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Décompositions dans certaines algèbres de Fréchet de fonctions holomorphes

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1. Introduction.

Dans ce travail, nous étudions le problème de décomposition suivant: Étant donnés deux ouverts bornés de $\mathbb{C}^p$, Ω_1 et Ω_2 (vérifiant certaines conditions) et étant donnée une matrice $A(z)$, carrée d'ordre n, dont les coefficients sont des fonctions holomorphes dans $\Omega_1 \cap \Omega_2$, ayant un prolongement C^∞ à l'adhérence $\overline{\Omega_1 \cap \Omega_2}$, peut-on trouver deux matrices $A_1(z)$, $A_2(z)$ holomorphes dans Ω_1 et Ω_2 respectivement et se prolongeant de manière C^∞ à $\overline{\Omega_1}$ et $\overline{\Omega_2}$ telles que sur $\Omega_1 \cap \Omega_2$ on ait

$$A = A_1 \, A_2 \, .$$

On connait l'importance de cette décomposition dans la classification des fibrés analytiques (voir par exemple [1], [8] et leurs bibliographies). Dans [8], une décomposition a déjà été obtenue en utilisant la structure topologique de l'espace $A^\infty(\Omega)$ des fontions holomorphes dans Ω qui se prolongent en des fonctions C^∞ sur $\overline{\Omega}$. Cet espace est une limite projective d'espaces de Banach E_n et grâce a un théorème de Mitagg-Leffler, on se ramène à établir la décomposition dans chacun des E_n, ce qui se fait à l'aide du théorème des fonctions implicites dans les espaces de Banach.

La métohode porposée ici est plus directe, nous travaillons directement dans les espaces de Fréchet en développant l'argument des fonctions implicites dans ces espaces. On met alors en évidence certaines propriétés intéressantes de l'espace de Frechét $A^\infty(\Omega)$ permettant l'introduction des opérateurs de régularisation de Nash [6] et de montrer un théorème des fontions implicites de type Nash-Moser. Nous suivons pour cela la présentation de R. S. Hamilton [4] et de S. Lojasiewicz et de E. Zehnder [5]. La présente méthode est antérieure à celle de [8] et peut sembler plus compliquée, mais elle est plus effective et peut s'appliquer à d'autres types de factorisation (de type Birkhoff par exemple, voir [7, Théorèmes 8.1.1 et 8.11.5]).

Les notations sont celles de [8] auquel nous revenons pour tous les résultats d'analyse complexe que nous utilisons dans le présent travail. Le second paragraphe est consacré à l'introduction des opérateurs de Nash et à poser le problème de la décomposition dans le cadre des matrices. Le troisième paragraphe traite du théorème des fontions implicites dont nous avons besoin pour la décomposition. Le quatrième paragraphe étudie la décomposition des sections de certains fibrès analytiques, certains calculs -à la formule de Campbell-Hausdorff- sont nécessaires, ils seront traités en appendice.

2. Cas de la propriété (P)

Dans cette section, nous considérons deux ouverts Ω_1 et Ω_2, pseudo -convexes, bornés et à bord lisse et C^∞ tels que

$$\overline{\Omega_1\backslash\Omega_2} \cap \overline{\Omega_2\backslash\Omega_1} = \varnothing\,,$$
$$\Omega_1 \cup \Omega_2 = \Omega \text{ est pseudo-convexe à bord lisse et } C^\infty.$$

Les bords de Ω_1, Ω_2 et Ω vérifient la propriété (P) suivante:

(P) Pour tout $M > 0$, $\lambda \in PSH(\Omega) \cap C^\infty(\overline{\Omega})$ (respectivement $PSH(\Omega_i) \cap C^\infty(\overline{\Omega}_i)$, $i = 1,2$), $0 \leq \lambda \leq 1$ telle que pour tout $z \in b\Omega$ (respectivement $z \in b\Omega_i, i = 1,2$)

$$\sum_{i,j=1}^{p} \frac{\partial^2 \lambda}{\partial z_i \partial \bar{z}_j}(z)\, t_i \bar{t}_j \geq M\,|t|^2\,.$$

L'intérêt de cette condition est formulé par un résultat de D. Catlin [2].

Théorème [2]. *Soit Ω un ouvert pseudo-convexe, borné, à bord lisse et C^∞ dont le bord $b\Omega$ vérifie la propriété (P), il existe une constante $C_m > 0$ telle que pour toute $(0,1)$-forme α, $\overline{\partial}$-fermée et à coefficients dans $H^m(\Omega)$, la solution u de $\overline{\partial} u = \alpha$, qui est orthogonale aux fonctions holomorphes satisfait à l'estimation*

$$\|u\|_m^2 \leq C_m \left(\|\alpha\|_m^2 + \|u\|^2\right),$$

où $\|u\|^2 = \|u\|^2_{L^2(\Omega)}$. En particulier, si α et dans $C^\infty_{(0,1)}(\overline{\Omega})$, u est dans $C^\infty(\overline{\Omega})$.

Dans le même travail [2], l'auteur montre que la propriété (P) est satisfaite par une classe très large d'ouverts de $\mathbb{C}^p$.

Suivant la terminologie de [9], on pose

Définition 2.1. *Un bon espace de Fréchet E est un espace dont la topologie est définie à l'aide d'une famille croissante de normes $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \cdots$ et où il existe une famille d'opérateurs $S_t : E \to E$, $t > 0$ tels que il existe $n_0 \in \mathbb{N}$, tel que pour tout $x \in E$*

$$\begin{aligned} \|S_t x\|_i &\leq C_{ij}\, t^{i-j+r} \|x\|_j\,, && i + n_0 \geq j\,, \\ \|S_t x - x\|_i &\leq C_{ij} t^{i-j+r} \|x\|_j\,, && i + n_0 \leq j\,. \end{aligned}$$

Les C_{ij} sont des constantes dépendant de i et j, mais non de x, ni de t.

Proposition 2.2. *Soit Ω un ouvert borné, pseudo-convexe à bord lisse et C^∞ et tel que $b\Omega$ satisfait à la propriété (P), l'espace de Fréchet $A^\infty(\Omega)$ est un bon espace de Fréchet.*

Nous avons besoin du résultat qui suit.

Lemme 2.3. *Soit $L^\infty(\mathbb{R}^n)$ l'espace des fonctions bornées sur $\mathbb{R}^n$ et $B(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) :\ D^\alpha f \in L^\infty(\mathbb{R}^n),\ \text{pour tout } \alpha \in \mathbb{N}^n\}$, on munit $B(\mathbb{R}^n)$ de la topologie définie à l'aide de la famille de semi-normes*

$$\|f\|_m = \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq m}} |D^\alpha f(x)|\,.$$

L'espace de Fréchet $B(\mathbb{R}^n)$ est un bon espace de Fréchet, en conséquence l'espace $C^\infty(\overline{\Omega})$ est un bon espace de Fréchet.

REMARQUE. Le fait que $C^\infty(\overline{\Omega})$ soit un bon espace de Fréchet résulte aussi de l'isomorphisme bien connu $C^\infty(\overline{\Omega}) \simeq S$ où S est l'espace des suites à décroissance rapide, qui est bon ([4]); ici, nous avons besoin d'inégalités explicites faisant intervenir directement le système de normes définies dans $C^\infty(\overline{\Omega})$.

PREUVE DU LEMME 2.3. On remarque que la construction de Nash [6] s'adapte. Soit φ une fonction C^∞, à support dans $B(0,1)$, boule centrée en 0 et de rayon 1, $\varphi \equiv 1$ sur $\overline{B(0,1/2)}$ et soit $\Psi = \widehat{\varphi}$ la transformée de Fourier de φ, alors $\Psi \in \mathcal{S}(\mathbb{R}^n)$ et $\int_{\mathbb{R}^n} P(x)\,\Psi(x)\,dx = P(0)$ pour tout polynôme sur $\mathbb{R}^n$. Pour $f \in B(\mathbb{R}^n)$ et $t > 0$, on pose

$$S_t f(x) = t^n \int \Psi(t(x-y))f(y)dy = \int \Psi(y)f\big(x - \frac{y}{t}\big)\,dt\,.$$

On vérifie, comme dans [6] que

$$\begin{aligned} \|S_t f\|_i &\le C_{ij}\, t^{i-j}\, \|f\|_j\,, \qquad i \ge j\,, \\ \|S_t f - f\|_i &\le C_{ij}\, t^{i-j}\, \|f\|_j\,, \qquad j \ge i\,, \end{aligned}$$

et donc $B(\mathbb{R}^n)$ est bon. Puisque Ω est à bord régulier, le Théorème de Seeley permet de trouver un opérateur linéaire $\mathcal{E}$,

$$C^\infty(\overline{\Omega}) \xrightarrow{\mathcal{E}} C_0^\infty(\widetilde{\Omega})$$

où $\widetilde{\Omega}$ est un ouvert borné de $\mathbb{R}^n$, $\overline{\Omega} \subset \widetilde{\Omega}$ et

$$\mathcal{E}(f)|_{\overline{\Omega}} = f, \quad \|\mathcal{E}(f)\|_{m,\widetilde{\Omega}} \le C_m \|f\|_{m,\Omega}, \quad f \in C^\infty(\widetilde{\Omega}), m \in \mathbb{N}\,.$$

Ainsi, on a une suite

$$C^\infty(\widetilde{\Omega}) \xrightarrow{\mathcal{E}} B(\mathbb{R}^n) \xrightarrow{R} C^\infty(\widetilde{\Omega})$$

où $R(f) = f|_{\overline{\Omega}}, f \in B(\mathbb{R}^n)$ et $R \circ \mathcal{E} = I_{C^\infty(\overline{\Omega})}$. Soit $(S_t'')_{t>0}$ la famille d'opérateurs dans $B(\mathbb{R}^n)$, définie précédemment, on pose

$$S_t' = R \circ S_t'' \circ \mathcal{E}\,.$$

$(S'_t)_{t>0}$ est une famille d'opérateurs dans $C^\infty(\overline{\Omega})$, vérifiant aussi, $f \in C^\infty(\overline{\Omega})$,

$$\begin{aligned} \|S'_t\|_i &\le C_{ij}\, t^{i-j}\, \|f\|_j\,, \qquad i \ge j\,,\\ \|S'_t f - f\|_i &\le C_{ij}\, t^{i-j}\, |f\|_j\,, \qquad i \le j\,, \end{aligned}$$

d'où le lemme.

Revenons à la Proposition 2.2. Puisque le bord de Ω vérifie la propriété (P), on peut considérer le projecteur de Bergman

$$P : C^\infty(\overline{\Omega}) \to A^\infty(\Omega)$$

alors $Pf = u - f$ où u est la solution de $\overline{\partial} u = \overline{\partial} f$, qui est orthogonale aux fonctions holomorphes, par le Théorème de D. Catlin: pour tout m entier,

$$\begin{aligned} \|Pf\|_{H^m(\Omega)} &\le \|u\|_{H^m(\Omega)} + \|f\|_{H^m(\Omega)}\\ &\le C_m\, \|f\|_{H^{m+1}(\Omega)}\,. \end{aligned}$$

(En effet : $\|u\|^2_{H^m(\Omega)} \le C_m\,(\|f\|^2_{H^{m+1}(\Omega)} + \|u\|^2_{L^2(\Omega)}) \le C_m\,(\|f\|_{H^{m+1}(\Omega)} + \|u\|_{L^2(\Omega)})^2$ on utilise ensuite $\|u\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \le \|f\|_{H^{m+1}(\Omega)}$).

Considérons à présent la suite

$$A^\infty(\Omega) \xrightarrow{\mathcal{J}} C^\infty(\overline{\Omega}) \xrightarrow{P} A^\infty(\Omega) \quad (\mathcal{J} \text{ injection canonique})$$

on a $P \circ \mathcal{J} = I_{A^\infty(\Omega)}$. Si l'on pose

$$\begin{cases} S_t : A^\infty(\Omega) \to A^\infty(\Omega)\,,\\ S_t = P \circ S'_t \circ \mathcal{J}\,. \end{cases}$$

On a grâce au lemme d'injection de Sobolev $H^{m+p}(\Omega) \to C^m(\overline{\Omega})$, que

$$\begin{aligned} \|S_t f\|_i &= \|P \circ S'_t \circ \mathcal{J}(f)\|_i\\ &\le C_i\, \|S'_t(f)\|_{H^{p+1}(\Omega)}\\ &\le C_i\, \|S'_t(f)\|_{p+i+1}\\ &\le C_{ij}\, t^{i-j}\, \|f\|_{j+p+1}\,, \end{aligned}$$

si $i \ge j$ et

$$\|S_t f - f\|_i = \|P \circ (S'_t - 1) \circ \mathcal{J}(f)\|_i$$

$$\begin{aligned}&\le C_i\,\|(S'_t-1)\circ \mathcal{J}(f)\|_{H^{p+i+1}(\Omega)}\\&\le C_i\,\|(S'_t-1)\circ \mathcal{J}(f)\|_{p+i+1}\\&\le C_{ij}\,t^{i-j}\,\|f\|_{j+p+1}\,.\end{aligned}$$

si $i \le j$. Les constantes intervenant dans ces inégalités changent à chaque étape mais ne dépendent que de i et de j (et aussi de p, qui est fixé, $p = \dim \mathbb{C}^p$), la proposition est démontrée.

REMARQUES 2.4. 1) Soient $\alpha \le \beta \le \gamma$ trois entiers et $f \in A^\infty(\Omega) \subset C^\infty(\overline{\Omega})$, f vérifie

$$\begin{aligned}\|f\|_\beta &\le \|S_t f - f\|_\beta + \|S_t f\|_\beta\\&\le C_{\beta,\gamma}\,t^{\beta-\gamma}\,\|f\|_{\gamma+p} + C_{\beta,\alpha}\,t^{\beta+\alpha}\,\|f\|_{\alpha+p}\,, \qquad t>0\,.\end{aligned}$$

En minimisant par rapport à t la fonction

$$t \to C_{\beta,\gamma}\,t^{\beta-\gamma}\,\|f\|_{\gamma+p} + C_{\beta,\gamma}\,t^{\beta-\gamma}\,\|f\|_{\alpha+p}$$

on a

$$\|f\|_\beta \le C_{\alpha,\beta,\gamma}\,\|f\|_{\alpha+p}^{(\beta-\gamma)/(\alpha-\gamma)}\,\|f\|_{\gamma+p}^{(\beta-\alpha)/(\gamma-\alpha)}$$

et si l'on considère f comme élément de $C^\infty(\overline{\Omega})$, il vient

$$\begin{aligned}\|f\|_\beta &\le \|S'_t f - f\|_\beta + \|S'_t f\|_\beta\\&\le C_{\beta,\gamma}\,t^{\beta-\gamma}\,\|f\|_\gamma + C_{\beta,\alpha}\,t^{\beta+\alpha}\,\|f\|_\alpha\\&\le C_{\alpha,\beta,\gamma}\,\|f\|_\alpha^{(\beta-\gamma)/(\alpha-\gamma)}\,\|f\|_\gamma^{(\beta-\alpha)/(\gamma-\alpha)}.\end{aligned}$$

Dans la toute la suite, on ne retiendra que la deuxième inégalité qui est plus précise.

2) La seule existence d'opérateurs $(S_t)_t$ sur $A^\infty(\Omega)$ montre, grâce à un théorème de Vogt [10] que $A^\infty(\Omega)$ est un sous-espace de S, espace des suites à décroissance rapide. En effet soit α_0 un entier quelconque fixé et $\beta \ge \alpha_0$ un autre entier, l'inégalité obtenue dans la Remarque 1) avec $\alpha_0 \le \beta \le 2\beta - \alpha_0$ devient $\|f\|_\beta^2 \le C_{\alpha,\beta,\gamma}\,\|f\|_{\alpha_0+p}\,\|f\|_{2\beta-\alpha_0+p}$, pour toute f dans $A^\infty(\Omega)$, ceci est exactement la propriété (D.N.) de [10].

3) L'existence de régularité globale pour les solutions u de $\overline{\partial} u = \alpha$ dans les ouverts Ω dont le bord vérifie une propriété telle que (P) fait de l'espace $A^\infty(\Omega)$ un quotient de S. En effet, dans [10], on caractérise les

espaces de Fréchet nucléaires qui sont quotients de S par la propriété -dite propriété (Ω)- suivante:

(Ω) Pour tout q, il existe q_0 tel que pour tout k, il existe n et une constante positive $C = C(q, q_0, k, n \in \mathbb{N})$ tels que pour tout $r > 0$

$$U_{q_0} \subset C\, r^n\, U_k + \frac{1}{r}\, U_q\, .$$

On montre dans [10] que $C^\infty(\overline{\Omega})$ vérifie la propriété (Ω), avec pour tout entier m, $U_m = \{f \in C^\infty(\overline{\Omega}) :\ \|f\|_m \leq 1\}$. Pour voir que $A^\infty(\Omega)$ vérifie aussi (Ω), fixons un entier q et soit q_0 l'entier associé à $q+p+1$ et écrivons l'inclusion précédente avec $k+p+1$ ($p = \dim \mathbb{C}^p$),

$$U_{q_0} \subset C'\, r^n\, U_{k+p+1} + \frac{1}{r}\, U_{q+p+1}\,, \qquad (r > 0)\,.$$

Soit $f \in U_{q_0} \cap A^\infty(\Omega)$, il y a donc $f_1' \in U_{k+p+1}$ et $f_2' \in U_{q+p+1}$ tels que $f = C' r^n\, f_1' + f_2'/r$. La $(0,1)$-forme $\omega = -C' r^n\, \overline{\partial} f_1' = \overline{\partial} f_2'/r$ est $\overline{\partial}$-fermée et à coefficients dans $C^\infty(\overline{\Omega})$, il existe $\alpha \in C^\infty(\overline{\Omega})$ telle que $\overline{\partial}\alpha = \omega$. On pose

$$f_1 = f_1' - \frac{1}{C' r^n}\, \alpha, \qquad f_2 = f_2' + r\alpha\,, \qquad f_1, f_2 \in A^\infty(\Omega)\,.$$

Soient $C_k, \ldots$ les constantes intervenant dans le lemme d'injection de Sobolev ou dans les majorations des normes $H^{k+p}(\Omega)$ par $\|\cdot\|_{k+p}$ on a

$$\begin{aligned}
\|f_1\|_k &= \left\| f_1' - \frac{1}{C' r^n}\, \alpha \right\|_k \\
&\leq C_k \left\| f_1' - \frac{1}{C'\, r^n}\, \alpha \right\|_{H^{k+p}(\Omega)} \\
&\leq C_k \left(\|f_1'\|_{H^{k+p}(\Omega)} + C^1\, r^n\, \frac{1}{C'\, r^n}\, \|f_1'\|_{H^{k+p+1}(\Omega)} \right) \\
&\leq C_k'\, \|f_1'\|_{k+p+1}\,,
\end{aligned}$$

de même

$$\|f_2\|_q \leq C_q'\, \|f_2'\|_{q+p+1}\,,$$

comme

$$f = C'\, r^n\, f_1 + \frac{1}{r}\, f_2 \in C_k\, r^n (U_k \cap A^\infty(\Omega)) + \frac{1}{r}\, C_q (U_q \cap A^\infty(\Omega))\,.$$

Il vient

$$U_{q_0} \cap A^\infty(\Omega) \subset C'(C'_q)^n C'_k\Big(\frac{r}{C'_q}\Big)^n U_k \cap A^\infty(\Omega) + \frac{C'_q}{r}(U_q \cap A^\infty(\Omega))$$

puisque r est positif quelconque, avec $C = C'\,(C'_q)^n\, C'_k$

$$U_{q_0} \cap A^\infty(\Omega) \subset C\, r^n\, (U_k \cap A^\infty(\Omega)) + \frac{1}{r}(U_q \cap A^\infty(\Omega))\,.$$

En résumé, on a montré la proposition suivante.

Proposition 2.5. *Si Ω est un ouvert pseudo-convexe borné, à bord lisse et C^∞ dont le bord vérifie une propriété telle que la propriété* (P), *l'espace de Fréchet nucléaire $A^\infty(\Omega)$ est à la fois un sous-espace et un quotient de l'espace S des suites complexes à décroissance rapide.*

(L'espace $A^\infty(\Omega)$ est nucléaire puisque c'est un sous espace fermé de $C^\infty(\overline{\Omega})$ qui l'est).

Dans la suite, nous aurons besoin du lemme simple suivant.

Lemme. *Soit X un espace métrique compact et soit E un bon espace de Fréchet, l'espace $C(X,E)$ des applications continues de X dans E est bon.*

Rappelons que la topologie de $C(X,E)$ est définie à l'aide des normes suivantes, pour $f \in C(X,E)$,

$$\|f\|_i = \sup_{x\in X} \|f(x)\|_i\,, \qquad i \in \mathbb{N}\,.$$

Si $(S'_t)_{t>0}$ est la famille d'opérateurs sur E, on définit sur $C(X,E)$ les opérateurs

$$S_t : C(X,E) \to C(X,E)$$
$$f \mapsto S_t f$$

où $(S_t f)(x) = S'_t(f(x))$, on en déduit aisément des inégalités sur S_t analogues à celles données sur S'_t.

Comme conséquence immédiate et pour un ouvert Ω dont le bord vérifie (P).

Corollaire 2.6. *Soit s un entier ≥ 2, $M(s; A^\infty(\Omega))$ l'algèbre des matrices $s \times s$ à coefficients dans $A^\infty(\Omega)$, si X est un métrique compact, l'espace $C(X, M(s, A^\infty(\Omega)))$ est un bon espace de Fréchet.*

Tout d'abord, remarquons que si E_1 et E_2 sont deux bons espaces de Fréchet avec deux familles d'opérateurs $(S_t^1)_{t>0}$ et $(S_t^2)_{t>0}$ correspondantes, l'espace produit $E_1 \times E_2$ est, lui aussi bon, il suffit de poser $S_t(x,y) = (S_t^1 x, S_t^2 y)$, $(x,y) \in E_1 \times E_2$ en munissant ce dernier, par exemple, du système de normes

$$(x,y) \in E_1 \times E_2 : \ \|(x,y)\|_i = \sup\{\|x\|_i, \|y\|_i\}, \qquad i \in \mathbb{N}.$$

Si bien que si $f \in M(s; A^\infty(\Omega)), f = (f_{ij})_{ij}$ et en posant encore: $\|f\|_n = \sup_{i,j} \|f_{ij}\|_n$. On a pour $f, g \in M(s, A^\infty(\Omega))$

$$\begin{aligned} \|f+g\|_n &\leq \|f\|_n + \|g\|_n , \\ \|fg\|_n &\leq s\,\|f\|_n\,\|g\|_n . \end{aligned}$$

Si $S_t'' : A^\infty(\Omega) \to A^\infty(\Omega)$, on définit

$$S_t' : \ M(s, A^\infty(\Omega)) \to M(s, A^\infty(\Omega))$$

par $(S_t' f)_{ij} = S_t''(f_{ij})$ et

$$S_t : \ C(X, M(s, A^\infty(\Omega)) \to C(X, M(s, A^\infty(\Omega))$$

par $(S_t f)(x) = S_t'(f(x))$, où $f \in C(X, M(s, A^\infty(\Omega))$ et $x \in X$. Ainsi

$$\begin{aligned} \|S_t f\|_i &\leq C_{ij}\, t^{i-j}\, \|f_a\|_{j+p+1} , \qquad i \geq j , \\ \|S_t f - f\|_i &\leq C_{ij}\, t^{i-j}\, \|f\|_{j+p+1} , \qquad i \leq j . \end{aligned}$$

Les C_{ij} sont des constantes ne dépendant que de i et de j. Si maintenant S est un fermé de X et E un espace de Fréchet, on note par: $C_S(X,E) = \{f : \ X \to E, \ f \text{ continue et } f(t) = 0, \ t \in S\}$, c'est aussi un espace de Fréchet, quand on le munit de la topologie induite par celle de $C(X,E)$. Il est aussi bon si $C(X,E)$ est bon, c'est-à-dire si E est bon. Avant d'énoncer le théorème principal de cette section, précisons la situation.

On considère deux ouverts Ω_1, Ω_2 de $\mathbb{C}^p$, pseudo-convexes, bornés et à bord lisse et C^∞ tels que

i) $\overline{\Omega_1 \backslash \Omega_2} \cap \overline{\Omega_2 \backslash \Omega_1} = \varnothing$,

ii) $\Omega_1 \cup \Omega_2$ est pseudo-convexe, à bord lisse et C^∞,

iii) les bords de Ω_1, Ω_2 et $\Omega = \Omega_1 \cup \Omega_2$ vérifient (P).

Dans ces conditions, on a le

Lemme 2.7. *Pour tout s entier ≥ 2, on a la suite exacte suivante*

$$\begin{aligned}0 &\to C_S(X, M(s, A^\infty(\Omega_1 \cup \Omega_2))) \\ &\xrightarrow{\mathcal{I}} C_S(X, M(s, A^\infty(\Omega_1))) \oplus C_S(X, M(s, A^\infty(\Omega_2)) \\ &\xrightarrow{R} C_S(X, M(s, A^\infty(\Omega_1 \cap \Omega_2))).\end{aligned}$$

avec de plus R inversible à droite et d'inverse linéaire et continu.

PREUVE. La suite $0 \to A^\infty(\Omega_1 \cup \Omega_2) \xrightarrow{i} A^\infty(\Omega_1) \oplus A^\infty(\Omega_2) \xrightarrow{\alpha} A^\infty(\Omega_1 \cap \Omega_2) \to 0$ est toujours exacte sous les seules hypothèses i) et ii). L'hypo - thèse iii) implique que de plus α est inversible à droite, d'inverse linéaire et continu, en effet si $\varphi \in C_0^\infty(\mathbb{C}^p)$, $\varphi \equiv 1$ sur $\overline{\Omega_1 \backslash \Omega_2}$ et $\varphi \equiv 0$ sur $\overline{\Omega_2 \backslash \Omega_1}$, la forme (0,1)-forme $\omega = f\ \overline{\partial}\varphi$ est $\overline{\partial}$-fermée et à coefficients dans $C^\infty(\overline{\Omega_1 \cup \Omega_2})$ et le fait que $\Omega_1 \cup \Omega_2$ ait un bord vérifiant (P) montre qu'il existe un opérateur $T_1 : C^\infty_{0,1}(\overline{\Omega_1 \cup \Omega_2}) \to C^\infty(\overline{\Omega_1 \cup \Omega_2})$ tel que $\overline{\partial}(T_1(f\overline{\partial}\varphi)) = f\overline{\partial}\varphi$, pour tout $f \in A^\infty(\Omega_1 \cap \Omega_2)$ et

$$\|T_1(f\overline{\partial}\varphi)\|_{H^m(\Omega_1 \cap \Omega_2)} \leq C_m\, \|f\overline{\partial}\varphi\|_{H^m(\Omega_1 \cup \Omega_2)}$$

ce qui montre, grâce au lemme d'injection de Sobolev et le fait que $f\overline{\partial}\varphi$ soit à coefficients dans $C^\infty(\overline{\Omega_1 \cup \Omega_2})$, que $T_1(f\overline{\partial}\varphi)$ est dans $C^\infty(\overline{\Omega_1 \cup \Omega_2})$. Evidemment, si $f_1 = (1-\varphi)f + T_1(f\overline{\partial}\varphi)$ et $f_2 = T(f\overline{\partial}\varphi) - f\varphi$, le couple $(f_1, f_2) \in A^\infty(\Omega_1) \times A^\infty(\Omega_2)$ dépend linéairement de f. Soit

$$\begin{aligned}\ell :\ A^\infty(\Omega_1 \cap \Omega_2) &\to A^\infty(\Omega_1) \times A^\infty(\Omega_2) \\ f &\mapsto (f_1, f_2)\end{aligned}$$

on a

$$\|\ell(f)\|_n \leq C_n\, \|f\|_{n+p}\,, \qquad n \in \mathbb{N}.$$

Si maintenant $f \in C(X, M(s; A^\infty(\Omega_1 \cap \Omega_2))$, on pose

$$\mathcal{L}(f)(t) = \ell(f(t))\,, \qquad t \in X\,,$$

alors

$$\mathcal{L}: \; C(X, M(s, A^\infty(\Omega_1 \cap \Omega_2)) \to C(X, M(s, A^\infty(\Omega_1)) \times C(X, m(s, A^\infty(\Omega_2))$$

est linéaire et $R \circ \mathcal{L}(f)(t) = \alpha$, $(\mathcal{L}(f)(t) = \alpha \circ \ell(f(t)) = f(t)$, $t \in X)$ c'est à dire

$$R \circ \mathcal{L} = I_{C(X, M(s; A^\infty(\Omega_1 \cap \Omega_2))}$$

on vérifie facilement que

$$\|\mathcal{L}(f)\|_n \leq C_n \|f\|_{n+p}, \qquad n \in \mathbb{N}.$$

Enfin, à cause de la linéarité de ℓ, l'application $\mathcal{L}$ induit une application notée encore $\mathcal{L}$

$$C_S(X, M(s; A^\infty(\Omega_1 \cap \Omega_2)) \to C_S(X, M(s, A^\infty(\Omega_1)) \times C_S(X, M(s, A^\infty(\Omega_2)).$$

avec

$$R \circ \mathcal{L} = I_{C_S(X, M(s; A^\infty(\Omega_1 \cap \Omega_2))}.$$

Le théorème principal de cette section est

Théorème 2.8. *Soient* $E_i = C_S(X, M(s; A^\infty(\Omega_i)), i = 1, 2$; $E = C_S(X, M(s; A^\infty(\Omega_1 \cap \Omega_2))$ *et* $U_i = \{f \in E : \|f\|_{i.1} \leq s^{-1}\}$. *Si* φ *est l'application*

$$\varphi : U_1 \times U_2 \subset E_1 \times E_2 \to E$$
$$(x, y) \mapsto xy - x - y$$

il existe $\delta > 0$ *et* $s_0 \in \mathbb{N}$ *tels que si* $V = \{y \in E : \|y\|_{s_0} < \delta\}$, *il existe une application continue* $\psi : V \to U_1 \times U_2$, $\varphi(\psi(y)) = y$, *pour tout* y *dans* V.

Avant de démontrer ce théorème signalons qu'il entraine un "lemme de matrices holomorphes" dans la classe A^∞. Si $z \in C(X, M(s, A^\infty (\Omega_1 \cap \Omega_2)))$, $z(t) = e$ pour tout $t \in S$ et $\|z - e\|_{s_0}, < \delta$, soit $z' = z - e$, z' est dans V, il va donc exister $x' \in U_1$ et $y' \in U_2$ tels que $x'y' - x' - y' = z'$. Soient $x = e - x'$ et $y = e - y'$ alors $xy = z$ et $x(t) = y(t) = e$ pour $t \in S$ où e est la matrice identité de $M(s, A^\infty(\Omega_i))$

$i = 1, 2$, de plus, si $\delta < s^{-1}$ (situation qu'on va réaliser en construisant le voisinage V), z est inversible. La norme de matrices considérées ici est $\|A\| \leq \sup_{1\leq i,j\leq s} |a_{ij}|$. Les antécédents x, y sont aussi inversibles. Enfin, le théorème ne donne de factorisation que pour les z qui sont voisins de la matrice identité dans $C(X, M(s, A^\infty(\Omega_1 \cap \Omega_2)))$ et $z(t) = e$, $t \in S$ mais si par exemple $X = \varnothing$ ou X est contractible en un point et $GL(s; A^\infty(\Omega_1 \cap \Omega_2))$ est connexe (par exemple si $\Omega_1 \cap \Omega_2$ est convexe) on montre facilement que $C(X, GL(X, A^\infty(\Omega_1 \cap \Omega_2)))$ est connexe par arcs, et en raisonnant comme dans la première partie, tout $z \in C(X, GL(s, A^\infty(\Omega_1 \cap \Omega_2)))$ s'écrit alors $z = xy$ où x (respectivement y) est dans $C(X, GL(s, A^\infty(\Omega_1)))$ (respectivement $C(X, GL(s, A^\infty(\Omega_2)).)$

Pour montrer le théorème, on a besoin de quelques inégalités.

Lemme 2.9. *Dans l'algèbre $C(X, M(s, A^\infty(\Omega)) = A$, Ω étant un ouvert à bord lisse (sans conditions supplémlentaires), on a les inégalités suivantes:*

a) Si $(\alpha, \beta, \gamma) \in \mathbb{N}^3$, $\alpha \leq \beta \leq \gamma$, $f \in A$,

$$\|f\|_\beta^{\gamma-\alpha} \leq C\,\|f\|_\gamma^{\beta-\alpha}\,\|f\|_\alpha^{\gamma-\beta}.$$

b) Si $(i, j) \in \mathbb{N} \times \mathbb{N}$ appartient au segment joignant (k, ℓ) et (m, n) et si $f, g \in A$,

$$\|f\|_i\,\|g\|_j \leq C\,(\|f\|_k\,\|f\|_\ell + \|f\|_m\,\|g\|_n),$$

c) Si $f, g \in A$ et n est un entier,

$$\|fg\|_n \leq C\,(\|f\|_n\,\|g\|_0 + \|f\|_0\,\|g\|_n).$$

Les constantes C intervenant dans ces inégalités dépendent naturellement des indices $\alpha, \beta, \ldots, i, j$ mais non des éléments $f \in A$. Les inégalités de b) et c) se déduisent de celles de a) par des méthodes habituelles. Pour montrer les inégalités de a), on se ramène facilement au cas où f, g sont dans $C^\infty(\overline{\Omega})$, le résultat se déduit alors de l'existence des opérateurs (S_t) sur $C^\infty(\overline{\Omega})$ comme on l'a fait dans les Remarques 2.4 (voir [4]).

Lemme 2.10. *Soit B un élément de $C(X, M(s, A^\infty(\Omega)))$ tel que $\|B - I\|_1 < \varepsilon$ où ε est assez petit, B est inversible dans $C(X, M(s, A^\infty(\Omega))$*

et pour tout entier $n \geq 1$ il existe une constante C_n ne dépendant que de n (et de s) tels que

$$\|B^{-1}\|_n \leq C_n \|B\|_n .$$

PREUVE. On procède essentiellement comme dans [8]. Remarquons d'abord que B^{-1} existe dans $C(X, M(s, A^0(\Omega))$ où $A^0(\Omega)$ est l'algèbre des fonctions holomorphes dans Ω et continues sur $\overline{\Omega}$ et à cause de la présence du déterminant, B^{-1} est en fait dans $C(X, M(s; A^\infty(\Omega))$. D'autre part, il existe une constante C, indépendante des éléments B vérifiant l'hypothèse du lemme telle que $\|B\|_0 \leq C$. En effet si $\varepsilon < s^{-1}$,

$$B^{-1} - I = \sum_{n\geq 1}(I - B)^n$$

et

$$\|B^{-1} - I\|_0 \leq \sum_{n\geq 1} s^n \|B - I\|_0^n \leq \frac{s\varepsilon}{1 - s\varepsilon}$$

et donc

$$\|B^{-1}\|_0 \leq \|I\|_0 + \frac{s\varepsilon}{1 - s\varepsilon} = C < +\infty .$$

Pour montrer les inégalités d'ordre supérieur ou égal à 1, on identifie B à une matrice $s \times s$ en la variable $z \in \overline{\Omega}$ et dépendant du paramètre $t \in X$. Pour t fixé et puisque $B^{-1}(t,z)B(t,z) = I$ pour tout opérateur différentiel du premier ordre D sans terme constant on a $DB^{-1}(t,z) = -B^{-1}(t,z)\,(DB(t,z))\,B^{-1}(t,z)$ donc

$$\|DB^{-1}(t,\cdot)\|_0 \leq C_1 \|DB(t,\cdot)\|_0 \quad \text{et} \quad \|DB^{-1}\|_0 \leq C_1 \|DB\|_0 ,$$

ou encore $\|B^{-1}\|_1 \leq C_1 \|B\|_1$

Supposons que l'on ait prouvé $\|B^{-1}\|_p \leq C_p \|B\|_p$, $p \leq n - 1$, $n \geq 2$.

Soit n' un multi-indice, $|n'| \leq n$ et $D^{n'}$ un opérateur différentiel d'ordre n'

$$\begin{aligned} O = D^{n'} I &= D^{n'}(B(t,z)\,B^{-1}(t,z)) \\ &= D^{n'}B(t,z)\,B^{-1}(t,z) + B(t,z)\,D^{n'}B^{-1}(t,z) \\ &\quad + \sum_{\substack{p+q=n' \\ |p|\leq n-1,\ |q|\leq n-1}} C_{pq}\, D^p B(t,z)\, D^q B^{-1}(t,z) . \end{aligned}$$

On en déduit, grâce aux hypothèses $\|B^{-1}\|_p \le C_p \|B\|_p$, $p \le n-1$, que

$$\|B(t,\cdot)\|_{|p|} \|B^{-1}(t,\cdot)\|_{|q|} \le \|B\|_{|p|} \|B^{-1}\|_{|q|} \le C \|B\|_{|p|} \|B\|_{|q|}$$

et par l'inégalité *a*) du Lemme 2.9,

$$\begin{aligned} \|B\|_p \|B\|_q &\le C \|B\|_{|n'|}^{|p|/|n'|} \|B\|_0^{1-|p|/|n'|} \|B\|_{|n'|}^{|q|/|n'|} \|B\|_0^{1-|q|/|n'|} \\ &\le C \|B\|_{|n'|} \le C \|B\|_n \end{aligned}$$

et comme

$$\|D^{n'} B(t,\cdot)\|_0 \le \|B(t,\cdot)\|_{|n'|} \le \|B\|_n$$

il vient que

$$\|B^{-1}\|_n \le C_n \|B\|_n \,.$$

On est en mesure de montrer que l'application φ du théorème principal est différentiable (au sens de Gâteaux), que sa différentielle est une "bonne" application linéaire, qu'elle est inversible à droite et que son inverse est aussi bon (au sens de [9]).

Proposition 2.11. *Soient* $E_i = C_S(X, M(s, A^\infty(\Omega_i)))$, $i = 1,2$, $E = C_S(X, M(s, A^\infty(\Omega_1 \cap \Omega_2)))$ *et* $U_i = \{x \in E_1 : \|x\|_{i,1} < s^{-1}\}$. *L'application* $\varphi : U_1 \times U_2 \to E$, $\varphi(x,y) = xy - x - y$ *est différentiable au sens Gâteaux et il existe une application linéaire* L,

$$\begin{aligned} L : U_1 \times U_2 \times E &\to E_1 \times E_2 \\ ((x,y),\eta) &\mapsto L(x,y)\eta \end{aligned}$$

telle que $d\varphi(x,y)L(x,y)\eta = \eta$; $\eta \in E$, $(x,y) \in U_1 \times U_2$.

Par définition

$$d\varphi(x,y)(h,k) = \lim_{t \to 0} \frac{\|\varphi((x,y) + t(h,k)) - \varphi(x,y)\|}{t} \,.$$

On en déduit que

$$d\varphi(x,y)(h,k) = h(y-e) + (x-e)k \,.$$

Afin de déterminer l'application L, on résoud en $(h,k) \in E_1 \times E_2$, l'équation $d\varphi(x,y)L(x,y)\eta = \eta$, c'est-à-dire que $h(y-e) + (x-e)k = \eta$.

Comme $x-e$ et $y-e$ sont inversibles si $(x,y) \in U_1 \times U_2$, il est équivalent de résoudre

$$(x-e)^{-1}h + k(y-e)^{-1} = (x-e)^{-1}\eta\,(y-e)^{-1}\,.$$

Mais si η est donné dans E, nous savons qu'il existe $(\xi_1, \xi_2) \in E_1 \times E_2$ tel que

$$\xi_1 + \xi_2 = (x-e)^{-1}\eta\,(y-e)^{-1}\,,$$

de plus le couple (ξ_1, ξ_2) dépend linéairement de $(x-e)^{-1}\eta(y-e)^{-1}$ lorsque η varie. On pose

$$h = (x-e)\,\xi_1 \in E_1\,, \qquad k = \xi_2\,(y-e) \in E_2\,.$$

Ceci définit une application

$$L : U_1 \times U_2 \times E \to E_1 \times E_2\,,$$
$$L(x,y)\eta = ((x-e)\xi_1, \xi_2(y-e))\,,$$

telle que $d\varphi(x,y)L(x,y)\eta = \eta$, d'où la proposition.

On désigne par $R(x,y)$ le reste dans le développement de φ,

$$R(x,y)(h,k) = \varphi((x,y)+(h,k)) - \varphi(x,y) = df(x,y)(h,k) = hk\,.$$

Dans la suite, nous aurons besoin des estimations simples suivantes.

Proposition 2.12. *Pour tout n, il existe $C_n > 0$ tels que si $(x,y) \in U_1 \times U_2$, $(h,k) \in E_1 \times E_2$ et $\eta \in E$*

a) $\|\varphi(x,y)\|_n \le C_n \|(x,y)\|_n$,

b) $\|d\varphi(x,y)(h,k)\|_n \le C_n (\|(h,k)\|_0 \|(x,y)\|_n + \|(h,k)\|_n)$,

c) $\|L(x,y)\eta\|_n \le C_n (\|(x,y)\|_{n+p} \|\eta\|_0 + \|\eta\|_{n+p})$,

d) $\|R(x,y)(h,k)\|_n \le C_n \|(h,k)\|_0 \|(h,k)\|_n$.

Ces inégalités résultent de ce qui précède, à titre d'exemple, montrons comment on obtient *c*). Par construction de L, si $(x,y) \in U_1 \times U_2$, $\eta \in E$,

$$L(x,y)\eta = ((x-e)\xi_1, \xi_2(y-e))\,, \qquad \xi_1 + \xi_2 = (x-e)^{-1}\eta(y-e)^{-1}\,,$$

avec de plus; $i = 1, 2$,

$$\|\xi_i\|_{H^m(\Omega_i)} \leq C_m \, \|(x-e)^{-1}\eta(y-e)^{-1}\|_{H^m(\Omega')} \,, \qquad m \in \mathbb{N} \,,$$

où on a noté pour simplifier $\Omega' = \Omega_1 \cap \Omega_2$. Par le lemme d'injection de Sobolev,

$$\|\xi_i\|_{n,\Omega_i} \leq C_n \, \|(x-e)^{-1}\eta(y-e)^{-1}\|_{n+p,\Omega'}$$

et par le Lemme 2.9.c)

$$\begin{aligned}\|(x-e)^{-1}\eta(y-e)^{-1}\|_{n+p,\Omega'} &\leq C_n \left(\|(x-e)^{-1}\eta\|_{0,\Omega'} \, \|(y-e)^{-1}\|_{n+p,\Omega'} \right. \\ &\qquad + \|(x-e)^{-1}\|_{n+p,\Omega'} \\ &\qquad \left. \cdot \|(y-e)^{-1}\|_{0,\Omega'}\right) \\ &\leq C_n \left(\|\eta\|_{0,\Omega'} \, \|(y-e)^{-1}\|_{n+p,\Omega'} \right. \\ &\qquad \left. + \|(x-e)^{-1}\eta\|_{n+p,\Omega'}\right) ,\end{aligned}$$

car si $(x, y) \in U_1 \times U_2$ $\|(x-e)^{-1}\|_0 \leq C$ et $\|(y-e)^{-1}\|_0 \leq C$.

En utilisant deux fois le même lemme, on trouve

$$\|(x-e)^{-1}\|_{n+p,\Omega'} \leq C_n \left(\|\eta\|_{n+p,\Omega'} + \|(x-e)^{-1}\|_{n+p,\Omega'} \, \|\eta\|_0\right)$$

et

$$\begin{aligned}\|(x-e)^{-1}\eta(y-e)^{-1}\|_{n+p,\Omega'} \leq C_n &\left(\|\eta\|_{0,\Omega'} \, \|(y-e)\|_{n+p,\Omega'} \right. \\ &+ \|\eta\|_{n+p,\Omega'} \\ &\left. + \|(x-e)\|_{n+p,\Omega'} \, \|\eta\|_{0,\Omega'}\right) ,\end{aligned}$$

comme

$$\|(x-e)\|_{n+p,\Omega'} \leq \|(x-e)\|_{n+p,\Omega_1} \quad \text{et} \quad \|(y-e)\|_{n+p,\Omega'} \leq \|(y-e)\|_{n+p,\Omega_2}$$

il vient enfin

$$\|\xi_i\|_{\Omega_i} \leq C_n \left(\|(x, y)\|_{n+p} \, \|\eta\|_0 + \|\eta\|_n\right) .$$

En recommançant de la même façon avec

$$\|(x-e)\xi_1\|_{n+p,\Omega_1} \quad \text{et} \quad \|\xi_2(y-e)\|_{n+p,\Omega_2}$$

on aboutit à l'inégalité c).

3. Preuve du théorème principal.

Dans ce paragraphe, on montre le théorème principal du Paragraphe 2, il s'agit d'un théorème des fonctions implicites dans les espaces de Fréchet. On sait qu'en toute généralité, un tel théorème ne peut être vrai. De nombreux contre-exemples se trouvent dans [4]. Cependant, on a pu dégager depuis [6], une classe d'espaces de Fréchet où un théorème des fonctions implicites est possible (voir [4], [5] et [6] et leurs bibliographies). L'espace $A^\infty(\Omega)$ -qui est limite projective d'espaces de Banach- fait partie de cette classe si le bord de l'ouvert Ω est régulier et vérifie une propriété telle que la propriété (P) du Paragraphe 2. Notre situation est voisine de celles déjà étudiées à un décalage d'un entier p près dans la définition des opérateurs S_t et dans les estimations portant sur l'inverse L de la différentielle. Ce décalage est dû aux estimations dont on dispose à l'heure actuelle sur le projecteur de Bergman.

La situation est formulée par le théorème suivant.

Théorème. *Soient E_1 et E_2 deux espaces de Fréchet, on suppose qu'il existe des opérateurs $S_t : E_1 \to E_1$ tels que pour p entier non nul*

$$\|S_t x\|_i \le C_{ij}\, t^{i-j}\, \|x\|_{j+p}\,, \qquad i \ge j\,,$$
$$\|S_t x - x\|_i \le C_{ij}\, t^{i-j}\, \|x\|_{j+p}\,, \qquad i \le j\,.$$

Soit U l'ouvert de E_1, $U = \{x \in E : \|x\|_1 \le a\}$ où a est un réel inférieur ou égal à 1 et soit $\varphi : U \to E_2$ une application différentiable au sens de Gâteaux, $\varphi(0) = 0$ et dont la différentielle $d\varphi$ est inversible à droite, d'inverse L, vérifiant en outre pour $x \in U$ et $y_1 \in E_1$, $y_2 \in E_2$ les inégalités

$$\|\varphi(x)\|_n \le C_n\,(1 + \|x\|_n)\,,$$
$$\|d\varphi(x) y_1\|_n \le C_n\,(\|x\|_n\, \|y_1\|_0 + \|y_1\|_n)\,,$$
$$\|L(x) y_2\|_n \le C_n\,(\|x\|_{n+p}\, \|y_2\|_0 + \|y_2\|_{n+p})\,,$$
$$\|R(x) y_1\|_n \le C_n\,(\|x\|_n\, \|y_1\|_0^2 + \|y\|_0\, \|y_1\|_n)\,.$$

Alors φ admet une section locale: il existe V, voisinage de 0 dans E_2 et une application ψ de V dans U telle que, pour tout y dans V, $\varphi(\psi(y))= y$.

PREUVE. On veut montrer que si y est dans un voisinage de 0 dans E_2, il y a un x dans U tel que $\varphi(x) = y$. Posons, comme dans [5], [9] et [4],

$$x_0 = 0\,, \qquad x_{q+1} = x_q + S_{t_q} L(x_p) z_q\,, \qquad z_q = y - \varphi(x_q)$$

et

$$\Delta x_q = x_{q+1} - x_q\,,$$

où $(t_q)_q$ est la suite suivante: Si $\tau = 3/2$, $t_q = 2^{\tau^q}$ de sorte que $t_{q+1} = t_q^\tau$. Avant de montrer que la suite (x_q) est convergente vers un élément x de V vérifiant $\varphi(x) = y$, on renforce sensiblement l'inégalité sur φ : $\|\varphi(x)\|_n \leq C'_n(1 + \|x\|_n)$ en utilisant le fait que $\varphi(0) = 0$ et que pour tout x dans U, y_1 dans E_1,

$$\|d\varphi(x)y_1\| \leq C_n\,(\|x\|_n\,\|y_1\|_0 + \|y_1\|_n)\,.$$

La formule de Taylor avec reste intégral appliquée à φ donne, U étant convexe

$$\varphi(x) = \varphi(0) + \int_0^1 d\varphi(tx)x\,dt = \int_0^1 d\varphi(tx)x\,dt\,.$$

Si $x \in U$,

$$\|\varphi(x)\|_n \leq \int_0^1 |d\varphi(tx)x\|_n\,dt \leq C'_n\,(\|x\|_n\,\|x\|_0 + \|x\|_n)\,.$$

Comme $\|x\|_0 \leq a \leq 1$, on a, avec une nouvelle constante $C_n = 2C'_n$,

$$\|\varphi(x)\|_n \leq C_n \|x\|_n\,, \qquad x \in U\,.$$

Dans toute la suite, c'est cette dernière estimation qu'on utilisera.

Lemme 3.1. *Si $\|y\|_1 \leq a$ et $\|x_j\|_1 \leq a$ pour $j = 1, \ldots, q-1$, pour tout entier n, il existe une constante K_n indépendante de q, telle que*

$$\|x_q\|_n \leq K_n\,t_q^{4p+1}\,\|y\|_n\,.$$

PREUVE. Commençons par remarquer que l'élément $z_j = y - \varphi(x_j)$ vérifie

$$\|z_j\|_n \leq \|y\|_n + \|\varphi(x_j)\|_n \leq \|y\|_n + C_n\,\|x_j\|_n\,.$$

Sans diminuer la généralité, on va supposer que, pour n entier donné, les constantes C_n intervenant dans les majorations du théorème précèdent sont égales à une même constante, notée encore C_n, supérieure ou égale à 1. Ainsi,

$$\|z_j\|_n \leq C_n\,(\|y\|_n + \|x_j\|_n)\,.$$

En particulier $\|z_j\|_1 \leq 2\,C_1$, dès que $\|x_j\|_1 \leq a$. Grâce aux estimations sur les opérateurs S_t, nous avons, si $n \geq 2p$,

$$\begin{aligned}\|\Delta x_j\|_n &= \|S_{t_j} L(x_j)z_j\|_n\, C_{n,n-2p}\, t_j^{2p}\, \|L(x_j)z_j\|_{n-p}\\ &\leq C_{n,n-2p}\, C_{n-p}\, t_j^{2p}\,(\|x_j\|_n\,\|z_j\|_0 + \|z_j\|_n)\\ &\leq 2\,C_1\, C_{n,n-2p}\, C_{n-p}\, t_j^{2p}\,\big(\|x_j\|_n + \|z_j\|_n\big)\,.\end{aligned}$$

D'autre part $\|x_j\|_n + \|z_j\|_n \leq 2C_n(\|x_j\|_n + \|y\|_n)$ donc

$$\|\Delta x_j\|_n \leq 4\,C_1\, C_n\, C_{n-p}\, C_{n,n-2p}\, t_j^{2p}\,\big(\|x_j\|_n + \|y\|_n\big)\,.$$

On va poser pour simplifier $A_n = 4C_1C_nC_{n-p}C_{n,n-2p}$. Revenons au lemme. Soit j, $1 \leq j \leq q-1$, puisque $x_j = x_{j-1} + \Delta x_j$, on a

$$\begin{aligned}\|x_j\|_n + \|y\|_n &\leq \|x_{j-1}\|_n + \|y\|_n + \|\Delta x_{j-1}\|_n\\ &\leq A_n\, t_{j-1}^{2p}\,(\|x_{j-1}\|_n + \|y\|_n) + (\|x_{j-1}\|_n + \|y\|_n)\\ &\leq 2A_n\, t_{j-1}^{2p}\,(\|x_{j-1}\|_n + \|y\|_n)\,.\end{aligned}$$

Car A_n et t_{j-1}^{2p} sont supérieurs à 1. En itérant, il vient

$$\begin{aligned}\|\Delta x_j\|_n &\leq (2A_n)^2\, t_j^{2p}\, t_{j-1}^{2p}\,(\|x_{j-1}\|_n + \|y\|_n)\\ &\ \ \vdots\\ &\leq (2A_n)^q\,(t_jt_{j-1}\cdots t_0)^{2p}\,\|y\|_n\,, \qquad (x_0 = 0)\,.\end{aligned}$$

Remarquons que si $j \leq q-1$

$$(t_jt_{j-1}\dots t_0)^{2p} \leq 2^{2p(\tau^q-1)/(\tau-1}} \leq 2^{4p\tau^q}$$

et que

$$x_q = (x_q - x_{q-1}) + \cdots + (x_1 - x_0) = \sum_{j=0}^{q-1} \Delta x_j$$

il vient alors

$$\|x_q\|_n \leq \sum_{j=0}^{q-1} \|\Delta x_j\|_n \leq q\,(2A_n)^q\, 2^{4p\tau^q}\, \|y\|_n \;.$$

On veut trouver K_n, indépendant de q tel que

$$q\,(2A_n)^q\, 2^{4p\tau^q}\, \|y\|_n \leq K_n\, t_q^{4p+1}\, \|y\|_n$$

pour cela on écrit $t_q^{4p+1} = 2^{(4p+1)\tau^q}$ et il suffit alors que

$$q\,(2A_n)^q \leq K_n\, 2^{\tau^q} \;.$$

Ce qui résulte facilement d'un passage à la limite après division par 2^{τ^q}.

Afin d'alléger quelque peu les calculs qui vont suivre on introduit

$$C' = C^4\, C_0\, C_{2p}\, C_{23p}\, C_{24p}\, C_{2p,22p}\, K_{24p} \;,$$
$$C'' = 2^2\, C_{2p}\, C_p^2\, C_{2p,0}^2 \;, \qquad C = C' + C'' \;, \qquad M = C + 2^{10p}$$

et on fixe δ, $0 < \delta < (a^{-1}M^3)^{-1}$.

Proposition 3.2. *Si $\|y\|_{24p} < \delta$, tous les x_q vérifient $\|x_q\|_{2p} \leq a$.*

PREUVE. On va prouver par récurrence sur q et sous l'hypothèse $\|y\|_{24p} < \delta$ les implications $(R_{q-1})_{q\geq 1}$ suivantes

$$(R_{q-1}) \qquad \|x_j\|_{2p} \leq a,\ 0 \leq j \leq q-1 \Rightarrow \|z_q\|_{2p} \leq M\, t_q^{-10p}\, \|y\|_{24p} \;.$$

Admettons pour un instant que $(R_{q'})$ est vraie pour $q' \geq 0$, on va prouver la proposition, par récurrence sur q. Si $q = 0$, $x_0 = 0$ et il n'y a rien à démontrer. Supposons que $\|x_j\|_{2p} \leq a$, $0 \leq j \leq q$. Cela veut dire en particulier que $R_{q'-1}$ est vraie pour tout $q' \leq q+1$. Pour j, $0 \leq j \leq q$, on a

$$\begin{aligned} \|\Delta x_j\|_{2p} &\leq \|S_{t_j}\, L(x_j) z_j\|_{2p} \\ &\leq C_{2p,0}\, t_j^{2p}\, \|L(x_j) z_j\|_p \\ &\leq C_{2p,0}\, C_p\, t_j^{2p}\, \left(\|x_j\|_{2p}\, \|z_j\|_0 + \|z_j\|_{2p}\right) \\ &\leq 2\, C_{2p,0}\, C_p\, t_j^{2p}\, \|z_j\|_{2p} \;. \end{aligned}$$

Comme R_{j-1} est vraie $\|z_j\|_{2p} \leq M\, t_j^{-10p}\, \|y\|_{24p}$, par suite

$$\begin{aligned}\|x_{q+1}\|_{2p} &\leq \sum_{j=0}^{q} \|\Delta x_j\|_{2p} \\ &\leq 2\, C_p\, C_{2p,0}\, M\, \|y\|_{24p} \sum_{j=0}^{q} t_j^{-8p} \\ &\leq 2\, C_p\, C_{2p,0}\, M\, \|y\|_{24p}\ ,\end{aligned}$$

car

$$\sum_{j=0}^{q} t_j^{-8p} \leq \sum_{j=0}^{+\infty} t_j^{-8p} = \sum_{j=0}^{+\infty} 2^{-8p\tau^j} \leq \sum_{j=0}^{\infty} 2^{-8p(1+j/2)} < 1\,.$$

Comme $\delta < (a^{-1}M^3)^{-1}$

$$\|x_{q+1}\|_{2p} \leq C''\, M\, \|y\|_{24p} \leq a\,.$$

La proposition sera donc démontrée si l'on prouve les implications $(R_{q-1})_{q\geq 1}$. Pour ce faire, en notant $\varphi'(x) = d\varphi(x)$, on établit d'abord une identité. On a, par définition de la suite $(x_j)_j$,

$$\varphi(x_{j+1}) = \varphi(x_j) + \varphi'(x_j)\, \Delta x_j + R(x_j)\, \Delta x_j$$

et donc

$$\begin{aligned}z_{j+1} = y - \varphi(x_{j+1}) &= z_j - \varphi'(x_j)\, \Delta x_j - R(x_j)\, \Delta x_j \\ &= z_j - \varphi'(x_j)\, S_{t_j}\, L(x_j) z_j - R(x_j)\, \Delta x_j \\ &= \varphi'(x_j) L(x_j) z_j - \varphi'(x_j)\, S_{t_j}\, L(x_j) z_j - R(x_j)\, \Delta x_j \\ &= \varphi'(x_j)(I - S_{t_j}) L(x_j) z_j - R(x_j)\, \Delta x_j\ .\end{aligned}$$

Cette identité, pour $j = 0$ donne,

$$\|z_1\|_{2p} \leq C_{2p} \|(I - S_{t_0}) L(x_0) z_0\|_{2p} + \|R(x_0) \Delta x_0\|_{2p}\ .$$

On majore successivement chacun des facteurs dans le second membre

$$\begin{aligned}C_{2p}\, \|(I - S_{t_0}) L(x_0) z_0\|_{2p} &\leq C_{2p}\, C_{2p,22p}\, t_0^{-20p}\, \|L(x_0) y\|_{23p} \\ &\leq C_{2p}\, C_{2p,22p}\, C_{23p}\, t_0^{-20p} \\ &\qquad \cdot \big(\|x_0\|_{23p}\, \|y\|_0 + \|y\|_{24p}\big)\end{aligned}$$

$$\leq C_{2p}\, C_{2p,22p}\, C_{23p}\, t_0^{-20p}\, \|y\|_{24p}\ , \quad (x_0 = 0)\,.$$

Le second facteur se majore comme suit

$$\begin{aligned}\|R(x_0)\Delta x_0\|_{2p} \leq C_{2p}\, \|\Delta x_0\|_{2p}^2 &\leq C_{2p}\, \|S_{t_0} L(x_0) y\|_{2p}^2 \\ &\leq C_{2p}\, (C_{2p,0}\, t_0^{2p}\, \|L(x_0)y\|_p)^2 \\ &\leq C_{2p}\, C_{2p,0}^2\, t_0^{4p}\, \|L(x_0)y\|_p^2 \\ &\leq C_{2p}\, C_{2p,0}\, C_p^2\, t_0^{4p}\, \|y\|_{2p}^2\ .\end{aligned}$$

En remarquant que $t_0^{10p} = 2^{10p} < M$ et $\|y\|_{2p} \leq \|y\|_{24p}$ il vient que

$$C_{2p}\, \|\Delta x_0\|_{2p}^2 \leq C_{2p}\, C_{2p,0}^2\, C_p^2\, M^2\, t_0^{-16p}\, \|y\|_{24p}^2\ .$$

En ajoutant les deux majorations obtenues, on a

$$\begin{aligned}\|z_1\|_{2p} \leq (&C_{2p}\, C_{2p,22p}\, C_{23p}\, t_0^{-20p} \\ &+ C_{2p}\, C_{2p,0}^2\, C_p^2\, M^2\, t_0^{-16p}\, \|y\|_{24p})\, \|y\|_{24p}\ .\end{aligned}$$

Or

$$\begin{aligned}C_{2p}\, C_{2p,22p}\, C_{23p}\, t_0^{-20p} &\leq C'\, t_0^{-20p} \\ &= C'\, t_1^{-20p\cdot(2/3)} = C'\, t_1^{-40p/3} \leq C'\, t_1^{-10p}\end{aligned}$$

et

$$\begin{aligned}C_{2p}\, C_{2p,0}^2\, C_p^2\, M^2\, t_0^{-16p} &\leq C''\, M^2\, t_0^{-16p} \\ &= C''\, M^2\, t_1^{-16p\cdot(2/3)} \leq C''\, M^2\, t_1^{-10p}\end{aligned}$$

donc

$$\begin{aligned}\|z_1\|_{2p} &\leq (C' + C''\, M^2\, \|y\|_{24p})\, t_1^{-10p}\, \|y\|_{24p} \\ &\leq C\,(1 + M^2\, \|y\|_{24})\, t_1^{-10p}\, \|y\|_{24p}\ .\end{aligned}$$

et d'après le choix des constantes C, M et δ

$$C\,(1 + M^2\, \|y\|_{24p}) \leq C\,(1 + M^2\delta) \leq M\,,$$

il vient finalement

$$\|z_1\|_{2p} \leq M\, t_1^{-10p}\, \|y\|_{24p}\ .$$

A présent, montrons que R_{q-1} implique R_q. Supposons que $\|x_j\|_{2p} \leq a$ pour j, $0 \leq j \leq q$, il s'agit de montrer que $\|x_{q+1}\|_2 p \leq M\, t_{q+1}^{-10p}\, \|y\|_{24p}$. On procède exactement comme pour z_1, en partant de

$$z_{q+1} = \varphi'(x_q)(I - S_{t_q})L(x_q)z_q - R(x_q)\Delta x_q$$

$$\begin{aligned}
\|\varphi'(x_q)(I - S_{t_q})L(x_q)z\|_{2p} &\leq C_{2p}\,\big(\|(I - S_{t_q})L(x_q)z_q\|_0\, \|x_q\|_{2p} \\
&\qquad + \|(I - S_{t_q})L(x_q)z_q\|_{2p}\big) \\
&\leq 2\, C_{2p}\, \|(I - S_{t_q})L(x_q)z_q\|_{2p} \\
&\leq 2\, C_{2p}\, C_{2p,22p}\, t_q^{-20p}\, \|L(x_q)z_q\|_{23p} \\
&\leq 2\, C_{2p}\, C_{2p,22p}\, C_{23p}\, t_q^{-20p} \\
&\quad \cdot \big(\|x_q\|_{24p}\, \|z_q\|_0 + \|z_q\|_{24p}\big)\,,
\end{aligned}$$

($\|x_q\|_{2p} \leq a \leq 1$), on a vu que

$$\|z_q\|_0 \leq \|y\|_0 + \|\varphi(x_q)\|_0 \leq \|y\|_0 + C_0\, \|x_q\|_0 \leq 2\, C_0$$

($\|y\|_0 \leq \delta \leq 1 \leq C_0$) par suite

$$\begin{aligned}
\|x_q\|_{24p}\, \|z_q\|_0 + \|z_q\|_{24p} &\leq 2\, C_0\big(\|x_q\|_{24p} + \|z_q\|_{24p}\big) \\
&\leq 2\, C_0\big(\|x_q\|_{24p} + \|y\|_{24p} + C_{24p}\, \|x\|_{24p}\big) \\
&\leq 2^2\, C_0\, C_{24p}\, \big(\|x_q\|_{24p} + \|y\|_{24p}\big) \\
&\leq 2^2\, C_0\, C_{24p}\, \big(K_{24p}\, t_q^{4p+1} + 1\big)\, \|y\|_{24p} \\
&\leq 2^3\, C_0\, C_{24p}\, K_{24p}\, t_q^{4p+1}\, \|y\|_{24p}\ ,
\end{aligned}$$

(Lemme 1). Si bien que

$$\begin{aligned}
\|\varphi'(x_q)(I - S_{t_q})L(x_q)z_q\|_{2p} &\leq 2^4\, C_0\, C_{2p}\, C_{23p}\, C_{24p} \\
&\quad \cdot C_{2p,22p}\, K_{24p}\, t_q^{-16p+1}\, \|y\|_{24p} \\
&\leq C'\, t_q^{-16p+1}\, \|y\|_{24p} \leq C'\, t_{q+1}^{-10p}\, \|y\|_{24p}\ .
\end{aligned}$$

Reste à majorer le terme $\|R(x_q)\Delta\ x_q\|_{2p}$,

$$\begin{aligned}
\|R(x_q)\Delta x_q\|_{2p} &\leq C_{2p}\,(\|x_q\|_{2p}\, \|\Delta x_q\|_0^2 + \|\Delta x_q\|_0\, \|\Delta x_q\|_{2p}) \\
&\leq 2\, C_{2p}\, \|\Delta x_q\|_{2p}^2 \\
&\leq 2\, C_{2p}\, \|S_{t_q} L(x_q)z_q\|_{2p}^2
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\,C_{2p}\left(C_{2p,0}\,t_q^{2p}\,\|L(x_q)z_q\|_p\right)^2\\
&\leq 2\,C_{2p}\,C_{2p,0}^2\,C_p^2\,t_q^{4p}\left(\|x_q\|_{2p}\,\|z_q\|_0+\|z_q\|_{2p}\right)\\
&\leq 2\,C_{2p}\,C_{2p,0}^2\,C_p^2\,2^2t_q^{4p}\,\|z_q\|_{2p}^2\\
&\leq 2\,C''\,t_q^{4p}\,\|z_q\|_{2p}^2\,,
\end{aligned}
$$

($\|x_q\|_{2p} \leq a \leq 1$ et $\|\Delta x_q\|_0 \leq \|\Delta x_q\|_{2p}$). Comme R_{q-1} est vraie $\|z_q\|_{2p} \leq M\,t_q^{-10p}\,\|y\|_{24p}$, par conséquent

$$
\|R(x_q)\Delta x_q\|_{2p} \leq 2\,C''\,M^2\,t_q^{-16p}\,\|y\|_{24p}^2 \leq 2\,C''\,M^2\,t_{q+1}^{-10p}\,\|y\|_{24p}^2\,.
$$

En définitive,

$$
\begin{aligned}
\|z_{q+1}\|_{2p} &\leq (C'+2\,C''M^2\,\|y\|_{24p})\,t_{q+1}^{-10p}\|_{24p}\\
&\leq C\,(1+2\,M^2\delta)\,t_{q+1}^{-10p}\,\|y\|_{24p}\\
&\leq M\,t_{q+1}^{-10p}\,\|y\|_{24p}\,.
\end{aligned}
$$

Ceci achève la preuve de la proposition.

Lemme 3.3. *Pour tout entier k, il existe une constante $C = C(k)$, un entier $n = n(k)$ tels que si $\|y\| < \delta$*

$$
\|z_q\|_{2p} \leq C\,t_q^{-k}\,\|y\|_{n(k)}\,, \qquad q \in \mathbb{N}\,.
$$

PREUVE. Si $k \leq 10p$, la proposition précédente affirme que $\|z_q\|_{2p} \leq M\,t_q^{-10p}\,\|y\|_{24p}$, on prend alors $C = M$ et $n(k) = 24p$. Supposons le lemme vrai pour un entier $k > 10\,p$ et montrons le pour $k+1$, posons pour simplifier $\alpha = 2(k+1)+6p+1$. On sait que pour $q \geq 0$.

$$
\|z_{q+1}\|_{2p} \leq C_{2p}\left(\|(I-S_{t_q})L(x_q)z_q\|_{2p}+\|R(x_q)\Delta x_q\|_{2p}\right).
$$

Mais

$$
\begin{aligned}
\|(I-S_{t_q})L(x_q)z_q\|_{2p} &\leq C_{2p,\alpha}\,t_q^{2p-\alpha}\,\|L(x_q)z_q\|_{\alpha+p}\\
&\leq C_{2p,\alpha}\,C_{\alpha+p},2\,C_0\,t_q^{2p-\alpha}\left(\|x_q\|_{\alpha+2p}+\|z_q\|_{\alpha+2p}\right),
\end{aligned}
$$

où, dans la dernière inégalité, on utilise $\|z_q\|_0 \leq 2C_0$. Comme

$$
\|z_q\|_{\alpha+2p} \leq C_{\alpha+2p}\left(\|y\|_{\alpha+2p}+\|x_q\|_{\alpha+2p}\right),
$$

on en déduit, grâce au Lemme 1 que

$$\begin{aligned}\|(I - S_{t_q})L(x_q)z_q\|_{2p} &\leq C(\alpha,p)\, t_q^{2p-\alpha} \left(\|x_q\|_{\alpha+2p} + \|y\|_{\alpha+2}\right)\\ &\leq C(\alpha,p)\, t_q^{4p+1+2p-\alpha}\, \|y\|_{\alpha+2p}\ ,\end{aligned}$$

avec une constante $C(\alpha,p)$, dépendant de α et de p, non de q. On remarque à présent, par définition de α, que $t_q^{6p+1-\alpha} \leq t_{q+1}^{-(k+1)}$. Ceci donne

$$C_{2p}\|(I - S_{t_q})L(x_q)z_q\|_{2q} \leq C_1(k)\, t_{q+1}^{-(k+1)}\, \|y\|_{2(k+1)+8p+1}\,.$$

Quand au terme $\|R(x_q)\Delta x_q\|_{2p}$, il se majore comme suit

$$\|R(x_q)\Delta x_q\|_{2p} \leq 2\, C_{2p}\, \|\Delta x_q\|_{2p}^2 \leq 2\, C'' t_q^{4p}\, \|z_q\|_{2p}^2\,.$$

Comme le lemme a été supposé vrai à l'ordre k, on a

$$\|z_q\|_{2p} \leq C(k)\, t_q^{-k}\, \|y\|_{n(k)}\ ,$$

c'est-à-dire,

$$\|R(x_q)\Delta x_q\|_{2p} \leq 2\, C'' C^2(k)\, t_q^{4p-2k}\, \|y\|_{n(k)}^2\,.$$

Le Lemme 2.9 entraine, par interpolation entre 0 et $2n(k)$,

$$\|y\|_{n(k)}^2 \leq C'(k)\, \|y\|_{2n(k)}\ ,$$

il vient finalement, avec une nouvelle constante $C_2(k)$,

$$\|R(x_q)\Delta x_q\|_{2p} \leq C_2(k)\, t_q^{4p-2k}\, \|y\|_{2n(k)}\,.$$

L'hypothèse $0 < 10p < k$ montre que $t_q^{4p-2k} \leq t_{q+1}^{-(k+1)}$. Donc

$$\|z_{q+1}\|_{2p} \leq C(k)\, t_{q+1}^{-k+1)}\, \|y\|_{n(k+1)}\ ,$$

où $C(k) = \max\{C_1(k), C_2(k)\}$ et $n(k+1) = \max\{2n(k), 2(k+1) + 8p + 1\}$.

Lemme 3.4. *Pour tout entier $n \geq 0$, pour tout entier $k \geq 0$, il existe une constante $C = C(n,k)$ et un entier $\sigma = \sigma(n,k)$ tels que si $\|y\|_{24p} \leq \delta$,*

$$\|\Delta x_q\|_n \leq C\, \|y\|_\sigma\, t_q^{-k} \qquad \textit{et} \qquad \|z_q\|_n \leq C\, \|y\|_\sigma\, t_q^{-k}\,.$$

PREUVE. Fixons deux entiers k et n, par le Lemme 2.9,

$$\|\Delta x_q\|_n \leq C\,\|\Delta x_q\|_0^{1/2}\,\|\Delta x_q\|_{2n}^{1/2}\,,$$

$$\begin{aligned}\|\Delta x_q\|_0 &\leq \|S_{t_q}L(x_q)z_q\|_0\\ &\leq C_{0,0}\,\|L(x_q)z_q\|_p\\ &\leq C_{0,0}\left(\|x_q\|_{2p}\,\|z_q\|_0+\|z_q\|_{2p}\right).\end{aligned}$$

La Proposition 3.2 implique que $\|x_q\|_{2p} \leq a \leq 1$ et alors

$$\|\Delta x_q\|_0 \leq 2\,C_{0,0}\,\|z_q\|_{2p}\,.$$

On applique le Lemme 3.3 avec $k' = 2k + 6p + 2$, il vient

$$\|\Delta x_q\|_0 \leq 2\,C_{0,0}\,C(k')\,t_q^{-k'}\,\|y\|_{n(k')}\,.$$

D'autre part $x_{q+1} = x_q + \Delta x_q$ et $\|\Delta x_q\|_{2n} \leq \|x_{q+1}\|_{2n} + \|x_q\|_{2n'}$ par le Lemme 1,

$$\|x_{q+1}\|_{2n} \leq K_{2n}\,t_{q+1}^{4p+1}\,\|y\|_{2n}\,, \qquad \|x_q\|_{2n} \leq K_{2n}\,t_q^{4p+1}\,\|y\|_{2n}\,.$$

et on a $\|\Delta x_q\|_{2n} \leq 2\,K_{2n}\,t_{q+1}^{4p+1}\,\|y\|_{2n}$ et

$$\|\Delta x_q\|_n \leq C(n,k)\,t_q^{-k'/2}\,t_{q+1}^{(4p+1)/2}\,\|y\|_{n(k')}^{1/2}\,\|y\|_{2n}^{1/2}$$

par le choix de $k' = 2k + 6p + 2$, on a

$$t_q^{-k'/2}\,t_{q+1}^{(4p+1)/2} = t_q^{-k'/2}\,t_q^{\tau(4p+1)/2} \leq t_q^{-k}$$

et si donc $\sigma = \sigma(n,k) = \sup\{2n, n(k')\}$,

$$\|y\|_{n(k')}^{1/2}\,\|y\|_{2n}^{1/2} \leq \|y\|_{\sigma(n,k)}$$

et finalement

$$\|\Delta x_q\|_n \leq C(n,k)\,t_q^{-k}\,\|y\|_{\sigma(n,k-)}\,.$$

Les inégalités sur $\|z_q\|_n$ se prouvent de la même manière en écrivant que

$$\|z_q\|_n \leq C_n\,\|z_q\|_0^{1/2}\,\|z_q\|_{2n}^{1/2}$$

par le Lemme 1: $\|z_q\|_{2n} \leq K_{2n}\, t_q^{4p+1}\, \|y\|_{2n}$, par le Lemme 2, $\|z_q\|_0 \leq \|z_q\|_{2p} \leq C\, t_q^{-k'}\, \|y\|_{n(k')}$ et on conclut de la même façon.

Conclusion: Dans le Lemme 3.4, on prend $k \geq 1$, comme $\sigma = \sigma(n, k)$ ne dépend pas de q, la suite (x_q) est de Cauchy dans E_1, on désigne par x sa limite. Le même Lemme 3.4 montre que la suite (z_q) tend vers 0 dans E_2, lorsque q tend vers plus l'infini et on a $y = \varphi(x)$, par continuité de φ. Enfin si y est dans E_2 vérifiant $\|y\|_{24p} \leq \sigma$, tous les x_q sont dans $U \subset E_1$. L'algorithme définissant la suite (x_q) a un sens et la limite x est nécessairement dans U. L'application Ψ définie sur $V = \{y \in E_2 :\ \|y\|_{24p} \leq \delta\}$, qui à y associe x répond aux exigences du théorème.

4. Fibrés vectoriels $\mathcal{A}^\infty$. Espaces des Sections.

Dans ce paragraphe, nous précisons quelques notions et établissons quelques lemmes techniques dans la catégorie des fibrés $\mathcal{A}^\infty$. Le résultat principal est le théorème dont voici l'énoncé: Soient Ω un ouvert borné, pseudo-convexe, à bord lisse et C^∞ vérifiant la propriété (P) et E un fibré vectoriel $\mathcal{A}^\infty$ sur $\overline{\Omega}$, l'espace des sections $\mathcal{A}^\infty$ de E est un bon espace de Fréchet.

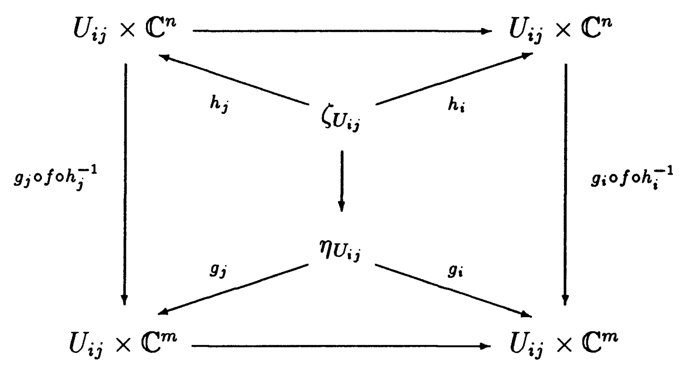
Définition 4.1. *Soit Ω un ouvert quelconque de $\mathbb{C}^p$ et $X = \overline{\Omega}$. Un fibré vectoriel topologique E, de rang n sur X est dit fibré $\mathcal{A}^\infty$ si le fibré restreint à Ω, $E|_\Omega$, est un fibré analytique tel que si $\{U_i, h_i\}$ est un atlas holomorphe de trivialisations locales, le cocycle associé*

$$g_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})\,,$$
$$(z, g_{ij}(z)v) = h_i h_j^{-1}(z, v)\,, \qquad z \in U_i \cap U_j\,, \qquad v \in \mathbb{C}^n\,,$$

est dans le groupe $GL(n, A^\infty(U_i \cap U_j))$ des matrices $n \times n$, inversibles et à coefficients dans $A^\infty(U_i \cap U_j)$.

Comme d'habitude, on note par: $E \xrightarrow{p} X$ de tels fibrés. Si ω est un ouvert de X et $u : \omega \to E$ est une section, u est dite section C^∞ (respectivement, $\mathcal{A}^\infty$) si les applications $h_i \circ u = u_i$ de $\omega \cap U_i$ dans $\mathbb{C}^n$ sont des éléments de $C^\infty(\overline{\omega \cap U_i}, \mathbb{C}^n) = C^\infty(\overline{\omega \cap U_i})^n$ (respectivement, $A^\infty(\omega \cap \Omega \cap U_i)^n$).

Soient ζ et η deux fibrés vectoriels $\mathcal{A}^\infty$ sur $\overline{\Omega}$ de rangs respectifs n et m, on peut trouver un recouvrement $\mathcal{U} = (U_i)_i$, ouvert, de $\overline{\Omega}$ et des homéomorphismes $h_i : \zeta_{U_i} = p_\zeta^{-1}(U_i) \to U_i \times \mathbb{C}^n$ et $g_i : \eta_{U_i} \to U_i \times \mathbb{C}^m$ tels que $p \circ h_i = p_\zeta$ et $p \circ g_i = p_\eta$ où $p : U \times \mathbb{C}^k \to U$, $k = m$ où $k = n$ est la première projection. Un morphisme $f : \zeta \to \eta$ est une application continue telle que $p_\zeta = p_\eta \circ f$ et pour tout i, $g_i \circ f \circ h_i^{-1} : U_i \times \mathbb{C}^n \to U_i \times \mathbb{C}^m$ soit de la forme $g_i \circ f \circ h_i^{-1}(z,t) = (z, A_i(z)t)$ où $A_i(z)$ est une matrice $m \times n$ à coefficients dans $A^\infty(U_i)$. Si U_j est une carte, $g_j \circ f \circ h_j^{-1} : U_j \times \mathbb{C}^n \to U_j \times \mathbb{C}^m$ l'application associée, par restriction à U_{ij}, on a le diagramme commutatif suivant



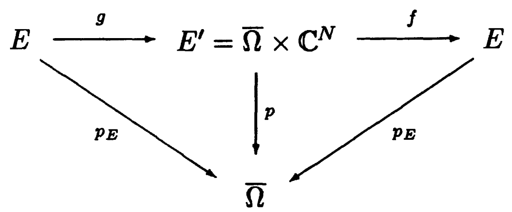
On désigne par $A^\infty(\Omega, E)$ l'espace des sections $\mathcal{A}^\infty$ d'un fibré vectoriel $\mathcal{A}^\infty$ E sur $\overline{\Omega}$ et si $f : \zeta \to \eta$ est un morphisme de fibrés $\mathcal{A}^\infty$, on note par $\Gamma(f) : A^\infty(\Omega, \zeta) \to A^\infty(\Omega, \eta)$ l'application entre les espaces des sections définie par $\Gamma(f)(s) = s \circ f,\ s \in A^\infty(\Omega, \zeta)$. On a d'après [8],

Lemme 4.2. *Soient $f, g : \zeta \to \eta$ deux morphismes tels que $\Gamma(f) = \Gamma(g)$, alors $f = g$. Et si $F : A^\infty(\Omega, \zeta) \to A^\infty(\Omega, \eta)$ est une application A^∞ linéaire, il existe un unique morphisme $f : \zeta \to \eta$ tel que $\Gamma(f) = F$.*

Et toujours d'après [8], on a le

Corollaire 4.3. *Soit E un fibré vectoriel $\mathcal{A}^\infty$ sur $\overline{\Omega}$, il existe un entier N assez grand et un épimorphisme $f : E' = \Omega \times \mathbb{C}^N \to E$. D'autre part il existe un morphisme $g : E \to E'$, $f \circ g = I_E$.*

On peut résumer cela dans le diagramme commutatif suivant



avec $f \circ g = I_E$, ceci donne une suite

$$A^\infty(\Omega, E) \overset{\Gamma(g)}{\to} A^\infty(\Omega)^N \overset{\Gamma(f)}{\to} A^\infty(\Omega, E)\,.$$

Comme $\Gamma(f \circ g) = \Gamma(f) \circ \Gamma(g)$, il vient que $\Gamma(f) \circ \Gamma(g) = I_{A^\infty(\Omega,E)}$. Disons maintenant un mot sur la topologie qu'on met sur $A^\infty(\Omega, E)$: on fixe un recouvrement fini $\mathcal{U} = (U_j)_{1\le j\le s}$ et ouvert de $\overline{\Omega}$, par des ouverts de trivialisations. Soient $\varphi_j : E|_{\overline{U}_j} \to \overline{U}_j \times \mathbb{C}^n$ les isomorphismes de trivialisation. Toute section s de E sur $\overline{\Omega}$ s'identifie à l'aide des applications φ_j à un système $(s_j)_{1\le j\le s}$ où pour tout j, s_j est un n-uple $(s_j^1, \dots, s_j^n)$ de fonctions $s_j^k \in A^\infty(U_j)$. Pour tout entier m et pour $s \in A^\infty(\Omega, E)$, on pose

$$\|s\|_m = \sup_{\substack{1\le j\le s \\ 1\le k\le n}} \|s_j^k\|_{m, A^\infty(U_j)}\,.$$

Les normes $\|\cdot\|_m$ dépendent du recouvrement $\mathcal{U}$ choisi, mais tout autre choix donne des normes équivalentes à celles-ci. Sur U_j l'application $\Gamma(g) : A^\infty(U_j, E) \to A^\infty(U_j)^N$ est de la forme

$$\Gamma(g)(s_j)(z) = A_j(z)\, s_j(z)\,, \qquad s_j \in A^\infty(U_j, E)\,,\ z \in U_j\,,$$

où $A_j(z)$ est une matrice $N \times n$, à coefficients dans $A^\infty(U_j)$. Donc

$$\begin{aligned}
\|\Gamma(g)s\|_{m,A^\infty(\Omega)^N} &\le \sup_{1\le j\le s} \|\Gamma(g)s\|_{m,A^\infty(U_j)^N} \\
&\le \sup_{1\le j\le s} \|A_j\, s_j\|_{m,A^\infty(U_i)^N} \\
&\le C_m \sup_{\substack{1\le j\le s \\ 1\le k\le n}} \|s_j^k\|_{m,A^\infty(U_j)} = C_m\, \|s\|_m\,.
\end{aligned}$$

L'application $\Gamma(g)$ est donc une bonne application linéaire continue entre les espaces de Fréchet $A^\infty(\Omega, E)$ et $A^\infty(\Omega)^N$, on procède de même pour $\Gamma(g)$, on a ainsi le

Théorème 4.4. *Si Ω est un ouvert borné, pseudo-convexe, à bord lisse C^∞, vérifiant* (P). *L'espace $A^\infty(\Omega, E)$ des sections $\mathcal{A}^\infty$ d'un fibré vectoriel $\mathcal{A}^\infty$ sur $\overline{\Omega}$ est un bon espace de Fréchet.*

On va préciser maintenant la notion de fibré A^∞ principal et prouver un "lemme de matrices holomorphes" pour certaines sections de ces fibrés, c'est le lemme fondamental dans la terminologie de Cartan [1].

Nous remplaçons l'argument d'équations différentielles de [1], non commode pour A^∞ par un argument de fonctions implicites qui utilise le Théorème 4.4 mais auparavant, nous énonçons une proposition relative aux fibrés vectoriels.

Proposition 4.5. *Soient Ω un ouvert pseudo-convexe dont le bord vérifie* (P) *et E un fibré vectoriel $\mathcal{A}^\infty$ sur $\overline{\Omega}$, on suppose que $\Omega = \Omega_1 \cup \Omega_2$, avec*

i) *Ω_1 et Ω_2 sont pseudo-convexe à bords vérifiant* (P),

ii) *$\Omega_1 \cap \Omega_2$ est à bord lisse et C^∞,*

iii) $\overline{\Omega_1 \setminus \Omega_2} \cap \overline{\Omega_2 \setminus \Omega_1} = \varnothing$.

Si $\mathcal{C}$ est un espace métrique compact, S un fermé de $\mathcal{C}$ et $\mathcal{F}$ est le faisceau des germes des sections $\mathcal{A}^\infty$ de E, on a l'exactitude de la suite

$$\mathcal{C} \to C_S(\mathcal{C}, \mathcal{F}(\Omega)) \xrightarrow{\alpha} C(\mathcal{C}, \mathcal{F}(\Omega_1)) \times C_S(\mathcal{C}, \mathcal{F}(\Omega_2)) \xrightarrow{\beta} C_S(\mathcal{C}, \mathcal{F}(\Omega_1 \cap \Omega_2)) \to 0\,,$$

où

$$\alpha(f)(t) = (f(t)|_{\Omega_1}, f(t)|_{\Omega_2})\,,$$
$$\beta(f_1, f_2)(t) = f_2(t)|_{\Omega_1 \cap \Omega_2} - f_1(t)|_{\Omega_1 \cap \Omega_2}\,,$$

pour t dans $\mathcal{C}$, f dans $C_S(\mathcal{C}, \mathcal{F}(\Omega))$, f_i dans $C_S(\mathcal{C}, \mathcal{F}(\Omega_i))$, $i = 1, 2$; $C_S(\mathcal{C}, \mathcal{F}(\cdot))$ étant l'espace des fonctions continues sur $\mathcal{C}$, à valeurs dans $\mathcal{F}(\cdot)$ et nulles sur S. De plus β est inversible à droite, d'inverse linéaire.

PREUVE. On procède essentiellement comme dans le Lemme 2.7 du Paragraphe 2. Puisque $\mathcal{F}$ est $\mathcal{A}^\infty$-cohérent ([8, Définition 2.1]), il résulte de la suite de Mayer-Vietoris et du Théorème B ([8]) l'exactitude de la suite

$$0 \to \mathcal{F}(\Omega_1 \cup \Omega_2) \xrightarrow{\alpha'} \mathcal{F}(\Omega_1) \times \mathcal{F}(\Omega_2) \xrightarrow{\beta'} \mathcal{F}(\Omega_1 \cap \Omega_2) \to 0$$

pour montrer la proposition, il suffit de montrer que β' est inversible à droite. On sait que la propriété (P) de $\Omega_1 \cup \Omega_2$ fait que l'application β'' dans la suite exacte

$$0 \to A^\infty(\Omega_1 \cup \Omega_2) \to A^\infty(\Omega_1) \times A^\infty(\Omega_2) \xrightarrow{\beta''} A^\infty(\Omega_1 \cap \Omega_2) \to 0$$

est inversible, d'inverse linéaire et continu (Lemme 2.7). Notons par $b'' = (b_1'', b_2'') : A^\infty(\Omega_1 \cap \Omega_2) \to A^\infty(\Omega_1) \times A^\infty(\Omega_2)$ cet inverse. Grâce au Théorème A ([8]), il existe $(G_\alpha)_{1 \leq \alpha \leq N}$: N sections de $\mathcal{F}$ sur $(\Omega_1 \cup \Omega_2)$ tel que pour tout f dans $\mathcal{F}(\Omega_1 \cap \Omega_2)$ s'écrit $f = \Sigma f_\alpha G_\alpha$, $f_\alpha \in A^\infty(\Omega_1 \cap \Omega_2)$ et avec les notations du Paragraphe 2, on a la commutativité du diagramme suivant,

$$\begin{array}{ccc}
A^{\infty N}(\Omega_1) \times A^{\infty N}(\Omega_2) & \xrightarrow{\beta'} & A^{\infty N}(\Omega \cap \Omega_2) \\
{\scriptstyle (\Gamma_1(f),\Gamma_2(f))} \big\downarrow & & \big\downarrow {\scriptstyle \Gamma(f)} \\
\mathcal{F}(\Omega_1) \times \mathcal{F}(\Omega_2) & \xrightarrow{\beta'} & \mathcal{F}(\Omega_1 \cap \Omega_2)
\end{array}$$

où $\Gamma_i(f) = \Gamma_{\Omega_i}(f)$, $i = 1, 2$ et $\Gamma(f) = \Gamma_{\Omega_1 \cap \Omega_2}(f)$. Si $\Gamma(g) : \mathcal{F}(\Omega_1 \cap \Omega_2) \to A^{\infty N}(\Omega_1 \cap \Omega_2)$ est l'application $\Gamma(g) = \Gamma_{\Omega_1 \cap \Omega_2}(g)$ vérifiant $\Gamma(f) \circ \Gamma(g) = I_{\mathcal{F}(\Omega_1 \cap \Omega_2)}$, on définit un inverse $b' : \mathcal{F}(\Omega_1 \cap \Omega_2) \to \mathcal{F}(\Omega_1) \times \mathcal{F}(\Omega_2)$ de β' en posant

$$\begin{aligned} b' &= (\Gamma_1(f), \Gamma_2(f)) \circ b'' \circ \Gamma(g) \\ &= (\Gamma_1(f) \circ b_1'' \circ \Gamma(g),\ \Gamma_2(f) \circ b_2'' \circ \Gamma(g)). \end{aligned}$$

La proposition en résulte.

Définition 4.6. *Soient G un groupe de Lie complexe, $\mathcal{G}$ son algèbre de Lie, U_0 un voisinage de 0 dans $\mathcal{G}$ telle que l'application exponentielle* *soit un biholomorphisme de U_0 sur un voisinage U_e de l'élément neutre e de G et V un autre voisinage de e, $V_e^2 \subset V_e \subset U_e$, on considère une suite (g_j) d'éléments de G, dense dans G et $\{V_j = g_j V_e, \varphi_j\}$ l'atlas de G correspondant à (g_j) et V_e. Une application f continue d'un ouvert U de $\mathbb{C}^p$ dans G est dans $A^\infty(U,G)$ si et seulement si, pour tout j, $\varphi_j \circ f$ est dans $A^\infty(U_{j,f})^m$, où $U_{j,f} = f^{-1}(V_j)$ et $m = \dim G$.*

REMARQUE 4.7. On peut munir $A^\infty(U,G)$ d'une distance invariante par translation pour laquelle le groupe topologique $A^\infty(U,G)$ devient métrisable et complet. Si $(L_\ell)_\ell$ est une suite exhaustive de compacts de $\overline{U}$ et $\log = (\exp)^{-1} : U_e \to U_0$, pour f, g deux éléments de $A^\infty(U,G)$ on pose

$$d(f,g) = \sum_{\ell,j,M} \frac{1}{2^{j+\ell+M}} \frac{\alpha_{j,\ell,M}}{1+\alpha_{j,\ell,M}} ,$$

$$\alpha_{j,\ell,M} = \sup_{z \in U_{j,f} \cap L_\ell} \ \sup_{|\alpha| \le M} |D^\alpha \log(g_j^{-1} f(z)\, g(z)^{-1} g_j)| .$$

Si maintenant $\mathcal{C}$ est un espace métrique compact, on munit l'espace $C(\mathcal{C}, A^\infty(U,G))$ des applications continues de $\mathcal{C}$ dans $A^\infty(U,G)$ de la topologie définie par la distance $\delta(f,g) = \sup_{t\in\mathcal{C}} d(f(t), g(t))$, $f, g \in C(\mathcal{C}, A^\infty(U,G))$. Les groupes topologiques $A^\infty(U,G)$ et $C(\mathcal{C}, A^\infty(U,G))$ est complets puisqu'ils admettent des voisinages des éléments neutres respectifs qui sont complets.

Définition 4.8. *Soit Ω un ouvert pseudo-convexe, à bord lisse et C^∞ (non nécessairement borné). La donnée d'un fibré $E \to X = \overline{\Omega}$, $\mathcal{A}^\infty$ et à fibre caractéristique un groupe de Lie complexe G est la donnée d'un recouvrement $\mathcal{U} = (U_i)_{i\in I}$ de X et pour tout i et tout j de $f_{ij} : (U_i \cap U_j) \times G \to G$, $f_{ij}(z,\cdot)$ est analytique dans G pour z dans $U_{ij} = U_i \cap U_j$ et pour tout g fixé dans G, $f_{ij}(\cdot, g)$ est dans $A^\infty(U_{ij}, G)$ satisfaisant en outre à*

i) $f_{ij}(z, f_{jk}(z,y)) = f_{ik}(z,y), \quad z \in U_i \cap U_j \cap U_k, \ y \in G,$

ii) *pour tout z dans U_{ij}, $f_{ij}(z,\cdot)$ est un automorphisme de G tels que E soit quotient de $\cup_i(U_i \times G)$ par la relation d'équivalence $\mathcal{R}$: si x est dans U_{ij}, on définie le point (x,y) de $U_j \times G$ au point $(x, f_{ij}(x,y))$ de $U_i \times G$.*

A un fibré $\mathcal{A}^\infty$ E, à fibre un groupe de Lie complexe G, on peut associer un fibré $\mathcal{A}^\infty$ vectoriel noté $Ad\,E$, dont les fibres sont les algèbres de Lie des fibres de E ([1, Section 7]). L'application exponentielle définie sur l'algèbre de Lie $\mathcal{G}$ de G, à valeurs dans G induit une application exp de $Ad\,E$ dans E. Il existe un voisinage $\Theta_{Ad\,E}$ de la section nulle de $Ad\,E$ et un voisinage Θ_E de la section neutre de E tels que exp soit un isomorphisme au sens des fibrés $\mathcal{A}^\infty$ de $\Theta_{Ad\,E}$ sur Θ_E. On note encore log, l'inverse de exp, qui est définie sur Θ_E.

Si ω est un ouvert de X, $\mathcal{A}^{\infty E}(\omega)$ (respectivement $\mathcal{A}^{\infty Ad\,E}(\omega)$) est l'espace des sections $\mathcal{A}^\infty$ de E (respectivement, $Ad\,E$) sur ω. De même si l'on remplace le faisceau $\mathcal{A}^\infty$ par C^∞. Si $\mathcal{C}$ est un métrique compact et S un fermé de $\mathcal{C}$, $A_{\mathcal{C}S}^{\infty E}(\omega)$ (respectivement, $\mathcal{A}_{\mathcal{C}S}^{\infty Ad\,E}(\omega)$) est l'espace des applications continues de $\mathcal{C}$ dans $\mathcal{A}^{\infty E}(\omega)$ (respectivement, $\mathcal{A}^{\infty Ad\,E}(\omega)$) neutres (respectivement, nulles) sur le fermé S.

Dorénavant, on prend $G = GL(n, \mathbb{C})$, $\mathcal{G} = M(n, \mathbb{C})$, l'application exponentielle de $\mathcal{G}$ dans G est l'exponentielle habituelle. Nous sommes en mesure d'énoncer le lemme fondammental de décomposition.

Lemme 4.9. *Soient Ω, Ω_1 et Ω_2 comme dans la Proposition 3.1. Il existe un voisinage Σ de l'application neutre dans $\mathcal{A}_{\mathcal{C}S}^{\infty E}(\Omega_1 \cap \Omega_2)$ et des applications*

$$\sigma_j \;:\; \Sigma \to \mathcal{A}_{\mathcal{C}S}^{\infty E}(\Omega_j)\,, \qquad j = 1, 2\,,$$

tels que $f = \sigma_1(f)\,\sigma_2(f)$ sur $\Omega_1 \cap \Omega_2$, pour tout f dans Σ.

PREUVE. Pour démontrer le lemme, il suffit de montrer que l'application

$$\Phi : \mathcal{A}_{\mathcal{C}S}^{\infty Ad\,E}(\Omega_1) \times \mathcal{A}_{\mathcal{C}S}^{\infty Ad\,E}(\Omega_2) \to \mathcal{A}_{\mathcal{C}S}^{\infty Ad\,E}(\Omega_1 \cap \Omega_2)$$

$$\Phi(f_1, f_2)(t) = \log e^{f_1(t)}\, e^{f_2(t)}\,, \qquad t \in C\,,$$

admet une section au voisinage de 0 dans $\mathcal{A}_{\mathcal{C}S}^{\infty Ad\,E}(\Omega_1 \cap \Omega_2)$. Puisque les espaces $\mathcal{A}^{\infty\, Ad\,E}(\Omega)$ sont des bons espaces de Fréchet (Théorème 4.4) nous utilisons notre théorème des fonctions implicites dans les espaces de Fréchet (Paragraphe 3, Théorème). En premier lieu, nous montrons que l'application Φ ci-dessus est différentiable au sens de Gâteaux, que sa différentielle est inversible en tout point $(x_0, y_0) \in \mathcal{A}_{\mathcal{C}S}^{\infty Ad\,E}(\Omega_1) \times \mathcal{A}_{\mathcal{C}S}^{\infty Ad\,E}(\Omega_2)$, voisin de $(0, 0)$. Les estimations nécessaires pour utiliser le théorème des fonctions implicites et qui nécessitent l'utilisation de formules à la Campbell-Hausdorff seront faites en appendice. Nous rappelons quelques propriétés utiles pour les calculs qui vont suivre,

a) Si $x \in \mathcal{G}$, $ad\, e^x = e^{ad\, x}$, où $ad\, x(y) = [x, y]$, $x, y \in \mathcal{G}$ et $ad\, x\ y = x\, y\, x^{-1}$ si $x \in G$.

b) $(\exp x)^{-1} \exp' x = g(ad\, x)$, $x \in \mathcal{G}$,

$$g(z) = \frac{1 - e^{-z}}{z} = \sum_{n \geq 0} (-1)^n \frac{z^n}{(n+1)!}, \qquad (z \in \mathbb{C},\ z \neq 0)$$

et $g(0) = 1$.

c) Si

$$\Psi(z) = \frac{z \log z}{z - 1} = z \sum_{n \geq 0} (-1)^n \frac{(z-1)^n}{n+1}, \qquad |z - 1| < 1,$$

on a pour $x \in \mathcal{G}$, voisin de 0: $g(x)\, \Psi(e^x) = 1 = \Psi(e^x)\, g(x)$.

Posons $\mathcal{G}_i = \mathcal{A}_{Cs}^{\infty Ad\, E}(\Omega_i)$, $i = 1, 2$, $\mathcal{G} = \mathcal{A}_{Cs}^{\infty Ad\, E}(\Omega)$ et établissons la

1) *Différentiabilité de* $\Phi : \mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}$, $\Phi(x, y) = \log e^x e^y$. (On note $e^x = \exp x$ pour simplifier). Soit $(x, y) \in \mathcal{G}_1 \times \mathcal{G}_2$, voisin de $(0, 0)$, on fixe $(h, k) \in \mathcal{G}_1 \times \mathcal{G}_2$ et $t \in \mathbb{R}$ suffisamment petit. Posons $H(t) = \log e^{x+th}\, e^{y+tk}$. Par dérivation de la fonction $t \to e^{H(t)}$

$$\exp' H(t)\, H'(t) = \exp'(x+th)\, h\, \exp(y+tk) + \exp(x+th) \exp'(t+tk)\, k\, .$$

En multipliant à gauche par $\exp(-H(t))$, il vient par le rappel *b)*

$$\begin{aligned} g(ad\, H(t))\, H'(t) &= \exp(-H(t))\, \exp'(H(t))\, H'(t) \\ &= \exp -(y + tk)\, \exp -(x + th) \\ &\quad \cdot \exp'(x + th)\, h\, \exp(y + tk) \\ &\quad + \exp -(y + tk)\, \exp'(y + tk) k \\ &= e^{-(y+tk)}\, g(ad(x + th))\, h\, e^{y+tk} + g(ad(y + tk))\, k \\ &= e^{-ad\,(y+tk)} g(ad\,(x + th))\, h + g(ad\,(y + tk))\, k\, . \end{aligned}$$

Le rappel *c)* permet d'écrire

$$H'(t) = \Psi(e^{ad\, H(t)}) \left(e^{-ad\,(y+tk)}\, g(ad\,(x + th))\, h + g(ad\,(y + tk))\, k\right).$$

En faisant $t = 0$ il vient

$$H'(0) = \Psi(e^{ad\,H(0)})\left(e^{-ad\,y}\,g(ad\,x)\,h + g(ad\,y)\,k\right).$$

Or

$$e^{ad\,H(0)} = ad\;e^{H(0)} = ad\;e^{x}\;e^{y} = e^{ad\,x}\,e^{ad\,y} = e^{ad\Phi(x,y)}$$

et

$$\Psi(e^{ad\,H(0)}) = \Psi(e^{ad\,\Psi(x,y)}) = \frac{ad\,\Phi(x,y)\,e^{ad\,\Phi(x,y)}}{e^{ad\,\Phi(x,y)}-1} = \frac{ad\,\Phi(x,y)}{1-e^{ad\,\Phi(x,y)}}\,.$$

Ceci donne

$$\begin{aligned} H'(0) &= \frac{e^{ad\,\Phi(x,y)}ad\,\Phi(x,y)}{e^{ad\,\Phi(x,y)}-1}\,e^{-ad\,y}\,\frac{1-e^{-ad\,x}}{ad\,x}\,h \\ &\quad + ad\,\Phi(x,y)\,\frac{e^{ad\,\Phi(x,y)}}{e^{ad\,\Phi(x,y)}-1}\,\frac{1-e^{-ad\,y}}{ad\,y}\,k \\ &= \frac{ad\,\Phi(x,y)}{e^{ad\,\Phi(x,y)}-1}\,\frac{e^{ad\,x}-1}{ad\,x}\,h + \frac{ad\,\Phi(x,y)}{1-e^{-ad\,\Phi(x,y)}}\,\frac{1-e^{-ad\,y}}{ad\,y}\,k\,. \end{aligned}$$

Cela montre que

$$\frac{\partial\Phi}{\partial x}(x,y) = \frac{ad\,\Phi(x,y)}{e^{ad\Phi(x,y)}-1}\,\frac{e^{ad\,x}-1}{ad\,x}\,,$$

$$\frac{\partial\Phi}{\partial y}(x,y) = \frac{ad\,\Phi(x,y)}{1-e^{-ad\,\Phi(x,y)}}\,\frac{1-e^{-ad\,y}}{ad\,y}\,.$$

2) *Surjectivité de la différentielle en* $(x,y) \in \mathcal{G}_1 \times \mathcal{G}_2$, *voisin de* $(0,0)$. Il s'agit de résoudre en (h,k) dans $\mathcal{G}_1 \times \mathcal{G}_2$ et pour η donné dans $\mathcal{G}$ l'équation

$$\frac{ad\,\Phi(x,y)}{e^{ad\,\Phi(x,y)}-1}\,\frac{e^{ad\,x}-1}{ad\,x}\,h + \frac{ad\,\Phi(x,y)}{1-e^{-ad\,\Phi(x,y)}}\,\frac{1-e^{-ad\,y}}{ad\,y}\,k = \eta$$

posons

$$\eta' = e^{-ad\,x}\,\frac{e^{ad,\Phi(x,y)}-1}{ad,\,\Phi(x,y)}\,\eta\,,$$

c'est un élément de $\mathcal{G}$ (Appendice), il existe, par la Proposition 4.5 un couple (η_1,η_2) dans $\mathcal{G}_1 \times \mathcal{G}_2$, $\eta_1 + \eta_2 = \eta'$. On pose

$$h = \frac{ad\,x}{1-e^{-ad\,x}}\,\eta_1\,, \qquad k = \frac{ad\,y}{e^{ad\,y}-1}\,\eta_2$$

et on a

$$\begin{aligned}\frac{\partial\Phi}{\partial x}(x,y)h &= \frac{ad\,\Phi(x,y)}{e^{ad\,\Phi(x,y)}-1}\,\frac{e^{ad\,x}-1}{ad\,x}\,\frac{ad\,x}{1-e^{-ad\,x}}\,\eta_1\\ &= \frac{ad\,\Phi(x,y)}{e^{ad\,\Phi(x,y)}-1}\,e^{ad\,x}\,\eta_1\,,\end{aligned}$$

$$\begin{aligned}\frac{\partial\Phi}{\partial y}(x,y)k &= \frac{ad\,\Phi(x,y)}{1-e^{-ad\,\Phi(x,y)}}\,\frac{1-e^{-ad\,y}}{ad\,y}\,\frac{ad\,y}{e^{ad\,y}-1}\,\eta_2\\ &= \frac{ad\,\Phi(x,y)}{e^{ad\,\Phi(x,y)}-1}\,e^{ad\,x}\,\eta_2\,,\end{aligned}$$

par suite

$$\frac{\partial\Phi}{\partial x}(x,y)h+\frac{\partial\Phi}{\partial y}(x,y)k=\eta\,.$$

On posera dans la suite $L(x,y)\eta=(h,k)$.

$L(x,y)$ est ainsi un inverse à droite de $d\Phi(x,y)$, pourvu que (x,y) soit voisin de $(0,0)$ dans $\mathcal{A}_{Cs}^{\infty Ad\,E}(\Omega_1)\times\mathcal{A}_{Cs}^{\infty Ad\,E}(\Omega_2)$. On verra après les estimations de l'Appendice, que si $U_i=\{x\in\mathcal{A}_{Cs(\Omega_i)}^{\infty Ad\,E}:\ \|x\|_{1,\Omega_i}\le a\}$, $i=1,2$, avec un réel a suffisamment petit et si Φ est l'application

$$\begin{aligned}&\Phi: U_1\times U_2\to\mathcal{A}_{Cs}^{\infty Ad\,E}(\Omega_1\cap\Omega_2)\,,\\ &\Phi(f_1,f_2)(t)=\log e^{f_1(t)}\,e^{f_2(t)}\,,\end{aligned}$$

alors l'image de Φ couvre un voisinage V de l'application nulle dans $\mathcal{A}_{Cs}^{\infty Ad\,E}(\Omega_1\cap\Omega_2)$. Si a est comme ci-dessus, on a le diagramme commutatif

$$\begin{array}{ccc}\mathcal{A}_{Cs}^{\infty Ad\,E}(\Omega_1)\times\mathcal{A}_{Cs}^{\infty Ad\,E}(\Omega_2) & \longrightarrow & \mathcal{A}_{Cs}^{\infty Ad\,E}(\Omega_1\cap\Omega_2)\\ {\scriptstyle\exp}\big\downarrow & & \big\downarrow{\scriptstyle\exp}\\ U_1\times U_2 & \xrightarrow{\ \Phi\ } & V\end{array}$$

On prend $\Sigma=\exp(V)$, ceci donne le lemme fondamental.

Soit $\mathcal{C}'$ un autre espace métrique compact, $\mathcal{C} \subset \mathcal{C}'$; pour tout ouvert U de X on note par $\mathcal{F}(U)$ le groupe des applications continues de $\mathcal{C}'$ dans $C^{\infty E}(\overline{U})$ telles que $f|_{\mathcal{C}}$ soient dans $\mathcal{A}^{\infty}_{\mathcal{C}S}(U)$.

Si (U_i) est une famille d'ouverts de trivialisation de E, $U_i \cap U \neq \varnothing$, on munit chaque groupe $C^{\infty E}(\overline{U \cap U_i})$ de la topologie indiquée dans la Remarque 4.7 et $\mathcal{F}(U)$ de la topologie produit de celles des $C^{\infty E}(\overline{U \cap U_i})$. Tous ces groupes topologiques sont alors métrisables et complets.

Voici maintenant le résultat de décomposition, pour la définition de la A^{∞}-convexité, on se referera à [8].

Corollaire 4.10. *Soient K_1, K_2 deux compacts de X, $A^{\infty}(\Omega)$-convexes tels que $K_1 \cup K_2$ soit $A^{\infty}(\Omega)$-convexe. Soit U_0 un voisinage de $K_1 \cap K_2$ dans X, si f est un élément de $\mathcal{F}(U_0)$ suffisamment voisin de l'élément neutre de $\mathcal{F}(U_0)$ il existe des voisinages U_i de K_i, $i = 1, 2$, $U_1 \cap U_2 \subset U_0$ et il existe des f_i dans $\mathcal{F}(U_i)$ tels que pour tout t dans $\mathcal{C}'$, on ait sur $\overline{U_1 \cap U_2}$*

$$f_1(t)\, f_2(t)^{-1} = f(t)\,.$$

Avant de donner la démonstration du Corollaire, on rappelle une généralisation du Théorème d'extension de Tietze, due à Dugundji ([3, Théorème 4.1]).

Lemme. *Soient X un espace métrique, A un fermé de X, L un espace vectoriel topologique localement convexe et f une application continues de A dans L. Il existe une extension continue F de f, de X dans L dont l'image $F(X)$ est contenue dans l'enveloppe convexe de $f(A)$.*

PREUVE DU COROLLAIRE 4.10. On raisonne comme dans [1]. Puisque K_1, K_2 et $K_1 \cap K_2$ sont $A^{\infty}(\Omega)$-convexes on peut trouver un voisinage U de $K_1 \cup K_2$ et des voisinages U'_i de K_i, $i = 1, 2$; U, U'_1 et U'_2 sont pseudo-convexes, à bords lisses et C^{∞}, bornés vérifiant (P), $U'_1 \cup U'_2 = U$, $U'_1 \cap U'_2$ est pseudo-convexe à bord lisse et C^{∞}, $\overline{U'_1 \backslash U'_2} \cap \overline{U'_2 \backslash U'_1} = \varnothing$ et $U'_1 \cap U'_2 \subset U_0$. Si f est suffisamment voisin de l'élément neutre dans $\mathcal{F}(U_0)$, par le lemme fondamental, il existe f'_i dans $A^{\infty E}_{\mathcal{C}S}(U'_i), i = 1, 2$ tels que $f(t) = f'_1(t)\, f'_2(t)^{-1}$ pour tout t dans $\mathcal{C}$. Si dans les données du théorème des fonctions implicites, on prend a suffisamment petit, on peut définir pour tout t dans $\mathcal{C}$ les sections $\log\, f'_i(t)$ éléments de $\mathcal{A}^{\infty Ad\, E}_{\mathcal{C}S}(U'_i)$. On dispose ainsi de deux applications $\log \circ f'_i$, définies sur $\mathcal{C}$ et à valeurs dans les espaces de Fréchet $\mathcal{A}_{Cs}^{\infty Ad\,E}(U_i')$ et qui sont voisines des applications nulles dans $C(\mathcal{C}, \mathcal{A}_{Cs}^{\infty Ad\,E}(U_i'))$, $i = 1, 2$. D'après le lemme ci-dessus, elle se prolongent en deux éléments g_i de $C(\mathcal{C}', \mathcal{A}_{Cs}^{\infty Ad\,E}(U_i'))$ en prenant $X = \mathcal{C}'$, $A = \mathcal{C}$ et $L = \mathcal{A}_{Cs}^{\infty Ad\,E}(U_i')$ et de plus les applications g_i sont aussi voisines des applications nulles. En considérant $f_i'' = \exp g_i$, les applications f_i'' sont définies sur $\mathcal{C}'$, à valeurs dans $\mathcal{A}_{Cs}^{\infty Ad\,E}(U_i')$, voisines des applications neutres dans $C(\mathcal{C}', \mathcal{A}_{Cs}^{\infty Ad\,E}(U_i'))$ et vérifient

$$f(t) = f_1''(t)\, f_2''(t)^{-1}$$

pour tout t dans $\mathcal{C}$. Considérons l'élément v de $C(\mathcal{C}', C_{Cs}^{\infty E}(\overline{U_1' \cap U_2'}))$, $v(t) = f_1''(t)\, f_2''(t)^{-1}\, f(t)^{-1}$ pour t dans $\mathcal{C}'$, en diminuant si besoin est le paramètre a du théorème des fonctions implicites et en prenant f suffisamment voisine de l'élément neutre dans $\mathcal{F}(U)$, on peut alors définir $\log v(t)$ pour tout t' dans $\mathcal{C}'$. On considère maintenant des voisinages U_i de K_i, $U_i \subset\subset U_i'$, et une fonction Ψ de classe C^∞, $\Psi \equiv 1$ sur $\overline{U_1 \cap U_2}$, $\Psi \equiv 0$ au voisinage du complémentaire de $\overline{U_1' \cap U_2'}$ et on définit une application w de $\mathcal{C}'$ dans $C_{Cs}^{\infty E}(U_1)$ par $w(t)(z) = \exp(\Psi(z) \log(v(t)(z)))$ et des éléments de $\mathcal{F}(U_i)$ en posant $f_1 = w^{-1} f_1''$, $f_2 = f_2''$, alors pour tout t dans $\mathcal{C}'$, on a sur $\overline{U_1 \cap U_2}$: $f(t) = f_1(t)\, f_2(t)^{-1}$, ce qui achève la preuve du corollaire.

Appendice.

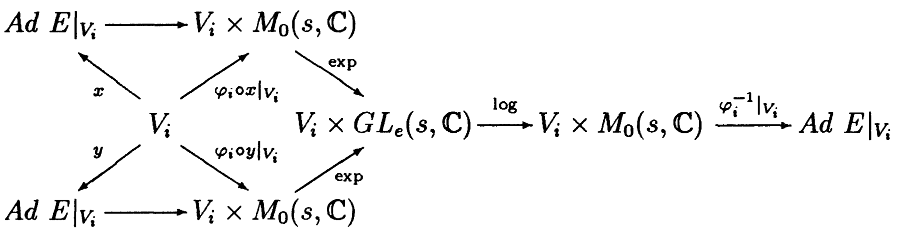
Pour terminer la preuve du lemme fondamental, il reste à voir que l'application Φ satisfait aux conditions du théorème des fonctions implicites (Paragraphe 3, Théorème).

La situation est décrite ainsi: les espaces $A^\infty(\Omega_i,\, Ad\,E)$ sont des bons espaces de Fréchet, on considère un voisinage U_i de la section nulle dans $A^\infty(\Omega_i, Ad\,E)$ donnée par: $U_i = \{x \in A^\infty(\Omega_i, Ad\,E) : \ \|x\|_{1,i} \leq a\}$ avec un réel a assez petit, afin que l'application

$$\Phi : U_1 \times U_2 \to A^\infty(\Omega_1 \cap \Omega_2, Ad\,E),$$

$$\Phi(x, y) = \log e^x e^y ,$$

soit définie et on veut prouver que l'image de $U_1 \times U_2$ par Φ contient un voisinage de la section nulle dans $A^\infty(\Omega_1 \cap \Omega_2, Ad\,E)$. On fixe une fois pour toute un recouvrement $\mathcal{U} = (U_i)$ de $\Omega_1 \cup \Omega_2$, constitué d'ouverts de trivialisation du fibré $Ad\,E$ (ou E). On a vu que la topologie de $A^\infty(\Omega, Ad\,E)$, lorsque Ω est borné, ne dépend pas du choix d'un recouvrement. Pour (x, y) élément de $U_1 \times U_2$ (a petit), on a au dessus de $V_i = \overline{U_i \cap \Omega_1 \cap \Omega_2}$ le diagramme



où exp : $V_i \times M_0(n, \mathbb{C}) \to V_i \times GL_e(n, \mathbb{C})$ est définie par $\exp(x; A) = (x, \exp A)$ et, dans le second membre, exp est l'exponentielle habituelle qui envoie un voisinage $M_0(n, \mathbb{C})$ de la matrice nulle sur un voisinage $GL_e(n, \mathbb{C})$ de la matrice identité. Au dessus de V_i, l'application Φ admet la représentation

$$\Phi(x|_{V_i}, y|_{V_i}) = \varphi_i^{-1}|_{V_i} \,(\log e^{\varphi_i|_V \circ x|_{V_i}} e^{\varphi_i|_V \circ y|_{V_i}}) .$$

Vu les normes définies dans les espaces des sections des fibrés vectoriels, on a

$$\|\Phi(x, y)\|_{n, V_i} = \|\log e^{\varphi_i \circ x} \, e^{\varphi_i \circ y}\|_{n, V_i}$$

et si l'on prouve que

$$\|\log e^{\varphi_i \circ x} \, e^{\varphi_i \circ y}\|_{n, V_i} \le C_n \, (1 + \|(\varphi_i \circ x, \varphi_i \circ y)\|_{n, V_i}) ,$$

où $\|(\zeta_1, \zeta_2)\|_n = \|\zeta_1\|_n + \|\zeta_2\|_n$, on aura

$$\|\Phi(x, y)\|_n \le C_n \, (1 + \|(x, y)\|_n) .$$

Le même raisonnement vaut pour la différentielle $d\Phi$, pour l'inverse L de $d\varphi$ et pour le reste $R(x, y)(h, k)$ dans le développement de Taylor de $\Phi(x + h, y + k)$. Le problème des estimations se ramène donc au cas des matrices, on peut même supposer que $\Omega_1 = \Omega_2$ dans les majorations sur $\Phi(x, y), d\Phi(x, y)(h, k)$ et $R(x, y)(h, k)$ (on s'occupera de l'application L un peut plus tard).

Lemme 1. *Soit s un entier, $s \geq 1$ et soit Ω un ouvert borné de $\mathbb{R}^n$, soit $\mathcal{U} = \{x \in M(s, C^\infty(\overline{\Omega})) : \ \|x\|_0 \leq \}$, l'application*

$$\Phi_0 : U \times U \to M(s, C^\infty(\overline{\Omega}))\,, \qquad \Phi_0(x,y) = \log e^x \, e^y$$

vérifie les estimations $\|\Phi_0(x,y)\|_n \leq C_n \,(1 + \|(x,y)\|_n)$, $(x,y) \in U \times U$, avec des constantes C_n indépendantes des x et des y.

PREUVE. On va utiliser les deus faits suivants.

a) Si $x \in U$, le spectre de chaque matrice scalaire $x(t), t \in \overline{\Omega}$ est contenu dans le disque $D(0, 1/2)$, de centre 0 et de rayon 1/2 et on a la représentation pour $t \in \overline{\Omega}$

$$e^{x(t)} = \frac{1}{2\pi i} \int_{C_{0,1}} (\lambda I - x(t))^{-1} \, e^\lambda \, d\lambda\,,$$

où, comme dans toute cette appendice $C_{a,r} = \{z \in \mathbb{C} : \ |z - a| = r\}$. Le terme $(\lambda I - x(t))^{-1}$ est une fonction C^∞ en t et si $\lambda \in C_{0,1}$, par le Lemme 2.10 (Paragraphe 2): $\|\lambda I - x)^{-1}\|_n \leq C_n \,(1 + \|x\|_n)$; par suite avec une nouvelle constante C_n, ne dépendant qu de n

$$\|e^x\|_n \leq C_n \,(1 + \|x\|_n)\,.$$

b) Si $x \in U$, $y \in U$, on a $\|e^x \, e^y - 1\|_0 < s^{-1}/2$ et alors $Sp(e^{x(t)} e^{y(t)}) \subset D(1, 1/2)$ pour tout t dans $\overline{\Omega}$, si $\log z$ est la détermination principale de la fonction log et si r est un réel $3/4 < r < 1$. On a

$$\log e^{x(t)} \, e^{y(t)} = \frac{1}{2\pi i} \int_{C_{1,r}} (\lambda I - e^{x(t)} e^{y(t)})^{-1} \log \lambda \ d\lambda$$

et donc $\|\log e^x e^y\|_n \leq C_n \,(1 + \|e^x e^y\|_n)$, par le Lemme 2.9 inégalité *c*) (Paragraphe 2) et grâce à *a*),

$$\begin{aligned} \|\log e^x e^y\|_n &\leq C_n \,(1 + \|e^x\|_n \, \|e^y\|_0 + \|e^x\|_0 \, \|e^y\|_n) \\ &\leq C_n \,(1 + \|x\|_n + |y\|_n) \\ &\leq C_n \,(1 + |(x,y)\|_n)\,. \end{aligned}$$

Lemme 2. *Si x, y, h, k sont des éléments de U et $R(\lambda, x) = (\lambda l - x)^{-1}$ on a*

i) $Sp(e^{x(t)+h(t)}e^{y(t)+k(t)}) \subset D(1, \frac{1}{2})$, *pour tout* $t \in \overline{\Omega}$,

ii)

$$e^{x+h} = e^x + \frac{1}{2\pi i}\int_{C_{0,1}} R(\lambda, x)\, h\, R(\lambda, x)\, e^\lambda\, d\lambda$$
$$+ \frac{1}{2\pi i}\int_{C_{0,1}} R(\lambda, x)\, h\, R(\lambda, x+h)\, h\, R(\lambda, x)\, e^\lambda\, d\lambda\,,$$

iii) *On désigne par* $D_1(x,h)$ *l'expression*

$$\frac{1}{2\pi i}\int_{C_{0,1}} R(\lambda, x)\, h\, R(\lambda, x)\, d\lambda\,.$$

Alors pour tout n*, on a avec une constante uniforme en* x, h

$$\|D_1(x,h)\|_n \leq C_n\,(\|x\|_n\,\|h\|_0 + \|h\|_n)\,.$$

PREUVE. Comme on va avoir besoin dans toute la suite des techniques de majorations nécessaires pour ce lemme, on va les faire avec quelques détails. Puisque la norme $\|\cdot\|_0$ vérifie $\|AB\|_0 \leq s\,\|A\|_0\,\|B\|_0$, $A, B \in M(s, C^\infty(\overline{\Omega}))$, on a

$$\|e^{x+h}e^{y+h} - I\|_0 \leq s\,\|e^{x+h} - I\|_0\,\|e^{y+k}\|_0 + \|e^{y+k} - I\|_0$$
$$\leq s\,(e^{s\|x+h\|_0} - 1)\, e^{s\|y+k\|_0} + e^{s\|y+k\|_0} - 1\,.$$

Comme x, y, h et k appartiennent à U

$$\|x+h\|_0 \leq \frac{\log(1+s^{-2})}{4s}\,, \qquad \|y+k\|_0 \leq \frac{\log(1+s^{-2})}{4s}\,,$$

et alors

$$s\,(e^{s\|x+h\|_0} - 1)\, e^{s\|y+k\|_0} + e^{s\|y+k\|_0} - 1 \leq s\,(e^{s(\|x+h\|_0+\|y+k\|_0)} - 1)$$
$$\leq s\,((1+s^{-2})^{1/2} - 1)$$
$$\leq \frac{s^{-1}}{2}\,,$$

d'où le point i).

Pour établir la formule ii) du lemme, on remarque ceci: Si $R(\lambda, x) = (\lambda I - x)^{-1}$, on a

$$R(\lambda, x+h) = R(\lambda, x) + R(\lambda, x)\, h\, R(\lambda, x+h)\,,$$
$$R(\lambda, x+h) = R(\lambda, x) + R(\lambda, x+h)\, h\, R(\lambda, x)\,.$$

Donc si f est analytique au voisinage de $\overline{D(a,r)}$ et si x et h sont deux éléments de U, avec $Sp(x+h) \subset\subset D(a,r)$, on a la formule

$$\begin{aligned} f(x+h) &= \frac{1}{2\pi i}\int_{C_{a,r}} (\lambda I - x - h)^{-1} f(\lambda)\, d\lambda \\ &= \frac{1}{2\pi i}\int_{C_{a,r}} R(\lambda,\ x+h)\, f(\lambda)\, d\lambda \end{aligned}$$

et donc

$$(*)\qquad \begin{aligned} f(x+h) = f(x) &+ \frac{1}{2\pi i}\int_{C_{a,r}} R(\lambda, x)\, h\, R(\lambda, x)\, f(\lambda)\, d\lambda \\ &+ \frac{1}{2\pi i}\int_{C_{a,r}} R(\lambda, x)\, h\, R(\lambda, x+h)\, h\, R(\lambda, x)\, f(\lambda)\, d\lambda\,. \end{aligned}$$

Cette dernière formule, avec $f(\lambda) = e^{\lambda}$ est la formule ii) cherchée. Prouvons maintenant le point iii) du lemme, on procède comme dans le Lemme 1

$$\|D_1(x,h)\|_n \le C_n \left((1 + \|x\|_n)\, \|h\|_0\, (1 + \|x\|_0) + (1 + \|x\|_0)^2\, \|h\|_n\right)$$

comme x est dans U, $1 + \|x\|_0 \le (1 + \|x\|_0)^2 \le C$, C est indépendant de x et alors

$$\begin{aligned} \|D_1(x,h)\|_n &\le C_n \left((1 + \|x\|_n)\, \|h\|_0 + \|h\|_n\right) \\ &\le C_n \left(\|x\|_n\, \|h\|_0 + \|h\|_n\right). \end{aligned}$$

Naturellement, les constantes C_n changent à chaque étape, mais restent indépendantes de x et h, éléments de U.

Lemme 3. *L'application $\Phi_0 : U \times U \to M(s, C^\infty(\overline{\Omega}))$ est différentiable au sens de Gâteaux et pour $(x,y) \in U \times U$ et $(h,k) \in M(s, C^\infty(\overline{\Omega}))^2$ et on a*

$$\|\frac{\partial \Phi_0}{\partial x}(x,y)(h,k)\|_n \le C_n \left(\|(x,y)\|_n\, \|h\|_0 + \|h\|_n\right),$$
$$\|\frac{\partial \Phi_0}{\partial y}(x,y)(h,k)\|_n \le C_n \left(\|(x,y)\|_n\, \|k\|_0 + \|k\|_n\right).$$

PREUVE. Les calculs de ce lemme vont donner des estimations et compléter ceux faits dans la preuve du lemme fondamental et qui permettaient de construire l'application $L(x,y)$, inverse à droite de $d\Phi(x,y)$. On conserve les notations des lemmes précédents. Posons

$$R_1(x,h) = \frac{1}{2\pi i}\int_{C_{0,1}} R(\lambda,x)\,h\,R(\lambda,x+h)\,h\,R(\lambda,x)\,e^{\lambda}\,d\lambda\,,$$

de manière que $e^{x+h} = e^x + D_1(x,h) + R_1(x,h)$, $x \in U$, $h \in U$. Si

$$D_2(y,k) = \frac{1}{2\pi i}\int_{C_{0,1}} R(\lambda,y)\,k\,R(\lambda,y)\,e^{\lambda}d\lambda\,,$$

$(y \in U, k \in U)$ et

$$R_2(y,k) = \frac{1}{2\pi i}\int_{C_{0,1}} R(\lambda,y)\,k\,R(\lambda,y+k)\,k\,R(\lambda,y)\,e^{\lambda}\,d\lambda\,,$$

$(y \in U,\ k \in U)$ on a aussi

$$e^{y+k} = e^y + D_2(y,k) + R_2(y,k)$$

et alors

$$e^{x+h}e^{y+k} = e^x e^y + D_1(x,h)\,e^y + e^x D_2(y,k) + R'((x,y))(h,k)\,,$$

avec

$$\begin{aligned} R'((x,y))(h,k) = e^x R_2(y,k) &+ R_1(x,y)\,e^y + D_1(x,h)\,D_2(y,k) \\ &+ D_1(x,h)\,R_2(y,k) + R_1(x,y)\,D_2(y,k) \\ &+ R_1(x,h)\,R_2(y,k)\,. \end{aligned}$$

On posera pour simplifier

$$T = T(x,y,h,k) = D_1(x,h)\,e^y + e^x D_2(y,k) + R'((x,y))(h,k)$$

et ainsi

$$e^{x+h}e^{y+k} = e^x e^y + T\,.$$

D'après le Lemme 2, $Sp(e^{x+h}e^{y+k}) \subset D(1,1/2)$ si x, y, h et k sont dans U, il en est de même de $Sp(e^x e^y)$ d'après le point b) du Lemme 1, par suite, avec r, $3/4 < r < 1$, et d'après la formule (*),

$$\begin{aligned}\log e^{x+h}e^{y+k} &= \log(e^x e^y + T)\\ &= \log e^x e^y + \frac{1}{2\pi i}\int_{C_{1,r}} R(\lambda, e^x e^y)\, T\, R(\lambda, e^x e^y)\, \log\lambda\, d\lambda\\ &\quad + \frac{1}{2\pi i}\int_{C_{1,r}} R(\lambda, e^x e^y)\, T\, R(\lambda, e^x e^y + T)\\ &\qquad\qquad \cdot T\, R(\lambda, e^x e^y)\, \log\lambda\, d\lambda\\ &= \log e^x e^y\\ &\quad + \frac{1}{2\pi i}\int_{C_{1,r}} R(\lambda, e^x e^y)\, D_1(x,h)\, e^y R(\lambda, e^x e^y)\, \log\lambda\, d\lambda\\ &\quad + \frac{1}{2\pi i}\int_{C_{1,r}} R(\lambda, e^x e^y)\, e^x D_1(y,k)\, R(\lambda, e^x e^y)\, \log\lambda\, d\lambda\\ &\quad + R((x,y))(h,k)\,.\end{aligned}$$

Cette dernière expression est le développement de Taylor de $\log e^{x+h}\, e^{y+k}$ avec les variables non commutatives x et y et $R((x,y))$ (h,k) vaut

$$\begin{aligned}&R((x,y))(h,k)\\ &= \frac{1}{2\pi i}\int_{C_{1,r}} R(\lambda, e^x e^y)\, R'((x,y))(h,k)\, R(\lambda, e^x e^y)\, \log\lambda\ d\lambda\\ &\quad + \frac{1}{2\pi i}\int_{C_{1,r}} R(\lambda, e^x e^y)\, T\, R(\lambda, e^x e^y + T)\, T\, R(\lambda, e^x e^y)\, \log\lambda\ d\lambda\,.\end{aligned}$$

Ces calculs donnent de nouvelles expressions des dérivées partielles de Φ_0, par exemple

$$\frac{\partial\Phi_0}{\partial x}(x,y)h = \frac{1}{2\pi i}\int_{C_{1,r}} R(\lambda, e^x e^y)\, D_1(x,h)\, e^y R(\lambda, e^x e^y)\, \log\lambda\ d\lambda$$

pour obtenir les estimations du Lemme 3, il suffit de les avoir pour les h dans U. Si h est quelconque, λh est dans U pour λ réel petit et on utilise l'homogénéité de degrés 1 en h, de l'expression $\partial\Phi_0(x,y)h/\partial x$. Il en est de même pour $\partial\Phi_0(x,y)k/\partial y$. On utilisera cette idée dans les estimations du reste $R((x,y))(h,k)$ qui est homogène de degrée 2 en (h,k). Or

$$\left\|\frac{\partial\Phi_0}{\partial x}(x,y)h\right\|_n \leq \frac{1}{2\pi}\int_{C_{1,r}} \|R(\lambda,e^x e^y)\,D_1(x,h)\,e^y R(\lambda,e^x e^y)\|_n \;|\log\lambda|\; d|\lambda|\,.$$

On utilise ensuite à plusieurs reprises l'inégalité c) du Lemme 2, l'inégalité sur $\|e^x e^y\|_n$ du Lemme 1 et celle sur $\|D_1(x,y)\|_n$ du Lemme 2, pour trouver finalement

$$\begin{aligned}\|\frac{\partial\Phi_0}{\partial x}(x,y)h\|_n &\leq C_n\left((1+\|(x,y)\|_n)\,\|h\|_0+\|h\|_n\right)\\ &\leq C_n\left(\|(x,y)\|_n\,\|h\|_0+\|h\|_n\right).\end{aligned}$$

De même

$$\left\|\frac{\partial\Phi_0}{\partial y}(x,y)k\right\|_n \leq C_n\left(\|(x,y)\|_n\,\|k\|_0+\|k\|_n\right)$$

et donc pour $(x,y)\in U\times U$ et $(h,k)\in M(s,C^\infty(\overline{\Omega}))^2$

$$\|d\Phi_0(x,y)(h,k)\|_n \leq C_n\left(\|(x,y)\|_n\,\|(h,k)\|_0+\|(h,k)\|_n\right).$$

Le Lemme 3 est démontré. Par les mêmes méthodes, on obtient aussi le

Lemme 4. *Si* $(x,y)\in U\times U,\ (h,k)\in E_1^2,$

$$\|R((x,y))(h,k)\|_n \leq C_n\left(\|(x,y)\|_n\,\|(h,k)\|_0^2+\|(h,k)\|_0\,\|(h,k)\|_n\right).$$

Nous passons maintenant aux estimations sur l'inverse $L(x,y)$ de $d\Phi(x,y)$ dont l'expression a été obtenue dans le lemme fondamental.

Si x et y sont voisins de la section nulle dans $\mathcal{A}^{\infty,Ad\,E}(\Omega_1)$ et $\mathcal{A}^{\infty Ad\,E}(\Omega_2)$ respectivement, si η appartient à $\mathcal{A}^{\infty Ad\,E}(\Omega_1\cap\Omega_2)$, on a

$$L(x,y)\,\eta=(h,k)\,,$$

avec

$$h=\frac{ad\,x}{1-e^{-ad\,x}}\,\eta_1\,,\qquad k=\frac{ad\,y}{e^{ad\,y}-1}\,\eta_2$$

et η_1, η_2, dont l'existence est assurée par la Proposition 4.5, vérifient

$$\eta_1 + \eta_2 = \eta' = e^{-ad\,x}\,\frac{e^{ad\,\Phi(x,y)} - 1}{ad\,\Phi(x,y)}\,\eta\,.$$

Il convient de préciser ce que l'on veut dire par x et y sont voisines de la section nulle dans $\mathcal{A}^{\infty, Ad\,E}(\Omega_1)$ et $\mathcal{A}^{\infty, Ad\,E}(\Omega_2)$ respectivement: on prend x dans $U_1 = \{x \in A^\infty(\Omega_1, Ad\,E) : \|x\|_{1,\Omega_1} \leq a\}$ et y dans $U_2 = \{y \in A^\infty(\Omega_2, Ad\,E) : \|y\|_{1,\Omega_2} \leq a\}$ avec un réel a suffisamment petit de sorte que pour chaque ouvert de trivialisation U_i, $U_i \cap \Omega_j \neq \varnothing$, $(j = 1, 2)$, si $\varphi_i : Ad\,E|_{\overline{U_i}} \to U_i \times M(s, \mathbb{C})$ est l'isomorphisme de trivialisation, la composée $\varphi_i \circ s|_{\overline{U_i \cap \Omega_j}}$ soit dans le voisinage de matrice nulle

$$\left\{x \in M(s, C^\infty(\overline{U_i \cap \Omega_j}) : \|x\|_{o,\overline{U_i \cap \Omega_j}} \leq \alpha \leq \frac{\log(1 + s^{-2})}{8s}\right\},$$

de même pour la section y. Ceci est possible puisque Ω_1 et Ω_2 sont bornés et le recouvrement $\mathcal{U} = (U_i)$, est fini. D'autre part, nous savons d'après la Proposition 2.12 que

$$\|\eta_i\|_{n,\Omega_j} \leq C_n\,\|\eta'\|_{n+p,\Omega_1 \cap \Omega_2} = C_n\left\|e^{-ad\,x}\,\frac{e^{ad\,\Phi(x,y)} - 1}{ad\,\Phi(x,y)}\,\eta\right\|_{n+p,\Omega_1 \cap \Omega_2},$$

où p est la dimension de l'espace. Pour estimer en norme $\|\cdot\|_n$ le couple (h, k) en fonction de la norme $\|\cdot\|_n$ de η, on estime h en fonction η_1, k en fonction de η_2 et

$$e^{-ad\,x}\,\frac{e^{ad\,\Phi(x,y)} - 1}{ad\,\Phi(x,y)}\,\eta$$

en fonction de η. Comme on l'a remarqué au dédut de cet appendice il suffit de faire les calculs dans le cas des matrices. Nous commençons par des observations simples:

a) Soit $A \in M(s, C(\overline{\Omega}))$, Ω est un ouvert borné dans $\mathbb{R}^n$, soit $Ad\,A$ l'opérateur linéaire

$$Ad\,A : M(s, C(\overline{\Omega})) \to M(s, C(\overline{\Omega}))\,,$$

$$Ad\,A(X) = [A, X] = AX - XA\,.$$

Alors

$$\|Ad\,A\|_0 \leq 2S\|A\|_0\,.$$

b) Soit $A \in M(s, C^\infty(\overline{\Omega}))$, $\|A\|_0 \leq 1/(4s)$. Si $\lambda \in \mathbb{C}$, $|\lambda| = 1$, pour toute matrice x dans $M(s, C^\infty(\overline{\Omega}))$ et tout entier n non nul

$$\|(\lambda - Ad\,A)^{-1}x\|_n \leq C_n\left(\|A\|_n\,\|x\|_0 + \|x\|_n\right).$$

En identifiant les matrices x à des vecteurs $\widetilde{x}$ à n^2 composantes, on peut trouver une matrice $\widetilde{A}$, $n^2 \times n^2$, à coefficients dans $C^\infty(\overline{\Omega})$: $Ad\,A\;x = \widetilde{\widetilde{x}}$ avec de plus $\|\widetilde{A}\| \leq 2\,\|A\|_n$ et alors

$$(\lambda - Ad\,A)^{-1}x = \lambda\sum_{p\geq 0}\frac{(Ad\,A)^p}{\lambda^p}\;x = \lambda\sum\frac{\widetilde{A}^p}{\lambda^p}\,\widetilde{x} = (\lambda - \widetilde{A})^{-1}\widetilde{x}\,,$$

donc

$$\|(\lambda - Ad\,A)^{-1}x\|_n \leq C_n\left(\|(\lambda - \widetilde{A})^{-1}\|_n\,\|\widetilde{x}\|_0 + \|(\lambda - \widetilde{A})^{-1}\|_0\,\|\widetilde{x}\|_n\right)$$

en tenant compte de $\|\widetilde{x}\|_k = \|x\|_k$ et le Lemme 2. On obtient

$$\begin{aligned}\|(\lambda - Ad\,A)^{-1}\|_n &\leq C_n\left((1 + \|\widetilde{A}\|_n)\,\|x\|_0 + \|\widetilde{x}\|_n\right)\\ &\leq C_n\left(\|A\|_n\,\|x\|_0 + \|x\|_n\right).\end{aligned}$$

c) Si $\eta_1, x \in M(s, C^\infty(\overline{\Omega}))$, $\|x\|_0 \leq \dfrac{\log(1+s^{-2})}{8s}$, on a

$$\left\|\frac{ad\,x}{1 - e^{-ad\,x}}\,\eta_1\right\|_n \leq C_n\left(\|x\|_n\,\|\eta_1\|_0 + \|\eta_1\|_n\right).$$

On écrit pour cela

$$\frac{ad\,x}{1 - e^{-ad\,x}}\,\eta_1 = \frac{1}{2\pi i}\int_{C_{0,1}}(\lambda I - ad\,x^{-1})\,\eta_1\,\frac{\lambda}{1 - e^{-\lambda}}\;d\lambda$$

et on utilise l'observation *b*) précédente.

d) Si $x, y \in M(s, C^\infty(\overline{\Omega}))$,

$$\|x\|_0 \leq \frac{\log(1+s^{-2})}{8s}\,, \qquad \|y\|_0 \leq \frac{\log(1+s^{-2})}{8s}\,,$$

et si $\eta \in M(s, C^\infty(\overline{\Omega}))$, on a

$$\|e^{-ad\,x}\frac{e^{ad\,\Phi(x,y)} - 1}{s\,ad\,\Phi(x,y)}\eta\|_{n+p} \leq C_n\left(\|(x,y)\|_{n+p}\,\|\eta\|_0 + \|\eta\|_{n+p}\right).$$

On écrit pour cela

$$e^{-ad\,x}\,\frac{e^{ad\,\Phi(x,y)}-1}{ad\,\Phi(x,y}\,\eta=\frac{1}{2\pi i}\int_{C_{0,1}}(\lambda I+ad\,x)^{-1}\,\frac{e^{ad\,\Phi(x,y)}-1}{ad\,\Phi(x,y)}\,\eta\,e^{\lambda}\,d\lambda\,.$$

Cela donne, par *b*)

$$\begin{aligned}\Big\|e^{-ad\,x}\,\frac{e^{ad\,\Phi(x,y)}-1}{ad\,\Phi(x,y)}\,\eta\Big\|_{n+p} &\le C_n\Big(\|x\|_{n+p}\,\Big\|\frac{e^{ad\,\Phi(x,y)}-1}{ad\,\Phi(x,y)}\,\eta\Big\|_0\\ &\qquad+\Big\|\frac{e^{ad\,\Phi(x,y)}-1}{ad\,\Phi(x,y)}\,\eta\Big\|_{n+p}\Big)\,,\end{aligned}$$

on utilise le Lemme 1, point *b*) pour estimer

$$\|\frac{e^{ad\,\Phi(x,y)}-1}{ad\,\Phi(x,y)}\eta\|_k$$

en fonction de $1+\|(x,y)\|_k$, on utilise ensuite que $\|x\|_{n+p}\le\|(x,y)\|_{n+p}$ et $\|\Phi(x,y)\|_0\le C$ à cause des hypothèses faites sur x et y.

On estime finalement

$$\|L(x,y)\eta\|_n=\Big\|\Big(\frac{ad\,x}{1-e^{-ad\,x}}\,\eta_1,\frac{ad\,x}{e^{ad\,y}-1}\,\eta_2)\|_n\,.$$

L'inégalité du point *c*) ci-dessus, ainsi q'une autre où x est remplacé par y et η_1 par η_2, permettent d'écrire

$$\begin{aligned}\|L(x,y)\eta\|_n &\le C_n\,(\|(x,y)\|_n\,\|(\eta_1,\eta_2)\|_0+\|(\eta_1,\eta_2)\|_n)\\ &\le C_n\,(\|(x,y)\|_n\,\|\eta'\|_0+\|\eta'\|_{n+p})\,,\end{aligned}$$

avec

$$\eta'=e^{-ad\,x}\frac{e^{ad\,\Phi(x,y)}-1}{ad\,\Phi(x,y)}\eta\,,$$

l'inégalité de *d*) donne enfin

$$\|L(x,y)\eta\|_n\le C_n\,(\|(x,y)\|_{n+p}\,\|\eta\|_0+\|\eta\|_{n+p})\,,$$

d'où les estimations voulues.

REMARQUES FINALES.

1) Nous venons de voir que si x est une section de $Ad\,E$ sur $\overline{\Omega}_1$ et y est une section de $Ad\,E$ sur $\overline{\Omega}_2$, toutes deux voisines de la section nulle, alors (Lemme 1)

$$\|\log e^x e^y\|_{n,\Omega_1\cap\Omega_2} \leq C_n\,(1+\|(x,y)\|_n)\,,$$
$$\|(x,y)\|_n = \|x\|_{n,\Omega_1} + \|y\|_{n,\Omega_2}\,.$$

Si maintenant $\mathcal{C}$ est un compact et $\mathcal{A}^{\infty,Ad\,E}(\Omega)$ l'espace des applications continues de $\mathcal{C}$ dans $\mathcal{A}^{\infty,Ad\,E}(\Omega)$ dont la topologie de Fréchet est définie par le système de normes

$$\|f\|_{n,\Omega} = \sup_{t\in\mathcal{C}} \|f(t)\|_{n,\Omega}\,, \qquad f\in\mathcal{A}_{\mathcal{C}}^{\infty,Ad\,E}(\Omega)\,,\ n\in\mathbb{N}\,,$$

alors pour toute application f_1 de $\mathcal{C}$ dans $\mathcal{A}_{\mathcal{C}}^{\infty,Ad\,E}(\Omega_1)$ et toute application f_2 de $\mathcal{C}$ dans $\mathcal{A}_{\mathcal{C}}^{\infty,Ad\,E}(\Omega_2)$ toutes deux voisines de l'application nulle, pour tout t dans $\mathcal{C}$

$$\|\log e^{f_1(t)}e^{f_2(t)}\|_{n,\Omega_1\cap\Omega_2} \leq C_n\,(1+\|(f_1(t),f_2(t))\|_n)\,,$$

donc

$$\begin{aligned}\|\Phi(f_1,f_2)\|_n &= \sup_{t\in\mathcal{C}} \|\log e^{f_1(t)}e^{f_2(t)}\|_{n,\Omega_1\cap\Omega_2}\\ &\leq C_n\,(1+\sup_{t\in\mathcal{C}}\|f_1(t),f_2(t)\|_n)\,.\end{aligned}$$

Il en est de même pour

$$\frac{\partial\Phi(x,y)}{\partial x}(h,k)\,,\quad \frac{\partial\Phi(x,y)}{\partial y}(h,k)\,,\quad L(x,y)\eta \quad\text{et}\quad R((x,y))(h,k)$$

et le lemme fondamental est démontré.

2) Nous aurions pu démontrer le théorème des matrices holomorphes du Paragraphe 2 en faisant intervenir l'application $(x,y)\mapsto\log e^x e^y$ pour des raisons de clarté, nous avons travaillé avec l'application $(x,y)\mapsto xy$ qui, du point de vue "calcul différentiel" est nettement plus simple.

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Calderón-type Reproducing Formula and the Tb Theorem

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Abstract. In this paper we use the Calderón-Zygmund operator theory to prove a Calderón type reproducing formula associated with a para-accretive function. Using our Calderón-type reproducing formula we introduce a new class of the Besov and Triebel-Lizorkin spaces and prove a Tb theorem for these new spaces.

Introduction.

Let ϕ be a function with the properties: $\phi \in \mathcal{S}(\mathbb{R}^n), \operatorname{supp} \widehat{\phi} \subseteq \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$, and $|\widehat{\phi}(\xi)| \geq c > 0$ if $3/5 \leq |\xi| \leq 5/3$. The classical Calderón Reproducing Formula can be stated as follows:

Theorem. (The Calderón Reproducing Formula) *Suppose that the function ϕ satisfies the properties above. Then there exists a function ψ satisfying the same properties as ϕ such that*

$$f = \sum_{k \in \mathbb{Z}} \psi_k * \phi_k * f \,,$$

where $\psi_k(x) = 2^{kn}\, \psi(2^k x)$ and the series converges in L^2 norm or in $\mathcal{S}'/\mathcal{P}$, the test functions modulo polynomials.

It is well known that the classical Calderón Reproducing Formula plays an important role in harmonic analysis and wavelets analysis as well. For instance, this formula can be used to study classical function spaces, namely the Besov and Triebel-Lizorkin spaces, and obtain atomic decompositions of these spaces, and prove the boundedness of Calderón-Zygmund operators, namely the $T1$ theorem for the Besov and Triebel-Lizorkin spaces. Further applications of this formula can be found in [C1], [C2], [CF], [FJ1], [FJ2], [FJW], [GM], [P] ,[R] and [U]. Since the classical Calderón Reproducing Formula is given by the action of convolution operators, the Fourier transform is the basic tool for proving such formula.

Our concern in this paper is to establish a Calderón-type reproducing formula associated to a para-accretive function introduced in [DJS], which are not convolution operators. The new idea to establish the Calderón-type reproducing formula associated to a para-accretive function is to use the Calderón-Zygmund operator theory. More precisely, we will introduce a class of "test functions" which will be said to be the strong b-smooth molecules, b is a para-accretive function, and a class of the Calderón-Zygmund operators whose kernels satisfy a strong smoothness condition. We then prove that the Calderón-Zygmund operators in the class above are bounded on "test functions", that is, these operators map the strong b-smooth molecules into the strong b-smooth molecules. Using the approximation to the identity associated to a para-accretive function introduced in [DJS] and a Coifman's idea (see [DJS]), we will construct a Calderón-Zygmund operator whose kernel satisfies the strong smoothness condition mentioned before and use this Calderón-Zygmund operator to establish our Calderón-type reproducing formula associated to a para-accretive function.

As an application of this reproducing formula we prove a Tb theorem. To be precise, suppose that T satisfies the hypotheses of the Tb theorem of [DJS], where $b_1 = b_2 = b$. Suppose also that $Tb = T^*b = 0$. The results of [L] and [HJTW] state that TM_b is bounded on $\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$ for $0 < \alpha < \varepsilon$ and $1 \leq p,q \leq \infty$, where ε is the regularity exponent of the kernel of T and M_b denotes the operator of multiplication by b. Hence T maps $b\dot{B}_p^{\alpha,q}$ into $\dot{B}_p^{\alpha,q}$ and $b\dot{F}_p^{\alpha,q}$ into $\dot{F}_p^{\alpha,q}$ for $0 < \alpha < \varepsilon$ and $1 \leq p,q \leq \infty$. Applying this to T^*, we obtain by duality that T maps $\dot{B}_p^{-\alpha,q}$ into $b^{-1}\dot{B}_p^{-\alpha,q}$ and $\dot{F}_p^{-\alpha,q}$ into $b^{-1}\dot{F}_p^{-\alpha,q}$ for $0 < \alpha < \varepsilon$ and $1 \leq p,q \leq \infty$. However, the results of [L] and [HJTW] can not be applied to the case where $\alpha = 0$. As in the case of $\mathbb{R}^n$, using our Calderón-type reproducing formula associated to a para-accretive function, we will introduce a new class of function and distribution spaces, namely the Besov and Triebel-Lizorkin spaces associated to a para-accretive function, and prove the Tb theorem for these new spaces, which includes the case where $\alpha = 0$.

The paper is organized as follows. In Section 1 we describe the notations, definitions and some known results to be used throughout and prove a boundedness result of a class of the Calderón-Zygmund operators. In Section 2 we establish the Calderón-type reproducing formulas associated to a para-accretive function. The Besov and Triebel-Lizorkin spaces will be introduced in Section 3 and a Tb theorem will be proved there. In the last section we make several remarks.

Section 1.

We begin by reviewing some basic facts about the Calderón-Zygmund operator theory.

Definition 1.1. *A singular integral operator T is a continuous linear operator from $\mathcal{D}(\mathbb{R}^n)$ into its dual that is associated to a kernel $K(x,y)$, a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$, satisfying the following conditions: for some constants $c > 0$ and $0 < \varepsilon \leq 1$,*

$$\text{(1.2.i)} \qquad |K(x,y)| \leq c\,|x-y|^{-n}\,, \qquad \text{for all } x \neq y\,,$$

$$\text{(1.2.ii)} \qquad |K(x,y) - K(x',y)| \leq c\,|x-x'|^{\varepsilon}|x-y|^{-n-\varepsilon}\,,$$

for all x, x' and y in $\mathbb{R}^n$ with $|x - x'| \leq |x-y|/2$, and

$$\text{(1.2.iii)} \qquad |K(x,y) - K(x,y')| \leq c\,|y-y'|^{\varepsilon}|x-y|^{-n-\varepsilon}\,,$$

for all x, y and y' in $\mathbb{R}^n$ with $|y - y'| \leq |x-y|/2$.

Moreover, the operator T can be represented by

$$\text{(1.3)} \qquad \langle Tf, g\rangle = \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} K(x,y)\,f(x)\,g(x)\,dx\,dy$$

for all $f, g \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$.

Definition 1.4. *An operator T is called weakly bounded if there exists a constant $c > 0$ such that for all f and $g \in \mathcal{S}$ supported in a cube $Q \subset \mathbb{R}^n$ with diameter at most $t > 0$,*

$$|\langle Tf, g\rangle| \le c\,t^n \left(\|f\|_\infty + t\,\|\nabla f\|_\infty\right)\left(\|g\|_\infty + t\,\|\nabla g\|_\infty\right). \tag{1.5}$$

It was shown in [DJS] that if T is a weakly bounded operator associated to a kernel satisfying (1.2.i) then T has a continuous extension from C_0^η into its dual, where C_0^η denotes the space of continuous functions f with compact support such that

$$\|f\|_\eta = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\eta} < +\infty\,,$$

and for such operator T the weak boundedness property in Definition 1.4 can be described as follows.

Definition 1.6. *Let T be a continuous operator from C_0^η into its dual for each $\eta > 0$. We say that T is weakly bounded if, for each $\eta > 0$, there is a constant $c > 0$ such that for all cubes Q with diameter at most $t > 0$ and all $f, g \in C_0^\eta$ supported in Q,*

$$|\langle Tf, g\rangle| \le c\,t^{1+2\eta/n}\,\|f\|_\eta \|g\|_\eta\,. \tag{1.7}$$

It was shown in [DJS] that if the kernel of T satisfies the condition (1.2.i), then (1.7) holds for all $\eta > 0$ whenever it holds for some $\eta > 0$. David and Journé gave a general criterion for the L^2 boundedness of singular integral operators defined in (1.1) ([DJ]).

Theorem 1.8. (The $T1$ Theorem of David-Journé) *Suppose that T is a singular integral operator defined in* (1.1). *Then T is bounded on L^2 if and only if: a) $T1 \in BMO$, b) $T^*1 \in BMO$, and c) T has the weak boundedness property defined in* (1.4).

Suppose that ϕ is a function with the properties as in the classical Calderón Reproducing Formula. The classical Besov spaces $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ for $\alpha \in R$ and $1 \le p, q \le \infty$ are the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \Big(\sum_{k\in Z} (2^{k\alpha}\|\phi_k * f\|_p)^q\Big)^{1/q} < +\infty\,,$$

and Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$ for $\alpha \in R$ and $1 \le p < \infty$, $1 \le q \le \infty$ are the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \|\big(\sum_{k\in Z}(2^{k\alpha}|\phi_k * f|)^q\big)^{1/q}\|_p < +\infty .$$

The $T1$ theorems for the classical Besov spaces $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$ were proved in [L] and [HJTW] , respectively.

Theorem 1.9. (The $T1$ Theorems for the Besov and Triebel-Lizorkin Spaces) *Suppose that T is a singular integral operator whose kernel satisfies the conditions* (1.2.i), (1.2.ii) *and* $T1 = 0$, *and T has the weak boundedness property. Then T is bounded on the Besov spaces $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$ for $0 < \alpha < \varepsilon, 1 \le p, q \le \infty$, where ε is the regularity exponent of the kernel of T.*

Replacing the function 1 in the $T1$ theorem by more general bounded function David, Journé and Semmes proved the Tb theorem ([DJS]). To state their Tb theorem we need the following definitions.

Definition 1.10. *A complex-valued bounded function b defined on $\mathbb{R}^n$ is said to be a para-accretive function if there exists a constant $c > 0$ such that for every cube $Q \subset \mathbb{R}^n$, there is a subcube $I \subseteq Q$ with*

$$\left|\frac{1}{|Q|}\int_I b(x)\,dx\right| \ge c > 0 . \tag{1.11}$$

Definition 1.12. *Suppose b_1 and b_2 are complex-valued functions whose inverse are also bounded. A singular integral operator is a continuous operator T from $b_1C_0^\eta$ into $(b_2C_0^\eta)'$ for all $\eta > 0$ for which there exists a kernel $K(x,y)$ satisfying the conditions* (i), (ii) *and* (iii) *of* (1.2) *such that for all $f, g \in C_0^\eta$ with* $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$,

$$\langle Tb_1 f, b_2 g\rangle = \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} g(x)\,b_2(x)\,K(x,y)\,b_1(y)\,f(y)\,dx\,dy .$$

David, Journé and Semmes proved the following Tb theorem.

Theorem 1.13. (The Tb Theorem of David-Journé-Semmes) *Suppose that b_1 and b_2 are para-accretive functions and T is a singular integral operator from $b_1C_0^\eta$ into $(b_2C_0^\eta)'$ defined in* (1.12). *Then T is bounded on L^2 if and only if: a) $Tb_1 \in BMO$, b) $T^*b_2 \in BMO$, and c) $M_{b_2}TM_{b_1}$ has the weak boundedness property defined in* (1.6).

We now introduce our "test functions".

Definition 1.14. *Fix two exponents $0 < \beta \leq 1$ and $\gamma > 0$. Suppose that b is a para-accretive function. A function f defined on $\mathbb{R}^n$ is said to be a strong b-smooth molecule of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ if f satisfies the following conditions:*

$$(1.15.\text{i}) \qquad |f(x)| \leq c\frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}},$$

$$(1.15.\text{ii}) \quad |f(x)-f(x')| \leq c\Big(\frac{|x-x'|}{d+|x-x_0|}\Big)^\beta \frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}},$$

for $|x-x'| \leq (d+|x-x_0|)/2$, and

$$(1.15.\text{iii}) \qquad \int_{\mathbb{R}^n} f(x)\,b(x)\,dx = 0.$$

This definition was first introduced in [M1] by considering the conditions (i) and (iii) of (1.15), and (ii) of (1.15) replaced by

$$(1.16) \quad \begin{aligned} |f(x)-f(x')| \leq{}& c\Big(\frac{|x-x'|}{d}\Big)^\beta \\ &\cdot\Big(\frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}}+\frac{d^\gamma}{(d+|x'-x_0|)^{n+\gamma}}\Big). \end{aligned}$$

We call such f a b-smooth molecule of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. The collection of all strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ will be denoted by $M^{(\beta,\gamma)}(x_0,d)$. If $f \in M^{(\beta,\gamma)}(x_0,d)$, the norm of f in $M^{(\beta,\gamma)}(x_0,d)$ is defined by

$$(1.17) \qquad \|f\|_{M^{(\beta,\gamma)}(x_0,d)} = \inf\{c \geq 0 : \text{ (1.9) (i), (ii) and (iii) hold }\}.$$

We denote $M^{(\beta,\gamma)}$ the class of all $f \in M^{(\beta,\gamma)}(0,1)$. It is easy to see that $M^{(\beta,\gamma)}$ is a Banach space under the norm $\|f\|_{M^{(\beta,\gamma)}} < +\infty$. We then introduce the dual space $(M^{(\beta,\gamma)})'$ consisting of all linear functionals $\mathcal{L}$ from $M^{(\beta,\gamma)}$ to $\mathbb{C}$ with the property that there exists a finite constant c such that for all $f \in M^{(\beta,\gamma)}$,

$$|\mathcal{L}(f)| \le c\,\|f\|_{M^{(\beta,\gamma)}}\,. \tag{1.18}$$

We denote $\langle h, f\rangle$ the natural pairing of elements $h \in (M^{(\beta,\gamma)})'$ and $f \in M^{(\beta,\gamma)}$. It is easy to check that for $x_0 \in \mathbb{R}^n$ and $d > 0$, $M^{(\beta,\gamma)}(x_0,d) = M^{(\beta,\gamma)}$ with equivalent norms. Thus, for all $h \in (M^{(\beta,\gamma)})'$, $\langle h, f\rangle$ is well defined for all $f \in M^{(\beta,\gamma)}(x_0,d)$ with $x_0 \in \mathbb{R}^n$ and $d > 0$.

We now state and prove the main result in this section.

Theorem 1.19. *Suppose that b is a para-accretive function and T is a singular integral operator from $C_0^\eta(\mathbb{R}^n)$ into its dual for all $\eta > 0$ such that T and $b^{-1}(T^*)M_b$ satisfy the hypotheses of Theorem 1.9 and further, $K(x,y)$, the kernel of T, satisfies the following strong smoothness condition*

$$\begin{aligned} &|(K(x,y)b^{-1}(y) - K(x',y)b^{-1}(y)) \\ &\quad - (K(x,y')b^{-1}(y') - K(x',y')b^{-1}(y'))| \\ &\qquad\qquad \le c\,|x-x'|^\varepsilon |y-y'|^\varepsilon |x-y|^{-n-2\varepsilon}\,, \end{aligned} \tag{1.20}$$

for all x, x', y and y' in $\mathbb{R}^n$ with $|x-x'| \le |x-y|/3$ and $|y-y'| \le |x-y|/3$.

Then T maps the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ to the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ for $0 < \beta, \gamma < \varepsilon$ where ε is the regularity exponent of the kernel of T. Moreover, denote $\|T\|$ the smallest constant in the estimates of the kernel of T, then there exists a constant $c > 0$ such that

$$\|Tf\|_{M^{(\beta,\gamma)}(x_0,d)} \le c\,\|T\|\;\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,. \tag{1.21}$$

In [M2] it was shown that if T satisfies the hypotheses of Theorem 1.19 except (1.20), then T maps b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ to b-smooth molecules of type (β',γ') centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$, which is not available for our purposes.

To prove Theorem 1.19, we follow Meyer's idea, [M2]. Fix a function $\theta \in \mathcal{D}$ with $\operatorname{supp}\theta \subset \{x \in \mathbb{R}^n : |x| \le 2\}$ and $\theta = 1$ on

$\{x \in \mathbb{R}^n : |x| \leq 1\}$. Suppose that f is a strong b-smooth molecule of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. We first prove that $T(f)(x)$ satisfies the size condition (i) of (1.15). To do this, consider first the case where $|x - x_0| \leq 5\,d$. Set $1 = \xi(y) + \eta(y)$ where $\xi(y) = \theta(y - x_0/(10\,d))$. Then, as in [M2],

$$\begin{aligned} Tf(x) &= \int K(x,y)\,(f(y) - f(x))\,\xi(y)\,dy \\ &\quad + \int K(x,y)\,f(y)\,\eta(y)\,dy \\ &\quad + f(x)\int K(x,y)\xi(y)\,dy = \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned}$$

Using lemmas 2 and 3 in [M2], we have

$$\begin{aligned} |\mathrm{I}| &\leq c \int_{|x-y|\leq 25\,d} |K(x,y)|\,|f(y) - f(x)|\,dy \\ &\leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int_{|x-y|\leq 25\,d} |x-y|^{-n}\,\frac{|x-y|^\beta}{d^{\beta+n}}\,dy \\ &\leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,d^{-n}\,, \end{aligned}$$

and

$$|\mathrm{III}| \leq c\,|f(x)| \leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,d^{-n}\,.$$

For the term II we have

$$\begin{aligned} |\mathrm{II}| &\leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int_{|y-x_0|>10d} |x-y|^{-n}\,\frac{d^\gamma}{|y-x_0|^{n+\gamma}}\,dy \\ &\leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,d^{-n}\,, \end{aligned}$$

since $|x - x_0| \leq 5\,d$. This shows that $Tf(x)$ satisfies (i) of (1.15) for the case $|x - x_0| \leq 5\,d$. Now consider the case where $|x - x_0| = R > 5\,d$. Set $1 = I(y)+J(y)+L(y)$ where $I(y) = \theta(8|x-y|/R)$, $J(y) = \theta(8|y-x_0|/R)$, and $f_1(y) = f(y)\,I(y)$, $f_2(y) = f(y)\,J(y)$, and $f_3(y) = f(y)\,L(y)$. Then it is easy to check the following estimates

$$|f_1(y)| \leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^{n+\gamma}}\,, \tag{1.22.a}$$

$$|f_1(y) - f_1(y')| \leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{|y-y'|^\beta}{R^\beta}\,\frac{d^\gamma}{R^{n+\gamma}}\,, \tag{1.22.b}$$

for all y and $y' \in \mathbb{R}^n$,

$$|f_3(y)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \cdot \frac{d^\gamma}{|y-x_0|^{n+\gamma}}\,\chi_{\{|y-x_0|>R/8\}}\,, \tag{1.22.c}$$

$$\int |f_3(y)|\,dy \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^\gamma}\,, \tag{1.22.d}$$

$$\begin{aligned}\Big|\int f_2(y)\,b(y)\,dy\Big| &\le c\Big(\int |f_1(y)|\,dy + \int |f_3(y)|\,dy\Big)\\ &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^\gamma}\,,\end{aligned} \tag{1.22.e}$$

since$\int f(y)b(y)dy = 0$. We now have

$$\begin{aligned}Tf_1(x) &= \int K(x,y)\,(f_1(y)-f_1(x))\,u(y)\,dy + f_1(x)\int K(x,y)\,u(y)\,dy\\ &= r_1(x) + r_2(x)\,,\end{aligned}$$

where $u(y) = \theta(4|x-y|/R)$. Applying the estimates in (1.22), we obtain

$$\begin{aligned}|r_1(x)| &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|x-y|\le R/2} |x-y|^{-n}\,\frac{|x-y|^\beta}{R^\beta}\,\frac{d^\gamma}{R^{n+\gamma}}\,dy\\ &\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^{n+\gamma}}\,,\end{aligned}$$

$$|r_2(x)| \le c\,|f_1(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,\frac{d^\gamma}{R^{n+\gamma}}\,.$$

For f_2 we have

$$\begin{aligned}Tf_2(x) = \int &\big(b^{-1}(y)K(x,y) - b^{-1}(x_0)K(x,x_0)\big)f_2(y)\,b(y)\,dy\\ &+ b^{-1}(x_0)\,K(x,x_0)\int f_2(y)\,b(y)\,dy\\ &= \sigma_1(x) + \sigma_2(x)\,.\end{aligned}$$

Using the estimates of the kernel of $b^{-1}T^*M_b$ and f_2 in (1.22),

$$|\sigma_1(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|y-x_0|\le R/4} \frac{|y-x_0|^{\varepsilon}}{R^{n+\varepsilon}} \frac{d^{\gamma}}{|y-x_0|^{n+\gamma}}\,dy$$
$$\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{d^{\gamma}}{R^{n+\gamma}},$$

since $\gamma < \varepsilon$, and

$$|\sigma_2(x)| \le c\,R^{-n} \left| \int f_2(y)\, b(y)\, dy \right| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{d^{\gamma}}{R^{n+\gamma}}.$$

Finally,

$$|Tf_3(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{\substack{|x-y|>R/8\\ |y-x_0|>R/8}} |x-y|^{-n} \frac{d^{\gamma}}{|y-x_0|^{n+\gamma}}\,dy$$
$$\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{d^{\gamma}}{R^{n+\gamma}}.$$

This proves that $Tf(x)$ satisfies (i) of (1.15) for $|x-x_0| \ge 5\,d$ and hence the estimate (i) of (1.15). It remains to prove that $Tf(x)$ satisfies the smoothness condition (ii) of (1.15). Set $|x-x_0| = R$ and $|x-x'| = \delta$. We consider only the case where $R > 5\,d$ and $\delta \le (d+R)/20$ (see the proof in [M2] for the case where $R \le 5\,d$). As in the above,

$$Tf_1(x) = \int K(x,y)\,(f_1(y) - f_1(x))\,\zeta(y)\,dy$$
$$+ \int K(x,y)\, f_1(y)\, \mu(y)\, dy$$
$$+ f_1(x) \int K(x,y)\,\zeta(y)\,dy,$$

where $1 = \zeta(y) + \mu(y)$ and $\zeta(y) = \theta(|x-y|/(2\,\delta))$. Denote the first term of right hand side above by $p(x)$ and the sum of the last two terms by $q(x)$. Then the size condition of K and the smoothness of f_1 in (1.22) yield

$$|p(x)| \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|x-y|\le 4\delta} |x-y|^{-n} \frac{|x-y|^{\beta}}{R^{\beta}} \frac{d^{\gamma}}{R^{n+\gamma}}\,dy$$
$$\le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^{\beta}}{R^{\beta}} \frac{d^{\gamma}}{R^{n+\gamma}}.$$

This estimate still holds with x replaced by x' for $|x - x'| = \delta$. Thus,

$$|p(x) - p(x')| \leq c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^\beta}{R^\beta} \frac{d^\gamma}{R^{n+\gamma}}\,.$$

For $q(x)$, using the condition that $T1 = 0$, we have

$$\begin{aligned} q(x) - q(x') &= \int \big(K(x,y) - K(x',y)\big)\big(f_1(y) - f_1(x)\big)\,\mu(y)\,dy \\ &\quad + \big(f_1(x) - f_1(x')\big) \int K(x',y)\,\zeta(y)\,dy \\ &= \mathrm{I} + \mathrm{II}\,. \end{aligned}$$

Again, using lemmas 2 and 3 in [M2], and the smoothness of f_1 in (1.22), we obtain

$$|\mathrm{II}| \leq c\,|f(x) - f(x')| \leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^\beta}{R^\beta} \frac{d^\gamma}{R^{n+\gamma}}\,.$$

Notice that

$$|f_1(y) - f_1(x)| \leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{|x-y|^\beta}{R^\beta} \frac{d^\gamma}{R^{n+\gamma}}\,,$$

for all $y \in \mathbb{R}^n$, term I is dominated by

$$\begin{aligned} &\int\limits_{2\delta \leq |x-y|} |K(x,y) - K(x',y)|\,|f_1(y) - f_1(x)|\,dy \\ &\qquad \leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \int\limits_{|x-y|\geq 2\delta} \frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}} \frac{|x-y|^\beta}{R^\beta} \frac{d^\gamma}{R^{n+\gamma}}\,dy \\ &\qquad \leq c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^\beta}{R^\beta} \frac{d^\gamma}{R^{n+\gamma}}\,, \end{aligned}$$

since $\beta < \varepsilon$. This shows that $Tf_1(x)$ satisfies the condition (ii) of (1.15) for the case $R > 5\,d$. Note that x and x' are not in the supports of f_2 and f_3 and $\delta \leq (d+R)/20 < R/16$. Using the strong smoothness condition of the kernel of T in (1.20) and the estimates of f_2 and f_3 in

(1.22), we then have

$$
\begin{aligned}
&|Tf_2(x) - Tf_2(x')| \\
&\quad = \Big| \int \big(K(x,y)b^{-1}(y) - K(x',y)b^{-1}(y)\big)\, f_2(y)\, b(y)\, dy \Big| \\
&\quad \le \Big| \int \big(\big(K(x,y)b^{-1}(y) - K(x',y)b^{-1}(y)\big) \\
&\qquad\qquad - \big(K(x,x_0)b^{-1}(x_0) - K(x',x_0)b^{-1}(x_0)\big)\big)\, f_2(y)\, b(y)\, dy \Big| \\
&\qquad + \big|\big(K(x,x_0) - K(x',x_0)\big)b^{-1}(x_0)\big| \, \Big| \int f_2(y)\, b(y)\, dy \Big| \\
&\quad \le c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \\
&\qquad\qquad \int\limits_{|y-x_0| \le R/4} \frac{|x-x'|^{\varepsilon} |y-x_0|^{\varepsilon}}{|x-x_0|^{n+2\varepsilon}} \frac{d^{\gamma}}{(d+|y-x_0|)^{n+\gamma}}\, dy \\
&\qquad + c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{|x-x'|^{\varepsilon}}{|x-x_0|^{n+\varepsilon}} \frac{d^{\gamma}}{R^{\gamma}} \\
&\quad \le c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^{\beta}}{R^{\beta}} \frac{d^{\gamma}}{R^{n+\gamma}},
\end{aligned}
$$

since $\beta, \gamma < \varepsilon$, and

$$
\begin{aligned}
|Tf_3(x) - Tf_3(x')| &= \Big| \int\limits_{|x-y| \ge R/8 > 2\delta} \big(K(x,y) - K(x',y)\big)\, f_3(y)\, dy \Big| \\
&\le c \int\limits_{|x-y| \ge R/8} \frac{|x-x'|^{\varepsilon}}{|x-y|^{n+\varepsilon}} |f_3(y)|\, dy \\
&\le c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \frac{\delta^{\varepsilon}}{R^{\varepsilon}} \frac{d^{\gamma}}{R^{n+\gamma}}.
\end{aligned}
$$

These estimates show that $T(f)(x)$ satisfies the condition (ii) in (1.15) for the case where $R > 5\,d$ and $\delta \le (d+R)/20$. The fact that $\int T(f)(x)\, b(x)\, dx = 0$ follows from the condition $T^*(b)(x) = 0$. This completes the proof of Theorem 1.19.

Section 2.

In this section we construct a Calderón-Zygmund operator whose kernel satisfies the strong smoothness condition (1.20) and use this operator to establish the Calderón-type reproducing formula associated to a para-accretive function. We first introduce the following definition (see [DJS]).

Definition 2.1. *A sequence $(S_k)_{k\in\mathbb{Z}}$ of operators is called to be an approximation to the identity associated to a para-accretive function b if $S_k(x,y)$, the kernel of S_k, are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}$ such that for all $k \in \mathbb{Z}$ and all x, x', y and y' in $\mathbb{R}^n$, and some $0 < \varepsilon \le 1$ and $c > 0$,*

(2.2.i)
$$S_k(x,y) = 0\,, \ \text{if } |x-y| \ge c\,2^{-k} \qquad \text{and} \qquad \|S_k\|_\infty \le c\,2^{kn}\,,$$

(2.2.ii)
$$|S_k(x,y) - S_k(x,y')| \le c\,2^{k(n+\varepsilon)}\,|y-y'|^\varepsilon\,,$$

(2.2.iii)
$$|S_k(x,y) - S_k(x',y)| \le c\,2^{k(n+\varepsilon)}\,|x-x'|^\varepsilon\,,$$

(2.2.iv)
$$\begin{aligned}|(S_k(x,y) - S_k(x',y)) - (S_k(x,y') - S_k(x',y'))|\\ \le c\,2^{k(n+2\varepsilon)}\,|x-x'|^\varepsilon\,|y-y'|^\varepsilon\,,\end{aligned}$$

(2.2.v)
$$\int_{\mathbb{R}^n} S_k(x,y)\,b(y)\,dy = 1\,, \qquad \text{for all } k \in \mathbb{Z} \text{ and } x \text{ in } \mathbb{R}^n\,,$$

(2.2.vi)
$$\int_{\mathbb{R}^n} S_k(x,y)\,b(x)\,dx = 1\,, \qquad \text{for all } k \in \mathbb{Z} \text{ and } y \text{ in } \mathbb{R}^n\,.$$

In [DJS] such operators were constructed and all conditions except for (iv) in (2.2) were checked. Note that in [DJS] S_k were given by $P_k^*\{P_k b\}^{-1}P_k$ where P_k satisfy the conditions (i), (ii), and (v) with $b(x) = 1$ in (2.2). We have

$$\begin{aligned}&\big(S_k(x,y) - S_k(x',y)\big) - \big(S_k(x,y') - S_k(x',y')\big)\\ &\quad = \int \big(P_k(z,x) - P_k(z,x')\big)\,(P_k b(z))^{-1}\,\big(P_k(z,y) - P_k(z,y')\big)\,dz\,.\end{aligned}$$

The condition (iv) in (2.2) then follows from simple calculation.

We can now state our Calderón-type reproducing formula.

Theorem 2.3. *Suppose that $(S_k)_{k\in\mathbb{Z}}$ is an approximation to the identity defined in (2.1). Set $D_k = S_k - S_{k-1}$. Then there exists a family of operators $(\widetilde{D}_k)_{k\in\mathbb{Z}}$ such that for all $f \in M^{(\beta,\gamma)}$,*

$$f = \sum_{k\in\mathbb{Z}} \widetilde{D}_k\, M_b\, D_k\, M_b(f)\,, \tag{2.4}$$

where the series converges in the norm of L^p, $1 < p < \infty$, and $M^{(\beta',\gamma')}$ with $\beta' < \beta$ and $\gamma' < \gamma$. Moreover, $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$, satisfy the following estimates: for $\varepsilon', 0 < \varepsilon' < \varepsilon$, where ε is the regularity exponent of S_k, there exists a constant $c > 0$ such that

$$|\widetilde{D}_k(x,y)| \le c\,\frac{2^{-k\varepsilon'}}{(2^{-k} + |x-y|)^{n+\varepsilon'}}\,, \tag{2.5.i}$$

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le c\left(\frac{|x-x'|}{2^{-k} + |x-y|}\right)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k} + |x-y|)^{n+\varepsilon'}}\,, \tag{2.5.ii}$$

for $|x - x'| \le (2^{-k} + |x-y|)/2$,

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y)\, b(x)\, dx = 0\,, \tag{2.5.iii}$$

for all $k \in \mathbb{Z}$ and y in $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y)\, b(y)\, dy = 0\,, \tag{2.5.iv}$$

for all $k \in \mathbb{Z}$ and x in $\mathbb{R}^n$.

The similar formula on spaces of homogeneous type for the case where $b(x) = 1$ was established in [HS2]. To prove theorem (2.3) we begin with a Coifman's idea. By non-degeneracy condition (v) and the size condition (i) in (2.2),

$$\mathrm{I} = \sum_{k\in\mathbb{Z}} D_k\, M_b \ \text{ in } L^2(\mathbb{R}^n)\,. \tag{2.6}$$

Coifman's idea is to rewrite (2.6) in the following way

$$
\begin{aligned}
\mathrm{I} &= \Big(\sum_{k\in\mathbb{Z}} D_k M_b\Big)\Big(\sum_{j\in\mathbb{Z}} D_j M_b\Big) \\
&= \sum_{|j|>N}\sum_{k\in\mathbb{Z}} D_{k+j} M_b D_k M_b \\
&\quad + \sum_{k\in\mathbb{Z}}\sum_{|j|\le N} D_{k+j} M_b D_k M_b = R_N + T_N
\end{aligned}
\tag{2.7}
$$

where

$$R_N = \sum_{|j|>N}\sum_{k\in\mathbb{Z}} D_{k+j} M_b D_k M_b$$

and

$$T_N = \sum_{k\in\mathbb{Z}} D_k^N M_b D_k M_b$$

with

$$D_k^N = \sum_{|j|\le N} D_{k+j},$$

and N is a fix positive integer. It was shown in [DJS] that $\lim_{N\to\infty} T_N = I$ in L^2 and hence T_N^{-1} is bounded on L^2 for large N. Our goal here is to show that T_N^{-1} maps the strong b-smooth molecules of type (β,γ) centered at $x_0\in\mathbb{R}^n$ with width $d>0$ to the strong b-smooth molecules of type (β,γ) centered at $x_0\in\mathbb{R}^n$ with width $d>0$. To be precise, we prove the following theorem.

Theorem 2.8. *Suppose that $(D_k)_{k\in\mathbb{Z}}$ is as in Theorem 2.3, and $T_N = \sum_{k\in\mathbb{Z}} D_k^N M_b D_k M_b$ where $D_k^N = \sum_{|j|\le N} D_{k+j}$ and N is a large positive integer. Then T_N^{-1} maps the strong b-smooth molecules of type (β,γ) centered at $x_0\in\mathbb{R}^n$ with width $d>0$ to the strong b-smooth molecules of type (β,γ) centered at $x_0\in\mathbb{R}^n$ with width $d>0$. More precisely, for $0<\beta,\gamma<\varepsilon$ there exists a constant $c>0$ such that if N is sufficiently large,*

$$\|T_N^{-1}(f)\|_{M^{(\beta,\gamma)}(x_0,d)} \le c\,\|f\|_{M^{(\beta,\gamma)}(x_0,d)}\,. \tag{2.9}$$

The proof of theorem (2.8) is based on the following technical lemma.

Lemma 2.10. *Suppose that the hypotheses of Theorem 2.8 are satisfied, and T_N and R_N are as in (2.7). Then for $0 < \varepsilon' < \varepsilon$ there exist a constant $c > 0$ and $\delta > 0$ such that*

$$|R_N(x,y)| \le c\, 2^{-N\delta}\, |x-y|^{-n}, \tag{2.11}$$

$$|R_N(x,y) - R_N(x',y)| \le c\, 2^{-N\delta}\, |x-x'|^{\varepsilon'}\, |x-y|^{-n-\varepsilon'}, \tag{2.12}$$

for $|x-x'| \le |x-y|/2$,

$$\begin{aligned} &\big|\big(R_N(x,y)b^{-1}(y) - R_N(x',y)b^{-1}(y)\big) \\ &\quad - \big(R_N(x,y')b^{-1}(y') - R_N(x',y')b^{-1}(y')\big)\big| \\ &\qquad \le c\, 2^{-N\delta}\, |x-x'|^{\varepsilon'}\, |y-y'|^{\varepsilon}\, |x-y|^{-n-2\varepsilon'}, \end{aligned} \tag{2.13}$$

for $|x-x'| \le |x-y|/3$ *and* $|y-y'| \le |x-y|/3$,

$$|\langle R_N f, g\rangle| \le c\, 2^{-N\delta}\, t^{1+2\eta/n}\, \|f\|_\eta\, \|g\|_\eta, \tag{2.14}$$

for all $f, g \in C_0^\eta(\mathbb{R}^n)$, $\eta > 0$, supported in Q with diameter at most $t > 0$.

Assuming Lemma 2.10 for the moment and applying the same proof of (2.10) to $b^{-1}(R_N)^* M_b$, and using the facts that $R_N(1) = 0$ and $(R_N)^*(b) = 0$, by Theorem 1.19,

$$\|R_N(f)\|_{M^{(\beta,\gamma)}(x_0,d)} \le c\, 2^{-N\delta} \|f\|\|_{M^{(\beta,\gamma)}(x_0,d)}, \tag{2.15}$$

for all $f \in M^{(\beta,\gamma)}(x_0,d)$. Using the fact that $T_N^{-1} = \sum_m (R_N)^m$, we obtain

$$\begin{aligned} \|T^{-1}{}_N(f)\|_{M^{(\beta,\gamma)}(x_0,d)} &\le \sum_{m=0} (c\, 2^{-N\delta})^m \|f\|_{M^{(\beta,\gamma)}(x_0,d)} \\ &\le c\, \|f\|_{M^{(\beta,\gamma)}(x_0,d)}, \end{aligned} \tag{2.16}$$

for a fixed sufficiently large integer N, which shows (2.9) and hence Theorem 2.8.

It remains to prove Lemma 2.10. In fact, we prove the following estimates: for $0 < \varepsilon'' < \varepsilon$ there exists a constant c such that

$$\begin{aligned} &|D_{k+j}\, M_b\, D_k\, M_b(x,y)| \\ &\qquad \le c\, 2^{-|j|\varepsilon} \frac{2^{-[(k+j)\wedge k]\varepsilon}}{(2^{-[(k+j)\wedge k]} + |x-y|)^{n+\varepsilon}}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &|D_{k+j}\, M_b\, D_k\, M_b(x,y) - D_{k+j}\, M_b\, D_k\, M_b(x',y)| \\ &\qquad \le c \left(\frac{|x-x'|}{2^{-[(k+j)\wedge k]}}\right)^{\varepsilon''} \frac{2^{-[(k+j)\wedge k]\varepsilon''}}{(2^{-[(k+j)\wedge k]} + |x-y|)^{n+\varepsilon''}}, \end{aligned} \tag{2.18}$$

for $|x-x'| \le |x-y|/2$,

$$\begin{aligned}
&\big|\big(D_{k+j}\, M_b\, D_k(x,y) - D_{k+j}\, M_b\, D_k(x',y)\big) \\
&\quad - \big(D_{k+j}\, M_b\, D_k(x,y') - D_{k+j}\, M_b\, D_k(x',y')\big| \\
&\qquad \le c\left(\frac{|x-x'|}{2^{-[(k+j)\wedge k]}}\right)^{\varepsilon''}\left(\frac{|y-y'|}{2^{-[(k+j)\wedge k]}}\right)^{\varepsilon''} \\
&\qquad\quad \cdot \frac{2^{-[(k+j)\wedge k]\varepsilon''}}{(2^{-[(k+j)\wedge k]}+|x-y|)^{n+\varepsilon''}},
\end{aligned} \tag{2.19}$$

for $|x-x'| \le |x-y|/3$ and $|y-y'| \le |x-y|/3$, where D_k are as in Lemma 2.10 and $a \wedge b$ denotes the minimum of a and b.

