the lemma to be false, one would have to have either $\mathrm{M}^k = (s \cdot \mathrm{M})^k$ or $\mathrm{M}'^k = (s \cdot \mathrm{M}')^k$. However, Σ' and Σ'' are non-empty. Therefore, by the discussion preceding the lemma, neither equality can occur since the two sides in each case have distinct prime ideal factorizations in L because s is in the center of G (since L is a CM-field) and takes any set of n of the $2n$ quantities in (33) onto its complement. (This argument occurs in Hecke's thesis [13], §12, for the case when k is a real quadratic field, but the idea is the same.)

6.3. Let $\xi \in \Xi_{\mathfrak{d}}(\mathfrak{O}, \tilde{\Sigma})$ so that $\mathfrak{d}\xi + \mathfrak{o}$ is a fractional ideal of the maximal order $\mathfrak{O}$ in K. (Cf. [3], §3.1) Since $\Pi \in \mathfrak{O}$, where Π is as in §6.2, we have

$$\begin{aligned} \Pi\xi &= \alpha\xi + \beta, & s\Pi \cdot \xi &= \alpha'\xi + \beta', \\ \Pi \cdot 1 &= \gamma\xi + \delta & s\Pi \cdot 1 &= \gamma'\xi + \delta', \end{aligned} \tag{34}$$

where α, δ, α', $\delta' \in \mathfrak{o}$, β, $\beta' \in \mathfrak{d}^{-1}$, γ, $\gamma' \in \mathfrak{d}$ are such that $\alpha\delta - \beta\gamma = \alpha'\delta' - \beta'\gamma' = N_{K/k}(\Pi) = \pi \gg 0$, while $g.c.d.(\alpha, \mathfrak{d}\beta, \mathfrak{d}^{-1}\gamma, \delta) = g.c.d.(\alpha', \mathfrak{d}\beta', \mathfrak{d}^{-1}\gamma', \delta') = (1)$. Let S_0, $S_0' \in R_{\mathfrak{d}}$ be defined by

$$S_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad S_0' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Then S_0, $S_0' \in T_{\mathfrak{d}}(\pi)$ (as defined in §3.1 of [3]) and (under linear fractional operation) we have

$$S_0 \cdot (\xi) = S_0' \cdot (\xi) = (\xi), \tag{35}$$

so that the image of (ξ) in $V_\omega = V_{\mathfrak{d}}$ is a fixed point of the Hilbert modular correspondence associated to $T_{\mathfrak{d}}(\pi)$. Moreover, $\mathfrak{U}(S) = (\pi)$ for all $S \in T_{\mathfrak{d}}(\pi)$.

So we let ϕ be a $\mathbf{Q}$-arithmetic modular form of weight $w \equiv 0 \bmod 2$ and f_ω be the holomorphic modular form with respect to Γ_ω it induces on each component X_ω, and P, p, Π, π, S_0, and S_0' be as just now described. As S runs over a set of representatives $S_1, \ldots, S_N$ of the right cosets of $\Gamma_{\mathfrak{d}}^+$ contained in

$$C_{\mathfrak{d}}(S_0) = C_{\mathfrak{d}}(S_0') = \Gamma_{\mathfrak{d}}^+ S_0 \Gamma_{\mathfrak{d}}^+, \tag{36}$$

then

$$\mu_S(z) = N(\mathfrak{U})^w((f_\omega \mid S)(z)/f_\omega(z)) \tag{37}$$

runs over the roots of $M_{f_\omega, S_0, \omega}(z)(X)$ as functions of $z \in \mathfrak{H}^n$, where we continue to assume f_ω is non-vanishing on all $\xi \in \Xi_{\mathfrak{d}}(\mathfrak{O})$. Let $\Xi = \Xi_{\mathfrak{d}}(\mathfrak{O})$ and $\Xi(\tilde{\Sigma}) = \Xi_{\mathfrak{d}}(\mathfrak{O}, \tilde{\Sigma})$. By definition, $f_\omega \mid S$ takes the form

$$(f_\omega \mid S)(z) = f_\omega(S \cdot z) j(S, z)^w,$$

hence, if $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mu_S(z)$ is equal to

$$\begin{aligned} &N(\mathfrak{U} \det(S))^w f_\omega(S \cdot z) f_\omega(z)^{-1} N(cz + d)^{-2w} = \\ &\qquad = N_{k/\mathbf{Q}}(\pi)^{2w} f_\omega(S \cdot z) f_\omega(z)^{-1} N(cz + d)^{-2w}. \end{aligned} \tag{38}$$

Let $f_1, \ldots, f_m \in \mathfrak{M}(\Gamma_{\mathfrak{v}}^+, \mathcal{O}, \{\varkappa\})$ be $\mathbf{Q}$-arithmetic modular functions generating the field of $\mathbf{Q}$-arithmetic modular functions with respect to $\Gamma_{\mathfrak{v}}^+$ such that each is holomorphic on

$$\Xi(S_0) = \Xi \cup T_{\mathfrak{v}}(S_0)\Xi,$$

($T_{\mathfrak{v}}(S_0)$ being the correspondence on V_{ω}^* associated to $C_{\mathfrak{v}}(S_0) = C_{\mathfrak{v}}(S_0')$), and such that if $\xi \in \Xi$ and $z \in \mathfrak{H}^n - \Xi$, then there is an index $1 \leqslant j \leqslant m$ such that f_j is holomorphic at z also and $f_j(z) \neq f_j(\xi)$. We may assume $f_1, \ldots, f_m$ are the affine coordinates on an affine $\mathbf{Q}$-open subset U of V_{ω}^* containing $\Xi(S_0)$ and all the cusps. Since V_{ω}^* is a $\mathbf{Q}$-normal projective variety, U is a $\mathbf{Q}$-normal affine variety and we may assume that if g is any $\mathbf{Q}$-arithmetic modular function regular at all $\xi \in \Xi(S_0)$ and at all cusps $\varkappa$, then g may be written as a quotient of polynomials in $f_1, \ldots, f_m$ with coefficients in $\mathbf{Q}$ such that the denominator does not vanish at any $\xi \in \Xi(S_0)$ or at any cusp $\varkappa$. Let $u_1, \ldots, u_M$ be indeterminates (at first), define

$$F_u(z) = \sum_{j=1}^{m} u_j f_j(z), \qquad z \in \mathfrak{H}^n,$$

and define the polynomial

$$G_{\pi,u}(X; \{f_j\}) = \prod_{i=1}^{N} (X - F_u \mid S_i),$$

where $S_1, \ldots, S_N$ are as above and, of course, $(F_u \mid S)(z) = F_u(Sz)$. $G_{\pi,u}$ is a polynomial in X, $u_1, \ldots, u_m$ whose coefficients are $\mathbf{Q}$-arithmetic modular functions holomorphic on Ξ, hence, expressible as rational functions of $f_1, \ldots, f_m$ with non-vanishing denominators on Ξ. By appropiate choice of ϕ, we may assume it does not vanish on $\Xi(S_0)$, hence μ_{S_i} is holomorphic at every point of Ξ. We then form another polynomial

$$\Phi_{\pi,u}(X, z) = \Phi_{\pi,u}(X) = \prod_{i=1}^{N} (X - (\mu_{S_i}(z) + F_u(S_i z)))$$

whose coefficients as a polynomial in X, $u_1, \ldots, u_m$ are $\mathbf{Q}$-arithmetic modular functions holomorphic on Ξ, hence expressible as rational functions in $f_1, \ldots, f_m$ with denominators that do not vanish for $z = (\xi) \in \Xi$.

For any $(\xi) \in \Xi$, say $(\xi) = (\xi^{\tilde{\sigma}_1}, \ldots, \xi^{\tilde{\sigma}_n})$, $\xi \in K - k$, we consider the system of equations

$$\Phi_{\pi,u}(X + F_u(S^*(\xi)), f_1(\xi), \ldots, f_m(\xi)) = 0 \tag{39}$$

depending on $u \in \mathbf{Q}^n$, with $S^* = S_0$ or S_0'. Now

$$S_0((\xi)) = S_0'((\xi)) = (\xi) \tag{40}$$

because of (35) (apply $\tilde{\sigma}_j \in \tilde{\Sigma}$ to (35) for $j = 1, \ldots, n$); therefore, $F_u((S_0(\xi))) = = F_u((S_0'(\xi))) = F_u((\xi))$. So consider the roots of the polynomial $\psi(X, u; (\xi))$ defined to be

$$\Phi_{\pi, u}(X + F_u(\xi), f_1(\xi), \ldots, f_m(\xi)).$$

For all complex $u = (u_1, \ldots, u_m)$, $\mu = \mu_{S_0}((\xi))$ and $\mu' = \mu_{S_0'}((\xi))$ are roots of it. Written out, we have

$$\Phi_{\pi, u}(X) = X^N + \sum_{\nu < N} P_\nu(u_1, \ldots, u_m; z) x^\nu,$$

where the coefficients of $P_\nu(u_1, \ldots, u_m; z)$ as a polynomial in $u_1, \ldots, u_m$ are $\mathbf{Q}$-arithmetic modular functions of $z \in \mathfrak{H}^n$ having no singularities on the set Ξ. The roots of $\Phi_{\pi, u}(X; (\xi))$ are $\mu_L((\xi)) + F_u((L\xi))$, $L \in C_{\mathfrak{d}}(S_0)$, which equals $\mu_L(\xi) + F_u(\xi)$ for all u if and only if $L(\xi)$ is in the $\Gamma_{\mathfrak{d}}^+$-orbit of (ξ). But suppose $L' \in C_{\mathfrak{d}}(S_0)$, so that $\det(L') = \eta'\pi \gg 0$ for some totally positive unit η' and so that L' is everywhere locally a *primitive* element of the $\mathfrak{o}$-lattice $R_{\mathfrak{d}}$. Suppose also that $L'(\xi) \in \Gamma_{\mathfrak{d}}^+(\xi)$, $L'(\xi) = \sigma^{-1}(\xi)$, $\sigma \in \Gamma_{\mathfrak{d}}^+$ or $L(\xi) = (\xi)$, where $L = \sigma L' = = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, say, and $\det L = \eta\pi \gg 0$, $\eta \in \mathfrak{o}_+^{\times}$. As before this implies there exists $M \in \mathbf{C}$ such that

$$\begin{aligned} M \cdot \xi &= \alpha\xi + \beta \\ M &= \gamma\xi + \delta. \end{aligned} \tag{41}$$

By assumption, $\xi \in \Xi = \Xi_{\mathfrak{d}}(\mathfrak{O})$, so that $M \in \mathfrak{O}$, and $M \cdot sM = \alpha\delta - \beta\xi = \eta\pi$ (where, as usual, s is complex conjugation). Then the prime factorization of M is of the form $\mathfrak{P}^b \cdot s\mathfrak{P}^c$, $b, c \geqslant 0$, $b + c = a$. But if $bc \neq 0$, M would be devisible by $\mathfrak{p}$, hence at the prime $\mathfrak{p}$, $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ would not be a primitive element of the lattice $R_{\mathfrak{d}\mathfrak{p}}$, contrary to definition of $C_{\mathfrak{d}}(S_0)$. Hence $(M) = \mathfrak{P}^a$ or $s\mathfrak{P}^a$ and so we may assume $M = \Pi$ or $s \cdot \Pi$, hence $L = S_0$ or S_0'; in other words, $L' \in \Gamma_{\mathfrak{d}}^+ \cdot S_0$ or $L' \in \Gamma_{\mathfrak{d}}^+ \cdot S_0'$. Then the roots of $\Psi(X; u; (\xi))$ as a polynomial in X are

$$\rho_L = \mu_L(\xi) + F_u(L(\xi)) - F_u(\xi), \qquad L \in C_{\mathfrak{d}}(S_0).$$

If $F_u(L\xi) - F_u(\xi)$ is not identically zero as a function of u, i.e., if $f_i(\xi) \neq f_i(L\xi)$ for some i, $1 \leqslant i \leqslant m$, then there is $u \in \mathbf{C}^m$ for which the above root ρ_L will not be equal to any root of

$$\Psi(X; 0; (\xi)) = M_{\pi, f_\omega}(X; (\xi)).$$

Hence the only common roots of $\Psi(X; u; (\xi))$ for all u are

$$\mu_1 = \rho_{S_0}, \qquad \mu_2 = \rho_{S_0'}. \tag{42}$$

The values of u for which other roots in common with $\Psi(X; 0; (\xi))$ exist then form a proper Zariski-closed subset $Z(\xi)$ of u-space depending on ξ. Therefore there exists $(u_0) = (u_1, \ldots, u_m)_0 \in \mathbf{Q}^m$ such that $\Psi(X; 0: (\xi))$ and $\Psi(X : u_0; (\xi))$ have monic *g.c.d.* of degree two having only the two roots μ_1, μ_2 in common. Since the number of $\Gamma_{\mathfrak{v}}^+$-orbits among such (ξ) is finite, we may assume u_0 is always the same. By the Weber-Perron theorem [32, 23], the monic *g.c.d.* of these two polynomials has coefficients which are rational functions of $f_1(\xi), \ldots, f_m(\xi)$ defined over $\mathbf{Q}$; thus,

$$\mu_1 + \mu_2 = P(f_1(\xi), \ldots, f_m(\xi))/Q(f_1(\xi), \ldots, f_m(\xi)), \tag{43}$$

where the polynomials P and Q belong to $\mathbf{Q}[X_1, \ldots, X_M]$ and are independent of $\xi \in \Xi$ and $Q(f_1(\xi), \ldots, f_m(\xi)) \neq 0$.

We now obtain expressions for the roots μ_1, μ_2. Take $\xi \in K - k$ as above such that $(\xi) \in \Xi$, $A = \mathfrak{v}\xi + \mathfrak{o}$ being a fractional $\mathfrak{O}$-ideal of K, and let $\Pi \in \mathfrak{O}$ be a fixed generator of $\mathfrak{P}^a$ as before. We use the relations (34) and the fact, just observed, that

$$g.c.d.(\alpha, \mathfrak{v}\beta, \mathfrak{v}^{-1}\gamma, \delta) = g.c.d.(\alpha', \mathfrak{v}\beta', \mathfrak{v}^{-1}\gamma', \delta') = (1).$$

Then $\Pi = \gamma\xi + \delta, s \cdot \Pi = \gamma' \cdot \xi + \delta'$, and according to (38) we have (since $(\xi) = (\xi^{\tilde{\sigma}_1}, \ldots, \xi^{\tilde{\sigma}_n})$ if $\tilde{\Sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ and $\xi \in \Xi(\tilde{\Sigma})$)

$$\begin{aligned}\mu_S(\xi) = N(\pi)^{2w} f_\omega(S \cdot \xi) f(\xi)^{-1} \prod_{\tilde{\sigma} \in \tilde{\Sigma}} (c^\sigma \xi^{\tilde{\sigma}} + d^\sigma)^{-2w} &= \\ = N_{k/\mathbf{Q}}(\pi)^{2w} N_{\tilde{\Sigma}}(\Pi)^{-2w} &= N_{s\tilde{\Sigma}}(\Pi)^{2w}\end{aligned}$$

because $\pi = \Pi \cdot s\Pi$. Similarly, $\mu_{S'}(\xi) = N_{\tilde{\Sigma}(\Pi)^{2w}}$; therefore,

$$\mu_1 + \mu_2 = N_{\tilde{\Sigma}}(\Pi)^{2w} + N_{s\tilde{\Sigma}}(\Pi)^{2w},$$

which depends only on $\tilde{\Sigma}$ and Π, and not on ξ.

As before, L is the Galois closure of K over $\mathbf{Q}$. Then $K^*(\tilde{\Sigma}) \subset L$ and it is known [25, §5] that L is a CM-field, too, and that the automorphism s of complex conjugation is in the center of $G = \mathrm{Gal}(L/\mathbf{Q})$. For $\eta \in K$, define $\theta(\eta) = \sum_{\tilde{\sigma} \in \tilde{\Sigma}} \eta^{\tilde{\sigma}}$. Then θ is just the trace of the representation $\bigoplus_{\tilde{\sigma} \in \tilde{\Sigma}} \tilde{\sigma}$ of the $\mathbf{Q}$-algebra K. Therefore $K^*(\tilde{\Sigma})$ contains all the determinants

$$N_{\tilde{\Sigma}}(\eta), \qquad \eta \in K,$$

of that representation. This shows that μ_1 and μ_2 both belong to $K^*(\tilde{\Sigma})$ Moreover, $\mu_2 = s\mu_1$, hence $\mu_1 + \mu_2 \in K^*(\tilde{\Sigma})_0$. Now we have shown that

$$\mu_1 + \mu_2 = R(f_1(\xi), \ldots, f_m(\xi))$$

and the left side is an element $A(\Pi, \tilde{\Sigma})$ of $K^*(\tilde{\Sigma})_0$. We also write $\mu_i = \mu_i(\tilde{\Sigma})$, $i = 1, 2$, for fixed Π.

Clearly $\Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma}) = \Xi_{\mathfrak{d}}(\mathcal{O}, s \cdot \tilde{\Sigma})$. Suppose now that $\tilde{\Sigma}'$ is another lifting of Σ such that $\Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma}')$ is non-empty and $\tilde{\Sigma}' \neq \tilde{\Sigma}$, $s\tilde{\Sigma}$.

Proposition 4. *If $\tilde{\Sigma}' \neq \tilde{\Sigma}$, $s \cdot \tilde{\Sigma}$, then $A(\Pi, \tilde{\Sigma}') \neq A(\Pi, \tilde{\Sigma})$.*

PROOF. This is the same statement as that of Lemma 3 in different notation. Consider then the equation

$$A(\Pi, \tilde{\Sigma}) = P(f_1(\xi), \dots, f_m(\xi))/Q(f_1(\xi), \dots, f_m(\xi)),$$

where $P(X_1, \dots, X_m)$ and $Q(X_1, \dots, X_m) \in \mathbf{Q}[X_1, \dots, X_m]$ and are independent of $\xi \in \Xi$, and $Q(f_1(\xi), \dots, f_m(\xi)) \neq 0$ for all $\xi \in \Xi$. Let

$$U_{\tilde{\Sigma}}(X_1, \dots, X_m) = P(X_1, \dots, X_m) - A(\Pi, \tilde{\Sigma})Q(X_1, \dots, X_m).$$

Then $U_{\tilde{\Sigma}}$ has coefficients in $K^*(\tilde{\Sigma})_0$, and the set of points where it vanishes cuts out a hypersurface section $V(\Pi, \tilde{\Sigma})$ on the $\mathbf{Q}$-open affine subset U of $V_{\mathfrak{d}}^*$ such that $V(\Pi, \tilde{\Sigma})$ is itself defined over $K^*(\tilde{\Sigma})_0$ and such that $V(\Pi, \tilde{\Sigma})$ intersects $A_{\mathfrak{d}}(\mathcal{O})$, the image of $\Xi_{\mathfrak{d}}(\mathcal{O})$ in $V_{\mathfrak{d}}^*$, in the image $A_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$ of $\Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$ because according to Proposition 4, $V(\Pi, \tilde{\Sigma}')$ cannot meet $A_{\mathfrak{d}}(\Pi, \tilde{\Sigma})$ if $\tilde{\Sigma}' \neq \tilde{\Sigma}, s\tilde{\Sigma}$. According to Theorem 4 of [3], $A_{\mathfrak{d}}(\mathcal{O})$ is a $\mathbf{Q}$-set (in the statement of that theorem, the first Ξ was mistakenly put in place of A). Therefore we have the following theorem, due to Hecke in the case of real quadratic k,

Theorem 1. *The finite zero-cycle $A_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$ on $V_{\mathfrak{d}}^*$ is defined over $K^*(\tilde{\Sigma})_0$. Therefore if f is any $K^*(\tilde{\Sigma})_0$-arithmetic modular function for $\Gamma_{\mathfrak{d}}^+$ which is holomorphic on $\Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$, then*

$$P_{\mathfrak{d}, \mathcal{O}, \tilde{\Sigma}, f}(X) \in K^*(\tilde{\Sigma})_0[X].$$

Now let $A(\mathcal{O}) = \bigcup_{\mathfrak{d}} A_{\mathfrak{d}}(\mathcal{O})$ and $A(\mathcal{O}, \tilde{\Sigma})$ be respectively the images of $\bigcup_{\mathfrak{d}} \Xi_{\mathfrak{d}}(\mathcal{O})$ and of $\bigcup_{\mathfrak{d}} \Xi_{\mathfrak{d}}(\mathcal{O}, \tilde{\Sigma})$ in $V_{\hat{G}(\mathbb{Z})}^* = \bigcup_{\mathfrak{d}} V_{\mathfrak{d}}^*$ and if $\mathfrak{n}$ is any integral ideal of k, let $A_{\mathfrak{n}}(\mathcal{O})$ and $A_{\mathfrak{n}}(\mathcal{O}, \tilde{\Sigma})$ denote respectively the corresponding images of the same sets in $V_{\mathbb{K}(\mathfrak{n})}^*$. Since $A_{\mathfrak{n}}(\mathcal{O})$ is the pre-image of $A(\mathcal{O})$ under the canonical morphism π: $V_{\mathbb{K}(\mathfrak{n})}^* \to V_{\hat{G}(\hat{\mathbb{Z}})}^*$ (defined via inclusion of double cosets), and π is defined over $\mathbf{Q}$ with respect to the $\mathbf{Q}$-structure defined earlier on $V_{\mathbb{K}(\mathfrak{n})}^*$, it follows that as an algebraic zero-cycle on $V_{\mathbb{K}(\mathfrak{n})}^*$, $A_{\mathfrak{n}}(\mathcal{O})$ is defined over $\mathbf{Q}$, while $A_{\mathfrak{n}}(\mathcal{O}, \tilde{\Sigma})$ is defined over $K^*(\tilde{\Sigma})_0$.

Let $L(\tilde{\Sigma})$ be the minimal field containing $K^*(\tilde{\Sigma})$ over which each of the points of $A(\mathcal{O}, \tilde{\Sigma})$ is rational and $L_{\mathfrak{n}}(\tilde{\Sigma})$ be the minimal field containing $K^*(\tilde{\Sigma})$ over which each of the points of $A_{\mathfrak{n}}(\mathcal{O}, \tilde{\Sigma})$ is rational. Each of these is a normal extension of $K^*(\tilde{\Sigma})$ because it is the splitting field of a polynomial over $K^*(\tilde{\Sigma})$. in fact, each is an abelian extension. Let $G(\tilde{\Sigma}) = \mathrm{Gal}(L(\tilde{\Sigma})/K^*(\tilde{\Sigma}))$ and $G_{\mathfrak{n}}(\tilde{\Sigma}) = \mathrm{Gal}(L_{\mathfrak{n}}(\tilde{\Sigma})/K^*(\tilde{\Sigma}))$. Then, following the same idea as utilized in

Hecke's thesis, one may see that $G(\tilde{\Sigma})$ is isomorphic to a certain subgroup of the group of those ideal classes C of K for which $N_{K/k}(C)$ lies in the narrow principal class of k. In [16] there is a generalization of the same idea for the classical (one-variable, $n = 1$) case (of elliptic modular functions) to the situation where one considers modular functions for principal congruence subgroups of the modular group and the extension of an imaginary quadratic field which their special values generate, which is closely related to Hasse's paper [12]; one of Karel's results [16] says that, without using the theory of elliptic functions as such, one may show such extensions are abelian (cf. §5 of [16]) with Galois group isomorphic to a subgroup of a certain ray class group. It is possible, using the results we have proved, and without using the theory of abelian varieties, to show that $G_{\mathfrak{n}}(\tilde{\Sigma})$ is also abelian and isomorphic to a subgroup of the ray class group mod $\mathfrak{n}$ of K. We intend to provide further details of this in a later paper.

One may also obtain the reciprocity law for the extension $L_{\mathfrak{n}}(\tilde{\Sigma})/K^*(\tilde{\Sigma})$, in form very similar to that of Shimura (with some possible modifications connected with the units of k). This is already indicated in Karel's paper. To deal with the reciprocity law we need a certain q-expansion principle to be proved in the next section. Other details will be supplied in a subsequent publication. We should like, however, to emphasize at this point the important influence on all these developments of Hecke's original ideas.

7. A q-expansion principle

We use the notation of [1]. In particular, Γ denotes an arithmetic group acting on a rational tube domain $\mathfrak{T}$, $V = \Gamma\backslash\mathfrak{T}$, and V^* is the Satake compactification of V. Let k be a number field and make the Assumptions 1′ and 2′, p. 649 of [1], namely that

(1′) $$\mathfrak{A}^{(d_0)}(\Gamma) = \mathfrak{A}_{k,\infty}^{(d_0)}(\Gamma) \otimes_k \mathbf{C}$$

for some positive integer d_0, where $\mathfrak{A}_{k,\infty}^{(d_0)}(\Gamma)$ denotes the graded algebra of modular forms for Γ of weights $\equiv 0 \bmod d_0$ having the coefficients of their Fourier expansions at ∞ in k, and

(2′) If $f \in \mathfrak{A}_{k,\infty}(\Gamma)$, then only finitely many primes divide the denominators of the coefficients of the Fourier expansion of f(*at* ∞).

Denote by $\mathfrak{o}$ the ring of integers of k. Then according to Theorem B, *loc. cit.*, there exist a positive integer d_0 and a finite set $\mathfrak{S}$ of prime ideals of $\mathfrak{o}$ such that if $F = \mathfrak{o}[\mathfrak{S}^{-1}]$, then the graded algebra $\mathfrak{A}_{R,\infty}^{(d_0)}(\Gamma)$ of modular forms (with respect to Γ) of weights $\equiv 0 \bmod d_0$ having all Fourier coefficients in R is finitely generated as graded algebra over R, and in fact by a finite set $(b_0, b_1, \ldots, b_M)$ of elements of weight d_0. Since the modular forms with Fourier coefficients in R actually span the graded algebra of all modular forms as a vector space, we may trivially replace R by any larger finitely generated subring of k.

Let $\xi \in \mathfrak{T}$ be a point where all k-arithmetic modular functions holomorphic at ξ take values in a fixed algebraic number field K/k (of finite degree over k). In other words, the image x of ξ in V^* is a K-rational point of V^*, V^* itself being defined over $k \subset K$.

If f is a k-arithmetic modular function, or, equivalently, a rational function on V^* defined over k, and if $y \in V^*(K)$ for some finite extension K of k, we say f is defined and finite mod $\mathfrak{p}$ at y, for a prime ideal $\mathfrak{p}$ of k if:

a) f is defined and finite at y in the usual sense (and then $f(y) \in K$); and

b) y belongs to a k-open affine subset U of V^* such that for some system $\alpha_1, \ldots, \alpha_M$ of affine coordinates on U, $\alpha_1(y), \ldots, \alpha_M(y)$ are integral over the valuation ring of $\mathfrak{p}$ in k and f may be expressed in the form

$$f = P(\alpha_1, \ldots, \alpha_M)/Q(\alpha_1, \ldots, \alpha_M),$$

where $P(X_1, \ldots, X_M)$ and $Q(X_1, \ldots, X_M)$ belong to $\mathfrak{o}[X_1, \ldots, X_M]$ and are such that $Q(\alpha_1(y), \ldots, \alpha_M(y))$, which belongs to K and is integral over the localization $\mathfrak{o}_{(\mathfrak{p})}$ of $\mathfrak{o}$ at $\mathfrak{p}$, is not divisible (locally) by any prime ideal $\mathfrak{P}$ extending $\mathfrak{p}$ to K. (In particular, $f(y)$ is integral over $\mathfrak{o}_{(\mathfrak{p})}$.)

According to [32, §§9-10; 27, 4(iii)], for every prime $\mathfrak{p}$ of k, V^* defines a $\mathfrak{p}$-variety. Let us fix some covering of V^* by a system of affine coordinate neighborhoods (in the Zariski topology). Then by Proposition 23 of [32], for almost all $\mathfrak{p}$ these provide a covering by affine coordinate systems of the $\mathfrak{p}$-variety associated to V^*. By the nature of the definition of $\mathfrak{p}$-variety, for f to be defined and finite mod $\mathfrak{p}$ at $y \in V^*(K)$, it does not matter which system of affine coordinates for the $\mathfrak{p}$-variety one uses for the criterion for f to be defined and finite mod $\mathfrak{p}$ at $y \in V^*(K)$. Then we have:

Theorem 2. *Let $x \in V^*(K)$ be the image of $\xi \in \mathfrak{T}$. Then there is a finite set $\mathfrak{S}'$ of primes $\mathfrak{p}$ of k with the following property: Let f be a k-arithmetic modular function for Γ. Suppose $\mathfrak{p}$ is a prime ideal of k, $\mathfrak{p} \notin \mathfrak{S}'$, and suppose f is defined and finite* mod $\mathfrak{p}$ *at x and defined and finite at the image $\bar{\infty}$ of the* *cusp at ∞ in V^*, and such that all the coefficients of the Fourier expansion of f at ∞ are in the maximal ideal of the valuation ring of $\mathfrak{p}$. Then*

$$f(x) \equiv 0 \bmod \mathfrak{p},$$

the congruence being taken in the intersection of the valuation rings of K containing $\mathfrak{p}$.

PROOF. With $b_0, \ldots, b_M \in \mathfrak{A}_{R,\infty}(\Gamma)_{d_0}$ being as before, we may assume, possibly after adding a finite set of primes to S, that $b_0 \equiv 1 \bmod \mathfrak{F}_L$ for some suitably large L and that $b_0(\xi) \neq 0$. (Cf. [1], Proposition 2.) The congruence means that, in a suitable ordering, all Fourier coefficients of b_0 corresponding to indices less than a certain bound (i.e., all the early terms in the Fourier series) are zero, except for the constant term which is 1. Then $\infty, \xi \in V^*(b_0)$, the affine open subset of V^* on which $b_0 \neq 0$.

Let $\alpha_j = b_j/b_0, j = 1, \ldots, M$, so that $\alpha_1, \ldots, \alpha_M$ is a system of affine coordinates on $V^*(b_0)$; then all the coefficients of the Fourier expansion of each α_j are in $R, j = 1, \ldots, M$. The system of affine coordinates $\alpha_1, \ldots, \alpha_M$ determines the structure of an affine R-scheme on $V^*(b_0)$, and for all but a finite number of primes $\mathfrak{p}$ their reductions mod $\mathfrak{p}$ are a system of affine coordinates on a neighborhood of any specialization ref $\mathfrak{p}$ of x on the reduction mod $\mathfrak{p}$ of $V^*(b_0)$. We may assume $\alpha_j(\xi)$, $\alpha_j(\infty)$ all belong to the integral closure of R in $\bar{Q}$. The statement that f is defined and finite mod $\mathfrak{p}$ at ξ means that

$$f = P(\alpha_1, \ldots, \alpha_M)/Q(\alpha_1, \ldots, \alpha_M) = \mathcal{P}/\mathcal{Q},$$

where P and Q are polynomials in M variables having $\mathfrak{p}$-integral coefficients in R such that $\mathcal{Q}(\xi) \not\equiv 0 \bmod \mathfrak{P}$ for every prime $\mathfrak{P}$ extending $\mathfrak{p}$ in the field generated over k by the coordinates $\{\alpha_j(\xi), \alpha_j(\infty) | j = 1, \ldots, M\}$. Then

$$Q(\alpha_1, \ldots, \alpha_M)f = P(\alpha_1, \ldots, \alpha_M) = b_0^{-N}P^*(b_0, b_1, \ldots, b_M),$$

where $P^*(X_0, X_1, \ldots, X_M)$ is a homogeneous polynomial of degree N in $M + 1$ variables. Thus

$$b_0^N Q(\alpha_1, \ldots, \alpha_M)f = P^*(b_0, \ldots, b_M),$$

which is a modular form of weight Nd_0. By hypothesis the Fourier coefficients at ∞ of $b_0^N Q(\alpha_1, \ldots, \alpha_M)$ all lie in R, and those of f in $\mathfrak{p}R$. By inverting a finite set of primes, we may assume R is a principal ideal domain ([20], Prop. 17, p. 22). Let π be a generator of $\mathfrak{p}R$: $\mathfrak{p}R = \pi \cdot R$. Then the Fourier coefficients of $\pi^{-1}P^*(b_0, \ldots, b_M)$ all lie in R. This means, by the choice of $b_0, \ldots, b_M$, that we may write

$$P^*(b_0, \ldots, b_M) = \pi \cdot P^{\#}(b_0, \ldots, b_M),$$

where $P^{\#}$ is a homogeneous polynomial of degree N with coefficients in R. Then

$$f = \pi \cdot b_0^{-N} P^{\#}(b_0, \ldots, b_M)/Q(\alpha_1, \ldots, \alpha_M) = \\ = \pi \cdot P_1(\alpha_1, \ldots, \alpha_M)/Q(\alpha_1, \ldots, \alpha_M),$$

and P_1 is a polynomial in M variables with coefficients in R. Therefore,

$$f(\xi) = \pi \cdot P_1(\alpha_1(\xi), \ldots, \alpha_M(\xi))/Q(\alpha_1(\xi), \ldots, \alpha_M(\xi)) = \\ \pi \cdot P_1(\alpha_1(\xi), \ldots, \alpha_M(\xi))/\mathfrak{Q}(\xi)$$

which is clearly an expression $\equiv 0 \bmod \mathfrak{p}$. Q.E.D.

8. As a Convenience to the Reader

We list here some minor corrections needed in [3] as a predecessor to this paper:

In the line immediately preceding equation (49) of 2.4 (of [3]), $M_2(\hat{\mathbb{Z}})$ should be $M_2(\hat{\mathfrak{o}})$.

On the next page after that in the fourth line of the proof of Lemma 2, following the last = sign there should be

$$\bigcup_{j=1}^{N} {}^{\omega\omega_1}\mathbb{K} S'^{-1}_{j\omega f} \cap G_+(\mathbf{Q})_f.$$

And on still the very next page after that, the second displayed equation, line eleven from the top, should read

$$a_{\nu\omega}(z) = \sigma_{N-\nu}(f_{\omega\omega_1}(S'^{-1}_{1\omega} \cdot z), \ldots, f_{\omega\omega_1}(S'^{-1}_{N\omega} \cdot z)).$$

These corrections are in addition to other needed corrections pointed out in the course of this paper.

References

[1] Baily, W. L., Jr. Theorems on the Finite Generation of Algebras of Modular Forms, *Amer. J. Math.* **104** (1982), 645-682.

[2] —. On the theory of Hilbert modular functions I., *J. Alg.* **90** (1984), 567-605.

[3] —. Arithmetic Hilbert Modular Forms, *Proc. of International Symposium on Automorphic Forms of Several Variables*, Katata (1983) Birkhäuser.

[4] —. Galois action on Eisenstein series and certain abelian extensions of number fields, *Pre-print*, 1980.

[5] Borel, A. Some Finiteness Properties of Adele Groups over Number Fields, *Publications I. H. E. S.* **16**, 101-126.

[6] —. Linear Algebraic Groups (notes by H. Bass), *W. A. Benjamin,* 1969.

[7] Borovoi, M. V. Langlands Conjecture on Conjugation of the Shimura Varieties, *Functional Anal. Appl.* 1983, 292-293.

[8] Deligne, P. Travaux de Shimura, Sem Bourbaki, 1970/71, exposé 389, *Lecture Notes in Mathematics, Springer Verlag* **244,** 1971.

[9] —. Variétés de Shimura: Interpretation modulaire et techniques de construction de modèles canoniques, *Proc. Symp. Pure Math.* **33-2**, AMS 1979, 247-290.

[10] Deuring, M. Algebraische Begründung der komplexen Multiplikation, *Hamb. Abh.* **16** (1946), 32-47.

[11] Hasse, H. Neue Begründung der komplexen Multiplikation, *Jour. Reine. Angew. Math.* **157** (1927), 115-139.

[12] —. Neue Begründung der komplexen Multiplikation II, *Jour. Reine Angew. Math.* **165** (1931), 64-88.

[13] Hecke, E. Höhere Modulfunktionen und ihre Anwendung auf die Zahlentheorie, *Math. Ann.* **71** (1912), 1-37.

[14] —. Über die Konstruktion Relative Abelscher Zahlkörper durch Modulfunktionen von zwei Variablen, *Math. Ann.* **74** (1913), 465-510.

[15] Karel, M. Eisenstein Series and Fields of Definition, *Compositio Math.,* **37** (1978), 121-169.

[16] —. Special Values of Elliptic Modular Functions, *Proc. of International Symposium on Automorphic Forms of Several Variables*, Katata (1983), Birkhäuser.

[17] Kazhdan, D. On arithmetic varieties, in *Lie Groups and their Representations, Adam Hilger,* London, 1975.

[18] —. On arithmetic varieties II.

[19] Klingen, H. Über den arithmetischen Charakter der Fourierkoeffizienten von Modulformen, *Math. Ann.* **147** (1962), 176-188.

[20] Lang, S. Algebraic Number Theory, *Addison Wesley,* 1970.

[21] Milne, J. The Action of an Automorphism of *C* on a Shimura Variety and its Special Points, *Progress in Math.* **35**, Birkhäuser (1981), 239-265.

[22] — and Shih, K.-Y. Automorphism Groups of Shimura Varieties and Reciprocity Laws, *Am. J. Math.* **103** (1981), 911-935.

[23] —. The action of complex conjugation on a Shimura variety, *Ann. Math.,* **113** (1981), 569-599.

[24] —. Conjugates of Shimura Varieties, in *Lecture Notes in Math.,* **900**, Springer-Verlag 1982.

[25] Miyake, K. Models of certain automorphic function fields, *Acta Math.,* **126** (1971), 245-307.

[26] Perron, O. Über den grösstengemeinsamen Teiler von zwei Polynomen, *Bayer. Akad. Ber.,* 1928.

[27] Shimura, G. Reduction of algebraic varieties with respect to a discrete valuation of the basic field, *Am. J. Math.,* **77** (1955), 134-176.

[28] —. Construction of class fields and zeta functions of algebraic curves, *Ann. Math.,* **85** (1967), 58-159.

[29] —. On canonical models of arithmetic quotients of bounded symmetric domains, *Ann. Math.,* **91** (1970), 144-22; II, **92** (1970), 528-549.

[30] —. On some arithmetic properties of modular forms of one and several variables, *Ann. Math.,* **102** (1975), 491-515.

[31] —. Introduction to the Arithmetic Theory of Automorphic Functions, *Iwanami Shoten Publishers and Princeton University Press*, 1971.
[32] —, and Taniyama, Y. Complex Multiplication of Abelian Varieties and its Applications to Number Theory, *Publ. Math. Soc. Japan,* **6**, 1961.
[33] Shih, K.-Y. Existence of certain canonical models, *Duke Math. Jour.,* **45** (1978) 63-66.
[34] Sugawara, M. Konstruktion gewisser algebraischer Zahlkörper durch die Modul funktionen zweier Variablen I, *Jap. Jour. Math.,* **11** (1934), 133-183.
[35] Taniyama, Y. Jacobian Varieties and Number Fields, *The Complete Works of Yutaka Taniyama, Yutaka Taniyama Complete Works Publication Society,* 1961, 1-68 (originally mimeographed notes, University of Tokyo, September 1955).
[36] Weber, H. Lehrbuch der Algebra (Kleine Ausgabe), Friedrich Vieweg & Sohn, Braunschweig, 1912.

Walter L. Baily, Jr.
University of Chicago
Chicago, Ill. 60637
U.S.A.

Research supported in part by NSF Grant MC8101571-AO1.

Revista Matemática Iberoamericana
Vol. 1, n.° 1, 1985

Polynomial Invariants of 2-component Links

K. Murasugi

1. Introduction

Let $L = X \cup Y$ be an oriented 2-component link in S^3. In this paper, we will define two different types of polynomials which are ambient isotopic invariants of L. One is associated with a cyclic cover branched along one of their components, and the other is associated with a metabelian cover of L. These invariants are defined for any link unless the linking number, $lk(X, Y)$, is ± 1.

The invariants a_n^*, h_n^* defined in [5] can be considered one of the special cases of our polynomial invariants. In fact, we can prove that a_n^* depends only on $lk(X, Y)$; therefore, for all n, a_n^* coincides for two links with the same linking number. (See Theorem 5.7.)

It should be noted that our metabelian representation of the link group differs completely from those studied in [2], [3], [9] or [10], where in most of the cases there exist only finitely many metabelian representations. We will prove in this paper that every 2-component link L with $lk(X, Y) \neq \pm 1$ has infinitely many metabelian coverings. In particular, if $lk(X, Y)$ is even, then the link group $G(L)$ has a representation on the dihedral group of order 2^{k+1} for each $k \geqslant 1$. (See Proposition 3.3 and Theorem 4.1.) Our polynomials are, in fact, the covering linkage invariants associated with these (infinite) sequences of coverings.

In this paper, some of the basic formulas involving Fox free differential calculus [1] will be used without proofs, since they have already been proved in [9] or are easy consequences of the results in [9].

2. Group Actions

Let F be a free group (of rank 2) freely generated by x and y.

Let $P = Z[[t]]$ be the ring of formal power series over the ring of integers Z. Denote by $Sym(S)$ the group of permutations on a set S.

Associated with an ordered sequence $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ of 1 or 2 is an action ϕ of F on P; that is, a homomorphism $\phi\colon F \to Sym(P)$ defined as follows:

$$\phi(x)\left(\sum_{i=0}^{\infty} a_i t^i\right) = a_0 + \sum_{i=1}^{\infty} (a_i + \delta(i)a_{i-1})t^i$$

$$\phi(y)\left(\sum_{i=0}^{\infty} a_i t^i\right) = a_0 + \sum_{i=1}^{\infty} (a_i + (1 - \delta(i))a_{i-1})t^i \tag{2.1}$$

where $\delta(i) = 1$ or 0 according to $j_i = 1$ or 2.

Throughout this paper, we do not distinguish between an action $\phi\colon F \times P \to P$ and the homomorphism $\phi\colon F \to Sym(P)$, associated with ϕ, and therefore the same symbol will be used.

EXAMPLE 2.1. Let $s = \{1, 2, 1, 2, \ldots\}$, where $j_k = 1$ if and only if k is odd. Then for $f(t) = \sum_{i=0}^{\infty} a_i t^i \in P$,

$$\phi(x)(f(t)) = f(t) + \sum_{i=0}^{\infty} a_{2i} t^{2i+1}$$

$$\phi(y)(f(t)) = f(t) + \sum_{i=0}^{\infty} a_{2i+1} t^{2i+2}.$$

Using Fox free derivative [1], we can now express the action ϕ more precisely. The following proposition is essentially Proposition 3.1 in [9].

Proposition 2.1. *Let $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ be an ordered sequence of* 1 *or* 2. *For $u \in F$, write $\phi(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i$. Then*

(1) $b_0 = a_0$,

(2) For $q \geqslant 1$,

$$b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial z_q}\right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial z_{q-1}\partial z_q}\right)^o + \ldots + a_0\left(\frac{\partial^q u}{\partial z_1 \ldots \partial z_q}\right)^o, \quad (2.2)$$

where z_i is x or y according to $j_i = 1$ or 2, and o denotes the trivializer.

For the proof, see Proposition 3.1 in [9].

Let $\hat{F}$ be the free group freely generated by $\{x_{f(t)}, y_{f(t)} \mid f(t) \in P\}$. Let $\{\mathfrak{D}_f(t)\}$ be the Reidemeister-Schreier rewriting function of $\hat{F}$ associated with the action ϕ ([4] or [9]). $\mathfrak{D}_{f(t)}: F \to \hat{F}$ is characterized by the following two properties: For any $f(t) \in P$ and $u, v \in F$,

(1) $\mathfrak{D}_{f(t)}(x) = x_{f(t)}$ and $\mathfrak{D}_{f(t)}(y) = y_{f(t)}$,

(2) $\mathfrak{D}_{f(t)}(uv) = \mathfrak{D}_{f(t)}u \cdot \mathfrak{D}_{\phi(u)f(t)}v$. (2.3)

The following properties will easily be proved from (2.3).

(1) $\mathfrak{D}_{f(t)}(u^{-1}) = (\mathfrak{D}_{\phi(u^{-1})f(t)}(u))^{-1}$

(2) If $\phi(uv^{-1}) = 1$, then $\mathfrak{D}_{f(t)}(uv^{-1}) = (\mathfrak{D}_{f(t)}u) \cdot (\mathfrak{D}_{f(t)}v)^{-1}$. (2.4)

Now let $\psi: \hat{F} \to P$ be a homomorphism defined by $\psi(x_{f(t)}) = 0$ and $\psi(y_{f(t)}) = = f(t)$.

Proposition 2.2. *Let $s = \{j_1, j_2, \ldots, j_k, \ldots\}$ be an ordered sequence of 1 or 2. For $u \in F$ and $f(t) = \sum_{i=0}^{\infty} a_i t^i \in P$, write $\psi\mathfrak{D}_{f(t)}u = \sum_{i=0}^{\infty} b_i t^i$. Then*

(1) $b_0 = a_0$,

(2) For $q \geqslant 1$,

$$b_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial z_q \partial y}\right)^o + \ldots$$

$$+ a_1\left(\frac{\partial^q u}{\partial z_2 \ldots \partial z_q \partial y}\right)^o + a_0\left(\frac{\partial^{q+1} u}{\partial z_1 \ldots \partial z_q \partial y}\right)^o, \quad (2.5)$$

where z_i is x or y according to $j_i = 1$ or 2.

For a proof, see Proposition 6.1 in [9].

In this paper we are particularly interested in the group action associated with a sequence $\{1, 1, \ldots, 1, \ldots\}$ or $\{1, 2, 2, \ldots, 2, \ldots\}$. Our approach, however, is different from what we did in [9] and hence, we obtain different representations of the link groups. To be more precise, we define two actions σ and τ of F on P.

Definition 2.1. *For $f(t) \in P$, define*

$$\begin{aligned} \sigma(x)f(t) &= (1+t)f(t) \\ \sigma(y)f(t) &= f(t) \end{aligned} \tag{2.6}$$

$$\begin{aligned} \tau(x)f(t) &= f(t) + f(0)t \\ \tau(y)f(t) &= (1+t)f(t) - f(0)t. \end{aligned} \tag{2.7}$$

Let P^* be the set of power series $f(t)$ for which $f(0) = 1$. Then σ and τ induce actions of F on P^*, since for any $u \in F$, $[\sigma(u)f(t)]_{t=0} = [\tau(u)f(t)]_{t=0} = 1$. Furthermore, let $q(t)$ be an element of P and $\langle q(t) \rangle$ the ideal of P generated by $q(t)$. Denote by R the quotient ring $P/\langle q(t) \rangle$.

Proposition 2.3.
(1) *σ induces an action $\bar{\sigma}$ of F on R.*
(2) *If $q(0) = 0$, then τ induces an action $\bar{\tau}$ of F on R.*

PROOF. It suffices (and is easy) to show that $\langle q(t) \rangle$ is closed under the actions σ and τ.

Remark 2.1. Let $q(t) = \sum_{i=0}^{\infty} s_i t^i$ and let s_m be the first non zero coefficient of $q(t)$. Then every element $f(t)$ in R has a unique representative $\bar{f}(t)$ of the form: $\bar{f}(t) = \bar{a}_0 + \bar{a}_1 t + \ldots + \bar{a}_k t^k + \ldots$, where $\bar{a}_0, \ldots, \bar{a}_{m-1}$ are integers, and if s_m is positive, then $\bar{a}_k (k \geqslant m)$ is a non-negative integer less than s_m, but if s_m is negative, $\bar{a}_k (k \geqslant m)$ is a non-positive integer greater than s_m. We call this unique representative $\bar{f}(t)$ the *normal* form of $\bar{f}(t)$.

EXAMPLE 2.2. Let $q(t) = 2 + 3t$. Then the normal form of $\bar{f}(t) = 3 + 6t - - t^2 - 3t^3 + 3t^4$ is $1 + t + t^3$.

Since Propositions 2.1 and 2.2 for $\phi = \sigma$ or τ will be used quite extensively in this paper, it will be convenient to state them as separate propositions.

Proposition 2.4. *For $u \in F$, write*

$$\sigma(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \tau(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} c_i t^i.$$

Then for any $q \geqslant 0$,

$$(1)\ b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial x}\right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial x^2}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} u}{\partial x^{q-1}}\right)^o + a_0\left(\frac{\partial^q u}{\partial x^q}\right)^o$$

$$(2)\ c_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial y}\right)^o + a_{q-2}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \ldots$$

$$+ a_1\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)^o + a_0\left(\frac{\partial^q u}{\partial x \partial y^{q-1}}\right)^o. \tag{2.8}$$

In particular, $a_0 = b_0 = c_0$.

Proposition 2.5. *Let* $\mathfrak{D}^{\sigma}_{f(t)}$ *and* $\mathfrak{D}^{\tau}_{f(t)}$ *be the Reidemeister-Schreier rewriting functions associated with the actions* σ *and* τ*, respectively. For*

$$f(t) = \sum_{i=0}^{\infty} a_i t^i \in P,$$

write

$$\psi\mathfrak{D}^{\sigma}_{f(t)}(u) = \sum_{i=0}^{\infty} b_i t^i \quad \textit{and} \quad \psi\mathfrak{D}^{\tau}_{f(t)}(u) = \sum_{i=0}^{\infty} c_i t^i.$$

Then for $q \geqslant 0$,

$$(1)\ b_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial x \partial y}\right)^o + \ldots + a_1\left(\frac{\partial^q u}{\partial x^{q-1}\partial y}\right)^o + a_0\left(\frac{\partial^{q+1} u}{\partial x^q \partial y}\right)^o,$$

$$(2)\ c_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right) + \ldots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o + a_0\left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o. \qquad (2.9)$$

3. Representations of a Free Group

For an integer n (positive, negative or 0), let

$$q_n(t) = \sum_{i=1}^{\infty} \binom{n}{i} t^i = (1+t)^n - 1.$$

As usual,

$$\binom{n}{i} \text{ denotes } \frac{n(n-1)\ldots(n-i+1)}{i!}.$$

Lemma 3.1. *If* $m \equiv 0 \pmod{n}$*, then* $q_m(t) \equiv 0 \pmod{q_n(t)}$.

A proof is easy.

Now let $R(n)$ be the quotient ring $P/\langle q_n(t)\rangle$, and let $R^*(n)$ be the set of elements $f(t)$ in $R(n)$ such that $f(0) = 1$. Since $q_n(0) = 0$, it follows from Propositions 2.3 and 2.4 that σ and τ, respectively, induce actions σ_n and τ_n of F on $R^*(n)$. Let $\Omega_\sigma(n)$ and $\Omega_\tau(n)$ denote the orbits of 1 in $R^*(n)$ under σ_n and τ_n respectively. Namely,

$$\Omega_\sigma(n) = \{\sigma_n(u)(1) \mid u \in F\} \quad \text{and} \quad \Omega_\tau(n) = \{\tau_n(u)(1) \mid u \in F\}.$$

σ_n and τ_n define homomorphisms

$$\sigma_n\colon F \to Sym(\Omega_\sigma(n)) \quad \text{and} \quad \tau_n\colon F \to Sym(\Omega_\tau(n)).$$

Proposition 3.2.
(1) *$\sigma_n(F)$ is a cyclic group of order $|n|$.*
(2) *$\tau_n(F)$ is a metabelian group.*

PROOF. (1) Since $(1 + t)^n \equiv 1 \pmod{q_n(t)}$, $\Omega_\sigma(n)$ consists of exactly $|n|$ elements $\{1, 1 + t, (1 + t)^2, \ldots, (1 + t)^{|n|-1}\}$. Since $\sigma_n(1 + t)^k = (1 + t)^{k+1}$, it follows that $\sigma(x^n) = 1$, but $\sigma(x^k) \neq 1$ for $1 \leqslant k < |n|$, and hence $\sigma_n(F)$ is a cyclic group of order $|n|$.

(2) Let $G = \tau_n(F)$. As a special case of Proposition 9.1 in [9], we see that $G'' = [G', G'] = 1$. Therefore, G is metabelian.

Generally, $\Omega_\tau(n)$ is not a finite set. Therefore, to obtain a finite representation of F, we need to "truncate" higher terms of $f(t)$. Let I_{k+1} be the ideal of P generated by t^{k+1} and $q_n(t)$. Let $R_k(n) = P/I_{k+1}$ and let $R_k^*(n)$ be the set of elements $f(t)$ in $R_k(n)$ such that $f(0) = 1$. An element of $R_k^*(n)$ is a polynomial of degree at most k, and it has the (unique) normal form of degree $\leqslant k$. (See Remark 2.1.) Obviouly, τ_n induces an action $\tau_{k,n}$ of F on $R_k^*(n)$. Let $\Omega_k(n)$ be the orbit of 1 under $\tau_{k,n}$; i.e., $\Omega_k(n) = \{\tau_{k,n}(u)(1) \mid u \in F\}$. $\tau_{k,n}$ defines a (transitive) homomorphism $\tau_{k,n}\colon F \to Sym(\Omega_k(n))$.

Proposition 3.3.
(1) *$\tau_{k,n}(F)$ is nilpotent of class at mos k.*
(2) *If n is a prime p, then $\tau_{k,n}(F)$ is a finite p-group.*
(3) *In particular, if $n = 2$, then $\tau_{k,n}(F)$ is the dihedral group of order 2^{k+1}.*

PROOF.
(1) follows from Proposition 3.2 in [9];
(2) since a proof will be done by an easy induction on k, the details will be omitted. Note that $(\tau_{k,p}(x))^{p^k} = 1$ and $(\tau_{k,p}(y))^p = 1$;
(3) denote $X = \tau_{k,n}(x)$ and $Y = \tau_{k,n}(y)$. Then a straight-forward calculation shows that for any $f(t) \in P^*$

$$\begin{aligned}
&(1)\ X^{2^k}(f(t)) \equiv f(t) \bmod I_{k+1}\\
&(2)\ Y^2(f(t)) \equiv f(t) \bmod I_{k+1}\\
&(3)\ (XY)^2(f(t)) \equiv f(t) \bmod I_{k+1}.
\end{aligned} \tag{3.1}$$

Therefore, $\tau_{k,2}(F)$ is a quotient group of the dihedral group

$$D_{2^k} = \langle A, B \mid A^{2^k} = B^2 = (AB)^2 = 1 \rangle.$$

But it is easy to see that they are, in fact, isomorphic. The details will be omitted.

Remark 3.1. We can prove, further, that for a prime p,

$$\begin{cases} \tau_{k,p}[x, y, x] = 1 \\ \tau_{k,p}[x, \underbrace{y, y, \dots, y}_{k \text{ times}}] = 1, \end{cases} \tag{3.2}$$

where $[u_1, u_2] = u_1 u_2 u_1^{-1} u_2^{-1}$ and $[u_1, u_2, \dots, u_m] = [[u_1, u_2, \dots, u_{m-1}], u_m]$.

In particular, $\tau_{2,p}$ is isomorphic to the group

$$M(p) = \langle x, y \mid x^p = y^p = [x, y]^p = 1,\ [x, y, x] = [x, y, y] = 1 \rangle.$$

Remark 3.2. p-group representations of F obtained in Proposition 3.3 (2) and (3) are quite different from those discussed in [9, §10] or [10, §§2-3].

4. Representations of Link Groups

Let $L = X \cup Y$ be an oriented 2-component link in S^3. In this section, we will define a homomorphism from the link group $G(L)$ onto the group $\sigma_n(F)$ or $\tau_{k,n}(F)$ for various n and k.

For the first group $\sigma_n(F)$, such a homomorphism $\Sigma_n: G(L) \to \sigma_n(F)$ always exists, since $\sigma_n(F)$ is cyclic of order $|n|$. In fact, let m_x and m_y be meridian elements of X and Y, respectively. Then for any integer n, a mapping $\Sigma_n: G(L) \to \sigma_n(F)$ defined by

$$\begin{cases} \Sigma_n(m_x) = \sigma_n(x) \\ \Sigma_n(m_y) = \mathrm{id} \end{cases} \tag{4.1}$$

gives a required homomorphism. However, it will be seen later that Σ_n is only interesting in our purpose when n divides $lk(X, Y)$, the linking number between X and Y.

On the other hand, the second group $\tau_{k,n}(F)$ is not an obvious group. In fact, $\tau_{k,n}(F)$ is metabelian, but not abelian. Nevertheless, for any k, we can find a homomorphism from $G(L)$ onto $\tau_{k,n}(F)$ when $lk(X, Y)$ is divisible by n.

In this section, we will prove the following theorem.

Theorem 4.1. *Let n be an integer. Suppose $lk(X, Y) \equiv 0 \pmod n$. Then for each $k \geqslant 1$, there is a homomorphism*

$$T_{k,n}: G(L) \to \tau_{k,n}(F) \subset Sym(\Omega_k(n))$$

such that $T_{k,n}(m_x) = \tau_{k,n}(x)$ *and* $T_{k,n}(m_y) = \tau_{k,n}(y)$. n *can be* 0 *only when* $lk(X, Y) = 0$.

PROOF. Since there is no essential difference in proving the theorem we may assume that n is a positive integer. Also we may assume *w.l.o.g.* that $lk(X, Y) \geqslant 0$.

Now, if $k = 1$, then $\tau_{k,n}(F)$ is a cyclic group of order n, and the theorem is trivially true. Therefore, we assume that $k \geqslant 2$.

Let $G(L) = \langle x_i, y_j \mid r_i, s_j \rangle$, $1 \leqslant i \leqslant \lambda$, $1 \leqslant j \leqslant \mu$, be a "modified" Wirtinger presentation of $G(L)$ in the following sense.

x_1 and y_1 correspond to prescribed meridian elements m_x and m_y, respectively, and relators are of the form

$$\begin{cases} r_i = u_i x_1 u_i^{-1} x_{i+1}^{-1}, & 1 \leqslant i \leqslant \lambda - 1 \\ r_\lambda = \eta x_1 \eta^{-1} x_1^{-1}, & \end{cases}$$

$$\begin{cases} s_j = v_j y_1 v_j^{-1} y_{j+1}^{-1}, & 1 \leqslant j \leqslant \mu - 1 \\ s_\mu = \xi y_1 \xi^{-1} y_1^{-1}, & \end{cases}$$

where u_i, v_j are words in $\{x_i, y_j\}$, and η and ξ represent longitudes of X and Y, respectively, so that $\{x_1, \eta\}$ and $\{y_1, \xi\}$ form peripheral subgroups of $G(L)$.

Let F^* be the free group freely generated by $\{x_i, y_j, 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu$. As before, F denotes the free group $\langle x, y \mid \rangle$. Let $\rho: F^* \to F^*$ and $\nu: F^* \to F$ be homomorphisms defined by

$$\begin{cases} \rho(x_1) = x \quad \text{and} \quad \rho(y_1) = y & \\ \rho(x_{i+1}) = u_i x_1 u_i^{-1}, & 1 \leqslant i \leqslant \lambda - 1 \\ \rho(y_{j+1}) = v_j v_1 v_j^{-1}, & 1 \leqslant j \leqslant \mu - 1. \end{cases}$$

$$\begin{cases} \nu(x_i) = x, & 1 \leqslant i \leqslant \lambda \\ \nu(y_j) = y, & 1 \leqslant j \leqslant \mu. \end{cases}$$

Using ρ and ν, we define the third homomorphism

$$\theta_{k+1} = \nu\rho^k : F^* \to F \quad \text{for} \quad k \geqslant 0.$$

(θ_{k+1} will be called the Chen-Milnor homomorphism.)

Let $T = \tau_{k,n}\theta_{k+1}$ be a homomorphism from F^* to $\tau_{k,n}(F)$. Then T will induce the homomorphism $T_{k,n}: G(L) \to \tau_{k,n}(F)$ if

$$T(r_i) = 1, \qquad 1 \leqslant i \leqslant \lambda, \quad \text{and} \quad T(s_j) = 1, \qquad 1 \leqslant j \leqslant \mu. \tag{4.2}$$

Now, Proposition 5.1 in [9] proves (4.2) except the last two relations $T(r_\lambda) = 1$ and $T(s_\mu) = 1$. Therefore, it only remains to show that

$$T[\eta, x_1] = 1 \quad \text{and} \quad T[\xi, y_1] = 1. \tag{4.3}$$

Since one of the relations in (4.2) is redundant, it is enough to show that $T[\eta, x_1] = 1$.

For simplicity, write $\theta_{k+1}(\eta) = h$. Since $\theta_{k+1}(x_1) = x$, it suffices to prove that

$$\tau_{k,n}(hx) = \tau_{k,n}(xh). \tag{4.4}$$

Denote $u = hx$ and $w = xh$, and write

$$\tau_{k,n}(u)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i \quad \text{and} \quad \tau_{k,n}(w)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} c_i t^i.$$

Then, since $a_0 = 1$, it follows from (2.8) (2) that for $q \geqslant 1$,

$$(1)\ b_q = a_q + a_{q-1}\left(\frac{\partial u}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q u}{\partial x \partial y^{q-1}}\right)^o \tag{4.5}$$

and

$$(2)\ c_q = a_q + a_{q-1}\left(\frac{\partial w}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} w}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

Now

$$\left(\frac{\partial^r u}{\partial y^r}\right)^o = \left(\frac{\partial^r w}{\partial y^r}\right)^o = \binom{m}{r}, \quad \text{where} \quad m = \left(\frac{\partial h}{\partial y}\right)^o = lk(X, Y),$$

[1], and $(\partial h/\partial x)^o = 0$. Further,

$$\left(\frac{\partial^q u}{\partial x \partial y^{q-1}}\right)^o = \left[\frac{\partial}{\partial x}\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)\right]^o = \left(\frac{\partial^q h}{\partial x \partial y^{q-1}}\right)^o$$

and

$$\left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o = \left[\frac{\partial}{\partial x}\left(x\frac{\partial^{q-1} h}{\partial y^{q-1}}\right)\right]^o = \left(\frac{\partial^{q-1} h}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q h}{\partial x \partial y^{q-1}}\right)^o.$$

Therefore, it follows from (4.5) that

$$b_0 = c_0 = 1, \qquad b_1 = \left(\frac{\partial u}{\partial x}\right)^o = 1 \quad \text{and} \quad c_1 = \left(\frac{\partial w}{\partial x}\right)^o = 1,$$

and hence,

$$\sum_{q=0}^{\infty} c_q t^q = \sum_{q=0}^{\infty} b_q t^q + \sum_{q=2}^{\infty} \left(\frac{\partial^{q-1} h}{\partial y^{q-1}}\right)^o t^q = \sum_{q=0}^{\infty} b_q t^q + \sum_{q=2}^{\infty} \binom{m}{q-1} t^q.$$

Since n devides m by the assumption, it follows from Lemma 3.1 that

$$\sum_{q=2}^{\infty} \binom{m}{q-1} t^q \equiv 0 \pmod{q_n(t)},$$

and hence

$$\sum_{q=0}^{\infty} c_q t^q \equiv \sum_{q=0}^{\infty} b_q t^q \pmod{I_{k+1}}.$$

This proves Theorem 4.1.

Now, let $g = \theta_{k+1}(\xi)$ and $v = gy$ and $z = yg$. Note that $(\partial g/\partial y)^o = 0$. Since $T_{k,n}[\xi, y_1] = 1$, we have $\tau_{k,n}(v) = \tau_{k,n}(z)$. Write

$$\tau_{k,n}(v)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} b_i t^i, \qquad a_0 = 1$$

and

$$\tau_{k,n}(z)\left(\sum_{i=0}^{\infty} a_i t^i\right) = \sum_{i=0}^{\infty} c_i t^i, \qquad a_0 = 1.$$

Then by (2.8) (2) we obtain

$$(1)\ a_0 = b_0 = c_0 = 1 \tag{4.6}$$

For $q \geqslant 1$,

$$(2)\ b_q = a_q + a_{q-1}\left(\frac{\partial v}{\partial y}\right)^o + \dots + a_1\left(\frac{\partial^{q-1} v}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q v}{\partial x \partial y^{q-1}}\right)^o$$

and

$$(3)\ c_q = a_q + a_{q-1}\left(\frac{\partial z}{\partial y}\right)^o + \dots + a_1\left(\frac{\partial^{q-1} z}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q z}{\partial x \partial y^{q-1}}\right)^o.$$

Since $(\partial v/\partial y)^o = (\partial z/\partial y)^o = 1$, it follows that for $q \geqslant 1$,

$$\left(\frac{\partial^q v}{\partial y^q}\right)^o = \left(\frac{\partial^q z}{\partial y^q}\right)^o = \binom{1}{q}$$

and hence

$$\left(\frac{\partial^q v}{\partial x \partial y^{q-1}}\right)^o = \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o + \left(\frac{\partial^{q-1} g}{\partial x \partial y^{q-2}}\right)^o$$

and

$$\left(\frac{\partial^q z}{\partial x \partial y^{q-1}}\right)^o = \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o.$$

Therefore, $\tau_{k,n}(v) = \tau_{k,n}(z)$ yields the following

Proposition 4.2. *Let $g = \theta_{k+1}(\xi)$. Then for $k \geqslant 1$,*

$$\left(\frac{\partial g}{\partial x}\right)^o t^2 + \left(\frac{\partial^2 g}{\partial x \partial y}\right)^o t^3 + \ldots + \left(\frac{\partial^q g}{\partial x \partial y^{q-1}}\right)^o t^{q+1} + \ldots \equiv 0 \qquad (\mathrm{mod}\, I_{k+1}). \tag{4.7}$$

Remark 4.1. A homomorphism $\Sigma_n: G(L) \to \sigma_n(F)$ is formally given as follows. First, define a homomorphism $\Sigma^*: F^* \to \sigma_n(F)$ by

$$\begin{cases} \Sigma^*(x_i) = \sigma_n(x) & \text{for} \quad i = 1, 2, \ldots, \lambda, \\ \Sigma^*(y_j) = id & \text{for} \quad j = 1, 2, \ldots, \mu. \end{cases} \tag{4.8}$$

Then $\Sigma^*(r_i) = \Sigma^*(s_j) = 1$ for any i, j. Therefore, Σ^* induces the homomorphism $\Sigma_n: G(L) \to \sigma_n(F)$. This rather obvious observation will be used in the next section.

5. Covering Space (I) Cyclic Covering

In the previous section we found representations Σ_n and $T_{k,n}$ of $G(L)$ on $\sigma_n(F)$ and $\tau_{k,n}(F)$.

To each finite representation ϕ we can associate a (unbranched) covering space M. Let $U(X)$ and $U(Y)$ denote tubular neighborhoods of X and Y in S^3, respectively. Then the covering space M associated with ϕ is a compact 3-manifold with boundary consisting of tori.

Suppose we have a homomorphism

$$\phi: \pi_1(M) \to A$$

from $\pi_1(M)$ to an abelian group A. Then ϕ induces the homomorphism $\hat{\phi}$ from $H_1(M)$ to A. The most characteristic element of $H_1(M)$ is a "longitude" $\tilde{\xi}$ of each boundary torus of M. In many cases, such a "longitude" can be realized as a "lift" of a longitude ζ of $\partial U(X)$ or $\partial U(Y)$, and then $\phi(\hat{\zeta})$ will be an invariant of the original link type L. By taking A as the polynomial ring $R_k(n)$, we will obtain our polynomial invariants.

In this section, we define such polynomial invariants for a finite representation Σ_n: $G(L) \to \sigma_n(F) \subset Sym(\Omega_\sigma(n))$.

Let M_σ be the (unbranched) covering space of $S^3 - L$ associated with Σ_n. M_σ is in fact the n-fold cyclic covering space of $S^3 - X$.

Let $\mathfrak{D}^{*\sigma}_{f(t)}$ be the Reidemeister-Schreier rewriting function associated with the action $\Sigma^*: F^* \to \sigma_n(F) \subset Sym(\Omega_\sigma(n))$, where Σ^* is defined in Remark 4.1. (4.8).

Now the set $S_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(x_i), \mathfrak{D}^{*\sigma}_{f(t)}(y_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_\sigma(n)\}$ generates a free group F^*_σ and $\pi_1(M_\sigma)$ has a presentation $\langle S_\sigma: R_\sigma, U_\sigma\rangle$ where $R_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(r_i), \mathfrak{D}^{*\sigma}_{f(t)}(s_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega(n)\}$ and $U_\sigma = \{\mathfrak{D}^{*\sigma}_{f(t)}(x_1), f(t) \in \Omega_\sigma(n)\}$.

Theorem 5.1. *Let n be an integer. Suppose* $lk(X, Y) \equiv 0 \pmod n$. *Then for* $k \geqslant 1$, *there exists a homomorphism* $\Phi_\sigma: \pi_1(M_\sigma) \to R_k(n)$ *such that for any* $f(t) \in \Omega_\sigma(n)$,

$$\Phi_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_1)) = 0 \quad \textit{and} \quad \Phi_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_1)) = f(t).$$

Remark 5.1. n can be 0 only if $lk(X, Y) = 0$, and then M_σ is an infinite cyclic covering space of $S^3 - X$.

PROOF OF THEOREM 5.1. To prove the theorem it suffices to define a homomorphism $\Phi^*_\sigma: F^*_\sigma \to R_k(n)$ such that $\Phi^*_\sigma(w) = 0$ for $w \in R_\sigma$ or U_σ.

Now let F_σ be the free group freely generated by a set $\{x_{f(t)}, y_{f(t)} \mid f(t) \in \Omega_\sigma(n)\}$, and let ψ_σ be a homomorphism from F_σ to $R_k(n)$ given by

$$\psi_\sigma(x_{f(t)}) = 0 \quad \text{and} \quad \psi_\sigma(y_{f(t)}) = f(t). \tag{5.2}$$

Using ψ_σ, we define, for $f(t) \in \Omega_\sigma(n)$ and for any i, j,

$$\begin{cases} \Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_i)) = \psi_\sigma\mathfrak{D}^{\sigma}_{f(t)}\theta_{k+1}(x_i) \\ \Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_j)) = \psi_\sigma\mathfrak{D}^{\sigma}_{f(t)}\theta_{k+1}(y_j). \end{cases} \tag{5.3}$$

Note that

$$\Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(x_1)) = \psi_\sigma\mathfrak{D}^{\sigma}_{f(t)}\theta_{k+1}(x_1) = \psi_\sigma\mathfrak{D}^{\sigma}_{f(t)}(x) = \psi_\sigma(x_{f(t)}) = 0,$$

and

$$\Phi^*_\sigma(\mathfrak{D}^{*\sigma}_{f(t)}(y_1)) = \psi_\sigma\mathfrak{D}^{\sigma}_{f(t)}\theta_{k+1}(y_1) = \psi_\sigma\mathfrak{D}^{\sigma}_{f(t)}(y) = \psi_\sigma(y_{f(t)}) = f(t),$$

and therefore, Φ_σ^* satisfies (5.1). We will prove further, for any $u \in F^*$,

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u) = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(u). \tag{5.4}$$

A proof will be done by induction on the length $l(u)$ of u.

If $l(u) = 1$, then $u = x_i^{\pm 1}$ of $y_j^{\pm 1}$. If $u = x_i$ or y_j, (5.4) is trivially true. Suppose $u = x_i^{-1}$. Since $\sigma_n(x_i) = \sigma_n(\theta_{k+1}(x_i))$, it follows from (2.4) (1) that

$$\begin{aligned}\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(x_i^{-1}) &= \Phi_\sigma^*[\mathfrak{D}_{\sigma_n(x_i^{-1})f(t)}^{*\sigma}(x_i)]^{-1}\\ &= \psi_\sigma[\mathfrak{D}_{\sigma_n(x_i^{-1})f(t)}^{\sigma}\theta_{k+1}(x_i)]^{-1}\\ &= \psi_\sigma[\mathfrak{D}_{\sigma_n(\theta_{k+1}(x_i)^{-1})f(t)}^{\sigma}\theta_{k+1}(x_i)]^{-1}\\ &= \psi_\sigma \mathfrak{D}_{f(t)}^\sigma(\theta_{k+1}(x_i))^{-1}\\ &= \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(x_i^{-1}).\end{aligned}$$

Similarly, (5.4) holds for $u = y_j^{-1}$.

Now suppose (5.4) holds for any element u with $l(u) < d$. Let w be an element of F^* with $l(w) = d$. Then $w = ux_i^{\pm 1}$ or $uy_j^{\pm 1}$ for some u with $l(u) = d - 1$.

Consider the case $w = ux_i$. Then

$$\begin{aligned}\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(w) &= \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(ux_i)\\ &= \Phi_\sigma^*[(\mathfrak{D}_{f(t)}^{*\sigma}u) \cdot (\mathfrak{D}_{\sigma_n(u)f(t)}^{*}(x_i))]\\ &= \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u) + \Phi_\sigma^* \mathfrak{D}_{\sigma_n(u)f(t)}^{*\sigma}(x_i)\\ &= \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(u) + \psi_\sigma \mathfrak{D}_{\sigma_n(u)f(t)}^\sigma \theta_{k+1}(x_i)\\ &= \psi_\sigma[\mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(u) \cdot \mathfrak{D}_{\sigma_n(u)f(t)}^\sigma \theta_{k+1}(x_i)].\end{aligned}$$

Since $\sigma_n(u) = \sigma_n(\theta_{k+1}(u))$, the last expression becomes

$$\psi_\sigma[\mathfrak{D}_{f(t)}^\sigma(\theta_{k+1}(u) \cdot \theta_{k+1}(x_i))] = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(ux_i). \tag{5.5}$$

Since similar computations provide the proofs for other cases, the details will be omitted.

Now we must show

$$\begin{aligned}&\Phi_\sigma^*(\mathfrak{D}_{f(t)}^{*\sigma} r_i) = 0, \qquad i = 1, 2, \ldots, \lambda\\ &\Phi_\sigma^*(\mathfrak{D}_{f(t)}^{*\sigma} s_j) = 0, \qquad j = 1, 2, \ldots, \mu.\end{aligned} \tag{5.6}$$

First consider $r_i = u_i x_1 u_i^{-1} x_{i+1}^{-1}$, $1 \leqslant i < \lambda$. Since $\theta_{k+1}(x_{i+1}) \equiv \theta_{k+2}(x_{i+1})$ $(\mathrm{mod}\, F_{k+2})$ by Proposition 5.1 in [9], Propositions 2.5 and (5.1) in [1] imply that

$$\psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(x_{i+1}) = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+2}(x_{i+1}).$$

Since $\theta_{k+2}(x_{i+1}) = \theta_{k+1}(u_i x_1 u_i^{-1})$ by the definition, we obtain

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(x_{i+1}) = \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(u_i x_1 u_i^{-1}),$$

and hence

$$\Phi_\sigma^*[(\mathfrak{D}_{f(t)}^{*\sigma}(x_{i+1}))(\mathfrak{D}_{f(t)}^{*\sigma}(u_i x_1 u_i^{-1}))^{-1}] = 0,$$

which is equal to

$$\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(x_{i+1}(u_i x_1 u_i^{-1})^{-1}) = 0,$$

by (2.4) (2).

Similarly, we can prove $\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(s_j) = 0$ for $j \neq \mu$.

Now it remains to show that

$$\begin{aligned} &(1)\ \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}[\eta, x_1] = 0, \quad \text{or} \\ &(2)\ \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}[\xi, y_1] = 0. \end{aligned} \tag{5.7}$$

Since $\sigma_n[\xi, y_1] = id$, (5.7) (2) is equivalent to

$$\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi y_1) = \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(y_1\xi). \tag{5.8}$$

To prove (5.8), we compute both sides separately. Note that $\theta_{k+1}(y_1) = y$ and $\sigma_n(y_1) = id$. Then

$$\begin{aligned} \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi y_1) &= \Phi_\sigma^*[\mathfrak{D}_{f(t)}^{*\sigma}(\xi)\cdot\mathfrak{D}_{\sigma_n(\xi)f(t)}^{*\sigma}(y_1)] \\ &= \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) + \Phi_\sigma^*\mathfrak{D}_{\sigma_n(\xi)f(t)}^{*\sigma}(y_1) \\ &= \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) + \psi_\sigma\mathfrak{D}_{\sigma_n(\xi)f(t)}^{\sigma}\theta_{k+1}(y_1) \\ &= \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) + \sigma_n(\xi)f(t). \end{aligned}$$

On the other hand,

$$\begin{aligned} \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(y_1\xi) &= \Phi_\sigma^*(\mathfrak{D}_{f(t)}^{*\sigma}y_1\cdot\mathfrak{D}_{\sigma_n(y_1)f(t)}^{*\sigma}\xi) \\ &= \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}y_1 + \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) \\ &= f(t) + \Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi). \end{aligned}$$

Therefore, it suffices to prove

$$f(t) \equiv \sigma_n(\xi)f(t) \qquad (\operatorname{mod} I_{k+1}). \tag{5.9}$$

Since $m = lk(X, Y) \equiv 0 \pmod n$, it follows from Lemma 4.1 that $q_m(t) \equiv 0$ $(\operatorname{mod} q_n(t))$. Also, since $\sigma_n(\xi) = \sigma_n(x^m)$, $\sigma_n(x^m)f(t) = (1+t)^m f(t) \equiv f(t)$ $(\operatorname{mod} q_n(t))$.

This now completes the proof of Theorem 5.1.

A straightforward computation of the relation $\Phi_\sigma^*\mathfrak{D}_{f(t)}^*[\eta, x_1] = 0$ yields the following

Corollary 5.2. *Let $h = \theta_{k+1}(\eta)$. Then*

$$\sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}h}{\partial x^q \partial y}\right)^o t^{q+1} \equiv 0 \qquad (\operatorname{mod} I_{k+1}). \tag{5.10}$$

Remark 5.2. (5.10) can be considered as the "dual" form to (4.7).

Now before our polynomial invariants are introduced, we need a few propositions.

Proposition 5.3. *For any $f(t) \in \Omega_\sigma(n)$,*

$$\Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\eta) = \Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\eta).$$

PROOF. Let $u = \theta_{k+1}(\eta)$. Then $\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\eta) = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma \theta_{k+1}(\eta) = \psi_\sigma \mathfrak{D}_{f(t)}^\sigma(u)$.

For $f(t) = \sum_{i=0}^{\infty} a_i t^i \in P^*$, write $\psi_\sigma \mathfrak{D}_1^\sigma(u) = \sum_{i=0}^{\infty} b_i t^i$ and $\psi_\sigma \mathfrak{D}_{f(t)}^\sigma(u) = \sum_{i=0}^{\infty} c_i t^i$.

Then Proposition 2.5 yields, since $a_0 = 1$,

$$(1)\ \sum_{i=0}^{\infty} b_i t^i = \sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}u}{\partial x^q \partial y}\right)^o t^q$$

$$(2)\ \sum_{i=0}^{\infty} c_i t^i = \sum_{j=1}^{\infty} a_j\left[\sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}u}{\partial x^q \partial y}\right)^o t^{q+j}\right] + \sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}u}{\partial x^q \partial y}\right)^o t^q. \tag{5.11}$$

By (5.10), for $j > 0$,

$$\sum_{q=0}^{\infty}\left(\frac{\partial^{q+1}u}{\partial x^q \partial y}\right)^o t^{q+j} \equiv 0 \qquad (\operatorname{mod} q_n(t))$$

and hence,

$$\sum_{i=0}^{\infty} b_i t^i \equiv \sum_{i=0}^{\infty} c_i t^i \qquad (\operatorname{mod} q_n(t)).$$

This proves Proposition 5.3.

Proposition 5.4. *For any $f(t) \in \Omega_\sigma(n)$,*

$$\Phi_\sigma^* \mathfrak{D}_{f(t)}^{*\sigma}(\xi) = f(t)[\Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\xi)].$$

PROOF. Since $f(t) \in \Omega_\sigma(n)$, $f(t)$ is of the form $(1+t)^r$ for some $0 \leqslant r < n$. (We assume that n is non-negative.)

Write

$$\Phi_\sigma^*\mathfrak{D}_{f(t)}^{*\sigma}(\xi) = \psi_\sigma\mathfrak{D}_{f(t)}^q\theta_{k+1}(\xi) = \sum_{i=0}^\infty c_i t^i \text{ and } \Phi_\sigma^*\mathfrak{D}_1^{*\sigma}(\xi) = \psi_\sigma\mathfrak{D}_1^q\theta_{k+1}(\xi) = \sum_{i=0}^\infty b_i t^i.$$

Denote $w = \theta_{k+1}(\xi)$. Then Proposition 2.5 yields again

$$(1)\ \sum_{i=0}^\infty b_i t^i = \sum_{q=0}^\infty \left(\frac{\partial^{q+1} w}{\partial x^q \partial y}\right)^o t^q, \quad \text{and}$$

$$(2)\ \sum_{i=0}^\infty c_i t^i = \sum_{j=0}^\infty a_j \left[\sum_{q=0}^\infty \left(\frac{\partial^{q+1} w}{\partial x^q \partial y}\right)^o t^{q+j}\right]. \tag{5.12}$$

Since

$$f(t) = (1+t)^r = \sum_{i=0}^\infty a_i t^i \quad \text{and} \quad \left(\frac{\partial w}{\partial y}\right)^o = 0,$$

it follows from (5.12) (1) (2) that

$$\begin{aligned} f(t)\sum_{i=0}^\infty b_i t^i &= \sum_{j=0}^\infty a_j t^j \left(\sum_{q=0}^\infty \left(\frac{\partial^{q+1} w}{\partial x^q \partial y}\right)^o t^q\right) \\ &= \sum_{j=0}^\infty a_j \left[\sum_{q=0}^\infty \left(\frac{\partial^{q+1} w}{\partial x^q \partial y}\right)^o t^{q+j}\right] \\ &= \sum_{i=0}^\infty c_i t^i. \end{aligned}$$

Definition 5.1. *Let n be an integer that divides $lk(X, Y)$. Then for any integer $k \geqslant 1$, define the polynomials $\eta_k^{(n)}(t)$ and $\xi_k^{(n)}(t)$ in $R_k(n)$ by*

$$\begin{aligned} \eta_k^{(n)}(t) &= \Phi_\sigma^*\mathfrak{D}_1^{*\sigma}(\eta), \quad \textit{and} \\ \xi_k^{(n)}(t) &= \Phi_\sigma^*\mathfrak{D}_1^{*\sigma}(\xi). \end{aligned} \tag{5.13}$$

Theorem 5.5. *For any $k \geqslant 1$, $\eta_k^{(n)}(t)$ and $\{f(t)\xi_k^{(n)}(t), f(t) \in \Omega_\sigma(n)\}$ are invariants of an oriented link type L.*

Remark 5.3. For any $u \in F$, $\theta_{k+1}(u) \equiv \theta_{k+2}(u) \bmod F_{k+2}$, and hence, for any $l \geqslant k$,

$$\begin{aligned} \eta_l^{(n)}(t) &\equiv \eta_k^{(n)}(t) \pmod{I_{k+1}} \\ \xi_l^{(n)}(t) &\equiv \xi_k^{(n)}(t) \pmod{I_{k+1}}. \end{aligned} \tag{5.14}$$

Since k can be taken arbitrarily large, these invariants are formal power series.

Corollary 5.6. *Suppose $lk(X, Y) = n \neq 0$. Let $\sum_{i=0}^{\infty} \bar{a}_i t^i$ be the normal form of $\eta_k^{(n)}$. Then for any $i \geqslant 0$, $\bar{a}_i = a_i^*$, where a_i^* is the invariant defined in* [5].

Now, as we did in [4] or [9], these invariants can be interpreted as the "linking number" between one component of the lifts of X or Y and the characteristic link defined in [9]. Using this geometric interpretation, we can obtain more information on $\eta_k^{(n)}(t)$.

Let $\tilde{X}$ and $\tilde{Y} = \tilde{Y}_0 \cup \ldots \cup \tilde{Y}_{n-1}$ be the lifts of X and Y, respectively, in the covering space M_σ associated with the homomorphism $\Sigma_n: G(L) \to \to \mathrm{Sym}(\Omega_\sigma(n))$.

Theorem 5.7. *Suppose $lk(X, Y) = rn$, $n > 0$. Then, for any $k \geqslant 1$,*

$$\eta_k^{(n)}(t) = r\left\{\binom{n}{1} + \binom{n}{2}t + \ldots + \binom{n}{n}t^{n-1}\right\}. \tag{5.15}$$

In particular, the invariant a_i^* defined in [5] *is completely determined by the linking number $lk(X, Y)$.*

PROOF. By [4] or [9], the characteristic link associated with the linking function or the homomorphism $\Phi_\sigma: \pi_1(M_\sigma) \to R_k(n)$ is a 1-cycle

$$\bar{Y} = \sum_{i=0}^{n-1} (1 + t)^i \tilde{Y}_i \quad \text{in} \quad H_1(\tilde{Y}; R_k(n)),$$

and $\bar{Y}$ bounds a 2-chain $\tilde{D}$ in $C_2(M_\sigma; R_k(n))$. Then by [4], $\eta_k^{(n)}(t) = \mathrm{Int}(\tilde{X}, \tilde{D})$, where Int denotes the intersection number. Since $\tilde{X}$ bounds a 2-chain $\tilde{C}$ in M_σ, $\mathrm{Int}(\tilde{X}, \tilde{D}) = lk(\tilde{X}, \bar{Y}) = \mathrm{Int}(\tilde{C}, \bar{Y})$. Obviously, $\mathrm{Int}(\tilde{C}, \tilde{Y}_i) = r$ and hence

$$\eta_k^{(n)}(t) = r \sum_{i=0}^{n-1} (1 + t)^i = r \sum_{j=1}^{n} \binom{n}{j} t^{j-1}.$$

This proves (5.15).

Corollary 5.8. *If $lk(X, Y) = 0$, then for any n and k, $\eta_k^{(n)}(t) = 0$.*

Corollary 5.9. *$\xi_k^{(n)}(0) = 0$ and $\eta_k^{(n)}(0) = lk(X, Y)$.*

PROOF. By Remark 5.3, $\xi_k^{(n)}(0) = \xi_0^{(n)}(0)$. Since $\theta_1(\xi) = x^m$, $m = lk(X, Y)$, we have

$$\xi_0^{(n)}(t) = \Phi_\sigma^* \mathfrak{D}_1^{*\sigma}(\xi) = \psi_\sigma \mathfrak{D}_1^\sigma \theta_1(\xi) = \psi_\sigma \mathfrak{D}_1^\sigma(x^m) = 0.$$

Therefore, $\xi_k^{(n)}(0) = 0$. The second part follows from (5.15).

Suppose $lk(X, Y) = 0$ and take $n = 0$. Then, since $q_n(t) = 0$, $\xi_k^{(0)}(t)$ is a rational function on t. Furthermore, if we let $s = 1 + t$ and express $\xi_k^{(0)}(t)$ as a Laurent polynomial on s, then it is essentially the η-function defined in [6]. In fact, we can prove the following theorem

Theorem 5.10. *Suppose that Y is contractible $S^3 - X$. Let $\eta(L, X, Y; s)$ be the polynomial defined in* [6]. *Write $\xi_k^{(0)}(t)$ as a Laurent polynomial $\tilde{\xi}_k(s)$ on $s = 1 + t$. Then for a sufficiently large k,*

$$\tilde{\xi}_k(s) \doteq \eta(L, X, Y; s),$$

where $A \doteq B$ means that A and B are equal up to a unit in $Z[s, s^{-1}]$.

A proof follows from the definition of $\xi_k^{(0)}(t)$ and Theorem 2 in [7].

6. Covering Space (II) Metabelian Covering

In this section, we consider the other representation $T_{k,n}: G(L) \to Sym(\Omega_k(n))$ and the covering space M_T of $S^3 - L$ associated with $T_{k,n}$.

Theorem 6.1. *Let n be an integer and suppose $lk(X, Y) \equiv 0 \pmod n$. Then for each $k \geqslant 1$, there exists a homomorphism*

$$\Phi_\tau: \langle S_\tau: R_\tau \rangle \to R_k(n)$$

such that, for any $f(t) \in \Omega_k(n)$,

$$\Phi_\tau(\mathfrak{D}^{*\tau}_{f(t)} x_1) = 0 \quad \textit{and} \quad \Phi_\tau(\mathfrak{D}^{*\tau}_{f(t)} y_1) = f(t), \tag{6.1}$$

where $\mathfrak{D}^{*\tau}_{f(t)}$ denotes the Reidemeister-Schreier rewriting function associated with *$T_{k,n}$ and $S_\tau = \{\mathfrak{D}^{*\tau}_{f(t)}(x_i), \mathfrak{D}^{*\tau}_{f(t)}(y_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_k(n)\}$ and $R_\tau = \{\mathfrak{D}^{*\tau}_{f(t)}(r_i), \mathfrak{D}^{*\tau}_{f(t)}(s_j) \mid 1 \leqslant i \leqslant \lambda, 1 \leqslant j \leqslant \mu, f(t) \in \Omega_k(n)\}$.*

PROOF. We can use the same argument employed in the proof of Theorem 5.1 using the Reidemeister-Schreier rewriting functions $\mathfrak{D}^{*\tau}_{f(t)}$, $\mathfrak{D}^{\tau}_{f(t)}$ and homomorphisms Φ^*_τ, ψ_τ instead of $\mathfrak{D}^{*\sigma}_{f(t)}$, $\mathfrak{D}^{\sigma}_{f(t)}$, Φ^*_σ, ψ_σ. What we need to prove here is the formula (6.2) below corresponding to (5.7) (1).

$$\Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(\eta x_1) = \Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(x_1 \eta). \tag{6.2}$$

First we compute both sides separately. The left hand side is

$$\begin{aligned}
\Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(\eta x_1) &= \Phi^*_\tau(\mathfrak{D}^{*\tau}_{f(t)}\eta \cdot \mathfrak{D}^{*\tau}_{T_{k,n}(\eta)f(t)} x_1) \\
&= \Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(\eta) + \Phi^*_\tau \mathfrak{D}^{*\tau}_{T_{k,n}(\eta)f(t)}(x_1) \\
&= \Phi^*_\tau \mathfrak{D}^{*\tau}_{f(t)}(\eta) \\
&= \psi_\tau \mathfrak{D}^{\tau}_{f(t)} \theta_{k+1}(\eta).
\end{aligned}$$

On the other hand, the right hand side is

$$\begin{aligned}\Phi_\tau^*\mathfrak{D}_{f(t)}^{*\tau}(x_1\eta) &= \Phi_\tau^*\mathfrak{D}_{f(t)}^{*\tau}(x_1) + \Phi_\tau^*\mathfrak{D}_{T_{k,n}(x_1)f(t)}^{*\tau}(\eta)\\ &= \Phi_\tau^*\mathfrak{D}_{T_{k,n}(x_1)f(t)}^{*\tau}(\eta)\\ &= \psi_\tau\mathfrak{D}_{\tau_{k,n}(x)f(t)}^{\tau}\theta_{k+1}(\eta).\end{aligned}$$

To compare these terms, let $u = \theta_{k+1}(\eta)$ and $f(t) = \sum_{i=0}^{\infty} a_i t^i$.

Write

$$\psi_\tau\mathfrak{D}_{f(t)}^{\tau}\theta_{k+1}(\eta) = \sum_{i=0}^{\infty} b_i t^i \quad\text{and}\quad \psi_\tau\mathfrak{D}_{\tau_{k,n}(x)f(t)}^{\tau}\theta_{k+1}(\eta) = \sum_{i=0}^{\infty} c_i t^i.$$

Then by (2.9) (2), we have, since $a_0 = 1$,

(1) $b_0 = \left(\dfrac{\partial u}{\partial y}\right)^o = m = lk(X, Y)$

(2) For $q \geqslant 1$,

$$b_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \dots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1} u}{\partial x\partial y^q}\right)^o. \tag{6.3}$$

Now $\tau_{k,n}(x) = f(t) + t = 1 + (1 + a_1)t + \sum_{i=2}^{\infty} a_i t^i$ and hence, (2.9) yields, again,

(1) $c_0 = \left(\dfrac{\partial u}{\partial y}\right)^o = m$

(2) $c_1 = (1 + a_1)\left(\dfrac{\partial u}{\partial y}\right)^o + \left(\dfrac{\partial^2 u}{\partial x\partial y}\right)^o$

(3) For $q \geqslant 2$,

$$\begin{aligned}c_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \dots + a_2\left(\frac{\partial^{q-1} u}{\partial y^{q-1}}\right)^o +\\ + (1 + a_1)\left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1} u}{\partial x\partial y^q}\right)^o.\end{aligned} \tag{6.4}$$

Therefore, for $q \geqslant 1$, $c_q - b_q = (\partial^q u/\partial y^q)^o$ and hence

$$\sum_{q=0}^{\infty} b_q t^q - \sum_{q=0}^{\infty} c_q t^q = \sum_{i=1}^{\infty}\left(\frac{\partial^i u}{\partial y^i}\right)^o t^i.$$

Since

$$\left(\frac{\partial^i u}{\partial y^i}\right)^o = \binom{(\partial u/\partial y)^o}{i} = \binom{m}{i}$$

for $i \geqslant 1$, it follows from Lemma 3.1 that

$$\sum_{i=1}^{\infty}\left(\frac{\partial^i u}{\partial y^i}\right)^o t^i = \sum_{i=1}^{\infty}\binom{m}{i} t^i = q_m(t) \equiv 0 \qquad (\operatorname{mod} q_n(t)),$$

and therefore,

$$\sum_{q=0}^{\infty} b_q t^q - \sum_{q=0}^{\infty} c_q t^q = 0.$$

This proves (6.2).

Proposition 6.2. *For any $f(t) \in \Omega_k(n)$,*

$$\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta). \tag{6.5}$$

PROOF. Let $f(t) = \sum_{i=0}^{\infty} a_i t^i$ and write $\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \sum_{i=0}^{\infty} b_i t^i$ and $\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta) = \sum_{i=0}^{\infty} c_i t^i$. Then by (2.9) (2) we have, for $q \geqslant 0$ and $u = \theta_{k+1}(\eta)$,

(1) $b_q = \left(\dfrac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o$ and

$$(2) \quad c_q = a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \ldots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o + \left(\frac{\partial^{q+1} u}{\partial x \partial y^q}\right)^o. \tag{6.6}$$

Therefore, $b_0 = c_0 = m = lk(X, Y)$ and

$$\begin{aligned}\sum_{q=0}^{\infty} c_q t^q - \sum_{q=0}^{\infty} b_q t^q &= \sum_{q=1}^{\infty}\left\{a_q\left(\frac{\partial u}{\partial y}\right)^o + a_{q-1}\left(\frac{\partial^2 u}{\partial y^2}\right)^o + \ldots + a_1\left(\frac{\partial^q u}{\partial y^q}\right)^o\right\} t^q \\ &= \sum_{j=1}^{\infty} a_j\left[\sum_{i=1}^{\infty}\binom{m}{i} t^{i+j-1}\right] \\ &\equiv 0 \quad (\operatorname{mod} q_n(t)),\end{aligned}$$

since for $j \geqslant 1$,

$$\sum_{i=1}^{\infty}\binom{m}{i} t^{i+j-1} \equiv 0 \quad (\operatorname{mod} q_n(t)).$$

Proposition 6.3. $T_{k,n}(\xi) = id$.

PROOF. Let $w = \theta_{k+1}(\xi)$ and write $T_{k,n}(\xi)(\sum_{i=0}^{\infty} a_i t^i) = \sum_{i=0}^{\infty} b_i t^i$, where $a_0 = 1$. Then by (2.8) (2),

$$b_q = a_q + a_{q-1}\left(\frac{\partial w}{\partial y}\right)^o + \ldots + a_1\left(\frac{\partial^{q-1} w}{\partial y^{q-1}}\right)^o + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

However,

$$\left(\frac{\partial^i w}{\partial y^i}\right)^o = \binom{(\partial w/\partial y)^o}{i} = 0, \quad \text{since} \quad \left(\frac{\partial w}{\partial y}\right)^o = 0.$$

Therefore,

$$b_q = a_q + \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o.$$

By Proposition 4.2 (4.7),

$$\sum_{q \geq 1} \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o t^q \equiv 0 \pmod{q_n(t)}$$

and hence

$$\sum_{q \geq 0} b_q t^q - \sum_{q \geq 0} a_q t^q = \sum_{q \geq 1} \left(\frac{\partial^q w}{\partial x \partial y^{q-1}}\right)^o t^q \equiv 0 \pmod{q_n(t)}.$$

Proposition 6.4. *For any $f(t) \in \Omega_k(n)$,*

$$\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\xi) = \Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\xi).$$

PROOF. The details will be omitted, since a proof can be obtained, using similar computations shown in the proofs of Propositions 6.2 and 6.3.

Definition 6.1. *Let n be an integer that divides $lk(X, Y)$. Then for any integer $k \geqslant 1$, define the polynomials $\bar{\eta}_k^{(n)}(t)$ and $\bar{\xi}_k^{(n)}(t)$ in $R_k(n)$ as follows:*

$$\begin{aligned} \bar{\eta}_k^{(n)}(t) &= \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta), \\ \bar{\xi}_k^{(n)}(t) &= \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\xi). \end{aligned} \tag{6.7}$$

Theorem 6.5. *For any $k \geqslant 1$, $\bar{\eta}_k^{(n)}(t)$ and $\bar{\xi}_k^{(n)}(t)$ are invariants for an oriented link type L.*

Remark 6.1. As is stated in Remark 5.3, for $l \geqslant k$, $\bar{\eta}_l^{(n)}(t) \equiv \bar{\eta}_k^{(n)}(t) \pmod{I_{k+1}}$ and $\bar{\xi}_l^{(n)}(t) \equiv \bar{\xi}_k^{(n)}(t) \pmod{I_{k+1}}$, and therefore, these invariants are formal power series.

Now, it follows from Proposition 7.1 in [4] that these invariants also can be interpreted as linking numbers between two cycles in the covering space M_τ associated with $T_{k,n}$. Since $T_{k,n}(\xi) = \text{id}$, every lift $\tilde{\xi}$ of a longitude ξ of Y in M_τ is a simple closed curve in M_τ and hence, $\bar{\xi}_k^{(n)}(t)$ is interpreted as the intersection number between $\tilde{\xi}$ and a 2-chain in $C_2(M_\tau; R_k(n))$ which bounds the characteristic link. On the other hand, $T_{k,n}(\eta)$ may not be an identity, and therefore, a lift $\tilde{\eta}$ of a longitude η of X in M_τ may not be a closed curve. Let r be the smallest positive integer such that $T_{k,n}(\eta^r) = \text{id}$. Then Proposition 6.2 shows that for any $f(t) \in \Omega_k(n)$, $\Phi_\tau^* \mathfrak{D}_{f(t)}^{*\tau}(\eta^r) = r\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta)$ and therefore, $\Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \bar{\eta}_t^{(n)}(t)$ can be considered the "linking number" between a "longitude" of a covering torus and the characteristic link in M_τ.

Corollary 6.6. *Suppose $lk(X, Y) = n \neq 0$. Let $\sum_{i=0}^{\infty} \bar{a}_i t^i$ be the normal form of $\bar{\eta}_k^{(n)}(t)$. Then $\bar{a}_0 = n$ and $\bar{a}_i = h_i^*$, where h_i^* is the invariant defined in* [5].

Corollary 6.7. *If the Alexander polynomial of L is* 0, *then all invariants $\eta(t)$, $\xi(t)$, $\bar{\eta}(t)$, $\bar{\xi}(t)$ vanish.*

Corollary 6.8. $\bar{\xi}_k^{(n)}(0) = 0$ *and* $\bar{\eta}_k^{(n)}(0) = lk(X, Y)$.

PROOF. The proof of the first part is similar to that of Corollary 5.9. On the other hand, $\bar{\eta}_k^{(n)}(0) = \bar{\eta}_0^{(n)}(0)$ and $\bar{\eta}_0^{(n)}(t) = \Phi_\tau^* \mathfrak{D}_1^{*\tau}(\eta) = \psi_\tau \mathfrak{D}_1^\tau \theta_1(\eta) = \psi_\tau \mathfrak{D}_1^\tau(y^m) = = \psi_\tau(y_1^m) = m$, since $\tau_{0,n}(y) = \text{id}$ and $\theta_1(\eta) = y^m$, where $m = lk(X, Y)$.

Finally, we study the behavior of these invariants under simple transformations of the link. The following two propositions are easy to prove, and therefore, the details will be omitted.

Proposition 6.9. *Let L' be the mirror image of an oriented link L. Then for any n and k,*

$$\begin{aligned} \eta_k^{(n)}(t)_{L'} &= -\eta_k^{(n)}(t)_L \\ \xi_k^{(n)}(t)_{L'} &= -\xi_k^{(n)}(t)_L \\ \bar{\eta}_k^{(n)}(t)_{L'} &= -\bar{\eta}_k^{(n)}(t)_L \\ \bar{\xi}_k^{(n)}(t)_{L'} &= -\bar{\xi}_k^{(n)}(t)_L. \end{aligned}$$

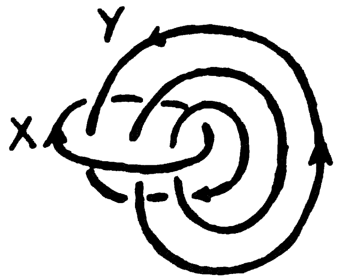
Proposition 6.10. *Let L^* be the link obtained from L by reversing the orientation of one component, X say. The n for any n and k,*

$$\begin{cases} \eta_k^{(n)}(t)_{L^*} = -\eta_k^{(n)}(t)_L \\ \xi_k^{(n)}(t)_{L^*} = \xi_k^{(n)}(t)_L. \end{cases}$$

7. Examples

In this section, we compute our invariants for two simple 2-component links.

Example 1. Torus link of type (6.2).



$G(L) = \langle x, y : [\eta, x] = 1,\ [\xi, y] = 1 \rangle$
where $\eta = x^{-1}yxyxyx^{-1}$ and
$\xi = xyxyxy^{-2}$.

$lk(X, Y) = 3$ and $\Delta(x, y) = 1 + xy + x^2y^2$.

Let $n = 3$ and $q_n(t) = 3t + 3t^2 + t^3$. Then for $k \geqslant 4$.

$$\begin{cases} \eta_k(t) = 3 + t^2 + 2t^3 + t^4 \\ \xi_k(t) = t^2 + 2t^3 + t^4 \end{cases}$$

$$\begin{cases} \bar{\eta}_k(t) = 3 + t^2 + 2t^3 + t^4 \\ \bar{\xi}_k(t) = t^2 + 2t^3 + t^4. \end{cases}$$

Example 2. Whitehead link.

$G(L) = \langle x, y : [\eta, x] = 1,\ [\xi, y] = 1 \rangle$
where $\eta = y^{-1}xyx^{-1}yxy^{-1}x^{-1}$ and
$\xi = x^{-1}yxy^{-1}xyx^{-1}y^{-1}$.

$lk(X, Y) = 0$ and $\Delta(x, y) = (1 - x)(1 - y)$.

Let $n = 3$ and $q_3(t) = 3t + 3t^3 + t^3$. Then for $k \geqslant 4$,

$$\begin{cases} \eta_k(t) = 0 \\ \xi_k(t) = t^2 + 2t^3 + t^4 \end{cases}$$

$$\begin{cases} \bar{\eta}_k(t) = t^2 + 2t^3 + t^4 \\ \bar{\xi}_k(t) = 0. \end{cases}$$

Let $n = 0$ and $q_0(t) = 0$. Then for any $k \geqslant 0$,

$$\begin{cases} \eta_k(t) = 0 \\ \xi_k(t) = (1 + t)^{-1} - 2 + (1 + t) \\ \qquad\ \ = t^2 - t^3 + t^4 - \dots . \end{cases}$$

References

[1] R. H. Fox. Free Differential Calculus, I, *Ann. of Math.* **57** (1953), 547-560.
[2] —. Metacyclic Invariants of Knots and Links, *Can. J. Math.* **22** (1970), 193-201.
[3] R. Hartley. Metabelian Representations of Knot Groups, *Pacific J. Math.* **82** (1979), 93-104.
[4] R. Hartley and K. Murasugi. Covering Linkage Invariants, *Can. J. Math.* **29** (1977) 1312-1339.
[5] R. Holmes and N. Smythe. Algebraic Invariants of Isotopy of Links, *Amer. J. Math.* **88** (1966) 646-654.
[6] S. Kojima and M. Yamasaki. Some New Invariants of Links, *Invent. Math.* **54** (1979), 213-228.
[7] T. Maeda and K. Murasugi. Covering Linkage invariants and Fox's Problem 13, *Contemporary Math.* **20** (1983) 271-283.
[8] J. Milnor. Isotopy of Links, *Lefschetz Symposium, Princeton Mathematical Series, 12* (1957) 280-306.
[9] K. Murasugi, Nilpotent coverings of Links and Milnor's Invariant, (to appear).
[10] —. On the Height of 2-Component Links, (to appear).

K. Murasugi
University of Toronto
Toronto M5S 1A1
Canada

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 1, N.º 1, 1985

The Concentration-Compactness Principle in the Calculus of Variations. The limit case, Part 1

P. L. Lions

Abstract

After the study made in the locally compact case for variational problems with some translation invariance, we investigate here the variational problems (with constraints) for example in $\mathbb{R}^N$ where the invariance of $\mathbb{R}^N$ by the group of dilatations creates some possible loss of compactness. This is for example the case for all the problems associated with the determination of extremal functions in functional inequalities (like for example the Sobolev inequalities). We show here how the concentration-compactness principle has to be modified in order to be able to treat this class of problems and we present applications to Functional Analysis, Mathematical Physics, Differential Geometry and Harmonic Analysis.

Key-words

Concentration-compactness principle, minimization problems, unbounded domains, dilatations invariance, concentration function, nonlinear field equations, Dirac masses, Morse theory, Sobolev inequalities, convolution, Yamabe problem, scalar curvature, conformal invariance, trace inequalities.

Subject AMS classification

49 A 22, 49 A 34, 49 F 15, 58 E 30, 58 E 20, 58 3 05, 46 E 35, 46 E 30.

Introduction

We have studied in the preceding parts (P. L. Lions [20], [21]) *variational problems* set in *unbounded domains*, where the unboundedness induces some possible *loss of compactness* (a classical example of such loss of compactness is the well-known fact that Rellich Teorem does not hold on unbounded domains like $\mathbb{R}^N$ for example). Roughly speaking we had to take care in [20], [21] of the difficulty caused by the invariance of $\mathbb{R}^N$ by the *non-compact group of translations.*

We want to study here, in a systematic way, variational problems where not only compactness may be lost because of translations but also because of the invariance of $\mathbb{R}^N$, say, by the *non-compact group of dilations*. This difficulty was absent from [20], [21] since we were interested there in the so-called locally compact case, while it is encountered there when studying the so-called limit-cases problems, or problems with limit exponents (see below for concrete examples).

Before giving examples and explaining the statements above, we would like to mention that most of the problems considered below have their origins in Geometry and in Mathematical Physics and have been studied by many authors. In particular we refer to the fundamental studies of T. Aubin [3] on the Yamabe problem; J. Sacks and K. Uhlenbeck [32], Y. T. Siu and S. T. Yau [34] on harmonic mappings; and of K. Uhlenbeck [41], [42], C. Taubes [36], [37], [38].

The dilations invariance of $\mathbb{R}^N$ is a typical difficulty in the study of the existence of *extremal functions* in *functional inequalities*; indeed if A is a linear bounded operator from a Banach space E into another Banach space F, one may consider the smallest positive constant C_0 such that the following inequality holds for all u in E;

$$||Au||_F \leqslant C_0||u||_E; \tag{1}$$

and one may ask whether the *best constant* C_0 is obtained for some u. Now if E, F are functional spaces, it is often the case that (1) is preserved if we perform a *scale change* that is if we replace $u(\cdot)$ by $u(\cdot\,/\sigma)$ for $\sigma > 0$. Of course the question concerning C_0 is equivalent to the solution of the following minimization problems:

$$\mathrm{Min}\{\,||u||_E/u \in E,\ ||Au||_F = 1\,\} \tag{2}$$

or

$$\mathrm{Min}\{\,-\,||Au||_F/u \in E,\ ||u||_E = 1\,\}; \tag{2$'$}$$

and the invariance of (1) by scale changes is often reflected by the invariance of $||\cdot||_E$ or $||\cdot||_F$ by changes such as:

$$u(\cdot) \to \sigma^{-\alpha}u\left(\frac{\cdot}{\sigma}\right)$$

where α depends on A, E, F. And this invariance will imply compactness defects on minimizing sequences of problems (2)-(2$'$).

Let us give a few examples of such situations:

EXAMPLE 1. *Sobolev inequalities.*

Let $1 \leqslant p < N/m$, $m \geqslant 1$ and let E be the Banach space consisting of functions in $L^q(\mathbb{R}^N)$ with $q = Np/(N - mp)$ such that all their derivatives of order m are in $L^p(\mathbb{R}^N)$; E is equipped for example with the norm $||D^m u||_{L^p(\mathbb{R}^N)}$. The so-called Sobolev embedding theorem (or Sobolev inequality) yields that E is continuously embedded in $F = L^q(\mathbb{R}^N)$. Therefore the question of extremal functions in the Sobolev inequealities

$$||u||_{L^q(\mathbb{R}^N)} \leqslant C_0||D^m u||_{L^p(\mathbb{R}^N)} \tag{3}$$

is an example of the above framework—A being the injection of E into F. The associated minimization problem is, for example,

$$\mathrm{Min}\left[\int_{\mathbb{R}^N} |D^m u|^p\,dx/u \in E,\ \int_{\mathbb{R}^N} |u|^q\,dx = 1\right]. \tag{4}$$

One then checks easily that if we replace u by $\sigma^{-N/q}u(\cdot\,/\sigma)$ for any $\sigma > 0$, the two functionals occuring in the above variational problem are preserved (this invariance being nothing else than the invariance of Sobolev inequalities (3) with respect to scale changes).

EXAMPLE 2. *Hardy-Littlewood-Sobolev inequalities.*

Let $0 < \mu < N$, $1 < p < (N/(N - \mu))$ and let q satisfy: $(1/p) + (\mu/N) = 1 + (1/q)$. The Hardy-Littlewood-Sobolev inequality then states

$$||K * u||_{L^q(\mathbb{R}^N)} \leqslant C_0 ||u||_{L^p(\mathbb{R}^N)}, \qquad \forall u \in L^p(\mathbb{R}^N) \tag{5}$$

where $K = 1/|x|^\mu$. The determination of the best C_0 is then equivalent to

$$\mathrm{Min}\Big[- \int_{\mathbb{R}^N} |K * u|^q \, dx / u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p \, dx = 1 \Big] \tag{6}$$

(that is (2′), with $E = L^p$, $F = L^q$, $Au = K * u$). Again the two functionals are invariant by the transformation: $u \to \sigma^{-N/q} u(\cdot/\sigma)$ for all $\sigma > 0$.

EXAMPLE 3. *Trace inequalities.*

Let $1 \leqslant p < N$, $N \geqslant 2$, $m \geqslant 1$ and let q be given by: $q = (N-1)p(N - mp)^{-1}$. It is well—known that there exists a bounded linear operator A—called the trace operator—from $E = \{u \in L^\alpha(\mathbb{R}^{N-1} \times \mathbb{R}_+), D^m u \in L^p(\mathbb{R}^{N-1} \times \mathbb{R}_+)\}$ with $\alpha = Np/(N - mp)$ equipped with the same norm as in Example 1 into $F = L^q(\mathbb{R}^{N-1})$ such that if $u \in \mathfrak{D}(\mathbb{R}^N)$, Au is the usual trace of u on $\mathbb{R}^{N-1} \times \{0\}$. For obvious reasons we still denote Au by u. In this context, problem (2) becomes

$$\mathrm{Min}\Big[\int_{\mathbb{R}^{N-1} \times \mathbb{R}_+} |D^m u|^p \, dx / u \in E, \int_{\mathbb{R}^{N-1}} |u(x', 0)|^q \, dx' = 1 \Big]. \tag{7}$$

And again both functionals are preserved if we replace u by $\sigma^{-(N-1)/q} u(\cdot/\sigma)$.

There are of course may more examples of this type (some are discussed in the following section). Let us now explain on these examples what we mean by loss of compactness induced by the dilations group (or the scale change invariance). This can be easily seen on the fact that, even if we know there exists a minimum in (2), (2′), (4), (6) or (7), the set of minima is not relatively compact in E: indeed if u is a minimum then $\sigma^{-\alpha} u(\cdot/\sigma) = u_\sigma$ would still be a minimum for all $\sigma > 0$ ($\alpha = N/q$ in Examples 1, 2, $\alpha = (N-1)/q$ in Example 3). Now if $\alpha \to 0$ or $\sigma \to \infty$, u_σ converges weakly to 0 (which is not a minimum) and the probability $|u_\sigma|^q$ (or $|u_\sigma|^p$) either converges weakly as $\sigma \to 0$ to a Dirac mass or spreads out (vanishing in Lemma I.1 of [20]) as $\sigma \to \infty$. This loss of compactness may be also seen on the fact that q in the various examples is a limit exponent and that if we consider only functions with support in a fixed bounded domain and if q is replaced by a smaller exponent, then the various minimization problems are standard consequences of the Rellich theorem. Notice also that the set of minima in (2), (2′), (4), (6) or (7) is also translation invariant therefore we also have the loss of compactness induced by the translation invariance, as we had in the problems studied in [20], [21].

We present here a *general method to solve variational problems* (with constraints) where such difficulties are encountered, that is problems with *limit exponents* or with a *scale change invariance* or problems like (2), (2') in functional spaces. In particular our methods enable us to prove that *any minimizing sequence of problems* (4), (6) *or* (7) *is relatively compact in E* up to a translation and a scale change(*). In particular there exists a minimum; this last assertion has been proved in Example 1 for the particular case of $m = 1$ by Rosen [31], G. Talenti [35], T. Aubin [4] and in Example 2 by E. H. Lieb [18] but all these works depend on the use of symmetrization and therefore cannot be extended to cover fully examples 1-3. Let us mention a few other applications of our methods.

EXAMPLE 4. *Yamabe problem in $\mathbb{R}^N$.*

An important problem in differential geometry is the so-called Yamabe conjecture or Yamabe problem (this problem will be explained in detail later on, see Yamabe [44], N. Trudinger [39], Eliasson [14] and T. Aubin [3]). We will come back below on the case when the problem is set on a compact manifold but here we restrict our attention to $\mathbb{R}^N$-prototype of a complete but non-compact manifold. We look for a positive function u in $\mathbb{R}^N$ solution of

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + k(x)u = K(x)u^{(N+2)/(N-2)} \text{ in } \mathbb{R}^N, \quad u > 0 \text{ in } \mathbb{R}^N \tag{8}$$

where a_{ij}, k, K are smooth functions, (a_{ij}) is symmetric definite positive. First, if we look for a solution which vanishes at infinity, our general method enables us to study completely the variational problems associated with (8).

Next, if we consider solutions which remain positive at infinity, we also solve similar variational problems where we look for functions which converge to a given positive constant at infinity. However in that case, we need severe restrictions on k, K. In [28], Ni proposed a different approach of (8) by the method of sub and supersolutions —that we recall in an appendix— which gives a very general existence result. Roughly speaking, one can find an interval $]0, \bar{\mu}[$ such that if $0 < \mu < \bar{\mu}$, *there exists a minimum solution $\underline{u}$* of (8) such that: $\underline{u}(x) \to \mu$ as $|x| \to \infty$. *We prove below* that under quite general assumptions, *there exists a second solution u of* (8) such that: $u(x) > \underline{u}(x)$ on $\mathbb{R}^N$, $u(x) \to \mu$ as $|x| \to \infty$. This is achieved in the appendix by looking at the problem satisfied by $(u - \underline{u})$ and by solving the associated variational problem by our concentration-compactness method.

(*) Of course if $p = 1$ in Examples 1,3; L^1 has to be replaced by the space of bounded measures.

EXAMPLE 5. *Nonlinear field equations.*

In various domains of Mathematical Physics one encounters the following nonlinear problem

$$-\Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N, \qquad u(x) \to 0 \quad \text{as} \quad |x| \to \infty \tag{9}$$

(here to simplify the presentation, we take scalar functions u). Of particular interest is the so-called ground state solution which, if it exists, is the minimum of the following problem (see for instance Coleman, Glazer and Martin [13], H. Berestycki and P. L. Lions [6])

$$I = \mathrm{Min}\Big[\int_{\mathbb{R}^N} |Du|^2\,dx \Big/ \int_{\mathbb{R}^N} F(u)\,dx = 1,\, u \in L^{2N/(N-2)}(\mathbb{R}^N), \\ Du \in L^2(\mathbb{R}^N), F(u) \in L^1(\mathbb{R}^N)\Big]. \tag{10}$$

where $F(t) = \int_0^t f(s)\,ds$, $N \geqslant 3$ (to simplify). In view of both known existence results on this problem (see the references above and their bibliographies), the behaviour of F at 0 and at ∞ is known to be determinant: more precisely in all known existence results of a minimum in (10), F is supposed to satisfy

$$\lim_{|t| \to 0_+} F(t)|t|^{-(2N/(N-2))} \leqslant 0, \qquad \lim_{|t| \to \infty} F(t)|t|^{-(2N/(N-2))} \leqslant 0.$$

Our method enables us to give a much more general condition for the existence of a minimum in (10) which will cover both the situation above and the case of the best Sobolev constant i.e. $F(t) = |t|^{2N/(N-2)}$. We assume that $F \in C(\mathbb{R})$, $F(0) = 0$ and

$$\exists \zeta \in \mathbb{R}, \qquad F(\zeta) > 0 \qquad \text{and} \tag{11}$$

$$\lim_{|t| \to 0_+} F^+(t)|t|^{-(2N/(N-2))} = \alpha \geqslant 0, \qquad \lim_{|t| \to \infty} F^+(t)|t|^{-(2N/(N-2))} = \beta \geqslant 0 \tag{12}$$

(of course if $\alpha, \beta > 0$, F^+ may be replaced by F); and we denote by

$$I^\infty = \mathrm{Min}\Big[\int_{\mathbb{R}^N} |Du|^2\,dx \Big/ \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\,dx = 1\Big];$$

(cf. Example 1 above). Then *we prove that any minimizing sequence* (u_n) *is relatively compact in* $L^{2N/(N-2)}(\mathbb{R}^N)$ (and $F(u_n)$ is relatively compact in $L^1(\mathbb{R}^N)$) *up to a translation if and only if*

$$I < \{\max(\alpha, \beta)\}^{-(N-2)/N} I^\infty \tag{13}$$

(if $\alpha = \beta = 0$, (13) holds automatically). We also prove that, if we allow the equality, (13) always holds, any minimizing sequence is always relatively compact in $L^{2N/(N-2)}(\mathbb{R}^N)$ up to a translation and a scale change (and we can analyse what happens exactly when the minimizing sequence is not compact up to a translation).

On those two examples, we see that the problem is not invariant under the action of the group of dilations; nevertheless the underlying invariance of $\mathbb{R}^N$ by dilations plays a crucial role in our solution of such problems. But even the full invariance of $\mathbb{R}^N$ by dilations is not needed, only the local part of it plays a role and this explains why our method also applies to problems set in regions different from $\mathbb{R}^N$. These regions may be compact —in which case the group of translations does not induce any more some form of loss of compactness— as is the case in the following typical example of such problems.

EXAMPLE 6. *Yamabe problem on compact manifolds.*

Let (M, g) be some N dimensional compact Riemannian manifold, a general open question is the determination of the class $\mathcal{C}$ of functions on M which can be achieved as scalar curvatures of metrics $\tilde{g}$ (pointwise) conformal to g. To solve this problem, one introduces for some positive function u on M a new metric given by: $\tilde{g} = u^{4/(N-2)}g$; we assume $N \geqslant 3$. Then if we denote by Δ the Laplace-Beltrami operator on (M, g) and by k the scalar curvature, one checks (see [3]) that the scalar curvature of $\tilde{g}$ is given by

$$\tilde{k} = \left\{ -\frac{4(N-1)}{N-2}\Delta u + ku \right\} u^{-(N+2)/(N-2)}.$$

Therefore K —a given function on M— belongs to $\mathcal{C}$ if there exists u solution of

$$-\frac{4(N-1)}{N-2}\Delta u + ku = Ku^{(N+2)/(N-2)} \quad \text{in} \quad M, u > 0 \quad \text{on} \quad M. \tag{14}$$

And up to some multiplicative constants such a u exists if we find a minimum of

$$I = \operatorname{Inf}\left[\int_M |\nabla u|^2 + ku^2 dV / u \in H^1(M), \int_M K|u|^{2N/(N-2)}\, dV = 1\right], \tag{15}$$

where k, K are given functions in $C(M)$.

Under natural assumptions on $(-\Delta + k)$ and K, we prove below that for any minimizing sequence (u_n) weakly convergent to some u then: either u is a minimum of (15) and (u_n) converges in H^1 to u, or $u \equiv 0$ and there exists $x_0 \in M$ such that

$$K(x_0) = \max_M K, \qquad |u_n|^{2N/(N-2)} \to \alpha\delta_{x_0}, \qquad |\nabla u_n|^2 \to \beta\delta_{x_0}$$

for some $\beta > 0$, and where $\alpha = 1/\max K$ (the above convergence is for the weak topology of bounded measures on M). This immediately yields the following result due to *T.* Aubin [3]: if we have

$$I < \left(\max_M K\right)^{-(N-2)/N} I^\infty \tag{16}$$

(where I^∞ is given as in Example 5), then there exists a minimum in (15) (actually we prove that any minimizing sequence is relatively compact in $H^1(M)$ if and only if (16) holds). And we refer to [3] for a sharp discussion of (16).

In fact, we present below more examples: in particular we will present the recent results of *H.* Brézis and J. M. Coron on the Rellich conjecture [8] and on harmonic maps [9] in the light of our systemtic tratment of such problems; and we will explain how it is possible to recover the results of Jacobs [15] on holomorphic functions by our general approach....

At that stage, we would like to explain the main lines of our method: roughly speaking in all the problems listed above, the main difficulty — created by the possible loss of compactness— is due to the fact that *some functional is not weakly continuous* and that *strong compactness is not a priori available.* Then, in the same spirit as in Parts 1 and 2 [20], [21] where we explained what were the two possible forms of "non compactness" due to unbounded domains, we investigate here what happens when passing to the limit on those functionals along weakly convergent sequences. We use basically some general compactness lemma which, roughly speaking, tells that weakly convergent sequences are converging strongly except possibly at "isolated" points where Dirac masses appear in the densities of the functionals. And this is of course a local property. A typical example is the following.

Lemma. *Let $(u_n)_n$ be a bounded sequence in $W^{m,p}(\Omega)^{(*)}$ for some $m > 0$, $p \in [1, N/m[$, and a bounded smooth domain Ω of $\mathbb{R}^N$. We may assume that u_n converges weakly in $W^{m,p}$ to some u and that $|u_n|^q$ converges weakly in the sense of measures to some ν, where $q = Np(N-mp)^{-1}$. Then there exist $(x_i)_{i\ge 1}$ in $\bar{\Omega}$, $(\nu_i)_{i\ge 1}$ in $[0, \infty[$ such that*

$$\nu = |u|^q + \sum_{i=1}^\infty \nu_i \delta_{x_i}, \qquad \sum_{i=1}^\infty \nu_i^{p/q} < \infty.$$

Actually we obtain more information on ν_i, x_i and we show that any such measure ν can be obtained as the weak limit of $|u_n|^q$ for some bounded sequence (u_n) in $W^{m,p}(\Omega)$ weakly convergent to u. In addition such a result is not at all restricted to Sobolev spaces but is based upon the underlying invariance by dilations.

Let us also emphasize that such a phenomenon of "energy" concentrations at points was first observed by J. Sacks and K. Uhlenbeck [32] in the study of harmonic mappings: see also Y. T. Siu and S. T. Yau [34]; S. Sedlacek [33]

If $p = 1$, we replace $L^1(\Omega)$ by the bounded measures on Ω.

for similar observations in the context of Yang-Mills equations and K. Uhlenbeck [43] for a general presentation. Let us only mention that this lemma is very simple and holds for *arbitrary* sequences (u_n).

With the help of such results, we are able to decide what happens to the functionals if the minimizing sequence (u_n) is not compact. Roughly speaking, u_n breaks in two parts u and $(u_n - u) = \tilde{u}_n$ which "concentrates around the isolated points x_i". Then this enables us to conclude that all minimizing sequences are relatively compact if and only if some strict subadditivity inequalities hold, exactly like in [20], [21]. Those inequalities with equalities allowed always hold and they involve, like in [20], [21], a notion of problem at infinity which is essentially obtained by using the dilation invariance of $\mathbb{R}^N$ (or the local invariance for other domains) and concentrating a test function around any fixed point of the domain.

In examples 1, 2, 3, those inequalities hold because of the homogeneity of the problem and the conclusion is reached, while in examples 4, 5, 6, only one of these collections of inequalities does not always hold and this explains the role of the strict inequalities that we mentioned in Examples 5, 6.

Despite the generality of the argument and of the approach, we postpone its general presentation until section III, while in section I we treat examples 1, 4, 5, in section II we treat examples 2, 3. Finally section IV is devoted to various problems in compact regions like example 6.

The results presented here were announced in [25], [26] and combined with those of P.L. Lions [20], [21] are the subject of lectures given at Collège de France for the Cours Peccot.

Finally, it is a pleasure to thank H. Brézis and J. M. Coron for several discussions and their interest in this work and to acknowledge that some of the questions treated here are motivated by E. H. Lieb's work [18].

Let us warn the reader that this work is divided in two parts: Part 1 consists of Section I, while the remainder is contained in Part 2. Notations are identical for both parts.

I. Sobolev inequalities and extremal functions

I.1 The main result

Let m be an integer (to simplify) $\geqslant 1$, let $p \in [1, \infty[$. If $N \geqslant 2$, $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we denote by $|D^m\varphi(x)|$ any product norm of all derivatives of order m at the point x. The classical Sobolev inequality states that if $p < (N/m)$, $q = Np(N - mp)^{-1}$ then there exists a positive constant C_0 such that for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |\varphi|^q \, dx\right)^{1/q} \leqslant C_0 \left(\int_{\mathbb{R}^N} |D^m\varphi|^p \, dx\right)^{1/p}. \tag{3}$$

We then denote by $\mathfrak{D}^{m,p}$ the completion of $\mathfrak{D}(\mathbb{R}^N)$ for the norm

$$||u|| = \left(\int_{\mathbb{R}^N} |D^m\varphi|^p \, dx\right)^{1/p};$$

actually for the special case $p = 1$, we consider directly $\mathfrak{D}^{m,1}$ as the space of u in $L^q(\mathbb{R}^N)$ such that $D^m u \in M_b(\mathbb{R}^N)$. The Sobolev inequality then holds for any $\varphi \in \mathfrak{D}^{m,p}$. In order to decide whether the best constant C_0 is achieved, we have to determine whether the following minimization problem has a minimum

$$I = \operatorname{Inf}\left(\int_{\mathbb{R}^N} |D^m u|^p \, dx / u \in \mathfrak{D}^{m,p}, \int_{\mathbb{R}^N} |u|^q \, dx = 1\right); \tag{4}$$

(we will also write $I = I_1$ and I_λ will be the value of the infimum of the same problem but with 1 replaced by λ).

Theorem I.1. *Every minimizing sequence $(u_n)_n$ of (4) is relatively compact in $\mathfrak{D}^{m,p}$ up to a translation and a dilation i.e. there exist $(y_n)_n$ in $\mathbb{R}^N$, $(\sigma_n)_n$ in $]0, \infty[$ such that the new minimizing sequence $\tilde{u}_n = \sigma_n^{-N/q} u_n(\cdot - y_n/\sigma_n)$ is relatively compact in $\mathfrak{D}^{m,p}$ for $p > 1$ and in L^q for $p = 1$ (in this case $|D^m u_n|^p$ is tight).*

In particular there exists a minimum of (4).

In the case when $m = 1$, this result implies easily the

Corollary I.1. *If $m = 1$, $p > 1$; any minimum u of (4) is given by*

$$u(x) = \sigma^{-N/q} u_1\left(\frac{\cdot - y}{\sigma}\right) \quad \textit{where} \quad y \in \mathbb{R}^N, \quad \sigma > 0$$

and $u_1(x) = \{1 + bx^{p/(p-1)}\}^{(p-N)/p}$, where $b > 0$ depends explicitly on p, N and I below. Moreover we have

$$I = \pi^{p/2} N\{(p-1)(N-p)^{-1}\}^{-(p-1)/p} \left\{\frac{\Gamma(1 + N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(1 + N - N/p)}\right\}^{p/N}.$$

Remark I.1. The value I and the fact that u, u_1 are minima were found by G. Rosen [31], G. Talenti [35], T. Aubin [4] and this was based upon a symmetrization argument and some optimal one-dimensional bounds discovered by G. A. Bliss [7].

Remark I.2. Of course Corollary I.1 holds with the norm $|Du|$ chosen to be the usual norm on $\mathbb{R}^N$.

We begin with the proof of Corollary I.1 using Theorem I.1: by Theorem I.1 we know there exists a minimum u of (4). Now by a simple use of Schwarz symmetrization we see that $v = u^*$ is also a minimum and thus $v_\sigma = \sigma^{-N/q} \cdot$ $\cdot\, v(x/\sigma)$ is a minimum for all $\sigma > 0$. In addition, v_σ being spherically symmetric, v solves the O.D.E. form of the Euler-Lagrange equation associated with (4) namely

$$\begin{cases} -(p-1)|v_\sigma'|^{p-2}v_\sigma'' - \dfrac{N-1}{r}|v_\sigma'|^{p-1} = Iv_\sigma^{q-1} \quad \text{for} \quad r > 0 \\ v_\sigma'(0) = 0, \qquad v_\sigma \geqslant 0, \qquad v_\sigma' \leqslant 0, \qquad v_\sigma(0) = \sigma^{-N/q}. \end{cases}$$

And remarking that for any constant I, there exists a constant b (which can be computed explicitly) such that the unique solution of this O.D.E. is given by

$$v_\sigma(r) = \sigma^{\mu}\{\sigma^{p/(p-1)} + br^{p/(p-1)}\}^{(p-N)/p}$$

where $\mu = (N-p)/(p(p-1))$. Computing the L^q norm of v (or v_σ) one then gets the values of b, I. Finally from the fact that both u and $v = u^*$ solve the same Euler equation, we conclude as in A. Alvino, P. L. Lions and G. Trombetti [1] there exists $y \in \mathbb{R}^N$, $u(y + \cdot) = u^*(\cdot)$.

Theorem I.1 is proved in the next section but we would like to explain the general scheme of proof here. First of all we saw in P.L. Lions [20], [21] that, if (u_n) is a minimizing sequence of (4), a crucial quantity is the concentration function of $|u_n|^q$. For technical reasons we have to consider the concentration function of

$$\rho_n = \sum_{j=0}^{m} |D^j u_n|^{q_j}$$

where $q_j = Np(N-(m-j)p)$. We denote by $L_n = \int_{\mathbb{R}^N} \rho_n\, dx$, of course: $L_n \geqslant$ $\geqslant \int_{\mathbb{R}^N} |u_n|^q + |D^m u_n|^p\, dx \geqslant 1 + I$, and L_n being bounded we may assume without loss of generality that $L_n \underset{n}{\to} L \geqslant 1 + I$.

In "locally compact" problems the occurrence of vanishing (see [20], [21] for more details) was easily avoided. On the other hand, here vanishing may occur since the concentration function $Q_n{}^\sigma$ of $u^\sigma(\cdot) = \sigma^{-N/q} u_n(\cdot/\sigma)$ is given by

$$Q_n^\sigma(t) = Q_n(t/\sigma) \quad \text{for} \quad t \geqslant 0,$$

(and playing with $\sigma = \sigma_n \underset{n}{\to} \infty$, one may build minimizing sequences for which vanishing occurs). We will avoid vanishing by choosing $(\sigma_n)_n$ in $[0, -\infty[$ such that

$$Q_n^{\sigma_n}(1) = 1/2 \tag{17}$$

(of course we could replace 1 by any $R \in]0, \infty[$, 1/2 by any $\theta \in]0, 1[$) indeed $Q_n^\sigma(1) = Q_n(1/\sigma)$ and Q_n is a non-decreasing continuous function such that

$$Q_n(0) = 0, \qquad 1 \leqslant \lim_{t \uparrow \infty} Q_n(t).$$

In what follows, we will still denote by u_n the new minimizing sequence $u_n^{\sigma_n}$ and by Q_n the associated concentration function, hence we have by (17): $Q_n(1) = 1/2$.

The proof given in the next section is organized as follows: *Step* 1: Using (17) and the concentration-compactness argument of [20] [21], we will show that ρ_n is up to a translation a tight sequence of bounded measures on $\mathbb{R}^N$; *Step* 2: Using again (17), we will check that u_n does not converge weakly to 0; *Step* 3: we conclude by proving that u_n converges weakly to u satisfying: $\int_{\mathbb{R}^N} |u|^q \, dx = 1$. Both Steps 2 and 3 will rely on a Lemma stated in section I.2 and proved in section I.3.

I.2. PROOF. In what follows we will denote by (u_n) all subsequences extracted from the original sequence (u_n).

Step 1. In view of (17), if $Q_n(t) \underset{n}{\to} Q(t)$ for some non-decreasing, non-negative function Q on $\mathbb{R}_+$, we have

$$0 < 1/2 = Q(1) \leqslant Q(t) \leqslant C, \qquad \forall 1 \leqslant t < +\infty.$$

Applying the method of [20], [21], in order to prove that *there exists* (y_n) *in* $\mathbb{R}^N$ *such that* $\rho_n(\cdot - y_N)$ *is tight on* $\mathbb{R}^N$, *we* just have to show that *dichotomy cannot occur.* In order to prove this claim, we assume that dichotomy occurs and we will reach a contradiction since: $I_\lambda = \lambda^{p/q} I$, thus

$$I = I_1 < I_\alpha + I_{1-\alpha}, \qquad \forall \alpha \in]0, 1[$$

(i.e. (S. 2) holds!). Therefore we assume that there exists $\bar{\alpha} \in]0, L[$ such that for all $\epsilon > 0$

$$\begin{cases} \exists y_n \in \mathbb{R}^N, \quad \exists R_0, R_n > 0, \quad R_n \succ R_0 \quad \text{and} \quad R_n \underset{n}{\to} \infty. \\ |\bar{\alpha} - \int_{y_n + B_{R_0}} \rho_n dx|, \quad \int_{R_0 \leq |x - y_n| \leq R_n} \rho_n \, dx \leqslant \epsilon. \end{cases} \tag{18}$$

Let $\xi, \eta \in C_b^\infty(\mathbb{R}^N)$ satisfying: $0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant 1$, $\xi = 1$ if $|x| \leqslant 1$, $\xi = 0$ if $|x| \geqslant 2$, $\eta = 1$ if $|x| \geqslant 1$, $\eta = 0$ if $|x| \leqslant 1/2$. We denote by $\xi_n = \xi((x - y_n)/R_1)$, $\eta_n = \eta((x - y_n)/R_n)$ where $R_1 \geqslant R_0$ is determined below. We then have

$$\left| \int_{\mathbb{R}^N} |D^m u_n|^p \, dx - \int_{\mathbb{R}^N} |D^m(\xi_n u_n)|^p \, dx - \int_{\mathbb{R}^N} |D^m(\eta_n u_n)|^p dx \right| \leqslant C(X_n^p + X_n) + \epsilon$$

provided n is large enough so that: $4R_1 \leqslant R_n$; and where

$$X_n = \left(\int_{\mathbb{R}^N} \sum_{j=0}^{m-1} \{ |D^{m-j}\xi_n|^p + |D^{m-j}\eta_n|^p \} |D^j u_n|^p \, dx \right)^{1/p}.$$

Using Hölder inequalities, we obtain

$$X_n^p \leqslant C \sum_{j=0}^{m-1} \left(\int_{\mathbb{R}^N} |D^{m-j}\xi_n|^{p_j} + |D^{m-j}\eta_n|^{p_j}\,dx\right)^{p/p_j} \cdot$$
$$\cdot \left(\int_{R_0 \le |x-y_n| \le R_n} |D^j u_n|^{q_j}\,dx\right)^{p/q_j}$$

where $p_j/p = (q_j/p)'$. We deduce in view of (18)

$$X_n^p \leqslant C\epsilon \sum_{j=0}^{m-1} \left(\int_{\mathbb{R}^N} |D^{m-j}\xi_n|^{p_j} + |D^{m-j}\eta_n|^{p_j}\,dx\right)^{p/p_j}$$

and

$$\int_{\mathbb{R}^N} |D^{m-j}\xi_n|^{p_j} + |D^{m-j}\eta_n|^{p_j}\,dx = \int_{\mathbb{R}^N} |D^{m-j}\xi|^{p_j} + |D^{m-j}\eta|^{p_j}\,dx$$

since $(m-j)p_j = N$. We obtain finally

$$\left|\int_{\mathbb{R}^N} |D^m u_n|^p\,dx - \int_{\mathbb{R}^N} |D^m u_n^1|^p\,dx - \int_{\mathbb{R}^N} |D^m u_n^2|^p\,dx\right| \leqslant C(\epsilon^{1/p} + \epsilon). \tag{19}$$

where $u_n^1 = \xi_n u_n$, $u_n^2 = \eta_n u_n$.

Without loss of generality we may assume that

$$\int_{\mathbb{R}^N} |u_n^1|^q\,dx \underset{n}{\rightarrow} \alpha, \qquad \int_{\mathbb{R}^N} |u_n^2|^q\,dx \underset{n}{\rightarrow} \beta$$

and $0 \leqslant \alpha, \beta \leqslant 1$, $|\beta - (1-\alpha)| \leqslant \epsilon$.

We claim that for all ϵ small enough $|D^m u_n^i|_{L^p}$ remains for $i = 1, 2$ bounded away from 0: indeed the above proof shows that

$$\left|\int \sum_{j=0}^{m} |D^j u_n^1|^{q_j}\,dx - \bar{\alpha}\right| \leqslant C(\epsilon^{1/p} + \epsilon)$$

$$\left|\int \sum_{j=0}^{m} |D^j u_n^2|^{q_j}\,dx - (L - \bar{\alpha})\right| \leqslant C(\epsilon^{1/p} + \epsilon)$$

and $\bar{\alpha} \in]0, L[$. Therefore let us denote by $\gamma > 0$ some constant such that for all ϵ small and for all n: $\gamma \leqslant |D^m u_n^i|_{L^p}^p$. Next, if for some sequence $\epsilon_k \underset{k}{\rightarrow} 0$, the constant $\alpha_k = \alpha(\epsilon_k)$ either goes to 0 or to 1, we deduce from (19)

$$I \geqslant I + \gamma - \delta(\epsilon_k)$$

where $\delta(t) \rightarrow 0$ as $t \rightarrow 0_+$; and this is not possible. On the other hand if $\alpha_k \underset{k}{\rightarrow} \alpha \in]0, 1[$, $\beta_k \underset{k}{\rightarrow} 1 - \alpha$ and we obtain from (19): $I \geqslant I_\alpha + I_{1-\alpha}$ and again this is not possible.

In conclusion we have proved that there exists (y_n) in $\mathbb{R}^N$ such that: $\forall \epsilon > 0$, $\exists R \in]0, \infty[$

$$\int_{|x-y_n| \geq R} \sum_{j=0}^{m} |D^j u_n|^{q_j}\,dx \leqslant \epsilon. \tag{20}$$

We still denote by $(u_n)_n$ the new minimizing sequence $(\tilde{u}_n)_n$ obtained by

$$\tilde{u}_n(x) = u_n(x + y_n), \qquad \forall x \in \mathbb{R}^N.$$

Without loss of generality we may assume that u_n converges weakly in $\mathfrak{D}^{m,p}$ and a.e. to some $u \in \mathfrak{D}^{m,p}$; and that $D^j u_n$ converges weakly and a.e. to $D^j u$ in $L^{q_j}(\mathbb{R}^N)$.

The next result —that we will call below the *second concentration compactness lemma*— is the crucial tool for the next two steps of the proof of Theorem I.1. Before stating this result let us observe that if $(u_n)_n \subset W^{m,p}(\Omega)$ for some smooth bounded region Ω of $\mathbb{R}^N$, by standard extension theorems we may assume without loss of generality that $(u_n)_n \subset W^{m,p}(\mathbb{R}^N)$ and $|u_n|^q$ is tight (even with some uniform compact support!).

Lemma I.1. *Let $(u_n)_n$ be a bounded sequence in $\mathfrak{D}^{m,p}$ converging weakly to some u and such that $|D^m u_n|^p$ converges weakly to μ and $|u_n|^q$ converges tightly to ν where μ, ν are bounded nonnegative measures on $\mathbb{R}^N$. Then we have:*

(i) *There exist some at most countable set J and two families $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$, $(\nu_j)_{j\in J}$ in $]0, \infty[$ such that*

$$\nu = |u|^q + \sum_{j\in J} \nu_j \delta_{x_j}. \tag{21}$$

(ii) *In addition we have*

$$\mu \geqslant |D^m u|^p + \sum_{j\in J} \mu_j \delta_{x_j} \tag{22}$$

for some $\mu_j > 0$ satisfying

$$\nu_j^{p/q} \leqslant \mu_j / I, \quad \textit{for all } j \tag{23}$$

hence

$$\sum_{j\in J} \nu_j^{p/q} < \infty.$$

(iii) *If $v \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$ and $|D^m(u_n + v)|^p$ converges weakly to some measure $\tilde{\mu}$, then $\tilde{\mu} - \mu \in L^1(\mathbb{R}^N)$; and therefore*

$$\tilde{\mu} \geqslant |D^m(u + v)|^p + \sum_{j\in J} \mu_j \delta_{x_j}.$$

(iv) *If $u \equiv 0$ and: $\left(\int d\mu\right) \leqslant I\left(\int d\nu\right)^{p/q}$; then J is a singleton and: $\nu = \gamma\delta_{x_0} = \mu(I\gamma^{p/q})^{-1}$ for some $\gamma > 0$, $x_0 \in \mathbb{R}^N$.*

The proof of this lemma is given in the next section.

Remark I.3. We claim that if $u \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$, J is an at most countable set, $(x_j)_{j\in J}$ are distinct points in $\mathbb{R}^N$ and $(\nu_j)_{j\in J}$ are positive numbers such that $\sum_{j\in J} \nu_j^{p/q} < \infty$, then the measure $\nu = |u|^q + \sum_{j\in J} \nu_j \delta_{x_j}$ is the tight limit of a sequence $|u_n|^q$ where u_n converges in $\mathfrak{D}^{m,p}$ to u. Hence, the above result completely characterizes the limits of $|u_n|^q$ for weakly convergent sequences of $\mathfrak{D}^{m,p}$. Of course u_n converges in L^q to u if and only if $\nu = |u|^q$; therefore the loss of compactness (for the Sobolev limit exponent) occurs at a countable number of points x_j (with weights ν_j such $\sum \nu_j^{p/q} < \infty$ and $p/q < 1$).

To prove the above claim, we consider $\varphi \in \mathfrak{D}(\mathbb{R}^N)$ with $\int |\varphi|^q\,dx = 1$ (say) — observe that we can take $\int |D^m\varphi|^p\,dx$ as close to I as we wish — . Then for any $x_0 \in \mathbb{R}^N$, $\varphi_n = \varphi_n = n^{N/q}\varphi(\cdot - x_0/n)$ satisfies

$$\begin{cases} \int |D^m\varphi_n|^p\,dx = \int |D^m\varphi|^p\,dx, & \int |\varphi_n|^q\,dx = 1 \\ |\varphi_n|^q \to \delta_{x_0}, & \varphi_n \underset{n}{\rightharpoonup} 0 \quad \text{in} \quad \mathfrak{D}^{m,p} \quad \text{weakly.} \end{cases}$$

Next for any finite subfamily J' of J, we consider for $n \geqslant n_0(J')$: $\psi_n = \sum_{j\in J'} \nu_j^{1/q}\varphi_n^{x_j}$, $\operatorname{Supp}\varphi_n^{x_j}$ are disjoint for $j \in J'$. Clearly we have

$$\begin{cases} \int |D^m\psi_n|^p\,dx = \left(\sum_{j\in J'} \nu_j^{p/q}\right)\int |D^m\varphi|^p\,dx \leqslant \left(\sum_{j\in J} \nu_j^{p/q}\right)\int |D^m\varphi|^p\,dx \\ \int |\psi_n|^q\,dx = \sum_{j\in J'} \nu_j, \qquad |\psi_n|^q \to \sum_{j\in J'} \nu_j\delta_{x_j}, \qquad \psi_n \underset{n}{\rightharpoonup} 0 \text{ in } \mathfrak{D}^{m,p} \text{ weakly.} \end{cases}$$

Increasing J' to J, we obtain by a diagonal procedure a sequence $\tilde{\psi}_n$ such that

$$\begin{cases} \int |D^m\tilde{\psi}_n|^p\,dx \leqslant \left(\sum_{j\in J} \nu_j^{p/q}\right)\int |D^m\varphi|^p\,dx \\ \int |\tilde{\psi}_n|^q\,dx \underset{n}{\to} \sum_{j\in J} \nu_j, \qquad \tilde{\psi}_n \underset{n}{\rightharpoonup} 0 \quad \text{in} \quad \mathfrak{D}^{m,p} \quad \text{weakly} \\ |\tilde{\psi}_n|^q \to \sum_{j\in J} \nu_j\delta_{x_j} \quad \text{tightly.} \end{cases}$$

We finally set: $u_n = u + \tilde{\psi}_n$, and one easily checks that u_n has the required properties. Actually one even checks that

$$|D^m u_n|^p \to |D^m u|^p + \left(\int |D^m\varphi|^p\,dx\right) \sum_{j\in J} \nu_j^{p/q}\delta_{x_j}.$$

We go back now to the proof of Theorem I.1:

Step 2. u, the weak limit of the minimizing sequence u_n, is not identically 0.

Indeed, in view of (20), we may apply Lemma I.1 (extracting if necessary some subsequences) and we know by (20)

$$\int_{\mathbb{R}^N} d\mu = I, \qquad \int_{\mathbb{R}^N} d\nu = 1. \tag{24}$$

Now if $u \equiv 0$, we may apply part iv) of lemma I.1 and we deduce: $\nu = \frac{1}{I}\mu = \delta_{x_0}$, for some $x_0 \in \mathbb{R}^N$.

On the other hand

$$\frac{1}{2} = Q_n(1) \geqslant \int_{B(x_0,1)} |u_n|^q \, dx \underset{n}{\to} 1;$$

this contradiction shows that $u \not\equiv 0$.

Step 3. u_n converges strongly to u.

Let us denote by $\alpha = \int_{\mathbb{R}^N} |u|^q \, dx$: by step 2 we know that $\alpha \in]0, 1]$ and we have to prove that $\alpha = 1$. Suppose that $\alpha \neq 1$, then applying Lemma I.1, we see

$$\begin{cases} \alpha = \int_{\mathbb{R}^N} |u|^q \, dx, & \sum_{j \in J} \nu_j = 1 - \alpha \\ \mu_j \geqslant I \nu_j^{p/q}, & \int_{\mathbb{R}^N} |D^m u|^p \, dx \leqslant I - \sum_{j \in J} \mu_j. \end{cases}$$

Hence, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |D^m u|^p \, dx &\leqslant I - \sum_{j \in J} \mu_j \\ &\leqslant I\Big(1 - \sum_{j \in J} \nu_j^{p/q}\Big) \\ &< I\Big(1 - \Big(\sum_{j \in J} \nu_j\Big)\Big)^{p/q} = I\alpha^{p/q} \end{aligned}$$

while $\int_{\mathbb{R}^N} |D^m u|^p \, dx \geqslant I_\alpha = I_\alpha = I\alpha^{p/q}$. The contradiction shows that $\alpha = 1$ and we conclude easily.

Remark I.4. We may rewrite the above argument in a way which clearly shows the role of sub-additivity inequalities like (S.2). Indeed

$$I = I_1 \geqslant \int_{\mathbb{R}^N} |D^m u|^p \, dx + \sum_{j \in J} \mu_j \geqslant I_\alpha + I \sum_{j \in J} \nu_j^{p/q} \geqslant I_\alpha + \sum_{j \in J} I_{\nu_j} > I_1 \qquad (!),$$

since we know that I_λ is strictly sub-additive and $\alpha + \sum_{j \in J} \nu_j = 1$.

I.3 The second concentration-compactness lemma

We now prove Lemma I.1: we *first treat the case when* $u \equiv 0$. The goal is to obtain some reversed Hölder inequality between ν and μ which will give the various informations contained in Lemma I.1 via Lemma I.2 below.

Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$, by Sobolev inequalities we have

$$\Big(\int_{\mathbb{R}^N} |\varphi|^q |u_n|^q \, dx\Big)^{1/q} I^{1/p} \leqslant \Big(\int_{\mathbb{R}^N} |D^m(\varphi u_n)|^p \, dx\Big)^{1/p}. \tag{25}$$

The left-hand side member of (25) goes to $\left(\int_{\mathbb{R}^N} |\varphi|^q \, d\nu\right)^{1/q} I^{1/p}$ as n goes to

∞. Now the right-hand side member is estimated as follows

$$\left|\left(\int_{\mathbb{R}^N} |D^m(\varphi u_n)|^p\, dx\right)^{1/p} - \left(\int_{\mathbb{R}^N} |\varphi|^p |D^m u_n|^p\, dx\right)^{1/p}\right| \leqslant$$
$$\leqslant C \sum_{j=0}^{m-1} \left(\int_{\mathbb{R}^N} |D^{m-j}\varphi|^p |D^j u_n|^p\, dx\right)^{1/p}.$$

And using the fact that φ has compact support and the Rellich theorem we see that this bound goes to 0 as n goes to ∞. Therefore, passing to the limit in (25), we obtain for all $\varphi \in \mathfrak{D}(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |\varphi|^q\, d\nu\right)^{1/q} \leqslant I^{-1/p}\left(\int_{\mathbb{R}^N} |\varphi|^p\, d\mu\right)^{1/q}. \tag{26}$$

And lemma I.1 is proved in the case $u \equiv 0$, by the application of

Lemma 1.2. *Let μ, ν be two bounded nonnegative measures on $\mathbb{R}^N$ satisfying for some constant $C_0 \geqslant 0$*

$$\left(\int_{\mathbb{R}^N} |\varphi|^q\, d\nu\right)^{1/q} \leqslant C_0\left(\int_{\mathbb{R}^N} |\varphi|^p\, d\mu\right)^{1/p}, \qquad \forall \varphi \in \mathfrak{D}(\mathbb{R}^N) \tag{26'}$$

where $1 \leqslant p < q \leqslant +\infty$. Then, there exist an at most countable set J, families $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$, $(\nu_j)_{j\in J}$ in $]0, \infty[$ such that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \mu \geqslant C_0^{-p} \sum_{j\in J} \nu_j^{p/q} \delta_{x_j}.$$

Thus, in particular

$$\sum_{j\in J} \nu_j^{p/q} < \infty.$$

If in addition: $\nu(\mathbb{R}^N)^{1/q} \geqslant C_0 \mu(\mathbb{R}^N)^{1/p}$, J reduces to a single point and $\nu = \gamma\delta_{x_0} = \gamma^{-p/q} C_0^p \mu$, for some $x_0 \in \mathbb{R}^N$ and for some $\gamma \geqslant 0$.

Lemma I.2 is proved below; we first conclude the proof of Lemma I.1. We now consider the general case of a weak limit u not necessarily 0. Of course (25) still holds and, if we denote by $v_n = u_n - u$, Brézis-Lieb lemma [10] yields for all $\varphi \in \mathfrak{D}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\varphi|^q |u_n|^q\, dx - \int_{\mathbb{R}^N} |\varphi|^q |v_n|^q\, dx \to \int_{\mathbb{R}^N} |\varphi|^q |u|^q\, dx$$

But, clearly, v_n is bounded in $\mathfrak{D}^{m,p}$ and $|v_n|^q$ is tight; therefore applying what we proved above we obtain the representation (21) of ν. Next, passing to the limit in (25) and using as before Rellich theorem we find for all $\varphi \in \mathfrak{D}(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |\varphi|^q\, d\nu\right)^{1/q} I^{1/p} \leqslant \left(\int_{\mathbb{R}^N} |\varphi|^p\, d\mu\right)^{1/p} + C \sum_{i=0}^{m-1} \left(\int_{\mathbb{R}^N} |D^{m-i}\varphi|^p |D^i u|^p\, dx\right)^{1/p}.$$

If φ satisfies: $0 \leqslant \varphi \leqslant 1$, $\varphi(0) = 1$, $\operatorname{Supp} \varphi = B(0, 1)$, $\varphi \in \mathcal{D}(\mathbb{R}^N)$; we apply the above inequality with $\varphi((x - x_j)/\epsilon)$ for $\epsilon > 0$ and where j is fixed in J. We obtain

$$\nu_j^{1/q} I^{1/p} \leqslant \mu(B(x_j, \epsilon))^{1/p} +$$
$$+ C \sum_{i=1}^{m-1} \left(\int_{B(x_j, \epsilon)} \epsilon^{-p(m-i)} \left| D^{m-i} \varphi\left(\frac{x - x_j}{\epsilon}\right) \right|^p |D^i u|^p dx \right)^{1/p}.$$

Now we may estimate each term of the sum by Hölder inequalities recalling that $D^i u \in L^{q_i}(\mathbb{R}^N)$ (by Sobolev inequalities)

$$\epsilon^{-p(m-i)} \int_{B(x_j, \epsilon)} \left| D^{m-i} \varphi\left(\frac{x - x_j}{\epsilon}\right) \right|^p |D^i u|^p \, dx \leqslant$$
$$\leqslant \left(\int_{B(x_j, \epsilon)} |D^i u|^{q_i} |D^i u|^{q_i} \, dx \right)^{p/q_i} \epsilon^{-p(m-i)} \left(\int_{\mathbb{R}^N} \left| D^{m-i} \varphi\left(\frac{x}{\epsilon}\right) \right|^{p_i} dx \right)^{(q_i - p)/q_i}$$

where $p_i = q_i p (q_i - p)^{-1}$, $(q_i - p)/q_i = (m - i)p/N$. Hence, we have

$$\nu_j^{1/q} I^{1/p} \leqslant \mu(B(x_j, \epsilon))^{1/p} + C \sum_{i=1}^{m-1} \left(\int_{B(x_j, \epsilon)} |D^i u|^{q_i} \, dx \right)^{p/q_i}.$$

This implies that $\mu(\{x_j\}) > 0$ and

$$\mu \geqslant \nu_j^{p/q} I \delta_{x_j}, \qquad \forall j \in J$$

and thus

$$\mu \geqslant \sum_{j \in J} I \nu_j^{p/q} \delta_{x_j} =: \mu_1.$$

Since by weak convergence we also have: $\mu \geqslant |D^m u|^p$ and since $|D^m u|^p$ and μ_1 are orthogonal, (22)-(23) are proved.

Finally to prove part iii) of Lemma I.1 we just observe that for all $\varphi \in C_b(\mathbb{R}^N)$, $\varphi \geqslant 0$

$$\left(\int_{\mathbb{R}^N} \varphi |D^m (u_n + v)|^p \, dx \right)^{1/p} - \left(\int_{\mathbb{R}^N} \varphi |D^m u_n|^p \, dx \right)^{1/p} \Big| \leqslant$$
$$\left(\int_{\mathbb{R}^N} \varphi |D^m v|^p \, dx \right)^{1/p}.$$

Passing to the limit in n, we find

$$\left| \left(\int_{\mathbb{R}^N} \varphi \, d\tilde{\mu} \right)^{1/p} - \left(\int_{\mathbb{R}^N} \varphi \, d\mu \right)^{1/p} \right| \leqslant \left(\int_{\mathbb{R}^N} \varphi h \, dx \right)^{1/p}$$

where $h \in L^1_+(\mathbb{R}^N)$. And this shows that the singular parts of $\tilde{\mu}$ and μ are the same; and we conclude.

We next turn to the proof of Lemma I.2. We first remark that (26′) holds by density for all φ bounded measurable. Therefore we see that in particular

ν is absolutely continuous with respect to μ i.e.: $\nu = f\mu$ where $f \in L^1_+(\mu)$. Since

$$\nu(A) \leqslant C_0 \mu(A)^{q/p}, \qquad \forall A \quad \text{Borel} \quad \subset \mathbb{R}^N$$

we have in fact $f \in L^\infty_+(\mu)$. Next, if $\mu = g\nu + \sigma$ where $g \in L^1_+(\nu)$, σ is a bounded nonnegative measure such that if $K = \operatorname{Supp} \varphi$, $\nu(K) = 0$; considering $\tilde{\mu} = 1_K \mu$ and taking φ in (26′) of the form $1_K \psi$ where ψ is bounded measurable, we see that without loss of generality we may assume that $\sigma \equiv 0$. We next denote by $\nu_k = g^\alpha 1_{(g \leq k)} \nu$, where $\alpha = q/(q-p)$. We are going to prove that ν_k is given by a finite number of Dirac masses; this will prove that $\nu 1_{(g \leq k)}$ is a finite number of Dirac masses for all $k < \infty$ and letting $k \to \infty$, the claim on ν will be proved (since $\nu(\{g = +\infty\}) = 0$).

To prove our claim on ν_k, we take in (26′) φ of the form

$$\varphi = g^{1/(q-p)} 1_{(g \leq k)} \psi$$

where ψ is an arbitrary bounded measurable function. We thus obtain for all ψ

$$\left(\int_{\mathbb{R}^N} |\psi|^q \, d\nu_k \right)^{1/q} \leqslant C_0 \left(\int_{\mathbb{R}^N} |\psi|^p \, d\nu_k \right)^{1/p}$$

(indeed: $g^{p/(q-p)} 1_{(g \leq k)} \mu = g^{q/(a-p)} 1_{(g \leq k)} \nu$).

This reversed Hölder inequality now yields our claim on ν_k: a short proof of this standard statement is the following. For any Borel set A the above inequality gives

$$\nu_k(A)^{1/q} \leqslant C_0 \nu_k(A)^{1/ap}.$$

Therefore either $\nu_k(A) = 0$, or $\nu_k(A) \geqslant \delta = C_0^{-\,p/(q-p)} > 0$. Since for each $x \in \mathbb{R}^N$, $\nu_k(\{x\}) = \lim_{\epsilon \downarrow 0} \downarrow \nu_k(B(x, \epsilon))$, we have for all $x \in \mathbb{R}^N$

$$\text{either} \quad \nu_k(\{x\}) \geqslant \delta, \quad \text{or} \quad \exists \epsilon > 0, \qquad \nu_k(B(x, \epsilon)) = 0.$$

Thus there exists a finite number of distinct points x_j in $\mathbb{R}^N$ such that

$$\begin{cases} \nu_k(\{x_j\}) \geqslant \delta & \forall i \leqslant j \leqslant m \\ \nu_k(B(x, \epsilon)) = 0 & \text{for some} \quad \epsilon = \epsilon(x) > 0, \qquad \forall x \notin \{x_j / 1 \leqslant j \leqslant m\}. \end{cases}$$

Let K be any compact set in $O = \{x / x \neq x_j \text{ for all } 1 \leqslant j \leqslant m\}$, we have by a finite covering of K by balls $B(x, \epsilon(x))$: $\nu_k(K) = 0$, therefore $\nu_k(O) = 0$; and our claim is proved.

At this point, we have proved the representation of ν and by (26′) we have

$$\mu(\{x_j\}) \geqslant C_0^{-p} \nu(\{x_j\})^{p/q}$$

Finally if $\nu(\mathbb{R}^N)^{1/q} \geqslant C_0 \mu(\mathbb{R}^N)^{1/p}$, taking $\varphi \equiv 1$ in (26′) we see that $\nu(\mathbb{R}^N)^{1/q} = C_0 \mu(\mathbb{R}^N)^{1/p}$; and using Hölder inequality we find for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |\varphi|^q \, d\nu \right)^{1/q} \leqslant C_0 \mu(\mathbb{R}^N)^\theta \left(\int_{\mathbb{R}^N} |\varphi|^q d\mu \right)^{1/p}$$

where $\theta = (q-p)/(pq)$. Observing that

$$\nu(\mathbb{R}^N) = C_0^q \mu(\mathbb{R}^N)^{q/p} = \{C_0 \mu(\mathbb{R}^N)^\theta\}^q \mu(\mathbb{R}^N)$$

we deduce from the above inequality: $\nu = \{C_0\mu(\mathbb{R}^N)^\theta\}^q \mu$. Therefore we have for all $\varphi \in \mathfrak{D}(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} |\varphi|^q \, d\nu\right)^{1/q} \leqslant \nu(\mathbb{R}^N)^{-\theta} \left(\int_{\mathbb{R}^N} |\varphi|^p \, d\nu\right)^{1/q}.$$

And the above proof already shows that: $\nu = \sum_{i=1}^m \nu_i \delta_{x_i}$, where $m \geqslant 1$, $(x_i)_i$ are m distinct points in $\mathbb{R}^N$ and $\nu_i > 0$.

We choose $\varphi \in \mathfrak{D}(\mathbb{R}^N)$ such that $\varphi(x_i) = \alpha_i > 0$; thus we find for all $\alpha_i > 0$

$$\left(\sum_{i=1}^m \alpha_i^q \nu_i\right)^{1/q} \left(\sum_{i=1}^m \nu_i\right)^{(q-p)/pq} \leqslant \left(\sum_{i=1}^m \alpha_i^p \nu_i\right)^{1/p}.$$

And this is possible if and only if $m = 1$.

Remark I.5. Lemma I.2 is of course valid in an arbitrary measure space and the various conclusions hold provided one replaces points in $\mathbb{R}^N$ by atoms...

I.4 Variants

We briefly mention here a few related problems and inequalities which can be treated in a similar way. In particular in all the cases mentioned below *all minimizing sequences are relatively compact up to a translation and a scale change*; and *the analogue of Lemma* I.1 *holds in each case*. The proofs being very similar to the previous ones, we skip them.

i) *Other norms in* $\mathfrak{D}^{m,p}(\mathbb{R}^N)$.

Of course we may replace the norm on $\mathfrak{D}^{m,p}$ by the following one

$$\begin{aligned} \mathcal{E}(u) &= |(-\Delta)^{m/2} u|^p_{L^p(\mathbb{R}^N)} && \text{if } m \text{ is even} \\ &= |\nabla(-\Delta)^{(m-1)/2} u|^p_{L^p(\mathbb{R}^N)} && \text{if } m \text{ is odd.} \end{aligned}$$

We could in fact take any norm in $\mathfrak{D}^{m,p}$ but the particular one chosen above is of interest since some additional information on extremal functions is available (see Corollary I.2 below) and since we have by easy integrations by parts

$$\mathcal{E}(u) = \sum_{|\alpha| = m} |D^\alpha u|^2_{L^2}. \tag{27}$$

if $p = 2$, $u \in \mathfrak{D}^{m,2}(\mathbb{R}^N)$ —and we recover the previous norm! The existence of extremal functions is determined by the following minimization problem

$$\operatorname{Inf}\{\mathcal{E}(u)/u \in \mathfrak{D}^{m,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^q \, dx = 1\}. \tag{28}$$

Corollary I.2. *Let* $(u_n)_n$ *be a minimizing sequence of* (28). *There exist* $(y_n)_n$ *in* $\mathbb{R}^N$, $(\sigma_n)_n$ *in* $]0,\infty[$ *such that the new minimizing sequence* $\tilde{u}_n = \sigma_n^{-N/q} u_n((\cdot - y_n)/\sigma_n)$ *is relatively compact in* $\mathfrak{D}^{m,p}$ *(for* $p > 1$, *and in* L^q *for* $p = 1$). *In particular the minimum is achieved. And if* $p > 1$, *for any minimum* u *of* (28), *there exists* $y \in \mathbb{R}^N$ *such that* $\tilde{u} = u(\cdot - y)$ *satisfies*

$$\begin{cases} (-\Delta)^\alpha \tilde{u} \text{ is spherically symmetric, nonnegative and decreasing} \\ \text{in } |x| \text{ for all } \alpha \in \mathbb{N} \text{ such that } \alpha \leqslant m/2. \end{cases}$$

The statement about the geometry of the minima is obtained as follows: let u be a minimum of (28). We set

$$f = (-\Delta)^{m/2} u \quad \text{if} \quad m \quad \text{is even,} = (-\Delta)^{(m-1)/2} u \quad \text{if} \quad m \quad \text{is odd.}$$

If m is even, $f \in L^p(\mathbb{R}^N)$ and if m is odd $f \in L^{\bar{p}}(\mathbb{R}^N)$ with $\bar{p} = Np/(N-p)$ (and $\nabla f \in L^p(\mathbb{R}^N)$). If we denote by φ^* the Schwarz symmetrization of φ, we introduce v solution of

$$(-\Delta)^\alpha v = f^* \quad \text{in} \quad \mathbb{R}^N, \qquad v \in L^q(\mathbb{R}^N),$$

with $\alpha = m/2$ if m is even, $\alpha = (m-1)/2$ if m is odd.

Smoothing and truncating f, we see that we may apply (α times) Talenti comparizon theorem on linear elliptic problems to deduce

$$u^* \leqslant v \qquad \text{a.e. in } \mathbb{R}^N.$$

In particular we have

$$\int_{\mathbb{R}^N} |u|^q dx = \int_{\mathbb{R}^N} |u^*|^q \, dx \leqslant \int_{\mathbb{R}^N} |v|^q \, dx$$

and

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} |f^*|^p \, dx = \int_{\mathbb{R}^N} |f|^p \, dx = \mathcal{E}(u) \quad \text{if} \quad m \quad \text{is even}$$

while

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} |\nabla f^*|^p \, dx \leqslant \int_{\mathbb{R}^N} |\nabla f|^p \, dx = \mathcal{E}(v) \quad \text{if} \quad m \quad \text{is odd.}$$

Hence v is also a minimum of (28) and all inequalities are equalities. We then conclude using the results and methods of A. Alvino, P. L. Lions and G. Trombetti [1] (in particular the method with Green functions).

ii) *Systems.*

Let $k \geqslant 1$, we want to consider here systems analogues of Sobolev inequalities (our motivation comes from the problem of nonlinear field equations —see section I.6 below). Let $u \in (\mathfrak{D}^{m,p}(\mathbb{R}^N))^k$; $u = (u^1, \ldots u^k)$ with $u^i \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$.

We denote by

$$\mathcal{E}(u) = \sum_{i=1}^{k} \int_{\mathbb{R}^N} |D^m u^i|^p \, dx.$$

Let $F \in C(\mathbb{R}^k)$ satisfy: $F(\zeta) > x > 0$ if $\zeta \neq 0$, F is homogeneous of degree q on $\mathbb{R}^k$. We deduce from Sobolev inequalities

$$\left(\int_{\mathbb{R}^N} F(u)\, dx\right)^{1/q} \leqslant C_0 \mathcal{E}(u)^{1/q} \tag{29}$$

And the existence of extremal functions is determined by the following minimization problem

$$\operatorname{Inf}\left\{ \mathcal{E}(u)/u \in (\mathcal{D}^{m,p}(\mathbb{R}^N))^k, \int_{\mathbb{R}^N} F(u)\, dx = 1 \right\}. \tag{30}$$

Exactly as before, any minimizing sequence is relatively compact up to a translation and a scale change, and there exists a minimum of (30). In addition the analogue of Lemma I.1 holds with $|u_n|^q$ replaced by $F(u_n)$. The only technical point we have to explain is why Brézis-Lieb lemma [10] still applies; and this in an application of the following remark:

Lemma I.3. *The nonlinearity F satisfies for all $a, b \in \mathbb{R}^k$*

$$|F(a+b) - F(a)| \leqslant \epsilon |a|^q + C_\epsilon(|b|^q + 1) \tag{31}$$

for all $\epsilon > 0$.

PROOF. Recall that F satisfies: $|F(t)| \leqslant C(1 + |t|^q)$ on $\mathbb{R}^k$. Hence to prove (31), we may assume without loss of generality that $|a| \geqslant 1$, $|a+b| \geqslant 1$ since if, for example, $|a+b| \leqslant 1$, we have

$$\begin{aligned} |F(a+b) - F(a)| &\leqslant C + C(1 + |a|^q) \\ &\leqslant C + C(1 + |b|^q) \end{aligned}$$

and (31) holds. Furthermore we may also assume that, for any $\delta > 0$ fixed: $|b| \leqslant \delta |a|$. Indeed if this is not the case we have

$$\begin{aligned} |F(a+b) - F(a)| &\leqslant C(1 + |a+b|^q) + C(1 + |a|^q) \\ &\leqslant C + C|a|^q + C|b|^q \\ &\leqslant C + (C\delta^{-q} + C)|b|^q. \end{aligned}$$

But if $|b| \leqslant \delta |a|$, we deduce

$$||a+b| - |a|| \leqslant \delta |a|, \frac{|a| - |a+b|}{|a+b|} \leqslant \frac{\delta}{1-\delta}, \frac{|b|}{|a+b|} \leqslant \frac{\delta}{1-\delta}.$$

And we obtain

$$|F(a+b)-F(a)| = \left| |a+b|^q F\left(\frac{a+b}{|a+b|}\right) - |a|^q F\left(\frac{a}{|a|}\right) \right|$$

$$\left| \frac{a+b}{|a+b|} - \frac{a}{|a|} \right| \leqslant \frac{|b|}{|a+b|} + \frac{||a| - |a+b||}{|a+b|} \leqslant \frac{2\delta}{1-\delta}.$$

Hence choosing δ small enough, we find for any fixed $\epsilon > 0$

$$\left| F\left(\frac{a+b}{|a+b|}\right) - F\left(\frac{a}{|a|}\right) \right| \leqslant \frac{\epsilon}{2},$$

$$\begin{aligned} |F(a+b)-F(a)| &\leqslant \left| F\left(\frac{a}{|a|}\right) \right| \left| |a+b|^q - |a|^q \right| + \frac{\epsilon}{2}|a|^q \\ &\leqslant C\left||a+b|^q - |a|^q\right| + \frac{\epsilon}{2}|a|^q \\ &\leqslant \epsilon|a|^q + C_\epsilon|b|^q. \end{aligned}$$

We next turn to some other extension to systems (again motivated by nonlinear field equations): let $k > 1$, let $q_i \in]0, q[$ for $1 \leqslant i \leqslant k$ be such that: $\sum_{i=1}^k q_i = q$. We denote by $\theta_i = q_i/q$. Clearly Hölder and Sobolev inequalities yield that for any $u = (u^1, \ldots, u^k) \in (\mathcal{D}^{m,p}(\mathbb{R}^N))^k$, we have

$$\left(\int_{\mathbb{R}^N} |u^1|^{q_1} \ldots |u^k|^{q_k}\, dx\right)^{1/q} \leqslant C_0 \mathcal{E}(u)^{1/p} \tag{32}$$

where

$$\mathcal{E}(u) = \prod_{i=1}^k \left(\int_{\mathbb{R}^N} |D^m u^i|^p\, dx\right)^{\theta_i}.$$

In view of the homogeneity of the problem in each u_i, the existence of extremal functions is equivalent to the existence of a minimum of

$$I = \operatorname{Inf}\Big\{ -\int_{\mathbb{R}^N} |u^1|^{q_1} \ldots |u^k|^{q_k}\, dx / u \in (\mathcal{D}^{m,p})^k, \\ \forall 1 \leqslant i \leqslant k, \int_{\mathbb{R}^N} |D^m u^i|^p\, dx = 1 \Big\}. \tag{33}$$

We denote by $I(\lambda_1, \ldots, \lambda_k)$ (for $\lambda_i > 0$) the value of the infimum where the constraints of norm 1 are replaced by

$$\int_{\mathbb{R}^N} |D^m u_i|^p\, dx = \lambda_i.$$

Clearly

$$I(\lambda_1, \ldots, \lambda_k) = \left(\prod_{i=1}^k \lambda_i^{\theta_i}\right)^{q/p} I < 0.$$

Therefore we have

$$I(1, \dots, 1) < I(\lambda_1, \dots, \lambda_k) + I(1 - \lambda_1, \dots, 1 - \lambda_k)$$

for all $\lambda_i \in (0, 1)$ such that

$$0 < \sum_{i=1}^{k} \lambda_i < k.$$

This strict sub-additivity inequality shows (cf P. L. Lions [21] and the arguments above) that any minimizing sequence $(u_n)_n$ is such that: $|D^m u_n^i|^p$ are tight, $\prod_{i=1}^{k} |u_n^i|^{q_i}$ is tight. And this enables us to argue as before, therefore any minimizing sequence is relatively compact up to a translation (the same for all u_n^i) and a scale change and a minimum of (33) exists.

Remark I.6. Of course in (28), (33) we may take any norm on $\mathcal{D}^{m,p}$ and the choice may depend on $i \in \{1, \dots, k\}$.

iii) *Fractional derivatives.*

We first recall that a norm on $W^{m,p}(\mathbb{R}^N)$ for $0 < m$, $1 \leqslant p < \infty$ is given by

$$||u||_{m,p}^p = \sum_{|\alpha| \leq \alpha_0} ||D^\alpha u||_{L^p}^p + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|D^{\alpha_0} u(x) - D^\alpha u(y)|^p}{|x - y|^{N+sp}} dx\, dy$$

where α_0 is the integer part of m and we assume: $\alpha_0 < m < \alpha_0 + 1$; and $s = (m - \alpha_0)$. The Sobolev inequality still holds

$$||u||_{L^q(\mathbb{R}^N)} \leqslant C_0 ||u||_{m,p}, \qquad \forall u \in W^{m,p}(\mathbb{R}^N);$$

where $q = Np/(N - mp)$. But if we replace u by $\sigma^{-N/q} u(\cdot/\sigma)$ in this inequality we find

$$||u||_{L^q(\mathbb{R}^N)} \leqslant C_0 \Big\{ \sum_{|\alpha| \leq \alpha_0} \sigma^{(m-\alpha)p} ||D^\alpha u||_{L^p}^p + \\ + \iint |D^{\alpha_0} u(x) - D^{\alpha_0} u(y)|^p |x - y|^{-(N+sp)} dx\, dy \Big\}^{1/p}.$$

Therefore sending σ to 0, we find for all $u \in W^{m,p}(\mathbb{R}^N)$

$$||u||_{L^q(\mathbb{R}^N)} \leqslant C_0 \mathcal{E}(u)^{1/p} \tag{34}$$

where

$$\mathcal{E}(u) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|D^{\alpha_0} u(x) - D^{\alpha_0} u(y)|^p}{|x - y|^{N+sp}} dx\, dy.$$

We then denote by $\mathcal{D}^{m,p}(\mathbb{R}^N)$ the space of functions u satisfying $u \in L^q$,

$\mathcal{E}(u) < \infty$; it is a reflexive Banach space equipped with the norm $\mathcal{E}(u)^{1/p}$. Exactly as before the best constant in (34) is achieved (and all minimizing sequences are compact up to translations and dilations). In lemma I.1 which still holds we have to replace $|D^m u_n|^p(x)$ by

$$\int |D^{\alpha_0} u_n(x) - D^{\alpha_0} u_n(y)|^p |x-y|^{-(N+sp)}\, dy.$$

Remark I.7. Of course we may replace $\mathcal{E}(u)^{1/p}$ by any norm on $\mathcal{D}^{m,p}(\mathbb{R}^N)$. For example if $p = 2$ and if $\hat{u}$ is the Fourier transform of u, we may take

$$\left\{\int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 |\xi|^{2m}\, d\xi\right\}^{1/2}.$$

In lemma I.2, $|D^m u_n|^2$ is to be replaced by

$$|F^{-1}(\xi^m \hat{u})|^2.$$

iv) *Convolution and Sobolev inequalities.*

The general Choquard-Pekar equations (cf. E. H. Lieb [19], P.L. Lions [22]) use the following limit embeddings

$$\left(\int\int_{\mathbb{R}^N \times \mathbb{R}^N} |u|^q(x) |u|^q(y) |x-y|^{-\alpha}\, dx\, dy\right)^{1/(2q)} \leqslant C_0 \| D^m u \|_{L^p}; \qquad (35)$$

where $m \geqslant 1$, $1 \leqslant p < \infty$, $0 < \alpha < N$ and q is given by

$$2q = (2N - \alpha)p(N - mp)^{-1}.$$

For example if $p = q = 2$, $m = 1$ then $\alpha = 4$ (and $N \geqslant 5$). All the results proved above adapt to this situation and in particular Lemma I.1 holds with $|u_n|^q$ replaced by

$$|u_n|^q(x) \cdot \int_{\mathbb{R}^N} |u_n|^q(y) |x-y|^{-\alpha}\, dy.$$

v) *Korn-Sobolev inequalities.*

To simplify we will consider only $N = 3$, $p = 2$, $m = 1$. Let $u \in H^1(\mathbb{R}^3)^3$, we denote by $\epsilon_{ij}(u)$ the linear deformations tensor; $\epsilon_{ij}(u) = \frac{1}{2}\{(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)\}$. A fundamental inequality in elasticity theory is the Körn inequality which yields the Korn-Sobolev inequalities: for all $u \in H^1(\mathbb{R}^3)^3$ we have

$$\| u \|_{L^6(\mathbb{R}^3)} \leqslant C\{ \| u \|_{L^2(\mathbb{R}^3)} + \| \epsilon(u) \|_{L^2(\mathbb{R}^3)} \}$$

where $\epsilon(u) = \{\sum_{i,j} |\epsilon_{ij}(u)|^2\}^{1/2}$. The same dimensional analysis ($u(\cdot) \to \sigma^{-1/2} u(\cdot/\sigma)$) shows that in fact

$$\| u \|_{L^6(\mathbb{R}^3)} \leqslant C_0 \| \epsilon(u) \|_{L^2(\mathbb{R}^3)} \qquad (36)$$

and thus (36) holds for all $u \in (\mathfrak{D}^{1,2}(\mathbb{R}^3))^3$ and $||\epsilon(u)||_{L^2}$ is an equivalent norm on $(\mathfrak{D}^{1,2}(\mathbb{R}^3))^3$. Therefore all the results given above still hold in that case.

vi) *Time-dependent problems.*

Let $Q = \mathbb{R}^N \times \mathbb{R}$. Then Sobolev inequalities give in that case

$$||u||_{L^q(Q)} \leqslant C_0||u||, \qquad \forall u \in \mathfrak{D}(Q)$$

where $||u||^p = ||u_t||^p_{L^p(Q)} + ||D_x^m u||^p_{L^p(Q)}$ (or any other equivalent norm, for example if $m = 2$, $p = 2$, we may choose: $||u|| = ||u_t - \Delta u||_{L^2(Q)}$). Here and below we have: $m \geqslant 1$, $1 < p < (N+m)/m$ and $q = (N+m)p(N+m-mp)^{-1}$. We then denote by $\mathfrak{D}^{m,1,p}(Q)$ the completion of $\mathfrak{D}(Q)$ for the norm $||\cdot||$. The existence of an extremal function is equivalent to the existence of a minimum of

$$\operatorname{Inf}\{||u||^p / u \in \mathfrak{D}^{m,1,p}(Q), \int_Q |u|^q \, dx \, dt = 1\} \tag{37}$$

Corollary I.3. *For any minimizing sequence* $(u_n)_n$ *of* (37), *there exist* $((y_n, t_n))_n$ *in* Q, $(\sigma_n)_n$ *in* $]0, \infty[$ *such that the new minimizing sequence*

$$\tilde{u}_n = \sigma_n^{-(N+m)/q} u_n\left(\frac{\cdot - y_n}{\sigma_n}, \frac{\cdot - t_n}{\sigma_n^m}\right)$$

is relatively compact in $\mathfrak{D}^{m,1,p}(Q)$. *In particular there exists a minimum.*

Of course there are many extensions that we skip such as: $D_t^k u \in L^q, D_x^m u \in L^p \ldots$

vii) *Nonlinear embeddings.*

We just give one example of many situations which can be treated by the methods described above. Let $u \in (\mathfrak{D}^{s,2}(\mathbb{R}^N))^N$ with $s = (N+2)/4$; then we have

$$|(u \cdot \nabla)u|_{L^2} = \left\{ \int_{\mathbb{R}^N} \sum_j \left(\sum_i u^i \frac{\partial u^j}{\partial x_i} \right)^2 dx \right\}^{1/2} \leqslant C_0 ||u||_{\mathfrak{D}^{s,2}} \tag{38}$$

—such norms have been defined above even if s is not an integer. Such equalities are interesting in the context of Navier-Stokes equations. Our methods yield the compactness up to translations and dilations of all minimizing sequences, the existence of extremal functions in (38) and informations on the weak convergence such as Lemma I.1 (one replaces $|u_n|^q$ by $|(u_n \cdot \nabla)u_n|^2 \ldots$).

Another application of these methods to the existence of extremal funtions for Sobolev-type inequalities is given in D. Jerison and J. M. Lee [16].

I.5 Yamabe problem in $\mathbb{R}^N$

We already explained in the Introduction the motivation for the study of the following equation

$$-\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + k(x)u = K(x)|u|^{4/(N-2)}u \quad \text{in} \quad \mathbb{R}^N \tag{39}$$

with $N \geqslant 3$. One is particularly interested in positive solutions of (39). We will assume (to simplify) all throughout the section

$$\begin{cases} a_{ij} = a_{ji} \in C_b(\mathbb{R}^N), \quad a_{ij} \to a_{ij}^{\infty} \quad \text{as} \quad |x| \to \infty \\ \exists \nu > 0, \quad \forall x \in \mathbb{R}^N, \quad (a_{ij}(x)) \geqslant \nu I_N \\ k, K \in C_b(\mathbb{R}^N), \quad k \to k^{\infty}, \quad K \to K^{\infty} \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{40}$$

And our first approach of (39) will require either

$$\begin{cases} \exists \alpha > 0, \quad \forall u \in \mathcal{D}(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x)\dfrac{\partial u}{\partial x_i}\dfrac{\partial u}{\partial x_j} + k(x)u^2\,dx \geqslant \alpha|Du|^2_{L^2} \\ \sup_{\mathbb{R}^N} K > 0, \quad k^{\infty} > 0 \quad \text{or} \quad k \in L^{N/2}(\mathbb{R}^N) \end{cases} \tag{41}$$

or

$$\begin{cases} \exists \alpha > 0, \quad K(x) \geqslant \alpha \quad \text{on} \quad \mathbb{R}^N \\ k^{\infty} > 0 \quad \text{or} \quad k \in L^{N/2}(\mathbb{R}^N) \end{cases} \tag{42}$$

If one is interested in *solutions of* (39) *which vanish at infinity*, then a minimum of the following minimization problem will provide such a solution

$$I = \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + k(x)u^2\,dx / u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \right.$$
$$\left. k(x)u^2 \in L^1(\mathbb{R}^N), \int_{\mathbb{R}^N} K(x)u^2\,dx = 1 \right\}. \tag{43}$$

Then if (41) or (42) holds, the class of minimizing functions is not empty and minimizing sequences are bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$; observe also that if (41) holds then $k^{\infty} \geqslant 0$ and that if $k^{\infty} > 0$, the minimizing class is included in $H^1(\mathbb{R}^N)$ and minimizing sequences are bounded in H^1.

We have seen in the previous sections that if a_{ij}, K are independent of x and if $k \equiv 0$, then, if u is a minimum, $\tilde{u} = \sigma^{-N/q}u(\cdot/\sigma)$ is still a minimum and nothing may prevent losses of compactness for minimizing sequences due to dilations (i.e. scale changes as above). Here of course, in general, the problem is not invariant by those scale changes anymore but we have to decide when and how the non-compactness in L^q of a minimizing sequence (for

$q = 2N/(N-2)$) —*i.e. when Dirac masses (as in Lemma I.1) do appear in the limits of* $|u_n|^q$— is avoided. Of course we have also to avoid the non-compactness due to translations and we know (cf. [20], [21]) that this is done using the problem at infinity

$$\bar{I} = \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + k^\infty u^2\, dx / u \in \mathcal{D}^{1,2}(\mathbb{R}^N), K^\infty \int_{\mathbb{R}^N} |u|^q\, dx = 1 \right\}$$

and $\bar{I} = +\infty$ if $K^\infty \leqslant 0$.

Here to avoid the non-compactness due to dilations we have to introduce a different notion of problem at infinity: to this end we denote by

$$\begin{cases} \mathcal{E}(u) = \displaystyle\int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + k(x) u^2\, dx \\ J(u) = \int_{\mathbb{R}^N} K(x)|u|^q\, dx. \end{cases}$$

Then for any fixed point y $\in \mathbb{R}^N$, we consider for $u \in \mathcal{D}^{1,2}$ or H^1

$$\mathcal{E}_y^\infty(u) = \lim_{\sigma \to 0} \mathcal{E}(\sigma^{-N/q} u((\cdot - y)/\sigma)) = \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(y) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x)\, dx, \tag{44}$$

$$J_y^\infty(u) = \lim_{\sigma \to 0} J(\sigma^{-N/q} u((\cdot - y)/\sigma)) = \int_{\mathbb{R}^N} K(y)|u|^q\, dx, \tag{45}$$

$$I_y^\infty = \operatorname{Inf}\{ \mathcal{E}_y^\infty(u) / u \in \mathcal{D}^{1,2}(\mathbb{R}^N), J_y^\infty(u) = 1 \} \tag{46}$$

and $I_y^\infty = +\infty$ if $K(y) \leqslant 0$. We could say that I_y^∞ is the value of the infimum of the problem "at infinity at y". We finally introduce

$$I^\infty = \operatorname{Inf}\{ I_y^\infty / y \in \mathbb{R}^N \} \tag{47}$$

Observe that $I_y^\infty \to \bar{I}$ as $|y| \to \infty$, and thus: $I^\infty \leqslant \bar{I}$.

In the particular situation at hand I_y^∞ and I^∞ may be computed using dilations, homogeneity and symmetry arguments

$$I_y^\infty = K^+(y)^{-2/q} \det(a_{ij}(y))^{1/N} I^0$$

where I^0 corresponds to the best Sobolev exponent

$$I^0 = \operatorname{Min}\{ \int_{\mathbb{R}^N} |\nabla u|^2\, dx / u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^q\, dx = 1 \}.$$

Therefore

$$I^\infty = \operatorname*{Inf}_{y \in \mathbb{R}^N} \{ K^+(y)^{-2/q} \det(a_{ij}(y))^{1/N} \} I^0. \tag{48}$$

The above construction of I^∞ easily yields

$$I \leqslant I^\infty \tag{49}$$

and if we denote by I_λ, I_λ^∞ the values of the infima of the same minimization problems but with 1 replaced by $\lambda > 0$, observing that

$$I_\lambda = \lambda^{2/q} I, \qquad I_\lambda^\infty = \lambda^{2/q} I^\infty;$$

we deduce from (49)

$$I = I_1 < I_\alpha + I_{1-\alpha}^\infty, \qquad \forall \alpha \in]0, 1[. \tag{50}$$

Therefore condition (S. 1) (of [20], [21]) holds if and only if $I < I^\infty$. By analogy with [20], [21], we expect the

Theorem I.2. *We assume* (40) *and* (41) *or* (42). *Let* $(u_n)_n$ *be a minimizing sequence of* (43).

i) *If* $I < I^\infty$, $(u_n)_n$ *is relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ (*and in* $H^1(\mathbb{R}^N)$, *if* $k^\infty > 0$). *In particular there exists a minimum and any minimum is, when* $I > 0$, *a positive solution of* (39) *up to a multiplicative constant.*

If $I = I^\infty$, *there exist minimizing sequences which are not relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$.

(ii) *If* $I = \bar{I} < I_y^\infty$ *for all* $y \in \mathbb{R}^N$, *and if* $(u_n)_n$ *is not relatively compact, there exist* $(y_n)_n$ *in* $\mathbb{R}^N$, $(\sigma_n)_n$ *in* $]0, \infty[$ *such that:* $|y_n| \underset{n}{\to} \infty$, $\sigma_n^{-N/q} u_n((\cdot - y_n)/\sigma_n)$ *is relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$. *In addition if* $k^\infty > 0$, $\sigma_n \underset{n}{\to} \infty$. *And there exist such sequences* $(u_n)_n$.

iii) *If* $I = I^\infty < \bar{I}$, u_n *converges weakly to* 0, $|u_n|^q$, $|Du_n|^2$ *are tight. And if we denote by* $C = \{y \in \mathbb{R}^N, I_y^\infty = I^\infty\}$ *and if* $|u_{n_k}|^q$ *converges weakly to some measure* ν, *we have*

$$\begin{cases} \nu = K(y)\delta_y \quad \textit{for some} \quad y \in C; \\ \exists \sigma_{n_k} \underset{k}{\to} \infty, \qquad \exists y_k, y_k/\sigma_{n_k} \underset{k}{\to} y \quad \text{and} \\ \sigma_{n_k}^{-N/q} u_{n_k}((\cdot + y_k)/\sigma_{n_k}) \text{ is relatively compact in } \mathfrak{D}^{1,2}(\mathbb{R}^N). \end{cases}$$

And such sequences $(u_n)_n$ *exist for any* $y \in C$.

iv) *If* $I = \bar{I} = I^\infty = I_y^\infty$ *for some* $y \in \mathbb{R}^N$ *then the conclusions of either* ii), *or* iii) *hold for subsequences. And both cases occur.*

We see that, even when compactness is not available, parts ii), iii), iv) describe exactly the phenomena involved. We will not explain here how to check the condition: $I < I^\infty$. Let us just mention that this is by no means easy and one may use the techniques of T. Aubin [3] (see also H. Brézis and L. Nirenberg [12]): this method is illustrated in the example following the proof of Theorem I.2. Let us also observe that if (42) holds and $k = \mu k_0$ for some $k_0 \leqslant 0$, $k_0 \not\equiv 0$, $k_0 \in L^{N/2}$, then $I < I^\infty$ for μ large. Of course if $I \leqslant 0$, then $I < I^\infty$!