It is easy to see (2.17). For instance, suppose $j \ge 0$, then

$$\begin{aligned}
&|D_{k+j}\, M_b\, D_k\, M_b(x,y)| \\
&\quad = \left|\int D_{k+j}(x,z)\, b(z)\, D_k(z,y)\, b(y)\, dz\right| \\
&\quad = \left|\int D_{k+j}\,(x,z)\, b(z)\,\big(D_k(z,y) - D_k(x,y)\big)\, b(y)\, dz\right| \\
&\quad \le c \int_{|z-x|\le 2^{-(k+j)}} 2^{(k+j)n}\, |z-x|^{\varepsilon}\, 2^{k(n+\varepsilon)}\, dz \\
&\quad \le c\, 2^{-j\varepsilon}\, 2^{kn}\, \chi_{\{|x-y|\le c\, 2^{-k}\}}
\end{aligned}$$

which shows (2.17) for the case $j \ge 0$. To see (2.18), suppose $j \ge 0$. Then there exists a constant c such that for $|x-x'| \le |x-y|/2$ and all α , $0<\alpha<\varepsilon$,

$$\begin{aligned}
&|D_{k+j}\, M_b\, D_k\, M_b(x,y) - D_{k+j}\, M_b\, D_k\, M_b(x',y)| \\
&\quad = \left|\int \big(D_{k+j}(x,z) - D_{k+j}(x',z)\big)\, b(z)\, D_k(z,y)\, b(y)\, dz\right| \\
&\quad = \left|\int \big(D_{k+j}(x,z) - D_{k+j}(x',z)\big)\, b(z)\right. \\
&\qquad\quad \left.\cdot \big(D_k(z,y) - D_k(x,y)\big)\, b(y)\, dz\right| \\
&\quad \le c \int_{|x-z|\le c\, 2^{-(k+j)}} 2^{(k+j)(n+\varepsilon)}\, |x-x'|^{\varepsilon}\, |z-x|^{\varepsilon}\, 2^{k(n+\varepsilon)}\, dz \\
&\qquad + c \int_{|x'-z|\le c 2^{-(k+j)}} |D_{k+j}(x,z) - D_{k+j}(x',z)| \\
&\qquad\qquad \cdot |z-x|^{\varepsilon}\, 2^{k(n+\varepsilon)}\, dz
\end{aligned}$$

$$
\begin{aligned}
&\le c\,|x-x'|^{\varepsilon}\,2^{k(n+\varepsilon)} \\
(2.20)\qquad &+c\int\limits_{|x'-z|\le c2^{-(k+j)}} |D_{k+j}(x,z)-D_{k+j}(x',z)| \\
&\qquad\qquad \cdot\big(|z-x'|^{\varepsilon}+|x-x'|^{\varepsilon}\big)\,2^{k(n+\varepsilon)}\,dz \\
&\le c\,|x-x'|^{\varepsilon}\,2^{k(n+\varepsilon)} \\
&+c\int\limits_{|x'-z|\le c2^{-(k+j)}} 2^{(k+j)(n+\alpha)}|x-x'|^{\alpha} \\
&\qquad\qquad \cdot|x-x'|^{\varepsilon}\,2^{k(n+\varepsilon)}\,dz \\
&\le c\,|x-x'|^{\varepsilon}\,2^{k(n+\varepsilon)}+c\,2^{\alpha}|x-x'|^{(\alpha+\varepsilon)}2^{k(n+\alpha+\varepsilon)}\,.
\end{aligned}
$$

Note that for $|x-x'|\le|x-y|/2$ the estimate of (2.17) implies

$$
(2.21)\quad |D_{k+j}\,M_b\,D_k\,M_b(x,y)-D_{k+j}\,M_b\,D_k\,M_b(x',y)|\le c\,2^{-j\varepsilon}\,2^{kn}\,.
$$

If choose α small enough, the geometric mean of (2.20) and (2.21) and the fact that the support of $D_{k+j}\,M_b\,D_k\,M_b(x,y)-D_{k+j}\,M_b\,D_k\,M_b(x',y)$ is contained in the set $\{|x-y|\le c\,2^{-k}\}\cup\{|x'-y|\le c\,2^{-k}\}$ yield (2.18) for the case $j\ge 0$. The proof of (2.18) for the case $j<0$ is similar but easier.

The proof of (2.19) is similar. Suppose $j\ge 0$. Then there exists a constant c such that for $|x-x'|\le|x-y|/3$, $|y-y'|\le|x-y|/3$ and all α , $0<\alpha<\varepsilon$,

$$
\begin{aligned}
&\Big|\big(D_{k+j}\,M_b\,D_k\,M_b(x,y)-D_{k+j}\,M_b\,D_k\,M_b(x',y)\big) \\
&\quad -\big(D_{k+j}\,M_b\,D_k\,M_b(x,y')-D_{k+j}\,M_b\,D_k\,M_b(x',y')\big)\Big| \\
&\qquad =\Big|\int\big(D_{k+j}(x,z)-D_{k+j}(x',z)\big) \\
&\qquad\qquad \cdot b(z)\,\big(D_k(z,y)-D_k(z,y')\big)\,b(y)\,dz\Big| \\
&\qquad =\Big|\int\big(D_{k+j}(x,z)-D_{k+j}(x',z)\big)\,b(z) \\
&\qquad\qquad \cdot\Big(\big(D_k(z,y)-D_k(z,y')\big) \\
&\qquad\qquad\quad -\big(D_k(x,y)-D_k(x,y')\big)\Big)\,b(y)\,dz\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le c \int_{|x-z|\le c\,2^{-(k+j)}} 2^{(k+j)(n+\varepsilon)}\,|x-x'|^{\varepsilon} \\
&\qquad\qquad \cdot |z-x|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)}\,dz \\
(2.22)\qquad &\quad + c \int_{|x'-z|\le c\,2^{-(k+j)}} |D_{k+j}(x,z)-D_{k+j}(x',z)| \\
&\qquad\qquad \cdot |z-x|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)}\,dz \\
&\le c\,|x-x'|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)} \\
&\quad + c \int_{|x'-z|\le c\,2^{-(k+j)}} 2^{(k+j)(n+\alpha)}\,|x-x'|^{\alpha} \\
&\qquad\qquad \cdot |x-x'|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)}\,dz \\
&\le c\,|x-x'|^{\varepsilon}\,|y-y'|^{\varepsilon}\,2^{k(n+2\varepsilon)} \\
&\quad + c\,2^{j\alpha}\,|x-x'|^{(\alpha+\varepsilon)}\,|y-y'|^{\varepsilon}\,2^{k(n+\alpha+2\varepsilon)}\,.
\end{aligned}
$$

Note that for $|x-x'|\le |x-y|/3$ and $|y-y'|\le |x-y|/3$ the estimates of (2.18) and (2.21) imply

(2.23)

$$
\begin{aligned}
&\big|\big(D_{k+j}\,M_b\,D_k\,M_b(x,y)-D_{k+j}\,M_b\,D_k\,M_b(x',y)\big) \\
&\quad -\big(D_{k+j}\,M_b\,D_k\,M_b(x,y')-D_{k+j}\,M_b\,D_k\,M_b(x',y')\big)\big| \\
&\qquad\qquad \le c\big(|x-x'|^{\varepsilon'}\,2^{k(n+\varepsilon')}\wedge 2^{-j\varepsilon}2^{kn}\big)\,.
\end{aligned}
$$

If choose α small enough, the geometric mean of (2.22) and (2.23) and the fact that the support of

$$
\begin{aligned}
&\big(D_{k+j}\,M_b\,D_k\,M_b(x,y)-D_{k+j}\,M_b\,D_k\,M_b(x',y)\big) \\
&\quad -\big(D_{k+j}\,M_b\,D_k\,M_b(x,y')-D_{k+j}\,M_b\,D_k\,M_b(x',y')\big)
\end{aligned}
$$

is contained in the set

$$
\{|x-y|\le c\,2^{-k}\}\cup\{|x'-y|\le c\,2^{-k}\}\cup\{|x-y'|\le c\,2^{-k}\}\cup\{|x'-y'|\le c\,2^{-k}\}
$$

yield (2.19) for the case $j\ge 0$. The proof of (2.19) for the case $j<0$ is similar but easier.

Considering the cases $j\ge 0$ and $j<0$ separately, and summing over k, and then, (2.17) implies (2.11), (2.18) and (2.19) imply (2.12) and (2.13) with constant $c\,2^{-N\delta}$ replaced by a constant c, respectively. By taking the geometric means with (2.11) we obtain (2.12) and (2.13). We leave these details to the reader. Finally, the estimate of (2.14) follows the following simple calculations.

$$\langle R_N f, g\rangle = \sum_{|j|>N}\sum_{k\in\mathbb{Z}} \iint D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, g(x)\, dy\, dx\,.$$

Since

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, dy\Big| \\
&\quad = \Big|\int D_{k+j}\, M_b\, D_k\, M_b(x,y)\, \big(f(y)-f(x)\big)\, dy\Big| \\
&\quad \le c \int_{\mathbb{R}^n} 2^{-|j|\varepsilon}\, 2^{[(k+j)\wedge k]n}\, \chi_{\{|x-y|\le c\,2^{-[(k+j)\wedge k]}\}}\, |f(y)-f(x)|\, dy \\
&\quad \le c\,2^{-|j|\varepsilon}\, 2^{-[(k+j)\wedge k]\eta}\, \|f\|_\eta\,,
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, dy\Big| \\
&\qquad \le c \int_{\mathbb{R}^n} 2^{-|j|\varepsilon}\, 2^{[(k+j)\wedge k]n}\, \chi_{\{|x-y|\le c\,2^{-[(k+j)\wedge k]}\}}\, |f(y)|\, dy \\
&\qquad \le c\,2^{-|j|\varepsilon}\, 2^{[(k+j)\wedge k]n}\, \|f\|_\infty\, |Q|\,,
\end{aligned}$$

and denote $|Q| = 2^{-k_0 n}$, we then, by the estimates above, have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, dy\Big| \\
&\qquad \le \begin{cases} c\,2^{-|j|\varepsilon}\, 2^{-([(k+j)\wedge k]-k_0)\eta}\, \|f\|_\eta\, |Q|^{\eta/n}\,; \\ c\,2^{-|j|\varepsilon}\, 2^{([(k+j)\wedge k]-k_0)n}\, \|f\|_\eta\, |Q|^{\eta/n}\,. \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|\iint D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, g(x)\, dy\, dx\Big| \\
&\qquad \le c\,2^{-|j|\varepsilon}\, 2^{-([(k+j)\wedge k]-k_0)\eta}\, \|f\|_\eta\, |Q|^{1+2\eta/n}\, \|g\|_\eta\,,
\end{aligned}$$

and

$$\Big|\iint D_{k+j}\, M_b\, D_k\, M_b(x,y)\, f(y)\, g(x)\, dy\, dx\Big|$$
$$\leq c\, 2^{-|j|\varepsilon}\, 2^{([(k+j)\wedge k]-k_0)n}\, \|f\|_\eta\, |Q|^{1+2\eta/n}\, \|g\|_\eta\,.$$

which yields (2.14). This shows Lemma 2.10.

Now we turn to the proof of Theorem 2.3. Let $\widetilde{D}_k = T_N^{-1} D_k^N$, where D_k^N is defined in (2.7) and N is a fixed large integer such that T_N^{-1} maps the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ to the strong b-smooth molecules of type (β,γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ by Theorem 2.8. It is easy to see that $D_k^N(x,y)$, the kernel of D_k^N, is a strong b-smooth molecule of type $(\varepsilon,\varepsilon)$ centered at y with width $2^{-k} > 0$. Thus, $\widetilde{D}_k(x,y) = T_N^{-1}[D_k^N(\cdot,y)](x)$, the kernel of $\widetilde{D}_k$, is a strong b-smooth molecule of type $(\varepsilon',\varepsilon')$ centered at y with width $2^{-k} > 0$ for $0 < \varepsilon' < \varepsilon$ by Theorem 2.8. This shows that $\widetilde{D}_k(x,y)$ satisfies the conditions (i), (ii) and (iii) of (2.5). The condition (iv) of (2.5) follows from the fact that $(D_k^N)(b) = 0$. All we need to do now is to prove that the series in (2.4) converges in the norm of L^p and $M^{(\beta',\gamma')}$. Suppose first that $f \in M^{(\beta,\gamma)}$. Then the convergence of the series in (2.4) in $M^{(\beta',\gamma')}$ is equivalent to

$$\lim_{M\to\infty} \| \sum_{|k|\leq M} \widetilde{D}_k\, M_b\, D_k\, M_b(f) - f\|_{M^{(\beta',\gamma')}} = 0\,, \tag{2.24}$$

for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$. Since

$$\begin{aligned}\sum_{|k|\leq M} \widetilde{D}_k\, M_b\, D_k\, M_b(f) &= T_N^{-1}\Big(\sum_{|k|\leq M} D_k^N\, M_b\, D_k\, M_b(f)\Big)\\ &= T_N^{-1}\Big(T_N - \sum_{|k|>M} D_k^N\, M_b\, D_k\, M_b(f)\Big)\\ &= f - \lim_{m\to\infty} R_N^m(f)\\ &\quad - T_N^{-1}\Big(\sum_{|k|>M} D_k^N\, M_b\, D_k\, M_b(f)\Big)\,,\end{aligned}$$

to show (2.24), it suffices to prove

$$\lim_{m\to\infty} \|R_N^m(f)\|_{M^{(\beta',\gamma')}} = 0\,, \tag{2.25}$$

$$\lim_{M\to\infty} \Big\|T_N^{-1}\Big(\sum_{|k|>M} D_k^N\, M_b\, D_k\, M_b(f)\Big)\Big\|_{M^{(\beta,'\gamma')}} = 0\,. \tag{2.26}$$

By (2.15),

$$\|R_N^m(f)\|_{M(\beta',\gamma')} \le (c\,2^{-N\delta})^m \,\|f\|_{M(\beta',\gamma')} \le (c\,2^{-N\delta})^m \,\|f\|_{M(\beta,\gamma)} \,,$$

since $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$, which gives (2.25). The proof of (2.26) is based on the following estimate

$$\Big\| \sum_{|k|>M} D_k^N \, M_b \, D_k \, M_b(f) \Big\|_{M(\beta',\gamma')} \le c\,2^{-M\sigma} \, \|f\|_{M(\beta,\gamma)} \,, \tag{2.27}$$

for all $0 < \beta' < \beta, 0 < \gamma' < \gamma$ and some $\sigma > 0$, and a constant c which is independent of f and M.

Assuming (2.27) for the moment, by Theorem 2.8, for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$,

$$\begin{aligned} \Big\|T_N^{-1} \sum_{|k|>M} D_k^N \, M_b \, D_k \, M_b(f)\Big\|_{M(\beta',\gamma')} & \\ &\le c \Big\| \sum_{|k|>M} D_k^N \, M_b \, D_k \, M_b(f)\Big\|_{M(\beta',\gamma')} \\ &\le c\,2^{-M\sigma} \, \|f\|_{M(\beta,\gamma)} \,, \end{aligned}$$

which gives (2.26).

To prove (2.27), it suffices to show that for $0 < \beta'' < \beta$ and $0 < \gamma' < \gamma$ there exist a constant c which is independent of f and M and some $\sigma > 0$ such that

$$\begin{aligned} &\Big| \sum_{|k|>M} D_k^N \, M_b \, D_k \, M_b(f)(x)\Big| \\ &\qquad \le c\,2^{-M\sigma}(1+|x|)^{-(n+\gamma')} \, \|f\|_{M(\beta,\gamma)} \,, \end{aligned} \tag{2.28}$$

$$\begin{aligned} &\Big| \sum_{|k|>M} D_k^N \, M_b \, D_k \, M_b(f)(x) - \sum_{|k|>M} D_k^N \, M_b \, D_k \, M_b(f)(x')\Big| \\ &\qquad \le c \Big(\frac{|x-x'|}{1+|x|}\Big)^{\beta''} \frac{1}{(1+|x|)^{n+\gamma'}} \, \|f\|_{M(\beta,\gamma)} \,, \end{aligned} \tag{2.29}$$

for $|x-x'| \le (1+|x|)/2$.

To see this, by taking the geometric average between (2.29) and the following estimate

$$\begin{aligned}\Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x) - \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x')\Big| \\ \le \Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x)\Big| + \Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x')\Big| \\ \le c\, 2^{-M\sigma}(1+|x|)^{-(n+\gamma')}\|f\|_{M^{(\beta,\gamma)}},\end{aligned}$$

for $|x-x'| \le (1+|x|)/2$, we have

$$(2.30)\qquad \begin{aligned}\Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x) - \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x')\Big| \\ \le c\, 2^{-M\sigma'}|x-x'|^{\beta'}(1+|x|)^{-(n+\gamma')}\, \|f\|_{M^{(\beta,\gamma)}},\end{aligned}$$

for $|x-x'| \le (1+|x|)/2$.

Now (2.28) and (2.30) together with the fact that

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{|k|>M} D_k^N M_b D_k M_b(f)(x)\, b(x)\, dx \\ = \int_{\mathbb{R}^n} \sum_{|k|>M} M_b D_k M_b(f)(x)\, (D_k^N)^*(b)(x)\, dx = 0,\end{aligned}$$

show that

$$\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x) \in M^{(\beta',\gamma')}$$

and

$$\Big\|\sum_{|k|>M} D_k^N M_b D_k M_b(f)\Big\|_{M^{(\beta',\gamma')}} \le c\, 2^{-M\sigma}\|f\|_{M^{(\beta,\gamma)}},$$

which gives (2.27).

Now we prove (2.28). Denote $E_k = D_k^N M_b D_k$. It is easy to check that $E_k(x,y)$, the kernel of E_k, satisfies the conditions (2.2.i), (2.2.ii), and (2.2.iii) with ε replaced by ε', $0<\varepsilon'<\varepsilon$, and $E_k(b)=0$. Consider first the case where $|x| \le 2$, then

$$\begin{aligned}
&\Big|\sum_{|k|>M} D_k^N M_b D_k M_b(f)(x)\Big| \\
&\quad = \Big|\sum_{|k|>M} E_k M_b(f)(x)\Big| \\
&\quad \le \Big|\sum_{k>M} \int_{\mathbb{R}^n} E_k(x,y)\, b(y)\, \big(f(y)-f(x)\big)\, dy\Big| \\
&\qquad + \Big|\sum_{k<-M} \int_{\mathbb{R}^n} E_k(x,y)\, b(y)\, f(y)\, dy\Big| \quad (\text{ since } E_k(b)=0) \\
&\quad \le c \sum_{k>M} 2^{-k\beta} \|f\|_{M(\beta,\gamma)} + c \sum_{k<-M} 2^{kn} \|f\|_{M(\beta,\gamma)} \\
&\quad \le c\,(2^{-M\beta} + 2^{-Mn}) \|f\|_{M(\beta,\gamma)} \\
&\quad \le c\, 2^{-M\sigma} (1+|x|)^{-(n+\gamma')} \|f\|_{M(\beta,\gamma)}\,, \qquad (\text{ since } |x| \le 2)
\end{aligned} \tag{2.31}$$

and, where $\sigma > 0$ is a constant and $0 < \gamma' < \gamma$.

This proves (2.28) for$|x| \le 2$. If $|x| > 2$, then

$$\begin{aligned}
\Big|\sum_{|k|>M} E_k M_b(f)(x)\Big| &\le \Big|\sum_{k>M} \int_{\mathbb{R}^n} E_k(x,y)\, b(y)\big(f(y)-f(x)\big)\, dy\Big| \\
&\quad + \Big|\sum_{k<-M} \int_{\mathbb{R}^n} E_k(x,y)\, b(y)\, f(y)\, dy\Big| = \mathrm{I} + \mathrm{II}\,.
\end{aligned}$$

Since $|x-y| \le c\,2^{-k} < c\,2^{-M}$ for $k > M$ and hence $|x-y| < 1$ if M is larger than $\log_2 c$. This gives that $|y| \ge |x| - |x-y| \ge |x|/2$ for $M > \log_2 c$ and term I is now bounded by a constant times

$$\begin{aligned}
&\sum_{k>M} \int_{\mathbb{R}^n} |E_k(x,y)|\, |x-y|^\beta \\
&\qquad \cdot \big((1+|y|)^{-(n+\gamma)} + (1+|x|)^{-(n+\gamma)}\big)\, dy\, \|f\|_{M(\beta,\gamma)} \\
&\quad \le c \Big(\sum_{k>M} 2^{-k\beta}\Big) (1+|x|)^{-(n+\gamma)} \|f\|_{M(\beta,\gamma)} \\
&\quad \le c\, 2^{-M\beta} (1+|x|)^{-(n+\gamma')} \|f\|_{M(\beta,\gamma)}\,.
\end{aligned} \tag{2.32}$$

To estimate the term II, by use of the fact that $\int_{\mathbb{R}^n} f(y)b(y)dy = 0$, we have

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^n} & E_k(x,y)\, b(y)\, f(y)\, dy\Big| \\
&= \Big|\int_{\mathbb{R}^n} \big(E_k(x,y) - E_k(x,0)\big)\, b(y)\, f(y)\, dy\Big| \\
&\le c \int\limits_{|y|\le |x|/2} |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy \\
&\quad + c \int\limits_{|x|/2<|y|<3|x|/2} |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy \\
&\quad + c \int\limits_{|y|\ge 3|x|/2} |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy\,.
\end{aligned}
\tag{2.33}
$$

Since

$$
|E_k(x,y) - E_k(x,0)| \le c\, 2^{k(n+\varepsilon)} \left(\frac{|y|}{|x|}\right)^{\varepsilon},
$$

the size condition of f yields

$$
\begin{aligned}
\int\limits_{|y|\le |x|/2} & |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy \\
&\le c \int\limits_{|y|\le |x|/2} 2^{k(n+\varepsilon)} \left(\frac{|y|}{|x|}\right)^{\varepsilon} \frac{1}{|y|^{n+\gamma}}\, dy \\
&\qquad \cdot \chi_{\{k:\ 2^k \le c|x|^{-1}\}}\, \|f\|_{M(\beta,\gamma)} \\
&\le c\, 2^{k\varepsilon}\, |x|^{-(n+\gamma)}\, \|f\|_{M(\beta,\gamma)}\,.
\end{aligned}
\tag{2.34}
$$

Similarly,

$$
\begin{aligned}
\int\limits_{|y|\ge 3|x|/2} & |E_k(x,y) - E_k(x,0)|\, |f(y)|\, dy \\
&\le c\, 2^{kn} \chi_{\{k:\ 2^k \le c|x|^{-1}\}}\, |x|^{-\gamma}\, \|f\|_{M(\beta,\gamma)} \\
&\le c\, 2^{k\sigma}\, |x|^{-(n+\gamma')}\, \|f\|_{M(\beta,\gamma)}
\end{aligned}
\tag{2.35}
$$

and

$$\begin{aligned}
&\int_{|x|/2<|y|<3|x|/2} |E_k(x,y) - E_k(x,0)|\,|f(y)|\,dy \\
(2.36)\qquad &\le c \int_{|x|/2<|y|<3|x|/2} \left(|E_k(x,y)| + |E_k(x,0)|\right) |f(y)|\,dy \\
&\le c\,2^{k\sigma}\,|x|^{-(n+\gamma')}\,\|f\|_{M^{(\beta,\gamma)}}\,,
\end{aligned}$$

where $\sigma = \gamma - \gamma' > 0$.

Combining (2.33), (2.34), (2.35) and (2.36) shows

$$\begin{aligned}
\mathrm{II} &\le c \sum_{k<-M} 2^{k\sigma}\,|x|^{-(n+\gamma')}\,\|f\|_{M^{(\beta,\gamma)}} \\
&\le c\,2^{-M\sigma}\,(1+|x|^{-(n+\gamma')}\,\|f\|_{M^{(\beta,\gamma)}}\,,
\end{aligned}$$

which together with (2.31) and (2.32) implies (2.28).

It remains to prove (2.29). We need only to check that

$$\sum_{|k|>M} D_k^N\,M_b\,D_k\,M_b \qquad \text{and} \qquad b^{-1}\Big(\sum_{|k|>M} D_k^N\,M_b\,D_k\,M_b\Big)^{*} M_b\,,$$

as operators, satisfy the hypotheses of Theorem 1.19 and the estimates of the kernels are independent of M. Since

$$b^{-1}\Big(\sum_{|k|>M} D_k^N\,M_b\,D_k\,M_b\Big)^{*} M_b = \sum_{|k|>M} D_k^*\,M_b\,(D_k^N)^*\,M_b\,,$$

and D_k and D_k^* satisfy the same conditions, so it suffices to check that $\sum_{|k|>M} D_k^N M_b D_k M_b$ satisfies the hypotheses of Theorem 1.19 with the constants independent of M. This follows from the simple computation. We leave these details to the reader.

Finally, to see that the series in (2.4) converges in L^p for $1 < p < \infty$, by the proof above, we only need to show that (2.25) and (2.26) still hold with the norm of $M^{(\beta',\gamma')}$ replaced by the norm of L^p for $1 < p < \infty$. The estimates in Lemma 2.10 show that $R_N b^{-1}$ is a Calderón-Zygmund operator with the operator norm at most $c\,2^{-N\delta}$ and hence R_N is bounded on L^p for $1 < p < \infty$ with the operator norm at most $c\,2^{-N\delta}$. This yields (2.25) and also implies that T_N^{-1} is bounded on L^p for $1 < p < \infty$. To see that (2.26) still holds with the norm of $M^{(\beta',\gamma')}$ replaced by the norm of L^p for $1 < p < \infty$, it suffices to show $\lim_{M\to\infty} \| \sum_{|k|>M} D_k^N M_b D_k M_b(f)\|_p = 0$ for $f \in L^p, 1 < p < \infty$. This can be proved by a result in [DJS]. More precisely,

$$\begin{aligned}
&\Big\| \sum_{|k|>M} D_k^N M_b D_k M_b(f)\Big\|_p \\
&= \sup_{\|g\|_{p'}\le 1} \langle \sum_{|k|>M} D_k^N M_b D_k M_b(f), g \rangle \\
&\le \sup_{\|g\|_{p'}\le 1} \Big\|\Big(\sum_{|k|>M} |D_k M_b(f)|^2\Big)^{1/2}\Big\|_p \Big\|\Big(\sum_{|k|>M} |M_b (D_k^N)^*(g)|^2\Big)^{1/2}\Big\|_{p'}
\end{aligned}$$

(by a result of [DJS])

$$\begin{aligned}
&\le c \sup_{\|g\|_{p'}\le 1} \Big\|\Big(\sum_{|k|>M} |D_k M_b(f)|^2\Big)^{1/2}\Big\|_p \|g\|_{p'} \\
&\le c \Big\|\Big(\sum_{|k|>M} |D_k M_b(f)|^2\Big)^{1/2} \|_p ,
\end{aligned}$$

where again by a result of [DJS] the last term tends to zero as M tends to infinity . This ends the proof of Theorem 2.3.

By an argument of duality we obtain the following Calderón-type reproducing formula on $(M^{(\beta,\gamma)})'$:

Theorem 2.37. *Suppose that $(D_k)_{k\in\mathbb{Z}}$ is as in Theorem 2.3. Then there exists a family of operators $(\widetilde{D}_k)_{k\in\mathbb{Z}}$ whose kernels satisfy the same properties as in Theorem 2.3 such that for all $f \in (M^{(\beta,\gamma)})'$,*

$$f = \sum_{k\in\mathbb{Z}} M_b \widetilde{D}_k M_b D_k(f) , \tag{2.38}$$

where the series converges in the sense that for all $g \in M^{(\beta',\gamma')}$ with $\beta' > \beta$ and $\gamma' > \gamma$,

$$\lim_{M\to\infty} \langle \sum_{|k|\le M} M_b \widetilde{D}_k M_b, D_k(f), g \rangle = \langle f, g\rangle . \tag{2.39}$$

We leave the details to the reader.

Section 3.

In this section we introduce a new class of the Besov and Triebel-Lizorkin spaces associated to a para-accretive function, which generalizes the classical Besove and Triebel-Lizorkin spaces on $\mathbb{R}^n$ and prove a Tb theorem on these spaces. We begin with the following proposition.

Proposition 3.1. *Suppose that $(S_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are approximations to the identity defined in* (2.1). *Set $D_k = S_k - S_{k-1}$ and $E_k = Q_k - Q_{k-1}$. Then for all $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$, where ε is the regularity exponent of the approximations to the identity, there exist two constants c_1 and $c_2 > 0$ such that*

$$
\begin{aligned}
c_1 \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|E_k(f)\|_p)^q \Big)^{1/q} &\le \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|D_k(f)\|_p)^q \Big)^{1/q} \\
&\le c_2 \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|E_k(f)\|_p)^q \Big)^{1/q},
\end{aligned} \tag{3.2}
$$

for $-\varepsilon < \alpha < \varepsilon$, $1 \le p, q \le \infty$,

$$
\begin{aligned}
c_1 \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |E_k(f)|)^q \Big)^{1/q} \Big\|_p &\le \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |D_k(f)|_p)^q \Big)^{1/q} \Big\|_p \\
&\le c_2 \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |E_k(f)|^q \Big)^{1/q} \Big\|_p,
\end{aligned} \tag{3.3}
$$

for $-\varepsilon < \alpha < \varepsilon$, $1 < p, q < \infty$.

PROOF. We first prove (3.2). Without loss of generality we may assume that

$$\Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|E_k(f)\|_p)^q \Big)^{1/q} < +\infty$$

and

$$\Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \|D_k(f)\|_p)^q \Big)^{1/q} < +\infty .$$

Since $D_k(\cdot,y) \in M^{(\varepsilon,\varepsilon)}$, by the Calderón-type reproducing formula in (2.37), there exists a family of operators $(\widetilde{E}_j)_{j\in Z}$ such that

$$\begin{aligned} D_k(f)(x) = \langle D_k(x,\cdot), f\rangle &= \langle D_k(x,\cdot), \sum_{j\in\mathbb{Z}} M_b\, \widetilde{E}_j\, M_b\, E_j(f)\rangle \\ &= \sum_{j\in\mathbb{Z}} D_k\, M_b\, \widetilde{E}_j\, M_b\, E_j(f)(x)\,. \end{aligned}$$

Thus,

$$\begin{aligned} \|D_k(f)\|_p &\le \sum_{j\in\mathbb{Z}} \|D_k\, M_b\, \widetilde{E}_j\, M_b\, E_j(f)\|_p \\ &\le \sum_{j\in\mathbb{Z}} \|D_k\, M_b\, \widetilde{E}_j\|_{p,p}\, \|M_b\, E_j(f)\|_p\,. \end{aligned} \tag{3.4}$$

The estimate in (2.17) still holds with D_k replaced by $\widetilde{E}_j$ and hence implies that $\|D_k\, M_b\, \widetilde{E}_j\|_{p,p} \le c\, 2^{-|k-j|\varepsilon''}$. Thus,

$$\begin{aligned} \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha}\, \|D_k(f)\|_p)^q\Big)^{1/q} &\le \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''}\, \|E_j(f)\|_p)^q\Big)^{1/q} \\ &\le c\Big(\sum_{k\in\mathbb{Z}} \big(\sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''+(k-j)\alpha}\big)^{q/q'} \\ &\qquad \cdot \sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''+(k-j)\alpha}\, (2^{j\alpha}\, \|E_j(f)\|_p)^q\Big)^{1/q} \\ &\le c\Big(\sum_{j\in\mathbb{Z}} (2^{j\alpha}\, \|E_j(f)\|_p)^q\Big)^{1/q} < +\infty\,, \end{aligned}$$

since we may choose $-\varepsilon'' < \alpha < \varepsilon''$ and hence

$$\sum_{k\in\mathbb{Z}} 2^{-|k-j|\varepsilon''+(k-j)\alpha} + \sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''+(k-j)\alpha} < +\infty\,.$$

The same proof can be applied to prove the other inequality in (3.2).

To prove (3.3), we will use the Fefferman-Stein vector-valued maximal function inequality. As in the proof above, we have

$$\begin{aligned} |D_k(f)(x)| &\le \sum_{j\in\mathbb{Z}} |D_k\, M_b\, \widetilde{E}_j\, M_b\, E_j(f)(x)| \\ &\le c\sum_{j\in\mathbb{Z}} 2^{-|k-j|\varepsilon''}\, M(M_b E_j(f))(x)\,, \end{aligned}$$

where M is the Hardy-Littlewood maximal function. This gives

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{k\alpha}\,|D_k(f)|)^q\Big)^{1/q}\Big\|_p$$

$$\leq c\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{k\alpha}\sum_{j\in\mathbb{Z}}2^{-|k-j|\varepsilon''}\,M(M_bE_j(f))^q\Big)^{1/q}\Big\|_p$$

$$\leq c\Big\|\Big(\sum_{k\in\mathbb{Z}}\big(\sum_{j\in\mathbb{Z}}2^{-|k-j|\varepsilon''+(k-j)\alpha}\big)^{q/q'}$$

$$\cdot\sum_{j\in\mathbb{Z}}2^{-|k-j|\varepsilon''+(k-j)\alpha}\big(2^{k\alpha}\,M(M_bE_j(f))^q\Big)^{1/q}\Big\|_p$$

$$\leq c\Big\|\sum_{j\in\mathbb{Z}}\big(2^{j\alpha}\,M(M_bE_j(f))^q\Big)^{1/q}\Big\|_p$$

(since $\sum_{k\in\mathbb{Z}}2^{-|k-j|\varepsilon''+(k-j)\alpha}+\sum_{j\in Z}2^{-|k-j|\varepsilon''+(k-j)\alpha}<+\infty$)

$$\leq c\Big\|\Big(\sum_{j\in\mathbb{Z}}2^{j\alpha}\,|M_bE_j(f)|^q\Big)^{1/q}\Big\|_p$$

(by the Fefferman-Stein vector-valued maximal function inequality for $1<p,q<\infty$)

$$\leq c\Big\|\Big(\sum_{j\in Z}2^{j\alpha}\,|E_j(f)|^q\Big)^{1/q}\Big\|_p$$

(since $b\in L^\infty(\mathbb{R}^n)$), which shows one inequality in (3.3). The other inequality in (3.3) can be proved by same manner.

We remark that if the kernels of E_k satisfy the conditions (i), (ii), (iii), and (iv) of (2.5), the first inequality in (3.2) and (3.3) still hold.

The proposition above allows us to introduce the following Besov and Triebel-Lizorkin spaces associated to a para-accretive function.

Definition 3.5. *Suppose that $(S_k)_{k\in\mathbb{Z}}$ is an approximations to the identity defined in* (2.1). *Set $D_k=S_k-S_{k-1}$. The Besov spaces $b\dot{B}_p^{\alpha,q}$, for $-\varepsilon<\alpha<\varepsilon$ and $1\leq p,q\leq\infty$, are the collection of $f\in(M^{(\beta,\gamma)})'$, for $0<\beta,\gamma<\varepsilon$, such that*

$$\|f\|_{b\dot{B}_p^{\alpha,q}}=\Big(\sum_{k\in\mathbb{Z}}(2^{k\alpha}\,\|D_k(f)\|_p)^q\Big)^{1/q}<+\infty\,. \tag{3.6}$$

The Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}$, *for* $-\varepsilon < \alpha < \varepsilon$ *and* $1 < p, q < \infty$, *are the collection of* $f \in (M^{(\beta,\gamma)})'$, *for* $0 < \beta, \gamma < \varepsilon$, *such that*

$$\|f\|_{b\dot{F}_p^{\alpha,q}} = \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |D_k(f)|)^q \Big)^{1/q} \Big\|_p < +\infty . \tag{3.7}$$

To prove the Tb theorem on the Besov spaces $b\dot{B}_p^{\alpha,q}$ and Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}$ we need the following proposition.

Proposition 3.8. *Suppose that* $f \in (M^{(\beta,\gamma)})'$ *with* $0 < \beta, \gamma < \varepsilon$ *and* $\|f\|_{b\dot{B}_p^{\alpha,q}} < \infty$ *for* $-\varepsilon < \alpha < \varepsilon$ *and* $1 \le p, q \le \infty$ *(respectively,* $\|f\|_{b\dot{F}_p^{\alpha,q}} < \infty$ *for* $-\varepsilon < \alpha < \varepsilon$ *and* $1 < p, q < \infty$*). Then there exists a sequence* $\{f_n\}_{n=1}^{\infty}, f_n \in bM^{(\varepsilon',\varepsilon')}$ *with* $0 < \varepsilon' < \varepsilon, n = 1, 2, \ldots,$ *such that*

$$\lim_{n\to\infty} \|f_n - f\|_{b\dot{B}_p^{\alpha,q}} = 0 \qquad (\textit{respectively, } \lim_{n\to\infty} \|f_n - f\|_{b\dot{F}_p^{\alpha,q}} = 0).$$

PROOF. Suppose $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ and $\|f\|_{b\dot{B}_p^{\alpha,q}} < +\infty$ (respectively, $\|f\|_{b\dot{F}_p^{\alpha,q}} < +\infty$). It suffices to show that for any $\delta > 0$, there exists a $g \in bM^{(\varepsilon',\varepsilon')}$ such that

$$\|g - f\|_{b\dot{B}_p^{\alpha,q}} < \delta \qquad (\text{respectively, } \|g - f\|_{b\dot{F}_p^{\alpha,q}} < \delta) .$$

To see this, it follows from the Calderón-type reproducing formula in (2.37) and the proof of (3.1) that

$$\Big\| \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) - f \Big\|_{b\dot{B}_p^{\alpha,q}} \le c \Big(\sum_{|j|>M} (2^{j\alpha} \|D_j(f)\|_p)^q \Big)^{1/q}$$

and hence

$$\lim_{M\to\infty} \Big\| \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) - f \Big\|_{b\dot{B}_p^{\alpha,q}} = 0$$

$$(\text{respectively, } \lim_{M\to\infty} \Big\| \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) - f \Big\|_{b\dot{F}_p^{\alpha,q}} = 0).$$

Now set

$$g_M(x) = \sum_{|j|\le M} \int_{|y|\le M} b(x) \widetilde{D}_j(x,y) b(y) D_j(f)(y)\, dy .$$

It is easy to check that $g_M \in bM^{(\varepsilon',\varepsilon')}$. All we need to do now is to show that for any given $\delta > 0$ there exists $M_0 > 0$ such that for all $M > M_0$

$$\Big\| g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) \Big\|_{b\dot{B}_p^{\alpha,q}} < \delta$$

$$\text{(respectively,} \quad \Big\| g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) \Big\|_{b\dot{F}_p^{\alpha,q}} < \delta).$$

To do this, we have

$$g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f)(x)$$
$$= - \sum_{|j|\le M} \int_{|y|>M} b(x) \widetilde{D}_j(x,y) b(y) D_j(f)(y)\, dy .$$

By the definition (3.5), then it follows from the proof of (3.1) that

$$\Big\| g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) \Big\|_{b\dot{B}_p^{\alpha,q}}$$
$$= \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \Big\| D_k \sum_{|j|\le M} \int_{|y|>M} b(x) \widetilde{D}_j(x,y) b(y) D_j(f)(y)\, dy \Big\|_p)^q \Big)^{1/q}$$

(respectively,

$$\Big\| g_M - \sum_{|j|\le M} M_b \widetilde{D}_j M_b D_j(f) \Big\|_{b\dot{F}_p^{\alpha,q}}$$
$$= \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} |D_k \sum_{|j|\le M} \int_{|y|>M} b(x) \widetilde{D}_j(x,y) b(y) D_j(f)(y)\, dy|)^q \Big)^{1/q} \Big\|_p)$$
$$\le \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \sum_{|j|\le M} \|D_k M_b \widetilde{D}_j\|_{p,p} \int_{|y|>M} |b(y) D_j(f)(y)|\, dy)^q \Big)^{1/q}$$

(respectively,

$$\le \Big\| \Big(\sum_{k\in\mathbb{Z}} (2^{k\alpha} \sum_{|j|\le M} 2^{-|k-j|\varepsilon'} M(\chi_M D_j(f)))^q \Big)^{1/q} \Big\|_p ,$$

where $\chi_M = \chi_{\{y:\ |y|\ge M\}}$)

$$\le c \Big(\sum_{|j|\le M} (2^{j\alpha} (\int_{|y|>M} |D_j(f)(y)|^p\, dy)^{1/p})^q \Big)^{1/q}$$

(respectively,

$$\leq c\Big\| \sum_{|j|\leq M} \big(2^{k\alpha}\,|\chi_M\, D_j(f)|^q\big)^{1/q}\Big\|_p).$$

Since $\|f\|_{b\dot{B}_p^{\alpha,q}} < +\infty$ (respectively, $\|f\|_{b\dot{F}_p^{\alpha,q}} < +\infty$), so that there exists $M_1 > 0$ such that for all $M \geq M_1$,

$$\Big(\sum_{|j|\geq M} \big(2^{j\alpha}\,\|D_j(f)\|_p\big)^q\Big)^{1/q} < \frac{1}{2c}\,\delta$$

$$\text{(respectively, } \Big\|\Big(\sum_{|j|\geq M} \big(2^{j\alpha}\,\|D_j(f))^q\Big)^{1/q}\Big\|_p < \frac{1}{2c}\,\delta\,).$$

It is easy to see that there exists $M_0 > M_1$ such that for all $M \geq M_0$

$$\Big(\sum_{|j|\leq M_0} \big(2^{j\alpha}\big(\int_{|y|>M} |D_j(f)(y)|^p\,dy\big)^{1/p}\big)^q\Big)^{1/q} < \frac{1}{2c}\,\delta$$

$$\text{(respectively, } \Big\|\Big(\sum_{|j|\leq M_0} \big(2^{j\alpha}\,|\chi_M\, D_j(f)|\big)^q\Big)^{1/q}\Big\|_p < \frac{1}{2c}\,\delta\,).$$

Thus, for $M \geq M_0$ we then have

$$\begin{aligned}
&\Big\|g_M - \sum_{|j|\leq M} M_b\,\widetilde{D}_j\,M_b\,D_j(f)\Big\|_{\dot{B}_p^{\alpha,q}(b)} \\
&\leq c\Big(\sum_{|j|\leq M} \big(2^{j\alpha}\big(\int_{|y|>M} |D_j(f)(y)|^p\,dy\big)^{1/p}\big)^q\Big)^{1/q} \\
&\leq c\Big(\sum_{|j|\geq M_0} \big(2^{j\alpha}\,\|D_j(f)\|_p\big)^q\Big)^{1/q} \\
&\quad + c\Big(\sum_{|j|\leq M_0} \big(2^{j\alpha}\big(\int_{|y|>M} |D_j(f)(y)|^p\,dy\big)^{1/p}\big)^q\Big)^{1/q} \\
&\leq c\,\frac{1}{2c}\,\delta + c\,\frac{1}{2c}\,\delta = \delta\,.
\end{aligned}$$

(respectively,

$$\begin{aligned}
\Big\|g_M - \sum_{|j|\le M} M_b\,\widetilde{D}_j\,M_b\,D_j(f)\Big\|_{b\dot F_p^{\alpha,q}}
&\le c\Big\|\Big(\sum_{|j|\le M}(2^{j\alpha}\,|\chi_M\,D_j(f)|)^q\Big)^{1/q}\Big\|_p\\
&\le c\Big\|\Big(\sum_{|j|\ge M_0}(2^{j\alpha}\,|D_j(f)|)^q\Big)^{1/q}\Big\|_p\\
&\quad + c\Big\|\Big(\sum_{|j|\le M_0}(2^{j\alpha}\,\chi_M\,D_j(f)|)^q\Big)^{1/q}\Big\|_p\\
&\le c\frac{1}{2c}\,\delta + c\frac{1}{2c}\,\delta = \delta\,).
\end{aligned}$$

This completes the proof of Proposition 3.8.

We now state the Tb theorem for $b\dot B_p^{\alpha,q}$ and $b\dot F_p^{\alpha,q}$. First notice that if T is a singular integral operator defined in (1.25), then T can be extended to a continuous linear operator from $bM^{(\beta,\gamma)}, 0<\beta,\gamma$, to $(bC_0^\eta)'$. To see this, let $f\in M^{(\beta,\gamma)}$ with $0<\beta,\gamma$ and $g\in bC_0^\eta(\mathbb{R}^n)$, and choose $\theta\in C_0^1(\mathbb{R}^n)$ with $\theta(x)=1$ for $x\in\operatorname{supp} g$, we then define

$$\langle Tbf, g\rangle = \langle T(bf\theta), g\rangle + \langle T(bf(1-\theta)), g\rangle\,. \tag{3.9}$$

Since $bf\theta\in bC_0^\eta(\mathbb{R}^n)$, so $\langle Tbf\theta, g\rangle$ is well defined. Using the facts that $bf(1-\theta)$ and g belong to $L^1(\mathbb{R}^n)$ and $\operatorname{supp} bf(1-\theta)\cap\operatorname{supp} g=\varnothing$, then

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} K(x,y)\,b(y)\,f(y)\,(1-\theta(y))\,g(x)\,dy\,dx$$

converges absolutely, and hence $\langle T(bf(1-\theta)), g\rangle$ is well defined. It is also easy to check that (3.9) is independent of the choice of θ.

Theorem 3.10. *Suppose that T is a singular integral operator defined in (1.25) and $T(b)=T^*(b)=0$, and M_bTM_b has the weak boundedness property. Then T can be extended to a bounded operator from $b\dot B_p^{\alpha,q}$ to $b^{-1}\dot B_p^{\alpha,q}$ for $-\varepsilon<\alpha<\varepsilon$, $1\le p,q\le\infty$, and from $b\dot F_p^{\alpha,q}$ to $b^{-1}\dot F_p^{\alpha,q}$ for $-\varepsilon<\alpha<\varepsilon$, $1<p,q<\infty$, where $f\in b^{-1}\dot B_p^{\alpha,q}$ if and only if*

$bf \in b\dot{B}_p^{\alpha,q}$, and $f \in b^{-1}\dot{F}_p^{\alpha,q}$ if and only if $bf \in b\dot{F}_p^{\alpha,q}$, and ε is the regularity exponent of the kernel of T.

PROOF. By Proposition 3.8, $bM^{(\varepsilon',\varepsilon')}$, $0 < \varepsilon' < \varepsilon$, is dense in $b\dot{B}_p^{\alpha,q}$ for $-\varepsilon' < \alpha < \varepsilon'$ and $1 \le p, q \le \infty$, and in $b\dot{F}_p^{\alpha,q}$ for $-\varepsilon' < \alpha < \varepsilon'$, $1 < p, q < \infty$. Thus, it suffices to show that for $f \in bM^{(\varepsilon',\varepsilon')} \cap b\dot{B}_p^{\alpha,q}$ with $-\varepsilon' < \alpha < \varepsilon'$ and $1 \le p, q \le \infty$,

$$\|Tf\|_{b^{-1}\dot{B}_p^{\alpha,q}} \le c\,\|f\|_{b\dot{B}_p^{\alpha,q}} \tag{3.11}$$

and for $f \in bM^{(\varepsilon',\varepsilon')} \cap b\dot{F}_p^{\alpha,q}$ with $-\varepsilon' < \alpha < \varepsilon'$ and $1 < p, q < \infty$,

$$\|Tf\|_{b^{-1}\dot{F}_p^{\alpha,q}} \le c\,\|f\|_{b\dot{F}_p^{\alpha,q}}\ , \tag{3.12}$$

where c is a constant which is independent of f.

To do this, since T can be extended to a continuous linear operator from $bM^{(\beta,\gamma)}, 0 < \beta, \gamma < \varepsilon$, to $(bC_0^\eta)'$, for $f \in bM^{(\varepsilon',\varepsilon')} \cap b\dot{B}_p^{\alpha,q}$ with $0 < \beta, \gamma < \varepsilon'$ we then have

$$Tf = \sum_{k\in\mathbb{Z}} T\,M_b\,D_k\,M_b\,\widetilde{D}_k(f) \qquad \text{in } \ (bC_0^\eta)'\,,$$

since $M_b^{-1}f \in M^{(\varepsilon',\varepsilon')}$ and hence, by Theorem 2.3,

$$M_b^{-1}f = \sum_{k\in\mathbb{Z}} D_k\,M_b\,\widetilde{D}_k\,M_b\,(M_b^{-1}f) = \sum_{k\in\mathbb{Z}} D_k\,M_b\,\widetilde{D}_k(f)$$

in the norm of $M^{(\beta,\gamma)}$ with $0 < \beta, \gamma < \varepsilon'$. Therefore,

$$\begin{aligned}\|Tf\|_{b^{-1}\dot{B}_p^{\alpha,q}} &= \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\,\|D_j\,M_b(Tf)\|_p\big)^q\Big)^{1/q}\\ &= \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\,\|D_j\,M_b \sum_{k\in\mathbb{Z}} T\,M_b\,D_k\,M_b\,\widetilde{D}_k(f)\|_p\big)^q\Big)^{1/q}\end{aligned}$$

(since $D_j(\cdot,y)b(y) \in bC_0^\eta$)

$$\le c\Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \sum_{k\in\mathbb{Z}} \|D_j\,M_b\,T\,M_b\,D_k\|_{p,p}\,\|\widetilde{D}_k(f)\|_p\big)^q\Big)^{1/q}$$

To estimate the last term above, we need the following lemma (see [HS1]).

Lemma 3.13. *Suppose that T satisfies the hypotheses of Theorem 3.10. Then for $0 < \varepsilon' < \varepsilon$ there exists a constant $c > 0$ such that $D_j\, M_b\, T\, M_b\, D_k(x, y)$, the kernel of $D_j\, M_b\, T\, M_b\, D_k$, satisfies the following estimate:*

$$|D_j\, M_b\, T\, M_b\, D_k(x,y)| \le c\, 2^{-|k-j|\varepsilon'} \frac{2^{-(k\wedge j)\varepsilon'}}{(2^{-(k\wedge j)} + |x-y|)^{n+\varepsilon'}} \, . \tag{3.14}$$

Assuming Lemma 3.13 for the moment, we have

$$\begin{aligned}
\|Tf\|_{b^{-1}\dot{B}_p^{\alpha,q}} &\le c \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \sum_{k\in\mathbb{Z}} 2^{-|k-j|\varepsilon} \, \|\widetilde{D}_k(f)\|_p\big)^q \Big)^{1/q} \\
&\le c \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha} \, \|\widetilde{D}_k(f)\|_p\big)^q \Big)^{1/q} \\
&\le c \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha} \, \|D_k(f)\|_p\big)^q \Big)^{1/q} = c\, \|f\|_{b\dot{B}_p^{\alpha,q}}
\end{aligned}$$

by the remark following (3.1). Similarly,

$$\begin{aligned}
\|Tf\|_{b^{-1}\dot{F}_p^{\alpha,q}} &\le c \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \sum_{k\in\mathbb{Z}} D_j\, M_b\, T\, M_b\, D_k\, M_b\, \widetilde{D}_k(f)|\big)^q \Big)^{1/q} \Big\|_p \\
&\le c \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha} \sum_{k\in\mathbb{Z}} 2^{-|k-j|\varepsilon}\, M(M_b\, \widetilde{D}_k(f))\big)^q \Big)^{1/q} \Big\|_p \\
&\le c \Big\| \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\, M(M_b\, \widetilde{D}_k(f))\big)^q \Big)^{1/q} \Big\|_p \\
&\le c \Big\| \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\, |b\, \widetilde{D}_k(f)|\big)^q \Big)^{1/q} \Big\|_p
\end{aligned}$$

(by the Fefferman-Stein vector-valued maximal function inequality)

$$\begin{aligned}
&\le c \Big\| \Big(\sum_{k\in\mathbb{Z}} \big(2^{k\alpha}\, |D_k(f)|\big)^q \Big)^{1/q} \Big\|_p \\
&= c\, \|f\|_{b\dot{F}_p^{\alpha,q}}
\end{aligned}$$

by the remark following (3.1).

All we need to do now is to show Lemma 3.13. We prove the estimate (3.14) in the crucial case where $j \geq k$ and $|x-y| \leq c\,2^{-k}$. The three remaining cases: $j \geq k$ and $|x-y| > c\,2^{-k}$, $j < k$ and $|x-y| \leq c\,2^{-j}$, $j < k$ and $|x-y| > c\,2^{-j}$, are similar or easier. Let $\eta_0 \in C^\infty(\mathbb{R}^n)$ be 1 on the unit ball and 0 outside its double. Set $\eta_1 = 1 - \eta_0$. Then, following the proof of Lemma 7 in Section 6 of [DJS], we have

$$\begin{aligned}
&D_j\, M_b\, T\, M_b\, D_k(x,y) \\
(3.15)\qquad &= \iint D_j(x,u)\, b(u)\, K(u,v)\, b(v)\, D_k(v,y)\, du\, dv \\
&= \iint D_j(x,u)\, b(u)\, K(u,v)\, b(v)\, \big(D_k(v,y) - D_k(x,y)\big)\, du\, dv
\end{aligned}$$

since $T(b) = 0$, so

$$\begin{aligned}
&D_j\, M_b\, T\, M_b\, D_k(x,y) \\
&\qquad = \iint D_j(x,u)\, b(u)\, K(u,v)\, b(v) \\
&\qquad\qquad \cdot \big(D_k(v,y) - D_k(x,y)\big)\, \eta_0\Big(\frac{v-x}{c\,2^{-j}}\Big)\, du\, dv \\
(3.16)\qquad &\qquad + \iint D_j(x,u)\, b(u)\, \big(K(u,v) - K(x,v)\big)\, b(v) \\
&\qquad\qquad \cdot \big(D_k(v,y) - D_k(x,y)\big)\, \eta_1\Big(\frac{v-x}{c\,2^{-j}}\Big)\, du\, dv \\
&\qquad = \mathrm{I} + \mathrm{II},
\end{aligned}$$

since $1 = \eta_0 + \eta_1$ and $D_j(b) = 0$. Now with $\psi(u) = D_j(x,u)$ and $\phi(v) = (D_k(v,y) - D_k(x,y))\eta_0((v-x)/c\,2^{-j})$,

$$\begin{aligned}
|\mathrm{I}| &= |\langle M_b\, T\, M_b \phi, \psi\rangle| \\
&\leq c\,2^{-j(n+2\eta)}\, \|\phi\|_{\mathrm{Lip}\eta}\, \|\psi\|_{\mathrm{Lip}\eta}
\end{aligned}$$

(by the weak boundedness property of $M_b T M_b$)

$$\begin{aligned}
&\leq c\,2^{-j(n+2\eta)}\big(2^{(k-j)\varepsilon}\, 2^{kn}\, 2^{j\eta}\big)\big(2^{-jn} 2^{\eta j}\big) \\
&\leq c\,2^{(k-j)\varepsilon}\, 2^{kn}\,,
\end{aligned}$$

which is dominated by the right side of (3.14) for the case where $j \geq k$ and $|x-y| \leq c2^{-k}$. Using the smoothness of $K(u,v)$ together with

$$|D_k(v,y) - D_k(x,y)| \leq c\,2^{k(n+\varepsilon)}\,|v-x|^{\varepsilon}\,\chi_{\{|v-y|\leq c\,2^{-k}\}\cup\{|x-y|\leq c\,2^{-k}\}}\,,$$

we have

$$\begin{aligned}
|\mathrm{II}| \leq c \iint\limits_{\substack{\{|v-y|\leq c\,2^{-k}\}\cup\{|x-y|\leq 2^{-k}\}\\ |v-x|\geq c\,2^{-j}}} & |D_j(x,u)|\,\frac{|u-x|^{\varepsilon}}{|v-x|^{n+\varepsilon}} \\
& \cdot 2^{k(n+\varepsilon)}\,|v-x|^{\varepsilon}\,du\,dv \\
\leq c\,2^{-j\varepsilon}\,2^{kn} & \int\limits_{|v-x|\geq c\,2^{-k}} |v-x|^{-(n+\varepsilon)}\,dv \\
+ c\,2^{(k-j)\varepsilon}\,2^{kn} & \int\limits_{c\,2^{-k}\geq|v-x|\geq c\,2^{-j}} |v-x|^{-n}\,dv \\
\leq c\,2^{(j-k)\varepsilon'}\,2^{kn}\,, &
\end{aligned}$$

which again is dominated by the right side of (3.14) for the case where $j \geq k$ and $|x-y| \leq c\,2^{-k}$. This proves (3.14) for this crucial case and completes the proof of Theorem 3.13.

Section 4.

We remark that our Calderón-type reproducing formula still holds if the conditions on the approximation to the identity are replaced by the following more general conditions:

(i) $$|S_k(x,y)| \leq c\,\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}\,;$$

(ii) $$\begin{aligned}
&|S_k(x,y) - S_k(x,y')| \\
&\quad \leq c\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}\,,
\end{aligned}$$

for $|y-y'| \le (2^{-k}+|x-y|)/2$;

$$\text{(iii)}\quad |S_k(x,y)-S_k(x',y)| \le c\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}},$$

for $|x-x'| \le (2^{-k}+|x-y|)/2$;

$$\text{(iv)}\quad \begin{aligned}&|(S_k(x,y)-S_k(x',y)-(S_k(x,y')-S_k(x',y'))|\\ &\le c\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}},\end{aligned}$$

for $|x-x'| \le (2^{-k}+|x-y|)/3$ and $|x-x'| \le (2^{-k}+|x-y|)/3$;

$$\text{(v)}\quad \int_{\mathbb{R}^n} S_k(x,y)\,b(y)\,dy = 1, \qquad \text{for all } k\in\mathbb{Z} \text{ and } x\in\mathbb{R}^n;$$

$$\text{(vi)}\quad \int_{\mathbb{R}^n} S_k(x,y)\,b(x)\,dx = 1, \qquad \text{for all } k\in\mathbb{Z} \text{ and } y\in\mathbb{R}^n.$$

Using our Calderón-type reproducing formula associated to a para-accretive function one can prove the atomic decomposition, duality, and interpolation for $b\dot{B}_p^{\alpha,q}$ and $b\dot{F}_p^{\alpha,q}$ as for the classical Besov and Triebel-Lizorkin spaces. Since the Fourier transform, translation and dilation are not used so all results in this paper can be generalized to spaces of homogeneous type introduced in [CW]. We will discuss these details elsewhere.

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The Hilbert transform and maximal function along nonconvex curves in the plane

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1. Introduction.

In this paper we study the Hilbert transform and maximal function related to a curve in $\mathbb{R}^2$. For $\Gamma(t) = (t, \gamma(t))$ with $\gamma(0) = 0$, we define the Hilbert transform associated to $\Gamma(t)$ by

$$\mathcal{H}_\Gamma f(x) = \int_{-1}^{1} f(x - \Gamma(t)) \frac{dt}{t} . \tag{1}$$

Similarly we define the maximal function by the formula

$$\mathcal{M}_\Gamma f(x) = \sup_{0<h\leq 1} \frac{1}{h} \int_0^h |f(x - \Gamma(t))| \, dt . \tag{2}$$

We are interested in obtaining L^p estimates of the form

$$\|\mathcal{H}_\Gamma f\|_p \leq A_p \, \|f\|_p , \tag{3}$$

and

$$\|\mathcal{M}_\Gamma f\|_p \leq A_p \, \|f\|_p . \tag{4}$$

A first stage in this study was completed in the 1970's due to the efforts of Nagel, Rivière, Stein and Wainger. Their work led to the following theorem (see [SW]).

Theorem A. *Suppose* $\Gamma(t)$ *is* C^∞ *and the curvature of* $\Gamma(t)$ *does not vanish to infinite order at the origin. Then*

$$\|\mathcal{H}_\Gamma f\|_p \le A_p \|f\|_p , \qquad 1 < p < \infty, \tag{5}$$

and

$$\|\mathcal{M}_\Gamma f\|_p \le A_p \|f\|_p , \qquad 1 < p \le \infty. \tag{6}$$

Much effort has been devoted to the study of $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$ without the assumption that the curvature of Γ does not vanish to infinite order. See for example, [CCVWW], [C1], [C2], [CCC], [DR], [NVWW1] and [NVWW2]. In particular, we have the following theorems (see [CCC]).

Theorem B. *Assume* $\gamma(t)$ *is convex for* $t > 0$. *If for some* $C > 1$, $\gamma'(C t) \ge 2\gamma'(t)$ *for* $t > 0$, *then*

$$\|\mathcal{M}_\Gamma f\|_p \le A_p \|f\|_p , \qquad 1 < p \le \infty.$$

If in addition, $\Gamma(t)$ *is even or odd, then*

$$\|\mathcal{H}_\Gamma f\|_p \le A_p \|f\|_p , \qquad 1 < p < \infty.$$

The hypothesis of the next theorem is expressed in terms of the functional $h(t) = t\gamma'(t) - \gamma(t)$ (see [NVWW1] and [NVWW2]).

Theorem C. *Assume* $\gamma(t)$ *is convex for* $t > 0$. *If for some* $C > 1$, $h(C t) \ge 2 h(t)$ *for* $t > 0$, *then*

$$\|\mathcal{M}_\Gamma f\|_2 \le A \|f\|_2 .$$

If in addition $\Gamma(t)$ *is odd, then*

$$\|\mathcal{H}_\Gamma f\|_2 \le A \|f\|_2 .$$

In this paper we wish to remove the convexity assumption. The fact that there should be some positive results is suggested by examples worked out by Wright, [W]. In [W], positive results are obtained if for example $\gamma(t)$ is $t^\alpha \sin(\log\log(1/t))$, $t^\alpha \sin(1/t^\beta)$ and $e^{-1/t^2}\sin(e^{1/t})$. These examples and Theorem B suggest the following provisional hypothesis. Let

$$u(t) = \sup_{0\le s\le t} |\gamma'(s)|$$

and assume for $t > 0$, $u(Ct) \ge 2\,u(t)$ for some $C > 1$. The fact that this hypothesis must be modified can be seen by considering certain "staircase" examples. That is, we define $\gamma(t) = 2^{-2^k}$ on $E_k = [2^{-k}, 2^{-k}(1+\delta_k)]$ and make γ linear on the complementary intervals, $F_k = [2^{-k}(1+\delta_k), 2^{-k+1}]$. Here $0 \le \delta_k \le 1$. These examples are a slight variant of examples considered in [SW].

For these examples we calculate $\mathcal{M}_\Gamma$ on the characteristic function of a rectangle with corners at $(-1,0)$, $(0,0)$, $(-1,-\varepsilon)$ and $(0,-\varepsilon)$. This calculation which is similar to that in [SW] shows that $\mathcal{M}_\Gamma$ is not bounded in L^p if $\sum \delta_k^p = +\infty$. It is easy to see that in these examples $u(2t) \ge 2\,u(t)$, and so the provisional hypothesis must be modified. Furthermore, it is not difficult to see that if $\sum \delta_k^p < +\infty$, $\mathcal{M}_\Gamma$ is bounded in L^p. In fact

$$\begin{aligned}\mathcal{M}_\Gamma f(x) &\le \sup_{0<h\le 1} \int_{\substack{0\\ t\in\cup E_k}}^h |f(x-\Gamma(t))|\,dt + \sup_{0<h\le 1} \frac{1}{h}\int_{\substack{0\\ t\in\cup F_k}}^h |f(x-\Gamma(t))|\,dt\\ &= M_1 f(x) + M_2 f(x)\,.\end{aligned}$$

M_1 can be shown to be bounded in L^p by using a square function argument as in [C2] while M_2 is bounded in L^p by arguments in [CCC]. In fact the argument shows that $\mathcal{M}_\Gamma$ is bounded in L^p no matter how Γ is defined on the intervals $\{E_k\}$. Thus these staircase examples suggest that we must add some hypothesis but that we need not require any hypothesis on a suitably small set E. The staircase examples further suggest that if $I_k = \{t :\ 2^{-k} \le t \le 2^{-k+1}\}$, then the correct assumption on the size of E should be that $\sum (2^k\,|I_k \cap E|)^p < +\infty$.

It is interesting to note that although $\sum(2^k\,|I_k\cap E|)^p < +\infty$ is the correct assumption on the size of E for the maximal function, it is not the correct size for the Hilbert transform. To see this let us consider the following variant of the above staircase example. That is, define $\gamma(t) = 9^{-2k}$ on $E_k = [2^{-k}, 2^{-k}(1+\delta_k)]$ and make γ linear on the complementary intervals $F_k = [2^{-k}(1+\delta_k), 2^{-k+1}]$. Extend γ as an odd function on $[-1,1]$ and write

$$\begin{aligned}\mathcal{H}_\Gamma f(x) &= \int_{\cup E_k^1} f(x-\Gamma(t))\,\frac{dt}{t} + \int_{\cup F_k^1} f(x-\Gamma(t))\,\frac{dt}{t}\\ &= H_1 f(x) + H_2 f(x)\,,\end{aligned}$$

where $E_k^1 = E_k \cup E_{-k}$ and $F_k^1 = F_k \cup F_{-k}$. As before H_2 is bounded in all L^p $(p>1)$ by arguments in [CCC]. On the other hand, H_1 (and thus $\mathcal{H}_\Gamma$) is bounded in all L^p $(p>1)$ if and only if $\sum \delta_k < +\infty$. It is easy to see that if $\sum \delta_k < +\infty$, then H_1 is bounded in all L^p $(p>1)$ by Minkowsky's inequality for integrals. However suppose that $\sum \delta_k = +\infty$ and consider the multiplier for H_1 ,

$$m(\xi,\eta) = \sum \int_{E_k} \sin(\xi t + \eta\,\gamma(t))\,\frac{dt}{t}\,.$$

Set $\xi = 0$ and $\eta = (\pi/2)\,9^{2N}$ for some large N and note that

$$m(0, 9^{2N}\,\frac{\pi}{2}) = \sum_{k<N} \int_{E_k} \sin(9^{2(N-k)}\,\frac{\pi}{2})\,\frac{dt}{t} + \sum_{k\geq N} \int_{E_k} \sin(9^{2(N-k)}\,\frac{\pi}{2})\,\frac{dt}{t}\,.$$

Since

$$\sum_{k\geq N} \int_{E_k} \sin(9^{2(N-k)}\,\frac{\pi}{2})\,\frac{dt}{t} \leq C\,9^{2N} \sum_{k\geq N} \delta_k\, 9^{-2k} \leq C\,9^{2N} \sum_{k\geq N} 9^{-2k} \leq C$$

and

$$\sum_{k<N} \int_{E_k} \sin(9^{2(N-k)}\,\frac{\pi}{2})\,\frac{dt}{t} = \sum_{k<N} \int_{E_k} \frac{dt}{t} \geq \frac{1}{2} \sum_{k<N} \delta_k\,,$$

we see that $m(\xi,\eta)$ is an unbounded function and so H_1 is not bounded in L^2 and hence unbounded in all L^p. This example therefore suggests that the correct assumption on the size of E for the Hilbert transform is $\sum 2^k\,|I_k \cap E| < +\infty$.

Since we want to impose no condition on $\gamma(t)$ in E, we modify $u(t)$ to

$$v(t) = \sup_{\substack{s<t\\ s\notin E}} |\gamma'(s)|\,.$$

If $\gamma(t)$ is convex, $\gamma'(t)$ is monotone and $\gamma(t) \leq t\,\gamma'(t)$. So if we set

$$\phi(t) = \sup_{\substack{s<t \\ s\notin E}} |\gamma(s)|\,,$$

we add two more provisional hypotheses, namely that outside of E,

$$\gamma'(t) \geq \varepsilon\, v(t) \qquad \text{and} \qquad \phi(t) \leq C\,t\,v(t)\,.$$

Thus one has modified the provisional hypothesis to the following:

A) There is an exceptional set E such that for the maximal function, $\sum (2^k\,|I_k \cap E|)^p < +\infty$ and for the Hilbert transform,

$$\sum 2^k\,|I_k \cap E| < +\infty\,.$$

B) Outside of E, $\phi(t) \leq C\,t\,v(t)$.

C) Outside of E, $v(\lambda t) \geq 2\,v(t)$ for some $\lambda > 1$.

D) Outside of E, $|\gamma'(t)| \geq \varepsilon\, v(t)$.

Unfortunately, for the example $\gamma(t) = t^k \sin(1/t)$, the hypothesis D) is not satisfied. So we replace D) by

D′) Outside of E, $|\gamma(t)| + t\,|\gamma''(t)| \geq \varepsilon\, v(t)$.

It turns out, as we shall see by an example later on, that A), B), C) and D′) do not suffice for the L^2 boundedness of the Hilbert transform (if Γ is extended to be an odd curve).

If one attemps to prove a positive result, one naturally divides I_k into various subintervals; subintervals which belong to E, subintervals which do not belong to E and $|\gamma'(t)| \geq \varepsilon\, v(t)$, and subintervals which do not belong to E and $|\gamma'(t)| < \varepsilon\, v(t)$ but γ'' is large. Thus I_k is partitioned into a possibly large number of subintervals. Our examples show that at least in the case of the Hilbert transform, our hypothesis must depend qualitatively on the number of such subintervals. If the number of subintervals into which we have divided I_k is N_k, we might then expect to modify B) to

B') On $I_k \setminus E$, $t\,v(t) \geq \varepsilon_0\, N_k\, \phi(t)$.

This latter assumption however is not satisfied for certain examples like $\gamma(t) = e^{-1/t^2} \sin e^{1/t}$. Examples such as this can be incorporated by modifying B') to $t\,v(t/2) \geq \varepsilon_0\, N_k\, \phi(t/2)$. It turns out that the proof requires one additional hypothesis, namely that the sequence $\{2^k N_k\}$ is sufficiently spread out. In fact we will assume that for all n,

$$\sum_{k\geq n} \frac{1}{2^k N_k} \leq C\, \frac{1}{2^n N_n} \tag{7}$$

where C is independent of n. (7) holds whenever $\{2^k N_k\}$ forms an increasing lacunary sequence. However condition (7) allows for situations where the N_k's might not be monotone. With the above remarks in mind, the following theorem seems reasonable.

Main Theorem. *Let* $\Gamma(t) = (t, \gamma(t)) \in C^2(0,1]$ *with* $\gamma(0) = \gamma'(0) = 0$. *Suppose*

$$I_k = E_k \cup F_k \cup G_k$$

is a disjoint union with F_k *and* G_k *each a union of at most* N_k *open intervals.*

Assume that for some $\varepsilon_0 > 0$,

$$v(\lambda t) \geq 2\, v(t) \qquad \text{for some} \ \ \lambda > 1 \ \ \text{on} \ \ I_k \setminus E_k \tag{8}$$

and

$$t\, v\big(\frac{t}{2}\big) \geq \varepsilon_0\, N_k\, \phi\big(\frac{t}{2}\big) \qquad \text{on} \ \ I_k \setminus E_k\,. \tag{9}$$

Suppose also that for some ε_1 *and* $\varepsilon_2 > 0$,

$$|\gamma'(t)| > \varepsilon_1\, v(t) \quad \text{and} \quad |t\,\gamma''(t)| < \varepsilon_2\, N_k\, v\big(\frac{t}{2}\big) \quad \text{on} \ \ F_k \tag{10}$$

and

$$|t\,\gamma''(t)| > \varepsilon_2\, N_k\, v\big(\frac{t}{2}\big) \quad \text{on} \ \ G_k\,. \tag{11}$$

Finally assume that (7) *holds. Then if* $\sum_k (2^k\, |E_k|)^p < +\infty$,

$$\|\mathcal{M}_\Gamma f\|_p \leq A_p\, \|f\|_p\,. \tag{12}$$

Also if $\Gamma(t)$ *is extended to the interval* $[-1, 1]$ *as an even or odd curve and* $\sum_k 2^k\, |E_k| < +\infty$,

$$\|\mathcal{H}_\Gamma f\|_p \leq A_p\, \|f\|_p\,, \qquad 1 < p < \infty\,. \tag{13}$$

We add six further remarks.

REMARK 1. In view of Theorem C we might be tempted to replace the hypothesis $v(Ct) \geq 2\,v(t)$ by $w(Ct) \geq 2\,w(t)$ (at least in the case $p = 2$) where

$$w(t) = \sup_{\substack{s \leq t \\ s \notin E}} |s\,\gamma'(s) - \gamma(s)|\,.$$

We shall show by an example that this can not be done.

REMARK 2. Examples show that the complement of the set $\{t \in I_k : |\gamma'(t)| < \varepsilon\, u(t)\}$ is too large to be contained in E_k.

REMARK 3. The staircase curves can be modified to show that for any p_0, $1 < p_0 < \infty$, there is a smooth curve $\Gamma(t)$ so that $\mathcal{M}_\Gamma$ is bounded in L^p for $p \geq p_0$ and unbounded for $p < p_0$. Other examples have been pointed out by M. Wierdl.

REMARK 4. We are not sure if the conclusion of the main theorem holds if $v(t)$ is replaced by $u(t)$ and the N_k's are omitted. However, the hypothesis of such a theorem would not be satisfied by staircase examples with very steep slopes in E, for which we know the conclusion is true.

REMARK 5. We do not know whether the quantitative hypothesis on N_k is necessary for the conclusions concerning the maximal function.

REMARK 6. For convex curves, the hypotheses of the main theorem are satisfied whenever γ' is infinitesimal doubling, *i.e.* $\gamma'(t) \leq C\,t\,\gamma''(t)$.

2. Proof of the main theorem.

We consider first the maximal function. Let us first reduce the problem to obtaining the L^p estimate for

$$Mf = \sup_k |M_k f|\,,$$

where

$$M_k f(x,y) = 2^k \int_{I_k \setminus E_k} f(x-t, y-\gamma(t))\,dt\,,$$

by using the square function argument alluded to above. In fact if μ_k denotes the positive measure such that

$$\mu_k(f) = \frac{1}{|E_k|} \int_{E_k} f(t, \gamma(t))\, dt\,,$$

then for $f \geq 0$, $\mathcal{M}_\Gamma f \leq C\left(Mf + \sup_k (2^k\,|E_k|\, f * \mu_k)\right)$. But

$$\begin{aligned}\|\sup_k (2^k\,|E_k|\, f * \mu_k)\|_p &\leq C\Big(\sum_k \|2^k\,|E_k|\, f * \mu_k\|_p^p\Big)^{1/p}\\ &\leq C\Big(\sum_k (2^k\,|E_k|)^p\Big)^{1/p} \|f\|_p \leq C\,\|f\|_p\,.\end{aligned}$$

Therefore it suffices to prove that M is bounded in L^p. In fact we will show that M is founded in L^p for all $p > 1$. This will be done by following ideas from Christ [C3] and Wright [W].

We will decompose M_k into a sum of four operators. To do this, let

$$R_k = \{\zeta = (\xi, \eta) \in \mathbb{R}^2 :\ v(2^{-k-N}) \leq |\xi/\eta| \leq v(2^{-k+N})\}\,,$$

where N is some large number to be determined later and define

$$S_k f = (\chi_{R_k} \cdot \hat{f})^{\vee}\,.$$

Also let

$$T_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix},$$

where

$$\alpha_k = \sum_{k \leq j} \frac{1}{2^j N_j} \qquad \text{and} \qquad \beta_k = \sum_{k \leq j} \frac{v(2^{-j-1})}{2^j N_j}\,.$$

Next choose $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\varphi(0) = 1$ and define $\Phi_k = (\varphi \circ T_k)^{\vee}$. Write

$$\begin{aligned}M_k &= \Phi_k * M_k + (\delta - \Phi_k) * (I - S_k) M_k + (\delta - \Phi_k) * S_k M_k\\ &\overset{\text{def}}{=} \Phi_k * M_k + M_k^1 + M_k^2\,,\end{aligned}$$

where δ denotes the Dirac mass at the origin. Since $\Phi_k * M_k$ will not in general be dominated by the usual maximal functions, we will also apply a g-functions argument to it by further writing $\Phi_k * M_k$ as a sum of two operators. To do this, let $\omega \in C_c^\infty(\mathbb{R})$ with $\omega(0) = 1$ and define K_k by

$$\hat{K}_k(\xi, \eta) = 2^k \omega(\beta_k \eta) \int_{I_k \setminus E_k} e^{i\xi t}\, dt\,.$$

Write $\sigma_k = \Phi_k * M_k - K_k$ so that $\Phi_k * M_k = K_k + \sigma_k$. Finally we have the desired decomposition,

$$M_k f = K_k * f + \sigma_k * f + M_k^1 f + M_k^2 f\,.$$

Note that $|K_k * f| \le C\,\mathcal{M}_s f$, where $\mathcal{M}_s$ denotes the strong maximal function, since K_k is dominated pointwise by

$$\frac{1}{2^{-k}\beta_k}\,\frac{1}{1+|2^k x|^2}\,\frac{1}{1+|y/\beta_k|^2}\,.$$

Thus we have

$$\begin{aligned} Mf \le\ & C\Big(\mathcal{M}_s f + \big(\sum |\sigma_k * f|^2\big)^{1/2} \\ & + \big(\sum |M_k^1 f|^2\big)^{1/2} + \big(\sum |M_k^2 f|^2\big)^{1/2}\Big)\,. \end{aligned} \tag{14}$$

By an argument used in [NSW], we will prove the L^p estimates for M by repeated applications of the following three lemmas.

Lemma 1. *M is bounded in L^2.*

Lemma 2. *If*

$$\|(\sum |M_k f_k|^2)^{1/2}\|_{p_0} \le C_{p_0}\, \|(\sum |f_k|^2)^{1/2}\|_{p_0}$$

for some $p_0 < 2$, then M is bounded in L^p for $p_0 < p \le 2$.

Lemma 3. *If M is bounded in L^{p_0} for some $p_0 \le 2$, then*

$$\|(\sum |M_k f_k|^2)^{1/2}\|_p \le C_p\, \|(\sum |f_k|^2)^{1/2}\|_p\,,$$

for

$$\frac{1}{p} \le \frac{1}{2}\Big(\frac{1}{p_0} + 1\Big)\,.$$

Lemma 3 follows from interpolation since the operators $\{M_k\}$ are positive and uniformly bounded in L^p, $1 \le p \le \infty$, as in [NSW]. The main estimates needed in the proofs of lemmas 1 and 2 are contained in the following two lemmas.

Lemma 4.

a) $|\hat{\sigma}_k(\zeta)| \le C\,|T_k\zeta|^{-1}$.

b) $|\hat{\sigma}_k(\zeta)| \le C\,|T_{k-3}\zeta|$.

PROOF. Recall that $\hat{\sigma}_k(\zeta) = \varphi(T_k\zeta)\hat{M}_k(\zeta) - \hat{K}_k(\zeta)$ where

$$\hat{K}_k(\xi,\eta) = 2^k\,\omega(\beta_k\eta)\int_{I_k\setminus E_k} e^{i\xi t}\,dt\,.$$

Note that

$$\left|2^k\int_{I_k\setminus E_k} e^{i\xi t}\,dt\right| \le C\,\frac{2^k N_k}{|\xi|} \le C\,\frac{1}{\alpha_k|\xi|}\,.$$

The last inequality follows from (7) since

$$\alpha_k = \sum_{k\le j}\frac{1}{2^j N_j} \le C\,\frac{1}{2^k N_k}\,. \tag{15}$$

Also $|\omega(\beta_k\eta)| \le C\,|\beta_k\eta|^{-1}$ and so $|\hat{K}_k(\zeta)| \le C\,|T_k\zeta|^{-1}$. Furthermore

$$|(\Phi_k * M_k)^\wedge(\zeta)| = |\varphi(T_k\zeta)\hat{M}_k(\zeta)| \le C\,|T_k\zeta|^{-1}\,,$$

which gives us part *a*). For *b*) note that

$$\begin{aligned}
\hat{\sigma}_k(\zeta) &= \varphi(T_k\zeta)\Big(2^k\int_{I_k\setminus E_k} e^{i(\xi t+\eta\,\gamma(t))}\,dt - 2^k\int_{I_k\setminus E_k} e^{i\xi t}\,dt\Big)\\
&\quad + 2^k\int_{I_k\setminus E_k} e^{i\xi t}\,dt\,\big(\varphi(T_k\zeta) - \omega(\beta_k\eta)\big)\\
&= \varphi(T_k\zeta)\Big(2^k\int_{I_k\setminus E_k}\big(e^{i(\xi t+\eta\,\gamma(t))} - e^{i\xi t}\big)\,dt\Big)\\
&\quad + 2^k\int_{I_k\setminus E_k} e^{i\xi t}\,dt\,\big((\varphi(T_k\zeta)-1) - (\omega(\beta_k\eta)-1)\big)\,,
\end{aligned}$$

and so

$$|\hat{\sigma}_k(\zeta)| \le C\left(2^k|\eta|\int_{I_k\setminus E_k}|\gamma(t)|\,dt + |T_k\zeta|\right)$$

$$\begin{aligned}
&\leq C\left(|\eta|\phi(t_*) + |T_k\zeta|\right)\\
&\leq C\left(|\eta|\frac{1}{N_{k-2}\,2^{k-2}}\,v(t_*) + |T_k\zeta|\right)\\
&\leq C\left(|\eta|\frac{1}{N_{k-2}\,2^{k-2}}\,v(2^{-k+2}) + |T_k\zeta|\right)\\
&\leq C\left(|\eta|\,\beta_{k-3} + |T_{k-3}\zeta|\right) \leq C\,|T_{k-3}\zeta|\,,
\end{aligned}$$

where $t_* \in I_{k-1}$ such that $2t_* \in I_{k-2} \setminus E_{k-2}$. The third inequality follows from applying (9) on $I_{k-2}\setminus E_{k-2}$. The second to last inequality uses the fact that the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are monotone and the estimate $2^j N_j \leq C\,2^k N_k$ for $j \leq k$ which follows from (7). This gives us *b*) and thus completes the proof of the lemma.

The next estimate is based on a lemma of Van der Corput whose proof can be found in [Z].

Van der Corput's lemma. *Let $f \in C^2[a,b]$ be a real-valued function such that $|f''(t)| \leq \lambda$ on $[a,b]$. Then*

$$\left|\int_a^b e^{i\,f(t)}\,dt\right| \leq C\,\frac{1}{\sqrt{\lambda}}\,,$$

where C is independent of f, a and b.

Lemma 5.

a) $|\hat{M}_k^1(\zeta)| \leq C\,|T_k\zeta|\,.$

b) $|\hat{M}_k^1(\zeta)| \leq C\,|T_k\zeta|^{-1/2}\,.$

PROOF. Note that

$$\hat{M}_k^1(\zeta) = \big(1-\varphi(T_k\zeta)\big)\big(1-\chi_{R_k}(\zeta)\big)\hat{M}_k(\zeta)\,, \tag{16}$$

where

$$\hat{M}_k = 2^k\int_{I_k\setminus E_k} e^{i(\xi t+\eta\,\gamma(t))}\,dt = 2^k\int_{F_k} e^{i\,f(t)}\,dt + 2^k\int_{G_k} e^{i\,f(t)}\,dt$$

and $f(t) = \xi t + \eta\,\gamma(t)$. The estimate in *a*) is clear from (16) since $\varphi(0) = 1$. We turn now to the proof of *b*). We may assume $\zeta \notin R_k$. We will consider two cases.

Case 1. $v(2^{-k+N}) \le |\xi/\eta|$. On $I_k \setminus E_k$,

$$
\begin{aligned}
|f'(t)| &\ge |\xi| - |\eta|\,\gamma'(t)| \\
&\ge |\xi| - |\eta|\,v(2^{-k+1}) \ge |\xi|\Big(1 - \frac{v(2^{-k+1})}{v(2^{-k+N})}\Big)\,.
\end{aligned} \tag{17}
$$

From assumption (8) and the fact that E is "thin", we see that

$$
\frac{v(2^{-k+1})}{v(2^{-k+N})} < \frac{1}{2} \tag{18}
$$

for N sufficiently large. So from (17) and (18) we see that

$$
|f'(t)| \ge \frac{|\xi|}{2} \tag{19}
$$

for N sufficiently large.

Since F_k and G_k are open sets, we will write

$$
F_k = \bigcup_{\ell=1}^{L_k}(a_\ell, b_\ell) \qquad \text{and} \qquad G_k = \bigcup_{\ell=1}^{M_k}(c_\ell, d_\ell)
$$

where L_k and M_k are at most N_k. Integration by parts gives

$$
\begin{aligned}
\Big|2^k \int_{F_k} e^{i\,f(t)}\,dt\Big| &= \Big|2^k \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} e^{i\,f(t)}\,dt\Big| \\
&\le C\Big(\frac{2^k N_k}{|\xi|} + 2^k \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{|f''(t)|}{f'(t)^2}\,dt\Big) \\
&\le C\Big(\frac{2^k N_k}{|\xi|} + \frac{2^k N_k\,|\eta|}{|\xi|^2} \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{v(t/2)}{t}\,dt\Big) \\
&\le C\Big(\frac{2^k N_k}{|\xi|} + \frac{2^k N_k}{|\xi|}\,\frac{v(2^{-k})}{v(2^{-k+N})} \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{dt}{t}\Big) \\
&\le C\,\frac{2^k N_k}{|\xi|} \le C\,\frac{1}{\alpha_k\,|\xi|} \le C\,|T_k\zeta|^{-1}\,.
\end{aligned}
$$

The first inequality uses (19) while the second inequality uses both (10) and (19). The second to last inequality follows from (15). To prove the last inequality, note that it suffices to prove that $\beta_k/\alpha_k \le C\,|\xi/\eta|$ in this case. But since $v(2^{-k+N}) \le |\xi/\eta|$,

$$\beta_k = \sum_{k\le j} \frac{v(2^{-j-1})}{2^j N_j} \le v(2^{-k+N}) \sum_{k\le j} \frac{1}{2^j N_j} \le \alpha_k \left|\frac{\xi}{\eta}\right|.$$

Next note that

$$\begin{aligned}
\left|2^k \int_{G_k} e^{i\,f(t)}\,dt\right| &= \left|2^k \sum_{\ell=1}^{M_k} \int_{c_\ell}^{d_\ell} e^{i\,f(t)}\,dt\right| \\
&\le C\left(\frac{2^k N_k}{|\xi|} + 2^k \sum_{\ell=1}^{M_k} \int_{c_\ell}^{d_\ell} \frac{|f''(t)|}{f'(t)^2}\,dt\right) \\
&= C\left(\frac{2^k N_k}{|\xi|} + 2^k \sum_{\ell=1}^{M_k} \left|\int_{c_\ell}^{d_\ell} \frac{|f''(t)|}{f'(t)^2}\,dt\right|\right) \\
&\le C\,\frac{2^k N_k}{|\xi|} \le C\,\frac{1}{\alpha_k\,|\xi|} \le C\,|T_k\zeta|^{-1}.
\end{aligned}$$

The second equality holds since (11) implies that $f''(t)$ is single-signed on each (c_ℓ, d_ℓ). The first and third to last inequalities follow from (19). The final inequality was already used in the treatment of the F_k. Thus $|\hat{M}_k^1(\zeta)| \le C\,|T_k\zeta|^{-1}$.

Case 2. $|\xi/\eta| \le v(2^{-k-N})$. On F_k,

$$\begin{aligned}
|f'(t)| &\ge |\eta\,\gamma'(t)| - |\xi| \ge \varepsilon_1\,|\eta|\,v(t) - |\xi| \\
&= \varepsilon_1\,|\eta|\,v(t)\Big(1 - \left|\frac{\xi}{\eta}\right| \frac{1}{\varepsilon_1\,v(t)}\Big) \ge \frac{\varepsilon_1}{2}\,|\eta|\,v(t)
\end{aligned} \tag{20}$$

for N large enough. Integration by parts shows

$$\begin{aligned}
\left|2^k \int_{F_k} e^{i\,f(t)}\,dt\right| &= \left|2^k \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} e^{i\,f(t)}\,dt\right| \\
&\le C\left(\frac{2^k N_k}{|\eta|\,v(2^{-k})} + 2^k \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{|f''(t)|}{f'(t)^2}\,dt\right) \\
&\le C\left(\frac{2^k N_k}{|\eta|\,v(2^{-k})} + \frac{2^k N_k\,|\eta|}{v^2(2^{-k})\,|\eta|^2} \sum_{\ell=1}^{L_k} \int_{a_\ell}^{b_\ell} \frac{v(t/2)}{t}\,dt\right)
\end{aligned}$$

$$\leq C\,\frac{2^k N_k}{|\eta|\,v(2^{-k})} \leq C\,|T_k\zeta|^{-1}\,.$$

The first inequality uses (20) while the second inequality uses both (10) and (20). To prove the last inequality, note that from (15) and $|\xi/\eta| \leq v(2^{-k-N})$, $\alpha_k\,|\xi| \leq C\,(2^k N_k)^{-1}|\xi| \leq C\,(2^k N_k)^{-1} v(2^{-k})\,|\eta|$. Also $\beta_k\,|\eta| \leq v(2^{-k})\,(2^k N_k)^{-1}|\eta|$. Hence the last inequality follows.

On G_k, $|f''(t)| \geq \varepsilon_2\,|\eta|\,N_k\,v(t/2)/t$ and so by Van der Corput's lemma,

$$\begin{aligned}\left|2^k \int_{G_k} e^{i\,f(t)}\,dt\right| &= \left|2^k \sum_{\ell=1}^{M_k} \int_{c_\ell}^{d_\ell} e^{i\,f(t)}\,dt\right| \\ &\leq C\,2^k N_k\,\frac{1}{\sqrt{2^k|\eta| N_k\,v(2^{-k-1})}} \\ &\leq C\,\sqrt{\frac{2^k N_k}{v(2^{-k-1})\,|\eta|}} \leq C\,|T_k\zeta|^{-1/2}\,.\end{aligned}$$

The proof of the last inequality is the same as in the treatment of F_k. Thus $|\hat{M}_k^1(\zeta)| \leq C\,|T_k\zeta|^{-1/2}$ in this case as well which finishes part *b*) and thus the lemma.

We turn now to the proofs of lemmas 1 and 2. First observe that our family $\{T_k\}$ satisfies the norm estimate

$$\|T_k^{-1}T_{k+1}\| \leq \alpha < 1\,. \tag{21}$$

In fact,

$$T_k^{-1}T_{k+1} = \begin{pmatrix} \alpha_{k+1}/\alpha_k & 0 \\ 0 & \beta_{k+1}/\beta_k \end{pmatrix}$$

and so to prove (21) it suffices to show that there is an $\alpha < 1$ such that for all k,

$$\frac{\alpha_{k+1}}{\alpha_k} \leq \alpha \qquad \text{and} \qquad \frac{\beta_{k+1}}{\beta_k} \leq \alpha\,.$$

This however follows easily from (7).

To show that Mf is bounded in L^2 we see from (14) that it suffices to prove that $(\sum|\sigma_k * f|^2)^{1/2}$, $(\sum|M_k^1 f|^2)^{1/2}$, and $(\sum|M_k^2 f|^2)^{1/2}$ are bounded in L^2. With the aid of Plancherel's Theorem, the L^2 estimates of the first two square functions reduce to showing that $\sum|\hat{\sigma}_k(\zeta)|^2$ and

$\sum |\hat{M}_k^1(\zeta)|^2$ are bounded functions of ζ. This easily follows from lemmas 4 and 5 together with (21). For the third square function, note that

$$M_k^2 f \le C\left(M_k S_k f + \mathcal{M}_s M_k S_k f\right) \le C\,\mathcal{M}_s M_k S_k f\,. \tag{22}$$

Then since $\sum \chi_{R_k}(\zeta) \le 2N$ (where N appears in the definition of the R_k's),

$$\begin{aligned} \|(\sum |M_k^2 f|^2)^{1/2}\|_2^2 &\le C \sum \int (\mathcal{M}_s M_k S_k f)^2 \le C \sum \int |M_k S_k f|^2 \\ &\le C \sum \int |S_k f|^2 = C \int \sum \chi_{R_k}(\zeta)\,|\hat{f}(\zeta)|^2\,d\zeta \\ &\le 2NC \int |\hat{f}(\zeta)|^2\,d\zeta = 2NC\,\|f\|_2^2\,. \end{aligned}$$

We have used the fact that the strong maximal function is bounded in L^2 and that the M_k's are uniformly bounded in L^2. Thus $(\sum |M_k^2 f|^2)^{1/2}$ and hence Mf is bounded in L^2. This completes the proof of Lemma 1.

We turn now to Lemma 2. Note that

$$\|(\sum |M_k f_k|^2)^{1/2}\|_p \le C_p\,\|(\sum |f_k|^2)^{1/2}\|_p\,, \qquad p_0 \le p \le 2\,,$$

by interpolation and so by (22),

$$\begin{aligned} \|(\sum |M_k^2 f|^2)^{1/2}\|_p &\le C_p\,\|(\sum |\mathcal{M}_s M_k S_k f|^2)^{1/2}\|_p \\ &\le C_p\,\|(\sum |M_k S_k f|^2)^{1/2}\|_p \\ &\le C_p\,\|(\sum |S_k f|^2)^{1/2}\|_p \\ &\le C_p\,\|f\|_p\,, \qquad p_0 \le p \le 2\,. \end{aligned}$$

We have used the fact that the strong maximal function satisfies vector-valued estimates, see [FS]. The last inequality follows from [NSW] and [CF] since the sequence $\{v(2^{-k})\}$ satisfies (18). Hence by (14), it suffices to prove that $(\sum |\sigma_k * f|^2)^{1/2}$ and $(\sum |M_k^1 f|^2)^{1/2}$ are bounded in L^p, $p_0 < p \le 2$. This is proved by an argument used in [CCVWW]. We will only sketch the argument here (the interested reader should consult [CCVWW] for more details). The argument is based on a general Littlewood-Paley decomposition developed in [CCVWW] and [CVWW].

Suppose that we have a family of invertible linear transformations on $\mathbb{R}^n$, $\{A_j\}_{j\in\mathbb{Z}}$, which satisfy the Rivière condition

$$\text{(R)} \qquad \|A_j^{-1}A_{j+1}\| \le \alpha < 1\,.$$

See [R]. Choose a smooth function $\phi(x)$ such that $\hat{\phi}(\xi) = 1$ for $|\xi| \le 1$ and $\hat{\phi}(\xi) = 0$ for $|\xi| \ge 2$. Now define the multipliers

$$m_j(\xi) = \hat{\phi}(A_{j+1}^*\xi) - \hat{\phi}(A_j^*\xi)$$

and the corresponding linear operators

$$P_jf(x) = (m_j\hat{f})^{\vee}(x)\,.$$

The following theorem can be found in [CVWW].

Theorem D. *Under the conditions stated above, we have*

$$\Big\|\big(\sum_{j\in\mathbb{Z}} |P_jf|^2\big)^{1/2}\Big\|_p \le C_p\, \|f\|_p\,, \qquad 1 < p < \infty\,,$$

and

$$\Big\|\sum_{j\in\mathbb{Z}} P_jf_j\Big\|_p \le C_p\, \Big\|\big(\sum_{j\in\mathbb{Z}} |P_jf|^2\big)^{1/2}\Big\|_p\,, \qquad 1 < p < \infty\,.$$

Also for each $\xi \ne 0$,

$$\sum_{j\in\mathbb{Z}} m_j(\xi) = 1\,.$$

We will use our family of invertible linear transformations on $\mathbb{R}^2$, $\{T_k\}$ and note that (R) is simply (21) in this case. To prove the L^p estimates, for say $(\sum |M_k^1 f|^2)^{1/2}$ (the same reasoning applies to $(\sum |\sigma_k * f|^2)^{1/2}$), we will decompose f with respect to the operators $\{P_j\}$. Write

$$M_k^1 f = M_k^1 \sum_{j\in\mathbb{Z}} P_{j+k} f = \sum_{j\in\mathbb{Z}} P_{j+k} M_k^1 f\,,$$

which implies

$$\big(\sum_k |M_k^1 f|^2\big)^{1/2} = \Big(\sum_k \Big|\sum_{j\in\mathbb{Z}} P_{j+k} M_k^1 f\Big|^2\Big)^{1/2}$$

$$\leq \sum_{j\in\mathbb{Z}} \Big(\sum_k |P_{j+k} M_k^1 f|^2 \Big)^{1/2} = \sum_{j\in\mathbb{Z}} G_j f \,,$$

where $G_j f = (\sum_k |P_{j+k} M_k^1 f|^2)^{1/2}$. We will prove that

a) $\|G_j f\|_{p_0} \leq C\,\|f\|_{p_0}$, and

b) $\|G_j f\|_2 \leq C\,2^{-\varepsilon\,|j|}\,\|f\|_2$,

for some $\varepsilon > 0$ and C independent of j. For *a)* note that

$$|M_k^1 f_k| \leq C\left(\mathcal{M}_s M_k f_k + \mathcal{M}_s S_k M_k f_k\right).$$

Therefore using the hypothesis of Lemma 2 that the family of operators $\{M_k\}$ satisfy an ℓ^2-valued L^{p_0} estimate, we obtain the same conclusion for the family $\{M_k^1\}$,

$$\begin{aligned}\|(\sum |M_k^1 f_k|^2)^{1/2}\|_{p_0} &\leq C\,\|(\sum |\mathcal{M}_s S_k M_k f_k|^2)^{1/2}\|_{p_0}\\ &\leq C\,\|(\sum |S_k M_k f_k|^2)^{1/2}\|_{p_0}\\ &\leq C\,\|(\sum |M_k f_k|^2)^{1/2}\|_{p_0}\\ &\leq C\,\|(\sum |f_k|^2)^{1/2}\|_{p_0}\,.\end{aligned}$$

Again we used the angular Littlewood-Paley theory developed in [NSW] and the fact that the strong maximal function satisfies vector-valued L^p estimates. Using Theorem D, we see that

$$\begin{aligned}\|G_j f\|_{p_0} &= \|(\sum_k |M_k^1(P_{j+k} f)|^2)^{1/2}\|_{p_0}\\ &\leq \|(\sum_k |P_{j+k} f|^2)^{1/2}\|_{p_0} \leq \|f\|_{p_0}\,.\end{aligned}$$

which gives *a)*. *b)* is proved by using Lemma 5 and (21). See [CCVWW] for details.

By interpolating the estimates in *a)* and *b)*, we see that $\|G_j f\|_p \leq C\,2^{-\varepsilon_p\,|j|}\,\|f\|_p$, $p_0 < p \leq 2$ for some $\varepsilon_p > 0$. Summing these L^p estimates for G_j gives us the desired L^p estimates for $(\sum |M_k^1 f|^2)^{1/2}$. This finishes the proof of Lemma 2. The treatment of the maximal function is now complete.

We turn to the Hilbert transform. Note that

$$\mathcal{H}_\Gamma f(x) = \int\limits_{\cup E_k^1} f(x-\Gamma(t))\,\frac{dt}{t} + \int\limits_{\cup E_k^2} f(x-\Gamma(t))\,\frac{dt}{t}$$

$$\stackrel{\text{def}}{=} H_E f(x) + H_G f(x)\,,$$

where $E_k^1 = E_k \cup (-E_k)$ and $E_k^2 = (I_k \setminus E_k) \cup (-(I_k \setminus E_k))$. The L^p estimates for $H_E f$ follow from Minkowski's inequality for integrals and the assumption that $\sum 2^k |E_k| < +\infty$. For $H_G f$, write

$$H_k f(z) = \int\limits_{E_k^2} f(z-\Gamma(t))\,\frac{dt}{t}$$

and note that $\sum_{k\geq 1} H_k f(z) = H_G f(z)$. We will decompose $H_k f$ into a sum of several operators. First let us consider a variant of the kernel K_k. Denote by B_k the function such that

$$b_k(\xi,\eta) = \hat{B}_k(\xi,\eta) = \omega(\beta_k \eta) \int\limits_{E_k^2} e^{i\xi t}\,\frac{dt}{t}$$

where β_k and ω are the same as in K_k. Note that $B_k = C_k - D_k$ where

$$\hat{C}_k(\xi,\eta) = \omega(\beta_k \eta) \int\limits_{I_k \cup (-I_k)} e^{i\xi t}\,\frac{dt}{t}$$

and

$$\hat{D}_k(\xi,\eta) = \omega(\beta_k \eta) \int\limits_{E_k^1} e^{i\xi t}\,\frac{dt}{t}\,.$$

Since the L^1 norm of D_k is no larger that $2^k |E_k|$, we see that the operator $\sum_k D_k * f$ is bounded in all L^p. Also

a) $|\,|C_k| * f\,| \leq C\,\mathcal{M}_s f,$ and

b) $|\hat{C}_k(\xi,\eta)| \leq C \min\{2^{-k}|\xi|, (2^{-k}|\xi|)^{-1}\}\,.$

We may apply Theorem D′ in [DR] to the operator $\sum_k C_k * f$ and find that it is bounded in all L^p, $p > 1$. Therefore it suffices to estimate the operator $\sum_k (H_k - B_k)$. Write

$$(H_k - B_k)f(x) = S_k(H_k - B_k)f(x) + (I - S_k)(H_k - B_k)f(x)$$

$$\stackrel{\text{def}}{=} H_k^1 f(x) + H_k^2 f(x)\,.$$

The L^p estimates of $\sum_k (H_k - B_k)$ will be based on the following lemma.

Lemma 6.

$$\Big\| \Big(\sum_{k\geq 1} |H_k f_k|^2 \Big)^{1/2} \Big\|_p \leq C_p \Big\| \Big(\sum_{k\geq 1} |f_k|^2 \Big)^{1/2} \Big\|_p, \qquad 1 < p < \infty,$$

and

$$\Big\| \Big(\sum_{k\geq 1} |B_k f_k|^2 \Big)^{1/2} \Big\|_p \leq C_p \Big\| \Big(\sum_{k\geq 1} |f_k|^2 \Big)^{1/2} \Big\|_p, \qquad 1 < p < \infty.$$

PROOF. We shall first derive the estimate for H_k. This will be proved by an interpolation argument. In fact we will prove that

$$\Big\| \Big(\sum_{k\geq 1} |H_k f_k|^q \Big)^{1/q} \Big\|_p \leq C_{p,q} \Big\| \Big(\sum_{k\geq 1} |f_k|^q \Big)^{1/q} \Big\|_p \tag{23}$$

for certain $1 \leq p, q \leq \infty$. Since the operators H_k are uniformly bounded in L^p, $1 \leq p \leq \infty$, we see that (23) holds for $p = q \geq 1$ and so in particular we have the lemma for $p = 2$. If γ is even, we have the pointwise estimate

$$\sup_{k\geq 1} |H_k f_k|(x,y) \leq C\Big(M\big(\sup_{k\geq 1} |f_k|\big)(x,y) + M\big(\sup_{k\geq 1} |f_k^1|\big)(-x,y) \Big) \tag{24}$$

where $f_k^1(x,y) = f_k(-x,y)$. Also if γ is odd, we have

$$\sup_{k\geq 1} |H_k f_k|(x,y) \leq C\Big(M\big(\sup_{k\geq 1} |f_k|\big)(x,y) + M\big(\sup_{k\geq 1} |f_k^2|\big)(-x,-y) \Big) \tag{25}$$

where $f_k^2(x,y) = f_k(-x,-y)$. Therefore from the L^p estimates for M, we see that (23) holds for $q = \infty$ and $p > 1$. Interpolating between $L^1(\ell^1)$ and $L^p(\ell^\infty)$, $p > 1$, establishes the lemma for $1 < p < 2$ and then duality gives us the full range. The argument for the operators $\{B_k\}$ is similar. We must only replace M by $\mathcal{M}_s$ in (24) and (25).

Lemma 6 is sufficient to give the L^p estimates for H_k^1. In fact for $1 < p < \infty$,

$$\begin{aligned}
\Big\| \sum_{k\geq 1} H_k^1 f \Big\|_p &= \Big\| \sum_{k\geq 1} S_k(H_k - B_k) f \Big\|_p = \Big\| \sum_{k\geq 1} S_k^2(H_k - B_k) f \Big\|_p \\
&\leq C_p \Big\| \Big(\sum_{k\geq 1} |(H_k - B_k) S_k f|^2 \Big)^{1/2} \Big\|_p \qquad (26) \\
&\leq C_p \Big\| \Big(\sum_{k\geq 1} |S_k f|^2 \Big)^{1/2} \Big\|_p \leq C_p \, \|f\|_p \,,
\end{aligned}$$

where we have used again the results of [NSW] and Lemma 6. For H_k^2, we will need the analogous estimates to Lemma 5 for the multiplier h_k^2 of the operator H_k^2. Note that $h_k^2(\zeta) = (1 - \chi_{R_k}(\zeta))(h_k(\zeta) - b_k(\zeta))$. Here h_k is the multiplier for H_k, *i.e.*,

$$h_k(\zeta) = \int_{E_k^2} e^{i(\xi t + \eta\, \gamma(t))} \, \frac{dt}{t} = h_k^+(\zeta) + h_k^-(\zeta)$$

where

$$h_k^+(\zeta) \stackrel{\text{def}}{=} \int_{I_k \setminus E_k} e^{i\zeta\, \Gamma(t)} \, \frac{dt}{t} \,, \qquad h_k^-(\zeta) \stackrel{\text{def}}{=} \int_{-(I_k \setminus E_k)} e^{i\zeta\, \Gamma(t)} \, \frac{dt}{t}$$

and b_k is the multiplier for B_k defined above. A direct consequence of Lemmas 4 and 5 is the following lemma.

Lemma 7.

a) $|h_k^2(\zeta)| \leq C\, |T_{k-3}\zeta| \,,$

b) $|h_k^2(\zeta)| \leq C\, |T_k\zeta|^{-1/2} \,.$

PROOF. Note that

$$h_k(\zeta) - b_k(\zeta) = \int_{E_k^2} \left(e^{i(\xi t + \eta\, \gamma(t))} - e^{i\xi t} \right) \frac{dt}{t} + (1 - \omega(\beta_k \eta)) \int_{E_k^2} e^{i\xi t} \, \frac{dt}{t}$$

and so as in Lemma 4,

$$|h_k(\zeta) - b_k(\zeta)| \leq C \left(2^k |\eta| \int_{I_k} \phi(t) \, dt + |\eta| \, \beta_k \right) \leq C\, |T_{k-3}\zeta| \,.$$

This gives us *a*). For *b*), integration by parts shows that Lemma 5 implies that *b*) holds for h_k^+. If γ is even, we see that $h_k^-(\xi,\eta) = h_k^+(-\xi,\eta)$ and if γ is odd, $h_k^-(\xi,\eta) = h_k^+(-\xi,-\eta)$. Therefore *b*) also holds for h_k^- and thus for h_k. Similarly for b_k and this completes the lemma.

Now if we follow the same arguments for the maximal function, using Lemma 7, we see that $\sum_k H_k^2$ is bounded in L^p, $p > 1$. This completes the proof of the main theorem.

3. Examples.

In this section, we will construct the two curves mentioned in the introduction. We will first construct an odd monotone curve $\Gamma(t) = (t,\gamma(t))$ on $[-1,1]$ which satisfies A), B), C) and D′) in the introduction but whose Hilbert transform is unbounded in L^2. We begin with a continuous piecewise linear curve. For $r \geq 0$, we will construct γ on $[2^{-2^{r+1}}, 2^{-2^r}]$. For $2^r + 1 \leq k \leq 2^{r+1}$, write

$$[2^{-k}, 2^{-k+1}] = \bigcup_{\ell=0}^{N_k-1} [a_\ell^k, b_\ell^k] \cup \bigcup_{\ell=0}^{N_k-1} [b_\ell^k, a_{\ell+1}^k] \stackrel{\text{def}}{=} F_k \cup E_k\,,$$

where $a_0^k = 2^{-k}$, $a_{N_k}^k = 2^{-k+1}$ and $N_k = 2^{2^{r+2}-2k}$ so that

$$\Delta_k = b_\ell^k - a_\ell^k = \frac{1}{2^k N_k}\Big(1 - \frac{1}{k^2}\Big)\,, \qquad \text{for } \ 0 \leq \ell \leq N_k - 1\,,$$

and

$$\delta_k = a_{\ell+1}^k - b_\ell^k = \frac{1}{2^k N_k\, k^2}\,, \qquad \text{for } \ 0 \leq \ell \leq N_k - 1\,.$$

On $[a_\ell^k, b_\ell^k]$, define

$$\gamma' = \frac{\pi}{2^{2^{r+2}} \Delta_k} \stackrel{\text{def}}{=} m_k \sim 2^{-k}\,.$$

On $[b_\ell^k, a_{\ell+1}^k]$, define

$$\gamma' = \frac{\pi}{2^{2^{r+2}} \delta_k} \stackrel{\text{def}}{=} M_k \sim k^2\, 2^{-k}\,.$$

Then

$$\gamma(b_\ell^k) - \gamma(a_\ell^k) = m_k \Delta_k = \frac{\pi}{2^{2^{r+2}}} \tag{27}$$

and

$$\gamma(a_{\ell+1}^k) - \gamma(b_\ell^k) = M_k \delta_k = \frac{\pi}{2^{2^{r+2}}}\,. \tag{28}$$

This will now define γ uniquely on $[0, 1/2]$ once we choose $\gamma(1/2)$. We will do so to make $\gamma(0) = 0$. Note that by (27) and (28),

$$\begin{aligned} \gamma(2^{-k+1}) - \gamma(2^{-k}) &= \sum_{\ell=0}^{N_k-1} \left(\gamma(a_{\ell+1}^k) - \gamma(b_\ell^k)\right) \\ &\quad + \sum_{\ell=0}^{N_k-1} \left(\gamma(b_\ell^k) - \gamma(a_\ell^k)\right) \\ &= \frac{\pi N_k}{2^{2^{r+2}}} + \frac{\pi N_k}{2^{2^{r+2}}} = \frac{2\pi}{2^{2k}}\,. \end{aligned} \tag{29}$$

Thus

$$\gamma\Big(\frac{1}{2}\Big) - \gamma\Big(\frac{1}{2^N}\Big) = \sum_{\ell=2}^{N} \left(\gamma(2^{-\ell+1}) - \gamma(2^{-\ell})\right) = \sum_{\ell=2}^{N} \frac{2\pi}{2^{2\ell}}$$

and so if $\gamma(1/2) = \pi/6$, $\gamma(0) = 0$. We will show that $\mathcal{H}_\Gamma$ is unbounded in L^2.

Since the Hilbert transform is a multiplier transformation, it suffices to show that the corresponding multiplier m is un unbounded function. Since γ is an odd function on $[-1, 1]$, the multiplier reduces to a sine integral,

$$m(\xi, \eta) = \int_0^1 \sin(f(t))\,\frac{dt}{t}\,,$$

where $f(t) = \xi t - \eta\,\gamma(t)$. We will take $\xi = 0$ and show that

$$\int_0^{1/2} \sin(\eta\,\gamma(t))\,\frac{dt}{t}$$

is unbounded as $\eta \to \infty$. Let us first note that

$$\begin{aligned} \Big| \int_{\cup E_k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} \Big| &\le C \sum_{k=2}^{\infty} \int_{E_k} \frac{dt}{t} \\ &\le C \sum_{k=2}^{\infty} 2^k |E_k| \le C \sum_{k=2}^{\infty} \frac{1}{k^2} \le C\,. \end{aligned}$$

Now let r be a large integer and set $\eta = 2^{2^{r+2}}$.

Claim 1.

$$\sum_{k=2}^{2^r} \int_{F_k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} + \sum_{k=2^{r+1}+1}^{\infty} \int_{F_k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} \stackrel{\text{def}}{=} \mathrm{I}(r) + \mathrm{II}(r)$$

remains bounded as $r \to \infty$. In fact,

$$\begin{aligned} |\mathrm{II}(r)| &\le C\,\eta \sum_{k\ge 2^r+1} \int_{F_k} \frac{\gamma(t)}{t}\,dt \le C\,\eta \sum_{k\ge 2^r+1} \gamma(2^{-k+1}) \\ &\le C\,\eta \sum_{k\ge 2^r+1} 2^{-2k} \le C\,\frac{\eta}{2^{2(2^r+1)}} \le C\,. \end{aligned}$$

The third inequality follows from (29) since

$$(30)\quad \gamma(2^{-k+1}) = \sum_{\ell=k}^{\infty} \left(\gamma(2^{-\ell+1}) - \gamma(2^{-\ell})\right) = 2\pi \sum_{\ell=k}^{\infty} 2^{-2\ell} = \frac{8\pi}{3}\,\frac{1}{2^{2k}}\,.$$

Also by integrating by parts,

$$\begin{aligned} |\mathrm{I}(r)| &\le C \sum_{k=2}^{2^r} \sum_{\ell=0}^{N_k-1} \frac{1}{\eta\, m_k\, a_\ell^k} \le C \sum_{k=2}^{2^r} \frac{N_k\, 2^{2k}}{\eta} \\ &\le C\,\frac{2^{2^{r+1}}}{\eta} \sum_{k=2}^{2^r} 1 \le C\,2^r\,\frac{2^{2^{r+1}}}{2^{2^{r+2}}} \le C\,. \end{aligned}$$

The third inequality holds since $N_k \le 2^{2^{r+1}-2k}$ for $k \le 2^r$ and this finishes the claim. For $2^r + 1 \le k \le 2^{r+1}$, write

$$\begin{aligned} \int_{F_k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} &= \sum_{\ell=0}^{N_k-1} \int_{a_\ell^k}^{b_\ell^k} \sin(\eta\,\gamma(t))\,\frac{dt}{t} \\ &= \sum_{\ell=0}^{N_k-1} \left(\frac{\cos(\eta\,\gamma(a_\ell^k))}{a_\ell^k} - \frac{\cos(\eta\,\gamma(b_\ell^k))}{b_\ell^k} \right) \frac{1}{\eta\, m_k} \\ &\quad - \sum_{\ell=0}^{N_k-1} \int_{a_\ell^k}^{b_\ell^k} \frac{\cos(\eta\,\gamma(t))}{t^2\,\eta\, m_k}\,dt \\ &\stackrel{\text{def}}{=} \mathrm{I}_k + \mathrm{II}_k\,. \end{aligned}$$

Since

$$|\mathrm{II}_k| \le C\,\frac{2^{2k}}{\eta\, m_k} \sum_{\ell=0}^{N_k-1} \int_{a_\ell^k}^{b_\ell^k} dt \le C\,\frac{2^{2k}}{\eta}\,,$$

we have

$$\sum_{k=2^r+1}^{2^{r+1}} |\mathrm{II}_k| \le C\,\frac{1}{\eta}\,2^{2(2^{r+1})} \le C\,.$$

Therefore it suffices to show that

$$\sum_{k=2^r+1}^{2^{r+1}} \mathrm{I}_k \quad \text{is unbounded as} \quad r \to \infty\,.$$

Claim 2. $\cos(\eta\,\gamma(a_\ell^k)) = -1/2$ for $0 \le \ell \le N_k - 1$. By (27) and (28), it suffices to show that $\cos(\eta\,\gamma(2^{-k+1})) = -1/2$. By (28) and (30),

$$\eta\,\gamma(2^{-k+1}) = \frac{8\pi}{3}\,2^{2^{r+2}-2k} = \frac{8\pi}{3}\,N_k\,.$$

Note that for $2^r + 1 \le k \le 2^{r+1}$, $2^{r+2} - 2k = 2\ell$ for some positive integer ℓ and so

$$\eta\,\gamma(2^{-k+1}) = \frac{\pi}{3}\,2^{2\ell+3} = \frac{\pi}{3}\,(2(2^{2(\ell+1)} - 1) + 2)\,.$$

Observe that if p is a positive integer, $2^{2p} - 1$ is a multiple of 3. In fact,

$$\begin{aligned} 3\,(1 + 2^2 + 2^4 + 2^6 + \cdots + 2^{2(p-1)}) &= (4-1)(1 + 4 + 4^2 + \cdots + 4^{p-1}) \\ &= 4^p - 1 = 2^{2p} - 1\,. \end{aligned}$$

Therefore,

$$\eta\,\gamma(2^{-k+1}) = \frac{\pi}{3}\,(2 \cdot 3n + 2) = 2\pi n + \frac{2\pi}{3}$$

for some positive integer n. This gives us the claim.

From (27) and (28), we see that $\cos(\eta\,\gamma(b_\ell^k)) = 1/2$ for $0 \le \ell \le N_k - 1$ and so

$$\mathrm{I}_k = -\,\frac{1}{\eta\, m_k} \sum_{\ell=0}^{N_k-1} \left(\frac{1}{a_\ell^k} + \frac{1}{b_\ell^k}\right) \le -\varepsilon\,\frac{N_k\, 2^{2k}}{\eta}$$

$$= -\varepsilon\,, \qquad \text{for some } \varepsilon > 0\,.$$

Thus

$$\Big|\sum_{k=2^r+1}^{2^{r+1}} \mathrm{I}_k\Big| \geq \varepsilon\, 2^r$$

and this finishes the proof that $\mathcal{H}_\Gamma$ is unbounded in L^2. By smoothing out γ on the exceptional set $\cup E_k$, we obtain a smooth curve whose Hilbert transform is still unbounded in L^2. Since in this case, $\phi(t) \sim t^2$ and $v(t) \sim t$, it is easy to see that A), B), C) and D′) are satisfied by this curve.

We will now construct a curve $\Gamma(t) = (t, \gamma(t))$ for $0 \leq t \leq 1$ where $v(t) = \sup_{s\leq t} |\gamma'(s)|$ is not doubling, $w(t) = \sup_{s\leq t} |s\,\gamma'(s) - \gamma(s)|$ is doubling and otherwise satisfies the conditions of the theorem (the set E is empty in this example). If γ is extended as an odd function on $[-1, 1]$, we will see that $\mathcal{M}_\Gamma$ and $\mathcal{H}_\Gamma$ are unbounded in every L^p, $p \geq 1$. We begin by considering a saw-toothed curve $\Gamma_1(t) = (t, \gamma_1(t))$, $0 \leq t \leq 1$. We simply require that Γ_1 be continuous, piecewise linear and for each $n \geq 1$, $\gamma_1(9^{-(n+1/2)}) = 0$ and $\gamma_1(9^{-n}) = 9^{-n}/n$. To see that $\mathcal{M}_{\Gamma_1}$ is unbounded in L^p, take a large integer N and let f_N be the characteristic function of the parallelogram

$$P_N = \Big\{(x,y) \in \mathbb{R}^2 :\ -2 \leq y \leq 0 \ \text{ and } \ \frac{2}{3}N\,y - 9^{-2N} \leq x \leq \frac{2}{3}N\,y\Big\}\,.$$

For $N \leq n \leq 2\,N$, consider smaller translated versions of P_N,

$$Q_n = \Big\{(x,y) \in \mathbb{R}^2 :\ -1 \leq y \leq 0, \frac{2}{3}N\,y - 9^{-(2N+1)} \leq x - 9^{-(n+1/2)} \leq \frac{2}{3}N\,y\Big\}\,.$$

Note that the Q_n's are disjoint. Also it is not hard to see that there is a positive δ and ε independent of N such that $\mathcal{M}_{\Gamma_1} f_N \geq \delta$ on each Q_n and $|Q_n| \geq \varepsilon |P_N|$, $N \leq n \leq 2\,N$. From this we see that $\|\mathcal{M}_{\Gamma_1} f_N\|_p^p \geq \varepsilon\,\delta^p\, N\,|P_N|$ whereas $\|f_N\|_p^p \leq |P_N|$ and so $\mathcal{M}_{\Gamma_1}$ is unbounded in L^p.

To see that $\mathcal{H}_{\Gamma_1}$ is unbounded in L^2, let us again consider the multiplier

$$m(\xi,\eta) = \int_0^1 \sin(f(t))\,\frac{dt}{t}\,,$$

where $f(t) = \xi\, t - \eta\, \gamma_1(t)$. Note that in $[9^{-(n+1)}, 9^{-n}]$,

$$\gamma_1(t) = -\frac{1}{2(n+1)}\, t + \frac{3}{2(n+1)}\, 9^{-(n+1)}\,, \tag{31.1}$$

if $9^{-(n+1)} \le t \le 9^{-(n+1/2)}$, and

$$\gamma_1(t) = \frac{3}{2n}\, t - \frac{1}{2n}\, 9^{-n}\,, \tag{31.2}$$

if $9^{-(n+1/2)} \le t \le 9^{-n}$.

Let n_0 be a large integer and set

$$\eta = \pi\, n_0^2\, 9^{n_0} \qquad \text{and} \qquad \xi = \frac{3}{2\, n_0}\, \eta = \frac{3}{2}\, \pi\, n_0\, 9^{n_0}\,.$$

Choose $n_1 > n_0$ such that $n_1\, 9^{n_1} \le n_0^2\, 9^{n_0} \le (n_1+1)\, 9^{n_1+1}$. Write

$$\int_0^{1/9} \sin(f(t))\, \frac{dt}{t} = \sum_{n=1}^{\infty} \int_{9^{-(n+1)}}^{9^{-(n+1/2)}} \sin(f(t))\, \frac{dt}{t} + \sum_{n=1}^{\infty} \int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\, \frac{dt}{t}\,.$$

We will show that the second sum is unbounded as $n_0 \to \infty$. The fact that the first sum is bounded as $n_0 \to \infty$ is somewhat easier.

Claim 1.

$$\sum_{n=n_1}^{\infty} \int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\, \frac{dt}{t} \qquad \text{and} \qquad \sum_{n=1}^{n_0-1} \int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\, \frac{dt}{t}$$

are bounded as $n_0 \to \infty$. To show that the first sum is bounded we need the following relationship between n_0 and n_1. Since

$$\frac{1}{9}\, \frac{n_0^2}{n_1+1} \le 9^{n_1-n_0} \le \frac{n_0^2}{n_1} \le n_0\,,$$

we have that

$$\frac{1}{9\log 9}\, \log \frac{n_0}{2} \le n_1 - n_0 \le \frac{\log n_0}{\log 9} \tag{32}$$

for n_0 large enough. The first inequality follows from the second inequality. Since

$$\Big|\int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\,\frac{dt}{t}\Big| \le \int_{9^{-(n+1/2)}}^{9^{-n}} \Big|\xi - \eta\,\frac{\gamma_1(t)}{t}\Big|\,dt$$
$$\le C\Big(\frac{n_0\,9^{n_0}}{9^n}\,\big(1-\frac{n_0}{n}\big) + \frac{n_0^2\,9^{n_0}}{n\,9^n}\Big),$$

we see that the first sum is bounded by a constant times

$$1 + \frac{n_0\,9^{n_0}}{n_1\,9^{n_1}}\,(n_1 - n_0).$$

This term is bounded as $n_0 \to \infty$. One can see this from (32) and the definition of n_1.

For the second sum, let us note that

$$\begin{aligned} f'(t) = \xi - \eta\,\gamma_1'(t) &= \frac{\pi\,n_0}{2}\,9^{n_0}\,\big(3 - 2\,n_0\,\gamma_1'(t)\big) \\ &= \frac{3}{2}\,\pi\,n_0\,9^{n_0}\big(1 - \frac{n_0}{n}\big) \end{aligned}$$

for $9^{-(n+1/2)} \le t \le 9^{-n}$. Thus

$$|f'(t)| \ge \frac{3\,\pi}{2}\,\frac{n_0 - n}{n}\,n_0\,9^{n_0}$$

and so

$$\Big|\int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\,\frac{dt}{t}\Big| \le C\,\frac{n\,9^n}{n_0\,9^{n_0}}\,\frac{1}{n_0 - n}$$

by integrating by parts. This shows that the second sum is bounded establishing the claim. Therefore it suffices to show that

$$\sum_{n=n_0}^{n_1-1} \int_{9^{-(n+1/2)}}^{9^{-n}} \sin(f(t))\,\frac{dt}{t} \stackrel{\text{def}}{=} \sum_{n=n_0}^{n_1-1} \mathrm{I}_n$$

is unbounded as $n_0 \to \infty$. From (31) we see that

$$f(t) = \xi\,t - \eta\,\gamma_1(t) = \frac{3}{2}\,\pi\,n_0\,9^{n_0}\Big(1 - \frac{n_0}{n}\Big)t + \frac{\pi}{2}\,\frac{n_0^2\,9^{n_0}}{n\,9^n}$$

for $9^{-(n+1/2)} \le t \le 9^{-n}$. Thus

$$\sum_{n=n_0}^{n_1-1} \left| \mathrm{I}_n - \int_{9^{-(n+1/2)}}^{9^{-n}} \sin\left(\frac{\pi\, n_0^2\, 9^{n_0}}{2n\, 9^n}\right) \frac{dt}{t} \right| \le C \sum_{n=n_0}^{n_1-1} n_0\, 9^{n_0-n}\Big(1-\frac{n_0}{n}\Big) \le C\,.$$

And

$$\Big| \sum_{n=n_0}^{n_1-1} \left(\sin\left(\frac{\pi\, n_0^2\, 9^{n_0}}{2n\, 9^n}\right) - \sin\left(\frac{\pi\, n_0\, 9^{n_0}}{2\, 9^n}\right)\right)\Big| \le C \sum_{n=n_0}^{n_1-1} n_0\, 9^{n_0-n}\Big(1-\frac{n_0}{n}\Big)$$
$$\le C\,.$$

Therefore it suffices to show that

$$\sum_{n=n_0}^{n_1-1} \sin\Big(\frac{\pi}{2}\, n_0\, 9^{n_0-n}\Big) = \sum_{k=0}^{n_1-n_0-1} \sin\Big(\frac{\pi}{2}\, n_0\, 9^{-k}\Big)$$

is unbounded as $n_0 \to \infty$. Take $n_0 = 9^N$ for some N and note that since $k \le n_1 - n_0$ implies that $k \le N$ by (32), we have

$$\sum_{k=0}^{n_1-n_0-1} \sin\Big(\frac{\pi}{2}\, 9^{N-k}\Big) = n_1 - n_0 \ge \varepsilon \log n_0$$

for some $\varepsilon > 0$ by (32) and this completes the proof that $\mathcal{H}_{\Gamma_1}$ is unbounded in L^2. It is easy to modify γ_1 to obtain a smooth γ whose maximal function and Hilbert transform is still unbounded in every L^p and such that $v(t) \sim -1/\log t$ does not have the doubling property, but $w(t) \sim -t/\log t$ is doubling.

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An inverse Sobolev lemma

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Abstract. We establish an inverse Sobolev lemma for quasiconformal mappings and extend a weaker version of the Sobolev lemma for quasiconformal mappings of the unit ball of $\mathbb{R}^n$ to the full range $0 < p < n$. As an application we obtain sharp integrability theorems for the derivative of a quasiconformal mapping of the unit ball of $\mathbb{R}^n$ in terms of the growth of the mapping.

1. Introduction.

Suppose that u belongs to the Sobolev space $W^{1,p}(B^n(1))$, $p > n$, where $B^n(1)$ is the unit ball of $\mathbb{R}^n$. Then the Sobolev imbedding theorem states that u is uniformly Hölder continuous in $B^n(1)$ with exponent $1 - n/p$, see [GT, 7.26]. Recently, in [AK1, 4.7] we established a partial converse to this imbedding.

Theorem A. *Let f be a K-quasiconformal mapping of $B^n(1)$ into $\mathbb{R}^n$. If f is uniformly Hölder continuous in $B^n(1)$ with exponent $0 < \alpha \leq 1$, then $f \in W^{1,p}(B^n(1))$ for some $p > n$, which depends only on K, n, α.*

Thus, for quasiconformal mappings the Sobolev imbedding admits a converse. Recall that a homeomorphism $f : D \to D'$ is K-quasiconformal if $f \in W^{1,n}_{\mathrm{loc}}(D)$ and

$$|f'(x)|^n \leq K \, J_f(x)$$

holds a.e. in D. Here $|f'(x)|$ is the operator norm of the formal derivative $f'(x)$ of f.

In this note, we study the invertibility of the Sobolev lemma, which states that if u belongs to the Sobolev space $W^{1,p}(B^n(1))$, $1 \le p < n$, then $|u|^{pn/(n-p)}$ is integrable over $B^n(1)$, *cf.* [GT, 7.26]. We prove the following inverse Sobolev lemma; see Corollary 4.6.

Theorem B. *Let $0 < p \le n$, and suppose that f is a K-quasiconformal mapping of $B^n(1)$ into $\mathbb{R}^n$. Then*

$$\int_{B^n(1)} |f'|^q \, dm < +\infty\,, \qquad \text{for all } \; 0 < q < p$$

if and only if

$$\int_{B^n(1)} |f|^s \, dm < +\infty\,, \qquad \text{for all } \; 0 < s < pn/(n-p)\,.$$

It should be observed that Theorem B extends a weaker version of the Sobolev lemma to the full range $0 < p < n$. We also point out that the inverse Sobolev lemma does not, in general, hold for Sobolev functions. It seems to be a special property of quasiconformal mappings. In fact, the inverse Sobolev lemma may even fail for non-injective mappings satisfying the above inequality and in particular for analytic functions. Indeed, there exist bounded analytic functions of the unit disc whose derivatives fail to be integrable [R].

We link the integrability of the derivative of a quasiconformal mapping to the integrability of the mapping itself by means of growth estimates for the mapping. As a handy tool we employ the notion of the *average derivative* of a quasiconformal mapping introduced by K.Astala and F.W.Gehring [AG1]. This substitute for the derivative has turned out to have a number of applications in questions related to boundary distortion of quasiconformal mappings, see [AG2], [AK2], [AK1], and [H]. Following Astala and Gehring we write

$$a_f(x) = \exp\Big(\int_{B_x} \log J_f(y) \, \frac{dm}{n|B_x|} \Big)\,,$$

where $|B_x|$ is the n-measure of B_x and B_x stands for $B(x, d(x, \partial D)/2)$.

In order to establish sharp integrability results (6.1), (6.2) in terms of the dilatation K we prove a quasiconformal analogue of Koebe type growth estimates for univalent functions. In the course of our study we provide new evidence to ensure that a_f plays the role of the derivative by generalizing some classical growth estimates on the derivative of a univalent function, *cf.* [Hy, 1.3, 1.9, 3.3], [P, 1.6].

2. Preliminaries.

Our notation and terminology conform with that of [V1]. In particular, $B^n(x,r)$ and $S^{n-1}(x,r) = \partial B^n(x,r)$ are the open ball and sphere of radius r centered at x. We abbreviate $B^n(0,1)$ to $B^n(1)$ and $S^{n-1}(0,r)$ to $S^{n-1}(r)$, and we write ω_{n-1} for the $(n-1)$-measure of $S^{n-1}(1)$. D and D' will always denote proper subdomains of the n-dimensional Euclidean space $\mathbb{R}^n$, and we apply the convention $B_x = B^n(x, d(x,\partial D)/2)$ for points x in D. We write $C = C(a,\dots)$ to indicate that C depends only on the parameters $a,\dots$ Finally, for any pair E, F of disjoint, closed sets in $\overline{D}$, $M(E,F;D)$ is the modulus of the family of curves joining E, F in D, and we abbreviate $M(E,F;\mathbb{R}^n)$ to $M(E,F)$ and $M(E,\partial D;D)$ to $M(E;D)$.

Next, we collect a number of results used in our proofs. First we state the following well known modulus estimate, see [G1].

Suppose that $E, F \subset \mathbb{R}^n$ are disjoint, non-degenerate, closed, connected sets with E bounded and F unbounded. Then

$$M(E,F) \geq \omega_{n-1}\Big(\log C\,\big(1+\frac{d(E,F)}{\operatorname{diam}(E)}\big)\Big)^{1-n}, \tag{2.1}$$

where $C = C(n)$.

We will frequently employ the following basic property of quasiconformal mappings; see [V1, 18.1], [V2, 2.4].

Lemma 2.2. *Let $f : D \to D'$ be K-quasiconformal. Then for any $0 < \lambda < 1$ there exist positive constants C_1, C_2 depending only on n, K, λ such that*

$$B^n(f(x), C_1 d') \subset f(B^n(x, C_2 d)) \subset B^n(f(x), \lambda d'),$$

where $d = d(x,\partial D)$ and $d' = d(f(x),\partial D')$. Moreover, there is a constant C_3 depending only on n, K such that

$$B^n(f(x), d'/C_3) \subset f(B_x) \subset B^n(f(x), C_3 d')$$

and

$$d(f(B_x), \partial D') \geq d'/C_3 .$$

Next, from Lemma 2.2, [G2, Lemma 4] and [IN, Theorem 2] we deduce

Lemma 2.3. *There exists a constant $C_1 = C_1(n, K) \geq 1$ such that if f is K-quasiconformal in D and $B = B^n(x, r) \subset D$ satisfies $r \leq d(x, \partial D)/C_1$, then for any $0 < p < n$*

$$\int_B |f'|^n dm \leq C_2\, r^{n(1-n/p)} \Big(\int_{C_1 B} |f'|^p\, dm \Big)^{n/p} ,$$

where $C_2 = C_2(n, K, p)$ and $C_1 B = B^n(x, C_1 r)$.

Furthermore, from [IN, Proposition 3] we have

Lemma 2.4. *For each $0 < p < \infty$ and any $K \geq 1$ there is a constant $C_2 = C_2(p, n, K)$ such that for all K-quasiconformal mappings f of D*

$$|f(x)| \leq C_2\, r^{-n/p} \Big(\int_{B^n(x,r)} |f|^p\, dm \Big)^{1/p}$$

whenever $B^n(x, C_1 r) \subset D$, where $C_1 = C_1(K, n)$.

We continue with a quasiconformal analogue [AG2, 1.8] of the Koebe distortion theorem.

Lemma 2.5. *Let $f : D \to D'$ be K-quasiconformal. There is a constant C, which depends only on n, K, so that for each $x \in D$*

$$\frac{1}{C}\, d(f(x), \partial D') \leq a_f(x)\, d(x, \partial D) \leq C\, d(f(x), \partial D')$$

and

$$\frac{1}{C} \Big(\int_{B_x} |f'(y)|^n \frac{dm}{|B_x|} \Big)^{1/n} \leq a_f(x) \leq C \Big(\int_{B_x} |f'(y)|^n \frac{dm}{|B_x|} \Big)^{1/n}$$

PROOF. First of our claims is [AG2, 1.8]. Moreover,

$$\Big(\int_{B_x} |f'(y)|^n \frac{dm}{|B_x|}\Big)^{1/n} \le \Big(K\frac{|fB_x|}{|B_x|}\Big)^{1/n}$$

and, by [V1, 34.5-6],

$$\Big(\int_{B_x} |f'(y)|^n \frac{dm}{|B_x|}\Big)^{1/n} \ge \Big(K^{1-n}\frac{|fB_x|}{|B_x|}\Big)^{1/n}.$$

Therefore, owing to Lemma 2.2, our second chain of inequalities is a consequence of the first.

We conclude with a lemma which will prove useful.

Lemma 2.6. *Let $f : D \to D'$ be K-quasiconformal. If $\gamma \subset D$ is a rectifiable curve with $l(\gamma) \ge d(\gamma, \partial D)$, then*

$$\operatorname{diam}(f\gamma) \le C\int_\gamma a_f(x)\, ds\,.$$

Here C depends only on n, K.

PROOF. Pick a cover $B_1, \dots, B_k$ of γ where each ball B_i is of the form $B_i = B_{x_i}$ with x_i on γ so that no point in D lies in more than $C = C(n)$ of these balls; this is possible by the Besicovitch covering theorem. Now Lemma 2.2 yields

$$\operatorname{diam}(f\gamma) \le \sum \operatorname{diam}(f(B_i)) \le C_1 \sum d(f(x_i), \partial D')\,,$$

where $C_1 = C_1(n, K)$. On the other hand, from the assumption on the length of γ, we deduce that for each i the one-dimensional measure of $\gamma \cap B_i$ cannot be less than $d(x_i, \partial D)/2$. So, appealing to Lemmas 2.2 and 2.5, we obtain

$$\int_\gamma a_f(x)\, ds \ge C_2 \sum d(x_i, \partial D)\, a_f(x_i) \ge C_3 \sum d(f(x_i), \partial D')\,,$$

where the constants C_2, C_3 depend only on n, K. The claim follows.

Observe that the behavior of the quasiconformal mapping $f(x) = x|x|^{-1/2}$ of $B^n(1)$ at the origin shows that some assumption on the diameter of the curve γ in Lemma 2.6 is necessary.

3. A local inverse Sobolev lemma.

Throughout this section f will be a K-quasiconformal mapping of $B^n(1)$, and we assume that $f(0) = 0$, $d(0, \partial f(B^n(1))) = 1$. We prove that integrability conditions over hyperbolic balls are characterized by the growth of f.

Theorem 3.1. *The following two conditions are equivalent for* $0<p<\infty$.

a) $\displaystyle\int_{B_x} |f|^p dm \le C_2$.

b) $|f(x)| \le C_1(1-|x|)^{-n/p}$.

The constants C_1, C_2 *depend only on* p, n, K *and on each other.*

PROOF. First note that, by Lemma 2.4, *a)* implies *b)*. For the converse, observe that $d(f(x), \partial f(B^n(1))) \le 1 + |f(x)|$; hence Lemma 2.2 gives

$$|f(y)| \le (1 + C_3)(1 + |f(x)|)$$

for all $y \in B_x$, where $C_3 = C_3(n, K)$. The desired implication follows.

Theorem 3.2. *Let* $0 < p \le n$, *fix* $x \in B^n(1)$, *and suppose that* $|f(x)| \le C_1(1-|x|)^{1-n/p}$. *Then*

$$\int_{B_x} |f'|^p \, dm \le C_2 \,,$$

where $C_2 = C_2(p, n, K, C_1)$.

PROOF. Observe from Lemma 2.2 that for all $y \in B_x$,

$$|f(y)| \le C_3\,(1-|x|)^{1-n/p} \,,$$

where $C_3 = C_3(n, K, C_1)$. Now

$$\int_{B_x} |f'|^n \, dm \le K\,|f(B_x)| \le C_4\,(1-|x|)^{(1-n/p)n} \,,$$

where $C_4 = C_4(p, n, K, C_1)$. Next, Hölder's inequality yields

$$\int_{B_x} |f'|^p \, dm \le |B_x|^{1-p/n} \Big(\int_{B_x} |f'|^n \, dm \Big)^{p/n} \,,$$

and the claim follows.

Theorems 3.1 and 3.2 are local in nature. It is not too difficult to produce examples where the converse to Theorem 3.2 fails. Nevertheless, we have a global version of the converse statement which completes the chain of implications.

Theorem 3.3. *Suppose that* $\int_{B_x} |f'|^p \, dm \leq C_1$ *for all* $x \in B^n(1)$ *where* $0 < p < n$. *Then*

$$|f(x)| \leq C_2 \, C_1^{1/p} \, (1 - |x|)^{1-n/p} \, ,$$

for all $x \in B^n(1)$, *where* $C_2 = C_2(p, n, K)$.

PROOF. We conclude from Lemma 2.3 that

$$\begin{aligned} \int_B |f'|^n \, dm &\leq C_4 \, (1 - |x|)^{n(1-n/p)} \Big(\int_{B_x} |f'|^p \, dm \Big)^{n/p} \\ &\leq C_5 \, (1 - |x|)^{n(1-n/p)} \, , \end{aligned}$$

where $B = B^n(x, (1-|x|)/C_3), C_3 = C_3(n, K)$, and $C_5 = C_4(p, n, K) C_1^{n/p}$. Since the inverse mapping of f is K^{n-1}-quasiconformal, Lemma 2.2 shows that

$$\int_B |f'|^n \, dm \geq \frac{1}{C_6} \, d(f(x), \partial f(B^n(1)))^n$$

with $C_6 = C_6(n, K)$, which permits us to deduce

$$\frac{d(f(x), \partial f(B^n(1)))}{1 - |x|} \leq C_7 \, C_1^{1/p} \, (1 - |x|)^{-n/p} \, ,$$

for all $x \in B^n(1)$, where $C_7 = C_7(p, n, K)$. Thus, by Lemma 2.5, we have

$$a_f(x) \leq C_8 \, C_1^{1/p} \, (1 - |x|)^{-n/p}$$

in $B^n(1)$, where $C_8 = C_8(p, n, K)$. Since $f(0) = 0$, the claim follows from Lemma 2.6 by integrating $a_f(x)$ along the line segment joining 0 to x.

Combining Theorems 3.1-3.3 we obtain

Theorem 3.4. *Let $0 < p < n$. Then the following conditions are equivalent.*

a) $|f(x)| \leq C_1(1-|x|)^{1-n/p}$ *in* $B^n(1)$.

b) $d(f(x), \partial f(B^n(1))) \leq C_2(1-|x|)^{1-n/p}$ *in* $B^n(1)$.

c) $a_f(x) \leq C_3\,(1-|x|)^{-n/p}$ *in* $B^n(1)$.

d) $\int_{B_x} |f'|^p\, dm \leq C_4$ *for all* $x \in B^n(1)$.

e) $\int_{B_x} |f|^{pn/(n-p)}\, dm \leq C_5$ *for all* $x \in B^n(1)$.

Here all constants depend only on p, n, K and each other.

PROOF. Conditions *a*), *d*), and *e*) are equivalent by Theorems 3.1-3.3. Furthermore, *b*) follows from *a*) by the triangle inequality, and *c*) from *b*) by Lemma 2.5. Finally, Lemma 2.6 enables us to deduce *a*) from *c*).

We point out that Theorem 3.4 gives the following somewhat surprising corollary.

Corollary 3.5. *Let $s > 0$. Then*

$$|f(x)| \leq C_1\,(1-|x|)^{-s} \quad \text{in } B^n(1)$$

if and only if

$$d(f(x), \partial f(B^n(1))) \leq C_2\,(1-|x|)^{-s} \quad \text{in } B^n(1)\,.$$

For completeness, let us comment on the Sobolev lemma for quasiconformal mappings in the case $p \geq n$. From [N, 1.4] and Lemma 2.5 we have

REMARK 3.6. The following conditions are equivalent.

a) $\int_{B_x} |f'|^n\, dm \leq C_1$ for all $x \in B^n(1)$.

b) $f \in \mathrm{BMO}(B^n(1))$.

c) $d(f(x), \partial f(B^n(1))) \leq C_2$ for all $x \in B^n(1)$.

Contrary to the case $0 < p < n$, one cannot characterize the integrability condition a) of Remark 3.6 by means of the growth of f. In fact, the argument of the proof of Theorem 3.4 gives the estimate $|f(x)| \le C \log(1/(1 - |x|))$, whereas there exist univalent functions of $B^2(1)$ of slower growth but not belonging to $\mathrm{BMO}(B^2(1))$. Examples of this type can easily be constructed with the help of the equivalence on a) and c) in Remark 3.6 and the methods employed in Section 5.

Finally, here is the case $p > n$.

REMARK 3.7. If f is uniformly Hölder continuous in $B^n(1)$ with some exponent $\alpha > 0$, then $\int_{B_x} |f'|^p \, dm \le C$ for all $x \in B^n(1)$, for an exponent $p = p(n, K, \alpha) > n$. Moreover, if f is conformal, one may take $p = n/(1 - \alpha)$. Conversely, if $\int_{B_x} |f'|^p \, dm \le C$ for some $p > n$ for all $x \in B^n(1)$, then f is uniformly Hölder continuous in $B^n(1)$ with exponent $1 - n/p$. Indeed, the assertion is a consequence of [AK1, 4.7] and [GM, 2.24].

4. The global case.

In this section we present global versions of the results of the preceding section. As earlier, we assume throughout this section that f is a K-quasiconformal mapping of $B^n(1)$ with $f(0) = 0$ and $d(0, \partial f(B^n(1)))$ $= 1$. We begin with an extension of the Sobolev lemma to the full range $0 < p < n$ for which record the following lemma due to K. Astala [AK2].

Lemma 4.1. *For each $p > 0$ and for all $1/2 < r < 1$*

$$\int_{S^{n-1}(r)} |f|^p \, d\sigma \le C \int_0^r M(t, f)^p \, (1 - t)^{n-2} \, dt \, ,$$

where $C = C(p, n, K)$ and $M(t, f) = \max_{|x|=t} |f(x)|$.

Theorem 4.2. *Suppose that $\int_{B_x} |f'|^p \, dm \le C$, $0 < p < n$, for all $x \in B^n(1)$. Then for all $0 < q < pn/(n - p)$*

$$\int_{B^n(1)} |f|^q \, dm \le C_1 \, C^{q/p} \, ,$$

where $C_1 = C_1(p, q, n, K)$.

PROOF. Notice first that by Lemmas 2.2 and 2.3 it suffices to establish the assertion with $B^n(1)$ replaced by $B = B^n(1) \setminus B^n(0, 1/2)$. Now Theorem 3.3 asserts that $M(t, f) \leq C_0\, C^{1/p}\,(1-t)^{1-n/p}$ for $0 < t < 1$, where $C_0 = C_0(p, n, K)$. Hence the claim follows by integrating the inequality in Lemma 4.1.

We point out that one cannot take $q = pn/(n-p)$ in Theorem 4.2; see Remarks 4.8 below. Next we establish an inverse Sobolev lemma for quasiconformal mappings of $B^n(1)$.

Theorem 4.3. *Let $0 < p < n$. Then for any $q > p$ there is a constant $C = C(p, q, n, K)$ such that*

$$\int_{B^n(1)} |f'|^p \, dm \leq C \Big(\int_{B^n(1)} |f|^{qn/(n-p)} \, dm \Big)^{(n-p)/n}$$

PROOF. Let $q > p$. It suffices to establish the integrability condition with $B^n(1)$ replaced by $B = B^n(1) \setminus B^n(0, 1/2)$. Now Hölder's inequality gives

$$\begin{aligned} \int_B |f'|^p \, dm &= \int_B |f'|^p |f|^{-q} |f|^q \, dm \\ &\leq \Big(\int_B |f'|^n |f|^{-nq/p} \, dm \Big)^{p/n} \Big(\int_B |f|^{qn/(n-p)} \, dm \Big)^{(n-p)/n} \end{aligned}$$

Next, the quasiconformality of f yields

$$\int_B |f'|^n |f|^{-nq/p} \, dm \leq K \int_{f(B)} |x|^{-nq/p} \, dm \, .$$

With the help of Lemma 2.2 we conclude that $f(B) \subset \mathbb{R}^n \setminus B^n(0, C_2)$. Hence

$$\int_{f(B)} |x|^{-nq/p} \, dm \leq C_3 \int_{C_2}^{\infty} t^{n-1-nq/p} \, dt \, .$$

The assertion follows from this string of inequalities because

$$n - 1 - \frac{nq}{p} < -1 \, .$$

From the proof of Theorem 4.3 we further deduce

Theorem 4.4. *Let* $0 < p < n$, *and suppose that*

$$\int_{f(B^n(1))\setminus B^n(1)} |x|^{-n}\, dm = C < +\infty\,.$$

Then

$$\int_{B^n(1)} |f'|^p\, dm \le C_1\, C^{p/n} \Big(\int_{B^n(1)} |f|^{pn/(n-p)}\, dm \Big)^{(n-p)/n}$$

Combining Theorems 3.4, 4.2 and 4.3 we have

Corollary 4.5. *Let* $0 < p < n$ *and suppose that* $\int_{B_x} |f'|^p\, dm \le M$ *in* $B^n(1)$ *or that* $\int_{B_x} |f|^{pn/(n-p)}\, dm \le M$ *in* $B^n(1)$. *Then*

$$\int_{B^n(1)} |f'|^q\, dm < \infty\,, \qquad \textit{for any}\ \ 0 < q < p\,.$$

Next, combining Theorems 3.4, 4.2, 4.3 and Corollary 4.5, we deduce

Corollary 4.6. *Let* $0 < p \le n$. *Then the following conditions are equivalent.*

a) $\displaystyle\int_{B^n(1)} |f'|^q\, dm < +\infty$, *for all* $0 < q < p$.

b) $\displaystyle\int_{B^n(1)} |f|^s\, dm < +\infty$, *for all* $0 < s < pn/(n-p)$.

c) $|f(x)| \le C_1\,(1-|x|)^{1-n/q}$ *in* $B^n(1)$, *for each* $0 < q < p$, *for some* C_1.

d) $\displaystyle\int_{B_x} |f'|^q\, dm \le C_2$, *for all* $x \in B^n(1)$, *for each* $0 < q < p$, *for some* C_2.

e) $\displaystyle\int_{B_x} |f|^s\, dm \le C_3$, *for all* $x \in B^n(1)$, *for each* $0 < s < pn/(n-p)$, *for some* C_3.

4.7. OPEN QUESTIONS.

(a) Suppose that $\int_{B^n(1)} |f'|^p\, dm < +\infty$ for some $0 < p < n$. Does it follow that $\int_{B^n(1)} |f|^q\, dm < +\infty$ with $q = pn/(n-p)$? By the Sobolev lemma this is the case for $1 \le p < n$, and from Theorem 4.2 we know that this integral converges for $0 < p < 1$ provided $0 < q < pn/(n-p)$.

(b) Suppose that $\int_{B_x} |f'|^p\, dm \le M$ for all $x \in B^n(1)$. If $0 < p \le n$, then Corollary 4.5 ensures that $\int_{B^n(1)} |f'|^q\, dm < +\infty$ for all $0 < q < p$. On the other hand, one can apply the example in [K] to show that this is not, in general, true for $p > n$. Is the conclusion nevertheless valid for analytic univalent functions of the unit disc for all $0 < p < \infty$? If this is the case, then Remark 3.7 would show that $\int_{B^2(1)} |f'|^q\, dm < +\infty$ for all $0 < q < 2/(1-\alpha)$ provided f is uniformly Hölder continuous in $B^2(1)$ with exponent $0 < \alpha < 1$. With some work one can show that this, in turn, would yield that the Hausdorff dimension of $\partial f(B^2(1))$ is at most $2/(1+\alpha)$.

REMARKS 4.8.

(a) Theorem 4.2 does not hold for $q = pn/(n-p)$ and Corollary 4.5 does not extend to the case $q = p$. Indeed, a simple counterexample is provided by the quasiconformal mapping $f(x) = (x-w)\,|x-w|^{-1-a}$, $a > 0$, where $w \in S^{n-1}(1)$. An appropriate modification of f shows that the assumption on $f(B^n(1))$ in Theorem 4.4 is necessary and that Theorem 4.3 fails for $q = p$.

(b) Corollary 4.6 shows that for $0 < p < n$ the global integrability of a quasiconformal mapping of $B^n(1)$ and that of its derivative are more or less completely characterized by the growth of the mapping.

(c) A look at the proof of Theorem 4.3 shows that we did not need the fact that the domain in consideration is a ball. Consequently, Theorem 4.3 extends to any domain D.

5. Koebe type distortion theorems.

Motivated by Corollary 4.6 we turn our attention towards distortion estimates for quasiconformal mappings. We establish the following distortion theorem that for plane univalent functions reduces to the classical results, *e.g.* [Hy, 1.3, 1.9], [P, 1.6]. Some parts of the theorem are apparently folklore, but we have not been able to locate these results in the literature except for the upper bound in (a), which is a special case of [FMV, 4.2].

Theorem 5.1. *Let f be K-quasiconformal in $B^n(1)$, and assume that $f(0) = 0$, $d(f(0), \partial f(B^n(1))) = 1$. Set $a = K^{1/(n-1)}$ and $b = (2K)^{1/(n-1)}$. Then*

a) $|f(x)| \geq |x|^a/C$ *and* $|f(x)| \leq C(1-|x|)^{-b}$.

b) $(1-|x|)^{b-1}/C \leq a_f(x) \leq C(1-|x|)^{-b-1}$.

c) If $fB^n(1)$ is convex, then

$$|f(x)| \leq C(1-|x|)^{-a}$$

and

$$(1-|x|)^{a-1}/C \leq a_f(x) \leq C(1-|x|)^{-a-1}.$$

d) For $n \geq 3$ there is $a' = a'(n,K)$ with $a' \to 1$ as $K \to 1$ such that c) holds without the convexity assumption if a is replaced with a'.

Here $C = C(n,K)$.

We divide the proof of Theorem 5.1 into several lemmas. To simplify our statements we assume in Lemmas 5.2-5.5 that $f : B^n(1) \to D = f(B^n(1))$ is K-quasiconformal, $f(0) = 0$, and $d(f(0), \partial D) = 1$.

Lemma 5.2. *We have*

$$|f(x)| \leq C(1-|x|)^{-b} \quad \text{and} \quad a_f(x) \leq C(1-|x|)^{-b-1},$$

where $b = (2K)^{1/(n-1)}$ and $C = (n,K)$.

PROOF. By Lemmas 2.2 and 2.5 we may assume that $|x| \geq 1/2$. For each such x set $E_x = \overline{B}_x$, and let $F = \overline{B}^n(0,1/5)$. Using a standard modulus argument, we deduce from (2.1) that

$$2\,M(E_x, F; B^n(1)) \geq \omega_{n-1}\left(\log\frac{C_1}{1-|x|}\right)^{1-n},$$

where $C_1 = C_1(n)$. On the other hand, (2.2) shows that $f(F) \subset B^n(C_2)$ and $|f(x)| \leq C_2|f(y)|$ whenever $|f(x)| \geq C_3$, where C_2, C_3 depend only on n, K. Hence

$$M(f(E_x), f(F); D) \leq \omega_{n-1}\left(\log(C_4\,|f(x)|)\right)^{1-n}$$

provided $|f(x)| \geq C_3$, where both constants C_3, C_4 depend only on n, K. By the quasiconformality of f

$$M(E_x, F; B^n(1)) \leq K\, M(f(E_x), f(F); D),$$

which permits as to infer that

$$\log(C_4\,|f(x)|) \leq b\,\log\big(\frac{C_1}{1-|x|}\big)$$

provided $|f(x)| \geq C_3$ and the proof for our first claim is complete.

Finally, the estimate for a_f follows from the first claim and Theorem 3.4.

Lemma 5.3. *We have*

$$|f(x)| \geq C\,|x|^a \quad \textit{and} \quad a_f(x) \geq C\,(1-|x|)^{b-1},$$

where $a = K^{1/(n-1)}$, $b = (2K)^{1/(n-1)}$, *and* $C = C(n, K)$.

PROOF. For each $0 < r < 1$, let $E_r = \overline{B}^n(r)$. Then $M(E_r; B^n(1)) = \omega_{n-1}\,(\log(1/r))^{1-n}$. On the other hand, if $\operatorname{diam}(f(E_r)) < 1/2$, then $M(f(E_r); D) \leq \omega_{n-1}\,(\log(1/\operatorname{diam}(f(E_r)))^{1-n}$. Since f is K-quasiconformal, we conclude that

$$\Big(\log\frac{1}{\operatorname{diam}(f(E_r))}\Big)^{n-1} \leq K\,\Big(\log\frac{1}{r}\Big)^{n-1}$$

Hence $\operatorname{diam}(f(E_r)) \geq C_1\,r^a$, where $a = K^{1/(n-1)}$ and $C_1 = C_1(n, K)$. The desired bound for $|f(x)|$ is now a consequence of Lemma 2.2.

Next we estimate $a_f(x)$. By Lemma 2.5 it suffices to show that

$$d(f(x), \partial D) \geq C_2\,(1-|x|)^b,$$

for each $x \in B^n(1)$ for some constant C_2 which only depends on K, n. Lemmas 2.2 and 2.5 permit us to assume that $|x| \geq 1/2$. Set again $E_x = \overline{B}_x$ for each such x and define $F = \overline{B}^n(0, 1/5)$. From Lemma 2.2 and the argument of the proof of Lemma 5.2 we conclude that it suffices to find constants C_3 and δ_1, which depend only on n, K, such that

$$M(f(E_x), f(F); D) \leq \omega_{n-1}\,\Big(\log\frac{C_3}{\operatorname{diam}(f(E_x))}\Big)^{1-n},$$

whenever $\operatorname{diam}(f(E_x)) \leq \delta_1$. Now Lemma 2.2 yields that $d(f(F), \partial D) \geq \delta_2 > 0$, where $\delta_2 = \delta_2(n, K)$. Applying again Lemma 2.2 we find a constant $\delta_1 = \delta_1(n, K, \delta_2) < \delta_2/2$ such that

$$f(F) \subset D \setminus B^n(f(x), \delta_2/2)$$

whenever $\operatorname{diam}(f(E_x)) \leq \delta_1$. Since $f(E_x) \subset B^n(f(x), \operatorname{diam}(f(E_x)))$, the desired modulus inequality follows with $C_3 = \delta_2/2$, and the proof is complete.

Lemma 5.4. *Suppose that D is convex. Then*

$$|f(x)| \leq C\,(1 - |x|)^{-a}$$

and

$$(1 - |x|)^{a-1}/C \leq a_f(x) \leq C\,(1 - |x|)^{-a-1}\,,$$

where $a = K^{1/(1-n)}$ and $C = C(n, K)$.

PROOF. Since D is convex, we have [V1, 7.7] for each $z \in \partial D$ that

$$2\,M(\overline{B}^n(z, r) \cap \overline{D}, \overline{D} \setminus B^n(z, R); D) \leq \omega_{n-1} \left(\log \frac{R}{r}\right)^{1-n} \tag{5.5}$$

whenever $0 < r < R$. So, using the notation of the proof of Lemma 5.2, we obtain

$$2\,M(f(E_x), f(F); D) \leq \omega_{n-1} \left(\log(C_5\,|f(x)|)\right)^{1-n},$$

whenever $|f(x)| \geq C_6$, where C_5, C_6 both depend only on n, K. The desired bound for $|f(x)|$ follows as in the proof of Lemma 5.2, and Theorem 3.4 yields the analogous bound for $a_f(x)$.

The lower bound for $a_f(x)$ is obtained using (5.5) in an appropriate step in the proof of Lemma 5.3.

Lemma 5.6. *For $n \geq 3$ there is $a' = a'(n, K)$ with $a' \to 1$ as $K \to 1$ and such that the estimates in Lemma 5.4 hold without the convexity assumption if a is replaced with a'.*

PROOF. As established in [AH, 1.2] and in [T], for $K \leq K_1(n)$, f has a

K'-quasiconformal extension $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$, where $K' = K'(n, K)$ satisfies $K' \to 1$ as $K \to 1$; see [V1, 13] for the definition of a quasiconformal mapping of $\overline{\mathbb{R}}^n$. Hence

$$M(E_x, F) \leq K' \, M(f(E_x), f(F)),$$

where E_x, F are as in the proof of Lemma 5.2. Thus, replacing K with K', the argument of the proof of Lemma 5.2 gives the desired upper bounds for $|f(x)|$ and $a_f(x)$. The lower bound for $a_f(x)$ follows by modifying the proof of Lemma 5.3.

EXAMPLE 5.7. Theorem 5.1 is sharp for each $K \geq 1$ for $n = 2$ and for convex images for general n.

Let $n = 2$, fix $K \geq 1$, and let k denote the Koebe function. Define $f(x) = x|x|^{K-1}$. Then f is K-quasiconformal and, consequently, $h = f \circ k$ is K-quasiconformal. Now a simple calculation shows that for $x = (0, t)$, $0 < t < 1$, we have $h(x) \geq C(1 - |x|)^{-2K}$, $a_h(x) \geq C(1 - |x|)^{-2K-1}$, and $a_h(-x) \leq C\,(1 - |x|)^{2K-1}$. Furthermore, $|f(x)| = |x|^K$. Finally, for the convex case set $g(x) = (x - w)|x - w|^{-1-K}$ for some $w \in S^1(1)$ (for $n > 2$ set $g(x) = (x - w)\,|x - w|^{-1-a}$ for some $w \in S^{n-1}(1)$, where $a = K^{1/(n/1)}$).

REMARK 5.8. The proofs of Lemmas 5.2 and 5.4 show that if $f(B^n(1))$ is contained in a half space, then $|f(x)| \leq C\,(1 - |x|)^{-a}$; the constant C will in this case also depend on the distance from the origin to the boundary of the half space.

6. Sharp integrability exponents.

The results of sections 4 and 5 yield sharp integrability exponents.

Theorem 6.1. *Suppose that f is a K-quasiconformal mapping of $B^n(1)$. Then*

$$\int_{B^n(1)} |f'|^p \, dm < +\infty,$$

for all $0 < p < n/(1 + (2K)^{1/(n-1)})$. Moreover, if $f(B^n(1))$ is contained in a half space, then $2K$ may be replaced with K. Furthermore,

$$\int_{B^n(1)} |f|^{pn/(n-p)} \, dm < +\infty$$

for the indicated values of p.

PROOF. The claim follows from Theorems 3.4, 4.3, 5.1 and Remark 5.8.

REMARKS 6.2.

(a) A result of Jerison and Weitsman [JW] implies that $|f|^q$ is integrable for some exponent $q = q(n, K)$, but the exponent obtained from their work is not sharp. The exponents in Theorem 6.1 are sharp for each $K \geq 1$ in the plane and for each $K \geq 1$ in $\mathbb{R}^n$, $n > 2$, for mappings into a half space. This follows via a simple calculation for the functions in Example 5.7. We refer the reader to [AK2] for the H^p-theory of quasiconformal mappings.

(b) We deduce the following from Theorem 6.1. If $D \subset \mathbb{R}^2$ is any simply connected domain and $f : D \to B^2(1)$ is K-quasiconformal, then $\int_D |f'(x)|^p dx < +\infty$ for all $2 - 2/(1+2K) < p \leq 2$; compare with [AK1, 4.10]. We point out that the standard factorization argument combined with the analogous result for univalent functions, *cf.* [B], due to F.W.Gehring and W.K.Hayman, fails to give this sharp bound.

(c) Suppose that f is K-quasiconformal in $\mathbb{R}^n \setminus \{0\}$. Then the arguments used in Section 5 apply to verify that $|f(x)| \leq C\,|x|^{-a}$ in $B^n(1)$, where $a = K^{1/(n-1)}$. Now integrating this estimate we observe that $\int_{B^n(1)} |f|^p \, dm < +\infty$, for $0 < p < n/K^{1/(n-1)}$. Hence, by and Theorem 4.3 and Remark 4.8, the analogous integrability result for $|f'|$ holds for $0 < p < n/(1 + K^{1/(n-1)})$. As easily seen, the above upper bounds for p are sharp.

We do not know whether the claim of Theorem 6.1 holds for all K-quasiconformal mappings of $B^n(1)$ in the case $n \geq 3$ if $2K$ is replaced with K. Next we produce an estimate which is asymptotically sharp as $K \to 1$; observe that if $w \in S^{n-1}(1)$, then $|f'|^{n/2}$ is not integrable over $B^n(1)$ for the Möbius transformation $f(x) = (x - w)\,|x - w|^{-2}$.

Theorem 6.3. *Let f be K-quasiconformal in $B^n(1)$, $n \geq 3$. Then*

$$\int_{B^n(1)} |f'|^p \, dx < +\infty\,, \qquad \textit{for all } \; 0 < p < p_0(n, K)\,,$$

where $p_0(n, K) \to n/2$ as $K \to 1$.

PROOF. First note that by Lemma 5.6

$$|f(x)| \leq C(1-|x|)^{-a}, \qquad x \in B^n(1),$$

with $a = a(n, K) \to 1$ as $K \to 1$. Then Theorem 3.4 and Corollary 4.5 yield the desired estimate.

ADDED IN PROOF. We have recently (Buckley, S. and Koskela, P., Sobolev-Poincaré inequalities for $0 < p < 1$) answered 4.7.(a) in the positive.

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Rough isometries and p-harmonic functions with finite Dirichlet integral

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1. Introduction.

Let G be an open subset of a Riemannian n-manifold M^n. A function $u \in C(G) \cap W^1_{p,\mathrm{loc}}(G)$, with $1 < p < \infty$, is called *p-harmonic in G* if it is a weak solution of

$$-\mathrm{div}\,\big(|\nabla u|^{p-2}\,\nabla u\big) = 0\,, \tag{1.1}$$

that is,

$$\int_G \big\langle |\nabla u|^{p-2}\,\nabla u, \nabla\varphi\big\rangle\, dm = 0 \tag{1.2}$$

for every $\varphi \in C_0^\infty(G)$. Equation (1.1) is the Euler-Lagrange equation of the variational integral

$$\int_G |\nabla u|^p\; dm\,.$$

We say that a Riemannian n-manifold M^n has the *Liouville D_p-property* if every p-harmonic function u on M^n with

$$\int_{M^n} |\nabla u|^p\; dm < +\infty$$

is constant. In this paper we study the invariance of the Liouville D_p-property under rough isometries between Riemannian manifolds; see Section 3 for the definition of a rough isometry. We prove that if M^n and N^ν are roughly isometric, and if both M^n and N^ν have bounded geometry, then M^n has the Liouville D_p-property if and only if so does N^ν (Theorem 5.13). Note that the dimension of M^n may differ from that of N^ν.

Our result is new also for harmonic functions ($p = 2$) even in 2-dimensional case. Indeed, in all previous results, excepting P. Pansu's result which will be discussed later, manifolds M^n and N^ν must be homeomorphic. It is known that the Liouville D_2-property is preserved under quasiconformal mappings between 2-dimensional Riemannian manifolds and under bilipschitz (sometimes also called quasi-isometric) maps in all dimensions $n \geq 2$. See, for instance, [SN, p. 405-411] where also slightly more general classes of maps are studied in this context. Note that rough isometries need not be continuous. Thus they form a very large class of maps which, however, have nice invariance properties. It is worth noting that a similar stability result is not true for positive (or bounded) harmonic functions even under bilipschitz maps. Indeed, Lyons [L] has constructed a manifold M and two metrics g and g', with $c^{-1}g' \leq g \leq c\,g'$ such that (M, g) has no non-constant positive harmonic functions but (M, g') carries a non-constant bounded harmonic function. It is an interesting open problem whether a similar unstability result holds for p-harmonic functions if $p \neq 2$. We remark that Pansu [P] has also studied the invariance of the Liouville D_p-property under rough isometries but under more restrictive assumptions on manifolds M^n and N^ν. He assumes that a global Sobolev inequality $\|u\|_q \leq c\,\|\nabla u\|_p$, with $q \geq p \geq 1$, holds for C_0^∞-functions of M^n and N^ν, and that cohomology groups $H^1(X, \mathbb{R})$, $X = M^n, N^\nu$, are trivial. With these additional requirements on M^n and N^ν, the same conclusion as in Theorem 5.13 can be made. He has informed the author that it is possible to obtain our result also by refining his arguments. However, our methods are different.

The proof of the result in this paper is based on ideas of A. A. Grigor'yan and M. Kanai. In [K2] Kanai showed that the positivity of 2-capacity at infinity, and so the existence of Green's function for the Laplace equation, is preserved under rough isometries between Riemannian manifolds of bounded geometry. On the other hand, Grigor'yan [G] has presented a criterion, which involves 2-capacities, for the existence of a non-constant harmonic function with L^2-integrable gradient on a Riemannian manifold. For the proof of our result, we first generalize Grigor'yan's criterion to the non-linear case at hand. Here we present somewhat shorter proofs than Grigor'yan's original ones which are not fully available in our setting. Then we show, by modifying Kanai's arguments, that the p-capacities in the criterion for the Liouville D_p-property remain essentially unchanged in rough isometries between manifolds of bounded geometry. The lack of injectivity of a rough isometry causes here some troubles which, however, can be solved by using a (semi)local Harnack inequality (Theorem 3.3). We want to emphasize that it is easy to obtain local Harnack's inequalities in the following form from known results in $\mathbb{R}^n$ by using suitable chart maps. Suppose $D \subset M^n$ is an open set and $C \subset D$ is compact. Then there exists a positive constant c such that

$$\sup_C u \leq c \inf_C u \tag{1.3}$$

whenever u is a positive p-harmonic function in D. The main disadvantage of (1.3) is that, with no assumptions on the geometry of M^n, the constant c depends not only on metric parameters of C and D but also on the location of D on M^n. Such an inequality is useless in the proof of the main result. In Section 3 we prove inequality (1.3) with $D = B(x,r)$, $C = \bar{B}(x,r/2)$, and with c independent of x if M^n has bounded geometry. Here $r \leq r_0 \leq 2(\text{inj } M^n)/3$ and c depends on r_0 but not on r. We think that this inequality may also have independent interest.

The main result is formulated for so called *$\mathcal{A}$-harmonic functions* which are continuous solutions of

$$-\text{div } \mathcal{A}(\nabla u) = 0\,,$$

where $\langle \mathcal{A}(\nabla u), \nabla u \rangle \approx |\nabla u|^p$, with $1 < p < \infty$. The precise assumptions on $\mathcal{A}$ are given in 2.16. In [H1-2] and [HR] we studied a classification of Riemannian manifolds based on the existence of non-constant $\mathcal{A}$-harmonic functions with various properties. By [H1, Section 5], there exists a non-constant bounded p-harmonic function v in M^n, with $\int_{M^n} |\nabla v|^p \, dm < +\infty$, if and only if M^n admits a non-constant $\mathcal{A}$-harmonic function u, with $\int_{M^n} |\nabla u|^p \, dm < +\infty$, for some, or, in fact, for every $\mathcal{A} \in \mathcal{A}_p(M^n)$. Thus it suffices to consider only bounded p-harmonic functions if we want to study whether a given manifold carries a non-constant $\mathcal{A}$-harmonic function with L^p-integrable gradient and $\mathcal{A}$ of type p.

Harmonic functions and rough isometries on graphs are studied in Markvorsen, S., McGuinness, S., Thomassen, C., "Transient random walks on graphs and metric spaces with applications to hyperbolic surfaces", *Proc. London Math. Soc.* (3) **64** (1992), 1-20.

2. Preliminaries.

2.1. Terminology.

Throughout the paper we assume that M^n is a non-compact, connected, and oriented Riemannian n-manifold, where $n \geq 2$, of class C^∞ equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$. The Riemannian distance and the volume form will be denoted by d and dm, respectively, and $|A| = \int_A dm$ stands for the volume of a measurable set $A \subset M^n$. Furthermore, if $|A| > 0$, we write

$$u_A = -\!\!\!\!\!\!\int_A u\, dm = \frac{1}{|A|} \int_A u\, dm$$

for the integral average of a measurable function u of A.

A vector field $X \in \operatorname{loc} L^1(G)$ is a (distributional) gradient of a function $u \in \operatorname{loc} L^1(G)$ if

$$\int_G u \operatorname{div} Y\, dm = -\int_G \langle X, Y \rangle\, dm$$

for all vector fields $Y \in C_0^1(G)$. The space of all functions $u \in L^1_{\mathrm{loc}}(G)$ whose distributional gradient ∇u belongs to $L^p(G)$, where $1 \leq p < \infty$, will be denoted by $L^1_p(G)$. The Sobolev space $W^1_p(G)$ consists of all functions $u \in L^1_p(G)$ which belong to $L^p(G)$, too. We equip $L^1_p(G)$ and $W^1_p(G)$ with the seminorm $\|\nabla u\|_p$ and with the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p\,,$$

respectively. The closures of $C_0^\infty(G)$ in $L^1_p(G)$ and in $W^1_p(G)$ are denoted by $L^1_{p,0}(G)$ and $W^1_{p,0}(G)$, respectively.

Throughout the paper $c, c_0, c_1, \ldots$ will be positive constants, and $c(a, b, \ldots)$ denotes a constant depending on $a, b, \ldots$ The actual value of c may vary even within a line.

Most of the time we assume that M^n is complete and has *bounded geometry* which, in this paper, means that the Ricci curvature of M^n

is uniformly bounded from below by $-(n-1)K^2$, with $K > 0$, and the injectivity radius of M^n, denoted by $\operatorname{inj} M^n$, is positive. The well-known comparison theorems [BC, p. 253-257] and [CGT, Section 4] then give estimates

$$|B(x,r)| \leq V_K(r) \quad \text{and} \quad \frac{|B(x,R)|}{|B(x,r)|} \leq \frac{V_K(R)}{V_K(r)} \tag{2.2}$$

for the volumes of geodesic balls for all $x \in M^n$ and $R \geq r > 0$. Here $V_K(r)$ is the volume of a geodesic ball of radius r in the simply connected complete Riemannian n-manifold of constant sectional curvature $-K^2$. This estimate holds without the assumption on the injectivity radius. By applying (2.2) to volumes of n-balls in $\mathbb{R}^n$, we obtain

$$\frac{V_K(r)}{r^n} \leq \frac{V_K(R)}{R^n} \tag{2.3}$$

for $R \geq r > 0$. Volumes of small geodesic balls in M^n have a lower bound

$$|B(x,r)| \geq v_0 r^n \tag{2.4}$$

for all $x \in M^n$ and for all $r \leq \operatorname{inj} M^n/2$, where v_0 is a positive constant depending only on n. This estimate is proved by C. B. Croke [Cr]. Another result of Croke which will be used in this paper is the following isoperimetric inequality

$$|D|^{(n-1)/n} \leq c \operatorname{area}(\partial D),$$

where $D \subset B(x,r)$ is a domain with smooth boundary, $r \leq \operatorname{inj} M^n/2$, and c depends only on n; see [Cr, Theorem 11] and [CGT, p. 16-17]. Hence

$$|D|^{(m-1)/m} \leq c\, |B(x,r)|^{1/n-1/m} \operatorname{area}(\partial D) \tag{2.5}$$

if $m \geq n$. It is well-known that the isoperimetric inequality (2.5) implies that

$$\begin{aligned} &\left(\int_{B(x,r)} |u|^{m/(m-1)}\, dm\right)^{(m-1)/m} \\ &\qquad\qquad \leq c\,|B(x,r)|^{1/n-1/m} \int_{B(x,r)} |\nabla u|\, dm\,, \end{aligned} \tag{2.6}$$

where c is the same constant as in (2.5) and $u \in C_0^\infty(B(x,r))$; see, for example [C]. We obtain a Sobolev estimate by applying (2.6) and Hölder's inequality to functions $v = |u|^\gamma$, where $u \in C_0^\infty(B(x,r))$ and γ is suitable, and approximating.

Lemma 2.7. *Suppose that M^n is a complete Riemannian n-manifold, with* $\operatorname{inj} M^n > 0$, *and that* $1 \le p < m$, *where* $m \ge n$. *Then there exists a constant* $c = c(n,m,p)$ *such that*

$$\left(\int_{B(x,r)} |u|^{pm/(m-p)}\,dm\right)^{(m-p)/m} \le c\,|B(x,r)|^{p/n-p/m} \int_{B(x,r)} |\nabla u|^p\,dm \tag{2.8}$$

for every $u \in W^1_{p,0}(B(x,r))$ *and* $r \le \operatorname{inj} M^n/2$.

The above estimate will be used in the proof of Harnack's inequality together with a Poincaré inequality. We recall Buser's isoperimetric inequality [B, Section 5]

$$\frac{\operatorname{area}(\partial\Omega \cap B)}{|\Omega|} \ge \frac{c^{1+Kr}}{r}, \tag{2.9}$$

where $B = B(x,r)$, Ω is an open subset of B with smooth boundary such that $|\Omega| \le |B|/2$, and $c < 1$ depends only on n. Note that r can be arbitrary large in this inequality. Buser normalized the metric so that the lower bound for the Ricci curvature is $-(n-1)$. By rescaling the metric back to our setting, we obtain (2.9). We rewrite the right hand side of (2.9) as $r^{-1}e^{-c_n(1+Kr)}$ where $c_n > 0$ depends only on n. The analytic counterpart of (2.9) is the following local Poincaré inequality

$$\int_B |u - u_B|\,dm \le r\,e^{c_n(1+Kr)} \int_B |\nabla u|\,dm, \tag{2.10}$$

where $c_n > 0$ and $u \in W^1_1(B)$; see [C], [K2] for deducing (2.10) from (2.9).

2.11. Rough isometries and nets on manifolds.

Following Kanai [K1-3], we say that a mapping $\varphi: X \to Y$ between metric spaces X and Y is a *rough isometry* if, for some $c > 0$, the c-neighborhood of φX coincides with Y, and if there exist constants $a \geq 1$ and $b \geq 0$ such that

$$a^{-1}\, d(x,y) - b \leq d\big(\varphi(x), \varphi(y)\big) \leq a\, d(x,y) + b \tag{2.12}$$

for all $x, y \in X$. Note that the mapping φ need not be continuous. Two metric spaces are said to be *roughly isometric* if there is a rough isometry between them. If $\varphi: X \to Y$ is a rough isometry satisfying (2.12) with the constants a and b, it is possible to construct a rough isometry $\psi: Y \to X$. Indeed, for any $y \in Y$, there exists at least one $x \in X$ such that $d(\varphi(x), y) < c$, where c is the constant in the definition. If we set $\psi(y) = x$, then ψ satisfies (2.12) with constants a and $a(b + 2c)$, and the $a(b + c)$-neighborhood of ψY coincides with X. Thus ψ is a rough isometry. It is called a *rough inverse* of φ. Furthermore, a composition $\psi \circ \varphi: X \to Z$ of rough isometries $\varphi: X \to Y$ and $\psi: Y \to Z$ is a rough isometry. Thus being roughly isometric is an equivalence relation.

A *net* is a countable set P with a family $\{N_p\}_{p\in P}$ of finite subsets N_p of P such that, for all $p, q \in P$, $p \in N_q$ if and only if $q \in N_p$. A sequence of points $p_0, p_1, \ldots, p_\ell$ in P is said to be a *path from p_0 to p_ℓ of length ℓ* if $p_k \in N_{p_{k-1}}$ for $k = 1, \ldots, \ell$. A net is *connected* if any two points of P can be joined by a path. For any two points p and q in a connected net P, we denote by $\delta(p, q)$ the minimum of the lengths of paths from p to q. Then δ satisfies the axioms of metric, and it is called the *combinatorial metric of P*. The *boundary* of a subset $S \subset P$ is the set $\{p \in P : \ \delta(p, S) = 1\}$ and it will be denoted by ∂S.

Suppose then that M^n is a Riemannian manifold. Let P be a maximal collection of κ-separated points, where $\kappa > 0$ is a fixed constant. Then P together with a net structure $\{N_p\}_{p\in P}$ of sets $N_p = \{q \in P : 0 < d(p,q) \leq 3\kappa\}$ is called a *κ-net on M^n*, or simply a net. Since M^n is assumed to be connected, it is easy to see that P is also connected. Next we show that a κ-net with the combinatorial metric δ is roughly isometric to M^n with no curvature assumptions on M^n; see [K1].

Lemma 2.13. *Let M^n be a Riemannian manifold and let P be a κ-net on M^n. Then (M, d) and (P, δ) are roughly isometric, and furthermore*

$$\frac{1}{3\kappa}\, d(p,q) \leq \delta(p,q) \leq \frac{1}{\kappa}\, d(p,q) + 1\,, \tag{2.14}$$

for all $p, q \in P$.

PROOF. We prove that the inclusion map $i : P \to M^n$, $i(p) = p$, is a rough isometry. Since P is a maximal κ-separated set, the κ-neighborhood of $P(= iP)$ coincides with M^n. To prove the left hand inequality, let p and q be two distinct points in P, and let $\delta(p,q) = \ell$. Then there exists a path $p_0 = p, p_1, \ldots, p_\ell = q$ of length ℓ. For each $i = 1, \ldots, \ell$, and $\varepsilon > 0$, there is a smooth curve from p_{i-1} to p_i of length at most $3\kappa + \varepsilon$. Thus there exists a piecewise smooth curve from p to q whose length is at most $3\kappa\ell + \ell\varepsilon$. Letting $\varepsilon \to 0$ we conclude that $d(p,q) \leq 3\kappa\delta(p,q)$. For the right hand inequality, let p and q, with $p \neq q$, be any points in P. Again there exists a curve γ from p to q of length $l(\gamma) \leq d(p,q) + \varepsilon$. Let ℓ be a positive integer such that $\kappa(\ell - 1) < l(\gamma) \leq \kappa\ell$. Now there are points $x_0 = p, x_1, \ldots, x_{\ell-1}, x_\ell = q$ on γ such that $d(x_{i-1}, x_i) \leq \kappa$ for all $i = 1, \ldots, \ell$. For each x_i, there exists a point $p_i \in P$ such that $d(x_i, p_i) < \kappa$ since P is a maximal κ-separated set. By the triangle inequality, $d(p_{i-1}, p_i) \leq 3\kappa$, and so $\delta(p_{i-1}, p_i) \leq 1$. Hence

$$\delta(p,q) \leq \ell \leq \frac{1}{\kappa}\, l(\gamma) + 1 \leq \frac{1}{\kappa}\,(d(p,q) + \varepsilon) + 1\,,$$

and the right hand inequality follows by letting $\varepsilon \to 0$. We see that $\kappa\delta(p,q) - \kappa \leq d(p,q) \leq 3\kappa\delta(p,q)$, and therefore the inclusion map satisfies (2.12) with $a = \max\{3\kappa, 1/\kappa\}$ and $b = \kappa$.

A net P is said to be *uniform* if $\sup\{\#N_p : p \in P\} < +\infty$. If P is a κ-net on a complete Riemannian n-manifold M^n whose Ricci curvature is bounded from below by $-(n-1)K^2$, then

$$\#\{p \in P : p \in B(x,r)\} \leq \mu(r)\,, \tag{2.15}$$

for every $x \in M^n$ and $r > 0$, where $\mu(r)$ depends only on r, n, K, and κ; see [K1]. In particular, such a net P is uniform.

2.16. $\mathcal{A}$-harmonic functions.

As we mentioned in the introduction, our result applies not only to p-harmonic functions but also to solutions of a wide class of equations modeled by the p-Laplace equation (1.1). Let $\mathcal{A}$ be a mapping

$\mathcal{A} : TM^n \to TM^n$ which satisfies the following assumptions for some numbers $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$:

$$\begin{aligned}&\text{the mapping } \mathcal{A}_x = \mathcal{A} \mid T_xM^n : T_xM^n \to T_xM^n \text{ is}\\ &\text{continuous for a.e. } x \in M^n, \text{ and}\\ &\text{the mapping } x \mapsto \mathcal{A}_x(X) \text{ is measurable}\\ &\text{for all measurable vector fields } X\end{aligned} \tag{2.17}$$

for a.e. $x \in M^n$ and for all $h \in T_xM^n$,

$$\langle \mathcal{A}_x(h), h \rangle \ge \alpha \, |h|^p \,, \tag{2.18}$$

$$|\mathcal{A}_x(h)| \le \beta \, |h|^{p-1} \,, \tag{2.19}$$

$$\langle \mathcal{A}_x(h) - \mathcal{A}_x(k), h - k \rangle > 0, \tag{2.20}$$

whenever $h \ne k$, and

$$\mathcal{A}_x(\lambda h) = |\lambda|^{p-2} \, \lambda \, \mathcal{A}_x(h) \tag{2.21}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$.

A mapping $\mathcal{A}$ which satisfies conditions (2.17)-(2.21) with the constant p is said to be of *type* p. The class of all $\mathcal{A}$ of type p will be denoted by $\mathcal{A}_p(M^n)$.

A function $u \in W^1_{p,\mathrm{loc}}(G)$ is a (weak) solution of the equation

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0 \tag{2.22}$$

in G if

$$\int_G \langle \mathcal{A}(\nabla u), \nabla \varphi \rangle \, dm = 0$$

for all $\varphi \in C_0^\infty(G)$. Continuous solutions of (2.22) are called $\mathcal{A}$-harmonic.

Perhaps the most important feature of $\mathcal{A}$-harmonic functions is the following *comparison principle.* If u and v are $\mathcal{A}$-harmonic functions in $G \Subset M^n$ with $u \ge v$ on ∂G, then $u \ge v$ in G. The comparison principle has made it possible to develop a non-linear potential theory for solutions of (2.22). For the basic results in the non-linear potential theory in the Euclidean n-space we refer to [GLM], [HK], and to a forthcoming book [HKM]. Finally, we remark that it follows directly from the properties of $\mathcal{A}$ that $\lambda u + \mu$ is $\mathcal{A}$-harmonic if u is $\mathcal{A}$-harmonic and λ and μ are constants.

3. Local Harnack's inequality.

In this section we prove a local Harnack inequality for positive $\mathcal{A}$-harmonic functions on a complete Riemannian n-manifold with bounded geometry. We need the result later in the paper. Harnack's inequalities are usually proved using the Moser iteration method where Sobolev and Poincaré inequalities are involved. We start by recalling the following Caccioppoli-type inequality from [H2]. We assume that $\mathcal{A} \in \mathcal{A}_p(M^n)$ satisfies conditions (2.17)-(2.21) with constants α and β, and that $G \subset M^n$ is open.

Lemma 3.1. *Let u be a positive $\mathcal{A}$-harmonic function in G, and let $v = u^{q/p}$, where $q \in \mathbb{R} \setminus \{0, p-1\}$ and $\mathcal{A}$ is of type p. Then*

$$\int_G \eta^p \, |\nabla v|^p \, dm \le \left(\frac{\beta \, |q|}{\alpha \, |q-p+1|} \right)^p \int_G v^p \, |\nabla \eta|^p \, dm \tag{3.2}$$

holds for every non-negative $\eta \in C_0^\infty(G)$.

The most important point in the following theorem is that the constant c_0 in (3.4) does not depend on x at all.

Theorem 3.3. *Suppose that M^n is a complete Riemannian n-manifold with bounded geometry, and let $\mathcal{A} \in \mathcal{A}_p(M^n)$. Then there exists, for each $0 < r_0 \le 2 \operatorname{inj} M^n/3$, a constant $c_0 = c_0(n, p, K, r_0, \beta/\alpha)$ such that*

$$\sup_{B(x,r/2)} u \le c_0 \inf_{B(x,r/2)} u \,, \tag{3.4}$$

for every positive $\mathcal{A}$-harmonic function u in a geodesic ball $B(x,r) \subset M^n$, where $r \le r_0$.

PROOF. The proof is similar to that in [H2] but we want to give it in detail to work out how c_0 depends on various parameters. Fix $r_0 \le 2 \operatorname{inj} M^n/3$, and let $r \le r_0$. Suppose that u is a positive $\mathcal{A}$-harmonic function in $B(x,r) \subset M^n$. Let $v = u^{q/p}$, where $q \in \mathbb{R} \setminus \{0, p-1\}$, let

$m = \max\{n, p+1\}$, and write $\lambda = m/(m-p)$. The Sobolev estimate (2.8) and the Caccioppoli inequality (3.2) imply that

$$
\begin{aligned}
&\left(\int_{B(x,3r/4)} |\eta v|^{p\lambda}\, dm\right)^{1/\lambda} \\
(3.5)\quad &\le c\,|B(x,3r/4)|^{p/n-p/m} \int_{B(x,3r/4)} \left(\eta^p\,|\nabla v|^p + v^p\,|\nabla\eta|^p\right) dm \\
&\le A\left(\left(\frac{|q|}{|q-p+1|}\right)^p + 1\right) \int_{B(x,3r/4)} v^p\,|\nabla\eta|^p\, dm
\end{aligned}
$$

for every non-negative $\eta \in C_0^\infty\big(B(x,3r/4)\big)$, where

$$A = c_1\,|B(x,3r/4)|^{p/n-p/m} \qquad \text{and} \qquad c_1 = c_1(n,p,\beta/\alpha)\,.$$

Let $r/2 \le t < t' \le 3r/4$, and write $t_i = t + (t'-t)2^{-i}$ and $B_i = B(x,t_i)$ for every $i = 0,1,\ldots$ Then $(t_i - t_{i+1})^{-p} = 2^{(i+1)p}(t'-t)^{-p}$, $B_0 = B(x,t')$, and $B(x,t) \subset B_i$ for every i. For each i, we choose a non-negative $\eta_i \in C_0^\infty\big(B(x,3r/4)\big)$ such that $\eta_i = 1$ in B_{i+1}, $\eta_i = 0$ outside B_i, and $|\nabla\eta_i| \le 2(t_i - t_{i+1})^{-1}$. Next we choose $q_0 \in \mathbb{R}\setminus\{0\}$ such that

$$|q_0\lambda^i - p + 1| \ge \frac{p\,(p-1)}{2m-p} \tag{3.6}$$

for every i. Applying (3.5) to η_i and to $q = q_0\lambda^i$ yields

$$
\begin{aligned}
&\left(\int_{B_{i+1}} u^{q_0\lambda^{i+1}}\, dm\right)^{1/\lambda} \\
&\qquad\le A\left(\left(\frac{|q_0\lambda^i|}{|q_0\lambda^i - p + 1|}\right)^p + 1\right) \frac{2^{(i+1)p}}{(t'-t)^p} \int_{B_i} u^{q_0\lambda^i}\, dm\,,
\end{aligned}
$$

and so

$$
\begin{aligned}
&\left(\int_{B_j} \left(u^{q_0}\right)^{\lambda^j} dm\right)^{1/\lambda^j} \\
&\qquad\le A^{S_j} \prod_{i=0}^{j-1} \left(\frac{|q_0\lambda^i|^p}{|q_0\lambda^i - p + 1|^p} + 1\right)^{1/\lambda^i} \frac{2^{pS_j'}}{(t'-t)^{pS_j}} \int_{B_0} u^{q_0}\, dm\,,
\end{aligned}
$$

where $S_j = \sum_{i=0}^{j} \lambda^{-i}$ and $S_j' = \sum_{i=0}^{j}(i+1)\lambda^{-i}$. The condition (3.6) implies that the product above has an upper bound which depends only on n and p (note that $m = \max\{n, p+1\}$). Letting $j \to \infty$ we get $S_j \to m/p$ and

$$\sup_{B(x,t)} u^{q_0} \leq \frac{c_2\, A^{m/p}\, |B(x,t')|}{(t'-t)^m} ⨍_{B(x,t')} u^{q_0}\, dm\,, \tag{3.7}$$

with $c_2 = c_2(n,p)$ provided that (3.6) holds. The condition (3.6) holds for every $q_0 < 0$. Moreover, for every $q > 0$, there can be at most one i such that

$$\left|q\lambda^i - p + 1\right| < \frac{p(p-1)}{2m-p}\,.$$

Thus every interval $[q/\lambda, q]$ contains a number q_0 which satisfies (3.6) for all i. To get rid of (3.6), suppose that $q \neq 0$. If $q < 0$, we set $q_0 = q$, otherwise, we choose $q_0 \in [q/\lambda, q]$ such that (3.6) holds for every i. Next we choose $c_3 = \max\{c_2, (2c_1^{1/p} v_0^{1/n})^{-m}\}$. Then

$$\frac{c_3\, A^{m/p}\, |B(x,t')|}{(t'-t)^m} \geq \frac{c_3\, c_1^{m/p} v_0^{m/n} (r/2)^m}{(r/4)^m} \geq 1$$

by (2.4). It follows from (3.7) that

$$\begin{aligned} \sup_{B(x,t)} u^q &= \Big(\sup_{B(x,t)} u^{q_0}\Big)^{q/q_0} \\ &\leq \left(\frac{c_3\, A^{m/p}\, |B(x,t')|}{(t'-t)^m}\right)^{q/q_0} \left(⨍_{B(x,t')} u^{q_0}\, dm\right)^{q/q_0} \qquad (3.8) \\ &\leq \frac{c_3^{\lambda}\, A^{m\lambda/p}\, |B(x,t')|^{\lambda}}{(t'-t)^{m\lambda}} ⨍_{B(x,t')} u^q\, dm\,. \end{aligned}$$

This holds for every $q \neq 0$ and $r/2 \leq t < t' \leq 3r/4$. Next we write $\mathcal{B}(s) = B(x, r/2 + sr/4)$ for $0 \leq s \leq 1$. Since $A = c_1\, |B(x, 3r/4)|^{p/n - p/m}$, we can write (3.8) as

$$\begin{aligned} \sup_{\mathcal{B}(s)} u^q &\leq c\left(\frac{|B(x,3r/4)|}{r^n}\right)^{m\lambda/n} (s'-s)^{-m\lambda} ⨍_{\mathcal{B}(s')} u^q\, dm \\ &\leq c\left(\frac{V_K(3r_0/4)}{r_0^n}\right)^{m\lambda/n} (s'-s)^{-m\lambda} ⨍_{\mathcal{B}(s')} u^q\, dm\,. \end{aligned}$$

Here we used volume estimates (2.2) and (2.3) to obtain first $|B(x,3r/4)| \le V_K(3r/4)$ and then $V_K(3r/4)r^{-n} \le V_K(3r_0/4)r_0^{-n}$. We have proved that

$$\sup_{B(s)} u \le \left(c\,(s'-s)^{m\lambda}\right)^{-1/q} \left(\fint_{B(s')} u^q\,dm\right)^{1/q},$$

and

$$\inf_{B(s)} u \ge \left(c\,(s'-s)^{m\lambda}\right)^{1/q} \left(\fint_{B(s')} u^{-q}\,dm\right)^{-1/q}$$

for all $q > 0$ and $0 \le s < s' \le 1$, where $c = c(n,p,\beta/\alpha,K,r_0)$. By the refined version of the John-Nirenberg Theorem [BG],

$$\sup_{B(x,r/2)} u \le \exp\bigl(c\,g(u)\bigr) \inf_{B(x,r/2)} u\,,$$

where

$$g(u) = \sup_{0\le s\le 1} \inf_{a\in\mathbb{R}} \fint_{B(s)} |\log u - a|\;dm$$

and $c = c(n,p,\beta/\alpha,K,r_0)$. To estimate $g(u)$, we first use the local Poincaré inequality (2.10) and Hölder's inequality

$$g(u) \le \frac{1}{|B(x,r/2)|} \inf_{a\in\mathbb{R}} \int_{B(x,3r/4)} |\log u - a|\;dm$$

$$\le \frac{r\exp\bigl(c_n(1+Kr)\bigr)}{|B(x,r/2)|} \int_{B(x,3r/4)} |\nabla \log u|\;dm$$

$$\le \frac{r\exp\bigl(c_n(1+Kr)\bigr)\,|B(x,3r/4)|^{1-1/p}}{|B(x,r/2)|}$$

$$\cdot \left(\int_{B(x,3r/4)} |\nabla \log u|^p\;dm\right)^{1/p}.$$

Furthermore, [HK, 2.24] implies that

$$\int_{B(x,3r/4)} |\nabla \log u|^p\;dm \le c\,(p,\beta/\alpha) \int_{B(x,r)} |\nabla \eta|^p\;dm \tag{3.9}$$

for every $\eta \in C_0^\infty(B(x,r))$ such that $\eta = 1$ in $B(x, 3r/4)$. We obtain an upper bound $c\,r^{-p}\,|B(x,r)|$ for the right hand side of (3.9) by choosing η such that $|\nabla\eta| \le 8/r$. Putting together these estimates yields

$$g(u) \le c \exp\big(c_n(1+Kr)\big)\frac{|B(x,3r/4)|}{|B(x,r/2)|}\left(\frac{|B(x,r)|}{|B(x,3r/4)|}\right)^{1/p}$$

$$\le c \exp\big(c_n(1+Kr_0)\big)\frac{V_K(3r/4)}{V_K(r/2)}\left(\frac{V_K(r)}{V_K(3r/4)}\right)^{1/p}$$

Finally, we apply (2.2) and (2.3) to volumes of n-balls in $\mathbb{R}^n$ to deduce first that $c\,r^n \le V_K(r/2)$ $\big(\le V_K(3r/4)\big)$, with $c = c(n)$, and then that

$$\frac{V_K(3r/4)}{V_K(r/2)} \le \frac{V_K(3r/4)}{c\,r^n} \le \frac{V_K(3r_0/4)}{c\,r_0^n}\,.$$

Similarly,

$$\frac{V_K(r)}{V_K(3r/4)} \le \frac{V_K(r_0)}{c\,r_0^n}\,.$$

Hence $g(u)$ has an upper bound which depends only on $n, p, \beta/\alpha, K$, and r_0. The theorem is proved.

As a consequence of the local Harnack inequality we obtain the following result.

Theorem 3.10. *Suppose that M^n is a complete Riemannian n-manifold with bounded geometry and that $\mathcal{A} \in \mathcal{A}_p(M^n)$. Let*

$$r_0 = \min\{1, \frac{2}{3}\operatorname{inj} M^n\}\ .$$

Then there exists a positive constant $c_4 = c_4(n, p, \beta/\alpha, K, r_0)$ such that

$$\text{(3.11)} \qquad d(x,y) > c_4\, r_0 \max\left\{\left|\log\frac{u(x)}{u(y)}\right|, \left|\log\frac{1-u(x)}{1-u(y)}\right|\right\} - r_0\ ,$$

whenever u is $\mathcal{A}$-harmonic in M^n, with $\inf_{M^n} u = 0$ and $\sup_{M^n} u = 1$.

PROOF. Let x and y be two points in M^n. We may assume that $u(x) > u(y)$. Suppose first that $d(x,y) \ge r_0$. Let γ be a minimal geodesic from x to y, and let $\ell \ge 2$ be an integer such that $(\ell -$

$1) r_0/2 < d(x,y) \le \ell r_0/2$. Then there are points $x_0 = x, x_1, \dots, x_\ell = y$ on γ such that $d(x_i, x_{i+1}) \le r_0/2$ for all $i = 0, 1, \dots, \ell-1$. Hence $B(x_i, r_0/2) \cap B(x_{i+1}, r_0/2) \ne \varnothing$ for all $i = 0, 1, \dots, \ell - 1$. The local Harnack inequality (3.4) implies that

$$\begin{aligned} u(x) &\le \sup_{B(x_0,r_0/2)} u \le c_0 \inf_{B(x_0,r_0/2)} u \\ &\le c_0 \sup_{B(x_1,r_0/2)} u \le c_0^2 \inf_{B(x_1,r_0/2)} u \le \cdots \\ &\le c_0^\ell \sup_{B(x_\ell,r_0/2)} u \le c_0^{\ell+1} \inf_{B(x_\ell,r_0/2)} u \le c_0^{\ell+1} u(y) . \end{aligned}$$

Hence $\ell + 1 \ge (\log c_0)^{-1} \log\bigl(u(x)/u(y)\bigr)$, and so

$$d(x,y) > c_4\, r_0 \log \frac{u(x)}{u(y)} - r_0 ,$$

with $c_4 = (2\log c_0)^{-1}$. If $d(x,y) < r_0$, there exists a point $z \in M^n$ such that $x, y \in B(z, r_0/2)$. Then $u(x) \le c_0\, u(y)$ by (3.4), and so $c_4\, r_0 \log\bigl(u(x)/u(y)\bigr) - r_0 \le -r_0/2$. The theorem follows by applying the same reasoning to the function $1 - u$.

4. A criterion for the Liouville D_p-property.

Manifolds which admit non-constant harmonic functions with bounded Dirichlet integral can be characterized by means of 2-capacities; see [G]. The purpose of this section is to generalize this criterion to the non-linear case (Theorem 4.6). It should be noted that M^n need not be of bounded geometry in this section.

A *condenser* is a triple $(F_1, F_2; G)$, where F_1 and F_2 are disjoint, non-empty, and closed sets in $\bar{G}$. Its *p-capacity* is the number

$$\operatorname{cap}_p(F_1, F_2; G) = \inf_u \int_G |\nabla u|^p \, dm ,$$

where the infimum is taken over all functions $u \in L^1_p(G)$ which are continuous in $G \cup F_1 \cup F_2$ with $u = 0$ in F_1 and $u = 1$ in F_2. Such a function is called *admissible* for $(F_1, F_2; G)$. If the class of admissible functions is empty, we set $\operatorname{cap}_p(F_1, F_2; G) = +\infty$.

Let $\{B_i\}_{i=1}^{\infty}$ be an exhaustion of M^n such that $B_i \Subset B_{i+1}$ for every i. We say that a set $A \subset M^n$ is unbounded if A has common points with $M^n \setminus B_i$ for every i. For an open set $\Omega \subset M^n$ and a compact set $F \subset \bar{\Omega}$, we define

$$\operatorname{cap}_p(F, \infty; \Omega) = \lim_{i \to \infty} \operatorname{cap}_p(F, \bar{\Omega} \setminus B_i; \Omega).$$

Note that the limit exists and is independent of the exhaustion since the assumption $B_i \Subset B_{i+1}$ implies that

$$\operatorname{cap}_p(F, \bar{\Omega} \setminus B_i; \Omega) \geq \operatorname{cap}_p(F, \bar{\Omega} \setminus B_{i+1}; \Omega).$$

Definition 4.1. *An unbounded open set $\Omega \subset M^n$ is called p-hyperbolic if there exists a compact set $F \subset \bar{\Omega}$ such that* $\operatorname{cap}_p(F, \infty; \Omega) > 0$.

We remark that any open set Ω' is p-hyperbolic if there exists a p-hyperbolic subset $\Omega \subset \Omega'$. We also observe that $\operatorname{cap}_p(F, \bar{\Omega} \setminus D; \Omega) \geq \operatorname{cap}_p(F, \infty; \Omega) > 0$ for each open $D \Subset M^n$ if Ω is p-hyperbolic and F is as in the definition.

Definition 4.2. *An unbounded open set $\Omega \subset M^n$, with $\partial\Omega \neq \varnothing$, is called D_p-massive if there exists a p-harmonic function u in Ω which is continuous in $\bar{\Omega}$, with $u = 0$ in $\partial\Omega$, $\sup_\Omega u = 1$, and*

$$\int_\Omega |\nabla u|^p \, dm < +\infty .$$

It is clear from the definition that the sets $\{x : u(x) < a\}$ and $\{x : u(x) > b\}$, and even all components of these sets, are D_p-massive if u is a non-constant bounded p-harmonic function in M^n, with $|\nabla u| \in L^p(M^n)$, and $\inf u < a < b < \sup u$.

Next we explain the connection between D_p-massive and p-hyperbolic sets.

Lemma 4.3. *Every D_p-massive set is also p-hyperbolic.*

Proof. Let Ω be D_p-massive, and let u be as in Definition 4.2. Suppose that $\{B_i\}_{i=1}^{\infty}$ is an exhaustion of M^n such that $B_i \Subset B_{i+1}$, and that

$\operatorname{cap}_p(F, \bar\Omega \setminus B_2; \Omega) > 0$, where $F = \bar B_1 \cap \partial\Omega \neq \varnothing$. Next we choose admissible functions $w_i \in W^1_p(\Omega \cap B_i)$, $i \geq 2$, for condensers $(F, \bar\Omega \setminus B_i; \Omega)$ such that $0 \leq w_i \leq 1$,

$$\int_{\Omega\cap B_i} |\nabla w_i|^p \, dm \leq \operatorname{cap}_p(F, \bar\Omega \setminus B_i; \Omega) + \frac{1}{i}, \tag{4.4}$$

and that $w_i \equiv 1$ in all those components of $\Omega \cap B_i$ whose closures do not intersect F. We choose these functions in the following way. Suppose that w_2 is chosen. Let v_2 be the unique p-harmonic function in $\Omega \cap B_2$ such that $v_2 - w_2 \in W^1_{p,0}(\Omega \cap B_2)$. We set $v_2 = 1$ in $\Omega \setminus B_2$. Then

$$\int_{\Omega\cap B_2} |\nabla v_2|^p \, dm \leq \int_{\Omega\cap B_2} |\nabla w_2|^p \, dm$$

and $v_2 \geq u$ in Ω. Next we choose w_3. Then the set $A = \{x \in \Omega : w_3(x) > v_2(x)\}$ is a subset of $\Omega \cap B_2$. If $A \neq \varnothing$,

$$\int_A |\nabla v_2|^p \, dm \leq \int_A |\nabla w_3|^p \, dm,$$

since v_2 is p-harmonic in A. We redefine w_3 by setting $w_3 = v_2$ in A. Clearly (4.4) still holds. By continuing similarly, we get a decreasing sequence of functions $\{v_i\}$ such that v_i is p-harmonic in $\Omega \cap B_i$, $v_i \geq u$, and that

$$\int_{\Omega\cap B_i} |\nabla v_i|^p \, dm \leq \int_{\Omega\cap B_i} |\nabla w_i|^p \, dm.$$

To finish the proof, suppose that Ω is not p-hyperbolic. Then $\operatorname{cap}_p(F, \bar\Omega \setminus B_i; \Omega) \to 0$, and so $\int_{\Omega\cap B_i} |\nabla v_i|^p \, dm \to 0$. Since $v_i \geq u$ and $\sup_\Omega u = 1$, the only possibility is that $v_i \to 1$. This is a contradiction since $\{v_i\}$ is decreasing. Hence Ω is p-hyperbolic.

Note that the assumption $\int_\Omega |\nabla u|^p \, dm < +\infty$ was not needed in the proof. The converse of Lemma 4.3 is not true, that is, there are p-hyperbolic sets which are not D_p-massive. Indeed, let $p < n$ and let $\Omega \subset \mathbb{R}^n$ be the upper half space $\{x : x_n > 0\}$. By symmetry, $\operatorname{cap}_p\big(\bar B^n(r) \cap \Omega, \infty; \Omega\big) = \operatorname{cap}_p\big(\bar B^n(r), \infty; \mathbb{R}^n\big)/2$. It is well-known that $\operatorname{cap}_p\big(\bar B^n(r), \infty; \mathbb{R}^n\big) = c r^{n-p} > 0$. Hence Ω is p-hyperbolic. On the other hand, Ω can not be D_p-massive. Otherwise, the lower half space would be D_p-massive by symmetry. But this implies that $\mathbb{R}^n$ does not have the Liouville D_p-property (see the end of the proof of Theorem 4.6) which leads to a contradiction with [H1, 5.9, 5.11]. The exact relation between D_p-massive and p-hyperbolic sets is given by Theorem 4.5. It says that D_p-massive sets are, in general, "broader" than p-hyperbolic sets. Indeed, a D_p-massive set Ω must contain a p-hyperbolic set Ω_1 such that $\operatorname{cap}_p(\partial\Omega, \partial\Omega_1; \Omega) < +\infty$. This is the meaning of Theorem 4.5, although we have formulated it in a slightly different way to avoid difficulties with boundary regularity.

Theorem 4.5. *An unbounded open set $\Omega \subset M^n$, with $\partial\Omega \neq \varnothing$, is D_p-massive if and only if there exists a p-hyperbolic $\Omega_1 \subset \Omega$ and a continuous function v in $\bar{\Omega}$ which is p-harmonic in $\Omega \setminus \bar{\Omega}_1$, with $v = 0$ in $\partial\Omega$, $v = 1$ in Ω_1, and $\int_\Omega |\nabla v|^p \, dm < +\infty$.*

PROOF. The idea of the proof comes from [G]. Suppose first that Ω is D_p-massive. Let u be as in Definition 4.2, and let $0 < \varepsilon < 1$. Then the set $\{x \in \Omega : \ u(x) > \varepsilon\}$ is D_p-massive, and hence p-hyperbolic. Furthermore, the function $v = \min\{u, \varepsilon\}/\varepsilon$ satisfies the assumptions of the claim.

To prove the converse, let $\{B_i\}_{i=1}^\infty$ be an exhaustion of M^n, with $B_i \Subset B_{i+1}$. For $i \geq 2$, we write

$$\Omega_i = \Omega_1 \setminus \bar{B}_i, \quad G_1 = \Omega \setminus \bar{\Omega}_1, \quad G_i = \Omega \setminus \bar{\Omega}_i, \quad \text{and} \quad G_i^k = G_i \cap B_k.$$

Let u_i^k be the unique p-harmonic function in G_i^k with boundary values $u_i^k - v \in W_{p,0}^1(G_i^k)$. We set $u_i^k = v$ in $\Omega \setminus G_i^k$. Now $0 \leq u_i^k \leq v$ and $u_{i+1}^k \leq u_i^k$ in Ω. Since the sequence $\{u_i^k\}_{k=1}^\infty$ is uniformly bounded, it is equicontinuous in G_i by the Hölder-continuity estimate [T, Theorem 2.2]. By Ascoli's theorem, there exists a subsequence, still denoted by $\{u_i^k\}_{k=1}^\infty$, which converges locally uniformly in G_i to a function u_i. We set $u_i = v$ in $\Omega \setminus G_i$. Then u_i is p-harmonic in G_i and the sequence $\{u_i\}_{i=1}^\infty$ is decreasing. By Harnack's principle [HK, 3.3], the limit function $u = \lim_{i\to\infty} u_i$ is p-harmonic in Ω. If we set $u = 0$ in $\partial\Omega$, then u is continuous in $\bar{\Omega}$ since $0 \leq u \leq v$ and $v \in C(\bar{\Omega})$, with $v = 0$ in $\partial\Omega$.

Next we shall show that u (multiplied by a suitable constant) satisfies the conditions in the definition of D_p-massiveness. First we observe that

$$\begin{aligned}\int_\Omega |\nabla u_i^k|^p \, dm &= \int_{G_i^k} |\nabla u_i^k|^p \, dm + \int_{\Omega\setminus G_i^k} |\nabla v|^p \, dm \\ &\leq \int_{G_i^k} |\nabla v|^p \, dm + \int_{\Omega\setminus G_i^k} |\nabla v|^p \, dm\end{aligned}$$

$$= \int_\Omega |\nabla v|^p \, dm < +\infty .$$

Passing to a subsequence we conclude that there exists a vector field $X \in L^p(\Omega)$ such that $\nabla u_i^k \to X$ weakly in $L^p(\Omega)$ as $k \to \infty$. But the convergence of u_i^k implies that $X = \nabla u_i$. Now $u_i - v \in L^1_{p,0}(\Omega)$ since $u_i^k - v \in L^1_{p,0}(\Omega)$. This in turn implies that

$$\begin{aligned}\int_\Omega |\nabla u_i|^p \, dm &= \int_{G_i} |\nabla u_i|^p \, dm + \int_{\Omega_i} |\nabla v|^p \, dm \\ &\le \int_{G_i} |\nabla v|^p \, dm + \int_{\Omega_i} |\nabla v|^p \, dm \\ &= \int_\Omega |\nabla v|^p \, dm < +\infty .\end{aligned}$$

By repeating the above reasoning, we get that $\int_\Omega |\nabla u|^p \, dm < +\infty$ and $u - v \in L^1_{p,0}(\Omega)$. It follows from Maz'ya's lemma [M, Lemma 2], which obviously holds in our situation, that

$$|\nabla u_i|^{p-2} \nabla u_i \to |\nabla u|^{p-2} \nabla u$$

weakly in $L^{p/(p-1)}(\Omega)$. It remains to show that $u \not\equiv 0$. Since Ω_1 is p-hyperbolic, there exists a compact set $F \subset \bar{\Omega}_1$ such that $\operatorname{cap}_p(F, \infty; \Omega_1) > 0$. Let $U \Subset M^n$ be a sufficiently large connected neighborhood of F so that $U \setminus \bar{\Omega}$ is non-empty. We write $\Omega_1' = \Omega_1 \cup U$ and $F_1 = \bar{U} \setminus \Omega$. Now Ω_1' is also p-hyperbolic, and $\operatorname{cap}_p(F_1, \infty; \Omega_1') > 0$ since F_1 and F lie in a same component of Ω_1'. For each i, u_i is admissible for the condenser $(\partial\Omega, \partial\Omega_i; G_i)$. Using this fact and well-known properties of capacities we get that

$$\begin{aligned}\int_\Omega |\nabla u_i|^p \, dm &\ge \operatorname{cap}_p(\partial\Omega, \partial\Omega_i; G_i) \\ &= \operatorname{cap}_p(M^n \setminus \Omega, \bar{\Omega}_i; M^n) \\ &\ge \operatorname{cap}_p(F_1, \bar{\Omega}_1' \setminus B_i; \Omega_1') \\ &\ge \operatorname{cap}_p(F_1, \infty; \Omega_1') > 0\end{aligned}$$

if i is large enough. Furthermore,

$$\begin{aligned}\int_\Omega |\nabla u_i|^p \, dm &= \int_\Omega \langle |\nabla u_i|^{p-2} \nabla u_i, \nabla v \rangle \, dm \\ &\to \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle \, dm ,\end{aligned}$$

and so ∇u can not vanish identically in Ω. We conclude that u is non-constant. Multiplying u by a suitable constant, if necessary, we get a function which satisfies all the conditions in the definition of D_p-massiveness. The theorem is proved.

An open set $G \Subset M^n$ is called *regular* if, for all functions $h \in C(\bar{G}) \cap W_p^1(G)$,

$$\lim_{x \to y} u(x) = h(y)$$

holds at every boundary point $y \in \partial G$ whenever u is the unique p-harmonic function in G with $u - h \in W_{p,0}^1(G)$. We refer to [M], [KM], and [LM] for the results concerning the boundary regularity. For example, all domains $\Omega \Subset M^n$ with C^1-boundaries are regular for all p.

Theorem 4.6. *A Riemannian n-manifold M^n admits a non-constant p-harmonic function u, with $\int_{M^n} |\nabla u|^p \, dm < +\infty$, if and only if there exist two p-hyperbolic sets Ω_1, $\Omega_2 \subset M^n$ such that* $\operatorname{cap}_p(\bar{\Omega}_1, \bar{\Omega}_2; M^n) < +\infty$.

PROOF. If M^n does not have the Liouville D_p-property, there exists a non-constant bounded p-harmonic function u in M^n, with $\int_{M^n} |\nabla u|^p dm < +\infty$. Let $\inf u < a < b < \sup u$. Then the sets $\Omega_1 = \{x : u(x) < a\}$ and $\Omega_2 = \{x : u(x) > b\}$ are D_p-massive, hence p-hyperbolic. Moreover,

$$\operatorname{cap}_p(\bar{\Omega}_1, \bar{\Omega}_2; M^n) \le \frac{1}{(b-a)^p} \int_{M^n} |\nabla u|^p \, dm < +\infty \,,$$

since the function

$$v = \max\left\{0, \min\left\{\frac{u-a}{b-a}, 1\right\}\right\}$$

is admissible for the condenser $(\bar{\Omega}_1, \bar{\Omega}_2; M^n)$.

Suppose then that Ω_1 and Ω_2 are p-hyperbolic, with $\operatorname{cap}_p(\bar{\Omega}_1, \bar{\Omega}_2; M^n) < +\infty$. Then there exists an admissible function w for the condenser $(\bar{\Omega}_1, \bar{\Omega}_2; M^n)$. By taking slightly larger open sets Ω_1' and Ω_2' with smooth boundaries and containing Ω_1 and Ω_2, respectively, such that $\Omega_1' \subset \{x : w(x) < 1/4\}$ and $\Omega_2' \subset \{x : w(x) > 3/4\}$, we obtain p-hyperbolic sets Ω_1' and Ω_2', with $\operatorname{cap}_p(\bar{\Omega}_1', \bar{\Omega}_2'; M^n) < +\infty$. Now there exists a continuous function u in M^n which is p-harmonic in

$M^n \setminus (\bar{\Omega}'_1 \cup \bar{\Omega}'_2)$ with $u = 0$ in $\bar{\Omega}'_1$, $u = 1$ in $\bar{\Omega}'_2$, and $\int_{M^n} |\nabla u|^p \, dm < +\infty$. By Theorem 4.5, the sets $\{x : u(x) > b\}$ and $\{x : u(x) < a\}$ are disjoint D_p-massive sets for $0 < a < b < 1$. Call them G_1 and G_2. Let $\{B_i\}$ be an exhaustion of M^n such that B_i is regular for every i. Let u_j, $j = 1, 2$, be a p-harmonic function in G_j satisfying the conditions in Definition 4.2. We extend u_j to M^n by setting $u_j = 0$ in $M^n \setminus G_j$. Let $v_i \in C(\bar{B}_i)$ be p-harmonic in B_i such that $v_i = u_1$ in ∂B_i. Then

$$u_1 \le v_i \le 1 - u_2$$

in B_i. Furthermore,

$$\int_{B_i} |\nabla v_i|^p \, dm \le \int_{B_i} |\nabla u_1|^p \, dm \le \int_{M^n} |\nabla u_1|^p \, dm < +\infty .$$

Thus there exists a subsequence, denoted again by v_i, which converges locally uniformly in M^n to a p-harmonic function v. Now $u_1 \le v \le 1 - u_2$ in M^n and $\int_{M^n} |\nabla v|^p \, dm < +\infty$. Since $\sup u_1 = \sup u_2 = 1$, v can not be constant. The theorem is proved.

EXAMPLE 4.7. We close this section by an example where the Liouville D_p-property essentially depends on p. Let $M^n = S^{n-1} \times \mathbb{R}$ be equipped with a metric

$$f^2 d\vartheta^2 + dt^2 ,$$

where $d\vartheta^2$ is the standard metric of the sphere S^{n-1} normalized so that $m_{n-1}(S^{n-1}) = 1$. We assume that f is a positive C^∞-function of M^n which depends only on t-coordinate of $(\vartheta, t) \in S^{n-1} \times \mathbb{R}$ and $f(\cdot\,, -t) = f(\cdot\,, t)$. We abbreviate $f(t) = f(\cdot\,, t)$. Then

$$m_{n-1}\big(\{(\vartheta, t) \in M^n : \ t = r\}\big) = f(r)^{n-1}$$

for every $r \in \mathbb{R}$. We claim that M^n has the Liouville D_p-property if and only if the integral

$$I = \int_1^\infty f(t)^{(1-n)/(p-1)} \, dt$$

diverges. To show this, suppose that $I < +\infty$. Let $\Omega = \{(\vartheta, t) \in M^n : t > 1\}$, $F = \{(\vartheta, t) \in M^n : \ t = 1\}$ $(= \partial\Omega)$, and $B_i = \{(\vartheta, t) \in M^n : |t| < i\}$ for $i = 2, 3, \ldots$ If u is an admissible function for $(F, \bar{\Omega} \setminus B_i; \Omega)$, we have

$$\int_{\gamma_\vartheta} |\nabla u| \, ds \ge 1$$

for each curve $\gamma_\vartheta : [1,i] \to M^n$, $\gamma_\vartheta(t) = (\vartheta, t)$, where $\vartheta \in S^{n-1}$. By Hölder's inequality,

$$\begin{aligned} 1 &\le \left(\int_{\gamma_\vartheta} |\nabla u|\, ds\right)^p \\ &= \left(\int_1^i |\nabla u(\vartheta,t)|\, f(t)^{(n-1)/p} f(t)^{(1-n)/p}\, dt\right)^p \\ &\le \left(\int_1^i |\nabla u(\vartheta,t)|^p f(t)^{n-1}\, dt\right)\left(\int_1^i f(t)^{(1-n)/(p-1)}\, dt\right)^{p-1}, \end{aligned}$$

and so

$$\int_1^i |\nabla u(\vartheta,t)|^p f(t)^{n-1}\, dt \ge \left(\int_1^i f(t)^{(1-n)/(p-1)}\, dt\right)^{1-p}$$

Integrating with respect to ϑ yields

$$\begin{aligned} \int_\Omega |\nabla u|^p\, dm &\ge \int_{S^{n-1}} \left(\int_1^i |\nabla u(\vartheta,t)|^p f(t)^{n-1}\, dt\right) d\vartheta \\ &\ge \left(\int_1^i f(t)^{(1-n)/(p-1)}\, dt\right)^{1-p} \ge I^{1-p} > 0 \end{aligned}$$

since we normalized $\int_{S^{n-1}} d\vartheta = 1$. Taking the infimum over u and then letting $i \to \infty$ yields $\operatorname{cap}_p(F, \infty; \Omega) > 0$, that is, Ω is p-hyperbolic. Similarly, $\Omega' = \{(\vartheta, t) \in M^n : t < -1\}$ is p-hyperbolic. Furthermore, $\operatorname{cap}_p(\bar\Omega, \bar\Omega'; M^n) < +\infty$, and therefore M^n has a non-constant p-harmonic function v with $\int_{M^n} |\nabla v|^p\, dm < +\infty$.

Conversely, suppose that the integral I diverges. For each $r > 0$, let $D(r) = \{(\vartheta, t) \in M^n : |t| < r\}$. Fix r and R such that $R > r > 0$. For each integer $k \ge 1$ and $i = 0, 1, \dots, k$, let $t_i = r + i(R-r)/k$. By, for instance [HKM, Section 2],

$$\begin{aligned} &\left(\operatorname{cap}_p(\bar D(r), M^n \setminus D(R); M^n)\right)^{1/(1-p)} \\ &\qquad \ge \sum_{i=0}^{k-1} \left(\operatorname{cap}_p(\bar D(t_i), M^n \setminus D(t_{i+1}); M^n)\right)^{1/(1-p)}. \end{aligned}$$

We get an estimate

$$\operatorname{cap}_p(\bar D(t_i), M^n \setminus D(t_{i+1}); M^n) \le (|D(t_{i+1})| - |D(t_i)|)\,(t_{i+1} - t_i)^{-p}$$

by choosing an admissible function u such that $u(\vartheta, t) = (t-t_i)/(t_{i+1} - t_i)$ in $D(t_{i+1}) \setminus D(t_i)$. Hence

$$
\begin{aligned}
&\left(\operatorname{cap}_p(\bar{D}(r), M^n \setminus D(R); M^n)\right)^{1/(1-p)} \\
&\qquad \geq \sum_{i=0}^{k-1} \left(\frac{|D(t_{i+1})| - |D(t_i)|}{t_{i+1} - t_i}\right)^{1/(1-p)} (t_{i+1} - t_i)\,.
\end{aligned}
\tag{4.8}
$$

Next we observe that

$$
\lim_{\varepsilon \to 0} \frac{|D(t+\varepsilon)| - |D(t)|}{\varepsilon} = m_{n-1}\big(\partial D(t)\big) = 2\, f(t)^{n-1}\,.
$$

Thus the right hand side of (4.8) tends to an integral

$$
2^{1/(1-p)} \int_r^R f(t)^{(1-n)/(p-1)}\, dt
$$

as $k \to \infty$. Hence $\lim_{R\to\infty} \operatorname{cap}_p(\bar{D}(r), M^n \setminus D(R); M^n) = 0$ if the integral I diverges. Since r is arbitrary, this implies that $\operatorname{cap}_p(C, \infty; M^n) = 0$ for every compact $C \subset M^n$. It follows from [H1, Section 5] that every p-harmonic function v on M^n with $\int_{M^n} |\nabla v|^p\, dm < +\infty$ is constant.

5. p-hyperbolic nets and the main result.

Throughout this section we assume that M^n and N^ν are complete Riemannian manifolds with bounded geometry, and that $P \subset M^n$ and $Q \subset N^\nu$ are κ-nets, with $\kappa \leq \min\{\operatorname{inj} M^n, \operatorname{inj} N^\nu\}/2$.

We shall define p-hyperbolicity on nets, and therefore we need a discrete counterpart for p-capacity. For each $q \in P$ (or Q) and a real-valued function u in N_q, we set

$$
|Du(q)| = \Big(\sum_{q' \in N_q} \big(u(q') - u(q)\big)^2\Big)^{1/2}
$$

In many occasions we use the fact that, for a uniform net P,

$$
\begin{aligned}
c_4^{-1} \sum_{q' \in N_q} |u(q') - u(q)|^p &\leq |Du(q)|^p \\
&\leq c_4^{p/2} \sum_{q' \in N_q} |u(q') - u(q)|^p\,,
\end{aligned}
\tag{5.1}
$$

where $c_4 = \sup\{\#N_q : q \in P\}$. Suppose that $S \subset P$ (or Q) is a connected infinite subnet. We say that S is ***p-hyperbolic***, with $1 < p < \infty$, if there exists a finite non-empty set $E \subset S$ such that

$$\operatorname{cap}_p(E, \infty; S) = \inf_u \sum_{q \in S} |Du(q)|^p > 0,$$

where the infimum is taken over all finitely supported functions u of $S \cup \partial S$, with $u = 1$ in E. Such functions are called admissible for $(E, \infty; S)$. Recall that $\partial S = \{q : \delta(q, S) = 1\}$.

Lemma 5.2. *Suppose that $S' \subset P$ is a connected subnet, $\Omega = \{x \in M^n : d(x, S' \cup \partial S') < 7\kappa\}$, and that $S = \{q \in P : d(q, \Omega) < \kappa\}$. Then Ω is a domain and S is a connected subnet.*

PROOF. Let x and y be any two points in Ω. Then there are points $q, q' \in S' \cup \partial S'$ such that $d(x, q) < 7\kappa$ and $d(y, q') < 7\kappa$. Since also $S' \cup \partial S'$ is connected, we can find a path in $S' \cup \partial S'$ from q to q'. Then the 7κ-neighborhood of this path is a connected subset of Ω which contains both x and y. This shows that Ω is connected and therefore a domain since clearly Ω is open. To show that S is connected, let q and q' be any two points of S. Then there are points $x, y \in \Omega$ such that $d(x, q) < \kappa$ and $d(y, q') < \kappa$. Since Ω is a domain, there exists a rectifiable curve which connects x and y in Ω. As in the proof of Lemma 2.13, we see that the κ-neighborhood of this curve contains a path in P, and hence in S, from q to q'. Thus S is connected.

Next we shall study how p-hyperbolicity of nets is related to p-hyperbolicity of open sets and vice versa. Although some parts of the following could be found in Kanai's paper [K2], we include all details for the convenience of the reader. We assume that S', Ω, and S are as in 5.2. First we attach to each continuous function $u \in W^1_{p,\mathrm{loc}}(\Omega)$ a function u^* of $S' \cup \partial S'$ by setting

$$u^*(q) = \fint_{B(q, 4\kappa)} u \, dm. \tag{5.3}$$

Then we have the following.

Lemma 5.4. *Let u and u^* be as above. Then there exists a constant $c = c(n, \kappa, K, p)$ such that*

$$\sum_{q \in S'} |Du^*(q)|^p \le c \int_\Omega |\nabla u|^p \, dm.$$

PROOF. In the following c will be a positive constant which is not necessarily the same at each occurence but, however, may depend only on n, κ, K, and p. For each $q \in S' \cup \partial S'$, the volume estimate $|B(q, 4\kappa)| \leq V_K(4\kappa)$, Hölder's inequality, and the local Poincaré inequality (2.12) imply that

$$\begin{aligned} V_K(4\kappa)^{p-1} \int_{B(q,4\kappa)} |\nabla u|^p \, dm &\geq \left(\int_{B(q,4\kappa)} |\nabla u| \, dm \right)^p \\ &\geq c \left(\int_{B(q,4\kappa)} |u(x) - u^*(q)| \, dm \right)^p . \end{aligned}$$

Let $q \in S'$ and $q' \in N_q$. Then $q' \in S' \cup \partial S'$, and by the previous estimate,

$$\begin{aligned} \int_{B(q,7\kappa)} |\nabla u|^p \, dm &\geq \frac{1}{2} \left(\int_{B(q,4\kappa)} |\nabla u|^p \, dm + \int_{B(q',4\kappa)} |\nabla u|^p \, dm \right) \\ &\geq c \Bigg(\Big(\int_{B(q,4\kappa)} |u(x) - u^*(q)| \, dm \Big)^p \\ &\qquad + \Big(\int_{B(q',4\kappa)} |u(x) - u^*(q')| \, dm \Big)^p \Bigg) \\ &\geq c \left(\int_{B(q,4\kappa) \cap B(q',4\kappa)} |u^*(q) - u^*(q')| \, dm \right)^p \\ &\geq c (v_0 \kappa^n)^p \, |u^*(q) - u^*(q')|^p . \end{aligned} \tag{5.5}$$

We recall that $\# N_q \leq c_4$, with c_4 independent of q. Using this fact and (5.1), we obtain from (5.5) that

$$|Du^*(q)|^p \leq c \int_{B(q,7\kappa)} |\nabla u|^p \, dm .$$

By (2.15), every point $x \in \Omega$ belongs to at most c balls $B(q, 7\kappa)$, where $q \in P$ and c is independent of x. Thus

$$\sum_{q \in S'} \int_{B(q,7\kappa)} |\nabla u|^p \, dm \leq c \int_{\Omega} |\nabla u|^p \, dm ,$$

and so

$$\sum_{q\in S'} |Du^*(q)|^p \le c \int_\Omega |\nabla u|^p \, dm$$

as claimed.

Conversely, for each function $\bar{v}$ of $S \cup \partial S$, we define a function $v \in C^\infty(\Omega)$ as follows. For each $q \in P$, we choose functions $\eta_q \in C_0^\infty(M^n)$ such that $0 \le \eta_q \le 1$, $\eta_q = 1$ in $B(q,\kappa)$, $\eta_q = 0$ outside $B(q, 3\kappa/2)$, and $|\nabla \eta_q| \le 4\kappa^{-1}$. For $x \in \Omega$, we set $P_x = P \cap B(x, 2\kappa)$. We remark that $\# P_x \le c_4 + 1$ since $P_x \subset N_q \cup \{q\}$ for some q. Then we define $v : \Omega \to \mathbb{R}$ by

$$v(x) = \frac{\sum_{q\in P_x} \bar{v}(q)\eta_q(x)}{\sum_{q\in P_x} \eta_q(x)}. \tag{5.6}$$

The function v will depend on the choice of η_q. Observe that $\sum_{q\in P_x} \eta_q(x) \ge 1$ since every x belongs to at least one $B(q,\kappa)$, with $q \in P_x$.

Lemma 5.7. *If $\bar{v}$ and v are as above, then $v \in C^\infty(\Omega)$ and there exists a constant $c = c(n,\kappa,K,p)$ such that*

$$\int_\Omega |\nabla v|^p \, dm \le c \sum_{q\in S} |D\bar{v}(q)|^p .$$

PROOF. Again c may vary even within a line but it can depend at most on n, κ, K, and p. Let $x \in \Omega$. First we note that $P_x \subset S \cup \partial S$, and thus $\bar{v}(q)$ is defined for every $q \in P_x$. This means that v is defined in Ω. To show that v is a C^∞-function, we observe that $d(y,q) \ge 3\kappa/2$ if $y \in \Omega \cap B(x, \kappa/2)$ and $q \in P_y \setminus P_x$. This implies that $\eta_q(y) = 0$, and therefore it is sufficient to consider only functions η_q, with $q \in P_x$, in the definition of $v(y)$ if $y \in \Omega \cap B(x,\kappa/2)$. Hence $v \in C^\infty(\Omega)$. To show the inequality in the claim, we abbreviate

$$\xi_q(x) = \frac{\eta_q(x)}{\sum_{q'\in P_x} \eta_{q'}(x)} .$$

Then

$$\begin{aligned}|\nabla \xi_q(x)| &\le |\nabla \eta_q(x)| \Big(\sum_{q'\in P_x} \eta_{q'}(x) \Big)^{-1} \\ &\quad + \eta_q(x) \Big(\sum_{q'\in P_x} \eta_{q'}(x) \Big)^{-2} \sum_{q'\in P_x} |\nabla \eta_{q'}(x)| \\ &\le 4\kappa^{-1}(1+c),\end{aligned}$$

where $c = \sup\{\#P_y : y \in M^n\}$. Suppose that $x \in B(q,\kappa) \cap \Omega$, where $q \in S$. Then

$$\begin{aligned} \nabla v(x) &= \sum_{q' \in P_x} \bar{v}(q')\, \nabla \xi_{q'}(x) \\ &= \sum_{q' \in N_q \cup \{q\}} \bar{v}(q')\, \nabla \xi_{q'}(x) \\ &= \sum_{q' \in N_q} \big(\bar{v}(q') - \bar{v}(q)\big) \nabla \xi_{q'}(x) . \end{aligned} \tag{5.8}$$

On the first line in (5.8) we used the facts that $P_x \subset N_q \cup \{q\}$ if $x \in B(q,\kappa)$, and that $\xi_{q'}(x) = 0$ if $q' \notin P_x$. For the last equality in (5.8), observe that

$$\sum_{q' \in N_q \cup \{q\}} \bar{v}(q)\, \nabla \xi_{q'}(x) = \bar{v}(q)\, \nabla \Big(\sum_{q' \in N_q \cup \{q\}} \xi_{q'}(x) \Big) = 0$$

since $\sum_{q' \in N_q \cup \{q\}} \xi_{q'}(x) = 1$. It follows from (5.8) and from the uniformness of P that, for every $x \in B(q,\kappa)$,

$$\begin{aligned} |\nabla v(x)|^p &\le c \Big(\sum_{q' \in N_q} |\bar{v}(q') - \bar{v}(q)| \Big)^p \\ &\le c \Big((\#N_q)^2 \max_{q' \in N_q} |\bar{v}(q') - \bar{v}(q)|^2 \Big)^{p/2} \\ &\le c \Big(\sum_{q' \in N_q} |\bar{v}(q') - \bar{v}(q)|^2 \Big)^{p/2} = c\, |D\bar{v}(q)|^p . \end{aligned}$$

Finally, $\Omega \subset \cup_{q \in S} B(q,\kappa)$ and $|B(q,\kappa)| \le V_K(\kappa)$, and therefore

$$\int_\Omega |\nabla v(x)|^p \, dm \le \sum_{q \in S} \int_{B(q,\kappa} |\nabla v(x)|^p \, dm \le c\, V_K(\kappa) \sum_{q \in S} |D\bar{v}(q)|^p .$$

The lemma is proved.

Lemma 5.9. *Let S', Ω, and S be as in Lemma* 5.2. *Then Ω is p-hyperbolic if S' is p-hyperbolic. Conversely, if Ω is p-hyperbolic, then S is p-hyperbolic.*

PROOF. Let $\{B_i\}$ be an exhaustion of M^n. Suppose first that S' is p-hyperbolic. Then there exists a finite non-empty set $E \subset S' \cup \partial S'$ such that $\operatorname{cap}_p(E, \infty; S') > 0$. We set $C = \cup_{q \in E} \bar{B}(q, 4\kappa)$. Let $u \in L^1_p(\Omega)$ be continuous in $\Omega \cup C \cup \bar{\Omega} \setminus B_i$ such that $u = 1$ in C and $u = 0$ in $\bar{\Omega} \setminus B_i$ for some (not fixed) i. We observe that $1 - u$ is admissible for $(C, \bar{\Omega} \setminus B_i)$. We define $u^* : S' \cup \partial S' \to \mathbb{R}$ by (5.3). Then u^* is admissible for $(E, \infty; S')$, that is, $u^* = 1$ in E and it has a finite support. By Lemma 5.4,

$$\int_\Omega |\nabla u|^p \, dm \geq c \sum_{q \in S'} |Du^*(q)|^p \geq c \operatorname{cap}_p(E, \infty; S').$$

Taking the infimum over all such functions u (and i) gives

$$\operatorname{cap}_p(C, \infty; \Omega) \geq c \operatorname{cap}_p(E, \infty; S') > 0,$$

and so Ω is p-hyperbolic.

For the proof of the second claim, we choose a compact set $C \subset \Omega$ such that $\operatorname{cap}_p(C, \infty; \Omega) > 0$. Let $E = \{q \in S \cup \partial S : \ d(q, C) < 2\kappa\}$. Then E is finite and non-empty. Let $\bar{v}$ be an admissible function for $(E, \infty; S)$. We define a function $v \in C^\infty(\Omega)$ by (5.6). Since $\bar{v}$ has a finite support, $v = 0$ in $\Omega \setminus K$ for some compact set $K \subset M^n$. For each $x \in C$,

$$v(x) = \frac{\sum_{q \in P_x} \bar{v}(q) \, \eta_q(x)}{\sum_{q \in P_x} \eta_q(x)} = 1$$

since $P_x \subset E$ and $\bar{v}(q) = 1$ in E. Hence $1 - v$ is admissible for $(C, \bar{\Omega} \setminus B_i; \Omega)$ whenever $K \subset B_i$. By Lemma 5.7,

$$\sum_{q \in S} |D\bar{v}(q)|^p \geq c \int_\Omega |\nabla v|^p \, dm \geq c \operatorname{cap}_p(C, \infty; \Omega) > 0.$$

Since this holds for all admissible functions $\bar{v}$ we get

$$\operatorname{cap}_p(E, \infty; S) \geq c \operatorname{cap}_p(C, \infty; \Omega) > 0.$$

This ends the proof.

For the next two lemmas, let $\psi: M^n \to N^\nu$ be a rough isometry. Then it induces a rough isometry $\varphi: P \to Q$ with respect to the combinatorial metrics of P and Q. Let a and b be the constants of φ in (2.12).

Lemma 5.10. *Let $S \subset P$ be connected, and let $S' = \{q \in Q : \delta(q, \varphi(S \cup \partial S)) \leq a + b\}$. Then S' is connected. Furthermore, let v be a function of $S' \cup \partial S'$ and $u = v \circ \varphi$. Then*

$$\sum_{x \in S} |Du(x)|^p \leq c \sum_{q \in S'} |Dv(q)|^p ,$$

where c is independent of v.

PROOF. Let q and q' be are any two points in S' and let x and y be points in $S \cup \partial S$ such that $\delta(q, \varphi(x)) \leq a + b$ and $\delta(q', \varphi(y)) \leq a + b$. Hence there exist paths in S' from q to $\varphi(x)$ and from q' to $\varphi(y)$, respectively. Since also $S \cup \partial S$ is connected, there exists a path $q_0 = x, q_1, \ldots, q_l = y$ in $S \cup \partial S$. For every $i = 0, 1, \ldots, l-1$, $\delta(\varphi(q_i), \varphi(q_{i+1})) \leq a + b$, and thus there is a path in S' from $\varphi(q_i)$ to $\varphi(q_{i+1})$ for every $i = 0, 1, \ldots, l-1$. Hence S' is connected. To prove the other part of the claim, let v be any function in $S' \cup \partial S'$. We abbreviate $c_6 = a + b$. Let $x \in S$ and $y \in S \cup \partial S$ be such that $\delta(x, y) = 1$. Then $\delta(\varphi(x), \varphi(y)) \leq c_6$. Thus there is a path $q_0 = \varphi(x), q_1, \ldots, q_\ell = \varphi(y)$ in S' of length $\ell \leq c_6$. Now $u(x) - u(y) = v(q_0) - v(q_1) + v(q_1) - \cdots + v(q_{\ell-1}) - v(q_\ell)$, and therefore

$$|u(x) - u(y)|^p \leq c_6^p \sum_{i=0}^{\ell-1} |v(q_i) - v(q_{i+1})|^p .$$

Since Q is uniform, the number of points $q \in Q$, with $\delta(q, q_0) \leq c_6$, is bounded by a constant which is independent of q_0. Hence we get, by also using (5.1), an estimate

$$|u(x) - u(y)|^p \leq c \max_{\delta(q, \varphi(x)) \leq c_6} |Dv(q)|^p .$$

Now the uniformness of P implies that

$$|Du(x)|^p \leq c_7 \max_{\delta(q, \varphi(x)) \leq c_6} |Dv(q)|^p . \tag{5.11}$$

Next we sum both sides of (5.11) over all $x \in S$. Then some terms $|Dv(q)|^p$ may appear several times on the right hand side. However, since φ is a rough isometry and P is uniform, there is a constant c_8 such that, for each $x \in S$, there can be at most c_8 points $x' \in S$ with $\delta(\varphi(x), \varphi(x')) \leq 2c_6$. Hence

$$\sum_{x \in S} |Du(x)|^p \leq c_7 \sum_{x \in S} \max_{\delta(q, \varphi(x)) \leq c_6} |Dv(q)|^p \leq c_7\, c_8 \sum_{q \in S'} |Dv(q)|^p .$$

Lemma 5.12. *Suppose that $\Omega \subset M^n$ is connected and p-hyperbolic. Let $S = \{q \in P : d(q, \Omega) < \kappa\}$. Then the set $\Omega' = \{y \in N^\nu : d(y, \varphi(S \cup \partial S)) < c_9\}$, where $c_9 = \max\{3\kappa(a+b), a+b\} + 7\kappa$, is a domain and p-hyperbolic.*

PROOF. Clearly Ω is open. To show that it is connected, let x and y be any points of Ω'. Then there are points q and q' in $S \cup \partial S$ such that $x \in B(\varphi(q), c_9)$ and $y \in B(\varphi(q'), c_9)$. Both of these balls are contained in Ω'. Futhermore, since S is a connected subnet by (the proof of) Lemma 5.2, so does $S \cup \partial S$. Thus there exists a path in $S \cup \partial S$, say $q_0 = q, q_1, \dots, q_\ell = q'$, from q to q'. By (2.14),

$$d\big(\varphi(q_i), \varphi(q_{i+1})\big) \le 3\kappa\, \delta\big(\varphi(q_i), \varphi(q_{i+1})\big) \le 3\kappa(a+b) < c_9\,,$$

and therefore $\cup_{i=0}^{\ell} B\big(\varphi(q_i), c_9\big)$ is a connected open subset of Ω' containing x and y. This implies that Ω' is a domain. It remains to prove that Ω' is p-hyperbolic. First we observe that S is p-hyperbolic by Lemma 5.9. Thus there exists a finite set $E \subset S \cup \partial S$ such that $\operatorname{cap}_p(E, \infty; S) > 0$. Let v be an admissible function in $S' \cup \partial S'$ for $(\varphi(E), \infty; S')$, that is, v has a finite support and $v = 1$ in $\varphi(E)$. For each $q \in S \cup \partial S$, we set $u(q) = v\big(\varphi(q)\big)$. Then $u = 1$ in E. Since the support of v is finite, there is a point $\tilde{q} \in S$ and $\delta_0 > 0$ such that $u(q) = v\big(\varphi(q)\big) = 0$ if $\delta\big(\varphi(\tilde{q}), \varphi(q)\big) \ge \delta_0$. Since φ is a rough isometry, there exists $\delta_1 > 0$ such that, $\delta\big(\varphi(\tilde{q}), \varphi(q)\big) \ge \delta_0$, and so $u(q) = 0$, if $\delta(\tilde{q}, q) \ge \delta_1$. The uniformness of P implies that there can be only finitely many points $q \in P$ with $\delta(\tilde{q}, q) < \delta_1$. Hence the support of u is finite and u is admissible for $(E, \infty; S)$. Lemma 5.10 then implies that

$$\sum_{q \in S'} |Dv(q)|^p \ge c \sum_{x \in S} |Du(x)|^p \ge c \operatorname{cap}_p(E, \infty; S) > 0\,.$$

This is true for every admissible v. Hence $\operatorname{cap}_p\big(\varphi(E), \infty; S'\big) > 0$ and S' is p-hyperbolic. It follows from Lemma 5.9 that the 7κ-neighborhood of $S' \cup S'$ is p-hyperbolic. Hence also Ω' is p-hyperbolic as a larger set.

We are now ready to prove the main theorem.

Theorem 5.13. *Let M^n and N^ν be complete Riemannian manifolds with bounded geometry and roughly isometric to each other. Then M^n has the Liouville D_p-property if and only if so does N^ν.*

PROOF. Fix $\kappa \leq \min\{\operatorname{inj} M^n/2, \operatorname{inj} N^\nu/2\}$. Let P and Q be κ-nets in M^n and in N^ν, respectively. Since M^n and N^ν are roughly isometric, there exists an induced rough isometry $\varphi\colon P \to Q$. Let $\psi\colon Q \to P$ be a rough inverse of φ. Suppose that M^n does not have the Liouville D_p-property. By [H1, Section 5], there exists a non-constant bounded p-harmonic function u in M^n with $\int_{M^n} |\nabla u|^p \, dm < +\infty$. We normalize u such that $\inf_{M^n} u = 0$ and $\sup_{M^n} u = 1$. Since being roughly isometric is an equivalence relation it is sufficient to prove that also N^ν admits a non-constant p-harmonic function with L^p-integrable gradient. For each $a, b \in]0,1[$, we denote by Ω_a and Ω^b any component of sets $\{x \in M^n : u(x) < a\}$ and $\{x \in M^n : u(x) > b\}$, respectively. Then Ω_a and Ω^b are p-hyperbolic domains. Let $0 < s < 1/4$ and $3/4 < t < 1$. We write $S_s = \{q \in P : d(q, \Omega_s) < \kappa\}$ and $S^t = \{q \in P : d(q, \Omega^t) < \kappa\}$. Then the sets $D_s = \{x \in N^\nu : d(x, \varphi(S_s \cup \partial S_s)) < c_9\}$ and $D^t = \{x \in N^\nu : d(x, \varphi(S^t \cup \partial S^t)) < c_9\}$ are p-hyperbolic by Lemma 5.12. We claim that, for some $0 < s < 1/4$ and $3/4 < t < 1$, $\operatorname{cap}_p(\bar{D}_1, \bar{D}^2; N^\nu) < +\infty$ which then proves the theorem by Theorem 4.6. Let

$$v = \max\left\{0, \min\{2(u - 1/4), 1\}\right\}.$$

Now $v = 0$ in $\Omega_{1/4}$ and $v = 1$ in $\Omega^{3/4}$. Then we set, for each $q \in P$,

$$v^* = \fint_{B(q,4\kappa)} v \, dm.$$

Next we define $\bar{w} : Q \to P$ by $\bar{w} = v^* \circ \psi$, where ψ is a rough inverse of φ. Finally, we attach to $\bar{w}$ a function $w \in C^\infty(N^\nu)$ as in (5.6). By lemmas 5.4, 5.7 and 5.10, we have

$$\int_{N^\nu} |\nabla w|^p \, dm \leq c \int_{M^n} |\nabla v|^p \, dm \leq 2^p c \int_{M^n} |\nabla u|^p \, dm < +\infty.$$

It remains to show that w is admissible for $(\bar{D}_s, \bar{D}^t; N^\nu)$ if s and t are properly chosen. Recall that

$$w(y) = \frac{\sum_{q \in Q_y} \bar{w}(q)\, \eta_q(y)}{\sum_{q \in Q_y} \eta_q(y)},$$

where $Q_y = Q \cap B(y, 2\kappa)$ and $\eta_q \in C_0^\infty(N^\nu)$ such that $\eta_q = 1$ in $B(q,\kappa)$ and $\eta_q = 0$ outside $B(q, 3\kappa/2)$. Since ψ is a rough inverse of φ, there exists a constant c_{10} depending only on a, b, and κ such that $d(x, \psi(\varphi(x))) \leq c_{10}$ for every $x \in P$. Let $q \in Q$ be such that $d(q, \bar{D}_s) \leq 2\kappa$. Then there is $y \in \bar{D}_s$, with $d(q, y) \leq 4\kappa$. Moreover, $d(y, \varphi(z)) \leq 2c_9$ for some $z \in S_s \cup \partial S_s$, and so $d(q, \varphi(z)) \leq 2c_9 + 4\kappa$. Since ψ is a rough isometry, $d(\psi(q), \psi(\varphi(z))) \leq c_{11}$, where c_{11} depends only on a, b, and κ. Hence $d(\psi(q), z) \leq c_{10} + c_{11}$. On the other hand, there is $z' \in S_s$ such that $d(z, z') \leq 3\kappa$. Finally, $d(z, x) \leq \kappa$ for some $x \in \Omega_s$. Hence, for every $y \in B(\psi(q), 4\kappa)$,

$$d(y, x) \leq c_{10} + c_{11} + 8\kappa \stackrel{\text{def}}{=} c_{12}\,, \tag{5.14}$$

where c_{12} is independent of q and x. Thus we can attach to each $q \in Q$, with $d(q, \bar{D}_s) \leq 2\kappa$, a point $x \in \Omega_s$ such that $d(y, x) \leq c_{12}$ whenever $y \in B(\psi(q), 4\kappa)$. By Theorem 3.10,

$$d(\partial\Omega_s, \partial\Omega_{1/4}) > c_4\, r_0 \log \frac{1}{4s} - r_0\,.$$

Hence we can choose $0 < s < 1/4$ such that

$$d(\partial\Omega_s, \partial\Omega_{1/4}) \geq 2\,c_{12}\,. \tag{5.15}$$

It follows from (5.14) and (5.15) that $B(\psi(q), 4\kappa) \subset \Omega_{1/4}$ whenever $q \in Q$, with $d(q, \bar{D}_s) \leq 2\kappa$. But this implies that $\bar{w}(q) = v^*(\psi(q)) = 0$ for such q, and so $w(x) = 0$ for every $x \in \bar{D}_s$. Similarly, we can choose $3/4 < t < 1$ such that $d(\partial\Omega^t, \partial\Omega^{3/4}) \geq 2c_{12}$. Then $B(\psi(q), 4\kappa) \subset \Omega^{3/4}$ if $q \in Q$ and $d(q, \bar{D}^t) \leq 2\kappa$. Hence $w(x) = 1$ for every $x \in \bar{D}^t$. We have showed that w is admissible for $(\bar{D}_s, \bar{D}^t; N^\nu)$ which then proves the theorem.

FINAL REMARK 5.16. In [H1] we proved that there exists Green's function for (2.22) on M^n, that is, a certain positive solution of

$$-\operatorname{div} \mathcal{A}(\nabla g) = \delta_y\,,$$

where $y \in M^n$ and $\mathcal{A}$ is of type p, if and only if $\operatorname{cap}_p(M^n, C) > 0$ for some compact set $C \subset M^n$. In the light of the previous consideration, it is clear that Kanai's Theorem [K2, Theorem 1] is true for every $1 < p < \infty$, that is, for a fixed $p \in]1, \infty[$, the existence of Green's function for equation (2.22) is preserved under rough isometries between Riemannian manifolds of bounded geometry.

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A Paley-Wiener theorem for step two nilpotent Lie groups

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1. Introduction.

The classical Paley-Wiener Theorem for the Fourier transform on $\mathbb{R}^n$ which characterises compactly supported functions in terms of their Fourier transforms plays an important role in many problems of Fourier analysis. It is therefore desirable to have analogues of the Paley-Wiener Theorem for compactly supported functions on general Lie groups whenever there is a group Fourier transform available. For the case of the spherical Fourier transform on semi simple Lie groups an analogue of the Paley-Wiener Theorem is known. In 1976, Ando [1] proved a Paley-Wiener type theorem for the Heisenberg group which is the simplest example of a nilpotent Lie group which is nonabelian. Recently, we have proved another Paley-Wiener theorem for the Heisenberg group, *cf.* [4]. In both papers the explicit form of the representations on the Heisenberg group has played an important role in formulating and proving Paley-Wiener theorems.

It is an interesting open problem to establish Paley-Wiener theorems for general nilpotent Lie groups. The aim of this paper is prove one such theorem for step two nilpotent Lie groups which is analogous to the Paley-Wiener theorem for the Heisenberg group proved in [4].

2. Some basic facts about nilpotent Lie groups.

In this section we briefly recall some basic results from the representation theory of nilpotent Lie groups. A general reference is the book [2] by Corwin and Greenleaf.

Let G be a nilpotent Lie group of dimension n and let $\mathfrak{g}$ be its Lie algebra. Denote the dual of the Lie algebra by $\mathfrak{g}^*$ and the centre of the enveloping algebra $u(\mathfrak{g})$ by $\zeta(\mathfrak{g})$. Every coadjoint orbit in $\mathfrak{g}^*$ is even dimensional. Let $2k$ be the maximal dimension which occurs and let $q = n - 2k$. Then there exists a nonempty Zariski-open subset Γ of $\mathbb{R}^q$ and for each $\lambda \in \Gamma$ there is a unitary irreducible representation π_λ of G realised on $L^2(\mathbb{R}^k)$. Moreover, there exists a rational function $R(\lambda)$ regular on Γ and unique up to multiplication by -1 such that the Plancherel Formula holds with $d\mu(\lambda) = |R(\lambda)| d\lambda_1 \cdots d\lambda_q$,

$$\int_G |f(g)|^2 \, dg = \int_\Gamma \|\pi_\lambda(f)\|_{HS}^2 \, d\mu(\lambda) \tag{2.1}$$

for $f \in L^1(G) \cap L^2(G)$. Here dg is the Haar measure on G and HS stands for the Hilbert-Schmidt norm.

We let K stand for the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^k)$ with the inner product $\langle T, S\rangle = \operatorname{Tr}(TS^*)$ and let $L^2(\Gamma, K)$ be the space of L^2 functions on Γ with values in K taken with respect to the measure $d\mu(\lambda)$. Then there is a unique bijective isometry, $\Phi : L^2(G) \longrightarrow L^2(\Gamma, K)$ such that for every f in $L^1(G) \cap L^2(G)$, $\Phi(f)$ is the function $\lambda \mapsto \pi_\lambda(f)$. Here and in (2.1) $\pi_\lambda(f)$ is the operator defined by

$$\pi_\lambda(f) = \int_G f(g)\, \pi_\lambda(g)\, dg . \tag{2.2}$$

The function Φ is called the group Fourier transform.

Now each representation π_λ of G defines a skew adjoint representation, also denoted by π_λ, of the Lie algebra $\mathfrak{g}$ by the formula

$$\pi_\lambda(X)\, \phi = \frac{d}{dt}\Big|_{t=0} \pi_\lambda(\exp tX)\phi , \tag{2.3}$$

where $\exp : \mathfrak{g} \longrightarrow G$ is the exponential map and ϕ is a C^∞ vector for π_λ. The skew adjointness of π_λ means that $\pi_\lambda(X)^* = -\pi_\lambda(X)$. Let

$X_1, X_2, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$. It then follows that if for each $\xi \in \mathbb{R}^n$ we define $U_\lambda(\xi)$ by

$$U_\lambda(\xi) = \exp\Big(-\sum_{j=1}^{n} \xi_j\, \pi_\lambda(X_j)\Big), \tag{2.4}$$

then $U_\lambda(\xi)$ becomes a unitary operator. This operator valued function is crucial for formulating a Paley-Wiener theorem.

We now specialise to the case of step two nilpotent Lie groups. In this case the group admits a dilation structure. By this we mean the existence of a family $\{\delta_r : r > 0\}$ of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_r = \exp(A \log r)$ where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues. In the case of step two nilpotent Lie groups the eigenvalues of A can be assumed to be 1 and 2. The maps $\exp \circ \delta_r \circ \exp^{-1}$ are group automorphisms of G and are called dilations of G. For facts about groups admitting dilations we refer to the monograph [3] of Folland and Stein. A function f defined on G is said to be homogeneous of degree α if $f(\delta_r g) = r^\alpha f(g)$ for all $r > 0$.

If G is a simply connected step two nilpotent Lie group then the exponential map is a global diffeomorphism. Using exponential coordinates we can identify G with $\mathbb{R}^{n-m} \times \mathbb{R}^m$ and the group law can be written in the form

$$(x,t)\cdot(y,s) = (x+y, t+s+F(x,y)), \tag{2.5}$$

where F is a bilinear form from $\mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^m$. Let Y_j be the right invariant vector fields agreeing with X_j at the origin. Recall that X_j and Y_j are defined by

$$X_j f(g) = \frac{d}{dt}\Big|_{t=0} f(g\cdot \exp tX_j), \tag{2.6}$$

$$Y_j f(g) = \frac{d}{dt}\Big|_{t=0} f(\exp tX_j \cdot g). \tag{2.7}$$

We then have the following lemma.

Lemma 2.1. *Let G be a step two nilpotent Lie group admitting dilations as above. Then there exists polynomials $P_{jk}(x)$, $j = 1,2,\ldots,n$, $k = 1,2,\ldots,m$ on $\mathbb{R}^{n-m}$ homogeneous of degree one such that*

$$(Y_j f - X_j f) = \sum_{k=1}^{m} P_{jk}(x)\,\frac{\partial f}{\partial t_k}, \qquad j = 1,2,\ldots,n. \tag{2.8}$$

PROOF. This lemma can be easily proved using the definitions (2.6) and (2.7). Expressions for $X_j f$ and $Y_j f$ have been obtained in [3] (see Proposition 1.26). That P_{jk} are independent of t follows from the fact that they are homogeneous of degree one. (For details see [3]). We also remark that for some j it may happen that $X_j = Y_j$.

The above lemma is very important for our purpose and its importance will become apparent soon. First we make some observations and a definition. Direct calculation shows that

$$\pi_\lambda(-X_j f) = \pi_\lambda(f)\,\pi_\lambda(X_j), \tag{2.9}$$

$$\pi_\lambda(Y_j f) = -\pi_\lambda(X_j)\,\pi_\lambda(f). \tag{2.10}$$

Together these two equations imply that

$$\pi_\lambda(Y_j f - X_j f) = [\pi_\lambda(f), \pi_\lambda(X_j)], \tag{2.11}$$

where $[T, S]$ stands for the commutator $TS - ST$. On the space of bounded operators on $L^2(\mathbb{R}^k)$ we can define n derivations $\delta_j(\lambda)$ by

$$\delta_j(\lambda)T = [T, \pi_\lambda(X_j)]. \tag{2.12}$$

Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ we define

$$\delta(\lambda)^\alpha = \delta_1(\lambda)^{\alpha_1} \cdots \delta_n(\lambda)^{\alpha_n}. \tag{2.13}$$

We say that an operator T is of class C^k if $\delta(\lambda)^\alpha T$ is bounded for all α with $|\alpha| \le k$.

Now from Lemma 2.1 and equation (2.11) we have the interesting formula

$$\pi_\lambda\Big(\sum_{k=1}^{n} P_{jk}(x)\,\frac{\partial f}{\partial t_k}\Big) = \delta_j(\lambda)\,\pi_\lambda(f). \tag{2.14}$$

The operations $\delta_j(\lambda)$ are derivations in the sense that

$$\delta_j(\lambda)(TS) = T\delta_j(\lambda)S + \delta_j(\lambda)TS.$$

The above formula connects multiplication by polynomials on the function side and derivations on the Fourier transform side and may be considered as the analogue of the formula

$$(-2\pi i x_j f)^\wedge(\xi) = \frac{\partial}{\partial \xi_j}\hat{f}(\xi) \tag{2.15}$$

for the euclidean Fourier transform. In the next section we formulate and prove a Paley-Wiener theorem using formula (2.14).

3. A Paley-Wiener theorem for smooth functions.

For the sake of simplicity we consider functions in the Schwartz class $\mathcal{S}(G)$. Let f be a function in $\mathcal{S}(G)$ and let $\tilde{f}$ stand for the partial Fourier transform in the central variable,

$$\tilde{f}(x,t) = \int_{\mathbb{R}^m} e^{-2\pi i t\cdot s} f(x,s)\, ds\,. \tag{3.1}$$

The formula (2.14) applied to $\tilde{f}$ takes the form

$$\delta_j(\lambda)\,\pi_\lambda(\tilde{f}) = -2\pi i\, \pi_\lambda((P_j f)^\sim)\,, \tag{3.2}$$

where the function $P_j(x,t)$ is defined by

$$P_j(x,t) = \sum_{k=1}^{m} P_{jk}(x)\, t_k\,. \tag{3.3}$$

We now define the modified Fourier transform of f in the following way. For each $\xi \in \mathbb{R}^n$, $\hat{f}(\xi)$ takes values in $L^2(\Gamma, K)$ and is given by

$$\hat{f}(\xi)(\lambda) = U_\lambda(\xi)\,\pi_\lambda(\tilde{f})\,U_\lambda(-\xi)\,. \tag{3.4}$$

Recalling the definition of $U_\lambda(\xi)$, taking derivative with respect to ξ_j and using (3.2) we obtain the interesting relation

$$\frac{\partial}{\partial \xi_j}\hat{f}(\xi) = -2\pi i\,(P_j f)^\wedge(\xi)\,, \tag{3.5}$$

which is the analogue of (2.15) for our modified Fourier transform on the group G.

The classical Paley-Wiener Theorem for the euclidean Fourier transform follows immediately from (2.15). If f is supported in $|x_j| \le B$, $j = 1,2,\dots,n$, then it follows that $|\partial_\xi^\alpha \hat{f}(\xi)| \le C\,(2\pi B)^{|\alpha|}$ and this leads directly to the extendability of $\hat{f}(\xi)$ as an entire function on $\mathbb{C}^n$ satisfying the estimate

$$|\hat{f}(\zeta)| \le e^{2\pi B|\mathrm{Im}\,\zeta|}\,. \tag{3.6}$$

In the same spirit we would like to set up an isomorphism between functions f supported in a set defined by the inequalities $|P_j(x,t)| \le B$, $j = 1, 2, \dots, n$ and a class of entire functions on $\mathbb{C}^n$.

To this end let $H_B(\mathbb{C}^n)$ stand for the space of entire functions $F(\zeta)$ taking values in the Hilbert space $E = L^2(\Gamma, K)$ which agrees with $\hat{f}(\xi)$ on $\mathbb{R}^n$ for some $f \in \mathcal{S}(G)$ and satisfies the estimate

$$\|F(\zeta)\|_E \le C\, e^{2\pi B|\mathrm{Im}\,\zeta|}. \tag{3.7}$$

Let G_B stand for the set defined by

$$G_B = \{(x,t) \in G : \ |P_j(x,t)| \le B,\ j = 1, 2, \dots, n\}. \tag{3.8}$$

Then we have the following theorem. Let $C^\infty(G_B)$ stand for the set of all smooth f supported in G_B.

Theorem 3.1. *The modified Fourier transform sets up an isomorphism between $\mathcal{S}(G) \cap C^\infty(G_B)$ and $H_B(\mathbb{C}^n)$.*

PROOF. The direct part of this theorem is easy. If $f \in \mathcal{S}(G) \cap C^\infty(G_B)$ then iteration of (3.5) gives us

$$\partial_\xi^\alpha \hat{f}(\xi) = (-2\pi i)^{|\alpha|}(P^\alpha f)^\wedge(\xi), \tag{3.9}$$

where $P^\alpha(x,t) = P_1(x,t)^{\alpha_1} \cdots P_n(x,t)^{\alpha_n}$. It then follows that

$$\|\partial_\xi^\alpha \hat{f}(\xi)\|_E^2 = (2\pi)^{2|\alpha|} \int_\Gamma \|\pi_\lambda(P^\alpha f)^\wedge\|_{HS}^2\, d\mu, \tag{3.10}$$

which by Plancherel Theorem gives the estimate

$$\begin{aligned} \|\partial_\xi^\alpha \hat{f}(\xi)\|_E^2 &= (2\pi)^{2|\alpha|} \int_G |P^\alpha(g)\, f(g)\|^2\, dg \\ &\le (2\pi B)^{2|\alpha|} \int_G |f(g)|^2\, dg. \end{aligned} \tag{3.11}$$

From these estimates it follows that the series

$$F(\zeta) = \sum \frac{\partial_\xi^\alpha \hat{f}(0)}{\alpha!}\, \zeta^\alpha \tag{3.12}$$

converges in the norm of E and represents an entire function. Moreover, the expansion

$$F(\zeta) = \sum \frac{\partial_\xi^\alpha \hat{f}(\xi)}{\alpha!} (i\eta)^\alpha, \qquad \zeta = \xi + i\eta, \tag{3.13}$$

gives the estimate

$$\|F(\zeta)\|_E \leq C\, e^{2\pi B|\mathrm{Im}\,\zeta|}, \tag{3.14}$$

This proves that $F(\zeta) \in H_B(\mathbb{C}^n)$.

We now turn to the converse. Let $F \in H_B(\mathbb{C}^n)$ and let f be the Schwartz class function such that $\hat{f}(\xi) = F(\xi)$ for $\xi \in \mathbb{R}^n$. We need to show that f is supported in G_B. This will follow immediately if we can show that

$$\int_G |P_j(x,t)|^{2k}\, |f(x,t)|^2\, dg \leq B^{2k}\, \|f\|_2^2 \tag{3.15}$$

for all k. Again in view of the equation (3.11) it is enough to show that

$$\left\| \left(\frac{\partial}{\partial \xi_j}\right)^k \hat{f}(0) \right\|_E^2 \leq (2\pi B)^{2k} \|f\|_2^2\,. \tag{3.16}$$

In order to establish this we proceed as follows.

Let $\theta \in C_0^\infty(\mathbb{R}^n)$ be a real valued function supported in $|x| \leq 1$ and $\int |\theta(x)|^2\, dx = 1$. For $\varepsilon > 0$ let $\theta_\varepsilon(x) = \varepsilon^{-n/2}\theta(x/\varepsilon)$ so that $\hat{\theta}_\varepsilon(\xi) = \varepsilon^{n/2}\hat{\theta}(\varepsilon\xi)$. (Here $\hat{\theta}$ stands for the euclidean Fourier transform on $\mathbb{R}^n$). By the classical Paley-Wiener Theorem we know that $\hat{\theta}_\varepsilon$ extends to an entire function which verifies the estimate

$$|\hat{\theta}_\varepsilon(\zeta)| \leq C_\varepsilon\, e^{2\pi\varepsilon|\mathrm{Im}\,\zeta|}\,. \tag{3.17}$$

As $\theta \in C_0^\infty$ we also know that $\hat{\phi}_\varepsilon \in L^2(\mathbb{R}^n)$.

We now consider the function $M_\varepsilon(\zeta) = \hat{\theta}_\varepsilon(\zeta)\, F(\zeta)$. This is an entire function taking values in E and satisfies

$$\|M_\varepsilon(\zeta)\|_E \leq C_\varepsilon\, e^{2\pi(B+\varepsilon)|\mathrm{Im}\,\zeta|}\,. \tag{3.18}$$

Moreover the calculation

$$\int_{\mathbb{R}^n} \|M_\varepsilon(\xi)\|_E^2\, d\xi = \|f\|_2^2 \int_{\mathbb{R}^n} |\hat{\theta}_\varepsilon(\xi)|^2\, d\xi = \|f\|_2^2 \tag{3.19}$$

shows that $M_\varepsilon \in L^2(\mathbb{R}^n, E)$. Now we can appeal to the classical Paley-Wiener Theorem to conclude that there is a function $T_\varepsilon \in L^2(\mathbb{R}^n, E)$ supported in $|x| \leq (B+\varepsilon)$ such that

$$M_\varepsilon(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} T_\varepsilon(x)\, dx \,. \tag{3.20}$$

Differentiating this k times with respect to ξ_j we get the relation

$$\left(\frac{\partial}{\partial \xi_j}\right)^k M_\varepsilon(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} (-2\pi i x_j)^k T_\varepsilon(x)\, dx \,. \tag{3.21}$$

By the euclidean Plancherel Theorem this gives the estimate

$$\begin{aligned} \int_{\mathbb{R}^n} \left\| \left(\frac{\partial}{\partial \xi_j}\right)^k M_\varepsilon(\xi) \right\|_E^2 d\xi &= \int_{\mathbb{R}^n} \|(2\pi x_j)^k T_\varepsilon(x)\|_E^2\, dx \\ &\leq (2\pi(B+\varepsilon))^{2k} \int_{\mathbb{R}^n} \|T_\varepsilon(x)\|_E^2\, dx \\ &= (2\pi(B+\varepsilon))^{2k} \int_{\mathbb{R}^n} \|M_\varepsilon(\xi)\|_E^2\, d\xi \\ &= (2\pi(B+\varepsilon))^{2k}\, \|f\|_2^2 \,. \end{aligned} \tag{3.22}$$

Finally, as $M_\varepsilon(\xi)$ is the product of $\hat\theta_\varepsilon(\xi)$ and $F(\xi)$ we have by Leibnitz Formula the relation

$$\partial_j^k M_\varepsilon(\xi) = \sum_{\ell=0}^{k} \binom{k}{\ell} \partial_j^\ell \hat\theta_\varepsilon(\xi)\, \partial_j^{k-\ell} \hat f(\xi) \,, \tag{3.23}$$

where ∂_j stands for $\partial/\partial\xi_j$. From the above relation we calculate

$$\begin{aligned} \|\partial_j^k M_\varepsilon(\xi)\|_{HS}^2 = \sum_{\ell=0}^{k}\sum_{i=0}^{k} \binom{k}{\ell}\binom{k}{i} \partial_j^\ell \hat\theta_\varepsilon(\xi) \partial_j^i \hat\theta_\varepsilon(\xi) \\ \cdot \operatorname{Tr}\left((\partial_j^{k-\ell}\hat f(\xi))(\partial_j^{k-i}\hat f(\xi))^*\right). \end{aligned} \tag{3.24}$$

But now

$$\operatorname{Tr}\left((\partial_j^{k-\ell}\hat f(\xi))(\partial_j^{k-i}\hat f(\xi))^*\right) = \operatorname{Tr}\left((\partial_j^{k-\ell}\hat f(0))(\partial_j^{k-i}\hat f(0))^*\right)$$

and we have the inequality

$$\begin{aligned}(3.25)\qquad &\sum_{\ell=0}^{k}\sum_{i=0}^{k}\binom{k}{\ell}\binom{k}{i}\Big(\int_{\mathbb{R}^n}\partial_j^\ell\hat\theta_\varepsilon(\xi)\partial_j^i\hat\theta_\varepsilon(\xi)\,d\xi\\ &\qquad\cdot\int_\Gamma \mathrm{Tr}\left((\partial_j^{k-\ell}\hat f(0))(\partial_j^{k-i}\hat f(0))^*\right)d\mu(\lambda)\\ &\qquad\qquad\le(2\pi(B+\varepsilon))^{2k}\|f\|_2^2\,.\end{aligned}$$

As $\hat\theta_\varepsilon(\xi)=\varepsilon^{n/2}\hat\theta(\varepsilon\xi)$ it follows that the integral

$$\int_{\mathbb{R}^n}\partial_j^\ell\hat\theta_\varepsilon(\xi)\,\partial_j^i\hat\theta_\varepsilon(\xi)\,d\xi=\varepsilon^{\ell+i}\int_{\mathbb{R}^n}(\partial_j^\ell\hat\theta)(\xi)\,(\partial_j^i\hat\theta)(\xi)\,d\xi$$

tends to 0 as $\varepsilon\to 0$ unless $\ell=i=0$. Therefore, if we let $\varepsilon\to 0$ in (3.25) the only surviving term is the one with $\ell=i=0$ and we get

$$(3.26)\qquad \int_\Gamma \mathrm{Tr}\left((\partial_j^k\hat f(0))\,(\partial_j^k\hat f(0))^*\right)d\mu(\lambda)\le(2\pi B)^{2k}\|f\|_2^2\,,$$

which proves (3.16).

This completes the proof of the theorem.

4. Some remarks and an example.

We have established a Paley-Wiener theorem for Schwartz class functions that are supported in sets of the form G_B and this class includes $C_0^\infty(G)$. The sets G_B are not compact and this is in sharp contrast with the classical case where one has to consider C_0^∞ functions for the holomorphic extendability of the Fourier transform. Nevertheless, we can say something more about the sets for a class of nilpotent Lie groups of step two which includes the famous Heisenberg groups.

Let G be a step two nilpotent Lie group with one dimensional centre so that $G=\mathbb{R}^{n-1}\times\mathbb{R}$. Then the polynomials $P_j(x,t)$ take the form

$$(4.1)\qquad P_j(x,t)=p_j(x)\,t\,,$$

where $p_j(x)$ are homogeneous of degree one. Let

$$(4.2)\qquad p_j(x)=\sum_{k=1}^{n-1}C_{jk}\,x_k$$

and further assume that the matrix (C_{kj}) is invertible. Under this assumption the conditions $|x_j| \le B$, $j = 1, 2, \ldots, n-1$, will be equivalent to $|p_j(x)| \le aB$, $j = 1, 2, \ldots, n-1$, for some $a > 0$. In this situation, though the set G_B is not compact its projection onto $\mathbb{R}^{n-1}$ is compact for each fixed t. Therefore, if $F \in H_B(\mathbb{C}^n)$ and $F(\xi) = \hat{f}(\xi)$ then for each fixed t, $f(x,t)$ will be compactly supported in a set of the form $|x_j| \le aB$, $j = 1, 2, \ldots, n-1$. But regarding the support of f as a function of t we could say nothing.

The above situation is well explained by the example of the Heisenberg group $H_n = \mathbb{R}^{2n} \times \mathbb{R}$. In this case the group law is given by

$$(x,t)(x',t') = (x + x', t + t' + F(x,x')) \tag{4.3}$$

with

$$F(x,x') = \frac{1}{2} \sum_{j=1}^{n} (x'_j\, x_{j+n} - x_j\, x'_{j+n})\,.$$

The left invariant vector fields are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\, x_{n+j}\, \frac{\partial}{\partial t}\,, \qquad j = 1, 2, \ldots, n\,, \tag{4.4}$$

$$X_{j+n} = -\frac{\partial}{\partial x_{n+j}} - \frac{1}{2}\, x_j\, \frac{\partial}{\partial t}\,, \qquad j = 1, 2, \ldots, n\,. \tag{4.5}$$

A calculation shows that

$$Y_j - X_j = x_{n+j}\, \frac{\partial}{\partial t}\,, \qquad j = 1, 2, \ldots, n\,, \tag{4.6}$$

$$Y_{j+n} - X_{j+n} = x_j\, \frac{\partial}{\partial t}\,, \qquad j = 1, 2, \ldots, n\,. \tag{4.7}$$

As $X_{2n+1} = Y_{2n+1} = \partial/\partial t$ we neglect the difference $X_{2n+1} - Y_{2n+1}$. Thus $p_j(x) = x_{n+j}$, $j = 1, 2, \ldots, n$, $p_j(x) = x_{j-n}$, $j = n+1, \ldots, 2n$, and we are in the above situation.

Another interesting feature of the Heisenberg group is the fact that $\pi_\lambda(x,t) = \pi_\lambda(x)\, e^{i\lambda t}$ and each $\pi_\lambda(x)$ defines a projective representation of $\mathbb{R}^{2n}$. Therefore, one could completely discard the variable t and consider functions on $\mathbb{R}^{2n}$ and define the so called Weyl transform. For the Weyl transform we have proved a Paley-Wiener theorem in [4] and there the isomorphism is between $C_0^\infty(\mathbb{R}^{2n})$ and a class of entire functions taking values in K.

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Convexity at infinity and Brownian motion on manifolds with unbounded negative curvature

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Harmonic functions on complete simply connected Riemannian manifolds with negative sectional curvatures have been extensively studied in the last two decades, and several basic questions in this field ([Dyn, p.19], [GW2, p.3], [GW1], [Yau]) are by now essentially solved, at least if the sectional curvatures are also assumed to be bounded from below : for such a manifold M, and with respect to its compactification $\widehat{M} = M \cup S_\infty(M)$ with the sphere at infinity $S_\infty(M)$ (for definitions see [EO]), the Dirichlet problem, the behavior at infinity of positive harmonic functions and, in probabilistic terms, the asymptotic behavior of the Brownian motion on M are all well understood ([Pra], [Kif1], [Cho], [And], [Sul], [AS], [Anc1], [Kif2], [Anc2]). In particular, the following property (P_M) holds: the Brownian path on M converges a.s. to some exit point $\xi \in S_\infty(M)$ ([Pra], [Sul]), and moreover, for each $\zeta \in S_\infty(M)$, the distribution of the exit point ξ converges to the Dirac measure δ_ζ when the starting point tends to ζ ([Sul]). In analytic terms, (P_M) means that the Dirichlet problem on M is solvable for each given continuous boundary datum $f \in C(S_\infty(M); \mathbb{R})$ ([And]; see also [AS], [Anc1]). By a theorem of Choi ([Cho]), (P_M) may also be deduced from the following convexity property (C_M): each point $\zeta \in S_\infty(M)$ has a fundamental system of neighborhoods V in $\widehat{M}$ such that $V \cap M$ is convex ([And]). In fact, [Cho] shows that it is sufficient to check the weaker property (C'_M): for each pair (ζ, ζ') of distinct points on $S_\infty(M)$, there is a neighborhood V of ζ in $\widehat{M}$ such that ζ' is not in the closure (in $\widehat{M}$) of the convex hull of $M \cap V$ (see the Appendix in Section 6 below). Note however, that in contrast to (C'_M), property (C_M) guarantee that the harmonic measure does not charge points (see Corollary 6.3).

In this paper, we consider the case of complete simply connected Riemannian manifolds whose sectional curvatures are negative (say bounded from above by -1) but not bounded from below, and we show that property (P_M) does not then hold in general (See [HM] for property (P_M) under a growth condition on the curvature).

Theorem A. *There is a complete simply connected Riemannian manifold M of dimension* 3, *with sectional curvatures ≤ -1, and a point $\zeta_0 \in S_\infty(M)$ such that*

i) *the Brownian motion B_s on M has a.s. an infinite lifetime,*

ii) *with probability* 1, *B_s exits from M at ζ_0.*

Clearly, for such an M, there is no non-trivial bounded harmonic function f which may be extended continuously on $\widehat{M} = M \cup S_\infty(M)$ (or even such that $\lim_{m \to \zeta_0} f(m)$ exists). A variant of the method gives also the following.

Theorem B. *There exists a complete simply connected Riemannian manifold M of dimension* 3, *with sectional curvatures ≤ -1 and such that*

i) *the Brownian motion B_s on M has a.s. an infinite lifetime,*

ii) *with probability* 1, *every point on the sphere at infinity $S_\infty(M)$ is a cluster point of B_s (when $s \to \infty$).*

It will be clear that we may as well construct examples such that the set of cluster points of the Brownian motion in M is a.s. a fixed continuum $K \subset S_\infty(M)$, the pair $(S_\infty(M), K)$ being prescribed up to topological equivalence. Also, both theorems extend to higher dimensions. See final remarks in Section 5.C.

In our framework (no lower bound assumption on the curvature), the basic tools available in the "bounded geometry" case collapse with the notable exception of Choi's Theorem. For example, if $\dim(M) = 2$, (C_M) evidently holds (each geodesic in M divides M in two convex regions) so that (P_M) is still true; for M rotationaly symmetric and $\dim(M) \geq 3$, (C_M) holds and thus also (P_M) ([Cho]). Here, from Choi's results (see Corollary 6.3 below), we have the following purely geometric consequence of Theorem A.

Corollary C. *There exists a complete simply connected Riemannian manifold M of dimension 3, with sectional curvatures ≤ -1, such that for some point $\zeta_0 \in S_\infty(M)$ and every neighborhood V of ζ_0 in $\widehat{M}$ the closed convex hull of $V \cap M$ is M.*

(Since a convex open subset of M is equal to the interior of its closure -the usual proof of the similar well-known statement in $\mathbb{R}^N$ is easily adapted-, one may in fact remove the word "closed" in the statement). Theorem B shows that we can also construct M with the above property for all $\zeta_0 \in S_\infty(M)$; we have stated Corollary C because, for sake of simplicity, we shall first give a direct and nonprobabilistic proof of this corollary.

AN EXAMPLE. It is interesting to observe that one may easily construct examples (of dimension ≥ 3) with the following property: there is a point $\zeta_0 \in S_\infty(M)$ such that the Brownian motion B_s on M has a (strictly) positive probability to converge to ζ_0 in $\widehat{M}$ when $s \to S$ (S being the lifetime of the Brownian particle). To see this, fix a complete simply connected Riemannian manifold (N, g) of dimension 2 whose sectional curvatures are ≤ -1 and whose Brownian motion has a.s. a finite lifetime; let $M = N \times \mathbb{R} = \{(\xi, u) : \xi \in N, u \in \mathbb{R}\}$ equipped with the metric

$$ds^2 = du^2 + e^{2u} g(d\xi, d\xi).$$

Then:

i) M is complete and its sectional curvatures are ≤ -1 (simple computations; M is in fact a special case of the warped products in [BO]),

ii) each region $\{u < a\}$ is a horoball in M at $\zeta_0 = \lim_{u \to -\infty}(\xi_0, u) \in S_\infty(M)$, ($\zeta_0$ is independent of $\xi_0 \in N$),

iii) from the standard description of the Brownian motion on M in terms of a Brownian motion with a drift on $\mathbb{R}$ and an independent

Brownian motion on N with a change of time, there is a strictly positive probability that u_s, the u-component of B_s, satisfies $|u_s| \leq 1$ for $s \leq 1$ and that the N-component explodes also before time $s = 1$ (so that $\lim_{s \to S} B_s = \zeta_0$ in $\widehat{M}$).

It follows that the distance of any fixed given point in M to the closed convex hull C_V of $V \cap M$, with V neighborhood of ζ_0 in $\widehat{M}$ is bounded (independently of V) (see Corollary 6.3); thus C_V contains some fixed point $z_0 \in M$ independent of V. However, as was pointed out to me by W. S. Kendall, (P_M) holds. We may for example observe that the sectional curvatures of M are bounded (from below) near each boundary point $\zeta \in S_\infty(M) \setminus \{\zeta_0\}$, so that by the constructions in [And, Section 2], there is for each $\zeta \in S_\infty(M) \setminus \{\zeta_0\}$ arbitrarily small neighborhoods V in $\widehat{M}$ such that $M \setminus V$ (or $M \cap V$) is convex. Hence, (C'_M) holds but not (C_M) and the harmonic measure has a non-trivial discrete part, though its support is the whole boundary. However, it seems difficult to deduce from this construction an example proving Theorem A (even if we drop there condition i)).

PROBLEMS WHICH REMAIN OPEN. We should also point out that our counter-examples leave open several related natural questions: does a Cartan-Hadamard manifold with sectional curvatures ≤ -1 always supports a non-trivial bounded (respectively, positive) harmonic function? Does the Martin boundary of such a manifold M always have dimension $n-1$ ($n = \dim(M)$)? ([Dyn, p.19]), or dimension $\geq n-1$? We do not know (even when the sectional curvatures are bounded from below) if positive (respectively, bounded) harmonic functions may fail to separate points in M!

PLAN OF THIS PAPER. Our proof of Theorem A will follow from a careful study of a class of metrics which is in some sense the simplest for which property (C'_M) is not clear. We show that a nice "convexity" property (connected to Choi's criterion) is related to the integrability along some rays of a positive function derived from the metric (Proposition 2.1); on the other hand it is shown that the curvature assumptions imply that this function in general tends to zero along the rays (Proposition 2.2), but in general not rapidly enough so as to ensure the integrability (Section 4). Using also some technical gluing lemmas (Section 3), we construct an example proving Corollary C. This then leads to examples proving Theorem A or B. Though the material in Section 2 is not really necessary for the proof of these theorems, it should bring some light on the problem and its difficulties, and could be useful for other questions. It would be interesting to know if Proposition 2.3 below could be in some way extended to all Cartan-Hadamard manifold of sectional curvatures ≤ -1.

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1. A class of Riemannian metrics.

1.1. In the sequel, we consider the manifold $M = \mathbb{R}^3 = \{(x,y,t) : x,y,t \in \mathbb{R}\}$ equipped with a Riemannian metric $\gamma = \gamma(g,h)$ in the form

$$ds_\gamma{}^2 = dt^2 + g(x,t)^2\, dx^2 + h(x,t)^2\, dy^2\,,$$

g and h being two smooth positive functions on $\mathbb{R}^2$ which are nondecreasing with respect to t and such that $\inf\{g(x,t) :\ t_0 < t < t_1\} > 0$ for all $t_0, t_1 \in \mathbb{R}$. It is easily checked that (M,γ) is then a complete Riemannian manifold.

It is also easy to compute the sectional curvatures of M on using the natural moving frame given by

$$e_1 = \frac{1}{g(x,t)}\,(1,0,0)\,, \quad e_2 = \frac{1}{h(x,t)}\,(0,1,0) \quad \text{and} \quad e_3 = (0,0,1)\,,$$

the corresponding Cartan forms being $\alpha_1 = g(x,t)\,dx$, $\alpha_2 = h(x,t)\,dy$, $\alpha_3 = dt$.

Since

$$d\alpha_1 = -\frac{g'_t}{g}\,\alpha_1 \wedge \alpha_3\,, \qquad d\alpha_2 = -\frac{h'_t}{h}\,\alpha_2 \wedge \alpha_3 + \frac{h'_x}{gh}\,\alpha_1 \wedge \alpha_2\,,$$

and $d\alpha_3 = 0$, the connexion matrix $\Omega = \{\omega_{ij}\}_{1\leq i,j\leq 3}$ is

$$\Omega = \begin{pmatrix} 0 & -\dfrac{h'_x}{hg}\,\alpha_2 & \dfrac{g'_t}{g}\,\alpha_1 \\ \dfrac{h'_x}{hg}\,\alpha_2 & 0 & \dfrac{h'_t}{h}\,\alpha_2 \\ -\dfrac{g'_t}{g}\,\alpha_1 & -\dfrac{h'_t}{h}\,\alpha_2 & 0 \end{pmatrix}$$

(Ω is skew symmetric and $d\alpha = -\Omega \wedge \alpha$); thus, the curvature matrix $\mathcal{K} = \Omega \wedge \Omega + d\Omega$ is the skew 3×3 matrix $\mathcal{K} = \{K_{ij}\}$ with

$$K_{12} = \Big(-\frac{g'_t h'_t}{gh} - \frac{h''_{xx}}{g^2 h} + \frac{h'_x g'_x}{g^3 h} \Big) \alpha_1 \wedge \alpha_2 + \Big(\frac{h''_{xt}}{gh} - \frac{h'_x g'_t}{g^2 h} \Big) \alpha_2 \wedge \alpha_3 \ ,$$

$$K_{13} = -\frac{g''_{tt}}{g} \alpha_1 \wedge \alpha_3 \ ,$$

and

$$K_{23} = -\frac{h''_{tt}}{h} \alpha_2 \wedge \alpha_3 + \Big(\frac{h''_{tx}}{gh} - \frac{g'_t h'_x}{g^2 h} \Big) \alpha_1 \wedge \alpha_2 \ .$$

In other words, the curvature tensor R is given by the formula

$$\begin{aligned} \langle R(u,v)u,v \rangle = & A\,(u_1 v_2 - v_1 u_2)^2 + B\,(u_1 v_3 - u_3 v_1)^2 \\ & + C\,(u_2 v_3 - u_3 v_2)^2 \\ & + 2\,D\,(u_1 v_2 - u_2 v_1)(u_2 v_3 - u_3 v_2)\,, \end{aligned}$$

where the u_i, v_j denote the coordinates of the vectors u and v with respect to the moving frame, and where

$$A = \Big(\frac{g'_t h'_t}{gh} + \frac{h''_{xx}}{g^2 h} - \frac{h'_x g'_x}{g^3 h} \Big), \qquad B = \frac{g''_{tt}}{g} \ ,$$

$$C = \frac{h''_{tt}}{h} \ , \qquad D = -\Big(\frac{h''_{xt}}{gh} - \frac{h'_x g'_t}{g^2 h} \Big) \ .$$

Since $|u \wedge v|^2 = \sum_{i<j} (u_i v_j - u_j v_i)^2$, it is then clear that γ has all its sectional curvatures $\leq -\alpha^2$, $\alpha \geq 0$, if and only if the following four inequalities hold for all $(x,t) \in \mathbb{R}^2$

$$\frac{g''_{tt}}{g} \geq \alpha^2 \ , \tag{1}$$

$$\frac{h''_{tt}}{h} \geq \alpha^2 \ , \tag{2}$$

$$\Big(\frac{g'_t h'_t}{gh} + \frac{h''_{xx}}{g^2 h} - \frac{h'_x g'_x}{g^3 h} \Big) \geq \alpha^2 \ , \tag{3}$$

$$\Big(\frac{h''_{xt}}{gh} - \frac{h'_x g'_t}{g^2 h} \Big)^2 \leq \Big(\frac{h''_{tt}}{h} - \alpha^2 \Big) \Big(\frac{g'_t h'_t}{gh} + \frac{h''_{xx}}{g^2 h} - \frac{h'_x g'_x}{g^3 h} - \alpha^2 \Big) \ . \tag{4}$$

Note that this set of inequalities expresses that the quadratic forms

$$q(X,Y,Z) = (A-\alpha^2)\,X^2 + (B-\alpha^2)\,Y^2 \\ + (C-\alpha^2)\,Z^2 + 2\,D\,X\,Z$$

are nonnegative (for all $(x,t) \in \mathbb{R}^2$). In the sequel, we shall say that a metric $\gamma(g,h)$ is of class $\gamma(-\alpha^2)$, $(\alpha \geq 0)$, if the above inequalities (1) to (4) hold throughout $\mathbb{R}^2$.

We note also for later use that for $g > 0$ nondecreasing with respect to t, (1) implies that

$$g(x,t) \geq g(x,s)\,\cosh(\alpha(t-s)) + \alpha^{-1}\,g'_t(x,s)\,\sinh(\alpha(t-s))$$

for $t \geq s$, $x \in \mathbb{R}$.

1.2. Assuming now that g and h define a $\gamma(-\alpha^2)$ metric $(\alpha > 0)$, it is easy to describe $S_\infty(M)$, the sphere at infinity of M (See [EO] for definitions and basic facts concerning the compactification $\widehat{M} = M \bigcup S_\infty(M)$ with the sphere at infinity, and for the basic relations of these objects with the geodesics in M). Clearly, the curves $\tau_{(x,y)}$: $t \mapsto (x,y,t)$ $(x, y \in \mathbb{R}$ fixed) are the unit-speed geodesics emanating from a point on $S_\infty(M)$; this point shall be denoted ∞_M. Denoting also $\zeta_{(x,y)}$ the end point (for $t \to +\infty$) of $\tau_{x,y}$ on $S_\infty(M)$, we have $S_\infty(M) = \{\zeta_{x,y} : x, y \in \mathbb{R}\} \cup \{\infty_M\}$, the mapping $(x,y) \mapsto \zeta_{(x,y)}$ being a homeomorphism from $\mathbb{R}^2$ on $S_\infty(M) \backslash \{\infty_M\}$. It is also easily seen that a basis $\{V_n\}_{n \geq 1}$ of neighborhoods of ∞_M can be obtained by setting $V_n = \widehat{M} \setminus \overline{W_n}$ and $W_n = \{(x,y,t) \in M : \ |x| \leq n, \ |y| \leq n, \ t \geq -n\}$, (here $\overline{W_n} = W_n \cup \{\zeta_{(x,y)} : \ |x| \leq n, \ |y| \leq n, \}$).

1.3. Finally, let us write the expression of the Laplace-Beltrami operator Δ induced by the metric $\gamma(g,h)$ with respect to the coordinates x, y, t; on using the standard formula (or direct computations), we have

$$\Delta = \frac{\partial^2}{\partial t^2} + \frac{1}{g^2}\,\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\,\frac{\partial^2}{\partial y^2} + \Big(\frac{g'_t}{g} + \frac{h'_t}{h}\Big)\,\frac{\partial}{\partial t} + \Big(\frac{h'_x}{hg^2} - \frac{g'_x}{g^3}\Big)\,\frac{\partial}{\partial x}\,.$$

In order to obtain our examples, we shall try below to construct metrics of class $\gamma(-\alpha^2)$ $(\alpha > 0)$ with g independent of x and such that the x-drift term $h'_x/(hg^2)$ is "large" compared to the t-drift term $g'_t/g+$

h'_t/h, and thus force the Brownian motion on M to move away from each geodesic $\tau_{(x,y)}$; in the next section, we study the ratio $h'_x/(h'_t g^2)$ in relation with the curvature assumptions and the Choi property C'_M.

2. Filling by convex hulls.

In this section, we assume that g and h define a $\gamma(-\alpha^2)$ metric ($\alpha > 0$), and that $g = g(t)$ is a function of t (*i.e.* g is independent of x). Of course, g is convex and increasing. Recall that M denotes the resulting manifold.

We denote by $M(\theta_0, a, b)$ the set $\{(x, y, t);\ t > \theta_0,\ a < x \leq b,\ y \in \mathbb{R}\}$ for given reals θ_0, a, b with $a < b$; let $N(\theta_0, a, b)$ denotes the projection of $M(\theta_0, a, b)$ on the (x, t)-plane, and let $F(\theta_0, b)$ be the "vertical" half plane $F(\theta_0, b) = \{(b, y, t) :\ t > \theta_0,\ y \in \mathbb{R}\}$.

Proposition 2.1. *Let t_0, a, b be three reals such that*

i) *$h(x,t)$ is nondecreasing in x (for each fixed t) on $M(\theta_0, a, b')$ for some $b' > b$ (*) and*

ii) $\displaystyle\int_{\theta_0}^{\infty} \delta(t)\, dt = +\infty$, *where*

$$\delta(t) = \inf\left\{\frac{h'_x(x,t)}{h'_t(x,t)\, g^2(t)} :\ a \leq x \leq b\right\}.$$

Then, the closed convex hull C of $F(\theta_0, b)$ in $M(\theta_0, a, b)$ (with respect to $\gamma(g, h)$) is $M(\theta_0, a, b)$.

We shall first establish the following lemma.

Lemma 2.2. *Let $\sigma : u \mapsto (x(u), y(u), t(u))$ be a unit speed geodesic in M with $\sigma(0) = m_0 = (x_0, y_0, t_0)$, and $\sigma'(0) = e_2 = h(x_0, y_0, t_0)^{-1}(0, 1, 0)$. The map $\tau : u \mapsto (x(u), t(u))$ is even, induces a global diffeomorphism of $]0, \infty)$ onto $\tau(]0, \infty))$, and the curve $\tau([0, +\infty))$ admits at $\tau(0)$ a tangent directed by the vector $V(x_0, t_0) = (h'_x(x_0, t_0), g(t_0)^2 h'_t(x_0, t_0))$. Also, the tangent to $\tau[0, +\infty)$ at $\tau(u)$ is (as a line) a continuous function of u, x_0, y_0, t_0.*

(*) The proposition is in fact also valid if we allow $b' = b$.

PROOF. Since the metric γ is invariant under the map $(x,y,t) \mapsto (x,-y,t)$, the points $\sigma(u)$ and $\sigma(-u)$ are symmetric with respect to the plane $y = y_0$ so that $\tau(u) = \tau(-u)$. Writing the Euler equations for the functional

$$F = \dot{t}^2 + g(t)^2\,\dot{x}^2 + h(x,t)^2\,\dot{y}^2\,,$$

we have the following three differential equations

$$\begin{aligned} t'' &= g\,g'\,x'^2 + h\,h'_t\,y'^2\,, \\ g^2\,x'' + 2\,g\,g'\,x'\,t' &= h\,h'_x\,y'^2\,, \\ h^2\,y' &= C\,, \end{aligned}$$

where C is a positive constant. Let $p_0 = (x_0, t_0)$. Since $t'(0) = x'(0) = 0$ and $y'(0) = (h(p_0))^{-1}$, it follows that $t''(0) = h(p_0)^{-1}\,h'_t(p_0)$ and $x''(0) = g(t_0)^{-2}\,h(p_0)^{-1}\,h'_x(p_0)$; also $h'_t(x_0,t_0) > 0$ ($t \mapsto h(x_0,t)$ is strictly convex and nondecreasing). Thus the third (and main) assertion of the lemma follows from the Taylor formula.

The first Euler equation above shows that t is a convex function of u, so that $t(u)$ is strictly increasing on $[0,\infty)$ and $t'(u) > 0$ for $u > 0$. Hence, τ is regular for $u > 0$ and injective on $[0,\infty[$. If we let $W(u) = (x'(u)/u, t'(u)/u) = \int_0^1 (x''(u\theta), t''(u\theta))\,d\theta$ for $u > 0$ and $W(0) = g(t_0)^{-2}\,h(p_0)^{-1}\,V(x_0,t_0)$, then $W(u)$ supports the tangent to τ at $\tau(u)$ and $W(u)$ depends continuously on $u \in [0,\infty[$, x_0, y_0, t_0 by the standard continuity theorems for solutions of differential equations.

REMARK. If $m_0 \in M(\theta_0, a, b')$, $x_0 < b'$, then $x(u)$ is increasing on any interval $J = [0, T[$ such that $\sigma(J) \subset M(\theta_0, a, b')$ (the second Euler equation above shows that $x'(u) > 0$ when $0 < u < T$).

PROOF OF PROPOSITION 2.1. C is invariant under translation in the y-variable so that $C = \{(x,y,t) : (x,t) \in \Phi\}$, Φ being a closed subset of $N(\theta_0, a, b)$.

We claim that the vector field $-V$ (V is defined in the previous lemma) is an inward vector field for Φ, which means that

$$\lim_{\substack{t>0 \\ t\to 0}} t^{-1} d(m - t\,V(m), \Phi) = 0$$

for each $m \in \Phi$.

To see this, fix $m_0 = (x_0, y_0, t_0) \in C$ and consider the (unit speed and oriented) geodesic arc σ_ε connecting $p(\varepsilon) = (x_0, y_0 - \varepsilon, t_0)$ to $p'(\varepsilon) =$

$(x_0, y_0 + \varepsilon, t_0)$ in M, and the projection τ_ε of σ_ε on the (x,t)-plane. Observe that the middle point $q(\epsilon)$ of σ_ε is in the plane $y = y_0$, with a tangent parallel to the y-axis; choose the parametrization of σ_ε such that $\sigma_\varepsilon(0) = q(\varepsilon)$, and let $\eta_\varepsilon > 0$ be the value of u for which $\tau_\varepsilon(u) = \tau_\varepsilon(-u) = (x_0, t_0)$. By the remark after Lemma 2.2, it is clear that for $\varepsilon > 0$ and small, one has $\sigma_\varepsilon \subset M(\theta_0, a, b)$ so that $\sigma_\varepsilon \subset C$ and $\tau_\varepsilon \subset \Phi$; also τ_ε is smooth at $m_1 = (x_0, t_0)$ and admits a tangent there, directed by $-\tau'_\varepsilon(\eta_\varepsilon)$.

It follows from the lemma above that when $\varepsilon \to 0$, the limit position of this tangent is the half-line emanating from m_1 and directed by $-V(m_1)$. This proves the claim, since a limit of Φ-inward vectors at fixed $m_1 \in \Phi$ is again a Φ-inward vector at m_1.

V being smooth (Lipschitz would be enough) and inward for Φ, it follows from a well-known theorem ([Bre]) that every curve $\gamma : [0, T[\to N(\theta_0, a, b)$ with $\gamma'(s) = -V(\gamma(s))$ for $s \in [0, T)$ and such that $\gamma(0) \in \Phi$, has all its image in Φ. Thus, to finish the proof it suffices now to observe that each maximal V-integral curve $\beta : J \to N(\theta_0, a, b)$ in $N(\theta_0, a, b)$ hits the line $x = b$. In fact, one has (letting $t(u)$ to denote the t-component of $\beta(u)$)

$$b - a \geq \int_\beta dx \geq \int_J \delta(t(u))\, dt(u) = \int_\beta \delta(t)\, dt$$

(both coordinates being increasing functions on J), so that our assumption on the function $\delta(t)$ implies that we must have $\sup\{t(u) :\ u \in J\} < +\infty$ and hence a hit with the line $x = b$ at the end point of β.

We shall see later that the assumptions of Proposition 2.1 may really occur. In this connection, it is interesting to note the following effect of the negative curvature for our class of metrics.

Proposition 2.3. *For every choice of functions $g(t)$ and $h(x,t)$ defining a $\gamma(-\alpha^2)$ metric ($\alpha > 0$) on M, and for every $a > 0$, we have $\lim_{t \to +\infty} \delta_1(t) = 0$, where*

$$\delta_1(t) = \sup\left\{ \left| \frac{h'_x(x,t)}{h'_t(x,t)\, g^2(t)} \right| :\ -a \leq x \leq a \right\}.$$

PROOF. The proof is based on the consideration of the level curves of h in $\mathbb{R}^2$. Let $x = \varphi(t)$, $t \in I$, be a maximal solution of $h'_x(\varphi(t), t)\, \varphi'(t) +$

$h'_t(\varphi(t),t) = 0$ with $h'_x(\varphi(t),t) \neq 0$ on I, say $h'_x(\varphi(t),t) < 0$ on I (*i.e.* φ is an increasing function). One immediately checks that

$$\varphi'' = \frac{1}{|h'_x|}\left(h''_{tt} + 2\,h''_{tx}\,\varphi' + h''_{xx}\,\varphi'^2\right)$$

on I. Because $\gamma(g,h)$ is of class $\gamma(0)$ (see the end of Section 1.1), we have (for all $(x,t) \in \mathbb{R}^2$)

$$h''_{tt}\,u^2 + 2\,\partial^2_{tx}\left(\frac{h}{g}\right)u\,v + \left(g^{-2}\,h''_{xx} + \frac{h'_t\,g'}{g}\right)v^2 \geq 0\,.$$

for all $u, v \in \mathbb{R}$.

On using $u = 1$ and $v = g\,\varphi'$ in this inequality, we derive from the expression of φ''

$$\varphi'' \geq -2\,\frac{g'}{g}\,\varphi' - g'\,g\,(\varphi')^3 \qquad \text{on } I\,,$$

where h has been eliminated. To "solve" this differential inequality, we first solve the differential equation

$$(E) \qquad \Psi'' = -2\,\frac{g'}{g}\,\Psi' - g'\,g\,(\Psi')^3\,.$$

If we let $\Psi'(t) = \lambda(t)\,g(t)^{-2}$, with $\lambda > 0$, (E) is equivalent to

$$\lambda' = -\frac{g'}{g^3}\lambda^3\,,$$

so that

$$\frac{1}{\lambda^2} = -\frac{1}{g^2} + C\,,$$

where C is some positive constant.

Whence the maximal solutions of (E)

$$\Psi(t) = \pm \int_t^{\infty} \frac{1}{g(s)\sqrt{Cg(s)^2 - 1}}\,ds + C'\,,$$

$t \in (t_0, +\infty)$, with $C = 1/g(t_0)^2$.

For each $t_1 \in I$, there exists a unique maximal E solution Ψ as above with $\Psi(t_1) = \varphi(t_1) = x_1$ and $\Psi'(t_1) = \varphi'(t_1)$, the corresponding t_0 and C being given by

$$C = \frac{1}{g(t_1)^2} + \frac{1}{\varphi'(t_1)^2 g(t_1)^4} \qquad \text{and} \qquad C\,g(t_0)^2 = 1$$

(see Figure 1).

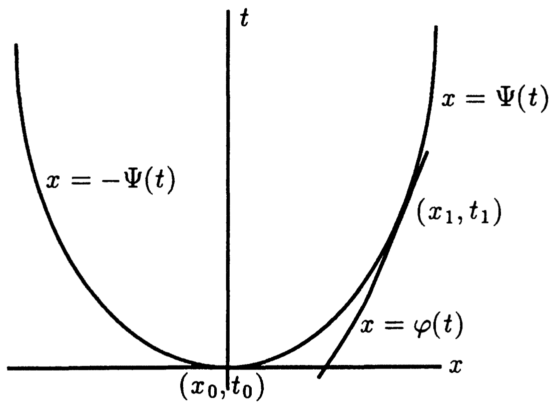

Figure 1

By the standard comparison theorem for first order differential equations, we see that when $t_0 < t \leq t_1$ and $t \in I$,

$$\varphi'(t) \leq \Psi'(t).$$

In particular, $(t_0, t_1] \subset I$ and for all $t \in [t_0, t_1]$, $\Psi(t) \leq \varphi(t) \leq \Psi(t_1)$.

We now make two observations. Firstly, we have the following lower bound of

$$\mu_\varphi(t_1) = \varphi'(t_1)\, g(t_1)^2 = \frac{g(t_1)^2 h'_t(x_1, t_1)}{|h'_x(x_1, t_1)|}$$

in terms of t_0; from the above expression of C and $C\, g(t_0)^2 = 1$, we get

$$(1) \qquad g(t_0)^2 = \frac{1}{C} = \frac{1}{\dfrac{1}{g(t_1)^2} + \dfrac{1}{g(t_1)^4 \varphi'(t_1)^2}} = \frac{\mu_\varphi^2}{1 + \dfrac{\mu_\varphi^2}{g(t_1)^2}} \leq \mu_\varphi^2 .$$

Also, since

$$\Psi(\infty) - \Psi(t) = \int_t^\infty \frac{g(t_0)}{\sqrt{g(s)^2 - g(t_0)^2}} \frac{ds}{g(s)},$$

and

$$\begin{aligned} g(s) &\geq g(t_0)\cosh(\alpha(s-t_0)) + \alpha^{-1} g'(t_0)\sinh(\alpha(s-t_0)) \\ &\geq g(t_0)\cosh(\alpha(s-t_0)), \end{aligned}$$

for $s \geq t_0$, we have, letting $t'_0 = \max\{t_0+1, 0\}$, $x'_0 = \varphi(t'_0)$ (and assuming $t'_0 \leq t_1$),

$$x_1 - x'_0 \leq \Psi(\infty) - \Psi(t'_0) \leq \frac{C_1}{g(t'_0)} \tag{2}$$

for some absolute constant C_1 depending only on α.

Suppose now that there is a sequence of points (x_j, t_j) with $t_j \to \infty$, $|x_j| \leq c$ for some fixed $c > 0$, and such that

$$\mu_j = g(t_j)^2 \frac{h'_t(x_j, t_j)}{|h'_x(x_j, t_j)|}$$

is bounded. Then, for the coresponding (E) curves through (x_j, t_j) and with the above notations we have (we omit the index j): i) t_0 is bounded from above by (1), so that t'_0 is bounded, and ii) x'_0 also stays bounded because of (2). Since, the level curve of h through (x_j, t_j) must hit $\{(x,t) : t = t'_0, |x - x'_0| \leq |\Psi(\infty) - \Psi(t'_0)|\}$ (Ψ depends on j), it is seen that this line meets a compact subset of $\mathbb{R}^2$ (independent of j), so that $h(x_j, t_j)$ stays bounded. But this is in contradiction with the exponential growth of h with respect to t in each strip $\{|x| \leq A\}$ ($A > 0$), and Proposition 2.3 is proved.

3. Two extension properties for γ-metrics.

In this section and the next, we let $g(t) = e^t$. If α is a (strictly) positive number, and if $h(x,t)$ is a smooth non-negative function on some region A of $\mathbb{R}^2$, we shall say that h is of class $\mathcal{H}(\alpha)$ on A if the following three inequalities hold on A:

$$h''_{tt} - \alpha h \geq 0,$$

$$h'_t + \frac{h''_{xx}}{g^2} - \alpha h \geq 0,$$

and

$$\left|\partial^2_{tx}\left(\frac{h}{g}\right)\right|^2 \leq (h''_{tt} - \alpha h)\left(h'_t + \frac{h''_{xx}}{g^2} - \alpha h\right),$$

and if $h(x,t)$ is nondecreasing with respect to t. If $A = \mathbb{R}^2$, $\alpha \leq 1$ and $h > 0$, this just means that g and h define a $\gamma(-\alpha)$ metric on $\mathbb{R}^3$.

We shall need the following two elementary observations.

3.1. If h_1 and h_2 are of type $\mathcal{H}(\alpha)$ on $A \subset \mathbb{R}^2$ and if $\lambda, \mu \in \mathbb{R}_+$, then $h = \lambda h_1 + \mu h_2$ is of type $\mathcal{H}(\alpha)$ on A. Observe that the $\mathcal{H}(\alpha)$ condition is equivalent to the non-negativity of a quadratic form on $\mathbb{R}^2$ whose coefficients depends linearly on h.

3.2. Let h be of type $\mathcal{H}(\alpha)$ on a region A such that $\inf\{t : (x,t) \in A\} > -\infty$ and let $\alpha' < \alpha$. If h admits a > 0 lower bound on A and if h_1 is any smooth bounded real function on A with bounded first and second order derivatives on A, then $h_1 + Ch$ is of type $\mathcal{H}(\alpha')$ on A provided that the constant C is chosen large enough.

In the following proposition, we denote by a_0 an absolute positive constant whose value is fixed before the statement of Lemma 3.4.

Proposition 3.3. *Let $h(x,t)$ be a smooth (strictly) positive function of type $\mathcal{H}(\alpha)$ (for some $\alpha \in (0,1)$) on the region $\omega(\varepsilon) = \{(x,t) : t_0 < t < t_1 + \varepsilon\} \cup \{(x,t) : |x| < x_0 + \varepsilon,\ t_1 \leq t < t_2 + \varepsilon\}$, where $t_1 = t_0 + 1$, $t_2 \geq t_1 + 2$, $\varepsilon > 0$, $x_0 > a_0 e^{-t_1}$, and assume that $h(x,t) = e^t$ for $t_0 < t < t_1 + \varepsilon$.*

Then, we may find a smooth positive function h_1 on $A = \{(x,t) : t_0 < t < \infty\}$ such that

i) *h_1 is of type $\mathcal{H}(\alpha)$ on A,*

ii) *$h_1(x,t) = e^t$ if $t \leq t_1$, or if t is large enough, and*

iii) *$h_1 = h$ for $|x| \leq x_0 - a_0 e^{-t_1}$ and $t_0 < t < t_2$.*

The proof will show that we may also require $h_1(x,t)$ to be for $|x|$ large a function of t only.

Before proving this proposition, we fix some notations. Let $\theta = f_0 * \varphi$ be a standard regularization of $f_0(x) = \inf\{x, 1\}$, with $\varphi \geq 0$ smooth, even, and such that $\operatorname{supp}(\varphi) = [-1/2, 1/2]$; observe that θ is concave, $\theta(x) = x$ for $x \leq 1/2$ and $\theta(x) = 1$ if and only if $x \geq 3/2$. We then choose and fix a positive number a, sufficiently large so that $-\theta''\theta + \theta'^2 \leq a^2$, and let $\Phi(x) = -\log(\theta(x/a))$ for $x > 0$. Clearly Φ is ≥ 0, convex non-increasing and smooth on $(0, +\infty)$, $\operatorname{supp}(\Phi) = (0, 3a/2]$

and $\Phi(x) = -\log(x/a)$ on $(0, a/2]$. We let $a_0 = 3a/2$ and observe the following.

Lemma 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth convex function with* $\operatorname{supp}(f) = [0, \infty)$, *and let $h(x,t) = g(t)f(t - \Phi(x))$ ($= 0$ if $x \leq 0$). Then h is smooth on $\mathbb{R}^2$, nondecreasing in t, $\{h > 0\} = \{(x,t) : x > 0,\ t > \Phi(x)\}$ and $h(x,t) = g(t)f(t)$ for $x > a_0$. Moreover h is of type $\mathcal{H}(1)$ on $\mathbb{R}^2$.*

PROOF. We have (with obvious notations and for $x > 0$)

$$\left|\partial^2_{tx}\left(\frac{h}{g}\right)\right|^2 = |f''\Phi'|^2,$$
$$h''_{tt} - h = e^t\,(2f' + f''),$$
$$h'_t\frac{g'}{g} + \frac{h''_{xx}}{g^2} - h = e^t\left(f'e^{-2t}(e^{2t} - \Phi'') + f''\,\Phi'^2 e^{-2t}\right).$$

Clearly, h is of type $\mathcal{H}(1)$ if $\Phi''(x) \leq e^{2t}$ for $t \geq \Phi(x)$, $0 < x < a_0$, or equivalently if $\Phi''(x) \leq \exp(2\,\Phi(x))$ for $0 < x \leq a_0$. But with our previous choices we have

$$\Phi''(x) = a^{-2}\,\theta\left(\frac{x}{a}\right)^{-2}\left(-\theta''\left(\frac{x}{a}\right)\theta\left(\frac{x}{a}\right) + \theta'\left(\frac{x}{a}\right)^2\right) \quad \text{and} \quad e^{2\Phi} = \theta^{-2},$$

so that the lemma follows from the choice of a.

REMARKS.

3.5. If $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and if we let $k(x,t) = h(e^{t_0}(x - x_0), t - t_0)$, h being the function in the previous lemma, we obtain a function k of type $\mathcal{H}(1)$ on $\mathbb{R}^2$ with $\{k > 0\} = \{(x,t) : t > t_0,\ x > x_0,\ t > t_0 + \Phi(e^{t_0}(x - x_0))\}$.

3.6. Let Ψ be the function : $\Psi(x) = \Phi(x)$ for $0 < x \leq a_1$ and $\Psi(x) = \Phi(2a_1 - x)$ for $a_1 \leq x < 2a_1$, where $a_1 \geq 3a_0/2$. Then, with f as above, the function $h_1(x,t) = g(t)f(t - \Psi(x))$ ($= 0$ if $x \notin (0, 2a_1)$) is of type $\mathcal{H}(1)$ and $\{h_1 > 0\} = \{(x,t) : 0 < x < 2a_1,\ t > \Psi(x)\}$.

PROOF OF PROPOSITION 3.3. We break the construction into three steps.

a) We first construct h_1 on the region $t_0 < t < t_2$. After multiplying h by a standard cut off function, we may assume that h is a smooth ≥ 0 function on $\mathbb{R}^2$ whose derivatives of order ≤ 2 are bounded, whose support is contained in $\omega'(\varepsilon) = \omega(\varepsilon) \cup \{(x,t) : \ t \leq t_0\}$, and which is of type $\mathcal{H}(\alpha)$ and > 0 on $\omega(2\varepsilon/3)$. Moreover $h = e^t$ if $t < t_1 + \varepsilon/2$.

Using the Remark 3.5 above, we construct a smooth and non-negative function $k_0(x,t)$ of type $\mathcal{H}(1)$ on $\mathbb{R}^2$ with $\{\, k_0 > 0 \,\} = \{(x,t) : x > x_0 - a_0 e^{-t_1},\ t - t_1 > \Phi(e^{t_1}(x - x_0) + a_0)\}$. If we let $k_1 = h + Ck_0$ where C is a large positive constant, then, by 3.2, k_1 is of type $\mathcal{H}(\alpha)$ on the set $B = \{(x,t) : \ x > x_0 + \varepsilon/2,\ t > t_1 + \varepsilon/2\}$.

On the other hand, for all positive value of C, $h + Ck_0$ is of type $\mathcal{H}(\alpha)$ on $\omega(\varepsilon/2)$. Thus, we obtain a function k_1 on $\mathbb{R}^2$ of type $\mathcal{H}(\alpha)$ on $\omega(\varepsilon/2) \cup B$.

On applying the similar procedure to k_1 for the region $B' = \{\, x < -x_0 - \varepsilon/2,\ t > t_1 + \varepsilon/2\}$, we obtain a smooth positive function k_2 on $\mathbb{R}^2$ which is of type $\mathcal{H}(\alpha)$ on the region $\{t_0 < t < t_2 + \varepsilon/2\}$ and which agrees with h for $|x| \leq x_0 - a_0 e^{-t_1}$, $t_0 < t < t_2$ and for $t_0 < t \leq t_1$. Note that for $|x| \geq x_0 + \varepsilon$ and $t > t_0$, $k_2(x,t)$ is a function of t only, which is increasing and such that $\partial_t^2 k_2(t,x) \geq k_2(t,x)$.

b) On multiplying k_2 by a cut-off function we may assume that k_2 has its support contained in $\{t \leq t_2 + \varepsilon/2\}$, is (strictly) positive of type $\mathcal{H}(\alpha)$ on $\{t_0 < t \leq t_2 + \varepsilon/4\}$, and that k_2 is for $|x| \geq x_0 + \varepsilon$ a function of t only. Let $k(t)$ be a smooth positive function on $\mathbb{R}$ with support $[t_2, \infty)$ and such that $k'' \geq k,\ k' \geq k$; we choose k in the form $k(t) = e^t\, \Psi(t)$ with Ψ convex, smooth, and $\operatorname{supp}(\Psi) = [t_2, \infty)$. Let $k_3(t,x) = k_2(t,x) + Ck(t)$. As before, if C is a large positive constant, then k_3 is of type $\mathcal{H}(\alpha)$ on $\{t_0 < t < +\infty\}$. Also, for $t > t_2 + \varepsilon$, $k_3 = Ce^t\Psi(t)$ is of type $\mathcal{H}(1)$ and is a function of t only.

c) To finish the proof, we observe that we can easily modify k_3 for $t > t_2' = t_2 + 2\varepsilon$ into a new function h_1 of type $\mathcal{H}(\alpha)$ on $\{t > t_2'\}$ in such a way that $h_1(t) = e^t$ for large t: if we let $\beta = \sqrt{\alpha}$, $u(t) = Ce^{(1-\beta)t}\Psi(t)$, u is convex increasing on $[t_2', \infty)$, and since $v(t) = e^{(1-\beta)t}$ is also convex and such that $\lim_{t\to\infty} t^{-1}v(t) = +\infty$, there exists a smooth convex function $w(t)$ on $[t_2', \infty)$ such that $w(t) = u(t)$ for $t_2' \leq t \leq t_2' + 1$, and $w(t) = e^{(1-\beta)t}$ for t large enough. So that $h_1(t) = e^{\beta t}w(t)$ agrees with k_3 on $[t_2', t_2' + 1]$, with e^t for large t, and verifies $\partial_t^2 h_1 \geq \beta^2 h_1$ for $t > t_2'$; thus, h_1 is of type $\mathcal{H}(\alpha)$. (We have used the following simple fact: if u and v are two smooth convex functions on $[0, \infty)$ and if $\lim_{t\to\infty} t^{-1}v(t) = +\infty$, there is a smooth convex function φ on $[0, +\infty[$ such that $\varphi(t) = u(t)$ if $0 \leq t \leq 1$ and $\varphi(t) = v(t)$ for large t).

We shall also use the following two variants of Proposition 3.3:

3.7. Replace in the statement $\omega(\varepsilon)$ by the region $\omega'(\varepsilon) = \{(x,t):\ t_0 < t < t_1+\varepsilon\} \cup \{(x,t):\ x < x_0+\varepsilon,\ t_1 \le t < t_2+\varepsilon\}$, where $x_0 \in \mathbb{R}$, $t_1 = t_0+1$, $t_2 \ge t_1+2$, and $\varepsilon > 0$. If h and its derivatives of order ≤ 2 are bounded (for large $|x|$), then the conclusions of Proposition 3.3 still hold, iii) being replaced by

iii)' $h_1 = h$ for $x \le x_0 - a_0 e^{-t_1}$ and $t_0 < t < t_2$.

Moreover, if the given function $h(t,x)$ is increasing with respect to x, h_1 may also be chosen increasing with respect to x.

3.8. One may more generally replace the region $\omega(\varepsilon)$ by a region $\omega'(\varepsilon) = \{(x,t):\ t_0 < t < t_1+\varepsilon\} \cup \{(x,t):\ t_1 < t < t_2+\varepsilon,\ x \in B\}$ where B is the union of a finite number of intervals. Then, assuming that h and its derivatives of order ≤ 2 are bounded, a simple adaptation of the proof above (using 3.4 and 3.6) shows that the conclusions of Proposition 3.3 hold, iii) being replaced by

iii)' $h_1 = h$ for $d(x, B^c) \ge 2a_0 e^{-t_1}$, $t_0 \le t \le t_2$.

If B is the union of two intervals $I =]-\infty, -a]$ and $J = -I$ where $a > 0$, and if h is even, increasing with respect to $x \in I$, then we may choose a function h_1, even with respect to x, and decreasing with respect to $x \in \mathbb{R}_+$.

We shall need another "pasting" lemma which says that given $t_0 \in \mathbb{R}$, $\alpha \in (0,1)$ and a function of type $\mathcal{H}(1)$ in the form $h(x,t) = e^t\, b(x)$, $x \in J$, where b is smooth convex and ≥ 1 on the open interval J (with $\|b'\|_\infty < +\infty$), we may construct on the region $\{(x,t):\ x \in J\}$ a function h_1 of type $\mathcal{H}(\alpha)$ equal to e^t when $t \le t_0$ and equal to h for t large enough. To state this lemma, we fix a smooth non-negative and non-increasing function φ on $\mathbb{R}$ such that $\varphi(t) = 0$ for $t \ge 1 - 1/16$, $\varphi(t) = 1$ for $t \le 3/4$; we also assume as we may that $\varphi(t)$ is convex for $t \ge 7/8$. Let $\Psi(t) = \varphi(1-t)$.

Lemma 3.9. *Let b be a smooth convex function on the open interval $J \subset \mathbb{R}$ such that $b \ge 1$ and $\|b'\|_\infty < +\infty$, and let t_0, ε, α be real numbers with $\varepsilon > 0$, $0 < \alpha < 1$. If $h(t,x) = e^t\big(\varphi(\varepsilon(t-t_0)) + \Psi(\varepsilon(t-t_0))\, b(x)\big)$, for $t \in \mathbb{R}$, $x \in J$, and $h(t,x) = e^t$ for all x and $t \le t_0 + 1/(16\varepsilon)$, then for ε small enough (depending only on α and*

$e^{-t_0}\|b'\|_\infty$), h is of type $\mathcal{H}(\alpha)$ on the region $\{(x,t):\ x \in J\} \cup \{(x,t):\ t < t_0 + 1/(16\varepsilon)\}$.

PROOF. We have

$$\begin{aligned}\frac{h''_{tt}}{h} - \alpha &= 1 - \alpha + 2\varepsilon\,\frac{\varphi' + b\Psi'}{\varphi + b\Psi} + \varepsilon^2\,\frac{\varphi'' + b\Psi''}{\varphi + b\Psi}\\ &\geq 1 - \alpha - 2\varepsilon\,\|(\varphi')^-\|_\infty - \varepsilon^2\left\|\left(\frac{\varphi''}{\varphi}\right)^-\right\|_\infty\end{aligned}$$

and

$$\begin{aligned}\frac{g'}{g}\,\frac{h'_t}{h} + \frac{h''_{xx}}{g^2 h} - \alpha &= 1 - \alpha + \varepsilon\,\frac{\varphi' + \Psi' b}{\varphi + \Psi\, b} + \frac{1}{g^2}\,\frac{\Psi\, b''}{\varphi + \Psi\, b}\\ &\geq 1 - \alpha - \varepsilon\|(\varphi')^-\|_\infty\end{aligned}$$

so that for ε small enough both quantities are greater than $(1-\alpha)/2$.

Observe next that if $t \geq t_0 + \varepsilon^{-1}/4$, or if $t \leq t_0 + \varepsilon^{-1}/16$, the mixed curvature term $h^{-1}\partial_t(h'_x/g)$ vanishes. For $t_0 + \varepsilon^{-1}/16 \leq t \leq t_0 + \varepsilon^{-1}/4$, and with obvious notations, we have

$$h^{-1}\left|\partial_t\left(\frac{h'_x}{g}\right)\right| = \varepsilon\left|\frac{\Psi'\, b'}{g(1+\Psi\, b)}\right| \leq \varepsilon\, e^{-t_0}\,\|\Psi'\|_\infty\,\|b'\|_\infty$$

and the lemma follows.

4. Proof of Corollary C.

We first exhibit a simple way of producing on a region of the type $U(t_1,t_2;J) = \{(x,y,t):\ t_1 < t < t_2,\ x \in J\}$ a $\gamma(-1)$ metric such that the integral $\int_{t_1}^{t_2} \delta_K(t)\,dt$ is very large, where we have let

$$\delta_K(t) = \inf\left\{\frac{h'_x(x,t)}{g(t)^2\,h'_t(x,t)}\ :\ x \in K\right\}.$$

Here, $J \subset \mathbb{R}$ denotes an interval with a finite upper bound, K is a compact subinterval, and g will simply be $g(t) = e^t$ as in the previous section, so that h should be positive and of type $\mathcal{H}(1)$ on $U(t_1,t_2;J)$.

We fix a smooth positive and convex function β on J such that $\beta'(x) > 0$, $\beta(x) \geq 1$ and

$$2\,\beta'(x)^2\Big(1+\frac{1}{2\beta(x)}\Big) \leq \beta(x)\,\beta''(x)\,, \tag{4.1}$$

for all $x \in J$. For example, we may take $\beta(x) = a + e^x$ with $a \geq 1$ sufficiently large depending on the upper bound of J.

Let $\Phi(t)$ be any smooth increasing function of t such that $\Phi'' + 2\Phi' \geq 0$ and $\Phi(t) \geq 1$ on (t_1, t_2). We then define

$$h(x,t) = e^t \exp(\Phi(t)\,\beta(x))\,, \qquad \text{for } x \in J\ ,\ t_1 < t < t_2\ , \tag{4.2}$$

and may easily check the following lemma.

Lemma 4.1. *The functions g and h define a $\gamma(-1)$ metric on the set $U(t_1, t_2; J)$.*

PROOF. Clearly, h is positive, increasing with respect to the t variable,

$$\begin{aligned}\frac{h''_{tt}(x,t)}{h(x,t)} - 1 &= 2\,\Phi'(t)\,\beta(x) + \beta(x)\,\Phi''(t) + \beta(x)^2\,\Phi'(t)^2\\ &= \beta(x)\,(\Phi''(t) + 2\,\Phi'(t)) + \beta(x)^2\,\Phi'(t)^2\,,\end{aligned}$$

$$\begin{aligned}\frac{h'_t(x,t)}{h(x,t)} + \frac{h''_{xx}(x,t)}{g^2(t)\,h(x,t)} - 1 &= \Phi'(t)\,\beta(x)\\ &\quad + \frac{1}{g(t)^2}\left(\Phi(t)^2\,\beta'(x)^2 + \Phi(t)\,\beta''(x)\right),\end{aligned}$$

and

$$\Big|\frac{1}{h}\,\partial_t\big(\frac{h'_x}{g}\big)\Big|^2 = \frac{1}{g(t)^2}\,\Phi'(t)^2\,\beta'(x)^2\,\left(1+\Phi(t)\,\beta(x)\right)^2.$$

Thus, it suffices to check that

$$\Phi'(t)^2\,\beta'(x)^2\,(1+\Phi(t)\,\beta(x))^2 \leq \beta(x)^2\,\Phi'(t)^2\left(\Phi(t)^2\,\beta'(x)^2+\Phi(t)\,\beta''(x)\right),$$

which is the same as

$$\Phi'(t)^2\,\beta'(x)^2\,(1+2\Phi(t)\,\beta(x)) \leq \Phi(t)\,\Phi'(t)^2\,\beta(x)^2\,\beta''(x)$$

or

$$2\,\beta'(x)^2\Big(1+\frac{1}{2\Phi(t)\beta(x)}\Big)\le\beta(x)\,\beta''(x)\,,$$

which follows from (4.1) since $\Phi(t)\ge 1$.

Lemma 4.2. *Let A be a positive number and $K\subset J$ be a compact interval. For each given t_1, we may choose $t_2>t_1+2$ and a function Φ as above such that*

i) $\displaystyle\int_{t_1}^{t_2}\delta_K(t)\,dt>A,$ *and*

ii) $\Phi(t)=1$ *for* $t_1<t<t_1+1/2$.

Here

$$\delta_K(t)=\inf\Big\{\frac{h'_x(x,t)}{g(t)^2\,h'_t(x,t)}\;:\;x\in K\Big\}\,.$$

PROOF. The inequality $\Phi''+2\,\Phi'\ge 0$ means that $\Phi'(t)\,e^{2t}$ is a nondecreasing function; thus, choosing

$$\Phi(t)=1+\int_{t_1}^{t}\varphi(s)\,e^{-2s}\,ds$$

with φ smooth, increasing on $(t_1,+\infty)$ and $\varphi(t)=0$ on $(t_1,t_1+1/2)$ guarantee property ii) above and the required differential inequality for Φ. Also, $\Phi(t)\ge 1$.

On the other hand, we have, for $t\ge t_1,\ x\in K$,

$$\frac{h'_x(x,t)}{g(t)^2\,h'_t(x,t)}=e^{-2t}\,\frac{\Phi(t)\beta'(x)}{1+\Phi'(t)\beta(x)}\ge c\,e^{-2t}\,\frac{\Phi(t)}{1+\Phi'(t)}\,,$$

where c is some positive constant (depending on β only). Now, assuming that $\varphi(t)$ is a (large) constant φ_0 on the interval $(t_1+3/4,T)$, we have the following lower bound (we let $t'_1=t_1+1$ and assume $T>t'_1+1$)

$$\int_{t'_1}^{T}\delta_K(t)\,dt\ge\frac{c}{4}\int_{t'_1}^{T}e^{-2t}\,\frac{\varphi_0\,e^{-2t'_1}}{1+e^{-2t}\varphi_0}\,dt\,.$$

So that, if moreover φ_0 is so large that $\varphi_0\,e^{-2T}\ge 1$,

$$\int_{t'_1}^{T}\delta_K(t)\,dt\ge\frac{c}{8}\,e^{-2t'_1}\,(T-t'_1)\,.$$

Finally, it is seen that if we choose $t_2 = T$ so large that

$$\frac{c}{8}\, e^{-2t'_1}\,(T - t'_1) \geq A$$

and then construct φ such that $\varphi(t) = \varphi_0$ on $(t_1 + 3/4, T)$, with φ_0 larger than e^{2T}, we obtain a number t_2 and a function Φ with all the required properties.

It is now easy to construct a function h on $\mathbb{R}^2$ which produces an example establishing Corollary C. Using propositions 3.3 (see Remark 3.7), 3.9 and the above lemmas, one constructs by induction a smooth positive function h on $\mathbb{R}^2$ and an increasing sequence $\{t_j\}_{j\geq 0}$ of reals such that (recall that $g(t) = e^t$ for all t)

i) $t_0 > 0$, $t_{j+1} - t_j \geq 1$,

ii) h is increasing with respect to each variable,

iii) $h(x,t) = g(t)$ for $t \leq t_0$ or $t_{4j+3} < t < t_{4(j+1)}$,

iv) g and h define a $\gamma(-1/4)$ metric on $\mathbb{R}^3$, and

v) $\displaystyle\int_{t_{4j+1}}^{t_{4j+2}} \delta_j(t)\, dt \geq 1$, where

$$\delta_j(t) = \inf\left\{\frac{h'_x(x,t)}{g(t)^2 h'_t(x,t)} : |x| \leq j\right\}, \qquad t_{4j+1} \leq t \leq t_{4j+2}\,.$$

Now, for $M = \mathbb{R}^3$ equipped with the corresponding γ metric, Proposition 2.1 shows that every neighborhood V of the point $\infty_M \in S_\infty(M)$ in the compactification $\widehat{M}$ (see Section 1.3) is such that the closed convex hull of $V \cap M$ in M is M itself; on the other hand, the sectional curvatures of M are all $\leq -1/4$.

5. The Brownian motion's behavior.

A. Let $g(t) = e^t$ and $h(x,t)$ define a $\gamma(-1/4)$ metric on $\mathbb{R}^3$ (*i.e.* h is smooth, positive and of type $\mathcal{H}(1/4)$ on $\mathbb{R}^2$) and, as before, let M to denote the corresponding Riemannian manifold. We assume once for all that on each region $\{(x,t) : t < a\}$, $a \in \mathbb{R}$, the partial derivatives of h of order ≤ 2 are bounded and that $h(x,t) = e^t$ for $t \leq 0$; thus, the sectional curvatures of $\gamma_{(g,h)}$ are bounded on each region $\{t < a\}$. Clearly, this is verified in the above construction in Section 4.

Let $\{\Omega,\, \mathcal{F},\, \{P_x\}_{x\in M},\, \{B_s\}_{s\geq 0}\}$ be the Brownian motion on M which we see as a continuous Markov process on $[0,+\infty]$ with value in the Alexandroff compactification $M \cup \{c(M)\}$ of M, $c(M)$ being the cemetery point (this is possible because M is transient). Denote by S the lifetime of B_s ($B_s = c(M)$ if and only if $s \geq S$), by $X(s)$, $Y(s)$ and $T(s)$ the components in $\mathbb{R}^3$ of B_s for $s < S$. We start with the following observations.

Lemma 5.1. *We have*

1) *Almost surely,* $\lim\limits_{s\to S} T_s(\omega) = +\infty$,

2) *Almost surely,* $\lim\limits_{s\to S} Y_s(\omega)$ *exists and is finite,*

3) *If h is nondecreasing with respect to x, then* $\lim\limits_{s\to S} X_s(\omega)$ *exists in* $(-\infty,+\infty]$ *almost surely.*

PROOF. The lemma follows from several application of the following basic (and standard) fact: if u is a continuous ≥ 0 supersolution on M (with respect to the Laplace-Beltrami operator Δ_M), then $\{u(B_s)\}_{s\geq 0}$ (with the usual convention that $u(c(M)) = 0$) is a nonnegative right-continuous supermartingale with respect to each probability measure P_x , so that almost surely, $s \mapsto u(B_s)$ admits a left-hand limit at every $s_0 \in (0,+\infty]$ ([DM, p.75 and p.79]). In particular, as $s \to S-0$, $u(B_s)$ has a finite limit almost surely. This being true for any Riemannian manifold, it is also seen that if v is a positive continuous Δ_M-superharmonic function on a region $\Omega \subset M$ such that $\sup\{s : B_s \in M\setminus\Omega\} < S$ a.s. (Ω is *absorbing*), then $\lim_{s\to S} v(B_s)$ exists and is finite a.s. Recall also from Section 1.3 that

$$\Delta_M = \frac{\partial^2}{\partial t^2} + \frac{1}{g^2}\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2}{\partial y^2} + \left(1+\frac{h'_t}{h}\right)\frac{\partial}{\partial t} + \frac{h'_x}{hg^2}\frac{\partial}{\partial x}\,.$$

It is clear that $u(x,y,t) = e^{-t}$ is Δ_M-superharmonic on M; it follows, by the above remark, that $T(s)$ admits a limit in $(-\infty,+\infty]$ a.s. On the other hand, since the sectional curvatures of M are bounded in each region $\{t < a\}$ the Brownian motion is a.s. bounded before leaving any such region, and the first assertion follows.

We then prove the third assertion in the lemma. From the form of Δ_M, and because h is nondecreasing in x, we see that $u(x,t)$ is superharmonic on M (or a region of M) if $u(x,t)$ is smooth nonincreasing in x and in t and is superharmonic with respect to to

$$L = \frac{\partial^2}{\partial t^2} + \frac{1}{g(t)^2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}.$$

L is the Laplace-Beltrami operator on N, the (x,t)-plane equipped with the hyperbolic metric $dt^2 + e^{2t}\,dx^2$. Thus, for each real x_0, if we let

$$u(x,t) = \begin{cases} 1, & \text{if } x \le x_0, \\ 1 - \frac{2}{\pi}\operatorname{Arctg}((x-x_0)e^t), & \text{otherwise}, \end{cases}$$

(u is the harmonic measure in N of the region $\{x \le x_0\}$), it is easily seen that u as a function on M is Δ_M-superharmonic. Since $u(B_s)$ converges a.s. when $s \to S$, and x_0 is chosen in a dense sequence of reals, and since $\lim_{s\to S} T(s) = +\infty$, it is clear that X_s converge a.s. in $[-\infty,+\infty]$ when $s \to S$. The value $-\infty$ is excluded by the supermartingale inequality

$$u(x) \ge E_x(\lim_{s\to S-0} u(B_s)) \ge P_x(\lim_{s\to S} X_s \ < x_0).$$

The second claim of the lemma may be proved similarly. Observe that $h(x,t) \ge e^{t/2}$ when $t \ge 0$, since $h''_{tt} \ge h/4$, and $h(x,t) = e^t$ for $t \le 0$, so that $h(x,t) \ge \cosh(t/2) + 2\sinh(t/2)$ for $t \ge 0$; thus, each positive function $u(y,t)$ which is nonincreasing in t, convex in y and superharmonic on the (y,t)-plane with respect to

$$L = \frac{\partial^2}{\partial t^2} + e^{-t}\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t}$$

is also Δ_M-superharmonic on the absorbing region $\Omega = \{t > 0\} \subset M$. It follows easily that for each real y_0, the function

$$u(x,y,t) = \begin{cases} 1, & \text{if } y \le y_0 \quad (\text{resp. } y \ge y_0), \\ 1 - \frac{2}{\pi}\operatorname{Arctg}(\frac{1}{2}\,|y-y_0|\,e^{\frac{t}{2}}), & \text{otherwise}, \end{cases}$$

is superharmonic on Ω. The claim follows then as above. (One could also use the convexity in M of the sets $C = \{y \le y_0\}$ and the corresponding superharmonic functions given by Proposition 6.1).

Lemma 5.2. *Let t_1, β and J be fixed as in Section 4, and let $K = [a,b] \subset J$; let η be some positive number and let τ to denote the first exit time of B_s out of $U = \{(x,y,t) : a < x < b,\ t_1'' = t_1 + 2 < t < t_2\}$. We may choose, in the statement of Lemma 4.2, Φ and $t_2 \geq t_1 + 10$ such that, if $h(x,t) = e^t \exp(\Phi(t)\,\beta(x))$ on U, then $P_m(T_\tau = t_2$ or $T_\tau = t_1'') \leq \eta$ for all $m = (x,y,t)$ such that $a < x < b$ and $t = (t_1'' + t_2)/2$.*

PROOF. It will suffice to use once again a supermartingale argument. We let $\sigma(m) = \sigma(x,y,t) = \exp(\varepsilon\, t - x)$ where ε will be chosen small (depending only on t_1, β and K) so that σ will be superharmonic on U if the constant φ_0 of the construction in Lemma 4.2 is taken large enough (depending on t_2). In fact, we have in U (with the notations of 4.2)

$$\sigma^{-1} \Delta_M(\sigma) = \varepsilon^2 + \varepsilon\,(2 + \beta(x)\,\varphi_0\, e^{-2t}) + e^{-2t} - e^{-2t}\,\Phi(t)\,\beta'(x)$$

and

$$\sigma^{-1} \Delta_M(\sigma) \leq \varepsilon^2 + 2\,\varepsilon + c_1\,\varepsilon\,\varphi_0\, e^{-2t} + e^{-2t} - c_2\, e^{-2t}\,\varphi_0\, e^{-2t_1}\,,$$

where the c_j are positive and depend only on β and K. We fix $\varepsilon > 0$ and small enough so that $c_1\,\varepsilon - c_2\, e^{-2t_1}/2 \leq 0$, and then may choose t_2 and φ_0 (in that order) so large that

$$\exp(b-a)\, \exp(-\frac{1}{2}\,\varepsilon\,(t_2 - t_1'')) \leq \frac{\eta}{2}\,, \qquad \exp(-\frac{1}{2}\,(t_2 - t_1'')) \leq \frac{\eta}{2}\,,$$

and $\Delta_M(s) \leq 0$ on U. Then, from the supermatingale inequality $E_m(\sigma(B_\tau)) \leq \sigma(m)$, we have

$$P_m(T_\tau = t_2) \leq \exp(b-a)\, \exp(-\frac{1}{2}\,\varepsilon\,(t_2 - t_1'')) \leq \frac{\eta}{2}\,,$$

if $m = (x,y,t)$, $t = (t_1'' + t_2)/2$, and $a < x < b$. On the other hand, by the superharmonicity of e^{-t}, we also have

$$P_m(t_\tau = t_1'') \leq \exp(-\frac{1}{2}\,(t_2 - t_1'')) \leq \frac{\eta}{2}\,.$$

We also need the following obvious lemma.

Lemma 5.3. *Assume that there is a sequence* $\{t_j\}$ *with* $\lim_{j\to\infty} t_j = +\infty$ *and such that* $h(t) = e^t$ *when* $t_j < t < t_j + 1$. *Then* $S = +\infty$ *almost surely.*

PROOF. Let $t'_j = t_j + 1/2$ and let τ_j to denote the first exit time from $\{t_j < t < t_j + 1\}$. Then, for $m = (x, y, t'_j) \in M$, $P_m(\tau_j \geq 1) = c$ where c is positive and independent of j, x, and y. The result then follows from the Borel-Cantelli Lemma (we may assume $t_j + 1 < t_{j+1}$).

With the above three lemmas, we may now derive Theorem A.

Proposition 5.4. *The construction performed in Section* 4 *can be achieved so that*

i) $S = +\infty$ *a.s.,* *and*

ii) $\lim_{s\to\infty} X_s(\omega) = +\infty$ *a.s.*

PROOF. It suffices to achieve the construction above (with a function h nondecreasing in x) in such a way that for a sequence of "boxes" $U_j = \{(x, y, t) : |x| < j,\ t_{4j+1} < t < t_{4j+2}\}$ ($\{t_j\}$ being a rapidly increasing sequence of reals, $t_0 > 0$) we have $P_m(|X_{\tau_j}| = j) \geq 1 - 2^{-j}$ when $m = (x, y, t)$ with $|x| < j$, $t = (t_{4j+1} + t_{4j+2})/2$, τ_j denoting the exit time from U_j, and $h(t) = e^t$ for $t_{4j+3} \leq t \leq t_{4j+4}$. The t_j may be chosen by induction, using Lemma 5.2 , Proposition 3.3 (and 3.7) above. By Lemma 5.1, the Markov property and the first Borel-Cantelli Lemma, $\lim_{s\to S} X_s = +\infty$ a.s. We may choose the gaps $t_{4j+4} - t_{4j+3}$ as large as we wish, whence i) by Lemma 5.3.

B. Let us now indicate the changes that should be made in order to construct an example proving Theorem B. We first notice that we may adapt the above construction in such a way that the Brownian motion converges a.s. to the end point (for $s \to +\infty$) on $S_\infty(M)$ of the geodesic $s \to (0, 0, s)$; however, the metric cannot be chosen among γ metrics.

First step. We construct a Riemannian manifold $\widetilde{M}$ in the following way: start with a $\gamma(-1/4)$ metric related to $g(t) = e^t$ and a function $h(x, t)$ such that

i) h is increasing with x for $x \leq 0$ and decreasing in x when $x \geq 0$,

ii) there is an increasing sequence $\{t_j\}_{j\geq 0}$ of positive numbers, with say $t_{j+1} > t_j + 10$, such that

$$ds^2 = dt^2 + e^{2t}\,dx^2 + e^{2t}\,dy^2$$

(the standard hyperbolic metric) for $t < t_0$ or $t_{4j+3} - 1 \leq t \leq t_{4j+4} + 1$.

$\widetilde{M}$ is $\mathbb{R}^3$ equipped with the metric d obtained from the metric $\gamma(g,h)$ by rotating the regions $\{t_{8j} \leq t \leq t_{8j+4}\}$ by $\pi/2$ around the $t-$axis, while the others regions $\{t < t_0\}$, $\{t_{8j+4} < t < t_{8j+8}\}$ are kept fixed. It is clear that $\widetilde{M}$ is a (smooth) Riemannian manifold with sectional curvatures $\leq -1/4$ and for which the description of the sphere at infinity in Paragraph 1.2 is still valid. Also, the Brownian motion $\{B_s\}_{s\geq 0}$ on $\widetilde{M}$ satisfies the following properties (X_s, Y_s, T_s denote the coordinates of B_s, S is the lifetime of B_s).

Lemma 5.5. *Almost surely,* $\lim_{s\to S} T_s(\omega) = +\infty$ *and both limits* $\lim_{s\to S} Y_s(\omega)$, $\lim_{s\to S} X_s(\omega)$ *exist and are finite.*

PROOF. The first point follows of course exactly as in Lemma 5.1. To prove the second claim, we first note that in N, the (x,t) plane equipped with the hyperbolic metric $ds^2 = dt^2 + e^{2t}\,dx^2$, the harmonic measure of $|x| = \pi/(2a)$ in the region $|x| < \pi/(2a)$ is explicitly given by

$$u_a(x,t) = 1 - \frac{2}{\pi}\operatorname{Arctg}\Big(\frac{\cos(ax)}{\sinh(ae^{-t})}\Big)\,.$$

Let $u_a(x,t) = 1$ when $|x| \geq \pi/(2a)$. u_a is convex with respect to to x on $[-\pi/(2a), \pi/(2a)]$, and is decreasing with respect to to t. It follows in particular that the function $f_a(x,t) = u(x/2, t/2)$ is superharmonic with respect to to

$$L = \frac{\partial^2}{\partial t^2} + e^{-t}\,\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\,.$$

Recall that

$$\Delta_N = \frac{\partial^2}{\partial t^2} + e^{-2t}\,\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\,.$$

From the construction of the metric of $\widetilde{M}$ and similar observations as in the proof of Lemma 5.1 (using in particular the monotonicity properties of h and u_a with respect to to x), it is then easily checked that the functions $(x,y,t) \mapsto f_a(x,t)$ are superharmonic on the region $\{t > 0\}$ of $\widetilde{M}$, and similarly for $(x,y,t) \mapsto f_a(y,t)$. The second assertion of the lemma follows from this and the supermartingale argument.

Now, by Lemma 5.2 and the extension lemmas of Section 3, for each given t_{4j} we may choose t_{4j+1}, t_{4j+2}, t_{4j+3} and the function $h(x,t)$ on $\{t_{4j} < t \leq t_{4j+3}\}$ in such a way that

a) $h = e^t$ if $t_{4j} \leq t \leq t_{4j} + 1$, or if $t_{4j+3} - 1 \leq t \leq t_{4j+3}$,

b) $h(x,t)$ is an even function of x which is decreasing on $\mathbb{R}_+$,

c) on $U_j = \{(t,x) :\ t_{4j+1} < t < t_{4j+2},\ x < -2a_0e^{-t_{4j}}\}$, h is as in Lemma 4.2 (with $J = (-\infty, -a_0e^{-t_{4j}}])$,

d) if we start B_s (the Brownian motion on $(M, \gamma(g,h))$) from (t'_0, x_0, y_0) , $t'_0 = (t_{4j+1}+t_{4j+2})/2$, $-j \leq x_0 \leq -2a_0e^{-t_{4j}}$, the probability to hit $\{x = -j\}$ or $\{x = -a_0\, e^{-t_{4j}}\}$ before $\{t = t_{4j+1}\}$ or $\{t = t_{4j+2}\}$ is larger than $1 - 2^{-j}$ (a_0 was defined in Section 3).

Thus, we may construct h with the above properties. By Lemma 5.5, it follows that, for the Brownian motion B_s on the corresponding Riemannian manifold $\widetilde{M}$, $\lim_{s\to S} X_s = \lim_{s\to S} Y_s = 0$ a.s. Since we may again choose the gaps $t_{4j+4} - t_{4j+3}$ very large, we may also realise $S = +\infty$ a.s.

Second step. We now consider $M = \{(x,y,t) :\ x,y,t \in \mathbb{R}\}$ equipped with a metric for which there is a sequence of regions $V_j = \{\rho_j \leq t \leq \rho_{j+1}\}$ $(j \geq 1)$ with $\{\rho_j\}$ rapidly increasing,

$$ds^2 = dt^2 + e^{2t}\, dx^2 + e^{2t}\, dy^2 \qquad \text{on } V_{2j}\ ,$$

the metric on V_{2j+1} being obtained by some translation $x \mapsto x+a_j$, $y \mapsto y + b_j$ from a metric of the type considered in the first step. Again, M has sectional curvatures $\leq -1/4$ and the description of $S_\infty(M)$ in Paragraph 1.2 holds. Choose and fix a dense sequence (a_j, b_j) in $\mathbb{R}^2$. By the first step, it is clear (and easy to prove) that one may successively choose the strips and the metric on these so that the Brownian motion $\{B_s\}$ on M starting from $m_0 = (0,0,0)$ hits the set $\{|x_j - a_j| + |y_j - b_j| \leq 4e^{-\rho_j},\ t = \rho_{j+1}\}$ with a probability $\geq 1 - 2^{-j}$; it is also clear that the lifetime of B_s is $+\infty$ a.s. The desired construction is then obtained and Theorem B is proved.

C. FINAL REMARKS.

1) We first sketch a more accurate variant of the construction. Let $\{a_j\}_{j\geq 0}$ be a sequence of real numbers and let M to denote the manifold $M = \{(x,y,t) : \ x,y,t \in \mathbb{R}\}$ equipped with a metric of the following type, for some rapidly increasing sequence $\{\theta_j\}_{j\geq 0}$ of positive numbers

$$ds^2 = dt^2 + e^{2t}\,dx^2 + e^{2t}\,dy^2\,,$$

when $t \leq \theta_0$,

$$ds^2 = dt^2 + e^{2t}\,dx^2 + h_{2j}(x - a_{2j}, t)^2\,dy^2\,,$$

when $\theta_{2j} \leq t \leq \theta_{2j+1}$, and

$$ds^2 = dt^2 + h_{2j+1}(y - a_{2j+1}, t)^2\,dx^2 + e^{2t}\,dy^2\,,$$

when $\theta_{2j+1} \leq t \leq \theta_{2j+2}$. Here $h_j(x,t)$ is a smooth positive and even function of type $\mathcal{H}(1/4)$ on $\mathbb{R}^2$ such that $h_j(x,t) = e^t$ when $t \leq \theta_j + \ell_j$ or $t \geq \theta_{j+1} - \ell_j$, ℓ_j being (much) smaller than $\theta_{j+1} - \theta_j$. We also require that $D^2 h_j$ is bounded on $\{t \leq \theta_{j+1}\}$.

Then, M is complete, its sectional curvatures are $\leq -1/4$, and again the Brownian motion $B_s = (X_s, Y_s, T_s)$ on M is such that $S = +\infty$ a.s., $\lim_{s\to\infty} T_s = \infty$ a.s. (S being the lifetime of Brownian motion). Moreover, for each given sequence $\{\varepsilon_j\}_{j\geq 0}$ of positive reals, we may (using a variation of the methods above) choose by induction the $(\ell_j, \theta_{j+1}, h_j)$ so that for each $m = (x, y, \theta_j + \ell_j)$, with $|x| + |y| \leq \varepsilon_j^{-1}$, and if τ_j denotes the first hitting time of B_s with $\{t = \theta_j\}$ or $\{t = \theta_{j+1} + \ell_{j+1}\}$, we have $P_m\{T_{\tau_j} = \theta_j\} \leq \varepsilon_j$,

$$P_m\{\,|X_{\tau_{2j}} - a_{2j}| + \sup_{s\leq \tau_{2j}} (d(X_s, [X_0, a_{2j}]) + |Y_s - Y_0|) \geq \varepsilon_j\} \leq 2^{-j}\,,$$

$$\begin{aligned} P_m\{&|Y_{\tau_{2j+1}} - a_{2j+1}| \\ &+ \sup_{s\leq \tau_{2j+1}} (d(Y_s, [Y_0, a_{2j+1}]) + |X_s - X_0|) \geq \varepsilon_j\} \leq 2^{-j}\,, \end{aligned}$$

and also

$$P_{m_0}\{|X_{\tau_j'}| + |Y_{\tau_j'}| \geq \varepsilon_{j+1}^{-1}\} \leq 2^{-j}\,,$$

where $m_0 = (0,0,0)$ and τ_j' is the first hitting time with $\{t = \theta_{j+1} + \ell_{j+1}\}$. Choosing the ε_j sufficiently small, it is then seen that the set of cluster values (for $s \to +\infty$) of $b_s = (X_s, Y_s)$ is a.s. equal to the set of cluster values C_Γ of the polygonal ray $\Gamma = \bigcup_{j\geq 1}[A_j, A_{j+1}]$, where $A_{2j} = (a_{2j}, a_{2j-1})$, $A_{2j+1} = (a_{2j}, a_{2j+1})$. Since any continuum in $\mathbb{R}^2$ may be realized as a set C_Γ, this explains the remark after the statement of Theorem B.

2) Fix an integer $m \geq 1$, let M be as in Section 5.A and let $\widetilde{M} = \{(x, y, z_1, \ldots, z_m, t) : \; x, y, t, \; z_j \in \mathbb{R}\}$ with metric

$$ds^2 = dt^2 + g(t)^2\, dx^2 + h(x,t)^2\, dy^2 + e^{2t} \sum_j dz_j^2 .$$

Then, simple direct computations show that $\widetilde{M}$ is a C.H. manifold with sectional curvatures $\leq -1/4$. Since

$$\begin{aligned} \Delta_{\widetilde{M}} = \frac{\partial^2}{\partial t^2} + \frac{1}{g^2}\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2}{\partial y^2} + \frac{1}{g^2}\sum_{j=1}^{m}\frac{\partial^2}{\partial {z_j}^2} \\ + \left(m + 1 + \frac{h'_t}{h}\right)\frac{\partial}{\partial t} + \frac{h'_x}{hg^2}\frac{\partial}{\partial x}, \end{aligned}$$

it is not difficult, using the argument in Section 5.A, to choose h such that Brownian motion on $\widetilde{M}$ has infinite lifetime and satisfies $\lim_{s\to\infty} X_s = +\infty$ ($X_s = x-$component of Brownian motion); thus Theorem A extends to all dimensions ≥ 3. It is also clearly seen how one may adapt the constructions above in Section 5.B and extend similarly Theorem B.

6. Appendix.

The following statement is essentially in [Cho]. That the smoothness assumptions in [Cho] may be dropped is already observed in [And] Theorem 1.4.

Proposition 6.1. *Let M be a complete simply connected Riemannian manifold whose sectional curvatures are ≤ -1, and let C be a closed convex subset of M, $C \neq \varnothing$. Set $u(m) = 1 - \tanh(d(m, C)/2)$, $m \in M$. Then, u is a superharmonic function on M.*

PROOF. If C is smooth, Theorem 4.3 in [Cho] says that u is superharmonic on $M \setminus C$. To settle the general case, we argue as follows. Let $m_0 \in M \setminus C$, let $m_1 = p_C(m_0)$ be the unique point in C with $d(m_0, m_1) = d(m_0, C)$, and let $C_0 = B(m_1, 1) \cap C$. By Lemma 6.2 below, C_0 is the limit of a decreasing sequence of smooth compact convex sets C_n. On the other hand, the projection map p_C is continuous, so that for m in some neighborhood V of m_0, $d(m, C) = d(m, C_0) = \lim_{n\to\infty} d(m, C_n)$. Thus, $u = \sup_{n\geq 1} u_n$ on V, with $u_n = 1 - \tanh(d(m, C_n)/2)$, and each u_n is superharmonic on V. Hence, u is continuous on M, superharmonic and ≤ 1 on $M \setminus C$, and equal to 1 on C. It is then clear that u is superharmonic on M.

Lemma 6.2. *Let M be a Cartan-Hadamard manifold and let K be a compact convex set in M. Then K is the intersection of a decreasing sequence $\{K_n\}_{n\geq 1}$ of smooth compact convex subsets of M.*

PROOF. Note that $F : m \mapsto d(m, K)$ is convex ([BO]) and that there is a smooth bounded function h on $U = \{m : d(m, K) < 2\}$ such that $\mathrm{Hess}_m(h) > cI$ for $m \in U$ and some $c > 0$ (*e.g.* $h(m) = |d(m, m_0)|^2$ with $m_0 \in M$). Approximating $F + \varepsilon h$ by smooth functions, it is seen that $F = \lim_{n\to\infty} F_n$ uniformly on $U' = \{m : d(m, K) < 1\}$, F_n being smooth and convex on $\bar{U}'$. For given $\varepsilon \in (0, 1)$ and large n, $K(n, \varepsilon) = \{F_n \leq \varepsilon + \max_K F_n\}$ is a compact neighborhood of K contained in $\{F < 2\varepsilon\}$, and $K(n, \varepsilon)$ is convex and smooth.

It follows from Proposition 6.1 (and the method of barriers) that property (C'_M) in the introduction (for a complete, simply connected, and negatively curved Riemannian manifold M) implies (P_M) ([Cho], [And]). From the probabilistic point of view, we have also the following simple corollary (by the usual supermatingale argument).

Corollary 6.3. *Let M, C and u be as in Proposition 6.1. Then the probability for the Brownian motion on M starting from $m_0 \in M$ to hit C is at most $u(m_0)$.*

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A characterization of 2-knots groups

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An n-knot group is the fundamental group of the complement of an n-sphere smoothly embedded in S^{n+2}.

Artin gave in 1925 ([A]) an algebraic characterization of 1-knot groups:

Theorem ([A]). *A group is a 1-knot group if and only if it has a presentation* $(x_1, \dots, x_n : x_j^{-1}\beta_j,\ \ 1 \leq j \leq n)$ *such that*

1) *For* $j = 1, \dots, n$ β_j *is conjugate to* $x_{\mu(j)}$ *in the free group* F *generated by* $x_1, \dots, x_n$,

2) $\prod_{j=1}^{n} \beta_j = \prod_{j=1}^{n} x_j$ *in* F, *and*

3) μ *is the permutation* $(1\ 2 \cdots n)$.

M. Kervaire gave in 1965 ([K]) an algebraic characterization of n-knot groups for $n \geq 3$.

Theorem ([K]). *Let* $n \geq 3$. *A group* G *is an* n*-knot group if and only if*

i) G *is finitely presented,*

ii) G *is normally generated by one element,*

iii) $H_1(G) = \mathbb{Z}$, *and*

iv) $H_2(G) = 0$.

We remark that if we drop the smoothness asumption in the definition of a knot, then their groups do not satisfy ii) in general. There are examples of wild 1-knots whose groups are free products $\mathbb{Z} * H$ that cannot be normally generated by one element (see [D] and [AF, example 2.1]). It is a conjecture, known to be true for $n = 1$, that (smooth) n-knot groups cannot be free products. Condition i) does not hold in general for groups of wild knots. In fact the group of a wild knot is a (smooth) 1-knot group if and only if it is finitely generated ([GHM]).

The problem of characterizing algebraically 2-knot groups has been posed several times (see for example [Su, Problem 4.7]). Ribbon 2-knot groups have been characterized algebraically by Yajima [Y].

We will give here a characterization of 2-knot groups in terms of presentations. It has the flavor of Artin's characterization of 1-knot groups. S. Kamada has independently, obtained another characterization of 2-knot groups ([Ka]).

A presentation $\mathfrak{G}$ of non positive deficiency $-h$ is *saddled* if it is of the form

$$(*) \quad \mathfrak{G} = \left\{x_1, \ldots, x_n : \; x_{2i-1}^{-1} x_{2i}, \; x_j^{-1}\beta_j, \; 1 \le i \le h, \; 1 \le j \le n\right\},$$

where

1) For $j = 1, \ldots, n$ β_j is conjugate to $x_{\mu(j)}$ in the free group F generated by $x_1, \ldots, x_n$,

2) $\prod_{j=1}^{n} \beta_j = \prod_{j=1}^{n} x_j$ in F.

If, in addition, the permutations μ and $\nu = \prod_{i=1}^{h}(2i-1, 2i)$ generate a transitive group of permutations of $\{1, 2, \ldots, n\}$ then we call $\mathfrak{G}$ *connected.*

The *genus* of the connected saddled presentation $\mathfrak{G}$ is $1 - (|\mu| - h + |\mu\nu|)/2$ where $|\pi|$ is the number of cycles of the permutation π.

The saddled presentation (*) is *unlinked* if

$$(x_1, \ldots, x_n : x_j^{-1}\beta_j \;\; 1 \le j \le n)$$

and

$$(x_1, \ldots, x_n : x_j^{-1}\beta_j' \;\; 1 \le j \le n)$$

present free groups, where

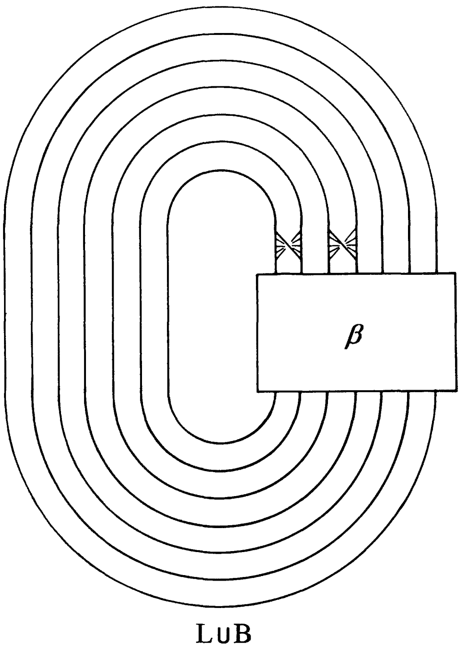
$$\beta'_j = \begin{cases} \beta_j\beta_{j+1}\beta_j^{-1}, & \text{if } j \text{ is odd and } j < 2h, \\ \beta_{j-1}, & \text{if } j \text{ is even and } j \le 2h, \\ \beta_j, & \text{if } j > 2h \end{cases}$$

(Notice these two presentations are saddled of deficiency 0. Also $\beta_j\beta_{j+1}\beta_j^{-1}$ can be replaced by $x_{j+1}\beta_{j+1}x_{j+1}^{-1}$).

Theorem 1. *A group is a 2-knot group if and only if it has a connected, unlinked, saddled presentation of genus* 0 (*c.u.s.p.* 0).

PROOF. A saddled presentation (*) determines, in a constructible way, a compact orientable 2-manifold S properly embedded in $S^3 \times [-1,1]$ as follows. First one can construct a (unique) braid β on n strings whose corresponding automorphism sends x_j to β_j. (See Birman's book [B, Corollary 1.8.3 and proof of theorem 1.9])

Let $L \subset S^3$ be the closed braid determined by β.



L∪B

Figure 1

Let B be the union of the bands joining the $(2i-1)$-string to the $2i$-string, $i = 1, \dots, h$ indicated in figure 1. Let $L' = \overline{L \cup \partial B - L \cap B}$, that is, L' is the closed braid determined by $\beta \prod_{i=1}^{h} \sigma_{2i-1}$.

Then $S = L \times [-1,0] \cup B \times \{0\} \cup L' \times [0,1] \subset S^3 \times [-1,1]$. The fundamental group of $S^3 \times [-1,1] - S$ is presented by $\mathfrak{G}$. S is connected if and only if $\mathfrak{G}$ is connected and if this is the case, the genus of S is the genus of $\mathfrak{G}$.

Every compact orientable 2-manifold properly and smoothly embedded in $S^3 \times [-1,1]$ with no elliptic points is isotopic to a surface determined by a saddled presentation. (The proof is similar to that of Alexander's Theorem ([B, Theorem 2.1]))

The group of $S^3 - L$ is presented by $(x_1, \dots, x_n : x_j^{-1}\beta_j \;\; j = 1, \dots, n)$ and the group of $S^3 - L'$ is presented by $(x_1, \dots, x_n : x_j^{-1}\beta'_j \;\; j = 1, \dots, n)$ so L and L' are trivial if and only if $\mathfrak{G}$ is unlinked. If this is the case then L bounds D, a union of disjoint disks in S^3, and L' bounds D', a union of disjoint disks in S^3, so that, if in addition, $\mathfrak{G}$ is connected and of genus 0, then

$$\Sigma^2 = D \times \{-1\} \cup S \cup D' \times \{1\} \subset S^3 \times [-1,1] \subset \partial(B^4 \times [-1,1]) = S^4$$

is a 2-knot whose group is still presented by $\mathfrak{G}$.

Every smooth 2-knot in S^4 is isotopic to one constructed as above. Hence G is a 2-knot group if and only if it has a c.u.s.p. 0.

Figure 2 describes a deformation of a description of the spun trefoil by a link with two bands so that a c.u.s.p. 0 can be read in Figure 3. Group generators are numbers, $\overline{n}$ denotes the inverse of n and $x^y = y^{-1}xy$.

It is easy to decide if a given finite presentation $\mathfrak{G}$ is saddled and connected and, if so, to compute its genus. Since one can decide whether a given link is trivial ([H],[S, Satz 4.1]), one can decide whether $\mathfrak{G}$ is unlinked. Hence the set of c.u.s.p. 0's is a recursive subset of the set of finite presentations.

Thus the set of smooth 2-knots is recursively enumerable (Markov's Theorem (see [B, Theorem 2.3]) helps to do the enumeration a little less inefficient). It is then possible to construct, from a given presentation of an 2-knot group G a 2-knot with that group: recursively enumerate all finite presentations of G until one finds a c.u.s.p. 0 $\mathfrak{G}$ and then construct the 2-knot determined by $\mathfrak{G}$.

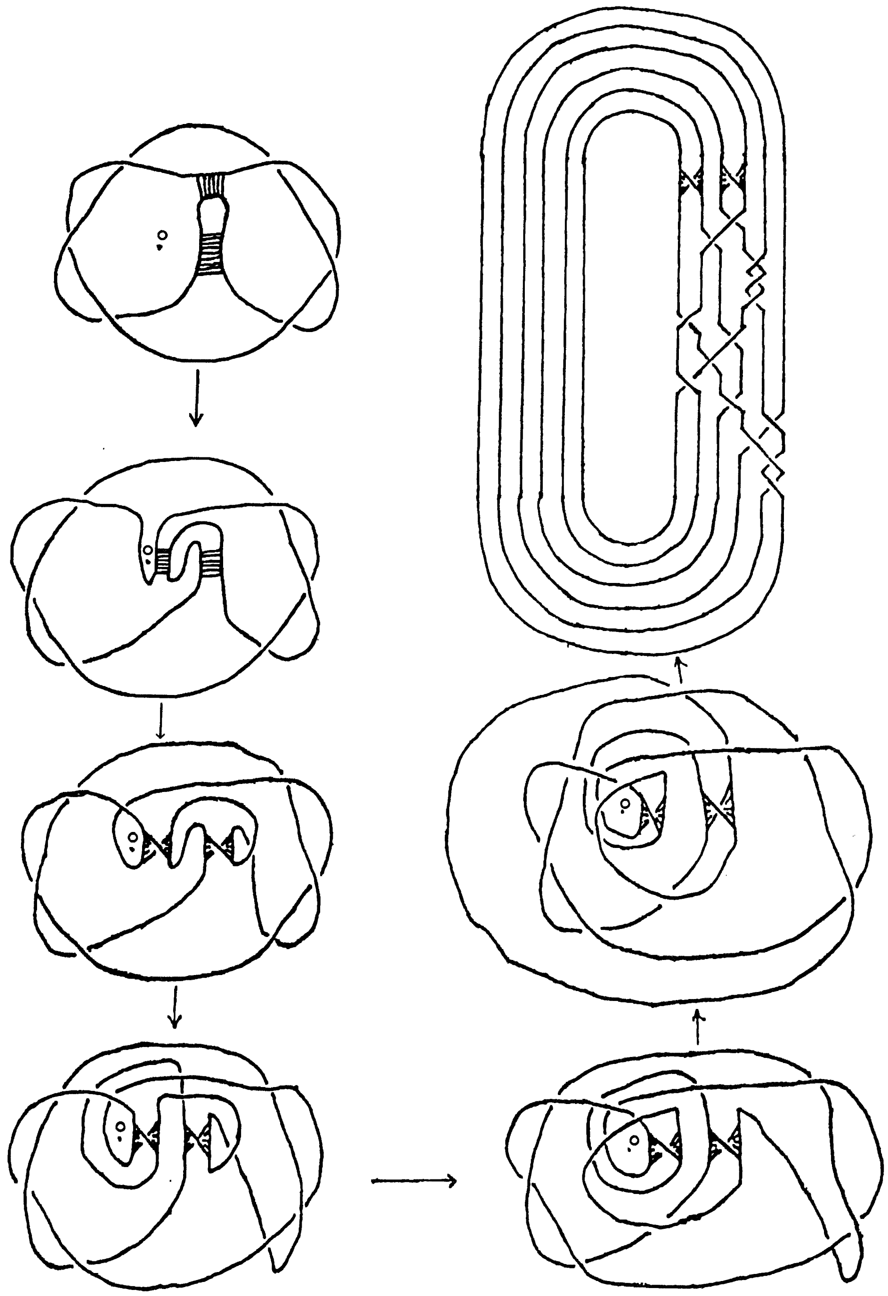


Figure 2

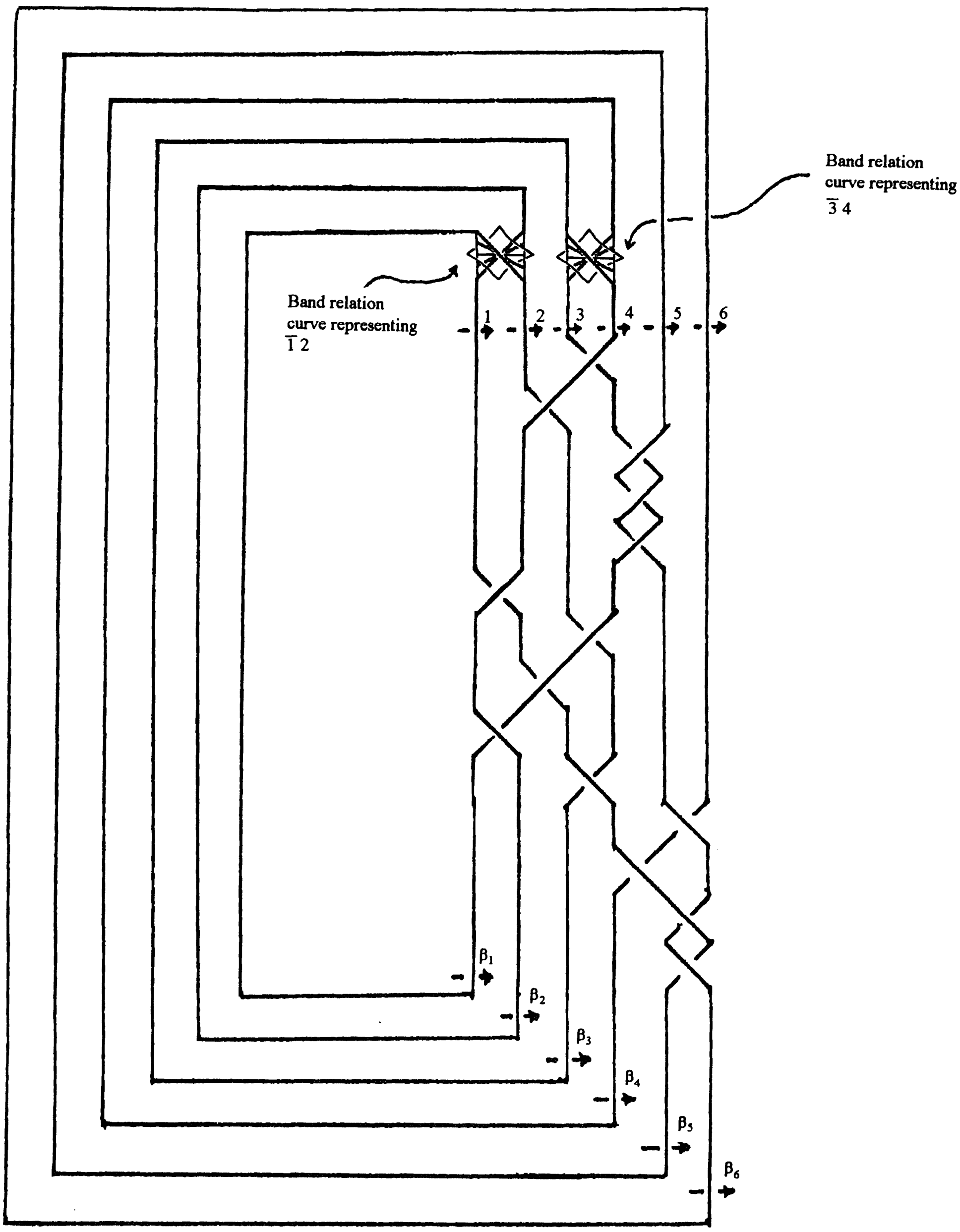

Band relation
curve representing
$\overline{3}\,4$
Band relation
curve representing
$\overline{1}\,2$
1
2
3
4
5
6
β_1
β_2
β_3
β_4
β_5
β_6

Figure 3

Theorem 1 can be generalized to treat the case of embeddings of a given compact orientable 2-manifold with empty boundary in S^4. Consider the saddled presentation (*) and the permutation

$$\nu = \prod_{i=1}^{h}(2i-1, 2i).$$

Denote by $T_1, \ldots, T_r$ the orbits of the elements of $\{1, \ldots, n\}$ under the action of the group generated by μ and ν. For $k = 1, \ldots, r$ let μ_k and ν_k be the restrictions of μ and ν to T_k and let h_k be the number of nontrivial cycles of ν_k. Write $g_k = 1-(|\mu_k|-h_k+|\mu_k\nu_k|)/2$. We call the unordered sequence $(g_1, \ldots, g_r)$ the *type* of the saddled presentation.

Theorem 2. *Let $M^2 = \coprod_{i=1}^{r} M_{g_i}$ be the 2-manifold with components $M_{g_1}, \ldots, M_{g_r}$ where M_g denotes a closed orientable surface of genus g. Then G is the group of the complement of a smooth submanifold of S^4 diffeomorphic to M^2 if and only if G has a saddled unlinked presentation of type $(g_1, \ldots, g_r)$.*

The proof is similar to that of Theorem 1.

Similar characterizations can be given to deal with groups of 2-manifolds M properly embedded in D^4. One would require the saddled presentation to be only "partially unlinked": $(x_1, \ldots, x_n : x_j^{-1}\beta_j \;\; 1 \leq j \leq n)$ should present a free group but $(x_1, \ldots, x_n : x_j^{-1}\beta_j' \;\; 1 \leq j \leq n)$ should present a free product $L * F_{|\mu\nu|-|\partial M|}$ the second factor being a free group on $|\mu\nu|-$(number of components of ∂M) generators.