PROOF OF THEOREM I.2. In all cases $(u_n)_n$ is bounded in $\mathcal{D}^{1,2}$ and if $k^\infty > 0$, $(u_n)_n$ is bounded in H^1. Depending whether $k^\infty = 0$ or $k^\infty > 0$, we consider $\rho_n \in L^1_+(\mathbb{R}^N)$ given by

$$\rho_n = |\nabla u_n|^2 + |u_n|^q \quad \text{or} \quad \rho_n = |\nabla u_n|^2 + |u_n|^q + u_n^2.$$

Applying the arguments of P. L. Lions [20], [21], we conclude that ρ_n is tight up to a translation if vanishing does not occur: indeed observe that we have

$$I = I_1 < I_\alpha + \bar{I}_{1-\alpha} \qquad \forall \alpha \in]0, 1[.$$

Now if vanishing occurs i.e.

$$\forall R < \infty, \qquad \operatorname*{Sup}_{y \in \mathbb{R}^N} \int_{y + B_R} \rho_n \, dx \underset{n}{\to} 0; \tag{51}$$

we have clearly

$$\left| \int_{\mathbb{R}^N} K(x)|u_n|^q \, dx - K^\infty \int_{\mathbb{R}^N} |u_n|^q \, dx \right| \leqslant C \sup_{|x| \geq R} |K(x) - K^\infty| + C \int_{B_R} |u_n|^q \, dx;$$

$$\left| \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx - \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}^\infty \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx \right| \leqslant$$

$$\leqslant C \sum_{i,j} \sup_{|x| \geq R} |a_{ij}(x) - a_{ij}^\infty| + C \int_{B_R} |\nabla u_n|^2 \, dx;$$

$$\left| \int_{\mathbb{R}^N} k(x) u_n^2 \, dx - \int_{\mathbb{R}^N} k^\infty u_n^2 \, dx \right| \leqslant C \sup_{|x| \geq R} |k(x) - k^\infty| + C \int_{B_R} u_n^2 \, dx \quad \text{if} \quad k^\infty > 0;$$

$$\leqslant C \|k\|_{L^{N/2}(\mathbb{R}^N - B_R)} + C_R \left(\int_{B_R} |u_n|^q \, dx \right)^{2/q}.$$

Therefore choosing R large and then n large, we see that vanishing implies: $I \geqslant \bar{I} = I^\infty$.

In a similar way if ρ_n (or a subsequence) is tight up to a translation y_n such that $|y_n| \underset{n}{\to} \infty$, then $I = \bar{I} = I^\infty$. Therefore if $I < I^\infty$, we have

$$\forall \epsilon > 0, \exists R < \infty, \forall n, \qquad \int_{|x| \geq R} \rho_n \, dx \leqslant \epsilon.$$

We now complete the proof of Part i) of Theorem I.2. We may now assume that $(u_n)_n$ converges weakly to some u (and a.e.): *We first show that* $u \not\equiv 0$: if it were the case we would have by Lemma I.1

$$|u_n|^q \to \sum_{k \in J} \nu_k \delta_{x_k}$$

for some at most countable set J of distinct points x_k in $\mathbb{R}^N$ and of positive real numbers ν_k. In addition we have the:

Lemma I.4. *Let $a_{ij} = a_{ji} \in C_b(\mathbb{R}^N)$ and assume $(a_{ij}) \geqslant 0$ on $\mathbb{R}^N$. If $u_n \underset{n}{\rightharpoonup} u$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $|u_n|^q$ is tight, we know by Lemma* I.1 *that: $|u_n|^q \to |u|^q + \sum_{k\in J} \nu_k \delta_{x_k}$. Extracting if necessary a subsequence, we may assume that*

$$\sum_{i,j} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \underset{n}{\rightharpoonup} \mu$$

for some positive bounded measure μ, then

$$\mu \geqslant \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{k\in J} \nu_k^{(N-2)/N} I^0 (\det a_{ij}(x_k))^{1/N} \delta_{x_k}.$$

This lemma is proved after the proof of Theorem I.2. Of course it is valid with various adaptations in any $\mathcal{D}^{m,p}$.

Now if we go back to the proof of Theorem I.2, we see that

$$\begin{cases} I \geqslant \sum_{k\in J} \nu_k^{(N-2)/N} I^0 (\det a_{ij}(x_k))^{1/N} + \lim_n \int_{\mathbb{R}^N} k(x) u_n^2 \, dx \\ 1 \leqslant \sum_{k\in J} K(x_k) \nu_k. \end{cases}$$

Next we claim that:

$$\int_{\mathbb{R}^N} |k| u_n^2 \, dx \underset{n}{\to} 0.$$

Indeed if $k^\infty > 0$, since $u_n \underset{n}{\to} 0$ in $L^2(B_R)$ strongly for all $R < \infty$ by Rellich theorem and ρ_n is tight, we see that $u_n \underset{n}{\to} 0$ in $L^2(\mathbb{R}^N)$ and our claim is proved.

On the other hand if $k^\infty = 0$ and thus $k \in L^{N/2}$, we remark

$$\int_{\mathbb{R}^N} |k| u_n^2 \, dx \leqslant \int_{B_R} |k| u_n^2 \, dx + C \| k \|_{L^{N/2}(\mathbb{R}^N - B_R)} \leqslant M \int_{B_R} u_n^2 \, dx + \\ + C \| (|k| - M)^+ \|_{L^{N/2}(B_R)} + C \| k \|_{L^{N/2}(\mathbb{R}^N - B_R)}$$

and we conclude choosing R large, then M large and finally n large.

Therefore (52) yields

$$\begin{cases} I \geqslant \sum_{k\in J} \nu_k^{(N-2)/N} I^0 (\det a_{ij}(x_k))^{1/N} \\ 1 \leqslant \sum_{k\in J} K(x_k) \nu_k. \end{cases}$$

On the other hand since $I \leqslant I^\infty$ and I^∞ is given by the formula (48), this implies that J reduces to a single point x_0 which is a minimum point of $K^{-2/q} \det(a_{ij})^{1/N}$ i.e. a minimum point of I_y^∞ and $I = I_{x_0}^\infty = I^\infty$. Of course if $I < I^\infty$, this is not possible and $u \not\equiv 0$.

We next conclude the proof of part i) of Theorem I.2 by showing that

$\int_{\mathbb{R}^N} K|u|^q\,dx = 1$. Let us denote by $\alpha = \int_{\mathbb{R}^N} K|u^q|\,dx$. By Lemma I.4 we know

$$I \geqslant \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx + \sum_{k\in J} \nu_k^{(N-2)/N} I^0(\det a_{ij}(x_k))^{1/N} + \frac{\lim}{n}\int_{\mathbb{R}^N} k(x)u_n^2\,dx$$

$$1 = \alpha + \sum_{k\in J} \nu_k K(x_k);$$

and exactly as before we prove that

$$\int_{\mathbb{R}^N} k(x)u_n^2\,dc \xrightarrow{n} \int_{\mathbb{R}^N} k(x)u^2\,dx.$$

Hence, we have

$$\begin{cases} I \geqslant \mathcal{E}(u) + \sum_{k\in J} \nu_k^{(N-2)/N} I^0(\det a_{ij}(x_k))^{1/N} \\ 1 - \alpha = \sum_{k\in J} \nu_k K(x_k). \end{cases}$$

Using (48), we deduce

$$\begin{aligned} I &\geqslant \mathcal{E}(u) + I^\infty \sum_{k\in J} \nu_k^{(N-2)/N} K^+(x_k)^{(N-2)/N} \\ &\geqslant \mathcal{E}(u) + I^\infty \Big\{\sum_{k\in J} \nu_k K^+(x_k)\Big\}^{(N-2)/N} = \mathcal{E}(u) + I_\beta^\infty \end{aligned}$$

where

$$\beta = \sum_{k\in J} \nu_k K^+(x_k) \geqslant 1 - \alpha.$$

If $\alpha \leqslant 0$, $\beta \geqslant 1$ and we obtain: $I \geqslant \mathcal{E}(u) + I^\infty > I^\infty$, a contradiction with the large inequality $I \leqslant I^\infty$ which always holds.

If $\alpha < 1$, we find: $I \geqslant \mathcal{E}(u) \geqslant I_\alpha > I$, another contradiction.

Finally if $\alpha \in]0,1[$, we find

$$I = I_1 \geqslant \mathcal{E}(u) + I_{1-\alpha}^\infty \geqslant I_\alpha + I_{1-\alpha}^\infty$$

and this contradicts (50). And part i) is proved.

To prove part ii), we observe that from the second part of the proof that, in the situation described in ii), either ρ_n "vanishes" or ρ_n is tight up to a translation y_n such that $|y_n| \xrightarrow[n]{} \infty$. In the first case $\rho_n(\cdot - y_n)$, for some arbitrary y_n satisfying $|y_n| \xrightarrow[n]{} \infty$, "vanishes" and by the arguments given above $\tilde{u}_n = u_n(\cdot - y_n)$ in both cases is a minimizing sequence of $\bar{I}$. If $k^\infty = 0$, we apply Theorem I.1 and we conclude. If $k^\infty > 0$, remarking that $\bar{I} = (K^\infty)^{-(N-2)/N} I_0$

$$k^\infty \int_{\mathbb{R}^N} \tilde{u}_n^2\,dx \xrightarrow[n]{} 0,$$

and thus $(K^\infty)^{-(N-2)/N}\tilde{u}_n$ is also a minimizing sequence of I_0. And applying

Theorem I.1 we find $(z_n)_n$ in $\mathbb{R}^N$, $(\sigma_n)_n$ in $]0, \infty[$ such that

$$\hat{u}_n = \sigma_n^{-N/q} \tilde{u}_n((\cdot - z_n)/\sigma_n)$$

is relatively compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. And this implies

$$0 < \delta \leqslant \int_{\mathbb{R}^N} (\hat{u}_n)^2 \, dx = \sigma_n^{-2N/q} \sigma_n^N \int_{\mathbb{R}^N} \tilde{u}_n^2 \, dx$$

therefore $\sigma_n \underset{n}{\rightarrow} +\infty$. And part ii) is proved.

Part iii) is easily deduced from the above arguments: we just need to observe that if $u_n \underset{n}{\rightharpoonup} 0$, $|u_n|^q \to K(y)\delta_y$, then $(u_n)_n$ is a minimizing sequence of I_y^∞ and by Theorem I.1 we conclude easily. Finally part iv) is a consequence of the proof already made.

Remark I.8. Of course if we know that there does not exist a minimum of I, then the conclusions of Parts ii), iii), iv) hold. In particular this is the case when

$$(a_{ij}(x)) \geqslant (a_{ij}^\infty), \qquad K(x) \leqslant K^\infty, \qquad k(x) \geqslant k^\infty.$$

PROOF OF LEMMA I.4. We take the notations of the proof of Lemma I.1 and we have for all fixed $k \in J$ and for all $\epsilon > 0$

$$\left| \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x) \frac{\partial}{\partial x_i} \left\{ \varphi\left(\frac{x - x_k}{\epsilon}\right) u_n \right\} \frac{\partial}{\partial x_j} \left\{ \varphi\left(\frac{x - x_k}{\epsilon}\right) u_n \right\} dx + \right.$$
$$\left. - \int_{\mathbb{R}^N} \varphi^2\left(\frac{x - x_k}{\epsilon}\right) \left\{ \sum_{i,j} a_{ij} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \right\} dx \right| \leqslant \delta(\epsilon)$$

where $\delta(\epsilon)$ denotes various quantities (ind. of n) which go to 0 as ϵ goes to 0. But $\operatorname{Supp} \varphi \subset B(0, 1)$ and a_{ij} is continuous, hence

$$\int_{\mathbb{R}^N} \varphi^2\left(\frac{x - x_k}{\epsilon}\right) \left\{ \sum_{i,j} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \right\} dx \geqslant$$
$$\geqslant -\delta(\epsilon) + \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(x_k) \frac{\partial}{\partial x_i} \left(\varphi\left(\frac{x - x_k}{\epsilon}\right) u_n \right) \frac{\partial}{\partial x_j} \left(\varphi\left(\frac{x - x_k}{\epsilon}\right) u_n \right) dx$$
$$\geqslant -\delta(\epsilon) + I_0 (\det a_{ij}(x_k))^{1/N} \left(\int_{\mathbb{R}^N} \left| \varphi\left(\frac{x - x_k}{\epsilon}\right) \right|^q |u_n|^q \, dx \right).$$

And sending n to ∞, we deduce

$$\mu(B(x_k, \epsilon)) \geqslant -\delta(\epsilon) + I^0 (\det a_{ij}(x_k))^{1/N} \nu_k^{2/q}.$$

We conclude letting $\epsilon \to 0$.

EXAMPLE. We want to mention a simple situation where $I < I^\infty$. We take $k \equiv 0$. Observe that if a_{ij} does not depend on x for all i, j, the minimum is achieved if and only if $K(x) \equiv K^\infty > 0$. Indeed if $K \equiv \sup K$, any minimum u of I will satisfy

$$\mathcal{E}(u) = I \leqslant I^\infty, \qquad (\sup K) \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx > 1$$

and this contradicts the choice of I^∞.

Now we take for example: $a_{ij}(x) = a(x)\delta_{ij}$, $0 < \alpha < a(x)$, $K(x) \leqslant 1$, $K(x) \to K^\infty$, $a(x) \to a^\infty$ as $|x| \to \infty$. We will assume $N \geqslant 5$ and we may always normalize K, a by assuming $K(0) = a(0) = 1$ (the choice of the origin is arbitrary).

In order to try to prove $I < I^\infty$, it is natural to use the extremal functions of $I^\infty = I^0$ i.e. $u_\epsilon(x) = (\epsilon^2 + |x|^2)^{-(N-2)/2}$. This method was first used by T. Aubin [3] (see also H. Brézis and L. Nirenberg [12]). We compute

$$\mathcal{E}(u_\epsilon) = (N-2)^2 \left(\int_{\mathbb{R}^N} a(\epsilon y) \frac{|y|^2}{(1 + |y|^2)^N} dy \right) \epsilon^{(N-2)}$$

$$J(u_\epsilon) = \epsilon^{-N} \int_{\mathbb{R}^N} K(\epsilon y)(1 + |y|^2)^N\, dy$$

And if a is twice differentiable at 0, we deduce easily (using symmetry arguments for the first expansion terms)

$$\mathcal{E}(u_\epsilon) = \epsilon^{-(N-2)} I^0 ||u_1||_{L^q} + \\ + \frac{(N-2)^2}{2} \epsilon^{4-N} \int_{\mathbb{R}^N} \Big(\sum_{i,j} a_{ij} y_i y_j \Big) |y|^2 (1 + |y|^2)^{-N} dy + o(\epsilon^{4-N})$$

$$J(u_\epsilon) = \epsilon^{-N} ||u_1||_{L^q}^q + \epsilon^{2-N} \int_{\mathbb{R}^N} \Big(\sum_{i,j} K_{ij} y_i y_j \Big) (1 + |y|^2)^{-N} dy + o(\epsilon^{2-N})$$

where

$$a_{ij} = \frac{\partial^2 a}{\partial x_i \partial x_j}, \qquad K_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}(O).$$

Since $I \leqslant \mathcal{E}(u_\epsilon) J(u_\epsilon)^{-(N-2)/N}$, *we conclude that* $I < I^\infty$ by choosing ϵ small enough provided

$$\int_{\mathbb{R}^N} \Big(\sum_{i,j} a_{ij} y_i y_j \Big) |y|^2 (1 + |y|^2)^{-N} dy < C_0 \int_{\mathbb{R}^N} \Big(\sum_{i,j} K_{ij} y_i y_j \Big) (1 + |y|^2)^{-N} dy$$

(recall that the origin is arbitrary !), where

$$C_0 = 2(N(N-2) ||u_1||_{L^q}^q)^{-1}.$$

It is possible to treat more general potentiels k: one possible extension relies on the following result. If $N \geqslant 3$, we have

$$\int_{\mathbb{R}^N} |u|^2 |x|^{-2}\, dx \leqslant \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx, \qquad \forall u \in \mathfrak{D}^{1,2}(\mathbb{R}^N) \tag{53}$$

(this classical inequality is sometimes called the "uncertainty principle"!). Hence if we consider for $\alpha > -((N-2)^2/4$ the coercive quadratic form

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + \alpha \frac{u^2}{|x|^2}\, dx,$$

we may study the question of the existence of an extremal function for the best constant of the Sobolev embedding when $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ is endowed with the norm $\mathcal{E}(u)^{1/2}$ i.e.

$$I_\alpha = \operatorname{Inf}\{ \mathcal{E}(u) / u \in \mathfrak{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx = 1 \}. \tag{54}$$

Theorem I.3. *For any minimizing sequence* $(u_n)_n$ of (54), *there exists* $(\sigma_n)_n$ *in* $]0, \infty[$ *such that the new minimizing sequence* $\tilde{u}_n = \sigma_n^{-(N-2)/N} u_n(\cdot / \sigma_n)$ *satisfies*:

i) *If* $\alpha < 0$, $\tilde{u}_n$ *is relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ *and thus a minimum of* (54) *exists.*

ii) *If* $\alpha \geqslant 0$, *there exists* $(y_n)_n$ *in* $\mathbb{R}^N$ *such that* $\tilde{u}_n(\cdot - y_n)$ *is relatively compact in* $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ *and if* $\alpha > 0$, $|y_n| \xrightarrow[n]{} \infty$. *In addition if* $\alpha > 0$, $I^\alpha = I^0$ *and no minimum exists.*

The proof of Theorem I.3 is very similar to the above proofs and we will skip it. Let us just mention that if $v_n(\cdot y_n)$ is relatively compact in $\mathfrak{D}^{1,2}(\mathbb{R}^N)$, and if $|y_n| \xrightarrow[n]{} \infty$ then

$$\int_{\mathbb{R}^N} |v_n|^2 |x|^{-2}\, dx \xrightarrow[n]{} 0,$$

Next, we explain without even stating a theorem, what is one possible extension of Theorem I.2. Take, to simplify, $a_{ij} \equiv \delta_{ij}$, $K \equiv 1$, and $k \in C_0(\mathbb{R}^N)$ satisfies

$$\lim_{|x| \to \infty} k(x)|x|^2 = \alpha > -\frac{(N-2)^2}{4}$$

Then we set: $I^\infty = I^\alpha$ (thus $I^\infty = I^0$ if $\alpha \geqslant 0$). With these notations we can prove that if $I < I^\infty$, all minimizing sequences are relatively compact while if $I = I^\infty$, there is a least one minimizing sequence which is not relatively compact —and we may analyse as in Theorem I.2 what are the possible losses of compactness.

At this point let us observe that *everything we did concerned positive solutions of* (39) *which vanish at infinity* and if we go back to the original motivation of the Yamabe equation (39) —given in the Introduction— it *is not clear that the new metric* —basically given by $|u|^{4/(N-2)}a_{ij}$— *is complete* (and in general it is not complete). On the other had *if we consider positive solutions of* (39) *such that*: $u(x) \geqslant \alpha$ on $\mathbb{R}^N$ for some $\alpha > 0$; the *new metric will be automatically complete.* This is why in the remainder of this section we will consider *bounded solutions of* (39) *positive uniformly on* $\mathbb{R}^N$. At this stage, let us mention the work of Ni [28] (see also [29], Kenig and Ni [17]) where, in the particular case $a_{ij}(x) \equiv \delta_{ij}$, $k \equiv 0$, general results are obtained by the elementary method of sub and supersolutions. This approach is recalled in the appendix where we also explain how it is possible to obtain twice more solutions —and this is done by variational arguments involving our general method. Here, we present still another approach which in the special case afore mentioned does not cover the full generality of Ni's results since more severe restrictions are made on K but on the other hand we obtain additional information on the solution and the approach also provides a general way to check the assumptions necessary in order to apply the method of sub and supersolutions of Ni.

We consider the following minimization problem

$$I_\lambda = \operatorname{Inf}\{\mathcal{E}(u)/u - \alpha \in \mathcal{D}^{1,2}(\mathbb{R}^N), J(u) = \lambda\} \tag{55}$$

where $\alpha > 0$, $N \geqslant 3$ are fixed. $\mathcal{E}, J$ still denote the same functionals and we assume

$$a_{ij} = a_{ji} \in C_b(\mathbb{R}^N); \qquad \exists \nu > 0 \qquad (a_{ij}(x)) \geqslant \nu I_N \tag{40'}$$

$$k, K \in L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N); \qquad K > 0 \quad \text{on} \quad \mathbb{R}^N \tag{56}$$

We define the energy at infinity exactly as before, but since $K^\infty = 0$, we have

$$I_\lambda^\infty = \operatorname*{Min}_{y \in \mathbb{R}^N} \{K^+(y)^{-2/q} \det\{a_{ij}(y)\}^{1/N}\} I^0. \tag{48'}$$

Theorem I.4. *We assume* (40′) *and* (56):

i) *Every minimizing sequence of* (55) *is relatively compact in* $X = \{\alpha + v, v \in \mathcal{D}^{1,2}(\mathbb{R}^N)\}$ *if and only if*

$$I_\lambda < I_\beta + I_{\lambda-\beta}^\infty, \qquad \forall \beta \in [0, \lambda[. \tag{S.1}$$

ii) *If we assume in addition*

$$\exists \gamma > 0, \forall \varphi \in \mathcal{D}(\mathbb{R}^N), \qquad \mathcal{E}(\varphi) \geqslant \gamma |D\varphi|^2_{L^2(\mathbb{R}^N)} \tag{57}$$

then there exists $\lambda_0 > 0$ *such that*: I_λ *is decreasing on* $[0, \lambda_0]$ *from* $-\infty$ to I_{λ_0}

and I_λ is increasing on $[\lambda_0, +\infty[$ from I_{λ_0} to $+\infty$. In addition there exists a unique $\varphi_1 \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap BUC(\mathbb{R}^N)$ satisfying

$$\begin{cases} -\sum_{i,j} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial \varphi_1}{\partial x_j}\right) + k(x)\varphi_1 = 0 \quad in \quad \mathbb{R}^N \\ \varphi_1 > 0 \quad in \quad \mathbb{R}^N, \qquad \varphi_1 \to \alpha \quad as \quad |x| \to \infty \end{cases} \tag{58}$$

and $\mathcal{E}(\varphi_1) = I_{\lambda_0}, \quad J(\varphi_1) = \lambda_0, \quad \varphi_1 - \alpha \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

iii) *Furthermore if $\lambda \in]0, \lambda_0[$,* (S.1) *holds and there exists a unique minimum of* (55), *$u_\lambda \in X$ which satisfies*

$$\begin{cases} -\sum_{i,j} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial u_\lambda}{\partial x_j}\right) + k(x)u_\lambda + \theta_\lambda K u_\lambda^{(N+2)/(N-2)} = 0 \quad in \quad \mathbb{R}^N \\ u_\lambda \in BUC(\mathbb{R}^N), \qquad u_\lambda > 0 \quad on \quad \mathbb{R}^N, \qquad u_\lambda \to \alpha \quad as \quad |x| \to \infty; \end{cases} \tag{59}$$

where θ_λ is a positive Lagrange multiplier. In addition u_λ is the unique solution of (59) *(in H^1_{loc} say) and θ_λ decreases continuously on $]0, \lambda_0[$ from $+\infty$ to* 0, *while u_λ increases continuously from* 0 *to φ_1 on $\mathbb{R}^N$.*

iv) *Finally there exists $\delta > 0$, such that for $\lambda \in]\lambda_0, \lambda_0 + \delta[$,* (S. 1) *holds. In particular there exists a minimum u_λ of* (55) *which solves*

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u_\lambda}{\partial x_j}\right) + k(x)u_\lambda = \theta_\lambda K u_\lambda^{(N+2)/(N-2)} \quad in \quad \mathbb{R}^N \tag{60}$$

and $u_\lambda \in X \cap BUC(\mathbb{R}^N)$, $u_\lambda \to \alpha$ as $|x| \to \infty$; where θ_λ is a positive Lagrange multiplier.

Remark I.9. Assumptions (56), (57) may be relaxed but the main assumption $k, K \in L^1(\mathbb{R}^N)$ subsists. In part i), we have: $I_0 = +\infty$ and $I_\alpha \to +\infty$ as $\alpha \to 0_+$, hence the strict inequality in (S.1) holds for α small.

Setting $v_\lambda = \theta_\lambda^{(N-2)/4} u_\lambda$ and using the variant of Ni's method given in the appendix, we find the

Corollary I.4. *We assume* (40′), (56), (57). *Then for any $\mu > 0$, there exists a unique solution u of*

$$\begin{cases} -\sum_{i,j} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial u}{\partial x_j}\right) + k(x)u + K(x)u^{(N+2)(N-2)} = 0 \quad in \quad \mathbb{R}^N \\ u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N), \quad u > 0 \quad on \quad \mathbb{R}^N, \quad u \to \mu \quad as \quad |x| \to \infty \end{cases} \tag{39'}$$

and u increases continuously in μ, $u < (\mu/\alpha)\varphi_1$ in $\mathbb{R}^N$, $u - \mu \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

There exists $\mu_0 > 0$, such that:

i) *for $\mu > \mu_0$, there does not exist a solution u of* (39) *such that:*

$$u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N), \qquad u \geqslant 0 \quad in \quad \mathbb{R}^N, \qquad u \to \mu \quad as \quad |x| \to \infty.$$

ii) *for* $0 < \mu \leqslant \mu_0$, *there exists a solution* u *of* (39) *in* $H^1_{\text{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$ *satisfying*: $u \to \mu$ *as* $|x| \to \infty$, $u > (\mu/\alpha)\varphi_1 > 0$ *on* $\mathbb{R}^N$, $u - \mu \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *and* u *is the minimum positive solution of* (39) *in* $H^1_{\text{loc}} \cap C_b$ *converging to* μ *at infinity. In addition* u *increases continuously with* μ *on* $\mathbb{R}^N$.

PROOF OF THEOREM I.4. We will prove part i) since it is a straightforward repetition of arguments given before provided we show that if $(u_n)_n$ is a minimizing sequence of (55) then $u_n - \alpha$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Indeed if $v_n = (u_n - \alpha)$, since $J(u_n) = \lambda$ and $K > 0$ on $\mathbb{R}^N$, we find:

$$||u_n||_{L^{2N/(N-2)}(B_R)} + ||v_n||_{L^{2N/(N-2)}(B_R)} \leqslant C_R, \qquad \forall R < \infty.$$

Next, in view of (40′):

$$\begin{aligned} \nu|Dv_n|^2_{L^2} &\leqslant C_\lambda + 2\int_{\mathbb{R}^N} |k|\alpha|v_n|\, dx + \int_{\mathbb{R}^N} |k| v_n^2\, dx \\ &\leqslant C + C||v_n||_{L^{2N/(N-2)}} + \int_{\mathbb{R}^N/B_R} |k| v_n^2\, dx \\ &\leqslant C + C||v_n||_{L^{2N/(N-2)}} + ||k||_{L^{N/2}(\mathbb{R}^N - B_R)} ||v_n||^2_{L^{2N/(N-2)}} \end{aligned}$$

and we conclude using Sobolev inequalities and choosing R large.

We next prove part ii): *we first show that* $I_\lambda \to +\infty$ as $\lambda \to 0_+$ or $\lambda \to +\infty$. Indeed if I_λ remains bounded when $\lambda \to 0_+$, the above argument shows that $\{v \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \mathcal{E}(\alpha + v) \leqslant I_\lambda + 1, J(\alpha + v) \in]0, 1[\}$ is bounded. Hence there exists v_n bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $J(\alpha + v_n) \underset{n}{\to} 0$. Since $K > 0$ in $\mathbb{R}^N$, this yields: $v_n \to -\alpha$ in measure locally, and this contradicts the boundedness of v_n in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ since $-\alpha \notin \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Next, if $\lambda \to +\infty$ and $I_\lambda \leqslant C$, there exists v_n in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$\mathcal{E}(\alpha + v_n) \leqslant C; \qquad J(\alpha + v_n) \underset{n}{\to} +\infty.$$

But

$$\mathcal{E}(\alpha + v_n) \geqslant \gamma|Dv_n|^2_{L^2} - C - C|Dv_n|_{L^2}$$

hence v_n is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and this contradicts the assumption on $J(\alpha + v_n)$.

Next, denoting by $\lambda_0 (> 0)$ an absolute minimum of the continuous function $(\lambda \to I_\lambda)$, we show the *monotonicity properties of* I_λ: we first observe that if $\lambda \in]0, \lambda_0[$ satisfies

$$I_\lambda = \min\{I_\mu / \mu \in]0, \bar\lambda]\}, \quad \text{for some} \quad \bar\lambda > \lambda,$$

then necessarily $\lambda = \lambda_0$. Indeed, clearly for such a λ, (S. 1) holds. By part i), there exists a minimum φ of I_λ which is a local minimum of $\mathcal{E}$ on X, therefore φ solves (58) and by standard regularity results $\varphi \in BUC(\mathbb{R}^N)$. We now prove the uniqueness of such a function φ proving thus the equality between λ and λ_0. Remark first that necessarily $\varphi > 0$ on $\mathbb{R}^N$.

Indeed for R large enough, denoting by

$$A = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + k$$

$$A\varphi = 0 \quad \text{in} \quad B_R, \qquad \varphi > 0 \quad \text{on} \quad \partial B_R \tag{60}$$

and (57) implies that the first eigenvalue of operator A on $H_0^1(B_R)$ is positive. We may thus apply the maximum principle to (60): $\varphi > 0$ on B_R.

Next if $\varphi, \psi \in H^1_{\text{loc}} \cap BUC$ solve: $A\varphi = A\psi = 0$ in $\mathbb{R}^N$, $\varphi, \psi \to \alpha$ as $|x| \to \infty$, we show that $\varphi \equiv \psi$. Indeed for R large enough we have

$$\alpha - \epsilon \leqslant \varphi, \psi \leqslant \alpha + \epsilon \quad \text{on} \quad \partial B_R$$

where $\epsilon = \epsilon(R) \to 0$ as $R \to \infty$.

Since (60) holds for φ, ψ and since we may apply the maximum principle, we obtain

$$\frac{\alpha+\epsilon}{\alpha-\epsilon}\varphi \geqslant \psi, \qquad \frac{\alpha+\epsilon}{\alpha-\epsilon}\psi \geqslant \varphi \quad \text{on} \quad \bar{B}_R;$$

and we conclude letting $R \to \infty$ ($\epsilon(R) \to 0$).

At this point we have proved that

$$I_{\lambda_0} < I_\lambda < I_\mu, \quad \text{if} \quad 0 < \mu < \lambda < \lambda_0.$$

Next if $\mu > \lambda \geqslant \lambda_0$, for each $\epsilon > 0$ fixed, there exists $u \in X$ such that

$$I_\lambda \leqslant \mathcal{E}(u) \leqslant I_\mu + \epsilon, \qquad J(u) = \mu$$

and considering $\tilde{u} = \theta u + (1-\theta)\varphi_1$ for $\theta \in [0, 1]$, we find $\theta \in [0, 1[$ such that: $J(\tilde{u}) = \lambda$. On the other hand since $u - \varphi_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\mathcal{E}(\tilde{u}) = \mathcal{E}(\varphi_1 + \theta(u - \varphi_1)) = \mathcal{E}(\varphi_1) + \theta^2\mathcal{E}(u - \varphi_1)$$

and $\mathcal{E}(\tilde{u})$ is strictly convex with respect to θ. Hence we find

$$I_\lambda \leqslant \mathcal{E}(\tilde{u}) < \theta\mathcal{E}(u) + (1-\theta)\mathcal{E}(\varphi_1) \leqslant \theta(I_\mu + \epsilon) + (1-\theta)I_{\lambda_0}$$

and this yields: $I_\lambda \leqslant I_\mu$. Next if $I_\lambda = I_\mu$, clearly $I_{\tilde{\lambda}} = I_\lambda = I_\mu$ for all $\tilde{\lambda} \in [\lambda, \mu]$ and thus (S. 1) holds for $\tilde{\lambda} \in]\lambda, \mu[$. Therefore for $\tilde{\lambda} \in]\lambda, \mu[$ fixed, there exists a minimum φ of $I_{\tilde{\lambda}}$ which is clearly a local minimum of $\mathcal{E}$ on X, hence $\varphi = \varphi_1, \hat{\lambda} = \lambda_0$. The contradiction shows: $I_{\lambda_0} \leqslant I_\lambda < I_\mu$ if $\lambda_0 \leqslant \lambda < \mu$; and part ii) is proved.

We now prove part iii): the properties of I_λ we proved imply easily that (S.1) holds for $\lambda \in]0, \lambda_0[$. Therefore there exists a minimum u of I_λ ($u \in X$) and observing that if for some $\varphi \in \mathfrak{D}_+(\mathbb{R}^N)$: $\int_{\mathbb{R}^N} K|u|^{4/(N-2)} u\varphi\, dx > 0$, then for t small enough

$$\mathcal{E}(u - t\varphi) \geqslant I_\mu > I_\lambda, \quad \text{for some} \quad \mu \in]0, \lambda[.$$

This shows that the Lagrange multiplier $(-\theta_\lambda)$ is strictly negative and thus u ($\in X \cap BUC(\mathbb{R}^N)$) solves (59). The remainder of part iii) is proved by showing that for each $\theta > 0$, there exists at most one $\bar{u} = \bar{u}_\theta$ solution of

$$A\bar{u} + \theta K|\bar{u}|^{4/(N-2)}\bar{u} = 0 \quad \text{in} \quad \mathbb{R}^N, \qquad \bar{u} \in X \cap BUC(\mathbb{R}^N). \tag{61}$$

and that $\bar{u}_\theta \geqslant \bar{u}_{\theta'}$, if $\theta \leqslant \theta'$. We first observe that $\bar{u} > 0$ on $\mathbb{R}^N$: indeed for R large enough: $\bar{u} \geqslant (\alpha/2)$ if $|x| \geqslant R$. Next we have

$$A\bar{u} + \theta K|\bar{u}|^{4/(N-2)}\bar{u} = 0 \quad \text{in} \quad B_R, \qquad \bar{u} \geqslant (\alpha/2) \quad \text{on} \quad \partial B_R$$

and $\lambda_1(A + \theta K|\bar{u}|^{4/(N-2)}, H_0^1(B_R)) > \lambda_1(A, H_0^1(B_R)) > 0$; therefore applying the maximum principle, we find $\bar{u} > 0$ in $\bar{B}_R$.

Next, let u, v be two solutions of (61) corresponding to $\theta > \theta' > 0$: we just need to prove that $u \leqslant v$ on $\mathbb{R}^N$. Indeed for R large enough: $0 < \alpha - \epsilon \leqslant \leqslant u, v \leqslant \alpha + \epsilon$ on ∂B_R, with $\epsilon = \epsilon(R) \to 0$ as $R \to \infty$. Let $w = ((\alpha + \epsilon)/(\alpha - \epsilon))v$, we have

$$\begin{aligned} Aw + \theta K w^{(N+2)/(N-2)} &\geqslant \frac{\alpha + \epsilon}{\alpha - \epsilon}(Av + \theta K v^{(N+2)/(N-2)}) \\ &\geqslant \frac{\alpha + \epsilon}{\alpha - \epsilon}(Av + \theta' K v^{(N+2)/(N-2)}) = 0 \quad \text{on} \quad B_R \end{aligned}$$

and $w \geqslant u$ on ∂B_R. Applying the maximum principle once more we conclude: $w \geqslant u$ on $\bar{B}_R$.

We finally prove part iv): we just have to prove that (S. 1) holds for $(\lambda - \lambda_0)$ small, positive. Let $\lambda > \lambda_0$, if (S. 1) does not hold there exists $\mu \in]0, \lambda[$ such that

$$I_\lambda = I_\mu + I^\infty_{\lambda - \mu}.$$

We first observe that $\mu \in]\lambda_0, \lambda[$: indeed $I_\lambda > I_{\lambda_0}$ and if $\mu \in]0, \lambda_0[$, $I_\mu + I^\infty_{\lambda-\mu} > > I_{\lambda_0} + I^\infty_{\lambda-\mu} > I_{\lambda_0} + I^\infty_{\lambda-\lambda_0} \geqslant I_\lambda$. Next we claim that (S. 1) holds for I_μ: indeed for $\bar{\mu} \in]0, \mu[$

$$(I_{\bar{\mu}} + I^\infty_{\mu-\bar{\mu}}) + I^\infty_{\lambda-\mu} > I_{\bar{\mu}} + I^\infty_{\lambda-\bar{\mu}} \geqslant I_\lambda = I_\mu + I^\infty_{\lambda-\mu}.$$

Therefore there exists a minimum u_μ of I_μ and one proves easily that $u_\mu > 0$ in $\mathbb{R}^N$ and $u_\mu - \alpha$ converges strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to $\varphi_1 - \alpha$ as $\mu \to \lambda_0$. Observing that $\mu \to \lambda_0$ when $\lambda \to \lambda_0$, we conclude the proof of part iv) as follows: we denote by $h = \lambda - \mu$ and we consider $\varphi \in \mathcal{D}_+(\mathbb{R}^N)$ such that

$$\frac{N+2}{N-2}\int_{\mathbb{R}^N} K\varphi_1^{(N+2)/(N-2)}\varphi\,dx = 1,$$

obviously $((N+2)/(N-2)\int_{\mathbb{R}^N} Ku_\mu^{(N+2)/(N-2)}\varphi\,dx = \theta_\mu \to 1$ as $\mu \to \lambda_0$. We introduce $v_\mu = u_\mu + h\theta_\mu^{-1}\varphi$, clearly $v_\mu \in X$ and

$$\begin{cases} J(v_\mu) \geqslant J(u_\mu) + \dfrac{N+2}{N-2}\displaystyle\int_{\mathbb{R}^N} Ku_\mu^{(N+2)/(N-2)}(h\theta_\mu^{-1}\varphi)\,dx = \mu + h = \lambda \\ \mathcal{E}(v_\mu) \leqslant \mathcal{E}(u_\mu) + Ch \end{cases}$$

for some C independent of $h \in]0,1[$. The properties of I_λ as a function of λ then yield: $I_\lambda \leqslant \mathcal{E}(v_\mu) \leqslant \mathcal{E}(u_\mu) + Ch = I_\mu + Ch$. On the other hand: $I_\lambda = I_\mu + I_h^\infty$; and we reach a contradiction for h small enough since $I_h^\infty = I_1^\infty h^{(N-2)/N}$.

I.6 Nonlinear field equations and limit exponents

As we explained in the introduction, one is interested in the so-called ground state solution of

$$-\Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N, \qquad u(x) \to 0 \quad \text{as} \quad |x| \to \infty; \tag{9}$$

where u is (for example) a scalar function. The ground state is determined through the minimum, if it exists, of the following minimization problem

$$I = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} |Du|^2\,dx \Big/ \int_{\mathbb{R}^N} F(u)\,dx = 1, u \in \mathcal{D}^{1,2}(\mathbb{R}^N), F(u) \in L^1(\mathbb{R}^N)\right\} \tag{10}$$

where $N \geqslant 3$, $F \in C(\mathbb{R})$, $F(0) = 0$. For more details concerning the relations between (9) and (10), we refer to H. Berestycki and P. L. Lions [6].

We assume

$$\exists \zeta \in \mathbb{R}, \qquad F(\zeta) > 0 \tag{11}$$

$$\lim_{|t| \to 0_+} F^+(t)|t|^{-2N/(N-2)} = \alpha \geqslant 0, \tag{12}$$

$$\lim_{|t| \to \infty} F^+(t)|t|^{-2N/(N-2)} = \beta \geqslant 0;$$

and we denote

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} |Du|^2\,dx, \qquad J(u) = \int_{\mathbb{R}^N} F(u)\,dx.$$

If $u_\sigma(\cdot) = \sigma^{-(N-2)/N} u(\cdot/\sigma)$, $\quad \mathcal{E}(u_\sigma) = \mathcal{E}(u)$ and

$$J(u_\sigma) = \sigma^N \int_{\mathbb{R}^N} F(\sigma^{-(N-2)/N} u(x))\, dx.$$

Therefore

$$J(u_\sigma) \to \beta \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx \quad \text{as} \quad \sigma \to 0_+,$$

while

$$J(u_\sigma) \to \alpha \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx \quad \text{as} \quad \sigma \to +\infty.$$

And this yields

$$I \leqslant I^\infty = \operatorname{Inf}\{\mathcal{E}(u)/\gamma \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx = 1, u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\}$$

with $\gamma = \max(\alpha, \beta)$; or

$$I \leqslant I^\infty = \gamma^{-(N-2)/N} I^0 \tag{62}$$

of course if $\gamma = 0$ i.e. $\alpha = \beta = 0$, $I^\infty = +\infty$ and the inequality is strict.

Theorem I.5. *Under assumptions* (11), (12), *any minimizing sequence* $(u_n)_n$ *of* (10) *is relatively compact in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *up to a translation if and only if*: $I < I^\infty$. *In particular if this strict inequality holds, there exists a minimum of* (10).

Remark I.10. If $\gamma = 0$, $I < I^\infty = +\infty$ and we recover the most general existence result —for the ground state— due to H. Berestycki and P. L. Lions [6], H. Brézis and E. H. Lieb [11]: in [6], this result was proved by a symmetrization argument which does not show that all minimizing sequences are relatively compact up to translations. Of course if $\gamma = 0$, we are in the locally compact case and the result of P. L. Lions [21] also applies to that particular situation. Let us also mention that the fact that minima of (10) yield ground states of (9) is due to Coleman, Glazer and Martin [13] —see also [6], [21]—. Except for a particular case (covered by the result above) due to F. V. Atkinson and L. A. Peletier [2] obtained by an O.D.E. method, the above result is the first where F is allowed to behave like $|t|^{2N/(N-2)}$ near 0 or at infinity.

Remark I.11. Combining the method below and those of P. L. Lions [21], we could treat as well x-dependent functionals or higher-order functionals. Let us also mention that the same result holds if u takes its value in $\mathbb{R}^m$ ($m \geqslant 1$), then we just need to assume (11) and

$$\begin{cases} \lim\limits_{|t| \to 0_+} F^+(t) F_0(t)^{-1} = \alpha \geqslant 0 \\ \lim\limits_{|t| \to \infty} F^+(t) F_1(t)^{-1} = \beta \geqslant 0 \end{cases} \tag{63}$$

where F_0, F_1 are continuous, positive on $\mathbb{R}^m - \{0\}$, homogeneous of degree $2N/(N-2)$; I^∞ is given in this situation by

$$I^\infty = \operatorname*{Min}_{i=0,1} [\operatorname{Inf}\{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx / u \in (\mathcal{D}^{1,2}(\mathbb{R}^N))^m, \int_{\mathbb{R}^N} F_i(u) \, dx = 1\}].$$

Observe that both infima are achieved by the results of section I.4.

Remark I.12. If $\alpha = \beta = \gamma > 0$, then by Theorem I.1, there exists u_0 such that

$$u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N), \mathcal{E}(u_0) = I^\infty, \int_{\mathbb{R}^N} |u_0|^{2N/(N-2)} \, dx = \gamma^{-1}, \qquad u \geqslant 0.$$

i) If $F(t) \geqslant \gamma |t|^{2N/(N-2)}$ and $F(t) \not\equiv \gamma |t|^{2N/(N-2)}$, then $I < I^\infty$: indeed we observe that, choosing by dilation u_0 the maximum of u_0 large enough, we may assume

$$\int_{\mathbb{R}^N} F(u_0) \, dx > 1, \mathcal{E}(u_0) = I^\infty.$$

Let $v_0(\cdot) = u_0(\cdot/\lambda)$ with

$$\lambda^{-N} = \int_{\mathbb{R}^N} F(u_0) \, dx > 1;$$

then $J(u_0) = 1$ and $I \leqslant \mathcal{E}(u_0) = \lambda^{N-2} \mathcal{E}(u_0) < I^\infty$.

ii) If $F(t) \leqslant \gamma |t|^{2N/(N-2)}$ and $F \not\equiv \gamma |t|^{2N/(N-2)}$, a similar argument shows that there does not exist a minimum of (10).

Remark I.13. If $\alpha = \gamma \geqslant \beta$, $\alpha > 0$ and if for some $t_0 > 0$

$$F(t) \geqslant \alpha |t|^{2N/(N-2)} \quad \text{for} \quad t \in [0, t_0] \qquad (\text{or } t \in [-t_0, 0])$$

and $F \not\equiv \alpha |t|^{2N/(N-2)}$ on $[0, t_0]$, considering $\sigma^{-(N-2)/N} u_0(\cdot/\sigma)$ with u_0 as in Remark I.12 and σ large enough, we deduce from the argument given above that $I < I^\infty$.

Remark I.14. By looking carefully at the proof below, we see that if $I = I^\infty$ and if $(u_n)_n$ is not relatively compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ then

i) if $\alpha < \beta$, there exist $(\sigma_n)_n$ in $]0, \infty[$, $(y_n)_n$ in $\mathbb{R}^N$ such that: $\sigma_n \underset{n}{\to} \infty$, $\tilde{u}_n = \sigma_n^{-(N-2)/N} u_n((\cdot - y_n)/\sigma_n)$ is relatively compact in $\mathcal{D}^{1,2}$ and the limits of its converging subsequences are minima of I^∞;

ii) if $\alpha > \beta$, the above still holds but with $\sigma_n \underset{n}{\to} 0$;

iii) if $\alpha = \beta$, the above still holds but $(\sigma_n)_n$ is arbitrary.

Remark I.15. Of course, if we are only interested in finding a (non trivial) solution of (9), we may use the maximum principle and assume: $f = F'$,

$F \in C^1(\mathbb{R})$ (for example), (11), and if $\zeta_\pm = \inf(\zeta > 0, F(\pm\zeta) > 0)$ we assume in addition

$$\lim_{|t| \to 0_+} F^+(t)|t|^{-2N/(N-2)} = \alpha \geqslant 0$$

$$\text{either } \exists \zeta'_+ > \zeta_+, f(\zeta'_+) \leqslant 0, \quad \text{or} \quad \lim_{|t| \to +\infty} F^+(t)|t|^{-2N/(N-2)} = \beta \geqslant 0$$

and

$$\text{either } \exists \zeta'_- > \zeta_-, f(-\zeta'_-) \geqslant 0, \quad \text{or} \quad \lim_{|t| \to \infty} F^+(t)|t|^{-2N/(N-2)} = \beta \geqslant 0.$$

We now turn to the proof of Theorem I.5: first we observe that $(u_n)_n$ is bounded in $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ and $F(u_n)$ is bounded in L^1, for any minimizing sequence $(u_n)_n$.

Indeed, since $\mathcal{E}(u_n)$ is bounded, ∇u_n is bounded in L^2 and u_n is bounded in $L^{2N/(N-2)}$. But (12) implies the existence of a constant $C \geqslant 0$ such that: $F^+(t) \leqslant C|t|^{2N/(N-2)}$; hence $F^+(u_N)$ is bounded in L^1 and using the constraint, the bounds claimed are proved. The proof below will use the concentration-compactness method of P. L. Lions [20], [21] with the sequence:

$$\rho_n = |\nabla u_n|^2 + |u_n|^{2N/(N-2)} + |F(u_N)|$$

We may of course assume that: $\int_{\mathbb{R}^N} \rho_n \, dx \underset{n}{\to} M > 0$. In what follows we will still denote by u_n all subsequences we extract. With these preliminaries, we prove below: Step 1, ρ_n does not vanish; Step 2, dichotomy does not occur; Step 3, weak limits are non trivial; Step 4, we conclude.

Step 1: ρ_n *does not vanish.*

Indeed if ρ_n vanishes i.e. if (in particular) there exists $R \in]0, \infty[$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{y + B_R} \rho_n \, dx \underset{n}{\to} 0$$

then we denote by $G(t) = (F(t) - \gamma|t|^{2N/(N-2)})^+$ and we claim that

$$G(u_n) \underset{n}{\to} 0 \quad \text{in} \quad L^1(\mathbb{R}^N).$$

To this end we just need to observe that G satisfies

$$\lim_{|t| \to 0_+} G(t)|t|^{-2N/(N-2)} = 0, \ \lim_{|t| \to \infty} G(t)|t|^{-2N/(N-2)} = 0.$$

Therefore, using Lemma II.2 of P. L. Lions [21], we deduce that $G(u_n) \underset{n}{\to} 0$ in $L^1(\mathbb{R}^N)$.

But this yields

$$\lim_n \gamma \int_{\mathbb{R}^N} |u_n|^{2N/(N-2)}\, dx \geqslant 1,\ \mathcal{E}(u_n) \underset{n}{\to} I$$

and this contradicts the strict inequality $I < I^\infty$. Hence if $I < I^\infty$, vanishing does not occur.

Step 2: *Dichotomy does not occur.*

In view of the concentration-compactness lemma (Lemma I.1 in P. L. Lions [20]), we check that dichotomy does not occur. To this end we first remark that if we replace 1 by $\lambda > 0$ in (10) and if we denote by I_λ the corresponding infimum, then

$$I_\lambda = \operatorname{Inf}\left\{ \mathcal{E}u\left(\frac{\cdot}{\lambda^{1/N}}\right) \middle/ J(u) = 1, u \in \mathcal{D}^{1,2}(\mathbb{R}^N), F(u) \in L^1 \right\} = \lambda^{(N-2)/N} I$$

therefore the subadditivity inequality (S. 2) holds

$$I < I_\alpha + I_{1-\alpha}, \qquad \forall \alpha \in]0, 1[. \tag{S. 2}$$

We now prove —as in P.L. Lions [20], [21], see also P.L. Lions [23], [24]— that dichotomy does not occur by contradiction. We will use a variant of the explicit dichotomy procedure of [20], [21] in order to cover the full generality of functions F: the idea of this variant was given to us by H. Brézis (see also H. Brézis and E.H. Lieb [11]). If dichotomy occurs we find $\alpha \in]0, M[$ such that for any fixed $\epsilon > 0$, there exist $(y_n)_n$ in $\mathbb{R}^N$, $0 < R_0 < \infty$, R_n in $]R_0, +\infty[$ satisfying

$$\begin{cases} \left| \int_{y_n + B_{R_0}} \rho_n\, dx - \alpha \right| \leqslant \epsilon, & \left| \int_{|x-y_n| \geq R_n} \rho_n\, dx - (M-\alpha) \right| \leqslant \epsilon \\ \int_{R_0 \leq |x-y_n| \leq R_n} v_n\, dx \leqslant \epsilon, & R_n \underset{n}{\to} \infty. \end{cases} \tag{64}$$

If we still denote by $(u_n)_n$ the minimizing sequence translated by y_n, we are going to "cut" u_n in two pieces such that both functionals $\mathcal{E}, J$ split in the sums of the corresponding functionals.

To this end, we introduce for $\lambda \geqslant 1$, $R \in]0, \infty[$ the mapping: $Tx = x$ if $|x| \leqslant R$, $Tx = \lambda x - (\lambda - 1)Rx|x|^{-1}$, and we set $u_n^1(x) = u_n(Tx)$. We now compute $J(u_n^1)$, $\mathcal{E}(u_n^1)$

$$\begin{aligned} J(u_n^1) &= \int_{|x| \leq R} F(u_n)\, dx + \int_{|x| > R} F(u_n(Tx))\, dx \\ &= \int_{|x| \leq R} F(u_n)\, dx + \int_{|y| > R} F(u_n(y))\phi(y)\, dy \end{aligned}$$

with $\phi(y)^{-1} = \lambda(\lambda + R|T^{-1}y|^{-1}(1-\lambda))^{N-1} \geqslant \lambda$ if $|y| \geqslant R$.

Hence we deduce

$$\left| J(u_n^1) - \int_{|x| \le R} F(u_n)\,dx \right| \leqslant \frac{C}{\lambda} \tag{65}$$

Next, we compute $\mathcal{E}(u_n^1)$

$$\mathcal{E}(u_n^1) = \int_{|x| \le R} |\nabla u_n|^2\,dx + \int_{|x| \ge R} |\nabla T \cdot \nabla u_n(Tx)|^2\,dx$$

and

$$T_{i,j} = \lambda \delta_{ij} + (\lambda - 1)\frac{R}{|x|^3}(x_i x_j - \delta_{ij}|x|^2),$$

therefore

$$\begin{aligned}\int_{|x| \ge R} |\nabla T \cdot \nabla u_n(Tx)|^2\,dx &\leqslant C\lambda \int_{|y| \ge R} |\nabla u_n(y)|^2 \phi(y)\,dy \\ &\leqslant C \int_{R_n \ge |y| \ge R} |\nabla u_n|^2\,dy + C(\lambda - (\lambda - 1)R/\bar{R}_n)^{1-N}\end{aligned}$$

if $R \leqslant R_n$, with $\bar{R}_n = \frac{1}{\lambda}(R_n + (\lambda - 1)R)$. Thus we obtain

$$\begin{cases} \left| \mathcal{E}(u_n^1) - \int_{|x| \le R} |\nabla u_n|^2\,dx \right| \leqslant C\epsilon + C(\lambda - (\lambda - 1)R/\bar{R}_n)^{1-N} \\ \text{if} \quad R_0 \leqslant R \leqslant R_n, \qquad \bar{R}_n = \frac{1}{\lambda}(R_n + (\lambda - 1)R). \end{cases}$$

And thus choosing $R = R_0$, λ large enough, we find for n large

$$\begin{cases} \left| J(u_n^1) - \int_{|x| \le R_0} F(u_n)\,dx \right| \leqslant \epsilon, \\ \left| \mathcal{E}(u_n^1) - \int_{|x| \le R_0} |\nabla u_n|^2\,dx \right| \leqslant C\epsilon. \end{cases} \tag{67}$$

We build u_n^2 in a similar way: we consider the mapping

$$Sx = \mu x \quad \text{if} \quad |x| \leqslant R, \qquad = x + (\mu - 1)Rx|x|^{-1} \quad \text{if} \quad |x| \geqslant R,$$

where $\mu \geqslant 1$, $R \geqslant R_0$; and we denote by: $u_n^2(x) = u_n(Sx)$. We have

$$J(u_n^2) = \mu^{-N} \int_{|y| \le \mu R} F(u_n)\,dy + \int_{|y| \ge \mu R} F(u_n(y))\psi(y)\,dy$$

with $\psi(y)^{-1} = (1 + (\mu - 1)R|S^{-1}y|^{-1})^{N-1}$. Therefore if $\mu R \leqslant R_n$

$$\begin{aligned}\left| J(u_n^2) - \int_{|y| \ge \mu R} F(u_n(y))\,dy \right| &\leqslant C\mu^{-N} + 2\int_{\mu R \le |y| \le R_n} |F(u_n)|\,dy + \\ &\quad + C|(1 + (\mu - 1)R/\tilde{R}_n)^{-(N-1)} - 1|;\end{aligned}$$

where $\tilde{R}_n = R_n - (\mu - 1)R$.

On the other hand, we have

$$\mathcal{E}(u_n^2) = \mu^{-(N-2)} \int_{|y| \le \mu R} |\nabla u_n|^2\,dy + \int_{|x| \le R} |\nabla S \cdot \nabla u_n(Sx)|^2\,dx$$

and

$$S_{i,j} = \delta_{ij} + (\mu - 1)\frac{R}{|x|^3}(|x|^2\delta_{ij} - x_i x_j).$$

Therefore if $\mu R \leqslant R_n$

$$\left|\int_{|x| \geq R} |\nabla S \cdot \nabla u_n(Sx)|^2 \, dx - \int_{|y| \geq \mu R} |\nabla u_n(y)^2| dy\right| \leqslant$$
$$\leqslant C\mu \int_{\mu R \leq |y| \leq R_n} |\nabla u_n(y)|^2 \, dy +$$
$$+ \int_{|y| \geq R_n} ||\nabla S(S^{-1}y) \cdot \nabla u_n(y)|^2 \psi(y) - |\nabla u_n|^2(y)| dy.$$
$$\leqslant C\mu \int_{\mu R \leq |y| \leq R_n} |\nabla u_n|^2 \, dy + C(\mu - 1)\frac{R}{\tilde{R}_n} + \int_{|y| \geq R_n} |\nabla u_n|^2 \{ |\psi(y) - 1| \} \, dy.$$

Finally we obtain if $R_0 \leqslant \mu R \leqslant R_n$

$$\left|\mathcal{E}(u_n^2) - \int_{|y| \geq \mu R} |\nabla u_n|^2 \, dy\right| \leqslant C\mu^{2-N} + C\epsilon\mu + C(\mu - 1)\frac{R}{\tilde{R}_n} +$$
$$+ C\{1 - (1 + (\mu - 1)R/\tilde{R}_n)^{-(N-1)}\}. \tag{69}$$

Combining (68), (69) and choosing $\mu = 1/\sqrt{\epsilon}$, $R = R_0$ we deduce finally that for n large enough

$$\begin{cases} \left|J(u_n^2) - \int_{|y| \geq R_n} F(u_n) \, dy\right| \leqslant C\epsilon, \\ \left|\mathcal{E}(u_n^2) - \int_{|y| \geq R_n} |\nabla u_n|^2 \, dy\right| \leqslant C\sqrt{\epsilon}. \end{cases} \tag{70}$$

We may now conclude: indeed if $J(u_n^1) \underset{n}{\rightarrow} \bar{\beta}$, we claim that $\bar{\beta}$ —which depends on ϵ— belongs to]0, 1[and remains bounded away from 0 or 1 as ϵ goes to 0. Indeed if $\bar{\beta} = \bar{\beta}_\epsilon \underset{\epsilon}{\rightarrow} \bar{\beta} = 0$, this means that:

$$\underline{\lim}_{\epsilon} \underline{\lim}_{n} \mathcal{E}(u_n^2) \geqslant I,$$

while (67) and (70) yield

$$\underline{\lim}_{\epsilon} \underline{\lim}_{n} \mathcal{E}(u_n^2) + \underline{\lim}_{\epsilon} \underline{\lim}_{n} \mathcal{E}(u_n^1) \leqslant I;$$

and we reach a contradiction since

$$\underline{\lim}_{\epsilon} \underline{\lim}_{n} \mathcal{E}(u_n) > 0.$$

If this were not the case we would have

$$\begin{aligned}\alpha = \underline{\lim}_{\epsilon}\,\underline{\lim}_{n} \int_{B_{R_0}} \rho_n\,dx &= \underline{\lim}_{\epsilon}\,\underline{\lim}_{n} \int_{B_{R_0}} |F(u_n)|\,dx \\ &= \underline{\lim}_{\epsilon}\,\underline{\lim}_{n} \int |F(u_n^2)|\,dx \\ &= \underline{\lim}_{\epsilon}\,\underline{\lim}_{n} \int -F(u_n^2) + 2F^+(u_n^2)\,dx = 0.\end{aligned}$$

If $\bar{\beta}_\epsilon \underset{\epsilon}{\to} \bar{\beta} < 0$, then for ϵ small and n large, $J(u_n^2) < 1$ and we deduce from (67) and (70): $J(u_n{}^2) < 1$ and $\mathcal{E}(u_n^2) < I$, and this is not possible.

Finally if $\bar{\beta}_\epsilon \underset{\epsilon}{\to} \bar{\beta} \geqslant 1$, we argue as before replacing u_n^2 by u_n^1. Thus we may assume that $\bar{\beta}_\epsilon \underset{\epsilon}{\to} \bar{\beta} \in]0, 1[$. We then deduce from (67) and (70) for ϵ small

$$I \geqslant \underline{\lim}_{n}\, \mathcal{E}(u_n^1) + \underline{\lim}_{n}\, \mathcal{E}(u_n^2) - C\sqrt{\epsilon} \geqslant I_{\bar{\beta}} = I_{1\bar{\beta}} - \delta(\epsilon)$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0_+$, and this contradicts (S. 2).

Therefore, from Step 1 and the above contradiction we deduce, using the concentration-compactness lemma of P. L. Lions [20], [21] that, if $I < I^\infty$, there exists $(y_n)_n$ in $\mathbb{R}^N$ such that $\rho_n(\cdot - y_n)$ is tight

$$\forall \epsilon > 0, \exists R < \infty, \forall n \geqslant 1, \int_{|x - y_n| \geq R} \rho_n\,dx \leqslant \epsilon. \tag{71}$$

We still denote by u_n the new minimizing sequence $u_n(\cdot - y_n)$. We may assume that u_n converges weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and a.e. on $\mathbb{R}^N$ to some $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ (and $F(u) \in L^1(\mathbb{R}^N)$ by Fatou's lemma).

Step 3: $u \not\equiv 0$ *if* $I < I^\infty$.

If $u \equiv 0$, then we claim that $G(u_n) = (F(u_n) - \gamma|u_n|^{2N/(N-2)})^+$ converges to 0 in $L^1(\mathbb{R}^N)$. Indeed since $F(t) \leqslant C|t|^{2N/(N-2)}$, we may find, using (71), R large enough such that

$$\int_{|x| \geq R} G(u_n)\,dx \leqslant \epsilon, \qquad \forall n \geqslant 1.$$

Now on B_R, we use the fact that $u_n \underset{n}{\to} 0$ in $L^1(B_R)$ and that

$$G(t) \leqslant \epsilon|t|^{2N/(N-2)} + C_\epsilon t, \qquad \forall t \in \mathbb{R}.$$

Therefore the claim is proved and we show exactly as in Step 1 that we reach a contradiction with $I < I^\infty$.

Step 4: *Conclusion.*

We first assume that $I < I^\infty$ and thus $u \not\equiv 0$. We just need to show that $J(u) = 1$. Of course $J(u) \leqslant 1$, since $\mathcal{E}(u) \leqslant I$.

By lemma I.1, we know there exist $(\nu_k)_{k\in K} \in]0, \infty[$, $(x_k)_{k\in K}$ in $\mathbb{R}^N$ —where K is at most countable and the points x_k are distinct— such that

$$|u_n|^{2N/(N-2)} \underset{n}{\rightharpoonup} |u|^{2N/(N-2)} + \sum_k \nu_k \delta_{x_k} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2N/(N-2)}\, dx \underset{n}{\rightarrow} \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx + \sum_k \nu_k.$$

We claim that

$$1 = \lim_n J(u_n) \leqslant J(u) + \beta \sum_k \nu_k. \tag{72}$$

We first choose R —using (71)— such that for all $n \geqslant 1$

$$\begin{cases} \int_{|x|\geq R} |F(u_n)| + |u_n|^{2N/(N-2)}\, dx \leqslant \epsilon, \\ \int_{|x|\geq R} |F(u)| + |u|^{2N/(N-2)}\, dx \leqslant \epsilon \\ \sum_{k:x_k \notin B_R} \nu_k \leqslant \epsilon. \end{cases}$$

To simplify the notations we assume that $x_k \in B_R$, $\forall k \in K$. We next apply Brézis-Lieb [10] to obtain

$$\int_{B_R} |F(u_n) - F(u) - F(u_n - u)|\, dx \underset{n}{\rightarrow} 0;$$

this is possible in view of the following observation: $\forall \epsilon > 0$, $\exists C_\epsilon \geqslant 0$

$$|F(a+b) - F(a)| \leqslant \epsilon |a|^{2N/(N-2)} + C_\epsilon(1 + |b|^{2N/(N-2)})$$

for all $a, b \in \mathbb{R}$. Then (72) is proved provided we show

$$\varliminf_n \int_{B_R} F(u_n - u)\, dx \leqslant \beta \sum_k \nu_k.$$

But we already know that: $|u_n - u|^{2N/(N-2)} \rightharpoonup \sum_k \nu_k \delta_{x_k}$, and by the same proof as in Step 3, we conclude

$$\varliminf \int_{B_R} F(u_n - u)\, dx \leqslant \varliminf \beta \int_{B_R} |u_n - u|^{2N/(N-2)}\, dx = \beta \sum_k \nu_k.$$

Using (72) and Lemma I.1, we finally obtain

$$\begin{cases} 1 \leqslant J(u) + \beta \sum_k \nu_k \leqslant J(u) + \gamma \sum_k \nu_k \\ I \geqslant \mathcal{E}(u) + I^0 \sum_k \nu_k^{(N-2)/N} \geqslant \mathcal{E}(u) + I^0\Big(\sum_k \nu_k\Big)^{(N-2)/N}. \end{cases}$$

If $J(u) \leqslant 0$, $\sum_k \nu_k \geqslant 1/\gamma$ and

$$I > I^0\Big(\sum_k \nu_k\Big)^{(N-2)/N} \geqslant I^0 \gamma^{-(N-2)/N} = I^\infty.$$

Hence $\alpha = J(u) \in]0, 1]$ and if $\alpha \in]0, 1[$

$$I \geqslant I_\alpha + I^0 \gamma^{-(N-2)/N}(1-\alpha)^{(N-2)/N} = I_\alpha + I^\infty_{1-\alpha}$$

But this contradicts the condition (S. 1) which holds here if $I < I^\infty$. Therefore $\alpha = 1$ and the compactness is proved.

On the other hand if $I = I^\infty$, we build a sequence $(u_n)_n$ such that $F(u_n) \in L^1$, $\int_{\mathbb{R}^N} F(u_n)\,dx \underset{n}{\to} 1$, $\mathcal{E}(u_n) \underset{n}{\to} I^\infty$ and u_n is not compact even up to a translation. Indeed if $\gamma = \alpha \geqslant \beta$, and $\gamma > 0$, we consider $u_0 \in C_0(\mathbb{R}^N)$ satisfying (cf. section I.1)

$$\alpha \int_{\mathbb{R}^N} |u_0|^{2N/(N-2)}\,dx = 1, \qquad \mathcal{E}(u_0) = I^\infty.$$

We set $u_n = n^{-(N-2)/N} u_0(\cdot/n)$ and we check easily the above properties. On the other hand if $\gamma = \beta > \alpha$ and $\beta > 0$, we take $(u_n)_n$ in $\mathcal{D}(\mathbb{R}^N)$, such that: Supp $u_n = \bar{B}_{1/n}$, $|\nabla u_n|^2 \to I^\infty \delta_0$, $\beta \int_{\mathbb{R}^N} |u_n|^{2N/(N-2)}\,dx = 1$. Again it is easy to check the above properties. And Theorem I.6 is proved.

Remark I.16. Using the particular quadratic structure of $\mathcal{E}$(and the x-independence of the functionals) one may give in Step 4 a slightly simpler argument: indeed if $u_n \underset{n}{\rightharpoonup} u$, then

$$\begin{cases} \mathcal{E}(u_n) - \mathcal{E}(u_n - u) = 2\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla u\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx \underset{n}{\to} \mathcal{E}(u) \\ J(u_n) - J(u_n - u) \underset{n}{\to} J(u) \end{cases}$$

and if $J(u) \in]0, 1[$ we simply use (S. 2) to conclude.

However this argument is very dependent on the special structures of $\mathcal{E}, J$ and fails completely if $\mathcal{E}$ is x-dependent or if

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx \quad \text{is replaced by} \quad \int_{\mathbb{R}^N} |\nabla u|^p\,dx \quad \text{for} \quad p \neq 2!$$

I.7 A remark on Moser's treatment of the limiting case of Sobolev inequalities

We want to discuss here a few properties of functions in $W_0^{1,N}(\Omega)$ where Ω is a bounded open set in $\mathbb{R}^N$ and $N \geqslant 2$. Clearly the Sobolev exponent becomes infinite but $W_0^{1,N}(\Omega)$ is not embedded in $L^\infty(\Omega)$. It is possible to check that if $|\nabla u|_{L^N(\Omega)} \leqslant 1$, there exists some $\alpha > 0$ (independent of u) such that

$$\int_\Omega \exp\{\alpha|u|^{N/(N-1)}\}\,dx \leqslant C.$$

This was proved by N. Trudinger [37] (see also S. I. Pohozaev [30], T. Aubin [5]...). This estimate was sharpened by J. Moser [27] who proved that if $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ where ω_k is the volume of S^k (for example $\alpha_2 = 4\pi$) then we have

$$\int_\Omega \exp\{\alpha_N|u|^{N/(N-1)}\}\,dx \leqslant C|\Omega| \tag{73}$$

and α_N is the best constant in the following sense: $\exp(\alpha|u|^{N/(N-1)}) \in L^1(\Omega)$ for any $\alpha > 0$ but α_N is the biggest constant such that $\exp(\alpha|u|^{N/(N-1)})$ is bounded in $L^1(\Omega)$ independently of u. In other words $W_0^{1,N}(\Omega)$ is embedded in the Orlicz space determined by $\phi(t) = \exp\{\alpha_N|t|^{N/(N-1)}\}$.

A natural question is then: is this embedding compact? The answer is no: indeed, if for example Ω is the unit ball, we consider $(u_n)_n$ defined by

$$u_n(x) = f_n(-N\operatorname{Log}|x|); \qquad f_n(t) = \left(\frac{n}{\alpha_N}\right)^{(N-1)/N}\frac{t}{n} \quad \text{if} \quad t \leqslant n,$$

$$= \left(\frac{n}{\alpha_N}\right)^{(N-1)/N} \quad \text{if} \quad t \geqslant n.$$

Clearly

$$\begin{aligned}|\nabla u_n|_{L^N}^N &= N^{N-1}\omega_{N-1}\int_0^\infty |f_n'(t)|^N\,dt \\ &= N^{N-1}\omega_{N-1}\int_0^n \left(\frac{n}{\alpha_N}\right)^{N-1}\left(\frac{1}{n}\right)^N dt = 1;\end{aligned}$$

and

$$\begin{aligned}|\exp\{\alpha_N|u_n|^{N/(N-1)}\}|_{L^1} &= \omega_{N-1}N^1\int_0^\infty \exp\{\alpha_N(f_n(t))^{N/(N-1)} - t\}\,dt \geqslant \\ &\geqslant \omega_{N-1}N^{-1}\int_n^\infty \exp\{n-t\}\,dt = \omega_{N-1}N^{-1}.\end{aligned}$$

Since $u_n \underset{n}{\rightarrow} 0$ a.e. and weakly in $W_0^{1,N}(\Omega)$, the embedding is not compact. Observe that $|\nabla u_n|^N \to \delta_0$ in $\mathcal{D}'(\Omega)$ and $\exp\{\alpha_N|u_n|^{N/(N-1)}\} \to c\delta_0$ for some $c > 0$.

The following result shows that this is the exceptional case

Theorem I.6. *Let* $(u_n)_n \subset W_0^{1,N}(\Omega)$ *satisfy*: $|\nabla u_n|_{L^N(\Omega)} \leqslant 1$. *Without loss of generality we may assume that* $u_n \underset{n}{\rightharpoonup} u$, $|\nabla u_n|^2 \underset{n}{\rightharpoonup} \mu$ *weakly. Then either* $\mu = \delta_{x_0}$ *for some* $x_0 \in \bar{\Omega}$ *and* $u_n \underset{n}{\rightharpoonup} 0$, $\exp(\alpha_N |u_n|^{N/(N-1)}) \underset{n}{\rightharpoonup} c\delta_{x_0}$ *for some* $c \geqslant 0$, *or there exists* $\alpha > 0$ *such that* $\exp\{(\alpha_N + \alpha)|u_n|^{N/(N-1)}\}$ *is bounded in* $L^1(\Omega)$ *and thus*

$$\exp\{\alpha_N |u_n|^{N/(N-1)}\} \underset{n}{\rightarrow} \exp\{\alpha_N |u|^{N/(N-1)}\} \quad in \quad L^1(\Omega).$$

In particular this is the case if $u \not\equiv 0$.

Remark I.17. We also deduce from this result that except for "small weak neighborhoods of 0" the embedding is compact and the best constant α_N may be improved.

PROOF OF THEOREM I.6. *We first treat the case when* $u \equiv 0$. Let $\xi \in C^1(\bar{\Omega})$, we have using Rellich theorem

$$|\nabla(\xi u_n)|_{L^N}^N = \int_\Omega |(\nabla \xi) u_n + \xi \nabla u_n|^N \, dx \underset{n}{\rightarrow} \int |\xi|^N \, d\mu$$

since $u_n \underset{L^N}{\rightarrow} 0$ (strongly). Without loss of generality we may assume that $|\nabla u_n|_{L^N} = 1$ (consider $v_n = u_n |\nabla u_n|_{L^N}^{-1}$), hence $\int d\mu = 1$ and $\operatorname{Supp} \mu \subset \bar{\Omega}$.

We first observe that if $\xi \in C^1(\bar{\Omega})$, $\xi \geqslant 0$ and $\int |\xi|^N d\mu = 1$, then $\exp\{\alpha_N |u_n|^{N(N-1)}\}$ is bounded in $L^p(\{\xi \geqslant 1 + \delta\})$ for $p = p_\delta > 1$, $\delta > 0$. In particular $\exp\{\alpha_N |u_n|^{N/(N-1)}\}$ converges to 1 in $L^1(\{\xi \geqslant 1 + \delta\})$ for any $\delta > 0$. Indeed $|\nabla(\xi u_n)|_{L^N} \underset{n}{\rightarrow} 1$ and

$$\int_\Omega \exp\{\alpha_N |\xi u_n|^{N/(N-1)} |\nabla(\xi u_n)|_{L^N}^{-N/(N-1)}\} \, dx \leqslant C.$$

thus for any $\gamma \in]1, (1 + \delta)^{N/(N-1)}[$ we have for n large enough

$$\int_{(\xi \geq 1 + \delta)} \exp\{\alpha_N \gamma |u_n|^{N/(N-1)}\} \, dx < C. \tag{74}$$

Next if $\mu = \delta_{x_0}$ for some $x_0 \in \bar{\omega}$, taking ξ with $\xi(x_0) = 1$, $\xi > 1$ on $\bar{\omega} - \{x_0\}$, and remarking that $\nabla u_n \rightarrow 0$ weakly in $L^N(\Omega)$ and thus $u_n \rightarrow 0$ weakly in $W_0^{1,N}(\Omega)$, we deduce

$$\exp\{\alpha_N |u_n|^{N/(N-1)}\} \rightarrow c\delta_{x_0}, \quad \text{for some} \quad c \geqslant 0.$$

On the other hand if μ is not a Dirac mass, we claim that we can find F_1, F_2 compact contained in $\bar{\Omega}$ such that

$$\mu(F_1), \mu(F_2) \in]0, 1[\quad \text{and} \quad F_1 \cup F_2 = \bar{\Omega}.$$

Indeed if μ is not a Dirac mass, there exists F compact contained in $\bar{\Omega}$ such that: $\mu(F) = \theta \in]0, 1[$. We denote by $O = \mathbb{R}^N - F$, $O_\epsilon = \{x \in \mathbb{R}^N, \operatorname{dist}(x, F) > \epsilon\}$. Clearly —considering μ as a measure on $\mathbb{R}^N$ supported in Ω— we have:

$\mu(O) = 1 - \theta = \lim_{\epsilon \downarrow 0} \uparrow \mu(O_\epsilon)$. Hence, there exists ϵ small enough such that: $\mu(O_\epsilon) \in]0, 1[$.

Then if $F_1 = \bar{\Omega} \cap O_\epsilon^c$, $F_2 = \bar{\Omega} \cap \bar{O}_\epsilon$, clearly $F_1 \cup F_2 = \bar{\Omega}$ and $\mu(F_1) = 1 - \mu(O_\epsilon) \in]0, 1[$, $\mu(O_\epsilon) \leqslant \mu(F_2) \leqslant \mu(O)$.

But we may now consider $\xi_1, \xi_2 \in C^1(\bar{\Omega})$ satisfying

$$\begin{cases} \xi_1, \xi_2 \geqslant 0, & \xi_1 = \frac{1}{2}(1 + \mu(F_1)^{-1}) \quad \text{on} \quad F_1, \\ & \xi_2 = \frac{1}{2}(1 + \mu(F_2)^{-1}) \quad \text{on} \quad F_2, \\ \int \xi_1 \, d\mu = 1, & \int \xi_2 \, d\mu = 1. \end{cases}$$

And using (74) we deduce that for some $\gamma > 1$, we have for n large enough and thus for all $n \geqslant 1$

$$\begin{cases} \int_{F_1} \exp\{\alpha_N \gamma |u_n|^{N/(N-1)}\} \, dx \leqslant C \\ \int_{F_2} \exp\{\alpha_N \gamma |u_n|^{N/(N-1)}\} \, dx \leqslant C \end{cases}$$

and we conclude.

We next consider the case when $u \not\equiv 0$ *and* $N = 2$: we claim that $v_n = \exp\{\alpha_2 |u_n|^2\}$ converges to $v = \exp\{\alpha_2 u^2\}$ in $L^p(\Omega)$ for

$$p < \bar{p} = (1 - |\nabla u|_{L^2}^2)^{-1} \qquad (\bar{p} = +\infty \quad \text{if} \quad |\nabla u|_{L^2} = 1).$$

Indeed we have

$$v_n = \exp\{\alpha_2[u^2 + 2u(u_n - u) + (u_n - u)^2]\} = v \bar{v}_n \tilde{v}_n$$

where

$$v = \exp\{\alpha_2 u^2\} \in L^q(\Omega) \qquad (\forall q < \infty), \qquad \bar{v}_n = \exp\{2\alpha_2 u(u_n - u)\}$$

converges to 1 in $L^q(\Omega)$ $(\forall q < \infty)$. Finally remarking that

$$C_n = \int_\Omega |\nabla(u_n - u)|^2 \, dx = 1 - 2 \int_\Omega \nabla u_n \cdot \nabla u \, dx + \int_\Omega |\nabla u|^2 \, dx \xrightarrow[n]{} \frac{1}{\bar{p}},$$

we obtain

$$|\exp\{\alpha_2 C_n^{-1}(u_n - u)^2\}|_{L^1} = |\tilde{v}_n^{1/C_n}|_{L^1} \leqslant C$$

and we conclude easily.

Finally if $u \not\equiv 0$ *and* $N \geqslant 3$; we claim that $v_n = \exp\{\alpha_N |u_n|^{N/(N-1)}\}$ converges to $v = \exp\{\alpha_N |u|^{N/(N-1)}\}$ in $L^p(\Omega)$ for $p < \bar{p} = (1 - |\nabla u|_{L^N})^{-1/(N-1)}$. Since $v_n \xrightarrow[n]{} v$ a.e., we just need to prove that for all $p < \bar{p}$

$$\int_\Omega \exp\{\alpha_N p |u_n|^{N/(N-1)}\} \, dx \leqslant C \qquad (\text{inf. of } n). \tag{75}$$

By standard symmetrization argument we may assume that Ω is a ball, u_n, u are spherically symmetric, non increasing with respect to $|x|$. Without loss of generality we may assume that Ω is the unit ball and we consider —following Moser [27]— f_n defined on $]0, \infty[$ by

$$u_n(x) = f_n(-N \operatorname{Log} |x|), \qquad u(x) = f(-N \operatorname{Log} |x|);$$

f_n, f are continuous, non decreasing and $f_n(0) = f(0) = 0$.

In addition we have for all $\alpha > 0$

$$\begin{cases} 1 = \int_\Omega |\nabla u_n|^N dx = N^{N-1}\omega_{N-1} \int_0^\infty |f_n'(t)|^N dt \\ \int_\Omega \exp\{\alpha |u_n|^{N/(N-1)}\} dx = N^{-1}\omega_{N-1} \int_0^\infty \exp\{\alpha |f_n(t)|^{N/(N-1)} - t\} dt. \end{cases}$$

We next consider $g_n(t) = (f_n')^*(t)$ the decreasing rearrangement of f_n' on $]0, \infty[$ and we set: $\tilde{f}_n(t) = \int_0^t g_n(s)\, ds$, $\tilde{u}_n = \tilde{f}_n(-N \operatorname{Log} |x|)$. Then we have

$$\begin{cases} \int_\Omega |\nabla \tilde{u}_n|^N dx = N^{N-1}\omega_{N-1} \int_0^\infty |\tilde{f}_n'(t)|^N dt = N^{N-1}\omega_{N-1} \int_0^\infty |f_n'(t)|^N dt = 1 \\ \tilde{u}_n(x) = \int_0^{-N \operatorname{Log}|x|} g_n(s)\, ds \geqslant \int_0^{-N \operatorname{Log}|x|} f_n'(s)\, \mathrm{ds} = u_n(x), \qquad \forall x \in \Omega. \end{cases}$$

In addition $f_n' \to f'$ weakly in $L^N(0, \infty)$ and thus we may assume that $g_n \to g$ weakly in $L^N(0, \infty)$. And if $\tilde{u}_n \underset{n}{\to} \tilde{u}$ weakly in $W_0^{1,N}(\Omega)$, then

$$\int_\Omega |\nabla \tilde{u}|^N dx = N^{N-1}\omega_{N-1} \int_0^\infty |g(t)|^N dt \geqslant N^{N-1}\omega_{N-1} \int_0^\infty |f'(t)|^N dt = \int_\Omega |\nabla u|^N dx$$

Hence, we just need to prove our claim for the new sequence $\tilde{u}_n$ i.e. we may assume without loss of generality that not only u_n, u are spherically symmetric, non increasing but f_n' is non increasing i.e.

$$u_n'' + \frac{1}{|x|} u_n' \leqslant 0 \quad \text{on} \quad]0, 1[.$$

But this yields that ∇u_n is relatively compact in L^P $(\epsilon \leqslant |x| \leqslant R - \epsilon)$ for all $p < \infty$, $\epsilon > 0$ —where R is the radius of Ω. Hence we may assume that $\nabla u_n \to$ $\to \nabla u$ a.e. in Ω.

All these reductions enable us to adapt the proof made below in the case $N = 2$. Indeed using Brézis-Lieb lemma [10], we deduce that

$$|\nabla(u_n - u)|^N_{L^N(\Omega)} \underset{n}{\to} 1 - |\nabla u|^N_{L^N(\Omega)}$$

thus for $\delta > 0$ small enough and for n large enough

$$\int_\Omega \exp\{\alpha_N p(1 + \delta)|u_n - u|^{N/(N-1)}\} dx \leqslant C$$

while $\exp\{\alpha_N |u|^{N/(N-1)}\} \in L^q(\Omega)$ for all $q < \infty$; and this proves (75) and the theorem is proved.

Remark I.18. In fact we have proved that:

i) if $u \not\equiv 0$, then $\exp\{\alpha|u_n|^{N/(N-1)}\}$ is bounded in $L^1(\Omega)$ for $0 < \alpha < \alpha_N(1 - |\nabla u|_{L^N}^N)^{-1/(N-1)}$.

ii) if μ is without atoms, then $\exp\{\alpha|u_n|^{N/(N-1)}\}$ is bounded in $L^1(\Omega)$ for all $\alpha > 0$.

In fact, we can prove by a close examination of the above proof that if we consider $\theta = \max_{x \in \Omega} \mu(\{x\})$ and if $\theta \in [0, 1[$ then $\exp\{\alpha|u_n|^{N/(N-1)}\}$ is bounded in $L^1(\Omega)$ for $0 < \alpha < \alpha_N(1 - \theta)^{-1/(N-1)}$.

References

[1] A. Alvino, P. L. Lions and G. Trombetti. A remark on comparizon results via symmetrization. *Preprint.*

[2] F. V. Atkinson and L.A. Peletier. Ground states of $-\Delta u = f(u)$ and the Emden-Fowler equation. *Preprint.*

[3] T. Aubin. Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pure Appl.,* **55** (1976), 269-296.

[4] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. *J. Diff. Geom.*, **11** (1976), 573-598; *announced in C. R. Acad. Sci. Paris*, **280** (1975), 279-282.

[5] T. Aubin. Sur la fonction exponentielle. C.R. Acad. Sci. Paris, **270** (1970), 1514-1517.

[6] H. Berestycki and P. L. Lions. Nonlinear scalar field equations. *Arch. Mech. Anal.*, **82** (1983), 313-376; *see also C. R. Acad. Sci. Paris*, **287** (1978), 503-506 and in *Bifurcation phenomena in Mathematical Physics and related topics*, C. Bardos and D. Bessis (eds.), Reidel, Dordrecht, 1980.

[7] G. Bliss. An integral inequality. *J. London Math. Soc.*, **5** (1930), 40-46.

[8] H. Brézis and J. M. Coron. Multiple solutions of H-systems and Rellich's conjecture. *To appear in Comm. Pure Appl. Math.*

[9] H. Brézis and J. M. Coron. Large solutions for harmonic maps in two dimensions. *Comm. Math. Phys.*, **92** (1983), 203-215.

[10] H. Brézis and E. H. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 1983.

[11] H. Brézis and E. H. Lieb. Minimum action solutions of some vector field equations. See also E. H. Lieb: Some vector field equations. In Proceedings International Conference on Differential Equations, Alabama, 1983.

[12] H. Brézis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, **36** (1983), 437-477.

[13] S. Coleman, V. Glazer and A. Martin. Action minima among solutions to a class of Euclidean scalar field equations. *Comm. Math. Phys.*, **58** (1978), 211-221.

[14] H. Eliasson. On variation of metrics. *Math. Scand.*, **29** (1971), 317-327.

[15] S. Jacobs. An isoperimetric inequality for functions analytic in multiply connected domains. *Report Mittag-Leffeler Institut*, 1970.

[16] D. Jerison and J. M. Lee. A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds. *Preprint.*

[17] C. Kenig and W. M. Ni. An exterior Dirichlet problem with applications to some nonlinear equations arising in geometry. *To appear in Amer. J. Math.*

[18] E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev inequality and related inequalities. *Ann. Math.,* **118** (1983), 349-374.

[19] E. H. Lieb. Existence and uniqueness of the minimizing solutions of Choquard's nonlinear equation. *Stud. Appl. Math.*, **57** (1977), 93-105.

[20] P. L. Lions. The concentration-compactness principle in the Calculus of Variations. The locally compact case, Part 1. *Ann. I. H., Anal. Nonlin.,* **1** (1984), 109-145.

[21] L. Lions. The concentration-compactness principle in the Calculus of Variations. The locally compact case, Part 2. *Ann. I. H., Anal. Nonlin.,* **1** (1984), 223-283.

[22] L. Lions. The Choquard equation and related questions. *Nonlin. Anal. T. M. A.,* **4** (1980), 1063-1073.

[23] L. Lions. Principe de concentration-compacité en Calcul des Variations. *C. R. Acad. Sci. Paris,* **294** (1982), 261-264.

[24] L. Lions. On the concentration-compactness principle. In *Contributions to Nonlinear Partial Differential Equations, Pitman, London,* 1983.

[25] L. Lions. Applications de la méthode de concentration-compacité à l'existence de fonctions extrêmales. *C. R. Acad. Sci. Paris,* **296** (1983), 645-648.

[26] L. Lions. La méthode de concentration-compacité en Calcul des Variations. In *Séminaire Goulaouic-Meyer-Schwartz 1982-1983, Ecole Polytechnique, Palaiseau.*

[27] J. Moser. A sharp form of an inequality of N. Trudinger. *Ind. Univ. Math. J.,* **20** (1971), 1077-1092.

[28] W. M. Ni. On the elliptic equation $\Delta u + K(x)u^{(N+2)/(N-2)} = 0$, its generalizations and applications in geometry. *Ind. Univ. Math. J.*, 1983.

[29] W. M. Ni. Conformal metrics with zero scalar curvature and a symmetrization process via maximum principle. In *Seminar on Differential Geometry, ed. S. T. Yau, Princeton Univ. Press, Princeton, N. J.,* 1982.

[30] S. I. Pohozaev. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Soviet Math. Dokl.,* **165** (1965), 1408-1412.

[31] G. Rosen. Minimum value for c in the Sobolev inequality $\|\phi\|_6 \leqslant c\|\nabla\phi\|_2$.*Siam J. Appl. Math.,* **21** (1971), 30-32.

[32] J. Sacks and K. Unlenbeck. The existence of minimal inmersions of 2-spheres. *Ann. Math.,* **113** (1981), 1-24.

[33] S. Sedlacek. A direct method for minimizing the Yang-Mills functional over 4-manifolds. *Comm. Math. Phys.,* 1983.

[34] Y. T. Siu and S. T. Yau. Compact Kähler manifolds of positive bisectional curvature. Invent. Math., **59** (1980), 189.

[35] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl.,* **110** (1976), 353-372.

[36] C. Taubes. The existence of a non-minimal solution to the SU(2) Yang-Mills-Higgs equations on $\mathbb{R}^3$. *Comm. Math. Phys.,* **86** (1982), 257-298 and 299-320.

[37] C. Taubes. Self-dual connections on nonself-dual 4-manifolds. J. Diff. Geom., **17**(1982), 139.

[38] C. Taubes. Path-connected Yang-Mills Modulispaces. To appear in J. Diff. Geom.

[39] N. S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Sc. Norm. Sup. Pisa,* **22** (1968), 265-274.
[40] N. S. Trudinger. On embedding into Orlicz spaces and some applications. *J. Math. Phys.,* **17** (1967), 473-484.
[41] K. Uhlenbeck. Removable singularities in Yang-Mills fields. *Comm. Math. Phys.,* **83** (1982), 11-29).
[42] K. Uhlenbeck. Connections with L^p bounds on curvature. *Comm. Math. Phys.,* **83** (1982), 31-42.
[43] K. Uhlenbeck. In *Seminar on Differential Geometry*, ed. S. T. Yau, Princeton Univ. Press, Princeton, N. J., 1982.
[44] H. Yamabe. On the deformation of Riemannian structures on compact manifolds. *Osaka Math. J.,* **12** (1960), 21-37.

P. L. Lions
CEREMADE
Université Paris IX-Dauphine
Place de Lattre de Tassigay
75775 Paris Cedex 16
France

Revista Matemática Iberoamericana
Vol. 1, n.° 2, 1985

A Littlewood-Paley Inequality for Arbitrary Intervals

José L. Rubio de Francia

1. Introduction

For every interval $I \subset \mathbb{R}$ we denote by S_I the partial sum operator: $(S_I f)\hat{} = \hat{f}\chi_I$. Given a sequence $\{I_k\}$ of disjoint intervals, we form the quadratic expression

$$\Delta f(x) = \Big(\sum_k |S_{I_k} f(x)|^2\Big)^{1/2} \tag{1.1}$$

We aim to prove here the following

Theorem 1.2. *For every p with $2 \leqslant p < \infty$, there exists $C_p > 0$ such that, for every sequence $\{I_k\}$ of disjoint intervals, the operator Δ defined by* (1.1) *satisfies*

$$\|\Delta f\|_p \leqslant C_p \|f\|_p \qquad (f \in L^p(\mathbb{R})). \tag{1.3}$$

Two particular cases of this result were previously known:

1.4. When $\{I_k\}$ is a lacunary sequence: $I_k = [a_{k-1}, a_k]$ with (say) $(a_{k+1} - a_k) \geqslant 2(a_k - a_{k-1})$, then (1.3) holds for all $1 < p < \infty$, and a converse inequality: $C_p \|\Delta f\|_p \leqslant \|\Delta f\|_p$ is also verified by every f such that supp

$(\hat{f}) \subset U_k I_k$. This is a classical theorem due to Littlewood and Paley [11], which is sometimes a good substitute for Plancherel's theorem in L^p, $p \neq 2$ (see [16], [5]).

1.5. When all the intervals I_k have the same length, then inequality (1.3) holds for $2 \leqslant p < \infty$, and this is best possible as it is shown by the example: $I_k = [k-1, k]$, $k = 1, 2, \ldots, N$ and $\hat{f} = \chi_{[0,N]}$ (with N large enough). This result was first proved by L. Carleson [1], and a different proof was given by A. Córdoba [3], who used it in order to obtain L^p estimates for Bochner-Riesz multipliers, [4].

In the proof presented below, we first reduce the problem to the case where the intervals $\{I_k\}$, after suitably dilated do not overlap too much. Once we are in this situation, it is possible to regularize the partial sum operators, obtaining, instead of Δf, its smooth version Gf, which is easier to handle as a vector valued singular integral. The estimates required for the kernel of G are a combination of classical Littlewood-Paley theory and the ones used in a simplified proof of the case (1.5), given in [14]. In the last three sections, we discuss some variants of the main result: weighted estimates, results in L^p with $p < 2$, and n-dimensional analogues.

This problem came to my knowledge through A. Córdoba, who was always firmly convinced of the truth of such a general statement. My finding the proof was greatly stimulated by conversations with L. Carleson, P. W. Jones, J. P. Kahane, M. Reimann, P. Sjögren and P. Sjölin, during a delightful stay in Sweden.

2. Reduction to the well-distributed case

All the intervals considered will be of finite length. For every interval I and $c > 0$, we denote by cI the interval with the same center as I and length: $|cI| = c|I|$.

Definition 2.1. *A sequence of intervals $\{I_k\}$ is well distributed if the doubles of the intervals have bounded overlapping, i.e.*

$$\sum_k \chi_{2I_k}(x) \leqslant C \qquad (x \in \mathbb{R})$$

Now, we define the Whitney decomposition $W(I)$ of an interval I as follows: First of all, the definition is invariant under translations and dilations, and if $I = [0, 1]$, then $W(I)$ consists of the intervals:

$$\left\{[a_{k+1}, a_k]\right\}_{k=0}^{\infty}; \quad \left[\frac{1}{3}, \frac{2}{3}\right]; \quad \left\{[1-a_k, 1-a_{k+1}]\right\}_{k=0}^{\infty}$$

where $a_k = 2^{-k}/3$. Observe that the intervals of $W(I)$ form a disjoint covering of I, and:

$$\begin{cases} 2H \subset I \quad \text{for every} \quad H \in W(I) \\ \sum_{H \in W(I)} \chi_{2H}(x) \leqslant 5 \qquad \text{for all } x \end{cases} \tag{2.2}$$

Lemma 2.3. *Given disjoint intervals* $\{I_k\}$, *let* $\Delta f(x)$ *be defined as in* (1.1), *and let*

$$\Delta_k f(x) = \Big(\sum_{H \in W(I_k)} |S_H f(x)|^2\Big)^{1/2}$$

Then for all $1 < p < \infty$, *we have the equivalence*

$$\|\Delta f\|_p \sim \Big\|\Big(\sum_k (\Delta_k f)^2\Big)^{1/2}\Big\|_p \qquad (f \in L^p)$$

PROOF: This is essentially known, and a more general (weighted) version of it will be given in 6.3 below. Here is however a short sketch of proof: The operators Δ_k are uniformly bounded in $L^2(w)$ if $w \in A_2$ (see [10]), from which it follows that

$$\Big\|\Big(\sum_k (\Delta_k f_k)^2\Big)^{1/2}\Big\|_p \leqslant C_p \Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_p \tag{2.4}$$

for all $1 < p < \infty$. When we choose $f_k = S_{I_k} f$ in (2.4), we obtain the inequality $\geqslant$ in the Lemma. Since there is equality of norms when $p = 2$, the usual duality argument proves the converse inequality $\leqslant$.

It follows that Theorem 1.2 holds for the sequence $\{I_k\}$ if and only if it holds for the sequence

$$\bigcup_k W(I_k) = \{H / H \in W(I_k) \quad \text{for some} \quad k\}$$

But this last sequence is well distributed according to (2.2), and we arrive at

Lemma 2.5. *In proving Theorem* 1.2, *it is no restriction to assume that the given sequence of intervals* $\{I_k\}$ *is well distributed.*

3. The smooth operator and the basic estimate

We start with a well distributed sequence, and we divide each interval I into seven consecutive intervals of equal length

$$I = I^{(1)} \cup I^{(2)} \cup \ldots \cup I^{(7)}, \qquad |I^{(i)}| = |I|/7$$

so that $8I^{(i)} \subset 2I$. It suffices to prove the theorem for each one of the families $\{I^{(i)} \mid I \in \text{initial sequence}\}$. Therefore, we can assume from the beginning that we are given a sequence $\mathbf{I}$ of disjoint intervals such that

$$\sum_{I \in \mathbf{I}} \chi_{8I}(x) \leqslant C \qquad (x \in \mathbb{R}) \tag{3.1}$$

It will be convenient to label the intervals of the sequence according to their length. Thus, for each integer k, let

$$\{I_k^j\}_j = \{I \in \mathbf{I} \mid 2^k \leqslant |I| < 2^{k+1}\}$$

For every k, j, let n_k^j be the first integer such that $n_k^j 2^k \in I_k^j$, and fix a Schwartz function $\psi(x)$ whose Fourier transform satisfies

$$\chi_{[-2,2]} \leqslant \hat{\psi} \leqslant \chi_{[-3,3]}$$

Then we define

$$\psi_k^j(x) = 2^k \psi(2^k x) \exp(2\pi i n_k^j 2^k x)$$

so that the Fourier transform of ψ_k^j is adapted to I_k^j, i.e.

$$(\psi_k^j)\hat{}\,(\xi) = \hat{\psi}(2^{-k}\xi - n_k^j) = \begin{cases} 1 & \text{if} \quad \xi \in I_k^j \\ 0 & \text{if} \quad \xi \notin 8I_k^j \end{cases} \tag{3.2}$$

Definition 3.3. *The smooth operator G associated to a sequence of intervals satisfying* (3.1) *is*

$$Gf(x) = \Big(\sum_{k \in \mathbb{Z}} \sum_j |\psi_k^j * f(x)|^2\Big)^{1/2} =$$
$$= \Big\{\sum_{k,j} \Big|\int 2^k \psi(2^k(x-y)) \exp(-2\pi i n_k^j 2^k y) f(y)\, dy\Big|^2\Big\}^{1/2}$$

It follows from (3.1) and (3.2) that $\sum_{k,j} |(\psi_k^j)\hat{}\,(\xi)|^2 \leqslant C$, which, by Plancherel's theorem, implies that Gf is well defined in $L^2(\mathbb{R})$ and satisfies

$$\|Gf\|_2 \leqslant C\|f\|_2 \tag{3.4}$$

Our objective is the corresponding L^p inequality, $2 < p < \infty$. This will be a consequence of the main estimate for Gf stated below. We denote by $(\cdot)^{\#}$ the sharp maximal operator of Fefferman and Stein [6], and also,

$$M_q f(x) = \{M(|f|^q)(x)\}^{1/q} \qquad (1 \leqslant q < \infty)$$

where $M = M_1$ stands for the Hardy-Litlewood maximal function. Then, we have for every $f \in L_C^\infty(\mathbb{R}) = \{\text{bounded functions with compact support}\}$

$$(Gf)^{\#}(x) \leqslant CM_2f(x) \qquad (x \in \mathbb{R}) \tag{3.5}$$

The next two sections will be devoted to the proof of (3.5). We wish to observe here that this will complete the proof of Theorem 1.2, since for all $f \in L_C^\infty$ and $2 < p < \infty$

$$\Big\| \Big(\sum_{k,j} |S_{I_k^j} f|^2 \Big)^{1/2} \Big\|_p \leqslant C_p \| Gf \|_p \leqslant C_p' \| (Gf)^{\#} \|_p \leqslant CC_p' \| M_2 f \|_p \leqslant C_p'' \| f \|_p$$

(the first inequality follows by the usual truncation argument which can be seen in [5], [16], [17], because $S_{I_k^j} f = S_{I_k^j}(\psi_k^j * f)$).

4. A lemma for vector-valued singular integrals

Let H be a separable Hilbert space, and let $K(x, y)$ be an H-valued function defined in $\mathbb{R}^2$ such that $\| K(x, \cdot) \|_H$ is locally integrable for each fixed $x \in \mathbb{R}$. Then

$$Tf(x) = \int f(y) K(x, y) \, dy$$

is well defined for every $f \in L_C^\infty(\mathbb{R})$. Given $x, z \in \mathbb{R}$, we denote

$$I_m(x, z) = \{ y \in \mathbb{R} : 2^m |x - z| < |y - z| \leqslant 2^{m+1} |x - z| \}$$

where m is an integer.

Lemma 4.1. *Suppose that T, defined as above, is a bounded operator from $L^2(\mathbb{R})$ to $L_H^2(\mathbb{R})$, and that the kernel $K(x, y)$ satisfies, for some $A > 0$, $\alpha > 1$, the condition*

$$\int_{I_m(x,z)} |\langle K(x, y) - K(z, y), \lambda \rangle|^2 \, dy \leqslant A^2 \frac{2^{-\alpha m} \| \lambda \|_H^2}{|x - z|} \tag{4.2}$$

for every $x, z \in \mathbb{R}$, $\lambda \in H$, and $m \geqslant 1$. Then, for the operator $Gf(x) = \| Tf(x) \|_H$ we have the estimate

$$(Gf)^{\#}(x) \leqslant C(A, \alpha) M_2 f(x) \qquad (f \in L_C^\infty)$$

PROOF. It is essentially a repetition of the argument in [6]. Given $x \in \mathbb{R}$ and an interval I centered at x, we define the vector

$$h_I = \int_{y \notin 2I} f(y) K(x, y) \, dy \in H$$

so that, if $\bar{f} = f \chi_{2I}$

$$Tf(z) - h_I = T\bar{f}(z) + \int_{y \notin 2I} f(y) [K(z, y) - K(x, y)] \, dy$$

Denoting by $g(z)$ an arbitrary H-valued function with $\|g(z)\|_H \leqslant 1$ for all $z \in I$, we can write

$$\frac{1}{|I|}\int_I \|Tf(z) - h_I\|_H\,dz \leqslant \frac{1}{|I|}\int_I \|T\bar{f}(z)\|_H\,dz +$$

$$+ \sup_g \frac{1}{|I|}\left|\int_I \langle g(z), \int_{y \notin 2I} f(y)[K(z,y) - K(x,y)]\,dy\rangle\,dz\right| = (1) + (2)$$

Now, the first term is easy to estimate

$$(1) \leqslant C\left(\frac{1}{|I|}\int_{2I} |f|^2\right)^{1/2} \leqslant C\sqrt{2}\,M_2f(x)$$

and in the second term, the value corresponding to each fixed g is majorized by

$$\frac{1}{|I|}\int_I \sum_{m=1}^{\infty} \int_{I_m(z,x)} |f(y)|\,|\langle g(z), K(z,y) - K(x,y)\rangle|\,dy\,dz \leqslant$$

$$\leqslant \sup_{z \in I} \sum_{m=1}^{\infty} \left(\int_{I_m(z,x)} |f(y)|^2\,dy\right)^{1/2} A2^{-\alpha m/2}|x - z|^{-1/2}$$

where we have used (4.2) and the fact that $\|g(z)\|_H \leqslant 1$. Thus,

$$(2) \leqslant 2A \sum_{m=1}^{\infty} 2^{(1-\alpha)m/2} M_2f(x)$$

and the series converges because $\alpha > 1$. Since

$$(Gf)^{\#}(x) \leqslant C \sup_I \frac{1}{|I|}\int_I \|Tf(z) - h_I\|_H\,dz$$

the proof is ended.

It is easy to formulate generalizations of this lemma: One can consider kernels defined in $\mathbb{R}^n \times \mathbb{R}^n$ with values $K(x,y) \in L(A,B)$, for some Banach spaces A, B, and replace the exponent 2 in our initial assumptions: $\|Tf\|_2 \leqslant C\|f\|_2$ and (4.2), by different exponents p, q. Some of these variants are considered in [15]. The simple case stated here is precisely what we need for our present problem.

5. Proof of the basic estimate

Here we shall use the preceding lemma in order to prove the pointwise estimate (3.5), thus finishing the proof of Theorem 1.2. We must therefore consider the l^2-valued kernel

$$K(x,y) = \{2^k\psi(2^kx - 2^ky)\exp(-2\pi i n_k^j 2^k y)\}_{k,j}$$

where ψ and n_k^j are defined in §3, and we must prove that $K(x,y)$ satisfies (4.2). It suffices to do so when $\lambda = \{\lambda_k^j\}_{k,j} \in l^2$ has unit norm, and for every such λ, we let

$$K_\lambda(x,y) = \langle K(x,y), \lambda\rangle = \sum_{k,j} \bar{\lambda}_k^j 2^k \psi(2^kx - 2^ky)\exp(-2\pi i n_k^j 2^k y) =$$
$$= \sum_k 2^k\psi(2^kx - 2^ky)q_k(2^ky)$$

where, for each $k \in \mathbb{Z}$, q_k is a 1-periodic function defined by its Fourier series

$$q_k(t) = \sum_j \bar{\lambda}_k^j \exp(-2\pi i n_k^j t)$$

Observe that $n_k^j \neq n_k^{j'}$ if $j \neq j'$, so that each q_k satisfies

$$\int_a^{a+1} |q_k(t)|^2\, dt \leqslant 1 \qquad (a \in \mathbb{R};\, k \in \mathbb{Z}) \tag{5.1}$$

and this is the only property of the functions q_k that we shall use, so that we disregard the fact that they also depend on λ. Our problem is then reduced to establishing the inequality

$$\int_{I_m(x,z)} |K_\lambda(x,y) - K_\lambda(z,y)|^2\, dy \leqslant A 2^{-\alpha m}|x - z|^{-1} \tag{5.2}$$

with $\alpha > 1$. We can assume that $z = 0$, since this amounts to translating q_k by 2^kz, so that (5.1) is preserved. On the other hand, replacing x by $2x$ does not change the inequality (5.2) at all, and thus, we can also assume that $1 \leqslant |x| < 2$. Writing $I_m(x,0) = I_m(x)$ we have by changing variables

$$\|K_\lambda(x,\cdot) - K_\lambda(0,\cdot)\|_{L^2(I_m(x))} \leqslant$$
$$\leqslant \sum_{k\in\mathbb{Z}} 2^{k/2}\left(\int_{I_{k+m}(x)} |\psi(2^kx - y) - \psi(-y)|^2 |q_k(y)|^2\, dy\right)^{1/2}$$
$$\leqslant \sum_k 2^{k/2}\left\{\sup_{y\in I_{k+m}(x)} |\psi(2^kx - y) - \psi(-y)|\right\}(2^{k+m+2} + 1)^{1/2}$$
$$= \sum_{k=-h}^{\infty} + \sum_{k=-\infty}^{-h-1}$$

where we choose $h = [2m/3]$. For the terms in the first sum we use the fact that $|\psi(y)| \leqslant C|y|^{-2}$, so that

$$\sup_{y \in I_{k+m}(x)} |\psi(2^k x - y) - \psi(-y)| \leqslant C 2^{-2k-2m}$$

and then,

$$\sum_{k=-h}^{\infty} (\ldots) \leqslant C \sum_{k=-h}^{\infty} 2^{-k-3m/2} \leqslant C 2^{-5m/6}$$

For the second sum we use the majorization

$$\sup_{y \in I_{k+m}(x)} |\psi(2^k x - y) - \psi(-y)| \leqslant C 2^k x \leqslant C 2^{k+1}$$

and we obtain (since $k + m < m - h \leqslant m/3$)

$$\sum_{k=-\infty}^{-h-1} (\ldots) \leqslant C \sum_{k=-\infty}^{-h-1} 2^{3k/2} 2^{m/6} \leqslant C 2^{-5m/6}$$

Combining everything, we have proved the desired inequality (5.2) with $\alpha = \frac{5}{3} > 1$.

Remarks. The initial computations involving λ_k^j*'s* are rather formal, and serious convergence problems may arise. However, everything becomes correct if we define a truncated smooth operator G_F by allowing only a finite set F of k*'s* and j*'s* in the definition. The final estimates are independent of the set F and so, a limiting argument proves the same result for the whole operator G.

A somewhat shorter computation is needed to show that

$$\int_{|y-z| \geq 2|x-z|} |K_\lambda(x,y) - K_\lambda(z,y)|\, dy \leqslant C \tag{5.3}$$

(instead of (5.2)). The analogue of Lemma 4.1 under this weaker assumption shows that $\|Gf\|_{BMO} \leqslant C\|f\|_\infty$, which is certainly weaker than (3.5) but still enough to prove our theorem, since interpolation with (3.4) gives $\|Gf\|_p \leqslant C_p \|f\|_p$, $2 < p < \infty$.

However, for the weighted analogues of Theorem 1.2 which we shall obtain in the next section, the full force of the basic estimate (3.5) is required.

6. Weighted inequalities

The following extension of the theorem just proved holds.

Theorem 6.1. *If* $2 < p < \infty$, *and if the weight* $w(x)$ *(in* $\mathbb{R}$*) belongs to the class* $A_{p/2}$, *then, the operator* Δ *defined by* (1.1) *for an arbitrary sequence of dis-*

joint intervals satisfies

$$\int [\Delta f(x)]^p w(x)\,dx \leqslant C_p(w) \int |f(x)|^p w(x)\,dx$$

PROOF. Let us consider first the smooth operator G associated to a sequence of intervals satisfying (3.1). Then, for all $w \in A_{p/2}$ $(2 < p < \infty)$ and f good enough

$$\int [Gf(x)]^p w(x)\,dx \leqslant C_{p,w} \int [(Gf)^{\#}(x)]^p w(x)\,dx \leqslant$$
$$\leqslant CC_{p,w} \int [M_2 f(x)]^p w(x)\,dx \leqslant C'_{p,w} \int |f(x)|^p w(x)\,dx$$

On the other hand, for arbitrary intervals $\{I_k\}$, the inequality

$$\Big\| \Big(\sum_k |S_{I_k} f_k|^2 \Big)^{1/2} \Big\|_{L^p(w)} \leqslant C_{p,w} \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_{L^p(w)} \tag{6.2}$$

$(w \in A_p,\ 1 < p < \infty)$ holds, because it holds for the Hilbert transform (see [10] for details). Thus, the usual truncation argument can be applied, i.e.: If $\{I_k\}$ is the given sequence of intervals, and the associated somooth operator is $Gf = \left(\sum_k |\psi_k * f|^2\right)^{1/2}$, with $\hat{\psi}_k = 1$ on I_k, then we define $f_k = \psi_k * f$ and use (6.2) to obtain

$$\int [\Delta f(x)]^p w(x)\,dx \leqslant C_{p,w} \int [Gf(x)]^p w(x)\,dx$$

$(w \in A_p;\ 1 < p < \infty)$. Putting everything together, the theorem is proved for well distributed sequences of intervals.

Now, for the reduction to the well-distributed case, we argue as in §2, and we only need to prove the weighted analogue of 2.3, namely

Lemma 6.3. *Given a sequence of disjoint intervals $\{I_k\}$, let $W(I_k)$ be the Whitney decomposition of each I_k. Then, for all $w \in A_p$, $1 < p < \infty$, we have the equivalence*

$$\Big\| \Big(\sum_k |S_{I_k} f|^2 \Big)^{1/2} \Big\|_{L^p(w)} \sim \Big\| \Big(\sum_k \sum_{H \in W(I_k)} |S_H f|^2 \Big)^{1/2} \Big\|_{L^p(w)}$$

for every $f \in L^p(w)$.

PROOF. Let Δf be defined as in (1.1), and let $\Delta_k f$ be the corresponding operator for the sequence $W(I_k)$. As we mentioned in Lemma 2.3, the operators Δ_k are uniformly bounded in $L^2(w)$ if $w \in A_2$, and more precisely (see [10]) if $\mathrm{supp}(\hat{f}) \subset I_k$

$$C_w^{-1} \int |f|^2 w \leqslant \int (\Delta_k f)^2 w \leqslant C_w \int |f|^2 w \qquad (w \in A_2)$$

with C_w independent of k. By the extrapolation theorem for A_p-weights (see [8], [13]) this implies

$$\Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_{L^p(w)} \sim \Big\|\Big(\sum_k [\Delta_k f_k]^2\Big)^{1/2}\Big\|_{L^p(w)}$$

for all $w \in A_p$ and $f_k \in L^p(w)$, $1 < p < \infty$, and taking $f_k = S_{I_k} f$ we get the desired equivalence

$$\|\Delta f\|_{L^p(w)} \sim \Big\|\Big(\sum_k [\Delta_k f]^2\Big)^{1/2}\Big\|_{L^p(w)}$$

which completes the proof of the lemma and the theorem.

The theorem is best possible for $p > 2$ in the sense that Δ cannot be bounded in $L^p_{(w)}$ for all $w \in A_q$ if $q > p/2$ (since this would imply that Δ is bounded in $L^{2-\epsilon}(\mathbb{R})$, which is false for some sequences $\{I_k\}$ of intervals). It is natural to expect, however, that

$$\int \sum_k |S_{I_k} f|^2 w \leqslant C_w \int |f|^2 w \qquad (w \in A_1) \tag{6.4}$$

for every sequence $\{I_k\}$ of disjoint intervals, since this is the limiting case of 6.1, and it is known to be true in the extremal cases considered in (1.4) and (1.5). It would suffice to obtain the same inequality for the smooth operator G, but the basic estimate: $(Gf)^{\#} \leqslant CM_2 f$ is not enough to prove it.

7. Some results in L^p, $p < 2$

Given a sequence $\{I_k\}$ of disjoint intervals, one may ask more generally for which values of p and q does the inequality

$$\Big\|\Big(\sum_k |S_{I_k} f|^q\Big)^{1/q}\Big\|_p \leqslant C\|f\|_p \tag{7.1}$$

hold. The example in (1.5) shows that a necessary condition (not only for arbitrary $\{I_k\}$, but even for equal length intervals) is: $q \geqslant \max(2, p')$. Thus, we have proved in Theorem 1.2 the best possible result for $2 \leqslant p < \infty$, and it is natural to expect that, for $1 < p < 2$, the best possible inequality is also true, namely.

Conjecture 7.2. *For arbitrary disjoint intervals $\{I_k\}$ and for $1 < p < 2$, the inequality*

$$\Big\|\Big(\sum_k |S_{I_k} f|^{p'}\Big)^{1/p'}\Big\|_p \leqslant C_p \|f\|_p$$

holds for *every $f \in L^p(\mathbb{R})$.*

As supporting evidence for this conjecture, apart from Theorem 1.2, we mention two partial results:

a) If $\{I_k\}$ is well distributed, $f \longrightarrow \left(\sum_k |S_{I_k}f|^{p'}\right)^{1/p'}$ is an operator of weak type (p,p), $1<p<2$.

b) If $1<p<2$ and $q>p'$ then (7.1) holds for arbitrary disjoint intervals $\{I_k\}$.

PROOF OF (*a*). The Hilbert transform H admits a vector valued extension: $\tilde{H}((f_k)) = (Hf_k)$ which is bounded in $L^p(l^q)$ for all $1<p,q<\infty$, and expressing every partial sum operator in terms of H (as in [16], for instance) we obtain

$$\left\|\left(\sum_k |S_{I_k}f_k|^q\right)^{1/q}\right\|_p \leqslant C_{p,q}\left\|\left(\sum_k |f_k|^q\right)^{1/q}\right\|_p \tag{7.3}$$

Now, we define ψ_k so that $\hat{\psi}_k$ is adapted to I_k, i.e.

$$\chi_{I_k} \leqslant \hat{\psi}_k \leqslant \chi_{2I_k}$$

Moreover, all ψ_k can be defined in terms of a fixed Schwartz function ψ, so that $|\psi_k(x)| = l_k|\psi(l_k x)|$ with $l_k = |I_k|$. Then, the operator

$$f \to (\psi_k * f)_{k\in\mathbb{N}}$$

is bounded from L^1 to weak-$L^1(l^\infty)$, because $\sup_k |\psi_k * f| \leqslant CMf$, and it is also bounded from L^2 to $L^2(l^2)$ due to the fact that the intervals $\{I_k\}$ are well distributed. By interpolation

$$\left\|\left(\sum_k |\psi_k * f|^{p'}\right)^{1/p'}\right\|_{p,\infty} \leqslant C_p\|f\|_p \qquad (1<p<2) \tag{7.4}$$

and we only have to apply (7.3) with $f_k = \psi_k * f$ and $q = p'$.

PROOF OF (*b*). We interpolate between the obvious inequality

$$\left\|\left(\sum_k |S_{I_k}f|^2\right)^{1/2}\right\|_2 \leqslant \|f\|_2 \qquad (f\in L^2)$$

and the following consequence of the Carleson-Hunt theorem ([2], [9])

$$\left\|\sup_k |S_{I_k}f|\right\|_{1+\varepsilon} \leqslant C_\varepsilon \|f\|_{1+\varepsilon} \qquad (f\in L^{1+\varepsilon};\ \varepsilon>0).$$

8. n-dimensional results

By an interval in $\mathbb{R}^n$, we shall mean the product of n one-dimensional intervals: $I = [a_1,b_1]\times[a_2,b_2]\times\ldots\times[a_n,b_n]$. We would like to state the analogue of Theorem 1.2 for an arbitrary sequence of disjoint intervals in $\mathbb{R}^n$, but in order to adapt the argument developed in sections §2 – §5, we should need a lemma similar to 4.1 for product-type vector-valued kernels. No such result seems to be known so far, though one may hope that the methods of [7] could be suitably modified to this end.

What one can prove by standard reiteration techniques is a theorem for «cross-partitions»: *A cross-partition of* $\mathbb{R}^n$ *is a family* $\{I_k\}_{k\in\mathbb{N}^n}$ *of disjoint n-dimensional intervals such that*

$$I_k = I^{(1)}_{k_1} \times I^{(2)}_{k_2} \times \ldots \times I^{(n)}_{k_n} \qquad (k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n)$$

where, for each $i = 1, 2, \ldots, n$, *the sequence of intervals* $\{I^{(i)}_j\}_{j\in\mathbb{N}}$ *form a partition of* $\mathbb{R}$.

Theorem 8.1. *If* $\{I_k\}_{k\in\mathbb{N}^n}$ *is a cross-partition of* $\mathbb{R}^n$ *and* $2 \leqslant p < \infty$, *then for all* $f \in L^p(\mathbb{R}^n)$

$$\Big\| \Big(\sum_k |S_{I_k} f|^2 \Big)^{1/2} \Big\|_p \leqslant C_p \|f\|_p$$

PROOF. For notational simplicity, we shall assume $n = 2$. Let $I_{j,k} = I'_j \times I''_k (j, k \in \mathbb{N})$ be the given family of intervals, and let $S_{j,k}$, S'_j and S''_k denote, respectively, the partial sum operators in $\mathbb{R}^2$ corresponding to the intervals $I_{j,k}$, $I'_j \times \mathbb{R}$ and $\mathbb{R} \times I''_k$. By the one-dimensional result and Fubini's theorem, we have

$$\Big\| \Big(\sum_j |S'_j f|^2 \Big)^{1/2} \Big\|_p \leqslant C_p \|f\|_p \qquad (f \in L^p; 2 \leqslant p < \infty) \tag{8.2}$$

and similarly for S''_k, $k \in \mathbb{N}$. Thus, the operator

$$S'' : f \to S''f = (S''_k f)_{k\in\mathbb{N}}$$

is bounded from $L^p(\mathbb{R}^2)$ to $L^p_H(\mathbb{R}^2)$, where $H = l^2$, and the theorem of Marcinkiewicz and Zygmund [12] (which is also valid for Hilbert space-valued functions) gives

$$\int \Big(\sum_j \|S'' f_j(x)\|_H^2 \Big)^{p/2} dx \leqslant C_p^p \int \Big(\sum_j |f_j(x)|^2 \Big)^{p/2} dx \tag{8.3}$$

Now, given $f \in L^p(\mathbb{R}^2)$, $2 \leqslant p < \infty$, we apply (8.3) with $f_j = S'_j f$ taking into account (8.2) and the fact that $S''_k S'_j f = S_{j,k} f$.

The same inequality holds in $L^p(w)$ if $w \in A^*_{p/2} = [A_{p/2}$ – weights with respect to all n-dimensional intervals], $2 < p < \infty$. Another partial result is the following.

Theorem 8.4. *Let* $\{Q_j\}$ *be a sequence of well distributed cubes (in the sense of* 2.1) *in* $\mathbb{R}^n$. *Then*

$$\left\| \left(\sum_j |S_{Q_j} f|^2 \right)^{1/2} \right\|_p \leqslant C_p \|f\|_p \qquad (2 \leqslant p < \infty)$$

The proof is a repetition of the arguments in §3, §4 and §5. More generally, if l_i, q_i are fixed positive numbers, one can prove the same result for a family of intervals $\{I_j\}$ such that I_j has dimensiones $l_1 \delta_j^{q_1} \times l_2 \delta_j^{q_n}$ for some $\delta_j > 0$. In this case, the definition of well distributed sequence is made in terms of the non-isotropic dilations: $\delta \cdot x = (\delta^{q_1} x_1, \delta^{q_2} x_2, \ldots, \delta^{q_n} x_n)$.

By putting both theorems together and using the general arguments of §2 (see also [14]), one can find a huge variety of configurations of intervals in $\mathbb{R}^n$ for which the inequality stated in 8.1 turns out to be true, but the general n-dimensional analogue of Theorem 1.2 seems to be still out of reach.

Added in proof. Since the result proved in this paper was known, several authors became interested in it making some contributions. Thus, another proof of the basic estimte (3.5) was given by P. Sjölin, and a different approach to the problem was found by J. Bourgain yielding, for a sequence of disjoint intervals covering $\mathbb{R}$, the inequality

$$\|f\|_p \leqslant \left\| \left(\sum_k |S_{I_k} f|^2 \right)^{1/2} \right\|_p \qquad (1 \leqslant p \leqslant 2)$$

(which, for $1 < p \leqslant 2$, is equivalent to theorem 1.2). Finally, J. L. Journé has been able to prove recently the general n-dimensional version of our theorem, namely, the analogue of theorem 8.1 for an arbitrary partition of $\mathbb{R}^n$ into n-dimensional intervals.

References

[1] Carleson, L. On the Littlewood-Paley theorem, *Inst. Mittag-Leffler Report*, 1967.

[2] —. On convergence and growth of partial sums of Fourier series, *Acta Math.*, **116** (1966), 135-157.

[3] Córdoba, A. Some remarks on the Littlewood-Paley theory, *Rend. Circ. Matem. Palermo*, II, **1** (1981) Suppl., 75-80.

[4] —. A note on Bochner-Riesz operators, *Duke Math. J.*, **46** (1979), 505-511.

[5] Edwards, R. E., Gaudry, G. I. Littlewood-Paley and multiplier theory, *Springer-Verlag*, Berlín, Heidelberg, New York, 1977.

[6] Fefferman, C., Stein, E. M. H^p-spaces of several variables, *Acta Math.*, **129** (1972), 137-193.

[7] Fefferman, R., Stein, E. M. Singular integrals on product spaces, *Adv. in Math.*, **45** (1982), 117-143.

[8] García-Cuerva, J. An extrapolation theorem in the theory of A_p weights, *Proc. Amer. Math. Soc.*, **83** (1983), 253-261.
[9] Hunt, R. On the convergence of Fourier series, (Proc. Conf. Orthogonal Expansions), *Southern Illinois Univ. Press*, Carbondale, Ill., (1968), 235-255.
[10] Kurtz, D. S. Littlewood-Paley and multiplier theorems on weighted L^p spaces, *Trans. Amer. Math. Soc.*, **259** (1980), 235-254.
[11] Littlewood, J. E., Paley, R. E. A. C. Theorems on Fourier series and power series, II, *Proc. London Math. Soc.*, **42** (1926), 52-89.
[12] Marcinkiewicz, J., Zygmund, A. Quelques inégalités pour les opérations linéaires, *Fund. Math.*, **32** (1939), 115-121.
[13] Rubio de Francia, J. L. Factorization theory and A_p weights, *Amer. J. Math.*, **106** (1984), 533-547.
[14] —. Estimates for some square functions of Littlewood-Paley type, *Publ. Sec. Mat. Univ. Autonoma Barcelona*, **27** (1983), 81-108.
[15] —. Ruíz, F. J., Torrea, J. L. Les opérateurs de Calderón-Zygmund vectoriels, *C. R. Acad. Sci. París, Sér.* I, **297** (1983), 477-480.
[16] Stein, E. M. Singular integrals and differentiability properties of functions, *Princeton Univ. Press*, Princeton, N. J., 1970.
[17] Zygmund, A. Trigonometric Series, I & II, *Cambridge Univ. Press,* London, New York, 1968.

José L. Rubio de Francia
División de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 1, N.º 2, 1985

Stable Planar Polynomial Vector Fields

J. Sotomayor

1. Introduction

A vector field in R^2 of the form

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$$

where $P = \sum a_{ij}x^iy^j$ and $Q = \sum b_{ij}x^iy^j$, $0 \leqslant i+j \leqslant n$, is called a *planar polynomial vector field of degree* $\leqslant n$. The $N = (n+1)(n+2)$ real numbers a_{ij}, b_{ij} are called the *coefficients of* X. The space of these vector fields, endowed with the structure of affine R^N-space in which X is identified with the N-tuple $(a_{00}, a_{10}, \dots, a_{0n}; b_{00}, \dots, b_{0n})$ of its coefficients, is denoted by χ_n.

The Poincaré compactification of $X \in \chi_n$ is defined to be the unique analytic vector field $\mathcal{P}(X)$ tangent to the sphere $S^2 = \{x^2+y^2+z^2=1\}$ and to the equator $S^1 = \{S^2, z=0\}$, whose restriction to the northern hemisphere $S^2_+ = \{S^2, z>0\}$ is given by $z^{n-1}p_*(X)$, where p is the central projection from R^2 to S^2_+, defined by $p(u,v) = (u,v,1)/(u^2+v^2+1)^{1/2}$. See 3 or [6] for a verification of the uniqueness and analyticity of $\mathcal{P}(X)$.

Definition 1.1. *a)* $X \in \chi_n$ is said to be topologically *stable* if there is a neighborhood V and a map h: $V \to \mathrm{Hom}(S^2, S^1)$ (homeomorphisms of S^2 which preserve S^1) such that $h_X = \mathrm{Id}$ and h_Y maps orbits of $\mathcal{P}(X)$ onto orbits of $\mathcal{P}(Y)$, for every $Y \in V$.

b) If furthermore, h can be chosen such that for each $x \in S^2$, $Y \to h_Y(x)$ is of class C^r, $r = 1, 2, \ldots, \infty, \omega$, then X is said to be *r-stable*.

Define by Σ_n (resp. Σ_n^r) the set of $X \in \chi_n$ defined in *a*) (resp. *b*)).

The *topological* (resp. *r*) *bifurcation set* in χ_n is defined by $\chi_n^1 = \chi_n - \Sigma_n$ (resp. $\chi_n^{r,1} = \chi_n - \Sigma_n^r$).

The point topological properties of Σ_n have been studied by Pugh [17] and dos Santos [19] Their works describe an open dense set of χ_n denoted here by $\mathcal{S}_n$, such that $\mathcal{S}_n \subset \Sigma_n$, which is defined by properly extending to elements of the form $\mathcal{P}(X)$ the conditions given by Andronov-Pontryagin [1] and Peixoto [14] for smooth vector fields on compact domains. These papers were preceded by the work of Gonzáles [6], devoted to the generic properties of elements of χ_n at infinity, i.e. on S^1.

Definition 1.2. Denote by $\mathcal{S}_n$ the set of $X \in \chi_n$ for which $\mathcal{P}(X)$ has *a*) all its singularities hyperbolic, *b*) all its periodic orbits hyperbolic and *c*) no saddle connection contained in $S^2 - S^1$.

The characterization of Σ_n depends on a delicate point, apparently overlooked in [19], which for future reference is formulated here as a problem.

Problem 1.1. Prove (or disprove) that the hyperbolicity of an attracting or repelling periodic orbit in $S^2 - S^1$ is necessary for topological stability in χ_n.

The main results of this paper, characterize Σ_n^r as $\mathcal{S}_n$ and establishes the simplest affine, analytical and measure theoretical meagerness properties of the bifurcation sets χ_n^1 and $\chi_n^{r,1}$ of χ_n. These meagerness properties have obvious thickness counterparts for Σ_n and Σ_n^r.

Theorem A. *a*) *The set of r-stable vector fields Σ_n^r, $r = 1, 2, \ldots, \omega$, coincides with $\mathcal{S}_n$.*

b) *Furthermore, $\chi_n^{r,1} = \chi_n - \mathcal{S}_n$, $r = 1, 2, \ldots, \omega$, is contained in the union of countably many one-to-one immersed analytic submanifolds of codimension $\geqslant 1$ in χ_n.*

Corollary 1.1. *$\chi_n^{r,1}$ and, therefore, χ_n^1 have null Lebesgue measure in χ_n.*

Corollary 1.2. *Let $\xi: \mathbb{R} \to \chi_n$ be a C^1 map. Call $G(\xi)$ the set of $V \in \chi_n$ such that $\xi + V$ meets $\mathcal{S}_n$ except at most in a countable set of points. Then $G(\xi)$ has total Lebesgue measure in χ_n.*

The null Lebesgue measure of the bifurcation set in compact plane regions was established by the author in [25].

Corolary 1.2 is a rather crude description of bifurcations of codimension one of χ_n. The actual geometry of the bifurcation phenomena of $\mathcal{P}(X)$ as the coefficients of X change along curves that meet $\chi_n^{r,1}$ transversally at regular points, has been studied in the simplest situations in [13].

2. Proof of Corollaries

Assume Theorem A, *b*).

Let $\chi_n - \mathcal{S}_n = US_j$, $j = 1, \ldots,$ where S_j are analytic submanifolds of codimension $\geqslant 1$.

The map $F(V, \cdot) = V + \xi(\cdot)$ is transversal of S_j if and only if V belongs to the set R_j of regular values of the projection of $F^{-1}(S_j)$ onto χ_n. Clearly $G(\xi) = R_j$. By Sard's Theorem [20], $G(\xi)$ has total Lebesgue measure. This argument applies to any map $\xi: R^k \to \chi_n$ of class C^k. It gives Corollary 1.1 if $k = 0$, and Corollary 1.2, if $k = 1$.

3. Proof of Theorem A

Take coordinates (θ, ρ), 2π-periodic in θ, defined by the covering map from $R \times (-1, 1)$ onto $S^2 - \{(0, 0, \pm 1)\}$, given by $(\theta, \rho) \to (x, y, z) = (1 + \rho^2)^{-1/2}$ $(\cos\theta, \sin\theta, \rho)$.

The expression for $z^{n-1}p_*(X)$, $X \in \chi_n$, in these coordinates is

$$(1 + \rho^2)^{(1-n)/2}\left[(\Sigma\rho^i A_{n-i}(\theta))\frac{\partial}{\partial\theta} - \rho(\Sigma\rho^i R_{n-i}(\theta))\frac{\partial}{\partial\rho}\right], \tag{3.1}$$

where $i = 0, 1, 2, \ldots, n$ and

$$\begin{aligned} A_k(\theta) &= A_k(X, \theta) = -P_k(\cos\theta, \sin\theta)\sin\theta + Q_k(\cos\theta, \sin\theta)\cos\theta \\ R_k(\theta) &= R_k(X, \theta) = P_k(\cos\theta, \sin\theta)\cos\theta + Q_k(\cos\theta, \sin\theta)\sin\theta, \end{aligned} \tag{3.2}$$

with $P_k = \sum a_{ij}x^iy^j$, $Q_k = \sum b_{ij}x^iy^j$, $i + j = k$.

This shows that $\mathcal{P}(X)$ must be given by (3.1), mod 2π, and is therefore analytic in S^2 and tangent to S^1.

Denote by $B(i)$ the set of $X \in \chi_n$ which to not satisfy condition $i = a$), *b*), *c*) of Definition 1.2. Theorem A, *b*) will follow from.

Proposition 3.1. *a*) *$B(a)$ is a semi-algebraic set in χ_n.*

b) *The set C of vector field of $\chi_n - B(a)$ with some graph of saddles and separatrices is closed in $\chi_n - B(a)$.*

c) *$B(b)$ is a closed semianalytic set in the open set $A = (\chi_n - B(a)) - C$.*

d) $B(c)$ is the union of finitely many one-to-one immersed analytic hypersurfaces in $\chi_n - B(a)$.

PROOF. *a*) Notice that $B(a)$ is the projection into χ_n of the union of the following semi-algebraic sets.

$$\{P = 0, Q = 0; \Delta = P_x Q_y - P_y Q_x = 0\},$$
$$\{P = 0, Q = 0; \Delta > 0, \sigma = P_x + Q_y = 0\},$$
$$\{A_n = 0, A'_n = 0\} \quad \text{and} \quad \{A_n = 0, R_n = 0\}.$$

The result follows from Tarski-Seindenberg Theorem [21].

b) If $X \to Y$ in $\chi_n - B(a)$, and Y does not have any graph, by continuation of all the saddle separatrices through saddle connections of Y one would arrive to separatrices whose limit sets are attractors or repellors.

By continuity, the same would hold for neighboring systems and, therefore, X could not have had graphs.

c) The following remaks will be needed.

Remark 3.1. If X has a periodic orbit at infinity, i.e. if S^1 is a periodic orbit of $\mathcal{P}(X)$, then it is hyperbolic if and only if

$$\mu = \int_0^{2\pi} R_n(X, \theta) A_n^{-1}(X, \theta)\, d\theta \neq 0.$$

Actually, the derivative $\Pi'(0)$ of the Poincaré return map Π associated to a transversal segment is given by

$$\log \Pi'(0) = (-1)\sigma\mu, \tag{3.3}$$

where σ denotes the sign of the orientation of the orbit relative to the canonical orientation of S^1.

In fact, from (3.1) the trajectories of $\mathcal{P}(X)$ near S^1 satisfy the following differential equation

$$\frac{d\rho}{d\theta} = \frac{-\rho(\Sigma \rho^i R_{n-i}(\theta))}{\Sigma \rho^i A_{n-i}(\theta)}, \qquad i = 0, 1, \ldots, n$$

Denote by $\rho = \rho(\rho_0, \theta)$ the solution of this equation, with initial condition $\rho(\rho_0, 0) = \rho_0$. The Poincaré return map is therefore given by $\Pi(\rho_0) = \rho(\rho_0, 2\pi)$. Therefore,

$$\Pi'(0) = \frac{\partial \rho}{\partial \rho_0}(0, 2\pi) = \exp\Big[-\int_0^{\sigma 2\pi} R_n A_n^{-1}\, d\theta\Big],$$

as follows from (3.3) and a well known formula for the derivative of solutions with respect to initial conditions. Since, in this case,

$$\int_0^{\sigma 2\pi} = \sigma \int_0^{2\pi},$$

this proves (3.3).

Remark 3.2. The derivative of μ in the direction of

$$V = T\frac{\partial}{\partial x} + U\frac{\partial}{\partial y} \in \chi_n$$

is given by

$$D\mu_X(V) = \int_0^{2\pi} (T_n Q_n - U_n P_n) A_n^{-2}(X, \theta)\, d\theta,$$

which is not null. In particular, if

$$V = (x^2 + y^2)^k\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \qquad n = 2k+1, \qquad D\mu(V) = \int_0^{2\pi} A_n^{-1}(\theta)\, d\theta \neq 0.$$

This shows that the space of vector fields in χ_n, $n = 2k+1$, with a non hyperbolic orbit at infinity, is an analytic hypersurface.

The proof of *c*) can be finished as follows.

For $X \in A$ take a neighborhood $V \subset A$, such that the Poincaré return map Π_i: $V \times L_i \to L_i$ of $\mathcal{P}(Y)$, $Y \in V$, is defined on a segment L_i-transversal to each periodic orbit γ_i of X. Write $0 = L_i \cap \gamma_i$.

Let n_i be the multiplicity of γ_i as a periodic orbit of X; that is, n_i is the order of the zero of $\Pi_i(X, x) - x$, at $0 \in L_i$. Using the Weierstrass Preparation Theorem [29], write $\Pi_i(Y, x) - x = U_i(Y, x)P_i(Y, x)$, where

$$P_i = x^{n_i} + a_{n_i-1}^{(i)}(Y)x^{n_i-1} + \ldots + a_0^{(i)}(Y),$$

with $a_j^{(i)}$ and U_i analytic functions, $U_i \neq 0$ in $V \times L_i$ and $a_j^{(i)}(X) = 0$.

There are two cases:

a) If γ_i is a periodic orbit on $S^2 - S^1$, $D_X a_0 \neq 0$. In fact, $a_0 = \Pi_i(\cdot, 0)U^{-1}(\cdot)$, and by [23; 1p. 383], if $V = T(\partial/\partial x) + U(\partial/\partial y)$,

$$D_X a_0(V) = U^{-1}(X, 0) \int_0^{\tau} \exp\left[-\int_0^t \operatorname{div} X\right](PU - QT)\, dt,$$

where τ is the period of γ_i.

b) If γ_i is the periodic orbit at infinity, $a_0 \equiv 0$, and $P_i = xQ_i$, where

$$Q_i = x^{n_i-1} + a_{n_i-1}(Y)x^{n_i-2} + \ldots + a_1^i(Y),$$

where n_i is odd and $a_1^{(i)}(Y) = U_i^{-1}(Y, 0)\{\exp[(-1)\sigma\mu(Y)] - 1\}$, according to Remark 3.1.

The non hyperbolic periodic orbits of $\mathcal{P}(Y)$ are near some γ_i. In case *a*) they intersect L_i at points where the quasi-polynomial $P_i(\cdot, x)$ has a multiple root. In case *b*), this happens when $a_1^{(i)}(Y) = 0$, for non hyperbolicity at infinity, and at multiple roots of $Q_i(\cdot, x)$, for non hyperbolicity on $S^2 - S^1$.

These sets, defined by the condition of having multiple roots are semianalytic. In fact, they are inverse inmages by the analytic maps $Y \to (a_{n_i-1}^{(i)}(Y), \ldots, a_0^{(i)}(Y))$, for case *a*), and $Y \to (a_{n_i-2}^{(i)}(Y), \ldots, a_1^{(i)}(Y))$, for case *b*), of the discriminant locus of the generic polynomials of correspondent degree, which is a semi-algebraic set [27].

d) The semi-algebraic set $\chi_n - B(a)$ has finitely many connected components $C_1, C_2, \ldots, C_l$ [28]. On each such component C, the saddle singular points $p_j(X)$ of $\mathcal{P}(X)$ as well as its four separatrices $S_i^j(X, s_i)$, $i = 1, 2, 3, 4$, parametrized by arc length s_i with origin in $p_j(X)$, are well defined analytic functions of the two variables. Take $S_1^j(Y, s_1)$ and $S_2^k(Y, s_2)$ two such separatrices, the first unstable and the second stable, which correspond to saddle points $p_j(Y)$ and $p_k(Y)$, which may be equal.

The set B_{jk} of $(Y, l) \in C \times R_+$ for which $S_1^j(Y)$ and $S_2^k(Y)$ form a saddle connection of lenght l is an analytic submanifold of dimension $N - 1$ in $C \times R_+$ whose projection into C is a one-to-one immersion.

In fact, for $(X, l_0) \in B_{jk}$, take a small segment L transversal to X through a point $p_0 = S_1^j(X, s_1(0)) = S_2^k(X, s_2(0))$. There are analytic functions $s_i(Y)$, $i = 1, 2$, implicitly defined by $S_1^j(Y, s_1(Y)) \in L$, $S_2^k(Y, s_2(Y)) \in L$ and such that $s_i(X) = s_i(0)$.

It was shown in [23], see also [2,18], that the derivative of the function $S = S_1^j(Y, s_1(Y)) - S_2^k(Y, s_2(Y))$ is given by

$$Ds_X(Z) = \int_{-\infty}^{\infty} \exp\left[-\int_0^t \operatorname{div} X\right](RT - QU)\,dt,$$

where $Z = T(\partial/\partial x) + U(\partial/\partial y)$, and the integral is computed on the saddle connection. Without loss of generality assume that the saddle connection does not contain $(0, 0, 1)$, and the coordinates (ρ, θ) of (3.1) can be used.

Writing $X = P(\partial/\partial x) + Q(\partial/\partial y)$ and $X^{\perp} = -Q(\partial/\partial x) + P(\partial/\partial y)$ in these coordinates and applying the above integral formula, one gets an expression of the form

$$DS_X(X^{\perp}) = \int_{-\infty}^{\infty} g(\rho, \theta)\rho\,dt,$$

where $g(\rho, \theta)$ is strictly positive. This shows that $DS_X \neq 0$.

Clearly this ends the proof of *d*). In fact, when $S(Y) = 0$, the lenght of the saddle connection is given by $l(Y) = s_1(Y) + s_2(Y)$ which is also analytic. Therefore, B_{jk} is an analytic manifold of dimension $N - 1$, which projects regularly into C. The set $B(c)$ in C is the union of finitely many images of such projections.

The proof of (A, b) follows from Proposition 3.1, by the stratification of semi-algebraic and semi-analytic sets into analytic manifolds. See Lojasiewicz [11] and Whitney [28].

The proof of Theorem A, a) is straight:

1) If $X \in \mathbb{S}_n$ the constructions of topological equivalences in [1, 7] all produce r-stability. Therefore, $X \in \Sigma_n^r$, $r = 1, \ldots, \omega$.

2) If $X \in \Sigma_n^r$, $r = 1, \ldots, \omega$, from A, b) X must be topologically equivalent to an element of $\mathbb{S}_n$ and therefore the singularities and periodic orbits of $\mathcal{P}(X)$ must be finite and there must not be saddle connections contained in $S^2 - S^1$. The r-stability condition forces the singularities and, particularly, the periodic orbits to be hyperbolic. Actually, for the hyperbolicity of singular points and infinite periodic orbits it is sufficient to impose topological stability.

4. Final Remarks

1) For the study of stable smooth vector fields on non compact domains, the reader is refered to Nitecky et al [8] and the references quoted in this work. Here, perturbations with compact support are allowed and stability is not a generic property.

2) The set $A = (\chi_n - B(a)) - C$ in Proposition 3.1 is related to the class of polynomial vector fields studied by Poincaré [15, Theorem 17], for which the finiteness of limit cycles was first proved.

For extensions and further developments of this finiteness Theorem, the reader is refered to Chicone-Shafer [3], Paterlini-Sotomayor [12], Iliashenko [9], Ye Yanquian [30], Pugh-Françoise [5] and references quoted in these works.

3) Using Thom's Transversality Theorem [26], it can be asserted that the generic one parameter family of elements in χ_n has at most countably many bifurcations. The idea of Corollary 1.2 was suggested to the author by his previous work [24] and by the reading of Pontrjagin [16].

4) Although Theorem A expresses the meagerness of the bifurcation set in analytical terms and implies, through Corollary 1.1, that a vector field in χ_n is probabilistically almost surely stable, i.e. on $\mathbb{S}_n$, it cannot be regarded as the ultimate result on this line of ideas. In fact, it does not give any estimate on the cost involved in deciding whether or not a given vector field in χ_n is stable, i.e. on $\mathbb{S}_n$, in the sense of complexity theory, a la Smale [22].

A key step for such estimate amounts to the study, in terms of n and R, of the volume of a tube of radius R of the set $\chi_n^{r,1} \cap S^{N-1}$ relative to the volume of the unitary sphere S^{N-1} of χ_n. The study can be done for the part of the tube around $B(a)$ in viev of the algebraic nature of this set, using ideas of integral geometry, as suggested by Smale [22], and results of Demmel [4]. The analysis for the part of the tube on $B(b)$ and $B(c)$ does not seem to be straight. The set $B(b)$ is not semi-algebraic, as follows from results of Illiashenko [10]. Also the set $B(c)$ is not semi-algebraic, as is easy to verify at least for n big. For $n = 2$, this is not known [30].

These remarks indicate that new different techniques and expectations should be devised in connection with the possibility of developing a complexity theory for the stability and bifurcations in χ_n.

5. Aknowledgement

The author aknowledges the hospitality of the University of California, Berkeley, where this paper was written and where he had stimulating conversations with S. Smale and C. Pugh.

References

[1] Andronov A. et al. Theory of bifurcations of Dynamic Systems on the plane, *J. Wiley*, New York, 1973.

[2] Bamón, R. A class of planar quadratic vector fields with a limit cycle surrounded by a saddle loop, *Proc. Amer. Math. Soc.*, **88** (1983), 719-724.

[3] Chicone, C., Shafer, D. Separatrix and limit cycles of quadratic systems and Dulac's Theorem, *Trans. Amer. Math. Soc.*, **278** (1983), 585-612.

[4] Demmel J. A numerical analyst's Jordan canonical form, *Thesis*, Berkeley, 1983.

[5] Françoise, J. P., Pugh, C. Deformation de cycles limites, *Prepint IHES*, M/82/62.

[6] Gonzáles Velasco, E. Generic properties of polynomial vector fields at infinity, *Trans. Amer. Math. Soc.*, **143**, (1969).

[7] Melo W., Palis, J. Geometric Dymical Systems, *Springer Verlag*, 1982.

[8] Nitecky, Z., et al. Global Structural stability of flows on open surfaces, *Memoirs Amer. Math. Soc.*, **261** (1982).

[9] Illiashenko, Ju. The origin of limit cycles under perturbations of equation $(dw/dz = -(Rz/Rw)$ where $R(z, w)$ is a polynomial, *Math. Sbornik*, **78** (1969) 353-364.

[10] Illiashenko, Ju. The non algebraic character of the manifold of differential equations with rational right hand sides and with multiple limit cycles, *Math USSR Sbornik*, **12** (1970) 453-456.

[11] Lojasiewicz, S. Ensembles Semianalitiques, *IHES*, (1969).

[12] Paterlini, R., Sotomayor, J. Quadratic Vector Fields with finitely many periodic orbits, *Springer Lect. Notes in Math.*, **1007**, (1983), 753-766.

[13] Paterlini, R., Sotomayor, J. Bifurcations of polynomial vector fields in the plane, to appear.

[14] Peixoto, M. Structural Stability on two dimensional manifolds. *Topology* 1, (1962), 101-120.

[15] Poincaré, H. Memoire sur les courbes definies par une equation differentielle, *J. Mathematiques*, **7** (1881), 375-422.
[16] Pontrjagin, L. Smooth manifolds and its applications to homotopy theory, *Amer. Math. Soc. Trans.*, **11** (1959).
[17] Pugh, C. Structural Stability and Hilbert's 16-th Problem, *Springer Lect. Notes in Math.*, **468** (1975).
[18] Robbin, J. An algebraic Kupka-Smale Theorem, *Springer Lect. Notes in Math.*, **898** (1981).
[19] Santos, G. Classification of generic quadratic vector fields with no limit cycles, *Springer Lect. Notes in Math.*, **597** (1977), 605-640.
[20] Sard, A. The measure of critical values of differentiable maps, *Bull. Amer. Math. Soc.*, **46** (1942).
[21] Seindenberg, A. A new decision method for elementary algebra, *Ann. of Math.*, **60** (1954), 365-374.
[22] Smale, S. The fundamental theorem of algebra and complexity theory, *Bull Amer. Math. Soc.*, **4** (1981), 1-36.
[23] Sotomayor, J. Estabilidade estructural de primeira ordem e variedades de Banach, *Thesis IMPA*, 1964.
[24] Sotomayor, J. Generic one parameter families of vector fields on two-dimensional manifolds, *Publ. Math. IHES*, **43** (1974).
[25] Sotomayor, J. La mesure de l'ensemble de bifurcation de champ de vecteurs dans le plan, *C. R. Acad. Sc. Paris*, **249** (1981).
[26] Thom, R. Quelques proprietes globales des varietes differentiables, *Comm. Math. Helv*, **28** (1954), 17-86.
[27] Varchenko, A. Theorems on the topological equisingularity of families of algebraic varieties and families of polynomial maps, *Math USSR Izvestija*, **6** (1972), 949-1008.
[28] Whitney, H. Elementary Structure of algebraic varieties, *Ann. of Math.*, **66** (1957), 545-556.
[29] Wall, T. C. T. The Weirstrass Preparation Theorem, *Springer Lect. Notes in Math.*, **192** (1971).
[30] Ye Yanqian Some problems in the qualitative theory of ordinary differential equations, *Jour. Diff. Eq.*, **46** (1982), 153-164.

J. Sotomayor
University of Toronto
Toronto, Canada, M551A1

Partially supported by CNPq, Brazil, and the J. S. Guggenheim Memorial Foundation.

Revista Matemática Iberoamericana
Vol. 1, n.° 2, 1985

Singular Integrals on Product H^P Spaces

R. Fefferman

1. Introduction

We shall begin describing some terminology and notation. By «Calderón-Zygmund space» we shall mean the class of all bounded operators, T, on $L^2(R^1)$ given by a kernel $k(x,y)$ so that $Tf(x) = \int_{R^1} k(x,y)f(y)\,dy$ and so that for each fixed $x \in R^1$, $k \in C^\infty(R^1/\{x\})$ as a function of y and satisfies

$$\left|\left(\frac{\partial}{\partial y}\right)^\alpha k(x,y)\right| \leqslant C_\alpha |x-y|^{-1-\alpha} \quad \text{for} \quad \alpha > 0. \tag{*}$$

We shall often identify the operator T with its kernel k. For a particular choice of a positive integer N, we define the norm of T in Calderón-Zygmund space $\|T\|_{CZ}$ by $\|T\|_{CZ} = \|T\|_{L^2,L^2} + \sum_{\alpha=1}^N C_\alpha$ where here C_α denotes the smallest constant for which (*) is valid.

Suppose $k(x)$ now stands for a kernel on R^1 with values in Calderón-Zygmund space satisfying:

(1) $|k(x)|_{CZ} \leqslant \dfrac{C}{|x|}$

(2) $\left|\left(\dfrac{d}{dx}\right)^j k(x)\right|_{CZ} \leqslant \dfrac{C}{|x|^{j+1}}, \qquad j \leqslant N$

and

$$(3) \int_{\alpha < |x| < \beta} k(x)\, dx = 0 \qquad \forall 0 < \alpha < \beta.$$

Then k defines an integral operator taking functions $f(x_1, x_2)$ on R^2 to functions $Hf(x_1, x_2)$ on R^2 as follows:

$$Hf(x_1, x_2) = \iint k_0(x_1, y_1, x_2, y_2) f(y_1, y_2)\, dy_1 dy_2 \qquad (\neq)$$

where

$$k_0(x_1, y_1, x_2, y_2) = k(x_1 - y_1) \cdot (x_2, y_2).$$

Another way of understanding how k gives rise to H is as follows: Suppose we associate to each function $f(x_1, x_2)$ on R^2, the function $\tilde{f}$ on R^1 taking its values in the space of functions (on R^1) of x_2 given by $\tilde{f}(x_1)(x_2) = f(x_1, x_2)$. Then

$$\widetilde{Hf}(x_1) = \int k(x_1 - y_1) \cdot \tilde{f}(y_1)\, dy_1.$$

Our theorem is then:

Theorem. Let $0 < p < 1$. Then there exists N so that if $k(x_1)$ is a kernel satisfying (1), (2) and (3) above and H is as in $(\neq)$, then H maps $H^P(R^2_+ \times R^2_+)$ boundedly to $L^P(R^2)$.

Recall that $f \in H^P(R^2_+ \times R^2_+)$, product H^P, means that f is a distribution on R^2 with the property that

$$\sup_{\delta_1, \delta_2 > 0} \left| f * \phi_{\delta_1, \delta_2}(x_1, x_2) \right| \in L^P(R^2)$$

for

$$\phi \in C_0^\infty(R^2),\ \phi_{\delta_1, \delta_2}(x_1, x_2) = \delta_1^{-1} \delta_2^{-1} \phi\left(\frac{x_1}{\delta_1}, \frac{x_2}{\delta_2}\right).$$

Before proving this theorem let us put it into some perspective. First of all, in case the values of $k(x)$ are convolution operators, then this theorem is already known. (See the article of E. M. Stein and the author [1]). The spirit of the proof we give here is along the lines of C. Fefferman's theorem on the maximal double Hilbert transform [2]. There, the product structures of the kernel plays a role, where here, we assume no such structure. The other main ingredient of the argument below is the atomic decomposition of H^P spaces on product domains ([3], [4], [5]). We shall assume that the reader is familiar with the properties of product H^P atoms.

Finally, we should mention the interesting work of J. L. Journé [6], where non convolution operators in the product setting are treated, and proven bounded on the L^P spaces and from L^∞ to product *BMO* (for the properties of product *BMO*, see [3]). This paper, we feel, should have simple generalizations to cover operators like Journé's bi-commutators, and the proof is probably very similar to the one given here.

PROOF OF THE THEOREM. In order to simplify things a little, we shall assume $p = 1$. The case $p < 1$ requires no major changes. We shall let $a(x_1, x_2)$ be an $H^1(R^2_+ \times R^2_+)$ atom supported in an open set Ω, which by dilation invariance, we may assume to have measure 1. For this atom a, we then show that if $\phi \in C^\infty(R^1)$, Ψ supported in $[-1, -1]$ is even and has its first N moments vanishing, then the corresponding square function in the first variable is in $L^1(R^2)$:

$$\left(\sum_{k=-\infty}^{+\infty} |\Psi_{2^k} *_1 (k * a)(x_1, x_2)|^2\right)^{1/2} \quad \in L^1(R^2).$$

This will prove the theorem.
(Here $\Psi_{2^k}(x_1) = 2^{-k}\Psi(x_1/2^k)$, $*_1$ refers to a convolution taken only in the first variable for each fixed x_2, and $k * a$ is the function such that $\widetilde{k * a}(x_1) = \int_{R^1} k(x_1 - t) \cdot \tilde{a}(t)\, dt$; see the introduction for an explanation of the $\sim$ notation).

We shall sometimes find it convenient to assume that Ψ has the form $\Psi^{(1)} * \Psi^{(1)}$ where $\Psi^{(1)}$ has properties similar to those listed for Ψ. Also it will suit our needs to define a cutoff function $\phi(x_1) \in C^\infty(R^1)$, which is even, is supported in $\frac{1}{4} < |x_1| < 4$ and so that $\sum_j \phi(x/2^j) \equiv 1$. Define $k_j(x) = k(x)\phi(x/2^j)$ and $k_{k,j}(x) = \Psi_{2k} * k_j(x)$.

Our proof will show that the norms $\|(\sum_k |k_{k,k+j} * a(x_1, x_2)|^2)^{1/2}\|_1$ decrease geometrically as $|j| \to \infty$. Then summing over j finishes the proof. To see what is going on let us first consider the case $j = 0$, and write $k_k = k_{k,k}$. To estimate $(\sum_k |k_k * a|^2)^{1/2}$ we next decompose a as follows: Using the notation of [3], [4] and [5], since a is an atom, it can be written as $a = \sum_{R \subset \Omega} e_R$ where the sum is taken over the dyadic subrectangles of Ω, and we are going to use this representation of a to do a splitting of a which depends upon the point (x_1, x_2). For an integer $r \geqslant 0$ and an integer l consider the following definitions: R_l is the collection of all dyadic subrectangles of Ω, $R = I \times J$ where $|I| = 2^l$ and where J is a maximal dyadic interval such that $I \times J \subset \Omega$. Split the rectangles in R_l into disjoint subclasses $R_{(x_1,x_2),l,r}$ by setting $R_{x,l,r}$ to be those rectangles of R_l, $R = I \times J$ where $2^{-r-1} < M(X_J)(x_2) \leqslant 2^{-r}$. Then for each fixed $(x_1, x_2) = x$,

$$\bigcup_{\substack{l \in Z \\ r \in Z, r \geq 0}} R_{x,l,r}$$

is precisely the collection of all dyadic subrectangles of Ω, $R \cong I \times J$ so that J is maximal.

Finally, we let

$$a_{l,r}^{(x_1,x_2)} = \sum_{R \in R_{x,l,r}} \Big(\sum_{S \subset R;\, x_1 \text{ length of } S = 2^l} e_S \Big),$$

so that

$$a = \sum_{l,r} a_{l,r}^x.$$

Now, we claim that $(\sum_k |k_k * a_{k+j,r}^x|^2)^{1/2}$ has an L^1 norm which is $0(c_1^{|j|} c_2^r)$ as $|j|, r \to \infty$ where the $c_i < 1$ for $i = 1, 2$. Summing these estimates on j and r finishes the argument that

$$\Big(\sum_k |k_k * a|^2\Big)^{1/2} \in L^1.$$

Again, we consider the special case $j = 0$ and show that

$$\Big\| \Big(\sum_k |k_k * a_{k,r}^x|^2\Big)^{1/2} \Big\|_{L^1} = 0(c^r), \qquad c < 1, \quad \text{as} \quad r \to \infty. \tag{0}$$

If $r \geqslant 1$, by the way we constructed $a_{k,r}^x$ it is clear that $k_k * a_{k,r}^x$ has support in $\{(x_1, x_2) \mid M_s(X_\Omega)(x_1, x_2) > \frac{1}{10 \cdot 2^r}\} = \tilde{\Omega}_r$. Since by the strong maximal theorem, $|\tilde{\Omega}_r| = 0(r2^r)$, by the Cauchy-Schwartz inqueality, to show (0) it will be sufficient to show

$$\Big\| \Big(\sum_k |k_k * a_{k,r}^x|^2\Big)^{1/2} \Big\|_{L^2} = 0(2^{-Nr})$$

We shall estimate $\|k_k * a_k^x\|_{L^2}$ by using the following trivial lemmas:

Lemma 1. Suppose $b(x)$ is a function on R^1 whose support lies in the union of the disjoint intervals I_k, and which has its first N moments vanishing over each I_k. Suppose a point $x \in R^1$ lies outside the union of the doubles of the I_k. Then for any Calderon-Zygmund operator T on R^1,

$$|Tb(x)| \leqslant C|T|_{CZ} \sup_k M(X_{I_k})^N(x) \cdot \mathcal{G}(b)(x),$$

where

$$\mathcal{G}(b)(x) = \big(\sum M^2(X_{I_k})\big)^{1/2} \cdot \big(\sum M^2(bX_{I_k})(x)\big)^{1/2}.$$

Lemma 2. Let $b(x_1, x_2)$ be a function on R^2 supported in an open set θ of finite measure. Denote by b_{x_1} the function given by $b_{x_1}(x_2) = b(x_1, x_2)$, and by

$L_k(x_1)$ the component intervals of $\theta_{x_1} = \{x_1 \mid (x_1, x_2) \in \theta\}$. Let $\mathcal{G}$ denote the operator which acts in the x_2 variable for each x_1, defined by

$$\mathcal{G}(b)(x_1, x_2) = \Big(\sum_k M^2(X_{I_k(x_1)}(x_2)\Big)^{1/2}\Big(\sum M^2(X_{I_k(x_1)}b_{x_1})(x_2)\Big)^{1/2}$$

Then $\|\mathcal{G}(b)\|_{L^2} \leqslant C|\theta|^{1/2}\|b\|_{L^2}$.

The point of these lemmas is that, as a function of x_2 for fixed x_1, $a^x_{k,r}(x_1, \cdot)$ has the properties of the function in lemma 1, so that $k_k(x_1 - t)$ as a singular integral in the x_2 variable applied to $a^x_{k,r}(t, \cdot)$ is dominated by

$$|k_k(x_1 - t)|_{CZ} \cdot \mathcal{G}[a^x_{k,r}(t, \cdot)] \cdot 2^{-Nr} \leqslant |k_k(x_1 - t)|_{CZ} \cdot \mathcal{G}[a_k(t, \cdot][a_k(t, \cdot)] \cdot 2^{-Nr}$$

where

$$a_k = \sum_{\substack{R \subset \Omega \\ R = I \times J \\ |I| = 2^k}} e_R.$$

We also have $\int_{R^1} |k_k(t)|_{CZ}\, dt \leqslant C$.

Therefore $|k_k * a^x_{k,r}(x_1, x_2)| \leqslant |k_k|_{CZ} *_1 \mathcal{G}(a_k)(x_1)(x_1, x_2) \cdot 2^{-Nr}$ and by lemma 2 we see that $\|\mathcal{G}(a_k)\|_2 \leqslant C\|a_k\|_2$. It follows that

$$\Big\|\Big(\sum_k |k_k * a^k_{k,r}|^2\Big)^{1/2}\Big\|_1 \leqslant 2^{-Nr}C\Big(\sum_k \|a_k\|_2^2\Big)^{1/2} \leqslant C'2^{-Nr}.$$

If $r = 0$ we estimate $\big(\sum_k |k_k * a^x_{k,0}|^2\big)^{1/2}$ by observing that this function is supported in the set $\{M_S(X_\Omega) > \frac{1}{10}\}$ which has measure $\leqslant C$. To estimate its L^1 norm we estimate its L^2 norm by observing that

$$\Big\|\Big(\sum |k_k * a^x_{k,0}|^2\Big)^{1/2}\Big\|_{L^2} \leqslant$$
$$\leqslant \sum_{r \geq 1} \Big\|\Big(\sum_k |k_k * a^x_{k,r}|^2\Big)^{1/2}\Big\|_{L^2} + \Big\|\Big(\sum_k |k_k * a_k|^2\Big)^{1/2}\Big\|_{L^2},$$

so we have only to estimate $\|(\sum |k_k * a_k|^2)^{1/2}\|_{L^2}$. But this is easy since $\int_{R^1} |k_k(t)|_{CZ}\, dt \leqslant C$, so $\|k_k * a_k\|_2 \leqslant C\|a_k\|_2$ and so $\sum_k \|k_k * a_k\|_2^2 \leqslant$ $\leqslant C\sum_k \|a_k\|_2^2 \leqslant C'$. Now, let us pass to the next case where $j > 0$ and $r > 0$. That is, we require an estimate of

$$\Big\|\Big(\sum_k |k_k * a^x_{k+j,r}|^2\Big)^{1/2}\Big\|_1 = 0(c^{j+r}) \text{ for some } c < 1.$$

Again, the support of $k_k * a^x_{k+j,r}$ is contained in $\tilde{\Omega}_r$ of measure $\leqslant Cr2^r$. So as above we need to estimate $\|k_k * a^x_{k+j,r}\|_2$. By using the fact that $\psi = \psi^{(1)} * \psi^{(1)}$ we may write $k_k * a^x_{k+j,r} = k'_k * (\psi^{(1)}_{2^k} *_1 a^x_{k+j,r})$ where k'_k has similar properties to k_k. Now we use the special form of $a^x_{k+j,r}$ to estimate $k'_k * (\psi^{(1)}_{2^k} *_1 a^x_{k+j,r})$. It turns out that essentially $a^x_{k+j,r}(x_1, x_2)$ over a dyadic interval in the x_1 variable of length 2^{k+j}, say the interval $[0, 2^{k+j}]$, is of the form

$\psi(x_1/2^{k+j}) \cdot \alpha_{x,r}(x_2)$. Then, for $x_1 \in [0, 2^{k+j}]$ we have $|\psi_{2^k}^{(1)} * [\psi(x_1/2^{k+j})]| \leqslant$ $\leqslant C2^{-jN}$ because of the vanishing moments of $\psi^{(1)}$. It follows that

$$|k_k' * (\psi_{2^k}^{(1)} *_1 a_{k+j,r}^x)(x_1, x_2)| \leqslant 2^{-jN-rN} \frac{1}{2^k} \int_{x_1-2^k}^{x_1+2^k} (\alpha_{k+j})(t, x_2)\, dt$$

where $\alpha_{k+j}(x_2)$ is the function such that $a_{k+j}(x_1, x_2) = \psi(x_1/2^{k+j}) \cdot \alpha_{k+j}(x_2)$. Again, from the lemmas, it follows that

$$\| k_k' * (\psi_{2^k}^{(1)} *_1 a_{k+j,r}^x) \|_2 \leqslant C2^{-jN-rN} \| a_{k+j} \|_2,$$

and this is the estimate we needed.
(Actually $a_{j+r,r}^x$ and a_{k+j} and a_{k+j} over dyadic intervals of length 2^{k+j} (say $[0, 2^{k+j}]$ are averages over $\tau \in [0, 2^{k+j}]$ of functions of the form $\psi(x_1 - \tau/2^{k+j})\alpha_\tau(x_2)$; this average does not, however, interfere with any of the estimates done above; see [5]). The case $r = 0$ and $j > 0$ follows from the cases $r > 0$, $j < 0$ as was carried out previously. If $r > 0$ and $j < 0$, then we estimate $\|(\sum_k |k_k * a_{k-|j|,r}^x|^2)^{1/2}\|_1$ by observing that the support of $k_k * a_{k-|j|,r}^x(x_1, x_2)$ is contained in $\tilde{\Omega}_{r+|j|}$ of measure $\leqslant C(r + |j|)2^{r+|j|}$. We estimate, as before, the L^2 norm of $k_k * a_{k-|j|,r}^x$ by using the fact that for each fixed x_2, $a_{k+|j|,r}^x(\cdot, x_2)$ has the following property: There exist disjoint intervals of length $2^{k-|j|}$ over which the first N moments of $a_{k-|j|,r}^x(\cdot, x_2)$ vanish. Just as for the familiar case of scalar valued kernels, we may take advantage of the smoothness of $k_k(x)$ and subtract off the correct Taylor approximation to k_k to produce $\widehat{k_k}$ such that $k_k * a_{k-|j|,r}^x = \widehat{k_k} * a_{k-|j|,r}^x$, but $|\widehat{k_k}(z)|_{CZ} \leqslant 2^{-k_2-|j|N}$. Now we use the fact that for fixed x_1, $a_{k-|j|,r}^x(x_1, \cdot)$ has N vanishing moments over the component intervals of $\{x_2 \mid (x_1, x_2) \in$ $\in \bigcup_{R \in R_{x,k-|j|,r}} R\}$ to dominate $\widehat{k_k}(t)$ acting on $a_{k-|j|,r}^x(x_1 - t, \cdot)$ by $2^{-Nr}\mathcal{G}(a_{k-|j|,r}^x(x_1 - t, \cdot))\, |\widehat{k_k}(t)|_{CZ}$ so that

$$|k_k * a_{k-|j|,r}^x(x_1, x_2)| \leqslant C2^{-Nr}|\widehat{k_k}| *_1 \mathcal{G}(a_{k-|j|})$$

and obviously

$$\| \widehat{k_k} *_1 \mathcal{G}(a_{k-|j|}) \|_1 \leqslant C2^{-|j|N} \| a_{k-|j|} \|_2,$$

which proves that

$$\left\| \left(\sum_k |k_k * a_{k-|j|,r}^x|^2 \right)^{1/2} \right\| \leqslant C2^{-N(r+|j|)}.$$

Passing to the estimate of $\|(\sum_k |k_k * a_{k-|j|,0}^x|^2)^{1/2}\|_1$ is routine and left to the reader.

Now we are almost finished. We have shown that $\|(\sum_k |k_k * a|^2)^{1/2}\|_1 \leqslant C$ and all that is left is to show that $\|(\sum_k |k_{k,k+j} * a|^2)^{1/2}\|_1$ tend to 0 geometrically. But this is immediate, since for say $j > 0$, $k_{k,k+j}(x)$ satisfies all the same estimates as $2^{-jN}k_{k+j}(x)$ and so by the estimates above,

$$\left\| \left(\sum_k |k_{k,k+j} * a|^2 \right)^{1/2} \right\|_1 = 0(2^{-jN}).$$

This proves our theorem.

References

[1] Fefferman, R. and Stein, E. M. Singular Integrals on Product Spaces, *Advances in Math.*, 45, **2** (1982), 117-143.

[2] Fefferman, C. Estimates for the Double Hilbert Transform, *Studia Math.*, 44 (1972), 1-15.

[3] Chang, S. Y. and Fefferman, R. A Continuous Version of the Duality of H^1 and BMO on the bi-disc, *Ann. Math.*, 112 (1980), 179-201.

[4] Chang, S. Y. and Fefferman, R. The Calderon-Zygmund Decomposition on Product Domains, *American Journal of Math.*, 104, **3**, 445-468.

[5] Fefferman, R. The Atomic Decomposition of H^1 in Product Spaces, to appear in Advances in Math.

[6] Journé, J. L. Calderon-Zygmund Operators on Product Spaces, to appear in Rev. Mat. Iberoamericana.

R. Fefferman
University of Chicago
Chicago, Illinois 60637

Revista Matemática Iberoamericana
Vol. 1, n.º 2, 1985

On the Boundary Values of Harmonic Functions

P. R. Garabedian

1. Introduction

Over the years many methods have been discovered to prove the existence of a solution of the Dirichlet problem for Laplace's equation. A fairly recent collection of proofs is based on representations of the Green's function in terms of the Bergman kernel function or some equivalent linear operator [3]. Perhaps the most fundamental of these aproaches involves the projection of an arbitrary function onto the class of harmonic functions in a Hilbert space whose norm is defined by the Dirichlet integral [5]. Here a problem has remained open concerning continuity at the boundary of the solution that is constructed by orthogonal projection. Past discussions of this question turned out to be successful in space of two or three dimensions, but failed for larger numbers of independent variables [2]. It is the purpose of the present note to remove any such restriction and simultaneously to give a concise treatment of the boundary condition that is applicable to other existence proofs.

Let D be a domain in n-dimensional space that has a smooth boundary ∂D. We introduce the Hilbert space H whose elements are the gradients of harmonic functions u with a finite Dirichlet integral

$$\|u\|^2 = (u, u) = \int_D |\nabla u|^2 \, d\tau.$$

That H is complete follows easily from the mean value theorem for the partial derivatives of u.

Let w stand for a continuous function on ∂D that can be extended inside D so that its Dirichlet integral there is finite. According to the Riesz representation theorem the bounded linear functional (u, w) can be expressed as the scalar product

$$(u, w) = (u, U)$$

of u with an element U of the Hilbert space H. In fact U may be viewed as the orthogonal projection of w onto H. It might be anticipated that U solves the Dirichlet problem for Laplace's equation in D with boundary values w assigned on ∂D. However, we shall not attempt to establish this directly.

Let us consider the auxiliary function

$$v = \left(w - U, \frac{1}{\omega_n r^{n-2}} \right),$$

where r stands for the distance from a point x in D to a variable point of integration ξ, and where $\omega_n/(n-2)$ is the surface area of a unit sphere. After a preliminary application of Green's theorem that resolves the singularity at $\xi = x$, differentiation under the sign of integration shows that v satisfies the partial differential equation

$$\Delta v = \Delta w$$

in D, since $1/(\omega_n r^{n-2})$ is a fundamental solution of Laplace's equation. To prove that $w - v$ solves the Dirichlet problem formulated above it therefore suffices to show that v vanishes continuously at the boundary ∂D. It is our intention to develop an elementary proof of this result in the next two sections of the paper.

2. Green's function of a nearly spherical domain

In Neumann's method of the arithmetic mean [4] the solution of the Dirichlet problem is sought as a double-layer potential

$$u = \frac{1}{\omega_n} \int_{\partial D} \mu \frac{\partial}{\partial \nu} \frac{1}{r^{n-2}} \, d\sigma,$$

where ν stands for the inner normal. A Fredholm integral equation

$$\frac{\mu}{2} + \frac{1}{\omega_n} \int_{\partial D} \mu \, d\omega = w$$

is found for the determination of the unknown density μ on ∂D, where

$$d\omega = \frac{\partial}{\partial \nu} \frac{1}{r^{n-2}} d\sigma$$

becomes, after division by $n-2$, the solid angle subtended from the point $x = x_0$ on ∂D by the surface element $d\sigma$. An exact solution may be obtained when D is a half-space, since in that case $d\omega = 0$, so $\mu = 2w$.

More generally, following Neumann, one can try to determine μ as the limit of a sequence of successive approximations μ_j defined by the formula

$$\mu_j = 2w - \frac{2}{\omega_n} \int_{\partial D} \mu_{j-1}\, d\omega.$$

A proof of convergence hinges on estimating the difference

$$\mu_{j+1} - \mu_j = -\frac{2}{\omega_n} \int_{\partial D} [\mu_j - \mu_{j-1}]\, d\omega.$$

For the moment let us suppose that ∂D consists of two pieces, one being the infinite section S_1 of an $(n-1)$-dimensional hyperplane that is cut out by a small cell S_2 of some convex surface, and the other being S_2 itself. Furthermore, let us assume that with reference to any point x_0 on S_1 or S_2 the solid angles subtended by either S_1 or S_2 are both less than $\epsilon/(n-2)$.

From the hypotheses we have formulated one can derive the estimate

$$|\mu_{j+1} - \mu_j| \leqslant \frac{4\epsilon}{\omega_n} \max|\mu_j - \mu_{j-1}|.$$

This follows because any line intersects the surfaces S_1 and S_2 in at most three points, so that the solid angle of integration $d\omega/(n-2)$ does not become multiplied by more than two. Therefore μ_j converges to a solution μ of the Fredholm equation provided that $\epsilon < \omega_n/4$. We conclude that the Dirichlet problem can be solved and the Green's function

$$G = \frac{1}{\omega_n r^{n-2}} + \cdots$$

for Laplace's equation exists in either of the two infinite domains bounded by S_1 and S_2.

Let us return to the case of a smooth surface ∂D bounding a finite domain D. To assess boundary values we require that each point x_0 of ∂D can be touched by a closed sphere located enterely in the exterior of D. An inversion mapping this sphere onto a half-space transforms ∂D into a surface that becomes convex in the neighborhood of the boundary point x_0. Consequently a convex surface element S_2 of ∂D enclosing x_0 can be cut out by a hyperplane whose outer section S_1 combines with S_2 to meet the hypotheses announced above. Thus we are assured that a Green's function G exists in the infinite region bounded by S_1 and S_2 that contains D.

In the next section we shall use G as a parametrix to estimate the boundary values of the auxiliary function v. To complete our discussion of Neumann's method here we observe that, coupled to the Kelvin transformation [4], it provides a convenient construction of the Green's function for a nearly spherical domain. Moreover, convergence of the Neumann series can be proved without the assumption of partial convexity that we have introduced as a matter of convenience.

3. Continuity at the boundary

We proceed to establish that the auxiliary function v defined in Section 1 approaches zero as its argument x approaches any point x_0 on the boundary surface ∂D. The analysis of Section 2 shows that it suffices to consider the case where x_0 lies on a convex surface element S_2 of ∂D which, together with the outer section S_1 of a corresponding hyperplane, bounds a domain containing D and possessing a well defined Green's function G.

Let us recall that $w - U$ is orthogonal to every harmonic functon u of the Hilbert space H in the sense that

$$(w - U, u) = 0.$$

Since the difference between G and the fundamental solution $1/(\omega_n r^{n-2})$ of Laplace's equation lies in H, it follows that v has the representation

$$v = (w - U, G).$$

Because G vanishes on S_2 we wish to draw a similar conclusion about v.

Given any number $\delta > 0$, the locus of points ξ where

$$G = G(x, \xi) \geqslant \delta$$

is seen to lie inside D when x is chosen sufficiently close to the boundary point x_0. The estimate of G required for the proof follows from a comparison with the Green's function of a half-space enclosing S_1 and S_2. In this situation Green's theorem yields the identity

$$v = -\int_{G>\delta} (G - \delta)\,\Delta w\, d\tau + \int_{G<\delta} (\nabla w - \nabla U)\cdot \nabla G\, d\tau.$$

As $x \to x_0$ the first integral on the right tends to zero with δ provided that w

has bounded second derivatives. The second integral, which is evaluated over the part of D where $G < \delta$, has the same property in view of the Schwarz inequality

$$\left[\int_{G<\delta} (\nabla w - \nabla U) \cdot \nabla G \, d\tau\right]^2 \leqslant \| w - U \|^2 \int_{G=\delta} G \frac{\partial G}{\partial \nu} d\sigma = \delta \| w - U \|^2 .$$

This completes the proof that v vanishes on ∂D. Thus $w - v$ solves the Dirichlet problem posed in Section 1, and our existence theorem is established.

A similar treatment of the boundary condition can be given for the solution of the Dirichlet problema constructed from Dirichlet's principle rather than from projection onto the Hilbert space of harmonic functions. The method succeeds for more general linear partial differential equations of the elliptic type, too [1]. The main requirement is that a fundamental solution can be found in the large. The proof can also be based on other principles of functional analysis, such as the Hahn-Banach theorem [2]. One advantage of the present approach, as we have already indicated, is that it applies in space of arbitrary dimension. On the other hand, a disadvantage is the restriction to domains with a smooth boundary.

References

[1] Fichera, G. Teorema d'esistenza per il problema bi-iperarmonico, *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.*, **5** (1948), 319-324.

[2] Garabedian, P. R. Partial Differential Equations, *Wiley,* New York, 1964.

[3] Garabedian, P. R., Schiffer M. M. On existence theorems of potential theory and conformal mapping, *Ann. Math.*, **52** (1950), 164-187.

[4] Kellogg, O. D. Foundations of Potential Theory, *Dover,* New York, 1953.

[5] Lax, P. D. A remark on the method of orthogonal projections, *Comm. Pure Appl. Math.,* **4** (1951), 457-464.

Paul R. Garabedian
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, N. Y. 10012

This work was supported by N.S.F. Grant DMS-8320430.

Revista Matemática Iberoamericana
Vol. 1, n.° 2, 1985

On Discrete Subgroups of Lie Groups and Elliptic Geometric Structures

Robert J. Zimmer

In this paper we continue the investigation of [7]-[10] concerning the actions of discrete subgroups of Lie groups on compact manifolds.

Let H be a connected semisimple Lie group with finite center and suppose that the $\mathbb{R}$-rank of every simple factor of H is at least 2. Let $\Gamma \subset H$ be a lattice subgroup and M^n a compact n-manifold with a volume density. Let $P \to M$ be a G-structure on M where G is a real algebraic group. More precisely, let $GL(n, \mathbb{R})^{(k)}$ be the k-jets of diffeomorphisms of $\mathbb{R}^n$ fixing 0, and $P^{(k)} \to M$ the principal $GL(n, \mathbb{R})^{(k)}$-bundle of k-frames on M([1], [13]). Then $GL(n, \mathbb{R})^{(k)}$ is a real algebraic group and the (k-th order) G-structure $P \to M$ is a reduction of $P^{(k)}$ to the real algebraic subgroup $G \subset GL(n, \mathbb{R})^{(k)}$. We shall let $\mathrm{Aut}(P) \subset \mathrm{Diff}(M)$ denote the subgroup of diffeomorphisms of M that preserve the G-structure.

In [6] we showed that under the above hypotheses any volume preserving action of H on M which preserves the G-structure is either trivial or implies the existence of a non-trivial Lie algebra homomorphism $L(H) \to L(G)$, or equivalently, a Lie algebra embedding $L(H') \to L(G)$ for some simple factor H' of H. In [7], [8] we put forward the following conjecture.

Conjecture. *With hypotheses as above, assume there is a smooth volume preserving action of* Γ *on* M *defining a homomorphism* $\Gamma \to \mathrm{Aut}(P)$. *Then either:*

a) There is Lie algebra embedding $L(H') \to L(G)$ *for some simple factor* H' *of* H; *or*

b) *there is a* Γ*-invariant Riemannian metric on* M.

We remark that, as explained in [7], the conjecture would imply in particular that actions of Γ on low dimensional manifolds are trivial on a subgroup of finite index.

In [8] this conjecture was proven under the additional assumptions that the G-structure is of finite type (in the sense of E. Cartan [3], or more generally in the sense of Tanaka [5]), and that the Γ-action is ergodic. (In [8] the existence of a Γ-invariant C^0-Riemannian metric is deduced. However the arguments of [10] show that in fact an invariant C^∞ metric exists.) In this paper we weaken the assumption of finite type to that of ellipticity at the expense of assuming that $\mathrm{Aut}(P)$ acts transitively on M. (However the ergodicity assumption is no longer needed.) We recall that the G-structure is elliptic if the infinitesimal automorphisms of P (i.e. the vector fields defined by 1-parameter subgroups of $\mathrm{Aut}(P)$) are characterized as those vector fields satisfying an elliptic partial differential equation. For first order structures, this is equivalent to the simple condition on G that the linear Lie algebra $L(G) \subset \mathfrak{Gl}(n, \mathbf{R})$ contains no matrices of rank 1 [3, Prop I.1.4]. (For higher order structures see [1, p. 71].) One of the salient features of an elliptic G-structure is that $\mathrm{Aut}(P)$ is a (finite dimensional) Lie group. The main result of this paper is the following.

Theorem 1. *With* H, Γ, M, G, P *as above, suppose that* $\Gamma \to \mathrm{Aut}(P)$ *is a volume preserving, G-structure preserving action of* Γ *on* M. *Assume that*

a) *Aut(P) is a Lie group (e.g., G elliptic), and*

b) *Aut(P) acts transitively on M.*

Then either

1. *there is a Lie algebra embedding* $L(H') \to L(G)$ *for some simple factor* H' *of* H; *or*
2. *there is a* Γ-invariant *Riemannian metric on* M.

We remark that if $\mathrm{Aut}(P)$ is almost connected (or more generally if a subgroup of Γ of finite index is mapped into the connected component of the identity, $\mathrm{Aut}(P)^0$) then Theorem 1 follows from the work of Margulis [4] combined with the result of [6] described above and Kazhdan's property for Γ [2], [12]. In general, if course, $\mathrm{Aut}(P)/\mathrm{Aut}(P)^0$ may be infinite.

There are two basic known results we need for the proof of Theorem 1. The first is that the above conjecture is true if conclusion (2) is weakened to asserting the existence of a Γ-invariant measurable Riemannian metric on M. (By a measurable Riemannian metric on a vector bundle we of course mean a measurable assignment of an inner product to each fiber of the bundle.) This is a consequence of the superrigidity theorem for cocycles [11], [12, Thm. 5.2.5]. More precisely, we have:

Lemma 2. (Cf. [7, sections 2, 3]). *Let $P \to M$ be a principal G-bundle where G is a real algebraic group. Let H be a connected semisimple Lie group with finite center such that the* $\mathbf{R}$*-rank of every simple factor of H is at least 2. Let $\Gamma \subset H$ be a lattice. Assume that every Lie algebra homomorphism $L(H) \to L(G)$ is trivial. Let V be a vector space on which G acts linearly (and smoothly), and $E \to M$ the associated vector bundle. If Γ acts by principal bundle automorphisms of P covering a finite volume preserving action on M, then there is a measurable Γ-invariant Riemannian metric on the vector bundle E.*

The second result we need, proved in [7] enables us to give an estimate for the integrability properties of the measurable invariant metric in lemma 2.

Lemma 3. [7, Theorem 4.1]. *Let Γ be a discrete Kazhdan group (i.e., group with Kazhdan's property T* [2], [12]), *and $\Gamma_0 \subset \Gamma$ a fixed finite generating set. Then there exists $K > 1$ with the following property. If (S, μ) is a standard Borel ergodic Γ-space with Γ-invariant probability measure, and $f\colon S \to \mathbb{R}$ is a measurable function satisfying $|f(s\gamma)| \leqslant K|f(s)|$ for almost all s and all $\gamma \in \Gamma_0$, then $f \in L^1(S)$.*

Now let V be a finite dimensional real vector space. If η, ξ are inner products on V, we set (as in [7, section 3])

$$M(\eta/\xi) = \max\{\, \|v\|_\eta / \|v\|_\xi \mid v \neq 0, v \in V\},$$

and if η_m, $\xi_m (m \in M)$ are measurable Riemannian metrics on a vector bundle $E \to M$ we let $M(\eta/\xi)\colon M \to \mathbf{R}$ be $M(\eta/\xi)(m) = M(\eta_m/\xi_m)$. Suppose Γ acts on E by vector bundle automorphisms, that M is compact, that η is a measurable Γ-invariant metric, and ξ is a smooth metric. Then for $\gamma \in \Gamma$, and $m \in M$,

$$\begin{aligned} M(\eta/\xi)(m\gamma) &= M(\gamma^*\eta/\gamma^*\xi)(m) = M(\eta/\gamma^*\xi)(m) \\ &\leqslant M(\eta/\xi)(m) M(\xi/\gamma^*\xi)(m). \end{aligned}$$

(Cf. [7, Cor. 4.2]). We thus deduce that there exists $B > 0$ such that $m \in M$ and $\gamma \in \Gamma_0$ implies $M(\eta/\xi)(m\gamma) \leqslant BM(\eta/\xi)(m)$. From these remarks and lemma 3, we obtain:

Lemma 4. *Let Γ be a Kazhdan group acting smoothly on a compact manifold M. Suppose Γ preserves a smooth probability measure μ on M. We* *let* $\mu = \int_E^{\oplus} \mu_t d\nu(t)$ *be an ergodic decomposition of* μ *under the* Γ*-action. (Thus* (E, ν) *is the space of ergodic components.) Suppose* η *is a measurable* Γ*-invariant Riemannian metric and that* ξ *is a smooth metric. Then for* q *sufficiently large, we have* $M(\eta/\xi) \in L^{2/q}(M, \mu_t)$ *for almost all* t.

We now assume the hypotheses of Theorem 1, and suppose that every Lie algebra homomorphism $L(H) \to L(G)$ is trivial. By lemma 2, there is a measurable Γ-invariant metric η on TM. Choose q as in lemma 4. If $f\colon M \to (0, \infty)$ is a measurable Γ-invariant function, then $f\eta$ is also a measurable Γ-invariant metric. There is clearly a measurable $h\colon E \to (0, \infty)$ such that $\int_E h(t)^{2/q}\left(\int M(\eta/\xi)^{2/q} d\mu_t\right) d\nu(t) < \infty$ and thus if we let $f = h \circ p$ where $p\colon M \to E$ is the map defining the decomposition into ergodic components we have that $f\eta$ is a measurable Γ-invariant Riemannian metric satisfying $M(f\eta/\xi) \in L^{2/q}(M, \mu)$. Thus, replacing η by $f\eta$, we shall assume $M(\eta/\xi) \in L^{2/q}(M, \mu)$. Let Y be the set of (globally defined) infinitesimal automorphisms of P, so that (by hypothesis (*a*) of Theorem 1) Y is a finite dimensional vector space of smooth sections of TM and (by hypothesis (*b*)), for each $m \in M$ the evaluation map $e_m\colon\ Y \to TM_m$ is surjective. For $F \in Y$, let $\Phi(F) = \int_M \|F(m)\|_{\eta m}^{1/q}\, d\mu$. Since $M(\eta/\xi) \in L^{2/q}(M)$, $0 \leqslant \Phi(F) < \infty$, and it is clear that $\Phi(F) = 0$ if and only if $F = 0$. Furthermore Φ is continuous. (To see this simply observe that

$$\begin{aligned} |\Phi(F)| &\leqslant \int |M(\eta/\xi)|^{1/q} \|F(m)\|_{\xi m}^{1/q} \\ &\leqslant \|M(\eta/\xi)^{1/q}\|_2 \left(\int \|F(m)\|_{\xi m}^{2/q}\right)^{1/2}. \end{aligned}$$

Thus, if $\max_{m \in M} \|F(m)\|_{\xi_m} \to 0$, we have $\Phi(F) \to 0$.) We also observe that $\Phi\colon Y \to [0, \infty)$ is homogeneous of degree $\frac{1}{q}$. It follows from these properties of Φ (and the fact that $\dim Y < \infty$) that $\{F \mid |\Phi(F)| < 1\}$ is a (non-empty) open set with compact closure. Since η is Γ-invariant, it is clear that Φ is also Γ-invariant, and the preceding sentence implies that the representation of Γ on Y is uniformly bounded. Since $\dim Y < \infty$, there is a Γ-invariant inner product on Y. Via the maps $\{e_m\}$ this defines a smooth metric on TM, and it is clear that Γ-invariance of the inner product on Y implies that this metric on TM is Γ-invariant. This proves Theorem 1.

Remarks (*a*). If k is a local field of characteristic 0, H an almost k-simple algebraic k-group, with k-rank$(H) > 2$, and $\Gamma \subset H_k$ is a lattice, then super-rigidity and Kazhdan's property hold for Γ [2], [12]. Thus, the above argument shows:

Theorem 10.5. *Let* M *be a compact manifold,* P *a* G*-structure on* M *such that* $Aut(P)$ *is a Lie group acting transitively on* M. *Let* $\Gamma \subset H_k$ *be as above,* and assume that Γ *acts on* M *so as to preserve a volume density and the G-structure* P. *Then there is a* Γ*-invariant Riemannian metric on* M.

(b) If $\mathrm{Aut}(P)$ is a Lie group which is not transitive on M but the globally defined infinitesimal automorphisms of P define a foliation of M (which is then of necessity Γ-invariant), then the above argument shows that there is a Γ-invariant smooth metric on the tangent bundle to the foliation.

References

[1] Guillemin, V., Sternberg, S. Deformation theory of pseudogroup structures, *Mem. Amer. Math. Soc.,* **64**, 1966.
[2] Kazhdan, D. Connection of the dual space of a group with the structure of its closed subgroups, *Funct. Anal. Appl.,* **1** (1967), 63-65.
[3] Kobayashi, S. Transformation groups in differential geometry, *Springer, New York*, 1972.
[4] Margulis, G. A. Discrete groups of motions of manifolds of non-positive curvature, *A. M. S. Translations,* **109** (1977), 33-45.
[5] Tanaka, N. On differential systems, graded Lie algebras and pseudogroups, *J. Math. Kyoto Univ.*, **10** (1970), 1-82.
[6] Zimmer, R. J. Semisimple automorphism groups of G-structures, *J. Diff. Geom.*, **19** (1984), 117-123.
[7] Zimmer, R. J. Volume preserving actions of lattices in semisimple groups on compact manifolds, *Publ. Math. I.H.E.S.*, **59** (1984), 5-33.
[8] Zimmer, R. J. Actions of lattices in semisimple groups preserving a G-structure of finite type, to appear, *Erg. Th. Dyn. Syst.*
[9] Zimmer, R. J. Kazhdan groups acting on compact manifolds, *Invent. Math.*, **75** (1984), 425-436.
[10] Zimmer, R. J. Lattices in semisimple groups and distal geometric structures, to appear, *Invent. Math.*
[11] Zimmer, R. J. Orbit equivalence and rigidity of ergodic actions of Lie groups, *Erg. Th. and Dyn. Sys.,* **1** (1981), 237-253.
[12] Zimmer, R. J. Ergodic Theory and Semisimple Groups, *Birkhauser Boston*, 1984.

Robert J. Zimmer
University of Chicago
Chicago, Illinois 60637

Research partially supported by NSF Grant DMS 8301882.

Revista Matemática Iberoamericana
Vol. 1, n.° 2, 1985

The Concentration-Compactness Principle in the Calculus of Variations. The Limit Case, Part. 2

P. L. Lions

Abstract. This paper is the second part of a work devoted to the study of variational problems (with constraints) in functional spaces defined on domains presenting some (local) form of invariance by a non-compact group of transformations like the dilations in $\mathbb{R}^N$. This contains for example the class of problems associated with the determination of extremal functions in inequalities like Sovolev inequalities, convolution or trace inequalities... We show how the concentration-compactness principle and method introduced in the so-called locally compact case are to be modified in order to solve these problems and we present applications to Functional Analysis, Mathematical Physics, Differential Geometry and Harmonic Analysis.

Key-words. Concentration-compactness principle, minimization problems, unbounded domains, dilation invariance, concentration function, nonlinear field equations, Dirac masses, Morse theory, Sobolev inequalities, convolution, weak L^p spaces, trace theorems, Yamabe problem, scalar curvature, conformal invariance.

Subject AMS classification. 49 A 22, 49 A 34, 49 F 15, 58 E 30, 58 E 20, 58 E 05, 46 E 35, 46 E 30.

Summary

1. Introduction

This paper is the second part of a work devoted to variational problems with compactness defects. We use the notations of Part 1 [65] and we refer the reader to Part 1 [65] for a general introduction to the problems studied here.

Finally, formula $(1-n)$ will mean formula (n) of Part 1 while formula (n) is the n^{th} formula of this paper.

2. Extremal functions in unbounded domains

2.1 *Hardy-Littlewood-Sobolev inequality*

In this section we are mainly concerned with the minimization problem associated to the «best constant» C_0 in the following classical inequality —called Hardy-Littlewood-Sobolev inequality—; see [39], [40], [32], [74]:

$$\|u * |x|^{-\lambda}\|_{L^q} \leqslant C \|y\|_{L^p}, \qquad \forall u \in L^p(\mathbb{R}^N) \tag{2.1}$$

for some C depending only on N, p, q, λ and where:

$$0 < \lambda < N, \quad 1 < p < \frac{N}{N-\lambda}, \qquad \frac{1}{p} + \frac{\lambda}{N} = 1 + \frac{1}{q} \cdot \tag{2.2}$$

Following E. H. Lieb [53], we consider the minimization problem:

$$= \operatorname{Inf}\left\{ -\int_{\mathbb{R}^N} |K * u|^q \, dx \, / \, u \in L^p(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\}; \tag{2.3}$$

we will also denote by I_μ the corresponding infimum where l is replaced by $\mu > 0$ so that: $I_\mu = \mu^{q/p} I < 0$. Of course we consider in this section $K(x) = |x|^{-\lambda}$ but our goal is to show how our general method applies in this example and gives the existence of a minimum (and compactness of minimizing sequences) without using the very particular properties of K namely the fact that K is spherically symmetric and decreasing. And this will enable us to treat general classes of potentials $K(x)$.

We prove here the following:

Theorem 2.1. *Under assumption* (2.3), *let* $(u_n)_n$ *be a minimizing sequence of problem* (2.3). *There exist* $(y_n)_n$ *in* $\mathbb{R}^N$, $(\sigma_n)_n$ *in* $]0, \infty[$ *such that the new minimizing sequence* $\tilde{u}_n$ *defined by*:

$$\tilde{u}_n(\cdot) = \sigma_n^{-N/p} u_n((\cdot - y_n)/\sigma_n)$$

is relatively compact in $L^p(\mathbb{R}^N)$. *In particular there exists a minimum of problem* (2.3).

Remark 2.1. The existence of a minimum is proved in E. H. Lieb [53], hence the above result is a minor extension of [53], but we emphasize the fact that the proof in [53] relies on symmetrization arguments which prevents any generality on the class of potentials K while our methods does not depend on the particular form of K.

Remark 2.2. Using —a posteriori— the symmetrization, one easily sees that any minimum of (2.3) is spherically symmetric, decreasing (up to a translation)— see [53] for more details. In addition if $p = q'$ or $p = 2$ or $q = 2$, the explicit values of I and of the minima are given in [53].

PROOF OF THEOREM 2.1. Let $(u_n)_n$ be a minimizing sequence of (2.3); since both functionals $\int_{\mathbb{R}^N} |K * u|^q \, dx$, $\int_{\mathbb{R}^N} |u|^p \, dx$ are invariant by the change: $\sigma^{-N/p} u(\cdot/\sigma)$, we have to get rid of the possibility of «vanishing» exactly as we did in Part 1 for Sobolev inequalities (Section 1). We thus consider a new minimizing sequence —that we still denote by u_n— obtained by dilating u_n such that:

$$Q_n(1) = \tfrac{1}{2} \tag{2.4}$$

where

$$Q_n(t) = \operatorname*{Sup}_{y \in \mathbb{R}^N} \int_{y + B_t} \rho_n(x) \, dx \quad (\forall t \geqslant 0), \qquad \rho_n = |u_n|^p.$$

Exactly as in [55], [56], [65], we exclude vanishing by (2.4) and dichotomy as in [55], [65] since $I_\lambda = \lambda^{q/p} I (<0)$ is strictly subadditive ((S.2) holds!) —here since $(u_n)_n$ is only in L^p, we do not have to use smooth cut-off functions to perform the dichotomy and the argument is exactly the one described in [55], [58]. In conclusion, there exists $(y_n)_n$ in $\mathbb{R}^N$ such that $\tilde{u}_n(\cdot) = u_n(\cdot + y_n)$, satisfies: $|\tilde{u}_n|^p$ is tight. In the remainder of the proof, we still denote by u_n the new minimizing sequence $\tilde{u}_n$. We may of course assume that: $u_n \to u$ weakly in $L^p(\mathbb{R}^N)$. Let us also observe that $|K * u_n|^q$ is tight and that $K * u_n \to K * u$ a.e. on $\mathbb{R}^N$: indeed for all $R < M < \infty$:

$$\begin{aligned}
\int_{|x| \geqslant M} |K * u_n|^q \, dx &\leqslant C_0 \|u_n\|^q_{L^p(|x| > R)} + \\
&\quad + \int_{|x| \geqslant M} \left| \int_{|y| \leqslant R} \frac{1}{|x - y|^\lambda} u_n(y) \, dy \right|^q dx \leqslant \\
&\leqslant \epsilon(R) + \int_{|x| \geq M} (|x| - R)^{-\lambda q} \, \mathrm{d}x \, \|u_n\|^q_{L^1(B_R)} \leqslant \\
&\leqslant \epsilon(R) + \delta_R(M)
\end{aligned}$$

where $\epsilon(R) \to 0$ if $R \to +\infty$ and $\delta_R(M) \to 0$ if $M \to +\infty$ for any fixed $R < \infty$. This shows the tightness of $|K * u_n|^q$.

Concerning the a.e. convergence of $K * u_n$, we just observe that we have the following series of inequalities:

$$\|K * (u_n 1_{B_R}) - K * u_n\|_{L^q} \leqslant \epsilon(R),$$

$$\|K * (u 1_{B_R}) - K * u\|_{L^q} \leqslant \epsilon(R)$$

where $\epsilon(R) \to 0$ if $R \to +\infty$;

$$K_d * (u_n 1_{B_R}) \to K_\delta * (u 1_{B_R}) \quad (\forall x \in \mathbb{R}^N, \quad \forall \delta > 0, \ \forall R < \infty)$$

where $K_\delta = 1_{|x| \geqslant \delta} K$;

$$\| K_\delta * (u_n 1_{B_R}) - K * (u_n 1_{B_R}) \|_m \leqslant \mu_R(\delta) \qquad (\text{ind}^t \text{ of } n)$$

where $\mu_R(\delta) \to 0$ as $\delta \to 0_+$, $m \in]N/\lambda, q[$.

And this yields the convergence in measure.

There just remains to prove that: $\int_{\mathbb{R}^N} |u|^p \, dx = 1$ (we will denote by $C_0 = -I$). To this end we adapt to our setting the method of sections I.2-I.3 of Part 1 [65]: a basic ingredient being the following lemma corresponding to lemma I.1 in [65]:

Lemma 2.1. *Let u_n converge weakly in $L^p(\mathbf{R}^N)$ to u and assume $|u_n|^p$ is tight. We may assume without loss of generality that $|K * u_n|^q$, $|u_n|^p$ converge weakly (or tightly) in the sense of measures to some bounded nonnegative measures ν, μ on $\mathbb{R}^N$. Then we have:*

iii) *There exist some at most countable set (possibly empty) and two families $(x_j)_{j \in J}$ of distinct points in $\mathbb{R}^N$, $(\nu_j)_{j \in J}$ in $]0, \infty[$ such that:*

$$\nu = |K * u|^q + \sum_{j \in J} \nu_j \delta_{x_j}. \tag{2.5}$$

ii) *In addition we have:*

$$\mu \geqslant |u|^p + \sum_{j \in J} \nu_j^{p/q} C_0^{-p/q} \delta_{x_j} \tag{2.6}$$

iii) *If $u \equiv 0$, $C_0 \mu(\mathbb{R}^N)^{q/p} \leqslant \nu(\mathbb{R}^N)$; then J is a singleton and $\nu = c_0 \delta_{x_0}$, $\mu = (c_0/C_0)^{+p/q} \delta_{x_0}$ for some $c_0 > 0$, $x_0 \in \mathbb{R}^N$.*

Remark 2.3. Exactly as in Remark I.3 ([65]), if ν is given by (2.5) with $u \in L^p(\mathbb{R}^N)$, $\sum_{j \in J} \nu_j^{p/q} < \infty$ then ν is the tight limit of $(|u_n|^p)_n$ where u_n converges weakly in $L^p(\mathbb{R}^N)$ to u.

Remark 2.4. Both lemma 2.1 and 2.1 have the same consequence: for example in the context of lemma 2.1, if $u_n \to u$ weakly in $L^p(\mathbb{R}^N)$ and if $|u_n|^p$ converges tightly to a measure μ without atoms then $K * u_n$ converges strongly in $L^p(\mathbb{R}^N)$ to $K * u$.

Using lemma 2.1, we may now conclude the proof of Theorem 2.1 following the scheme of the proof of section 1.2: first, if $u \equiv 0$ and if $|K * u_n|^q$, $|u_n|^p$ converge tightly to some bounded nonnegative measures ν, μ, we have

$$\nu(\mathbb{R}^N) = C_0, \qquad \mu(\mathbb{R}^N) = 1.$$

Hence we may apply part (iii) of lemma 2.1: $\mu = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^N$. And we obtain a contradiction with (2.4).

Next, if $\alpha = \int_{\mathbb{R}^N} |u|^p \, dx \in]0,1[$, we observe:

$$I_1 = I = -\int_{\mathbb{R}^N} |K * u|^q \, dx - \sum_{j \in J} \nu_j.$$

In view of the homogeneity of $(\lambda \to I_\lambda)$ we deduce:

$$I_1 \geqslant I_\alpha + \sum_{j \in J} I_{\mu_j}, \quad \text{with} \quad \mu_j = (\nu_j / C_0)^{p/q}$$

Since $\sum_{j \in J} \mu_j \leqslant 1 - \alpha$ by (2.6), we finally obtain:

$$I_1 \geqslant I_\alpha + I_{1-\alpha}$$

and this contradicts the fact that (S.2) holds for all $\mu > 0$:

$$I_\mu < I_\alpha + I_{\mu - \alpha}, \qquad \forall \alpha \in]0, \mu[. \tag{S.2}$$

Therefore $\alpha = 1$ and we conclude.

PROOF OF LEMMA 2.1. Many of the arguments below are identical to those introduced in the proof of Lemma 1.1 [65]; only technical details differ!

We first observe that since $(K * u_n)$ converges a.e. to $K * u$ and $(|K * u_n|^q)_n$ is tight, applying the Brézis-Lieb lemma [21] we just need to prove (2.5) in the case when $u \equiv 0$. Using Lemma 1.2 [21], we only have to prove that:

$$\int |\varphi|^q \, d\nu \leqslant C_0 \Big(\int |\varphi|^p \, d\mu \Big)^{q/p}, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^N). \tag{2.7}$$

This inequality will then prove i) and iii).

To show (2.7), we first remark that for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$:

$$\int |K * (\varphi u_n)|^q \, dx \leqslant C_0 \Big(\int |\varphi|^p |u_n|^p \, dx \Big)^{q/p}.$$

Then (2.7) is deduced from the following claim:

$$\Big| \int |K * (\varphi u_n)|^q \, dx - \int |\varphi|^q |K * u_n|^q \, dx \Big| \to 0.$$

Using the argument we already made on the tightness of $|K * u_n|^q$ we just have to show that for all $M < \infty$: $K * (\varphi u_n) - \varphi(K * u_n)$ converges to 0 in $L^q(B_M)$. But for almost all x we have:

$$K * (\varphi u_n)(x) - \varphi(x)(K * u_n)(x) =$$
$$= \int_{|y| \leqslant R} \frac{1}{|x - y|^\lambda} (\varphi(y) - \varphi(x)) u_n(y) \, dy + \varphi K * (u_n 1_{C^R})$$

where $C^R = \{ y \in \mathbb{R}^N, |y| > R \}$.

Since $|u_n 1_{C^R}|_{L^p} \leqslant \epsilon(R)$ with $\epsilon(R) \to 0$ if $R \to +\infty$; we just have to bound for any $R < \infty$:

$$\left| \int_{|y| \leqslant R} \frac{1}{|x-y|^\lambda} (\varphi(y) - \varphi(x)) u_n(y)\, dy \right| = v_n(x).$$

Denoting by $R(x,y) = (\varphi(y) - \varphi(x))|x-y|^{-\lambda}$ and observing that $R(x,y)1_{|y| \leqslant R} \in L^r(\mathbb{R}^N)$ for each x where $r < \frac{N}{\lambda - 1}$ if $\lambda > 1$, $r \leqslant +\infty$ if $\lambda \leqslant 1$, we see that: $v_n \to 0$ a.e. on $\mathbb{R}^N$. Finally for some $s > q$

$$\| v_n \|_{L^s(B_M)} \leqslant C(M,R) \| u_n \|_{L^p} = C(M,R)$$

and thus $v_n \to 0$ in $L^q(B_M)$; and (2.7) is proved.

We next show part ii) of Lemma 2.1: since $\mu \geqslant |u|^p$, we just have to show that for each fixed $j \in J$:

$$\mu(\{x_j\}) \geqslant (\nu_j / C_0)^{p/q}.$$

Let $\sigma_\epsilon = \left(\frac{\cdot - x_j}{\epsilon}\right)$, where $\varphi \in \mathfrak{D}_+(\mathbb{R}^N)$, $\varphi(0) = 1$, $\varphi \leqslant 1$ and $\operatorname{Supp} \varphi \subset B_1$. We have:

$$\int |K * (\varphi_\epsilon u_n)|^q \, dx \leqslant C_0 \left(\int \varphi_\epsilon^p |u_n|^p \, dx \right)^{q/p}. \tag{2.8}$$

We fix ϵ and we let n go to $+\infty$: we estimate the left-hand side of (2.8) as follows:

$$K * (\varphi_\epsilon u_n) - (K * u_n)\varphi_\epsilon = K * (\varphi_\epsilon u_n 1_{C^R}) - [K * (u_n 1_{C^R})]\varphi_\epsilon + \psi$$

where $\psi = \psi(\epsilon, n, R)$ satisfies: $\| \psi \|_{L^q} \leqslant \delta(R) \to 0$ as $R \to +\infty$. In addition exactly as we did before:

$$\| K * (\varphi_\epsilon u_n) - (K * u_n)\varphi_\epsilon \|_{L^q(|x| > M)} \leqslant \mu(M) \to 0 \quad \text{as} \quad M \to \infty.$$

Finally by easy arguments identical to those given to prove (2.7), we show:

$$K * (\varphi_\epsilon u_n) - (K * u_n)\varphi_\epsilon \to K * (\varphi_\epsilon u) - (K * u)\varphi_\epsilon \quad \text{in} \quad L^q(B_M).$$

This together with (2.8) yields:

$$\begin{aligned} \left(\int |\varphi_\epsilon|^q \, d\nu \right)^{1/q} &\leqslant C_0^{1/q} \left(\int |\varphi_\epsilon|^p \, d\mu \right)^{1/p} + \\ &+ \delta(R) + \mu(M) + \| A_\epsilon^R u \|_{L^q(B_M)} + \| \{ K * (u 1_{C^R}) \} \varphi_\epsilon \|_{L^q(B_M)} \end{aligned} \tag{2.9}$$

where $A_\epsilon^R v = K * (v 1_{C^R} \varphi_\epsilon)$. Since A_ϵ^R is a family of uniformly bounded operators from L^p to $L^q(B_M)$, in order to show that $A_\epsilon^R u$ converges in $L^q(B_M)$ to 0 as ϵ goes to 0, we just need to check it for $v \in \mathfrak{D}(\mathbb{R}^N)$ and this is then obvious since $\varphi_\epsilon v_\epsilon \to 0$ in $L^p(\mathbb{R}^N)$. Therefore using the fact that $\varphi_\epsilon(x_j) = 1$, $\operatorname{Supp} \varphi_\epsilon \subset B(x_j, \epsilon)$ and (2.9), we obtain letting ϵ go to 0, then R, M go to $+\infty$:

$$\nu(\{x_j\})^{1/q} = \nu_j^{1/q} \leqslant C_0^{1/q} \mu(\{x_j\})^{1/p}$$

and this yields (2.6).

Remark 2.5. Of course, we also have the analogue of part iii) of Lemma 1.1: namely under the assumptions of Lemma 2.1, and if $v \in L^p(\mathbb{R}^N)$, $|v + u_n|^p$ converges weakly to some measure $\tilde{\mu}$ then $\tilde{\mu} - \mu \in L^1(\mathbb{R}^N)$ and

$$\tilde{\mu} \geqslant |u + v|^p + \sum_{j \in J} (\nu_j / C_0)^{p/q} \delta_{x_j}.$$

Remark 2.6. Another proof of (2.6) consists in using Brézis-Lieb lemma [21] to deduce

$$C_0 \left(\int |\varphi_\epsilon|^p \, d\mu \right)^{q/p} \geqslant \int |K * (\varphi_\epsilon u)|^q \, dx + \lim_n \int |K * (\varphi_\epsilon v_n)|^q \, dx$$

where $v_n (u_n - u) \to 0$. Thus in view of the proof of part i) we deduce:

$$\begin{cases} \lim \int |K * (\varphi_\epsilon v_n)|^q \, dx = \int |\varphi_\epsilon|^q \, d\tilde{\nu} \\ \tilde{\nu} = \sum_{j \in J} \epsilon_j \delta_{x_j}. \end{cases}$$

Therefore we have:

$$C_0 \, \mu(B(x_j, \epsilon))^{q/p} \geqslant \nu_j.$$

2.2 *Other potentials*

In this section we consider various questions related to problem (2.3) where K is now a general potential. To simplify the presentation (see the remarks below) we consider only the following situation:

$$K(x) = \varphi(x) \bar{K}(x) + \psi(x) \tag{2.10}$$

where

$$\varphi \in C_b(\mathbb{R}^N), \quad \varphi(x) \to \beta \quad \text{as} \quad |x| \to \infty, \quad \psi(x) \in L^{N/\lambda}(\mathbb{R}^N) \tag{2.11}$$

$$t^{-\lambda} \bar{K}(x) = \bar{K}(tx), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^N - \{0\},$$
$$\bar{K} \in C(\mathbb{R}^N - \{0\}), \quad \bar{K} > 0 \quad \text{on} \quad \mathbb{R}^N - \{0\}. \tag{2.12}$$

We will denote by $\alpha = \varphi(0)$.

Clearly enough, except in the case when φ is constant ($\neq 0$) and $\psi \equiv 0$, (2.3) is no more invariant by dilations. But, still, the invariance of $\mathbb{R}^N$ by dilations induces possible loss of compactness; to understand this possibility we compute for any $u \in L^p(\mathbb{R}^N)$, $\sigma > 0$:

$$\int_{\mathbb{R}^N} \left| K * \left[\sigma^{-N/p} u\left(\tfrac{\cdot}{\sigma}\right)\right]\right|^q dx = \int_{\mathbb{R}^N} |K_\sigma * u|^q \, dx$$

with $K_\sigma(x) = \sigma^\lambda K(\sigma x) = \varphi(\sigma x)\bar{K}(x) + \sigma^\lambda \psi(\sigma x)$.

Therefore the value I of the infimum is not changed if we replace K by K_σ for all $\sigma > 0$ and letting $\sigma \to +\infty$, or $\sigma \to 0$ we deduce:

$$I \leqslant \operatorname{Inf}\left\{ -\int_{\mathbb{R}^N} |\beta \bar{K} * u|^q \, dx / u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\}$$

or

$$I \leqslant \operatorname{Inf}\left\{ -\int_{\mathbb{R}^N} |\alpha \bar{K} * u|^q \, dx / u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\}$$

and we denote by I^∞ the minimum of these two upper bounds i.e. if $\gamma = \max(|\alpha|, |\beta|)$:

$$I^\infty \leqslant \operatorname{Inf}\left\{ -\int_{\mathbb{R}^N} |\gamma \bar{K} * u|^q \, dx / u \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\}. \tag{9.13}$$

Denoting by I_μ, $I_\mu{}^\infty$ the values of the infima in (2.3), (2.13) where 1 replaced by $\mu > 0$ and observing that $I_\mu = \mu^{q/p} I$, $I_\mu^\infty = \mu^{q/p} I^\infty$ with $I, I^\infty < 0$ we conclude these considerations by observing that we have proved:

$$I_\mu \leqslant I_\mu^\infty, \qquad \forall \mu > 0; \quad I \leqslant I^\infty \tag{2.14}$$

$$I_\mu < I_\alpha + I_{\mu-\alpha} \leqslant I_\alpha + I_{\mu-\alpha}^\infty, \qquad \forall \alpha \in [0, \mu[. \tag{2.15}$$

Therefore, we expect the:

Theorem 2.2. *We assume* (2.2), (2.11), (2.12).

i) *If* $\varphi \equiv \beta \neq 0$, $\psi \equiv 0$, *then every minimizing sequence* $(u_n)_n$ *of* (2.3) *is relatively compact up to a dilation* $(\sigma_n)_n$, *and a translation* $(y_n)_n$ *in* $L^p(\mathbb{R}^N)$ *i.e.* $\sigma_n^{-N/p} u_n((\cdot - y_n)/\sigma_n)$ *is relatively compact in* $L^p(\mathbb{R}^N)$ *for some* y_n *in* $\mathbb{R}^N$, σ_n *in* $]0, \infty[$. *And* (2.3) *has a minimum.*

ii) *Any minimizing sequence of* (2.3) is relat*ively compact in* $L^p(\mathbb{R}^N)$ *up to a translation if and only if*:

$$I < I^\infty. \tag{2.16}$$

Remark 2.7. We first observe that if $\alpha = \beta = 0$ i.e. $K \in L^{N/\lambda}(\mathbb{R}^N)$ then (2.16) automatically holds since $I^\infty = 0$ and all minimizing sequences are compact up to translations. But as it will be observed in Step 1 of the proof, this is due to an easy compactness argument which shows that this case is actually treated by the concentration-compactness method in the locally compact case [55], [56]. In fact this compactness property still holds for Lorentz spaces i.e. when $K \in L^{N/\lambda, \alpha}(\mathbb{R}^N)$ for any $1 < \alpha < \infty$ (same proof as below).

Remark 2.8. Condition (2.16) clearly holds if, for example,

$$\varphi(x) \geqslant \alpha = \beta > 0, \qquad \varphi \not\equiv \alpha, \qquad \psi \geqslant 0$$

Indeed let u_0 be a minimum of I^∞ (which exists by case i) of the above result, we may assume that $u_0 \geqslant 0$ (replace u_0 by $|u_0|$) then:

$$\begin{aligned} I \leqslant -\int_{\mathbb{R}^N} |K * u_0|^q \, dx &\leqslant -\int_{\mathbb{R}^N} |\varphi \bar{K}) * u_0|^q \, dx \\ &< -\int_{\mathbb{R}^N} |\bar{K} * u_0|^q \, dx \leqslant I^\infty \end{aligned}$$

On the other hand the same type of argument shows that if

$$0 \leqslant \varphi(x) \leqslant \gamma = \alpha = \beta, \qquad \varphi \not\equiv \gamma, \qquad \gamma > 0; \qquad \psi \equiv 0$$

not only (2.16) does not hold i.e. $I = I^\infty$ but (2.3) does not have a minimum. This class of K contains the example mentioned in [2.53].

Remark 2.9. In fact, the method of proof enables us to treat much more general potentials K. First of all in (2.12), the condition that $\bar{K} > 0$ may be replaced by $\bar{K} \not\equiv 0$; next we could treat

$$K = \sum_{i=1}^{\infty} \varphi_i(x) \bar{K}_i(x - x_i) + \psi(x)$$

with $\psi \in L^{N/\lambda}(\mathbb{R}^N)$, $\varphi_i \in C_b(\mathbb{R}^N)$ and $\sum_{i=1}^\infty |\varphi_i(x)| \in C_b(\mathbb{R}^N)$; $\bar{K}_i \in C(\mathbb{R}^N - \{0\})$, $\bar{K}_i(tx) = t^{-\lambda} \bar{K}_i(x)$ $\forall t > 0$, $\forall x \neq 0$, $|\bar{K}_i(x)| \leqslant (C/|x|^\lambda)$; $(x_i)_i \geqslant 1$ is a family of distinct points in $\mathbb{R}^N$. Denoting by $\alpha_i = \varphi_i(0)$, $\beta_i = \lim_{|x| \to \infty} \varphi_i(x)$ (which we assume exists). Theorem 2.2 still holds provided we define I^∞ by

$$I^\infty = \operatorname{Min}\left(\inf_i I_i^\infty, I_\infty^\infty\right)$$

with I_i^∞, I_∞^∞ corresponding to the potentials $\alpha_i \bar{K}_i$, $\sum_i \beta_i \bar{K}_i$.

Other technical extensions of (2.10) are possible and we will skip them.

Remark 2.10. Another possible extension is to replace $K * u$ by some $\int_{\mathbb{R}^N} K(x, y) u(y) \, dy$. For instance if we consider $K(x, y) = R(x, y) \bar{K}(x - y)$ where $\bar{K}$ satisfies (2.12), $R(x, y) \in C_b(\mathbb{R}^N \times \mathbb{R}^N)$ and $R(x, y) \to \beta$ if $|x - y| \to \infty$. Then part (ii) of Theorem 2.2 still holds if we replace I^∞ by

$$I^\infty = \operatorname{Min}\left\{ \operatorname*{Inf}_{y \in \mathbb{R}_N} I_y^\infty, I_\infty^\infty \right\}$$

where I_y^∞, I_∞^∞ correspond to (2.3) with the potentials $R(y,y)\bar{K}$, $\beta\bar{K}$.

Remark 2.11. Even if we may extend the classes of K for which we may analyse completely problem (2.3) (see also Corollary 2.1 bellow) we are unable to treat (2.3) for an arbitrary $K \in M^{N/\lambda}(\mathbb{R}^N) = L^{N/\lambda,\infty}(\mathbb{R})^N$. This is due to the fact that Lemma 2.1 which still holds for potentials like (2.10) is not true in general for arbitrary $K \in M^{N/\lambda}(\mathbb{R}^N)$. Indeed consider:

$$K(x) = |x_1|^{-\alpha}\varphi(x_2), \qquad x_1 \in \mathbb{R}^n, \quad x_2 \in \mathbb{R}^m$$

and (for example) $0 < \alpha < n$, $\varphi \in D_+(\mathbb{R}^m)$ $(\varphi \not\equiv 0)$, $x = (x_1, x_2)$. Then $K \in M^{N/\lambda}$ if $N/\lambda = n/\alpha$. In this example, one remarks that if $(u_n)_n$ converges weakly in $L^p(\mathbb{R}^N)$ to u, if $|u_n|^p$ is tight and if we choose $u_n(x_1, u_2) = v_n(x_1)w_n(x_2)$ where v_n converges weakly to v in $L^p(\mathbb{R}^n)$, w_n converges weakly to w in $L^p(\mathbb{R}^m)$ then denoting by μ, ν the tight limits of the measures $|u_n|^p$, $|k * u_n|^q$ (or subsequences) we have:

$$\nu = |K * u|^q + \sum_{i \in J} \nu_j \delta_{x_1^j} \otimes (\varphi * w)^q$$

$$\mu \geqslant |u|^p + \sum_{j \in J} (\nu_j/C)^{p/q} \delta_{x_1^j} \otimes \tilde{\mu}$$

for some at most countable family J, distinct points x_1^j in $\mathbb{R}^n$ bounded nonegative measure $\tilde{\mu}$ on $\mathbb{R}^m$, $C > 0$.

We now turn to the *proof of Theorem* 2.2:

Step 1: *Preliminary reductions*

We first explain why ψ may be assumed to be 0: indeed we just need to observe that if

$$u_n \to u \quad \text{weakly in} \quad L^p(\mathbb{R}^N), \qquad (|u_n|^p)_n \quad \text{is tight} \tag{2.17}$$

then $\psi * u_n \to_n \psi * u$ strongly in $L^p(\mathbb{R}^N)$.

By the density of $D(\mathbb{R}^N)$ in $L^{N/\lambda}(\mathbb{R}^N)$, we may without loss of generality assume that $\psi \in D(\mathbb{R}^N)$ since

$$\|\psi * u_n - \tilde{\psi} * u_n\|_{L^q} \leqslant C\|\psi - \tilde{\psi}\|_{L^{N/\lambda}};$$

But if $\psi \in D(\mathbb{R}^N)$, $\psi * u_n$ converges a.e. to $\psi * u$ and $\psi * u_n$ is bounded in $L^p \cap L^\infty$. Finally since $|u_n|^p$ is tight, $|\psi * u_n|^q$ is tight and we conclude easily.

This easy observation indicates that ψ creates no difficulty in the argument below, hence we assume from now on: $\psi \equiv 0$. Next, if we still denote by Q_n the concentration function of $|u_n|^p$, where u_n is a minimizing sequence of (3) and if we denote by $u_n^\sigma = \sigma^{-N/p} u_n(\cdot/\sigma)$ we observe that the concentration function of $|u_n^\sigma|^p$ satisfies:

$$Q_n^\sigma(t) = Q_n(t/\sigma), \qquad \forall t \geqslant 0.$$

Therefore there exists $(\sigma_n)_n$ in $]0, \infty[$ such that (4) holds. We denote by $\tilde{u}_n = u_n^{\sigma_n}$. Observe that:

$$\int |K * u_n|^q \, dx = \int |K_n * \tilde{u}_n|^q \, dx$$

where $K_n = \varphi(x/\sigma_n)\bar{K}(x)$ and recall that we already saw that for each $n \geqslant 1$ the value of the infimum (2.3) is not changed if we replace K by K_n.

We now apply the standard concentration compactness method ([58], [59], [55]): vanishing is ruled out by (2.4). If dichotomy occurs we find $\alpha \in]0, 1[$ such that for all $\epsilon > 0$, there exist R_0, R_n, y_n satisfying:

$$\left| \int |\tilde{u}_n^1|^p \, dx - \alpha \right| \leqslant \epsilon, \qquad \int |\tilde{u}_n^2|^p \, dx \to 1 - \alpha, \qquad R_n \to \infty,$$
$$\tilde{u}_n^1 = \tilde{u}_n \cdot \chi_{\{|x - y_n| \leq R_0\}}, \qquad \tilde{u}_n^2 = \tilde{u}_n \cdot \chi_{\{|x - y_n| \leq R_n\}}.$$

Let $v_n = \tilde{u}_n - (\tilde{u}_n^1 + \tilde{u}_n^2)$, we have clearly:

$$\begin{cases} \left| \int |K_n * \tilde{u}_n|^q \, dx - \int |K_n * (\tilde{u}_n^1 + \tilde{u}_n^2)|^q \, dx \right| \leqslant C\epsilon \\ \int |K^n * \tilde{u}_n^1|^q \, dx \geqslant I_{\alpha - \epsilon}, \qquad \underline{\lim}_{n} \int |K_n * \tilde{u}_n^2|^q \, dx \geqslant I_{1-\alpha}. \end{cases}$$

Since (S.2) holds (cf. (2.15)), we reach a contradiction since

$$\left| \int |K_n * (\tilde{u}_n^1 + \tilde{u}_n^2)|^q \, dx - \int |K_n * \tilde{u}_n^1|^q \, dx - \int |K_n * \tilde{u}_n^2|^q \, dx \right| \leqslant$$
$$\leqslant C \int \| \bar{K} * |\tilde{u}_n^1| \| \, |\bar{K} * |\tilde{u}_n^2| \|^{q-1} + |\bar{K} * |\tilde{u}_n^2| \| \, |\bar{K} * |\tilde{u}_n^1| \|^{q-1} \, dx.$$

To conclude we prove that this integral goes to 0; and since both terms are basically equivalent, we will only treat the first one: first

$$\int_{|x| > M} |\bar{K} * \tilde{u}_n^1| \, |\bar{K} * \tilde{u}_n^2|^{q-1} \, dx \leqslant C \int_{|x| > M} |\bar{K} * |\tilde{u}_n^1| |^q \, dx.$$

Translating if necesary $\tilde{u}_n$, we may assume $y_n = 0$. Then $\tilde{u}_n^1$ has its support in a fixed ball B_{R_0} and we deduce as in section 2.1 that the above integral is bounded by $\delta(M) \to 0$ as $M \to \infty$ (ind. of n). Next we consider:

$$\int_{|x|<M} |\bar{K} * |\tilde{u}_n^1|| \, |\bar{K} * |\tilde{u}_n^2||^{q-1} \, dx =$$
$$= \int_{|x|<M} \Big| \int_{|x-y| \leq R_0+M} \bar{K}(x-y) |\tilde{u}_n^1(y)| \, dy \cdot$$
$$\cdot \Big| \int_{|y| \geq R_n} \bar{K}(x-y) |\tilde{u}_n^2(y) \, dy|^{q-1} \, dx \Big| \geqslant$$
$$\leqslant R_n^{-\epsilon} \int_{|x|<M} z_n(x) \Big| \int \bar{K}(x-y) |x-y|^{\epsilon} |\tilde{u}_n^2(y)| dy \Big|^{q-1} dx$$

where $0 < \epsilon < \lambda$, $z_n = (\bar{K} \cdot \chi_{\{|x| \leqslant R_0 - M\}}) * |\tilde{u}_n^1|$: z_n is bounded in $L^1 \cap L^q$. We conclude observing that

$$|(\bar{K}|x|^{\epsilon}) * |\tilde{u}_n^2||^{q-1} \quad \text{is bounded in} \quad L^{r_\epsilon}, \quad \text{with} \quad r_\epsilon = q_\epsilon/(q-1)$$

and q_ϵ is given by: $\frac{1}{p} + \frac{\lambda - \epsilon}{N} = 1 + \frac{1}{q_\epsilon}$ (choose ϵ small enough), hence $q_\epsilon > q$, $r_\epsilon > q'$ and $r'_\epsilon \in [1, q]$.

Therefore dichotomy does not occur and we conclude: there exists $(y_n)_n$ in $\mathbb{R}^N$ such that $|\tilde{u}_n(\cdot + y_n)|^p$ is tight. We still denote by $\tilde{u}_n$ this translated sequence. We may assume that $\tilde{u}_n$ converges weakly to $\tilde{u} \in L^p(\mathbb{R}^N)$, that $\sigma_n \to \sigma \in [0, \infty]$. We denote by $C = -I$, $\bar{C} = -\bar{I}$ where $\bar{I}$ corresponds to (2.3) with $K = \alpha\bar{K}$, $K^\infty = \beta\bar{K}$ if $\sigma = 0$, $= \alpha\bar{K}$ if $\sigma = +\infty$, $= \varphi(\sigma x)\bar{K}$ if $\sigma \in]0, \infty[$.

Step 2. *A variation of Lemma* 2.1.

Lemma 2.2. *Let $\tilde{u}_n$ converge weakly in $L^p(\mathbb{R}^N)$ to $\tilde{u}$ and assume $|\tilde{u}_n|^p$ is tight. We may assume that $|K_n * \tilde{u}|^q$, $|\tilde{u}|^p$ converge weakly to some measures ν, μ. Then part* i) *of Lemma* 2.1 *holds with K replaced by K^∞ in* (2.5); *and we have:*

$$\begin{cases} \mu \geqslant |\tilde{u}|^p + \sum_{j \in J} (\nu_j/\bar{C})^{p/q} \delta_{x_j} & \text{if} \quad \sigma \in]0, \infty[\\ \mu \geqslant |\tilde{u}|^p + \sum_{j \in J} (\nu_j/C)^{p/q} \delta_{x_j} & \text{if} \quad \sigma = 0. \end{cases} \tag{2.18}$$

And if $\tilde{u} = 0$ and $C_1 \mu(\mathbb{R}^N)^{q/p} \leqslant \nu(\mathbb{R}^N)$ with $C_1 = \bar{C}$ if $\sigma \in]0, \infty]$, $C_1 = C$ if $\sigma = 0$; then J is a singleton and $\nu = c_0 \delta_{x_0}$, $\mu = (c_0/C_1)^{p/q} \delta_{x_0}$ for some $x_0 \in \mathbb{R}^N$, $c_0 > 0$.

PROOF. The proof is very similar to the one of Lemma 2.1, and we will only sketch it. Since $|K_n| \leqslant C|x|^{-\lambda}$; it is clear that $(|K_n * \tilde{u}_n|^q)_n$ is tight. In addition $(K_n * \tilde{u}_n) \to K^\infty * \tilde{u}$ a.e. in $\mathbb{R}^N$, and $K_n * \tilde{u} - K^\infty * \tilde{u} \to 0$ in $L^q(\mathbb{R}^N)$. Furthermore if $\sigma > 0$, $(K_n - K^\infty) * \tilde{u} \to 0$ in $L^q(\mathbb{R}^N)$ and this proves the above result if $\sigma > 0$. In the case when $\sigma = 0$, we just go through the proof of Lemma 2.1 and we find if $u \equiv 0$:

$$\Big(\int_{\mathbb{R}^N} |\xi|^q \, d\nu\Big) \leqslant C\Big(\int_{\mathbb{R}^N} |\xi|^p \, d\mu\Big)^{q/p}, \qquad \forall \xi \in \mathcal{D}(\mathbb{R}^N).$$

And this reverse Hölder inequality allows us to conclude.

Step 3. *$\tilde{u}_n$ is compact in $L^p(\mathbb{R}^N)$.*

If $\tilde{u} \equiv 0$, by Lemma 2.2 since $\nu(\mathbb{R}^N) = -I = C$, $\mu(\mathbb{R}^N) = 1$, $\nu = C\delta_{x_0}$, $\mu = C\delta_{x_0}$ for some $x_0 \in \mathbb{R}^N$. But this contradicts (2.4). Hence $\tilde{u} \not\equiv 0$; now let $\theta = \int_{\mathbb{R}^N} |\tilde{u}|^p \, dx$. If $\theta \in]0, 1[$ we argue as follows: first of all if $\sigma = -\infty$, then by (2.18) and (2.5):

$$\begin{cases} 1 \geqslant \theta + \sum_{j \in J} \mu_j \quad \text{with} \quad \mu_j = (\nu_j/\bar{C})^{p/q} \\ I = I_1 \geqslant \bar{I}_\theta + \sum_{j \in J} \bar{I}_{\mu_j} \geqslant I_\theta + I^\infty_{1-\theta} \end{cases}$$

and we reach a contradiction in view of (2.15).

On the other hand if $\sigma \in]0, \infty[$, still by (2.18) and (2.5):

$$\begin{cases} 1 \geqslant \theta + \sum_{j \in J} \mu_j \\ I = I_1 \geqslant I_\theta + \sum_{j \in J} \bar{I}_{\mu_j} \geqslant I_\theta + I^\infty_{1-\theta} \end{cases}$$

and again we conclude.

Finally if $\sigma = 0$, we again use (2.18) and (2.5):

$$\begin{cases} 1 \geqslant \theta + \sum_{j \in J} \mu_j \\ I = I_1 \geqslant I^\infty_\theta + I_{1-\theta}; \end{cases}$$

and we conclude: $\theta = 1$ i.e. $\tilde{u}_n$ converges to $\tilde{u}$ in $L^p(\mathbb{R}^N)$.

Step 4. *Conclusión.*

If we had $\sigma = +\infty$, then $(K_n - \alpha\bar{K}) * \tilde{u} \to 0$ in $L^q(\mathbb{R}^N)$; indeed $|\tilde{u}_n|^p$ and $\big||x|^{-\lambda} * \tilde{u}_n\big|^q$ are tight hence we may restrict the integrals on $|x - y| \leqslant K$. But $\varphi\left(\frac{x-y}{\sigma_n}\right)$ converges uniformly to α if $\sigma_n \to +\infty$ on such a set and we conclude. Now this would imply:

$$I = \lim_n \int |K_n * \tilde{u}_n|^q \, dx \geqslant \bar{I} \geqslant I^\infty;$$

and if (2.16) holds this is not possible.

On the other hand if we had $\sigma = 0$, then we claim that

$$(K_n - \beta\bar{K}) * \tilde{u}_n \to 0 \quad \text{in} \quad L^q(\mathbb{R}^N)$$

And again (2.16) would rule out this possibility. To prove the claim we just have to prove for any $R < \infty$ that

$$\int_{|x| \leqslant R} \left| \int_{|y| \leqslant R} \left| \varphi\left(\frac{x-y}{\sigma_n}\right) - \beta \right| \bar{K}(x-y)|\tilde{u}_n(y)| \, dy \right|^q dx \to 0.$$

Taking subsequences if necessary, we may assume that $|\tilde{u}_n| \leqslant \bar{u}$ which belongs to $L^p(\mathbb{R}^N)$, and thus the above integral is estimated by:

$$C \int_{|x| \leqslant R} \left| \int_{|y| \leqslant R} \chi_{\{|x-y| \leqslant \delta\}} \cdot |x-y|^{-\lambda} \bar{u}(y)\, dy \right|^q dx + \epsilon_n^\delta$$

where $\epsilon_n^\delta \to 0$, for any fixed $\delta > 0$. And we conclude since the first integral converges to 0 as $\delta \to 0_+$.

Therefore $\sigma \in]0, +\infty[$, but this is equivalent to the compactness of u_n.

We have actually proved the:

Corollary 2.1. *We assume* (2.2), (2.11), (2.12) *and we denote by* I_1^∞, I_2^∞ *the infima given by* (2.3) *where K is repalced by* $\alpha\bar{K}$, $\beta\bar{K}$. *Let* $(u_n)_n$ *be a minimizing sequence of* (2.3), *then there exist* $(\sigma_n)_n$ *in* $]0, \infty[$, $(y_n)_n$ *in* $\mathbb{R}^N$ *such that* $\tilde{u}_n(\cdot) = \sigma_n^{-N/p} u_n((\cdot - y_n)/\sigma_n)$ *is relatively compact in* $L^p(\mathbb{R}^N)$. *In addition, if* $I = I_1^\infty < I_2^\infty$, *all limit points of* (σ_n) *lie in* $]0, -\infty]$, *and there exists* $(u_n)_n$ *such that* $\sigma_n \to +\infty$; *while if* $I = I_2^\infty < I_1^\infty$, *all limit points of* (σ_n) *lie in* $[0, \infty[$ *and there exists* $(u_n)_n$ *such that* $\sigma_n \to 0$. *Finally if* $I = I_1^\infty = I_2^\infty$ *both cases occur.*

2.3 *Trace inequalities*

We first recall the well-known trace theorems (see for example Amdas [1]): let $u \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$ with $p \in [1, N/m[$, m integer $\geqslant 1$ (for example!), $N \geqslant 2$, then there exists a bounded linear operator γu mapping $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^{N-1})$ —where q is given by: $q = (N-1)p(N-mp)^{-1}$— such that if u es smooth, then γu is the restriction of u on $\mathbb{R}^{N-1} \times \{0\}$. For obvious reasons we will still denote by u the trace operator γu.

The minimization problem associated with the question of the attainability of the norm of γ is of course:

$$I = \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} |D^m u|^p \, dx / u \in \mathfrak{D}^{m,p}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^{N-1}} |u(x', 0)|^q \, dx' = 1 \right\} \quad (2.19)$$

here $\left(\int |D^m u|^p\right)^{1/p}$ is just any norm on $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ which is «scale invariant» like for example:

$$\left(\sum_{|\alpha| = \alpha = m} \|D^\alpha u\|_{L^p}^r \right)^{1/r} \text{(for any } r \in [1, \infty],$$

$\|\Delta^{m/2} u\|_{L^p}$ if m is even, $\|\nabla(\Delta^k u)\|_{L^p}$ if m is odd...

It is clear that both functionals are invariant under the change $\left\{ u \to \sigma^{-(N-1)/q} u\left(\frac{\cdot}{\sigma}\right) \right\}$ for any $\sigma > 0$; and that if I_λ denotes the infimum given by (2.19) where 1 is replaced by λ: $I_\lambda = \lambda^{p/q} I_1 = \lambda^{p/q} I$; and thus (S.2) holds.

Theorem 2.3. *Let $(u_n)_n$ be a minimizing sequence of* (2.19), *then there exist* $(\sigma_n)_n$ *in* $]0, \infty[$, $(y'_n)_n$ *in* $\mathbb{R}^{N-1}$ *such that the new minimizing sequence* $\tilde{u}_n$ *given by*:

$$\tilde{u}_n(x', x_N) = \sigma_n^{-(N-1)/q} u_n((x' - y'_n)/\sigma_n, x_N/\sigma_n), \qquad \forall x' \in \mathbb{R}^{N-1}, \quad \forall x_N \in \mathbb{R}$$

is relatively compact in $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ (*). *In particular* (2.19) *has a minimum.*

Remark 2.12. Just as in section 1.4 the above result admits many variants like: m non integer, Korn-trace inequalities, convolution-trace inequalities, «time-dependent» spaces, limit cases ($mp = N$)... Let us also mention the following extension of Theorem 2.3, we may instead consider the trace of u on $\mathbb{R}^k$ for $1 \leqslant k \leqslant N-1$ (i.e. on $\mathbb{R}^k \times \{0\}$) then $q = kp(N - mp)^{-1}$ and the above result still holds with $y'_n \in \mathbb{R}^k$ (provided $q > q$ i.e. $p > (N-k)/m$.

Remark 2.13. If $m = 1$ and $|Du|$ is the usual norm on $\mathbb{R}^N$, then if u is minimum of (2.19), the Steiner symmetrization of u —that we denote by u^*— is still a minimum of (2.19): u^* is spherically symmetric in $x' \in \mathbb{R}^{N-1}$, non increasing with respect to $|x'|$, and even in x_N, non increasing for $x_N \geqslant 0$.

Remark 2.14. We could of course replace $W^{m,p}(\mathbb{R}^N)$ by $W^{m,p}(Q)$ where $Q = \mathbb{R}^{N-1} x\,]0, \infty[$, then Theorem 2.3 still holds. If $m = 1$ the corresponding value of the infimum $\tilde{I}$ is given by:

$$\tilde{I} = \tfrac{1}{2} I_2 = 2^{p/q-1} I.$$

PROOF. We are going to apply the concentration compactness method to the bounded measures $(P_n)_n$:

$$P_n = \sum_{j=0}^{m} |D^j u_n|^{q_j} + |u_n|^q (x', 0) \otimes \delta_0(x_N)$$

where $q_j = Np/(N - (m-j)p)$, and where $(u_n)_n$ is a miminizing sequence. Hence, we consider:

$$Q_n(t) = \sup_{y \in \mathbb{R}^N} P_n(y + B_t), \qquad \forall t \geqslant 0.$$

Remarking that if we replace u_n by $\sigma^{-(N-1)/q} u_n\left(\frac{\cdot}{\sigma}\right)$, $Q_n(t)$ is replaced by $Q_n\left(\frac{t}{\sigma}\right)$, we may always assume choosing $\sigma = \sigma_n$ conveniently:

$$Q_n(1) = \tfrac{1}{2}.$$

Such a choice prevents vanishing from occuring while, as usual, dichotomy does not occur (cf. sections 1, 2.1-2). Therefore there exists $y_n = (y'_n, y''_n) \in$

$\in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $P_n(\cdot + y_n)$ is tight i.e.;

$$\forall \epsilon > 0, \quad \exists R < \infty, \quad \forall n \geqslant 1, \qquad P_n(\mathbb{R}^N - (y_n + B_R)) \leqslant \epsilon \tag{20}$$

We next claim that we may choose $y_n'' = 0$; indeed if $\epsilon < 1$ then $|y_n''| \leqslant R$ since if $|y_n''| > R$

$$y_n + \bar{B}_R \subset \mathbb{R}^{N-1} \times \mathbb{R}^*, \quad \text{thus} \quad \int_{x_N=0} |u_n|^q \, dx \leqslant \epsilon$$

and this contradicts the constraint. Therefore taking $\tilde{y}_n = (y_n', 0)$, (2.20) still holds if we replace R by $2R$; and we may thus assume $y_n'' = 0$.

The remainder of the proof is then an easy adaptation of arguments given in the sections above in view of the

Lemma 2.3. *Let $(u_n)_n$ be a bounded sequence in $\mathcal{D}^{m,p}(\mathbb{R}^N)$ such that $(|D^m u_n|^p)$ is tight. We may assume u_n converges a.e. to $u \in \mathcal{D}^{m,p}$, $|D^m u_n|^p$, $|u_n|^q(x', 0) \otimes \delta_0(x_N)$ converges weakly to some bounded, nonnegative measures on $\mathbb{R}^N$ μ, ν —and* $\operatorname{Supp} \nu \subset \{x_N = 0\}$.

i) *Then we have for some at most countable family J, for some families $(x_j)_{j \in J}$ fo distinct points in $\mathbb{R}^{N-1} \times \{0\}$, $(\nu_j)_{j \in J}$ in $]0, \infty[$*

$$\nu = |u|^q(x', 0) \otimes \delta_0(x_N) + \sum_{j \in J} \nu_j \delta_{x_j} \tag{2.21}$$

$$\mu \geqslant |D^m u|^p + \sum_{j \in J} I \nu_j^{p/q} \delta_{x_j} \tag{2.22}$$

ii) *If $u \equiv 0$ and $\mu(\mathbb{R}^N) \leqslant I \nu(\mathbb{R}^N)^{p/q}$ then $J = \{x_0\}$ for some $x_0 \in \mathbb{R}^{N-1} \times \{0\}$ and $\nu = c_0 \delta_{x_0}$, $\mu = I c_0^{p/q} \delta_{x_0}$ for some $c_0 > 0$.*

We skip the proof of this lemma which is totally similar to the one of Lemma 1.1 (or Lemma 2.1).

2.4. *Singular inequalities*

Let us first recall the following inequality

$$\int_{\mathbb{R}^N} |u|^p |x|^{-p} \, dx \leqslant C \int_{\mathbb{R}^N} |\nabla u|^p \, dx \tag{2.23}$$

for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, with $1 \leqslant p < N$ —this inequality is easily proved by the use of Schwarz symmetrization and standard one dimensional inequalities

$$\int_0^\infty |u|^p t^\beta \, dt \leqslant C(p, \beta) \int_0^\infty |u'|^p t^{\beta + p} \, dt \tag{2.24}$$

for $1 \leqslant p < \infty$, $\beta \in \mathbb{R}$, $u \in \mathcal{D}(]0, \infty[)$.

We next want to observe that there exists a general class of inequalities like (2.23) namely

$$\int_{\mathbb{R}^N} |u|^p |x|^{-mp}\, dx \leqslant C \int_{\mathbb{R}^N} |D^m u|^p\, dx, \qquad \forall u \in \mathfrak{D}^{m,p}(\mathbb{R}^N) \tag{2.25}$$

where $m \geqslant 1$, $1 \leqslant p < (N/m)$, and $\left(\int_{\mathbb{R}^N} |D^m u|^p\, dx\right)^{1/p}$ is any norm on $\mathfrak{D}^{m,p}$ which is «scale-invariant». In particular to prove (2.25), we will choose the norm

$$\|(-\Delta)^{m/2} u\|_{L^p(\mathbb{R}^N)} \text{ if } m \text{ is even}, \quad \|\nabla(-\Delta)^{(m-1)/2} u\|_{L^p(\mathbb{R}^N)} \text{ if } m \text{ is odd}.$$

By density we may consider only $u \in \mathfrak{D}(\mathbb{R}^N - \{0\})$. We then observe that if $f = (-\Delta)^{m/2} u$ or $(-\Delta)^{(m-1)/2} u$ depending on the parity of m, and if we denote by v $(\in \mathfrak{D}^{m,p}(\mathbb{R}^N))$ the solution of

$$(-\Delta)^k v = f^* \quad \text{in} \quad \mathbb{R}^N \quad \left(k = \tfrac{m}{2} \text{ if } m \text{ is even, } k = \tfrac{m-1}{2} \text{ if } m \text{ is odd}\right)$$

where φ^* denotes the Schwarz symmetrization of φ, then by Talenti comparizon theorem [77]: $u^* \leqslant v$ a.e. on $\mathbb{R}^N$, and thus

$$\|(-\Delta)^{m/2} u\|_{L^p} = \|(-\Delta)^{m/2} v\|_{L^p} \quad \text{if } m \text{ is even}$$

$$\|\nabla(-\Delta)^{(m-1)/2} u\|_{L^p} = \|\nabla f\|_{L^p} \geqslant \|\nabla f^*\|_{L^p} = \|\nabla(-\Delta)^{(m-1)/2} v\|_{L^p} \quad \text{if } m \text{ is odd}$$

$$\int_{\mathbb{R}^N} |u|^p |x|^{-mp}\, dx \leqslant \int_{\mathbb{R}^N} |u^*|^p |x|^{-mp}\, dx \leqslant \int_{\mathbb{R}^N} |v|^p |x|^{-mp}\, dx$$

and thus it is enough to prove (2.25) for spherically symmetric functions.

Now for spherically symmetric functions v we may assume by density that $v \in \mathfrak{D}(\mathbb{R}^N - \{0\})$ and we remark using (2.24)

$$\int_{\mathbb{R}^N} |v|^p |x|^{\beta}\, dx \leqslant C(p, \beta) \int_{\mathbb{R}^N} |\nabla v|^p |x|^{\beta + p}\, dx$$

for v spherically symmetric, $\in \mathfrak{D}(\mathbb{R}^N - \{0\})$, $\beta \in \mathbb{R}$, $p \in [1, \infty[$. Then we obtain

$$\int_{\mathbb{R}^N} |v|^p |x|^{-pm}\, dx \leqslant C_1 \int_{\mathbb{R}^N} |Dv|^p |x|^{-p(m-1)}\, dx \leqslant$$
$$\leqslant C_2 \int_{\mathbb{R}^N} |D^2 v|^p |x|^{-p(m-2)}\, dx \leqslant C \int_{\mathbb{R}^N} |D^m v|^p\, dx$$

and (2.25) holds. Another proof (communicated to us by H. Brézis) uses Lorentz spaces: if $u \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$ then $u \in L^{q,p}(\mathbb{R}^N)$ and thus $|u|^p \in L^{q/p,1}(\mathbb{R}^N)$ while $|x|^{-mp} \in L^{N/(mp),\infty}(\mathbb{R}^N)$. This proves the claim since $(q/p)' = N/(mp)$.

In addition if we combine (2.25) with Hölder and Sobolev inequalities we find

$$\left(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^r}\,dx\right)^{1/q} \leqslant C_0\left(\int_{\mathbb{R}^N} |D^m u|^p\,dx\right)^{1/p}, \qquad \forall u \in \mathcal{D}^{m,p}(\mathbb{R}^N) \tag{2.26}$$

where $p < q < Np/(N-mp)$, $1 \leqslant p < N/m$, $m \geqslant 1$; and r is given by

$$\frac{N-r}{q} = \frac{N-mp}{p} \quad \text{or} \quad r = N - q(N-mp)/p. \tag{2.27}$$

The associated minimization problem is then

$$I = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} |D^m u|^p\,dx / u \in \mathcal{D}^{m,p}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^q |x|^{-r}\,dx = 1\right\} \tag{2.28}$$

Observe that this minimization problem is *not* invariant by translations and is invariant by dilations or more precisely by the change

$$u \to \sigma^{-(N-r)/q} u\left(\frac{\cdot}{\sigma}\right), \qquad \forall \sigma > 0$$

Let us also remark that if I_λ denotes the infimum corresponding to the constraint where 1 is replaced by $\lambda > 0$

$$I_\lambda = \lambda^{p/q} I_1 = \lambda^{p/q} I, \quad \forall \lambda > 0$$

and thus (S.2) holds.

Theorem 2.4. *Any minimizing sequence* $(u_n)_n$ *of* (2.28) *is relatively compact in* $\mathcal{D}^{m,p}(\mathbb{R}^N)$ *up to a dilation i.e. there exists* $(\sigma_n)_n$ *in* $]0, \infty[$ *such that the new minimizing sequence* $\tilde{u}_n(\cdot) = \sigma_n^{-(N-r)/q} u_n\left(\frac{\cdot}{\sigma_n}\right)$ *is relatively compact in* $\mathcal{D}^{m,p}(\mathbb{R}^N)$. *In particular there exists a minimum in* (2.28).

Remark 2.15. Exactly as in section 1.4, Remark 2.12, there are many variants and extensions of the above inequalities and results in particular we may replace $|x|^{-r}$ by various potentials K satisfying for example

$$\lim_{x\to 0} K(x)|x|^{-r} = \alpha \geqslant 0, \qquad \lim_{|x|\to\infty} K(x)|x|^{r} = \beta > 0.$$

Remark 2.16. If $m = 1$, by a symmetrization argument and an O.D.E. analysis one may compute explicitely the expression of I and of any minimum. The existence of a minimum and these explicit expressions are given in Glaser, Martin, Grosse and Thirring [38], E.H. Lieb [53].

Remark 2.17. Clearly if $p = q$, $I_\lambda = \lambda I$ and (S.2) fails; and neither does our method continue to apply, but also —at least if $m = 1$— there does not exist a minimum of (2.28).

PROOF OF THEOREM 2.4. Again the proof follows the general scheme of our method: if $(u_n)_n$ is a minimizing sequence and if

$$\rho_n = \sum_{j=0}^{m} |D^j u_n|^{p_j}$$

where $p_j = Np/(N - (m - j)p)$, we may choose $\sigma_n > 0$ such that, if we still denote by $(u_n)_n$ the new minimizing sequence $[\sigma_n^{-(N-r)/q} u_n(\cdot/\sigma_n)]$, we have

$$Q_n(1) = \frac{I}{2}, \quad \text{with} \quad Q_n(t) = \operatorname*{Sup}_{y \in \mathbb{R}^N} \int_{y + B_t} \rho_n \, dx, \qquad \forall t \geqslant 0.$$

Since (S.2) holds we prove easily that u_n is tight up to a translation i.e. there exists $(y_n)_n$ in $\mathbb{R}^N$ such that

$$\forall \epsilon > 0, \qquad \exists R < \infty, \qquad \int_{|x - y_n| \geq R} \rho_n(\lambda) \, dx \leqslant \epsilon.$$

We claim that $(y_n)_n$ remains bounded and we argue by contradiction: $|y_n|$ (or a subsequence) goes to $+\infty$ as $n \to \infty$. Then let $\xi \in \mathfrak{D}_+(\mathbb{R}^N)$, $\xi \equiv 1$ on B_1, $0 \leqslant \xi \leqslant 1$, Supp $\xi \subset B_2$ and let us denote by $\xi_n = \xi((\cdot - y_n)/R)$. The above inequality easily yields

$$\int_{\mathbb{R}^N} |D^m(u_n - v_n)|^p \, dx \leqslant \delta(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0$$

where $v_n = \xi_n u_n$. Therefore for ϵ small enough

$$\int_{\mathbb{R}^N} |v_n|^q |x|^{-r} \, dx \geqslant \tfrac{1}{2}.$$

On the other hand

$$\int_{\mathbb{R}^N} |v_n|^q |x|^r \, dx \leqslant \int_{|x - y_n| \leqslant 2R} |u_n|^q |x|^{-r} \, dx \leqslant C(|y_n| - 2R)^{-r}, \quad \text{for } n \text{ large}$$

and we reach a contradiction which proves our claim. Hence $(y_n)_n$ is bounded and we may as well take $y_n = 0$.

The remainder of the proof is then a repetition of arguments made above and in Part 1 [65] in view of the following lemma —which is also proved by similar methods as before.

Lemma 2.4. *Let $(u_n)_n$ be a bounded sequence in $\mathfrak{D}^{m,p}(\mathbb{R}^N)$ such that $|D^m u_n|^p$ is tight. We may assume that u_n converges a.e. to $u \in \mathfrak{D}^{m,p}(\mathbb{R}^N)$ and that $|D^m u_n|^p$, $|u_n|^q |x|^{-r}$ converge weakly to some measures μ, ν. Then we have:*

i) $$\nu = |u|^q |x|^{-r} + \nu_0 \delta_0 \quad \textit{with} \quad \nu_0 \geqslant 0;$$

ii) $$\mu \geqslant |D^m u^p| + I \nu_0^{p/q} \delta_0$$

Remark 2.15. If $|u_n|^{q^*}$, $|u_n|^p |x|^{-mp}$ converge weakly to some measures ν^*, ν^0 where $q^* = Np(N - mp)$, we have

$$\begin{cases} \nu^0 = |u|^p|x|^{-mp} + \nu_1\delta_0 \\ \nu^* = |u|^{q^*} + \sum_{j\in J} \nu_j\delta_{x_j} \quad \text{and if} \quad \nu_0 > 0, \quad 0 \in \{x_j/j\in J\} \\ \nu^*(0)^{(1-\theta)/q^*}\nu_1^{\theta/mp} \geqslant \nu_0 \end{cases}$$

Remark 2.16. The fact that only δ_0 occurs is clear: since u_n is bounded in $L^{q^*}(\mathbb{R}^N)$, $|u_n|^q|x|^{-r}$ is bounded in $L^{\alpha}_{\text{loc}}(\mathbb{R}^N - \{0\})$ for some $\alpha > 1$ (and part i) above is obvious!).

2.5. *Nonlinear problems in unbounded domains*

We want to give in this section a few examples of nonlinear problems in unbounded domains which possess a variational structure and that we treat by our concentration-compactness method.

We begin with a model problem namely the Yamabe equation in infinite streps: let $N \geqslant 1$, $\Omega = 0 \times \mathbb{R}^p$ where 0 es a bounded domain in $\mathbb{R}^m$ and $m + p = N$. We consider positive, nontrivial solutions (vanishing at infinity) of

$$-\Delta u - \lambda u = u^{\frac{N+2}{N-2}} \quad \text{in} \quad \Omega,\ u > 0 \quad \text{in} \quad \Omega,\ u \in H_0^1(\Omega) \tag{2.29}$$

where $\lambda > 0$. This problem —somewhat related to the Yamabe problem— was investigated by H. Brézis and L. Nirenberg [23] in the case when Ω is bounded— see also sections 4.1-2 below.

In view of the homogeneity of the nonlinearity, we obtain a solution of (2.29) if we solve the following minimization problem

$$I = \operatorname{Inf}\left\{\int_\Omega |\nabla u|^2 - \lambda u^2\, dx\Big/\int_\Omega |u|^{\frac{2N}{N-2}} dx = 1\right\}, \quad u \in H_0^1(\Omega) \tag{2.30}$$

and we denote by

$$I^\infty = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} |\nabla u|^2\, dx\Big/\int_{\mathbb{R}^N} |u|^{2N/(N-2)} dx = 1\right\}\cdot$$

We denote by λ_1 the first eigenvalue of $-\Delta$ in $H_0^1(0)$ (λ_1 is also the infimum of the spectrum of $-\Delta$ in $H_0^1(\Omega)$). The methods of Part 1 and the sections above immediately yield:

Theorem 2.5. *For any minimizing sequence* $(u_n)_n$ *of* (2.30), *there exists* $(y_n)_n \subset \{0\} \times \mathbb{R}^p$ *such that* $(u_n(\cdot + y_n))_n$ *is relatively compact in* $H_0^1(\Omega)$ *if and only if* (2.16) *holds*

$$I < I^\infty \tag{2.16}$$

In particular if (2.16) *holds, there exists a minimum of* (2.30) *and a solution of* (2.29). *In addition* (2.16) *holds if* $N \geqslant 4$ *or if* $N = 3$ *and* $\lambda \in]\bar{\lambda}_1, \lambda_1[$ *where* $\bar{\lambda}_1 \in [0, \lambda_1[$.

Remark 2.17. The result —as long as the existence of a minimum and (2.16) are concerned— is very much the same as in *H.* Brézis and L. Nirenberg [23]. And the quick discussion of (2.16) we mention above is deduced from [23]: indeed if B_R is a ball in $\mathbb{R}^p$ of radius R we have

$$I \leqslant I_R = \operatorname{Inf}\left\{\int_{OxB_R} |\nabla u|^2 - \lambda u^2\, dx \Big/ \int_{OxB_R} |u|^{\frac{2N}{N-2}} dx = 1, \quad u \in H_0^1(OxB_R)\right\}$$

and in [23] it is proved that: $I_R < I^\infty$ if $N \geqslant 4$, $I_R < I^\infty$ if $N = 3$ and $\lambda \in]\bar{\lambda}_1^R, \lambda_1[$ for some $\bar{\lambda}_1^R$. Clearly $\bar{\lambda}_1^R \downarrow \bar{\lambda}_1$ as $R \uparrow +\infty$ and we do not know if $\bar{\lambda}_1 > 0$ or $\bar{\lambda}_1 = 0$.

Remark 2.18. The above problem and result is only an example of our method: we could as well treat general minimization problems (combining the methods of P.L. Lions [55], [56] and of Part 1 [65]) such as

$$I = \operatorname{Inf}\left\{\int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + c(x) u^2\, dx \Big/ \int_\Omega F(x, u)\, dx = 1\right\}$$

where (a_{ij}) is uniformly elliptic and a_{ij}, c, $F(x, t)$ satisfy various assumptions and where Ω is an arbitrary unbounded domain (strip, halfspace, exterior domain...). In particular this could allow us to study the Yamabe equation

$$\begin{cases} -\sum_{i,j} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x) \dfrac{\partial u}{\partial x_j}\right) + c(x) u = K(x) u^{\frac{N+2}{N-2}} & \text{in } \Omega \\ u \in \mathfrak{D}^{1,2}(\Omega), \quad u > 0 \quad \text{in} \quad \Omega,\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Remark 2.19. Concerning semilinear equations in infinite strips

$$-\Delta u = f(u) \quad \text{in} \quad \Omega, \qquad u > 0 \quad \text{in} \quad \Omega, \qquad u \in H_0^1(\Omega)$$

where $\Omega = \mathfrak{D} \times \mathbb{R}^p$, O bounded domain in $\mathbb{R}^m$, $N \geqslant 3$ and $f \in C^1(\mathbb{R})$, $f(0) = 0$, $f'(0) > -\lambda_1$.

Such problems have been studied in M. J. Esteban [34]; C. J. Amick and J. F. G. Toland [3]; J. Bona, D. K. Bose and R. E. L. Turner [15]; P. L. Lions [56] in the «locally compact» case. If we assume, for instance, that f is odd and

$$0 \leqslant \frac{f(t)}{t} - f'(0) \leqslant \theta\{f'(t) - f'(0)\}, \quad \forall t \in \mathbb{R}, \tag{2.31}$$

for some $\theta \in]0, 1[$;

$$\lim_{|t| \to \infty} f(t)|t|^{-\frac{N+2}{N-2}} = \alpha \geqslant 0; \tag{2.32}$$

then the above problem will be solved if we find a minimum of

$$I = \operatorname{Inf}\{\mathcal{E}(u), u \in H_0^1(\Omega), \quad u \not\equiv 0, \quad J(u) = 0\} \tag{2.33}$$

where

$$\mathcal{E}(u) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 - F(u)\,dx, \qquad J(u) = \int_\Omega |\nabla u|^2 - f(u)u\,dx,$$

and

$$F(t) = \int_0^t f(s)\,ds.$$

To this end we introduce

$$I^\infty = \operatorname{Inf}\{\mathcal{E}^\infty(u), \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u \not\equiv 0, \quad J^\infty(u) = 0\}$$

with

$$\mathcal{E}^\infty(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - \alpha\frac{N-2}{2N}\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}dx,$$

$$J^\infty = \int_{\mathbb{R}^N}|\nabla u|^2 - \alpha|u|^{\frac{2N}{N-2}}dx \qquad (\text{if } \alpha = 0, \quad I^\infty = +\infty).$$

If $\alpha > 0$, by an easy scaling argument I^∞ is also given by

$$I^\infty = I_0^{N/2}\alpha^{-(N-2)/2}\frac{N+2}{2N},$$

$$I_0 = \operatorname{Inf}\left\{\int_{\mathbb{R}^N}|\nabla u|^2dx\Big/\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}dx = 1\right\}\cdot$$

Then any minimizing sequence of (2.33) converges up to a translation (of the form $(0, \bar{y}_n)$) if and only if $I < I^\infty$.

Sketch of the proof of Theorem 2.5. We apply the general scheme of proof we used before: in particular we use the first concentration compactness lemma ([58], [55]) with the density

$$\rho_n = |\nabla u_n|^2 + u_n^2 + |u_n|^{\frac{2N}{N-2}}.$$

And we just have to explain how we avoid i) vanishing of ρ_n, ii) that the weak limit u of u_n is not trivial if μ_n is tight.

First, if ρ_n vanishes i.e. if

$$\sup_{y\in\mathbb{R}^N}\int_{y+B_R}\rho_n\,dx \to 0, \qquad \forall R < \infty;$$

where ρ_n is defined on $\mathbb{R}^N$ by extending u_n by 0—then we know (cf. P. L. Lions

[55], [56]) that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < \frac{2N}{N-2}$. Thus for all $\delta > 0$, $|u_n| \quad \delta \to 0$ 0 in $L^p(\mathbb{R}^N)$ for $2 < p < \infty$. Let $v_n = (|u_n| - \delta)^+$, we have

$$\begin{cases} \operatorname{meas}\{v_n > 0\} = \operatorname{meas}\{|u_n| > \delta\} \leqslant \dfrac{1}{\delta^2} \displaystyle\int_{\mathbb{R}^N} |u_n|^2 \, dx \leqslant \dfrac{C}{\delta^2} \\ \displaystyle\int v_n^2 \, dx \leqslant C \left(\int v_n^p \, dx \right)^{2/p} \quad \text{for all} \quad p > 2 \end{cases}$$

and thus $v_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 \leqslant p < \frac{2N}{N-2}$. Therefore

$$\begin{aligned} I &= \varliminf_n \int |\nabla u_n|^2 - \lambda u_n^2 \, dx \geqslant \\ &\geqslant \varliminf_n \int_{\mathbb{R}^N} |\nabla v_n|^2 + \varliminf_n \int_{\mathbb{R}^N} |\nabla w_n|^2 - w_n^2 \, dx \geqslant \\ &\geqslant \varliminf_n \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx, \qquad \text{where } w_n = |u_n| \wedge \delta \end{aligned}$$

and $\int_{\mathbb{R}^N} |v_n|^{\frac{2N}{N-2}} dx \to 1$. Hence $I \geqslant I^\infty$ and this contradicts (2.16).

Next, if ρ_n is tight and if u_n converges weakly and a.e. to some $u \in H_0^1(\Omega)$, we want to check that $u \not\equiv 0$. But if $u \equiv 0$, since u_n^2 is tight, $u_n \to 0$ in $L^2(\Omega)$ thus

$$I = \varliminf_n \int |\nabla u_n|^2 \, dx \geqslant I^\infty$$

and again this contradicts (16).

The remainder of the proof consists then of straightforward adaptations of previous arguments.

We now turn to a nonlinear problem involving nonlinear boundary conditions: this problem —in the locally compact case-was investigated in P. L. Lions [56] and we refer to [56] for various considerations on its solutions— may be formulated as follows

$$I_\lambda = \operatorname{Inf}\left\{ \int_\Omega |\nabla u|^2 \, dx \Big/ \int_{\partial\Omega} F(u) \, ds = \lambda \right\} \tag{2.31}$$

where $\lambda > 0$, u belongs to $\mathfrak{D}^{1,2}(\Omega)$ (closure of $C^1_{\text{comp}}(\bar{\Omega})$ for the seminorm $|\nabla u|_{L^2(\Omega)}$), F is a given nonlinearity and Ω is and unbounded domain (smooth) of $\mathbb{R}^N$. To simplify the presentation only, we will consider two examples

$$\Omega = \{x_N > 0\} \tag{2.32}$$

$$\Omega = \mathbb{R}^N - \bar{O}, \text{ where } O \text{ is a smooth bounded open set in } \mathbb{R}^N. \tag{2.33}$$

We will assume that $N \geqslant 3$ and that F satisfies

$$F \in C(\mathbb{R}), \qquad F(0) = 0 \tag{2.34}$$

$$\lim_{|t| \to 0} F^+(t)|t|^{-q} = \alpha \geqslant 0 \tag{2.35}$$

$$\lim_{|t| \to \infty} F^+(t)|t|^{-q} = \beta \geqslant 0 \tag{2.36}$$

(if α or $\beta > 0$, we may replace F^+ by F) and where $q = \frac{2(N-1)}{N-2}$. We denote by

$$I_0 = \operatorname{Inf}\left\{ \int_{(x_N > 0)} |\nabla u|^2 \, dx / u \in \mathfrak{D}^{1,2}(x_N > 0), \int_{x_N = 0} |u|^q \, dx' = 1 \right\}.$$

(recall that this problem was solved in section 2.3).

Theorem 2.6. *If* Ω *is given by* (2.32), *we assume* (2.34), (2.35), (2.36) *and we denote by* $I_\lambda^\infty = (\max(\alpha, \beta)\lambda^{-1})^{-2/q} I_0$ ($= +\infty$ *if* $\alpha = \beta = 0$) *while if* Ω *is given by* (2.33), *we assume* (2.34), (2.36) *and we denote by* $I_\lambda^\infty = (\beta\lambda^{-1})^{-2/q} I_0$. *If* (2.32) *holds, every minimizing sequence is relatively compact in* $\mathfrak{D}^{1,2}(\Omega)$ *up to a translation of the form* $(y_n', 0)$ *if and only if*

$$I_\lambda < I_\lambda^\infty \tag{2.37}$$

If (2.33) *holds then every minimizing sequence is relatively compact in* $\mathfrak{D}^{1,2}(\Omega)$ *if and only if*

$$I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty, \qquad \forall \alpha \in [0, \lambda[. \tag{S.1}$$

Remark 2.20. By an obvious argument, if Ω is given by (2.32), $I_\lambda = \lambda^{\frac{N-2}{N-1}} I_1$ and thus (2.37) is equivalent to (S.1). On the other hand if Ω is given by (2.33), since $\partial\Omega$ is bounded, the problem at infinity (for the translations group) disappears and thus there only remains the problem at infinity obtained by focussing u at a boundary point via dilations.

Sketch of the proof of Theorem 2.6. We first explain why the large inequalities always hold (i.e. $I_\lambda \leqslant I_\lambda^\infty$ in the first case, $I_\lambda \leqslant I_\alpha + I_{\lambda-\alpha}^\infty$ $\forall \alpha \in [0, \lambda[$ in the second case). If (2.32) holds, we introduce $u_n \in \mathfrak{D}(\mathbb{R}^N)$ satisfying: Supp $u_n \subset B(0, 1/n)$, $\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to I_0$, $\int_{x_N = 0} |u_n|^q \, dx' = 1$; we consider $v_n = \beta^{-1/q}\lambda^{1/q} u_n$ and if $\beta > 0$, we deduce

$$\int_{x_N = 0} F(v_n) \, dx' \to 1, \int_{(x_N > 0)} |\nabla v_n|^2 \, dx \to \beta^{-2/q}\lambda^{2/q} I_0.$$

In a similar way if $\alpha > 0$, we may choose $u_n \in \mathfrak{D}(\mathbb{R}^N)$ satisfying

$$\max_{\mathbb{R}^N} |u_n| = 1, \int_{x_N = 0} \alpha |u_n|^q \, dx' = \lambda,$$

$$\int_{x_N > 0} |\nabla u_n|^2 \, dx \to \alpha^{-2/q}\lambda^{2/q} I_0$$

and we let $v_n = \left(\frac{1}{n}\right)^{\frac{N-2}{2}} u_n(n\,\cdot)$. Then we find

$$\int_{x_N=0} F(v_n)\,dx' \to \lambda, \qquad \int_{x_N>0} |\nabla v_n|^2\,dx \to \alpha^{-2/q}\lambda^{2/q} I_0.$$

Hence $I_\lambda \leqslant I_\lambda^\infty$ and if $I_\lambda = I_\lambda^\infty$, there exists a minimizing sequence which is not relatively compact even up to a translation.

In the second case —i.e. if (2.33) holds— let $\alpha \in [0, \lambda[$ and let $(u_n^1)_n$ be a minimizing sequence of I_α. On the other hand let u^2 be a minimum of I_0, we may always assume that $0 \in \partial\Omega$ and that $e_N = (0, \dots, 0, 1)$ is the unit inward normal to $\partial\Omega$ at 0. We then set

$$u_n^2 = \sigma_n^{-\frac{N-2}{2}} (\lambda - \alpha)^{2/q} \beta^{-2/q} u^2(\cdot/\sigma_n)$$

where $\sigma_n \to 0$ is to be determined. We finally set: $u_n = u_n^1 + u_n^2$. Observing that we may take u_n^1 in $\mathfrak{D}(\mathbb{R}^N)$ if we wish, it is easy to check that we may choose $(\sigma_n)_n$ in such a way that

$$\begin{aligned}
&\left| \int_{\partial\Omega} F(u_n)\,ds - \int_{\partial\Omega} F(u_n^1)\,ds - \int_{\partial\Omega} \beta |u_n^2|^q\,ds \right| \to 0,\\
&\int_{\partial\Omega} \beta |u_n^2|\,ds \to \lambda - \alpha,\\
&\int_\Omega |\nabla u_n|^2\,dx - \int_\Omega |\nabla u_n^1|^2\,dx - \int_\Omega |\nabla u_n^2|^2\,dx \to 0,\\
&\int_\Omega |\nabla u_n^2|^2\,dx \to I_{\lambda-\alpha}^\infty.
\end{aligned}$$

Therefore $(u_n)_n$ satisfies: $\int_\Omega |\nabla u_n|^2\,dx \to I_\alpha + I_{\lambda-\alpha}^\infty$, $\int_{\partial\Omega} F(u_n)\,ds \to \lambda$ and $u_n - u_n^1 \to 0$ weakly in $\mathfrak{D}^{1,2}(\Omega)$. And this proves the large inequalities and the fact that strict inequalities are necessary for the compactness of all minimizing sequences.

The proof of the sufficiency of the conditions (2.37) of (S.1) is then very similar to proofs made in Part 1 and before. We will only explain how we conclude in the case when (2.33) holds once we know that $|\nabla u_n|^2 + |u_n|^{\frac{2N}{N-2}}$ is tight. By arguments similar to those made in section 1.6, we obtain:

Lemma 2.3. *Assume Ω is given by* (2.33), *that u_n converges weakly in $\mathfrak{D}^{1,2}(\Omega)$ to u and that $\rho_n = |\nabla u_n|^2 + |u_n|^{\frac{2N}{N-2}}$ is tight. We may assume that $|\nabla u_n|^2$, ν_n converge weakly to some measures μ, ν where* ν_n is the measure on $\bar\Omega$ *supported by $\partial\Omega$ such that: $\forall \varphi \in C_b(\bar\Omega)$, $\int \varphi\,d\nu_n = \int_{\partial\Omega} \varphi F(u_n)\,ds$. Then there exist J at a most countable set* (*possibly empty*) of $(x_j)_{j\in J}$ *distinct points of* $\partial\Omega$, $(\nu_j)_{j\in J} \in]0, \infty[$ *such that*

$$\nu = \nu_\infty + \beta \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \mu \geqslant |\nabla u|^2 + I_0 \sum_{j\in J} \nu_j^{2/q} \delta_{x_j}$$

where ν_∞ is defined by

$$\int \varphi \, d\nu_\infty = \int_{\partial\Omega} \varphi F(u)\, ds, \qquad \forall \varphi \in C_b(\bar{\Omega}).$$

Remark 2.21. Similar results hold of course for sequences in $\mathfrak{D}^{m,p}(\Omega)$.

We skip the proof of the lemma since it is very similar to arguments given in Part 1 and before: let us just observe that

$$\int_{\partial\Omega} |F(u_n - u) - \beta |u_n - u|^q|\, ds \to 0$$

and that the fact that the best constant I_0 (for half-spaces) occurs in the estimate for μ is due to a localization argument. Indeed if we follow the proof of Lemma I.1 ([55]) or Lemma II.1 we see that the lower bound on $\mu(\{x_j\})$ is obtained by multiplying u_n by some convenient cut-off function $\varphi\left(\frac{x-x_j}{\epsilon}\right)$. Thus all computations take place in the ball $B(x_j, \epsilon)$ and using local charts we may actually argue as if we were in a half-space.

We next conclude this section with another problem —motivated by geometric considerations, see Cherrier [25] and section 4.2 below—; we will consider it in an unbounded domain Ω, we look for positive solutions of

$$-\Delta u = f(u) \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial n} = g(u) \quad \text{on} \quad \partial\Omega, \quad u > 0 \quad \text{on} \quad \bar{\Omega} \tag{2.38}$$

where $f, g \in C(\mathbb{R})$, $f(0) = g(0) = 0$; n is the unit outward normal. Denoting by $F(t) = \int_0^t f(s)\, ds$, $G(t) = \int_0^t g(s)\, ds$ and assuming for example that f, g are odd, one way to solve problems "like" (2.38) is to consider the following minimization problem

$$I_\lambda = \operatorname{Inf}\left\{ \int_\Omega |\nabla u|^2 \, dx \Big/ \int_\Omega F(u)\, dx + \int_{\partial\Omega} G(u)\, ds = \lambda \right\}. \tag{2.39}$$

However a solution of (2.39) leads only to a solution of (2.38) where f, g are multiplied by a Lagrange multiplier which can be taken care of only if F, G are homogeneous of the same degree —and this case is not really interesting— or if Ω is a half-space,

$$f(u) = |u|^{q_1 - 2}u, \qquad g(u) = |u|^{q_2 - 2}u \quad \text{with} \quad q_1 = \frac{2N}{N-2}, \qquad q_2 = \frac{2(N-1)}{N-2}$$

This is why we will not consider (2.39) —that we may analyse easily with our methods.

Instead, we will use the artificial constraint method (see for example C. V. Coffman [26], [27], P. L. Lions [56]) which will requiere the following structure conditions on f, g

$$f(t) = f_0(t) - mt, \qquad m \geqslant 0, \qquad 0 \leqslant f_0(t)t^{-1} \leqslant \theta f_0'(t) \qquad \forall t \in \mathbb{R} \tag{2.40}$$

$$g(t) = g_0(t) - \mu t, \qquad \mu \geqslant 0, \qquad 0 \leqslant g_0(t)t^{-1} \leqslant \theta g_0'(t) \qquad \forall t \in \mathbb{R} \tag{2.41}$$

$$\begin{cases} \lim_{|t| \to \infty} f_0(t)|t|^{2-q_1}t^{-1} = \beta_1 \geqslant 0 \\ \lim_{|t| \to \infty} g_0(t)|t|^{2-q_2}t^{-1} = \beta_2 \geqslant 0 \end{cases} \tag{2.42}$$

$$\text{if} \quad m = 0, \qquad \lim_{t \to 0_+} f_0(t)t^{-(q_1-1)} = \alpha_1 \geqslant 0 \tag{2.43}$$

$$\text{if} \quad \mu = 0, \qquad \lim_{t \to 0_+} g_0(t)t^{-(q_2-1)} = \alpha_2 \geqslant 0. \tag{2.44}$$

We consider now the following minimization problem

$$I = \mathrm{Inf}\{\mathcal{E}(u)/F(u) \in L^1(\Omega), \quad G(u) \in L^1(\partial\Omega), \quad J(u) = 0\} \tag{2.45}$$

where

$$\mathcal{E}(u) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 - F(u)dx - \int_{\partial\Omega} G(u)ds,$$

$$J(u) = \int_\Omega |\nabla u|^2 - f(u)u\,dx - \int_{\partial\Omega} g(u)u\,ds.$$

Using (2.40)-(2.41), it is easy to check that a minimum of (2.45) is indeed a solution of (2.38).

To simplify the presentation, we will consider only the cases when Ω is given either by (2.32), or by (2.33). We need to introduce the following quantities

$$I^{\infty,i} = \mathrm{Inf}\{\mathcal{E}^{\infty,i}(u)/J^{\infty,i}(u) = 0, \quad u \not\equiv 0\}, \qquad i = 1, 2, 3$$

$$\mathcal{E}^{\infty,1}(u) = \int_{\mathbb{R}^N} \tfrac{1}{2}|\nabla u|^2 - F(u)dx, \quad J^{\infty,1}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 - f(u)u\,dx$$

$$\mathcal{E}^{\infty,2}(u) = \int_{(x_N>0)} \tfrac{1}{2}|\nabla u|^2 - (\beta_1/q_1)|u|^{q_1}dx - \int_{(x_N=0)} (\beta_2/q_2)|u|^{q_2}dx'$$

$$J^{\infty,2}(u) = \int_{(x_N>0)} |\nabla u|^2 - \beta_1|u|^{q_1}dx - \int_{(x_N=0)} \beta_2|u|^{q_2}dx'$$

$$\mathcal{E}^{\infty,3}(u) = \int_{(x_N>0)} \tfrac{1}{2}|\nabla u|^2 - (\alpha_1/q_1)|u|^{q_1}dx - \int_{(x_N=0)} (\alpha_2/q_2)|u|^{q_2}dx'$$

$$J^{\infty,3}(u) = \int_{(x_N>0)} |\nabla u|^2 - \alpha_1|u|^{q_1}dx - \int_{(x_N=0)} \alpha_2|u|^{q_2}dx'$$

Of course if m (resp. μ) > 0 we set $\alpha_1 = 0$ (resp. $\alpha_2 = 0$) and if $\alpha_1 = \alpha_2 = 0$ (or $\beta_1 = \beta_2 = 0$) we set $I^{\infty,3} = +\infty$ (or $I^{\infty,2} = +\infty$).

To motivate the introduction of these various functionals, let us explain that $I^{\infty,1}$ corresponds to the "problem at $x_N^\infty = +\infty$" obtained by the action of the translation group if for example Ω is given by (2.32), while $I^{\infty,2}$ is obtained by "concentrating u" at a boundary point by the action of the dilation groups ("concentrating u" at an interior point is not necessary here since it is contained in $I^{\infty,1}$), and finally $I^{\infty,3}$ is obtained by "scaling out" u

($u \to \sigma^{-(N-2)/2}u(\cdot/\sigma)$ with $\sigma \to +\infty$) again by the action of the dilation group. The next two results correspond to the two domains Ω we consider:

Theorem 2.7. *We assume* (2.32), (2.40)-(2.44).

1) *If* $m = \mu = 0$, $f_0 = \gamma_1|t|^{q_1-1}t$, $g_0 = \gamma_2|t|^{q_2-1}t$ *with* $\gamma_1, \gamma_1 > 0$ *then any minimizing sequence of* (2.45) *is relatively compact in* $\mathfrak{D}^{1,2}(\Omega)$ *up to a scale change* ($\sigma \to \sigma^{-(N-2)/2}u(\cdot/\sigma)$ *and a translation of the form* $(y_n', 0)$. *In particular there exists a minimum.*

We denote $I^\infty = \mathrm{Min}(I^{\infty,1}, I^{\infty,2}, I^{\infty,3})$. *Then the condition*

$$I < I^\infty \tag{2.16}$$

is necessary and sufficient for the compactness of all minimizing sequences up to a translation of the form $(y_n', 0)$.

Theorem 2.8. *We assume* (2.33), (2.40)-(2.42) *and we denote by* $I^\infty = \mathrm{Min}(I^{\infty,1}, I^{\infty,2})$. *Then* (2.16) *is necessary and sufficient for the compactness of all minimizing sequences of* (2.45).

Remark 2.22. We could treat as well arbitrary unbounded domains such that: $\forall R < \infty$, $y \in \Omega$, $B(y, R) \subset \Omega$, or strip-like domains... Combining the methods of P. L. Lions [55], [56] and of Part 1 [65], we may treat exactly as below x-dependent problems and in particular

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left[a_{ij}(x)\dfrac{\partial u}{\partial x_j}\right] + k(x)u = K(x)u^{(N+2)/(N-2)} \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial v^A} + k'u = K'u^{N/(N-2)} \quad \text{on} \quad \partial\Omega, \qquad u > 0 \quad \text{in} \quad \bar{\Omega} \end{cases}$$

where a_{ij}, k, k', K, K' are given functions having limits as $|x| \to \infty$, $x \in \Omega$, $(a_{ij}(x))$ is uniformly elliptic, K, K' are not everywhere nonpositive and are nonnegative at ∞ and the quadratic form associated with the linear part of the problem is positive on $\mathfrak{D}^{1,2}(\Omega)$. Of course v^A is the conormal associated with $(a_{ij}(x))$ i.e. $v_i^A = a_{ij}n_j$ $\forall i$.

Remark 2.23. In fact our method not only shows Theorems 2.7-8 but also explains how compactness may be lost if $I = I^\infty$: for example if $I = I^{\infty,2} < I^{\infty,1} \wedge I^{\infty,3}$, a noncompact minimizing sequence $(u_n)_n$ will satisfy: $|\nabla u_n|^2 \to \delta_{x_0}$, $\beta_1|u_n|^{q_1} + \beta_2|u_n|^{q_2} \to \delta_{x_0}$ for some $x_0 \in \mathbb{R}^{N-1}x\{0\}$ and there exist $\sigma_n \to \infty$, $y_n = (y_n', 0)$, $-y_n/\sigma_n \to x_0$ such that $\sigma_n^{-(N-2)/2}u_n((\cdot - y_n)/\sigma_n)$ converges to a minimun of $I^{\infty,2}$ (up to subsequences...). And there exists such a sequence $(u_n)_n$.

Remark 2.24. We will not discuss here conditions (2.16): this strict inequality may be analyzed as in P. L. Lions [55], [56], [65], T. Aubin [6], H. Brézis and L. Nirenberg [23]... Let us only observe that by a symmetry argument similar to the one used below we have if Ω is given by (2.32): $I \leqslant \frac{1}{2} I^{\infty,1}$ and thus $I^{\infty} = \mathrm{Min}(I^{\infty,2}, I^{\infty,3})$.

Sketch of the proof of Theorems 2.7-8. First of all, in the case when (2.32) holds and $m = \mu = 0$, $f_0 = \gamma_1 |t|^{q_1 - 1} t$, $g_0 = \gamma_2 |t|^{q_2 - 1} t$, $\gamma_1, \gamma_2 > 0$, the minimization problem (2.45) is "scale invariant" and invariant by translations of the form $(y', 0)$. We claim that $I < I^{\infty,1}$. To check this strict inequality, we just observe that there exists $u \in \mathfrak{D}^{1,2}(\mathbb{R}^N)$ symmetric with respect to x_N (actually radial) such that

$$\begin{cases} I^{\infty,1} = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - (\gamma_1/q_1) |u|^{q_1} \, dx \\ \int_{\mathbb{R}^N} |\nabla u|^2 - \gamma_1 |u|^{q_1} \, dx = 0. \end{cases}$$

Thus: $\int_{(x_N > 0)} |\nabla u|^2 - \gamma_1 |u|^{q_1} \, dx = 0$; and there exists $\theta \in]0, 1[$ such that if $v = \theta u$

$$\int_{(x_N > 0)} |\nabla v|^2 - \gamma_1 |v|^{q_1} \, dx - \int_{(x_N = 0)} \gamma_2 |v|^{q_2} \, dx' = 0.$$

Therefore we have denoted by $\alpha = \int_{(x_N > 0)} |\nabla u|^2 = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$

$$\begin{aligned} I &= \theta^2 \frac{\alpha}{2} - \theta^{q_1} \frac{\alpha}{q_1} - \frac{1}{q_2} (\theta^2 \alpha - \theta^{q_1} \alpha) \\ &= \theta^2 \alpha \left(\frac{1}{2} - \frac{1}{q_2} \right) + \theta^{q_1} \alpha \left(\frac{1}{q_2} - \frac{1}{q_1} \right) \leqslant \alpha \left(\frac{1}{2} - \frac{1}{q_1} \right) = \frac{1}{2} I^{\infty,1}. \end{aligned}$$

Thus, by a convenient choice of the scaling and of a translation, we may assume that any minimizing sequence satisfies

$\rho_n = |\nabla u_n|^2 + \gamma_1 |u_n|^{q_1} + \gamma_2 |u_n|^{q_2} \otimes \delta_0(x_N)$ is tight and $\mathrm{Sup}_{y \in \mathbb{R}^N} \int_{y + B_1} d\rho_n = L$ where $L < \mathrm{Inf}_n \int_{\mathbb{R}^N} d\rho_n$. Indeed vanishing is ruled out by the scaling, dichotomy as in [55] and the tightness cannot be obtained through a sequence (y^n) such that y_N^n is unbounded because of the strict inequality: $I < I^{\infty,1}$. Assuming that u_n converges weakly and a.e. to u, we have to show that $u \not\equiv 0$: If this is the case, we conclude easily adapting arguments given before (or in Part 1) and in [56]. Now if $u \equiv 0$, we may assume that $|\nabla u_n|^2$, $\gamma_1 |u_n|^{q_1}$, $\gamma_2 |u_n|^{q_2} \otimes \delta_0(x_N)$ converge weakly to some measures μ, ν_1, ν_2: we already know that ν_1, ν_2 are given by countable sums of Dirac masses and that μ charges any point charged by $\nu_1 + \nu_2$. We claim that μ, ν_1, ν_2 are given by *one* Dirac mass contradicting thus the constraint on ρ_n.

Indeed if $x^0 \in \{x_N \geqslant 0\}$ is such that

$$\mu(\{x^0\}) - \nu^1(\{x^0\}) - \nu^2(\{x^0\}) < 0$$

then we may find $\xi \in \mathfrak{D}(\mathbb{R}^N)$ supported in a small enough ball centered at x^0 such that

$$\begin{cases} J(\xi u_n) \to -\alpha < 0 \\ \left(\frac{1}{2} - \frac{1}{q_1}\right)\gamma_1 \int_{(x_N>0)} |\xi u_n|^{q_1}\,dx + \left(\frac{1}{2} - \frac{1}{q_2}\right)\gamma_2 \int_{(x_N=0)} |\xi u_n|^{q_2}\,dx' \to \beta < I \end{cases}$$

Indeed observe that

$$I = \left(\frac{1}{2} - \frac{1}{q_1}\right)\int d\nu^1 + \left(\frac{1}{2} - \frac{1}{q_2}\right)\int d\nu^2.$$

It is then easy to reach a contradiction as in [56]. Therefore for each point x^j in the support of $\nu_1 + \nu_2$, we find

$$\mu(\{x^j\}) - \nu^1(\{x^j\}) - \nu^2(\{x^j\}) \geqslant 0$$

and thus

$$0 \leqslant \sum_{j\in J} \mu(\{x^j\}) - \nu^1(\{x^j\}) - \nu^2(\{x^j\}) \leqslant \int d\mu - \int d\nu^1 - \int d\nu^2 = 0.$$

Hence

$$\mu = \sum_{j\in J} \mu_j \delta_{x^j}, \qquad \nu^1 = \sum_{j\in J} \nu_j^1 \delta_{x^j}, \qquad \nu^2 = \sum_{j\in J} \nu_j^2 \delta_{x^j}$$

and

$$\mu_j = \nu_j^1 + \nu_j^2 > 0, \quad x^j \in \{x_N \geqslant 0\}, \quad \mu_j \geqslant c_1(\nu_j^1)^{1/q_1} + c_2(\nu_j^2)^{2/q_2}.$$

This last inequality yields that J is finite (since $\mu_j > 0$, $\forall j \in J$). In addition choosing for each $j \in J$ a cut-off function ξ supported in a small ball centered at x^j, we see that

$$I \leqslant \left(\frac{1}{2} - \frac{1}{q_1}\right)\nu_j^1 + \left(\frac{1}{2} - \frac{1}{q_2}\right)\nu_j^2, \qquad I = \sum_{j\in J} \left(\frac{1}{2} - \frac{1}{q_1}\right)\nu_j^1 + \left(\frac{1}{2} - \frac{1}{q_2}\right)\nu_j^2$$

and this only possible if J is singleton. Hence: $\mu = \mu_0 \delta_{x^0}$, $\nu^1 = \nu_0^1 \delta_{x^0}$, $\nu^2 = \nu_0^2 \delta_{x^0}$ where $\mu_0 > 0$, $\nu_0^1 + \nu_0^2 = \mu_0$, $\nu_0^1 \geqslant 0$, $\nu_0^2 \geqslant 0$, $x^0 \in \{x_N \geqslant 0\}$. (If $x_N^0 > 0$, $\nu_0^2 = 0$, $\nu_0^1 = \mu_0$ and we would have: $I = I^{\infty,}$. Therefore $x_N^0 = 0$, $\nu_0^2 > 0$, $\nu_0^1 > 0$).

In the general case, we apply the arguments of P. L. Lions [55], [56] to deduce that $\rho_n = |\nabla u_n|^2 + |u_n|^{q_1} + m u_n^2 + (|u_n|^{q_2} + \mu u_n^2) \otimes \delta_0(x_N)$ —if (32) holds; if (33) holds we consider $|\nabla u_n|^2 + |u_n|^{q_1} + m u_n^2$— is tight: in particular we use the strict inequality $I < I^{\infty,1}$ to obtain that if $\rho_n(\cdot + y^n)$ is tight then y_N^n is bounded if (32) holds, or y^n is bounded if (33) holds. Then if u_n converges weakly and a.e. to u, we have to check that $u \not\equiv 0$ and the remainder of the proof is then a combination of arguments of P. L. Lions [56] and of those given in Part 1 and above.

Let us check that $u \not\equiv 0$: if $u \equiv 0$, we observe

$$I = \lim \epsilon(u_n) = \lim \int_\Omega \frac{1}{2} f(u_n)u_n - F(u_n)dx + \int_{\partial\Omega} \frac{1}{2} g(u_n)u_n - G(u_n)ds.$$

And since ρ_n is tight, we deduce easily

$$\left| \int_\Omega \frac{1}{2} f(u_n)u_n - F(u_n)dx - \left(\frac{1}{2} - \frac{1}{q_1}\right) \int_\Omega \beta_1 |u_n|^{q_1} dx \right| \to 0$$

$$\left| \int_{\partial\Omega} \frac{1}{2} g(u_n)u_n - G(u_n)dx - \left(\frac{1}{2} - \frac{1}{q_2}\right) \int_{\partial\Omega} \beta_2 |u_n|^{q_2} dx \right| \to 0.$$

Similarly, we have

$$J(u_n) - J^{\infty,2}(u_n) \to 0, \quad \text{thus} \quad J^{\infty,2}(u_n) \to 0.$$

This shows that

$$I \geqslant \inf\left\{ \int_\Omega \left(\frac{1}{2} - \frac{1}{q_1}\right) \beta_1 |u|^{q_1} dx + \int_{\partial\Omega} \left(\frac{1}{2} - \frac{1}{q_2}\right) \beta_2 |u|^{q_2} ds \Big/ u \in \mathfrak{D}^{1,2}(\Omega), J^{\infty,2}(u) = 0 \right\} = I^{\infty,2}$$

and this contradicts (16).

3. The General Principle

3.1 *Heuristic derivation*

In this section, we want to explain the common features of the problems and methods introduced in Part 1 and here exactly as we did in the locally compact case in P. L. Lions [58], [55], [56]. By no means, the claims below concerning the equivalence between certain compactness results and the subadditivity inequalities (S.1), (S.2) are to be understood as rigorous results: they are indications on what are the crucial inequalities to be checked and on a general scheme of proof.

We begin with the general case (the case of invariance by translations or dilations being treated below) and we keep the setting used in [58], [55]: let

H be a functions space on $\mathbb{R}^N$ (more general situations are considered below) and let $J, \mathcal{E}$ be functionals defined on H (or on a subdomain of H) of the following type

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} e(x, Au(x))dx; \quad J(u) = \int_{\mathbb{R}^N} j(x, Bu(x))dx$$

where $e(x,p)$, $j(x,q)$ are real-valued functions defined respectively on $\mathbb{R}^N \times \mathbb{R}^m$, $\mathbb{R}^N \times \mathbb{R}^n$ and j is nonnegative; A, B are operators (possibly nonlinear) from H into E, F (functions spaces defined on $\mathbb{R}^N$ with values in $\mathbb{R}^m$, $\mathbb{R}^n$) which commute with the translations group of $\mathbb{R}^N$. We assume $\mathcal{E}(0) = J(0) = 0$. We want to study the following minimization problem

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, \quad J(u) = 1\} \tag{3.46}$$

and we embed this problem in a one parameter family of problems

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, \quad J(u) = \lambda\} \tag{3.47}$$

where $\lambda > 0$; of course $I = I_1$.

As we saw in the examples we have treated in sections 1 and 2, we have to evaluate the effects of the non-compactness of the translations and dilations group. This is why (to simplify) we assume

$$e(x,p) \to e_\infty^\infty(p), \quad j(x,q) \to j_\infty^\infty(q) \quad \text{as} \quad |x| \to \infty \tag{3.48}$$

(the precise meaning of the above convergence has to be worked out in all examples) and we set

$$I_\lambda^{\infty,\infty} = \operatorname{Inf}\{\mathcal{E}^{\infty,\infty}(u)/u \in H, J^{\infty,\infty}(u) = \lambda\} \tag{3.49}$$

where

$$\mathcal{E}^{\infty,\infty}(u) = \int_{\mathbb{R}^N} e_\infty^\infty(Au)dx, \quad J^{\infty,\infty}(u) = \int_{\mathbb{R}^N} j_\infty^\infty(Bu)\,dx.$$

Next, to take care of the dilations group, we assume to simplify that there exists a critica power $\alpha > 0$ such that $T_{\sigma,y}u = \sigma^{-\alpha}u((\cdot - y)/\sigma) \in H$ if $u \in H$ and if $y \in \mathbb{R}^N$, and we assume

$$\mathcal{E}(T_{\sigma,0}u) \to \tilde{\mathcal{E}}^{\infty,\infty}(u), \quad J(T_{\sigma,0}u) \to \tilde{J}^{\infty,\infty}(u) \quad \text{if} \quad \sigma \to +\infty \tag{3.50}$$

$$\mathcal{E}(T_{\sigma,y}u) \to \tilde{\mathcal{E}}^{\infty,y}(u), \quad J(T_{\sigma,y}u) \to \tilde{J}^{\infty,y}(u) \quad \text{if} \quad \sigma \to 0_+ \tag{3.51}$$

and we introduce for all $y \in \mathbb{R}^N$

$$\tilde{I}_\lambda^{\infty,\infty} = \operatorname{Inf}\{\tilde{\mathcal{E}}^{\infty,\infty}(u)/u \in H, \tilde{J}^{\infty,\infty}(u) = \lambda\} \tag{3.52}$$

$$I_\lambda^\infty = \operatorname{Min}\left(I_\lambda^{\infty,\infty}, \tilde{I}_\lambda^{\infty,\infty}, \operatorname*{Inf}_{y \in \mathbb{R}^N} I_\lambda^{\infty,y}\right). \tag{3.54}$$

We may now state a heuristic principle (which holds in all the examples treated before and below) that we call the concentrantion-compactness principle. To be rigorous, the following claims need many structure conditions (a priori bounds on minimizing sequences which insures in particular the finiteness of I_λ, the continuity of I_λ with respect to $\lambda \ldots$; convexity or weak l.s.c. properties of the «main» terms in $\mathcal{E}, J \ldots$) and it seems very difficult to give a unique framework covering the variety of the examples we treat.

We first claim (this part being easy to justify by the very way I_λ^∞ was defined and the arguments of [55]) that we *always have the large subadditivity inequalities*

$$I_\lambda \leqslant I_\alpha + I^\infty{}_{\lambda-\alpha}, \qquad \forall \alpha \in [0, \lambda]. \tag{3.55}$$

Next, we «claim» that, for a fixed $\lambda > 0$, all *minimizing sequences* of (47) *are compact if and only if*

$$I_\lambda < I_\alpha + I^\infty{}_{\lambda-\alpha}, \qquad \forall \alpha \in [0, \lambda[. \tag{S.1}$$

Indeed we first «prove» the «tightness» of any minimizing seqeunce $(u_n)_n$ by applying the first concentration-compactness lemma (see [58], [55]): since (S.1) implies

$$\begin{cases} I_\lambda < I_\alpha + I_{\lambda-\alpha}^{\infty,\infty} & \forall \alpha \in]0, \lambda[\\ I_\lambda < \operatorname{Min}(I_\lambda^{\infty,\infty}, \tilde{I}_\lambda^{\infty,\infty}) \end{cases}$$

dichotomy, vanishing and tightness up to an unbounded translation cannot happen. Next if $(u_n)_n$ «converges weakly» to some $u \in H$ we claim that $u \not\equiv 0$: if u were 0, then the effect of the «almost dilations invariance» ($u \to T_{\sigma,0}u$) would be that u_n concentrates around at most a countable number of points. But since $(u_n)_n$ is a minimizing sequence, we claim that u_n concentrates around a *single point* (up to subsequences); one way to understand this claim is to argue as follows, isolate one concentration point x^0 and split u_n into two parts: the part concentrating at x^0 and the part concentrating around the other points. If this were to happen, we would have for some $\alpha \in]0, \lambda[$

$$I_\lambda \geqslant I_\alpha + \operatorname*{Inf}_{y \in \mathbb{R}^N} I_{\lambda-\alpha}^{\infty,y} \geqslant I_a + I_{\lambda-\alpha}^\infty$$

contradicting (S.1). Hence, (u_n) concentrates at a single point x^0 and we deduce

$$I_\lambda \geqslant I_\lambda^{\infty,x^0} \geqslant \inf_{y \in \mathbb{R}^N} I_\lambda^{\infty,y} \geqslant I_\lambda^\infty$$

again contradicting (S.1). Therefore $u \not\equiv 0$. Finally if $J(u) = \alpha \in]0, \lambda[$ we split u_n into two parts: basically u and $u_n - u$ (this is only a rough idea — cf.

precise arguments in sections 1-2). Again $u_n - u$ concentrates at a countable number of points and we deduce

$$I_\lambda \geqslant I_\alpha + \inf_{y \in \mathbb{R}^N} I^{\infty,y}_{\lambda-\alpha} \geqslant I_\alpha + I^{\infty}_{\lambda-\alpha}$$

The contradiction shows that u_n converges to u, a minimun of (47).

This heuristic argument not only shows that (S.1) is *a necessary and sufficient condition* for *the compactness of all minimizing sequences* of (47) but also enables us to analyse what are the *possible losses of compactness if* (S.1) *fails*. For example if we know that

$$I_\lambda < I_\alpha + I^{\infty}_{\lambda-\alpha}, \qquad \forall \alpha \in]0, \lambda[\tag{3.56}$$

then (S.1) is equivalent to

$$I_\lambda < I^{\infty}{}_{\lambda}. \tag{16}$$

And if $I_\lambda = I^{\infty,\infty}_\lambda < \mathrm{Min}(\tilde{I}^{\infty,\infty}_\lambda, I^{\infty,y}_\lambda)$, $\forall y \in \mathbb{R}^N$, we obtain that any noncompact minimizing sequence is compact up to a translation y_n such that $|y_n| \to \infty$. Similarly if $I_\lambda = I^{\infty,y}_\lambda < \mathrm{Min}(I^{\infty,\infty}_\lambda, \tilde{I}^{\infty,\infty}_\lambda)$ for some $y \in \mathbb{R}^N$; then any noncompact minimizing sequence concentrates at an infimum point y^0 of $\mathrm{Inf}_{y\in\mathbb{R}^N} I^{\infty,y}_\lambda$ (and conveniently rescaled is compact, converging to a minimun point of $I^{\infty,y}_\lambda$ if $I^{\infty,y}_\lambda$ satisfies (S.2) below !)...

Next, we explain that the above ideas still carry out to cover more general situations where *j is not nonnegative*, or $\mathbb{R}^N$ *is replaced by* and unbounded region Ω such that

$$\forall R < \infty, \quad \exists y \in \Omega, \quad y + B_R \subset \Omega.$$

Indeed if j is negative somewhere, in general we still have to consider only $\alpha \in [0, \lambda[$: this is basically due to the fact that $J(0) = \mathcal{E}(0) = 0$ and with the above notations if $\mu = J(u) > \lambda$, we would have: $I_\mu \leqslant I_\lambda$ and this is not possible in general.

And when $\mathbb{R}^N$ is replaced by Ω, we assume (48) for $|x| \to \infty$, $x \in \bar{\Omega}$ and we replace in (54) the infimum over $y \in \mathbb{R}^N$ by the infimum over $y \in \bar{\Omega}$.

We now turn to problems with a complete or partial invariance: first of all we consider problems which are invariant by the changes $I_{\sigma,y}$ for all $\sigma > 0$, $y \in \mathbb{R}^N$ i.e.: $\mathcal{E}(T_{\sigma,y}u) = \mathcal{E}(u)$, $J(T_{\sigma,y}u) = J(u) \quad \forall u \in H$, $\forall \sigma > 0$, $\forall y \in \mathbb{R}^N$. In this case by similar arguments to the ones given above *any minimizing sequence $(u_n)_n$ is relatively compact up to a change T_{σ_n,y_n} if and only if (S.2) holds*: in particular if (S.2) holds, then there exist $(\sigma_n)_n$ in $]0, \infty[$, $(y_n)_n$ in $\mathbb{R}^N$ such that $T_{\sigma_n,y_n}u_n$ is relatively compact.

Next, we may consider *problems* which are *invariant by translations but not by dilations*: in this case we set

$$I_\lambda^\infty = \mathrm{Min}(\tilde{I}_\lambda^{\infty,\infty}, I_\lambda^{\infty,0}$$

(observe that $I_\lambda^{\infty,y} = I_\lambda^{\infty,0}$, $\forall y \in \mathbb{R}^N$); and *any minimizing sequence is compact up to a translation if and only if* (S.1) *holds*. Conversely, we may have to solve *problems invariant by dilations but not by all translations*: for example problems in a half space $\{x_N \geqslant 0\}$ invariant by dilations and by translations of the form $(y', 0)$. In this situation, we define $I_\lambda^{\infty,\infty}$ as before by considering

$$\mathcal{E}^{\infty,\infty}(u) = \lim_{|y|\to\infty} \mathcal{E}(u(\cdot + y)), \quad J^{\infty,\infty}(u) = \lim \mathcal{E}(u(\cdot + y))$$

(in this example above we only take: $\lim\limits_{y_N \to +\infty}$) and we set $I_\lambda^\infty = I_\lambda^{\infty,\infty}$. And all *minimizing sequences are compact up to a dilation* (and a translation of the form $(y_n', 0)$ in the example) if and only if (S.1) holds. Similar variants exist if the problem —or the domain— has only a *restricted* number of *translation invariance* (exs.: strips, half-spaces...).

We now turn to the important particular case of a *compact region* Ω of $\mathbb{R}^N$ (or a *N-dimensionnal compact Riemannian manifold*). If the problem is «set in Ω», it is clear that the translations do not play anymore any role and similarly for $T_{\sigma,y}u$ when $\sigma \to +\infty$. Thus the only «*non-compactness*» remaining concerns the *action* of $T_{\sigma,y}u$ as $\sigma \to 0_+$ for any $y \in \bar{\Omega}$; hence we just assume (51) for $y \in \bar{\Omega}$ and we set for all $\lambda > 0$

$$I_\lambda^\infty = \operatorname*{Inf}_{y \in \bar{\Omega}} I_\lambda^{\infty,y}. \tag{3.57}$$

In this very particular case, the above principle reduces to the following ideas: *(S.1) is a necessary and sufficient condition for the compactness of all minimizing sequences*. In addition if (56) holds and thus (S.1) *is equivalent to* (16), then we have:

i) *if* (16) *holds, any minimizing sequence is compact*, ii) *if* (16) *does not hold i.e.* $I_\lambda = I_\lambda^\infty$ then *there exists a noncompact minimizing sequence* and *any such sequence converges weakly to* 0, *concentrating at a minimum point* y_0 *of* (57) (up to subsequences). In addition *if* I_λ^∞ *satisfies* (S.2), *there exist* $(\sigma_n)_n$ *in* $]0, \infty[$, (y_n) *in* $\mathbb{R}^N$ *such that* $T_{\sigma_n,y_n} u_n$ *is compact and converges to a minimum of* I_λ^{∞,y_0}, and $\sigma_n \to \infty$, $-y_n/\sigma_n \to y$ up to (subsequences). Let us also point out that when Ω is a compact manifold, the action $T_{\sigma,y}$ is not well defined but since we want to concentrate u at the point y only the local properties of Ω near y matter and via local charts and the tangent space $T_y\Omega$, we may still define $I_\lambda^{\infty,y}$ as a problem on the tangent space i.e. $\mathbb{R}^N$ if Ω is N-dimensional.

We next want to make several remarks: i) we may treat as well problems with *multiple constraints*

$$I(\lambda_1, \ldots, \lambda_m) = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, J_i(u) = \lambda_i\}$$

then one defines exactly as we did $I^\infty(\lambda_1, \ldots, \lambda_m)$ and (S.1) is to be replaced by

$$I(\lambda_1, \ldots, \lambda_m) < I(\alpha_1, \ldots, \alpha_m) + I^\infty(\lambda_1 - \alpha_1, \ldots, \lambda_m - \alpha_m)$$

for all $\alpha_i \in [0, \lambda_i]$, $\sum_i \alpha_i < \sum \lambda_i$.

ii) It may be important to treat the following type of constraints

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, J(u) = 0, u \not\equiv 0\}$$

—see [56] and section II—. Then denoting by I_λ the infimum corresponding to $J(u) = \lambda$ for $\lambda \in \mathbb{R}$, (S.1) is to be replaced by

$$I < I_\lambda + I^\infty_{-\lambda} \quad \forall \lambda \neq 0, \quad I < I^\infty$$

where I^∞ is defined as before. Very often the first series of inequalities hold easily (notice also that I is not modified if we replace $\mathcal{E}$ by $\mathcal{E} + \mu J \ldots$).

iii) In the locally compact case (cf. P. L. Lions [55], [56]) we refind the fact that the action of $T_{\sigma,y}$ does not play any role observing that $\tilde{\mathcal{E}}^{\infty,\infty}$ or $\tilde{J}^{\infty,\infty}$ and $\mathcal{E}^{\infty,y}$ or $J^{\infty,y}$ are trivial in this case and thus I^∞_λ reduces to $I^{\infty,\infty}_\lambda$.

iv) If J has a completely indefinite sign, it may happen that (S.1) has to be replaced by

$$I_\lambda < I_\alpha + I^\infty_{\lambda-\alpha}, \quad \forall \alpha \in \mathbb{R} - \{\lambda\} \tag{S.1'}$$

Remark III.1. In order to illustrate (at last !) the above discussion we wish to indicate briefly a list of the various types of problems encountered and the corresponding results in Part 1 and here:

1. Invariance by dilations and translations: Theorem I.1; Corollary I.2; Problem (1.35); Theorem II.1.
2. Invariance by translations, not by dilations: Problem (1.30); Theorem I.5; Theorem II.2; Theorem II.7 ii).
3. Invariance by dilations, not by translations: Theorem I.3; Theorem II.3; Theorem II.4; Theorem II.7 i).
4. Restricted invariance by translations: Theorem II.3; Theorem II.5; Theorem II.6; Theorem II.7.
5. Nonisotropic dilation invariance: Corollary I.3.
6. General situations: Theorem I.2; Theorem I.4; Theorem II.6; Theorem II.8.

7. Multiple constraints: Problem (1.33)
8. Constraint $J(u) = 0$: Theorem II.7; Theorem II.8.
9. Problems in compact domains: section IV.

We next would like to explain what we mean by concentrates around a point: this means that the densities of the functionals or of related norms — which are bounded L^1_+ functions— converge weakly to Dirac masses (cf. Lemma I.1; sections I.4 ii), iii), iv), vi), vii); Lemma I.4; Theorem I.6; Lemma II.1; Lemma II.2; Lemma II.3; lemma II.4...).

Finally, we want to conclude this section by emphasizing that (S.1), (S.2) are necessary and sufficient conditions for the compactness of all minimizing sequences but that there might exist a minimum even if (S.1) (or (S.2) fails) see P. L. Lions [ ] for such an example in the locally compact case. In addition (S.1), (S.2) may be difficult to check (but anyway one has to check them!): in particular when (S.1) reduces to (16) and $I^\infty_\lambda = \operatorname{Inf}_y I^{\infty,y}_\lambda$, in order to check (16), it is natural to try as a test function: $\bar{u}_\varepsilon = T_{\varepsilon,y_0}\bar{u}$ where y_0 is a minimum point of (57), $\bar{u}$ a minimum of I^{∞,y_0}_λ and ε goes to 0. Indeed observe that any noncompact minimizing sequence will be very similar to $\bar{u}_\varepsilon$ (if I^∞_λ satisfies (S.2)) if $I_\lambda = I^\infty_\lambda$. This motivates the choice of $\bar{u}_\varepsilon$ in order to analyse (16): this choice was first considered by T. Aubin [6], see also H. Brézis and L. Nirenberg [23], H. Brézis and J. M. Coron [19], [20], P. L. Lions [65].

We make two final remaks on (S.1) and (S.2) —that will be developed further elsewhere—: first of all, if $I_\lambda < I^\infty_\lambda$, I^∞_μ satisfies (S.2) for all $\mu \in]0,\lambda]$ and (S.1) does not hold there exists $\alpha \in]0,\lambda[$ such that

$$I_\lambda = I_\alpha + I^\infty_{\lambda-\alpha}$$

We then claim that I_α satisfies (S.1): indeed if we had

$$I_\alpha = I_\beta + I^\infty_{\alpha-\beta} \quad \text{with} \quad \beta \in [0,\alpha[$$

this would imply

$$I_\lambda = I_\beta + I^\infty_{\alpha-\beta} + I^\infty_{\lambda-\alpha} > I_\beta + I^\infty_{\lambda-\beta} \geqslant I_\lambda;$$

a contradiction. In addition $\{\alpha \in]0,\lambda], \text{ (S.1) holds for } I_\alpha\}$ is open if $I^\infty_\mu \mu^{-1} \to +\infty$ when $\mu \to 0_+$: indeed if (S.1) holds for I_{α_0}, then for α near α_0, $I_\alpha < I^\infty_\alpha$ and if (S.1) fails for I_α there exists $\beta \in]0,\alpha[$ such that

$$I_\beta + I^\infty_{\alpha-\beta} = I_\alpha, \quad \beta \to \alpha_0 \quad \text{as} \quad \alpha \to \alpha_0.$$

But there exists a minimum for I_β (cf. the argument above) which converges to a minimum of I_{α_0} as $\beta \to \alpha_0$. Hence it is easy to show that: $I_\alpha \leqslant I_\beta + C(\alpha-\beta)$ for α near α_0; in other words $I^\infty_{\alpha-\beta} \leqslant C(\alpha-\beta)$ for α near α_0, if (S.1) fails. Thus (S.1) holds in a neighborhood of α_0.

3.2. *The role of symmetries*

In this section, we want to explain how *the invariance of functionnals by symmetries* (orthogonal transformations of $\mathbb{R}^N$) fits in the general picture of minimization problems and the concentration-compactness principle. To motivate what follows let us recall that it was observed in W. Strauss [75] (and developped in H. Berestycki and P. L. Lions [13]) that the embedding from $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for $2 < p < 2N/(N-2)$ $(N \geqslant 3)$ is compact when restricted to spherically symmetric functions. This was used in [75], [13] to solve various minimization problems by restricting a priori (or a posteriori via symmetrization) the infimum to spherically symmetric functions (see also P. L. Lions [62]). Such compactness arguments are extended to more general symmetries in P. L. Lions [62], [63]. In addition in those compactness results one proves that if $H_r^1(\mathbb{R}^N)$ is the subspace of $H^1(\mathbb{R}^N)$ consisting of spherically symmetric functions then $H_r^1(\mathbb{R}^N) \hookrightarrow L^\infty(|x| > \delta)$ for any $\delta > 0$ (see Appendix 2 for more general results of this type) hence on the domain $(|x| > \delta)$ the limit exponent $2N/(N-2)$ is meaningless for $H_r^1(\mathbb{R}^N)$ and compactness is available (see M. J. Esteban and P. L. Lions [35] for an application of this fact).

We want here to explain these observations by the help of an extension of the *concentration-compactness principle*, taking into account the *invariance of the functionals by a group of orthogonal transformations* of $\mathbb{R}^N$. Let us also mention that we were led to the heuristic principle which follows by the study due to C.V. Coffman and Markus [28] and that the analysis below will be developped further elsewhere.

We still consider the general setting of the preceding section where $\mathbb{R}^N$ is replaced by a domain Ω. We assume that Ω, $\mathcal{E}$, J are *invariant* under the *action of a group G* of orthogonal transformations of $\mathbb{R}^N$ (of course if Ω is a compact N-dimensionnal manifold we adapt the notion of such a group in a straightforward way...) and we consider for $\lambda > 0$

$$\bar{I}_\lambda = \operatorname{Inf}\{\mathcal{E}(u)/u \in H, \quad u \text{ is } \quad G\text{-invariant}, J(u) = \lambda\}$$

where G-invariant means: $u(x) = u(g \cdot x)$, $\forall x \in \bar{\Omega}$, $\forall g \in G$.

We need now to define *the problems at infinity*: first of all we define $\tilde{I}_\lambda^{\infty,\infty}$ exactly as before adding to the set of minimizers the constraint that u is G-invariant.

Next, observing that if «u is concentrated at y» and if u is G-invariant u is also concentrated at every point $z = g \cdot y$ for some $g \in G$, we consider the *equivalence class*: $\omega(y) = \{z = g \cdot y/g \in G\}$, and we denote by $s(y) = \#\,\omega(y)$. If $s(y) < \infty$, we define

$$\bar{I}_\lambda^{\infty,y} = \sum_{z\in\omega(y)} I_{\lambda/s(y)}^{\infty,z} = s(y) I_{\lambda/s(y)}^{\infty,y}; \tag{59}$$

(Of course if $0 \in \bar{\Omega}$, $\omega(0) = \{0\}$ and $s(0) = 1$) $\bar{I}_\lambda^{\infty,y}$ does not really depend of y but on its equivalence class $\omega(y)$. Next if $s(y) = +\infty$, we set: $\bar{I}_\lambda^{\infty,y} = \lim_{n\to\infty} nI_{\lambda/n}^{\infty,y} \in]-\infty, +\infty]$ (The fact that the limit exists is an easy exercise, since the function $\varphi(t) = \bar{I}_t^{\infty,y}$ is subadditive on $[0, \lambda]$). In many cases this limit is trivial (either 0 or $+\infty$).

Finally, to take into account the effect of the translations (if Ω satisfies: $\forall R < \infty$, $\exists y \in \Omega$, $B(y, R) \subset \Omega$) we consider

$$s_R = \inf [\# \omega(x)/|x| \geqslant R, \quad x \in \bar{\Omega}] \leqslant +\infty.$$

For R large s_R is constant and we denote by s its value. We then set

$$\begin{cases} \bar{I}_\lambda^{\infty,\infty} = sI_{\lambda/s}^{\infty,\infty} & \text{si} \quad s < \infty \\ \bar{I}_\lambda^{\infty,\infty} = \lim_{n\to\infty} nI_{\lambda/n}^{\infty,e} & \text{if} \quad s = +\infty \end{cases} \tag{60}$$

The same heuristic considerations of the preceding section show that *the strict sub-additivity inequality.*

$$\bar{I}_\lambda < \bar{I}_\alpha + \bar{I}_{\lambda-\alpha}^\infty, \qquad \forall \alpha \in [0, \lambda[\tag{S.3}$$

is still *necessary and sufficient for the compactness of all minimizing sequences* of (58). And we have the same adaptations, extensions, variations as before for problems invariant by dilations, (some) translations. Furthermore if (S.3) fails, we know how compactness is lost on noncompact minimizing sequences.

In particular, in *the locally compact case*, $\bar{I}_\mu^\infty$ reduces to $I_\mu^{\infty,\infty}$ defined by (60); while if $\bar{\Omega}$ *is compact*, $\bar{I}_\lambda^{\infty,\infty}$, $\bar{I}_\lambda^{\infty,\infty}$ disappear and $\bar{I}_\lambda^\infty$ reduces to $\operatorname{Inf}_{y\in\bar{\Omega}} \bar{I}_\lambda^{\infty,y}$.

Before giving briefly two examples below (more may be found in section IV and in a future study), we would like to point out that in some vague sense symmetries may help to find a solution of the Euler equation associated with (58) or equivalently with (47) (if $\mathcal{E}$, J are $C^1, \ldots$) since $n\varphi\left(\frac{\lambda}{n}\right) \geqslant \varphi(\lambda)$ if φ is subadditive and since $n\varphi\left(\frac{\lambda}{n}\right)$ is «essentially nondecreasing» with respect to n (at least along multiples...) therefore $\bar{I}_\lambda^\infty$ essentially increases if s, $\inf_{y\in\bar{\Omega}} s(y)$ increase.

Another way to see this improvement of the conditions (S.1) – (S.2) is to observe that if $s = +\infty$, the first concentration-compactness lemma yields that we have either vanishing, or compactness. And recalling that if $\rho_n = u_n^2 + |\nabla u_n|^2$ vanishes then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2N/(N-2)$ (cf. P. L. Lions [55], [56]), we find back the compactness results of W. Strauss [75], P. L. Lions [63] in a very direct way. Similarly if $\operatorname{Inf}\{s(y) \mid y \in \bar{Q}\} = +\infty$, *no Dirac masses may form since J* would contain an infinite set $\omega(y)$ on which each Dirac mass δ_z has a fixed intensity thus contradicting the summability of the measure!

Example 3.1. Let $N \geqslant 3$, consider the functionals on $H^1(\mathbb{R}^N)$

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2} V(|x|)u^2 \, dx - \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|} \, dx \, dy$$

$$J(u) = \int_{\mathbb{R}^N} u^2 \, dx;$$

where $V \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ with $\frac{N}{2} \leqslant p, q < \infty$. If we do not use the spherical symmetry of V i.e. we only consider

$$I = \operatorname{Inf}\{\mathcal{E}(u)/J(u) = 1, \quad u \in H^1(\mathbb{R}^N)\}$$

then —cf. P. L. Lions [55], [59]— all minimizing sequences are relatively compact if and only if

$$I < I^\infty = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 \, dx - \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|} \, dx \, dy / J(u) = 1\right\}.$$

And if $V \geqslant 0$, $V \not\equiv 0$, there is *no* minimum.

On the other hand (this was observed in P. L. Lions [57] and it is clear in view of the above arguments) if we consider for $\lambda > 0$

$$\bar{I}_\lambda = \operatorname{Inf}\{\mathcal{E}(u)/J(u) = \lambda, \quad u \in H^1(\mathbb{R}^N), \quad u \text{ spherically symmetric}\},$$

then $\bar{I}_\lambda < 0$ and all minimizing sequences are compact and a minimum exists (thus $\bar{I}_1 > I$!). This is also clear in view of our arguments above: since (we are in the locally compact case) $\bar{\bar{I}}_\lambda^{\infty,\infty} = 0$, $\bar{I}_\lambda^{\infty,y} = 0$, $\forall y$ and $\bar{I}_\lambda^{\infty,\infty} = \lim n I_{\lambda/n}^\infty = 0$ and thus (S.3) is equivalent to $\bar{I}_\lambda < \bar{I}_\alpha$, $\forall \alpha \in]0, \lambda[$, and this is easily checked since $\bar{I}_\lambda < 0$.

Example 3.2. Let $N \geqslant 3$, consider the functionals defined on $\mathcal{D}^{1,2}(\Omega)$ —where $\Omega = \{x \in \mathbb{R}^N, \quad |x| > 1\}$— by

$$\mathcal{E}(u) = \int_\Omega a(|x|)|\nabla u|^2 \, dx, \qquad J(u) = \int_\Omega K(|x|)|u|^{2N/(N-2)} \, dx$$

where a, K are positive continuous, $a, K \to a^\infty, K^\infty > 0$ as $|x| \to \infty$. We then consider

$$\bar{I}_\lambda = \operatorname{Inf}|\mathcal{E}(u)/u \in D^{1,2}(\Omega), \quad J(u) = \lambda, \quad u \text{ spherically symmetric}\}.$$

We compute easily: $\bar{I}_\lambda^{\infty,\infty} = +\infty$, $\bar{I}_\lambda^{\infty,y} = +\infty$, $\forall y \in \bar{\Omega}$ and

$$\bar{I}_\lambda^\infty = \mathrm{Inf}\Big\{a^\infty \int_{\mathbb{R}^N} |\nabla u|^2\, dx / u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\quad u \text{ spherically symmetric},\\ K^\infty \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx = \lambda\Big\}.$$

And (S.3) reduces (since $\bar{I}_\lambda = \lambda^{(N-2)/N}\bar{I}_1$, $\bar{I}_\lambda^\infty = \lambda^{(N-2)/N}\bar{I}_1^\infty$) to

$$I_\lambda{}^\infty < \bar{I}_\lambda^\infty$$

If this condition holds, all minimizing sequences are compact and a minimum exists, while if $I_\lambda{}^\infty = \bar{I}_\lambda^\infty$ there exists a minimizing sequence which is not compact and any such sequence $(u_n)_n$ satisfies: $\exists(\sigma_n)_n \in]0, \infty[$ such that $\sigma_n \to 0$, $\sigma_n^{-(N-2)/2} u_n\left(\frac{\cdot}{\sigma_n}\right)$ is relatively compact and its limit points are minima of $\bar{I}_\lambda^\infty$.

4. Problems in compact regions

4.1 *Yamabe problem*

Our main goal in this section is to explain T. Aubin's results on Yamabe problem in the light of our general arguments. We first recall the nature of the problem.

Let (M, g) be a C^∞ N-dimensionnal Riemannian manifold. We denote by Δ the Laplace-Beltrami operator on (M, g); in local coordinates this operator is given by

$$\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x_j}\right)$$

where $\sum_{i,j} g_{ij}\, dx^i\, dx^j$ is the metric, $g^{ij} = (g_{ij})^{-1}$, $g = \det(g_{ij})$.

Let k be the scalar curvature of (M, g). One is interested in the determination of all functions K which can be realized as the scalar curvature of a metric which is pointwise conformal to g i.e. of a metric $\tilde{g}$ obtained by multiplying g by a positive function on M. Now if we introduce the unknown function u (positive on M) such that

$$\tilde{g} = u^{4/(N-2)} g,$$

the above condition on K is equivalent to the so-called Yamabe equation (see H. Yamabe [84], T. Aubin [9]; H. Eliasson [33] for the detailed computations)

$$-4\frac{N-1}{N-2}\Delta u + ku = Ku^{(N+2)/(N-2)} \quad \text{in } M, \qquad u > 0 \quad \text{in } M \tag{Y}$$

—where of course $N \geqslant 3$. In fact, H. Yamabe considered in [84] only the case when K is constant and claimed that in this case the problem could always be solved. As it was remarked by N. Trudinger [79], the argument in [84] was not complete and the case when $K \equiv 1$ is still an open question (at least for $3 \leqslant N \leqslant 5$).

Let us mention at this point that related questions concerning scalar curvature and deformations on variations of metrics are considered in J. L. Kazdan and F. W. Warner [(47], [48], [49], [50], J. L. Kazdan [46]; A. E. Fischer and J. E. Marsden [37]; J. P. Bourguignon and J. P. Ezin [17]; J. P. Bourguignon [16].

Let λ_1 denote the first eigenvalue of the operator

$$-4\frac{N-1}{N-2}\Delta + k$$

on $H^1(M)$; it is easily seen that:

i) if $\lambda_1 > 0$ and $K \leqslant 0$, no solution of (Y) exists;

ii) if $\lambda_1 = 0$: no solutions exists if $K \not\equiv 0$, $K \leqslant 0$ or $K \geqslant 0$; trivial solutions exists if $K \equiv 0$ (and are unique up to a multiplicative constant);

iii) if $\lambda_1 < 0$: no solution exists if $K \geqslant 0$ while if $K \leqslant 0$, $K \not\equiv 0$ it is a standard exercize on semilinear elliptic equations to show that (Y) has a unique positive solution (one can also make a few remarks of the same spirit if K has both signs). We refer to T. Aubin [9] for a brief exposition of these facts.

In view of these remarks, it is natural to assume

$$\lambda_1 > 0; \qquad \operatorname*{Max}_M K > 0. \tag{61}$$

In this case, one way of finding (possibly) solutions of (Y) is to look at the following minimization problem

$$I = \operatorname{Inf}\{\,\mathcal{E}(u)/u \in H^1(M), \quad J(u) = 1\,\} \tag{62}$$

where

$$\mathcal{E}(u) = \int_M 4\frac{N-1}{N-2}|\nabla u|^2 + ku^2, \qquad J(u) = \int_M K|u|^{2N/(N-2)}.$$

Then any minimum of (62) is, up to a change of sign and multiplication by a positive constant, a solution of (Y). Let us emphasize that the converse may be false! Let us also mention that, as long as (62) is concerned, it is not necessary to consider only a function k which is the scalar curvature and in what follows, k, K are *arbitrary functions* in $C(M)$ *satisfying* (61).

At this stage, we want to explicit the condition (S.1) (which, as it should be, will be the critical condition needed to solve (62)): first of all since

$I_\lambda = \lambda^{(N-2)/N} I > 0$ (where I_λ is the same infimum as in (62) with $J(u) = 1$ replaced by $J(u) = \lambda$), we see that (62) reduces to

$$I < I^\infty \tag{16}$$

and we have to compute

$$I^\infty = \operatorname*{Inf}_{y \in M} I^{\infty, y};$$

$$I^{\infty,y} = \operatorname{Inf}\{\mathcal{E}^{\infty,y}(u)/u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad J^{\infty,y}(u) = 1\}$$

$\mathcal{E}^{\infty,y}(u)$, $J^{\infty,y}(u)$ being obtained by concentrating u in $\mathbb{R}^N$ at 0 by the dilations $\left(\sigma^{-(N-2)/2}u\left(\frac{\cdot}{\sigma}\right),\ \varphi \to 0_+\right)$ and bringing it back at y on M by a local chart. Remarking that if $a_{ij} = a_{ij} > 0$

$$I(a,\lambda) = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx/u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\,dx = \lambda\right\}$$
$$= \det(a_{ij})^{1/N}\lambda^{N/(N-2)}I_0$$

where

$$I_0 = \operatorname{Inf}\left\{\int_{\mathbb{R}^N} |\nabla u|^2\,dx/u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\,dx = 1\right\}$$

(i.e. $I_0^{-1/2}$ is the best constant for Sobolev inequalities, cf. Part 1 [65]); we deduce easily that

$$\begin{aligned} I^{\infty,y} &= 4\frac{N-1}{N-2}K(y)^{-(N-2)/N}I_0 \qquad &&\text{if } K(y) > 0 \\ &= +\infty &&\text{if } K(y) \leqslant 0. \end{aligned}$$

Therefore we have

$$I^\infty = 4\frac{N-1}{N-2}\left(\max_M K\right)^{-(N-2)/N}I_0; \tag{63}$$

and we already know that $I \leqslant I^\infty$.

Theorem 4.1. *We assume* (61). *If $I \leqslant I^\infty$, any minimizing sequence of* (62) *is relatively compact in $H^1(M)$ and a minimum exists. If $I = I^\infty$, there exist minimizing sequences which are not compact and any such sequence $(u_n)_n$ satisfies* (*up to subsequences*)

$$\begin{cases} u_n \to 0 \quad \text{weakly in } H^1(M), \\ |u_n|^{2N/(N-2)} \to \left(\max_M K\right)^{-1}\delta_{x_0}, \quad |\nabla u_n|^2 \to I^\infty\delta_{x_0} \qquad (\text{in } \mathcal{D}'(M)) \\ K(x_0) = \max_M K. \end{cases} \tag{64}$$

Let us immediately mention that this result is a small extension of a *result due to T. Aubin* [6] (see also [9], [11], [12]) where it is proved that, if $I < I^\infty$, a minimum exists by a different method. In addition in T. Aubin [6], (16) is discussed in details and in particular if k is the scalar curvature, $K \equiv 1$ (*Yamabe original problem*) it is proved that (16) *holds for «most» manifolds M if $N \geqslant 6$.*

Remark 4.1 In addition to (64), we may prove that one can find cut-off functions $\xi_n \in C^\infty(M)$ supported in $B(x_0, \epsilon_n)$ with $\epsilon_n \to 0$ such that $v_n = \xi_n u_n \to 0$ in $H^1(M)$ strongly, and if w_n is the sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ obtained from v_n by local charts, there exists $(\sigma_n)_n$ in $]0, \infty[$, $(y_n)_n$ in $\mathbb{R}^N$ such that

$$\begin{cases} \sigma_n \to \infty, \quad y_n/\sigma_n \to 0, \quad \sigma_n^{-(N-2)/2} w_n((\cdot + y_n)/\sigma_n) \text{ is relatively} \\ \text{compact in } \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{and its limit points are minima of } I^\infty. \end{cases}$$

The proof of Theorem 4.1 is an easy adaptation of arguments given before (in particular Theorem 1.2 in Part 1 [65]): we just observe that M being compact the tightness of $\rho_n = |\nabla u_n|^2 + |u_n|^{2N/(N-2)}$ is automatic and that we have the:

Lemma 4.1. *Let $(u_n)_n$ converge weakly to u in $H^1(M)$. We may assume that $|\nabla u_n|^2$, $|u_n|^{2N/(N-2)}$ converge weakly to some measures μ, ν. Then we have*

$$\nu = |u|^{2N/(N-2)} + \sum_{j \in J} \nu_j \delta_{x_j} \tag{65}$$

$$\mu \geqslant |\nabla u|^2 + I_0 \sum_{j \in J} \nu_j^{(N-2)/N} \delta_{x_j} \tag{66}$$

for some at most countable family J, and where $\nu_j > 0$, x_j are distinct points of M.

We skip the proof of this lemma since it is totally similar to the proof of Lemma 1.1: let us only point out that I_0 in (66) corresponds to $\mathbb{R}^N$ $(\simeq T_{x_j}M)$ and that this comes from the localization procedure around x_j used in the proof of Lemma 1.1.

Remark 4.2. Of course similar results hold for $(u_n)_n$ bounded sequence in $W^{m,p}(M)$.

Remark 4.3. The proof of T. Aubin [6] concerning the existence of a minimum if (16) holds uses heavily the «best constant» C_λ of the Sobolev inequality on M

$$\left(\int_M |u|^{2N/(N-2)}\right)^{(N-2)/2N} \leqslant C_\lambda \left(\int_M |\nabla u|^2 + \lambda u^2\right)^{1/2}$$

and the fact proved in T. Aubin [7] that $C_\lambda \downarrow C_0$ as $\lambda \uparrow \infty$ where C_0 is the best constant of the Sobolev inequality in $\mathbb{R}^N$. In fact our methods also prove this elementary fact easily: indeed consider

$$I^\lambda = \operatorname{Inf}\left\{ \int_M |\Delta u|^2 + \lambda u^2 \Big/ \int_M |u|^{2N/(N-2)} = 1, u \in H^1(M) \right\}.$$

Applying Theorem IV.1 we see that $I^\lambda \leqslant I_0$, $\forall \lambda > 0$, $I^\lambda \downarrow$ as $\lambda \uparrow$, I^λ is achieved if $I^\lambda < I_0$ (and this happens for λ small since $I^\lambda \downarrow 0$ as $\lambda \downarrow 0_+$). Therefore either $I^\lambda = I_0$ for $\lambda \leqslant \lambda_0 > 0$ (and this is a very interesting situation where the best constant C_0 is achieved on M!), or $I^\lambda < I_0$. We claim that in this case $I^\lambda \uparrow I_0$ as $\lambda \uparrow +\infty$. If this is not the case, denoting by u_λ the minimum corresponding to I^λ, we have

$$\int_M |\nabla u_\lambda|^2 \underset{\lambda \to \infty}{\longrightarrow} \alpha I_0, \qquad u_\lambda \underset{\lambda \to \infty}{\longrightarrow} 0 \quad \text{in} \quad L^2(M).$$

Thus

$$|\nabla u_\lambda|^2 \to \mu \geqslant I_0 \sum_{j \in J} \nu_j^{(N-2)/N} \delta_{x_j},$$

$$|u_j|^{2N/(N-2)} \to \nu = \sum_{j \in J} \nu_j \delta_{x_j},$$

and this would give

$$I_0 > \alpha \geqslant I_0 \sum_{j \in J} \nu_j^{(N-2)/N} \geqslant I_0 \Big(\sum_{j \in J} \nu_j \Big)^{(N-2)/N} = I_0.$$

Therefore we have proved not only that $I^\lambda \uparrow I_0$ as $\lambda \uparrow \infty$ but also that either $I^\lambda = I_0$ for λ large or I^λ is achieved and any corresponding minimum u_λ satisfies (up to subsequences)

$$\int_M |\nabla u_\lambda|^2 \underset{\lambda}{\to} I_0, \qquad \int_M u_\lambda^2 \underset{\lambda}{\to} 0$$

$$|\nabla u_\lambda|^2 \underset{\lambda}{\to} I_0 \delta_{x_0}, \qquad |u_\lambda|^{2N/(N-2)} \underset{\lambda}{\to} \delta_{x_0}$$

for some $x_0 \in M$.

We next present some new existence results concerning (Y) using «symmetries». Assume that (M, g) is embedded in $\mathbb{R}^p$ (for some $p > N$) and that (M, g), $\mathcal{E}$, J are invariant under the action of a group G of one to one transformations of $\mathbb{R}^p$. In particular we have: $\forall h \in G$

$$\mathcal{E}(u(h \cdot)) = \mathcal{E}(u(\cdot)), \quad J(u(h \cdot)) = J(u), \qquad \forall u \in C^\infty(M).$$

Our typical example is S^N with the usual metric, then we may take for G any subgroup of the group of orthogonal transformations $O(p)$ and our assumption just means that K is invariant by G.

Any minimum of the following minimization problem is still a solution of (Y) (invariant by G, giving a new metric invariant by G)

$$\bar{I} = \operatorname{Inf}\{\mathcal{E}(u)/J(u) = 1, \quad u \in H^1(M), \quad u \text{ is } G\text{-invariant}\}. \tag{65}$$

We denote by: $\omega(y) = \{h \cdot y/h \in G\}$, $s(y) = \#\omega(y)$ and we set

$$\begin{cases} \bar{I}^{\infty,y} = K(y)^{-(N-2)/N} s(y)^{2/N} I_0 & \text{if} \quad K(y) > 0, \quad s(y) < \infty \\ \bar{I}^{\infty,y} = +\infty & \text{if} \quad K(y) \leqslant 0, \quad s(y) = +\infty. \end{cases} \tag{66}$$

$$\bar{I} = \operatorname*{Inf}_{y \in M} \bar{I}^{\infty,y} \tag{67}$$

(notice that $s(y)$, $\bar{I}^{\infty,y}$ are not, in general, continuous on M).

We have immediately the following:

Theorem 4.2. *We assume* (61). *Then $\bar{I} \leqslant \bar{I}^\infty$ and if $\bar{I} < \bar{I}^\infty$ all minimizing sequences of* (65) *are relatively compact in $H^1(M)$ and there exists a minimum. While if $\bar{I} = \bar{I}^\infty$, there exists a minimizing sequence which is not relatively compact and any such sequence $(u_n)_n$ satisfies*

$$u_n \to 0 \quad \textit{weakly in} \quad H^1(M); \qquad |\nabla u_n|^2 \to \frac{\bar{I}^\infty}{s(x_0)} \sum_{z \in \omega(x_0)} \delta_z;$$
$$|u_n|^{2N/(N-2)} \to K(x_0)^{-1} \frac{1}{s(x_0)} \sum_{z \in \omega(x_0)} \delta_z$$

for some x_0 satisfying

$$K(x_0) > 0, \quad s(x_0) < \infty, \qquad K(x_0)^{-(N-2)/N} s(x_0)^{2/N} = \operatorname*{Inf}_{y \in M} K(y)^{-(N-2)/N} s(y)^{2/N}$$

Remark 4.4. If $K \equiv 1$ and $\operatorname*{Inf}_{y \in M} s(y) = p > 1$ then for p large $\bar{I} < \bar{I}^\infty$: indeed take $u \equiv 1$ in (65), $\bar{I} \leqslant \int_M k < p^{2/M} I_0$ for p large.

Remark 4.5. If $M = S^N$ and K is invariant by a subgroup G of $O(N+1)$ such that $\operatorname*{Inf}_{y \in M} s(y) = p$. Then

$$\bar{I}^\infty \geqslant p^{2/N} (\max K)^{-(N-2)/N} I_0.$$

If (for example) $M = S^2$, if $K(x_1, x_2, x_3) = K(x_3)$ (for $x = (x_1, x_2, x_3) \in S^2$) and $K(\pm 1) \leqslant 0$ or if $K(x_1, x_2, x_3) = K(x_2, x_3)$ and $K \leqslant 0$ for $x_2^2 + x_3^2 = 1$, then $\bar{I}^{\infty,y} = +\infty$, $\forall y \in M$ and a minimum exists. These last examples may also be obtained by symmetrization and results corresponding to Appendix 2.

4.2 *Related problems*

In this section, we consider various problems strongly related to Yamabe equation: the first one concerns Yamabe equation in a bounded open set Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions and we will present some variants of results due to H. Brézis and L Nirenberg [23] and also some new results when symmetries occur. We will also briefly consider the case of Newmann conditions. The second class of problems we consider is the problem introduced in Cherrier [25] which extends the Yamabe problem to manifolds with boundary.

We thus begin with the following problem: let Ω be a bounded open set of $\mathbb{R}^N$ with $N \geqslant 3$, let $a_{ij}(x)$, $c(x)$, $K(x)$ be continuous functions of $\bar{\Omega}$ satisfying

$$(a_{ij}(x)) = (a_{ji}(x)) \geqslant \nu I_N, \qquad \forall x \in \bar{\Omega}, \quad \text{for some} \quad \nu > 0 \tag{68}$$

$$\forall \varphi \in \mathcal{D}(\Omega), \qquad \int_\Omega \sum_{i,j} a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + c\varphi^2 \, dx \geqslant \alpha \| \varphi \|^2_{H_0^1(\Omega)}$$

for some $\alpha > 0$

$$\max_\Omega K > 0. \tag{70}$$

We want to solve the following equation

$$-\sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + cu = K u^{(N+2)/(N-2)} \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0, \quad \text{in } \Omega, \tag{71}$$

and we thus consider

$$I = \operatorname{Inf} \{ \mathcal{E}(u) / u \in H_0^1(\Omega), \quad J(u) = 1 \}. \tag{72}$$

where

$$\mathcal{E}(u) = \int_\Omega \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + cu^2 \, dx, \qquad J(u) = \int_\Omega K |u|^{2N/(N-2)} \, dx.$$

In view of the homogeneity of the problem ($I_\lambda = \lambda^{(N-2)/N} I$) (S.1) once more reduces to

$$I < I^\infty \tag{16}$$

where $I^\infty = \operatorname{Min}_{y \in \Omega} I^{\infty, y}$ and

$$I^{\infty,y} = \operatorname{Min} \left\{ \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(y) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \, dx / u \in \mathcal{D}^{1,2}, \right.$$
$$\left. \int_{\mathbb{R}^N} K(y) |u|^{2N/(N-2)} \, dx = 1 \right\}$$
$$= \det(a_{ij}(y))^{1/N} K(y)^{-(N-2)/N} I_0, \quad \text{if } K(y) > 0;$$

$$I^{\infty,y} = +\infty \qquad \text{if } K(y) \leqslant 0.$$

The strict analogue of Theorem 4.1 is the following result (and we skip its proof):

Corollary 4.1. *We assume* (68) – (70). *If* (16) *holds, any minimizing sequence is relatively compact in* $H_0^1(\Omega)$ *and there exists a minimum of* (72) *and a solution of* (71). *If* (16) *does not hold i.e.* $I = I^\infty$, *there exist non-compact minimizing sequences and any such sequence* $(u_n)_n$ *satisfies (up to subsequences)*

$$\left\{\begin{array}{l} u_n \to 0 \quad \textit{weakly in} \quad H_0^1(\Omega); \quad \sum_{i,j} a_{ij}\dfrac{\partial u_n}{\partial x_i}\dfrac{\partial u_n}{\partial x_j} \to (\det a_{ij}(x^0))^{1/N} I_0 \delta_{x_0}, \\ |u_n|^{2N/(N-2)} \to K(x_0)^{-1}\delta_{x_0}, \quad \textit{for some} \quad x^0 \in \bar\Omega \textit{ satisfying} \\ (\det a_{ij}(x^0))^{1/N} K(x^0)^{-(N-2)/N} = \operatorname{Min}_{x\in\Omega} (\det a_{ij}(x))^{1/N} K(x)^{-(N-2)/N} \\ \sigma_n \to +\infty, \quad \exists y_n \in \mathbb{R}^N, \quad y_n/\sigma_n \to x^0, \quad \sigma_n^{-(N-2)/N} u_n((\cdot + y_n)/\sigma_n) \to \bar u \\ \hfill \textit{minimum of } I^{\infty,x^0} \end{array}\right. \tag{73}$$

Of course we know explicitly the minima of I^{∞,x_0} (cf. Part 1 [65]).

Remark 4.6. In H Brézis and L. Nirenberg [23], the case when $a_{ij}(x)$, c, K are independent of x is treated: not only the fact that (16) implies the compactness of minimizing sequences is proved but also discussed in details (following their argument we discuss below (16)). But it is worth pointing out that the method used in [23] to pass to the limit on minimizing sequences cannot work as such in our general setting: indeed the main point is to avoid the weak convergence of $(u_n)_n$ to 0. In [23], one simply says that if $u_n \to 0$ then

$$\mathcal{E}(u_n) - \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}\,dx \to 0$$

but this is not enough to use (16)! The loss may be seen on the fact that we give criteria below which show that (16) holds for $c(x) \equiv \lambda \geqslant 0$ under appropiate conditions on a_{ij}, K.

Let us also mention that the difference on the methods may be seen on the following («artificial») problem

$$I = \operatorname{Inf}\left\{\int_\Omega |\nabla u|^p\,dx - \lambda\int_\Omega |u|^p\,dx / u \in W_0^{1,p}(\Omega), \quad \int_\Omega |u|^q\,dx = 1\right\} \tag{74}$$

where $1 < p < N$, $q = Np(N-p)^{-1}$, $0 < \lambda < \lambda_1^p$ (λ_1^p is the largest constant $\mu > 0$ such that: $\mu\int_\Omega |u|^p\,dx \leqslant \int_\Omega |\nabla u|^p\,dx$). Then if $p \neq 2$, the arguments of H. Brézis and L. Nirenberg [23] do not apply anymore while we may still prove that if $I < I^\infty$, (74) is «well-posed».

Remark 4.7. In order to analyse (16), we may follow the choice of T. Aubin [6], H. Brézis and L. Nirenberg [23] (this choice was explained in section III) and we find, if $N \geqslant 5$ for example, that (16) holds provided there exists a minimum point x^0 of $(\det(a_{ij}))^{1/N}K^{-(N-2)/N}$ on $\bar{\Omega}$ which lies in Ω and such that

$$\int \sum_{i,j,k,l} a_{ij,kl}(x^0) y_i y_j y_k y_l (1 + |y|^2)^{-N} dy$$
$$< C_N^1 \lambda + C_N^2 \frac{(\det a_{ij}(x^0))^{1/N}}{K(x^0)} \cdot \int \sum_{i,j} K_{ij}(x^0) y_i y_j (1 + |y|^2)^{-N} dy$$

for some explicit positive constants C_N^1, C_N^2. Here we took $c \equiv -\lambda$. Observe that a_{ij}, K are independent of x, this condition holds automatically and it may hold even if $\lambda \leqslant 0$.

We now turn to a problem which is somewhat similar to the previous ones: we assume that Ω is smooth, (68), (70) and

$$\forall \varphi \in H^1(\Omega), \qquad \mathcal{E}(\varphi) \geqslant \alpha \|\varphi\|_{H'}^2 \tag{69'}$$

for some $\alpha > 0$. And we consider

$$I = \operatorname{Inf}\{\mathcal{E}(u)/u \in H^1(\Omega), \quad J(u) = 1\} \tag{74}$$

Again (16) is the key assumption and we have to compute I^∞ i.e. $I^{\infty,y}$

$$\begin{cases} \text{if} \quad y \in \bar{\Omega}, & K(y) \leqslant 0, \quad I^{\infty,y} = +\infty \\ \text{if} \quad y \in \Omega, & K(y) > 0, \quad I^{\infty,y} = (\det a_{ij}(y))^{1/N} K(y)^{-(N-2)/N} I_0 \\ \text{if} \quad y \in \partial\Omega, & K(y) > 0, \quad I^{\infty,y} = 2^{-2/N} (\det a_{ij}(y))^{1/N} K(y)^{-(N-2)/N} I_0 \end{cases}$$

(observe that K is l.s.c. on $\bar{\Omega}$), and $I^\infty = \operatorname{Min}_{y \in \Omega} I^{\infty,y}$. The value of $I^{\infty,y}$ if $y \in \partial\Omega$, $K(y) > 0$ comes from the fact that, if we concentrate u at y on $\partial\Omega$, extending u evenly across $\partial\Omega$, we obtain at the limit, since $\partial\Omega$ is smooth, the above value of $I^{\infty,y}$ that is

$$I^{\infty,y} = \frac{1}{2} \operatorname{Inf} \left\{ \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(y) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx / u \in \mathcal{D}^{1,2}, \int_{\mathbb{R}^N} K(y) |u|^{2N/(N-2)} dx = 2 \right\}.$$

And exactly as before we have:

Corollary 4.2. *We assume* (68), (69′), (70) *and that* Ω *is smooth. If* (16) *holds, any minimizing sequence is relatively compact in* $H^1(\Omega)$ *and there exists a minhere exists a minimum of* (74).

Remark 4.7. If $I = I^\infty$, we may analyze (as in (73) what happens for the non-compact minimizing sequences. Let us also mention that we could treat as well problems where i) the nonlinearity $K(x)|u|^{2N/(N-2)}$ is replaced by a general nonlinearity $F(x, t)$ such that

$$\lim_{|t|\to\infty} F^+(x,t)|t|^{-2N/(N-2)} = K_+(x), \text{ uniformly for } x \in \bar{\Omega};$$

ii) we consider nonlinear terms on the boundary for example

$$J(u) = \int_{\partial\Omega} G(x, u)\, dS$$

with $\lim_{|t|\to\infty} G^+(x,t)|t|^{-2(N-1)((N-2)} = K'^+(x)$, uniformly for $x \in \partial\Omega$.

Example 4.1. Let $a_{ij}(x) \equiv \delta_{ij}$, $K(x) \equiv 1$, $c(x) \equiv \lambda$; (69′) is equivalent to $\lambda > 0$. Choosing $\bar{u} \equiv \operatorname{meas}(\Omega)^{-(N-2)/2N}$, we see that (16) holds if $\lambda \in]0, \lambda_0[$ with $\lambda_0 = I^\infty \operatorname{meas}(\Omega)^{-2/N} = 2^{-2/N} I_0 \operatorname{meas}(\Omega)^{-2/N}$. In addition for λ small the minimum is unique and is $\bar{u}$ (easy consequence of the implicit function theorem). We do not know much more information except that if $\lambda_2 - \frac{4}{N-2}\lambda_0 < 0$ —where λ_2 is the second eigenvalue of $-\Delta$ on $H^1(\Omega)$— then for λ in $]0, \lambda_0 + \delta[$ where $\delta > 0$, (16) holds and any minimum is constant.

Indeed, if for $\lambda = \lambda_0$, $\bar{u}$ were a minimum, writing the second-order condition for the minimality of $\bar{u}$, we would find

$$\forall \varphi \in H^1(\Omega), \qquad \int_\Omega |\nabla\varphi|^2 - \frac{4\lambda_0}{N-2}\varphi^2\, dx + \frac{4\lambda_0}{(N-2)} \operatorname{meas}(\Omega)^{-1}\left(\int_\Omega \varphi\right)^2 \geqslant 0$$

and this is impossible if $\lambda_2 - \frac{4}{N-2}\lambda_0 < 0$. Let us mention that it is easy to find examples of sets Ω for which this inequality is true.

Remark 4.7. It is interesting to observe that, for $y \in \partial\Omega$, the quantity $I^{\infty,y}$ depends on the regularity of $\partial\Omega$ at y. For example if Ω is given by

$$\Omega = \Omega_1 \times, \ldots, \times \Omega_m, \quad \Omega_i \text{ smooth region of } \mathbb{R}^{n_i}$$

then

$$\text{for } \; y \in \Omega_1 \times \Omega_2 \times \ldots \times \Omega_m,\; I_y^\infty = (\det a_{ij}(y))^{1/N} K^+(y)^{-(N-2)/N} I_0$$

while

$$\text{if } \; y \in \partial\Omega_1 \times \ldots \times \Omega_m, \quad I_y^\infty = m^{-2/N} (\det a_{ij}(y))^{1/N} K^+(y)^{-(N-2)/N} I_0$$

i.e.

$$I_y^\infty = \frac{1}{m} \operatorname{Inf}\left\{ \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}(y) \frac{\partial u}{\partial x_i} \frac{u}{\partial x_j} dx \Big/ \int_{\mathbb{R}^N} K^+(y)|u|^{2N/(N-2)}\, dx = m \right\}$$

We conclude these considerations on Yamabe-type problems by considering the effect of symmetries on the existence of solutions to

$$-\Delta u = u^{(N+2)/(N-2)} \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega. \tag{75}$$

It is a well-known result due to S. Pohozaev [70] that if Ω is star-shaped with respect to, say, 0 then (75) has no solution. On the other hand if Ω is an annulus ($\Omega = \{x \in \mathbb{R}^N, r < |x| < R\}$, for some $r, R > 0$), it is well-known that (75) has a radial solution (see for example Kazdan and Warner [51]). If we wish the understand the implications of these two observations, we need to introduce the following setting: assume that $0 \notin \bar{\Omega}$, that Ω is invariant by a group of orthogonal transformations of $\mathbb{R}^N$ and set

$$\begin{aligned} \bar{I} &= \operatorname{Inf}\left\{ \int_\Omega |\nabla u|^2 \, dx \Big/ \int_\Omega |u|^{2N/(N-2)} \, dx = 1, \quad u \in H_0^1(\Omega), \quad u \text{ is } G\text{-invariant} \right\}; \\ s &= \operatorname*{Inf}_{y \in \bar{\Omega}} s(y), \quad s(y) = \#\{a = g \cdot y, \quad g \in G\}. \end{aligned} \tag{76}$$

Our methods immediately yield:

Corollary 4.3. *Let $\bar{I}^\infty = s^{2/N} I_0$ if $s < \infty$, $\bar{I}^\infty = +\infty$ if $s = +\infty$. Then if $\bar{I} < \bar{I}^\infty$, all minimizing sequences of* (76) *are relatively compact in $H_0^1(\Omega)$ and there exists a minimum of* (76) *and a solution of* (75).

Remark 4.8. if $\Omega = \Omega_1 \times \Omega_2$ with Ω_1 arbitrary in $\mathbb{R}^{N_1}$ ($N_1 \geqslant 0$) and $\Omega_2 = \{x_2 \in \mathbb{R}^{N-N_1}, r < |x_2| < R\}$ for some $r, R > 0$, then $s = +\infty$ and (75) has a solution (this can also be seen on the results of Appendix 2). If $0 \notin \bar{\Omega}$ and $s \geqslant 2$ ($s \geqslant 2$ if Ω is symmetric with respect to 0!), and if $T_r^R = \{x \in \mathbb{R}^N / r < |x| < R\} \subset \Omega$ for some $0 < r < R$, then clearly

$$\begin{aligned} \bar{I} \leqslant I(r, R) = \operatorname{Min}\Big\{ \int_{T_r^R} |\nabla u|^2 \, dx \Big/ \int_{T_r^R} |u|^{2N/(N-2)} \, dx &= 1, \\ u \in H_0^1(T_r^R), \quad u \text{ is spherically symmetric} \Big\}&. \end{aligned}$$

By the dilation invariance $I(r, R) = I\left(\frac{r}{R}, 1\right) = \lambda\left(\frac{r}{R}\right)$ and λ is a continuous, increasing function on $]0, 1[$ such that

$$\lim_{t \to 0+} \lambda(t) = I_0, \qquad \lim_{t \to 1-} \lambda(t) = +\infty.$$

Hence there exists a unique d_s such that

$$\lambda(d_s) = s^{2/N} I_0, \qquad d_s \to 1 \quad \text{if} \quad \to +\infty$$

and *thus if $\frac{r}{R} < d_s$, the condition $\bar{I} < \bar{I}^\infty$ is satisfied.*

Recently, by a critical point argument (instead of a minimization argument) J. M. Coron [30] —using arguments of section 4.6— was able to prove the existence of a solution of (75) without using symmetries if we assume $T_r^R \subset \Omega$ and $\frac{r}{R} < d_2$.

We conclude this section by a last example of applications of our arguments namely the problem introduced in P. Cherrier [25]: let $(\bar{M}, g)$ be some N dimensionnal compact Riemannian manifold with boundary, let N be its boundary endowed with the Riemannian structure induced by $\bar{M}$ and let $M = \bar{M} \backslash N$, we assume that $\bar{M}$ is orientable and we denote by $\frac{d}{dn}$ the derivation with respect to the vector field of outward unitary vectors (for the metric g) normal to N (in $\bar{M}$). If we look for a new metric $\tilde{g}$ pointwise conformal to g: $\tilde{g} = u^{4/(N-2)} g$ for some $u > 0$ on $\bar{M}$ such that the new scalar curvature and the new mean curvature are prescribed functions K, K', we are led to the following equation —see P. Cherrier [25]

$$\begin{cases} -\Delta u + ku = K u^{(N+2)/(N-2)} & \text{in } M \\ \dfrac{\partial u}{\partial n} + k'u = K' u^{N/(N-2)} & \text{on } M \end{cases} \tag{77}$$

where $k, k', K, K' \in C(\bar{M})$ and where we assume (to simplify)

$$K, K' \geqslant 0 \quad \text{on} \quad \bar{M}; \quad \max_{\bar{M}} (K + K') > 0; \quad \lambda_1 > 0 \tag{78}$$

where λ_1 is the first eigenvalue of the operator $(-\Delta + k)$ on $H^1(M)$ with the boundary condition $\left(\frac{\partial u}{\partial n} + k'u = 0\right)$ i.e.

$$\lambda_1 = \operatorname{Min}\left\{ \int_M |\nabla u|^2 + ku^2 + \int_N k'u^2 / u \in H^1 M), \quad \int_M u^2 = 1 \right\}.$$

In [25], conditions are given for a solvability of (77) with K, K' replaced by $\theta K, \theta K'$ qhere θ is some Lagrange multiplier. Using our method we may extend the results of [25] (for $N \geqslant 3$) but we prefer to solve directly the exact problem. To this end, we consider the artificial constraint method i.e.

$$I = \operatorname{Inf}\{ \mathcal{E}(u) / u \in H^1(M), \quad J(u) = 0, \quad u \not\equiv 0 \} \tag{79}$$

where

$$\mathcal{E}(u) = \int_M |\nabla u|^2 + ku^2 - \frac{N-2}{N} K |u|^{2N/(N-2)} + \\ + \int_N k'u^2 - \frac{N-1}{N} K' |u|^{2(N-1)/(N-2)},$$

$$J(u) = \langle \mathcal{E}'(u), u \rangle = \int_M |\nabla u|^2 + ku^2 - K|u|^{2N/(N-2)} + \\ + \int_N k'u^2 - K'|u|^{2(N-1)/(N-2)}.$$

We next compute $I^{\infty,y}$:

i) if $y \in \bar{M}$, $K(y) = 0$ or if $y \in N$, $K(y) = K'(y) = 0$ then $I^{\infty,y} = +\infty$

ii) if $y \in M$, $K(y) > 0$, $I^{\infty,y}$ is given by

$$I^{\infty,y} = \operatorname{Min}\left\{ \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N-2}{N} K(y)|u|^{2N/(N-2)}\, dx / u \in \mathfrak{D}^{1,2}(\mathbb{R}^N), \right.$$

$$\left. u \not\equiv 0, \quad \int_{\mathbb{R}^N} |\nabla u|^2\, dx = \int_{\mathbb{R}^N} K(y)|u|^{2N/(N-2)}\, dx \right\}$$

$$= \frac{2}{N} I_0^{N/2} K(y)^{-(N-2)/2}$$

iii) if $y \in N$, $K(y) + K'(y) > 0$, $I^{\infty,y}$ is given by

$$I^{\infty,y} = \operatorname{Min}\left\{ \int_{(x_N > 0)} |\nabla u|^2 - \frac{N-2}{N} K(y)|u|^{2N/(N-2)}\, dx + \right.$$

$$+ \frac{N-2}{N-1} \int_{(x_N = 0)} K'(y)|u|^{2(N-1)/(N-2)}\, dx' /$$

$$/ u \in \mathfrak{D}^{1,2}(x_N > 0),\ u \not\equiv 0, \quad \int_{(x_N > 0)} |\nabla u|^2 - K(y)|u|^{2N/(N-2)}\, \mathrm{d}x =$$

$$\left. = \int_{(x_N = 0)} K'(y)|u|^{2(N-1)/(N-2)}\, dx' \right\}.$$

And we let $I^\infty = \operatorname{Min}_{y \in \bar{M}} I^{\infty,y}$. We obtain as before the

Theorem 4.3. *We assume* (78). *If $I < I^\infty$, any minimizing sequence is relatively compact in $H^1(M)$ and there exist a minimum of* (79) *and a solution of* (77). *If $I = I^\infty$, there exist non compact minimizing sequences and any such sequence $(u_n)_n$ satisfies* (*up to subsequences*)

$$\begin{cases} u_n \to 0 \quad \text{weakly in} \quad H^1(M); \quad |\nabla u_n|^2 \to \alpha\delta_{x^0}, \quad |u_n|^{2N/(N-2)} \to \beta\delta_{x^0} \\ \mu_n \to \gamma\delta_{x^0} \quad \text{for some} \quad x^0 \in \bar{M} \quad \text{which minimizes} \quad I^{\infty,x} \quad \text{on} \quad \bar{M}; \\ \text{if} \quad x^0 \in M, \quad \alpha = \dfrac{N}{2} I^\infty, \quad \beta = \dfrac{N}{2} K(x^0)^{-1} I^\infty, \quad \gamma = 0, \\ \text{if } x^0 \in N, \quad \alpha - \dfrac{N-2}{N} K(x^0)\beta - \dfrac{N-2}{N-1} K'(x^0)\gamma = I^\infty, \\ \qquad \alpha - K(x^0)\beta - K'(x^0)\gamma = 0 \end{cases}$$

where μ_n is the measure on $\bar{M}$, supported in N such that

$$\forall \varphi \in C(\bar{M}), \qquad \int \varphi \, d\mu_n = \int_N |u_n|^{2(N-1)/(N-2)} \varphi .$$

4.3. *An inequality for holomorphic functions*

In this section, we want to discuss some inequalities for holomorphic functions. Let Ω be a smooth domain of C: if $\Gamma = \partial\Omega$, we consider the space $E^p(\Omega)$ (for $p \geqslant 1$) of holomorphic functions in Ω with traces on Γ in L^p i.e. (for example) holomorphic functions f such that

$$\overline{\lim_{\epsilon \to 0}} \int_{\Gamma_\epsilon} |f(z)|^p \, |dz| < \infty$$

where $\Gamma_\epsilon = \{ G(z, z_0) = \epsilon \}$, if $z_0 \in \Omega$, $G(z, z_0)$ is the Green's function. The notation

$$\int_\Gamma |f(z)|^p |dx| = \overline{\lim_{\epsilon \to 0}} \int_{\Gamma_\epsilon} |f(z)|^p |dz|$$

will be used everywhere below. Then if $f \in E^P(\Omega)$, the following inequality holds

$$\left(\int_\Omega |f(z)|^{2p} \, dx \, dy \right)^{1/2p} \leqslant C_p \left(\int_\Gamma |f(z)|^p |dz| \right)^{1/p} \tag{80}$$

We will see below that if $p > 1$, this inequality is very easy to prove. If $p = 1$, it was proved by Carleman [24], Aronszajn [4] for simply connected domains Ω and by S. Jacobs [43] for arbitrary domains.

If $p > 1$, one just needs to observe that $|f(z)| = u(x, y)$ is subharmonic and thus by the maximum principle: $|f(z)| \leqslant w(x, y)$ where

$$-\Delta w = 0 \quad \text{in} \quad \Omega, \qquad w = |f| \quad \text{on} \quad \Gamma$$

this boundary value problem may be solved by duality one finds

$$\| w \|_{W^{1/p, p}(\Omega)} \leqslant C \| f \|_{L^p(\Gamma)}$$

and we obtain (80) using Sobolev inequalities. It is worth pointing out that such an argument is false for $p = 1$.

In S. Jacobs [4] the question of the best constant C_1 was solved for $p = 1$ for arbitrary domains. If Ω is simple connected $C_1 = (4\pi)^{-1/2}$ and this best constant is achieved for the Bergman kernel function. For a multiply connected domain, the problem to solve is

$$I = \operatorname{Inf} \left\{ \int_\Gamma |f|^p |dz| \Big/ f \in E^p(\Omega), \quad \int_\Omega |f|^{2p} \, dx \, dy = 1 \right\} \tag{81}$$

(if $p = 1$, Ω is simply connected, the result recalled above just says: $I = (4\pi)^{1/2}$).

The underlying dilatations invariance is

$$f \to \sigma^{-1/p} f\left(\frac{\cdot}{\sigma}\right)$$

Notice also that I is not changed if we replace Ω by Ω' provided Ω and Ω' are conformally equivalent, and that the functionals are preserved by conformal self maps of Ω.

The main result we want to discuss is the:

Theorem 4.4. (S. Jacobs [41]). *Let $p = 1$, Ω be multiply connected then $I < (4\pi)^{1/2}$ and any minimizing sequence of* (81) *is relatively compact in $L^2(\Omega)$. In particular there exists a minimum of* (81).

Our goal here is to interprete this result as an example of application of our method (thus providing a simpler existence proof). To this end we have to understand why $I^\infty = (4\pi)^{1/2}$: first of all it is clear that $I^{\infty,y} = +\infty$ if $y \in \Omega$ (local compactness of holomorphic functions...), next if $y \in \partial\Omega$ the concentration around y shows that we are led to the problem in small neighborhoods of 0 in an halfspace or by the conformal equivalence to the *some problem* but in the *unit disc* and thus $I^\infty = (4\pi)^{1/2}$ if $p = 1$; if $p > 1$

$$I^\infty = \operatorname{Inf}\left\{ \int_T |f|^p |dz| \Big/ \int_D |f|^{2p} |dz| = 1 \right\}.$$

where D is the unit disc and $T = \partial\Delta$.

Then the compactness of minimizing sequences in the above result in immediately deduced from the analogue of Lemma 1.1 (Lemma 2.1,...) that we give below. Notice that our method also yields that if $I < I^\infty$ (for any $p > 1$), then minimizing sequences are compact and the infimum is achieved.

Lemma 4.2. *Let $(f_n)_n$ be bounded in $E^p(\Omega)$, assume that f_n converges weakly in $L^{2p}(\Omega)$ to some f, where $p \in [1, \infty[$. We may assume that $|f_n|^{2p}$ converges weakly to a measure ν on $\bar{\Omega}$ and that the measure μ_n given by*

$$\forall \varphi \in C(\bar{\Omega}), \qquad \int \varphi \, d\mu_n = \int_\Gamma \varphi |f_n|^p |dz|$$

converges to some measure μ. Then we have

i) $\nu = |f|^{2p} + \sum_{j \in J} \nu_j \delta_{z_j}$

ii) $\mu \geqslant \mu_1 + \sum_{j \in J} \nu_j^{1/2} I^\infty \delta_{z_j}$

for some at most countable family J, constants ν_j in $]0, \infty[$, point z_j on Γ.

The proof of this lemma is again a repetition of arguments given several times before: if $f \equiv 0$, then clearly ν is supported on Γ and we find that for all φ holomorphic in Ω, continuous on $\bar{\Omega}$

$$\left(\int_\Gamma |\varphi|^{2p}\, d\nu\right)^{1/2p} \leqslant C \int_\Gamma |\varphi|^p\, d\mu$$

By a density result, this inequality actually holds for all $\varphi \in C(\Gamma)$ and we deduce that ν is an at most countable sum of Dirac masses (and $\sum_j \nu_j^{1/2} < \infty$) —cf. Lemma 1.2 of Part 1 [63]—.

Indeed we claim that $\{|\varphi|_{|\Gamma}, \varphi$ holomorphic in Ω, continuous on $\bar{\Omega}\}$ is dense in $C_+(\Gamma)$: we need to prove this claim only when Ω is $(|z| < 1)$.

A short proof of this claim (which was indicated to us by J. M. Lasry) is as follows: any $\varphi \in C_+(\Gamma)$ may be approximated by a real nonnegative trigonometric polynomial

$$P = \sum_{n=-N}^{+N} a_n e^{in\theta},$$

then we consider the following holomorphic function (on $\mathbb{C}$) $\tilde{\varphi}$

$$\tilde{\varphi}(z) = \sum_{n=0}^{2N} a_{n-N} z^n$$

son that $\tilde{\varphi}(e^{i\theta}) = e^{iN\theta} P$ and $|\tilde{\varphi}|_{|\Gamma} = \mathrm{P}$.

The general representation of ν (part i) is then deduced as in the previous cases from the a.e. convergence of f_n to f (cf. Lemma 1.1).

Finally part ii) is obtained as before (usign the same density result as above) observing that if z_j is fixed, we may always find a simple connected domain $\omega \subset \Omega$ such that points of $\partial\Omega$ near z_j belong to $\partial\omega$ and essentially work in ω instead of Ω.

We conclude observing that i) and ii) imply

$$\varliminf_n \left\{\int_\Gamma |f_n|^p |dz| - I^\infty \left(\int_\Omega |f_n - f|^{2p}\right)^{1/2}\right\} \geqslant \int_\Gamma |f|^p |dz|$$

and if $p = 1$, this was the crucial lemma in [41] for the proof of the existence of a minimum.

4.4 *A remark on some isoperimetric inequalities*

We want to discuss here some properties of the following isopermetric inequality

$$|Q(v)|^{2/3} \leqslant C_0 \int_{\mathbb{R}^2} |\nabla v|^2\, dx\, dy, \qquad \forall v \in \mathfrak{D}(\mathbb{R}^2; \mathbb{R}^3) \tag{82}$$

where $C_0 = (32\pi)^{-1/3}$, and Q is the functional defined by

$$Q(v) = \int_{\mathbb{R}^2} v \cdot (v_x \wedge v_y)\, dx\, dy. \tag{83}$$

The proof of (82) —which is an isoperimetric inequality for the graph of v— may be found in Wente [83]. By density (82) still holds for $v \in H^1(\mathbb{R}^2; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^3)$ and C_0 is achieved for

$$\bar{v}(x, y) = (1 + x^2 + y^2)^{-1}(x, y, 1)$$

(see Wente [83], H. Brézis and J. M. Coron [18] for more details).

Finally let us mention that Q may be defined actually on functions in $H^1(\mathbb{R}^2; \mathbb{R}^3)$ with compact support and that $v_x \wedge v_y$ is not only meaningful in L^1 but also in H^{-1} for such functions V.

We next want to observe that (82) is invariant by translations and dilations and that Q, $\mathcal{E}$ are invariant by

$$v \to v\left(\frac{\cdot}{\sigma}\right), \qquad \forall \sigma > 0$$

where

$$\mathcal{E}(v) = \int_{\mathbb{R}^2} |\nabla v|^2\, dx\, dy.$$

Therefore to check if we may apply our method to such functionals we need to see if the analogues of Lemma 1.1 still hold true here: let us thus consider a sequence $(v^n)_n$ bounded in $H_0^1(\Omega; \mathbb{R}^3)$ where Ω is bounded in $\mathbb{R}^2$ (so that v^n extended by 0 is in $H^1(\mathbb{R}^2; \mathbb{R}^3)$). We next want to define a distribution T^n given by: $T^n = v^n \cdot (v_x^n \wedge v_y^n)$, T^n will be supported in $\bar{\Omega}$ and is defined by

$$\langle T^n, \varphi \rangle = \langle \varphi v^n, v_x^n \wedge v_y^n \rangle, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^2)$$

and thus T^n is (for example) bounded in $W^{-1,p'}(\mathbb{R}^2)$, $\forall p > 2$. In addition: $Q(v^n) = \langle T^n, \chi \rangle$, for any $\chi \in \mathcal{D}(\mathbb{R}^2)$, $\chi \equiv 1$ on $\bar{\Omega}$.

We may now state the

Lemma 4.3. *With the above observations, we may assume that $|\nabla u_n|^2$, T_n converge in $\mathcal{D}'(\mathbb{R}^2)$ to μ, T and that v^n converges weakly to v. Then we have*

i) $$T = T_0 + \sum_{j \in J} \nu_j \delta_{x_j}$$

where T_0 is defined through v as T_n is defined through v^n, J is some at most countable set, $(\nu_j)_{j \in J} \in \mathbb{R} - \{0\}$, $(x_j)_{j \in J}$ are points of $\bar{\Omega}$.

ii) $$\mu \geqslant |\nabla v|^2 + (1/C_0) \sum_{j \in J} |\sigma_j|^{2/3}\, \delta_{x_j}.$$

iii) *If $u \in H_0^1(\Omega; \mathbb{R}^3)$ and if we denote by $\tilde{T}_n$ the distributions associated with $v^n + u$ as T_n is associated to v^n the (up to subsequences)*

$$\begin{cases} |\nabla(v^n + u)|^2 \rightharpoonup \tilde{\mu}, & \tilde{\mu} - \mu \in L^1(\mathbb{R}^2) \\ \tilde{T}_n \rightharpoonup \tilde{T}_0 + \sum_{j \in J} \nu_j \delta_{x_j} \end{cases}$$

where $\tilde{T}_0$ corresponds to $u + v$.

iv) *In particular*

$$\varliminf_n \mathcal{E}(v_n) - \frac{1}{C_0} |Q(v_n - v)|^{2/3} \geqslant \mathcal{E}(v).$$

The proof of lemma 4.3 is totally similar to the proofs of the corresponding results we proved before: considering first the case when $v \equiv 0$, we obtain for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$

$$|\langle T^n, \varphi^3 \rangle| = |Q(\varphi v^n)| \leqslant C_0^{3/2} \Big(\int_{\mathbb{R}^2} |\nabla(\varphi v^n)|^2 \Big)^{3/2}$$

and thus passing to the limit

$$|\langle T, \varphi^3 \rangle|^{2/3} \leqslant C_0 \Big(\int \varphi^2 \, d\mu \Big), \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^2).$$

But this implies easily that T is a signed measure on $\mathbb{R}^2$; and the reaminder of the proof is then totally similar to the proof of Lemma 1.1 (and of the other related results...).

This observation enables us to apply the general concentration-compactness arguments and we may now give in interpretation of the results of H. Brézis and J. M. Coron [18] on *the existence of a second solution to H-systems*: we refer the reader to [18] for the motivations of the introduction of the following *minimization problem* which, if solved, yields the *existence of a second solution* to the Dirichlet problem for H systems (a similar analysis works also for the Plateau problem). We thus consider

$$I = \operatorname{Inf} \{ \mathcal{E}(u) \,/\, v \in H_0^1(\Omega; \mathbb{R}^3), \quad Q(v) = 1 \}. \tag{84}$$

where Ω is bounded in $\mathbb{R}^2$, $\mathcal{E}$ is given by

$$\mathcal{E}(v) = \int_\Omega |\nabla v|^2 + 4H \int_\Omega u \cdot (v_x \wedge v_y)$$

and u is a given function in $H^1(\Omega; \mathbb{R}^3)$ —for example—, H is a given positive constant—, in [18], u is in fact the «first» solution of the H system, solution obtained by Hildebrandt [41], [42]. In order to have a non trival minimization problem, we assume

$$\exists \alpha > 0, \quad \forall v \in H_0^1(\Omega; \mathbb{R}^3), \qquad \mathcal{E}(v) \geqslant \alpha |\nabla v|_{L^2}^2. \tag{85}$$

Clearly enough we have for all $y \in \bar{\Omega}$

$$I^\infty = I^{\infty, y} = \operatorname{Inf}\{\mathcal{E}(u) / u \in \mathcal{D}(\mathbb{R}^2; \mathbb{R}^3), \quad Q(u) = 1\} = \frac{1}{C_0}; \tag{86}$$

and by homogeneity (81) reduces to

$$I < I^\infty \tag{16}$$

Therefore we deduce from our general arguments (and Lemma 4.3) that *if* (16) *holds, any minimizing sequence of* (84) *is relatively compact in* $H_0^1(\Omega; \mathbb{R}^3)$. To conclude our interpretation of the results of [17], we recall that in [17] it is proved that if $u \in C^2_{\text{loc}} \cap L^\infty$ then (16) holds if and only if u is not constant on Ω. We emphasize the fact that we did not prove here any new result but we only show one *needs* to compare I with $\frac{1}{C_0} = (32\pi)^{1/2} (= I^\infty)$ and this again is a consequence or our general method.

4.5 *Harmonic maps*

As in the preceding two sections, we will not prove any new result but we will just explain in the light of our systematic treatment the solution of some minimization problem associated with the question of harmonic maps. We will thus follow the presentation of H. Brézis and J. M. Coron [18] (see also J. Jost [43] for related results). By no means, the remakrs which follow pretend to cover the subjetct of harmonic maps and we refer the interested reader to the deep work of J. Sacks and K. Uhlenbeck [72], R. Schasen and S. T. Yau [73], Y. T. Siu and, S. T. Yau [75]. To simplify we will consider only harmonic maps from the unit ball Ω of $\mathbb{R}^2$ into S^2 with a precribed boundary condition

$$u = \gamma \quad \text{on} \quad \partial\Omega$$

(where γ is, of course, the restriction to $\partial\Omega$ —the trace— of some function v *in* $H^1(\Omega: S^2)$ i.e. $v \in H^1(\Omega; \mathbb{R}^3)$, $v \in S^2$ a.e. in Ω).

Harmonic maps from Ω into S^2 with the above boundary condition are critical points of the functional

$$\mathcal{E}(u) = \int_\Omega |\nabla u|^2$$

«restricted to the set» $A = \{u \in H^1(\Omega; S^2);\ u = \gamma \text{ on } \partial\Omega\}$. Clearly $\mathcal{E}$ achieves its minimum on A: let u_0 be such a minimum.

Following [18], we consider for $u \in H^1(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ the functional

$$Q(u) = \frac{1}{4\pi} \int_\Omega u \cdot (u_x \wedge u_y)$$

and we recall (see [18] for more details) that if $u_1, u_2 \in A$ then

$$Q(u_1) - Q(u_2) \in \mathbb{Z}$$

(identifying Ω with the northern hemisphere of S^2, and «reflecting» u_2 we may consider (u_1, u_2) as a map from S^2 to S^2 and $Q(u_1) - Q(u_2)$ is the degree of this map).

We set

$$J(u) = Q(u) - Q(u_0), \qquad \forall u \in A;$$

so that J is integer-value on A (and $J(A) = \mathbb{Z}$).

Then let $k \neq 0$, if we find a minimum of

$$I_k = \operatorname{Inf}\{\mathcal{E}(u) \,/\, u \in A, \quad \mathrm{J}(u) = k\} \tag{87}$$

then such a minimum will be a local minimum and thus a critical point of $\mathcal{E}$ on A.

We may now apply the concentration-compactness argument: we then need to define the *problem at infinity* (the underlying *scaling invariance* is: $u \to u\left(\frac{\cdot}{\sigma}\right)$ for $\sigma > 0$)

$$I_\mu^\infty = \operatorname{Inf}\left\{ \int_{\mathbb{R}^2} |\nabla\varphi|^2 \,/\, \varphi \in C^\infty(\mathbb{R}^2; S^2), \quad \varphi \text{ constant near infinity}, \right.$$
$$\left. \int_{\mathbb{R}^2} \varphi \cdot (\varphi_x \wedge \varphi_y) = 4\pi\mu \right\}.$$

Using the above remark on the degree and the value of C_0 in the preceding constant, we find

$$\begin{cases} I_\mu^\infty = +\infty \quad \text{if} \quad \mu \notin \mathbb{Z}, \qquad I_\mu^\infty = |\mu| I_1^\infty \quad \text{if} \quad \mu \in \mathbb{Z} \\ I_1^\infty = 8\pi. \end{cases}$$

Then *in this setting,* (S.1) *reduces to*

$$I_k < I_l + I_{k-l}^\infty, \qquad \forall l \in \mathbb{Z} - \{k\}; \tag{88}$$

with, in fact,

$$I_{k-l}^\infty = 8\pi|k - l|; \qquad I_0 = \operatorname*{Inf}_A \mathcal{E} \leqslant I_k, \qquad \forall k \in \mathbb{Z} - \{0\}.$$

And it is now a straightforward application of our arguments to show that (88) *is a necessary and sufficient condition for the compactness of all minimizing sequences of* (87). But in addition the very especial form of I_μ^∞ enables us to make the following remarks: if $l \geqslant 2k$ and $k > 0$ (or $l \leqslant 2k$, $k < 0$)

$$I_l + I_{k-l}^\infty = I_l + 8\pi(l - k) \geqslant I_0 + 8\pi k = I_0 + I_k^\infty$$

therefore (88) *is equivalent to*

$$I_k < I_l + I^{\infty}_{k-l} = I_l + 2\pi|k-l|, \qquad \forall l \text{ between } 0 \text{ and } 2k, \quad l \neq k, \tag{89}$$

In particular *if* $k = \pm 1$, (89) *reduces to*

$$I_k < I_0 + 8\pi \tag{90}$$

And *we recover the crucial inequality* of [18] (inequality (3), Lemma 2) as a very particular case of (S.1); in [18], it is proved that if γ is not constant, (90) holds either for $k = 1$, or for $k = -1$. In both cases this yields the existence of a local minimum different from u_0. Of course (90) is the *major difficulty* in the proof of the existence of such a second critical point (let us just mention that the method folowed in [18] to check (90) follows the empirical rule given in section III) but our goal here is to show that (90) is natural and had to be expected!

4.6 *Morse theory*

We want to explain on the example of Yamabe type equations what informations the results such as Lemma 1.1 (and the related weak convergence results) imply on the possibility of using Morse theory on functionals associated with the preceding problems.

To simplify the presentation, we will only present our results in the case of Yamabe equations even if they apply to *all* the situations considered before (convolution, trace, H-systems, holomorphic functions, harmonic maps...). We will thus consider a sequence $(u_n)_n$ in $H^1_0(\Omega)$ —where Ω is a bounded open set in $\mathbb{R}^n$— satisfying

$$\begin{cases} -\Delta u_n = |u_n|^{4/(N-2)}u_n + f_n \quad \text{in} \quad \Omega, \qquad f \to f \quad \text{in} \quad H^{-1}(\Omega) \\ S_n(u_n) \to c \end{cases} \tag{91}$$

where $c \in \mathbb{R}$ is fixed,

$$S_n(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - \frac{N-2}{2N}|v|^{2N/(N-2)}\,dx - \langle f_n, v\rangle.$$

We denote by $S^\infty(v)$, $S(v)$ the functionals corresponding to $f_n = 0$, $f_n = f$. The reasons for considering (91) come from the (P.S.) condition which is the crucial condition for the application of critical point hteory. The following result is an obvious application of Lemma 1.1:

Corollary 4.4. *Assume that* $(u_n)_n$ *satisfies* (91), *then* u_n *is bounded in* $H_0^1(\Omega)$ *and assuming that* u_n, $|\nabla u_n|^2$ *converge weakly to* $u \in H_0^1(\Omega)$, μ *bounded nonnegative measure on* $\bar{\Omega}$ *we have:*

i) *u solves*:

$$-\Delta u = |u|^{4/(N-2)}u + f \quad \text{in} \quad \Omega, \qquad u \in H_0^1(\Omega) \tag{92}$$

ii)
$$\mu = |\nabla u|^2 + \sum_{i=1}^{m} \mu_i \delta_{x_i}$$

where $m \geqslant 0$; $x_1, \ldots, x_m$ *are m distinct points of* $\bar{\Omega}$ *and* $(\mu_i)_i$ *satisfies*

$$\mu_i \geqslant I_0^{N/2} \tag{93}$$

where

$$I_0 = \mathrm{Min}\left\{\int_{\mathbb{R}^N} |\nabla u|^2\, dx \,/\, u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx = 1\right\}.$$

iii) $|u_n|^{2N/(N-2)}$ *converges weakly to*: $|u|^{2N/(N-2)} + \sum_{i=1}^{m} \mu_i \delta_{x_i}$.

iv) $c = S(u) + \frac{1}{N} \sum_{i=1}^{m} \mu_i$.

Remark 4.9. The fact that compactness is lost at most at a finite number of points was first observed by J. Sacks and K Uhlenbeck [72] in the study of harmonic maps; see also Y. T. Siu and S. T. Yau [75], K. Uhlenbeck [85, 86], C. Tanbes [78].

Remark 4.10. Take $f = 0$, then (92) implies that $S(u) \geqslant 0$ and thus $c \geqslant \frac{m}{N} I_0^{N/2}$. Hence critical point theory (or Morse theory) may be applied on level sets below c: this was used in H. Brézis and L. Nirenberg [23]; see also C. Taubes [78] for related considerations.

Notice also that if $c < \frac{2I_0^{N/2}}{N}$, only one point (one Dirac mass) may occur; similarly if u_n is nonnegative and $c \in \left]\frac{1}{N} I_0^{N/2}, \frac{2}{N} I_0^{N/2}\right[$ no Dirac mass may appear and u_n converges in H^1 to u. This observation is used in J. M. Coron [30].

Remark 4.11. If one had a complete description of solutions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ of

$$-\Delta u = |u|^{4/(N-2)}u \quad \text{in} \quad \mathbb{R}^N;$$

one would be able to obtain (in a straightforward way) a much more precise behavior of u_n nearby each point x_i. This programa was recently completed (in great details) by H. Brézis and J. M. Coron [20] in the case of H-systems; and that should be a general phenomenon.