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Weak-type estimates for the Riesz transforms associated with the gaussian measure

Eugene B. Fabes, Cristian E. Gutiérrez and Roberto Scotto

1. Introduction.

In this paper, we will study the behavior of the Riesz transforms associated with the Gaussian measure $\gamma(x)\,dx = e^{-|x|^2}\,dx$ in the space $L^1_\gamma(\mathbb{R}^n)$. These transformations are defined by

$$\mathcal{R}_j f(y) = \lim_{\varepsilon \to 0} \int_{|y-z|>\varepsilon} k_j(y,z)\, f(z)\, dz\,,$$

where

$$k_j(y,z) = \int_0^1 \Big(\frac{1-r^2}{-\log r}\Big)^{1/2} \frac{z_j - r\,y_j}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr\,,$$

$j = 1, \dots, n$.

The study of the boundedness properties of $\mathcal{R}_j$ in the spaces $L^p_\gamma(\mathbb{R}^n)$ began with the work of B. Muckenhoupt [Mu], when the dimension is $n = 1$. He proved the boundedness of this transformation (in this case it is only one operator) when $p > 1$ and the weak-type (1,1). In higher dimensions the L^p-boundedness, $p > 1$, was first proved by

P. A. Meyer [Me], by using probabilistic methods. The same result was also proved by several authors, [Gn], [Gt], [Pi], and [U]. The proof in [Gn] is probabilistic, the others are analytic. Also, all proofs except the one in [U] give strong-type constants bounded independently of the dimension n.

The purpose of this paper is to show that $\mathcal{R}_j$ are of weak-type (1,1) in any dimension. The proof uses analytic methods, and it is carried out by decomposing the kernel in several pieces and by studying each piece in appropriate regions. Some of the ideas we use here have been developed by P. Sjögren in [Sj].

We begin by explaining the notion of Riesz's transforms for the Gaussian measure. Let L be the differential operator defined by

$$L = \frac{1}{2}\Delta - x \cdot \text{grad},$$

and consider the set of eigenvalues λ of the problem

$$Lu = \lambda\, u,$$

with boundary conditions

$$u(x) = O(|x|^k), \qquad \text{for some } k \geq 0 \text{ as } |x| \to +\infty.$$

This set is discrete, the eigenvalues are of the form $-m$, m non-negative integer, and the corresponding eigenfunctions are the multidimensional Hermite polynomials $H_\alpha(x)$, defined below, $\alpha = (\alpha_1, \dots, \alpha_n)$, with $|\alpha| = m$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The one-dimensional Hermite polynomials are defined by

$$H_0(x) = 1, \qquad H_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad n \geq 1.$$

They have the following basic properties

$$\int_{-\infty}^{+\infty} H_n(x)^2 e^{-x^2}\, dx = 2^n\, n!\, \sqrt{\pi}, \qquad n = 0, 1, \dots,$$

$$\int_{-\infty}^{+\infty} H_0(x)\, e^{-x^2}\, dx = \int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi},$$

and

$$\int_{-\infty}^{+\infty} H_n(x)\, e^{-x^2}\, dx = 0, \qquad \text{for } n \geq 1.$$

Also

$$H'_{n+1}(x) = -2\,(n+1)\,H_n(x)\,,$$
$$H_{n+1}(x) + 2\,xH_n(x) + 2\,nH_{n-1}(x) = 0\,, \qquad n \geq 0\,,$$
$$H_{-1}(x) = 0\,,$$

and

$$H''_n(x) - 2\,xH'_n(x) + 2\,nH_n(x) = 0\,.$$

The multidimensional Hermite polynomials are defined by taking products of one-dimensional Hermite polynomials. Indeed, if $\alpha = (\alpha_1, \ldots, \alpha_n)$, with α_j non-negative integers, and $x = (x_1, \ldots, x_n)$, then we define

$$H_\alpha(x) = H_{\alpha_1}(x_1) \ldots H_{\alpha_n}(x_n)\,,$$

where $H_{\alpha_j}(x_j)$ are one-dimensional Hermite polynomials in the variable x_j.

The differential operator L is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup T_t defined by

$$T_t f(x) = \int_{\mathbb{R}^n} k(t,x,y)\, f(y)\, dy\,,$$

where

$$k(t,x,y) = \frac{1}{\pi^{n/2}(1-e^{-2t})^{n/2}} \exp\left(-\frac{|e^{-t}x - y|^2}{1-e^{-2t}}\right)\,,$$

$t > 0, x \in \mathbb{R}^n$. This means that if we set $u(x,t) = T_t f(x)$ then u is a solution of the equation

$$u_t = \frac{1}{2}\,\Delta_x u - x \cdot \mathrm{grad}_x u\,.$$

By using the properties of the Hermite polynomials mentioned above it is easy to see that

$$L\,H_\alpha(x) = -|\alpha|\,H_\alpha(x)\,,$$

and

$$T_t\,H_\alpha(x) = e^{-|\alpha| t}\,H_\alpha(x)\,.$$

The measure $\gamma(x)\,dx$ makes the operator L self-adjoint; therefore, it is the natural measure to study the boundedness properties of the operators associated with L.

In this frame the Riesz transforms are defined as follows. Given j, $1 \le j \le n$, and $H_\alpha(x)$ a multidimensional Hermite polynomial, the j-th Riesz transform of H_α is defined by

$$\mathcal{R}_j(H_\alpha)(x) = -\frac{1}{\sqrt{|\alpha|}}\,\frac{\partial}{\partial x_j}\,H_\alpha(x) = \frac{2\alpha_j}{\sqrt{|\alpha|}}\,H_{\alpha-e_j}(x)\,,$$

where e_j are the coordinate vectors in $\mathbb{R}^n$. By linearity the definition of $\mathcal{R}_j$ extends to any polynomial in $\mathbb{R}^n$.

We now show that this definition formally coincides, except for a multiplicative constant, with the one given at the beginning of the section. In fact, let H_α be a multidimensional Hermite polynomial, we have

$$\begin{aligned}\int k_j(y,z)\,H_\alpha(z)\,dz &= \int_0^1 \Big(\frac{1-r^2}{-\log r}\Big)^{1/2}\,\frac{1}{2r}\,\frac{1}{(1-r^2)^{(n+1)/2}} \\ &\qquad\cdot \frac{\partial}{\partial y_j}\Big(\int H_\alpha(z)\,e^{-|ry-z|^2/(1-r^2)}\,dz\Big)\,dr \\ &= \int_0^1 \frac{1}{2\,r\,(-\log r)^{1/2}}\,\frac{\partial}{\partial y_j}\,(T_{-\log r}\,H_\alpha(y))\,dr \\ &= \int_0^1 \frac{1}{2\,r\,(-\log r)^{1/2}}\,\frac{\partial}{\partial y_j}\,\Big(e^{|\alpha|\log r}\,H_\alpha(y)\Big)\,dr \\ &= \frac{\partial}{\partial y_j}\,H_\alpha(y)\int_0^1 \frac{1}{2\,r\,(-\log r)^{1/2}}\,e^{|\alpha|\log r}\,dr\,.\end{aligned}$$

By making the change of variables $r = e^{-t^2/|\alpha|}$, the last integral equals to $\sqrt{\pi}/(2\sqrt{|\alpha|})$, and the desired conclusion follows.

Instead of studying the operators $\mathcal{R}_j$ it is enough to consider the operator

$$K^*f(y) = \sup_{\varepsilon>0}\,\Big|\int_{|y-z|>\varepsilon} K(y,z)\,f(z)\,dz\Big|\,,$$

with kernel

$$K(y,z) = \int_0^1 \frac{z_j - r y_j}{(1-r^2)^{(n+3)/2}}\,e^{-|ry-z|^2/(1-r^2)}\,dr\,. \tag{1.2}$$

The kernels k_j and K have basically the same behavior. In fact, we shall show that the absolute value of its difference gives an integral operator of weak-type (1,1) with respect to $\gamma\,dx$ (see Remark at the end of Section 2). Also, by symmetry it is enough to study the case when $j = 1$.

Given $R > 0$ let

$$N_R = \big\{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n : |y| \le R \text{ and } |z| \le R, \\ \text{or } |z| \ge R/2 \text{ and } |y-z| \le R/|z|\big\},$$

and $N_R^y = \{z : (y,z) \in N_R\}$. We define the operators

$$K^* f(y) = \sup_{\varepsilon>0} \Big| \int_{|y-z|>\varepsilon} K(y,z)\, f(z)\, dz \Big|,$$

$$K_1^* f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz \Big|,$$

$$K_2^* f(y) = \int_{\mathbb{R}^n \setminus N_R^y} |K(y,z)|\, |f(z)|\, dz\,.$$

We clearly have

$$K^* f(y) \le K_1^* f(y) + K_2^* f(y)\,.$$

We shall show that K_i^*, i=1,2 are of weak-type (1,1) with respect to γ.

The organization of the paper is as follows. In Section 2 we prove estimates of K_1^* and that it is of weak-type (1,1). This done by showing that K_1^* can be pointwise controlled in terms of certain maximal and singular integral operators appropriatly truncated. In Section 3 estimates of K_2^* are shown as well as the weak-type (1,1). The proofs requiere precise estimations of the size of various integrals in different regions. In order to make the paper comprehensible we give most of the details.

2. The estimate of $\mathbf{K_1^*}$.

We begin by introducing the following operators.

Let $b \ge a > 0$, we define the maximal operator

$$M_{a,b} f(y) = \sup_{0<r<(a\wedge b/|y|)} \frac{1}{\gamma(B_r(y))} \int_{B_r(y)} |f(z)|\, \gamma(z)\, dz\,,$$

where $B_r(y)$ denotes the Euclidean ball with radius r and centered at y. Also, given a function $f \in L^1_\gamma(\mathbb{R}^n)$ and a Calderón-Zygmund convolution kernel k we define

$$(2.0) \qquad K_\gamma f(y) = \sup_{\varepsilon>0} \left| \int_{\varepsilon<|y-z|\leq(a\wedge b/|y|)} k(y-z)\, f(z)\, dz \right| .$$

By $\chi_E(y)$ we denote the characteristic function of the set E.

We have the following

Lemma 1. *The operator K_γ is of weak-type* (1,1) *with respect to the measure $\gamma\, dz$, i.e. there exists a constant $C = C(n,a,b)$ such that*

$$\int_{E_\lambda} \gamma(y)\, dy \leq \frac{C}{\lambda} \|f\|_{L^1_\gamma} ,$$

where $E_\lambda = \{y : K_\gamma f(y) > \lambda\}$, for every $\lambda > 0$.

PROOF. We first construct a countable family of balls $\mathcal{F}$ with bounded overlapping, whose union is $\mathbb{R}^n$ and on each ball $B \in \mathcal{F}$ all values of $\gamma(x)$ are equivalent. Given $\alpha \geq 1$ we define the following sequence

$$x_1 = \alpha , \qquad x_{k+1} = x_k + \frac{1}{x_k} , \qquad k \geq 1 .$$

The sequence $\{x_k\}$ is strictly increasing and $x_k \to +\infty$ as $k \to \infty$. Set $l_0 = x_1$ and $l_k = x_{k+1} - x_k$, $k \geq 1$, then $l_{k+1} < l_k < 2\, l_{k+1}$. Let

$$R_j = \{x \in \mathbb{R}^n : x_j \leq |x| < x_{j+1}\} , \qquad j \geq 1 ,$$

the width of R_j is l_j. Let $B_1^j, \ldots, B_N^j$ be a maximal disjoint family of balls contained in R_j and such that the diameter of B_k^j is l_j for all k, $1 \leq k \leq N$. If y_k^j is the center of B_k^j then we have $|y_k^j| = (x_{j+1} + x_j)/2$. It is easy to see that $\bigcup_{k=1}^N 2B_k^j \supset R_j$, where $2B$ denotes the ball with the same center as B but twice the radius. Let us define $\tilde{B}_k^j = 2B_k^j$.

The family $\mathcal{F}$ is the collection of all balls $\tilde{B}_k^j$ and the ball $B(0, x_1)$. It is obvious that the union is $\mathbb{R}^n$. We show that $\mathcal{F}$ has bounded overlaps. If $x \in \bigcap_{k=1}^l \tilde{B}_{i_k}^j$ then $l \leq 4^n$. This is because

$$B(x, 2\, l_j) \supset \bigcup_{k=1}^{l} \tilde{B}_{i_k}^j .$$

Let

$$\tilde{R}_j = \left\{ x \in \mathbb{R}^n : \ x_j - \frac{1}{2\,x_j} \le |x| < x_{j+1} + \frac{1}{2\,x_j} \right\}, \qquad j \ge 1 .$$

Then $\bigcup_{k=1}^{N} \tilde{B}_k^j \subset \tilde{R}_j$. We have that

$$\tilde{R}_k \cap \tilde{R}_j = \varnothing , \qquad \text{for} \ \ k > j+2 ,$$

which follows from the fact that

$$x_{j+1} + \frac{1}{2\,x_j} < x_k - \frac{1}{2\,x_k}$$

by the construction of x_j. It remains to show that on each $B \in \mathcal{F}$ all the values of γ are equivalent. In fact, for $B(0, x_1)$ this is obvious. If $B \in \mathcal{F}$ then $B = B(y_k, 1/x_j)$ for some j and $|y_k| = (x_{j+1} + x_j)/2$. Consequently $B \subset B(y_k, 2/|y_k|)$ and in the last ball all values of γ are equivalent.

Take $\alpha = b/a \ge 1$ in the construction above and define

$$Tf(y) = \sup_{\varepsilon' > \varepsilon > 0} \left| \int\limits_{\varepsilon < |y-z| < \varepsilon'} k(y-z)\, f(z)\, dz \right| .$$

We write

$$E_\lambda = \bigcup_{B \in \mathcal{F}} E_\lambda \cap B .$$

Suppose $B = B(0, \alpha)$, if $y \in E_\lambda \cap B(0, \alpha)$ then the integration in $K_\gamma f$ is over the set $|y - z| < a$ and consequently

$$K_\gamma f(y) \le T(\chi_{B_{a+b/a}(0)} f)(y) .$$

If $|y| > \alpha$ then the integration in K_γ is over $|y-z| < b/|y|$. If we assume that $B = B(y_k, 1/x_j)$ with $|y_k| = (x_{j+1} + x_j)/2$ and $y \in B(y_k, 1/x_j)$ then

$$|z - y_k| \le |z - y| + |y - y_k| \le \frac{b}{|y|} + \frac{1}{x_j} \le \frac{c(a,b)}{|y_k|} .$$

This follows because since

$$x_j > \alpha \quad \text{and} \quad \frac{1}{x_j} < \left(1 - \left(\frac{a}{b}\right)^2\right) |y_k|$$

we have

$$|y| \geq |y_k| - \frac{1}{x_j} \geq \left(1 - \left(\tfrac{a}{b}\right)^2\right) |y_k| .$$

Analogously, $(1-(a/b)^2)\,|y_k| < x_j$. Therefore

$$K_\gamma f(y) \leq T(\chi_{B(y_k, c/|y_k|)} f)(y) ,$$

for $y \in B(y_k, 1/x_j)$, and $|y| > \alpha$.

Consequently, by the weak-type (1,1) of T with respect to Lebesgue measure (see [St]), we have

$$\begin{aligned}
\gamma(E_\lambda \cap B(0,\alpha)) &\leq |E_\lambda \cap B(0,\alpha)| \\
&\leq |\{y :\ T(\chi_{B_{a+b/a}(0)} f)(y) > \lambda\}| \\
&\leq \frac{c}{\lambda} \int_{B_{a+b/a}(0)} |f(z)|\, dz \\
&\leq \frac{c_a}{\lambda} \int_{B_{a+b/a}(0)} |f(z)|\, \gamma(z)\, dz .
\end{aligned}$$

Also,

$$\begin{aligned}
&\gamma(E_\lambda \cap \{|y| > \alpha\} \cap B(y_k, 1/x_j)) \\
&\qquad \leq c\,\gamma(y_k)\, |E_\lambda \cap \{|y| > \alpha\} \cap B(y_k, 1/x_j)| \\
&\qquad \leq c\,\gamma(y_k)\, |\{y :\ T(\chi_{B(y_k, c/|y_k|)} f)(y) > \lambda\}| \\
&\qquad \leq c\,\gamma(y_k) \frac{1}{\lambda} \int_{B(y_k, c/|y_k|)} |f(z)|\, dz \\
&\qquad \leq \frac{c}{\lambda} \int_{B(y_k, c/|y_k|)} |f(z)|\, \gamma(z)\, dz .
\end{aligned}$$

By adding up and using the fact that the family of balls $B(0,\alpha)$, $B(y_k, c/|y_k|)$ has bounded overlaps the lemma follows.

REMARK. Since $M_{a,b}$ is pointwise dominated by the Hardy-Littlewood maximal function defined with the measure $\gamma(x)\,dx$, it follows from the Besicovitch covering lemma that $M_{a,b}$ is of weak-type (1,1) with respect to that measure.

We define the following operators

$$T_1 f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq 2R/|y|}} k(y-z)\, f(z)\, dz \Big| \chi_{B_R^c(0)}(y),$$

$$T_2 f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k(y-z)\, f(z)\, dz \Big| \chi_{B_R(0)}(y),$$

$$T_3 f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{|z|\geq R \\ \varepsilon<|y-z|\leq 1}} k(y-z)\, f(z)\, dz \Big| \chi_{B_R(0)}(y).$$

These operators are of the form (2.0).

We have the following

Theorem 1. *Let $n \geq 2$, $R \geq 4$ and $k(z) = z_1/|z|^{n+1}$. There exist a constant $C = C(n,R)$ and kernels $k_1(y,z), k_2(y,z)$ satisfying*

$$|k_1(y,z)| \leq C \frac{|y|^{1/2}}{|y-z|^{n-1/2}}, \qquad \text{for } |y| > R \text{ and } |y-z| \leq 2R/|y|,$$

$$|k_2(y,z)| \leq C \frac{1}{|y-z|^{n-1}}, \qquad \text{for } |y| \leq R \text{ and } |y-z| \leq 2R,$$

and such that

$$K_1^* f(y) \leq C \Big(\sum_{i=1}^{6} T_i f(y) + M_{2,2R} f(y) + M_{1,R} f(y) \Big),$$

where T_i, $i = 4,5,6$ are defined by

$$T_4 f(y) = \Big(\int_{\substack{|z|\geq R/2 \\ |y-z|\leq R/|z|}} |k_1(y,z)|\, |f(z)|\, dz \Big) \chi_{B_R^c(0)}(y),$$

$$T_5 f(y) = \Big(\int_{\substack{|z|\leq R \\ |y-z|\leq 2R}} |k_2(y,z)|\, |f(z)|\, dz \Big) \chi_{B_R(0)}(y),$$

and

$$T_6 f(y) = \Big(\int\limits_{\substack{|z|>R \\ |y-z|\le R/|z|}} |k_2(y,z)|\,|f(z)|\,dz \Big) \chi_{B_R(0)}(y)\,.$$

Corollary. *Let $n \geq 2$ and $R \geq 4$. The operator K_1^* is of weak-type (1,1) with respect to the measure $\gamma\,dz$.*

PROOF OF THEOREM 1. Assume $|y| > R$ and $\varepsilon < |y-z| \leq \dfrac{2R}{|y|}$. We write

$$\begin{aligned} K(y,z) &= (z_1 - y_1)\int_0^1 \frac{1}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\,dr \\ &\quad + y_1 \int_0^1 (1-r)\frac{1}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\,dr \qquad (2.1)\\ &= K_1(y,z) + K_2(y,z)\,. \end{aligned}$$

Note that

$$\frac{|y-z|}{|y|} \leq \frac{2R}{|y|^2} < \frac{2}{R} < 1\,, \qquad \text{for } R > 2\,.$$

Then we have

$$\begin{aligned} |K_2(y,z)| &\leq |y_1| \int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-|ry-z|^2/(2(1-r))}\,dr \\ &= |y_1| \int_0^1 \frac{1}{r^{(n+1)/2}}\, e^{-|(1-r)y-z|^2/(2r)}\,dr \\ &\leq |y| \Bigg(\int_0^{|y-z|/|y|} + \int_{|y-z|/|y|}^1 \frac{1}{r^{(n+1)/2}}\, e^{-|y-z|^2/(2r)} \\ &\qquad\qquad \cdot e^{-r|y|^2/2}\,dr \Bigg) e^{y\cdot(y-z)} \\ &\leq e^{2R}|y| \Bigg(\int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|y-z|^2/(2r)}\,dr \\ &\qquad + \int_{|y-z|/|y|}^{\infty} \frac{1}{r^{(n+1)/2}}\,dr \Bigg) \\ &\leq e^{2R}|y| \Bigg(\frac{2^{(n+1)/2}}{|y-z|^{n-1}} \int_{(|y-z|\,|y|/2)^{1/2}}^{\infty} u^{n-2}\, e^{-u^2}\,du \end{aligned}$$

$$+\frac{2}{n-1}\Big(\frac{|y|}{|y-z|}\Big)^{(n-1)/2}\Big)$$

$$\le C_n(R)\left(\frac{|y|}{|y-z|^{n-1}}+\frac{|y|^{(n+1)/2}}{|y-z|^{(n-1)/2}}\right).$$

We also write

$$\begin{aligned}K_1(y,z)&=(z_1-y_1)\int_0^{1-|y-z|/|y|}\frac{1}{(1-r^2)^{(n+3)/2}}\,e^{-|ry-z|^2/(1-r^2)}\,dr\\&\quad+(z_1-y_1)\int_{1-|y-z|/|y|}^1\frac{1}{(1-r^2)^{(n+3)/2}}\,e^{-|ry-z|^2/(1-r^2)}\,dr\\&=K_3(y,z)+K_4(y,z).\end{aligned}$$

As in the estimate of K_2 we get

$$\begin{aligned}|K_3(y,z)|&\le e^{2R}\,|z_1-y_1|\int_{|y-z|/|y|}^1\frac{1}{r^{(n+3)/2}}\,e^{-|y-z|^2/(2r)}\,dr\\&\le C_n(R)\,\frac{|z_1-y_1|}{|y-z|^{n+1}}\int_{|y-z|/\sqrt{2}}^{(|y|\,|y-z|/2)^{1/2}}u^ne^{-u^2}\,du\\&\le C_n(R)\,\frac{|y|^{1/2}|y-z|^{1/2}}{|y-z|^n}\\&=C_n(R)\,\frac{|y|^{1/2}}{|y-z|^{n-1/2}}\,.\end{aligned}$$

Let

$$\psi(t)=\frac{1}{(2-t)^{(n+3)/2}}\,e^{-|(1-r)y-z|^2/((2-t)r)}\,,\qquad 0\le t<2\,. \tag{2.2}$$

Since $|y-z|/|y|<1$, for $R>2$ we write

$$\begin{aligned}K_4(y,z)&=(z_1-y_1)\int_0^{|y-z|/|y|}\frac{1}{r^{(n+3)/2}}\,\psi(0)\,dr\\&\quad+(z_1-y_1)\int_0^{|y-z|/|y|}\frac{1}{r^{(n+3)/2}}\,(\psi(r)-\psi(0))\,dr\\&=K_5(y,z)+K_6(y,z)\,.\end{aligned}$$

Observe that for $0 \leq t \leq r \leq 1$

$$|\psi'(t)| \leq c_n \left(1 + \frac{|(1-r)y - z|^2}{r}\right) e^{-|(1-r)y-z|^2/((2-t)r)}$$
$$\leq c_n \, e^{-|(1-r)y-z|^2/(4r)} .$$

This follows since there exists $c > 0$ such that

$$(1+s)\, e^{-s/(2-t)} \leq c\, e^{-s/4}, \qquad \text{for } \ 0 \leq t \leq 1 .$$

Hence

$$|K_6(y,z)| \leq c_n \, |z_1 - y_1| \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}} \, e^{-|(1-r)y-z|^2/(4r)} \, dr$$
$$\leq C_n(R) \int_0^1 \frac{1}{r^{(n+1)/2}} \, e^{-|y-z|^2/(4r)} \, dr$$
$$\leq C_n(R) \frac{1}{|y-z|^{n-1}} .$$

To estimate K_5 we write

$$K_5(y,z) = c_n \, (z_1 - y_1) \, e^{y\cdot(y-z)} \int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}} \, e^{-r|y|^2/2} \cdot \frac{1}{r^2} \, e^{-|y-z|^2/(2r)} \, dr$$
$$= 2\, c_n \frac{z_1 - y_1}{|y-z|^2} \, e^{y\cdot(y-z)} \int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}} \, e^{-r|y|^2/2} \cdot \frac{d}{dr}\left(e^{-|y-z|^2/(2r)}\right) dr .$$

By integrating by parts we get

$$K_5(y,z) = 2\, c_n \frac{z_1 - y_1}{|y-z|^2} \, e^{y\cdot(y-z)} \, e^{-|y|\,|y-z|} \left(\frac{|y|}{|y-z|}\right)^{(n-1)/2}$$
$$+ c_n \frac{z_1 - y_1}{|y-z|^2} \, |y|^2 \, e^{y\cdot(y-z)} \cdot \int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}} \, e^{-|y-z|^2/(2r)} \, e^{-r|y|^2/2} \, dr$$

$$+ (n-1)\, c_n \frac{z_1 - y_1}{|y-z|^2} \cdot \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|(1-r)y-z|^2/(2r)}\, dr$$
$$= K_7(y,z) + K_8(y,z) + K_9(y,z)\,.$$

We have

$$|K_7(y,z)| \le c_n \frac{|y|^{(n-1)/2}}{|y-z|^{(n+1)/2}}\,,$$

and

$$|K_8(y,z)| \le c_n\, e^{2R} \frac{|z_1 - y_1|}{|y-z|^2}\, |y|^2 \int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}}\, e^{-|y-z|^2/(2r)}\, dr$$
$$\le c_n \frac{|y|^{1/2}}{|y-z|^{n-1/2}}\,.$$

In the last estimate we have considered the cases $n = 2$ and $n > 2$ separately and used the fact that

$$\int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-4}\, e^{-u^2}\, du \le \frac{c}{(|y|\,|y-z|)^{1/2}}\,,$$

for $n > 2$.

To estimate K_9 we define

$$\phi(t) = e^{-|(1-t)y-z|^2/(2r)}\,, \qquad 0 \le t < 1\,, \tag{2.3}$$

and write

$$K_9(y,z) = (n-1)\, c_n \frac{z_1 - y_1}{|y-z|^2} \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, \phi(0)\, dr$$
$$+ (n-1)\, c_n \frac{z_1 - y_1}{|y-z|^2} \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, (\phi(r) - \phi(0))\, dr$$
$$= K_{10}(y,z) + K_{11}(y,z)\,.$$

Observe that

$$\phi'(t) = -\frac{1}{r}\, ((1-t)y - z) \cdot y\, \phi(t)\,,$$

and consequently for $0 < t \le r < 1$ and $|y-z| \le 2R/|y|$ we have

$$|\phi'(t)| \le |y| \frac{|(1-t)y - z|}{r} \phi(t) \le C(R) \frac{|y|}{\sqrt{r}} e^{-|y-z|^2/(4r)} .$$

Hence

$$\begin{aligned} |K_{11}(y,z)| &\le C_n(R) \frac{|y|}{|y-z|} \int_0^{|y-z|/|y|} \frac{1}{r^{n/2}} e^{-|y-z|^2/(4r)}\, dr \\ &\le C_n(R) \frac{|y|}{|y-z|^{n-1}} \int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-3} e^{-u^2}\, du \\ &\le C_n(R) \frac{|y|^{1/2}}{|y-z|^{n-1/2}} . \end{aligned}$$

Now we write

$$\begin{aligned} K_{10}(y,z) &= c'_n \frac{z_1 - y_1}{|y-z|^{n+1}} \int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-2} e^{-u^2}\, du \\ &= c''_n \frac{z_1 - y_1}{|y-z|^{n+1}} \\ &\quad - c'_n \frac{z_1 - y_1}{|y-z|^{n+1}} \int_0^{(|y-z|\,|y|/2)^{1/2}} u^{n-2} e^{-u^2}\, du \\ &= c'_n \frac{z_1 - y_1}{|y-z|^{n+1}} + K_{12}(y,z) , \end{aligned}$$

and we obtain

$$|K_{12}(y,z)| \le c_n \frac{1}{|y-z|^n} (|y-z|\,|y|)^{1/2} = c_n \frac{|y|^{1/2}}{|y-z|^{n-1/2}} .$$

It is easy to see that in the region $|y| > R$ and $|y-z| \le \frac{2R}{|y|}$ the kernels

$$\frac{|y|^{(n-1)/2}}{|y-z|^{(n+1)/2}} , \quad \frac{1}{|y-z|^{n-1}} , \quad \frac{|y|}{|y-z|^{n-1}} , \quad \frac{|y|^{(n+1)/2}}{|y-z|^{(n-1)/2}} ,$$

are all dominated by

$$\frac{|y|^{1/2}}{|y-z|^{n-1/2}} .$$

Consequently, in case $|y| > R$ and $\varepsilon < |y-z| \le 2R/|y|$ we obtain that

$$K(y,z) = c_n\, k(y-z) + k_1(y,z) \tag{2.4}$$

where

$$k_1 = K_1 + \cdots + K_{12}\ ,$$

and

$$|k_1(y,z)| \le C_n(R) \frac{|y|^{1/2}}{|y-z|^{n-1/2}}\ .$$

We now assume $|y| \le R$, and $\varepsilon < |y-z| \le 2R$. By (2.1)

$$K(y,z) = K_1(y,z) + K_2(y,z)\ ,$$

and

$$\begin{aligned} |K_2(y,z)| &\le R \int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-|ry-z|^2/(2(1-r))}\, dr \\ &\le R\, e^{2R^2} \int_0^1 \frac{1}{r^{(n+1)/2}}\, e^{-|y-z|^2/(2r)}\, dr \\ &\le C_n(R) \frac{1}{|y-z|^{n-1}}\ . \end{aligned}$$

We write

$$\begin{aligned} K_1(y,z) &= (z_1 - y_1) \int_0^1 \frac{1}{r^{(n+3)/2}}\, \psi(0)\, dr \\ &\quad + (z_1 - y_1) \int_0^1 \frac{1}{r^{(n+3)/2}}\, (\psi(r) - \psi(0))\, dr \\ &= \overline{K}_3(y,z) + \overline{K}_4(y,z)\ , \end{aligned}$$

where $\psi(t)$ is defined by (2.2).

As in the estimate of $K_6(y,z)$ we obtain

$$|\overline{K}_4(y,z)| \le C_n(R) \frac{1}{|y-z|^{n-1}}\ .$$

If ϕ is defined by (2.3) then

$$\begin{aligned}\overline{K}_3(y,z) &= c_n\,(z_1-y_1)\int_0^1 \frac{1}{r^{(n+3)/2}}\,\phi(0)\,dr \\ &\quad + c_n\,(z_1-y_1)\int_0^1 \frac{1}{r^{(n+3)/2}}\,(\phi(r)-\phi(0))\,dr \\ &= c_n\,\frac{z_1-y_1}{|y-z|^{n+1}}\int_{|y-z|/\sqrt{2}}^{+\infty} u^n\,e^{-u^2}\,du + \overline{K}_6(y,z) \\ &= \tilde{c}_n\,\frac{z_1-y_1}{|y-z|^{n+1}} \\ &\quad - c_n\,\frac{z_1-y_1}{|y-z|^{n+1}}\int_0^{|y-z|/\sqrt{2}} u^n\,e^{-u^2}\,du + \overline{K}_6(y,z) \\ &= \tilde{c}_n\,\frac{z_1-y_1}{|y-z|^{n+1}} + \overline{K}_5(y,z) + \overline{K}_6(y,z)\,.\end{aligned}$$

We have

$$|\overline{K}_5(y,z)| \le \frac{C_n}{|y-z|^{n-1}}\,,$$

and

$$\begin{aligned}|\overline{K}_6(y,z)| &\le C_n(R)\,|z_1-y_1|\,|y|\int_0^1 \frac{1}{r^{(n+2)/2}}\,e^{-|y-z|^2/(4r)}\,dr \\ &\le C_n(R)\,\frac{|z_1-y_1|}{|y-z|^n}\,|y|\int_{|y-z|/2}^{+\infty} u^{n-1}\,e^{-u^2}\,du \\ &\le \frac{C_n(R)}{|y-z|^{n-1}}\,.\end{aligned}$$

Therefore, in case $|y|\le R$, and $|y-z|\le 2R$, we obtain

$$K(y,z) = k(y-z) + k_2(y,z)\,, \tag{2.5}$$

with

$$k_2 = \overline{K}_1(y,z)+\cdots+\overline{K}_6(y,z)\,,$$

and

$$|k_2(y,z)| \le \frac{C_n}{|y-z|^{n-1}}\,.$$

If $|y| > R$ then $(y, z) \in N_R$ if and only if $|z| \geq R/2$ and $|y - z| \leq R/|z|$. Hence $|y| \approx |z|$, *i.e.*

$$\Big(1 + \frac{4}{R}\Big)^{-1} |z| \leq |y| \leq \Big(1 + \frac{4}{R}\Big) |z| \,,$$

in particular $|y| \leq 2R$ for $R \geq 4$. Then by (2.4)

$$\int\limits_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz = c_n \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} k(y-z)\, f(z)\, dz + \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} k_1(y,z)\, f(z)\, dz \,.$$

Therefore

$$\sup_{\varepsilon>0} \Big| \int\limits_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz \Big| \leq c_n \sup_{\varepsilon>0} \Big| \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} k(y-z)\, f(z)\, dz \Big| + \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} |k_1(y,z)|\, |f(z)|\, dz \,.$$

Since $R \geq 4$ then $|z|/2 \leq |y| \leq 2|z|$ and so

$$\frac{R}{2|y|} \leq \frac{R}{|z|} \leq \frac{2R}{|y|} \,.$$

We have

$$\int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq R/|z|}} k(y-z)\, f(z)\, dz = \int\limits_{\substack{|z|\geq R/2 \\ \varepsilon<|y-z|\leq 2R/|y|}} k(y-z)\, f(z)\, dz \int\limits_{\substack{|z|\geq R/2 \\ R/|z|<|y-z|\leq 2R/|y|}} k(y-z)\, f(z)\, dz \,.$$

To estimate the second integral on the right hand side we write

$$
\begin{aligned}
I_1 &= \Big| \int\limits_{\substack{|z|\geq R/2\\ R/|z|<|y-z|\leq 2R/|y|}} k(y-z)\,f(z)\,dz\Big| \\
&\leq \int\limits_{R/2|y|<|y-z|\leq 2R/|y|} |k(y-z)|\,|f(z)|\,dz \\
&\leq c \int\limits_{R/2|y|<|y-z|\leq 2R/|y|} \frac{1}{|y-z|^n}\,|f(z)|\,dz \\
&\leq \frac{c}{\left(\frac{R}{2|y|}\right)^n} \int\limits_{|y-z|\leq 2R/|y|} |f(z)|\,dz\,.
\end{aligned}
$$

It easy to see that $e^{-|z|^2} \approx e^{-|y|^2}$ for $|y-z| \leq 2R/|y|$ and $|z| \leq 2|y|$. Therefore

$$
\begin{aligned}
I_1 &\leq \frac{c}{\gamma(B_{2R/|y|}(y))} \int_{B_{2R/|y|}(y)} |f(z)|\,\gamma(z)\,dz \\
&\leq c\,M_{2,2R}f(y)\,,
\end{aligned}
$$

since $|y| > R$.

Consequently if $|y| > R$ then

$$
\begin{aligned}
\sup_{\varepsilon>0} \Big| \int\limits_{\substack{N_R^y\\ |y-z|>\varepsilon}} K(y,z)\,f(z)\,dz\Big| \leq c_n \sup_{\varepsilon>0} \Big| &\int\limits_{\substack{|z|\geq R/2\\ \varepsilon<|y-z|\leq 2R/|y|}} k(y-z)\,f(z)\,dz\Big| \\
&+ M_{2,2R}f(y) \\
&+ \int\limits_{\substack{|z|\geq R/2\\ \varepsilon<|y-z|\leq R/|z|}} |k_1(y,z)|\,|f(z)|\,dz\,.
\end{aligned}
$$

Let us consider the case $|y| \leq R$. We write

$$
\int\limits_{\substack{N_R^y\\ |y-z|>\varepsilon}} K(y,z)\,f(z)\,dz = \int\limits_{\substack{|z|\leq R\\ |y-z|>\varepsilon}} K(y,z)\,f(z)\,dz
$$

$$+ \int_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} K(y,z)\, f(z)\, dz$$

and $|z| \leq R$ implies $|y-z| \leq 2R$. Consequently we may apply (2.5) and get

$$\int_{\substack{|z|\leq R \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz = c_n \int_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k(y-z)\, f(z)\, dz + \int_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k_2(y,z)\, f(z)\, dz\,.$$

Thus,

$$\sup_{\varepsilon>0} \Big| \int_{\substack{|z|\leq R \\ |y-z|>\varepsilon}} K(y,z)\, f(z)\, dz \Big| \leq c_n \sup_{\varepsilon>0} \Big| \int_{\substack{|z|\leq R \\ \varepsilon<|y-z|\leq 2R}} k(y-z)\, f(z)\, dz \Big| + \int_{\substack{|z|\leq R \\ |y-z|\leq 2R}} |k_2(y,z)|\, |f(z)|\, dz\,.$$

If $|z| > R$ and $|y-z| \leq R/|z|$ then $|y-z| < 1$ and since $|y| \leq R$ we may apply (2.5) to write

$$\int_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} K(y,z)\, f(z)\, dz = \int_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} k(y-z)\, f(z)\, dz + \int_{\substack{|z|>R \\ \varepsilon<|y-z|\leq R/|z|}} k_2(y,z)\, f(z)\, dz$$

$$= J_1 + J_2\,.$$

We have

$$|J_2| \leq \int_{\substack{|z|>R \\ |y-z|\leq R/|z|}} |k_2(y,z)|\, |f(z)|\, dz\,.$$

Since $R/|z| < 1$ we write

$$J_1 = \int_{\substack{|z|>R \\ \varepsilon<|y-z|<1}} k(y-z)\,f(z)\,dz - \int_{\substack{|z|>R \\ R/|z|<|y-z|<1}} k(y-z)\,f(z)\,dz$$
$$= J_3 - J_4\,.$$

In the region $|z| > R$ and $R/|z| \le |y-z| \le 1$ we have that $|z| \le 1 + R$. Since $|y| \le R$ we have $e^{-|z|^2} \approx e^{-|y|^2}$, for $z \in B_1(y)$. Consequently

$$|J_4| \le C\left(\frac{1+R}{R}\right)^n \frac{1}{\gamma(B_1(y))}\int_{B_1(y)} |f(z)|\,\gamma(z)\,dz$$
$$\le C_R\, M_{1,R}f(y)\,.$$

Therefore

$$\sup_{\varepsilon>0}\left|\int_{\substack{|z|>R \\ \varepsilon<|y-z|\le R/|z|}} K(y,z)\,f(z)\,dz\right|$$
$$\le c_n \sup_{\varepsilon>0}\left|\int_{\substack{|z|>R \\ \varepsilon<|y-z|\le 1}} k(y-z)\,f(z)\,dz\right| + \int_{\substack{|z|>R \\ |y-z|\le R/|z|}} |k_2(y,z)|\,|f(z)|\,dz + M_{1,R}f(y)\,.$$

Then for $|y| \le R$ we obtain

$$\sup_{\varepsilon>0}\left|\int_{\substack{N_R^y \\ |y-z|>\varepsilon}} K(y,z)\,f(z)\,dz\right| \le c_n\left(\sup_{\varepsilon>0}\left|\int_{\substack{|z|\le R \\ \varepsilon<|y-z|\le 2R}} k(y-z)\,f(z)\,dz\right| + \sup_{\varepsilon>0}\left|\int_{\substack{|z|<R \\ \varepsilon<|y-z|\le 1}} k(y-z)\,f(z)\,dz\right|\right.$$

$$+ \int_{\substack{|z|\leq R\\ |y-z|\leq 2R}} |k_2(y,z)|\,|f(z)|\,dz$$

$$+ \int_{\substack{|z|>R\\ |y-z|\leq R/|z|}} |k_2(y,z)|\,|f(z)|\,dz$$

$$\left. + M_{1,R}f(y)\right) .$$

This completes the proof of Theorem 1.

PROOF OF THE COROLLARY. By Lemma 1, the operators T_1, T_2 and T_3 are of weak-type (1,1) with respect to the Gaussian measure.

We shall show that the operators T_4, T_5 and T_6 are bounded in $L^1_\gamma(\mathbb{R}^n)$, and consequently of weak-type (1,1) with respect to the Gaussian measure. Let us first consider T_5 and T_6. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} T_5 f(y)\, e^{-|y|^2}\, dy &= \int_{|y|\leq R} e^{-|y|^2} \int_{\substack{|z|\leq R\\ |y-z|\leq 2R}} |k_2(y,z)|\,|f(z)|\,dz\,dy \\
&\leq e^{R^2} c_n(R) \int_{|y|\leq R} \int_{|y-z|\leq 2R} \frac{|f(z)|}{|y-z|^{n-1}}\, e^{-|z|^2}\,dz\,dy \\
&\leq c_n(R) \int_{\mathbb{R}^n} |f(z)|\, e^{-|z|^2} \int_{|y-z|\leq 2R} \frac{1}{|y-z|^{n-1}}\,dy\,dz \\
&\leq C_n(R)\, \|f\|_{L^1_\gamma} .
\end{aligned}$$

If $|y| \leq R < |z|$, and $|y-z| \leq R/|z|$ then $|z|^2 \leq 1 + 2R + |y|^2$ and consequently

$$\int_{\mathbb{R}^n} T_6 f(y)\, e^{-|y|^2}\, dy \leq e^{1+2R} c_n(R) \int_{|y|\leq R} e^{-|y|^2} \int_{|y-z|\leq 1} \frac{|f(z)|}{|y-z|^{n-1}}\, e^{|y|^2-|z|^2}\,dz\,dy$$

$$\leq c_n(R) \int_{\mathbb{R}^n} |f(z)|\, e^{-|z|^2} \int_{|y-z|\leq 1} \frac{1}{|y-z|^{n-1}}\, dy\, dz$$

$$\leq C_n(R) \|f\|_{L^1_\gamma}\,.$$

Let us now consider T_4. If $|y| > R$, $|z| \geq R/2$, and $|y-z| \leq R/|z|$, then $|y| \approx |z|$, and by (2.4) we have

$$\int_{\substack{|z|\geq R/2\\ |y-z|\leq R/|z|}} |k_1(y,z)|\,|f(z)|\, dz \leq C_n(R) \int_{\substack{|z|\geq R/2\\ |y-z|\leq R/|z|}} \frac{|y|^{1/2}}{|y-z|^{n-1/2}} |f(z)|\, dz$$

$$= \tilde{T}_4 f(y)\,.$$

We have

$$\int_{|y|>R} \tilde{T}_4 f(y)\, e^{-|y|^2}\, dy \leq c_n(R) \int_{|y|>R} e^{-|y|^2} \int_{|y-z|\leq R/|z|} |z|^{1/2} \frac{|f(z)|}{|y-z|^{n-1/2}}\, e^{|y|^2-|z|^2}\, dz\, dy$$

$$\leq c_n(R) \int_{\mathbb{R}^n} |f(z)|\, e^{-|z|^2} |z|^{1/2} \int_{|y-z|\leq R/|z|} \frac{1}{|y-z|^{n-1/2}}\, dy\, dz$$

$$\leq C_n(R)\, \|f\|_{L^1_\gamma}\,.$$

This ends the proof of the corollary.

REMARK. We show here that the kernels k_j and K have basically the same behavior when $j = 1$. The remaining values of j are treated in a similar way. We set

$$\varphi(r) = \left(\frac{1-r^2}{-\log r}\right)^{1/2}.$$

The function φ is increasing in $(0,1)$, $\varphi(1)=\sqrt{2}$, and the inequality

$$-\log r \le \frac{1-r^2}{2\,r^2}$$

valid for $0<r<1$ implies that $\varphi(r)\ge\sqrt{2}\,r$ and consequently

$$\varphi(1)-\varphi(r)\le\sqrt{2}\,(1-r).$$

We write

$$\begin{aligned} k_1(y,z) &= k_1(y,z)-\sqrt{2}\,K(y,z)+\sqrt{2}\,K(y,z)\\ &= H(y,z)+\sqrt{2}\,K(y,z). \end{aligned}$$

We have

$$\begin{aligned} -H(y,z) = (z_1-y_1)&\int_0^1 \frac{\varphi(1)-\varphi(r)}{(1-r^2)^{(n+3)/2}}\,e^{-|z-ry|^2/(1-r^2)}\,dr\\ &+ y_1\int_0^1 \frac{\varphi(1)-\varphi(r)}{(1-r^2)^{(n+3)/2}}\,(1-r)\,e^{-|z-ry|^2/(1-r^2)}\,dr\\ = H_1(y,z)&+H_2(y,z). \end{aligned}$$

We have

$$\begin{aligned} |H_1(y,z)| &\le c\,|z_1-y_1|\int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\,e^{-|z-ry|^2/(1-r^2)}\,dr\\ &= \tilde{H}_1(y,z), \end{aligned}$$

and

$$\begin{aligned} |H_2(y,z)| &\le c\,|y_1|\int_0^1 \frac{1}{(1-r)^{(n-1)/2}}\,e^{-|z-ry|^2/(1-r^2)}\,dr\\ &= \tilde{H}_2(y,z). \end{aligned}$$

We now proceed as in the proof of Theorem 1. We first assume $|y|>R$ and $\varepsilon<|y-z|\le 2R/|y|$. We begin by estimating $\tilde{H}_2$. As in the estimate of K_2 in Theorem 1 we get

$$
\begin{aligned}
\tilde{H}_2(y,z) &\le |y| \left(\int_0^{|y-z|/|y|} + \int_{|y-z|/|y|}^1 \frac{1}{r^{(n-1)/2}}\, e^{-|y-z|^2/(2r)} \right. \\
&\qquad \left. \cdot\, e^{-r|y|^2/(2)}\, dr \right) e^{y\cdot(y-z)} \\
&\le e^{2R}\, |y| \left(\int_0^{|y-z|/|y|} \frac{1}{r^{(n-1)/2}}\, e^{-|y-z|^2/(2r)}\, dr \right. \\
&\qquad \left. + \int_{|y-z|/|y|}^1 \frac{1}{r^{(n-1)/2}}\, dr \right) \\
&\le e^{2R}\, |y| \left(\frac{c_n}{|y-z|^{n-3}} \int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-4}\, e^{-u^2}\, du \right. \\
&\qquad \left. + \left(\frac{|y|}{|y-z|} \right)^{(n-3)/2} \right) \\
&\le C_n(R)\, |y| \left(\frac{c}{|y-z|^{n-3}}\, \frac{1}{(|y||y-z|)^{1/2}} \right. \\
&\qquad \left. + \left(\frac{|y|}{|y-z|} \right)^{(n-3)/2} \right).
\end{aligned}
$$

We now estimate $\tilde{H}_1$. We write

$$
\begin{aligned}
\tilde{H}_1(y,z) &= |z_1 - y_1| \int_0^{1-|y-z|/|y|} \frac{1}{(1-r^2)^{(n+1)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\
&\quad + |z_1 - y_1| \int_{1-|y-z|/|y|}^1 \frac{1}{(1-r^2)^{(n+1)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\
&= H_3(y,z) + H_4(y,z).
\end{aligned}
$$

We estimate H_3 in the same way we estimated K_3 and get

$$
H_3(y,z) \le c_n(R)\, \frac{|y|^{1/2}}{|y-z|^{n-5/2}}\,.
$$

We also have

$$
H_4(y,z) \le |z_1 - y_1| \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|z-(1-r)y|^2/(2r)}\, dr
$$

$$\leq e^{2R}\,|z_1 - y_1| \int_0^{|y-z|/|y|} \frac{1}{r^{(n+1)/2}}\, e^{-|z-y|^2/(2r)}\, dr$$
$$\leq c\, e^{2R}\, \frac{|z_1 - y_1|}{|z-y|^{n-1}} \int_{(|y-z|\,|y|/2)^{1/2}}^{+\infty} u^{n-2}\, e^{-u^2}\, du$$
$$\leq C(R)\, \frac{1}{|z-y|^{n-2}}\,.$$

In the region $|y| > R$ and $|y-z| < 2R/|y|$ the last kernel is dominated by

$$C(R)\, \frac{|y|^{1/2}}{|y-z|^{n-1/2}}\,.$$

In the region $|y| \leq R$ and $\varepsilon < |y-z| \leq 2R$ we proceed as in the proof of Theorem 1 to get

$$H_1(y,z) \leq C_n(R)\, \frac{1}{|y-z|^{n-1}}\,,$$

and

$$H_2(y,z) \leq C_n(R)\, \frac{1}{|y-z|^{n-3}}\,.$$

Now by arguing as in the proof of Theorem 1 we obtain that

$$H_1^* f(y) = \sup_{\varepsilon>0} \Big| \int_{\substack{N_R^y \\ |y-z|>\varepsilon}} H(y,z)\, f(z)\, dz \Big| \leq C \sum_{i=4}^{6} T_i f(y)\,,$$

and therefore H_1^* is of weak-type (1,1).

We also define

$$H_2^* f(y) = \int_{\mathbb{R}^n \setminus N_R^y} |H(y,z)|\, |f(z)|\, dz\,,$$

and since $\varphi(r) \leq \sqrt{2}$, we have that

$$|H(y,z)| \leq c \int_0^1 \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr\,.$$

The right hand-side of last inequality is the kernel of the operator considered in Lemma 2, Section 3, which restricted to the region $\mathbb{R}^n \setminus N_R^y$ gives an integral operator of weak-type (1,1) for R sufficiently large. This will be proved in Theorem 2.

3. The estimate of K_2^*.

For $z \neq 0$ set $\eta = |z|$. Given $y \in \mathbb{R}^n$ there exist unique ξ and v such that

$$y = \xi \frac{z}{\eta} + v\,.$$

Given $y, z \in \mathbb{R}^n$ by $\alpha(y,z)$ we denote the angle between y, z, $0 \leq \alpha(y,z) \leq \pi$. We shall show that the set $\mathbb{R}^n \times \mathbb{R}^n \setminus N_R$ can be written as a disjoint union of the following sets:

$$D_1 = \{(y,z) \notin N_R : \ \xi \leq \eta, \text{ and } \alpha(y,z) \geq \pi/4\}\,,$$

$$D_2 = \{(y,z) \notin N_R : \ \xi > \eta, \text{ and } |y-z| \geq \beta\,(|y| \vee |z|)\}\,,$$

$$D_3 = \{(y,z) \notin N_R : |y-z| < \beta\,(|y| \vee |z|), \\ \text{or both } \xi \leq \eta \text{ and } \alpha(y,z) < \pi/4\}\,,$$

provided $\beta > 0$ and sufficiently small.

We set

$$D_3^1 = \{(y,z) \notin N_R : \ |y-z| < \beta\,(|y| \vee |z|)\}\,,$$

and

$$D_3^2 = \{(y,z) \notin N_R : \ \xi \leq \eta \text{ and } \alpha(y,z) < \pi/4\}\,.$$

We clearly have that $D_3 = D_3^1 \cup D_3^2$, $D_1 \cap D_2 = \varnothing$, $D_2 \cap D_3 = \varnothing$, and $D_1 \cap D_3^2 = \varnothing$. Observe that if $|y-z| < \beta\,(|y| \vee |z|)$ then $|y| \approx |z|$, with constant in the equivalence only depending on β. Therefore

$$\{(y,z) : \ |y-z| < \beta\,(|y| \vee |z|)\} \subset \{(y,z) : \ |y-z| \leq C_\beta\,|z|\}\,.$$

If β is sufficiently small then $C_\beta < 1/2$ and we have

$$\{(y,z) : \ |y-z| < \beta\,(|y| \vee |z|)\} \subset \{(y,z) : \ \alpha(y,z) < \pi/4\}\,.$$

Then by taking $\beta > 0$ sufficiently small we obtain that $D_1 \cap D_3^1 = \varnothing$, and $\mathbb{R}^n \times \mathbb{R}^n \setminus N_R = D_1 \cup D_2 \cup D_3$.

Lemma 2. *Let K be the kernel defined by* (1.2). *Given $\beta > 0$ sufficiently small, $B_1 > 0$, and $B_2 \geq 2$, there exist $R > 0$ sufficiently large and a constant C depending only on β, R, B_1, B_2 and n such that the following estimates hold.*

a) *If* $(y,z) \in D_1$ *then*

$$|K(y,z)| \le C\, e^{\xi_+^2-\eta^2}, \qquad \xi_+ = \xi \vee 0 .$$

b) *If* $(y,z) \in D_2$ *then*

$$|K(y,z)| \le C .$$

c.1) *If* $(y,z) \in D_3$ *and* $|y-z| \ge \beta\,(|y| \vee |z|)$ *then*

$$|K(y,z)| \le C\, e^{\xi^2-\eta^2} .$$

c.2) *If* $(y,z) \in D_3$, $|y-z| < \beta\,(|y| \vee |z|)$, *and* $|v| < B_1/\eta$ *then*

$$|K(y,z)| \le C\,\eta^n \left(1 \wedge e^{\xi^2-\eta^2}\right) .$$

c.3) *If* $(y,z) \in D_3$, $|y-z| < \beta\,(|y| \vee |z|)$, *and* $|v| \ge B_2/\eta$ *then*

$$|K(y,z)| \le C\, \frac{1}{|v|^n} \left(1 \wedge e^{\xi^2-\eta^2}\right) .$$

PROOF. We define

$$K_1(y,z) = \int_0^1 \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr .$$

Clearly $|K(y,z)| \le K_1(y,z)$ where $K(y,z)$ is defined by (1.2).

Case a). Suppose first $y \cdot z \le 0$. This implies $\xi \le 0$, and consequently $\xi_+ = 0$. We also have

$$|y-z|^2 \ge |y|^2 + |z|^2 - 2y\cdot z \ge (|z|^2 + |y|^2) \ge (\beta\,(|y| \vee |z|))^2 ,$$

and since $(y,z) \notin N_R$, we obtain that $|y-z| \ge \beta R$. In this case we also have $y \cdot (y-z) \ge 0$.

We write

$$\begin{aligned} K_1(y,z) &= \int_0^{1/2} \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\ &\quad + \int_{1/2}^1 \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}}\, e^{-|ry-z|^2/(1-r^2)}\, dr \\ &= K_2(y,z) + K_3(y,z) . \end{aligned}$$

Since $y \cdot z \leq 0$, we have

$$\begin{aligned}
K_2(y,z) &= \int_0^{1/2} \frac{|z_1 - r y_1|}{(1-r^2)^{(n+3)/2}} \\
&\qquad \cdot e^{-(r^2|y|^2 + r^2|z|^2 - 2r y\cdot z)/(1-r^2)}\, dr\, e^{-\eta^2} \\
&\leq \Big(\frac{4}{3}\Big)^{(n+3)/2} \Big(|z_1| \int_0^{1/2} e^{-(r z_1)^2/2}\, dr \\
&\qquad + |y_1| \int_0^{1/2} e^{-(r y_1)^2/2}\, dr \Big) e^{-\eta^2} \\
&\leq c_n \Big(\int_0^{+\infty} e^{-u^2}\, du \Big) e^{-\eta^2} .
\end{aligned}$$

Also, since $y \cdot (y - z) \geq 0$, we have

$$\begin{aligned}
K_3(y,z) &= \int_0^{1/2} \frac{|z_1 - (1-r) y_1|}{((2-r)r)^{(n+3)/2}}\, e^{-|(1-r)y - z|^2/((2-r)r)}\, dr \\
&\leq |z_1 - y_1| \int_0^{1/2} \frac{1}{((2-r)r)^{(n+3)/2}} \\
&\qquad \cdot e^{-|y-z|^2/((2-r)r)}\, e^{2\, y\cdot(y-z)/(2-r)}\, dr \\
&\quad + \int_0^{1/2} \frac{|y_1|}{((2-r))^{(n+3)/2}}\, e^{-r|y|^2/(2-r)}\, \frac{1}{r^{(n+1)/2}} \\
&\qquad \cdot e^{-|y-z|^2/((2-r)r)}\, e^{2\, y\cdot(y-z)/(2-r)}\, dr \\
&\leq 2\, e^{(4/3)\, y\cdot(y-z)} \Big(|z_1 - y_1| \int_0^{1/2} \frac{1}{((2-r)r)^{n/2}} \\
&\qquad \cdot e^{-|y-z|^2/((2-r)r)} \\
&\qquad \cdot \frac{(1-r)}{((2-r)r)^{3/2}}\, dr \\
&\quad + \int_0^{1/2} \frac{1}{((2-r)r)^{(n-1)/2}} \\
&\qquad \cdot e^{-|y-z|^2/((2-r)r)}\, \frac{(1-r)}{((2-r)r)^{3/2}}\, dr \Big) .
\end{aligned}$$

By making the change of variables

$$u = \frac{|y-z|}{((2-r)r)^{1/2}}$$

the last expression equals to

$$\begin{aligned} C\, e^{(4/3)\, y\cdot(y-z)} \Big(\frac{|z_1 - y_1|}{|y-z|^{n+1}} \int_{2\,|y-z|/\sqrt{3}}^{+\infty} u^n\, e^{-u^2}\, du \\ + \frac{1}{|y-z|^n} \int_{2\,|y-z|/\sqrt{3}}^{\infty} u^{n-1}\, e^{-u^2}\, du \Big) \\ \le C\, e^{(4/3)\, y\cdot(y-z)} \frac{1}{|y-z|^n}\, e^{-4\,|y-z|^2/3} \\ \cdot \big(P_{n-1}(|y-z|) + P_{n-2}(|y-z|) \big)\,, \end{aligned}$$

where P_{n-1} and P_{n-2} are polynomials of degree $n-1$ and $n-2$ respectively. Therefore

$$\begin{aligned} K_3(y,z) &\le C_n(R)\, e^{(4/3) y\cdot(y-z)}\, e^{-4\,|y-z|^2/3} \\ &= C_n(R)\, e^{(4/3) y\cdot z} e^{-4\,|z|^2/3} \\ &\le C_n(R)\, e^{-\eta^2}\,, \end{aligned}$$

since $|y-z| \ge \beta R$.

Second, we assume $y \cdot z > 0$. In this case we have $0 < \xi \le \eta$. Since $\alpha(y,z) \ge \pi/4$, we also have $\xi \le |y|/\sqrt{2}$. In addition, for $0 < \xi \le \eta$ and $0 \le r \le 1$, we have $|\eta - r\xi| = \eta - r\xi \le \eta$.

We write

$$\begin{aligned} K_1(y,z) = \int_0^1 &\frac{\big|(\eta - r\xi)\frac{z_1}{\eta} - r v_1\big|}{(1-r^2)^{(n+3)/2}}\, e^{-(r|v|)^2/(1-r^2)} \\ &\cdot e^{-|\xi - r\eta|^2/(1-r^2)}\, dr\; e^{\xi^2-\eta^2} \\ = I_1\, e^{\xi^2-\eta^2}\,. \end{aligned}$$

We shall show that the integral I_1 is bounded by a constant independent of ξ, η and v.

In fact, let us first assume that $\eta < R/2$. Since $(y,z) \notin N_R$, $|y| > R$, consequently

$$|v| = \Big| y - \xi \frac{z}{\eta} \Big| \ge |y| - \xi > \Big(1 - \frac{1}{\sqrt{2}}\Big) R\,.$$

Therefore

$$
\begin{aligned}
I_1 &\le \eta \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, dr \\
&\quad + \int_0^1 \frac{|rv_1|}{(1-r^2)^{1/2}}\, \frac{1}{(1-r^2)^{(n+2)/2}}\, e^{-(r|v|)^2/(1-r^2)}\, dr \\
&\le C\,(1+R) \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, dr \\
&\le C\,(1+R)\bigg(\int_0^{1/2} \frac{1}{(1-r)^{(n+3)/2}}\, dr \\
&\qquad\qquad + \int_{1/2}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|v|^2/(16(1-r))}\, dr \bigg) \\
&\le C_n(R) \left(1 + |v|^{-(n+1)} \int_0^\infty u^n\, e^{-u^2}\, du \right) \\
&\le C_n(R)\,.
\end{aligned}
$$

In case $\eta \ge R/2$, we have

$$
I_1 \le C\,\eta \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr\,.
$$

If ξ is near to η, *i.e.* $(1-\beta/2)\eta < \xi \le \eta$, then $\eta - \xi < \beta\,\eta/2$, and we claim that

$$
|y - z| \ge \beta\,\eta\,,
$$

for $\beta > 0$ sufficiently small. In fact, we obviously have $|y-z| \ge |y| - \eta$, and by the assumptions $y\cdot z > 0$ and $\alpha(y,z) \ge \pi/4$ we have $\xi \le |y|/\sqrt{2}$. Then

$$
|y - z| \ge \left[\sqrt{2}\left(1 - \frac{\beta}{2}\right) - 1 \right] \eta\,.
$$

If

$$
\beta \le \frac{2(\sqrt{2}-1)}{2+\sqrt{2}}
$$

then the quantity in brackets is greater than or equal to β, and the claim follows. Consequently

$$
|v| \ge |y - z| - |\eta - \xi| > \left(\beta - \frac{\beta}{2}\right)\eta = \beta\,\frac{\eta}{2} > \beta\,\frac{R}{2}\,.
$$

Hence,

$$\frac{(r|v|)^2}{4(1-r)} \geq \frac{\beta^2}{4}\frac{\eta^2}{4}\frac{r^2}{1-r},$$

therefore

$$\begin{aligned} I_1 &\leq C\eta \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(\beta r\eta)^2/(16(1-r))}\, dr \\ &\leq C\, 2^{(n+3)/2}\eta \int_0^{1/2} e^{-(\beta r\eta)^2/(8)}\, dr \\ &\quad + C\eta \int_{1/2}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(\beta\eta)^2/(64(1-r))}\, dr \\ &\leq \frac{C_n}{\beta}\left(1+\frac{\eta}{\eta^{n+1}}\right) \\ &\leq C_n(\beta, R)\,. \end{aligned}$$

If $0 < \xi \leq (1-\beta/2)\eta$ then $\eta - \xi \geq \beta\eta/2$, consequently

$$\begin{aligned} I_1 &\leq C\eta \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\ &= C\eta \int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\ &\quad + C\eta \int_{1-(\eta-\xi)/(2\eta)}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\ &= \mathrm{I} + \mathrm{II}\,. \end{aligned}$$

If $0 < r < 1 - (\eta-\xi)/(2\eta)$, then $1 - r > (\eta-\xi)/(2\eta) > \beta/4$. Hence,

$$\mathrm{I} \leq C\left(\frac{4}{\beta}\right)^{(n+3)/2} \eta \int_0^1 e^{-(\xi - r\eta)^2/2}\, dr \leq C_n(\beta)\,.$$

If $1-(\eta-\xi)/(2\eta) < r < 1$, then $r\eta - \xi > (\eta+\xi)/2 - \xi > \beta\eta/4$. Therefore

$$\mathrm{II} \leq C\eta \int_0^1 \frac{1}{r^{(n+3)/2}}\, e^{-(\beta\eta)^2/(32\, r)}\, dr \leq C_n(\beta)\frac{1}{\eta^n} \leq C_n(\beta, R)\,.$$

This completes the proof of case *a*).

Case b). Since $(y,z) \in D_2$, it follows that $\eta < \xi \le |y|$ and $|y-z| \ge \beta(|y| \vee |z|) = \beta\,|y|$. Also, $(y,z) \notin N_R$ implies $|y-z| \ge \beta R$. So,

$$\frac{\beta}{2} \le \frac{|y-z|}{2|y|} \le \frac{1}{2} + \frac{\eta}{2|y|} \le 1\,.$$

Then we have

$$\begin{aligned} K_1(y,z) &\le C \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|ry-z|^2/(4(1-r))}\, dr \\ &= C \int_0^1 \frac{1}{r^{(n+2)/2}}\, e^{-|(1-r)y-z|^2/(4r)}\, dr \\ &= \frac{4C}{|y-z|^2}\, e^{(1/2)\, y\cdot(y-z)} \int_0^1 \frac{e^{-r|y|^2/4}}{r^{(n-2)/2}} \\ &\qquad\qquad \cdot \frac{d}{dr}\Big(e^{-|y-z|^2/(4r)}\Big)\, dr \\ &= \frac{4C}{|y-z|^2}\, e^{(1/2)\, y\cdot(y-z)} \bigg(\frac{e^{-r|y|^2/4}}{r^{(n-2)/2}}\, e^{-|y-z|^2/(4r)} \bigg|_{r=0}^{r=1} \\ &\qquad\qquad - \int_0^1 \frac{d}{dr}\Big(\frac{e^{-r|y|^2/4}}{r^{(n-2)/2}}\Big)\, e^{-|y-z|^2/(4r)}\, dr \bigg)\,. \end{aligned}$$

For $n = 2$ the last expression equals to

$$\begin{aligned} &\frac{4C}{|y-z|^2}\, e^{(1/2)\, y\cdot(y-z)}\, e^{-|y|^2/4}\, e^{-|y-z|^2/4} \\ &\quad + C \frac{|y|^2}{|y-z|^2} \int_0^1 e^{-|(1-r)y-z|^2/(4r)}\, dr \le C \frac{1+|y|^2}{|y-z|^2} \\ &\quad \le C(\beta, R)\,, \end{aligned}$$

and in case $n > 2$ equals to

$$\begin{aligned} &\frac{4C}{|y-z|^2}\, e^{-|z|^2/4} + \frac{C}{|y-z|^2}\, |y|^2 \int_0^1 \frac{1}{r^{(n-2)/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr \\ &\quad + \frac{2C(n-2)}{|y-z|^2} \int_0^1 \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr \\ &\qquad \le \frac{C}{|y-z|^2} \Big(1 + (1+|y|^2) \int_0^1 \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr \Big)\,. \end{aligned}$$

To estimate the last integral we write

$$\int_0^1 \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr = \int_0^{|y-z|/(2|y|)} + \int_{|y-z|/(2|y|)}^1$$
$$\le \int_0^{|y-z|/(2|y|)} \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr + \left(\frac{2\,|y|}{|y-z|}\right)^{n/2}$$

If $0 < r < |y-z|/(2|y|)$ then $|y-z-ry| \ge |y-z| - r|y| \ge |y-z|/2$, and consequently

$$\int_0^{|y-z|/(2|y|)} \frac{1}{r^{n/2}}\, e^{-|y-z-ry|^2/(4r)}\, dr \le \int_0^1 \frac{1}{r^{n/2}}\, e^{-|y-z|^2/(16\,r)}\, dr \le \frac{C_n}{|y-z|^{n-2}}\,.$$

Therefore for $n > 2$

$$K_1(y,z) \le C_n(\beta)\left(\frac{1}{|y-z|^2} + \frac{1+|y|^2}{|y-z|^2} + \frac{1+|y|^2}{|y-z|^n}\right) \le C_n(\beta, R)\,.$$

This completes the proof of case b).

If $(y,z) \in D_3$ then $\eta \ge R/2$ (otherwise $D_3^z = \varnothing$), and since $(y,z) \notin N_R$, we have $|y-z| > R/\eta$.

Case $c.1$). Since $(y,z) \in D_3$, we have $\xi \le \eta$ and $\alpha(y,z) < \pi/4$. So, $|y-z| \ge \beta\eta$, $0 < \xi \le \eta$, and $\eta > R/2$. Then by arguing as in case a) when $y \cdot z > 0$, and $\eta > R/2$ we obtain

$$K_1(y,z) \le C_n(\beta)\, e^{\xi^2-\eta^2}\,.$$

Case $c.2$). Let us first assume that $\xi > \eta$. We claim that there exists $R = R(B_1)$ large enough, such that if $\xi > \eta$ then

$$\eta < \xi \le |y| < \frac{\eta}{1-\beta}\,, \tag{3.1}$$

and

$$\xi - \eta > \frac{1}{\eta}\,. \tag{3.2}$$

In fact, $|y| \leq \beta|y| + \eta$ implies (3.1). Next, suppose that $\xi - \eta \leq 1/\eta$ then

$$|v| \geq |y-z| - (\xi-\eta) \geq \frac{R-1}{\eta} \geq \frac{B_1}{\eta}\,.$$

If $R \geq B_1 + 1$ we get a contradiction with the fact that $|v| < B_1/\eta$.

We write

$$\begin{aligned}
K_1(y,z) &= \int_0^1 \frac{|(\eta - r\xi)\frac{z_1}{\eta} - rv_1|}{(1-r^2)^{(n+3)/2}}\, e^{-(r|v|)^2/(1-r^2)}\, e^{-|r\xi-\eta|^2/(1-r^2)}\, dr \\
&\leq \int_0^1 \frac{|(\eta - r\xi)|}{(1-r^2)^{1/2}} \frac{1}{(1-r^2)^{(n+2)/2}} \\
&\qquad \cdot e^{-(r|v|)^2/(1-r^2)}\, e^{-|r\xi-\eta|^2/(1-r^2)}\, dr \\
&\quad + \int_0^1 \frac{r|v|}{(1-r^2)^{1/2}} \frac{1}{(1-r^2)^{(n+2)/2}} \\
&\qquad \cdot e^{-(r|v|)^2/(1-r^2)}\, e^{-|r\xi-\eta|^2/(1-r^2)}\, dr \\
&\leq c \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|r\xi-\eta|^2/(4(1-r))}\, dr \\
&= c\left(\int_0^{1-(\xi-\eta)/(2\xi)} \frac{1}{(1-r)^{(n+2)/2}}\, dr \right. \\
&\quad \left. + \int_{1-(\xi-\eta)/(2\xi)}^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|r\xi-\eta|^2/(4(1-r))}\, dr \right) \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

We have

$$\mathrm{I} \leq c_n \left(\frac{\xi}{\xi - \eta} \right)^{n/2} \leq c_n(\beta)\, \eta^n\,.$$

If $1 - (\xi-\eta)/(2\xi) < r < 1$ then $r\xi - \eta > (\xi+\eta)/2 - \eta = (\xi - \eta)/2$. Hence

$$\mathrm{II} \leq \int_0^{(\xi-\eta)/(2\xi)} \frac{1}{r^{(n+2)/2}}\, e^{-|\xi-\eta|^2/(16\, r)}\, dr \leq \frac{C_n}{|\xi-\eta|^n} \leq C_n\, \eta^n\,.$$

Therefore in case $\xi > \eta$ we obtain

$$K_1(y,z) \leq C_n\, \eta^n\,.$$

Second, suppose now that $\xi \leq \eta$. We claim that for R sufficiently large, *i.e.* $R \geq B_1 + 1$ and

$$\frac{R^2}{R-1} > 8\left(\frac{1-\beta}{1-2\beta}\right),$$

we have

$$\eta - \xi < \frac{\eta}{2(1-\beta)}, \tag{3.3}$$

and

$$\eta - \xi > \frac{1}{\eta}. \tag{3.4}$$

In fact, first observe that in the region $|y-z| < \beta\,(|y| \vee |z|)$ we have $\xi > 0$ for $\beta < 1/4$. Then

$$\eta = |z| \leq |y-z| + |y| \leq \frac{\beta}{1-\beta}\,\eta + \xi + |v| \leq \frac{\beta}{1-\beta}\,\eta + \xi + \frac{R-1}{\eta},$$

and consequently

$$\begin{aligned}
\frac{1-2\beta}{1-\beta}\,\eta &\leq \xi + \frac{R-1}{\eta} \\
&\leq \xi + 2R\,\frac{R-1}{R^2} \\
&\leq \xi + \frac{R}{2}\,\frac{1}{2}\,\frac{1-2\beta}{1-\beta} \\
&< \xi + \frac{1}{2}\,\frac{1-2\beta}{1-\beta}\,\eta,
\end{aligned}$$

and (3.3) follows. The proof of (3.4) is similar to that of (3.2). Now we have

$$\begin{aligned}
K_1(y,z) \leq c\Big((\eta-\xi)&\int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\,dr \\
&+ \xi \int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\,dr \\
&+ \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\,dr\Big)\, e^{\xi^2-\eta^2} \\
= (\mathrm{III} + \mathrm{IV} + \mathrm{V})\, e^{\xi^2-\eta^2}.&
\end{aligned}$$

We have

$$\begin{aligned}\mathrm{III} \le (\eta-\xi)\int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+3)/2}}\,dr \\ + (\eta-\xi)\int_{1-(\eta-\xi)/(2\eta)}^1 \frac{1}{(1-r)^{(n+3)/2}}\,e^{-|\xi-r\eta|^2/(2(1-r))}\,dr\,.\end{aligned}$$

If $0 < r < 1-(\eta-\xi)/(2\eta)$ then $\eta-\xi < 2\eta(1-r)$. Also, if $1-(\eta-\xi)/(2\eta) < r < 1$ then $r\eta-\xi = \eta-\xi-(1-r)\eta > (\eta-\xi)/2$. Therefore

$$\begin{aligned}(\eta-\xi)\int_0^{1-(\eta-\xi)/(2\eta)} &\frac{1}{(1-r)^{(n+3)/2}}\,dr \\ &\le 2\eta\int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+1)/2}}\,dr \\ &\le c_n\,\eta\left(\frac{\eta}{\eta-\xi}\right)^{(n-1)/2} \\ &\le C_n\,\eta^n\,,\end{aligned}$$

and

$$\begin{aligned}(\eta-\xi)\int_{1-(\eta-\xi)/(2\eta)}^1 &\frac{1}{(1-r)^{(n+3)/2}}\,e^{-|\xi-r\eta|^2/(2(1-r))}\,dr \\ &\le (\eta-\xi)\int_0^{(\eta-\xi)/(2\eta)} \frac{e^{-(\xi-\eta)^2/(8r)}}{r^{(n+3)/2}}\,dr \\ &\le c_n\,\frac{\eta-\xi}{(\eta-\xi)^{n+1}} \\ &\le c_n\,\eta^n\,.\end{aligned}$$

Also

$$\begin{aligned}\mathrm{IV} &\le \eta\int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+1)/2}}\,dr \\ &\quad + \eta\int_{1-(\eta-\xi)/(2\eta)}^1 \frac{1}{(1-r)^{(n+1)/2}}\,e^{-(\xi-\eta)^2/(8(1-r))}\,dr \\ &\le c_n\left(\eta\left(\frac{\eta}{\eta-\xi}\right)^{(n-1)/2} + \frac{\eta}{(\eta-\xi)^{n-1}}\right) \\ &\le c_n\,\eta^n\,,\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{V} &\le \int_0^{1-(\eta-\xi)/(2\eta)} \frac{1}{(1-r)^{(n+2)/2}}\, dr \\
&\quad + \int_{1-(\eta-\xi)/(2\eta)}^{1} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(\xi-\eta)^2/(8(1-r))}\, dr \\
&\le c_n \left(\left(\frac{\eta}{\eta-\xi}\right)^{n/2} + \frac{1}{(\eta-\xi)^n} \right) \\
&\le c_n\, \eta^n .
\end{aligned}
$$

This completes the proof of case $c.2)$.

Case $c.3)$. We shall first assume that $|v| > B_2|\xi-\eta|$ and consider two cases: $\eta < \xi$ and $\eta \ge \xi$. Let us first consider the case $\eta < \xi$ and $|v| > B_2(\xi-\eta)$. We claim that

$$
\frac{|v|}{\xi} < \frac{2\beta}{1-\beta},
$$

in particular by taking $0 < \beta < 1/5$, we obtain $|v|/\xi < 1/2$. In fact,

$$
\eta < \xi \le |y| < \frac{1}{1-\beta}\,\eta < \frac{1}{1-\beta}\,\xi
$$

which implies

$$
|v| \le |y-z| + |\xi-\eta| < \beta\,(|y|+\xi) \le 2\,\beta\,|y| \le \frac{2\beta}{1-\beta}\,\eta \le \frac{2\beta}{1-\beta}\,\xi,
$$

and the claim is proved.

We have

$$
\begin{aligned}
K_1(y,z) &\le c \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, e^{-|r\xi-\eta|^2/(4(1-r))}\, dr \\
&= c \left(\int_0^{1-|v|/\xi} + \int_{1-|v|/\xi}^{1} \right) \\
&= \mathrm{VI} + \mathrm{VII}.
\end{aligned}
$$

If $0 < r < 1 - |v|/\xi$, then

$$\eta - r\xi = (1-r)\xi - (\xi - \eta) > |v| - (\xi - \eta) > \left(1 - \frac{1}{B_2}\right)|v|,$$

(we take $B_2 > 1$), therefore

$$\mathrm{VI} \le \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-c|v|^2/(1-r)}\, dr \le c_n \frac{1}{|v|^n}\,.$$

On the other hand, since $1 - |v|/\xi > 1/2$ we obtain

$$\mathrm{VII} \le \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|v|^2/(16(1-r))}\, dr \le c_n \frac{1}{|v|^n}\,.$$

Second, let us consider the case that $\xi \le \eta$ and $|v| > B_2(\eta - \xi)$. Then we have

$$\frac{|v|}{\eta} < \frac{2\beta}{1-\beta}\,.$$

In fact,

$$|v| \le |y - z| + |\xi - \eta| < \frac{\beta}{1-\beta}\eta + (\eta - \xi) \le \frac{\beta}{1-\beta}\eta + \frac{1}{B_2}|v|\,.$$

If we choose $B_2 \ge 2$ then

$$\frac{|v|}{2} < \frac{\beta}{1-\beta}\eta\,.$$

Consequently

$$\begin{aligned} K_1(y,z) \le \Bigg(&\int_0^1 \frac{|\eta - r\xi|}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\ &+ \int_0^1 \frac{|rv_1|}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))} \\ &\qquad \cdot e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \Bigg) e^{\xi^2 - \eta^2} \\ \le C \Bigg(&(\eta - \xi) \int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))} \\ &\qquad \cdot e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \end{aligned}$$

$$+\xi\int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-(r|v|)^2/(2(1-r))} \cdot e^{-|\xi-r\eta|^2/(2(1-r))}\, dr$$

$$+\int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))} \cdot e^{-|\xi-r\eta|^2/(2(1-r))}\, dr\Big)\, e^{\xi^2-\eta^2},$$

and

$$(\eta-\xi)\int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr$$
$$\le (\eta-\xi)\Big(\int_0^{1-|v|/\eta} \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr + \int_{1-|v|/\eta}^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, dr\Big).$$

If $0<r<1-|v|/\eta$ then

$$\xi - r\,\eta = (1-r)\,\eta - (\eta-\xi) > \Big(1-\frac{1}{B_2}\Big)\,|v|\,.$$

Also, $1-|v|/\eta > 1/2$ by taking $\beta < 1/5$. Hence, both summands on the right hand side of the last inequality are bounded by

$$(\eta-\xi)\int_0^1 \frac{1}{(1-r)^{(n+3)/2}}\, e^{-|v|^2/(c(1-r))}\, dr \le c_n\,\frac{\eta-\xi}{|v|^{n+1}} \le c_n\,\frac{1}{|v|^n}\,.$$

The conditions $0<r<1-|v|/\eta$, and $|v| > B_2(\eta-\xi)$ imply that $\eta-\xi < ((1-r)\eta))/B_2$, therefore

$$\xi - r\,\eta = (1-r)\,\eta - (\eta-\xi) > \Big(1-\frac{1}{B_2}\Big)\,(1-r)\,\eta \ge \frac{(1-r)\eta}{2}\,,$$

since $B_2 \ge 2$. Then we have

$$\xi\int_0^1 \frac{1}{(1-r)^{(n+1)/2}}\, e^{-(r|v|)^2/(2(1-r))}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr$$
$$\le \eta\int_0^{1-|v|/\eta} \frac{1}{(1-r)^{(n+1)/2}}\, e^{-(1-r)\,\eta^2/8}\, dr$$

$$\begin{aligned}
&+\eta \int_{1-|v|/\eta}^{1} \frac{1}{(1-r)^{(n+1)/2}}\, e^{-|v|^2/(8(1-r))}\, dr \\
&\le \eta\, e^{-\eta|v|/8} \int_{|v|/\eta}^{1} \frac{1}{r^{(n+1)/2}}\, dr \\
&\quad + c_n \frac{\eta}{|v|^{n-1}} \int_{\sqrt{\eta|v|}/4}^{\infty} u^{n-2}\, e^{-u^2}\, du \\
&\le C_n \left(\eta \left(\frac{\eta}{|v|} \right)^{(n-1)/2} e^{-\eta|v|/8} + \frac{\eta}{|v|^{n-1}} \frac{e^{-\eta|v|/8}}{\sqrt{\eta|v|}} \right) \\
&\le C_n \left((\eta|v|)^{(n+1)/2} + (\eta|v|)^{1/2} \right) \frac{e^{-\eta|v|/8}}{|v|^n} \\
&\le C_n \frac{1}{|v|^n} \,.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_0^1 &\frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr \\
&\le \int_0^{1-|v|/\eta} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr \\
&\quad + \int_{1-|v|/\eta}^{1} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, dr \\
&\le C \int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-c|v|^2/(1-r)}\, dr \\
&\le c_n \frac{1}{|v|^n} \,.
\end{aligned}$$

Therefore, if $|v| > B_2|\xi-\eta|$ we then have

$$K_1(y,z) \le C_n \frac{1}{|v|^n}\, e^{\xi^2-\eta^2} \,.$$

From now on we may assume $|v| \le B_2|\xi-\eta|$. We may also assume that

$$|\xi-\eta| > \frac{1}{\eta} \qquad \text{and} \qquad |v| > \frac{B_2}{\eta} \,.$$

In fact, if $|\xi-\eta| \le 1/\eta$ then $|v| > B_2\,|\xi-\eta|$, which falls into the case previously considered.

Let us first assume $\xi-\eta > 1/\eta$ and $|v| > B_2/\eta$. We have $\xi-\eta < \beta\,\xi$, this is because

$$|y| \le |y-z| + |z| \le \beta\,(|y| \vee |z|) + |z| = \beta\,|y| + \eta$$

and so $|y| < (1-\beta)^{-1}\eta$ which implies $\eta < \xi < |y| < (1-\beta)^{-1}\eta$. If $0 < r < 1/2$ then

$$\eta - r\,\xi = (1-r)\,\xi - (\xi-\eta) > \Big(\frac{1}{2}-\beta\Big)\xi > \frac{1}{B_2}\Big(\frac{1}{2}-\beta\Big)\,|v|\,,$$

and by arguing as in the case when $\eta \le \xi$ and $|v| > B_2(\xi-\eta)$, we obtain

$$\begin{aligned} K_1(y,z) &\le C\int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, e^{-|r\xi-\eta|^2/(4(1-r))}\,dr \\ &\le C\int_0^{1/2} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|r\xi-\eta|^2/(4(1-r))}\,dr \\ &\quad + C\int_{1/2}^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\,dr \\ &\le C\int_0^1 \frac{1}{r^{(n+2)/2}}\, e^{-c\,|v|^2/(r)}\,dr \\ &\le C_n\,\frac{1}{|v|^n}\,. \end{aligned}$$

Second, let us now assume $\eta-\xi > 1/\eta$ and $|v| > B_2/\eta$. We have

$$\eta - \xi \le |y-z| < \frac{\beta}{1-\beta}\,\eta$$

(this follows because $|y-z| \le \beta\,(|y| \vee |z|)$ and by analizing $|y| > \eta$ or $|y| \le \eta$) which implies

$$\frac{\eta-\xi}{\eta} < \frac{\beta}{1-\beta} < \frac{1}{4}\,,$$

by taking $\beta < 1/3$. We write

$$\begin{aligned} K_1(y,z) \le C\Big(\int_0^1 \frac{|\eta - r\xi|}{(1-r)^{(n+3)/2}}\, & e^{-|\xi-r\eta|^2/(2(1-r))} \\ & \cdot e^{-(r|v|)^2/(2(1-r))}\,dr \end{aligned}$$

$$+\int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))} \cdot e^{-(r|v|)^2/(4(1-r))}\, dr\Big) e^{\xi^2-\eta^2}$$

$$= C\Big(\int_0^{1-((1-\beta)/(2\beta))(\eta-\xi)/\eta} + \int_{1-((1-\beta)/(2\beta))(\eta-\xi)/\eta}^{1-\beta(\eta-\xi)/\eta} + \int_{1-\beta(\eta-\xi)/\eta}^{1} \frac{|\eta-r\xi|}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))} \cdot e^{-(r|v|)^2/(2(1-r))}\, dr$$

$$+\int_0^{1/2} + \int_{1/2}^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))} \cdot e^{-(r|v|)^2/(4(1-r))}\, dr\Big) e^{\xi^2-\eta^2}.$$

If $0 < r < 1/2$ then

$$\xi - r\,\eta = (1-r)\,\eta - (\eta-\xi) > \frac{\eta}{2} - \frac{\beta}{1-\beta}\,\eta > \frac{\eta}{4} > \frac{|v|}{4B_2},$$

therefore

$$\int_0^{1/2} \frac{1}{(1-r)^{(n+2)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, dr + \int_{1/2}^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-(r|v|)^2/(4(1-r))}\, dr$$

$$\le C\int_0^1 \frac{1}{(1-r)^{(n+2)/2}}\, e^{-c|v|^2/(1-r)}\, dr \le \frac{C_n}{|v|^n}.$$

If

$$0 < r < 1 - \frac{1-\beta}{2\beta}\frac{\eta-\xi}{\eta}$$

then

$$|\eta - r\xi| = \eta - \xi + (1-r)\,\xi < \Big(1 + \frac{2\beta}{1-\beta}\Big)(1-r)\,\eta,$$

and

$$\xi - r\,\eta = (1-r)\,\eta - (\eta - \xi) > \left(1 - \frac{2\beta}{1-\beta}\right)(1-r)\,\eta\,.$$

Therefore

$$\begin{aligned}
&\int_0^{1-\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta} \frac{|\eta - r\xi|}{(1-r)^{(n+3)/2}}\, e^{-|\xi - r\eta|^2/(2(1-r))}\, dr \\
&\quad\le C\,\eta \int_0^{1-\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta} \frac{1}{(1-r)^{(n+1)/2}}\, e^{-c(1-r)\eta^2}\, dr \\
&\quad\le C\,\eta \int_{\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta}^{1} \frac{1}{r^{(n+1)/2}}\, dr\; e^{-c(\eta-\xi)\eta} \\
&\quad\le C_n\,\eta \left(\frac{\eta}{\eta-\xi}\right)^{(n-1)/2} e^{-c(\eta-\xi)\eta} \\
&\quad\le \frac{C_n}{(\eta-\xi)^n} \\
&\quad\le \frac{C_n}{|v|^n}\,.
\end{aligned}$$

The case

$$1 - \frac{1-\beta}{2\beta}\,\frac{\eta-\xi}{\eta} < r < 1 - \beta\,\frac{\eta-\xi}{\eta}$$

is equivalent to

$$\beta\,\frac{\eta-\xi}{\eta} < 1 - r < \frac{1-\beta}{2\beta}\,\frac{\eta-\xi}{\eta}\,.$$

Then

$$|\eta - r\xi| = \eta - \xi + (1-r)\,\xi \le \left(\frac{1}{\beta} + 1\right)(1-r)\,\eta\,,$$

and consequently

$$\int_{1-\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta}^{1-\beta(\eta-\xi)/\eta} \frac{|\eta - r\xi|}{(1-r)^{(n+3)/2}} \cdot e^{-|\xi-r\eta|^2/(2(1-r))}\, e^{-(r|v|)^2/(2(1-r))}\, dr$$

$$\le C\,\eta \int_{1-\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta}^{1-\beta(\eta-\xi)/\eta} \frac{1}{(1-r)^{(n+1)/2}} \cdot e^{-|\xi-r\eta|^2/(2(1-r))}\,dr\; e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\le C\,\eta \left(\frac{\eta}{\eta-\xi}\right)^{(n+1)/2} \cdot \int_{\beta(\eta-\xi)/\eta}^{\left((1-\beta)/(2\beta)\right)(\eta-\xi)/\eta} e^{-c\,\eta|r\eta-(\eta-\xi)|^2/(\eta-\xi)}\,dr\; e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\le C \left(\frac{\eta}{\eta-\xi}\right)^{(n+1)/2} \cdot \int_{-(1-\beta)(\eta-\xi)}^{\left((1-\beta)/(2\beta)-1\right)(\eta-\xi)} e^{-c\,\eta u^2/(\eta-\xi)}\,du\; e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\le C \left(\frac{\eta}{\eta-\xi}\right)^{(n+1)/2} \left(\frac{\eta-\xi}{\eta}\right)^{1/2} e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\le C \left(\frac{\eta}{\eta-\xi}\right)^{n/2} e^{-c\,\eta|v|^2/(\eta-\xi)}$$

$$\le \frac{C_n}{|v|^n}\,.$$

If $1-\beta\,(\eta-\xi)/\eta < r < 1$ then

$$|\eta - r\xi| = \eta - r\xi = \eta - \xi + (1-r)\,\xi \le (1+\beta)(\eta-\xi)$$

and

$$r\eta - \xi = \eta - \xi - (1-r)\,\eta > (1-\beta)(\eta-\xi)\,.$$

Therefore

$$\int_{1-\beta\,(\eta-\xi)/\eta}^{1} \frac{|\eta-r\xi|}{(1-r)^{(n+3)/2}}\, e^{-|\xi-r\eta|^2/(2(1-r))}\, e^{-(r|v|)^2/(2(1-r))}\,dr$$

$$\le (1+\beta)\,(\eta-\xi) \int_{1-\beta\,(\eta-\xi)/\eta}^{1} \frac{1}{(1-r)^{(n+3)/2}}\, e^{-c(\xi-\eta)^2/(1-r)}\,dr$$

$$\le \frac{C_n}{(\eta-\xi)^n}$$

$$\le \frac{C_n}{|v|^n}\,.$$

Hence

$$K_1(y,z) \le \frac{C_n}{|v|^n}\, e^{\xi^2-\eta^2}\,.$$

This completes the proof of the case *c*.3) and therefore Lemma 2 is complete.

Theorem 2. *The operator K_2^* is of weak-type* (1,1) *with respect to the measure $\gamma(z)\,dz$, provided that R is sufficiently large.*

PROOF. We decompose K_2^* in the following way

$$\begin{aligned} K_2^* f(y) &= \int_{\mathbb{R}^n\setminus N_R^y} |K(y,z)|\,|f(z)|\,dz \\ &= \int_{D_1^y} |K(y,z)|\,|f(z)|\,dz + \int_{D_2^y} |K(y,z)|\,|f(z)|\,dz \\ &\quad + \int_{D_3^y} |K(y,z)||f(z)|\,dz \\ &= K_R^1 f(y) + K_R^2 f(y) + K_R^3 f(y)\,. \end{aligned}$$

We shall first show that K_R^i, $i=1,2$, are bounded operators in $L^1_\gamma(\mathbb{R}^n)$.
Let $f\in L^1_\gamma(\mathbb{R}^n)$, we may assume $f\ge 0$. We have

$$\|K_R^1 f\|_{L^1_\gamma} = \int_{\mathbb{R}^n} f(z) \int_{D_1^z} |K(y,z)|\,\gamma(y)\,dy\,dz\,,$$

and by Lemma 2, part *a*)

$$\begin{aligned} \int_{D_1^z} |K(y,z)|\,\gamma(y)\,dy &\le C_n \int_{\{y:\ \xi\le\eta,\alpha(y,z)\ge\pi/4\}} e^{\xi_+^2-\eta^2}\, e^{-|y|^2}\,dy \\ &\le C_n\, e^{-\eta^2}\Big(\int_{\{y:\ \xi\le 0\}} e^{-|y|^2}\,dy \\ &\qquad + \int_{\{y:\ 0<\xi\le|y|/\sqrt{2}\}} e^{\xi^2}\, e^{-|y|^2}\,dy\Big) \\ &\le C_n e^{-|z|^2}\,. \end{aligned}$$

Hence K_R^1 is bounded in $L^1_\gamma(\mathbb{R}^n)$. Also

$$\begin{aligned}\|K_R^2 f\|_{L^1_\gamma} &= \int_{\mathbb{R}^n} f(z) \int_{D_2^z} |K(y,z)|\,\gamma(y)\,dy\,dz \\ &= \int_{|z|\le R} f(z) \int_{D_2^z} |K(y,z)|\,\gamma(y)\,dy\,dz \\ &\quad + \int_{|z|>R} f(z) \int_{D_2^z} |K(y,z)|\,\gamma(y)\,dy\,dz\,,\end{aligned}$$

and by Lemma 2, part b)

$$\begin{aligned}\int_{D_2^z} |K(y,z)|\,\gamma(y)\,dy &\le C \int_{\{y:\ \xi>\eta\}} e^{-|y|^2}\,dy \\ &= C \int_{\mathbb{R}^{n-1}} e^{-|v|^2} \int_\eta^\infty e^{-\xi^2}\,d\xi\,dv \\ &\le C\left(\frac{1}{R}\chi_{B_R^c(0)}(z) + e^{R^2}\chi_{B_R(0)}(z)\right) e^{-|z|^2}\,.\end{aligned}$$

Therefore K_R^2 is bounded in $L^1_\gamma(\mathbb{R}^n)$.

To show that K_R^3 is of weak-type (1,1) we shall dominate its kernel by a kernel which is constant on certain cubes. We split $\mathbb{R}^n$ into a non-overlapping sequence of open cubes Q_i with center x_i, $i = 1, 2, \dots$ such that

$$c_n\left(1 \wedge \frac{1}{|x_i|}\right) \le \operatorname{diam}(Q_i) \le 1 \wedge \frac{1}{|x_i|}\,.$$

The sequence of cubes can be chosen such that $\{|x_i|\}$ is a non-decreasing sequence. We set

$$K_3(y,z) = |K(y,z)|\,\chi_{D_3}(y,z)\,,$$

and we define

$$\overline{K}_3(y,z) = \sum_{i=1}^\infty \chi_{Q_i}(y) \sup_{y'\in Q_i} K_3(y',z)\,.$$

We claim that $\overline{K}_3(y,z)$ satisfies the estimates of Lemma 2, part $c.i$), $i = 1, 2, 3$, with new constants.

In fact, first observe that for $y, y' \in Q_i$ we have

$$y = \xi \frac{z}{\eta} + v, \qquad y' = \xi' \frac{z}{\eta} + v',$$

with

$$|\xi - \xi'| \leq |y - y'| \leq \operatorname{diam}(Q_i) \leq \left(1 \wedge \frac{1}{|x_i|}\right),$$

and

$$|\xi + \xi'| \leq |y| + |y'| \leq 2\left(|x_i| + \left(1 \wedge \frac{1}{|x_i|}\right)\right).$$

Therefore,

$$|\xi'^2 - \xi^2| \leq 2\left[\left(1 \wedge \frac{1}{|x_i|}\right)^2 + |x_i|\left(1 \wedge \frac{1}{|x_i|}\right)\right] \leq 4,$$

and consequently

$$e^{\xi'^2 - \eta^2} \leq e^4 e^{\xi^2 - \eta^2}.$$

For given β and R we set $D_3 = D_3(\beta, R)$, where D_3 is defined at the beginning of this section. Observe that if $\beta_1 \leq \beta_2$ then $D_3(\beta_1, R) \subset D_3(\beta_2, R)$; and if $R_1 \leq R_2$ then $D_3(\beta, R_2) \subset D_3(\beta, R_1)$. The fact $(y, z) \in D_3$ implies that $|z| \geq R/2$. We want to show that $\overline{K}_3(y, z)$ satisfies an estimate like in Lemma 2, part c.1). Suppose first that $(y, z) \in D_3(\beta, R)$, $|y - z| \geq \beta(|y| \vee |z|)$, and $y, y' \in Q_i$. In case $|y' - z| \geq \beta(|y'| \vee |z|)$ we may apply Lemma 2, part c.1), to obtain

$$K_3(y', z) \leq c\, e^{\xi'^2 - \eta^2}. \tag{3.1}$$

In case $|y' - z| < \beta(|y'| \vee |z|)$ we shall show that for R sufficiently large we have

$$|y' - z| \geq (1 - \beta)\left(\beta - \frac{2}{R}\right)(|y'| \vee |z|). \tag{3.2}$$

In fact, in such case we have

$$(1 - \beta)\,|y'| < |z| < \frac{1}{1 - \beta}\,|y'|,$$

and

$$|y' - z| \geq |y - z| - |y - y'| \geq \beta(|y| \vee |z|) - \frac{2}{R}\,|z| \geq \left(\beta - \frac{2}{R}\right)|z|.$$

Now by analizing the cases $|y'| > |z|$ and $|y'| \le |z|$ we have that

$$\left(\beta - \frac{2}{R}\right)|z| > \left(\beta - \frac{2}{R}\right)(1-\beta)\,(|y'| \vee |z|)$$

and (3.2) follows. Consequently, if $(y', z) \in D_3((\beta - 2/R)(1-\beta), R)$ then we may apply Lemma 2, part c.1), to obtain (3.1). If $(y', z) \notin D_3((\beta - 2/R)(1-\beta), R)$ then by taking β small and R large we get

$$(y', z) \in D_1((\beta - \frac{2}{R})(1-\beta), R) \cup D_2((\beta - \frac{2}{R})(1-\beta), R)\,.$$

If $(y', z) \in D_1((\beta - 2/R)(1-\beta), R)$, since $|y' - z| < \beta\,(|y'| \vee |z|)$, we then have $\xi' > 0$, and by Lemma 2, part a), we get (3.1). If $(y', z) \in D_2((\beta - 2/R)(1-\beta), R)$ then $\xi' > \eta$ and by Lemma 2, part b), we obtain (3.1).

Next, let us assume that $(y, z) \in D_3(\beta, R)$, $|y - z| < \beta\,(|y| \vee |z|)$, and $y, y' \in Q_i$. In this case we have

$$(1-\beta)\,|z| < |y| < \frac{1}{1-\beta}\,|z|\,,$$

and since $|z| > R/2$, we have $|y| > (1-\beta)R/2 > 3$ for R sufficiently large. This implies that if $y \in Q_i$ then $|x_i| > \sqrt{2}$, and consequently $|y| \approx |x_i|$. It is easy to see that

$$|y' - z| < \left(2\beta + \frac{2}{R}\right)(|y'| \vee |z|)\,.$$

Therefore $(y', z) \in D_3(2\beta + 2/R, R)$. If $|v| < B/\eta$ then

$$|v'| \le |y - y'| + |v| \le \frac{c}{\eta} + |v| < \frac{B + c}{\eta} = \frac{B_1}{\eta}\,.$$

Then taking β small and R large, by Lemma 2, part c.2), we have

$$|K(y', z)| \le C\,\eta^n \left(1 \wedge e^{\xi'^2 - \eta^2}\right) \le \overline{C}\,\eta^n \left(1 \wedge e^{\xi^2 - \eta^2}\right)\,.$$

If $|v| \ge B/\eta$ then $|v'| \ge |v| - |y - y'| \ge |v| - c/\eta$ and so

$$|v'| \ge \frac{B - c}{\eta} = \frac{B_2}{\eta} \qquad \text{and} \qquad |v'| \ge \left(1 - \frac{c}{B}\right)|v|\,.$$

By Lemma 2, part $c.3$), we obtain

$$|K(y',z)| \leq \frac{C_n}{|v'|^n}\left(1 \wedge e^{\xi'^2-\eta^2}\right) \leq \frac{\overline{C}_n}{|v|^n}\left(1 \wedge e^{\xi^2-\eta^2}\right).$$

This proves the claim.

Clearly

$$K_R^3 f(y) \leq \int_{D_3^y} \overline{K}_3(y,z)\, f(z)\, dz = \overline{K}_R^3 f(y),$$

and $\overline{K}_R^3 f(y)$ is constant on every Q_i.

In order to prove that $\overline{K}_R^3$ is of weak-type (1,1), given $\lambda > 0$ we shall construct a set E with the following properties:

a) $E \subset E_\lambda = \{y \in \mathbb{R}^n : \overline{K}_R^3 f(y) > \lambda\}$.

b) $\displaystyle\int_{E_\lambda} e^{-|y|^2}\, dy \leq c \int_E e^{-|y|^2}\, dy$.

c) If $U(z) = \displaystyle\int_E \overline{K}_3(y,z)\, e^{-|y|^2}\, dy$, then $U(z) \leq c\, e^{-|z|^2}$ in $\mathbb{R}^n$.

The last two properties imply the weak-type inequality, in fact

$$\begin{aligned}\int_{E_\lambda} e^{-|y|^2}\, dy &\leq c \int_E e^{-|y|^2}\, dy \\ &\leq \frac{c}{\lambda}\int_E \overline{K}_R^3\, f(y) e^{-|y|^2}\, dy \\ &= \frac{c}{\lambda}\int_E \int_{\mathbb{R}^n} \overline{K}_3(y,z)\, f(z)\, dz\; e^{-|y|^2}\, dy \\ &= \frac{c}{\lambda}\int_{\mathbb{R}^n} f(z)\, U(z)\, dz \\ &\leq \frac{c}{\lambda}\, \|f\|_{L^1_\gamma}.\end{aligned}$$

The construction of the set E is done as in [Sj]. We recall the construction here.

Given a positive integer j we define the cone

$$K_j = \{x : \alpha(x,y) \leq \pi/4 \text{ for some } y \in Q_j\}.$$

To each cube Q_j we associate a forbidden region F_j defined by

$$F_j = \bigcup \{Q_i : \ i > j \ \text{ and } \ Q_i \cap (Q_j + K_j) \neq \varnothing\} .$$

The set E is constructed as follows. Let Q_{i_1} the first cube that intersects E_λ. Since $\overline{K}^3_R f(y)$ is constant on each cube Q_i we have that $Q_{i_1} \subset E_\lambda$. Next we pick Q_{i_2}, $i_1 < i_2$, the first cube that intersects E_λ and is not contained in the forbidden region F_{i_1}. Continuing in this way, Q_{i_j} is the first cube that intersects E_λ and is not contained in the forbidden regions F_{i_k} for any Q_{i_k} already selected. The set E is by definition the union of the cubes Q_{i_j}, $j = 1, 2, \dots$ Property *a*) above is obvious. The proof of property *b*) can be found in [Sj].

Let us prove *c*). Let S_z denote the support of $\overline{K}_3(\cdot, z)$. The set S_z consists of the union of those Q_i that intersect D_3^z. Let l_v be the line parallel to z and passing through v, with $v \perp z$. We have

$$\begin{aligned} U(z) &= \int_{\mathbb{R}^n} \overline{K}_3(y,z)\, e^{-|y|^2}\, dy \\ &= \int_{\mathbb{R}^{n-1}} e^{-|v|^2} \int_{l_v \cap E \cap S_z} \overline{K}_3(s\frac{z}{\eta} + v, z)\, e^{-s^2}\, ds\, dv , \end{aligned}$$

and we set

$$I(v) = \int_{l_v \cap E \cap S_z} \overline{K}_3(s\frac{z}{\eta} + v, z)\, e^{-s^2}\, ds .$$

Let $w \in l_v \cap E \cap S_z$, then there exists a unique i such that $w \in Q_i \cap D_3^z$. The angle between w and z is less than $\pi/4$, and therefore $z \in K_i$, K_i being the cone defined before. In fact, if $\alpha(w,z) > \pi/4$ and $(w,z) \in D_3$ then $|w - z| < \beta\,(|w| \vee |z|)$ and for β small this implies $\alpha(w,z) \leq \pi/4$.

Every element of l_v is of the form $s\, z/\eta + v$, and if $Q_i \cap l_v \neq \varnothing$ then every cube Q_j with $j > i$ that intersects l_v is included in F_i. Therefore, for every fixed v, $l_v \cap E \cap S_z$ is the line segment I determined by the intersection between Q_i and the line l_v. We shall estimate the size of I. Let y, y' be the endpoints of such segment, with

$$y = \xi\, \frac{z}{\eta} + v\, , \qquad y' = \xi'\, \frac{z}{\eta} + v\, , \qquad \text{and} \qquad 0 < \xi < \xi' .$$

Hence

$$I = \{s\, \frac{z}{\eta} + v : \ \xi < s < \xi'\} , \qquad |\xi - \xi'| \leq \left(1 \wedge \frac{1}{|x_i|}\right) ,$$

where x_i is the center of Q_i. If $|x_i| > 2$ then

$$\xi \le |y| \le |y - x_i| + |x_i| \le \frac{5}{4}|x_i|,$$

and

$$\xi' \le \xi + \frac{1}{|x_i|} \le \xi + \frac{5}{4\xi}.$$

If $|x_i| \le 2$ then $\xi \le |y - x_i| + 2 \le 3$ and $\xi' \le \xi + 1 \le \xi + 3/\xi$. Therefore, I is included in the line segment

$$J = \left\{ s\frac{z}{\eta} + v : \ \xi \le s \le \xi + c\Big(1 \wedge \frac{1}{\xi}\Big) \right\}.$$

Now we shall estimate $I(v)$. Let us first assume that $w \in I$ and $|w - z| \ge \beta(|w| \vee |z|)$. We have

$$\overline{K}_3(w,z) = \overline{K}_3(y,z) \le \overline{C}\, e^{\xi^2 - \eta^2}.$$

If Q_i intersects part of the circles

$$|y - z| \le \beta\eta \qquad \text{or} \qquad |y - z| \le \frac{\beta}{1-\beta}\eta,$$

then one can get the same estimates, *i.e.* $\overline{C}\, e^{\xi^2-\eta^2}$, by applying lemma 2, part *c*.1), with $|y - z| > \delta\eta$ and $\delta < \beta$.

Hence

$$I(v) \le C\left(1 \wedge \frac{1}{\xi}\right) e^{\xi^2-\eta^2} e^{-\xi^2} \le e^{-|z|^2}.$$

Second, we assume $w \in I$, $|w - z| < \beta(|w| \vee |z|)$ and $|v| < B/\eta$. We have

$$\overline{K}_3(w,z) = \overline{K}_3(y,z) \le C\,\eta^n \left(1 \wedge e^{\xi^2-\eta^2}\right).$$

Consequently

$$I(v) \le \frac{c}{\xi}\eta^n \left(1 \wedge e^{\xi^2-\eta^2}\right) e^{-\xi^2} \le c\,\eta^{n-1} e^{-\eta^2},$$

since $\xi > \eta$ (see proof of Lemma 2, part *c*.1)).

Third, we assume $w \in I$, $|w - z| < \beta(|w| \vee |z|)$ and $|v| \ge B/\eta$. Then

$$\overline{K}_3(w,z) = \overline{K}_3(y,z) \le \frac{C}{|v|^n}\left(1 \wedge e^{\xi^2-\eta^2}\right).$$

Consequently

$$I(v) \le \frac{c}{\xi}\,\frac{1}{|v|^n}\left(1 \wedge e^{\xi^2-\eta^2}\right) e^{-\xi^2} \le c\,\frac{c}{\eta}\,\frac{1}{|v|^n}\, e^{-\eta^2},$$

since $\xi > \eta$ (see proof of Lemma 2, part $c.3$)).

Therefore, in the first case above we have

$$U(z) \le c\left(\int_{\mathbb{R}^{n-1}} e^{-|v|^2}\, dv\right) e^{-|z|^2},$$

and in the second and third cases above we obtain

$$\begin{aligned} U(z) &= \int\limits_{|v|_{n-1}<B/\eta} e^{-|v|^2}\, I(v)\, dv + \int\limits_{|v|_{n-1}\ge B/\eta} e^{-|v|^2}\, I(v)\, dv \\ &\le c_n\left(\eta^{n-1} \int\limits_{|v|_{n-1}<B/\eta} dv + \frac{1}{\eta} \int\limits_{|v|_{n-1}\ge B/\eta} \frac{1}{|v|^n}\, dv\right) e^{-|z|^2} \\ &\le c_n\, e^{-|z|^2}. \end{aligned}$$

This completes the proof of property c), therefore, Theorem 2 is complete.

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Projecteurs invariants, matrices de dilatation, ondelettes et analyses multi-résolutions

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Résumé. On montre que les bases d'ondelettes (bi-orthogonales) associées à une matrice de dilatation compatible avec les translations entières proviennent en général d'analyses multi-résolutions. La démonstration se fait à l'aide de l'étude des projecteurs qui commutent avec les translations entières.

Abstract. We show that (bi-orthogonal) wavelet bases associated to a dilation matrix which is compatible with integer shifts are generally provided by a multi-resolution analysis. The proof is done by studying the projectors which commute with integer shifts.

Introduction.

Les premières bases d'ondelettes [3], [10], [14], [17] étaient des bases orthonormées de $L^2(\mathbb{R}^n)$ $(\psi_{\varepsilon,j,k})_{1\leq\varepsilon\leq 2^n-1,\, j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ engendrées par dilatation et translation dyadiques

$$\psi_{\varepsilon,j,k}(x) = 2^{jn/2}\psi_\varepsilon(2^j x - k) \tag{1}$$

à partir d'un ensemble fini de fonctions ψ_ε.

On note W_j l'espace des ondelettes d'échelle $1/2^j$ (W_j est le sous-espace vectoriel fermé de $L^2(\mathbb{R}^n)$ engendré par les $\psi_{\varepsilon,j,k}$, $1 \leq \varepsilon \leq 2^n-1$, $k \in \mathbb{Z}^n$). W_j est alors invariant par translation par un facteur $k/2^j$ ($k \in \mathbb{Z}^n$): $f \in W_j$ si et seulement si $f(x - k/2^j) \in W_j$. On note alors V_0 l'espace des grandes échelles

$$V_0 = \overline{\bigoplus_{j<0} W_j} .$$

Comme V_0 est orthogonal à $\oplus_{j\geq 0} W_j$, on voit que V_0 est invariant par translations entières: si $k \in \mathbb{Z}^n$, $f \in V_0$ si et seulement si $f(x - k) \in V_0$. En fait, dans la plupart des bases d'ondelettes, V_0 a une base orthonormée de la forme $(\varphi(x - k))_{k\in\mathbb{Z}^n}$. On est alors dans le cadre d'une *analyse multi-résolution* au sens de S. Mallat [18], [19] et φ est appelée *fonction d'échelle* associée aux *ondelettes* ψ_ε.

Cette première génération de bases d'ondelettes a été développée dans les années 1985-88 et les principales constructions et propriétés en ont été exposées dans le livre de Y. Meyer paru en 1990, [19]. Depuis 1989, de nouvelles notions de base d'ondelettes ont été développées, essentiellement pour des raisons pratiques (filtres à phase linéaire pour les bases bi-orthogonales [11], maillage en quinconce pour les matrices de dilatation [6], bases d'ondelettes à support compact polynomiales par morceaux pour les analyses multi-résolution multiples [1]).

Les bases d'ondelettes que nous allons considérer dans cet article seront des bases bi-orthogonales

$$(\psi_{\varepsilon,j,k})_{1\leq\varepsilon\leq E,\, j\in\mathbb{Z},\, k\in\mathbb{Z}^n}\,, \qquad (\psi^*_{\varepsilon,j,k})_{1\leq\varepsilon\leq E,\, j\in\mathbb{Z},\, k\in\mathbb{Z}^n}\,,$$

engendrées par translation et par l'opération d'une matrice de dilatation A. Une *matrice de dilatation* est un opérateur linéaire A de $\mathbb{R}^n$ dans $\mathbb{R}^n$ dont toutes les valeurs propres sont de module strictement plus grand que 1. Les fonctions $\psi_{\varepsilon,j,k}$ et $\psi^*_{\varepsilon,j,k}$ sont alors définies par

$$(2) \qquad \begin{aligned} \psi_{\varepsilon,j,k}(x) &= |\det A|^{j/2}\, \psi_\varepsilon(A^j x - k)\,,\\ \psi^*_{\varepsilon,j,k}(x) &= |\det A|^{j/2}\, \psi^*_\varepsilon(A^j x - k)\,, \end{aligned}$$

et doivent définir *deux bases biorthogonales* de $L^2(\mathbb{R}^n)$, c'est-à-dire que l'on demande que les $\psi_{\varepsilon,j,k}$ et les $\psi^*_{\varepsilon,j,k}$ vérifient

(3.1) *complétude*: la famille $(\psi_{\varepsilon,j,k})_{1\leq\varepsilon\leq E,\, j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ est totale dans $L^2(\mathbb{R}^n)$

(3.2) *bi-orthogonalité*: $\langle \psi_{\varepsilon,j,k}, \psi^*_{\varepsilon',j',k'} \rangle = \delta_{\varepsilon,\varepsilon'}\, \delta_{j,j'}\, \delta_{k,k'}$ pour le produit scalaire de $L^2(\mathbb{R}^n)$

(3.3) *presque-orthogonalité*: il existe une constante C telle que pour toute suite presque nulle $(\lambda_{\varepsilon,j,k})$ on ait

$$\Big\| \sum_{\varepsilon=1}^{E} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \lambda_{\varepsilon,j,k}\, \psi_{\varepsilon,j,k} \Big\|_2^2 \le C \sum_{\varepsilon=1}^{E} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\lambda_{\varepsilon,j,k}|^2$$

$$\Big\| \sum_{\varepsilon=1}^{E} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \lambda_{\varepsilon,j,k}\, \psi^*_{\varepsilon,j,k} \Big\|_2^2 \le C \sum_{\varepsilon=1}^{E} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\lambda_{\varepsilon,j,k}|^2 .$$

L'application

$$(\lambda_{\varepsilon,j,k}) \to \sum_{\varepsilon=1}^{E} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \lambda_{\varepsilon,j,k}\, \psi_{\varepsilon,j,k}$$

est alors un isomorphisme de $\ell^2(\{1,\ldots,E\} \times \mathbb{Z} \times \mathbb{Z}^n\}$ sur $L^2(\mathbb{R}^n)$ et on a

$$\langle \sum_{\varepsilon=1}^{E} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \lambda_{\varepsilon,j,k}\, \psi_{\varepsilon,j,k} \,,\, \psi^*_{\varepsilon',j',k'} \rangle = \lambda_{\varepsilon',j',k'} \ .$$

Un cas important est lorsque la dilatation A et les translations par des éléments de $\mathbb{Z}^n$ vérifient la *relation de compatibilité*

$$A\mathbb{Z}^n \subset \mathbb{Z}^n \,. \tag{4}$$

En effet, on note à nouveau W_j l'espace engendré par les $\psi_{\varepsilon,j,k}$ et on note W^*_j celui engendré par les $\psi^*_{\varepsilon,j,k}$. De même, on note

$$V_0 = \overline{\bigoplus_{j<0} W_j} \qquad \text{et} \qquad V^*_0 = \overline{\bigoplus_{j<0} W^*_j} \,.$$

Alors V_0 et V^*_0 sont invariants par translations entières: $f \in V_0$ si et seulement si $f(x-k) \in V_0$ (lorsque $k \in \mathbb{Z}^n$) et $f \in V^*_0$ si et seulement si $f(x-k) \in V^*_0$; en effet on a

$$V_0^{\perp} = \overline{\bigoplus_{j\ge 0} W^*_j} \qquad \text{et} \qquad V_0^{*\perp} = \overline{\bigoplus_{j\ge 0} W_j} \,,$$

or W^*_j et W_j sont invariants par translation par un facteur $A^{-j}k$; comme $A^j\mathbb{Z}^n \subset \mathbb{Z}^n$ lorsque $j \ge 0$, on voit que les W_j et W^*_j sont invariants par translation entière pour $j \geq 0$, d'où V_0 et V_0^* également. Cela peut s'exprimer à l'aide de l'opérateur de sommation partielle P_0 défini par

$$P_0 f = \sum_{\varepsilon=1}^{E} \sum_{j<0} \sum_{k \in \mathbb{Z}^n} \langle f, \psi^*_{\varepsilon,j,k} \rangle \, \psi_{\varepsilon,j,k} \, . \tag{5}$$

P_0 est le projecteur de $L^2(\mathbb{R}^n)$ sur V_0 parallèlement à $(V_0^*)^\perp$ et l'invariance de V_0 et V_0^* par translations entières est équivalente à la commutation de P_0 avec les translations entières

$$P_0\big(f(x-k)\big) = \big(P_0 f\big)(x-k) \, , \tag{6}$$

pour tout $k \in \mathbb{Z}^n$ et pour tout $f \in L^2$. En général, V_0 a une base de Riesz de la forme $(\varphi_\delta(x-k)_{1 \leq \delta \leq D,\, k \in \mathbb{Z}^n}$; on parle alors d'*analyse multi-résolution de multiplicité* D et la relation entre le nombre E d'ondelettes ψ_ε et le nombre D de fonctions d'échelle φ_δ est la suivante:

$$E = (|\det A| - 1) D \, . \tag{7}$$

En particulier E est un multiple de $|\det A| - 1$.

Le but de cet article est triple:

i) on cherchera à donner des critères simples pour qu'un projecteur P_0 de $L^2(\mathbb{R}^n)$ commutant avec les translations entières vérifie que son image V_0 admet une base de Riesz de la forme $(\varphi_\delta(x-k))_{1 \leq \delta \leq D,\, k \in \mathbb{Z}^n}$;

ii) on donnera un critère pour établir la propriété de presque-orthogonalité (3.3); ce critère s'appliquera plus généralement à des *vaguelettes*. C'est-à-dire qu'on considérera une famille $\psi_{j,k}$ de fonctions de $L^2(\mathbb{R}^n)$ telles que $(\det A)^{-j/2} \psi_{j,k}(A^{-j}(x+k))$ soit dans un ensemble fixé B ; on parlera alors de B-vaguelettes et le critère portera sur l'ensemble B. L'intérêt de ce lemme est qu'un opérateur qui transforme une base d'ondelettes en une famille de B-vaguelettes est automatiquement continu sur $L^2(\mathbb{R}^n)$ si les B-vaguelettes sont presque-orthogonales (*cf.* [20, Chapitre VIII]).

iii) on donnera un critère simple pour qu'une base d'ondelettes provienne d'une analyse multi-résolution.

Les points ii) et iii) sont traités dans le cadre des matrices de dilatation et demandent une étude préalable de la géométrie attachée à cette dilatation. Cette étude est classique en analyse harmonique (elle se rattache par exemple aux groupes de Lie nilpotents de Folland et Stein [12] ou aux espaces de nature homogène de Coifman et Weiss [9]). La nouveauté des résultats que nous présentons ici ne résulte donc pas de l'utilisation des matrices de dilatation: le point iii) est déjà nouveau dans le cadre des bases d'ondelettes "traditionnelles".

Le plan de l'article est le suivant:

NOTATIONS. La transformée de Fourier $\hat{f}$ d'une fonction $f \in L^1(\mathbb{R}^n)$ est donnée par

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix\xi}\, dx\,.$$

I. Projecteurs invariants par translations entières.

On considère P_0 un projecteur de $L^2(R^n)$ invariant par translations entières, c'est-à-dire un opérateur linéaire continu de $L^2(\mathbb{R}^n)$ dans lui-même qui vérifie

i) $P_0 \circ P_0 = P_0\,,$

ii) pour tout $k \in \mathbb{Z}^n$ et pour tout $f \in L^2(\mathbb{R}^n)$,

$$P_0\big(f(x-k)\big) = \big(P_0 f\big)(x-k)\,.$$

On note $V_0 = \operatorname{Im} P_0$ et $V_0^* = (\operatorname{Ker} P_0)^{\perp}$.

On rappelle qu'une famille $(f_k)_{k\in K}$ de fonctions de $L^2(\mathbb{R}^n)$ est une *base de Riesz* d'un sous-espace fermé V de $L^2(\mathbb{R}^n)$ si elle vérifie

- $(f_k)_{k\in K}$ engendre V: les combinaisons linéaires des f_k sont denses dans V,

- il existe deux constantes strictement positives A et B telles que pour toute suite presque nulle $(\lambda_k)_{k\in K}$ on ait

$$A\sum_{k\in K}|\lambda_k|^2 \leq \Big\|\sum_{k\in K}\lambda_k f_k\Big\|_2^2 \leq B\sum_{k\in K}|\lambda_k|^2. \tag{8}$$

Si $(f_k)_{k\in K}$ est une base de Riesz de V_0, il existe une base de Riesz $(g_k)_{k\in K}$ de V_0^* telle que $\langle f_k, g_{k'}\rangle = \delta_{k,k'}$ et on a alors

$$P_0 f = \sum_{k\in K}\langle f, g_k\rangle f_k\ .$$

La base $(g_k)_{k\in K}$ est appelée la *base duale* de la base $(f_k)_{k\in K}$ dans V_0^*. Si V_0 a une base de Riesz de la forme $(\varphi_{\delta,k} = \varphi_\delta(x-k)_{1\leq\delta\leq D,\, k\in\mathbb{Z}^n}$ alors la base duale $\varphi^*_{\delta,k}$ est de même type: $\varphi^*_{\delta,k}$ est défini comme l'unique élément de V_0^* solution de

$$\{\langle\varphi^*_{\delta,k}, \varphi_{\delta',k'}\rangle = \delta_{\delta,\delta'}\delta_{k,k'} \quad \text{pour } 1\leq\delta'\leq D \text{ et } k'\in\mathbb{Z}^n\}\ ;$$

il est donc évident que

$$\varphi^*_{\delta,k}(x) = \varphi^*_{\delta,0}(x-k)\,.$$

Il n'est pas vrai en général qu'un sous-espace fermé V de $L^2(\mathbb{R}^n)$ invariant par translations entières ait une base de Riesz elle-même invariante par translations entières (c'est-à-dire de la forme $(\varphi_\delta(x-k))_{\delta\in\Delta,\, k\in\mathbb{Z}^n}$). Un contre-exemple simple est

$$V = \{f\in L^2(\mathbb{R}^n):\ \operatorname{supp}\hat f \subset [-\frac{\pi}{2}, +\frac{\pi}{2}]^n\}$$

(*cf.* Contre-exemple numéro 1).

Nous allons donner un critère simple pour que V_0 admette une telle base: le projecteur P_0 que nous allons considérer aura un noyau $p(x,y)$ localement intégrable et suffisamment décroissant loin de la diagonale.

Definition 1. *Soit P_0 un opérateur linéaire continu sur $L^2(\mathbb{R}^n)$ tel que P_0 est invariant par translations entières, et soit ω un poids symétrique sur $L^2(\mathbb{R}^n)$, c'est-à-dire une fonction localement intégrable strictement positive et telle que $\omega(x) = \omega(-x)$ p.p. Alors P_0 aura un noyau ω-localisé $p(x,y)$ si $p(x,y)$ est une fonction localement intégrable sur $\mathbb{R}^n \times \mathbb{R}^n$ telle que*

$$\langle P_0 f, g\rangle = \iint p(x,y)\, f(y)\, \bar{g}(x)\, dx\, dy\,, \tag{9}$$

pour tout $f, g \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{x\in[0,1]^n} \int_{y\in\mathbb{R}^n} \omega(x-y)|p(x,y)|^2\, dx\, dy < +\infty\,, \tag{10}$$

et

$$\int_{y\in[0,1]^n} \int_{x\in\mathbb{R}^n} \omega(x-y)|p(x,y)|^2\, dx\, dy < +\infty\,. \tag{11}$$

Le choix du domaine d'intégration dans (10) et (11) s'explique par le fait que

$$\omega(x+k-y-k)\,|p(x+k,y+k)|^2 = \omega(x-y)\,|p(x,y)|^2 \quad \text{p.p.}$$

Le noyau $p(x,y)$ est entièrement caractérisé par (9) (d'après le théorème des noyaux-distributions de L. Schwartz par exemple).

Les poids que nous allons utiliser seront les poids introduits par A. Beurling dans [4].

Definition 2. *Une fonction $\omega : \mathbb{R}^n \to \mathbb{R}$ est un poids de Beurling si elle vérifie les quatre conditions suivantes pour deux constantes C et M strictement positives:*

i) $\dfrac{1}{C} \le \omega(x) \le C\,(1+\|x\|)^M$,

ii) $\displaystyle\int_{\mathbb{R}^n} \frac{dx}{\omega(x)} < +\infty$,

iii) $\dfrac{1}{\omega} * \dfrac{1}{\omega} \le C\,\dfrac{1}{\omega}$,

iv) $\omega(x+y) \le C\,\omega(x)\,\omega(y)$.

Les principales propriétés des poids de Beurling que nous aurons à utiliser dans cet article sont décrites dans l'Annexe A.

Notre théorème principal est alors le suivant

Théorème 1. *Soit P_0 un opérateur linéaire continu sur $L^2(\mathbb{R}^n)$ tel que*

i) *P_0 est un projecteur: $P_0 \circ P_0 = P_0$,*

ii) *P_0 est invariant par translations entières:*

$$P_0\big(f(x-k)\big) = \big(P_0 f\big)(x-k),$$

pour tout $f \in L^2$ et pour tout $k \in \mathbb{Z}^n$.

On note $V_0 = \operatorname{Im} P_0$ et $V_0^* = (\operatorname{Ker} P_0)^\perp$. Soit enfin ω un poids de Beurling symétrique (i.e. $\omega(x) = \omega(-x)$). Alors

a) Si V_0 a une base de Riesz de la forme $(\varphi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$ avec les $\varphi_\delta \in L^2(\omega\, dx)$ et si V_0^* a une base de Riesz de la forme $(\psi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$ avec les $\psi_\delta \in L^2(\omega\, dx)$ alors la base duale $(\varphi_\delta^*(x-k))$ des $(\varphi_\delta(x-k))$ dans V_0^* vérifie que $\varphi_\delta^* \in L^2(\omega\, dx)$ et l'opérateur P_0 a un noyau ω-localisé $p(x,y)$.

b) Inversement, si P_0 a un noyau ω-localisé alors V_0 a une base de Riesz de la forme $(\varphi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$. Le nombre D ne dépend pas du choix de la base de Riesz, mais se calcule à l'aide du "périodisé" $\tilde{p}(x,y)$ de $p(x,y)$ par les formules suivantes

$$\tilde{p}(x,y) = \sum_{k\in\mathbb{Z}^n} p(x, y-k), \tag{12}$$

$$D = \iint_{[0,1]^n\times[0,1]^n} \tilde{p}(x,y)\,\tilde{p}(y,x)\,dy\,dx. \tag{13}$$

(On montrera en particulier que $\tilde{p}(x,y)$ est une fonction $\mathbb{Z}^n \times \mathbb{Z}^n$-périodique de carré intégrable sur $[0,1]^n \times [0,1]^n$).

c) Si $n = 1$, on peut de plus choisir la base $(\varphi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$ avec $\varphi_\delta \in L^2(\omega\, dx)$ si $p(x,y)$ est ω-localisé. Cela est faux en général pour $n \ge 2$ mais si V_0 a une base de Riesz

$$(\varphi_\delta(x-k))_{1\le\delta\le D,\ k\in\mathbb{Z}^n}$$

avec $\varphi_\delta \in L^2(\omega\, dx)$ et si $p(x,y)$ est ω-localisé alors la base duale $(\varphi_\delta^*(x-k))$ des $(\varphi_\delta(x-k))$ dans V_0^* vérifie $\varphi_\delta^* \in L^2(\omega\, dx)$.

d) En particulier, si un sous-espace V_0 de L^2 a une base de Riesz de la forme $(\varphi_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}^n}$ avec $\varphi_\delta \in L^2(\omega\, dx)$ alors il a une base orthonormée $(\psi_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}^n}$ avec $\psi_\delta \in L^2(\omega\, dx)$.

Nous traiterons le cas $n = 1$ dans la section consacrée au cas de la dimension 1. Nous donnerons également un contre-exemple dans le cas $n = 2$ dans le Contre-exemple numéro 2.

Avant de démontrer le Théorème 1, nous allons démontrer une série de lemmes sur les bases de Riesz invariantes par translations entières.

I.1. Familles de Riesz invariantes par translations entières.

Pour deux fonctions f et g de $L^2(\mathbb{R}^n)$ nous définissons la *fonction de corrélation* $C(f,g)$ par

$$C(f,g)(\xi) = \sum_{k\in\mathbb{Z}^n} \hat{f}(\xi + 2k\pi)\,\bar{\hat{g}}(\xi + 2k\pi)\,. \tag{14}$$

La fonction $C(f,f)$ est appelée *fonction d'auto-corrélation* de f; la série qui la définit converge presque-partout vers une fonction $2\pi\mathbb{Z}^n$-périodique intégrable sur $[0,2\pi]^n$ et

$$\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} C(f,f)(\xi)\,d\xi = \|f\|_2^2\,.$$

Comme

$$\sum_{k\in\mathbb{Z}^n} |\hat{f}(\xi+2k\pi)|\,|\hat{g}(\xi+2k\pi)| \leq \sqrt{C(f,f)(\xi)}\,\sqrt{C(g,g)(\xi)}$$

par Cauchy-Schwarz, on voit que $C(f,g)$ est définie presque-partout et est une fonction $2\pi\mathbb{Z}^n$-périodique intégrable sur $[0,2\pi]^n$.

Lemme 1. *La famille $(f(x-k))_{k\in\mathbb{Z}^n}$ est presque-orthogonale si et seulement si $C(f,f) \in L^\infty$.*

Le lemme est immédiat: pour $(\lambda_k)_{k\in\mathbb{Z}^n}$ une suite presque-nulle on a

$$\Big\| \sum_{k\in\mathbb{Z}^n} \lambda_k\, f(x-k)\Big\|_2^2 = \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} \Big| \sum_{k\in\mathbb{Z}^n} \lambda_k\, e^{-ik\xi}\Big|^2 C(f,f)(\xi)\, d\xi\,.$$

Demander que $(f(x-k))$ soit presque orthogonale revient à demander que $\sqrt{C(f,f)}$ soit un multiplicateur de $L^2([0,2\pi]^n)$, et donc que $C(f,f)$ soit essentiellement borné.

Une *famille de Riesz* $(f_k)_{k\in K}$ de $L^2(\mathbb{R}^n)$ est une famille (f_k) qui est une base de Riesz du sous-espace fermé qu'elle engendre dans L^2.

Lemme 2. *La famille* $(f_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$ *est une famille de Riesz si et seulement si la matrice d'autocorrélation*

$$M(\xi) = (C(f_\delta, f_{\delta'}))_{1\le\delta,\delta'\le D}$$

vérifie

i) *les coefficients* $C(f_\delta, f_{\delta'})$ *sont essentiellement bornés,*

ii) *il existe* $C>0$ *tel que* $|\det M(\xi)| \ge C$ *p.p.*

Les conditions i) et ii) reviennent à dire que $M \in M_D(L^\infty)$ et que M est inversible dans $M_D(L^\infty)$, où $M_D(L^\infty)$ est l'algèbre des matrices $D\times D$ à coefficients dans $L^\infty([0,2\pi]^n)$.

Le lemme est immédiat. En effet, si $(f_\delta(x-k))$ est la base de Riesz d'un espace W et si $(f^*_\delta(x-k))$ est la base duale de $f_\delta(x-k))$ dans W, on a

$$f^*_\delta = \sum_k \sum_{\delta'} \langle f^*_\delta, f^*_{\delta'}(x-k)\rangle f_{\delta'}(x-k)\,,$$

d'où

$$\begin{aligned}\widehat{f^*_\delta}(\xi) &= \sum_{\delta'} \Big(\sum_k \langle f^*_\delta, f^*_{\delta'}(x-k)\rangle e^{-ik\xi}\Big) \widehat{f_{\delta'}}(\xi)\\ &= \sum_{\delta'} C(f^*_\delta, f^*_{\delta'})(\xi)\, \hat f_{\delta'}(\xi)\end{aligned}$$

d'où

$$C(f^*_\delta, f_{\delta'}) = \sum_{\delta''} C(f^*_\delta, f^*_{\delta''})\, C(f_{\delta''}, f_\delta)\,.$$

Comme

$$C(f^*_\delta, f_{\delta'}) = \sum_k \langle f^*_\delta, f_{\delta'}(x-k)\rangle e^{-ik\xi} = 1\,,$$

on voit que $M(\xi)$ a pour inverse la matrice d'auto-corrélation M^* des f^*_δ. Comme les f_δ et les f^*_δ engendrent des familles presque-orthogonales, leurs matrices d'auto-corrélation sont à coefficients dans L^∞ et en particulier

$$\det M(\xi) \geq \frac{1}{\|\det M^*\|_\infty}\,.$$

Inversement, si $M(\xi)$ vérifie $M \in M_D(L^\infty)$ et $\inf \operatorname{ess} \det M > 0$, alors presque partout $M(\xi)$ est une matrice hermitienne définie positive et donc

$$(\lambda_1, \ldots, \lambda_d)\, M(\xi) \begin{pmatrix} \bar\lambda_1 \\ \vdots \\ \bar\lambda_1 \end{pmatrix} \geq \gamma(\xi) \sum |\lambda_i|^2$$

pour $\gamma(\xi)$ la plus petite valeur propre de $M(\xi)$. Or on a

$$\gamma(\xi) \geq \frac{\det M(\xi)}{C(\xi)^{D-1}}$$

où $C(\xi)$ est la plus grande valeur propre de $M(\xi)$; on a

$$C(\xi) \leq D^{1/2} \Big(\sum_\delta C(f_\delta, f_\delta)(\xi)\Big)^{1/2}$$

$\gamma(\xi)$ est donc minoré indépendamment de ξ par une constante $\gamma > 0$ et on a, en posant $v_\delta = \sum_k \lambda_{k,\delta}\, e^{-ik\xi}$,

$$\begin{aligned} \Big\| \sum_{k\in\mathbb{Z}^n} \sum_\delta \lambda_{k,\delta}\, f_\delta(x-k)\Big\|_2^2 &= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} (v_1, \ldots, v_D)\, M \begin{pmatrix} \bar v_1 \\ \vdots \\ \bar v_1 \end{pmatrix} d\xi \\ &\geq \frac{1}{(2\pi)^n}\gamma \int_{[0,2\pi]^n} \sum_\delta |v_\delta(\xi)|^2\, d\xi \\ &= \frac{1}{(2\pi)^n}\, \gamma \sum_\delta \sum_k |\lambda_{k,\delta}|^2\,. \end{aligned}$$

La famille $(f_\delta(x-k))$ est donc bien une famille de Riesz.

Pour deux familles de fonctions $(f_\delta)_{1\le\delta\le D}$ et $(g_\varepsilon)_{1\le\varepsilon\le E}$, nous définissons la *matrice de corrélation* $M((f_\delta),(g_\varepsilon))$ comme la matrice

$$(C(f_\delta, g_\varepsilon))_{1\le\delta\le D,\,1\le\varepsilon\le E}\,.$$

Enfin deux sous-espaces fermés V et W de L^2 seront dit *en dualité* si on a $L^2 = V \oplus W^\perp$; cela revient à demander qu'il existe un projecteur continu de L^2 sur V parallèlement à $W^\perp$.

Lemme 3. *Soit V et W deux sous-espaces fermés de $L^2(\mathbb{R}^n)$ tels que V a une base de Riesz $(f_\delta(x-k))_{1\le\delta\le D}$ et W une base de Riesz $(g_\varepsilon(x-k)_{1\le\varepsilon\le E}$. Soient par ailleurs $(\varphi_\eta)_{1\le\eta\le H}$ une famille de fonctions de V et $(\psi_\theta)_{1\le\theta\le T}$ une famille de fonctions de W. Alors*

a) $M((\varphi_\eta),(\psi_\theta)) = M((\varphi_\eta),(f_\delta^*))M((f_\delta),(g_\varepsilon))M((g_\varepsilon^*),(\psi_\theta))$ *où* $(f_\delta^*(x-k))$ *est la base duale des* $(f_\delta(x-k))$ *dans V et* $(g_\varepsilon^*(x-k))$ *la base duale des* $(g_\varepsilon(x-k))$ *dans W.*

b) $(\varphi_\eta(x-k))_{1\le\eta\le H,\,k\in\mathbb{Z}^n}$ *une base de Riesz de V si et seulement si $D = H$,* $M((\varphi_\eta),(f_\delta^*)) \in M_D(L^\infty)$ *et est inversible dans* $M_D(L^\infty)$.

c) Si $N(\xi) = (N_{\delta,\delta'})$ *est définie par* $N(\xi) = (M(f_\delta),(f_\delta)))^{-1/2}$ *et si*

$$\hat{\varphi}_\delta = \sum_{\delta'} N_{\delta,\delta'}(\xi)\,\hat{f}_{\delta'}(\xi)\,,$$

alors les $(\varphi_\delta(x-k))_{k\in\mathbb{Z}^n}$ *forment une base orthonormée de V, ("Orthonormalisation de Gram").*

d) V et W sont en dualité si et seulement si $D = E$ et $M(f_\delta),(g_\varepsilon))$ *est inversible dans* $M_D(L^\infty)$. *De plus la base duale* $(\gamma_\delta^*(x-k))$ *des* $(f_\delta(x-k))$ *dans W se calcule par* $\hat{\gamma}_\delta^* = \sum_\varepsilon N_{\delta,\varepsilon}(\xi)\,\hat{g}_\varepsilon$ *où la matrice N vérifie* $M((f_\delta),(g_\varepsilon))^t\bar{N} = Id$, *ou encore* $N = M((g_\varepsilon),(f_\delta))^{-1}$.

Le lemme est immédiat. Le point *a*) provient des identités

$$\hat{\varphi}_\eta = \sum_\delta C(\varphi_\eta, f_\delta^*)\,\hat{f}_\delta$$

et

$$\hat{\psi}_\theta = \sum_\varepsilon C(\psi_\theta, g_\varepsilon^*)\,\hat{g}_\varepsilon = \sum_\varepsilon \overline{C(g_\varepsilon^*, \psi_\theta)}\,\hat{g}_\varepsilon\,.$$

Si les $(\varphi_\eta(x-k))$ forment une base de Riesz de V, le rang de $M((\varphi_\eta),(\varphi_\eta))$ est H p.p.; or d'après la formule *a*) il est $\le D$ p.p.

puisqu'on peut factoriser $M((f_\delta),(f_\delta))$. On obtient alors $H = D$. De plus le calcul du déterminant de $M((\varphi_\eta),(\varphi_\eta))$ donne

$$\det M((\varphi_\eta),(\varphi_\eta)) = \det M((f_\delta),(f_\delta))\,|\det M((\varphi_\eta),(f_\delta^*))|^2\,,$$

ce qui prouve que $|\det M((\varphi_\eta),(f_\delta^*))|$ se minore p.p. par une constante > 0. $M((\varphi_\eta),(f_\delta^*))$ s'inverse donc dans $M_D(L^\infty)$. La réciproque est immédiate par le Lemme 2 et *b*) est démontré.

Pour vérifier *c*), on remarque qu'il est immédiat que les $(\varphi_\delta(x-k))$ forment une base de Riesz de V puisque $M((\varphi_\delta),(f_\delta^*)) = N(\xi)$ et donc

$$\det M((\varphi_\delta),(f_\delta^*)) = \frac{1}{\sqrt{\det M((f_\delta),(f_\delta))}}\,.$$

Que $N(\xi)$ soit à coefficients L^∞ est évident: on a

$$N(\xi) = \frac{2}{\pi}\int_0^{+\infty} (I + t^2 M(\xi))^{-1}\,dt\,.$$

Comme $M(\xi) \in M_D(L^\infty)$ et que

$$(1 + t^2\lambda(\xi))^D \le \det I + t^2 M(\xi) \le (1 + t^2\Lambda(\xi))^D$$

où $\lambda(\xi)$ et $\Lambda(\xi)$ sont les plus petites et plus grandes valeurs propres de $M(\xi)$, on obtient immédiatement

$$\|(I + t^2 M(\xi))^{-1}\|_{M_D(L^\infty)} \le C\,\frac{1}{1+t^2}$$

et donc $N(\xi) \in M_D(L^\infty)$. Pour vérifier que la famille $(\varphi_\delta(x-k))$ est orthonormée, il faut et il suffit de vérifier que $M((\varphi_\delta),(\varphi_\delta)) = I_D$; or par *a*)

$$M((\varphi_\delta),(\varphi_\delta)) = N(\xi)\,M((f_\delta),(f_\delta))\,N(\xi) = I\,.$$

Le point *c*) est donc démontré.

Enfin *d*) est évident car V et W sont en dualité si et seulement si W a une base de Riesz $(\gamma_\delta^*(x-k))_{1\le\delta\le D,\,k\in\mathbb{Z}^n}$ telle que $M((f_\delta),(\gamma_\delta^*)) = I_D$.

I.2. Projecteurs ω-localisés.

Nous pouvons maintenant passer à la démonstration du Théorème 1.

a) Cas où V_0 et V_0^* ont des bases de Riesz invariants par translations entières et dans $L^2(\omega\, dx)$.

Si V_0 a une base de Riesz $(\varphi_\delta(x-k))_{1\leq\xi\leq D,\, k\in\mathbb{Z}^n}$ et V_0^* une base de Riesz $(\psi_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}^n}$ avec φ_δ et ψ_δ dans $L^2(\omega\, dx)$, il est immédiat que la base duale $(\varphi_\delta^*(x-k))$ des $(\varphi_\delta(x-k))$ dans V_0^* vérifie également que $\varphi_\delta^* \in L^2(\omega\, dx)$: il suffit de remarquer que la matrice de corrélation $M((\varphi_\delta),(\psi_\delta))$ s'inverse dans $M_D(K_\omega)$ d'après le lemme de Wiener (démontré dans l'Annexe A.3).

Maintenant P_0 se calcule par

$$P_0 f = \sum_{k\in\mathbb{Z}^n} \sum_{\delta=1}^{D} \langle f, \varphi_\delta^*(x-k)\rangle\, \varphi_\delta(x-k) \tag{15}$$

et il suffit de vérifier que lorsque φ et φ^* sont dans $L^2(\omega\, dx)$ la fonction

$$q(x,y) = \sum_{k\in\mathbb{Z}^n} |\varphi^*(y-k)|\, |\varphi(x-k)|$$

est localement intégrable dans $\mathbb{R}^n \times \mathbb{R}^n$ et appartient à

$$L^2([0,1]^n \times \mathbb{R}^n, \omega(x-y)\, dx\, dy)\,.$$

Le noyau $p(x,y)$ de P_0 se calcule alors comme

$$\sum_{k\in\mathbb{Z}^n} \sum_{\delta=1}^{D} \varphi_\delta^*(y-k)\, \varphi_\delta(x-k)$$

et sera ω-localisé. Or l'estimation sur $q(x,y)$ est immédiate

$$\iint_{\substack{x\in[0,1]^n\\ y\in\mathbb{R}^n}} \omega(x-y) \Big| \sum_k |\varphi^*(y-k)|\, |\varphi(x-k)| \Big|^2 dx\, dy$$

$$\leq \iint_{\substack{x\in[0,1]^n\\ y\in\mathbb{R}^n}} \Big(\sum_k |\varphi^*(y-k)|^2 \omega(y-k)\, |\varphi(x-k)|^2 \omega(x-k) \Big) \cdot \Big(\omega(x-y) \sum_k \frac{1}{\omega(x-k)} \frac{1}{\omega(y-k)} \Big)\, dx\, dy$$

$$\leq C \iint_{\substack{x\in[0,1]^n\\ y\in\mathbb{R}^n}} \sum_k |\varphi^*(y-k)|^2 \omega(y-k)\, |\varphi(x-k)|^2 \omega(x-k)\, dx\, dy$$

$$= C \int |\varphi^*(y)|^2 \omega(y)\, dy \int |\varphi(x)|^2 \omega(x)\, dx\,.$$

Le point i) du Théorème 1 est donc démontré.

b) Cas où P_0 a un noyau ω-localisé.

On commence par remarquer que la fonction $\tilde{p}(x,y)$ définie par

$$\tilde{p}(x,y) = \sum_{k\in\mathbb{Z}^n} p(x,y-k)$$

(qui est $\mathbb{Z}^n \times \mathbb{Z}^n$ périodique: cela est évident en y; pour la variable x il suffit de remarquer que $p(x+k,y) = p(x,y-k)$ par invariance de P_0) est de carré intégrable sur $L^2([0,1]^n \times [0,1]^n)$. Cela est immédiat puisque

$$\begin{aligned}
&\iint_{[0,1]^n\times[0,1]^n} \Big|\sum_{k\in\mathbb{Z}^n} |p(x,y-k)|\Big|^2 dx\,dy \\
&\qquad \le \iint_{[0,1]^n\times[0,1]^n} \sum_{k\in\mathbb{Z}^n} |p(x,y-k)|^2\omega(x-y+k) \\
&\qquad\qquad\qquad \cdot \sum_{k\in\mathbb{Z}^n} \frac{1}{\omega(x-y+k)}\, dx\,dy \\
&\qquad \le C \iint_{[0,1]^n\times[0,1]^n} \sum_{k\in\mathbb{Z}^n} |p(x,y-k)|^2\omega(x-y+k)|\, dx\,dy \\
&\qquad = C \int_{[0,1]^n}\int_{\mathbb{R}^n} |p(x,y)|^2\omega(x-y)\, dx\,dy < +\infty
\end{aligned}$$

par hypothèse. En particulier, on peut définir un opérateur continu $\tilde{P}$ sur $L^2([0,1]^n)$ par

$$\tilde{P}f = \int_0^1 \tilde{p}(x,y)\, f(y)\, dy\,. \tag{16}$$

Lemme 4.

i) *Si* $f \in L^2(\omega\, dx)$ *alors* $\tilde{f} = \sum_{k\in\mathbb{Z}^n} f(x-k) \in L^2([0,1]^n)$.

ii) *Si* $f \in L^2(\omega\, dx)$ *alors* $P_0 f \in L^2(\omega\, dx)$.

iii) *Si* $f \in L^2([0,1]^n)$*on a* $(P_0 f)^\sim = \tilde{P}(f)$.

iv) $\tilde{P}$ *est un projecteur de* $L^2([0,1]^n)$ *sur*

$$\tilde{V}_0 = \{\tilde{f} :\ f \in V_0 \cap L^2(\omega\, dx)\}\,.$$

Le lemme est facile à démontrer. En effet

$$\begin{aligned}
\int_{[0,1]^n} &| \sum_{k\in\mathbb{Z}^n} f(x-k)|^2\,dx \\
&\le \int_{[0,1]^n} \sum_{k\in\mathbb{Z}^n} |f(x-k)|^2\omega(x-k) \sum_{k\in\mathbb{Z}^n} \frac{1}{\omega(x-k)}\,dx \\
&\le C \sum_{k\in\mathbb{Z}^n} \frac{1}{\omega(k)} \|f\|^2_{L^2(\omega\,dx)}\,,
\end{aligned}$$

ce qui prouve i).

Pour vérifier le point ii), on écrit

$$\begin{aligned}
\int \omega(x) &\Big| \int_p (x,y)\, f(y)\,dy \Big|^2 dx \\
&\le \int \omega(x) \Big(\sum_k \Big(\int_{y\in k+[0,1]^n} |p(x,y)|^2 dy \Big)^{1/2} \\
&\qquad \cdot \Big(\int_{y\in k+[0,1]^n} |f(y)|^2\,dy \Big)^{1/2} \Big)^2 dx \\
&\le \int \Big(\sum_k \int_{y\in k+[0,1]^n} |p(x,y)|^2 \omega(x-k) \\
&\qquad \cdot \int_{y\in k+[0,1]^n} |f(y)|^2 dy\, \omega(k) \Big) \Big(\sum_k \frac{\omega(x)}{\omega(k)\,\omega(x-k)} \Big) dx \\
&\le C \sum_k \int_{y\in k+[0,1]^n} |f(y)|^2 dy\, \omega(k) \\
&\qquad \cdot \int_{x\in\mathbb{R}^n} \int_{y\in k+[0,1]^n} |p(x-k,y-k)|^2 \omega(x-k)\,dy\,dx \\
&\le C' \int_{x\in\mathbb{R}^n} \int_{y\in[0,1]^n} |p(x,y)|^2 \omega(x-y)\,dy\,dx \\
&\qquad \cdot \int |f(y)|^2 \omega(y)\,dy < +\infty\,.
\end{aligned}$$

Pour vérifier le point iii), on remarque que $L^2([0,1]^n)$ se plonge dans $L^2(\omega\,dx)$ en prolongeant les fonctions de $L^2([0,1]^n)$ par 0 en dehors de $[0,1]^n$; si $f \in L^2([0,1]^n)$ on a alors $P_0 f \in L^2(\omega\,dx)$ et $(P_0 f)^\sim \in$

$L^2([0,1]^n)$. De plus, on a presque partout

$$\begin{aligned}\tilde{P}(f)(x) &= \int_{[0,1]^n} \tilde{p}(x,y)\, f(y)\, dy = \sum_k \int_{[0,1]^n} p(x-k,y)\, f(y)\, dy \\ &= \sum_k \int_{\mathbb{R}^n} p(x-k,y)\, f(y)\, dy = \sum_k P_0 f(x-k)\,,\end{aligned}$$

ce qui prouve iii).

Le point iv) est alors évident: si $f \in L^2([0,1]^n)$ $\tilde{P}(f) = (P_0 f)^\sim \in \tilde{V}_0$ et il suffit de vérifier que si $f \in V_0 \cap L^2(\omega\, dx)$ alors $\tilde{P}(\tilde{f}) = \tilde{f}$. Or

$$\begin{aligned}\tilde{P}(\tilde{f})(x) &= \int_{[0,1]^n} \sum_k p(x, y-k) \sum_p f(y-p)\, dy \\ &= \int_{[0,1]^n} \sum_k p(x+k, y) \sum_p f(y-p)\, dy \\ &= \sum_{p\in\mathbb{Z}^n} \int_{[0,1]^n} \sum_k p(x' + k - p, y-p)\, f(y-p)\, dy \\ &= \sum_{p\in\mathbb{Z}^n} \int_{[0,1]^n} \sum_k p(x+k, y-p)\, f(y-p)\, dy \\ &= \sum_k \int_{\mathbb{R}^n} p(x+k,y)\, f(y)\, dy \\ &= \sum_k (P_0 f)(x+k) \\ &= \tilde{f}(x)\,.\end{aligned}$$

Lemme 5. *$\tilde{V}_0$ est de dimension finie D où D est donnée par* (13).

En effet, comme $\tilde{p}(x,y) \in L^2([0,1]^n \times [0,1]^n)$, l'opérateur $\tilde{P}$ est un opérateur de Hilbert-Schmidt et donc compact. La boule-unité $\tilde{B}$ de $\tilde{V}_0$ est bornée, donc $\tilde{P}(\tilde{B}) = \tilde{B}$ est relativement compacte; cela implique que $\dim \tilde{V}_0 < +\infty$. Si $(f_\delta)_{1\le\delta\le D}$ est une base de $\tilde{V}_0$ et (f_δ^*) la base duale de (f_δ) dans $(\operatorname{Ker}\tilde{P})^\perp$, alors on a

$$\tilde{p}(x,y) = \sum_{\delta=1}^{D} \bar{f}_\delta^*(y)\, f_\delta(x) \quad \text{p.p.} \tag{17}$$

et donc

$$\iint_{[0,1]^n \times [0,1]^n} \tilde{p}(x,y)\,\tilde{p}(y,x)\,dx\,dy = \sum_{\delta=1}^{D} \sum_{\delta'=1}^{D} \langle f_\delta, f_{\delta'} \rangle^2$$
$$= \sum_{\delta} \sum_{\delta'} \delta_{\delta,\delta'} = D\,.$$

Le Lemme 5 est donc démontré.

Bien entendu le Lemme 5 est insuffisant pour conclure que V_0 a une base de Riesz de la forme $(\varphi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$. Cela peut être éclairé par la remarque suivante:

Lemme 6. *Soient $f_1, \ldots, f_D \in L^2(\omega\,dx)$. Alors la matrice de Gram des vecteurs $(\tilde{f}_\delta)_{1\le\delta\le D}$ est la matrice d'auto-corrélation des f_δ $M((f_\delta),(f_\delta))$ en $\xi = 0$.*

En effet, si on calcule le produit scalaire de $\tilde{f}_\delta$ et de $\tilde{f}_{\delta'}$ on obtient

$$\int_{[0,1]^n} \tilde{f}_\delta(x)\,\bar{\tilde{f}}_{\delta'}(x)\,dx = \int_{[0,1]^n} \Big(\sum_k f_\delta(x-k)\Big)\bar{\tilde{f}}_{\delta'}(x)\,dx$$
$$= \int_{\mathbb{R}^n} f_\delta(x)\,\bar{\tilde{f}}_{\delta'}(x)\,dx$$
$$= \int_{\mathbb{R}^n} f_\delta(x) \sum_k \bar{f}_{\delta'}(x-k)\,dx$$
$$= \sum_{k\in\mathbb{Z}^n} \langle f_\delta, f_{\delta'}(x-k)\rangle = C(f_\delta, f_{\delta'})|_{\xi=0}\,.$$

Ne considérer que $\tilde{V}_0$ ne renseigne donc sur les matrices de corrélation des éléments de V_0 qu'en 0, alors qu'on a besoin d'un renseignement sur tout $[0,2\pi]^n$. On va donc faire "varier" $\tilde{V}_0$. Plus précisément on note V_ξ l'espace $V_\xi = \{e^{-ix\xi} f \,:\, f \in V_0\}$, $V_\xi^* = \{e^{-ix\xi} f \,:\, f \in V_0^*\}$, P_ξ le projecteur sur V_ξ parallèlement à $(V_\xi^*)^\perp$, p_ξ son noyau (qui est ω-localisé puisque $p_\xi(x,y) = p(x,y)e^{i\xi(y-x)}$) et enfin $\tilde{V}_\xi = \{\tilde{f} \,:\, f \in V_\xi \cap L^2(\omega\,dx)\}$.

Lemme 7. *La dimension de $\tilde{V}_\xi$ vaut D pour tout ξ.*

En effet P_ξ est un projecteur invariant par translations entières à noyau ω-localisé. La dimension de $\tilde{V}_\xi$ se calcule alors comme

$$\dim \tilde{V}_\xi = \iint\limits_{[0,1]^n\times[0,1]^n} \tilde{p}_\xi(x,y)\,\tilde{p}_\xi(y,x)\,dx\,dy\,.$$

Par convergence dominée, on vérifie immédiatement que cette intégrale est une fonction continue du paramètre ξ. Comme elle prend des valeurs entières, elle est constante.

Nous pouvons maintenant montrer que V_0 a une base de Riesz de la forme $(\varphi_\delta(x-k))_{1\le\delta\le D,\ k\in\mathbb{Z}^n}$. En effet, nous savons que pour chaque $\xi_0 \in [0,2\pi]^n$, $\dim \tilde{V}_{\xi_0} = D$; il existe donc D fonctions $f_1^{\xi_0},\dots,f_D^{\xi_0}$ dans $V_0 \cap L^2(\omega\,dx)$ telles que $((e^{-ix\xi_0} f_\delta^{\xi_0})^\sim)_{1\le\delta\le D}$ soit une base de $\tilde{V}_\xi$; or la matrice de Gram des $(e^{-ix\xi_0} f_\delta^{\xi_0})^\sim$ n'est autre que la matrice d'autocorrélation des $(f_\delta^{\xi_0})$ en $\xi=\xi_0$. Cette matrice est à coefficients dans $A(\mathbb{T}^n)$; si son déterminant est non nul en $\xi=\xi_0$, il reste non nul et minoré par une constante $\gamma(\xi_0)>0$ et les coefficients de la matrice restent bornés par $1/\gamma(\xi_0)$ sur une boule $B(\xi_0,r(\xi_0))$. Comme $[0,2\pi]^n$ est compact, on peut extraire une famille finie $B(\xi_\alpha,r(\xi_\alpha))_{1\le\alpha\le A}$ qui recouvre $[0,2\pi]^n$. On note

$$B_\alpha = B(\xi_\alpha,r(\xi_\alpha))\cap[0,2\pi]^n\,,$$

$$C_\alpha = B_\alpha \setminus \bigcup_{\beta<\alpha} B_\beta$$

et

$$D_\alpha = \bigcup_{k\in 2\pi\mathbb{Z}^n} C_\alpha + 2k\pi\,.$$

Nous allons montrer que la famille $(\varphi_\delta)_{1\le\delta\le D}$ définie par

$$\hat{\varphi}_\delta(\xi) = \sum_{\alpha=1}^{A} \hat{f}_\delta^{\xi_\alpha}\chi_{D_\alpha}(\xi)$$

convient

- Si $\widehat{\varphi_{\delta,\alpha}} = \hat{f}_\delta^{\xi_\alpha}\chi_{D_\alpha}(\xi)$ alors $\varphi_{\delta,\alpha}\in V_0$ et donc $\varphi_\delta \in V_0$: d'abord $\hat{\varphi}_{\delta,\alpha}\in L^2$ car $\chi_{D_\alpha}\in L^\infty$; ensuite si $g\in (V_0)^\perp$ alors

$$C(\varphi_{\delta,\alpha},g) = \chi_{D_\alpha}(\xi)\,C(f_\delta^{\xi_\alpha},g)(\xi) = 0$$

car

$$C(f_\delta^{\xi_\alpha}, g)(\xi) = \sum_{k \in \mathbb{Z}^n} \langle f_\delta^{\xi_\alpha}(x+k), g\rangle\, e^{-ik\xi} \quad \text{p.p.}$$

et g est orthogonal à tous les $f_\delta^{\xi_\alpha}(x+k)$. Comme

$$C(\varphi_{\delta,\alpha}, g) = \sum_{k \in \mathbb{Z}^n} \langle \varphi_{\delta,\alpha}(x+k), g\rangle\, e^{-ik\xi} \quad \text{p.p.},$$

on conclut en particulier que $\langle \varphi_{\delta,\alpha}, g\rangle = 0$, et donc $\varphi_{\delta,\alpha} \in V_0$.

$$(\bullet) \qquad \begin{aligned} C(\varphi_\delta, \varphi_{\delta'}) &= \sum_\alpha \sum_\beta \chi_{D_\alpha}\, \chi_{D_\beta}\, C(f_\delta^{\xi_\alpha}, f_{\delta'}^{\xi_\beta}) \\ &= \sum_\alpha \chi_{D_\alpha}\, C(f_\delta^{\xi_\alpha}, f_{\delta'}^{\xi_\alpha}) . \end{aligned}$$

On en conclut que les coefficients de la matrice d'auto-corrélation des (φ_δ) restent majorés par $\max_{1\le\alpha\le A} 1/\gamma(\xi_\alpha)$ et que son déterminant reste minoré par $\inf_{1\le\alpha\le A}\gamma(\xi_\alpha)$. La famille $(\varphi_\delta(x-k))_{1\le\delta\le D,\ k\in\mathbb{Z}^n}$ est donc une famille de Riesz de V_0 .

• Il reste à vérifier que les $\varphi_\delta(x-k)$ engendrent tout V_0. Or si $f \in V_0 \cap L^2(\omega\, dx)$, $(e^{-i\xi x} f)^\sim$ s'exprime comme une combinaison linéaire des $(e^{-i\xi x} f_\delta^{\xi_\alpha})^\sim$ si $\xi \in C_\alpha$ avec les coefficients bornés: ces coefficients $r_{\delta,\alpha}(\xi)$ sont solution de

$$M\left((f_\delta^{\xi_\alpha}), (f_\delta^{\xi_\alpha})\right)\Big|_\xi \begin{pmatrix} \bar r_{1,\alpha}(\xi) \\ \vdots \\ \bar r_{D,\alpha}(\xi) \end{pmatrix} = \begin{pmatrix} C(f_1^{\xi_\alpha}, f) \\ \vdots \\ C(f_D^{\xi_\alpha}, f) \end{pmatrix} .$$

Si

$$R_{\delta,\alpha}(\xi) = \sum_{k \in 2\pi\mathbb{Z}^n} r_{\delta,\alpha}(\xi + 2k\pi) ,$$

alors on a

$$\hat f \chi_{D_\alpha} = \sum_\delta R_{\delta,\alpha}(\xi)\, \chi_{D_\alpha}(\xi)\, \hat f_\delta^{\xi_\alpha} ,$$

et en fin de compte on obtient

$$\hat f = \sum_\delta \left(\sum_\alpha R_{\delta,\alpha}(\xi)\, \chi_{D_\alpha}(\xi) \right) \hat\varphi_\delta ,$$

ce qui prouve que f se décompose sur les $\varphi_\delta(x-k)$. Comme $V_0 \cap L^2(\omega\, dx)$ est dense dans V_0 (puisque $V_0 \cap L^2(\omega\, dx) = P_0(L^2(\omega\, dx))$ et que $L^2(\omega\, dx)$ est dense dans L^2), les $(\varphi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$ forment une base de Riesz de V_0.

Le point ii) du Théorème 1 est donc démontré.

c) Fin de la démonstration. Rappelons que nous renvoyons à des sections ultérieures la démonstration du cas $n=1$ et le contre-exemple. Le point *d*) est démontré dans l'Annexe A.3. De même, le Lemme 16 nous apprend que si V_0 a une base de Riesz $(\varphi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$ avec $\varphi_\delta \in L^2(\omega\, dx)$ alors la base duale $(\psi_\delta(x-k))$ des $(\varphi_\delta(x-k))$ dans V_0 vérifie également $\psi_\delta \in L^2(\omega\, dx)$. Or la base duale $(\varphi_\delta^*(x-k))$ des $(\varphi_\delta(x-k))$ dans V_0^* se calcule par la formule $\varphi_\delta^* = P_0^*\psi_\delta$,

$$\langle \varphi_\delta^*, \varphi_{\delta'}(x-k)\rangle = \langle P_0^*\psi_\delta, \varphi_{\delta'}(x-k)\rangle = \langle \psi_\delta, P_0(\varphi_{\delta'}(x-k)\rangle = \delta_{\delta,\delta'}\delta_{k,0}\,.$$

Si P_0 a un noyau ω-localisé, il en va de même pour P_0^* (qui a pour noyau $q(x,y) = \bar{p}(y,x)$) et donc $P_0^*(\psi_\delta) \in L^2(\omega\, dx)$ d'après le Lemme 4.

II. Le lemme des vaguelettes.

Nous allons donner ici un lemme des vaguelettes associé à une matrice de dilatation sur $\mathbb{R}^n$. La formulation est déjà nouvelle dans le cas classique des dilatations dyadiques.

Proposition 1. *Soit* $(f_{j,k})_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ *une famille de fonctions dans* $L^2(\mathbb{R}^n)$ *telles que, pour un* $\varepsilon > 0$ *et un* $\alpha > 0$, *on ait*

i) $f_{j,k} \in L^2((1+\|x\|)^{n+\varepsilon}dx)$ *et*

$$\int |f_{j,k}(x)|^2\,(1+\|x\|)^{n+\varepsilon}\,dx \le 1\,.$$

ii) $f_{j,k} \in H^\alpha$ *(espace de Sobolev) et*

$$\int (1+|\xi|^2)^\alpha\,|\hat{f}_{j,k}(\xi)|^2\,d\xi \le 1\,.$$

iii) $\displaystyle\int f_{j,k}\,dx = 0\,.$

Alors la famille $(\psi_{j,k})_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ *définie par*

$$\psi_{j,k}(x) = 2^{jn/2}\, f_{j,k}(2^j x - k)$$

est presque-orthogonale dans $L^2(\mathbb{R}^n)$. *Plus précisément, il existe une constante* $C(\varepsilon,\alpha)$ *ne dépendant que* ε *et de* α *(mais pas des* $f_{j,k}$*) telle que pour toute suite presque nulle* $(\lambda_{j,k})$ *on ait*

$$\Big\| \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \lambda_{j,k}\, \psi_{j,k} \Big\|_2 \le C(\varepsilon,\alpha) \Big(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\lambda_{j,k}|^2 \Big)^{1/2}. \tag{18}$$

Le lemme des vaguelettes "traditionnel" impose les conditions plus fortes aux $f_{j,k}$

$$\|(1+\|x\|)^{n+\varepsilon'} f_{j,k}\|_\infty \le 1 \qquad \text{et} \qquad \sup_{x\neq y} \frac{|f_{j,k}(x) - f_{j,k}(y)|}{|x-y|^{\alpha'}} \le 1$$

pour un ε' et un $\alpha' > 0$ (et $\int f_{j,k} dx = 0$). Il est facile de vérifier que si les $f_{j,k}$ sont de telles vaguelettes, alors il existe ε, α et $\gamma > 0$ tel que les $\gamma f_{j,k}$ vérifient i) et ii). Le lemme traditionnel ne suffit pas pour traiter le cas de bases d'ondelettes discontinues: le système de Haar par exemple, mais également des bases d'ondelettes bi-orthogonales à support compact qui peuvent être irrégulières au sens de Hölder mais sont toujours mieux que L^2 au sens Sobolev [7].

La Proposition 1 est un cas particulier du Théorème 2 que nous allons décrire ci-dessous. Nous considérerons une matrice de dilatation A au lieu de la multiplication par 2. Nous allons remplacer la localisation $f \in L^2((1+\|x\|)^{n+\varepsilon} dx)$ (où la norme $\|\cdot\|$ est homogène pour la multiplication par 2: $\|2x\| = 2\|x\|$) par une condition adaptée à l'opération de A.

Definition 3. *Soit* A *une matrice de dilatation sur* $\mathbb{R}^n$. *Une pseudo-norme sur* $(\mathbb{R}^n, A)$ *est une fonction* ρ *définie sur* $\mathbb{R}^n$ *telle que*

a) ρ *est* C^∞ *sur* $\mathbb{R}^n \setminus \{0\}$ *et continue en* 0,

b) pour tout $x \neq 0$, $\rho(x) > 0$ *et* $\rho(x) = \rho(-x)$,

c) pour tout $x \in \mathbb{R}^n$, $\rho(Ax) = |\det A|\, \rho(x)$.

Dans l'Annexe B, l'existence et les propriétés des pseudo-normes sont démontrées, en particulier les propriétés suivantes:

- *Unicité*: si ρ' est une autre pseudo-norme sur $\mathbb{R}^n$, alors on a pour deux constantes C et C' strictement positives et pour tout $x \in \mathbb{R}^n$: $C\rho(x) \leq \rho'(x) \leq C'\rho(x)$.

- *Comparaison avec* $\|\cdot\|$: il existe deux constantes α_0 et α_1 $(0 < \alpha_0 < \alpha_1)$ et une constante $M_0 > 0$ telles que

$$\frac{1}{M_0}\|x\|^{\alpha_1} \leq \rho(x) \leq M_0\|x\|^{\alpha_0} \qquad \text{si } \|x\| \leq 1\,, \tag{19}$$

$$\frac{1}{M_0}\|x\|^{\alpha_0} \leq \rho(x) \leq M_0\|x\|^{\alpha_1} \qquad \text{si } \|x\| \geq 1\,. \tag{20}$$

- *Inégalité triangulaire*: il existe une constante C_1 telle que

$$\rho(x,y) \leq C_1(\rho(x) + \rho(y))\,. \tag{21}$$

- *Croissance de la norme*: il existe une constante C_2 telle que pour tout $x \in \mathbb{R}^n$ et pour tout $\lambda \in [0,1]$,

$$\rho(\lambda x) \leq C_2\,\rho(x)\,. \tag{22}$$

La pseudo-norme $\rho(x)$ se comporte comme $\|x\|^n$. En particulier, on a

- $\displaystyle\int_{\rho(x)\leq 1} \frac{1}{\rho(x)^\varepsilon}\,dx < +\infty$ si et seulement si $\varepsilon < 1$.
- $\displaystyle\int_{\rho(x)\geq 1} \frac{1}{\rho(x)^\varepsilon}\,dx < +\infty$ si et seulement si $\varepsilon > 1$.
- Pour $\varepsilon > 0$, $\omega(x) = (1+\rho(x))^{1+\varepsilon}$ est un poids de Beurling sur $\mathbb{R}^n$.

Enfin comme la transformation de Fourier transforme l'opération de A en l'opération de $\tilde{A}^{-1}$ où $\tilde{A}$ est la transposée de A, il est naturel également d'associer à A une pseudo-norme $\tilde{\rho}$ sur $(\mathbb{R}^n, \tilde{A})$. Le Lemme des vaguelettes est alors le suivant:

Théorème 2 (Lemme des vaguelettes). *Soit A une matrice de dilatation sur $\mathbb{R}^n$, ρ une pseudo-norme sur $(\mathbb{R}^n, A)$ et $\tilde{\rho}$ une pseudo-norme sur $(\mathbb{R}^n, \tilde{A})$. Soit $(f_{j,k})_{j\in\mathbb{Z},\,k\in\mathbb{Z}^n}$ une famille de fonctions de $L^2(\mathbb{R}^n)$ telles que, pour un $\varepsilon > 0$ et un $\alpha > 0$, on ait*

i) $f_{j,k} \in L^2((1+\rho(x))^{1+\varepsilon}dx)$ *et*

$$\int |f_{j,k}(x)|^2(1+\rho(x))^{1+\varepsilon}\,dx \le 1\,,$$

ii) $\widehat{f_{j,k}} \in L^2((1+\tilde\rho(\xi))^{\alpha}d\xi)$ *et*

$$\int |\widehat{f_{j,k}}(\xi)|^2(1+\tilde\rho(\xi))^{\alpha}\,d\xi \le 1\,,$$

iii) $\int f_{j,k}\,dx = 0\,.$

Alors la famille $(\psi_{j,k})_{j\in\mathbb{Z},\,k\in\mathbb{Z}^n}$ *définie par*

$$\psi_{j,k}(x) = |\det A|^{j/2} f_{j,k}(A^j x - k)$$

est presque-orthogonale dans $L^2(\mathbb{R}^n)$*. Plus précisément, il existe une constante* $C(\varepsilon,\alpha)$ *ne dépendant que de* ε *et de* α *telle que pour toute suite presque nulle* $\lambda_{j,k}$*) on ait*

$$\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\lambda_{j,k}\,\psi_{j,k}\Big\|_2 \le C(\varepsilon,\alpha)\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}|\lambda_{j,k}|^2\Big)^{1/2}. \tag{23}$$

DÉMONSTRATION. Le principe de la démonstration est très simple. On pose

$$F_j = \sum_{k\in\mathbb{Z}^n}\lambda_{j,k}\,f_{j,k}(x-k)$$

et

$$G_j = \sum_{k\in\mathbb{Z}^n}\lambda_{j,k}\psi_{j,k}\,.$$

Alors il est clair que $\|G_j\|_2 = \|F_j\|_2$ et que $\|F_j\|_2 \le C\big(\sum_{k\in\mathbb{Z}^n}|\lambda_{j,k}|^2\big)^{1/2}$ du fait que $(1+\rho(x))^{1+\varepsilon}$ est un poids de Beurling. Le Lemme des vaguelettes est donc un lemme de presque-orthogonalité entre échelles j. En fait, on va montrer qu'il existe α' et $\varepsilon' > 0$ tels que, en notant $I_{\alpha'}$ l'opérateur d'"intégration fractionnaire" $\widehat{I_{\alpha'}f}(\xi) = \tilde\rho(\xi)^{-\alpha'}\tilde f(\xi)$ et $D_{\alpha'}$ l'opérateur de "dérivation fractionnaire" $\widehat{D_{\alpha'}f}(\xi) = \tilde\rho(\xi)^{+\alpha'}\tilde f(\xi)$, on ait

- $I_{\alpha'}f_{j,k} \in L^2((1+\rho(x))^{1+\varepsilon'})$ et $\int |I_{\alpha'}\,f_{j,k}(x)|^2(1+\rho(x))^{1+\varepsilon'}dx \le C,$

- $D_{\alpha'} f_{j,k} \in L^2((1+\rho(x))^{1+\varepsilon'})$ et $\displaystyle\int |D_{\alpha'} f_{j,k}(x)|^2 (1+\rho(x))^{1+\varepsilon'} dx \le C$,

où la constante C ne dépend que de α, ε, α' et ε' (et de n et de A). On obtiendra alors

$$\|I_{\alpha'} F_j\|_2 \le C\Big(\sum_{k\in\mathbb{Z}^n} |\lambda_{j,k}|^2\Big)^{1/2}$$

et de même

$$\|D_{\alpha'} F_j\|_2 \le C\Big(\sum_{k\in\mathbb{Z}^n} |\lambda_{j,k}|^2\Big)^{1/2},$$

du fait que

$$I_{\alpha'}\big(f_{j,k}(x-k)\big) = \big(I_{\alpha'} f_{j,k}\big)(x-k)$$

et que

$$D_{\alpha'}\big(f_{j,k}(x-k)\big) = \big(D_{\alpha'} f_{j,k}\big)(x-k)$$

et que $(1+\rho(x))^{1+\varepsilon'}$ est encore un poids de Beurling. Maintenant il suffit de constater que les opérateurs $I_{\alpha'}$ et $D_{\alpha'}$ sont homogènes par rapport à la dilatation A

$$I_{\alpha'}(f(Ax)) = |\det A|^{-\alpha'} (I_{\alpha'} f)(Ax)\,, \tag{24}$$

$$D_{\alpha'}(f(Ax)) = |\det A|^{\alpha'} (D_{\alpha'} f)(Ax)\,. \tag{25}$$

On calcule alors, si $j \ge j'$, $\langle G_j, G_{j'}\rangle$ comme $\langle G_j, G_{j'}\rangle = \langle I_{\alpha'}\, G_j, D_{\alpha'}\, G_{j'}\rangle$ et donc

$$|\langle G_j, G_{j'}\rangle| \le C\,|\det A|^{-\alpha'|j-j'|}\Big(\sum_{k\in\mathbb{Z}^n} |\lambda_{j,k}|^2\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}^n} |\lambda_{j',k}|\Big)^{1/2}.$$

Cette dernière majoration suffit à assurer (23) puisque

$$\sum_j \sum_{j'} |\det A|^{-\alpha'|j-j'|} |\lambda_j|\,|\lambda_{j'}| \le \Big(1 + 2\frac{|\det A|^{-\alpha'}}{1-|\det A|^{-\alpha'}}\Big) \sum_j |\lambda_j|^2\,.$$

La démonstration du Théorème 2 est donc ramenée à celle du lemme suivant

Lemme 8. *Pour tout ε et $\alpha > 0$ il existe ε' et $\alpha' > 0$ et une constante $C > 0$ tels que*

i) *si* $f \in L^2((1+\rho(x))^{1+\varepsilon}dx)$ *et* $\tilde{f} \in L^2((1+\tilde{\rho}(\xi))^{\alpha}d\xi)$ *alors*

$$\begin{aligned}(26)\quad &\int |D_{\alpha'} f(x)|^2 (1+\rho(x))^{1+\varepsilon'}\,dx \\ &\leq C\left(\int |f(x)|^2(1+\rho(x))^{1+\varepsilon}\,dx + \int |\hat{f}(\xi)|^2(1+\tilde{\rho}(\xi))^{\alpha}\,d\xi\right),\end{aligned}$$

ii) *si* $f \in L^2((1+\rho(x))^{1+\varepsilon}dx)$ *et* $\int f\,dx = 0$ *alors*

$$(27)\quad \int |I_{\alpha'} f(x)|^2(1+\rho(x))^{1+\varepsilon'}\,dx \leq C\int |f(x)|^2(1+\rho(x))^{1+\varepsilon}\,dx\,.$$

Pour démontrer le lemme, on introduit les espaces fonctionnels L^2_η et H_η pour $\eta \geq 0$

$$L^2_\eta = L^2((1+\rho(x))^\eta dx)$$

et

$$H_\eta = \{f \in L^2 :\ \text{il existe}\ \ g \in L^2_\eta,\ f = \hat{g}\}$$

muni des normes

$$\|f\|_{L^2_\eta} = \left(\int (1+\rho(x))^\eta |f(x)|^2\,dx\right)^{1/2}$$

et

$$\|f\|_{H_\eta} = \|g\|_{L^2_\eta} \qquad (f = \hat{g})\,.$$

Pour $\eta > 1$, H_η est une algèbre de Beurling et on note M_η l'espace de ses multiplicateurs.

Désignons par $\tilde{\sigma}$ une fonction C^∞ sur $\mathbb{R}^n$ qui coïncide avec $\tilde{\rho}$ lorsque $\tilde{\rho}(x) \geq 1$ et qui vérifie $\tilde{\sigma}(x) > 0$ pour tout $x \in \mathbb{R}^n$ (y compris 0). On note $\tilde{\sigma}_\gamma$ la fonction $\tilde{\sigma}_\gamma(x) = \tilde{\sigma}(x)^\gamma$. Si $z \in \mathbb{C}$ vérifie $\mathrm{Re}\, z = 1$ alors $\tilde{\sigma}_{\alpha z}\hat{f}$ appartient à $L^2 = H_0$ si $\hat{f} \in L^2((1+\tilde{\rho}(\xi))^\alpha d\xi)$ et

$$\|\tilde{\sigma}_{\alpha z}\hat{f}\|_{H_0} \leq C\,\|(1+\tilde{\rho}(\xi))^{\alpha/2}\hat{f}\|_2\,.$$

Si $\mathrm{Re}\, z = 0$, on vérifie facilement que $\tilde{\sigma}_{\alpha z} \in M_{1+\varepsilon}$ et que

$$\|\tilde{\sigma}_{\alpha z}\|_{M_{1+\varepsilon}} \leq C\,(1+|z|)^N\,, \qquad \text{pour un } N \in \mathbb{N}\,;$$

il sisffit de remarquer que $\tilde{\sigma}_{\alpha z}$ est bornée ainsi que toutes ses dérivées et d'appliquer le critère d'appartenance à $M_{1+\varepsilon}$ démontré dans l'Annexe

A.2; pour contrôler les dérivées de $\tilde{\sigma}_{\alpha z}$, on remarque que pour $\tilde{\rho}(x) \leq |\det A|^2$ on a un contrôle immédiat de $|\partial^\beta \tilde{\sigma}_{\alpha z}(x)/\partial x^\beta|$ par $C(1+|z|)^{|\beta|}$; pour $\tilde{\rho}(x) > |\det A|^2$ on remarque que si $|\det A|^{j-1} < \tilde{\rho}(x) < |\det A|^{j+1}$, alors $\tilde{\sigma}_{\alpha z}(x) = |\det A|^{\alpha z j} \tilde{\sigma}_{\alpha z}(A^{-j}x)$, or les coefficients des matrices A^{-j} se majorent indépendamment de j pour $j \geq 3$ de sorte que les dérivées de $\tilde{\sigma}_{\alpha z}$ sur $\{\xi : \ |\det A|^{j-1} < \tilde{\rho}(\xi) < |\det A|^{j+1}\}$ se majorent indépendamment de j (pour $j \geq 3$) par les bornes des dérivées de $\tilde{\sigma}_{\alpha z}$ sur $\{\xi : \ \tilde{\rho}(x) \leq |\det A|^2\}$. On a donc pour $\operatorname{Re} z = 0$,

$$\|\tilde{\sigma}_{\alpha z}\hat{f}\|_{H_{1+\varepsilon}} \leq C\,(1+|z|)^N \|\hat{f}\|_{H_{1+\varepsilon}} .$$

On en conclut que pour $0 < \theta < 1$, $\tilde{\sigma}_{\alpha\theta}\hat{f} \in [H_{1+\varepsilon}, H_0]_{[\theta]} = H_{(1-\theta)(1+\varepsilon)}$. (Il est immédiat de déterminer l'interpolé complexe $[H_{1+\varepsilon}, H_0]_{[\theta]}$, puisque par transformation de Fourier on sa ramène à l'interpolé complexe $[L^2_{1+\varepsilon}, L^2_0]_{[\theta]}$, c'est-à-dire l'interpolé d'espaces L^2 à poids qui est donc bien connu). Si θ est suffisamment petit, $(1-\theta)(1+\varepsilon) > 1$.

On a donc montré que si η est suffisamment petit, $\tilde{\sigma}_\eta \hat{f} \in H_{1+\varepsilon(\eta)}$ avec $\varepsilon(\eta) > 0$ et

$$\|\tilde{\sigma}_\eta \hat{f}\|_{H_{1+\varepsilon(\eta)}} \leq C(\eta)\left(\|f\|_{L^2_{1+\varepsilon}} + \|(1+\tilde{\rho}(\xi))^{\alpha/2}\hat{f}\|_2\right).$$

On va montrer que $\widehat{D_\eta f} \in H_{1+\varepsilon(\eta)}$ quitte à diminuer $\varepsilon(\eta)$. En effet, on fixe φ et ψ C^∞ à support compact avec $\varphi \equiv 1$ pour $\tilde{\rho}(x) \leq 1$ et $\psi \equiv 1$ au voisinage du support de φ. Alors on a

$$\tilde{\rho}(\xi)^\eta \hat{f} = \tilde{\rho}(\xi)^\eta \varphi(\xi)\,\frac{1}{\tilde{\sigma}_\eta(\xi)}\,\psi(\xi)\,\tilde{\sigma}_\eta\,\hat{f} + (1-\varphi(\xi))\,\tilde{\sigma}_\eta(\xi)\,\hat{f}.$$

Les fonctions $\psi/\tilde{\sigma}_\eta$ et $1-\varphi$ sont bornées ainsi que toutes leurs dérivées et sont donc des multiplicateurs de l'algèbre de Beurling $H_{1+\varepsilon(\eta)}$. Par ailleurs la fonction $\tilde{\rho}(\xi)^\eta \varphi(\xi)$ est la transformée de Fourier d'une fonction Γ_η étudiée dans l'Annexe B.2: on montre que $|\Gamma_\eta(x)| \leq C(1+\rho(x))^{-1-\eta}$ de sorte que $\Gamma_\eta \in L^2_{1+\varepsilon(\eta)}$ si $\varepsilon(\eta)$ est choisi strictement inférieur à η. On obtient donc $\tilde{\rho}(\xi)^\eta \hat{f} \in H_{1+\varepsilon(\eta)}$ si $\eta = \theta\alpha$ avec $\theta < \varepsilon/(1+\varepsilon)$ et $\varepsilon(\eta) < \inf\{\eta, \varepsilon - \theta(1+\varepsilon)\}$. L'inégalité (26) est donc démontré avec $\alpha' < \varepsilon\alpha/(1+\varepsilon)$ et $\varepsilon' < \inf\{\alpha', \varepsilon - \alpha'(1+\varepsilon)/\alpha\}$.

Pour estimer $I_{\alpha'}f$, on remarque d'abord que quel que soit $\eta > 0$, $\tilde{\sigma}(\xi)^{-\eta}$ est un multiplicateur de $H_{1+\varepsilon}$ (*i.e.* $\tilde{\sigma}(\xi)^{-\eta} \in M_{1+\varepsilon}$) puisque c'est une fonction C^∞ bornée ainsi que toutes ses dérivées. On fixe à nouveau $\varphi \in C^\infty$ à support compact telle que $\varphi(\xi) = 1$ pour $\tilde{\rho}(\xi) \leq 1$. On a alors

$$\tilde{\rho}(\xi)^{-\eta} \hat{f} = \tilde{\rho}(\xi)^{-\eta} \varphi(\xi) \hat{f} + (1 - \varphi(\xi)) \tilde{\sigma}(\xi)^{-\eta} \hat{f} .$$

Pour $\eta < 1$, on définit la fonction Δ_η par $\hat{\Delta}_\eta = \tilde{\rho}(\xi)^{-\eta}\varphi$; enfin on désigne par g la fonction définie par $\hat{g} = (1 - \varphi)\tilde{\sigma}^{-\eta}\hat{f}$. Il est clair que $I_\eta f = \Delta_\eta * f + g$, où $g \in L^2((1 + \rho(x))^{1+\varepsilon} dx)$. Pour étudier $\Delta_\eta * f$, on remarque que, puisque $\int f\, dx = 0$, on a

$$\Delta_\eta * f = I_1(x) + I_2(x)$$

avec

$$I_1(x) = \int_{\rho(y) < \rho(x)/2C_1} (\Delta_\eta(x - y) - \Delta(x) f(y)\, dt$$

et

$$I_2(x) = \int_{\rho(y) > \rho(x)/2C_1} (\Delta_\eta(x - y) - \Delta(x)) f(y)\, dy .$$

C_1 est la constante introduite dans l'inégalité triangulaire (21). De même α_0 sera l'exposant introduit dans les inégalités (19) et (20). Alors l'étude de la fonction Δ_η menée dans l'Annexe B.2 permet de majorer, pour tout $\gamma \leq \alpha_0$, par

$$|I_1(x)| \leq C \int \frac{1}{(1 + \rho(x))^{1+\gamma-\eta}} \rho(y)^\gamma\, |g(y)|\, dy .$$

Or $\rho(y)^\gamma g(y) \in L^2((1 + \rho(x))^{1+\varepsilon-2\gamma}) \subset L^1(\mathbb{R}^n)$ si $\gamma < \varepsilon/2$; si η est tel que $\eta < \inf\{1, \alpha_0, \varepsilon/2\}$, alors $I_1(x) \in L^2((1 + \rho(x))^{1+\varepsilon(\eta)} dx)$ pour tout $\varepsilon(\eta) < \inf\{(\varepsilon - 2\eta)/3, \alpha_0 - \eta\}$.