PROOF OF COROLLARY 4.4. If $u_n - u = v_n$ and $|\nabla v_n|^2$, $|v_n|^{2N/(N-2)}$ converge weakly to μ_0, ν_0, Corollary 4.4 will be proved if we show that $\mu_0 = \nu_0$. Indeed by Lemma 4.1, we know

$$\mu_0 \geqslant \sum_{j \in J} I_0 \nu_j^{(N-2)/N} \delta_{x_j}, \qquad \nu_0 = \sum_{j \in J} \nu_j \delta_{x_j}, \qquad \mu - \mu_0 \in L^1(\Omega)$$

nad $\mu_0 = \nu_0$ would imply that J is finite and parts ii) – iv) of Corollary 4.4 (part i) is an exercize on weak limits).

To prove that $\mu_0 = \nu_0$, we observe that for all $\varphi \in C^1(\bar{\Omega})$

$$\int_\Omega \varphi |\nabla v_n|^2 + v_n (\nabla\varphi, \nabla v_n)\, dx =$$
$$= \int_\Omega \{ |u + v_n|^{4/(N-2)}(u + v_n) - |u|^{4/(N-2)} u \} \cdot v_n \varphi\, dx + \langle f_n - f, v_n \varphi \rangle$$

and passing to the limit as n goes to ∞, we obtain

$$\int_\Omega \varphi\, d\mu_0 = \int_\Omega \varphi\, d\nu_0, \qquad \forall \varphi \in C^1(\bar{\Omega}).$$

Appendix 1. Existence of two solutions of the Yamabe problem in $\mathbb{R}^N$

We want here to present a few results concerning the existence of solutions of

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + ku = K u^{(N+2)/(N-2)} \quad \text{in } \mathbb{R}^N \tag{A.1}$$

$$u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N); \qquad \exists c_0 > 0, \quad u \geqslant c_0 \quad \text{on } \mathbb{R}^N \tag{A.2}$$

where $k, K \in C_b(\mathbb{R}^N)$ (for example), $a_{ij} = a_{ji} \in C_b(\mathbb{R}^N)$ and

$$\forall R < \infty, \quad \exists \nu > 0, \quad \forall x \in \bar{B}_R, \qquad (a_{ij}) \geqslant \nu I_N.$$

We will first present some results due to W. M. Ni [68] (see also [69], Kenig and Ni [52]): the main assumption for the application of Ni's method is the following

$$\exists \varphi_1 \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N), \quad A\varphi_1 = 0 \quad \text{in } \mathbb{R}^N, \qquad \varphi_1 \geqslant c_1 > 0 \quad \text{on } \mathbb{R}^N; \tag{A.3}$$

where A is the linear operator given by the left side of (A.1).

Of course (A.3) holds if $a_{ij}(x) \equiv \delta_{ij}$, $k \equiv 0$ (this corresponds to the usual metric on $\mathbb{R}^N$) or under convenient decay assumptions at infinity (cf. Kenig and Ni [52]). We also gave in section 1.5 conditions which ensures that (A.3) holds (and they may be easily extended...). Notice that (A.3) implies that the first eigenvalue of A in $H^1_0(\Omega)$ is positive for any bounded open set Ω.

The result which follows is an adaptation of the method of Ni:

Theorem A.1. (cf. [68], [69]). *We assume* (A.3). *If we assume*

$$\exists \bar{u} \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \qquad A\bar{u} \geqslant |K| \quad \textit{in } \mathbb{R}^N \tag{A.4}$$

then there exists a sequence of solutions $(u_n)_n$ *of* (A.1) – (A.2) *satisfying:* $\sup_{\mathbb{R}^N} u_n \to 0$. *In addition if K has a constant sign,* (A.4) *is necessary for the the existence of a solution of* (A.1) – (A.2).

PROOF. First, if $K \geqslant 0$, (A.4) is clearly necessary and if (A.4) holds we may assume that: $\inf_{\mathbb{R}^N} \bar{u} > 0$ (if it is not the case, we consider $\bar{u} + \mu\varphi_1$ for μ large). Then for $\delta > 0$ small, $\delta\bar{u}$ is a supersolution of (A.1) while $\lambda\varphi_1$ is a subsolution for all $\lambda \geqslant 0$; and the method of sub and supersolutions immediately yields the above results. Next, if $K \leqslant 0$ and if u solves (A.1) then we have

$$A(-u) = (-K)u^{(N+2)/(N-2)} \geqslant c_0^{(N+2)/(N-2)}|K|$$

and (A.4) holds. On the other hand if (A.4) holds, replacing, if necessary, $\bar{u}$ by $\bar{u} - \mu\varphi_1$ for μ large, we may assume: $\inf \bar{v} > 0$, where $\bar{v} = -\bar{u}$. Again, for δ small, $\delta\bar{v}$ is a subsolution of (A.1) while $\lambda\varphi_1$ is a supersolution for all $\lambda \geqslant 0$ and we conclude.

Finally for some arbitrary K, we consider u_n, v_n solutions of (A.1) – (A.2) with respectively $|K|$, $-|K|$ such that $\sup_{\mathbb{R}^N} u_n$, $\sup_{\mathbb{R}^N} v_m \to 0$; u_n is a supersolution and v_m is a subsolution. And we conclude.

Let us mention that the above proof actually shows the existence of an uncountable infinity of solution of (A.1) – (A.2).

In order to have a more precise description of the solutions of (A.1) – (A.2) at infinity, we will assume instead of (A.3).

$$\begin{gathered} \exists \varphi_1 \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N), \qquad A\varphi_1 = 0 \quad \text{in } \mathbb{R}^N, \\ \varphi_1 > 0 \quad \text{in } \mathbb{R}^N, \qquad \varphi_1 \to 1 \quad \text{as} \quad |x| \to \infty. \end{gathered} \tag{A.5}$$

Observe that if φ_1 exists, φ_1 is unique. Similarly, we will strengthen (A.4) to

$$\exists \bar{u} \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N), \qquad A\bar{u} \geqslant |K| \quad \text{in } \mathbb{R}^N; \tag{A.6}$$

(and this implies that $\bar{u} \geqslant 0$ in $\mathbb{R}^N$ and the existence of $\tilde{u}$ such that

$$A\tilde{u} = |K| \quad \text{in } \mathbb{R}^N, \qquad \tilde{u} \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N), \qquad \tilde{u} \geqslant 0 \quad \text{in } \mathbb{R}^N).$$

We may now replace (A.2) by

$$\exists u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N), \qquad u \to \mu \quad \text{as} \quad |x| \to \infty \tag{A.7}$$

where $\mu > 0$ is given. And we have the:

Theorem A.2. *We assume* (A.5) – (A.6).

i) *If* $K \leqslant 0$, *for any* $\mu > 0$, *there exists a unique solution* u_μ *of* (A.1) – (A.7). *In addition:* $u_\mu \leqslant \mu\varphi_1$ *on* $\mathbb{R}^N$ *and* u_μ *is increasing in* μ.
ii) *If* $K \geqslant 0$, *there exists* $\mu_0 \in]0, \infty[$, $\mu_0 < \infty$ *if* $K \not\equiv 0$ *such that for* $\mu > \mu_0$, *there does not exist a solution of* (A.1) – (A.7) *and for* $\mu \in]0, \mu_0[$, *there exists a minimum solution* u_μ *of* (A.1) – (A.7). *In addition:* u_μ *is increasing in* μ, $u_\mu \geqslant \mu\varphi_1$. *Finally under the assumptions of Corollary* 1.4, $u_\mu - \mu \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *and as* $\mu \uparrow \mu_0$, u_μ *increases to* u_{μ_0} *the minimum solution of* (A.1) – (A.7) (*for* $\mu = \mu_0$).
iii) *If* K *is arbitrary, there exists* $\mu_0 \in]0, \infty[$ *such that for* $\mu \in]0, \mu_0[$ *there exists a solution of* (A.1) – (A.7).

PROOF. *We first prove part* (*i*). We remark that $\mu\varphi_1$ is a supersolution of (A.1) which satisfies (A.7). Next if $\tilde{u}$ satisfies

$$A\tilde{u} = -K \quad \text{in } \mathbb{R}^N, \qquad \tilde{u} \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N), \qquad \tilde{u} \geqslant 0 \quad \text{in } \mathbb{R}^N$$

we set $\underline{u}_\mu = (\mu\varphi_1 - \lambda\tilde{u})^+$ for some $\lambda > 0$. We then have by standard results

$$A\underline{u}_\mu \leqslant 1_{(\underline{u}_\mu \geq 0)}\lambda K \leqslant K\underline{u}_\mu^{(N+2)/(N-2)} \quad \text{on } \mathbb{R}^N$$

if λ is chosen such that

$$\underline{u}_\mu^{(N+2)/(N-2)} \leqslant (\mu\varphi_1)^{(N+2)/(N+2)} \leqslant \lambda.$$

Thus $\underline{u}_\mu$ is a subsolution of (A.1) satisfying (A.7) and the existence part is complete.

The various uniqueness and comparison results are deduced from the following claim: let $v, w \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$ satisfy

$$\begin{cases} Av + |K|v^{(N+2)/(N-2)} \leqslant 0 \quad \text{in } \mathbb{R}^N, \quad v \geqslant 0 \quad \text{in } \mathbb{R}^N, \quad \limsup_{|x| \to \infty} v \leqslant \mu \\ Aw + |K|w^{(N+2)/(N-2)} \geqslant 0 \quad \text{in } \mathbb{R}^N, \quad w \geqslant 0 \quad \text{in } \mathbb{R}^N, \quad \liminf_{|x| \to \infty} w \geqslant \mu. \end{cases}$$

then $v \leqslant w$ on $\mathbb{R}^N$. Indeed for all $\epsilon > 0$, we may find R large enough such that

$$v \leqslant (1 + \epsilon)w = w_\epsilon \quad \text{for } |x| \geqslant R.$$

since we have on B_R

$$A(w_\epsilon - v) - \frac{N+2}{N-2}|K|w^{4/(N-2)}(w_\epsilon - v) \geqslant$$

$$\geqslant A(w_\epsilon - v) + |K|(w_\epsilon^{(N+2)/(N-2)} - v^{(N+2)/(N-2)}) \geqslant 0$$

and since the first eigenvalue of A and thus of

$$A + \frac{N+2}{N-2}|K|w_\epsilon^{4/(N-2)}$$

is positive (on $H_0^1(B_R)$), we conclude: $w_\epsilon \geqslant v$ in $\mathbb{R}^N$.

Observe also that part iii) is easily deduced from parts i) and ii). *We finally prove part* ii) and the arguments which follow are very much the same than those used in the study of semilinear elliptic problems with convex increasing nonlinearities in bounded domains (see for example M. G. Crandall and P. H. Rabinowitz [31]; F. Mignot and J. P. Puel [66]: D. G. De Figueiredo, P. L. Lions and R. D. Nussbaum [36]; P. L. Lions [64]). By the proof of Theorem A.1 we already know that for μ small there exists a solution of (A.1) – (A.7). We then let $\mu_0 = \sup\{\mu > 0/\exists v$ supersolution of (A.1), v *satisfies* (A.7)$\}$; so that $\mu_0 \in]0, \infty]$. If $\mu \in]0, \mu_0[$, we set $u^0 = \mu\varphi_1$ and we define by induction u^n as follows

$$Au^n = K(u^{n-1})^{(N+2)/(N-2)} \text{ in } \mathbb{R}^N, \quad u^n \to \mu \text{ as } |x| \to \infty, \qquad u^n \in H^1_{\text{loc}} \cap C_b$$

then observing that $v \geqslant u^0$, we deduce that u^n increases (strictly if $K \not\equiv 0$) to u_μ solution of (A.1) – (A.7). By arguments similar to those used above, we also check that any supersolution v of (A.1) satisfying (A.7) actually satisfies: $v \geqslant \mu\varphi_1$; and thus u_μ is the minimum solution.

We next claim that if $K \not\equiv 0$, $\mu_0 < \infty$: indeed if u_μ solves (A.1) – (A.7), sience $u_\mu \geqslant \mu\varphi_1$, we have on a fixed ball B_R (such that $K \not\equiv 0$ on B_R)

$$Au_\mu \geqslant K(\mu\varphi_1)^{4/(N-2)}u_\mu \quad \text{in} \quad B_R, \qquad u_\mu > 0 \quad \text{on } \bar{B}_R$$

and thus the first eigenvalue of $A - K(\mu\varphi_1)^{4/(N-2)}$ is positive and this is not possible for μ large.

Next, we claim that if $\mu \in]0, \mu_0[$ the first eigenvalue of

$$A - \frac{N+2}{N-2}Ku_\mu^{4/(N-2)}$$

is positive on $H_0^1(B_R)$ for all $R < \infty$ and thus

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} \sum_{i,j} a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j} + k\varphi^2\,dx \geqslant \frac{N+2}{N-2}\int_{\mathbb{R}^N} Ku_\mu^{4/(N-2)}\varphi^2\,dx.$$

Indeed if we denote by $v^n = u_\mu - u^n$ (assuming that $K \not\equiv 0$), we have

$$\begin{cases} Av^N \geqslant \dfrac{N+2}{N-2}K(u^{n-1})^{4/(N-2)}v^{n-1} \geqslant \dfrac{N+2}{N-2}K(u^{n-1})^{4/(N-2)}v^n & \text{in } \mathbb{R}^N \\ v^n > 0 \quad \text{on } \mathbb{R}^N \end{cases}$$

and thus the above claim is proved (observe that if the first eigenvalue in B_R, is nonnegative, it is positive in B_R for $R < R'$).

Now if we assume the conditions of Corollary 1.4, observing that $u_\mu - \mu\varphi_1$ (with the above notations) belongs to $\mathfrak{D}^{1,2}(\mathbb{R}^N)$, we deduce multiplying (A.1) by $u_\mu - \mu\varphi_1$ that we have for some $\delta > 0$

$$\delta \int_{\mathbb{R}^N} |\nabla(u_\mu - \mu\varphi_1)|^2 \, dx + (1 + \delta) \int_{\mathbb{R}^N} K u_\mu^{4/(N-2)} (u_\mu - \mu\varphi_1)^2 \, dx \leqslant$$
$$\leqslant \int_{\mathbb{R}^N} K u_\mu^{(N+2)/(N-2)} (u_\mu - \mu\varphi_1) \, dx.$$

and we conclude easily (using the properties of K) that $u_\mu - \mu\varphi_1$ is bounded in $\mathfrak{D}^{1,2}$.

The analogy we have used above of (A.1) – (A.7) with semilinear problems strongly suggests of seeking a second solution above u_μ for $\mu \in]0, \mu_0[$. This is what we prove below (under convenient assumptions). To simplify the presentation, we will assume from now on that $a_{ij} \equiv \delta_{ij}$, $k \equiv 0$, $K \geqslant 0$ (so that $\varphi_1 = 1$). Our main assumption on K will be

$$K \in L^p(\mathbb{R}^N) \cap C_b(\mathbb{R}^N), \quad \text{for some} \quad p \in [1, \tfrac{N}{2}[\tag{A.8}$$

(it is possible to extend this assumption by a careful inspection of the proof below).

Notice that this insures that (A.6) holds and thus there exists $\mu_0 > 0$ such that for $0 < \mu < \mu_0$, there exists a minimum solution u_μ of (A.1) – (A.7) (which is increasing in μ).

In order to find a second solution of (A.1) – (A.7) above u_μ we are going to apply the Mountain Path lemma of Ambrosetti and Rabinowitz as in [31], [36] on the translated problem

$$-\Delta v = f(x, v) \quad \text{in } \mathbb{R}^N, \qquad v \in \mathfrak{D}^{1,2}(\mathbb{R}^N) \tag{A.9}$$

where $f(x,t) = K(x)(u_\mu(x) + t^+)^{(N+2)/(N-2)} - K(x)u_\mu(x)^{(N+2)/(N-2)}$. Hence we consider the functional

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla v|^2 - F(x, v) \, dx,$$

where

$$F(x,t) = \int_0^t f(x,s) \, ds = \frac{N-2}{2N} K(x)(u_\mu(x) + t^+)^{2N/(N-2)} -$$
$$- \frac{N-2}{2N} K(x) \, u_\mu(x)^{2N/(N-2)} - K(x) u_\mu(x)^{(N+2)/(N-2)} t^+ .$$

But two difficulties occur: first of all we have to prove that 0 is a «local strict minimum» of $\mathcal{E}$; next in order to apply the Mountain Path lemma, we need to check Palais-Smale condition which in view of [23] or section 4.6 holds provided we check that the tentative critical value is below $\frac{1}{N} I_0^{N/2}$. The first step is thus:

Theorem A.3. *Under assumption* (A.8), *either there exists a second solution* $\bar{u}_\mu$ *of* (A.1) – (A.7) *satisfying:* $\bar{u}_\mu > u_\mu$ *on* $\mathbb{R}^N$, *or there exists* $\delta_0 > 0$ *such that*

$$\mathcal{E}(v) \geqslant 0, \qquad \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \qquad \|v\| \leqslant \delta_0 \tag{A.10}$$

$$\operatorname{Inf}\{\mathcal{E}(v) / v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \ \|v\| = \delta\} > 0, \qquad \forall \delta \in]0, \delta_0]. \tag{A.11}$$

PROOF. We first show (A.10) assuming that there does not exist a second solution of (A.1) – (A.7) such that $\bar{u}_\mu > u_\mu$ on $\mathbb{R}^N$. To this end we set $I_\delta = \operatorname{Inf}\{\mathcal{E}(v) / v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \|v\| \leqslant \delta\}$. We argue by contradiction and we thus assume: $I_\delta < 0$. If we show that the infimum is achieved for $v = v_\delta$, our nonexistence assumption yields that $\|v_\delta\| = \delta$ for δ small and thus there exists $\theta_\delta > 0$ such that

$$-(1 + \theta_\delta)\Delta v_\delta = f(x, v_\delta) \quad \text{in } \mathbb{R}^N, \qquad v_\delta > 0 \quad \text{in } \mathbb{R}^N, \qquad \|v_\delta\|_{\mathcal{D}^{1,2}} = \delta.$$

Recalling that in the proof of Theorem A.2 we have proved

$$\forall \varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} |\nabla\varphi|^2 - \frac{N+2}{N-2} K u_\mu^{(N+2)/(N-2)} \varphi^2 \, dx \geqslant 0; \tag{A.12}$$

it is easy to deduce that $\theta_\delta \to 0$ as $\delta \to 0_+$.

Then one shows by standard regularity results that $v_\delta \to 0$ in $L^\infty(\mathbb{R}^N)$ as $\delta \to 0_+$: hence for δ small $u_\mu + v_\delta < u_{\mu'}$, for some $\mu' \in]\mu, \mu_0[$. Observing that $u_\mu + v_\delta$ is a subsolution of (A.1) – (A.7), we deduce the existence of a second solution of (A.1) – (A.7) between $u_\mu + v_\delta$ and $u_{\mu'}$. The contradiction proves our claim.

There just remains to show that the infimum of I_δ is achieved for δ small if $I_\delta < 0$: we apply the concentration-compactness arguments and we set $\rho_n = |\nabla u_n|^2 + |u_n|^{2N/(N-2)}$ where $(u_n)_n$ is a minimizing sequence. If ρ_n vanishes, because of (A.8), $\lim \mathcal{E}(u_n) \geqslant 0$; while dichotomy or tightness up to an unbounded sequence cannot occur still because of (A.8) since, for instance, if ρ_n is tight up to y_n and $|y_n| \to \infty$.

$$\mathcal{E}(u_n) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^N} F(x, u_n) \, dx \geqslant -\int_{\mathbb{R}^N} F(x, u_n) \, dx$$

$$0 \leqslant \int_{\mathbb{R}^N} F(x, u_n) \, dx \leqslant \epsilon + \int_{|x - y_n| \leq R_\epsilon} F(x, u_n) \, dx$$

and the last integral goes to 0 in view of (A.8) and since $|y_n| \to \infty$.

Now if ρ_n is tight and u_n converges weakly and a.e. to u, we observe that

$$\left| \int_{\mathbb{R}^N} F(x, u_n) - F(x, u) - \frac{N-2}{2N} K(x) |u_n - u|^{2N/(N-2)} \, dx \right| \to 0.$$

And this yield observing that $\|u\| \leqslant \delta$ and denoting by $v_n = (u_n - u)$

$$I_\delta \geqslant I_\delta + \underline{\lim}_n \int_{\mathbb{R}^N} \frac{1}{2} |\nabla v_n|^2 - \frac{N-2}{2N} |v_n|^{2N/(N-2)} \, dx$$

$$\geqslant I_\delta + \underline{\lim}_n \frac{1}{2} \|v_n\|^2 - c_N \|v_n\|^{2N/(N-2)}$$

and since $\|v\| \leqslant 2\delta$, we deduce that for δ small $v_n \to 0$ strongly and u is a minimum of I_δ.

To prove (A.11), we first observe that without loss of generality we may assume that: $\mathcal{E}(v) > 0$ for $\|v\|$ small, $v \not\equiv 0$. Next if for all $\delta > 0$

$$\bar{I}_\delta = \operatorname{Inf}\{ \mathcal{E}(v) \,/\, v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad \|v\|^2 = \delta \} = 0$$

we may prove exactly as above that $\bar{I}_\delta$ is achieved and we reach a contradiction (notice that (S.1) holds since $\bar{I}_\delta^\infty = \frac{\delta'}{2}$, $\bar{I}_{\delta'} = 0$ for δ' small).

The second step is given by:

Theorem A.4. *Under assumption* (A.8)*, and if there exists a path γ i.e. a continuous map γ from* [0, 1] *into* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *such that*

$$\max_{t \in [0,1]} \mathcal{E}(\gamma(t)) < \frac{1}{N} \Big(\sup_{\mathbb{R}^N} K \Big)^{-(N-2)/2} I_0^{N/2}, \qquad \mathcal{E}(\gamma(1)) \leqslant 0; \tag{A.13}$$

then there exists a second solution $\bar{u}_\mu$ of (A.1) – (A.7).

Of course (A.13) holds of there exists v_1 in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$\max_{t \geq 0} \mathcal{E}(tv_1) < \frac{1}{N} \Big(\sup_{\mathbb{R}^N} K \Big)^{-(N-2)/2} I_0^{N/2}.$$

And this strict inequality may be checked with the method of H. Brézis and L. Nirenberg [23] and we find, for example, that *if there exists a maximum point x^0 of K such that: $D^j K(x^0)$* for $1 \leqslant j \leqslant [(N-2)/2]$ (where $[x]$ denotes the integer part of x, and where of course K is assumed to be nearby x^0) *then* (A.13) *holds* for $N \geqslant 4$ (notice that if $N = 4,5$, this condition is automatically satisfied).

PROOF OF THEOREM A.4. We may assume that (A.10), (A.11) and (A.13) hold. We then set $u_1 = \gamma(1)$,

$$c = \inf_{\bar\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{E}(\bar\gamma(t))$$

where $\Gamma = \{\bar\gamma \in C([0,1], \mathcal{D}^{1,2}(\mathbb{R}^N)), \bar\gamma(0) = 0, \bar\gamma(1) = u_1\}$. In view of (A.10), (A.11): $c > 0$. We need to check Palais-Smale condition i.e. if $(u_n)_n$ satisfies

$$\begin{cases} -\Delta u_n = f(x, u_n) + \epsilon_n \ \text{ in } \mathcal{D}'(\mathbb{R}^N), & \epsilon_n \to 0 \ \text{ in } \ (\mathcal{D}^{1,2}(\mathbb{R}^N))' \\ \mathcal{E}(u_n) \to c & \end{cases}$$

we have to show that u_n is relatively compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

First of all observe that for δ small (A.11) yields

$$0 < c \leqslant \mathcal{E}(v) \leqslant \int_{\mathbb{R}^N} \frac{1}{2}|\nabla v|^2 - \frac{N+2}{N-2} K u_\mu^{4/(N-2)} v^2 \, dx$$

if $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $v \geqslant 0$, $\|v\|_{\mathcal{D}^{1,2}} = \delta$. Therefore we have in fact

$$\exists \nu > 0, \qquad \int_{\mathbb{R}^N} \frac{1}{2} |\nabla v|^2 - \frac{N+2}{N-2} K u_\mu^{4/(N-2)} v^2 \, dx \geqslant \nu \int_{\mathbb{R}^N} |\nabla v|^2 \, dx.$$

The enables us to prove that $(u_n)_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$: indeed we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx &= \int_{\mathbb{R}^N} f(x, u_n) u_n \, dx + \langle \epsilon_n, u_n \rangle \geqslant \\ &\geqslant \int_{\mathbb{R}^N} \frac{N+2}{N-2} K u_\mu^{4/(N-2)} (u_n^+)^2 \, dx + \\ &+ \gamma \int_{\mathbb{R}^N} F(x, u_n) - \frac{1}{2} \frac{N+2}{N-2} K u_\mu^{4/(N-2)} (u_n^+)^2 \, dx - \|\epsilon_n\| \, \|u_n\| \end{aligned}$$

where $\gamma \in \left]2, \frac{2N}{N-2}\right[$ is arbitrary. And usign (A.14) it is easy to show that $(u_n)_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

If u_n converges weakly to $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, using (A.8) one may prove that $(|\nabla u_n|^2 + |u_n|^{2N/(N-2)})$ is tight and thus $u_n - u$ (by Lemma 1.1) concentrates at some points $(x_j)_{j \in J}$ and we have (see section 4.6 for more details)

$$\begin{cases} -\Delta u = f(x, u) \ \text{ in } \ \mathbb{R}^N, & u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ c = \mathcal{E}(u) + \sum_{j \in J} \frac{1}{N} \mu_j, & \mu_j K(x_j) \geqslant I_0 \mu_j^{(N-2)/N} \end{cases}$$

hence

$$\mu_j \geqslant I_0^{N/2} K(x_j)^{-(N-2)/2} \geqslant I_0^{N/2} \left(\sup_{\mathbb{R}^N} K \right)^{-(N-2)/2}.$$

And we reach a contradiction with (A.13) since

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - F(x,u)\,dx = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - \frac{3}{2}\frac{N+2}{N-2}Ku_\mu^{4/(N-2)}(u^+)^2\,dx +$$

$$- \int_{\mathbb{R}^N} F(x,u) - \frac{1}{2}\frac{N+2}{N-2}Ku_\mu^{4/(N-2)}(u^+)^2\,dx \geqslant$$

$$\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N+2}{N-2}Ku_\mu^{4/(N-2)}(u^+)^2\,dx +$$

$$- \frac{1}{\gamma}\int_{\mathbb{R}^N} f(x,u)u - \frac{N+2}{N-2}Ku_\mu^{4/(N-2)}(u^+)^2\,dx \geqslant 0$$

since $\gamma > 2$.

Appendix 2. Improved Sobolev inequalities by symmetries

We want to collect here a few easy remarks on classes of functions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ presenting i) symmetry properties, ii) support properties. Roughly speaking if those functions possess enough symmetries and if fixed points of the symmetries do not lie in their supports, Sobolev inequalities may be improved. The easiest example is the following: let H be the space of functions u in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ for some $1 \leqslant p < N$ such that i) u is spherically symmetric, ii) $\operatorname{Supp} u \subset \{|x| \geqslant \delta\}$ for some fixed $\delta > 0$. Then $H \hookrightarrow L^q(\mathbb{R}^N)$ for $Np/(N-p) \leqslant q \leqslant \infty$.

To simplify the presentation we will only treat the following situation: let Ω be an open set like $\Omega = \omega \times O_1 \times \ldots \times O_m$ where $m \geqslant 1$, ω is a bounded open set of $\mathbb{R}^{N_0}$ (possibly empty), $O_1, \ldots, O_m$ are given by

$$O_i = \{x_i \in \mathbb{R}^{N_i} / |x_i| \geqslant \delta_i\}, \qquad \forall i \in \{1, \ldots, m\};$$

where $N_i \geqslant 2$, $\delta_i > 0$. Clearly $N = \sum_{i=0}^m N_i$. We will denote by $x = (x_0, x_1, \ldots, x_m)$ a generic point of Ω. Let E be the subspace of $\mathcal{D}^{1,p}(\mathbb{R}^{N_0} \times \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_m})$ consisting of functions which are spherically symmetric with respect to each $x_i \in \mathbb{R}^{N_i}$ and let $F = E \in \mathcal{D}_0^{1,p}(\Omega)$.

We begin with the case when $N_0 + m > p$: then let $u \in \mathcal{D}(\mathbb{R}^N) \cap E$ and let v be defined on $\mathbb{R}^{N_0} \times (]0,\infty[)^m$ by

$$v(x_0, t_1, \ldots, t_m) = \prod_{i=1}^m t_i^{(N_i-1)/p} u(x_0, x_1, \ldots, x_m), \quad \text{with} \quad |x_i| = t_i.$$

Then $v \in \mathcal{D}_0^{1,p}(Q)$ with $Q = \mathbb{R}^{N_0} = (]0,\infty[)^m$ and if $N_i > p$ of all i

$$\|v\|_{\mathcal{D}_0^{1,p}(Q)} \leqslant C\|u\|_E + C\sum_{i=1}^{m}\left\|\frac{u}{|x_i|}\right\|_{L^p} \leqslant C\|u\|_E + C\sum_{i=1}^{M}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p}$$
$$\leqslant C\|u\|_E$$

therefore: $\|v\|_{L^{\bar{q}}(Q)} \leqslant C\|u\|_E$, where $\bar{q} = (N_0 + m)p/(N_0 + m - p)$.

And we find in conclusion that *if* $N_0 + m > p$, $p < N_i$, $\forall i \in \{1, \ldots, m\}$

$$\left(\int_{\mathbb{R}^N} P(x)|u|^{\bar{q}}\,dx\right)^{1/\bar{q}} \leqslant C\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}, \qquad \forall u \in E \tag{B.1}$$

where

$$P = \prod_{i=1}^{m} |x_i|^{\theta_i},$$

with $\theta_i = (N_i - 1)(\bar{q} - p)/p$. Using Sobolev and Hölder inequalities we also find if $q^* = (Np)/(N - p)$

$$\left(\int_{\mathbb{R}^N} P^{\theta}|u|^{q}\,dx\right)^{1/q} \leqslant C\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}, \qquad \forall u \in E \tag{B.2}$$

for all $q \in [q^*, \bar{q}]$, where $\theta = (\bar{q}/q)(q - q^*)(\bar{q} - q^*)^{-1}$. And thus we have

$$\|u\|_{L^q(\Omega)} \leqslant C\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}, \qquad \forall u \in F, \qquad \forall q \in [q^*, \bar{q}]. \tag{B.3}$$

Next, *if* $p > N_0 + m$, $p < N_i$, $\forall i \in \{1, \ldots, m\}$; the same proof shows

$$|u| \leqslant C\left\{\inf_{i \leq j \leq m} |x_j|^{1 - n/p}\right\}\left\{\prod_{i=1}^{m} |x_i|^{-(N_i - 1)/p}\right\}|\nabla u|_{L^p}, \qquad \forall u \in E \tag{B.4}$$

where $n = N_0 + m$. In particular, we find

$$\|u\|_{L^q(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega)}, \qquad \forall u \in F \tag{B.5}$$

for all $q^* \leqslant q \leqslant \infty$. The same result holds if $N_0 = 0$, $m = 1$, $p = 1$.

Similar results may be obtained for all $u \in E$ in the remaining cases ($p < N_0 + m$, $\exists i$, $p \geqslant N_i$, or $p = N_0 + m$; or $p > N_0 + m$. $\exists i$, $p \geqslant N_i$) but we will skip them. Now for $u \in F$, we indicate that if $p < N_0 + m$ then (B.3) *still holds, while if* $p \geqslant N_0 + m$, (B.5) *holds for* $q \in [q^*, \infty]$ *if* $p > N_0 + m$, *for* $q \in [q^*, \infty[$ *if* $p = N_0 + m$.

Indeed the above proofs are easily adapted for $u \in F$.

References

[1] Adams, R. Sobolev spaces, *Academic Press*, New-York, 1975.

[2] Alvino, A., Lions, P. L. and Trombetti, G. A remark on comparizon results via symmetrization. To appear.

[3] Amick, C. J. and Toland, J. F. Nonlinear elliptic eigenvalue problems on a infinite strip: global theory of bifurcation and asymptotic bifurcation, *Math. Ann.*
[4] Aronsjan, N. Theory of reproducing kernels. T.A.M.S., **68**, 337.
[5] Atkinson, F. V. and Peletier, L. A. Ground states of $-\Delta u = f(u)$ and the Emden-Fowler equation, *Preprint.*
[6] Aubin, T. Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pure Appl.,* **55** (1976), 269-296.
[7] Aubin, T. Problèmes isopérimétriques et spaces de Sobolev, *J. Diff. Geom.,* (1976), 573-598; announced in *C. R. Acad. Sci.* Paris, **280** (1975) 279-282.
[8] Aubin, T. Sur la fonction exponentielle, *C. R. Acad. Sci.,* Paris, **270** (1970), 1514-1517.
[9] Aubin, T. Nonlinear analysis on manifolds. Monge-Ampère equations, *Springer*, New-York, 1982.
[10] Aubin, T. Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire, *J. Funct. Anal.,* **32** (1979), 148-174.
[11] Aubin, T. The scalar curvature. In Differential Geometry and relativity, Cohen and Flato eds., Reidel Publishing Company, 1976.
[12] Aubin, T. Best constants in the Sobolev imbedding theorem: the Yamabe problem. In Seminar on Differental Geometry , ed. S. T. Yau, Princeton Univ. Press, Princeton N. J., 1982.
[13] Berestycki, H. and Lions, P. L. Nonlinear scalar field equations. *Arch. Rat. Mech. Anal.,* **82** (1983), 313-376; see also *C. R. Acad. Sci.* Paris, **287** (1978), 503-506, and in Bifurcation phenomena in Mathematical Physics and related topics, C. Bardos and D. Bessis (eds.), Reidel, Dordrecht, 1980.
[14] Bliss, G. An integral inequality, *J. London Math. Soc,* **5** (1930), 40-46.
[15] Bona, J. Bose, D. K., and Turner, R. E. L. Finite amplitude steady waves in stratified fluids, *J. Math. Pures Appl.*
[16] Bourghignon, J. P. Sur une condition d'intégrabilité d'origine géométrique pour une famille d'équations aux dérivées partielles non linéaires sur la sphére. In Séminaire Goulaouic-Meyer-Schwartz 1981-1982, Ecole Polytechnique, Palaiseau, 1982.
[17] Bourguignon, J. P. and Ezin, J. P. Functions courbure scalaire dans une classe conforme et groupe conforme des variétés riemaniennes compactes. To appear.
[18] Brézis, H. and Coron, J. M. Multiple solutions of H-systems and Rellich's conjecture. To appear in *Comm. Pure Appl. Math.*
[19] Brézis, H. and Coron, J. M. Large solutions for harmonic maps in two dimensions. *Comm. Math Phys.,* **92** (1983) 203-215.
[20] Brézis, H. and Coron, J. M. Convergence de solutions de H-systèmes et applications aux surfaces à courbure moyenne constante. Comptes-Rendus *Acad. Sci.* Paris, 1984.
[21] Brézis, H. and Lieb, E. H. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.,* 1983.
[22] Brézis, H. and Lieb, E. H. To appear in *J. Funct. Anal.* See also E. H. Lieb: Some vector-valued field equations. In *Proc. Intern. Conf. Diff. Eqs.*, Alabama 1983, to appear.
[23] Brézis, H. and Nirenberg, L. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.,* **36** (1983), 437-477.

[24] Carleman, T. Zur theorie der minimalflächen, *Math. Zeit.,* **9,** 154.
[25] Coffman, C. V. A minimum-maximum principle for a class of nonlinear integral equations, *J. Anal. Math.,* **22** (1969), 391-419.
[27] Coffman, C. V. On a class of nonlinear elliptic boundary value problems, *J. Math. Mech.,* **19** (1970), 351-356.
[28] Coffman, C. V. and Markus. To appear.
[29] Coleman, S., Glazer, V. and Martin, A. Action minima among solutions to a class of Euclidean scalar field equations, *Comm. Math. Phys.,* (1978) 211-221.
[30] Coron, J. M. Topologie et cas limite des injections de Sobolev, *Cr. Acad. Sci.,* Paris, 1984.
[31] Crandall, M. G. and Rabinowitz, P. H. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, *Arch. Rat. Mech. Anal.,* **58** (1975), 207-218.
[32] Du Plessis, N. Some theorems about the Riesz fractional integral, *Trans. Amer. Math. Soc.,* **80** (1955), 124-134.
[33] Elaisson, H. On variation of metrics. *Math. Scand.,* **29** (1971), 317-327.
[34] Esteban, M. J. Nonlinear elliptic problems in strip-like domains; symmetry of positive vortex rings, *Nonlinear Anal. T.M.A.,* **7** (1983), 365-379.
[35] Esteban, M. J., and Lions, P. L. Existence and nonexistence results for semilinear elliptic problems in unbounded domains, *Proc. Roy. Soc.,* Edimburgh, **82** (1983), 1-14.
[36] de Figueiredo, D. G., Lions, P. L. and Nussbaum, R. D. A priori estimates and existence of positive solutions of semilinear elliptic equations, *J. Maths Pures Appl.,* **61** (1982), 41-63.
[37] Fisher, A. E., and Marsden, J. E. Deformations of the scalar curvature, *Duke Math. J.*, **4** (1975), 519-547.
[38] Glazer, V., Martin, A., Grosse, H., and Thirring, W. A family of optimal conditions for the absence of bound states in a potential, In Studies of Mathematical Physics; E. H. Lieb, B. Simon and A. S. Wightman, eds., Princeton Univ. Press, 1976.
[39] Hardi, G. H., and Littlewood, J. E. Some properties of fractional integrals (1), *Math. Zeitschr.,* **27** (1928), 565-606.
[40] Hardy, G. H., and Littlewood, J. E. On certain inequalities connected with the calculus of variations, *J. London Math. Soc.,* **5** (1930), 34-39.
[41] Hildebrandt, S. On the Plateau problem for surfaces of constant mean curvature, *Comm. Pure Appl. Math.,* **23** (1970), 97-114.
[42] Hildebrandt, S. Nonlinear elliptic systems and harmonic mappings, *Proc. Beijing Symp. Diff. Geom.* and *Diff. Eq.*, Beijing, To appear.
[43] Jacobs, S. An isoperimetric inequality for functions analytic in multiply connected domains, *Report Mittag-Leffler Institut*, 1970.
[44] Jerison, D., and Lee, J. M. A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds, Preprint.
[45] Jost, J. The Dirichlet problem for harmonic maps from a surface with boundary onto a 2-sphere with non constant boundary values, *Invent. Math.,* 1984.
[46] Kazdan, J. Gaussian and scalar curvature, and update, In *Seminar on Differential Geometry*, S. T. Yau ed., Princeton Univ. Press, Princeton, N. J., 1982.
[47] Kazdan, J. and Warner, F. W. Scalar curvature and conformal deformation of Riemannian structure, *J. Diff. Geom.,* **10** (1975), 113-134.
[48] Kazdan, J. and Warner, F. W. Existence and conformal deformation of

metrics with prescribed Gaussian and scalar curvature, *Ann. of Math.*, **101** (1975), 317-331.

[49] Kazdan, J., and Warnar, F. W. Prescribing curvatures, *Proc., Symp. Pure Math.*, vol. 27, 1975, *Amer. Math. Soc.*

[50] Kazdan, J. and Warner, F. W. A direct approach to the determination of Gaussian and scalar curvature functions, *Invent. Mqath.*, **28** (1975), 227-230.

[51] Kazdan, J. and Warner, F. W. Remarks on some quasilinear elliptic equations, *Comm. Pure Appl. Math.*, **16** (1974), 103-125.

[52] *Kenig, C.* and Ni, W. M. An exterior Dirichlet problem with applications to some nonlinear equations arising in geometry. To appear in *Amer. J. Math.*.

[53] Lieb, E. H. Sharp constants in the Hardy-Littlewood-Sobolev inequality and related inequalities, *Ann. Math.*, **118** (1983), 349-374.

[54] Lieb, E. H. Existence and uniqueness of the minimizing solutions of Choquard's nonlinear equations, *Stud. Appl. Math.*, **57** (1977), 93-105.

[55] Lions, P. L. The concentration-compactness principle in the Calculus of Variations. The locally compact case, Part 1, *Ann. I.H.P., Anal. Nonlin.*, **1** (1984), 109-145.

[56] Lions, P. L. The concentration-compactness principle in the Calculus of Variations. The locally compact case, Part 2, *Ann. I.H.P., Anal. Nonlin.*, 1984.

[57] Lions, P. L. The Choquard equation and related questions, *Nonlin. Anal. T.M.A.*, **4** (1980), 1063-1073.

[58] Lions, P. L. Principe de concentration-compacité en Calcul des Variations, *C.R. Acad. Sci.*, Paris, **294** (1982), 261-264.

[59] Lions, P. L. On the concentration-compactness principle. In Contributions to Nonlinear Partial Differential Equations, Pitman, London, 1983.

[60] Lions, P. L. Applications de la méthode de concentration-compacité à l'existence de fonctions extrêmales, *C.R. acad. Sci.*, Paris, **296** (1983), 645-648.

[61] Lions, P. L. La méthode de concentration-compacité en Calcul des Variations. In Séminaire Goulaouic-Meyer-Schwartz 1982-1983. Ecole Polytechnique, Palaiseau.

[62] Lions, P. L. Minimization problems in $L^1(\mathbb{R}^N)$, *J. Funct. Anal.*, **41** (1981), 236-275.

[63] Lions, P. L. Symmetry and compactness in Sobolev spaces, *J. Funct. Anal.*, **49** (1982), 315-334.

[64] Lions, P. L. On the existence of positive solutions for semilinear elliptic equations. SIAM Review, **24** (1982), 442-467.

[65] Lions, P. L. The concentration-compactness principle in the Calculus in Variations. The limit case, Part 1, To appear.

[66] Mignont, F., and *Puel, J. P.* Sur une classe de problèmes nonlinéaires avec nonlinéarité positive, croissante, convexe, *Comm. P.D.E.*

[67] Moser, J. A sharp form of an inequality of N. Trudinger, *Ind. Univ. Math. J.*, **20** (1971), 1077-1092.

[68] Ni, W. M. On the elliptic equation $\Delta u + K(x)u^{(N+2)/(N-2)} = 0$, its generalization and applications in geometry, *Ind. Univ. Math. J.*, 1983.

[69] Ni, W. M. Conformal metrics with zero scalar curvature and a symmetrization process via maximum principle, In *Seminar on Differential Geometry*, ed., S. T. Yau, Princeton Univ. Press, Princeton, N. J., 1982.

[70] Pohozaev, S. I. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet math. Dokl.*, **165** (1965), 1408-1412.

[71] Rosen, G. Minimum value for c in the Sobolev inequality $\|\phi\|_6 \leqslant c\|\nabla\phi\|_2$, *SIAM J. Appl. Math.*, **21** (1971), 30-32.
[72] Sacks, J., and Uhlenbeck, K. The existence of minimal immersions of 2-spheres, *Ann. Math.*, **113** (1981), 1-24.
[73] Schoen, R. and Yau, S. T. Existence of incompressible minimal surfaces and the topology of manifolds inth nonnegative scalar curvature, *Ann. if Math.*, **110** (1975) 127-142.
[74] Sedlacek, S. A direct method for minimizing the Yang-Mills functional over 4-manifolds, *Comm. Math. Phys.*, 1983.
[75] Siu, Y. T., and Yau, S. T. Compact Kahler manifolds of positive bisectional curature, *Invent. Math.*, **59** (1980), 183.
[76] Sobolev, S. L. On a theorem of function analysis, *Mat. Sbornik*, **4** (1938), 471-479, (A.M.S. Transl. Ser. 2, **34** (1963), 39-68).
[77] Strauss, W. Existence of solitary waves in higher dimensions, *Comm. Math. Phys.*, **55**, (1977), 149-162.
[78] Talenti, G. Best constant in Sobolev inequality, *Ann. Math. Pura Appl.*, **110** (1976), 353-372.
[79] Talenti, G. Elliptic equations and rearrangements, *Ann. Scuola Norm. Sup.*, Pisa, **3** (1976), 697-718.
[80] Taubes, C. The existence of a non-minimal solution to the SU(2) Yang-Mills-Higgs equations on $\mathbb{R}^3$, *Comm. Math. Phys.*, **86** (1982), 257-298, 299-320.
[81] Taubes, C. Self-dual connections on non self dual 4-anifolds, *J. Diff. Geom.*, **17** (1972), 135.
[82] Taubes, C. Path-cannected Yang-Mills moduli spaces. To appear in *J. Diff. Geom.*
[83] Trudinger, N. S. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Sc. Norm. Sup.*, Pisa, **22** (1968), 265-274.
[84] Trudinger, N. S. On imbedding into Olitz spaces and some applications, *J. Math. Phys.*, **17** (1967) 473-484.
[85] Uhlenbeck, K. Removable singularities in Yang-Mills fields, *Comm. Math. Phys.*, **83** (1982), 11-29.
[86] Uhlenbeck, K. Connections with L^p bounds on curvature, *Comm. Math. Phys.*, **83** (1982), 31-42.
[87] Uhlenbeck, K. In *Seminar on Differential Geometry*, ed. S. T. Yan, Princeton Univ. Press, Princeton N. J., 1972.
[88] Wente, H. The Dirichlet problem with a volume constraint, *Manuscripta Math.*, **11** (1974), 141-157.
[89] Yamabe, H. On the deformation of Riemannian structures on compact manifolds, *Osaka Math. J.*, **12** (1960), 21-37.

P. L. Lions
CEREMADE
Université Paris IX-Dauphine
Place de Lattre de Tassigny
75775 Paris, Cedex 16

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 1, N.° 3, 1985

On the Eisenstein Series of Hilbert Modular Groups

Goro Shimura

Introduction

Throughout the paper, we let F denote a totally real algebraic number field of degree n, and $\mathbf{a}$ the set of all archimedean primes of F. Given a set X, we denote by $X^{\mathbf{a}}$ the product of $\mathbf{a}$ copies of X, that is, the set of all indexed elements $(x_v)_{v\in\mathbf{a}}$ with $x_v \in X$. If $y \in X^{\mathbf{a}}$, y_v will detnote its v-component. Putting $H = \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$, we let $SL_2(F)$ act on $H^{\mathbf{a}}$ through the injection of $SL_2(F)$ into $SL_2(\mathbf{R})^{\mathbf{a}}$. For $\sigma \in \mathbf{Z}^{\mathbf{a}}$ and $v \in \mathbf{a}$, we define a differential operator L_v^{σ} on $H^{\mathbf{a}}$ by

$$L_v^{\sigma} = -4y_v^{2-\sigma_v}(\partial/\partial z_v)y_v^{\sigma_v}(\partial/\partial \bar{z}_v),$$

where z_v is the variable on the v-factor of $H^{\mathbf{a}}$ and $y_v = \mathrm{Im}(z_v)$.

Given a congruence subgroup Γ of $SL_2(F)$ and $\lambda \in \mathbf{C}^{\mathbf{a}}$, we denote by $\mathcal{A}(\sigma, \lambda, \Gamma)$ the set of all C^{∞}-functions f on $H^{\mathbf{a}}$ such that

i) $f(\gamma(z)) = \prod_{v\in\mathbf{a}}(c_v z_v + d_v)^{\sigma_v} f(z)$ for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,
ii) $L_v^{\sigma} f = \lambda_v f$ for every $v \in \mathbf{a}$,
iii) f is slowly increasing at every cusp.

Further we let $\mathcal{S}(\sigma, \lambda, \Gamma)$ denote the set of all cusp forms, defined as usual, belonging to $\mathcal{A}(\sigma, \lambda, \Gamma)$, and $\mathcal{N}(\sigma, \lambda, \Gamma)$ the orthogonal complement of $\mathcal{S}(\sigma, \lambda, \Gamma)$ in $\mathcal{A}(\sigma, \lambda, \Gamma)$.

Now the main purpose of this paper is to show that $\mathcal{N}(\sigma, \lambda, \Gamma)$ can be spanned, in most cases, by certain Eisenstein series, which are functions $E(z, s; \rho)$ of the variable z on $H^{\mathbf{a}}$, a complex parameter s, and another discrete parameter $\rho \in \mathbf{C}^{\mathbf{a}}$. Namely, given σ, λ, and Γ, we can choose s_0 and ρ so that a suitable finite set of $E(z, s_0; \rho)$ spans $\mathfrak{N}(\sigma, \lambda, \Gamma)$ (Theorem 7.3). If $F = \mathbf{Q}$, the parameter ρ indicates nothing but the weight σ, but if $F \neq \mathbf{Q}$, ρ involves a variable in $\mathbf{R}^{\mathbf{a}}$ which parametrizes the archimedean factors of Hecke (Grössen-) characters of F. There are two cases in which Eisenstein series by themselves cannot generate $\mathfrak{N}(\sigma, \lambda, \Gamma)$. In fact, if $4\lambda_v = (1 - \sigma_v)^2$ for every $v \in \mathbf{a}$, we need $\partial E/\partial s$ (Theorem 7.8); in the other case, we need the residues of the $E(z, s)$ (Theorem 7.9). These theorems are valid also for eigenforms of half-integral weight, which can be defined by making suitable modifications in the above definition. Our results are not complete in the sense that we have to exclude the case of «multiple» λ, which occurs only when $F \neq \mathbf{Q}$, $(1 - \sigma_v)^2 \leqslant 4\lambda_v \in \mathbf{R}$ for all $v \in \mathbf{a}$, and $(1 - \sigma_v)^2 < 4\lambda_v$ for at least one v. We believe, however, that our technique is applicable even to multiple λ, and therefore no serious difficulties are expected in the task of extending our results to the most general case.

As an application, we shall show that every holomorphic Hilbet modular form is a sum of a holomorphic cusp form and a holomorphic Eisenstein series. This holds for all integral and half-integral weights $\geqslant \frac{1}{2}$ (Theorems 8.3, 8.4, and formula (8.3)). The explicit Fourier expansions of certain Eisenstein series obtained in our previous papers [12] and [13] play an essential role in the proof of this result as well as in that of the theorems on $\mathfrak{N}(\sigma, \lambda, \Gamma)$.

Another application concerns an interpretation of the zeros of L-functions of F in the critical strip. To explain the idea, let us assume $F = \mathbf{Q}$ for simplicity. Given $\xi = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Q})$ and $f \in \mathfrak{N}(\sigma, \lambda, \Gamma)$ with $\lambda \in \mathbf{C}$, $\sigma \in \mathbf{Z}$ and $\Gamma \subset SL_2(\mathbf{Z})$, we can speak of a Fourier expansion of f at the cusp $\xi(\infty)$, which has the form

$$(cz + d)^{-\sigma} f(\xi(z)) = a_\xi y^{s_0 - \sigma/2} + a'_\xi y^{1 - s_0 - \sigma/2} + \sum_{n \in \mathbf{Z}} b_\xi(n, y) e^{2\pi i n x/N}$$

with $0 < N \in \mathbf{Z}$, constants a_ξ and a'_ξ, and a complex number s_0 such that $\lambda = (s_0 - \sigma/2)(1 - s_0 - \sigma/2)$. Now we call f a *cyclopean form of exponent* $1 - s_0 - \sigma/2$ if $0 < \mathrm{Re}(s_0) < \frac{1}{2}$ and $a_\xi = 0$ for every $\xi \in SL_2(\mathbf{Q})$. Then we shall show that a nonzero cyclopean form exists if and only if there exists a Dirichlet character ψ such that $L(2s_0, \psi) = 0$ and $\psi(-1) = (-1)^\sigma$. The same type of assertion can be made also for $F \neq \mathbf{Q}$ (Theorem 9.1). This result is tautological if $\Gamma = SL_2(\mathbf{Z})$, in the sense that it follows immediately from the well-known Fourier expansion of the Eisenstein series of $SL_2(\mathbf{Z})$. The assertion in the general case, however, is nontrivial, even when $F = \mathbf{Q}$. In fact, the L-functions involve Euler-products and Dirichlet (or Hecke) characters ψ while our definition of cyclopean forms does not require any such multiplicative structure at least on the surface, which is why we think that the fact deserves a statement as we present here.

Let us conclude the introduction by mentioning the previous investigations. The eigenforms were first studied by Maass in [3] and [4] for the congruence subgroups of $SL_2(\mathbf{Z})$. In particular, he proved a certain bilinear relation of the coefficients of the constant terms of eigenforms and showed that $\mathfrak{N}(\sigma, \lambda, \Gamma)$ can be spanned by Eisenstein series when $\sigma = 0$, $\lambda \geqslant \frac{1}{4}$ and $\Gamma \subset SL_2(\mathbf{Z})$. In [7], Roelcke generalized these to the eigenforms of an arbitrary weight with respect to an arbitrary Fuchsian group. The present paper owes much to their ideas in those papers; in fact, one of the key points in our treatment is a generalization of their bilinear relations.

In the holomorphic case, the fact that an elliptic modular form of integral weight is the sum of a cusp form and an Eisenstein series was proved by Hecke [1]. This was extended by Kloosterman [2] to the Hilbert modular forms of weight $\geqslant 2$. The case of weight 1 was proved recently by Shimizu [8]. As for the forms of half-integral weight, Petersson [6] obtained a corresponding result for weight $\geqslant \frac{5}{2}$ when $F = \mathbf{Q}$. Recently the case of weight $\frac{3}{2}$ with $F = \mathbf{Q}$ was settled by Pei [5].

1. Congruence subgroups and factors of automorphy

The symbols F, n, $\mathbf{a}$, $X^{\mathbf{a}}$, and H we used in the introduction will have the same meaning throughout the paper. In addition, we let $\mathbf{f}$ denote the set of all nonarchimedean primes of F, $\mathfrak{g}$ the maximal order of F, $\mathfrak{g}^{\times}$ the group of all units of F, and $\mathfrak{d}$ the different of F. Each element of $\mathbf{a}$ will be viewed as an injection of F into $\mathbf{R}$. Then $F \otimes_{\mathbf{Q}} \mathbf{R}$ and $F \otimes_{\mathbf{Q}} \mathbf{C}$ can be identified naturally with $\mathbf{R}^{\mathbf{a}}$ and $\mathbf{C}^{\mathbf{a}}$, respectively, through the map $a \otimes b \mapsto (a_v b)_{v \in \mathbf{a}}$ for $a \in F$ and $b \in \mathbf{R}$ (or $\mathbf{C}$), where a_v denotes the image of a under v. We write $a \gg 0$ for $a \in \mathbf{R}^{\mathbf{a}}$ if $a_v > 0$ for all v. For two elements c and x of $\mathbf{C}^{\mathbf{a}}$, we put

$$c^x = \prod_{v \in \mathbf{a}} c_v^{x_v} \tag{1.1}$$

whenever each factor is well-defined (according to the context). We denote by u the identity element of the ring $\mathbf{C}^{\mathbf{a}}$. We have then

$$c^{su} = \prod_{v \in \mathbf{a}} c_v^s \quad \text{for} \quad s \in \mathbf{C}. \tag{1.2}$$

Given an associative ring R with identity element, we denote by $R^{\times}$ the group of all invertible elements of R, and by $M_2(R)$ the ring of all 2×2-matrices with entries in R, and put $SL_2(R) = \{\xi \in M_2(R) \mid \det(\xi) = 1\}$ when R is commutative. For $\xi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$, we write $a = a_\xi$, $b = b_\xi$, $c = c_\xi$, and $d = d_\xi$. For $\alpha \in SL_2(\mathbf{R})$ and $z \in \mathbf{C}$, we put

$$\alpha(z) = (a_\alpha z + b_\alpha)/(c_\alpha z + d_\alpha), \quad j(\alpha, z) = c_\alpha z + d_\alpha. \tag{1.3}$$

Further, for $\alpha = (\alpha_v)_{v \in \mathbf{a}} \in SL_2(\mathbf{R})^{\mathbf{a}}$ and $z = (z_v)_{v \in \mathbf{a}} \in \mathbf{C}^{\mathbf{a}}$, we put

$$\text{(1.4}a) \qquad \alpha(z) = (\alpha_v(z_v))_{v\in\mathbf{a}}, \quad j_v(\alpha, z) = j(\alpha_v, z_v),$$

$$\text{(1.4}b) \qquad j_\alpha(z) = j(\alpha, z) = (j_v(\alpha, z))_{v\in\mathbf{a}} \quad (\in \mathbf{C}^{\mathbf{a}}).$$

With u as in (1.2), we have $x^u = N_{F/\mathbf{Q}}(x)$ for $x \in F$ and also

$$j_\alpha(z)^u = \prod_{v\in\mathbf{a}} j_v(\alpha, z).$$

We define our basic group G and its parabolic subgroup P by

$$\text{(1.5)} \qquad G = SL_2(F), \qquad P = \{\alpha \in G \mid c_\alpha = 0\}.$$

We identify $M_2(F) \otimes_{\mathbf{Q}} \mathbf{R}$ with $M_2(\mathbf{R})^{\mathbf{a}}$ and embed $M_2(F)$ and G into $M_2(\mathbf{R})^{\mathbf{a}}$ and $SL_2(\mathbf{R})^{\mathbf{a}}$; then we let G act on $H^{\mathbf{a}}$ (or even on $\mathbf{C}^{\mathbf{a}}$) through this embedding.

Given an integral ideal $\mathfrak{z}$ and fractional ideals $\mathfrak{x}$ and $\mathfrak{y}$ in F such that $\mathfrak{x}\mathfrak{y}$ is integral, we define a subring $\mathfrak{o}[\mathfrak{x}, \mathfrak{y}]$ of $M_2(F)$ and subgroups $\Gamma[\mathfrak{x}, \mathfrak{y}]$ and $\Gamma[\mathfrak{z}]$ of G by

$$\text{(1.6)} \qquad \mathfrak{o}[\mathfrak{x}, \mathfrak{y}] = \{\alpha \in M_2(F) \mid a_\alpha \in \mathfrak{g}, d_\alpha \in \mathfrak{g}, b_\alpha \in \mathfrak{x}, c_\alpha \in \mathfrak{y}\},$$

$$\text{(1.7}a) \qquad \Gamma[\mathfrak{x}, \mathfrak{y}] = \mathfrak{o}[\mathfrak{x}, \mathfrak{y}] \cap G,$$

$$\text{(1.7}b) \qquad \Gamma[\mathfrak{z}] = \{\alpha \in G \mid a_\alpha \equiv d_\alpha \equiv 1, b_\alpha \equiv c_\alpha \equiv 0 \pmod{\mathfrak{z}}\}.$$

A subgroup Γ of G is called a *congruence subgroup* of G if it contains $\Gamma[\mathfrak{z}]$ as a subgroup of finite index for some $\mathfrak{z}$.

We are going to consider automorphic forms of integral and half-integral weights with respect to congruence subgroups of G. A *weight* will be an element σ of $(1/2)\mathbf{Z}^{\mathbf{a}}$ such that $2\sigma_v \pmod 2$ is independent of v. Our treatment will be divided into two cases according to the parity: Case I for $\sigma \in \mathbf{Z}^{\mathbf{a}}$ (*integral weight*) and Case II for $\sigma \notin \mathbf{Z}^{\mathbf{a}}$ (*half-integral weight*). We consider the group $\mathfrak{G}_\sigma$ consisting of all couples (α, l) formed by $\alpha \in G$ and a holomorphic function l on $H^{\mathbf{a}}$ such that $l(z)^2 = t j_\alpha(z)^{2\sigma}$ with a root of unity t, the group-law being defined by

$$\text{(1.8)} \qquad (\alpha, l)(\alpha', l') = (\alpha\alpha', l(\alpha'(z))l'(z)).$$

In Case I, $\mathfrak{G}_\sigma$ is obviously isomorphic to the direct product of G and the group of all roots of unity. For $\xi = (\alpha, l) \in \mathfrak{G}_\sigma$, we write $\alpha = \mathrm{pr}(\xi)$, $l = l_\xi$, $a_\xi = a_\alpha$, $b_\xi = b_\alpha$, $c_\xi = c_\alpha$, $d_\xi = d_\alpha$, and put $\xi(z) = \alpha(z)$ for $z \in H^{\mathbf{a}}$. The group $\mathfrak{G}_\sigma$ is introduced for the purpose of dealing with Case II. We consider it even in Case I, simply in order to make our exposition uniform.

Let F_v denote the v-completion of F for each $v \in \mathbf{a} \cup \mathbf{f}$. If $\mathfrak{x}$ is a fractional ideal in F and $v \in \mathbf{f}$, we denote by $\mathfrak{x}_v$ its closure in F_v. We put $G_v = SL_2(F_v)$ and define the adelization $G_{\mathbf{A}}$ and $P_{\mathbf{A}}$ of G and P as usual. We denote by $G_{\mathbf{a}}$ and $G_{\mathbf{f}}$ the archimedean and nonarchimedean factors of $G_{\mathbf{A}}$; we identify G with its diagonal embedding into $G_{\mathbf{A}}$, and $SL_2(\mathbf{R})^{\mathbf{a}}$ with $G_{\mathbf{a}}$. For $\mathfrak{x}$ and $\mathfrak{y}$ as in (1.6), we put

$$(1.9a)\qquad D[\mathfrak{x}, \mathfrak{y}] = \prod_{v \in \mathbf{f} \cup \mathbf{a}} D_v[\mathfrak{x}, \mathfrak{y}],$$

$$(1.9b)\qquad D_v[\mathfrak{x}, \mathfrak{y}] = \begin{cases} SO(2) = \{x \in G_v \mid {}^t xx = 1\} & (v \in \mathbf{a}), \\ \mathfrak{o}[\mathfrak{x}, \mathfrak{y}]_v \cap G_v & (v \in \mathbf{f}), \end{cases}$$

where $\mathfrak{o}[\mathfrak{x}, \mathfrak{y}]_v$ is the closure of $\mathfrak{o}[\mathfrak{x}, \mathfrak{y}]$ in $M_2(F_v)$. We observe that $\Gamma[\mathfrak{x}, \mathfrak{y}] = G \cap D[\mathfrak{x}, \mathfrak{y}] G_{\mathbf{a}}$. There is another important subset

$$(1.10)\qquad W = G \cap P_{\mathbf{A}} \cdot D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$$

of $G_{\mathbf{A}}$. Obviously $P \cdot W \cdot \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}] = W$. In [13, Proposition 3.2], we assigned, to each $\beta \in W$, a holomorphic function h_β on $H^{\mathbf{a}}$ that satisfies the following conditions:

(1.11*a*) $h_\beta(z)^4 = j_\beta(z)^{2u}$ (and hence $(\beta, h_\beta) \in \mathcal{G}_{u/2}$);

(1.11*b*) $h_{\alpha\beta\gamma}(z) = h_\alpha(z) h_\beta(\gamma(z)) h_\gamma(z)$ if $\alpha \in P$, $\beta \in W$, and $\gamma \in \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$;

(1.11*c*) $h_\alpha(z) = |d_\alpha|^{u/2}$ if $\alpha \in P$;

(1.11*d*) $h_\gamma^2 / j_\gamma^u = (d_\gamma / |d_\gamma|)^u \left(\dfrac{F(\sqrt{-1})/F}{d_\gamma \mathfrak{g}} \right)$ if $\gamma \in \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$.

As for the last two properties, see [13, Proposition 1.2 and (3.13)]. We then define, in Case II, a map $\Lambda_\sigma^k \colon W \to \mathcal{G}_\sigma$ for each odd integer k by

$$(1.12)\qquad \Lambda_\sigma^k(\beta) = (\beta, h_\beta^k j_\beta^{\sigma - (k/2)u}) \qquad (\beta \in W).$$

Then (1.11*b*) implies that

$$(1.13)\qquad \Lambda_\sigma^k(\alpha\beta\gamma) = \Lambda_\sigma^k(\alpha)\Lambda_\sigma^k(\beta)\Lambda_\sigma^k(\gamma) \quad \text{for} \quad \alpha,\ \beta,\ \gamma \quad \text{as in (1.11b)}.$$

In Case I, we define an injection $\Lambda_\sigma^0 \colon G \to \mathcal{G}_\sigma$ by

$$(1.14)\qquad \Lambda_\sigma^0(\beta) = (\beta, j_\beta^\sigma) \qquad (\beta \in G).$$

Given an integral ideal $\mathfrak{c}$, we put

$$(1.15)\qquad \Delta_\sigma^k[\mathfrak{c}] = \begin{cases} \Lambda_\sigma^0(\Gamma[\mathfrak{c}]) & \text{(Case I)}, \\ \Lambda_\sigma^k(\{\beta \in \Gamma[2\mathfrak{c}\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}] \mid a_\beta - 1 \in \mathfrak{c}\}) & \text{(Case II)}, \end{cases}$$

assuming that $\mathfrak{c} \subset 4\mathfrak{g}$ in Case II. Here and henceforward, we understand that $k = 0$ in Case I. There is a congruence subgroup Γ of G such that

$$(1.16)\qquad \Lambda_\sigma^k(\gamma) = \Lambda_\sigma^1(\gamma) \quad \text{for every} \quad \gamma \in \Gamma.$$

This follows from (1.11*d*) and [10, Lemma 7.4].

Now, by a *congruence subgroup of* $\mathcal{G}_\sigma$, we understand a subgroup Δ of $\mathcal{G}_\sigma$ satisfying the following two conditions:

(1.17*a*) pr *gives an isomorphism of* Δ *onto a subgroup of* G;

(1.17*b*) Δ *contains* $\Delta_\sigma^k[\mathfrak{a}]$ *as a subgroup of finite index for some* $\mathfrak{a}$ *and* k, *where* k *should be* 0 *in Case I.*

If Δ is a congruence subgroup of $\mathcal{G}_\sigma$, then so is $\xi\Delta\xi^{-1}$ for every $\xi \in \mathcal{G}_\sigma$. This is trivial in Case I, and is proved in [13, Proposition 1.3] in Case II.

2. Automorphic eigenforms

For a function f: $H^{\mathbf{a}} \to \mathbf{C}$ and $\alpha \in \mathcal{G}_\sigma$, we define $f \parallel \alpha$: $H^{\mathbf{a}} \to \mathbf{C}$ by

$$(f \parallel \alpha)(z) = l_\alpha(z)^{-1} f(\alpha(z)). \tag{2.1}$$

From now on, we always put $y_v = \operatorname{Im}(z_v)$, $y = (y_v)_{v \in \mathbf{a}}$, and view y as an $\mathbf{R}^{\mathbf{a}}$-valued function on $H^{\mathbf{a}}$. Then we have

$$y^p \parallel \alpha = l_\alpha^{-1} |j_\alpha|^{-2p} y^p \qquad (p \in \mathbf{R}^{\mathbf{a}}, \alpha \in \mathcal{G}_\sigma). \tag{2.2}$$

For $v \in \mathbf{a}$ and $\sigma \in \mathbf{R}^{\mathbf{a}}$, we define differential operators ϵ_v, δ_v^σ, and L_v^σ acting on C^∞-functions f on $H^{\mathbf{a}}$ by

$$\epsilon_v f = -y_v^2 \cdot \partial f/\partial \bar{z}_v, \tag{2.3a}$$

$$\delta_v^\sigma f = y_v^{-\sigma_v} \cdot \partial(y_v^{\sigma_v} f)/\partial z_v, \tag{2.3b}$$

$$L_v^\sigma f = 4\delta_v^{\sigma'}\epsilon_v, \qquad \sigma_v' = \sigma_v - 2. \tag{2.3c}$$

We have then

$$\begin{aligned} L_v^\sigma &= -4y_v^2\partial^2/\partial z_v \partial \bar{z}_v + 2i\sigma_v y_v \partial/\partial \bar{z}_v \\ &= -\sigma_v + 4\epsilon_v \delta_v^\sigma. \end{aligned} \tag{2.4}$$

It can easily be seen, for every $\xi \in \mathcal{G}_\sigma$, that

$$\delta_v^\sigma(f \parallel \xi) = (\delta_v^\sigma f) \parallel \xi^* \quad \text{with} \quad \xi^* = (\operatorname{pr}(\xi), j_\xi^{2v} l_\xi), \tag{2.5a}$$

$$\epsilon_v(f \parallel \xi) = (\epsilon_v f) \parallel \xi_* \quad \text{with} \quad \xi_* = (\operatorname{pr}(\xi), j_\xi^{-2v} l_\xi), \tag{2.5b}$$

$$L_v^\sigma(f \parallel \xi) = (L_v^\sigma f) \parallel \xi. \tag{2.5c}$$

Furthermore, if $p \in \mathbf{R}^{\mathbf{a}}$, we have

$$L_v^\sigma y^p = p_v(1 - \sigma_v - p_v) y^p. \tag{2.6}$$

Let Δ be a congruence subgroup of $\mathcal{G}_\sigma$. By an *automorphic eigenform* with respect to Δ, we understand a real-analytic function f on $H^{\mathbf{a}}$ satisfying the following three conditions:

(2.7*a*) $f \| \alpha = f$ *for every* $\alpha \in \Delta$;

(2.7*b*) $L_v^\sigma f = \lambda_v f$ *with* $\lambda_v \in \mathbf{C}$ *for every* $v \in \mathbf{a}$;

(2.7*c*) *for every* $\xi \in \mathcal{G}_\sigma$, *there exist positive numbers* A, B. *and* c (*depending on* f *and* ξ) *such that* $y^{\sigma/2}\,|(f \| \xi)(x+iy)| \leqslant Ay^{cu}$ *if* $y^u > B$.

We denote by $\mathcal{Q}(\sigma, \lambda, \Delta)$ the set of all such f, and by $\mathcal{Q}(\sigma, \lambda)$ the union of $\mathcal{Q}(\sigma, \lambda, \Delta)$ for all congruence subgroups Δ of $\mathcal{G}_\sigma$. Condition (2.7*c*) concerns, in essence, only finitely many elements ξ of $\mathcal{G}_\sigma$. In fact, put

$$\mathcal{P}_\sigma = \{\xi \in \mathcal{G}_\sigma \mid \mathrm{pr}(\xi) \in P\}. \tag{2.8}$$

Then $\Delta\backslash\mathcal{G}_\sigma/\mathcal{P}_\sigma$ is a finite set as will be seen in Section 3. The inequality of (2.7*c*) is true for all $\xi \in \mathcal{G}_\sigma$ if it is true for the members of a complete set of representatives of $\Delta\backslash\mathcal{G}_\sigma/\mathcal{P}_\sigma$.

Given two continuous functions f and g satisfying (2.7*a*), we define their inner product $\langle f, g\rangle$ by

$$\langle f, g\rangle = \mu(\Phi)^{-1}\int_\Phi \bar{f} g y^\sigma \, d\mu(z) \qquad (\Phi = \Delta\backslash H^{\mathbf{a}}), \tag{2.9}$$

where

$$\mu(\Phi) = \int_\Phi d\mu(z) \quad \text{and} \quad d\mu(z) = y^{-2u}\prod_{v\in\mathbf{a}} dx_v\, dy_v.$$

This does not depend on the choice of Δ. We see easily that

$$\langle f, g\rangle = \langle f\|\alpha, g\|\alpha\rangle \quad \text{for every} \quad \alpha \in \mathcal{G}_\sigma. \tag{2.10}$$

To study the Fourier expansion of an eigenform, let us put

$$\mathbf{e}(w) = e^{2\pi i w} \qquad \text{for} \quad w \in \mathbf{C}, \tag{2.11a}$$

$$\mathbf{e}_{\mathbf{a}}(z) = \mathbf{e}\Big(\sum_{v\in\mathbf{a}} z_v\Big) \quad \text{for} \quad z \in \mathbf{C}^{\mathbf{a}}. \tag{2.11b}$$

If $f \in \mathcal{Q}(\sigma, \lambda, \Delta)$, f has a Fourier expansion of the form

$$f(x+iy) = \sum_{h\in\mathfrak{m}} b(h, y)\mathbf{e}_{\mathbf{a}}(hx) \tag{2.12}$$

with a lattice $\mathfrak{m}$ in F, where $hx = (h_v x_v)_{v\in\mathbf{a}}$ (i.e., the product in the algebra $\mathbf{C}^{\mathbf{a}}$). We can find a subgroup U of $\mathfrak{g}^\times$ of finite index such that

$$\Lambda_\sigma^k(\mathrm{diag}[a, a^{-1}]) \in \Delta \quad \text{for all} \quad a \in U.$$

Then

$$f(a^2z) = |a|^{-(k/2)u}a^{-\sigma+(k/2)u}f(z) \quad \text{for every} \quad a \in U, \tag{2.13a}$$

$$b(a^2h, y) = |a|^{(k/2)u}a^{\sigma-(k/2)u}b(h, a^2y) \quad \text{for every} \quad a \in U. \tag{2.13b}$$

Now (2.7*b*) implies that $b(h, y)$ as a function in y_v satisfies

$$(y_v^2\partial^2/\partial y_v^2 + \sigma_v y_v \partial/\partial y_v - 4\pi^2 h_v^2 y_v^2 + 2\pi h_v \sigma_v y_v + \lambda_v)b = 0. \tag{2.14}$$

If $h_v \neq 0$, the solutions are given by Whittaker functions. To present them in a normalized form, we introduce a function $V(g; \alpha, \beta)$, defined for $0 < g \in \mathbf{R}$ and $(\alpha, \beta) \in \mathbf{C}^2$, which has an expression

$$V(g; \alpha, \beta) = e^{-g/2}g^{\beta}\Gamma(\beta)^{-1}\int_0^\infty e^{-tg}(1+t)^{\alpha-1}t^{\beta-1}\,dt \tag{2.15}$$

for $\mathrm{Re}(\beta) > 0$. This can be continued as a holomorphic function in (α, β) (and real-analytic in (g, α, β)) to the whole $\mathbf{C}^2$, and satisfies

$$V(g; 1-\beta, 1-\alpha) = V(g; \alpha, \beta), \tag{2.16}$$

$$\lim_{g\to\infty} e^{g/2}V(g; \alpha, \beta) = 1. \tag{2.17}$$

These facts are well known. For the reader's convenience, we give in Section 10 a self-contained treatment of Whittaker functions of this type including the proofs of these and other properties of V.

Now, given $\sigma \in \mathbf{C}^{\mathbf{a}}$ and $\lambda \in \mathbf{C}^{\mathbf{a}}$, we take α_v and β_v so that

$$\sigma_v = \alpha_v - \beta_v, \qquad \lambda_v = \beta_v(1-\alpha_v), \tag{2.18}$$

and define a function W_v for $t \in \mathbf{R}^\times$ by

$$W_v(t; \sigma, \lambda) = \begin{cases} V(4\pi t; \alpha_v, \beta_v) & \text{if} \quad t > 0, \\ |4\pi t|^{-\sigma_v}V(-4\pi t; \beta_v, \alpha_v) & \text{if} \quad t < 0. \end{cases} \tag{2.19}$$

If (α_v, β_v) is a solution of (2.18), then the other solution is $(1-\beta_v, 1-\alpha_v)$ (which may be equal to (α_v, β_v)). In view of (2.16), W_v is well-defined. Now it can be verified that $W_v(h_v y_v; \sigma, \lambda)$ as a function of y_v satisfies (2.14); it is $O(y_v^c)$ with $c \in \mathbf{R}$ when $y_v \to \infty$, as can be seen from (2.17); moreover, such a solution of (2.14) is unique up to constant factors (see Proposition 10.1).

We now put, for $t \in (\mathbf{R}^\times)^{\mathbf{a}}$, $\sigma \in \mathbf{R}^{\mathbf{a}}$, and $\lambda \in \mathbf{C}^{\mathbf{a}}$,

$$W(t; \sigma, \lambda) = \prod_{v\in\mathbf{a}} W_v(t_v; \sigma, \lambda). \tag{2.20}$$

Then conditions (2.7*b*, *c*) imply that $b(h, y)$ is a constant multiple of $W(hy; \sigma, \lambda)$, and hence

$$f(x+iy) = b_0(y) + \sum_{0 \neq h \in \mathfrak{m}} b_h W(hy; \sigma, \lambda)\mathbf{e}_{\mathbf{a}}(hx) \tag{2.21}$$

with a function $b_0(y)$ and $b_h \in \mathbf{C}$. The nature of b_0 will be examined in the next section. We call b_0 *the constant term of* f, and understand, by a *cusp form*, an element f of $\mathcal{A}(\sigma, \lambda)$ such that the constant term of $f \| \xi$ is 0 for every $\xi \in \mathcal{G}_\sigma$. We denote by $\mathcal{S}(\sigma, \lambda)$ the set of all such forms and put $\mathcal{S}(\sigma, \lambda, \Delta) = \mathcal{A}(\sigma, \lambda, \Delta) \cap \mathcal{S}(\sigma, \lambda)$.

Proposition 2.1. *Let f and b_h be as in* (2.21). *Then*:

(1) *There exist constants $p > 0$ and $q \geqslant 0$ such that $|b_h| \leqslant p|h|^{qu+\sigma/2}$ for all* $h \in \mathfrak{m}$, $\neq 0$.

(2) *There exist positive constants A, B, and C such that*

$$y^{\sigma/2} \sum_{0 \neq h \in \mathfrak{m}} |b_h W(hy; \sigma, \lambda)| \leqslant A \cdot \exp(-By^{u/n}) \quad \text{if} \quad y^u \geqslant C.$$

(3) *$\langle f, g \rangle$ is meaningful if either f or g is a cusp form.*

(4) *If f is a cusp form, we can take $q = 0$ in* (1).

Proposition 2.2. *Put $\Delta^* = \{\xi^* \mid \xi \in \Delta\}$, $\Delta_* = \{\xi_* \mid \xi \in \Delta\}$ with ξ^* and ξ_* of* (2.5*a*, *b*). *Then*

$$\epsilon_v \mathcal{A}(\sigma, \lambda, \Delta) \subset \mathcal{A}(\sigma - 2v, \lambda - (\sigma_v - 2)v, \Delta_*),$$

$$\delta_v^\sigma \mathcal{A}(\sigma, \lambda, \Delta) \subset \mathcal{A}(\sigma + 2v, \lambda + \sigma_v v, \Delta^*),$$

where v is viewed as the element of $\mathbf{C}^{\mathbf{a}}$ of which the v-component is 1 *and all other components are* 0.

Proposition 2.3. *$\mathcal{A}(\sigma, \lambda, \Delta)$ is finite-dimensional over* $\mathbf{C}$.

These propositions will be proved in Section 11.

Define subsets $\mathcal{N}(\sigma, \lambda)$ and $\mathcal{N}(\sigma, \lambda, \Delta)$ of $\mathcal{A}(\sigma, \lambda)$ by

$$\mathcal{N}(\sigma, \lambda) = \{g \in \mathcal{A}(\sigma, \lambda) \mid \langle f, g \rangle = 0 \quad \text{for all} \quad f \in \mathcal{S}(\sigma, \lambda)\}, \tag{2.22a}$$

$$\mathcal{N}(\sigma, \lambda, \Delta) = \{g \in \mathcal{A}(\sigma, \lambda, \Delta) \mid \langle f, g \rangle = 0 \quad \text{for all} \quad f \in \mathcal{S}(\sigma, \lambda, \Delta)\}. \tag{2.22b}$$

Then we see easily that

$$\mathcal{A}(\sigma, \lambda, \Delta) = \mathcal{S}(\sigma, \lambda, \Delta) \oplus \mathcal{N}(\sigma, \lambda, \Delta), \tag{2.23a}$$

$$\mathcal{A}(\sigma, \lambda) = \mathcal{S}(\sigma, \lambda) \oplus \mathcal{N}(\sigma, \lambda), \tag{2.23b}$$

$$\mathcal{N}(\sigma, \lambda, \Delta) = \mathcal{N}(\sigma, \lambda) \cap \mathcal{A}(\sigma, \lambda, \Delta). \tag{2.24}$$

The inclusion $\mathfrak{N}(\sigma, \lambda, \Delta) \subset \mathfrak{N}(\sigma, \lambda)$ is not completely trivial. To see this, let $g \in \mathfrak{N}(\sigma, \lambda, \Delta)$; take any normal subgroup Δ' of Δ. Then $g = f + h$ with $f \in$ $\in \mathcal{S}(\sigma, \lambda, \Delta')$ and $h \in \mathfrak{N}(\sigma, \lambda, \Delta')$. For $\gamma \in \Delta$, we have $g = f\|\gamma + h\|\gamma$. Observing that $f\|\gamma \in \mathcal{S}(\sigma, \lambda, \Delta')$ and $h\|\gamma \in \mathfrak{N}(\sigma, \lambda, \Delta')$. We obtain $f\|\gamma = f$ and $h\|\gamma = h$ and hence $f \in \mathcal{S}(\sigma, \lambda, \Delta)$ and $h \in \mathfrak{N}(\sigma, \lambda, \Delta)$. Therefore $g = h \in \mathfrak{N}(\sigma, \lambda, \Delta')$, which shows that $g \in \mathfrak{N}(\sigma, \lambda)$.

Proposition 2.4. *For each* $v \in \mathbf{a}$, *we have*

$$(2.25) \qquad \langle \epsilon_v f, g \rangle = \langle f, \delta_v^\sigma g \rangle \quad \text{for} \quad f \in \mathcal{A}(\sigma + 2v, \lambda), \qquad g \in \mathcal{A}(\sigma, \lambda'),$$

if either f *or* g *is a cusp form*;

$$(2.26) \qquad \langle L_v f, g \rangle = \langle f, L_v g \rangle \quad \text{for} \quad f \in \mathcal{S}(\sigma, \lambda), \qquad g \in \mathcal{A}(\sigma, \lambda').$$

These formulas are actually true for C^∞-functions f and g satisfying only (2.7*a*, *c*), under a suitable condition on the convergence, as will be shown in Section 6; (2.26) follows from (2.25), since

$$\langle L_v f, g \rangle = 4\langle \delta_v^{\sigma - 2v} \epsilon_v f, g \rangle = 4\langle \epsilon_v f, \epsilon_v g \rangle = \langle f, L_v g \rangle.$$

(Formula (2.25) was proved also in [14, Lemma 2.3].) This shows also that $\langle f, L_v f \rangle = 4\langle \epsilon_v f, \epsilon_v f \rangle \geqslant 0$. Therefore $\mathcal{S}(\sigma, \lambda) \neq \{0\}$ only if $0 \leqslant \lambda_v \in \mathbf{R}$ for every $v \in \mathbf{a}$.

Proposition 2.5. *Every holomorphic function on* $H^{\mathbf{a}}$ *satisfying* (2.7*a*, *c*) *belongs to* $\mathcal{A}(\sigma, 0, \Delta)$. *Moreover, every element of* $\mathcal{S}(\sigma, 0)$ *is holomorphic on* $H^{\mathbf{a}}$.

PROOF. A function f on $H^{\mathbf{a}}$ is holomorphic if and only if $\epsilon_v f = 0$ for every $v \in \mathbf{a}$. Thus the first assertion is obvious. If $f \in \mathcal{S}(\sigma, 0)$, we have $4\langle \epsilon_v f, \epsilon_v f \rangle =$ $= \langle f, L_v f \rangle = 0$, so that $\epsilon_v f = 0$, which proves the second assertion.

3. The constant term of an eigenform

Let U be a subgroup of $\mathfrak{g}^\times$ of finite index. We call an element τ of $\mathbf{R}^{\mathbf{a}}$ *U-admissible* if

$$(3.1) \qquad |x|^{i\tau} = 1 \quad \text{for all} \quad x \in U \quad \text{and} \quad \sum_{v \in \mathbf{a}} \tau_v = 0,$$

and denote by T_U the set of all U-admissible τ. We call τ *admissible* if it is U-admissible for some U. We can easily prove

$$\{p \in \mathbf{C}^{\mathbf{a}} \mid |x|^p = 1 \quad \text{for all} \quad x \in U\} = iT_U \oplus \mathbf{C}u. \tag{3.2}$$

Now let $b(y)$ be the constant term of an element of $\mathcal{A}(\sigma, \lambda)$. Putting $h = 0$ in (2.14), we have

$$(y_v^2\partial^2/\partial y_v^2 + \sigma_v y_v \partial/\partial y_v + \lambda_v)b = 0. \tag{3.3}$$

A pair of independent solutions of this equation can be given as follows:

(3.4*a*) y_v^p and y_v^q with the roots p and q of $X^2 - (1 - \sigma_v)X + \lambda_v$ if $4\lambda_v \neq (1 - \sigma_v)^2$,

(3.4*b*) y_v^q and $y_v^q \log y_v$ with $q = (1 - \sigma_v)/2$ if $4\lambda_v = (1 - \sigma_v)^2$.

Therefore $b(y)$ is a linear combination of the products of these functions for all $v \in \mathbf{a}$. However, not every product can appear. In fact, in view of (2.13*b*), we can find a subgroup U of $\mathfrak{g}^\times$ of finite index whose elements are all totally positive and such that

$$b(a^2 y) = a^{-\sigma} b(y) \quad \text{for every} \quad a \in U. \tag{3.5}$$

This imposes a nontrivial condition on the combination of the solutions of (3.4*a*, *b*). To be precise, we have:

Proposition 3.1. *The constant term $b(y)$ of an element of $\mathcal{A}(\sigma, \lambda)$ has one of the following forms*:

(i) *If* $4\lambda_v = (1 - \sigma_v)^2$ *for all* $v \in \mathbf{a}$, *then* $b(y) = a_1 y^q + a_2 y^q \log y^u$ *with* $a_i \in \mathbf{C}$ *and* $q = (u - \sigma)/2$.

(ii) *If* $4\lambda_v \neq (1 - \sigma_v)^2$ *for some* $v \in \mathbf{a}$, *then* $b(y) = \sum_p a_p y^p$ *with* $a_p \in \mathbf{C}$ *and* $p \in \mathbf{C}^{\mathbf{a}}$. *Each p must satisfy the following two conditions*:

$$\lambda_v = p_v(1 - \sigma_v - p_v) \quad \textit{for all} \quad v \in \mathbf{a}; \tag{3.6a}$$

$$p = su - (\sigma - i\tau)/2 \ \textit{with} \ s \in \mathbf{C} \ \textit{and} \ \tau \in T_U. \tag{3.6b}$$

PROOF. Put $\mathbf{b} = \{v \in \mathbf{a} \mid 4\lambda_v = (1 - \sigma_v)^2\}$ and $(\log y)^{\mathbf{d}} = \prod_{v \in \mathbf{d}} \log y_v$ for $\mathbf{d} \subset \mathbf{b}$. Then $b(y) = \sum_{\mathbf{d} \subset \mathbf{b}} \sum_p A_{p,\mathbf{d}} y^p (\log y)^{\mathbf{d}}$ with constants $A_{p,\mathbf{d}}$ and p_v as in (3.6*a*). Take a maximal subset $\mathbf{d}$ of $\mathbf{b}$ such that $A_{p,\mathbf{d}} \neq 0$ for some p. Fix such a p. Then (3.5) implies that $a^{\sigma + 2p} = 1$ for $a \in U$; hence $\sigma + 2p = i\tau + 2su$ with $s \in \mathbf{C}$ and $\tau \in T_U$ by (3.2). Thus p must be as in (3.6*b*). Suppose $\mathbf{d} \neq \varnothing$ and let $\mathbf{d} = \mathbf{e} \cup \{w\}$ with an arbitrarily fixed w. Then (3.5) implies that $\sum A_{p,\mathbf{x}} \log a_v^2 = 0$ for every $a \in U$, where the sum is taken over all $(v, \mathbf{x})$ such that $\{v\} \cup \mathbf{e} = \mathbf{x} \subset \mathbf{b}$. Since $A_{p,\mathbf{d}} \neq 0$, this can happen only when $\mathbf{e} = \varnothing$ and $\mathbf{b} = \mathbf{a}$. Then $p = (u - \sigma)/2$, $b(y) = Ay^p + \sum_{v \in \mathbf{a}} A_v y^p \log y_v$, and $\sum_v A_v \log a_v^2 = 0$ for all $a \in U$, and hence we obtain (i). If $A_{p,\mathbf{d}} = 0$ for all $\mathbf{d} \neq \emptyset$, then we obtain (ii).

Thus, given σ and λ, $b(y)$ belongs to a two-dimensional space if $4\lambda_v = (1 - \sigma_v)^2$ for every $v \in \mathbf{a}$, and to a 2^n-dimensional space otherwise. The latter space can actually be reduced to a 2-dimensional space in most cases. In fact, take p, τ, and s as in (3.6a, b). Let $q = u - \sigma - p$. Then $q = (1 - s)u - (\sigma + i\tau)/2$, and hence q, $-\tau$, and $1 - s$ satisfy (3.6a, b). Since $4\lambda_v \neq (1 - \sigma_v)^2$ for some v, we have $p \neq q$, and therefore y^p and y^q form a two-dimensional vector space. Now our question is whether y^r, with r different from p and q, can occur. Suppose it can, and let $r = tu - (\sigma - i\varkappa)/2$ with $t \in \mathbf{C}$ and $\varkappa \in T_U$. Then, for each v, r_v must coincide with p_v or q_v. Decompose $\mathbf{a}$ into the disjoint union of three subsets $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ so that $r_v = p_v = q_v$ for $v \in \mathbf{b}$, $r_v = p_v \neq q_v$ for $v \in \mathbf{c}$, and $r_v = q_v \neq p_v$ for $v \in \mathbf{d}$. Then $\mathbf{c} \neq \emptyset$ and $\mathbf{d} \neq \emptyset$; $t + i\varkappa_v/2 = s + i\tau_v/2$ for $v \in \mathbf{b} \cup \mathbf{c}$ and $t + i\varkappa_v/2 = 1 - s - i\tau_v/2$ for $v \in \mathbf{b} \cup \mathbf{d}$. Hence $\mathrm{Re}(s) = \mathrm{Re}(t) = \mathrm{Re}(1 - s)$, so that $\mathrm{Re}(s) = 1/2$. Therefore $\mathrm{Re}(p) = (u - \sigma)/2$. Observing that $(1 - \sigma_v)^2 \leqslant 4\lambda_v \in \mathbf{R}$ if and only if $\mathrm{Re}(p_v) = (1 - \sigma_v)/2$, we obtain

Proposition 3.2. *The constant term $b(y)$ of an element of $\mathfrak{A}(\sigma, \lambda)$, for fixed σ and λ, belongs to a two-dimensional vector space unless the following condition is satisfied*:

(3.7) $F \neq \mathbf{Q}$, $(1 - \sigma_v)^2 \leqslant 4\lambda_v \in \mathbf{R}$ *for all* $v \in \mathbf{a}$, *and* $(1 - \sigma_v)^2 < 4\lambda_v$ *for at least one* v.

If this is satisfied and if p is as in (3.6a, b), *then* $\mathrm{Re}(p) = (u - \sigma)/2$.

We call λ *critical* if $4\lambda_v = (1 - \sigma_v)^2$ for all $v \in \mathbf{a}$; otherwise we call λ *noncritical*. We call λ *simple* if either λ is critical or λ is noncritical and there are only two p's satisfying (3.6a, b). In the latter case, if p is one, the other is $u - \sigma - p$. We call λ *multiple* if it is not simple. Any p as in (3.6a, b) is called an *exponent attached to* λ. In Remark 5.5 below, we shall give an example of multiple λ.

Hereafter we fix a weight σ and write simply $\mathcal{G}$ and $\mathcal{P}$ for $\mathcal{G}_\sigma$ and $\mathcal{P}_\sigma$, where $\mathcal{P}_\sigma$ is defined by (2.8). Given an admissible τ, we put, throughout the rest of the paper,

$$\rho = (\sigma - i\tau)/2 \quad (\in \mathbf{C}^{\mathbf{a}}). \tag{3.8}$$

Then, for λ and p of (3.6a, b), we have

$$\lambda_v = (s - \rho_v)(1 - s - \bar{\rho}_v), \tag{3.9a}$$

$$p = su - \rho, \qquad u - \sigma - p = (1 - s)u - \bar{\rho}. \tag{3.9b}$$

Let Δ be a congruence subgroup of $\mathcal{G}$ and let $\Gamma = \mathrm{pr}(\Delta)$. Then pr gives a bijective map of $\mathcal{P}\backslash\mathcal{G}/\Delta$ onto $P\backslash G/\Gamma$, which is a finite set corresponding bijectively to $\Gamma\backslash(F \cup \{\infty\})$ via the map $\alpha \mapsto \alpha^{-1}(\infty)$. Therefore we call a coset $\mathcal{P}\xi\Delta$ with $\xi \in \mathcal{G}$ a *cusp-class* of Δ. Given a cusp-class $\mathcal{P}\xi\Delta$, we call it *ρ-regular* if

$$(3.10a) \qquad y^{-\rho}\|\gamma = y^{-\rho} \quad \text{for every} \quad \gamma \in \mathcal{P} \cap \xi\Delta\xi^{-1}.$$

or equivalently,

$$(3.10b) \qquad l_\gamma^{-1}|d_\gamma|^{\sigma - i\tau} = 1 \quad \text{for every} \quad \gamma \in \mathcal{P} \cap \xi\Delta\xi^{-1}.$$

It can easily be seen that (3.10*a*) is in fact a condition on $\mathcal{P}\xi\Delta$, independent of the choice of a representative ξ. The meaning of this condition is explained by:

Proposition 3.3. *Let ρ, λ, and p be as above; let $f \in \mathcal{Q}(\sigma, \lambda, \Delta)$ and $\xi \in \mathcal{G}$. Then y^p or $y^p \log y^u$ can appear nontrivially in the constant term of $f\|\xi^{-1}$ only if $\mathcal{P}\xi\Delta$ is ρ-regular.*

This follows immediately from our definition.

If all cusp-classes of Δ are ρ-regular, then the same is true for every congruence subgroup of Δ. Such a Δ indeed exists because of

Proposition 3.4. *Given $\rho = (\sigma - i\tau)/2$ as above, there exists an integral ideal $\mathfrak{a}$ such that all cusp-classes of $\Delta_\sigma^k[\mathfrak{a}]$ are ρ-regular.*

PROOF. Take an integral ideal $\mathfrak{b}$ so that $x \gg 0$ and $x^{i\tau} = 1$ if $x \in \mathfrak{g}^\times$ and $x - 1 \in \mathfrak{b}$. In Case II, choose $\mathfrak{b}$ so that $\mathfrak{b} \subset 4\mathfrak{g}$. Write simply $\Delta[\mathfrak{a}]$ for $\Delta_\sigma^k[\mathfrak{a}]$. In either case, we have $\mathcal{G} = \mathcal{P}Z\Delta[\mathfrak{b}]$ with a finite subset Z of $\mathcal{G}$. We can find an integral ideal $\mathfrak{a} \subset \mathfrak{b}$ such that $\Delta[\mathfrak{a}] \subset \bigcap_{\zeta \in Z} \zeta^{-1}\Delta[\mathfrak{b}]\zeta$. Obviously $\mathcal{G} = \mathcal{P}X\Delta[\mathfrak{a}]$ with a suitable subset X of $Z\Delta[\mathfrak{b}]$. If $\xi \in \zeta\Delta[\mathfrak{b}]$ with $\zeta \in Z$, then $\xi\Delta[\mathfrak{a}]\xi^{-1} = \zeta\Delta[\mathfrak{a}]\zeta^{-1} \subset \Delta[\mathfrak{b}]$. Let $\gamma \in \mathcal{P} \cap \Delta[\mathfrak{b}]$. Then, in Case II, we have $l_\gamma = |d_\gamma|^{ku/2} d_\gamma^{\sigma - (k/2)u}$ by (1.11*c*) and (1.12). Hence our choice of $\mathfrak{b}$ implies (3.10*b*) for $\Delta = \Delta[\mathfrak{a}]$. The same can be verified in Case I in a similar way.

4. Eisenstein series

Given a congruence subgroup Δ of $\mathcal{G}$, we define its Eisenstein series by

$$(4.1) \qquad E(z, s) = E(z, s; \rho, \Delta) = \sum_{\alpha \in (\mathcal{P} \cap \Delta)\backslash\Delta} y^{su - \rho}\|\alpha.$$

Here $s \in \mathbf{C}$ and $\rho = (\sigma - i\tau)/2$ with an admissible τ. To make the sum meaningful, we have to assume that $y^{-\rho} \| \gamma = y^{-\rho}$ for every $\gamma \in \mathcal{P} \cap \Delta$, that is, $\mathcal{P}\Delta$ is ρ-regular. The series is covergent for $\mathrm{Re}(s) > 1$, and can be continued as a meromorphic function in s to the whole s-plane. (See Theorem 4.2 below for a precise statement.) Assuming this result, we have obviously $E(z, s) \| \gamma = E(z, s)$ for every $\gamma \in \Delta$, and moreover, by (2.6) and (2.5c),

$$L_v^\sigma E(z, s) = \lambda_v E(z, s) \quad \text{with} \quad \lambda_v = (s - \rho_v)(1 - s - \bar{\rho}_v) \tag{4.2}$$

for every $v \in \mathbf{a}$. Therefore, if $E(z, s)$ is finite at s, it satisfies (2.7a, b) as a function in z, and in fact belongs to $\mathcal{A}(\sigma, \lambda, \Delta)$ as (2.7c) can be shown in our later discussion.

From our definition, we can easily derive a relation

$$[\mathcal{P} \cap \Delta : \mathcal{P} \cap \Delta']E(z, s; \rho, \Delta) = \sum_{\gamma \in \Delta' \backslash \Delta} E(z, s; \rho, \Delta') \| \gamma \tag{4.3}$$

for every congruence subgroup $\Delta' \subset \Delta$. For each ρ-regular cusp-class $\mathcal{P}\xi\Delta$ (see (3.10a, b)), we put

$$E(z, s; \rho, \xi, \Delta) = E(z, s; \rho, \xi\Delta\xi^{-1}) \| \xi. \tag{4.4}$$

Then we see easily that

$$E(z, s; \rho, \xi, \Delta) \| \gamma = E(z, s; \rho, \xi, \Delta) \quad \text{for every} \quad \gamma \in \Delta, \tag{4.5}$$

$$E(z, s; \rho, \alpha\xi\gamma, \Delta) = l_\alpha^{-1} |d_\alpha|^{2\rho - 2su} E(z, s; \rho, \xi, \Delta) \quad \text{if} \quad \alpha \in \mathcal{P} \quad \text{and} \quad \gamma \in \Delta. \tag{4.6}$$

Thus, ignoring elementary factors, we associate with Δ exactly as many Eisenstein series as its ρ-regular cusp-classes.

We now introduce another type of Eisenstein series, which is attached to an integral ideal $\mathfrak{c}$ in F and a Hecke character ψ: $F_\mathbf{A}^\times / F^\times \to \mathbf{C}^\times$. We assume

(4.7a) $\quad 4\mathfrak{g} \supset \mathfrak{c}$ in Case II;

(4.7b) $\quad |\psi| = 1;$

(4.7c) the finite part of the conductor of ψ divides $\mathfrak{c}$;

(4.7d) $\quad \psi(x) = |x|^{i\tau}(x/|x|)^{\sigma'} \quad \text{for} \quad x \in F_\mathbf{a}^\times,$

where

$$\sigma' = \begin{cases} \sigma & \text{(Case I)}, \\ \sigma - (k/2)u & \text{(Case II)}, \end{cases} \tag{4.8}$$

$F_\mathbf{A}^\times$ denotes the idele group of F, and $F_\mathbf{a}^\times$ its archimedean factor. We fix $\mathfrak{c}$, assume $\mathfrak{c} \subset 4\mathfrak{g}$ in Case II, and put

$$D = \begin{cases} D[\mathfrak{g}, \mathfrak{c}] & \text{(Case I)}, \\ D[2\mathfrak{b}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{b}] & \text{(Case II)}, \end{cases} \tag{4.9a}$$

$$\Gamma_0(\mathfrak{c}) = G \cap D G_{\mathbf{a}}. \tag{4.9b}$$

Writing simply Γ for $\Gamma_0(\mathfrak{c})$, take a complete set of representatives B of $P\backslash(G \cap P_{\mathbf{A}} D)/\Gamma$. Take, for each $\beta \in B$, a complete set of representatives R_β of $(P \cap \beta\Gamma\beta^{-1})\backslash\beta\Gamma$. Then we put

$$E_k(z, s; \rho, \psi, \mathfrak{c}) = \sum_{\beta \in B} N(\mathfrak{a}_\beta)^{2s} \sum_{\alpha \in R_\beta} \psi(d_\alpha \mathfrak{a}_\beta^{-1}) \psi_{\mathbf{a}}(d_\alpha) y^{su-\rho} \| \Lambda_\sigma^k(\alpha), \tag{4.10}$$

where $\mathfrak{a}_\beta = c_\beta \mathfrak{g} + d_\beta \mathfrak{g}$ in Case I and $\mathfrak{a}_\beta = 2c_\beta \mathfrak{b}^{-1} + d_\beta \mathfrak{g}$ in Case II, and $\psi_{\mathbf{a}}$ is the archimedean part of ψ; we use the same letter ψ for the ideal character attached to ψ; we understand that $\psi(d_\alpha \mathfrak{a}_\beta^{-1})\psi_{\mathbf{a}}(d_\alpha) = \psi(\mathfrak{a}_\beta^{-1})$ if $\mathfrak{c} = \mathfrak{g}$. The right-hand side of (4.10) is convergent for $\mathrm{Re}(s) > 1$ and satisfies (4.2). In Case I, we have $k = 0$, and so we write simply E for E_k. As for the relation between the series of type (4.10) and that of (4.1), see (4.24) and Proposition 5.3 below.

Define the L-function $L(s, \psi)$ of ψ as usual and put

$$L_{\mathfrak{c}}(s, \psi) = L(s, \psi) \prod_{\mathfrak{p} | \mathfrak{c}} [1 - \psi(\mathfrak{p}) N(\mathfrak{p})^{-s}], \tag{4.11}$$

where $\mathfrak{p}$ denotes a prime ideal in F.

Theorem 4.1. *The series $E_k(z, s; \rho, \psi, \mathfrak{c})$ can be continued as a meromorphic function to the whole s-plane. More precisely, put*

$$D(z, s) = \begin{cases} \prod_{v \in \mathbf{a}} \Gamma(s + (|\sigma_v| + i\tau_v)/2) L_{\mathfrak{c}}(2s, \psi) E(z, s; \rho, \psi, \mathfrak{c}) & \text{(Case I)}, \\ \prod_{v \in \mathbf{a}} \Gamma_v(s + i\tau_v/2) L_{\mathfrak{c}}(4s - 1, \psi^2) E_k(z, s; \rho, \psi, \mathfrak{c}) & \text{(Case II)}, \end{cases}$$

where Γ_v in Case II is defined by

$$\Gamma_v(s) = \Gamma(s + (\theta_v/2) - (1/4)) \cdot \begin{cases} \Gamma(s + (\sigma_v/2)) & \text{if} \quad 2\sigma_v \geqslant -1, \\ \Gamma(s - (\sigma_v/2)) & \text{if} \quad 2\sigma_v < -1, \end{cases}$$

with the smallest nonnegative integer θ_v that is congruent modulo 2 to $\sigma_v - 1/2$ or $\sigma_v + 1/2$ according as $2\sigma_v \geqslant -1$ or $2\sigma_v < -1$. Then there is a real analytic function on $H^{\mathbf{a}} \times \mathbf{C}$ that is holomorphic in s and that coincides with $s(s-1)D(z, s)$ in Case I and with $(s - 3/4)D(z, s)$ in Case II for $\mathrm{Re}(s) > 1$. (*Thus we are able to speak of possible simple poles at $s = 0$, 1,* or $3/4$.) *In Case I, the pole at $s = 0$ occurs if and only if $\mathfrak{c} = \mathfrak{g}$, $\psi = 1$, and $\sigma = \tau = 0$; the pole at $s = 1$ occurs if and only if $\psi = 1$ and $\sigma = \tau = 0$. In Case II, the pole at $s = 3/4$ occurs if and only if $\psi^2 = 1$ and, for every $v \in \mathbf{a}$, $\sigma_v - 1/2$ is either an even nonnegative integer or an odd negative integer.*

The result in Case II is merely a paraphrase of [13, Corollary 6.2]. In fact, the symbols k, ρ, μ, and τ there correspond to k, σ', τ, and θ here. If we denote by $E^*(z,s)$ the function $E(z,s;k/2,\rho,\psi,\mathfrak{c})$ there, then, comparison of (4.10) with [13, (4.7c)] shows that

$$E_k(z,s;\rho,\psi,\mathfrak{c}) = y^{(k/4)u-\rho}E^*(z,s-k/4), \tag{4.12}$$

and hence our assertion follows immediately from [13, Corollary 6.2].

The result in Case I can be obtained by modifying the formulation of [12]. To be more specific, take $m=1$ in [12]; using the same notation, we define a function f on $G_{\mathbf{A}}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin P_{\mathbf{A}}D, \\ \psi_{\mathfrak{f}}(d_p)^{-1}\psi_{\mathfrak{c}}(d_w)^{-1}J(x,\mathbf{i})^{-1} & \text{if } x = pw \text{ with } p \in P_{\mathbf{A}} \text{ and } w \in D, \end{cases} \tag{4.13}$$

where

$$\psi_{\mathfrak{f}}(a) = \prod_{v\in\mathbf{f}} \psi(a_v), \qquad \psi_{\mathfrak{c}}(a) = \prod_{v|\mathfrak{c}} \psi(a_v),$$

and

$$J(x,\mathbf{i}) = \prod_{v\in\mathbf{a}} j(x_v,i)^{\sigma_v}|j(x_v,i)|^{i\tau_v-\sigma_v}.$$

We then define a series $E_{\mathbf{A}}$ on $G_{\mathbf{A}}$ by

$$E_{\mathbf{A}}(x,s) = \sum_{\alpha\in P\backslash G} f(\alpha x)\epsilon(\alpha x)^{-2s} \qquad (x\in G_{\mathbf{A}},\ s\in\mathbf{C}) \tag{4.14}$$

with ϵ of [12, (2.11)]. We can easily verify that

$$E(z,s;\rho,\psi,\mathfrak{c}) = y^{-\rho}E_{\mathbf{A}}(x,s)J(x,\mathbf{i}) \tag{4.15}$$

if $x \in G_{\mathbf{a}}$ and $x_v(i) = z_v$ for $v \in \mathbf{a}$. Put

$$E'(z,s) = E(z,s;\rho,\psi,\mathfrak{c})\,\|\,\Lambda_\sigma^0(\eta), \qquad \eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.16}$$

This has a Fourier expansion of the form

$$E'(z,s) = \sum_{h\in\mathfrak{b}} a(h,y,s)\mathbf{e}_{\mathbf{a}}(hx) \tag{4.17}$$

with $\mathfrak{b} = (\mathfrak{c}\mathfrak{d})^{-1}$. Applying the methods of [12] to E' with obvious modifications, we find that if $\mathfrak{c} \neq \mathfrak{g}$,

$$a(h,y,s) = N(\mathfrak{b})^{-1/2}N(\mathfrak{c})^{-1}a_{\mathfrak{f}}(h,s)y^{su-\rho}\prod_{v\in\mathbf{a}} \xi(y_v,h_v;s+\bar{\rho}_v,s-\rho_v), \tag{4.18a}$$

$$a_{\mathfrak{f}}(h,s) = \prod_{v\in\mathbf{f},\,v\nmid\mathfrak{c}} \alpha(v,h,\psi(\pi_v)q_v^{-2s}), \tag{4.18b}$$

where ξ, α, π_v, and q_v are defined by [12, (3.18), (3.22), and (3.23)] (in the one-dimensional case). In particular

$$\xi(g, h; \alpha, \beta) = \int_{\mathbf{R}} \mathbf{e}(-hx)(x+ig)^{-\alpha}(x-ig)^{-\beta}\,dx. \tag{4.19}$$

As for $a_{\mathfrak{f}}$, we have

$$L_{\mathfrak{c}}(2s, \psi)a_{\mathfrak{f}}(h, s) = \begin{cases} L_{\mathfrak{c}}(2s-1, \psi) & \text{if} \quad h = 0, \\ \Sigma\psi(\mathfrak{a})N(\mathfrak{a})^{1-2s} & \text{if} \quad h \neq 0, \end{cases} \tag{4.20}$$

where $\mathfrak{a}$ runs over all integral ideals in F prime to $\mathfrak{c}$ and dividing $h\mathfrak{c}\mathfrak{d}$. This result for $h = 0$ is already given in [12, Theorem 7.7, I]. If $h \neq 0$, the Euler v-factor for $v \nmid h\mathfrak{c}\mathfrak{d}$ is determined by [12, Proposition 4.6]. The «bad factors» can be determined by the methods of [13, §6]. In fact, the present case is easier than [13, §6]. Thus after an easy calculation, we obtain a final result as given in (4.20). As for ξ of (4.19), we have

$$\prod_{v \in \mathbf{a}} \xi(y_v, 0; \alpha_v, \beta_v) = \tag{4.21a}$$

$$= i^{-\{\sigma\}}(2\pi)^n(2y)^{u-\alpha-\beta} \prod_{v \in \mathbf{a}} \Gamma(\alpha_v + \beta_v - 1)\Gamma(\alpha_v)^{-1}\Gamma(\beta_v)^{-1},$$

$$y^{\beta} \prod_{v \in \mathbf{a}} \xi(y_v, h_v; \alpha_v, \beta_v) = \tag{4.21b}$$

$$= (-2i)^{\{\sigma\}}\pi^{\{\alpha\}}|h|^{\alpha-u}W(hy; \sigma, \lambda) \prod_{v \in \mathbf{a}} \Gamma(\gamma_v)^{-1}.$$

where σ and λ are determined by (2.18), $\{\alpha\} = \sum_{v \in \mathbf{a}} \alpha_v$, and $\gamma_v = \alpha_v$ or β_v according as $h_v > 0$ or $h_v < 0$ (see [11, (1.31), (4.34K)]). Therefore we obtain our assertion on D in Case I by examining the local behavior of each Fourier coefficient of E', provided that $\mathfrak{c} \neq \mathfrak{g}$. To treat the case $\mathfrak{c} = \mathfrak{g}$, we first observe that if $\mathfrak{c} \supset \mathfrak{e}$, we have (in both cases $\mathfrak{c} \neq \mathfrak{g}$ and $\mathfrak{c} = \mathfrak{g}$)

$$E(z, s; \rho, \psi, \mathfrak{c}) = \sum_{\gamma \in \Gamma_0(\mathfrak{e}) \backslash \Gamma_0(\mathfrak{c})} \psi_{\mathfrak{c}}(d_\gamma)^{-1} j_\gamma^{-\sigma} E(\gamma(z), s; \rho, \psi, \mathfrak{e}), \tag{4.22}$$

which can be proved in the same manner as in [12, Proposition 2.4, (ii)]. Suppose $\mathfrak{c} = \mathfrak{g}$. Take an arbitrary $\mathfrak{e} \neq \mathfrak{g}$. Then our result on $E(z, s; \rho, \psi, \mathfrak{e})$ shows that the poles can occur only at $s = 0$ and $s = 1$; the pole at $s = 0$ is produced by the difference of $L(s, \psi)$ from $L_{\mathfrak{e}}(s, \psi)$. To see that these poles do occur when $\psi = 1$ and $\rho = 0$, we apply the method of [12] to $E(z, s)$ (instead of $E'(z, s)$) to find that

$$E(z, s; \rho, \psi, \mathfrak{c}) = y^{su-\rho} + \sum_{h \in \mathfrak{b}^{-1}} b(h, y, s)\mathbf{e}_{\mathbf{a}}(hx),$$

with Fourier coefficients b which are similar to but somewhat more complicated than the above $a(h,y,s)$. It is easy, however, to see that $b(h,y,s) = a(h,y,s)$ if $\mathfrak{c} = \mathfrak{g}$, and hence the poles at $s = 0$ and $s = 1$ occur if $\psi = 1$ and $\rho = 0$. This completes the proof of Theorem 4.1.

Now observe that $E(z,s;\rho,\Delta_\sigma^k[\mathfrak{c}])$ is meaningful if and only if

$$(4.23)\qquad |x|^{i\tau}(x/|x|)^{\sigma'} = 1 \quad \text{for every} \quad x \in \mathfrak{g}^\times \quad \text{such that} \quad x \equiv 1 \pmod{\mathfrak{c}}.$$

This holds if and only if a Hecke character ψ satisfying (4.7*b*, *c*, *d*) exists. Assuming (4.23), let Ψ be the set of all such characters ψ with fixed $\mathfrak{c}$ and ρ, and $|\Psi|$ the number of elements in Ψ. Then we have

$$(4.24)\qquad |\Psi|E(z,s;\rho,\Delta_\sigma^k[\mathfrak{c}]) = \sum_{\psi\in\Psi} E_k(z,s;\rho,\psi,\mathfrak{c})$$

for the same reason as in [12, Proposition 2.4].

Theorem 4.2. *$E(z,s;\rho,\Delta)$ can be continued as a meromorphic function in s to the whole plane in the sense that there exist a nonzero holomorphic function $A(s)$ and a real analytic function $B(z,s)$ on $H^{\mathbf{a}} \times \mathbf{C}$, holomorphic in s such that $A(s)E(z,s;\rho,\Delta) = B(z,s)$ for $Re(s) > 1$. Moreover, $E(z,s;\rho,\Delta)$ is holomorphic in s except at the following points*:

(1) $0 \leqslant \mathrm{Re}(s) < 1/2$ *in Case I and* $1/4 \leqslant \mathrm{Re}(s) < 1/2$ *in Case II*;
(2) *a possible simple pole at $s = 1$ in Case I, which occurs only if $\rho = 0$*;
(3) *a possible simple pole at $s = 3/4$ in Case II, which occurs only if $\tau = 0$ and σ is given as at the end of Theorem* 4.1;
(4) *possible poles at the roots of a polynomial $T_\rho(s)$ given by*

$$T_\rho(s) = \prod_{v\in\mathbf{a}} \Gamma(s + (|\sigma_v| + i\tau_v)/2)/\Gamma(s + (\delta_v + i\tau_v)/2) \qquad (\textit{Case I}),$$

$$T_\rho(s) = \prod_{v\in\mathbf{a}} \Gamma_v(s + i\tau_v/2)/\big[\Gamma\big(s + (i\tau_v/2) - \tfrac{1}{4}\big)\Gamma\big(s + (i\tau_v/2) + \tfrac{1}{4}\big)\big] \qquad (\textit{Case II}),$$

where $\delta_v = 0$ or 1 according as σ_v is even or odd, and Γ_v is as in Theorem 4.1.

PROOF. In view of (4.3), it is sufficient to prove our theorem when $\Delta = \Delta_\sigma^k[\mathfrak{c}]$. Let D_ψ denote the function D defined in Theorem 4.1, and put

$$R_\psi(s) = \begin{cases} \prod_{v\in\mathbf{a}} \Gamma(s + (\delta_v + i\tau_v)/2)L_{\mathfrak{c}}(2s,\psi) & \text{(Case I)},\\ \prod_{v\in\mathbf{a}} \Gamma\big(s + (i\tau_v/2) - \tfrac{1}{4}\big)\Gamma\big(s + (i\tau_v/2) + \tfrac{1}{4}\big)L_{\mathfrak{c}}(4s - 1,\psi^2) & \text{(Case II)}.\end{cases}$$

Then $R_\psi(s) \neq 0$ except at the points of (1). By (4.24), we have

$$(4.25)\qquad |\Psi|E(z,s;\rho,\Delta_\sigma^k[\mathfrak{c}]) = \sum_{\psi\in\Psi} D_\psi(z,s)/[T_\rho(s)R_\psi(s)].$$

Observing that $T_\rho(s)$ is indeed a polynomial in s, we obtain our assertions from Theorem 4.1.

The polynomial $T_\rho(s)$ has no zero when $\mathrm{Re}(s) \geqslant 1/2$. Therefore $E(z, s; \rho, \Delta)$ *is holomorphic in s if* $\mathrm{Re}(s) \geqslant 1/2$ *except for a possible simple pole described in* (2) *or* (3) *of the above theorem.* The pole at $s = 1$ does occur if $\rho = 0$. In fact we have

Proposition 4.3. *For a congruence subgroup* Γ *of G, let* $r(\Gamma)$ *be the residue of* $E(z, s; 0, \Lambda_0^0(\Gamma))$ *at* $s = 1$. *Then* $r(\Gamma)$ *is a positive number with the following properties*:

(i) $r(\Gamma)/r(\Gamma') = [\Gamma:\Gamma']/[P\cap\Gamma:P\cap\Gamma']$ *if* $\Gamma' \subset \Gamma$;

(ii) $r(SL_2(\mathfrak{g})) = 2^{n-2}\pi^n D_F^{-1}\zeta_F(2)^{-1}R_F$, *where* D_F *is the discriminant of F,* ζ_F *is the zeta function of F, and* R_F *is the regulator of F*;

(iii) $r(\Gamma)\mu(\Gamma\backslash H^{\mathbf{a}}) = \mu_K(P\cap\Gamma)\backslash K)$, *where* $K = \{z \in H^{\mathbf{a}} \mid \mathrm{Im}(z)^u = 1\}$, *furnished with a certain invariant measure* μ_K (*see the proof below*).

PROOF. Assertion (ii) follows immediately from (4.24) and the explicit Fourier expansion given in the proof of Theorem 4.1. This together with (4.3) proves that $r(\Gamma)$ is a positive number satisfying (i). As for (iii), we give here only a sketch of the proof. Put $A = \{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}$ and $B = \{y \in A \mid y^u = 1\}$. Then every $y \in A$ can be written uniquely $y = t^{1/n}y'$ with $0 < t \in \mathbf{R}$ and $y' \in B$. Let $d^\times y = y^{-u}\,dy$ with the Euclidean measure dy on $\mathbf{R}^{\mathbf{a}}$. Then $d^\times y = t^{-1}\,dt\,dy'$ with a Haar measure dy' on B. Since $K = \mathbf{R}^{\mathbf{a}} \times B$, we can determine a measure μ_K on K by $d\mu_K(x + iy) = dx\,dy'$. Now take $\Gamma = SL_2(\mathfrak{g})$ and put $U = \{a^2 \mid a \in \mathfrak{g}^\times\}$. By a well known principle, we have

$$\int_{A/U} \varphi(y^u)d^\times y = 2^{n-1}R_F\int_0^\infty \varphi(t)t^{-1}\,dt \tag{4.26}$$

for a continuous function φ on A. In particular, this implies

$$\mu_K((P\cap\Gamma)\backslash K) = 2^{n-1}D_F^{1/2}R_F. \tag{4.27}$$

Take $\varphi(t) = e^{-t}t^s$. Then

$$2^{n-1}D_F^{1/2}R_F\Gamma(s) = \int_{(P\cap\Gamma)\backslash H^{\mathbf{a}}} \exp(-y^u)y^{(s+1)u}\,d\mu(z) = \int_{\Gamma\backslash H^{\mathbf{a}}} M(z, s)\,d\mu(z),$$

where

$$M(z, s) = \sum_{\alpha \in (P\cap\Gamma)\backslash\Gamma} \exp(-\mathrm{Im}(\alpha(z))^u)\,\mathrm{Im}(\alpha(z))^{(s+1)u}.$$

Since $1 - t \leqslant e^{-t} \leqslant 1$, we have, for $1 < s \in \mathbf{R}$,

$$E(z, s + 1) - E(z, s + 2) \leqslant M(z, s) \leqslant E(z, s + 1),$$

where $E(z, s) = E(z, s; 0, \Lambda_0^0(\Gamma))$. Then we see that $\lim_{s\to 0} sM(z, s) = r(\Gamma)$, and hence $r(\Gamma)\mu(\Gamma\backslash H^{\mathbf{a}}) = 2^{n-1}D_F^{1/2}R_F$, which proves (iii) when $\Gamma = SL_2(\mathfrak{g})$. The general case can be proved in a similar way; alternatively, it follows from the special case by virtue of (i).

Combining (ii), (iii), and (4.27), we find that

$$\mu(SL_2(\mathfrak{g})\backslash H^{\mathbf{a}}) = 2\pi^{-n}D_F^{3/2}\zeta_F(2), \tag{4.28}$$

which is classical.

Proposition 4.4. *Let Q be a finite set of functions of the form $E(z, s)\|\alpha$ with $\alpha \in G$ and E of type* (4.1) *or* (4.10), *and let $g(z, s) = \sum_{q\in Q} f_q(s)q(z, s)$ with meromorphic functions f_q on $\mathbf{C}$. Then, for every $s_0 \in \mathbf{C}$, there exists an integer m and a neighborhood V of s_0 such that $(s - s_0)^m g(z, s)$ is a real analytic function on $H^{\mathbf{a}} \times V$ that is holomorphic in s. If in particular, g is finite at $s = s_0$, then $g(z, s_0)$ is an element of $\mathfrak{A}(\sigma, \lambda)$ with $\lambda_v = (s_0 - \rho_v)(1 - s_0 - \bar{\rho}_v)$.*

This will be proved in Section 11.

5. The constant terms of Eisenstein series

Lemma 5.1. *Let $\mathfrak{x}$ be a lattice in F and $\mathfrak{y}$ its dual lattice defined by*

$$\mathfrak{y} = \{\, b \in F \mid \mathrm{Tr}_{F/\mathbf{Q}}(b\mathfrak{x}) \subset \mathbf{Z} \,\}.$$

Further let

$$S(z, \mathfrak{x}; \alpha, \beta) = \sum_{a\in\mathfrak{x}} (z + a)^{-\alpha}(\bar{z} + a)^{-\beta} \qquad (z \in H^{\mathbf{a}}; \alpha, \beta \in \mathbf{C}^{\mathbf{a}}).$$

Then this is convergent and real analytic (*at least*) *on*

$$H^{\mathbf{a}} \times \{\, (\alpha, \beta) \in \mathbf{C}^{\mathbf{a}} \times \mathbf{C}^{\mathbf{a}} \mid \mathrm{Re}(\alpha_v) > \tfrac{1}{2}, \mathrm{Re}(\beta_v) > \tfrac{1}{2} \quad \text{for every} \quad v \in \mathbf{a} \,\},$$

and has a Fourier expansion

$$\mu(\mathbf{R}^{\mathbf{a}}/\mathfrak{x})S(x + iy, \mathfrak{x}; \alpha, \beta) = \sum_{h\in\mathfrak{y}} \mathbf{e}_{\mathbf{a}}(hx)\xi(y, h; \alpha, \beta),$$

$$\xi(y, h; \alpha, \beta) = \prod_{v\in\mathbf{a}} \xi(y_v, h_v; \alpha_v, \beta_v)$$

with ξ of (4.19).

This can be proved in the same fashion as in [11, (1.32), Lemma 1.4] (see the last sentence of [11, §1]).

Proposition 5.2. *Let Y be a complete set of representatives of ρ-regular cusp-classes of* Δ in the sense that every ρ-regular cusp-class is gi*ven as $\mathcal{P}\xi\Delta$ with exactly one $\xi \in Y$. Let E_ξ, for $\xi \in Y$, denote $E(z, s; \rho, \xi, \Delta)$ with fixed ρ and Δ. Then, for $(\xi, \eta) \in Y \times Y$, we have*

$$E_\xi \| \eta^{-1} = \delta_{\xi\eta} y^{su-\rho} + f_{\xi\eta}(s) y^{u-su-\bar{\rho}} + \sum_{0 \neq h \in \mathfrak{y}} g_{\xi\eta}(h, s, y)\mathbf{e}_{\mathbf{a}}(hx),$$

where $f_{\xi\eta}$ and $g_{\xi\eta}$ are meromorphic in s, $\delta_{\xi\eta}$ is Kronecker's delta, and $\mathfrak{y}$ is a lattice in F.

PROOF. The point of our assertion is merely in the shape of the constant term. Fix one ξ and put $\Delta_\xi = \xi\Delta\xi^{-1}$. Put $r(a) = \Lambda_\sigma^k\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for $a \in F$. Then $r(F) \cap \Delta_\xi = r(\mathfrak{x})$ with a lattice $\mathfrak{x}$ in F. Take a subset Φ of Δ so that $1 \notin \Phi$ and $\{1\} \cup \Phi$ is a complete set of representatives of $(\mathcal{P} \cap \Delta_\xi)\backslash\Delta_\xi/r(\mathfrak{x})$. Then 1 and the elements $\varphi r(a)$ with $\varphi \in \Phi$ and $a \in \mathfrak{x}$ represent $(\mathcal{P} \cap \Delta_\xi)\backslash\Delta_\xi$ without overlap. Therefore

$$E_\xi \| \xi^{-1} = y^{su-\rho} + \sum_{\varphi \in \Phi} \sum_{a \in \mathfrak{x}} y^{su-\rho} \| \varphi r(a).$$

Fix one $\varphi \in \Phi$ and put $c = c_\varphi$, $d = d_\varphi$. Then $c \neq 0$, and

$$\sum_{a \in \mathfrak{x}} y^{su-\rho} \| \varphi r(a) = t y^{su-\rho} c^{-\alpha-\beta} \sum_{a \in \mathfrak{x}} (z + c^{-1}d + a)^{-\alpha}(\bar{z} + c^{-1}d + a)^{-\beta}$$

with $\alpha = su + \bar{\rho}$, $\beta = su - \rho$, and a constant t such that $|t| = 1$. By Lemma 5.1, this has an expansion of the form

$$t y^{su-\rho} c^{-\alpha-\beta} \mu(\mathbf{R}^{\mathbf{a}}/\mathfrak{x})^{-1} \sum_{h \in \mathfrak{y}} \mathbf{e}_{\mathbf{a}}(h(x + c^{-1}d))\xi(y, h; \alpha, \beta).$$

By (4.21*a*), the term $h = 0$ produces a function of the form $tc^{-\alpha-\beta}p(s)y^{u-su-\bar{\rho}}$ with a meromorphic fuction p independent of φ. Taking the sum over all $\varphi \in \Phi$, we obtain the Fourier expansion of $E_\xi \| \xi^{-1}$ in the form stated in our proposition. The meromorphic continuation of $E_\xi \| \xi^{-1}$ implies that of $f_{\xi\xi}$ and $g_{\xi\xi}$ to the whole s-plane.

Next let $\xi \neq \eta \in Y$. Let Z be a complete set of representatives of $(\mathcal{P} \cap \Delta_\xi)\backslash\xi\Delta\eta^{-1}/[r(F) \cap \Delta_\eta]$. Then the elements $\zeta r(a)$ with $\zeta \in Z$ and $r(a) \in r(F) \cap \Delta_\eta$ represent $(\mathcal{P} \cap \Delta_\xi)\backslash\xi\Delta\eta^{-1}$ without overlap. Therefore the same argument as above establishes the Fourier expansion of $E_\xi \| \eta^{-1}$; the only difference is that $y^{su-\rho}$ doesn't appear this time.

Proposition 5.3. *Let* $\Gamma = \Gamma_0(\mathfrak{c})$, $\Gamma' = \{\gamma \in \Gamma \mid d_\gamma - 1 \in \mathfrak{c}\}$, $\Delta = \Lambda_\sigma^k(\Gamma)$, *and* $\Delta' = \Lambda_\sigma^k(\Gamma')$. *Assume* (4.23). *Let D, B,* and $\mathfrak{a}_\beta$ *be as in* (4.9*a*) *and* (4.10). *Then $\mathcal{P}\alpha\Delta'$ is ρ-regular for every $\alpha \in \Lambda_\sigma^k(G \cap P_{\mathbf{A}}D)$. Moreover, if T_β is a complete* *set of representatives of* $(P\cap\beta\Gamma\beta^{-1})\backslash\beta\Gamma/\Gamma'$, *then*

$$E_k(z, s; \rho, \psi, \mathfrak{c}) = \sum_{\beta\in B} N(\mathfrak{a}_\beta)^{2s} \sum_{\xi\in T_\beta} \psi(d_\xi\mathfrak{a}_\beta^{-1})\psi_{\mathfrak{a}}(d_\xi)\cdot E(z, s; \rho, \Lambda_\sigma^k(\xi), \Delta').$$

Proof. Put $\tilde{\alpha} = \Lambda_\sigma^k(\alpha)$ for $\alpha\in G\cap P_{\mathbf{A}}D$. Observe that

$$\mathcal{P}\cap\tilde{\alpha}\Delta'\tilde{\alpha}^{-1} = \Lambda_\sigma^k(P\cap\alpha\Gamma'\alpha^{-1}).$$

Then the first assertion can easily be verified. Let S_ξ be a complete set of representatives of $(P\cap\xi\Gamma'\xi^{-1})\backslash\xi\Gamma'$. Then the S_ξ for all $\xi\in T_\beta$ form a disjoint union, which gives a complete set of representatives of $(P\cap\beta\Gamma\beta^{-1})\backslash\beta\Gamma$. Taking this union as R_β of (4.10), we obtain our formula.

Proposition 5.4. *Let* E_ψ *denote the function of* (4.10), *and let* $\zeta\in\mathcal{G}$. *Then the constant term of* $E_\psi\|\zeta^{-1}$ *contains* $y^{su-\rho}$ *nontrivially if and only if* $\mathrm{pr}(\zeta)\in$ $\in G\cap P_{\mathbf{A}}D$ *with* D *of* (4.9a). *Moreover, the term involving* $y^{su-\rho}$ *has the form* $ab^sy^{su-\rho}$ *with* $a\in\mathbf{C}$ *and* $0<b\in\mathbf{R}$.

Proof. Let $\alpha = \mathrm{pr}(\zeta)$. By Propositions 3.3, 5.2, and 5.3, $y^{su-\rho}$ appears nontrivially in $E_\psi\|\zeta^{-1}$ only if $\alpha\in P\beta\Gamma$ for some $\beta\in B$, that is, only if $\alpha\in G\cap P_{\mathbf{A}}D$. Conversely, if $\alpha\in P\beta\Gamma$ with $\beta\in B$, such a β is unique, and $\alpha\in P\xi\Gamma'$ with a unique $\xi\in T_\beta$. Therefore, Propositions 5.2 and 5.3 show that $y^{su-\rho}$ appears nontrivially in $E_\psi\|\zeta^{-1}$ in the form as claimed.

Remark 5.5. To show that the exceptional case of Proposition 3.2 can happen, take $[F:\mathbf{Q}] = 2$ and set $\mathbf{a} = \{v, w\}$. Take θ so that $T_U = \mathbf{Z}\theta$. Then $\theta_v = -\theta_w$. Let $p = su - (\sigma - im\theta)/2$ and $r = tu - (\sigma - in\theta)/2$ with $s = (1 + in\theta_v)/2$, $t = (1 + im\theta_v)/2$, and $m,\ n\in\mathbf{Z}$. Suppose $|m|\neq|n|$. Then y^p, $y^{u-\sigma-p}$, y^r, $y^{u-\sigma-r}$ belong to the same set of eigenvalues $\{\lambda_v, \lambda_w\}$, where $4\lambda_v = (1-\sigma_v)^2 +$ $+ (m+n)^2\theta_v^2$ and $4\lambda_w = (1-\sigma_w)^2 + (m-n)^2\theta_v^2$. If we put $\rho = (\sigma - im\theta)/2$ and $\rho' = (\sigma - in\theta)/2$, then y^p, $y^{u-\sigma-p}$, y^r, and $y^{u-\sigma-r}$ can appear nontrivially in $E(z, s; \rho, \Delta)$, $E(z, 1-s; \bar{\rho}, \Delta)$, $E(z, s; \rho', \Delta)$, and $E(z, 1-t; \bar{\rho}', \Delta)$, respectively, for a sufficiently small Δ.

6. Bilinear relations

Let Δ be a congruence subgroup of $\mathcal{G}$ and let $\Gamma = \mathrm{pr}(\Delta)$. Take a minimal finite subset X of $\mathcal{G}$ so that $\mathcal{G} = \bigcup_{\xi\in X}\mathcal{P}\xi\Delta$. For each $\xi\in X$, let $Q_\xi = P\cap\mathrm{pr}(\xi\Delta\xi^{-1})$. We consider a group

$$\Theta = \left\{\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix} \,\middle|\, a\in U_1, b\in\mathfrak{m}\right\} \tag{6.1}$$

with a fractional ideal $\mathfrak{m}$ of F and a subgroup U_1 of $\mathfrak{g}^\times$ of finite index. We choose U_1 and $\mathfrak{m}$ so that $\Lambda_\sigma^k(\Theta) \subset \xi\Delta\xi^{-1}$ for all $\xi \in X$ and $a \gg 0$ for every $a \in U_1$.

Let $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ and $g \in \mathfrak{A}(\sigma, \lambda', \Delta)$. Assuming that both λ and λ' are noncritical, put, for each $\xi \in X$,

$$f \| \xi^{-1} = \sum_p a_{p,\xi} y^p + \dots, \tag{6.2a}$$

$$g \| \xi^{-1} = \sum_q b_{q,\xi} y^q + \dots \tag{6.2b}$$

with constants $a_{p,\xi}$, $b_{q,\xi}$ and $p \in \mathbf{C}^{\mathbf{a}}$ as given in Proposition 3.1, where... indicates the nonconstant terms of the Fourier expansions. If both λ and λ' are critical, we put

$$f \| \xi^{-1} = a_\xi y^p + a_\xi' y^p \log y^u + \dots, \tag{6.3a}$$

$$g \| \xi^{-1} = b_\xi y^p + b_\xi' y^p \log y^u + \dots \tag{6.3b}$$

with $p = (u - \sigma)/2$.

Theorem 6.1. *Suppose $\lambda' = \bar{\lambda}$ and λ is noncritical and simple. Fix one exponent p attached to λ and put $q = u - \sigma - p$. Then*

$$\sum_{\xi \in X} \nu_\xi (\bar{a}_{p,\xi} b_{\bar{q},\xi} - \bar{a}_{q,\xi} b_{\bar{p},\xi}) = 0,$$

where $\nu_\xi = [Q_\xi\{\pm 1\} : \Theta\{\pm 1\}]^{-1}$. If $\lambda = \lambda'$ and λ is critical, one has

$$\sum_{\xi \in X} \nu_\xi (\bar{a}_\xi b_\xi' - \bar{a}_\xi' b_\xi) = 0.$$

PROOF. For $0 < r \in \mathbf{R}$, put

$$T_r = \{ z \in H^{\mathbf{a}} \mid y^u > r \}, \qquad M_r = \{ z \in H^{\mathbf{a}} \mid y^u = r \}.$$

We can find an r such that the sets $\xi^{-1}(Q_\xi \backslash T_r)$ can be embedded into $\Gamma \backslash H^{\mathbf{a}}$ without overlap. For each ξ, take a positive number $r(\xi) > r$. Also take a union J of small neighborhoods of elliptic fixed points on $\Gamma \backslash H^{\mathbf{a}}$. Let K be the complement of $\bigcup_{\xi \in X} \xi^{-1}(Q_\xi \backslash T_{r(\xi)}) \cup J$ in $\Gamma \backslash H^{\mathbf{a}}$. Then K is a compact manifold with boundary, and

$$\partial K = \sum_{\xi \in X} \xi^{-1}(Q_\xi \backslash M_{r(\xi)}) - \partial J.$$

Let φ be a Γ-invariant C^∞-form on $H^{\mathbf{a}}$ of codegree 1. Then

$$\int_K d\varphi = \int_{\partial K} \varphi = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} \varphi \cdot \xi^{-1} - \int_{\partial J} \varphi, \tag{6.4}$$

where $B_\xi = \Theta_\xi \backslash M_{r(\xi)}$ (with a natural orientation). We fix one $v \in \mathbf{a}$ and put

$$\omega = y^{-2u} \prod_{v \in \mathbf{a}} dx_v \wedge dy_v,$$

$$\zeta_v = (i/2) y^{-2u} d\bar{z}_v \wedge \prod_{w \neq v} dx_w \wedge dy_w,$$

and $\varphi = \bar{f} h y^\sigma \zeta_v$ with two C^∞-functions f and h on $H^{\mathbf{a}}$ satisfying (2.7a) with Δ and Δ_* (of Proposition 2.2), respectively. Then it is easy to see that

$$d\varphi = \bar{f} \delta_v^\tau h y^\sigma \omega - \overline{\epsilon_v f} \cdot h y^\tau \omega \qquad (\tau = \sigma - 2v).$$

Applying (6.4) to this and taking the limit when $r \to \infty$, we find

$$\langle f, \delta_v^\tau h \rangle = \langle \epsilon_v f, h \rangle \tag{6.5}$$

provided that these inner products are meaningful, and that f or h is rapidly decreasing in the sense that the inequality of (2.7c) holds for *every* $c \in \mathbf{R}$. This proves (2.25). Now take $h = \epsilon_v g$ with g of weight σ. Then

$$d\varphi = \tfrac{1}{4} \bar{f} L_v^\sigma g \cdot y^\sigma \cdot \omega - \overline{\epsilon_v f} \cdot \epsilon_v g \cdot y^\tau \omega.$$

Putting similarly $\varphi' = \bar{g} \epsilon_v f \cdot y^\sigma \zeta_v$, we find that

$$d\varphi - \overline{d\varphi'} = \tfrac{1}{4} (\bar{f} L_v^\sigma g - \overline{L_v^\sigma f} \cdot g) y^\sigma \omega.$$

Applying (6.4) to this form, we obtain

$$\tfrac{1}{4} \int_K (\bar{f} L_v^\sigma g - \overline{L_v^\sigma f} \cdot g) y^\sigma \omega = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} (\varphi - \bar{\varphi}') \circ \xi^{-1} - \int_{\partial J} (\varphi - \bar{\varphi}').$$

We now assume that f and g are eigenfunctions with expansions as in (6.2a, b). Then

$$\varphi \circ \xi^{-1} = -\tfrac{i}{2} \sum_{p,q} q_v \bar{a}_{p,\xi} b_{q,\xi} y^{\bar{p}+q+v+\sigma} \zeta_v + \dots ,$$

$$\bar{\varphi}' \circ \xi^{-1} = \tfrac{i}{2} \sum_{p,q} \bar{p}_v \bar{a}_{p,\xi} b_{q,\xi} y^{\bar{p}+q+v+\sigma} \bar{\zeta}_v + \dots .$$

Here the unwritten terms contain some contributions to the «constant terms» of the Fourier expansions, but they tend to zero in our later limit process. (This can easily be shown by virtue of (2) of Proposition 2.1.) Put $U = \{a^2 \mid a \in U_1\}$. Then $\Theta \backslash M_r$ may be viewed as the product of $\mathbf{R}^{\mathbf{a}}/\mathfrak{m}$ and $\{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0, y^u = r\}/U$. Then we can easily prove

Lemma 6.2. *Let $t = su + i\tau \in \mathbf{C}^{\mathbf{a}}$ with $s \in \mathbf{C}$ and τ in the set T_U of* (3.1). *Then*

$$\int_{\Theta \backslash M_r} y^{t+v} \zeta_v = (-i/2) \mu(\mathbf{R}^{\mathbf{a}}/\mathfrak{m}) R_U r^{s-1},$$

where R_U is the regulator of U defined by $R_U = R_F[\mathfrak{g}^\times : U\{\pm\}]$ with the regulator R_F of F.

Applying this to the first terms of $(\varphi - \bar{\varphi}') \circ \xi^{-1}$, we find that

$$(\lambda'_v - \bar{\lambda}_v)\int_K \bar{f}gy^\sigma\omega + 4\int_{\partial_J}(\varphi - \bar{\varphi}')$$
$$= \mu(\mathbf{R}^{\mathbf{a}}/\mathfrak{m})R_U \sum_{\xi\in X} \nu_\xi \sum_{p,q} (\bar{p}_v - q_v)\bar{a}_{p,\xi}b_{q,\xi}r_\xi^{e(p,q)} + \ldots$$

where $e(p, q) = \sum_{w\in\mathbf{a}}(\bar{p}_w + q_w + \sigma_w - 1)/[F: \mathbf{Q}]$. Suppose that $\lambda' = \bar{\lambda}$ and λ is simple. Then, with one exponent p fixed, the sum $\sum_{p,q}$ can be written as

$$(2\bar{p}_v - 1 + \sigma_v)(\bar{a}_{p,\xi}b_{u-\sigma-\bar{p},\xi} - \bar{a}_{u-\sigma-p,\xi}b_{\bar{p},\xi}).$$

Since λ is not critical, $2\bar{p}_v - 1 + \sigma_v \neq 0$ for at least one v. Therefore, taking the limit when $J \to \varnothing$ and $r_\xi \to \infty$, we obtain the first assertion of Theorem 6.1. The second one can be proved in a similar way.

If λ is not simple, $e(p, q)$ can be a pure imaginary number which is not necessarily equal to 0. Therefore we obtain certain linear relations even for multiple λ, whose nature is somewhat different from that for simple λ.

7. Construction of $\mathfrak{N}(\sigma, \lambda, \Delta)$ by Eisenstein series

We are going to show that the space $\mathfrak{N}(\sigma, \lambda, \Delta)$ of (2.23*a*) is generated by the series of type (4.1), their derivatives, and their residues. Given σ and λ, we are interested in the case where $\mathcal{A}(\sigma, \lambda, \Delta) \neq \mathcal{S}(\sigma, \lambda, \Delta)$, that is, the case in which nontrivial constant terms appear. Then λ must be given as in Proposition 3.1. We assume throughout that λ is simple. Then

$$\lambda_v = (s_0 - \rho_v)(1 - s_0 - \bar{\rho}_v), \qquad \rho = (\sigma - i\tau)/2 \tag{7.1}$$

with $s_0 \in \mathbf{C}$ and an admissible $\tau \in \mathbf{R}^{\mathbf{a}}$. Notice that (s_0, ρ) may be changed for $(1 - s_0, \bar{\rho})$ without changing (σ, λ). Notice also that (7.1) includes critical λ as a special case. In fact λ is critical if and only if $s_0 = \frac{1}{2}$ and $\tau = 0$; then $\rho = \frac{\sigma}{2}$. This is so if and only if $s_0u - \rho = u - s_0u - \bar{\rho}$.

In this section, we fix a complete set of representatives X of $\mathcal{P}\backslash\mathcal{G}/\Delta$, and also a subset Y of X that represents all ρ-regular cusp-classes of Δ; we then denote by $\varkappa$ the number of elements of Y. Further we let $\mathcal{E}[\rho, \Delta]$ denote the complex vector space spanned by the functions $E(z, s; \rho, \xi, \Delta)$ for all $\xi \in Y$. For a complex number s_0, we denote by $\mathcal{E}[s_0, \rho, \Delta]$ the subspace of $\mathcal{E}[\rho, \Delta]$ consisting of all functions $g(z, s)$ that are finite at s_0, and by $\mathcal{E}(s_0, \rho, \Delta)$ the vector space consisting of $g(z, s_0)$ for all $g \in \mathcal{E}[s_0, \rho, \Delta]$. Similarly we denote by $\mathcal{E}^*[s_0, \rho, \Delta]$ the set of elements of $\mathcal{E}[\rho, \Delta]$ that have at most a simple pole at s_0 and by $\mathcal{E}^*(s_0, \rho, \Delta)$ the residues at s_0 of all elements of $\mathcal{E}^*[s_0, \rho, \Delta]$.

Proposition 7.1. *Both* $\mathcal{E}(s_0, \rho, \Delta)$ *and* $\mathcal{E}^*(s_0, \rho, \Delta)$ *are contained in* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *with* λ *of* (7.1).

PROOF. The spaces in question are contained in $\mathfrak{A}(\sigma, \lambda, \Delta)$ by virtue of Proposition 4.4. To prove that they are orthogonal to cusp forms, take a congruence subgroup $\Delta' \subset \Delta$ so that

$$\mathcal{P} \cap \Delta' = \left\{ \Lambda_\sigma^k\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) \mid a \in U_1, b \in \mathfrak{b} \right\}$$

with an ideal $\mathfrak{b}$ and a subgroup U_1 of $\mathfrak{g}^\times$ of finite index consisting of totally positive units. Then $(\mathcal{P} \cap \Delta')\backslash H^{\mathbf{a}}$ can be represented by $B \times A$, with $B = \mathbf{R}^{\mathbf{a}}/\mathfrak{b}$ and $A = \{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}/U$ where $U = \{a^2 \mid a \in U_1\}$. We now consider an integral

$$\int_A \int_B \overline{f(x+iy)}\, dx \cdot y^{(s-2)u+\bar{\rho}}\, dy$$

for $f \in \mathcal{S}(\sigma, \lambda', \Delta)$ with any fixed λ'. Since the constant term of f is 0, this is obviously 0. If $\mathrm{Re}(s)$ is sufficiently large and $\Phi = \Delta'\backslash H^{\mathbf{a}}$, the integral can be transformed to

$$\int_\Phi \left\{ \sum_{\gamma \in (\mathcal{P} \cap \Delta')\backslash \Delta'} (y^{su+\bar{\rho}} \bar{f}) \circ \gamma \right\} d\mu(z) = \mu(\Phi)\langle f, E(z, s; \rho, \Delta')\rangle.$$

Therefore $\langle f, E(z, s; \rho, \Delta')\rangle = 0$ for sufficiently large $\mathrm{Re}(s)$. The same holds with Δ instead of Δ', by virtue of (4.3). Then the desired orthogonality can easily be shown by analytic continuation.

Proposition 7.2. (i) $\dim \mathcal{E}[\rho, \Delta] = \varkappa$.

(ii) *The map* $g(z, s) \to g(z, s_0)$ *gives an isomorphism of* $\mathcal{E}[s_0, \rho, \Delta]$ *onto* $\mathcal{E}(s_0, \rho, \Delta)$ *provided that* λ *of* (7.1) *is noncritical.*

PROOF. Assertion (i) follows immediately from Proposition 5.2. Let $g = \sum_{\xi \in Y} a_\xi E(z, s; \rho, \xi, \Delta) \in \mathcal{E}[s_0, \rho, \Delta]$. Then we have

$$g(z, s_0) \| \eta^{-1} = a_\eta y^{s_0 u - \rho} + \left(\sum_{\xi \in Y} a_\xi f_{\xi\eta}\right)(s_0) y^{u - s_0 u - \bar{\rho}} + \dots \tag{7.2}$$

for every $\eta \in Y$. If λ is noncritical, we have $s_0 u - \rho \neq u - s_0 u - \bar{\rho}$, and therefore, if $g(z, s_0) = 0$, we have $a_\eta = 0$ for all $\eta \in Y$, so that $g = 0$. This proves (ii).

Theorem 7.3. *With* λ, s_0, *and* ρ *as in*(7.1), *suppose* λ *is noncritical and simple; suppose also that* $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta]$ *and* $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}_0, \bar{\rho}, \Delta]$. *Then* $\mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}(s_0, \rho, \Delta)$, *and* $\mathfrak{A}(\sigma, \lambda, \Delta)$ *is the direct sum of* $\mathcal{S}(\sigma, \lambda, \Delta)$ *and* $\mathcal{E}(s_0, \rho, \Delta)$.

PROOF. Put $p = s_0 u - \rho$ and $q = u - \sigma - p$. Let Y' be the set of all $\xi \in X$ such that $\mathcal{P}\xi\Delta$ is $\bar{\rho}$-regular, and $\varkappa'$ the number of elements of Y'. Given $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ and $g \in \mathfrak{A}(\sigma, \bar{\lambda}, \Delta)$, we consider expansions

$$f \| \xi^{-1} = a_\xi y^p + a'_\xi y^q + \dots \tag{7.3a}$$

$$g \| \xi^{-1} = b_\xi y^{\bar{p}} + b'_\xi y^{\bar{q}} + \dots \tag{7.3b}$$

for each $\xi \in X$. By Proposition 3.3 and Theorem 6.1, we have

$$\sum_{\xi \in Y} \nu_\xi \bar{a}_\xi b'_\xi - \sum_{\xi \in Y'} \nu_\xi \bar{a}'_\xi b_\xi = 0. \tag{7.4}$$

Moreover, the map

$$f \mapsto ((a_\xi)_{\xi \in Y}, (a'_\xi)_{\xi \in Y'}) \tag{7.5}$$

gives an injection of $\mathfrak{A}(\sigma, \lambda, \Delta)/\mathcal{S}(\sigma, \lambda, \Delta)$ into $\mathbf{C}^\mu$ with $\mu = \varkappa + \varkappa'$; a similar statement holds with g and $\bar{\lambda}$ instead of f and λ. By Proposition 7.2 and our assumption, $\mathcal{E}(s_0, \rho, \Delta)$ is $\varkappa$-dimensional, and $\mathcal{E}(\bar{s}_0, \bar{\rho}, \Delta)$ is $\varkappa'$-dimensional. Each g in the latter space produces a linear relation of type (7.4), and hence the image of the map of (7.5) is at most $\varkappa$-dimensional. This combined with (2.23a) completes the proof.

Remark 7.4. (1) If $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$, then, by Theorem 4.2, $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta]$ except when $s_0 = 1$ and $\rho = 0$ in Case I and $s_0 = \frac{3}{4}$, $\tau = 0$, and σ is as in Theorem 4.1 in Case II. If $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta]$ and $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$, then we have automatically $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}_0, \bar{\rho}, \Delta]$, since $\bar{s}_0 = s_0$ and $\bar{\rho} = \rho$ in those exceptional cases.

(2) The pair (σ, λ) corresponds to (s_0, ρ) and $(1 - s_0, \bar{\rho})$. Therefore, changing (s_0, ρ) for $(1 - s_0, \bar{\rho})$ if necessary, we can take s_0 such that $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$ without changing λ.

Proposition 7.5. *The number of* ρ-regular cusp-classes of Δ *is equal to the number of $\bar{\rho}$-regular cusp-classes of* Δ.

PROOF. Given ρ and Δ, we can find s_0 so that $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta] = \mathcal{E}[1 - \bar{s}_0, \rho, \Delta]$, $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}_0, \bar{\rho}, \Delta] = \mathcal{E}[1 - s_0, \bar{\rho}, \Delta]$, and λ of (7.1) is noncritical and simple (cf. Proposition 3.2). Then we have $\mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}(s_0, \rho, \Delta) = \mathcal{E}(1 - s_0, \bar{\rho}, \Delta)$, which proves our proposition.

Proposition 7.6 *Suppose* $\mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}(s_0, \rho, \Delta)$, *and* λ *is noncritical and simple. For* $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ *and* $\xi \in Y$, *put*

$$f \| \xi^{-1} = a_\xi y^p + a'_\xi y^q + \dots$$

with $p = s_0u - \rho$ and $q = u - \sigma - p$. If $a_\xi = 0$ for all $\xi \in Y$, then f is a cusp form.

PROOF. Let $f = g(z, s_0) + h$ with $g \in \mathcal{E}[s_0, \rho, \Delta]$ and $h \in \mathcal{S}(\sigma, \lambda, \Delta)$. Writing g as in the proof of Proposition 7.2, we see that the assumption $a_\xi = 0$ implies that $g = 0$.

Theorem 7.7. *Suppose every* ρ-regular cusp-class of Δ *is also $\bar{\rho}$-regular. Define a $\mathbf{C}^Y$-valued meromorphic function $\mathbf{E}_\Delta(z, s, \rho)$ by*

$$\mathbf{E}_\Delta(z, s, \rho) = (E(z, s; \rho, \xi, \Delta))_{\xi \in Y}. \tag{7.6}$$

Then there exists an $\mathrm{End}(\mathbf{C}^Y)$*-valued meromorphic function $\Phi_\Delta(s, \rho)$ on $\mathbf{C}$ such that*

$$\mathbf{E}_\Delta(z, s, \rho) = \Phi_\Delta(s, \rho)\mathbf{E}_\Delta(z, 1 - s, \bar{\rho}), \tag{7.7a}$$

$$\Phi_\Delta(1 - s, \bar{\rho})\Phi_\Delta(s, \rho) = 1. \tag{7.7b}$$

Moreover, there is a diagonal matrix A, depending only on Δ and Y, whose diagonal entries are positive integers such that

$$\overline{\Phi_\Delta(s, \rho)}A \cdot {}^t\Phi_\Delta(1 - \bar{s}, \rho) = A. \tag{7.7c}$$

PROOF. Put $\lambda_v(s) = (s - \rho_v)(1 - s - \bar{\rho}_v)$, $p = su - \rho$, and $q = u - \sigma - p$. Suppressing the symbols z and Δ for simplicity, we have

$$E(s, \rho, \xi)\|\eta^{-1} = \delta_{\xi\eta}y^p + f_{\xi\eta}(s)y^q + \dots, \tag{7.8a}$$

$$E(1 - s, \bar{\rho}, \xi)\|\eta^{-1} = \delta_{\xi\eta}y^q + g_{\xi\eta}(1 - s)y^p + \dots \qquad (\xi, \eta \in Y) \tag{7.8b}$$

with meromorphic $f_{\xi\eta}$ and $g_{\xi\eta}$. Then, for every $\zeta \in Y$, we have

$$\begin{aligned}\Big\{E(1 - s, \bar{\rho}, \xi) - \sum_{\eta \in Y} g_{\xi\eta}(1 - s)E(s, \rho, \eta)\Big\}\Big\|\zeta^{-1} &= \\ = 0y^p + \Big\{\delta_{\xi\zeta} - \sum_{\eta \in Y} g_{\xi\eta}(1 - s)f_{\eta\zeta}(s)\Big\}y^q + \dots .&\end{aligned} \tag{7.9}$$

Now we can find a nonempty open subset W of $\mathbf{C}$ such that $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s, \rho, \Delta] = \mathcal{E}[1 - \bar{s}, \rho, \Delta]$, $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}, \bar{\rho}, \Delta] = \mathcal{E}[1 - s, \bar{\rho}, \Delta]$, and that $\lambda(s)$ is noncritical and simple for every $s \in W$. (As to simple λ, see Proposition 3.2.) Now the left-hand side of (7.9) without $\|\zeta^{-1}$ belongs to $\mathfrak{N}(\sigma, \lambda(s), \Delta)$ for $s \in W$. By Proposition 7.6, we have

$$E(1 - s, \bar{\rho}, \xi) = \sum_{\eta \in Y} g_{\xi\eta}(1 - s)E(s, \rho, \eta),$$

$$\delta_{\xi\zeta} = \sum_{\eta} g_{\xi\eta}(1 - s)f_{\eta\zeta}(s).$$

Writing $\Phi(s,\rho)$ for the matrix $(f_{\xi\eta}(s))$, we obtain (7.7a, b). Now

$$E(1-\bar{s},\rho,\xi)\parallel\eta^{-1}=\delta_{\xi\eta}y^{\bar{q}}+f_{\xi\eta}(1-\bar{s})y^{\bar{p}}+\ldots,$$

and $E(1-\bar{s},\rho,\xi)$ belongs to $\mathfrak{A}(\sigma,\bar{\lambda},\Delta)$ for $s\in W$. By (7.4), we have

$$\sum_{\eta\in Y}\nu_\eta[\delta_{\xi\eta}\delta_{\zeta\eta}-\overline{f_{\xi\eta}(s)}f_{\zeta\eta}(1-\bar{s})]=0.$$

Denoting by D the diagonal matrix whose diagonal elements are ν_η, we obtain $D=\overline{\Phi(s,\rho)}D\cdot{}^t\Phi(1-\bar{s},\rho)$, which proves (7.7$c$).

By Remark 7.4, (1), $E(z,s;\rho,\xi,\Delta)$ is finite at $s=\frac{1}{2}$ and hence $\Phi_\Delta(s,\rho)$ is finite at $s=\frac{1}{2}$. Moreover, we have $\Phi_\Delta\left(\frac{1}{2},\rho\right)^2=1$ if $\tau=0$.

Theorem 7.8. *Suppose* λ *is critical* (*and hence* $\rho=\frac{\sigma}{2}$). *Let* $\mathcal{E}'\left(\frac{1}{2},\rho,\Delta\right)$ *denote the space spanned by* $(\partial g/\partial s)\left(z,\frac{1}{2}\right)$ *for all* $g\in\mathcal{E}[\rho,\Delta]$, *and* $\mathcal{E}_0'\left(\frac{1}{2},\rho,\Delta\right)$ *the subspace of* $\mathcal{E}'\left(\frac{1}{2},\rho,\Delta\right)$ *consisting of* $(\partial g/\partial s)\left(z,\frac{1}{2}\right)$ *for all such g satisfying* $g\left(z,\frac{1}{2}\right)=0$. *Further let* ν_+ (*resp.* ν_-) *the multiplicity of* 1 (*resp.* -1) *in the eigenvalues of* $\Phi\left(\frac{1}{2},\rho\right)$. *Then* $\varkappa=\nu_++\nu_-$, $\dim\mathcal{E}\left(\frac{1}{2},\rho,\Delta\right)=\nu_+$, $\dim\mathcal{E}_0'\left(\frac{1}{2},\rho,\Delta\right)=\nu_-$, *and*

$$\mathcal{E}'\left(\tfrac{1}{2},\rho,\Delta\right)\subset\mathfrak{N}(\sigma,\lambda,\Delta)=\mathcal{E}\left(\tfrac{1}{2},\rho,\Delta\right)\oplus\mathcal{E}_0'\left(\tfrac{1}{2},\rho,\Delta\right).$$

Moreover, $\mathcal{E}\left(\frac{1}{2},\rho,\Delta\right)$ *consists of the elements of* $\mathfrak{N}(\sigma,\lambda,\Delta)$ *that do not involve* $y^{(u-\sigma)/2}\log y^u$.

Proof. For simplicity, let us suppress the symbols ρ and Δ occasionally. That an element of $\mathcal{E}'\left(\frac{1}{2}\right)$ satisfies (2.7a, b) can be verified immediately. That it satisfies (2.7c) is shown in the proof of Proposition 4.4 in Section 11, and hence $\mathcal{E}'\left(\frac{1}{2}\right)\subset\mathfrak{A}(\sigma,\lambda)$. The orthogonality with cusp forms can also be seen, because the integral expressing $\langle f,g(z,s)\rangle$ is uniformly convergent in a neighborhood of $s=\frac{1}{2}$ for every fixed cusp form f. Thus $\mathcal{E}'\left(\frac{1}{2}\right)\subset\mathfrak{N}(\sigma,\lambda)$. Put $p=(u-\sigma)/2$. From (7.8a) we obtain

$$E\left(\tfrac{1}{2},\rho,\xi\right)\Big\|\,\eta^{-1}=\left[\delta_{\xi\eta}+f_{\xi\eta}\left(\tfrac{1}{2}\right)\right]y^p+\ldots,$$

$$(\partial E/\partial s)\left(\tfrac{1}{2},\rho,\xi\right)\Big\|\,\eta^{-1}=\left[\delta_{\xi\eta}-f_{\xi\eta}\left(\tfrac{1}{2}\right)\right]y^p\log y^u+(df_{\xi\eta}/ds)\left(\tfrac{1}{2}\right)y^p+\ldots.$$

For $g(z,s)=\sum_\xi c_\xi E(s,\rho,\xi)$ with $c=(c_\xi)_{\xi\in Y}\in\mathbf{C}^Y$, we have $g\left(z,\frac{1}{2}\right)=0$ if and only if ${}^t\Phi\left(\frac{1}{2}\right)c=-c$. Hence $\dim\mathcal{E}\left(\frac{1}{2}\right)=\nu_+$. If ${}^t\Phi\left(\frac{1}{2}\right)c=-c$, we have $(\partial g/\partial s)\left(z,\frac{1}{2}\right)\Big\|\,\eta^{-1}=2c_\eta y^p\log y^u+\ldots$, which shows that $\dim\mathcal{E}_0'\left(\frac{1}{2}\right)=\nu_-$. Since no element of $\mathcal{E}\left(\frac{1}{2}\right)$ involves $y^p\log y^u$, we see that $\mathcal{E}\left(\frac{1}{2}\right)$ and $\mathcal{E}_0'\left(\frac{1}{2}\right)$ form a direct sum of dimension $\varkappa$. Now Theorem 6.1 shows that $\mathfrak{N}(\sigma,\lambda,\Delta)$ has dimension $\leqslant\varkappa$. Therefore we obtain all the remaining assertions.

Theorem 7.9. *With* λ, s_0, and ρ *as in* (7.1), *suppose that* λ is real, noncritical, and simple. Suppose $\mathcal{E}[\rho, \Delta] = \mathcal{E}^*[s_0, \rho, \Delta]$ *and a cusp-class of* Δ *is* ρ*-regular if and only if it is* $\bar{\rho}$*-regular. Then* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *has dimension* $\varkappa$, *and is the direct sum of* $\mathcal{E}(s_0, \rho, \Delta)$ *and* $\mathcal{E}^*(s_0, \rho, \Delta)$.

PROOF. Define $R\colon \mathcal{E}[\rho, \Delta] \to \mathcal{E}^*(s_0, \rho, \Delta)$ by $R(g) = \mathrm{Res}_{s_0} g(z, s)$. Then $\mathcal{E}[s_0, \rho, \Delta] = \mathrm{Ker}(R)$, so that, by Proposition 7.2,

$$\dim \mathcal{E}(s_0, \rho, \Delta) + \dim \mathcal{E}^*(s_0, \rho, \Delta) = \varkappa.$$

Let $h \in \mathcal{E}(s_0, \rho, \Delta) \cap \mathcal{E}^*(s_0, \rho, \Delta)$. Then $h(z) = r(z, s_0) = R(g)$ with $r \in \mathcal{E}[s_0, \rho, \Delta]$ and $g \in \mathcal{E}[\rho, \Delta]$. Put

$$r = \sum_{\xi \in Y} a_\xi E(z, s; \rho, \xi, \Delta), \qquad g = \sum_{\xi \in Y} b_\xi E(z, s; \rho, \xi, \Delta) \tag{7.10}$$

with $a_\xi, b_\xi \in \mathbf{C}$. Then, for $\eta \in Y$, we have

$$\begin{aligned} h \,\|\, \eta^{-1} &= a_\eta y^p + \left(\textstyle\sum_\xi a_\xi f_{\xi\eta}\right)(s_0) y^q + \dots \\ &= 0 y^p + \left(\textstyle\sum_\xi b_\xi \mathrm{Res}_{s_0} f_{\xi\eta}\right) y^q + \dots, \end{aligned}$$

where $p = s_0 u - \rho$ and $q = u - \sigma - p$. Hence $a_\eta = 0$ for all η, so that $h = 0$. Thus $\mathcal{E}(s_0, \rho, \Delta)$ and $\mathcal{E}^*(s_0, \rho, \Delta)$ form a direct sum of dimension $\varkappa$. Consider again the map of (7.5) of $\mathfrak{A}(\sigma, \lambda, \Delta)/\mathcal{S}(\sigma, \lambda, \Delta)$ into $\mathbf{C}^{2\varkappa}$. Now the relation of Theorem 6.1 shows that the image of the map has dimension at most $\varkappa$. This completes the proof.

Remark 7.10. Given σ and λ, we can take s_0 and τ so that $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$ and (7.1) is satisfied. By Theorem 4.2, we have $\mathcal{E}[\rho, \Delta] = \mathcal{E}^*[s_0, \rho, \Delta]$ if $\mathrm{Re}(s_0) \geqslant \frac{1}{2}$; moreover, the pole occurs only when $s_0 = 1$ or $= \frac{3}{4}$, and $\rho = \bar{\rho}$. Theorem 7.9 is applicable to such cases.

Remark 7.11. If $\rho = 0$ and $s_0 = 1$, we see that $\mathcal{E}^*(1, 0, \Delta)$ consists of the constants, as shown in Proposition 4.3. Therefore we obtain

$$\dim \mathcal{E}(s_0, \rho, \Delta) = \varkappa - 1 \quad \text{if} \quad \rho = 0 \quad \text{and} \quad s_0 = 1. \tag{7.11}$$

Combining Theorems 7.3, 7.8, 7.9 and Remarks 7.4, 7.10, we obtain

Theorem 7.12. *If* λ *is simple,* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *has dimension* $\varkappa$.

In this section, we treated $\mathfrak{N}(\sigma, \lambda, \Delta)$ only for simple λ. If λ is multiple, $\mathfrak{N}(\sigma, \lambda, \Delta)$ is probably generated by Eisenstein series with several different (s_0, ρ), as Remark 5.5 suggests. The proof of this fact does not seem very difficult, though the author has no complete result.

8. Applications to holomorphic forms

Let $\mathcal{H}(\sigma, \Delta)$ denote the set of all holomorphic functions on $H^{\mathbf{a}}$ satisfying (2.7*a*, *c*), and $\mathcal{H}(\sigma)$ the union of $\mathcal{H}(\sigma, \Delta)$ for all congruence sugroups Δ of $\mathcal{G}_\sigma$. (It is well known that (2.7*c*) follows from (2.7*a*) and the holomorphy if $F \neq \mathbf{Q}$.) If $f \in \mathcal{H}(\sigma)$, it has an expansion

$$f(z) = b_0 + \sum_{0 \ll h \in \mathfrak{m}} b_h \mathbf{e}_{\mathbf{a}}(hz) \tag{8.1}$$

with a lattice $\mathfrak{m}$ in F and complex coefficients b_0 and b_h. Given a subfield K of $\mathbf{C}$, we denote by $\mathcal{H}(\sigma, K)$ and $\mathcal{H}(\sigma, \Delta, K)$ the subsets of $\mathcal{H}(\sigma)$ and $\mathcal{H}(\sigma, \Delta)$ consisting of all f such that the coefficients b_0 and b_h belong to K. We shall be especially interested in the case where K is the maximal abelian extension of $\mathbf{Q}$ which we denote by $\mathbf{Q}_{ab}$.

Proposition 8.1.
(i) $\mathcal{S}(\sigma, 0, \Delta) \subset \mathcal{H}(\sigma, \Delta) \subset \mathcal{A}(\sigma, 0, \Delta)$;
(ii) $\mathcal{S}(\sigma, 0, \Delta) = \mathcal{H}(\sigma, \Delta)$ *if* $\sigma \notin \mathbf{Q}u$.

PROOF. Assertion (i) is a restatement of Proposition 2.5. If $f \in \mathcal{H}(\sigma, \Delta)$, $\xi \in \mathcal{G}$, and c_0 is the constant term of $f \| \xi$, then (2.13*b*) shows that $c_0 = a^\sigma c_0$ for every a in a subgroup of $\mathfrak{g}^\times$ of finite index. Therefore $c_0 = 0$ if $\sigma \notin \mathbf{Q}u$, which proves (ii).

In order to study the holomorphic elements of $\mathcal{N}(\sigma, 0, \Delta)$, put

$$\mathcal{N}\mathcal{H}(\sigma, \Delta) = \mathcal{H}(\sigma, \Delta) \cap \mathcal{N}(\sigma, 0, \Delta). \tag{8.2}$$

From (2.23*a*) and the above (i), we obtain

$$\mathcal{H}(\sigma, \Delta) = \mathcal{S}(\sigma, 0, \Delta) \oplus \mathcal{N}\mathcal{H}(\sigma, \Delta). \tag{8.3}$$

The main purpose of this section is to show that $\mathcal{N}\mathcal{H}(\sigma, \Delta)$ can be obtained from Eisenstein series. By (ii) of the above proposition, the problem concerns only the case $\sigma \in \mathbf{Q}u$.

Proposition 8.2 *Let $E(z, s)$ denote any series of type* (4.1), (4.4), *or* (4.10) *with* $2\rho = \sigma = tu$, $0 < t \in \left(\frac{1}{2}\right)\mathbf{Z}$. *Suppose $k = 2t$ in Case II. Then the following assertions hold*:

(i) *E is finite at $s = t/2$.*
(ii) *If $t > 2$ or $t = 1$, $E(z, t/2)$ belongs to $\mathcal{H}(tu, \mathbf{Q}_{ab})$.*
(iii) *Suppose $t = 2$ or $t = 3/2$; suppose also $F \neq \mathbf{Q}$. Then $E(z, t/2)$ belongs to $\mathcal{H}(tu, \mathbf{Q}_{ab})$.*

(iv) *Suppose* $F = \mathbf{Q}$ *and* $t > 1$. *Then* $E_k(z, t/2; tu/2, \psi, \mathfrak{c})$ belongs to $\mathcal{H}(tu, \mathbf{Q}_{ab})$ *except in the following two cases*; (*A*) $t = 2$ *and* $\psi = 1$; (*B*) $t = 3/2$ *and* $\psi^2 = 1$.

(v) *Suppose* $t = 1/2$. *Then* $E(z, s)$ *has at most a simple pole at* $s = 3/4$ *and the residue is* $\pi^{-n}R_F$ *times an element of* $\mathcal{H}(tu, \mathbf{Q}_{ab})$, *where* R_F *is the regulator of* F.

PROOF. The assertions in Case II are included in [13, Theorem 2.3]. In Case I, the results are essentially due to Hecke [1] when $F = \mathbf{Q}$, and to Kloosterman [2] and Klingen in the case $[F:\mathbf{Q}] > 1$, though our formulation is different from theirs. In the present formulation, the assertions in Case I are included in [12, Theorem 7.1] as special cases.

Theorem 8.3. *Let* $2\rho = \sigma = tu$ *with* $0 < t \in \left(\frac{1}{2}\right)\mathbf{Z}$. *If* $t > \frac{1}{2}$, *one has*

$$\mathfrak{N}\mathcal{H}(\sigma, \Delta) = \mathcal{E}\left(\tfrac{t}{2}, \rho, \Delta\right) \cap \mathcal{H}(\sigma, \Delta). \tag{8.4}$$

Moreover

$$\mathfrak{N}\mathcal{H}(\sigma, \Delta) = \mathcal{E}\left(\tfrac{t}{2}, \rho, \Delta\right) \tag{8.5}$$

except in the following three cases: (i) $t = \frac{1}{2}$; (ii) $t = \frac{3}{2}$ *and* $F = \mathbf{Q}$; (iii) $t = 2$ *and* $F = \mathbf{Q}$.

PROOF. The last assertion follows from (8.4) and Proposition 8.2. Now Proposition 3.2 shows that λ is simple if $\lambda = 0$. Moreover, λ is critical if and only if $t = 1$. Therefore, putting $s_0 = \frac{t}{2}$ with $t > 1$ in Theorem 7.3, we obtain

$$\mathfrak{N}(\sigma, 0, \Delta) = \mathcal{E}\left(\tfrac{t}{2}, \rho, \Delta\right) \quad \text{if} \quad t > 1, \tag{8.6}$$

which proves (8.4). If $t = 1$, the last part of Theorem 7.8 proves (8.4).

As for the case $t = \frac{1}{2}$, we have

Theorem 8.4. $\mathfrak{N}\mathcal{H}\left(\frac{u}{2}, \Delta\right) = \mathcal{E}^*\left(\frac{3}{4}, \frac{u}{4}, \Delta\right)$.

PROOF. By Proposition 8.2, (v), $\mathcal{E}^*\left(\frac{3}{4}, \frac{u}{4}, \Delta\right) \subset \mathcal{H}\left(\frac{u}{2}, \Delta\right)$. In view of Theorem 7.9, it is sufficient to prove that 0 is the only holomorphic element of $\mathcal{E}\left(\frac{3}{4}, \frac{u}{4}, \Delta\right)$. To see this, let $r \in \mathcal{E}\left[\frac{3}{4}, \frac{u}{4}, \Delta\right]$ and express r as in (7.10). Then we see that

$$r\left(z, \tfrac{3}{4}\right) \| \eta^{-1} = a_\eta y^{u/2} + c_\eta + \dots$$

with $c_\eta \in \mathbf{C}$ for every $\eta \in Y$. If $r\left(z, \frac{3}{4}\right)$ is holomorphic, we have $a_\eta = 0$ for every η, so that $r = 0$, which proves the desired fact.

Remark 8.5. The result of [13, Proposition 6.4] tog·ther with (4.3), (4.12), and (4.24) shows that the elements of $\mathcal{E}^*\left(\frac{3}{4}, \frac{u}{4}, \Delta\right)$ are theta series.

As to the previous investigations on $\mathfrak{N}\mathfrak{H}(\sigma, \Delta)$, the reader is referred to the papers mentioned in the introduction.

9. Cyclopean forms

We call an element f of $\mathfrak{A}(\sigma, \lambda)$ a *cyclopean form* (or simply a *cyclops*) *of exponent* q, if the following conditions (9.1a, b, c) are satisfied:

(9.1a) $f \in \mathfrak{N}(\sigma, \lambda)$;

(9.1b) *for every* $\xi \in \mathcal{G}$, *the constant term of* $f \| \xi$ *is of the form* $c_\xi y^q$ *with* $c_\xi \in \mathbf{C}$; *that is, it has no term of the form* by^p *with* p *other than* q;

(9.1c) $$(1 - \sigma_v)/2 < \operatorname{Re}(q_v) < \begin{cases} (2 - \sigma_v)/2 & \\ (3 - 2\sigma_v)/4 & \end{cases} \textit{for every} \quad v \in \mathbf{a} \quad \begin{matrix} (\textit{Case I}), \\ (\textit{Case II}). \end{matrix}$$

By Proposition 3.2, (9.1c) implies that λ is noncritical and simple. By (3.6b), we can put $q = (1 - s_0)u - \bar{\rho}$ and $\rho = (\sigma - i\tau)/2$ with $s_0 \in \mathbf{C}$ and an admissible τ. Then (9.1c) is equivalent to

(9.2) $$\tfrac{1}{2} > \operatorname{Re}(s_0) > \begin{cases} 0 & \text{(Case I)}, \\ \frac{1}{4} & \text{(Case II)}. \end{cases}$$

We also note that

(9.3) $$\lambda_v = q_v(1 - \sigma_v - q_v) = (s_0 - \rho_v)(1 - s_0 - \bar{\rho}_v).$$

Put $p = s_0 u - \rho$. If $f \in \mathfrak{A}(\sigma, \lambda)$, we have, for $\xi \in \mathcal{G}$,

$$f \| \xi = b_\xi y^p + c_\xi y^q + \dots .$$

Thus (9.1b) means that $b_\xi = 0$ for *every* $\xi \in \mathcal{G}$.

Theorem 9.1. *Let* $\rho = (\sigma - i\tau)/2$ *and* $q = (1 - s_0)u - \bar{\rho}$ *with* $s_0 \in \mathbf{C}$ *and an admissible* τ. *In Case II, let* k *be an arbitrarily fixed odd integer. If there exists a nonzero cyclopean form of* $\mathfrak{A}(\sigma, \lambda)$ *of exponent* q, *then there exists a Hecke character* ψ *of* F *such that*

(9.4a) $L(2s_0, \psi) = 0$ $\quad$ (*Case I*),

(9.4b) $L(4s_0 - 1, \psi^2) = 0$ $\quad$ (*Case II*),

(9.5) $\psi(x) = |x|^{\pm i\tau}(x/|x|)^{\sigma'}$ *for* $x \in F_{\mathbf{a}}^\times$, *where* $\sigma' = \sigma$ *in Case I and* $\sigma' = \sigma - ku/2$ *in Case II.*

Conversely, suppose there exist a Hecke character ψ *of* F *and a complex number* s_0 *satisfing* (9.2), (9.4*a* or *b*), *and* (9.5). *Then there exists a nonzero cyclops of* $\mathfrak{A}(\sigma, \lambda)$ *of exponent* q. *More explicitly,*

$$[L(2s, \psi)E(z, s; \rho, \psi, \mathfrak{c})]_{s=s_0} \qquad (\textit{Case I}),$$

$$[L(4s-1, \psi^2)E_k(z, s; \rho, \psi, \mathfrak{c})]_{s=s_0} \qquad (\textit{Case II})$$

are cyclopes, for every multiple $\mathfrak{c}$ *of the conductor of* ψ *that is divisible by* 4 *in Case II.*

PROOF. We prove this only in Case II; Case I can be treated in a similar way. Suppose $L(4s_0 - 1, \psi^2) \neq 0$ for every ψ of type (9.5). Then $\bar{s}_0$ has the same property. Let f be a cyclops of exponent q belonging to $\mathfrak{N}(\sigma, \lambda, \Delta)$. Theorem 4.1 together with (4.3) and (4.24) shows that $\mathcal{E}[\rho, \Delta] = \mathcal{E}[s_0, \rho, \Delta]$ and $\mathcal{E}[\bar{\rho}, \Delta] = \mathcal{E}[\bar{s}_0, \bar{\rho}, \Delta]$. By Theorem 7.3, we have $f(z) = h(z, s_0)$, $h = \sum_{\xi \in Y} a_\xi E_k(z, s; \rho, \xi, \Delta)$ with $a_\xi \in \mathbf{C}$. Putting $p = s_0 u - \rho$ and employing the notation of Proposition 5.2, we have

$$f \| \eta^{-1} = a_\eta y^p + \Big(\sum_{\xi \in Y} a_\xi f_{\xi\eta}\Big)(s_0) y^q + \dots$$

for $\eta \in Y$. Hence $a_\eta = 0$ for all $\eta \in Y$, so that $f = 0$, a contradiction.

Conversely, suppose $L(4s_0 - 1, \psi^2) = 0$ for s_0 and ψ satisfying (9.2) and (9.5). Take any common multiple $\mathfrak{c}$ of 4 and the conductor of ψ, and put

$$g(z, s) = L_\mathfrak{c}(4s - 1, \psi^2)E_k(z, s; \rho, \psi, \mathfrak{c}).$$

By Theorem 4.1, g is finite at s_0. Hence $g(z, s_0)$ belongs to $\mathfrak{N}(\sigma, \lambda)$ by Propositions 7.1 and 5.3. Now, for every $\zeta \in \mathfrak{G}$, we have, by Proposition 5.4.

$$g(z, s) \| \zeta = ac^s L_\mathfrak{c}(4s - 1, \psi^2) y^{su - \rho} + \dots$$

with $a \in \mathbf{C}$ and $0 < c \in \mathbf{R}$. Therefore $g(z, s_0)$ satisfies (9.1*b*). To show that $g(z, s_0) \neq 0$, we consider an element η_0 of G as in [13, (4.10)]. Then the Fourier coefficients of $g \| \Lambda_\sigma^k(\eta_0)$ has been determined in [13, §6]. In particular, its constant term at s_0 is a nonzero constant times $L_\mathfrak{c}(4s_0 - 2, \psi^2)y^q$. Since $-1 < 4s_0 - 2 < 0$, this term is nonvanishing. This completes the proof, since $L_\mathfrak{c}/L$ is nonvanishing for this value.

Proposition 9.2. *Let* s_0 *be a complex number satisfying* (9.2). *Define* Φ_Δ *as in Theorem* 7.7 *for each* Δ *such that a cusp-class of* Δ *is* ρ*-regular if and only if it is* $\bar{\rho}$*-regular. Then a Hecke character* ψ *of* F *satisfying* (9.4*a* or *b*) *and* (9.5) *exists if and only if* $\det \Phi_\Delta(s, \rho)$ *has a pole at* s_0 *for some* Δ. *Moreover, the maximum number of linearly independent cyclopes in* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *with* λ *of* (9.3) *is* $\varkappa - \operatorname{rank} \Phi_\Delta(1 - s_0, \bar{\rho})$.

Proof. By Theorem 7.3 and Remark 7.4, (1), we have $\mathfrak{N}(\sigma, \lambda, \Delta) = \mathcal{E}(1 - s_0, \bar{\rho}, \Delta)$. Given (a row vector) $c \in \mathbf{C}^Y$, we have

$$\sum_{\xi} c_\xi E(1 - s_0, \bar{\rho}, \xi) \Big\| \eta^{-1} = c_\eta y^q + \sum_{\xi} c_\xi g_{\xi\eta}(1 - s_0) y^p + \ldots$$

with the same notation as in (7.8b). This gives a nontrivial cyclops of exponent q if and only if $c \neq 0$ and $c\Phi_\Delta(1 - s_0, \bar{\rho}) = 0$, which proves the last assertion. The first assertion follows from this fact, Theorem 9.1, (7.7b), and Proposition 3.4.

10. Appendix I: Whittaker functions

For $y > 0$ and $(\alpha, \beta) \in \mathbf{C}^2$, we put

$$\tau(y, \alpha, \beta) = \int_0^\infty e^{-yt}(1 + t)^{\alpha - 1} t^{\beta - 1}\, dt. \tag{10.1}$$

This is convergent if $\mathrm{Re}(\beta) > 0$. We have obviously

$$\left(\frac{\partial}{\partial y}\right)\tau(y, \alpha, \beta) = -\tau(y, \alpha, \beta + 1). \tag{10.2}$$

Since $(1 + t)^\alpha = (1 + t)^{\alpha - 1}(1 + t)$, we obtain

$$\tau(y, \alpha + 1, \beta) = \tau(y, \alpha, \beta) + \tau(y, \alpha, \beta + 1). \tag{10.3}$$

Integration by parts shows

$$\beta\tau(y, \alpha + 1, \beta) = y\tau(y, \alpha + 1, \beta + 1) - \alpha\tau(y, \alpha, \beta + 1). \tag{10.4}$$

From these formulas, we obtain easily

$$\left\{ y\left(\frac{\partial}{\partial y}\right)^2 + (\alpha + \beta - y) \cdot \frac{\partial}{\partial y} - \beta \right\} \tau(y, \alpha, \beta) = 0. \tag{10.5}$$

Let us now put

$$V(y, \alpha, \beta) = e^{-y/2} y^\beta \Gamma(\beta)^{-1} \tau(y, \alpha, \beta). \tag{10.6}$$

From (10.3), we obtain

$$V(y, \alpha + 1, \beta) = V(y, \alpha + 1, \beta + 1) - \alpha y^{-1} V(y, \alpha, \beta + 1). \tag{10.7}$$

This shows that V can be continued as a holomorphic function in (α, β) to the whole $\mathbf{C}^2$. Now we have

$$y^\beta \tau(y, \alpha, \beta) = \int_0^\infty e^{-t}(1 + y^{-1}t)^{\alpha - 1} t^{\beta - 1}\, dt.$$

Therefore we see, at least for $\mathrm{Re}(\beta) > 0$, that

$$\lim_{y\to\infty} e^{y/2} V(y, \alpha, \beta) = 1. \tag{10.8}$$

Since this is consistent with (10.7), we can easily verify that (10.8) holds uniformly for (α, β) in any compact subset of $\mathbf{C}^2$.

We now consider a differential equation

$$y^2 f''(y) + \sigma y f'(y) + (\lambda + A\sigma y - A^2 y^2) f(y) = 0 \tag{10.9}$$

with $A \in \mathbf{R}^\times$, $(\sigma, \lambda) \in \mathbf{C}^2$, and $0 < y \in \mathbf{R}$.

Proposition 10.1. *Let α and β be complex numbers such that $\alpha - \beta = \sigma$ and $\beta(1 - \alpha) = \lambda$. For fixed α, β, and A, define a function f_A by*

$$f_A(y) = \begin{cases} V(2Ay, \alpha, \beta) & \text{if } A > 0, \\ |2Ay|^{-\sigma} V(-2Ay, \beta, \alpha) & \text{if } A < 0. \end{cases}$$

Then f_A is a solution of (10.9). *Moreover, if f is a solution of* (10.9) *and $f(y) = O(y^B)$ with $B \in \mathbf{R}$ when $y \to \infty$, then f is a constant multiple of f_A.*

PROOF. That f_A is a solution of (10.9) follows from (10.5) in a straightforward way. Let f be a solution of (10.9) such that $f(y) = O(y^B)$. Then

$$(y^\sigma f')' = y^\sigma(f'' + \sigma y^{-1} f') = y^\sigma(A^2 - A\sigma y^{-1} - \lambda y^{-2}) f = O(y^C)$$

with $C \in \mathbf{R}$ when $y \to \infty$. It follows that $y^\sigma f'$, as well as f', is $O(y^D)$ with $D \in \mathbf{R}$. Now put $h = f_A f' - f_A' f$. Then $h' = f_A f'' - f_A'' f = -\sigma y^{-1} h$, and hence $h = ay^{-\sigma}$ with a constant a. Since both f_A and f_A' are $O(e^{-|A|y/2})$ as can easily be seen from (10.8) and (10.2), we see that $a = 0$. Therefore f is a constant multiple of f_A.

In Proposition 10.1, we can change (α, β) for $(1 - \beta, 1 - \alpha)$ without changing σ and λ. Therefore $V(2Ay, 1 - \beta, 1 - \alpha)$ for $A > 0$ is a solution of (10.9), and hence must be a constant multiple of f_A. In view of (10.8), we thus obtain

$$V(y, 1 - \beta, 1 - \alpha) = V(y, \alpha, \beta). \tag{10.10}$$

We note also that, given a compact subset K of $\mathbf{C}^2$, there exist two positive constants B and C depending only on K such that

$$|V(y, \alpha, \beta)| \leqslant Ce^{-y/2}(1 + y^{-B}) \qquad \text{for } y > 0 \text{ and } \qquad (\alpha, \beta) \in K. \tag{10.11}$$

This can be proved in an elementary way by means of (10.1) and (10.8); for details, see [11, pp. 282-283].

With σ, λ, A, and f_A as in Proposition 10.1, define a function φ_A on H by

$$\varphi_A(x+iy,\sigma,\lambda) = e^{iAx}f_A(y). \tag{10.12}$$

Further define operators ϵ and δ^σ on H by $\epsilon f = -y^2\partial f/\partial\bar{z}$ and $\delta^\sigma f = = y^{-\sigma}\partial(y^\sigma f)/\partial z$. Then we can easily verify, employing (10.2), (10.3), and (10.4), that

$$\epsilon\varphi_A(z,\sigma,\lambda) = \begin{cases} \lambda(4Ai)^{-1}\varphi_A(z,\sigma-2,\lambda+2-\sigma) & \text{if } A>0,\\ (4Ai)^{-1}\varphi_A(z,\sigma-2,\lambda+2-\sigma) & \text{if } A<0, \end{cases} \tag{10.13a}$$

$$\delta^\sigma\varphi_A(z,\sigma,\lambda) = \begin{cases} iA\varphi_A(z,\sigma+2,\lambda+\sigma) & \text{if } A>0,\\ (\lambda+\sigma)iA\varphi_A(z,\sigma+2,\lambda+\sigma) & \text{if } A<0. \end{cases} \tag{10.13b}$$

11. Appendix II: Proofs of Propositions 2.1, 2.2, 2.3, and 4.4

Throughout this section, we put $U_F = \{a \in \mathfrak{g}^\times \mid a \gg 0\}$, $\mu(y) = \operatorname{Min}\{y_v \mid v \in \mathbf{a}\}$ for $y \in \mathbf{R}^\mathbf{a}$, $|z| = (|z_v|)_{v\in\mathbf{a}}$ and $\{z\} = \sum_{v\in\mathbf{a}} z_v$ for $z \in \mathbf{C}^\mathbf{a}$. For example, we have $\mathbf{e}_\mathbf{a}(i|h|y) = \exp(-2\pi\{|h|y\})$ for $h \in F$ and $0 \ll y \in \mathbf{R}^\mathbf{a}$.

Lemma 11.1 *Let $\mathfrak{a}$ be a fractional ideal of F, and β an element of $\mathbf{R}^\mathbf{a}$. Then there exist positive constants A, B, and C such that*

$$\sum_{0\neq h\in\mathfrak{a}} |h|^\beta \mathbf{e}_\mathbf{a}(i|h|y) \leqslant A(1+\mu(y)^{-B})\exp(-Cy^{u/n}).$$

for $0 \ll y \in \mathbf{R}^\mathbf{a}$.

Proof. Let $\|h\| = \{h^2\}^{1/2}$. If $c \geqslant 0$, then $|h_v|^c \leqslant \|h\|^c$, and

$$|h_v|^{-c} = \left|h^{-u}\prod_{w\neq v} h_w\right|^c \leqslant N(\mathfrak{a})^{-c}\|h\|^{(n-1)c} \quad \text{for} \quad 0 \neq h \in \mathfrak{a}.$$

Therefore $|h|^\beta \leqslant A\|h\|^b$ for $0 \neq h \in \mathfrak{a}$ with positive constants A and b. Now $\{|h|y\} \geqslant n|hy|^{u/n} \geqslant nN(\mathfrak{a})^{1/n}y^{u/n}$ for such h. Put $C = \pi nN(\mathfrak{a})^{1/n}$. Then $2\pi\{|h|y\} \geqslant \pi\{|h|y\} + Cy^{u/n} \geqslant \pi\mu(y)\|h\| + Cy^{u/n}$. Therefore we have $\sum_h |h|^\beta\exp(-2\pi\{|h|y\}) \leqslant A\cdot\exp(-Cy^{u/n})\sum_h \|h\|^b\exp(-\pi\mu(y)\|h\|)$. Since there are only finitely many h's in $\mathfrak{a}$ such that $\|h\| < 1$, we may assume, changing A for a larger constant, that b is a positive integer. For $0 < m \in \mathbf{Z}$, let p_m be the number of elements h of $\mathfrak{a}$ such that $m-1 < \|h\| \leqslant m$. Then $p_m \leqslant Dm^{n-1}$ with a constant D, and the last sum $\sum_h$ is majorized by $D\sum_{m=1}^\infty m^{b+n-1}e^{t-mt}$ with $t = \pi\mu(y)$. This is $\leqslant E(1+t^{-b-n})$ with a constant E, which completes the proof.

Lemma 11.2. *Let Δ be a congruence subgroup of $\mathfrak{G}$, and f a continuous function on $H^{\mathbf{a}}$ satisfying* (2.7*a*, *c*). *Then there exist two positive constants A and B such that*

$$(11.1) \qquad |y^{\sigma/2}f(x+iy)| \leqslant A(y^{Bu}+y^{-Bu}) \quad \textit{for all} \quad x+iy \in H^{\mathbf{a}}.$$

Proof. With a compact fundamental domain M of $\mathbf{R}^{\mathbf{a}}/\mathfrak{g}$ and $0 < c \in \mathbf{R}$, put

$$(11.2) \qquad T_c = \{x+iy \mid x \in M, \mu(y) > c\}.$$

Then we can take a finite subset X of $\mathfrak{G}$ so that $H^{\mathbf{a}} = \bigcup_{\beta\in\Delta,\,\xi\in X}\beta\xi(T_c)$. By (2.7*c*), we can find two positive constants A and B such that $|y^{\sigma/2}(f\,\|\,\xi)(x+iy)| \leqslant Ay^{Bu}$ if $\mu(y) > c$ and $\xi \in X$. Given $z = x+iy \in H^{\mathbf{a}}$, take $\beta \in \Delta$ and $\xi \in X$ so that $z = \beta\xi(z')$ with $z' = x'+iy' \in T_c$. Let $\mathrm{pr}(\xi^{-1}) = \begin{pmatrix} * & * \\ p & q \end{pmatrix}$ and $\mathrm{pr}(\beta\xi)^{-1} = \begin{pmatrix} * & * \\ r & s \end{pmatrix}$. To prove our lemma, we may assume that $\mathrm{pr}(\Delta) \subset SL_2(\mathfrak{g})$. Then $r, s \in p\mathfrak{g} + q\mathfrak{g}$. Let D be the smallest of $N(p\mathfrak{g}+q\mathfrak{g})$ for all $\xi \in X$. If $r \neq 0$, we have $|r^u| \geqslant D$. Now $y'^u = y^u|rz+s|^{-2u} \leqslant D^{-2}y^{-u}$, and hence $|y^{\sigma/2}f(z)| = |y'^{\sigma/2}(f\,\|\,\beta\xi)(z')| \leqslant Ay'^{Bu} \leqslant A(D^2y^u)^{-B}$ if $r \neq 0$. When $r = 0$, we have $y'^u = s^{-2u}y^u \leqslant D^{-2}y^u$, so that $|y^{\sigma/2}f(z)| \leqslant A(D^{-2}y^u)^B$. This proves our lemma.

Proof of Proposition 2.1. Given $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$, define b_h as in (2.21). By Lemma 11.2, we see easily that

$$(11.2') \qquad |y^{\sigma/2}b_h W(hy; \sigma, \lambda)| \leqslant A'(y^{Bu}+y^{-Bu})$$

with positive constants A' and B independent of y and h. let U be a subgroup of U_F of finite index such that $\Lambda_\sigma^k(\mathrm{diag}[a, a^{-1}]) \subset \Delta$ for every $a \in U$. Now we can find two positive constants c_1 and c_2 with the following property: given $0 \ll y \in \mathbf{R}^{\mathbf{a}}$, there exists an element a of U such that $c_1y^{u/n} \leqslant a_v^2y_v \leqslant c_2y^{u/n}$ for every $v \in \mathbf{a}$. Hereafter c_m for $m = 3, 4, \ldots$ will denote constants independent of h and y. Given $0 \neq h \in F$, take $a \in U$ so that $c_1|h|^{u/n} \leqslant |a_vh_v| \leqslant c_2|h|^{u/n}$. By (10.8), we can find a constant $d > 1$ so that $|V(g; \alpha_v, \beta_v)| \geqslant 2^{-1}e^{-g/2}$ and $|V(g; \beta_v, \alpha_v)| \geqslant 2^{-1}e^{-g/2}$ if $g \geqslant d$. Put $t = c_1^{-1}d|h|^{-u/n}$. Then $ta_v|h_v| \geqslant d$, so that

$$(11.3) \qquad |(tah)^{\sigma/2}W(tah; \sigma, \lambda)| \geqslant c_3|tah|^{\sigma'/2}e^{-2\pi t\{|ah|\}},$$

where $\sigma'_v = \mathrm{sgn}(h_v)\sigma_v$. Taking ta to be y in (11.2′), we find that

$$|(ta)^{\sigma/2}b_hW(tah; \sigma, \lambda)| \leqslant A'(t^{nB}+t^{-nB}),$$

which together with (11.3) shows that

$$|h^{-\sigma/2}b_h| \leqslant c_4(t^{nB}+t^{-nB})|tah|^{-\sigma'/2}e^{2\pi t\{|ah|\}}.$$

Since $t = c_1^{-1}d|h|^{-u/n}$ and $|a_v h_v| \leqslant c_2|h|^{u/n}$, we have $|(tah)_v| \leqslant dc_2/c_1$, and hence $|h^{-\sigma/2}b_h| \leqslant c_5|h|^{Bu}$. This proves (1) of Proposition 2.1. Next, we see from (10.11) that

$$|W(hy;\sigma,\lambda)| \leqslant c_6 \sum_{s\in S} |hy|^s \mathbf{e}_{\mathbf{a}}(ihy) \tag{11.4}$$

with a finite subset S of $\mathbf{R}^{\mathbf{a}}$. Hence, by Lemma 11.1, we obtain

$$y^{\sigma/2} \sum_{h\neq 0} |b_h W(hy;\sigma,\lambda)| \leqslant c_7 \sum_{s\in S} y^s(1+\mu(y)^{-B})\exp(-Cy^{u/n}) \tag{11.5}$$

with constants B and C independen of s. Now the left-hand side is invariant under $y \to a^2y$ with $a \in U$. Given y, take $a \in U$ so that $c_1 y^{u/n} \leqslant (a^2y)_v \leqslant c_2 y^{u/n}$ for every $v \in \mathbf{a}$. Then $\mu(a^2y) \geqslant c_1 y^{u/n}$ and $(a^2y)^s \leqslant c_8(y^{Du} + y^{-Du})$ with D independent of s. Hence, substituting a^2y for y in (11.5), we obtain

$$y^{\sigma/2} \sum_{h\neq 0} |b_h W(hy;\sigma,\lambda)| \leqslant c_9(y^{Eu} + y^{-Eu})\exp(-Cy^{u/n}) \tag{11.6}$$

with a constant E, which proves (2) of Proposition 2.1. Assertion (3) is now an easy consequence of (2) and (2.7c). To prove (4), take $f \in \mathcal{S}(\sigma,\lambda,\Delta)$ and take X as in the proof of Lemma 11.2. Applying (2) to $f\|\xi$ for each $\xi \in X$, we see that $y^{\sigma/2}f$ is bounded on the whole $H^{\mathbf{a}}$. Therefore we can take $B = 0$ in (11.1) and also in (11.2′). Repeating the proof of (1) with $B = 0$, we can conclude that $h^{-\sigma/2}b_h$ is bounded. This completes the proof.

Lemma 11.3. *Let f be a C^∞-function of form* (2.21) *satisfying* (2.7a, b). *Suppose $|b_h| \leqslant p|h|^{qu+\sigma/2}$ for $0 \neq h \in \mathfrak{m}$ with positive constants p and q. Then f satisfies* (2.7c).

PROOF. Applying the above proof of (2) to $f - b_0(y)$, we obtain, from (11.6) that

$$y^{\sigma/2}|f - b_0(y)| \leqslant A(y^{Eu} + y^{-Eu})\exp(-Cy^{u/n}).$$

Since $b_0(y)$ is a linear combination of the functions of Proposition 3.1, we have $y^{\sigma/2}|f(z)| \leqslant A'(y^{Ju} + y^{-Ju})$ on $H^{\mathbf{a}}$ with constants A' and J. Then (2.7c) can easily be verified.

PROOF OF PROPOSITION 2.2. Let $g \in \mathcal{A}(\sigma,\lambda,\Delta)$. It is straightforward to see that $\epsilon_v g$ and $\delta_v^\sigma g$ satisfy (2.7a, b) with modified σ, λ, and Δ as stated in the proposition. To verify (2.7c), take $\xi \in \mathcal{G}_\sigma$. Then $(\epsilon_v g)\,\|\,\xi_* = \epsilon_v(g\|\xi)$ by (2.5b). Since $g\|\xi \in \mathcal{A}(\sigma,\lambda)$, it has an expansion of type (2.21) with b_h as in (1) of Proposition 2.1. By (10.13a), we see that

$$(\epsilon_v g)\,\|\,\xi_* = \epsilon_v b_0 + \sum_{0\neq h} c_h W(hy;\sigma - 2v, \lambda + (2-\sigma_v)v)$$

with c_h satisfying (1) of Proposition 2.1 with $\sigma - 2v$ instead of σ. Therefore, by Lemma 11.3, $\epsilon_v g$ satisfies (2.7c). The assertion for $\delta^\sigma{}_v g$ can be proved in a similar way.

PROOF OF PROPOSITION 2.3. We first note that given two positive integers a and p, and a positive real number $r < 1$, one has

$$\sum_{m=p}^{\infty} m^a x^m \leqq C(a, r) p^a x^p \quad \text{for} \quad 0 \leqslant x \leqslant r \tag{11.7}$$

with a constant $C(a, r)$ independent of p and x. In fact,

$$\sum_{m=p}^{\infty} m^a x^{m-p} = \sum_{n=0}^{\infty} (n+p)^a x^n \leqslant \sum_{i=0}^{\infty} \binom{a}{i} p^{a-i} \sum_{n=0}^{\infty} n^i r^n.$$

Now take X as in the proof of Lemma 11.2. For $f \in \mathcal{S}(\sigma, \lambda, \Delta)$ and $\xi \in X$, put $M_f = \mathrm{Max}|y^{\sigma/2} f|$ and

$$f\|\xi = \sum_h b_{h,\xi} W(hy; \sigma, \lambda)\mathbf{e}_{\mathbf{a}}(hx).$$

Since $|y^{\sigma/2}(f\|\xi)| \leqslant M_f$, we have $|b_{h,\xi}| \leqslant AM_f|h|^{\sigma/2}$ with a constant A independent of f, as can be seen from the proof of (1) and (4) of Proposition 2.1. Fix an integer $p > 1$ and suppose $b_{h,\xi} = 0$ for all $\xi \in X$ and all h such that $\|h\| < p$. Then, by (11.4), we have

$$|y^{\sigma/2} f\|\xi| < BM_f \sum_{s \in S} |hy|^{\sigma/2+s} \mathbf{e}_{\mathbf{a}}(i|h|y)$$

with a constant B independent of f. The same reasoning as in the proof of Lemma 11.1 shows that, for any fixed $q > 0$, we have

$$|y^{\sigma/2} f\|\xi| \leqslant CM_f \sum_{m=p}^{\infty} m^a e^{-\pi m \mu(y)} \quad \text{if} \quad \mu(y) > q$$

with a constant C and a positive integer a independent of f. By (11.7), we have $|y^{\sigma/2} f\|\xi| \leqslant D_q M_f p^a e^{-\pi p \mu(y)}$ for $\mu(y) > q$ with a constant D_q independent of f and p. Take q to be c of (11.2). For every $z \in H^{\mathbf{a}}$, take $\beta \in \Delta$ and $\xi \in X$ so that $z = \beta\xi(z')$ with $z' = x' + iy' \in T_c$. Then

$$|y^{\sigma/2} f(z)| = |y^{\sigma/2} f(\beta\xi(z'))| = |y'^{\sigma/2}(f\|\beta\xi)(z')|,$$

and hence $M_f \leqslant D_q M_f p^a e^{-\pi pq}$. If p is sufficiently large, we obtain $M_f = 0$. This shows that $f = 0$ if $b_{h,\xi} = 0$ for $\|h\| < p$ and for all $\xi \in X$. Thus $\mathcal{S}(\sigma, \lambda, \Delta)$ is finite-dimensional. Now the constant term of an element of $\mathcal{A}(\sigma, \lambda)$ belongs to a 2^n-dimensional space as shown in Section 3, and hence $\mathcal{A}(\sigma, \lambda, \Delta)/\mathcal{S}(\sigma, \lambda, \Delta)$ is finite-dimensional. This completes the proof.

PROOF OF PROPOSITION 4.4. By (4.3) and (4.24), we may restrict E to the functions of type (4.10). Then our first assertion follows immediately from Theorem 4.1. Suppose g is finite at s_0. By (4.4) and analytic continuation, we see that $L_v^\sigma g(z, s_0) = \lambda_v g(z, s_0)$. In order to verify (2.7c) for $g(z, s_0)$, we consider a function D' which is obtained from D of Theorem 4.1 by replacing E by E', where E' is defined by (4.16) in Case I and by [13, (4.10)] in Case II. Then we take a Fourier expansion

$$l(s)D'(z, s) = a_0(s, y) + \sum_{h \neq 0} a_h(s) W(hy; \sigma, \lambda)\mathbf{e_a}(hx), \tag{11.8}$$

where $l(s)$ is the polynomial 1, $s(s-1)$, or $s - \frac{3}{4}$ that cancels the pole(s) of D'. Then for every compact subset K of $\mathbf{C}$, we have $|a_h(s)| \leqslant A|h|^{\sigma/2 + Bu}$ for $s \in K$ with positive constants A and B depending only on D' and K. This follows from the explicit form of $a_h(s)$ given by (4.20) in Case I and by [13, Theorem 6.1] in Case II. Then Lemma 11.3 shows that $l(s)D'(z, s)$ satisfies (2.7c). Put $q(z, s) = l(s)D'(z, s)\|\xi$ with any $\xi \in \mathcal{G}$. Then $q(z, s)$ belongs to $\mathfrak{A}(\sigma, \lambda)$ and satisfies (2.7c) uniformly on K. By a well-known principle, the same type of estimate holds for $\partial^m q/\partial s^m$ for every m. Now we consider a finite linear combination $\sum f_q(s)q(z, s)$ with meromorphic functions f_q on $\mathbf{C}$. We observe that if it is finite at s_0, it satisfies (2.7c). This completes the proof.

References

[1] Hecke, E. Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik, *Abh. Math. Sem. Hamburg*, **5**(1927), 199-224 (Werke, No. 24).

[2] Kloosterman, H. D. Theorie der Eisensteinschen Reihen von mehreren Veränderlichen, *Abh. Math. Sem. Hamburg*, **6** (1928), 163-188.

[3] Maass, H. Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.*, **121** (1949), 141-183.

[4] Maass, H. Die Differentialgleichungen in der Theorie der elliptischen Modulfunktionen, *Math. Ann.*, **125** (1953), 235-263.

[5] Pei, T-y. Eisenstein series of weight 3/2: I, II, *Trans. Amer. Math. Soc.*, **274** (1982), 573-606, **283** (1984), 589-603.

[6] Petersson, H. Über die Entwicklungskoeffizienten der ganzen Modulformen und ihre Bedeutung für die Zahlentheorie, *Abh. math. Sem. Hamburg*, **8** (1931), 215-242.

[7] Roelcke, W. Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene: I, II, *Math. Ann.*, **167** (1966), 292-337, **168** (1967), 261-324.

[8] Shimizu, H. A remark on the Hilbert modular forms of weight 1, *Math. Ann.*, **265** (1983), 457-472.

[9] Shimura, G. The special values of the zeta functions associated with Hilbert modular forms, *Duke Math. J.*, **45** (1978), 637-679.

[10] Shimura, G. The arithmetic of certain zeta functions and automorphic forms on orthogonal groups, *Ann. of Math.,* **111** (1980), 313-375.
[11] Shimura, G. Confluent hypergeometric functions on tube domains, *Math. Ann.,* **260** (1982), 269-302.
[12] Shimura, G. On Eisenstein series, *Duke Math. J.,* **50** (1983), 417-476.
[13] Shimura, G. On Eisenstein series of half-integral weight, *Duke Math. J.,* **52** (1985), 281-314.
[14] Shimura, G. On certain zeta functions attached to two Hilbert modular forms: I. The case of Hecke characters, *Ann. of Math.,* **114** (1981), 127-164.

Goro Shimura
Princeton University
Fine Hall, Washington Road
Princeton, New Jersey 08544
U.S.A.

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 1, N.° 3, 1985

Sur la corrélation des fonctions de Piltz

Etienne Fouvry et Gérald Tenenbaum

1. Résultats et méthodes

Pour k et l entiers supérieurs à 2, on s'intéresse ou nombre $\mathfrak{T}_{k,l}(x)$ de solutions de l'équation

$$m_1 \dots m_l - n_1 \dots n_k = 1 \tag{1}$$

où les m_i et les n_i sont des entiers $\geqslant 1$ vérifiant

$$n_1 \dots n_k \leqslant x.$$

En notant $\tau_k(n)$ la $k^{\text{ième}}$ fonction de Piltz, *i.e.* le nombre de représentations de l'entier n en produit de k entiers $\geqslant 1$, on a facilement la relation

$$\mathfrak{T}_{k,l}(x) = \sum_{n \leq x} \tau_k(n)\tau_l(n+1)$$

La question, non encore résolue, de trouver, pour x tendant vers l'infini, un équivalent de $\mathfrak{T}_{k,l}(x)$ pour $k \geqslant l \geqslant 3$ est un problème très difficile de théorie des nombres, bien que la technique de Vinogradov et Takhtadzhyan [12] laisse espérer y parvenir, dans le cas $k = l = 3$, par le biais de séries d'Eisenstein sur $SL(3, \mathbb{R})$.

Pour $l = 2$, la situation se simplifie; après différents travaux ([5], [8] entre autres), on est parvenu à

$$\mathfrak{T}_2(x) = xP_2(\log x) + O_\epsilon(x^{\frac{2}{3}+\epsilon}) \qquad (\epsilon > 0 \quad \text{quelconque}) \tag{2}$$

(on a posé $\mathfrak{T}_k(x) = \mathfrak{T}_{k,2}(x)$ et P_k est un polynôme de degré k). L'égalité (2), due à Deshouillers et Iwaniec [4], est basée sur la technique de Fourier et des majorations en moyenne de sommes de Kloosterman [3]. Cette méthode se prolonge à $k = 3$, donnant, par conséquent, la relation [2]:

$$\mathfrak{T}_3(x) = xP_3(\log x) + O(x^{1-\delta}) \quad (\delta \text{ constante} > 0).$$

Signalons qu'on peut aborder aussi cette question par l'étude de la répartition de la fonction τ_3 dans les progressions arithmétiques, en combinant la méthode de Burgess et la majoration individuelle de sommes de Kloosterman de dimension 2 —conséquence de la résolution, par Deligne, de la conjecture de Weil [7]. Mais ces méthodes semblent s'éteindre pour $k \geqslant 4$.

En (1), apparaît un problème additif binaire, pour lequel la méthode de dispersion, due à Linnik, est efficace. Elle donne ainsi, pour $k \geqslant 2$, l'estimation [9]

$$\mathfrak{T}_k(x) = xP_k(\log x) + O_\epsilon(x(\log x)^\epsilon). \tag{3}$$

Le terme d'erreur de (3) a été réduit à

$$O(x(\log\log x)^{c'_k}(\log x)^{-1})$$

(c'_k ne dépend que de k) par Mothashi [10], qui en plus de la dispersion, se sert de la méthode du cercle et du grand crible (il obtient ainsi une extension du théorème de Bombieri-Vinogradov au produit de convolution de deux suites arithmétiques).

L'objet de cet article est de donner la démonstration du théorème suivant, annoncé dans ([6], corollaire 4):

Théorème 1. *Pour $k \geqslant 4$, il existe $c = c(k) > 0$, tel qu'on ait l'égalité*

$$\mathfrak{T}_k(x) = xP_k(\log x) + O(x \exp(-c(\log x)^{\frac{1}{2}})).$$

La démonstration se scinde en deux parties; dans la première, on montre l'égalité

$$\mathfrak{T}_k(x) = \mathfrak{T}'_k(x) + O(x \exp(-c\mathfrak{L}^{\frac{1}{2}})) \tag{4}$$

avec $\mathfrak{L} = \log x$ et $\mathfrak{T}'_k(x) = 2 \sum_{q \le \sqrt{x}} \frac{1}{\varphi(q)} \sum_{\substack{q^2 \le n \le x \\ (n,q)=1}} \tau_k(n)$.

Le terme principal $\mathfrak{T}'_k(x)$ sera évalué dans la deuxième partie, par des techniques d'analyse complexe (Théorème 2).

2. Démonstration de (4)

Soit $\delta(n)$ la fonction caractéristique de l'ensemble des carrés. On a

$$\begin{aligned}\mathfrak{T}_k(x) &= \sum_{n \le x} \tau_k(n)\tau_2(n+1) \\ &= \sum_{n\le x} \tau_k(n)\Bigg(2 \sum_{\substack{q \mid n+1 \\ q^2 \le n}} 1 + \delta(n+1)\Bigg) \\ &= 2 \sum_{q \le \sqrt{x}} \sum_{\substack{q^2 \le n \le x \\ n \equiv -1[q]}} \tau_k(n) + O_\epsilon(x^{\frac{1}{2}-\epsilon}).\end{aligned}$$

La démonstration de (4) se ramène à la majoration

$$S_k(x) = O(x \exp(-c\mathfrak{L}^{\frac{1}{2}})) \tag{5}$$

avec

$$S_k(x) = \sum_{q \le \sqrt{x}} \Bigg(\sum_{n_1 \dots n_k \equiv -1[q]} 1 - \frac{1}{\varphi(q)} \sum_{(n_1 \dots n_k, q) = 1} 1 \Bigg) \tag{6}$$

où les variables $n_1 \dots n_k$ satisfont la condition

$$q^2 \leqslant n_1 \dots n_k \leqslant x. \tag{7}$$

La quantité $S_k(x)$ se place naturellement dans le cadre des problèmes liés aux extensions du théorème de Bombieri-Vinogradov; nous utiliserons des résultats de [6], créés dans le but d'améliorer le terme d'erreur dans le problème des diviseurs de Titchmarsh (ces résultats ont été aussi trouvés, de façon indépendante, par Bombieri, Friedlander et Iwaniec [1]).

a) Préparation des variables $q, n_1, \dots, n_k$

On se propose de rendre les variables indépendantes dans la condition (7); on effectue pour cela un découpage des intervalles de variation par une technique classique (voir [1], [6] par exemple). On pose

$$\Delta = 1 + \exp(-\eta\mathfrak{L}^{\frac{1}{2}}) \qquad (\eta > 0 \quad \text{sera fixé plus tard})$$

et on note $(N_1, \dots, N_k)$ un k-uplet de nombres de la forme

$$\Delta^{\nu_i} \quad (\nu_i = 0, 1, 2, \dots; 1 \leqslant i \leqslant k)$$

vérifiant $N_1 \dots N_k \leqslant x\Delta^{-k}$. Il y a ainsi $O(\exp(2k\eta\mathfrak{L}^{\frac{1}{2}}))$ tels k-uplets qui donnent la décomposition

$$S_k(x) = \sum_{(N_1, \dots, N_k)} S_k(N_1, \dots, N_k) + O\bigg(x \exp\Big(-\frac{\eta}{2}\mathfrak{L}^{\frac{1}{2}}\Big)\bigg)$$

où $S_k(N_1, \ldots, N_k)$ est défini par la formule (6), à la différence près que la condition (7) devient

$$q^2 \leqslant n_1 \ldots n_k, \qquad N_i \leqslant n_i < N_i\Delta \qquad (1 \leqslant i \leqslant k).$$

Le terme d'erreur provient de la contribution des $(n_1, \ldots, n_k)$ vérifiant

$$n_1 \ldots n_k \leqslant x, \qquad \Delta^{\nu_i} \leqslant n_i < \Delta^{\nu_i + 1} \qquad (1 \leqslant i \leqslant k) \quad \text{et} \quad \Delta^{\nu_1 + \ldots + \nu_k + k} > x,$$

ces conditions entraînent l'inégalité

$$x \geqslant n_1 \ldots n_k \geqslant x\Delta^{-k} = x(1 - O(\exp(-\eta\mathcal{L}^{\frac{1}{2}})))$$

qui permet d'appliquer le lemme suivant.

Lemme 1 ([9] page 24). *Pour tout k, il existe k_1 tel qu'on ait l'estimation*

$$\sum_{\substack{X - Y < n \leq X \\ n \equiv a[q]}} \tau_k(n) = O_k(Yq^{-1}(\log X)^{k_1})$$

uniformément pour $(a, q) = 1$, $q \leqslant Y^{0,8}$ et $X^{0,9} \leqslant Y \leqslant X$.

On fait de même pour la variable q, pour parvenir finalement à

$$S_k(x) = \sum_{(N_1, \ldots, N_k)} \tilde{S}_k(N_1, \ldots, N_k) + O\left(x \exp\left(-\frac{\eta}{2}\mathcal{L}^{\frac{1}{2}}\right)\right)$$

avec

$$\tilde{S}_k(N_1, \ldots, N_k) = \sum_{q \leq (N_1 \ldots N_k)^{1/2}} \left(\sum_{n_1 \ldots n_k \equiv -1[q]} 1 - \frac{1}{\varphi(q)} \sum_{(n_1 \ldots n_k, q) = 1} 1\right)$$

où les n_i vérifient

$$N_i \leqslant n_i < N_i\Delta \qquad (1 \leqslant i \leqslant k).$$

Pour démontrer (5), il suffit de montrer qu'il existe $\eta_1 > 0$ tel qu'on ait la majoration

$$\tilde{S}(N_1, \ldots, N_k) = O(x \exp(-\eta_1\mathcal{L}^{\frac{1}{2}})) \tag{8}$$

et de choisir $\eta = \dfrac{\eta_1}{4k}\cdot$

Remarquons que le Lemme 1 entraîne qu'on peut se restreindre à

$$N_1 \ldots N_k \geqslant x^{0,99}. \tag{9}$$

La démonstration est différente, suivant qu'il y a, ou non, un N_i dans l'intervalle $\mathcal{I} = [x^{1/1000k}, x^{0,3}]$.

b) S'il y a un N_i dans $\mathcal{I}$

La démonstration du lemme suivant ([6] proposition 1′) repose sur un calcul de dispersion et des majorations en moyenne de sommes de Kloosterman ([3]). On montre:

Lemme 2. *Soient a un entier non nul, $\varkappa$ un entier $\geqslant 1$, $\epsilon > 0$, M et N deux réels vérifiant: $M, N \geqslant 2$ et $M^{\epsilon} \leqslant N \leqslant M^{\frac{1}{2}-\epsilon}$. On désigne aussi par $\mathfrak{M}$ et $\mathfrak{N}$ deux intervalles inclus dans $[M, 2M[$ et $[N, 2N[$ et par (α_m) une suite de réels vérifiant $|\alpha_m| \leqslant \tau_{\varkappa}(m)$.*

Sous ces conditions, il existe $c_1 = c_1(\varkappa, \epsilon) > 0$ tel qu'on ait la majoration

$$\sum_{\substack{q \leq Q \\ (q,a)=1}} \left(\sum_{\substack{m\in\mathfrak{M},\, n\in\mathfrak{N} \\ mn \equiv a[q]}} \alpha_m - \frac{1}{\varphi(q)} \sum_{\substack{m\in\mathfrak{M},\, n\in\mathfrak{N} \\ (mn,q)=1}} \alpha_m \right) = O_{\varkappa,\epsilon,a} \quad (MN \exp(-c_1(\log MN)))$$

dès qu'on a

$$Q \leqslant M^{\frac{3}{4}-\epsilon}.$$

En donnant à N la valeur du N_i appartenant à $\mathcal{I}$ et à M celle du produit des N_j restants, le Lemme 2 fournit directement la majoration (8).

c) Si aucun des N_i ne se trouve dans $\mathcal{I}$

La définition de $\mathcal{I}$ et la restriction (9) impliquent qu'un, deux ou trois des N_i sont supérieurs à $x^{0,3}$. Le cas le plus difficile est celui où ils sont au nombre de trois; on peut alors supposer

$$N_3 \geqslant N_2 \geqslant N_1 > x^{0,3}.$$

On écrit $\tilde{S}_k(N_1, \ldots, N_k)$ sous la forme

$$\tilde{S}_k(N_1, \ldots, N_k) = \sum_{q \leq (N_1 \ldots N_k)^{1/2}} \left(\sum_{mn_1n_2n_3 \equiv -1[q]} \alpha_m - \frac{1}{\varphi(q)} \sum_{(mn_1n_2n_3, q)=1} \alpha_m \right)$$

avec $|\alpha_m| \leqslant \tau_k(m)$, $(N_4 \ldots N_k) \leqslant m < (N_4 \ldots N_k)\Delta^{k-3}$ et

$$N_i \leqslant n_i < N_i\Delta \qquad (1 \leqslant i \leqslant 3).$$

On est ainsi ramené à la situation traitée dans [6], §4, qu'il est donc inutile de reprendre intégralement mais qu'on se contente de rappeler. On commence par rendre les variables q, n_1, n_2, n_3 «lisses» grâce aux fonctions $b_{l,\Delta'}$ du Lemme 2 de [6] (avec $\Delta' = 1 + \exp(-100\eta\mathcal{L}^{\frac{1}{2}})$), puis on applique la technique de Fourier à la grande variable n_3 et on fait appel aux majorations de sommes de Kloosterman déjà évoquées pour profiter des compensations sur q, n_1 et n_2 (la variable m est inférieure à $x^{1/1000}$). On obtient encore l'inégalité (8), ce qui termine la démonstration de (5).

3. Le terme principal

Dans cette section, nous nous proposons d'évaluer le terme principal $\mathfrak{T}'_k(x)$ de (4). Nous établissons le théorème suivant.

Theoreme 2. *Il existe un polynôme de degré k, P_k, tel que l'on ait pour $k \geqslant 1$ et $\epsilon > 0$*

$$\mathfrak{T}'_k(x) = xP_k(\log x) + O_{k,\epsilon}(x^{1-\beta_k+\epsilon}) \tag{10}$$

avec

$$\beta_k = \min\left(\frac{1}{4}, \frac{1}{2k}\right).$$

Compte tenu de (4), cela suffira à établir le Théorème 1.

Nous n'avons pas cherché à optimiser la valeur de β_k en fonction de k. Les coefficients du polynôme P_k sont explicitement calculables à partir des formules (17), (18), (20), (24), (25).

Le principe de la démonstration peut être exposé comme suit. On a

$$\mathfrak{T}'_k(x) = 2\sum_{n\leq x} \tau_k(n)F(n)$$

avec

$$F(n) = \sum_{\substack{q\leq\sqrt{n}\\(q,n)=1}} \frac{1}{\varphi(q)}.$$

Nous établirons par intégration complexe (Lemme 5) la formule asymptotique

$$F(n) = hg(n)\left[\tfrac{1}{2}\log n + \lambda(n) + h'\right] + O_\epsilon(n^{-\frac{1}{4}+\epsilon}) \tag{11}$$

où g est une fonction multiplicative, λ une fonction additive, et h, h' des constantes, qui seront explicitées. Nous montrerons ensuite que, pour chaque fonction multiplicative Ψ d'une certaine classe $\mathcal{C}$, la somme $\sum_{n \leq x} \Psi(n)\tau_k(n)$ posséde un développement asymptotique du type (10) avec un polynôme dépendant explicitement de Ψ, de degré $k-1$. Les fonctions $g(n)\exp\{z\lambda(n)\}$, $z \in \mathbb{C}$, $|z| \leqslant 1$, sont dans $\mathcal{C}$. Cela permet de traiter les trois termes principaux apparaissant dans (11): le terme en $g(n)$ découle du cas général, le terme en $g(n)\log n$ relève d'une intégration par parties, et, le terme en $g(n)\lambda(n)$, interpréte comme la dérivée en $z = 0$ de $g(n)\exp\{z\lambda(n)\}$, est calculé par la formule de Cauchy.

Avant d'aborder la démonstration, fixons quelques notations et conventions.

La lettre p est exclusivement réservée pour dénoter un nombre premier. La lettre s désigne une variable complexe et σ, t, sont implicitement définis par $s = \sigma + it$.

Soit $\alpha_j, j \geqslant 0$, la dérivée d'ordre j à l'origine de $s\zeta(s+1)$. Ainsi $\alpha_0 = 1$, $\alpha_1 = \gamma$ (la constante d'Euler) et

$$\alpha_j = (-1)^j \int_1^\infty \{t\}(\log t)^{j-1} \frac{dt}{t^2}, \qquad (j \geqslant 2).$$

Nous notons $A_j(k)$ la dérivée d'ordre j en $s = 0$ de $\{s\zeta(s+1)\}^k$, *i.e.*

$$A_j(k) = \sum_{j_1 + \cdots + j_k = j} \binom{j}{j_1, j_2, \ldots, j_k} \alpha_{j_1} \ldots \alpha_{j_k}, \qquad (j, k \geqslant 1).$$

On introduit les séries de Dirichlet

$$f(n;s) = \sum_{\substack{q=1 \\ (q,n)=1}}^{\infty} \varphi(q)^{-1} q^{-s} = \prod_{p \nmid n} \left(1 + \frac{p^{-s}}{(p-1)(1-p^{-s-1})}\right)$$

$$h(s) = \prod_p \left(1 + \frac{p^{-s}}{p(p-1)}\right), \quad g(n;s) = \prod_{p \mid n} \left(1 + \frac{p^{-s}}{(p-1)(1-p^{-s-1})}\right)^{-1}$$

de sorte que

$$f(n;s) = g(n;s)h(s)\zeta(s+1) \tag{12}$$

On remarque que $h(s)$ est holomorphe et uniformément bornée pour $\sigma \geqslant \sigma_0 > -1$ et que $g(n;s)$ est, pour chaque n, méromorphe sur $\mathbb{C}$.

On pose

$$h = h(0) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = 1{,}9436\ldots, \qquad h' = \gamma + \frac{h'(0)}{h(0)},$$

et l'on définit les fonctions arithmétiques g et λ par

$$\left.\begin{aligned} g(n) = g(n;0) &= \prod_{p|n}\left(1-\left(p-1+\frac{1}{p}\right)^{-1}\right), \\ \lambda(n) = \frac{g'(n;0)}{g(n;0)} &= \sum_{p|n}\frac{p^2(p^2-p+1)}{(p-1)^5}\log p. \end{aligned}\right\} \quad (n \geqslant 1)$$

Lemme 3. *Pour chaque* $\epsilon > 0$, *il existe une constante* $\beta = \beta(\epsilon) > 0$ *telle que l'on ait*

$$|g(n;s)| \leqslant \prod_{p|n}(1+\beta p^{-\sigma-1}), \qquad (\sigma \geqslant -1+\epsilon, n \geqslant 1).$$

DÉMONSTRATION. Posons $G_p(z) = 1 + \dfrac{z}{(p-1)(1-z/p)}$, de sorte que

$$g(n;s) = \prod_{p|n} G_p(p^{-s})^{-1}, \qquad (n \geqslant 1).$$

Le seul zéro de $G_p(z)$ est $z = -p(p-1)$. il existe donc une fonction $G(p)$, dépendant de ϵ, telle que

$$|G_p(z)| \geqslant G(p), \qquad (|z| \leqslant p^{1-\epsilon}). \tag{13}$$

De plus, on a pour $|z| \leqslant p^{1-\epsilon}$, $G_p(z) = 1 + O_\epsilon(z/p)$. Cela implique l'existence d'un $p_0 = p_0(\epsilon)$ et d'un $\beta = \beta(\epsilon)$ tels que

$$|G_p(z)| \geqslant (1+\beta|z|/p)^{-1}, \qquad (p > p_0, |z| \leqslant p^{1-\epsilon}) \tag{14}$$

On déduit de (13) et (14) que l'on a

$$|g(n;s)| \leqslant \prod_{\substack{p|n\\ p\leq p_0}} G(p)^{-1} \prod_{\substack{p|n\\ p>p_0}} (1+\beta p^{-\sigma-1}), \qquad (n \geqslant 1).$$

Si β a été choisi suffisamment grand, cela implique la majoration souhaitée.

Le lemme suivant est une variante de la formule de Perron effective prouvée dans [11; Lemma 3.12]. Nous laissons la démonstration au lecteur.

Lemme 4. *Pour* $n \geqslant 1$, *posons* $n_1 = [\sqrt{n}] + \frac{1}{2}$, $\chi = \frac{1}{\log 2n}$. *On a uniformément pour* $T \geqslant 1$

$$F(n) = \frac{1}{2i\pi}\int_{\chi-iT}^{\chi+iT} f(n;s)\frac{n_1^s}{s}\,ds + O\left(\frac{\log n}{T} + \frac{\log\log n}{\sqrt{n}}\right).$$

Lemme 5. *Soit* $\epsilon > 0$. *On a*

$$F(n) = hg(n)\left[\tfrac{1}{2}\log n + \lambda(n) + h'\right] + O_\epsilon(n^{-\frac{1}{4}+\epsilon}). \tag{15}$$

DÉMONSTRATION. On applique le lemme 4 en déplaçant la droite d'intégration vers la gauche jusqu'à $\sigma = -1 + \epsilon$. Le résidu en $s = 0$ fournit le terme principal à $O(n^{-\frac{1}{2}})$ près. L'intégrale sur le contour déformé est majorée en utilisant l'estimation

$$h(s)\zeta(s+1) \ll_\epsilon 1 + |t|^{-\sigma+\epsilon}, \quad (\sigma \geqslant -1 + \epsilon)$$

et le lemme 3. Le choix optimal $T = n^{\frac{1}{4}}$ fournit le résultat annoncé.

Corollaire. *Soit* $\epsilon > 0$. *On a*

$$\mathfrak{T}'_k(x) = 2h \sum_{n \leq x} g(n)\tau_k(n)\left[\tfrac{1}{2}\log n + \lambda(n) + h'\right] + O_{\epsilon,k}(x^{\frac{3}{4}+\epsilon}).$$

Lemme 6. *Soit* $\delta > 0$ *fixé et soit* Ψ *une fonction arithmétique fortement multiplicative satisfaisant à*

$$|\Psi(p) - 1| \leqslant Cp^{-\delta}$$

pour tout p, *avec* $C > 0$. *Alors la fonction* $\theta_k(n)$ *définie par*

$$\theta_k = \Psi\tau_k * \tau_{-k}$$

satisfait

$$|\theta_k(n)| \leqslant C^{\omega(n)}\tau_k(n)n^{-\delta/k}, \quad (n \geqslant 1).$$

En particulier, la série de Dirichlet $\Sigma\theta(n)n^{-s}$ *est absolument convergente pour* $\sigma > 1 - \delta/k$.

DÉMONSTRATION. On a (formellement)

$$\begin{aligned}
\sum_{n=1}^{\infty} \theta_k(n)n^{-s} &= \zeta(s)^{-k} \prod_p \left(1 + \Psi(p)\sum_{\nu=1}^{\infty}\binom{k+\nu-1}{\nu}p^{-\nu s}\right)\\
&= \prod_p (1-p^{-s})^k(1 + \Psi(p)[(1-p^{-s})^{-k} - 1])\\
&= \prod_p ((1-p^{-s})^k + \Psi(p)[1 - (1-p^{-s})^k])\\
&= \prod_p \left(1 + (1-\Psi(p))\sum_{\nu=1}^{k}(-1)^\nu\binom{k}{\nu}p^{-\nu s}\right)
\end{aligned}$$

θ_k est donc la fonction multiplicative définie par

$$\theta_k(p^\nu) = (1-\Psi(p))(-1)^\nu\binom{k}{\nu}, \quad (\nu \geqslant 1, p \geqslant 2),$$

d'où:

$$|\theta_k(p^\nu)| \leqslant Cp^{-\delta}\binom{k}{\nu} \leqslant Cp^{-\delta\nu/k}\binom{k+\nu-1}{\nu} = Cp^{-\delta\nu/k}\tau_k(p^\nu).$$

La seconde inégalité utilise le fait que $\theta_k(p^\nu) = 0$ pour $\nu > k$. Cela implique la majoration souhaitée.

Lemme 7. *Sous les hypothèses du Lemme 6, posons*

$$\eta = \delta/(1+\delta), \quad et \quad F_k(s) = \sum_{n=1}^{\infty} \theta_k(n) n^{-s}.$$

On a pour tout $\epsilon > 0$

$$\sum_{n \leq x} \Psi(n)\tau_k(n) = xQ_{k-1}(\log x) + O_{\epsilon,k}(x^{1-\eta/k+\epsilon}) \tag{16}$$

où $Q_{k-1}(\xi)$ *est le polynôme de degré* $(k-1)$ *défini par*

$$Q_{k-1}(\xi) = \frac{1}{(k-1)!} \sum_{j+r+m+q=k-1} (-1)^q q! \binom{k-1}{j,r,m,q} A_j(k) F_k^{(r)}(1) \xi^m. \tag{17}$$

DÉMONSTRATION. On a

$$\sum_{n=1}^{\infty} \Psi(n)\tau_k(n) n^{-s} = \zeta(s)^k F_k(s)$$

où, d'après le lemme 6, $F_k(s)$ est une série de Dirichlet absolument convergente pour $\sigma > 1 - \delta/k$. On peut donc appliquer la formule de Perron effective [11; Lemma 3.12] sous la forme

$$\sum_{n \leq x} \Psi(n)\tau_k(n) = \frac{1}{2i\pi}\int_{c-iT}^{c+iT} \frac{\zeta(s)^k F_k(s)}{s} x^s\, ds + O_{\epsilon,k}\left(\frac{x^{1+\epsilon}}{T}\right),$$

avec $c = 1 + 1/\log x$.

Le résidu en $s = 1$ est égal à $xQ_{k-1}(\log x)$. On obtient le résultat annoncé en déplaçant le segment d'intégration vertical vers la gauche jusqu'à $\sigma = 1 - \delta/k + \epsilon$, en utilisant la majoration classique de $|\zeta(s)|$ dans la bande critique, et, en choisissant finalement la valeur optimale $T = x^{\eta/k}$. Nous omettons les détails de cette manipulation classique.

Corollaire 1. *Soit* U_{k-1} *le polynôme de degré* $k-1$ *obtenu en choisissant dans* (17) $\Psi = g$. *On a*

$$\sum_{n \leq k} g(n)\tau_k(n) = xU_{k-1}(\log x) + O_{\epsilon,k}(x^{1-1/(2k)+\epsilon}). \tag{18}$$

C'est immédiat puisque g satisfait les hypothèses du Lemme 6 avec $\delta = 1$, $C = 2$.

Corollaire 2. *Il existe un polynôme de degré k, V_k, tel que l'on ait*

$$\sum_{n \le x} g(n)\tau_k(n) \log n = xV_k(\log x) + O_{\epsilon,k}(x^{1-1/(2k)+\epsilon}). \tag{19}$$

DÉMONSTRATION. Soit $\Lambda(x)$ le membre de gauche de (18). Le membre de gauche de (19) vaut

$$\int_1^x \log y \, d\Lambda(y) = \Lambda(x)\log x - \int_1^x \Lambda(y)\frac{dy}{y}\cdot$$

On obtient donc le résultat annoncé avec

$$V_k(\xi) = \xi U_{k-1}(\xi) - e^{-\xi}\int_0^\xi e^\tau U_{k-1}(\tau)\,d\tau \tag{20}$$

Lemme 8. *Pour chaque nombre complexe z de module $\leqslant 1$, il existe un polynôme $Q_{k-1}(\xi; z)$ de degré $k-1$, dont les coefficients dépendent analytiquement de z, tel que l'on ait uniformément en z, $|z| \leqslant 1$,*

$$\sum_{n \le x} g(n)e^{z\lambda(n)}\tau_k(n) = xQ_{k-1}(\log x; z) + O_{\epsilon,k}(x^{1-1/(2k)+\epsilon}). \tag{21}$$

DÉMONSTRATION. La fonction $\Psi(n) = g(n)e^{z\lambda(n)}$ satisfait les hypothèses du Lemme 6 avez $\delta = 1 - \epsilon$, et $C = C(\epsilon)$ pour tout $\epsilon > 0$. Cela permet d'appliquer le Lemme 7, avec un reste majoré uniformément par rapport à z. On a

$$Q_{k-1}(\xi; z) = \frac{1}{(k-1)!}\sum_{j+r+m+q=k-1}(-1)^q q!\binom{k-1}{j, r, m, q}A_j(k)\rho_{k,r}(z)\xi^m \tag{22}$$

où $\rho_{k,r}(z)$ est la dérivée r-ième en $s = 1$ de

$$\rho_k(s; z) = \prod_p (1 + (1 - g(p)e^{z\lambda(p)})((1 - p^{-s})^k - 1)).$$

En particulier, les $\rho_{k,r}(z)$ sont donc des fonctions holomorphes de z.

Corollaire. *On a*

$$\sum_{n \le x} g(n)\lambda(n)\tau_k(n) = xW_{k-1}(\log x) + O_{k,\epsilon}(x^{1-1/(2k)+\epsilon}) \tag{23}$$

où W_{k-1} est le polynôme de degré $k-1$ défini par

$$W_{k-1}(\xi) = \frac{\partial}{\partial z}Q_{k-1}(\xi; 0). \tag{24}$$

DÉMONSTRATION. On multiplie la formule (21) par z^{-2} et on l'intègre sur le cercle $|z| = 1$.

Fin de la démonstration du Théorème 2. Compte tenu du corollaire au lemme 5, on obtient la formule annoncée avec

$$P_k(\xi) = 2h\left[\tfrac{1}{2} V_k(\xi) + W_{k-1}(\xi) + h' U_{k-1}(\xi)\right]. \tag{25}$$

Bibliographie

[1] Bombieri, E., Friedlander, J. and Iwaniec, H. Primes in arithmetic progressions to large moduli, *Acta Math.*, (à paraître).

[2] Deshouillers, J. M. Majorations en moyenne de sommes de Kloosterman. *Séminaire Théorie des Nombres de Bordeaux* (1980-81), exposé n.° 3.

[3] Deshouillers, J.-M. and Iwaniec, H. Kloosterman sums and the Fourier coefficients of cusp forms. *Invent. Math.,* **70** (1982), 219-288.

[4] Deshouillers, J.-M. and Iwaniec, H. An additive divisor problem. *J. London Math. Soc.* (2) **26** (1982), 1-14.

[5] Esterman, T. Über die Darstellung einer Zahl als Differenz von zwei Produkten, *J. reine angew. Math.,* **164** (1931), 173-182.

[6] Fouvry, E. Sur le problème des diviseurs de Titchmarsh, *J. reine angew. Math.,* **357** (1985), 51-76.

[7] Friedlander, J. and Iwaniec, H. Incomplete Kloosterman Sums and a Divisor Problem, *Annals of Math.*, **121** (1985), 319-350.

[8] Heath-Brown, D. R. The fourth power moment of the Riemann Zeta function, *Proc. London Math. Soc.,* (3) **38** (1979), 385-422.

[9] Linnik, Ju. V. The dispersion method in binary additive problems, *A.M.S. Transl. of Math. Monographs,* n.° 4, Providence, Rhode Island (1963).

[10] Motohashi, Y. An asymptotic series for an additive divisor problem, *Math. Zeit.,* **170** (1980), 43-63.

[11] Titchmarsh, E. C. *The Theory of the Riemann Zêta function.* Oxford at the Clarendon Press (1951).

[12] Vinogradov, A. I. and Takhtadzhyan, L. A. Theory of Eisenstein series for the group $SL(3, \mathbb{R})$ and its application to a binary problem. *J. Soviet. Math.,* **18** (1982), 293-324.

Etienne Fouvry
U.E.R. de Mathématiques et d'Informatique
Université de Bordeaux I
351, cours de la Libération
F-33405 Talence Cedex

Gérald Tenenbaum
Départament de Mathématiques
Université de Nancy I
B.P. 239
F-54506 Vandoeuvre Cedex

Revista Matemática Iberoamericana
Vol. 1, n.° 3, 1985

Calderón-Zygmund Operators on Product Spaces

Jean-Lin Journé

1. Introduction

In their well-known theory of singular integral operators, Calderón and Zygmund [3] obtained the boundedness of certain convolution operators on $\mathbb{R}^d$ which generalize the Hilbert transform H in $\mathbb{R}^1$, defined for $f \in C_0^\infty(\mathbb{R}^1)$ by

$$(0.1) \qquad Hf(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} \frac{f(y)}{x-y}\, dy.$$

Typical examples of such operators are the Riesz transforms R_j, $j \in [1, d]$, defined for $f \in C_0^\infty(\mathbb{R}^d)$ by

$$(0.2) \qquad R_j f(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy.$$

Their program can be decomposed into two steps. In the first one they prove L^2-boundedness using Plancherel's theorem. In the second step they use the smoothness and size properties of the kernel and the L^2-boundedness to prove L^p-boundedness for $p \in]1, +\infty[$ as well as the a.e. convergence of the r.h.s. of (0.2) for $f \in L^p$, $p \in]1, +\infty[$. Peetre [14] has shown that these operators are also bounded from $\mathrm{BMO}(\mathbb{R}^d)$ to $\mathrm{BMO}(\mathbb{R}^d)$.

The theory has been generalized in two ways.

In the first extension, one considers non-convolution operators associated to a kernel in the following sense. Let Δ be the diagonal set of $\mathbb{R}^d \times \mathbb{R}^d$ and let K be a locally bounded function defined from $\mathbb{R}^d \times \mathbb{R}^d \backslash \Delta$ to $\mathbb{C}$. Let T: $C_0^\infty(\mathbb{R}^d) \to [C_0^\infty(\mathbb{R}^d)]'$ be a linear operator defined in the weakest possible sense. Then K is the kernel of T if for $f, g \in C_0^\infty(\mathbb{R}^d)$ with disjoint supports, $\langle g, Tf \rangle$ is given by $\iint g(x)K(x,y)f(y)\,dx\,dy$. Suppose moreover that K satisfies some smoothness and size properties analogous to those enjoyed by the kernels of the Riesz transforms. Of course one cannot conclude that T is bounded on L^2 and if T is not a convolution operator one usually cannot use Plancherel's theorem. However it was observed that if the operator is known to be bounded on L^2 the second part of the program of Calderón and Zygmund can be carried out and one obtains a variety of results as in the convolution case. See [8] or [12]. In addition these operators are bounded from L^∞ to BMO, the obstruction for boundedness on BMO being purely algebraic; that is, they are bounded on BMO if and only if they are well defined on BMO, as for instance, in the convolution case. The most famous non-convolution operator of this kind is the Cauchy-operator on Lipschitz curves T_a defined for $a \in L^\infty_{\mathbb{C}}(\mathbb{R})$, $\|a\|_\infty < 1$, $f, g \in C_0^\infty(\mathbb{R})$ by

$$\langle g, T_a f \rangle = \iint \frac{g(x)f(y)}{(x-y) + \int_x^y a(u)\,du}\,dx\,dy. \tag{0.3}$$

This example also illustrates the weakness of the theory since it leaves open the question of the L^2-boundedness of such operators. See however [2] and [7] for the Cauchy kernel. This gap has been recently filled, up to a certain extent, by the so-called $T1$-theorem [9] which asserts that under a very weak regularity condition, T is bounded on L^2 if and only if $T1$ and T^*1, defined appropriately, both lie on BMO.

The second extension is due to R. Fefferman and E. Stein [11]. They study convolution operators which satisfy certain quantitative properties enjoyed by tensor products of operators of Calderón-Zygmund type, as for instance the double Hilbert transform defined for $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ by

$$[(H_1 \otimes H_2)f](x_1, x_2) = \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \iint_{\substack{|x_1 - y_1| > \epsilon_1 \\ |x_2 - y_2| > \epsilon_2}} \frac{f(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}\,dy_1\,dy_2. \tag{0.4}$$

For such tensor products the L^p-boundedness for $p \in]1, +\infty[$ is a trivial consequence of Fubini's theorem but for the more general Fefferman-Stein operators a new machinery is built in [11] which unfortunately uses at each step that the operators under consideration are convolution operators. Moreover it ignores «the BMO aspect of things» in which we shall be mostly interested, while it gives sharp results on maximal operators, which we cannot handle.

Our purpose is to unify up to a certain extent these two generalizations and to define on a product of n Euclidean spaces a class of singular integraloperators which coincides with the extended Calderón-Zygmund class in the case $n = 1$ and coincides in the convolution case with the Fefferman-Stein class when $n = 2$. Actually we extend the non-convolution-Calderón-Zygmund class, and then proceed by induction for $n > 2$. The basic example of an operator considered in this setting is the «n^{th}-Cauchy operator» associated to the kernel K_a, defined for $a \in L^\infty_{\mathbb{C}}(\mathbb{R}^n)$ and $\|a\|_\infty < 1$ by

$$K_a(x, y) = \frac{1}{\prod_{i=1}^{n} (x_i - y_i) + \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} a(u_1, \dots, u_n)\, du}.$$

As in the case $n = 1$, this kernel K_a can be expanded in the sum $\sum_{j \in N} L_a^j$ of «commutators» where

$$L_a^j(x, y) = \left[\prod_{i=1}^{n} (x_i - y_i) \right]^{-j-1} \left[\int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} a(u)\, du \right]^j.$$

Let $\bar{L}_a^j$ be the operator associated to L_a^j. Then we show $\|\bar{L}_a^j\|_{2,2} \leqslant C_n^j \|a\|_\infty^j$.
Thus we can sum the series and obtain

$$\|\bar{K}_a\|_{2,2} \leqslant \frac{1}{1 - C_n \|a\|_\infty} \quad \text{for} \quad \|a\|_\infty < \frac{1}{C_n}.$$

The general case $\|a\|_\infty < 1$ remains open.

The connection between L^2 and BMO, emphasized by the $T1$-Theorem and its proof, turns out to be extremely useful in this setting too. The BMO-space to be considered is the space of Chang-Fefferman studied in [5] which takes into account the product structure of the underlying space. As in the classical situation one makes two kinds of size and smoothness assumptions (integral or pointwise) on the kernel according to whether the associated operator is known to be bounded on L^2 or not. In the first case we show under rather weak assumptions on the kernel that the operator is also bounded from L^∞ to BMO and therefore on all L^p's for $p \in]1, +\infty[$ and under somewhat stronger assumptions the boundedness on BMO, if there is no algebraic obstruction. In the second case we show a $T1$-theorem in the spirit of the classical one. In the case where T is given by a kernel K antisymmetric in each pair $(x_i, y_i)_{1 \le i \le n}$ as K_a or L_a^j for instance, the the $T1$-theorem reduces to: T is bounded on L^2 if and only if $T1 \in \text{BMO}$.

In Sections 1 and 2 we recall some basic notations on singular integrals and Calderón-Zygmund operators in the classical situation and on BMO and Carleson measures on product spaces. The class of operators we wish to study is presented in Section 3, together with their more immediate properties.

In Section 4 we reduce the implication «L^2-boundedness $\to L^\infty$-BMO-boundedness» to a geometric lemma which we prove in Section 5. This lemma may be thought of as a substitute for the Whitney decomposition in the setting of product spaces. In Section 6 we state our «$T1$-theorem» and reduce its proof to two technical points which are studied in Section 7 and 8. Section 9 deals with a special property of antisymmetric kernels, which is new even when $n = 1$ and which is applied to the study of the kernel K_a for $\|a\| \leqslant \epsilon_n$. Finally we apply in Section 10 the geometric lemma of Section 5 to extend a result of J. L. Rubio de Francia on a Littlewood-Paley inequality of arbitrary intervals of $\mathbb{R}$ to the n-dimensional setting.

It is a real pleasure to express my gratitude to Raphy Coifman, Guy David, Yves Meyer, and Stephen Semmes for reading the manuscript and suggesting several improvements, to J. L. Rubio de Francia for bringing to my attention the conjecture solved in Section 10 and to Peter Jones for telling me it was obviously true.

1. Classical singular integral operators and Calderón-Zygmund operators on R^d

The definitions we shall adopt are standard. However, the terminology will be slightly different than usual ([8], [12]).

Let $\Omega = \mathbb{R}^d \times \mathbb{R}^d \backslash \Delta$, where $\Delta = \{(x,y), x = y\}$, and let $\delta \in]0,1[$.

Definition 1. *Let K be a continuous function defined on Ω and taking its values in a Banach space B. This function K is a B-δ-standard kernel if the following are satisfied, for some constant $C > 0$.*

For all (x,y) in Ω,

$$|K(x,y)|_B \leqslant \frac{C}{|x-y|^d}. \tag{1.1}$$

For all (x,y) in Ω, and x' in $\mathbb{R}^d$ such that $|x-x'| < \dfrac{|x-y|}{2}$, (1.2)

$$|K(x,y) - K(x',y)|_B \leqslant C\frac{|x-x'|^\delta}{|x-y|^{d+\delta}} \quad \text{and}$$

$$|K(y,x) - K(y,x')|_B \leqslant C\frac{|x-x'|^\delta}{|x-y|^{d+\delta}}.$$

The smallest constant C for which (1.1) and (1.2) hold is denoted by $|K|_{\delta,B}$. We shall omit the subscript B when it creates no ambiguity.

Definition 2. *Let T: $C_0^\infty(\mathbb{R}^d) \to [C_0^\infty(\mathbb{R}^d)]'$ be a continuous linear mapping. T*

is a singular integral operator (SIO) *if, for some* $\delta \in]0, 1[$, *there exists a* C-δ-*standard kernel* K *such that for all functions* f, g *in* $C_0^\infty(\mathbb{R}^d)$ *having disjoint supports,*

$$\langle g, Tf\rangle = \iint g(x)K(x, y)f(y)\, dx\, dx. \tag{1.3}$$

Here $\langle g, Tf\rangle$ denotes the action of the distribution Tf on the function g. We shall also say that T is a δ-SIO.

Definition 3. *Let* T *be a* δ-SIO. *It is a* δ-*Calderón-Zygmund operator (*δ-CZO) *if it extends boundedly from* L^2 *to itself.*

The following theorem gives necessary and sufficient conditions for a δ-SIO to be a δ-CZO. The statement of these conditions is explained afterwards.

Theorem 1 [9]. *Let* T *be a* δ-SIO. *It is a* δ-CZO *if and only if*

$$T1 \in \mathrm{BMO} \tag{1.4}$$

$$T^*1 \in \mathrm{BMO} \tag{1.5}$$

$$T \text{ has the weak-boundedness property} \tag{1.6}$$

In order to give a meaning to (1.4) we must show how T acts on bounded C^∞ functions. The meaning of (1.5) will then be clear since T^*, defined by $\langle g, T^*f\rangle = \langle f, Tg\rangle$ for all $f, g \in C_0^\infty(\mathbb{R}^d)$, is also a δ-SIO if T is.

The action of an SIO, T on $C_b^\infty(\mathbb{R}^d)$, the set of bounded C^∞ functions, is described the following way ([8], [9]). For $f \in C_b^\infty(\mathbb{R}^d)$, Tf will be a distribution acting on $C_{00}^\infty(\mathbb{R}^d)$, the subspace of $C_0^\infty(\mathbb{R}^d)$ of functions g such that $\int g\, dx = 0$. Let g be such a function and let $h \in C_0^\infty(\mathbb{R}^d)$ be equal to f on a neighborhood of $\operatorname{supp} g$, so that g and $f - h$ have disjoint supports.

If f has compact support,

$$\langle g, Tf\rangle = \langle g, Th\rangle + \langle g, T(f-h)\rangle, \qquad \text{where, by (1.3),}$$

$$\langle g, T(f-h)\rangle = \iint g(x)K(x, y)[f(y) - h(y)]\, dx\, dy.$$

Since g has mean value 0, this is also equal to

$$\iint g(x)[K(x, y) - K(x_0, y)]\,[f(y) - h(y)]\, dx\, dy,$$

where x_0 is any point of $\operatorname{supp} g$. Notice that by (1.2), this integral is absolutely convergent even if $(f - h)$ has non-compact support, and is independent of x_0. This integral can therefore serve as a definition of $\langle g, T(f-h)\rangle$. Obviously $\langle g, Th\rangle + \langle g, T(f-h)\rangle$ does not depend on the choice of h. Hence we can set

$$\langle g, Tf\rangle = \langle g, T(f-h)\rangle + \langle g, Th\rangle,$$

and this defines the desired extension.

This description yields immediately an effective method for computing Tf when $f \in C_b^\infty(R^d)$.

Lemma 1. *Let θ be in $C_0^\infty(R^d)$ and equal to 1 on $\{x, |x| < 1\}$. Let θ_q be defined for $q \in \mathbb{N}$ by $\theta_q(x) = \theta\left(\frac{x}{q}\right)$, and for f on $C_b^\infty(R^d)$ let $f_q = f\theta_q$. Then for all g in $C_{00}^\infty(R^d)$,*

$$\langle g, Tf\rangle = \lim_{q \to +\infty} \langle g, Tf_q\rangle. \tag{1.7}$$

We shall now give the meaning of (1.6). See [9].

Definition 4. *Let T be a δ-SIO. It has the **weak boundedness property** if for any bounded subset B of $C_0^\infty(\mathbb{R}^d)$ there exists $C_B > 0$ such that for any pair (η, ξ) of elements of B and any (x, t) in $\mathbb{R}_{d+1}^+$,*

$$|\langle \eta_t^x, T\xi_t^x\rangle| \leqslant c_B t^{-d}, \tag{1.8}$$

where ξ_t^x is defined by $\xi_t^x(y) = \frac{1}{t^d}\xi\left(\frac{y-x}{t}\right)$ and η_t^x similarly.

We shall also write that T has the WBP.

Note that any operator T bounded on L^2 has the WBP since there exists a constant C'_B such that $\|\xi_t^x\|_2 \leqslant C'_B t^{-d/2}$ for all (x, t) in $\mathbb{R}_{d+1}^+$ and ξ in B.

It is easy to show that T has the WBP if there exists a constant C and an integer N such that for all cubes Q of length $\delta(Q)$ and all functions f and g supported in Q, $|\langle g, Tf\rangle| \leqslant C|Q|P(N, g, Q)P(N, f, Q)$, where

$$P(N, g, Q) = \sum_{|\alpha| \leq N} |\delta(Q)|^\alpha \left\| \frac{\partial^\alpha}{\partial x^\alpha} g \right\|_\infty. \tag{1.9}$$

It is well known that CZO's are bounded from L^∞ to BMO. However, there exist conditions much weaker than (1.2) that will ensure that an operator T, bounded on L^2, associated in the sense of (1.3) to a kernel K, is bounded from L^∞ to BMO. The weakest of the known conditions is

$$\int_{|x-y| > 2|x-x'|} |K(x, y) - K(x', y)|\, dy < C$$

and is due to Calderón and Zygmund. For our purposes it will be best to assume something slightly stronger

$$\int_{|x-y| > 2^k|x-x'|} |K(x, y) - K(x', y)|\, dy \leqslant C2^{-k\epsilon} \tag{1.10}$$

for some $\epsilon > 0$ and all $k \in \mathbb{N}$.

Definition 5. *A locally integrable function K satisfying to* (1.10) *is an ϵ-kernel. An operator T bounded on L^2 and associated to an ϵ-kernel is a*

Calderón-Zygmund operator of type ϵ (CZ_ϵ). ***If K takes its values in a normed space V, then it is a V-ϵ-kernel.***

We denote by $|K|_{\epsilon, V}$ the smallest C for which (1.10) holds.

This distinction between pointwise conditions like (1.2) and integral conditions like (1.10) becomes crucial when the operator T maps functions of $C_0^\infty(\mathbb{R})$ into Hilbert-space valued distributions, that is, distributions acting of functions taking their values in a Hilbert space H. In this case the kernel K takes its values in H and there are two possible ways to extend (1.10) in this setting, namely

$$\int_{|x-y|>2^k|x-x'|} \|K(x,y) - K(x',y)\|_H \, dy < C2^{-k\epsilon}$$

or, for all $\lambda \in H$ such that $\|\lambda\|_H = 1$,

$$\int_{|x-y|>2^k|x-x'|} |\langle \lambda, K(x,y) - K(x',y)\rangle_H| \, dy < C2^{-k\epsilon}. \tag{1.11}$$

Observe that an operator T bounded from L^2 to L_H^2 associated to a kernel K satisfying (1.11) is bounded from L^∞ to BMO_H and therefore from L^p to L_H^p for all $p \in]2, +\infty[$, [15].

A slightly stronger version of (1.11) appears in the proof of the following theorem of J. L. Rubio de Francia.

Theorem 2 [15]. *Let $\{I_k\}_{k\in N}$ be a collection of disjoint intervals of $\mathbb{R}$ and let S_{I_k} be the Fourier multiplier with symbol χ_{I_k}. Finally let Δ be defined on L^2 by $\Delta f = [\Sigma(S_{I_k}f)^2]^{1/2}$. Then Δ is bounded on L^p for all $p \in [2, +\infty[$.*

We shall conclude this section with a lemma of Coifman and Meyer, some notations and a remark.

The letter φ will always denote a C_0^∞ radial function supported in the unit ball and such that $\int \varphi \, dx = 1$. Let us define φ_t by $\varphi_t(y) = \frac{1}{t^d} \varphi\left(\frac{y}{t}\right)$. Then P_t is the convolution with φ_t.

The letter ψ will denote a radial C_{00}^∞ function supported in the unit ball and **such that, for all $\xi \in R^d$, (1.12) $\int_0^{+\infty} |\hat{\psi}(t\xi)|^2 t^{-1} \, dt = 1$.** We define ψ_t and Q_t like **φ_t and P_t.**

Lemma 2. *Let T be a δ-*SIO *having the* WBP. *For all bounded subsets B of $C_0^\infty(R^d)$ and $\eta, \xi \in B$ such that $\int \eta \, dx = 0$ or $\int \xi \, dx = 0$,*

$$|\langle \eta_t^x, T\xi_t^y \rangle| \leqslant C_B \omega_{\delta,t}(x-y), \tag{1.13}$$

where $\omega_{\delta,t}(x-y) = \dfrac{t^\delta}{t^{d+\delta} + |x-y|^{d+\delta}}$.

Conversely every continuous operator T: $C_0^\infty(R^d) \to [C_0^\infty(R^d)]'$ having the WBP *and satisfying* (1.13) *is a δ'-*SIO *for all $\delta' < \delta$.*

We omit the proof of this lemma, which is elementary. For the converse part one uses the decomposition of T as $-\int_0^\infty \frac{\partial}{\partial t}(P_t T P_t)\,dt$.

This lemma suggests the following convention. In order to unify (1.1), (1.2) and (1.8) in an inequality analogous to (1.13) we shall remove the assumption «$\int \eta\,dx = 0$ or $\int \xi\,dx = 0$» when $x = y$. In the rest of the paper and without explicit mention we shall assume that if two functions η_t^x and ξ_t^y appear in an inequality of type (1.13) and $x \neq y$, then $\int \eta\,dx = 0$ or $\int \xi\,dx = 0$.

Finally let us observe that if a function f is, say, in L^2 and T is a $CZ\epsilon$, then Tf is a L^2-function and $Q_t Tf$ is a C^∞ function. Let $x \in \mathbb{R}^d$ and suppose $f(z) = 0$ when $|x - z| \leqslant 2t$. Then we can write

$$Q_t Tf(x) = \int_{|x-y|>2t} (Q_t T)_{xz} f(z)\,dz \tag{1.14}$$

where $(Q_t T)_{xz} = \int \psi_t(x-y)[K(y,z) - K(x,z)]\,dy$.

By (1.10), we have

$$\int_{|x-z|>2^k t} |(Q_t T)_{xz}|\,dz \leqslant C 2^{-k\epsilon}. \tag{1.15}$$

As a consequence, the following inequality holds for all $u \in \mathbb{R}_+$

$$\int_{\substack{r \leq u \\ 2t \vee u \leq |x-z|}} |(Q_t T)_{xz}|\,dz \frac{dt}{t} \leqslant C_{\epsilon, T}. \tag{1.16}$$

2. Carleson measures and BMO on product spaces

Let Ω be an open subset in $\mathbb{R} \times \mathbb{R}$. $S(\Omega)$ is the subset of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ of (x_1, t_1, x_2, t_2)'s such that $]x_1 - t_1, x_1 + t_1[\times]x_2 - t_2, x_2 + t_2[\subseteq \Omega$.

Definition 6 [4]. *A Carleson measure on* $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ *is a measure* $d\mu(x_1, t_1, x_2, t_2) = d\mu(x, t)$ *such that for all* Ω

$$\int_{S(\Omega)} d\mu(x,t) \leqslant C_\mu |\Omega|.$$

Definition 7. *A function b is in* BMO($\mathbb{R} \times \mathbb{R}$) *if it can be written as* $a_0 + H_1 a_1 + H_2 a_2 + H_1 H_2 a_3$, *with* $\sum_{i=0}^3 \|a_i\|_\infty < +\infty$ *and where the* H_j's, $j \in \{1, 2\}$ *are the partial Hilbert transforms. Moreover,* $[\inf \sum_{i=0}^3 \|a_i\|_\infty]$, *where the* inf *is taken over all possible decompositions of b, is a norm that makes* BMO($\mathbb{R} \times \mathbb{R}$) *a Banach space.*

Let Q_{t_1} be defined on $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ by $Q_{t_1}[f \otimes g] = [Q_{t_1} f] \otimes g$ and similarly for Q_{t_2}. Clearly Q_{t_1} and Q_{t_2} extend by linearity to $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. A. Chang and R. Fefferman have proved the following.

Theorem A [5]. *A function* b *in* L^2_{loc} *is in* BMO *if and only if* $([Q_{t_1} Q_{t_2} b](x_1, x_2))^2 \, dx_1 \, dx_2 \, (t_1 t_2)^{-1} \, dt_1 \, dt_2$. *Is a Carleson measure on* $\mathbb{R}^2_+ \times \mathbb{R}^2_+$.

Theorem B [6]. *A linear operator* T *bounded from* L^2 *to* L^2 *and from* $L^\infty(\mathbb{R}^2)$ *to* BMO$(\mathbb{R} \times \mathbb{R})$ *is bounded on all* L^p*'s for* $p \in]2, +\infty[$.

It is a routine exercise to rewrite these definitions and theorems when $\mathbb{R} \times \mathbb{R}$ is replaced by $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_n}$ and $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ is replaced by $\mathbb{R}^{d_1+1}_+ \times \times \mathbb{R}^{d_2+1}_+ \times \ldots \times \mathbb{R}^{d_n+1}_+$. Moreover Theorems A and B remain valid if the functions under consideration are Hilbert-space valued. This will be used without mention in Section 10. In order to avoid minor technical complication we shall suppose from now on that all the d_i's are equal to 1.

3. Extension of the definitions of Section 1 in the setting of product spaces

Let T_1 and T_2 be two classical δ-SIO's on $\mathbb{R}$ and let $T = T_1 \otimes T_2$. This operator T is a priori defined from $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ to its algebraic dual by the formula

$$\langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle = \langle g_1, T_1 f_1 \rangle \langle g_2, T_2 f_2 \rangle.$$

Let L_1 and L_2 be the kernels of T_1 and T_2. If g_1 and f_1 have disjoint supports, we can write

$$\langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle = \int g_1(x) L_1(x, y) f_1(y) \langle g_2, T_2 f_2 \rangle \, dx \, dy.$$

Let us put on the set of δ-CZO's the norm $\| \ \|_{\delta CZ}$ defined by $\|S\|_{\delta CZ} = = \|S\|_{2,2} + |K|_\delta$ where K is the kernel of S. This makes the set of δ-CZO's a Banach space which we denote by δCZ. Let $K_1(x, y) = L_1(x, y) T_2$. Then K_1 is a δCZ-valued function and is actually a δCZ-δ-standard kernel and one has

$$\langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle = \iint g_1(x) \langle g_2, K_1(x, y) f_2 \rangle f_1(y) \, dx \, dy.$$

We can define $K_2(x, y)$ in a similar fashion. Now we forget that T is a tensor product and set the following definition.

Definition 8. *Let* T: $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ *be a continuous linear mapping. It is a* δ-SIO *on* $\mathbb{R} \times \mathbb{R}$ *if there exists a pair* (K_1, K_2) *of* δCZ-δ*-standard kernels so that, for all* $f, g, h, k \in C_0^\infty(\mathbb{R})$, *with* $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$,

$$\langle g \otimes k, Tf \otimes h \rangle = \iint g(x) \langle k, K_1(x, y) h \rangle f(y) \, dx \, dy, \tag{3.1}$$

$$\langle k \otimes g, Th \otimes f \rangle = \iint g(x) \langle k, K_2(x, y) h \rangle f(y) \, dx \, dy \tag{3.2}$$

Let $\tilde{T}$ be defined by

$$\langle g \otimes k, \tilde{T} f \otimes h \rangle = \langle f \otimes k, Tg \otimes h \rangle.$$

It is readily seen that $\tilde{T}$ is a δ-SIO if T is. Its kernels $\tilde{K}_1$ and $\tilde{K}_2$ will be given by $\tilde{K}_1(x, y) = K_1(y, x)$ and $\tilde{K}_2(x, y) = [K_2(x, y)]^*$.

Definition 9. *A* δ-SIO T *on* $\mathbb{R} \times \mathbb{R}$ *is a* δ-CZO *if* T *and* $\tilde{T}$ *are bounded on* L^2.

The role of $\tilde{T}$ becomes clear in Section 6.

We can again put a norm on the set of δ-CZO's on $\mathbb{R} \times \mathbb{R}$ by setting

$$\| T \|_{\delta CZ(\mathbb{R} \times \mathbb{R})} = \| T \|_{2,2} + \| \tilde{T} \|_{2,2} + \sum_{i=1}^{2} |K_i|_{\delta, \delta CZ(\mathbb{R})}.$$

Using this remark one can easily define δ-CZO's on a product space with an arbitrary number of factors, by induction on this number.

We can repeat the same procedure to define $CZ\epsilon$'s on product spaces. However for $CZ\epsilon$'s there is no need to consider the partial adjoints as for δ-CZO's.

Let T be a $CZ\epsilon$ on $\mathbb{R}$ and K its kernel. We define $\| T \|_{CZ\epsilon}$ as $\| T \|_{2,2} + |K|_\epsilon$. A $CZ\epsilon$ T on $\mathbb{R} \times \mathbb{R}$ will be a bounded operator on L^2 associated in the sense of Definition 8 to a pair fo $CZ\epsilon$-ϵ-kernels and we shall put $\| T \|_{CZ\epsilon} = \| T \|_{2,2} + \sum_{i=1}^{2} |K_i|_{\epsilon, CZ\epsilon}$.

In order to state an analogue of Theorem 1 in the product setting we need to observe that a δ-SIO on $\mathbb{R} \times \mathbb{R}$ has a natural extension from $C_b^\infty \otimes C_b^\infty$ to $[C_{00}^\infty \otimes C_{00}^\infty]'$. This can be shown by an iteration of the argument sketched in Section 1. It also follows that Lemma 1 can be extended, using the same notations.

Lemma 3. *For all* $g_1, g_2 \in C_{00}^\infty(\mathbb{R})$ and $f_1, f_2 \in C_b^\infty(\mathbb{R})$,

$$\begin{aligned} \lim_{q \to +\infty\, q' \to +\infty} \langle g_1 \otimes g_2, T[(f_1)_q \otimes (f_2)_{q'}] \rangle &= \lim_{q \to +\infty} \langle g_1 \otimes g_2, T[(f_1)_q \otimes f_2] \rangle \\ &= \langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle. \end{aligned}$$

In order to extend the definition of the WBP in the product setting it is convenient to introduce the following notations.

Let T be a δ-SIO on $\mathbb{R} \times \mathbb{R}$ and $f, g \in C_0^\infty(\mathbb{R})$. The operator $\langle g, T^1 f \rangle : C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R})]'$ is defined by

$$\langle h, \langle g, T^1 f \rangle k \rangle = \langle g \otimes h, Tf \otimes k \rangle.$$

It is easy to see that $\langle g, T^1 f \rangle$ is a δ-SIO on $\mathbb{R}$ with kernel $\langle g, T^1 f \rangle(x, y) = \langle g, K_2(x, y) f \rangle$. One defines $\langle g, T^2 f \rangle$ similarly. The notation $T^1 f = 0$ simply means $\langle g, T^1 f \rangle = 0$ for all g. Notice that all this makes sense if $f \in C_b^\infty(\mathbb{R})$ and

$g \in C_{00}^{\infty}(\mathbb{R})$. In particular $T^1 1 = 0$ is equivalent to $\langle k, T^2 h\rangle 1 = 0$ for all $k, h \in C_0^{\infty}(\mathbb{R})$. Similarly $T^1{}^* 1 = 0$ means $\langle k, T^2 h\rangle^* 1 = 0$ in the same conditions. Exchanging the role of indices we obtain the meaning of $T^2 1 = 0$ or $T^2{}^* 1 = 0$.

In the following, the notations are those of Definition 4.

Definition 10. *Let T be a δ-*SIO *on $\mathbb{R} \times \mathbb{R}$. T has the* WBP *if for $i \in \{1, 2\}$*

$$\| \langle \eta_t^x, T^i \xi_t^x \rangle \|_{CZ\delta} \leqslant C_B t^{-1}. \tag{3.3}$$

It is easy to see that a δ-CZO on $\mathbb{R} \times \mathbb{R}$ has the WBP.

Next we indicate the extension of Lemma 2 in the product setting.

Lemma 4. *Let T be a δ-*SIO *with the* WBP. *Then for all B, $(\eta, \xi) \in B \times B$, $(x, y) \in \mathbb{R} \times \mathbb{R}$, $t > 0$ and $i \in \{1, 2\}$,*

$$\| \langle \eta_t^x, T^i \xi_t^y \rangle \|_{CZ\delta} \leqslant C_B \omega_{\delta, t}(x - y). \tag{3.4}$$

Conversely every bounded operator T defined from $C_0^{\infty} \otimes C_0^{\infty}$ to its dual satisfying to (3.4) *is a δ'-SIO having the* WBP *for all $\delta' < \delta$.*

In this statement we made use of the convention of Section 1. The proof of this lemma is routine and we omit the details. Of course lemmas 3 and 4 extend in the setting of an arbitrary product of copies of $\mathbb{R}$.

To conclude this section, we shall give the analogue of (1.14), (1.15) and (1.16) in the product setting.

Suppose first that there are only two factors in the product. Let T be a $CZ\epsilon$ on $\mathbb{R} \times \mathbb{R}$ and $f \in L^2(\mathbb{R}^2)$. Then $(Q_{t_1} Q_{t_2} Tf)(x_1, x_2)$ is a C^{∞} function of (x_1, x_2). If x_1, t_1 are fixed and $f(z_1, z_2) = 0$ for $|x_1 - z_1| \leqslant 2t_1$, then we can write

$$[Q_{t_1} Q_{t_2} Tf](x_1, x_2) = \int Q_{t_2} [(Q_{t_1} T)_{x_1 \bar{z}_1} f(\bar{z}_1, z_2)](x_2)\, d\bar{z}_1, \tag{3.5}$$

where $(Q_{t_1} T)_{x_1 \bar{z}_1}$ is a $CZ\epsilon$ acting on functions of z_2, and given by

$$(Q_{t_1} T)_{x_1 \bar{z}_1} = \int \psi_{t_1}(x_1 - y_1)[K_1(y_1, \bar{z}_1) - K_1(x_1, \bar{z}_1)]\, dy_1.$$

Here K_1 is the first kernel of T and the symbol « $^-$ » over z_1 simply means that z_1 has become a parameter in (3.5). It is not clear that the integral in (3.5) converges absolutely. However by (3.6) below, that will be the case if $f(\bar{z}_1, z_2)$ is uniformly in $L^2(dz_2)$, in particular if f is bounded with compact support.

The definition of a $CZ\epsilon$ on $\mathbb{R} \times \mathbb{R}$ immediately yields the following generalization of (1.15),

$$\int_{|x_1 - z_1| \geq 2^k t_1} \| (Q_{t_1} T)_{x_1 z_1} \|_{CZ\epsilon}\, dz_1 \leqslant C 2^{-k\epsilon}. \tag{3.6}$$

The case of a product of three spaces or more is very similar.

For all $I \subseteq [1,n]$, $(x_i, i \in I) \in \mathbb{R}^I$ and $(t_i, i \in I) \in (\mathbb{R}_+)^I$ and $(z_i, i \in I) \in \mathbb{R}^I$ such that for all $i \in I$, $|z_i - x_i| \geqslant 2t_i$ (we write also $|z_I - x_I| \geqslant 2t_I$) the symbol $[Q_{t_I} T]_{x_I z_I}$ denotes a $CZ\epsilon$ acting on $L^2(\mathbb{R}^J)$, where $J = [1,n] \backslash I$. This $CZ\epsilon$ is defined by induction on $|I|$. If $I = \{i\}$ and K_i is the kernel of T in the variable i, then

$$[Q_{t_I} T]_{x_I z_I} = \int \psi_{t_i}(x_i - y_i)[K_i(y_i, z_i) - K_i(x_i, z_i)]\, dy_i.$$

Now if $[Q_{t_I} T]_{x_I z_I}$ is defined and $I' = I \cup \{i\}$ we define $[Q_{t_{I'} z_{I'}}] = [Q_{t_i}[Q_{t_I} T]_{x_I} \mp z_I]_{x_i z_i}$. This makes sense since $[Q_{t_I} T]_{x_I z_I}$ is itself a $CZ\epsilon$ and has a kernel in the i-variable. On the other hand it is readily seen that $[Q_{t_{I'}} T]_{x_{I'} z_{I'}}$ depends only on $t_{I'}, x_{I'}$ and $z_{I'}$ and not on the decomposition of I' as $I \cup \{i\}$. So the notation is consistent.

Let $I \subseteq [1,n]$ and $J = [1,n] \backslash I$ and let $f \in L^\infty(\mathbb{R}^n)$ have compact support and suppose $f(z) = 0$ if $|x_i - z_i| \leqslant 2t_i$ for some $i \in I$. Then with obvious notations we write

$$[Q_t Tf](x) = \int Q_{t_J}[[Q_{t_I} T]_{x_I \bar{z}_I}] f(\bar{z}_I, z_J)](x_J)\, d\bar{z}_I. \tag{3.7}$$

From (3.10) below it follows that this integral is absolutely convergent. Indeed (1.15) and (1.16) extend easily to the following, where $i \notin I$ and $I' = I \cup \{i\}$:

$$\int_{|x_i - z_i| \geq 2^{k_i} t_i} \| [Q_{t_{I'}} T]_{x_{I'} z_{I'}} \|_{CZ\epsilon}\, dz_i \leqslant C 2^{-k_i \epsilon} \| [Q_{t_I} T]_{x_I z_I} \|_{CZ\epsilon}, \tag{3.8}$$

and for $u \in \mathbb{R}_+$

$$\int_{\substack{t_i \leq u \\ 2t_i \vee u \leq |x_i - z_i|}} \| [Q_{t_{I'}} T]_{x_{I'} z_{I'}} \|_{CZ\epsilon}\, dz_i \frac{dt_i}{t_i} \leqslant C_\epsilon \| [Q_{t_I} T]_{x_I z_I} \|_{CZ\epsilon}. \tag{3.9}$$

Moreover it follows from (3.8),

$$\int_{|x_I - z_I| \geq 2^{k_I} t_I} \| (Q_{t_I} T)_{x_I z_I} \|_{CZ\epsilon}\, dz_I \leqslant C_\epsilon \| T \|_{CZ\epsilon} \times 2^{-\epsilon \Sigma_{i \in I} k_i}. \tag{3.10}$$

4. L^∞-BMO boundedness of $CZ\epsilon$'s on $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \mathbb{R}^n$

We wish to show the following.

Theorem 3. *Let T be a $CZ\epsilon$ on $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$. Then T admits a bounded extension from $L^\infty(\mathbb{R}^n)$ to* BMO$(\mathbb{R} \times \ldots \times \mathbb{R})$.

By interpolation it follows that T is bounded on all L^p's for $p \in]2, +\infty[$ and if T^* is also a $CZ\epsilon$, then T is bounded on all L^p's for $p \in]1, +\infty[$. This situation occurs automatically in the convolution case where we can conclude the following.

Corollary. *Let T be a $CZ\epsilon$ on $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ and a convolution operator. Then T admits a bounded extension from* BMO($\mathbb{R}^n$) *to itself.*

To prove the corollary we use the H^1-BMO duality [5] and an argument of [10], p. 150. Since L^2 is dense in H^1 (this is a trivial consequence of the atomic decomposition for H^1 [6]) it is enough to show that for all $f \in L^2 \cap H^1$, $\|T^*f\|_{H^1} \leqslant C\|f\|_{H^1}$, or equivalently that T^*f, H_1T^*f, H_2T^*f and $H_1H_2T^*f$ are all in L^1 with a norm less that $C\|f\|_{H^1}$. But as functions of L^2 these four functions are equal to T^*f, T^*H_1f, T^*H_2f and $T^*H_1H_2f$ which are in L^1 since by Theorem 3 T^* maps H^1 in L^1 and f, H_1f, H_2f, and H_1H_2f are all in H^1. The corollary is proved.

There are other $CZ\epsilon$'s which are candidates for being bounded on BMO, namely those defined on BMO. In the case $n = 2$ to be defined on BMO is equivalent to the conditions $T^1 1 = 0 = T^2 1$. It turns out that one can still prove that T is then bounded on BMO but the assumptions on the kernels of the $CZ\epsilon$'s have to be strengthened in order to know that TH_1, TH_2 and TH_1H_2 are also $CZ\epsilon$'s if T is and satisfies $T^1 1 = 0 = T^2 1$. We omit the details.

We now turn to the proof of Theorem 3. In order to use the induction hypothesis it is convenient to have the following formulation of Theorem 3, which is clearly equivalent by Theorem A.

Theorem 3′. *There exists $C_{n,\epsilon} > 0$ such that for all bounded open subsets Ω of $\mathbb{R}^n$, all $b \in L^\infty(\mathbb{R}^n)$ with compact support and all $T \in CZ\epsilon(\mathbb{R} \times \ldots \times \mathbb{R})$,*

$$\int_{S(\Omega)} |Q_t Tb(x)|^2 \, dx \frac{dt}{t} \leqslant C_{n,\epsilon} \|T\|^2_{CZ\epsilon} \|b\|^2_\infty |\Omega|. \tag{4.1}$$

We shall need the following lemma.

Lemma 5. *There exists a constant $C_{n,\epsilon}$ such that for all bounded open subsets Ω of $\mathbb{R}^n$ there exists n functions $T_1, \ldots, T_n$ defined from $S(\Omega)$ to $\mathbb{R}_+$ such that $T_i(x,t) \geqslant 2t_i$ and with the following properties*:

$$\text{If} \quad \Omega_n = \bigcup_{(x,t)\in S(\Omega)} \prod_{1 \leq i \leq n}]x_i - T_i, x_i + T_i[, \tag{4.2}$$

then $|\Omega_n| \leqslant C_n |\Omega|$.

For all $T \in CZ\epsilon(\mathbb{R} \times \ldots \times \mathbb{R})$, all $I \subseteq [1,n]$, $I \neq \phi$, let

$$E_{x_I t_I z_I} = \bigcup_{T_I(x,t) < |x_I - z_I|} \prod_{j \notin I}]x_j - t_j, x_j + t_j[.$$

Then

$$\int |E_{x_I t_I z_I}| \, \|[Q_{t_I} T]_{x_I z_I}\|_{CZ\epsilon} \frac{dt_I}{t_I} \, dz_I \, dx_I \leqslant C_{n,\epsilon} |\Omega| \, \|T\|_{CZ\epsilon}. \tag{4.3}$$

Of course when $I = [1, n]$ (4.3) has to be interpreted the following way: $[Q_{t_I}T]_{x_I z_I}$ is a real number and $|E_{x_I t_I z_I}| = 1$ if $T_i(x, t) \leqslant |x_i - z_i|$ for all $i \in [1, n]$ and $|E_{x_I t_I z_I}| = 0$ otherwise.

We postpone the proof of Lemma 5 to the next section.

Let Ω and Ω_n be as in Lemma 5, $b \in L^\infty(\mathbb{R}^n)$ with compact support and $\|b\|_\infty \leqslant 1$ and let $T \in CZ\epsilon(\mathbb{R}^n)$ with $\|T\|_{CZ\epsilon} \leqslant 1$. We want to prove

$$\int_{S(\Omega)} |Q_t Tb(x)|^2 \frac{dx\,dt}{t} \leqslant C_{n,\epsilon}|\Omega|. \tag{4.4}$$

Using (4.2) we immediately reduce to the case where b is supported out of Ω_n. Just write $b = b\chi_{\Omega_n} + b\chi_{\Omega_n^c}$ and observe that

$$\int_{S(\Omega)} |(Q_t Tb\chi_{\Omega_n})(x)|^2 \frac{dx\,dt}{t} \leqslant C_n \|b\chi_{\Omega_n}\|_2^2 \leqslant C_n|\Omega_n| \leqslant C_n|\Omega|.$$

Suppose from now on that b is supported out of Ω_n. Then, for each $(x, t) \in S(\Omega)$ and $z \in \operatorname{supp} b$, $|z_i - x_i| \geqslant T_i$ for at least one index i. This yields the following decomposition for b:

$$b = \sum_{\substack{I \subseteq [1,n] \\ I \neq \phi}} (-1)^{|I|-1} b_{x,t,I},$$

where

$$b_{x,t,I}(z) = b(z) \prod_{i \in I} \chi_{\{z_i,\, |x_i - z_i| \geq T_i(x,t)\}}.$$

Thus

$$(Q_t Tb)(x) = \sum_{\substack{I \subseteq [1,n] \\ I \neq \phi}} (-1)^{|I|-1} (Q_t Tb_{x,t,I})(x).$$

Therefore, to prove (4.4) it is enough to prove for all $I \subseteq [1, n]$, $I \neq \phi$,

$$\int_{S(\Omega)} |Q_t Tb_{x,t,I}(x)|^2 \frac{dx\,dt}{t} \leqslant C_{n,\epsilon}|\Omega|. \tag{4.5}$$

Since $T_i(x, t) \geqslant 2t_i$, we can use (3.7), which reads

$$(Q_t Tb_{x,t,I})(x) = \int_{\bar{z}_I} [Q_{t_J}[Q_{t_I}T]_{x_I \bar{z}_I} b(\bar{z}_I, z_J)](x_J) \chi_{|x_I - \bar{z}_I| \geq T_I} d\bar{z}_I. \tag{4.6}$$

For x_I, t_I fixed, let $E_{x_I t_I} = \cup \prod_{j \notin I}]x_j - t_j, x_j + t_j[$, the union being over the t_J's such that $(x_I, x_J, t_I, t_J) \in S(\Omega)$. Minkowski's inequality and (4.6) yield

$$\int_{S(E_{x_I t_I})} |Q_t T b_{x,t,I}(x)|^2 \, dx_J \frac{dt_J}{t_J} \leqslant$$

$$\leqslant \left[\int_{|z_I - x_I| \geq 2t_I} \left(\int_{S(E_{x_I t_I})} [Q_{t_J}[Q_{t_I} T]_{x_I \bar{z}_I} b(\bar{z}_I, z_J)](x_J)|^2 \times \right. \right.$$
$$\left. \left. \times \chi_{|x_I - z_I| \geq T_I} \frac{dx_J \, dt_J}{t_J} \right)^{1/2} d\bar{z}_I \right]^2 .$$

Now let x_I, t_I, z_I be fixed and $E_{x_I t_I z_I}$ as defined in Lemma 5. If $(x_J, t_J) \in S(E_{x_I t_I})$ and $T_I(x, t) \leqslant |x_I - z_I|$, then $(x_J, t_J) \in S(E_{x_I t_I z_I})$. Therefore we need only to dominate

$$\left[\int_{|z_I - x_I| \geq 2t_I} \left[\int_{S(E_{x_I t_I z_I})} |[Q_{t_J}[Q_{t_I} T]_{x_I \bar{z}_I} b(\bar{z}_I, z_J)](x_J)|^2 \frac{dx_J \, dt_J}{t_J} \right]^{1/2} d\bar{z}_I \right]^2 .$$

The induction hypothesis under the form (4.1) yields the following majorant

$$\left[\int_{|z_I - x_I| \geq 2t_I} |E_{x_I t_I z_I}|^{1/2} \| [Q_{t_I} T]_{x_I z_I} \|_{CZ_\epsilon} \, dz_I \right]^2 .$$

By (3.10) and Cauchy-Schwarz, this is less than

$$C_{n,\epsilon} \left[\int_{|z_I - x_I| \geq 2t_I} |E_{x_I t_I z_I}| \, \| [Q_{t_I} T]_{x_I z_I} \|_{CZ_\epsilon} \, dz_I \right] .$$

It remains to integrate against $dx_I dt_I / t_I$ and use (4.3). In the case where $I = [1, n]$ some minor modifications in notations are needed. They are left to the reader. The proof is therefore reduced to showing Lemma 5.

5. Proof of Lemma 5

When $n = 1$ this lemma is trivial. Let Ω be a bounded open subset of $\mathbb{R}$ and for $x \in \Omega$, let $I(x)$ be the connected component of x in Ω. Then simply set $T(x, t) = |I(x)|$ for $(x, t) \in S(\Omega)$. Clearly $T(x, t) \geqslant 2t$ since $]x - t, x + t[\subseteq I(x)$. Moreover $]x - I(x), x + I(x)[\subseteq 3I(x)$ which implies (4.2) with $C_1 = 3$. Finally (4.3) reduces to

$$\text{(5.1)} \qquad \int_{S(\Omega)} \int_z |(Q_t T)_{x,z}| \chi_{|x-z| > |I(x)|} \, dz \, dx \frac{dt}{t} \leqslant C_\epsilon |\Omega| \, \| T \|_{CZ_\epsilon},$$

which follows trivially from (3.9) with $u = |I(x)|/2$. This observation will permit us to illustrate in a simple case one point of the strategy of the proof.

Lemma 6. *Suppose we have built* $T_1 \dots T_n$ *such that* (4.3) *holds for* $I = [2, n]$. *Then if* $T_1 \geqslant |I_{x_I t_I}(x_1)|$, (4.3) *holds for* $I' = [1, n]$.

Here $I_{x_I t_I}(x_1)$ denotes the connected component of x_1 in $E_{x_I t_I}$ as defined in Section 4.

Let x_I, t_I, z_I be fixed. To deduce (4.3) for I' form (4.3) for I, it is enough to show that

$$\int |E_{x_{I'} t_{I'} z_{I'}}| \, |[Q_{t_{I'}} T]_{x_{I'} z_{I'}}| \frac{dt_1}{t_1} dz_1 \, dx_1 \leqslant C |E_{x_I t_I z_I}| \, \| [Q_{t_I} T]_{x_I z_I} \|_{CZ\epsilon},$$

and then integrate against $dx_I \dfrac{dt_I}{t_I} dz_I$.

This inequality actually means

$$\int_{(x_1, z_1, t_1)/|x_{I'} - z_{I'}| \geq T_{I'}} |[Q_{t_{I'}} T]_{x_{I'} z_{I'}}| \frac{dt_1}{t_1} dz_1 \, dx_1 \leqslant C |E_{x_I t_I z_I}| \, \| [Q_{t_I} T]_{x_I z_I} \|_{CZ\epsilon}.$$

Thanks to the formula

$$[Q_{t_{I'}} T]_{x_{I'} z_{I'}} = [Q_{t_1} [Q_{t_I} T]_{x_I z_I}]_{x_1 z_1},$$

we are almost in position to use (5.1). We only need that the conditions on (x_1, z_1, t_1) imply

i) $(x_1, t_1) \in S(E_{x_I t_I z_I})$

ii) $|x_1 - z_1| \geqslant |I_{x_I t_I z_I}(x_1)|$, where $I_{x_I t_I z_I}(x_1)$ is the component of x_1 in $E_{x_I t_I z_I}$.

i) follows from the definition of $E_{x_I x_I z_I}$ and from the condition $|x_I - z_I| \geqslant$ $\geqslant T_I$. ii) follows from the fact that $E_{x_I t_I z_I} \subseteq E_{x_I t_I}$ for all z_I. Therefore $|x_1 - z_1| \geqslant T_1 \geqslant |I_{x_I t_I}(x_1)| \geqslant |I_{x_I t_I z_I}(x_1)|$. This implies ii) and lemma 6 is proved.

In general one point in the strategy will be to define the T_i*'s* by induction on i in such a way that if I is a set of indices and $i_0 < \inf I$, and if the T_i*'s* are such that (4.3) holds for I, then it holds for $\{i_0\} \cup I$, almost independently of the choice of the T_i*'s* for $i > i_0$.

Lemma 7. *Let* $\Omega \subseteq \mathbb{R}^n$, $(x_1, t_1) \in \mathbb{R}^2_+$ *and for* $(x_2, \dots, x_n) \in E_{x_1 t_1}$, *let* $\tau(x_1, t_1, x_2, \dots, x_n) = 2t_1 \vee \inf \left\{ \alpha, (\chi_{E_{x_1 t_1} \setminus E_{x_1 \alpha}})^*(x_2, \dots, x_n) \geqslant \frac{1}{2} \right\}$. *For* $(x, t) \in S(\Omega)$, *let* $\tau(x, t) = \sup \tau(x_1, t_1, y_2, \dots, y_n)$, *the* sup *being over those* $(y_i)_{2 \leq i \leq n}$ *such that* $|x_i - y_i| \leqslant t_i$ *for all* $i \in [2, n]$. *For* $z_1 \in \mathbb{R}$ *such that* $|z_1 - x_1| \geqslant 2t_1$ *let*

$$\tilde{E}_{x_1 t_1 z_1} = \bigcup_{\tau(x, t) < |x_1 - z_1|} \prod_{i \leq 2}]x_i - t_i, x_i + t_i[.$$

Then, for $T \in CZ\epsilon(\mathbb{R} \times \ldots \times \mathbb{R})$ with $\|T\|_{CZ_\epsilon} \leqslant 1$

$$\int_{|x_1 - z_1| \geq 2t_1} |\tilde{E}_{x_1 t_1 z_1}| \, \|(Q_{t_1} T)_{x_1 z_1}\|_{CZ_\epsilon} \, dx_1 \frac{dt_1}{t_1} dz_1 \leqslant C_n |\Omega|. \tag{5.2}$$

Moreover, if

$$\Omega' = \bigcup_{(x,t) \in S(\Omega)}]x_1 - \tau, x_1 + \tau[\times \prod_{i \geq 2}]x_i - t_i, x_i + t_i[,$$

then

$$\chi_{\Omega'} \leqslant \tfrac{1}{2}(\chi_\Omega)^*, \tag{5.3}$$

where $*$ *is the strong Hardy-Littlewood maximal operator.*

In order to prove (5.3), it is enough to prove that, for all $(x, t) \in S(\Omega)$,

$$\frac{\left|]x_1 - \tau, x_1 + \tau[\times \prod_{i \geq 2}]x_i - t_i, x_i + t_i[\cap \Omega\right|}{2^n \tau_1 \times \prod_{i \geq 2} t_i} \geqslant \frac{1}{2}.$$

If $\tau = 2t_1$ this is obvious since $(x, t) \in S(\Omega)$. If $\tau > 2t_1$, we can choose β such that $\tau > \beta > 2t_1$ and $(y_i)_{2 \leq i \leq n}$ such that $|x_i - y_i| \leqslant t_i$ for $i \in [2, n]$, and $\tau_1(x_1, t_1, y_2, \ldots, y_n) > \beta$. Therefore $(\chi_{E_{x_1 t_1} \setminus E_{x_1 \beta}})^*(y_2, \ldots, y_n) < \frac{1}{2}$ and in particular

$$\frac{\left|\prod_{i \geq 2}]x_i - t_i, x_i + t_i[\cap E_{x_1 t_1} \setminus E_{x_1 \beta}\right|}{2^{n-1} \prod_{i \geq 2} t_i} < \frac{1}{2}.$$

Since $\prod_{i \geq 2}]x_i - t_i, x_i + t_i[\subseteq E_{x_1 t_1}$, this is equivalent to

$$\frac{\left|\prod_{i \geq 2}]x_i - t_i, x_i + t_i[\cap E_{x_1 \beta}\right|}{2^{n-1} \prod_{i \geq 2} t_i} > \frac{1}{2}.$$

Since $]x_1 - \beta, x_1 + \beta[\times E_{x_1 \beta} \subseteq \Omega$, this implies

$$\frac{\left|]x_1 - \beta, x_1 + \beta[\times \prod_{i \geq 2}]x_i - t_i, x_i + t_i[\cap \Omega\right|}{2^n \left[\beta \times \prod_{i \geq 2} t_i\right]} > \frac{1}{2}.$$

Letting β tend to τ, we obtain the desired inequality and (5.3).

To prove (5.2) observe that

$$\begin{aligned}\tilde{E}_{x_1 t_1 z_1} &\subseteq \{(y_2, \ldots, y_n) \in \mathbb{R}^{[2,n]},\ \tau_1(x_1, t_1, y_2, \ldots, y_n) < |\chi_1 - z_1|\} \subseteq \\ &\subseteq \left\{(y_2, \ldots, y_n), (\chi_{E_{x_1 t_1} \setminus E_{x_1 |x_1 - z_1|}})^*(y_2 \ldots y_n) \geqslant \tfrac{1}{2}\right\}.\end{aligned}$$

This latter inclusion follows from the trivial fact that $(\chi_{E_{x_1t_1}\backslash E_{x_1\alpha}})^*(y_2, \ldots, y_n)$ is an increasing function of α. These inclusions imply $|\tilde{E}_{x_1t_1z_1}| < C_n|E_{x_1t_1}\backslash E_{x_1|x_1-z_1|}|$. At this point we need the following.

Lemma 8. *Let* $x_1 \in \mathbb{R}$, $T \in CZ\epsilon(\mathbb{R} \times \ldots \times \mathbb{R})$ *with* $\|T\|_{CZ\epsilon} \leqslant 1$, *and let* F: $\mathbb{R}_+ \to \mathbb{R}_+$ *be a decreasing function vanishing for t large. Then*

$$(5.4) \quad \int_{|x_1-z_1|\geq 2t_1} [F(t_1) - F(|x_1 - z_1|)]\,\|(Q_{t_1}T)_{x_1z_1}\|_{CZ\epsilon}\frac{dt_1}{t_1}dz_1 \leqslant C_\epsilon F(0^+).$$

It is easy to reduce to the case where F is C^1. In this case write $F(t_1) - F(|x_1 - z_1|) = -\int_{t_1}^{|x_1-z_1|} F'(u)\,du$. Using (3.9) with $I = \phi$ and $I' = \{1\}$, we obtain, since $-F' > 0$, a domination of the L.H.S. of (5.4) by $-\|T\|_{CZ\epsilon} \times \int_0^{+\infty} F'(u)\,du$, which proves Lemma 8.

To prove (5.3) we apply (5.4) with $F(t) = E_{x_1t}$. The restriction $|x_1 - z_1| \geqslant 2t_1$ is irrelevant since otherwise $\tilde{E}_{x_1t_1z_1} = \phi$. An application of (5.4) and the inequality $|\tilde{E}_{x_1t_1z_1}| \leqslant C(|E_{x_1t_1}| - |E_{x_1|x_1-z_1|}|)$ yield

$$\int |\tilde{E}_{x_1t_1z_1}|\,\|(Q_{t_1}T)_{x_1z_1}\|_{CZ\epsilon}\frac{dt_1}{t_1}dz_1 \leqslant C_\epsilon|E_{x_10^+}|, \quad \text{where} \quad E_{x_10^+} = \bigcup_{t_1>0} E_{x_1t_1}.$$

An integration in x_1 yields $C_\epsilon \int |E_{x_10^+}|\,dx_1$ as a majorant of the l.h.s. of (5.3). But this is exactly $C_\epsilon|\Omega|$, and Lemma 7 is proved.

We shall use Lemma 7 with many indices playing the role of index 1 and with many sets instead of Ω; we shall specify which index and which set are considered, e.g. $\tau_1(x_1, t_1, x_2, t_2, \ldots, x_n, t_n, \Omega)$.

A direct consequence of Lemma 7 is the following. If $T_1(x, t) \geqslant \tau_1(x, t, \Omega)$, then $E_{x_1t_1z_1} \subseteq \tilde{E}_{x_1t_1z_1}$ and (5.2) implies (4.3) for $I = \{1\}$. Now we define the T_i*'s* by induction on i. The letter ω will denote an open subset of $\mathbb{R}^k$ for some $k \in [1, n]$ which will be specified by the context. We shall use the notation $E_{x_It_I}$ as in Section 4 but we shall specify the set under consideration, e.g. $E_{x_It_I}(\Omega)$. Finally $I_{xt} = I_{x_It_I}(x_1)$ with the notations of Lemma 6.

We set

$$T_1 = |I_{xt}| \vee \sup_{\substack{I \subseteq [2,n] \\ J = [2,n]\backslash I \\ \omega \subseteq E_{x_It_I}(\Omega) \\ (x_1, t_1, x_J, t_J) \in S(\Omega)}} \tau_1(x_1, t_1, x_J, t_J, \omega),$$

$$\Omega_1 = \bigcup]x_1 - T_1, x_1 + T_1[\times \prod_{i\geq 2}]x_i - t_i, x_i + t_i[,$$

the union being taken over $(x, t) \in S(\Omega)$,

$$T_2 = \sup_{\substack{I \subseteq [3, n] \\ J = [3, n] \setminus I \\ \omega \subseteq E_{x_I t_I}(\Omega_1) \\ (x_1, T_1, x_2, t_2, x_J, t_J) \in S(\omega)}} \tau_2(x_1, T_1, x_2, t_2, x_J, t_J, \omega),$$

and $\Omega_2 = \cup]x_1 - T_1, x_1 + T_1[x]x_2 - T_2, x_2 + T_2[\times \prod_{i \geq 3}]x_i - t_i, x_i + t_i[$.

Suppose $T_1 \ldots T_{i-1} T_i$ are already defined and let

$$\Omega_i = \cup \prod_{j \geq i}]x_j - T_j, x_j + T_j[\times \prod_{k \geq i}]x_k - t_k, x_k + t_k[.$$

We define T_{i+1} as follows

$$T_{i+1} = \sup_{\substack{I \subseteq [i+2, n] \\ J = [i+2, n] \setminus I \\ \omega \subseteq E_{x_I t_I}(\Omega_i) \\ (x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_J, t_J) \in S(\omega)}} \tau_{i+1}(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_J, t_J, \omega).$$

Finally let

$$\Omega_{n-1} = \cup \Big(\prod_{i \leq n-1}]x_i - T_i, x_i + T_i[\Big) \times]x_n - t_n, x_n + t_n[,$$

and let $T_n = \tau_n(x_1, T_1, \ldots, x_{n-1}, T_{n-1}, x_n, t_n, \Omega_{n-1})$.

The property (4.2) will be a trivial consequence of the following.

Lemma 9. *For all $i \in [1, n-1]$, $(\chi_{\Omega_i})^* \geqslant \frac{1}{2} \chi_{\Omega_{i+1}}$.*

If $i = n - 1$, this is an immediate consequence of (5.3) applied with index n and set Ω_{n-1}.

If $i < n - 1$, let $(x, t) \in S(\Omega)$, $\alpha > 0$ be such that $t_{i+1} < \alpha < T_{i+1}$. There exists $I \subseteq [i+2, n]$ and $\omega \subseteq E_{x_I t_I}(\Omega_i)$ such that $(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_J, t_J) \in S(\omega)$ and $\tau_{i+1}(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_J, t_J, \omega) > \alpha$. The proof of Lemma 7 shows that

$$\prod_{j \leq i}]x_j - T_j, x_j + T_j[\times]x_{i+1} - \tau_{i+1}, x_{i+1} + \tau_{i+1}[\times]x_J - t_J, x_J + t_J[$$

has at least half of its volume in ω. Since $\omega \subseteq E_{x_I t_I}(\Omega_i)$, $\omega \times]x_I - t_I, x_I + t_I[\subseteq$ $\subseteq \Omega_i$. Hence

$$\prod_{j\leq i}\,]x_j - T_j, x_j + T_j[\, \times \,]x_{i+1} - \tau_{i+1}, x_{i+1} + \tau_{i+1}[\, \times \prod_{j>i+1}\,]x_j - t_j, x_j + t_j[$$

has half of its volume in Ω_i. Let α tend to T_{i+1} and the same is proved for T_{i+1} instead of τ_{i+1}. Finally we have proved that Ω_{i+1} is the union of rectangles that have at least half of their volume in Ω_i. This implies the lemma. Actually we have skipped the case where $i = 1$ and $T_1 = I_{xt}$, but then the argument is trivial.

We are left with proving (4.3). To do so we replace $E_{x_I t_I z_I}$ by a larger set $F_{x_I t_I z_I}$ defined as follows. Let $i_0 = \inf I$. Then

$$F_{x_I t_I z_I} = \bigcup_{T_I(x,t) < |x_I - z_I|} \prod_{j<i_0}\,]x_j - T_j, x_j + T_j[\, \times \prod_{\substack{j>i_0\\ j\notin I}}\,]x_j - t_j, x_j + t_j[.$$

Now we shall prove by induction on $|I|$ that

$$\int |F_{x_I z_I t_I}|\, \|[Q_{t_I} T]_{x_I z_I}\|_{CZ\epsilon} \frac{dt_I}{t_I}\, dz_I\, dx_I \leqslant C_{n\epsilon} |\Omega|\, \|T\|_{CZ\epsilon}. \tag{5.5}$$

This will be sufficient since $t_j < T_j$ for all j and $E_{x_I t_I z_I} \subseteq F_{x_I t_I z_I}$. Also, by Lemma 6 it is enough to consider the case $|I| < n$.

If I has a single element i, then (5.5) is a direct consequence of (5.2) applied with the set Ω_i and the index i, since $T_i(x, t) \geqslant \tau_i(x_1, T_1, \ldots, x_{i-1}, T_{i-1}, x_i, t_i, \ldots \ldots, x_n, t_n, \Omega_i)$ and $|\Omega_i| \leqslant C_n|\Omega|$ by Lemma 9.

If I has more than a single element let $K = I\backslash\{i_0\}$, and let $G^{i_0}_{x_K t_K z_K}$ be defined as

$$\bigcup_{T_K \leq |x_K - z_K|} \prod_{j<i_0}\,]x_j - T_j, x_j + T_j[\, \times \,]x_{i_0} - t_{i_0}, x_{i_0} + t_{i_0}[\, \times \prod_{\substack{j>i_0\\ j\notin I}}\,]x_j - t_j, x_j + t_j[.$$

Clearly $G^{i_0}_{x_K t_K z_K} \subseteq F_{x_K t_K z_K}$. Moreover we have the following.

Lemma 10.

$$\int |F_{x_I t_I z_I}|\, \|[Q_{t_I} T]_{x_I z_I}\|_{CZ\epsilon} \frac{dt_{i_0}}{t_{i_0}}\, dz_{i_0} \leqslant C_{n,\epsilon} |(Q_{t_K} T)_{x_K z_K}\|_{CZ\epsilon} |G^{i_0}_{x_K t_K z_K}|.$$

With Lemma 10, one deduces immediately (5.5) for I from (5.5) for K. Therefore the induction, and the proof of Lemma 5, reduce to Lemma 10 which we now prove. To do so we shall apply (5.2) with the set $G^{i_0}_{x_K t_K z_K}$, the index i_0 and the operator $(Q_{t_K} T)_{x_K z_K}$. Let $\tilde{F}_{x_I t_I z_I} = \bigcup \prod_{j\notin I}]y_j - s_j, y_j + s_j[$, where the union is taken over those $(y_j, s_j)_{j\notin I}$ such that

$$\tau_{i_0}(x_{i_0}, t_{i_0}, y, s, G^{i_0}_{x_K t_K z_K}) \leqslant |x_{i_0} - z_{i_0}| \quad \text{and} \quad (x_{i_0}, t_{i_0}, y, s) \in S(G^{i_0}_{x_K t_K z_K}).$$

Then (5.2) reads as follows:

$$\int |\tilde{F}_{x_I t_I z_I}| \, \|(Q_{t_I}T)_{x_I z_I}\|_{CZ\epsilon} \, dx_{i_0} \frac{dt_{i_0}}{t_{i_0}} \, dz_{i_0} \leqslant C_n \|(Q_{t_K}T)_{x_K z_K}\|_{CZ\epsilon} \|G^{i_0}_{x_K t_K z_K}\|.$$

Therefore we need only to prove $F_{x_I t_I z_I} \subseteq \tilde{F}_{x_I t_I z_I}$. In other words we must show that if $T_I(x, t) < |x_I - z_I|$, that is $T_K(x, t) < |x_K - z_K|$ and $T_{i_0}(x, t) < |x_{i_0} - z_{i_0}|$, then $((x_j, T_j)_{j < i_0}, (x_{i_0}, t_{i_0}), (x_j, t_j)_{j > i_0, j \notin I}) \in S(G^{i_0}_{x_K t_K z_K})$ and the associated $\tau_{i_0}(\,\cdot\,, G^{i_0}_{x_K t_K z_K}) \leqslant |x_{i_0} - z_{i_0}|$. The first assertion follows from the definition of T_{i_0}. Indeed $G^{i_0}_{x_K t_K z_K} \subseteq E_{x_K t_K}(\Omega_{i_0 - 1})$ (with $\Omega_0 = \Omega$), and therefore $T_{i_0} \geqslant$ $\geqslant \tau_{i_0}(\,\cdot\,, G^{i_0}_{x_K t_K z_K})$ where « · » means $((x_j, T_j)_{j < i_0}, (x_{i_0}, t_{i_0}), (x_j, t_j)_{j > i_0, j \notin I})$. Now $\tau_{i_0}(\,\cdot\,, G^{i_0}_{x_K t_K z_K}) \leqslant T_{i_0}(x, t) \leqslant |x_{i_0} - z_{i_0}|$ and the lemma is proved.

6. A «$T1$-theorem» in the product setting

If T is a δ-SIO on $\mathbb{R} \times \mathbb{R}$ and has the WBP, the conditions $T1 = 0$ and $T^*1 = 0$ do not imply that T is bounded on L^2. This is why we introduced in Section 3 the partial adjoint $\tilde{T}$. Now if $T1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0$, then T is bounded on L^2. Moreover the following is true.

Theorem 4. *Let T be a δ-SIO on $\mathbb{R} \times \mathbb{R}$ having the* WBP *and such that* $T1$, T^*1, $\tilde{T}1$ and $\tilde{T}^*1$ *lie in* BMO($\mathbb{R} \times \mathbb{R}$). *Then T extends boundedly from L^2 to L^2.*

Let us consider an example. Let $(a_{k_1,k_2})_{\mathbb{Z} \times \mathbb{Z}}$ be a bounded real-valued sequence on $\mathbb{Z} \times \mathbb{Z}$ and let $\tilde{a} = \sum \sum a_{k_1 k_2} e^{i2^{k_1}x} e^{i2^{k_2}y}$ be the tempered distribution such that $\langle \tilde{a} | \psi \rangle = \sum \sum a_{k_1 k_2} \hat{\psi}(-2^{k_1}, -2^{k_2})$ for all $\psi \in S(\mathbb{R}^2)$. Let $\psi_0 \in S(\mathbb{R}^1)$ be such that $\hat{\psi}_0(0) = 0$ and $\sum_{k \in \mathbb{Z}} \hat{\psi}_0(2^{-k}\xi) = 1$ for $\xi = 0$ and let E_k be the Fourier multiplier of symbol $\hat{\psi}_0(2^{-k}\xi)$. Let $T_a : C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ be defined by $\langle g_1 \otimes g_2, T_a f_1 \otimes f_2 \rangle = \sum_{k_1} \sum_{k_2} a_{k_1 k_2} \langle \Delta^*_{k_1} g_1, e^{i2^{k_1}} f_1 \rangle \langle g_2, e^{i2^{k_2}} \Delta_{k_2} f_2 \rangle$. It is easy to show that this series is absolutely convergent. Moreover, if the rows and columns of the matrix $((a_{k_1 k_2}))$ are uniformly bounded, then T_a is a 1-SIO, satisfies $T_a 1 = T^*_{\tilde{a}} 1 = \tilde{T}^*_{\tilde{a}} 1 = 0$ and $\tilde{T}_a 1 = \tilde{a}$, and T_a has the WBP. Finally T_a is bounded if and only if $((a_{k_1 k_2}))$ is bounded on $l^2(\mathbb{Z})$ and $\tilde{T}_a 1$ is in BMO only if $((a_{k_1 k_2}))$ is Hilbert-Schmidt.

This example shows that $\tilde{T}1$ and $\tilde{T}^*1$ have to be taken into account in order to obtain L^2-boundedness but the conditions $\tilde{T}1$ and $\tilde{T}^*1 \in$ BMO are not necessary.

From Theorems 3 and 4 applied to T and $\tilde{T}$ we obtain the following.

Corollary. *Let T be a δ-*SIO *on* $\mathbb{R} \times \mathbb{R}$. *It is a δ-*CZO *if and only if* $T1$, T^*1, $\tilde{T}1$ *and* $\tilde{T}^*1$ *lie in* BMO *and it has the* WBP.

To avert the suspicion of vain aesthetism, we shall now explain why we require $\tilde{T}$ to be bounded on L^2 in the definition of a CZO. This is not merely to have a nice characterization of CZO's but to have statements which extend in the setting of an arbitrary finite product of copies of $\mathbb{R}$. It is a very good exercise to extend the proof of Theorem 4 that we shall give below and an opportunity to see why one needs to take into account $\tilde{T}$ in the definition of $\| \ \|_{CZ\delta(\mathbb{R}\times\mathbb{R})}$. We shall leave it to the reader and stick from now on to the case $n = 2$ (except in Section 10).

The proof of Theorem 4 can be decomposed in three steps.

In the first step, one simply observes that if T satisfies $T^1 1 = T^{1*}1 = 0$ and has the WBP, then if can be viewed as a classical vector valued SIO, $\bar{T}$ acting on $C_0^\infty(\mathbb{R}) \otimes H$, where $H = L^2(\mathbb{R}, dx_2)$, and for which $\bar{T}1 = \bar{T}^*1 = 0$. The proof of the L^2-boundedness of such an operator is the hilbertian version of the proof of [9] based on the Cotlàr-Knapp-Stein lemma.

The second step is the decomposition of an operator T having the WBP, such that $T1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0$ as the sum of two operators S and $T - S$ having the WBP and such that $S^2 1 = S^{*2}1 = 0$ and $(T - S)^1 1 = (T - S)^{*1}1 = 0$. The L^2-boundedness of T is then a consequence of the first step. The construction of S is given in Section 7.

The last step is, as in the classical situation, to construct for all functions $b \in \mathrm{BMO}$, a CZO V_b such that $V_b 1 = b$ and $V_b^* 1 = \tilde{V}_b 1 = \tilde{V}_b^* 1 = 0$. Now if T satisfies the assumptions of the theorem and b_1, b_2, b_3 and b_4 are $T1$, T^*1, $\tilde{T}1$ and $\tilde{T}^*1$ respectively, the operator $T - V_{b_1} - V_{b_2}^* - \tilde{V}_{b_3} - \tilde{V}_{b_4}^*$ is of the type studied in the second step, so that T is bounded on L^2. The operator V_b is described in Section 8.

7. Decomposition of T when $T1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0$

Let $\beta \in \mathrm{BMO}(\mathbb{R})$ and let $U_\beta: C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R})]'$ be defined by $\langle g, U_\beta f\rangle = \int_0^{+\infty} \langle (Q_t g), (Q_t\beta)(P_t f)\rangle \, dt/t$. It is classical that this integral is absolutely convergent and that U_β is a 1-CZO with $\|U_\beta\|_{1CZ} \leqslant C\|\beta\|_{\mathrm{BMO}}$. Moreover $U_\beta 1 = \beta$ and $U_\beta^* 1 = 0$ ([9]).

Now let T be a δ-SIO on $\mathbb{R} \times \mathbb{R}$ such that $T1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0$. We define the operator N as follows.

For all $f_1, f_2, g_1, g_2 \in C_0^\infty(\mathbb{R})$

$$\langle g_1 \otimes g_2, N f_1 \otimes f_2\rangle = \langle g_1, U_{\{\langle g_2, T^2 f_2\rangle 1\}} f_1\rangle. \tag{7.1}$$

Lemma 11. *The operator N is a δ'-SIO having the WBP for all $\delta' < \delta$. Moreover $N^2 1 = (N^2)^* 1 = (N^1)^* 1 = 0$ and $(T - N)^1 1 = 0$.*

In order to prove Lemma 11 we shall need the following.

Lemma 12. *Let $T\colon C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ be a continuous linear mapping. Suppose that for every bounded subset B of $C_0^\infty(\mathbb{R})$, there exists a constant C_B such that:*

i) *for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, all $t_1, t_2 \in \mathbb{R}_+$, and all $\eta_1, \xi_1, \eta_2, \xi_2 \in B$*

$$|\langle \eta_{1t_1}^{x_1} \otimes \eta_{2t_2}^{x_2}, T\xi_{1t_1}^{y_1} \otimes \xi_{2t_2}^{y_2} \rangle| \leqslant C_B \omega_{\delta, t_1}(x_1 - y_1)\omega_{\delta, t_2}(x_2 - y_2);$$

ii) *for all $(x, y) \in \mathbb{R}$, $t > 0$, $i \in \{1, 2\}$ and $\eta, \xi \in B$,*

$$\| \langle \eta_t^x, T^i \xi_t^y \rangle 1 \|_{\mathrm{BMO}} \leqslant C_B \omega_{\delta, t}(x - y) \text{ and } \| \langle \eta_t^x, T^i \xi_t^y \rangle^* 1 \|_{\mathrm{BMO}} \leqslant C_B \omega_{\delta, t}(x - y).$$

Then T is a δ'-SIO on $\mathbb{R} \times \mathbb{R}$ and has the WBP for all $\delta' < \delta$. Conversely any δ-SIO having the WBP satisfies i) and ii).

This lemma is an immediate consequence of Lemma 2 and of Theorem 1.

Let us apply it to N. Applying the converse part of Lemma 12 to T, using (7.1) and the properties of the operators U_β for $\beta \in \mathrm{BMO}(\mathbb{R})$, one obtains easily the property i) for N as well as the property ii) for $i = 2$. From the formula $\langle f, N^2 g \rangle = U_{\{\langle f, T^2 g \rangle 1\}}$ we also conclude $(T - N)^1 1 = 0$ and $(N^1)^* 1 = 0$. We are left with showing that N satisfies ii) with $i = 1$. In fact we shall prove $N^2 1 = (N^2)^* 1 = 0$, or, in other words $\langle g, N^1 f \rangle 1 = \langle g, N^1 f \rangle^* 1 = 0$ for all $f, g \in C_0^\infty(\mathbb{R})$. For this we shall use the assumptions $T1 = \tilde{T}1 = 0$.

To show $\langle g, N^1 f \rangle 1 = 0$, it is enough by Lemma 1 to show that for all $h \in C_{00}^\infty(\mathbb{R})$, $\lim_{q \to +\infty} \langle h, \langle g, N^1 f \rangle \theta_q \rangle = 0$, which means $\lim_{q \to +\infty} \langle g, U_{\{\langle h, T^2 \theta_q \rangle 1\}} f \rangle = 0$, where θ_q is defined in Lemma 1. This is immediate from the two following lemmas.

Lemma 13. *Let $(\beta_q)_{q \in \mathbb{N}}$ be a bounded sequence taking its values in $\mathrm{BMO}(\mathbb{R})$. If $\lim_{q \to +\infty} \beta_q = 0$ for $\sigma^*(H^1, \mathrm{BMO})$, then for all $f, g \in C_0^\infty(\mathbb{R})$, $\lim_{q \to +\infty} \langle g, U_{\beta_q} f \rangle = 0$.*

Lemma 14. *Let T be a δ-SIO on $\mathbb{R} \times \mathbb{R}$ such that $T1 = 0$. Then for all $h \in C_{00}^\infty(\mathbb{R})$ and for $i \in \{1, 2\}$, the sequence $(\langle h, T^i \theta_q \rangle 1)_{q > q_0}$ satisfies the hypothesis of Lemma 13 for q_0 big enough.*

To prove Lemma 13, observe that the integrals $\iint (Q_t g)(x)(Q_t \beta_q)(x) \cdot (P_t f)(x) t^{-1} \, dx \, dt$ are uniformly absolutely convergent since $\| Q_t \beta_q \|_\infty \leqslant C$, $\int_0^{+\infty} \| Q_t g \| t^{-1} \, dt < +\infty$ and $\sup_{t > 0} \| P_t f \|_2 < +\infty$. Therefore we can take the limit under the integral sign. Since by assumption $\lim_{q \to +\infty} (Q_t \beta_q)(x) = 0$ for all $(x, t) \in \mathbb{R}_+^2$, Lemma 13 is proved.

To prove Lemma 14 we pick a function $k \in C_0^\infty(\mathbb{R})$ and we want to prove that $|\langle k, \langle h, T^i\theta_q\rangle 1\rangle|$ is less than $C\|k\|_{H^1}$ for $q > q_0$, q_0 and C being independent of k, and that $\lim_{q\to+\infty} |\langle k, \langle h, T^i\theta_q\rangle 1\rangle| = 0$. This latter fact follows from $T1 = 0$ and from Lemma 3 in Section 3. To prove the first fact it is enough to prove that for $q > q_0$ and $q' > q_0$

$$|\langle k, \langle h, T^i(\theta_q - \theta_{q'})\rangle 1\rangle| \leqslant C\|k\|_{H^1}, \tag{7.2}$$

and then take the limit when $q' \to +\infty$. Notice now that if $\operatorname{supp} h \cap \cap \operatorname{supp}(\theta_q - \theta_{q'}) = \phi$, then $\langle h, T^i(\theta_q - \theta_{q'})\rangle = \iint h(x)K_i(x, y)(\theta_q - \theta_{q'})(y)\,dx\,dy$. This will be true if q_0 is chosen large enough and in this case a straightforward computation (using $\int h\,dx = 0$) yields $\|\langle h, T^i(\theta_q - \theta_{q'})\rangle\|_{\delta CZ} \leqslant C$, which implies (7.2).

We have proved $N^2 1 = 0$. The proof of $N^{2*}1 = 0$ is identical. One just has to use $\tilde{T}1 = 0$ instead of $T1 = 0$. This proves Lemma 11.

We also need another operator M, similar to N, defined by

$$\langle g_1 \otimes g_2, Mf_1 \otimes f_2\rangle = \langle g_1, U^*_{\{\langle g_2, T^2 f_2\rangle^* 1\}} f_1\rangle.$$

This operator M is also an SIO and has the WBP. Moreover, $M^2 1 = M^{2*}1 = M^1 1 = 0$ and $(T - M)^{1*}1 = 0$. This can be shown using the same arguments as for N.

We now set $S = M + N$ so that S has the WBP, $S^2 1 = S^{2*}1 = 0$ and $(T - S)^1 1 = (T - S)^{1*}1 = 0$.

8. Construction of the operator V_b

The construction of V_b is inspired by the construction of the operators U_β, $\beta \in \in \mathrm{BMO}$ of Section 7; see [9].

The family of operators $(P_t)_{t>0}$ is defined as in Section 2, but now Q_t denotes $-t\frac{\partial}{\partial t}P_t$, so that $\int Q_t \frac{dt}{t} = I$ and $\int Q_t^2 \frac{dt}{t} = C_0 I$, where C_0 is not necessarily 1.

Let $b \in \mathrm{BMO}(\mathbb{R} \times \mathbb{R})$ and let W_b: $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ be defined by

$$\langle f_1 \otimes f_2, W_b g_1 \otimes g_2\rangle =$$
$$= \iint \langle Q_{t_1} f_1 \otimes Q_{t_2} f_2, (Q_{t_1} Q_{t_2} b) P_{t_1} g_1 \otimes P_{t_2} g_2\rangle \frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

The L^2-boundedness of W_b is, as in the classical situation [9], a consequence of the fact that $(Q_{t_1} Q_{t_2} b)^2\,dx_1 dx_2\,(t_1 \cdot t_2)^{-1}\,dt_1\,dt_2$ is a Carleson measure [5] and of the properties of such measures on product spaces [4]. On the other hand one sees easily that W_b is a 1-SIO whose kernels take their values in

$\{U_\beta, \beta \in \mathrm{BMO}(\mathbb{R})\}$. Moreover, an application of Lemma 3 shows that $W_b 1 = C_0 b$, $W_b^* 1 = 0$, $\tilde{W}_b 1 = 0$ and $\tilde{W}_b^* 1 = 0$. It remains to show that $\tilde{W}_b$ is bounded on L^2. In order to do this, it is enough to show that $\tilde{W}_b$ maps L^∞ into BMO. Similarly $\tilde{W}_b^*$ will map L^∞ into BMO so that the L^2-boundedness of W_b will follow by interpolation between $H^1 \to L^1$ and $L^\infty \to \mathrm{BMO}$ [13].

We want the estimate $\|\tilde{W}_b f\|_{\mathrm{BMO}} \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_\infty$. Consider the operator T_f: $b \to \tilde{W}_b f$. We need to show that it maps BMO to itself. Observe that T_f, which is given by

$$\langle h_1 \otimes h_2, T_f b_1 \otimes b_2\rangle = \iint \langle P_{t_1} h_1 \otimes Q_{t_2} h_2, (Q_{t_1} P_{t_2} f) Q_{t_1} b_1 \otimes Q_{t_2} b_2\rangle \frac{dt_1}{t_1}\frac{dt_2}{t_2},$$

is itself a 1-SIO, and satisfies $T_f^2 1 = T_f^{2*} 1 = T_f^1 1 = 0$. Therefore we already know that T_f maps L^2 to L^2. From Theorem 3 it follows that T_f maps L^∞ to BMO. To show that T_f maps BMO to itself, we observe that $T_f H_1$, $T_f H_2$ and $T_f H_1 H_2$ are SIO's, because the kernel of $Q_t H$ satisfies the same estimates as the kernel of P_t. Since these operators are bounded on L^2 as well as T_f, they also map L^∞ to BMO. Hence T_f is bounded on BMO.

The proof of Theorem 4 is complete.

9. Bicommutators of Calderón-Coifman type

In the classical situation, a standard kernel K is antisymmetric if $K(x, y) = -K(y, x)$ for all $(x, y) \in \Omega$. Such a kernel induces automatically and SIO T defined for all $f, g \in C_0^\infty(\mathbb{R})$ by

$$\langle g, Tf\rangle = \lim_{\epsilon \to 0} \iint_{|x-y|>\epsilon} g(x)K(x,y)f(y)\,dx\,dy. \tag{9.1}$$

The existence of the limit is a consequence of the antisymmetry of the kernel K and of the smoothness of f and g. Actually,

$$\langle g, Tf\rangle = \tfrac{1}{2}\iint K(x,y)[g(x)f(y) - f(x)g(y)]\,dx\,dy, \tag{9.2}$$

so that $|\langle g, Tf\rangle| \leqslant C\,(\mathrm{diam}\,[\mathrm{supp}\, g \cup \mathrm{supp} f])^3 \|g'\|_\infty \|f'\|_\infty$.

This clearly implies that T has the WBP. Since $T = -T^*$, T is bounded on L^2 if and only if $T1 \in \mathrm{BMO}$, by Theorem 1.

The best known examples of CZO's generated by antisymmetric kernels in the manner just described are the Calderón commutators associated to the kernels $[(A(x) - A(y))/(x-y)]^k \cdot (x-y)^{-1}$ where $k \geqslant 0$ and A: $\mathbb{R} \to \mathbb{C}$ satisfies $A' = a \in L^\infty$. Calderón proved in [2] that $\|T_k\|_{\mathrm{CZ1}} \leqslant C^k$. This estimate which has been improved in [7], can be easily obtained from Theorem 1 ([9]).

Actually, this can also be derived from a more general result on antisymmetric kernels.

Let K be an antisymmetric standard kernel and A: $\mathbb{R} \to \mathbb{C}$ be such that $A' = a \in L^\infty$, and let K_a be defined by $K_a(x, y) = K(x, y)\,[A(x) - A(y)] \cdot \cdot (x - y)^{-1}$ for all $(x, y) \in \Omega$. Clearly K_a is also an antisymmetric standard kernel and defines an SIO T_a having the WBP.

Proposition 1. *If T is a* CZO, *then T_a is a* CZO, *and for all $\delta \in]0, 1]$ there exists $C_\delta > 0$ such that*

$$\| T_a \|_{\mathrm{CZ}\delta} \leqslant C_\delta \| a \|_\infty \| T \|_{\mathrm{CZ}\delta}. \tag{9.3}$$

This propostion can be generalized to the product setting.

Let L: $\Omega \times \Omega \to \mathbb{C}$ be a function such that for all (x_1, y_1) and $(x_2, y_2) \in \Omega$

$$|L(x_1, y_1, x_2, y_2)| \leqslant \frac{C}{|x_1 - y_1|\,|x_2 - y_2|}. \tag{9.4}$$

If L is antisymmetric in each couple it defines a continuous operator T: $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ by

$$\langle g_1 \otimes g_2, Tf_1 \otimes f_2 \rangle =$$
$$= \lim_{\epsilon_1 \to 0\, \epsilon_2 \to 0} \iiiint\limits_{|x_1 - y_1| > \epsilon_1,\, |x_2 - y_2| > \epsilon_2} g_1(x_1) g_2(x_2) L(x_1, y_1, x_2, y_2) f_1(y_1) f_2(y_2)\, dx_1\, dx_2\, dy_1\, dy_2$$

for all $f_1, f_2, g_1, g_2 \in C_0(\mathbb{R})$.

As in the classical situation, the existence of this limit is a consequence of the antisymmetry of L, of (9.4) and of the smoothness of f_1, f_2, g_1 and g_2. It is easy to see that T has two kernels K_1 and K_2 in the sense of Definition 8, in Section 3. These are given by

$$\langle g, K_1(x) f \rangle = \lim_{\epsilon \to 0} \iint\limits_{|u - v|} g(u) K(x, y, u, v) f(v)\, du\, dv$$

and

$$\langle g, K_2(x) f \rangle = \lim_{\epsilon \to 0} \iint\limits_{|u - v|} g(u) K(u, v, x, y) f(v)\, du\, dv$$

for all $f, g \in C_0^\infty(\mathbb{R})$.

Let A: $\mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a function such that

$$\frac{\partial^2 A}{\partial_{x_1} \partial_{x_2}} = a \in L^\infty$$

(in the distributional sense) and let $\tilde{A}$: $\Omega \times \mathrm{A} \to \mathbb{C}$ be defined by

$$\tilde{A}(x_1, y_1, x_2, y_2) = \frac{A(x_1, x_2) + A(y_1, y_2) - A(y_1, x_2) - A(x_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}$$

for all (x_1, y_1) and $(x_2, y_2) \in \Omega$. If L is antisymmetric and satisfies (9.4), then $L\tilde{A}$ has the same properties. Hence $L\tilde{A}$ defines an operator T_a in the same manner as L defines T. The first kernel $K_{a,1}$ of T_a is defined by

$$\langle g, K_{a,1}(x, y) f \rangle = \lim_{\epsilon \to 0} \iint_{|u-v| > \epsilon} g(u) L(x, y, u, v) \tilde{A}(x, y, u, v) f(v)\, dv\, du.$$

Notice now that because T is a δ-SIO, so is T_a. This is an immediate consequence of Proposition 1 and the fact that for x and y fixed, $\tilde{A}(x, y, u, v)$ is of the form $[B(u) - B(v)]/(u - v)$, with $\|B'\|_\infty \leqslant \|a\|_\infty$ and $\frac{\partial \tilde{A}}{\partial x}(x, y, u, v)$ is of the form $[C(u) - C(v)]/(u - v)$, with $\|C'\|_\infty \leqslant \|a\|_\infty/(x - y)$.

Proposition 2. *If T is a* CZO, *then T_a is a* CZO, *and for all $\delta \in]0, 1]$ there exists $C_\delta > 0$ such that*

$$\|T_a\|_{\mathrm{CZ}\delta} \leqslant C_\delta \|a\|_\infty \|T\|_{\mathrm{CZ}\delta}.$$

In particular the kernel $[(x_1 - y_1)(x_2 - y_2)]^{-1} \cdot [\tilde{A}]^k$ defines a CZO T_k of norm less than $C^k \|a\|_\infty^k$. The L^2-boundedness of T_1 was first proved by J. Aguirre in [1].

We now turn to the proofs and start with Proposition 1. For simplicity we shall assume $\delta = 1$. We know that it is enough to show that for $a \in L^\infty$, $T_a 1 \in \mathrm{BMO}$ and $\|T_a 1\|_{\mathrm{BMO}} \leqslant C\|a\|_\infty$. To show this inequality we are going to exhibit a CZO S such that $T_a 1 = Sa$, that is, for all $g \in C_{00}^\infty(\mathbb{R}) \langle g, Sa \rangle = \langle g, T_a 1 \rangle$. This equality determines $\langle g, Sa \rangle$ for all $g \in C_{00}^\infty(\mathbb{R})$ and $a \in C_0^\infty(\mathbb{R})$. But since $|K_a(x, y)| \leqslant C|x - y|^{-2}$ when $a \in C_0^\infty(\mathbb{R})$, $T_a 1$ acts not only on $C_{00}^\infty(\mathbb{R})$ but on $C_0^\infty(\mathbb{R})$, so that $\langle g, Sa \rangle$ is well defined for all $a \in C_0^\infty(\mathbb{R})$ and $g \in C_0^\infty(\mathbb{R})$. Moreover,

$$\langle g, Sa \rangle = \lim_{\epsilon \to 0} \iint_{|x-y| > \epsilon} g(x) K(x, y) \frac{A(x) - A(y)}{x - y}\, dx\, dy.$$

Let g and a have disjoint supports. Then if $x \in \operatorname{Supp} g$ and $|y - x| \leqslant \operatorname{dist}(\operatorname{Supp} a, \operatorname{Supp} g)$, $A(x) = A(y)$ so that the integral defining $\langle g, Sa \rangle$ is absolutely convergent. This permits us to compute the kernel of S, namely $K_s(x, u) = \int_{-\infty}^{u} K(x, y) \frac{1}{x - y}\, dy$ if $x > u$ and $K_s(x, u) = \int_u^{+\infty} K(x, y) \frac{1}{y - x}\, dy$ if $x < u$. This kernel K_s is clearly a standard kernel and because of that we can apply Theorem 1 to show that S is a CZO. We first notice that $S1 = T1$ and therefore lies in BMO. This is formally obvious, since when $a = 1$, $[A(x) - A(y)] \cdot \cdot (x - y)^{-1} dx\, dy$, but it can be proved rigorously using Lemma 1. Next, we compute S^*1. For $a \in C_0^\infty(\mathbb{R})$, $A(x) = 0$ for x large enough. Moreover, since $\langle g, Sa \rangle$ can be rewritten as $\frac{1}{2}\iint [g(x) - g(y)]K(x, y)\,[A(x) - A(y)](x - y)^{-1}\, dx\, dy$ an application of Lemma1 shows easily that $S^*1 = 0$. Finally, to prove that S has the WBP we choose g and $a \in C_0^\infty(\mathbb{R})$ and suppose that the supports of a and g are contained in some interval $]x_0 - t, x_0 + t[$. We decompose the integral as $I_1 + I_2 + I_3$, where

$$I_1 = \frac{1}{2} \iint_{x, y \in]x_0 - 2t, x_0 + 2t[} [g(x) - g(y)]K(x, y) \frac{A(x) - A(y)}{x - y}\, dx\, dy,$$

$$I_2 = \frac{1}{2} \iint_{\substack{x \in]x_0 - 2t, x_0 + 2t[\\ y \notin]x_0 - 2t, x_0 + 2t[}} g(x)K(x, y) \frac{A(x) - A(y)}{x - y}\, dx\, dy,$$

and $I_3 = I_2$ because of the antisymmetry of K. Clearly $|I_1| \leqslant C\|g'\|_\infty \|a\|_\infty t^2$ and $|I_2| \leqslant C\|g\|_\infty \|a\|_\infty t$. These estimates imply that S has the WBP. Theorem 1 can be applied to S, which is a CZO. This proves Proposition 1.

We shall denote by U the linear mapping that sends a CZO T defined by an antisymmetric kernel to the operator S we have just considered. From the proof of Proposition 1 it follows that (9.6) $\|U(T)\|_{CZ\delta} \leqslant C\|T\|_{CZ\delta}$.

The proof of Proposition 2 follows the same lines as the proof of Proposition 1. Notice first that an SIO T defined from an antisymmetric kernel by (9.5) has the WBP. This can be seen exactly as in the classical situation. Moreover, such an operator satisfies $T1 = T^*1 = -\tilde{T}1 = -\tilde{T}^*1$. Hence, to prove that it is bounded on L^2, it is enough to show that $T1 \in \mathrm{BMO}$, by Theorem 4, and this is necessary by Theorem 3.

We now wish to prove that if an antisymmetric kernel L defines a CZO, T then the SIO T_a defined by $L\tilde{A}$ satisfies $T_a 1 \in \mathrm{BMO}$. To do this we consider the operator W: $L^\infty(\mathbb{R} \times \mathbb{R}) \to [C_{00}^\infty(\mathbb{R}) \otimes C_{00}^\infty(\mathbb{R})]'$ defined by $Wa = T_a 1$. If $a \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ then $T_a 1$ is actually an element of $[C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ because of the decay properties of $L\tilde{A}$ at ∞. Hence $\langle g_1 \otimes g_2, Wa_1 \otimes a_2 \rangle$ can be defined for $g_1, g_2, a_1, a_2 \in C_0^\infty(\mathbb{R})$ by

$$\lim_{\epsilon_1 \to 0\, \epsilon_2 \to 0} \iiiint_{\substack{|x_1 - y_1| > \epsilon_1 \\ |x_2 - y_2| > \epsilon_2}} g_1(x_1)g_2(x_2)L(x_1, y_1, x_2, y_2)\tilde{A}(x_1, y_1, x_2, y_2)\, dx_1\, dy_1\, dx_2\, dy_2.$$

We are left with proving that W is a CZO. To compute the kernels X_1 and X_2 of W we notice that if $a = a_1 \otimes a_2$.

$$\tilde{A}(x_1, y_1, x_2, y_2) = \frac{A_1(x_1) - A_1(y_1)}{x_1 - y_1} \, \frac{A_2(x_2) - A_2(y_2)}{x_2 - y_2},$$

where $A_1' = a_1$ and $A_2' = a_2$. The computation we did to compute K_s shows that if $x_1 > u_1$ and $g_2, a_2 \in C_0^\infty(\mathbb{R})$, then

$$\langle g_2, X_1(x_1, u_1)a_2 \rangle =$$

$$= \lim_{\epsilon_2 \to 0} \int_{-\infty}^{u_1} \iint_{|x_2 - y_2| > \epsilon_2} g_2(x_2) \frac{L(x_1, y_1, x_2, y_2)}{x_1 - y_1} \, \frac{A_2(x_2) - A_2(y_2)}{x_2 - y_2} \, dx_2 \, dy_2 \, dy_1.$$

This is also equal to

$$\left\langle g_2, \left(U\left[\int_{-\infty}^{u_1} K_1(x_1, y_1) \frac{1}{x_1 - y_1} \, dy_1 \right] \right) a_2 \right\rangle.$$

Similar formulas hold for the case $x_1 < u_1$ and for X_2. From (9.6) it follows that X_1 and X_2 are both 1CZ-1-standard kernels. The WBP of W can also be checked easily. Indeed, for $g_1, a_1 \in C_0^\infty(\mathbb{R})$,

$$\langle g_1, W^1 a_1 \rangle = U\left[\lim_{\epsilon \to 0} \iint_{|x_1 - y_1| > \epsilon} g_1(x_1) K_1(x_1, y_1) \frac{A_1(x_1) - A_1(y_1)}{x_1 - y_1} \, dx_1 \, dy_1 \right],$$

so that the proof in the classical case extends immediately. Moreover, this equality implies that for $a_1, g_1 \in C_0^\infty(\mathbb{R}) \langle g_1, W^1 a_1 \rangle^* 1 = 0$, or equivalently, $W^{2*}1 = 0$. Similarly $W^{1*}1 = 1$. By Lemma 3 this implies $W^*1 = \tilde{W}1 = = \tilde{W}^*1 = 0$. Finally, $W1 = T1$ can be proved using Lemma 3. Therefore W is a CZO and Proposition 2 is proved.

Of course this result extends to an arbitrary finite product of copies of $\mathbb{R}$, by a simple induction on the number of factors. We omit the details.

10. A Littlewood-Paley inequality for arbitrary rectangles

We wish to prove the following extension of Theorem 2.

Theorem. 5. *Let $\{R_k\}_{k \in \mathbb{N}}$ be a collection of disjoint rectangles in $\mathbb{R}^n$ with sides parallel to the axes and let S_k be the Fourier multiplier of symbol χ_{R_k}. Let $\bar{\Delta}$ be defined on L^2 by $|\bar{\Delta} f| = \left[\sum_k (S_k f)^2\right]^{1/2}$. Then $\bar{\Delta}$ is bounded on L^p for all $p \in [2, +\infty[$.*

We shall assume the reader to be familiar with [15] where this theorem is proved in the case $n = 1$. In this paper it is shown that the theorem for $n = 1$ is a consequence of the following.

Lemma 15. *Let ψ be fixed in $S(\mathbb{R})$ son that $\chi_{[-2,2]} \leqslant \hat{\psi} \leqslant \chi_{[-3,3]}$ and let Ψ_k^j be the convolution operator of symbol $\hat{\psi}\left(\frac{\xi}{2^k} - j\right)$. Let χ: $\mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ be such that the operator T_χ is bounded on L^2 where $T_\chi f(x) = \chi(j, k)[\Psi_k^j f(x)]_{j,k}$ takes its values in $l^2(\mathbb{Z} \times \mathbb{Z})$. Then T_χ is bounded from L^∞ to $\mathrm{BMO}_{l^2(\mathbb{Z}\times\mathbb{Z})}$.*

Observe that, by Plancherel's theorem, the L^2-boundedness of T_χ is equivalent to

$$\sum_{j,k} \chi(j, k)|\hat{\psi}(\xi 2^{-k} - j)|^2 \in L^\infty(d\xi).$$

The reduction of Theorem 2 to Lemma 15 is done by means of standard Littlewood-Paley theory, A_2 weights and interpolation between $L^2 \to L^2$ and $L^\infty \to \mathrm{BMO}$. All these arguments go through in the n-dimensional setting without any problem. Finally the main ingredient in the proof of Lemma 15 is the following.

Lemma 16. *Let $(x, t) \in \mathbb{R}_+^2$ and $a \in L^2{}_{\mathrm{loc}}$ be supported out of $]x - 2t, x + 2t[$. Then for all $\eta > 0$, there exists $C_\eta > 0$ such that*

$$\sum_{k,j} |(Q_t \Psi_k^j a)(x)|^2 \leqslant C_\eta \int a^2(z) \left(\frac{t}{|x - z|}\right)^{5/3 - \eta} \frac{dz}{t}. \tag{10.1}$$

Note that the summation is over all $(k, j) \in \mathbb{Z} \times \mathbb{Z}$. This lemma is actually a reformulation of the Lemma 4.1 of [15] and we leave the translation to the reader. From (10.1) it follows by a standard argument that if $a \in L^\infty$ then

$$\sum_{k,j} \chi(j, k)|(Q_t \Psi_k^j a)^2(x)|^2 \frac{dt}{t} dx$$

is a Carleson measure, or equivalently that $[\chi(j, k)\Psi_k^j(a)]_{k,j}$ lies in $\mathrm{BMO}_{l^2(\mathbb{Z}\times\mathbb{Z})}(\mathbb{R})$. Thus Lemma 16 implies Lemma 15.

As we said, Theorem 5 follows from an appropriate analogue of Lemma 15 in the n-dimensional context.

Lemma 17. *Let Ψ and Ψ_k^j be as in Lemma 15 and let χ: $[\mathbb{Z}^n]^2 \to \{0, 1\}$ be such that the operator T_χ: $L^2(\mathbb{R}^n\, dx) \to L^2_{l^2(\mathbb{Z}^{2n})}(\mathbb{R}^n\, dx)$ is bounded, where $T_\chi f(x)$ takes its vlalues in $L^2(\mathbb{Z}^{2n})$ and is given by $[\chi(j, k)([\Psi_{k_1}^{j_1} \otimes \ldots \otimes \Psi_{k_n}^{j_n}]f)(x)]_{(j,k)}$. Then T_χ is bounded from L^∞ to $\mathrm{BMO}_{l^2(\mathbb{Z}^{2n})}$, or equivalently,*

$$\sum_{j,k} \chi(j, k)[Q_{t_1}\Psi_{k_1}^{j_1} \otimes \ldots \otimes Q_{t_n}\Psi_{k_n}^{j_n} a(x)]^2\, dx \frac{dt}{t} \tag{10.2}$$

is a Carleson measure on $[\mathbb{R}_+^2]^n$ if $a \in L^\infty$.

To avoid any convergence problems we suppose that χ is finitely supported but we shall obtain estimates independent of this assertion. We shall use a variant of Lemma 5. It can be shown that (5.3) remains true if $\|[Q_{t_I}T]_{x_I z_I}\|_{CZ\epsilon}$ is replaced by $[t_I^\epsilon/(x_I - z_I)^{1+\epsilon}]$, the point being that Lemma 8 remains true if in (5.4) $\|Q_{t_1}T)_{x_1z_1}\|_{CZ\epsilon}$ is replaced by $[t_1^\epsilon/(x_1 - z_1)^{1+\epsilon}]$. Let us rewrite the variant of (4.3):

$$\int |E_{x_I t_I z_I}| \frac{t_I^{\epsilon - 1}}{(x_1 - z_I)^{1+\epsilon}} dz_I \, dt_I \, dx_I \leqslant C_{n,\epsilon}|\Omega|. \tag{10.3}$$

For technical reasons we need to assume that the T_i's constructed in Section 5 take their values in the set $\{2^k, K \in \mathbb{Z}\}$. This is of course not a restriction. Replacing T_i by $\inf\{2^k, T_i \leqslant 2^k\}$ yields (4.3) and (10.3) a fortiori and (4.2) with the constant $2^n C_n$. We shall use this familly of functions $\{T_i, i \in [1,n]\}$ with various sets playing the role of Ω and even various dimensions. Let I be a set of indices in $[1,n]$ and ω a bounded open subset of $\mathbb{R}^I$. Then $\{T_i(x,t,\omega), i \in I, (x,t) \in S(\omega)\}$ will refer to the family of functions constructed in dimension $|I|$ with ω playing the role of Ω.

Let Ω be an open subset of $\mathbb{R}^n$ and let Ω_n, $T_i(x,t,\Omega)$, $1 \leqslant i \leqslant n$ be as in Sections 4 and 5. By the same argument as in Section 4, and with the same notations, we are reduced to proving an estimate similar to (4.5), namely for $a \in \mathbb{L}^\infty(\mathbb{R}^n)$ and $\|a\|_\infty < 1$

$$\int_{S(\Omega)} \Sigma \chi(k,j) \, |Q_t \Psi_k^j a_{x,t,I}(x)|^2 \frac{dx\,dt}{t} \leqslant C_n |\Omega|. \tag{10.4}$$

This inequality will be a consequence of the following.

Lemma 18. *Suppose that the function a_{xt} is of the form $a\chi_{E_{xt}}$ where $E_{xt} \subseteq \mathbb{R}^n$ is defined by the following set of conditions.*

Let $i \in [1,n]$. For all $j \in [1,n]$, x_j, z_j, let $l_j \in \mathbb{Z}$, be such that $2^{l_j} \leqslant |x_j - z_j| < 2^{l_j+1}$. Let $S_1(x,t), S_2(x,t,l_1), \ldots, S_i(x,t,l_1,l_2,\ldots,l_{i-1})$ be i functions taking their values in the set $\{2^k, k \in \mathbb{Z}\}$ and larger than $2t_1, 2t_2, \ldots, 2t_i$ respectively. Let $F_{x_1t_1x_2t_2\ldots x_it_i}$ be a subset of $\mathbb{R}^n$. Then $(z_1,\ldots,z_n) \in E_{xt}$ if and only if

$$(z_1, z_2, \ldots, z_n) \in F_{x_1t_1\ldots x_it_i} = F_{x_I t_I}$$

$$H \begin{cases} 2^{l_1+1} > |x_1 - z_1| \geqslant 2^{l_1} \geqslant S_1(x,t) \geqslant 2t_1 \\ 2^{l_2+1} > |x_2 - z_2| \geqslant 2^{l_2} \geqslant S_2(x,t,l_1) \geqslant 2t_2 \\ 2^{l_i+1} > |x_i - z_i| \geqslant 2^{l_i} \geqslant S_i(x,t,l_1,\ldots,l_{i-1}) \geqslant 2t_i. \end{cases}$$

Let $I = [1,i]$ and x_I, t_I, l_I be fixed, and let $D_{x_I t_I l_I} = \bigcup \prod_{q>i}]x_q - t_q, x_q + t_q[$, where the union is extended to those $(x_{i+1}, t_{i+1}, \ldots, x_n, t_n)$ such that

$2^{l_1} \geqslant S_1(x, t), \ldots, 2^{l_i} \geqslant S_i(x, t, l_1, \ldots, l_{i-1})$. *If for all* $\epsilon > 0$

$$\iiint_{|x_I - z_I| \geq 2t_I} |D_{x_I t_I l_I}| \frac{dx_I\, dt_I\, dz_I}{t_I^{1-\epsilon}|x_I - z_I|^{1+\epsilon}} \leqslant C_\epsilon |\Omega|, \tag{10.5}$$

then

$$\int_{S(\Omega)} \Sigma \chi(k, j) \, |Q_t \Psi_k^j a_{x,t}(x)|^2 \frac{dx\, dt}{t} \leqslant C|\Omega|. \tag{10.6}$$

Let us see first why Lemma 18 implies (10.4). Observe that it is enough to prove (10.4) when I is of the form $[1, i]$. Indeed, if the construction of the T_i's is non-symmetric in the various indices, the properties of the T_i's which are used are expressed symmetrically and therefore we can reorder the coordinates in such a way that I is of the form $[1, i]$. Now we apply Lemma 18 with $S_1(x, t) = T_1(x, t, \Omega), S_2(x, t, \Omega_1) = T_2(x, t, \Omega) \ldots S_i(x, t, l_1 \ldots l_{i-1}) = T_i(x, t, \Omega)$ and $F_{x_1 t_1 \ldots x_n t_n} = \mathbb{R}^n$. In this case $D_{x_I t_I l_I} = E_{x_I t_I z_I}$ (with the notations of Lemma 5). Indeed if the T_j's take their values in $\{2^k, k \in \mathbb{Z}\}$, then $|x_j - z_j| \geqslant T_j \Leftrightarrow 2^{l_j} \geqslant T_j$. Thus (10.5) coincides with (10.3) and (10.6) coincides with (10.4). We are left with showing Lemma 18.

The proof of Lemma 18 uses a backward induction on i. We start with the case $i = n$. Then we can use the following.

Lemma 19. *Let* $(x, t) \in \mathbb{R}_+^2$ *and* $b_{x,t} \in L^2_{\text{loc}}$ *be such that* $b_{x,t}(z) = 0$ *if* $|x_i - z_i| \leqslant 2t_i$ *for some* $i \in [1, n]$. *Then for all* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$\sum_{(l,j) \in [\mathbb{Z}^n]^2} |(Q_t \Psi_k^j b_{x,t})(x)|^2 \leqslant \int [b_{x,t}(z)]^2 \frac{\left[\prod_{1 \leq i \leq n} t_i\right]^{2/3 - \eta}}{\left[\prod_{1 \leq i \leq n} |x_i - z_i|\right]^{5/3 - \eta}} dz. \tag{10.7}$$

This lemma is the n-dimensional analogue of Lemma 16 and its proof is nothing but the n-th fold application of Lemma 16 successively in each coordinate.

As for Lemma 5, (10.5) has to be interpreted differently when $I = [1, n]$. In this case it reads

$$\iiint_{|x_I - z_I| \geq 2t_I,\, H \text{satisfied}} \frac{dx_I\, dt_I\, dz_I}{t_I^{1-\epsilon}|x_I - z_I|^{1-\epsilon}} \leqslant C_\epsilon |\Omega|, \tag{10.8}$$

the restriction $z \in F_{x_I t_I}$ being irrelevant. Now to obtain (10.6), choose $\eta = \frac{1}{3}$, $b_{x,t} = a_{xt}$ and apply (10.7). Then integrate against $dx_I\, dt_I / b_I$ and (10.8) with $\epsilon = \frac{1}{3}$.

We turn to the general case and choose $i < n$.

Let $(a_{xt}, (x, t) \in S(\Omega))$ be a family of functions satisfying the hypothesis of the lemma. We decompose a_{xt} as $\sum_{l_I} a_{xtl_I}$ where $l_I \geqslant 2t_I$ and

$$a_{xtl_I} = a_{xt} \prod_{s \in I} \chi_{2^{l_s} \leq |x_s - z_s| < 2^{l_s + 1}}.$$

Now for x_I, t_I, l_I fixed and $D_{x_I t_I l_I}$ as defined in Lemma 18 we apply Lemma 5 in dimension $(n - |I|)$. This yields $(n - |I|)$ functions $\tilde{T}_j$, $j > 1$, where $\tilde{T}_j(x_J, t_J) = T_j(x_J, t_J, D_{x_I t_I l_I})$ for $(x_J, t_J) \in S(D_{x_I t_I l_I})$. Let

$$\tilde{D}_{x_I t_I l_I} = \bigcup_{x_J, t_J} \prod_{j > i}]x_j - \tilde{T}_j, x_j + \tilde{T}_j[.$$

From (4.2) we conclude $|\tilde{D}_{x_I t_I l_I}| \leqslant C|D_{x_I t_I l_I}|$. We define $\tilde{a}_{xtl_I} = a_{xtl_I} \chi_{\tilde{D}_{x_I t_I l_I}}(z_{i+1} \dots z_n)$ and $\tilde{a}_{xt} = \sum_{l_I} \tilde{a}_{xtl_I}$.

Let $(x, t) \in S(\Omega)$, $(k, j) \in [\mathbb{Z}^n]^2$ and $\alpha > 0$ be given. By Cauchy-Schwarz,

$$|Q_t \Psi_k^j \tilde{a}_{xt}(x)|^2 \leqslant \left[\sum_{l_I} 2^{-\Sigma_{1 \leq s \leq i} l_s \alpha} \right] \left[\sum_{l_I} 2^{\Sigma_s l_s \alpha} |Q_t \Psi_k^j \tilde{a}_{xtl_I}(x)|^2 \right]$$

which is less than

$$\left[\prod_{1 \leq s \leq i} \frac{1}{t_s} \right]^\alpha \left[\sum_{l_I} 2^{\Sigma_{1 \leq s \leq i} l_s \alpha} |Q_t \Psi_k^j \tilde{a}_{xtl_I}(x)|^2 \right]. \tag{10.9}$$

We wish to show (10.6) with $\tilde{a}_{xt}$ instead of a_{xt}. Observe that the L^2-boundedness of T_χ is equivalent to

$$\sum_{j,k} \chi(j, k) \left[\prod_{1 \leq r \leq n} \hat{\psi}(\xi_r 2^{-k_r} - j_r) \right]^2 \leqslant C.$$

Fix $(j_I, k_I) \in (\mathbb{Z}^I)^2$ and set $\xi_I = j_I 2^{k_I}$. Since $\hat{\psi}(0) = 1$, we obtain

$$\sum_{j_J, k_J} \chi(j,k) \left[\prod_{i < r \leq n} \hat{\psi}(\xi_r 2^{-k_r} - j_r) \right]^2 \leqslant C,$$

where $j = (j_I, j_J)$ and $k = (k_I, k_J)$. This implies that for (j_I, k_I) fixed, the operator $T_{j_I k_I}$ defined by $T_{j_I k_I} f(x) = [\chi(j, k) \Psi_{kJ}^j f(x)]_{j_J, k_J}$, is bounded from $L^2(\mathbb{R}^J)$ to $L^2_{l^2(\mathbb{Z}^J \times \mathbb{Z}^J)}(\mathbb{R}^J)$.

In order to estimate the $l \cdot h \cdot s$ of (10.6) with $\tilde{a}_{xt}$ we rewrite it as

$$\int \sum_{j_I k_I} \left[\int \sum_{j_J, k_J} \chi(k, j) |Q_t \Psi_k^j \tilde{a}_{x,t}(x)|^2 \frac{dx_J dt_J}{t_J} \right] \frac{dx_I dt_I}{t_I}, \tag{10.10}$$

and estimate first the part between brackets. By (10.9) this is less than

$$\frac{1}{(t_I)^\alpha} \sum_{l_I} 2^{\Sigma_{1 \leq s \leq i} l_s \alpha} \int \sum_{k_J, j_J} \chi(k, j) \, |Q_t \Psi_k^j \tilde{a}_{xtl_I}(x)|^2 \frac{dx_J dt_J}{t_J}.$$

Observe that $\tilde{a}_{xtl_I}(z)$ is of the form $a(z)\chi_{G_{x_I t_I l_I}}(z)$ where $G_{x_I t_I l_I}$ is some subset of $\mathbb{R}^n$ depending only on x_I, t_I, and l_I. Now for x_I, t_I, l_I, j_I, k_I fixed we can apply the boundedness of the operator $T_{j_I k_I}$ to the function of z_J

$$[Q_{t_I}\Psi^{j_I}_{k_I}\tilde{a}_{xtl_I}](x_I, z_J)$$

since this does not depend on x_J and t_J. We obtain a majorization of the previous integral by

$$\frac{1}{(t_I)^{\alpha}}\sum_{l_I} 2^{\sum_{1\leq s\leq i} l_s \alpha}\int_{\mathbb{R}^J} |[Q_{t_I}\Psi^{j_I}_{k_I}\tilde{a}_{xtl_I}](x_I, z_J)|^2\, dz_J.$$

To estimate (10.10) we must sum in (j_I, k_I), then in l_I, and finally integrate against $dx_I\, dt_I/t_I$. We fix x_I, t_I, l_I and z_J. observe that the function $\tilde{a}_{xtl_I}(z_I, \bar{z}_J)$ vanishes if $|x_s - z_s| \leqslant 2t_s$ for some $s \in [1, i] = I$. Therefore we can apply Lemma 19 in dimension i and obtain:

$$\sum_{j_I, k_I} |[Q_{t_I}\Psi^{j_I}_{k_I}\tilde{a}_{xtl_I}](x_I, \bar{z}_J)|^2 \leqslant \int_{\mathbb{R}^I} |\tilde{a}_{xtl_I}(z)|^2\, \frac{\left[\prod_{1\leq s\leq i} t_s\right]^{2/3-\eta}}{\left[\prod_{1\leq s\leq i} |x_s - z_s|\right]^{5/3-\eta}}\, dz_I.$$

Now we integrate in z_J and sum over l_I keeping in mind that $2^{l_I} \leqslant |x_I - z_I| \leqslant 2^{l_I+1}$. We are then reduced to integrating the following against $dx_I\, dt_I/t_I$:

$$\int_{\mathbb{R}^n} |\tilde{a}_{xtl_I}(z)|^2\, \frac{[t_I]^{2/3-(\eta+\alpha)}}{[|x_I - z_I|]^{5/3-(\eta+\alpha)}}\, dz.$$

But this is less than

$$\|a\|_\infty^2 \int_{\mathbb{R}^I} |\tilde{D}_{x_I t_I l_I}|\, \frac{[t_I]^{2/3-(\eta+\alpha)}}{[|x_I - z_I|]^{5/3-(\eta+\alpha)}}\, dz_I.$$

Now we use $|\tilde{D}_{x_I t_I l_I}| \leqslant C|D_{x_I t_I l_I}|$, then integrate against $dx_I\, dt_I/t_I$ using (10.5) with $\eta = \alpha = \epsilon = \frac{2}{9}$, and we obtain the desired estimate for the expression (10.10).

To complete the proof of (10.6) we must prove it also when $a_{x,t}$ is replaced by $a_{x,t} - \tilde{a}_{x,t}$ in the $l \cdot h \cdot s$. This is where we are going to use the induction hypothesis, namely that Lemma 18 is true for $k \in]i, n]$. Recall that $a_{xt} - \tilde{a}_{xt}$ is given by

$$[a_{xt} - \tilde{a}_{xt}](z_I, z_J) = \sum_{l_I} a_{xtl_I}(z_I, z_J)\chi_{\tilde{D}^c_{x_I t_I l_I}}(z_J).$$

By the definition of $\tilde{D}_{x_I t_I l_I}$, we can write, if $z_J \in \tilde{D}^c_{x_I t_I l_I}$,

$$1 = \sum_{\substack{K \subseteq J \\ K \neq \phi}} \prod_{r \in K} \chi_{|x_r - z_r| > \tilde{T}r}.$$

Therefore

$$a_{xt} - \tilde{a}_{xt} = \sum_{\substack{K \subseteq J \\ K \neq \phi}} \bar{a}_{x,t,K},$$

where

$$\bar{a}_{x,t,K} = \left[\sum_{l_I} a_{xtl_I} \chi_{\tilde{D}^c_{x_I t_I l_I}}(z_J) \prod_{r \in K} \chi_{|x_r - z_r| > \tilde{T}_r} \right].$$

Now we apply the induction hypothesis to each function $\bar{a}_{x,t,K}$. It is enough to show that we can do so when K is of the form $[i+1, k]$, the general case being deduced by a reordering of the coordinate indices. Let $k \geqslant i+1$ be fixed and $K = [i+1, k]$. Then $\bar{a}_{x,t,K}$ satisfies the assumptions of Lemma 18 for k, with $S_1 \ldots S_i$ as before, $S_{i+1} = T_{i+1}(x_J, t_J, D_{x_I t_I l_I}) \ldots$ $S_k = T_k(x_J, t_J, D_{x_I t_I l_I})$. The set $F_{x_{I \cup K} t_{I \cup K}}$ is equal to

$$F_{x_I t_I} \cap \cup_{l_I} \left[\left[\prod_{s \in I} \{z_s, 2^{l_s} \leqslant |x_s - z_s| < 2^{l_s + 1}\} \right] \times \tilde{D}^c_{x_I t_I l_I} \right].$$

Finally (10.5) with $I \cup K$ instead of I is a consequence of Lemma 5 applied to $D_{x_I t_I l_I}$ and more particularly of (10.3) which in this case says that

$$\int |D_{x_{I \cup K} t_{I \cup K} l_{I \cup K}}| \frac{t_k^{\epsilon - 1}}{|x_K - z_K|} dz_K \, dt_K \, dx_K \leqslant C_{n,\epsilon} |D_{x_I t_I l_I}|,$$

$D_{x_{I \cup K} t_{I \cup K} l_{I \cup K}}$ being defined as in Lemma 18. The conclusion is that we can indeed apply the induction hypothesis to all the functions $\bar{a}_{x,t,K}$ for $K \subseteq [1, n] \backslash I$ and $K \neq \phi$ and therefore we obtain (10.6) for $a_{xt} - \tilde{a}_{xt}$. Thus Lemma 18 is proved, from which follows Lemma 17. Theorem 5 can now be proved by the same arguments as developed in [15] to deduce Theorem 2 from Lemma 15. We omit the details.

This proof shows the limits of the underlying philosophy of this paper, also implicitly contained in [11]: take a good class of operators, look at the tensor products of them, write all the quantitatives properties you can about those tensor products and look at the class of all operators that satisfy the same quantitative properties; then you can work on this new class. From what we just did, it seems that working with the class obtained by starting from vector-valued singular integral operators satisfying (1.11) is not so simple. Indeed to prove Theorem 5 we had to use the very special structure of the operator under consideration, in particular that the summation in Lemma 16 could be taken over all $(k, j) \in \mathbb{Z}^2$ independently of the function χ and that the operator

T_X is a «local tensor product», which corresponds to the fact that it could be written under the form $(\Psi_{k_I}^{j_I} \otimes T_{k_I j_I})_{k_I, j_I}$, with the $T_{k_I j_I}$ essentially of the same form that T_X.

Let us conclude with a remark along the same lines. Starting with a class of symbols $S^0_{\rho\delta}$ on $\mathbb{R} \times \mathbb{R}$, one can do the same «tensor product manipulation» and define a class of symbols $[S^0_{\rho\delta}]^n$ on $\mathbb{R}^n \times \mathbb{R}^n$ by the conditions

$$\left| \frac{\partial^{\alpha+\beta}}{\partial \xi^\alpha \, \partial x^\beta} \sigma(x, \xi) \right| \leqslant C_{\alpha,\beta} \prod_{1 \le i \le n} [1 + |\xi_i|]^{\delta\beta_i - \rho\alpha_i}.$$

Are the corresponding ψdO*'s* bounded when $0 \leqslant \delta < \rho \leqslant 1$ or when $0 \leqslant \delta = \rho < 1$, for instance on L^2 or on some L^p? A partial answer is the following: if $\rho = 1$ then the corresponding ψdO*'s* are CZO's in our sense. This can be seen by the same arguments as in [12]. Otherwise the problem seems entirely open.

References

[1] Aguirre, J. Multilineal pseudo-differential operators and paraproducts, *Thesis, Washington University*, 1981.

[2] Calderón, A. P. Cauchy integrals on Lipschitz curves and related operators, *Proc. Nat. Acad. Sc. USA*, **74** (1977), 1324-1327.

[3] Calderón, A. P., and Zygmund, A. On the existence of certain singular integrals, *Acta Math.*, **88** (1952), 85-139.

[4] Chang, S. Y. A. Carleson measure on the bi-disc, *Ann. of Math.*, **109** (1979), 613-620.

[5] Chang, S. Y. A., and Fefferman, R. A continuous version of duality of H^1 with BMO on the bi-disc, *Ann. of Math.*, **112** (1980), 179-201.

[6] Chang, S. Y. A., and Fefferman, R. The Calderón-Zygmund decomposition on product domains, *Am. J. of Math.*, Vol. 104, **3**, 455-468.

[7] Coifman, R. R., McIntosch, A., and Meyer, Y. L'intégrale de Cauchy definit un operateur borné sur L^2 pour les courbes lipschitziennes, *Ann. of Math.*, **116** (1982), 361-387.

[8] Coifman, R. R., and Meyer, Y. Au delà des opérateurs pseudo-differentiels, *Asterisque*, No. 57.

[9] David, G. and Journé, J. L. A boundedness criterion for generalized Calderón-Zygmund operators, *Ann. of Math.,* **120** (1984),. 371-397.

[10] Fefferman, C. and Stein, E. M. H^p spaces of several variables. *Acta. Math*, **129** (1972), 137-193.

[11] Fefferman, R and Stein, E. M. Singular integrals on product spaces, *Adv. in Math.*, Vol. 45, No. 2, 117-143.

[12] Journé, J. L. Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón, L. N. 994, *Springer-Verlag.*
[13] Lin, K. C. Harmonic Analysis on the Bidisc. Thesis U.C.L.A., 1984.
[14] Peetre, J. On convolution operators leaving $L^{p,\lambda}$ spaces invariant. *Ann. Math. Pura Appl.,* **72** (1966), 295-304.
[15] Rubio de Francia, J. L. A Littlewood-Paley inequality for arbitrary intervals. *Revista Ibero-Americana,* **2** (1985).

Université de Strasbourg
Département de Mathématique
7 rue René Descartes
67 Strasbourg, France

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 1, N.° 3, 1985

Special Positions for Surfaces Bounded by Closed Braids

Lee Rudolph*

A «braid» is an algebraic datum —an element of a certain group. A «closed braid» is a geometric construct from that datum— a knot or link in a certain sort of special position in the 3-sphere. By a theorem of Alexander, every (tame, oriented) link can be moved into that special position. In this way the algebra of braids has been brought to bear on various aspects of the geometrical theory of knots and links.

Now, every link is the boundary of surfaces of various kinds (e.g., embedded surfaces in S^3; ribbon-immersed surfaces in S^3; surfaces embedded in more or less restricted ways in D^4), and these surfaces are of interest not only for what they tell about their boundaries but also in themselves. It is natural to ask whether surfaces bounded by a closed braid can themselves be put into any sort of special position, which might or might not be constructible from some kind of algebraic data. These notes are concerned with various such constructions.

Here is a rough outline of the paper. In §1, I define closed braids and recall from [Rudolph 1] the notion of a *braided surface* in $D^2 \times D^2$ bounded by a closed braid in $S^1 \times D^2$. A braided surface in $D^2 \times D^2$ is essentially the same thing as a *ribbon surface* in D^4, and §2 gives a fairly detailed account of ribbon surfaces in D^4 and their relationship to *ribbon-immersed surfaces* in S^3. In §3, I use various simple branched coverings (first of $\mathbb{C}$ by $\mathbb{C}$, given by a complex polynomial of degree n with $n - 1$ distinct critical values; then of $S^1 \times \mathbb{C}$ by $S^1 \times \mathbb{C}$; finally of S^3 by S^3, branched over a trivial link of unknot-

* Research supported by the Fonds National Suisse.

ted circles) to establish «representation theorems» for braids and closed braids. (Sample: Proposition 3.10 shows that every closed braid on n strings is the inverse image in S^3 of a suitable unknot in S^3, *via* the covering alluded to above.) In §4, I continue this program, constructing *braided surfaces in* S^3which are ribbon-immersed (and correspond by «pushing into D^4» to the earlier braided surfaces), from *preband representations* of braids in the free *prebraid group*. The methods throughout are geometric, emphasizing «multivalued functions», although the surfaces constructed all have convenient algebraic descriptions.

In §5, I introduce the reader to the *Markov surfaces* which [Bennequin] has recently contributed to the study of closed braids, with signal success. An important subclass of the Markov surfaces (which includes all the incompressible ones), which I have named *Bennequin surfaces*, includes within it those braided surfaces in S^3 which are embedded (rather than simply ribbon-immersed). I show that, conversely, there is a formal sense in which the theory of Bennequin surfaces can be reduced to the theory of embedded braided surfaces in S^3.

In §6, I quote without proof Markov's Theorem and the important new *Inequality of Bennequin.* Using Markov's Theorem, I show that there is also a formal reduction of the theory of general (smooth, oriented —in short, *slice*) surfaces in D^4, bounded by a closed braid $\hat{\beta}$, $\beta \in B_n$, to the theory of band representations of the various usual injections $\beta^{(k)}$ of β into B_{n+k} (i.e., adding k extra trivial strings). I end with a discussion of the possibility that Bennequin's Inequality, which he has proved for the standard (Seifert) genus of a closed braid, might hold also for the slice (or Murasugi) genus.

I am very grateful to my friend José Montesinos, Chair of Topology at the University of Zaragoza, for inviting me to Zaragoza, where in April 1984 I gave four lectures of which the content has expanded into these notes; and more grateful that he urged me to write them down. It is a pleasure to acknowledge the support of the Fonds National Suisse, and of the Section de Mathématiques, Université de Genève—particular thanks being due to Professors Gerhard Wanner (Président de la Section) and Claude Weber, and to Michel Boileau for his helpful criticism of my preliminary treatment of the «slice-Bennequin conjecture».

Geneva, May 1984

CONTENTS

§1. Preliminaries; closed braids, braided surfaces, and band representations

Throughout, we will view the real plane $\mathbb{R}^2$ in its guise as the complex line $\mathbb{C}$. In particular, complex structures (plus the «outward normal» convention) orient such spaces as $D^2, D^4, \dots$.

1.1. *Notation.* The *round* 4-*ball* is $D^4 = \{(z, w) \in \mathbb{C}^2: N(z, w) \leqslant 2\}$, where $N(z, w) = |z|^2 + |w|^2$ is the *squared-norm* function. The *round* 3-*sphere* is $S^3 = \partial D^4$. (Note that D^4 and S^3 have radius $\sqrt{2}$. We will drop the adjective «round» most of the time.) We will write rD^2 for $\{z \in \mathbb{C}: |z| \leqslant r\}$, rS^1 for $\partial(rD^2)$, and drop the r if it equals 1. A *bidisk* is a product $r_1D^2 \times r_2D^2 \subset \mathbb{C}^2$ $(r_1, r_2 > 0)$. *We will always take* $r_1 = 1$, and write D_r for $D^2 \times rD^2$. The *unit bidisk* is $D = D_1$.

The boundary of a bidisk is a 3-sphere with corners. Let there be fixed, once for all, a smoothing homeomorphism h: $D \to D^4$ such that $h|S^1 \times S^1$ is the identity on $S^1 \times S^1 = S^3 \cap \partial D$, and $h|(D - S^1 \times S^1)$ is a diffeomorphism between the smooth manifolds-with-boundary $D - S^1 \times S^1$ and $D^4 - S^1 \times S^1$. We will write $\partial_1 D_r = S^1 \times rD^2$ and $\partial_2 D_r = D^2 \times rS^1$ for the two solid tori into which $S^1 \times rS^1$ splits ∂D_r. The smoothing h gives fixed product structures to the two solid tori $h(\partial_1 D)$ and $h(\partial_2 D)$ in S^3. We will use H, composed with a homothety of w, to smooth any bidisk boundary ∂D_r.

Fix an integer $n \geqslant 1$. The *n-fold symmetric product* of $\mathbb{C}$ is the quotient $\mathbb{C}^n/S_n$, where the symmetric group S_n of all permutations of $\{1, \dots, n\}$ acts on the space $\mathbb{C}^n$ of ordered n-tuples of points of $\mathbb{C}$ (i.e., maps $\{1, \dots, n\} \to \mathbb{C}$) naturally (by «permuting coordinates»). Let $E_n \subset \mathbb{C}[T]$ denote the complex affine space of dimension n consisting of the monic polynomials of degree n. By the Fundamental Theorem of Algebra, the map from $\mathbb{C}^n/S_n$ to E_n induced by $(z_1, \dots, z_n) \to (T - z_1) \dots (T - z_n)$ is a bijection (in fact, it is a homeomorphism if the symmetric product is given the quotient topology). *Henceforth we identify* $\mathbb{C}^n/S_n$ *and* E_n *by this map.* In particular, we give the symmetric product a smooth structure. We think of elements of $\mathbb{C}^n/S_n = E_n$ indifferently as unordered n-tuples of complex numbers (counting multiplicities), n-element subsets of $\mathbb{C}$ (counting multiplicities), or monic polynomials of degree n, as suits our convenience. The subset Δ_n, or simply Δ, of n-tuples with some multiplicity at least 2 (alternatively, monic polynomials with some repeated root) is called the *discriminant locus*; its complement $E_n - \Delta$ is the *configuration space* (*of n distinct points of* $\mathbb{C}$).

1.2 *Facts.* The discriminant locus is a complex algebraic hypersurface in the affine space E_n. It is irreducible (being the image of the hyperplane $\{z_1 = z_2\}$ in $\mathbb{C}^n$) but for $n \geqslant 3$ it is singular (along its subset of n-element multisubsets of $\mathbb{C}$ supported by $n-2$ or fewer distinct points). The isomorphism class of the fundamental group of the configuration space does not depend on the choice of basepoint (because Δ is of real codimension 2 and sufficiently well-behaved), and for any choice of basepoint this group is the normal closure of a single element (namely, the boundary of a 2-disk which meets Δ transversely at precisely one point of its dense, open, connected subset of regular points). There are exactly two (mutually inverse) conjugacy classes of such elements.

1.3 Definitions. The *n-string braid group* B_n (*based at* $* \in E_n - \Delta$) is the fundamental group $\pi_1(E_n - \Delta; *)$ of the configuration space. A *band* in B_n is an element represented by a loop bounding a disk in E_n which meets Δ transversely in a single regular point; a band is *positive* or *negative* according to the sign of its linking number with Δ (which, at least along its regular points, has a natural orientation because it is a complex algebraic set). Bands will be one of our principal technical tools in what follows: the name will be justified then.

1.4 *Conventions.* E_0 is a one-point set $\{0\}$, Δ_0 is empty, and B_0 is the one-element group with identity denoted o. The identity of B_n, $n \geqslant 1$, will be denoted $o^{(n)}$. Note that $B_1 = \{o^{(1)}\}$ is isomorphic to B_0 but not identical to it; more generally, each pair of groups B_n, $B_m (n \neq m)$ is *disjoint* (they are, after all, groups of homotopy classes of paths in disjoint spaces). This apparent pedantry will, I believe, be seen to pay off later.

1.5 Definitions. An oriented closed 1-manifold L (briefly, a *link*) embedded in a bidisk boundary ∂D_r is a *closed braid on n strings* if $L \subset \partial_1 D_r$ and $pr_1|L: L \to S^1$ is an orientation-preserving covering map of degree n. Closed braids in S^3 are defined *via* the smoothing h. A compact oriented 2-manifold-with-boundary S smoothly embedded in D_r is a *braided surface of degree n* if $pr_1|S: (S, \partial S) \to (D^2, S^1)$ is an orientation-preserving branched cover of degree n. Braided surfaces in D^4 are defined *via h*. (Neither «cover» nor «branched cover» is intended to imply that the total space is connected.) Note that the boundary of a braided surface is a closed braid.

If X is any set, an *n-valued complex function* on X is a function $f: X \to E_n$. The *graph* of an n-valued function f is $grf = \{(x, z): x \in X, z \in f(x)\} \subset X \times \mathbb{C}$.

1.6 Proposition. *A closed braid L on n strings is the graph of a* (*continuous*) *n-valued function on S^1 with values in the configuration space $E_n - \Delta \subset E_n$. A braided surface S of degree n is the graph of a* (smooth) *n-valued function on D^2.*

PROOF. Clear (the smoothness of the function associated to S comes from the implicit function theorem and the Fundamental Theorem of Algebra). □

Conversely, if f is any continuous function from S^1 to the configuration space, its n values are uniformly bounded in size by some constant, so its graph is a closed braid in some bidisk boundary ∂D_r.

1.7 *Notation.* If $\beta \in B_n$ then any closed braid on n strings which is the graph of a (based) loop in the homotopy class β will be called a *closure* of β, and denoted $\hat{\beta}$.

1.8 Proposition. *The set of closed braids on n strings, modulo the equivalence relation of isotopy through closed braids, is naturally in bijection with the set of conjugacy classes in B_n.*

PROOF. Both sets are naturally in bijection with the set of free homotopy classes of loops in the configuration space. □

The situation is less simple for braided surfaces, however. If g is an arbitrary continuous (even smooth) function from D^2 to E_n, then its graph certainly lies in all sufficiently large bidisks, but it need not be a braided surface—for indeed it need not be even topologically embedded.

1.9 *Example.* The function $D^2 \to E_2: z \to T^2 - z^2$ is smooth (in fact, complex analytic); its graph is the union of two copies of D^2 with the origins identified. Note that this function is not transverse to Δ_2.

The function $D^2 \to E_3: x + iy \to T^3 - 3(x^2 + y^2)T + x(1 + iy)$ is smooth, though not transverse to Δ_3. Its graph is not smooth at $(0, 0)$ though it is p.l. locally flat there.

The function $D^2 \to E_3: z \to T^3 - z^2$ is smooth, not transverse to Δ_3, and has a graph which is a topologically embedded disk which is not p.l. locally flat at one point.

The function $D^2 \to E_3: z \to T^3 - z$ is smooth and has a smooth graph even though it is not transverse to Δ_3.

To say precisely which functions into E_n have smooth graphs is a non-trivial problem. (It would involve having an explicit understanding of which closed braids are of the isotopy type, in the 3-sphere, of some completely split link of unknots.) However, for our purposes the generic situation suffices.

1.10 Definition. A braided surface S is *simple* if the branched covering $pr_1|S$ is simple (that is, the critical points are locally like either $z \to z^2$ or $z \to \bar{z}^2$ near $0 \in \mathbb{C}$, and the critical values are distinct).

The following is clear.

1.11 Proposition. *A simple braided surface S of degree n in D_r is the graph of a smooth n-valued function on D^2, transverse to Δ_n, which has all its values bounded in absolute value by r; and conversely. Every braided surface of degree n can be arbitrarily closely approximated by simple braided surfaces of degree n.* □

1.12 Definitions. A *band representation of length $l \geqslant 0$ in B_n* is an ordered l-tuple $\underline{\underline{b}} = (b(1), \ldots, b(l))$ where each $b(j)$ is a band, positive or negative, in B_n (Def. 1.3.). We write $l(\underline{\underline{b}})$ for l. The *braid of* $\underline{\underline{b}}$ is $\beta(\underline{\underline{b}}) = b(1)b(2)\ldots$ $\ldots b(l) \in B_n$; the *closed braid of* $\underline{\underline{b}}$ is $\hat{\beta}(\underline{\underline{b}})$, the closure of $\beta(\underline{\underline{b}})$. (If $l = 0$ then $\underline{\underline{b}}$ is the empty tuple, with braid $o^{(n)}$ and closed braid $\hat{o}^{(n)}$, the simplest closed braid representing the completely split link of n unknots.)

1.13 Proposition. *To each band representation $\underline{\underline{b}}$ of length l in B_n can be associated a simple braided surface $S(\underline{\underline{b}})$ of degree n with l branch points of $pr_1|S(\underline{\underline{b}})$ and $\partial S(\underline{\underline{b}}) = \hat{\beta}(\underline{\underline{b}})$. Up to isotopy though simple braided surfaces (covering an isotopy of D^2) every simple braided surface is some such $S(\underline{\underline{b}})$. The various band representations $\underline{\underline{b}}$, such that a given simple braided surface can be so isotoped to $S(\underline{\underline{b}})$, are all related to each other in a reasonable way.*

Outline of proof (for more details, consult [Rudolph 1]): The ordered composition of loops in the configuration space representing the bands $b(1), \ldots, b(l)$ is a loop which extends to a map of D^2 into E_n which meets Δ_n transversely in l points, each corresponding to one of the bands in the composition. A small perturbation of this map is smooth everywhere and its graph is a surface $S(\underline{\underline{b}})$ with the desired properties.

Conversely, given a smooth map $g: D^2 \to E_n$ transverse to Δ_n in l points, any system of arcs in D^2 joining the points of $g^{-1}(\Delta_n)$ to the basepoint of S^1, and disjoint except for that common basepoint, provides one with a band representation $\underline{\underline{b}}$ of the homotopy class of $g|S^1$, of length l, with gr g a particular $S(\underline{\underline{b}})$. Two different such systems of arcs differ by an autohomeomorphism of D^2 which fixes S^1 pointwise and $g^{-1}(\Delta_n)$ setwise. The group of such autohomeomorphisms (which is, as a matter of fact, isomorphic to B_l) acts on the set of band representations of length l; this group is generated by *slides* (or, in [Moishezon]'s language, *elementary transformations*) $(b(1), \ldots, b(i), b(i+1), \ldots, b(l)) \to (b(1), \ldots, b(i)b(i+1)b(i)^{-1}, b(i), \ldots, b(l))$. (Note that the conjugate of a band is of course again a band.) □

1.14 *Remarks.* The boundary of a braided surface, as we have defined it, is a smooth closed braid. We did not require that a closed braid be smooth. But no generality would be lost if we did: for pr_1 induces a normal bundle for any closed braid L, and L is therefore tame, and can be isotoped (even though closed braids) to be smooth.

Or course, every smooth closed braid bounds simple braided surfaces. They are not unique, for indeed, one may always increase the number of branch points by two. On the level of band representations, this corresponds to replacing $(b(1), \ldots, b(l))$ by the *elementary expansion* $(b(1), \ldots, b(l), w, w^{-1})$, for any band w. It is shown in [Rudolph 1] that any two band representations of a given braid in B_n may be joined by a sequence of elementary expansions, slides, and *elementary contractions* (inverses, when possible, of elementary expansions).

We conclude this section with a digression—a proof in the language of multivalued functions of a well-known and interesting fact.

1.15 *Scholium. The configuration space $E_n - \Delta$ is an Eilenberg-MacLane space (that is, its higher homotopy groups $\pi_k(E_n - \Delta)$, $k \geqslant 2$, all vanish).*

PROOF BY INDUCTION ON n. Clearly $E_1 - \Delta = E_1 = \mathbb{C}$ is contractible. Let n be greater than 1, and let $f\colon S^k \to E_n - \Delta$ be a continuous map of a k-sphere, $k > 1$, into the configuration space. We will show that f is freely homotopic to a constant map, which will prove the theorem. Consider $\operatorname{gr} f$ in $S^k \times \mathbb{C}$. This is a covering space of S^k; because S^k is simply connected, it is a trivial covering space, i.e., $\operatorname{gr} f$ is the union of n disjoint graphs of 1-valued functions $f_1, \ldots, f_n$. Clearly f is homotopic to f' in $E_n - \Delta$, where $\operatorname{gr} f'$ is the union of the graphs of the n functions $f_1 - f_n, f_2 - f_n, \ldots, 0$, and for $i = 1, \ldots, n-1$, the function $f_i - f_n$ is nowhere zero on S^k. Then each $f_i - f_n$ lifts to a function g_i on S^k with $\exp g_i = f_i - f_n$; since the graphs of the $f_i - f_n$ are pairwise disjoint, so are the graphs of $g_1, \ldots, g_{n-1}$, and thus their union is the graph of a continuous $(n-1)$-valued function g on S^k. A homotopy of g to a constant gives a homotopy of f', and thus of f, to a constant. □

§2. Ribbon surfaces and braided surfaces

Recall that N is the squared-norm function from D^4 to $[0, 2]$.

2.1 Definitions. A smooth function ω from a compact 2-manifold-with-boundary S to $[0, 2]$ is *topless* if $\omega^{-1}(2) = \partial S$, ω has no critical points in a collar of ∂S, and no critical point of ω (in $\operatorname{Int} S$) is a local maximum of ω. (Note that no non-degeneracy assumptions are put on the critical points of ω; but if ω is, say, real-analytic—and presumably in general—then suitable arbitrarily small perturbations of ω are both topless and Morse. In particular, if S supports a topless function then S is itself *topless* in the sense that it has a handle decomposition without 2-handles; alternatively, S has no closed components.) A surface S embedded smoothly and *properly* (i.e., $\partial S = S \cap S^3$) in D^4 is *ribbon-embedded* if $N|S$ is topless, and S is *ribbon* if it is ambient isotopic to a ribbon-embedded surface.

We define ribbon-embedded and ribbon surfaces in D *via* the smoothing homeomorphism h.

2.2 *Remarks.* Ribbon-embedded surfaces arise in nature. One class of examples comes from complex analytic geometry. Let U be an open set in $\mathbb{C}^2$ containing D^4, and let $\Gamma \subset H$ be a non-singular complex analytic curve (i.e., locally in U, Γ is the zero-set of a complex-analytic function with non-zero gradient), so that Γ intersects S^3 transversely. Then the surface $S = \Gamma \cap D^4$ is ribbon-embedded. (This may be proved directly by using a local parametrization of S. [More generally, the composition of M with the resolution of a singular piece of complex curve will be topless too.] Or one may appeal to the much more general theorem in [Milnor] on Stein manifolds.)

2.3 *Example.* Let $U = \mathbb{C}^2$, and let Γ be defined by $4zw = 1$. Then S is an annulus naturally parametrized by $A = \{\zeta \in \mathbb{C}: 4 - \sqrt{15} \leqslant |\zeta|^2 \leqslant 4 + \sqrt{15}\}$ under the map $f: \zeta \to (\zeta/2, 1/2\zeta)$. Considering $N \circ f$, we see that the critical points of $N|S$ are all degenerate—they form a circle of local minima. Nonetheless, with our definition the surface is ribbon-embedded as it should be. Of course, an arbitrarily small linear perturbation of S will replace it with an equivalently embedded ribbon-embedded annulus on which N has a single minimum and a single saddlepoint, both non-degenerate. (This is a special case of an observation of Nomizu and Cecil.)