Pour contrôler I_2, on écrit

$$|I_2| \leq \int_{\rho(y) < \rho(x)/2C_1} (1 + \rho(x))^{-1+\eta}\, |f(y)|\, dy + \int_{\substack{\rho(y) > \rho(x)/2C_1 \\ \rho(x-y) < \rho(x)/2C_1}} (1 + \rho(x - y))^{-1+\eta}\, |f(y)|\, dy = J_1 + J_2 .$$

Pour J_1, on remarque que si $\gamma < \varepsilon/2$ alors $\rho(y)^\gamma |f(y)| \in L^2((1 + \rho(x))^{1+\varepsilon-2\gamma} dx) \subset L^1$ et donc

$$J_1 \leq C\, (1 + \rho(x))^{-1+\eta-\gamma} ,$$

de sorte que si $\eta < \varepsilon/2$ alors $J_1 \in L^2((1+\rho(x))^{1+\varepsilon(\eta)}dx)$ pour tout $\varepsilon(\eta) < (\varepsilon - 2\eta)/3$. Quant à J_2, on le majore par

$$J_2 \leq \int (1+\rho(x-y))^{-1+\eta-\gamma}(1+\rho(y))^{\gamma}\,|f(y)|\,dy$$

puisque $\rho(x-y) < \rho(y)$, et à nouveau

$$J_2 \in L^2((1+\rho(x))^{1+\varepsilon(\eta)}dx)$$

pour $\eta < \varepsilon/2$ et $\varepsilon(\eta) < (\varepsilon - 2\eta)/3$. (27) est donc démontré pour

$$\alpha' < \inf\left\{1, \alpha_0, \frac{\varepsilon}{2}\right\} \qquad \text{et} \qquad \varepsilon' < \inf\left\{\frac{\varepsilon - 2\alpha'}{3}, \alpha_0 - \alpha'\right\}.$$

Le Lemme 8 et le Théorème 2 sont donc démontrés.

III. Bases d'ondelettes et analyses multi-résolutions.

Nous pouvons maintenant énoncer le théorème d'existence des analyses multi-résolution.

Théorème 3. *Soit A une matrice de dilatation sur $\mathbb{R}^n$ et*

$$(\psi_{\varepsilon,j,k})_{1\leq\varepsilon\leq E,\, j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$$

une base de Riesz de $L^2(\mathbb{R}^n)$ de base duale $(\psi^*_{\varepsilon,j,k})$ telles que

i) $\psi_{\varepsilon,j,k}(x) = |\det A|^{j/2}\,\psi_\varepsilon(A^j x - k)$,

ii) $\psi^*_{\varepsilon,j,k}(x) = |\det A|^{j/2}\,\psi^*_\varepsilon(A^j x - k)$,

iii) *il existe $\eta > 1$ tel que $\psi_\varepsilon \in L^2((1+\rho(x))^\eta dx)$ (où ρ est une pseudonorme sur $(\mathbb{R}^n, A)$) et tel que $\psi^*_\varepsilon \in L^2((1+\rho(x))^\eta dx)$,*

iv) *il existe $\alpha > 0$ tel que $\hat{\psi}_\varepsilon \in L^2((1+\tilde{\rho}(x))^\alpha dx)$ (où $\tilde{\rho}$ est une pseudo-norme sur $(\mathbb{R}^n, \tilde{A})$) et tel que $\hat{\psi}^*_\varepsilon \in L^2((1+\tilde{\rho}(x))^\alpha dx)$,*

v) $A\mathbb{Z}^n \subset \mathbb{Z}^n$.

Alors

a) $\int \psi_\varepsilon\,dx = \int \psi^*_\varepsilon\,dx = 0$ (de sorte que les $\psi_{\varepsilon,j,k}$ satisfont les hypothèses du lemme des vaguelettes).

b) Le projecteur de sommes partielles P_0 défini par

$$P_0 f = \sum_{j<0} \sum_{\varepsilon=1}^{E} \sum_{k\in\mathbb{Z}^n} \langle f, \psi^*_{\varepsilon,j,k}\rangle \psi_{\varepsilon,j,k} \tag{28}$$

est invariant par translations entières et a un noyau ω-localisé pour le poids de Beurling $\omega = (1+\rho(x))^\eta$.

c) En particulier, $V_0 = \operatorname{Im} P_0$ a une base de Riesz de la forme $(\varphi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$, où le nombre D vérifie

$$E = D(|\det A| - 1)\,. \tag{29}$$

REMARQUES.

i) Si $n = 1$, on peut choisir $\varphi_\delta \in L^2(\omega\, dx)$ d'après le Théorème 1.

ii) La conclusion c) est fausse sous l'hypothèse $\psi_\varepsilon \in L^2((1+\rho(x))^\eta dx)$ pour un $\eta < 1$ comme le montrera le Contre-exemple numéro 3.

DÉMONSTRATION.

a) La nullité des intégrales $\int \psi_\varepsilon\, dx$ et $\int \psi^*_\varepsilon\, dx$ provient du lemme suivant:

Lemme 9. *Soit $f \in L^2(\mathbb{R}^n)$ telle que la famille*

$$(f_{j,k} = |\det A|^{j/2} f(A^j x - k))_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$$

soit presque-orthogonale. Alors, si $f \in L^1$, $\int f\, dx = 0$.

On raisonne par l'absurde. Si $\int f\, dx \neq 0$, il existe R_0, tel que

$$\int_{\rho(y)>R_0} |f(y)|dy \le \frac{1}{4}\left|\int f\, dx\right|.$$

On note $B = \{y : \ \rho(y) \le 1\}$ et χ_B sa fonction caractéristique. On a

$$\langle \chi_B, f_{j,k}\rangle = |\det A|^{-j/2} \int_{\rho(y+k)\le|\det A|^j} f(y)\, dy\,.$$

Si $\rho(k) \leq |\det A|^j/2C_1$ et si j est suffisamment grand pour que $R_0C_1 \leq |\det A|^j/2$, on obtient

$$|\langle \chi_B, f_{j,k}\rangle| \geq \frac{3}{4}\left|\int f\,dx\right| |\det A|^{-j/2}.$$

On obtient donc

$$\sum_j \sum_k |\langle \chi_B, f_{j,k}\rangle|^2 \geq \frac{9}{16}\left|\int f\,dx\right|^2 \sum_{j\geq j_0} |\det A|^{-j}\,\#\{k \in \mathbb{Z}^n : \ \rho(k) \leq \frac{1}{2C_1}|\det A|^j\}.$$

Maintenant si y vérifie $\rho(y) \leq |\det A|^j/4C_1^2$, et si $k \in \mathbb{Z}^n$ est tel que $y \in k + [0,1]^n$, on a

$$\rho(k) \leq C_1\left(\rho(y) + \rho(k-y)\right) \leq \frac{1}{4C_1}|\det A|^j + C,$$

de sorte que si j est assez grand pour que $C \leq |\det A|^j/4C_1$, on a $\rho(k) \leq |\det A|^j/2C_1$. Cela prouve que pour j assez grand

$$\begin{aligned}\#\{k \in \mathbb{Z}^n : \ \rho(k) \leq \frac{1}{2C_1}|\det A|^j\} &= \Big|\sum_{\substack{k\in\mathbb{Z}^n \\ \rho(k)\leq|\det A|^j/2C_1}} k + [0,1]^n\Big| \\ &\geq \left|\{y : \ \rho(y) \leq \frac{1}{4C_1^2}|\det A|^j\}\right| \\ &= |\det A|^j \left|\{y : \ \rho(y) \leq \frac{1}{4C_1^2}\}\right|.\end{aligned}$$

On obtient donc

$$\sum_j \sum_k |\langle \chi_B, f_{j,k}\rangle|^2 \geq \frac{9}{16}\left|\int f\,dx\right|^2 \left|\{y : \ \rho(y) \leq \frac{1}{4C_1^2}\}\right| \sum_{j\geq j_1} |\det A|^{-j}\,|\det A|^j = +\infty$$

ce qui contredit la presque-orthogonalité des $f_{j,k}$.

b) On note Q_j le projecteur

$$Q_j f = \sum_{\varepsilon=1}^{E} \sum_{k\in\mathbb{Z}^n} \langle f, \psi^*_{\varepsilon,j,k}\rangle\, \psi_{\varepsilon,j,k},$$

de sorte que $I = \sum_j Q_j$ et $P_0 = \sum_{j<0} Q_j = I - \sum_{j\geq 0} Q_j$. Or Q_j est invariant par translation par $A^{-j}\mathbb{Z}^n$ de manière évidente. En particulier pour $j \geq 0$, $A^{-j}\mathbb{Z}^n \supset \mathbb{Z}^n$ et donc Q_j est invariant par translations entières. Comme $P_0 = I - \sum_{j\geq 0} Q_j$ on voit que P_0 est invariant par translations entières.

On note $q_j(x,y)$ le noyau de $Q_j(x,y)$. Comme $(1+\rho(x))^\eta$ est un poids de Beurling, on sait que

$$\int_{[0,1]^n}\int_{\mathbb{R}^n} |q_0(x,y)|^2 \rho(x-y)^\eta \, dx\, dy < +\infty .$$

Par ailleurs $q_j(x,y) = |\det A|^j \, q_0(A^j x, A^j y)$ de sorte que pour $j \geq 0$

$$\int_{[0,1]^n}\int_{\mathbb{R}^n} |q_j(x,y)|^2 \rho(x-y)^\eta \, dx\, dy$$
$$= \int_{A^j[0,1]^n}\int_{\mathbb{R}^n} |q_0(x,y)|^2 \rho(x-y)^\eta \, |\det A|^{j(-\eta)} \, dx\, dy .$$

Or la fonction $x \mapsto \int_{\mathbb{R}^n} |q_0(x,y)|^2 \rho(x-y)^\eta \, dy$ est $\mathbb{Z}^n$-périodique et

$$\sum_{k\in\mathbb{Z}^n} \chi_{A^j[0,1]^n}(x-k) = \sum_{k\in\mathbb{Z}^n} \chi_{[0,1]^n}(A^{-j}x - A^{-j}k)$$
$$= \sum_{r\in A^{-j}\mathbb{Z}^n/\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} \chi_{[0,1]^n}(A^{-j}x - k - r)$$
$$= |\det A|^j$$

(puisque $\mathbb{Z}^n \subset A^{-j}\mathbb{Z}^n$) de sorte que

$$\int_{[0,1]^n}\int_{\mathbb{R}^n} |q_j(x,y)|^2 \rho(x-y)^\eta \, dx\, dy$$
$$= |\det A|^{j(1-\eta)} \int_{[0,1]^n}\int_{\mathbb{R}^n} |q_0(x,y)|^2 \rho(x-y)^\eta \, dx\, dy .$$

Comme $\eta > 1$, on obtient

$$\sum_{j\geq 0} \|q_j(x,y)\|_{L^2([0,1]^n\times\mathbb{R}^n, \rho(x-y)^\eta \, dx\, dy)} < +\infty ,$$

de sorte que le noyau $p(x,y)$ de P_0 est en dehors de la diagonale $x = y$ une fonction de carré localement intégrable telle que

$$(30) \qquad \int_{[0,1]^n}\int_{\mathbb{R}^n} |p(x,y)|^2 \rho(x-y)^\eta \, dx\, dy < +\infty .$$

Il ne nous reste plus qu'à estimer $\int_{[0,1]^n}\int_{[-1,2]^n}|p(x,y)|^2\,dx\,dy$ pour conclure que P_0 a un noyau ω-localisé. On commence par remarquer qu'il existe $r_0>2$ tel que $q_0(x,y)$ soit localement de puissance r_0-ième intégrable. En effet on sait qu'il existe α' et $\eta'>0$ tel que $D_{\alpha'}\psi_\varepsilon\in L^2((1+\rho(x))^{1+\eta'}dx)$. On considère alors la famille de fonctions $\psi_{\varepsilon,z}$ définie par $\hat{\psi}_{\varepsilon,z}=(\tilde{\sigma}^{\alpha'}/\tilde{\sigma}^{\theta z})\hat{\psi}_\varepsilon$, où $\theta>1/2$ (où $\tilde{\sigma}$ a été définie dans la preuve du Lemme 8). Pour $\operatorname{Re} z=1$ on a $\hat{\psi}_{\varepsilon,z}\in L^1$ et donc $\psi_{\varepsilon,z}\in L^\infty((1+\rho(x))^{1+\eta'}dx)$; pour $\operatorname{Re} z=0$ on a $\hat{\psi}_{\varepsilon,z}\in H_{1+\eta'}$ ou encore $\psi_{\varepsilon,z}\in L^2((1+\rho(x))^{1+\eta'}dx)$. Par interpolation complexe, on a $\psi_\varepsilon=\psi_{\varepsilon,\alpha'/\theta}$ (θ étant choisi $>\alpha'$) et donc $\psi_\varepsilon\in L^r((1+\rho(x))^{1+r'}dx)$ avec $r=2\theta/(\theta-\alpha')>2$. On fixe alors $r_0\le r$, $r_0>2$ tel que $r_0/2<1+\eta'$. On a, pour un compact K fixé,

$$\iint_{K\times K}|\psi_\varepsilon(x-k)|^{r_0}|\psi_\varepsilon^*(y-k)|^{r_0}\,dx\,dy$$
$$\le C\frac{1}{(1+\rho(k))^{2+2\eta'}}\|\psi_\varepsilon\|^{r_0}_{L^{r_0}((1+\rho(x))^{1+\eta'}dx)}\|\psi_\varepsilon^*\|^{r_0}_{L^{r_0}((1+\rho(x))^{1+\eta'}dx)}$$

de sorte que

$$\sum_k\|\psi_\varepsilon(x-k)\,\psi_\varepsilon^*(y-k)\|_{L^{r_0}(K\times K)}\le C\sum_k\frac{1}{(1+\rho(k))^{2(1+\eta')/r_0}}<+\infty\,,$$

puisque $2(1+\eta')/r_0>1$.

Comme $q_j(x,y)=|\det A|^jq_0(A^jx,A^jy)$, on a pour tout compact K

$$\iint_{K\times K}|q_j(x,y)|^2\,dx\,dy=\iint_{A^{+j}K\times A^{+j}K}|q_0(x,y)|^2\,dx\,dy$$
$$\le\Big(\iint_{A^{+j}K\times A^{+j}K}|q_0(x,y)|^r\,dx\,dy\Big)^{2/r_0}$$
$$\cdot\,|\det A|^{j(1-2/r_0)}|K\times K|^{1-2/r_0}\,.$$

Pour $j\le0$, $A^jK\times A^jK$ reste compris dans un compact $\tilde{K}\times\tilde{K}$ fixe et l'intégrabilité de $|q_0|^{r_0}$ sur $\tilde{K}\times\tilde{K}$ donne donc pour $j\le0$

$$\|q_j\|_{L^2(K\times K)}\le C\,|\det A|^{j(r_0-2)/(2r_0)}$$

ce qui entraîne que $p_0=\sum_{j\le0}q_j$ est localement de carré intégrable.

Le projecteur P_0 est donc bien ω-localisé avec $\omega = (1+\rho(x))^\eta$.

c) Ce point est une simple application du Théorème 1. La relation entre D et E s'établit de la manière suivante. On pose $V_1 = \{f(Ax) : f \in V_0\}$. Alors V_1 dispose de deux bases de Riesz invariantes par translations entières:

- les $(\varphi(\delta(Ax-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$ engendrées par translations entières à partir des $|\det A|\, D$ fonctions $(\varphi_\delta(Ax-r)_{1\le\delta\le D,\, r\in\mathbb{Z}^n/A\mathbb{Z}^n}$
- les $(\varphi_\delta(x-k))_{1\le\delta\le D,\, k\in\mathbb{Z}^n}$ et les $(\psi_\varepsilon(x-k))_{1\le\varepsilon\le E,\, k\in\mathbb{Z}^n}$ engendrées à partir de $D+E$ fonctions.

Comme le nombre de fonctions génératrices ne dépend pas du choix de la base, on a $D+E = D\,|\det A|$.

IV. Le cas de la dimension 1.

Le cas de la dimension 1 se traite par le lemme suivant:

Lemme 10. *Soit ω un poids de Beurling sur $\mathbb{R}$ et $f_1,\dots,f_N$, N fonctions de $L^2(\omega\,dx)$ telles que pour tout $\xi \in [0,2\pi]$ il existe $i \in \{1,\dots,N\}$ avec $C(f_i,f_i)(\xi) \neq 0$. Alors il existe une suite presque nulle $(\lambda_{i,k})_{1\le i\le N,\, k\in\mathbb{Z}}$ telle que la fonction $f = \sum_{i=1}^N \sum_k \lambda_{i,k} f_i$ vérifie pour tout ξ, $C(f,f)\neq 0$.*

DÉMONSTRATION. Il s'agit de montrer qu'il existe N polynômes trigonométriques $P_1,\dots,P_N$ (avec $P_i = \sum_{k\in\mathbb{Z}} \lambda_{i,k}\, e^{-ik\xi}$) tels que

$$(31)\qquad (P_1(\xi),\dots,P_N(\xi))M(f_1,\dots,f_N)(\xi)\begin{pmatrix}\bar P_1(\xi)\\ \vdots \\ \bar P_N(\xi)\end{pmatrix} \neq 0,$$

pour tout $\xi \in [0,2\pi]$. Or la matrice $M(f_1,\dots,f_N)(\xi)$ est hermitienne positive

$$(\alpha_1,\dots,\alpha_N)\,M(f_1,\dots,f_N)(\xi)\begin{pmatrix}\bar\alpha_1\\ \vdots \\ \bar\alpha_N\end{pmatrix} \ge 0,$$

pour tout $(\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N$, ce qui entraîne:

$$\left| (\beta_1, \ldots, \beta_N) M(f_1, \ldots, f_N)(\xi) \begin{pmatrix} a_1 \\ \vdots \\ \alpha_N \end{pmatrix} \right|^2$$

$$\leq \left| (\beta_1, \ldots, \beta_N) M(f_1, \ldots, f_N)(\xi) \begin{pmatrix} \bar\beta_1 \\ \vdots \\ \bar\beta_N \end{pmatrix} \right|$$

$$\cdot \left| (\alpha_1, \ldots, \alpha_N) M(f_1, \ldots, f_N)(\xi) \begin{pmatrix} \bar\alpha_1 \\ \vdots \\ \bar\alpha_N \end{pmatrix} \right|$$

pour tout $\xi \in [0, 2\pi]$, de sorte que (31) est équivalent à

$$M(f_1, \ldots, f_N)(\xi) \begin{pmatrix} \bar P_1(\xi) \\ \vdots \\ \bar P_N(\xi) \end{pmatrix} \neq 0,$$

ou encore

$$\sum_{i=1}^N \bar P_i(\xi) \begin{pmatrix} C(f_1, f_i)(\xi) \\ \vdots \\ C(f_N, f_i)(\xi) \end{pmatrix} \neq \vec 0.$$

Par densité des polynômes trigonométriques dans l'espace des fonctions continues 2π-périodiques, on voit que le Lemme 10 se réécrit en

Lemme 11. *Soit $\vec u_1, \ldots, \vec u_N$, N fonctions continues 2π-périodiques de $\mathbb{R}$ dans $\mathbb{C}^N$. Si pour tout $\xi \in [0, 2\pi]$ il existe i avec $\vec u_i(\xi) \neq \vec 0$, alors il existe $\lambda_1, \ldots, \lambda_N$, N fonctions continues 2π-périodiques de $\mathbb{R}$ dans $\mathbb{C}$ telles que pour tout $\xi \in [0, 2\pi]$, $\sum_i \lambda_i(\xi) \vec u_i(\xi) \neq \vec 0$.*

En effet, quitte à réindexer (avec répétition ...) les $\vec u_i$, on peut supposer que $[0, 2\pi] = \bigcup_{i=1}^N [t_i, t_{i+1}]$ avec $t_1 = 0$, $t_{N+1} = 2\pi$, $t_i < t_{i+1}$ et $\vec u_i$ sans zéro sur $[t_i, t_{i+1}]$, et donc sur $[t_i - \alpha_i, t_{i+1} + \beta_i]$ pour des nombres α_i et $\beta_i > 0$. On fixe alors $\varphi_i \in C^\infty$, 2π-périodique tel que $\varphi_i \equiv 1$ sur $[t_i, t_{i+1}]$, $0 \leq \varphi_i < 1$ en dehors de $\bigcup_{k \in \mathbb{Z}} ([t_i, t_{i+1}] + 2k\pi)$ et φ_i portée par

$$\left[t_i - \frac{\alpha_i}{2}, t_{i+1} + \frac{\beta_i}{2} \right] \cap \left[\frac{t_{i-1} + t_i}{2}, \frac{t_{i+1} + t_{i+2}}{2} \right]$$

modulo-2π (où on pose $t_{-1} = t_{N-1}$ et $t_{N+2} = t_1$). Enfin ω_i, $1 \leq i \leq N$, sont des fonctions C^∞ et 2π-périodiques à valeurs réelles que nous fixerons ensuite.

On pose $\lambda_j = e^{i\omega_j(x)}\varphi_j(x)/\|\vec{u}_j\|$. On voit alors que sur $]t_j, (t_j + t_{j-1})/2]$

$$\sum \lambda_k \vec{u}_k = \frac{\vec{u}_j}{\|\vec{u}_j\|} e^{i\omega_j} + \varphi_{j-1}(x) \frac{\vec{u}_{j-1}}{\|\vec{u}_{j-1}\|} e^{i\omega_{j-1}}$$

avec $\varphi_{j-1}(x) < 1$, de sorte que $\sum \lambda_k \vec{u}_k \neq 0$, et de même pour $[(t_j + t_{j+1})/2, t_{j+1}[$. Les seuls zéros possibles sont donc les points t_j où on a:

$$\sum \lambda_k \vec{u}_k(t_j) = \frac{\vec{u}_j}{\|\vec{u}_j\|} e^{i\omega_j(t_j)} + \frac{\vec{u}_{j-1}}{\|\vec{u}_{j-1}\|} e^{i\omega_{j-1}(t_j)}.$$

On impose $\omega_j(t_j) = 0$; la valeur de $\omega_j(t_{j+1})$ est alors arbitraire en dehors éventuellement de la valeur $\theta_j + 2\pi\mathbb{Z}$ où

$$\frac{\vec{u}_{j+1}}{\|\vec{u}_{j+1}\|} + e^{i\theta_j} \frac{\vec{u}_j}{\|\vec{u}_j\|} \neq 0.$$

On choisit une telle fonction ω_j et les lemmes 10 et 11 sont prouvés.

On a alors la proposition suivante

Proposition 2. *Soit V_0 un sous-espace fermé de $L^2(\mathbb{R})$ invariant par translations entières et tel que $V_0 \cap L^2(\omega\, dx)$ soit dense dans V_0 pour un poids de Beurling ω. Pour $\xi \in [0, 2\pi]$, on note*

$$V_\xi = \{g \in L^2([0,1]) : \textit{il existe } f \in V_0 \cap L^2(\omega\, dx),\ g = \sum_{k\in\mathbb{Z}} e^{-i\xi(x-k)} f(x-k)\}.$$

Alors les assertions suivantes sont équivalentes

i) *V_0 a une base orthonormée de la forme $(f_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}}$ avec $f_\delta \in L^2(\omega\, dx)$.*

ii) *Pour tout $\xi \in [0, 2\pi]$, $\dim V_\xi = D$.*

DÉMONSTRATION. i) implique ii) est immédiat car

$$g_{\delta,\xi} = \sum_{k\in\mathbb{Z}^n} e^{-i(x-k)\xi} f_\delta(x-k)$$

est alors une base orthonormée de V_ξ.

ii) implique i) se démontre par récurrence sur D. Comme pour la démonstration du Théorème 1 en dimension n, on remarque que l'hypothèse ii) entraîne qu'il existe un recouvrement ouvert fini $(V_\alpha)_{\alpha\in A}$ de $[0,2\pi]$ et des fonctions $(f_{\alpha,\delta})_{1\le\delta\le D}$ dans $V_0 \cap L^2(\omega\, dx)$ telles que la matrice $M(f_{\alpha,1},\dots,f_{\alpha,N})$ soit inversible sur V_α. En particulier, $C(f_{\alpha,1},f_{\alpha,1})$ ne s'annule pas sur V_α. Le Lemme 10 implique alors qu'il existe $g_1 \in V_0 \cap L^2(\omega\, dx)$ telle que $C(g_1,g_1)$ ne s'annule en aucun point de $[0,2\pi]$. On note W_1 l'espace engendré par les $g_1(x-k)$, $k \in \mathbb{Z}$. Il admet comme base de Riesz la famille $(g_1(x-k))_{k\in\mathbb{Z}}$ avec $g_1 \in L^2(\omega\, dx)$. On sait alors qu'il admet une base orthonormée $(f_1(x-k))_{k\in\mathbb{Z}}$ avec $f_1 \in L^2(\omega\, dx)$. On pose alors $V_1 = V_0 \cap W_1^\perp$. On a les propriétés suivantes sur V_1 (en notant Q_1 le projecteur orthogonal sur W_1):

- $V_1 = (I-Q_1)V_0$; en particulier $V_1 \cap L^2(\omega\, dx)$ est dense dans V_1,
- pour $\xi \in [0,2\pi]$, $V_{1,\xi}$ est le complémentaire orthogonal de $W_{1,\xi}$ dans $V_{0,\xi}$:

- $W_{1,\xi} = \mathbb{C}\, f_{1,\xi}$ et $\int_0^1 |f_{1,\xi}(x)|^2\, dx = 1$,
- le projecteur orthogonal de $L^2([0,1])$ sur $W_{1,\xi}$ est donné par

$$Q_{1,\xi}(g) = \langle g, f_{1,\xi}\rangle_{L^2([0,1])}\, f_{1,\xi} = \langle \tilde g, e^{-i\xi x} f_1\rangle_{L^2(\mathbb{R})}\, f_{1,\xi}$$

en prolongeant g en $\tilde g$ par périodicité;

- en particulier, en notant $f_\xi = \sum_k e^{-i\xi(x-k)} f(x-k)$

$$Q_{1,\xi}(f_\xi) = \Big(\sum_k e^{i\xi k}\langle f(x-k), f_1\rangle\Big) f_{1,\xi}$$

d'où $Q_{1,\xi}(V_{1,\xi}) = 0$

- il est clair que, puisque $V_0 = V_1 \oplus W_1$ et que $Q_1(L^2(\omega\, dx)) \subset L^2(\omega\, dx)$, que $V_{0,\xi} = V_{1,\xi} + W_{1,\xi}$; comme $Q_{1,\xi}(V_{1,\xi}) = 0$, cette somme est directe et $V_{0,\xi} = V_{1,\xi} \overset{\perp}{\oplus} W_{1,\xi}$, ce qui prouve que $\dim V_{1,\xi} = D-1$.

On peut alors appliquer l'hypothèse de récurrence à V_1 et exhiber une base orthonormée $(f_\delta(x-k))_{2\le\delta\le D,\, k\in\mathbb{Z}}$ de V_1 avec $f_\delta \in L^2(\omega\, dx)$. La Proposition 2 est donc démontrée.

Le Théorème 1 en dimension 1 est alors immédiat, puisque nous savons démontrer dans ce cas que $\dim V_\xi = \mathrm{C^{te}}$. De plus, nous pouvons le préciser de la manière suivante

Proposition 3. *a) Soient V_0 et V_0^* deux sous-espaces fermés de $L^2(\mathbb{R})$ invariants par translations entières. On suppose que $L^2 = V_0 \oplus (V_0^*)^\perp$ et on note P_0 le projecteur de L^2 sur V_0 parallèlement à $(V_0^*)^\perp$ et $p(x,y)$ le noyau distribution de P_0. Alors*

• *Pour un poids de Beurling ω symétrique, V_0 admet une base de Riesz de la forme $(\varphi_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}}$ de base duale $(\varphi_\delta^*(x-k))$ dans V_0^* avec φ_δ et φ_δ^* dans $L^2(\omega\, dx)$ si et seulement si p est localement de carré intégrable et*

$$\int_{x\in[0,1]}\int_{y\in\mathbb{R}} \omega(x-y)\,(|p(x,y)|^2 + |p(y,x)|^2)\,dx\,dy < +\infty\,.$$

• *V_0 admet une base de Riesz $(\varphi_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}}$ de base duale $(\varphi_\delta^*(x-k))$ dans V_0^* telle que pour tout $k\in\mathbb{N}$, $x^k\varphi_\delta \in L^2$ et $x^k\varphi_\delta^* \in L^2$ si et seulement si p est localement de carré intégrable et*

$$\int_{x\in[0,1]}\int_{y\in\mathbb{R}} |x-y|^k(|p(x,y)|^2 + |p(y,x)|^2)\,dx\,dy < +\infty\,,$$

pour tout $k\in\mathbb{N}$.

• *V_0 admet une base de Riesz $(\varphi_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}}$ de base duale $(\varphi_\delta^*(x-k))$ dans V_0^* telle que il existe $\varepsilon>0$, $e^{\varepsilon|x|}\varphi_\delta \in L^2$ et $e^{\varepsilon|x|}\varphi_\delta^* \in L^2$ si et seulement si p est localement de carré intégrable et il existe $\alpha>0$ telle que*

$$\int_{x\in[0,1]}\int_{y\in\mathbb{R}} e^{\alpha|x-y|}\,(|p(x,y)|^2 + |p(y,x)|^2)\,dx\,dy < +\infty\,.$$

• *V_0 admet une base de Riesz $(\varphi_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}}$ de base duale $(\varphi_\delta^*(x-k))$ dans V_0^* telle que φ_δ et φ_δ^* sont à support compact si et seulement si p est localement de carré intégrable et il existe $M>0$ telle que*

$$p(x,y) = 0\,, \qquad si\ |x-y| > M\,.$$

b) Mêmes conclusions dans le cas orthogonal ($V_0 = V_0^*$, $p(x,y) = \overline{p(y,x)}$) avec des bases orthonormées ($\varphi_\delta = \varphi_\delta^*$).

DÉMONSTRATION. Hormis le cas du support compact, tous les cas sont traitables par le Lemme 10 et en remarquant que ces conditions d'intégrabilité sont stables par orthogonalisation. Le cas du support compact est traité dans [16].

La Proposition 3 entraîne immédiatement l'existence de fonctions d'échelles φ_δ, φ_δ^* associée à des ondelettes bi-orthogonales ψ_ε, ψ_ε^*:

Proposition 4. *Soit A un entier ≥ 2 et*

$$(\psi_{\varepsilon,j,k} = A^{j/2}\psi_\varepsilon(A^j x - k))_{1\leq\varepsilon\leq E,\, j,k\in\mathbb{Z}}$$

une base d'ondelettes bi-orthogonales associée à A. Alors

i) *Si pour un $\alpha > 0$ les ondelettes ψ_ε et ψ_ε^* sont dans $L^2(|x|^{1+\alpha}dx)$ et dans l'espace de Sobolev H^α, alors E est divisible par $A-1$: $E = D(A-1)$ et l'espace V_0 associé à $(\psi_{\varepsilon,j,k})$ admet une base de Riesz de la forme $(\varphi_\delta(x-k))_{1\leq\delta\leq D,\, k\in\mathbb{Z}}$.*

ii) *De plus on peut choisir les fonctions d'échelles φ_δ et φ_δ^**

- *dans $L^2(|x|^{1+\beta}dx)$ si ψ_ε et ψ_ε^* sont dans $L^2(|x|^{1+\beta}dx)$ $(\beta > 0)$,*
- *à décroissance rapide si les ψ_ε et ψ_ε^* sont à décroissance rapide,*
- *à décroissance exponentielle si les ψ_ε et ψ_ε^* sont à décroissance exponentielle,*
- *à support compact si les ψ_ε et les ψ_ε^* sont à support compact.*

Ces résultats restent valables pour des bases orthonormées ($\varphi_\delta = \varphi_\delta^*$ pour $\psi_\varepsilon = \psi_\varepsilon^*$).

C'est une conséquence directe de la Proposition 3. L'existence des fonctions d'échelle a d'abord été démontré en 1991 dans le cas de la décroissance exponentielle ou du support compact en utilisant l'analyticité des transformées de Fourier [15], [16]. Ce résultat a été étendu à une base orthonormée d'ondelettes $(\psi_{j,k} = 2^{j/2}\psi(2^j x - k))$ générale par P. Auscher en 1992, [2], sous des hypothèses différentes de celles que nous avons choisies ($\hat{\psi}$ continue, $\hat{\psi} = O(|\xi|^\alpha)$ au voisinage de 0 avec $\alpha > 0$, $\hat{\psi} = O(|\xi|^{-1/2-\alpha})$ au voisinage de l'infini).

En particulier, les bases d'ondelettes orthonormées

$$(\psi_{j,k} = 2^{j/2}\,\psi(2^j x - k)_{j,k\in\mathbb{Z}}$$

ont une structure remarquable:

Théorème 4. *Soit $\psi \in L^2(\mathbb{R})$. Alors les assertions suivantes sont équivalentes:*

a) Les fonctions $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$, $j,k \in \mathbb{Z}$, forment une base orthonormée de $L^2(\mathbb{R})$ et de plus ψ est à décroissance rapide (pour tout $k \in \mathbb{N}$, $x^k\psi \in L^2$) et il existe $\varepsilon > 0$ tel que $\psi \in H^\varepsilon$.

b) Il existe une fonction 2π-périodique m_0 de classe C^∞ telle que

- $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$,
- $m_0(0) = 1$,
- *m_0 vérifie la condition d'Albert Cohen: il existe un compact K tel que $\cup_{k\in\mathbb{Z}} K + 2k\pi = \mathbb{R}$ et tel que pour tout $\xi \in K$, pour tout $j \geq 1$, $m_0(\xi/2^j) \neq 0$,*

et il existe un entier $N \in \mathbb{Z}$ tels que

$$\hat{\psi}(\xi) = e^{-i(N+1/2)\xi}\, \bar{m}_0\Big(\frac{\xi}{2}+\pi\Big) \prod_{j=2}^{+\infty} m_0\Big(\frac{\xi}{2^j}\Big). \tag{32}$$

Le nombre N est déterminé par $N + 1/2 = \int x\, |\psi(x)|^2\, dx$. La fonction m_0 est unique à un facteur $e^{iM\xi}$ près (avec $M \in \mathbb{Z}$).

DÉMONSTRATION. *a)* implique *b)*: Nous savons qu'il existe une fonction d'échelle (orthogonale) φ_1 associée à ψ. On peut supposer $\hat{\varphi}_1(0) = 1$ et alors $\hat{\varphi}_1(\xi) = \prod_{j=1}^{\infty} m_1(\xi/2^j)$ où m_1 est la fonction C^∞ et 2π-périodique

$$m_1(\xi) = \sum_{k\in\mathbb{Z}} \langle 2\,\varphi(2x), \varphi(x-k)\rangle\, e^{-ik\xi}.$$

On dispose d'une autre ondelette associée à φ_1, ψ_1 définie par

$$\hat{\psi}_1(\xi) = e^{-i\xi/2}\, \bar{m}_1((\xi/2)+\pi)\hat{\varphi}_1(\xi/2).$$

On a alors

$$\hat{\psi}(\xi) = U(\xi)\,\hat{\psi}_1(\xi) \quad \text{avec } U \in C^\infty,\ 2\pi\text{-périodique et } |U(\xi)| = 1. \tag{33}$$

La fonction U s'écrit alors $U(\xi) = e^{i\theta(\xi)}$ où θ est C^∞ et vérifie

$$\theta(\xi+2\pi) - \theta(\xi) = 2M\pi, \qquad \text{avec } M \in \mathbb{Z}.$$

Il existe une fonction γ, C^∞, 2π-périodique, à valeurs réelles telle que

$$\gamma(2\xi) - \gamma(\xi) - \gamma(\xi+\pi) = \theta(2\xi) - 2M\xi.$$

En effet, on écrit

$$\theta(\xi) = M\xi + \sum_{k\in\mathbb{Z}} \theta_k\, e^{-ik\xi} \qquad \text{et} \qquad \gamma(\xi) = \sum_{k\in\mathbb{Z}} \gamma_k\, e^{-ik\xi}.$$

Or on a alors $\theta_k = \gamma_k - 2\gamma_{2k}$, d'où

$$\gamma_k = \sum_{j=0}^{\infty} 2^j \theta_{2^j k}$$

si $k \neq 0$ et $\gamma_0 = -\theta_0$. (En particulier, γ est unique). On pose $V(\xi) = e^{i\gamma(\xi)}$. On a alors

$$U(2\xi) = V(2\xi)\,\bar{V}(\xi)\,\bar{V}(\xi+\pi)\,e^{2iM\xi}.$$

On obtient

$$\begin{aligned}\hat{\psi}(\xi) &= e^{-i\xi/2}\,e^{2iM\xi}\,\bar{m}_1(\frac{\xi}{2}+\pi)\,\bar{V}(\frac{\xi}{2}+\pi)\,V(\xi)\,\bar{V}(\frac{\xi}{2})\prod_{j=2}^{\infty} m_1(\frac{\xi}{2^j}) \\ &= e^{-i\xi/2}e^{2iM\xi}\,\bar{m}_1(\frac{\xi}{2}+\pi)\,\frac{\bar{V}(\xi/2+\pi)}{\bar{V}(\xi)}\prod_{j=2}^{\infty} m_1(\frac{\xi}{2^j})\,\frac{V(\xi/2^j)}{V(2\xi/2^j)}\,.\end{aligned}$$

On pose donc

$$m_0(\xi) = m_1(\xi)\,\frac{V(\xi)}{V(2\xi)}\,,$$

de sorte que

$$m_0(\xi+\pi) = m_1(\xi+\pi)\,\frac{V(\xi+\pi)}{V(2\xi)}$$

et on obtient

$$\hat{\psi}(\xi) = e^{-i\xi/2}\,e^{2iM\xi}\,\bar{m}_0(\frac{\xi}{2}+\pi)\prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})\,.$$

La fonction m_0 vérifie toutes les conditions de $b)$ parce que m_1 les vérifie (elle provient d'une fonction d'échelle φ_1) et que $|m_0(\xi)| = |m_1(\xi)|$. Cela prouve l'existence de m_0 et N.

Si m_1, N_1 est une autre solution de (32), les fonctions φ_1 définies par

$$\hat{\varphi}_1 = \prod_{j=1}^{\infty} m_1(\frac{\xi}{2^j})$$

et φ_0 définie par

$$\prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j}) = \hat{\varphi}_0$$

sont deux fonctions d'échelle associées à ψ. On a donc $\hat{\varphi}_1 = V(\xi)\,\hat{\varphi}_0$ avec V 2π-périodique, C^∞ et $|V(\xi)| = 1$. On obtient alors

$$m_1(\xi)\,V(\xi) = V(2\xi)\,m_0(\xi)$$

d'une part en écrivant $\hat{\varphi}_1(2\xi)$ de deux manières différentes, et d'autre part

$$e^{-i\xi}\,e^{-2iN_1\xi}\,\bar{m}_1(\xi+\pi)\,V(\xi) = e^{-i\xi}\,e^{-2iN\xi}\,\bar{m}_0(\xi+\pi)$$

en écrivant $\hat{\psi}(2\xi)$ de deux manières différentes.

On a alors

$$V(2\xi)\,\bar{V}(\xi)\,\bar{V}(\xi+\pi)\,m_0(\xi) = \bar{V}(\xi+\pi)\,m_1(\xi) = e^{-2i(N_1-N)\xi}\,m_0(\xi)$$

et

$$V(2\xi)\,\bar{V}(\xi)\,\bar{V}(\xi+\pi)\,m_0(\xi+\pi) = e^{-2i(N_1-N)\xi}\,m_0(\xi+\pi)$$

d'où

$$V(2\xi)\,\bar{V}(\xi)\,\bar{V}(\xi+\pi) = e^{-2i(N_1-N)\xi}\,.$$

En écrivant $V(\xi) = e^{-iM\xi}e^{i\theta(\xi)}$ où θ est C^∞, 2π-périodique à valeurs réelles, on obtient

$$\theta(2\xi) - \theta(\xi) - \theta(\xi+\pi) = -2(N_1 - N)\xi + 2K\pi$$

d'où nécessairement $N_1 = N$ et $\theta(\xi) = -2K\pi$, de sorte que $V = e^{-iM\xi}$ et $m_1 = e^{-iM\xi}m_0$.

L'unicité est donc prouvée. Changer m_0 en $m_0\,e^{-iM\xi}$ revient à changer φ_0 en $\varphi_0(x - M)$, ce qui est bien un "invariant" de la base $(\psi(x-k))_{k\in\mathbb{Z}}$. Le fait que $\int x\,|\psi(x)|^2\,dx = N + 1/2$ est alors bien connu (*cf.* [22] par exemple).

b) implique *a*) est classique: le fait que ψ est alors l'ondelette d'une base orthonormée $(\psi_{j,k})$ est assuré par le Théorème d'Albert Cohen [5], le fait que ψ est C^∞ à décroissance rapide est démontré par exemple dans [19] et le fait que ψ est H^ε pour un $\varepsilon > 0$ peut être trouvé dans [13].

V. Contre-exemples.

Contre-exemple n° 1.

$$V = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset [-\frac{\pi}{2}, +\frac{\pi}{2}]^n\}$$

est stable par translation (puisque $\widehat{f(x-y)}(\xi) = e^{-iy\xi}\hat{f}(\xi))$, et donc en particulier par translation entière. Cependant il ne contient pas de fonction f telle que la famille $(f(x-k))_{k\in\mathbb{Z}^n}$ soit une famille de Riesz. En effet, il est clair que $C(f,f) = \sum_{k\in\mathbb{Z}^n} |\hat{f}(\xi+2k\pi)|^2$ s'annule sur $[-\pi,\pi]^n \backslash [-\pi/2, +\pi/2]^n$. A fortiori V n'a pas de base de Riesz de la forme $(\varphi_\delta(x-k))_{\delta\in\Delta,\, k\in\mathbb{Z}^n}$.

On remarquera que la projecteur orthogonal sur V est l'opérateur P défini par $\widehat{Pf} = \chi_{[-\pi/2,\pi/2]^n}\hat{f}$ et donc

$$Pf(x) = p(x) * f(x)\,, \qquad \text{avec } p(x) = \prod_{i=1}^{n} \frac{\sin \pi x_i/2}{\pi x_i}\,.$$

En particulier

$$p(x) \in L^2\left((1+|x_1|)^{\alpha_1} \dots (1+|x_n|)^{\alpha_n}\right) dx$$

si $\alpha_1, \dots, \alpha_n$ sont tous < 1. Si on avait eu

$$p(x) \in L^2\left((1+|x_1|)^{\alpha_1} \dots (1+|x_n|)^{\alpha_n}\right) dx$$

avec $\alpha_1, \dots, \alpha_n > 1$, le poids $\omega = \prod_1^N (1+|x_i|)^{\alpha_i}$ serait un poids de Beurling et V aurait eu une base de Riesz invariante par translations entières.

Contre-exemple n° 2.

On considère l'ondelette ψ et la fonction d'échelle associée φ de P. G. Lemarié et Y. Meyer [17]: φ et ψ sont dans la classe de Schwartz $\mathcal{S}(\mathbb{R}^2)$, $\operatorname{supp} \hat{\varphi} \subset [-4\pi/3, 4\pi/3]$, $\hat{\varphi} = 1$ sur $[-2\pi/3, 2\pi/3]$ et est > 0 sur $]-4\pi/3, 4\pi/3[$ et enfin $\operatorname{supp} \hat{\psi} = [-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]$.

On considère alors la base d'ondelettes orthonormées de $L^2(\mathbb{R}^2)$

$$(\psi_{\varepsilon,j,k} = 2^j \psi_\varepsilon(2^j x - k))_{1\le\varepsilon\le 3,\, j\in\mathbb{Z},\, k\in\mathbb{Z}^2}$$

avec $\psi_1 = \psi \otimes \varphi$, $\psi_2 = \varphi \otimes \psi$ et $\psi_3 = \psi \otimes \psi$. La fonction $\varphi \otimes \varphi$ est une fonction d'échelle associée à cette base.

Maintenant on considère l'opérateur unitaire U défini par

$$\widehat{Uf}(\xi,\eta) = \frac{\xi + i\eta}{|\xi + i\eta|}\,\hat{f}(\xi,\eta)\,. \tag{34}$$

U est un opérateur unitaire de L^2 qui vérifie: $U\big(f(x-h)\big) = \big(Uf\big)(x-h)$ pour tout $f \in L^2$ et tout $h \in \mathbb{R}^2$, et $U\big(f(\lambda x)\big) = \big(Uf\big)(\lambda x)$ pour tout $f \in L^2$ et tout $\lambda > 0$. En particulier, les fonctions $U\psi_\varepsilon$ sont les ondelettes d'une base d'ondelettes orthonormées de $L^2(\mathbb{R}^2)$ et $U(\varphi\otimes\varphi)$ en est une fonction d'échelle associée.

Y. Meyer a montré que, bien que les fonctions $U\psi_\varepsilon$ soient dans la classe de Schwartz $\mathcal{S}(\mathbb{R}^2)$ des fonctions C^∞ à décroissance rapide ainsi que toutes leurs dérivées, elles n'admettent pas de fonction d'échelle associée Φ avec $\Phi \in L^1$.

En effet, si une telle fonction Φ existait, on aurait que $\hat{\Phi}$ est continue et que

$$\hat{\Phi} = m(\xi,\eta)\,\frac{\xi + i\eta}{|\xi + i\eta|}\,\hat{\varphi}(\xi)\,\hat{\varphi}(\eta) \qquad \text{p.p.}$$

avec

$$m(\xi,\eta) = \sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \Omega(\xi + 2k\pi, \eta + 2\ell\pi)$$

en posant

$$\Omega(\xi,\eta) = \hat{\Phi}(\xi,\eta)\,\frac{\xi - i\eta}{|\xi + i\eta|}\,\hat{\varphi}(\xi)\,\hat{\varphi}(\eta)\,.$$

La fonction m devrait vérifier, pour que Φ soit fonction d'échelle,

$$\inf\operatorname{ess}|m(\xi,\eta)| > 0\,.$$

On pose alors

$$\theta(\xi,\eta) = \frac{m(\pi,\pi)}{m(\pi,\eta)\,m(\xi,\pi)}\,\frac{\hat{\Phi}(\xi,\eta)}{\hat{\varphi}(\xi)\,\hat{\varphi}(\eta)}\,. \tag{35}$$

Alors

i) θ est continue de $[-\pi,\pi]^2$ dans $\mathbb{C}$: cela vient de ce que m est continue en dehors de $2\pi\mathbb{Z}^2$ et que $\hat{\varphi}$ ne s'annule pas sur $[-\pi,\pi]$.

ii) θ ne s'annule pas sur $[-\pi,\pi]^2$: pour tout $(\xi,\eta)\in[-\pi,\pi]^2$,

$$\theta(\xi,\eta)\neq 0\,.$$

iii) $\theta(\xi,\eta)=\xi+i\eta/|\xi+i\eta|$ pour (ξ,η) sur le bord de $[-\pi,\pi]^2$.

Or il est facile de vérifier que les points i), ii) et iii) contredisent le Théorème de Brouwer: si

$$\gamma(\xi,\eta)=\frac{\xi+i\eta}{|\xi+i\eta|}\sup\left\{\frac{|\xi|}{\pi},\frac{|\eta|}{\pi}\right\},$$

γ est un homéomorphisme de $[-\pi,\pi]^2$ sur le disque unité $\mathbb{D}$ de $\mathbb{C}$; on pose alors

$$\tilde\theta(z)=-\frac{\theta(\gamma^{-1}(z))}{|\theta(\gamma^{-1}(z))|}\,;$$

c'est une fonction continue de $\mathbb{D}$ dans $\partial\mathbb{D}$ qui vérifie $\tilde\theta(z)=-z$ sur $\partial\mathbb{D}$; elle ne peut donc pas avoir de point fixe, en contradiction avec le Théorème de Brouwer.

Il n'y a donc pas de fonction d'échelle intégrable associée aux $U\psi_\varepsilon$.

Contre-exemple n° 3.

L'idée de ce contre-exemple est due à J.-L. Journé. On définit

$$E=[-\frac{8\pi}{7},-\frac{4\pi}{7}]\cup[\frac{4\pi}{7},\frac{6\pi}{7}]\cup[\frac{24\pi}{7},\frac{32\pi}{7}]$$

et ψ définie par $\hat\psi=\chi_E(\xi)$. On vérifie immédiatement que

$$\sum_{k\in\mathbb{Z}}\hat\psi(\xi+2k\pi)=1\qquad\text{p.p.}$$

Il est alors immédiat de vérifier que $(2^{j/2}\psi(2^jx-k))_{j,k\in\mathbb{Z}}$ est une base orthonormée de L^2.

Si V_0 est l'espace engendré par les $\psi_{\ell,k}$, $\ell<0$, il est immédiat qu'une fonction de V_0 a son support contenu dans l'adhérence de

$$\bigcup_{\ell<0}\operatorname{supp}\hat\psi\Big(\frac{\xi}{2^\ell}\Big)\,,$$

et donc dans

$$[-\frac{4\pi}{7}, -\frac{4\pi}{7}] \cup [\frac{6\pi}{7}, \frac{8\pi}{7}] \cup [\frac{12\pi}{7}, \frac{16\pi}{7}] = F .$$

Or si $\operatorname{supp} \hat{f} \subset F$, on a

$$C(f,f) = \sum_{k \in 2\pi\mathbb{Z}} |\hat{f}(\xi + 2k\pi)|^2$$

nulle sur $[8\pi/7, 10\pi/7]$, de sorte que V_0 ne contient pas de famille de Riesz de la forme $(f(x-k))_{k\in\mathbb{Z}}$, et a fortiori n'a pas de base de Riesz invariante par translation entière.

On remarquera que $x^k \hat{\psi}$ est de carré intégrable pour tout k, tandis que $|x|^\varepsilon \psi$ n'est de carré intégrable que si $\varepsilon < 1/2$. Si $|x|^\varepsilon \psi$ avait été de carré intégrable pour un $\varepsilon > 1/2$, nos résultats auraient impliqué l'existence d'une fonction d'échelle φ associée à ψ. Notre critère est donc optimal.

Annexe A: Poids et algébres de Beurling.

Nous présentons dans cette annexe quelques résultats classiques sur les poids de Beurling.

Rappelons qu'un *poids de Beurling* [4] est une fonction positive ω sur $\mathbb{R}^n$ telle que pour deux constantes C et M positives on ait

i) pour tout $x \in \mathbb{R}^n$, $1C \le \omega(x) \le C\,(1 + \|x\|)^M$,

ii) $\displaystyle\int_{\mathbb{R}^n} \frac{dx}{\omega(x)} < +\infty$,

iii) pour tout $x \in \mathbb{R}^n$, $\displaystyle\int_{\mathbb{R}^n} \frac{1}{\omega(x-y)}\,\frac{1}{\omega(y)}\,dy \le C\,\frac{1}{\omega(x)}$,

iv) pour tous $x, y \in \mathbb{R}^n$, $\omega(x+y) \le C\,\omega(x)\,\omega(y)$.

On commencera par remarquer qu'on peut discrétiser ces conditions de la manière suivante:

Lemme 12. *ω est un poids de Beurling si et seulement si elle vérifie pour deux constantes C et M strictement positives*

j) *pour tout $k \in \mathbb{Z}^n$, $0 < \omega(k) \le C\,(1 + \|k\|)^M$,* jj) $\sum_{k \in \mathbb{Z}^n} \frac{1}{\omega(k)} < +\infty$,

jjj) *pour tout* $k \in \mathbb{Z}^n$, $\sum_{p \in \mathbb{Z}^n} \frac{1}{\omega(k-p)} \frac{1}{\omega(p)} \leq C \frac{1}{\omega(k)}$,

jv) *pour tout* $y \in [0,1]^n$, *pour tout* $k \in \mathbb{Z}^n$,

$$\frac{1}{C} \omega(k) \leq \omega(k+y) \leq C\, \omega(k).$$

Ce lemme est immédiat. Une suite $(\omega(k))_{k \in \mathbb{Z}^n}$ qui vérifie les conditions j) à jjj) sera appelée un *poids de Beurling sur* $\mathbb{Z}^n$.

A.1: Les algèbres de Beurling.

A un poids de Beurling ω sur $\mathbb{R}^n$ on associe les espaces fonctionnels suivants

- $L^2_\omega = L^2(\omega\, dx)$ muni de la norme $\|f\|_{L^2_\omega} = \|\sqrt{\omega} f\|_2$,
- $H_\omega = \{f \in L^2 :$ il existe $g \in L^2_\omega,\ f = \hat{g}\}$ muni de la norme

$$\|f\|_{H_\omega} = \|g\|_{L^2_\omega} \qquad (f = \hat{g}),$$

- $M_\omega = \{m \in L^\infty :$ pour tout $f \in H_\omega,\ mf \in H_\omega\}$ muni de la norme

$$\|m\|_{H_\omega} = \sup_{\|f\|_{H_\omega} \leq 1} \|mf\|_{H_\omega} .$$

Si $(\omega(k))_{k \in \mathbb{Z}^n}$ est un poids de Beurling sur $\mathbb{Z}^n$, on lui associe l'espace fonctionnel suivant (en notant $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$):

- $K_\omega = \{f \in L^2(\mathbb{T}^n) :\ f = \sum_{k \in \mathbb{Z}^n} a_k e^{-ikx},\ \sum_{k \in \mathbb{Z}^n} |a_k|^2 \omega(k) < +\infty\}$,

muni de la norme

$$\|\sum_k a_k e^{-ikx}\|_{K_\omega} = \Big(\sum_k |a_k|^2 \omega(k)\Big)^{1/2}.$$

Enfin on note $C^0(\mathbb{R}^n)$ l'espace des fonctions continues bornées sur $\mathbb{R}^n$, muni de la norme $\|\cdot\|_\infty$. C'est une algèbre de Banach pour la multiplication ponctuelle des fonctions. Les *algèbres de Wiener* $A(\mathbb{R}^n)$ et $A(\mathbb{T}^n)$ sont les sous-algèbres de $C^0(\mathbb{R}^n)$ définies par

- $A(\mathbb{R}^n) = \{f \in C^0(\mathbb{R}^n) :$ il existe $g \in L^1,\ f = \hat{g}\}$

muni de la norme

$$\|f\|_{A(\mathbb{R}^n)} = \|g\|_1 \ ,$$

- $A(\mathbb{T}^n) = \{f = \sum_{k\in\mathbb{Z}^n} a_k e^{-ikx} : \sum_{k\in\mathbb{Z}^n} |a_k| \leq +\infty\}$

muni de la norme

$$\|\sum_k a_k e^{-ikx}\|_{A(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} |a_k| \ .$$

Proposition 5.

a) Les espaces L^2_ω, H_ω, M_ω ou K_ω sont des espaces de Banach.

b) L^2_ω s'injecte continûment dans $L^1(\mathbb{R}^n)$ et c'est une sous-algèbre de $L^1(\mathbb{R}^n)$ pour la convolution.

c) H_ω s'injecte continûment dans $A(\mathbb{R}^n)$ et c'est une sous-algèbre de $A(\mathbb{R}^n)$ pour la multiplication ponctuelle.

d) M_ω s'injecte continûment dans $C^0(\mathbb{R}^n)$ et c'est une sous-algèbre de $C^0(\mathbb{R}^n)$ pour la multiplication ponctuelle.

e) K_ω s'injecte continûment dans $A(\mathbb{T}^n)$ et c'est une sous-algèbre de $A(\mathbb{T}^n)$ pour la multiplication ponctuelle.

f) Si ω est un poids de Beurling sur $\mathbb{R}^n$, alors $K_\omega \subset M_\omega$ et les normes $\|\cdot\|_{K_\omega}$ et $\|\cdot\|_{M_\omega}$ sont équivalentes sur K_ω.

DÉMONSTRATION. *a)* Ce point est évident pour L^2_ω, H_ω et K_ω (qui sont des espaces de Hilbert). Pour le cas de M_ω, on commence par vérifier que si $m \in M_\omega$ alors d'une part $\|m\|_{M_\omega} < +\infty$ de sorte que $\|\cdot\|_{M_\omega}$ définit bien une norme sur M_ω et d'autre part que M_ω s'injecte continûment dans $C^0(\mathbb{R}^n)$.

En effet, si $m \in M_\omega$ alors l'application $f \mapsto mf$ est continue sur H_ω d'après le Théorème du Graphe Fermé: si $f_n \mapsto f$ dans H_ω et $mf_n \mapsto g$ dans H_ω alors $mf_n \mapsto mf$ dans L^2 et $mf_n \to g$ dans L^2, de sorte que $g = mf$ dans L^2 et donc dans H_ω. Cela montre que $\|m\|_{M_\omega} < +\infty$. Par ailleurs, si $\varphi \in C^\infty_C$ et $\varphi(0) = 1$, il suffit de remarquer que $\varphi(x - x_0) \in H_\omega$ quel que soit x_0, que $\|\varphi(x - x_0)\|_{H_\omega} = \|\varphi\|_{H_\omega}$, que

$L^2_\omega \subset L^1$ (*cf.* *b*)) et donc que $H_\omega \subset A(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$ pour conclure que

$$\begin{aligned}|m(x_0)| = |m(x_0)\varphi(0)| &\le C\|m(x)\varphi(x-x_0)\|_{H_\omega} \\ &\le C\|\varphi(x-x_0)\|_{H_\omega}\|m\|_{M_\omega} \\ &\le C\|\varphi\|_{H_\omega}\|m\|_{M_\omega}\end{aligned}$$

et donc que M_ω s'injecte continûment dans $C^0(\mathbb{R}^n)$.

Pour vérifier que M_ω est complet, il suffit de vérifier que pour toute suite (m_n) de fonctions telle que $\sum_{m\in\mathbb{N}}\|m_n\|_{M_\omega} < +\infty$ la série $\sum_n m_n$ converge dans M_ω. D'après les remarques précédentes, on voit que $\sum_n m_n$ converge dans $C^0(\mathbb{R}^n)$; soit m sa limite dans $C^0(\mathbb{R}^n)$. Si $f \in H_\omega$, on a $mf = \sum_n m_n f$ avec

$$\sum_n \|m_n f\|_{H_\omega} \le \|f\|_{H_\omega}\sum_{n\in\mathbb{N}}\|m_n\|_{M_\omega}\,.$$

Cela suffit pour conclure que $m \in M_\omega$ et que $\sum m_n$ converge dans M_ω.

b) Par Cauchy-Schwartz

$$\|f\|_1 \le \|f\|_{L^2_\omega}\Big(\int\frac{dx}{\omega(x)}\Big)^{1/2}$$

et donc $L^2_\omega \subset L^1$. Par ailleurs

$$\begin{aligned}&\int\Big|\int f(x-y)\,g(y)\,dy\Big|^2\omega(x)\,dx \\ &\qquad\le \iint |f(x-y)|^2\,\omega(x-y)\,|g(y)|^2\,\omega(y)\,dy \\ &\qquad\qquad\cdot\int\frac{1}{\omega(x-y)}\,\frac{1}{\omega(y)}\,dy\;\omega(x)\,dx \\ &\qquad\le C\iint |f(x-y)|^2\,\omega(x-y)\,|g(y)|^2\,\omega(y)\,dy\,dx \\ &\qquad\le C\,\|f\|^2_{L^2_\omega}\,\|g\|^2_{L^2_\omega}\end{aligned}$$

et donc $f*g \in L^2_\omega$ si $f,g \in L^2_\omega$.

c) est immédiat, les propriétés de H_ω se déduisant de celles de L^2_ω par trasformation de Fourier.

d) est immédiat puisque l'inclusion $M_\omega \subset C^0(\mathbb{R}^n)$ a déjà été établie et que M_ω est stable par multiplication ponctuelle par définition même de M_ω.

e) se montre de manière analogue à *b*):

$$\sum_k |a_k| \le \Big(\sum_k |a_k|^2\omega(k)\Big)\Big(\sum_k \frac{1}{\omega(k)}\Big)^{1/2}$$

de sorte que $K_\omega \subset A(\mathbb{T}^n)$. De plus,

$$\Big(\sum_k a_k\, e^{-ikx}\Big)\Big(\sum_k b_k\, e^{-ikx}\Big) = \sum_k \Big(\sum_p a_{k-p}\, b_p\Big) e^{-ikx}$$

et

$$\begin{aligned}\sum_k \Big|\sum_p a_{k-p}\, b_p\Big|^2 \omega(k) &\le \sum_k \Big(\sum_p |a_{k-p}|^2\omega(k-p)\, |b_p|^2\omega(p)\Big)\\ &\quad\cdot\Big(\sum_k \frac{1}{\omega(k-p)}\,\frac{1}{\omega(p)}\Big)\omega(k)\\ &\le C\sum_k \Big(\sum_p |a_{k-p}|^2\omega(k-p)\, |b_p|^2\omega(p)\Big)\\ &\le \Big(\sum_k |a_k|^2\,\omega(k)\Big)\Big(\sum_k |b_k|^2\,\omega(k)\Big)\end{aligned}$$

de sorte que $fg \in K_\omega$ si $f, g \in K_\omega$.

Enfin *f*) est facile à vérifier. En effet, si $f \in L^2_\omega$ et $g = \sum_k a_k\, e^{-ikx} \in K_\omega$ il s'agit de vérifier que la fonction h, définie par $\hat{h}(\xi) = g(\xi)\hat{f}(\xi)$, est dans L^2_ω. On commence par remarquer que $\|f\|_{L^2(\omega)}$ est équivalente à

$$\Big(\sum_{k\in\mathbb{Z}^n}\Big(\int_{k+[0,1]^n} |f(y)|^2\, dy\Big)\,\omega(k)\Big)^{1/2}$$

puisque $\omega(k+y) \approx \omega(k)$ pour $y \in [0,1]^n$. Or $h(x) = \sum_{p\in\mathbb{Z}^n} a_p\, f(x-p)$, de sorte que par Minkowski:

$$\Big(\int_{k+[0,1]^n} |h(y)|^2\, dy\Big)^{1/2} \le \sum_{p\in\mathbb{Z}^n} |a_p|\Big(\int_{k-p+[0,1]^n} |f(y)|^2\, dy\Big)^{1/2}.$$

La même démonstration qu'au point e) permet donc de conclure que $h \in L^2_\omega$. On obtient ainsi que K_ω s'injecte continûment dans M_ω.

Il reste alors à vérifier que la norme $\|\cdot\|_{K_\omega}$ se contrôle par la norme $\|\cdot\|_{M_\omega}$. Pour cela, on note χ la fonction caractéristique de $[0,1]^n$. On a $\chi \in L^2_\omega$ de sorte que si $\sum a_k e^{-ikx} \in K_\omega$ on a

$$\begin{aligned}\sum_k |a_k|^2 \omega(k) &\leq C \, \| \sum_k a_k \, \chi(x-k) \|^2_{L^2_\omega} \\ &\leq C \, \|\chi\|^2_{L^2_\omega} \, \| \sum_k a_k \, e^{-ikx} \|^2_{M_\omega}\end{aligned}$$

et la proposition est donc entièrement démontrée.

A.2: Un critère d'appartenance à $\mathbf{M}_\omega$.

Nous allons donner un critère simple d'appartenance à l'espace des multiplicateurs M_ω. Nous utiliserons pour cela la caractérisation de M_ω donnée dans le livre de Coifman et Meyer ([8, Chapitre I, Proposition 1]).

Lemme 13. *Soit ω un poids de Beurling sur $\mathbb{R}^n$ et φ une fonction C^∞ à support compact dans $\mathbb{R}^n$ telle que $\sum_{k \in \mathbb{Z}^n} \varphi(x-k) = 1$. Alors pour une fonction m bornée les assertions suivantes sont équivalentes:*

i) $m \in M_\omega$.

ii) *Pour tout $k \in \mathbb{Z}^n$,*

$$m(x)\,\varphi(x-k) \in H_\omega \qquad \textit{et} \qquad \sup_{k \in \mathbb{Z}^n} \|m(x)\,\varphi(x-k)\|_{H_\omega} < +\infty .$$

De plus les normes $\|m\|_{M_\omega}$ et $\sup_{k \in \mathbb{Z}^n} \|m(x)\,\varphi(x-k)\|_{H_\omega}$ sont équivalentes.

Le critère que nous utiliserons sera alors le suivant:

Lemme 14. *Si M est un exposant tel que $\omega(x) \leq C\,(1 + \|x\|)^M$ et si N est un nombre entier tel que $N > (n+M)/2$, alors une condition suffisante pour qu'une fonction m appartienne à M_ω est que m soit bornée ainsi que toutes ses dérivées jusqu'à l'ordre N. De plus, on a*

$$\|m\|_{M_\omega} \leq C \sum_{|\alpha| \leq N} \left\| \frac{\partial^\alpha m}{\partial x^\alpha} \right\|_\infty . \tag{36}$$

DÉMONSTRATION. Il s'agit d'estimer $\sup_k \|m(x)\,\varphi(x-k)\|_{H_\omega}$; or on a

$$\|m(x)\,\varphi(x-k)\|_{H_\omega} = \|m(x+k)\,\varphi(x)\|_{H_\omega} \,.$$

Si on pose $f_k \in L^2$ comme $\hat{f}_k = m(x+k)\,\varphi(x)$, on a $\operatorname{supp} \hat{f}_k \subset B(0,R)$ pour un R indépendant de k et de m et

$$\sum_{|\alpha|\le N} \left\|\frac{\partial^\alpha \hat{f}_k}{\partial x^\alpha}\right\|_\infty \le C \sum_{|\alpha|\le N} \left\|\frac{\partial^\alpha m}{\partial x^\alpha}\right\|_\infty$$

pour une constante C indépendante de k et de m.

On obtient alors par transformation de Fourier inverse

$$|f_k(x)| \le C\,(1+\|x\|)^{-N} \sum_{|\alpha|\le N} \left\|\frac{\partial^\alpha m}{\partial x^\alpha}\right\|_\infty$$

et le majorant est dans L^2_ω dès que $2N - M > n$.

A.3. Le Lemme de Wiener pour l'algèbre de Beurling $\mathbf{K}_\omega$.

Nous rappelons ici un calcul classique pour les sous-algèbres de $A(\mathbb{T}^n)$. Nous empruntons notre démonstration au livre de Rudin [21, Chapitre 11].

Lemme 15 (Lemme de Wiener). *Soit B un espace de Banach complexe de fonctions continues sur $[0,2\pi]^n$ tel que*

i) $B \subset A(\mathbb{T}^n)$ *et* $\|f\|_{A(\mathbb{T}^n)} \le C\,\|f\|_B$,

ii) *pour tout* $f \in B$, *pour tout* $g \in B$, $fg \in B$,

iii) *Les polynômes trigonométriques sont denses dans* B,

iv) *Ils existent* C, M, *telles que pour tout* $k \in \mathbb{Z}^n$,

$$\|e^{ikx}\|_B \le C\,(1+\|k\|)^M \,.$$

Dans ces conditions, si $f \in B$ *et si pout tout* $x \in [0,2\pi]^n$, $f(x) \neq 0$ *alors* $1/f \in B$.

DÉMONSTRATION. On commence par remarquer que B est une algèbre de Banach commutative. En effet, à g fixé, l'application $f \mapsto fg$ est continue d'après le Théorème du Graphe Fermé (si $f_n \to f$ dans B et $f_n g \to h$ dans B, alors $f_n g \to fg$ dans $A(\mathbb{T}^n)$ et $f_n g \to h$ dans $A(\mathbb{T}^n)$, d'où $f = fg$ dans $A(\mathbb{T}^n)$ donc dans B); comme l'application $g \mapsto fg$ est également continue, on en conclut que

$$\sup\{\|fg\|_B : \ \|g\|_B \leq 1\} < +\infty$$

pour tout f et donc que les applications $f \mapsto fg$, $g \in B$ et $\|g\|_B \leq 1$ sont équicontinues d'après le Théorème de Banach-Steinhaus. Cela nous permet de conclure que, pour une constante C indépendante de f et de g,

$$\|fg\|_B \leq C\,\|f\|_B\,\|g\|_B \,.$$

D'après un théorème général sur les algèbres de Banach complexes commutatives, on sait que f est inversible dans B si et seulement si pour tout homomorphisme complexe h de B dans $\mathbb{C}$ on a $h(f) \neq 0$. Le lemme de Wiener se ramène alors à montrer que tout homomorphisme h est de la forme $h(f) = f(x_0)$ pour un $x_0 \in [0, 2\pi]^n$.

Pour cela, on note g_r l'exponentielle $g_r(x) = e^{ix_r}$ $(x = (x_1, \dots, x_n))$ pour $r = 1, \dots, n$. On a

$$|h(g_r)|^n = |h(g_r^n)| \leq C\,\|g_r^n\|_B \leq C'(1+n)^M\,;$$

cela entraîne que $|h(g_r)| \leq 1$; on a de même $|h(1/g_r)| \leq 1$ et donc $|h(g_r)| = 1$. Il existe alors $y_r \in [0, 2\pi]$ tel que $h(g_r) = e^{iy_r}$. Si on pose $x_0 = (y_1, \dots, y_n)$ on a $h(g_r) = g_r(x_0)$ pour $r = 1, \dots, n$. Cette propriété passe aux polynômes trigonométriques pour des raisons algébriques puis à toute fonction de B par densité des polynômes trigonométriques dans B. Le Lemme de Wiener est alors démontré.

Le Lemme de Wiener s'applique en particulier à l'algèbre $B = K_\omega$, où ω est un poids de Beurling sur $\mathbb{Z}^n$. En corollaire, on obtient

Lemme 16. *Soit ω un poids de Beurling sur $\mathbb{R}^n$ tel que $\omega(x) = \omega(-x)$. Soit $(f_\delta)_{1 \leq \delta \leq D}$ une famille de fonctions dans L^2_ω. Alors*

i) *la famille $(f_\delta(x-k))_{1 \leq \delta \leq D,\, k \in \mathbb{Z}^n}$ est presque-orthogonale dans $L^2(\mathbb{R}^n)$ et la matrice d'auto-corrélation de f_δ, $1 \leq \delta \leq D$, est à coefficients dans K_ω,*

ii) *si la famille* $(f_\delta(x-k))_{1\leq\delta\leq D,\,k\in\mathbb{Z}^n}$ *est une base de Riesz d'un sous-espace* V *de* $L^2(\mathbb{R}^n)$ *alors la base duale* $(f_\delta^*(x-k))$ *de* $(f_\delta(x-k))_{1\leq\delta\leq D,\,k\in\mathbb{Z}^n}$ *dans* V *et la base orthonormée* $(\psi_\delta(x-k))$ *de* V *obtenue à partir de* $(f_\delta(x-k))$ *par le procédé d'orthonormalisation de Gram du Lemme 3 vérifient* $f_\delta^*\in L^2_\omega$ *et* $\psi_\delta\in L^2_\omega$.

DÉMONSTRATION. Pour vérifier le point i), on note

$$\varepsilon_p(f)=\left[\int_{p+[0,1]^n}|f(x)|^2\,dx\right]^{1/2}$$

et on remarque comme précédemment que la norme $\|f\|_{L^2_\omega}$ est équivalente à

$$\Big(\sum_{p\in\mathbb{Z}^n}\varepsilon_p(f)^2\,\omega(p)\Big)^{1/2}.$$

Or on a

$$\begin{aligned}|\langle f_\delta,f_{\delta'}(x-k)\rangle| &\leq \sum_{p\in\mathbb{Z}^n}\int_{p+[0,1]^n}|f_\delta(x)|\,|f_{\delta'}(x-k)|\,dx\\ &\leq \sum_{p\in\mathbb{Z}^n}\varepsilon_p(f_\delta)\,\varepsilon_{p-k}(f_{\delta'})\,.\end{aligned}$$

Comme les coefficients de la matrice d'auto-corrélation sont donnés par

$$C(f_\delta,f_{\delta'})=\sum_{k\in\mathbb{Z}^n}\langle f_\delta,f_{\delta'}(x-k)\rangle\,e^{-ik\xi}\,,$$

on voit que ces coefficients sont bien dans K_ω (d'après la démonstration du point e) de la Proposition 5). En particulier, ils sont bornés et la famille $(f_\delta(x-k))_{1\leq\delta\leq D,\,k\in\mathbb{Z}^n}$ est presque-orthogonale dans $L^2(\mathbb{R}^n)$.

Le point ii) est alors immédiat. La matrice d'auto-corrélation M des f_δ est à coefficients dans l'algèbre K_ω; de plus son déterminant s'inverse dans K_ω d'après le Lemme de Wiener. On en conclut que l'inverse de cette matrice est à coefficients dans K_ω. On vérifie de même que la racine carrée inverse de M est à coefficients dans K_ω, puisque

$$M^{-1/2}=\frac{2}{\pi}\int_0^{+\infty}(I+t^2M)^{-1}\,dt\,.$$

Comme K_ω est inclus dans M_ω, on obtient finalement $f_\delta^* \in L^2_\omega$ et $\psi_\delta \in L^2_\omega$. Le lemme est alors démontré.

Annexe B: Géométrie des dilatations.

Pour étudier la géométrie introduite par l'opération d'une dilatation A sur $\mathbb{R}^n$, nous avons introduit la notion de pseudo-norme; rappelons que selon la Définition 3 une *pseudo-norme* sur $(\mathbb{R}^n, A)$ est une fonction ρ définie sur $\mathbb{R}^n$ telle que

a) ρ est C^∞ sur $\mathbb{R}^n\backslash\{0\}$ et continue en 0,

b) pour tout $x \neq 0$, $\rho(x) > 0$ et $\rho(x) = \rho(-x)$,

c) $x \in \mathbb{R}^n$, $\rho(Ax) = |\det A|\, \rho(x)$.

Nous commençons par vérifier que les pseudo-normes existent et sont uniques à un ordre de grandeur près de sorte qu'on pourra parler de "la" pseudo-norme de $(\mathbb{R}^n, A)$.

Lemme 17.

i) *Il existe des pseudo-normes sur* $(\mathbb{R}^n, A)$.

ii) *Si* ρ_1 *et* ρ_2 *sont deux pseudo-normes sur* $(\mathbb{R}^n, A)$, *il existe une constante* $C > 0$ *telle que pour tout* $x \in \mathbb{R}^n$ *on ait*

$$\frac{1}{C}\, \rho_1(x) \leq \rho_2(x) \leq C\, \rho_1(x)\,.$$

En effet, si $1 < \alpha < \beta$ sont tels que les valeurs propres de A aient leurs modules dans l'intervalle $]\alpha, \beta[$, il existe une constante $C_0 > 0$ telle que pour tout $x \in \mathbb{R}^n$ et pour tout $k \geq 0$,

$$\frac{1}{C_0}\, \alpha^k \|x\| \leq \|A^k x\| \leq C_0\, \beta^k \|x\|\,. \tag{37}$$

Si $k \leq 0$, on a alors

$$\frac{1}{C_0}\, \beta^k \|x\| \leq \|A^k x\| \leq C_0\, \alpha^k \|x\|\,.$$

On fixe une fonction $\varphi \in C_c^\infty(\mathbb{R}^n)$ telle que $\varphi(x) = \varphi(-x)$,

$$\operatorname{supp}\varphi \subset \{x :\ D_1 \leq \|x\| \leq D_2\}$$

et $\varphi \equiv 1$ sur $\{x : \ C_1 \le \|x\| \le C_2\}$ où $C_2 > \beta C_1 C_0$.

La fonction

$$\rho(x) = \sum_{k \in \mathbb{Z}^n} |\det A|^{-k} \varphi(A^k x)$$

est alors une somme localement finie sur $\mathbb{R}^n \backslash \{0\}$ de fonctions C^∞ (car $\lim_{k\to-\infty} A^k x = 0$ et $\lim_{k\to+\infty} A^k x = +\infty$ d'après (37)), de sorte qu'elle est C^∞ sur $\mathbb{R}^n\backslash\{0\}$; elle est continue en 0 car elle est normalement sommable au voisinage de 0. Il est immédiat que $\rho(Ax) = |\det A|\, \rho(x)$. Elle vérifie $\rho(x) = \rho(-x)$. Il ne reste qu'à vérifier que $\rho(x) \neq 0$. Si $x \neq 0$, il existe k tel que $\|A^k x\| \le C_1$ puisque $A^k x \to 0$ quand $k \to -\infty$, et pour k assez grand $\|A^k x\| > C_1$ puisque $A^k x \to +\infty$ quand $k \to +\infty$. Si

$$k_0 = \max\{k : \ \|A^k x\| \le C_1\},$$

on a

$$C_1 < \|A^{k_0+1} x\| \le C_0\, \beta \|A^{k_0} x\| \le C_0\, \beta\, C_1 < C_2$$

et donc

$$\rho(x) \ge |\det A|^{-k_0-1} > 0.$$

Si ρ_1 est une autre pseudo-norme sur $(\mathbb{R}^n, A)$, on remarque que

$$\{x \in \mathbb{R}^n : \ 1 \le \rho(x) \le |\det A|\}$$

est fermé (puisque ρ est continue) et borné (puisque si $\|x\| > \beta^2 C_1 C_0$ on a $\|A^{-2}x\| > C_1$ de sorte que $k_0(x) \le -3$ et $\rho(x) \ge |\det A|^2$) donc compact; en particulier puisque ρ_1 est continue et > 0 sur ce compact, on a pour $1 \le \rho(x) \le |\det A|$,

$$\frac{1}{C} \le \rho_1(x) \le C,$$

d'où si x est quelconque $\neq 0$ et si k est tel que $|\det A|^k \le \rho(x) \le |\det A|^{k+1}$:

$$\begin{aligned}\frac{1}{C}\,\rho(x) &= \frac{1}{C}\,|\det A|^k \rho(A^{-k}x)\\ &\le |\det A|^{k+1} \rho_1(A^{-k}x)\\ &= |\det A| \rho_1(x)\end{aligned}$$

et

$$\begin{aligned}\rho_1(x) &= |\det A|^k \rho_1(A^{-k}x)\\ &\leq C\,|\det A|^k\\ &\leq C\,|\det A|^k \rho(A^{-k}x)\\ &= C\,\rho(x)\,.\end{aligned}$$

Le Lemme 17 est donc démontré.

EXEMPLE 1. Si $A = \lambda I$ avec $\lambda > 1$, la pseudo-norme est $\rho(x) = \|x\|^n$.

B.1. Propriétés de la pseudo-norme.

- Comparaison avec la norme usuelle:

Lemme 18. *Il existe deux constantes α_0 et α_1 telles que $0 < \alpha_0 < \alpha_1$ et une constante M_0 telle que*

i) $\dfrac{1}{M_0}\,\|x\|^{\alpha_1} \leq \rho(x) \leq M_0\,\|x\|^{\alpha_0}$ si $\|x\| \leq 1$,

ii) $\dfrac{1}{M_0}\,\|x\|^{\alpha_0} \leq \rho(x) \leq M_0\,\|x\|^{\alpha_1}$ si $\|x\| \geq 1$.

En effet, sur $K = \{x :\ 1 \leq \|x\| \leq \beta\}$ la norme ρ est bornée et atteint ses bornes $m \leq \rho(x) \leq M$. De plus pour tout $x \in \mathbb{R}^n\backslash\{0\}$ il existe k_0 tel que $\|A^{k_0}x\| \leq 1$ et $\|A^{k_0+1}x\| > 1$ ($k_0 = \max\{k :\ \|A^kx\| \leq 1\}$) et alors

$$\|A^{k_0+1}x\| \leq \beta\,\|A^{k_0}x\| \leq \beta\,,$$

de sorte que l'on a

$$\begin{aligned}m\,|\det A|^{-k_0-1} &\leq |\det A|^{-k_0-1}\rho(A^{k_0+1}x)\\ &= \rho(x)\\ &\leq M\,|\det A|^{-k_0-1}\,.\end{aligned}$$

Si $k_0 \geq 0$, on a

$$\|A^{k_0+1}x\| \in \left[\frac{1}{C_0}\,\alpha^{k_0+1}\|x\|\,,\ C_0\,\beta^{k_0+1}\|x\|\right]$$

et donc

$$\frac{1}{C_0}\,\alpha^{k_0+1}\|x\| \leq \beta \qquad \text{et} \qquad C_0\,\beta^{k_0+1}\|x\| \geq 1\,,$$

d'où

$$m\left(\frac{1}{\beta C_0}\|x\|\right)^{(\log|\det A|)/\log\alpha} \leq \rho(x) \leq M\,(C_0\,\|x\|)^{(\log|\det A|)/\log\beta}$$

tandis que si $k_0 \leq -1$,

$$\|A^{k_0+1}x\| \in \left[\frac{1}{C_0}\,\beta^{k_0+1}\|x\|\,,\ C_0\,\alpha^{k_0+1}\|x\|\right]$$

et donc

$$\frac{1}{C_0}\,\beta^{k_0+1}\|x\| \leq \beta \qquad \text{et} \qquad C_0\,\alpha^{k_0+1}\|x\| \geq 1\,,$$

d'où

$$m\left(\frac{1}{\beta C_0}\,\|x\|\right)^{(\log|\det A|)/\log\beta} \leq \rho(x) \leq M\,(C_0\,\|x\|)^{(\log|\det A|)/\log\alpha}$$

et le lemme est démontré avec

$$\alpha_0 = \frac{\log|\det A|}{\log\beta} \qquad \text{et} \qquad \alpha_1 = \frac{\log|\det A|}{\log\alpha}\,.$$

- Intégrale de Riemann:

Lemme 19.

a) $\displaystyle\int_{\rho(x)\geq 1}\frac{dx}{\rho(x)^\varepsilon} < +\infty$ *si et seulement si* $\varepsilon > 1$.

b) $\displaystyle\int_{\rho(x)\leq 1}\frac{dx}{\rho(x)^\varepsilon} < +\infty$ *si et seulement si* $\varepsilon < 1$.

En effet

$$\int_{|\det A|^{k_0}\leq\rho(x)\leq|\det A|^{k_1+1}}\frac{dx}{\rho(x)^\varepsilon} = \sum_{k_0}^{k_1}\int_{|\det A|^{k}\leq\rho(x)\leq|\det A|^{k+1}}\frac{dx}{\rho(x)^\varepsilon}$$
$$= \left(\sum_{k_0}^{k_1}|\det A|^{(1-\varepsilon)k}\right)\int_{1\leq\rho(x)\leq|\det A|}\frac{dx}{\rho(x)}\,.$$

- Inégalité triangulaire:

Lemme 20. *Il existe une constante C_1 telle que pour tous $x, y \in \mathbb{R}^n$,*

(38) $$\rho(x+y) \leq C_1\,(\rho(x)+\rho(y))\,.$$

En effet supposons $\rho(x) \geq \rho(y)$ et $|\det A|^k \leq \rho(x) \leq |\det A|^{k+1}$. Alors

$$\rho(x+y) = |\det A|^k \rho(A^{-k}x + A^{-k}y) \leq \rho(x)\,\rho(A^{-k}x + A^{-k}y).$$

Or

$$\rho(A^{-k}x) \leq |\det A| \qquad \text{et} \qquad \rho(A^{-k}y) \leq \rho(A^{-k}x) \leq |\det A|.$$

Comme $K = \{(z,w):\ \rho(z) \leq |\det A| \text{ et } \rho(w) \leq |\det A|\}$ est compact, ρ est majorée sur K par une constante C_1. On obtient donc

$$\rho(x+y) \leq C_1 \rho(x) \leq C_1\,(\rho(x) + \rho(y)).$$

- Croissance de la norme:

Lemme 21. *Il existe une constante C_2 telle que pour tout $x \in \mathbb{R}^n$ et pour tout $\lambda \in [0,1]$,*

$$\rho(\lambda x) \leq C_2\,\rho(x). \tag{39}$$

A nouveau, on fixe k tel que $|\det A|^k \leq \rho(x) \leq |\det A|^{k+1}$; alors

$$C_2 = \sup\{\rho(\lambda z):\ \rho(z) \leq |\det A|,\ 0 \leq \lambda \leq 1\};$$

C_2 est bien défini puisque

$$\{z:\ \rho(z) \leq |\det A|\} \times [0,1]$$

est compact.

- Poids de Beurling:

Lemme 22. *Si $\varepsilon>1$, $\omega(x)=(1+\rho(x))^\varepsilon$ est un poids de Beurling sur $\mathbb{R}^n$.*

En effet, $1 \leq \omega(x)$ et $\omega(x) \leq C\,(1+\|x\|)^{\alpha_1 \varepsilon}$ d'après le Lemme 18. $\int(1/\omega)\,dx < +\infty$ d'après le Lemme 14. D'après le Lemme 20, on a

$$\omega(x+y) \leq C\,(\omega(x) + \omega(y)) \leq 2\,C\,\omega(x)\omega(y).$$

Enfin on a

$$\inf\left\{\frac{1}{\omega(y)}, \frac{1}{\omega(x-y)}\right\} \leq 2\,C\,\frac{1}{\omega(x)},$$

et donc
$$\int \frac{1}{\omega(y)}\frac{1}{\omega(x-y)}\,dy \leq 4\,C\int \frac{dy}{\omega(y)}\,\frac{1}{\omega(x)}\,.$$

B.2. Pseudo-norme et transformation de Fourier.

Nous rappelons quelques propriétés classiques de la transformée de Fourier des distributions homogènes par dilatation. Remarquons d'abord que la transformation de Fourier transforme l'action de la matrice de dilatation A en celle de la transposée $\tilde{A}$ de A

$$\widehat{f(Ax)}(\xi) = |\det A|^{-1}\hat{f}(\tilde{A}^{-1}\xi)\,. \tag{40}$$

$\tilde{A}$ est également une matrice de dilatation et nous appellerons $\tilde{\rho}$ une pseudo-norme associée à $\tilde{A}$.

Une fonction f est homogène de degré λ pour la dilatation A si $f(Ax) = |\det A|^{\lambda} f(x)$. Une distribution T est homogène de degré λ si on a pour tout $\varphi \in C_c^{\infty}$,

$$\langle T, |\det A|^{-1}\varphi(A^{-1}x)\rangle = |\det A|^{\lambda}\langle T,\varphi\rangle\,.$$

On voit par (21) que la transformée de Fourier d'une distribution homogène de degré λ pour A est une distribution homogène de degré $-1-\lambda$ pour $\tilde{A}$ (et réciproquement, A et $\tilde{A}$ jouant des rôles symétriques).

Nous fixons maintenant une fonction $\varphi \in C_c^{\infty}(\mathbb{R}^n)$ valant 1 au voisinage de 0. Pour $\varepsilon \in]0,1[$ nous notons $\gamma_\varepsilon(x) = \tilde{\rho}(x)^{\varepsilon}\,\varphi(x)$ et $\delta_\varepsilon(x) = \tilde{\rho}(x)^{-\varepsilon}\,\varphi(x)$. Ce sont deux fonctions intégrables sur $\mathbb{R}^n$ et on note Γ_ε et Δ_ε leurs transformées de Fourier inverses.

Notons α_0 et α_1 les exposants associés à ρ par le Lemme 18 et C_1 et C_2 les constantes associées à ρ par les Lemmes 20 et 21. On a alors le résultat suivant:

Lemme 23. *Pour tout $\varepsilon \in]0,1[$, il existe une constante C_ε telle que*

i) *pour tout $x \in \mathbb{R}^n$, $|\Gamma_\varepsilon(x)| \leq C_\varepsilon\,(1+\rho(x))^{-1-\varepsilon}$,*

ii) *pour tout $x \in \mathbb{R}^n$, $|\Delta_\varepsilon(x)| \leq C_\varepsilon\,(1+\rho(x))^{-1+\varepsilon}$,*

iii) *pour tous $x, y \in \mathbb{R}^n$ tels que $\rho(y) \leq \rho(x)/2C_1$*

$$|\Delta_\varepsilon(x+y) - \Delta_\varepsilon(x)| \leq C_\varepsilon\,\rho(y)^{\alpha_0}(1+\rho(x))^{-1+\varepsilon-\alpha_0}\,.$$

DÉMONSTRATION. Les estimations i), ii) et iii) sont évidentes lorsque $\rho(x) \leq 1$. En effet, Γ_ε et Δ_ε sont des fonctions C^∞ bornées ainsi que toutes leurs dérivées (parce qu'elles sont transformées de Fourier de fonctions intégrables à support compact) de sorte que $|\Gamma_\varepsilon(x)| \leq \|\Gamma_\varepsilon\|_\infty$, $|\Delta_\varepsilon(x)| \leq \|\Delta_\varepsilon\|_\infty$ et

$$|\Delta_\varepsilon(x+y) - \Delta_\varepsilon(x)| \leq M_0\, \rho(y)^{\alpha_0} \|\vec{\nabla}\Delta_\varepsilon\|_\infty$$

pour $\|y\| \leq 1$.

Pour vérifier i) avec $\rho(x) > 1$, on introduit la suite $F(N)$, $N \geq 0$, définie par

$$F(N) = |\det A|^{N(1+\varepsilon)} \sup\{|\Gamma_\varepsilon(x)| : \ |\det A|^{N-1} \leq \rho(x) \leq |\det A|^N\}.$$

Montrer i) équivaut à montrer que $F(N)$ reste bornée quand $N \to +\infty$. Or on a la propriété de "presque-homogénéité"

$$\Gamma_\varepsilon(Ax) = |\det A|^{-1-\varepsilon}\Gamma_\varepsilon(x) + \theta_\varepsilon(Ax),$$

avec

$$\theta_\varepsilon(x) = \frac{1}{2\pi} \int \tilde{\rho}(z)^\varepsilon \, (\varphi(z) - \varphi(\tilde{A}z))\, e^{izx}\, dz.$$

θ_ε est la transformée de Fourier d'une fonction C^∞ à support compact; c'est donc une fonction C^∞ à décroissance rapide et on a pour tout $Q > 0$

$$|\Gamma_\varepsilon(Ax)| \leq |\det A|^{-1-\varepsilon}|\Gamma_\varepsilon(x)| + C_Q\, \rho(Ax)^{-Q},$$

d'où si $Q > 1 + \varepsilon$

$$F(N+1) \leq F(N) + C_Q\, |\det A|^{N(1+\varepsilon-Q)}$$

ce qui entraîne

$$F(N) \leq F(0) + C_Q \frac{1}{1 - |\det A|^{1+\varepsilon-Q}}.$$

On a donc montré i).

La majoration de $|\Delta_\varepsilon(x)|$ par $C_\varepsilon\,(1 + \rho(x))^{-1+\varepsilon}$ se démontre de manière analogue.

Si $\rho(y) \leq 2C_1\rho(x)$, on a

$$\rho(x) \leq C_1\,(\rho(x+y) + \rho(y)) \leq C_1\, \rho(x+y) + \frac{1}{2}\, \rho(x)$$

de sorte que

$$|\Delta_\varepsilon(x+y)| \leq C_\varepsilon\,(1+\rho(x+y))^{-1-\varepsilon} \leq C'_\varepsilon\,(1+\rho(x))^{-1-\varepsilon}\,.$$

iii) se déduit donc immédiatement de ii) si $\rho(y)$ n'est pas trop petit,

$$\rho(y) \geq \frac{1}{2C_1C_2|\det A|}\,\rho(x)\,,$$

par exemple. Nous allons donc démontrer iii) en supposant de plus que

$$\rho(y) \leq \frac{1}{2C_1C_2|\det A|}\,\rho(x)\,.$$

Pour démontrer iii), on fixe N tel que $|\det A|^{N-1} \leq \rho(x) \leq |\det A|^N$ et on pose $v = A^{-N}y$; alors

$$\rho(v) = |\det A|^{-N}\rho(y) \leq \frac{\rho(y)}{\rho(x)} \leq \frac{1}{2C_1C_0|\det A|}\,.$$

L'inégalité

$$|\Delta_\varepsilon(x+y) - \Delta_\varepsilon(x)| \leq C_\varepsilon\,\frac{\rho(y)^{\alpha_0}}{\rho(x)^{\alpha_0}}\,(1+\rho(x))^{-1+\varepsilon}$$

se réécrit en

$$|\Delta_\varepsilon(x+A^Nv) - \Delta_\varepsilon(x)| \leq C'_\varepsilon\,\rho(v)^{\alpha_0}(1+\rho(x))^{-1+\varepsilon}\,,$$

et c'est cette égalité-là que nous allons démontrer.

Plus précisément, pour $v \in \mathbb{R}^n$ et $N \geq 0$, on pose

$$F(N,v) = |\det A|^{N(1-\varepsilon)} \sup\big\{|\Delta_\varepsilon(x+A^Nv) - \Delta_\varepsilon(x)| : \\ |\det A|^{N-1} \leq \rho(x) \leq |\det A|^N\big\}\,.$$

Il s'agit de démontrer qu'il existe une constante D_ε telle que, pour tout $N \geq 0$ et tout v tel que

$$\rho(v) \leq \frac{1}{2C_1C_2|\det A|}\,,$$

on a

$$|F(N,v)| \leq D_\varepsilon\,\rho(v)^{\alpha_0}\,.$$

On sait déjà que $|F(0,v)| \leq C\,\rho(v)^{\alpha_0}$. Par ailleurs, on sait que

$$\Delta_\varepsilon(Ax) = |\det A|^{-1+\varepsilon}\Delta_\varepsilon(x) + \zeta_\varepsilon(Ax)$$

avec

$$\zeta_\varepsilon(x) = \frac{1}{2\pi}\int \tilde{\rho}(z)^{-\varepsilon}\,(\varphi(z) - \varphi(\tilde{A}z))\,e^{izx}\,dz\,.$$

On a donc pour tout $Q > 0$

$$\begin{aligned}|\zeta_\varepsilon(Ax + A^{N+1}v) - \zeta_\varepsilon(Ax)|&\\ &\leq \|A^{N+1}v\|\,C_Q \sup_{\lambda\in[0,1]} \rho(Ax + \lambda A^{N+1}v)^{-Q}\,.\end{aligned}$$

D'après le Lemme 13, on a

$$\begin{aligned}\|A^{N+1}v\| &\leq M_0 \sup\{\rho(A^{N+1}v)^{\alpha_0}, \rho(A^{N+1}v)^{\alpha_1}\}\\ &\leq M_0\,|\det A|^{(N+1)\alpha_1}\rho(v)^{\alpha_0}.\end{aligned}$$

Par ailleurs, si $\rho(Ax) \geq |\det A|^N$, on a

$$\begin{aligned}\rho(Ax) &\leq C_1\,(\rho(Ax + \lambda A^{N+1}v) + \rho(\lambda A^{N+1}v))\\ &\leq C_1\,\rho(Ax + \lambda A^{N+1}v) + C_1\,C_2\,|\det A|^{N+1}\rho(v)\\ &\leq C_1\,\rho(Ax + \lambda A^{N+1}v) + \frac{1}{2}\,\rho(Ax)\end{aligned}$$

de sorte que si $Q > 1 + \alpha_1 - \varepsilon$ on obtient

$$F(N+1,v) \leq F(N,v) + D\,\rho(v)^{\alpha_0}|\det A|^{N(1+\alpha_1-\varepsilon-Q)}$$

et enfin

$$F(N,v) \leq F(0,v) + D\,\rho(v)^{\alpha_0}\frac{1}{1 - |\det A|^{1+\alpha_1-\varepsilon-Q}} \leq D_\varepsilon\,\rho(v)^{\alpha_0}$$

et le lemme est donc démontré.

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Bases orthonormées de paquets d'ondelettes

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1. Notations et énoncé du théorème principal.

Soit H un espace de Hilbert, muni d'une base hilbertienne $(e_k)_{k\in\mathbb{Z}}$. On considère deux filtres miroirs en quadrature, de fonctions de transfert

$$m_0(\theta) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h_k \, e^{ik\theta} \in C^\infty(\mathbb{R}, \mathbb{C})$$

et

$$m_1(\theta) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} g_k \, e^{ik\theta} = e^{-i\theta} \, \overline{m_0(\theta+\pi)} \, .$$

On suppose que m_0 vérifie les conditions usuelles,

$$m_0(0) = 1 \, , \qquad m_0(\theta) \neq 0 \, , \quad \text{si } |\theta| \leq \frac{\pi}{2} \, ,$$

et

$$|m_0(\theta)|^2 + |m_0(\theta+\pi)|^2 = 1 \, .$$

On pose

$$e_k^0 = \sum_{\ell\in\mathbb{Z}} h_{2k-\ell} \, e_\ell \, , \qquad e_k^1 = \sum_{\ell\in\mathbb{Z}} g_{2k-\ell} \, e_\ell \, .$$

On sait (voir [CMW]) que $(e_k^0)_{k\in\mathbb{Z}}$ et $(e_k^1)_{k\in\mathbb{Z}}$ sont les bases hilbertiennes de deux sous-espaces orthogonaux H^0 et H^1, et que $H = H^0 \oplus H^1$.

On préfère noter $H = H_{[0,1]}$, $H^0 = H_{[0,1/2]}$, $H^1 = H_{[1/2,1]}$.

La décomposition de H en H^0 et H^1 est alors associée à la décomposition $[0,1] = [0,1/2] \cup [1/2,1]$. On note $\mathcal{I}$ l'ensemble des intervalles dyadiques de $[0,1]$, c'est-à-dire du type

$$I = \left[\frac{\varepsilon_1}{2} + \cdots + \frac{\varepsilon_j}{2^j}\,,\ \frac{\varepsilon_1}{2} + \cdots + \frac{\varepsilon_j}{2^j} + \frac{1}{2^j}\right] .$$

En itérant le procédé de décomposition décrit plus haut, on obtient une famille $(H_I)_{I\in\mathcal{I}}$ d'espaces de Hilbert munis chacun d'une base $(e_{k,I})_{k\in\mathbb{Z}}$, avec la propriété suivante

- Si I^0 et I^1 sont les moitiés gauche et droite d'un intervalle $I \in \mathcal{I}$, alors $H_I = H_{I^0} \oplus H_{I^1}$, et la somme est orthogonale.

Une conséquence de cette propriété est que, si $(I_\alpha)_{\alpha\in A}$ est une partition finie de $[0,1]$ constituée d'intervalles dyadiques, alors

$$H = H_{[0,1]} = \bigoplus_{\alpha\in A} H_{I_\alpha} ,$$

et la somme est orthogonale.

Il est naturel d'essayer d'étendre cette décomposition au cas de partitions infinies.

On note $\pi_I : H \to H_I$ la projection orthogonale de H sur H_I.

Pour $x \in H$, $\|x\| = 1$, on pose $\mu_x(I) = \|\pi_I(x)\|^2$. On sait (voir [CMW]) que μ_x s'étend en une mesure de probabilité *continue* sur les Boréliens de $[0,1]$.

En choisissant, par exemple,

$$\sigma = \frac{1}{3}\sum_{k\in\mathbb{Z}} \frac{\mu_{e_k}}{2^{|k|}}$$

on obtient immédiatement le résultat suivant

Théorème 1. *Il existe une mesure de probabilité continue σ sur $[0,1]$ telle que, si $(I_n)_{n\geq 0}$ est une suite d'intervalles dyadiques deux à deux disjoints, les deux propriétés suivantes sont équivalentes*

i) $H = \overline{\bigoplus_{n\geq 0} H_{I_n}}$, *et la somme est orthogonale,*

ii) $\sigma\Big([0,1] \setminus \bigcup_{n\geq 0} I_n\Big) = 0$.

Bien sûr, le choix de σ n'est pas unique, et toute mesure $d\sigma' = g\, d\sigma$ avec $g \in L^1([0,1], d\sigma)$ et $g > 0$ σ-p.p. conviendrait.

Le but de ce travail est de démontrer, dans le cas des filtres d'Ingrid Daubechies, un résultat plus fort, conjecturé par Yves Meyer.

On suppose désormais (voir [D], [M]) que m_0 est la fonction de transfert d'un filtre à coefficients réels, de longueur finie $2N$, $N \geq 2$, c'est-à-dire que

$$\begin{cases} \sqrt{2}\, m_0(\theta) = h_0 + h_1\, e^{i\theta} + \cdots + h_{2N-1}\, e^{i(2N-1)\theta}, \\ \sqrt{2}\, m_1(\theta) = h_0\, e^{-i\theta} - h_1\, e^{-2i\theta} + \cdots + (-1)^k\, h_k\, e^{-i(k+1)\theta} \\ \qquad\qquad + \cdots - h_{2N-1}\, e^{-2Ni\theta}, \end{cases}$$

et que $\hat{\varphi}(\theta) = m_0(\theta/2) \cdots m_0(\theta/2^j) \cdots$ est la transformée de Fourier d'une fonction φ de classe C^r, où

$$\gamma_1 N \leq r \leq \gamma_2 N\,, \qquad \gamma_2 > \gamma_1 > 0\,.$$

Ces filtres sont ceux qui sont utilisés dans la pratique en compression du signal.

L'énoncé du théorème principal de ce travail est alors le suivant

Théorème 2. *Si l'on travaille avec des filtres d'I. Daubechies, il existe une mesure de probabilité continue $d\sigma$ sur $[0,1]$, et un isomorphisme isométrique $J : H \longrightarrow L^2([0,1], d\sigma)$ ayant la propriété suivante*

- *Si $J(x) = f$, alors $J(\pi_I(x)) = f\,\chi_I$, où χ_I est la fonction indicatrice de l'intervalle I, et $\pi_I : H \longrightarrow H_I$ est l'opérateur de projection orthogonale.*

REMARQUE 1. Les propriétés des filtres d'I. Daubechies qui seront utilisées dans la démonstration du Théorème 2 sont

- Le fait que (h_k) et (g_k) sont de longueur finie $\leq 2N$.

- Le fait que

$$2^j\, e^{i2^j k\xi} m_0(\xi)\cdots m_0(2^{j-1}\xi) \underset{j\to\infty}{\rightharpoonup} 2\pi\,\varphi(k) \sum_{\ell\in\mathbb{Z}} \delta(\xi - 2\pi\ell)\,,$$

où la convergence se fait au sens des distributions 2π-périodiques, à k fixé.

REMARQUE 2. Pour des filtres infinis, la question de savoir si J existe est encore ouverte.

Si, à la place des filtres d'I. Daubechies, on met des filtres de Haar, définis par

$$\begin{cases} m_0(\theta) = \dfrac{1+e^{i\theta}}{\sqrt{2}}\,, \\[2mm] m_1(\theta) = \dfrac{1-e^{i\theta}}{\sqrt{2}}\,, \end{cases}$$

on peut écrire pour tout $I \in \mathcal{I}$

$$\begin{cases} \pi_I(e_0) = 2^{|I|/2}\, e_{0,I}\,, \\ \pi_I(e_{-1}) = \pm\, 2^{|I|/2}\, e_{-1,I}\,, \end{cases}$$

et donc

$$\begin{cases} \|\pi_I(e_0)\| = \|\pi_I(e_{-1})\|\,, \\ \langle \pi_I(e_0), \pi_I(e_{-1})\rangle_H = 0\,. \end{cases}$$

Si le Théorème 2 était vrai pour les filtres de Haar, on aurait donc

$$\begin{cases} |J(e_0)| = |J(e_{-1})|\,, & \sigma\text{-p.p.} \\ J(e_0)\, J(e_{-1}) = 0\,, & \sigma\text{-p.p.} \end{cases}$$

d'où $J(e_0) = J(e_{-1}) = 0$ σ-p.p., ce qui est absurde. Donc dans le cas des filtres de Haar, le Théorème 2 est faux.

Pour éviter cet écueil, on considère l'espace de Hilbert H^+ engendré par la base $(e_n)_{n\geq 0}$.

H^+ est stable par les projections π_I. On note $H_I^+ = \pi_I(H^+)$.

Yves Meyer a fait l'observation suivante

Théorème 2 bis. *Si l'on travaille avec les filtres de Haar, le Théorème 2 est vrai en remplaçant H par H^+ et H_I par H_I^+ dans son énoncé. De plus, la mesure $d\sigma$ est la mesure de Lebesgue.*

Ce résultat s'obtient par une *construction explicite* de l'isomorphisme J. Yves Meyer considère la suite $(w_n)_{n\geq 0}$ des fonctions de Walsh, dont la première est $w_0 = \chi_{[0,1]}$.

Ces fonctions forment une base de $L^2([0,1],\mathbb{R})$, et vérifient les relations

$$\begin{cases} w_n(2x) = \dfrac{1}{2}\left(w_{2n}(x) + w_{2n+1}(x)\right), \\ w_n(2x-1) = \dfrac{1}{2}\left(w_{2n}(x) - w_{2n+1}(x)\right). \end{cases}$$

Il suffit alors de poser $J(e_n) = w_n$, et les relations ci-dessus permettent de vérifier les conclusions du Théorème 2 bis.

On serait tenté par une construction analogue dans le cas des filtres d'I. Daubechies. Malheureusement, la tâche semble difficile. Elle est - entre autres - compliquée par le fait de la mesure *$d\sigma$ n'est pas la mesure de Lebesgue* dans plusieurs cas.

L'auteur de cet article a en effet démontré, dans un travail non encore publié, que les filtres d'I. Daubechies de longueur inférieure ou égale à 10 ont une mesure $d\sigma$ associée singulière. On peut d'ailleurs conjecturer que c'est le cas pour tous les filtres d'I. Daubechies.

La démonstration du Théorème 2 que nous allons donner ne passera donc pas par une construction explicite de l'isomorphisme J. Elle reposera sur l'idée suivante

Pour $\xi \in [0,1]$ et $x \in H$, on essaie d'étudier la limite $l(\xi,x) \in \mathbb{R}^{\mathbb{Z}}$ des coefficients sur la base $(e_{k,I})_{k\in\mathbb{Z}}$ de $\pi_I(x)/\|\pi_I(x)\|$, lorsque $(|I| \to 0\ ,\ \xi \in I)$.

Si l'on pose $f = J(x)$, l'information contenue dans $f\,\chi_I$ se réduit à un scalaire lorsque $|I| \to 0$. On peut donc s'attendre, si le Théorème 2 est vrai, à ce que $l(\xi,x)$, lorsqu'elle existe, soit indépendante de x (au signe près). Réciproquement, on peut espérer que l'existence de J se déduira des propriétés de l .

En fait, l'étude de $l(\xi, x)$ ne semble possible que pour ξ de la forme $k/2^j$, et x de la forme $\sum_{k\in K} \alpha_k e_k$, où K est une partie finie de $\mathbb{Z}$. Cette restriction sur les valeurs de ξ rendra un peu délicat le passage du local (étude de l) au global (existence de J).

2. Plan de la démonstration du Théorème 2.

Pour démontrer le Théorème 2, il nous suffira de prouver le résultat suivant

Théorème 3. *Soit $T : H \longrightarrow H$ un opérateur continu. Supposons que pour tout intervalle dyadique I, on ait $T \circ \pi_I = \pi_I \circ T$.*

Alors il existe une suite bornée $T_j : H \longrightarrow H$ d'opérateurs de la forme $T_j = \sum_{I\in E_j} \lambda_I \pi_I$, telle que $\langle T_j f, g\rangle \to \langle Tf, g\rangle$ pour tous $f, g \in H$.

Ici, les E_j sont des parties finies de $\mathcal{I}$, et les λ_I sont des scalaires.

En effet, d'après le Théorème 1, la famille (π_I) engendre une mesure spectrale $\tilde{\pi}$, c'est-à-dire une application de l'ensemble des Boreliens de $[0,1]$ dans celui des projecteurs orthogonaux de H, dénombrablement additive et telle que $\tilde{\pi}([0,1]) = 1_H$. De plus, le projecteur $\tilde{\pi}(\{a\})$ associé à un singleton $\{a\}$ est toujours nul, par continuité de σ. Le Théorème 3 implique que tout projecteur orthogonal commutant avec l'image de $\tilde{\pi}$ est lui-même dans cette image. Ceci entraîne le Théorème 2, d'après la théorie de la multiplicité spectrale dans les espaces de Hilbert (voir par exemple [H, Chapitre III]).

La convergence des T_j souhaitée étant une convergence faible, on va se ramener, pour démontrer le Théorème 3, à l'étude de $\langle T_j f, g\rangle$ pour $f, g \in H^{(j)}$, les $H^{(j)}$ étant une suite croissante d'espaces de dimension finie, avec $\overline{\bigcup_{j\geq 0} H^{(j)}} = H$.

Plus précisément, nous définissons $H^{(j)}$ comme l'espace engendré par $(e_k)_{-2^{j-1}<k\leq 2^{j-1}}$. Pour tout intervalle dyadique $I \subset [0,1]$, nous définissons de même l'espace $H_I^{(j)}$ engendré par $(e_{k,I})_{-2^{j-1}<k\leq 2^{j-1}}$.

La propriété remarquable des $H_I^{(j)}$, lorsqu'on travaille avec des filtres d'Ingrid Daubechies, est qu'il existe $j_0 \geq 0$ tel que pour $j \geq j_0$, on ait toujours $\pi_I(H^{(j)}) \subset H_I^{(j)}$.

On vérifie très simplement cette propriété par récurrence sur i, avec $2^{-i} = |I|$, pour un filtre $(h_k)_{0 \leq k \leq 2N-1}$, en choisissant $2^{j_0} \geq 2N$. *C'est ici qu'on utilise le fait que les filtres sont finis.*

Maintenant, pour établir le Théorème 3, il suffira de démontrer la proposition suivante

Proposition 1. *Pour tous $j \geq j_0$ et $\varepsilon > 0$, il existe une partie finie $A_{j,\varepsilon}$ de $\mathcal{I}$, composée d'intervalles deux à deux disjoints, telle que, pour tout $x \in H^{(j)}$, on ait*

$$\Big\| x - \sum_{I \in A_{j,\varepsilon}} \langle x, \zeta_I \rangle \, \zeta_I \Big\| \leq \varepsilon \, \|x\| \,,$$

où l'on a posé

$$\zeta_I = \Big(\sum_{k \in \mathbb{Z}} \varphi(k)^2 \Big)^{-1/2} \Big(\sum_{k \in \mathbb{Z}} \varphi(k) \, e_{k,I} \Big) \,.$$

Nous prouverons la Proposition 1 au Paragraphe 3. Vérifions auparavant que cette proposition implique bien le Théorème 3.

On suppose donc la Proposition 1 vraie, et on considère T tel que (pour tout $I \in \mathcal{I}$) $T \circ \pi_I = \pi_I \circ T$. T laisse stable les espaces H_I, et on a toujours, par construction, $\zeta_I \in H_I$.

Par conséquent, si I, I' sont deux intervalles distincts (donc disjoints) de $A_{j,\varepsilon}$, on a $\langle T\zeta_I \,, \zeta_{I'} \rangle = 0$.

On pose maintenant $T_j = \sum_{I \in A_{j,1/j}} \lambda_I \pi_I$, avec $\lambda_I = \langle T\zeta_I, \zeta_I \rangle$. On a bien sûr $\|T_j\| \leq \|T\|$, et pour $j \geq j_0$, $f, g \in H^{(j)}$, on écrit

$$\begin{aligned}
&|\langle Tf, g \rangle - \langle T_j f, g \rangle| \\
&\quad \leq \Big| \langle Tf, g \rangle - \Big\langle T\Big(\sum_{I \in A_{j,1/j}} \langle \zeta_I, f \rangle \, \zeta_I \Big), \sum_{I \in A_{j,1/j}} \langle \zeta_I, g \rangle \, \zeta_I \Big\rangle \Big| \\
&\qquad + \Big| \langle T_j f, g \rangle - \Big\langle T_j\Big(\sum_{I \in A_{j,1/j}} \langle \zeta_I, f \rangle \, \zeta_I \Big), \sum_{I \in A_{j,1/j}} \langle \zeta_I, g \rangle \, \zeta_I \Big\rangle \Big| \\
&\quad \leq \frac{4\,\|T\|}{j} \, \|f\| \, \|g\| \,.
\end{aligned}$$

L'union des $H^{(j)}$ étant dense dans H, la Proposition 1 implique donc bien le Théorème 3.

3. Preuve de la Proposition 1.

Une idée "naïve" pour construire l'isomorphisme J, serait d'associer à tout vecteur de base e_k la fonction $e^{ik\theta} \in L^2((0,2\pi),\mathbb{C})$.

On obtient ainsi un isomorphisme

$$U : H_{\mathbb{C}} = \overline{\bigoplus_{k\in\mathbb{Z}} \mathbb{C}\, e_k} \longrightarrow L^2(\,(0,2\pi),\mathbb{C}\,)\ .$$

Les $e_{k,I}$ deviennent $2^{j/2}\, e^{ik2^j\theta}\, m_I(\theta)$, avec

$$m_I(\theta) = m_{\varepsilon_1}(\theta)\, m_{\varepsilon_2}(2\theta)\cdots m_{\varepsilon_j}(2^{j-1}\theta)\,,$$

et

$$I = \left[\frac{\varepsilon_1}{2} + \cdots + \frac{\varepsilon_j}{2^j}\,,\ \frac{\varepsilon_1}{2} + \cdots + \frac{\varepsilon_j}{2^j} + \frac{1}{2^j}\right]\ .$$

L'espace $H^{(j)}$ devient l'espace des polynômes trigonométriques à coefficients réels $\sum_{-2^{j-1}<k\leq 2^{j-1}} c_k\, e^{ik\theta}$.

Malheureusement, la réalisation U de l'espace H ainsi obtenue diffère beaucoup de l'isomorphisme J cherché. Tout d'abord, les espaces naturels de départ et d'arrivée de U sont complexes, ceux de J sont réels. Mais surtout, les espaces $U(H_I)$ ne sont pas constitués de fonctions supportées par I. Cela vient du fait que m_I *n'est pas* la fonction indicatrice de I.

Ceci, outre le fait, déjà mentionné au Paragraphe 1, que les mesures $d\sigma$ sont souvent singulières, nous indique que *J a peu de parenté avec la transformée de Fourier.* Cependant, l'isomorphisme U va nous être très utile dans la démonstration de la Proposition 1.

Nous utiliserons en fait la réalisation relative U_J consistant à associer, pour $J \in \mathcal{I}$ fixé, $e^{ik\theta}$ au vecteur $e_{k,J}$.

On aura alors $U_J(e_{k,I}) = e^{ik2^{\tilde{j}}\theta}\, m_{\tilde{I}}(\theta)\, 2^{\tilde{j}/2}$, pour $I \subset J$, $\tilde{I}$ et $\tilde{j} = |\tilde{I}|$ étant définis par $I = \alpha\tilde{I} + \beta$, α et β étant tels que $J = \alpha[0,1] + \beta$.

Nous démontrons d'abord le

Lemme 1. *Si $x \in H_J^{(j)}$, et si l'on pose $U_J(x) = f(\theta)$, et $U_J \circ \pi_I(x) = f_I(\theta)$ pour*

$$\begin{cases} |I| = 2^{-j}|J|\,, \\ I \subset J\,, \end{cases}$$

alors

$$f(\theta) = \sum_{\substack{I \subset J \\ |I|=2^{-j}|J|}} 2^{j/2} M_{\bar{I}}(\theta)\, f_I(\theta 2^j)$$

et l'on a $\|x\|^2 = \sum |f_I(0)|^2$.

PREUVE. Puisque $f \in \mathcal{P}^{(j)}$, on a

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\, d\theta \;=\; 2^{-j} \sum_{0 \le k < 2^j} |f(2k\pi\, 2^{-j})|^2\,.$$

Maintenant, nous remarquons que la théorie des filtres QMF peut s'appliquer en considérant le sous-groupe discret $\mathcal{U}^{(j)}$ des racines 2^j-ièmes de l'unité, au lieu du groupe $\mathcal{U} = \{z \in \mathbb{C} : \ |z| = 1\}$. Ainsi, le fait que la matrice

$$\begin{pmatrix} m_0(\theta) & m_1(\theta) \\ m_0(\theta+\pi) & m_1(\theta+\pi) \end{pmatrix}$$

soit unitaire pour $e^{i\theta} \in \mathcal{U}^{(j)}$ entraîne que

$$2^{-j} \sum_{0 \le k < 2^j} |f(2k\pi\, 2^{-j})|^2 \;=\; \sum_{|I|=2^{-j}|J|} |f_I(0)|^2\,.$$

Cette égalité est la version discrétisée de la formule

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\, d\theta \;=\; \sum_{|I|=2^{-j}|J|} \frac{1}{2\pi}\int_0^{2\pi} |f_I(2^j\theta)|^2\, d\theta\,.$$

Le Lemme 1 est donc démontré.

Le Lemme 1 nous fournit un isomorphisme isométrique

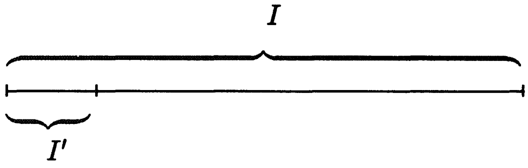
$$\begin{aligned} b_J^{(j)} : H_J^{(i)} &\rightarrow \mathbb{R}^{2^j} \\ x &\mapsto (\xi_I)_{\{I \subset J :\, |I|=2^{-j}|J|\}} \end{aligned}$$

avec

$$\xi_I = f_I(0) = \sum_{k\in\mathbb{Z}} \langle x, e_{k,I}\rangle \, .$$

Pour relier cet isomorphisme à l'approximation cherchée de la Proposition 1, nous démontrons le

Lemme 2. *On considère un intervalle dyadique $I \subset [0,1]$, de longueur 2^{-m}. On désigne par $I' \subset I$ l'intervalle dyadique de longueur 2^{-n_0-m} inclus dans I, et situé le plus à gauche possible dans I.*



On pose $\alpha = \left(\sum \varphi(k)^2\right)^{1/2}$, et on reprend les vecteurs $\zeta_{I'}$ définis dans la Proposition 1. Alors, pour tout $x \in H_I^{(j)}$ avec $j \geq j_0$, on a

$$\left\| \pi_{I'}(x) - \alpha\, 2^{-n_0/2} \Big(\sum_{k\in\mathbb{Z}} \langle x, e_{k,I}\rangle \Big) \zeta_{I'} \right\| \leq \eta(j, n_0)\, 2^{-n_0/2}\, \|x\| \, ,$$

où, pour tout j fixé, $\lim_{n_0 \to +\infty} \eta(j, n_0) = 0$, et où $\eta(j, n_0)$ ne dépend pas de J, ni de m.

Avant de démontrer le Lemme 2, observons que le facteur de normalisation $2^{-n_0/2}$ est naturel, puisqu'il exprime la répartition moyenne de l'énergie $\|x\|^2$ entre les 2^{n_0} sous-intervalles dyadiques de I de longueur $2^{-n_0}|I|$. Quant au coefficient α, il peut, s'il est différent de 1, fausser cette répartition. Cet argument sera développé dans un prochain article, et donnera des critères permettant d'affirmer qu'une mesure $d\sigma$ trouvée par le Théorème 1 ou 2 est singulière.

Démonstration du Lemme 2. Le fait que les filtres sont finis sera fondamental ici.

On va travailler avec la réalisation U_I. On identifie donc

$$x = \sum_{2^{j-1} < q \leq 2^j} \alpha_q \, e_{q,I}$$

à $f(\theta) = \sum \alpha_q \, e^{iq\theta}$.

Dans cette réalisation, $e_{q,I'}$ devient

$$2^{n_0/2}\, e^{ik2^{n_0}\theta}\, m_0(\theta)\cdots m_0(2^{n_0-1}\theta) \;=\; 2^{-n_0/2}\, G_{n_0,k}(\theta)\,.$$

On remarque alors que

$$G_{n_0,k}(\theta) \underset{n_0\to+\infty}{\longrightarrow} 2\pi\,\varphi(k)\sum_{l\in\mathbb{Z}}\delta(\theta-2\pi l)\,,$$

où la convergence se fait au sens des distributions 2π-périodiques, pour k fixé.

Dans notre réalisation, $\pi_{I'}(x)$ est devenu

$$2^{-n_0}\sum_{-2^{j-1}<k\le 2^{j-1}}\langle f, G_{n_0,k}\rangle\, G_{n_0,k}\,,$$

où le crochet $\langle\cdot,\cdot\rangle$ est le crochet de dualité entre fonctions C^∞, 2π-périodiques et distributions 2π-périodiques.

Par ailleurs, $\zeta_{I'}$ est devenu

$$\frac{1}{\alpha}\sum_{-2^{j-1}<k\le 2^{j-1}}\varphi(k)\, 2^{-n_0/2}\, G_{n_0,k}\,.$$

Il est alors immédiat de calculer

$$\left\|\pi_{I'}(x)-\alpha\, 2^{-n_0/2}\, f(0)\,\zeta_{I'}\right\|^2 = \sum_{k=-2^{j-1}+1}^{2^{j-1}} 2^{-n_0}\left|\langle f, G_{n_0,k}\rangle - f(0)\,\varphi(k)\right|^2.$$

Pour terminer la preuve du Lemme 2, on remarque que, pour j fixé, et $x\in H_I^{(j)}$, $\|x\|\le 1$, f est un polynôme de $\mathcal{P}^{(j)}$, avec $\|f\|\le 1$: l'ensemble de ces polynômes est un compact dans $C^\infty(0,2\pi)$, d'où la convergence

$$\langle f, G_{n_0,k}\rangle \underset{n_0\to+\infty}{\longrightarrow} f(0)\,\varphi(k)$$

uniforme sur cet ensemble. On peut donc construire $\eta(j,n_0)$ vérifiant les conclusions du Lemme 2.

Enonçons maintenant un corollaire évident des lemmes 1 et 2.

Corollaire. *Avec les notations du Lemme 2, on a*

$$\left\|\pi_{I'}(x) - \langle x, \zeta_{I'}\rangle\, \zeta_{I'}\right\| \;\leq\; \eta\, 2^{-n_0/2}\, \|\pi_I(x)\|\,,$$

pour tout $x \in H^{(j)}$. *De plus,*

$$\sum_{|I|=2^{-j}} \|\pi_{I'}(x)\|^2 \;\geq\; c\, 2^{-n_0}\, \|x\|^2\,,$$

où $c > 0$ *est une constante indépendante de* j, ε.

La première inégalité résulte du Lemme 2, la deuxième résulte du Lemme 1 combiné au Lemme 2. (On peut choisir par exemple $c = \alpha^2/2$).

Ce corollaire nous montre qu'on peut "manger" une partie de x, de norme $2^{-n_0/2}\sqrt{c}\,\|x\|$, approximer cette partie avec la précision $\eta\, 2^{-n_0/2}\|x\|$, par un vecteur de type $\sum\langle x, \zeta_I\rangle\zeta_I$. On est alors tenté d'appliquer de nouveau ce procédé à la partie restante, et d'arriver finalement à approximer x tout entier, par itérations.

PREUVE DE LA PROPOSITION 1 PAR ITÉRATIONS. On définit trois suites E_m, F_m, G_m d'intervalles dyadiques, par récurrence,

$E_0 = [0,1]$, F_0 est l'ensemble des 2^j intervalles dyadiques de type $\left[k/2^j\,,\,(k+1)/2^j\right] = J$,

G_0 est l'ensemble des 2^j intervalles de type $\left[k/2^j\,,\,(k+2^{-n_0})/2^j\right] = J'$ associés aux intervalles J.

Supposons maintenant E_m, F_m, G_m construits, et que G_m est constitué d'intervalles J' associés aux intervalles J de F_m (c'est-à-dire que $J' \subset J$, $|J'| = 2^{-n_0}|J|$, et J' le plus à gauche possible).

On décompose alors chaque intervalle $J \setminus J'$ en n_0 intervalles dyadiques. On appelle E_{m+1} la collection des intervalles dyadiques ainsi obtenus; on subdivise chaque intervalle de E_{m+1} en 2^j intervalles dyadiques égaux, et on obtient la collection F_{m+1}; enfin, la collection G_{m+1} est constituée des intervalles J' associés aux intervalles J de F_{m+1}.

Il résulte de cette construction que la famille $\bigcup_{m\geq 0} G_m$ est constituée d'intervalles dyadiques deux à deux disjoints. On va poser $A_{j,\varepsilon} = \bigcup_{m=0}^{m_0} G_m$, pour m_0 bien choisi.

On peut écrire

$$\begin{cases} x &= x_0 + y_1 \,, \\ y_1 &= x_1 + y_2 \,, \\ &\vdots \\ y_{m_0} &= x_{m_0} + y_{(m_0+1)} \,, \end{cases}$$

où $x_m = \sum_{J' \in G_m} \pi_{J'}(x)$ et $y_m = \sum_{I \in E_m} \pi_I(x)$.

D'après le Corollaire, on a $\|x_m\|^2 \geq c\, 2^{-n_0} \|y_m\|^2$. Par ailleurs, on a $\|y_{m+1}\|^2 = \|y_m\|^2 - \|x_m\|^2$. Donc, par récurrence, $\|y_m\|^2 \leq (1 - c\, 2^{-n_0})^m \|x\|^2$.

On peut donc écrire $x = x_0 + \cdots + x_{m_0} + \text{Reste}$, avec

$$\|\text{Reste}\|^2 \leq (1 - c\, 2^{-n_0})^{m_0+1} \|x\|^2 \,.$$

Maintenant, on applique la première partie du Corollaire à chaque y_m, ce qui permet d'écrire

$$\|\pi_{I'}(y_m) - \langle y_m, \zeta_{I'} \rangle \, \zeta_{I'}\| \leq \eta(j, n_0)\, 2^{-n_0/2} \, \|\pi_I(y_m)\| \,,$$

pour chaque $I \in F_m$.

Cette inégalité entraîne, par orthogonalité des H_I,

$$\begin{aligned} \Big\| x_m - \sum_{I' \in G_m} \langle x, \zeta_{I'} \rangle \zeta_{I'} \Big\| &\leq \eta(j, n_0)\, 2^{-n_0/2} \, \|y_m\| \\ &\leq \frac{\eta(j, n_0)}{\sqrt{c}} \, \|x_m\| \,. \end{aligned}$$

Comme ces erreurs sont localisées dans les espaces $\bigoplus_{J' \in G_m} H_{J'}$, deux à deux orthogonaux pour deux valeurs de m distinctes, et orthogonaux au Reste, on a

$$\begin{aligned} &\Big\| x_m - \sum_{I' \in \cup_{m=0}^{m_0} G_m} \langle x, \zeta_{I'} \rangle \, \zeta_{I'} \Big\|^2 \\ &\qquad\qquad \leq \left(\frac{\eta(j, n_0)^2}{c} + (1 - c\, 2^{-n_0})^{m_0+1} \right) \|x\|^2 \,. \end{aligned}$$

On conclut en prenant n_0 tel que

$$\frac{\eta(j,n_0)^2}{c} \leq \frac{\varepsilon^2}{2},$$

puis en choisissant m_0 tel que $(1-c\,2^{-n_0})^{m_0+1} \leq \varepsilon^2/2$. La Proposition 1 est démontrée, le Théorème principal est donc vrai.

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A weighted version of Journé's Lemma

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In this paper we discuss a weighted version of Journé's covering lemma, a substitute for the Whitney decomposition of an arbitrary open set in $\mathbb{R}^2$ where squares are replaced by rectangles. We then apply this result to obtain a sharp version of the atomic decomposition of the weighted Hardy spaces $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ and a description of their duals when p is close to 1.

A nonnegative locally integrable function $w(x, y)$ on $\mathbb{R}^2$ is called a *weight.* A weight w is said to satisfy *Muckenhoupt's* $A_p(\mathbb{R}\times\mathbb{R})$ *condition on rectangles*, or plainly the A_p *condition*, $1 < p < \infty$, provided that

$$\sup_R \Big(\frac{1}{|R|}\iint_R w(x,y)\,dx\,dy\Big)\Big(\frac{1}{|R|}\iint_R w(x,y)^{-1/(p-1)}dx\,dy\Big)^{p-1} \le c\,,$$

where R runs over all rectangles with sides parallel to the coordinate axes. When $p = 1$ this condition reduces to

$$\frac{1}{|R|}\iint_R w(x,y)\,dx\,dy \le c \operatorname*{ess\,inf}_{(x,y)\in R} w(x,y)\,, \qquad \text{all } R\,.$$

We say that w satisfies the $A_\infty(\mathbb{R}\times\mathbb{R})$ *condition* if it satisfies the A_p condition for some $p < \infty$. The constant c that appears on the right-hand side in the inequalities above is called *the* A_p *constant of* w, and a property is said to be *independent in* A_p provided it depends on c, and not on the particular weight w in A_p involved. By the Lebesgue differentiation theorem it readily follows that if w satisfies the A_p condition, then $w(x,\cdot)$ satisfies Muckenhoupt's $A_p(\mathbb{R})$ condition, uniformly for a.e. x, with constant $\le c$, the A_p constant for w; similarly for $w(\cdot,y)$.

The same holds for A_∞: an A_∞ weight w is uniformly in $A_\infty(\mathbb{R})$ for a.e. x, or y, fixed. By well-known properties of A_∞ weights, if $w(x,\cdot)$ is an $A_\infty(\mathbb{R})$ weight uniformly in x, then the following holds: given $x \in \mathbb{R}$ and $0 < \varepsilon < 1$, there exists η, $0 < \eta < 1$, such that if $A \subseteq I$ and

$$(1) \qquad \frac{w(x,A)}{w(x,I)} > \eta\,, \qquad \text{then} \quad \frac{w(x',A)}{w(x',I)} > \varepsilon \quad \text{for a.e. } x' \in \mathbb{R}\,.$$

It is clear that we may always choose $\eta \ge 1/2$ above, and we do so.

Under the assumption that w is uniformly A_∞ for a variable fixed and uniformly doubling for the other variable fixed, the weighted strong maximal operator $M_{S,w}f(x,y)$ given by

$$M_{S,w}f(x,y) = \sup_{(x,y)\in R} \frac{1}{w(R)} \int\!\!\int_R |f(u,v)|\, w(u,v)\, du\, dv\,,$$

is known to be bounded in $L^2_w(\mathbb{R}^2)$, say, *cf.* [JT] and [F1].

Given a bounded open set $\Omega \subset \mathbb{R}^2$, $x \in \mathbb{R}$ and $t > 0$, following [J], let

$$E_{x,t} = \{y \in \mathbb{R} :\ [x-t, x+t] \times \{y\} \subseteq \Omega\}\,.$$

Each $E_{x,t}$ is open, because Ω is open, and, for each x, $E_{x,t}$ is decreasing in t.

Let $E_{x,t} = \bigcup_k J^k_{x,t}$ denote the decomposition of $E_{x,t}$ into open interval components, and let $t(k,x)$ be the infimum over those $\tau \ge t$ such that

$$(2) \qquad w\big(x, J^k_{x,t} \cap E_{x,\tau}\big) \le \eta\, w\big(x, J^k_{x,t}\big)\,,$$

where $1/2 \le \eta < 1$ corresponds to the value $\varepsilon = 1/2$ above.

Proposition 1. *Given a bounded open set Ω, let*

$$\hat{\Omega} = \bigcup_{x,t,k} \big(x - t(k,x), x + t(k,x)\big) \times J^k_{x,t}\,,$$

and assume that the weight $w(x,y)$ is uniformly $A_\infty(\mathbb{R})$ for a variable fixed, and uniformly doubling for the other variable fixed. Then $w(\hat{\Omega}) \le c\, w(\Omega)$, where c is independent of Ω.

PROOF. As it is readily seen by the containment relation between the sets involved, we have

$$(3)\quad w\big(((x-s,x+s)\times J^k_{x,t})\cap\Omega\big)\geq w\big((x-s,x+s)\times (J^k_{x,t}\cap E_{x,s})\big).$$

Now, if $s < t(k,x)$, from (2) and (1) it follows that

$$(4)\qquad w(x', J^k_{x,t}\cap E_{x,s}) > \frac{1}{2}\, w(x', J^k_{x,t}), \qquad \text{a.e. } x'\in\mathbb{R}.$$

Thus, integrating (4) over $(x-s, x+s)$, combining the resulting expression with (3), and setting $R=(x-s,x+s)\times J^k_{x,t}$, we obtain

$$(5)\qquad \iint_R \chi_\Omega(x,y)\, w(x,y)\, dx\, dy > \frac{1}{2}\iint_R w(x,y)\, dx\, dy.$$

Now, if $(x',y')\in\hat{\Omega}$, there exist x,t,k such that $x'\in(x-t(k,x), x+t(k,x))$, and also $s<t(k,x)$ so that $(x',y')\in(x-s,x+s)\times J^k_{x,t}=R$. Whence, by (5),

$$\hat{\Omega}\subseteq\Big\{M_{S,w}(\chi_\Omega)>\frac{1}{2}\Big\},$$

and by the continuity of $M_{S,w}$ in $L^2_w(\mathbb{R}^2)$,

$$w(\hat{\Omega})\leq c\, w(\Omega),$$

with c independent of Ω.

Proposition 2. *Suppose Ω and w are as in Proposition 1, and that ϕ is a nondecreasing function with $\phi(0)=0$. Then*

$$\int_0^{+\infty}\int_{\mathbb{R}}\sum_k\int_{J^k_{x,t}}\phi\Big(\frac{t}{t(k,x)}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t}\leq c\, w(\Omega)\int_0^1\phi(s)\,\frac{ds}{s},$$

where c is a constant independent of Ω and ϕ.

PROOF. From (2) it readily follows that

$$w(x, J^k_{x,t})\leq\frac{1}{1-\eta}\, w(x, J^k_{x,t}\setminus E_{x,t(k,x)}).$$

Thus, save for the factor $1/(1-\eta)$, the left-hand side of the above expression does not exceed

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{+\infty}\sum_k \chi_{J^k_{x,t}\setminus E_{x,t(k,x)}}(y)\,\phi(\frac{t}{t(k,x)})\,w(x,y)\,\frac{dt}{t}\,dy\,dx$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} B(x,y)\,w(x,y)\,dx\,dy\,,$$

say. We want to show that

$$B(x,y) \le c\,\chi_\Omega(x,y)\int_0^1 \phi(s)\,\frac{ds}{s}\,.$$

Clearly if $(x,y)\notin\Omega$, then $B(x,y)=0$. Also, if $(x,y)\in\Omega$, at most one summand in the above sum does not vanish, the one corresponding to the index k, say. Thus,

$$B(x,y) = \chi_\Omega(x,y)\int_0^{+\infty} \chi_{J^k_{x,t}\setminus E_{x,t(k,x)}}(y)\,\phi(\frac{t}{t(k,x)})\,\frac{dt}{t}\,.$$

Let $T(x,y) = \sup\{s : \ [x-s,x+s]\times\{y\}\subseteq\Omega\}$ Since $J^k_{x,t}$ is an interval component of $E_{x,t}$, from this definition it readily follows that $t\le T(x,y)$. We may also assume that $T(x,y)\le t(k,x)$, for if $t(k,x)<T(x,y)$, then it follows that $y\in E_{x,t(k,x)}$, and the integrand above vanishes. Whence

$$\begin{aligned} B(x,y) &\le \chi_\Omega(x,y)\int_0^{T(x,y)} \chi_{J^k_{x,t}\setminus E_{x,t(k,x}}(y)\,\phi(\frac{t}{t(k,x)})\,\frac{dt}{t} \\ &\le \chi_\Omega(x,y)\int_0^{T(x,y)} \phi(\frac{t}{T(x,y)})\,\frac{dt}{t} \\ &= \chi_\Omega(x,y)\int_0^1 \phi(s)\,\frac{ds}{s}\,. \end{aligned}$$

Replacing this in the expression above gives the desired estimate.

Now we pass to discuss the discrete version of Journé's covering lemma. For Ω as before, let $M_2(\Omega)$ denote the collection of those rectangles (dyadic) $R = I\times J$ so that I, J are dyadic and J is maximal with respect to the inclusion property in Ω.

Given arbitrary intervals I, J, not necessarily dyadic, let

$$J^I = \{y \in J : \ I \times \{y\} \subseteq \Omega\}.$$

If by rI we denote the interval concentric with I with sidelength r times that of I, we define $\hat{I}$ as follows: it is the smallest interval I' concentric with I, $I' \supset (1/8)I$, such that

$$w(x, J^{I'}) \leq \frac{1}{2}\, w(x, J) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Proposition 3. *Suppose the open set Ω, weight w and the function ϕ are as in Proposition 2. Then*

$$\sum_{R \in M_2(\Omega)} w(R)\, \phi\Big(\frac{|I|}{|\hat{I}|}\Big) \leq c \left(\int_0^1 \phi(8s)\, \frac{ds}{s} \right) w(\Omega)\,.$$

PROOF. Let $\mathcal{I}_n$ denote the collection of those dyadic intervals I such that $R = I \times J \in M_2(\Omega)$ for some dyadic interval J, and $|I| = 2^n$, $n = 0, \pm 1, \pm 2, \ldots$ Then, since $J^I \supset J$ for $R = I \times J \in M_2(\Omega)$, the sum we want to estimate does not exceed

$$\sum_n \sum_{I \in \mathcal{I}_n} \int_I \sum_{J^I} \int_{J^I} \phi\Big(\frac{|I|}{|\hat{I}|}\Big)\, w(x,y)\, dx\, dy$$
$$\leq \sum_n \int_{2^{n-3}}^{2^{n-2}} \sum_{I \in \mathcal{I}_n} \int_I \sum_{J^I} \int_{J^I} w(x,y)\, dx\, dy\, \phi\Big(\frac{|I|}{|\hat{I}|}\Big)\, \frac{dt}{t}\,.$$

Fix now n, and $I \in \mathcal{I}_n$. Let $S = \{x \in I : \ [x - t, x + t] \in I\}$, and note that for $t \in (2^{n-3}, 2^{n-2})$, since $|I| = 2^n$, $2S \supseteq I$. Thus by the uniform doubling property of $w(\cdot, y)$, the above expression does not exceed

$$c \sum_n \int_{2^{n-3}}^{2^{n-2}} \sum_{I \in \mathcal{I}_n} \int_S \sum_{J^I} \int_{J^I} \phi\Big(\frac{|I|}{|\hat{I}|}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t}\,.$$

Furthermore, since $t \geq 2^{n-3} = |(1/8)I|$, and since $x \in S$, it readily follows that $y \in J^k_{x,t}$, one of the components of $E_{x,t}$, and the above expression is dominated by

$$c \sum_n \int_{2^{n-3}}^{2^{n-2}} \sum_{I \in \mathcal{I}_n} \int_S \sum_k \int_{J^k_{x,t}} \phi\Big(\frac{|I|}{|\hat{I}|}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t}\,.$$

Since in the above expression $|I| \leq 8t$, and since $[x-t, x+t] \subseteq I$ and consequently $J = J^I \subseteq J^k_{x,t}$, we see from the definitions of $t(k,x)$ and $|\hat{I}|$ (recall that $1/2 \leq \eta < 1$) that these quantities are essentially the same. Moreover, since in the definition of $t(k,x)$ the right-hand side is larger, so must be the left-hand side, and consequently $t(k,x) \leq |\hat{I}|$. Thus we may continue our estimation by

$$
\begin{aligned}
c \sum_n \int_{2^{n-3}}^{2^{n-2}} \sum_{I \in \mathcal{I}_n} \int_S \sum_k \int_{J^k_{x,t}} \phi\Big(\frac{8t}{t(k,x)}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t} \\
\leq c \sum_n \int_{2^{n-3}}^{2^{n-2}} \int_{\mathbb{R}} \sum_k \int_{J^k_{x,t}} \phi\Big(\frac{8t}{t(k,x)}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t} \\
\leq c \int_0^{+\infty} \int_{\mathbb{R}} \sum_k \int_{J^k_{x,t}} \phi\Big(\frac{8t}{t(k,x)}\Big)\, w(x,y)\, dy\, dx\, \frac{dt}{t}\,.
\end{aligned}
$$

Then the proof proceeds exactly as that of Proposition 2.

Proposition 4. *Under the conditions of Proposition 3, we have*

$$
w\Big(\bigcup_{R \in M_2(\Omega)} \hat{I} \times J\Big) \leq c\, w(\Omega)\,, \qquad c \text{ independent of } \Omega\,.
$$

Because the proof is similar to that of Proposition 1 it is omitted.

As a first application of the weighted version of Journé's lemma we discuss the atomic decomposition of the weighted Hardy spaces $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $0 < p \leq 1$.

Given a smooth function ψ supported in $(-1,1)$ with nonvanishing integral, put

$$
\psi_{s,t}(x,y) = \frac{1}{s}\,\psi\Big(\frac{x}{s}\Big)\,\frac{1}{t}\,\psi\Big(\frac{y}{t}\Big)\,, \qquad s,t > 0\,,
$$

and for a distribution f in $\mathbb{R}^2$, let

$$
f^*(x,y) = \sup_{\varepsilon_1,\varepsilon_2 > 0} |f * \psi_{\varepsilon_1,\varepsilon_2}(x,y)|\,.
$$

Then $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ consists of those distributions f such that $f^* \in L^p_w(\mathbb{R}^2)$, and we set $\|f\|_{H^p_w} = \|f^*\|_{L^p_w}$. We would like to discuss the so-called *atomic decomposition* of elements of these spaces when $w \in A_r(\mathbb{R} \times \mathbb{R})$, $1 \le r \le 2$.

A function $a(x,y)$ is called a (p,w)-*atom*, if

a) the set where $a(x,y) \neq 0$ is contained in a set Ω, with

$$\|a\|_{L^2_w} \le w(\Omega)^{1/2-1/p} < +\infty ,$$

b) $a = \sum a_R$, where the subatoms a_R have the following properties:

i) if $a_R(x,y) \neq 0$, then $(x,y) \in \tilde{R} = 3I \times 3J$, and $\tilde{R} \subseteq \Omega$,

ii) $R = I \times J$ is a dyadic rectangle, and no rectangle is repeated,

iii) for all integers $\alpha \le [r/p - 1]$,

$$\int_{\tilde{I}} x^\alpha \, a_R(x,y)\, dx = \int_{\tilde{J}} y^\alpha a_R(x,y)\, dy = 0 ,$$

iv) $\left(\sum \|a_R\|^2_{L^2_w}\right)^{1/2} \le w(\Omega)^{1/p-1/2}$.

The atomic decomposition states that $f \in H^p_w(\mathbb{R} \times \mathbb{R})$ if and only if $f = \sum_i \lambda_i a_i$, where the a_i's are (p,w)-atoms, the sum is taken in the sense of distributions and in the norm sense, and $\sum_i \lambda_i^p \le c\|f\|^p_{H^p_w}$. That $f \in H^p_w$ can be decomposed into such sum is very similar to the unweighted case considered by R. Fefferman in [F2], and the proof is not discussed here.

Thus, we propose to prove the following result

Proposition 5. *Suppose that $w \in A_r$ and that a is a (p,w)-atom. Then $\|a\|_{H^p_w} \le c$, where c is independent of a and independent in A_r.*

PROOF. Given $R = I \times J \subseteq \Omega$, let $\hat{I}$ now denote the interval which is the largest between $\hat{I}$ from Journé's lemma and $2I$; and similarly for $\hat{J}$. Let $\hat{R} = (\hat{I} \times J) \bigcup (I \times \hat{J}) = \hat{I} \times \hat{J}$. If

$$\hat{\Omega} = \bigcup_{R \subseteq \Omega} \hat{R} ,$$

then by Proposition 4 above, $w(\hat{\Omega}) \le c\, w(\Omega)$, where c is independent of Ω and w.

In order to estimate $\|a\|_{H^p_w} = \|a^*\|_{L^p_w}$, we break up the integral that gives the L^p_w norm into $\hat{\Omega}$ and $\mathbb{R}^2 \setminus \hat{\Omega}$. The contribution over $\hat{\Omega}$ is readily handled: indeed, if M_S denotes the strong maximal function, then since $w \in A_2(\mathbb{R} \times \mathbb{R})$, and $a^*(x,y) \le c\, M_S a(x,y)$, by Propositon 4 it follows that

$$\begin{aligned}
&\int_{\hat{\Omega}} a^*(x,y)^p\, w(x,y)\, dx\, dy \\
&\qquad \le c \int_{\hat{\Omega}} M_S a(x,y)^p\, w(x,y)^{p/2}\, w(x,y)^{1-p/2}\, dx\, dy \\
&\qquad \le c \left(\int_{\hat{\Omega}} M_S a(x,y)^2\, w(x,y)\, dx\, dy \right)^{p/2} w(\hat{\Omega})^{1-p/2} \\
&\qquad \le c\, \|a\|^p_{L^2_w}\, w(\hat{\Omega})^{1-p/2} \\
&\qquad \le c\, w(\Omega)^{p(1/2-1/p)} w(\Omega)^{1-p/2} \\
&\qquad \le c\,.
\end{aligned}$$

Next, if $a = \sum_r a_R$, we consider each subatom a_R separately; by translation if necessary we may assume that a_R is centered at the origin, and if $R = I \times J$, we estimate the larger expression

$$\int_{\mathbb{R}\setminus\hat{I}} \int_{\mathbb{R}} a_R{}^*(x,y)^p\, w(x,y)\, dx\, dy\,.$$

For this purpose we show that the following two estimates hold:

$$\int_{\mathbb{R}\setminus\hat{I}} \int_{\mathbb{R}\setminus\hat{J}} a_R{}^*(x,y)^p\, w(x,y)\, dx\, dy \le c \left(\frac{|R|}{|\hat{R}|} \right)^p, \tag{6}$$

and

$$\int_{\mathbb{R}\setminus\hat{I}} \int_{\hat{J}} a_R{}^*(x,y)^p\, w(x,y)\, dx\, dy \le c \left(\frac{|I|}{|\hat{I}|} \right)^p. \tag{7}$$

We do (6) first. Let $p_N(\psi,\cdot)$ denote the Taylor expansion of degree N of ψ. By the moment condition on a_R it readily follows that

$$\begin{aligned}
&|a_R * \psi_{\varepsilon_1,\varepsilon_2}(x,y)| \\
&\quad \le \frac{1}{\varepsilon_1\varepsilon_2} \iint_{\tilde{R}} |\psi(\frac{x-u}{\varepsilon_1}) - p_N(\psi, -\frac{u}{\varepsilon_1})| \\
&\qquad\qquad \cdot |\psi(\frac{y-v}{\varepsilon_2}) - p_N(\psi, -\frac{v}{\varepsilon_2})|\, |a_R(u,v)|\, du\, dv \\
&\quad \le \frac{c}{\varepsilon_1\varepsilon_2} \iint_{\tilde{R}} \left(\frac{|u|}{\varepsilon_1}\right)^{N+1} \left(\frac{|v|}{\varepsilon_2}\right)^{N+1} |a_R(u,v)|\, du\, dv\,.
\end{aligned}$$

Notice that if $x \notin 2I$ and $u \in I$, then $|x|/2 \le |x-u| \le 2|x|$, so that if $\varepsilon_1 \le |x|/2$, then $\psi_{\varepsilon_1}(x-u) = 0$. We may thus assume that $\varepsilon_1 \ge |x|/2$, and likewise that $\varepsilon_2 \ge |x_2|/2$. Therefore, since $|u\,v| \le |\tilde{R}|$, the above expression does not exceed

$$\frac{c|\tilde{R}|^{N+1}}{(|x|\,|y|)^{N+2}} \iint_{\tilde{R}} |a_R(u,v)|\, w(u,v)^{1/2} w(u,v)^{-1/2} du\, dv$$
$$\le \frac{c|R|^{N+1}}{(|x|\,|y|)^{N+2}} \|a_R\|_{L^2_w} \left(\iint_{\tilde{R}} w(u,v)^{-1} du\, dv \right)^{1/2}.$$

Now, by the bound on a_R, and since $w \in A_2(\mathbb{R}\times\mathbb{R})$, this expression does not exceed

$$c \frac{|R|^{N+1}}{(|x|\,|y|)^{N+2}} \frac{1}{w(\Omega)^{1/p-1/2}} \frac{|\tilde{R}|}{w(\tilde{R})^{1/2}}.$$

Thus

$$\text{(8)} \quad \begin{aligned} &\int_{\mathbb{R}\setminus\tilde{I}} \int_{\mathbb{R}\setminus\tilde{J}} a_R^*(x,y)^p\, w(x,y)\, dx\, dy \\ &\quad \le c \frac{|R|^{(N+2)p}}{w(\Omega)^{1-p/2} w(\tilde{R})^{p/2}} \int_{\mathbb{R}\setminus\tilde{I}} \int_{\mathbb{R}\setminus\tilde{J}} \frac{w(x,y)}{(|x|\,|y|)^{(N+2)p}}\, dx\, dy. \end{aligned}$$

In order to estimate the integral in (8) note that if $w \in A_r(\mathbb{R}\times\mathbb{R})$, then by the choice of N, $N(p+2) \ge r$; the argument proceeds now using well-known estimates in the case of the line, *cf.* [T, Proposition IX, 4.5 (iv)], and the fact that the restrictions of w are uniformly in $A_r(\mathbb{R})$ for each variable fixed. Indeed, the expression in question does not exceed

$$\begin{aligned} &c \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{w(x,y)}{(|x|+|\hat{I}|)^{(N+2)p} (|y|+|\hat{J}|)^{(N+2)p}}\, dx\, dy \\ &\quad \le c \int_{\mathbb{R}} \frac{1}{(|y|+|\hat{J}|)^{(N+2)p}} \int_{\mathbb{R}} \frac{w(x,y)}{(|x|+|\hat{I}|)^{(N+2)p}}\, dx\, dy \\ &\quad \le c \frac{1}{|\hat{I}|^{(N+2)p}} \int_{\mathbb{R}} \frac{1}{(|y|+|\hat{J}|)^{(N+2)p}} \int_{\hat{I}} w(x,y)\, dx\, dy \\ &\quad = c \frac{1}{|\hat{I}|^{(N+2)p}} \int_{\hat{I}} \int_{\mathbb{R}} \frac{w(x,y)}{(|y|+|\hat{J}|)^{(N+2)p}}\, dy\, dx \end{aligned}$$

$$\leq c\frac{1}{|\hat{I}|^{(N+2)p}}\frac{1}{|\hat{J}|^{(N+2)p}}\int_{\hat{I}}\int_{\hat{J}} w(x,y)\,dx\,dy$$
$$= c\frac{w(\hat{R})}{(|\hat{I}|\,|\hat{J}|)^{(N+2)p}}\,.$$

Thus, replacing this estimate in the right-hand side of (8), and by Proposition 4, we obtain that the left-hand side there does not exceed

$$c\frac{|R|^{(N+2)p}}{w(\Omega)^{1-p/2}w(R)^{p/2}}\frac{w(\hat{R})}{(|\hat{I}|\,|\hat{J}|)^{(N+2)p}}$$
$$\leq c\left(\frac{|R|}{|\hat{R}|}\right)^{(N+1)p}\left(\frac{|R|}{|\hat{I}|\,|\hat{J}|}\right)^{p}\left(\frac{w(R)}{w(\Omega)}\right)^{1-p/2}\leq c\left(\frac{|R|}{|\hat{R}|}\right)^{p},$$

which, of course, gives (6).

We show now estimate (7). By the moment condition on a_R we get

$$|a_R*\psi_{\varepsilon_1,\varepsilon_2}(x,y)|\leq\frac{1}{\varepsilon_1\varepsilon_2}\left|\int_{\tilde{I}}\int_{J}(\psi(\frac{x-u}{\varepsilon_1})-p_N(\psi,-\frac{u}{\varepsilon_1}))\cdot\psi(\frac{y-v}{\varepsilon_2})\,a_R(u,v)\,du\,dv\right|$$
$$\leq\frac{c}{\varepsilon_1}\int_{\tilde{I}}\left(\frac{|u|}{\varepsilon_1}\right)^{N+1}M^2a_R(u,y)\,du\,,$$

where M^2 denotes the Hardy maximal operator in the second variable only. Thus

$$a_R^*(x,y)\leq c\frac{|\tilde{I}|^{N+1}}{|x|^{N+2}}\int_{\tilde{I}}M^2a_R(u,y)\,du\,,$$

and consequently,

$$\int_{\mathbb{R}\setminus\hat{I}}\int_{\hat{J}}a_R^*(x,y)\,w(x,y)\,dx\,dy$$
$$\leq c\,|I|^{(N+1)p}\int_{\mathbb{R}\setminus\hat{I}}\frac{1}{|x|^{(N+2)p}}\int_{\hat{J}}\left(\int_{\tilde{I}}M^2a_R(u,y)\,du\right)^p w(x,y)\,dy\,dx$$
$$\leq c\,|I|^{(N+2)p}\int_{\hat{J}}\left(\frac{1}{|\tilde{I}|}\int_{\tilde{I}}M^2a_R(u,y)\,du\right)^p\int_{R^1\setminus\hat{I}}\frac{w(x,y)}{|x|^{(N+2)p}}\,dx\,dy$$
$$\leq c\,|I|^{(N+2)p}\int_{\hat{J}}M^1(M^2a_R)(x,y)^p\int_{\mathbb{R}\setminus\hat{I}}\frac{w(x,y)}{|x|^{(N+2)p}}\,dx\,dy\,,$$

where M^1 denotes the Hardy maximal operator in the first variable only.

As before, by the usual $A_r(\mathbb{R})$ properties it follows that for y-a.e.

$$\int_{\mathbb{R}\setminus\hat{I}} \frac{w(x,y)}{|x|^{(N+2)p}}\,dx \le \frac{c}{|\hat{I}|^{(N+2)p}} \int_{\hat{I}} w(x,y)\,dx\,,$$

and consequently,

$$\int_{\mathbb{R}\setminus\hat{I}} \int_{\hat{J}} a_R^*(x,y)^p\, w(x,y)\,dx\,dy$$
$$\le c\left(\frac{|I|}{|\hat{I}|}\right)^{(N+2)p} \int_{\hat{I}}\int_{\hat{J}} M^1(M^2 a_R)(x,y)^p\, w(x,y)\,dy\,dx\,.$$

Note that the above integral looks similar to the first expression we estimated, and, in fact, since $w \in A_2(\mathbb{R}\times\mathbb{R})$, it does not exceed

$$c\,\|a_R\|_{L^2_w}^p\, w(\hat{R})^{1-p/2} \le c\, w(\Omega)^{p/2-1} w(\hat{R})^{1-p/2} \le c\,,$$

which completes the proof of (7).

We would like now to improve on these estimates; following R. Fefferman, put

$$b_R(x,y) = \frac{w(R)^{1/2-1/p}}{\|a_R\|_{L^2_w}}\, a_R(x,y)\,,$$

and observe that $b_R(x,y)$ is an atom supported on R, and that the above estimate applied to b_R gives

$$\int_{\mathbb{R}\setminus\hat{I}}\int_{\mathbb{R}} b_R^*(x,y)^p\, w(x,y)\,dy\,dx \le c\left(\frac{|I|}{|\hat{I}|}\right)^p + c\left(\frac{|R|}{|\hat{R}|}\right)^p \le c\left(\frac{|I|}{|\hat{I}|}\right)^p.$$

Thus, replacing b_R by its expression in terms of a_R, it readily follows that

$$\int_{\mathbb{R}\setminus\hat{I}}\int_{\mathbb{R}} a_R^*(x,y)^p\, w(x,y)\,dx\,dy \le c\,\|a_R\|_{L^2_w}^p\, w(R)^{1-p/2}\left(\frac{|I|}{|\hat{I}|}\right)^p.$$

This is all we need, as we are now ready to sum over the collection of all the maximal dyadic rectangles R contained in Ω. In fact, by Hölder's inequality and the properties of atoms, it follows that

$$\sum_R \int_{\mathbb{R}\setminus\hat{I}} \int_{\mathbb{R}} a_R^*(x,y)^p\, w(x,y)\, dx\, dy$$

$$\leq c \sum_R \|a_R\|_{L_w^2}^p\, w(R)^{1-p/2} \left(\frac{|I|}{|\hat{I}|}\right)^p$$

$$\leq c \left(\sum_R \|a_R\|_{L_w^2}^2\right)^{p/2} \left(\sum_R w(R)^{(1-p/2)(2/p)'} \left(\frac{|I|}{|\hat{I}|}\right)^{p(2/p)'}\right)^{1/(2/p)'}$$

$$\leq c\, w(\Omega)^{p/2-1} \left(\sum_R w(R) \left(\frac{|I|}{|\hat{I}|}\right)^{p(2-p)/2}\right)^{1-p/2}$$

We now invoke Journé's lemma with $\phi(s) = s^{p(2-p)/2}$, and note that the above expression is then dominated by

$$c\, w(\Omega)^{p/2-1}\, w(\Omega)^{1-p/2} \leq c\,,$$

and the proof is complete.

To complete the results discussed here we consider a description of the duals to the Hardy spaces $H_w^p(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$, $r/2 < p \leq 1$, when $w \in A_r(\mathbb{R}\times\mathbb{R})$; by known properties of weights the case $H_w^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ when $w \in A_2(\mathbb{R}\times\mathbb{R})$ is included.

Given a real-valued function b on $\mathbb{R}^2$, and a weight $v \in A_r(\mathbb{R}\times\mathbb{R})$, consider the following expression: if Ω is a bounded open set in $\mathbb{R}^2$, and R runs over the collection of the maximal dyadic rectangles contained in Ω, then set

$$\|b\|_{\eta,v} = \sup_{\Omega} \inf_{b_R} \left(\frac{1}{w(\Omega)^\eta} \sum_R \|b - b_R\|_{L_v^2(R)}^2\right)^{1/2},$$

where b_R runs over the family of functions of the form

$$b_R(x,y) = c_1\, b_1(y) + c_2\, b_2(x)\,,$$
$$\operatorname{supp} b_1 \subseteq J\,, \quad \operatorname{supp} b_2 \subseteq I\,, \quad R = I \times J\,.$$

We then have

Proposition 6. *$H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)^*$, the dual of the Hardy space $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $r/2 < p \leq 1$ can be identified with $B_{2/p-1,1/w}(\mathbb{R}\times\mathbb{R})$, the collection of those square integrable functions b such that $\|b\|_{2/p-1,1/w} < +\infty$.*

PROOF. We begin by showing that each $b \in B_{2/p-1,1/w}(\mathbb{R}\times\mathbb{R})$ induces a bounded linear functional on $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, with norm less than or equal to $c\,\|b\|_{2/p-1,1/w}$.

Suppose, then, that $a = \sum_R a_R$ is a (p,w)-atom in $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, and let $b \in B_{2/p-1,1/w}(\mathbb{R}\times\mathbb{R})$. Then, by the properties of atoms, a judicious choice of the b_R's, and Cauchy's inequality,

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^2} a(x,y)\, b(x,y)\, dx\, dy \right| \\
&\leq \sum_R \left| \iint_R a_R(x,y)\, b(x,y)\, dx\, dy \right| \\
&\leq \sum_R \iint_R |a_R(x,y)|\, |b(x,y) - b_R(x,y)|\, w(x,y)^{1/2}\, w(x,y)^{-1/2}\, dx\, dy \\
&\leq \Big(\sum_R \|a_R\|^2_{L^2_w} \Big)^{1/2} \Big(\sum_R \|b - b_R\|^2_{L^2_{1/w}} \Big)^{1/2} \\
&\leq w(\Omega)^{(1-2/p)/2} \Big(\sum_R \|b - b_R\|^2_{L^2_{1/w}} \Big)^{1/2} \\
&\leq \|b\|_{2/p-1,1/w}\,.
\end{aligned}$$

Next, if $f \in H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, then it admits an atomic decomposition $f = \sum_j \lambda_j a_j$, where the a_j's are (p,w)-atoms and $\|f\|_{H^p_w} \sim (\sum_j |\lambda_j|^p)^{1/p}$. Thus,

$$\begin{aligned}
\left| \iint_{\mathbb{R}^2} f(x,y)\, b(x,y)\, dx\, dy \right| &\leq \sum_j |\lambda_j| \left| \iint_{\mathbb{R}^2} a_j(x,y)\, b(x,y)\, dx\, dy \right. \\
&\leq \Big(\sum_j |\lambda_j|^p \Big)^{1/p} \|b\|_{2/p-1,1/w}\,,
\end{aligned}$$

and the assertion follows.

Conversely, suppose that $L \in H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)^*$. Then on a dense subset there, consisting of smooth functions, L can be represented by

$b(x,y)$ in the form

$$L(f) = \iint_{\mathbb{R}^2} f(x,y)\, b(x,y)\, dx\, dy\,.$$

Let now Ω be a bounded open set in $\mathbb{R}^2$, and suppose that $\Omega = \bigcup_R R$, where the R's are the maximal dyadic rectangles contained in Ω. Now, given a function $g \in L^2_w(\mathbb{R}^2)$ and $R = I \times J$, set

$$g_R(x,y) = \frac{1}{|I|}\int_I g(u,y)\, du + \frac{1}{|J|}\int_J g(x,v)\, dv - \frac{1}{|R|}\iint_R g(u,v)\, du\, dv\,.$$

Then

$$\int_I (g(x,y) - g_R(x,y))\, dx = \int_J (g(x,y) - g_R(x,y))\, dy = 0\,,$$

and

$$\|g - g_R\|_{L^2_w(R)} \le c\, \|g\|_{L^2_w(R)}\,.$$

The first assertion is readily verified, and to see the second we consider the first term in g_R, the others being handled analogously. Note that since $w(\cdot, y) \in A_2(\mathbb{R})$ uniformly in y,

$$\begin{aligned}
&\int_J \int_I \left(\frac{1}{|I|}\int_I g(u,y)\, du\right)^2 w(x,y)\, dx\, dy \\
&\le \int_I \int_J \left(\frac{1}{|I|}\int_I g(u,y)^2 w(u,y)\, du\right)\left(\frac{1}{|I|}\int_I \frac{1}{w(u,y)}\, du\right) w(x,y)\, dx\, dy \\
&= \int_J \int_I g(u,y)^2 w(u,y) \left(\frac{1}{|I|}\int_I \frac{1}{w(u,y)}\, du\right)\left(\frac{1}{|I|}\int_I w(x,y)\, dx\right) du\, dy\,.
\end{aligned}$$

Now, since $w(\cdot, y) \in A_2(\mathbb{R})$, uniformly in y, the above expression involving the inner integrals does not exceed the A_2 constant of w, and the whole expression is less than or equal to $c\,\|g\|_{L^2_w}$, as claimed. The other terms are dealt with in a similar fashion.

Suppose now that Ω is a bounded open subset in $\mathbb{R}^2$, and that $f \in L^2(\Omega)$ is such that

$$\left(\sum_R \|f\|^2_{L^2_w(R)}\right)^{1/2} = 1\,.$$

Then, by the above remark, there is a constant c such that

$$a(x,y) = c \frac{1}{w(\Omega)^{1/p-1/2}} \sum_R (f(x,y) - f_R(x,y)) \chi_R(x,y),$$

is a (p,w)-atom of norm 1, and consequently,

$$\begin{aligned} \|L\| &\geq \|L(a)\| \\ &= c \frac{1}{w(\Omega)^{1/p-1/2}} \sum_R \iint_R (f(x,y) - f_R(x,y)) \, b(x,y) \, dx \, dy \\ &= c \frac{1}{w(\Omega)^{1/p-1/2}} \iint_R f(x,y) \left(b(x,y) - b_R(x,y) \right) dx \, dy . \end{aligned}$$

Since this estimate holds for all such f's, by duality it readily follows that

$$c \left(\frac{1}{w(\Omega)^{2/p-1}} \sum_R \|b - b_R\|^2_{L^2_{1/w}(R)} \right)^{1/2} \leq \|L\| ,$$

which is precisely what we wanted to show.

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Hilbert transforms and maximal functions along rough flat curves

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Introduction.

Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a continuous curve of class $C^1(\mathbb{R}\backslash\{0\})$ with $\Gamma(0) = 0$. It is a question of considerable interest to find necessary and/or sufficient conditions on Γ so that the operators $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$ defined by

$$\mathcal{H}_\Gamma f(x) = \text{p.v.} \int_{-\infty}^{+\infty} f(x - \Gamma(t)) \, \frac{dt}{t}$$

and

$$\mathcal{M}_\Gamma f(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x - \Gamma(t))| \, dt$$

are bounded on $L^p(\mathbb{R}^n)$ for certain $1 < p < \infty$. The case when Γ is well-curved at the origin (*i.e.* $\{\Gamma'(0), \dots, \Gamma^{(k)}(0)\}$ spans $\mathbb{R}^n$ for some k with $k \geq n$) is by now very well understood (see [SW]) and when $n = 2$ and Γ is flat and convex (or "biconvex") a great deal is known too, (see [CCC...], [CCVWW], [NVWW1], [NVWW2]). However, in higher dimensions, the case of flat curves is much less well-understood (for known results see [NVWW3], [CVWW], [Z2]) even to the extent that it is not clear which basic class(es) of curves one should be studying. In the three above-mentioned papers, the following substitute notion for convexity was proposed: the curve $\Gamma(t)$ should be of the form

$(t, \gamma_2(t), \dots, \gamma_n(t))$ where each γ_j is of class $C^n(0, +\infty)$, and for each $j = 2, \dots, n$ the determinant

$$D_j = \det \begin{pmatrix} 1 & \gamma_2' & \dots & \gamma_j' \\ 0 & \gamma_2'' & \dots & \gamma_j'' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j)} & \dots & \gamma_j^{(j)} \end{pmatrix}$$

should be positive on $(0, +\infty)$. (A similar condition would be supposed on $(-\infty, 0)$.) In two dimensions this reduces to $\gamma_2'' > 0$, that is, convexity of $(t, \gamma(t))$. Associated with curves in this class there are auxiliary functions h_j, $(j = 1, \dots, n)$ defined by

$$h_j(t) = \frac{1}{D_{j-1}(t)} \det \begin{pmatrix} t & \gamma_2(t) & \dots & \gamma_j(t) \\ 1 & \gamma_2'(t) & \dots & \gamma_j'(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j-1)}(t) & \dots & \gamma_j^{(j-1)}(t) \end{pmatrix}$$

where $D_0 \equiv 1$. In [NVWW3] it is shown that this convexity condition is equivalent to the positivity of h_j and h_j' for $1 \le j \le n$. (Note that when $n = 2$, $h_1(t) = t$ and $h_2(t) = t\gamma_2'(t) - \gamma_2(t)$.) It is also shown in [NVWW3] that a necessary and sufficient condition for *odd* curves of this class to have $\mathcal{H}_\Gamma$ bounded on L^2 is the bounded doubling of each of the functions h_j, that is the existence of a constant $C > 1$ so that $h_j(Ct) \ge h_j(t)$. In [Z2] it was shown that doubling of the h_j's is sufficient for L^2 boundedness of $\mathcal{M}_\Gamma$, while in [CVWW] it was shown that a slightly stronger condition (h_j's "infinitesimally doubling") is sufficient for L^p boundedness of $\mathcal{H}_\Gamma$ (for Γ odd) and $\mathcal{M}_\Gamma$ for $1 < p < \infty$. Thus a reasonably complete theory is available for curves of this class. Nevertheless the theorem of Nagel, Stein and Wainger [NSW] concerning differentiation in lacunary directions implies that if Γ is the polygonal curve obtained by joining points on (t, t^2, t^3) of the form $\pm 2^j$, $(j \in \mathbb{Z})$, by straight line segments, then the associated $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$ are bounded on L^p, $1 < p < \infty$. These Γ do not fall under the scope of the theory of the curves considered in [CVWW]. The purpose of this paper is to provide a theory which includes these curves as a special case. It will turn out that while our theory does handle these curves, it does not handle curves which near the origin behave like $(t, e^{-1/|t|}\operatorname{sgn} t,\ e^{-1/|t|}\log(1/|t|)\operatorname{sgn} t)$ in which the ratio of derivatives of coordinates varies slowly. Such curves do fall under the scope of

[CVWW]. It should be pointed out that in 2-dimensions a result of Ziesler [Z1] includes both results. It turns out that the analysis in the present paper is much simpler than that in [CVWW]. Probably the reason is that for curves like $(t, e^{-1/|t|}\operatorname{sgn} t,\ e^{-1/|t|}\log(1/|t|)\operatorname{sgn} t)$ the intrinsic geometry just is more complicated.

Let us say that a curve $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ is of class (K) if Γ is continuous, $\Gamma \in C^1(\mathbb{R}\backslash\{0\})$, $\Gamma(0) = 0$, and if $\Gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, then

$$|\gamma_1'|, \quad \left|\frac{\gamma_{j+1}'}{\gamma_j'}\right|, \qquad (j = 1, \dots, n-1)$$

are increasing for $t > 0$ and decreasing for $t < 0$. We say that Γ is *balanced* if for some $C \geq 1$, all $1 \leq j \leq n$, and all $t > 0$ we have

$$\left|\frac{\gamma_j(-t)}{\gamma_j(Ct)}\right| \leq 1, \qquad \left|\frac{\gamma_j(t)}{\gamma_j(-Ct)}\right| \leq 1.$$

Theorem. *Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a balanced curve of class* (K). *If Γ also satisfies*

(D)
$$\left|\frac{\gamma_{j+1}'(\lambda t)}{\gamma_j'(\lambda t)}\right| \geq 2\left|\frac{\gamma_{j+1}'(t)}{\gamma_j'(t)}\right|$$

for $1 \leq j \leq n-1$, some $\lambda > 1$ and all $t \neq 0$, then $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$ are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

REMARKS.

1. When $n = 2$ and $\gamma_1(t) = t$, this theorem is in [CCC...]; see also [CóRdeF]; note that class (K) in this case reduces to monotonicity and single-signedness of γ_2' on $(-\infty, 0)$ and on $(0, +\infty)$. Thus class (K) gives an alternative variant of convexity in higher dimensions.

2. The behaviour of Γ for $t < 0$ is irrelevant for $\mathcal{M}_\Gamma$. So in (K) and (D) no assumptions are necessary for $t < 0$, and balance is dropped when considering $\mathcal{M}_\Gamma$.

3. The C^1 assumption in (K) can be relaxed to allow, for example, the piecewise linear curves discussed above. That the theory applies to flat curves is clear since we require no more than one derivative to exist.

4. Even when $n = 2$ there are curves satisfying the hypotheses of our theorem for which it is *not* true that the map $t \mapsto \xi\,\Gamma'(t)$ has

boundedly many changes of monotonicity in any dyadic interval. For example, let $\Gamma(t) = (t^2/2, t^3/3 + \int_0^t s^7 \sin(s^{-5}/6)\, ds)$. We thank Jim Wright for pointing this out to us.

5. As in [CóRdeF], balance is necessary for a curve γ in class (K) to have $\mathcal{H}_\Gamma$ bounded on any L^p. Indeed, if $\mathcal{H}_\Gamma$ is bounded on $L^p(\mathbb{R}^n)$ and π is a projection onto any subspace of $\mathbb{R}^n$, then $\mathcal{H}_{\pi\Gamma}$ is bounded on $L^p(\pi\mathbb{R}^n)$ and hence on $L^2(\pi\mathbb{R}^n)$. In particular,

$$\left| \int_{-\infty}^{+\infty} e^{i\xi_k\, \gamma_k(t)} \frac{dt}{t} \right| \leq C\,, \qquad (1 \leq k \leq n)\,.$$

But if $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, satisfies $\rho(0) = 0$ and $|\rho'|$ is increasing for $t > 0$ and decreasing for $t < 0$, and if ρ also satisfies

$$\left| \int_{-\infty}^{+\infty} e^{i\xi\, \rho(t)} \frac{dt}{t} \right| \leq C\,,$$

then for any $s, t > 0$ such that $|\rho(t)/\rho(-s)| = 1$, we must have $|\log(s/t)| \leq 8 + C$; see [CóRdeF] (where $\rho'(0) = 0$ was assumed but not used). An additional feature of this present paper is the clarification of the role of balance as being the condition necessary to ensure compatibility of the two Calderón-Zygmund theories naturally associated to the two halves of the curve corresponding to $t > 0$ and $t < 0$.

6. If $\gamma_1, \gamma_3, \ldots$ are odd and $\gamma_2, \gamma_4, \ldots$ are even, then (D) is necessary for a curve of class (K) with $\gamma'_{j+1}(0)/\gamma'_j(0) = 0$, $j = 1, \ldots, n-1$ to have $\mathcal{H}_\Gamma$ bounded on any L^p. Indeed it is enough to see that if $\Gamma(t) = (\gamma_1(t), \gamma_2(t))$ in $\mathbb{R}^2$ with γ_1 odd and γ_2 even, $\gamma'_2(0)/\gamma'_1(0) = 0$, γ'_1, γ'_2/γ'_1 increasing on $(0, +\infty)$, then doubling of γ'_2/γ'_1 is necessary for $\mathcal{H}_\Gamma$ to be bounded on $L^p(\mathbb{R}^2)$. When $\gamma_1(t) = t$, this was done in [NVWW1]; see also [CCC...]. The argument of the latter paper easily adapts in the present situation.

7. If $\Gamma : [0, +\infty) \longrightarrow \mathbb{R}^n$ is convex in the sense that its D_j's are positive and also Γ satisfies certain normalisation conditions at the origin, it is not difficult to see (*cf.* [Z1, Lemmas 3.3 and 3.4]) that after applying an appropriate lower triangular matrix with ones on the diagonal to Γ, (not affecting its convexity) we may assume that γ'_j is positive and increasing for $j = 1, \ldots, n$ and that γ'_{j+1}/γ'_j is increasing for $j = 1, \ldots, n-1$. So (K) is satisfied by this modified curve. If this modified curve is extended to be odd, doubling of the γ'_{j+1}/γ'_j ($j = 1, \ldots, n-1$) now implies doubling of the h_j's for the modified (and

hence the original) curve since this latter condition is equivalent to L^2 boundedness of $\mathcal{H}_\Gamma$ for such curves.

2. Proof of Theorem.

We define the measures σ_k and μ_k by

$$\int f\,d\mu_k = \frac{1}{2^k}\int_{2^k}^{2^{k+1}} f(\Gamma(t))\,dt\,,$$

$$\int f\,d\sigma_k = \int\limits_{2^k\le|t|\le 2^{k+1}} f(\Gamma(t))\,\frac{dt}{t}\,.$$

We decompose $\mathcal{H}_\gamma f$ as

$$\mathcal{H}_\Gamma f = \sum_{k\in\mathbb{Z}} \sigma_k * f$$

and majorize $\mathcal{M}_\Gamma f$ by

$$\mathcal{M}_\Gamma f \le C\sup_k \mu_k * |f|\,.$$

Notice that

$$\hat{\mu}_k(\xi) = \frac{1}{2^k}\int_{2^k}^{2^{k+1}} e^{2\pi i\xi\cdot\Gamma(t)}\,dt$$

and

$$\hat{\sigma}_k(\xi) = \int\limits_{2^k\le|t|\le 2^{k+1}} e^{2\pi i\xi\cdot\Gamma(t)}\,\frac{dt}{t}\,.$$

In keeping with [CCVWW] and [CVWW] we shall introduce to the problem a family of dilation matrices which allows us to normalize the measures μ_k and σ_k so that the Fourier transforms of the normalized measures have uniform decay estimates (save for certain exceptional sets of directions which we shall handle separately, as in [CCC...], [CóRdeF], [NVWW2], [Z2].) Indeed, a combination of the ideas of [CCVWW], [CCC...], [DRdeF], and [Ch] yields the following proposition, which may be found essentially in [Z0, Theorem 2.4.1, p. 22].

Proposition. *Let $n \geq 2$. Suppose that $\{A_k\}_{k\in\mathbb{Z}}$ is a family of matrices in $GL(n,\mathbb{R})$ satisfying*

$$\|A_{k+1}^{-1} A_k\| \leq \alpha < 1\,. \tag{1}$$

Suppose $\{\nu_k\}_{k\in\mathbb{Z}}$ is a family of measures such that

$$A_{k+1}^{-1}\operatorname{supp}\nu_k \subseteq B \tag{2}$$

(where B is some fixed ball). Suppose that

$$\int d\nu_k = 0 \tag{3}$$

and that

$$|\hat{\nu}_k(\xi)| \leq C\,|A_k^*\xi|^{-1}\,, \quad \text{except when } \xi \text{ belongs to a cone } C_k\,. \tag{4}$$

Letting $\widehat{(T_k f)}(\xi) = \chi_{C_k}(\xi)\,\hat{f}(\xi)$, we further suppose that

$$\Big\|\Big(\sum_k |T_k f|^2\Big)^{1/2}\Big\|_p \leq C_p\,\|f\|_p\,, \qquad (1 < p < \infty)\,. \tag{5}$$

Then $f \mapsto \sum \nu_k * f$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

The differences between this proposition and Theorem 2.4.1 of [Z0] are that in [Z0], the conclusion was stated directly in terms of $\mathcal{H}_\Gamma$ (for Γ odd) and $\mathcal{M}_\Gamma$, and that an auxiliary Littlewood-Paley inequality

$$\Big\|\sum_k T_k f_k\Big\|_p \leq C\,\Big\|\Big(\sum_k |T_k f_k|^2\Big)^{1/2}\Big\|_p$$

was required there also. However, when T_k corresponds to the characteristic function of a cone, this inequality follows from (5) as may be seen by taking $S_k = T_k$ in the following standard lemma whose proof is omitted.

Lemma 1. *Assume $\Big\|\Big(\sum_k |T_k f|^2\Big)^{1/2}\Big\|_{p'} \leq C\,\|f\|_{p'}$ for all $f \in L^{p'}$. Then*

$$\Big\|\sum_k T_k S_k f_k\Big\|_p \leq C\,\Big\|\Big(\sum_k |S_k f_k|^2\Big)^{1/2}\Big\|_p\,.$$

Corollary. *If*

$$\Big\|\Big(\sum_k |T_k f|^2\Big)^{1/2}\Big\|_p \le C_p \|f\|_p$$

and

$$\Big\|\Big(\sum_k |S_k f|^2\Big)^{1/2}\Big\|_p \le C \|f\|_p$$

for all $f \in L^p$, $1 < p < \infty$, *then*

$$\Big\|\Big(\sum_k |T_k S_k f|^2\Big)^{1/2}\Big\|_p \le C'_p \|f\|_p \,, \qquad 1 < p < \infty .$$

PROOF. Take $f_k = \pm f$ in the lemma and average over the choice of $\pm$ in the usual way.

Notice that if Φ is a fixed C^∞ function of compact support in $\mathbb{R}^n$ then the measures $\Phi_k(x)\,dx = (\det A_k)^{-1}\Phi(A_k^{-1}x)\,dx$ satisfy (2) and (4) (with no exceptional set of directions).

To handle the maximal function we set $A_k = A(2^k)$ where, for $t > 0$,

$$A(t) = \begin{pmatrix} \gamma_1(t) & 0 & \dots & 0 \\ 0 & \gamma_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_n(t) \end{pmatrix}$$

(in keeping with the diagonally invariant nature of the problem), and $\nu_k = \pm[\mu_k - \Phi_k\,dx]$ where Φ is normalized so that $\int d\nu_k = 0$ and so (3) holds. Then (1) and (2) follow from the fact that each $|\gamma_j'|$ is increasing.

Let $M > 1$ be large and fixed and let

$$C_k = \bigcup_{1 \le i < j \le n} C_k^{ij}$$

where

$$C_k^{ij} = \left\{\xi :\ \frac{1}{M}\left|\frac{\gamma_j'(2^k)}{\gamma_i'(2^k)}\right| \le \left|\frac{\xi_i}{\xi_j}\right| \le M\left|\frac{\gamma_j'(2^{k+1})}{\gamma_i'(2^{k+1})}\right|\right\}.$$

We shall show that if $\xi \notin C_k$, then

$$|\hat{\mu}_k(\xi)| \le \frac{C}{|A_k^*\xi|}\,,$$

(*i.e.* (4) holds for μ_k, see Lemma 3 below), and that (5) holds for $\{C_k\}$.
Assuming we have done this we have then

$$\begin{aligned}\mathcal{M}_\Gamma f &\le \sup_k \mu_k * |f| \\ &\le \Big(\sum_k |\mu_k * f - \Phi_k * f|^2\Big)^{1/2} + \sup_k |\Phi_k * f|.\end{aligned}$$

the first term is controlled on L^p by averaging the conclusion of the Proposition over all choices of arbitrary $\pm$ in the standard way, while the second is controlled by the Hardy-Littlewood maximal function (defined with respect to the dilations $A(t)$) as in [NVWW2] or [CCVWW]. Thus $\mathcal{M}_\Gamma$ is bounded on L^p.

Before proceeding to prove (4) and (5), let us see what differences are involved when we instead consider $\mathcal{H}_\Gamma$. We repeat the above arguments, but now with $\sigma_k^\pm$ in place of μ_k where

$$\int f\, d\sigma_k^+ = \int_{2^k \le t \le 2^{k+1}} f(\Gamma(t))\, \frac{dt}{t}$$

and $\sigma_k^- = \sigma_k^+ - \sigma_k$. For σ_k^+ we have matrices A_k^+ and for σ_k^- we have A_k^-; the same arguments as for $\mathcal{M}_\Gamma$ allow us to conclude that both

$$\sum (\sigma_k^+ * f - \Phi_k^+ * f)$$

and

$$\sum (\sigma_k^- * f - \Phi_k^- * f)$$

are bounded on L^p, $1 < p < \infty$, once the estimates corresponding to (4) and (5) have been established. Hence

$$\sum \sigma_k * f = \sum (\sigma_k^+ - \sigma_k^-) * f$$

differs from an L^p bounded operator $(1 < p < \infty)$ by

$$\sum (\Phi_k^+ - \Phi_k^-) * f.$$

This latter operator is easily seen to be a Calderón-Zygmund operator (with respect to the dilations $A(t)$, as in [CCVWW]) and so is L^p

bounded ($1 < p < \infty$) and of weak type 1 if and only if it is L^2 bounded; this occurs if and only if

$$\Big|\sum_k \hat{\Phi}_k^+(\xi) - \hat{\Phi}_k^-(\xi)\Big| = \Big|\sum_k \hat{\Phi}(A_k^{+*}\xi) - \hat{\Phi}(A_k^{-*}\xi)\Big|$$

defines a bounded function on $\mathbb{R}^n$.

Lemma 2. *If there is an $r \geq 0$ so that $j - k \geq r$ implies that*

$$\|A_j^{+^{-1}} A_k^-\| \leq 1 \qquad \text{and} \qquad \|A_j^{-^{-1}} A_k^+\| \leq 1$$

for all j, then

$$\sup_{\xi\in\mathbb{R}^n} \Big|\sum_k \hat{\Phi}(A_k^{+*}\xi) - \hat{\Phi}(A_k^{-*}\xi)\Big|$$

is finite.

The easy proof is left to the reader. (Note that this lemma provides a compatibility condition between the two halves of the curve so that the two Calderón-Zygmund theories generated should be consistent.) In our present case, the hypothesis of this lemma reduces to the condition that Γ should be balanced.

So, the proof of the theorem will be finished once we have established the estimates (4) and (5).

Lemma 3. *If $\xi \notin C_k$ then*

$$|\hat{\mu}_k(\xi)| \leq \frac{C}{|A_k^*\xi|}\,.$$

PROOF. We first observe that $\xi \notin C_k$ implies the existence of an m, $1 \leq m \leq n$, such that for $2^k \leq t \leq 2^{k+1}$ and $j \neq m$, we have

$$\left|\frac{\gamma_j'(t)}{\gamma_m'(t)}\right| \leq \frac{1}{M}\left|\frac{\xi_m}{\xi_j}\right|. \tag{6}$$

Indeed, $\xi \notin C_k$ implies that if $1 \leq i < j \leq n$, then either

$$\left|\frac{\gamma_j'(t)}{\gamma_i'(t)}\right| \geq \left|\frac{\gamma_j'(2^k)}{\gamma_i'(2^k)}\right| \geq M\left|\frac{\xi_i}{\xi_j}\right|,$$

for all t such that $2^k \leq t \leq 2^{k+1}$, or

$$\left|\frac{\gamma_j'(t)}{\gamma_i'(t)}\right| \leq \left|\frac{\gamma_j'(2^{k+1})}{\gamma_i'(2^{k+1})}\right| \leq \frac{1}{M}\left|\frac{\xi_i}{\xi_j}\right|,$$

for all t such that $2^k \leq t \leq 2^{k+1}$.

Thus of the functions $t \mapsto |\xi_j||\gamma_j'(t)|$ $(1 \leq j \leq n)$, one exceeds all the others by a factor of at least M uniformly on $[2^k, 2^{k+1}]$. If this is the function $|\xi_m|\,|\gamma_m'(t)|$, (6) follows. Now, with $\xi \notin C_k$ and m fixed and satisfying (6) we can write

$$\begin{aligned}\hat{\mu}_k(\xi) &= \frac{1}{2^k}\int_{2^k}^{2^{k+1}} e^{2\pi i\xi\cdot\Gamma(t)}\,dt \\ &= \frac{1}{2^k}\int_{\gamma_m(2^k)}^{\gamma_m(2^{k+1})} e^{2\pi i\xi\cdot\Gamma\circ\gamma_m^{-1}(s)}\,\frac{ds}{\gamma_m'\circ\gamma_m^{-1}(s)}\,.\end{aligned}$$

Letting

$$\phi(s) = 2\pi\xi\cdot\Gamma\circ\gamma_m^{-1}(s)$$

we see that

$$\phi'(s) = 2\pi\sum_{j=1}^{n}\xi_j\,\frac{\gamma_j'(\gamma_m^{-1}(s))}{\gamma_m'(\gamma_m^{-1}(s))}\,,$$

which, by virtue of (6) satisfies

$$|\phi'(s)| \geq 2\pi\,|\xi_m|\left(1-\frac{n-1}{M}\right) \geq C\,|\xi_m|$$

(if M is sufficiently large), for all $\gamma_m(2^k) \leq s \leq \gamma_m(2^{k+1})$. We now set

$$w(s) = \frac{e^{i\phi(s)}}{\phi'(s)\,\gamma_m'\circ\gamma_m^{-1}(s)}\,,$$

so that

$$\begin{aligned}w'(s) = &\frac{ie^{i\phi(s)}}{\gamma_m'\circ\gamma_m^{-1}(s)} + \frac{e^{i\phi(s)}}{\phi'(s)}\left(\frac{1}{\gamma_m'\circ\gamma_m^{-1}}\right)'(s) \\ &- \frac{e^{i\phi(s)}\,\phi''(s)}{(\phi'(s))^2\,\gamma_m'\circ\gamma_m^{-1}(s)}\end{aligned}$$

and

$$\begin{aligned}|\hat{\mu}_k(\xi)| &\leq \frac{1}{2^k}\left|w(\gamma_m(2^{k+1})) - w(\gamma_m(2^k))\right| \\ &\quad + \frac{C}{2^k}\frac{1}{|\xi_m|}\int_{\gamma_m(2^k)}^{\gamma_m(2^{k+1})}\left|\left(\frac{1}{\gamma_m' \circ \gamma_m^{-1}}\right)'\right| ds \\ &\quad + \frac{C}{2^k}\frac{1}{|\xi_m|^2}\int_{\gamma_m(2^k)}^{\gamma_m(2^{k+1})}\frac{|\phi''(s)|}{|\gamma_m' \circ \gamma_m^{-1}(s)|}\, ds \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III},\end{aligned}$$

by the Fundamental Theorem of Calculus and the fact that $|\phi'(s)| \geq C|\xi_m|$.