2.4 *Remarks* (*continued*). Another class of naturally occuring ribbon-embedded surfaces (which in fact includes the complex curves) consists of smooth minimal surfaces in D^4, in the sense of differential geometry. In fact, [Hass] proves a converse: every isotopy class of ribbon surfaces in D^4 contains minimal surfaces.

(It is important, by the way, to understand that Hass's result concerns the *round* ball—of course, the value of the radius is irrelevant—with the *flat* metric induced from $\mathbb{C}^2$. Presumably such other nice metrics as those of constant, non-zero curvature could also be used. But it is easy to find, cf. [Rudolph 4], for any smooth topless orientable S in D^4, a smooth embedding i of D^4 in $\mathbb{C}^2$ which carries S onto a surface in the non-round ball $i(D^4)$

minimal with respect to its flat metric; or, alternatively, which pulls back that flat metric to a non-flat metric on the round ball, in which S is minimal. Yet, as we shall recall shortly, there are surfaces in D^4 isotopic to no ribbon surface.)

2.5 *Question.* Of course there are non-orientable ribbon surfaces, while every complex curve is naturally oriented. But: *does every isotopy class of orientable (oriented?) ribbon surfaces in D^4 contain a piece of complex curve?* (The answer is presumably «no» but I know of no proof.)

2.6 *Remarks (concluded).* Not every smooth, properly embedded, topless surface in D^4 is ribbon. For relative Morse theory shows that if S is a ribbon with tubular neighborhood u in D^4, then the exterior $D^4 - u$ of S can be built from a collar of the exterior $S^3 - u$ of ∂S in S^3 by attaching handles of index 2, 3, and 4 only. In particular the inclusion-induced homomorphism $\pi_1(S^3 - \partial S) \to \pi_1(D^4 - S)$ is onto. Yet there are, for instance, many smooth 2-disks in D^4 bounded by an unknot and having a larger group than $\mathbb{Z}$ as fundamental group of the complement. Such disks are not ribbon disks.

2.7 Proposition. *A braided surface is a ribbon surface.*

PROOF. Comparing $M \circ h|S$ with $|pr_1|S|^2$, we see that—perhaps after an initial vertical isotopy to make $pr_2|S$ uniformly very nearly zero—the evident toplessness of the latter imposes toplessness on the former. □

If S is a simple braided surface of degree n with l branch points of $pr_1|S$, the proof shows S is isotopic to a ribbon-embedded suface on which $N \circ h$ is Morse with n local minima and l saddles.

The following theorem is proved in [Rudolph 1]; a variant on the proof (which needs only minor modifications, along the lines of the first proof, to cover the general case) is presented in [Rudolph 2].

2.8 Theorem. *Every oriented ribbon surface is ambient-isotopic to a braided surface.* □

2.9 *Remark.* The isotopy constructed in the cited proof(s) is generally «large», and cannot be expected to be «conservative»—that is, the isotopy cannot usually be taken to be relative to a part of S on which pr_1 already happens to be a branched cover of its image. For instance, [Morton] gives an example of a 4-string closed braid L which is unknotted in ∂D and thus certainly bounds ribbon disks in D; his proof that L is «irreducible» (in a certain sense) actually shows more, namely, that any braided surface bounded by L has genus at least 1 (read the proof in conjunction with Example 5.2 of [Rudolph 1]), so an isotopy of a ribbon disk bounded by L into braided position must move L quite far (across $\partial_2 D$, in fact).

Although we will give no proof of Theorem 2.8, it may be remarked that the proof is essentially 3-, rather than 4-, dimensional, and makes heavy use of the notion of ribbon-immersed surfaces in S^3, which we now introduce for other purposes.

2.10 Definition. Let S be a topless (not necessarily oriented, or orientable), surface. A mapping f: $S \to S^3$ is a *ribbon immersion* if it has the following properties:

1. f is a smooth immersion without triple points;
2. in the domain S of f, the double points consist of $2r$ pairwise disjoint closed arcs $A'_1, A''_1, \ldots, A'_r, A''_r$ with $f(A'_k) = f(A''_k)$, $k = 1, \ldots, r$, such that each A'_k is contained in $\operatorname{Int} S$ *and each* A''_k has both endpoints (and no other points) on ∂S; and
3. along the r arcs $A_k = f(A'_k)$ of double points of f in the range, the two sheets of $f(S)$ cross transversely.

Of course the arcs A_k may be quite twisted, but there is an ambient isotopy of S^3 carrying them onto short «straight» (e.g., geodesic) arcs, and after such an isotopy a ribbon immersion looks, locally in domain and range, like figure 1.

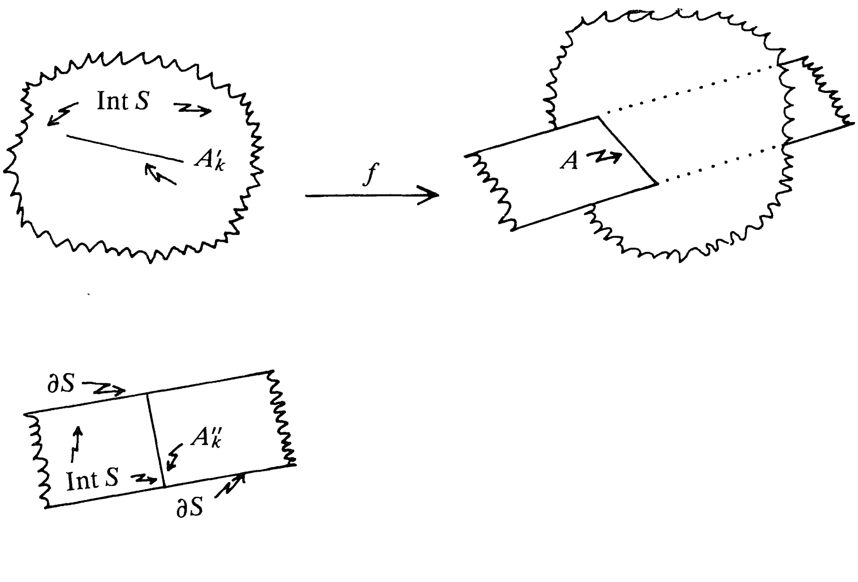


Fig. 1

2.11 Definition. Let f, S, $A'_1, \ldots, A''_r$ be as above. Then a topless Morse function ω on S is *adapted* to f if is strictly positive and for each $j = 1, \ldots, r$, $\omega(A'_j) \cap \omega(A''_j) = \varnothing$. A topless handle decomposition $S = h^0_1 \cup \ldots \cup h^0_s \cup h^1_1 \cup \ldots \ldots \cup h^1_t$ is *adapted* to f if each A'_i is interior to some h^0_j and each A''_i is proper in some h^1_k.

It is easy to see that, given f and S, there do indeed exist both adapted topless Morse functions and adapted handle decompositions, and indeed that the many-many correspondence of functions and decompositions preserves adaptation to f.

2.12 *Construction.* Let f be a ribbon immersion of S in S^3. Then for any topless Morse function ω adapted to f, the map $\frac{1}{2}\omega f$: $S \to D^4$: $x \to \frac{1}{2}\omega(x)f(x)$ is a ribbon embedding of S in D^4. (The factor $\frac{1}{2}$ is due to our convention that D^4 has radius $\sqrt{2}$.) We will call it *the push-in of f by the factor $\omega/2$.*

PROOF. The push-in is an embedding because the only possible double points in the range are separated by the radial coordinate, by definition of adaptation to f. It is ribbon because on the image the functions ω and N are essentially the same. □

2.13 Proposition. *Each ambient-isotopy class of ribbon surfaces in D^4 contains a ribbon-embedded surface which is the push-in of a ribbon immersion in S^3 by an appropriate factor. Different push-ins of the same ribbon immersion are ambient isotopic.*

Idea of proof (see, e.g., [Tristram] for more details, in a different language): Start with a ribbon-embedded surface in the given isotopy class, not containing $(0, 0)$. By isotopy, this surface may be assumed to have all its local minima for the restriction of N in the interval $]0, 1[$ and all its saddle values in $]1, 2[$. At this stage, an application of relative Morse theory yields a topless handle decomposition of the surface, and a ribbon immersion to which that decomposition is adapted, such that the push-in of that immersion by the appropriate adapted factor (half the square root of the restriction of N to the surface) is isotopic to the surface by an isotopy leaving N invariant. The second statement is proved similarly. □

2.14 *Remark.* Given f and S, an adapted Morse function on S may well need to have more critical points than the minimal number for a topless, but not adapted, Morse function. On the other hand, an obvious construction of adapted handle decomposition —which makes each A''_i a transverse arc of a different 1-handle, and engulfs each A'_i by a different 0-handle, and uses as many more handles as necessary to get adaptation— is likely to use far too many handles.

Figure 2 depicts a ribbon-immersed disk (bounded by the square knot), together with immersions onto it from (*a*) a disk with an adapted handle decomposition displayed, (*b*) a disk with some level sets of an adapted Morse function drawn in.

In §4, we will see how to go from a band representation $\underline{\underline{b}}$ in B_n (more correctly, from a «preband» representation that maps onto $\underline{\underline{b}}$) directly to a ribbon immersion of (the abstract surface) $S(\underline{\underline{b}})$ in S^3, in such a way that a natural push-in of this immersion recovers $S(\underline{\underline{b}})$ as braided surface. (See also 5.22.)

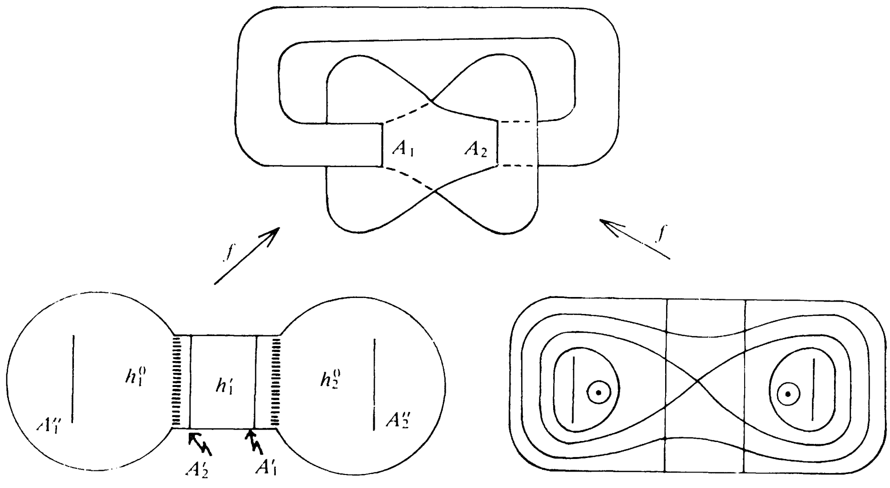


Fig. 2

§3. Prebraids and «standard generators» of the braid group

Let $V = \{v_1, \dots, v_{n-1}\}$ be a set of $n - 1$ (pairwise distinct) complex numbers, v_0 a basepoint in $\mathbb{C} - V$. Let F_{n-1} denote the group $\pi_1(\mathbb{C} - V; v_0) = \pi_1((\mathbb{C} \cup \cup \{\infty\}) - (V \cup \{\infty\}); v_0)$. Of course F_{n-1} is a free group of rank $n - 1$. More specifically: there are $n - 1$ arcs I_j' in $\mathbb{C}$, such that I_j' has endpoints v_0 and v_j, $I_j' \cap I_k' = \{v_0\}$ if $j \neq k$, and the counterclockwise cyclic order of the I_j' at their common endpoint v_0 is $I_1', \dots, I_{n-2}', I_{n-1}', I_1'$; and there are closed 2-cells N_j', with $N_j' \cap N_k' = \{v_0\}$ if $j \neq k$, and I_j' embedded in N_j' like a radius in D^2. Then the elements $x_1, \dots, x_{n-1}$ of F_{n-1}, where x_j is represented by the boundary of N_j' traversed once counterclockwise, are free generators of F_{n-1}, which we call *the standard generators of* F_{n-1} *with respect to the star* $\bigcup_{j=1}^{n-1} I_j'$. (The generators don't depend on the N_j', only on the star; in fact, only on the star up to ambient isotopy fixing $V\cup\{v_0\}$ at all times. Differently embedded stars, however, do give different sets of standard generators—which are of course related by easily understood moves. We will not be concerned with this.)

Represent F_{n-1} on S_n (the permutations of $\{1,\ldots,n\}$) by sending x_j to the transposition $(j\,j+1)$, for each j. In the usual way, the regular covering space of $(\mathbb{C}\cup\{\infty\})-(V\cup\{\infty\})$ which corresponds to this representation has a unique completion to a branched covering space $p: X\to\mathbb{C}\cup\{\infty\}$ branched over $V\cup\{\infty\}$. The map p has *simple* critical values at the points of V (two sheets come together) and complete branching (all sheets coming together in an n-cycle) over ∞, and X is a compact, connected surface.

By calculating Euler characteristics, one finds that X is homeomorphic to a sphere. The map p induces on X a unique complex structure for which p is complex analytic, so with that structure X must be biholomorphic to the Riemann sphere $\mathbb{C}\cup\{\infty\}$ itself. But now p must be a rational map. As only one point of X maps to ∞, there are coordinates for X in which p is a polynomial. Now we drop the notation X and forget about the points at infinity. We have proved the following.

3.1 Proposition. *With V as above, there is a polynomial $p: \mathbb{C}\to\mathbb{C}$ of degree n with simple critical points, and critical values V, so that p realizes the branched covering of $\mathbb{C}$ which corresponds to the given representation of F_{n-1}. (We may take p to be monic.)* □

3.2 *Remark.* A critical point of a polynomial is one where the derivative vanishes; it is simple iff the second derivative is non-zero there. Of course it is a generic property of polynomials to have all critical points simple and all critical values distinct.

3.3 *Example.* Let $n=3$, $V=\{2,-2\}$, $I_1'=[0,2]$, $I_2'=[-2,0]$. Then we can take $p(z)=z^3-3z$. The critical points of p are 1 and -1, with corresponding critical values -2 and 2 as desired.

Consider $p^{-1}(N_j')$ in the domain of p. Since N_j' contains a single, simple critical value of p, this inverse image has $n-1$ components. Let N_j denote that one which contains a critical point of p, so $p|N_j: N_j\to N_j'$ is a 2-sheeted cyclic branched cover of a disk by a disk. Let $I_j=N_j\cap p^{-1}(I_j')$, so I_j is embedded in N_j like a diameter in D^2. Evidently the endpoints of I_j are points of $p^{-1}(v_0)$. In fact, we can number the points of $p^{-1}(v_0)$ as $z_1,\ldots,z_n$ in such a way that I_j has endpoints z_j and z_{j+1}, $j=1,\ldots,n-1$. The union $I=\bigcup_{j=1}^{n-1} I_j$ is an arc (which p folds onto the star used to define the standard generators x_j of F_{n-1}) with endpoints z_1 and z_n, containing all the z_j in order.

3.4 Definition. Let $J \subset \mathbb{C}$ be an arc with endpoints w_1 and w_n containing, in linear order, n distinct points $w_1, \ldots, w_n$. Let Q_j be a closed 2-cell in $\mathbb{C}$ which intersects J along J_j, its subarc with endpoints w_j and w_{j+1}, in such a way that J_j is embedded in Q_j like a diameter and $Q_j \cap Q_k$ contains either one point or none, depending on whether $|j-k| = 1$ or $|j-k| > 1$. Realize the n-string braid group B_n as $\pi_1(E_n - \Delta; \{w_1, \ldots, w_n\})$. Then for $j = 1, \ldots, n-1$, the *standard generator* σ_j *of* B_n (*with respect to the given basepoint and given arc* J) is the homotopy class of the loop l_j: $(S^1, 1) \to (E_n - \Delta, \{w_1, \ldots, w_n\})$: $\exp i\theta \to \{s(\theta), t(\theta), w_1, \ldots, \hat{w}_j, \hat{w}_{j+1}, \ldots, w_n\}$, where $s(\theta), t(\theta)$ are the preimages of $\exp i\theta$ by a fixed (for instance, by using arc length if ∂Q_j is rectifiable) double cover of S^1 by ∂Q_j such that $1 \in S^1$ is covered by $\{w_j, w_{j+1}\}$ and the cover respects orientations.

3.5 *Remarks.* The notation σ_j makes no reference to n. Since (cf. 1.4) the groups B_n are disjoint, this—though hallowed by use, justified by algebra, and undoubtedly convenient—is geometrically unfortunate. ...Nor does the notation indicate the arc J. Clearly, choices of J which differ by isotopies fixing each w_j at all times give the same «standard generators» (nor do the choices of Q_j, etc., matter); but differently embedded arcs give different sets of generators, which, however, differ by understandable moves. Abstractly, of course, all such sets of generators are identical, in that they differ by automorphisms of B_n (induced by homeomorphisms of $\mathbb{C}$ fixing each w_j).

3.6 Proposition. *Each* σ_j *is a positive band in* B_n. *The set* $\sigma_1, \ldots, \sigma_{n-1}$ *of standard generators of* B_n *is, indeed, a set of generators of* B_n.

PROOF. The first phrase follows from the observation that the double cover of S^1 by ∂Q_j used to define σ_j extends to a 2-sheeted cyclic branched cover of D^2 by Q_j, and that the 2-valued inverse to this, extended to be n-valued by the $n-2$ constants w_i $(i \neq j, j+1)$, is a map of D^2 into E_n which—with a minimal amount of care—is transverse to Δ which it meets in one point, positive by orientation arguments.

The rest of the proposition is due to [Artin]. A proof may be given along these lines: take $w_j = j$, $J = [1, n]$, Q_j the round 2-disk of radius 1/2 centered at $j + \frac{1}{2}$. With the right choice of double cover $Q_j \to S^1$, the real parts of the n values of the loop l_j look as drawn in Figure 3A (where S^1 has been cut open into $[0, 2\pi]$). By isotopies (through closed braids, respecting a basepoint) any closed braid can be first put into general position with respect to projection of its n values onto their real parts, then moved until it is a composition of (appropriately rescaled) pictures like that in Figure 3A; cf. Figures 3B, 3C. □

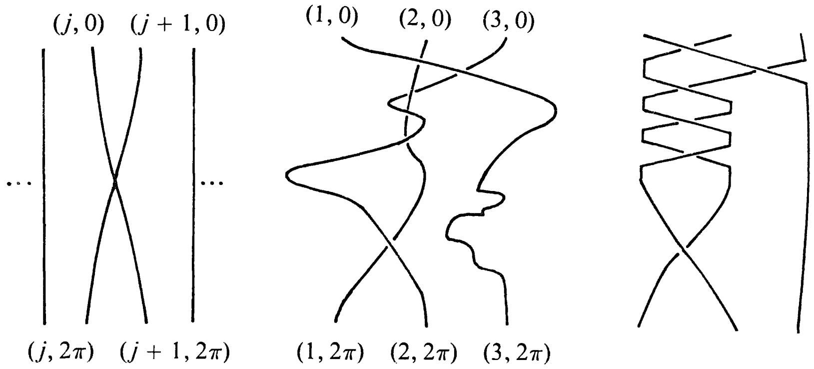


Fig. 3

3.7 *Remarks.* The standard generators satisfy some geometrically obvious relations, namely, for $i = 1, \dots, n-2$, R_i: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and for $1 \leqslant i < j-1 \leqslant n-1$, R_{ij}: $\sigma_i\sigma_j = \sigma_j\sigma_i$. We will not scruple to use these. It is somewhat less obvious that all relations in B_n are consequences of these; since we will not use this fact, we refer the reader to various published proofs, in [Birman] and sources cited there (beginning with [Artin]).

(In the light of what we will do next, an alternative proof of 3.6. and 3.7. can presumably be given by carefully following through the details of a Lefschetz-style analysis of the fundamental group of an algebraic surface and its plane sections.)

We return to the polynomial map p: $\mathbb{C} \to \mathbb{C}$ of degree n.

3.8 Proposition. *The n-valued map* $p^{-1}: \mathbb{C} \to E_n$ *maps* $\mathbb{C} - V$ *into the configuration space* $E_n - \Delta$. *The induced homomorphism on* π_1, $(p^{-1})_*: F_{n-1} \to B_n$ (*where we use* $p^{-1}(v_0)$ *as basepoint of* B_n), *is given by* $x_j \to \sigma_j, j = 1, \dots, n-1$.

Proof. A choice of arc J is implicit in the notation $\sigma_1, \dots, \sigma_{n-1}$. Take $J = I$ (as defined before 3.4.), $Q_j = N_j$. The composition $(p^{-1}) \circ m_j$, where m_j is a loop representing x_j, isn't quite a loop l_j of the sort required in 3.4., but it differs only by inessential activity in the $n-2$ 2-cells other than N_j which lie in $p^{-1}(N_j')$.

(The parenthesis at the end of 3.7. is based on the observation that p^{-1} is a *linear* map into E_n. Indeed, writing $w = p(z)$, we have $p^{-1}(w) = \{z \in \mathbb{C}: p(z) = = w\} = \{z: p(z) - w = 0\}$, so p^{-1} linearly parametrizes a line on which only the constant term of the monic polynomial changes. If this line is in general position with respect to Δ—and it is—then Lefschetz tells us that F_{n-1} maps onto B_n. Further, [Zariski] and [van Kampen] tell us that moving the line around appropriate loops of lines gives all relations.)

3.9 Definitions. With V, p, etc., as above, the *prebraid group* is F_{n-1}. A *closed prebraid* is the graph in $S^1 \times \mathbb{C} - V$ of a loop $f: S^1 \to \mathbb{C} - V$. (Note that a closed prebraid, as a subset of an appropriately large $S^1 \times rD^2$, is a 1-string closed braid in ∂D_r and in particular an unknot.) A *criticized closed prebraid* is the union of a closed prebraid and $S^1 \times V$; appropriately oriented, a criticized closed prebraid is a closed n-string braid of a very special sort, in sufficiently big bidisk boundaries.

3.10 Proposition. *The set in $S^1 \times \mathbb{C}$, which is the inverse image by $id_{S^1} \times p$ of a closed prebraid, is a closed n-string braid (in every sufficiently big bidisk boundary). Every isotopy class of closed braids on n strings contains such covers of prebraids.*

PROOF. The first statement is evident. The second is the geometric counterpart of surjectivity of $(p^{-1})_*$ (3.6., 3.8.). $\square$

In §§4-5 we shall see how the use of prebraids can simplify various constructions of surfaces bounded by closed braids.

3.11 *Remark.* All the material in this section can be somewhat generalized, as follows. Instead of representing F_{n-1} in S_n by $x_j \to (j\, j+1)$, take some other representation in which each x_j goes to a transposition and the product $x_1 \ldots x_{n-1}$ goes to the same n-cycle $(1\, n\, n-1 \ldots 3\, 2)$ as before. Then, again, the corresponding simple covering can be taken to be a polynomial p of degree n with simple critical points, and critical values $v_1, \ldots, v_{n-1}$; now, however, the interesting part of the preimage of the star $\bigcup_{j=1}^{n-1} I_j$ is a tree T which is not necesarily an arc. (It is combinatorially equivalent to the tree on vertices $1, \ldots, n$ with an edge for each transposition which is the image of an x_j.) As before, T can be thickened into a «cactus» on which p is 2:1 onto a neighborhood of the original star; and one can read off from the tree certain generators (represented by motions of points inside the cactus) of B_n which might be called T-standard and which are the images by $(p^{-1})_*$ of the x_j. For $n \geqslant 4$, however, these generators do *not* have to be equivalent by automorphism to the standard generators.

3.12 *Example.* If T is a triod Y, we can express a set of Y-standard generators of B_4 in terms of the standard standard generators as σ_1, $\sigma_2\sigma_3\sigma_2^{-1}$, σ_2. No automorphism of B_4 carries these three elements to σ_1, σ_2, σ_3 in any order (consider the standard relations, and the calculation $\sigma_i(\sigma_2\sigma_3\sigma_2^{-1})\sigma_i^{-1}(\sigma_2\sigma_3\sigma_2^{-1})^{-1} \neq$ $\neq o^{(4)}$ for $i = 1, 2$).

3.13 *Remark, concluded.* The construction in [Rudolph 3] of the fibration of the complement of a closed strictly positive braid works equally well for braids that are strictly positive (or more generally, strictly homogeneous) in any fixed set of T-standard generators.

§4. Prebands, tadpoles, and ribbon immersions

As in §3, $p: \mathbb{C} \to \mathbb{C}$ is a polynomial of degree n with $n - 1$ distinct critical values $v_1, \ldots, v_{n-1}$ which form the set V, v_0 is a basepoint in $\mathbb{C} - V$, and F_{n-1} is the (free) prebraid group $\pi_1(\mathbb{C} - V; v_0)$, with standard generators $x_1, \ldots, x_{n-1}$.

4.1 Definition. A *positive* (resp., *negative*) *preband in* F_{n-1} is a conjugate of a standard generator (resp., the inverse of a standard generator). Note that $(p^{-1})_*: F_{n-1} \to B_n$ maps each preband to a band (of the same sign), but for $n \geqslant 3$ the preimage of any band contains both prebands and prebraids that are not prebands.

4.2 Definition. A (smooth) map τ: $(D^2, 1) \to (\mathbb{C}, v_0)$ is a *standard tadpole* if it has the following properties:

(1) for $\operatorname{Re} z$, the real part of z, non-negative, $\tau(z) = \tau(\operatorname{Re} z)$;
(2) $\tau|(\operatorname{Int} D^2 \cap \{z: \operatorname{Re} z < 0\})$ is a diffeomorphism onto an open set $U(\tau)$; the closure $\overline{U(\tau)}$ is disjoint from v_0 and includes precisely one point of V, namely $v_{i(\tau)}$, which is $\tau(-1/2)$;
(3) $\tau|[0, 1]$ is an immersion in general position (i.e., it has no triple points and only finitely many double points, at which tangent directions are distinct) into the complement of $\overline{U(\tau)} \cup V$, and $\tau^{-1}(v_0) = 1$;
(4) $\tau|(S^1 - \{1\})$ is an immersion.

A *tadpole* is a composition $\tau \circ \delta^{-1}$, where δ is a diffeomorphism of $(D^2, 1)$ with some other smooth 2-cell and τ is a standard tadpole.

We note that (1) forces τ/S^1 to hve tangent vector zero at 1, so we can't strengthen (4); nor is $\tau \mid (S^1 - \{1\})$ in general position, for again by (1) the entire right semicircle consists of (at best) double points of this immersion. From (2) and (4) we deduce that $\overline{U(\tau)}$ is a homeomorph of D^2, with boundary smooth except at the single point which is the image by τ of the diameter of D^2 on the imaginary axis; there, the boundary has a (generalized, real) «cusp», i.e., there is a one-sided tangent line.

We call $\overline{U(\tau)}$ the *head* of τ, $v_{i(\tau)}$ the *eye* of τ, and the immersed arc $\tau([0, 1])$ the *tail* of τ. The head of τ and τ itself are called *positive* or *negative* depending on whether the orientation induced on $U(\tau)$ by the range agrees or disagrees with that induced by the domain. If we abuse language slightly and call the image of τ, rather than the map τ, a tadpole, no harm is done so long as we remember orientation.

The following lemma is clear, and clearly the motivation for these definitions.

4.3 Lemma. *A prebraid is a preband if and only if it is represented by the boundary of a tadpole.* $\square$

4.4 Definition. A tadpole is *embedded* if its tail is embedded. A preband is *embedded* if it can be represented by the boundary of an embedded tadpole.

4.5 *Construction.* Fix, once for all, a homeomorphism α: $[-1, 1] \to [0, 2]$ such that $\alpha \mid [-1, 0]$: $x \to (1 - x^2)^{1/2}$, α is C^∞ on $]-1, 1[$, and α^{-1} is C^∞ at 1 and has all derivatives 0 there. (If we are willing to work with C^1 surfaces and braids—and there is no real reason not to—one could take $\alpha \mid [0, 1]$: $x \to 2 - (1 - x^2)^{1/2}$.) Call the region $T = \{z \in \mathbb{C}\colon \operatorname{Re} z \in [-1, 1], |\operatorname{Im} z| \leqslant \alpha(\operatorname{Re} z)\}$ the *tongue*; the tongue contains D^2 and any standard tadpole τ has a unique extension, which we shall also denote τ, over T which preserves property (1). Note that T has a boundary which is smooth except for two corners, at $1 \pm 2i$, where it has (infinitely flat) cusps.

The function $\mathbb{C} - \{3\} \to S^1\colon z \to (z - 3)/|z - 3|$ restricts to a map $\eta_{I_0}\colon T \to S^1$ with image a closed subinterval $I_0 = \exp i\left[\frac{3}{4}\pi, \frac{5}{4}\pi\right]$, which sends the corners of T to the endpoints of the interval and has each other level set a closed line segment in T. Write η_I for the composition η_{I_0} with the direct similarity (in the affine structure of $S^1 \approx \mathbb{R}/2\pi\mathbb{Z}$) which carries I_0 onto I, any other (non-trivial) closed subinterval of S^1. A function η_I is a *height* for T.

Finally, given a (standard) tadpole τ (as extended to T), we construct the map $R(\tau, I)\colon T \to S^1 \times \mathbb{C}\colon z \to (\eta_I(z), \tau(z))$, and call it (or, abusively, its image) a (*standard*) *geometric preband.*

4.6 Proposition. *A geometric preband is a ribbon immersion of T into $S^1 \times \mathbb{C}$. It is an embedding if and only if the tadpole involved is embedded.*

(In 2.10, ribbon immersions were defined with target S^3, and on manifolds without corners, but it is clear what should be meant.)

PROOF. On the right half of the tongue, $R(\tau, I)$ is, essentially, the restriction of the product of an embedding η_I and τ, an immersion on each level set of η_I; so $R(\tau, I)$ is an immersion there. On the left half of the tongue, τ itself is an immersion, so $R(\tau, I)$ is there, too. Now we need only check the double points. They are all due to double points of the tail of τ. If $\tau(s) = \tau(t)$, $0 < s < t < 1$, then each of the intervals I, $\eta_I^{-1}(\{\mathrm{Re} z = t\})$, $\eta_I^{-1}(\{\mathrm{Re}\, z = s\})$ contains the next in its interior. Write $A'' = \{z \in T: \mathrm{Re}\, z = s\}$, $A' = \{z \in T: \mathrm{Re}\, z = t, \eta_I(z) \in \eta_I(A'')\}$, $A = R(\tau, I)(A')$. Then also $A = R(\tau, I)(A'') = \eta_I(A'') \times \{\tau(s)\}$; A'' is a proper arc in T, and A' is an interior arc; and all such pairs of arcs A', A'' exhaust the double points of $R(\tau, I)$ and enjoy the properties required in Definition 2.10. □

4.7 Definition. (Compare with 1.12.) *A preband representation of length l in F_{n-1}* is an l-tuple $\underline{\underline{x}} = (x(1), \ldots, x(l))$ of prebands in F_{n-1}. The **prebraid** of $\underline{\underline{x}}$ is $\rho(\underline{\underline{x}}) = x(1) \ldots x(l)$. The *closed prebraid of* $\underline{\underline{x}}$ is $\hat{\rho}(\underline{\underline{x}}) \subset S^1 \times (\mathbb{C} - V)$.

The following lemma is easily proved, and provides one way to get around a slight technical awkwardness in Construction 4.9.

4.8 Lemma. *Let $q: \mathbb{C} \to \mathbb{C}$ be a polynomial of degree n with leading monomial $q_0(z) = az^n$, $a \neq 0$. Then for every $\epsilon > 0$, there exists a radius $r > 0$ for which the following are true.*

(1) *The critical values of q lie in $\frac{r}{4}D^2$.*
(2) $q^{-1}\left(\frac{r}{4}D^2\right) \subset q_0^{-1}\left(\frac{r}{2}D^2\right)$.
(3) *There is a smooth branched covering $\tilde{q}: \mathbb{C} \to \mathbb{C}$, uniformly within ϵ of q in the chordal metric of $S^2 = \mathbb{C} \cup \{\infty\}$, such that $\tilde{q}$ is equal to q on $q_0^{-1}\left(\frac{r}{2}D^2\right)$ and $\tilde{q}$ is equal to q_0 on $q_0^{-1}\left(\frac{3}{4}rD^2\right)$.* □

4.9 *Construction.* We use our polynomial p to construct related branched coverings. *It will now be convenient to write w for its variable, rather than z.* Let r be such as provided by Lemma 4.8., for $p = q$ (for any ϵ), and $\tilde{p}$ likewise. The map $\mathbb{C}^2 \to \mathbb{C}^2$: $(z, w) \to (z, p(w))$ is a branched covering, with $n - 1$ complex lines of critical points, and critical values $\mathbb{C} \times V$. So is its approximation $(z, w) \to (z, \tilde{p}(z, w))$, and this latter map has the advantage—since $\tilde{p}$ is a monomial «near infinity»—that what covers a sufficiently large bidisk is itself exactly a bidisk. In fact, let $\tilde{r} = |\tilde{p}^{-1}(r)|$, then this map covers D_r by $D_{\tilde{r}}$. We will use the letter P to denote, indifferently, the restriction of $(z, w) \to (z, \tilde{p}(z, w))$ to $D_{\tilde{r}}$, or to $\partial D_{\tilde{r}}$. Somewhat more abusively (but to our immense convenience) we shall *also* denote by P the branched covering $D^4 \to D^4$ induced by P *via* our standard smoothings of bidisks, as well as its restriction $S^3 \to S^3$.

4.10 *Remarks.* Of course (from (1) of Lemma 4.8, and taking minimal care with the basepoint) the induced homomorphisms $(p^{-1})_*$ and $(\tilde{p}^{-1})_*$ from F_{n-1} to B_n are identical.

On S^3, P actually decomposes—as we have set things up—into a covering of $h(\partial_1 D)$ by itself, and a covering of $h(\partial_2 D)$ by *it*self. All the action happens in the former, where each meridional disk $h(\{\exp i\theta\} \times D^2)$ covers itself by a simple n-sheeted cover; in $h(\partial_2 D)$, P is unbranched, the product of the identity on D^2 with the n-sheeted cover of S^1 by itself.

In fact (after giving them a natural orientation) the critical values of P on $D_{\tilde{r}}$ or on D^4 are a braided surface of degree $n-1$, none other than a particular $S(\phi)$ (where ϕ is the band representation of length zero in B_{n-1}); likewise, the critical values of P on $\partial D_{\tilde{r}}$ or on S^3 are a closed braid $\hat{o}^{(n-1)}$. And the same is true of the critical points (in the covering spaces). Proposition 3.10 can be sharpened to say that every isotopy class of closed n-string braids in S^3 is represented by a closed braid $P^{-1}(\hat{\rho})$, where $\hat{\rho} \cup \hat{o}^{(n-1)}$ is a closed braid (i.e., $\hat{\rho}$ is a closed prebraid criticized by the critical values of P in S^3).

4.11 *Construction.* Let the polynomial p and radius r be as above. Henceforth we demand of each tadpole τ that it satisfy these extra hypotheses:

(5) the head of τ lies in $\frac{r}{4}D^2$;

(6) the part of the tail of τ in the annulus $\frac{3r}{4}D^2 - \operatorname{Int}\frac{r}{4}D^2$ is a straight radial line segment.

Thus we have put the basepoint v_0 on $\frac{3r}{4}S^1$, which is no loss of generality. Certainly Lemma 4.3 still holds for these restricted tadpoles.

Now, let $\underline{\underline{x}}$ be a preband representation in F_{n-1}; let $I(1), \ldots, I(l(\underline{\underline{x}}))$ be disjoint closed intervals of $S^1 - \{1\}$, occurring in the order of their indices; let τ_j be a tadpole representing $x(j)$. We construct a subset of ∂D_r from this data; denoted $\Sigma(\underline{\underline{x}})$, it is the union of the (images of the) geometric prebands $R(\tau_j, I(j))(T)$, together with the annulus $S^1 \times \left[1, \frac{4}{3}\right]v_0 \subset S^1 \times rD^2$, together with the disk $D^2 \times \left\{\frac{4}{3}v_0\right\} \subset D^2 \times rS^1$. The following lemma is evident.

4.12 Lemma. *The set $\Sigma(\underline{\underline{x}})$ is a smooth, ribbon-immersed disk in ∂D_r (with corners along the corners of ∂D_r). It intersects the critical values of P transversally in $l(\underline{\underline{x}})$ points (one in each geometric preband). It intersects $\partial_2 D_r$ in a single meridional disk. The boundary $\partial\Sigma(\underline{\underline{x}})$ is a closed prebraid in ∂D_r, of type $\hat{\rho}(\underline{\underline{x}})$, and is criticized by the critical values of P. The map $\Sigma(\underline{\underline{x}}) \cap \cap \delta_1 D \to S^1$ gotten by restricting pr_1 has no critical points.* □

The reader may formulate a notion of equivalence of surfaces with the properties enunciated in the lemma, so that the various examples of $\Sigma(\underline{\underline{x}})$ produced by varying the choices of tadpoles, etc., are equivalent.

4.13 Definition. A *prebraided disk* is any such $\Sigma(\underline{\underline{x}})$. If $\Sigma(\underline{\underline{x}})$ is a prebraided disk, let $\Sigma'(\underline{\underline{x}})$ temporarily denote its «resolution», that is, the canonical smooth disk which ribbon-immerses onto $\Sigma(\underline{\underline{x}})$ (abstractly, an identification space obtained by glueing together a disk, an annulus, and several tongues).

4.14 Construction. Let $\underline{\underline{x}}$ be a preband representation. Denote by $S(\underline{\underline{x}})$ the subset of S^3 obtained as the image by smoothing of $P^{-1}(\Sigma(\underline{\underline{x}})) \subset \partial SD_{\bar{r}}$ (or alternatively, as P^{-1} of the image in S^3, by smoothing, of $\Sigma(\underline{\underline{x}}) \subset \partial D_r$). We suppose h to have been chosen sensibly so that $S(\underline{\underline{x}})$ is smoothly embedded near $S^1 \times S^1$.

4.15 Proposition. *Let $\underline{\underline{x}}$ be a preband representation in F_{n-1}, $\underline{\underline{b}} = (p^{-1})_*$ ${}_*(x(1)), \ldots, (p^{-1})_*(x(l)))$ the corresponding band representation in B_n. Then $S(\underline{\underline{x}})$ is a ribbon-immersed surface in S^3, and there is a push-in of it into D^4* which is the braided surface $S(\underline{\underline{b}})$. *In particular, $\partial S(\underline{\underline{x}}) = \hat{\beta}(\underline{\underline{b}})$.*

Sketch of proof. The covering P induces a covering of $\Sigma'(\underline{\underline{x}})$, call it $S'(\underline{\underline{x}})$, which is a smooth surface, and evidently ribbon-immerses onto $S(\underline{\underline{x}})$.

We can actually produce a push-in back at the level of the prebraided disk $\Sigma(\underline{\underline{x}})$, which pushes it (rather, its smoothed image in S^3) into D^4 to be a braided surface of degree 1, transverse to the critical values of P. It suffices to find a topless Morse function of $\Sigma'(\underline{\underline{x}})$, with a single local minimum of value (say) 1, and constantly 2 on the boundary, which is adapted to the ribbon-immersion onto $\Sigma(\underline{\underline{x}})$. To find one, consider the geometric preband associated to a tadpole; thanks to evident properties of the height function, of the two components A', A'' of the preimage of a double arc in the range, it is always the case that the proper arc separates the interior arc from the head of the tadpole; so we can construct a Morse function that «engulfs» the interior arc before it touches the proper arc, and this is what is wanted.

When we lift such a push-in factor back to $S'(\underline{\underline{x}})$, the single local minimum becomes n local minima, and each of the $l(\underline{\underline{x}})$ intersections of $\Sigma(\underline{\underline{x}})$ with the critical values of P creates a saddlepoint. $\square$

4.16 *Remarks.* This proposition gives a *practical* justification for the notation $S(\underline{\underline{x}})$, which in any case is not in *formal* conflict with the notation $S(\underline{\underline{b}})$ as introduced in 1.13, since $\underline{\underline{x}}$ and $\underline{\underline{b}}$ are objects of two different types.

When $\underline{\underline{x}}$ is embedded (4.4), or rather when the tadpoles chosen to represent the prebands are all embedded, $S(\underline{\underline{x}})$ is embedded in S^3. It is the (essentially) what was called an *O-braided surface in* S^3 in [Rudolph 2]: here O is the unknot $h(\{0\} \times S^1)$ thought of as a fibred knot in S^3; that component of P^{-1} of the image of an embedded geometric preband, which contains a critical point of P, is a (geometric) *band* in the sense of [Rudolph 2].

4.17 Definition. Such a surface as $S(\underline{\underline{x}})$ is a *braided surface in* S^3. (This, again, is not a formal conflict, nor should it be one in practice since the braided surfaces $S(\underline{\underline{b}})$ previously defined are in 4-dimensional ambient spaces.) This is sharpening of the usage in [Rudolph 1], where $S(\underline{\underline{b}})$ was used indiferently for $S(\underline{\underline{x}})$ and $S((p^{-1})_*(\underline{\underline{x}}))$: the new notation seems preferable because it actually indicates *the arrangement of the singularities* in S^3.

Figure 4 illustrates two prebraided disks and corresponding brainded surfaces. (Only the parts in $h(\delta_1 D)$, cut open, are shown.)

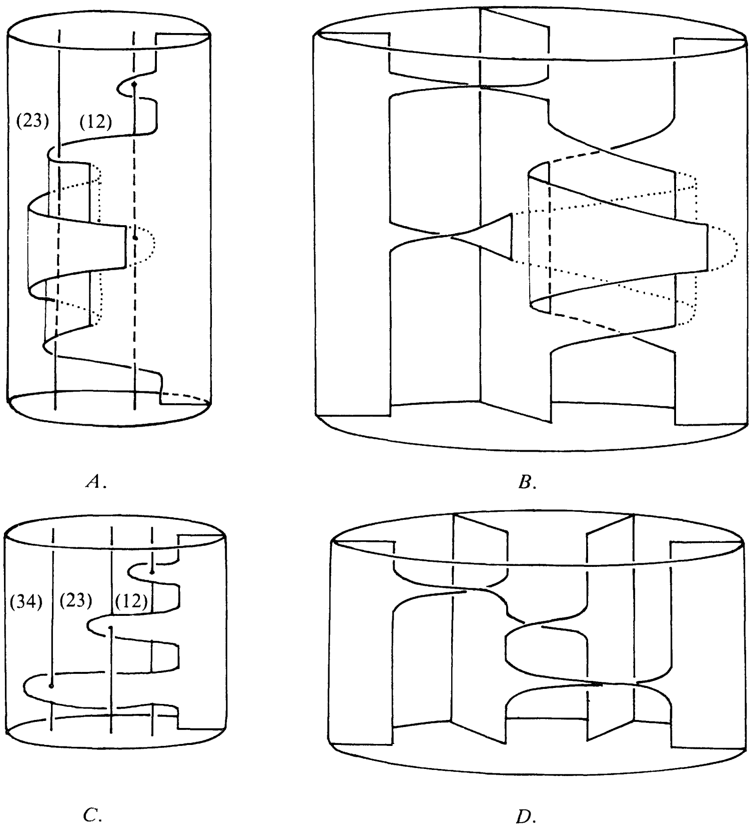


Fig. 4 *A* & *B*: $\Sigma(\underline{x})$ and $S(\underline{x})$, $\underline{x} = (x_1^{-1}, x_2^2 x_1^{-1} x_2^{-2})$ in F_2 (immersed).
C & *D*: $\Sigma(\underline{x})$ and $S(\underline{x})$, $\underline{x} = (x_1, x_2^{-1}, x_{23})$ in F_3 (embedded).

4.18 Remark. The reader is referred to [Rudolph 1] for an exposition, and examples, of the «calculus of (pre-)bands» (use of slides, expansions, and contractions, cf. 1.13-1.14, above) and the techniques of picturing braided surfaces in S^3.

§5. Markov surfaces and Bennequin surfaces

Recently, [Bennequin] proved some new and interesting facts about closed braids, and obtained new proofs of some old results, as a preliminary stage in his investigation of exotic contacts structures on S^3. His proofs involved the introduction and exploitation of a class of specially positioned Seifert surfaces. I propose to reinterpret these surfaces in the context of braided surfaces which we have established.

5.1 *Reminder*. A surface S in S^3 is a **Seifert surface** (*for* the link ∂S) if it is smooth, compact, oriented, and topless. A surface S in S^3 is *incompressible* if, for every simple closed curve $C \subset S$ such that C is the boundary of a smooth disk with interior disjoint from S, there is a disk contained in S with boundary C. It is a fact that every (smooth) link in S^3 is the boundary of some incompressible Seifert surface. In particular, if S is a Seifert surface which has maximal Euler characteristic among all Seifert surfaces with the same boundary, then S is incompressible. (There are, however, links with incompressible Seifert surfaces of arbitrarily high genus.)

5.2 Notation. If L is a link in S^3, let $X(L)$ be the maximum of the integers $\chi(S)$, S a Seifert surface for L.

For L a knot, or more generally a link which admits only connected Seifert surfaces (e.g., a fibred link), the *genus* $g(L)$ is unambiguously defined, and of course $g(L) = 1 - \frac{1}{2}(X(L) + rkH_0(L))$. But X is easier to calculate with, here.

Recall (1.1) that S^3 contains $S^1 \times S^1$ and is split by this torus into the two solid tori we have called $h(\partial_1 D)$ and $h(\partial_2 D)$, where $h(\partial_1 D) = \{(z, w) \in \mathbb{C}^2: |z|^2 + |w|^2 = 2, |z| \geqslant |w|\}$. These solid tori are equipped, *via* h, with fixed product structures. In particular, we can identify the universal cover of $h(\partial_1 D)$ with $\mathbb{R} \times D^2$, and treat projection on the first factor as a multivalued function θ: $h(\partial_1 D) \to \mathbb{R}$ (that is, θ is some fixed branch of $\frac{1}{i} \log \circ pr_1 \circ h^{-1}|h(\partial_1 D))$. We write D_t^2 for the meridional disk $\theta^{-1}(t) \subset h(\partial_1 D)$. Similarly, write ϕ for the «angular coordinate» in $h(\delta_2 D)$.

5.3 Definition. A Seifert surface $S \subset S^3$ is a *Markov surface* if it has the following properties (1) – (4):

(1) ∂S is a closed braid in $\operatorname{Int} h(\partial_1 D)$;

(2) $S \cap h(\partial_2 D)$ is the union of finitely many meridional disks $\phi^{-1}(s_j)$ (disregarding orientations);

(3) θ_S, the restriction of θ to $S \cap h(\partial_1 S)$, has no degenerate critical points, and distinct critical points of θ_S have distinct critical values (modulo 2π);

(4) each critical point of θ_S is a saddlepoint.

5.4 *Remark.* Taken together, (3) and (4) mean, geometrically, that all tangencies of S with leaves of the product foliation of $\operatorname{Int} h(\partial_1 D)$ by fibres of θ are non-degenerate saddles. Now, in [Bennequin] (which after all appears in a volume dedicated to Georges Reeb), Markov surfaces are defined similarly but in terms not of this foliation but rather of a Reeb foliation of $\operatorname{Int} h(\partial_1 D)$. But I claim the definitions are essentially equivalent: for, granting non-degeneracy of the tangencies, there are only finitely many; they all happen, therefore, inside some *closed* solid torus interior to $h(\partial_1 D)$, on which the foliation induced by the Reeb foliation and the foliation by meridional disks are isotopic. Thus the two species of Markov surface differ only by an inessential change of coordinates.

Thus (see also 5.7) all the pictures in [Bennequin] can be understood in terms of the present definition: in fact, more readily than for the original, since what is drawn is a disk but on the original interpretation it has to be understood as a plane (a leaf of the Reeb foliation) together with a circle at infinity.

5.5 Definition. Let S be a Seifert surface satisfying properties (1), (2), and (3) of Definition 5.3. Then both the level set $\theta_S^{-1}(t)$ and the pair $(D_t^2, \theta_S^{-1}(t))$ will be called *the t-section of S.* If t is not a critical value of θ_S, the t-section is a smooth, oriented 1-manifold-with-boundary, containing some number (perhaps zero) of simple closed curves as components, together with n arcs joining a point of S_t^1 to a point of $\operatorname{Int} D_t^2$ and k arcs joining two points of S_t^1 to each other: following [Bennequin], we call the former arcs *free* and the latter *tied.* (The integer $n \geqslant 1$ is of course the number of strings of the closed braid ∂S; and $n + 2k$ is the number of meridional disks $\phi^{-1}(s_j)$ of $h(\partial_2 D)$ contained in S.) A *critical section,* $\theta_S^{-1}(t)$ for t a critical value, has precisely one singular point, which is either an isolated point of $\operatorname{Int} D_t^2$ (if it is a local extremum of θ_S) or a point at which the section is an immersed 1-manifold with two transverse branches (a saddlepoint). The critical sections also have well-defined oriented boundaries. A saddle section may have zero, one, or two simple closed curves (through the singular point) which are properly contained in a component of $\theta_S^{-1}(t)$.

5.7. We leave to the reader to formulate and prove converses to the assertions of 5.6, to the effect that any (suitably smoothly changing) family of «abstract» t-sections actually fits together into the part of a surface with properties (1), (2), and (3) which lies in $h(\partial_1 D)$. (Be cautious: it might not be a Seifert surface without extra hypotheses, i.e., a closed component could appear.) Also, note that, given S, if $I \subset S^1$ is an interval without critical values of θ_S, then $S \cap \theta_S^{-1}(I)$ is the trace of an isotopy between the two sections of S at the endpoints of I, which is suitably unique, and that therefore a surface with properties (1), (2), and (3) can be adequately pictured (as in [Bennequin, pp. 109-110 ff.], or at the lower corners of the pages in [Douady]) by drawing sections «just on each side of» the singular sections (and, if there are closed components, making it clear how they move about in sections with more than one).

5.8 Theorem. *Let $\hat{\beta} \subset h(\partial_1 D) \subset S^3$ be a closed braid, and $S \subset S^3$ an incompressible Seifert surface for $\hat{\beta}$. Then S is ambient isotopic, with $\hat{\beta}$ fixed, to a Markov surface.*

(This slightly strengthens the statement of Théorème 4 of [Bennequin]; the proof is basically the same.)

PROOF. We achieve properties (1) – (4) of Definition 5.3 by a sequence of isotopies, each of which leaves intact those properties already acquired; we keep calling the surface S.

Property (1) is a hypothesis.

An arbitrarily small isotopy not only puts S transverse to the core circle $h(\{0\} \times S^1)$ of $h(\partial_2 D)$, but makes its intersections with an infnitesimal solid torns $h(\epsilon D^2 \times S^1)$ all into (not necessarily correctly oriented) meridional disks. A radial expansion of this torus achieves property (2).

Again, an arbitrarily small isotopy (supported in $\operatorname{Int} h(\partial_1 D)$) achieves property (3).

We are left with the task of eliminating local extrema of θ_S. First, without changing their number, we rearrange them, as follows. Let the t_1-section of S be non-singular. Among its simple closed curves (if any), consider one of those (if any) which were born at a local minimum of θ_S with value $t_0 \in]t_1 - 2\pi, t_1[$: that is, if possible, take $C \subset \beta_S^{-1}(t_1)$ which bounds a disk (its life history) on S with interior disjoint from $D^2_{t_1}$ and on which θ_S has a single critical point (a local minimum). Such a curve C also bounds a disk in $D^2_{t_1}$ (possibly with interior points in S). The two disks together make a 2-sphere, which bounds a 3-cell, which guides an isotopy («pushing from bottom to top») which decreases by at least one the number of simple closed curves in the t_1-section of S which were born at minima —so we may assume there are none such.

Now let $t_0 \in]t_1 - 2\pi, t_1[$ be the greatest minimum value of θ_S less that t_1 (if there are any local minima). The simple closed curve born at the isolated point of $\theta_S^{-1}(t_0)$ does not survive to the t_1-section, so it dies at some intermediate critical level. It doesn't die at a local maximum (or S would contain a 2-sphere), so it dies at a saddlepoint. If it dies by absorption (i.e., the number of simple closed curves in the sections decreases by one as you pass up through the saddle section), the life history of the 2-cell it bounds in D_t is a 3-cell with interior disjoint from S, in position for an «embedded handle cancellation» —an isotopy which reduces the total number of critical points of θ_S by two (the minimum and the saddlepoint). If, on the contrary, the curve dies by splitting into two disjoint simple closed curves C_1 and C_2 (in the regular section $\theta_S^{-1}(t_2)$ «just above» the saddle), then certainly one of C_1 or C_2 (and perhaps both) bounds a 2-cell in $D_{t_2}^2$ with interior disjoint from S (for —numbering so C_1 does not enclose C_2— nothing but C_1 could possibly get inside C_2, and nothing at all could get inside C_1 since we have controlled births): now we use incompressibility to conclude that C_1 (say) also bounds a disk contained in S, on which θ_S necessarily has at least one local *maximum*. Now an isotopy cancels (at least) two critical points of θ_S (namely, the saddle-point where the minimum died, and the maximum).

Thus after isotopies which don't create new critical points, we can assume θ_S has no local minima. Turning the procedure upside down, we can eliminate local maxima the same way, achieving (4), and S is now Markov. □

5.9 Definition. A *Bennequin* surface is a Markov surface for which no section contains a simple closed curve.

5.10 Theorem (Bennequin). *Let $\hat{\beta}$ be a closed braid in S^3. Then $\hat{\beta} = \partial S$ for some Bennequin surface S. If F is a Seifert surface for $\hat{\beta}$ with $\chi(F) = X(\hat{\beta})$ then F is ambient isotopic, with $\hat{\beta}$ fixed, to a Bennequin surface.*

Proof. Let S be a Markov surface. If $C \subset \theta_S^{-1}(t)$ is a simple closed curve then C does not bound on S (for if it did, θ_S would have to have a local extremum on the subsurface of S bounded by C), much less bound a disk on S. Yet if C is an innermost such curve in D_t^2, it does bound a disk with interior disjoint from S. So an incompressible Markov surface is a Bennequin surface. In particular, an incompressible Seifert surface (for instance, one of maximal Euler characteristic for its boundary) is ambient isotopic, with boundary fixed, to a Markov surface which is *ipso facto* Bennequin. □

5.11 Definition. Let S be a Markov surface, with boundary $\hat{\beta}$ a closed braid on n strings. Each meridional disk $\phi^{-1}(s_j)$ in $S \cap h(\partial_2 D)$ inherits an orientation from S, which it passes on to its boundary (oriented counterclockwise), a circle $h(S^1 \times \{\exp is_j\}) \subset S^1 \times S^1$. If this orientation agrees with the orientation by increasing t on S^1, call the circle and the disk *positive*, otherwise *negative*: so there are $n + k$ positive and k negative disks. Let S^+ be the (essentially unique) Markov surface obtained from S by removing from S each of its negative disks $\phi^{-1}(s_j)$ together with a collar of $\partial(\phi^{-1}(s_j))$ in $h(\partial_1 D)$ on which θ_S has no critical points (either interior to the collar or as restricted to its boundary circles).

5.12 Theorem. *Let $\underline{\underline{x}}$ be a preband representation in F_{n-1} with each preband $x(i)$ embedded. Then the braided surface $S(\underline{\underline{x}}) \subset S^3$ is a Bennequin surface* *with no negative disks. Up to isotopy through surfaces of the same sort, every Bennequin surface with no negative disks is a braided surface* $S(\underline{\underline{x}})$.

PROOF. *Via* h, we can consider the prebraided disk $\Sigma(\underline{\underline{x}}) \subset S^3$ as a very special Bennequin surface, containing a single (positive) meridional disk of $h(\partial_2 D)$, and such that $\theta_{\Sigma(\underline{\underline{x}})}$ has no critical points —so each section is a single free arc. Then the preimage $S(\underline{\underline{x}}) = P^{-1}(\Sigma(\underline{\underline{x}}))$ in S^3 is manifestly a Bennequin surface with n positive disks (and no negative ones); the section $\theta_{S(\underline{\underline{x}})}^{-1}(t)$ is singular if and only if $\theta_{\Sigma(\underline{\underline{x}})}^{-1}(t)$ passes through a critical value of P in D_t^2; then, as the branching of P is simple, the singular point of $\theta_{S(\underline{\underline{x}})}^{-1}(t)$ is a crossing of two transverse branches, i.e., t is a saddle value for $\theta_{S(\underline{\underline{x}})}$, and we have derived as a bonus that the number of critical values of $\theta_{S(\underline{\underline{x}})}$ is the length of $\underline{\underline{x}}$.

As to the converse, we prove (what appears to be) more than stated (see the Remark following), namely, that up to isotopy through Bennequin surfaces with no negative disks, every such Bennequin surface is $S(\underline{\underline{x}})$ for some preband representation $\underline{\underline{x}}$ in which each $x(j)$ is an *elementary* embedded preband: one of the $n(n-1)$ prebands $x_{uv}^{\pm 1} = (x_u x_{u+1} \ldots x_{v-1}) x_v^{\pm 1} (x_u \ldots x_{v-1})^{-1}$, $1 \leqslant$ $\leqslant u \leqslant v \leqslant n-1$ (so $x_{uu} = x_u$ is a standard generator).

In fact, by Constructions 4.9, 4.11, and 4.13, all $S(\underline{\underline{x}})$, $\underline{\underline{x}}$ a preband representation in F_{n-1}, meet $h(\partial_2 D)$ in precisely the same set of n positive meridional disks, and these may be naturally numbered (cf. 3.3) from 1 to n so that they appear in that cyclic order and so that, for instance, $S((x_j))$ has $n-1$ components, one of which contains disks j and $j+1$. Let S be a given Bennequin surface with no negative disks. By isotopy we may assume its n positive disks are these n canonical disks. Let t_j, $0 < t_1 < \ldots < t_l$ be the critical values of θ_S. The component of $\theta_S^{-1}(t_j)$ containing the singular point has two boundary points on ∂S and two others on two of the n canonical positive disks, let them be numbered $u(j)$ and $v(j)+1$, $u(j) \leqslant v(j)$. Also, there is a sign $\epsilon(j) = \pm 1$ naturally associated to the critical point of θ_S in the t_j-section, determined not just by the section but by local behavior on either side of it; the sign of the critical point of $S((x_j))$ is $+1$. Put $x(j) = x_{u(j), v(j)}^{\epsilon(j)}$, $\underline{\underline{x}} = (x(1), \ldots, x(l))$. I claim that S is isotopic (by an isotopy through Bennequin surfaces, supported in $h(\partial_1 D)$) to $S(\underline{\underline{x}})$. This claim is a refinement of the claim in 5.7, and like that, is left to the reader to prove. (See also 5.22). □

5.13 *Remarks*. A consequence of Theorem 5.12 is that there is a «retraction», call it $x \to x'$, of the set of all embedded prebands onto its subset of elementary embedded prebands, for which $(p^{-1})_*(x) = (p^{-1})_*(x')$; and this latter band is always one of the bands $\sigma_{uv}^{\pm 1} = (\sigma_u \sigma_{u+1}, \ldots, \sigma_{v-1}) \sigma_v^{\pm 1} (\sigma_u, \ldots, \sigma_{v-1})^{-1}$ which were called embedded bands in [Rudolph 1]. In that paper's construction of models of braided surfaces $S(\underline{\underline{b}})$ in S^3 (as ribbon immersions), only these bands —*expressed as given as words in the standard generators*— could be used if double points were not to be created (essentially because not only the positive disks, but also their collars in $h(\partial_1 D)$, were held fixed). The present construction, using surfaces $S(\underline{\underline{x}})$ associated to preband representations, is thus at once more precise and less general, for *not every ribbon-immersed surface* called $S(\underline{\underline{b}})$ in the earlier paper occurs as $S(\underline{\underline{x}})$ in the current terminology (the point being that $S(\underline{\underline{x}})$ is, as it were, p-equivariant). However, all embedded surfaces do so occur.

For $n \geqslant 3$ there are always non-elementary embedded prebands. One source of many such (perhaps all?) is the following. Let f: $\frac{r}{4}D^2 \to \frac{r}{4}D^2$ be an autohomeomorphism fixing $\frac{r}{4}S^1$ pointwise and V setwise. Then f_* is an automorphism of the prebraid group F_{n-1}. If f_* leaves invariant the representation on S_n associated to p (equivalently, if f is covered by an autohomeomorphism g of $p^{-1}\left(\frac{r}{4}D^2\right)$, then f_* will carry any embedded preband, in particular an elementary one, onto an embedded preband that will generally not be elementary.

For example: with V, N_j', etc., as at the beginning of §3, write W_{ij} for some 2-cell which is the union (not disjoint) of N_i', N_j', and a small disk centered at v_0 which is disjoint from V. Let σ: $W_{ij} \to W_{ij}$ be an autohomeomorphism which is conjugate, *via* a homeomorphism of $(W_{ij}, \{v_i, v_j\})$ with $(2D^2, \{-1, 1\})$, to $r\exp\sqrt{-1}\theta \to r\exp\sqrt{-1}\,(\theta + (2-r)\pi)$. Then if $j > i+1$, the two-fold composition $f = \sigma \circ \sigma$ is such an f as described above, while if $j = i+1$, the three-fold composition $f = \sigma \circ \sigma \circ \sigma$ is. In the first case, from the standard generators x_i and x_j are produced embedded prebands $(x_i x_j) x_i (x_i x_j)^{-1}$ and $x_i x_j x_i^{-1}$, respectively; in the second case we get $(x_i x_{i+1} x_i) x_{i+1} (x_i x_{i+1} x_i)^{-1}$ and $(x_i x_{i+1}) x_i (x_i x_{i+1})^{-1}$; other generators aren't touched.

It seems likely that further analysis of this situation would pay off. Perhaps it would lead to a geometric derivation of the relations in the standard presentation of the braid group. But we will not pursue the topic further at this time.

To continue our study of the connection between braided surfaces and Bennequin surfaces, we must hark back to the braid groups.

5.14 Definition The *usual injection* $\bar{u}_{n,n+k}$ of the prebraid group F_{n-1} into F_{n+k-1}, $k \geqslant 0$, takes the standard generator x_j of $F_{n-1}(j = 1, \ldots, n-1)$ to the standard generator of the same name in F_{n+k-1}. The *usual injection* $u_{n,n+k}$: $B_n \to B_{n+k}$ likewise is defined on standard generators, taking $\sigma_j \in B_n$ to $\sigma_j \in B_{n+k}$ $(j = 1, \ldots, n-1)$, cf. 1.4. That $u_{n,n+k}$ is a homomorphism follows from the fact that it actually is the homomorphism induced by such a map of configuration spaces $E_n - \Delta_n \to E_{n+k} - \Delta_{n+k}$ as $\{z_1, \ldots, z_n\} \to$ $\to \{z_1, \ldots, z_n, 1 + \sum_1^n |z_j|, 2 + \sum_1^n |z_j|, \ldots, k + \sum_1^n |z_j|\}$. (But note the adhockery of this map, which in any case could never be chosen to be a complex polynomial.) That $u_{n,n+k}$ is injective is less evident. But that (and the fact that it is a homomorphism) follows from the known presentations of the groups (cf. [Birman]), so we take it for granted. It must be emphasized again, however, that (for $k > 0$) the usual injection is NOT an inclusion, for the domain and range are disjoint groups; nor is the injection canonical, which is why I have named it merely «usual».

We also define F_{-1} and F_0 to be (distinct) 1-element groups, and $\bar{u}_{0,k}$ and $u_{0,k}$ in the only possible way. Then for $n, m, k \geqslant 0$, always $u_{n,n+k+m} = u_{n+k,n+k+m} \circ u_{n,n+k}$ and similarly with $\bar{u}$ for u. Also the canonical surjections $F_{n-1} \to B_n$ and the usual injections make up various commutative squares.

Now, the notations $u_{n,n+k}$ and $\bar{u}_{n,n+k}$ are cumbersome. We intend to avoid them, whenever practical, as follows.

5.15 *Notation.* If $\beta \in B_n$, let $\beta^{(k)} = u_{n,n+k}(\beta) \in B_{n+k}$. Similarly for prebraids. Thus $o^{(k)}$ is the identity of B_k (recall that o is the unique element of B_0), consistent with the earlier notation 1.4. (We could also resolve the ambiguous notation for standard generators by a convention that, for $n \geqslant 2$, $\sigma_{n-1} \in B_n$, thus making the standard generators of B_n carry the names $\sigma_1^{(n-2)}, \ldots, \sigma_j^{(n-1-j)}, \ldots, \sigma_{n-1}^{(0)} = \sigma_{n-1}$. We will not carry out this plan in this paper, however.)

Further, we denote the closure of $\beta^{(k)}$ by $\hat{\beta}^{(k)}$, rather than trying to stretch the roof over the whole complex symbol. Not only is this kind to typographers, but it is consistent with the following useful convention: if L is a link, then $L^{(k)}$ denotes the *split sum* of L with k unknots (that is, the union of L with the boundary of k smooth disks, pairwise disjoint and disjoint from L-out of the context of closed braids, this could also be denoted by $L \# O$, adapting the notation for boundary connected sum of pairs to the case of submanifolds without boundary). In particular (and this is a bit of misfortune), $\hat{o} = \varnothing$ is the empty link, $\hat{o}^{(1)}$ is the unknot.

We amend our conventions in the case of prebraids: by $\hat{x}^{(k)}$ we will denote the *criticized* closure of the prebraid $\bar{u}_{n,n+k}(x)$ (without the critical points of the covering, we couldn't see any difference between the two prebraids), where $x \in F_{n-1}$. So $\hat{x}^{(0)} \neq \hat{x}$ (the first is an n-string closed braid, the latter a 1-string closed braid).

Note that the consistency of the usual injections means that always $\beta^{(k)(m)} = \beta^{(k+m)}$.

5.16 Theorem. *Let S be a Bennequin surface with boundary $\hat{\beta}$, $\beta \in B_n$, and k negative disks. Then S^+ is a Bennequin surface with $\partial(S^+) = \hat{\beta}^{(k)}$.*

PROOF. Obvious. □

5.17 *Warning*. The conclusion of 5.16 must be interpreted with care. Suppose one has chosen a way to cut open $h(\partial_1 D)$ along a meridional disk, and project it onto a rectangle, in such a way that ∂S projects onto a braid diagram in the usual way. Thus the crossings in the diagram correspond to standard generators (and their inverses) of B_n (with some basepoint of the configuration space implicit). Then, very likely, $\partial(S^+)$ will *not project to (what you think ought to be) a braid diagram for* $\beta^{(k)}$. For the choice of projection (of the solid cylinder onto a rectangle) imposes (by the conventions for reading braid diagrams) an injection of B_n into B_{n+k} which is probably not $u_{n,n+k}$. By moving the k «new» points «behind and to the right of» the n «old» points (at the top and bottom of the cylinder, and then straight all the way down) —a move which can be effected by a diffeomorphism of D^2 (times the identity on S^1)— the boundary of S^+ can, indeed, be made to look right. But generally the standard, narrow collars of the negative disks on S^+ will be carried into broad, flop ones. (See 5.19, for an example.)

5.18 Theorem. *Up to isotopy through Bennequin surfaces, every Bennequin surface bounded by a closed n-string braid $\hat{\beta}$ is obtained from a braided surface $S(\underline{\underline{x}})$, where $\underline{\underline{x}}$* is an embedded preband representation in F_{n+k-1} *which maps to $\beta^{(k)}$ in B_{n+k}, by attaching k collared negative disks to $\hat{\beta}^{(k)} - \hat{\beta}$.*

PROOF. Immediate from 5.12 and 5.16. □

5.19 *Warning*. It is *not* true that, if $\underline{\underline{x}}$ is an embedded preband representation in F_{n+k-1} mapping to $\beta^{(k)} \in B_{n+k}$, $k \geqslant 1$, then necessarily it is possible to attach k collared negative disks to $\hat{\beta}^{(k)} - \hat{\beta}$ in the complement of $S(\underline{\underline{x}})$, to obtain a Bennequin surface for $\hat{\beta}$. Two distinct problems arise.

First, if $\underline{\underline{x}}$ is already the image by (the obvious map on preband representations associated to) the usual injection $\bar{u}_{n+m,n+k}$ of a preband representation in F_{n+m-1}, for some $1 \leqslant m < k$, then attaching collared negative disks to *all* the components of $\hat{\beta}^{(k)} - \hat{\beta}$ would produce $k - m > 1$ 2-spheres in the resulting surface. (For instance, if $\underline{\underline{x}} = (x_{12})$ in F_3, then the obvious projection of $\partial S(\underline{\underline{x}})$ doesn't look like $\hat{o}_1^{(2)}, \sigma_1^{(2)} \in u_{2,4}(B_2) \subset B_4$, *viz.* 5.17; yet it is, and attempting to attach two disks to the last two components brings trouble.) Of course this could be handled by convention.

More seriously, there are cases like $\underline{\underline{x}} = (x_1, x_1, x_1^{-1}, x_1^{-1})$ in F_1, a preband representation for the braid $o^{(2)} = o^{(1)(1)}$ in B_2. Here, *each* component of $\hat{o}^{(2)}$ has non-zero linking number with a suitable (simple closed) curve on $S(\underline{\underline{x}})$, so no disk at all (let alone a collared negative disk) can be attached to $S(\underline{\underline{x}})$ along either boundary component.

Note, however, that the surface just constructed is (very) compressible. In fact, we have the following converse to 5.18.

5.20 Theorem. *Let $S(\underline{\underline{x}})$ be an incompressible embedded braided surface with boundary $\hat{\beta}^{(k)}$. Then there is a smoothly embedded surface S in S^3 with boundary $\hat{\beta}$ which is the union along $\hat{\beta}^{(k)} - \hat{\beta}$ of $S(\underline{\underline{x}})$ and k 2-disks, such that the topless components of S are a Bennequin surface.*

PROOF. Let us say that an oriented 2-disk G^- embedded in S^3 is a *floppily collared negative disk* if $G = -G^-$, the same disk with opposite orientation, is a Bennequin surface for a 1-string closed braid and G has only one (positive) meridional disk of $h(\partial_2 D)$ in it. It is clear that if we can find k floppily collared negative disks, pairwise disjoint and with boundaries the k components of $\hat{\beta}^{(k)} - \hat{\beta}$, and interiors disjoint from $S(\underline{\underline{x}})$, then (suitably smoothed along $\hat{\beta}^{(k)} - \hat{\beta}$) the union of $S(\underline{\underline{x}})$ and these disks is such an S as we require.

We begin by finding F, a union of pairwise disjoint floppily collared negative disks with $\partial F = \hat{\beta}^{(k)} - \hat{\beta}$ such that $\operatorname{Int} F$ is disjoint from $\hat{\beta}$ and transverse to $\operatorname{Int} S(\underline{\underline{x}})$. (For instance, one can realize $S(\underline{\underline{x}})$ so its boundary really looks like $\hat{\beta}^{(k)}$ in a braid diagram, then take F to be the union of obvious collared, and *a fortiori* floppily collared, negative disks; naturally the transversality is no problem.) We will modify the given F, staying in the class of unions of k floppily collared negative disks, until $\operatorname{Int} F \cap S(x) = \emptyset$, at which point we will be done.

If $\operatorname{Int} F \cap S(\underline{\underline{x}}) \neq \emptyset$, by transversality it is a union of simple closed curves. Let C_1 be one of them which is «innermost» on F (that is, bounds a 2-cell in F with interior disjoint from $S(\underline{\underline{x}})$). By incompressibility, C_1 bounds a 2-cell on $S(\underline{\underline{x}})$. Let C_2 be an innermost curve in this 2-cell (possible C_1 itself). Then C_2 is not necessarily innermost on F, but we don't care. Let $E \subset S(\underline{\underline{x}})$ and $E' \subset F$ be the 2-cells bounded by C_2. I claim that if we remove a 2-cell slightly larger than E' from F, and replace it by a 2-cell with the same boundary which lies parallel and close to E in the complement of $S(\underline{\underline{x}})$, then the revised F is still a union of pairwise disjoint floppily collared negative disks with boundary $\hat{\beta}^{(k)} - \hat{\beta}$. In fact, by the transversality of the intersection and property (2) of the Markov surfaces $S(\underline{\underline{x}})$ and $-F$, either E' contains a negative disk of $-F$, or it lies in $h(\partial_1 D)$; since $\partial E' = \partial E = C_2$, consideration of linking numbers shows that whichever alternative holds for E' also holds for E. In each case, we see that «replacing E' by E» (as we essentially have done) preserves the desired properties of F. The operation also, of course, decreases the number of intersections of $\operatorname{Int} F$ and $S(\underline{\underline{x}})$. When this reaches zero we are done. □

5.21 *Remark*. Theorems 5.12, 5.16, 5.18, and 5.20 in a sense reduce Bennequin surface theory to braided surface theory, and thence (*via* the calculus alluded to in 4.18) to the algebra and combinatorics of band and preband representations. It might, for instance, be possible to prove Bennequin's Inequality (see the next section) purely within the context of braided surfaces, though to date I have not succeeded in doing so.

An interesting practical question that arises when one considers Theorem 5.20 (and one which might have an answer of independent interest, given the interest in incompressible surfaces among 3-manifold topologists) is, *How can one tell from* $\underline{\underline{x}}$ whether or not $S(\underline{\underline{x}})$ *is incompressible?*

5.22 *Addendum.* Here is a direct path from a Bennequin surface $S \subset S^3$ without negative disks to a braided surface $S(\underline{\underline{b}}) \subset D$ which is essentially a push-in of S. (Presumably, with a suitable definition of «Bennequin ribbon-immersed surface in S^3» one could obtain all $S(\underline{\underline{b}})$ this way.) First, by isotopy, arrange S so that in each critical section the arc-lengths of the four arms of the singular component of $S_t \subset D^2_t$ are of equal arc-length. Next, define r: $S \cap h(\partial_1 D) \to [1/2, 1]$ by requiring $r|\partial S$ to be identically equal to 1, $r|(S \cap S^1 \times S^1)$ to be identically 1/2, and $r|A$ to be an affine function of arc-length from ∂S for A a component of any section S_t. Clearly r is smooth and well-defined (and takes the value 3/4 at the singular points of the critical sections). Map $S \cap h(\partial_1 D)$ into $\left(D^2 - \operatorname{Int}\frac{1}{2}D^2\right) \times D^2$ by sending a point $x = h(\exp it, w)$ to $(r(x)\exp it, w)$. Then $S \cap S^1 \times S^1$ maps to the union of n circles $\frac{1}{2}S^1 \times \{\exp is_j\}$, and we extend our map to $S \cap h(\partial_2 D)$ by sending $x = h(z, \exp is_j)$ to $\left(\frac{z}{2}, \exp is_j\right)$. Evidently, the map constructed embeds S in D with image a braided surface $S(\underline{\underline{b}})$ of degree n, and $h(\partial S(\underline{\underline{b}})) = \partial S$ by construction.

§6. Markov's Theorem, Bennequin's Inequality, and some conjectural generalizations

In this section I will state, without proof, two major results in the application of braids to knot theory: the reader is referred to [Bennequin] for proofs of both (or to [Birman] for Markov's Theorem: however, the differential-topological approach of [Bennequin] is perhaps closer to the spirit of the present paper than the combinatorial-topological approach of [Birman]). I will then discuss various generalizations, all conjectural, which are suggested when one thinks in terms of braided surfaces.

6.1 Markov's Theorem. *Let* $\beta \in B_n$, $\gamma \in B_p$ *be two braids such that the closed braids* $\hat{\beta}, \hat{\gamma} \subset S^3$ *are ambient isotopic in* S^3. *Then there is a finite sequence* $\beta(j) \in B_{n(j)}$ *of braids,* $j = 1, \ldots, N$ *with* $\beta(1) = \beta$, $\beta(N) = \gamma$, *such that for each* $j = 1, \ldots, N-1$, *one of the following three cases holds:*

(1) $n(j+1) = n(j)$ *and for some* $w(j) \in B_{n(j)}$, *we have* $\beta(j+1) = w(j)\beta(j)w(j)^{-1}$; *or,*

(2) *$n(j+1) = n(j) + 1$ and for $\epsilon = +1$ or $\epsilon = -1$, we have $\beta(j+1) = \beta(j)^{(1)}\sigma_{n(j)}$; or,*

(3) *$n(j+1) = n(j) - 1$ and for $\epsilon = +1$ or $\epsilon = -1$, we have $\beta(j) = \beta(j+1)^{(1)}\sigma_{n(j+1)}$.*

Conversely, if two braids are joined by such a sequence, then their closures are of the same ambient isotopy type. □

6.2 *Definition.* In case (1) [resp., (2); (3)] of Markov's Theorem, we say that $\beta(j+1)$ is obtained from $\beta(j)$ by a *Markov move* of *type* (1) *with conjugator* w [resp., of *type* (2^ϵ); of *type* (3^ϵ)].

6.3 Bennequin's Inequality. *Let $\beta \in B_n$. Let $e(\beta) \in \mathbb{Z}$ denote its exponent sum* (*see Remark* 6.4), *$X(\hat{\beta})$ the maximum Euler characteristic of a Seifert surface for $\hat{\beta}$* (cf. 5.2). *Then we have*

$$IB(\beta): \quad n - |e(\beta)| \geqslant X(\hat{\beta})$$

(*which I will call «Bennequin's Inequality for β»*). □

6.4 *Remark* Recall that $e\colon B_n \to \mathbb{Z}$ is abelianization, normalized to send a positive band to $+1$. Consequently $|e(\beta)|$ is certainly a lower bound for the number of bands needed to represent β in B_n, so $n - |e(\beta)|$ is an upper bound for the Euler characteristic of a braided surface (in D) with boundary $\hat{\beta}$.

6.5 Definition. A *slice surface* in D^4 is a compact, topless, smooth 2-manifold-with-boundary properly embedded in D^4. If L is a smooth, oriented link in S^3, define invariants $X_r(L)$, $X_s(L)$ by putting $X_r(L) = \max\{\chi(S)\colon S \subset D^4$ is an oriented ribbon surface with $\partial S = L\}$, $X_s L = \max\{\chi(S)\colon S \subset D^4$ is an oriented slice surface with $\partial S = L\}$. Then (since any Seifert surface in S^3 is, in particular, a ribbon-immersed surface without singularities, and can thus be pushed into D^4 to become a ribbon; and any ribbon surface is slice) we have, for every L, $X(L) \leqslant X_r(L) \leqslant X_s(L)$. It is well-known that the first inequality can be strict (existence of non-trivial «ribbon knots», e.g., $\partial S((\sigma_1, \sigma_2^3\sigma_1\sigma_2^{-3})))$; it is an open question whether the second inequality is ever strict, even in the case $X_s(L) = 1$, L a knot.

6.6 Ribbon-Bennequin Conjecture. *For every n and every $\beta \in B_n$, we have*

$$rIB(\beta): \quad n - |e(\beta)| \geqslant X_r(\hat{\beta}).$$

(*which I will call the «ribbon-Bennequin inequality for β»*).

6.7 Slice-Bennequin Conjecture. *For every n and every $\beta \in B_n$, we have*

$$sIB(\beta): \quad n - |e(\beta)| \geqslant X_s(\hat{\beta})$$

(which I will call the «slice-Bennequin inequality for β»).

6.8 *Remarks.* (1) Of course, for every β, $sIB(\beta) \to rIB(\beta) \to IB(\beta)$.

(2) There are various β for which $sIB(\beta)$ is known to hold with equality. For example, the various positive braids $\sigma_1^k \in B_2$ $(k \geqslant 1)$, $(\sigma_1\sigma_2)^k \in B_3(1 \leqslant k \leqslant 5)$, $\sigma_1^{2k+1}(\sigma_2\sigma_1\sigma_3\sigma_2)^3 \in B_4$ all have «total signature», i.e., positive-definite Siefert form (they occur as links of so-called «simple» singularities of complex plane curves), so the embedded braided surfaces $S((\sigma_1, \ldots, \sigma_1))$, etc., corresponding to the given braid words (read as band representations), of Euler characteristic $n - e(\beta)$ in each case, actually are of maximal Euler characteristic among all slice surfaces for the closed braids.

(3) I know of no counterexample to the Slice-Bennequin Conjecture. On the other hand, suppose one defines a *topological slice surface* in D^4 to be a compact, topless 2-manifold with boundary properly embedded in D^4 which, though not necessarily smooth, has a neighborhood in D^4 which is homeomorphic to the product of the surface and $\operatorname{Int} D^2$. Then (using a deep result of [Freedman] on knots with Alexander polynomial 1) I have shown that the natural «topological-slice-Bennequin Conjecture» is *false*: for every $n \geqslant 5$, there are braids $\beta \in B_n$ such that $\hat{\beta}$ bounds some topological slice surface of Euler characteristic strictly greater than $n - |e(\hat{\beta})|$, cf. [Rudolph 6]. (Though not remarked in that paper, it is in fact the case that «many» such braids exist —e.g., any positive braid with «summit power» at least 2, cf. [Birman]). Of course, such a topological slice surface must be expected (if it is not smoothable) to have horrible behavior, somewhere, with respect to those smooth functions (N, for the round ball D^4; pr_1, for the bidisk D) in terms of which we have gained some understanding of ribbon surfaces, braided surfaces, and even (as we shall shortly see) smooth slice surfaces.

6.9 Definition. Let $S \subset D^4$ be a compact, smoothly embedded 2-manifold-with-boundary with $\partial S = S \cap S^3$ (but not necessarily topless), in general position with respect to the squared-norm function N. Let the Morse function $N|S$ have exactly $m \geqslant 0$ local maxima in $\operatorname{Int} S$, and let $G_1, \ldots, G_m$ be disjoint closed smooth 2-disks embedded in $\operatorname{Int} S$ such that $N|G_j$ is constant on ∂G_j and has a single critical point in $\operatorname{Int} G_j$, a local maximum, $j = 1, \ldots, m$. There is an isotopy of S in $\mathbb{C}^2$ which fixes the points of S outside the G_j and replaces S by $S' = \left(S - \bigcup_{j=1}^{m} G_j\right) \cup \left(\bigcup_{j=1}^{m} G_j'\right)$, where G_j' $(j = 1, \ldots, m)$ is a disk on which $N|G_j'$ has a single interior critical point, a local maximum with value *greater than* 2 (the value of N on S^3). Let $S^{(m)}$ denote $S' \cap D^4$.

By construction, $S^{(m)}$ is a ribbon-embedded surface in D^4, and $\partial(S^{(m)}) = (\partial S)^{(m)}$ (Notation 5.15), so we may unambiguously write $\partial S^{(m)}$. It may be seen that $S^{(m)}$ is well-defined up to isotopy. We call $S^{(m)}$ the *decapitation* of S.

More generally, with S as above, let $q \geqslant m$. By a small isotopy, S may be perturbed to a surface S_q with $N|S_q$ Morse, having $2(q - m)$ more critical points than $N|S$ in pairs of cancelling saddlepoints and local maxima. (If S is connected, S_q is essentially well-defined; in general, one should specify the partition of the q new maxima among the components.) We let $S^{(q)}$ denote $S_q^{(q)}$. If, in particular, S is ribbon-embedded and $q \geqslant 0$, a surface $S^{(q)}$ will be called the result of *punching q holes in S.*

We use the smoothing h to transfer all these notions and notations to the bidisk.

6.10 Example. Let $\underline{\underline{b}}$ be a band representation in B_n. Then (extending Notation 5.15) by $\underline{\underline{b}}^{(q)}$ we denote the band representation in B_{n+q} with $b^{(q)}(j) = b(j)^{(q)}$, $j = 1, \ldots, l(\underline{\underline{b}})$. Of course $\beta(\underline{\underline{b}}^{(q)}) = \beta(\underline{\underline{b}})^{(q)}$. Denote concatenation of lists by C (e.g., $(A_1, A_2)C(A_3, A_4) = (A_1, A_2, A_3, A_4)$). Then for $q \geqslant 0$ one easily sees that $S(\underline{\underline{b}})^{(q)}$ (rather, the particular type of $S(\underline{\underline{b}})^{(q)}$ obtained by punching all q holes in a certain single component of $S(\underline{\underline{b}})$) can be braided as $S(\underline{\underline{b}}^{(q)}C(\sigma_n, \sigma_n^{-1}, \sigma_{n+1}, \sigma_{n+1}^{-1}, \ldots, \sigma_{n+q-1,n+q-1}^{-1}))$. (If $S(\underline{\underline{b}})$ isn't connected, the various types of $S(\underline{\underline{b}})^{(q)}$ could all be represented similarly, using suitable embedded bands in place of standard generators.)

6.11 Proposition. *Let $\hat{\beta} \subset \partial D$ be a closed braid. If $S \subset D^4$ is an oriented slice surface with boundary $h(\hat{\beta})$, then there is some $q \geqslant 0$ and some band representation $\underline{\underline{b}}$ of $\beta^{(q)}$ such that $h(S(\underline{\underline{b}})) = S^{(q)}$. (In words: any oriented slice surface for a closed braid can have holes punched in it until it can be realized as a braided surface for the original closed braid with trivial strings added.)*

PROOF. By Theorem 2.8 (proved in [Rudolph 1]), the ribbon surface $S^{(m)}$ obtained by decapitating S is isotopic to a braided surface $h(S(\underline{\underline{c}}))$, for some band representation $\underline{\underline{c}}$. Now, as an oriented link in ∂D, $\partial S(\underline{\underline{c}})$ is of the same isotopy type as $\hat{\beta}^{(m)}$. Then the proposition is a consequence of the following lemma (take $q = m + k$, k as provided by the lemma). □

6.12 Lemma. *Let α and δ be braids with $\hat{\alpha}$ ambient isotopic to $\hat{\delta}$. Then there is an integer $k \geqslant 0$ such that, for any band representation $\underline{\underline{d}}$ of δ, there is a band representation $\underline{\underline{a}}$ of $\alpha^{(k)}$ with $S(\underline{\underline{a}})$ ambient isotopic to $S(\underline{\underline{d}})^{(k)}$.*

PROOF. Let $\underline{\underline{d}}$ be a band representation of δ.

If δ is obtained from α by a Markov move of type (1) with conjugator w, then $w^{-1}\,\underline{\underline{d}}\,w$ (in the obvious sense) will do for $\underline{\underline{a}}$, with $k = 0$. If δ is obtained from α by a Markov move of type (2^ϵ), $\delta = \alpha^{(1)}\sigma_n^\epsilon(\alpha \in B_n)$, then let $k = 1$ and put $\underline{\underline{a}} = \underline{\underline{d}}\,C(\sigma_n^{-\epsilon})$. If δ is obtained from α by a Markov move of type (3^ϵ), $\alpha = \delta^{(1)}\sigma_n^\epsilon(\delta \in B_n)$, again let $k = 0$, $\underline{\underline{a}} = \underline{\underline{d}}^{(1)}C(\sigma_n^\epsilon)$. Each time, $S(\underline{\underline{a}})$ is isotopic to $S(\underline{\underline{d}})^{(k)}$.

For general α and δ with isotopic closures, Markov's Theorem 6.1 says that a finite sequence of Markov moves joins α and δ. We see that the lemma is true, with k the minimum number (over all such sequences) of moves of type (2^ϵ) required. $\square$

6.13 *Remark.* Proposition 6.11 is a strong form of the observation made in the final paragraph of the body of [Rudolph 1], pp. 30-31. It allows a partial answer (Theorem 6.15) to the question raised there, whether the method of band representations can give any information about «slice genus» (essentially, X_s). It also suggests a method of attack on the problem of whether every slice knot is a ribbon knot, or more generally, whether $X_s = X_r$: namely, find very well controlled braided surfaces isotopic to arbitrary oriented slice surfaces with holes punched in them; then manipulate these surfaces until, along the lines of 5.20, the holes can be filled back in to produce *ribbon* surfaces. The problem is to find the right manipulations....

Similarly, if we apply 6.11 to the empty closed braid, we see that in some sense the whole theory of smooth, oriented, compact surfaces in D^4 without boundary «reduces» to the study of band representations of the trivial braids $o^{(k)}$, $k = 1, 2, 3, \ldots$; however it remains to be seen whether this «reduction» is useful.

6.14 Examples. Let $\beta = \sigma_3^2\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-3}\sigma_2\sigma_1^{-1}\sigma_2 \in B_4$. This braid, found by [Morton], has closure an unknot, yet $\hat{\beta}$ is the boundary of no braided disk in D. There is, of course, a sequence of Markov moves connecting $\beta \in B_4$ to $o^{(1)} \in B_1$, and [Morton] gives an explicit and straightforward such sequence in which there figures a single move of type (2). Then Lemma 6.12 says that there must be a once-punctured braided disk (i.e., braided annulus) in D with boundary $\hat{\beta}^{(1)}$. In fact, one discovers the quite complicated surface $S(\underline{\underline{b}})$, where $\underline{\underline{b}}$ is the band representation in B_5 given by $w(\sigma_4, \sigma_2^{-1}, \sigma_3^{-1}, u\sigma_1u^{-1})w^{-1}C(\sigma_4^{-1})$, where $w = \sigma_3\sigma_4^{-1}\sigma_2^{-1}\sigma_1\sigma_3$ and $u = \sigma_3^{-2}\sigma_2^2$. This can be simplified somewhat by the calculus of slides, but (to date) I have not succeeded in putting into a nice form. Note that this braided annulus certainly does not *appear* to be the push-in to D^4 of an embedded braided annulus $S(\underline{\underline{x}})$ in S^3 (though a calculation of $\pi_1(D^4 - S(\underline{\underline{b}}))$, along the lines of [Rudolph 1], yields $\mathbb{Z}$ and so does not rule out the possibility). Since $\hat{\beta}$ is an unknot, certainly Theorem 5.18 says that there is an embedded k-punctured disk $S(\underline{\underline{x}}) \subset S^3$ with boundary $\hat{\beta}^{(k)}$ for some $k \geqslant 1$; but the difficulties of putting the theorem on a constructive, practical, footing seem insurmountable.

In [Rudolph 1], Example 4.3 is a braided annulus of degree 4 which is the decapitation of a knotted 2-sphere (the 2-twist spun trefoil). The reader is urged to find a band representation of some $o^{(k)}$ which yields this annulus with $k-2$ more punctures.

6.15. There is an interesting consequence of Proposition 6.11. To state it conveniently, we recall that a braid in B_n is *quasipositive* [Rudolph 1, 2, 3, 5] if it is a product of positive bands. The quasipositive braids in B_n form a subsemigroup, strictly larger (for $n \geqslant 3$) than the subsemigroup of *positive* braids (a braid is positive if it is a product of standard generators). The property of being (quasi)positive is preserved by the usual injections. A particularly important positive braid in B_n is $(\sigma_1\sigma_2 \ldots \sigma_{n-1})^n$; it is usually called Δ^2 (cf. [Birman]), but I will call it ∇ here, or ∇_n for greater precision. It is easy to prove the following.

Lemma. (1) *For every n, for every $\beta \in B_n$, there is an integer $Q \geqslant 0$ such that the product $\beta \nabla_n^Q$ is quasipositive, and an integer $P \geqslant Q$ such that the product $\beta\nabla_n^P$ is positive.* (2) *For every n, for every quasipositive $\pi \in B_n$, there is an integer $k \geqslant 0$ and a factorization $\nabla_{n+k} = \pi^{(k)}\varkappa$ in B_{n+k} with $\varkappa$ quasipositive.* (3) *The closure $\hat{\nabla}_n$ is the link of n consistently oriented fibres of the Hopf fibration $S^3 \to S^2$, e.g., the intersection of $S^3 \subset \mathbb{C}^2$ with $\{(z, w)\colon z^N = w^n\}$.* $\square$

6.16 Proposition. *The slice-Bennequin Conjecture 6.7 is true if (and only if) the Slice-Bennequin inequality $sIB(\beta)$ holds for every quasipositive braid β, if (and only if) $sIB(\nabla_n)$ holds for all sufficiently large n.*

PROOF. We show that, starting from any counterexample to the conjecture, one can produce n such that ∇_n is a counterexample.

Thus, assume $sIB(\beta)$ fails for some $\beta \in B_n$, that is, $X_s(\hat{\beta}) > n - |e(\beta)|$. Then also $sIB(\beta^{-1})$ fails, so we can assume that $e(\beta)$ is non-negative. Let S be a slice surface for $\hat{\beta}$, oriented, with $\chi(S) = X_s(\hat{\beta})$. By Proposition 6.11, there is some $q \geqslant 0$ and a band representation $\underline{\underline{b}}$ of $\beta^{(q)}$ in B_{n+q} with $S(\underline{\underline{b}})$ isotopic to $S^{(q)}$. By Lemma 6.14, there is $Q \geqslant 0$ with $\beta\nabla_n^Q \in B_n$ quasipositive. Let $\underline{\underline{c}}$ be the band representation of $(\beta\nabla_n^Q)^{(q)}$ which is $\underline{\underline{b}}$ followed by Qn repetitions of the braid word (band representation with each band a standard generator) $(\sigma_1, \sigma_2, \ldots, \sigma_{n-1})^{(q)}$. Then $S(\underline{\underline{c}})$ is $S(\underline{\underline{b}})$ with $n(n-1)Q$ extra 1-handles attached. Also, $S(\underline{\underline{c}})$ is isotopic to $F^{(q)}$ for some F, a slice surface with boundary the closure of $\beta\nabla_n^Q$. I claim that $sIB(\beta\nabla_n^Q)$ fails: for we have $X_s((\beta\nabla_n^Q)^\wedge) \geqslant \chi(F) = q + \chi(S(c)) = q + \chi(S(b)) - n(n-1)Q = \chi(S) - n(n-1)Q = S_s(\hat{\beta}) - n(n -$

$-1)Q > n - |e(\beta)| - n(n-1)Q = n - e(\beta) - n(n-1)Q = n - e(\beta\nabla_n^Q) =$ $= n - |e(\beta\nabla_n^Q)|$. We have shown that if there is a counterexample in B_n, then there is a quasipositive counterexample in B_n.

But using (2) of Lemma 6.15, we see that if there is a quasipositive counterexample $\pi \in B_n$, then for some k there is a factorization $\nabla_{n+k} = \pi^{(k)}\varkappa$ in B_{n+k} with $\varkappa$ quasipositive. The same trick as was just used (attaching positive bands —in this case, those of a quasipositive band representation of k in B_{n+k}— to a counterexamplifying slice surface for $\hat{\pi}$) shows that $sIB(\nabla_{n+k})$ doesn't hold.

Of course, the «only if» statements are trivial. □

6.17 Corollary. *If the slice-Bennequin conjecture is false, then so is the «Thom Conjecture».*

PROOF. The so-called «Thom Conjecture» asserts that a non-singular complex algebraic curve in $\mathbb{CP}^2$ has the minimal possible genus among all smoothly embedded 2-manifolds in its homology class in $H_2(\mathbb{CP}^2; \mathbb{Z})$. (Cf. [Boileau-Weber]). If the slice-Bennequin conjecture is false, let $S \subset D^4$ be a slice surface of unexpectedly high Euler characteristic for the d-component Hopf link $\tilde{\nabla}_d$. Using (3) of Lemma 6.15, one can replace, on an algebraic curve $\{(z, w)\colon z^d = w^d - \epsilon\}$, ϵ sufficiently small and nonzero, a piece bounded by this link (and having the expected Euler characteristic) by S. The resulting smooth «surgered» surface is homologous to the curve and has smaller genus. □

6.18 *Remark*. The corollary (and its proof) may be summarized in the slogan, «If you can't slice Bennequin, you can surger Thom».

Note that the Thom Conjecture might be false, but not by surgery; then there would be no reason to conclude that the slice-Bennequin conjecture is false too.

Index of notations

A_k, A'_k, A''_k	Double arcs of a ribbon immersion (Def. 2.10).
$B_n (n \geqslant 0)$	Braid group on n strings ($B_0 = \{o\}$) (Def. 1.3).
$\underline{\underline{b}}$; $b(j)$	A band representation; the j^{th} band in $\underline{\underline{b}}$ (Def. 1.12).
$\beta(\underline{\underline{b}})$; $\hat{\beta}(\underline{\underline{b}})$	The braid of $\underline{\underline{b}}$; the closed braid of $\underline{\underline{b}}$ (Def. 1.12).
$\beta^{(k)}$; $\hat{\beta}^{(k)}$	The usual injection of $\beta \in B_n$ into B_{n+k}; the closure of $\beta^{(k)}$, which is the split sum of $\hat{\beta}$ and $\hat{o}^{(k)}$ (Def. 5.14, Notation 5.15).

$D; D_r; D^4$	The unit bidisk; the bidisk of biradius $(1, r)$; the round ball of radius $\sqrt{2}$, all in $\mathbb{C}^2$ (Def. 1.1).
$D^2; rD^2; D_t^2$	The unit disk in $\mathbb{C}$; the disk of radius r in $\mathbb{C}$; the disk $\{\exp it\} \times D^2$ (Definitions 1.1, 5.2).
Δ_n $(n \geqslant 0)$; Δ	The discriminant locus in E_n $(\Delta_0 = \varnothing)$; any Δ_n (Def. 1.1).
$\partial_i D$ $(i = 1, 2)$	Half the boundary of D (a solid torus) (Def. 1.1).
E_n $(n \geqslant 0)$	Complex affine space of monic polynomials in T of degree n $(E_0 = \{0\})$, identified with the n-fold symmetric product of $\mathbb{C}$ (Def. 1.1).
e	Exponent sum of a braid (Rmk. 6.4).
F_{n-1} $(n \geqslant 0)$	The prebraid group, a free group of rank $n - 1$ (F_0 and F_{-1} are distinct trivial groups) (Def. 3.0).
gr	Graph (of a multi-valued function) (Def. 1.5).
$h; h(\partial_i D)$	A fixed smoothing of the bidisk boundary; a fixed solid torus in S^3 (Definitions 1.1, 5.2)
η_I	A height for the tongue T (Construction 4.5).
$I'_j; I_j; I$	An arc used to define a standard generator of the prebraid group (Def. 3.0); a related arc (after Ex. 3.3); the union of the I_j.
J	Any arc (the end-to-end union of $J_1, \ldots, J_{n-1}$) in $\mathbb{C}$ used to define a set of «standard generators» of the braid group B_n (for example, I) (Def. 3.4).
$l(\underline{\underline{b}}); l(\underline{\underline{x}})$	Length of a band representation $\underline{\underline{b}}$ or preband representation $\underline{\underline{x}}$ (Defs. 1.12, 4.7).
N	The squared-norm function (Def. 1.1)
$N'_j; N_j$	Certain 2-cells containing I'_j and I_j (Def. 3.0).
O	The unknot (*passim*).
$o; o^{(k)}, k > 0$	The identity of B_0; the identity of B_k (Convention 1.4).
P	A certain simple branched cover $S^3 \to S^3$ (Def. 4.9).
$p; p^{-1}; \tilde{p}$	A certain simple branched cover of $\mathbb{C}$ by $\mathbb{C}$, given by a complex polynomial of degree n; the inverse of p, as an n-valued map, or a map $\mathbb{C} \to E_n$; an approximation to p which is a monomial near ∞ (Props. 3.1 and 3.8, Construction 4.9).
$(p^{-1})_*$	The surjection of F_{n-1} onto B_n induced by p^{-1} (Prop. 3.8).
Q_j	A certain 2-cell containing $J_j \subset J$ (Def. 3.4).
$R(\tau, I)$	A geometric preband (Construction 4.5).
$r; \tilde{r}$	Radii related to p, $\tilde{p}$ (Construction 4.9).
$\rho(\underline{\underline{x}}); \hat{\rho}(\underline{\underline{x}})$	The prebraid, resp., closed prebraid, of a preband representation $\underline{\underline{x}}$ (Def. 4.7).
S^+	A Markov surface with its negative disks removed.
$S(\underline{\underline{b}}); S(\underline{\underline{x}})$	The braided surface in D or D^4 constructed from a band representation $\underline{\underline{b}}$ (Prop. 1.13); the braided surface in S^3

	constructed from a preband representation $\underline{\underline{x}}$ (Construction 4.14).
S^3	The round sphere of radius $\sqrt{2}$ (Def. 1.1).
S_n	The symmetric group on n letters (Def. 1.1).
$\Sigma(\underline{\underline{x}})$	The prebraided disk in S^3 constructed from a preband representation $\underline{\underline{x}}$ (Def. 4.13).
σ_j; $\sigma_{uv}^{\pm 1}$	The standard generators ($j = 1, \ldots, n-1$) of B_n (Def. 3.4); the embedded bands ($1 \leqslant u \leqslant v \leqslant n-1$) in B_n (Rmk. 5.13).
T	The tongue (Construction 4.5).
τ	A tadpole (Def. 4.2).
$\overline{U(\tau)}$	The head of the tadpole τ (Def. 4.2).
$\bar{u}_{m,n}$; $u_{m,n}$	The usual injection of F_{m-1} into F_{n-1}, or of B_m into B_n ($m \geqslant n$) (Def. 5.14).
V	The critical values of p (Prop. 3.1).
v_0; $v_i, i > 0$	The basepoint of F_{n-1}; the elements of V (Def. 3.0).
$X(L)$	The maximal Euler characteristic of a Seifert surface for the link L (Notation 5.2).
X_r, X_s	the ribbon and slice analogues of X (Def. 6.5).
x_i ($1 \leqslant i \leqslant n-1$)	The standard generators of F_{n-1} (Def. 3.0).
$\underline{\underline{x}}$; $x(i)$	A preband representation; the i^{th} preband in $\underline{\underline{x}}$ (Def. 4.7).
$x_{uv}^{\pm 1}$	The elementary embedded prebands (Thm. 5.12).
ϕ	The angular coordinate in $h(\partial_2 D)$ (Notation 5.2).
θ; θ_S	The angular coordinate in $h(\partial_1 D)$; its restriction to the part of a surface S in $h(\partial_1 D)$.
$\hat{\ }$	Operation of closure applied to a braid or prebraid (Defs. 1.12, 4.7).
$\_^{(k)}$	When applied to a braid, shorthand for a usual injection; when applied to a link, split sum with a trivial link of k unknotted components (Notation 5.15).
∇_n	An element of B_n usually denoted Δ_n^2 (Def. 6.14).

When a space is given explicitly as a Cartesian product of two or more factors, the notation pr_i denotes projection onto the i^{th} factor. The restriction of a mapping f to a subset M of its domain is denoted $f|M$.

References

Artin, E. Theory of braids, *Ann. of Math.,* **48** (1947), 101-126.

Bennequin, D. Entrelacements et équations de Pfaff, *Astérisque,* **107-108** (1983), 87-161.

Birman, Joan S. Braids, Links, and Mapping Class Groups, *Ann. of Math.*, Studies No. 82.

Boileau, M. and Weber, C. Le problème de J. Milnor sur le nombre gordien des noeuds algébriques, in *Noeuds, Tresses, et Singularités,* Monographie No. 31 de L'Enseignement Mathématique, Genève (1983).

Douady, A. Noeuds et structures de contact en dimension 3 (d'après D. Bennequin), Seminaire Bourbaki No. 604 (1982).

Freedman, M. The disk theorem, to appear in Proceedings of the International Congress of Mathematicians, Warsaw, 1983.

Hass, J. The geometry of the slice-ribbon problem, *Math. Proc. Camb. Phil. Soc.*, (1983), 101-108.

Milnor, J. W. Morse Theory, *Ann. of Math.,* Studies No. 51.

Morton, H. An irreducible 4-string braid with unknotted closure, Math. Proc. Camb. Phil. Soc., **93** (1983), 259-261.

Rudolph, Lee Braided surfaces and Seifert ribbons for closed braids, *Comment. Math. Helvetici,* **58** (1983), 1-37.

— Constructions of quasipositive knots and links, I, in *Noeuds, Tresses et Singularités,* Monographie No. 31 de L'Enseignement Mathématique, Genève (1983), 233-245.

— Some Knot theory of complex plane curves, *ibid.*, 99-122.

— Plane curves in fancy balls, to appear in *L'Enseignement Mathématique*, 1985.

— Algebraic functions and closed braids, *Topology* **22** (1983), 191-202.

— Some topologically locally flat surfaces in the complex projective plane, to appear in *Comment. Math. Helvetici*, 1984.

Tristram, A. G. Some cobordism invariants for links, *Proc. Camb. Phil. Soc.,* **66** (1969), 251-264.

van Kampen, E. R. On the fundamental group of an algebraic curve, *Am. J. Math.,* **55** (1933), 255-260.

Zariski, O. On the problem of existence of algebraic functions of two variables possessing a given branch curve, *Am. J. Math.* **51** (1929), 305-328.

Lee Rudolph
P. O. Box 251
Adamsville, R. I. 02801
U.S.A.

Revista Matemática Iberoamericana
Vol. 1, n.º 4, 1985

Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation

Dédié à Alberto P. Calderón

G. David, J. L. Journé, et S. Semmes

Introduction

Calderón et Zygmund ont introduit une classe d'opérateurs intégraux singuliers généralisant la transformée de Hilbert et les transformées de Riesz [CZ]. Cette classe s'est progressivement enrichie jusqu'à contenir des opérateurs qui ne sont pas de convolution. Dans l'étude de ces opérateurs, le problème central est celui de la continuité sur L^2, pour lequel en général Plancherel n'est pas suffisant. Ce problème a suscité de très belles pages d'analyse ([C1], [C2], [CMM1], [CM1]) où des cas particuliers sont traités: les commutateurs de Calderón, le noyau de Cauchy sur les courbes lipschitziennes, les opérateurs multilinéaires de Coifman et Meyer. Ce n'est que récemment qu'un critère général de continuité sur L^2 est apparu [DJ]. Ce critère, appelé le «Théorème $T1$», dit qu'un opérateur d'intégrale singulière est borné sur L^2 si et seulement si il est faiblement borné (voir la Définition 3) et lui-même et son adjoint envoient la fonction 1 dans BMO.

Un premier motif de frustation est que ce théorème ne permet pas de montrer directement la continuité sur L^2 du plus célèbre des opérateurs de Calderón-Zygmund, le noyau de Cauchy sur les graphes lipschitziens [CMM1]. Un autre motif de frustation est le suivant. Depuis les papiers [CW1] et [CW2] de Coifman-Weiss, il est reconnu que le cadre naturel de la théorie des intégrales singulières est celui des espaces de type homogène. On voudrait avoir une théorie des opérateurs aussi souple que celle des espaces de nature homogène, et en particulier invariante par changement de variable bilipschitzien. Or le Théorème $T1$ ne l'est pas. Coifman a réglé ce problème en démontrant le Théorème $T1$ dans le cadre des espaces de nature homogène. Dans le cas particulier d'un espace euclidien muni de la mesure $b(x)\,dx$, on obtient l'énoncé suivant:

«Soit b une fonction positive bornée telle que b^{-1} soit également bornée, et soit M_b l'opérateur de multiplication ponctuelle par b. Un opérateur d'intégrale singulière T est borné sur L^2 si et seulement si $M_b T M_b$ est faiblement borné, Tb et $T^* b$ sont dans BMO».

Le principal outil de sa démonstration, non publiée, est le lemme de Cotlar, Knapp et Stein [KS], réputé depuis sa naissance comme l'outil le plus prometteur pour les problèmes de continuité sur L^2.

Peu après, Y. Meyer remarquait qu'il suffirait de pouvoir remplacer, dans l'énoncé précédent, l'hypothèse «b est positive» para l'hypothèse d'accrétivité $Re\, b \geqslant \delta > 0$ pour pouvoir traiter directement la continuité sur L^2 de l'opérateur de Cauchy sur les graphes lipschitziens. En collaboration avec A. McIntosh, il obtint ce résultat dans le cas où T et son transposé envoient b sur 0, ce qui est suffisant pour beaucoup d'applications, dont le noyau de Cauchy [MM]. Leur démonstration repose sur un résultat d'interpolation provenant de la solution du problème de Kato en dimension 1 [CMM1], et semble limitée au cadre des espaces euclidiens.

L'objet de ce texte est de donner des conditions très générales sur des fonctions b et c pour que l'énoncé suivant soit vrai: «un opérateur d'intégrale singulière T est borné sur L^2 si et seulement si $M_c T M_b$ est faiblement borné, $Tb \in$ BMO, et ${}^tTc \in$ BMO», où ${}^tT$ désigne le transposé de T. On est amené à introduire une classe de fonctions, que nous appellerons para-accrétives, contenant les fonctions accétives et les dérivées des paramétrisations normales de courbes de Lavrentiev (aussi appelées courbes corde-arc). L'énoncé écrit plus haut est vrai lorsque b et c sont para-accrétives, et cette classe de fonction est, en un certain sens, optimale.

La démonstration du théorème est écrite dans les espaces euclidiens, mais peut facilement être généralisée aux espaces de nature homogène en utilisant [A], [CW1], [CW2], [MS1], [MS2]. C'est dans cet esprit que nous avons remplacé l'espace C_0^∞ des fonctions test par l'espace C_0^η des fonctions hölderiennes d'exposant η á support compact, et, naturellement, que nous nous sommes interdit l'usage de la transformée de Fourier.

Au paragraphe 1, nous faisons quelques rappels sur les intégrales singulières. Au paragraphe 2, nous énonçons les théorèmes de Coifman et McIntosh Meyer, et nous démontrons le théorème de Coifman dans le cas particulier de

$(\mathbb{R}^d, b(x)\,dx)$. Les fonctions para-accrétives sont introduites au paragraphe 3, et nous démontrons au paragraphe 4 deux lemmes de théorie des opérateurs destinés à remplacer le lemme de Cotlar-Knapp-Stein. Le théorème principal est énoncé au paragraphe 5, et il est montré au paragraphe 6 que la para-accretivité est une condition nécessaire pour que le Théorème *Tb* soit vrai. Quelques extensions du Théorème *Tb* sont mentionnées au paragraphe 7; le Théorème *Tb* est utilisé au paragraphe 8 pour prouver directement la continuité de l'opérateur de Cauchy sur les courbes de Lavrentiev, et pour construire un calcul fonctionnel holomorphe en plusieurs variables pour certains opérateurs différentiels du premier ordre à coefficients peu réguliers. On montre au paragraphe 9 que l'espace d'interpolation complexe à mi-chemin entre l'espace de Sobolev B^{-s} $(s > 0)$ et bB^s est L^2 si et seulement si la fonction b est para-accrétive.

Nous souhaitons vivement remercier R. Coifman pour nous avoir réunis cet automne et, bien sûr, pour nous avoir expliqué ses idées. La visite de A. McIntosh et Y. Meyer fut le moment le plus excitant de ce séjour, et aussi le plus fructueux. Quoique plus occasionnels, T. Fack, J. St-Raymond et J. C. Sikorav ont été de précieux collaborateurs, que nous tenons à remercier. La rédaction de cet article a bénéficié de nombreuses suggestions de L. Schwartz et de M. Herman dont nous leur sommes très reconnaissants.

I. OPÉRATEURS DE CALDERÓN-ZYGMUND

1. Opérateurs d'integrale singulière

Un opérateur d'intégrale singulière est habituellement un opérateur défini et continu de $C_0^\infty(\mathbb{R}^d)$ dans son dual et associé, en un sens à préciser, à un noyau vérifiant des estimations dites «standard» (voir [CM1]) que nous rappelons.

Définition 1. *Soit K une fonction continue définie sur $\Omega = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d, x \neq y\}$ et soit $0 < \delta \leqslant 1$. La fonction K est un noyau δ-standard s'il existe une constante $C > 0$ telle que*

(1.1) *pour tout (x,y) dans Ω,* $|K(x,y)| \leqslant \dfrac{C}{|x-y|^d}$

(1.2) *pour tout* (x,y) *dans* Ω *et tout* $x' \in \mathbb{R}^d$ *tel que* $|x-x'| < \dfrac{|x-y|}{2}$,

$$|K(x,y) - K(x',y)| \leqslant C \frac{|x-x'|^\delta}{|x-y|^{d+\delta}} \quad et$$

$$|K(y,x) - K(y,x')| \leqslant C \frac{|x-x'|^\delta}{|x-y|^{d+\delta}}.$$

La plus petite constante C pour laquelle (1.1) *et* (1.2) *sont satisfaits sera notée* $|K|_\delta$.

Définition 2. *Soit* $K\colon \Omega \to \mathbb{C}$ *une fonction vérifiant* (1.1) *et soit* $T\colon C_0^\infty(\mathbb{R}^d) \to$ $\to [C_0^\infty(\mathbb{R}^d)]'$ *un opérateur linéaire continu. La fonction K est associée à T si, pour tous f et g* C^∞ *à supports compacts disjoints,*

$$\langle g, Tf \rangle = \int g(x)K(x,y)f(y)\,dy\,dx, \tag{1.3}$$

où l'on note $\langle g, Tf \rangle$ *l'action de la distribution Tf sur la fonction g.*

Dans le cas où la fonction K est antisymétrique, elle est automatiquement associée à un opérateur $\tilde{K}$ défini par

$$\langle g, \tilde{K}f \rangle = \frac{1}{2} \iint \{g(x)f(y) - g(y)f(x)\} K(x,y)\,dy\,dx, \tag{1.4}$$

g et f étant deux fonctions C^∞ à support compact. La régularité de f et g entraîne la convergence de l'intégrale; plus précisément, si le diamètre de $\operatorname{supp} f \cup \operatorname{supp} g$ est inférieur à t,

$$|\langle g, \tilde{K}f \rangle| \leqslant Ct^{d+2} \|\nabla f\|_\infty \|\nabla g\|_\infty, \tag{1.5}$$

où C ne dépend que de la dimension et de la constante apparaissant dans (1.1). L'inégalité (1.5) est une conséquence de l'estimation

$$|g(x)f(y) - g(y)f(x)| \leqslant A|x-y|,$$

où $A = C(\|g\|_\infty \|\nabla f\|_\infty + \|f\|_\infty \|\nabla g\|_\infty) \leqslant Ct(\|\nabla f\|_\infty \|\nabla g\|_\infty)$. L'inégalité (1.5) entraîne que $\tilde{K}$ est faiblement borné. Rappelons ce que ceci signifie (voir aussi [DJ]).

Définition 3. *Un opérateur linéaire continu* $T\colon C_0^\infty(\mathbb{R}^d) \to [C_0^\infty(\mathbb{R}^d)]'$ *est **faiblement borné** s'il existe* $N \geqslant 0$ *et* $C > 0$ *tels que pour tout cube Q et toutes fonctions f et g* C^∞ *et supportées dans Q,*

$$|\langle g, Tf \rangle| \leqslant C|Q|P(N,f,Q)P(N,g,Q), \tag{1.6}$$

où $P(N,f,Q)$ est la semi-norme

$$P(N,f,Q) = \sum_{|\alpha| \leq N} |Q|^{|\alpha|/d} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_\infty .$$

Les méthodes de variable réelle de Calderón, Zygmund et leur école permettent de montrer qu'un opérateur d'intégrale singulière s'étend en un opérteur continu sur L^2 si et seulement si on peut choisir $N = 0$ dans la définition précédente, ou en d'autres termes si

$$|\langle g, Tf \rangle| \leqslant C|Q| \, \|g\|_\infty \|f\|_\infty . \tag{1.7}$$

Une démonstration peut être trouvée dans [J2], pp. 43 et 49. Nous reviendrons plus tard sur les hypothèses qu'il faut ajouter à (1.6) pur déduire (1.7). Pour l'instant, notons que si T est un opérateur borné associé à un noyau K vérifiant (1.1), on peut, dans la Définition 3, choisir $N = 1$, ou même mieux encore.

Proposition 1. *Soit T un opérateur faiblement borné associé à un noyau K vérifiant* (1.1). *Alors, pour tout* $0 < \eta \leqslant 1$, *il existe* $C_\eta > 0$ *tel que, pour toutes fonctions $C^\infty f$ et g supportées dans un cube Q,*

$$|\langle f, Tf \rangle| \leqslant C_\eta |Q|^{1 + (2\eta/d)} \|f\|_\eta \|g\|_\eta , \tag{1.8}$$

où

$$\|f\|_\eta = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta} .$$

Notons que (1.6) implique que T peut être étendu en un opérateur continu de $C_0^N(\mathbb{R}^d)$ dans son dual. La proposition 1 nous dit que T peut même être étendu en un opérateur continu de λ^η dans son dual, où λ^η est la clôture de $C_0^\infty(\mathbb{R}^d)$ pour la norme $\| \ \|_\eta$.

Remarquons que (1.8) est d'autant plus fort que η est petit, et (1.7) équivaut à dire que C_η ne dépend pas de η.

Rappelons le lemme de commutation de Meyer [M].

Lemme 1.1. *Soit T: $C_0^\infty(\mathbb{R}^d) \to [C_0^\infty(\mathbb{R}^d)]'$ un opérateur linéaire faiblement borné (au sens de la définition 3) associé à un noyau K vérifiant* (1.1). *Alors, si f, g et h sont trois fonctions de $C_0^\infty(\mathbb{R}^d)$,*

$$\langle f, T(gh) \rangle - \langle fg, Th \rangle = \iint f(x)(g(y) - g(x))K(x,y)h(y)\,dx\,dy. \tag{1.9}$$

Les hypothèses du lemme entraînent immédiatement que cette intégrale est absolument convergente, ce qui reste d'ailleurs vrai lorsque f et h sont bornées à support compact et $\|g_\eta\| < +\infty$ pour un $\eta > 0$.

Démontrons le lemme. L'égalité (1.9) est vraie quand T est donné par intégration contre un noyau K défini et localement borné sur $\mathbb{R}^d \times \mathbb{R}^d$. Nous allons donc nous ramener à ce cas. Soit $\varphi \in C^\infty(\mathbb{R}^d)$ une fonction radiale, symétrique, d'intégrale 1, supportée dans la boule unité, et soit P_t l'opérateur de convolution avec la fonction φ_t déterminée par $\varphi_t(x) = (1/t^d)\varphi(x/t)$. L'opérateur P_t est continu de $C_0^\infty(\mathbb{R}^d)$ dans lui-même et est auto-adjoint. L'opérateur T_t défini par $\langle g, T_t f\rangle = \langle (P_t g), T(P_t f)\rangle$ est donc bien défini et continu de $C_0^\infty(\mathbb{R}^d)$ dans son dual. De plus, T_t est donné par intégration contre un noyau localement borné que nous allons maintenant expliciter.

Soient f et g deux fonctions de $C_0^\infty(\mathbb{R}^d)$, et $t_0 > 0$. Soit Q un cube contenant les supports des fonctions $\varphi_t(x - \cdot)g(\cdot)$ et $\varphi_t(y - \cdot)f(\cdot)$ pour tous $x, y \in \mathbb{R}^d$ et $t \leqslant t_0$. Pour tout $z \in \mathbb{R}^d$, soit

$$\varphi_t^z(x) = \varphi_t(x - z) = \frac{1}{t^d}\varphi\left(\frac{x - z}{t}\right).$$

Pour $N \geqslant 1$, soit C_Q^N l'espace de Banach des fonctions N fois continûment différentiable supportées dans Q, muni de la norme

$$\|f\|_{N,Q} = \sum_{|\alpha| \leq N} \left\|\frac{\partial^\alpha f}{\partial x^\alpha}\right\|_\infty.$$

L'intégrale $\int g(z)\varphi_t^z dz$ est absolument convergente dans C_Q^N et sa somme est $P_t g$. D'autre part, la continuité faible de T entraîne que, pour un certain $N \geqslant 0$, T est continu de C_Q^N dans son dual. On a donc

$$\begin{aligned}\langle g, T_t f\rangle &= \langle P_t q, T(P_t f)\rangle \\ &= \left\langle \int g(z)\varphi_t^z dz, T\int f(w)\varphi_t^w dw\right\rangle \\ &= \iint g(z)\langle \varphi_t^z, T\varphi_t^w\rangle f(w)\, dz\, dw.\end{aligned}$$

Le noyau de T_t ets donc $K_t(z, w) = \langle \varphi_t^z, T\varphi_t^w\rangle$, qui est une fonction localement bornée de (z, w), et même bornée par $C_\varphi t^{-d}$ en raison de l'hypothèse (1.1) sur le noyau de T et de la continuité faible de T.

Pour conclure la démonstration du Lemme 1.1, on fait tendre t vers 0. Il est clair que $\lim_{t\to 0} P_t f = f$ dans C_Q^N, de sorte que $\langle g, Tf\rangle = \lim_{t\to 0}\langle g, T_t f\rangle$. Appliquant l'égalité (1.9) à T_t, on est amené à montrer que

$$\lim_{t\to 0}\iint f(x)\{g(y) - g(x)\}K_t(x, y)h(y)\, dx\, dy =$$

$$= \iint f(x)\{g(y) - g(x)\}K(x, y)h(y)\, dx\, dy.$$

Par convergence dominée il suffit de montrer que $\lim_{t\to 0} K_t(x, y) = K(x, y)$ p.p. pour $x \neq y$ et que $|K_t(x, y)| \leqslant C/|x - y|^d$. Or, si $x \neq y$, les supports de

φ_t^x et φ_t^y sont disjoints pour $t < |x-y|/2$ et dans ce cas

$$K_t(x,y) = \langle \varphi_t^x, T\varphi_t^y \rangle = \iint \varphi_t(x-u)K(u,v)\varphi_t(v-y)\,dy\,dx,$$

de sorte que, par le théorème de différentiation de Lebesgue, $\lim_{t\to 0} K_t(x,y) = K(x,y)$ p.p. sur Ω. De plus, (1.1) entraîne que $|\langle \varphi_t^x, T\varphi_t^y\rangle| \leqslant C/|x-y|^d$; lorsque $t \geqslant |x-y|/4$ cette inégalité découle de (1.6).

Le Lemme 1.1 est démontré; nous passons à la démonstration de la proposition. Comme l'énoncé de la Proposition 1 est invariant par translations et dilatations, il suffit de la montrer lorsque Q est le cube unité. Soit $\theta \in C_0^\infty(\mathbb{R}^d)$ une fonction égale à 1 sur le cube unité. Le Lemme 1.1 entraîne $|\langle g, Tf\rangle| = |\langle g, T(f\theta)\rangle| \leqslant |\langle gf, T\theta\rangle| + \iint |f(x)|\,|g(x)-g(y)|\,|K(x,y)|\,|\theta(y)|\,dx\,dy$.

L'intégrale est clairement dominée par $C_\eta \|f\|_\infty \|g\|_\eta$, donc par $C_\eta \|f\|_\eta \|g\|_\eta$, de sorte qu'il reste à montrer que $|\langle gf, T\theta\rangle| \leqslant C_\eta \|f\|_\eta \|g\|_\eta$.

Comme $\|gf\|_\eta \leqslant \|g\|_\infty \|f\|_\eta + \|f\|_\infty \|g\|_\eta \leqslant C_\eta \|g\|_\eta \|f\|_\eta$, il suffit en fait de montrer que pour tout h,

$$(1.10) \qquad |\langle h, T\theta\rangle| \leqslant C_\eta \|h\|_\eta .$$

Soient P_t comme dans la démonstration du Lemme 1.1 et $Q_t = -t(\partial/\partial t)P_t$. D'après (1.6),

$$(1.11) \qquad |\langle P_1 h, T\theta\rangle| \leqslant C\|h\|_\infty \leqslant C_\eta \|h\|_\eta .$$

D'autre part, si $\tilde{Q}$ est un cube contenant le support de θ dans son intérieur, $\lim_{t\to 0} P_t h = h$ dans $C_{\tilde{Q}}^N$ pour tout $N \geqslant 0$. Il suffit donc de montrer que pour tout $t < 1$, $|\langle P_t h, T\theta\rangle| \leqslant C_\eta \|h\|_\eta$ et de passer à la limite en utilisant la continuité de T de $C_{\tilde{Q}}^N$ dans son dual. Compte tenu de (1.11), il suffit de vérifier que $|\langle (P_1 - P_t)h, T\theta\rangle| \leqslant C_\eta \|h\|_\eta$, ou encore $\int_t^1 |\langle Q_s h, T\theta\rangle|\,ds/s \leqslant C_\eta \|h\|_\eta$. Cette inégalité résultera de

$$(1.12) \qquad |\langle Q_s h, T\theta\rangle| \leqslant C_{\eta'} \|h\|_\eta s^{\eta'}$$

pour tout $s \in]0,1]$ et tout $\eta' < \eta$.

Pour vérifier (1.12), on choisit à nouveau φ comme dans la démonstration du Lemme 1.1 et on note $\varphi_s^x(y) = (1/s^d)\varphi[(y-x)/s]$. Alors

$$Q_s h(y) = \int Q_s h(y)\varphi_s^x(y)\,dx \quad \text{et} \quad \theta(y) = \int \theta(y)\varphi_s^z(y)\,dz,$$

ces égalités ayant lieu dans $C_{\tilde{Q}}^N$ pour tout $N \geqslant 0$. Donc

$$|\langle Q_s h, T\theta\rangle| \leqslant \iint |\langle (Q_s h)\varphi_s^x, T\theta\varphi_s^z\rangle|\,dx\,dz.$$

On remarque maintenant que

$$\|Q_s h\|_\infty = \sup_{x \in \mathbb{R}^d} \left| \int \frac{1}{s^d} \psi\left(\frac{x-y}{s}\right) h(y)\, dy \right|,$$

où ψ est une fonction radiale de moyenne nulle, C^∞ et supportée dans la boule unité. Par conséquent

$$\begin{aligned}\|Q_s h\|_\infty &\leqslant \sup_x \left| \int \frac{1}{s^d} \psi\left(\frac{x-y}{s}\right)(h(y) - h(x))\, dy \right| \\ &\leqslant C \sup_{|x-y| \leq 2s} |h(y) - h(x)| \leqslant C s^\eta \|h\|_\eta.\end{aligned}$$

Cette inégalité permet de majorer $\iint_{|x-z| \geq 3s} |\langle (Q_s h)\varphi_s^x, T\theta\varphi_s^z \rangle|\, dx\, dz$ par $Cs^\eta \|h\|_\eta \iint_{\substack{|x-z| \geq 3s \\ |x|+|z| \leq C}} |x-z|^{-d}\, dx\, dz$, soit encore par $Cs^\eta \|h\|_\eta \left(\log \frac{1}{s} + 1\right)$. Pour conclure la démonstration de (1.12), et par conséquent de la Proposition 1, il ne reste qu'à vérifier que

$$\iint_{|x-z| \leq 3s} |\langle (Q_s h)\varphi_s^x, T\theta\varphi_s^z \rangle|\, dx\, dz \leqslant C s^\eta \|h\|_\eta. \tag{1.13}$$

Notons que pour tout α,

$$\left\| \frac{\partial^\alpha}{\partial x_\alpha} Q_s h \right\|_\infty \leqslant C s^{\eta - |\alpha|} \|h\|_\eta,$$

ce qui se voit exactement comme dans le cas $|\alpha| = 0$. Comme pour $|x - z| \leqslant 3s$, $(Q_s h)\varphi_s^x$ et $\theta\varphi_s^z$ sont supportés dans un cube de côté $5s$, on déduit de (1.6) que $|\langle (Q_s h)\varphi_s^x, T\theta\varphi_s^z \rangle| \geqslant C s^{d+\eta} \|h\|_\eta$. Il suffit pour obtenir (1.13) d'intégrer cette inégalité sur le domaine $\{|x - z| \leqslant 3s \text{ et } |x| \leqslant C\}$.

La Proposition 1 est démontrée. Elle signifie que l'on peut remplacer (1.6) par (1.8) dans la Définition 3 sans la changer, à condition que T soit associé à un noyau K vérifiant (1.1). Elle suggère également une modification de la définition d'un opérateur d'intégrale singulière, motivée par le fait suivant. Nous aurons l'occasion de travailler sur des espaces de nature homogène abstraits. Or sur ces espaces, la notion de fonction C^∞ n'existe pas; par contre Macias et Segovia ont montré [MS1] [MS2] que, sur tout espace de nature homogène, la notion de fonction höldérienne existe pour tout exposant suffisamment proche de 0. Il est donc naturel de définir un opérateur d'intégrale singulière comme a priori défini sur les fonctions à support compact höldériennes d'un exposant $\eta \in]0, 1]$.

Nous aurons besoin d'une généralisation supplémentarie. La lettre b désignera dans toute la suite une foction à valeurs complexes bornée ainsi que son inverse, et M_b désignera l'opérateur de multiplication ponctuelle par b. Soit

$C_0^\eta(\mathbb{R}^d)$ l'espace des fonctions f à support compact telles que

$$\|f\|_\eta = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\eta} < +\infty,$$

et soit $bC_0^\eta(\mathbb{R}^d)$ l'image de $C_0^\eta(\mathbb{R}^d)$, munie de la topologie image.

Définition 4. *Soient b_1 et b_2 deux fonctions bornées ainsi que leurs inverses. Un **opérateur d'intégrale singulière** (SIO) est un opérateur linéaire continu $T\colon b_1C_0^\eta(\mathbb{R}^d) \to [b_2C_0^\eta(\mathbb{R}^d)]'$ pour tout $\eta > 0$, associé à un noyau standard K au sens que, si f et g sont dans $C_0^\eta(\mathbb{R}^d)$ et ont des supports disjoints,*

$$\langle b_2 g, T(b_1 f)\rangle = \iint g(x)b_2(x)K(x,y)b_1(y)f(y)\,dy\,dx.$$

Remarquons que $M_{b_2}TM_{b_1}$ est défini de $C_0^\eta(\mathbb{R}^d)$ dans son dual et a pour noyau $b_2(x)K(x,y)b_1(y)$, qui vérifie (1.1). On dira que $M_{b_2}TM_{b_1}$ est faiblement borné s'il vérifie (1.8) pour un $\eta > 0$.

Comme dans le cas où $b_1 = b_2 = 1$, un SIO a une extensión naturelle à l'espace $b_1C_b^\eta(\mathbb{R}^d)$ des produits par b_1 de fonctions bornées localement höldériennes d'exposant η. Soit $\{b_2C_0^\eta(\mathbb{R}^d)\}_0$ le sous-espace de $b_2C_0^\eta(\mathbb{R}^d)$ constitué des fonctions d'intégrale nulle. Pour définir $\langle g, Tf\rangle$ lorsque $f\in b_1C_b^\eta(\mathbb{R}^d)$ et $g\in\{b_2C_0^\eta(\mathbb{R}^d)\}_0$, on choisit $h\in C_0^\eta(\mathbb{R}^d)$ égale à 1 sur un voisinage de $\operatorname{supp} g$ et on écrit

$$\langle g, Tf\rangle = \langle g, T(fh)\rangle + \langle g, T[f(1-h)]\rangle$$

où, pour un $x_0\in\operatorname{supp} g$,

$$\langle g, T[f(1-h)]\rangle = \iint g(x)(K(x,y) - K(x_0,y))f(y)(1-h(y))\,dy\,dx.$$

L'inégalité (1.2) entraîne que cette intégrale est convergente à l'infini. De plus, g et $1-h$ ayant des supports disjoints, le terme à intégrer est borné, de sorte que l'intégrale est absolument convergente. La propriété $\int g(x)\,dx = 0$ entraîne qu'elle ne dépend pas du choix de x_0; il est aussi facile de voir que $\langle g, Tf\rangle$, ainsi défini, ne dépend pas du choix de h et que cette nouvelle définition coincide avec l'ancienne lorsque $f\in b_1C_0^\eta(\mathbb{R}^d)$. Remarquons que, pour $f\in b_1C_b^\eta(\mathbb{R}^d)$, Tf est défini par dualité contre les fonctions de moyenne nulle, donc à une constante additive près.

Nous allons maintenant énoncer un lemme qui permet de calculer Tf pour $f\in b_1C_b^\eta(\mathbb{R}^d)$. Une suite $(T_m)_{m\in\mathbb{N}}$ de SIO de noyaux K_m est dite bornée si les T_m sont équicontinus de $b_1C_0^\eta(\mathbb{R}^d)$ dans $[b_2C_0^\eta(\mathbb{R}^d)]'$ pour chaque $\eta > 0$ et s'il existe $\delta\in\,]0,1]$ et $C > 0$ tels que $|K_m|_\delta \leqslant C$ pour tout $m\in\mathbb{N}$.

Lemme 1.2. *Soit $(T_m)_{m\in\mathbb{N}}$ une suite bornée de* SIO *et $T: b_1 C_0^\eta(\mathbb{R}^d) \to \{b_2 C_0^\eta(\mathbb{R}^d)\}'$ un opérateur linéaire continu tel que pour tous $f \in b_1 C_0^\eta(\mathbb{R}^d)$ et $g \in b_2 C_0^\eta(\mathbb{R}^d)$, $\lim_{m\to+\infty} \langle g, T_m f\rangle = \langle g, Tf\rangle$. Si les noyaux K_m convergent uniformément sur tout compact de Ω vers une limite K, alors T est un* SIO *de noyau K. De plus, pour tout $f \in b_1 C_b^\eta(\mathbb{R}^d)$ et tout $g \in \{b_2 C_0^\eta(\mathbb{R}^d)\}_0$, $\lim_{m\to\infty} \langle g, T_m f\rangle = \langle g, Tf\rangle$.*

La première assertion est triviale. Pour vérifier la seconde, on choisit g, f et h comme avant l'énoncé du lemme. Alors $\langle g, T(fh)\rangle = \lim_{m\to\infty} \langle g, T_m(fh)\rangle$ par hypothèse, et $\langle g, Tf(1-h)\rangle = \lim_{m\to\infty} \langle g, T_m[f(1-h)]\rangle$ d'après le théorème de convergence dominée. Le Lemme 1.2 est démontré.

Pour terminer ces préliminaires, nous donnons la démonstration dans ce contexte d'un théorème de Lemarié. L'étape principale est le lemme suivant.

Lorsque φ et ψ sont deux fonctions à support compact, on notera $\Delta\varphi$ le diamètre du support de φ, $\Delta\psi$ le diamètre du support de ψ, et $\Delta\varphi\psi$ le diamètre de $\operatorname{supp}\varphi \cup \operatorname{supp}\psi$.

Lemme 1.3. *Soit $T: b_1 C_0^\eta(\mathbb{R}^d) \to [b_2 C_0^\eta(\mathbb{R}^d)]'$ un* SIO *tel que $M_{b_2} T M_{b_1}$ soit faiblement borné, et dont le noyau K est δ-standard pour un $\delta > \eta$. On suppose de plus que $Tb_1 = 0$. Soient $\varphi \in C_0^\eta(\mathbb{R}^d)$ et $\psi \in \{b_2 C_0^\eta(\mathbb{R}^d)\}_0$. Alors*

$$(1.14) \qquad |\langle \psi, Tb_1\varphi\rangle| \leqslant C \frac{(\Delta\psi)^\eta (\Delta\varphi + \Delta\psi)^{d+\eta}}{(\Delta\varphi\psi)^{d+\eta}} \|\psi\|_{L^1} \|\varphi\|_\eta .$$

Nous pouvons nous contenter de démontrer le lemme lorsque $\Delta\psi \leqslant \Delta\varphi$ (autrement, on remplace le support de φ par une boule de rayon $\Delta\psi$, ce qui ne change pas significativement $[(\Delta\varphi + \Delta\psi)/\Delta\varphi\psi]^{d+\eta}$).

Commençons par le cas où $\Delta\varphi\psi \geqslant 3\Delta\varphi$. Dans ce cas la distance entre $\operatorname{supp}\varphi$ et $\operatorname{supp}\psi$ est supérieure à $\Delta\varphi\psi/3$. Soit $x_0 \in \operatorname{supp}\psi$; en utilisant $\int \psi(x)\,dx = 0$ et la régularité en x de $K(x,y)$, on obtient

$$\begin{aligned}
|\langle \psi, Tb_1\varphi\rangle| &= \left| \iint \psi(x) K(x,y) b_1(y) \varphi(y)\,dy\,dx \right| \\
&= \left| \iint \psi(x)(K(x,y) - K(x_0,y)) b_1(y)\varphi(y)\,dy\,dx \right| \\
&\leqslant C \|\psi\|_{L^1} \|b_1\|_\infty \|\varphi\|_{L^1} \frac{(\Delta\psi)^\delta}{|\Delta\varphi\psi|^{\delta+d}} \\
&\leqslant C \frac{(\Delta\psi)^\delta (\Delta\varphi)^{d+\eta}}{(\Delta\varphi\psi)^{d+\delta}} \|\psi\|_1 \|\varphi\|_\eta ,
\end{aligned}$$

ce qui est mieux que (1.14) car $\delta > \eta$.

Nous supposons maintenant que $\Delta\varphi\psi \leqslant 3\Delta\varphi$. En utilisant à nouveau l'invariance par translation et dilation de (1.14), nous pouvons supposer que $\Delta\varphi\psi = 1$ et $\operatorname{supp}\varphi \cup \operatorname{supp}\psi \subset Q_0$, où Q_0 est le cube unité. On veut montrer que

$$|\langle \psi, Tb_1\varphi\rangle| \leqslant V(\Delta\psi)^\eta \|\psi\|_{L^1}\|\varphi\|_\eta.$$

Soit $\theta \in C_0^\eta(\mathbb{R}^d)$ une fonction égale à 1 sur $2Q_0$, de sorte que $\langle \psi, Tb_1\varphi\rangle = \langle \psi, Tb_1\varphi\theta\rangle$. Nous décomposons $\langle \psi, Tb_1\varphi\theta\rangle$ en $\langle \psi, Tb_1(\varphi - \varphi(x_0))\theta\rangle + \langle \psi, Tb_1\varphi(x_0)\theta\rangle$, où $x_0 \in \operatorname{supp}\psi$. Le deuxième terme est égal, en vertu de $Tb_1 = 0$, à $\varphi(x_0)\langle w, Tb_1(\theta - 1)\rangle$. Comme $\int \psi(x)\,dx = 0$ et $\operatorname{supp}\psi \cap \operatorname{supp}(\theta - 1) = \emptyset$, nous obtenons

$$\begin{aligned}|\langle \psi, Tb_1\varphi(x_0)\theta\rangle| &= |\varphi(x_0)|\left|\iint \psi(x)(K(x,y) - K(x_0,y))(\theta(y) - 1)\,dx\,dy\right|\\ &\leqslant C\|\varphi\|_\infty\|\psi\|_{L^1}\int_{|y|>1/2}\frac{(\Delta\psi)^\delta}{|y|^{d+\delta}}\,dy\\ &\leqslant C(\Delta\psi)^\delta\|\psi\|_{L^1}\|\varphi\|_\eta.\end{aligned}$$

Nous décomposons maintenant le premier terme en

$$\langle \psi, Tb_1(\varphi - \varphi(x_0))\tilde\theta\rangle + \langle \psi, Tb_1(\varphi - \varphi(x_0))(\theta - \tilde\theta)\rangle,$$

où $\tilde\theta$ est une fonction de $C_0^\eta(\mathbb{R}^d)$ telle que $\tilde\theta(u) = 1$ lorsque la distance de u à $\operatorname{supp}\psi$ est inférieure à $\Delta\psi$, telle que le diamètre de $\operatorname{supp}\tilde\theta$ soit inférieur à $4\Delta\psi$, et que $\|\tilde\theta\|_\eta \leqslant C(\Delta\psi)^{-\eta}$. Le teme $\langle \psi, Tb_1(\varphi - \varphi(x_0))(\theta - \tilde\theta)\rangle$ est alors estimé en utilisant $\int \psi(x)\,dx = 0$ et $\operatorname{supp}\psi \cap \operatorname{supp}(\theta - \tilde\theta) = \phi$, exactement comme on a estimé $\langle \psi, Tb_1\varphi(x_0)\theta\rangle$, ce qui fournit le majorant

$$C\|\psi\|_1\int_{|y-x_0|\geq\Delta\psi}\frac{(\Delta\psi)_\delta}{|x - x_0|^{d+\delta}}|y - x_0|^\eta\|\varphi\|_\eta\,dy,$$

ou encore $C(\Delta\psi)^\eta\|\psi\|_{L^1}\|\varphi\|_\eta$, car $\delta > \eta$.

Il reste à etimer $\langle \psi, Tb_1(\varphi - \varphi(x_0))\tilde\theta\rangle$.

Nous pouvons remplacer θ par $\tilde\theta^2$, qui vérifie les mêmes conditions. De plus, grâce au Lemme 1.1, nous pouvons remplacer $\langle \psi, Tb_1(\varphi - \varphi(x_0))\tilde\theta^2\rangle$ par $\langle \psi(\varphi - \varphi(x_0))\theta, Tb_1\tilde\theta\rangle$ modulo un terme d'erreur dominé par

$$\|(\varphi - \varphi(x_0))\tilde\theta\|_\eta\iint|\psi(x)|\frac{1}{|x - y|^{d+\eta}}|\tilde\theta(y)|\,|b_1(y)|\,dx\,dy,$$

donc par $\|(\varphi - \varphi(x_0))\tilde\theta\|_\eta\|\psi\|_{L^1}(\Delta\psi)$, soit encore, puisque

$$\|(\varphi - \varphi(x_0))\tilde\theta\|_\eta \leqslant \|\varphi\|_\eta\|\tilde\theta\|_\infty + \|(\varphi - \varphi(x_0))\chi_{\operatorname{supp}\tilde\theta}\|_\infty\|\tilde\theta\|_\eta \leqslant C\|\varphi\|_\eta,$$

par $C\|\varphi\|_\eta\|\psi\|_{L^1}(\Delta\psi)^\eta$.

Pour estimer $\langle \psi(\varphi - \varphi(x_0))\tilde{\theta}, Tb_1\tilde{\theta}\rangle$, nous choisissons une fonction h, avec un support de la même taille que celui de ψ, telle que $\int h(x)\,dx = \int \psi(x)(\varphi(x) - \varphi(x_0))\tilde{\theta}(x)\,dx$ et telle que $\|h/b_2\|_\eta \leqslant C(\Delta\psi)^{-d}\|\psi\|_{L^1}\|\varphi\|_\eta$. If suffit maintenant d'estimer séparément $\langle h, Tb_1\tilde{\theta}\rangle$ et $\langle \psi(\varphi - \varphi(x_0))\tilde{\theta} - h, Tb_1\tilde{\theta}\rangle$. Pour le premier terme on obtient, en appliquant (1.8) à $M_{b_2}TM_{b_1}$ sur un cube de diamètre $\leqslant C\Delta\psi$, une majoration par

$$C(\Delta\psi)^{d+2\eta}\left\|\frac{h}{b_2}\right\|_\eta \|\theta\|_\eta \leqslant C(\Delta\psi)^\eta \|\psi\|_{L^1}\|\varphi\|_\eta.$$

Pour le deuxième terme on utilise $\int \{\psi(\varphi - \varphi(x_0))\tilde{\theta} - h\}\,dx = 0$ et $Tb_1 = 0$, et on obtient

$$\begin{aligned}|\langle \psi(\varphi - \varphi(x_0))\tilde{\theta} - h, Tb_1\tilde{\theta}\rangle| &= |\langle \psi(\varphi - \varphi(x_0))\tilde{\theta} - h, Tb_1(\tilde{\theta} - 1)\rangle| = \\ = \left|\iint [\psi(x)(\varphi(x) - \varphi(x_0))\tilde{\theta}(x) - h(x)][K(x,y) - K(x_0,y)]b_1(y) - 1)\,dx\,dy\right| &\leqslant \\ \leqslant C(\Delta\psi)^\eta \|\psi\|_{L^1}\|\varphi\|_\eta.\end{aligned}$$

Le Lemme 1.3 est donc complètement démontré. Il montre en particulier que si T vérifie les hypothèses du Lemme 1.3 et si $\varphi \in C_0^\eta(\mathbb{R}^d)$, alors $Tb_1\varphi$ définit pour chaque cube Q une forme linéaire continue sur l'espace des fonctions $\psi \in L^1(Q)$ d'intégrale nulle. Autrement dit, $Tb_1\varphi$ est, localement, une fonction bornée définie à une constante additive près. Si $x \neq y$, on peut choisir un cube Q de diamètre $\leqslant C|x-y|$, et (1.14) entraîne que $|\langle \psi, Tb_1\varphi\rangle| \leqslant C|x-y|^\eta \|\varphi\|_\eta \|\psi\|_{L^1}$ pour toute ψ d'intégrale nulle supportée dans Q. Autrement dit,

$$|Tb_1\varphi(x) - Tb_1\varphi(y)| \leqslant C\|\varphi\|_\eta |x-y|^\eta.$$

Nous noterons, pour $0 < \eta < 1$, $\lambda^\eta(\mathbb{R}^d)$ le complété de $C_0^\eta(\mathbb{R}^d)$ pour la norme $\|\ \|_\eta$.

Nous venons de voir que, si $\varphi \in C_0^\eta(\mathbb{R}^d)$, $\|Tb_1\varphi\|_\eta \leqslant C\|\varphi\|_\eta$; de plus $Tb_1\varphi(x)$ peut être défini à l'aide du noyau pour x assez grand, et

$$\lim_{|x|+|y|\to+\infty} \frac{|Tb_1\varphi(x) - Tb_1\varphi(y)|}{|x-y|^\eta} = 0,$$

de sorte que $Tb_1\varphi \in \lambda^\eta(\mathbb{R}^d)$. On en déduit le résultat suivant.

Théorème 1. [L2]: *Soient* $0 < \eta < \delta \leqslant 1$, *et soit* $T\colon b_1C_0^\eta(\mathbb{R}^d) \to \{b_2C_0^\eta(\mathbb{R}^d)\}'$ *un* SIO *tel que* $M_{b_2}TM_{b_1}$ *soit faiblement borné, et associé à un noyau* δ-*standard. Si* $Tb_1 = 0$, *alors* TM_{b_1} *admet une extension continue sur* $\lambda^\eta(\mathbb{R}^d)$.

2. Opérateurs de Calderón-Zygmund et critères de continuité-L^2

Un opérateur de Calderón-Zygmund est un opérateur d'intégrale singulière qui s'étend en un opérateur borné sur L^2. Les méthodes de variable réelle permettent alors de montrer qu'un tel opérateur est continu sur L^p pour tout $p \in]1, +\infty[$. De plus, on peut définir son action sur L^∞ para le procédé décrit au paragraphe 1 pour définir l'action des SIO sur $b_1 C_b^\infty(\mathbb{R}^d)$. Nous le ferons en détail lors de la démonstrtion du Lemme 2.7.

Un problème important est donc de trouver des critères généraux de continuité-L^2 pour les opérateurs d'intégrale singulière. Lorsque ceux-ci sont des opérateurs de convolution, la transformation de Fourier permet de donner une condition nécessaire et suffisante très simple portant sur le noyau. Le primerier critère s'appliquant à des opérateurs qui ne soient pas de convolution fut le suivant [DJ].

Rappelons que si $T: b_1 C_0^\eta(\mathbb{R}^d) \to [b_2 C_0^\eta(\mathbb{R}^d)]'$ est un SIO, alors ${}^tT$ défini par $\langle g, {}^tTf \rangle = \langle f, Tg \rangle$ est également un SIO, défini de $b_2 C_0^\eta(\mathbb{R}^d)$ dans $[b_1 C_0^\eta(\mathbb{R}^d)]'$.

Théorème 1. *Un* SIO $T: C_0^\eta(\mathbb{R}^d) \to [C_0^\eta(\mathbb{R}^d)]'$ *est un opérateur de Calderón-Zygmund si et seulement si T est faiblement borné,* $T1$ *et* ${}^tT1$ *sont dans* BMO.

Ce théorème a été étendu par R. Coifman dans le cadre des espaces de nature homogène dans un travail non publié. Sans entrer dans le détail signalons le cas particulier suivant.

Si $b: \mathbb{R}^d \to \mathbb{R}_+$ est bornée ainsi que son inverse, alors $(\mathbb{R}^d, b\,dx)$ muni de la distance euclidienne ordinaire est un espace de nature homogène; de plus les noyaux d'opérateurs d'intégrale singulière sur cet espace sont exactement les mêmes que sur $(\mathbb{R}^d, dx)$. Enfin, le Théorème-$T1$ sur $(\mathbb{R}^d, b\,dx)$ équivaut à l'énoncé suivant:

Théorème 2. *Soit* $b: \mathbb{R}^d \to \mathbb{R}_+$ *une fonction bornée ainsi que son inverse. Un* SIO $T: bC_0^\eta(\mathbb{R}^d) \to [bC_0^\eta(\mathbb{R}^d)]'$ *est un opérateur de Calderón-Zygmund si et seulement si $M_b T M_b$ est faiblement borné, Tb et ${}^tTb$ sont dans* BMO.

Cet énoncé est à rapprocher du suivant, qui est une reformulation d'un théorème récent de McIntosh et Meyer [MM]. Une fonction b bornée à valeurs complexes est accrétive s'il existe $\epsilon > 0$ tel que $\operatorname{Re} b \geqslant \epsilon$ p.p.

Théorème 3. *Soit* $b: \mathbb{R}^d \to \mathbb{C}$ *une fonction bornée accrétive. Un SIO* $T = bC_0^\eta(\mathbb{R}^d) \to [bC_0^\eta(\mathbb{R}^d)]'$ *est un opérateur de Calderón-Zygmund si $M_b T M_b$ est faiblement borné, Tb et ${}^tTb$ sont nuls.*

Si cette dernière hypothèse était remplacée par «Tb et ${}^tTb$ sont dans BMO», alors le Théorème 3 pourrait se dire: «Soit b une fonction accrétive. Le Théorème-$T1$ est vrai sur $(\mathbb{R}^d, b\,dx)$».

Cet ensemble de théorèmes suggère la question suivante. Pour quelles fonctions b bornées ainsi que leur inverse le Théorème-$T1$ est-il vrai sur $(\mathbb{R}^d, b\,dx)$? En particulier, le théorème de McIntosh et Meyer indique que ces fonctions n'ont pas besoin d'être à valeurs réelles.

La réponse à cette question sera l'objet de notre résultat principal, énoncé au paragraphe 5. Pour l'instant nous allons analyser les démonstrations du Théorème-$T1$ et du Théorème 2 en soulignant les arguments et idées qui nous seront utiles dans la suite.

La première étape dans la démonstration du Théorème-$T1$ est la réduction au cas où $T1 = {}^tT1 = 0$. Pour ce faire, il suffit pour toute fonction $\beta \in$ $\in$ BMO de construire un opérateur de Calderón-Zygmund U_β tel que $U_\beta 1 = \beta$ et ${}^tU_\beta 1 = 0$. En effet, soit T vérifiant les hypothèses du théorème, $\beta = T1$ et $\gamma = {}^tT1$ el soit $\tilde{T} = T - U_\beta - {}^tU_\gamma$. Alors $\tilde{T}1 = {}^t\tilde{T}1 = 0$ et $\tilde{T}$ est faiblement borné. De plus T est un opérateur de Calderón-Zygmund si et seulement si $\tilde{T}$ l'est.

La construction des opérateurs U_β est donnée dans [ ]. Elle repose sur l'existence d'une fonction $\psi \in [C_0^\infty(\mathbb{R}^d)]$ radiale, telle que

$$\int \psi\, d\xi = 0 \quad \text{et} \quad \int_0^{+\infty} |\psi(t\xi)|^2 \frac{dt}{t} = 1$$

pour tout $\xi \in \mathbb{R}^d \bigcup \{0\}$. On en déduit que, si Q_t est l'opérateur de convolution de symbole $\psi(t\theta)$, alors $I = \int_0^{+\infty} Q_t^2 \frac{dt}{t}$, l'intégrale convergeant au sens de la topologie forte des opérateurs sur L^2, et faiblement sur BMO. La continuité-L^2 de $\tilde{T}$, originalement traitée par le lemme de Cotlar-Knapp-Stein [KS] que nous rappellerons, peut également être traitée en utilisant cette même décomposition de l'identité [M]. Toutefois une telle décomposition fait défaut dans les espaces de nature homogène abstraits. On est alors amené à utiliser un lemme d'approximation dû à Coifman, reposant sur le lemme de Cotlar-Knapp-Stein que nous rappelons:

Lemme C.K.S. [KS]. *Soit $(T_j)_{j\in\mathbb{Z}}$ une suite d'opérateurs bornés sur un espace de Hilbert H. On suppose que pour tous j, $k \in \mathbb{Z}$,*

(2.1) $$\|T_j T_k^*\| \leqslant \omega(j-k),$$

et

(2.2) $$\|T_j^* T_k\| \leqslant \omega(j-k),$$

où $j \to \omega(j)$ est une suite telle que $\sum_j \omega(j)^{1/2} < +\infty$. Alors, pour tout sous-ensemble fini I de $\mathbb{Z}$,

$$\left\| \sum_{i \in I} T_i \right\| \leqslant \sum_{j \in \mathbb{Z}} \omega(j)^{1/2}. \tag{2.3}$$

Rappelons qu'une série $\sum_j u_j$ à valeurs dans un espace de Hilbert est sommable si et seulement si ses sommes finies sont uniformément bornées. En particulier, pour tout $x \in H$, la série $\sum_{j \in \mathbb{Z}} T_j x$ est sommable, ou, en d'autres termes, la série $\sum_{j \in \mathbb{Z}} T_j$ est fortement sommable et sa somme T vérifie $\| T \| \leqslant$ $\leqslant \sum_j \omega(j)^{1/2}$.

En situation concrète, les hypothèses de presque-orthogonalité (2.1) et (2.2) sont souvent une conséquence du lemme suivant.

Lemme 2.1. *Soient U et V deus opérateurs donnés par des noyaux* $u(x, y)$ et $v(x, y)$ *vérifiant, pour des constantes $C > 0$, $\alpha > 0$, $A > 0$, $j \in \mathbb{Z}$ et $k \in \mathbb{Z}$, $k < j$,*

$$u(x, y) = 0 \quad \textit{pour} \quad |x - y| \geqslant C2^{-j}, \tag{2.4}$$

$$\int |u(x, y)| \, dy \leqslant A \quad \textit{pour tout} \quad x, \tag{2.5}$$

$$\int u(x, y) \, dy = 0 \quad \textit{pour tout} \quad x, \tag{2.6}$$

$$v(x, y) = 0 \quad \textit{pour} \quad |x - y| \geqslant C2^{-k}, \tag{2.7}$$

$$|v(x', y) - v(x, y)| \leqslant C|x' - x|^{\alpha} 2^{k(d + \alpha)}. \tag{2.8}$$

Alors

$$\| UV \|_{p,p} \leqslant CA2^{(k-j)\alpha} \quad \textit{pour tout} \quad p \in [1, +\infty].$$

Pour démontrer le Lemme 5 il suffit de vérifier que le noyau $K(x, z) =$ $= \int u(x, y) v(y, z) \, dy$ de UV, qui est nul pour $|x - z| \geqslant C2^{-k}$ d'après (2.4) et (2.7), satisfait à $|K(x, z)| \leqslant CA2^{(k-j)\alpha} 2^{kd}$. Ainsi on aura, pour $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, $|UVf(x)| \leqslant CA2^{(k-j)\alpha} 2^{kd} \int_{|x-y| < C2^{-k}} |f(y)| \, dy$, ce qui entraînera la conclusion désirée.

Or, d'après (2.6) et (2.8),

$$\begin{aligned} |K(x, z)| &= \left| \int u(x, y) v(y, z) \, dy \right| = \\ &= \left| \int u(x, y)(v(y, z) - v(x, z)) \, dy \right| \leqslant \\ &\leqslant \int |u(x, y)| \, |y - x|^{\alpha} 2^{k(d + \alpha)} \, dy, \end{aligned}$$

ce qui d'après (2.4) et (2.5) vaut moins que $CA2^{kd} 2^{(k-j)\alpha}$.

Le Lemme 2.1 est démontré.

Nous revenons au lemme d'approximation de Coifman, que nous énonçons dans le cas de l'espace de type homogène $(\mathbb{R}^d, b(x)\,dx)$, muni de la distance euclidienne classique.

Lemme 2.2. *Soit b une fonction positive bornée ainsi que son inverse. Il existe une suite $(S_k)_{k\in\mathbb{Z}}$ d'opérateurs donnés par des noyaux s_k définis sur $\mathbb{R}^d \times \mathbb{R}^d$ et vérifiant, pour un $C > 0$ et un $\alpha > 0$,*

$$|s_k(x,y)| \leqslant C2^{kd} \quad \textit{pour tous} \quad x, y \in \mathbb{R}^d \tag{2.9}$$

$$s_k(x,y) = 0 \quad \textit{pour} \quad |x-y| \geqslant C2^{-k} \tag{2.10}$$

$$s_k(x,y) = s_k(y,x) \quad \textit{pour tous} \quad x, y \in \mathbb{R}^d, \tag{2.11}$$

$$|s_k(x',y) - s_k(x,y)| \leqslant C|x'-x|^{\alpha}2^{k(d+\alpha)} \quad \textit{pour tous} \quad x, x', y \in \mathbb{R}^d, \tag{2.12}$$

$$\int s_k(x,y)b(y)\,dy = 1 \quad \textit{pour tout} \quad x. \tag{2.13}$$

Alors, si $\Delta_k = S_k - S_{k-1}$ et, pour tout N, $\Delta_k^N = \sum_{|j-k|\leq N}\Delta_j$, la série $V_N = \sum_{k\in\mathbb{Z}}\Delta_k M_b \Delta_k^N$ est fortement convergente dans L^2 et définit un opérateur de Calderón-Sygmund dont le noyau est δ-standard pour tout $\delta < \alpha$. Enfin, $\lim_{N\to\infty} V_N = M_b^{-1}$ en norme $CZ\delta$.

Rappelons que la norme $CZ\delta$ est définie sur les opérateurs de Calderón-Zygmund T don le noyau K est δ-standard par $\|T\|_{CZ\delta} = \|T\|_{2,2} + |K|_\delta$.

Commençons par vérifier l'existence d'une telle suite S_k. Soit M_k l'opérateur défini par $M_k f(x) = c2^{kd}\int_{|x-y|<2^{-k}} f(x)\,dx$ pour $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, et où c est choisi pour que $M_k 1 = 1$. Remarquons que $M_k b$ est lipschitzienne et a une dérivée majorée par $C2^k$, et que $\inf b \leqslant M_k b(x) \leqslant \sup b$ pour tout x. On choisit $S_k = M_k\{M_k b\}^{-1}M_k$, où $\{M_k b\}^{-1}$ est l'opérateur de multiplication poctuelle par $[M_k b(x)]^{-1}$. La propriété (2.12) est satisfaite parce que $M_k b$ est lipschitzienne, et les propriétés (2.9), (2.10), (2.11) et (2.13) évidentes.

Pour prouver que la série définissant V_N converge, nous allons vérifier que les opérateurs $(M_b^{1/2}\Delta_j M_b^{1/2})_{j\in\mathbb{Z}}$ satisfont aux hypothèses du Lemme C.K.S. Nous noterons $t_j(x,y)$ le noyau de Δ_j. Comme b est réelle, $(M_b^{1/2}\Delta_j M_b^{1/2})^* = M_b^{1/2}\Delta_j^* M_b^{1/2}$ a les mêmes propriétés que ${M_b}^{1/2}\Delta_j {M_b}^{1/2}$, il suffit en fait de montrer (2.1), et l'on peut même supposer $j \geqslant k$. Le noyau de $\Delta_j M_b \Delta_k^*$ étant $\int t_j(x,y)b(y)\bar{t}_k(z,y)\,dy$, sa norme sur $L^2(\mathbb{R}^n)$ est estimée en appliquant le Lemme 2.1 avec $u(x,y) = t_j(x,y)b(y)$ et $v(x,y) = \overline{t_k(y,x)}$. L'hypothèse (2.6) découle de (2.13), et les autres hypothèses du Lemme 2.1 sont trivialement vérifiées. On obtient donc $\|\Delta_j M_b \Delta_k^*\|_{2,2} \leqslant C2^{(k-j)\alpha}$.

Soit, pour tout $n \in \mathbb{Z}$, $W_n = \sum_k \Delta_k M_b \Delta_{k-n}$, de sorte que $V_N = \sum_{|n|\leq N} W_n$. En vertu du calcul précédent, les opérateurs $M_b^{1/2}\Delta_k M_b \Delta_{k+n} M_b^{1/2}$ vérifient (2.1) et (2.2) avec la suite ω_n définie par $\omega_n(j) = C\min(2^{-2|n|\alpha}, 2^{-|k-j|\alpha})$. Par conséquent, la série $\sum_k \Delta_k M_b \Delta_{k+n}$ est fortement convergente, toutes ses sommes partielles ont une norme inférieure à $C|n|2^{-|n|\alpha}$, et en particulier $\| W_n \| \leqslant C|n|2^{-|n|}$. La série définissant V_N est donc également fortement sommable, et de plus la suite V_N converge en norme d'opérateur quand $N \to +\infty$.

Pour identifier $\lim_{N\to\infty} V_N$, montrons d'abord que pour tout $f \in L^2$, la série $\sum\sum_{k,j\in\mathbb{Z}} \Delta_k M_b \Delta_j f$ est sommable. Or une série à valeurs dans un espace de Hilbert est sommable si et seulement si ses sommes finies sont uniformément bornées. Soient $f \in L^2$ et I un ensemble fini de $\mathbb{Z} \times \mathbb{Z}$; alors

$$\Big\| \sum\sum_{(k,j)\in I} \Delta_k M_b \Delta_j f \Big\| \leqslant \sum_n \Big\| \sum_{(k,j)\in I \text{ et } j-k=n} \Delta_k M_b \Delta_j \Big\| \leqslant$$
$$\leqslant C \sum_n |n| 2^{-|n|\alpha} \leqslant C.$$

Donc $\sum\sum_{k,j\in\mathbb{Z}} \Delta_k M_b \Delta_j f$ peut être calculé en regroupant des termes, et $\sum\sum_{k,j} \Delta_k M_b \Delta_j = \sum_k \big(\sum_j \Delta_k M_b \Delta_j\big)$.

Or $\lim_{j\to+\infty} S_j = \lim_{j\to+\infty} M_j\{M_j b\}^{-1} M_j = M_b^{-1}$ et $\lim_{j\to-\infty} S_j = 0$ pour la topologie forte des opérateurs (il suffit de vérifier que $\lim_{j\to+\infty} S_j bf = f$ et $\lim_{j\to-\infty} S_j bf = 0$ pour $f \in C_0^\alpha(\mathbb{R}^d)$, ce qui est une conséquence de (2.9), (2.10) et (2.13)) on en déduit que

$$\sum_j \Delta_k M_b \Delta_j = \Delta_k M_b M_b^{-1} = \Delta_k \quad \text{et} \quad \sum\sum_{k,j} \Delta_k M_b \Delta_j = \sum_k \Delta_k = M_b^{-1},$$

de sorte que $\lim_{N\to\infty} V_N = M_b^{-1}$.

Nous montrons maintenant que le noyau K_n de W_n vérifie, pour tout $\delta < \alpha$,

$$|K_n|_\delta \leqslant C 2^{|n|(\delta-\alpha)}. \tag{2.14}$$

Cela entraînera que $\lim_{N\to\infty} V_N$ existe en norme $CZ\delta$ et cette limite sera nécessairement M_b^{-1}. Le Lemme 2.2 sera alors complètement démontré.

Nous pouvons supposer que n est positif. Notons que le noyau $K_{n,k}$ de $\Delta_k M_b \Delta_{k-n}$ vérifie, d'après la démonstration du Lemme 2.1,

$$|K_{n,k}(x,z)| \leqslant C 2^{-n\alpha} 2^{(k-n)d} \tag{2.15}$$

et

$$K_{n,k}(x,z) = 0 \quad \text{si} \quad |x-z| \leqslant C 2^{-(k-n)}.$$

On en déduit que

$$|K_n(x,z)| \leqslant \sum_k |K_{n,k}(x,z)| \leqslant C 2^{-n\alpha} \frac{1}{|x-z|^d}. \tag{2.16}$$

Nous allons maintenant vérifier que si $|x' - x| \leqslant \frac{1}{2}|x - z|$,

$$|K_{n,k}(x', z) - K_{n,k}(x, z)| \leqslant C|x' - x|^{\alpha} 2^{(k-n)(d+\alpha)} \tag{2.17}$$

et

$$|K_{n,k}(z, x') - K_{n,k}(z, x)| \leqslant C|x' - x|^{\alpha} 2^{(k-n)(d+\alpha)}, \tag{2.18}$$

en commençant par (2.17).

Lorsque $|x' - x| \geqslant 2^k$, (2.17) est une conséquence de (2.15). Lorsque $|x' - x| \leqslant 2^k$, on utilise le fait que, d'après (2.12), $\int |t_k(x', y) - t_k(x, y)|\, dy \leqslant \leqslant C|x' - x|^{\alpha} 2^{k(d+\alpha)}$. On en déduit que

$$\begin{aligned} |K_{n,k}(x', z) - K_{n,k}(x, z)| &= \left| \int (t_k(x', y) - t_k(x, y)) b(y) t_{k-n}(y, z)\, dy \right| = \\ &= \left| \int (t_k(x', y) - t_k(x, y)) b(y) (t_{k-n}(y, z) - \right. \\ &\quad \left. - t_{k-n}(x, z))\, dy \right| \leqslant \\ &\leqslant C|x' - x|^{\alpha} 2^{k(d+\alpha)} 2^{-k\alpha} 2^{(k-n)(d+\alpha)} 2^{-kd} = \\ &= C|x' - x|^{\alpha} 2^{(k-n)(d+\alpha)}. \end{aligned}$$

Pour montrer (2.18), on remarque à nouveau que si $|x' - x| \geqslant 2^k$, (2.18) découle de (2.15). Lorsque $|x' - x| \leqslant 2^k$, on a

$$\begin{aligned} |K_{n,k}(z, x') - K_{n,k}(z, x)| &= \left| \int t_k(z, y) b(y) (t_{k-n}(y, x') - t_{k-n}(y, x))\, dy \right| \leqslant \\ &\leqslant C|x' - x|^{\alpha} 2^{(k-n)(d+\alpha)}. \end{aligned}$$

On déduit des inégalités (2.17) et (2.18) que, pour $|x' - x| < |x - z|/2$,

$$|K_n(x', x) - K_n(x, z)| \leqslant \frac{C|x' - x|^{\alpha}}{|x - z|^{d+\alpha}} \tag{2.19}$$

et

$$|K_n(z, x') - K_n(z, x)| \leqslant \frac{C|x' - x|^{\alpha}}{|x - z|^{d+\alpha}}. \tag{2.10}$$

On déduit (2.14) de (2.16), (2.19) et (2.20). Le Lemme 2.2 est démontré.

Revenons aux problèmes que nous avons énoncés dans le cas $b = 1$: étant donnée une fonction b réelle positive bornée ainsi que son inverse, nous voulons montrer que si $\tilde{T}: bC_0^{\eta}(\mathbb{R}^d) \to \{bC_0^{\eta}(\mathbb{R}^d)\}'$ est un SIO tel que $M_b T M_b$ soit faiblement borné et tel que $Tb = {}^tTb = 0$, alors T est borné sur $L^2(\mathbb{R}^d)$. Nous voulons également, pour toute fonction $\beta \in \mathrm{BMO}$, construire un opérateur de Calderón-Zygmund $U_{\beta,b}$ tel que $U_{\beta,b}b = \beta$ et ${}^tU_{\beta,b}b = 0$.

Le premier problème sera résolu au moyen des lemmes suivants, où les notations sont celles du Lemme 2.2.

Lemme 2.3. *Pour tout k, l'opérateur Δ_k est continu de $bC_0^\alpha(\mathbb{R}^d)$ dans $C_0^\alpha(\mathbb{R}^d)$.*

Ce lemme est trivial. Il permet, pour tout opérateur T satisfaisant aux hypothèses précédentes et tout couple $(j_1, j_2) \in \mathbb{Z} \times \mathbb{Z}$, de définir un opérateur $\tilde{T}_{j_1,j_2}$: $bC_0^\alpha(\mathbb{R}^d) \to \{bC_0^\alpha(\mathbb{R}^d)\}'$ par la formule $\langle g, \tilde{T}_{j_1,j_2} f\rangle = \langle M_b\Delta_{j_2}g, \tilde{T}(M_b\Delta_{j_1}f)\rangle$. On a alors le lemme suivant.

Lemme 2.4. *L'opérateur $\tilde{T}_{j_1,j_2}$ est continu sur L^2 et vérifie $\|\tilde{T}_{j_1,j_2}\|_{2,2} \leqslant$ $\leqslant C_\nu 3^{-\nu|j_1-j_2|}$ pour tout $\nu < \min(\alpha, \delta)$.*

Par un argument déjà employé dans la démonstration du Lemme 1.1, T_{j_1,j_2} est donné par intégration contre le noyau $K_{j_1,j_2}(x, y) = \langle bt_{j_1}^x, T(bt_{j_2}^y)\rangle$, où l'on a noté $t_j^x(z) = t_j(x, z) = s_j(x, z) - s_{j-1}(x, z)$. En raison de la symétrie des hypothèses, nous pouvons supposer $j_1 \geqslant j_2$. Une application du Lemme 1.3 avec $\eta = \nu$, $\psi = bt_{j_1}^x$, et $\varphi = t_{j_2}^y$ donne

$$\begin{aligned}|K_{j_1,j_2}(x, y)| &\leqslant C\frac{2^{-j_1\nu}2^{-j_2(d+\nu)}}{|x-y|^{d+\nu} + 2^{-j_2(d+\nu)}}\,2^{j_2(d+\nu)}\\ &= C\frac{2^{-j_1\nu}}{|x-y|^{d+\nu}2^{-j_2(d+\nu)}}.\end{aligned}$$

Il s'ensuit que $\tilde{T}_{j_1,j_2}$ est borné sur L^∞, L^1, et donc L^2 avec une norme majorée par $C2^{-|j_1-j_2|\nu}$. Le Lemme 2.4 est démontré.

Fixons un entier N tel que $\|V_NM_b - I\|_{2,2} \leqslant \frac{1}{2}$, ce qui est possible d'après le Lemme 2.2. Si J_1 et J_2 sont deux sous-ensembles finis de $\mathbb{Z}$, nous définissons un opérateur $\tilde{T}_{N,J_1,J_2}$: $bC_0^\alpha(\mathbb{R}^d) \to \{bC_0^\alpha(\mathbb{R}^d)\}'$ par

$$\langle g, \tilde{T}_{N,J_1,J_2}f\rangle = \sum_{j_1\in J_1}\sum_{j_2\in J_2}\langle M_b\Delta_{j_2}M_b\Delta_{j_2}^N g, \tilde{T}M_b\Delta_{j_1}M_b\Delta_{j_1}^N f\rangle.$$

Lemme 2.5. *L'opérateur $\tilde{T}_{N,J_1,J_2}$ est borné sur $L^2(\mathbb{R}^d)$, et $\|\tilde{T}_{N,J_1,J_2}\|_{2,2} \leqslant$ $\leqslant CN^2$, oú la constante C ne dépend ni de J_1, ni de J_2.*

Notons que la continuité sur L^2 de l'opérateur $\sum_{j\in\mathbb{Z}}\Delta_j^*M_b\Delta_j$, qui se démontre comme celle de W_0, entraîne l'estimation quadratique suivante: pour tout $f \in L^2(\mathbb{R}^d)$, $\sum_{j\in\mathbb{Z}}\|\Delta_jf\|_2^2 \leqslant C\|f\|_2^2$. Il en résulte que $\sum_{j\in\mathbb{Z}}\|\Delta_j^Nf\|_2^2 \leqslant$ $\leqslant CN^2\|f\|_2^2$. On en déduit que, pour $f, g \in bC_0^\alpha(\mathbb{R}^d)$,

$$\langle g, \tilde{T}_{N,J_1,J_2}f\rangle = \left|\sum_{j_1\in J_1}\sum_{j_2\in J_2}\langle M_b\Delta_{j_2}^N g, \tilde{T}_{j_1,j_2}(M_b\Delta_{j_1}^N f)\rangle\right| \leqslant$$
$$\leqslant \sum_{j_1}\sum_{j_2}\|\Delta_{j_2}^N g\|_2\,\|\tilde{T}_{j_1,j_2}\|_{2,2}\,\|\Delta_{j_1}^N f\|_2,$$

ce qui, d'après le Lemme 2.4, est dominé par

$$\sum_{j_1}\sum_{j_2}\|\Delta_{j_2}^N g\|_2 2^{-\nu|j_1-j_2|}\|\Delta_{j_1}^N f\|_2 \leqslant CN^2\|f\|_2\|g\|_2$$

car la matrice de terme général $2^{-\nu|j_1-j_2|}$ définit un opérateur borné sur l^2. Le Lemme 2.5 est démontré.

Nous voulons en conclure que $\tilde{T}$ a une extension bornée sur $L^2(\mathbb{R}^d)$. Nous savons déjà que $\tilde{T}M_b$ a une extension continue sur $\lambda^\nu(\mathbb{R}^d)$. Nous allons montrer l'existence de deux espaces $\mathcal{E}$ et $\mathfrak{D}$ possédant les propriétés suivantes:

(2.21) $\mathcal{E}$ est un sous-espace dense de $\lambda^\nu\cap L^2$,
(2.22) $\mathfrak{D}\subset(\lambda^\nu)'\cap L^2$ et $\mathfrak{D}$ est dense dans L^2,
(2.23) pout tout $(f,g)\in\mathcal{E}\times\mathfrak{D}$, $|\langle g,\tilde{T}M_bf\rangle|\leqslant C\|f\|_2\|g\|_2$.

Ces trois hypothèses entraînent que $\tilde{T}$ a une extension bornée sur L^2. En effet, la restriction de $\tilde{T}M_b$ à $\mathcal{E}$ admet une extension continue sur L^2 (notons-la $T'M_b$). Comme $\mathcal{E}$ est dense dans $\lambda^\nu\cap L^2$, $T'M_b$ coincide avec $\tilde{T}M_b$ sur $\lambda^\nu\cap L^2$, et en particulier sur $C_0^\nu(\mathbb{R}^d)$. Donc T' est une extension bornée de $\tilde{T}$.

Prenons $\mathcal{E}=V_NM_bC_0^\alpha(\mathbb{R}^d)$ et $\mathfrak{D}=M_bV_NL_{00}^\infty(\mathbb{R}^d)$, où $L_{00}^\infty(\mathbb{R}^d)$ est le sous-espace de $L^\infty(\mathbb{R}^d)$ constitué des fonctions à support compact et d'intégrale nulle, et où N est assez grand pour que V_NM_b soit un isomorphisme de L^2 et de λ^ν (c'est possible d'après le Lemme 2.2 et le Théorème 1). On en déduit (2.21) car $C_0^\alpha(\mathbb{R}^d)$ est un sous-espace dense de $\lambda^\nu\cap L^2$, et (2.22) car $L_{00}^\infty\subset(\lambda^\nu)'\cap L^2$ et L_{00}^∞ est dense dans L^2. La démonstration de (2.23) utilisera le lemme suivant.

Lemme 2.6. *Si $f\in C_0^\alpha(\mathbb{R}^d)$, la série $\sum_j\Delta_jM_bf$ converge normalement dans λ^ν. Si $g\in L_{00}^\infty(\mathbb{R}^d)$, la série $\sum M_b\Delta_jg$ converge normalement dans $(\lambda^\nu)'$.*

Pour démontrer la première assertion, remarquons que pour $f\in C_0^\alpha(\mathbb{R}^d)$ et $j\in\mathbb{Z}$, $\|\Delta_jM_bf\|_\alpha\leqslant C_f2^{j(d+\alpha)}$ d'après (2.12), et Δ_jM_bf est à support compact. Donc

$$\|\Delta_jM_bf\|_\nu\leqslant C_f\|\Delta_jM_bf\|_\alpha\leqslant C_f2^{j(d+\alpha)},$$

ce qui prouve que la série $\sum_{j=-\infty}^0\Delta_jM_bf$ converge normalement. Pour estimer $\|\Delta_jM_bf\|_\nu$ lorsque $j\geqslant0$, on utilise la régularité de f et $\Delta_jb=0$. On a

$$\begin{aligned}\|\Delta_j M_b f\|_\infty &= \sup_x \left| \int t_j(x,y) f(y) b(y)\, dy \right| = \\ &= \sup_x \left| \int t_j(x,y)(f(y) - f(x)) b(y)\, dy \right| \leqslant \\ &\leqslant C_f 2^{-\alpha j}.\end{aligned}$$

Si $|x' - x| \geqslant 2^{-j}$,

$$\frac{|\Delta_j M_b f(x') - \Delta_j M_b f(x)|}{|x' - x|^\nu} \leqslant C 2^{(\nu - \alpha) j}.$$

Il reste donc à estimer $|\Delta_j M_b f(x') - \Delta_j M_b f(x)|/|x' - x|^\nu$ lorsque $|x' - x| \leqslant \leqslant 2^{-j}$. Dans ce cas,

$$\begin{aligned}|\Delta_j M_b f(x') - \Delta_j M_b f(x)| &\leqslant \int |t_j(x', y) - t_j(x, y)|\, |b(y)|\, |f(y) - f(x)|\, dy \leqslant \\ &\leqslant C|x' - x|^\alpha \int_{|x-y| \leq C2^{-j}} 2^{j(d+\alpha)} |y - x|^\alpha\, dy \leqslant \\ &\leqslant C|x' - x|^\alpha \leqslant C 2^{(\nu - \alpha) j} |x' - x|^\nu.\end{aligned}$$

Par conséquent, si $f \in C_0^\infty(\mathbb{R}^d)$, $\sum_j \|\Delta_j M_b f\|_\nu < +\infty$.

Soit maintenant $g \in L_{00}^\infty(\mathbb{R}^d)$, et soit $h \in C_0^\nu(\mathbb{R}^d)$ tel que $\|h\|_\nu \leqslant 1$.

Si $j \geqslant 0$,

$$|\langle M_b \Delta_j g, h \rangle| \leqslant C_g \|\Delta_j M_b h\|_\infty \leqslant C_g \sup_{|x'-x| \leq C2^{-j}} |h(x') - h(x)| \leqslant C_g 2^{-j\nu}.$$

Si $j \leqslant 0$ et $x_0 \in \operatorname{supp} g$,

$$\begin{aligned}|\langle M_b \Delta_j g, h \rangle| &= \left| \iint g(x)(t_j(x,y) - t_j(x_0, y)) b(y) h(y)\, dy\, dx \right| = \\ &= \left| \iint g(x)(t_j(x,y) - t_j(x_0, y)) b(y)(h(y) - h(x_0))\, dy\, dx \right| \leqslant \\ &\leqslant C\|g\|_\infty \iint_{\substack{|y - x_0| \leq C2^{-j} + C_g \\ |x - x_0| \leq C_g}} |x - x_0|^\alpha 2^{j(d+\alpha)} |y - x_0|^\nu\, dy\, dx \leqslant \\ &\leqslant C_g 2^{j(\alpha - \nu)}.\end{aligned}$$

Par conséquent, $\sum_j \|M_b \Delta_j g\|_{(\lambda^\nu)'} < +\infty$, et le Lemme 2.6 est démontré.

Comme pour tout $j \in \mathbb{Z}$, $\Delta_j M_b$ est continu sur λ^ν (c'est un cas particulier trivial du Théorème 1), on déduit du Lemme 2.6 que pour $f \in C_0^\alpha(\mathbb{R}^d)$, la série $\sum_j \Delta_j M_b \Delta_j^N M_b f$ converge normalement dans λ^ν; de même, si $g \in L_{00}^\infty(\mathbb{R}^d)$, la série $\sum_j M_b \Delta_j M_b \Delta_j^N g$ converge normalement dans $(\lambda^\nu)'$. De plus, les sommes valent $V_N M_b f$ et $M_b V_N g$ respectivement, car les séries convergent aussi dans L^2. On en déduit que la série double $\sum_{j_1} \sum_{j_2} < M_b \Delta_{j_2} M_b \Delta_{j_2}^N g, \tilde{T} M_b \Delta_{j_1} M_b \Delta_{j_1}^N M_b f \rangle$ converge absolument et vaut $\langle M_b V_N g, \tilde{T} M_b V_N M_b f \rangle$.

Le Lemme 2.5 entraîne alors que

$$|\langle M_b V_N g, \tilde{T} M_b V_N M_b f\rangle| \leqslant CN^2 \|f\|_2 \|g\|_2 \leqslant CN^2 \|V_N M_b f\|_2 \|M_b V_N g\|_2,$$

ce qui est précisément (2.23). Nous avons donc montré que $\tilde{T}$ a une extension continue sur $L^2(\mathbb{R}^d)$.

Il nous faut maintenant contruire, pour tout $\beta \in \mathrm{BMO}$ et toute b positive bornée ainsi que son inverse, un operáteur de Calderón-Zygmund $U_{\beta,b}$ tel que $U_{\beta,b} b = \beta$ et ${}^t U_{\beta,b} b = 0$. Pour cela, nous aurons besoin d'un résultat qui quoique classique et facile, ne semble pas être dans la littérature.

Lemme 2.7. *Soit T un opérateur borné sur $L^2(\mathbb{R}^d)$ associé à un noyau K vérifiant, pour un certain $\delta > 0$,*

$$|K(x', y) - K(x, y)| \leqslant C \frac{|x' - x|^\delta}{|x - y|^{d+\delta}} \tag{2.24}$$

pour tous x, x', $y \in \mathbb{R}^d$ tels que $|x - y| > 2|x' - x|$.

Alors T admet une extension continue de $L^\infty(\mathbb{R}^d)$ dans BMO$(\mathbb{R}^d)$. *Si de plus $T1 = 0$, alors T admet une extension continue de* BMO$(\mathbb{R}^d)$ *dans* BMO$(\mathbb{R}^d)$.

Rappelons la définition de l'extensión d'un tel opérateur à L^∞. Soient $f \in L^\infty$, et Q un cube de $\mathbb{R}^n$. On écrit $f = f_1 + f_2$, où $f_1 = f\chi_{2Q}$, et on définit Tf sur Q, à une constante additive près, par la formule

$$(Tf)_Q(x) = Tf_1(x) + \int (K(x, y) - K(x_0, y)) f_2(y)\, dy,$$

où x_0 est le centre du cube Q. Il est clair que si Q' et une cube contenant Q, et si x_0' est son centre, la restriction de $(Tf)_{Q'}$ à Q est égale à $Tf(Q)$ modulo la constante $\int_{\mathbb{R}^d \setminus 2Q'} (K(x_0', y) - K(x_0, y)) f(y)\, dy - \int_{2Q' \setminus 2Q} K(x_0, y) f(y)\, dy$. Donc cette définition de $(Tf)_Q$ est cohérente, et définit une fonction modulo les constantes sur tout $\mathbb{R}^d$. De plus,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |(Tf)_Q|\, dx \leqslant \left[\frac{1}{|Q|}\int_Q |Tf_1|^2\, dx\right]^{1/2} &+ \\ &+ \frac{1}{Q}\iint_{\substack{x \in Q \\ y \in \mathbb{R}^d \setminus 2Q}} |K(x, y) - K(x_0, y)|\, |f(y)|\, dy\, dx \leqslant \\ &\leqslant C\|f\|_\infty,\end{aligned}$$

d'après la continuité de T sur L^2 et l'hypothèse (2.24). Ceci montre que l'extension construite est continue de L^∞ dans BMO.

Si $T1 = 0$, l'action de T peut être étendue à l'espace des fonctions bornées modulo les constantes, et même à BMO. En effect, $T1 = 0$ permet de choisir un représentat de f tel que $\int f_1(x)\,dx = 0$, et les propriétés des fonctions de BMO entraînent alors [JN], [J2]

$$\left[\frac{1}{|Q|}\int_{2Q}|f(x)|^2\,dx\right]^{1/2} \leqslant C\|f\|_{\mathrm{BMO}} \tag{2.25}$$

et

$$\frac{1}{|Q|}\iint_{\substack{x\in Q\\ y\in\mathbb{R}^d\setminus 2Q}} \frac{|x-x_0|^\delta}{|x-y|^{n+\delta}}|f(y)|\,dy \leqslant C\|f\|_{\mathrm{BMO}}. \tag{2.26}$$

On en déduit

$$\frac{1}{|Q|}\int |(Tf)_Q|\,dx \leqslant C\|f\|_{\mathrm{BMO}},$$

à nouveau d'après la continuité de T sur L^2 et (2.24). Le Lemme 2.7 est démontré.

Ce lemme s'applique à l'opérateur $V_N M_b$. En effet, $V_N M_b$ est continu sur L^2, son noyau vérifie (2.24), et (2.13) entraîne que $V_N(b) = V_N M_b 1 = 0$. On déduit du Lemme 2.4 que $\lim_{N\to\infty} V_N M_b = I$ en norme d'opérateur sur BMO. On choisira N de sorte que $V_N M_b$ soit inversible aussi sur BMO, et l'on notera R son inverse sur BMO.

Nous pouvons maintenant donner la forme de l'opérateur $U_{\beta,b}$. Ce sera la limite faible des opérateurs $U_{j,\beta,b}$ définis par

$$U_{j,\beta,b} = \sum_{k=-j}^{j} \Delta_k M_b\{\Delta_k^N M_b R\beta\} S_k,$$

où $\{\Delta_k^N M_b R\beta\}$ désigne l'opérateur de multiplication ponctuelle par $\Delta_k^N M_b R\beta$. Remarquons que $\Delta_k^N b = 0$, et par conséquent $\|\Delta_k^N M_b R\beta\|_\infty \leqslant C\|\beta\|_{\mathrm{BMO}}$, de sorte que les $U_{j,\beta,b}$ résultera du lemme suivant.

Lemme 2.8. *Soient* $\gamma\in \mathrm{BMO}(\mathbb{R}^d)$ *et* $f\in L^2(\mathbb{R}^d)$. *Alors*

$$\sum_k \|(\Delta_k^N M_b\gamma)(S_k f)\|_2^2 \leqslant C\|\gamma\|_{\mathrm{BMO}}^2\|f\|_2^2.$$

Soit $C_0 > 0$ fixé. Rappelons que, pour qu'une suite $(\mu_k)_{k\in\mathbb{Z}}$ de mesures positives sur $\mathbb{R}^d$ soit telle que, pour tout $f\in L^2$,

$$\sum_k \|f_k\|_{L^2(\mathbb{R}^d,\,d\mu_k)}^2 \leqslant C_1\|f\|_2^2,$$

où $f_k(x) = 2^{kd}\int_{|x-y| \leq C_0 2^{-k}} f(y)\,dy$, et il faut et il suffit que pour tout $x \in \mathbb{R}^d$ et tout $k \in \mathbb{Z}$,

$$\sum_{j \leq k} \mu_j(B(x, 2^{-k})) \leqslant C_2 2^{-kd}, \tag{2.27}$$

où $B(x, 2^{-k})$ est la boule de rayon 2^{-k} et de centre x.

Ce que nous venons d'énoncer est une formulation «en temps discret» d'un résultat de Carleson [CM1], [J2].

Pour vérifier que ce résultat s'applique pour démontrer le Lemme 2.8, notons que, si C_0 est assez grand, $|S_k f(x)| \leqslant C|f_k(x)|$, de sorte que le Lemme 2.8 découlera du résultat suivant, pour tout $x \in \mathbb{R}^d$ et tout $k \in \mathbb{Z}$,

$$\int_{|x-y| \leq 2^{-k}} \sum_{j=k}^{+\infty} |\Delta_j^N M_b \gamma(x)|^2\,dx \leqslant C 2^{-kd}. \tag{2.28}$$

It est facile de voir que (2.28) découle de l'inégalité $\sum_0^\infty \|\Delta_j^N f\|_2^2 \leqslant C\|f\|_2^2$, valable pour tout $f \in L^2$, des propriétés (2.25) et (2.26) des fonctions de BMO, et des propriétés des opérateurs Δ_j induites par (2.9), (2.10), (2.12) et (2.13). Nous omettons les détails.

Nous pouvons maintenant montrer que les $U_{j,\beta,b}$ sont uniformément bornés sur $L^2(\mathbb{R}^d)$. Soient f et $g \in L^2$;

$$\begin{aligned}
|\langle g, U_{j,\beta,b} f\rangle| &\leqslant \sum_{k=-j}^{j} |\langle \Delta_k g, M_b\{\Delta_k^N M_b R\beta\} S_k f\rangle| \leqslant \\
&\leqslant C\left[\sum_{k=-j}^{j} \Delta_k g_2^2\right]^{1/2} \left[\sum_{k=-j}^{j} \|(\Delta_k^N M_b R\beta)(S_k f)\|_2^2\right]^{1/2} \leqslant \\
&\leqslant C\|g\|_2 \|\beta\|_{\mathrm{BMO}} \|f\|_2.
\end{aligned}$$

Ceci montre également que les $U_{j,\beta,b}$ convergent faiblement sur L^2 lorsque $j \to +\infty$. Soit $U_{\beta,b}$ leur limite. Des calculs analogues à ceus faits pour montrer (2.16), (2.19) et (2.20) permettent de voir que les $U_{j,\beta,b}$ satisfont aux hypothèses du Lemme 1.2. Comme ${}^t U_{j,\beta,b} b = 0$ pour tout j, on en déduit, d'après ce lemme, que ${}^t U_{\beta,b} b = 0$. Il nous reste à montrer que $U_{\beta,b} b = \beta$. Pour cela nous avons besoin d'une variante du Lemme 1.2.

Lemme 2.9. *Soit $(T_j)_{j \in \mathbb{N}}$ une suite d'opérteurs vérifiant uniformément les hypothèses du lemme 2.7 et tels que $T_j 1 = 0$ pour tout j. Si les noyaux K_j convergent uniformément sur tout compact de Ω vers un noyau K et si les T_j convergent faiblement vers un opérateur T comme opérateurs bornés sur $L^2(\mathbb{R}^d)$, alors pour tout $f \in \mathrm{BMO}(\mathbb{R}^d)$ et tout g borné, à support compact et d'intégrale nulle,*

$$\langle g, Tf \rangle = \lim_{m \to \infty} \langle g, T_m f \rangle.$$

Ce lemme découle clairement de la démonstration du Lemme 2.7.

Pour démontrer que $U_{\beta, b} b = \beta$, il suffit, d'après le Lemme 1.2, de montrer que pour tout $g \in \{bC_0^\eta(\mathbb{R}^d)\}_0$, $\lim_{j \to +\infty} \langle g, U_{j,\beta,b} b \rangle = \langle g, \beta \rangle$. Or

$$\langle g, U_{j,\beta,b} b \rangle = \left\langle g, \sum_{k=-j}^{j} \Delta_k M_b \{\Delta_k^N M_b R\beta\} S_k b \right\rangle =$$

$$= \left\langle g, \sum_{k=-j}^{j} \Delta_k M_b \Delta_k^N M_b R\beta \right\rangle.$$

D'après le Lemme 2.9 appliqué a la suite $\sum_{k=-j}^{j} \Delta_k M_b \Delta_k^N M_b$, et la définition de V_N,

$$\lim_{j \to +\infty} \langle g, U_{j,\beta,b} b \rangle = \langle g, V_N M_b R\beta \rangle = \langle g, \beta \rangle,$$

d'après la définition de R. L'opérateur $U_{\beta, b}$ possède donc toutes les propriétés requises.

Ceci achève la démonstration du Théorème $T1$ et du Théorème 2.

II. FONCTIONS PARA-ACCRÉTIVES ET THÉORÈME *Tb*

3. Fonctions para-accrétives

Nous avons déjà remarqué que les théorèmes $T1$, 2 et 3 suggèrent la question suivante: quelles conditions une fonction b bornée ainsi que son inverse doit-elle satisfaire pour que le Théorème $T1$ soit vrai sur $(\mathbb{R}^d, b(x)\,dx)$? Le Théorème 3 suggère que b ne doit pas nécessairement être à valeurs réelles. De plus, un examen attentif de la démonstration du Thérème 2 que nous nous avons donnée amène aux constatations suivantes. Le fait que b soit réelle n'a servi que pour permettre d'affirmer que les Δ_k^* satisfaisaient également la propriété (2.13), ce qui était essentiel pour permettre d'utiliser le Lemme 2.1 et vérifier les hypothèses de propre-orthogonalité du lemme CKS. Celui-ci n'a servi qu'à montrer les deux faits suivants:

(3.1) Il existe $C > 0$ et $\epsilon > 0$ tels que, pour toute partie finie A de $\mathbb{Z}$,

$$\left\| \sum_{j \in A} \Delta_j M_b \Delta_{j+n} \right\|_{2,2} \leqslant C 2^{-\epsilon |n|};$$

(3.2) Il existe $C > 0$ tel que pour tout $f \in L^2(\mathbb{R}^d)$,

$$\sum_{j \in \mathbb{Z}} \|\Delta_j f\|_2^2 \leqslant C \|f\|_2^2.$$

Rappelons que (3.1) et les conditions (2.9)-(2.13) entraînent que la série double $\sum_{j_1} \sum_{j_2} \Delta_{j_1} M_b \Delta_{j_2}$ est fortement sommable, que sa somme est M_b^{-1}, et que de plus $\lim_{N \to \infty} V_N M_b = I$ en norme CZδ. L'estimation quadratique (3.2) est utilisée dans le Lemme 2.5 pour montrer que les opérateurs $\tilde{T}_{N, J_1, J_2}$ sont bornés.

De ces renarques, on peut conclure que le Théorème $T1$ sera vrai sur $(\mathbb{R}^d, b\,dx)$ pour toute fonction b bornée ainsi que son inverse vérifiant les propriétés suivantes:

(3.3) Il existe une suite S_k d'opérateurs donnés par des noyaux s_k vérifiant (2.9)...(2.13);

(3.4) on peut choisir les S_k de manière que, si $\Delta_k = S_k - S_{k-1}$, alors les propriétés (3.1) et (3.2) son vérifiées.

Nous montrerons au paragraphe 6 que la condition (3.3) est en fait nécessaire pour que le Théorème $T1$ soit vrai sur $(\mathbb{R}^d, b\,dx)$.

De plus, nous verrons que si (3.3) est satisfaite, alors (3.4) l'est aussi, de sorte que (3.3) est aussi une condition suffisante, ce que nous démontrerons au paragraphe 5.

Définition 5. *Une fonction b: $\mathbb{R}^d \to \mathbb{C}$, bornée ainsi que son inverse, est dite para-accrétive s'il existe une suite $(s_k)_{k \in \mathbb{Z}}$ de fonctions de $\mathbb{R}^d \times \mathbb{R}^d$ dans $\mathbb{C}$ vérifiant les conditions* (2.9)...(2.13).

Remarquons que la construction des s_k donnée dans la démonstration du Lemme 2.2 pour des fonctions réelles positives reste valable pour les fonctions accrétives, de sorte que toute fonction accrétive est para-accrétive. La propriété cruciale des fonctions para-accrétives est justement que l'on peut construire les s_k d'une façon analogue.

Proposition 2. *Soit b: $\mathbb{R}^d \to \mathbb{C}$ une fonction bornée ainsi que son inverse. Les propriétés suivantes sont équivalentes.*

(A) *b est para-accrétive;*

(B) *il existe $\epsilon_1 > 0$ et $N_1 > 0$ tels que pour tout $k \in \mathbb{Z}$ et tout cube dyadique Q de côté 2^{-k}, il existe un cube dyadique $\tilde{Q}$ de même taille tel que la distance de $\tilde{Q}$ à Q soit inférieure à $N_1\, 2^{-k}$ et* $|1/|\tilde{Q}| \int_{\tilde{Q}} b(x)\, dx| \geqslant \epsilon_1$;

(*C*) *il existe* $C_1 > 0$, $\delta_1 > 0$ *et pour tout* $k \in \mathbb{Z}$ *une fonction* $u_k = \mathbb{R}^d \times \mathbb{R}^d \to C$ *telle que*

$$|u_k(x,y)| \leqslant C_1 2^{kd} \quad \textit{pour tout} \quad x, y \in \mathbb{R}^d \tag{3.5}$$

et

$$u_k(x,y) = 0 \quad si \quad |x-y| \geqslant C_1 2^{-k};$$

(3.6) *pour tous* x, y *et* $y' \in \mathbb{R}^d$,

$$|u_k(x,y') - u_k(x,y)| \leqslant C_1 2^{k(d+\delta_1)} |y'-y|^{\delta_1};$$

(3.7) *pour tout* $x \in \mathbb{R}^d$, $\dfrac{1}{C_1} \leqslant \left| \int u_k(x,y) b(y)\, dy \right| \leqslant C_1;$

(*D*) *il existe* $C_2 > 0$, $\delta_2 > 0$, *et pour tout* $k \in \mathbb{Z}$ *une fonction* v_k: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *possédant les propriétes (3.5), (3.6) et (3.7) avec* $C_1 = C_2$ *et* $\delta_1 = \delta_2$, *et telle que*

(3.8) *pour tout* $x \in \mathbb{R}^d$, $\int v_k(x,y)\, dy = 1;$

(3.9) *pour tout* $y \in \mathbb{R}^d$, $\int v_k(x,y)\, dx = 1;$

(3.10) *pour tout* $y \in \mathbb{R}^d$, *la fonction* $v_k(\cdot\,, y)$ *est constante sur chaque cube dyadique de volume* 2^{-kd}.

Avant de démontrer cette proposition, indiquons comment l'implication $A \Rightarrow D$ nous permettra de faire un choix privilégié parmi les suites de fonctions s_k de la Définition 5. Soit V_k l'opérateur de noyau $v_k(x,y)$ et $\{V_k b\}$ l'opérateur de multiplication ponctuelle par $V_k b$. D'après (3.7) cet opérateur est inversible. On pose $S_k = {}^t V_k \{V_k b\}^{-1} V_k$; on vérifie aisément que les noyaux s_k des S_k satisfont (2.9)... (2.13).

Nous venons de démontrer que (*D*) implique (*A*). Comme (*A*) entraîne (*C*) trivialement, il nous suffit de prouver les implications $(C) \Rightarrow (B)$ et $(B) \Rightarrow (D)$.

La démonstration de $(C) \Rightarrow (B)$ repose essentiellement sur une intégration par parties. Soient $(u_k)_{k \in \mathbb{Z}}$, C_1 et δ_1 comme dans (*C*), et fixons $x \in \mathbb{R}^d$ et $n_0 \in \mathbb{N}$. Ecrivons

$$\int u_k(x,y) b(y)\, dy = \sum_{Q \in \mathfrak{D}_{k+n_0}} \int_Q u_k(x,y) b(y)\, dy,$$

où $\mathfrak{D}_{k+n_0}$ est l'ensemble des cubes dyadiques de volume $2^{-d(k+n_0)}$. Pour tout cube $Q \in \mathfrak{D}_{k+n_0}$

$$\int_Q u_k(x,y)b(y)\,dy = \int_Q [u_k(x,y) - m_Q u_k(x,\cdot)]b(y)\,dy + |Q|(m_Q u(x,\cdot))m_Q b.$$

On en déduit

$$\left|\int_Q u_k(x,y)b(y)\,dy\right| \leqslant C\|b\|_\infty C_1 2^{-\delta_1(k+n_0)}2^{k(d+\delta_1)}|Q| + C_1 2^{kd}|m_Q b|\,|Q|.$$

Comme, en vertu de (3.5), il y a au plus $C\,C_1^d 2^{n_0 d}$ cubes Q de $\mathfrak{D}_{k+n_0}$ tels que $\int_Q u_k(x,y)b(y)\,dy \neq 0$, il découle

$$\begin{aligned} C_1^{-1} &\leqslant \left|\int u_k(x,y)b(y)\,dy\right| \\ &\leqslant CC_1^d 2^{n_0 d}\{C_1 2^{-\delta_1 n_0}2^{dk} + C_1 2^{kd}\sup|m_Q b|\}2^{-d(k+n_0)} \leqslant \\ &\leqslant CC_1^{d+1}\{2^{-\delta_1 n_0} + \sup|m_Q b|\}, \end{aligned}$$

où le sup est pris sur tous les cubes $Q \in \mathfrak{D}_{k+n_0}$ tels que la distance de x à Q soit inférieure à $C2^{-k}$.

En choisissant n_0 suffisamment grand, on conclut que $\sup|m_Q b| \geqslant \epsilon(C_1)$. La propriété (B) est donc vérifiée avec $\epsilon_1 = \epsilon(C_1)$ et $N_1 = C2^{n_0}$, ce qui démontre $(C) \Rightarrow (B)$.

Nous montrons maintenant que (B) implique (D). L'idée est de former pour tout k une partition de $\mathbb{R}^d$ en ensembles E_J tels que $|m_{E_J} b| \geqslant \epsilon$ pour un $\epsilon > 0$ fixe, pus de poser $w_k(x,y) = \frac{1}{|E_J|}$ si x et y sont dans le même ensemble E_J et $w_k(x,y) = 0$ sinon. Les propriétés (3.8) et (3.9) seront automatiquement satisfaites, ainsi que (3.10) si chaque E_J est une union de cubes dyadiques de volume 2^{-kd}. Pour avoir (3.5), ils suffira alors que les E_J aient un diamètre inférieur à $C2^{-k}$. Ensuite on construira les v_k en régularisant les w_k en y, ce qui permettra d'avoir (3.6) sans affecter les autres propriétés.

Nous donnons maintenant les détails. Soient ϵ_1 et N_1 tels que (B) soit satisfaite, et soit K une grande constante, dont la valeur sera précisée plus tard. Fixons $k \in \mathbb{Z}$. Soit $\mathfrak{D}_k$ la collection des cubes dyadiques de volume 2^{-kd}. On construit une application $L\colon \mathfrak{D}_k \to \mathfrak{D}_k$ de la façon suivante. Si $|m_Q b| \geqslant \epsilon_1/KN_1^d$, alors $L(Q) = Q$. Si $|m_Q b| \leqslant \epsilon_1/KN_1^d$, alors $L(Q)$ est un cube tel que la distance de Q à $L(Q)$ soit inférieure à $N_1 2^{-k}$ et $|m_{L(Q)} b| \geqslant \epsilon_1$. Pour tout $J \in \mathfrak{D}_k$ tel que $|m_J b| \geqslant \epsilon_1/KN_1^d$, on note E_J l'union de ses antécédents par L. Si $|m_J b| < \epsilon_1$, $E_J = J$; si $|m_J b| \geqslant \epsilon_1$, E_J est l'union de J et d'au plus $C_d N_1^d$ cubes Q tels que $|m_Q b| \leqslant \epsilon_1/KN_1^d$. En choisissant K suffisamment grand, on aura $\left|\int_{E_J} b(y)\,dy\right| \geqslant \frac{\epsilon_1}{2}2^{-kd}$, et $|m_{E_J} b| \geqslant \epsilon_1/2C_d N_1^d$. On obtient donc une partition de $\mathbb{R}^d$ en ensembles E_J tels que $|m_{E_J} b| \geqslant \epsilon$, où $\epsilon > 0$ ne dépend que de ϵ_1, N_1 et d. On définit w_k comme indiqué précédemment. Il est clair que le volume de E_J est supérieur à 2^{-k}, et que son diamètre est dominé par $C2^{-k}$. Par conséquent, w_k satisfait toutes les conditions requises, sauf (3.6).

Nous voulons maintenant régulariser en y la fonction $w_k(x,y)$. Soit φ une fonction positive, C^∞ à support compact, et d'intégrale 1. On pose

$$v_k(x,y) = \int w_k(x,z)(2^k M)^d \varphi(2^k M(z-y))\, dz.$$

Les propriétés (3.5), (3.8), (3.9) et (3.10) sont encore vérifiées par $v_k(x,y)$. De plus, (3.6) est maintenant satisfaite. Il reste donc à voir que (3.7) n'est pas affecté par cetté régularisation, si M est choisi assez grand. on peut choisir M de sorte que

$$\int \left| \left\{ \int w_k(x,z)(2^k M)\varphi(2^k M(z-y))\, dz \right\} - w_k(x,y) \right| dy \leqslant \tfrac{\epsilon}{2} \|b\|_\infty^{-1}.$$

En effet, pour tout y, l'intégrand est dominé par $\|w_k\|_\infty$, et de plus il est nul pour tous les y qui son à une distance $\geqslant C/2^k M$ de la frontière d'un des E_J. Si x est dans l'ensemble E_J,

$$\left| \int w_k(x,y) b(y)\, dy \right| = \frac{1}{|E_J|} \left| \int_{E_J} b(y)\, dy \right| \geqslant \epsilon,$$

de sorte que $\left|\int v_k(x,y)b(y)\,dy\right| \geqslant \frac{\epsilon}{2}$. Comme $\left|\int v_k(x,y)b(y)\,dy\right| \leqslant \|b\|_\infty$, v_k satisfait aussi (3.7), et nous avons fini la démonstration de la Proposition 2.

La condition (3.10) exprime une certaine régularité en x de $v_k(x,y)$. Nous allons maintenant voir comment on peut l'utiliser.

Lemme 3.1. *Soit H opérateur continu sur L^1 donné par intégration contre un noyau $h(x,y)$ tel que*

(3.11) *pour un certain $t \geqslant 0$, $h(x,y) = 0$ dès que $|x-y| \geqslant t$,*
(3.12) *pour tout $x \in \mathbb{R}^d$, $\int h(x,y)\, dy = 0$.*

Alors HV_k est continu sur L^1 avec une norme inférieure à $Ct2^k \|H\|_{1,1}$.

Notons que, d'après (3.5), V_k est borné sur L^1 avec une norme uniformément bornée. Le lemme n'a donc d'intérêt que lorsque $t \leqslant 2^{-k}$.

Supposons donc $t \leqslant 2^{-k}$; l'operateur HV_k est donné par intégration contre le noyau

$$H_k(x,z) = \int h(x,y) v_k(y,z)\, dy.$$

Nous voulons majorer $\sup_z \int |H_k(x,z)|\, dx$. *Si* $z \in E_J$, et si $d(x, \partial E_J) > t$, alors $v_k(\cdot, z)$ est constante sur $\{y, |x-y| \leqslant t\}$. On en déduit que

$$H_k(x,z) = \int h(x,y) v_k(y,z)\, dy = 0$$

à cause de (3.12). Par conséquent

$$\begin{aligned}\int\left|\int h(x,y)v_k(y,z)\,dy\right|dx &\leqslant C2^{kd}\iint_{d(x,\partial E_j)\leq t}|h(x,y)|\,dy\,dx \leqslant\\ &\leqslant C2^{kd}\int_{d(y,\partial E_j)\leq 2t}\left\{\int|h(x,y)|\,dx\right\}dy\leqslant\\ &\leqslant Ct2^k\sup_y\left\{\int|h(x,y)|\,dx\right\}\leqslant\\ &\leqslant Ct2^k\|H\|_{1,1}.\end{aligned}$$

Le Lemme 3.1 est démontré. Notons que seules les propriétés (3.5) et (3.10) des v_k ont été utilisées.

Nous avons indiqué après la Proposition 2 comment on choisit une famille S_k d'opérateurs dont les noyaux vérifient (2.9)... (2.13). Pour vérifier que les S_k satisfont également (3.1) et (3.2), nous devrons nous appuyer sur des lemmes autres que le Lemme CKS, car la fonction b n'est plus nécessairement réelle.

4. Deux lemmes à la Cotlar

Dans ce paragraphe, H sera le complexifié d'un espace de Hilbert réel, et l'on notera $\langle\cdot,\cdot\rangle$ le produit bilinéaire sur H. Tous les opérateurs considérés agiront sur H.

Lemme 4.1. *Soit $(A_j)_{j\in\mathbb{Z}}$ une suite d'opérateurs uniformément bornés telle que $\lim_{j\to\infty}A_j = I$ et $\lim_{j\to-\infty}A_j = 0$ au sens de la convergence forte. On suppose l'existence de constantes $\epsilon>0$, $C_0>0$ et $C_1>0$ telles que, si $B_j = A_j - A_{j-1}$ et $n\in\mathbb{N}$,*

(4.1) $$\|B_jB_{j+n}\| + \|B_{j+n}B_j\| \leqslant C_0 2^{-\epsilon n}$$

et

(4.2) $$\text{pour tout } x\in H,\qquad \sum_{j\in\mathbb{Z}}\{\|B_jx\|^2 + \|{}^tB_jx\|^2\}\leqslant C_1^2\|x\|^2.$$

Soit $n\in\mathbb{N}$, et soit $B_j^n = \sum_{j-n}^{j+n}B_k$. D'après (4.2), l'opérateur $T_n = \sum_{j=-\infty}^{\infty}B_jB_j^n$ est défini comme série faiblement convergente d'opérateurs. De plus, $\lim_{n\to\infty}T_n = I$ en norme d'opérateur et la série double $\sum\sum_{j,k\in\mathbb{Z}}B_jB_k$ est fortement sommable. Enfin $\|I-T_n\|\leqslant\frac{1}{2}$ pour $n\geqslant C_0'(1+|\mathrm{Log}\,C_1|)$, où C_0' ne dépend que C_0, ϵ, et $\sup_j\|A_j\|$.

La lettre C désignera les constantes qui ne dépendent que de C_0, ϵ, et $\sup_j\|A_j\|$. Nous commençons par démontrer le lemme sous l'hypothèse

$\sum_{j\in\mathbb{Z}} \|B_j\| < +\infty$. Toutes les séries que nous écrirons seront donc automatiquement convergentes.

Soient $m \in \mathbb{N}$, et $F \subset \mathbb{Z}\times\mathbb{Z}\backslash\{(j,k), |j-k| \leqslant m\}$; soit $S_F = \sum\sum_{(j,k)\in F} B_j B_k$. Nous voulons estimer $\|T_n S_F T_n\|$ pour $n \in \mathbb{N}$. Soient x et y dans H; on majore $|\langle x, T_n S_F T_n y\rangle|$ par

$$\sum_r \sum_s |\langle x, B_s B_s^n S_F B_r B_r^n y\rangle| = \sum_r \sum_s |\langle {}^t B_s^n x, B_s S_F B_r B_r^n y\rangle| =$$
$$= \sum_r \sum_s \|{}^t B_s^n x\| \, \|B_s S_F B_r\| \, \|B_r^n y\|.$$

Comme $\left(\sum_r \|B_r^n y\|^2\right)^{1/2} \leqslant (2n+1)\left(\sum_r \|B_r y\|^2\right)^{1/2} \leqslant (2n+1)C_1\|y\|$ et pareillement $\left(\sum_s \|{}^t B_s^n x\|^2\right)^{1/2} \leqslant (2n+1)C_1\|x\|$, on voit que

$$|\langle x, T_n S_F T_n y\rangle| \leqslant (2n+1)^2 C_1^2 \|x\| \, \|y\| M_F.$$

où M_F est la norme de la matrice de terme général $a_{s,r} = \|B_s S_F B_r\|$, agissant sur $l^2(\mathbb{Z})$. Donc, $\|T_n S_F T_n\| \leqslant (2n+1)^2 C_1^2 M_F$. Por estimer M_F, on majore $a_{s,r}$ par $\sum_j \sum_{k, \, |j-k|>m} \|B_s B_j B_k B_r\|$. Or (4.1) entraîne que $\|B_s B_j B_k B_r\|$ est inférieur à $C2^{-\epsilon|s-j|}2^{-\epsilon|k-r|}$, et aussi à $C2^{-\epsilon|j-k|}$, donc à $C2^{-(\epsilon/2)(|s-j|+|j-k|+|k-r|)}$. Si l'on somme cette majoration par rapport à r, puis k, puis j, on obtient $\sum_r a_{s,r} \leqslant C2^{-\epsilon m/2}$ pour tout s. Si l'on somme en s, j, et k, on obtient $\sum_s a_{s,r} \leqslant C2^{-\epsilon m/2}$ pour tout r. Il en découle, par interpolation (ou en utilisant l'inégalité de Schwarz) que $M_F \leqslant C2^{-\epsilon m/2}$. Donc $\|T_n S_F T_n\| \leqslant Cn^2 C_1^2 2^{-\epsilon m/2}$. Comme on a supposé que $\sum_j \sum_k \|B_j B_k\| < +\infty$, la série $\sum_j \sum_k B_j B_k$ est normalement sommable, et sa somme est l'identité. Il existe donc un $n \in \mathbb{N}$ tel que $\|I - T_n\| \leqslant \frac{1}{2}$. Soit n_0 le plus petit entier tel que $\|I - T_{n_0}\| \leqslant \frac{1}{2}$. Si $n_0 > 2$, choisissons $m = n_0 - 1$ et $F = \{(j,k), |j-k| > n_0 - 1\}$, de sorte que $S_F = I - T_{n_0-1}$ et $\|S_F\| > \frac{1}{2}$ par définition de n_0. Comme $\|T_{n_0}^{-1}\| \leqslant 2$, on obtient

$$\frac{1}{8} \leqslant \frac{1}{\|T_{n_0}^{-1}\|} \|S_F\| \frac{1}{\|T_{n_0}^{-1}\|} \leqslant \|T_{n_0} S_F T_{n_0}\| \leqslant Cn_0^2 C_1^2 2^{-(\epsilon/2)(n_0-1)},$$

ce qui entraîne que $n_0 \leqslant C(1 + |\mathrm{Log}\, C_1|)$. Il en découle que pour m et F arbitraires,

$$\|S_F\| \leqslant \|T_{n_0}^{-1}\|^2 \|T_{n_0} S_F T_{n_0}\| \leqslant Cn_0^2 C_1^2 2^{-\epsilon m/2} \leqslant CC_1^2(1 + |\mathrm{Log}\, C_1|)^2 2^{-\epsilon m/2}.$$

Le Lemme 4.1 est donc démontré lorsque $\sum_j \|B_j\| < +\infty$. Dans le cas général, on modifie une suite A_j vérifiant les hypothèses de la façon suivante: on remplace A_j par I lorsque j est très grand, et par 0 lorsque j est très petit. Si F est un sous-ensemble fini donné de $\mathbb{Z}\times\mathbb{Z}$, on peut faire en sorte que l'opérateur S_F correspondant à la suite modifiée soit le même que pour la suite originale. Comme la suite tronquée vérifie les hypothèses du lemme avec les constantes $\epsilon' = \epsilon$, $C_0' = [1/(1 - 2^{-\epsilon})]C_0$ et $C_1' = C_1 + 4\sup_j \|A_j\|$, et de plus est telle que $\sum \|B_j\|^2 < +\infty$, on en déduit que l'inégalité $\|S_F\| \leqslant CC_1^2(1 + |\mathrm{Log}\, C_1|)^2 2^{-\epsilon m/2}$ reste valable pour tout F fini. Il s'ensuit que pour tout $x \in H$, la série double $\sum_j \sum_k B_j B_k x$ est sommable car ses sommes finies sont uniformément majorées. De plus, $\|I - T_m\| \leqslant CC_1^2(1 + |\mathrm{Log}\, C_1|)^2 2^{-\epsilon m/2}$ tend vers 0 quand $m \to +\infty$, ce qui conclut la démonstration du Lemme 4.1.

Il se trouve que, dans les situations où nous appliquerons ce lemme, l'existence de C_0 et de ϵ sera claire, tandis que nous aurons besoin du lemme suivant pour établir celle de C_1.

Lemme 4.2. *Soit $(A_j)_{j\in\mathbb{Z}}$ une suite d'opérateurs uniformément bornés sur H telle que $\lim_{j\to+\infty} A_j = I$ et $\lim_{j\to-\infty} A_j = 0$ au sens de la convergence forte. On suppose l'existence de $C_0 > 0$ et $\epsilon > 0$ tels que si $B_j = A_j - A_{j-1}$,*

$$\|B_j B_{j+n}\| + \|B_{j+n} B_j\| \leqslant C_0 2^{-\epsilon n}$$

pour tout $n \in \mathbb{N}$. On suppose en outre que A_j admet la factorisation $A_j = D_j E_j$, où $(D_j)_{j\in\mathbb{Z}}$ et $(E_j)_{j\in\mathbb{Z}}$ sont deux suites d'opérateurs uniformément bornés vérifiant les propriétés

(4.3) *si $G_j = E_j - E_{j-1}$, $\sum \|G_j x\|^2 \leqslant C_0 \|x\|^2$ pour tout $x \in H$,*

et, si $F_j = D_j - D_{j-1}$,

(4.4) $\|E_j B_{j+n}\| \leqslant C_0 2^{-\epsilon n}$ *pour tout $j \in \mathbb{Z}$ et tout $n \in \mathbb{N}$,*
(4.5) $\|F_j E_{j-n}\| \leqslant C_0 2^{-\epsilon n}$ *pour $j \in \mathbb{Z}$ et $n \in \mathbb{N}$, et*
(4.6) $\|F_j E_{j-n} B_{j-n-n'}\| \leqslant C_0 2^{-\epsilon(n+n')}$ *pour $j \in \mathbb{Z}$ et $n, n' \in \mathbb{N}$.*

On suppose enfin que la suite $({}^tA_j)_{j\in\mathbb{Z}}$ satisfait à toutes les hypothèses faites sur la suite (A_j).

La conclusion est l'existence d'une constante C_1 telle que pout tout $x \in H$,

$$\sum_{j\in\mathbb{Z}} \|B_j x\|^2 + \|{}^tB_j x\|^2 \leqslant C_1^2 \|x\|^2. \tag{4.7}$$

Comme pour le lemme précédent, il suffit de démontrer qu'il existe une constante C_1, ne dépendant que des hypothèses du lemme, telle que (4.7) soit satisfaite dès que $\sum_j \|B_j\| + \sum_j \|F_j\| < +\infty$. En effet, pour tout $N < +\infty$, on aura $\sum_{j=-N}^{N} \|B_j x\|^2 + \|{}^tB_j x\|^2 \leqslant C_1^2 \|x\|^2$ pour tout x, en appliquant (4.7) à une suite tronquée remplaçant (A_j), et la constante C_1 sera indépendante de N.

Supposons donc que $\sum_j \|B_j\| + \sum_j \|F_j\| < +\infty$, et décomposons B_j de la façon suivante:

$$\begin{aligned} B_j &= D_jE_j - D_{j-1}E_{j-1} = \\ &= D_j(E_j - E_{j-1}) + (D_j - D_{j-1})E_{j-1} = \\ &= D_jG_j + F_jE_{j-1}. \end{aligned}$$

Comme les D_j sont uniformément bornés, (4.3) entraîne $\sum_j \|D_jG_jx\|^2 \leqslant \leqslant C\|x\|^2$ pour tout $x \in H$. Soit m un entier positif, dont la valeur sera décidée plus tard. On majore $\left(\sum_j \|F_jE_{j-1}x\|^2\right)^{1/2}$ par

$$\Big(\sum_j \|F_jE_{j-m}x\|^2\Big)^{1/2} + \sum_{l=2}^{m} \Big[\sum_j \|F_j(E_{j-l+1} - E_{j-l})x\|^2\Big]^{1/2}.$$

En utilisant à nouveau (4.3) et le fait que les F_j sont uniformément bornés, on voit que

$$\Big[\sum_j \|B_jx\|^2\Big]^{1/2} \leqslant \Big[\sum_j \|F_jE_{j-m}x\|^2\Big]^{1/2} + Cm\|x\|. \tag{4.8}$$

Pour estimer $\sup_{x,\, \|x\| \le 1} \left[\sum_j \|F_jE_{j-m}x\|^2\right]$, il suffit d'estimer

$$\sup_{x,y,\, \|x\| \le 1,\, \|y\| \le 1} \Big|\Big\langle y, \sum_j E^*_{j-m}F^*_jF_jE_{j-m}x\Big\rangle\Big|.$$

Comme nous avons supposé que $\sum_j \|B_j\| < +\infty$, nous savons qu'il existe une constante $C_1 > 0$, et nous voulons seulement obtenir un contrôle de C_1. D'après le Lemme 4.1, il existe $n_0 \leqslant C(1 + |\mathrm{Log}\, C_1|)$ tel que $\|I - T_{n_0}\| < \frac{1}{2}$, de sorte que $\|T^{-1}_{n_0}\| \leqslant 2$. Il suffit donc d'estimer

$$\Big|\Big\langle y, \Big(\sum\sum_{|u-v| \le n_0} B_vB_u\Big)^*\Big(\sum_j E^*_{j-m}F^*_jF_jF_{j-m}\Big)\Big(\sum\sum_{|s-r| \le n_0} B_sB_r\Big)x\Big\rangle\Big|$$

pour $\|x\| \leqslant 1$ et $\|y\| \leqslant 1$.

Par le même raisonnement que dans la démonstration du lemme précédent, et en utilisant les estimations quadratiques

$$\Big\{\sum_u \Big\|\sum_{u=v-n_0}^{v+n} B_uy\Big\|^2\Big\}^{1/2} \leqslant (2n_0 + 1)C_1\|y\| \quad \text{et}$$

$$\Big\{\sum_s \Big\|\sum_{r=s-n_0}^{s+n_0} B_rx\Big\|^2\Big\}^{1/2} \leqslant (2n_0 + 1)C_1\|x\|,$$

on voit que

$$\sup_{x,\, \|x\| \le 1} \Big[\sum_j \|F_jE_{j-m}x\|^2\Big] \leqslant Cn_0^2C_1^2M, \tag{4.9}$$

où M est la norme de la matrice de terme géneral

$$b_{v,s} = \sum_j \|B^*_vE^*_{j-m}F^*_jF_jE_{j-m}B_s\|.$$

Lorsque $s \leqslant j - m$, $\|F_j E_{j-m} B_s\| \leqslant C_0 2^{-\epsilon(j-s)}$ à cause de (4.6); lorsque $s \geqslant j - m$,

$$\|F_j E_{j-m} B_s\| \leqslant C \|F_j E_{j-m}\|^{1/2} \|E_{j-m} B_s\|^{1/2} \leqslant C 2^{-\epsilon m/2} 2^{-\epsilon(s-j+m)}$$

à cause de (4.5) et (4.4). Dans les deux cas,

$$\|F_j E_{j-m} B_s\| \leqslant C 2^{-\epsilon m/2} 2^{-\epsilon|s-j+m|/2}.$$

Il en découle $b_{v,s} \leqslant C 2^{-\epsilon m} \sum_{l \in \mathbb{Z}} 2^{-\epsilon(|s-l|+|v-l|)} \leqslant C 2^{-\epsilon m}(1 + |s - v|) 2^{-\epsilon|s-v|/2}$. Par conséquent, $M \leqslant C 2^{-\epsilon m}$.

Soit $C_2 = \sup_{\|x\| \leq 1} \left(\sum_j \|B_j X\|^2\right)^{1/2}$. D'après (4.8), (4.9) et ce que nous venons de voir,

$$C_2 \leqslant C n_0 C_1 M^{1/2} + Cm \leqslant C(1 + |\mathrm{Log}\, C_1|) C_1 2^{-\epsilon m/2} + Cm.$$

On majore pareillement $\sup_{\|x\| \leq 1} \left(\sum_j \|{}^t B_j x\|^2\right)^{1/2}$ en utilisant la symétrie des hypothèses sur les A_j et les ${}^t A_j$, de sorte que $C_1 \leqslant C(1 + |\mathrm{Log}\, C_1|) C_1 2^{-\epsilon m/2} + Cm$, où la constance C est indépendante de m. On choisit pour m la partie entière de $(2 \,\mathrm{Log}\, C_1)/(\epsilon \,\mathrm{Log}\, 2)$, on obtient $C_1 \leqslant C(1 + |\mathrm{Log}\, C_1|)$, ce qui entraîne $C_1 \leqslant C$. Nous avons donc le contrôle de C_1 recherché, et le Lemme 4.2 est démontré.

5. Le théorème *Tb*

Nous allons utiliser les résultats des paragraphes précédents pour démontrer le théorème suivant, qui généralise les théorèmes 2 et 3.

Théorème Tb. *Soient b_1 et b_2 deux foctions para-accrétives sur $\mathbb{R}^d$. Soit $T: b_1 C_0^\eta(\mathbb{R}^d) \to [b_2 C_0^\eta(\mathbb{R}^d)]'$ un* SIO. *Alors T admet une extension continue sur $L^2(\mathbb{R}^d)$ si et seulement si $M_{b_2} T M_{b_1}$ est faiblement borné, $Tb_1 \in$* BMO, *et ${}^t Tb_2 \in$* BMO.

Rappelons que les fonctions para-accrétives sont définies et étudiées au paragraphe 3; les autres termes de l'énoncé sont définis au paragraphe 1.

Nous ne démontrerons le théorème que lorsque $b_1 = b_2$, les changements à faire dans le cas géneral étant triviaux. Signalons que la méthode de McIntosh et Meyer pour démontrer le Théorème 3 s'étend aussi au cas de deux fonctions accrétives distinctes b_1 et b_2. Un exemple d'opérateur lié au théorème de McIntosh et Meyer, dans le cas où $b_2 = 1$ et b_1 est une fonction accrétive définie sur $\mathbb{R}$, est précisément l'opérateur de Kato $\sqrt{Db_1 D}$ étudié dans [KM].

Nous passons maintenant à la démonstration du théorème. Soit $(S_j)_{j \in \mathbb{Z}}$ la suite d'opérateurs définis après l'énoncé de la Proposition 2, paragraphe 3.

Comme nous l'avons remarqué au début du paragraphe 3, le théorème Tb sera démontré dès que nous aurons prouvé que cette suite S_k satisfait les propriétés (3.1) et (3.2). Pour ce faire, nous allons vérifier que la suite $A_j = S_j M_b$ vérifie les hypothèses du Lemme 4.2. Dans ce cas, (4.7) implique que $\sum_j \|B_j f\|^2 = \sum_j \|\Delta_j M_b^{-1} f\|_2^2 \leqslant C\|f\|_2^2$, ce qui est équivalent à (3.2). D'autre part, si les hypothèses du Lemme 4.2 sont satisfaites, d'après le Lemme 4.2 celles du Lemme 4.1 le sont aussi, et il résulte de la démonstration du Lemme 2 que, si F est une partie finie de $\mathbb{Z} \times \mathbb{Z} \backslash \{(j,k), |j-k| \leqslant m\}$, alors $\left\| \sum\sum_{j,k \in F} B_j B_k \right\| \leqslant C 2^{-\epsilon m/2}$. En particulier, si A est une partie finie de $\mathbb{Z}$,

$$\left\| \sum_{j \in A} B_j B_{j+m+1} \right\| = \left\| \sum_{j \in A} \Delta_j M_b \Delta_{j+m+1} M_b \right\| \leqslant C 2^{-\epsilon m/2},$$

ce qui est la condition (3.1).

Vérfifions que les $S_j M_b$ vérifient toutes les hypothèses du Lemme 4.2.

Comme les $S_j M_b$ sont uniformément bornés sur L^2, l'hypothèse $\lim_{j \to +\infty} S_j M_b = I$ au sens fort découle de ce que pour toute $f \in C_0^\alpha(\mathbb{R}^d)$, $\lim_{j \to +\infty} S_j M_b f = f$, ce qui résulte des propriétés (2.9) et (2.13) de $s_j(x,y)$. De même, $\lim_{j \to -\infty} S_j M_b = 0$ au sens fort.

L'inégalité $\|\Delta_j M_b \Delta_{j+n}\| + \|\Delta_{j+n} M_b \Delta_j\| \leqslant C_0 2^{-\epsilon n}$ est une conséquence du Lemme 2.1 et des propriétés (2.9)... (2.13), que nous avons d'ailleurs déjà vue au Lemme 2.2.

La factorisation de $A_j = S_j M_b = {}^t V_j \{V_j b\}^{-1} V_j M_b$ est donnée par $D_j = {}^t V_j$ et $E_j = \{V_j b\}^{-1} V_j M_b$. Les D_j et E_j sont clairement uniformément bornés. Pour vérifier (4.3), on écrit

$$\begin{aligned} G_j &= \{V_j b\}^{-1} V_j M_b - \{V_{j-1} b\}^{-1} V_{j-1} M_b = \\ &= -\{V_j b\}^{-1} \{V_{j-1} b\}^{-1} (\{V_j b\} - \{V_{j-1} b\}) V_j M_b + \\ &\quad + \{V_{j-1} b\}^{-1} (V_j - V_{j-1}) M_b. \end{aligned}$$

Soit $T_j = (V_j^* - V_{j-1}^*)(V_j - V_{j-1})$. On déduit de (3.5), (3.6), (3.8), et du Lemme 2.1 que $\|T_j T_k^*\| + \|T_j^* T_k\| \leqslant C 2^{-\alpha|j-k|}$. On peut donc appliquer le Lemme CKS, ce qui donne

$$\left| \left\langle f, \sum_j T_j f \right\rangle \right| = \sum_j \|(V_j - V_{j-1}) f\|_2^2 \leqslant C \|f\|_2^2$$

et par conséquent

$$\sum_j \|\{V_{j-1} b\}^{-1} (V_j - V_{j-1}) M_b f\|_2^2 \leqslant C \|f\|_2^2.$$

De plus, l'inégalité quadratique $\sum_j \|(V_j - V_{j-1}) f\|_2^2 \leqslant C \|f\|_2^2$, appliquée à $f = b\chi_{B(x, C2^{-k})}$, montre que la suite de mesures $\mu_j = |(V_j - V_{j-1}) b|^2 \, dx$ vérifie la condition de Carleson (2.27). On en déduit, comme on l'avait fait au paragraphe 2, que pour tout $f \in L^2$;

$$\sum_j \left\| \frac{1}{(V_j b)(V_{j-1} b)} [(V_j - V_{j-1})b](V_j M_b f) \right\|_2^2 \leqslant C \| f \|_2^2,$$

ce qui entraîne (4.3).

Pour prouver (4.4), on applique le Lemme 2.1 avec $U = {}^t\Delta_{j+n} M_b$ et $V = {}^t V_j$. On obtient $\| {}^t\Delta_{j+n} M_b {}^t V_j \| \leqslant C 2^{-\epsilon n}$, d'où

$$\| E_j B_{j+n} \| = \| \{ V_j b \}^{-1} V_j M_b \Delta_{j+n} M_b \| \leqslant C 2^{-\epsilon n}.$$

Pour vérifier (4.5), montrons que

$$\| ({}^t V_j - {}^t V_{j-1}) \{ V_{j-n} b \}^{-1} V_{j-n} M_b \|_{1,1} \leqslant C 2^{-n}. \tag{5.1}$$

Comme ces opérateurs sont uniformément bornés sur L^∞ à cause de (3.5), il en résultera par interpolation que

$$\| F_j E_{n-j} \|_{2,2} \leqslant C 2^{-n/2}, \tag{5.2}$$

ce qui est (4.5).

Pour prouver (5.1), nous utilisons la remarque suivant la démonstration du Lemme 3.1, qui nous dit que nous pouvons remplacer dans ce lemme l'opérateur V_k par $\{ V_k b^{-1} \} V_k$, dont le noyau vérifie également (3.5) et (3.10). Le Lemme 3.1, appliqué avec $H = {}^t V_j - {}^t V_j$, $t = C 2^{-j}$ et $k = j - n$, fournit aussitôt (5.1).

Pour montrer (4.6), et compte-tenu de (5.2), il suffit de montrer que $\| F_j E_{j-n} B_{j-n-n'} \| \leqslant C 2^{-\alpha n'}$. Nous écrivons $F_j E_{j-n} B_{j-n-n'} = UV$, avec $U = ({}^t V_j - {}^t V_{j-1}) \{ V_{j-n} b \}^{-1} V_{j-n} M_b$ et $V = \Delta_{j-n-n'}$. Comme $U1 = 0$, on peut appliquer le Lemme 2.1 qui donne $\| UV \|_{2,2} \leqslant C 2^{-\alpha n'}$.

Nous avons fini de vérifier que les Lemmes 4.1 et 4.2 s'appliquent à notre situation. Le théorème *Tb* est donc complètement démontré.

6. Nécessité de la para-accrétivité

La proposition suivante est une forme de réciproque au théorème *Tb*.

Proposition 3. *Soit b: $\mathbb{R}^d \to \mathbb{C}$ une fonction bornée. Si la conclusion du théorème Tb est vraie avec $b_1 = b_2 = b$, alors b est para-accrétive.*

Pour démontrer la Proposition 3 il suffit, pour toute fonction b qui n'est pas para-accrétive, de construire une suite $(T_n)_{n \in \mathbb{N}}$ d'opérateurs vérifiant uniformément les hypothèses du Théorème Tb , mais qui ne sont pas uniformément bornés sur $L^2(\mathbb{R}^d)$, et qui sont donnés par intégration contre des noyaux positifs. Il est alors facile de choisir une suite sommable $(\alpha_n)_{n\in\mathbb{N}}$ de nombres réeles positifs telle que $\lim_{n\to\infty}\sup \|\alpha_n T_n\|_{2,2} = +\infty$. Comme les T_n sont donnés par des noyaux positifs, l'opérateur $\sum \alpha_n T_n$, qui vérifie les hypothèses du Théorème Tb, est alors non borné sur $L^2(\mathbb{R}^d)$.

Supposons donc que b n'est pas para-accrétive. D'après la Proposition 2, pour tout $n > 0$ il existe un cube dyadique Q (soit 2^{-kd} son volume) tel que pour tout cube dyadique $\tilde{Q}$ de même volume vérifiant $d(Q, \tilde{Q}) \leqslant 2^n 2^{-k}$, on ait $|m_{\tilde{Q}} b| \leqslant 1/n$.

Soit X la fonction caractéristique de $\{x \in \mathbb{R}^d, d(x, Q) \leqslant 2^{n-1}2^{-k}\}$, M_X l'opérateur de multiplication ponctuelle par X, et, pour $t > 0$, P_t l'opérateur de noyau $(1/t^d)\mathbb{1}_{\{|x-y|\leq t\}}$.

Lemme 6.1. *L'opérateur*

$$T_n = \int_{2^{-k}}^{2^{n-k-1}} P_t M_X P_t \frac{dt}{t}$$

est un opérateur d'intégrale singulière dont le noyau satisfait les estimations standard de manière uniforme. De plus, $\|T_n\|_{2,2} \leqslant [(n-1)/4] \operatorname{Log} 2$.

Le calcul du noyau de T_n, et la vérification des estimations standard (1.1) et (1.2) sont faciles et nous les laissons au lecteur. Pour $t \leqslant 2^{n-k-1}$, $P_t X \geqslant \frac{1}{2}X$, de sorte que

$$T_n X \geqslant \int_{2^{-k}}^{2^{n-k-1}} \frac{1}{4} X \frac{dt}{t} \geqslant \frac{(n-1)}{4} \operatorname{Log} 2X,$$

ce qui démontre le Lemme 6.1.

Nous aurons également besoin du lemme suivant.

Lemme 6.2. *Il existe $C > 0$ tel que pour tout $T \in [2^{-k}, 2^{n-k-1}]$,*

$$|M_X P_t b| \leqslant \frac{C 2^{-k}}{t} + \frac{1}{n}. \tag{6.1}$$

Pour démontrer ce lemme, on remarque que si $d(x, Q) \leqslant 2^{n-k-1}$ et si $t \in [2^{-k}, 2^{n-k-1}]$, la boule de centre x et de rayon t est telle que tout cube dyadique $\tilde{Q}$ de volume 2^{-kd} qui la touche vérifie $|m_{\tilde{Q}} b| \leqslant 1/n$. On en déduit que $|P_t b(x)| \leqslant (1/n) + C(2^{-k}/t)$ en approximant la boule de centre x et de rayon t par une union de cubes dyadiques de volume 2^{-kd}, ce qui démontre le Lemme 6.2.

Nous déduisons de ce lemme que $\| T_n b \|_\infty \leqslant C$; comme ${}^tT = T$ il ne nous reste plus qu'à montrer que les $M_b T M_b$ sont uniformément faiblement bornés. Il suffit en fait de voir que si $f \in C_0^1(\mathbb{R}^d)$ a un support de diamètre s, alors

$$\| T_n b \|_\infty \leqslant C(\| f \|_\infty + s \| \nabla f \|_\infty). \tag{6.2}$$

Notons que si $d(x, Q) \leqslant 2^{n-k-1}$ et $f \in C_0^1(\mathbb{R}^d)$,

$$M_X P_t(bf)(x) = P_t(bf)(x) = f(x) P_t b(x) + P_t[b(f - f(x))](x).$$

D'après (6.1), $\| M_X P_t(bf) \|_\infty \leqslant C\{(2^{-k}/t) + (1/n)\} \| f \|_\infty + Ct \| \nabla f \|_\infty$. nous retiendrons cette majoration si $t \leqslant s$. Si $t \geqslant s$, on a $\| P_t(bf) \|_\infty \leqslant C(s/t)^d \| f \|_\infty$. On en déduit que

$$\begin{aligned} \| T_n b f \|_\infty &\leqslant C \int_s^{+\infty} \left(\frac{s}{t}\right)^d \| f \|_\infty \frac{dt}{t} + C \int_0^s t \| \nabla f \|_\infty \frac{dt}{t} + \\ &\quad + \int_{2^{-k}}^{2^{n-k-1}} \left(\frac{C 2^{-k}}{t} + \frac{1}{n}\right) \| f \|_\infty \frac{dt}{t} \leqslant \\ &\leqslant C(\| f \| + s \| \nabla f \|_\infty). \end{aligned}$$

Nous avons donc une suite T_n comme souhaité, et la Proposition 3 est démontrée.

Une variante légèrement plus élaborée du contre-exemple précédent permet de montrer la proposition suivante.

Proposition 4. *Soit b_1 une fonction para-accrétive sur $\mathbb{R}^d$. Si b_2 est une fonction bornée telle que la Théorème Tb soit vrai pour le couple (b_1, b_2), alors b_2 est para-accrétive.*

Nous omettons la démonstration de cette proposition.

Il est possible que l'on puisse montrer que b_1 et b_2 sont para-accrétives si le Théorème *Tb* est vrai pour le couple (b_1, b_2), mais nous n'avons pas su le faire. Cela montrerait que le Théorème *Tb* est en un certain sens optimal. Toutefois, il es clair qu'il existe une extension du Théorème *Tb* où les modules des fonctions b_1 et b_2 seraient des poids de Muckenhoupt. L'absence pour l'instant d'applications pour une telle extension suggère toutefois de différer son étude.

Signalons que toute fonction bornée ainsi que son inverse n'est pas nécessairement para-accrétive. Les exponentielles imaginaires nous fournissent une infinité non dénombrable de contre-exemples.

III EXTENSIONS ET APPLICATIONS

7. Extensions

Nous allons illustrer la généralité de la méthode employée dans la démonstration du Théorème *Tb* para quelques extensions.

A. Espaces de nature homogène

Rappelons qu'un espace de nature homogène (X, d, μ) est un espace topologique X, muni d'une quasi-distance d et d'une mesure de Radon positive μ, pour lequel il existe une constante C telle que pour tout $x \in X$ et tout $r > 0$, $\mu(B(x, 2r)) \leqslant C\mu(B(x, r))$, où $B(x, r)$ est la boule $\{y \in X, d(x, y) \leqslant r\}$.

Nous verrons que l'on peut définir, sur chaque espace de nature homogène, une classe d'opérateurs d'intégrale singulière comme nous l'avons fait dans le cas euclidien. Dans le cas de certains groupes de Lie nilpotents, cette classe géneralise les opérateurs de convolution considérés dans [KS]. Enfin, modulo certaines hypothèses innocentes sur l'espace (X, d, μ), le Théorème *Tb* se généralise.

Le point de départ est le résultat suivant de Macias et Segovia.

Théoréme 4 [MS1]. *Soit (X, d, μ) un espace de nature homogène. Il existe une quasi-distance δ topologiquement équivalente à d verifiant*

(i) $\delta(x, y) \approx \inf_B \mu(B)$, *l'inf étant pris sur toutes les boules B contenant x et y, et*
(ii) *il existe $C > 0$ et $\alpha > 0$ tels que, pour tous x, y, $z \in X$,*

$$|\delta(x, z) - \delta(y, z)| \leqslant C\delta(x, y)^{\alpha}(\delta(x, z) + \delta(y, z))^{1-\alpha}.$$

Au vu de ce résultat, on peut remplacer l'espace (X, d, μ) par (X, δ, μ). L'avantage est que, sur l'espace (X, δ, μ), la fonction δ est localement höldérienne, et que, si $C_0^{\eta}(X, \delta)$ est l'espace des fonctions höldériennes d'exposant η à support compact, $C_0^{\eta}(X, \delta)$ est dense dans $L^2(X, d\mu)$ si η est assez pettit [MS1].

Une fonction $K(x, y)$, définie pour $x \neq y$, est un noyau standard sur (X, δ, μ) s'il existe $C > 0$ et $\eta > 0$ tels que

$$(7.1) \qquad |K(x, y)| \leqslant C\delta(x, y)^{-1} \quad \text{pour} \quad x \neq y,$$

et

$$(7.2)\quad |K(x',y) - K(x,y)| + |K(y,x') - K(y,x)| \leqslant C \frac{\delta(x',x)^\eta}{(\delta(x',y) + \delta(x,y))^{1+\eta}}.$$

Il est facile de voir que, si l'on prend $\delta(x,y) = |x-y|^d$ sur $(\mathbb{R}^d, dx)$, cette définition d'un noyau standard coincide avec celle du paragraphe 1.

Soient b_1 et b_2 dans $L^\infty(X,\mu)$. Comme dans le cas euclidien, un opérateur d'intégrale singulière est un opérateur continu $T: b_1 C_0^\eta(X,\delta) \to b_2 C_0^\eta(X,\delta)'$ associé à un noyau standard $K(x,y)$ au sens que, pour f et g dans $C_0^\eta(X,\delta)$ à supports disjoints.

$$\langle g, Tf \rangle = \iint g(x) b_2(x) K(x,y) b_1(y) f(y)\, d\mu(y)\, d\mu(x).$$

La démonstration du Théorème $T1$ sur les espaces de nature homogène suit de très près la démonstration du Théorème 2 donnée au paragraphe 2. Le point crucial est l'existence d'une «bonne» approximation de l'identité, destinée à remplacer les opérateurs S_k du Lemme 2.2. On a besoin d'une suite $(s_k)_{k\in\mathbb{Z}}$ de fonctions, définies de $X \times X$ dans $\mathbb{R}^+$, telles que, pour un $\eta > 0$ et pour tout $k \in \mathbb{Z}$,

$$(7.3)\qquad s_k(x,y) = 0 \quad \text{si} \quad \delta(x,y) \geqslant C2^{-k} \quad \text{et} \quad \|s_k\|_\infty \leqslant C2^k;$$

$$(7.4)\qquad |s_k(x,y) - s_k(x',y)| \leqslant C2^{k(1+\eta)}\delta(x,x')$$

pour x, x' et y dans X:

$$(7.5)\qquad s_k(x,y) = s_k(y,x) \quad \text{pour tous} \quad x,\ y \in X;$$

$$(7.6)\qquad \int_X s_k(x,y)\, d\mu(y) = 1 \quad \text{pour tout} \quad x \in X.$$

Pour construire les s_k, on a besoin des deux hypothèses supplémentaires suivantes sur (X,δ,μ):

$$(7.7)\qquad \mu(X) = +\infty$$

$$(7.8)\ \text{pour tout } x \in X, \qquad \mu(\{x\}) = 0.$$

On choisit une fonction h dérivable: $\mathbb{R}^+ \to \mathbb{R}^+$, égale à 1 sur $[0, 1/2]$ et à 0 sur $[2, +\infty[$. On appelle T_k l'opérateur défini par le noyau $2^{-k}h(\delta(x,y)/2^k)$. La propriété d'espace homogène et les hypothèses (7.7) et (7.8) entraînent que $T_k 1 \approx 1$. Soient M_k l'opérateur de multiplication par $1/T_k 1$, W_k l'opérateur de multiplication par $\{T_k(1/T_k 1)\}^{-1}$, et $S_k = M_k T_k W_k T_k M_k$. Les propriétés (7.3), (7.5) et (7.6) son immédiates, et (7.4) provient de la propriété (ii) de δ.

On a de plus $\lim_{k\to+\infty} S_k = I$ et $\lim_{k\to-\infty} S_k = 0$, la convergence ayant lieu pour la topologie forte des opérateurs bornés sur L^2. Le reste de la démonstration du Théorème 2 donnée plus haut est encore valable dans le contexte des espaces de nature homogène.

Pour démontrer la Théorème Tb dans le cadre des espaces de nature homogène, il nous faut en outre un substitut à la Proposition 2. La propriété (B) est modifiée en remplaçant les cubes dyadiques par des boules. Pour construire des fonctions v_k comme dans $(B) \Rightarrow (D)$, on décompose, pour tout k, X en use union disjointe d'ensembles $E_{k,J}$ dont la mesure est $\geqslant C2^{-k}$ et le diamètre pour la quasi-distance δ est inférieur à $C2^{-k}$. Il faut encore, pour que la condition «pour tout $y \in X$, la fonction $v_k(\cdot, y)$ est constante sur chaque $E_{k,J}$» entraîne assez de régularité en x pour pouvoir être utilisée dans le Lemme 3.1, que la mesure de l'ensemble des points de X dont la distance à la frontière d'un des $E_{k,J}$ est inférieure à t décroisse suffisamment vite avec t.

On est amené à faire l'hypothèse supplémentaire suivante sur (X, δ, μ):

(7.9) Il est possible de choisir la quasi-distance du théorème de Macias et Segovia de telle sorte que, en plus de (i) et de (ii), on ait

(iii) il existe $\gamma > 0$ et $C > 0$ tels que, pour tout $x \in X$ et tous $r > s > 0$, $\mu(B(x, r)\backslash B(x, s)) \leqslant C(r - s)^{\gamma} r^{1-\gamma}$.

Avec les hypothèses supplémentaires (7.7), (7.8) et (7.9), on peut démontrer une proposition semblable à la Proposition 2. Le reste de la démonstration du Théorème Tb se déroule alors à peu près comme dans le cas euclidien, de sorte que le Théorème Tb est vrai sur (X, δ, μ). Nous omettons les détails de la démonstration.

B. Opérateurs agissant sur des fonctions à valeurs matricielles

Le Théorème Tb s'étend aussi sans grande modification au cas où l'opérateur T envoie des fonctions à valeurs dans $\mathbb{R}^n$ dans des fonctions à valeurs dans $\mathbb{R}^n$. Le noyau $K(x, y)$ est alors une matrice $n \times n$ dont les coefficients sont notés $K_{i,j}(x, y)$, $1 \leqslant i, j \leqslant n$. On dira que K est un noyau standard si les $K_{i,j}$ sont des noyaux standard. Il est commode de considérer l'opérateur T comme agissant sur les fonctions définies sur $\mathbb{R}^d$ et à valeurs dans $\mathfrak{M}_n(\mathbb{C})$ (l'espace des matrices $n \times n$ à coefficients complexes), au lieu de $\mathbb{R}^n$.

Nous définissons la dualité sur les fonctions à valeurs matricielles à l'aide de la forme bilinéaire

$$\langle G, F \rangle = \int \tilde{G}(x) F(x)\, dx, \tag{7.10}$$

où $\tilde{G}(x)$ est pour tout $x \in \mathbb{R}^d$ la matrice transposée de $G(x)$. Notons que le produit $\langle G, F \rangle$ est une matrice constante.

On notera $C_0^\infty = C_0^\infty(\mathbb{R}^d, \mathfrak{M}_n(C))$ l'espace des fonctions C^∞ à support compact définies sur $\mathbb{R}^d$ et à valeurs dans $\mathfrak{M}_n(\mathbb{C})$; si B est une fonction à valeurs matricielles, BC_0^∞ sera l'image de C_0^∞ pour l'opérateur de multiplication à gauche par $B(x)$.

On se donne donc deux fonctions $B_1(x)$ et $B_2(x)$, à valeurs dans $\mathfrak{M}_n(\mathbb{C})$, telles que $B_1^{-1}(x)$ et $B_2^{-1}(x)$ existent et soient bornées. Un SIO est un opérateur linéaire continu $T: B_1 C_0^\infty \to (B_2 C_0^\infty)'$ (où le dual est défini à l'aide du produit bilinéaire ci-dessus), pour lequel il existe un noyau standard $K(x, y) = ((K_{i,j}(x, y))$ tel que, si F et G sont dans C_0^∞ et ont des supports disjoints,

$$\langle B_2 G, TB_1 F\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{G}(x)\tilde{B}_2(x)K(x, y)B_1(y)F(y)\,dy\,dx.$$

Définissons la transposition des opérateurs par la formule $\langle G, {}^tTF\rangle = \langle F, TG\rangle^{\sim}$. Ainsi, le transposé de l'opérateur de multiplication à gauche par $B(x)$ est l'opérateur de multiplication à gauche par $B(x)$. D'autre part, si T est un SIO, alors ${}^tT$ est un SIO, et son noyau est $K(y, x)$.

Avec ces notations, on a le théorème suivant:

Théorème TB. *Soient B_1 et B_2 deux fonctions bornées de $\mathbb{R}^d$ dans $\mathfrak{M}_n(\mathbb{C})$, et telles que B_1^{-1} et B_2^{-1} existent et soient bornées. On suppose que B_1 et B_2 satisfont à la condition (D) de la Proposition 2, où (3.7) est remplacée par*

(3.7)′ *pour tout $x \in \mathbb{R}^d$, $\int u_k(x, y)B(y)\,dy$ est inversible, et la norme de son inverse est inférieure à C_1.*

Soit $T: B_1 C_0^\infty(\mathbb{R}^d, \mathfrak{M}_n(\mathbb{C})) \to [B_2 C_0^\infty(\mathbb{R}^d, \mathfrak{M}_n(\mathbb{C}))]'$ un SIO. *Alors T admet une extension continue sur $L^2(\mathbb{R}^d, \mathfrak{M}_n(\mathbb{C}))$ si et seulement si $\mathfrak{M}_{\tilde{B}_2} TM_{B_1}$ est faiblement borné, $TB_1 \in$* BMO, et *${}^tTB_2 \in$* BMO.

Les définitions de TB_1, ${}^tTB_2$, et de la bornitude faible de $M_{\tilde{B}_2} TM_{B_1}$ sont les mêmes que dans le cas scalaire.

La démonstration de ce théorème est la même que celle du Théorème *Tb*; nous en omettons les détails.

La Proposition 2 ne semble pas se généraliser aux fonctions à valeurs matricielles. Dans la démonstration de $(B) \Rightarrow (D)$, pour pouvoir prouver que l'intégrale de b sur chaque E_J est assez grande, il faudrait que si deux matrices A et B sont telles que $\|A^{-1}\| \leqslant \frac{1}{\epsilon}$ et $\|B^{-1}\| \geqslant \frac{C}{\epsilon}$, alors $\|(A - B)^{-1}\| \leqslant \frac{2}{\epsilon}$. Une telle propriété est naturellement fausse en général, et nous ne savons pas démontrer le Théorème *TB* pour une fonction à valeurs matricielles B qui satisferait à la condition (B) de la Proposition 2.

C. Les nombres de Clifford

Rappelons que l'algèbre de Clifford $C_n(\mathbb{R})$ est l'algèbre sur $\mathbb{R}$ engendrée par une unité $e_0 = 1$ et n éléments $e_1, \ldots, e_n$, avec les relations $e_i^2 = -1$ pour $1 \leqslant i \leqslant n$ et $e_i e_j = -e_j e_i$ pour $1 \leqslant i \neq j \leqslant n$. Si $A = (i_1, \ldots, i_k)$ est une suite finie strictement croissante de $\{1, \ldots, n\}$, on pose $e_A = e_{i_1}, \ldots, e_{i_k}$; par convention, $e_\phi = e_0$. Les e_A forment une base de $C_n(\mathbb{R})$, qui est donc de dimension 2^n sur $\mathbb{R}$.

Les nombres de Clifford sont le sous-espace vectoriel de dimension $n + 1$ engendré par $e_0, \ldots, e_n$. Si $x = x_0 + x_1 e_1 + \cdots + x_n e_n$ est un nombre de Clifford non nul, et si $|x|^2 = x_0^2 + x_1^2 + \cdots + x_n^2$, alors x est inversible, et $x^{-1} = = (2/|x|^2)(x_0 - x_1 e_1 - \cdots - x_n e_n)$. En particulier, $|1/x| = 1/|x|$, et la propriété qui nous faisait défaut pour les matrices est vraie. La Proposition 2 est donc encore vraie pour les fonctions b bornées définies sur $\mathbb{R}^d$ et à valeurs dans les nombres de Clifford.

Como l'algèbre de Clifford opère sur elle-même (par exemple par multiplication à gauche), on déduit du Théorème TB et la remarque précédente que le Théorème Tb reste valable lorsque T est associé à un noyau $K(x, y)$ à valeurs dans $C_n(\mathbb{R})$, et b_1, b_2 sont deux fonctions para-accrétives définies sur $\mathbb{R}^d$ et à valeurs dans les nombres de Clifford.

8. Opérateur de Cauchy et calcul fonctionnel holomorphe en plusieurs variables

Une conséquence facile du Théorème Tb est la continuité sur L^2 de l'opérateur défini par le noyau de Cauchy sur une courbe corde-arc. Rappelons qu'une courbe rectifiable Γ du plan complexe, admettant la paramétrisation par la longueur d'arc $s \to z(s)$, définie pour $s \in \mathbb{R}$, est dite corde-arc si, pour une constante $C \geqslant 1$ et tout $(s, t) \in \mathbb{R}^2$, $|z(s) - z(t)| \geqslant \frac{1}{C}|s - t|$.

Si Γ est une courbe corde-arc, alors le noyau $K(x, y) = 1/(z(x) - z(y))$ est un noyau standard, et de plus antisymétrique. Nous avons vu au paragraphe 1 (formules (1.4) et (1.5)) que $K(x, y)$ définit un SIO $T: z'C_0^\infty(\mathbb{R}) \to z'C_0^\infty(\mathbb{R})$ tel que $M_{z'}TM_{z'}$ soit faiblement borné. On vérifie sans peine que $Tz' = {}^tTz' = 0$ (dans BMO) en utilisant la formule de Cauchy. De plus, la fonction z' vérifie clairement la propriété (B) de la Proposition 2, de sorte que l'on peut appliquer le Théorème Tb et qu'on obtient la continuité sur L^2 de l'opérateur T.

Rappelons que ce résultat est une conséquence facile du Théorème de Coifman-McIntosh-Meyer [CMM1] (voir [CDM]). Notons aussi que, lorsque Γ est le graphe d'une fonction lipschitzienne, ce résultat découle, par le même argument, du Théorème de McIntosh et Meyer cité au paragraphe 2.

L'extension du Théorème Tb aux nombres de Clifford permet de démontrer directement que le noyau de Cauchy-Clifford associé au graphe d'une fonction lipschitzienne: $\mathbb{R}^n \to \mathbb{R}$ définit un opérateur borné sur $L^2(\mathbb{R}^n)$. Comme ce résultat est, de toute façon, une conséquence facile du Théorème de Coifman, McIntosh et Meyer, nous ne donnons pas les détails (voir [DJS]).

Nous nous proposons d'étendre en plusieurs dimensions la construction par Coifman et Meyer [CM2] d'un calcul fonctionnel holomorphe en $(1/(1+i\varphi'))(d/dx)$, où φ est une fonction lipschitzienne: $\mathbb{R} \to \mathbb{R}$. Rappelons de quoi il s'agit.

Soit H_α l'espace des fonctions bornées de $\mathbb{R}$ dans $\mathbb{C}$ admettant un prolongement holomorphe borné sur le secteur $S_\alpha = \{z \in \mathbb{C}, |\operatorname{Im} z| < \alpha |\operatorname{Re} z|\}$, et soit $\Gamma \subset \mathbb{C}$ le graphe d'une fonction lipschitzienne φ vérifiant $\|\varphi'\|_\infty < \alpha$, *c-à-d.* $\Gamma = \{x + y, y = \varphi(x)\}$. Soit $m \in H_\alpha$. On fait les hypothèses a priori que φ est à support compact et qu'il existe $C > 0$ et $a > 1$ tels que, pour tout $z \in S_\alpha$, $\operatorname{Log} |m(z)| \leqslant C - (1/C)|z|^a$.

Sous ces hypothèses, Coifman et Meyer [CM2] associent au couple (m, Γ) un opérateur M_Γ défini par la formule

$$M_\Gamma f(z) = \int_{\mathbb{R}} \int_\Gamma e^{2i\pi t(z-w)} f(w) m(t)\, dw\, dt,$$

où $f \in L^1(\Gamma, ds)$ a un support compact, et $z \in \Gamma$.

Lorsque $\Gamma = \mathbb{R}$, M_Γ est, à une constante près, le multiplicateur de Fourier de symbole m; lorsque Γ est quelconque, et $m(t) = sgn(t)$, M_Γ n'est autre que l'opérateur de Cauchy sur la courbe Γ. Coifman et Meyer démontrent le théorème suivant.

Théorème 5 [CM2]. *L'opérateur M_Γ est borné sur $L^2(\Gamma, ds)$ avec une norme qui ne dépend que de $\|m\|_\alpha = \sup_{z \in S_\alpha} |m(z)|$ et de $\|\varphi'\|_\infty$.*

Signalons que, lorsque Coifman et Meyer démontrèrent ce théorème, la continuité sur L^2 de l'opérateur de Cauchy sur les courbes lipschitziennes n'était connue que sous la restriction de Calderón $\|\varphi'\|_\infty \leqslant \delta_0$, qui était donc ajoutée aux hypothèses.

Vérifions que ce théorème peut se démontrer en utilisant le Théorème de McIntosh et Meyer cité au paragraphe 2. On considère l'opérateur T_Γ, défini de $(1 + i\varphi')C_0^\infty(\mathbb{R})$ dans son dual par $\langle g, T_\Gamma f \rangle = \int_\Gamma \tilde{g}(z) M_\Gamma \tilde{f}(z)\, dz$, où $\tilde{f}(x + i\varphi(x)) = (1 + i\varphi'(x))^{-1} f(x)$ et $\tilde{g}(x + i\varphi(x)) = (1 + i\varphi'(x))^{-1} g(x)$. Nous voulons appliquer le Théorème 3, avec $b = 1 + i\varphi'$, à l'opérateur T_Γ.

Le noyau de T_Γ est

$$K(x, y) = \int_{\mathbb{R}} e^{2i\pi t(x + i\varphi(x) - y - i\varphi(y))} m(t)\, dt.$$

L'hypothèse a priori faite sur m entraîne que la fonction L définie par $L(z) = \int_{\mathbb{R}} e^{2i\pi tz} m(t)\,dt$ est analytique et bornée sur S_α. De plus, en utilisant un changement de contour et en faisant tourner l'axe des x jusqu'à ce que tz soit réel, on obtient le résultat suivant: pour tout $\gamma < \alpha$, il existe une constante $C_\gamma > 0$, ne dépendant que de γ, α, et $\|m\|_\alpha$, telle que

$$|L(z)| \leqslant \frac{C_\gamma}{|z|} \quad \text{et} \quad |L'(z)| \leqslant \frac{C_\gamma}{|z|^2} \quad \text{sur } S_\gamma.$$

Ces inégalités entraînent que K est un noyau 1-standard. Les égalités $T_\Gamma(1 + i\varphi') = {}^tT_\Gamma(1 + i\varphi') = 0$ découlent de la formule de Cauchy, et il ne reste qu'à vérifier que $M_{1+i\varphi'} T_\Gamma M_{1+i\varphi'}$ est faiblement borné. Ce sera une conséquence immédiate du lemme suivant.

Lemme 8.1. *Il existe une constante $C > 0$, ne dépendant que α et de $\|\varphi'\|_\infty$, telle que pour tout $f \in C_0^\infty(\mathbb{R})$ supportée par un intervalle I,*

$$\|T_\Gamma M_{1+i\varphi'} f\|_\infty \leqslant C \|m\|_\alpha \{ \|f\|_\infty + |I|\, \|f'\|_\infty \}.$$

Par homogénéité, on peut se contenter de démontrer le lemme lorsque $|I| = 1$. Choisissons $a > 1$ assez petit pour que $\operatorname{Re} z^a > 0$ dans $\{z \in \mathbb{C}, |\operatorname{Im} z| < < \alpha \operatorname{Re} z\}$. Pour tout $x \in \mathbb{R}$, on définit une fonction g_x de la façon suivante. On convient que $z^a = \exp(a \operatorname{Log} z)$ pour $\operatorname{Re} z > 0$ et $z^a = (-z)^a$ pour $\operatorname{Re} z < 0$, et l'on pose $g_x(y) = f(x) \exp[-(x + i\varphi(x) - y - i\varphi(y))^a]$. On majore $|T_\Gamma M f(x)|$ par $|T_\Gamma M(f - g_x)(x)| + |T_\Gamma M g_x(x)|$.

Un changement de contour montre que

$$\begin{aligned} T_\Gamma M_{1+i\varphi'} g_x(x) &= \int_{\mathbb{R}} L(x + i\varphi(x) - y - i\varphi(y))(1 + i\varphi'(y)) g_x(y)\,dy = \\ &= \int_{\Gamma_x = \{x + i\varphi(x) - y - i\varphi(y),\, y \in \mathbb{R}\}} L(z) f(x) \exp(-z^a)\,dz = \\ &= f(x) \int_{\mathbb{R}} L(u) \exp(-u^\alpha)\,du. \end{aligned}$$

Avec notre définition, $u^\alpha = |u|^\alpha$ pour u réel, et la transformée de Fourier de $\exp(-u^\alpha)$ est intégrable, de sorte que, en utilisant Plancherel, on obtient

$$|T_\Gamma M_{1+i\varphi'} g_x(x)| \leqslant C_a \|m\|_{L^\infty(\mathbb{R})}.$$

Le terme $T_\Gamma M_{1+i\varphi'}(f - g_x)(x)$ est majoré par

$$\|m\|_\alpha \int_{\mathbb{R}} |L(x + i\varphi(x) - y - i\varphi(y))(1 + i\varphi'(y))(f(y) - g_x(y))|\,dy.$$

En vertu de la régularité et de la décroissance de $f - g_x$, le second terme est donc dominé par $C\|m\|_\alpha(|I|\, \|f'\|_\infty + \|f\|_\infty)$.

Le Lemme 8.1 est démontré, et le théorème de McIntosh et Meyer s'applique.

Rappelons comment l'opérateur $T_\Gamma M_{1+i\varphi'}$ peut être interprété en termes de calcul fonctionnel. Soit T le multiplicateur de Fourier de symbole m, que l'on peut noter $m(D)$, où $D = (1/i)(d/dx)$. L'opération $m \to m(D)$ est un homorphisme d'algèbres de Banach entre L^∞ et l'ensemble des opérateurs bornés sur L^2, et définit un calcul fonctionnel pour D. Soient h un difféomorphisme croissant bilipschitzien de $\mathbb{R}$ dans $\mathbb{R}$, et V_k l'opérateur défini par $V_h f = f \circ h$. L'opération $m \to V_h m(D) V_n^{-1}$ définit un calcul fonctionnel pour l'opérateur $V_h D V_h^{-1} = M_{h'}^{-1} D$.

D'autre part, le noyau de $m(M_{h'}^{-1}D) = V_h m(D) V_h^{-1}$ est égal à $L(h(x) - h(y))h'(y)$. Si l'on remplace $h(x)$ par $x + i\varphi(x)$, on obtient le noyau de l'opérateur $T_\Gamma M_{1+i\varphi'}$. Plus précisément, le théorème de Coifman et Meyer permet de montrer que $V_h T V_h^{-1}$ est une fonction analytique réelle de h', définie dans l'ouvert $\{u \in L^\infty, \inf u > 0\}$ de $L^\infty_{\mathbb{R}}(\mathbb{R})$. Cette fonction analytique réelle admet un prolongement holomorphe dans l'ouvert $\{u \in L^\infty_{\mathbb{C}}, \inf |u| > 0$ et $|Im\, u| < \gamma \operatorname{Re} u\}$, pour un certain $\gamma > 0$. L'opérateur $T_\Gamma M_{1+i\varphi'}$ est précisément la valeur de ce prolongement au point $u = 1 + i\varphi'$. On peut aussi montrer que $m \to$ $\to T_\Gamma M_{1+i\varphi'}$ définit un calcul fonctionnel holomorphe pour l'opérateur $M_{1+i\varphi'}^{-1} D$, défini pour $m \in H_\alpha$. Nous renvoyons à [J1] pour les détails concernant le prolongement complexe de $V_h T V_{h^{-1}}$.

Nous nous intéressons maintenant au problème analogue en dimension $d > 1$. Soient m une fonction bornée définie sur $\mathbb{R}^d$, et T le multiplicateur de symbole m. Pour tout difféomorphisme bi-lipschitzien h de $\mathbb{R}^d$ sur lui-même, soient $J_h(x)$ la matrice jacobienne de h au point x et V_h l'opérateur défini par $V_h f = f \circ h$. Nous voulons étudier $V_h T V_h^{-1}$ en tant que fonction de J_h. Plus précisément, soient J le sous-espace fermé de $L^\infty(\mathbb{R}, \mathfrak{M}_d(\mathbb{R}))$ constitué des fonctions matrices jacobiennes des fonctions lipschitziennes de $\mathbb{R}^d$ dans $\mathbb{R}^d$, et $\mathfrak{U}$ l'ouvert de J correspondant aux difféomorphismes bi-lipschitziens.

Coifman et Meyer ont posé la question suivante: quelles hypothèses doit-on faire sur m pour que $V_h T V_h^{-1}$ soit une fonction analytique de $J_h \in \mathfrak{U}$, et que peut-on dire dans ce cas de son prolongement analytique complexe? Ce problème est étudié dans [J1], où il est démontré qu'une condition nécessaire est l'existence d'une extension holomorphe de m bornée sur un secteur de $\mathbb{C}^d$ du type

$$S_\alpha = \left\{ z \in \mathbb{C}^d, \sum_{1 \leq i \leq d} |\operatorname{Im} z_i|^2 \leqslant \alpha \sum_{1 \leq i \leq d} |\operatorname{Re} z_i|^2 \right\}.$$

Yves Meyer a remarqué que si cette propriété est vraie pour un $\alpha > 1$, alors m est constante. On ne s'intéressera donc qu'au cas où $\alpha < 1$.

Nous allons montrer que cette condition sur m est également suffisante.

Soit H_α l'espace des fonctions définies sur $\mathbb{R}^d$ et admettant une extension holomorphe bornée dans S_α, et, pour $m \in H_\alpha$, $\|m\|_\alpha = \sup_{z \in S_\alpha} |m(z)|$. Soit $J_\mathbb{C}$ le sous-espace fermé de $L^\infty(\mathbb{R}^d, \mathfrak{M}_n(\mathbb{C}))$ constitué par les fonctions matrices jacobiennes des fonctions lipschitziennes définies de $\mathbb{R}^d$ dans $\mathbb{C}^d$, et soit $\mathcal{U}_{\alpha,\epsilon}$ l'ouvert de $J_\mathbb{C}$ correspondant aux applications $v = (v_i)_{1 \leq i \leq d}$ telles que, pour un $\epsilon' < \epsilon$ et un $\alpha' < \alpha$, on ait

$$\epsilon'|x-y|^2 \leqslant \sum_{1 \leq i \leq d} |v_i(x) - v_i(y)|^2 \leqslant \frac{1}{\epsilon'}|x-y|^2$$

et

$$\sum_{1 \leq i \leq d} |\operatorname{Im} v_i(x) - \operatorname{Im} v_i(y)|^2 \leqslant \alpha' \sum_{1 \leq i \leq d} |\operatorname{Re} v_i(x) - \operatorname{Re} v_i(y)|^2$$

pour tous $x, y \in \mathbb{R}^d$.

Théorème 6. *Soit $m \in H_\alpha$. La fonction définie de $\mathcal{U}$ dans l'espace des opérateurs bornés sur $L^2_\mathbb{C}(\mathbb{R}^d)$ qui à J_h associe $V_h T V_h^{-1}$ est analytique réelle dans $\mathcal{U}$ et admet un prolongement analytique complexe borné sur chaque ouvert $U_{\gamma,\epsilon}$, où $\gamma < \alpha$ et $\epsilon > 0$.*

Remarquons que la condition $\alpha < 1$ nous permet, en multipliant m par $\exp\left(-C\sum_i z_i^2\right)$, qui est dominé sur S_α par $\exp\left(C'\sum z_i^2\right)$, de faire l'hypothèse qualitative qu'il existe C et $C' > 0$ tels que $|m(z)| \leqslant C\exp\left(-C'\sum_i |z_i|^2\right)$. Le cas géneral en découle alors par passage à la limite.

La démonstration de ce théorème a la même structure que celle du théorème de Coifman-Meyer utilisant le théorème de McIntosh-Meyer.

Le noyau de $V_h T V_h^{-1}$ est $L(h(x) - h(y)) \det J_h(y)$, où L est défini sur $\mathbb{R}^d$ par

$$L(x) = \int_{\mathbb{R}^d} exp\left(2\pi i \sum_j x_j \xi_j\right) m(\xi)\, d\xi.$$

L'hypothèse a priori sur m entraîne que L se prolonge en une fonction entière sur $\mathbb{C}^d$. De plus, comme en dimension 1, pour tout $\gamma < \alpha$ on a

$$|L(z)| \leqslant C\|m\|_\alpha \left(\sum_j |z_j|^2\right)^{-d/2}$$

et

$$|\operatorname{grad} L(z)| \leqslant C\|m\|_\alpha \left(\sum_j |z_j|^2\right)^{-(d+1)/2} \quad \text{pour tout} \quad z \in S_\gamma,$$

où la constante C ne dépend que de γ et de α. On en conclut que le noyau $L(v(x) - v(y))$ est uniformément 1-standard lorsque J_v reste dans l'ouvert

$\mathcal{U}_{\gamma,\epsilon}$. Comme L est entière, ce noyau définit sans ambiguïté un opérateur $T_{m,v}$. Il reste à montrer que $T_{m,v}$ est borné sur $L^2(\mathbb{R}^d)$, avec une norme inférieure à $C(\alpha,\gamma,\epsilon)\|m\|_\alpha$. Le théorème en découlera car $T_{m,v}$ est une fonction holomorphe de v.

Des arguments du même type que ceux utilisés en dimension 1 permettent de montrer que $T_{m,v}(\det J_v) = {}^tT_{m,v}(\det J_v) = 0$ et que $M_vT_{m,v}M_v$ est faiblement borné, où M_v est l'opérateur de multiplication ponctuelle par $\det J_v$. Toutefois, le théorème de McInstosh-Meyer ne peut s'utiliser lorsque la fonction $\det J_v$ n'est pas accrétive, ce qui est en général le cas lorsque α n'est pas proche de 0. Neanmoins, le Théorème Tb et le lemme suivant permettent de conclure.

Lemme 8.2. *Sous les hypothèses du Théorème 6, la fonction J_v est para-accrétive.*

Pour démontrer le lemme, observons que pour tout $t > 0$ et tout $J_v \in \mathcal{U}_{\gamma,\epsilon}$, un changement de contour donne

$$\int_{\mathbb{R}^d} t^d \exp\left\{-t^2 \sum_j (v_j(x) - v_j(y))^2\right\} \det J_v(y)\,dy = C_0,$$

où $C_0 = \int_{\mathbb{R}^d} \exp(-|x|^2)\,dx$.

En prenant $t = 2^k$ et en posant

$$u_k(x,y) = 2^{kd}\left(\exp\left[-2^{2k}\sum_j (v_j(x) - v_j(y))^2\right]\right)\varphi\left(\frac{(x-y)2^k}{M}\right),$$

où $\varphi \in C_0^\infty(\mathbb{R}^d)$ est égale à 1 au voisinage de l'origine, et où M est assez grand, on voit que $\det J_v$ vérifie la condition (C) de la Proposition 2. Ceci démontre le Lemme 8.2 et le Théorème 6.

Comme en dimension 1, le Théorème 6 peut être interprété en termes de calcul fonctionnel holomorphe. Soit $v\colon \mathbb{R}^d \to \mathbb{C}^d$ tel que $J_v \in \mathcal{U}_{\gamma,\epsilon}$ pour un $\gamma < \alpha$ et un $\epsilon < 0$, et notons J_v^{-1} grad le système de champs de vecteurs $(L_j)_{1\le j\le d}$ définis par $L_j(x) = \sum_k a_{j,k}(x)(\partial/\partial x_k)$, où les $a_{j,k}(x)$ sont les coefficients de la matrice $J_v^{-1}(x)$. On pose, pour $m \in H_\alpha$, $m\left(\frac{1}{i}J_v^{-1}\,\mathrm{grad}\right) = T_{m,v}M_v$ (le Théorème 6 permet de donner un sens à $T_{m,v}M_v$, même quand m n'est pas dominé par $\exp\left(-C'\sum |z_j|^2\right)$).

L'application ϕ qui à m associe $m\left(\frac{1}{i}J_v^{-1}\,\mathrm{grad}\right)$ est une application linéaire bornée de H_α dans l'espace des opérateurs bornés sur L^2, qui a les propriétés suivantes:

(i) si m_1 et m_2 sont dans H_α, alors

$$\phi(m_1m_2) = \phi(m_1)\phi(m_2);$$

(ii) si $(m_j)_{j\in\mathbb{N}}$ est une suite dans H_α, avec $\|m_j\| \leqslant C$ et qui converge vers $m \in H_\alpha$ uniformément sur tout compact de S_α, alors $\lim_{j\to\infty} \phi(m_j) = \phi(m)$ pour la topologie forte des opérateurs;

(iii) si $|m(z)| \leqslant C\exp(-C'|z_i|^2)$ pour un C et un $C' > 0$, alors $\phi(m)$ est l'opérateur donné par intégration contre le noyau $L(v(x) - v(y)) \det J_v(y)$, où L est la fonction entière définie sur $\mathbb{C}^d$ par

$$L(z) = \int_{\mathbb{R}^d} \exp\left(2\pi i \sum_{j=1}^{d} z_j \xi_j\right) m(\xi)\, d\xi.$$

Notons que les propriétés (ii) et (iii) déterminent uniquement le calcul fonctionnel qui à m associe $m\left(\frac{1}{i} J_v^{-1}\,\mathrm{grad}\right)$. La propriété (iii) est vraie par définition; la partie (ii) ainsi que le fait que $\phi(m)$ est un opérateur borné découlent du théorème. La propriété d'homomorphisme est vraie quand v est réelle, car alors $\phi(m) = V_h T_h^{-1}$, où $T = m\left(\frac{1}{i}\mathrm{grad}\right)$ est le multiplicateur de symbole m; elle reste vraie pour v complexe par prolongement analytique.

Nous allons formuler ce résultat de façon un peu différente.

Théorème 7. *Soit $(L_j)_{1\leq j\leq d}$ un système de d champs de vecteurs à coefficients bornés, continûment différentiables et à coefficients dans $\mathbb{C}^d$. Soient $(a_{j,k})_{1\leq k\leq d}$ les coefficients de L_j, de sorte que $L_j = \sum_k a_{i,k}(\partial/\partial x_k)$. On suppose que les champs L_j commutent deux à deux, et que la matrice $((a_{j,k}))$ possède un inverse $B = ((b_{j,k}))$ uniformément bornée et se factorisant sous la forme $B = (I + iU)V$, où U et V sont des matrices réelles, et $\|U\| \leqslant \eta$ pour un certain $\eta \in]0, 1[$. Alors, pour tout $\alpha \in]\eta^2, 1[$, on a un calcul fonctionnel $m \xrightarrow{\phi} m\left(\frac{1}{i}L_1, \ldots, \frac{1}{i}L_d\right)$, défini sur H_α, et à valeurs dans les opérateurs bornés sur $L^2(\mathbb{R}^d)$.*

Nous allons montrer que, sous les hypothèses du théorème, les L_j proviennent d'une application lipschitzienne $v: \mathbb{R}^d \to \mathbb{C}^d$, telle que $J_v \in \mathfrak{U}_{\gamma,\epsilon}$. Par calcul fonctionnel, nous entendons une fonction ϕ, qui vérifie les conditions (i), (ii) et (iii) ci-dessus. Le Théorème 7 sera donc une conséquence de ce qui a éte dit plus haut et du lemme suivant.

Lemme 8.2. *Si les L_j sont comme dans le Théorème 7, il existe $v: \mathbb{R}^d \to \mathbb{C}^d$, de classe C^2, et $\epsilon > 0$ tels que, pour tout $\gamma > \eta^2$, $J_v \in \mathfrak{U}_{\gamma,\epsilon}$ et $J_v^{-1}(x) = ((a_{j,k}(x)))$ pour tout x.*

Monstrons d'abord qu'il existe $v: \mathbb{R}^d \to \mathbb{C}^d$ tel que $J_v^{-1} = ((a_{j,k}))$. Soit $((b_{j,k})) = ((a_{j,k}))^{-1}$. Pour conclure que $((b_{j,k}))$ est une matrice jacobienne, il nous faut montrer que $(\partial b_{j,k}/\partial x_l) = (\partial b_{l,k}/x_j)$ pour tous j, k, l.

Les relations $L_i L_j = L_j L_i$ entraînent, en identifiant les coefficients $\partial/\partial x_l$ dans les produits $L_i L_j$ et $L_j L_i$, que $L_i a_{j,l} = L_j a_{i,l}$ pour tous i, j, l. Autrement dit,

$$\sum_m a_{i,m} \frac{\partial}{\partial x_m} a_{j,l} = \sum_n a_{j,n} \frac{\partial}{\partial x_n} a_{i,l}.$$

On considère ces égalités (pour $i = 1, \ldots, d$ et (j, l) fixé) comme un système d'équations en les $(\partial/\partial x_m) a_{j,l}$, que l'on résoud en inversant la matrice $((a_{i,m}))$. On obtient

$$\frac{\partial}{\partial x_m} a_{j,l} = \sum_i b_{m,i} \sum_n a_{j,n} \frac{\partial}{\partial x_n} a_{i,l},$$

pour tous m, j, l.

On résoud maintenant les équations

$$\sum_n a_{j,n} \left(\sum_i b_{m,i} \frac{\partial}{\partial x_n} a_{i,l} \right) = \frac{\partial}{\partial x_m} a_{j,l}$$

pour $1 \leqslant j \leqslant n$ en inversant la matrice $((a_{j,n}))$. On obtient

$$\sum_i b_{m,i} \frac{\partial}{\partial x_n} a_{i,l} = \sum_j b_{n,j} \frac{\partial}{\partial x_m} a_{j,l}$$

pour tous m, n, l.

Comme $\sum_i b_{m,i} a_{i,l}$ et $\sum_j b_{n,j} a_{j,l}$ sont constants, on obtient

$$-\sum_i \left(\frac{\partial}{\partial x_m} b_{m,i} \right) a_{i,l} = -\sum_j \left(\frac{\partial}{\partial x_m} b_{n,j} \right) a_{j,l}$$

pour tous m, n, l. La matrice $((a_{i,l}))$ étant inversible, cela entraîne que $(\partial/\partial x_n) b_{m,j} = (\partial/\partial x_m) b_{n,j}$ pour touts j, m, et b. Donc la matrice $((b_{j,k}))$ est bien, localement la matrice jacobienne d'une fonction v. Par simple connexité, on peut trouver une fonction v de classe C^2 telle que $((b_{j,k})) = J_v$.

Il nous reste à montrer que l'on peut trouver $\epsilon > 0$ tel que $J_v \in \mathfrak{U}_{\gamma,\epsilon}$ pour tout $\gamma \in]\eta^2, \alpha[$. Par hypothèse, on peut écrire $J_v = (I + iU)V$, où $V = \operatorname{Re} J_v$ est une matrice bornée ainsi que son inverse.

Lemme 8.3. *L'application $\mathbb{R}^d \to \mathbb{R}^d$ qui à x associe $h(x) = \operatorname{Re} v(x)$ est un difféomorphisme bi-lipschitzien de $\mathbb{R}^d$.*

Admettons que nous ayons démontré le lemme. La matrice jacobienne de l'application qui à x associe $\operatorname{Im} v(h^{-1}(x))$ est $UV(h^{-1}(x))V^{-1}(h^{-1}(x)) = U(h^{-1}(x))$. Par conséquent,

$$\sum_i |\operatorname{Im} v_i(h^{-1}(x) - \operatorname{Im} v_i(h^{-1}(y))|^2 \leqslant \eta^2 \sum_i |x_i - y_i|^2$$

pour tout $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. On en déduit en composant avec h que

$$\sum_i |\operatorname{Im} v_i(s) - \operatorname{Im} v_i(t)|^2 \leqslant \eta^2 \sum_i |\operatorname{Re} v_i(s) - \operatorname{Re} v_i(t)|^2$$

pour s, $t \in \mathbb{R}^d$. Comme $h = \operatorname{Re} v$ est bi-lipschitzienne, il s'ensuit que J_v est dans $\mathfrak{U}_{\gamma,\epsilon}$ pour tout $\gamma \in]\eta^2, \alpha[$ et un certain $\epsilon > 0$.

La démonstration du Lemme 8.3 nous a été communiquée par J. C. Sikorav. L'application h est, localement, un difféomorphisme bi-lipschitzien car $V = J_h$ est borné ainsi que son inverse. Nous devons montrer que le résultat global en découle. Soit $\gamma: [0, 1] \to \mathbb{R}^d$ un arc rectifiable tel que $\gamma(0)$ soit dans $h(\mathbb{R}^d)$. Montrons qu'il existe un arc rectifiable $\tilde{\gamma}: [0, 1] \to \mathbb{R}^d$, tel que $\gamma(t) = h(\tilde{\gamma}t))$ pour $0 \leqslant t \leqslant 1$. En effet, si l'on peut trouver une arc continu $\tilde{\gamma}$, défini sur $[0, t_0[$ pour un $t_0 < 1$, et tel que $\gamma(t) = h(\tilde{\gamma}(t))$ pour $t < t_0$, alors $\tilde{\gamma}$ est rectifiable, et sa longueur d'arc est inférieure à une constante fois la longueur de l'arc γ. Par conséquent, $\lim_{t \to t_0} \tilde{\gamma}(t)$ existe, et le résultat local permet de prolonger $\tilde{\gamma}$ à un intervalle $[0, t_0 + \epsilon]$. Par compacité, on peut définir $\tilde{\gamma}$ sur $[0, 1]$. De plus, l'arc $\tilde{\gamma}$ est une fonction continue de γ si, par exemple, $\gamma(0)$ et $\tilde{\gamma}(0)$ sont deux points fixes. On en déduit que h est surjective. De plus, si h n'était pas injective, on pourrait trouver un arc de $\mathbb{R}^d$, joignant deux points distincts A et B, dont l'image par h soit une boucle. En déformant cette boucle en un point, et en choisissant un relèvement de chaque boucle intermédiaire qui joigne A et B, on obtiendrait des arcs joignant A et B, et dont la longueaur tendrait vers 0. Donc, h est bijective, et comme l'image d'un arc est un arc de longueur comparable, il s'ensuit que h est bi-lipschitzienne.

Signalons pour conclure que l'hypothèse faite dans le Théorème 6 sur la matrice $((a_{j,k}))$, qui est invariante par changement de varible bi-lipschitzien de $\mathbb{R}^d$, est vérifié automatiquement dès que $\|I - ((a_{j,k}))\| < \eta'$ pour un $\eta' < \frac{1}{3}$. En effect, dans ce cas $\|I - ((b_{j,k}))\| < \eta''$ avec $\eta'' < \frac{1}{2}$, d'où il découle que $\|((\operatorname{Im} b_{j,k}))\| < \eta''$ et $\|((\operatorname{Re} b_{jk}))^{-1}\| < 1/(1 - \eta'')$. ce qui entraîne l'inégalité $\|\mathfrak{U}\| \leqslant \eta$ avec $\eta = \eta''/(1 - \eta'') < 1$.

9. Applications a l'interpolation

La démonstration du Théorème $T1$ dans le cas où $T1 = {}^tT1 = 0$ esquissée au paragraphe 2 montre le lien entre les critères généraux de continuité sur L^2

d'opérateurs d'intégrale singulière et l'interpolation. Un autre exemple de ce lien est donné par le théorème suivant, démontré par P. G. Lemarié.

Pour $0 < s < d/2$, on note B^s l'espace de Sobolev d'exposant s, c'est-à-dire le complété de $C_0^\infty(\mathbb{R}^d)$ pour la norme $\|f\|_{B^s} = \left(\int_{\mathbb{R}^d} |f(\xi)|^2 |\xi|^{2s}\, d\xi\right)^{1/2}$.

Théorème 8 [L2] [M]. *Soit $T = C_0^\infty(\mathbb{R}^d) \to C_0^\infty(\mathbb{R}^d)'$ un opérateur linéaire faiblement borné, associé à un noyau K vérifiant* (1.1) *et*

$$|K(x,y) - K(x',y)| \leqslant \frac{C|x'-x|^\delta}{|x-y|^{d+\delta}} \tag{9.1}$$

pour $|x'-x| < |x-y|/2$. Si de plus $T1 = 0$, alors T s'étend en un opérateur continu sur B pour $0 < s < \delta$.

Nous noterons dans la suite $[X, Y]_\theta$, $0 \leqslant \theta \leqslant 1$, les espaces d'interpolation complexe entre deux espaces X et Y.

Le Théorème $T1$, dans le cas où $T1 = {}^tT1 = 0$, se déduit du Théorème 8 par dualité et interpolation, en utilisant le fait que L^2 est l'espace d'interpolation complexe $[B^s, (B^s)')]_{1/2}$.

Le Théorème de McIntosh et Meyer cité au paragraphe 2 a été démontré de la même manière, à ceci près qu'on a besoin d'un résultat d'interpolation plus difficile, à savoir que $L^2 = [bB^s, (B^s)']_{1/2}$, où b est une fonction accrétive [CMM1], [KM].

Réciproquement, on peut utiliser la démonstration du Théorème Tb pour prouver des résultats d'interpolation. En aprticulier, nous verrons que si b est para-accrétive, $[bB^s, (B^s)']_{1/2} = L^2$. Nous verrons également que la para-accrétivité est une condition nécesaire pour que ce résultat d'interpolation ait lieu. Ceci suggère que l'intérêt de cette classe de fonctions existe peut-être indépendamment de la théorie des intégrales singulières.

Rappelons maintenant la définition des espaces de Sobolev L^p_α. Soit $\mathcal{S}(\mathbb{R}^d)$ la classe Schwartz. Pour $\alpha > 0$, Δ^α est défini sur $\mathcal{S}(\mathbb{R}^d)$ par $\Delta^\alpha f(\xi) = |\xi|^{2\alpha} f(\xi)$.

La définition de $\Delta^{-\alpha}$ nécessite quelques précautions. On choisit l'entier $k = 2\alpha - 1$ pour 2α entier, et $k = [2\alpha]$ autrement. Soit $\mathcal{P}_k$ l'espace vectorial des polynômes de degré $\leqslant k$, et soit

$$\begin{aligned}\mathcal{S}_k &= \left\{f \in \mathcal{S}, \int \hat{f}(x) x^\beta\, dx = 0 \quad \text{pour tout multi-indice } \beta \text{ tel que } |\beta| \leqslant k\right\}\\ &= \{f \in \mathcal{S}, D^\beta f(0) = 0 \quad \text{pour tout } \beta \text{ tel que } |\beta| \leqslant k\}.\end{aligned}$$

L'operateur $\Delta^{-\alpha}$ peut être défini sur $\mathcal{S}_k$, et $\Delta^{-\alpha}(\mathcal{S}_k)$ est composé de fonctions C^∞ qui décroissent à l'infini comme $|x|^{-d-(k+1)+2\alpha}$, donc au moins comme $|x|^{-d}$. Par dualité, on peut définit $\Delta^{-\alpha}$ de $L^1((1+|x|)^{-d}dx)$ dans $(\mathcal{S}_k)' = \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}_k$. en particulier, on peut définir $\Delta^{-\alpha}$ sur L^p, $1 < p < +\infty$.

Si $1 < p < +\infty$ et $\alpha \in \mathbb{R}$, on définit l'espace de Sobolev L^p_α par $L^p_\alpha = \Delta^{-\alpha/2}L^p$. Notons que, lorsque $\alpha > 0$, L^p_α est un espace de distributions définies modulo des polynômes.

Les propriétés d'interpolation des L^p_α sont classiques. Nous voulons savoir si elles restent vraies lorsqu'on perturbe les espaces de Sobolev à l'aide de fonctions bornées.

Soit $b \in L^\infty(\mathbb{R}^d)$ telle que $1/b$ soit aussi bornée. On définit des espaces X^p_α par

$$X^p_\alpha = bL^p_\alpha \quad \text{si} \quad \alpha > 0 \tag{9.2}$$

et

$$X^p_\alpha = L^p_\alpha \quad \text{si} \quad \alpha < 0.$$

Quand $\alpha > 0$, X^p_α est défini modulo $b\,\mathcal{P}_k$, où $k = \alpha - 1$ si α est entier et $k = [\alpha]$ autrement. Notons aussi que $X^p_\alpha = b\Delta^{-\alpha/2}L^p$ pour $\alpha \geqslant 0$ et $X^p_\alpha = \Delta^{-\alpha/2}bL^p$ pour $\alpha < 0$. On peut se demander à quelle condition sur b l'espace d'interpolation complexe entre deux X^p_α est un X^p_α. Le théorème suivant permet de répondre à cette question.

Théorème 9. *On se donne $b \in L^\infty_{\mathbb{C}}(\mathbb{R}^d)$, telle que $1/b$ soit aussi bornée. Pour $1 < p_0, p_1 < +\infty, \alpha_0, \alpha_1 \in \mathbb{R}$ et $0 \leqslant \theta \leqslant 1$, on définit p et α par*

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1} \quad \text{et} \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1. \tag{9.3}$$

Si b est para-accrétive, alors, pour tous les choix de p_0, p_1, α_0, α_1, et θ,

$$[X^{p_0}_{\alpha_0}, X^{p_1}_{\alpha_1}]_\theta = X^p_\alpha, \quad \text{avec équivalence de normes.} \tag{9.4}$$

Réciproquement, s'il existe p_0, p_1, α_0, α_1 avec $\alpha_0 > 0$ et $\alpha_1 < 0$ tels que (9.4) *soit satisfaite pour tout $\theta \in [0, 1]$, alors b est para-accrétive.*

Nous nous contentons d'esquisser la démonstration du Théorème 9 (on peut trouver une démonstration plus précise dans [DJS]).

Le théorème est classique lorsque $b = 1$, et (9.4) découle du cas classique lorsque α_0 et α_1 ont le même signe. Le théorème de réitération de T. Wolff [W] nous permet alors de nous restreindre au cas où α_0 et α_1 sont petits et de signes opposés.

L'idée de la démonstration de la partie directe est, comme dans le cas classique, de prouver l'équivalence de la norme de X^p_α avec la norme L^p d'une sorte de fonction d'aire, ce qui permet de réduire l'interpolation des X^p_α à celle des L^p à valeurs vectorielles.

Les fonctions d'aires, au lieu d'être construites à l'aide de l'opérateur de convolution par une fonction ψ_t de moyenne nulle, seront construites à l'aide des opérateurs Δ_k définis au paragraphe 3 (après l'énoncé de la Proposition 2). Les estimations importantes sont les suivantes:

$$\left\| \left\{ \sum_k |\Delta_k f(x)|^2 2^{2\alpha k} \right\}^{1/2} \right\|_{L^p} \leqslant C_{\alpha,p} \| f \|_{X^p_\alpha}; \tag{9.5}$$

(9.6) $\sum_{k=-\infty}^{\infty} M_b \Delta_k M_b \Delta_k^N - I$ tend vers 0 en norme d'opérateur sur X^p_α quand $N \to +\infty$, où l'on a noté $\Delta_k^N = \sum_{j=-N}^{N} \Delta_{k+j}$;

$$\frac{1}{C_{\alpha,p}} \| f \|_{X^p_\alpha} \leqslant \left\| \left\{ \sum_k |\Delta_k f(x)|^2 2^{2\alpha k} \right\}^{1/2} \right\|_{L^p}. \tag{9.7}$$

La démonstration de (9.5), (9.6) et (9.7) est un peu fastidieuse, et nous la passons sous silence. Le cas important est celui où $\alpha = 0$ et $p = 2$, où (9.5) et (9.6) découlent facilement des estimations prouvées aux paragraphes 2 et 4; on en déduit (9.5) et (9.6) dans le cas $\alpha = 0$, $1 < p < +\infty$, puis dans le cas général, en utilisant des arguments d'opérateurs de Calderón-Zygmund à valeurs vectorielles. L'estimation (9.7) est alors une conséquence facile de (9.5) et (9.6) par un argument de dualité.

On note A^p_α l'espace des fonctions mesurables $F(x, k)$, définies sur $\mathbb{R}^d \times \mathbb{Z}$, et telles que $\left\{ \sum |F(x, k)|^2 2^{2\alpha k} \right\}^{1/2}$ soit dans $L^p(\mathbb{R}^d)$. On montre sans peine que $[A^{p_0}_{\alpha_0}, A^{p_1}_{\alpha_1}]_\theta = A^p_\alpha$, ce qui va nous permettre de déduire (9.4) de (9.5), (9.6) et (9.7).

En effet, on peut définir ϕ par

$$F(x, k) \xrightarrow{\phi} f(x) = \sum_{k=-\infty}^{\infty} M_b \Delta_k F(\cdot, k)(x).$$

Comme (9.6) implique que $\sum_k M_b \Delta_k M_b \Delta_k^N$ est inversible sur les X^p_α pour N assez grand et (9.5) implique que $F(x, k) = M_b \Delta_k^N f \in A^p_\alpha$ quand $f \in L^p_\alpha$, on en déduit que ϕ envoie les A^p_α surjectivement dans les X^p_α. Par interpolation, ϕ envoie $[A^{p_0}_{\alpha_0}, A^{p_1}_{\alpha_1}]_\theta = A^p_\alpha$ dans $[X^{p_0}_{\alpha_0}, X^{p_1}_{\alpha_1}]_\theta$, qui contient dont X^p_α.

D'autre part, l'application ψ définie par $f(x) \xrightarrow{\psi} F(x, k) = \Delta_k f(x)$ envoie les X^p_α dans les A^p_α à cause de (9.5), et par conséquent envoie $[X^{p_0}_{\alpha_0}, X^{p_1}_{\alpha_1}]_\theta$ dans $[A^{p_0}_{\alpha_0}, A^{p_1}_{\alpha_1}]_\theta = A^p_\theta$. Soit $f \in [X^{p_0}_{\alpha_0}, X^{p_1}_{\alpha_1}]$; f est dans la somme $X^{p_0}_{\alpha_0} + X^{p_1}_{\alpha_1}$, et de plus $\Delta_k f$ est dans A^p_α. De même $F(x, k) = M_b \Delta_k^N f(x)$ est dans A^p_α et $\phi(F) = \sum_k M_b \Delta_k M_b \Delta_k^N f$ est dans X^p_α, où l'on a choisi N assez grand pour que $\sum_k M_b \Delta_k M_b \Delta_b^N$ soit inversible sur $X^{p_0}_{\alpha_0} + X^{p_1}_{\alpha_1}$ et sur X^p_α. On en déduit que f est dans X^p_α, et donc que $[X^{p_0}_{\alpha_0}, X^{p_1}_{\alpha_1}] \subset X^p_\theta$, ce qui termine la démonstration de la partie directe.

Pour la partie réciproque, on commence par utiliser le théorème de réitération classique pour se réduire au cas où $\alpha_0 > 0 > \alpha_1$ sont tous deux très petits en valeur absolue, ce qui permet de ne pas avoir à définir $X^{p_1}_{\alpha_1}$ modulo des polynômes. Si b n'est pas para-accrétive, on choisit θ tel que α soit 0 (de sorte que $A^p_\alpha = L^p$), et on construit des opérateurs K qui envoient $X^{p_i}_{\alpha_i}$ dans $A^{p_i}_{\alpha_i}$ pour $i = 0$, 1, mais ont une norme aussi grande qu'on veut de X^p_α dans A^p_α. Nous n'entrons pas dans les détails.

Bibliographie

[A] Aguirre, J. Multilinear pseudo-differential operators and paraproducts, Ph. D Thesis, *Washington University*, (1981).

[BDS] Brackx, F., Delanghe, R., et Sommen, F. Clifford analysis, Pitman, (1982).

[C1] Calderón, A. P. Commutators of singular integral operators, *Proc. Nat. Acad. Sci. USA*, **53** (1965), 1092-1099.

[C2] Calderón, A. P. Cauchy integrals of Lipschitz curves and related operators, *Proc. Nat. Acad. Sci. USA*, **74** (1977), 1324-1327.

[CDM] Coifman, R. R., David, G. et Meyer, Y. La solution des conjectures de Calderón, *Adv. in Math.*, **48** (1983), 144-148.

[C De M] Coifman, R. R., Deng, P. G. et Meyer, Y. Domaine de la racine carrée de certains opérateurs différentiels accrétifs, *Ann. Inst. Fourier 33,* **2** (1983), 123-134.

[CMM1] Coifman, R. R., McIntosh, A. et Meyer, Y. L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, *Ann. of Math.*, **116** (1982), 361-388.

[CMM2] Coifman, R. R., McIntosh, A. et Meyer Y. The Hilbert transform on Lipschitz curves, *Proc. of the Centre for Mathematical Analysis, Miniconference on P.D.E.'s*, Canberra, July 9-10, 1981.

[CM1] Coifman, R. R. et Meyer Y. Au-delà des opérateurs pseudo-différentiels, Astérisque 57, S.M.F. (1978).

[CM2] Coifman, R. R. et Meyer, Y. Fourier analysis of multilinear convolutions, Calderón's theorem, and analysis on Lipschitz curves, *Lect. Notes in Math., Springer-Verlag,* **779** (1979), 109-122.

[CW1] Coifman, R. R. et Weiss, G. Analyse harmonique non-commutative sur certains espaces homogènes, *Lect. Notes in Math., Springer Verlag,* **242** (1971).

[CW2] Coifman, R. R. et Weiss, G. Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.*, **83** (1977), 569-645.

[CZ] Calderón, A. P. et Zygmund, A. On the existence of certain singular integrals, *Acta Math.*, **88** (1952), 85-139.

[DJ] David, G. et Journé, J. L. A boundedness criterion for generalized Calderón-Zygmund operators, *Annals of Math.*, **120** (1984), 371-397.

[DJS] David, G., Journé, J. L. et Semmes, S. Calderón-Zygmund operators, para-accretive functions, and interpolation, preprint.

[FJK] Fabes, E., Jerison, D. et Kenig, C. Multilinear Littlewood-Paley estimates with applications to partial differential equations, *Proc. Nat. Acad. Sci. USA*, **79** (1982), 5746-5750.

[FS] Fefferman, C. et Stein, E. H^p-spaces of several variables, *Acta Math.*, **129** (1972), 137-193.

[JN] John, F. et Nirenberg, L. On functions of bounded mean oscillation, *Comm. Pure Appl. Math.*, **14** (1961), 415-426.

[J1] Journé, J. L. Thèse de 3ème cycle, Orsay n.° 2955 (1981).

[J2] Journé, J. L. Calderón-Zygmund operators, pseudo-differential operators, and the Cauchy integral of Calderon, *Lect. Notes in Math., Springer-Verlag*, **994** (1983).

[KM] Kenig, C. et Meyer, Y. Kato's square roots of accretive operators and Cauchy kernels on Lipschitz curves are the same, Mittag-Leffler report n.° 4, (1983).

[KS] Knapp, A. W. et Stein, E. M. Intertwining opeators on semi-simple Lie groups, *Ann. Math.*, **93** (1971), 489-578.

[L1] Lemarié, P. G. Algèbres d'opérateurs et semi-groupes de Poisson sur un espace de nature homogène. Thèse de 3ème cycle, Orsay, 1984.

[L2] Lemarié, P. G. Continuité sur les espaces de Besov des opérateurs définis par des intégrales singulières, à paraître aux *Annales de l'Institut Fourier.*

[MS1] Macias, R. A. et Segovia, C. Lipschitz functions on spaces of homogeneous teype, *Adv. in Math.*, **33** (1979), 257-270.

[MS2] Macias, R. A. et Segovia, C. A decomposition into atoms of distributions on spaces of homogeneous type, *Adv. in Math.*, **33** (1979), 271-309.

[MM] McIntosh, A. et Meyer, Y. Algèbre d'opérateurs définis par des intégrales singulières, à paraître dans les *C. R. Acad. Sci. Paris.*

[M] Meyer, Y. Les nouveaux opérateurs de Calderón-Zygmund, à paraître aux Actes du Colloque Schwartz, Astérisque.

[Mu] Murray, M. A. Ph. D Thesis, *Yale University*, (1983).

[P] Peetre, J. On convolution operators leaving $L^{p,\lambda}$ spaces invariant, *Ann. Math. Pura Appl.*, **72** (1966), 295-304.

[S] Stein, E. M. Singular integrals and differentiability properties of functions, *Princeton University Press*, 1970.

[W] Wolff, T. A note on interpolation spaces, *Lect. Notes in Math.*, **908** (1981), 199-204. Harmonic analysis (edited by F. Ricci and G. Weiss), *Springer-Verlag.*

Guy David
Centre de Mathématiques
Ecole Polytechnique
91128 Palaiseau CEDEX France

Jean-Lin Journé
Université de Strasbourg
I.R.M.A.
7 Rue Descartes
67 Strasbourg, France

Stephen Semmes
Yale University Box 2155
New Haven, CT 06520
U.S.A.

Revista Matemática Iberoamericana
Vol. 1, n.° 4, 1985

Oscillations of Anharmonic Fourier Series and the Wave Equation

A. Haraux V. Komornik

Introduction

Let Ω be a bounded domain in R^n, $n \geqslant 1$ and consider the usual wave equation with Dirichlet boundary conditions on $\Gamma = \partial\Omega$

$$(1) \qquad \begin{cases} u_{tt} - \Delta u = 0, & (t,x) \in R \times \Omega \\ u_{|\Gamma} = 0, & t \in R. \end{cases}$$

It is well-known that for any «initial data» $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists one and only one solution u of (1) in the functional class $C(R, H_0^1(\Omega)) \cap C^1(R, L^2(\Omega))$ such that $u(0,x) = u_0(x)$ and $u_t(0,x) = v_0(x)$. Moreover, for any such data, the vector $U(t) = (u(t,\cdot),\ u_t(t,\cdot))$ is almost periodic as a function: $R \to H_0^1(\Omega) \times L^2(\Omega)$ and the «anharmonic Fourier series» for u is given by the (generally formal) expansion formula

$$(2) \qquad u(t,x) = \sum_{n \geq 0} u_n \cos(t\sqrt{\lambda_n} + \alpha_n)\varphi_n(x)$$

where $(\lambda_n)_{n \geq 1}$ is the sequence of eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$, $\varphi_n(x)$ is an orthonormal (in $L^2(\Omega)$) associated sequence of eigenfunctions, $\{u_n\}$ and $\{\alpha_n\}$ are two sequences of real numbers which can be computed in terms of u_0, v_0 and n.

It is well-known (cf. for example [5]) that formula (2) does not define in general an absolutely convergent series for $x \in \Omega$ fixed. However, formula (2) makes sense pointwise if the initial data (u_0, v_0) lie in $H_0{}^m(\Omega) \times H_0^{m-1}(\Omega)$ with $m > \frac{n}{2}$, for example. In such a case, it becomes reasonable to ask about the behavior of the *sign* of $u(t, x_0)$ on a given interval $J \subset R$. Indeed, since $u(t, x_0)$ is then almost periodic with mean-value equal to 0, it is clear (cf. for example [3]) that $u(t, x_0)$ cannot keep a constant sign on an infinite interval unless $u(t, x_0) \equiv 0$ for $t \in R$.

In case $n = 1$, $\Omega =]0, l[$, $l > 0$, it is immediate that either $u(t, x_0) \equiv 0$, or $u(t, x_0)$ takes both positive and negative values on J as soon as $|J| \geqslant 2l$. This property has been generalized in [2] to a class of semi-linear wave equations.

In case $n > 1$, we know that $u(t, x)$ cannot remain $\geqslant 0$ in Ω for all $t \in J$ with $|J| > \pi/\sqrt{\lambda_1}$ (cf. [2]). However the *local* behavior of $u(t, x)$ is difficult to study already if $n = 2$ and Ω is a rectangle, for the usual wave equation (1).

In this paper, we have collected some partial results on the sign of $u(t, x)$ where u is a (sufficiently regular) solution of

$$\text{(3)} \qquad \begin{cases} u_{tt} + (-1)^m \Delta^m u = 0 & (t, x) \in R \times \Omega \\ u_{|\Gamma} = \cdots = \Delta^{m-1} u_{|\Gamma} = 0 & t \in R. \end{cases}$$

These results rely on a study of the sign of almost periodic functions of a special form restricted to a bounded interval J.

1. Construction of positive functions orthogonal to some subspaces of $C([0, T])$

In this section, we consider a linear subspace X of the vector space AP, o of all (continuous) real-valued *almost periodic functions on R with mean-value* 0.

We try to answer the following question: find a function $p \in L^1(0, T)$ $(T > 0)$ such that

$$\text{(1.2)} \qquad \begin{cases} p(t) > 0 & \text{a.e. on} \quad]0, T[\\ \forall f \in X, & \int_0^T p(t) f(t)\, dt = 0 \end{cases}$$

Our motivation for doing this is the following.

Proposition 1.1. *Let $T > 0$ be such that there exists $p \in L^1(0, T)$ satisfying* (1.1) *and* (1.2). *Then for any $f \in X$ we have the following alternative*

(*a*) either $f(t) = 0$, $\forall t \in [0, T]$;

(*b*) or there exists t_1, t_2 in $[0, T]$ with $f(t_1) > 0$ and $f(t_2) < 0$.

PROOF. Assume for example that $f(t) \geqslant 0$ on $[0, T]$. Then from (1.1) and (1.2) we deduce $p(t)f(t) = 0$ a.e. on $]0, T[$. Since $p(t) \neq 0$ a.e. on $]0, T[$, we conclude that $f(t) = 0$, $\forall t \in [0, T]$. □

The following simple result, although it will not be used in this paper, seems to be interesting in itself.

Proposition 1.2. *Assume that* $\dim(X) < +\infty$. *Then there exists* T_0 *such that for all* $T \geqslant T_0$, *there exists* $p \in C([0, T])$ *satisfying* (1.1) *and* (1.2).

PROOF. Let $\{f_j\}$ $1 \leqslant j \leqslant n$ be a basis of X. We can assume (as a consequence of Schmidt orthogonalization procedure) that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f_i(t) f_j(t)\, dt = \delta_{i,j} \tag{1.3}$$

Let E_T be the vector subspace of $L^2(0, T)$ generated by $\{f_{j|[0,T]}\}_{j \in \{1,\ldots,n\}}$. We denote by v_T the (orthogonal) projection of the constant function 1 on E_T in the Hilbert space $L^2(0, T)$. We have

$$v_T(t) = \sum_{j=1}^{n} v_j(T) f_j(t), \qquad \forall t \in [0, T]$$

and the property: $1 - v_T \in (E_T)^\perp$ yields

$$\int_0^T f_j(t)\, dt = \sum_{i \neq j} v_i(T) \int_0^T f_i(t) f_j(t)\, dt + v_j(T) \int_0^T |f_j(t)|^2\, dt$$

On dividing by $T > 0$:

$$\frac{1}{T} \int_0^T f_j(t)\, dt = \sum_{i \neq j} \left\{ \frac{1}{T} \int_0^T f_i(t) f_j(t)\, dt \right\} v_i(T) + \left\{ \frac{1}{T} \int_0^T |f_j(t)|^2\, dt \right\} v_j(T)$$

Since

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f_j(t)\, dt = 0$$

and as a consequence of the orthonormality conditions (1.3) we deduce

$$\operatorname*{Sup}_{1 \leq j \leq n} |v_j(T)| \to 0 \quad \text{as} \quad T \to +\infty.$$

As an immediate consequence, for all $T \geqslant T(\epsilon)$ we have $\|v_T\|_\infty \leqslant \epsilon$. Hence $p = 1 - v_T$ satisfies (1.2) and $p(t) \in C([0, T])$ with $1 - \epsilon \leqslant p(t) \leqslant 1 + \epsilon$ on $[0, T]$ for all $T \geqslant T(\epsilon)$.

Remarks 1.3. (*a*) It follows from the proof of Proposition 1.2 that $p(t)$ can be taken in the same regularity class as the vector function $F(t) = (f_1(t), \dots, f_n(t))$.

(*b*) For any $T \geqslant T_0$, and any vector $a \in R^n$, the real-valued function $f(t) = a \cdot F(t)$ satisfies the alternative described in Proposition 1.1. It is also possible to show this last result *directly* by working in $H = \mathrm{Vect}(F(R)) \neq \{0\}$. Indeed, if a_k is a sequence of vectors in H with $\|a_k\| = 1$ and $a_k \cdot F(t) \geqslant 0$ on $[0, k]$, any limiting point a of $\{a_k\}$ satisfies $\|a\| = 1$ and $a \cdot F(t) \geqslant 0$ on $R^+ \Rightarrow a \cdot F(t) \equiv 0$ on R and $a \cdot a = 0$, which is absurd.

(*c*) A variant of Hahn-Banach theorem shows that the *converse* of Proposition 1.1 is true if $\dim(X) < +\infty$. However

—If $\dim(X) = +\infty$, the converse is not true in general.

—If we used point (*b*) above to show Proposition 1.2, we would only have found $p \in L^2(0, T)$.

Now let $\tau > 0$ be arbitrary: we define

$$X_\tau = \left\{ u \in C(R), u(t+\tau) = u(t) \quad \text{and} \quad \int_0^\tau u(t)\,dt = 0 \right\}$$

We also set, *by definition* $X_0 = \{0\}$. The main result of this section is the following.

Theorem 1.4. *Let* $\{\tau_j\}_{1 \leq j < +\infty}$ *be a non-increasing sequence of* $\geqslant 0$ *numbers such that*

$$\tau_2 > 0 \quad \text{and} \quad \sum_{j=1}^{\infty} \tau_j = T < +\infty.$$

There exists a function $h\colon R \to R$ *such that*

$$\forall (x, y) \in R \times R, \quad |h(x) - h(y)| \leqslant |x - y| \tag{1.4}$$

$$\forall x \in]0, T[, \quad h(x) > 0 \tag{1.5}$$

$$\forall x \in R \backslash]0, T[, \quad h(x) = 0 \tag{1.6}$$

$$\forall j \in N \backslash \{0\}, \quad \forall \varphi \in X_{\tau_j}, \quad \int_R h(x)\varphi(x)\,dx = 0 \tag{1.7}$$

In addition, we have

$$\forall x \in R, \quad h(T - x) = h(x) \tag{1.8}$$

$$x \leqslant y \leqslant \tfrac{T}{2} \Rightarrow h(x) \leqslant h(y). \tag{1.9}$$

PROOF. We define inductively a sequence of functions $h_n\colon R \to R$ as follows

$$h_1(x) = \begin{cases} \tau_2 & \text{if} \quad x \in]0, \tau_1[\\ 0 & \text{if} \quad x \notin]0, \tau_1[\end{cases} \tag{1.10}$$

$$(1.11)\qquad h_n(x) = \begin{cases} \dfrac{1}{\tau_n}\displaystyle\int_0^{\tau_n} h_{n-1}(x-t)\,dt & \text{if } n \geqslant 2, \quad \tau_n > 0 \\ h_{n-1}(x) & \text{if } n \geqslant 3, \quad \tau_n = 0 \end{cases}$$

Lemma 1.5. *For any $n \in N$, $n \geqslant 2$ the function $h_n(x)$ is such that*

$$(1.12)\qquad \forall (x, y) \in R \times R, \quad |h_n(x) - h_n(y)| \leqslant |x - y|$$

$$(1.13)\qquad \forall x \in \left]0, \sum_1^n \tau_j\right[, \quad h_n(x) > 0$$

$$(1.14)\qquad \forall x \in R \Big\backslash \left]0, \sum_1^n \tau_j\right[, \quad h_n(x) = 0$$

$$(1.15)\qquad \forall \varphi \in \bigcup_1^n X_{\tau_j}, \quad \int_R h_n(x)\varphi(x)\,dx = 0$$

$$(1.16)\qquad \forall x \in R, \quad h_n\left(\sum_1^n \tau_j - x\right) = h_n(x)$$

$$(1.17)\qquad x \leqslant y \leqslant \frac{1}{2}\sum_1^n \tau_j \Rightarrow h_n(x) \leqslant h_n(y)$$

$$(1.18)\qquad \int_R h_n(x)\,dx = \tau_1\tau_2$$

$$(1.19)\qquad 0 < \epsilon \leqslant \frac{1}{2}\sum_1^{n-1} \tau_j \Rightarrow \int_0^{\epsilon} h_n(x)\,dx \geqslant \int_0^{\epsilon - \tau_n} h_{n-1}(x)\,dx$$

PROOF OF LEMMA 1.5. The proofs of (1.12)-(1.18) are by induction on n. The properties (1.13) $\rightarrow$ (1.18) are obviously satisfied for $n = 1$. Property (1.12) is true for $n = 2$, since

$$h_2(x) = \frac{1}{\tau_2}\int_0^{\tau_2} h_1(x-t)\,dt = \frac{1}{\tau_2}\int_{x-\tau_2}^{x} h_1(y)\,dy,$$

hence

$$h_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant \tau_2 \\ \tau_2 & \text{if } \tau_2 \leqslant x \leqslant \tau_1 \\ \tau_1 + \tau_2 - x & \text{if } \tau_1 \leqslant x \leqslant \tau_1 + \tau_2 \\ 0 & \text{if } x > \tau_1 + \tau_2 \end{cases}$$

The inductive argument from $n-1$ to n is trivial if $\tau_n = 0$. If $\tau_n > 0$, we proceed as follows:

$$(1.12)\qquad |h_n(x) - h_n(y)| \leqslant \frac{1}{\tau_n}\int_0^{\tau_n} |h_{n-1}(x-t) - h_{n-1}(y-t)|\,dt \leqslant$$
$$\leqslant \frac{1}{\tau_n}\int_0^{\tau_n} |x-y|\,dt = |x-y|$$

(1.13) and (1.14): obvious from (1.11).

$$(1.15)\qquad \int_R h_n(x)\varphi(x)\,dx = \int_R \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x-t)\,dt\varphi(x)\,dx =$$
$$= \frac{1}{\tau_n}\int_0^{\tau_n}\int_R h_{n-1}(x-t)\varphi(x)\,dx\,dt =$$
$$= \frac{1}{\tau_n}\int_0^{\tau_n}\int_R h_{n-1}(u)\varphi(u+t)\,du\,dt =$$
$$= \frac{1}{\tau_n}\int_R h_{n-1}(u)\left\{\int_0^{\tau_n} \varphi(u+t)\,dt\right\} du.$$

If $\varphi \in \bigcup_1^{n-1} X_{\tau_j}$, then $\int_R h_{n-1}(u)\varphi(u+t)\,du = 0$ for $t \in R$, and we deduce $\int_R h_n(x)\varphi(x)\,dx = 0$.

If $\varphi \in X_{\tau_n}$, then $\int_0^{\tau_n} \varphi(u+t)\,dt = 0$ for $u \in R$, and the result follows by integrating in u.

From now on, we use the notation:

$$T_n = \sum_{j=1}^{n} \tau_j, \qquad \forall n \geqslant 1.$$

$$(1.16)\qquad h_n(T_n - x) = \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(T_n - x - t)\,dt =$$
$$= \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x + t - \tau_n)\,dt =$$
$$= \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x-u)\,du = h_n(x)$$

for all $n \geqslant 2$.

(1.17) This property is obviously true for $n = 2$. If > 2, we remark that

$$\tau_n h_n(x) = \int_{x-\tau_n}^{x} h_{n-1}(y)\,dy, \qquad \forall x \in R \Rightarrow h_n \in C^1(R)$$

and

$$\tau_n h_n'(x) = h_{n-1}(x) - h_{n-1}(x - \tau_n)$$

Hence $h_n' \geqslant 0$ on $\left]-\infty, \frac{1}{2}T_{n-1}\right]$. Moreover, if $x \in \left[\frac{1}{2}T_{n-1}, \frac{1}{2}T_n\right]$, we have $T_{n-1} - x \leqslant \frac{1}{2}T_{n-1}$ and $T_{n-1} - x \geqslant x - \tau_n$, hence $\tau_n h_n'(x) = h_{n-1}(T_{n-1} - x) - h_{n-1}(x - \tau_n) \geqslant 0$. Finally, h_n is non-decreasing on $\left]-\infty, \frac{1}{2}T_n\right]$.

(1.18)
$$\int_R h_n(x)\,dx = \int_R \frac{1}{\tau_n}\int_{x-\tau_n}^{x} h_{n-1}(u)\,du\,dx =$$
$$= \int_R h_{n-1}(u)\left[\int_u^{u+\tau_n} \frac{1}{\tau_n}dx\right]du = \int_R h_{n-1}(u)\,du$$

(1.19) As a consequence of (1.14) and (1.17) we have

$$\int_0^\epsilon h_n(x)\,dx = \int_0^\epsilon \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x-t)\,dt\,dx \geqslant \int_0^\epsilon \frac{1}{\tau_n}\int_0^{\tau_n} h_{n-1}(x-\tau_n)\,dt\,dx =$$
$$= \int_0^\epsilon h_{n-1}(x-\tau_n)\,dx = \int_{-\tau_n}^{\epsilon-\tau_n} h_{n-1}(u)\,du = \int_0^{\epsilon-\tau_n} h_{n-1}(u)\,du. \ \square$$

END OF PROOF OF THEOREM 1.4. If $\tau_{n_0} = 0$ for some $n_0 \geqslant 2$, there is nothing left to prove. If $\tau_n > 0$ for all $n \geqslant 1$, we remark that

$$\forall n \geqslant 2, \quad \forall x \in R,$$
$$|h_n(x) - h_{n-1}(x)| \leqslant \frac{1}{\tau_n}\int_0^{\tau_n} |h_{n-1}(x-t) - h_{n-1}(x)|\,dt \leqslant$$
$$\leqslant \frac{1}{\tau_n}\int_0^{\tau_n} t\,dt = \frac{1}{2}\tau_n.$$

Since $\sum_{n=1}^{\infty} \tau_n < +\infty$, $\{h_n(x)\}$ is a Cauchy sequence in $C_B(R)$.
Let

$$h(x) = \lim_{n \to +\infty} h_n(x), \qquad \forall x \in R.$$

We claim that h satisfies (1.4) $\to$ (1.9). Since the sequence h_n satisfies (1.12), from (1.13) $\to$ (1.18) we deduce easily all the properties required on h *except* (1.5). Now if (1.5) is not satisfied, then for some $\epsilon > 0$ we have $h(x) = 0$ on $[0, \epsilon]$. We pick $m \in N$ large enough so that

$$\epsilon \leqslant \frac{1}{2}\sum_1^m \tau_k \ \ \textit{and} \ \ \sum_{m+1}^{\infty} \tau_k < \epsilon.$$

For all $n \in N$, $n > m$ we find as a consequence of (1.19):

$$\int_0^\epsilon h_n(x)\,dx \geqslant \int_0^{\epsilon - \tau_n} h_{n-1}(x)\,dx \geqslant \cdots \geqslant \int_0^{\epsilon - \sum_{m+1}^{n} \tau_j} h_m(x)\,dx.$$

Hence:

$$\int_0^\epsilon h_n(x)\,dx \geqslant \int_0^{\epsilon - \sum_{m+1}^{\infty} \tau_j} h_m(x)\,dx > 0.$$

By letting $n \to +\infty$ we find $\int_0^\epsilon h(x)\,dx > 0$.

This contradiction with $h \equiv 0$ on $[0, \epsilon]$ shows that in fact $h > 0$ on $]0, T[$.

Corollary 1.6. *Let* $\{\tau_j\}_{1 \leq j < +\infty}$ *be as in the statement of Theorem* 1.4 *and let* $f_j \in X\tau_j$, $\forall j \in \{1, 2, \ldots, \}$ *be such that* $\sum_{j=1}^{+\infty} \|f_j\|_\infty < +\infty$. *We set*

$$f(t) = \sum_{j=1}^{+\infty} f_j(t), \qquad \forall t \in R.$$

Then for any interval $J \subset R$ such that $|J| \geqslant T$, *we have either* $f(t) \equiv 0$ *on* J, *or* $\exists (t_1, t_2)$ *in* J *with* $f(t_1) > 0$ *and* $f(t_2) < 0$.

PROOF. Let X be the closure in $C_B(R)$ of the *algebraic sum* $\sum_{j=1}^{\infty} X\tau_j$. Clearly, $X \subset AP, o$ and X is translation-invariant, i.e. $f(t + \alpha) \in X$ for all $f \in X$, $\alpha \in R$. Assume that $f \in X$ and $f \geqslant 0$ on J with $|J| \geqslant T$. Let $a \in R$ be such that $[a, a + T] \subset J$. Then $g(t) = f(t + a) \in X$.

As a consequence of Proposition 1.1 and Theorem 1.4, we obtain $g \equiv 0$ on $[0, T]$, hence $f \equiv 0$ on $[a, a + T]$. Since a is arbitrary in $[\operatorname{Inf} J, \operatorname{Sup} J - T]$ we conclude that $f \equiv 0$ on J.

2. Oscillation length and pseudo-analyticity measure

Let X be as in section 1. We define three nonnegative numbers, possibly infinite, which play an important role in the study of oscillation properties.

Definition 2.1. *The oscillation* legth *of* X *is the number* $l_1(X) = \inf\{l > 0, \forall a \in R, \forall f \in X, f \geqslant 0$ *on* $[a, a + l] \Rightarrow f \equiv 0$ *on* $[a, a + l]\}$.

The pseudo-analyticity measure of X *is* $l_2(X) = \inf\{l > 0, \forall a \in R, \forall f \in X, f \equiv 0$ *on* $[a, a + l] \Rightarrow f \equiv 0$ *on* $R\}$.

We also define $l_3(X) = \inf\{l > 0, \forall a \in R, \forall f \in X, f \geqslant 0$ *on* $[a, a + l] \Rightarrow$ $\Rightarrow f \equiv 0$ *on* $R\}$.

Proposition 2.2. *We have*

$$l_3(X) = \operatorname{Sup}\{l_1(X), l_2(X)\}$$

PROOF. This is an obvious consequence of the definitions of the numbers $l_i(X)$.

Remark 2.3. If $X = AP, o$ we have $l_1(X) = l_2(X) = +\infty$.

We have $l_1(X) > 0$ as soon as $X \neq \{0\}$.

In contrast with this property of l_1, it is clear that if $X \subset$ {real analytic functions}, then $l_2(X) = 0$.

It is impossible to compare *in general* the values of $l_1(X)$ and $l_2(X)$. Indeed, if $\{0\} \neq X \subset$ {real analytic functions}, we have $0 = l_2(X) < l_1(X)$. On the other hand, it is not difficult to find $f \in AP, o$ such that f is 1-periodic, with $f = 0$ on $[0, 1 - \epsilon]$, $f \neq 0$ (hence $l_2(Rf) \geqslant 1 - \epsilon$) and $f(t)$ takes positive and negative values *in any neighbourhood of* $1 - \epsilon$ *and* 1. Hence if $f(t)$ has a constant sign on some interval J, we must have either $J \subset [m - 1, m - \epsilon]$ or $J \subset [m - \epsilon, m]$ for some $m \in Z$. In particular, if $|J| > \epsilon$ we deduce $f \equiv 0$ on J. This obviously implies that $l_1(RF) \leqslant \epsilon$.

A major result of this section is the following.

Theorem 2.4. *Let* $\{\tau_j\}_{1 \leq j \leq n}$ *be a finite sequence of positive numbers. Then*

$$l_3\left(\sum_{j=1}^{n} X\tau_j\right) \leqslant \sum_{j=1}^{n} \tau_j.$$

PROOF. Let $X = \sum_{j=1}^{n} X\tau_j$. It follows from Corollary 1.6 that $l_1(X) \leqslant \sum_{j=1}^{n} \tau_j$. Hence Theorem 2.4 will be proved as soon as we establish the following lemma.

Lemma 2.5. *Let* $a \in R$ *be arbitrary and* $f \in X$ *be such that* $f \equiv 0$ *on* $J = \left[a, a + \sum_{j=1}^{n} \tau_j\right]$. *Then* $f \equiv 0$ *on* R.

PROOF. By induction on n. The result is obviously true if $n = 1$. Assume that we have the result for $n - 1$ with $n \geqslant 2$. Let $f = \sum_{j=1}^{n} f_j$ with $f_j \in X\tau_j$ and $f \equiv 0$ on J. Then

$$g(t) = f(t + \tau_n) - f(t) = \sum_{j=1}^{n-1} \{f_j(t + \tau_n) - f_j(t)\} \in \sum_{j=1}^{n-1} X_{\tau_j}$$

and $g \equiv 0$ on $J^* = \left[a, a + \sum_{j=1}^{n-1} \tau_j\right]$. By the induction hypothesis, $g \equiv 0 \Rightarrow f$ is τ_n-periodic. The result follows immediately. $\square$

Remark 2.6. In our applications to hyperbolic equations of the second order in t, Lemma 2.5 will not be very useful since the results that we shall obtain will follow by taking each «harmonic oscillation» in a different $X\tau_j$, so that for an infinite number of harmonics we get nothing, while when the harmonics are in finite number we have analycity in t!. Therefore, the following extension of Theorem 2.4 will in fact reveal essential for our purpose.

Theorem 2.7. *Let $\{\tau_j\}_{1 \le j < +\infty}$ be an infinite sequence of positive numbers. We set $Y = \{f \in AP$, o, $\exists \{f_j\}_{j \ge 1}$ such that $\sum_{j=1}^{\infty} \|f_j\|_\infty < +\infty$ and $f(t) = \sum_{j=1}^{\infty} f_j(t)$ on $R\}$. Then:*

$$l_3(Y) \leqslant \sum_{j=1}^{\infty} \tau_j = T.$$

PROOF. If $T = +\infty$, there is nothing to prove. If $T < +\infty$, we know already that $l_1(Y) \leqslant T$ as a consequence of Corollary 1.6. Therefore to have the result it is sufficient to prove the following lemma.

Lemma 2.8. *Let $a \in R$ be arbitrary and $f \in Y$ be such that $f \equiv 0$ on $[a, a + T] = J$. Then $f \equiv 0$ on R.*

PROOF. Since Y is translation-invariant, it suffices to consider the case $a = 0$. let $f(t) = \sum_{j=1}^{\infty} f_j(t)$, $f_j \in X\tau_j$, $\sum_1^{\infty} \|f_j\| < +\infty$. We assume $T < +\infty$ and we set $\epsilon_k = \sum_k^{\infty} \tau_j$. Let $p_k = R \to R$ be a continuous function such that

$$\begin{aligned}
&\operatorname{Supp}(p_k) \subset [0, \epsilon_k] \\
&p_k > 0 \quad \text{on} \quad]0, \epsilon_k[\\
&p_k(\epsilon_k - t) = p_k(t) \\
&\int_0^{\epsilon_k} p_k(s)\varphi(s)\, ds = 0, \qquad \forall \varphi \in \bigcup_k^{\infty} X\tau_j.
\end{aligned}$$

We introduce

$$g_k(t) = \int_0^{\epsilon_k} f(t+s) p_k(s)\, ds \quad \text{for all} \quad t \in R.$$

We have

$$g_k(t) = \int_0^{\epsilon_k} \sum_{j=1}^{\infty} f_j(t+s) p_k(s)\, ds = \sum_{j=1}^{\infty} \int_0^{\epsilon_k} f_j(t+s) p_k(s)\, ds =$$

$$= \sum_{j=1}^{k-1} \int_0^{\epsilon_k} f_j(t+s) p_k(s)\, ds,$$

therefore $g_k \in \sum_1^{k-1} X\tau_j$ for all $k \in N$, $k \geqslant 2$. From $f \equiv 0$ on $[0, T]$ we deduce $g_k = 0$ on $[0, T - \epsilon_k]$, hence as a consequence of Lemma 2.5 (note that $T - \epsilon_k = \sum_1^{k-1} \tau_j$) we have $g_k \equiv 0$ on R. Now let

$$\lambda_k = \int_0^{\epsilon_k} p_k(s)\, ds > 0 \quad \text{and} \quad \mu_k(t) = \frac{1}{\lambda_k} p_k(t), \quad t \in R.$$

Because of the properties of p_k, it is immediate to check that $\mu_k \to \delta_0$, the Dirac mass at 0 for the weak-star topology of $M_B(-1, 1)$ (say) as $k \to +\infty$. We deduce immediatly:

$$\forall t \in R, \qquad \lim_{k \to +\infty} \frac{1}{\lambda_k} g_k(t) = f(t).$$

Since $g_k \equiv 0$, this convergence clearly implies that in fact $f \equiv 0$ on R. Hence the proof of Lemma 2.8 is completed. □

3. Optimality of the results in sections 1 and 2

In this section, X_τ is defined as previously. We also use the following subspaces of X_τ:

$$\hat{X}_{\tau,k} = \left\{ u \in X_\tau,\quad \exists \{u_j\} \in R^k,\quad \exists \{\alpha_j\} \in R^k,\quad u(t) = \sum_{j=1}^{k} u_j \cos\left(2j \frac{\pi t}{\tau} + \alpha_j\right) \right\}$$

$$\hat{X}_\tau = \bigcup_{k=1}^{+\infty} \hat{X}_{\tau,k}.$$

a) On the optimality of Theorem 1.4

It will appear as an easy consequence of the following density result.

Theorem 3.1. *Let $n \geqslant 1$ be an integer, $\tau_1, \ldots, \tau_n$ some positive numbers such that $\tau_j/\tau_i \notin Q$ if $i \neq j$, and T such that $0 < T < \sum_{j=1}^n \tau_j$. Then the restrictions to $[0, T]$ of functions in $\sum_{j=1}^n \hat{X}_{\tau_j}$ are dense in $C([0, T])$.*

PROOF. It has been pointed out to us by Y. Meyer that Theorem 3.1 can be derived as an easy consequence of general results from the theory of meanperiodic functions (cf. [6]). For completeness we will give below a more direct proof based on a density result for the «limiting case» $T = \sum_{j=1}^n \tau_j$.

Theorem 3.2. *Let n and $\{\tau_j\}_{1\leq j\leq n}$ be as in the statement of Theorem 3.1. We denote by $\mathcal{P}_n$ the set of polynomial functions of degree $\leqslant n$. The restrictions to $\left[0, \sum_{j=1}^n \tau_j\right]$ of functions in $\sum_{j=1}^n \hat{X}_{\tau_j} + \mathcal{P}_n$ are dense in $C\left(\left[0, \sum_{j=1}^n \tau_j\right]\right)$.*

PROOF. We rely on the following two simple lemmas concerning the map $\mathcal{C}: C(R) \to C(R)$ defined by $(\mathcal{C}f)(t) = f(t+\tau) - f(t)$. [$\tau > 0$ is given].

Lemma 3.3. $\forall k \geqslant 1$, $\mathcal{C}(\mathcal{P}_k) = \mathcal{P}_{k-1}$.

Lemma 3.4. $\forall \sigma > 0$ *with* $\tau/\sigma \notin Q$, *we have* $\mathcal{C}(\hat{X}_\sigma) = \hat{X}_\sigma$.

The proof of lemma 3.3 is obvious. To prove lemma 3.4, it is sufficient to check that $\forall k \in N$, $k \geqslant 1$, $\mathcal{C}(\hat{X}_{\sigma,k}) = \hat{X}_{\sigma,k}$. But obviously $\mathcal{C}(\hat{X}_{\sigma,k}) \subset \hat{X}_{\sigma,k}$ and if we denote by $\mathcal{C}_k$ the restriction of $\mathcal{C}$ to $\hat{X}_{\sigma,k}$, we have $\mathcal{C}_k^{-1}(0) = \{0\}$ because $\tau/\sigma \notin Q$. Since $\hat{X}_{\sigma,k}$ is finite dimensional, the result of lemma 3.4 is now obvious.

PROOF OF THEOREM 3.2 CONTINUED. We proceed by induction on n.

—For $n = 1$ the result is obviously true since

$$C([0, \tau_1]) = X_{\tau_1} + \mathcal{P}_1$$

and $\hat{X}_{\tau_1}$ is dense in X_{τ_1} for the topology of $C([0, \tau_1])$.

—For $n \geqslant 1$, we consider an arbitrary function $f \in C([0, T])$ with

$$T = \sum_{j=1}^{n} \tau_j.$$

We define

$$\tilde{f}(t) = f(t + \tau_n) - f(t), \qquad \forall t \in [0, T - \tau_n]. \tag{3.1}$$

By the induction hypothesis, for any $\delta > 0$ there exists $\tilde{f}_j \in \hat{X}_{\tau_j}$ for $1 \leqslant j \leqslant n-1$ and $\tilde{p} \in \mathcal{P}_{n-1}$ such that

$$\left\| \tilde{f} - \tilde{p} - \sum_{j=1}^{n-1} \tilde{f}_j \right\|_{C([0, T-\tau_n])} \leqslant \delta. \tag{3.2}$$

As a consequence of lemmas 3.3 and 3.4, we may assume for all j as above

$$\tilde{f}_j(t) = f_j(t + \tau_n) - f_j(t), \qquad \forall t \in R; \quad f_j \in \hat{X}_{\tau_j} \tag{3.3}$$

$$\tilde{p}(t) = p(t + \tau_n) - p(t), \qquad \forall t \in R; \quad p \in \mathcal{P}_n. \tag{3.4}$$

Also by the case $n = 1$ we can find $f_n \in \hat{X}_{\tau_n}$ and $q \in \mathcal{P}_1$ such that

$$\left\| f - (p + q) - \sum_{j=1}^{n} f_j \right\|_{C([0, \tau_n])} \leqslant \delta. \tag{3.5}$$

Clearly, $\tilde{q}$ is a constant and we have

$$|\tilde{q}(0)| \leqslant 2\delta + \left| \left(\tilde{f} - \tilde{p} - \sum_{j=1}^{n-1} f_j \right)(0) \right| \leqslant 3\delta.$$

Finally, let

$$h: = f - (p + q) - \sum_{j=1}^{n} f_j \quad \text{on} \quad [0, T].$$

Then we have

$$\| h \|_{C([0, \tau_n])} \leqslant \delta \tag{3.6}$$
$$|h(t + \tau_n) - h(t)| \leqslant 4\delta, \qquad \forall t \in [0, T - \tau_n]. \tag{3.7}$$

From (3.6) and (3.7) it is immediate to deduce

$$\left\| f - \sum_{j=1}^{n} f_j - (p + q) \right|_{C([0, T])} \leqslant \left(1 + \frac{4T}{\tau_n} \right) \delta. \tag{3.8}$$

Since δ can be taken arbitrarily small and $f_j \in X_{\tau_j}$, $p + q \in \mathcal{P}_n$, the induction step is achieved, and the proof of Theorem 3.2 is completed. □

PROOF OF THEOREM 3.1. Let $\mathfrak{M}$ be the space of bounded measures on R which are supported by $\left[0, \sum_{j=1}^{n} \tau_j\right]$ and consider

$$Z = \left\{ \mu \in \mathfrak{M}, \quad \forall f \in \sum_{j=1}^{n} \hat{X}_{\tau_j}, \quad \mu(f) = 0 \right\}.$$

As a consequence of Theorem 3.2, we have

$$\dim(Z) \leqslant n + 1 < +\infty.$$

Let now $0 < T < \sum_{j=1}^{n} \tau_j$ and consider a bounded measure ν on R with $\operatorname{supp}(\nu) \subset [0, T]$, such that $\nu \in Z$.

For $a \in \left[0, \sum_{j=1}^{n} \tau_j - T\right]$, the translated measure $\nu(\,\cdot\, + a) = \nu_a$ is also in Z. On the other hand, if $\nu \neq 0$, it is obvious to show, by looking at the supports, that all the measures ν_a are linearly independent. Since Z is finite-dimensional, we *must* have $\nu = 0$. Hence for $0 < T < \sum_{j=1}^{n} \hat{X}_{\tau_j}$, we obtain that $\sum_{j=1}^{n} \hat{X}_{\tau_j}$ is dense in $C([0, T])$.

This density result immediately implies the following. □

Corollary 3.3. *If all the τ_j such that $\tau_j \neq 0$ are pairwise incommensurable, the result of Theorem* 1.4 *and Corollary* 1.6 *are optimal in the sense that the number $T = \sum_{j=1}^{\infty} \tau_j$ cannot be replaced by any number $T' < T$.*

b) On the optimal character of Theorem 2.4 *when $n = 2$*

The result of Theorem 2.4 (just like Theorem 1.4) is not optimal *in general* for $n = 2$. Indeed, if $\tau_1 \in N\tau_2$, the conclusions of both Theorem 1.4 and 2.4 are still valid with $T = \tau_1 + \tau_2$ replaced by $T' = \tau_1 < T$. On the other hand, the following result shows that Theorem 2.4 is optimal when $n = 2$ and $\tau_1/\tau_2 \notin Q$.

Theorem 3.4. *Let $0 < \tau_2 < \tau_1$ with $\tau_2/\tau_1 \notin Q$, and $0 < T < \tau_1 + \tau_2$. Then there exists $u_1 \in X_{\tau_1}$ and $u_2 \in X_{\tau_2}$ such that $u_1 \neq 0$ and $u_1 + u_2 \equiv 0$ on $[0, T]$.*

The proof of Theorem 3.4 relies on the following.

Lemma 3.5. *Let J be a closed interval such that $|J| < \tau_2$. For any $p \in N$, there exists a* finite *set $F \subset Z$, such that $[-p, p] \cap Z \subset F$ and having the following property: setting $X = F\tau_1 + Z\tau_2$, for all $t \in J$ we have*

$$t \in X \Leftrightarrow t + \tau_1 \in X. \tag{3.9}$$

PROOF. Since $\tau_2/\tau_1 \notin Q$, $N\tau_1 - N\tau_2$ and $N\tau_2 - N\tau_1$ are everywhere dense in R. We set $J = [a, b]$.

1. There exists l, s in N with $l > p$ and such that $b - \tau_2 < l\tau_1 - s\tau_2 < a$. As a consequence, for any $m \in Z$ we have either $m > -s$ and $l\tau_1 + m\tau_2 > b$, or $m \leqslant -s$ and $l\tau_1 + m\tau_2 < a$.
   This implies in particular $(l\tau_1 + Z\tau_2) \cap J = \emptyset$.
2. There exists k, r in N with $k > p$ and such that
   $$b - \tau_2 + \tau_1 < -k\tau_1 + r\tau_2 < a + \tau_1.$$
   This implies $(-k\tau_1 + Z\tau_2) \cap (J + \tau_1) = \emptyset$.
3. We consider $F = \{-k, -k+1, \ldots, l\}$.
   —If $t \in J$ and $t = n\tau_1 + m\tau_2$ with $-k \leqslant n \leqslant l$, then we have in fact $n \leqslant l - 1$, hence $t + \tau_1 \in X$.
   —If $t \in J$ and $t + \tau_1 = n\tau_1 + m\tau_2$ with $-k \leqslant n \leqslant l$, then in fact $n \geqslant -k + 1$, hence $t \in X$. $\square$

PROOF OF THEOREM 3.4. The result is obvious if $T \leqslant \tau_1$. If $T > \tau_1$, we fix $\delta > 0$ small enough so that $|J| < \tau_2$ with $J = [-\delta, T - \tau_1 + \delta]$.

Let X be as in Lemma 3.6: then

$$\alpha = \operatorname{Inf}\{|x-y|, x \in X, y \in X, x \neq y\} > 0$$

We choose ρ such that $0 < \rho < \frac{1}{2}\inf\{\alpha, \delta\}$ and a function $\varphi \not\equiv 0$, $\varphi \in \mathfrak{D}\,(]0, +\infty[)$ such that $\operatorname{Supp}(\varphi) \subset [0, \rho]$.

Let $w(t) = \varphi(\operatorname{dist}(t, X))$, $\forall t \in R$. Clearly, $w \in C^\infty(R) \cap Y_{\tau_2}$, and $w \not\equiv 0$, since the function $\operatorname{dist}(t, X)$ takes at least all values of $[0, \rho]$ as t ranges over R. We now show that $w(t + \tau_1) = w(t)$ for all $t \in [0, T - \tau_1]$. Indeed:

(*a*) If $\operatorname{dist}(t, X) \geqslant \rho$, we cannot have $\operatorname{dist}(t + \tau_1, X) < \rho$: assuming this inequality, since $\rho < \frac{1}{2}\alpha$ there would exist a unique point $x \in X$ such that $|x - (t + \tau_1)| < \rho$. Because $\rho < \delta$, we deduce $x - \tau_1 \in J \cap (X - \tau_1) \Rightarrow x - \tau_1 \in X$, a contradiction since $|t - (t - \tau_1)| < \rho$. Hence we *must* have $\operatorname{dist}(t + \tau_1, X) \geqslant \rho$. In this case we have $w(t) = w(t + \tau_1) = 0$.

(*b*) If $\operatorname{dist}(t, X) < \rho$, let $x \in X$ be such that $|x - t| = \operatorname{dist}(t, X)$. Because $t \in [0, T - \tau_1]$ we have $x \in J$, hence $x \in J \cap X \Rightarrow x + \tau_1 \in X$. Now $\operatorname{dist}(t + \tau_1, X) = |x - t|$, because there is at most one point $y \in X$ such that $|t + \tau_1 - y| \leqslant \rho$, and $y = x + \tau_1$ precisely fulfills this condition with $|t + \tau_1 - y| = |x - t|$.

We conclude that $\operatorname{dist}(t + \tau_1, X) = \operatorname{dist}(t, X) \Rightarrow w(t + \tau_1, X) = w(t, X)$. Finally, let $u(t) = w'(t)$: we have $u \in X_{\tau_2}$ and $u \not\equiv 0$ since w is not constant. We finally have $u + v \equiv 0$ on $[0, T]$ where $v(t)$ is the unique τ_1-periodic function such that $v(t) = -w'(t)$ on $[0, T]$. Clearly $v \in X_{\tau_1}$, hence the proof of Theorem 3.4 is completed. □

4. Applications to some hyperbolic problems of the second order with respect to t

a) *An abstract oscillation theorem*

Let H be a real Hilbert space and $A: D(A) \subset H \to H$ be a (possibly unbounded) linear operator such that $A = A^* \geqslant 0$ and A is strongly positive with A^{-1} *compact*.

If we set $V = D(A^{1/2})$, it is well-known that for any $(u_0, v_0) \in V \times H = \mathcal{H}$, the abstract second-order equation

$$u'' + Au(t) = 0, \qquad t \in R \tag{4.1}$$

has a unique solution $u \in C(R, V) \cap C^1(R, H)$ such that $u(0) = u_0$ and $u'(0) = v_0$.

Moreover, the equation (4.1) generates a group of isometries $T(t)$ on $\mathcal{H}$ endowed with the norm $\|(u, v)\|_{\mathcal{H}} = \{|A^{1/2}u|_H^2 + |v|_H^2\}^{1/2}$.

Therefore, for all $(u_0, v_0) \in \mathcal{H}$, the function $t \to (u(t), u'(t)) \in \mathcal{H}$ is almost periodic. (Cf. for example [4], lecture 24, Proposition 9.) On the other hand u' is bounded: $R \to V'$ and it follows that $u(t) = -A^{-1}u''(t)$ has mean-value 0 in V.

As a consequence, for any $\zeta \in V'$, the function $t \to <\zeta, u(t)>$ cannot remain nonnegative on an infinite interval except if $<\zeta, u(t)> \equiv 0$ on R. The results of section 1 now allow us to state a more precise property.

Theorem 4.1. *In addition to the above hypotheses on A, assume that the eigenvalues of A on H, denoted by $\{\lambda_n\}_{n \in N \setminus \{0\}}$ and repeated according to their multiplicity order, are such that*

$$\sum_{n=1}^{+\infty} \frac{2\pi}{\sqrt{\lambda_n}} = T < +\infty. \tag{4.2}$$

Then, for any $\zeta \in [D(A^{1/4})]'$ and any solution u of (4.1), we have the following alternative: either, $<\zeta, u(t)> \equiv 0$ on R, or for any interval J of R with $|J| \geqslant T$, there exists τ_1 and τ_2 in J such that

$$<\zeta, u(\tau_1)> > 0 \quad \textit{and} \quad <\zeta, u(t_2)> < 0.$$

PROOF. Let $\{\varphi_n\}_{n \geq 1}$ be an orthonormal (in H) sequence of eigenfunctions relative to $\{\lambda_n\}_{n \geq 1}$.

We set $\tau_n = 2\pi/\sqrt{\lambda_n}$ for $n \geqslant 1$, and we consider first the case where

$$u_0 = \sum_{j=1}^{k} u_j \varphi_j, \qquad v_0 = \sum_{j=1}^{k} v_j \varphi_j$$

In this case, $u(t)$ is given by

$$u(t) = \sum_{j=1}^{k} \left\{ u_j \cos(\sqrt{\lambda_j}\, t) + \frac{v_j}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) \right\} \varphi_j$$

Hence for any $\zeta \in V'$, $<\zeta, u(t)> \in \sum_{j=1}^{k} X_{\tau_j}$.

We claim that *in general*, the series defining $u(t)$ is in fact absolutely convergent in $D(A^{1/4})$: as a consequence we shall have, for any $\zeta \in [D(A^{1/4})]'$. $<\zeta, u(t)> = f(t) \in Y = \{f \in AP,\ \exists f_n \in X_{\tau_n}$ such that $\sum_{n=1}^{\infty} \|f_n\|_\infty < +\infty$ and $f = \sum_{n=1}^{\infty} f_n\}$. Indeed, let $u_0 = \sum_{j=1}^{\infty} u_j \varphi_j$, $v_0 = \sum_{j=1}^{n} v_j \varphi_j$ be the Fourier expansions of the initial data (u_0, v_0) and

$$u_0^n = \sum_{j=1}^{n} u_j \varphi_j, \qquad v_0^n = \sum_{j=1}^{n} v_j \varphi_j.$$

It is clear that

$$u(t) = \lim_{n \to +\infty} \sum_{j=1}^{n} \left\{ u_j \cos(\sqrt{\lambda_j}\, t) + \frac{v_j}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) \right\} \varphi_j$$

in $C_B(R, V)$.

Also we have

$$\sum_{j=1}^{\infty} \left\| \left\{ u_j \cos(\sqrt{\lambda_j}\, t) + \frac{v_j}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) \right\} \varphi_j \right\|_{D(A^{1/4})} \leqslant \sum_{j=1}^{\infty} \left\{ |u_j| + \frac{|v_j|}{\sqrt{\lambda_j}} \right\} \| \varphi_j \|_{D(A^{1/4})} =$$

$$= \sum_{j=1}^{\infty} \{ \lambda_j^{1/4} |u_j| + \lambda_j^{-1/4} |v_j| \} \leqslant$$

$$\leqslant \left(\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \right)^{1/2} \left(\sum_{j=1}^{\infty} \lambda_j |u_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \right)^{1/2} \left(\sum_{j=1}^{\infty} |v_j|^2 \right)^{1/2} =$$

$$= \left(\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \right)^{1/2} (\| u_0 \|_V + \| v_0 \|_H) < +\infty.$$

Hence the claim is proved and for any $\zeta \in [D(A^{1/4})]'$ we have $<\zeta, u(t)> \in Y$ as explained above. The conclusion of Theorem 4.1 is now an immediate consequence of Theorem 2.7.

b) Examples of application

Let us start with a one-dimensional case.

Example 4.2. Let $\Omega =]0, l[$, $l > 0$ and $h(x) \in L^\infty(\Omega)$. We consider the equation

$$(4.3) \qquad \begin{cases} u_{tt} + u_{xxxx} + h(x)u = 0, & t \in R, x \in \Omega \\ u(t,0) = u(t,l) = u_{xx}(t,0) = u_{xx}(t,l) = 0, & t \in R \end{cases}$$

We set $H = L^2(\Omega)$ and

$$D(A) = \{ u \in H_0^1(\Omega) \cap H^4(\Omega), \qquad u_{xx} \in H_0^1(\Omega) \}$$
$$Au = u_{xxxx} + hu \quad \text{for all} \quad u \in D(A).$$

Let $A_0 u = u_{xxxx}$ for $u \in D(A)$. The eigenvalues of A_0 are given by

$$\lambda_n^0 = \left(\frac{n\pi}{l} \right)^4, \qquad \forall n \in N \backslash \{0\}.$$

As a consequence, the eigenvalues of A are such that $\lambda_n \geqslant \left(\frac{n\pi}{l}\right)^4 - \|h^-\|_{L^\infty(\Omega)}$. Hence, if we assume $\|h^-\|_{L^\infty(\Omega)} < \left(\frac{\pi}{l}\right)^4$, we have A strongly positive with compact inverse in H. Also in this case, $D(A^{1/4}) = H_0^1(\Omega)$. From Theorem 4.1, we obtain that for any $\zeta \in H^{-1}(\Omega)$ and any solution $u \in C(R, H^2 \cap H_0^1(\Omega)) \cap \cap C^1(R, L^2(\Omega))$ of (4.3), we have the alternative

—either $\langle \zeta, u(t) \rangle \equiv 0$ on R

—or for any interval J of R such that

$$|J| \geqslant \sum_{n=1}^{+\infty} \frac{2\pi}{\sqrt{\left(\frac{n\pi}{l}\right)^4 - \|h^-\|_\infty}} = T,$$

there exists t_1 and $t_2 \in J$ such that $\langle \zeta, u(t_1) \rangle > 0$ and $\langle \zeta, u(t_2) \rangle < 0$. As a particular case, for any $x_0 \in]0, l[$, we have $\delta_{x_0} \in H^{-1}(\Omega)$: hence the function $u(t, x_0)$ is either $\equiv 0$ on R, or must take >0 and <0 values on each interval J such that $|J| \geqslant T$, for any weak solution u of (4.3).

EXAMPLE 4.3. Let Ω be any bounded domain of R^n, $n \geqslant 1$ with sufficiently regular boundary $\Gamma = \partial\Omega$. We consider the equation

$$\text{(4.4)} \quad \begin{cases} u_{tt} + (-1)^m \Delta^m u = 0 & t \in R, \quad x \in \Omega \\ \Delta^s u(t, x) = 0 & \text{for} \quad s \in \{0, 1, \ldots, m-1\}, \quad t \in R, \quad x \in \Gamma \end{cases}$$

where $m \in N$ is such that $m > n$.

In the case where $\Omega = (]0, \pi[)^n$, the eigenvalues of $A = (-1)^m \Delta^m u$ with $D(A) = \{u \in H^{2m}(\Omega), \Delta^s u = 0 \text{ on } \Gamma \text{ for all } s \in \{0, 1, \ldots, m-1\}\}$ are given by the formula

$$\lambda_{j_1, j_2, \ldots, j_n} = (j_1^2 + j_2^2 + \cdots + j_n^2)^m$$

By using the variational characterisation of the eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$, it is easy to show for any Ω the existence of two constants $c(\Omega)$, $C(\Omega)$ with $0 < c(\Omega) < C(\Omega) < +\infty$, such that

$$\frac{c(\Omega)}{(j_1^2 + \cdots + j_n^2)^{m/2}} \leqslant \frac{2\pi}{\sqrt{\lambda_{j_1, \ldots, j_n}}} \leqslant \frac{C(\Omega)}{(j_1^2 + \cdots + j_n^2)^{m/2}}$$

where $\lambda_{j_1, \ldots, j_n}$ are the eigenvalues of A associated with Ω.

As a consequence:

$$\sum_{j_1, \ldots, j_n} \frac{2\pi}{\sqrt{\lambda_{j_1, \ldots, j_n}}} < +\infty \Leftrightarrow \sum_{j_1, \ldots, j_n} \frac{1}{(j_1^2 + \cdots + j_n^2)^{m/2}} < +\infty \Leftrightarrow m > n.$$

Under the *same condition*, we have $D(A^{1/4}) \hookrightarrow C(\bar{\Omega})$.

Hence for any $x_0 \in \Omega$, the map $u \in D(A^{1/4}) \to u(x_0)$ is well-defined and can be considered as an element of $[D(A^{1/4})]'$. As a consequence of Theorem 4.1, we obtain that for some $T < +\infty$ (increasing, in fact, with the diameter of Ω), the following property holds: for any u solution of (4.4) and any $x_0 \in \Omega$, we have either $u(t, x_0) \equiv 0$, or for any interval J with length $> T$, there exists t_1, t_2 in J with $u(t_1, x_0)\, u(t_2 x_0) < 0$.

c) A counterexample

The example 4.3 does not include the wave equation (case $m = 1$) in *any* dimension and even for $n = 1$. This clearly means that our method cannot give always the best possible result, since for $n = 1$ the solutions *do* oscillate, for a seemingly quite special reason (namely the periodicity of solutions in t). According to this remark, it becomes essential to decide whether in fact the oscillation property is (or is not) always true for the wave equation, at least for C^∞ solutions, say.

The following construction shows that *it is not the case*, therefore one should be careful while attempting to generalize our example 4.3 under weaker conditions on m.

Theorem 4.4. *Let a, b be positive and such that $b^2/a^2 \notin Q$. Let $\Omega =]0, a[\times]0, b[\subset R^2$ and $(x_0, y_0) \in \Omega$ be any point such that $x_0/a \notin Q$, $y_0/b \notin Q$.*

Then for any $T > 0$, there exists a solution $u \in C^\infty(R \times \bar{\Omega})$ of

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} = 0 & (t; x, y) \in R \times \Omega \\ u(t, x, y) = 0 & (t; x, y) \in R \times \partial\Omega \\ u(t, x_0, y_0) \geqslant 1 & \text{on} \quad [0, T] \end{cases}$$

PROOF. Let $u_{m,n}$ be a double sequence of real numbers, $m \geqslant 1$, $n \geqslant 1$ and such that $u_{m,n} = 0$ for $m > m_0$ or $n > n_0$.

We set

$$u(t, x, y) = \sum_{m,n} u_{m,n} \frac{\cos\left(\dfrac{m\pi t}{a}\sqrt{n^2 + \dfrac{a^2}{b^2} + \alpha_{m,n}}\right)}{\sin\left(\dfrac{mn\pi x_0}{a}\right)\sin\left(\dfrac{m\pi y_0}{b}\right)} \sin\left(\dfrac{mn\pi x}{a}\right)\sin\left(\dfrac{m\pi y}{b}\right)$$

for $(t, x, y) \in R \times \Omega$.

It is easy to check that $u \in C(R \times \bar{\Omega})$ and $u_{tt} - u_{xx} - u_{yy} = 0$ in $R \times \Omega$, with $u(t, x, y) = 0$ on $R \times \partial\Omega$.

Moreover, we have

$$u(t, x_0, y_0) = \sum_{m,n} u_{m,n} \cos\left(\frac{m\pi t}{a}\sqrt{n^2 + \frac{a^2}{b^2}} + \alpha_{m,n}\right) =$$

$$= \sum_n \left\{\sum_m u_{m,n} \cos\left(\frac{m\pi t}{a}\sqrt{n^2 + \frac{a^2}{b^2}} + \alpha_{m,n}\right)\right\}$$

For $n \in N$ fixed and m_0 ranging over N, the function

$$\varphi_n(t) = \sum_{m=1}^{m_0} u_{m,n} \cos\left(\frac{m\pi t}{a}\sqrt{n^2 + \frac{a^2}{b^2}} + \alpha_{m,n}\right)$$

can be taken equal to *any* element of the space $\hat{X}_{\tau_n}$ with $\tau_n = 2ab(a^2 + n^2b^2)^{-1/2}$.

Also for $n_1 \neq n_2$, the numbers τ_{n_1} and τ_{n_2} are incommensurable since $b^2/a^2 \notin Q$.

Finally, we have

$$\sum_{n \in N} \tau_n = +\infty.$$

Now we pick n_0 such that

$$\sum_1^{n_0} \tau_n > T.$$

As a consequence of Theorem 3.1, there exists

$$f(t) \in \sum_{n=1}^{n_0} \hat{X}_{\tau_n}$$

such that $f \geqslant 1$ on $[0, T]$.

As a consequence of the remarks above, we can choose first m_0 large enough, and then the coefficients $u_{m,n}$ such that $u(t, x_0, y_0) = f(t)$.

This concludes the proof of Theorem 4.4.

References

[1] Amerio, L., Prouse, G. Almost periodic functions and functional equations. *Van Nostrand*, New-York (1971).

[2] Cazenave, T., Haraux, A. Propriétés oscillatoires des solutions de certaines équations des ondes semi-linéaires, *C.R.A.S. Paris,* Série I, **298** (1984), 449-452.

[3] Cazenave, T., Haraux, A. to appear in «Research Notes in Mathematics». Brézis, H. & Lions Editors, J. L. Nonlinear partial differential equations and their applications, *Collège de France Seminar*, Pitman (1984-85).

[4] Haraux, A. Nonlinear Evolution Equations-Global behavior of Solutions, Lecture Notes in Maths. No. 841, *Springer* (1981).

[5] Meyer, Y. Nombres premiers et vibrations, *Séminaire Delange -Poitou- Pisot* (1972).

[6] Meyer, Y. Algebraic numbers and harmonic analysis, *North-Holland publishing Co.*, (1970).

A. Haraux
Université Pierre et Marie Curie
Laboratoire d'Analyse Numérique
(U. A. 189)
Tour 55-65-5ème étage
4, place Jussieu
75230 Paris Cedex 05 (France)

V. Komornic
Université Eötvös Loránd
Département de Mathématiques
Chaire d'Analyse
Budapest, 8
Pf: 323—1445 (Hungrie)

Revista Matemática Iberoamericana
Vol. 1, n.° 4, 1985

A Convolution Inequality Concerning Cantor-Lebesgue Measures

Michael Christ

It is known that there exist positive measures μ on the circle group $T = \mathbb{R}/2\pi Z$, totally singular with respect to Lebesgue measure, for which there exist exponents $1 < p < q < \infty$ such that $\|f*\mu\|_q \leqslant C\|f\|_p$ for all f. Thus convolution with μ is smoothing in a weak sense. For instance, Stein [4] has pointed out that any measure satisfying $|\hat{\mu}(n)| = 0(|n|^{-\epsilon})$, $\epsilon > 0$, has this property, and Bonami [2] has shown that certain Riesz products, whose Fourier coefficients do not tend to zero, do also. Let μ_λ denote the Cantor-Lebesgue measure associated with the Cantor set of constant ratio of dissection $\lambda > 2$. Then Oberlin [3] proved that μ_3 has the property in question, and by building in part on his work Beckner, Janson and Jerison [1] proved the same for all rational $\lambda > 2$. In the present note a simple technique for the treatment of questions of this type will be introduced and applied to the μ_λ, for irrational λ as well. The technique is rather imprecise but flexible, and applies to Riesz products as well as to a certain class of multiplier operators. It rests on Littlewood-Paley theory and iteration, together with knowledge of the Fourier coefficients of the μ_λ.

To fix the notation let dx denote Lebesgue measure on T, normalized to be a probability measure. $\|f\|_p = \left(\int |f(x)|^p\,dx\right)^{1/p}$. If $m: Z \to \mathbb{C}$ and $1 < p \leqslant q < \infty$, its multiplier norm is defined to be

$$\|m\|_{p,q} = \sup_f \|(m\hat{f})^\vee\|_q / \|f\|_p.$$

Both the function and the associated operator will be denoted by m. Thus mf denotes $(m\hat{f})^{\vee}$ and $m_1 m_2$ denotes both the product of two functions and the composition of two operators. Let $2 < \lambda \in \mathbb{R}$. The Cantor set E_λ of constant ratio of dissection λ is the subset of $T = [-\pi, \pi)$ defined as follows: Delete from T the interval of length $2\pi(1 - 2\lambda^{-1})$ centered at 0. From each of the two intervals remaining delete a centered interval whose length is $(1 - 2\lambda^{-1})$ times the length of the interval. Continue indefinitely and let E_λ be the set of all points not eventually deleted. Associated to E_λ in a natural way is a totally singular probability measure μ_λ. We refer to Zygmund [5] for the precise definition and for the formula

$$\hat{\mu}(n) = (-1)^n \prod_{j=1}^{\infty} \cos(\pi(\lambda - 1)\lambda^{-j} n).$$

Theorem. *For any real $\lambda > 2$ and any $p \in (1, \infty)$ there exists $q(p, \lambda) > p$ such that $\|f * \mu_\lambda\|_q \leqslant \|f\|_q$ for all $f \in L^p$.*

It suffices to demonstrate the existence of $q > 2$ and $B < \infty$ such that $\|f * \mu_\lambda\|_q \leqslant B\|f\|_2$ for all f. For the general result of Beckner, Janson and Jerison then implies the existence of $r(B, q) > 2$ for which convolution with μ_λ is actually a contraction from L^2 to L^r. Alternatively, our argument could be refined slightly to yield $B = 1$ directly. Since convolution with any probability measure of mass one is a contraction on L^1 and L^∞, the Riesz-Thörin interpolation theorem then establishes our therorem for all p. One advantage of the case $p = 2$ is the next remark, taken from [1]: If $m_1, m_2 \colon Z \to \mathbb{C}$ are multipliers, $q \geqslant 2$ and $|m_1(n)| \leqslant |m_2(n)|$ for all $n \in Z$, then $\|m_1\|_{2,q} \leqslant \|m_2\|_{2,q}$. For m_1 may be expressed as $m_0 m_2$ where $\|m_0\|_{l^\infty} \leqslant 1$, and hence $\|m_1\|_{2,q} \leqslant \leqslant \|m_0\|_{2,2} \; \|m_2\|_{2,q} \leqslant \|m\|_{2,q}$.

We say that a strictly increasing sequence $\{n_j : j \geqslant 0\} \subset Z$ is σ-lacunary if $\sigma > 1$ and $(n_{j+1} - n_j) \geqslant \sigma(n_j - n_{j-1})$ for all $j \geqslant 1$. Given such a sequence define multiplier operators Δ_j by

$$(\Delta_j f)^\wedge(n) = \begin{cases} \hat{f}(n) & \text{if} \quad n_j \leqslant n < n_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. *If $1 < p \leqslant 2 \leqslant q < \infty$ and $\sigma > 1$, there exists $A_1(p, q, \sigma) < \infty$ such that for any σ-lacunary sequence $\{n_j\}$ and any $m \colon Z \to \mathbb{C}$ satisfying $m(n) = 0$ for all $n < n_0$,*

$$\|m\|_{p,q} \leqslant A_1 \sup_j \|\Delta_j m\|_{p,q}.$$

If σ is fixed then $A_1 \to 1$ as $p, q \to 2$.

PROOF. By Littlewood-Paley theory $\|(\Sigma|\Delta_j f|^2)^{1/2}\|_p \leqslant C_1\|f\|_p$ for all $p \in (1, \infty)$, where $C_1 = C_1(p, \sigma) \to 1$ as $p \to 2$. Moreover if $\hat{f}(n) = 0$ for all $n < n_0$ then $\|f\|_p \leqslant C_2\|(\Sigma|\Delta_j f|^2)^{1/2}\|_p$, where again $C_2 \to 1$ as $p \to 2$. Hence

$$\begin{aligned}\|mf\|_q &\leqslant C_2\|(\Sigma|\Delta_j mf|^2)^{1/2}\|_q \\ &\leqslant C_2(\Sigma\|\Delta_j mf\|_q^2)^{1/2} \\ &\leqslant C_2 \sup \|\Delta_j m\|_{p,q} \cdot (\Sigma\|\Delta_j f\|_p^2)^{1/2} \\ &\leqslant C_2 \sup \|\Delta_j m\|_{p,q}\|(\Sigma|\Delta_j f|^2)^{1/2}\|_p \\ &\leqslant C_1 C_2 \sup \|\Delta_j m\|_{p,q}\|f\|_p.\end{aligned}$$

Minkowski's inequality plus the hypotheses $p \leqslant 2 \leqslant q$ imply the second and fourth inequalities.

This clarification of the author's original proof is due to E. M. Stein. The lemma fails for all other pairs of exponents p, q. An elementary variant will also be useful below. Suppose $\{I_j: 1 \leqslant j \leqslant N\}$ are disjoint intervals. Let $m_j = m \cdot \chi_{I_j}$ and suppose that $m = \Sigma m_j$.

Lemma 2. *For any* $1 < p \leqslant q < \infty$ *there exists* $A_2(p, q, N) < \infty$ *such that* $\|m\|_{p,q} \leqslant A_2 \max \|m_j\|_{p,q}$. *If* N *is fixed then* $A_2 \to 1$ *as* $p, q \to 2$.

The proof involves only the boundedness of the Hilbert transform and the Riesz-Thörin theorem.

Fix λ and let $\delta > 0$ be a small number, depending on λ, to be specified momentarily. By an interval we henceforth mean a subinterval I of $\mathbb{R}$, neither of whose endpoints lie in Z. Though only the intersection of I with Z will actually be relevant, it is convenient to work in $\mathbb{R}$.

Lemma 3. *For any* $\lambda > 1$ *there exists* $\delta > 0$ *such that for any* $k \geqslant 1$ *and any interval* I *of length* $|I| \leqslant \lambda^k/2(\lambda - 1)$, *there exists a subinterval* $J \subset I$ *so that*

$$|J| \leqslant \frac{\lambda^{k-1}}{2(\lambda - 1)}$$

$$|I \backslash J| \leqslant \frac{(\lambda^k - \lambda^{k-1})}{2(\lambda - 1)} = \frac{\lambda^{k-1}}{2}$$

$$|\cos(\pi(\lambda - 1)\lambda^{-k}\xi)| \leqslant 1 - \delta \quad \text{for} \quad \xi \in I \backslash J$$

and so that each endpoint of J *either coincides with an endpoint of* I *or lies at distance greater than* $\delta\lambda^k$ *from the boundary of* I.

This holds by homogeneity and the fact that $\cos(\pi\xi)$ has at most one quarter of a full period on any interval of length $1/2$, hence has absolute value equal to one at most once. This final conclusion is purely technical in significance.

Finally we turn to the Cantor-Lebesgue measures. Let $q = q(\lambda)$ be slightly larger than two. Set $m_k(\xi) = \prod_{j=1}^{k} \cos(\pi(\lambda - 1)\lambda^{-k}\xi)$.

We show by induction that $\|m_k \chi_I\|_{2,q} \leqslant B$ for any interval I of length at most $\lambda^k/2(\lambda - 1)$, with q and B indpendent of I and k. Since $|\mu_\lambda(\xi)| \leqslant |m_k(\xi)|$, the theorem then follows via the remark preceding Lemma 1 and an easy passage to the limit.

Given such an I, fix a subinterval $J_k \subset I$ satisfying the conclusions of Lemma 3. Partition $I \backslash J_k$ into at most $\lambda + 3$ subintervals of lengths at most $\lambda^{k-1}/2(\lambda - 1)$. By induction on k the multiplier norm of the restriction of m_{k-1} to each subinterval is at most B, and hence $\|m_k \chi_{I \backslash J_k}\|_{2,q} \leqslant (1 - \delta)\|m_{k-1}\chi_{I \backslash J_k}\|_{2,q} \leqslant$ $\leqslant (1 - \delta)A_2 B$ by Lemma 2 and the remark preceding Lemma 1.

Since $|J_k| \leqslant \lambda^{k-1}/2(\lambda - 1)$, Lemma 3 may be applied repeatedly to construct $J_1 \subset J_2 \subset \cdots \subset J_k$ where $|J_i| \leqslant \lambda^{i-1}/2(\lambda - 1)$, so that all conclusions of that lemma hold at each step. By induction and the reasoning of the last paragraph $\|m_k \chi_{J_{i+1} \backslash J_i}\|_{2,q} \leqslant (1 - \delta)A_2 B$. Let $\{n_j\}$ denote the finite sequence of distinct right endpoints of the intervals J_i, in ascending order, and let R and L be those portions of I lying to the right and left of J_1, respectively. If it were true that $\{n_j\}$ must be σ-lacunary, then we could conclude by Lemma 1 that $\|m_k \chi_R\|_{2,q} \leqslant (1 - \delta)A_1 A_2 B$. Since J_1 contains at most one integer and $\|m_k\|_{l^\infty} \leqslant 1$, certainly $\|m_k \chi_{J_1}\|_{2,q} \leqslant 1$. Treating L in the same fashion as R and applying Lemma 2 yields $\|m_k \chi_I\|_{2,q} \leqslant A_2 \max(1, (1 - \delta)A_1 A_2 B)$. Fix any B strictly larger than one. Then $A_2 \max(1, (1 - \delta)A_1 A_2 B) \leqslant B$ provided q is sufficiently close to two. Thus the inductive step would be complete.

Unfortunately $\{n_j\}$ need not quite be lacunary. But let N be the least integer such that $\delta\lambda^N \leqslant 1$. Then $\{n_j : j \equiv 0 \bmod N\}$ is σ-lacunary, with $\sigma = 2(\lambda - 1) > 1$, provided δ is small. Indeed the worst case occurs when a large number of the J_i share one right endpoint, so that $n_{Nj} - n_{N(j-1)} = \lambda^k/2(\lambda - 1)$ for some large k. But if $n_{N(j+1)} > n_{Nj}$ then by the final clause of Lemma 3 $n_{N(j+1)} - n_{Nj} \geqslant$ $\geqslant \delta\lambda^{k+N} \geqslant \lambda^k$, so $\sigma \geqslant \lambda^k/(\lambda^k/2(\lambda - 1)) = 2(\lambda - 1)$. By first Lemma 2 and then Lemma 1, $\|m_k \chi_R\|_{2,q} \leqslant (1 - \delta)A_1 A_2^2 B$, and the proof is concluded as above.

Remarks

1. The Theorem holds for Cantor-Lebesgue measures with variable ratios of dissection as well. Suppose $2 < A < \infty$ and let $2 < \lambda_j \leqslant A$ for each $j \geqslant 1$. Then

$$\hat{\mu}(n) = (-1)^n \prod_{j=1}^{\infty} \cos\left(\pi(\lambda_j - 1) \prod_{i \leq j} \lambda_i^{-1} n\right)$$

is the sequence of Fourier coefficients of a probability measure μ [5], and the above arguments apply equally well to μ.

2. If $\lambda \leqslant 2$ then the construction of the Cantor-Lebesgue measure μ_λ breaks down. But the formula for $\hat{\mu}_\lambda$ still makes sense, and by the same reasoning defines a bounded multiplier from L^2 to L^q for some $q > 2$, provided $\lambda > 1$.

3. Our techniques produce examples of weighted norm inequalities for Fourier series which fall outside the scope of the general theory presently known. If convolution with μ is bounded from L^p to L^2 then $(\Sigma|\hat{f}(n)|^2 w(n))^{1/2} \leqslant C\|f\|_p$ where $w(n) = |\hat{\mu}(n)|^2$. More general sequences w may be constructed by iterating Littlewood-Paley decompositions of Z as in our proof.

4. The simplest examples [4] of singular measures μ with the property in question are those for which $\hat{\mu}(n) \to 0$ at a geometric rate as $|n| \to \infty$. Riesz products and Cantor-Lebesgue measures are interesting in part because their Fourier coefficients do not tend to zero. However the main point in our argument is that their Fourier coefficients actually do tend to zero, as n «tends to infinity» in a rather nonstandard sense reminiscent of p-adic analysis.

5. Our argument is closely related to the theory of $\Lambda(p)$ sets.

That μ_λ has the L^p-improving property for all rational $\lambda > 2$ was established independently and almost simultaneously by Ritter [6] and Beckner, Janson and Jerison; Ritter's proof appears to have been the first.

References

[1] Beckner, W., Janson, S., and Jerison, D. Convolution inequalities on the circle, *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, W. Beckner et. al. editors, Wadsworth, Belmont, Ca., 1983.

[2] Bonami, A. Étude des coefficients de Fourier des fonctions de $L^p(G)$, *Ann. Inst. Fourier*, Grenoble, **20** (1970), 335-402.

[3] Oberlin, D. A convolution property of the Cantor-Lebesgue measure, *Colloq. Math.*, **67** (1982), 113-117.

[4] Stein, E. M. Harmonic analysis on $\mathbb{R}^n$, *Studies in Harmonic Analysis,* M. A. A. Studies Series 13, J. M. Ash ed., 1976, 97-135.

[5] Zygmund, A. Trigonometric Series, 2nd edition, Volume 1, Cambridge U. Press, 1959.

[6] Ritter, D. L. Some singular measures on the circle which improve L^p spaces, Ph. D. dissertation, *Louisiana State Uni*$\rightarrow^{\epsilon\varrho s\iota\tau\xi}$, 1981.

[7] Ritter, D. L. Some singular measures on the circle which improve L^p spaces, to appear in Colloq. Math.

Department of Mathematics
Princeton University
Princeton, N. J. 08544
U.S.A.

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 1, N.º 4, 1985

Poincaré-Cartan Forms in Higher Order Variational Calculus on Fibred Manifolds

Jaime Muñoz Masqué

Introduction

The aim of the present work is to present a geometric formulation of higher order variational problems on arbitrary fibred manifolds. The problems of Engineering and Mathematical Physics whose natural formulation requires the use of second order differential invariants are classic, but it has been the recent advances in the theory of «integrable» non-linear partial differential equations and the consideration in Geometry of invariants of increasingly higher orders that has highlighted the interest of being able to work with a general formalism for higher order variational problems (see for instance [5], [7], [8]).

As in the case of first order problems, the central point of the theory lies in the construction of the Poincaré-Cartan form associted to a Lagrangian density. The method followed here for such a construction has been to analyse the reiterated process of integration by parts classically employed in the local deduction of Euler-Lagrange equations. The conclusion reached is that if we wish to carry out this process for an arbitrary fibred manifold $p\colon Y \to X$, this depends essentially on the choice of a derivation law ∇ on the vertical bundle $V(Y)$ and on a linear connection ∇_0 of the manifold X. By means of the derivation laws ∇, ∇_0 it is possible to define an operator L, called the total Lie derivative, which will allow us to globally perform this process and through which the diverse differential forms of the theory are expressed with intrinsic and explicit formulas. Furhtermore, the process itself, understood in this way, gives rise to the definition of the Poincaré-Cartan form Θ associated to an r-order Lagrangian density $\mathcal{L}v$ (Proposition 5.1 and Theorem 5.2). In fact, from a methodological point of view, this work could be considered as an extension to higher order problems of the method used in [2] for first order problems. These are the main characteristics of our method for the construction of higher order Poincaré-Cartan forms. Other different procedures to obtain such forms have also been presented recently ([1], [9] and [12]).

In the classical cases (i.e. when either $r \leqslant 2$ and n is arbitrary, or $n = 1$ and r is arbitrary) the form Θ depends neither on ∇_0 nor on ∇ and its expression coincides with that obtained by different authors using different methods (see [2], [6], [7], [14], [16]). In the general case (i.e. when $r > 2$ and $n > 1$) Θ does not depend on ∇ but it does depend on ∇_0; that is, we have a family of Poincaré-Cartan forms $\Theta(\nabla_0, \mathcal{L}v)$ which are parametrized by the linear connection ∇_0. Regarding this, it should be noted that the expression in local coordinates of the Poincaré-Cartan form for higher order variational problems which appears in certain works, even recent ones (and which in our construction corresponds to the form associated to the flat connection determined by a coordinate system) *is not covariant* in the general case; that is, when $r > 2$ and $n > 1$. Hence, the results obtained by using the aforementioned form are strictly local. In a more geometric sense, we could say that the only global results of higher order variational calculus are precisely those which remain convariant with respect to the linear connection ∇_0 which parametrizes the Poincaré-Cartan forms.

According to the procedure in [2] and by an adequate differential characterization of the notion of infinitesimal contact transformations, we formulate an r-order variational problem as a problem of invariance of the functional defined by an r-order Lagrangian density with respect to the Lie algebra of the infinitesimal contact transformations of the fibred manifold Y. This geometric presentation of variational problems (which is sufficient for the aims of this work) has the advantage of showing from the start the fundamental group of transformations which plays a part in the theory and which allows us to give a strictly differential treatment to it. In this context, the variational formula of Lagrangian density is expressed by an equation in the bundle $J^{2r-1}(Y)$ among the diverse differential forms of the theory. Naturally, when this equation is integrated along a holonomic section we obtain the expression in terms of the Euler-Lagrange operator well-known in the functional formulation of variational calculus (Proposition 9.1).

It is interesting to make the observation that, from this point of view, our construction of Poincaré-Cartan forms is invariant with respect to the group of vertical automorphisms of the fibred manifold Y. Indeed, if ψ is a vertical automorphism of Y, we have $J^{2r-1}(\psi)^*\Theta(\nabla_0, \mathcal{L}v) = \Theta(\nabla_0, J^r(\psi)^*(\mathcal{L}v))$. In fact, this formula is a particular case of Theorem 10.1 in which the behaviour of $\Theta(\nabla_0, \mathcal{L}v)$ is analised with respect to an arbitrary automorphism of Y (not necessarily vertical). This theorem also has an important repercussion in the definition of the Poisson brackets on the space of Noether invariants (Proposition 10.5).

Both the variation formula of Lagrangian density and the principal results of §9 are based on an explicit formula for the exterior differential of the form Θ obtained in §7. This formula also contains good information about the geometric properties of the variational problem defined by $\mathcal{L}v$. For example, it allows us to decide when the form Θ (defined on $J^{2r-1}(Y)$) is projectable to $J^{2r-h}(Y)$ for $h = 2, \ldots, r$ (Corollary 7.7). This is an important question in that since Θ is defined in a jet bundle other than that of $\mathcal{L}v$, the notion of regularity for higher order problems in Field Theory has an aspect very different from its usual one. This problem will be dealt with in a later work.

A lot of the material contained in sections §1-§7 was part of the author's PH.D. thesis at the University of Salamanca and their main formulas were anticipated (without proof) in [3]. I should this like to reiterate my thanks to Professor P. L. García for his interest and encouragement. Nevertheless, other results are new (for example, the independence of the products $(\,,\,)_{(k,l)}$ for $k + l = 2r + 1$ in Theorem 7.2 and Theorems 8.1 and 10.1). Such results complete the development of the theory.

1. Preliminaries and notations

Let $p\colon Y \to X$ be a fibred manifold (i.e., p is a surjective submersion). We shall use the notation $V(Y)$ for the sub-bundle of $T(Y)$ of vertical vectors over X. If $f\colon X' \to X$ is a differentiable mapping, we denote by $f^*(Y) \to X'$ the induced fibred manifold; we shall also write $Y_{X'} = f^*(Y)$ (specially when f is an open immersion). If $E \to X$ is a vector bundle, we denote by $\Gamma(E)$ the $C^\infty(X)$-module of differentiable sections of E over X; for any open set $U \subset X$ we write $\Gamma(U, E) = \Gamma(E_U)$. If $E_i \to X_i$ is a vector bundle and $f_i\colon X \to X_i$ is a differentiable mapping, with $i = 1, 2$, we shall denote by $E_1 \otimes_X E_2$, $\mathrm{Hom}_X(E_1, E_2)$ the vector bundles $f_1^*(E_1) \otimes f_2^*(E_2)$, $\mathrm{Hom}\,(f_1^*E_1, f_2^*E_2)$, respectively. Let E be a vector bundle over X and ω_q a E-valued q-form. We recall that by pulling ω_q back via a differentiable map $f\colon X' \to X$, we obtain a $f^*(E)$-valued q-form $f^*(\omega_q)$ over X'; we shall often denote this form by $\omega_{q|f}$ as well (specially when $f\colon X' \to X$ is a submanifold). All the definitions and results concerning valued differential calculus have been taken form [11]. The interior product and the Lie derivative (relative to a derivation law) of a valued q-form ω_q with respect to a vector field D will be denoted by $i_D\omega_q$ and $L_D\omega_q$, respectively. The exterior product of valued forms with respect to a bilinear map of vector bundles $B: E_1 \times_X E_2 \to E_3$ will simply be denoted by $\omega_q \wedge \eta_r$, but we shall also write $\omega_q \wedge_{(B)} \eta_r$ when we need to specify the bilinear map under consideration. The interior product of a E_2-valued q-form with respect to an E_1-valued vector field relative to the bilinear map B can also be defined by imposing that $i_{D\otimes e_1}(\omega_q \otimes e_2) = (i_D\omega_q) \otimes B(e_1, e_2)$.