Now

$$\begin{aligned}\mathrm{I} &\leq \frac{C}{2^k|\xi_m|}\frac{1}{|\gamma_m'(2^k)|} \\ &\leq \frac{C}{2^k\sum_{j=1}^n|\xi_j|\,|\gamma_j'(2^k)|} \qquad \text{(by (6) with } t = 2^k\text{)} \\ &\leq \frac{C}{\sum_{j=1}^n|\xi_j|\,|\gamma_j(2^k)|} \leq \frac{C}{|A_k^*\xi|}\end{aligned}$$

(since each $|\gamma_j'|$ is monotonic).

Similarly

$$\mathrm{II} \leq \frac{C}{2^k|\xi_m|}\frac{1}{|\gamma_m'(2^k)|} \leq \frac{C}{|A_k^*\xi|}$$

since $\gamma_m' \circ \gamma_m^{-1}$ is monotonic on $[2^k, 2^{k+1}]$.

Finally, for III, it is enough to show that for each $j \neq m$,

$$\frac{|\xi_j|}{|\xi_m|}\int_{\gamma_m(2^k)}^{\gamma_m(2^{k+1})}\left|\left(\frac{\gamma_j' \circ \gamma_m^{-1}}{\gamma_m' \circ \gamma_m^{-1}}\right)'(s)\right|\frac{ds}{|\gamma_m' \circ \gamma_m^{-1}(s)|} \leq \frac{C}{|\gamma_m'(2^k)|}$$

and then argue as for I. But since $|\gamma_m' \circ \gamma_m^{-1}(s)| \geq |\gamma_m'(2^k)|$ and $(\gamma_j'/\gamma_m') \circ \gamma_m^{-1}$ is monotonic, this estimate reduces to

$$\frac{|\xi_j|}{|\xi_m|}\left|\frac{\gamma_j'}{\gamma_m'}(2^{k+1}) - \frac{\gamma_j'}{\gamma_m'}(2^k)\right| \leq C,$$

which follows from (6). This completes the proof of the Lemma.

Finally, to establish (5), we first note that

$$\chi_{C_k} = \sum_{i<j} \chi_{C_k^{ij}} + \sum_{\substack{i<j \\ \ell<m \\ (i,j)\neq(l,m)}} \chi_{C_k^{ij}} \chi_{C_k^{lm}} + \dots \pm \prod_{i<j} \chi_{C_k^{ij}} .$$

Setting $\widehat{(T_k^{ij} f)}(\xi) = \chi_{C_k^{ij}}(\xi) \hat{f}(\xi)$, we see that, using the Corollary to Lemma 1, it suffices to prove

$$\left\| \left(\sum_k |T_k^{ij} f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p , \qquad 1 < p < \infty .$$

But this is now a two-dimensional inequality, and as such follows from the following lemma and assumption (D).

Lemma 4. *If $M > 1$, $\rho > 1$ and $\lambda_{k+1} \geq \lambda_k \rho$ ($k \in \mathbb{Z}$), and we define for $\xi \in \mathbb{R}^2$*

$$\widehat{(R_k f)}(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } \dfrac{\lambda_k}{M} \leq \dfrac{|\xi_1|}{|\xi_2|} \leq M \lambda_{k+1} \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\left\| \left(\sum_k |R_k f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p , \qquad 1 < p < \infty .$$

PROOF. It is a standard exercise, using the Marcinkiewicz multiplier theorem and the theorem of Nagel, Stein and Wainger on Differentiation in Lacunary Directions, see [NSW]. The proof of our main theorem is complete.

3. Concluding Remarks.

1. By projection on the first co-ordinate, the following assertion is contained in the theorem: if $\gamma(t)$ satisfies $|\gamma'(t)|$ increasing for $t > 0$ and

decreasing for $t < 0$, $\gamma(0) = 0$, and $|\gamma(-t)/\gamma(Ct)|$, $|\gamma(t)/\gamma(-Ct)| \leq 1$ (all $t > 0$, some C) then

$$f \mapsto \int_{-\infty}^{+\infty} f(x-\gamma(t))\,\frac{dt}{t}$$

is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$. Indeed more is true: this operator is of weak type 1. (The corresponding fact for maximal functions, and the L^p boundedness of this operator are well-known -see [W] or [CóRdeF] and [W] respectively.) Since we already know that the operator is bounded on L^2 it is enough to see that the convolution kernel satisfies an appropriate variant of the Hörmander condition. As in the proof of the main theorem, the balance condition allows us to reduce this to verification of a Hörmander condition for $t > 0$. Now, formally

$$\int_0^{+\infty} f(x-\gamma(t))\,\frac{dt}{t} = K * f(x),$$

where

$$K(x) = \frac{1}{\gamma'(\gamma^{-1}(x))\,\gamma^{-1}(x)},$$

and we may clearly assume without loss of generality that γ' is positive and increasing on $(0, +\infty)$ so that K is decreasing. Although we have no estimates on the derivative of K, monotonicity is enough to allow us to apply the Hörmander criterion in the form

$$\sup_{j\in\mathbb{Z}}\ \sup_{0<y\leq 3\,\gamma(2^j)} \int_{x\geq 5\,\gamma(2^j)} |K(x) - K(x-y)|\,dy \leq C$$

(see [CCVWW, Theorem 2.3]). Indeed, for j fixed and $y \leq 3\,\gamma(2^j)$,

$$\begin{aligned}\int_{x\geq 5\,\gamma(2^j)} |K(x)-K(x-y)|\,dy &= \int_{5\,\gamma(2^j)-y}^{+\infty} K(x)\,dx - \int_{5\,\gamma(2^j)}^{+\infty} K(x)\,dx \\ &\leq \int_{2\,\gamma(2^j)}^{5\,\gamma(2^j)} K(x)\,dx \\ &= \log\frac{t_1}{t_0},\end{aligned}$$

where $\gamma(t_0) = 2\,\gamma(2^j)$, and $\gamma(t_1) = 5\,\gamma(2^j)$. But γ' increasing implies γ doubles, so $\log(t_1/t_0)$ is bounded.

2. Finally, we ask the following question: is there a condition involving at most one derivative of an odd curve Γ which gives L^2 boundedness of $\mathcal{H}_\Gamma$ but not L^p boundedness for all $1 < p < \infty$? (In $\mathbb{R}^2$, γ' doubling is strictly stronger than h doubling for convex curves.)

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Heat kernel upper bounds on a complete non-compact manifold

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Introduction.

Let M be a smooth connected non-compact geodesically complete Riemannian manifold, Δ denote the Laplace operator associated with the Riemannian metric, $n \geq 2$ be the dimension of M. Consider the heat equation on the manifold

$$u_t - \Delta u = 0\,, \tag{1.1}$$

where $u = u(x,t)$, $x \in M$, $t > 0$. The heat kernel $p(x,y,t)$ is by definition the smallest positive fundamental solution to the heat equation which exists on any manifold (see [Ch], [D]).

The purpose of the present work is to obtain uniform upper bounds of $p(x,y,t)$ which would clarify the behaviour of the heat kernel as $t \to +\infty$ and $r \equiv \operatorname{dist}(x,y) \to +\infty$. In the Euclidean space $\mathbb{R}^n$ the heat kernel is given by the following well known formula

$$p(x,y,t) = \frac{1}{(4\pi t)^{n/2}} \exp\Big(-\frac{r^2}{4t}\Big) \tag{1.2}$$

so that it decreases as $C\,t^{-n/2}$ as $t \to +\infty$ and its behaviour for large r is determined by the Gaussian term $\exp(-r^2/4t)$. In the n-dimensional hyperbolic space $\mathbb{H}^n_k$ (of constant curvature $-k^2 < 0$) the heat kernel is known as well. It takes the simplest form in the case of the dimension 3,

$$p(x,y,t) = \frac{1}{(4\pi t)^{3/2}} \frac{kr}{\sinh(kr)} \exp\left(-\frac{r^2}{4t} - k^2 t\right). \tag{1.3}$$

Its the most significant difference from the Euclidean heat kernel is an exponential decay in t given by the factor $\exp(-k^2 t)$.

The first question to be discussed here is which geometric properties of the manifold ensure decreasing of the heat kernel as $t \to +\infty$ with a prescribed speed? More precisely, when the heat kernel satisfies the following estimate

$$p(x,y,t) \leq f(t), \tag{1.4}$$

for all $x, y \in M$, $t > 0$ where $f(t)$ is a monotonically decreasing function on the positive real semi-axis. This kind of a heat kernel estimate is often referred to as an on-diagonal estimate because (1.4) follows from the same inequality for $x = y$ (see the Proposition 2.1 bellow). It is well known and due to Nash [N] (see also [CKS]) that a heat kernel on-diagonal upper bound is deduced from a suitable isoperimetric inequality. Consider, for example, a special case when $f(t) = C t^{-n/2}$ which takes place in the Euclidean space. Then the corresponding inequality

$$p(x,y,t) \leq C t^{-n/2} \tag{1.5}$$

can be proved whenever we know, that for any smooth function v with a compact support the following Sobolev inequality holds

$$\int_M |\nabla v|^2 \geq c \left(\int_M |v|^{2n/(n-2)} \right)^{(n-2)/n} \tag{1.6}$$

(of course we have to assume here that $n > 2$). This inequality is close to the classical isoperimetric inequality between the volume of a bounded region and the area of its boundary

$$\operatorname{Area}(\partial\Omega) \geq c \operatorname{Vol}(\Omega)^{(n-1)/n},$$

for any bounded domain Ω with a smooth boundary $\partial\Omega$. Namely, the isoperimetric inequality above implies (1.6) (converse is not true -see [CL]).

N.Varopoulos [V85] proved that the Sobolev inequality (1.6) is not only sufficient but a necessary condition as well for the upper bound (1.5) to be valid. On the other hand, Carlen, Kusuoka and Stroock [CKS] found an alternative form of this phenomenon: the upper bound (1.5) is equivalent to the Nash inequality

$$\left(\int_M f^2\right)^{1+2/n} \leq C \int_M |\nabla f|^2 \left(\int_M |f|\right)^{4/n} \tag{1.6'}$$

supposed to hold for any $f \in C_c^\infty(M)$. This theorem has the advantage that it does not require the hypothesis $n > 2$. Therefore, the upper estimate (1.5) is equivalent to either Sobolev and Nash inequality.

The Gaussian factor in heat kernel upper bounds on Riemannian manifolds appeared in the paper of Cheng, Li and Yau [ChLY] but they considered the heat kernel behaviour only in a finite time interval. For the whole range of time the on-diagonal bound (1.5) was shown by several authors to imply the following Gaussian off-diagonal correction

$$p(x, y, t) \leq C\, t^{-n/2} \exp\left(-c\frac{r^2}{t}\right) . \tag{1.7}$$

Apparently for the first time it was proved by Ushakov [U] in 1980. He derived (1.7) from (1.5) by means of pure analytical tools without appealing to geometry. Although he treated the case of unbounded regions in Euclidean space his proof can be carried over to abstract manifolds too. Another proof was obtainded by Davies and Pang [DP] (see also the preceding work [D87]) who used a logorithmic Sobolev inequality which is equivalent to both estimates (1.5) and (1.7) (where constant c can be taken arbitrarily close to $1/4$). Thus, we have that each the of relations (1.5), (1.6), (1.6′), (1.7) is equivalent to the others. The following questions arise:

1. To obtain an isoperimetric property which would be equivalent to estimate (1.4) for a general function f without the polynomial or other restrictions on the behaviour of f at infinity;
2. To obtain the corresponding estimate with the Gaussian off-diagonal correction term.

If the function f has at least a polynomial decay (which means in this context that it satisfies the condition

$$f(t) \leq C\, f(a\,t)$$

for all $t > 0$ and $a \in [1,2]$), for example

$$f(t) = C \begin{cases} t^{-n/2}, & \text{if } t < 1, \\ t^{-m/2}, & \text{if } t \geq 1, \end{cases}$$

then as it was proved by Davies (see [D89], [DP]) (1.4) implies

$$p(x,y,t) \leq C_\varepsilon f(t) \exp\left(-\frac{r^2}{(4+\varepsilon)t}\right)$$

(where $\varepsilon > 0$ is arbitrary). For example, this bound holds for the Riemannian product $M = K \times \mathbb{R}^m$, K being a $(n-m)$-dimensional compact manifold.

There are some other works which treat the polynomial case. In a series of papers of A. K. Gushchin (see, for example, [G], [GMM]) heat kernel bounds are proved for the case of unbounded region in Euclidean space with the Neumann boundary condition provided some isoperimetric inequality is valid. He obtained the exaustive results but also for the case when the heat kernel has a polynomial decay as $t \to +\infty$.

At the same time there exist many important classes of manifolds whose heat kernel decreases faster than polynomially as $t \to +\infty$. For hyperbolic space and for a wide class of negatively curved manifolds the heat kernel has an exponential decay. There are examples of manifolds -covering manifolds with a deck transformation group being a polycyclic one- for which the heat kernel decreases as $t \to +\infty$ subexponentially but superpolynomially (see [V91], [A] for discrete counterparts).

In the present paper we adduce a new approach of obtaining the heat kernel upper bounds which enables us to cover all results mentioned above, and, moreover, to get the corresponding estimates for a larger variety of manifolds.

The theorem to be formulated below establishes equivalence in a rather general situation between the on-diagonal estimate (1.4), the corresponding Gaussian estimate and some inequality of Faber-Krahn type which we use in place of the Sobolev and Nash inequalities mentioned above.

Let $\Lambda(v)$ be a positive continuous monotonically decreasing function on the positive real semi-axis.

Definition 1.1. *Let us say that a Λ-isoperimetric inequality is valid for a region $\Omega \subset M$, if for any sub-region $D \subset \Omega$ the first Dirichlet* *eigenvalue* $\lambda_1(D)$ *is controlled below by* $\Lambda(\operatorname{Vol} D)$,

$$\lambda_1(D) \geq \Lambda(\operatorname{Vol} D). \tag{1.8}$$

It is very natural to use Λ-isoperimetric inequality for evaluating the heat kernel. Indeed, the heat kernel of a bounded region $\Omega \subset M$ decreases as $t \to +\infty$ as $C\exp(-\lambda_1(\Omega)t)$. If $\lambda_1(M) \equiv \lim_{\Omega\to M} \lambda_1(\Omega) > 0$ then one can hope that the heat kernel $p(x,y,t)$ behaves itself as $C\exp(-\lambda_1(M)t)$ for large t. Otherwise, if $\lambda_1(M) = 0$ one may expect that the order of decay of p as $t \to +\infty$ depends on the speed of convergence $\lambda_1(\Omega)$ to 0 on expanding of Ω.

Note that the value $\lambda_1(M)$ is referred to as the spectral radius of the manifold M and coincides with the bottom of the spectrum in $L^2(M)$ of $-\Delta$.

A Λ-isoperimetric inequality can be easily deduced from the following inequality between the volume and the area

$$\operatorname{Area}(\partial D) \geq g(\operatorname{Vol} D) \tag{1.8'}$$

with some function g. Namely, if $g(v)/v$ is a decreasing function then (1.8′) implies (1.8) where Λ is expressed through g by

$$\Lambda(v) = \frac{1}{4}\left(\frac{g(v)}{v}\right)^2$$

(this is a consequence of the well-known inequality of Cheeger -see Proposition 2.4 below). Conversely, at least for a polynomial function $\Lambda(v) = C\,v^{-\nu}$, $\nu > 0$ and for a manifold of non-negative Ricci curvature the Λ-isoperimetric inequality implies the isoperimetric inequality (1.8′) with the function $C\,g(v)$ -see [C].

For example, in the Euclidean space we have

$$\Lambda(v) = c_n\,v^{-2/n}, \tag{1.9}$$

while for the hyperbolic space $\mathbb{H}^n_k$

$$\Lambda(v) = \max\{c_n\,v^{-2/n},\, \lambda\} \tag{1.9'}$$

where $\lambda = (n-1)^2k^2/4$.

For a class of covering manifolds Coulhon and Saloff-Coste [CS] proved the isoperimetric inequality (1.8′) which implies (1.8) with the following function Λ having an intermediate magnitude

$$\Lambda(v) = C\,(\log v)^{-\nu}\,, \tag{1.10}$$

for large v where ν is a positive constant.

Let us define a function $V(t)$ by means of the following relation

$$t = \int_0^{V(t)} \frac{dv}{v\,\Lambda(v)}\,. \tag{1.11}$$

Of course, we have to suppose that the integral here converges at 0. This is not a strong restriction because for small values v the function $\Lambda(v)$ is expected to be as in Euclidean space. Obviously, $V(t)$ is an increasing function on $(0, +\infty)$.

The following theorem is one of the main results to be presented here.

Theorem 1.1. *Consider the following hypotheses,*

1. *Λ-isoperimetric inequality holds on M, i.e. for any pre-compact region $\Omega \subset M$ we have*

$$\lambda_1(\Omega) \geq \Lambda(|\Omega|)\,.$$

2. *For all $x, y \in M$, and $t > 0$*

$$p(x,y,t) \leq \frac{C}{V(ct)} \exp\left(-\frac{r^2}{Dt}\right)\,.$$

3. *For all $x \in M$, and $t > 0$*

$$p(x,x,t) \leq \frac{C}{V(ct)}\,.$$

4. *For any pre-compact region $\Omega \subset M$ we have*

$$\lambda_k(\Omega) \geq c\,\Lambda\big(C\,\frac{|\Omega|}{k}\big)\,, \qquad \textit{for all } k = 1, 2, \ldots$$

and suppose that the function $V(t)$ is regular in some sense (see below). We claim that

$$1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4$$

where the constants c, C are positive and can be different in the different items; $D > 4$ can be taken arbitrarily close to 4.

Let us note that the implication 2$\Longrightarrow$3 is obvious and included into the theorem for the sake of completeness. This theorem means that Λ-isoperimetric inequality is equivalent to either the on-diagonal bound and the off-diagonal Gaussian estimate up to the constant multiples. Indeed, the item 4 of the Theorem 1.1 implies for $k = 1$ the Λ-isoperimetric inequality which differs from that of the item 1 only by the multiples c, C. As a consequence we see that the isoperimetric inequality for the first eigenvalue implies the corresponding inequalities for the higher eigenvalues.

The regularity conditions mentioned in the Theorem 1.1 are the following. The implication 1$\Longrightarrow$2 is proved in the Theorem 5.1 in Section 5 under the hypothesis that for some $T \in]0, +\infty]$ the function $tV'(t)/V(t)$ is increasing for $t > T$ and bounded for $t \leq 2T$ by some constant. This hypothesis restricts, first, the behaviour of the function $V(t)$ as $t \to 0$ -it may be equal to t^ν, $\nu > 0$ but may not be equal to $e^{-1/t}$, and, second, the function $V(t)$ may not have too flat parts on its graph if it grows superpolynomially as $t \to +\infty$. For example, for large t the function $V(t)$ may be equal to $(\log t)^\alpha t^\beta \exp(t^\gamma)$ with the arbitrary non-negative constants α, β, γ.

The implication 3$\Longrightarrow$4 requires that, first, the function $V(t)$ is obtained by the transformation (1.11) from a function Λ (see Proposition 2.2 in Section 2 for the explicit conditions) and, second, its logorithmic derivative $V'(t)/V(t)$ has at most a polynomial decay in the sense mentioned in the discussion above. This condition holds, for example, provided $V(t) = t^\nu$, $\nu > 0$ for small t and the derivatives $V'(t)$ makes no jumps for large t, for instance, the function $(\log t)^\alpha t^\beta \exp(t^\gamma)$ satisfies it.

We see that the regularity assumptions does not restrict the rate of increase of $V(t)$ as $t \to +\infty$. Note that the function $V(t)$ defined from (1.11) cannot increase faster than exponentially -the fastest growth corresponds to the case when $\Lambda(v)$ is a positive constant for large v. This conforms to the fact that the heat kernel cannot decrease in t quicker than exponentially that follows from the same property of the heat kernel in a bounded region.

Let us also note that the implication 1$\Longrightarrow$3 does not require any additional condition and this is the reason why we distinguish this part of the Theorem 1.1 as a separate Theorem 2.1.

EXAMPLES. 1. Let us set

$$\Lambda(v) = C \begin{cases} v^{-2/n}, & \text{if } v < 1, \\ v^{-2/m}, & \text{if } v \geq 1. \end{cases} \tag{1.12}$$

Then by the Theorem 1.1 we have the following estimate

$$p(x,y,t) \leq \frac{C_D}{\min\{t^{n/2},\ t^{m/2}\}} \exp\left(-\frac{r^2}{Dt}\right).$$

Taking here $m = n$ we see that the heat kernel estimate (1.7) of a Euclidean type is equivalent to the Λ-isoperimetric inequality (1.9). On the other hand as was mentioned above (1.7) is equivalent to the Sobolev inequality (1.6). Hence, the Sobolev inequality (1.6) and the Λ-isoperimetric inequality (1.9) are equivalent. Of course, that can be proved directly too, see [C].

2. If Λ is given by the formula (1.10) for large v (and is Euclidean one for small v) then by the Theorem 1.1 we have for large t

$$p(x,y,t) \leq C \exp\left(-c_\nu\, t^{1/(\nu+1)} - \frac{r^2}{Dt}\right). \tag{1.13}$$

In particular, if $\nu = 2$ as it takes place on a covering manifold with a deck transformation group of the exponential growth (see [CS]), then the heat kernel decreases for large t at least as fast as $\exp(-ct^{1/3})$. This is a sharp order of the heat kernel decay as has been shown in [A].

3. Let Λ be the function (1.9′) with some $\lambda > 0$ as it holds on a simply connected manifolds with a strictly negative curvature, then by the Theorem 1.1

$$p(x,y,t) \leq C\, t^{-n/2} \exp(-c\,\lambda\, t - \frac{r^2}{D\,t}).$$

This estimate will be improved in Section 5 for t bounded away from 0, for example $t > 1$

$$p(x,y,t) \leq C \left(1 + \frac{r^2}{t}\right)^{1+n/2} \exp(-\lambda\, t - \frac{r^2}{4\,t}). \tag{1.14}$$

The coefficient λ at the exponent is sharp as can be seen in the case of the hyperbolic space. Note that the largest possible value of λ here is the spectral radius of the manifold $\lambda_1(M)$.

A part of the Theorem 1.1 which relates to behaviour of the heat kernel in time variable is proved in Section 2 below (see the Theorems 2.1 and 2.2 which treat the cases 1$\Longrightarrow$3 and 3$\Longrightarrow$4 respectively). We apply Λ-isoperimetric inequality in order to derive some inequality of Nash type which we use in place of the Sobolev inequality.

The main efforts are directed to get estimates containing the Gaussian factor. For this purpose we consider some weighted integral of p^2 over the entire manifold (in place of the usual integral of p^2 over an exterior of a ball -see Section 4 for details) which happens to decrease in t. This property simplifies considerably the proof and makes our approach very flexible. To estimate this integral (Theorems 4.1 and 4.2 in Section 4) we apply some mean-value type theorem which relies upon Λ-isoperimetric inequality and is of independent interest (Theorems 3.1 and 3.2 in Section 3).

Having the integral estimates it is easy to pass to pointwise heat kernel upper bounds using the semigroup property of the heat kernel (see Section 5). The Theorem 5.1 completes the remaining part 1$\Longrightarrow$2 of the Theorem 1.1. The method of proving of the Theorem 5.1 enables us to get heat kernel estimates not only in the case when a Λ-isoperimetric inequality holds on the entire manifold but also when a Λ-isoperimetric inequality holds with different functions Λ on different parts of the manifold. We consider two kinds of such a situation.

1. Suppose that Λ-isoperimetric inequality is known to be true in any geodesic ball but, possibly, with its own function Λ. Under this hypothesis we obtain a heat kernel upper bound (see Theorem 5.2) which is applicable, for example, to an arbitrary manifold and yields the estimate (1.14) above where the C has to depend on x, y. Another example of applications of the Theorem 5.2 is the heat kernel estimate on a manifold of a non-negative Ricci curvature

$$p(x,y,t) \le \frac{C}{\operatorname{Vol} B^x_{\sqrt{t}}} \exp\left(-\frac{r^2}{(4+\varepsilon)t}\right), \qquad \varepsilon > 0$$

(where B^x_R denotes the geodesic ball with the centre $x \in M$ and of the radius R) obtained first by P. Li and S.-T. Yau [LY]. A similar result of B. Davies [D88] for a manifold of a bounded below Ricci curvature is covered by our approach too.

2. On the other hand we are able to get an extra information concerning the heat kernel behaviour whenever in addition to a global Λ-isoperimetric inequality there is a stronger isoperimetric inequality in

a neighbourhood of infinity. The Theorem 5.3 appears to be an example of such a statement and yields on a Cartan-Hadamard manifold the following estimate

$$p(x,y,t) \le \frac{C}{t^{n/2}} \exp\left(-c\left(\frac{r^2}{4t} - \lambda_1(M)t - rk(r/2)\right)\right),$$

where $-k^2(R)$ denotes the supremum of the sectional curvature in the exterior of the ball B^y_R. What is new here is the third term at the exponent

$$\exp\left(-c\, r\, k(r/2)\right),$$

which associates with the term

$$\frac{kr}{\sinh(kr)}$$

in the case of the heat kernel (1.3) of the hyperbolic space. It can cause the heat kernel to decrease as $r \to +\infty$ faster than predicted by the Gaussian term provided the curvature $-k^2(R)$ outside the ball of the radius R approaches to $-\infty$ fast enough as $R \to +\infty$.

The results of this paper were partially announced in [G87b], [G87c], [G87a], [G88] and [G91a].

NOTATIONS.

$C_{a,b,\dots}$ -a positive constant, depending on $a, b, \dots$, maybe different on different occasions;
$\operatorname{dist}\{x,y\}$ -the geodesic distance between the points $x, y \in M$;
B^x_R -an open geodesic ball of the radius R centered at the point $x \in M$;
$\operatorname{meas}_k A$ -k-dimensional Riemannian measure of set A;

$$|A| \equiv \operatorname{Vol} A \equiv \begin{cases} \operatorname{meas}_n A, & \text{if } A \subset M, \\ \operatorname{meas}_{n+1} A, & \text{if } A \subset M \times \mathbb{R}. \end{cases}$$

$\lambda_k(\Omega)$ -the k-th eigenvalue of the Dirichlet boundary value problem in Ω.

2. Decay of the heat kernel in time.

The heat kernel on a manifold can be constructed by means of the following process. For any relatively compact subset $\Omega \subset M$ with a smooth boundary one can define the heat kernel $p_\Omega(x, y, t)$ as the Green function of mixed the problem the for heat equation in $\Omega \times (0, +)\infty$ with vanishing Dirichlet boundary values. Let us denote by $\varphi_k(x)$ the k-th eigenfunction of the Dirichlet boundary value problem in Ω, $k = 1, 2, \ldots$ so that the sequence $\{\varphi_k\}$ is an orthonormal basis in the space $L^2(\Omega)$, then the following eigenfunction expansion holds

$$p_\Omega(x, y, t) = \sum_{k=1}^{\infty} \exp\left(-\lambda_k(\Omega)t\right) \varphi_k(x)\, \varphi_k(y)\,. \tag{2.1}$$

A proof of this expansion as well as a justification of other properties of p_Ω to be used in this Section can be found in [D] and [Cha].

The maximum principle implies that p_Ω is non-negative and monotone in Ω: it increases on expansion of Ω. At the same time the integral of the heat kernel remains bounded,

$$\int_\Omega p_\Omega(x, y, t)\, dy \leq 1\,,$$

which implies that p_Ω has a finite limit as $\Omega \to M$ where $\Omega \to M$ denotes an exhaustion of M by a sequence of relatively compact domains Ω. The limit

$$p(x, y, t) = \lim_{\Omega \to M} p_\Omega(x, y, t)$$

is obviously the smallest positive fundamental solution to the heat equation *i.e.* the heat kernel on M. The function $p(x, y, t)$ inherits from $p_\Omega(x, y, t)$ the following properties:

1. $p(x, y, t) > 0$ for all $x, y \in M, t > 0$ and satisfies the heat equation with respect to x, t for any fixed y;

2. $p(x, y, t) \to \delta_y(x)$ in sense of distributions as $t \to 0$;

3. symmetry: $p(x, y, t) = p(y, x, t)$;

4. boundedness of the entire heat flow

$$\int_M p(x, y, t)\, dy \leq 1\,; \tag{2.2}$$

5. semi-group property

$$\int_M p(x,y,t)\,p(y,z,s)\,dy = p(x,z,t+s)\;; \tag{2.3}$$

6. if $v \in L^p(M), 1 \le p < \infty$ it follows that the following operator

$$T_t v(x) = \int_M p(x,y,t)\,v(y)\,dy \tag{2.4}$$

defines a contraction semi-group in $L^p(M)$ and the function $u(x,t) = T_t v(x)$ is a solution to the Cauchy problem for the heat equation (1.1) with an initial function $v(x)$ (the latter means that

$$\lim_{t\to 0} \|u(x,t) - v(x)\|_{L^p(M)} = 0\,,$$

see [S]).

It is standard now that to obtain pointwise upper bounds of the heat kernel. One proves first a suitable integral estimate and then applies the semi-group property (2.3). In the simplest setting this idea is illustrated by the following proposition.

Proposition 2.1. *The following inequalities are equivalent for any fixed* $t > 0$.

1. $p(x,y,t) \le f(t)$, *for all* $x,y \in M$.
2. $p(x,x,t) \le f(t)$, *for all* $x \in M$.
3. $\int_M p^2(x,y,t)\,dy \le f(2t)$, *for all* $x \in M$.

This proposition is not a new one, nonetheless we shall prove it for convenience of the reader.

$1 \Longrightarrow 2$. Evident.

$2 \Longrightarrow 3$. According to the semi-group property (2.3) we have for $x = z$

$$\int_M p^2(x,y,t)\,dy = p(x,x,2t) \le f(2t)\,.$$

This argument implies, in particular, that the function $p(x,y,t)$ as a function of y lies in $L^2(M)$.

$3 \Longrightarrow 1$. According to the symmetry property of the heat kernel and by the Cauchy-Schwarz inequality we have

$$\begin{aligned} p(x,y,t) &= \int_M p(x,\xi,\frac{t}{2})\,p(\xi,y,\frac{t}{2})\,d\xi \\ &\le \left(\int_M p^2(x,\xi,\frac{t}{2})\,d\xi\right)^{1/2} \left(\int_M p^2(y,\xi,\frac{t}{2})\,d\xi\right)^{1/2} \\ &\le (f(t)\,f(t))^{1/2} = f(t) \end{aligned}$$

Let us note that , in addition, each of conditions 1-3 is equivalent to each of the following hypotheses

4. $\|T_t v\|_\infty \le f(t)\,\|v\|_{L^1(M)}$, for all $v \in L^1(M)$.
5. $\|T_t v\|_\infty \le f(2t)\,\|v\|_{L^2(M)}$, for all $v \in L^2(M)$.

Next we shall assume that a Λ-isoperimetric inequality holds on the manifold under consideration, the function $\Lambda(v)$ being positive, continuous and monotonically decreasing on $(0, +\infty)$. In addition, we suppose that the function Λ satisfies the condition

$$\int_{0+} \frac{dv}{v\,\Lambda(v)} < +\infty. \tag{2.5}$$

This relation holds, for example, when $\Lambda(v) = C\,v^{-\nu}$ for small values of v where $\nu > 0$, as it takes place in the Euclidean space.

Let us denote the set of such functions $\Lambda(v)$ by $\mathcal{L}$. For any $\Lambda \in \mathcal{L}$ we define a function $V(t)$ as follows

$$t = \int_0^{V(t)} \frac{dv}{v\,\Lambda(v)}. \tag{2.6}$$

Equivalently, $V(t)$ is a positive solution to the Cauchy problem

$$V'(t) = V\,\Lambda(V), \qquad V(0) = 0. \tag{2.7}$$

Since Λ is decreasing, then

$$\int_0^{+\infty} \frac{dv}{v\,\Lambda(v)} = +\infty,$$

which implies that $V(t)$ is defined on the entire interval $(0, +\infty)$.

A class of functions $V(t)$ obtained by (2.6) (or (2.7)) will be denoted by $\mathcal{V}$. More explicit discription of this class will be given below. The mapping from $\mathcal{L}$ onto $\mathcal{V}$ given by (2.6) will be referred to as *V-transformation.*

Theorem 2.1. *If the Λ-isoperimetric inequality holds on manifold M with a function $\Lambda \in \mathcal{L}$ then for all $x, y \in M$, and $t > 0$ the following heat kernel estimate is valid*

$$p(x,y,t) \le \frac{2}{\delta\, V((1-\delta)t)}\,, \tag{2.8}$$

where $\delta > 0$ is arbitrary.

Note that this theorem contains the part $1 \Longrightarrow 3$ of Theorem 1.1.

PROOF. Let Ω be an open pre-compact subset of M with a smooth boundary. It suffices to proof that for all $y \in \Omega$, $t > 0$ and $\delta \in (0,1)$,

$$\int_\Omega p_\Omega^2(x,y,t)\,dx \le \frac{2}{\delta\, V((2-2\delta)t)} \tag{2.8$'$}$$

Indeed, the integral in (2.8′) is equal to $p_\Omega(y,y,2t)$. Passing to the limit as $\Omega \to M$ we get from (2.8′) a similar inequality for $p(y,y,2t)$ which implies, in its turn, pointwise estimate (2.8) as it follows from Proposition 2.1.

The proof of (2.8′) relies upon the following lemma.

Lemma 2.1. *For any non-negative function $v(x) \in C_c^\infty(\Omega)$ and for any $\delta > 0$ the following inequality is valid*

$$\int_\Omega |\nabla v|^2 \ge (1-\delta)\, B\, \Lambda\!\left(\frac{2A^2}{\delta B}\right), \tag{2.9}$$

where

$$A = \int_\Omega v\,, \qquad B = \int_\Omega v^2\,.$$

PROOF OF THE LEMMA. The proof follows the arguments of A. K. Gushchin [Gu]. For any positive τ the following inequality is evidently true

$$v^2 \le (v-\tau)_+^2 + 2\,\tau\, v\,. \tag{2.10}$$

Integrating (2.10) over Ω we get

$$\int_\Omega v^2 \le \int_{\{v>\tau\}} (v-\tau)^2 + 2\tau \int_\Omega v \,. \tag{2.11}$$

By the minimax property of the first eigenvalue in the region $\{v > \tau\}$ and according to the Λ-isoperimetric inequality we have

$$\frac{\displaystyle\int_\Omega |\nabla v|^2}{\displaystyle\int_{\{v>\tau\}} (v-\tau)^2} \ge \lambda_1(\{v>\tau\}) \ge \Lambda(\operatorname{Vol}\{v>\tau\}) \,.$$

Since

$$\operatorname{Vol}\{v>\tau\} \le \frac{1}{\tau}\int_\Omega v = \frac{A}{\tau} \,,$$

it follows that

$$\int_{\{v>\tau\}} (v-\tau)^2 \le \frac{\displaystyle\int_\Omega |\nabla v|^2}{\Lambda(\frac{A}{\tau})} \,.$$

Substituting it into (2.11) we get

$$B \le \frac{\displaystyle\int_\Omega |\nabla v|^2}{\Lambda(\frac{A}{\tau})} + 2\tau A$$

and

$$\int_\Omega |\nabla v|^2 \ge (B - 2\tau A)\,\Lambda(\frac{A}{\tau}) \,.$$

Taking here $\tau = \delta B/(2A)$ we obtain (2.9).

To proceed with the proof of Theorem 2.1 let us fix some $y \in M$ and introduce the notations

$$u(x,t) = p_\Omega(x,y,t) \,, \qquad I(t) = \int_\Omega u(x,t)^2\,dx \,.$$

Taking into account that $\int_\Omega u(x,t)dx \le 1$ and applying Lemma 2.1 we obtain

$$\int_\Omega |\nabla u|^2 \ge (1-\delta)\, I(t)\, \Lambda(\frac{2}{\delta\, I(t)}) \,. \tag{2.12}$$

Note that although the function $u(\cdot,t)$ is not in $C_c^\infty(\Omega)$ as required by the Lemma 2.1, this lemma is nonetheless applicable because u vanishes on $\partial\Omega$ and, thereby, can be approximated in $W^{1,2}(\Omega)$ by an element of $C_c^\infty(\Omega)$.

On the other hand differentiating $I(t)$ with respect to t we get

$$I'(t) = 2\int_\Omega u_t\, u = 2\int_\Omega u\,\Delta u = -2\int_\Omega |\nabla u|^2 .$$

Substituting it into (2.12) we obtain a differential inequality

$$I'(t) \le -2\,(1-\delta)\, I(t)\,\Lambda\Big(\frac{2}{\delta\, I(t)}\Big). \tag{2.13}$$

Integrating this inequality we have

$$\int_{I(t_0)}^{I(t)} \frac{dI}{I\,\Lambda(2/(\delta I))} \le -2\,(1-\delta)\int_{t_0}^{t} dt = -2\,(1-\delta)\,(t-t_0),$$

where $t > t_0 > 0$. Changing a variable $v = 2/(\delta\, I)$ we get

$$\int_0^{2/(\delta\, I(t))} \frac{dv}{v\,\Lambda(v)} \ge \int_{2/(\delta\, I(t_0))}^{2/(\delta\, I(t))} \frac{dv}{v\,\Lambda(v)} \ge 2\,(1-\delta)\,(t-t_0). \tag{2.14}$$

Letting here $t_0 \to 0$ and applying the definition of the function $V(t)$ we finally obtain

$$\frac{2}{\delta\, I(t)} \ge V(2\,(1-\delta)\,t), \qquad I(t) \le \frac{2}{\delta\, V(2\,(1-\delta)\,t)},$$

which is equivalent to (2.8′).

Now we are going to show that a Λ-isoperimetric inequality is also a necessary condition for the heat kernel upper bounds of kind $p(x,y,t) \le f(t)$. Let us start with the following observation.

Proposition 2.2. *The class $\mathcal{V}$ contains all positive functions $V(t) \in C^1(0,+\infty)$ such that*

1. $V'(t) > 0$,
2. $V(0) = 0, \quad V(+\infty) = +\infty$,

3. $\dfrac{V'(t)}{V(t)}$ *is monotonically decreasing.*

There are no other functions in $\mathcal{V}$.

Indeed, the fact that any function from $\mathcal{V}$ satisfies 1-3 is a simple consequence of the definition (2.7). For example, the property 3 follows from the monotone decreasing of Λ. Inversely, let $V(t)$ be a function satisfying 1-3, then we can define $\Lambda(v)$ by means of the relation

$$\Lambda(V(t)) = \frac{V'(t)}{V(t)} \tag{2.15}$$

which is nothing but transformed (2.7). Since the function $V(t)$ is a bijection by 1-3 it follows that (2.15) determines a unique function $\Lambda \in \mathcal{L}$.

Thus, the V-transformation has an inverse one define by (2.15) which will be referred to as a Λ-transformation.

Definition 2.1. *A positive function $f(t)$ defined on $(0, +\infty)$ is said to be of a polynomial decay if for some positive constant α the following inequality holds for all $t > 0$, $a \in [1, 2]$*

$$f(at) \geq \alpha f(t)\,. \tag{2.16}$$

Note that any monotone increasing function as well as a decreasing function $f(t) = t^{-N}$ satisfies this definition. It is easy to see that the condition (2.16) holds whenever we have the following differential inequality

$$f'(t) \geq -N\frac{f(t)}{t}\,, \tag{2.16'}$$

where $\alpha = 2^{-N}$.

The next theorem is converse in some sense to Theorem 2.1.

Theorem 2.2. *Let $V(t) \in \mathcal{V}$ and suppose that*

$$\text{the function } \left(\log V(t)\right)' \text{ is of polynomial decay.} \tag{2.17}$$

Suppose that for all $x \in M$ and $t > 0$ we have the estimate

$$p(x,x,t) \le \frac{1}{V(t)} \,. \tag{2.18}$$

Then for any precompact open set $\Omega \subset M$

$$\lambda_k(\Omega) \ge C_\alpha \, \Lambda\Big(\frac{|\Omega|}{k}\Big), \qquad k = 1,2,3,\dots \tag{2.19}$$

where function the Λ is the Λ-transformation of V.

REMARK. The condition (2.17) does not restrict the rate of increase of $V(t)$ as $t \to +\infty$. In any case such standard functions as

$$V(t) = \exp(t^\nu), t^\nu, (\log t)^\nu, (\log\log t)^\nu, \quad \text{etc},$$

where $\nu > 0$, satisfy (2.17). Let us notice also that Theorem 2.2 coincides with the part $3 \Longrightarrow 4$ of the main Theorem 1.1.

Corollary 2.1. *Suppose that under the hypotheses of the Theorem 2.1 the following estimate holds in place of* (2.18)

$$p(x,x,t) \le \frac{a}{V(bt)} \,, \tag{2.18'}$$

then we have instead of (2.19)

$$\lambda_k(\Omega) \ge C_\alpha \, b\Lambda\Big(a\frac{|\Omega|}{k}\Big), \qquad k = 1,2,3,\dots \tag{2.19'}$$

Indeed, this is a simple consequence of the following proposition which, in turn, follows obviously from the definition of the V-transformation.

Proposition 2.3. *Suppose that a function $V(t)$ is the V-transformation of $\Lambda \in \mathcal{L}$, then the function $b\Lambda(a\,v)$ has the V-transformation $a^{-1}V(bt)$, where a, b are arbitrary positive numbers.*

Corollary 2.2. *Let a Λ-isoperimetric inequality hold on the manifold where $\Lambda \in \mathcal{L}$ and its V-transformation $V(t)$ satisfies the condition* (2.17), *then for any pre-compact open set $\Omega \subset M$,*

$$\lambda_k(\Omega) \ge C_{\alpha,\gamma} \, \Lambda\Big(\gamma\frac{|\Omega|}{k}\Big), \qquad k = 1,2,3,\dots \tag{2.20}$$

where $\gamma > 2$ can be chosen arbitrarily.

Indeed, by Theorem 2.1 we have

$$p(x,x,t) \le \frac{2}{\delta\, V((1-\delta)\,t)}\,.$$

Since the Λ-transformation of the function $(\delta/2)\,V((1-\delta)t)$ is the function $(1-\delta)\,\Lambda((2/\delta)\,v)$ (see Proposition 2.3) it follows from Theorem 2.2 that

$$\lambda_k(\Omega) \ge C_\alpha\,(1-\delta)\,\Lambda\big(\frac{2}{\delta}\,\frac{|\Omega|}{k}\big)\,,$$

which implies (2.20) with $\gamma = 2/\delta$.

It would be interesting to find out to what extent the constants γ and $C_{\alpha,\gamma}$ in (2.20) can be optimized.

Finally, concluding the discussion around Theorem 2.2 let us notice that property (2.17) of V can be derived from the following one of Λ

$$\Lambda\, D^2\Lambda \ge N^{-1}(D\Lambda)^2\,, \tag{2.21}$$

where $D \equiv d/d(\log v)$

PROOF OF THEOREM 2.2. An idea behind the proof is close to that of [ChL]. Let $\{\varphi_k(x)\}$ be an orthonormal basis consisting of eigenfunctions of the Dirichlet boundary value problem in $\Omega \subset M$. According to the eigenfunction expansion (2.1) we have for $x = y$

$$p_\Omega(x,x,t) = \sum_{k=1}^{\infty} \exp\big(-\lambda_k(\Omega)t\big)\,\varphi_k(x)^2\,.$$

Integrating over Ω we obtain

$$\int_\Omega p_\Omega(x,x,t)\,dx = \sum_{k=1}^{\infty} \exp\big(-\lambda_k(\Omega)t\big)\,. \tag{2.22}$$

On the other hand as it follows from (2.18)

$$\int_\Omega p_\Omega(x,x,t)\,dx \le \frac{|\Omega|}{V(t)}\,.$$

Combining this with (2.22) and noting that for any $k \geq 1$

$$\sum_{m=1}^{\infty} \exp(-\lambda_m(\Omega)t) > k \exp(-\lambda_k(\Omega)t)$$

we get

$$k \exp(-\lambda_k(\Omega)t) \leq \frac{|\Omega|}{V(t)},$$

i.e.

$$\lambda_k(\Omega) \geq \frac{1}{t} \log \frac{k\, V(t)}{|\Omega|}. \tag{2.23}$$

This inequality is true for all $t > 0$ and we can choose t arbitrarily. Let us denote for the sake of simplicity $v = |\Omega|/k$, find τ from equation $V(\tau) = v$ and take $t = 2\tau$. For this t we get from (2.23)

$$\lambda_k(\Omega) \geq \frac{1}{2\tau}(\log V(2\tau) - \log V(\tau)) = \frac{1}{2} f(\theta),$$

where

$$f(t) \equiv \frac{d}{dt} \log V(t),$$

$\theta \in (\tau, 2\tau)$ being a mean value. Since f has a polynomial decay it follows that

$$f(\theta) \geq \alpha f(\tau).$$

Hence, $\lambda_k(\Omega) \geq \alpha f(\tau)/2$. Since

$$f(\tau) = \frac{V'(\tau)}{V(\tau)} = \Lambda(V(\tau)) = \Lambda(v) = \Lambda(\frac{|\Omega|}{k})$$

it follows that

$$\lambda_k(\Omega) \geq \frac{\alpha}{2} \Lambda(\frac{|\Omega|}{k}),$$

which was to be proved.

In the conclusion of this section we present a sufficient condition for the transience property of the Brownian motion on M which follows from the arguments of the proof of Theorem 2.1. The Brownian motion is transient if for some (and hence for all) pairs of distinct points $x, y \in M$,

$$\int_0^{+\infty} p(x,y,t)\,dt < +\infty\,, \tag{2.24}$$

which means that there exists a positive fundamental solution $E(x,y)$ of the Laplace equation which is just given by the integral $E(x,y) = \int_0^{+\infty} p(x,y,t)\,dt$. Of course, one can take the integral in (2.24) from some positive t_0 rather than from 0 because for $x \neq y$ function $p(x,y,t)$ is bounded on any bounded time interval. Since

$$p(x,y,t) \le \sqrt{p(x,x,t)\,p(y,y,t)}$$

(see the proof of the Proposition 2.1) it follows that the Brownian motion is transient provided for some $t_0 > 0$ and any $y \in M$,

$$\int_{t_0}^{+\infty} p(y,y,t)\,dt < +\infty\,. \tag{2.24'}$$

As soon as we have an upper estimate for the heat kernel we can check whether (2.24′) holds. In fact we need for this purpose an upper bound only for large t. That leads us to the following theorem.

Theorem 2.3 *Suppose that for any open pre-compact region $D \subset M$ with the volume $\operatorname{Vol} D > V_0$ (where V_0 is some positive number) the isoperimetric inequality* (1.8) *holds with an arbitrary continuous decreasing function Λ, then the Brownian motion is transient provided*

$$\int^{+\infty} \frac{dv}{v^2\,\Lambda(v)} < +\infty\,. \tag{2.25}$$

PROOF. Let us extend the function $\Lambda(v)$ into the interval $(0, V_0)$ as a constant: $\Lambda(v) \equiv \Lambda(V_0)$. Since $\lambda_1(D)$ decreases on expansion D we can claim that the Λ-isoperimetric inequality is valid now for all domains D. We cannot apply directly Theorem 2.1 because $\Lambda \notin \mathcal{L}$ but we can apply formula (2.14) obtained in the course of the proof of Theorem 2.1 without using $\Lambda \in \mathcal{L}$. Putting there $\delta = 1/2$ and noting that $I(t) = p_\Omega(y,y,2t)$ we obtain that for any region Ω and for any $y \in \Omega,\ t > t_0$,

$$\int_{4/p_\Omega(y,y,2t_0)}^{4/p_\Omega(y,y,2t)} \frac{dv}{v\,\Lambda(v)} \ge t - t_0\,. \tag{2.14'}$$

Let us introduce the function $v(t)$ from the relation

$$t - t_0 = \int_{v_0}^{v(t)} \frac{dv}{v\,\Lambda(v)}, \tag{2.26}$$

where $v_0 = 4/p_\Omega(y, y, 2t_0)$. As it follows from (2.14′)

$$\frac{4}{p_\Omega(y, y, 2t)} \geq v(t), \qquad p_\Omega(y, y, 2t) \leq \frac{4}{v(t)}.$$

Hence, in order to estimate the integral (2.24′) from above it suffices to obtain a uniform in Ω bound of $\int_{t_0}^{+\infty} dt/v(t)$. Changing a variable in the integral we have

$$\begin{aligned}\int_{t_0}^{+\infty} \frac{1}{v(t)}\,dt &= \int_{v_0}^{+\infty} \frac{1}{v}\frac{dt}{dv}\,dv \\ &= \int_{v_0}^{+\infty} \frac{1}{v^2\,\Lambda(v)}\,dv = \int_{4/p_\Omega(y,y,2t_0)}^{+\infty} \frac{1}{v^2\,\Lambda(v)}\,dv.\end{aligned}$$

Thus, collecting all these relations together, we obtain

$$\int_{t_0}^{+\infty} p_\Omega(y, y, 2t)\,dt \leq 4 \int_{4/p_\Omega(y,y,2t_0)}^{+\infty} \frac{1}{v^2\,\Lambda(v)}\,dv.$$

Letting $\Omega \to M$ we see that the integral (2.24′) is finite, which was to be proved.

Let us compare this result with a similar one obtained earlier in [G85]. Namely, the theorem of [G85] establishes transience provided the following isoperimetric inequality holds for any bounded region $D \subset M$ with a smooth boundary:

$$\text{Area}\,(\partial D) \geq g(\text{Vol}\,D) \tag{2.27}$$

and the function g satisfies the inequality

$$\int^{+\infty} \frac{dv}{g(v)^2} < +\infty. \tag{2.28}$$

Recall first that as was mentioned already in Section 1 inequality (2.27) implies some Λ-isoperimetric inequality . Indeed, the following is true.

Proposition 2.4. *Suppose that for any bounded region D with a smooth boundary such that $\bar{D} \subset \Omega$ the inequality (2.27) holds, where Ω is an arbitrary Riemannian manifold and the function g is such that $g(v)/v$ is a decreasing one. Then the Λ-isoperimetric inequality holds in Ω where Λ is given by the relation*

$$\Lambda(v) = \frac{1}{4}\left(\frac{g(v)}{v}\right)^2 . \tag{2.29}$$

Indeed, let ω be some sub-region in Ω and the closure of D lie inside ω. By the hypothesis (2.27)

$$\operatorname{Area}(\partial D) \geq g(|D|) = |D| \,\frac{g(|D|)}{|D|} \geq |D| \,\frac{g(|\omega|)}{|\omega|} .$$

Thus, the Cheeger's isoperimetric constant (see [Che]),

$$h(\omega) \equiv \inf_{D \subset \omega} \frac{\operatorname{Area}(\partial D)}{|D|}$$

satisfies the following estimate

$$h(\omega) \geq \frac{g(|\omega|)}{|\omega|} .$$

By the Cheeger's theorem we have

$$\lambda_1(\omega) \geq \frac{1}{4}\, h(\omega)^2 ,$$

which implies (2.29).

We are left to notice that (2.25) is transformed to (2.28) when substituting g from (2.29).

3. Mean-value type theorem.

In this Section we deal with a mean-value type theorem. We call so a theorem which establishes a relation between a value which a solution of heat equation takes at some point and an integral of the solution over some neighbourhood of this point. This theorem will enable us to obtain a dependence on distance function when estimating the heat kernel. In fact, the theorem is alredy proved in its the most important part in [G91]. Here we are going to simplify the final result.

Let us introduce the following notations. Let Λ be a function of class $\mathcal{L}$. Since the function $v/\Lambda(v)$ is strictly monotone on $(0,+\infty)$ and has a range $(0,+\infty)$ it follows that it is inversible. Let us denote the inverse function by ω and define the functions $\widetilde{V}(t)$, $\widetilde{W}(r)$ for $t > 0$, $r > 0$ as follows

$$t = \int_0^{\widetilde{V}(t)} \frac{d\xi}{\omega(\xi)}, \qquad r = \int_0^{\widetilde{W}(r)} \frac{d\xi}{\sqrt{\xi\omega(\xi)}}. \tag{3.1}$$

The following theorem was proved in [G91].

Theorem 3.1. *Suppose that the Λ-isoperimetric inequality holds in some ball B_R^z where the function Λ is such that the integrals in (3.1) converge at zero and the functions $\widetilde{V}$, $\widetilde{W}$ are defined on $[0,+\infty[$. Let $C \equiv B_R^z \times (0,T)$, $T > 0$ and suppose that a function $u(x,t) \in C^\infty(\bar{C})$ satisfies in C the inequality*

$$u_t - \Delta u \leq 0. \tag{3.2}$$

Then

$$u(z,T)_+^2 \leq \frac{4}{\min\{\widetilde{V}(cT), \widetilde{W}(cR)\}} \int_C u_+^2, \tag{3.3}$$

where $c > 0$ is an absolute constant, for example, $c = 0.0001$.

EXAMPLES. 1. If $\Lambda = a\, v^{-2/n}$ then

$$\widetilde{V}(t) = C_n\, a^{n/2}\, t^{(n+2)/2}, \qquad \widetilde{W}(r) = C_n\, a^{n/2}\, r^{n+2}$$

and (3.3) acquires the form

$$u(z,T)_+^2 \leq \frac{C_n\, a^{-n/2}}{\min\{\sqrt{T}, R\}^{n+2}} \int_C u_+^2. \tag{3.4}$$

In the Euclidean space $\mathbb{R}^n$ this inequality was proved by Moser [M].

2. If $\Lambda = \max\{av^{-2/n},\ A\}$, $A > 0$, then

$$\widetilde{V}(t) \asymp a^{n/2} \min\{t,\ A^{-1}\}^{(n+2)/2} \exp(c\,A\,t)\,,$$
$$\widetilde{W}(r) \asymp a^{n/2} \min\{r^2,\ A^{-1}\}^{(n+2)/2} \exp(c\,\sqrt{A}\,r)\,,$$

where $\asymp$ means "is in finite ratio with" and the constants bounding the ratio of the right and left-hand sides in these relations depend only on n.

3. Let us set

$$\Lambda(v) = \frac{b}{R^2}\left(\frac{\operatorname{Vol} B_R^z}{v}\right)^\beta$$

with some positive constants b, β. The Λ-isoperimetric inequality with this function Λ holds in any ball B_R^z on a manifold with a non-negative Ricci curvature with the constants $\beta = 2/n$, $b = C_n$ (see [G91] for the proof). This Λ-isoperimetric inequality is in some sense a natural one when obtaining the heat kernel upper bound of the following kind: $p(x,y,t) \le C/\operatorname{Vol} B^x_{\sqrt{t}}$ -see Section 5 below for details. But when staying inside the ball B_R^z the Λ-isoperimetric inequality in question is essentially the same as that of the Example 1 provided we take $a = (b/R^2)(\operatorname{Vol} B_R^z)^\beta$. Substituting into (3.4) and noting that $T \operatorname{Vol} B_R^z = \operatorname{Vol} \mathcal{C}$, we obtain

$$u(z,T)_+^2 \le \frac{C_n}{\min\left\{(T/R^2)^{1/\beta},\ R^2/T\right\}} \frac{1}{\operatorname{Vol}\mathcal{C}} \int_{\mathcal{C}} u_+^2\,. \tag{3.5}$$

If $T = R^2$ then the first fraction in front of the integral in (3.5) does not depend on T and R at all and (3.5) means that the value of the function u at the top point is estimated from above by the L^2-norm of u over the cylinder $\mathcal{C}$. This is why we call the Theorem 3.1 as a mean-value type theorem.

The purpose of this section is to replace the functions $\widetilde{V}, \widetilde{W}$ from the statement of the Theorem 3.1 by other functions which are more convenient for applications. Next we assume that the function Λ under consideration lies in the class $\mathcal{L}$ and, moreover, $\sqrt{\Lambda}$ is also in $\mathcal{L}$, then we can consider the functions $V(t)$ and $W(r)$, being the V-transformations of Λ and $\sqrt{\Lambda}$ respectively. The latter means that $W(r)$ is defined by the following relation

$$r = \int_0^{W(r)} \frac{dv}{v\sqrt{\Lambda(v)}}\,. \tag{3.6}$$

Proposition 3.1. *If* Λ *and* $\sqrt{\Lambda}$ *belong to* $\mathcal{L}$ *it follows that the functions* $\widetilde{V}(t)$ *and* $\widetilde{W}(r)$ *defined from* (3.1) *exist for all* $t > 0$, $r > 0$ *and satisfy the estimates*

$$\widetilde{V}(t) \geq \frac{1}{4}\, t\, V(\frac{1}{4}\, t)\,, \tag{3.7}$$

$$\widetilde{W}(r) \geq \frac{1}{64}\, r^2\, W(\frac{1}{8}\, r)\,. \tag{3.8}$$

REMARK. The inequalities of the opposite direction are valid too

$$\widetilde{V}(t) \leq t\, V(t)\,, \qquad \widetilde{W}(r) \leq \frac{1}{4}\, r^2\, W(r)\,,$$

but we do not use them.

First we prove the following lemma.

Lemma 3.1. *Under the assumptions made above the functions* $1/V(t)$ *and* $1/\sqrt{W(r)}$ *are convex.*

PROOF. Consider first the former of these functions. It suffices to prove that t as a function of the argument $1/V$ is convex. Let us consider the derivative

$$\frac{dt}{d(1/V)} = -\frac{V^2}{V'} = -\frac{V}{\Lambda(V)}\,,$$

where we have applied that $V' = V\,\Lambda(V)$. When $1/V$ is increasing V is decreasing, $\Lambda(V)$ increasing, $V/\Lambda(V)$ decreasing. Hence, $dt/d(1/V)$ is increasing and t as a function of $1/V$ is convex.

Similarly one can prove that r as a function of $2/\sqrt{W}$ is convex because

$$\frac{dr}{d(2/\sqrt{W})} = -\sqrt{\frac{W}{\Lambda(W)}}\,.$$

PROOF OF (3.7). By definition the function ω satisfies the relation

$$\xi = \frac{\omega(\xi)}{\Lambda(\omega(\xi))}\,. \tag{3.9}$$

According to (3.1) we have

$$t=\int_0^{\widetilde{V}(t)}\frac{d\xi}{\omega(\xi)}=\int_0^{\omega(\widetilde{V})}\frac{1}{\omega}\frac{d\xi}{d\omega}d\omega$$
$$=\int_0^{\omega(\widetilde{V})}\frac{1}{\omega}\frac{\Lambda d\omega-\omega d\Lambda}{\Lambda^2}$$
$$=\int_0^{\omega(\widetilde{V})}\frac{1}{\omega}\frac{d\omega}{\Lambda(\omega)}-\int_\infty^{\Lambda(\omega(\widetilde{V}))}\frac{d\Lambda}{\Lambda^2},$$

$$t=\int_0^{\omega(\widetilde{V})}\frac{d\omega}{\omega\,\Lambda(\omega)}+\frac{1}{\Lambda(\omega(\widetilde{V}))}\,. \tag{3.10}$$

This implies, in particular, that the integral in the definition (3.1) of the function $\widetilde{V}$ converges at 0. Moreover, since $\Lambda(\omega)$ is a monotone decreasing function it follows that

$$\int_0^{+\infty}\frac{d\omega}{\omega\,\Lambda(\omega)}=+\infty\,,$$

which means that the function $\widetilde{V}(t)$ is defined for all positive t. Let us consider two cases. Let first the integral on the right-hand side of (3.10) be at least as large as the second summand, then

$$\frac{t}{2}\le\int_0^{\omega(\widetilde{V})}\frac{d\omega}{\omega\Lambda(\omega)}\,,$$

whence $\omega(\widetilde{V}(t))\ge V(t/2)$ follows. Together with (3.9) and (2.7) this implies

$$\widetilde{V}(t)\ge\frac{V(t/2)}{\Lambda(V(t/2))}=\frac{V^2(t/2)}{V'(t/2)}\,. \tag{3.11}$$

We are left to show that for all $\tau>0$

$$\frac{V^2(\tau)}{V'(\tau)}\ge\frac{1}{2}\tau\,V(\frac{1}{2}\tau) \tag{3.12}$$

(it is evident that (3.12) for $\tau=t/2$ and (3.11) imply (3.7)).

In order to prove (3.12) consider the function $v(\tau) \equiv 1/V(\tau)$. Due to Lemma 3.1 this function is a convex one which means, in particular, that

$$v'(\tau) \geq \frac{v(\tau) - v(\tau/2)}{\tau/2} \geq -\frac{v(\tau/2)}{\tau/2},$$

Substituting here $v = 1/V$ we obtain (3.12).

Consider now the second case when the former summand on the right-hand side of (3.12) is less than the latter, then

$$\frac{t}{2} < \frac{1}{\Lambda(\omega(\widetilde{V}))}, \qquad \frac{2}{t} > \Lambda(\omega(\widetilde{V})).$$

Since the range of the function Λ covers the interval $[\Lambda(\omega(\widetilde{V})), +\infty[$ it follows that $2/t$ lands into the range of Λ. Denote by $u(t)$ the smallest number u for which $\Lambda(u) = 2/t$, then $\Lambda(u) > \Lambda(\omega(\widetilde{V}))$ and, hence, $u(t) < \omega(\widetilde{V}(t))$. Using (3.9) we get

$$\widetilde{V}(t) > \frac{u}{\Lambda(u)} = \frac{t\,u(t)}{2}. \tag{3.13}$$

It follows from (2.6) that

$$t = \int_0^{V(t)} \frac{dv}{v\,\Lambda(v)} > \int_{V(t)/2}^{V(t)} \frac{dv}{v\,\Lambda(v)} \geq \frac{V(t)/2}{V(t)\,\Lambda(V(t)/2)} = \frac{1}{2\,\Lambda(V(t)/2)}.$$

Replacing here t by $t/4$ we obtain

$$\Lambda(\frac{1}{2}\,V(\frac{1}{4}\,t)) > \frac{2}{t}.$$

It yields together with $2/t = \Lambda(u(t))$ and (3.13)

$$\frac{1}{2}\,V(\frac{1}{4}\,t) \leq u(t) < \frac{2}{t}\,\widetilde{V}(t),$$

which coincides with (3.7).

The inequality (3.8) is proved in the same way. Let us sketch briefly the main points of the proof. It follows from the definition (3.1) of the function $\widetilde{W}$ that

$$r = \int_0^{\omega(\widetilde{W})} \frac{dv}{v\sqrt{\Lambda(v)}} + \frac{2}{\sqrt{\Lambda(\omega(\widetilde{W}))}}. \tag{3.14}$$

Suppose first that the integral on the right-hand side of (3.14) is greater than or equal to the second summand, then $\omega(\widetilde{W}) \geq W(r/2)$ which implies together with (3.9) and $W' = W\sqrt{\Lambda(W)}$ that

$$\widetilde{W}(r) \geq \frac{W^3(r/2)}{W'^2(r/2)} . \tag{3.15}$$

Next we use the facts that the function $v(r) \equiv 2/\sqrt{W(r)}$ is by the Lemma 3.1 a convex one and for such a function

$$v'(\tau) \geq -\frac{v(\tau/2)}{\tau/2} .$$

Hence, we obtain

$$\frac{W^{3/2}(\tau)}{W'(\tau)} \geq \frac{\tau}{4}\sqrt{W(\frac{\tau}{2})} .$$

Taking here $\tau = r/2$ and substituting into (3.15) we obtain (3.8).

Suppose now the integral in (3.13) is smaller than the second summand, then we have

$$\Lambda(\omega(\widetilde{W})) < \frac{16}{r^2} , \qquad \omega(\widetilde{W}) > u(r) , \qquad \widetilde{W}(r) > \frac{r^2\, u(r)}{16} , \tag{3.16}$$

where the function $u(r)$ is defined from the relation $\Lambda(u(r)) \equiv 16/r^2$.

On the other hand (3.6) yields

$$r > \frac{1}{2\sqrt{\Lambda(W(r)/2)}}$$

or, replacing r by $r/8$

$$\Lambda(\frac{1}{2}\, W(\frac{1}{8}\, r)) > \frac{16}{r^2} ,$$

which implies $W(r/8)/2 \leq u(r)$. Collecting all these inequalities together we obtain finally (3.8).

Now the Theorem 3.1 can be reformulated as follows.

Theorem 3.2. *Suppose that the Λ-isoperimetric inequality holds in some ball $B_R^z \subset M$ with the function $\Lambda(v)$ such that $\sqrt{\Lambda} \in \mathcal{L}$. Let*

$\mathcal{C} \equiv B^z_R \times (0,T)$, $T > 0$ and suppose that a function $u(x,t) \in C^\infty(\bar{\mathcal{C}})$ satisfies in $\mathcal{C}$ the inequality

$$u_t - \Delta u \leq 0 .$$

Then

$$u(z,T)^2_+ \leq \frac{C}{\min\{T\,V(cT),\, R^2\,W(cR)\}} \int_{\mathcal{C}} u^2_+ \,, \tag{3.17}$$

where $c > 0$, C are absolute constants.

4. Integral estimates with Gaussian term.

We start this Section with some auxiliary properties of the Λ-transformation which are going to be used in obtaining upper bounds of the heat kernel containing the factor $\exp(-c\,r^2/t)$. Let us assume that Λ is a function of class $\mathcal{L}$ and, moreover, $\sqrt{\Lambda} \in \mathcal{L}$. Let $V(t)$ and $W(r)$ be V-transformations of Λ and $\sqrt{\Lambda}$ respectively. Let us denote by $\mathcal{R}(t)$ the function $W^{-1} \circ V$, *i.e.*

$$\mathcal{R}(t) \equiv \int_0^{V(t)} \frac{dv}{v\sqrt{\Lambda(v)}} \,. \tag{4.1}$$

Lemma 4.1. *For all $t_2 > t_1 > 0$ the following inequality holds*

$$\frac{(\mathcal{R}(t_2) - \mathcal{R}(t_1))^2}{t_2 - t_1} \leq \log \frac{V(t_2)}{V(t_1)} \,. \tag{4.2}$$

PROOF. Let $V_i = V(t_i)$, then (4.1) implies

$$\begin{aligned}(\mathcal{R}(t_2) - \mathcal{R}(t_1))^2 &= \left(\int_{V_1}^{V_2} \frac{dv}{v\sqrt{\Lambda(v)}}\right)^2 \\ &\leq \int_{V_1}^{V_2} \frac{dv}{v\,\Lambda(v)} \int_{V_1}^{V_2} \frac{dv}{v} \\ &= (t_2 - t_1) \log \frac{V_2}{V_1} \,,\end{aligned}$$

whence (4.2) follows.

Proposition 4.1. *Let $T > 0$ and $\delta \in (0,1)$ be given.*

i) *If the function*

$$\frac{V(t)}{V(\delta t)} \tag{4.3}$$

is bounded above by a constant N for $t \leq T$ (where T may be equal to infinity) then for any $t \leq T$

$$\frac{\mathcal{R}^2(t)}{t} \leq C_{N,\delta}\,. \tag{4.4}$$

ii) *Suppose that the function $V(t)/V(\delta t)$ is monotonically increasing for $t \geq T$, then for any $t > T$ the following inequality holds*

$$\frac{\mathcal{R}^2(t)}{t} \leq C_\delta \log\frac{V(t)}{V(\delta t)} + C_{T,\delta}\,, \tag{4.5}$$

where $C_\delta = (1-\delta)/(1-\sqrt{\delta})^2$.

PROOF OF i). Let us consider a sequence

$$t_k = t\,\delta^k\,, \qquad k = 0,1,2,\ldots \tag{4.6}$$

By the Lemma 4.1 we have

$$\mathcal{R}(t_k) - \mathcal{R}(t_{k+1}) \leq \sqrt{t_k - t_{k+1}}\Big(\log\frac{V(t_k)}{V(t_{k+1})}\Big)^{1/2} \tag{4.7}$$

or, taking into account the hypothesis i)

$$\mathcal{R}(t_k) - \mathcal{R}(t_{k+1}) \leq \delta^{k/2}\sqrt{(1-\delta)\,t\log N}\,.$$

Adding these inequalities over all k we obtain

$$\mathcal{R}(t) \leq C_{N,\delta}\sqrt{t}\,,$$

which coincides with (4.4).

PROOF OF ii). Consider again the sequence (4.6). Denote by K the biggest integer for which $t_K > T$ is still valid. This means in particular, that $t_K \in [T, T/\delta]$. Let us denote for the sake of brevity $V_i = V(t_i)$, $r_i = R(t_i)$ and consider the sequence

$$\frac{r_k^2}{t_k}, \qquad k = 0, 1, 2, \ldots, K. \tag{4.8}$$

Suppose first that this sequence is a monotone increasing one, then we have

$$\frac{\mathcal{R}^2(t)}{t} \leq \frac{r_K^2}{t_K} \leq \sup_{\tau \in [T, T/\delta]} \frac{\mathcal{R}^2(\tau)}{\tau} = C_{T,\delta}, \tag{4.9}$$

which implies, of course, (4.5).

Consider now the second case when there is a term of the sequence (4.8), say, t_m such that

$$\frac{r_m^2}{t_m} > \frac{r_{m+1}^2}{t_{m+1}}. \tag{4.10}$$

We may assume that m is the smallest number for which (4.10) is true. It follows from (4.10) that

$$r_{m+1} < r_m \sqrt{\delta}.$$

Applying the Lemma 4.1 we obtain

$$\frac{(r_m - r_m\sqrt{\delta})^2}{(1-\delta)\, t_m} \leq \log \frac{V(t_m)}{V(\delta t_m)}. \tag{4.11}$$

Since the function $V(t)/V(\delta t)$ is increasing it follows that

$$\log \frac{V(t_m)}{V(\delta t_m)} \leq \log \frac{V(t)}{V(\delta t)}.$$

Due to the choice of m we have

$$\frac{\mathcal{R}^2(t)}{t} \leq \frac{r_m^2}{t_m}.$$

Substituting these relations into (4.11) we obtain finally

$$\frac{\mathcal{R}^2(t)}{t} \leq \frac{1-\delta}{(1-\sqrt{\delta})^2} \log \frac{V(t)}{V(\delta t)},$$

whence (4.5) follows.

The following lemma reduces the question whether the function $V(t)/V(\delta t)$ satisfies the hypotheses of the Proposition 4.1 to a simpler one.

Lemma 4.2. *Suppose that $\delta \in (0,1)$, $\infty \geq T' > T/\delta \geq 0$.*

i) *If the function*

$$\frac{d\log V(t)}{d\log t} \tag{4.12}$$

is bounded by a constant N for $t \in]T,T']$, then the function $V(t)/V(\delta t)$ is bounded by the constant δ^{-N} on $]T/\delta, T']$.

ii) *If the function (4.12) is monotone increasing on $]T,T']$ it follows that the function $V(t)/V(\delta t)$ is monotone increasing on $]T/\delta, T']$.*

PROOF. We have for $t \in]T,T']$

$$\log\frac{V(t)}{V(\delta t)} = \log V(t) - \log V(\delta t) = \int_{\log \delta t}^{\log t} \frac{d\log V(\tau)}{d\log \tau}\, d\log\tau\,. \tag{4.13}$$

Note that the integral is taken over an interval of a constant length $-\log\delta$ lying in $]\log T,\ \log T']$. If the function to be integrated is bounded by N then the left-hand side in (4.13) does not exceed $N\log\delta$. Else if the function is increasing then the left-hand side of (4.13) is increasing too because the interval of integration is moving to right as t is getting larger.

Now we proceed to our main estimates of some integrals of the heat kernel. The next theorem does not still make use of the foregoing properties of the Λ-transformation -they will simplify the further applications of this theorem.

Theorem 4.1. *Let the Λ-isoperimetric inequality hold in some ball B_R^z, $\sqrt{\Lambda}$ being a function of class $\mathcal{L}$. Let $V(t)$ and $W(r)$ be V-transformations of Λ and $\sqrt{\Lambda}$ respectively, then for all $\rho \in (0,R]$ and $0 < \tau \leq t$ the inequality holds*

$$\int_M p^2(x,z,t)\exp\left(\frac{d^2(x)}{2t}\right)dx \leq \frac{C}{\min\left\{V(c\tau),\, (\rho^2/\tau)W(c\rho)\right\}}\,, \tag{4.14}$$

where $d(x) \equiv \operatorname{dist}\{x, B^z_\rho\} \equiv \{\operatorname{dist}\{x,z\} - \rho\}_+$ *and* c, C *are some absolute positive constants* (c *is the same as in the Theorem* 3.2).

REMARK. Of course, one could replace in the statement τ by t and ρ by R -that would simplify the formulating, but, generally speaking, these are not the optimal choice of τ, ρ.

PROOF. The idea behind the proof is similar to that of [G87c]. Let $\varphi(x) \in C^\infty_c(M)$ be an arbitrary but fixed function and $\Omega \subset M$ be an arbitrary pre-compact region with a smooth boundary containing $\operatorname{supp}\varphi$ and B^z_R. Set

$$u_\Omega(x,t) \equiv \int_\Omega p_\Omega(x,y,t)\,\varphi(y)\,dy$$

and apply to this function the Theorem 3.2 in the cylinder $B^z_\rho \times (t, t-\tau)$:

$$u_\Omega(z,t)^2 \le K \int_{t-\tau}^{t} \int_{B^z_\rho} u^2(x,s)\,dx\,ds$$

where

$$K \equiv \frac{C}{\min\left\{\tau\, V(c\,\tau),\ \rho^2 W(c\,\rho)\right\}}\,.$$

Note that the function being integrated here can be multiplied by the following term

$$\exp\left(-\frac{d^2(x)}{2\,(t-s)}\right),$$

which is equal identically to 1 in the ball B^z_ρ and, besides, the domain of integrating can be extended to the entire region Ω. Thus, we obtain

$$u_\Omega(z,t)^2 \le K \int_{t-\tau}^{t} \int_\Omega u^2_\Omega(x,s)\exp\left(-\frac{d^2(x)}{2\,(t-s)}\right) dx\,ds\,. \tag{4.15}$$

Next we need the following lemma.

Lemma 4.3 (Integral maximum principle). *Let* Ω *be some pre-compact region in* M *with a smooth boundary and* $w(x,s)$ *be a solution to the heat equation* $w_s - \Delta w = 0$ *in* $\Omega \times (T_0, T)$ *vanishing on the boundary* $\partial\Omega \times (T_0,T)$. *Suppose that* $\xi(x,s)$ *is a Lipschitz function in* $\Omega \times (T_0,T)$ *such that*

$$\xi_s + \frac{1}{2}\left|\nabla\xi\right|^2 \le 0\,, \tag{4.16}$$

then the following integral

$$I(s) \equiv \int_\Omega w^2(x,s)\, e^{\xi(x,s)}\, dx \tag{4.17}$$

is a decreasing function in $s \in (T_0, T)$*. Moreover, if* $0 \le \lambda \le \lambda_1(\Omega)$ *it follows that the function*

$$I(s)\, \exp(2\,\lambda\, s)$$

decreases in s *as well.*

REMARK. We use the derivatives of the function $\xi(x,s)$ although it may be not differentiable. But this function is assumed to be Lipschitz and, thereby, is locally differentiable in a weak sense. The inequality (4.16) is also understood in the sense of distributions.

The proof of the first part of this Lemma is well-known (see, for example, [ChLY], [PE]) and consists of checking that $I'(s) \le 0$. The expression on the left-hand side of (4.16) appears as a discriminant of some quadratic polynomial which is to be non-negative. For the second part including the exponential decay of $I(s)$, see [G92].

Let us apply this Lemma to the functions u_Ω and

$$\xi(x,s) = -\frac{d^2(x)}{2\,(t-s)}\ .$$

The distance function $d(x)$ is evidently Lipschitz and $|\nabla d| \le 1$ whence the validity of (4.16) follows. We obtain that the integral over Ω in (4.15) does not exeed that for $s = 0$. Therefore

$$u_\Omega(z,t)^2 \le K\,\tau \int_\Omega \varphi^2(x)\, \exp\left(-\frac{d^2(x)}{2\,t}\right) dx\,.$$

Letting here $\Omega \to M$ we deduce that the same estimate is valid for the function

$$u(x,t) = \lim_{\Omega\to M} u_\Omega(x,t) = \int_M p(x,y,t)\, \varphi(y)\, dt$$

which implies

$$u(z,t)^2 \le K\,\tau \int_M \varphi^2(x)\, \exp\left(-\frac{d^2(x)}{2\,t}\right) dx\,. \tag{4.18}$$

Consider now a mapping $\Phi : L^2(M) \longrightarrow \mathbb{R}$ given by the rule

$$\Phi(\eta) \equiv \int_M p(x,z,t) \exp\left(\frac{d^2(x)}{4t}\right) \eta(x)\, dx\,, \tag{4.19}$$

t being fixed. Let us first explain why this mapping is defined, *i.e.* the integral converges. For a function η of the form $\varphi(x)\exp(-d^2(x)/4t)$, where $\varphi \in C_c^\infty(M)$ we have

$$\Phi(\eta) = \int_M p(x,z,t)\, \varphi(x)\, dx\,.$$

This integral here coincides with $u(z,t)$ and by (4.18) we have

$$\Phi(\eta)^2 \le K\tau \int_M \varphi^2(x) \exp\left(-\frac{d^2(x)}{2t}\right) dx = K\tau \int_M \eta^2(x)\, dx\,,$$

whence an estimate follows

$$\Phi(\eta)^2 \le K\tau\, \|\eta\|_2^2\,.$$

Thus, the mapping Φ is bounded on the set of functions η of the form under consideration. Since this set is dense in $L^2(M)$ it follows that

$$\|\Phi\|^2 \le K\tau\,. \tag{4.20}$$

On the other hand the definition (4.19) of Φ implies that

$$\|\Phi\|^2 = \int_M p^2(x,z,t) \exp\left(\frac{d^2(x)}{2t}\right) dx\,.$$

Combining this with (4.20) we get (4.14) which was to be proved.

Corollary 4.1. *Under hypotheses of the Theorem* 4.1 *the following estimate is valid for any* $D > 2$

$$\begin{aligned} E_D(z,t) &\equiv \int_M p^2(x,z,t) \exp\left(\frac{r^2}{Dt}\right) dx \\ &\le \frac{C \exp\left(\frac{\rho^2}{(D-2)t}\right)}{\min\left\{V(c\tau),\, \frac{\rho^2}{\tau} W(c\rho)\right\}}\,, \end{aligned} \tag{4.21}$$

where $r = \operatorname{dist}\{x, z\}$.

PROOF. It is easy to check that for any $D > 2$ a quadratic inequality holds

$$d(x)^2 = (r - \rho)_+^2 \geq \frac{2}{D} r^2 - \frac{2}{D-2} \rho^2 ,$$

which implies together with (4.14) inequality (4.21).

REMARK. For $D = 2$ one cannot hope to estimate E_D because E_2 can be equal to infinity as it happens in the Euclidean space.

The expression $E_D(z, t)$ is of much importance for us. As we have seen it is estimated above via an isoperimetric property of a manifold. On the other hand, it will enable us to obtain pointwise estimates of the heat kernel in the next section. The following property of E_D is also very useful.

Proposition 4.2. *On an arbitrary manifold M for any $D > 2$ the quantity $E_D(x, t)$ is finite for all $x \in M$ and $t > 0$; moreover, the function*

$$E_D(x, t) \exp(2\lambda t) \tag{4.22}$$

is decreasing in t provided $0 \leq \lambda \leq \lambda_1(M)$.

PROOF. For any point $x \in M$ there exists a small positive R such that the ball B_R^x is diffeomorphic to a Euclidean one and the Euclidean metric is finite proportional to the Riemannian one, *i.e.* the ball B_R^x is quasi-isometric to the Euclidean one. We claim that in the ball B_R^x a Λ-isoperimetric inequality holds with a Euclidean function *i.e.* for any region $\Omega \subset B_R^x$

$$\lambda_1(\Omega) \geq C (\operatorname{Vol} \Omega)^{-2/n} .$$

Indeed, the first Dirichlet eigenvalue $\lambda_1(\Omega)$ is defined as the infimum of the ratio of the integrals $\int_\Omega |\nabla\varphi|^2$ and $\int_\Omega \varphi^2$ over all test functions $\varphi \in C_c^\infty(\Omega)$. Either integral is altered under a quasi-isometric transformation at most by a constant factor and, thereby, the same is happening with their ratio. Hence, the first eigenvalue is changed at most by a constant multiple too and the relationship between it and the volume of Ω remains as that in the Euclidean space up to a constant multiple.

The fact of the presence of a local Λ-isoperimetric inequality allows us to apply the Corollary 4.1 which ensures the finiteness of $E_D(x, t)$.

To prove the second assertion let us fix some $x \in M$ and consider a pre-compact region $\Omega \subset M$, $x \in \Omega$ with a smooth boundary, then by the Lemma 4.3 the function

$$\exp(2\lambda t)\int_{\Omega}\exp\left(\frac{r^2}{Dt}\right)p_{\Omega}^2(x,y,t)\,dy \tag{4.23}$$

decreases in t whenever $0 \le \lambda \le \lambda_1(M)$ (note that $\lambda_1(M) \le \lambda_1(\Omega)$). Let $\Omega \to M$ then the function in (4.23) tends to $\exp(2\lambda t)\,E_D(x,t)$ whence the decreasing of the function (4.22) follows.

Our next step is to simplify the right-hand side of (4.21). For this purpose we have to impose some restriction on function the $V(t)$ which, however, will not affect the rate of increase of $V(t)$ as $t \to +\infty$. Namely, suppose that for some $T \in]0,+\infty]$ and $N > 0$,

$$\begin{cases} \dfrac{d\log V(t)}{d\log t} & \text{is increasing for } t > T\,, \\ \dfrac{d\log V(t)}{d\log t} \le N\,, & \text{for } t \le 2T\,. \end{cases} \tag{4.24}$$

This condition needs some comments. For all reasonable applications of the Theorem 4.1 we have for small values of t that $V(t) = C\,t^{\nu}$, $\nu > 0$. Therefore for small t the function $d\log V(t)/d\log t$ is bounded above. If this function remains bounded at infinity then (4.24) is satisfied for $T = +\infty$. Otherwise the function in question is unbounded and we may assume it to be increasing at a neighbourhood of infinity (this restriction causes no troubles for applications). If this is the case then condition (4.24) is satisfied for some finite value of T. Let us note that the first case (*i.e.* $T = +\infty$) takes place for a polynomial function $V(t)$ whereas the second one (when T is finite) holds for a function $V(t)$ of superpolynomial growth.

Theorem 4.2. *Suppose that as in the Theorem* 4.1 *the* Λ*-isoperimetric inequality holds in some ball* B_R^z *and that* $\sqrt{\Lambda} \in \mathcal{L}$. *Let* $V(t)$ *and* $W(r)$ *be as above* V*-transformations of* Λ *and* $\sqrt{\Lambda}$ *respectively and, in addition, suppose* $V(t)$ *satisfies the condition* (4.24). *Suppose also that for some* $\tau > 0$

$$\tau \le t\,, \qquad \tau \le R^2\,, \qquad \mathcal{R}(c\tau) \le cR\,, \tag{4.25}$$

where $\mathcal{R}(\cdot)$ *is defined by* (4.1), *then for any* $D > 2$

$$E_D(z,t) \le \frac{C_{N,T}}{\delta\, V(\widetilde{c}\,\delta\,\tau)}\,, \tag{4.26}$$

where $\widetilde{c} = c^2/12$, $\delta = \min\{D-2, 6/c\}$ *and* c *is the same as in the Theorem* 4.1.

PROOF. Let us first observe that for $D > 6/c$ the right-hand side of (4.26) does not depend on D while the left-hand side (*i.e.* the quantity E_D) is decreasing with respect to D. Hence, the estimate (4.26) for $D > 6/c$ will follow from that of $D = 6/c$. This is why we assume from now on that $D \le 6/c$ and, thereby, $\delta = D - 2$. Similarly, due to the monotone decreasing of $E_D(z,t)$ in t it suffices to consider the case $t = \tau$. Let us apply the Corollary 4.1 for $\tau = t$ and for

$$\rho = \max\{\sqrt{\varepsilon\, t},\ c^{-1}\mathcal{R}(\varepsilon\, c\, t)\}\,, \tag{4.27}$$

where $\varepsilon \le 1$ is to be chosen later, then according to (4.27) and (4.25) $\rho \le R$ and by (4.21) we have

$$E_D(z,t) \le \frac{C\,\exp\left(\dfrac{\rho^2}{\delta\, t}\right)}{\min\{V(c\,t),\ \dfrac{\rho^2}{t}\,W(c\,\rho)\}}\,. \tag{4.28}$$

Since $\rho^2/t \ge \varepsilon$ and $W(c\,\rho) \ge W(\mathcal{R}(\varepsilon\, c\, t) = V(\varepsilon\, c\, t)$ it follows that the denominator in (4.28) is at least as large as $\varepsilon\, V(\varepsilon\, c\, t)$. To estimate the numerator first note that due to (4.24) and the Lemma 4.2 (applied for $\delta = 1/2$ -this is not the δ from the Theorem 4.2!) the function $V(t)/V(t/2)$ is bounded above by C_N in $]0,2T]$ and increasing in $[2T, +\infty[$. Therefore, by Proposition 4.1 we have for all $t > 0$

$$\frac{\mathcal{R}^2(t)}{t} \le 6\,\log\frac{V(t)}{V(t/2)} + C\,,$$

where $C = C(N,T)$. Replacing here t by $\varepsilon\, c\, t$ we obtain

$$\frac{\mathcal{R}^2(\varepsilon\, c\, t)}{\varepsilon\, c\, t} \le 6\,\log\frac{V(\varepsilon\, c\, t)}{V(\varepsilon\, c\, t/2)} + C$$

or

$$\frac{(c^{-1}\mathcal{R}(\varepsilon\, c\, t))^2}{\delta\, t} \leq \frac{6\,\varepsilon}{\delta\, c} \log \frac{V(\varepsilon\, c\, t)}{V(\varepsilon\, c\, t/2)} + \frac{\varepsilon}{\delta\, c}\, C\,.$$

Let us take

$$\varepsilon = \frac{c\,\delta}{6}$$

and note that the hypothesis $D \leq 2 + 6/c$ implies $\delta \leq 6/c$ and $\varepsilon \leq 1$. We obtain thereby

$$\frac{(c^{-1}\mathcal{R}(\varepsilon\, c\, t))^2}{\delta\, t} \leq \log \frac{V(\varepsilon\, c\, t)}{V(\varepsilon\, c\, t/2)} + C\,.$$

Since

$$\frac{(\sqrt{\varepsilon\, t})^2}{\delta\, t} = \frac{\varepsilon}{\delta} \leq \frac{c}{6}$$

it follows from (4.27) that

$$\frac{\rho^2}{\delta\, t} \leq \log \frac{V(\varepsilon\, c\, t)}{V(\varepsilon\, c\, t)} + C + \frac{c}{6}\,,$$

whence we obtain

$$\exp\left(\frac{\rho^2}{\delta\, t}\right) \leq C_{N,T}\, \frac{V(\varepsilon\, c\, t)}{V(\varepsilon\, c\, t/2)}\,.$$

This is a desired estimation of the numerator in (4.28). Substituting it into (4.28) together with the obtained above estimate of the denominator we get finally

$$E_D(z,t) \leq \frac{C_{N,T}}{\varepsilon V(\varepsilon\, c\, t)}\, \frac{V(\varepsilon\, c\, t)}{V(\varepsilon\, c\, t/2)} = \frac{C_{N,T}}{\delta\, V(\tilde{c}\,\delta\, t)}\,,$$

as it was to be proved.

Corollary 4.2. *Suppose that in addition to the hypotheses of Theorem 4.2 the function $\Lambda(v)$ has a polynomial decay (see Definition 2.1), in particular*

$$\Lambda(2\, v) \geq \alpha\, \Lambda(v)\,, \tag{4.29}$$

for some $\alpha > 0$ and in place of the condition (4.25) we have

$$\tau \leq t\,, \qquad \mathcal{R}(c\, \tau) \leq c_1\, R\,, \tag{4.30}$$

where $c_1 = c_1(\alpha)$, *then the estimate* (4.26) *continues to hold.*

REMARK. If the function $\Lambda(v)$ is obtained from the relation (2.29) of the Proposition 2.4 then it satisfies (4.29) automatically with $\alpha = 1/4$ because the function $g(v)$ in (2.29) is increasing. That means that the hypothesis (4.29) holds in all reasonable cases whenever a Λ-isoperimetric inequality takes place.

PROOF OF THE COROLLARY. As in the proof of the Theorem 4.2 we can take $\tau = t$. The constant c_1 is to be chosen in the course of the proof. First impose the restriction $c_1 \le c$. Then (4.30) implies $\mathcal{R}(ct) \le ct$ so we have the second half of condition (4.25). To obtain its first half it suffices to prove that for all $t > 0$

$$\mathcal{R}(t) \ge \beta\sqrt{t}, \tag{4.31}$$

where $\beta = \beta(\alpha) > 0$. Indeed, as soon as we have proved (4.31) we obtain from (4.30)

$$t \le \left(\frac{c_1}{\beta}\right)^2 R^2,$$

whence the first of inequalities (4.25) follows provided $c_1 \le \beta$. Thus, we can take $c_1 = \min\{c, \beta\}$ and apply Theorem 4.2.

To prove (4.31) let us use definition (4.1) of $\mathcal{R}(t)$ and definition (2.6) of $V(t)$. The inequality (4.31) is transformed to the form

$$\left(\int_0^V \frac{dv}{v\sqrt{\Lambda(v)}}\right)^2 \ge \beta^2 \int_0^V \frac{dv}{v\,\Lambda(v)}. \tag{4.32}$$

Consider a sequence of points $V_i = V/2^i$, $i = 0, 1, 2, \ldots$, then

$$\begin{aligned}
\int_0^V \frac{dv}{v\sqrt{\Lambda(v)}} &= \sum_{i=0}^{\infty} \int_{V_{i+1}}^{V_i} \frac{dv}{v\sqrt{\Lambda(v)}} \\
&\ge \sum_{i=0}^{\infty} \int_{V_{i+1}}^{V_i} \frac{dv}{v\sqrt{\Lambda(V_{i+1})}} \\
&\ge \sum_{i=0}^{\infty} \log\frac{V_i}{V_{i+1}} \frac{1}{\sqrt{\Lambda(V_{i+1})}} \\
&\ge \sum_{i=0}^{\infty} \log 2\sqrt{\frac{\alpha}{\Lambda(V_i)}},
\end{aligned}$$

where we have used $\Lambda(V_{i+1}) \leq \Lambda(V_i)/\alpha$. Applying a simple inequality

$$\left(\sum X_i\right)^2 \geq \sum X_i^2$$

we obtain

$$\left(\int_0^V \frac{dv}{v\sqrt{\Lambda(v)}}\right)^2 \geq \sum_{i=0}^{\infty} \frac{\alpha \log^2 2}{\Lambda(V_i)} .$$

On the other hand

$$\begin{aligned}\int_0^V \frac{dv}{v\,\Lambda(v)} &= \sum_{i=0}^{\infty} \int_{V_{i+1}}^{V_i} \frac{dv}{v\,\Lambda(v)} \\ &\leq \sum_{i=0}^{\infty} \int_{V_{i+1}}^{V_i} \frac{dv}{v\,\Lambda(V_i)} = \sum_{i=0}^{\infty} \frac{\log 2}{\Lambda(V_i)} .\end{aligned}$$

Finally we get

$$\left(\int_0^V \frac{dv}{v\sqrt{\Lambda(v)}}\right)^2 \geq \alpha \log 2 \int_0^V \frac{dv}{v\,\Lambda(v)}$$

and $\beta = \sqrt{\alpha \log 2}$.

5. Pointwise estimates with the Gaussian term.

The following statement plays the main role in obtaining pointwise estimates of the heat kernel from the integral estimates. As in the previous Section we shall use the notation

$$E_D(x,t) = \int_M \exp\left(\frac{r^2}{Dt}\right) p^2(x,y,t)\,dy\,, \tag{5.1}$$

where $r = \mathrm{dist}\{x,y\}$.

Proposition 5.1. *On an arbitrary manifold M the following inequality holds for all $x, y \in M$, $t > 0$ and $D > 2$*

$$p(x,y,t) \leq \exp\left(-\frac{r^2}{2Dt}\right) \sqrt{E_D(x,\frac{t}{2})\,E_D(y,\frac{t}{2})}\,, \tag{5.2}$$

where $r = \operatorname{dist}\{x, y\}$, *and, besides*

$$(5.3)\quad p(x,y,t) \le \exp\left(-\frac{r^2}{2Dt} - \lambda t\right) \exp(\lambda t_0) \sqrt{E_D(x, \frac{t_0}{2})\, E_D(y, \frac{t_0}{2})}$$

provided $t \ge t_0 > 0,\ 0 \le \lambda \le \lambda_1(M)$.

PROOF. Let us denote by r_1, r_2 the distances from an arbitrary point $z \in M$ to x, y. Applying the semi-group property of the heat kernel and an elementary inequality $r_1^2 + r_2^2 \ge r^2/2$ we obtain

$$\begin{aligned} p(x,y,2t) &= \int_M p(x,z,t)\, p(z,y,t)\, dz \\ &\le \exp\left(-\frac{r^2}{4Dt}\right) \int_M p(x,z,t) \exp\left(\frac{r_1^2}{2Dt}\right) \\ &\qquad\qquad \cdot p(z,y,t) \exp\left(\frac{r_2^2}{2Dt}\right) dz \\ &\le \exp\left(-\frac{r^2}{4Dt}\right) \left(\int_M p^2(x,z,t) \exp\frac{r_1^2}{Dt}\, dz\right)^{1/2} \\ &\quad \cdot \left(\int_M p^2(y,z,t) \exp\frac{r_2^2}{Dt}\, dz\right)^{1/2}, \end{aligned}$$

whence (5.2) follows.

In order to prove (5.3) note that as it follows from Proposition 4.2

$$\exp(2\lambda t_0)\, E_D(x, \frac{t_0}{2})\, E_D(y, \frac{t_0}{2}) \ge \exp(2\lambda t)\, E_D(x, \frac{t}{2})\, E_D(y, \frac{t}{2}),$$

whence

$$\exp(-\lambda(t - t_0)) \sqrt{E_D(x, \frac{t_0}{2})\, E_D(y, \frac{t_0}{2})} \ge \sqrt{E_D(x, \frac{t}{2})\, E_D(y, \frac{t}{2})},$$

which together with (5.2) imply (5.3).

Theorem 5.1. *Suppose that the* Λ*-isoperimetric inequality holds on the manifold with a function* $\Lambda \in \mathcal{L}$. *Assume also that its* V*-transformation* $V(t)$ *satisfies the condition* (4.26) *with some constants* T, N, *then for all* $x, y \in M$, $t > 0$ *and any* $D > 2$

$$(5.4)\qquad p(x,y,t) \le \frac{C_{D,N,T}}{V(\hat{c}t)} \exp\left(-\frac{r^2}{2Dt}\right),$$

where $\hat{c} = \hat{c}(D) = \min\{D-2,\ 6/c\}c^2/24$ *and* c *is the constant from the Theorem* 3.2.

Indeed, by the Proposition 5.1 we have the inequality (5.2) whence we obtain (5.4) by estimating the quantities E_D applying the Theorem 4.2 for $R = +\infty$ and $\tau = t$.

EXAMPLES. In all the following examples the function $\Lambda(v)$ is equal to $C\,v^{-2/n}$ for small values v, say, for $v < v_0$, so that $V(t) = C_n\,t^{n/2}$ for $t < t_0$ and by the Theorem 5.1 for these values of t we have the estimate

$$p(x,y,t) \le \frac{C_D}{t^{n/2}} \exp\left(-\frac{r^2}{2\,D\,t}\right). \tag{5.5}$$

Consider now different possibilities of behaviour of the function $\Lambda(v)$ for large values of the argument v. Note that the following identity is useful for checking the condition (4.26)which is satisfied in all the following examples:

$$\frac{d\log V(t)}{d\log t} = t\,\frac{V'(t)}{V(t)} = t\,\Lambda(V(t)). \tag{5.6}$$

1. Let for $v > v_0$

$$\Lambda(v) = C\,v^{-\nu}, \qquad \nu > 0.$$

Then for $t > t_0$

$$V(t) \ge C_n\,t^{1/\nu}$$

and by Theorem 5.1

$$p(x,y,t) \le \frac{C_D}{t^{1/\nu}} \exp\left(-\frac{r^2}{2\,D\,t}\right). \tag{5.7}$$

2. Suppose that for large v

$$\Lambda(v) = C\,(\log v)^{-\nu}.$$

then for large values of t we get

$$V(t) \ge C_\nu \exp\left(C_\nu\,t^{1/(\nu+1)}\right)$$

and the corresponding estimate

$$p(x,y,t) \le C_\nu^{-1} \exp\left(- C_\nu\, t^{1/(\nu+1)} - \frac{r^2}{2Dt}\right) . \tag{5.8}$$

3. Let $\Lambda(v) \equiv \lambda$, λ a positive constant, for $v > v_0$, that is the manifold has a positive spectral radius. Theorem 5.1 yields

$$p(x,y,t) \le C^{-1} \exp\left(- C\,\lambda\, t - \frac{r^2}{2Dt}\right) . \tag{5.9}$$

In the situation with a positive spectral radius one can obtain a sharper information about the rate of decay of the heat kernel as $t \to +\infty$. Namely, C in front of λt is a superfluous term. The following theorem treats this case.

Theorem 5.2. *Suppose that in any ball B_R^x of a fixed radius $R > 0$ the Λ-isoperimetric inequality holds with the function $\Lambda = \Lambda_{x,R}$ defined as follows*

$$\Lambda_{x,R}(v) = a(x,R)\, v^{-\nu} , \tag{5.10}$$

where $a(x,R) > 0$, $\nu > 0$. Then for all $x, y \in M$, $t > t_0 > 0$,

$$p(x,y,t) \le \frac{C_\nu \left(1 + \frac{r^2}{t}\right)^{1+1/\nu} \exp\left(-\frac{r^2}{4t} - \lambda\, t\right) \exp(\lambda\, t_0)}{\min\left\{t_0, R^2\right\}^{1/\nu} \left(a\,(x,R)\; a\,(y,R)\right)^{1/2\nu}} , \tag{5.11}$$

where $\lambda = \lambda_1(M)$, $r = \operatorname{dist}\{x,y\}$.

REMARK. Note that the isoperimetric inequality (5.10) takes place on any complete manifold for $\nu = 2/n$ which follows from compactness arguments. Therefore, the estimate (5.11) holds also on any manifold and gives the precise speed of decay of the heat kernel as $t \to +\infty$. If, in addition, one knows that all the coefficients $a(x,R)$ are uniformly bounded away from 0, then the estimate (5.11) takes the following form

$$p(x,y,t) \le C \left(1 + \frac{r^2}{t}\right)^{1+1/\nu} \exp\left(-\frac{r^2}{4t} - \lambda\, t\right) \tag{5.12}$$

provided $t > t_0$.

A similar inequality (without the term $-\lambda t$) is obtained in [D90] under the hypothesis of "weak bounded geometry" which is stronger than our uniform local Λ-isoperimetric inequality.

PROOF OF THEOREM 5.2. Computing the V-transformations of $\Lambda_{x,R}$ and $\sqrt{\Lambda_{x,R}}$ one obtains

$$V(t) = (\nu\, a\, t)^{1/\nu}, \qquad W(r) = (\frac{1}{4}\,\nu^2 a\, r^2)^{1/\nu}, \tag{5.13}$$

whence it follows that

$$\mathcal{R}(t) = 2\sqrt{t/\nu}\,.$$

Let us note also that the function $t\Lambda(V(t)) \equiv 1/\nu$ is bounded. Applying the Theorem 4.2 for $\tau = \min\{t,\, C_\nu R^2\}$ and $D \in (2, 2 + 6/c)$ we get

$$E_D(x,t) \le \frac{C_\nu}{\delta^{1+1/\nu}\, a(x,R)^{1/\nu}\, \min\{t, R^2\}^{1/\nu}}\,,$$

which together with (5.3) implies

$$p(x,y,t) \le \frac{C_\nu}{\delta^{1+1/\nu}} \exp\left(-\frac{r^2}{2\,(2+\delta)\,t} - \lambda\, t\right) A\,, \tag{5.14}$$

where

$$A = \frac{\exp(\lambda\, t_0)}{(a(x,R)\, a(y,R))^{1/2\nu} \min\{t_0, R^2\}^{1/\nu}}$$

and $\delta = D - 2 < 6/c$. Taking here $\delta = \min\{6/c,\, t/r^2\}$ and noting that for this δ

$$\frac{r^2}{2t} - \frac{r^2}{(2+\delta)\,t} = \frac{\delta\, r^2}{2\,(2+\delta)\,t} \le \frac{1}{4}\,,$$

we obtain finally (5.11).

Theorem 5.2 can give a non-trivial information also in the case $\lambda_1(M) = 0$. Next we suppose that the Λ-isoperimetric inequality holds in any ball $B_R^x \in M$ with the following function $\Lambda = \Lambda_{x,R}$

$$\Lambda_{x,R}(v) = \frac{b}{R^2}\,(\mathrm{Vol}\, B_R^x)^{2/n}\, v^{-2/n}\,, \tag{5.15}$$

where b is a positive constant. Here n is the dimension of the manifold M but formally this is not necessarily: n may be any positive number.

For example, this is true with $n = \dim M$ for a manifold M of a non-negative Ricci curvature (see [G91] for the proof).

Proposition 5.2. *Under the hypothesis above the heat kernel admits the following estimate for all $x, y \in M$ and $t > 0$*

$$p(x,y,t) \le C_{n,b} \left(1 + \frac{r^2}{t}\right)^{3(n+1)/4} \frac{\exp\left(-\frac{r^2}{4t}\right)}{\operatorname{Vol} B^x_{\sqrt{t}}}. \tag{5.16}$$

Besides, for any two balls B^x_R and $B^y_\rho \subset B^x_R$ we have

$$\frac{\operatorname{Vol} B^x_R}{\operatorname{Vol} B^y_\rho} \le C_{n,b} \left(\frac{R}{\rho}\right)^n. \tag{5.17}$$

Inversely, suppose that a manifold M is known to have the heat kernel satisfying the inequality

$$p(x,x,t) \le \frac{C}{\operatorname{Vol} B^x_{\sqrt{t}}}, \tag{5.18}$$

for all $x \in M$ and $t > 0$ and assume in addition that for any couple of concentric balls B^x_ρ, B^x_R where $\rho \le R$ we have

$$\frac{\operatorname{Vol} B^x_R}{\operatorname{Vol} B^x_\rho} \le C \left(\frac{R}{\rho}\right)^n. \tag{5.19}$$

Then in any ball the Λ-isoperimetric inequality holds with the function $\Lambda = \Lambda_{x,R}$ defined by (5.15).

REMARK. The statement of the theorem means that the Λ-isoperimetric inequality in any ball given by (5.15) is simply equivalent to the conjuncture of the heat kernel estimate (5.16) and the volume ratio estimate (5.17). On a manifold with a non-negative Ricci curvature a similar estimate of the heat kernel was proved first by Li and Yau [LY]. In the view of the sharp results of B. Davies (see [D88], [DP]) the order $3(n+1)/4$ of the polynomial correction term in (5.16) seems not to be optimal. The advantage of our approach is that the result is stable under a quasi-isometric transformation of the Riemannian metric. Indeed, it is easy to see that the Λ-isoperimetric inequality in question is a quasi-isometric invariant. It can be considered as a replacement of the notion of a non-negative Ricci curvature when dealing with a manifold whose metric is not smooth enough to have a curvature at all.

PROOF. Let us prove first (5.17) applying the result of Carron [C] that a Λ-isoperimetric inequality implies some lower bounds for the volume of a geodesic ball. Indeed, the Proposition 2.4 from [C] states the following:

If the Λ-isoperimetric inequality holds in a region $\Omega \subset M$ with the function $\Lambda(v) = a\, v^{-2/n}$, then any ball $B^y_\rho \subset \Omega$ admits the volume estimate

$$\operatorname{Vol} B^y_\rho \geq C_n\, a^{n/2} \rho^n .$$

Applying this result to the set $\Omega = B^x_R$ and taking

$$a = \frac{b}{R^2} (\operatorname{Vol} B^x_R)^{2/n}$$

we get (5.17).

To prove the heat kernel estimate (5.16) let us apply Theorem 5.2 with the function

$$a(x, R) = \frac{b}{R^2} (\operatorname{Vol} B^x_R)^{2/n}$$

and with $t_0 = t,\ R = \sqrt{t}$. We obtain

$$p(x, y, t) \leq \frac{C_n \left(1 + r^2/t\right)^{1+n/2} \exp\left(-\frac{r^2}{4t}\right)}{\left(\operatorname{Vol} B^x_{\sqrt{t}} \operatorname{Vol} B^y_{\sqrt{t}}\right)^{1/2}} . \tag{5.20}$$

The expression on the right-hand side of (5.20) is going to be simplified to get rid of $\operatorname{Vol} B^y_{\sqrt{t}}$. It is known how to do that on a non-negatively curved manifold (see [LY]): one should estimate the ratio of the volumes $\operatorname{Vol} B^x_{\sqrt{t}}$ and $\operatorname{Vol} B^y_{\sqrt{t}}$ via the distance r between points x, y using the volume comparison theorem for such a manifold. Here we apply the inequality (5.17) in place of the comparison theorem. Namely, we have

$$\begin{aligned} \operatorname{Vol} B^x_{\sqrt{t}} &\leq \operatorname{Vol} B^y_{r+\sqrt{t}} \\ &\leq C_{n,b} \left(\frac{r + \sqrt{t}}{\sqrt{t}}\right)^n \operatorname{Vol} B^y_{\sqrt{t}} \\ &\leq C_{n,b} \left(1 + \frac{r^2}{t}\right)^{n/2} \operatorname{Vol} B^y_{\sqrt{t}} . \end{aligned} \tag{5.21}$$

Replacing the volume $\operatorname{Vol} B^y_{\sqrt{t}}$ in (5.20) by its lower bound from (5.21) we get (5.16).

Let us turn to the proof of the converse statement. The idea is the same as in Theorem 2.2 but first we observe that the hypothesis (5.19) implies that for any two intersecting balls B^x_R and B^y_ρ such that $\rho \le R$ the following holds

$$C^{-1}\left(\frac{R}{\rho}\right)^{n_1} \le \frac{\operatorname{Vol} B^x_R}{\operatorname{Vol} B^y_\rho} \le C\left(\frac{R}{\rho}\right)^{n}, \tag{5.22}$$

where $n_1 > 0$ depends on n and on the constant in (5.19).

This is proved in [G91], see Theorem 1.1 there. We only mention that the right inequality in (5.22) follows evidently from (5.19) whereas the left one exploits (5.19) as well as the non-compactness of the manifold under consideration.

Hence, we can claim that for any two intersecting ball B^x_R and B^y_ρ the following relation holds without any restriction on the radii R, ρ

$$\frac{\operatorname{Vol} B^x_R}{\operatorname{Vol} B^y_\rho} \le C f\left(\frac{R}{\rho}\right), \tag{5.23}$$

where

$$f(\xi) = \begin{cases} \xi^{n_1}, & \text{if } \xi < 1, \\ \xi^{n}, & \text{if } \xi \ge 1. \end{cases} \tag{5.24}$$

Indeed, if $R \ge \rho$ then right inequality (5.22) is applicable. Otherwise we apply the left inequality (5.22) exchanging R and ρ.

To prove the Λ-isoperimetric inequality in a given ball B^x_R with the function (5.15) let us apply the eigenfunction expansion as was done in the course of the proof of Theorem 2.2 and obtain for any region Ω lying in a ball B^x_R and for any value of time $t > 0$ the estimate

$$\exp(-\lambda_1(\Omega)t) \le \int_\Omega p(y,y,t)\,dy \le C\int_\Omega \frac{dy}{\operatorname{Vol} B^y_{\sqrt{t}}}\,.$$

We have according to (5.23) that

$$\frac{\operatorname{Vol} B^x_R}{\operatorname{Vol} B^y_{\sqrt{t}}} \le C\, f\left(\frac{R}{\sqrt{t}}\right),$$

which implies

$$\exp(-\lambda_1(\Omega)t) \leq C\, f(\frac{R}{\sqrt{t}})\, \frac{\operatorname{Vol}\Omega}{\operatorname{Vol} B^x_R}\,.$$

Now we choose the time t in order to get at most e^{-1} on the right-hand side above

$$C\, e\, f(\frac{R}{\sqrt{t}}) = \frac{\operatorname{Vol} B^x_R}{\operatorname{Vol}\Omega}\,, \tag{5.25}$$

whence it follows that

$$\lambda_1(\Omega) \geq \frac{1}{t}\,. \tag{5.26}$$

We can evaluate t using (5.25) and the following property of the function f which reflects its polynomial nature and follows obviously from the definition (5.24): for any $\gamma > 0$ there exists a positive constant c_γ such that for all $\xi > 0$

$$\gamma\, f(\xi) \leq f(c_\gamma\, \xi)\,.$$

Therefore, taking here $\gamma = C\, e$ we have

$$\gamma\, f(\frac{R}{\sqrt{t}}) \leq f(c_\gamma\, \frac{R}{\sqrt{t}})\,,$$

which together with (5.25) yields

$$f(c_\gamma\, \frac{R}{\sqrt{t}}) \geq \frac{\operatorname{Vol} B^x_R}{\operatorname{Vol}\Omega}\,.$$

Let us note that the ratio on the right-hand side here is at least as much as 1 whence it follows that

$$c_\gamma\, \frac{R}{\sqrt{t}} \geq \left(\frac{\operatorname{Vol} B^x_R}{\operatorname{Vol}\Omega}\right)^{1/n}\,.$$

Evaluating t from this inequality and substituting into (5.26) we get finally the desired Λ-isoperimetric inequality .

A similar heat kernel estimate can be obtained on a manifold of a Ricci curvature bounded from below by some constant $-K$. For such a manifold it was proved in [V89] that there is some small positive constant $\rho = \rho(K)$ such that in any ball B^x_ρ of radius ρ the Λ-isoperimetric inequality is satisfied with the function

$$\Lambda_x(v) = C_K \left(\operatorname{Vol} B^x_\rho\right)^{2/n} v^{-2/n}.$$

Applying Theorem 5.2 for $R = \rho,\ t_0 = \rho^2$ we obtain

$$p(x,y,t) \le C_{K,n,\lambda} \frac{\left(1 + \frac{r^2}{t}\right)^{1+n/2} \exp\left(-\frac{r^2}{4t} - \lambda t\right)}{\left(\operatorname{Vol} B^x_\rho \operatorname{Vol} B^y_\rho\right)^{1/2}}, \tag{5.27}$$

provided $t > \rho^2,\quad \lambda = \lambda_1(M)$.

In all the foregoing examples the heat kernel bounds contain two main multiples: a multiple which is responsible for behaviour of heat kernel in time and the term $\exp\left(-cr^2/t\right)$ which, in fact, controls a decay of the heat kernel when $r \to +\infty$ although the Gaussian factor appears on a non-geometric ground. The rest of this Section is devoted to a situation when one more factor emerges depending on r. This term is expected on a quickly expanding manifold for the following reason. As it was mentioned in Section 2, we have always

$$\int_M p(x,y,t)\,dy \le 1. \tag{5.28}$$

If the volume of a ball of the radius R is growing up faster than $\exp(CR^2)$ as $R \to +\infty$ then the heat kernel $p(x,y,t)$ has to decrease, generally speaking, faster than $\exp(-Cr^2)$ so that the integral (5.28) is balanced. This is why one more factor is expected with a quick decay as $r \to +\infty$.

Let us fix some point $z \in M$ and consider the function

$$\lambda(R) \equiv \lambda_1(M \setminus \overline{B^z_R}), \tag{5.29}$$

where the exterior of the ball is regarded as a submanifold and $R \in [0, +\infty)$. Obviously, $\lambda(R)$ is an increasing function of R and $\lambda(0)$ is nothing but the spectral radius $\lambda_1(M)$.

For example, if M is a Cartan-Hadamard manifold (*i.e.* its sectional curvature is non-positive and it is simply connected) then

$$\lambda(R) \ge \frac{1}{4}(n-1)^2 k^2(R), \tag{5.30}$$

where $-k^2(R)$ is equal to the supremum of the sectional curvature in the exterior of the ball B_R^z.

Theorem 5.3. *Suppose that the Λ-isoperimetric inequality holds on the manifold with the function*

$$\Lambda(v) = a\,v^{-2/n} \tag{5.31}$$

with some $a > 0$. Then for all $x \in M$ such that $r \equiv \operatorname{dist}\{x, z\} > \sqrt{t}$ the inequality is valid

$$p(x,z,t) \le \frac{C_{a,n,\gamma}}{t^{n/2}} \exp\left(-\gamma\frac{r^2}{4t} - \gamma\,\lambda(0)\,t - \bar{c}\,r\sqrt{\lambda(\gamma r)}\right), \tag{5.32}$$

where $0 < \gamma < 1$ is arbitrary, $\bar{c} = \bar{c}(\gamma) > 0$.

REMARK For small values of r and even for all $x, z \in M$, $t > 0$ we have, by Theorem 5.2, the estimate (5.11) for $R = +\infty$, $a(x,R) \equiv a$ or, by Theorem 5.1, the estimate (5.5).

Theorem 5.3 is applicable for a Cartan-Hadamard manifold because the Euclidean isoperimetric inequality holds for such a manifold. The third term $\exp(-\bar{c}\,r\sqrt{\lambda(\gamma r)})$ acquires on a Cartan-Hadamard manifold the form

$$\exp\left(-\bar{c}\,\frac{1}{2}\,(n-1)\,k(\gamma r)\,r\right). \tag{5.33}$$

Let us compare the estimate (5.32) in this setting with the exact formula (1.3) of the heat kernel of $\mathbb{H}_k^3$. The multiple (5.33) corresponds to the term

$$\frac{k\,r}{\sinh(k\,r)}$$

whose decay is similar to that of (5.33) up to the constant $\bar{c}$.

Of course, if the curvature $-k^2(R)$ is growing (to the negative side) fast enough then the term (5.33) can play the main role in the heat kernel behaviour as $r \to +\infty$.

PROOF OF THEOREM 5.3. In the exterior of any ball B_R^z the Λ-isoperimetric inequality holds with the function

$$\Lambda_R(v) = \max\{a\, v^{-2/n},\ \lambda(R)\}\,. \tag{5.34}$$

Let us denote by $V_R(t)$ and $W_R(r)$ V-transformations of Λ and $\sqrt{\Lambda}$ respectively. One can compute that

$$V_R(t) \geq C_{a,n}\, t^{n/2} \exp\left(\frac{1}{2}\,\lambda(R)\, t\right) \tag{5.35}$$

and

$$W_R(t) \geq C_{a,n}\, r^n \exp\left(\frac{1}{2}\,\sqrt{\lambda(R)} r\right)\,. \tag{5.27}$$

Of course, 1/2 in the exponent is too rough but all the same we will apply further Theorem 3.2 which does not give a sharp constant at the corresponding place.

Let us fix some ball B_ρ^x of radius $\rho < r = \operatorname{dist}\{x, z\}$ and put $R = r - \rho$. Since the ball B_ρ^x lies in the exterior of B_R^z it follows that the Λ-isoperimetric inequality holds in B_ρ^x with the function $\Lambda = \Lambda_R$. Applying Theorem 3.2 to the function $p(\cdot\,, z, \cdot)$ in the cylinder $B_\rho^x \times (t/2, t)$ we get

$$\begin{aligned} p(x,z,t)^2 &\leq \frac{C}{\min\left\{\frac{t}{2}\, V_R(c\frac{t}{2}),\ \rho^2\, W_R(c\,\rho)\right\}} \\ &\quad \cdot \int_{t/2}^{t} \int_{B_\rho^x} p^2(y,z,s)\, dy\, ds\,. \end{aligned} \tag{5.37}$$

On the other hand by Theorem 4.2 we have for $D > 2$

$$\begin{aligned} \int_{B_\rho^x} p^2(y,z,s)\, dy &\leq \int_{M\setminus B_R^z} p^2(y,z,s)\, dy \\ &\leq \exp\left(-\frac{R^2}{D\,s}\right) \int_M p^2(y,z,s) \exp\left(\frac{r^2}{D\,s}\right) dy \\ &\leq \frac{C_{D,a,n}}{s^{n/2}} \exp\left(-\frac{R^2}{D\,s}\right)\,. \end{aligned}$$

Substituting into (5.37) and noting that $t/2 \le s \le t$ we obtain

$$p^2(x,z,t) \le \frac{C_{D,a,n} \exp\left(-\dfrac{R^2}{Dt}\right)}{t^{n/2} \min\left\{V_R(c\frac{t}{2}),\ W_R(c\rho)\frac{\rho^2}{t}\right\}}$$

or, estimating V_R and W_R from (5.35) and (5.36) respectively,

$$\begin{aligned} p^2(x,z,t) \le & \frac{C_{D,a,n}}{t^n \min\left\{1,\ \frac{\rho^2}{t}\right\}^{1+n/2}} \\ & \cdot \exp\left(-\frac{R^2}{Dt} - \frac{1}{4} c \min\left\{\lambda(R)t,\ \rho\sqrt{\lambda(R)}\right\}\right). \end{aligned} \tag{5.38}$$

Now we apply the following elementary inequality

$$\varkappa^2 X^2 - 2\varkappa XY + Y\min\{X,Y\} \ge 0,$$

which is valid for all $X, Y \ge 0$ and $0 < \varkappa < 1/2$. Taking here $X = \rho\sqrt{\lambda(R)}$, $Y = \lambda(R)t$ we have

$$\min\left\{\rho\sqrt{\lambda(R)},\ \lambda(R)t\right\} \ge 2\varkappa\rho\sqrt{\lambda(R)} - \varkappa^2\frac{\rho^2}{t}.$$

Therefore (5.38) implies

$$\begin{aligned} p^2(x,z,t) \le & \frac{C_{D,a,n}}{t^n \min\left\{1,\ \frac{\rho^2}{t}\right\}^{1+n/2}} \\ & \cdot \exp\left(-\frac{R^2}{Dt} - \frac{1}{2} c\varkappa\rho\sqrt{\lambda(R)} + c\varkappa^2\frac{\rho^2}{4t}\right). \end{aligned}$$

Let us set $R = \gamma r$ and, correspondingly, $\rho = (1-\gamma)r$ where $0 < \gamma < 1$. Since by the hypothesis $r > \sqrt{t}$ it follows that $\rho^2/t \ge C_\gamma$. Taking $\varkappa$ small enough and increasing a bit D we get rid of the summand $c\varkappa^2\rho^2/(4t)$

$$p^2(x,z,t) \le \frac{C_{D,a,n,\gamma}}{t^n} \exp\left(-\frac{\gamma^2 r^2}{Dt} - \frac{1}{2}\gamma\varkappa c\sqrt{\lambda(\gamma r)}\,r\right). \tag{5.39}$$

Finally we apply estimate (5.11) which yields for $t_0 = (1-\gamma)t$

$$p(x,z,t) \le \frac{C_{D,a,n,\gamma}}{t^{n/2}} \exp\left(-\frac{r^2}{2Dt} - \gamma\lambda(0)t\right) \tag{5.40}$$

Multiplying (5.40) and (5.39) to powers γ and $(1-\gamma)/2$ respectively and taking D being close enough to 2 we obtain (5.32) (one has to replace γ^2 by γ in the final expression).

Corollary 5.1. *If the Λ-isoperimetric inequality holds on a manifold with the function $\Lambda(v) = a\,v^{-2/n}$, $a > 0$, then for all $z \in M$*

$$\begin{aligned}\limsup_{r\to\infty} \frac{1}{r} \sup_{t>0} \log p(x,z,t) \\ \le -C\left(\sqrt{\lambda_1(M)} + \sqrt{\lambda_{\text{ess}}(M)}\right),\end{aligned} \tag{5.41}$$

where $r = \operatorname{dist}\{x,z\}$, $\lambda_{\text{ess}}(M)$ is the bottom of the essential spectrum of $-\Delta$ in $L^2(M)$, and $C > 0$ is an absolute constant.

First note that for $r > 1$ we can give up the factor $t^{n/2}$ in (5.32). Indeed, if $t > 1$ this is evident, otherwise this term is majorized by $\exp(cr^2/t)$. Thus, taking $\gamma = 1/2$ in (5.32) we have for $r > 1$

$$\log p(x,z,t) \le -C\left(\frac{r^2}{t} + \lambda(0)t + r\sqrt{\lambda(r/2)}\right) + C_{a,n}\,.$$

Since

$$\frac{r^2}{t} + \lambda(0)t \ge 2r\sqrt{\lambda(0)},$$

it follows that

$$\sup_{t>0} \log p(x,z,t) \le -C\left(r\sqrt{\lambda(0)} + r\sqrt{\lambda(r/2)}\right) + C_{a,n}\,.$$

Dividing this inequality by r and passing to limit as $r \to +\infty$ we get (5.31) because $\lambda(0) = \lambda_1(M)$ and

$$\lim_{r\to+\infty} \lambda(r) = \lambda_{\text{ess}}(M)$$

(see [DL]).

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The sharp Poincaré inequality for free vector fields: an endpoint result

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1. Introduction.

The main purpose of this paper is to show a Poincaré type inequality of the following form:

$$\begin{aligned}\Big(\frac{1}{|B(r)|}\int_{B(r)} |f - f_{B(r)}|^q\Big)^{1/q} \\ \le C\, r \Big(\frac{1}{|B(r)|}\int_{B(r)} \Big(\sum_{i=1}^m |X_i f|\Big)^p\Big)^{1/p}\,,\end{aligned} \tag{1.1}$$

for all $f \in C^\infty(\overline{B(r)})$ with $q = pQ/(Q-p)$, $1 < p < Q$, where Q is a positive integer which will be specified later, and $X_1, \dots, X_m$ are C^∞ vector fields on $\mathbb{R}^d$ satisfying Hörmander's condition, $B(r)$ denotes a metric ball of radius r associated to the natural metric induced by the vector fields,

$$f_{B(r)} = \frac{1}{|B(r)|}\int_{B(r)} f\,.$$

The first Poincaré type inequality for the general vector fields satisfying Hörmander's condition was derived by D. Jerison. In 1986, D. Jerison [J] proved the above type inequality in the case $q = p$. When $q = p = 2$, this inequality is equivalent to finding a lower bound $C^{-1}r^{-2}$ on the least nonzero eigenvalue in the Newman problem for $L = \sum_{i=1}^{m} X_i^* X_i$ on the ball $B(r)$. Soon after, D. Jerison and A. Sánchez-Calle in [JS] proved, among other things, the Poincaré inequality associated to the subelliptic operators.

Recently, the author has shown in [L1] the weighted Poincaré type inequalities for vector fields. One of the main ingredients of [L1] is the pointwise estimate for functions (without compact support) over the metric balls controlled by the fractional integral of certain maximal function (see (1.2) below). A nonweighted Poincaré type inequality (1.1) was also obtained, as a byproduct, in [L1] for all $q < pQ/(Q-p)$ except the endpoint $q = pQ/(Q-p)$.

Jerison's work [J] deserves some more explaination here. In [J] (see also [JS]), he first showed an inequality of the following form:

$$\int_B |f - f_B|^p \leq C\rho(B)^p \int_{2B} \left(\sum_{i=1}^{m} |X_i f| + |f| \right)^p ,$$

where C is independent of f and B, $2B$ is the double of the metric ball B. Then he got rid of 2 in the limit of the integral on the right side (replacing $2B$ by B) and obtained the desired result by a covering argument based on the Whitney decomposition. This argument was motivated by an argument due to R. Kohn [K] in the case of the bilipschitzian image of a ball. This Jerison-Kohn type of covering argument is by now fairly well-known.

In [L1], we were able to prove a pointwise estimate for any $f \in C^\infty(\overline{B})$ of the following type (for any $\xi \in B$):

$$|f(\xi) - C(f,B)| \leq \int_{cB} \frac{M(\sum_{i=1}^{m} |\tilde{X}_i f| + |f|)}{\rho(\xi,\eta)^{Q-1}} d\eta , \tag{1.2}$$

where $\tilde{X}_i$ are the lifted vector fields of X_i by the Rothschild-Stein lifting theorem [RS], and B is the metric ball associated to $\{\tilde{X}_i\}_{i=1}^{m}$ in the space with extra variables, and M is the Hardy-Littlewood maximal operator in this metric space with metric $\rho(\xi,\eta)$, and $c \geq 1$ is an absolute constant. Having obtained this pointwise estimate, we showed in [L1] for $f \in C^\infty(\overline{cB})$ that

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |f(x) - f_B|^q dx \right)^{1/q} \\ &\qquad \leq c\rho(B) \left(\frac{1}{|B|} \int_{cB} \left(\sum_{i=1}^{m} |X_i f| + |f| \right)^p \right)^{1/p} , \end{aligned} \tag{1.3}$$

for all $1 \le q \le pQ/(Q-p)$, $1 < p < Q$, where B is the ball associated to the original vector fields $X_1, \ldots, X_m$ and $\rho(B)$ is the radius of B. The advantage of such pointwise estimates is that it also leads to weighted Poincaré inequalities and sharper unweighted inequality (see [L1]).

The covering lemma argument in [J] also applies to the case $p < q$ (see [L1]). Thus by employing this covering lemma argument we were able to get rid of the constant c in the limit of the integral on the right hand side of (1.3) (replacing c by 1) and we proved in [L1]

$$\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q dx\right)^{1/q} \le c\,\rho(B)\left(\frac{1}{|B|}\int_B \left(\sum_{i=1}^m |X_i f|\right)^p\right)^{1/p},$$

for small $\rho(B)$ and $1 \le q < pQ/(Q-p)$. However, it seems that such covering lemma argument does not lead to the endpoint result for $q = pQ/(Q-p)$ (see the discussion in [L1] and the remark at the end of this paper). Thus the endpoint result still remains unproved though it is expected to be true.

Thus the purpose of this paper is two fold. One is to show the endpoint result for $q = pQ/(Q-p)$, the other is that we carry out a different covering argument from the one in [J]. We like to point out that the method we used here is motivated by B. Bojarski's work [B] in which he proved the Sobolev embedding theorems on domains satisfying certain chain condition in the setting of euclidean space. This argument can be extended to the weighted version for doubling weights, see for example, Chua [C].

The following remarks are in order. First of all, when the vector fields are free (see [RS] or below for definition), *e.g.* on graded nilpotent group like the Heisenberg group, this endpoint result is sharp. Our nonweighted Poincaré inequality proved in [L1] for all $1 < p < Q$ and $1 \le q < pQ/(Q-p)$, and the sharp form for $q = pQ/(Q-p)$ here in this special case have recently been used to prove a compensated compactness result on the Heisenberg group by Grafakos-Rochberg [GR]. Secondly, as Professor M. Christ pointed out to us, for non-free vector fields, even the exponent $q = pQ/(Q-p)$ may not be the optimal one. We like to thank Professor M. Christ for bringing this to our attention. However, we do not discuss further here. Thirdly, we should mention that the Sobolev type inequality (for functions with compact support) for vector fields can be obtained easily (see [L1]) thanks to the fundamental solution estimates of the sums of squares of vector fields by Sánchez-Calle [Sa] and Nagel-Stein-Wainger [NSW] (for the more general case by C. Fefferman and A. Sánchez-Calle, see [FeS]), see also [L1] for the weighted versions of Sobolev embedding theorem and [L2] for the Rellich-Kondrachov compact embedding theorem with applications to the estimates for the fundamental solutions of degenerate subelliptic operators. The weighted Poincaré-Sobolev inequalities have been used in [L1] to establish the Harnack inequalities for degenerate subelliptic operators of divergence form, and as a continuation in [L4], for strongly degenerate Schrödinger's operators which contains the result in [CGL]. Fourthly, when $1 < p < \infty$, both weighted Poincaré and Sobolev inequalities hold for $1 \leq q \leq pQ/(Q-1) + \delta_p$ for certain $\delta_p > 0$ in the case of equal weights in the A_p class (see Theorem B in [L1]). Thus in the case of no weights, Poincaré and Sobolev inequalities hold as well with such exponents $p < q$ as a special case.

2. Some preliminaries and the statement of the theorem.

Let Ω be a bounded, open and pathconnected domain in $\mathbb{R}^n$, and let $X_1, \ldots, X_m$ be a collection of C^∞ real vector fields defined in a neighbourhood of the closure $\overline{\Omega}$ of Ω. For a multi-index $\alpha = (i_1, \ldots, i_k)$, denote by X_α the commutator $[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]] \ldots]$ of length $k = |\alpha|$. Throughout this paper we assume that the vector fields satisfy Hörmander's condition: there exists some positive integer s such that $\{X_\alpha\}_{|\alpha| \leq s}$ span the tangent space of $\mathbb{R}^d$ at each point of Ω. We can define a metric as follows: An admissible path γ is a Lipschitz curve $\gamma : [a,b] \to \Omega$ such that there exist functions $c_i(t)$, $a \leq t \leq b$, satisfying $\sum_{i=1}^m c_i(t)^2 \leq 1$ and $\gamma'(t) = \sum_{i=1}^m c_i(t) X_i(\gamma(t))$ for almost every $t \in [a,b]$. Then a natural metric on Ω associated to $X_1, \ldots, X_m$ is defined by

$$\begin{aligned} \varrho(\xi, \eta) = \min\{b \geq 0 : &\text{ there exists an admissible path} \\ &\gamma : [0,b] \longrightarrow \Omega \text{ such that} \\ &\gamma(0) = \xi\,, \text{ and } \gamma(b) = \eta\}\,. \end{aligned}$$

The metric ball is defined by $B(\xi, r) = \{\eta : \varrho(\xi,\eta) < r\}$. This metric is equivalent to the various other metrics defined in the work of Nagel-Stein-Wainger [NSW]. Note that the Lebesgue measure is doubling with respect to the metric balls as shown in [NSW]. Thus (Ω, ϱ) is a homogeneous space.

By the Rothschild-Stein lifting theorem (see [RS]), the vector fields $\{X_i\}_{i=1}^m$ on $\Omega \subset \mathbb{R}^d$ can be lifted to vector fields $\{\tilde{X}_i\}_{i=1}^m$ in $\tilde{\Omega} = \Omega \times T \subset$

$\mathbb{R}^d \times \mathbb{R}^{N-d}$, where T is the unit ball in $\mathbb{R}^{N-d}$, by adding extra variables so that the resulting vector fields are free, *i.e.*, the only linear relation between the commutators of order less than or equal to s at each point of $\tilde{\Omega}$ are the antisymmetric and Jacobi's identity. Let $\mathcal{G}(m,s)$ be the free Lie algebra of steps with m generators, that is the quotient of the free Lie algebra with m generators by the ideal generated by the commutators of order at least $s+1$. Then $\{X_\alpha\}_{|\alpha|\le s}$ are free if and only if $d = \dim \mathcal{G}(m,s)$. We also define $Q = \sum_{j=1}^{s} j m_j$ where m_j is the number of linearly independent commutators of length j.

Here is the statement of the main theorem.

Theorem 2.1. *There exist positive constants r_0 and μ such that for any $f \in C^\infty(\overline{B(r)})$, and $1 < p < Q$, $1 \le q \le pQ/(Q-p)$, we have the following*

$$\left(\frac{1}{|B(r)|}\int_{B(r)} |f - f_{B(r)}|^q\right)^{1/q} \le C\,r\left(\frac{1}{|B(r)|}\int_{B(r)}\Big(\sum_{i=1}^{m}|X_i f|\Big)^p\right)^{1/p},$$

provided $\mu B(r) \subset \Omega$ and $r \le r_0$.

2. Boman chain condition for the metric space (Ω, ϱ).

We first introduce the notion of the so-called "Boman chain condition" in the context of homogeneous space. This condition seems slightly different from the corresponding version in the euclidean space. However, it suffices for our purpose here.

Definition. *Let (X,ϱ) be a homogeneous space in the sense of Coifman - Weiss. An open set E in X is said to satisfy the Boman chain condition if there exists a positive constant μ and a family $\mathcal{F}$ of disjoint metric balls B such that*

i) $E = \bigcup_{B\in\mathcal{F}} 2B$.

ii) $\sum_{B\in\mathcal{F}} \chi_{10B}(x) \le M\,\chi_\Omega(x)$ *for all* $x \in X$.

iii) *There is a so-called "central ball" $B_0 \in \mathcal{F}$ such that each ball $B \in \mathcal{F}$ can be connected to B_0 by a finite chain of balls $B_0, \dots, B_{k(B)} =$*

B in such a way that $2B_j \bigcap 2B_{j+1} \neq \varnothing$ *and* $4B_j \bigcap 4B_{j+1}$ *contains a metric ball* D_j *whose volume is comparable to those of both* B_j *and* B_{j+1}.

iv) *Moreover,* $B \subset \mu B_j$ *for all* $j = 0, 1, \ldots, k(B)$.

The above E is called a Boman chain domain. The explicit numbers $2, 4$ and 10 are not essential here. We define in such a way just for the simplicity.

In the case $X = \mathbb{R}^n$ and ϱ is the euclidean metric, it is the standard chain condition. It is known that any euclidean cubes, balls, John's domain, bounded Lipschitz domains and (ϵ, ∞) domains are all Boman chain domains. In the general homogeneous space, it will be hard to verify if some domain is a Boman chain domain. However, we will show below any metric ball in our (Ω, ϱ) associated to the vector fields $X_i, \ldots, X_m$ is indeed a Bomain chain domain.

Lemma 3.1. *Let* $B = B(\xi_1, r_1) \subset \Omega$ *be a metric ball. Then* B *is a Boman chain domain.*

We like to point out that Lemma 3.1 is implicit in Jerison's work [J]. Indeed, the only thing we need to check is iv) in the definition above. And, it also follows from Jerison's work. For the completeness of the presentation, we will show the details. We first need the following Whitney decomposition whose proof can be found in Folland-Stein's book [FS] (see also [J]).

Lemma 3.2. *Let* $E = B(\xi_1, r_1)$, *then there is a pairwise disjoint family of balls* $\mathcal{F}$ *and a constant* M *depending only on the doubling constant of the Lebesgue measure with respect to the metric balls such that*

i) $E = \bigcup_{B \in \mathcal{F}} 2B$,

ii) $B \in \mathcal{F}$ *implies that* $10^2 \rho(B) \le \rho(B, \partial E) \le 10^3 \rho(B)$,

iii) $\#\{B \in \mathcal{F} : \ \eta \in 10B\} \le M$.

Here $\rho(B, \partial E)$ *is the distance, in the metric* ϱ, *from* B *to* ∂E. $\#S$ *denotes the number of elements in the set* S.

Lemma 3.2 already provides more or less the first three conditions in the Boman chain condition. For $B \in \mathcal{F}$, define γ_B as an admissible path from the center η_B of B to ξ_1 (the center of E) of length less or equal than r_1. Denote the subset of E defined by the image of γ_B by γ_B as well. This path may not be unique, but will be fixed throughout this paper. Denote $\mathcal{F}(B) = \{A \in \mathcal{F} : 2A \bigcap \gamma_B \neq \varnothing\}$. The following has been proved by Jerison [J].

Lemma 3.3. *Let $B \in \mathcal{F}$, then there are no elements of $\mathcal{F}(B)$ of radius less than $\rho(B)/100$.*

PROOF OF LEMMA 3.1. We select a central ball $B_0 \in \mathcal{F}$ such that $\xi_1 \in 2B_0$ and will fix it throughout the proof. As proved in [J], $\#\mathcal{F}(B)$, which is equal to the number of elements in $\mathcal{F}(B)$, is finite and with a upper bound $C\log(r_1/\rho(B))$ though it is not uniformly bounded on $B \in \mathcal{F}$. We then order the elements of $\mathcal{F}(B)$ as $\mathcal{F}(B) = \{A_1, \dots, A_{k(B)}\}$ such that $A_1 = B_0$ and $A_{k(B)} = B$, and $2A_k \bigcap 2A_{k+1} \neq \varnothing$ for all k. Thus by the construction in Lemma 3.2, $4A_k \bigcap 4A_{k+1}$ contains a ball D_k whose volume is comparable to those of both A_k and A_{k+1}. Thus, the first three conditions in the definition of chain condition are already verified. We will show $B \subset \mu A_k$, for some $\mu > 0$ and for all k. Now let η_B be the center of the ball B and η_k be the center of the ball A_k. Then

$$\begin{aligned}\varrho(\eta_B, \eta_{A_k}) &\le \varrho(\eta_B, \partial E) + \varrho(\eta_{A_k}, \partial E)\\ &\le 10^3\rho(B) + 10^3\rho(A_k) \le 10^6\rho(A_k),\end{aligned}$$

for all k, by Lemma 3.2. Thus $B \subset 10^8 A_k$, by Lemma 3.2 again. Therefore, we can take $\mu = 10^8$.

4. Proof of the main theorem.

We will need two technical lemmas.

Lemma 4.1. *Given $1 \le p < \infty$. Let $\{B_\alpha\}$ be an arbitrary family of open metric balls in (Ω, ϱ) with $\mu B_\alpha \subset \Omega$ and $\{a_\alpha\}_{\alpha \in I}$ be nonnegative numbers. Then*

$$\Big\| \sum_\alpha a_\alpha \chi_{\mu B_\alpha} \Big\|_{L^p(\Omega)} \le C \Big\| \sum_\alpha a_\alpha \chi_{B_\alpha} \Big\|_{L^p(\Omega)},$$

where C is independent of $\{a_\alpha\}$ and $\{B_\alpha\}$.

The proof is standard. For completeness we include a detailed proof.

PROOF. Let $\phi \in L^{p}(\Omega)$, with $1/p + 1/p' = 1$, $1 < p' < \infty$. Let

$$M\phi(\xi) = \sup_{\xi \in B} \frac{1}{|B|} \int_B |\phi(\eta)|\, d\eta$$

be the Hardy-Littlewood maximal function of $\phi(\xi)$ with respect to the metric balls in Ω and the supremum is taken over all balls B including ξ. Then it is known that $\|M\phi\|_{L^{p'}(\Omega)} \leq C\, \|\phi\|_{L^{p'}(\Omega)}$. Hence

$$\begin{aligned}
\Big| \int_\Omega \sum_\alpha a_\alpha \chi_{\mu B_\alpha}(\xi)\, \phi(\xi)\, d\xi \Big| &= \Big| \sum_\alpha |\mu B_\alpha| \frac{1}{|\mu B_\alpha|} \int_{\mu B_\alpha} \phi(\xi)\, d\xi \Big| \\
&\leq C \Big| \sum_\alpha a_\alpha \int_{B_\alpha} M\phi(\xi)\, d\xi \Big| \\
&\leq C \int_\Omega \sum_\alpha a_\alpha \chi_{B_\alpha}(\xi)\, M\phi(\xi)\, d\xi \\
&\leq C \Big\| \sum_\alpha a_\alpha \chi_{B_\alpha} \Big\|_{L^p(\Omega)} \|M\phi\|_{L^{p'}(\Omega)} \\
&\leq C \Big\| \sum_\alpha a_\alpha \chi_{B_\alpha} \Big\|_{L^p(\Omega)} \|\phi\|_{L^{p'}(\Omega)} .
\end{aligned}$$

Thus the lemma follows.

Lemma 4.2. *Assume $p > 1$, then for any metric balls I and B with $I \subset B \subset \Omega$ we have*

$$\left(\frac{\rho(I)}{\rho(B)}\right) \left(\frac{|I|}{|B|}\right)^{1/q - 1/p} \leq C,$$

provided that $1 \leq q \leq p\,Q/(Q-p)$ and $\rho(B) \leq r_0$ for some $r_0 > 0$.

This lemma is proved in [L1]. It is Lemma 6.12 in [L1].

We also note that by an easy covering argument we can reduce (1.3) to (this is just for simplicity)

$$\left(\frac{1}{|B|} \int_B |f(x) - f_B|^q dx\right)^{1/q} \leq c\,\rho(B) \left(\frac{1}{|B|} \int_{2B} \Big(\sum_{i=1}^m |X_i f| + |f|\Big)^p\right)^{1/p} \tag{4.3}$$

We note that (4.3) is equivalent to

$$(4.4)\quad \int_B |f - f_B|^q \le C\,\rho(B)^q\,|B|^{1-q/p}\left(\int_{2B}\Big(\sum_{i=1}^m |X_i f| + |f|\Big)^p\right)^{q/p},$$

for all $1 \le q \le pQ/(Q-p)$.

We now start to prove Theorem 2.1. Fix the central ball B_0 as in Lemma 3.1. We have

$$(4.5)\quad \begin{aligned} \|f - f_{2B_0}\|^q_{L^q(E)} &\le 2^{q-1}\sum_{B\in\mathcal{F}} \|f - f_{2B}\|^q_{L^q(2B)} \\ &\quad + 2^{q-1}\sum_{B\in\mathcal{F}} \|f_{2B} - f_{2B_0}\|^q_{L^q(2B)} \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Replacing f by $f - f_{2B_0}$ in the inequality (4.4) we will get

$$(4.6)\quad \begin{aligned} &\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q dx\right)^{1/q} \\ &\quad \le c\,\rho(B)\left(\frac{1}{|B|}\int_{2B}\Big(\sum_{i=1}^m |X_i f| + |f - f_{2B_0}|\Big)^p\right)^{1/p}, \end{aligned}$$

for any given $B \in \mathcal{F}$. Now fix temporarily $B \in \mathcal{F}$ and consider the chain $\mathcal{F}(B) = \{A_1, \dots, A_{k(B)}\}$ constructed in the proof of Lemma 3.1. Thus

$$\begin{aligned} \|f_{2B} - f_{2B_0}\|_{L^q(2B)} &\le C \sum_{j=1}^{k(B)-1} \|f_{2A_j} - f_{2A_{j+1}}\|_{L^q(2B)} \\ &\le C \sum_{j=1}^{k(B)-1} \Big(\frac{|B|}{|4A_j \bigcap A_{j+1}|}\Big)^{1/q} \\ &\qquad \cdot \|f_{2A_j} - f_{2A_{j+1}}\|_{L^q(4A_j \cap 4A_{j+1})} \\ &\le C \sum_{j=1}^{k(B)-1} \Big(\frac{|B|}{|A_j|}\Big)^{1/q} \|f - f_{2A_j}\|_{L^q(4A_j)} \\ &\quad + C \sum_{i=1}^{k(B)} \Big(\frac{|B|}{|A_{j+1}|}\Big)^{1/q} \|f - f_{2A_{j+1}}\|_{L^q(4A_{j+1})} \end{aligned}$$

$$\leq 2C \sum_{j=1}^{k(B)-1} \left(\frac{|B|}{|A_j|}\right)^{1/q} \|f - f_{2A_j}\|_{L^q(4A_j)} .$$

We observe that

$$\|f - f_{2A_j}\|_{L^q(4A_j)} \leq \|f - f_{4A_j}\|_{L^q(4A_j)} + \|f_{4A_j} - f_{2A_j}\|_{L^q(4A_j)} ,$$

and

$$\begin{aligned}\|f_{4A_j} - f_{2A_{j+1}}\|_{L^q(4A_j)} &\leq |4A_j|^{1/q} \left(\frac{1}{|2A_j|} \int_{2A_j} |f - f_{4A_j}|\right) \\ &\leq C \left(\int_{4A_j} |f - f_{4A_j}|^q\right)^{1/q} .\end{aligned}$$

Therefore, we get

$$\|f_{2B} - f_{2B_0}\|_{L^q(2B)} \leq C \sum_{j=1}^{k(B)-1} \left(\frac{|B|}{|A_j|}\right)^{1/q} \|f - f_{4A_j}\|_{L^q(4A_j)} .$$

Since, by the chain condition, $B \subset \mu A_j$ for each $A_j \in \mathcal{F}(B)$, we then have

$$\begin{aligned}\|f_{2B} - f_{2B_0}\|_{L^q(2B)} \frac{\chi_{2B}(\xi)}{|B|^{1/q}} &\leq C \sum_{A\in\mathcal{F}} \left(\frac{1}{|A|}\right)^{1/q} \|f - f_{4A}\|_{L^q(4A)}\, \chi_{2\mu A}(\xi) \\ &= C \sum_{A\in\mathcal{F}} a_A\, \chi_{2\mu A} .\end{aligned}$$

In the above expression, a_A is notationally defined in an obvious way. For the term II in (4.5), we have

$$\mathrm{II} \leq C \sum_{B\in\mathcal{F}} \int_\Omega \|f_{2B} - f_{2B_0}\|^q_{L^q(2B)} \frac{\chi_{2B}(\xi)}{|B|} .$$

Since $\sum_{B\in\mathcal{F}} \chi_{2B}(\xi) \leq C$, we derive

$$\mathrm{II} \leq C \int_\Omega \Big| \sum_{A\in\mathcal{F}} a_A\, \chi_{2\mu A} \Big|^q .$$

By lemma (4.1), we then get

$$\mathrm{II} \le C \int_\Omega \Big| \sum_{A\in\mathcal{F}} a_A \chi_A \Big|^q .$$

Since $\sum_{A\in\mathcal{F}} \chi_A(\xi) \le C$, we will have

$$\mathrm{II} \le C \sum_{A\in\mathcal{F}} a_A^q \int_\Omega \chi_A(\xi) \le C \sum_{A\in\mathcal{F}} \|f - f_{4A}\|^q_{L^q(4A)} \,.$$

Therefore, by the inequality (4.4)

$$\begin{aligned}
\mathrm{II} &\le C \sum_{A\in\mathcal{F}} |A|^{1-q/p} \rho(A)^q \left(\int_{8A} \Big(\sum_{i=1}^m |X_i f| + |f - f_{2B_0}| \Big)^p \right)^{q/p} \\
&\le C\, |E|^{1-q/p} \rho(E)^q \sum_{A\in\mathcal{F}} \left(\int_{8A} \Big(\sum_{i=1}^m |X_i f| + |f - f_{2B_0}| \Big)^p \right)^{q/p} \\
&\le C\, |E|^{1-q/p} \rho(E)^q \left(\int_{E} \Big(\sum_{i=1}^m |X_i f| + |f - f_{2B_0}| \Big)^p \right)^{q/p} .
\end{aligned}$$

In the last inequality we used the fact $q \ge p$, $8A \subset E$ and $\sum_{A\in\mathcal{F}} \chi_{8A}(\xi) \le C$, and in the one next to the last we used Lemma 4.2.

For the term I in (4.5), the estimate is the same by replacing $4A$ by $2A$ in the estimate of II. Therefore we have shown the main theorem if we use the Minkowski inequality on the right side above and letting $\rho(E)$ be very small in order to bootstrap the second term on the right.

Finally, we remark out that if we use the covering argument in [J] (see [L1]), there will be a factor like

$$\log \left(\frac{\rho(E)}{\rho(B)} \right) \left(\frac{\rho(B)}{\rho(E)} \right)^q \left(\frac{|B|}{|E|} \right)^{1-q/p} .$$

In order to bound this term uniformly independent of $B \subset E$, we need to require that $q < pQ/(Q-p)$. This will cause to lose the endpoint. However, the argument in this paper does not produce the above factor (without the logarithm term).

Added in proof: The first draft of the paper was done and circulated in September of 1992. After this paper was accepted for publication, there have been some new embedding theorems established on various function spaces associated with the general vector fields of Hörmander type. Here are the details:

1) We mention here that when $p = 1$ and $q = Q/(Q-1)$ the nonweighted Poincaré inequality has been proved by Franchi, Wheeden and the author [FLW]. It was proved by Jerison [J] when $p = q = 1$. Weighted inequalities are also obtained in [FLW] which improve the previous two-weighted results of the author in [L1].

2) When $p = Q$, besides the Poincaré inequality for $1 \leq q \leq pQ/(Q-1)+\delta_p$, we can even show that the exponential integrability of the function (without assuming the compact support for the function). This has been done by the author [L3] together with the embedding theorems on the Campanato-Morrey spaces $M^{p,\lambda}$ (for $1 < p < \lambda \leq Q$). The embedding theorems on the Campanato-Morrey spaces allow the gap larger than that in the Poincaré inequality, *i.e.* $1/Q$.

3) When $p > Q$, then we can show the function itself belongs to the local Lischitz spaces $C^{0,\gamma}_{\text{loc}}$ for $\gamma = 1 - Q/p$. When $\sum_{i=1}^{m} |X_i f|$ is in certain Morrey spaces $M^{p,\lambda}$ (for $p \geq 1$), we can also show the function itself in certain local Lipschitz spaces or of exponential integrability. These have been shown by the author in [L5].

4) The Sobolev inequality for $p = 1$ and $q = Q/(Q-1)$ for functions with compact support has been proved in [FGaW] for general vector fields. Weighted Sobolev inequalities for $p = 1$ are also proved in [FGaW]. The Sobolev type inequality has been established on the group earlier, see [VSCC] and numerous references therein.

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On singular integrals of Calderón-type in $\mathbb{R}^n$, and BMO

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Abstract. We prove L^p (and weighted L^p) bounds for singular integrals of the form

$$\text{p.v.} \int_{\mathbb{R}^n} E\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy ,$$

where $E(t) = \cos t$ if Ω is odd, and $E(t) = \sin t$ if Ω is even, and where $\nabla A \in \text{BMO}$. Even in the case that Ω is smooth, the theory of singular integrals with "rough" kernels plays a key role in the proof. By standard techniques, the trigonometric function E can then be replaced by a large class of smooth functions F. Some related operators are also considered. As a further application, we prove a compactness result for certain layer potentials.

1. Introduction.

In this note we extend to $\mathbb{R}^n$ some 1-dimensional results of T. Murai (see, *e.g.* [Mu1], [Mu2], [Mu3]). We are concerned with n-dimensional singular integral of "Calderón-type", defined by

$$(1.1) \qquad T[A]f(x) = \text{p.v.} \int_{\mathbb{R}^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy ,$$

where F is a suitably smooth function defined on the real line, A is real-valued and belongs to the BMO Sobolev space $I_1(\text{BMO})$ (I_1 denotes the usual fractional integral operator of order 1, suitably defined on BMO), and Ω is homogeneous of degree zero and bounded on the sphere. In applications Ω is usually smooth, but one of the main themes of this paper is that even in the smooth case, singular integrals with rough kernels will arise in a natural way when one extends to $\mathbb{R}^n$ certain 1-dimensional perturbation techniques of G. David (as described, for example, in the survey article of Coifman and Meyer [CM]), and Murai. We recall that BMO is the Banach space of locally integrable functions modulo constants with norm

$$\|b\|_* = \frac{1}{|Q|}\int_Q |b - m_Q(b)| \approx \Big(\frac{1}{|Q|}\int_Q |b - m_Q(b)|^q\Big)^{1/q},$$

where $1 \le q < \infty$,

$$m_Q(b) = \frac{1}{|Q|}\int_Q b,$$

and where the comparability of the various L^q means is a very well known result of John and Nirenberg. Then $A \in I_1(\text{BMO})$ if and only if A is a continuous function with a locally integrable gradient (in the weak sense) and $\nabla A \in \text{BMO}$ (see Strichartz [Stz] for more on the BMO Sobolev spaces). We remark that in fact one can show that A is absolutely continuous in the sense of Tonelli, so that ∇A exists a.e. Since the kernels that we shall consider are anti-symmetric, the case that A is Lipschitz can be reduced to the 1-dimensional setting by the method of rotations, but of course this method is not applicable for $A \in I_1(\text{BMO})$. Furthermore, for at least one of the applications that we have in mind, it will be necessary to prove weighted estimates which can only be obtained by an intrinsically n-dimensional approach.

Statement of results.

We define an operator T by

$$(1.2) \qquad Tf(x) = \text{p.v.}\int_{\mathbb{R}^n} E\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n}\,f(y)\,dy,$$

where Ω is homogeneous of degree zero, essentially bounded, and either odd or even; $A \in I_1(\text{BMO})$, and $E(t) = \cos t$ (if Ω is odd) or $E(t) = \sin t$ (if Ω is even). Our main result is the following

Theorem 1.3. *There exists an absolute constant $\mu > 0$ (we shall in fact observe that we may take $\mu = 1$) such that the operator T defined in (1.2) with Ω, A and E as stated, satisfies the norm inequality*

$$\|Tf\|_{p,w} \leq C(n,p,A_p)\,(1+\|\nabla A\|_*)^{\mu}\,\|\Omega\|_\infty\,\|f\|_{p,w}\,, \tag{1.4}$$

for all $1 < p < \infty$ and $w \in A_p$, with constants $C(n,p,A_p)$ depending only on n, p and the A_p constant of w.

Here we shall interpret the principal value in the following weak sense -we shall prove that all (double) truncations of T satisfy (1.4) with a uniform constant independent of the truncation. For anti-symmetric kernels, the principal value limit of the mapping $T : \mathcal{D} \longrightarrow \mathcal{D}'$ exists by a well known device.

A variant of (1.2) which can be treated by the same techniques and which is useful in applications is

$$\begin{aligned}\tilde{T}f(x) = \text{p.v.} \int_{\mathbb{R}^n} (B(x)-B(y))\, E\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \\ \cdot \frac{\Omega(x-y)}{|x-y|^{n+1}}\, f(y)\, dy\,,\end{aligned} \tag{1.5}$$

where B is Lipschitz, $A \in I_1(\mathrm{BMO})$, $\Omega \in L^\infty(S^{n-1})$, but now $E(t) = \cos t$ if Ω is even and $E(t) = \sin t$ if Ω is odd. We have the following

Theorem 1.6. *For $\tilde{T}$ defined in (1.5) (again we make the same weak interpretation of the principal value as in Theorem 1.3), there exists an absolute constant $\tilde{\mu} > 0$ such that for all $1 < p < \infty$, $w \in A_p$, we have*

$$\|\tilde{T}f\|_{p,w} \leq C(n,p,A_p)\,(1+\|\nabla A\|_*)^{\tilde{\mu}}\,\|\nabla B\|_\infty\,\|\Omega\|_\infty\,\|f\|_{p,w}\,. \tag{1.7}$$

Given Theorems 1.3 and 1.6, we will then be able to obtain, by rather standard methods, the following corollaries. The first will be an easy consequence of Theorem 1.3.

Theorem 1.8. *Let μ be the same as in Theorem 1.3, and let $T[A]$ be defined as in (1.1) where $\Omega \in L^\infty(S^{n-1})$ and Ω is odd if F is even or vice versa. Furthermore, we assume that $A \in I_1(\mathrm{BMO})$, that $F \in C^{\mu+2}(\mathbb{R})$ and that F and its first $\mu+2$ derivatives belong to L^1 (so that $\hat{F}(\xi) \leq C\,(1+|\xi|^{-(\mu+2)})$. Then*

$$\|T[A]f\|_{p,w} \leq C\,(1+\|\nabla A\|_*)^{\mu}\,\|\Omega\|_\infty\,\|f\|_{p,w}\,, \qquad 1<p<\infty\,, \tag{1.9}$$

for all $w \in A_p$, where C depends on dimension, p, F, and the A_p constant of w.

A corollary of Theorems 1.3, 1.6 and 1.8 is the following

Theorem 1.10. *Let $\nu = \max\{\mu, \tilde{\mu}\}$ (with the same μ, $\tilde{\mu}$ as in Theorems 1.3 and 1.6). Set*

$$T_*[A,B]f(x) = \sup_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} \big(B(x) - B(y) - \nabla B(y)\cdot(x-y)\big) \cdot F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y)\,dy\Big| ,$$

where $A, B \in I_1(\mathrm{BMO})$, $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ for some $0 < \alpha \le 1$, $F \in C^{\nu+2}$, with F and its first $\nu + 2$ derivaties belongig to L^1, and F and Ω are either both odd or both even. Furthermore, we now suppose that $F(t) \le C(1+|t|)^{-1}$. Then for all $1 < p < \infty$, $w \in A_p$, we have

$$\|T_*[A,B]f\|_{p,w} \le C(n,p,A_p,F,\Omega)\, \|\nabla B\|_* \,(1 + \|\nabla A\|_*)^\nu\, \|f\|_{p,w} \, .$$

REMARK. We point out that were we interested only in proving Theorem 1.10, we could have done so without invoking Theorems 1.3, 1.6 and 1.8. The point is that if A and B are Lipschitz, then the operator norm of $T_*[A,B]$ is not changed if A and B are perturbed by a linear function. For B this is obvious, and for A it is a consequence of the method of rotations. The perturbation techniques of David can then be used to extend to the case that $A, B \in I_1(\mathrm{BMO})$, because for $\Omega \in \mathrm{Lip}_\alpha$, and for F satisfying the mild decay assumption, the kernel is almost (although not quite) a "standard" kernel. We are motivated to prove the sharper results given here (especially those for the trigonometric kernels, Theorems 1.3 and 1.6) by analogy to the 1-dimensional work of Murai.

A special case of particular interest is the double layer potential for Laplace's equation. The boundedness of the trace of this operator on the boundary of a BMO_1 domain (*i.e.* a domain whose boundary is locally the graph $(x, A(x))$ of a function $A \in I_1(\mathrm{BMO})$) is a special case of results for singular integrals on surfaces (see, *e.g.*, the papers of Semmes [Se], David [D], or David and Jerison [DJe], but it can also be obtained as a very easy corollary of Theorem 1.10. In fact the boundary double layer potential in (local) graph coordinates equals

$$Kf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{A(x) - A(y) - \nabla A(y) \cdot (x-y)}{\Big(1 + \Big(\dfrac{A(x) - A(y)}{|x-y|}\Big)^2\Big)^{(n+1)/2}} \, \frac{f(y)}{|x-y|^{n+1}} \, dy \, .$$

Thus, if we set $\Omega = 1$, $A = B$, $F(t) = (1+t^2)^{-(n+1)/2}$, we may deduce that

$$\|Kf\|_{L^p(\Gamma)} \le C(n,p,\|\nabla A\|_*) \, \|f\|_{L^p(\Gamma)} \, , \tag{1.11}$$

where Γ is the hypersurface $(x, A(x))$, and where $C(n,p,\|\nabla A\|_*) \to 0$ as $\|\nabla A\|_* \to 0$. The inequality (1.11) is an immediate corollary of Theorem 1.10 and the following observation:

Lemma 1.12. *Surface measure on Γ equals an A_p weight times Lebesgue measure, if $\nabla A \in$ BMO, i.e., the weight $w = \sqrt{1+|\nabla A|^2}$ belongs to $\bigcap_{1<p<\infty} A_p$, and furthermore, the A_p constant of w is nor larger than $C(1 + \|\nabla A\|_*)$.*

PROOF OF THE LEMMA. Since $\omega \approx 1 + |\nabla A|$, it is enough to show that

$$\begin{aligned} \sup_Q \Big(\frac{1}{|Q|}\int_Q (1+|b|)\Big) \Big(\frac{1}{|Q|}\int_Q (1+|b|)^{-1/(p-1)}\Big)^{p-1} \\ \le C\,(1 + \|b\|_*) \, , \end{aligned} \tag{1.13}$$

for all $1 < p < \infty$ and $b \in$ BMO.

We set $b_Q = \frac{1}{|Q|}\int_Q b$. The left side of (1.13) is no larger than

$$\begin{aligned} \Big(\frac{1}{|Q|}\int_Q (1+|b-b_Q|)\Big) \Big(\frac{1}{|Q|}\int_Q (1+|b|)^{-1/(p-1)}\Big)^{p-1} \\ + \Big(\frac{1}{|Q|}\int_Q \Big(\frac{|b_Q|}{1+|b|}\Big)^{1/(p-1)}\Big)^{p-1} . \end{aligned} \tag{1.14}$$

The first term in (1.14) is no larger than $1 + \|b\|_*$. We split the $1/(p-1)$ power of the second term in (1.14) into

$$\frac{1}{|Q|} \int_{\substack{Q \\ |b|>|b_Q|/2}} + \frac{1}{|Q|} \int_{\substack{Q \\ |b|\le|b_Q|/2}} = \text{I} + \text{II} .$$

Trivially, $\mathrm{I} \leq 2^{1/(p-1)}$. Now II equals

$$(1.15) \qquad \frac{1}{|Q|} \int_{\substack{Q \\ |b| \leq |b_Q|/2}} \left(\frac{|b_Q|}{1+|b|}\right)^{1/(p-1)} .$$

But the restriction on the domain of integration implies that $|b - b_Q| \approx |b_Q|$, so (1.15) is no larger than a constant times

$$\frac{1}{|Q|} \int_Q |b - b_Q|^{1/(p-1)} ,$$

and the lemma follows.

As another corollary of Theorem 1.10, we will use the techniques of Fabes, Jodeit and Riviere [FJR], to prove the compactness of the boundary double layer potential on "VMO_1" domains. That is, domains whose boundary is given in local coordinates by the graph of a function whose gradient belongs to VMO. We define the space VMO $(\mathbb{R}^n)$ by the property that $v \in$ VMO if and only if there exist continuous v_j with compact support such that $\|v - v_j\| \to 0$ (*i.e.* VMO is the BMO closure of the continuous functions with compact support). Let D be bounded VMO_1 domain in $\mathbb{R}^{n+1}$ and let Γ be its boundary. The boundary double layer potential on Γ is defined by

$$KF = \lim_{\varepsilon \to 0} K_\varepsilon f ,$$

where

$$(1.16) \qquad K_\varepsilon f(P) = C_n \int_{\{|P-Q|>\varepsilon\}\cap\Gamma} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^{n+1}} f(Q)\, d\sigma(Q) ,$$

and N_Q is the unit outer normal. We will prove the following

Theorem 1.17. *Let $D \subset R^{n+1}$ be a bounded* VMO_1 *domain with boundary Γ. Let K_ε be defined as in* (1.16). *Then $Kf = \lim_{\varepsilon\to 0} K_\varepsilon f$ exists almost everywhere on Γ and in $L^p(\Gamma)$ norm, and K is a compact operator on $L^p(\Gamma)$, $1 < p < \infty$.*

The paper is organized as follows. In the next Section we state some known results which be used in the sequel. In Section 3 we prove our main result Theorem 1.3. The proof of Theorem 1.6 is virtually identical to that of Theorem 1.3 and is left to the interested reader. In Section 4 we prove Theorems 1.8 and 1.10 and in Section 5 we discuss the compactness of the double layer potential on VMO_1 domains (Theorem 1.17).

2. Some useful known results.

We first recall a well known corollary, via the method of rotations, of the corresponding theorem in dimension one. Let T be the singular integral in (1.2), *i.e.*

$$(2.1) \qquad Tf(x) = \text{“p.v.”} \int E\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy\,,$$

with E and Ω as in (1.2), but where we now take A to be Lipschitz. Let $\tilde{T}$ be the singular integral defined in (1.5), again with A Lipschitz. We then have

Theorem 2.2. *For T and $\tilde{T}$ as above, and for all $1 < p < \infty$, the following norm inequalites hold:*

$$\text{(i)} \qquad \|Tf\|_p \le C(n,p)\,(1 + \|\nabla A\|_\infty)^\mu\, \|\Omega\|_\infty\, \|f\|_p$$

and

$$\text{(ii)} \qquad \|\tilde{T}f\|_p \le C(n,p)\,(1 + \|\nabla A\|_\infty)^{\tilde{\mu}}\, \|\nabla B\|_\infty\, \|\Omega\|_\infty\, \|f\|_p\,,$$

where μ and $\tilde{\mu}$ are absolute constants.

PROOF. The method of rotations. By invoking the 1-dimensional result of Murai [Mu2], we may take $\mu = 1$ in (i). Alternatively, one could give an intrinsically n-dimensional treatment by invoking the results in [H2] and following the argument in Section 3 below to bootstrap the Lipschitz constant. We also remark that in the Lipschitz case, the principal value exists almost everywhere, but we shall actually use only the fact that Theorem 2.2 holds for all truncations of T and $\tilde{T}$, with bound independent of the truncation.

We shall also use a $T1$ theorem for rough singular integrals proved by the author in [H], although in the present paper we shall require a less general version than that in [H]. In order to state this theorem, we first need to set some notation. Let $\psi \in C_0^\infty(|x| \le 1)$ be radial, non-trivial, real-valued, and have mean value zero, and set (with slight abuse of notation) $\Psi_s(|x|) = s^{-n}\psi(|x|/s)$. We assume that ψ is normalized so that

$$\int_0^{+\infty} \hat{\psi}(s)^2 \frac{ds}{s} = 1 .$$

If $Q_s f = \psi_s * f$, then Q_s satisfies the "Calderón-reproducing formula"

$$\int_0^{+\infty} Q_s^2 \frac{ds}{s} = I , \tag{2.3}$$

where the operator-valued integral converges in the strong operator topology on L^2, as may be verified by Plancherel's Theorem. Choose a non-negative $\varphi \in C_0^\infty(1/4, 1)$ so that φ defines a smooth partition of unity $\sum_{j=-\infty}^{+\infty} \varphi(2^{-j}r) = 1$, $r > 0$. Set

$$K_j(x,y) = K(x,y)\,\varphi(2^{-j}|x-y|) , \tag{2.4}$$

and define

$$T_j f(x) = \int K_j(x,y)\, f(y)\, dy ,$$

where K satisfies the size condition

$$|K(x,y)| \le C_1\, |x-y|^{-n} . \tag{2.5}$$

We also impose the following weak smoothness condition: assume that for all Q_s as above and $s \le 2^j$, we have, for some $\varepsilon > 0$

$$\|Q_s T_j 1\|_\infty \le C_2\, \|\psi\|_1 (2^{-j}s)^\varepsilon \tag{2.6}$$

and

$$\|Q_s T_j\|_{\mathrm{op}} \le C_3\, \|\psi\|_1 (2^{-j}s)^\varepsilon , \tag{2.7}$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the $L^2 \longrightarrow L^2$ operator norm. We then have

Theorem 2.8. *Suppose that $K(x,y)$ is anti-symmetric (i.e. $K(x,y) = -K(y,x)$), that $K(x,y)$ satisfies the size condition* (2.5) *and that T_j satisfies* (2.6) *and* (2.7). *We define truncated operators*

$$T_{N,M} = \sum_{j=N}^{M} T_j ,$$

and assume that $\|T_{N,M}1\|_* \leq C_4$ *(uniformly in* N *and* M*). Then for all* $1 < p < \infty$*, and* $w \in A_p$*, we have*

$$\|T_{N,M}f\|_{p,w} \leq C(n,p,A_p)\,(C_1 + C_2 + C_3 + C_4)\,\|f\|_{p,w}\,.$$

In particular, the bound is independent of the truncation.

A much more general version of this theorem is proved in [H] (see also [H2], where these ideas were implicit) but the above will be sufficient for our purposes.

For the safe of self-containment, we will sketch the proof of the special case used here. We will argue formally, and refer the reader to [H] for the details. Using the Calderón reproducing formula, and the partition of unity as above, we write

$$T = T_{N,M} = \int_0^{+\infty}\int_0^{+\infty}\sum_{j=N}^{M} Q_s^2 T_j Q_t^2\,\frac{ds}{s}\,\frac{dt}{t}\,.$$

It is enough to consider the case $s \leq t$ (the other case is dual to this one). By the extrapolation Theorem for A_p weights (see, *e.g.* [GR]), it is enough to consider the case $p = 2$. It suffices to show that for all $w \in A_2$ and $f, g \in C_0^\infty$ we have

$$\Big|\int_0^{+\infty}\int_0^{t}\sum_{j=N}^{M}\langle Q_s T_j Q_t^2 f, Q_s g\rangle\,\frac{ds}{s}\,\frac{dt}{t}\Big| \leq C(n,A_2)\,(C_1 + C_2 + C_3 + C_4)\,\|f\|_{2,w}\,\|g\|_{2,1/w}\,,$$

In the left side of this last inequality, we split the sum into $\sum_1 + \sum_2$, where $\sum_1$ runs over j: $2^j \geq s^\theta t^{1-\theta}$, and $0 < \theta < 1$ is to be chosen. By an idea of [DR], it can be shown that (2.5), (2.7) and an interpolation argument imply a weighted version of (2.7) with a smaller ε, *i.e.*

$$\|Q_s T_j f\|_{2,w} \leq C(n,A_2)\,(C_1 + C_3)\,(2^{-j}s)^\varepsilon\,\|f\|_{2,w}\,.$$

An application of Schwarz's inequality and weighted Littlewood-Payley theory now yield the desired estimate for $\sum_1$.

Next, to handle $\sum_2$, we consider

$$\mathrm{I} + \mathrm{II} + \mathrm{III} = \int_0^{+\infty}\int_0^{t}\langle Q_s \textstyle\sum_2 (T_j - T_j 1) Q_t^2 f, Q_s g\rangle\,\frac{ds}{s}\,\frac{dt}{t}$$

$$+\int_0^{+\infty}\int_0^t \langle Q_s T_{N,M}1\, Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t}$$
$$-\int_0^{+\infty}\int_0^t \langle Q_s \textstyle\sum_1 T_j 1\, Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t}.$$

The term III can be handled by a straightforward application of (2.6), Schwarz, and weithted Littlewood-Payley theory. In II, we use that

$$P_s = \int_s^{+\infty} Q_t^2 \frac{dt}{t}$$

is "nice", so the fact that $T_{N,M}1 \in \mathrm{BMO}$, combined with (weighted) Carleson measure theory yield the desired estimate in this case. Finally, to treat I, we write the kernel of $Q_s \sum_2 (T_j - T_j 1)$ as

$$\iint \psi_s(x-z)\sum\nolimits_2 K_j(z,u)\,(\psi_t(u-y)-\psi_t(x-y))\,du\,dz\,,$$

which we claim is dominated in absolute value by

$$C_n C_1 \left(\frac{s}{t}\right)^{\varepsilon} t^{-n} \chi_{\{|x-y|\le C_n t\}}$$

(so that the corresponding operator is controlled by $(s/t)^{\varepsilon}$ times the Maximal function, and the theorem follows. To prove the claim, we first observe that by definition of $\sum_2$, we have $|z-u| \le s^{\theta}t^{1-\theta}$. Thus, the integrand is unchanged if we multiply it by a smooth radial cut-off function $\eta(|x-u|/(s^{\theta}t^{1-\theta}))$, where $\eta = 1$ on $\{|x| < 10\}$ and vanishes if $|x| > 11$. Furthermore, it is well known (see, *e.g,* [DJo]), that (2.5) plus anty-symmetry imply the Weak Boundedness property

$$\text{(WBP)}\quad |\langle h, Tg\rangle| \le C\,C_1\,R^n(\|h\|_\infty + R\,\|\nabla h\|_\infty)\,(\|g\|_\infty + R\,\|\nabla g\|_\infty)\,,$$

for all $h, g \in C_0^\infty$ with support in any ball of radius R. The claim then follows by (WBP), with

$$\theta = \frac{n+1+\varepsilon}{n+2}\,, \qquad 0 < \varepsilon < 1\,,$$

and

$$h(z) = \psi_s(x-z)\,, \qquad g(u) = (\psi_t(u-y) - \psi_t(x-y))\,\eta\Big(\frac{|x-u|}{s^{\theta}t^{1-\theta}}\Big)\,.$$

There is one more result which we shall find useful in the sequel. It is an unpublished theorem of Mary Weiss, and the proof can be found in a paper of C. Calderón [CC, Lemma 1.4]. We define a maximal operator

$$D_*A(x) = \sup_{h\neq 0} \frac{|A(x+h)-A(x)|}{|h|} \,. \tag{2.9}$$

We have the following

Lemma 2.10 (M. Weiss). *Suppose that $q > n$, and $\nabla A \in L^q_{\mathrm{loc}}$ (the gradiend being defined in the weak sense). Then, for all $\gamma > 1$,*

$$\text{(i)} \quad \frac{|A(x)-A(y)|}{|x-y|} \leq C_{q,\gamma} \Big(\frac{1}{|x-y|^n} \int\limits_{|x-z|\leq \gamma |x-y|} |\nabla A(z)|^q \, dz \Big)^{1/q},$$

and

$$\text{(ii)} \quad \|D_*A\|_q \leq C_q \, \|\nabla A\|_q \,.$$

Remark. Since BMO $\subseteq L^q_{\mathrm{loc}}$, a standard argument involving Lemma 2.10.ii shows that if $A \in I_1(\mathrm{BMO})$, then ∇A exists almost everywhere.

Proof of Theorem 1.3.

Using the same notation as in Theorem 2.8, we set

$$K(x,y) = \cos\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^n} \,, \tag{3.1}$$

where Ω is odd and bounded, so that the size condition (2.5) and the anti-symmetry condition $K(x,y) = -K(y,x)$ are immediate (we will prove explicitly only the case $E(t) = \cos t$, Ω odd, as the proof in the other case is identical). We will prove that the truncated operators $T_{N,M}$ satisfy (1.4) with a bound independent of the truncation. Without loss of generality we may take $\|\Omega\|_\infty = 1$, so by Theorem 2.8 it is enough to verify the smoothness conditions (2.6) and (2.7) with constants $C_2, C_3 \leq C_n(1 + \|\nabla A\|_*)$, and to show that $T_{N,M}1 \in$ BMO with

$$\|T_{N,M}1\|_* \leq C_n \, (1 * \|\nabla A\|_*)^\mu \,, \tag{3.2}$$

uniformly in N and M, with the same μ as in Lemma 2.2.i. The proof of (2.7) will be deferred until the end of this section. We note in passing that in contrast to the case that A is Lipschitz, the kernel $K(x,y)$ need not, in general, satisfy "standard" smoothness estimates when $A \in I_1(\mathrm{BMO})$, even if $\Omega \in C^\infty$. It turns out that when one perturbs A by an appropriate linear term in order to get nice (local) estimates for A, one introduces a factor that is homogeneous of degree zero and which may have uncontrollably large regularity estimates. We will be forced then, to prove (2.7) with a bound depending only on the size of Ω. We shall return to this point below.

We now proceed to prove (3.2), and in the process we will also verify (2.6).

In order to prove (3.2), we first recall a characterization of BMO which appears in a paper of Stromberg [Sbg, Lemma 3.1 and its corollary] where the idea is attributed to F. John.

Lemma 3.3 (John-Stromberg). *Let b be measurable and assume that there exist $\alpha > 0$ and $0 < \gamma \leq 1/2$ such that for every cube Q there is a constant C_Q with*

$$|\{x \in Q : \; |b(x) - C_Q| > \alpha\}| \leq \gamma\, |Q| .$$

Then $b \in \mathrm{BMO}$ and $\|b\|_* \leq C_\gamma\, \alpha$.

We will prove (3.2) by verifying the conditions of Lemma 3.3 for $b = T_{N,M}1$, and with $\alpha = C\,(1 + \|\nabla A\|_*)^\mu$, α, γ independent of N, M. We note that the use of Lemma 3.3 to treat $T1$ via bootstrapping has appeared previously in [CM]. We fix a cube Q with center x_0 and side length s. Let $\eta \in C_0^\infty[-11\sqrt{n}, 11\sqrt{n}]$, and suppose $\eta = 1$ on $[-10\sqrt{n}, 10\sqrt{n}]$. We write

$$\begin{aligned} T_{N,M}1 = & \int_{\mathbb{R}^n} \cos\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n} \\ & \qquad \cdot \Phi_N^M(|x-y|)\,\Big(1-\eta\Big(\frac{|x-y|}{s}\Big)\Big)\,dy \\ & + \int_{\mathbb{R}^n} \cos\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n} \qquad (3.4) \\ & \qquad \cdot \Phi_N^M(|x-y|)\,\eta\Big(\frac{|x-y|}{s}\Big)\,dy \\ = & \; U(x) + V(x) , \end{aligned}$$

where $\Phi_N^M(r) = \sum_{j=N}^M \varphi(2^{-j}r)$, with the same φ as in the definition of K_j (see (2.4)).

We treat U first. Let Q_j be a cube with center x_0 and side length $10 \cdot 2^j$. Set

$$A_j(x) = A(x) - \vec{a}_j \cdot x\,,$$

where $\vec{a}_j = \frac{1}{|Q_j|}\int_{Q_j} \nabla A$. By an elementary trigonometric identity, $U(x)$ equals (in polar coordinates)

$$\begin{aligned}(3.5)\qquad &\sum_{j=N}^{M} \int_{S^{n-1}} \Omega_j(\theta) \int_0^{+\infty} \varphi(2^{-j}r)\\ &\qquad \cdot \cos\left(\frac{A_j(x) - A_j(x-r\theta)}{r}\right)\left(1-\eta\left(\frac{r}{s}\right)\right)\frac{dr}{r}\,d\theta\,,\end{aligned}$$

where $\Omega_j(\theta) = \cos(\vec{a}_j \cdot \theta)\,\Omega(\theta)$, minus another term with sine in place of cosine in (3.5) and in the definition of Ω_j. We discuss only (3.5), as the other term can handle by the same argument, and for simplicity we again designate (3.5) as $U(x)$. We now claim that

$$(3.6)\qquad \int_Q |U(x) - U(x_0)|\,dx \le C_n\,\|\nabla A\|_*\,|Q|\,.$$

Note that the integrand in (3.5) is zero unless $2^j \ge 10\sqrt{n}\,s$. But by Lemma 2.10, for $r \approx 2^j$ and $|x - x_0| \le \sqrt{n}\,s < 2^j/10$, we have for any $q > n$,

$$\begin{aligned}&|A_j(x) - A_j(x_0)| + |A_j(x-r\theta) - A_j(x_0 - r\theta)|\\ &\qquad \le C_n\,|x-x_0|\left(|x-x_0|^{-n}\int_{Q_j} |\nabla A|^q\right)^{1/q}\\ &\qquad \le C_n\,s^{1-n/q}\,2^{jn/q}\,\|\nabla A\|_*\,.\end{aligned}$$

The claim now follows by a straightforward computation (the reader should bear in mind that the j-th summand is vacuous unless $2^j \ge C_n\,s$; the same argument also proves (2.6) -in that case we treat what is essentially a single j term in (3.5)).

Thus, by Tchebychev's inequality and (3.6), we have

$$(3.7)\qquad |\{x \in Q:\ |U(x) - U(x_0)| > \beta_1\,(1 + \|\nabla A\|_*)^\mu\}| \le \frac{1}{10}\,|Q|\,,$$

for β_1 large enough and depending only upon dimension.

We now consider $V(x)$. For $x \in Q$, we write

$$\begin{aligned} V(x) &= \int \cos\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\frac{\Omega(x-y)}{|x-y|^n}\,\Phi_N^M(|x-y|)\,\chi_{10\,Q}(y)\,dy \\ &\quad + \int \cos\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\frac{\Omega(x-y)}{|x-y|^n} \\ &\qquad \cdot \Phi_N^M(|x-y|)\Big(\eta\Big(\frac{|x-y|}{s}\Big)-\chi_{10\,Q}(y)\Big)\,dy \\ &= W(x)+Y(x)\,. \end{aligned}$$

The domain of integration in $Y(x)$ is contained in an annulus centered at x with inner and outer radii comparable to s, so

$$|Y(x)| \le \beta_2 = \beta_2(n)\,, \tag{3.8}$$

uniformly in $x \in Q$.

The term W will be treated by a variant of the perturbation technique of G. David. We set

$$A_Q(x) = A(x) - \vec{a}_Q \cdot x\,,$$

where

$$\vec{a}_Q = \frac{1}{20\,Q}\int_{20\,Q}\nabla A\,.$$

Then

$$K(x,y) = \cos\Big(\frac{A_Q(x)-A_Q(y)}{|x-y|}\Big)\frac{\Omega_{1,Q}(x-y)}{|x-y|^n} = K_{1,Q}(x,y)\,, \tag{3.9}$$

where $\Omega_{1,Q}(x) = \cos(\vec{a}_Q \cdot x/|x|)\,\Omega(x)$, minus another term $K_{2,Q}$ with sine in place of cosine. This splitting of K into $K_{1,Q} - K_{2,Q}$ give rise to a splitting of W into $W_1 - W_2$. Since each term can be handled in the same way, we concentrate on

$$W_1(x) = \int K_{1,Q}(x,y)\,\Phi_N^M(|x-y|)\,\chi_{10\,Q}(y)\,dy\,.$$

Since $x \in Q$ and $y \in 10\,Q$, we can replace A_Q in this last expression by a function A_1 which agrees (modulo and additive constant) with A_Q on $10\,Q$, is supported in $20\,Q$, and satisfies

$$\int |\nabla A_1|^q \le C_{n,q}\,\|\nabla A\|_*^q \tag{3.10}$$

(see Cohen [Co, p. 698] for the details of the construction of such an A_1; see also $\tilde{A}$ in Corollary 5.6 below). Now by (3.10) and Lemma 2.10, we have that for $q > n$ and L a large number to be chosen

$$
\begin{aligned}
|\{x \in (10\,Q)^\circ :\ D_* A_1(x) > L\,\|\nabla A\|_*\}| & \\
&\le \frac{1}{(L\,\|\nabla A\|_*)^q} \int |D_* A_1|^q \\
&\le \frac{C_n\,|Q|}{L} \le \frac{6^{-n}|Q|}{100}\,,
\end{aligned}
\tag{3.11}
$$

for $L = L_n$ large enough and depending only on dimension. The Set

$$G = \{x \in (10\,Q)^\circ :\ D_* A_1 > L\,\|\nabla A\|_*\}$$

is open, and $A_1|_{G^c}$ is Lipschitz with constant $L\,\|\nabla A\|_*$. We can therefore make a Whitney extension [S, Chapter 6] of A_1, call it $\tilde{A}_1$, such $\tilde{A}_1$ and A_1 agree on G^c and $\tilde{A}$ is Lipschitz on $\mathbb{R}^n$ with constant $C_n\,L\,\|\nabla A\|_*$. We set

$$\tilde{K}_{1,Q} = \cos\Big(\frac{\tilde{A}(x) - \tilde{A}(y)}{|x-y|}\Big)\,\Omega_{1,Q}(x-y)$$

($\Omega_{1,Q}$ as in (3.9)), define

$$\tilde{W}_1(x) = \int \tilde{K}_{1,Q}(x,y)\,\Phi_N^M(|x-y|)\,\chi_{10\,Q}(y)\,dy\,, \tag{3.12}$$

and let $\tilde{W}_2$ be the analogous term with sine in place of cosine (the cosine in $\Omega_{1,Q}$ is of course also replaced by a sine). Then

$$
\begin{aligned}
\tilde{W}_1(x) &= \tilde{T}_1(\chi_{10\,Q})(x)\,,\\
\tilde{W}_2(x) &= \tilde{T}_2(\chi_{10\,Q})(x)\,,
\end{aligned}
$$

for appropriate (truncated) singular integral operators $\tilde{T}_1$, $\tilde{T}_2$, each with L^2 operator norm no larger than $C_n\,(1 + \|\nabla A\|_*)^\mu$ (see Theorem 2.2.i).