The k-jet bundle of local sections of $p: Y \to X$ is denoted by $p_k: J^k = J^k(Y/X) \to X$ and $p_k^h: J^h \to J^k$, $h \geqslant k$, stands for the projection $p_k^h(j_x^h x) = j_x^k s$; we set $n = \dim X$, $n + m = \dim Y$. Any fibred chart for Y with local coordinates (x_j, y_i), $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant m$, induces a fibred chart for J^k with local coordinates (x_j, y_α^i) defined by: $y_0^i = y_i$, $y_\alpha^i(j_x^k s) = (\partial^{|\alpha|}/\partial x^\alpha)(y_i \circ s)(x)$, $1 \leqslant |\alpha| \leqslant k$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $f: Y \to Y'$ is a morphism of fibred manifolds whose induced mapping on the base manifolds $\bar{f}: X \to X'$ is a local diffeomorphism, we can define a map $J^k(f): J^k(Y/X) \to J^k(Y'/X')$ by $J^k(f)(j_x^k s) = j_{x'}^k s'$, where $x' = \bar{f}(x)$ and $s' = f \circ s \circ \bar{f}^{-1}$.

As is well-known ([2], [3], [6], [13]), the k-jet bundle J^k is endowed with a canonical $V(J^{k-1})_{J^k}$-valued 1-form θ^k, called the *structure form of order k*, whose local expression is

$$(1.1) \qquad \theta^k = \sum_i \sum_{|\alpha| < k} \theta_\alpha^i \otimes (\partial/\partial y_\alpha^i), \qquad \theta_\alpha^i = dy_\alpha^i - \sum_j y_{\alpha+(j)}^i dx_j,$$

where (j) stands for the multi-index $(j)_k = \delta_{jk}$ (δ_{jk} being the Kronecker index). For a geometric definition of the form θ^k by means of the notion of vertical differential of a section, one may consult [13]. We recall two basic properties of the structure forms:

(1.2) A section $\bar{s}$ of $p_k: J^k \to X$ is the k-jet prolongation of a section s of p (i.e., $\bar{s} = j^k s$) if and only if $\theta^k{}_{|\bar{s}} = 0$.

(1.3) The structure form θ^k is a section of the vector sub-bundle $\mathrm{Hom}_{J^k}(T(J^{k-1}), V(J^{k-1}))$ of $\mathrm{Hom}(T(J^k), V(J^{k-1})_{J^k}) = T^*(J^k) \otimes_{J^k} \otimes_{J^k} V(J^{k-1})$ and it determines a retract of the injection of $V(J^{k-1})_{J^k}$ into $T(\partial^{k-1})_{J^k}$. Thus, the exact sequence

$$0 \to V(J^{k-1})_{J^k} \to T(J^{k-1})_{J^k} \xrightarrow{(p_{k-1})_*} T(X)_{J^k} \to 0$$

splits canonically. We shall denote by $b_k: T(X)_{J^k} \to T(J^{k-1})_{J^k}$ the section of $(p_{k-1})_*$ associated to the retract determined by θ^k; that is, $b_k(z, D_x) = (j^{k-1}s)_*(D_x)$ with $z = j_x^k s$, $D_x \epsilon T_x(X)$.

Let $p: Y \to X$ be a fibred manifold and $E \to Y$ be a vector bundle. We set: $T_h^i(X, E) = T_h^i(X) \otimes_Y E$, $T_h^i(X) = (\otimes^h T^*(X)) \otimes (\otimes^i T(X))$, and similarly for the symmetric powers: $S_h^i(X,E) = S_h^i(X) \otimes_Y E$, $S_h^i(X) = S^h T^*(X) \otimes S^i T(X)$. Any bilinear mapping of vector bundles $B\colon E_1 \times_Y E_2 \to E_3$ induces a bilinear map on the symmetric powers

$$B_{h,h'}^{i,i'}\colon S_h^i(X,E_1) \times_Y S_{h'}^{i'}(X,E_2) \to S_{h+h'}^{i+i'}(X,E_3)$$

by the formula

$$(1.4)\qquad B_{h,h'}^{i,i'}(T_h \otimes T^i T_h \otimes T^i \otimes_{l_1} \otimes e_1, \bar{T}_{h'} \otimes \bar{T}^{i'} \otimes e_2) = \\ = \frac{h!h'!}{(h+h')!}(T_h \cdot \bar{T}_{h'}) \otimes (T^i \cdot \bar{T}^{i'}) \otimes B(e_1,e_2),$$

where the dot on the right hand-side stands for the symmetric product.

Let l be a now-negative integer such that $l \leqslant h$, $l \leqslant i$ and let $j\colon \{1,2,\ldots,l\} \to$ $\to \{1,2,\ldots,h\}$, $k\colon \{1,2,\ldots,k\} \to \{1,2,\ldots,i\}$ be two injective mappings. We write $j = (j_1,\ldots,j_l)$, $k = (k_1,\ldots,k_l)$. We denote by $c_j^k\colon T_h^i(X,E) \to$ $\to T_{h-l}^{i-l}(X,E)$ the contraction of the covariant indices j with the contravariant indices k; that is,

$$(1.5)\qquad c_j^k(w_1 \otimes \cdots \otimes w_h \otimes D_1 \otimes \cdots \otimes D_i \otimes e) = \\ = w_{j_1}(D_{k_1}) \cdots w_{j_l}(D_{k_l})(w_{j'_1} \otimes \cdots \otimes w_{j'_{h-l}} \otimes D_{k'_1} \otimes \cdots \otimes D_{k'_{i-l}} \otimes e),$$

where $j'_1 < \cdots < j'_{h-l}$, $k'_1 < \cdots < k'_{i-l}$ are the complementary sequences of $\{j_1,\ldots,j_l\}$, $\{k_1,\ldots,k_l\}$ in $\{1,\ldots,h\}$, $\{1,\ldots,i\}$, respectively and $w_1,\ldots,w_h \in T_x^*(X)$, $D_1,\ldots,D_i \in T_x(X)$, $e \in E_y$, with $p(y) = x$.

It is easily verified that c_j^k maps $S_h^i(X,E)$ onto $S_{h-l}^{i-l}(X,E)$ and also that the restriction of c_j^k to $S_h^i(X,E)$ does not depend on the indices j,k chosen. Thus, we can define a homomorphism $c_{h,l}^i\colon S_h^i(X,E) \to S_{h-l}^{i-l}(X,E)$ such that,

$$(1.6)\qquad c_{h,l}^i(w_1 \cdots w_h \otimes D_1 \cdots D_i \otimes e) = \\ = \sum_{j,k} (w_{j_1} \cdots w_{j_l})(D_{k_1} \cdots D_{k_l})(w_{j'_1} \cdots w_{j'_{h-l}} \otimes D_{k'_1} \cdots D_{k'_{i-l}} \otimes e),$$

where the indices j,k on the right hand-side run over all the sequences such that $1 \leqslant j_1 < \cdots < j_l \leqslant h$, $1 \leqslant k_1 < \cdots < k_l \leqslant i$, and $j'_1 < \cdots < j'_{h-l}$, $k'_1 < \cdots < k'_{i-l}$ are as above. In other words, $c_{h,l}^i$ is the contraction of l covariant indices with l contravariant ones in the vector sub-bundle of totally symmetric tensors of type (h,i).

The homomorphisms $c_{h,l}^i$ satisfy the following properties:

$$(1.7)\qquad c_{h-l,l'}^{i-l} \circ c_{h,l}^i = c_{h,l+l'}^i, \quad \text{if} \quad l+l' \leqslant h \quad \text{and} \quad l+l' \leqslant i.$$

(1.8) Let B: $E_1 \times_Y E_2 \to E_3$ be a bilinear map and ω_q, η'_r, ω'_q, η_r differential forms taking values in $S_h^i(X,E_1)$, $S_h^i(X,E_2)$, $S_{i-h}^0(X,E_1)$, $S_{i-h}^0(X,E_2)$, respectively. Then

$$(a)\quad c^{i-l}_{i-l,i-l}((c^i_{h,l}\omega_q)\wedge\eta_r) = c^i_{i,i}(\omega_q\wedge\eta_r),\quad \text{for}\quad l\leqslant h\leqslant i,$$
$$(b)\quad c^{i-l}_{i-l,i-l}(\omega'_q\wedge(c^i_{h,l}\eta'_r)) = c^i_{i,i}(\omega'_q\wedge\eta'_r),\quad \text{for}\quad l\leqslant h\leqslant i,$$

where the exterior products are taken with respect to the bilinear mappings induced by B according to (1.4).

The proof follows from a simple computation and will thus be omitted.

2. Total lie derivative

Let $p\colon Y\to X$ be a submersion and E a vector bundle over Y. The *total contraction* is the homomorphism

$$c\colon \Lambda^q T^*(J^{k-1})\otimes_{J^k} S^i_h(X,E)\to\Lambda^{q-1}T^*(J^{k-1})\otimes_{J^k}S^i_{h+1}(X,E)$$

given by

$$(c\omega_q)(D_2,\ldots,D_q;D'_0,\ldots,D'_h,w_1,\ldots,w_i) =$$
$$=\frac{1}{h+1}\sum_{j=0}^{h}\omega_q(b_kD'_j,D_2,\ldots,D_q;D'_0,\ldots,\hat{D}'_j,\ldots,D'_h,w_1,\ldots,w_i),$$

where $b_k\colon T(X)_{J^k}\to T(J^{k-1})_{J^k}$ is the section defined in (1.3).

Proposition 2.1. *The total contraction satisfies the following conditions*:

(*a*) $c\circ c=0$.

(*b*) $c(\eta_q\wedge\eta'_r)=(c\eta_q)\wedge\eta'_r+(-1)^q\eta_q\wedge(c\eta'_r)$, *where* η_q,η'_r *are differential forms with values in* $S^i_h(X,E_1)$, $S^{i'}_{h'}(X,E_2)$, *respectively, and the exterior products are taken with respect to the mappings induced by a bilinear map* $B\colon E_1\times_Y E_2\to E_3$.

(*c*) *On* $\Lambda^qT^*(J^{k-1})\otimes_{J^k}S^{i+1}_0(X,E)$, *we have* $c^i_{1,1}\circ c\circ c^{i+1}_{1,1}\circ c=0$.

PROOF. Condition (*a*) is an immediate consequence of the definition. First we shall prove (*b*) when η_q is an ordinary (non-valued) differential form and $B\colon(Y\times\mathbb{R})\times_Y E_2\to E_2$ is the natural bilinear mapping. We proceed by induction on q. If ω is an ordinary one-form, we have

$$c(\omega\wedge\eta'_r)(D_1,\ldots,D_r;D'_0,\ldots,D'_h,w_1,\ldots,w_i) =$$
$$=\frac{1}{h+1}\sum_{j=0}^{h}i_{(b_kD'_j)}(\omega\wedge\eta'_r)(D_1,\ldots,D_r;D'_0,\ldots,\hat{D}'_j,\ldots,D'_h,w_1,\ldots,w_i) =$$

$$= \frac{1}{h+1} \sum_{j=0}^{h} \omega(b_k D'_j)\eta'_r(D_1, \ldots, D_r; D'_0, \ldots, \hat{D}'_j, \ldots, D'_h, w_1, \ldots, w_i) -$$
$$- \frac{1}{h+1} \sum_{j=0}^{h} \sum_{l=1}^{r} (-1)^{l-1} \omega(D_l)\eta'_r(b_k D'_j, D_1, \ldots, \hat{D}_l, \ldots, D_r;$$
$$D'_0, \ldots, \hat{D}'_j, \ldots, D'_h, w_1, \ldots, w_i)$$
$$= (c\omega \wedge \eta'_r)(D_1, \ldots, D_r; D'_0, \ldots, D'_h, w_1, \ldots, w_i)$$
$$- \omega \wedge c\eta'_r)(D_1, \ldots, D_r; D'_0, \ldots, D'_h, w_1, \ldots, w_i).$$

proving (*b*) in this case. Now, if ω_q is an ordinary q-form, according to the induction hypothesis,

$$\begin{aligned} c((\omega \wedge \omega_q) \wedge \eta'_r) &= c(\omega \wedge (\omega_q \wedge \eta'_r)) = \\ &= c\omega \wedge (\omega_q \wedge \eta'_r) - \omega \wedge (c\omega_q \wedge \eta'_r + (-1)^q \omega_q \wedge c\eta'_r) = \\ &= c(\omega \wedge \omega_q) \wedge \eta'_r + (-1)^{q+1} (\omega \wedge \omega_q) \wedge c\eta'_r. \end{aligned}$$

In the general case,

$$\eta_q = \omega_q \otimes w_1 \cdots w_h \otimes D'_1 \cdots D'_i \otimes e,$$
$$\eta'_r = \omega'_r \otimes w'_1 \cdots w'_{h'} \otimes D''_1 \cdots D''_{i'} \otimes e'$$

and by setting $D_j = b_k(\partial/\partial x_j)$, from the above result we obtain

$$c(\eta_q \wedge \eta'_r) = \frac{h!h'!}{(h+h'+1)!} \sum_j i_{D_j}(\omega_q \wedge \omega'_r) \otimes dx_j \cdot w_1 \cdots w_h \cdot w'_1 \cdots w'_{h'} \otimes$$
$$\otimes D'_1 \cdots D'_i \cdot D''_1 \cdots D''_{i'} \otimes B(e, e')$$
$$(c\eta_q) \wedge \eta'_r = \frac{h!h'!}{(h+h'+1)} \sum_j ((i_{D_j}\omega_q) \wedge \omega'_r) \otimes dx_j \cdot w_1 \cdots w_h \cdot w'_1 \cdots w'_{h'} \otimes$$
$$\otimes D_1 \cdots D_i \cdot D''_1 \cdots D''_{i'} \otimes B(e, e')$$

and similarly for $\eta_q \wedge (c\eta'_r)$. Hence $c(\eta_q \wedge \eta'_r) = (c\eta_q) \wedge \eta'_r + (-1)^q \eta_q \wedge (c\eta'_r)$, and thus (*b*) is proved. Finally,

$$(c^i_{1,1} \circ c \circ c^{i+1}_{1,1} \circ c)(\omega_q \otimes D'_0 \cdots D'_i \otimes e) =$$
$$= \sum_{a \neq h} \sum_{j,l} (D'_h x_j)(D'_a x_l)(i_{D_l} i_{D_j} \omega_q) \otimes D'_0 \cdots \hat{D}'_a \cdots \hat{D}'_h \cdots D'_i \otimes e = 0.$$

In what follows we shall consider a derivation law ∇_0 in $T(X)$ (i.e., a linear connection of X) and a derivation law ∇ in the vector bundle $E \to Y$. As is well known, ∇_0 induces a derivation law ∇^*_0 in the vector bundle $p^*T(X) = T(X)_Y$. We define an operator

$$L: \Gamma(\Lambda^q T^*(J^{k-1}) \otimes_{J^{k-1}} S^i_h(X, E)) \to \Gamma(\Lambda^q T^*(J^k) \otimes_{J^k} S^i_{h+1}(X, E)),$$

called the *total Lie derivative* associated to the pair of derivation laws (Δ_0, ∇), by the formula

$$L = c \circ d + d \circ c,$$

where d is the exterior differential with respect to the derivation law induced by ∇_0^* and ∇ in the symmetric powers $S_h^i(X, E)$.

It is clear that the operator L is $\mathbb{R}$-linear and commutes with c, that is,

$$L \circ c = c \circ L.$$

Furthermore, L satisfies the formal property of a derivation. Namely, we have

Proposition 2.2. *Let ∇_i be a derivation law in E_i, with $i = 1, 2, 3$, and $B: E_1 \times {}_Y E_2 \to E_3$ a bilinear map compatible with these derivation laws. Then*

$$L(\eta_q \wedge \eta_r') = (L\eta_q) \wedge \eta_r' + \eta_q \wedge (L\eta_r'),$$

where η_q, η_r' are differential forms taking values in $S_h^i(X, E_1)$, $S_{h'}^{i'}(X, E_2)$, respectively, and the exterior products are taken with respect to the bilinear mappings $B_{h,h'}^{i,i'}$.

PROOF. This follows from (b) of Propostion 2.1 and the properties of the exterior differential for valued forms.

3. Structure forms associated to a pair of derivation laws

Given a linear connection ∇_0 of X and a derivation law ∇ in the vertical bundle $V(Y)$, we define a sequence of differential forms $(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(k)}, \ldots)$ by the following recurrence relations:

(3.1) $\theta^{(1)} = \theta^1$, the structure form on J^1; $\theta^{(k)} = L\theta^{(k-1)}$, for $k > 1$.

Thus, $\theta^{(k)}$ is a $(S^{k-1}T^*(X) \otimes_{J^k} V(Y))$-valued one-form on J^k which is called the *structure form of order k associated to the pair* (∇_0, ∇).

We shall now determine the local expression of the forms $\theta^{(k)}$. Let (x_j, y_i) be a fibred coordinate system for Y; we set

$$\nabla_{0(\partial/\partial x_j)}(\partial/\partial x_k) = \sum_l \mathring{\Gamma}_{jk}^l (\partial/\partial x_l), \tag{3.2}$$

$$\nabla_{(\partial/\partial x_j)}(\partial/\partial y_i) = \sum_a \Gamma_{ji}^a (\partial/\partial y_a), \qquad \nabla_{(\partial/\partial y_h)}(\partial/\partial y_i) = \sum_a \bar{\Gamma}_{hi}^a (\partial/\partial y_a). \tag{3.3}$$

For the sake of simplicity, we shall henceforth denote by $\mathbb{D}_j$ the derivation of the ring $A = \varinjlim C^\infty(J^k)$ defined by the formula

$$\mathbb{D}_j = \partial/\partial x_j + \sum_i \sum_{|\alpha|=0}^{\infty} y^i_{\alpha+(j)}(\partial/\partial y^i_\alpha). \tag{3.4}$$

Note that $\mathbb{D}_j(f) = b_k(\partial/\partial x_j)(f)$ if $f \epsilon C^\infty(J^{k-1})$, or, in other words, $(\mathbb{D}_j f) \circ j^k s = (\partial/\partial x_j)(f \circ j^{k-1} s)$ for every local section s of $p: Y \to X$. Moreover, $(dx)^\alpha$ stands for the symmetric product

$$(dx)^\alpha = dx_1^{\alpha_1} \cdots dx_n^{\alpha_n} = \overset{(\alpha_1}{dx_1 \cdots dx_1} \cdots \overset{(\alpha_n}{dx_n \cdots dx_n}, \tag{3.5}$$

and similarly for $(\partial/\partial x)^\alpha$. Hence, with the notations of (1.1) we have

Proposition 3.1. *There exist unique functions $\mathring{a}_{\beta\alpha} \epsilon C^\infty(X)$, $|\beta| \leqslant |\alpha|$, $a^{hi}_\alpha \epsilon C^\infty(J^{|\alpha|})$ such that:*

$$L^u \theta^h_0 = \frac{1}{u!} \sum_{|\beta| \le u} \sum_{|\alpha| = u} \mathring{a}_{\beta\alpha} \theta^h_\beta \otimes (dx)^\alpha, \tag{3.6}$$

$$L^v(\partial/\partial y_h) = \frac{1}{v!} \sum_i \sum_{|\alpha| = v} a^{hi}_\alpha (dx)^\alpha \otimes (\partial/\partial y_i). \tag{3.7}$$

These functions are completely determined by the following recurrence relations:

$$\mathring{a}_{00} = 1 \tag{3.8}$$

$$\begin{aligned} \mathring{a}_{\beta\alpha} &= \sum_j [\partial \mathring{a}_{\beta,\alpha-(j)}/\partial x_j + \mathring{a}_{\beta-(j),\alpha-(j)}] - \\ &\quad - \sum_{j,k,l} (1 + \alpha_l - \delta_{jl} - \delta_{kl}) \mathring{\Gamma}^l_{jk} \mathring{a}_{\beta,\alpha+(l)-(j)-(k)} \end{aligned} \tag{3.9}$$

$$a^{hi}_0 = \delta_{hi}, \qquad a^{hi}_{(j)} = \Gamma^i_{jh} + \sum_v y^v_{(j)} \bar{\Gamma}^i_{vh} \tag{3.10}$$

$$\begin{aligned} a^{hi}_\alpha &= \sum_j \Big[\mathbb{D}_j a^{hi}_{\alpha-(j)} + \sum_g a^{hg}_{\alpha-(j)} a^{gi}_{(j)} \Big] - \\ &\quad - \sum_{j,k,l} (1 + \alpha_l - \delta_{jl} - \delta_{kl}) \mathring{\Gamma}^l_{jk} a^{hi}_{\alpha+(l)-(j)-(k)} \end{aligned} \tag{3.11}$$

Finally, by setting

$$A^{hi}_{\beta\sigma} = \sum_{\alpha \le \sigma, |\beta| \le |\alpha|} \binom{|\sigma|}{|\alpha|} \mathring{a}_{\beta\alpha} a^{hi}_{\sigma-\alpha}, \tag{3.12}$$

we obtain

$$\theta^{(k)} = \frac{1}{(k-1)!} \sum_{h,i} \sum_{|\beta| \le k-1} \sum_{|\sigma| = k-1} A^{hi}_{\beta\sigma} \theta^h_\beta \otimes (dx)^\sigma \otimes (\partial/\partial y_i). \tag{3.13}$$

PROOF. Formulas (3.6) and (3.7) can easily be proved by recurrence on u and v, respectively, using the formal properties of the operator L. In fact, assuming (3.6), we have

$$L^{u+1}\theta_0^h = L(L^u\theta_0^h) = \frac{1}{u!}\sum_{|\beta|\le u}\sum_{|\alpha|=u}\left[\sum_j \frac{1}{u+1}(\mathbb{D}_j\mathring{a}_{\beta\alpha})\theta_\beta^h \otimes (dx)^{\alpha+(j)} + \right.$$

$$\left. + \sum_j \frac{1}{u+1}\mathring{a}_{\beta\alpha}\theta_{\beta+(j)}^h \otimes (dx)^{\alpha+(j)} + \mathring{a}_{\beta\alpha}\theta_\beta^h \otimes L(dx)^\alpha\right]$$

$$= \frac{1}{(u+1)}\sum_{|\beta|\le u}\sum_{|\alpha|=u}\left[\sum_j((\mathbb{D}_j\mathring{a}_{\beta\alpha})\theta_\beta^h + \mathring{a}_{\beta\alpha}\theta_{\beta+(j)}^h)\otimes(dx)^{\alpha+(j)} - \right.$$

$$\left. - \sum_{j,k,l}\alpha_l\mathring{\Gamma}^l_{jk}\mathring{a}^h_{\beta\alpha}\theta_\beta^h \otimes (dx)^{\alpha-(l)+(j)+(k)}\right],$$

and similarly for (3.7). From that, (3.9) and (3.11) also follow. On the other hand, we have

$$\theta^{(k)} = L^{k-1}\theta^1 = \sum_h L^{k-1}[\theta_0^h \otimes (\partial/\partial y_h)] =$$

$$= \sum_h \sum_{u+v=k-1}\binom{k-1}{u}(L^u\theta_0^h)\wedge(L^v(\partial/\partial y_h)) =$$

$$= \sum_{h,i}\sum_{u=0}^{k-1}\sum_{|\beta|\le u}\sum_{|\alpha|=u}\sum_{|\alpha'|=k-1-u}\frac{1}{u!(k-1-u)!}\mathring{a}_{\beta\alpha}a^{hi}_{\alpha'}\theta^h_\beta \otimes$$

$$\otimes (dx)^{\alpha+\alpha'}\otimes(\partial/\partial y_i) =$$

$$= \frac{1}{(k-1)!}\sum_{h,i}\sum_{|\beta|\le k-1}\sum_{|\sigma|=k-1}\left[\sum_{\alpha\le\sigma,|\beta|\le|\alpha|}\binom{|\sigma|}{|\alpha|}\mathring{a}_{\beta\alpha}a^{hi}_{\sigma-\alpha}\right]\theta^h_\beta\otimes(dx)^\sigma\otimes(\partial/\partial y_i),$$

thus proving formula (3.13).

Remark. It follows from (3.12) that

$$A^{hi}_{\beta\sigma}\epsilon C^\infty(J^{|\sigma|-|\beta|}). \tag{3.14}$$

Moreover, from (3.9) we obtain by induction on $|\alpha|$,

$$\mathring{a}_{\beta\alpha} = \delta_{\beta\alpha}|\alpha|!/\alpha! \quad \text{for} \quad |\beta| = |\alpha|. \tag{3.15}$$

Then, from (3.10) and (3.12) we have

$$A^{hi}_{\beta\sigma} = \delta_{hi}\delta_{\beta\sigma}|\sigma|!/\sigma! \quad \text{for} \quad |\beta| = |\sigma|. \tag{3.16}$$

Corollary 3.2. *The homomorphism* P_r: $\oplus_{k=0}^{r} S^k T(X) \otimes_{J^r} V^*(Y) \to V^*(J^r)$ *mapping* $(f_0, \ldots, f_r)$ *into the restriction of* $\theta^{(1)} \circ f_0 + \theta^{(2)} \circ f_1 + \cdots + \theta^{(r+1)} \circ f_r$ *to* $V(J^r)$ *is an isomorphism of vector bundles.*

PROOF. First, note that the definition of P_r makes sense because locally $\theta^{(r+1)} \circ f_r$ belongs to the submodule generated over $C^\infty(J^r)$ by $(\theta_\beta^h)_{|\beta| \leq r}$, as follows from (3.13) and (3.14). Since the vector bundles $V^*(J^r)$ and $\oplus_{k=0}^{r} S^k T(X) \otimes_{J^r} V^*(Y)$ have the same rank, it will be sufficient to prove that P_r is injective. We proceed by induction on r. The case $r = 0$ is trivial. If $P_r(f_0, \ldots, f_r) = 0$ we have $\theta^{(1)}(D) \circ f_0 + \cdots + \theta^{(r+1)}(D) \circ f_r = 0$ for every vertical tangent vector D in J^r. In particular, by taking $D = \partial/\partial y_\beta^h$ with $|\beta| = r$, from (3.13) and (3.16) we obtain $[(dx)^\beta \otimes (\partial/\partial y_h)](f_r) = 0$; that is, $f_r = 0$. Hence, $P_{r-1}(f_0, \ldots, f_{r-1}) = 0$ and according to the induction hypothesis, $f_0 = 0, \ldots, f_{r-1} = 0$.

Corollary 3.3. *With the above notations, we have* $(\theta^{(1)}, \ldots, \theta^{(r)}) = P_{r-1}^* \circ \theta^r$. *Consequently a section* $\bar{s}$ *of* p_r: $J^r \to X$ *is the jet prolongation of a section* s *of* p: $Y \to X$ *if and only if* $\theta^{(k)}|_{\bar{s}}$ *vanishes for* $k = 1, \ldots, r$ (cf. [13, Proposition 2]).

PROOF. It follows from formula (3.13) that $\ker(\theta^{(1)}, \ldots, \theta^{(r)}) = \ker \theta^r$. Therefore it is sufficient to see that $(\theta^{(1)}, \ldots, \theta^{(r)})(\partial/\partial y_\beta^h) = (P_{r-1}^* \circ \theta^r)(\partial/\partial y_\beta^h)$, with $|\beta| \leqslant r - 1$. However, it is easily checked, by using (3.13) and the definition of P_{r-1}, that both sides of the preceding equation give the same result when applied to $(\partial/\partial x)^\sigma \otimes dy_i$ for $|\sigma| < r$.

4. Higher order variational problems

A vector field D in J^r is an *infinitestimal contact transformation of order r* if for any derivation law ∇ in $V(J^{r-1})$ there exists an endomorphism ϕ of the vector bundle $V(J^{r-1})_{J^r}$ such that $L_D \theta^r = \phi \circ \theta^r$, where the Lie derivative is taken with respect to the derivation law induced by ∇. Indeed, if the previous condition is fulfilled for a derivation law ∇, it is automatically verified for any other derivation law. We now recall some basic facts concerning higher order infinitesimal contact transformations. For the proof of these results and further information one may consult [13].

(4.1) For *any* vector field D in Y (not necessarily p-projectable) there exists a unique infinitesimal contact transformation $D_{(r)}$ of order r projectable onto D.

(4.2) Moreover if $k > 0$ and $\bar{D}$ is an arbitrary infinitesimal contact transformation of order k, for every $r > k$ there exists a unique infinitesimal contact transformation $\bar{D}_{(r)}$ of order r projectable onto $\bar{D}$. In particular, if D is a vector field in Y, it follows that $D_{(r)}$ is projectable onto $D_{(k)}$ for every $r > k$.

For any open set $U \subset X$ we denote by $T^r(U)$ the space of all the infinitesimal contact transformations of order r corresponding to the induced fibred manifold Y_U; we denote by $T^r_c(U)$ the set of vector fields in $T^r(U)$ whose support has compact image in U. Then

(4.3) $T^r(U)$ is a Lie algebra with respect to the Lie bracket of vector fields, and $T^r_c(U)$ is an ideal of $T^r(U)$. Furthermore, the map $D \to D_{(r)}$ is an injection of Lie algebras.

(4.4) Let τ_t be the local 1-parameter group of local transformations generated by a vector field D in Y. If D is p-projectable, each transformation τ_t defines an automorphism of the fibred manifold Y and $J^r(\tau_t)$ is the local 1-parameter group generated by the vector field $D_{(r)}$.

Proposition 4.1. *A vector field D in J^r is an infinitesimal contact transformation if an only if there exist homomorphisms*

$$\phi^k_l \colon S^{l-1}T^*(X) \otimes_{J^r} V(Y) \to S^{k-1}T^*(X) \otimes_{J^r} V(Y), \qquad 1 \leqslant k, \quad l \leqslant r,$$

such that

$$L_D\theta^{(k)} = \phi^k_1 \circ \theta^{(1)} + \cdots + \phi^k_r \circ \theta^{(r)} \quad \textit{for} \quad k = 1, \ldots, r.$$

PROOF. This is immediate from Corollary 3.3.

From now on we shall assume that the base manifold is orientable. Once a volume element v on X has been fixed, we can associate a functional $\mathbb{L} \colon S(U) \to \mathbb{R}$ to each function $\mathcal{L} \in C^\infty(J^r)$ by the formula

$$\mathbb{L}(s) = \int_{j^r s} \mathcal{L} v = \int_U (j^r s)^* (\mathcal{L} v),$$

where $U \subset X$ is an open set and $S(U) \subset \Gamma(Y/U)$ is the space of those sections for which the above integral exists. For any section $s \in \Gamma(Y/U)$ we also define a linear form $\delta_s \mathbb{L} \colon T^r_c(U) \to \mathbb{R}$ by the formula

$$(\delta_s \mathbb{L})(D) = \int_{j^r s} L_D(\mathcal{L} v).$$

According to the definition of infinitesimal contact transformations given above, the linear functional $\delta\mathbb{L}$ represents the infinitesimal variation of the functional $\mathbb{L}$ on the space of generalized infinitesimal transformations of J^r induced by the infinitesimal automorphisms of the fibred structure $p \colon Y \to X$. We shall say that a section s is *critical* for the Lagrangian density $\mathcal{L} v$ when the linear functional $\delta\mathbb{L}$ has no variation at s; or in other words, when $\delta_s \mathbb{L} = 0$.

A basic problem in the Calculus in Variations is to characterize critical sections as solutions of a differential system defined on an appropiate jet bundle. In the following three sections we shall construct the Poincaré-Cartan forms associated to a higher order variational problem and examine their main properties. As will be shown below, these forms are the fundamental tools which will allow us to obtain not only the characterization of the critical sections but also the geometric properties of the manifold of solutions of a variational problem.

5. Poincaré-Cartan forms

Let $\mathcal{L}$ be a differentiable function on J^r. We denote by $\overline{d\mathcal{L}}$ the restriction of $d\mathcal{L}$ to $V(J^r)$. According to Corollary 3.2 there exist unique sections f_k of $S^kT(X) \otimes_{J^r} V^*(Y)$, $0 \leqslant k \leqslant r$, such that: $P_r(f_0, \ldots, f_r) = \overline{d\mathcal{L}}$; or in other words, $\theta^{(1)}(D) \circ f_0 + \cdots + \theta^{(r+1)}(D) \circ f_r = (d\mathcal{L})(D)$ for every vertical vector field D in J^r. Let v be a volume element on X. We define

$$\omega_k = c^k_{1,1} c(v \otimes f_k) \quad \text{for} \quad k = 1, \ldots, r. \tag{5.1}$$

In what follows we shall consider ω_k as a $S^{k-1}T(X) \otimes_{J^r} V^*(Y)$-valued $(n-1)$-form on J^r.

Remark. The forms ω_k only depend on the Lagrangian density $\mathcal{L}v$. In fact, in v' is another volume element on X and $\mathcal{L}v = \mathcal{L}'v'$, we have $v' = \rho v'$, $\mathcal{L}' = \bar{\rho}^1 \mathcal{L}$ for an invertible function $\rho \in C^\infty(X)$. Then $\overline{d\mathcal{L}'} = \bar{\rho}^1\, \overline{d\mathcal{L}}$ and $f'_k = \bar{\rho}^1 f_k$, $0 \leqslant k \leqslant r$. Hence $\omega'_k = c^k_{1,1} c(v' \otimes f'_k) = c^k_{1,1} c(v \otimes f_k) = \omega_k$.

Locally, we set $f_k = \sum_i \sum_{|\alpha| = k} f^i_\alpha (\partial/\partial x)^\alpha \otimes dy_i$. From the definition of f_k and formula (3.13) we obtain

$$|\beta|!\, f^h_\beta + \sum_i \sum_{|\alpha| = |\beta| + 1}^{r} \alpha!\, A^{hi}_{\beta\alpha} f^i_\alpha = \frac{\partial \mathcal{L}}{\partial y^h_\beta}, \quad \text{for} \quad |\beta| < r, \tag{5.2}$$

$$r!\, f^h_\beta = \frac{\partial \mathcal{L}}{\partial y^h_\beta}, \quad \text{for} \quad |\beta| = r. \tag{5.3}$$

Such equations determine the sections f_k by descending recurrence on k. By choosing the coordinates (x_j) so that $v = dx_1 \wedge \cdots \wedge dx_n$, we have

$$\omega_k = \sum_{i,j} \sum_{|\alpha| = k-1} (-1)^{j-1}(1 + \alpha_j) f^i_{\alpha + (j)} v_j \otimes (\partial/\partial x)^\alpha \otimes dy_i, \quad 1 \leqslant k \leqslant r, \tag{5.4}$$

as follows from (5.1), where $v_j = dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$.

We shall now deal with the exterior product of a $S^kT^*(X) \otimes V(Y)$-valued form η and a $S^kT(X) \otimes V^*(Y)$-valued form η' with respect to the bilinear map induced by duality. This product may be factorized as follows. With the same notations as in §1, let

$$B: S^kT^*(X) \otimes V(Y) \times_Y S^kT(X) \otimes V^*(Y) \to Yx_xS_k^k(X))$$

be the bilinear map canonically induced by duality on the vertical bundle. Then, we have

$$\eta \wedge \eta' = c_{k,k}^k\Big(\eta \underset{(B)}{\wedge} \eta'\Big).$$

Proposition 5.1. *Let* $\Theta_0 = \theta^{(1)} \wedge \omega_1 + \cdots + \theta^{(r)} \wedge \omega_r + \mathfrak{L}v$.

We have

$$\begin{aligned} d\Theta_0 = & -[\theta^{(1)} \wedge (d\omega_1 - v \otimes f_0) + \theta^{(2)} \wedge d\omega_2 + \cdots + \theta^{(r)} \wedge d\omega_r] + \\ & + \sum_{k=1}^{r} c_{k,k}^k c\Big(d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k)\Big), \end{aligned} \tag{5.5}$$

where the exterior differentials on the right hand-side are taken with respect to the derivation laws induced by ∇_0, ∇.

PROOF. Since $(\theta^{(1)} \circ f_0 + \cdots + \theta^{(r+1)} \circ f_r - d\mathfrak{L})(D) = 0$ for every vertical vector field D, it is clear that $\sum_{k=0}^{r} \theta^{(k+1)} \circ f_k - d\mathfrak{L}$ is a section of the vector sub-bundle $T^*(X)_{J^r}$. Hence,

$$\left(\sum_{k=0}^{r} \theta^{(k+1)} \circ f_k - d\mathfrak{L}\right) \wedge v = 0.$$

That is,

$$d\mathfrak{L} \wedge v = \sum_{k=0}^{r} (\theta^{(k+1)} \circ f_k) \wedge v = (\theta^{(1)} \circ f_0) \circ v + \sum_{k=1}^{r} (L\theta^{(k)} \circ f_k) \wedge v.$$

Then

$$d\Theta_0 = -\left[\theta^{(1)} \wedge (d\omega_1 - v \otimes f_0) + \sum_{k=2}^{r} \theta^{(k)} \wedge d\omega_k\right] +$$

$$+ \sum_{k=1}^{r} [(L\theta^{(k)} \circ f_k) \wedge v + d\theta^{(k)} \wedge \omega_k].$$

Since the total contraction c vanishes on the structure forms $\theta^{(k)}$, we have

$$(L\theta^{(k)} \circ f_k) \wedge v = (cd\theta^{(k)} \circ f_k) \wedge v = c_{k,k}^k\Big[cd\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k)\Big].$$

Moreover, from (*b*) of (1.8) and (5.1) we obtain:

$$d\theta^{(k)} \wedge \omega_k = c^{k-1}_{k-1,k-1}\Big[d\theta^{(k)} \underset{(B)}{\wedge} c^k_{1,1} c(v \otimes f_k)\Big]$$

$$= c^k_{k,k}\Big[d\theta^{(k)} \underset{(B)}{\wedge} c(v \otimes f_k)\Big]$$

The result follows from these three formulas.

Now, the central point of the theory is to reduce all the forms $\theta^{(k)}$ in formula (5.5) to the first, $\theta^{(1)}$, using the fundamental recurrence relationship (3.1). In this formalism the procedure constitutes the intrinsic version of the well-known classical method of reiterated integration by parts which is used in deducing higher order Euler-Lagrange equations. Furthermore, this procedure will lead to the definition of the Poincaré-Cartan form for Lagrangian densities of arbitrary order, starting from the form Θ_0 defined in the previous proposition.

Theorem 5.2 (fundamental). *With the above hypotheses and notations we have:*

$$d\Theta = \theta^{(1)} \wedge E + \Phi, \tag{5.6}$$

where we have set:

$$\Theta = \Theta_0 + \sum_{k=1}^{r-1} \theta^{(k)} \wedge \left(\sum_{i=1}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i} \right) \tag{5.7}$$

$$E = v \otimes f_0 - \sum_{h=0}^{r-1} (-1)^h \prod_{i=0}^{h-1} (c^{i+1}_{1,1} L)\, d\omega_{h+1} \tag{5.8}$$

$$\Phi = \sum_{k=1}^{r-1} c^k_{k,k} c d\left(\theta^{(k)} \underset{(B)}{\wedge} \left(\sum_{h=1}^{r-k} (-1)^h \prod_{i=1}^{h-1} (c^{k+i}_{1,1} L)\, d\omega_{k+h} \right)\right) + \tag{5.9}$$

$$+ \sum_{k=1}^{r} c^k_{k,k} c\Big(d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k)\Big).$$

We shall call the ordinary n-form Θ *the* Poincaré-Cartan form *associated to the Lagrangian density* $\mathfrak{L}v$ *relative to the derivation laws* ∇_0, ∇, *and we shall call the* $V^*(Y)$*-valued n-form* E *the* Euler-Lagrange form.

Remark. In §7 we shall see that the $(n+1)$-form Φ is a 2-contact form. We also note that the forms Θ, E and Φ are defined on J^{2r-1}, because operators c and L are applied $r-1$ times, at most, in the preceding formulas.

PROOF. First we shall prove by recurrence on $k = 0, \dots, r-1$ the following formula:

$$(*)\quad d\left[\Theta_0 + \sum_{h=1}^{k} \theta^{(r-h)} \wedge \left(\sum_{i=1}^{h} (-1)^i c_{1,1}^{r-h} c \prod_{j=1}^{i-1} (c_{1,1}^{r-h+j} L)\, d\omega_{r-h+i}\right)\right] =$$

$$= \theta^{(1)} \wedge (v \otimes f_0) - \sum_{h=1}^{r-k-1} \theta^{(h)} \wedge d\omega_h - \theta^{(r-k)} \wedge$$

$$\wedge \left(\sum_{h=0}^{k} (-1)^h \prod_{i=0}^{h-1} (c_{1,1}^{r-k+i} L)\, d\omega_{r-k+h}\right) +$$

$$+ \sum_{h=1}^{k} c_{r-h,r-h}^{r-h} c d\left[\theta^{(r-h)} \underset{(B)}{\wedge} \left(\sum_{i=1}^{h} (-1)^i \prod_{j=1}^{i-1} (c_{1,1}^{r-h+j} L)\, d\omega_{r-h+i}\right)\right] +$$

$$+ \sum_{h=1}^{r} c_{h,h}^{h} c\left(d\theta^{(h)} \underset{(B)}{\wedge} (v \otimes f_h)\right),$$

where we assume that the sums vanish and the products are the identity when the lower index is greater than the upper one.

Note that (*) for $k = 0$ reduces to formula (5.5). We set

$$\eta^{(k)} = \sum_{h=0}^{k} (-1)^h \prod_{i=0}^{h-1} (c_{1,1}^{r-k-i} L)\, d\omega_{r-k+h}.$$

By using part (*b*) of (1.8) and the fact that the exterior differential commutes with contractions, we have

$$\begin{aligned}
\theta^{(r-k)} \wedge \eta^{(k)} &= c_{r-k-1,r-k-1}^{r-k-1}\left(L\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) \\
&= c_{r-k-1,r-k-1}^{r-k-1}\left[L\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) - \theta^{(r-k-1)} \underset{(B)}{\wedge} L\eta^{(k)}\right] \\
&= c_{r-k-1,r-k-1}^{r-k-1} c d\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) + \\
&\quad + d c_{r-k-1,r-k-1}^{r-k-1} c\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) - \\
&\quad - c_{r-k-2,r-k-2}^{r-k-2}\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} c_{1,1}^{r-k-1} L\eta^{(k)}\right)
\end{aligned}$$

Moreover, since the total contraction is an anti-derivation which vanishes on the structure forms $\theta^{(k)}$, we have

$$\begin{aligned}
c_{r-k-1,r-k-1}^{r-k-1} c\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\right) &= -c_{r-k-1,r-k-1}^{r-k-1}\left(\theta^{(r-k-1)} \underset{(B)}{\wedge} c\eta^{(k)}\right) \\
&= -\theta^{(r-k-1)} \wedge c_{1,1}^{r-k-1} c\eta^{(k)}
\end{aligned}$$

Thus, finally:

$$\theta^{(r-k)} \wedge \eta^{(k)} = c^{r-k-1}_{r-k-1,r-k-1} c d\Big(\theta^{(r-k-1)} \underset{(B)}{\wedge} \eta^{(k)}\Big) - \\ - d(\theta^{(r-k-1)} \wedge c^{r-k-1}_{1,1} c\eta^{(k)}) - \\ - \theta^{(r-k-1)} \wedge c^{r-k-1}_{1,1} L\eta^{(k)}$$

Then, sustituting this expression in (*) we find the corresponding formula for $k+1$. Thus, the proof of formula (*) is complete. In particular, for $k = r-1$ we obtain formula (5.6) of the statement.

We define valued $(n-1)$-forms $\Omega_1, \ldots, \Omega_r$ on J^{2r-1} by setting

$$\Omega_k = \omega_k + \sum_{i=1}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i} \qquad (1 \leqslant k \leqslant r). \tag{5.10}$$

We also define a $V^*(J^{r-1})$-valued $(n-1)$-form Ω on J^{2r-1}, which will be called the *Legendre form*, by the following formula:

$$\Omega = P_{r-1} \circ (\Omega_1, \ldots, \Omega_r), \tag{5.11}$$

with the same notations as in Corollary 3.2. Then, Poincaré-Cartan form can be rewritten as follows:

$$\Theta = \theta^{(1)} \wedge \Omega_1 + \cdots + \theta^{(r)} \wedge \Omega_r + \mathcal{L}v = \theta^r \wedge \Omega + \mathcal{L}v. \tag{5.12}$$

6. Recurrence relations for $\Omega_1, \ldots, \Omega_r$ and local expression of Poincaré-Cartan forms

Proposition 6.1. *The forms $\Omega_1, \ldots, \Omega_r$ satisfy the following conditions*:

(6.1) $\Omega_k = \omega_k - c^k_{1,1} c d\Omega_{k+1}$ *for* $k = 1, \ldots, r-1$, *and* $\Omega_r = \omega_r$.

(6.2) $c^{k-1}_{1,1} c\Omega_k = c^{k-1}_{1,1} c\omega_k = 0$ *for* $k = 2, \ldots, r$.

(6.3) $\Omega_k = \omega_k - c^k_{1,1} L\Omega_{k+1}$ *for* $k = 1, \ldots, r-1$.

PROOF. It follows form (*c*) of Proposition 2.1 and (5.10) that

$$\Omega_k = \omega_k - c^k_{1,1} c d\omega_{k+1} + \sum_{i=2}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i}$$

$$= \omega_k - c^k_{1,1} c d\Bigg(\Omega_{k+1} - \sum_{i=1}^{r-k-1} (-1)^i c^{k+1}_{1,1} c \prod_{j=1}^{i-1} (c^{k+1+j}_{1,1} L)\, d\omega_{k+1+i}\Bigg) + \\ + \sum_{i=2}^{r-k} (-1)^i c^k_{1,1} c \prod_{j=1}^{i-1} (c^{k+j}_{1,1} L)\, d\omega_{k+i} =$$

$$\Omega_k = \omega_k - c_{1,1}^k cd\Omega_{k+1} +$$
$$+ \sum_{i=1}^{r-k-1} (-1)^i c_{1,1}^k c c_{1,1}^{k+1}(dc + cd) \prod_{j=1}^{i-1} (c_{1,1}^{k+1+j} L)\, d\omega_{k+1+i} +$$
$$+ \sum_{i=2}^{r-k} (-1)^i c_{1,1}^k c \prod_{j=1}^{i-1} (c_{1,1}^{k+j} L)\, d\omega_{k+i}$$
$$= \omega_k - c_{1,1}^k cd\Omega_{k+1},$$

which proves (6.1).

On the other hand, by again using (*c*) of Proposition 2.1 and the definition of ω_k, we obtain $c_{1,1}^{k-1} c\omega_k = c_{1,1}^{k-1} c c_{1,1}^k c(v \otimes f_k) = 0$, thus proving the second part of (6.2). The first part is obtained from (6.1) by descending recurrence on k. Finally, (6.3) is a direct consequence of (6.1) and (6.2).

The forms $\Omega_1, \dots, \Omega_r$ can be obtained in the same way as the forms $\omega_1, \dots, \omega_r$ were derived. Namely, we have

Proposition 6.2. *There exist unique sections F_k, $1 \leqslant k \leqslant r$, of*

$$S^k T(X) \otimes_{J^{2r-k}} V^*(Y)$$

such that

$$\Omega_k = c_{1,1}^k c(v \otimes F_k).$$

Thus, Ω_k is a section of the vector sub-bundle

$$\Lambda^{n-1} T^*(X) \otimes_{J^{2r-k}} S^{k-1} T(X) \otimes_Y V^*(Y).$$

PROOF. The uniqueness part is easily verified. In order to prove the existence of such sections we proceed locally by setting $F_k = \sum_i \sum_{|\alpha| = k} F_\alpha^i (\partial/\partial x)^\alpha \otimes \otimes dy_i$.

Then, the proof is by descending recurrence on k. If $k = r$, it is sufficient to take $F_r = f_r$, because $\Omega_r = \omega_r$. Let us assume that the formula is also true for $k, k+1, \dots, r$ with $k > 1$. That is, there exist sections F_l such that

$$\Omega_l = c_{1,1}^l c(v \otimes F_l) = \sum_{i,j} \sum_{|\alpha| = l-1} (-1)^{j-1}(1 + \alpha_j) F_{\alpha+(j)}^i v_j \otimes (\partial/\partial x)^\alpha \otimes dy_i, \quad k \leqslant l \leqslant r,$$

with the same notations as in (5.4). Then it follows from a simple (but rather long) computation in local coordinates that

$$\Omega_{k-1} = \omega_{k-1} - c_{1,1}^{k-1} cd\Omega_k = \sum_{i,j} \sum_{|\alpha| = k-2} (-1)^{j-1}(1 + \alpha_j) F_{\alpha,j}^i v_j \otimes (\partial/\partial x)^\alpha \otimes dy_i,$$

where the coefficient $F_{\alpha,j}^i$ is given by

$$F^i_{\alpha,j} = f^i_{\alpha+(j)} - \sum_l (1 + \alpha_l + \delta_{jl})\left[\mathbb{D}_l F^i_{\alpha+(j)+(l)} - \sum_h a^{ih}_{(l)} F^h_{\alpha+(j)+(l)} \right] -$$

$$- \sum_{l,q,u} (1 + \alpha_l + \delta_{jl} - \delta_{lq})(1 + \alpha_u + \delta_{lu} - \delta_{qu} + \delta_{ju}) \overset{\circ}{\Gamma}{}^q_{ul} F^i_{\alpha+(j)+(l)+(u)-(q)}$$

Here $\mathbb{D}_l$ stands for the vector field defined in (3.4). The above formula shows that $F^i_{\alpha,j}$ only depends on $\alpha + (j)$, or in other words, that if $\alpha + (j) = \alpha' + (j')$, then $F^i_{\alpha,j} = F^i_{\alpha',j'}$. We can therefore define F^i_σ for $|\sigma| = k - 1$ by setting $F^i_\sigma = F^i_{\alpha,j}$, where $\sigma = \alpha + (j)$ is an arbitrary decomposition of the multi-index σ. This proves the existence of F_{k-1} and completes the proof.

The previous formula can be rewritten as follows:

$$F^i_\sigma = f^i_\sigma - \sum_j (1 + \sigma_j)\left[\mathbb{D}_j F^i_{\sigma+(j)} - \sum_h a^{ih}_{(j)} F^h_{\sigma+(j)} \right] - \tag{6.4}$$

$$- \sum_{j,k,l} (1 + \sigma_k - \delta_{kl})(1 + \sigma_j + \delta_{jk} - \delta_{jl}) \overset{\circ}{\Gamma}{}^l_{jk} F^i_{\sigma+(j)+(k)-(l)} \quad (|\sigma| = 1, \ldots, r-1).$$

Furthermore, since $F_r = f_r$, from (5.3) we obtain

$$F^i_\sigma = f^i_\sigma = \frac{1}{r!}(\partial\mathcal{L}/\partial y^i_\sigma), \qquad |\sigma| = r. \tag{6.5}$$

Formulas (6.4) and (6.5) together with (5.2) determine the sections $F_1, \ldots, F_r$ by descending recurrence. They can also be used to obtain the local expression of Poincaré-Cartan forms. In fact, by (5.12) we have

$$\Theta = \sum_{h,j} \sum_{|\beta|=0}^{r-1} (-1)^{j-1} \lambda^h_{\beta j} \theta^h_\beta \wedge v_j + \mathcal{L}v \tag{6.6}$$

and

$$\lambda^h_{\beta j} = \sum_i \sum_{|\alpha|=|\beta|}^{r-1} (\alpha + (j))! A^{hi}_{\beta\alpha} F^i_{\alpha+(j)}, \qquad |\beta| = 0, \ldots, r-1. \tag{6.7}$$

In particular

$$\lambda^h_{\beta j} = \frac{1 + \beta_j}{r}(\partial\mathcal{L}/\partial y^h_{\beta+(j)}), \qquad |\beta| = r-1, \tag{6.8}$$

as follows from (6.7), (3.16) and (6.5). Note also that

$$\lambda^h_{\beta j} \in C^\infty(J^{2r-1-|\beta|}), \qquad |\beta| = 0, \ldots, r-1. \tag{6.9}$$

Proposition 6.3. *With the above notations there exist functions $\mu^h_{\beta j}$ such that:*

$$\lambda_{\beta j}^{h} = \sum_{|\sigma|=0}^{r-1-|\beta|} C_{\beta j}^{\sigma} \mathbb{D}^{\sigma}(\partial \mathcal{L}/\partial y_{\beta+(j)+\sigma}^{h}) + \mu_{\beta j}^{h}, \tag{6.10}$$

where the coefficient is given by $C_{\beta j}^{\sigma} = (-1)^{|\sigma|}(1+\beta_j)|\beta|!\,|\sigma|!\binom{\beta+(j)+\sigma}{\sigma}/ /|\beta+(j)+\sigma|!$ *and* $\mathbb{D}^{\sigma} = \mathbb{D}_1^{\sigma_1} = \mathbb{D}_n^{\sigma_n}$, $\mathbb{D}_j$ *being the vector field introduced in* (3.4). (Note that $[\mathbb{D}_j, \mathbb{D}_k] = 0$, which justifies the notation employed.)

$$\mu_{\beta j}^{h} \in C^{\infty}(J^{2r-2-|\beta|}) \tag{6.11}$$

and $\mu_{\beta j}^{h}$ *vanishes when* ∇_0 *and* ∇ *are the flat derivation laws associated to the coordinate system* (x_j, y_i).

PROOF. First note that all the functions $A_{\beta\alpha}^{hi}$ for $|\beta| < |\alpha|$ vanish when ∇_0 and ∇ are the flat derivation laws. Next, by descending recurrence on $|\sigma|$ and using (5.2), (5.3), (6.4), (6.5) it is not difficult to prove that there exist functions $G_{\sigma}^{i} \in C^{\infty}(J^{2r-|\sigma|-1})$, which vanish when ∇_0 and ∇ are the flat derivation laws, such that:

$$F_{\sigma}^{i} = \sum_{|\alpha|=0}^{r-|\sigma|} (-1)^{|\alpha|} \left[\frac{\binom{\sigma+\alpha}{\alpha}|\alpha|!}{|\sigma+\alpha|!} \right] \mathbb{D}^{\alpha}\left(\frac{\partial \mathcal{L}}{\partial y_{\sigma+\alpha}^{i}} \right) + G_{\sigma}^{i}. \tag{6.12}$$

The result now follows from (6.7).

7. A more explicit formula for $d\Theta$

Proposition 7.1. *Let E be the Euler-Lagrange form associated to an r-order Lagrangian density relative to the derivation laws* ∇_0, ∇. *If D, D′ are vector fields in* J^{2r-1} *vertical over X, then* $i_D i_{D'} E = 0$.

PROOF. For $h = 0, \dots, r-1$, let M_k^h be the module of $S^hT(X) \otimes V^*(Y)$--valued n-forms η on J^k such that $i_D i_{D'}\eta = 0$ for all vertical vector fields D, D'. Locally, M_k^h is spanned by the valued forms $v \otimes (\partial/\partial x)^{\alpha} \otimes dy_i$, $dy_{\beta}^{i} \wedge v_j \otimes \otimes (\partial/\partial x)^{\alpha} \otimes dy_{\alpha}$ ($|\alpha| = h$, $|\beta| \leqslant k$). On the other hand, we note that, with the same notations as in (3.2) and (3.4), for any ordinary one-form w on J^k the following formula holds true:

$$L(w) = \sum_{l} \Big(L_{\mathbb{D}_l} w - \sum_{j,\nu} \mathring{\Gamma}_{jl}^{\nu} w(\mathbb{D}_{\nu})\, dx_j \Big) \otimes dx_l. \tag{7.1}$$

Taking in particular $w = dy_{\beta}^{i}$ and $w = dx_{\mu}$ it is easily seen that

$$(c_{1,1}^{h+1}L)M_k^{h+1} \subset M_{k+1}^{h}. \tag{7.2}$$

Moreover, since

$$\prod_{i=0}^{h} (c_{1,1}^{i+1}L)(\eta) = \prod_{i=0}^{h-1} (c_{1,1}^{i+1}L)[(c_{1,1}^{h+1}L)(\eta)],$$

by induction on h we have that $\eta \in M_k^h$ implies

$$\prod_{i=0}^{h-1} (c_{1,1}^{i+1}L)(\eta) \in M_{k+h}^0.$$

The result now follows from (5.8) and (5.4), by setting $k = r$, $\eta = d\omega_{h+1}$.

Theorem 7.2. *Let E, Θ be the Euler-Lagrange and Poincaré-Cartan forms, respectively, associated to an r-order Lagrangian density $\mathcal{L}v$ with respect to the pair ∇_0, ∇. There exist unique bilinear mappings*

$$(\ ,\ )_{(k,l)}\colon (S^{k-1}T(X) \otimes V(Y)) \times_{J^{2r-1}} (S^{l-1}T(X) \otimes V(Y)) \to T(X)_{J^{2r-1}}$$

$$((k, l) \in I_r = \{(k, l) \in \mathbb{N} \times \mathbb{N};\quad 1 \leqslant k \leqslant l \leqslant 2r - 1, k \leqslant r, k + l \leqslant 2r + 1\})$$

which are alternating when $k = l$, such that:

$$d\Theta = \theta^{(1)} \wedge E + \sum_{(k,l)\in I_r} \left(\theta^{(k)} \underset{(k,l)}{\wedge} \theta^{(l)}\right) \cdot v, \tag{7.3}$$

where the form $(\theta^{(k)} \wedge_{(k,l)} \theta^{(l)})$. v is defined by the formula

$$\left(\left(\theta^{(k)} \underset{(k,l)}{\wedge} \theta^{(l)}\right) \cdot v\right)(D_0, \ldots, D_n) = \sum_{i<j} (-1)^{i+j-1} v\left(\left(\theta^{(k)} \underset{(k,l)}{\wedge} \theta^{(l)}\right)(D_i, D_j), D_0, \ldots, \hat{D}_i, \ldots, \hat{D}_j, \ldots, D_n\right).$$

Furthermore, the bilinear mappings $(\ ,\ )_{(k,l)}$ for $k + l = 2r + 1$ do not depend on the derivation laws chosen ∇_0 and ∇.

PROOF. First we shall prove that the form Φ of formula (5.6) locally belongs to the submodule M spanned by the forms: $\theta_\alpha^h \wedge \theta_\beta^i \wedge v_j$ ($|\alpha| < r$, $|\beta| < 2r - 1$, $|\alpha| + |\beta| < 2r$).

According to (5.9) it will be sufficient to prove that the forms

$$c_{k,k}^k c\left(d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k)\right) \qquad (k = 1, \ldots, r)$$

and

$$c_{k,k}^k c d\left(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\right) \qquad (k = 1, \ldots, r-1)$$

belong to M, where

$$\eta_k = \sum_{h=1}^{r-k} (-1)^h \prod_{i=1}^{h-1} (c_{1,1}^{k+i} L)\, d\omega_{k+h} \qquad (k = 1, \dots, r-1).$$

For the first group of these forms we proceed directly. Since dx_j, $\theta^i_\alpha(|\alpha| < k)$ is a local basis of $T^*(J^{k-1})_{j^k}$, it follows from (3.13) and (3.14) that there exist sections $S^{hi}_{\alpha\beta}$ of $S^k_{k-1}(X)_{j^k}$ such that

$$d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k) = \sum_{h,i} \sum_{|\alpha|<k} \sum_{|\beta|<k} \theta^h_\alpha \wedge \theta^i_\beta \wedge v \otimes S^{hi}_{\alpha\beta}.$$

This equation, when c and $c^k_{k,k}$ are applied, yields

$$c^k_{k,k} c\Big(d\theta^{(k)} \underset{(B)}{\wedge} (v \otimes f_k)\Big) = \frac{1}{k} \sum_{h,i} \sum_j \sum_{|\alpha|<k} \sum_{|\beta|<k} (-1)^{j-1} c^k_{k,k}(dx_j \cdot S^{hi}_{\alpha\beta}) \theta^h_\alpha \wedge \theta^i_\beta \wedge v_j.$$

thus proving our assertion in this case.

Next, we shall consider the second group of forms. We have

$$(*) \qquad c^k_{k,k} cd\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big) = c^k_{k,k} L\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big) - c^k_{k,k} dc\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big).$$

But from (5.10), (5.1) and Proposition 6.2, we conclude that

$$c^k_{1,1} c\eta_k = \Omega_k - \omega_k = c^k_{1,1} c(v \otimes (F_k - f_k)).$$

Thus, by using (b) of (1.8) and the fact that d commutes with contractions, the last term in (*) can be transformed as follows:

$$\begin{aligned}
-c^k_{k,k} dc\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big) &= dc^k_{k,k}\Big(\theta^{(k)} \underset{(B)}{\wedge} c\eta_k\Big) = dc^{k-1}_{k-1,k-1}\Big(\theta^{(k)} \underset{(B)}{\wedge} (c^k_{1,1} c\eta_k)\Big) = \\
&= dc^{k-1}_{k-1,k-1}\Big(\theta^{(k)} \underset{(B)}{\wedge} c^k_{1,1} c(v \otimes (F_k - f_k))\Big) = \\
&= dc^k_{k,k}\Big(\theta^{(k)} \underset{(B)}{\wedge} c(v \otimes (F_k - f_k))\Big) = -c^k_{k,k} dc\Big(\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) = \\
&= -c^k_{k,k} L\Big(\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) + c^k_{k,k} cd\Big(\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) = \\
&= -c^k_{k,k} L\Big(\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) + c^k_{k,k} c\Big(d\theta^{(k)} \underset{(B)}{\wedge} v \mathbin{\dot{=}} (F_k - f_k)\Big) - \\
&\quad - (-1)^n c^k_{k,k} c\Big(\theta^{(k)} \underset{(B)}{\wedge} v \wedge d(F_k - f_k)\Big).
\end{aligned}$$

Upon substituting this expression into Eq. (*), we obtain

$$c^k_{k,k}cd\Big(\theta^{(k)} \underset{(B)}{\wedge} \eta_k\Big) = c^k_{k,k}L\Big(\theta^{(k)} \underset{(B)}{\wedge} (\eta_k - v \otimes (F_k - f_k))\Big) +$$
$$+ c^k_{k,k}c\Big(d\theta^{(k)} \underset{(B)}{\wedge} v \otimes (F_k - f_k)\Big) - (-1)^n c^k_{k,k}c\Big(\theta^{(k)} \underset{(B)}{\wedge} v \wedge d(F_k - f_k)\Big).$$

We shall now show that η_k belongs to the submodule M^k_{2r-k-1} introduced in the proof of Proposition 7.1. In fact, this in an immediate consequence of (7.2) and the following recurrence relations

$$\eta_{r-1} = -d\omega_r, \qquad \eta_k = -d\omega_{k+1} - c^{k+1}_{1,1}L\eta_{k+1} \qquad (k = 1, \ldots, r-2),$$

which follow directly from the definition of η_k. Therefore, η_k can be expressed as $\eta_k = v \otimes S^k + \sum_{i,j} \sum_{|\beta| < 2r-k} dy^i_\beta \wedge v_j \otimes S^{ki}_{\beta j}$, for certain local sections S^k, $S^{ki}_{\beta j}$ of $S^kT(X) \otimes_{J^{2r-k-1}} V^*(Y)$. Or equivalently,

$$\eta_k = v \otimes \Big(S^k + \sum_{i,j} \sum_{|\beta| < 2r-k} (-1)^{j-1} y^i_{\beta+(j)} S^{ki}_{\beta j}\Big) + \sum_{i,j} \sum_{|\beta| < 2r-k} \theta^i_\beta \wedge v_j \otimes S^{ki}_{\beta j}.$$

Hence, the relation $c^k_{1,1}c(\eta_k - v \otimes (F_k - f_k)) = 0$ obtained before, implies:

(i) $$F_k - f_k = S^k + \sum_{i,j} \sum_{|\beta| < 2r-k} (-1)^{j-1} y^i_{\beta+(j)} S^k_{\beta j}$$

and

(ii) $$(-1)^{j+l-1} c^k_{1,1}(dx_j \otimes S^{ki}_{\beta l}) + c^k_{1,1}(dx_l \otimes S^{ki}_{\beta j}) = 0.$$

From (i) we deduce:

$$\eta_k - v \otimes (F_k - f_k) = \sum_{i,j} \sum_{|\beta| < 2r-k} \theta^i_\beta \wedge v_j \otimes S^{ki}_{\beta j}.$$

Hence,

$$c^k_{k,k}L\Big(\theta^{(k)} \underset{(B)}{\wedge} (\eta_k - v \otimes (F_k - f_k))\Big) = \sum_{i,j} \sum_{|\beta| < 2r-k} \theta^{(k+1)} \wedge (\theta^i_\beta \wedge v_j \otimes S^k_{\beta j}) +$$
$$+ \sum_i \sum_{j,l} \sum_{|\beta| < 2r-k} \theta^{(k)} \wedge (\theta^i_{\beta+(l)} \wedge v_j \otimes (dx_l \otimes S^{ki}_{\beta j})) + \sum_{i,j} \sum_{|\beta| < 2r-k} \theta^{(k)} \wedge$$
$$\wedge\, \theta^i_\beta \wedge L(v_j \otimes S^{ki}_{\beta j}),$$

where we have used the equality $L(\theta^i_\beta) = \sum_j \theta^i_{\beta+(j)} \otimes dx_j$.

Using (7.1) with $w = dx_\mu$, it is easily checked that $c^k_{k,k}L(\theta^{(k)} \wedge_{(B)} (\eta_k - v \otimes \otimes (F_k - f_k)))$ belong to M when $k > 1$ (i.e. when $k = 2, \ldots, r-1$). In addition, $v \wedge d(F_k - f_k)$ can be written as a linear combination of the forms $\theta^i_\sigma \wedge v \otimes (\partial/\partial x)^\alpha \otimes dy_a$ $(|\sigma| \leqslant 2r - k, |\alpha| = k)$. Thus, the term $c^k_{k,k}c(\theta^{(k)} \wedge_{(B)} v \wedge \wedge d(F_k - f_k))$ also belongs to M when $k > 1$. On the other hand, by means of the same argument considered in the first part of the proof, it is easily seen that the term $c^k_{k,k}c(d\theta^{(k)}\wedge_{(B)} v\otimes(F_k-f_k))$ belongs to M even for $k=1$. Therefore, it only remains to prove that

$$c^1_{1,1}L\Big(\theta^{(1)}\underset{(B)}{\wedge}(\eta_1-v\otimes(F_1-f_1))\Big)-(-1)^n c^1_{1,1}c\Big(\theta^{(1)}\underset{(B)}{\wedge}v\wedge d(F_1-f_1)\Big)$$

also belongs to M. To this end, we first separate the highest order terms in the above form, obtaining

$$\begin{aligned}
&c^1_{1,1}L\Big(\theta^{(1)}\underset{(B)}{\wedge}(\eta_1-v\otimes(F_1-f_1))\Big)-(-1)^n c^1_{1,1}c\Big(\theta^{(1)}\underset{(B)}{\wedge}v\wedge d(F_1\ -f_1)\Big)=\\
&\quad=\sum_{h,i}\sum_{j,l}\sum_{|\beta|=2r-2}[(-1)^{j+l-1}B(\partial/\partial y_h\otimes c^1_{1,1}(dx_j\otimes S^{1i}_{\beta l}))+\\
&\qquad+B(\partial/\partial y_h\otimes c^1_{1,1}(dx_l\otimes S^{1i}_{\beta j}))]\theta^h_0\wedge\theta^i_{\beta+(l)}\wedge v_j+\\
&\qquad+\sum_{i,j}\sum_{|\beta|<2r-2}(\theta^{(2)}\wedge(\theta^i_\beta\wedge v_j\otimes S^{1i}_{\beta j})+\\
&\qquad+\sum_l\theta^{(1)}\wedge(\theta^i_{\beta+(l)}\wedge v_j\otimes dx_l\otimes S^{1i}_{\beta j}))+\\
&\qquad+\sum_{i,j}\sum_{|\beta|\le 2r-2}\theta^{(1)}\wedge(\theta^i_\beta\wedge L(v_j\otimes S^{1i}_{\beta j}))+(-1)^n\theta^{(1)}\wedge c(v\wedge dS^1)+\\
&\qquad+(-1)^n\theta^{(1)}\wedge c\Big(\sum_{i,j}\sum_{|\beta|<2r-2}(-1)^{j-1}v\wedge\theta^i_{\beta+(j)}\otimes S^{1i}_{\beta j}\Big)+\\
&\qquad+(-1)^n\theta^{(1)}\wedge c\Big(\sum_{i,j}\sum_{|\beta|\le 2r-2}(-1)^{j-1}y^i_{\beta+(j)}v\wedge dS^{1i}_{\beta j}\Big).
\end{aligned}$$

The first term on the right-hand side vanishes by virtue of (ii), while all the other terms lie in the submodule M. We have thus completed the proof of our first statement.

We shall now consider the uniqueness of the bilinear maps $(\,,\,)_{(k,l)}$. Locally, each one of these mappings determines n bilinear forms $(\,,\,)^j_{(k,l)}$ given by

$$(\,,\,)_{(k,l)}=\sum_j(\,,\,)^j_{(k,l)}\partial/\partial x_j.$$

Then, as $(\,,\,)_{(k,k)}$ is alternating, applying both sides of formula (7.3) to $(\partial/\partial y^h_\alpha,\partial/\partial y^i_\beta,\mathbb{D}_1,\dots,\hat{\mathbb{D}}_j,\dots,\mathbb{D}_n)$ ($|\alpha|<r$, $|\beta|<2r-1$, $|\alpha|\leqslant|\beta|$, $|\alpha|+|\beta|<$ $<2r$), we obtain:

$$\begin{aligned}
(7.4)\quad&\Phi(\partial/\partial y^h_\alpha,\partial/\partial y^i_\beta,\mathbb{D}_1,\dots,\hat{\mathbb{D}}_j,\dots,\mathbb{D}_n)=\\
&=(-1)^{j-1}\epsilon_{\alpha\beta}((dx)^\alpha\otimes\partial/\partial y_h,(dx)^\beta\otimes\partial/\partial y_i)^j_{(|\alpha|+1,|\beta|+1)}+\\
&\quad+\sum_{(k,l)\epsilon I_{\alpha\beta}}\Big(\Big(\theta^{(k)}\underset{(k,l)}{\wedge}\theta^{(l)}\Big)\cdot v\Big)(\partial/\partial y^h_\alpha,\partial/\partial y^i_\beta,\mathbb{D}_1,\dots,\hat{\mathbb{D}}_j,\dots,\mathbb{D}_n),
\end{aligned}$$

where we have set,

$$\epsilon_{\alpha\beta} = 1/\alpha!\beta! \quad \text{for} \quad |\alpha| < |\beta| \quad \text{and} \quad \epsilon_{\alpha\beta} = 2/\alpha!\beta! \quad \text{for} \quad |\alpha| = |\beta|,$$

and

$$I_{\alpha\beta} = \{(k,l)\epsilon I_r;\, |\alpha| < k,\, |\beta| < l,\, (k,l) \neq (|\alpha| + 1, |\beta| + 1)\}.$$

In particular,

$$(7.5) \quad \Phi(\partial/\partial y^h_\alpha, \partial/\partial y^i_\beta, \mathbb{D}_1, \ldots, \hat{\mathbb{D}}_j, \ldots, \mathbb{D}_n) =$$
$$= (-1)^{j-1}\epsilon_{\alpha\beta}((dx)^\alpha \otimes \partial/\partial y_h, (dx)^\beta \otimes \partial/\partial y_i)^j_{(|\alpha|+1,|\beta|+1)} \quad (|\alpha| + |\beta| = 2r - 1).$$

It is now clear that formulas (7.4) and (7.5) completely determine, by descending recurrence on $|\alpha| + |\beta|$, the bilinear mapping in question.

Because of the uniqueness of such mappings, in order to prove their existence it will be sufficient to give a local definition of them so that Eq. (7.3) will be fulfilled locally. First we use the above formulas to define $(\,,\,)_{(k,l)}$ by descending recurrence on $k + l$. Next, we note that $(\theta^{(k)} \wedge_{(k,l)} \theta^{(l)}) \cdot v$ belongs to the submodule M when $(k,l)\epsilon I_r$. Thus, we only need to check that forms Φ and $\sum_{(k,l)\epsilon I_r}(\theta^{(k)} \wedge_{(k,l)} \theta^{(l)}) \cdot v$ coincide when applied to

$$(\partial/\partial y^h_\alpha, \partial/\partial y^i_\beta, \mathbb{D}_1, \ldots, \hat{\mathbb{D}}_j, \ldots, \mathbb{D}_n),$$
$$(|\alpha| < r, |\beta| < 2r - 1, |\alpha| \leqslant |\beta|, |\alpha| + |\beta| < 2r).$$

However, this condition leads us to Eq. (7.4), which is fulfilled by the very definition of the bilinear mapping $(\,,\,)_{(k,l)}$.

Finally, we shall prove the independence of the products $(\,,\,)_{(k,l)}$, $k + l = 2r + 1$, by a method which will provide further information.

Let ψ: $V(J^{r-1}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^{r-1}) \to T(X)_{J^{2r-1}}$ be the bilinear mapping given by the formula

$$\psi(D, D') = \tilde{v}^{-1}(i_{D'} i_D d\Theta),$$

where $\tilde{v}$: $T(X) \to \Lambda^{n-1}T^*(X)$ is the isomorphism induced by the volume element: $\tilde{v}(D) = i_D v$. Note that the definition makes sense, because formulas (6.6) and (6.7) imply that $i_{D'}d\Theta = \sum_{h,j}\sum_{|\beta|=0}^{r-1}(-1)^{j-1}D'(\lambda^h_{\beta j})\theta^h_\beta \wedge v_j$ for every vector field D' in J^{2r-1} vertical over J^{r-1}. Hence, $i_{D'}d\Theta$ is a section of the vector sub-bundle $\Lambda^n T^*(J^{r-1})_{J^{2r-1}}$. Moreover, from this we also obtain the local expression of ψ:

$$\psi(D, D') = -\sum_{h,j}\sum_{|\beta|=0}^{r-1}(D'\lambda^h_{\beta j})(Dy^h_\beta)(\partial/\partial x_j).$$

Then, since $\lambda^h_{\beta j}$ is a function on $J^{2r-1-|\beta|}$ (formula (6.9)), we have:

(*a*) The bilinear mapping ψ vanishes on the vector sub-bundle

$$T(J^{r-1}/J^{2r-2-k}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^k) \quad \text{of} \quad V(J^{r-1}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^{r-1})$$
$$\text{for} \quad r-1 \leqslant k \leqslant 2r-2.$$

Let us fix an index k such that $r-1 \leqslant k \leqslant 2r-2$, and let ψ_k be the restriction of ψ to $V(J^{r-1}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^k)$. According to (*a*), ψ_k induces a bilinear mapping on the quotient $\bar{\psi}_k$: $V(J^{2r-2-k}) \otimes_{J^{2r-1}} T(J^{2r-1}/J^k) \to T(X)_{J^{2r-1}}$. Let $\bar{\psi}'_k$ be the restriction of $\bar{\psi}_k$ to the vector sub-bundle $S^{2r-2-k}T^*(X) \otimes \otimes V(Y) \otimes_{J^{2r-1}} T(J^{2r-1}/J^k)$. (Recall that $T(J^k/J^{k-1})$ is canonically isomorphic to $S^kT^*(X) \otimes_{J^k} V(Y)$.) Then, as above, we have:

(*b*) The mapping $\bar{\psi}'_k$ vanishes on the vector sub-bundle

$$S^{2r-2-k}T^*(X) \otimes V(Y) \otimes_{J^{2r-1}} T(J^{2r-1}/J^{k+1}).$$

Thus, $\bar{\psi}'_k$ finally induces a bilinear map

$$B_k\colon (S^{2r-2-k}T^*(X) \otimes_{J^{2r-1}} V(Y)) \otimes (S^{k+1}T^*(X) \otimes_{J^{2r-1}} V(Y)) \to T(X)_{J^{2r-1}}$$
$$(r-1 \leqslant k \leqslant 2r-2)$$

given locally by

$$(7.6) \quad B_k((dx)^\alpha \otimes (\partial/\partial y_h), (dx)^\beta \otimes (\partial/\partial y_i)) = -\alpha!\beta! \sum_j (\partial\lambda^h_{\alpha j}/\partial y^i_\beta)(\partial/\partial x_j) =$$
$$= \alpha!\beta!\tilde{v}^{-1}(i_{(\partial/\partial y^i_\beta)} i_{(\partial/\partial y^h_\alpha)}\, d\Theta) \quad (|\alpha| = 2r-2-k,\ |\beta| = k+1)$$

We shall now compare (7.6) with (7.5). Let us fix two multi-indices α, β such that $|\alpha| < r$, $|\beta| < 2r-1$, $|\alpha| \leqslant |\beta|$, $|\alpha| + |\beta| = 2r-1$. We set $k = |\beta| - 1$, so that $|\alpha| = 2r-2-k$ and $r-1 \leqslant k \leqslant 2r-3$. Since $|\alpha| > 0$, $|\beta| > 0$ in this case, from Proposition 7.1 and formula (5.6) we obtain $i_{(\partial/\partial y^i_\beta)} i_{(\partial/\partial y^h_\alpha)}\Phi = = i_{(\partial/\partial y^i_\beta)} i_{(\partial/\partial y^h_\alpha)}(d\Theta)$, and (7.5) becomes

$$((dx)^\alpha \otimes (\partial/\partial y_h), (dx)^\beta \otimes (\partial/\partial y_i))^j_{(2r-1-k,k+2)} = -\alpha!\beta!(\partial\lambda^h_j/\partial y^i_\beta).$$

Hence,

$$(7.7) \qquad B_k = (\ ,\ )_{(2r-1-k,k+2)} \qquad (r-1 \leqslant k \leqslant 2r-3)$$

Lemma 7.3. *The bilinear mappings B_k, $r-1 \leqslant k \leqslant 2r-2$, do not depend on the derivation laws chosen. Actually, they only depend on the Hessian metric of the Lagrangian $\mathcal{L}$.*

PROOF OF THE LEMMA. According to (6.10) and (6.11) we have

$$\partial\lambda^h_{\alpha j}/\partial y^i_\beta = \sum_{|\sigma|=0}^{r-1-|\alpha|} C^\sigma_{\alpha j}(\partial/\partial y^i_\beta)\mathbb{D}^\sigma(\partial\mathcal{L}/\partial y^h_{\alpha+(j)+\sigma})$$

$$(|\alpha| = 2r-2-k,\ |\beta| = k+1,\ r-1 \leqslant k \leqslant 2r-2).$$

On the other hand, we note that for every $f \in C^\infty(J^k)$ the following formulas hold true:

$$\begin{aligned} &\partial/\partial y^i_\beta(\mathbb{D}^\sigma f) = 0, && \text{if } |\sigma| + k < |\beta| \\ &\partial/\partial y^i_\beta(\mathbb{D}^\sigma f) = \partial f/\partial y^i_{\beta-\sigma}, && \text{if } |\sigma| + k = |\beta|. \end{aligned}$$

(This is proved by induction on $|\sigma|$ using the identity $[\partial/\partial y^i_\beta, \mathbb{D}_j] = \partial/\partial y^i_{\beta-(j)}$.) Hence,

$$\partial\lambda^h_{\alpha j}/\partial y^i_\beta = \sum_{|\sigma|=r-1-|\alpha|} C^\sigma_{\alpha j}(\partial^2\mathcal{L}/\partial y^i_{\beta-\sigma}\partial y^h_{\alpha+(j)+\sigma}). \tag{7.8}$$

As $|\beta - \sigma| = |\alpha + (j) + \sigma| = r$, the lemma follows and the proof of the theorem is complete.

Corollary 7.4. *For every vector field D in J^{2r-1} vertical over J^{2r-2} the valued $(n-1)$-form $i_D E$ does not depend on the derivation laws chosen.*

PROOF. Since D lies in $T(J^{2r-1}/J^{2r-2})$, from (7.3) we obtain $i_D d\Theta = -\theta^{(1)} \wedge (i_D E)$. Locally, there exist ordinary n-forms such that $E = \sum_h w_h \otimes dy_h$, and by applying $i_{(\partial/\partial y_h)}$ to the first equation, by virtue of Proposition 7.1, we have $i_D w_h = -i_{(\partial/\partial y_h)} i_D d\Theta$. Thus, $i_D w_h$ is a section of $\Lambda^{n-1}T^*(X)_{J^{2r-1}}$. Therefore, it is completely determined by $\tilde{v}^{-1}(i_D w_h) = B_{2r-2}(\partial/\partial y_h, D)$. The result now follows from the previous lemma.

Corollary 7.5. *Let E, E' be the Euler-Lagrange forms associated to an r-order Lagrangian density with respect to the derivation laws (∇_0, ∇), (∇'_0, ∇'), respectively. Then, there exists a unique $Hom_{J^{2r-1}}(V(J^{2r-2}), V^*(Y))$-valued $(n-1)$-form η on J^{2r-1} horizontal over X such that*

$$E' - E = \theta^{2r-1} \wedge \eta.$$

In particular, for every local section s of Y, the valued form $E_{|j^{2r-1}s}$ does not depend on the derivation laws chosen.

PROOF. We set $E = \sum_i w_i \otimes dy_i$, $E' = \sum_i w'_i \otimes dy_i$ and $G_i = w'_i(\mathbb{D}_1, \ldots, \mathbb{D}_n) - w_i(\mathbb{D}_1, \ldots, \mathbb{D}_n)$. From (7.3) we obtain

$$i_{\mathbb{D}_n}\cdots i_{\mathbb{D}_1}(d\Theta)=\sum_{j=1}^{n}(-1)^{j+n}L_{\mathbb{D}_j}\left(\sum_{k=1}^{r}\theta^{(k)}\circ\Omega_k(\mathbb{D}_1,\ldots,\hat{\mathbb{D}}_j,\ldots,\mathbb{D}_n)\right)+$$
$$+\,i_{\mathbb{D}_n}\cdots i_{\mathbb{D}_1}(d\mathfrak{L}\wedge v)=(-1)^n\theta^{(1)}\circ(i_{\mathbb{D}_n}\cdots i_{\mathbb{D}_1}E).$$

Writing down the corresponding equation for E' and subtracting, we have

$$\sum_i G_i\theta_0^i=\sum_j L_{\mathbb{D}_j}\Big(\sum_i\sum_{|\alpha|<r}g_{\alpha j}^i\theta_\alpha^i\Big)=\sum_{i,j}\sum_{|\alpha|<r}(\mathbb{D}_jg_{\alpha j}^i)\theta_\alpha^i+\sum_{i,j}\sum_{|\alpha|<r}g_{\alpha j}^i\theta_{\alpha+(j)}^i,$$

for certain differentiable functions $g_{\alpha j}^i$ on J^{2r-1}. Therefore,

(*a*) $\quad G_i=\sum_j(\mathbb{D}_jg_{0j}^i).$

(*b*) $\quad \sum_j(\mathbb{D}_jg_{\alpha j}^i)+\sum_{\beta+(j)=\alpha}g_{\beta j}^i=0\quad(0<|\alpha|<r).$

(*c*) $\quad \sum_{\beta+(j)=\alpha}g_{\beta j}^i=0\quad(|\alpha|=r).$

From (*b*) it is verified that the function $G_i^k=(-1)^k\sum_j\sum_{|\beta|=k}\mathbb{D}^{\beta+(j)}g_{\beta j}^i$ does not depend on the index $k=0,\ldots,r-1$. But $G_i^0=G_i$ and $G_i^{r-1}=0$, as follows from (*a*) and (*c*), respectively. Hence, $G_i=0$. Therefore, by Proposition 7.1, we can write

$$E'-E=\sum_{h,i}\sum_j\sum_{|\alpha|\le 2r-1}\eta_{\alpha j}^{hi}\theta_\alpha^h\wedge v_j\otimes dy_i\qquad(\eta_{\alpha j}^{hi}\in C^\infty(J^{2r-1}))$$

and, by virtue of the preceding corollary, the coefficients $\eta_{\alpha j}^{hi}$ for $|\alpha|=2r-1$ must vanish. Thus, $\eta=\sum_{h,i}\sum_j\sum_{|\alpha|<2r-1}\eta_{\alpha j}^{hi}v_j\otimes dy_\alpha^h\otimes dy_i$ is the unique form fulfilling the conditions of the statement.

Corollary 7.6. *Let Θ, Θ' be the Poincaré-Cartan forms associated to an r-order Lagrangian density with respect to the derivation laws (∇_0,∇), (∇_0',∇'), respectively. There exists a $\mathrm{Hom}_{J^{2r-1}}(V(J^{2r-2}),V^*(J^{r-1}))$-valued $(n-1)$-form $\bar\eta$ on J^{2r-1} (not necessarily unique) such that*

$$d\Theta'-d\Theta=\theta^r\wedge(\theta^{2r-1}\wedge\bar\eta).$$

PROOF. This is an immediate consequence of formula (7.3) and the previous corollary.

Corollary 7.7. *The Poincaré-Cartan form Θ is projectable to J^{2r-h} for $h=2,\ldots,r$ if and only if B_k vanishes for $k=2r-h,\ldots,2r-2$. Thus, since the bilinear mapping B_k do not depend on the derivation laws chosen, if the form Θ corresponding to the pair ∇_0, ∇ is projectable to J^{2r-h}, then it is also true for the form Θ' corresponding to any other pair of derivation laws.*

PROOF. This follows from the first equality in formula (7.6).

Remark. According to formula (7.8) a sufficient condition for the form Θ to be projectable to J^r is that $\mathcal{L}: J^r \to \mathbb{R}$ must be an affine function over J^{r-1}. This condition is also necessary if $\dim X = n = 1$. Note that in this case formula (7.8) reads $\partial\lambda^h_{\alpha 1}/\partial y^i_\beta = (-1)^{r-1-\alpha}(\partial^2\mathcal{L}/\partial y^h_r y^i_r)$.

8. Analysis of how Poincaré-Cartan forms depend on the derivation laws chosen

Theorem 8.1. *The Poincaré-Cartan form Θ associated to an r-order Lagrangian density $\mathcal{L}v$ with respect to a pair of derivation laws ∇_0, ∇ does not depend on the vertical derivation law ∇. In fact, the value taken by Θ at a point $j^{2r-1}_x(s)$ only depends on $j^{r-2}_x(\mathrm{sym}\,\nabla_0)$, where sym ∇_0 means the symmetric connection associated to ∇_0.*

PROOF. Let Θ' be the Poincaré-Cartan form constructed with the same linear connection ∇_0 of the manifold X and another derivation law ∇' in the vertical bundle. According to formula (6.6), locally we have

$$\Theta = \sum_{h,j}\sum_{|\beta|=0}^{r-1}(-1)^{j-1}\lambda^h_{\beta j}\theta^h_\beta \wedge v_j + \mathcal{L}v \quad \text{and} \quad \Theta' = \sum_{h,j}\sum_{|\beta|=0}^{r-1}(-1)^{j-1}\lambda'^h_{\beta j}\theta^h_\beta \wedge v_j + \mathcal{L}v.$$

We set:

$$G^h_{\alpha j} = \sum_i \sum_{\alpha\le\sigma,\,|\sigma|\le r-1} (\sigma + (j))!\binom{|\sigma|}{|\alpha|} a^{hi}_{\sigma-\alpha} F^i_{\sigma+(j)} \qquad (0 \leqslant |\alpha| \leqslant r-1),$$

and similarly for the form Θ'. Hence, from formulas (6.7) and (3.12) we obtain

$$\lambda^h_{\beta j} = \sum_{|\alpha|=|\beta|}^{r-1} \mathring{a}_{\beta\alpha} G^h_{\alpha j} \qquad (0 \leqslant |\beta| \leqslant r-1).$$

Functions $G^h_{\alpha j}$ satisfy the following property: If $\alpha + (j) = \alpha' + (j')$, then $G^h_{\alpha j} = G^h_{\alpha' j'}$. Actually, if $\alpha + (j) = \alpha' + (j')$, we have $\alpha = \tau + (j')$, $\alpha' = \tau + (j)$ for a certain multi-index τ, and, thus, from the definition of $G^h_{\alpha j}$ we obtain

$$G^h_{\alpha j} = G^h_{\tau+(j'),j} = \sum_i \sum_{\substack{|\sigma|\le r-1 \\ \tau\le\sigma-(j')}} (\sigma + (j))!\binom{|\sigma|}{|\tau|+1} a^{hi}_{\sigma-(j')-\tau} F^i_{\sigma+(j)} =$$

(by setting $\sigma' = \sigma - (j')$)

$$G^h_{\alpha j} = \sum_i \sum_{\substack{|\sigma'| \leq r-2 \\ \tau \leq \sigma'}} (\sigma' + (j) + (j'))! \binom{|\sigma'| + 1}{|\tau| + 1} a^{hi}_{\sigma' - \tau} F^i_{\sigma' + (j) + (j')} =$$

(by setting $\sigma'' = \sigma' + (j)$)

$$= \sum_i \sum_{\substack{|\sigma''| \leq r-1 \\ \tau \leq \sigma'' - (j)}} (\sigma'' + (j'))! \binom{|\sigma''|}{|\tau| + 1} a^{hi}_{\sigma'' - (j) - \tau} F^i_{\sigma'' + (j')} = G^h_{\tau + (j), j'} =$$

$$= G^h_{\alpha' j'}.$$

We can therefore define functions $\bar{G}^h_\alpha$ $(1 \leqslant |\alpha| \leqslant r)$ such that $G^h_{\alpha j} = \bar{G}^h_{\alpha + (j)}$, for $|\alpha| = 0, \ldots, r - 1$.

On the other hand, from formula of Corollary 7.6 we deduce that

$$i_{\mathbb{D}_1} \cdots i_{\mathbb{D}_n} d(\Theta' - \Theta) = 0.$$

Hence,

$$(a) \quad \sum_j \mathbb{D}_j(\lambda'^h_{0j} - \lambda^h_{0j}) = 0$$

$$(b) \quad \sum_i \mathbb{D}_j(\lambda'^h_{\beta j} - \lambda^h_{\beta j}) + \sum_i (\lambda'^h_{\beta - (j), j} - \lambda^h_{\beta - (j), j}) = 0 \qquad (0 < |\beta| < r)$$

$$(c) \quad \sum_j (\lambda'^h_{\beta - (j), j} - \lambda^h_{\beta - (j), j}) = 0 \qquad (|\beta| = r).$$

We shall now prove by descending recurrence on $k = 0, \ldots, r - 1$ that

$$(*) \qquad \lambda^h_{\beta j} = \lambda'^h_{\beta j} \quad \text{and} \quad G^h_{\alpha j} = G'^h_{\alpha j} \quad (|\alpha| = |\beta| = k).$$

For $k = r - 1$, it follows from (6.8) that $\lambda^h_{\beta j} = \lambda'^h_{\beta j}$ $(|\beta| = r - 1)$, and from the definition of $G^h_{\alpha j}$ we obtain $G^h_{\alpha j} = (\alpha!/|\alpha|!)\lambda^h_{\alpha j}$ when $|\alpha| = r - 1$. Hence, in this case $G^h_{\alpha j} = G'^h_{\alpha j}$. Let us assume that conditions (*) are fulfilled for $r - 1, r - 2, \ldots, k > 0$. Thus, equation (b) for $|\beta| = k$ becomes,

$$0 = \sum_j (\lambda'^h_{\beta - (j), j} - \lambda^h_{\beta - (j), j}) = \sum_j \sum_{|\alpha| = k-1} \mathring{a}_{\beta - (j), \alpha} (\bar{G}'^h_{\alpha + (j)} - \bar{G}^h_{\alpha + (j)}) =$$

$$= \left(\sum_j \frac{(k-1)!}{(\beta - (j))!} \right) (\bar{G}'^h_\beta - \bar{G}^h_\beta) = (k!/\beta!)(\bar{G}'^h_\beta - \bar{G}^h_\beta).$$

Therefore, $\bar{G}'^h_\beta = \bar{G}^h_\beta$; or in other words, $G^h_{\alpha j} = G'^h_{\alpha j}$ for $|\alpha| = k - 1$. Furthermore, for $|\beta| = k - 1$ we have

$$\lambda'^h_{\beta j} = \sum_{|\alpha| = |\beta|}^{r-1} \mathring{a}_{\beta\alpha} G'^h_{\beta j} = \frac{(k-1)!}{\beta!} G'^h_{\beta j} + \sum_{|\alpha| = k}^{r-1} \mathring{a}_{\beta\alpha} G^h_{\alpha j} =$$

$$= \frac{(k-1)!}{\beta!} G^h_{\beta j} + \sum_{|\alpha| = k}^{r-1} \mathring{a}_{\beta\alpha} G^h_{\alpha j} = \sum_{|\alpha| = |\beta|}^{r-1} \mathring{a}_{\beta\alpha} G^h_{\alpha j} = \lambda^h_{\beta j}.$$

Thus, (*) is proved for $|\alpha| = |\beta| = k - 1$. In particular, we conclude that the corresponding coefficients of forms Θ and Θ' coincide. Hence, $\Theta = \Theta'$.

Since Θ does not depend on ∇, this form can be calculated using the flat vertical derivation law associated to the local basis $\partial/\partial x_j$, $\partial/\partial y_i$ (i.e., by taking $\Gamma^a_{ji} = \bar{\Gamma}^a_{hi} = 0$). In this case, we have $a^{hi}_\alpha = 0$ when $\alpha > 0$ and $A^{hi}_{\beta\alpha} = \delta_{hi}\mathring{a}_{\beta\alpha}$. Consequently, equations (5.2), (6.4) and (6.7) can now be written respectively as follows:

$$(8.1) \qquad |\beta|!f^h_\beta + \sum_{|\alpha| = |\beta|+1}^{r} \alpha!\mathring{a}_{\beta\alpha}f^h_\alpha = \sum_{|\alpha| = |\beta|}^{r} \alpha!\mathring{a}_{\beta\alpha}f^h_\alpha = \partial\mathcal{L}/\partial y^h_\beta \qquad (0 \leqslant |\beta| \leqslant r-1).$$

$$(8.2) \qquad F^i_\sigma = f^i_\sigma - \sum_j (1+\sigma_j)(\mathbb{D}_j F^i_{\sigma+(j)}) - \sum_{j,k,l}(1+\sigma_k-\delta_{kl})(1+\sigma_j+\delta_{jk}-\delta_{jl})\mathring{\Gamma}^l_{jk}F^i_{\sigma+(j)+(k)-(l)}, \qquad (|\sigma| = 1, \dots, r-1).$$

$$(8.3) \qquad \lambda^h_{\beta j} = \sum_{|\alpha|=|\beta|}^{r-1} (\alpha+(j))!\mathring{a}_{\beta\alpha}F^h_{\alpha+(j)} = \sum_{|\sigma|=|\beta|+1}^{r} \sigma!\mathring{a}_{\beta,\sigma-(j)}F^h_\sigma \qquad (0 \leqslant |\beta| \leqslant r-1).$$

Moreover, a direct computation shows that formulas (8.2) and (3.9) remain true when the components $\mathring{\Gamma}^l_{jk}$ are substituted by the functions $\mathring{\Gamma}^{*l}_{jk} = \frac{1}{2}(\mathring{\Gamma}^l_{jk} + \mathring{\Gamma}^l_{kj})$, which obviously proves tht Θ only depends on sym (∇_0) (see [10; Proposition 7.9 of Chapter III]).

Finally, we shall prove our last assertion of the statement. First we note that $\mathring{a}_{\beta\alpha}(x)$ for $|\beta| < |\alpha|$ only depends on $j_x^{|\alpha|-|\beta|-1}(\nabla_0)$, as is easily checked by induction on $|\alpha|$ using the recurrence relations for the functions $\mathring{a}_{\beta\alpha}$ (formula (3.9)). Formula (8.1) thus implies that f^h_β only depends on $j^{r-1-|\beta|}(\nabla_0)$ for $|\beta| = 0, \dots, r-1$. Similarly, from (8.2) we derive that F^i_σ only depends on $j^{r-1-|\sigma|}(\nabla_0)$ for $|\sigma| = 1, \dots, r-1$. Thus, from (8.3) we conclude that $\lambda^h_{\beta j}$ only depends on $j^{r-2-|\beta|}(\nabla_0)$ for $|\beta| = 0, \dots, r-2$. As the coefficients $\lambda^h_{\beta j}$ ($|\beta| = r-1$) do not depend on ∇_0, the proof of the theorem is complete.

Corollary 8.2. *The Legrende form Ω associated to an r-order Lagrangian density with respect to a pair of derivation laws ∇_0, ∇ does not depend on the vertical derivation law ∇.*

Remark. The forms $\Omega_1, \dots, \Omega_{r-1}$, defined in (5.10) do depend on ∇.

As an example we shall now compute the first group of coefficients of Poincaré-Cartan form Θ depending on the linear connection ∇_0, for an arbitrary

r-order variational problem; that is, $\lambda^h_{\beta j}$ ($|\beta| = r - 2$). According to the above three formulas, we have

$$\lambda^h_{\beta u} = \frac{1+\beta_u}{r-1}(\partial\mathcal{L}/\partial y^h_{\beta+(u)}) - \frac{1+\beta_u}{r(r-1)}\sum_j(1+\beta_j+\delta_{ju})\mathbb{D}_j(\partial\mathcal{L}/\partial y^h_{\beta+(j)+(u)}) -$$

$$-\frac{1+\beta_u}{r!(r-1)}\sum_{|\alpha|=r}\alpha!\mathring{a}_{\beta+(u),\alpha}(\partial\mathcal{L}/\partial y^h_\alpha) + \sum_{|\alpha|=r}(\alpha!/r!)\mathring{a}_{\beta,\alpha-(u)}(\partial\mathcal{L}/\partial y^h_\alpha) +$$

$$-\frac{1+\beta_u}{r(r-1)}\sum_{j,k,l}(1+\beta_k+\delta_{ku}-\delta_{kl})(1+\beta_j+\delta_{ju}+$$

$$+\delta_{jk}-\delta_{jl})\mathring{\Gamma}^l_{jk}(\partial\mathcal{L}/\partial y^h_{\beta+(u)+(j)+(k)+(l)})$$

Moreover, using the recurrence formulas for the functions $\mathring{a}_{\beta\alpha}$ we find

$$\mathring{a}_{\beta+(u),\alpha} = \sum_j \mathring{a}_{\beta+(u)-(j),\alpha-(j)} -$$

$$-\sum_{j,k,l}(1+\alpha_l-\delta_{jl}-\delta_{kl})\mathring{\Gamma}^l_{jk}\mathring{a}_{\beta+(u),\alpha+(l)-(j)-(k)},$$

and finally,

$$(8.4)\quad \lambda^h_{\beta u} = \frac{1+\beta_u}{r-1}(\partial\mathcal{L}/\partial y^h_{\beta+(u)}) -$$

$$-\frac{1+\beta_u}{r(r-1)}\sum_j(1+\beta_j+\delta_{ju})\mathbb{D}_j(\partial\mathcal{L}/\partial y^h_{\beta+(j)+(u)}) -$$

$$-\sum_{|\alpha|=r}(\alpha!/r!)\left[\frac{1+\beta_u}{r-1}\sum_j\mathring{a}_{\beta+(u)-(j),\alpha-(j)} - \mathring{a}_{\beta,\alpha-(u)}\right](\partial\mathcal{L}/\partial y^h_\alpha)$$

$$(|\beta| = r-2).$$

Proposition 8.3. *For variational problems of order $r \leqslant 2$ on an arbitrary fibred manifold and for variational problems of arbitrary order on a fibred manifold with a 1-dimensional base manifold the Poincaré-Cartan form does not depend on the linear connection ∇_0.*

PROOF. For first order variational problems the result is well-known and, in fact, follows from formula (6.8). For second order variational problems the result follows from the above formula, since in this case the last summand on the right hand-side vanishes. Let us now consider the case $\dim X = n = 1$. Dropping the corresponding index to the base manifold in the local expressions, formulas (3.9), (8.1), (8.2) and (8.3) read, respectively, as follows:

$$\mathring{a}_{\beta\alpha} = \partial\mathring{a}_{\beta,\alpha-1}/\partial x + \mathring{a}_{\beta-1,\alpha-1} - (\alpha-1)\mathring{\Gamma}\mathring{a}_{\beta,\alpha-1},$$

$$\beta! f^h_\beta + \sum_{\alpha=\beta+1}^{r} \alpha!\mathring{a}_{\beta\alpha} f^h_\alpha = \partial\mathcal{L}/\partial y^h_\beta$$

$$F^i_\sigma = f^i_\sigma - (\sigma+1)(\mathbb{D}F^i_{\sigma+1}) - \sigma(\sigma+1)\mathring{\Gamma}F^i_{\sigma+1},$$

$$\lambda^h_\beta = \sum_{\sigma=\beta+1}^{r} \sigma!\mathring{a}_{\beta,\sigma-1}F^h_\sigma$$

It is then easily checked that the following recurrence relation holds true,

$$\lambda^i_{\beta-1} = \partial\mathcal{L}/\partial y^i_\beta - \mathbb{D}\lambda^i_\beta,$$

thus proving the independence of form Θ in this case.

Remark. The above proposition can also be proved without calculation using corollary 7.6 and the system $(a) - (b) - (c)$ in the proof of the previous theorem (see [4], [16]).

We shall now calculate the coefficients of the Poincaré-Cartan form for a third order variational problem. From formula (6.8) we derive in particular,

$$\lambda^h_{(kl)j} = \frac{1}{3}(1 + \delta_{jk} + \delta_{jl})\left(\frac{\partial\mathcal{L}}{\partial y^h_{(jkl)}}\right).$$

The values of the intermediate coefficients are directly deduced from the general formula (8.4). In fact, we have

$$\lambda^h_{(u)u} = \partial\mathcal{L}/\partial y^h_{(uu)} - \frac{1}{6}\sum_j (juu)!\mathbb{D}_j(\partial\mathcal{L}/\partial y^h_{(juu)})$$

$$\lambda^h_{(v)u} = \frac{1}{2}(\partial\mathcal{L}/\partial y^h_{(uv)}) - \frac{1}{6}\sum_j (juv)!\mathbb{D}_j(\partial\mathcal{L}/\partial y^h_{(juv)}) +$$

$$+ \frac{1}{12}\sum_{j,k}(jkv)!\mathring{\Gamma}^u_{jk}(\partial\mathcal{L}/\partial y^h_{(jkv)}) - \frac{1}{12}\sum_{j,k}(jku)!\mathring{\Gamma}^v_{jk}(\partial\mathcal{L}/\partial y^h_{(jku)}) \qquad (u \neq v).$$

In all these formulas we have used the following notations: $(jk) = (j) + (k)$, $(jkl) = (j) + (k) + (l)$, etc. Finally, the first group of coefficients may be calculated by reiterating the above method. We obtain

$$\lambda^h_{0u} = \partial\mathcal{L}/\partial y^h_{(u)} - \frac{1}{2}\sum_j (ju)!\mathbb{D}_j(\partial\mathcal{L}/\partial y^h_{(ju)}) +$$

$$+ \frac{1}{6}\sum_{j,k}(1+\delta_{ju})(1+\delta_{jk}+\delta_{ku})\mathbb{D}_j\mathbb{D}_k(\partial\mathcal{L}/\partial y^h_{(jku)}) +$$

$$+ \frac{1}{12}\sum_{j,k,l}(jkl)!\mathbb{D}_l\{\mathring{\Gamma}^u_{jk}(\partial\mathcal{L}/\partial y^h_{(jkl)})\} -$$

$$- \frac{1}{12}\sum_{j,k,l}(jku)!\mathbb{D}_l\{\mathring{\Gamma}^l_{jk}(\partial\mathcal{L}/\partial y^h_{(jku)}\}.$$

9. Variation formula of Lagrangian density: characterization of critical sections

Proposition 9.1 (Variation formula of Lagrangian density.) *Let Θ be a Poincaré-Cartan form associated to an r-orden Lagrangian density $\mathcal{L}v$. For every infinitesimal contact transformation $\bar{D}$ in J^{2r-1} there exists a $V^*(J^{2r-2})$-valued $(n-1)$-form ξ on J^{2r-1} such that,*

$$L_{\bar{D}}(\mathcal{L}v) = \theta^{(1)}(\bar{D})\circ E + d(i_{\bar{D}}\Theta) + \theta^{2r-1}\wedge\xi. \tag{9.1}$$

The linear functional $\delta_s\mathbb{L}$ defined in §4 is thus given by the following formula:

$$(\delta_s\mathbb{L})(D) = \int_{j^r s} L_D(\mathcal{L}v) = \int_{j^{2r-1}s}\theta^{(1)}(D_{(2r-1)})\circ E \qquad (D\in T^r_c(U)). \tag{9.2}$$

PROOF. According to formula (5.12) and the definition of infinitesimal contact transformations, there exists an endomorphism ϕ of the vertical vector bundle so that,

$$L_{\bar{D}}(\mathcal{L}v) = L_{\bar{D}}\Theta - (L_{\bar{D}}\theta^r)\wedge\Omega - \theta^r\wedge(L_{\bar{D}}\Omega) =$$
$$= i_{\bar{D}}d\Theta + d(i_{\bar{D}}\Theta) - \theta^r\wedge(\phi^*\circ\Omega + L_{\bar{D}}\Omega).$$

On the other hand, decomposition (7.3) implies in particular that there exists $\mathrm{Hom}_{J^{2r-1}}(V(J^{2r-2}), V^*(J^{r-1}))$-valued $(n-1)$-form $\tilde{\eta}$ on J^{2r-1} such that

$$d\Theta = \theta^{(1)}\wedge E + \theta^r\wedge(\theta^{2r-1}\wedge\tilde{\eta}).$$

Hence,

$$i_{\bar{D}}d\Theta = \theta^{(1)}(\bar{D})\circ E - \theta^{(1)}\wedge(i_{\bar{D}}E) + \theta^r(\bar{D})\circ(\theta^{2r-1}\wedge\tilde{\eta}) -$$
$$- \theta^r\wedge(\theta^{2r-1}(\bar{D})\circ\tilde{\eta}) + \theta^r\wedge(\theta^{2r-1}\wedge(i_{\bar{D}}\tilde{\eta})).$$

Thus, in order to obtain formula (9.1) it is sufficient to take

$$\xi = -\phi^*\circ\Omega - L_{\bar{D}}\Omega - i_{\bar{D}}E + \theta^r(\bar{D})\circ\tilde{\eta} - \theta^{2r-1}(\bar{D})\circ\tilde{\eta} - \theta^r\wedge(i_{\bar{D}}\tilde{\eta}).$$

Since the support of D has compact image in U, formula (9.2) follows directly from Stokes' theorem.

Theorem 9.2. (First Characterization.) *Let E be the Euler-Lagrange form associated to an r-order Lagragian density $\mathfrak{L}v$ on the fibred manifold Y with respect to a pair of derivation laws ∇_0, ∇. A section s of Y is critical for the variational problem defined by $\mathfrak{L}v$ if and only if:*

$$E_{|j^{2r-1}s} = 0$$

Furthermore, this condition does not depend on the pair of derivation laws chosen. Thus, the valued differential system on J^{2r-1} given by

$$(\theta^{(k)}, E) \qquad (k = 1, \ldots, 2r-1)$$

constitutes a global and intrinsic version of the Euler-Lagrange equations for higher order variational problems.

PROOF. A section s is critical if and only if $(\delta_s \mathbb{L})(D) = 0$ for every $D \in T^r_c(D)$, where U is the domain of s. By virtue of formula (9.2), this is equivalent to the following condition

$$\int_{j^{2r-1}s} \theta^{(1)}(D_{(2r-1)}) \circ E = 0 \qquad (D \in T^r_c(U)).$$

Since this equation must hold for all vector fields of $T^r_c(U)$, we conclude that $E_{|j^{2r-1}s} = 0$, and conversely. The independence of the derivation laws ∇_0, ∇ follows immediately from Corollary 7.5. Moreover, according to the same corollary, in order to compute $E_{|j^{2r-1}s}$ we can locally use the flat derivation laws. Then, with the notations of (5.2) and (5.4), we have $f_0 = \sum_i (\partial\mathfrak{L}/\partial y_i)\, dy_i$, and

$$\omega_k = \frac{1}{k!} \sum_{i,j} \sum_{|\alpha| = k-1} (-1)^{j-1}(1 + \alpha_j)\left(\frac{\partial \mathfrak{L}}{\partial y^i_{\alpha+(j)}}\right) v_j \otimes \left(\frac{\partial}{\partial x}\right)^{\alpha} \otimes dy_i \, (1 \leqslant k \leqslant r).$$

Thus, from formula (5.8) we obtain

$$(9.3)\, E = \sum_i \left\{ \frac{\partial \mathfrak{L}}{\partial y_i} v + \sum_{h=0}^{r-1} (-1)^{h+1} \sum_j \sum_{|\alpha| = h} (-1)^{j-1} \frac{1 + \alpha_j}{1 + h} d\left(\mathbb{D}^{\alpha} \frac{\partial \mathfrak{L}}{\partial y^i_{\alpha+(j)}} \right) \wedge \right.$$
$$\left. \wedge\, v_j \right\} \otimes dy_i.$$

Hence,

$$E_{|j^{2r-1}s} = \sum_i \left\{ \sum_{k=0}^{r} \sum_{|\beta| = k} (-1)^k \left(\frac{\partial^{|\beta|}}{\partial x^{\beta}}\right)\left(\frac{\partial \mathfrak{L}}{\partial y^i_{\beta}} \circ j^r s\right)\right\} v \otimes dy_i.$$

Theorem 9.3. (Second characterization.) *Let Θ be a Poincaré-Cartan form associated to an r-order Lagrangian density $\mathcal{L}v$ on the fibred manifold Y. A section s of Y is critical for the variational problem defined by $\mathcal{L}v$ if an only if:*

$$(i_D d\Theta)_{|j^{2r-1}s} = 0 \quad \textit{for all vector fields } D \textit{ in } \quad J^{2r-1}.$$

Furthermore, this condition does not depend on the particular Poincaré-Cartan form chosen.

PROOF. From formula (7.3) we deduce that for every vector field D in J^{2r-1}, we have $(i_D d\Theta)_{|j^{2r-1}s} = (\theta^{(1)}(D) \circ E)_{|j^{2r-1}s}$. The result now follows from Theorem 9.2 and Corollary 7.6.

For every open set $U \subset X$, we denote by $\Gamma(U, \mathcal{V})$ the set of critical sections of the variational problem determined by $\mathcal{L}v$ which are defined on the domain U. Since $\Gamma(U, \mathcal{V})$ is the «set of solutions» of a globally defined differential operator, it is clear that $\mathcal{V}$ is a sheaf of sets over the manifold X.

A section $\bar{s}$ (not necesarily holonomic) of the canonical projection $p_{2r-1}: J^{2r-1} \to X$ is said to be a *Hamilton extremal* of the variational problem defined by $\mathcal{L}v$ with respect to the linear connection ∇_0 if $(i_D d\Theta)|_{\bar{s}}$ vanishes for all vector fields D in J^{2r-1}, where Θ is the Poincaré-Cartan form associated with ∇_0. We shall denote by $\bar{\mathcal{V}}(\nabla_0)$ the sheaf of Hamiltonian extremals with respect to the linear connection ∇_0.

According to the second characterization of the critical sections, the jet prolongation $s \to j^{2r-1}s$ induces an injection of $\mathcal{V}$ into each $\bar{\mathcal{V}}(\nabla_0)$.

Remark. Note that condition $(i_D d\Theta)|_{\bar{s}} = 0$ can also be viewed as a differential equation on the linear connection ∇_0.

EXAMPLE. Let us consider the third order variational problem defined on the canonical projection $p: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ by the Lagrangian density $\mathcal{L}v = \frac{1}{2} y_{(3,0)}^2 \, dx_1 \wedge dx_2$.

The Poincaré-Cartan form corresponding to ∇_0 is given by the formula

$$\Theta = y_{(3,0)} \, dy_{(2,0)} \wedge dx_2 - y_{(4,0)} \, dy_{(1,0)} \wedge dx_2 - \frac{1}{2} \mathring{\Gamma}^2_{11} y_{(3,0)} \, dy_{(1,0)} \wedge dx_1 -$$

$$- \frac{1}{2} \mathring{\Gamma}^2_{11} y_{(3,0)} \, dy_{(0,1)} \wedge dx_2 + \left[y_{(5,0)} - \frac{1}{2} \frac{\partial \mathring{\Gamma}^2_{11}}{\partial x_2} y_{(3,0)} - \right.$$

$$\left. - \frac{1}{2} \mathring{\Gamma}^2_{11} y_{(3,1)} \right] dy \wedge dx_2 - \frac{1}{2} \left[\frac{\partial \mathring{\Gamma}^2_{11}}{\partial x_1} y_{(3,0)} + \mathring{\Gamma}^2_{11} y_{(4,0)} \right] dy \wedge dx_1 +$$

$$+\left[y_{(2,0)}y_{(4,0)} - y_{(1,0)}y_{(5,0)} - \frac{1}{2}y^2_{(3,0)} - \frac{1}{2}\frac{\partial\mathring{\Gamma}^2_{11}}{\partial x_1}y_{(0,1)}y_{(3,0)} + \right.$$

$$\left. + \frac{1}{2}\frac{\partial\mathring{\Gamma}^2_{11}}{\partial x_2}y_{(1,0)}y_{(3,0)} - \frac{1}{2}\mathring{\Gamma}^2_{11}y_{(0,1)}y_{(4,0)} + \frac{1}{2}\mathring{\Gamma}^2_{11}y_{(1,0)}y_{(3,1)}\right] dx_1 \wedge dx_2,$$

and the differential system which determines Hamiltonian extremals $\bar{s} = (s_\alpha)_{|\alpha| \le 5}$, is the following:

$$s_{(1,0)} = \frac{\partial s_0}{\partial x_1}, \qquad s_{(2,0)} = \frac{\partial s_{(1,0)}}{\partial x_1}, \qquad s_{(4,0)} = \frac{\partial s_{(3,0)}}{\partial x_1},$$

$$s_{(5,0)} = \frac{\partial s_{(4,0)}}{\partial x_1} + \frac{1}{2}\mathring{\Gamma}^2_{11}\left(s_{(3,1)} - \frac{\partial s_{(3,0)}}{\partial x_2}\right)$$

$$s_{(3,0)} = \frac{\partial s_{(2,0)}}{\partial x_1} + \frac{1}{3}\mathring{\Gamma}^2_{11}\left(\frac{\partial s_{(1,0)}}{\partial x_2} - \frac{\partial s_{(0,1)}}{\partial x_1}\right) + \frac{1}{2}\frac{\partial\mathring{\Gamma}^2_{11}}{\partial x_1}\left(\frac{\partial s_0}{\partial x_2} - s_{(0,1)}\right)$$

$$\frac{\partial s_{(5,0)}}{\partial x_1} = \frac{1}{2}\mathring{\Gamma}^2_{11}\left(\frac{\partial s_{(3,1)}}{\partial x_1} - \frac{\partial s_{(4,0)}}{\partial x_2}\right) + \frac{1}{2}\frac{\partial\mathring{\Gamma}^2_{11}}{\partial x_1}\left(s_{(3,1)} - \frac{\partial s_{(3,0)}}{\partial x_2}\right) +$$

$$+ \frac{1}{2}\frac{\partial\mathring{\Gamma}^2_{11}}{\partial x_2}\left(\frac{\partial s_{(3,0)}}{\partial x_1} - s_{(4,0)}\right)$$

Note that even for sections which are holonomic up to third order the above system depends on the connection chosen; only for 4-holonomic sections does the system become independent of ∇_0.

10. Functoriality of Poincaré-Cartan forms, infinitesimal symmetries and Noether invariants

In order to emphasize the dependence on the linear connection ∇_0, in this section we shall denote by $\Theta(\nabla_0, \mathcal{L}v)$ the Poincaré-Cartan form associated to the Lagrangian density $\mathcal{L}v$ constructed with the connection ∇_0. As we have seen in Theorem 8.1, $\Theta(\nabla_0, \mathcal{L}v)$ only depends on $j^{r-2}(\nabla_0)$. Let us denote by $K \to X$ the affine bundle of linear connections of the manifold X. We can thus define an ordinary n-form $\Theta(\mathcal{L}v)$ on the manifold $Z = J^{r-2}(K) \times_X J^{2r-1}(Y)$ by the formula

$$\Theta(\mathcal{L}v)_{(j^{r-2}_x\nabla_0, j^{2r-1}_x s)} = \Theta(\nabla_0, \mathcal{L}v)_{(j^{2r-1}_x s)}.$$

We shall call the form $\Theta(\mathcal{L}v)$ the *universal Poincaré-Cartan form* associated to the Lagrangian density $\mathcal{L}v$.

Remark. It follows from the previous definition that $j^{r-2}(\nabla_0)^*\Theta(\mathcal{L}v) = \Theta(\nabla_0, \mathcal{L}v)$. Note also that $\Theta(\mathcal{L}v)$ is horizontal over $J^{2r-1}(Y)$. The local expression of the form $\Theta(\mathcal{L}v)$ is also given by formulas (8.1), (8.2) and (8.3); but functions $\lambda^h_{\beta j}$ must now be considered as differentiable functions on $J^{r-2}(K)$.

Theorem 10.1. (Functoriality of Poincaré-Cartan forms.) *Let ψ be an automorphism of the fibred manifold Y; that is,*

$$\begin{array}{ccc} Y & \xrightarrow{\ \psi\ } & Y \\ p \downarrow & & \downarrow p \\ X & \xrightarrow[\ \bar{\psi}\ ]{} & X \end{array}$$

The automorphism $\tilde{\psi}\colon L(X) \to L(X)$ of the bundle of linear frames induced by $\bar{\psi}$ maps ∇_0 into a connection $\bar{\nabla}_0 = \tilde{\psi}(\nabla_0)$ (see [10, pp. 79 and 226]). *Then*

$$J^{2r-1}(\psi)^*(\Theta(\bar{\nabla}_0, \mathcal{L}v)) = \Theta(\nabla_0, J^r(\psi)^*(\mathcal{L}v)).$$

In particular, if ψ is a vertical automorphism of Y,

$$J^{2r-1}(\psi)^*(\Theta(\nabla_0, \mathcal{L}v)) = \Theta(\nabla_0, J^r(\psi)^*(\mathcal{L}v)).$$

PROOF. Let (x_j, y_i) be a fibred coordinate system for Y. We define a new fibred coordinate system $(\bar{x}_j, \bar{y}_i)$ by setting $\bar{x}_j = x_j \circ \bar{\psi}^{-1}$, $\bar{y}_i = y_i \circ \psi^{-1}$, and denote by $(\bar{y}^i_\alpha)$ the corresponding coordinate system induced on the jet bundles. We also denote by $\bar{\theta}^i_\alpha$ the components of the structure forms in the coordinate system $(\bar{y}^i_\alpha)$. Let $\bar{\nabla}$ be the flat derivation law associated to $(\bar{x}_j, \bar{y}_i)$. According to (3.6), there exist unique functions $\mathring{\bar{a}}_{\beta\alpha} \in C^\infty(X)$ such that

$$\bar{L}^u\bar{\theta}^h_0 = \frac{1}{u!}\sum_{|\beta| \le u}\sum_{|\alpha| = u} \mathring{\bar{a}}_{\beta\alpha}\bar{\theta}^h_\beta \otimes (d\bar{x})^\alpha,$$

where the total Lie derivative $\bar{L}$ is taken with respect to $(\bar{\nabla}_0, \bar{\nabla})$. Functions $\mathring{\bar{a}}_{\beta\alpha}$ fulfil the following conditions $\mathring{\bar{a}}_{\beta\alpha} \circ \bar{\psi} = \mathring{a}_{\beta\alpha}$, where $\mathring{a}_{\beta\alpha}$ stand for the functions associated to ∇_0 and the falt derivation law ∇ determined by (x_j, y_i). This is easily verified by induction on $|\alpha|$ using the recurrence relations for these functions and the fact that the components of the linear connection $\bar{\nabla}_0$ with respect to $(\bar{x}_j)$ are $\mathring{\bar{\Gamma}}^k_{ij} = \mathring{\Gamma}^k_{ij} \circ \bar{\psi}^{-1}$. On the other hand, let $\bar{\lambda}^h_{\beta j}$ be the coefficients of the Poincaré-Cartan form $\Theta(\bar{\nabla}_0, \mathcal{L}v)$ in the coordinate system $(\bar{x}_j, \bar{y}^i_\alpha)$. We shall also use the obvious notations for the sections $\bar{f}_k$, $\bar{F}_k$ associated to this form with respect to the derivation laws $\bar{\nabla}_0$ and $\bar{\nabla}$.

We can check by induction on $|\beta|$ that the following formula holds true,

$$\bar{y}^h_\beta \circ J^k(\psi) = y^h_\beta \qquad (|\beta| \leqslant k).$$

Hence, $dy^h_\beta = J^k(\psi)^*(d\bar{y}^h_\beta)$, and consequently,

$$J^r(\psi)^*\bar{\theta}^h_\beta = \theta^h_\beta \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \bar{y}^h_\beta} \circ J^r(\psi) = \frac{\partial}{\partial y^h_\beta}(\mathcal{L} \circ J^r(\psi)).$$

Let $\mathcal{L}' \in C^\infty(J^r)$ be the unique function such that $J^r(\psi)^*(\mathcal{L}v) = \mathcal{L}'v$; or equivalently, $\mathcal{L}' = \rho(\mathcal{L} \circ J^r(\psi))$, where ρ is defined by the condition $\bar{\psi}^* v = \rho v$. Let us denote by $\lambda^h_{\beta j}$ the coefficients of the Poincaré-Cartan form $\Theta(\nabla_0, \mathcal{L}'v)$ in the coordinate system (x_j, y^i_α). The components (in the same coordinate system) of the sections f_k, F_k associated to this form with respect to the derivation laws ∇_0, ∇ will be denoted by f^h_β, F^i_σ, respectively. Then, from the last formula above and Eq. (8.1) for $\bar{f}_k$ we derive $f^h_\beta = \bar{f}^h_\beta \circ J^r(\psi)$. Now, by recurrence on $|\sigma|$, it follows from formula (8.2) that $F^i_\sigma = \bar{F}^i_\sigma \circ J^{2r-1}(\psi)$. Thus, we obtain $\lambda^h_{\beta j} = \bar{\lambda}^h_{\beta j} \circ J^{2r-1}(\psi)$, and therefore

$$J^{2r-1}(\psi)^*(\Theta(\bar{\nabla}_0, \mathcal{L}v)) = \sum_{h,j} \sum_{|\beta|=0}^{r-1} (-1)^{j-1}\lambda^h_{\beta j}\theta^h_\beta \wedge (\bar{\psi}^*\bar{v}_j) + J^r(\psi)^*(\mathcal{L}v) = \\ = \Theta(\nabla_0, \mathcal{L}'v),$$

since $\bar{\psi}^*\bar{v}_j = v_j$. This completes the proof of the theorem.

Corollary 10.2. (Infinitesimal functoriality of the universal Poincaré-Cartan form.) *Let D be a p-projectable vector field in the fibred manifold Y. We denote by $\bar{D}$ its projection on X. Let $\tilde{D}$ be the vector field induced by $\bar{D}$ in the affine bundle of linear connections of X. Then*

$$L_{(\tilde{D}_{(r-2)}, D_{(2r-1)})}(\Theta(\mathcal{L}v)) = \Theta(L_{D_{(r)}}(\mathcal{L}v)). \tag{10.1}$$

In particular, if D is p-vertical,

$$L_{D_{(2r-1)}}(\Theta(\mathcal{L}v)) = \theta(L_{D_{(r)}}(\mathcal{L}v)).$$

PROOF. Let τ_t, $\bar{\tau}_t$, $\tilde{\tau}_t$ be the local 1-parameter groups generated by D, $\bar{D}$, $\tilde{D}$, respectively. The 1-parameter group generated by $(\tilde{D}_{(r-2)}, D_{(2r-1)})$ is $(J^{r-2}(\tilde{\tau}_t), J^{2r-1}(\tau_t))$, as was pointed out in (4.4). Thus, formula (10.1) is equivalent to the following

$$(J^{r-2}(\tilde{\tau}_t), J^{2r-1}(\tau_t))^*\Theta(\mathcal{L}v) = \Theta(J^r(\tau_t)^*(\mathcal{L}v)). \tag{a}$$

Moreover, since $\Theta(\mathcal{L}v)$ is horizontal over $J^{2r-1}(Y)$, in order to verify (b) it will be sufficient to prove that for every linear connection ∇_0 of X, we have:

$$j^{r-2}(\nabla_0)^*(J^{r-2}(\tilde{\tau}_t), J^{2r-1}(\tau_t))^*\Theta(\mathcal{L}v) = J^{2r-1}(\tau_t)^* j^{r-2}(\tilde{\tau}_t(\nabla_0))^*\Theta(\mathcal{L}v) = \\ = j^{r-2}(\nabla_0)^*\Theta(J^r(\tau_t)^*(\mathcal{L}v)).$$

But according to the previous remark this is equivalent to

$$J^{2r-1}(\tau_t)^*(\Theta(\tilde{\tau}_t(\nabla_0), \mathcal{L}v)) = \Theta(\nabla_0, J^r(\tau_t)^*(\mathcal{L}v)).$$

The result follows immeditely from the preceding theorem.

A p-projectable vector field D in the manifold Y is said to be an *infinitesimal symmetry* of an r-order Lagrangian density $\mathcal{L}v$ if $L_{D_{(r)}}(\mathcal{L}v) = 0$.

For every open set $V \subset Y$ we denote by $\Gamma(V, \mathfrak{D})$ the set of infinitesimal symmetries of the Lagrangian density $\mathcal{L}v$. It follows directly from the above definition that $\mathfrak{D}$ is a sheaf of Lie algebras over Y. Moreover, we denote by $\mathfrak{D}^v$ the ideal of $\mathfrak{D}$ determined by the p-vertical infinitesimal symmetries. We have thus an exact sequence of sheaves over Y,

$$0 \to \mathfrak{D}^v \to \mathfrak{D} \to p^{-1}(\mathrm{Der}_X) \to 0.$$

Corollary 10.3. *A p-projectable vector field D in Y is an infinitesimal symmetry of $\mathcal{L}v$ if and only if,*

$$L_{(\tilde{D}_{(r-2)}, D_{(2r-1)})}(\Theta(\mathcal{L}v)) = 0.$$

In particular, p-vertical infinitesimal symmetries are characterized by the condition

$$L_{D_{(2r-1)}}(\Theta(\mathcal{L}v)) = 0.$$

If D is an infinitesimal symmetry of $\mathcal{L}v$, then the ordinary $(n-1)$-form $i_{D_{(2r-1)}}\Theta(\mathcal{L}v)$ will be called the *Noether invariant* corresponding to D. Note that $i_{D_{(2r-1)}}\Theta(\mathcal{L}v)$ is a differential form on the manifold $Z = J^{r-2}(K) \times_X J^{2r-1}(Y)$. The Noether invariant corresponding to D with respect to the connection ∇_0 is, by definition, the ordinary $(n-1)$-form on the manifold $J^{2r-1}(Y)$,

$$j^{r-2}(\nabla_0)^*(i_{D_{(2r-1)}}\Theta(\mathcal{L}v)) = i_{D_{(2r-1)}}\Theta(\nabla_0, \mathcal{L}v).$$

Note also that the Noether invariant corresponding to D really only depends on $D_{(r)}$.

For a different approach to the theory of Noether invariants one may consult [15].

Proposition 10.4. *If D is an infinitesimal symmetry of $\mathcal{L}v$, for every critical section s we have*:

$$d[(i_{D_{(2r-1)}}\Theta(\nabla_0, \mathcal{L}v))_{|j^{2r-1}s}] = 0.$$

Thus, once a linear connection ∇_0 has been fixed, each Noether invariant defines a function on v with values in the space of closed $(n-1)$-forms of X by the formula $f_D(s) = (i_{D_{(2r-1)}}\Theta(\nabla_0, \mathcal{L}v))_{|j^{2r-1}s}$.

PROOF. Since D is an infinitesimal symmetry of $\mathfrak{L}v$, we have

$$L_{D_{(2r-1)}}\Theta(\nabla_0, \mathfrak{L}v) = L_{D_{(2r-1)}}(\theta^r \wedge \Omega) = (L_{D_{(r)}}\theta^r) \wedge \Omega + \theta^r \wedge (L_{D_{(2r-1)}}\Omega)$$
$$= \theta^r \wedge (\phi^* \circ \Omega + L_{D_{(2r-1)}}\Omega).$$

Hence,

$$(L_{D_{(2r-1)}}\Theta(\nabla_0, \mathfrak{L}v))_{|j^{2r-1}s} = d[(i_{D_{(2r-1)}}\Theta(\nabla_0, \mathfrak{L}v))_{|j^{2r-1}s}] + $$
$$+ (i_{D_{(2r-1)}} d\Theta(\nabla_0, \mathfrak{L}v))_{|j^{2r-1}s} = 0,$$

and the result follows from the second characterization of critical sections.

Proposition 10.5. *The mapping τ which associates to each infinitesimal symmetry $D \in \mathfrak{D}$ its Noether invariant is $\mathbb{R}$-linear and* Ker τ *is an ideal of $\mathfrak{D}$. According to this, we can translate by τ the Lie algebra structure of $\mathfrak{D}$ to the set J of Noether invariants. J will be called the* Poisson algebra *associated to the variational problem under consideration. Furthermore, $J^v = \tau(\mathfrak{D}^v)$ is an ideal of the Poisson algebra.*

PROOF. Let D, D' be two infinitesimal symmetries of $\mathfrak{L}v$. Assume that $D \in \operatorname{Ker} \tau$ (i.e., $i_{D_{(2r-1)}}\Theta(\mathfrak{L}v) = 0$). Then, from Corollary 10.3 we obtain,

$$i_{[D,D']_{(2r-1)}}\Theta(\mathfrak{L}v) = i_{D_{(2r-1)}}L_{D'_{(2r-1)}}\Theta(\mathfrak{L}v) = -i_{D_{(2r-1)}}L_{\bar{D}'_{(r-2)}}\Theta(\mathfrak{L}v) +$$
$$= -L_{\bar{D}'_{(r-2)}}(i_{D_{(2r-1)}}\Theta(\mathfrak{L}v)) = 0.$$

Hence, $[D, D'] \in \operatorname{Ker} \tau$.

References

[1] Ferraris, M. Fibered connections and Global Poincaré-Cartan forms in higher-order Calculus of Variations, Proc. of the Conference on Differential Geometry and its Aplications, Nové Mésto na Moravé, Sept. 1983, Vol. II. Aplications, *Univerzita Karlova*, Praga, (1984), 61-91.

[2] García, P. L. The Poincaré-Cartan Invariant in the Calculus of Variations, Symposia Mathematica 14, *Academic Press,* London, (1974), 219-246.

[3] García, P. L., Muñoz, J. On the Geometrical Structure of Higher Order Variational Calculus, Proc. IUTAM-ISIMM Sympos. on Modern Developments in Analyt. Mech. Turin, June 1982, Vol I.-Geometrical Dynamics, *Tecnoprint*, Bologna, (1983), 127-147.

[4] García, P. L., Muñoz, J. Higher order Analytical Dynamics, Proc. of the VII Escuela Latinoamericana de Matemáticas, Caracas, July, (1984), (to appear).

[5] Gel'Fand, I. M., Dikii, L. A. Asymptotic behaviour of the resolvent of Sturm Liouville equations and the algebra of Korteweg-De Vries equations, *Russian Math. Surveys,* **30:5** (1975), 77-113.

[6] Goldschmidt, H., Sternberg, S. The Hamilton-Cartan Formalism in the Calculus of Variations, *Ann. Inst. Fourier,* **23** (1973), 203-267.
[7] Griffiths, P. A. Exterior Differential Systems and the Calculus of Variations, *Birkhäuser*, Boston, (1983).
[8] Hermann, R. A Geometric Variational Formalism for the Theory of Nonlinear Waves (Geometric Methods in Mathematical Physics, Proceedings, 1979), Lecture Notes in Math., **775** (1980), 145-161. *Springer-Verlag*, Berlin.
[9] Horák, M., Kolár, I. On the higher order Poincaré-Cartan forms, *Cz. Math. J.,* 33 (108), **3** (1983), 467-475.
[10] Kobayashi, S., Nomizu, K. Foundations of Differential Geometry, Vol. I, *Interscience*, New York, (1963).
[11] Koszul, J. L. Lectures on fibre bundles and differential geometry, *Institute of Fundamental Research*, Bombay, (1960).
[12] Krupka, D. Lepagean Forms in Higher Order Variational Theory, Proc. IUTAM-ISIMM Sympos. on Modern Developments in Analyt. Mech., Turin, June 1982, Vol. I. Geometrical Dynamics, *Tecnoprint*, Bologna, (1983), 197-238.
[13] Muñoz, J. Formes de structure et transformtions infinitésimales de contact d'ordre supérieur, *C.R. Acad. Sc.,* Paris, 298, Série I, **8** (1984), 185-188.
[14] Muñoz, J. An axiomatic characterization of the Poincaré-Cartan form for second order variational problems, Proc. of the XII International Conference on Differential Geometric Methods in Theroretical Physics, Clausthal, August, (1983), (to appear).
[15] Muñoz, J. Pre-simplectic Structure for higher order variational problems, Proc. of the Conference on Differential Geometry and its Applications, Nové Mésto na Moravé, Sept. 1983, Vol. II-Applications, Univerzita Karlova, Praga, (1984), 191-205.
[16] Sternberg, S. Some Preliminary Remarks on the Formal Variational Calculus of Gel'fand and Dikii (Differential Geometrical Methods in Mathematical Physics, II, Proceedings, 1977), Lecture Notes in Math. 676, (1978), 399-407. *Springer-Verlag*, Berlin.

Sección de Matemáticas
Universidad de Salamanca
Plaza de la Merced, 1-4
37008 Salamanca.
Spain

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 1, N.° 4 1985

Local Entropy Moduli and Eigenvalues of Operators in Banach Spaces

Bernd Carl **Thomas Kühn**

Introduction

In the paper local entropy moduli of operators between Banach spaces are introduced. They constitute a generalization of entropy numbers and moduli, and localize these notions in an appropriate way. Many results regarding entropy numbers and moduli can be carried over to local entropy moduli.

We investigate relations between local entropy moduli and s-numbers, spectral properties, eigenvalues, absolutely summing operators. As applications local entropy moduli of identical and diagonal operators between l_p-spaces can be estimated. It is shown, that in general «local» and «global» degree of compactness considerably differ, but under certain type assumptions on the underlying Banach spaces they coincide. Finally, the results are applied to obtain (optimal) estimates for eigenvalues of certain integral operators.

0. Preliminaries

Throughout the paper all Banach spaces, $X, Y, Z, \ldots$, are complex. The dual and the closed unit ball of X are denoted by X' and B_X, respectively. For the class of all (bounded linear) operators from X into Y we shall write $\mathcal{L}(X, Y)$, and for $\mathfrak{L}(X, X)$ simply $\mathfrak{L}(X)$. Concerning (quasi-normed) operator ideals we refer to the monograph [19]. We shall use mainly the ideals (Π_p, π_p) and $(\Pi_{p,2}, \pi_{p,2})$ of p- and $(p, 2)$-absolutely summing operators.

An important role will play the notion of type, see [18] for more informations. A Banach space X is of *(Rademacher) type p*, $1 \leqslant p \leqslant 2$, if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ the inequality

$$\mathbb{E}\left\| \sum_1^n x_i \epsilon_i \right\| \leqslant c\left(\sum_1^n \|x_i\|^p \right)^{1/p}$$

holds, where (ϵ_i) is a sequence of independent random variables, each taking the values $+1$ and -1 with probability $\frac{1}{2}$. The type p constant of X is then defined as $T_p(X) = \inf c$. Replacing the sequence (ϵ_i) by a sequence of independent standard Gaussian variables one can define Banach spaces of Gaussian type p, the Gaussian type p constant of X will be denoted by $\tilde{T}_p(X)$. A Banach space is of Gaussian type p iff it is of Rademacher type p, therefore we will not distinginsh between these two notions in the sequel, but only between the constants $T_p(X)$ and $\tilde{T}_p(X)$. As examples let us mention that the function spaces L_p (over arbitrary σ-finite measure spaces) are of type $\min(p, 2)$ if $1 \leqslant p < \infty$.

Moreover, we shall use the concept of s-numbers of operators, which also may be found in [19]. Here we only state the definitions of some s-numbers, for their properties see [19] and [20].

Given an operator $S \in \mathfrak{L}(X, Y)$, the n^{th} *approximation number* is defined by

$$a_n(S) = \inf\{\|S\text{-}L\| : L \in \mathfrak{L}(X, Y),\ \operatorname{rank} L < n\},$$

the n^{th} *Gelfand number* by

$$c_n(S) = \inf\{\|S J_M^X\| : M \subseteq X, \operatorname{codim} M < n\},$$

where J_m^X is the embedding from M into X,

the n^{th} *Kolmogorov number* by

$$d_n(S) = \inf\{\|Q_N^Y S\| : N \subseteq Y, \dim N < n\},$$

where Q_N^Y is the quotient map from Y onto Y/N, the n^{th} *Hilbert number* by

$$h_n(S) = \sup\{a_n(BSA) : \|A : l_2 \to X\| \leqslant 1, \|B : Y \to l_2\| \leqslant 1\},$$

the n^{th} *Weyl number* by

$$x_n(S) = \sup\{a_n(SA) : \|A : l_2 \to X\| \leqslant 1\}$$

and the n^{th} *dual Weyl number* by

$$y_n(S) = \sup\{a_n(BS) \colon \|B \colon Y \to l_2\| \leqslant 1\}.$$

Given two sequences of positive real numbers (a_n) and (b_n) we shall write $a_n = 0(b_n)$ if $a_n \leqslant cb_n$ for some constant $c > 0$ and all $n \in \mathbb{N}$. The symbol $a_n \sim b_n$ means $a_n = 0(b_n)$ and $b_n = 0(a_n)$.

1. Entropy quantities

Let us start by defining the entropy quantities we are going to use in the sequel. For entropy numbers see e.g. [19], entropy moduli were introduced in [5], while local entropy moduli are considered here for the first time.

Given an operator $S \in \mathcal{L}(X, Y)$, the n^{th} *entropy number* is

$$\epsilon_n(S) = \inf\left\{\epsilon > 0 \colon \exists y_1, \ldots, y_n \in Y \quad \text{such that} \quad S(B_X) \subseteq \bigcup_1^n (y_i + \epsilon B_Y)\right\},$$

the n^{th} *dyadic entropy number* is

$$e_n(S) = \epsilon_{2^{n-1}}(S),$$

the n^{th} *entropy modulus* is

$$g_n(S) = \inf\{k^{1/2n}\epsilon_k(S) \colon k \in \mathbb{N}\}$$

and the n^{th} *local entropy modulus* is

$$G_n(S) = \sup\{g_n(Q_N^Y S) \colon N \subseteq Y, \operatorname{codim} N \leqslant n\}.$$

By the definition of compactness of operators, $S \in \mathcal{L}(X, Y)$ is compact, iff $\epsilon_n(S) \to 0$ (or, equivalently, $e_n(S) \to 0$) as $n \to \infty$.

Thus entropy numbers quantify in a certain sense the notion of compactness. The «degree of compactness» of an operator can be characterized by the asymptotic behaviour of its (dyadic) entropy numbers. Another important point is the eigenvalue inequality [9]

$$\left(\prod_{i=1}^n |\lambda_i(S)|\right)^{1/n} \leqslant \inf_{k \in \mathbb{N}} k^{1/2n}\epsilon_k(S), \qquad S \in \mathcal{L}(X), \quad n \in \mathbb{N},$$

which motivated the introduction of entropy moduli. Finally, local entropy moduli are a local version of the concept of entropy moduli. The remarkable difference is that $G_n(S) \to 0$ as $n \to \infty$ not necessarily implies the compactness of S. But, on the other hand, the eigenvalue inequality for entropy moduli

$$\left(\prod_{i=1}^{n} |\lambda_i(S)|\right)^{1/n} \leqslant g_n(S), \qquad S \in \mathfrak{L}(X), \qquad n \in \mathbb{N}$$

remains true for local entropy moduli, too. More detailed results will be stated later on in section 3.

Algebraic (and other) properties of entropy numbers and moduli can be found in [19] and [5], respectively. Therefore we want to list here only properties of local entropy moduli. Let $S \in \mathfrak{L}(X, Y)$, $T \in \mathfrak{L}(Y, Z)$ and $n \in \mathbb{N}$

(i) $\|S\| = G_1(S) \geqslant G_n(S)$.

(ii) $G_n(TS) \leqslant G_n(T)G_n(S)$ (multiplicativity).

(iii) $G_n(S) = 0$ whenever $\operatorname{rank} S < n$ and $G_n(I_X) = 1$ whenever $\dim X \geqslant n$, where I_X is the identity on X.

(iv) $G_n(S) = g_n(S)$ if $\dim Y = n$.

(v) $G_n(SQ) = G_n(S)$ for every metric surjection $Q \in \mathfrak{L}(Z, X)$ (surjectivity).

(vi) Let (X_0, X_1) be an interpolation couple of Banach spaces and X be an intermediate space of K-type θ, $0 < \theta < 1$, and let $S \in \mathfrak{L}(X_0 + X_1, Y)$. Then

$$G_n(S: X \to Y) \leqslant 2G_{[n/(1-\theta)]}(S: X_0 \to Y)^{1-\theta} \cdot G_{[n/\theta]}(S: X_1 \to Y)^{\theta}.$$

For interpolation theory see e.g. Bergh/Löfström [2].

Since all these properties can be easily derived from those for entropy numbers [19], proofs are omitted.

Moreover, for each of the quantities $s = \alpha, c, d, h, x, y, e, g, G$ we introduce the notion

$$s(S) = \inf \{s_n(S): n \in \mathbb{N}\} \quad \text{for} \quad S \in \mathfrak{L}$$

and, for $0 < p \leqslant \infty$, the classes

$$\mathfrak{L}^{(s)}_{p,t} = \left\{S \in \mathfrak{L}: \sum_{n=1}^{\infty} n^{(t/p)-1} s_n(S)^t < \infty\right\}, \qquad 0 < t < \infty,$$

$$\mathfrak{L}^{(s)}_{p,\infty} = \left\{S \in \mathfrak{L}: \sup_{n \in \mathbb{N}} n^{1/p} s_n(S) < \infty\right\} \qquad \text{and}$$

$$\mathfrak{L}^{(s)}_{p,\omega} = \left\{S \in \mathfrak{L}: \lim_{n \to \infty} n^{1/p} s_n(S) = 0\right\}.$$

2. s-Numbers

In this section we shall investigate realtions between entropy quantities, especially local entropy moduli, and s-numbers. We start with Hilbert numbers.

Proposition 1. *If $0 < p, t \leqslant \infty$, then*

$$\mathfrak{L}_{p,t}^{(G)} \subseteq \mathfrak{L}_{p,t}^{(h)}.$$

Moreover, $h_n(S) \leqslant G_n(S)$ for every $S \in \mathfrak{L}$.

PROOF. Given $S \in \mathfrak{L}(X, Y)$, $n \in \mathbb{N}$ and $\epsilon > 0$ one can find [19, 11.4.3.] operators $A \in \mathfrak{L}(l_2^n, X)$, $B \in \mathfrak{L}(Y, l_2^n)$ with $\|A\| \leqslant 1$, $\|B\| \leqslant 1$ and $BSA = (1-\epsilon)h_n(S)I_n$, where I_n is the identity in l_2^n. By the properties of local entropy moduli, $(1-\epsilon)h_n(S) = G_n(BSA) \leqslant \|B\| G_n(S) \|A\|$, which implies $h_n(S) \leqslant G_n(S)$. The inclusion $\mathfrak{L}_{p,t}^{(G)} \subseteq \mathfrak{L}_{p,t}^{(h)}$ is an immediate consequence of this inequality. $\square$

Next let us state without proofs two lemmata from [6], that will be frequently used in what follows.

Lemma 2. *Let $s = c$ or d and $n \in \mathbb{N}$. Then for every $S \in \mathfrak{L}(X, Y)$*

$$\left(\prod_{i=1}^{n} s_i(S) \right)^{1/n} \leqslant n \sup \{ G_n(BSA) \colon \|A \colon l_1^n \to X\| \leqslant 1, \ \|B \colon Y \to l_\infty^n\| \leqslant 1 \}.$$

Lemma 3. *There are absolute constants $c_1, c_2 > 0$ such that for every p, $1 \leqslant p \leqslant 2$, and $n \in \mathbb{N}$ the inequalities*

(i) $G_n(A) \leqslant c_1 n^{1/p-1} T_p(X) \|A\|$ *for* $A \in \mathfrak{L}(l_1^n, X)$

and

(ii) $G_n(B) \leqslant c_2 n^{1/p-1} T_p(Y') \|B\|$ *for* $B \in \mathfrak{L}(Y, l_\infty^n)$

hold.

A simple combination of these two lemmata yields the following relation between Gelfand, Kolmogorov and local entropy classes.

Proposition 4. *Let $s = c$ or d, let $1 \leqslant p, q \leqslant 2$ and $0 < r, t \leqslant \infty$ such that $\frac{1}{w} = 1 + \frac{1}{r} - \frac{1}{p} - \frac{1}{q} > 0$. If X and Y are Banach spaces such that X is of type p and Y' of type q, then*

$$\mathfrak{L}_{r,t}^{(G)}(X, Y) \subseteq \mathfrak{L}_{w,t}^{(s)}(X, Y).$$

Supposing that X and Y' are even of type 2, Gordon, König and Schütt [11] showed that

$$a_n(T) \sim c_n(T) \sim d_n(T)$$

for all operators $T \in \mathfrak{L}(X, Y)$.

Using this fact and Proposition 4 we derive

Proposition 5. *Let X and Y' be of type* 2, *and let* $0 < r, t \leqslant \infty$. *Then all classes* $\mathfrak{L}_{r,t}^{(s)}(X, Y)$ *with* $s \in \{a, c, d, e, g, G\}$ *coincide.*

Remark. In this special situation «local» and «global» degree of compactness are the same. As shown in Proposition 19, in general there is a big gap between them.

Finally let us consider Weyl and dual Weyl numbers.

Proposition 6. *Let* $1 \leqslant p \leqslant 2$ *and* $0 < r, t \leqslant \infty$ *such that* $\frac{1}{s} = \frac{1}{r} + \frac{1}{2} - \frac{1}{p} > 0$. *Let X and Y be Banach spaces. If Y' has type p, then*

$$\mathfrak{L}_{r,t}^{(G)}(X, Y) \subseteq \mathfrak{L}_{s,t}^{(x)}(X, Y),$$

and if X has type p, then

$$\mathfrak{L}_{r,t}^{(G)}(X, Y) \subseteq \mathfrak{L}_{s,t}^{(y)}(X, Y).$$

Moreover, there is a constant $c > 0$ such that for all $S \in \mathfrak{L}(X, Y)$, all $1 \leqslant p \leqslant 2$ and $n \in \mathbb{N}$,

$$x_n(S) \leqslant cn^{(1/p)-(1/2)} T_p(Y') G_n(S)$$

and

$$y_n(S) \leqslant cn^{(1/p)-(1/2)} T_p(X) G_n(S).$$

PROOF. Given $A \in \mathfrak{L}(l_2, X)$, $\|A\| \leqslant 1$, and $B \in \mathfrak{L}(Y, l_2)$, $\|B\| \leqslant 1$ we conclude from Lemma 2 and Lemma 3

$$a_n(SA) = c_n(SA) \leqslant cn^{(1/p)-(1/2)} T_p(Y') G_n(S),$$

and

$$a_n(BS) = d_n(BS) \leqslant cn^{(1/p)-(1/2)} T_p(X) G_n(S),$$

where we used that

$$a_n(T) = c_n(T) \quad \text{for} \quad T \in \mathfrak{L}(H, X) \quad \text{and}$$
$$a_n(T) = d_n(T) \quad \text{for} \quad T \in \mathfrak{L}(Y, H),$$

H being a Hilbert space, see [19, 11.5.2. and 11.6.2.]. These inequalities imply the desired estimates for $x_n(S)$ and $y_n(S)$, from which the inclusions, stated in the first part of the proposition, easily follow. □

3. Spectral properties and eigenvalues

In this section we want to describe spectral properties and the eigenvalue behaviour of operators in $\mathcal{L}(X)$ in terms of their local entropy moduli.

First of all let us briefly explain the notations we are going to use. Given an operator $S \in \mathcal{L}(X)$ consider the coset $S + \mathcal{K}(X)$ as an element of the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the ideal of compact operators in the algebra $\mathcal{L}(X)$. The spectral radius of this element is called the *essential spectral radius* of S, $r_{ess}(S)$. Let $\sigma(S)$ denote the usual spectrum of S, then for every $r > r_{ess}(S)$ the set

$$\{\lambda \in \mathbb{C}: \lambda \in \sigma(S), |\lambda| \geqslant r\}$$

consists only of finitely many points, each being an eigenvalue of S of finite algebraic multiplicity. Thus we can order all eigenvalues λ of S with $|\lambda| > r_{ess}(S)$ in such a way that

$$|\lambda_1(S)| \geqslant |\lambda_2(S)| \geqslant \cdots,$$

where each eigenvalue is counted according to its algebraic multiplicity. If there are only n eigenvalues λ with $|\lambda| > r_{ess}(S)$, then put $\lambda_{n+1}(S) = \lambda_{n+2}(S) = \cdots = r_{ess}(S)$. So we have assigned to every $S \in \mathcal{L}(X)$ the sequence $(\lambda_n(S))$. For more details we refer to Zemanek [24] and the references given therein.

An operator $S \in \mathcal{L}(X, Y)$ is called *strictly cosingular*, if $Q_N^Y S$ is never a surjection, whenever N is an infinite dimensional subspace of Y, see [19, 1.10.2.].

Proposition 7. *Every operator in $\mathcal{L}_{\infty,\omega}^{(G)}$ is strictly cosingular.*

PROOF. Let $S \in \mathcal{L}(X, Y)$ with $\lim_{n\to\infty} G_n(S) = 0$. Assuming S being not strictly consingular one could find an infinite codimensional subspace N of Y that $Q_N^Y S$ is a surjection. Hence the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad S \quad} & Y \\ {\scriptstyle Q_M^X}\downarrow & & \downarrow{\scriptstyle Q_N^Y} \\ X/M & \xrightarrow{\quad S_0 \quad} & Y/N \end{array}$$

where M is the kernel of $Q_N^Y S$ and S_0 is an isomorphism. Now, by the surjectivity of local entropy moduli, for every $n \in \mathbb{N}$,

$$1 = G_n(I_{X/M}) = G_n(S_0^{-1}S_0) \leqslant \|S_0^{-1}\| G_n(S_0 Q_M^X) = \\ = \|S_0^{-1}\| G_n(Q_N^Y S) \leqslant \|S_0^{-1}\| G_n(S).$$

Letting $n \to \infty$ this yields a contradiction, thus proving the proposition. □

Next we need the notion of *Riesz operators*. These are operators $S \in \mathcal{L}(X)$ such that for every complex number λ the operator $I - \lambda S$ has finite dimensional kernel and finite codimensional closed range, see [19, 26.].

The essential spectral radius of Riesz operators is always equal to zero, hence all their non-zero eigenvalues have finite multiplicities and can be arranged in non-decreasing order. Since every strictly cosingular operator in $\mathcal{L}(X)$ is Riesz [19, 26.6.10.], we get

Corollary 8. *For every Banach space X the class $\mathcal{L}^{(G)}_{\infty,\omega}(X, X)$ consists of Riesz operators only.*

The essential spectral radius can be computed by local entropy moduli, as the following result shows.

Proposition 9. *If $S \in \mathcal{L}(X)$, then*

$$r_{ess}(S) = \lim_{N\to\infty} G(S^N)^{1/N}.$$

PROOF. It is known (see e.g. [24]), that

$$r_{ess}(S) = \lim_{N\to\infty} a(S^N)^{1/N} = \lim_{N\to\infty} e(S^N)^{1/N}.$$

By Lemma 2 $c_n(S) \leqslant nG_n(S)$ for arbitrary $n \in \mathbb{N}$.

Combining this with

$$a_n(S) \leqslant 2n^{1/2}c_n(S) \qquad [19,\ 11.12.2.]$$

and the monotonicity of approximation numbers we obtain

$$a(S) \leqslant 2n^{3/2}G_n(S) \quad \text{for} \quad n \in \mathbb{N}.$$

Observing that $G_n(T) \leqslant \sqrt{2}\, e_n(T)$ for $T \in \mathcal{L}$ we get for all $n, k, N \in \mathbb{N}$

$$a(S^{Nk})^{1/Nk} \leqslant (2n^{3/2})^{1/Nk} G_n(S^{Nk})^{1/Nk} \leqslant$$
$$\leqslant (2n^{3/2})^{1/Nk} G_n(S^N)^{1/N} \leqslant (2n^{3/2})^{1/Nk} 2^{1/2N} e_n(S^N)^{1/N}.$$

Letting now $k \to \infty$ yields

$$r_{ess}(S) \leqslant G_n(S^N)^{1/N} \leqslant 2^{1/2N} e_n(S^N)^{1/N}.$$

Taking then the infimum over all $n \in \mathbb{N}$ and letting finally $N \to \infty$ the assertion follows. $\square$

Let us now turn to eigenvalue means.

Proposition 10. *Let $S \in \mathcal{L}(X)$ and $n \in \mathbb{N}$. Then*

$$\left(\prod_{i=1}^{n} |\lambda_i(S)| \right)^{1/n} = \lim_{N \to \infty} G_n(S^N)^{1/N}.$$

PROOF. There is an n-dimensional invariant subspace X_0 of X such that the restriction S_0 of S onto X_0 has exactly the eigenvalues $\lambda_1(S), \ldots, \lambda_n(S)$. If P is any projection from X onto X_0 and J is the canonical embedding from X_0 into X, then for all $N \in \mathbb{N}$, $S_0^N = PS^N J$. This implies

$$g_n(S_0^N) = G_n(PS^N J) \leqslant \|P\| G_n(S^N) \leqslant \|P\| g_n(S^N).$$

Now the result of Makai and Zemanek [17]

$$\left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = \lim_{N \to \infty} g_n(T) \quad \text{for} \quad T \in \mathcal{L}(X)$$

and

$$\left(\prod_{i=1}^{n} |\lambda_i(S)| \right)^{1/n} = \left(\prod_{i=1}^{n} |\lambda_i(S_0)| \right)^{1/n}$$

yield the assertion. □

4. Absolutely summing operators

The goal of this section is to establish some relationships between p- and $(p, 2)$-absolutely summing operators, Weyl numbers and local entropy moduli for operators acting in Banach spaces having certain (Rademacher) type.

Before doing that one more notion is required. Let (g_i) denote a sequence of independent standard Gaussian random variables. Then for any operator $T \in \mathcal{L}(l_2^n, X)$ one sets

$$l(T) = \left(\mathbb{E} \left\| \sum_{1}^{n} Te_i g_i \right\|^2 \right)^{1/2},$$

and for $T \in \mathcal{L}(H, X)$, H being an arbitrary Hilbert space, let

$$l(T) = \sup \{ l(TP) \colon P \in \mathcal{L}(l_2^n, H) \text{ unitary}, \quad n \in \mathbb{N} \}.$$

This so-called γ-summing norm was introduced by Linde and Pietsch [16]. We need the following relations to 2-absolutely summing operators (cf. [10]) and entropy numbers (due to Sudakov, see e.g. [14]), which we state as

Lemma 11. *There is a constant $c > 0$ such that for all $T \in \mathfrak{L}(H, X)$, where H is any Hilbert space, X any Banach space,*

(i) $l(T) \leqslant \tilde{T}_2(X)\pi_2(T')$ *and*

(ii) $\sup_{n \geq 1} (\ln n)^{1/2}\epsilon_n(T') \leqslant cl(T)$.

Now we are prepared to prove the

Theorem 12. *Let $1 < p \leqslant 2 \leqslant q < \infty$ such that $\frac{1}{r} = \frac{1}{q} + \frac{1}{2} - \frac{1}{p} > 0$. If X is a Banach space whose dual is of type p, then for all Banach spaces Y,*

$$\Pi_{q,2}(X, Y) \subseteq \mathfrak{L}^{(G)}_{r,\infty}(X, Y).$$

Moreover, there is some constant $c > 0$ (neither depending on p nor on q) such that for all $S \in \mathfrak{L}(X, Y)$

$$G_n(S) \leqslant cn^{-1/r}\pi_{q,2}(S)\tilde{T}_p(X').$$

PROOF. Given any operator $S \in \mathfrak{L}(X, Y)$ and any n-codimensional subspace N of Y one has the following factorization diagram

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "S_0"'] & Y \arrow[d, "Q_N^Y"] \\ l_2^n \arrow[r, "A"] & Y/N \end{tikzcd}$$

where $\pi_2(Q_N^Y S) = \pi_2(S_0)$ and $\|A\| = 1$.

Setting $Z = S_0'(l_2^n)$, which is an at most n-dimensional subspace of X', we obtain from the preceding lemma

$$\begin{aligned} n^{1/2}g_n(Q_N^Y S) &\leqslant (2n)^{1/2}\epsilon_{2n}(Q_N^Y S) \leqslant (2n)^{1/2}\epsilon_{2n}(S_0)\|A\| \leqslant \\ &\leqslant 2^{1/2}cl(S_0') \leqslant 2^{1/2}c\pi_2(S_0)\tilde{T}_2(Z). \end{aligned}$$

By the results of Tomczak-Jaegermann [23],

$$\pi_2(S_0) = \pi_2(Q_N^Y S) \leqslant 2n^{(1/2)-(1/q)}\pi_{q,2}(Q_N^Y S) \leqslant 2n^{(1/2)-(1/q)}\pi_{q,2}(S)$$

and

$$\tilde{T}_2(Z) \leqslant 2n^{(1/p)-(1/2)}\tilde{T}_p(Z) \leqslant 2n^{(1/p)-(1/2)}\tilde{T}_p(X'),$$

yielding the estimate

$$G_n(S) \leqslant 2^{5/2}cn^{-1/r}\pi_{q,2}(S)\tilde{T}_p(X').$$

This inequality immediately implies the inclusion stated in the first part of the theorem. □

We state now the most important special case as

Corollary 13. *Let $2 \leqslant q < \infty$. If the dual of the Banach space X is of type 2, then for all Banach spaces Y*

$$\Pi_{q,2}(X, Y) \subseteq \mathfrak{L}_{q,\infty}^{(G)}(X, Y).$$

Next we want to investigate the inverse problem: Under which conditions on the underlying Banach spaces local entropy classes do consist of $(q, 2)$-summing operators only? To answer this question we need the following result by Pietsch [20] concerning Weyl classes:

$$\mathfrak{L}_{p,p}^{(x)} \subseteq \Pi_{p,2} \quad \text{for} \quad p > 2 \quad \text{and} \quad \mathfrak{L}_{2,1}^{(x)} \subseteq \Pi_2.$$

Combining this with Proposition 6 one can derive

Corollary 14. *Given $1 \leqslant p \leqslant 2 < q < \infty$ and $0 < r < \infty$ such that $\frac{1}{q} = \frac{1}{r} - \frac{1}{p} + \frac{1}{2} > 0$, the inclusions*

$$\mathfrak{L}_{r,q}^{(G)}(X, Y) \subseteq \Pi_{q,2}(X, Y) \qquad \text{and}$$
$$\mathfrak{L}_{p,1}^{(G)}(X, Y) \subseteq \Pi_2(X, Y)$$

hold for all Banach spaces X and Y, provided that Y' is of type p.

Corollaries 13 and 14 imply now a result similar to that for Weyl numbers [20].

Corollary 15. *Let X and Y be Banach spaces whose duals are of type 2. Then*

$$\mathfrak{L}_{q,q}^{(G)}(X, Y) \subseteq \Pi_{q,2}(X, Y) \subseteq \mathfrak{L}_{q,\infty}^{(G)} \quad \textit{for} \quad 2 < q < \infty$$

and

$$\mathfrak{L}_{2,1}^{(G)}(X, Y) \subseteq \Pi_2(X, Y) \subseteq \mathfrak{L}_{2,\infty}^{(G)}(X, Y).$$

Remark. As shown in [7] the result for $q = 2$ is valid also for entropy numbers and moduli instead of local entropy moduli.

The next result is devoted to p-absolutely summing operators.

Theorem 16. *If H is a Hilbert space and X a Banach space, then*

$$\Pi_p(H, X) \subseteq \mathfrak{L}_{2,\infty}^{(G)}(H, X) \quad \textit{for} \quad 0 < p < \infty.$$

Moreover, there is a constant $c > 0$ such that for all $2 \leqslant p < \infty$, $n \in \mathbb{N}$ and $S \in \mathcal{L}(H, X)$

$$G_n(S) \leqslant cp^{1/2}n^{-1/2}\pi_p(S).$$

PROOF. Let $S \in \mathcal{L}(H, X)$, $2 \leqslant p < \infty$, $n \in \mathbb{N}$ and $N \subseteq X$ with $\operatorname{codim} N \leqslant n$. Denote by Q the quotient map from X onto X/N, let H_0 be the orthogonal complement of the kernel of QS, and let J be the embedding from H_0 into H and P the orthogonal projection from H onto H_0. Then clearly $QS = QSJP$. By the Pietsch factorization theorem [19] we have the following commutative diagram

$$\begin{array}{ccccc} H & \xrightarrow{QS} & X/N & \xrightarrow{J_0} & Y \\ {\scriptstyle A}\downarrow & & & & \downarrow{\scriptstyle B} \\ L_\infty(K,\gamma) & & \xrightarrow{\quad I \quad} & & L_p(K,\gamma) \end{array}$$

where J_0 is a metric injection into an appropriate Banach space Y (possessing the metric extension property), γ is a probability measure on a compact Hausdorff space K, I is the identity and

$$\pi_p(QS) = \|A\| \, \|B\|.$$

Let $m := \dim H_0$. Then by a result of Kashin (see Szarek and Tomczak-Jaegermann [22]), there is an m-dimensional subspace E of l_1^{2m} with $d(E, l_2^m) < 4e$. (Here $d(X, Y)$ denotes the Banach-Mazur distance of two isomorphic Banach spaces X and Y.)

Hence also $d(E, H_0) < 4e$ and one can find an isomorphism $T \in \mathcal{L}(E, H_0)$ with $\|T\| \, \|T^{-1}\| \leqslant 4e$. Now we have the commutative diagram

$$\begin{array}{ccccccc} H & \xrightarrow{T^{-1}P} & E & \xrightarrow{JT} & H & \xrightarrow{J_0QS} & Y \\ & & {\scriptstyle J_1}\downarrow & & {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle B} \\ & & l_1^{2m} & \xrightarrow{A_0} & L_\infty(K,\gamma) & \xrightarrow{I} & L_p(K,\gamma), \end{array}$$

where J_1 is the canonical injection and A_0 is an extension of AJT with $\|A_0\| = \|AJT\|$. Such A_0 exists, since $L_\infty(K, \gamma)$ has the metric extension property. By Lemma 3 we have (see [6])

$$g_m(IA_0) \leqslant aT_2(L_p(K,\gamma))m^{-1/2}\|IA_0\|$$

with some absolute constant $a > 0$. Observing that $T_2(L_p(K, \gamma)) \leqslant p^{1/2}$ we get from the monotonicity and injectivity of entropy moduli [5]

$$\begin{aligned} g_n(QS) &\leqslant g_m(QS) \leqslant 2g_m(J_0QSJP) = 2g_m(BIA_0J_1T^{-1}P) \leqslant \\ &\leqslant 2g_m(IA_0)\|B\|\,\|J_1T^{-1}P\| \\ &\leqslant 2ap^{1/2}\|A\|\,\|T\|\,\|B\|\,\|T^{-1}\|m^{-1/2} \leqslant \\ &\leqslant 8eap^{1/2}\pi_p(QS)n^{-1/2}. \end{aligned}$$

Since N was arbitrarily chosen this finally yields the desired estimate

$$G_n(S) \leqslant cp^{1/2}\pi_p(S)n^{-1/2} \quad \text{with} \quad c = 8ea.$$

Hence, also

$$\Pi_p(H, X) \subseteq \mathfrak{L}^{(G)}_{2,\infty}(H, X) \quad \text{for} \quad 0 < p < \infty. \qquad \square$$

5. Identical and diagonal operators

As concrete examples we consider in this section identical and diagonal operators between l_p-spaces, and determine the exact asymptotic behaviour of their local entropy moduli. As an application we shall see that there are non-compact operators whose local entropy moduli tend to zero, in contrast to the situation for entropy numbers or moduli. We start with identity operators.

Proposition 17. *Let* $1 < p < q < \infty$ *and* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. *Then* $G_n(I\colon l_p \to l_q) \sim n^{-1/r}$, *hence* $I \in \mathfrak{L}^{(G)}_{r,\infty}(l_p, l_q)$.

PROOF. The *estimate from below* is quite simple. Given $n \in \mathbb{N}$ one has the relation

$$1 = G_n(I\colon l_q^n \to l_q^n) \leqslant \|I\colon l_q^n \to l_p^n\|\, G_n(I\colon l_p^n \to l_q^n) \leqslant n^{(1/p)-(1/q)} G_n(I\colon l_p \to l_q),$$

hence $G_n(I\colon l_p \to l_q) \geqslant n^{-1/r}$.

Now let us turn to the *estimate from above*.

First we consider the *case* $1 < p < q \leqslant 2$.

By Bennett [1] or [3] it holds $I \in \Pi_{u,1}(l_p, l_q)$ with $\frac{1}{u} = \frac{1}{p} - \frac{1}{q} + \frac{1}{2}$. Moreover, there is a constant $c_0 > 0$ such that for all p, q with $1 < p < q \leqslant 2$ the estimate $\pi_{u,1}(I\colon l_p \to l_q) \leqslant c_0$ holds. This implies

$$\pi_{r,2}(I\colon l_p \to l_q) \leqslant c_0$$

and by Theorem 12

$$G_n(I\colon l_p \to l_q) \leqslant c_1 n^{-1/r} \tilde{T}_2(l_{p'}) \pi_{r,2}(I) \leqslant c_2 (p')^{1/2} n^{-1/r}.$$

Note, that c_2 is independent of p and q. Next let us treat the *case* $2 = p < < q < \infty$.

Let N be an arbitrary n-codimensional subspace of l_q, and let $Q: l_q \to l_q/N$ be the canonical quotient map. Our aim is to estimate $\pi_q(QI)$ from above. Denoting the orthogonal projection from l_2 onto the orthogonal complement of the kernel of QI by P, and the injection of this space into l_2 by J, one has obviously

$$QI = QIJP.$$

Therefore, using the inequality

$$\pi_q(S) \leqslant \pi_q(S') \quad \text{for all} \quad S \in \mathfrak{L}(l_2, l_q)$$

(see Pietsch [19, 19.5.2.]) we obtain

$$\pi_q(QI) \leqslant \pi_q(IJ) \leqslant \pi_q(J'I') \leqslant \pi_2(J'I').$$

Since $J'I'$ has rank at most n, by Tomczak-Jaegermann [23],

$$\pi_2(J'I') \leqslant 2n^{(1/2)-(1/r)}\pi_{r,2}(I': l_{q'} \to l_2) \leqslant 2c_0 n^{(1/2)-(1/r)},$$

where c_0 is the constant from the previous case. As shown in the proof of Theorem 16, there is a constant $c_3 > 0$ such that

$$g_n(QI) \leqslant c_3 q^{1/2} n^{-1/2} \pi_q(QI) \leqslant c_4 q^{1/2} n^{-1/2},$$

where again c_4 does not depend on q. This finally gives the desired estimate

$$G_n(I) \leqslant c_4 q^{1/2} n^{-1/r}.$$

Combining these two cases we get the result in the *case* $1 < p < 2 < q < \infty$.

It holds with some absolute constant $c_5 > 0$

$$\begin{aligned} G_n(I: l_p \to l_q) &\leqslant G_n(I: l_p \to l_2) G_n(I: l_2 \to l_q) \leqslant \\ &= c_5 (p')^{1/2} n^{(1/2)-(1/p)} q^{1/2} n^{(1/q)-(1/2)} = \\ &= c_5 (p'q)^{1/2} n^{-1/r}. \end{aligned}$$

Finally, the remaining *case* $2 < p < q < \infty$ can be treated by interpolation. Let $0 < \theta < 1$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q}$. Since l_p is then of K-type θ with respect to the interpolation couple (l_q, l_2), we get the estimate

$$\begin{aligned} G_n(I: l_p \to l_q) &\leqslant 2G_{[n/(1-\theta)]}(I: l_q \to l_q)^{1-\theta} G_{[n/\theta]}(I: l_2 \to l_q)^{\theta} \leqslant \\ &\leqslant 2c_4^{\theta} \left[\frac{n}{\theta}\right]^{\theta[(1/q)-(1/2)]} \leqslant c_6 n^{(1/q)-(1/p)} = c_6 n^{-1/r}, \end{aligned}$$

where c_6 only depends on θ but not on n. $\square$

As a useful consequence let us state

Corollary 18. *There is a constant* $c > 0$ *such that for all* $n \geqslant 8$ $n^{-1} \leqslant G_n(I\colon l_1 \to l_\infty) \leqslant c(\ln n)n^{-1}$, *therefore* $I \in \mathfrak{L}_{p,t}^{(G)}(l_1, l_\infty)$ *for all* $p > 1$, $0 < t \leqslant \infty$.

PROOF. The estimate from below can be proved in the same way as in Proposition 14. The estimate from above follows from the factorization

$$G_n(I\colon l_1 \to l_\infty) \leqslant \|I\colon l_1 \to l_p\| \, G_n(I\colon l_p \to l_q) \|I\colon l_q \to l_\infty\| \leqslant c_5(p'q)^{1/2} n^{(1/q)-(1/p)},$$

where $1 < p < 2 < q < \infty$ are arbitrary, and c_5 is the constant from the previous proposition. Specifying now p and q as $p' = q = \ln n$ we get

$$G_n(I\colon l_1 \to l_\infty) \leqslant c_5 q n^{2/q-1} = c_5(\ln n)n^{-1} n^{2/\ln n} = c(\ln n)n^{-1} \quad \text{with} \quad c = c_5 e^2.$$

The inclusion $I \in \mathfrak{L}_{p,t}^{(G)}$ for $p > 1$, $0 < t \leqslant \infty$, is a consequence of the estimate from above. □

Now let us return to the question, how «local» and «global» degree of compactness are related to each other. The following result shows that there is in general a big difference between them. This supplements Proposition 5. Let $\mathfrak{K}$ denote the ideal of compact operators.

Proposition 19. *Let* $0 < p, t < \infty$. *Then*

$$\mathfrak{L}_{p,t}^{(G)} \subseteq \mathfrak{K} \quad \text{iff} \quad p \leqslant 1.$$

PROOF. Since the (clearly non-compact) identity $I \in \mathfrak{L}(l_1, l_\infty)$ belongs to all classes $\mathfrak{L}_{p,t}^{(G)}$ with $p > 1$, it remains to prove the «if» part. Let $S \in \mathfrak{L}_{1,t}^{(G)}$ for some $0 < t < \infty$. Assuming S being non-compact one had $\inf_{n \in \mathbb{N}} c_n(S) > 0$, and the inequality (Lemma 2)

$$c_n(S) \leqslant n G_n(S)$$

would imply the contradiction

$$\infty = \sum_{n=1}^{\infty} \frac{c_n(S)^t}{n} \leqslant \sum_{n=1}^{\infty} (n^{1-(1/t)} G_n(S))^t < \infty.$$

hence $S \in \mathfrak{K}$, and the proof is finished. □

Remark. The exact asymptotic behaviour of $G_n(I\colon l_1 \to l_\infty)$ is not known, but the conjecture $G_n(I\colon l_1 \to l_\infty) \approx n^{-1}$ seems reasonable. The validity of this conjecture would imply, that even the class $\mathcal{L}_{1,\infty}^{(G)}$ contains non-compact operators. Obviously, $\mathcal{L}_{1,\omega}^{(G)} \subseteq \mathcal{K}$.

Similar estimates as for the identity $I \in \mathcal{L}(l_1, l_\infty)$, involving certain logarithmic terms, can be derived also for the identities $I \in \mathcal{L}(l_p, l_q)$, where either $p = 1$ or $q = \infty$.

Now let us turn to diagonal operators. The diagonal operator D_σ, generated by a given sequence $\sigma = (\sigma_n) \in l_\infty$, and acting between appropriate Banach sequence spaces, is defined by $D_\sigma(\zeta_n) = (\sigma_n \zeta_n)$.

Proposition 20. *Let* $1 < p, q < \infty$, $0 < r < \infty$, $0 < t \leqslant \infty$ *and* $\frac{1}{s} = \frac{1}{p} + \frac{1}{r} - \frac{1}{q} > 0$. *Then* $D_\sigma \in \mathcal{L}_{s,t}^{(G)}(l_p, l_q)$ *iff* $\sigma \in l_{r,t}$.

PROOF. If $\sigma \in l_{r,t}$, then by [4], $D_\sigma \in \mathcal{L}_{s,t}^{(e)}(l_p, l_q) \subseteq \mathcal{L}_{s,t}^{(G)}(l_p, l_q)$. It remains to prove the «only if» part. Without loss of generality let us assume that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots > 0$. Let D_n be the operator D_σ restricted to the first n coordinates.

In the case $p \leqslant q$ Proposition 17 implies

$$\begin{aligned} n^{(1/q)-(1/p)} &\leqslant G_n(I: l_p^n \to l_q^n) \leqslant G_n(D_n: l_p^n \to l_q^n)\|D_n^{-1}: l_q^n \to l_q^n\| \leqslant \\ &\leqslant G_n(D_\sigma: l_p \to l_q)\sigma_n^{-1}. \end{aligned}$$

if $p > q$, we proceed as follows, again using Proposition 17,

$$\begin{aligned} 1 &= G_n(I: l_p^n \to l_p^n) \leqslant \\ &\leqslant G_n(D_n: l_p^n \to l_q^n)\|D_n^{-1}: l_q^n \to l_q^n\| G_n(I: l_q^n \to l_p^n) \leqslant \\ &\leqslant cG_n(D_\sigma)\sigma_n^{-1} n^{(1/q)-(1/p)}. \end{aligned}$$

In both cases, $G_n(D_\sigma) \geqslant c_1 \sigma_n n^{1/q - 1/p}$, hence $D_\sigma \in \mathcal{L}_{s,t}^{(G)}$ implies the desired assertion $\sigma \in l_{r,t}$. $\square$

6. Eigenvalues of integral operators

In this section we apply the results obtained till now to certain integral operators, namely to Hille-Tamarkin and weakly singular integral operators. For the latter ones we only consider the critical case where the order of the singularity is half of the dimension of the domain on which the operator acts.

Throughout the section let (Ω, Σ, μ) be any σ-finite measure space. By a kernel we mean a $\mu \times \mu$-measurable function $K: \Omega \times \Omega \to \mathbb{C}$. To every kernel K we assign the integral operator T_K, defined as

$$T_K f(s) = \int_\Omega K(s, t) f(t)\, d\mu(t), \qquad s \in \Omega$$

for measurable functions f, provided the integral exists. We shall pose such assumptions on K, that T_K acts as a bounded operator between appropriately chosen function spaces. Thus let us introduce the classes $(L_q)L_p = (L_q(\Omega, \Sigma, \mu))L_p(\Omega, \Sigma, \mu)$, $1 \leqslant p, q < \infty$, consisting of all kernels K with

$$\|K\|_{(L_q)L_p} := \left(\int_\Omega \left(\int_\Omega |K(s,t)|^q \, d\mu(s)\right)^{p/q} d\mu(t)\right)^{1/p} < \infty.$$

These kernels are called *Hille-Tamarkin kernels.*

Theorem 21. *Let $1 \leqslant p < \infty$, $1 < q \leqslant 2$ and $K \in (L_q)L_p$. Then*

$$T_K \in \mathfrak{L}^{(G)}_{q',\infty}(L_{p'}, L_q).$$

Moreover, for all $n \in \mathbb{N}$

$$G_n(T_K) \leqslant c n^{(1/q)-1} \|K\|_{(L_q)L_p}$$

with some constant c, depending on p and q but not on the underlying measure space.

PROOF. Put

$$g(t) = \left(\int_\Omega |K(s,t)|^q \, d\mu(s)\right)^{1/q} \quad \text{for} \quad t \in \Omega,$$

and for $s, t \in \Omega$ set

$$\tilde{K}(s,t) := \begin{cases} K(s,t)/g(t) & \text{if} \quad g(t) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in L_p$ and $\int_\Omega |\tilde{K}(s,t)|^q \, d\mu(s) \leqslant 1$ for every $t \in \Omega$. This implies $M_g \in \mathfrak{L}(L_{p'}, L_1)$, where $M_g f = gf$, and for $f \in L_1$

$$\begin{aligned} \|T_{\tilde{K}} f\|_{L_q} &= \left(\int_\Omega \left|\int_\Omega \tilde{K}(s,t) f(t) \, d\mu(t)\right|^q d\mu(s)\right)^{1/q} \leqslant \\ &\leqslant \int_\Omega \left(\int_\Omega |\tilde{K}(s,t)|^q \, d\mu(s)\right)^{1/q} |f(t)| \, d\mu(t) \leqslant \|f\|_{L_1}. \end{aligned}$$

Hence $T_{\tilde{K}} \in \mathfrak{L}(L_1, L_q)$, and $T_K = T_{\tilde{K}} M_g$.

If $2 \leqslant p < \infty$, then L_p has type 2. By Kwapién [15] one has

$$\mathfrak{L}(L_1, L_q) = \pi_{q',2}(L_1, L_q)$$

and with some $c_1 > 0$

$$\pi_{q',2}(S) \leqslant c_1 \|S\| \quad \text{for all} \quad S \in \mathfrak{L}(L_1, L_q).$$

Now Corollary 13 implies for $n \in \mathbb{N}$

$$G_n(T_K) \leqslant c_2 n^{-1/q'} \|T_K\| \leqslant c_2 n^{-1/q'} \|K\|_{(L_q)L_p},$$

where the constant c_2 depends only on p, q, but not on the measure space.

In the case $1 < p < 2$ one can determine $2 < r < \infty$ such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$. Then g can be splitted as $g = g_1 g_2$, where $g_1 \in L_2$ and $g_2 \in L_r$. Therefore $M_{g_1} \in \mathfrak{L}(L_2, L_1)$, $M_{g_2} \in \mathfrak{L}(L_{p'}, L_2)$ and $M_g = M_{g_2} M_{g_1}$. Hence, similar as in the first case one can conclude

$$G_n(T_k) \leqslant \|M_{g_2}\| G_n(T_{\tilde{K}} M_{g_1}) \leqslant \|g_2\|_{L_r} c_3 n^{-1'/q} \|\tilde{K}\| \, \|g_1\|_{L_2} = c_3 n^{-1/q'} \|K\|_{(L_q)L_p}.$$

Thus, in both cases,

$$T_K \in \mathfrak{L}^{(G)}_{q',\infty}(L_{p'}, L_q). \qquad \square$$

Next we apply this result in order to get estimates for eigenvalues of Hille-Tamarkin operators with kernels from the space $(L_{p'})L_p$, $2 \leqslant p < \infty$.

These results are already known, see [20], but the approach via local entropy moduli is new. In [8] entropy numbers were used to obtain similar results. But there some additional restrictions (e.g. finiteness of the underlying measure space) had to be posed, which can be omitted now.

Theorem 22. *Let $2 \leqslant p < \infty$ and $K \in (L_{p'})L_p$. Then $(\lambda_n(T_K)) \in l_p$.*

PROOF. Without loss of generality we may and do assume that $\|K\|_{(L_{p'})L_p} \leqslant 1$. Then, by the proof of the preceding theorem, there is a constant $c > 0$ only depending on p, but not on the underlying measure space, such that

$$G_n(T_K) \leqslant c n^{-1/p} \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence, by Corollary 8, T_K is a Riesz operator, and by Proposition 10,

$$|\lambda_n(T_K)| \leqslant G_n(T_K) \leqslant c n^{-1/p} \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence for arbitrary $r > p$

$$\sum_{n=1}^{\infty} |\lambda_n(T_K)|^r \leqslant \frac{cr}{r-p}. \tag{*}$$

Now we proceed in an analogous way as it was done in Pietsch [21]. Define the kernel $\bar{K}$ on $\Omega^2 \times \Omega^2$ by $\bar{K}(s_1, s_2, t_1, t_2) = K(s_1, t_1) K(s_2, t_2)$. Then $\bar{K}$ belongs to the space $(L_{p'})L_p$ over the product measure space $(\Omega, \Sigma, \mu) \times (\Omega, \Sigma, \mu)$, and again $\|\bar{K}\|_{(L_{p'})L_p} \leqslant 1$. As in [21] it can be shown, that if λ, μ are eigenvalues of T_K with algebraic multiplicities l and m, then $\lambda\mu$ is an eigenvalue of $T_{\bar{K}}$ having multiplicity at least $l \cdot m$. Applying now inequality (*) to $T_{\tilde{K}}$ instead of T_K we get

$$\left(\sum_{n=1}^{\infty} |\lambda_n(T_K)|^r\right)^2 = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} |\lambda_{n_1}(T_K)\lambda_{n_2}(T_K)|^r \leqslant \sum_{n=1}^{\infty} |\lambda_n(T_{\tilde{K}})|^r \leqslant \frac{cr}{r-p}.$$

Thus (*) holds even with constant $\left(\frac{cr}{r-p}\right)^{1/2}$. Iterating this argument it follows

$$\sum_{n=1}^{\infty} |\lambda_n(T_K)|^r \leqslant 1 \quad \text{for every} \quad r > p,$$

which implies

$$\sum_{n=1}^{\infty} |\lambda_n(T_k)|^p \leqslant 1,$$

hence the proof is finished. □

Finally let us consider *weakly singular integral operators*. Here we want to restrict our attention to the border case where the order of singularity is half of the dimension of the domain on which the operator acts. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, N any positive integer, and let $\Delta = \{(x, x): x \in \Omega\}$. Suppose the measurable kernel $K: \Omega^2 \backslash \Delta \to \mathbb{C}$ is of the form

$$K(x, y) = \frac{L(x, y)}{|x - y|^{N-\alpha}}, \qquad 0 > \alpha \geqslant N,$$

with $L \in L_\infty(\Omega^2)$. (Here $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$.) Then the operator T_K is compact in $L_\infty(\Omega)$. As shown by König, Retherford and Tomczak-Jaegermann [13, Proposition 12] its eigenvalues are square summable whenever $\alpha > N/2$ and satisfy

$$\lambda_n(T_K) = 0(n^{-\alpha/N}) \quad \text{if} \quad 0 < \alpha < N/2.$$

In both cases the result is optimal. The conjecture

$$\lambda_n(T_K) = 0(n^{-1/2}) \quad \text{in the border case} \quad \alpha = N/2$$

fails, as was proved by König [12]. The optimal asymptotic behaviour is

$$\lambda_n(T_K) = 0\left(\left(\frac{\ln n}{n}\right)^{1/2}\right).$$

In order to illustrate the usefulness of the concept of local entropy moduli we want to give a proof of the last mentioned result via local entropy moduli. We replace the condition $L \in L_\infty(\Omega^2)$ of König [12] by a weaker one.

Theorem 23. *Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, and let $\Delta = \{(x,x): x \in \Omega\}$. Given a measurable kernel $K: \Omega^2 \backslash \Delta \to \mathbb{C}$ of the form*

$$K(x,y) = \frac{L(x,y)}{|x-y|^{N/2}}$$

with

$$l(y) := \sup_{x \in \Omega} |L(x,y)| \in L_p(\Omega), \qquad 2 < p < \infty,$$

one has

$$\lambda_n(T_K) = 0\left(\left(\frac{\ln n}{n}\right)^{1/2}\right).$$

PROOF. Put

$$\tilde{K}(x,y) = \begin{cases} \dfrac{K(x,y)}{l(y)} & \text{if} \quad l(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then T_K as operator in $L_{p'}(\Omega)$ can be factorized as follows

$$\begin{array}{ccccc} L_{p'} & & \xrightarrow{\quad T_K \quad} & & L_{p'} \\ {\scriptstyle M_1}\downarrow & & & & \uparrow{\scriptstyle I_1} \\ L_1 & \xrightarrow{T_{\tilde{K}}} & L_{2,\infty} & \xrightarrow{I_0} & L_s \end{array},$$

where s is any real with $p' < s < 2$, and I_0, I_1 are the respective identities. Since L_p (the dual of $L_{p'}$) is of type 2 and since every $S \in \mathfrak{L}(L_1, L_s)$ is $(s', 2)$-absolutely summing with $\pi_{s',2}(S) \leqslant \rho \|S\|$, for some constant $\rho > 0$ not depending on S and s, Corollary 13 implies with some constans $c_1, c_2 > 0$ independent of s,

$$\begin{aligned} G_n(T_K) &\leqslant G_n(I_0 T_{\tilde{K}} M_1)\|I_1\| \leqslant \\ &\leqslant c_1 T_2(L_p) n^{-1/s'} \pi_{s',2}(I_0 T_K M_1)\|I_1\| \leqslant \\ &\leqslant c_2 n^{-1/s'} \|I_0 T_K M_1\| \, \|I_1\|. \end{aligned}$$

Observing that

$$\|I_0\| \leqslant \left(\frac{2}{2-s}\right)^{1/s} \mu(\Omega)^{(1/s)-(1/2)},$$

$$\|I_1\| \leqslant \mu(\Omega)^{(1/p')-(1/s)}, \qquad \|M_1\| \leqslant \|l\|_{L_p(\Omega)} \quad \text{and}$$

$$\|T_{\tilde{K}}\| \leqslant \|k\|_{L_{2,\infty}(\Omega-\Omega)} < \infty, \quad \text{where} \quad k(x) = |x|^{-N/2},$$

we obtain the estimate

$$G_n(T_K) \leqslant c_3 n^{-1/s'}\left(\frac{2}{2-s}\right)^{1/s},$$

with $c_3 = c_3(\Omega, p, l)$ not depending on s. For n large enough one can specify now s such that $\frac{2}{2-s} = \ln n$, $p' < s < 2$. Then $n^{-1/s'}\left(\frac{2}{2-s}\right)^{1/s} \leqslant c_4\left(\frac{\ln n}{n}\right)^{1/2}$, and we finally get $G_n(T_K) \leqslant c_5\left(\frac{\ln n}{n}\right)^{1/2}$ for n large enough. Via Proposition 10 this implies the desired assertion. □

References

[1] Bennett, G. Inclusion mappings between l^p-spaces, *J. Functional Analysis,* **13** (1973), 20-27.

[2] Bergh, J., Löfström, J. Interpolation spaces, *Berlin-Heidelberg- New York,* 1976.

[3] Carl, B. Absolut $(p, 1)$-summierende identische Operatoren von l_u nach l_v, *Math. Nachr.,* **63** (1974), 353-360.

[4] Carl, B. Entropy numbers of diagonal operators with an application to eigenvalue problems, *J. Approximation Theory,* **32** (1981), 135-150.

[5] Carl, B. Entropy numbers, entropy moduli, *s*-numbers, and eigenvalues of perators in Banach spaces (to appear).

[6] Carl, B. Inequalities of Bernstein-Jackson-type and the degree of compactness of operators in Banach spaces, *Ann. Inst. Fourier,* **35** (1985).

[7] Carl, B. On the entropy of absolutely summing operators, *Arch. Math.,* **43** (1984), 183-186.

[8] Carl, B., Kühn, T. Entropy and eigenvalues of certain integral operators, *Math. Ann.,* **268** (1984), 127-136.

[9] Carl, B., Triebel, H. Inequalities between eigenvalues, entropy numbers, and related quantities of compact operators in Banach spaces, *Math. Ann.,* **251** (1980), 129-133.

[10] Chevet, S., Chobanjan, S. A., Linde, W., Tarieladze, V. I. Caractérisation de certaines classes d'espaces de Banach par le mesures gaussiennes, *C. R. Acad. Sc., Paris,* 285, Sér. A, (1977), 793-796.

[11] Gordon, Y., König, H., Schütt, C. Geometric and probabilistic estimates for entropy and approximation numbers of operators, *J. Approximation Theory,* (1986).

[12] König, H. Some remarks on weakly singular integral operators, *Integral Equations, Operator Theory,* **3** (1980), 397-407.

[13] König, H., Retherford, J. R., Tomczak-Jaegermann, N. On the eigenvalues of $(p, 2)$-summing operators and constants associated with normed spaces, *J. Functional Analysis,* **37** (1980), 88-126.

[14] Kühn, T. γ-radonifying operators and entropy ideals, *Math. Nachr.,* **107** (1982), 53-58.

[15] Kwapień, S. Some remarks on (p, q)-absolutely summing operators in l_p-spaces, *Studia Math.,* **29** (1968), 327-337.

[16] Linde, W., Pietsch, A. Mappings of Gaussian measures of cylindrical sets in Banach spaces, *Teor. Verojatn. i Primenen.*, **19** (1974), 472-487 (Russian).
[17] Makai, Jr., E., Zemánek, J. Geometrical means of eigenvalues, *J. Operator Theory*, **7** (1982), 173-178.
[18] Maurey, B., Pisier, G. Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, *Studia Math.*, **58** (1976), 45-90.
[19] Pietsch, A. Operator ideals, Berlin, 1978.
[20] Pietsch, A. Weyl numbers and eigenvalues of operators in Banach spaces, *Math. Ann.*, **247** (1980), 149-168.
[21] Pietsch, A. Eigenvalues of absolutely r-summing operators, Aspects of mathematics and its applications (Ed. J. A. Barroso), *Elevier Sci. Publ.*, Amsterdam, 1986.
[22] Szarek, S., Tomczak-Jaegermann, N. On nearly Euclidean decompositions for some classes of Banach spaces, *Compositio Math.*, **40** (1980), 367-385.
[23] Tomczak-Jaegermann, N. Computing 2-summing norm with few vectors, *Ark. Mat.* **17** (1979), 273-277.
[24] Zemánek, J. The essential spectral radius and the Riesz part of the spectrum, *Coll. Math. Soc. J. Bolyai,* **35**, Functions, Series, Operators, Budapest, 1980; North-Holland, Amsterdam, 1983.

Friedrich-Schiller-Universität Jena
DDR-6900 Jena
German Democratic Republic