Let $G = \cup I_i$ be a truncated Whitney decomposition of G into non-overlapping cubes I_i with

$$\operatorname{diam} I_i \le \operatorname{dist}\{I_i, G^c\} \le 4\operatorname{diam} I_i\,,$$

and set $G^* = \cup 6I_i$. For $x \in G^{*c} \cap Q$, we have that

$$W_1(x) - \tilde{W}_1(x) = \sum_i \int_{I_i} \Big(\cos\Big(\frac{A_1(x) - A_1(y)}{|x-y|}\Big) - \cos\Big(\frac{A_1(x) - \tilde{A}(y)}{|x-y|}\Big)\Big) \cdot \frac{\Omega_{1,Q}(x-y)}{|x-y|^n}\, \Phi_N^M(|x-y|)\, dy\,,$$

where we have used that $A_1 = \tilde{A}$ on G^c. By the Whitney construction, we can select $y_i \in G^c$ such that $\operatorname{dist}\{y_i, I_i\} \leq 4 \operatorname{diam} I_i$, so that the first term in the integral is bounded in absolute value by

$$\begin{aligned} \frac{|A_1(y) - \tilde{A}(y)|}{|x-y|} &\leq \frac{|A_1(y) - A(y_i)|}{|x-y|} + \frac{|\tilde{A}(y_i) - \tilde{A}(y)|}{|x-y|} \\ &\leq (D_* A_1(y) + C_n L \|\nabla A\|_*) \frac{\operatorname{diam} I_i}{|x-y|}\,. \end{aligned} \tag{3.13}$$

If we set

$$R_1 = |W_1 - \tilde{W}_1|\,,$$

then by (3.12), for $x \in G^{*c}$, we have

$$R_1(x) \leq \sum_i \frac{d_i}{(d_i + |x-y_i|)^{n+1}} \int_{I_i} (C_n L \|\nabla A\|_* + D_* A_1(y))\, dy\,,$$

where $d_i = \operatorname{diam} I_i$, and we have used that for $x \in (6I_i)^c$, we have $|x-y| \approx |x-y_i|$. Then

$$\begin{aligned} \int_{G^{*c}\cap Q} R_1(x)\, dx &\leq C_n \int_{\cup I_i} (L \|\nabla A\|_* + D_* A_1(y))\, dy \\ &\leq C_n \Big(L \|\nabla A\|_* |Q| + \int_{10\,Q} D_* A_1(y)\, dy \Big)\,. \end{aligned}$$

By Hölder's inequality, (3.10) and Lemma 2.10, we have

$$\int_{G^{*c}\cap Q} R_1(x)\, dx \leq C_n L \|\nabla A\|_* |Q|\,, \tag{3.14}$$

and the same estimate holds for $\|R_2\|_{L^1(G^{*c}\cap Q)}$, where $R_2 = |W_2 - \tilde{W}_2|$.

By the definition of G^* and (3.11), we have that $|G^*| \leq |Q|/100$. We now take $C_Q = U(x_0)$ in Lemma 3.3. Then, by (3.7), for β_3 to be chosen, and with β_1, β_2 as in (3.7) and (3.8) respectively,

$$\begin{aligned} &|\{x \in Q : \; |T_N^M 1 - U(x_0)| \geq (\beta_1 + \beta_2 + \beta_3)(1 * \|\nabla A\|_*^\mu\}| \\ &\leq \frac{|Q|}{100} + \frac{|Q|}{10} + |\{x \in Q \cap G^{*c} : \; |\tilde{W}_1(x)| + |\tilde{W}_2(x)| \\ &\qquad\qquad + |R_1(x)| + |R_2(x)| > \beta_3(1 + \|\nabla A\|_*^\mu\}| \,. \end{aligned}$$

If we take β_3 large enough and depending only on dimension, the conclusion of Lemma 3.3 will follow by Tchebychev's inequality, (3.14) and its equivalent for R_2, and the fact that for $i = 1, 2$, $\tilde{W}_i = \tilde{T}_i(\chi_{10\,Q})$ where $\|\tilde{T}_i\|_{\text{op}} \leq C_n (1 + \|\nabla A\|_*)\mu$. The details are left to the reader. This concludes the proof of (3.2).

To finish the proof of Theorem 1.3, we need to verify (2.7); *i.e.* with Q_s as in (2.3), we will prove that for $s \leq 2^j$ and for some $\varepsilon > 0$,

$$\|Q_s T_j\|_{\text{op}} \leq C_n \, \|\psi\|_1 \, (2^{-j} s)^\varepsilon \, (1 + \|\nabla A\|_*) \,, \tag{3.15}$$

where we recall that $T_j f(x) = \int K_j(x, y) \, f(y) \, dy$, and

$$K_j(x, y) = \cos\Big(\frac{A(x) - A(y)}{|x - y|}\Big) \, \frac{\Omega(x - y)}{|x - y|^n} \, \varphi(2^{-j}|x - y|) \,.$$

The proof of (3.15) follows some ideas of Christ and Journé [CJ, estimate (5.4) and its proof, and also personal communication]. More generally, we shall consider kernels $K(x, y)$ which have the following property: we assume that for every cube I (with sides parallel to the coordinates axes), the kernel has the following interpretation:

$$K(x, y) = \frac{\Omega_I(x - y)}{|x - y|^n} \, \sigma_I(x, y) \,, \qquad (x, y) \in I \times I \,, \tag{3.16}$$

where Ω_I is homogeneous of degree zero and essentially bounded on the sphere and where σ_I has compact support and belongs to the Sobolev space $L^2_\varepsilon(\mathbb{R}^n \times \mathbb{R}^n)$, for some $\varepsilon > 0$ and with L^2_ε norm $C\,|I|^{1-\varepsilon/n}$. We observe that for $(x, y) \in I \times I$, the kernels of Theorem 1.3 can be written in the form (3.16) as a difference of two terms, the first of which has

$$\Omega_I(\theta) = \cos(\vec{a}_I \cdot \theta) \, \Omega(\theta) \,,$$

and

$$\sigma_I(x,y) = \cos\Big(\frac{A_I(x) - A_I(y)}{|x-y|}\Big)\,\eta_I(x,y)\,, \tag{3.17}$$

where

$$\vec{a}_I = \frac{1}{|I|}\int_I |\nabla A|\,, \qquad A_I(x) = A(x) - \vec{a}_I \cdot x\,,$$

and where for $(x,y) \in I \times I$ we are permitted to multiply by a smooth cut-off function η_I which equals 1 on $I \times I$ and vanishes on the complement of $2I \times 2I$. The second term is the same as this one but with sines in place of cosines. By Lemma 2.10, it is routine to verify that the particular σ_I in (3.17) satisfies, for all $0 < \varepsilon < 1$,

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\sigma_I(x+u,y+v) - \sigma_I(x,y)|^2}{(|u|+|v|)^{2n+2\varepsilon}}\,dx\,dy\,du\,dv \\ \leq C_\varepsilon\,|I|^{2(1-\varepsilon/n)}(1 + \|\nabla A\|_*)^2\,, \end{aligned} \tag{3.18}$$

with a constant C_ε independent of I (as long as η_I is defined in terms of translates and dilates of some fixed "mother" η). We remark that in particular, (3.18) holds if

$$\begin{aligned} \iiint\limits_{\substack{x,y,h\in I\\ |h|<s}} (|\sigma_I(x+h,y) - \sigma_I(x,y)|^2 + |\sigma_I(x,y+h) - \sigma_I(x,y)|^2)\,dx\,dy\,dh \\ \leq C\,|I|^2\Big(\frac{s}{|I|^{1/n}}\Big)^{2\delta}\,, \end{aligned}$$

for some $\delta > \varepsilon$ and C independent of I. The latter estimate is [CJ, (5.4)]. It is now enough to prove the following

Lemma 3.19. *Let (as usual) $K_j(x,y) = K(x,y)\,\varphi(2^{-j}|x-y|)$, where for all I, $K(x,y)$ has the representation* (3.16), *with $\|\Omega_I\|_\infty \leq 1$, and*

$$\|\sigma_I\|_{L^2_\varepsilon(\mathbb{R}^n\times\mathbb{R}^n)} \leq C_0\,|I|^{1-\varepsilon/n}\,,$$

for some $\varepsilon > 0$ and C_0 independent of I. Then, for

$$T_j f(x) = \int K_j(x,y)\,f(y)\,dy\,,$$

we have that for some $\tilde{\varepsilon} > 0$,

$$\|Q_s T_j\|_{\mathrm{op}} \le C_n C_0 \|\psi\|_1 \Big(\frac{s}{2^j}\Big)^{\tilde{\varepsilon}}, \qquad s < 2^j .$$

Proof of Lemma 3.19. By dilation invariance, it is enough to consider the case $j = 0$. We decompose $\mathbb{R}^n$ into a mesh of non-overlapping unit cubes, $\mathbb{R}^n = \cup I_i$, so that $f = \sum_i f \chi_{I_i}$ almost everywhere. Then for $s \le 1$, $Q_s T_0 f \chi_{I_i}$ is supported in $10\sqrt{n}\, I_i$, so that the terms $Q_s T_0 f \chi_{I_i}$ have "bounded overlaps". Thus we have the orthogonality property

$$\int \Big| \sum_i Q_s T_0 f \chi_{I_i} \Big|^2 \le C_n \sum_i \int |Q_s T_0 f \chi_{I_i}|^2 .$$

It is therefore enough to prove that for f supported in any unit cube I_0

$$\int |Q_s T_0 f|^2 \le C_n C_0 \|\psi\|_1 s^{\tilde{\varepsilon}} \int |f|^2 .$$

We now fix such a unit cube I_0, assume that f is supported there, and write $K_0(x,y) = K(x,y)\varphi(|x-y|)$, where $K(x,y)$ has the representation (3.16) with $I = 5\sqrt{n}\, I_0$. By the Sobolev space estimate for σ_I, we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{\sigma}(\xi,\tau)|^2 (1 + |\xi| + |\tau|)^{2\varepsilon} \, d\xi \, d\tau \le C_0^2 , \tag{3.20}$$

where for simplicity of notation we will now write $\sigma = \sigma_I$, since I is fixed. We can then decompose $\sigma(x,y) = g(x,y) + h(x,y)$, where

$$g(x,y) = (\chi_{\{|\xi|+|\tau| > s^{-\delta}\}} \hat{\sigma}(\xi,\tau))\check{}\,(x,y)$$

and

$$h(x,y) = (\chi_{\{|\xi|+|\tau| \le s^{-\delta}\}} \hat{\sigma}(\xi,\tau))\check{}\,(x,y) ,$$

$\delta > 0$ to be chosen. then by Plancherel and (3.24),

$$\|g\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \le C_0 \, s^{\delta\varepsilon} . \tag{3.21}$$

We are now ready to estimate $\|Q_s T_0 f\|_2^2$. By Schwarz, the part of this expression with g in place of σ is dominated by

$$\text{(3.22)}\qquad \int\Big(\iint |\psi_s(x-z)|\,|g(z,u)|^2\,dz\,du\Big) \cdot \Big(\iint |\psi_s(x-z)|\,|f(u)|^2\,du\,dz\Big)\,dx\,.$$

The second factor in brackets equals $\|\psi\|_1\|f\|_2^2$. By (3.21), we then have that (3.22) is no larger than $C\,C_0\|\Omega\|_\infty\|\psi\|_1^2\,s^{2\delta\epsilon}\,\|f\|_2^2$, which is the desired estimate for this term.

We now consider the part of $\|Q_s T_0 f\|_2$ with h in place of σ. By the definition of h and Minkowski's inequality, this term is bounded by

$$\text{(3.23)}\qquad \iint_{|\xi|+|\tau|\le s^{-\delta}} |\hat\sigma(\xi,\tau)|\,\Big(\int\Big|\iint \psi_s(x-z)\,e^{-2\pi i z\cdot\xi}k(z-u) \cdot e^{-2\pi i u\cdot\tau}f(u)\,du\,dz\Big|^2 dx\Big)^{1/2} d\xi\,d\tau\,,$$

where

$$\text{(3.24)}\qquad k(x)=\frac{\Omega(x)}{|x|^n}\,\varphi(|x|)\,.$$

(Here we have taken $\Omega=\Omega_I$, since I is fixed). We write

$$\text{(3.25)}\qquad \begin{aligned}\int \psi_s(x-z)\,e^{-2\pi i z\cdot\xi}k(z-u)\,dz &= \int \psi_s(x-z)\,(e^{-2\pi i z\cdot\xi}-e^{-2\pi i x\cdot\xi})\,k(z-u)\,dz\\ &\quad + e^{-2\pi i x\cdot\xi}\psi_s * k(x-u)\,.\end{aligned}$$

By Fourier transform estimates of Duoandikoetxea and Rubio de Francia [DR, Section 4], the L^2 operator norm of $f\mapsto \psi_s * k * f$, for k of the form (3.24), is dominated by

$$C_\alpha\,\|\psi\|_1\,s^\alpha\,,\qquad 0<\alpha<1\,.$$

We note that this last estimate does not require that Ω have mean value zero. The desired estimate for the part of (3.23) corresponding to the second term in (3.25) then follows easily if we take δ small enough.

The first term in (3.25) is bounded in absolute value by the pointwise estimate

$$C\,|\xi|\,s\,\|\psi\|_1 \chi_{\{|x-u|\leq 2\}}\,,$$

and a routine computation yields the conclusion of Lemma 3.19.

4. Proofs of Theorems 1.8 and 1.10.

We prove Theorem 1.8 first, by means of a well known technique (see, *e.g.* [CM, pp. 33-34]). We consider only the case that F is even and Ω is odd, the proof in the other case being virtually identical.

Once again, we interpret the principal value in a weak sense, treating truncated operators and obtaining bounds independent of the truncation, and without loss of generality we take $\|\Omega\|_\infty = 1$. We set

$$T_{N,M}[A] = \sum_{j=N}^{M} T_j\,,$$

where T_j has kernel

$$K_j(x,y) = F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n}\,\varphi(2^{-j}|x-y|)\,.$$

Following [CM], we write

$$F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) = C\int_{-\infty}^{+\infty}\hat{F}(\xi)\,\cos\Big(\xi\,\frac{A(x)-A(y)}{|x-y|}\Big)\,d\xi\,,$$

where in the Fourier inversion formula we have used that F is even. But by Theorem 1.3, the operator $T_{N,M,\xi}$ defined by

$$T_{N,M,\xi}f(x) = \sum_{j=N}^{M}\int\cos\Big(\xi\,\frac{A(x)-A(y)}{|x-y|}\,\frac{\Omega(x-y)}{|x-y|^n}\Big)\varphi(2^{-j}|x-y|)\,f(y)\,dy$$

satisfies, for all $1<p<\infty$ and $w\in A_p$,

$$\|T_{N,M,\xi}f\|_{p,w} \leq C(n,p,A_p)\,(1+\|\nabla A\|_*|\xi|)^\mu\,\|f\|_{p,w}\,. \tag{4.1}$$

The Theorem then follows by a straightforward argument involving Fubini's Theorem, Minkowski's integral inequality, (4.1) and the fact that we have imposed enough regularity upon F that $|\hat{F}(\xi)| \leq C\,(1 + |\xi|)^{-\mu-2}$.

PROOF OF THEOREM 1.10. We first state a lemma which can be deduced from Theorem 1.6 in exactly the same way that Theorem 1.8 followed from Theorem 1.3. The proof is left to the reader. We recall that $\Phi_N^M(r) = \sum_{j=N}^M \varphi(2^{-j}r)$.

Lemma 4.2. *Let $\tilde{T}_{N,M}[A,B]$ be defined by*

$$\tag{4.3} \begin{aligned} \tilde{T}_{N,M}[A,B]\,f(x) = \int (B(x) - B(y))\,F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) \\ \cdot \frac{\Omega(x-y)}{|x-y|^{n+1}}\,\Phi_N^M(|x-y|)\,f(y)\,dy\,, \end{aligned}$$

where B is Lipschitz, $A \in I_1(\mathrm{BMO})$, and $\Omega \in L^\infty(S^{n-1})$. We also assume that F and Ω are either both odd, or both even and that $F \in C^{\tilde{\mu}+2}$ where F and its first $\tilde{\mu}+2$ derivatives belong to L^1 (for the same $\tilde{\mu}$ as in Theorem 1.6). Then for all $1 < p < \infty$ and $w \in A_p$,

$$\tag{4.4} \begin{aligned} &\|\tilde{T}_{N,M}[A,B]\,f\|_{p,w} \\ &\qquad \leq C(n,p,F,A_p)\,\|\nabla B\|_\infty\,(1 + \|\nabla A\|_*)^{\tilde{\mu}}\,\|\Omega\|_\infty\,\|f\|_{p,w}\,, \end{aligned}$$

uniformly in N and M.

In order to apply G. David "good-λ" techniques to prove Theorem 1.10, we shall want to control certain appropriate maximal singular integrals. We let $T_*[A]$ be the maximal singular integral corresponding to $T[A]$ in (1.1), *i.e.*

$$\tag{4.5} \begin{aligned} T_*[A]f(x) = \sup_{N,M\in\mathbb{Z}} \Big| \int F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y}{|x-y|^n} \\ \cdot\,\Phi_N^M(|x-y|)\,f(y)\,dy\Big|\,, \end{aligned}$$

where we now suppose that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, that $F(t) \leq C\,(1-|t|)^{-1}$ and otherwise A, F and Ω are as in Theorem 1.8. We also define

$$\tag{4.6} \tilde{T}_*[A,B]\,f = \sup_{N,M} \big|\tilde{T}_{N,M}[A,B]\,f\big|\,,$$

with $\tilde{T}_{N,M}[A,B]$ as in (4.3) but again with $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ and $F(t) \leq C\,(1-|t|)^{-1}$. Even though the kernels of these operators are not quite standard (since A need not be Lipschitz) the decay of F at infinity still permits us to prove

Lemma 4.7. *With $T_*[A]$ and $\tilde{T}_*[A,B]$ defined in (4.5) and (4.6) respectively, and now assuming that $\Omega \in Lip_\alpha(S^{n-1})$ for some $0 < \alpha \leq 1$, and $F(t) \leq (1+|t|)^{-1}$ we have, for all $1 < p < \infty$,*

$$(4.8) \qquad \|T_*[A]\,f\|_p \leq C(n,p,F,\Omega)\,(1+\|\nabla A\|_*)^\mu\,\|f\|_p$$

and

$$(4.9) \qquad \|T_*[A,B]\,f\|_p \leq C(n,p,F,\Omega)\,\|\nabla B\|_\infty\,(1+\|\nabla A\|_*)^{\tilde{\mu}}\,\|f\|_p\,.$$

PROOF OF LEMMA 4.7. We prove only (4.8) as the other inequality has the same proof. The proof will be based on a well known inequality of Cotlar, which can be obtained by a very slight modification of the usual arguments for standard kernels. We begin by observing that for $f \in L^p$, the limit

$$T_N[A]\,f = \lim_{M\to\infty} T_{N,M}[A]\,f$$

exists pointwise as an almost everywhere convergent integral, and by Fatou's Lemma and Theorem 1.8,

$$(4.10) \qquad \|T_N[A]\,f\|_p \leq C(n,p)\,(1+\|\nabla A\|_*)^\mu\,\|\Omega\|_\infty\,\|f\|_p\,,$$

even without imposing the smootness assumption on Ω or the decay condition $F(t) \leq C\,(1+|t|)^{-1}$. Furthermore, the difference between sharp and smooth truncations is controlled by the maximal function, so it is enough to consider the maximal singular integral

$$T_*f = \sup_{\varepsilon>0} |T_\varepsilon f|\,,$$

where

$$(4.11) \qquad T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n}\,f(y)\,dy\,.$$

Now, by (4.10), T_ε is uniformly bounded on L^p, so by Banach-Alaouglu, there exists a subsequence $\varepsilon_j \to 0$ (depending on f), such that $T_{\varepsilon_j} f$

converges in the weak-$*$ sense to something which we call Tf, and furthermore Tf satisfies the bound (4.10). With this definition of Tf, we claim that the following Cotlar inequality holds almost everywhere:

$$\tag{4.12} \begin{aligned} T_*f(x) \le C(n,\delta,\Omega,F)\Big(&MTf(x) \\ &+ (1+\|\nabla A\|_*)^\mu \big(M(|f|^{1+\delta})(x)\big)^{1/(1+\delta)}\Big), \end{aligned}$$

for all $\delta > 0$. In fact (4.12) can be proved by a small modification of the argument in [Jo, pp. 56-57], once we establish the following lemma. The lemma says essentially that in the present setting our kernels are almost standard.

Lemma 4.13. *Let*

$$K(x,y) = F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)}{|x-y|^n},$$

with F, A and Ω as in Lemma 4.7. Then

$$\begin{aligned} \int\limits_{|x-y|>2\,|x-x'|} &|K(x,y)-K(x',y)|\,|f(y)|\,dy \\ &\le C(n,\delta,F,\Omega)\,(1+\|\nabla A\|_*)\,\big(M(|f|^{1+\delta})(x)\big)^{1/(1+\delta)}, \end{aligned}$$

for all $\delta > 0$.

PROOF OF LEMMA 4.13. We set $r = |x-x'|$, and split the integral into

$$\sum_{j=1}^{\infty} \int\limits_{2^j r<|x-y|\le 2^{j+1} r} = \sum_{j=1}^{\infty}\int_{R_j}$$

It suffices to show that

$$\begin{aligned} \int_{R_j} &|K(x,y)-K(x',y)|\,|f(y)|\,dy \\ &\le C(n,\delta,F,\Omega)\,(1+\|\nabla A\|_*)\big(M(|f|^{1+\delta})(x)\big)^{1/(1+\delta)}\,2^{-j\theta}, \end{aligned}$$

for some $\theta > 0$ and depending on δ. Let B be the ball of radius $3r$ and center x, and set

$$m_B(\nabla A) = \frac{1}{|B|}\int_B \nabla A.$$

There are three cases.

Case 1: $|m_B(\nabla A)| \leq 2^{j\varepsilon}(\|\nabla A\|_* + 1)$, where we have fixed $0 < \varepsilon < 1$. In this case, Lemma 2.10 implies that, for $q > n$,

$$\begin{aligned} |A(x) - A(x')| &\leq C_{q,n} \, |x - x'|^{1-n/q} \, r^{n/q} \\ &\quad \cdot \Big(r^{-n} \int_B |\nabla A - m_B(\nabla A) + m_B(\nabla A)|^q \Big)^{1/q} \\ &\leq C_{q,n} \, r \, (1 + 2^{j\varepsilon}) \, (1 + \|\nabla A\|_*) \, , \end{aligned}$$

and also

$$\begin{aligned} |A(x) - A(y)| &\leq C_{q,n} \, |x - y| \\ &\quad \cdot \Big(\frac{1}{(2^j r)^n} \int_{2^j B} |\nabla A - m_{2^j B}(\nabla A) + m_{2^j B}(\nabla A)|^q \Big)^{1/q} \\ &\leq C_{q,n} \, |x - y| \, (1 + j + 2^{j\varepsilon}) \, (1 + \|\nabla A\|_*) \, , \end{aligned}$$

where in the last inequality we have used a well known property of BMO to obtain the bound

$$\begin{aligned} m_{2^j B}(\nabla A) &= m_{2^j B}(\nabla A) - m_B(\nabla A) + m_B(\nabla A) \\ &\leq C_n \, j \, \|\nabla A\|_* + 2^{j\varepsilon}(1 + \|\nabla A\|_*) \, . \end{aligned}$$

The claim follows in the present case with $\theta = \min\{1 - \varepsilon, \alpha)$ and $\delta = 0$, by a standard argument using the smoothness of Ω and the fact that F' is bounded (since $\xi \, \hat{F}(\xi) \in L^1$).

Case 2. $|m_B(\nabla A) \geq 2^{j\varepsilon}(1 + \|\nabla A\|_*)$ and

$$\Big| \frac{x - y}{|x - y|} \cdot \frac{m_B(\nabla A)}{|m_B(\nabla A)|} \Big| \geq 2^{-j\varepsilon/2} \, .$$

In this case

$$\Big| \frac{x - y}{|x - y|} \cdot m_B(\nabla A) \Big| \geq 2^{j\varepsilon/2}(1 + \|\nabla A\|_*) \, .$$

But by Lemma 2.10 ,

$$\begin{aligned} \frac{|A(x) - A(y) - m_B(\nabla A) \, (x - y)|}{|x - y|} &\leq C_n \, j \, \|\nabla A\|_* \\ &\ll \Big| \frac{x - y}{|x - y|} \cdot m_B(\nabla A) \Big| \, . \end{aligned} \tag{4.15}$$

Thus

$$\frac{|A(x)-A(y)|}{|x-y|} > C_n\, 2^{j\varepsilon/2}(1+\|\nabla A\|_*)\,.$$

The same estimate holds also for $(A(x')-A(y))/|x-y|$, and by the same argument, since

$$(4.16)\qquad \begin{aligned} \Big|\frac{m_B(\nabla A)}{|m_B(\nabla A)|}\cdot\Big(\frac{x-y}{|x-y|}-\frac{x'-y}{|x'-y|}\Big)\Big| &\le C_n\, 2^{-j} \\ &\ll \Big|\frac{x-y}{|x-y|}\cdot\frac{m_B(\nabla A)}{|m_B(\nabla A)|}\Big| \end{aligned}$$

if $\varepsilon < 1$. Thus, by the decay assumption of F,

$$\Big|F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\Big| + \Big|F\Big(\frac{A(x')-A(y)}{|x'-y|}\Big)\Big| \le \frac{C_n}{2^{j\varepsilon/2}(1+\|\nabla A\|_*)}\,.$$

We then obtain (4.14) in the present case, with $\delta = 0$ and $\theta = \varepsilon/2$.

Case 3. $\Big|\dfrac{x-y}{|x-y|}\cdot\dfrac{m_B(\nabla A)}{|m_B(\nabla A)|}\Big| \le 2^{-j\varepsilon/2}.$

In this case, with $\vec{w}_0 = m_B(\nabla A)/|m_B(\nabla A)|$, by Hölder's inequality and a change to polar coordinates we have

$$\begin{aligned} &\int_{R_j\cap\{|\frac{x-y}{|x-y|}\cdot\vec{w}_0|\le 2^{-j\varepsilon/2}} |K(x,y)\, f(y)|\, dy \\ &\qquad \le C_n\Big(\frac{1}{2^{jn}r^n}\int_{|\theta\cdot\vec{w}_0|\le 2^{-j\varepsilon/2}} d\theta \int_{2^j r}^{2^{j+1}r} \rho^{n-1}\, d\rho\Big)^{\delta/(1+\delta)} \\ &\qquad\quad \cdot \|F\|_\infty\, M\big(|f|^{1+\delta}\big)^{1/(1+\delta)}(x)\,. \end{aligned}$$

By the first inequality in (4.16), a slight variation of this argument can be used to consider $K(x',y)$. This concludes the proof of Lemma 4.13, and in the present case $\theta = \varepsilon\,\delta/(1+\delta)$.

The Cotlar inequality (4.12) can now be deduced by a rather straightforward adaptation of the argument in [Jo, pp. 56-57]. We leave the details to the reader. Lemma 4.7 then follows immediately.

With the maximal singular integrals under control, we are now in position to apply the "good-λ" perturbation techniques of G. David in a rather straightforward way to establish Theorem 1.10. Since the ideas are familiar, we shall be as brief as possible.

As usual we perform a Whitney decomposition of the set

$$E_\lambda = \{T_*[A,B]\, f > \lambda\} = \cup Q_j\,,$$

where the Q_j's are non-overlapping and

$$\operatorname{diam} Q_j \le \operatorname{dist}\{Q_j, E_\lambda^c\} \le 4 \operatorname{diam} Q_j\,.$$

Here $T_*[A,B]$ is defined in Theorem 1.10, with $A, B \in I_1(\mathrm{BMO})$. We fix $Q = Q_j$, and it is enough to prove

$$\begin{aligned}(4.17)\qquad |\{x \in Q:\ T_*[A,B]\, f > 3\,\lambda,\ M(|f|^{1+\delta})^{1/(1+\delta)} \le \gamma\,\lambda\}| \\ \le C(n,\Omega,F,\delta)\,\varepsilon_0\,|Q|\,,\end{aligned}$$

for any fixed $\delta > 0$ and for some suitably small, fixed ε_0, and γ to be chosen depending upon ε_0, where without loss of generality we take $\|\nabla B\|_* = 1$.

We may assume there is an $x_0 \in Q$ such that

$$M(|f|^{1+\delta})^{1/(1+\delta)}(x_0) \le \gamma\,\lambda\,,$$

or else the left side of (4.17) is zero. For an appropriate $\tilde{x} \in E_\lambda^c$, we let $\tilde{\beta}$ be the ball with center $\tilde{x}$ and radius $10 \operatorname{diam} Q$. We write $f = f_1 + f_2$, where

$$f_1 = f\chi_{\tilde{\beta}}\,, \qquad f_2 = f\chi_{\tilde{\beta}^c}\,.$$

If $\tilde{x} \in E_\lambda^c$ is chosen so that

$$\operatorname{dist}\{\tilde{x}, Q\} \le 4 \operatorname{diam} Q\,,$$

then, by essentially the same argument as that used to prove Lemma 4.13,

$$\begin{aligned}(4.18)\qquad &|T_\varepsilon[A,B]\, f_2(x) - T_\varepsilon[A,B]\, f_2(\tilde{x})| \\ &\quad \le C_n\, M(|f|^{1+\delta})^{1/(1+\delta)}(x_0) \\ &\qquad + C(n,\delta,F,\Omega)\,(1 + \|\nabla A\|_*)\, M(|f|^{1+\delta})^{1/(1+\delta)}(x_0)\,,\end{aligned}$$

because x, $\tilde{x}$ and x_0 are all far from the support of f_2. The details are left to the reader, but we do point out that when following the proof of Lemma 4.13, we write

$$B(x)-B(y)-\nabla B(y)\cdot(x-y)=(B(x)-B(y)-m_{\tilde{Q}}(\nabla B)\cdot(x-y) \\ +(m_{\tilde{Q}}(\nabla B)-\nabla B(y))\cdot(x-y),$$

where $\tilde{Q}$ is an appropriate dilate of Q. The first term on the right side of the las expression is "locally standard", and the term $(m_{\tilde{Q}}(\nabla B)-\nabla B(y))\,f(y)$ can be handled by Hölder's inequality. We also mention that when treating (4.18), there is an error term, controlled by $M(|f|^{1+\delta})^{1/(1+\delta)}(x_0)$, which arises when integrating over an appropriate symmetric difference. Since $\tilde{x}\in E_\lambda^c$,

$$T_*[A,B]\,f_2(x)\le\lambda+(4.18)\le\lambda\,(1+\gamma\,C\,(1+\|\nabla A\|_*)). \tag{4.19}$$

To handle f_1, we set

$$B_Q(x)=B(x)-\Big(\frac{10^{-n}}{|Q|}\int_{10\,Q}\nabla B\Big)\cdot x=B(x)-m_{10\,Q}(\nabla B)\cdot x$$

and we write

$$\begin{aligned} T_\varepsilon[A,B]\,f_1(x)=&\int\limits_{|x-y|>\varepsilon}(B_Q(x)-B_Q(y))\,F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\\ &\qquad\cdot\frac{\Omega(x-y)}{|x-y|^{n+1}}\,f_1(y)\,dy\\ &+\int\limits_{|x-y|>\varepsilon}F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\,\frac{\Omega(x-y)\,(x-y)}{|x-y|^{n+1}}\\ &\qquad\cdot\big(m_{10\,Q}(\nabla B)-\nabla B(y)\big)\,f_1(y)\,dy\\ =&\,\tilde{T}_\varepsilon[A,B_Q]\,f_1(x)+T_\varepsilon[A]\,\big((m_{10\,Q}(\nabla B)-\nabla B)\,f_1\big)(x). \end{aligned}\tag{4.20}$$

By a direct application of (4.8), the supremum in ε of the absolute value of the second term in (4.20) is bounded in L^p norm by the $1/p$ power of

$$\begin{aligned} &C(n,p,F,\Omega)\,(1+\|\nabla A\|_*)^{\mu p}\int_{10\,Q}(|m_{10\,Q}(\nabla B)-\nabla B|\,|f|)^p\\ &\le C(n,p,F,\Omega)(1+\|\nabla A\|_*)^{\mu p}|Q|\,\|\nabla B\|_*^p\,M(|f|^{1+\delta})^{1/(1+\delta)}(x_0), \end{aligned}\tag{4.21}$$

for p chosen so that $1 < p < 1+\delta$.

To handle the first term in (4.20), we repeat the argument used in the proof of Theorem 1.3 to approximate B_Q by a Lipschitz function $\tilde{B}$ with

$$\|\nabla \tilde{B}\|_\infty \le C_n\, L\, \|\nabla B\|_* = C_n\, L\,,$$

where L depends only on ε_0 and dimension, and such that

$$|\{x \in 10\,Q:\ \tilde{B} \neq B_Q\}| \le \frac{6^{-n}|Q|}{100}\,\varepsilon_0\,.$$

(See (3.12) and the related discussion). Then, for x in a subset G^{*c} of Q with measure at least $(100-\varepsilon_0)|Q|/100$, we have

$$\sup_\varepsilon |\tilde{T}_\varepsilon[A, B_Q]\, f_1(x)| \le Rf(x) + \tilde{T}_*[A, \tilde{B}]\, f_1(x)\,,$$

where by (4.9)

$$\int_Q (\tilde{T}_*[A,\tilde{B}]\, f_1)^{1+\delta} \le C(n,p,F,\Omega)\,(1+\|\nabla A\|_*)^{\tilde{\mu}(1+\delta)} L\,|Q|\, M(|f|^{1+\delta})(x_0)\,,$$

and, for $x \in Q \cap G^{*c}$,

$$Rf(x) \le C(n, \|F\|_\infty) \sum_i \frac{d_i}{d_i + |x-y|^{n+1}} \cdot \int_{I_i} (C_n\, L + D_* B_Q(y))\, |f(y)|\, dy\,,$$

where $d_i = \operatorname{diam} I_i$, and $\cup I_i$ is the Whitney decomposition of the set

$$G = \{x \in 10\,Q:\ \tilde{B} \neq B_Q\}\,.$$

Thus

$$\int_{Q\cap G^{*c}} Rf(x)\, dx \le C_n\, L\, \|F\|_\infty\, M(|f|^{1+\delta})^{1/(1+\delta)}(x_0)\, |Q|\,. \tag{4.23}$$

By combining the estimates (4.19), (4.21), (4.22) and (4.23), and using the fact that $M(|f|^{1+\delta})^{1/(1+\delta)}(x_0) \le \gamma\,\lambda$, we can deduce (4.17) by standard arguments; we take

$$\gamma = \frac{\varepsilon_0}{(1+\|\nabla A\|_*)^\nu\, L}$$

for a suitable ε_0.

5. Compactness of the Boundary Double Layer Potential on bounded VMO_1 domains: Proof of Theorem 1.17.

Given Theorem 1.10, the proof is a relatively straightforward modification of the techniques of Fabes, Jodeit and Riviere [FJR]. We will essentially follow their argument, except for some small technical differences which arise in the VMO_1 case.

We begin by proving a series of elementary lemmas, which say essentially that a VMO_1 function A can be approximated (locally) by $C_0^{1,\alpha}$ functions, $0 < \alpha < 1$. Here $C_0^{1,\alpha} = \{$compactly supported $f \in L^1$ with $|\nabla f(x) - \nabla f(y)| \leq C\,|x-y|^\alpha\}$. Although by definition there exists a sequence of compactly supported continuous vector fields converging to ∇A in BMO norm, and each term in this sequence is in turn uniformly approximable by C_0^∞ functions, it is not immediately evident that these smooth vector fields are conservative. That is why we take this more circuitous route.

Lemma 5.1. *Let $b \in \mathrm{BMO}$. Fix a cube I and choose a smooth function η such that $\eta = 1$ on I, $\eta = 0$ on $(2I)^c$, $0 \leq \eta \leq 1$ and $\|\nabla \eta\|_\infty \leq C_n\,(\operatorname{diam} I)^{-1}$. Then $\eta\,(b - m_I b) \in \mathrm{BMO}$, with*

$$\|\eta\,(b - m_I b)\|_* \leq C_n\,\|b\|_*\,,$$

where $m_I b = |I|^{-1} \int_I b$.

PROOF. Let Q be a cube which meets $2I$. There are two cases. The first is trivial: if $|Q| \geq 2^n |I|$, then

$$\frac{1}{|Q|}\int_Q |\eta\,(b - m_I b)| \leq \frac{1}{|2I|}\int_{2I} |b - m_I b| \leq C_n\,\|b\|_*\,.$$

Case 2. $|Q| < 2^n |I|$. We set $b_I = b - m_I b$. Then

$$\frac{1}{|Q|}\int_Q |\eta\, b_I - m_Q(\eta\, b_I)| \leq \frac{1}{|Q|}\int_Q |\eta\,(b_I - m_Q b_I)| + \frac{1}{|Q|}\int_Q |\eta\, m_Q b_I - m_Q(\eta\, b_I)|\,.$$

Since $m_Q b_I = m_Q b - m_I b$, and $0 \leq \eta \leq 1$, the first term in the last expression is trivially bounded by $\|b\|_*$. We re-write the second term as

$$\frac{1}{|Q|}\int_Q \big|(\eta\, m_Q b_I - (m_Q\eta)(m_Q b_I)) + ((m_Q\eta)(m_Q b_I) - m_Q(\eta\, b_I))\big| . \tag{5.2}$$

The absolute value of the first expression in brackets is equal to

$$\Big|\frac{1}{|Q|}\int_Q (\eta(x)-\eta(y))\,dy\,(m_Q b - m_I b)\Big| \le C_n\,\frac{\operatorname{diam} Q}{\operatorname{diam} I}\,\log\frac{|I|}{|Q|}\,\|b\|_* ,$$

where in the inequality we have used our assumption about $\|\nabla\eta\|_*$ and also a well known property of BMO. Since $|Q| < 2^n|I|$, the desired estimate follows for this part of (5.2). The second expression in brackets is dominated in absolute value by

$$\frac{1}{|Q|}\int_Q |\eta(y)|\,|m_Q b - m_I b - (b - m_I b)| \le \|b\|_* .$$

Corollary 5.3. *Let I and η be as in Lemma* 5.1. *Then* $v \in \mathrm{VMO}$ *implies that* $\eta(v - m_I v) \in \mathrm{VMO}$.

PROOF. Choose v_j continuous such that $v_J \to v$ in BMO norm. Now apply Lemma 5.1 to $\eta((v - v_j) - m_I(v - v_j))$.

Corollary 5.4. *Let* $v \in \mathrm{VMO}$, *and let η, I, be as in Lemma* 5.1. *Then there exists* $\{u_j\}$, $u_j \in C_0^\infty$, *such that* $\|\eta(v - m_I v) - u_j\|_* \to 0$.

PROOF. By the previous corollary, $\eta(v - m_I v)$ can be approximated in BMO norm by continuous functions with compact support, which in turn are uniformly approximable by functions in C_0^∞.

Corollary 5.5. *Let η, I, v and u_j be as in Corollary* 5.4. *Let S be any standard Calderón-Zygmund type convolution singular integral operator with a smooth kernel. Then*

$$\|S u_j - S(\eta\,(v - m_I v))\|_* \to 0 .$$

PROOF. Immediate by standard Calderón-Zygmund theory and the previous corollary.

Corollary 5.6. *Suppose* $\nabla A \in \mathrm{VMO}(\mathbb{R}^n)$. *Let* I *and* η *be as above, so that* $\eta(\nabla A - m_I(\nabla A)) \in \mathrm{VMO}$. *Set*

$$A_I(x) = A(x) - m_I(\nabla A) \cdot x\,.$$

Following [Co, p. 698], *let* x_0 *be a point on the boundary of* $5\sqrt{m}\,I$, *and set*

$$\tilde{A} = \eta(x)\,(A_I(x) - A_I(x_0))\,.$$

Then there exists a sequence $\{A_j\} \subset I_1(\mathrm{BMO})$ *such that* $\nabla A_j \in \mathrm{Lip}_\alpha$, *for any given* $0 < \alpha < 1$, *and* $\|\nabla A_j - \nabla A\|_* \to 0$.

PROOF. Let $\vec{R} = (R_1, R_2, \ldots, R_n)$, where R_j denotes the j-th Riesz transform. A well known classical identity says that

$$\tilde{A} = -\vec{R} \cdot \vec{R}\tilde{A} = -I_1(\vec{R} \cdot \nabla \tilde{A})\,.$$

By definition,

$$\begin{aligned}\nabla \tilde{A}(x) &= \nabla \eta(x)\,(A_I(x) - A_I(x_0)) + \eta(x)\,(\nabla A(x) - m_I(\nabla A)) \\ &= \vec{a}(x) + \vec{b}(x)\,.\end{aligned}$$

By Corollary 5.5, there exist $\vec{u}_j \in C_0^\infty$ such that $\|S\vec{u}_j - S\vec{b}\|_* \to 0$, for any classical convolution type singular integral with a smooth kernel. Furthermore $S\vec{u}_j \in \mathrm{Lip}_\alpha$, $0 < \alpha < 1$, for all such S (see, *e.g.* Taibleson [T]) (here $\mathrm{Lip}_\alpha = \{f : |f(x) - f(y)| \le C\,|x-y|^\alpha\}$). By Lemma 2.10,

$$\|\vec{a}\|_\infty \le C_n\,\|\nabla A\|_*\,,$$

and also, for $x, y \in 2I$, and for all $q > n$,

$$|A_I(x) - A_I(y)| \le C_q\,|x-y|^{1-n/q}|I|^{1/q}\|\nabla A\|_*\,.$$

Thus $\vec{a}$ is continuous with compact support and belongs to Lip_α, $0 < \alpha < 1$, so again $S\vec{a} \in \mathrm{Lip}_\alpha$ by [T]. We now define

$$A_j = -I_1(\vec{R} \cdot (\vec{a} + \vec{u}_j))\,.$$

The conclusion of the lemma then follows by Corollary 5.5, the result of [T] and the identity

$$\frac{\partial}{\partial x_j} I_1 \vec{R} = R_j \vec{R}\,,$$

where $S = R_j\vec{R}$ is a classical singular integral.

REMARK. We have thus shown that $\tilde{A}$ can be approximated by $C^{1,\alpha}$ functions. The improvement to $C_0^{1,\alpha}$ will arise in the proof of Theorem 1.17.

We are now in a position to follow [FJR] and prove the compactness of

$$Kf = \lim_{\varepsilon\to 0} K_\varepsilon f\,,$$

where

$$K_\varepsilon f(P) = \int\limits_{\{|P-Q|>\varepsilon\}\cap\Gamma} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^{n+1}} f(Q)\,d\sigma(Q)\,.$$

(Here we have dropped the dimensional constant).

By a partition of unity argument we may change to local graph coordinates and treat the Euclidean operator

$$\tilde{K}f = \lim_{\varepsilon\to 0}\tilde{K}_\varepsilon f\,, \tag{5.7}$$

where

$$\tilde{K}_\varepsilon f(x) = \int\limits_{|x-y|^2+(A(x)-A(y))^2>\varepsilon^2} \frac{A(x)-A(y)-\nabla A(y)\cdot(x-y)}{(|x-y|^2+(A(x)-A(y)^2)^{(n+1)/2}} f(y)\,dy\,, \tag{5.8}$$

with $\nabla A \in \mathrm{VMO}(\mathbb{R}^n)$. Since surface measure is an A_p weight $(1 < p < \infty)$ times Lebesgue measure (see Lemma 1.12), and since we have localized, it is enough to show that $\tilde{K}$ is compact as an operator $L^p_w(I)$, for a cube I and $w \in A_p$. The almost everywhere existence of the principal value in (5.7) will be shown in the course of the proof. As a first approximation, we consider

$$T[A,A]f = \lim_{\varepsilon\to 0} T_\varepsilon[A,A]f\,, \tag{5.9}$$

where

$$T_\varepsilon[A,B]f(x) = \int\limits_{|x-y|>\varepsilon} \frac{B(x)-B(y)-\nabla B(y)\cdot(x-y)}{(|x-y|^2+(A(x)-A(y)^2)^{(n+1)/2}} f(y)\,dy\,, \tag{5.10}$$

Theorem 5.11. *Fix a cube $I \subseteq \mathbb{R}^n$, let $A \in I_1(\mathrm{BMO})$, with $\nabla A \in \mathrm{VMO}$. Then for $\operatorname{supp} f \subset I$, the principal value $T[A,A]f$ exists almost everywhere in I and in $L^p_w(I)$ norm, and furthermore $T[A,A]$ is a compact operator on $L^p_w(I)$, $1 < p < \infty$, $w \in A_p$.*

PROOF. Suppose $\operatorname{supp} f \subseteq I$, and let $x, y \in I$. Then for $\tilde{A}$ defined as in Corollary 5.6, we have

$$A(x) - A(y) - \nabla A(y) \cdot (x-y) = \tilde{A}(x) - \tilde{A}(y) - \nabla \tilde{A}(y) \cdot (x-y).$$

It is therefore enough to consider

$$T[A, \tilde{A}]\, f = \lim_{\varepsilon \to 0} T_\varepsilon[A, \tilde{A}]\, f ,$$

where $T_\varepsilon[A,B]$ is defined in (5.10). We will use the techniques of [FJR], but with Theorem 1.10 in place of Calderón's Theorem.

We begin by observing that for $B \in C_0^{1,\alpha}$, $0 < \alpha < 1$, the pointwise existence of $T[A,B]\, f = \lim_{\varepsilon \to 0} T_\varepsilon[A,B]\, f$ is trivial, because the extra smoothness of B weakens the singularity. Furthermore, we claim that for $B \in C_0^{1,\alpha}$, $T_\varepsilon[A,B]$ converges to $T[A,B]$ in the operator norm of L^p_w. To see this, we use the smoothness of B to write

$$|T[A,B]\, f(x) - T_\varepsilon[A,B]\, f(x)| \le C \int\limits_{|x-y|>\varepsilon} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \le C\, \varepsilon^\alpha M f(x),$$

where the last inequality is implied by a well-known result for approximate identities (see [S, pp. 61-63]). The claim follows. Thus the compactness of $T[A,B]$ on $L^p_w(I)$, for $B \in C_0^{1,\alpha}$ follows immediately from the compactness of $T_\varepsilon[A,B]$ for each $\varepsilon > 0$. But the latter fact may be deduced by a standard argument (see, *e.g.* [Tor, pp. 429-430]) and the fact that

$$\int\limits_{|x-y|>\varepsilon} |x-y|^{-n} |f(y)|\, dy \le C_{p,w,\varepsilon}\, \|f\|_{p,w}$$

(see [GR, p. 416]). The details are left to the reader.

We now proceed to prove the compactness of $T[A, \tilde{A}]$. By Corollary 5.6, there exists a sequence $A_j \in I_1(\mathrm{BMO})$ with $\nabla A_j \in \mathrm{Lip}_\alpha$, and such that $\nabla A_j \to \nabla \tilde{A}$ in BMO norm. If we define $\tilde{A}_j$ in the same way as $\tilde{A}$

(see the statement of Corollary 5.6), then $\tilde{A}_j \in C_0^{1,\alpha}$, and for $x \in I$ and $\operatorname{supp} f \subset I$,

$$T[A, A_j]\, f(x) = T[A, \tilde{A}_j]\, f(x)\,.$$

By our previous remarks, the principal value operator $T[A, \tilde{A}_j]$ exists and is compact on $L^p_w(I)$, so the same holds for $T[A, A_j]$. Let us assume for the moment that the principal value $T[A, \tilde{A}]$ exists. The compactness of $T[A, \tilde{A}]$ is then a consequence of the fact that $T[A, A_j] \to T[A, \tilde{A}]$ in the operator norm of L^p_w, and the latter fact may be deduced by writing

$$T[A, \tilde{A}] - T[A, A_j] = T[A, \tilde{A} - A_j]\,,$$

applying Theorem 1.10 with $B = \tilde{A} - A_j$, and using the fact that $\|\nabla \tilde{A} - \nabla A_j\|_* \to 0$.

To see that

$$T[A, \tilde{A}]\, f(x) = \lim_{\varepsilon \to 0} T_\varepsilon[A, \tilde{A}]\, f(x)$$

exists almost everywhere in I, for $\operatorname{supp} f \subset I$, we write

$$T_\varepsilon[A, \tilde{A}]\, f(x) = T_\varepsilon[A, \tilde{A} - A_j]\, f(x) + T_\varepsilon[A, A_j]\, f(x)\,.$$

But we have already observed that $\lim_{\varepsilon \to 0} T_\varepsilon[A, A_j]$ exists, and by Theorem 1.10, the operator

$$T_*[A, \tilde{A} - A_j]\, f = \sup_{\varepsilon > 0} |T_\varepsilon[A, \tilde{A} - A_j]\, f|$$

goes to zero in operator norm on L^p_w.

The almost everywhere (and norm) convergence of $T_\varepsilon[A, \tilde{A}]\, f$ as $\varepsilon \to 0$ now follows by standard arguments. This concludes the proof of Theorem 5.11.

We now finish the proof of Theorem 1.17. Let $\operatorname{supp} f \subseteq I$. We write, for $x \in I$,

$$\tilde{K}_\varepsilon f(x) = (\tilde{K}_\varepsilon f(x) - T_\varepsilon[A, A]\, f(x)) + T_\varepsilon[A, A]\, f(x)\,. \tag{5.12}$$

By Theorem 5.11, $T[A, A]\, f = \lim_{\varepsilon \to 0} T_\varepsilon[A, A]\, f$ exists almost everywhere and defines a compact operator on $L^p_w(I)$. Thus, it is enough to show that $\tilde{K}_\varepsilon f - T_\varepsilon[A, A]\, f \to 0$ almost everywhere and in $L^p_w(I)$ operator norm, as $\varepsilon \to 0$.

To control this error term, we will consider a more general expression. Set

$$R_\varepsilon[A,B]\,f(x) = \int\limits_{|x-y|^2+(A(x)-A(y))^2\geq\varepsilon^2\geq|x-y|^2} \frac{B(x)-B(y)-\nabla B(y)\cdot(x-y)}{(|x-y|^2+(A(x)-A(y))^2)^{(n+1)/2}}\,f(y)\,dy\,.$$

Lemma 5.13. *Let* $R_*[A,B]\,f = \sup_{\varepsilon>0}|R_\varepsilon[A,B]\,f|$. *Then for all* $\delta>0$

$$R_*[A,B]\,f(x) \leq C_{n,\delta}\,\|\nabla B\|_*\big(M(|f|^{1+\delta})(x)\big)^{1/(1+\delta)}$$

PROOF. Let $Q=Q(\varepsilon,x)$ be the cube with center x and side length ε. Then in R_ε we may replace B by B_Q, where

$$B_Q(x) = B(x) - \Big(\frac{1}{|Q|}\int_Q \nabla B\Big)\cdot x = B(x) - m_Q(\nabla B)\cdot x\,.$$

By Lemma 2.10,

$$|B_Q(x)-B_Q(y)| \leq C_n\,|x-y|\,\log\frac{\varepsilon}{|x-y|}\,\|\nabla B\|_*\,.$$

Thus, with $U_\varepsilon = \{y : |x-y|^2 \leq \varepsilon^2 < |x-y|^2+(A(x)-A(y))^2\}$, we have

$$\begin{aligned}\sup_{\varepsilon>0}\Big|\int_{U_\varepsilon}&\frac{B_Q(x)-B_Q(y)}{(|x-y|^2+(A(x)-A(y))^2)^{(n+1)/2}}\,f(y)\,dy\Big| \\ &\leq \sup_{\varepsilon>0}\Big(\frac{C_n}{\varepsilon^n}\int\limits_{|x-y|<\varepsilon}\|\nabla B\|_*\,\log\frac{\varepsilon}{|x-y|}\,|f(y)|\,dy\Big) \\ &\leq C_n\,\|\nabla B\|_*\,Mf(x)\,,\end{aligned}$$

where the last inequality follows by [S, pp. 61-63]. By Hölder's inequality

$$\Big|\int_{U_\varepsilon}\frac{\nabla B_Q(y)\cdot(x-y)}{(|x-y)^2+(A(x)-A(y))^2)^{(n+1)/2}}\,f(y)dy\Big|$$

$$\leq \Big(\frac{1}{\varepsilon^n} \int\limits_{|x-y|\leq \varepsilon} |\nabla B_Q(y)|^{(1+\delta)/\delta}\Big)^{\delta/(1+\delta)} \Big(\frac{1}{\varepsilon^n} \int\limits_{|x-y|\leq \varepsilon} |f(y)|^{1+\delta}\Big)^{1/(1+\delta)},$$

and the lemma follows.

We now return to the matter of showing that $R_\varepsilon[A,A]\,f \to 0$ almost everywhere and in operator norm. As it was the case for $T[A,A]$ in the proof of Theorem 5.11, we have that as an operator on $L^p_w(I)$, $R_\varepsilon[A,A] = R_\varepsilon[A,\tilde{A}]$, with $\tilde{A}$ as in Corollary 5.6. Furthermore there exist $A_j \in C^{1,\alpha}$ with $\|\nabla A_j - \nabla \tilde{A}\|_* \to 0$. Now

$$R_\varepsilon[A,\tilde{A}] = R_\varepsilon[A,\tilde{A} - A_j] + R_\varepsilon[A,A_j]\,.$$

Since $R_\varepsilon[A,A_j] = R_\varepsilon[A,\tilde{A}_j]$ (as operators on $L^p_w(I)$), where $\tilde{A}_j \in C^{1,\alpha}_0$, it is easy to see that for each j,

$$R_\varepsilon[A,A_j]\,f(x) \to 0\,, \qquad \text{a.e. in } I, \text{ as } \varepsilon \to 0\,,$$

and in $L^p_w(I)$ operator norm, by virtue of Hölder's continuity of ∇A_j (in fact, for almost everywhere x,

$$|R_\varepsilon[A,A_j]\,f(x)| \leq C\,\|\nabla A_j\|_{\mathrm{Lip}_\alpha}\,\varepsilon^\alpha\,Mf(x)\,)\,.$$

Finally, by Lemma 5.13,

$$\|R_*[A,\tilde{A} - A_j]\,f\|_{p,w} \leq C(n,p,A_p)\,\|\nabla\tilde{A} - \nabla A_j\|_*\|f\|_{p,w}\,,$$

and Theorem 1.17 follows by letting $j \to \infty$.

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Reconstructing a neural net from its output

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Introduction.

Neural nets were originally introduced as highly simplified models of the nervous system. Today they are widely used in technology and studied theoretically by scientists from several disciplines. (See *e.g.* [N]). However, they remain little understood.

Mathematically, a (feed-forward) neural net consists of

(1) A finite sequence of positive integers $(D_0, D_1, \dots, D_L)$,

(2) A family of real numbers (ω_{jk}^{ℓ}) defined for $1 \le \ell \le L$, $1 \le j \le D_\ell$, $1 \le k \le D_{\ell-1}$, and

(3) A family of real numbers (θ_j^ℓ) defined for $1 \le \ell \le L$, $1 \le j \le D_\ell$.

The sequence $(D_0, D_1, \dots, D_L)$ is called the *architecture* of the neural net, while the ω_{jk}^ℓ are called *weights* and the θ_j^ℓ *thresholds*.

Neural nets are used to compute non-linear maps from $\mathbb{R}^N$ to $\mathbb{R}^M$ by the following construction. We begin by fixing a nonlinear function $\sigma(x)$ of one variable. Analogy with the nervous system suggests that we take $\sigma(t)$ asymptotic to constants as t tends to $\pm\infty$; a standard choice, which we adopt throughout this paper, is $\sigma(x) = \tanh(x/2)$. Given an "input" $(t_1, \dots, t_{D_0}) \in \mathbb{R}^{D_0}$, we define real numbers x_j^ℓ for $0 \le \ell \le L$, $1 \le j \le D_\ell$ by the following induction on ℓ.

(4) If $\ell = 0$ then $x_j^\ell = t_j$.

(5) If the $x_k^{\ell-1}$ are known, $1 \le \ell \le L$, then we set

$$x_j^\ell = \sigma\Big(\sum_{1 \le k \le D_{\ell-1}} \omega_{jk}^\ell x_k^{\ell-1} + \theta_j^\ell \Big), \qquad \text{for } 1 \le j \le D_\ell .$$

Here $x_1^\ell, \dots, x_{D_\ell}^\ell$ are interpreted as the outputs of D_ℓ "neurons" in the ℓ^{th} "layer" of the net. The *output map* of the net is defined as the map

$$\Phi: (t_1, \dots, t_{D_0}) \longmapsto (x_1^L, \dots, x_{D_L}^L). \tag{6}$$

In practical applications, one tries to pick the neural net

$$[(D_0, D_1, \dots, D_L),\ (\omega_{jk}^\ell),\ (\theta_j^\ell)]$$

so that the output map Φ approximates a given map about which we have only imperfect information. The main result of this paper is that under generic conditions, perfect knowledge of the output map Φ uniquely specifies the architecture, the weights and the thresholds of a neural net, up to obvious symmetries. More precisely, the obvious symmetries are as follows. Let $(\gamma_0, \gamma_1, \dots, \gamma_L)$ be permutations, with

$$\gamma_\ell: \{1, \dots, D_\ell\} \to \{1, \dots, D_\ell\};$$

and let $\{\varepsilon_j^\ell : 0 \le \ell \le L,\ 1 \le j \le D_\ell\}$ be a collection of ± 1's. Assume that γ_ℓ is the identity and $\varepsilon_j^\ell = +1$ whenever $\ell = 0$ or $\ell = L$. Then one checks easily that the neural nets

(7) $[(D_0, D_1, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ and

(8) $[(D_0, D_1, \dots, D_L), (\widetilde{\omega}_{jk}^\ell), (\widetilde{\theta}_j^\ell)]$

have the same output map if we set

$$\widetilde{\omega}_{jk}^\ell = \varepsilon_j^\ell\, \omega_{[\gamma_\ell j][\gamma_{\ell-1} k]}^\ell\, \varepsilon_k^{\ell-1} \quad \text{and} \quad \widetilde{\theta}_j^\ell = \varepsilon_j^\ell\, \theta_{[\gamma_\ell j]} . \tag{9}$$

This reflects the facts that the neurons in layer ℓ are interchangeable, $1 \le \ell \le L-1$, and that the function $\sigma(x)$ is odd. The nets (7) and (8) will be called *isomorphic* if they are related by (9). Note in particular that isomorphic neural nets have the same architecture. Our main theorem asserts that, under generic conditions, any two neural nets with the same output map are isomorphic.

We discuss the generic conditions which we impose on neural nets. We have to avoid obvious counterexamples such as

(10) Suppose all the weights ω^{ℓ}_{jk} are zero. Then the output map Φ is constant. The architecture and thresholds of the neural net are clearly not uniquely determined by Φ.

(11) Fix ℓ_0, j_1, j_2 with $1 \le \ell_0 \le L-1$ and $1 \le j_1 < j_2 \le D_{\ell_0}$. Suppose we have $\theta^{\ell_0}_{j_1} = \theta^{\ell_0}_{j_2}$ and $\omega^{\ell_0}_{j_1 k} = \omega^{\ell_0}_{j_2 k}$ for all k. Then (5) gives $x^{\ell_0}_{j_1} = x^{\ell_0}_{j_2}$. Therefore, the output depends on $\omega^{\ell_0+1}_{jj_1}$ and $\omega^{\ell_0+1}_{jj_2}$ only through the sum $\omega^{\ell_0+1}_{jj_1} + \omega^{\ell_0+1}_{jj_2}$. So the output map does not uniquely determine the weights.

Our hypotheses are more than adequate to exclude these counterexamples. Specifically, we assume that

(12) $\theta^{\ell}_{j} \neq 0$, and $|\theta^{\ell}_{j}| \neq |\theta^{\ell}_{j'}|$ for $j \neq j'$.

(13) $\omega^{\ell}_{jk} \neq 0$; and for $j \neq j'$, the ratio $\omega^{\ell}_{jk}/\omega^{\ell}_{j'k}$ is not equal to any fraction of the form p/q with p, q integers and $1 \le q \le 100\, D^2_{\ell}$.

Evidently, these conditions hold for generic neural nets. The precise statement of our main theorem is as follows. *If two neural nets satisfy* (12), (13) *and have the same output, then the nets are isomorphic.* In Section I we give a slightly different but clearly equivalent statement of our main result. It would be interesting to replace (12), (13) by minimal hypotheses, and to study functions $\sigma(x)$ other than $\tanh(x/2)$.

We now sketch the proof of our main result, sacrificing accuracy for simplicity. After a trivial reduction, we may assume $D_0 = D_L = 1$. Thus, the outputs of the nodes $x^{\ell}_{j}(t)$ are functions of one variable, and the output map of the neural net is $t \mapsto x^{L}_{1}(t)$. The key idea is to continue the $x^{\ell}_{j}(t)$ analytically to complex values of t, and to read off the structure of the net from the set of singularities of the x^{ℓ}_{j}. Note that $\sigma(x) = \tanh(x/2)$ is meromorphic, with poles at the points of an arithmetic progression $\{(2m+1)\pi i \colon m \in \mathbb{Z}\}$. This leads to two crucial observations.

(14) When $\ell = 1$, the poles of $x^{\ell}_{j}(t)$ form an arithmetic progression Π^{1}_{j}, and

(15) When $\ell > 1$, every pole of any $x^{\ell-1}_{k}(t)$ is an accumulation point of poles of any $x^{\ell}_{j}(t)$.

In fact, (14) is immediate from the formula $x_j^1(t) = \sigma(\omega_{j1}^1 t + \theta_j^1)$, which is merely the special case $D_0 = 1$ of (5). We obtain

$$\Pi_j^1 = \left\{ \frac{(2m+1)\pi i - \theta_j^1}{\omega_{j1}^1} : \, m \in \mathbb{Z} \right\}. \tag{16}$$

To see (15), fix ℓ, j, $\bar{k}$, and assume for simplicity that $x_{\bar{k}}^{\ell-1}(t)$ has a simple pole at t_0, while $x_k^{\ell-1}(t)$, $k \neq \bar{k}$, is analytic in a neighborhood of t_0. Then

$$x_{\bar{k}}^{\ell-1}(t) = \frac{\lambda}{t - t_0} + f(t), \tag{17}$$

with f analytic in a neighborhood of t_0.

From (17) and (5), we obtain

$$x_j^\ell(t) = \sigma(\omega_{j\bar{k}}^\ell \lambda (t - t_0)^{-1} + g(t)), \tag{18}$$

with

$$g(t) = \omega_{j\bar{k}}^\ell f(t) + \sum_{k \neq \bar{k}} \omega_{jk}^\ell \, x_k^{\ell-1}(t) + \theta_j^\ell \tag{19}$$

analytic in a neighborhood of t_0.

Thus, in a neighborhood of t_0, the poles of $x_j^\ell(t)$ are the solutions $\tilde{t}_m$ of the equation

$$\frac{\omega_{j\bar{k}}^\ell \lambda}{\tilde{t}_m - t_0} + g(\tilde{t}_m) = (2m+1)\pi i, \qquad m \in \mathbb{Z}. \tag{20}$$

There are infinitely many solutions of (20), accumulating at t_0. Hence, t_0 is an accumulation point of poles of $x_j^\ell(t)$, which completes the proof of (15).

In view of (14), (15), it is natural to make the following definitions. The *natural domain* of a neural net is the largest open subset of the complex plane to which the output map $t \mapsto x_1^L(t)$ can be analytically continued. For $\ell \geq 0$ we define the ℓ^{th} *singular set* $\text{Sing}(\ell)$ by setting

$\text{Sing}(0)$ = complement of the natural domain in $\mathbb{C}$, and

$\text{Sing}(\ell + 1)$ = the set of all accumulation points of $\text{Sing}(\ell)$.

These definitions are made entirely in terms of the output map, without reference to the structure of the given neural net. On the other hand, the sets Sing(ℓ) contain nearly complete information on the architecture, weights and thresholds of the net.

This will allow us to read off the structure of a neural net from the analytic continuation of its output map. To see how the sets Sing(ℓ) reflect the structure of the net, we reason as follows.

From (14) and (15) we expect that

(21) For $1 \le \ell \le L$, Sing$(L-\ell)$ is the union over $j = 1, \dots, D_\ell$ of the set of poles of $x_j^\ell(t)$, together with their accumulation points (which we ignore here), and

(22) For $\ell \ge L$, Sing(ℓ) is empty.

Immediately, then, we can read off the "depth" L of the neural net; it is simply the smallest ℓ for which Sing(ℓ) is empty.

We need to solve for D_ℓ, ω_{jk}^ℓ, θ_j^ℓ. We proceed by induction on ℓ.

When $\ell = 1$, (14) and (21) show that Sing$(L-1)$ is the union of arithmetic progressions Π_j^1, $j = 1, \dots, D_1$. Therefore, from Sing$(L-1)$ we can read off D_1 and the Π_j^1. (We will return to this point later in the introduction.) In view of (16), Π_j^1 determines the weights and thresholds at layer 1, modulo signs. Thus, we have found D_1, ω_{jk}^1, θ_j^1.

When $\ell > 1$, we may assume that

(23) The $D_{\ell'}$, $\omega_{jk}^{\ell'}$, $\theta_j^{\ell'}$ are already known, for $1 \le \ell' \le \ell$.

Our task is to find D_ℓ, ω_{jk}^ℓ, θ_j^ℓ. In view of (23), we can find a pole t_0 of $x_{\bar{k}}^{\ell-1}(t)$ for our favorite $\bar{k}$. Assume for simplicity that t_0 is a simple pole of $x_{\bar{k}}^{\ell-1}(t)$, and that the $x_k^{\ell-1}(t)$, $k \ne \bar{k}$, are analytic in a neighborhood of t_0. Then $x_{\bar{k}}^{\ell-1}(t)$ is given by (17) in a neighborhood of t_0, with λ already known by virtue of (23). Let U be a small neighborhood of t_0.

We will look at the image Y of $U \cap$ Sing$(L-\ell)$ under the map $t \mapsto \lambda/(t-t_0)$. Since λ, t_0 and Sing$(L-\ell)$ are already known, so is Y. On the other hand, we can relate Y to D_ℓ, ω_{jk}^ℓ, θ_j^ℓ as follows. From (21) we see that Y is the union over $j = 1, \dots, D_\ell$ of

(24) $Y_j =$ image of $U \cap \{$ Poles of $x_j^\ell(t)\}$ under $t \longmapsto \lambda/(t-t_0)$.

For fixed j, the poles of $x_j^\ell(t)$ in a neighborhood of t_0 are the $\tilde{t}_m$ given by (20). We write

$$(25)\qquad \frac{\omega^{\ell}_{j\bar{k}}\lambda}{\tilde{t}_m - t_0} = \left[\frac{\omega^{\ell}_{j\bar{k}}\lambda}{(\tilde{t}_m - t_0)} + g(\tilde{t}_m)\right] + \left[g(t_0) - g(\tilde{t}_m)\right] .$$

Equation (20) shows that the first expression in brackets in (25) is equal to $(2m+1)\pi i$. Also, since $\tilde{t}_m \to t_0$ as $|m| \to +\infty$ and g is analytic in a neighborhood of t_0, the second expression in brackets in (25) tends to zero. Hence,

$$\frac{\omega^{\ell}_{j\bar{k}}\lambda}{\tilde{t}_m - t_0} = (2m+1)\pi i - g(t_0) + o(1), \qquad \text{for large } m .$$

Comparing this with the definition (24), we see that Y_j is asymptotic to the arithmetic progression

$$(26)\qquad \Pi^{\ell}_j = \left\{ \frac{(2m+1)\pi i - g(t_0)}{\omega^{\ell}_{j\bar{k}}} : \ m \in \mathbb{Z} \right\} .$$

Thus, the known set Y is the union over $j = 1, \dots, D_\ell$ of sets Y_j, with Y_j asymptotic to the arithmetic progression Π^{ℓ}_j. From Y, we can therefore read off D_ℓ and the Π^{ℓ}_j. (We will return to this point in a moment). We see at once from (26) that $\omega^{\ell}_{j\bar{k}}$ is determined up to sign by Π^{ℓ}_j. Thus, we have found D_ℓ and $\omega^{\ell}_{j\bar{k}}$. With more work, we can also find the θ^{ℓ}_j, completing the induction on ℓ.

The above induction shows that the structure of a neural net may be read off from the analytic continuation of its output map. We believe that the analytic continuation of the output map will lead to further consequences in the study of neural nets.

Let us touch briefly on a few points which we glossed over above. First of all, suppose we are given a set $Y \subset \mathbb{C}$, and we know that Y is the union of sets $Y_1, \dots, Y_D$, with Y_j asymptotic to an arithmetic progression Π_j. We assumed above that $\Pi_1, \dots, \Pi_D$ are uniquely determined by Y. In fact, without some further hypothesis on the Π_j, this need not be true. For instance, we cannot distinguish $\Pi_1 \cup \Pi_2$ from Π_3 if $\Pi_1 = \{\text{odd integers}\}$, $\Pi_2 = \{\text{even integers}\}$, $\Pi_3 = \{\text{all integers}\}$. On the other hand, we can clearly recognize $\Pi_1 = \{\text{all integers}\}$ and $\Pi_2 = \{m\sqrt{2} : m \text{ an integer}\}$ from their union $\Pi_1 \cup \Pi_2$. Thus, irrational numbers enter the picture. The rôle of our generic hypothesis (13) is to control the arithmetic progressions that arise in our proof.

Secondly, suppose $x_{\bar{k}}^{\ell}(t)$ has a pole at t_0. We assumed for simplicity that $x_k^{\ell}(t)$ is analytic in a neighborhood of t_0 for $k \neq \bar{k}$. However, one of the $x_k^{\ell}(t)$, $k \neq \bar{k}$, may also have a pole at t_0. In that case, the $x_j^{\ell+1}(t)$ may all be analytic in a neighborhood of t_0, because the contributions of the singularities of the x_k^{ℓ} in $\sigma\big(\sum_k \omega_{jk}^{\ell+1} x_k^{\ell} + \theta_j^{\ell+1}\big)$ may cancel. Thus, the singularity at t_0 may disappear from the output map. While this circumstance is hardly generic, it is not ruled out by our hypotheses (12), (13). Because singularities can disappear, we have to make technical changes in our description of Sing(ℓ). For example, in the discussion following (23), Y need not be the union of the sets Y_j. Rather, Y is their "approximate union", in a sense to be made precise in (II.A.1) below.

Next, we should point out that the signs of the weights and thresholds require some attention, even though we have some freedom to change signs by applying isomorphisms. (See (9).) In effect, we introduce in Section IV.A an extra induction on the number of neurons in the net, in order to show that the signs come out correctly. The induction comes into play in the substantial Lemma IV.B.16 below.

Finally, in the definition of the natural domain, we have assumed that there is a unique maximal open set to which the output map continues analytically. This need not be true of a general real-analytic function on the line -for instance, take $f(t) = (1 + t^2)^{1/2}$. Fortunately, Lemma III.A.1 below shows that the natural domain is well-defined for any function that continues analytically to the complement of a countable set. The defining formula (5) lets us check easily that the output map continues to the complement of a countable set, so the natural domain makes sense. This concludes our overview of the proof of our main theorem.

Both the uniqueness problem and the use of analytic continuation have already appeared in the neural net literature. In particular, it was R. Hecht-Nielson who pointed out the rôle of isomorhisms and posed the uniqueness problem. His paper with Chen and Lu [CLH] on "equioutput transformations" on the space of all neural nets influenced our work. E. Sontag [So] and H. Sussman [Su] proved sharp uniqueness theorems for one hidden layer. The proof in [Su] uses complex variables.

At this stage, few non-trivial results are known for neural nets with more than one hidden layer, *i.e.* with $L > 1$. However, a recent paper of Macintyre and Sontag [MS] proves finiteness of the VC dimension, a measure of the computing power of a neural net.

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I. Statement of the Main Results.

A. Definitions.

A *neural net* consists of the following:

(1) A finite sequence of positive integers $(D_0, D_1, \ldots, D_L)$ with $L \geq 1$.

(2) A collection of real numbers (ω^ℓ_{jk}), defined for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$, $1 \leq k \leq D_{\ell-1}$.

(3) A collection of real numbers (θ^ℓ_j), defined for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$.

Here, L is called the *depth* of the net, and $(D_0, D_1, \ldots, D_L)$ is called the *architecture* of the net. The ω^ℓ_{jk} are called *weights*, while the θ^ℓ_j are called *thresholds.*

Thus, a neural net has the form $[(D_0, \ldots, D_L), (\omega^\ell_{jk}), (\theta^\ell_j)]$. We denote neural nets by $\mathcal{N}$, $\mathcal{N}'$, $\widetilde{\mathcal{N}}$, etc.

For $\mathcal{N} = [(D_0, \ldots, D_L), (\omega^\ell_{jk}), (\theta^\ell_j)]$, we define functions $x^\ell_j(t_1, \ldots, t_{D_0}, \mathcal{N})$ by the following induction on ℓ.

(4) $$x^0_j(t_1, \ldots, t_{D_0}, \mathcal{N}) = t_j \quad \text{for} \quad 1 \leq j \leq D_0 .$$

(5) $$x^\ell_j(t_1, \ldots, t_{D_0}, \mathcal{N}) = \sigma\left(\sum_{k=1}^{D_{\ell-1}} \omega^\ell_{jk} x^{\ell-1}_k(t_1, \ldots, t_{D_0}, \mathcal{N}) + \theta^\ell_j \right),$$

for $1 \leq j \leq D_\ell$, where

(6) $$\sigma(x) = \tanh\left(\frac{x}{2}\right) .$$

We call $(t_0, t_1, \ldots, t_{D_0}) \in \mathbb{R}^{D_0}$ the *input* to the neural net; we call $(x^L_j(t_1, \ldots, t_{D_0}, \mathcal{N}))_{1 \leq j \leq D_L} \in \mathbb{R}^{D_L}$ the *output*, or the *function computed by the neural net*; and we call $x^\ell_j(t_1, \ldots, t_{D_0}, \mathcal{N})$ the *function computed by the j^{th} node of the ℓ^{th} layer.*

When it is clear which neural net we are talking about, we may write $x^\ell_j(t_1, \ldots, t_{D_0})$ in place of $x^\ell_j(t_1, \ldots, t_{D_0}, \mathcal{N})$. Also, when $D_0 = 1$, we may write $x^\ell_j(t)$ or $x^\ell_j(t, \mathcal{N})$ in place of $x^\ell_j(t_1), x^\ell_j(t_1, \mathcal{N})$.

The *size* of a neural net $\mathcal{N} = [(D_0, \ldots, D_L), (\omega^{\ell}_{jk}), (\theta^{\ell}_j)]$ is defined simply as $D_0 + D_1 + \cdots + D_L$.

Next we discuss isomorphisms of neural nets. Let

$$\mathcal{N} = [(D_0, \ldots, D_L), (\omega^{\ell}_{jk}), (\theta^{\ell}_j)] \tag{7}$$

be a neural net. Then let

$$\gamma_\ell : \{1, \ldots, D_\ell\} \to \{1, \ldots, D_\ell\} \tag{8}$$

be permutations. Finally, let ε^{ℓ}_j be a collection of signs,

$$\varepsilon^{\ell}_j = \pm 1, \qquad \text{for } \ 0 \le \ell \le L, \ 1 \le j \le D_\ell . \tag{9}$$

In terms of $\mathcal{N}$, (γ_ℓ), (ε^{ℓ}_j), we define a new neural net

$$\widetilde{\mathcal{N}} = [(D_0, \ldots, D_L), (\widetilde{\omega}^{\ell}_{jk}), (\widetilde{\theta}^{\ell}_j)], \tag{10}$$

where

$$\widetilde{\omega}^{\ell}_{jk} = \varepsilon^{\ell}_j \, \omega^{\ell}_{(\gamma_\ell j)(\gamma_{\ell-1} k)} \, \varepsilon^{\ell-1}_k \tag{11}$$

and

$$\widetilde{\theta}^{\ell}_j = \varepsilon^{\ell}_j \, \theta^{\ell}_{(\gamma_\ell j)} . \tag{12}$$

An easy induction on ℓ shows that

$$x^{\ell}_j(\tilde{t}_1, \ldots, \tilde{t}_{D_0}, \widetilde{\mathcal{N}}) = \varepsilon^{\ell}_j \, x^{\ell}_{(\gamma_\ell j)}(t_1, \ldots, t_{D_0}, \mathcal{N}), \tag{13}$$

provided $(\tilde{t}_1, \ldots, \tilde{t}_{D_0})$ and $(t_1, \ldots, t_{D_0})$ are related by

$$\tilde{t}_j = \varepsilon^0_j \, t_{(\gamma_0 j)}, \qquad \text{for } \ 1 \le j \le D_0 . \tag{14}$$

In particular, if we assume

(15) $\varepsilon^{\ell}_j = 1$ when $\ell = 0$ or L and γ_0, γ_L are the identity permutation,

then (13), (14) show that

$$x^{L}_j(t_1, \ldots, t_{D_0}, \widetilde{\mathcal{N}}) = x^{L}_j(t_1, \ldots, t_{D_0}, \mathcal{N}), \qquad \text{for } \ 1 \le j \le D_L . \tag{16}$$

Thus, the neural nets $\widetilde{\mathcal{N}}$ and $\mathcal{N}$ compute the same function. We say that the nets $\mathcal{N}$, $\widetilde{\mathcal{N}}$ are *isomorphic* if they are related by (7), ..., (12) for some choice of (γ_ℓ), (ε_j^ℓ) satisfying (15). For fixed

$$[(\gamma_\ell)_{0 \le \ell \le L},\ (\varepsilon_j^\ell)_{0 \le \ell \le L,\ 1 \le j \le D_\ell}]$$

satisfying (15), the map $\mathcal{N} \mapsto \widetilde{\mathcal{N}}$ given by (7), (10), (11), (12) is is called the *isomorphism induced by* $[(\gamma_\ell), (\varepsilon_j^\ell)]$. One checks easily that compositions and inverses of isomorphisms are again isomorphisms. Note that any two isomorphic neural nets $\mathcal{N}$, $\widetilde{\mathcal{N}}$ have the same architecture.

It is useful to pick out a single representative from an isomorphism class of neural nets. Thus, we say that $\mathcal{N} = [(D_0, \ldots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ is in *standard order* if for each ℓ, $1 \le \ell < L$, we have

$$(17) \qquad 0 < \theta_1^\ell < \theta_2^\ell < \cdots < \theta_{D_\ell}^\ell .$$

The proof of the following observation is left to the reader.

(18) **Lemma.** *Every neural net* $\mathcal{N} = [(D_0, \ldots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ *satisfying the generic condition*

$$(19) \qquad \theta_j^\ell \ne 0, \qquad |\theta_j^\ell| \ne |\theta_{j'}^\ell| \quad \text{for } j \ne j',$$

is isomorphic to one and only one neural net in standard order.

B. The Main Theorems.

The main result of this paper is as follows.

(1) **Uniqueness Theorem.** *Let* $\mathcal{N} = [(D_0, \ldots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ *and* $\widetilde{\mathcal{N}} = [(\widetilde{D}_0, \ldots, \widetilde{D}_{\widetilde{L}}), (\widetilde{\omega}_{jk}^\ell), (\widetilde{\theta}_j^\ell)]$ *be neural nets in standard order, satisfying the following generic conditions.*

$$(2) \qquad \omega_{jk}^\ell \ne 0 \quad \text{and} \quad \left|\frac{\omega_{jk}^\ell}{\omega_{j'k}^\ell}\right| \ne \frac{p}{q},$$

for $j \ne j'$, $p, q \in \mathbb{Z}$, $1 \le q \le 100\, D_\ell^2$,

$$(3) \qquad \widetilde{\omega}_{jk}^\ell \ne 0 \quad \text{and} \quad \left|\frac{\widetilde{\omega}_{jk}^\ell}{\widetilde{\omega}_{j'k}^\ell}\right| \ne \frac{p}{q},$$

for $j \neq j'$, $p, q \in \mathbb{Z}$, $1 \le q \le 100\, \widetilde{D}_\ell^2$. *Assume* $D_0 = \widetilde{D}_0$, $D_L = \widetilde{D}_{\widetilde{L}}$, *and*

$$x_j^L(t_1, \dots, t_{D_0}, \mathcal{N}) = x_j^{\widetilde{L}}(t_1, \dots, t_{D_0}, \widetilde{\mathcal{N}}), \tag{4}$$

for all $(t_1, \dots, t_{D_0}) \in \mathbb{R}^{D_0}$ *and all* j, $1 \le j \le D_L$.

Then the nets $\mathcal{N}$ *and* $\widetilde{\mathcal{N}}$ *are identical:*

(5) $L = \widetilde{L}$,

(6) $D_\ell = \widetilde{D}_\ell$ *for* $0 \le \ell \le L$,

(7) $\omega_{jk}^\ell = \widetilde{\omega}_{jk}^\ell$ *for* $1 \le \ell \le L$, $1 \le j \le D_\ell$, $1 \le k \le D_{\ell-1}$,

(8) $\theta_j^\ell = \widetilde{\theta}_j^\ell$ *for* $1 \le \ell \le L$, $1 \le j \le D_\ell$.

The Uniqueness Theorem 1 reduces immediately to the special case. $D_0 = 1$, $D_L = 1$. To see this, we fix j_0, $1 \le j_0 \le D_L$, and k_0, $1 \le k_0 \le D_0$. Then we restrict attention to the j_0^{th} outputs $x_{j_0}^L(\cdot, \widetilde{\mathcal{N}})$ for inputs of the form $(0, \dots, 0, t, 0, \dots, 0)$, where the t occurs in the k_0^{th} coordinate. Thus we obtain functions $x^L(t, \mathcal{N})$, $x^{\widetilde{L}}(t, \widetilde{\mathcal{N}})$ of a single variable t. These functions are computed by neural nets $\mathcal{N}_{\text{reduced}}$, $\widetilde{\mathcal{N}}_{\text{reduced}}$ obtained from $\mathcal{N}$, $\widetilde{\mathcal{N}}$ by deleting irrelevant input and output nodes. The special case $D_0 = D_L = 1$ of Theorem (1), applied to $\mathcal{N}_{\text{reduced}}$ and $\widetilde{\mathcal{N}}_{\text{reduced}}$, shows that $\mathcal{N}_{\text{reduced}}$ and $\widetilde{\mathcal{N}}_{\text{reduced}}$ are identical. Since j_0 and k_0 were arbitary, it follows that $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are identical. Thus, Theorem (1) is reduced to the special case $D_0 = D_L = 1$.

From now on, *we change the definition of neural nets* to include the requirement $D_0 = D_L = 1$. Thus, a neural net computes a single function of one variable. In view of the elementary Lemma A.18, our uniqueness theorem is reduced to the following statement.

(9) **Uniqueness Theorem.** *Let* $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ *and* $\widetilde{\mathcal{N}} = [(\widetilde{D}_0, \dots, \widetilde{D}_{\widetilde{L}}), (\widetilde{\omega}_{jk}^\ell), (\widetilde{\theta}_j^\ell)]$ *be neural nets satisfying the generic conditions*

$$\omega_{jk}^\ell \neq 0, \qquad \textit{and} \qquad \left| \frac{\widetilde{\omega}_{jk}^\ell}{\widetilde{\omega}_{j'k}^\ell} \right| \neq \frac{p}{q}, \tag{10}$$

for $j \neq j'$, $p, q \in \mathbb{Z}$, $1 \le q \le 100\, D_\ell^2$, *and*

$$\widetilde{\omega}_{jk}^\ell \neq 0, \qquad \textit{and} \qquad \left| \frac{\widetilde{\omega}_{jk}^\ell}{\widetilde{\omega}_{j'k}^\ell} \right| \neq \frac{p}{q}, \tag{11}$$

for $j \neq j'$, $p, q \in \mathbb{Z}$, $1 \le q \le 100\, \widetilde{D}_\ell^2$.

If $x_1^L(t, \mathcal{N}) = x_1^{\widetilde{L}}(t, \widetilde{\mathcal{N}})$ *for all real* t*, then* $\mathcal{N}$ *and* $\widetilde{\mathcal{N}}$ *are isomorphic.*

The rest of this paper is devoted to the proof of Theorem 9.

C. A Small Technical Lemma.

The following observation on isomorphic neural nets will be used much later, in the proof of Theorem B.9.

(1) **Lemma.** *Let* $\mathcal{N} = [(D_0, \dots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$ *and* $\widetilde{\mathcal{N}} = [(D_0, \dots, D_L), (\widetilde{\omega}_{jk}^\ell), (\widetilde{\theta}_j^\ell)]$ *be isomorphic neural nets. Assume that*

(2) $\omega_{jk}^\ell \neq 0$ *for all* ℓ, j, k, $1 \le \ell \le L$, $1 \le j \le D_\ell$, $1 \le k \le D_{\ell-1}$,

(3) $|\omega_{jk}^\ell| \neq |\omega_{j'k}^\ell|$ *for all* ℓ, $j \neq j'$, k, $1 \le \ell \le L$, $1 \le j, j' \le D_\ell$, $1 \le k \le D_{\ell-1}$,

(4) $\omega_{jk}^\ell = \widetilde{\omega}_{jk}^\ell$ *for* $1 \le \ell \le L-1$, $1 \le j \le D_\ell$, $1 \le k \le D_{\ell-1}$.

(*Note: We do not assume* (4) *for* $\ell = L$).

Then $\mathcal{N}$ *and* $\widetilde{\mathcal{N}}$ *are identical.*

PROOF. Since $\mathcal{N}$, $\widetilde{\mathcal{N}}$ are isomorphic, there are permutations γ_ℓ and signs ε_j^ℓ such that

$$\omega_{jk}^\ell = \varepsilon_j^\ell\, \widetilde{\omega}_{(\gamma_\ell j)(\gamma_{\ell-1} k)}^\ell\, \varepsilon_k^{\ell-1}, \tag{5}$$

$$\theta_j^\ell = \varepsilon_j^\ell\, \widetilde{\theta}_{(\gamma_\ell j)}^\ell, \tag{6}$$

$$\gamma_0 = \text{identity},\ \gamma_L = \text{identity},\ \varepsilon_1^0 = 1,\ \varepsilon_1^L = 1. \tag{7}$$

Since $\widetilde{\omega}_{jk}^\ell = \omega_{jk}^\ell$ for $\ell \le L-1$, (5) implies

$$\omega_{jk}^\ell = \varepsilon_j^\ell\, \omega_{(\gamma_\ell j)(\gamma_{\ell-1} k)}^\ell\, \varepsilon_k^{\ell-1}, \qquad \text{for } 1 \le \ell \le L-1, \tag{8}$$

so that

$$|\omega_{jk}^\ell| = |\omega_{(\gamma_\ell j)(\gamma_{\ell-1} k)}^\ell|, \qquad \text{for } 1 \le \ell \le L-1. \tag{9}$$

From (7) we have $\gamma_0 =$ identity. By (3) and (9), $\gamma_{\ell-1} =$ identity implies $\gamma_\ell =$ identity for $1 \le \ell \le L-1$. Hence $\gamma_\ell =$ identity for all $\ell \le L-1$. Since $\gamma_L =$ identity by (7), we know that all the $\gamma_\ell =$ identity. Thus, (8) becomes

$$(10) \qquad \omega^\ell_{jk} = \varepsilon^\ell_j \, \omega^\ell_{jk} \, \varepsilon^{\ell-1}_k \,, \qquad \text{for } 1 \le \ell \le L-1 \,.$$

From (7) we have $\varepsilon^0_k = 1$, since $D_0 = 1$. By (2) and (10), $\varepsilon^{\ell-1}_k = 1$ (all k) implies $\varepsilon^\ell_j = 1$ (all j) for $1 \le \ell \le L-1$. Hence $\varepsilon^\ell_j = 1$ whenever $\ell \le L-1$. Since also (7) gives $\varepsilon^L_j = 1$ because $D_L = 1$, we know that $\varepsilon^\ell_j = 1$ for all ℓ, j, $0 \le \ell \le L$, $1 \le j \le D_\ell$. Since $\varepsilon^\ell_j = 1$ and $\gamma_\ell =$ identity, (5) and (6) show that the nets $\mathcal{N}$, $\widetilde{\mathcal{N}}$ are identical.

II. Approximate Arithmetic Progressions.

A. Preliminaries.

(1) **Definition.** *Let E, $E_1, \dots, E_n \subset \mathbb{C}$ be given. We say that E is the approximate union of $E_1, \dots, E_n$ if the following conditions hold.*

(2) *$E \subset E_1 \cup \cdots \cup E_n$, and*

(3) *Any point belonging to exactly one of the $E_1, \dots, E_n$ belongs to E.*

(4) **Definition.** *For $\omega, \beta \in \mathbb{C}$ with $\omega \neq 0$, define $\Pi(\omega,\beta) = \{\omega k + \beta : k \in \mathbb{Z}\}$. We say that $E \subset \mathbb{C}$ approximates $\Pi(\omega,\beta)$ if for every $\varepsilon > 0$ the following conditions hold.*

(5) *All but finitely many points of E lie within distance ε of some point in $\Pi(\omega,\beta)$, and*

(6) *All but finitely many points of $\Pi(\omega,\beta)$ lie within distance ε of some point in E.*

Note that

(7) $\Pi(\omega,\beta) = \Pi(\omega',\beta')$ if and only if $\omega' = \pm\,\omega$ and $\beta' = \beta + \omega\, m$ for some $m \in \mathbb{Z}$.

(8) **Definition.** *Let H be a set of integers. We define the upper and lower densities $\Delta^*(H)$, $\Delta_*(H)$ by setting*

$$(9)\qquad \Delta^*(H) = \limsup_{\substack{N\to\infty \\ M\to-\infty}} \frac{\text{Number of integers in } [M,N]\cap H}{N-M}$$

and

$$(10)\qquad \Delta_*(H) = \liminf_{\substack{N\to\infty \\ M\to-\infty}} \frac{\text{Number of integers in } [M,N]\cap H}{N-M}.$$

If $\Delta^*(H) = \Delta_*(H)$, then we write $\Delta(H)$ for their common value, and we say that H has density $\Delta(H)$.

We will need the following special case of H. Weyl's Theorem on the equidistribution mod1 of arithmetic progressions (See [W]).

(11) **Theorem.** *Suppose $\theta \in \mathbb{R}$ is irrational and $0 < \varepsilon < 1/2$. Let $\beta \in \mathbb{R}$. Then $\Delta(\{k \in \mathbb{Z}\colon |\theta k + \beta - m| < \varepsilon$ for some $m \in \mathbb{Z}\}) = 2\varepsilon$.*

(12) **Corollary.** *Let ω, ω', β, β' be complex numbers, with $\omega, \omega' \neq 0$. Assume that ω'/ω is real and irrational. Then we can make the density*

$$\Delta^*(\{k \in \mathbb{Z}\colon \operatorname{dist}\{\omega k + \beta, \Pi(\omega', \beta')\} < \varepsilon\})$$

arbitrarily small, by taking $\varepsilon > 0$ small enough.

B. The Deconstruction Lemma.

Suppose $E \subset \mathbb{C}$ is the approximate union of sets $E_1, \ldots, E_D$; and suppose that each E_j approximates an arithmetic progression $\Pi(\omega_j, \beta_j)$. We want to know that the progressions $\Pi(\omega_j, \beta_j)$ are uniquely determined by E. Also, for each j_0, we want to pick out infinitely many points $(x_\nu)_{\nu \geq 1}$ that belong to E_{j_0} but not to any E_j, $j \neq j_0$. The following result provides this information.

Deconstruction Lemma. *Let E, $E_1, \ldots, E_D$, $\widetilde{E}_1, \ldots, \widetilde{E}_{\widetilde{D}}$ be subsets of $\mathbb{C}$, and let $\Pi(\omega_1, \beta_1), \ldots, \Pi(\omega_D, \beta_D)$, $\Pi(\widetilde{\omega}_1, \tilde{\beta}_1), \ldots, \Pi(\widetilde{\omega}_{\widetilde{D}}, \tilde{\beta}_{\widetilde{D}})$ be arithmetic progressions. Assume the following conditions.*

(1) *E is the approximate union of $E_1, \ldots, E_D$.*

(2) *Each E_j approximates the arithmetic progression $\Pi(\omega_j, \beta_j)$.*

(3) *For $j \neq j'$ and p, q integers with $1 \leq q \leq 100\, D^2$, we have $|\omega_j/\omega_{j'}| \neq p/q$.*

(4) *E is the approximate union of $\widetilde{E}_1, \ldots, \widetilde{E}_{\widetilde{D}}$.*

(5) *Each $\widetilde{E}_j$ approximates the arithmetic progression $\Pi(\widetilde{\omega}_j, \tilde{\beta}_j)$.*

(6) *For $j \neq j'$, and p, q integers with $1 \leq q \leq 100\, \widetilde{D}^2$, we have $|\widetilde{\omega}_j/\widetilde{\omega}_{j'}| \neq p/q$.*

Then $D = \widetilde{D}$, and there is a permutation

$$\gamma: \{1, \ldots, D\} \to \{1, \ldots, D\}$$

with the following properties:

(7) *$\Pi(\omega_j, \beta_j) = \Pi(\widetilde{\omega}_{\gamma j}, \tilde{\beta}_{\gamma j})$ for $1 \leq j \leq D$.*

(8) *Given j_0, $1 \leq j_0 \leq D$, there is a sequence $(x_\nu)_{\nu \geq 1}$ in $\mathbb{C}$ such that*

(9) *$|x_\nu| \to \infty$ as $\nu \to \infty$,*

(10) *Each x_ν belongs to E_{j_0} but not to E_j for $j \neq j_0$,*

(11) *Each x_ν belongs to $\widetilde{E}_{\gamma j_0}$, but not to $\widetilde{E}_j$ for $j \neq \gamma\, j_0$.*

C. Preparation for the proof of the Deconstruction Lemma.

We begin with a definition. We say that $\Pi(\omega, \beta)$ *fits into* $E \subset \mathbb{C}$ if for any $\varepsilon > 0$ we have

$$(1) \qquad \Delta_*(\{k \in \mathbb{Z}\colon \operatorname{dist}\{\omega k + \beta, E\} < \varepsilon\}) \geq \frac{9}{10}\,.$$

Note that (1) is phrased in terms of ω and β, but in fact depends only on $\Pi(\omega, \beta)$. (See (A.7)).

(2) **Lemma.** *If E, E_j, $\Pi(\omega_j, \beta_j)$ are as in the Deconstruction Lemma, then $\Pi(\omega_j, \beta_j)$ fits into E.*

PROOF. Fix $j' \neq j$. For small enough $\varepsilon > 0$ we will estimate

$$\Delta_\varepsilon(j,j') = \Delta^*(\{k \in \mathbb{Z}\colon \operatorname{dist}\{\omega_j k + \beta_j,\ \Pi(\omega_{j'},\beta_{j'})\} < \varepsilon\}). \tag{3}$$

Let ℓ_j,$\ell_{j'}$ denote the lines $\omega_j\mathbb{R} + \beta_j$, $\omega_{j'}\mathbb{R} + \beta_{j'}$ in $\mathbb{C}$. We distinguish several cases.

CASE 1: $\ell_j \neq \ell_{j'}$. Then $\operatorname{dist}\{x, \ell_{j'}\}$ is bounded below by a positive constant as $x \in \ell_j$ tends to infinity. Hence, $\Delta_\varepsilon(j,j') = 0$ for $\varepsilon > 0$ small enough.

CASE 2: $\ell_j = \ell_{j'}$ and $\omega_j/\omega_{j'}$ is irrational. Then by (A.12), we can make $\Delta_\varepsilon(j,j')$ arbitrarily small by taking $\varepsilon > 0$ small enough.

CASE 3: $\ell_j = \ell_{j'}$ and $\omega_j/\omega_{j'} = p/q$ in lowest terms, with $p, q \in \mathbb{Z}$ and $q > 0$. In view of (B.3), we have $q > 100D^2$.

Assume we are given distinct integers $k_1, k_2 \in \mathbb{Z}$ with

$$\operatorname{dist}\{\omega_j k_1 + \beta_j,\ \Pi(\omega_{j'},\beta_{j'})\} < \varepsilon, \tag{4}$$

and

$$\operatorname{dist}\{\omega_j k_2 + \beta_j,\ \Pi(\omega_{j'},\beta_{j'})\} < \varepsilon. \tag{5}$$

Thus, for integers m_1 and m_2, we have

$$|(\omega_j k_1 + \beta_j) - (\omega_{j'} m_1 + \beta_{j'})| < \varepsilon, \tag{6}$$

and

$$|(\omega_j k_2 + \beta_j) - (\omega_{j'} m_2 + \beta_{j'})| < \varepsilon. \tag{7}$$

Subtracting (6) from (7), and recalling that $\omega_j/\omega_{j'} = p/q$, we get

$$\left|\frac{p}{q} - \frac{m_2 - m_1}{k_2 - k_1}\right| < \frac{2\varepsilon}{|\omega_{j'}|\,|k_2 - k_1|}. \tag{8}$$

If $p/q \neq (m_2 - m_1)/(k_2 - k_1)$, then

$$\left|\frac{p}{q} - \frac{m_2 - m_1}{k_2 - k_1}\right| \geq \frac{1}{q\,|k_2 - k_1|},$$

which contradicts (8) provided we take $\varepsilon < |\omega_{j'}|/(2q)$. Hence, $p/q = (m_2 - m_1)/(k_2 - k_1)$. Since p/q is in lowest terms, it follows that $k_2 - k_1$

is a multiple of q. So we have shown that (4), (5) imply $k_2 \equiv k_1 \bmod q$. It follows at once that $\Delta_\varepsilon(j, j') \leq 1/q$, for $\varepsilon > 0$ small enough.

Since $q > 100\,D^2$ in *CASE* 3, our analysis of the above cases gives $\Delta_\varepsilon(j, j') < 1/(100\,D^2)$ for $j' \neq j$, if $\varepsilon > 0$ is small enough.

Summing over all $j' \neq j$ and recalling (3), we get

$$\Delta^*(\{k \in \mathbb{Z}\colon \operatorname{dist}\{\omega_j k + \beta_j, \Pi(\omega_{j'}, \beta_{j'})\} < \varepsilon \text{ for some } j' \neq j\}) < \frac{1}{100\,D}\,. \tag{9}$$

Now suppose $k \in \mathbb{Z}$ satisfies

$$\operatorname{dist}\{\omega_j k + \beta_j,\ \Pi(\omega_{j'}, \beta_{j'})\} > \varepsilon\,, \qquad \text{for all } j' \neq j\,. \tag{10}$$

Since E_j approximates $\Pi(\omega_j, \beta_j)$, we can find $x_k^j \in E_j$ for all but finitely many k so that

$$|x_k^j - (\omega_j k + \beta_j)| \leq \frac{\varepsilon}{10}\,. \tag{11}$$

In particular, $|x_k^j| \to \infty$ as $k \to \infty$.

On the other hand, since $E_{j'}$ approximates $\Pi(\omega_{j'}, \beta_{j'})$, we have $E_{j'} \subset F_{j'} \cup \{z \in \mathbb{C}\colon \operatorname{dist}\{z,\ \Pi(\omega_{j'}, \beta_{j'})\} < \varepsilon/10\}$ with $F_{j'}$ finite. Hence, (10) implies

$$\operatorname{dist}\{\omega_j k + \beta_j, E_{j'}\} \geq \frac{9}{10}\,\varepsilon\,, \qquad \text{for all } j' \neq j\,, \tag{12}$$

for all but finitely many k. Comparing (11) and (12), we see that $x_k^j \notin E_{j'}$, $j' \neq j$. Thus, all but finitely many k satisfying (10) have the property $x_k^j \in E_j \setminus \cup_{j' \neq j} E_{j'}$. Since E is the approximate union of $E_1, \ldots, E_D$, it follows that $x_k^j \in E$. Hence, (11) implies

$$\operatorname{dist}\{\omega_j k + \beta_j, E\} < \varepsilon\,. \tag{13}$$

So (13) holds for all but finitely many of the k that satisfy (10). Therefore, by (9), we have $\Delta_*(\{k \in \mathbb{Z}\colon \operatorname{dist}\{\omega_j k + \beta_j, E\} < \varepsilon\}) \geq 1 - 1/(100\,D)$, which shows that $\Pi(\omega_j, \beta_j)$ fits into E.

(14) **Lemma.** *Let* $\Pi(\omega, \beta)$ *and* $\Pi(\omega', \beta')$ *be arithmetic progressions, with* $\Pi(\omega, \beta) \not\subset \Pi(\omega', \beta')$. *Fix* $D > 0$, *and define*

$$q(\omega, \omega') = \begin{cases} q\,, & \text{if } \dfrac{\omega}{\omega'} = \dfrac{p}{q} \text{ in lowest terms, with } p, q \in \mathbb{Z},\ q \geq 2\,. \\[2mm] 100\,D^2\,, & \text{if } \dfrac{\omega}{\omega'} \text{ is irrational, an integer, or non-real.} \end{cases} \tag{15}$$

Then

$$\Delta^*(\{k \in \mathbb{Z}: \operatorname{dist}\{\omega k + \beta, \Pi(\omega', \beta')\} < \varepsilon\}) \leq \frac{1}{q(\omega, \omega')}, \tag{16}$$

for $\varepsilon > 0$ *small enough.*

PROOF. As in the previous lemma, we set $\ell = \omega\mathbb{R} + \beta$, $\ell' = \omega'\mathbb{R} + \beta'$, and we distinguish several cases.

CASE 1: $\ell \neq \ell'$. As in the proof of the previous lemma, the left-hand side of (16) is equal to zero if ε is small enough.

CASE 2: $\ell = \ell'$ and ω/ω' irrational. As in the proof of the previous lemma, we can make the left-hand side of (16) arbitrarily small by taking ε small enough.

CASE 3: $\ell = \ell'$ and $\omega/\omega' = p/q$ in lowest terms, with $p, q \in \mathbb{Z}$ and $q \geq 2$. As in the proof of the previous lemma,

$$\operatorname{dist}\{\omega k_1 + \beta, \Pi(\omega', \beta')\} < \varepsilon, \quad \operatorname{dist}\{\omega k_2 + \beta, \Pi(\omega', \beta')\} < \varepsilon,$$

imply $k_2 = k_1 \bmod q$, so that (16) is obvious.

CASE 4: $\ell = \ell'$ and $\omega/\omega' = p$ for some integer p. Then $\beta' - \beta$ is not a multiple of ω', since $\Pi(\omega, \beta) \not\subset \Pi(\omega', \beta')$.
Take $\varepsilon < \min_{k\in\mathbb{Z}} |\beta' - \beta - k\omega'|$. Then for all $k, m \in \mathbb{Z}$ we have

$$|(\omega k + \beta) - (\omega' m + \beta')| = |\beta - \beta' - (m - pk)\omega'| > \varepsilon,$$

so that $\operatorname{dist}\{\omega k + \beta, \Pi(\omega', \beta')\} > \varepsilon$, and the left-hand side of (16) equals zero.

(17) **Lemma.** *Assume the hypotheses of the Deconstruction Lemma, and suppose*

$$|\omega_1| < |\omega_2| < \cdots < |\omega_D|. \tag{18}$$

Fix an integer s, $1 \leq s \leq D$. *Let* $\Pi(\omega, \beta)$ *be an arithmetic progression with the following properties:*

(19) $\Pi(\omega, \beta)$ *fits into* E,

(20) $\omega \neq p\omega_j/q$ *for* $p, q \in \mathbb{Z}$, $1 \leq q \leq 10\,s$, *if* $j < s$.

Then either $\Pi(\omega,\beta)=\Pi(\omega_s,\beta_s)$, *or else* $|\omega|>|\omega_s|$.

PROOF. Assume the lemma is false. Thus,

$$\Pi(\omega,\beta)\neq\Pi(\omega_s,\beta_s)\,, \tag{21}$$

$$|\omega|\leq|\omega_s|\,. \tag{22}$$

Suppose for the moment that $\Pi(\omega,\beta)\subset\Pi(\omega_j,\beta_j)$ for some j, $1\leq j\leq D$. Then

$$\omega=p\,\omega_j\,,\qquad \text{for an integer } p\neq 0\,. \tag{23}$$

If $j<s$, then (23) contradicts (20). Hence, $j\geq s$ and (18), (23) yield

$$|\omega|=|p|\,|\omega_j|\geq|\omega_j|\geq|\omega_s|\,. \tag{24}$$

Moreover, at least one of the inequalities in (24) will be strict, unless $p=\pm1$ and $j=s$. Therefore, (22) yields $p=\pm1$ and $j=s$. Since $\Pi(\omega,\beta)\subset\Pi(\omega_j,\beta_j)=\Pi(\omega_s,\beta_s)$ and $\omega=p\,\omega_j=\pm\,\omega_s$, it follows that $\Pi(\omega,\beta)=\Pi(\omega_s,\beta_s)$, contradicting (21). This contradiction proves that

$$\Pi(\omega,\beta)\not\subset\Pi(\omega_j,\beta_j)\,,\qquad \text{for } 1\leq j\leq D\,. \tag{25}$$

Next, we apply (19) and (1) to conclude that

$$\Delta_*(\{k\in\mathbb{Z}\colon \operatorname{dist}\{\omega k+\beta,E\}<\varepsilon\})\geq\frac{9}{10}\,,\qquad \text{for any } \varepsilon>0\,. \tag{26}$$

Since E is the approximate union of $E_1,\ldots,E_D$, we have $E\subset E_1\cup\cdots\cup E_D$, so that $\operatorname{dist}\{\omega k+\beta,E\}<\varepsilon$ implies $\operatorname{dist}\{\omega k+\beta,E_j\}<\varepsilon$ for some j. Hence, (26) yields

$$\sum_{j=1}^{D}\Delta^*(\{k\in\mathbb{Z}\colon\ \operatorname{dist}\{\omega k+\beta,E_j\}<\varepsilon\})\geq\frac{9}{10}\,,\quad \text{for any } \varepsilon>0\,. \tag{27}$$

Moreover, each E_j approximates $\Pi(\omega_j,\beta_j)$. Hence, given $\varepsilon>0$, we have

$$E_j\subset F_j\cup\{z\in\mathbb{C}\colon\ \operatorname{dist}\{z,\Pi(\omega_j,\beta_j)\}<\varepsilon\}$$

with F_j finite. So, for large integers k, $\operatorname{dist}\{\omega k+\beta,E_j\}<\varepsilon$ implies $\operatorname{dist}\{\omega k+\beta,\Pi(\omega_j,\beta_j)\}<2\varepsilon$. Thus, (27) implies

$$\sum_{j=1}^{D}\Delta^*(\{k\in\mathbb{Z}\colon\ \operatorname{dist}\{\omega k+\beta,\Pi(\omega_j,\beta_j)\}<2\,\varepsilon\})\geq\frac{9}{10}\,, \tag{28}$$

for any $\varepsilon > 0$. Taking ε small enough, and using (25) to apply Lemma (14), we obtain

$$(29) \qquad \Delta^*(\{k \in \mathbb{Z}\colon \operatorname{dist}\{\omega k + \beta, \Pi(\omega_j, \beta_j)\} < 2\varepsilon\}) \le \frac{1}{q(\omega, \omega_j)},$$

where

$$(30) \quad q(\omega, \omega_j) = \begin{cases} q, \text{ if } \dfrac{\omega}{\omega_j} = \dfrac{p}{q} \text{ in lowest terms, with } p, q \in \mathbb{Z},\ q \ge 2, \\ 100\, D^2, \text{ otherwise.} \end{cases}$$

From (28), (29), we obtain

$$(31) \qquad \sum_{j=1}^{D} \frac{1}{q(\omega, \omega_j)} \ge \frac{9}{10}.$$

On the other hand, we can prove an upper bound for the left-hand side of (31). Immediately from (20) and (30), we have $q(\omega, \omega_j) \ge 10\, s$ for $j < s$, so that

$$(32) \qquad \sum_{1 \le j < s} \frac{1}{q(\omega, \omega_j)} \le \frac{1}{10}.$$

To control $q(\omega, \omega_j)$ for $j \ge s$, we prove

$$(33) \qquad q(\omega, \omega_j) \ge 2,$$

and

$$(34) \qquad q(\omega, \omega_j) \le 10\, D, \qquad \text{for at most one } j \ge s.$$

In fact, (33) is immediate from (30). If (34) were false, then we would have

$$(35) \qquad \frac{\omega}{\omega_j} = \frac{p}{q}, \qquad \frac{\omega}{\omega_{j'}} = \frac{p'}{q'},$$

with $p, q, p', q' \in \mathbb{Z}$, $j, j' \ge s$, $j \ne j'$, $1 \le q, q' \le 10\, D$. From (18), (22), (35) we have $|\omega| \le |\omega_s| \le |\omega_j|$ so that $|p| \le |q|$; and similarly, $|p'| \le |q'|$. In particular,

$$(36) \qquad 0 < |p|,\ |p'|,\ |q|,\ |q'| \le 10\, D,$$

by another application of (35). A final application of (35) gives

$$\frac{\omega_j}{\omega_{j'}} = \frac{q\,p'}{p\,q'} \equiv \frac{P}{Q}\,. \tag{37}$$

Since $1 \le |Q| \le 100\,D^2$ by (36), equation (37) contradicts hypothesis (B.3) of the Deconstruction Lemma. This contradiction proves (34). Immediately from (33), (34), we obtain

$$\sum_{s \le j \le D} \frac{1}{q(\omega,\omega_j)} \le \frac{1}{2} + \frac{D-s}{10\,D} \le \frac{1}{2} + \frac{1}{10}\,. \tag{38}$$

Together, (32) and (38) yield

$$\sum_{j=1}^{D} \frac{1}{q(\omega,\omega_j)} \le \frac{7}{10}\,,$$

contradicting (31). Thus, assuming our lemma to be false, we arrived at a contradiction.

(39) **Lemma.** *Assume the hypotheses of the Deconstruction Lemma. Then there is no arithmetic progression* $\Pi(\omega,\beta)$ *with the following properties:*

(40) $\Pi(\omega,\beta)$ *fits into* E,

(41) $\omega \ne \frac{p}{q}\,\omega_j$ *for* $p, q \in \mathbb{Z}$, $1 \le q \le 10\,D$, *whenever* $1 \le j \le D$.

PROOF. Assume $\Pi(\omega,\beta)$ satisfies (40), (41). By (40) and (1), we have $\Delta_*(\{k \in \mathbb{Z}\colon \operatorname{dist}\{\omega k + \beta, E) < \varepsilon\}\} \ge 9/10$, for any $\varepsilon > 0$. As in the proof of the previous lemma, this implies that

$$\sum_{j=1}^{D} \Delta^*(\{k \in \mathbb{Z}\colon\ \operatorname{dist}\{\omega k + \beta, \Pi(\omega_j,\beta_j)\} < 2\,\varepsilon\}) \ge \frac{9}{10}\,, \tag{42}$$

for any $\varepsilon > 0$. Moreover, (41) shows that $\Pi(\omega,\beta) \not\subset \Pi(\omega_j,\beta_j)$ for any j, $1 \le j \le D$, so that Lemma (14) applies. We obtain from (14) and (42) the estimate

$$\sum_{j=1}^{D} \frac{1}{q(\omega,\omega_j)} \ge \frac{9}{10}\,, \tag{43}$$

with

$$(44)\quad q(\omega,\omega_j) = \begin{cases} q\,, \text{ if } \dfrac{\omega}{\omega_j} = \dfrac{p}{q} \text{ in lowest terms, with } p, q \in \mathbb{Z},\ q \geq 2\,, \\ 100\,D^2\,, \text{ otherwise.} \end{cases}$$

On the other hand, (41) and (44) imply $q(\omega, \omega_j) \geq 10\,D$ for all j, so that

$$\sum_{j=1}^{D} \frac{1}{q(\omega,\omega_j)} \leq \frac{1}{10}\,,$$

contradicting (43).

D. Proving the Deconstruction Lemma.

Let E, $E_1, \dots, E_D$, $\widetilde{E}_1, \dots, \widetilde{E}_{\widetilde{D}}$, $\Pi(\omega_j, \beta_j)$, $\Pi(\widetilde{\omega}_j, \tilde{\beta}_j)$ be as in the Deconstruction Lemma. Hypothesis (B.3) shows that the $|\omega_j|$ are all distinct. Without loss of generality, we may therefore permute the E_j and $\Pi(\omega_j, \beta_j)$ to reduce matters to the case

$$(1)\qquad |\omega_1| < \cdots < |\omega_D|\,.$$

Similarly, we may assume

$$(2)\qquad |\widetilde{\omega}_1| < \cdots < |\widetilde{\omega}_{\widetilde{D}}|\,.$$

Also, we may assume

$$(3)\qquad D \leq \widetilde{D}\,.$$

For the rest of the proof, we will assume (1), (2), (3). We will prove that

$$(4)\qquad \Pi(\omega_j, \beta_j) = \Pi(\widetilde{\omega}_j, \tilde{\beta}_j)\,, \qquad \text{for } 1 \leq j \leq D\,.$$

To see this, fix s, $1 \leq s \leq D$, and suppose

$$(5)\qquad \Pi(\omega_j, \beta_j) = \Pi(\widetilde{\omega}_j, \tilde{\beta}_j)\,, \qquad \text{for } 1 \leq j < s\,.$$

(This assumption is vacuous for $s = 1$). We will see that (5) implies

$$(6)\qquad \Pi(\omega_s, \beta_s) = \Pi(\widetilde{\omega}_s, \tilde{\beta}_s)\,.$$

In fact, (5) and (A.7) show that

$$\omega_j = \pm\widetilde{\omega}_j \quad \text{for} \quad 1 \le j < s\,. \tag{7}$$

The analogue of Lemma C.2 for the $\widetilde{E}_j$ and $\Pi(\widetilde{\omega}_j, \tilde{\beta}_j)$ shows that

$$\Pi(\widetilde{\omega}_s, \tilde{\beta}_s) \quad \text{fits into } E\,. \tag{8}$$

Since $s \le \min\{D, \widetilde{D}\}$, equation (7) and hypothesis (B.6) show that

$$\widetilde{\omega}_s \ne \frac{p}{q}\,\omega_j\,, \qquad \text{for } 1 \le j < s,\ p, q \in \mathbb{Z},\ 1 \le q \le 10\,s\,. \tag{9}$$

Conditions (8), (9) are the hypotheses of Lemma (C.17), which tells us that

$$\text{either} \quad \Pi(\widetilde{\omega}_s, \tilde{\beta}_s) = \Pi(\omega_s, \beta_s)\,, \quad \text{or else} \quad |\widetilde{\omega}_s| > |\omega_s|\,. \tag{10}$$

The same argument works with the rôles of the $\Pi(\omega_j, \beta_j)$ and $\Pi(\widetilde{\omega}_j, \tilde{\beta}_j)$ interchanged, so we have also

$$\text{either} \quad \Pi(\omega_s, \beta_s) = \Pi(\widetilde{\omega}_s, \tilde{\beta}_s)\,, \quad \text{or else} \quad |\omega_s| > |\widetilde{\omega}_s|\,. \tag{11}$$

Since at least one of the inequalities $|\omega_s| > |\widetilde{\omega}_s|$, $|\widetilde{\omega}_s| > |\omega_s|$ must be false, (10) and (11) imply (6). Thus, (5) implies (6), completing the proof of (4).

Next we show that

$$D = \widetilde{D}\,. \tag{12}$$

If (12) were false, then by (3) we would have

$$\widetilde{D} \ge D + 1\,. \tag{13}$$

By the analogue of Lemma C.2 for the $\widetilde{E}_j$ and $\Pi(\widetilde{\omega}_j, \tilde{\beta}_j)$, we know that

$$\Pi(\widetilde{\omega}_{D+1}, \tilde{\beta}_{D+1}) \quad \text{fits into } E\,. \tag{14}$$

Also, by (4) and (A.7) we have

$$\widetilde{\omega}_j = \pm\,\omega_j\,, \qquad \text{for } 1 \le j \le D\,. \tag{15}$$

Hence, hypothesis (B.6) shows that

$$\widetilde{\omega}_{D+1} \neq \frac{p}{q}\,\omega_j\,, \qquad \text{for } 1 \le j \le D,\ p,q \in \mathbb{Z},\ 1 \le q \le 10\,D\,. \tag{16}$$

Together, conditions (14) and (16) contradict Lemma C.39. This contradiction completes the proof of (12).

Next, fix j_0, $1 \le j_0 \le D$. We will construct a sequence $(x_\nu)_{\nu \ge 1}$ such that

$$|x_\nu| \to \infty\,, \tag{17}$$

$$x_\nu \in E_{j_0} \setminus \bigcup_{j \neq j_0} E_j\,, \tag{18}$$

$$x_\nu \in \widetilde{E}_{j_0} \setminus \bigcup_{j \neq j_0} \widetilde{E}_j\,. \tag{19}$$

To construct (x_ν) with these properties, we first note that

$$\Pi(\omega_{j_0}, \beta_{j_0}) \not\subset \Pi(\omega_j, \beta_j)\,, \qquad \text{for } j \neq j_0\,, \tag{20}$$

since ω_{j_0} is not an integer multiple of ω_j. Hence for $\varepsilon > 0$ small enough, Lemma C.14 shows that

$$\Delta^*(\{k \in \mathbb{Z}\colon\ \operatorname{dist}\{\omega_{j_0} k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} < \varepsilon\}) \le \frac{1}{q(\omega_{j_0}, \omega_j)}\,, \tag{21}$$

where

$$q(\omega_{j_0}, \omega_j) = \begin{cases} q\,, \text{ if } \dfrac{\omega_{j_0}}{\omega_j} = \dfrac{p}{q} \text{ in lowest terms, with } p,q \in \mathbb{Z},\ q \ge 2\,, \\ 100\,D^2\,, \text{ otherwise}\,. \end{cases}$$

Hypothesis (B.3) shows that $q(\omega_{j_0}, \omega_j) \ge 100D^2$ for $j \neq j_0$, so that (21) implies

$$\Delta^*(\{k \in \mathbb{Z}\colon\ \operatorname{dist}\{\omega_{j_0} k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} < \varepsilon\}) \le \frac{1}{100\,D^2}\,, \quad \text{for } j \neq j_0\,.$$

Summing over j, we obtain

$$\begin{aligned} &\Delta^*(\{k \in \mathbb{Z}\colon\ \operatorname{dist}\{\omega_{j_0} k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} < \varepsilon\,,\ \text{for some } j \neq j_0\}) \\ &\qquad \le \frac{1}{100\,D}\,. \end{aligned} \tag{22}$$

Let $K = \{k \in \mathbb{Z}\colon \operatorname{dist}\{\omega_{j_0}k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} \geq \varepsilon$ for all $j \neq j_0\}$. Recall that E_j approximates $\Pi(\omega_j, \beta_j)$, so that

$$E_j \subset F_j \cup \{z \in \mathbb{C}\colon \operatorname{dist}\{z, \Pi(\omega_j, \beta_j)\} < \varepsilon/3\}$$

with F_j finite. Therefore, for all but finitely many $k \in \mathbb{Z}$ we have

$$\operatorname{dist}\{\omega_{j_0}k + \beta_{j_0}, E_j\} < \frac{2\varepsilon}{3} \quad \text{implies} \quad \operatorname{dist}\{\omega_{j_0}k + \beta_{j_0}, \Pi(\omega_j, \beta_j)\} < \varepsilon .$$

It follows that

$$\operatorname{dist}\{\omega_{j_0}k + \beta_{j_0}, E_j\} \geq \frac{2}{3}\varepsilon, \qquad \text{for all } j \neq j_0, \tag{23}$$

for all but finitely many $k \in K$.

Similarly, $\widetilde{E}_j$ approximates $\Pi(\widetilde{\omega}_j, \tilde{\beta}_j) = \Pi(\omega_j, \beta_j)$ by (4) and (12), so the proof of (23) yields also

$$\operatorname{dist}\{\omega_{j_0}k + \beta_{j_0}, \widetilde{E}_j\} \geq \frac{2}{3}\varepsilon, \qquad \text{for all } j \neq j_0, \tag{24}$$

for all but finitely many $k \in K$.

On the other hand, E_{j_0} approximates $\Pi(\omega_{j_0}, \beta_{j_0})$. Hence, for all but finitely many $k \in K$ we can find

(25) $\hat{x}_k \in E_{j_0}$ satisfying

(26) $|\hat{x}_k - (\omega_{j_0}k + \beta_{j_0})| < \frac{\varepsilon}{3}$.

Comparing (26) with (23), (24), we conclude that all but finitely many $k \in K$ satisfy

(27) $\hat{x}_k \notin E_j$ for $j \neq j_0$, and

(28) $\hat{x}_k \notin \widetilde{E}_j$ for $j \neq j_0$.

Since E is the approximate union of $E_1, \ldots, E_D$, we know from (25) and (27) that $\hat{x}_k \in E$. This in turn gives $\hat{x}_k \in \widetilde{E}_1 \cup \cdots \cup \widetilde{E}_{\widetilde{D}}$, since E is the approximate union of the $\widetilde{E}_j$. In view of (28), we obtain

$$\hat{x}_k \in \widetilde{E}_{j_0}, \tag{29}$$

for any k satisfying (25)-(28).

Thus, for all but finitely many $k \in K$, the following hold:

(30) $|\hat{x}_k - (\omega_{j_0} k + \beta_{j_0})| < \varepsilon$,

(31) $\hat{x}_k \in E_{j_0} \setminus \bigcup_{j \neq j_0} E_j$,

(32) $\hat{x}_k \in \widetilde{E}_{j_0} \setminus \bigcup_{j \neq j_0} \widetilde{E}_j$.

Finally, let $(x_\nu)_{\nu \geq 1}$ be an enumeration of the $\hat{x}_k$ for $k \in K$ that satisfy (30), (31), (32). Estimate (22) and the definition of K show that there are indeed infinitely many such $\hat{x}_k$, so we get an infinite sequence. Estimate (30) shows that $|x_\nu| \to \infty$ as $\nu \to \infty$. Hence, (17), (18), (19) follow at once from (30), (31), (32). We have proven the conclusions of the Deconstruction Lemma, with $\gamma =$ identity.

We conclude this section with a simple special case of the Deconstruction Lemma.

(33) **Corollary.** *Suppose E_j approximates $\Pi(\omega_j, \beta_j)$ for $1 \leq j \leq D$. Assume that $|\omega_j/\omega_{j'}| \neq p/q$ for $j \neq j'$, $p, q \in \mathbb{Z}$, $1 \leq q \leq 100\, D^2$. Then for each j_0, $1 \leq j_0 \leq D$, we can find a sequence $(x_\nu)_{\nu \geq 1}$ of complex numbers, such that*

(34) *$|x_\nu| \to \infty$ as $\nu \to \infty$, and*

(35) *$x_\nu \in E_{j_0} \setminus \bigcup_{j \neq j_0} E_j$ for each ν.*

PROOF. Set $\widetilde{E}_j = E_j$, $\Pi(\widetilde{\omega}_j, \tilde{\beta}_j) = \Pi(\omega_j, \beta_j)$, $E = E_1 \cup \cdots \cup E_D$. Then the Deconstruction Lemma applies, and it gives a sequence $(x_\nu)_{\nu \geq 1}$ satisfying (34) and (35).

III. Analytic Continuation of Neural Nets.

A. Preliminaries.

Let $\mathcal{N} = [(D_0, \ldots, D_L), (\omega^\ell_{jk}), (\theta^\ell_j)]$ be a neural net. We will show that the functions $x^\ell_j(t, \mathcal{N})$, defined initially for t real, continue analytically to an open subset of $\mathbb{C}$ with countable complement. We will analyze the largest domain Ω to which we can analytically continue the output $x^L_1(t, \mathcal{N})$. The point-set topology of Ω leads us to define a hierarchy of singular sets $\text{Sing}(\ell, \mathcal{N})$ in the complex plane. The sets $\text{Sing}(\ell, \mathcal{N})$ are defined entirely in terms of the output function $t \mapsto x_1^L(t, \mathcal{N})$ $(t \in \mathbb{R})$, yet they carry a lot of information on the architecture, weights and thresholds of $\mathcal{N}$.

We begin our discussion with a simple, general result on analytic functions defined in the complement of a countable set.

(1) **Lemma.** *Let $f(t)$ be a function on $\mathbb{R}$, and suppose that f continues analytically to an open set $\Omega \subset \mathbb{C}$, with $\mathbb{R} \subset \Omega$ and $\mathbb{C} \setminus \Omega$ countable. Then there is one and only one open set $\Omega_* \subset \mathbb{C}$ with the following properties:*

(2) $\mathbb{R} \subset \Omega_*$,

(3) $\mathbb{C} \setminus \Omega_*$ *is countable,*

(4) *Let $\Omega' \subset \mathbb{C}$ be any connected open set that meets $\mathbb{R}$. Then f continues analytically into Ω' if and only if $\Omega' \subset \Omega_*$.*

We call Ω_* the natural domain of f.

PROOF. We start with the following remark.

(5) Suppose $\Omega_1, \Omega_2 \subset \mathbb{C}$ are open sets, with Ω_1 connected and $\mathbb{C} \setminus \Omega_2$ countable. Let F_1, F_2 be analytic on Ω_1, Ω_2 respectively, and assume $F_1 = F_2$ to infinite order at some point of $\Omega_1 \cap \Omega_2$. Then $F_1 = F_2$ on all of $\Omega_1 \cap \Omega_2$.

Indeed, (5) is immediate from the fact that $\Omega_1 \cap \Omega_2$ is the complement of a countable set in Ω_1, and thus $\Omega_1 \cap \Omega_2$ is connected.

Now let W be the collection of all open sets $\Omega' \subset \mathbb{C}$ such that $\mathbb{R} \subset \Omega'$, $\mathbb{C} \setminus \Omega'$ is countable, and f continues analytically to Ω'. If Ω', $\Omega'' \in W$, and if F, G denote the analytic continuations of f to Ω', Ω'' respectively, then $F = G$ in $\Omega' \cap \Omega''$ by (5). It follows that f continues analytically to $\Omega_* = \cup_{\Omega' \in W} \Omega'$. Since $\Omega \in W$ by hypothesis, properties (2) and (3) are obvious, and we know that

(6) f continues analytically to any open set $\Omega' \subset \Omega_*$.

Next, suppose $\Omega' \subset \mathbb{C}$ is open and connected, and meets $\mathbb{R}$; and assume f continues analytically to an analytic function G on Ω'. Let F denote the analytic continuation of f to Ω_*. Then $F = G$ on $\Omega' \cap \Omega_*$ by (5), so that f continues analytically to $\Omega' \cup \Omega_*$. We have shown that

(7) If $\Omega' \subset \mathbb{C}$ is open, connected and meets $\mathbb{R}$, and if f continues analytically to Ω', then $\Omega' \subset \Omega_*$.

Assertions (6) and (7) complete the proof of (4). It remains only to prove the uniqueness of Ω_*. Thus, suppose Ω_*^1 and Ω_*^2 both have properties (2), (3), (4). Then Ω_*^1 is an open, connected set that meet $\mathbb{R}$, and f continues analytically to Ω_*^1. Since Ω_*^2 has property (4), it follows that $\Omega_*^1 \subset \Omega_*^2$. Similarly, $\Omega_*^2 \subset \Omega_*^1$, which proves that Ω_* is unique.

(8) **Definition.** *Let f be a function on $\mathbb{R}$. If f continues analytically to an open set with countable complement, then we define the sets* $\operatorname{Sing}(\ell, f) \subset \mathbb{C}$ *for $\ell \geq 0$ by the following induction:*

(9) $\operatorname{Sing}(0, f)$ *is the complement of the natural domain of f,*

(10) $\operatorname{Sing}(\ell+1, f)$ *is the set of accumulation points of* $\operatorname{Sing}(\ell, f)$.

We will take f to be the output of a neural net. The next two lemmas help us to show that f continues analytically to an open set with countable complement, and to understand the sets $\operatorname{Sing}(\ell, f)$.

(11) **Lemma.** *Let F be analytic on a connected, open set $\Omega \subset \mathbb{C}$. Let $\Pi(\omega, \beta)$ be an arithmetic progression. Suppose that F is either nonconstant, or else identically equal to a constant not belonging to $\Pi(\omega, \beta)$. Then the set $E = \{t \in \Omega\colon F(t) \in \Pi(\omega, \beta)\}$ has no accumulation points in Ω. In particular, E is countable.*

PROOF. Suppose $t_\nu \to t_*$ as $\nu \to \infty$, with $t_\nu \in E$ and $t_* \in \Omega$. Then $F(t_\nu) \to F(t_*)$ and $F(t_\nu) \in \Pi(\omega, \beta)$. It follows that $F(t_\nu)$ is eventually constant: $F(t_\nu) = b$ for all $\nu \geq \nu_0$, with $b \in \Pi(\omega, \beta)$. Since $\{t_\nu\}$ accumulate at t_* and Ω is connected, it follows in turn that $F(t) = b$ for all $t \in \Omega$, contradicting our hypothesis on F. Thus, E has no accumulation points in Ω. This implies that $E_N = \{t \in E\colon |t| \leq N \text{ and } \operatorname{dist}\{t, \mathbb{C} \setminus \Omega\} \geq 1/N\}$ is a bounded set without accumulation points. Hence E_N is finite, so that $E = \cup_{N \geq 1} E_N$ is countable.

(12) **Lemma.** *Let U be a disc centered at $z_0 \in \mathbb{C}$. Suppose Φ is meromorphic and Ψ analytic in a neighborhood of $\bar{U}$. Assume Φ has a single pole at z_0 (not necessarily simple). Let $\Pi(\omega, \beta)$ be an arithmetic progression. Then the set*

$$E = \{\Phi(t) + \Psi(t)\colon\ t \in U \setminus \{z_0\} \quad \textit{and} \quad \Phi(t) \in \Pi(\omega, \beta)\} \tag{13}$$

approximates the arithmetic progression

$$\Pi(\omega, \beta + \Psi(z_0)). \tag{14}$$

PROOF. Suppose that Φ has a pole of order m at z_0. Then for large enough $\zeta \in \mathbb{C}$, the solutions of

$$\Phi(z) = \zeta \tag{15}$$

are given by a Puiseux expansion. That is, the solutions to (15) are

$$z = H(\zeta^{-1/m}), \tag{16}$$

where H is analytic in a neighborhood of the origin, and $\zeta^{-1/m}$ runs over all the m^{th} roots of ζ^{-1}. Also, $H(0) = z_0$ and $H'(0) \neq 0$ (see [H]).

Let $\varepsilon > 0$ be given. We have to verify

(17) All but finitely many $\xi \in E$ lie within distance ε of $\Pi(\omega, \beta + \Psi(z_0))$, and

(18) All but finitely many $\xi \in \Pi(\omega, \beta + \Psi(z_0))$ lie within distance ε of E.

Pick $\delta > 0$ so small that

$$|z - z_0| \leq \delta \quad \text{implies} \quad z \in U \text{ and } |\Psi(z) - \Psi(z_0)| < \varepsilon. \tag{19}$$

To verify (17), suppose

$$0 < |z - z_0| \leq \delta \quad \text{and} \quad \Phi(z) \in \Pi(\omega, \beta). \tag{20}$$

Then dist $\{\Phi(z) + \Psi(z), \Pi(\omega, \beta + \Psi(z_0))\} \leq |\Psi(z) - \Psi(z_0)| < \varepsilon$. On the other hand, Φ is analytic on a neighborhood of the closure of $\widehat{U} = \{z \in U: |z - z_0| > \delta\}$, and Φ is non-constant on $\widehat{U}$ since Φ has a pole at z_0. Hence, Φ is bounded on $\widehat{U}$, and $\{z \in \widehat{U}\colon \Phi(z) = \xi\}$ is finite for any ξ. It follows that $\{z \in \widehat{U}\colon \Phi(z) \in \Pi(\omega, \beta)\}$ is finite. Thus, all but finitely many points of E arise as $\xi = \Phi(z) + \Psi(z)$ for some z satisfying (20), and therefore satisfy dist $\{\xi, \Pi(\omega, \beta + \Psi(z_0))\} < \varepsilon$. This proves (17).

To verify (18), let $\xi \in \Pi(\omega, \beta + \Psi(z_0))$ be sufficiently large, and let $z = H((\xi - \Psi(z_0))^{-1/m})$ for any choice of the m^{th} root. Thus, $z \in U \setminus \{z_0\}$ and $\Phi(z) = \xi - \Psi(z_0) \in \Pi(\omega, \beta)$, so that $\zeta = \Phi(z) + \Psi(z) = \xi - \Psi(z_0) + \Psi(z)$ belongs to E. Moreover, if ξ is large enough, then $(\xi - \Psi(z_0))^{-1/m}$ will be so small that $|z - z_0| = |H((\xi - \Psi(z_0))^{-1/m}) - H(0)| < \delta$. Therefore, $|\Psi(z) - \Psi(z_0)| < \varepsilon$ by (19), so that $|\zeta - \xi| = |\Psi(z) - \Psi(z_0)| < \varepsilon$, and thus dist $\{\xi, E\} < \varepsilon$. This completes the proof of (18).

B. Continuing the Output to the Complement of a Countable Set.

Fix a neural net $\mathcal{N} = [(D_0, \ldots, D_L), (\omega^{\ell}_{jk}), (\theta^{\ell}_j)]$. Recall that $D_0 = D_L = 1$. By induction on ℓ $(0 \le \ell \le L)$ we will define for $1 \le j \le D_\ell$ a set $\Omega^\ell_j \subset \mathbb{C}$ and a function $x^\ell_j(t, \mathcal{N})$ on Ω^ℓ_j. For $\ell = 0$, we set

(1) $\Omega^0_1 = \mathbb{C}$, and

(2) $x^0_1(t, \mathcal{N}) = t$.

Assume we have defined the $\Omega^{\ell-1}_j$ and $x^{\ell-1}_j(t, \mathcal{N})$ for a fixed ℓ, $1 \le \ell \le L$. Then set

$$\Omega^{\ell-1}_* = \bigcap_{1 \le j \le D_{\ell-1}} \Omega^{\ell-1}_j, \tag{3}$$

$$E^\ell_j = \left\{ t \in \Omega^{\ell-1}_* \colon \sum_{k=1}^{D_{\ell-1}} \omega^\ell_{jk} x^{\ell-1}_k(t, \mathcal{N}) + \theta^\ell_j \in \Pi(2\pi i, \pi i) \right\}, \tag{4}$$

$$\Omega^\ell_j = \Omega^{\ell-1}_* \setminus E^\ell_j, \tag{5}$$

and

$$x^\ell_j(t, \mathcal{N}) = \sigma\left(\sum_{k=1}^{D_{\ell-1}} \omega^\ell_{jk} x^{\ell-1}_k(t, \mathcal{N}) + \theta^\ell_j \right), \qquad \text{for } t \in \Omega^\ell_j. \tag{6}$$

Note that (6) makes sense because we need not evaluate $\sigma(\,\cdot\,)$ at one of its poles. The poles of σ are precisely $\Pi(2\pi i, \pi i)$. Note also that $\mathbb{R} \subset \Omega^\ell_j$, since $\sum_{k=1}^{D_{\ell-1}} \omega^\ell_{jk} x^{\ell-1}_k(t, \mathcal{N}) + \theta^\ell_j$ is real for t real, so that $\mathbb{R} \cap E^\ell_j = \varnothing$. For real t, our formulas (2) and (6) agree with the definition of $x^\ell_j(t, \mathcal{N})$ given in Section I. Hence we have extended the outputs of the nodes from $\mathbb{R}$ to subsets Ω^ℓ_j of the complex plane.

Note that (3) leaves Ω^L_* undefined. We make the natural definition

$$\Omega^L_* = \Omega^L_1 \tag{7}$$

(recall that $D_L = 1$ and compare with (3)). Our definitions have the obvious consequences

$$\Omega^\ell_* = \Omega^{\ell-1}_* \setminus \bigcup_{j=1}^{D_\ell} E^\ell_j, \qquad 1 \le \ell \le L, \tag{8}$$

and

$$\mathbb{C} \setminus \Omega_*^\ell = \bigcup_{1 \le \ell' \le \ell} \bigcup_{j=1}^{D_{\ell'}} E_j^{\ell'} , \qquad 1 \le \ell \le L. \tag{9}$$

(10) **Lemma.** *The following properties hold for* $1 \le \ell \le L$, $1 \le j \le D_\ell$.

(11) Ω_j^ℓ *is open, and* $\mathbb{C} \setminus \Omega_j^\ell$ *is countable.*

(12) E_j^ℓ *is a countable subset of* $\Omega_*^{\ell-1}$, *with no accumulation points in* $\Omega_*^{\ell-1}$.

(13) $x_j^\ell(t, \mathcal{N})$ *is analytic on* Ω_j^ℓ.

(14) $x_j^\ell(t, \mathcal{N})$ *has poles at the points of* E_j^ℓ.

PROOF. We use induction on ℓ. Fix ℓ, $1 \le \ell \le L$, and assume

(15) $\Omega_j^{\ell-1}$ is open, and $\mathbb{C} \setminus \Omega_j^{\ell-1}$ is countable, $1 \le j \le D_{\ell-1}$, and

(16) $x_j^{\ell-1}(t, \mathcal{N})$ is analytic on $\Omega_j^{\ell-1}$, $1 \le j \le D_{\ell-1}$.

Note that (15), (16) are obvious for $\ell = 1$ by (1), (2).

We will show that (15), (16) imply (11)-(14). This will imply Lemma (10). From (15), (16) and (3), we see that $\Omega_*^{\ell-1}$ is open, that $\mathbb{C} \setminus \Omega_*^{\ell-1}$ is countable, and that

$$X_j^\ell(t) = \sum_{k=1}^{D_{\ell-1}} \omega_{jk}^\ell \, x_j^{\ell-1}(t, \mathcal{N}) + \theta_j^\ell , \qquad 1 \le j \le D_\ell , \tag{17}$$

is analytic on $\Omega_*^{\ell-1}$. Moreover, $\mathbb{R} \subset \Omega_*^{\ell-1}$, and $X_j^\ell(t)$ is real for $t \in \mathbb{R}$. Hence, if $X_j^\ell(t)$ is constant on $\Omega_*^{\ell-1}$, then that constant is real. In particular, $X_j^\ell(t)$ is either non-constant on $\Omega_*^{\ell-1}$, or else identically equal to a constant not in $\Pi(2\pi i, \pi i)$. Therefore, by Lemma A.11 and (4), the set $E_j^\ell \subset \Omega_*^{\ell-1}$ is countable and has no accumulation points in $\Omega_*^{\ell-1}$. This proves (12), from which (11) follows at once by virtue of (5), since $\Omega_*^{\ell-1}$ is open and has countable complement. Assertion (13) follows from the formula $x_j^\ell(t, \mathcal{N}) = \sigma(X_j^\ell(t))$, since $X_j^\ell(t) \notin \Pi(2\pi i, \pi i)$ for $t \notin E_j^\ell$. To verify (14), let $t_0 \in E_j^\ell \subset \Omega_*^{\ell-1}$. By (12), we can find a disc $U_\delta = \{t \in \mathbb{C}: |t - t_0| < \delta\} \subset \Omega_*^{\ell-1}$ such that $U_\delta \setminus \{t_0\}$ does not meet E_j^ℓ. Thus, $U_\delta \setminus \{t_0\} \subset \Omega_*^\ell$, so $X_j^\ell(t) \notin \Pi(2\pi i, \pi i)$ and $x_j^\ell(t, \mathcal{N}) = \sigma(X_j^\ell(t))$ for $t \in U_\delta \setminus \{t_0\}$. Moreover, $X_j^\ell(t)$ is analytic on $\Omega_*^{\ell-1}$, hence on U_δ;

and we have $X_j^\ell(t_0) \in \Pi(2\pi i, \pi i)$ since $t_0 \in E_j^\ell$. These remarks show that $x_j^\ell(t, \mathcal{N})$ has a pole at t_0, proving (14). The proof of (11)-(14) is complete.

Lemma (10) shows in particular that the output of the neural net $t \mapsto x_1^L(t, \mathcal{N})$ continues analytically from $\mathbb{R}$ to an open subset of $\mathbb{C}$ with countable complement. Hence, the natural domain of $x_1^L(t, \mathcal{N})$ and the sequence of singular sets $\operatorname{Sing}(\ell, x_1^L(t, \mathcal{N}))$ are well-defined. We write $\operatorname{Sing}(\ell, \mathcal{N})$ for $\operatorname{Sing}(\ell, x_1^L(t, \mathcal{N}))$, and note that

(18) The sets $\operatorname{Sing}(\ell, \mathcal{N})$, $\ell \geq 0$, are determined completely by the output $t \mapsto x_1^L(t, \mathcal{N})$ $(t \in \mathbb{R})$ of the neural net $\mathcal{N}$.

In a similar spirit, we see at once from the definitions (1)-(7) that

(19) For each ℓ, $1 \leq \ell \leq L$, the sets Ω_j^ℓ, Ω_*^ℓ, E_j^ℓ and the functions $x_j^\ell(t, \mathcal{N})$ are determined completely by the $D_{\ell'}$, $\omega_{jk}^{\ell'}$, $\theta_j^{\ell'}$ with $1 \leq \ell' \leq \ell$.

C. The Structure of the Singular Sets.

In this section, we will study the sets $\operatorname{Sing}(\ell, \mathcal{N})$ associated to a neural net $\mathcal{N} = [(D_0, \ldots, D_L), (\omega_{jk}^\ell), (\theta_j^\ell)]$, in terms of the sets Ω_*^ℓ, Ω_j^ℓ, E_j^ℓ defined in the previous section.

(1) **Lemma.** $E_1^L \subset \operatorname{Sing}(0, \mathcal{N}) \subset \bigcup_{\ell=1}^{L} \bigcup_{j=1}^{D_\ell} E_j^\ell$.

PROOF. Let Ω_* be the natural domain of $x_1^L(t, \mathcal{N})$ and let $X(t)$ be the analytic continuation of $x_1^L(t, \mathcal{N})$ to Ω_*. Lemma B.10 shows that $x_1^L(t, \mathcal{N})$ continues analytically from $\mathbb{R}$ to Ω_1^L. Hence the defining property (A.4) for the natural domain tells us that

(2) $\Omega_1^L \subset \Omega_*$, and

(3) $X(t) = x_1^L(t, \mathcal{N})$ for $t \in \Omega_1^L$.

From (2), (B.7), (B.9) and (A.9), we get

$$\operatorname{Sing}(0, \mathcal{N}) = \mathbb{C} \setminus \Omega_* \subset \mathbb{C} \setminus \Omega_1^L = \mathbb{C} \setminus \Omega_*^L = \bigcup_{1 \leq \ell \leq L} \bigcup_{j=1}^{D_\ell} E_j^\ell ,$$

which is half of Lemma 1.

To verify the other half of Lemma 1, suppose $t_0 \in E_1^L \cap \Omega_*$. Since $E_1^L \subset \Omega_*^{L-1}$ and E_1^L has no accumulation points in Ω_*^{L-1}, we know that a small enough disc

$$U_\delta = \{t \in \mathbb{C}\colon\ |t - t_0| < \delta\}$$

is contained in Ω_*^{L-1}, and that $U_\delta \setminus \{t_0\}$ does not meet E_1^L. Then (3) shows that $X(t) = x_1^L(t, \mathcal{N})$ in $U_\delta \setminus \{t_0\}$, while (B.14) gives

$$\lim_{\substack{t \to t_0 \\ t \neq t_0}} |x_1^L(t, \mathcal{N})| = \infty\,.$$

Hence, $\lim_{t \to t_0,\ t \neq t_0} |X(t)| = \infty$, which contradicts the fact that $X(t)$ is analytic on an open set Ω_* containing t_0. Therefore, $E_1^L \cap \Omega_*$ is empty, *i.e.* $E_1^L \subset \mathbb{C} \setminus \Omega_* = \operatorname{Sing}(0, \mathcal{N})$.

Next we study the E_j^ℓ, as well as

$$\mathring{E}_j^\ell = E_j^\ell \setminus \bigcup_{j' \neq j} E_{j'}^\ell\,. \tag{4}$$

To do so, we impose the following hypothesis on the weights (ω_{jk}^ℓ).

(5) **Assumption.** *$\omega_{jk}^\ell \neq 0$, and for $j \neq j'$, the ratio $\omega_{jk}^\ell / \omega_{j'k}^\ell$ is not equal to any fraction of the form p/q with p, q integers and $1 \le q \le 100\, D_\ell^2$.*

Immediately from (B.1)-(B.4), we see that E_j^1 consists of all $t \in \mathbb{C}$ such that $\omega_{j1}^1 t + \theta_j^1 \in \Pi(2\pi i, \pi i)$. In other words,

$$E_j^1 = \Pi\left(\frac{2\pi i}{\omega_{j1}^1}, \frac{\pi i - \theta_j^1}{\omega_{j1}^1}\right), \qquad 1 \le j \le D_1\,. \tag{6}$$

From (4), (5), (6) and Corollary II.D.33, we get

(7) $\mathring{E}_j^1$ is infinite. $1 \le j \le D$.

The following lemma shows how $E_j^{\ell+1}$ and $\mathring{E}_j^{\ell+1}$ look near a point of $\mathring{E}_j^{\ell}$.

(8) **Lemma.** *Fix* $t_0 \in \mathring{E}^{\ell}_{k_0}$, $1 \le \ell \le L-1$, $1 \le k_0 \le D_\ell$. *For* $\delta > 0$, *set*

$$U_\delta = \{t \in \mathbb{C}\colon\ |t - t_0| < \delta\}. \tag{9}$$

If δ *is small enough, then the following properties hold.*

(10) $U_{2\delta} \subset \Omega^{\ell-1}_*$.

(11) $U_{2\delta} \setminus \{t_0\} \subset \Omega^{\ell}_{k_0}$; *thus,* $x^{\ell}_{k_0}(t, \mathcal{N})$ *is analytic on* $U_{2\delta} \setminus \{t_0\}$, *with a pole at* t_0.

(12) $U_{2\delta} \subset \Omega^{\ell}_k$ *for* $k \ne k_0$; *thus* $x^{\ell}_k(t, \mathcal{N})$ *is analytic on* $U_{2\delta}$.

$$\begin{aligned}&(U_\delta \setminus \{t_0\}) \cap E^{\ell+1}_j \\ &\quad = \left\{ t \in U_\delta \setminus \{t_0\} \colon \sum_{k=1}^{D_\ell} \omega^{\ell+1}_{jk} x^{\ell}_k(t, \mathcal{N}) + \theta^{\ell+1}_j \in \Pi(2\pi i, \pi i) \right\},\end{aligned} \tag{13}$$

for $1 \le j \le D_{\ell+1}$.

(14) *The set* $F^{\ell}_j = \{x^{\ell}_{k_0}(t, \mathcal{N})\colon t \in (U_\delta \setminus \{t_0\}) \cap E^{\ell+1}_j\}$ *approximates the arithmetic progression* $\Pi\left(2\pi i/\omega^{\ell+1}_{jk_0}, \beta^{\ell+1}_{jk_0}\right)$ *for some complex number* $\beta^{\ell+1}_{jk_0}$, $1 \le j \le D_{\ell+1}$.

(15) *For each* j_0, $1 \le j_0 \le D_{\ell+1}$, t_0 *is an accumulation point of* $\mathring{E}^{\ell+1}_{j_0}$.

PROOF. We know that $t_0 \in \mathring{E}^{\ell}_{k_0} \subset E^{\ell}_{k_0} \subset \Omega^{\ell-1}_*$ by (B.12), so (10) holds simply because $\Omega^{\ell-1}_*$ is open. Another application of (B.12) shows that $U_{2\delta} \setminus \{t_0\}$ meets none of the E^{ℓ}_k $1 \le k \le D_\ell$, if δ is small. Since $t_0 \in \mathring{E}^{\ell}_{k_0}$, it follows that $U_{2\delta} \setminus \{t_0\} \subset \Omega^{\ell}_{k_0}$ and $U_{2\delta} \subset \Omega^{\ell}_k$ for $k \ne k_0$, by (B.5) and (10). Therefore, (11) and (12) follow from (B.13), (B.14).

Next note that (11), (12) yield $U_{2\delta} \setminus \{t_0\} \subset \Omega^{\ell}_*$. Hence (13) follows at once from the definition (B.4).

We set $U = U_\delta$,

$$\Phi(t) = \sum_{k=1}^{D_\ell} \omega^{\ell+1}_{jk}\, x^{\ell}_k(t, \mathcal{N}) + \theta^{\ell+1}_j,$$

$$\Psi(t) = -\sum_{\substack{1 \le k \le D_\ell \\ (k \ne k_0)}} \omega^{\ell+1}_{jk}\, x^{\ell}_k(t, \mathcal{N}) - \theta^{\ell+1}_j,$$

$z_0 = t_0$, $\Pi(\omega, \beta) = \Pi(2\pi i, \pi i)$. Then (11) and (12) show that the hypotheses of Lemma A.12 are satisfied. In view of (13), that lemma shows that $\{\omega_{jk_0}^{\ell+1} x_{k_0}^{\ell}(t, \mathcal{N}) \colon t \in (U \setminus \{t_0\}) \cap E_j^{\ell+1}\}$ approximates an arithemtic progression of the form $\Pi(2\pi i, \beta_j)$. This yields (14) at once.

It remains to verify (15). By (5), (14) and Corollary II.D.33, we can find a sequence $(x_\nu)_{\nu \geq 1}$ satisfying

(16) $|x_\nu| \to \infty$, as $\nu \to \infty$,

(17) $x_\nu \in F_{j_0}$,

(18) $x_\nu \notin F_j$, for $j \neq j_0$, $1 \leq j \leq D_{\ell+1}$.

By definition of F_j, (17) means that

(19) $x_\nu = x_{k_0}^{\ell}(t_\nu, \mathcal{N})$, with

(20) $t_\nu \in (U_\delta \setminus \{t_0\}) \cap E_{j_0}^{\ell+1}$.

If we had $t_\nu \in E_j^{\ell+1}$ for some $n \neq j_0$, then (19), (20) would imply $x_\nu \in F_j$, contradicting (18). Hence $t_\nu \notin E_j^{\ell+1}$, $j \neq j_0$, so that (20) can be sharpened to

$$t_\nu \in (U_\delta \setminus \{t_0\}) \cap \mathring{E}_{j_0}^{\ell+1}. \tag{21}$$

Also, (11), (19) and (16) show that $t_\nu \to t_0$ as $\nu \to \infty$. Therefore, (21) shows that t_0 is an accumulation point of $\mathring{E}_{j_0}^{\ell+1}$, which is (15).

(22) **Corollary.** *The set $\mathring{E}_j^{\ell}$ is infinite, for $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$.*

PROOF. We use induction on ℓ. For $\ell = 1$, the Corollary is already known (see (7)). If $\mathring{E}_{k_0}^{\ell}$ is non-empty, then (15) shows that $\mathring{E}_j^{\ell+1}$ must be infinite, completing the induction.

(23) **Corollary.** *The output function $x_1^L(t, \mathcal{N})$ is non-constant.*

PROOF. If $x_1^L(t, \mathcal{N})$ were constant, its natural domain would be all of $\mathbb{C}$, so that $\mathrm{Sing}(0, \mathcal{N})$ would be empty. However, we know that $\mathring{E}_1^L = E_1^L \subset \mathrm{Sing}(0, \mathcal{N})$ by Lemma C.1, and $\mathring{E}_1^L$ is infinite, by the preceding corollary.

(24) **Corollary.** *All the functions* $x^{\ell}_{j}(t,\mathcal{N})$, $1 \le \ell \le L$, $1 \le j \le D_{\ell}$, *are non-constant.*

PROOF. Fix $\bar{\ell}$, $\bar{\jmath}$. Then $x^{\bar{\ell}}_{\bar{\jmath}}(t,\mathcal{N})$ is the output of a simpler neural net $\mathcal{N}_{\#} = [(D^{\#}_{0},\dots,D^{\#}_{L_{\#}}),\omega^{\#\ell}_{jk},(\theta^{\#\ell}_{j})]$, defined by

$$\begin{gathered} L_{\#} = \bar{\ell},\ \ D^{\#}_{\ell} = D_{\ell}\,, \qquad \text{for } \ell < L_{\#}\,,\ D_{L_{\#}} = 1\,,\\ \omega^{\#\ell}_{jk} = \omega^{\ell}_{jk} \quad\text{and}\quad \theta^{\#\ell}_{j} = \theta^{\ell}_{j}\,, \qquad \text{for } \ell < L_{\#}\,,\\ \omega^{\#\bar{\ell}}_{1k} = \omega^{\bar{\ell}}_{\bar{\jmath}k} \quad\text{and}\quad \theta^{\#\bar{\ell}}_{1} = \theta^{\bar{\ell}}_{\bar{\jmath}}\,. \end{gathered}$$

The net $\mathcal{N}_{\#}$ again satisfies (5), so Corollary (23) applies to $\mathcal{N}_{\#}$. Thus $x^{L_{\#}}_{1}(t,\mathcal{N}_{\#})$ is non-constant, and we observed that $x^{\bar{\ell}}_{\bar{\jmath}}(t,\mathcal{N}) = x^{L_{\#}}_{1}(t,\mathcal{N}_{\#})$.

Next, we relate $\mathrm{Sing}(\ell,\mathcal{N})$ for $\ell \ge 1$ to the sets $E^{\ell'}_{j}$.

(25) **Lemma.** $\mathring{E}^{\ell}_{j} \subset \mathrm{Sing}(L-\ell,\mathcal{N})$ *for* $1 \le \ell \le L$, $1 \le j \le D_{\ell}$.

PROOF. We use downward induction on ℓ. When $\ell = L$, (25) is contained in (1). For the induction step, fix ℓ $(1 \le \ell \le L-1)$, and assume

$$\mathring{E}^{\ell+1}_{j} \subset \mathrm{Sing}(L-\ell-1,\mathcal{N})\,. \tag{26}$$

We shall prove that

$$\mathring{E}^{\ell}_{k_0} \subset \mathrm{Sing}(L-\ell,\mathcal{N})\,, \qquad \text{for } 1 \le k_0 \le D_{\ell}\,. \tag{27}$$

In fact, (26) and (15) show that every point of $\mathring{E}^{\ell}_{k_0}$ is an accumulation point of $\mathrm{Sing}(L-\ell-1,\mathcal{N})$ and thus belongs to $\mathrm{Sing}(L-\ell,\mathcal{N})$ by Definition A.10. Hence, (26) implies (27), completing the induction.

(28) **Lemma.** $\mathrm{Sing}(L-\bar{\ell},\mathcal{N}) \subset \bigcup_{1 \le \ell \le \bar{\ell}} \bigcup_{1 \le j \le D_{\ell}} E^{\ell}_{j}$, *for* $1 \le \bar{\ell} \le L$.

PROOF. Again we use downward induction on $\bar{\ell}$. When $\bar{\ell} = L$, (28) is contained in (1). For the induction step, fix $\bar{\ell}$, $2 \le \bar{\ell} \le L$, and assume

$$\mathrm{Sing}(L-\bar{\ell},\mathcal{N}) \subset \bigcup_{1 \le \ell \le \bar{\ell}} \bigcup_{j=1}^{D_{\ell}} E^{\ell}_{j}\,. \tag{29}$$

We shall prove that

$$\text{(30)} \qquad \operatorname{Sing}(L-\bar{\ell}+1,\mathcal{N}) \subset \bigcup_{1\le \ell<\bar{\ell}} \bigcup_{j=1}^{D_\ell} E_j^\ell \,.$$

In fact, (B.9) shows that the right-hand side of (29) is a closed set. Hence, any accumulation point of $\operatorname{Sing}(L-\bar{\ell},\mathcal{N})$ is again contained in the right-hand side of (29). By definition (A.10), this implies that

$$\operatorname{Sing}(L-\bar{\ell}+1,\mathcal{N}) \subset \bigcup_{1\le \ell\le\bar{\ell}} \bigcup_{j=1}^{D_\ell} E_j^\ell \,.$$

Therefore, (30) will follow if we can prove

(31) No point of $E_{j_0}^{\bar{\ell}}$ is an accumulation point of $\bigcup_{1\le \ell\le\bar{\ell}} \bigcup_{j=1}^{D_\ell} E_j^\ell$, $1\le j_0 \le D_{\bar{\ell}}$.

Assertion (31) is equivalent to

(32) $E_{j_0}^{\bar{\ell}}$ contains no accumulation points of E_j^ℓ $1\le\ell\le\bar{\ell}$, $1\le j_0\le D_{\bar{\ell}}$, $1\le j\le D_\ell$.

Thus, (30) follows from (32). To prove (32), we distinguish two cases.

CASE 1: $\ell<\bar{\ell}$. From (B.9) and (B.12) we see that

$$\mathbb{C}\setminus\Omega_*^{\bar{\ell}-1} = \bigcup_{1\le\ell<\bar{\ell}} \bigcup_{j=1}^{E_\ell} E_j^\ell$$

and that $E_{j_0}^{\bar{\ell}} \subset \Omega_*^{\bar{\ell}-1}$. Since $\Omega_*^{\bar{\ell}-1}$ is open, these remarks imply (32) for $\ell<\bar{\ell}$.

CASE 2: $\ell=\bar{\ell}$. Then (B.12) shows that $E_{j_0}^{\bar{\ell}} \subset \Omega_*^{\bar{\ell}-1}$, and that $E_j^{\bar{\ell}}$ has no accumulation points in $\Omega_*^{\bar{\ell}-1}$. These remarks imply (32) for $\ell=\bar{\ell}$.

Thus, (32) holds in either case, which completes the proof of (30). We have shown that (29) implies (30), completing the downward induction.

(33) **Corollary.** $\operatorname{Sing}(\ell,\mathcal{N})$ *is empty for* $\ell\ge L$.

PROOF. Lemma (28) yields $\mathrm{Sing}(L-1,\mathcal{N}) \subset \bigcup_{j=1}^{D_\ell} E_j^1$. From (6) we see that $\bigcup_{j=1}^{D_1} E_j^1$ has no accumulation points. Hence, $\mathrm{Sing}(L,\mathcal{N})$ is empty, from which (33) is obvious.

(34) **Lemma.** *Fix* $t_0 \in \overset{\circ}{E}{}_{k_0}^\ell$, $1 \le \ell \le L-1$, $1 \le k_0 \le D_\ell$, *and set*

$$U_\delta = \{t \in \mathbb{C}\colon\ |t-t_0| < \delta\}\,, \qquad \textit{for}\ \ \delta > 0\,. \tag{35}$$

If δ *is small enough, then* $\mathrm{Sing}(L-\ell-1,\mathcal{N}) \cap (U_\delta \smallsetminus \{t_0\})$ *is the approximate union of the sets*

$$E_j^{\ell+1} \cap (U_\delta \smallsetminus \{t_0\})\,, \qquad j = 1,\dots,D_{\ell+1}\,.$$

PROOF. We must prove two assertions:

$$\mathrm{Sing}(L-\ell-1,\mathcal{N}) \cap (U_\delta \smallsetminus \{t_0\}) \subset \bigcup_{j=1}^{D_{\ell+1}} E_j^{\ell+1} \cap (U_\delta \smallsetminus \{t_0\}) \tag{36}$$

and

(37) Any point belonging to exactly one of the sets $E_j^{\ell+1} \cap (U_\delta \smallsetminus \{t_0\})$, $1 \le j \le D_{\ell+1}$, belongs also to $\mathrm{Sing}(L-\ell-1,\mathcal{N}) \cap (U_\delta \smallsetminus \{t_0\})$.

However, (37) is immediate from (25), so it remains only to prove (36). From (28) we have
(38)

$$\mathrm{Sing}(L-\ell-1,\mathcal{N}) \cap (U_\delta \smallsetminus \{t_0\}) \subset \bigcup_{1 \le \ell' \le \ell+1} \bigcup_{j=1}^{D_{\ell'}} \left(E_j^{\ell'} \cap (U_\delta \smallsetminus \{t_0\})\right).$$

On the other hand, since $t_0 \in E_{k_0}^\ell$, (32) shows that $E_j^{\ell'} \cap (U_\delta \smallsetminus \{t_0\})$ is empty if $\ell' \le \ell$ and δ is small enough. Therefore, (38) implies (36).

(39) **Lemma.** $\mathrm{Sing}(L-1,\mathcal{N})$ *is the approximate union for the sets* E_j^1 *for* $j = 1,\dots,D_1$.

PROOF. Immediate from Lemmas 25 and 28.

D. Summary.

Let $\mathcal{N} = [(D_0, \dots, D_L), (\omega^\ell_{jk}), (\theta^\ell_j)]$ be a neural net. We make the following

(1) **Assumption.** *$\omega^\ell_{jk} \neq 0$, and for $j \neq j'$, the ratio $\omega^\ell_{jk}/\omega^\ell_{j'k}$ is not equal to any fraction of the form p/q with p, q integers and $1 \leq q \leq 100\, D^2_\ell$.*

(2) **Lemma.** *For each ℓ $(1 \leq \ell \leq L)$, the sets Ω^ℓ_j, Ω^ℓ_*, E^ℓ_j, $\mathring{E}^\ell_j$ and the functions $x^\ell_j(t, \mathcal{N})$ are determined entirely by the $D_{\ell'}$, $\omega^{\ell'}_{jk}$ and $\theta^{\ell'}_j$ for $\ell' \leq \ell$ (see* (B.19)).

(3) **Lemma.** *For each $\ell \geq 0$, the set* $\mathrm{Sing}\,(\ell, \mathcal{N})$ *is determined entirely by the output $t \mapsto x^L_1(t, \mathcal{N})$ $(t \in \mathbb{R})$ of the neural net (see* (B.18)).

(4) **Lemma.** *For $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$, the function $x^\ell_j(t, \mathcal{N})$ is analytic on Ω^ℓ_j, with poles at the points of E^ℓ_j (see* (B.13) *and* (B.14)).

(5) **Lemma.** $\mathrm{Sing}\,(\ell, \mathcal{N})$ *is empty for $\ell \geq L$ (see* (C.33)).

(6) **Lemma.** $\mathrm{Sing}\,(L-1, \mathcal{N})$ *is the approximate union of the arithmetic progressions $\Pi\left(2\pi i/\omega^1_{j1}, (\pi i - \theta^1_j)/\omega^1_{j1}\right)$ for $j = 1, \dots, D_1$ (see* (C.6) *and* (C.39)).

(7) **Lemma.** *For $1 \leq \ell \leq L$, $1 \leq j \leq D_\ell$, the set $\mathring{E}^\ell_j$ is infinite (see* (C.22)).

(8) **Lemma.** *Fix $t_0 \in \mathring{E}^\ell_{k_0}$, $1 \leq \ell \leq L-1$, $1 \leq k_0 \leq D_\ell$. For $\delta > 0$, set*

(9) $$U_\delta = \{t \in \mathbb{C} \colon |t - t_0| < \delta\}\,, \quad V_\delta = U_\delta \setminus \{t_0\}\,.$$

Then the following properties hold if δ is small enough.

(10) *$V_{2\delta} \subset \Omega^\ell_{k_0}$ (see* (C.11)).

(11) *$U_{2\delta} \subset \Omega^\ell_k$ for $k \neq k_0$ $(1 \leq k \leq D_\ell)$ (see* (C.12)).

(12) *$\mathrm{Sing}(L-\ell-1,\mathcal{N})\cap V_\delta$ is the approximate union of the sets $E_j^{\ell+1}\cap V_\delta$ for $j=1,\dots,D_{\ell+1}$, (see* (C.34)).

(13) $$E_j^{\ell+1}\cap V_\delta = \left\{ t\in V_\delta\colon \sum_{k=1}^{D_\ell} \omega_{jk}^{\ell+1} x_k^\ell(t,\mathcal{N}) + \theta_j^{\ell+1} \in \Pi(2\pi i,\pi i) \right\},$$

$1\le j\le D_{\ell+1}$, *(see* (C.13)).

(14) *The set $F_j=\{x_{k_0}^\ell(t,\mathcal{N})\colon t\in E_j^{\ell+1}\cap V_\delta\}$ approximates an arithmetic progression of the form $\Pi\left(2\pi i/\omega_{jk_0}^{\ell+1},\ \beta_{jk_0}^{\ell+1}\right)$ for some complex number $\beta_{jk_0}^{\ell+1}$, (see* (C.14)).

(15) **Lemma.** *For $1\le\ell\le L$, $1\le j\le D_\ell$, the function $t\mapsto x_j^\ell(t,\mathcal{N})$ $(t\in\mathbb{R})$ is non-constant (see* (C.24)).

IV. Proof of the Uniqueness Theorem.

A. Setting up the Induction.

In this section, we start the proof of Theorem I.B.9. We begin with some preliminary remarks. Let $\mathcal{N}=[(D_0,\dots,D_L),\ (\omega_{jk}^\ell),\ (\theta_j^\ell)]$ be a neural net satisfying condition (I.B.10). By (III.D.6) and Corollary II.D.33, the set $\mathrm{Sing}(L-1,\mathcal{N})$ is non-empty. On the other hand, (III.D.5) shows that $\mathrm{Sing}(\ell,\mathcal{N})$ is empty for $\ell\ge L$. Hence, the depth L of the neural net can be inferred from a knowledge of the sets $\mathrm{Sing}(\ell,\mathcal{N})$, $\ell\ge 0$. Lemma III.D.3 therefore shows that L can be inferred from knowledge of the output function $t\mapsto x_1^L(t,\mathcal{N})$. Thus, if two neural nets produce the same output, then they have the same depth.

Now let $\mathcal{N}=[(D_0,\dots,D_L),\ (\omega_{jk}^\ell),\ (\theta_j^\ell)]$, $\widetilde{\mathcal{N}}=[(\widetilde{D}_0,\dots,\widetilde{D}_{\widetilde{L}}),\ (\widetilde{\omega}_{jk}^\ell),\ (\widetilde{\theta}_j^\ell)]$ be neural nets satisfying the hypotheses of Theorem I.B.9. We must show that $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are isomorphic. If Theorem I.B.9 were false, then we could find a counterexample with $\max\{\mathrm{Size}(\mathcal{N}),\ \mathrm{Size}(\widetilde{\mathcal{N}})\}$ as small as possible. (Recall that the size of $\mathcal{N}$ is defined as the sum $D_0+\cdots+D_L$.) Thus, we may assume that

(1) $\max\{\mathrm{Size}(\mathcal{N}),\ \mathrm{Size}(\widetilde{\mathcal{N}})\}=S$, and that

(2) Theorem I.B.9 holds for any two neural nets $\mathcal{N}'$, $\widetilde{\mathcal{N}}$ of size strictly less than S.

Also, from the preceding paragraph, we know that

$$L = \widetilde{L}\,. \tag{3}$$

By induction on $\bar{\ell}$, $1 \le \bar{\ell} \le L$, we will prove that by subjecting $\mathcal{N}$, $\widetilde{\mathcal{N}}$ to isomorphisms we can achieve

$$D_\ell = \widetilde{D}_\ell\,, \qquad \text{for } \ell \le \bar{\ell}\,, \tag{4}$$

$$\omega^\ell_{jk} = \widetilde{\omega}^\ell_{jk}\,, \qquad \text{for } 1 \le \ell \le \bar{\ell},\ 1 \le j \le D_\ell\,,\ 1 \le k \le D_{\ell-1} \tag{5}$$

$$\theta^\ell_j = \widetilde{\theta}^\ell_j\,, \qquad \text{for } 1 \le \ell \le \bar{\ell},\ 1 \le j \le D_\ell\,. \tag{6}$$

If we can prove this for $\bar{\ell} = L$, then Theorem I.B.9 is established.

We will prove (4), (5), (6) by induction on $\bar{\ell}$. In this section, we treat the case $\bar{\ell} = 1$, while the next section gives the induction step. Thus, suppose $\bar{\ell} = 1$. Lemmas III.D.3 and III.D.6 show that the set $\text{Sing}\,(L-1, \mathcal{N})$ is the approximate union of the arithmetic progressions $\Pi\left(2\pi i/\omega^1_{j1}, (\pi i - \theta^1_j)/\omega^1_{j1}\right)$, $1 \le j \le D_1$, and also the approximate union of the progressions $\Pi\left(2\pi i/\widetilde{\omega}^1_{j1}, (\pi i - \widetilde{\theta}^1_j)/\widetilde{\omega}^1_{j1}\right)$, $1 \le j \le \widetilde{D}_1$. The Deconstruction Lemma therefore tells us that $D_1 = \widetilde{D}_1$, and that

$$\Pi\left(\frac{2\pi i}{\omega^1_{j1}}, \frac{\pi i - \theta^1_j}{\omega^1_{j1}}\right) = \Pi\left(\frac{2\pi i}{\widetilde{\omega}^1_{(\gamma j)1}}, \frac{\pi i - \widetilde{\theta}^1_{(\gamma j)}}{\widetilde{\omega}^1_{(\gamma j)1}}\right)\,, \qquad 1 \le j \le D_1\,,$$

for a permutation $\gamma\colon \{1, \ldots, D_1\} \to \{1, \ldots, D_1\}$. Remark I.A.7 yields

$$\omega^1_{j1} = \varepsilon_j\, \widetilde{\omega}^1_{(\gamma j)1}\,, \quad \theta^1_j = \varepsilon_j\, \widetilde{\theta}^1_{(\gamma j)} + 2\pi i\, m_j\,, \qquad 1 \le j \le D_1\,, \tag{7}$$

for $\varepsilon_j = \pm 1$ and integers m_j. Since θ^1_j and $\widetilde{\theta}^1_j$ are real, we must have $m_j = 0$. Also, by subjecting $\widetilde{\mathcal{N}}$ to an isomorphism that permutes the nodes of layer 1, we can achieve $\gamma = \text{identity}$ in (7). Thus,

$$\omega^1_{j1} = \varepsilon_j\, \widetilde{\omega}^1_{j1}\,, \quad \theta^1_j = \varepsilon_j\, \widetilde{\theta}^1_j\,, \quad \varepsilon_j = \pm 1\,, \qquad 1 \le j \le D_1\,. \tag{8}$$

If $L > 1$, then we can subject $\widetilde{\mathcal{N}}$ to an isomorhism that changes the signs of the nodes at layer 1, to achieve $\varepsilon_j = 1$, $1 \le j \le D_1$, in (8).

Thus, we have achieved (4), (5), (6) with $\bar{\ell} = 1$, unless $L = 1$. If $L = 1$, then there is no isomorphism that changes the signs at layer 1, since layer 1 is the output layer. In this case we argue as follows. For $L = 1$, the outputs of the nets $\mathcal{N}$, $\widetilde{\mathcal{N}}$ are

$$(9) \qquad x_1^L(t, \mathcal{N}) = \sigma(\omega_{11}^1 t + \theta_1^1) \qquad x_1^L(t, \widetilde{\mathcal{N}}) = \sigma(\widetilde{\omega}_{11}^1 t + \widetilde{\theta}_1^1)\,.$$

Equation (8) says that

$$(10) \qquad \omega_{11}^1 = \varepsilon\,\widetilde{\omega}_{11}^1\,, \quad \theta_1^1 = \varepsilon\,\widetilde{\theta}_1^1\,, \quad \varepsilon = \pm 1\,.$$

From (9), (10) we get $x_1^L(t, \mathcal{N}) = \varepsilon x_1^L(t, \widetilde{\mathcal{N}})$ for $t \in \mathbb{R}$. On the other hand, hypothesis of Theorem I.B.9 gives $x_1^L(t, \mathcal{N}) = x_1^L(t, \widetilde{\mathcal{N}})$, and $x_1^L(t, \mathcal{N})$ is not identically zero. Hence, $\varepsilon = +1$, so that we have achieved (4), (5), (6).

B. The Inductive Step.

Suppose $\bar{\ell}$ is given, $1 \le \bar{\ell} \le L - 1$, and the nets $\mathcal{N}$, $\widetilde{\mathcal{N}}$ in the previous section satisfy

$$(1) \qquad D_\ell = \widetilde{D}_\ell\,, \qquad \text{for } 0 \le \ell \le \bar{\ell},$$

$$(2) \qquad \omega_{jk}^\ell = \widetilde{\omega}_{jk}^\ell\,, \qquad \text{for } 1 \le \ell \le \bar{\ell},\ 1 \le j \le D_\ell,\ 1 \le k \le D_{\ell-1}\,,$$

$$(3) \qquad \theta_j^\ell = \widetilde{\theta}_j^\ell\,, \qquad \text{for } 1 \le \ell \le \bar{\ell},\ 1 \le j \le D_\ell\,.$$

Then we will prove that

$$(4) \qquad D_{\bar{\ell}+1} = \widetilde{D}_{\bar{\ell}+1}\,,$$

and that we can subject $\mathcal{N}$, $\widetilde{\mathcal{N}}$ to isomorphisms to achieve

$$(5) \qquad \omega_{jk}^\ell = \widetilde{\omega}_{jk}^\ell\,, \qquad \text{for } 1 \le \ell \le \bar{\ell} + 1,\ 1 \le j \le D_\ell,\ 1 \le k \le D_{\ell-1}\,,$$

$$(6) \qquad \theta_j^\ell = \widetilde{\theta}_j^\ell\,, \qquad \text{for } 1 \le \ell \le \bar{\ell},\ 1 \le j \le D_\ell\,.$$

This inductive step will complete the proof of Theorem I.B.9.

Let E_j^ℓ, $\overset{\circ}{E}{}_j^\ell$, Ω_j^ℓ, Ω_*^ℓ be the sets constructed from $\mathcal{N}$ in Section III, and let $\widetilde{E}_j^\ell$, $\overset{\circ}{\widetilde{E}}{}_j^\ell$, $\widetilde{\Omega}_j^\ell$, $\widetilde{\Omega}_*^\ell$ be the analogous sets arising from $\widetilde{\mathcal{N}}$. By (1), (2), (3) and Lemma III.D.2, we have

$$(7) \qquad E_j^\ell = \widetilde{E}_j^\ell\,, \quad \overset{\circ}{\widetilde{E}}{}_j^\ell = \overset{\circ}{E}{}_j^\ell\,, \quad \widetilde{\Omega}_j^\ell = \Omega_j^\ell\,, \quad \widetilde{\Omega}_*^\ell = \Omega_*^\ell\,, \quad \text{for } \ell \le \bar{\ell},$$

and

$$(8)\qquad x_j^\ell(t,\mathcal{N}) = x_j^\ell(t,\widetilde{\mathcal{N}}), \qquad \text{for } \ell \le \bar{\ell},\ t \in \Omega_j^\ell .$$

Set

$$(9)\qquad k_0 = D_{\bar{\ell}} .$$

Lemma III.D.7 shows that $\overset{\circ}{E}{}^{\bar{\ell}}_{k_0} = \overset{\circ}{\widetilde{E}}{}^{\bar{\ell}}_{k_0}$ is infinite. Fix any $t_0 \in \overset{\circ}{E}{}^{\bar{\ell}}_{k_0}$. For $\delta > 0$, set

$$(10)\qquad U_\delta = \{t \in \mathbb{C}\colon\ |t - t_0| < \delta\}, \qquad V_\delta = U_\delta \setminus \{t_0\},$$

and take δ so small that (III.D.10)-(III.D.14) hold with $\bar{\ell}$ in place of ℓ, both for $\mathcal{N}$ and $\widetilde{\mathcal{N}}$. Lemma III.D.3 gives

$$(11)\qquad \operatorname{Sing}(L - \bar{\ell} - 1, \mathcal{N}) = \operatorname{Sing}(L - \bar{\ell} - 1, \widetilde{\mathcal{N}}).$$

We will check that

$$F = \{x^{\bar{\ell}}_{k_0}(t,\mathcal{N})\colon\ t \in V_\delta \cap \operatorname{Sing}(L - \bar{\ell} - 1, \mathcal{N})\}$$

is the approximate union of the sets F_j defined in (III.D.14). This amounts to showing that

(12) $F \subset \bigcup_{j=1}^{D_{\bar{\ell}+1}} F_j$, and

(13) Any point x belonging to exactly one of the F_j must belong to F.

To see (12), let $x \in F$. Then $x = x^{\bar{\ell}}_{k_0}(t,\mathcal{N})$ with $t \in V_\delta \cap \operatorname{Sing}(L - \bar{\ell} - 1, \mathcal{N})$, so that $t \in V_\delta \cap E_j^{\bar{\ell}+1}$ for some j, by (III.D.12). Since $x = x^{\bar{\ell}}_{k_0}(t,\mathcal{N})$ with $t \in V_\delta \cap E_j^{\bar{\ell}+1}$, we have $x \in F_j$, proving (12). To check (13), suppose x belongs to F_{j_0} but not to any other F_j. Then since $x \in F_{j_0}$, we have $x = x^{\bar{\ell}}_{k_0}(t,\mathcal{N})$ with $t \in E_{j_0}^{\bar{\ell}+1} \cap V_\delta$. If $t \in E_j^{\bar{\ell}+1}$ for some $j \ne j_0$, then it would follow that $x \in F_j$, contradicting our assumption. Hence, t belongs to $E_{j_0}^{\bar{\ell}+1}$ but not to $E_j^{\bar{\ell}+1}$ for $j \ne j_0$. Since also $t \in V_\delta$, (III.D.12) implies $t \in V_\delta \cap \operatorname{Sing}(L - \bar{\ell} - 1, \mathcal{N})$, so that $x = x^{\bar{\ell}}_{k_0}(t,\mathcal{N})$ belongs to F. This completes the proof of (13), and shows that F is the approximate union of the F_j, $1 \le j \le D_{\bar{\ell}+1}$. In view of (11), an analogous argument shows that F is also the approximate union of the

$\widetilde{F}_j$, $1 \le j \le \widetilde{D}_{\bar{\ell}+1}$, where $\{\widetilde{F}_j\}$ are the analogues of the $\{F_j\}$ arising from $\widetilde{\mathcal{N}}$. Moreover, F_j approximates $\Pi\left(2\pi i/\omega_{jk_0}^{\bar{\ell}+1}, \beta_j\right)$ for suitable β_j, while $\widetilde{F}_j$ approximates $\Pi\left(2\pi i/\widetilde{\omega}_{jk_0}^{\bar{\ell}+1}, \tilde{\beta}_j\right)$ for suitable $\tilde{\beta}_j$, by (III.D.14).

Since also the $(\omega_{jk_0}^{\bar{\ell}+1})$ and $(\widetilde{\omega}_{jk_0}^{\bar{\ell}+1})$ satisfy (I.B.10) and (I.B.11), the Deconstruction Lemma applies. Hence, (4) holds, and for some permutation $\gamma: \{1, \ldots, D_{\bar{\ell}+1}\} \to \{1, \ldots, D_{\bar{\ell}+1}\}$ we have

$$\Pi\left(\frac{2\pi i}{\omega_{jk_0}^{\bar{\ell}+1}}, \beta_j\right) = \Pi\left(\frac{2\pi i}{\widetilde{\omega}_{(\gamma j)k_0}^{\bar{\ell}+1}}, \tilde{\beta}_{(\gamma j)}\right), \qquad 1 \le j \le D_{\bar{\ell}+1}.$$

In particular,

$$(14) \qquad \omega_{jk_0}^{\bar{\ell}+1} = \varepsilon_j \widetilde{\omega}_{(\gamma j)k_0}^{\bar{\ell}+1}, \quad 1 \le j \le D_{\bar{\ell}+1}, \qquad \text{with } \varepsilon_j = \pm 1.$$

By subjecting $\widetilde{\mathcal{N}}$ to an isomorphism that permutes the nodes of layer $(\bar{\ell}+1)$, we can preserve (1)-(4) and (7), (8), (11), and bring about $\gamma =$ identity in (4). Thus we may assume

$$(15) \qquad \omega_{jk_0}^{\bar{\ell}+1} = \varepsilon_j \widetilde{\omega}_{jk_0}^{\bar{\ell}+1}, \qquad 1 \le j \le D_{\bar{\ell}+1},$$

with $\varepsilon_j = \pm 1$ and $\gamma =$ identity. Recall that $k_0 = D_{\bar{\ell}}$ (see (9)).

The next step is to establish the following result.

(16) **Lemma.** *$\omega_{jk}^{\bar{\ell}+1} = \varepsilon_j \widetilde{\omega}_{jk}^{\bar{\ell}+1}$, $1 \le j \le D_{\bar{\ell}+1}$, $1 \le k \le D_{\bar{\ell}}$, and $\theta_j^{\bar{\ell}+1} = \varepsilon_j \widetilde{\theta}_j^{\bar{\ell}+1}$, $1 \le j \le D_{\bar{\ell}+1}$.*

Note that the proof of (15) applies to other k, not just k_0, and shows that $\omega_{jk}^{\bar{\ell}+1} = \varepsilon_{jk} \widetilde{\omega}_{(\gamma' j)k}^{\bar{\ell}+1}$ with $\varepsilon_{jk} = \pm 1$, and γ' depending on k. However, (16) gives sharper restrictions on the ω's and $\widetilde{\omega}$'s.

PROOF OF LEMMA 16. We return to the Deconstruction Lemma applied to F, F_j, $\widetilde{F}_j$, $\Pi\left(2\pi i/\omega_{jk_0}^{\bar{\ell}+1}, \beta_j\right)$, $\Pi\left(2\pi i/\widetilde{\omega}_{jk_0}^{\bar{\ell}+1}, \tilde{\beta}_j\right)$. Since $\gamma =$ identity, the Deconstruction Lemma yields for each fixed j_0, $1 \le j_0 \le D_{\bar{\ell}+1}$, a sequence $(x_\nu)_{\nu \ge 1}$ with the properties

$$(17) \qquad |x_\nu| \to \infty, \qquad \text{as } \nu \to \infty,$$

$$(18) \qquad x_\nu \in F_{j_0} \setminus \bigcup_{j \ne j_0} F_j,$$

$$(19) \qquad x_\nu \in \widetilde{F}_{j_0} \setminus \bigcup_{j \ne j_0} \widetilde{F}_j.$$

Since $x_\nu \in F_{j_0}$, we have

$$x_\nu = x^{\bar{\ell}}_{k_0}(t_\nu, \mathcal{N}), \tag{20}$$

with

$$t_\nu \in V_\delta \cap E^{\bar{\ell}+1}_{j_0}. \tag{21}$$

Observe that

$$t_\nu \notin V_\delta \cap E^{\bar{\ell}+1}_{j}, \qquad \text{for } j \neq j_0. \tag{22}$$

In fact, if (22) were false, then (20) would show that $x_\nu \in F_j$ with $j \neq j_0$, contradicting (18). Similarly, we know from (19) that

$$t_\nu \notin V_\delta \cap \widetilde{E}^{\bar{\ell}+1}_{j}, \qquad \text{for } j \neq j_0. \tag{23}$$

From (21), (22) and (III.D.12), we see that $t_\nu \in V_\delta \cap \mathrm{Sing}(L - \bar{\ell} - 1, \mathcal{N})$. Hence also $t_\nu \in V_\delta \cap \mathrm{Sing}(L - \bar{\ell} - 1, \widetilde{\mathcal{N}})$ by (11), so that another application of (III.D.12) yields $t_\nu \in V_\delta \cap \widetilde{E}^{\bar{\ell}+1}_{j_1}$ for some j_1. In view of (23), we must have $j_1 = j_0$. Hence,

$$t_\nu \in V_\delta \cap \widetilde{E}^{\bar{\ell}+1}_{j_0}. \tag{24}$$

From (17), (20), $t_\nu \in V_\delta$, and (III.D.4), (III,D.10), we see that

$$t_\nu \to t_0, \qquad \text{as } \nu \to \infty, \quad t_\nu \neq t_0. \tag{25}$$

From (21), (24) and (III.,D.13), we get

$$t_\nu \in V_\delta, \qquad \sum_{k=1}^{D_{\bar{\ell}}} \omega^{\bar{\ell}+1}_{j_0 k} x^{\bar{\ell}}_k(t_\nu, \mathcal{N}) + \theta^{\bar{\ell}}_{j_0} \in \Pi(2\pi i, \pi i) \tag{26}$$

and

$$t_\nu \in V_\delta, \qquad \sum_{k=1}^{D_{\bar{\ell}}} \widetilde{\omega}^{\bar{\ell}+1}_{j_0 k} x^{\bar{\ell}}_k(t_\nu, \widetilde{\mathcal{N}}) + \theta^{\bar{\ell}+1}_{j_0} \in \Pi(2\pi i, \pi i). \tag{27}$$

In view of (15), (26), (27), we obtain

$$\left(\sum_{k=1}^{D_{\bar\ell}-1} \omega_{j_0 k}^{\bar\ell+1}\, x_k^{\bar\ell}(t_\nu,\mathcal{N}) + \theta_{j_0}^{\bar\ell+1}\right) - \left(\sum_{k=1}^{D_{\bar\ell}-1} (\varepsilon_{j_0}\, \widetilde\omega_{j_0 k}^{\bar\ell+1})\, x_k^{\bar\ell}(t_\nu,\widetilde{\mathcal{N}}) + (\widetilde\theta_{j_0}^{\bar\ell+1}\varepsilon_{j_0})\right) \in \Pi(2\pi i, 0)\,.$$

That is,

$$F(t_\nu) \in \Pi(2\pi i, 0)\,, \tag{28}$$

with

$$F(t) = \left(\sum_{k=1}^{D_{\bar\ell}-1} \omega_{j_0}^{\bar\ell+1}\, x_k^{\bar\ell}(t,\mathcal{N}) + \theta_{j_0}^{\bar\ell+1}\right) - \left(\sum_{k=1}^{D_{\bar\ell}-1} (\varepsilon_{j_0}\, \widetilde\omega_{j_0 k}^{\bar\ell+1})\, x_k^{\bar\ell}(t,\widetilde{\mathcal{N}}) + (\varepsilon_{j_0}^{\bar\ell+1}\widetilde\theta_{j_0}^{\bar\ell+1})\right). \tag{29}$$

Since $t_0 \in \mathring{E}_{k_0}^{\bar\ell}$ with $k_0 = D_{\bar\ell}$, (III.D.4) and (III.D.11) show that $F(t)$ is analytic on U_δ. Also, (25) yields $F(t_\nu) \to F(t_0)$ as $\nu \to \infty$. This shows that $F(t_\nu)$ is eventually constant, by (28). Another application of (25) shows that $F(t)$ is constant on U_δ. In view of (28), there is an integer m such that $F(t) = 2\pi i m$ for all $t \in U_\delta$. However, $F(t)$ is analytic on $\Omega_*^{\bar\ell}$ (by (7), (29) and (IIII.D.4)). Since $\Omega_*^{\bar\ell}$ contains V_δ (by (III.D.10), (III,D.11)), and since $\Omega_*^{\bar\ell}$ is connected, it follows by analytic continuation that $F(t) = 2\pi i m$ for all $t \in \Omega_*^{\bar\ell}$. In particular, $F(t) = 2\pi i m$ for $t \in \mathbb{R}$. A glance at (29) shows that $F(t)$ is real for $t \in \mathbb{R}$. Hence, $m = 0$, so that $F(t) = 0$ for t real, *i.e.*

$$\sum_{k=1}^{D_{\bar\ell}-1} \omega_{j_0 k}^{\bar\ell+1}\, x_k^{\bar\ell}(t,\mathcal{N}) + \theta_{j_0}^{\bar\ell+1} = \sum_{k=1}^{D_{\bar\ell}-1} (\widetilde\omega_{j_0 k}^{\bar\ell+1}\,\varepsilon_{j_0})\, x_k^{\bar\ell}(t,\widetilde{\mathcal{N}}) + (\widetilde\theta_{j_0}^{\bar\ell+1}\varepsilon_{j_0})\,, \tag{30}$$

for all $t \in \mathbb{R}$.

To complete the proof of (16), we distinguish two cases.

CASE 1: $D_{\bar\ell} = 1$. Then already (15) shows that $\omega_{j_0 k}^{\bar\ell+1} = \varepsilon_{j_0}\,\widetilde\omega_{j_0 k}^{\bar\ell+1}$ for $1 \le k \le D_{\bar\ell}$, and (30) shows that $\theta_{j_0}^{\bar\ell+1} = \varepsilon_{j_0}\widetilde\theta_{j_0}^{\bar\ell+1}$. Since j_0 is arbitrary, $1 \le j_0 \le D_{\bar\ell+1}$, the proof of (16) is complete in *Case* 1.

CASE 2: $D_{\bar{\ell}} > 1$. Then the left and right-hand sides of (30), composed with $\sigma(\,\cdot\,)$, are the outputs of auxiliary neural nets $\widehat{\mathcal{N}}$ and $\check{\mathcal{N}}$ respectively. Specifically, we set

$$(31)\quad \widehat{\mathcal{N}} = [(D_0, D_1, \ldots, D_{\bar{\ell}-1}, D_{\bar{\ell}} - 1, 1), (\widehat{\omega}^{\ell}_{jk}), (\widehat{\theta}^{\ell}_j)], \quad \text{with}$$

$$(32)\quad \widehat{\omega}^{\ell}_{jk} = \omega^{\ell}_{jk},\ \widehat{\theta}^{\ell}_j = \theta^{\ell}_j \quad \text{for} \quad \ell \le \bar{\ell}, \quad \text{and}$$

$$(33)\quad \widehat{\omega}^{\bar{\ell}+1}_{1k} = \omega^{\bar{\ell}+1}_{j_0 k},\ \widehat{\theta}^{\bar{\ell}+1}_1 = \theta^{\bar{\ell}+1}_{j_0}; \quad \text{and we set}$$

$$(34)\quad \check{\mathcal{N}} = [(D_0, D_1, \ldots, D_{\bar{\ell}-1}, D_{\bar{\ell}} - 1, 1), (\check{\omega}^{\ell}_{jk}), (\check{\theta}^{\ell}_j)], \quad \text{with}$$

$$(35)\quad \check{\omega}^{\ell}_{jk} = \widetilde{\omega}^{\ell}_{jk},\ \check{\theta}^{\ell}_j = \widetilde{\theta}^{\ell}_j \quad \text{for} \quad \ell \le \bar{\ell}, \quad \text{and}$$

$$(36)\quad \check{\omega}^{\bar{\ell}+1}_{1k} = \varepsilon_{j_0}\widetilde{\omega}^{\bar{\ell}+1}_{j_0 k},\ \check{\theta}^{\bar{\ell}+1}_1 = \varepsilon_{j_0}\widetilde{\theta}^{\bar{\ell}+1}_{j_0}.$$

That is, $\widehat{\mathcal{N}}$ is made from $\mathcal{N}$ by deleting the following nodes:

(a) Node k_0 at level $\bar{\ell}$;

(b) All nodes except node j_0 at level $\bar{\ell}+1$;

(c) All nodes at levels higher than $\bar{\ell}+1$.

For the surviving nodes in $\widehat{\mathcal{N}}$, the weights and thresholds are the same as those of $\mathcal{N}$. Thus, $x^{\bar{\ell}+1}_{j_0}(t, \mathcal{N})$ is the output of the net $\widehat{\mathcal{N}}$.

Similarly, $\check{\mathcal{N}}$ is made from $\widetilde{\mathcal{N}}$ by deleting the same nodes as in (a), (b), (c) above. For the surviving nodes in $\check{\mathcal{N}}$, the weights and thresholds are the same as those of $\widetilde{\mathcal{N}}$, except that we multiply the weights and thresholds at the output level of $\check{\mathcal{N}}$ by ε_{j_0}. Thus, $\varepsilon_{j_0} x^{\bar{\ell}+1}_{j_0}(t, \widetilde{\mathcal{N}})$ is the output of the net $\check{\mathcal{N}}$. Note that our assumption $D_{\bar{\ell}} > 1$ was needed to define $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$ as neural nets.

Equation (30) shows that the nets $\widehat{\mathcal{N}}$ and $\check{\mathcal{N}}$ produce the same output. Also, $\widehat{\mathcal{N}}$ and $\check{\mathcal{N}}$ satisfy (I.B.10) and (I.B.11). Moreover, the size of $\widehat{\mathcal{N}}$ is strictly less than that of $\mathcal{N}$, and the size of $\check{\mathcal{N}}$ is strictly less than that of $\widetilde{\mathcal{N}}$. (For, one node at level $\bar{\ell}$, and possible additional nodes, are deleted from $\mathcal{N}$, $\widetilde{\mathcal{N}}$ to make $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$). Hence, by (A.1) and (A.2), the uniqueness Theorem I.B.9 applies to $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$. Therefore, $\widehat{\mathcal{N}}$ is isomorphic to $\check{\mathcal{N}}$. Also, by definition of $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$ and by (1), (2), (3), the nets $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$

are identical below their output level, *i.e.* $\widehat{\omega}^{\ell}_{jk} = \check{\omega}^{\ell}_{jk}$, $\widehat{\theta}^{\ell}_{j} = \check{\theta}^{\ell}_{j}$ for $\ell \le \bar{\ell}$. Also, for fixed ℓ, k, we have $\widehat{\omega}^{\ell}_{jk} \neq 0$ and $|\widehat{\omega}^{\ell}_{jk}| \neq |\widehat{\omega}^{\ell}_{j'k}|$ for $j \neq j'$, by (32) and hypothesis (I.B.10). Therefore, Lemma I.C.1 applies, and shows that the nets $\widehat{\mathcal{N}}$, $\check{\mathcal{N}}$ are identical. In particular,

$$(37)\quad \omega^{\bar{\ell}+1}_{j_0} = \varepsilon_{j_0}\, \widetilde{\omega}^{\bar{\ell}+1}_{j_0 k} \quad \text{for} \quad 1 \le k \le D_{\bar{\ell}} - 1\,, \quad \text{and} \quad \theta^{\bar{\ell}+1}_{j_0} = \varepsilon_{j_0}\, \widetilde{\theta}^{\bar{\ell}+1}_{j_0}\,.$$

Since j_0 , $1 \le j_0 \le D_{\bar{\ell}+1}$, was arbitrary, Lemma 16 follows from (9), (15) and (37).

To finish the proof of Theorem I.B.9, we distinguish two cases.

CASE 1: $L > \bar{\ell} + 1$. Then by subjecting $\widetilde{\mathcal{N}}$ to an isomorphism that changes the signs of the nodes at level $\bar{\ell} + 1$, we can achieve (5) and (6). (That is obvious from (1), (2), (3), (16).) We already proved (4), so we have completed the inductive step in the proof of Theorem I.B.9 in Case 1.

CASE 2: $L = \bar{\ell} + 1$. Then (1), (2), (3), (16) show that $D_\ell = \widetilde{D}_\ell$, $0 \le \ell \le L$, and that

$$(38)\qquad \omega^{\ell}_{jk} = \widetilde{\omega}^{\ell}_{jk}\,, \quad \theta^{\ell}_{j} = \widetilde{\theta}^{\ell}_{j}\,, \qquad \text{if } \ell < L\,,$$

$$(39)\qquad \omega^{L}_{1k} = \varepsilon\, \widetilde{\omega}^{L}_{1k}\,, \quad \theta^{L}_{1} = \varepsilon \widetilde{\theta}^{L}_{1}\,, \qquad \text{with } \varepsilon = \pm 1\,.$$

It follows at once that $x^L_1(t, \mathcal{N}) = \varepsilon\, x^L_1(t, \widetilde{\mathcal{N}})$ for all $t \in \mathbb{R}$. However, from the hypothesis of Theorem I.B.9, we know that $x^L_1(t, \mathcal{N}) = x^L_1(t, \widetilde{\mathcal{N}})$ since $L = \widetilde{L}$. Since also $x^L_1(t, \mathcal{N})$ is a non-constant function (see (III.D.15)), it follows that $\varepsilon = +1$, hence (38), (39) show that the nets $\mathcal{N}$, $\widetilde{\mathcal{N}}$ are identical. In particular, we have achieved (4), (5), (6), completing the inductive step in the proof of theorem (I.B.9) in *Case* 2.

The proof of Theorem I.B.9 is complete.

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Compacité par compensation pour une classe de systèmes hyperboliques de $p \geq 3$ lois de conservation

Sylvie Benzoni-Gavage et Denis Serre

Abstract. We are concerned with a strictly hyperbolic system of conservation laws $u_t + f(u)_x = 0$, where u runs in a region Ω of $\mathbb{R}^p$, such that two of the characteristic fields are genuinely non-linear whereas the other ones are of Blake Temple's type. We begin with the case $p = 3$ and show, under some more or less technical assumptions, that the approximate solutions $(u^\varepsilon)_{\varepsilon>0}$ given either by the vanishing viscosity method or by the Godunov scheme converge to weak entropy solutions as ε goes to 0. The first step consists in using techniques from the Blake Temple systems lying in the separate works of Leveque-Temple and Serre. Then we apply a compensated compactness method and the theory of Di Perna on 2×2 genuinely non-linear systems. Eventually the proof is extended to the general case $p > 3$.

1. Introduction.

L'étude qui suit a été inspirée par des modèles d'écoulements diphasiques gaz/liquide en recherche pétrolière [1] constitués de systèmes hyperboliques 3×3 ayant deux champs caractéristiques vraiment non-linéaires et un champ de Blake Temple. L'exemple le plus simple est le système suivant

$$\begin{cases} \rho_t + (\rho\, v)_x = 0\,, \\ (\rho\, c)_t + (\rho\, c\, v)_x = 0\,, \\ (\rho\,(1+c)\, v)_t + (\rho\,(1+c)\, v^2 + p)_x = 0\,, \end{cases}$$

où la fonction $p = p(\rho, c)$ est donnée par $p = a^2\, c\, \rho/(1-\rho)$, avec a constante. Dans ces lois de conservation, ρ représente la fraction massique de liquide ($0 < \rho < 1$), v la vitesse du gaz (supposée égale dans ce modèle à celle du liquide), et c s'apparente à une concentration: c'est le rapport

$$\frac{(\text{fraction massique} \times \text{densité})_{\text{gaz}}}{(\text{fraction massique} \times \text{densité})_{\text{liquide}}}\,.$$

On observe, grâce à la première équation, que la seconde s'écrit sous forme quasilinéaire

$$c_t + v\, c_x = 0\,,$$

ce qui signifie que c est un invariant de Riemann fort associé à la valeur propre v. De plus les surfaces de niveau de c sont clairement des plans en variables conservatives $(\rho, \rho\, c, \rho\,(1+c)\, v)$: le champ associé à v est donc de Temple. On vérifie enfin sans peine que les deux autres champs sont vraiment non-linéaires.

Ces propriétés sont a priori de deux types complètement différents. Cependant on sait que, d'une part, les systèmes 2×2 vraiment non-linéaires et, d'autre part, les systèmes dont tous les champs sont de Blake Temple admettent des solutions faibles entropiques obtenues comme limites de solutions approchées uniformément bornées dans L^∞ : le premier résultat est dû à Di Perna [3] grâce à la compacité par compensation et le second a été obtenu séparément par Leveque-Temple [6] et Serre [7] par une estimation a priori de la variation totale. L'objet de cet article est de démontrer, en combinant les différentes méthodes, un résultat semblable concernant les systèmes avec deux champs vraiment non-linéaires et les autres de Temple. Par souci de clarté le cas des systèmes 3×3 (donc avec un seul champ Temple) est traité intégralement avant de donner l'extension au cas général.

Soit un tel système

$$(1) \qquad u_t + f(u)_x = 0\,, \qquad t > 0\,,\ x \in \mathbb{R}\,,\ u(x,t) \in \mathbb{R}^3\,.$$

Il s'agît d'étudier l'existence de solutions faibles entropiques pour le problème de Cauchy associé à (1) avec une condition initiale $u(x,0) = u_0(x)$ où $u_0 \in VB(\mathbb{R})$.

On considère pour cela une suite solutions approchées $(u^\varepsilon)_{\varepsilon>0}$ obtenue soit par la régularisation parabolique

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon\, u_{xx}^\varepsilon\,, \qquad t > 0\,,\ x \in \mathbb{R}\,, \tag{2}$$

soit par le schéma de Godunov [4] (avec $\varepsilon = \max\{\Delta x, \Delta t\}$).

Le problème principal est, dans les deux cas, de montrer que le passage à la limite est autorisé (à l'extraction d'une sous-suite près) malgré la non-linéarité. Comme rien ne permet ici d'affirmer que l'on aura des domaines invariants, on *doit supposer* au départ que la suite $(u^\varepsilon)_{\varepsilon>0}$ est bornée dans $L^\infty(\mathbb{R} \times [0,T])$. C'est cette hypothèse, essentielle, qui est un obstacle au traitement des systèmes diphasiques par la même méthode: car il faudrait que les solutions approchées restent dans un compact du domaine diphasique $0 < \rho < 1$, ce qu'on ne sait pas. On envisage donc des systèmes moins singuliers que les systèmes diphasiques, espérant que cette hypothèse soit satisfaite. De là on va montrer, lorsqu'il existe une entropie fortement convexe, que $(u^\varepsilon)_{\varepsilon>0}$ admet une sous-suite convergeant presque partout vers une solution faible entropique du système (1).

La démarche est la suivante: dans un premier temps, à l'aide d'estimations en variation totale de $w^\varepsilon = w(u^\varepsilon)$ où w est un invariant de Riemann fort associé au champ de Blake Temple, on extrait une sous-suite fortement convergente de $(w^\varepsilon)_{\varepsilon>0}$. Le système est alors partiellement découplé. On montre ensuite que la méthode de compacité par compensation [11], qui ne s'applique a priori qu'aux entropies (ce qui est particulièrement intéressant dans le cas d'un système riche [8]), s'étend ici à certains couples d'objets (E, F) où E n'est pas nécessairement une entropie. Ces objets sont construits en sorte que l'étude puisse être finalement ramenée à celle d'un système 2×2 vraiment non-linéaire et en nombre suffisant pour conclure d'après l'analyse de Di Perna.

Précisons les hypothèses, classiques [5], concernant le système (1): il sera supposé strictement hyperbolique, c'est-à-dire admettant trois valeurs propres réelles distinctes $\lambda_1 < \lambda_2 < \lambda_3$, avec de plus un flux f assez régulier pour que ses éléments propres soient au moins de classe

C^2. On supposera par ailleurs qu'il existe une entropie de classe C^2 fortement convexe (généralement issue d'un principe physique).

L'hypothèse spécifique de notre étude est, comme annoncé plus haut, que deux des champs caractéristiques sont vraiment non-linéaires tandis que l'autre est de Blake Temple. Pour fixer les idées on supposera que le champ de Temple est le second. Toutefois la preuve ne dépend pas de l'ordre des valeurs propres. L'hypothèse concernant ce champ signifie que, si l désigne le vecteur propre à gauche associé, la famille des droites affines $u + \mathbb{R}\, l(u)$ est orthogonale à une famille à un paramètre d'(hyper)plans affines. Ces hyperplans définissent alors un feuilletage dans l'espace des états $\Omega \subset \mathbb{R}^3$ (imposé par la physique) et les bords de ce dernier sont constitués de deux feuilles et de composante(s) connexe(s) tranverse(s) au feuilletage. Si le paramètre décrivant ces hyperplans est bien choisi, il fournit un invariant de Riemann fort w de classe C^2 et l'on peut ainsi prendre $l(u) = dw(u)$ pour tout $u \in \Omega$.

2. Approximation parabolique.

2.1. Découplage.

Lemme 2.1. *Soit $T > 0$. Si la solution du problème de Cauchy associé à (2), avec comme condition initiale $u^\varepsilon(x,0) = u_0(x)$, reste dans un compact K de Ω, indépendant de ε, pour $t \leq T$, alors on a les estimations a priori suivantes*

i) *la suite $(u^\varepsilon)_{\varepsilon>0}$ est bornée dans $L^\infty(\mathbb{R} \times [0,T])$,*

ii) *la suite $(\sqrt{\varepsilon}\, u^\varepsilon_x)_{\varepsilon>0}$ est bornée dans $L^2(\mathbb{R} \times [0,T])$,*

iii) *la suite $(w^\varepsilon)_{\varepsilon>0}$ est bornée dans $L^\infty([0,\infty[; VB(\mathbb{R}))$, où $w^\varepsilon = w \circ u^\varepsilon$,*

iv) *la suite $(w^\varepsilon_t)_{\varepsilon>0}$ est bornée dans $L^1(0,T; H^{-1}_{\text{loc}}(\mathbb{R}))$.*

DÉMONSTRATION. Notons que le système parabolique (2) est régularisant: la solution u^ε est de classe C^∞.

Le point i) n'est que la traduction de l'hypothèse en termes d'espaces fonctionnels.

Le point ii) provient d'une estimation d'énergie: celle-ci se déduit de façon très classique [3] de l'équation vérifiée par l'entropie fortement convexe. Cela implique évidemment, par composition, une estimation analogue pour $(\sqrt{\varepsilon}\, w_x^\varepsilon)_{\varepsilon>0}$.

Quant à iii), c'est un résultat général concernant les invariants de Riemann associés à des champs Temple [7]. Cette estimation signifie en particulier que $(w^\varepsilon)_{\varepsilon>0}$ est bornée dans $L^\infty(\mathbb{R}\times[0,T])$ et que $(w_x^\varepsilon)_{\varepsilon>0}$ est bornée dans $L^\infty([0,\infty[;L^1(\mathbb{R}))$. Notons que, dans le cas où tous les champs sont de Temple, on obtient ainsi la même propriété pour (u^ε) et on en déduit facilement une estimation sur (u_t^ε) (plus forte que iv)). Ce n'est pas le cas ici. Mais on montre l'estimation sur (w_t^ε) donnée au iv) comme suit.

Puisque dw est un vecteur propre à gauche de df, on a d'après (2),

$$w_t^\varepsilon + \lambda(u^\varepsilon)w_x^\varepsilon = \varepsilon\, dw(u^\varepsilon)\cdot u_{xx}^\varepsilon\,.$$

De plus, comme les surfaces de niveau de w sont affines, il existe une application $m:\Omega\longrightarrow(\mathbb{R}^3)'$ continue telle que $d^2w(u)=m(u)\otimes dw(u)$ pour tout $u\in\Omega$. On a donc

$$w_t^\varepsilon = -\lambda(u^\varepsilon)w_x^\varepsilon - \varepsilon\, m(u^\varepsilon)\cdot u_x^\varepsilon\, w_x^\varepsilon + \varepsilon\, w_{xx}^\varepsilon\,. \tag{3}$$

Or on a des estimations a priori pour chacun de ces trois termes. En effet, grâce à i) et iii), on constate que le premier terme est borné dans $L^\infty([0,\infty[;L^1(\mathbb{R}))$ tandis que i), ii) montrent que le second est borné dans $L^1(\mathbb{R}\times[0,T])$ et le dernier dans $L^2(0,T;H^{-1}_{\mathrm{loc}}(\mathbb{R}))$. Des injections classiques, en particulier $L^1(\mathbb{R})\hookrightarrow H^{-1}_{\mathrm{loc}}(\mathbb{R})$, permettent alors de conclure.

Théorème 2.1. *Soit $T>0$. Si la solution du système parabolique* (2) *reste dans un compact K de Ω pour $t\le T$, alors la suite $(w^\varepsilon)_{\varepsilon>0}$ admet une sous-suite convergeant presque partout dans $\mathbb{R}\times[0,T]$ lorsque $\varepsilon\to 0$.*

DÉMONSTRATION. Cela résulte des estimations du lemme précédent et du théorème de compacité suivant [10]:

Si X, B, Y sont des espaces de Banach tels que $X\hookrightarrow B$ avec injection compacte et $B\hookrightarrow Y$, si $\mathcal{F}\subset\mathcal{D}'(0,T;Y)$ est borné dans $L^p(0,T;X)$ pour un $p\in[1,\infty[$ avec $\partial_t\mathcal{F}=\{f_t : f\in\mathcal{F}\}$ borné dans $L^1(0,T;Y)$, alors $\mathcal{F}$ est relativement compact dans $L^p(0,T;B)$.

Si I est un intervalle borné de $\mathbb{R}$, ce théorème s'applique avec $X=VB(I)$, $B=L^1(I)$, $Y=H^{-1}(I)$ et $\mathcal{F}=\{w^\varepsilon : \varepsilon>0\}$: en effet, l'injection de X dans B est compacte d'après le théorème de Helly sur les fonctions à variation bornée et $\mathcal{F}$ satisfait l'hypothèse pour $p = \infty$ donc a fortiori pour tout p fini. Ceci permet d'extraire une sous-suite de $(w^\varepsilon)_{\varepsilon>0}$ convergeant presque partout dans $I \times [0,T]$.

Par le procédé diagonal on obtient ainsi une sous-suite convergeant presque partout dans $\mathbb{R} \times [0,T]$, que l'on désignera encore par $(w^\varepsilon)_{\varepsilon>0}$. On notera $\overline{w}$ sa limite: on a évidemment $\overline{w} \in w(K)$ presque partout.

2.2. Compacité par compensation.

Rappelons [2] que toute suite (u^ε) bornée dans $L^\infty(\mathbb{R} \times [0,T])$ admet une sous-suite, encore notée (u^ε), pour laquelle il existe pour presque tout $(x,t) \in \mathbb{R}\times[0,T]$ une mesure de probabilité $\nu_{(x,t)}$ à support compact telle que

$$g(u^\varepsilon) \rightharpoonup \langle \nu, g\rangle$$

dans L^∞-faible-$\star$, pour toute application continue $g : \Omega \longrightarrow \mathbb{R}$. La méthode de compacité par compensation consiste à étudier la mesure ν (dite mesure de Young) afin de démontrer, si possible, qu'elle est presque partout réduite à une masse de Dirac. L'outil de base pour cela est le lemme divergence-rotationnel [11]:

Lemme 2.2. *Si on a des applications continues $E_i, F_i : \Omega \longrightarrow \mathbb{R}$, $i = 1,2$ satisfaisant les estimations a priori*

$$E_i(u^\varepsilon)_t + F_i(u^\varepsilon)_x \in \textit{compact de } H^{-1}_{\mathrm{loc}}(\mathbb{R} \times [0,T])$$

alors elles vérifient la relation

$$\langle \nu, E_1F_2 - E_2F_1\rangle = \langle \nu, E_1\rangle \langle \nu, F_2\rangle - \langle \nu, E_2\rangle \langle \nu, F_1\rangle .$$

Pour obtenir des informations sur ν, il s'agît donc d'exhiber des objets *suffisamment nombreux* satisfaisant les *estimations a priori* du lemme. Pour les systèmes 2×2 les couples entropie/flux conviennent généralement car il y en a un espace de dimension infinie. Ici on va utiliser une notion plus faible qui fait l'objet de la définition suivante.

Definition 2.1. *On dira qu'une application $E : \Omega \longrightarrow \mathbb{R}$, de classe C^2, est une sous-entropie de* (1) *s'il existe $F : \Omega \longrightarrow \mathbb{R}$, de classe C^2, et $h : \Omega \longrightarrow \mathbb{R}$, continue, telles que*

$$dF - dE\, df = h\, dw .$$

REMARQUES.

1. Les couples sous-entropie/flux (E, F) ainsi définis sont caractérisés par

$$dF(u) - dE(u)\, df(u) \in \mathbb{R}\, l_2(u)\,, \qquad \text{pour tout } u \in \Omega\,,$$

ce qui équivaut du fait de la stricte hyperbolicité de (1) à

$$(dF - dE\, df)\, r_j \equiv 0\,, \qquad j = 1, 3\,.$$

Il y a donc autant d'équations que d'inconnues: cela permet d'espérer obtenir suffisamment de solutions.

2. De manière plus précise on devrait dire "sous-entropie relative au second champ caractéristique". Car il y en a d'autres, associées à chaque champ caractéristique. Mais ce sont celles associées au champ de Temple qui sont utiles: on va en effet voir qu'elles satisfont l'hypothèse du Lemme 2.2.

Lemme 2.3. *Pour tout couple sous-entropie/flux (E, F) on a, sous les hypothèses du Lemme* 2.1 *concernant l'approximation parabolique*

$$E(u^\varepsilon)_t + F(u^\varepsilon)_x \in \textit{compact de } H^{-1}_{\text{loc}}(\mathbb{R} \times [0, T])\,.$$

DÉMONSTRATION. Si $dF - dE\, df = h\, dw$ on déduit immédiatement de (2)

$$(4) \quad E(u^\varepsilon)_t + F(u^\varepsilon)_x = \varepsilon\, E(u^\varepsilon)_{xx} - \varepsilon\, d^2E(u^\varepsilon) \cdot (u^\varepsilon_x, u^\varepsilon_x) + h(u^\varepsilon)\, w^\varepsilon_x\,.$$

Les estimations i), ii) et iii) du Lemme 2.1 montrent alors que $E(u^\varepsilon)_t + F(u^\varepsilon)_x \in$ compact de $H^{-1}_{\text{loc}}(\mathbb{R} \times [0, T])$ + borné de $L^1(\mathbb{R} \times [0, T])$.

Comme de plus $E(u^\varepsilon)_t + F(u^\varepsilon)_x$ est dans un borné de $W^{-1,\infty}_{\text{loc}}$, cela suffit d'après le lemme de Murat [11] pour montrer que $E(u^\varepsilon)_t + F(u^\varepsilon)_x$ reste dans un compact de H^{-1}_{loc}.

On a ainsi démontré le résultat:

Théorème 2.2. *La mesure de Young $\nu_{(x,t)}$ vérifie, pour presque tout $(x, t) \in \mathbb{R} \times [0, T]$ et pour tous couples sous-entropies/flux (E_i, F_i), $i = 1, 2$, l'équation de Tartar*

$$\langle \nu, E_1F_2 - E_2F_1 \rangle = \langle \nu, E_1 \rangle \langle \nu, F_2 \rangle - \langle \nu, E_2 \rangle \langle \nu, F_1 \rangle\,.$$

2.3. Conclusion.

Théorème 2.3. *Si la solution du système parabolique* (2) *reste dans un compact* K *de* Ω *pour* $t \leq T$, *alors la suite* $(u^\varepsilon)_{\varepsilon > 0}$ *admet une sous-suite convergeant presque partout dans* $\mathbb{R} \times [0,T]$ *lorsque* $\varepsilon \to 0$ *et sa limite est solution faible entropique de* (1).

DÉMONSTRATION. La première partie de l'énoncé repose sur un théorème qui sera démontré au Paragraphe 4.

D'après le Théorème 2.1, on a une sous-suite de (w^ε) convergeant p.p. et d'après le Théorème 2.2, $\nu_{(x,t)}$ vérifie p.p. l'équation de Tartar pour tous les couples sous-entropies/flux. On verra que cela implique $\nu_{(x,t)} = \delta_{u(x,t)}$ p.p. et par conséquent $u^\varepsilon \longrightarrow u$ dans L^1_{loc}.

Le fait que u soit alors une solution faible entropique de (1) est très classique : puisque $g(u^\varepsilon) \rightharpoonup g(u)$ dans L^∞-faible-$\star$ dès que g est continue, le passage à la limite dans les intégrales contre une fonction test est autorisé. Notons que, s'il y a unicité de la solution faible entropique, c'est toute la suite qui converge vers u.

3. Schéma de Godunov.

On aimerait maintenant démontrer un résultat analogue pour une approximation *numérique* de (1). Le schéma se prêtant le mieux à ce genre de considérations est le schéma de Godunov. Comme il utilise de façon essentielle la résolution de problèmes de Riemann, on doit renforcer l'hypothèse sur le champ Temple: on le supposera ou bien linéairement dégénéré ou bien vraiment non-linéaire [5]. De plus, afin d'exploiter au mieux la structure de la solution numérique, on supposera que la valeur propre associée est de signe constant.

On désigne par Δx et Δt des pas d'espace et de temps respectivement. Sous la condition de Courant-Friedrichs-Lewy,

$$\frac{\Delta t}{\Delta x} \sup\{|\lambda_k(u)| : \ k = 1,2,3, \ u \in K\} \leq \frac{1}{2},$$

la solution numérique est construite comme suit.

On définit les valeurs initiales par

$$u_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\, dx\,, \qquad j \in \mathbb{Z}\,,$$

où $x_{j+1/2} = (j + 1/2)\Delta x$.

Puis, pour $(x,t) \in [j\Delta x, (j+1)\Delta x[\times[n\Delta t, (n+1)\Delta t[$, $j \in \mathbb{Z}$, $n \in \{0,\dots,N-1\}$,

$$u^\varepsilon(x,t) = U_R\Big(\frac{x - x_{j+1/2}}{t - n\Delta t}\ ;\ u_j^n,\ u_{j+1}^n\Big)\,,$$

où l'application $(y,\tau) \longmapsto U_R(y/\tau; u_g, u_d)$ est la solution autosimilaire du problème de Riemann d'états à gauche et à droite respectivement u_g et u_d et, pour $x \in [x_{j-1/2}, x_{j+1/2}[$,

$$u^\varepsilon(x,(n+1)\Delta t) = u_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u^\varepsilon(x,(n+1)\Delta t-)\, dx\,.$$

3.1. Découplage.

Lemme 3.1. *Soit $T = N\Delta t > 0$. Pour la solution numérique u^ε définie ci-dessus, on a*

i) *la suite $(w^\varepsilon)_{\varepsilon>0}$ est bornée dans $L^\infty([0,\infty[; VB(\mathbb{R}))$, où $w^\varepsilon = w \circ u^\varepsilon$.*

ii) *la suite $(w_t^\varepsilon)_{\varepsilon>0}$ est bornée dans $M_b(\mathbb{R}\times]0,T[)$.*

DÉMONSTRATION. Par construction u^ε est régulière par morceaux: dans les bandes $[\ n\Delta t,\ (n+1)\Delta t\ [$ elle est composée d'états constants séparés par des ondes élémentaires de discontinuité (ou de détente); elle est aussi discontinue aux instants $n\Delta t$ à cause du procédé de moyenne.

On commence par montrer le point i) à l'aide des propriétés spécifiques aux champs Temple: en particulier w ne peut varier qu'à travers les 2-ondes et cela impose à $VT(w^\varepsilon)$ d'être constante dans les bandes $[n\Delta t, (n+1)\Delta t[$, [7]; la question est de savoir si le procédé de moyenne est susceptible de faire augmenter cette variation totale. Le calcul suivant montre que c'est impossible: par construction, u_j^{n+1} appartient au convexe compris entre les plans d'équation

$$w(u) = \inf\{w^\varepsilon(x,(n+1)\Delta t-) :\ x \in [x_{j-1/2}, x_{j+1/2}[\,\}$$

et

$$w(u) = \sup\{w^\varepsilon(x,(n+1)\Delta t-) : \ x \in [x_{j-1/2}, x_{j+1/2}[\,\}\,.$$

Si par exemple le signe de λ_2, supposé constant, est positif, les valeurs extremales de w^ε dans $[x_{j-1/2}, x_{j+1/2}[$ sont $w_{j-1}^n = w(u_{j-1}^n)$ et $w_j^n = w(u_j^n)$. Donc il existe $\theta_j \in [0,1]$ tel que $w_j^{n+1} = \theta_j\, w_j^n + (1-\theta_j)\, w_{j-1}^n$, de sorte que

$$w_{j+1}^{n+1} - w_j^{n+1} = \theta_{j+1}\,(w_{j+1}^n - w_j^n) + (1-\theta_j)\,(w_j^n - w_{j-1}^n)\,.$$

Par suite

$$\begin{aligned} VT(w^\varepsilon(x,(n+1)\Delta t)) &= \sum_{j\in\mathbb{Z}} |w_{j+1}^{n+1} - w_j^{n+1}| \\ &\le \sum_{j\in\mathbb{Z}} |w_{j+1}^n - w_j^n| = VT(w^\varepsilon(x, n\Delta t))\,. \end{aligned}$$

Si $\lambda_2 < 0$ on a $w_j^{n+1} = \theta_j\, w_j^n + (1-\theta_j)\, w_{j+1}^n$ et le calcul est tout à fait analogue. On a donc, par récurrence, $VT(w^\varepsilon) \le VT(w \circ u_0)$.

Cela implique en particulier que $(w^\varepsilon)_{\varepsilon>0}$ est bornée dans $L^\infty(\mathbb{R} \times [0,T])$ et que $(w_x^\varepsilon)_{\varepsilon>0}$ est bornée dans $M_b(\mathbb{R}\times]0,T[)$: sa masse totale est

$$|||w_x^\varepsilon||| = \sum_{n=1}^{N-1} \Delta t \sum_{j\in\mathbb{Z}} |w_{j+1}^n - w_j^n| \le T\, VT(w\circ u_0)\,.$$

On obtient par la même méthode l'estimation en temps de ii), sans avoir recours à une estimation d'énergie:

$$|||w_t^\varepsilon||| = \sum_{n=1}^{N-1} \Delta x \sum_{j\in\mathbb{Z}} |w_j^{n+1} - w_j^n| = \sum_{n=1}^{N-1} \Delta x \sum_{j\in\mathbb{Z}} (1-\theta_j)\, |w_{j-1}^n - w_j^n|\,,$$

si $\lambda_2 > 0$. Dans tous les cas:

$$|||w_t^\varepsilon||| \le T\,\frac{\Delta x}{\Delta t}\, VT(w\circ u_0)\,,$$

d'où le résultat si les pas de temps et d'espace sont choisis en sorte que le rapport $\Delta x/\Delta t$, déjà minoré par la condition Courant-Friedrichs-Lewy, soit aussi majoré.

REMARQUE. Il est intéressant d'étudier le cas d'un système avec conditions aux limites afin de voir s'il y a encore de telles estimations.

Prenons par exemple un domaine spatial de la forme $]-\infty, b]$ où $b = (J+1/2)\Delta x$. En affectant à la maille fictive $[x_{J+1/2}, x_{J+3/2}[$ la valeur $a_n = a(n\Delta t)$, où a est la condition au bord que l'on imposerait dans le système parabolique, on peut aussi calculer la valeur de u_J^{n+1} par le schéma de Godunov.

Notons que pour le système parabolique les estimations en variation totale sont toujours plus délicates à obtenir. Mais avec le schéma de Godunov on peut faire un calcul très proche de celui du lemme. On note $u_{J+1/2}^n = U_R(0; u_J^n, a_n)$ et l'on étudie

$$V_{n+1} = \sum_{j<J} |w_{j+1}^{n+1} - w_j^{n+1}| + |w_J^{n+1} - w(u_{J+1/2}^n)| \, .$$

La discussion est ici quelque peu différente selon le signe de λ_2. En effet, si λ_2 est positif, la deuxième caractéristique est sortante et w ne sera pas affecté par la condition au bord, alors qu'il le sera si λ_2 est négatif.

Plus précisément si $\lambda_2 > 0$ on a

$$w_j^{n+1} = \theta_j \, w_j^n + (1-\theta_j) \, w_{j-1}^n \, , \qquad j \leq J \, ,$$

et

$$w(u_{J+1/2}^n) = w_J^n \, .$$

On en déduit aisément

$$V_{n+1} \leq \sum_{j<J} |w_{j+1}^n - w_j^n| \leq V_n \, .$$

Tandis que si $\lambda_2 < 0$ on a

$$w_j^{n+1} = \theta_j \, w_j^n + (1-\theta_j) \, w_{j+1}^n \, , \qquad j < J \, ,$$

mais

$$w_J^{n+1} = \theta_J \, w_J^n + (1-\theta_J) \, w(a_n) \, ,$$

puisque $w(u_{J+1/2}^n) = w(a_n)$.

On en déduit alors

$$V_{n+1} \leq \sum_{j<J} |w_{j+1}^n - w_j^n| + |w_J^n - w(a_n)| \, ,$$

puis finalement

$$V_{n+1} \leq V_n + |w(a_{n-1}) - w(a_n)| \,.$$

On a donc le résultat suivant, quel que soit le signe de λ_2.

Proposition 3.1. *Si $w \circ u_0 \in VB(]-\infty, b])$ et $w \circ a \in VB(0,T)$, alors la solution numérique u^ε fournie par le schéma de Godunov appliqué à (1) dans $]-\infty, b] \times [0,T]$, avec u_0 comme condition initiale et a comme condition au bord, vérifie*

$$VT(w \circ u^\varepsilon) \leq VT(w \circ u_0) + VT(w \circ a)\,.$$

Revenons au cas sans conditions aux limites. On déduit facilement du Lemme 3.1 la compacité voulue pour la suite (w^ε).

Théorème 3.1. *Soit $T = N\Delta t > 0$. La suite $(w^\varepsilon)_{\varepsilon>0}$ admet une sous-suite convergeant presque partout dans $\mathbb{R} \times [0,T]$ lorsque $\varepsilon \to 0$.*

DÉMONSTRATION. En effet le théorème de Helly s'applique à chaque instant dans un intervalle borné I de l'espace. On conclut comme dans le cas parabolique par le procédé diagonal. On notera également $(w^\varepsilon)_{\varepsilon>0}$ la sous-suite et $\overline{w}$ sa limite.

3.2. Compacité par compensation.

Il s'agît maintenant de montrer que les couples sous-entropie/flux satisfont l'hypothèse du Lemme 2.2 pour la solution numérique. La démonstration est un peu plus technique que dans le cas parabolique du fait de la présence de discontinuités. Elle est fondée sur deux résultats préliminaires.

Le premier est relativement classique [3] et regroupe les estimations tirées de l'entropie fortement convexe. Si celle-ci est choisie positive (ce qui est toujours possible quitte à lui ajouter une fonction affine) et notée $\mathcal{E}$, de flux $\mathcal{F}$, on montre

Lemme 3.2. *Soit $T = N\Delta t > 0$. Si u_0 est à support compact et si la solution numérique définie ci-dessus reste dans un compact K de Ω*

pour $t \leq T$ *alors il existe deux constantes* C *et* C' *telles que, pour tout* $\varepsilon > 0$,

$$\sum_{n=1}^{N-1} \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} |u^\varepsilon(x, n\Delta t-) - u_j^n|^2 \, dx \leq C \,, \tag{5}$$

$$0 \leq \int_0^T \sum_{\text{sauts}} (\sigma[\mathcal{E}(u^\varepsilon)] - [\mathcal{F}(u^\varepsilon)]) \, dt \leq C' \,, \tag{6}$$

où $[\cdot]$ *désigne la valeur du saut et* σ *sa vitesse.*

Le second étudie $|\sigma[E(u^\varepsilon)] - [F(u^\varepsilon)]|$ pour les couples sous-entropie/flux (E, F) en fonction de la nature de la discontinuité.

Lemme 3.3. *Si la solution numérique reste dans un compact* K *de* Ω *et si* (E, F) *est un couple sous-entropie/flux alors il existe deux constantes* C_1 *et* C_2 *telle qu'à travers toute discontinuité de vitesse* σ *on ait*

$$|\sigma[E(u^\varepsilon)] - [F(u^\varepsilon)]| \leq C_1 \, (\sigma[\mathcal{E}(u^\varepsilon)] - [\mathcal{F}(u^\varepsilon)]) \,,$$

si cette discontinuité est un 1*-choc ou un* 3*-choc,*

$$|\sigma[E(u^\varepsilon)] - [F(u^\varepsilon)]| \leq C_2 \, |[w^\varepsilon]| \,,$$

si cette discontinuité est associée au second champ (Temple).

DÉMONSTRATION. La première estimation est vraie pour les entropies classiques. Sachant que w est constant à travers les 1-ondes et les 3-ondes, on va montrer par un développement limité que les sous-entropies se comportent en fait comme des entropies à travers ces discontinuités. Par contre, le long des 2-ondes, le terme prépondérant est précisément $|[w^\varepsilon]|$.

Soit en effet un état $u_g \in K$ et $\sigma \mapsto u_d(\sigma)$ la courbe de discontinuité issue à droite de u_g, associée à une valeur propre λ: on note $\sigma_0 = \lambda(u_g)$.

On étudie $G : \sigma \mapsto F(u_d(\sigma)) - F(u_g) - \sigma \, (E(u_d(\sigma)) - E(u_g))$. On a $G(\sigma_0) = 0$.

De plus, si $dF - dE \, df = h \, dw$, on obtient, en utilisant la relation de Rankine-Hugoniot

$$f(u_d) - f(u_g) - \sigma \, (u_d - u_g) \equiv 0$$

et par dérivations successives

$$G'(\sigma_0) = \Big(h \circ u_d \, \frac{d(w \circ u_d)}{d\sigma}\Big)(\sigma_0)\,,$$
$$G''(\sigma_0) = \frac{d}{d\sigma}\Big(h \circ u_d \, \frac{d(w \circ u_d)}{d\sigma}\Big)(\sigma_0)\,,$$
$$G^{(3)}(\sigma_0) = \frac{d^2}{d\sigma^2}\Big(h \circ u_d \, \frac{d(w \circ u_d)}{d\sigma}\Big)(\sigma_0) + d^2E(u_g)(r(u_g), r(u_g))\,.$$

Dans le cas d'un 1-choc ou d'un 3-choc, on a l'identité $w \circ u_d \equiv w(u_g)$. Donc

$$G'(\sigma_0) = 0\,, \qquad G''(\sigma_0) = 0\,,$$
$$G^{(3)}(\sigma_0) = d^2E(u_g)(r(u_g), r(u_g))\,.$$

Il en est de même pour l'entropie fortement convexe. On en déduit les inégalités, pour $\sigma \in \lambda(K)$,

$$|G(\sigma) - G(\sigma_0)| \leq c\,\|r(u_g)\|^2 |\sigma - \sigma_0|^3 \leq C_1\,(\mathcal{G}(\sigma_0) - \mathcal{G}(\sigma))\,,$$

où $\mathcal{G}(\sigma) = \mathcal{F}(u_d(\sigma)) - \mathcal{F}(u_g) - \sigma\,(\mathcal{E}(u_d(\sigma)) - \mathcal{E}(u_g)) \leq \mathcal{G}(\sigma_0) = 0$ par définition d'un choc, et C_1 dépend seulement de $E, \mathcal{E}$ et K.

Dans le cas d'une discontinuité associée au champ de Temple, on a

$$G'(\sigma_0) = -h(u_g)\,dw(u_g) \cdot r(u_g)$$

avec $dw(u_g) \cdot r(u_g) \neq 0$, d'où, pour $\sigma \in \lambda(K)$,

$$\begin{aligned} |G(\sigma) - G(\sigma_0)| &\leq c\,|dw(u_g) \cdot r(u_g)|\,|\sigma - \sigma_0| \\ &\leq C_2\,|w \circ u_d(\sigma) - w(u_g)|\,. \end{aligned}$$

On peut alors affirmer

Lemme 3.4. *Sous les hypothèses du Lemme 3.2, on a, pour tout couple sous-entropie/flux (E, F),*

$$E(u^\varepsilon)_t + F(u^\varepsilon)_x \in \textit{compact de } H^{-1}_{\mathrm{loc}}(\mathbb{R} \times [0, T])\,.$$

DÉMONSTRATION. Sachant que $E(u^\varepsilon)_t + F(u^\varepsilon)_x = h(u^\varepsilon)\, w^\varepsilon_x$ en dehors des discontinuités, on décompose la mesure $\theta^\varepsilon = E(u^\varepsilon)_t + F(u^\varepsilon)_x$ en trois termes. On a, pour tout $\varphi \in \mathcal{D}(\mathbb{R} \times [0,T])$,

$$\langle \theta^\varepsilon, \varphi \rangle = V(\varphi) - \Sigma(\varphi) - L(\varphi)\,,$$

où

$$L(\varphi) = \sum_{n=1}^{N-1} \int_{\mathbb{R}} \varphi(x, n\Delta t)\,(E^\varepsilon(x, n\Delta t-) - E^\varepsilon(x, n\Delta t))\, dx$$

est traité exactement comme pour les vraies entropies. On montre [3], grâce à l'inégalité (5), que $L = L_1 + L_2$ avec $L_1 \in$ borné de $M_b(\mathbb{R}\times]0,T[)$ et $L_2 \in$ compact de $W^{-1,q}_{\text{loc}}$ pour un $q < 2$. L'estimation des termes

$$\Sigma(\varphi) = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \sum_{1 \le k \le K_n} \varphi(x_k(t), t)\,(\sigma_k [E(u^\varepsilon)]_k - [F(u^\varepsilon)]_k)\, dt$$

et

$$V(\varphi) = \sum_{n=0}^{N-1} \sum_{1 \le k \le K_n - 1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{x_k(t)}^{x_{k+1}(t)} \varphi\, h(u^\varepsilon)\, w^\varepsilon_x\, dx\, dt\,,$$

où l'on a noté $t \longmapsto x_k(t)$, $1 \le k \le K_n$, les courbes de discontinuité dans la bande $[n\Delta t, (n+1)\Delta t[$, est obtenue grâce au Lemme 3.3 et à l'inégalité (6). Elle s'écrit

$$|V(\varphi) - \Sigma(\varphi)| \le c\, \|\varphi\|_\infty \Big(\int_0^T VT(w^\varepsilon)\, dt + C' \Big)\,,$$

d'où, avec le Lemme 3.1, $(V - \Sigma) \in$ borné de $M_b(\mathbb{R}\times]0,T[)$.

Donc $\theta^\varepsilon \in$ borné de M_b + compact de $W^{-1,q}_{\text{loc}}$. Comme on a aussi $\theta^\varepsilon \in$ compact de $W^{-1,\infty}_{\text{loc}}$, le Lemme de Murat permet de conclure.

Par conséquent on a

Théorème 3.2. *La mesure de Young $\nu_{(x,t)}$ associée à (u^ε) vérifie, pour presque tout $(x,t) \in \mathbb{R} \times [0,T]$ et pour tous couples sous-entropies/flux (E_i, F_i), $i = 1, 2$, l'équation de Tartar*

$$\langle \nu, E_1 F_2 - E_2 F_1 \rangle = \langle \nu, E_1 \rangle \langle \nu, F_2 \rangle - \langle \nu, E_2 \rangle \langle \nu, F_1 \rangle\,.$$

3.3. Conclusion.

Théorème 3.3. *Soit $T = N\Delta t > 0$. Si u_0 est à support compact et si la solution numérique de Godunov reste dans un compact K de Ω pour $t \leq T$, alors la suite $(u^\varepsilon)_{\varepsilon>0}$ admet une sous-suite convergeant presque partout dans $\mathbb{R} \times [0,T]$ lorsque $\varepsilon \to 0$ et sa limite est solution faible entropique de* (1).

DÉMONSTRATION. On conclut à la convergence, comme dans le cas parabolique, à l'aide des Théorèmes 3.1 et 3.2. On montre ensuite aisément, par des intégrations par parties semblables à celles faites dans l'étude de θ^ε, que la limite u est solution faible entropique de (1).

4. Réduction de ν.

L'objet de ce paragraphe est de démontrer le théorème utilisé plus haut, à savoir

Théorème 4.1. *Si une suite $(u^\varepsilon)_{\varepsilon>0}$ de solutions approchées de* (1), *restant dans un compact K de Ω, est telle que*

$$w(u^\varepsilon) \to \overline{w}, \qquad p.p.,$$

et sa mesure de Young ν vérifie presque partout

$$\langle \nu, E_1F_2 - E_2F_1 \rangle = \langle \nu, E_1 \rangle \langle \nu, F_2 \rangle - \langle \nu, E_2 \rangle \langle \nu, F_1 \rangle$$

quels que soient les couples sous-entropies/flux (E_i, F_i), $i = 1, 2$, alors on a en fait pour presque tout (x,t)

$$\nu_{(x,t)} = \delta_{u(x,t)} ,$$

où $u(x,t) \in K$.

La méthode va consister à faire correspondre les sous-entropies de (1) aux entropies d'un système 2×2 vraiment non-linéaire dépendant régulièrement d'un paramètre.

Mais d'abord précisons ce qu'implique l'hypothèse sur $w(u^\varepsilon)$ pour la mesure de Young ν.

Lemme 4.1. *Sous les hypothèses du Théorème 4.1, on a*

$$\operatorname{supp} \nu_{(x,t)} \subset \{u \in K : \ w(u) = \overline{w}(x,t)\}\,, \qquad p.p.$$

DÉMONSTRATION. C'est une conséquence facile de la définition de ν. En effet, si $\varphi \in \mathcal{D}(\mathbb{R} \times [0,T])$ on a en décomposant les termes

$$\int_0^T \int_{\mathbb{R}} (w(u^\varepsilon)(x,t) - \overline{w}(x,t))^2 \varphi(x,t)\, dx\, dt \\ \longrightarrow \int_0^T \int_{\mathbb{R}} \langle \nu_{(x,t)}, (w - \overline{w}(x,t))^2 \rangle\, \varphi(x,t)\, dx\, dt\,.$$

Or on a aussi, par le Théorème de Lebesgue

$$\int_0^T \int_{\mathbb{R}} (w(u^\varepsilon)(x,t) - \overline{w}(x,t))^2 \varphi(x,t)\, dx\, dt \ \longrightarrow\ 0\,.$$

Il s'ensuit

$$\langle \nu_{(x,t)}, (w - \overline{w}(x,t))^2 \rangle = 0\,, \qquad \text{p.p.}\,,$$

d'où le résultat puisque l'argument est positif.

Soit maintenant $\Phi : u \longmapsto (v_1, v_3, w)$ un changement de variables de classe C^2. La mesure image de $\nu_{(x,t)}$ est pour presque tout (x,t) de la forme

$$\Phi^*(\nu_{(x,t)}) = \mu_{(x,t)} \otimes \delta_{\overline{w}(x,t)}\,.$$

Le problème revient donc à montrer que $\mu_{(x,t)}$ est presque partout une masse de Dirac. On va voir ce que signifie pour μ l'équation de Tartar en ν.

Pour cela on note $\dot{}$ la différentiation par rapport à w, δ celle par rapport à $v = (v_1, v_3)$ et l'on étudie la structure des sous-entropies dans ces nouvelles variables.

Proposition 4.1. *Il existe une application $B : \Phi(\Omega) \longrightarrow \mathcal{M}_2(\mathbb{R})$ telle que, pour tout w, $B(\cdot, w)$ a ses deux champs caractéristiques vraiment non-linéaires et, pour des applications $E, F : \Omega \longrightarrow \mathbb{R}$ de classe C^2, les assertions suivantes sont équivalentes, avec $\eta = E \circ \Phi^{-1}, q = F \circ \Phi^{-1}$,*

i) *L'application E est une sous-entropie de* (1) *associée au flux F.*

ii) *L'application $\eta(\cdot, w)$ est une entropie de $B(\cdot, w)$ associée au flux $q(\cdot, w)$, c'est-à-dire,*

$$\delta q - \delta\eta\, B \equiv 0\,.$$

DÉMONSTRATION. Cela résulte de quelques calculs élémentaires. Tout d'abord on remarque que, puisque $dw\, df = \lambda_2\, dw$,

$$dF - dE\, df = \delta q\, dv - \delta\eta\, dv\, df + (\dot{q} - \lambda_2\, \dot{\eta})\, dw\,.$$

Or $I = \delta u\, dv + \dot{u}\, dw$, d'où

$$dv\, df = dv\, df\, \delta u\, dv + dv\, df\, \dot{u}\, dw\,.$$

Soit alors $B = dv\, df\, \delta u$. On a

$$dF - dE\, df = (\delta q - \delta\eta\, B)\, dv + (\dot{q} - \lambda_2\, \dot{\eta} - \delta\eta\, dv\, df\, \dot{u})\, dw\,.$$

On en déduit l'équivalence entre i) et ii).

Il reste à montrer que B est vraiment-non linéaire à w fixé. Comme $dw\, r_j \equiv 0$, $j = 1, 3$, on déduit de la définition de B que

$$B\, dv\, r_j = dv\, df\, r_j = \lambda_j\, dv\, r_j\,, \qquad j = 1, 3\,,$$

avec un abus de notation évident, c'est-à-dire que les vecteurs propres de B sont $e_j = dv\, r_j$, $j = 1, 3$, associés aux valeurs propres $\mu_j = \lambda_j \circ \Phi^{-1}$, $j = 1, 3$. Or $d\lambda_j = \delta\mu_j\, dv + \dot{\mu}_j\, dw$. Donc

$$\delta\mu_j\, e_j = d\lambda_j\, r_j\,, \qquad j = 1, 3\,.$$

La vraie non-linéarité des premier et troisième champs de (1) impose donc aux deux champs de B d'être aussi vraiment non-linéaires.

Sachant que pour presque tout (x, t) on a d'une part $\Phi^*(\nu_{(x,t)}) = \mu_{(x,t)} \otimes \delta_{\overline{w}(x,t)}$ et d'autre part l'équation de Tartar pour $\nu_{(x,t)}$ et toutes les sous-entropies, on se place désormais en un point où ces deux propriétés ont lieu. On note $w_0 = \overline{w}(x, t)$.

On s'intéresse alors au problème inverse par rapport à la proposition précédente: est-il possible, à partir d'un couple entropie/flux (η_0, q_0) de $B(\cdot, w_0)$, de reconstruire un couple sous-entropie/flux (E, F) de (1) tel qu'en variables (v, w) on ait $\eta(\cdot, w_0) = \eta_0$ et $q(\cdot, w_0) = q_0$?

Un tel relèvement s'obtient assez facilement lorsque le changement de variables Φ est bien choisi. En effet, comme w est à la fois un invariant de Riemann *au sens de Lax* pour le premier *et* le troisième champ, on peut prendre pour v_1 un autre 3-invariant de Riemann au sens de Lax, indépendant de w, et pour v_3 un 1-invariant de Riemann au sens de Lax également indépendant de w. Ceci fournit bien un changement de variables car on a

$$\begin{pmatrix} dv_1 \\ dv_3 \\ dw \end{pmatrix} \begin{pmatrix} r_1 & r_3 & r_2 \end{pmatrix} = \begin{pmatrix} e_1 & e_3 & * \\ & 0 & 1 \end{pmatrix},$$

où les vecteurs e_1 et e_3 valent respectivement $\begin{pmatrix} dv_1 \cdot r_1 \\ 0 \end{pmatrix}$ et $\begin{pmatrix} 0 \\ dv_3 \cdot r_3 \end{pmatrix}$ par définition de v; ils ne s'annulent pas sinon dv_1 ou dv_3 serait liée à dw en un tel point, ce qui est contraire à la construction de v. Avec ce choix la matrice $B(v,w)$ est diagonale en tout point et vaut $\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_3 \end{pmatrix}$.

Par suite les couples (η, q) correspondant en variables (v,w) aux couples sous-entropie/flux (E,F) sont les solutions régulières par rapport à w du système linéaire

$$\text{(7)} \qquad \left\{ \begin{aligned} \frac{\partial q}{\partial v_1} &= \mu_1(v,w) \frac{\partial \eta}{\partial v_1}, \\ \frac{\partial q}{\partial v_3} &= \mu_3(v,w) \frac{\partial \eta}{\partial v_3}. \end{aligned} \right.$$

Le lemme suivant répond finalement à la question.

Lemme 4.2. *Soient (η_0, q_0) un couple entropie/flux de classe C^2 de $B(\cdot, w_0)$. Alors il existe (η, q) de classe C^2, vérifiant le système* (7) *pour tout w, tel que $\eta(\cdot, w_0) = \eta_0$ et $q(\cdot, w_0) = q_0$.*

DÉMONSTRATION. Par hypothèse on a

$$\left\{ \begin{aligned} \frac{\partial q_0}{\partial v_1} &= \mu_1(v,w_0) \frac{\partial \eta_0}{\partial v_1}, \\ \frac{\partial q_0}{\partial v_3} &= \mu_3(v,w_0) \frac{\partial \eta_0}{\partial v_3}. \end{aligned} \right.$$

Notons que l'élimination de q dans (7) conduit à l'équation pour η

$$\text{(8)} \qquad \frac{\partial^2 \eta}{\partial v_1 \partial v_3} = \frac{1}{\mu_3 - \mu_1} \left(\frac{\partial \mu_1}{\partial v_3} \frac{\partial \eta}{\partial v_1} - \frac{\partial \mu_3}{\partial v_1} \frac{\partial \eta}{\partial v_3} \right).$$

Cette équation est satisfaite par η_0 lorsque $w = w_0$ et ses coefficients sont de classe C^1 en v et de classe C^2 en w.

Le *problème de Goursat* associé à (8) avec les mêmes conditions sur les caractéristiques que celles vérifiées par η_0 admet donc une solution globale unique η de classe C^2 en w *et* en v, [1]. On a en particulier $\eta(\cdot, w_0) = \eta_0$.

On obtient alors q par quadrature, également de classe C^2 et tel que $q(\cdot, w_0) = q_0$.

Corollaire 4.1. *La mesure $\mu_{(x,t)}$ et, par suite, $\nu_{(x,t)}$ sont réduites à des masses de Dirac.*

DÉMONSTRATION. Soient deux couples entropies/flux (η_0^i, q_0^i), $i = 1, 2$, de $B(\cdot, w_0)$ où, rappelons-le, $w_0 = \overline{w}(x,t)$. Soient $(E_i, F_i) = (\eta^i \circ \Phi^{-1}, q^i \circ \Phi^{-1})$ où (η^i, q^i) sont associés à (η_0^i, q_0^i) par le lemme précédent.

La mesure $\nu_{(x,t)}$ vérifie l'équation de Tartar avec (E_i, F_i). Donc $\mu_{(x,t)}$ vérifie l'équation de Tartar avec (η_0^i, q_0^i).

D'après DiPerna [3] ceci impose $\mu_{(x,t)} = \delta_{v_0}$, où $v_0 \in \Phi(\{u \in K : w(u) = w_0\})$. On a ainsi $\nu_{(x,t)} = \delta_{u_0}$, où $u_0 = \Phi^{-1}(v_0, w_0)$.

Le Théorème 4.1 est alors démontré.

5. Cas des systèmes à p équations.

Soit un système strictement hyperbolique de p lois de conservation $(p > 3)$,

$$(9) \qquad u_t + f(u)_x = 0, \qquad t > 0,\ x \in \mathbb{R},\ u(x,t) \in \Omega \subset \mathbb{R}^p,$$

ayant deux champs vraiment non-linéaires et les $(p-2)$ autres de B. Temple. Soient $w_2, \ldots, w_{p-1}$ les invariants de Riemann forts associés. Pour chacun d'entre eux, si (9) admet une entropie fortement convexe, on a les estimations a priori du Lemme 2.1 pour l'approximation parabolique et du Lemme 3.1 pour le schéma de Godunov. La suite de solutions approchées, si elle reste dans un compact K de Ω, admet donc une sous-suite $(u^\varepsilon)_{\varepsilon>0}$ telle que $(w_j^\varepsilon)_{\varepsilon>0}$, $j = 2, \ldots, p-1$ convergent presque partout dans $\mathbb{R} \times [0, T]$ vers $\overline{w}_j$. D'autre part la généralisation directe de la notion de sous-entropie est la suivante

Definition 5.1. *Deux applications $E, F : \Omega \longrightarrow \mathbb{R}$ de classe C^2 forment un couple sous-entropie/flux si et seulement si*

$$dF(u) - dE(u)\, df(u) \in \bigoplus_{j=2}^{p-1} \mathbb{R}\, l_j(u)\,, \qquad \textit{pour tout } u \in \Omega\,.$$

Ceci équivaut encore au système de deux équations à deux inconnues

$$(dF - dE\, df)\, r_j \equiv 0\,, \qquad j = 1, p\,.$$

Dans les Lemmes 2.3 et 3.4 le terme en w est remplacé par une somme de termes en w_j, $j = 2, \ldots, p-1$ de même nature. Par conséquent on a aussi

$$E(u^\varepsilon)_t + F(u^\varepsilon)_x \in \text{compact de } H^{-1}_{\text{loc}}(\mathbb{R} \times [0, T])\,,$$

d'où l'équation de Tartar pour la mesure de Young associée à $(u^\varepsilon)_{\varepsilon > 0}$

$$\langle \nu_{(x,t)}, E_1 F_2 - E_2 F_1 \rangle = \langle \nu_{(x,t)}, E_1 \rangle \, \langle \nu_{(x,t)}, F_2 \rangle - \langle \nu_{(x,t)}, E_2 \rangle \, \langle \nu_{(x,t)}, F_1 \rangle\,,$$

pour presque tout $(x, t) \in \mathbb{R} \times [0, T]$ et pour tous couples sous-entropie/flux (E_i, F_i), $i = 1, 2$.

La réduction de ν peut alors se faire exactement comme au paragraphe 4, grâce au changement de variables

$$\Phi : u \longmapsto (v_1, v_p, w_2, \ldots, w_{p-1})\,,$$

où v_1 est un p-invariant de Riemann au sens de Lax indépendant de $w_2, \ldots, w_{p-1}$ et v_p est un 1-invariant de Riemann au sens de Lax également indépendant de $w_2, \ldots, w_{p-1}$. La mesure $\mu_{(x,t)}$ définie par

$$\Phi^*(\nu_{(x,t)}) = \mu_{(x,t)} \otimes \delta_{\overline{w}_2(x,t)} \otimes \cdots \otimes \delta_{\overline{w}_{p-1}(x,t)}$$

vérifie en effet l'équation de Tartar associée aux couples entropie-flux d'un système 2×2 vraiment non-linéaire et elle est donc réduite à une masse de Dirac.

Finalement on a démontré le théorème

Théorème 5.1. *Soit* $(u^\varepsilon)_{\varepsilon>0}$ *une suite de solutions approchées de* (9) *obtenues par la régularisation parabolique* (2) *ou le schéma de Godunov. Si* $u^\varepsilon(x,t) \in K \subset \Omega$ *p.p. dans* $\mathbb{R} \times [0,T]$, *où* K *est un compact indépendant de* ε, *alors* $(u^\varepsilon)_{\varepsilon>0}$ *admet une sous-suite convergeant p.p. dans* $\mathbb{R} \times [0,T]$ *vers une solution faible entropique de* (9).

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Interpolation of infinite order entire functions

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Introduction.

In [2] and [3], Berenstein and Struppa studied the relation between Dirichlet series and solutions of convolution equations. In [3], they considered the equation

$$\mu * f = 0\,,$$

where f is an analytic function on the upper half plane and μ is an analytic functional on the complex plane $\mathbb{C}$. It turns out that if μ satisfies a "slowly decreasing" condition, then f can be represented by a series

$$f(z) = \sum_{k=1}^{\infty} \sum_{l=1}^{J_k} P_{k,l}(z)\, e^{a_{k,l}\, z}\,,$$

where $\hat{\mu}(a_{k,l}) = 0$, $P_{k,l}$ are polynomials and $\hat{\mu}$ represents the Fourier transform of μ. This is a representation of a generalized Dirichlet series. Under certain conditions, "gap" theorems similar to the Fabry gap theorem for Dirichlet series may be proven. We refer the reader to [3] for further remarks.

In [2], they consider the case where f is holomorphic in a cone Γ contained in the right half plane with vertex at the origin and μ can be described by integration against a measurable function with compact support in that cone. In this case, with μ slowly decreasing, f can be represented as a Dirichlet series, in its simplest form

$$f(w) = \sum_{k=1}^{\infty} c_k(w)\, e^{-z_k w}, \qquad w \in \Gamma,$$

where c_k is a polynomial of degree less than m_k and $\hat{\mu}$ vanishes at z_k with multiplicity m_k. We again refer the interested reader to [2] for further details.

The point of this approach to Dirichlet series is that, while the clasical Fabry gap theorem requires that the frequencies

$$0 < |z_1| < |z_2| < \dots$$

satisfy the addititional finite density condition

$$\overline{\lim_{k\to\infty}} \frac{k}{|z_k|} < +\infty,$$

and $m_k = 1$, and other convergence theorems require finite density, in [2] and [3], one can alow for the existence of a constant $\alpha > 0$ such that

$$\overline{\lim_{k\to\infty}} \frac{k}{|z_k|^\alpha} < +\infty,$$

(as well as no assumption on the z_k being real or $m_k = 1$). Clearly, this condition is weaker than the previous one when $\alpha > 1$. It leads to the study of interpolation problems for holomorphic functions of finite order α.

On the other hand, a sequence like $z_k = \ln k$ will not satisfy such hypotheses for any $\alpha > 0$. Nevertheless, such sequences are of great interest since the family of ordinary Dirichlet series includes the Riemann ζ-function

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}.$$

In order to study such problems, one must have some knowledge of infinite order functions. This article, which is based on the author's Thesis, provides the framework necessary in order to extend the theorems of [2] and [3] to the infinite order case.

Let f be an entire function which vanishes at the points z_k with multiplicity m_k and nowhere else. Then, given a doubly indexed sequence of complex numbers $\{a_{k,l}\}_{k\geq 1,\, 0\leq l<m_k}$ satisfying

$$\sum_{l=0}^{m_k-1} |a_{k,l}| \leq A\, e^{B\rho(z_k)}, \qquad A, B > 0,$$

when does there exist an entire function $\lambda(z)$ such that

$$|\lambda(z)| \leq A_1\, e^{B_1\rho(z)}, \qquad A_1, B_1 > 0,$$

and

$$\frac{\lambda^{(l)}(z_k)}{l!} = a_{k,l}\,?$$

Here ρ is a subharmonic function satisfying other conditions to be explained in the paper. The new contribution is for the case when ρ grows fast enough so that λ is of infinite order.

The plan of this paper is to extend the results of [1] to the infinite order case. One possibility is to allow an extra constant inside of ρ, *i.e.* let

$$|f(z)| \leq A\, e^{B\rho(Cz)}, \qquad A, B, C > 0.$$

It turns out here that the proofs are the same and only the statements are presented here in Section 2. In Section 1, the main part of this work, we do not allow the extra constant. Theorem 1.1 is the major result, which precisely calculates orders of infinite order functions. The rest of the section presents ramifications of this result which should lead to methods to attack the problems mentioned above.

Throughout this paper, we let $\mathbb{N}$ denote the natural numbers, $M_f(r)$ denote the maximum modulus of f on a circle of radius r, and $D(z;r)$ denote the circle of radius r centered at z.

1. Interpolation for infinite order functions.

Let $\{z_k\}_{k=1}^{\infty}$ be a divergent sequence of complex numbers and $\{m_k\}_{k=1}^{\infty}$ be a sequence of positive integers. We start with the following definition.

Definition. *$V = \{(z_k, m_k)\}$ is the multiplicity variety for an entire function f if it vanishes precisely at the points z_k, $k \geq 1$, with multiplicity m_k. We write $V = V(f)$ when V is the multiplicity variety* *for f. More generally, $V = V(f_1, \ldots, f_m)$ is a multiplicity variety for $f_1, \ldots, f_m$ if $\{z_k\}$ is the set of common zeros of the functions $f_1, \ldots, f_m$ and the functions vanish at those points with multiplicity at least m_k, and one of them with multiplicity exactly m_k.*

A well known theorem is the following [1, Theorem 3].

Theorem 1.A. *Let $V = \{(z_k, m_k)\}_{k \geq 1}$ be a multiplicity variety. Let $\{a_{k,l}\}$ be any sequence of complex numbers, where $k \geq 1$ and $0 \leq l \leq m_k - 1$. Then there exists an entire function λ such that*

$$\frac{\lambda^{(l)}(z_k)}{l!} = a_{k,l}\,.$$

By Theorem 4 in [1], in order to study the interpolation problem, we first study the corresponding problem of zeros.

We are interested in putting growth restrictions on the sequence $\{a_{k,l}\}$ and the function λ in Theorem 1.A. We make the following

Definition. *Let f be an entire function. For $r > 0$, $n_f(r)$ is defined to be the number of zeros, counted with multiplicity, of f in the circle of radius r, excluding those at the origin. When the function being considered is clear, we will drop the subscript.*

The growth of f depends upon the growth of n_f. The relationship is provided by Lemma 1.A. First, let us review what happens in the case of finite order functions and discuss the differences and difficulties encountered for infinite order functions. Our source for this discussion is the book of Levin [4, Chap. 1].

Let $\{a_n\}_{n=1}^{\infty}$ be the set of zeros of an entire function (excluding those at the origin), arranged in order of increasing modulus and repeated according to multiplicity. The starting point is the infinite sum

$$\sum_{n=1}^{\infty} \left| \frac{z}{a_n} \right|^{p_n+1}, \tag{1.1}$$

where p_n is a sequence of non-negative integers chosen so that the sum (1.1) converges uniformly on compact sets. Then we can consider the infinite product

$$E(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\sum_{i=1}^{p_n} \frac{1}{i}\left(\frac{z}{a_n}\right)^i\right). \tag{1.2}$$

In the case of finite order functions, the numbers a_n satisfy the following suplemental condition: there exists a positive number λ such that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda}$$

converges. In this case, let p denote the smallest integer for which

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} \tag{1.3}$$

converges. Setting $p_n = p$ for all n in (1.1) will be enough to assure convergence of (1.1). Here, then, is the first complication for infinite order functions: no such simplification (1.3) is posible.

The first step in finding an upper bound for the product (1.2) is finding an upper bound for each term. Taking logarithms of both sides of (1.2) will turn the infinite product into an infinite sum. We will obtain an upper bound for each term in the sum. Adding together all the upper bounds will result in another sum which can be written as a Stieltjes integral. It is this integral which must be evaluated to obtain a formula for the growth of (1.2).

The upper bound for each term is given by Lemma 2 in Chapter 1 of Levin [4, p. 11]. We modify it for our purposes and restate it as Lemma 2.A.

Lemma 2.A. *For $p_n \geq 1$ and all complex numbers z,*

$$\ln\left|\left(1-\frac{z}{a_n}\right)\exp\left(\sum_{i=1}^{p_n}\frac{1}{i}\left(\frac{z}{a_n}\right)^i\right)\right| \leq A_{p_n}\frac{\left|\frac{z}{a_n}\right|^{p_n+1}}{1+\left|\frac{z}{a_n}\right|}, \tag{1.4}$$

where

$$A_{p_n} = 3\,e\,(2+\ln p_n).$$

If $p_n = 0$, the sum is empty, so we have

$$\ln\left|\left(1-\frac{z}{a_n}\right)\right| \leq \ln\left(1+\left|\frac{z}{a_n}\right|\right).$$

Summing (1.4) over all n gives

$$\ln|E(z)| \le \sum_{n=1}^{\infty} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n}(|a_n|+r)}, \tag{1.5}$$

for $|z| = r$.

In the case of finite order functions, the numerator is independent of n and may be taken outside the sum. Then, writing (1.5) as a Stieltjes integral, we obtain

$$\ln|E(z)| \le A_p\, r^{p+1} \int_0^{+\infty} \frac{dn(t)}{t^p\,(t+r)}. \tag{1.6}$$

Integration by parts then gives the formula in Levin [4, p. 12] for finite order functions

$$\ln|E(z)| \le k_p\, r^p \left(\int_0^r \frac{n(t)}{t^{p+1}}\, dt + r \int_r^{+\infty} \frac{n(t)}{t^{p+2}}\, dt \right), \tag{1.7}$$

for $k_p = 3\,e\,(p+1)\,(2+\ln p)$ for $p \ge 1$, $k_0 = 1$, and $|z| = r$.

In the infinite order case, the Stieltjes integral becomes

$$\int_0^{+\infty} \frac{A_{p(t)}\, r^{p(t)+1}\, dn(t)}{t^{p(t)}\,(t+r)}, \tag{1.8}$$

where $p(t)$ is a continuous function such that $p(t) = p_n$ at $t = n$.

Such a simple integration by parts procedure is now impossible. There is also the problem of convergence of (1.8) and chosing the correct $p(t)$ which will minimize the integral. We will see that it is easy to chose the correct $p(t)$. The integral is a Laplace-type integral in that most of the contribution to the integral takes place in the region around $t = r$. This leads us to using the Laplace method to evaluate the integral.

Another diference between finite and infinite order functions can be noted here. Inequality (1.7) shows that the order of $E(z)$ is no larger than one of $n(r)$ (although, the type, of course, may be infinite). For example, we know

$$|\sin(\pi z)| \le A\, e^{\pi r}, \qquad \text{for } |z| = r,$$

and

$$\left|\frac{1}{\Gamma(z)}\right| \le A\, e^{Br\ln r}, \qquad \text{for } |z| = r.$$

Here $n_{\sin(\pi z)}(r) = 2r$ and $n_{1/\Gamma(z)}(r) = r$.

This behavior leads us to believe that a function with zeros at $\ln n$ should be bounded by e^{e^r}. However, this is not the case. We will see that infinite order functions can grow much faster than $n(r)$. The reason for this is that the integral (1.8) contains the term $dn(t)$. Very loosely speaking, (1.6) and (1.8) show functions grow at the rate $r\,dn(r)$. Of course, for finite order functions,

$$r\,dn(r) = C\,n(r),$$

for the appropriate constant C. For infinite order functions, $dn(r)$ can be much larger than $n(r)$. The dividing line is at $n(r) = e^r$, in which case $n(r) = dn(r)$. The growth rate of the infinite order functions that we will consider can be somewhat sharpened using the Laplace method.

Before continuing, we first show that the choice of $p(t)$ does not cause any problems with convergence of (1.1).

Lemma 1.1. *Convergence of* (1.8) (*and, hence, of* (1.5)) *implies convergence of* (1.1).

Proof. Assume p_n is chosen so that (1.5) converges. Then, for large enough $n_0 \geq 1$, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n}(|a_n|+r)} &\geq \sum_{n=1}^{\infty} \frac{r^{p_n+1}}{|a_n|^{p_n}(2|a_n|)} \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1}.\end{aligned}$$

Roughly speaking, our conditions for Theorem 1.1 require $\ln n'(r)$ to be convex (and therefore, of course, $n(r)$) and

$$n(r) \leq e^{r^\beta},$$

for some $\beta > 0$.

Theorem 1.1. *Let $f(z)$ be an entire function of infinite order with prescribed zeros at $z = a_n$, excluding possible zeros at the origin. Let $n(r)$ be a majorant of the number of zeros in the annulus $0 < |z| < r$ such that*

$$\begin{aligned} dn_f[r_k, r_{k+1}) &= n_f(r_{k+1}) - n_f(r_k) \\ &\leq n(r_{k+1}) - n(r_k) \qquad (1.9)\\ &= dn[r_k, r_{k+1}),\end{aligned}$$

for every $k \geq 1$ *where* $0 < r_1 < r_2 < \dots$ *is the sequence of increasing moduli of the zeros not at the origin. Assume for large enough* $r \geq r_0$,

$$\begin{aligned}
&(1.10) \qquad \ln n'(r) \text{ is increasing and convex,} \\
&(1.11) \qquad \ln r = o(\ln n'(r)) \quad \text{as } r \to \infty\,, \\
&(1.12) \qquad (\ln n')''((1 \pm \delta)r) \leq B_1 (\ln n')''(r)\,, \\
&(1.13) \qquad |(\ln n')'''((1 \pm \delta)r)| \leq B_2 |(\ln n')'''(r)|\,,
\end{aligned}$$

where (1.12) *and* (1.13) *hold for any* δ *such that* $0 < \delta < 1$ *and for some constants* B_1, $B_2 > 0$.

Then there exists an infinite product E *asociated with those zeros such that*

$$\ln |E(z)| \leq \frac{C \ln \left(r \dfrac{n'(r)}{n''(r)} \right) n'(r)}{\left(\dfrac{1}{r} \dfrac{n''(r)}{n'(r)} + \dfrac{n'(r)\, n'''(r) - (n''(r))^2}{(n'(r))^2} \right)^{1/2}}\,,$$

for $|z| = r$ *and large enough* $r > 0$.

PROOF. We will use the following notations and lemma for the proof of Theorem 1.1.

$$\begin{aligned}
p(t) &= t\, (\ln n')'(t)\,, \\
\psi(t) &= (\ln r - \ln t)\, p(t) + \ln n'(t)\,, \\
\varphi(t) &= \frac{6\, e + 3\, e \ln(p(t))}{t + r}\,.
\end{aligned}$$

Before we proceed, we establish a technical lemma. The conditions (1.10), (1.11), (1.12) and (1.13) of Theorem 1.1 imply

Lemma 1.2. *For the* δ *chosen in the proof of the theorem, and for large enough* $r \geq r_0$,

a) *For every* $0 < C_1 < 1$, *there exists* $0 < C_2 < (1 + C_1)/2$ *such that*

$$\ln n'(C_1 r) \leq C_2 \ln n'(r)\,.$$

b) *For some* $\varepsilon > 0$ *and for every* $t > e^2 r$, $\psi(t) \leq -\varepsilon \ln n'(t)$.

c) *For* $|t - r| \leq \delta r$, $|(\ln n')'(t) - (\ln n')'(r)| \leq \delta^{1/2} B_1^2\, |(\ln n')'(r)|$, *for some* $B_1 > 0$.

d) *For* $|t-r| \le \delta r$, $|(\ln n')''(t) - (\ln n')''(r)| \le \delta^{1/2} B_2^2\, |(\ln n')'(r)|$, *for some* $B_2 > 0$.

e) ψ *is increasing on* $(r_0, r]$ *and decreasing on* $[r, +\infty)$.

f) $(\ln n')''(r) < O(r^N)$ *and* $(\ln n')'(r) < O(r^N)$, *for some* $N > 0$.

g) $\displaystyle\int_0^{e^2 r} \varphi(t)\, dt = O(1)$.

h) $\varphi(t) \in L^\infty(2r, +\infty)$.

i) $\displaystyle\sup_{|t-r|\le\delta r} \varphi(t) = O(\ln p(r))$.

j) $\displaystyle\frac{r}{(n'(r))^\eta} = O\,(1-\psi''(r))$, *for any* $\eta > 0$.

PROOF. a) Follows from (1.9).

b) We have

$$\begin{aligned}
\psi(t) &\le (\ln r - \ln e^2 r)\, t\, (\ln n')'(t) + \ln n'(t) \\
&\le -2t\, (\ln n')'(\xi) + \ln n'(t) \\
&= -\frac{2}{\delta}\, (\ln n'(t) - \ln n'(t-\delta t) + \ln n'(t) \\
&= \Big(1 - \frac{2}{\delta}\Big) \ln n'(t) + \frac{2}{\delta}\, \ln n'(t-\delta) \\
&\le \Big(1 - \frac{2}{\delta}\Big) \ln n'(t) + \frac{2}{\delta}\, C\, \ln n'(t) \\
&= \Big(1 - \frac{2}{\delta}\Big) \ln n'(t) + \frac{2}{\delta}\, \Big(\frac{2-\delta}{2} - \varepsilon\Big) \ln n'(t) \\
&= -\frac{2\varepsilon}{\delta}\, \ln n'(t).
\end{aligned}$$

The second line is the mean value theorem. The fifth and sixth lines follow from a) with $\varepsilon = (1 + C_1)/2 - C_2$.

c) We have

$$\begin{aligned}
|(\ln n')'(t) - (\ln n')'(r)| &= |(t-r)\, (\ln n')''(r \pm \delta_1 r)| \\
&\le \delta\, r\, B_1\, |(\ln n')''(r \pm \delta_2 r)| \\
&= \delta^{1/2} B_1\, |(\ln n')'(r \pm \delta^{1/2} r) - (\ln n')'(r)| \\
&\le \delta^{1/2} B_1^2\, |(\ln n')'(r)|.
\end{aligned}$$

The first and third lines follow from the mean value theorem. The second and fourth lines follow from (1.12).

d) The proof is identical to a), using (1.13) instead of (1.12).

e) Obvious.

f) Condition (1.12) implies that $(\ln n')''(r) < O(r^N)$, for some $N \in \mathbb{N}$. Then

$$\begin{aligned} O(r^N) &> (1-\delta)\, r \max_{s\in[(1-\delta)r,r]} (\ln n')''(s) \qquad \text{(by (1.12))} \\ &\geq \int_{(1-\delta)r}^{r} (\ln n')''(t)\, dt\,. \end{aligned}$$

Evaluating the integral,

$$\begin{aligned} \int_{(1-\delta)r}^{r} (\ln n')''(t)\, dt &= (\ln n')'(r) - (\ln n')'(1-\delta)r) \\ &\geq (\ln n')'(r) - C\,(\ln n')'(r) \\ &= (1-C)\,(\ln n')'(r)\,. \end{aligned}$$

The second line follows from integrating (1.12) for $C = B_1/(1+B_1)$. The proof is complete since $1 - C > 0$.

g) We have

$$\begin{aligned} \int_0^{e^2 r} \varphi(t)\, dt &< C \int_0^{e^2 r} \frac{\ln t}{t+r}\, dt \qquad \text{(from part f))} \\ &\leq C_1\, r\,. \end{aligned}$$

h) For $t \geq 2r$,

$$|\varphi(t)| \leq \left|\frac{C \ln r}{r}\right| \leq C\,.$$

i) We have

$$\int_{r-\delta r}^{r+\delta r} |\varphi(t) - \varphi(r)|\, dt \leq \delta\, C\, r\, \varphi(r) \leq \delta\, C\, \ln p(r)\,.$$

j) We have

$$\begin{aligned} n'(r) &\geq C\, r^{\nu_0(r)} && \text{(by (1.1))} \\ &> C\, r^N && \text{(for any } N \in \mathbb{N}) \\ &\geq (-\psi''(r))^{1/(2\eta)} && \text{(for some } \eta > 0 \text{ and part f))} \\ &\geq \left(\frac{(-\psi''(r))^{1/2}}{\ln p(r)}\right)^{1/\eta}. \end{aligned}$$

The desired conclusion follows upon rearranging.

Note that (1.10) implies that both $(\ln n')'(r)$ and $(\ln n')''(r)$ are non-negative. By (1.5),

$$\ln|E(z)| \le \sum_{n=1}^{\infty} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n}(r+|a_n|)}. \tag{1.14}$$

We write (1.14) as two sums

$$\sum_{n=1}^{n_0-1} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n}(r+|a_n|)} + \sum_{n_0-1}^{\infty} \frac{A_{p_n} r^{p_n+1}}{|a_n|^{p_n}(r+|a_n|)},$$

where n_0 is the first integer such that $|a_{n_0}| \ge r_0$. The first sum is finite so we may chose $\varphi_n \equiv 1$. Then, for $r > 2|a_{n_0}|$, the sum is bounded above by

$$C \sum_{n=1}^{n_0-1} \left|\frac{r}{a_n}\right| \le C \sum_{n=1}^{\infty} \left|\frac{r}{a_n}\right|^{p_n+1}$$

The second sum can be written as a Stieltjes integral

$$\int_{|a_{n_0}|}^{+\infty} \frac{A_{p(t)}\, r^{p(t)+1}\, dn_f(t)}{t^{p(t)}(t+r)}.$$

This is bounded above by

$$\int_{|a_{n_0}|}^{+\infty} \frac{A_{p(t)}\, r^{p(t)+1}\, dn(t)}{t^{p(t)}(t+r)} \tag{1.15}$$

by (1.9). We rewrite (1.15) in the form

$$r \int_{|a_{n_0}|}^{+\infty} \frac{A_{p(t)}\, e^{(\ln r - \ln t)p(t) + \ln n'(t)}}{t+r}\, dt \tag{1.16}$$

and we ned to minimize this integral. Treat the exponent as a function of two variables, $p(t)$ and t, and let

$$\psi_0(p(t), t) = (\ln r - \ln t)\, p(t) + \ln n'(t).$$

Taking partials,

$$\frac{\partial\psi_0}{\partial p(t)} = \ln r - \ln t\,, \qquad \frac{\partial\psi_0}{\partial t} = -\frac{1}{t}\,p(t) + \frac{n''(t)}{n'(t)}\,.$$

The partial derivatives simultaneously vanish at

$$t = r\,, \qquad p(t) = t\,\frac{n''(t)}{n'(t)}\,.$$

By the second derivative test for partials, we find that this critical point is a saddle point. Accordingly, we use a generalization of the Laplace method to evaluate the integral (1.16). (See [11, p. 27] for a detailed proof of the Laplace method).

We need to evaluate

$$\int_{|a_{n_0}|}^{+\infty} \varphi(t)\, e^{\psi(t)}\, dt = e^{\psi(r)} \int_{|a_{n_0}|}^{+\infty} \varphi(t)\, e^{\psi(t)-\psi(r)}\, dt = e^{\psi(r)} I\,.$$

Then

$$\begin{aligned} I &= \Big(\int_{|a_{n_0}|}^{r-\delta r} + \int_{r-\delta r}^{r+\delta r} + \int_{r+\delta r}^{e^2 r} + \int_{e^2 r}^{+\infty} \Big)\, \varphi(t)\, e^{\psi(t)-\psi(r)}\, dt \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where $0 < \delta < 1/2$.

For I_1, by Hölder's inequality,

$$\begin{aligned} I_1 &= \int_{|a_{n_0}|}^{r-\delta r} \varphi(t)\, e^{\psi(t)-\psi(r)}\, dt \\ &\le \|\varphi(t)\|_1\, \|e^{\psi(t)-\psi(r)}\|_\infty \\ &\le C\, r\, (e^{\psi(r-\delta r)-\psi(r)})\,, \qquad \text{(by Lemma 1.2.g))}\,. \end{aligned}$$

Now

$$\begin{aligned} \psi(r-\delta r) - \psi(r) &= (\ln r - \ln(r-\delta r))\,(r-\delta r)\,\frac{n''(r-\delta r)}{n'(r-\delta r)} \\ &\quad + \ln n'(r-\delta r) - \ln n'(r) \\ &\le \ln\Big(\frac{1}{1-\delta}\Big)\,(1-\delta)\, r\, \frac{n''(\xi)}{n'(\xi)} \end{aligned}$$

$$\begin{aligned}&+\ln n'(r-\delta r)-\ln n'(r)\\&=(1-\delta)\ln\Big(\frac{1}{1-\delta}\Big)\,r\,\frac{\ln n'(r)-\ln n'(r-\delta r)}{\delta r}\\&+\ln n'(r-\delta r)-\ln n(r)\,.\end{aligned}$$

The ξ in the fourth line is obtained from the mean value theorem, and using the fact that $(\ln n')'(r)$ is increasing. It follows that

$$\begin{aligned}\psi(r-\delta r)-\psi(r)=&\Big(\frac{1-\delta}{\delta}\ln\Big(\frac{1}{1-\delta}\Big)-1\Big)\\&\cdot(\ln n'(r)-\ln n'(r-\delta r))\,.\end{aligned}$$

Hence

$$\begin{aligned}\psi(r-\delta r)-\psi(r)&\le-\eta_1\,(\ln n'(r)-\ln n'(r-\delta r))\\&\le-\eta_1\,(1-C)\,\ln n'(r)\,.\end{aligned}$$

In the last two lines, η_1 and C are constants such that $\eta_1>0$ and $0<C<1$. The last line follows from Lemma 1.2.a) and e).

The above argument used the fact that

$$\frac{1-\delta}{\delta}\ln\Big(\frac{1}{1-\delta}\Big)<1\,.$$

This follows directly from the inequality $\ln x<x-1$ for $x>0$, $x\neq 1$. Thus, by Lemma 1.2.j), we have

$$I_1=O\Big(\frac{\ln p(r)}{(-\psi''(r))^{1/2}}\Big)\,.$$

In order to make our calculations more precise, we will further split I_2.

$$\begin{aligned}I_2&=\int_{r-\delta r}^{r+\delta r}\varphi(t)\,e^{\psi(t)-\psi(r)}\,dt\\&=\varphi(r)\int_{r-\delta r}^{r+\delta r}e^{\psi(t)-\psi(r)}\,dt+\int_{r-\delta r}^{r+\delta r}(\varphi(t)-\varphi(r))\,e^{\psi(t)-\psi(r)}\,dt\\&=I_2'+I_2''\,.\end{aligned}$$

From Taylor's formula with remainder, for $|t-r| \le \delta\, r$,

$$\psi(t)-\psi(r)=\psi'(t')\,\frac{(t-r)^2}{2}\,,$$

where $t'=t'(t)$, and $|t'-r|\le \delta\, r$. Using c) and d) in Lemma 1.2 above, we can make the following calculations.

$$\begin{aligned}
|\psi'(t)-\psi'(r)| &= \Big|(\ln r-\ln t)\,p''(t)-\frac{1}{t}\,(\ln n')'(t)-(\ln n')''(t)\\
&\qquad +\frac{1}{r}\,(\ln n')'(r)+(\ln n')''(r)\Big|\\
&= \big|2\,(\ln r-\ln t)\,(\ln n')''(t)+(\ln r-\ln t)\,t\,(\ln n')'''(t)\\
&\qquad -\frac{1}{t}\,(\ln n')'(t)-(\ln n')''(t)\\
&\qquad +\frac{1}{r}\,(\ln n')'(r)-(\ln n')''(r)\big|\,.
\end{aligned}$$

This can be estimated by

$$\begin{aligned}
|\psi'(t)-\psi'(r)| &\le \Big|2\,\ln\Big(\frac{1}{1+\delta}\Big)\,(\ln n')''(t)\Big|\\
&\qquad +\Big|\ln\Big(\frac{1}{1+\delta}\Big)\,t\,(\ln n')'''(t)\Big|\\
&\qquad +|(\ln n')''(t)-(\ln n')''(r)|\\
&\qquad +\Big|\frac{1}{r}\,((\ln n')'(t)-(\ln n')'(r))\Big|\\
&\le \delta\, C\,(\ln n')''(r)+\varepsilon_3\,(\ln n')''(r)\\
&\qquad +\varepsilon_1\,\frac{1}{r}\,(\ln n')'(r)+\varepsilon_2\,(\ln n')''(r)\\
&\le \varepsilon\,\Big|\frac{1}{r}\,(\ln n')'(r)+(\ln n')''(r)\Big|\\
&= \varepsilon\,|\psi''(r)|\,.
\end{aligned} \tag{1.17}$$

In (1.17), ε_3 was obtained using the same method of proof in Lemma 1.2.c). Also, in the first term, the fact that

$$\Big|\ln\frac{1}{1+\delta}\Big| \approx \frac{\delta}{1+\delta}$$

was used. For I_2', let $\varepsilon > 0$ be arbitrary. By the argument above, we may choose δ such that

$$(1.18) \qquad |\psi''(t) - \psi''(r)| < \varepsilon\,|\psi''(r)|, \qquad \text{for } |t-r| < \delta\, r.$$

So by (1.18),

$$\varphi(r)\int_r^{r+\delta r} e^{(1+\varepsilon)\,\psi''(r)\,(t-r)^2/2}\,dt \le I_2' \le \varphi(r)\int_r^{r+\delta r} e^{(1-\varepsilon)\,\psi''(r)\,(t-r)^2/2}\,dt.$$

Using the fact that $\int_0^{+\infty} e^{-u^2}\,du = \sqrt{\pi}/2$, we conclude that

$$I_2' \sim \frac{\varphi(r)}{-\psi''(r)^{1/2}}.$$

We will now consider I_2''.

$$\begin{aligned} |I_2''| &\le \sup_{|t-r|\le\delta r} |\varphi(t)-\varphi(r)| \int_{r-\delta r}^{r+\delta r} e^{\psi(t)-\psi(r)}\,dt \\ &\le C\,\ln p(r)\int_{r-\delta r}^{r+\delta r} e^{(1-\varepsilon)\psi''(r)\,(t-r)^2/2}\,dt, \end{aligned}$$

by Lemma 1.2.i), the mean value theorem, and (1.18).

Making the substitution

$$u = \left(\frac{-\psi''(r)\,(1-\varepsilon)}{2}\right)^{1/2}(t-r),$$

we obtain

$$|I_2''| \le \frac{C\,\ln p(r)}{(-\psi''(r))^{1/2}(1-\varepsilon)^{1/2}}\int_0^{+\infty} e^{-u^2}\,du.$$

Since the integral converges, we have

$$|I_2''| = O\left(\frac{\ln p(r)}{(-\psi''(r))^{1/2}}\right).$$

The estimate of I_3 is similar to I_1, with ψ decreasing on $[r, +\infty)$, so

$$I_3 = O\left(\frac{\ln p(r)}{(-\psi''(r))^{1/2}}\right).$$

For I_4, by Hölder's inequality,

$$e^{\psi(r)} I_4 \leq \int_{e^2 r}^{+\infty} \varphi(t)\, e^{\psi(t)}\, dt \leq \|\varphi(t)\|_\infty \int_{e^2 r}^{+\infty} e^{-\eta \ln n'(t)}\, dt,$$

for some $\eta > 0$, by Lemma 1.2.b). Thus $e^{\psi(r)} I_4$ converges by Lemma 1.2.h) and (1.1) above.

Putting everything together, we have

$$\ln |E(z)| \leq C \frac{\ln p(r)}{(-\psi''(r))^{1/2}}\, n'(r).$$

Corollary 1.1. *The conditions of Theorem* 1.1 *imply that* $\ln n'(r) = O(r^N)$ *for some* $N \in \mathbb{N}$. *Hence*

$$n'(r) \leq e^{r^N}, \qquad \textit{for some } N \in \mathbb{N}.$$

PROOF.

$$\begin{aligned}
O(r^N) &= \frac{1}{2} r\, (\ln n')'(r) && \text{(by Lemma 1-2 f))}\\
&\geq \int_{r/2}^{r} (\ln n')'(t)\, dt && \text{(since } (\ln n')'(t) \text{ is non-decreasing)}\\
&= \ln n'(r) - \ln n'\left(\frac{r}{2}\right)\\
&\geq \ln n'(r) - C \ln n'(r) && \text{(for some } 0 < C < 1,\\
& && \text{using Lemma 1.2.a))}\\
&= (1 - C) \ln n'(r).
\end{aligned}$$

The proof is complete since $1 - C > 0$.

EXAMPLE 1.1. Let $f_0(z)$ be the function with zeros at $a_n = \ln n$ for each $n \in \mathbb{N}$. Then $n_{f_0}(r) = e^r$. So

$$\ln n'_{f_0}(r) = r, \qquad (\ln n'_{f_0})'(r) = 1, \qquad (\ln n'_{f_0})''(r) = 0.$$

It is clear that the conditions of Theorem 1.1 are satisfied. Thus

$$\ln|f_0(z)| \leq \frac{C\,\ln r\,e^r}{r^{-1/2}} = C\,r^{1/2}\ln r\,e^r\,.$$

Note that the only estimation of any kind occurred in the use of Lemma 1.A. The integral in Theorem 1.1 was evaluated fairly explicitly (*i.e.* up to the multiplicative constant term). Thus, the growth obtained should be best possible.

EXAMPLE 1.2. Theorem 1.1 is independent of the argument of the zeros. Let $g_0(z)$ be the function with zeros at $a_n = (\ln n)^{1/\alpha}\,e^{i\theta_n}$ with $\alpha > 1$, $0 \leq \theta_n < 2\pi$ and $n \in \mathbb{N}$. Then $n_{g_0}(r) = e^{r^\alpha}$. It follows that

$$\begin{aligned}\ln n'_{g_0}(r) &= \ln(\alpha\,r^{\alpha-1}) + r^\alpha\,,\\ (\ln n'_{g_0})'(r) &= \frac{\alpha-1}{r} + \alpha\,r^{\alpha-1}\,,\\ (\ln n'_{g_0})''(r) &= \frac{1-\alpha}{r^2} + \alpha(\alpha-1)\,r^{\alpha-2}\,.\end{aligned}$$

Notice that for all $\alpha > 1$, $(\ln n'_{g_0})'(r)$ is eventually positive, so $\ln n'_{g_0}(r)$ is eventually convex. The rest of the conditions of Theorem 1.1 hold, so

$$\begin{aligned}\ln|g_0(z)| &\leq \frac{C\,\ln(r\ln(\alpha\,r^{\alpha-1}) + r^{\alpha+1})\,\alpha\,r^{\alpha-1}\,e^{r^\alpha}}{\left(\dfrac{\alpha-1}{r^2} + \alpha\,r^{\alpha-2} + \dfrac{1-\alpha}{r^2} + \alpha(\alpha-1)\,r^{\alpha-2}\right)^{1/2}}\\ &\leq C\,r^{\alpha/2}\,e^{r^\alpha}\ln r\,.\end{aligned}$$

Let $\rho(z)$ be a radial non-negative subharmonic function. Then we make the following definition.

Definition. *A_ρ is the space of all entire functions f satisfying the growth condition*

$$|f(z)| \leq A\,e^{B\rho(z)}\,, \qquad A = A_f,\ B = B_f > 0\,.$$

In order to further develop the interpolation theory (see [1]), we now turn to the question of minimum modulus. Again there is a complication. We can no longer obtain the lower bound

$$|f(z)| \geq \varepsilon\,e^{-A\rho(r)}\,, \qquad \text{for } |z| = r\,,$$

outside some family of circles, as was done for finite order functions. The reason is simple. This was obtained in the case of finite order functions by [4, p. 21] and using the fact that $\rho(2r) = O(\rho(r))$. This is no longer true for infinite order functions (see Example 1.1).

To handle this, and in order to provide the necessary Fréchet space with which to study the interpolation theory, we make the following definition.

Definition. *Define $h(r)$ to be some non-increasing function which satisfies the condition*

$$\rho(r + h(r)) = O(\rho(r)). \tag{1.19}$$

We will also assume that $\lim_{r\to\infty} h(r)/r = 0$. This condition is not a restriction for our purposes (*i.e.* infinite order functions) since $h(r) = C$ for some $C > 0$, implies $f(z)$ is a finite order function and this case has been dealt with already (see [1], [7] and [8]). As an example,

$$\rho(r) = \mu(r)\, e^{r^\beta} \quad \text{implies} \quad h(r) = \frac{1}{r^{\beta-1}}$$

and assume that $\mu(r)$ is non-decreasing, $\mu(2r) = O(\mu(r)$, and $\mu(r) = O(r^N)$ for some $N \in \mathbb{N}$. This is easily seen. First, it is clear that

$$\mu\Big(r + \frac{1}{r^{\beta-1}}\Big) = O(\mu(r))$$

because of the upper bound on $\mu(r)$. Since $(1+x)^\beta = 1 + O(x)$, as $x \to 0$, we have that

$$\Big(r + \frac{1}{r^{\beta-1}}\Big)^\beta = r^\beta\Big(1 + \frac{1}{r^\beta}\Big)^\beta = r^\beta\Big(1 + O\Big(\frac{1}{r^\beta}\Big)\Big) = r^\beta + O(1).$$

Thus

$$e^{(r+r^{1-\beta})^\beta} \le e^{r^\beta + C} = O(e^{r^\beta}).$$

This example leads to

Corollary 1.2. *Assumming the hypotheses of Theorem 1.1,*

$$h(r) \ge \frac{1}{r^{\beta-1}}, \qquad \text{for some } \beta > 0. \tag{1.20}$$

PROOF. Follows from Corollary 1.1 and the above calculation.

Let $\tilde{\rho}(r) = r\,\rho(r)/h(r)$. Clearly we also have

$$\tilde{\rho}(r + h(r)) = O(\tilde{\rho}(r)) \tag{1.21}$$

by (1.19) and (1.20) since $r/h(r)$ is non-decreasing and $1 \le r/h(r) \le r^{\beta}$.

Further analogs of results in [1] require $f^{m_k}(a_k)/m_k! \le A\,e^{B\rho(r_k)}$. Since it is easily possible that $h(r) < 1$, Cauchy's formula provides the following bound on the multiplicities m_k. We have

$$\left|\frac{f^{(m_k)}(a_k)}{m_k!}\right| = \left|\frac{1}{2\pi i}\int\limits_{|z-z_k|=h(r_k)} \frac{f(z)\,dz}{(z-a_k)^{m_k+1}}\right| \le \frac{A\,e^{B\rho(r_k)}}{h(r_k)^{m_k}}\,.$$

If $h(r) < 1$, we must have

$$\left(\frac{1}{h(r_k)}\right)^{m_k} \le A\,e^{B\rho(r_k)}$$

or

$$m_k \ln\frac{1}{h(r_k)} \le C + D\,\rho(r_k)\,,$$

which implies

$$m_k \le \frac{C + D\,\rho(r_k)}{-\ln h(r_k)}\,. \tag{1.22}$$

We refer to (1.22) as the *automatic bound.*

We make the following definition of a slowly decreasing function in A_ρ.

Definition. *A function $f \in A_\rho$ is called slowly decreasing if the following two conditions hold:*

i) *There exist $\varepsilon > 0$, $A > 0$ such that each connected component S_α of the set*

$$S(f;\varepsilon,A) = \{z:\ |f(z)| < \varepsilon\,e^{-A\tilde{\rho}(z)}\}$$

is relatively compact.

ii) *There exists a constant* $B > 0$ *independent of* α *such that*

$$\tilde{\rho}(\zeta) \le B\,\tilde{\rho}(z) + B\,, \qquad \text{for any } z, \zeta \in S_\alpha, \text{ any } \alpha\,.$$

Before we proceed, we need a lemma which gives an upper bound on $n(r)$. This is a modification of the proof given in [4, p. 15] to the case of infinite order functions. Recall $\tilde{\rho}(r) = r\,\rho(r)/h(r)$.

Lemma 1.3. *Let* $f \in A_\rho$ *of infinite order. Assume*

a) $|f(0)| \ge 1$,

b) $\rho(r + h(r)) = O(\rho(r))$,

c) $\displaystyle\lim_{r\to\infty} \frac{h(r)}{r} = 0$.

Then

$$n_f(r) = O(\tilde{\rho}(r))\,.$$

PROOF.

$$\begin{aligned}
C\,\rho(r + h(r)) &\ge \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r + h(r))\,e^{i\theta})|\,d\theta \\
&\ge \int_r^{r+h(r)} \frac{n(t)}{t}\,dt && \text{(by Jensen's inequality)} \\
&\ge n(r)\int_r^{r+h(r)} \frac{dt}{t} && \text{(since } n(t) \text{ is increasing)} \\
&= n(r)\,(\ln(r + h(r)) - \ln r) \\
&= n(r)\,\ln\left(1 + \frac{h(r)}{r}\right) \\
&\sim \frac{h(r)}{r}\,n(r) && \text{(by condition c))}
\end{aligned}$$

which implies the statement of the lemma.

Notice that since $\tilde{\rho}(r+h(r)) = O(\tilde{\rho}(r))$, we also have $n_f(r+h(r)) = O(\tilde{\rho}(r))$.

The following theorem also makes use of a theorem of Momm's [6] for lower bounds of an entire function.

Theorem 2.A. *Let f be entire with $f(0) = 1$. Then, for each $0 < r < r + h(r)$, there is a Jordan curve Γ in $r < |z| < r + h(r)$ around the origin such that*

$$(1.23)\quad \ln|f(z)| \geq \frac{-C}{h(r)}\left(\int_0^{r+h(r)} \sqrt{\frac{\ln M_f(t)}{r+h(r)-t}}\, dt\right)^2, \quad \text{for } z \in \Gamma.$$

Theorem 1.2. *Let $f \in A_\rho$ satisfy the conditions of Theorem 1.1 and assume $f(0) = 1$. Then f is slowly decreasing and (f), the ideal generated by f in $A_{\tilde{\rho}}$ is closed in the space $A_{\tilde{\rho}}$ (i.e. g/f entire implies that $g/f \in A_\rho$).*

PROOF. To prove slowly decreasing, it suffices to show that in every annulus $r \leq |z| \leq r + h(r)$, there exists a Jordan curve around the origin in that annulus such that on that curve, $f(z)$ attains the appropriate minimum modulus.

Let $\sigma(r) = \ln \rho(r)$. Consider the integral in (1.23). With our notation, it is estimated from above by

$$\int_0^{r+h(r)} \frac{e^{\sigma(t)/2}}{\sqrt{r+h(r)-t}}\, dt.$$

Upon integrating by parts, we have

$$\begin{aligned}
\int_0^{r+h(r)} \frac{e^{\sigma(t)/2}}{\sqrt{r+h(r)-t}}\, dt &= -2\sqrt{r+h(r)-t}\, e^{\sigma(t)/2}\Big|_0^{r+h(r)} \\
&\quad + \int_0^{r+h(r)} \sqrt{r+h(r)-t}\, \sigma'(t)\, e^{\sigma(t)/2}\, dt \\
&\leq \sqrt{r+h(r)} \int_0^{r+h(r)} \sigma'(t)\, e^{\sigma(t)/2}\, dt \\
&= 2\sqrt{r+h(r)}\, e^{\sigma(t)/2}\Big|_0^{r+h(r)} \\
&\leq 2\sqrt{r+h(r)}\, e^{\sigma(r+h(r))/2} \\
&\leq 2\sqrt{(r+h(r))\,\rho(r+h(r))} \\
&\leq B\sqrt{r\,\rho(r)}.
\end{aligned}$$

Plugging into (1.23), we obtain

$$\ln|f(z)| \geq \frac{-Cr}{h(r)}\,\rho(r).$$

This proves that f is slowly decreasing in $A_{\tilde{\rho}}$. By Proposition 3 of [1], (f) is closed in $A_{\tilde{\rho}}$.

REMARK. Notice that both the minimum modulus theorem for analytic functions with no zeros and Lemma 1.3 was used in the proof. Independently, a multiplicative factor of $r/h(r)$ appeared in both cases. This leads us to believe that $A_{\tilde{\rho}}$ is the correct space in which to study interpolation theory for infinite order functions.

EXAMPLE 1.3. Consider the $f_0(z)$ in Example 1.1. By Theorem 1.2, f_0 is slowly decreasing and (f_0) is closed in the space

$$\{f(z):\ \ln|f(z)| \leq C\, r^{3/2} \ln r\, e^r\}.$$

EXAMPLE 1.4. Consider the $g_0(z)$ in Example 1.2. By Theorem 1.2, g_0 is slowly decreasing and (g_0) is closed in the space

$$\{g(z):\ \ln|g(z)| \leq C\, r^{3\alpha/2} e^{r^\alpha} \ln r\}.$$

As a consequence, in view of [1, Proposition 3], if f is slowly decreasing in A_ρ, it is only invertible in the space $A_{\tilde{\rho}}$.

More generally, any theorem in [1] that mentions

$$|f(z)| \geq \varepsilon\, e^{-A\rho(r)}, \qquad |z| = r,$$

must be changed in the infinite order case to

$$|f(z)| \geq \varepsilon\, e^{-A\tilde{\rho}(r)}$$

to reflect the above facts.

Asociated to a multiplicity variety V in $\mathbb{C}$, there is a unique closed ideal in $A(\mathbb{C})$,

$$I = I(V) = \{F \in A(\mathbb{C}) : F \text{ vanishes at } z_k \text{ with multiplicity} \geq m_k\}.$$

Two functions g and h in $A(\mathbb{C})$ can be identified modulo I if and only if

$$\frac{g^{(l)}(z_k)}{l!} = \frac{h^{(l)}(z_k)}{l!} = a_{k,l}, \qquad 0 \leq l \leq m_k - 1,\ k = 1, 2, \ldots \tag{1.24}$$

Hence Theorem 1.A above states that the quotient space $A(\mathbb{C})/I$ can be identified to the space of al sequences $\{a_{k,l}\}$. We will describe them as *analytic functions on* V and denote that space by $A(V)$. The map $\varrho_V = \varrho$,

$$\varrho : A(\mathbb{C}) \to A(V),$$

which takes $g \in A(\mathbb{C})$ to $\{a_{k,l}\} \in A(V)$ via (1.24) above, is called the *restriction map.*

Before we proceed, we need some definitions. In what follows, let $h_1(r) = \min\{h(r), 1\}$.

Definition. *Let* $V = \{(z_k, m_k)\}$ *be a multiplicity variety. Then* $A_\rho(V)$ *is the space of all functions* $\{a_{k,l}\} \in A(V)$ *such that for some constants* $A,\ B > 0$

$$(1.25) \qquad \sum_{l=0}^{m_k-1} |a_{k,l}|\, h_1(r_k)^l \le A\, e^{B\rho(r_k)}, \qquad k \ge 1,\ 0 \le l \le m_k .$$

Note that when $m_k = O(e^{B\rho(r_k)})$, then (1.25) is equivalent to

$$(1.26) \qquad |a_{k,l}|\, h_1(r_k)^l \le A_1\, e^{B_1\rho(r_k)}, \qquad k \ge 1,\ 0 \le l \le m_k .$$

Notice that because of the automatic bound, we may take (1.26) as the definition for $A_\rho(V)$ whenever $h(r) < 1$.

We now define $\rho(r; s) = \rho(r + h_1(r)s)$.

Definition. *The space* $A_{\rho,\infty}(V)$ *consists of those* $\{a_{k,l}\} \in A(V)$ *such that for some* $A,\ B > 0$ *and all* $s \ge 1$

$$(1.27) \qquad \sum_{l=0}^{m_k-1} |a_{k,l}|\, (h_1(r_k)s)^l \le A\, e^{B\rho(r_k; s)}, \qquad k \ge 1,\ 0 \le l \le m_k .$$

Note that because of the definition of $h_1(r)$, we can be more precise about what is meant by $\rho(r_k; s)$. By repeating (1.19) $[s]+1$ times, where $[\cdot]$ denotes greatest integer, we obtain

$$\rho(r_k; s) \le A\, e^{Bs}(\rho(r_k)),$$

for some constants $A,\ B > 0$.

Definition. *If ϱ maps A_ρ onto $A_\rho(V)$, we will say that V is an interpolating variety for A_ρ. If ϱ maps A_ρ onto $A_{\rho,\infty}$, then we will say that V is a weak interpolating variety for A_ρ.*

Squires, in his Thesis, [7, Theorem 2] and [8, Theorem 3], provides a purely geometric condition for a finite order function to be interpolating. That is, whether or not $f \in A_\rho$ interpolated depended only upon the geometry of $V(f)$. That theorem has an analog in the infinite order case, and we present that here.

Theorem 1.3. *Let $f \in A_\rho$, where $n_f(r)$ satisfies the conditions of Theorem 1.1 and assume $V = V(f)$. Then V is an interpolating variety in $A_{\tilde\rho}$ if and only if there exists constants C, $D > 0$ such that*

$$m_k \leq \frac{C\,\tilde\rho(|a_k|) + D}{|\ln h(|a_k|)|}, \tag{1.28}$$

$$\int_0^{h(|a_k|)} \frac{n(a_k, t, V)}{t}\,dt \leq C\,\tilde\rho(|a_k|) + D. \tag{1.29}$$

PROOF. By [1, Theorem 4], if V is an interpolating variety, then there exist constants $\varepsilon, C > 0$ such that

$$\frac{|f^{(m_k)}(a_k)|}{m_k!} \geq \varepsilon e^{-C\,\tilde\rho(|a_k|)}. \tag{1.30}$$

(Recall f is only invertible in the space $A_{\tilde\rho}$). Now (1.28) follows from Cauchy's formula. (If $h(r) < 1$, this is precisely the automatic bound.) We first show that (1.30) implies

$$\prod_{\substack{|a_n - a_k| < h(|a_k|)\\ a_n \neq a_k}} \frac{|a_n - a_k|}{|a_k|} \geq \varepsilon\, e^{-C\tilde\rho(|a_k|)}. \tag{1.31}$$

Let $f(z) = (z - a_k)^{m_k} g(z)$ and write $1 = |g(a_k)|/|g(a_k)|$.

For the numerator, note $f^{(m_k)}(a_k) = m_k!\, g(a_k)$. For the denominator, write the canonical factorization of $g(a_k)$. We have

$$\prod_{\substack{|a_n - a_k| < h(|a_k|)\\ a_n \neq a_k}} \frac{|a_n - a_k|}{a_k} = \frac{1}{P_1 P_2 P_3} \left| \frac{f^{(m_k)}(a_k)}{m_k!} \right|$$

with

$$P_1 = \prod_{|a_n - a_k| \geq h(|a_k|)} \left| \left(1 - \frac{a_k}{a_n}\right) E_{p_n}\left(\frac{a_k}{a_n}\right) \right|,$$

$$P_2 = \prod_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \left| E_{p_n}\left(\frac{a_k}{a_n}\right) \right|,$$

$$P_3 = \prod_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \left| \frac{a_k}{a_n} \right|.$$

Here, E_{p_n} represents the exponential part of the n^{th} term in the factorization of $f(z)$ from Theorem 1.1.

We need only obtain appropriate upper bounds for the three products in the denominator. Note that $P_1 = f_k(a_k)$, where

$$f_k(z) = \prod_{|a_n - a_k| \leq h(|a_k|)} \left(1 - \frac{a_k}{a_n}\right) E_{p_n}\left(\frac{a_n}{a_k}\right).$$

We show that $f(z)$ has the same growth as $f_k(z)$. First, it is clear that $dn_{f_k}(t) = dn_f(t)$. Second, letting $\{b_j\}_{j=1}^{\infty} = Z(f_k)$, we have $|b_j| \geq |a_j|$. Then

$$\sum_{j=1}^{\infty} \left| \frac{z}{b_j} \right|^{p_k} \leq \sum_{j=1}^{\infty} \left| \frac{z}{a_j} \right|^{p_k},$$

so by the argument in Theorem 1.1, $|f_k(a_k)| \leq A\, e^{B\rho(|a_k|)}$.

For the second product, we have

$$\begin{aligned}
\ln P_2 &\leq \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \sum_{j=1}^{p_n} \frac{1}{j} \left| \frac{a_k}{a_n} \right|^j \\
&\leq \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \sum_{j=1}^{p_n} \frac{1}{j} \left(\frac{|a_k| + h(|a_k|)}{|a_n|} \right)^j.
\end{aligned}$$

It follows that

$$\ln P_2 \leq \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} C \ln p(|a_n|) \left(\frac{|a_k| + h(|a_k|)}{|a_n|} \right)^{p(|a_n|)}$$

$$
\begin{aligned}
&\le \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \ne a_k}} C \ln p(|a_n|) \Big(\frac{|a_k| + h(|a_k|)}{|a_n|}\Big)^{p(|a_n|)+1} \\
&\le C\,\rho(|a_k| + h(|a_k|)) \\
&\le C\rho(|a_k|)\,,
\end{aligned}
$$

since $|a_n| < |a_k| + h(|a_k|)$ and using the arguments in Theorem 1.1 and the assumptions on ρ.

For the last product, we have

$$
\begin{aligned}
\prod_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \ne a_k}} \frac{|a_n|}{a_k} &\le \Big(\frac{|a_k|}{|a_k| - h(|a_k|)}\Big)^{n_f(|a_k| + h(|a_k|))} \\
&\le \Big(\frac{|a_k|}{|a_k| - h(|a_k|)}\Big)^{C\,|a_k|\rho(|a_k|)/h(|a_k|)} \\
&\le \Big(1 + \frac{h(|a_k|)}{|a_k| - h(|a_k|)}\Big)^{C\,|a_k|\rho(|a_k|)/h(|a_k|)} \\
&\le \Bigg(\Bigg(1 + \frac{h(|a_k|)}{\frac{|a_k|}{h(|a_k|)} - 1}\Bigg)^{|a_k|/h(|a_k|)}\Bigg)^{C\rho(|a_k|)} \\
&\le e^{2\,C\,\rho(|a_k|)}\,.
\end{aligned}
$$

We have just shown (1.31). Let $n(a_k, t, V)$ be the number of points in V in a disk of radius t centered at a_k, excluding a_k itself. Inequality (1.31) also implies

$$
\prod_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \ne a_k}} \frac{|a_n - a_k|}{a_k} \ge \varepsilon\, e^{-C\,\tilde{\rho}(|a_k|)}\,.
$$

Taking logarithms,

$$
\sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \ne a_k}} \ln|a_n - a_k| - n(a_k, h(|a_k|), V)\,\ln|a_k| \ge -C\,\tilde{\rho}(|a_k|) - D\,. \tag{1.32}
$$

Upon writing a Stieltjes integral and integrating by parts, we have

$$
\int_0^{h(|a_k|)} \frac{n(a_k, t, V)}{t}\,dt = \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \ne a_k}} \ln \frac{h(|a_k|)}{|a_n - a_k|}
$$

$$
\begin{aligned}
(1.33) \qquad & = n(a_k, h(|a_k|), V) \\
& \quad - \sum_{\substack{|a_n - a_k| < h(|a_k|) \\ a_n \neq a_k}} \ln |a_n - a_k| \,.
\end{aligned}
$$

Now, (1.31) and (1.32) together give

$$
\int_0^{h(|a_k|)} \frac{n(a_k, t, V)}{t} \, dt \leq C \, \tilde{\rho}(|a_k|) + D + n(a_k, h(|a_k|), V) \ln \left(\frac{h(|a_k|)}{|a_k|} \right),
$$

which gives (1.29) for $|a_k| > h(|a_k|)$.

Since f attains the desired minimum modulus on some Jordan curve around the origin in every annulus $r \leq |z| \leq r + h(|a_k|)$, by Jensen's Theorem,

$$
\begin{aligned}
\ln \left| \frac{f^{(m_k)}(a_k)}{m_k!} \right| &= \frac{1}{2\pi} \int_0^{2\pi} \ln |f(a_k + h(|a_k|) \, e^{i\theta})| \, d\theta \\
& \quad - \int_0^{h(|a_k|)} \frac{n(a_k, t, V)}{t} \, dt - m_k \ln h(|a_k|) \\
& \geq -C \, \tilde{\rho}(|a_k|) - D - \int_0^{h(|a_k|)} \frac{n(a_k, t, V)}{t} \, dt \\
& \quad - m_k \ln h(|a_k|) \,.
\end{aligned}
$$

Conditions (1.28) and (1.29) then imply

$$
\ln \left| \frac{f^{(m_k)}(a_k)}{m_k!} \right| \geq -C \, \tilde{\rho}(|a_k|) - D \,.
$$

By [1, Theorem 4], V is an interpolating variety.

It should be aparent that the concept of a slowly decreasing function is very important in interpolation theory. Since the definition of slowly decreasing means that $f(z)$ attains a minimum modulus, that implies that groups of zeros of f are "well" separated, in some sense. The idea now is that convexity of $n(r)$ should already imply that individual zeros of f are well separated, since the growth of $n(r)$ is so regular. That is, we should be able to show that, under certain conditions, if $n(r)$ is convex, then not only is f slowly decreasing, but each component of $S(f; \varepsilon, A)$ contains only one zero of f.

As a starting point, consider the diference quotient

$$\frac{n(r_{k+1}) - n(r_k)}{r_{k+1} - r_k},$$

where $\{r_k\}$ is the increasing sequence of the moduli of the zeros. The numerator is simply the multiplicity of the zero at $z = a_k$. The idea is that if the denominator has a "nice"lower bound, then each distinct zero of the function f should be trapped in its own component of $S(f; \varepsilon, A)$. To take a concrete example, consider the function $f_0(z)$ in Example 1.1. There,

$$r_{k+1} - r_k = \ln(k+1) - \ln k = \ln\left(1 + \frac{1}{k}\right) \sim \frac{1}{k} \sim \frac{1}{n'_{f_0}(r_k)}.$$

This is the kind of condition we would like for Theorem 1.4.

We introduce the following notation for Theorem 1.4:

$$\nu(r) = \frac{\rho(r)}{n'(r)},$$

i.e., the factor multiplied by $n'(r)$ in Theorem 1.1,

$$\tilde{\nu}(r) = \frac{\nu(r)\, r}{h(r)\, \ln(r\, n'(r))},$$

$$\tilde{h}(r) = \min\{h(r), \tilde{\nu}(r)\}.$$

Notice that since $\tilde{h}(r)$ is always bounded above by $h(r)$, (1.25) holds, with $\tilde{h}(r)$ replacing $h(r)$. Also, (1.25) holds with $n'(r)$ replacing $\rho(r)$ since $n'(r)$ is bounded above by e^{r^β} for some $\beta > 0$.

Lemma 1.4. *For $n(r)$ satisfying the conditions of Theorem* 1.1,

$$\tilde{\nu}(r) \geq \frac{C}{r^N},$$

for some constants C, $N > 0$.

PROOF. Since $h(r) \leq r$, we have

$$\begin{aligned}
\tilde{\nu}(r) &= \frac{\nu(r)\, r}{h(r)\, \ln(r\, n'(r))} \\
&\geq \frac{\nu(r)}{\ln(r\, n'(r))} \\
&= \frac{C\, \ln(r\, (\ln n')'(r))}{\left(\frac{1}{r}(\ln n')'(r) + (\ln n')''(r)\right)^{1/2} \ln(r\, n'(r))}.
\end{aligned}$$

For the numerator,

$$\ln(r(\ln n')'(r)) \geq C\,, \tag{1.34}$$

since $\ln n'(r)$ is increasing by (1.10). For the denominator,

$$\left(\frac{1}{r}(\ln n')'(r) + (\ln n')''(r)\right)^{1/2} \leq C\,r^N\,, \tag{1.35}$$

for some $N > 0$ by Lemma 1.2.e); and

$$\ln(r\,n'(r)) \leq C\,r^N\,, \tag{1.36}$$

by Corollary 1.1 for some (possibly different) $N > 0$. Now, (1.34), (1.35) and (1.36) imply

$$\tilde{\nu}(r) \geq \frac{C}{r^N}\,.$$

Theorem 1.4. *Let $f(z) \in A_\rho$, where $n_f(r)$ satisfies the conditions of Theorem 1.1. Assume*

$$m_k \leq r_k^m\,, \qquad \textit{for some uniform } m > 0\,, \tag{1.37}$$

$$\frac{1}{n'(r_{k+1})} \leq r_{k+1} - r_k \leq \frac{r_k^m}{n'(r_k)}\,, \qquad \textit{for all } k\,. \tag{1.38}$$

Here r_k is the modulus of a zero of f and m_k its multiplicity. Then there exists a family of circles C_k, each circle centered at a zero of $f(z)$ such that

$$C_j \cap C_k = \varnothing\,, \qquad \textit{for } j \neq k\,, \tag{1.39}$$

$$\ln|f(z)| \geq -C\,\tilde{\rho}(r) - D\,, \tag{1.40}$$

for $|z| = r$ and some constants C, $D > 0$ for al z outside the set of exceptional circles $\cup_k C_k$.

PROOF. Recall we are assumming that the moduli of distinct zeros of f are separated per the discussion prior to this theorem. We must establish the circles satisfying (1.39). We first show that the definition of $\tilde{h}(r)$ is reasonable, in the sense that it is possible to have several zeros of $f(z)$ in any circle of radius $2\,\tilde{h}(r)$.

From (1.10), we have

$$\frac{1}{n'(r)} \le \frac{1}{r^{\nu_0(r)}}, \tag{1.41}$$

for some $\nu_0(r)$ such that $\lim_{r\to\infty} \nu_0(r) = \infty$. Thus it will be sufficient if we can show

$$\tilde{h}(r) \ge \frac{C}{r^\beta}, \tag{1.42}$$

for some $\beta > 0$ and $C > 0$. And $\tilde{h}(r)$ satisfies (1.42) since $h(r)$ and $\tilde{\nu}(r)$ do, by Corollary 1.2 and Lemma 1.4, respectively.

Now (1.38), (1.41) and (1.42) imply

$$r_{k+1} - r_k \le C\,\tilde{h}(r_k). \tag{1.43}$$

Writing $r_{k+1} = r_k + (r_{k+1} - r_k)$ and using (1.43), (1.38) implies

$$r_{k+1} - r_k \ge \frac{B_1}{n'(r_k)}, \qquad \text{for some } B_1 > 0. \tag{1.44}$$

Then, a circle of radius, for example, $B_1/(3\,n'(r_k))$ around r_k will guarantee that (1.39) holds. Fix $B_0 = B_1/3$ for the rest of the proof.

The proof of (1.40) is a modification of that given in [4, p. 125]. Assume $|z| = r$ and that z does not lie in the set of exceptional circles just found. Let

$$F(z) = \prod_{|z-a_n|\le 2\,\tilde{h}(r)} \left(1 - \frac{z}{a_n}\right). \tag{1.45}$$

Notice that, since $|z - a_n| \le 2\,\tilde{h}(r)$ and $|a_n| \ge r - 2\,\tilde{h}(r)$, we have, for r big enough,

$$\left|\frac{z-a_n}{a_n}\right| \le \frac{2\,\tilde{h}(r)}{r - 2\,\tilde{h}(r)} < 1$$

and therefore

$$\ln|F(z)| < 0. \tag{1.46}$$

Now, for z outside the exceptional circles,

$$\begin{aligned} |F(z)| &= \prod_{|z-a_n|\leq 2\,\tilde{h}(r)} \left|\frac{z-a_n}{a_n}\right| \\ &\geq \prod_{|z-a_n|\leq 2\,\tilde{h}(r)} \left(\frac{B_0}{(r-2\,\tilde{h}(r))\,n'(r)}\right) \quad \text{(by (1.44))} \\ &\geq \left(\frac{C}{r\,n'(r)}\right)^{n(z,V,2\,\tilde{h}(r))} \end{aligned}$$

so

$$\ln|F(z)| \geq -C\,n(z,V,2\,\tilde{h}(r))\,\ln(r\,n'(r)) - D\,. \tag{1.47}$$

We next obtain an upper bound for $n(z,V,2\tilde{h}(r))$. Notice that, since (1.47) was obtained independent of the argument of the zeros, it must remain true even if all the zeros lie on the same diameter of $D(z;2\,\tilde{h}(r))$, the circle of center z and radius $2\,\tilde{h}(r)$. Thus, by (1.39), the total of the diameters of the excluded circles in $D(z;2\tilde{h}(r))$ must be less than the diameter of $D(z;2\,\tilde{h}(r))$. The total diameter of the excluded circles in $D(z;2\,\tilde{h}(r))$ is bounded below by $n(z,V,2\,\tilde{h}(r))$ times the smallest diameter of the excluded circles in $D(z;2\,\tilde{h}(r))$. The smallest diameter is bounded below by

$$\frac{2B_0}{n'(r+2\,\tilde{h}(r))} \geq \frac{C_0}{n'(r)}\,.$$

Since the diameter of $D(z;2\,\tilde{h}(r)) = 4\,\tilde{h}(r)$,

$$\frac{C_0\,n(z,V,2\,\tilde{h}(r))}{n'(r)} \leq C\,\tilde{h}(r)\,.$$

(The arbitrary constant C accounts for the fact that some of the excluded circles may be only partly inside $D(z;2\,\tilde{h}(r))$. Thus, there exists a constant $B > 0$ such that

$$n(z,V,2\,\tilde{h}(r)) \leq B\,\tilde{h}(r)\,n'(r)\,.$$

Now, since $\tilde{h}(r) \leq \tilde{\nu}(r)$, we have

$$n(z,V,2\,\tilde{h}(r)) \leq B\,\tilde{\nu}(r)\,n'(r) = \frac{B\,r\,\nu(r)\,n'(r)}{h(r)\,\ln(r\,n'(r))} = \frac{B\,\tilde{\rho}(r)}{\ln(r\,n'(r))}\,.$$

Plugging into (1.47), we get

$$\ln|F(z)| \geq -C\,\tilde{\rho}(r) - D\,. \tag{1.48}$$

Let

$$F(z;\zeta) = \prod_{|z-a_n|\leq \tilde{h}(r)} \left(1 - \frac{z+\zeta}{a_n}\right).$$

Fixing z and thinking of $F(z;\zeta)$ as a function of the complex variable ζ, it is clear that (1.48) still holds for $|\zeta| \leq \tilde{h}(r)$. Note $F(z;0) = F(z)$.

Let $f(z;\zeta) = f(z+\zeta)$ and write

$$\varphi(z;\zeta) = \frac{f(z;\zeta)}{F(z;\zeta)}, \qquad \text{for } |\zeta| \leq \tilde{h}(r)\,.$$

By Theorem 1.2, there exists a r_0, $0 \leq r_0 \leq \tilde{h}(r)$, such that

$$\ln|f(z+\zeta)| \geq -C\,\tilde{\rho}(r) - D\,, \qquad \text{for every } |\zeta| = r\,.$$

This remains true even if $\tilde{h}(r) = \tilde{\nu}(r)$. Just repeat the proof with $\tilde{\nu}(r)$ replacing $h(r)$, and use

$$n(z, V, \tilde{\nu}(r)) \leq n(z, V, h(r)) \leq C\,\tilde{\rho}(r)\,.$$

Then, using (1.46), we obtain that

$$\ln|\varphi(z;\zeta)| = \ln|f(z;\zeta)| - \ln|F(z;\zeta)| \geq -C\,\tilde{\rho}(r) - D\,. \tag{1.49}$$

Now, $\varphi(z;\zeta)$ has no zeros in $|\zeta| \leq r_0$ and by the minimum modulus principle, it takes its minimum modulus on $|\zeta| = r_0$. In particular,

$$\ln|\varphi(z;0)| = \ln|f(z)| - \ln|F(z)| \geq -C\,\tilde{\rho}(r) - D\,.$$

Finaly, (1.48) and (1.49) together give

$$\ln|f(z)| \geq -C\,\tilde{\rho}(r) - D\,.$$

We close this chapter with propositions regarding the space $A_{\rho,\infty}(V)$.

Proposition 1.1. $A_{\rho,\infty}(V) \subseteq A_\rho(V)$.

PROOF. Let $s = 1$ in (1.27).

Proposition 1.2. *The restriction map* $\varrho : A(\mathbb{C}) \to A(V)$ *maps* A_ρ (*with* m_k *satisfying the automatic bound*) *into* $A_{\rho,\infty}(V)$.

PROOF. From Cauchy's formula, for $0 \le j \le m_k - 1$,

$$\frac{f^{(j)}(z_k)}{j\,!} = \int_{|\zeta - z| = h_1(r_k)s} \frac{f(\zeta)}{(\zeta - z)^{j+1}}\, d\zeta\,.$$

Then, taking the sum,

$$\begin{aligned}\sum_{j=0}^{m_k-1} \frac{|f^{(j)}(z_k)|}{j\,!} s^j &\le m_k \left(\frac{1}{h_1(r_k)}\right)^{m_k} A\, e^{B\,\rho(r_k;s)} \\ &\le \frac{A_1\,\rho(r_k) + B_1}{-\ln r_k} A_2\, e^{B_2\,\rho(r_k)} A\, e^{B\,\rho(r_k;s)} \\ &\le A\, e^{B\,\rho(r_k;s)}\,.\end{aligned}$$

REMARK. (1.27) and the automatic bound give the estimate

$$|a_{k,l}|\, h_1(r_k)^l \le \frac{A\, e^{B\,\rho(r_k;s)}}{s^l}\,.$$

Compare with (1.26). Then $A_{\rho,\infty}(V) = A_\rho(V)$ if and only if

$$m_k = O\left(\inf_{s\ge 1} \frac{1 + B\,\rho(r_k;s) - \rho(r_k)}{\ln s}\right), \tag{1.50}$$

for some constant $B > 0$.

To see this, first assume (1.50). Note that because of Proposition 1.1, we need only show $A_\rho(V) \subseteq A_{\rho,\infty}(V)$. This condition follows directly, taking into account that (1.50) implies $s^{m_k} \le A\, e^{B\,(\rho(r_k;s) - \rho(r_k))}$. Since $A_\rho(V) \subseteq A_{\rho,\infty}(V)$, we must have

$$s^{m_k} A\, e^{B\,\rho(r_k)} \le A_1\, e^{B_1\,\rho(r_k;s)}\,.$$

Solving for m_k gives (1.50).

2. The Spaces $A_{\boldsymbol{\rho}}$.

We will now establish the theory regarding the spaces $A_{\boldsymbol{\rho}}$. The proofs will be omited, as they are basically unchanged from those in [1].

Let $\rho(z)$ be a subharmonic function satisfying the following two conditions:

(2.1.i) $\rho(z) \geq 0$ and $\log(1+|z|^2) = O(\rho(z))$;

(2.1.ii) there exist constants B_0, C_0, $D_0 > 0$ such that $|\zeta - z| \leq 1$ implies $\rho(\zeta) \leq B_0\,\rho(C_0 z) + D_0$.

We now define the spaces $A_{\boldsymbol{\rho}}$. The bold face $\boldsymbol{\rho}$ is to emphasize the constant inside the argument of ρ.

Definition. *Let f be entire. We say that $f \in A_{\boldsymbol{\rho}}$ if*

$$|f(z)| \leq A\, e^{B\,\rho(Cz)}, \quad \textit{for some } A,\ B,\ C > 0\,.$$

Condition (2.1.i) provides a minimum growth for the function f, guaranteeing that all polynomials belong to $A_{\boldsymbol{\rho}}$ for any ρ. We note that the only entire functions that grow slower than a polynomial are the constants, and nothing interesting happens there. Condition (2.1.ii) controls the growth of $\rho(z)$ in small discs. We also show below that (2.1.ii) guarantees that $A_{\boldsymbol{\rho}}$ is closed under differentiation for any ρ. Note that if ρ is radial, then subharmonicity implies that ρ is convex. Furthermore, since (2.1.i) implies that ρ approaches infinity, we can assume that ρ is increasing. If ρ is not radial, we can assume that ρ is eventually increasing on any line.

There are two reasons why we would like to study such spaces.

EXAMPLE 2.1. One reason why we may need to consider the space $A_{\boldsymbol{\rho}}$ instead of the case where we do not allow the constant inside the argument of ρ is that the zeros of the function are to dense, causing the function to grow too fast for the later case. Consider a function with simple zeros at $z = \ln\ln n$ for al $n \in \mathbb{N}$. In this case,

$$n(r) = e^{e^r}\,,$$

which grows faster than the infinite order functions considered earlier.

EXAMPLE 2.2. The other reason that we may need to use the space A_p is that while the zeros themselves may not be dense, the multiplicities of the zeros may be too large. Consider the function with zeros at $z = \ln n$, for $n \in \mathbb{N}$ and "pile up"the multiplicity in the following manner:

1) At $z = 1$, place a zero with multiplicity $[e^e]$ (where $[\,\cdot\,]$ represents the integer part).

2) At $z = e$, place a zero with multiplicity $[e^{e^e}]$.

3) In general, place the zero a_k at $e^{|a_{k-1}|}$ with multiplicity $[e^{e^{|a_k|}}]$.

Here there are very few distinct zeros. However, the multiplicity is so high - $m_k = e^{e^{|a_k|}}$ for every k -that we are forced to deal with the space A_p.

Note that arbitrary derivatives belong to A_p for any ρ (*i.e.* the space A_p is closed under differentiation)

$$\begin{aligned}\frac{|f^{(n)}(z)|}{n!} &= \left|\frac{1}{2\pi i}\int_{|z-\zeta|=1/C}\frac{f(\zeta)\,d\zeta}{(\zeta-z)^{n+1}}\right| \\ &\le A\,e^{B\,\max_{|z-\zeta|=1/C}\rho(C\zeta)}\,C^{n+1} \\ &\le A\,e^{B\,\rho(Cz)}.\end{aligned}$$

Since we are only interested in the case where $\rho(Cz) \neq O(\rho(z))$, most of the time, conditions (2.1.i) and (2.1.ii) will always hold. In fact, if $\rho(z) = \rho(|z|)$, it is easy to see that not only are we dealing with infinite order functions, but (2.1.i) holds for arbitrarily rapidly growing functions.

Our question is to study, for given ρ, the range of $A_p(\mathbb{C})$ under the restriction map ϱ. The addition of the extra constant in the argument of ρ allows for a lot of "room to maneuver". Putting it another way, we are not putting a very precise growth rate on the functions in A_p. However, this does have the advantage of simplifying the calculations to the point that the proofs in this chapter vary only slightly from those in [1]. In Section 2, we will study some clases of infinite order functions in more detail.

We will begin with the A_p version of the Semi-Local Interpolation Theorem.

Definition. *For* $f_1, \ldots, f_m \in A_p$, $\|\mathbf{f}\| = (\sum_{i=1}^m |f_i|^2)^{1/2}$,

$$S(\mathbf{f};\varepsilon,B,C) = \{z \in \mathbb{C} :\ |\mathbf{f}(z)| < \varepsilon\,e^{-B\,\rho(Cz)}\}.$$

We will use a bold $\mathbf{f}$ when we are dealing with a collection of functions $f_1, \ldots, f_m$.

Semi-Local Interpolation Theorem. *Let $\tilde{\lambda}(z)$ be analytic on*

$$S(\mathbf{f}; \varepsilon, B, C)$$

and satisfy

$$|\tilde{\lambda}(z)| \leq A' e^{B' \rho(C' z)}, \qquad \text{for } z \in S(\mathbf{f}; \varepsilon, B, C).$$

Then there exists an entire function $\lambda(z) \in A_{\boldsymbol{\rho}}$, constants ε_1, B_1, $C_1 > 0$ and functions $\alpha_1, \ldots, \alpha_m$ analytic on $S(\mathbf{f}; \varepsilon_1, B_1, C_1)$ such that for all $z \in S(\mathbf{f}; \varepsilon_1, B_1, C_1)$

$$\lambda(z) = \tilde{\lambda}(z) + \sum_{i=1}^{m} \alpha_i(z)\, f_i(z), \quad \text{and} \quad \frac{\lambda^{(l)}(z_k)}{l!} = \frac{\tilde{\lambda}^{(l)}(z_k)}{l!}$$

and $|\alpha_i(z)| \leq A\, e^{B\rho(Cz)}$ for some new constants A, B, and $C > 0$.

Following [1], we now define the space of analytic functions with growth conditions on a multiplicity variety V.

Definition. *Let $V = \{(z_k, m_k)\}$ be a multiplicity variety. Then $A_{\boldsymbol{\rho}}(V)$ is the space of all functions $\{a_{k,l}\} \in A(V)$ such that for some constants A, B, $C > 0$,*

$$\sum_{l=0}^{m_k - 1} |a_{k,l}| \leq A\, e^{B\, \rho(C z_k)}, \qquad k \in \mathbb{N}. \tag{2.2}$$

If $m_k = O(e^{B\, \rho(C z_k)})$ then (2.2) is equivalent to

$$|a_{k,l}| \leq A\, e^{B\, \rho(C z_k)}. \tag{2.3}$$

We show below that $\varrho(A_{\boldsymbol{\rho}}) \subseteq A_{\boldsymbol{\rho}}(V)$. In general $A_{\boldsymbol{\rho}}(V)$ is much bigger than $\varrho(A_{\boldsymbol{\rho}})$. We give an example here to show what can happen.

EXAMPLE 2.3. To take an extreme case, let $n_f(z)$ satisfy the conditions of Theorem 1.1 with all the multiplicities one, $a_k = \ln k$ and $a_{k,0} = a_k =$

0. Then (2.2) is certainly satisfied for any ρ, but we have seen (Example 1.1) that such a function f must satisfy

$$|f(z)| \le e^{r^{1/2} \ln r \, e^r}, \qquad \text{for } |z| = r .$$

One reason is that the growth (2.2) depends on purely local conditions. Allowing more global conditions will give a sharper bound which in some cases may be more appropriate. Therefore, we now let

$$\rho(C; z; r) = \max_{|\zeta| \le r} \rho(C\,(z + \zeta))$$

and make the following definition.

Definition. *The space $A_{\boldsymbol{p},\infty}(V)$ consists of those $\{a_{k,l}\} \in A(V)$ such that for some A, B, $C > 0$ and al $r \ge 1$*

$$\sum_{l=0}^{m_k - 1} |a_{k,l}|\, r^l \le A\, e^{B\,\rho(C; z_k; r)} . \tag{2.4}$$

We then have the following propositions.

Proposition 2.1. $A_{\boldsymbol{p},\infty}(V) \subseteq A_{\boldsymbol{p}}(V)$.

Proposition 2.2. *$\varrho : A(\mathbb{C}) \to A(V)$ maps $A_{\boldsymbol{p}}$ into $A_{\boldsymbol{p},\infty}(V)$.*

We now seek growth conditions on $\{a_{k,l}\}$ for $A_{\boldsymbol{p},\infty}(V)$ analogous to (2.3).

Proposition 2.3. *If $\rho(z) = \rho(|z|)$ with $r = |z|$, then $\rho(C; z; D_1\, r + D_2) \le \rho(C'; r)$ for some C', D_1, $D_2 > 0$.*

EXAMPLE 2.4. Let $\kappa(r)$ be some function of r tending toward infinity faster than any linear function. Then (2.4), with $r = \kappa(r)$, gives the estimate

$$|a_{k,l}| \le \frac{A\, e^{B\rho(C; z_k; \kappa(r))}}{\kappa(r)^l} . \tag{2.5}$$

Compare with (2.3). Then $A_{\boldsymbol{p},\infty}(V) = A_{\boldsymbol{p}}(V)$ if and only if

$$m_k = O\left(\inf_{r \ge 1} \frac{1 + B\, \rho(C; z_k; \kappa(r) - \rho(D z_k)}{\ln \kappa(r)} \right) , \tag{2.6}$$

for some $B > 0$. To see this, first assume (2.6). Note that by Proposition 2.1, we need only show that $A_p(V) \subseteq A_{p,\infty}(V)$. Assume (2.2). Since (2.2) holds for the sum, it certainly holds for each term. (2.6) then implies

$$A\,e^{B\,\rho(Cr)} \leq \frac{A'\,e^{B'\,\rho(C';z_k;\kappa(r))}}{(\kappa(r))^{m_k}} \leq \frac{A'\,e^{B'\,\rho(C;z_k;\kappa(r))}}{(\kappa(r))^{l}}\,.$$

So,

$$m_k|a_{k,l}|\,\kappa(r)^l \leq m_k\,A\,e^{B\,\rho(C;z_k;\kappa(r))}\,. \tag{2.7}$$

Since (2.6) implies $m_k = O(e^{B\,\rho(C;z_k;\kappa(r))})$, and (2.7) is true for each l, we conclude

$$\sum_{l=0}^{m_k-1} |a_{k,l}|\,\kappa(r)^l \leq A\,e^{B\,\rho(C;z_k;\kappa(r))}\,.$$

Now assume (2.2). Then

$$\sum_{l=0}^{m_k-1} |a_{k,l}|\,\kappa(r)^l \leq \kappa(r)^{m_k} A\,e^{B\,\rho(Cz_k)}\,.$$

Since $A_p(V) \subseteq A_{p,\infty}(V)$, we must also have

$$\kappa(r)^{m_k}\,A\,e^{B\,\rho(Cz_k)} \leq A_1\,e^{B_1\,\rho(C_1;z_k;\kappa(r))}\,.$$

Solving for m_k gives (1.10).

We now make the following definitions.

Definition. *If ϱ maps A_p onto $A_p(V)$, we will say that V is an interpolating variety for A_p. If ϱ maps A_p onto $A_{p,\infty}(V)$, then we will say that V is a weak interpolating variety for A_p.*

Definition. *If $f_1, \ldots, f_m \in A_p$ then*

$$I_{\mathrm{loc}}(f_1, \ldots, f_m)\,,$$

the local ideal generated by $(f_1, \ldots, f_m)$, is the set of all functions $g \in A_p$ such that, for any $z \in \mathbb{C}$, there is an open neighborhood U of z and functions $g_1, \ldots, g_m \in A(U)$ with the property

$$g = \sum_{j=1}^{m} f_j\,g_j \quad \textit{in } U$$

If $V = V(f_1, \dots, f_m)$ is the variety of common zeros of $f_1, \dots, f_m$ (with multiplicity) then $I_{\mathrm{loc}}(f_1, \dots, f_m) = I(V) \cap A_\rho$. Since $I(V)$ is closed in $A(\mathbb{C})$ (with the topology of uniform convergence on compacta) and $A_\rho \hookrightarrow A(\mathbb{C})$ is continuous, it follows that $I_{\mathrm{loc}}(f_1, \dots, f_m)$ is closed in A_ρ. By $(f_1, \dots, f_m)$ we will denote the *ideal generated in* A_ρ by those same functions.

Definition. *We say that* $f_1, \dots, f_m$ *as above are jointly invertible if*

$$I_{\mathrm{loc}}(f_1, \dots, f_m) = (f_1, \dots, f_m).$$

For a single function f, we say that f is invertible if $I_{\mathrm{loc}}(f) = (f)$; in particular, the principal ideal generated by f is closed. In general, we do not expect these two ideals to coincide. We will see later that if $\rho(z) = \rho(|z|)$ then (f) is always closed. Se also [8, Theorem 7.1].

Hence, f invertible in A_ρ implies that if $g \in A_\rho$ and $g/f \in A(\mathbb{C})$ then $g/f \in A_\rho$. It also implies that (f) is closed and, consequently, the map $g \to fg$ is an open map from A_ρ onto (f).

Theorem 2.1. *Let* $f_1, \dots, f_m \in A_\rho$ *and* $V = V(f_1, \dots, f_m)$. *If, for some* ε, B, $C > 0$, *we have for all* $(z_k, m_k) \in V$

$$\sum_{j=1}^{m} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \geq \varepsilon\, e^{-B\,\rho(Cz)}, \tag{2.8}$$

then V *is an interpolating variety. In the converse direction, if* V *is an interpolating variety and the functions* $f_1, \dots, f_m$ *are jointly invertible, then* (2.8) *holds for some* ε, B, $C > 0$ *at every point* $(z_k, m_k) \in V$.

From the proof of Theorem 2.1, we obtain

Corollary 2.1. *With the same hypotheses of joint invertibility as in Theorem* 2.1, *the multiplicity variety* $V = V(f_1, \dots, f_m)$ *is an interpolating variety if and only if there exist* ε, B, $C > 0$ *such that*

i) *each* $z_k \in V$ *is contained in a bounded component of* $S(\mathbf{f}; \varepsilon, B, C)$ *with diameter at most* 1.

ii) *No two points of* V *lie in the same component.*

We now consider analogous results for weak interpolating varieties. Recall that

$$\rho(C;z;r) = \max_{|\zeta|\le r} \rho(C\,(z+\zeta)).$$

For $B > 0$, $0 \le l \le m_k - 1$, and $\{(z_k, m_k)\} = V$, let

$$\gamma_{k,l} = \gamma_{k,l}(B) = \inf_{r>0} \frac{e^{B\,\rho(C;z;r)}}{r^l}$$

and

$$\gamma_k = \gamma_k(B) = \gamma_{k,m_k-1}(B)\,. \tag{2.9}$$

Notice the $\gamma_{k,l}$ come basically from (2.4), *i.e.*

$$\{a_{k,l}\} \in A_{\rho,\infty} \quad \text{implies} \quad |a_{k,l}| \le A\,\gamma_{k,l}(B)\,,$$

for some A, $B > 0$.

The following theorem gives a necessary condition for $V = V(f_1, \ldots, f_m)$ to be a weak interpolating variety when $f_1, \ldots, f_m$ are jointly invertible.

Theorem 2.2. *Let $V = V(f_1, \ldots, f_m)$, $f_j \in A_\rho$. Supose that V is a weak interpolating variety and that $f_1, \ldots, f_m$ are jointly invertible. Then for each $B > 0$, there exist constants ε, C_1, $C_2 > 0$ such that*

$$\sum_{j=1}^{m} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \ge \varepsilon\,\gamma_k(B)\,e^{-C_1\,\rho(C_2 z_k)}\,.$$

Next we give sufficient conditions. For each $B > 0$, let $R_k = R_k(B) \ge 1$ denote a point at which

$$\frac{e^{B\,\rho(C;z_k;R_k)}}{R_k^{m_k-1}} \le 2\,\gamma_k\,,$$

(recall (2.9), the definition of γ_k).

Theorem 2.3. *Let $f_1, \ldots, f_m \in A_\rho$ and $V = V(f_1, \ldots, f_m)$. Suppose that for each $B > 0$, there exist constants ε_1, C_1, C_2, C_3, C_4, $C_5 > 0$ such that for all $(z_k, m_k) \in V$,*

i) $m_k \leq C_1\,\rho(C_2 z_k) + C_3\,,$

ii) $\rho(z; 2R_k) \leq C_1\,\rho(C_2 z_k) + C_3\,,$ *for all* $|z - z_k| \leq 2R_k\,,$

iii) $\displaystyle\sum_{j=1}^{m} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \geq \varepsilon_1\,\gamma_k(B)\,e^{-C_4\,\rho(C_5 z_k)}\,.$

Then V is a weak interpolating variety.

There is also an analogue to Corollary 2.1, with the same notation.

Corollary 2.2. *If the hypotheses of Theorem 2.3 hold, then for some constants ε, B, C, C_1, C_2, C_3, $C_4 > 0$ we have*

i) *Each $z_k \in V$ belongs to a bounded component of $S(\mathbf{f}; \varepsilon, B, C)$ and $\rho(z)$ satisfies*

$$\rho(C_1 z) \leq C_2\,\rho(C_3 \zeta) + C_4\,,$$

for any z,ζ of that component.

ii) *No two distinct points of V lie in the same bounded component of $S(\mathbf{f}; \varepsilon, B, C)$.*

Since $A(V)$ is generally much larger than the range of the restriction map $\varrho : A_\rho \to A(V)$, the next question is to try and find a description of the subspace of $A(V)$ which is the range of ϱ. We start with the concept of a slowly decreasing function.

Definition. *A function $f \in A_\rho$ is caled slowly decreasing if the following two conditions hold.*

i) *There exist constants ε, B, $C > 0$ such that each connected component S_α of the set*

$$S(f; \varepsilon, B, C) = \{z :\ |f(z)| < \varepsilon\, e^{-B\,\rho(Cz)}\}$$

is relatively compact.

ii) *There exist constants D_1, D_2, D_3, $D_4 > 0$, independent of α, such that*

$$\rho(D_1 z) \leq D_2\,\rho(D_3 \zeta) + D_4\,,$$

for any z,$\zeta \in S_\alpha$ and any α.

Proposition 2.4. *If f is slowly decreasing, then f is invertible.*

Proposition 2.5. *If $\rho(z) = \rho(|z|)$, then any $f \in A_{\rho}$, not identically zero, is slowly decreasing.*

We need the following lemma. Most of the proofs are immediate consequences of the definition of slowly decreasing.

Lemma 2.3. *If $f \in A_{\rho}$ is slowly decreasing, then there are rectifiable Jordan curves Γ_α with the following properties:*

a) *The sets U_k are pairwise disjoint and $V = V(f) \subset \bigcup_\alpha U_\alpha$ where $U_\alpha = \operatorname{int}\Gamma_\alpha$.*

b) *For some constant $A > 0$, we have, for all α,*

$$|f(z)| \geq \frac{1}{A}\, e^{-A\,\rho(Az)}\,,$$

for $z \in \Gamma_\alpha$.

c) *For some constants B, $B_1 > 0$, we have, for any α, and any pair $z,\zeta \in \overline{U}_\alpha$,*

$$\rho(B_1 z) \leq B\,\rho(B\zeta) + B\,.$$

d) *If d_α is the diameter of Γ_α, then for some constant $C > 0$, we have*

$$d_\alpha \leq C\, e^{C\,\rho(Cz)}\,,$$

for any $z \in \overline{U}_\alpha$.

e) *For some constant $D > 0$,*

$$\operatorname{length}(\Gamma_\alpha) \leq D\, e^{D\,\rho(Dz)}\,,$$

for any $z \in \overline{U}_\alpha$.

f) *If n_α denotes the number of points in $V_\alpha = V \cap U_\alpha$, counted with multiplicity, then*

$$n_\alpha \leq N\, e^{N\,\rho(Nz)}\,,$$

for some constant $N > 0$ and any $z \in \overline{U}_\alpha$.

For the U_α obtained in Lemma 2.3 we make the following definition.

Definition. *Let $\{a_{k,l}^{(\alpha)}\} = \{a_{k,l}\} \cap A(V_\alpha)$. Then $A_{\rho,g}(V)$ consists of those functions $\{a_{k,l}\} \in A(V)$ such that, for $\varphi \in A(U_\alpha)$ and*

$$\|a_{k,l}^{(\alpha)}\|_\alpha = \inf\{\|\varphi\|_\infty :\ \varphi \in A(U_\alpha) \text{ and } \varrho_{V_\alpha}(\varphi) = \{a_{k,l}^{(\alpha)}\}\}\,,$$

there exist constants C_1, C_2, $C_3 > 0$ *independent of* α *such that*

$$\|a_{k,l}^{(\alpha)}\|_\alpha \le C_1\, e^{C_2\, \rho(C_3 z)}\,, \qquad \text{for any } z \in U_\alpha\,. \tag{2.10}$$

See also [5] for a discussion of these spaces.

Theorem 2.4. *If the function* $f \in A_{\boldsymbol{\rho}}$ *is slowly decreasing, the map* ϱ_V *induces a linear topological isomorphism between the spaces* $A_{\boldsymbol{\rho}}/((f))$ *and* $A_{\boldsymbol{\rho},g}(V)$ *for* $V = V(f)$.

Theorem 2.4 shows that even though $A_{\boldsymbol{\rho},g}(V)$ was defined in terms of a specific family of curves Γ_α, $A_{\boldsymbol{\rho},g}(V)$ is actually a subspace of $A(V)$ independent of the family $\{\Gamma_\alpha\}$.

We can obtain a characterization of $A_{\boldsymbol{\rho},g}(V)$ in terms of polynomial interpolation if the following lemma holds.

Lemma 2.4. *If* $\rho(z) = \rho(|z|)$ *then* c) *in Lemma* 2.3 *can be replaced by*

c′) *Let* $W_\alpha = \{z \in \mathbb{C} :\ \mathrm{dist}\,\{z, U_\alpha\} \le 2d_\alpha\}$, *then for some constant* $B > 0$, *we have*

$$\rho(\zeta) \le B\,\rho(Bz) + B\,,$$

for any α *and any* $z, \zeta \in W_\alpha$.

To continue, we first restate some facts about the Newton interpolation formula and divided differences (see [10, p. 326]).

Let $\zeta_1, \dots, \zeta_n$, $n = n_\alpha$ stand for the points in V_α, counted with multiplicity. Then the polynomials

$$\begin{aligned} P_0 &= 1\,,\\ P_1(z) &= z - \zeta_1\,,\\ &\ \ \vdots\\ P_{n-1}(z) &= \prod_{j=1}^{n-1}(z-\zeta_j)\,, \end{aligned}$$

form a basis of the space of polynomials of degre $n-1$. There is a unique polynomial $Q = Q_\alpha$ of degre at most $n-1$ such that

$$\varrho_{V_\alpha}(Q_\alpha) = \{a_{k,l}^{(\alpha)}\}$$

and it can be written as

$$Q(z) = \sum_{j=0}^{n-1} \Delta^{(j)} P_j(z) .$$

$Q(z)$ is called the *Newton interpolation polynomial.* The coefficients

$$\Delta^{(j)} = \Delta^{(j)}(\{a_{k,l}^{(\alpha)}\})$$

are the j^{th} divided differences of the $a_{k,l}^{(\alpha)}$'s. They can be computed recursively. For example, if $\zeta_1 = z_k$ then $\Delta^{(0)} = a_{k,0}$. If $\zeta_1, \ldots, \zeta_m$ are distinct points, then

$$\Delta^{(m)} = \sum_{k=1}^{m} \frac{a_k}{\prod_{j\neq k}(\zeta_k - \zeta_j)} .$$

Higher multiplicities are handled by taking appropriate limits. For example, if

$$\zeta_1 = \cdots = \zeta_l = z_k ,$$

then $\Delta^{(l)} = a_{k,l}$. If $Q(z)$ satisfies

$$|Q(z)| \le K_1 \, e^{K_2 \, \rho(K_3 z)} \tag{2.11}$$

for some K_1, K_2, $K_3 > 0$, the above discussion shows that (2.10) holds and hence there exists a function $\varphi \in A_{\boldsymbol{p}}$ such that $\varrho_{V_\alpha}(\varphi) = \{a_{k,l}^{(\alpha)}\}$. This can be done even if we know the estimate for a single α since $\{b_{k,l}\} \in A(V)$, defined by

$$b_{k,l} = \begin{cases} a_{k,l} , & \text{if } z_k \in V_\alpha , \\ 0 , & \text{if } z_k \notin V_\alpha , \end{cases}$$

is in $A_{\boldsymbol{p},g}(V)$. We can estimate the $\Delta^{(j)}$ by the following lemma [10, p. 329].

Lemma 2.A. *Let φ be holomorphic in the open set $W \subseteq \mathbb{C}$, $|\varphi(z)| \le M$ in W, and $\zeta_1, \ldots, \zeta_n$ be given such that for some $\delta > 0$, $\bigcup_{j=1}^{n} D(\zeta_j; \delta) \subseteq W$, then*

$$|\Delta^{(j)}| \le \left(\frac{2}{\delta}\right)^j M , \qquad 0 \le j \le n-1 .$$

(Here the $\Delta^{(j)}$ are computed with respect to $\varrho_V(Q)$, V the multiplicity variety associated to $\zeta_1, \ldots, \zeta_n$).

Hence, assumming c′) holds, if either $Q = Q_\alpha$ satisfies (2.1) or (2.10) holds, we have, for some constants A_1, B_1, $C_1 > 0$,

$$\|\{a_{k,l}^{(\alpha)}\}\|'_\alpha = \max_{0\le j\le n-1} |\Delta^{(j)}(\{a_{k,l}^{(\alpha)}\})\, d_\alpha^j| \le A_1\, e^{B_1\, \rho(C_1 z)}\,, \tag{2.12}$$

for all $z \in U_\alpha$, $n = n_\alpha$. This follows from Lemma 2.A with $W = W_\alpha$ and $\delta = 2\,d_\alpha$. In particular, if

$$\{a_{k,l}\} \in A_{\mathbf{p},g}(V)\,,$$

then (2.12) holds for every α with the constants independent of α. Conversely, if (2.12) holds for a given α, then it is obvious from the definition of the polynomials P_j and Q_α that, for every $z \in U_\alpha$ and some new constants A_2, B_2, $C_2 > 0$,

$$\begin{aligned} |Q_\alpha(z)| &\le A_1\, e^{B_1\, \rho(C_1 z)} \sum_{j=0}^{n-1} \frac{|(z-\zeta_1)\cdots(z-\zeta_j)|}{d_\alpha^j} \\ &\le n_\alpha\, A_1\, e^{B_1\, \rho(C_1 z)} \\ &\le A_2\, e^{B_2\, \rho(C_2 z)}\,. \end{aligned}$$

The last inequality follows from Lemma 2.3.f). Hence, if (2.12) holds with constants independent of α, then $\{a_{k,l}\} \in A_{\mathbf{p},g}(V)$. We collect these remarks in Theorem 2.5.

Theorem 2.5. *Let $f \in A_{\mathbf{p}}$ be slowly decreasing and the norms $\|a_{k,l}^{(\alpha)}\|'_\alpha$ of $\{a_{k,l}\} \in A(V)$ be defined with respect to some grouping $\{\Gamma_\alpha\}$ satisfying* a)-c′)-f) *of Lemmas* 2.3 *and* 2.4. *Then $A_{\mathbf{p}}/((f))$ is isomorphic under the restriction map ϱ to the subspace of $A(V)$ of those $\{a_{k,l}\}$ such that* (2.12) *holds for some constants A_1, B_1, $C_1 > 0$ independent of α.*

We close this section with some remarks on when each V_α contains only one point of V.

Proposition 2.6. *If f is slowly decreasing and there is a grouping $\{\Gamma_\alpha\}$ for $V = V(f)$ such that every V_α contains a single point of V, then $A_{\mathbf{p},g}(V) = A_{\mathbf{p},\infty}(V)$.*

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On the resolvents of dyadic paraproducts

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1. Introduction.

We consider the boundedness of certain singular integral operators that arose in the study of Sobolev spaces on Lipschitz curves, [P1]. The standard theory available (David and Journé's $T1$ Theorem, for instance; see [D]) does not apply to this case because the operators are not necessarily Calderón-Zygmund operators, [Ch]. One of these operators gives an explicit formula for the resolvent at $\lambda = 1$ of the dyadic paraproduct, [Ch].

The operators in question can be thought of as multipliers for a wavelet basis, [M]. In particular, given a function in $L^p(\mathbb{R}^n)$, we can write its decomposition in the Haar basis. If we want to perturb the coefficients, we know that we can multiply by a bounded sequence of numbers and preserve the norm. Suppose instead that we multiply each Haar coefficient by a function. What are the necessary and sufficient conditions for such an operator to be bounded in $L^p(\mathbb{R}^n)$?

Our model operator will be

$$Tf(x) = \sum_j \omega_j(x)\, \Delta_j f(x)\,,$$

where $\Delta_j f$ denotes the Haar decomposition of f at level j, and the ω_j's are functions.

Given a doubling dyadic weight ω (*i.e.* a positive locally integrable function such that $\omega(\tilde{Q}) \leq C\,\omega(Q)$, for every dyadic cube Q, $\tilde{Q}$ its parent, and where $\omega(Q) = \int_Q \omega)$. Consider the non trivial examples

$$\omega_j(x) = \frac{\omega(x)}{E_j\omega(x)} \qquad \text{or} \qquad \omega_j(x) = \frac{\omega(x)}{E_{j+1}\omega(x)}\,,$$

where we define $E_j\omega(x) = \omega(Q)/|Q|$, $x \in Q \in \mathcal{D}_j$, and $\mathcal{D}_j$ denotes the dyadic cubes of side length 2^{-j}, $|Q|$ denotes the Lebesgue measure of Q . Denote by T_ω and P_ω the operators corresponding to the first and second sequence of weights respectively. In Section 3 we give necessary and sufficient conditions for the boundedness of such operators, more precisely, we will prove

Theorem I. *Given a dyadic doubling weight* ω, $1 < p < \infty$, $1/p+1/q = 1$, *the following properties are equivalent,*

i) T_ω *is bounded in* $L^p(\mathbb{R}^n)$,

ii) P_ω *is bounded in* $L^p(\mathbb{R}^n)$,

iii) $\omega \in RH_p^d(\mathbb{R}^n)$,

iv) M_ω *is bounded in* $L^q(\mathbb{R}^n)$,

v) S_ω *is bounded in* $L^q(\mathbb{R}^n)$,

where M_ω *and* S_ω *are weighted maximal and square functions, namely, for* $E_j^\omega f = E_j f\omega/E_j\omega$, *and* $\Delta_j^\omega = E_{j+1}^\omega - E_j^\omega$, *let*

$$M_\omega f = \sup_j E_j^\omega |f| \qquad \textit{and} \qquad S_\omega f = (\sum_j |\Delta_j^\omega f|^2)^{1/2}\,,$$

and $\omega \in RH_p^d(\mathbb{R}^n)$ *(dyadic reverse Hölder* p*) means that for each dyadic cube* Q,

$$\frac{1}{|Q|}\int_Q \omega^p(x)dx \le C\left(\frac{1}{|Q|}\int_Q \omega\right)^p.$$

Notice that there are doubling weights in some RH_p^d classes that are not in all of them. For those weights, P_ω and T_ω are bounded in some L^p spaces but not in all of them. Therefore they are not dyadic Calderón-Zygmund operators (C-Z).

Operating formally with P_ω we get the Neumann series for $I - \Pi_b$, where Π_b is the dyadic paraproduct (see [Ch]), and $b \in \text{BMO}$ (we do not use the standard correspondence $\omega = e^b$ but a different one first used in [KFP] and fully developed by S. Buckley in his Ph.D. Thesis).

The RH_p^d condition on ω is not enough to imply that the paraproduct is a contraction (if it were, the Neumann series would trivially converge to the inverse of $I - \Pi_b$), but it does guarantee the convergence of P_ω.

Loosely speaking, given a locally integrable function b, the weight ω, associated to it under this correspondence, is dyadic doubling if and only if there exists a constant $0 < \varepsilon < 1$ such that for all $j \in \mathbb{Z}$, $|\Delta_j b| \leq 1 - \varepsilon$. We will say that such a function b is of A_∞^d-type if the weight associated to b under the correspondence mentioned before, is in the Muckenhoup class of weights A_∞^d (see [GC-RF]). Similarly for RH_p^d-type. In particular these sets of functions are subsets of dyadic BMO.

There is an operator, denoted by P_b, that can be identified with P_ω, for ω and b related under this correspondence. We will show that whenever P_b is well defined and is a bounded operator, then it is the inverse of $I - \Pi_b$.

As a consequence of Theorem I and the previous remarks, we can show that,

Theorem II. *Given a locally integrable function b and $0 < \varepsilon < 1$ such that for all $j \in \mathbb{Z}$, $|\Delta_j b| \leq 1 - \varepsilon$. Then the operator $I - \Pi_b$ has a bounded inverse in some $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if b is of A_∞^d-type. In that case, $(I - \Pi_b)^{-1} = P_b$.*

Moreover the necessary and suficient condition for having a bounded inverse in $L^p(\mathbb{R}^n)$ is that b is of RH_p^d-type.

Since the paraproduct is a bilinear operator in b and f, then the existence of a bounded inverse on $L^p(\mathbb{R}^n)$ for $I - \lambda \Pi_b$ depends on the function λb satisfying the hypothesis of the previous theorem. If it does then $\mu = 1/\lambda$ belongs to the resolvent of Π_b.

We remark that we are dealing here with operators that are not necessarily compact operators. See [R] for some resuls on compact paraproducts.

By previous observations, although $I - \Pi_b$ is a dyadic C-Z operator for $b \in \text{BMO}$, the inverse is not necessarily a dyadic C-Z operator.

The representation P_b really goes beyond the contractive case.

Theorem III. *Given $-1 \leq \lambda < 0$, there exist $1 < p < \infty$, $0 < \varepsilon < 1$, and a locally integrable function b such that for all $j \in \mathbb{Z}$, $|\Delta_j b| \leq 1 - \varepsilon$ and b is of RH_p^d-type but λb is not of RH_p^d-type. In particular, for that b, $I - \Pi_b$ is invertible but $\|\Pi_b\|_{p,p} \geq 1/|\lambda|$.*

The paper is organized as follows. First we will study the invertibility of $I - \Pi_b$, and prove Theorem II up to the boundedness of P_b. In the second part we will study the operators P_ω and T_ω. Finally we will clarify the correspondence $b \leftrightarrow \omega$ and the identification of P_b and P_ω, and we will produce the non-contractive examples of Theorem III.

These results are, in their first version $n = 1$, $p = 2$, part of my Ph.D. Thesis. I would like to thank my advisor, P.W. Jones for suggesting the initial problem and guiding me through the completion of this work. I extend my thanks to R.R. Coifman for very helpful conversations.

2. Inverting Paraproducts.

2.1. Preliminaries.

2.1.1. Dyadic grid and Haar basis in $\mathbb{R}^n$.

Throughout this paper, we will use "C" to indicate a constant that depends only on p and the dimension n. $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ indicates the set of all dyadic cubes in $\mathbb{R}^n$. Denote by $\mathcal{D}_k$ the k th generation of $\mathcal{D}$, consisting of those dyadic cubes with side length 2^{-k}.

Let us define the n^{th}-dimensional Haar basis.

To each dyadic cube $Q = I_1 \times \cdots \times I_n$ we associate $2^n - 1$ functions indexed by $A_n = \{e = (e_1, \ldots, e_n) : e_i = 1, 0;\ e \neq 0\}$. For $\mathbf{x} = (x_1, \ldots, x_n)$ we set

$$\psi_Q^e(\mathbf{x}) = \psi_{I_1}^{e_1}(x_1) \cdots \psi_{I_n}^{e_n}(x_n)\,,$$

where

$$\psi_I^0(x) = \frac{1}{|I|^{1/2}}\, \chi_I(x) \qquad \text{and} \qquad \psi_I^1(x) = h_I(x)\,;$$

here χ_I and h_I denote respectively the characteristic and the Haar functions associated with the interval I; more precisely, h_I is $\pm 1/|I|^{1/2}$ depending if you are on the right or the left half of I, and zero otherwise.

These n-dimensional Haar funtions are a basis of $L^2(\mathbb{R}^n)$.

2.1.2. Expectation and difference operators.

Let us define for $k \in \mathbb{Z}$ the operators expectation E_k, and difference Δ_k by

$$E_k f(x) = \frac{1}{|Q|} \int_Q f(t)\, dt\,, \qquad x \in Q \in \mathcal{D}_k\,,$$
$$\Delta_k f(x) = E_{k+1} f(x) - E_k f(x)\,.$$

Since $E_k \to I$ (here I denotes the identity operator) as $k \to +\infty$, and $E_k \to 0$ as $k \to -\infty$, both limits are in the L^p sense; then the following equalities hold in the $L^p(\mathbb{R}^n)$ sense, $E_k = \sum_{j<k} \Delta_j$, and $I = \sum_{j \in \mathbb{Z}} \Delta_j$.

Finally, it is not hard to check that

$$\Delta_k f(x) = \sum_{Q \in \mathcal{D}_k} \sum_{e \in A_n} \langle f, \psi_Q^e \rangle\, \psi_Q^e(x)\,.$$

This proves that the Haar system is complete.

2.1.3. Paraproducts and BMO.

Define formally the *paraproduct* associated to a locally integrable function b by

$$\Pi_b f = \sum_{k \in \mathbb{Z}} E_k f\, \Delta_k b\,.$$

Let us compute formally its adjoint Π_b^* ($\int Tf\, g = \int f\, T^* g$, $f \in L^p$, $g \in L^q$, $1/p + 1/q = 1$),

$$\Pi_b^* f = \sum_{k \in \mathbb{Z}} \Delta_k f\, \Delta_k b\,.$$

A locally integrable function b is in dyadic $\mathrm{BMO}_d(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q |b(x) - m_Q b|\, dx \le C\,,$$

where $m_Q b = \int_Q b/|Q|$.

A remarkable fact, due to John and Niremberg (see [G, Chapter VI]), is that for all $1 < p < \infty$,

$$\|b\|_* \sim \sup_{Q\in\mathcal{D}} \left(\frac{1}{|Q|}\int_Q |b(x) - m_Q b|^p \, dx\right)^{1/p} . \tag{1}$$

If $b \in \mathrm{BMO}_d(\mathbb{R}^n)$ then the paraproduct is a bounded operator on $L^p(\mathbb{R}^n)$, for a proof see [M, p. 273]. Moreover $\|\Pi_b f\|_p \le C\,\|b\|_* \|f\|_p$.

Recall the following lemma,

Carleson's Lemma. *Let $\{a_Q\}_{Q\in\mathcal{D}}$ be a sequence of positive numbers, that satisfies a Carleson condition, i.e*

$$\sum_{Q\in\mathcal{D}(Q_0)} a_Q \le C\,|Q_0|\,, \qquad \textit{for all } Q_0 \in \mathcal{D}.$$

Then for any sequence of positive numbers λ_Q, $Q \in \mathcal{D}$,

$$\sum_{Q\in\mathcal{D}} a_Q \lambda_Q \le \int \lambda^*(x)\, dx\,, \qquad \textit{where } \lambda^*(x) = \sup_{x\in Q} \lambda_Q\,.$$

For a proof see [M, p. 273].

Actually, for our dyadic version of the paraproduct, boundedness in $L^2(\mathbb{R}^n)$ is a consequence of Carleson's lemma and the boundedness of the dyadic Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{j\in\mathbb{Z}} E_j|f|(x)$$

in $L^2(\mathbb{R}^n)$. Just observe that $b \in \mathrm{BMO}_d$ implies that the sequence $\{|Q|b_Q^2\}$ is a Carleson sequence (by (1)), for $p = 2$), where $b_Q = \Delta_j b(x_Q)$, $Q \in \mathcal{D}_{j+1}$, and $x_Q \in Q$. Set $\lambda_Q = (E_j f(x_Q))^2$, Q as before; then $\lambda^*(x) \le Mf(x)^2$. By Littlewood-Paley theory (see Section 3.1.2), and since by (5), $\Delta_j(\Pi_b f) = E_j f\, \Delta_j b$, then,

$$\|\Pi_b f\|_2^2 \sim \|(\sum_j |E_j f \Delta_j b|^2)^{1/2}\|_2^2\,.$$

This is equal to,

$$\sum_j \sum_{Q\in\mathcal{D}_{j+1}} |Q|\,|\Delta_j b(x_Q)|^2 |E_j f(x_Q)|^2 = \sum_{Q\in\mathcal{D}} |Q|\, b_Q^2\, \lambda_Q\,.$$

The right hand side is bounded by $C\int(Mf)^2$ by Carleson's lemma, and this in turn is bounded by $C\|f\|_2^2$.

2.2. Inverting $I - \Pi_b$.

Let $b \in \mathrm{BMO}_d(\mathbb{R}^n)$. Given $g \in L^p(\mathbb{R}^n)$ we want to solve the following equation for $f \in L^p(\mathbb{R}^n)$,

$$(I - \Pi_b)f = g\,.$$

If the paraproduct is a contraction (*i.e.* $\|\Pi_b\|_{p,p} < 1$) we can certainly solve the equation. The solution will be given by the Neumann series of $I - \Pi_b$, more precisely,

$$f = \sum_{n=0}^{\infty} \Pi_b^n g\,.$$

We would like to find a solution that goes beyond the contractive case.

Suppose there exist $f, g \in L^p(\mathbb{R}^n)$ satisfying the equation

$$(I - \Pi_b)f = g\,.$$

Then $f = g + \Pi_b f$. We can look to its Haar decomposition at level m and get, with the notation in Section 2.1.2, $\Delta_m f = \Delta_m g + \Delta_m(\Pi_b f)$. Since $\Delta_m f = E_{m+1}f - E_m f$, and $\Delta_m(\Pi_b f) = \Delta_m b\, E_m f$, we get the following *recurrence equation*,

$$E_{m+1}f = \Delta_m g + (1 + \Delta_m b)\, E_m f\,.$$

Solving this equation we conclude that

$$E_{m+1}f = \sum_{k<m} \Delta_k g \prod_{j=k+1}^{m} (1 + \Delta_j b) + \Delta_m g\,. \tag{2}$$

Since $f \in L^p(\mathbb{R}^n)$, the right hand side converges to f in the L^p sense, so it does the left hand side.

Define the operators P_b^m applied to a function $g \in L^p(\mathbb{R}^n)$ by the formula on the left hand side of (2). And define P_b as the limit (if it exists) in the L^p sense of the P_b^m, as $m \to +\infty$. With this in mind, at least formally, we can write the formula,

$$P_b g = \sum_{k\in\mathbb{Z}} \Delta_k g \prod_{j=k+1}^{\infty} (1+\Delta_j b) . \tag{3}$$

Notice that P_b will be bounded in $L^p(\mathbb{R}^n)$ if the operators P_b^m are uniformly bounded. By construction if $P_b g$ is well defined it must produce a solution $f = P_b g$ to the equation $(I - \Pi_b)f = g$. We will prove the following theorem,

Theorem 1. *The operator $I - \Pi_b$ has a bounded inverse in $L^p(\mathbb{R}^n)$ if and only if the operator P_b is bounded in $L^p(\mathbb{R}^n)$. Moreover in that case*

$$(I - \Pi_b)^{-1} = P_b \, .$$

Now we want to understand the operator P_b. We will find necessary and sufficient conditions for its boundedness in $L^p(\mathbb{R}^n)$, under some mild smallness assumption on the function b.

Theorem 2. *Given a locally integrable function b and $0 < \varepsilon < 1$, such that for all $j \in \mathbb{Z}$, $|\Delta_j b| \leq 1 - \varepsilon$, then the operator P_b is bounded in $L^p(\mathbb{R}^n)$ if and only if the function b is of reverse Hölder p type (RH_p^d-type).*

The condition on b will be described in Section 4, for now let us just say that such b needs to belong to $\mathrm{BMO}(\mathbb{R}^n)$. We say that b is of A_∞^d-type if there exists a $1 < p < \infty$ so that b is of RH_p^d-type.

As an immediate consequence of the previous theorems we get

Theorem II. *For functions b as before. The operator $I - \Pi_b$ has a bounded inverse in $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, if and only if b of A_∞^d-type. In that case $(I - \Pi_b)^{-1} = P_b$.*

This representation of $I - \Pi_b^{-1}$ really goes beyond the contractive case. The representation P_b captures cancellation between the terms of the Neumann series that is otherwise neglected.

Theorem III. *Given $-1 \leq \lambda < 0$, there exist $1 < p < \infty$, $0 < \varepsilon < 1$, and a locally integrable function b such that for all $j \in \mathbb{Z}$, $|\Delta_j b| \leq 1 - \varepsilon$*

and b is of RH_p^d-type but λb is not of RH_p^d-type. In particular, for that b, $I - \Pi_b$ is invertible but $\|\Pi_b\|_{p,p} \geq 1/|\lambda|$.

REMARK. For a suitable function $b \in \mathrm{BMO}_d(\mathbb{R}^n)$, the operator $I - \Pi_b$ provides an example of a dyadic C-Z operator (see [Ch]) whose inverse is bounded in some L^{p_0} but not in every L^p, $1 < p < \infty$, therefore $(I - \Pi_b)^{-1}$ is not a dyadic C-Z operator. To produce such examples it is enough to observe that there exists b of $RH_{p_0}^d$-type which is not of RH_p^d-type for some other $1 < p < \infty$.

Examples like these have been produced by Ph. Tchamitchian, see [M. p. 300].

In the next section we will prove Theorem 1, in Section 3 we will prove Theorem 2, and in the last section we will construct the examples for Theorem 3.

2.3. Proof of Theorem 1.

We want to find the inverse of $I - \Pi_b$. We can formally write down the power series

$$\sum_{j=0}^{\infty} \Pi_b^j g \qquad (\textit{paraseries}).$$

This series will certainly converge in $L^p(\mathbb{R}^n)$ if the paraproduct is a contraction, *i.e.* $\|\Pi_b\|_{p,p} < 1$. This will happen if b has small $\mathrm{BMO}_d(\mathbb{R}^n)$ norm. In this case we can compute the inverse $I - \Pi_b^{-1}$ and it will coincide with the *paraseries*.

Operating formally on $P_b g$ we get that $P_b g = \sum_{j=0}^{\infty} \Pi_b^j g$.

We would like to show that whenever P_b is bounded in $L^p(\mathbb{R}^n)$ then

$$P_b f = (I - \Pi_b)^{-1} f, \qquad \text{for all } f \in L^p(\mathbb{R}^n).$$

The rest of the discussion will concentrate on trying to make this argument rigorous.

It is enough to prove a local version of the result. By local we mean to replace $\mathbb{R}^n$ by any fixed dyadic cube Q_0. Because of the scaling invariance we can choose the unit cube $[0,1]^n$. In this case, just consider the Haar functions associated to the dyadic cubes $Q \subset Q_0$, and let us assume that the L^p functions we are dealing with have mean value zero.

In this setting the operators

$$P_b^N g = \sum_{k=0}^{N-1} \Delta_k g \prod_{j=k+1}^{N} (1 + \Delta_j b) + \Delta_N g \,,$$

are well defined for locally integrable functions g and b, since they are just finite sums of finite products; hence we are free to operate with them.

2.3.1. E_k, Δ_k algebra.

Fix $Q_0 = [0,1]^n$ once and for all. Let $\mathcal{D} = \mathcal{D}(Q_0)$ denote the dyadics contained in Q_0 and let $\mathcal{D}_k(Q_0)$ denote the k^{th} generation, consisting of those cubes in $Q \in \mathcal{D}(Q_0)$ with sidelength $l(Q) = 2^{-k} l(Q_0)$. Let $L_0^p(Q_0)$ denote the subspace of $L^p(Q_0)$ of functions with mean value zero on Q_0 ($\int_{Q_0} f = 0$).

The expectation and difference operators E_k, Δ_k, are defined as before except that now $k \geq 0$ and the dyadic cubes involved are those $Q \in \mathcal{D}(Q_0)$. The paraproduct and its adjoint are defined similarly. Just for the record, and remembering that we will be working with functions with mean value zero on the base cube Q_0, *i.e.* $E_0 f = 0$, we have $E_k = \sum_{j=0}^{k-1} \Delta_j$, and

$$\Pi_b f = \sum_{k=0}^{\infty} E_k f \, \Delta_k b \,,$$

(4)
$$\Pi_b^* f = \sum_{k=0}^{\infty} \Delta_k f \, \Delta_k b \,.$$

Notice that in both the local and global versions

(5)
$$\Delta_k(\Pi_b f) = E_k f \, \Delta_k b \,.$$

The following *composition* and *product* rules hold (see [Ga] for similar results on more general martingales),

(6)
$$\begin{aligned} E_k \Delta_j &= \begin{cases} \Delta_j \,, & \text{if } j < k \,, \\ 0 \,, & \text{otherwise}, \end{cases} \\ \Delta_k \Delta_j &= \begin{cases} \Delta_j \,, & \text{if } k = j \,, \\ 0 \,, & \text{otherwise}, \end{cases} \\ E_k \, E_j &= E_{\min\{k,j\}} \,, \\ \Delta_k f \times \Delta_j g &= \Delta_k (f \times \Delta_j g) \qquad \text{when } k > j \,. \end{aligned}$$

In particular if $k \leq M$, and $0 \leq i_1 < i_2 < \cdots < i_s = M$, then

$$E_k(\Delta_{i_1} f_1 \times \cdots \times \Delta_{i_s} f_s) = 0\,, \tag{7}$$

$$E_k\Big(\sum_{i \geq M} \Delta_i\Big) = 0\,. \tag{8}$$

The operator P_b becomes with this notation

$$P_b g = \sum_{k=0}^{\infty} \Delta_k g \prod_{j=k+1}^{\infty} (1 + \Delta_j b)\,.$$

2.3.2. Theorem 1 - Local version.

Let $f_N = \sum_{k=0}^{N} \Delta_k f$ (projection onto low frequency space). Recall that in this setting, we define

$$P_b^N g = \sum_{k=0}^{N-1} \Delta_k g \prod_{j=k+1}^{N} (1 + \Delta_j b) + \Delta_N g\,.$$

For $g \in L_0^p(Q_0)$, we define $P_b g$ as the limit (if it exists) as $N \to \infty$ in the L^p sense of $P_b^N g$.

Then clearly $P_b^N g = P_{b_N} g_N$. The main formula says that on the subspace of step functions $f = f_N$, the operator P_{b_N} is the inverse of the operator $I - \Pi_{b_N}$.

Lemma 1 (Main formula). *If g has mean value zero on Q_0 then for all positive integers N,*

$$P_{b_N}(I - \Pi_{b_N})\, g_N = (I - \Pi_{b_N}) P_{b_N} g_N = g_N\,.$$

This last statement is a purely algebraic issue. We will prove it later. Let us assume it is true for a moment.

If we consider step functions b_N, clearly when the operator P_{b_N} acts on frequencies higher than N, it behaves like the identity. In particular $P_{b_N}(g - g_N) = g - g_N$.

Notice also that by (8), the paraproduct associated to b_N annihilates frequencies higher than N; in particular, $\Pi_{b_N}(g - g_N) = \sum_{k=0}^{N} \Delta_k b\, E_k(g - g_N) = 0\,.$

After these remarks and using the main formula, we conclude that $(I-\Pi_{b_N})P_{b_N}g = P_{b_N}(I-\Pi_{b_N})g = g$, for all $g \in L_0^p(Q_0)$.

So we found that on $L_0^p(Q_0)$ then

$$P_{b_N} = (I-\Pi_{b_N})^{-1}, \qquad \text{for all } N \geq 0. \tag{10}$$

With a limiting argument we can show

Theorem 1′. *The operator $I-\Pi_b$ has a bounded inverse in $L_0^p(Q_0)$ if and only if P_b is a bounded operator on $L^p(Q_0)$. Moreover $P_b = (I-\Pi_b)^{-1}$ on $L_0^p(Q_0)$*

PROOF. It is clear that if there exists a bounded inverse it will have to coincide with P_b, by construction. We have to prove that if P_b is bounded then there is a bounded inverse and they coincide.

It is enough to show that

$$\int g\,f = \int P_b(I-\Pi_b)g\,f = \int (I-\Pi_b)P_b g\,f\,,$$

for all $f \in L^q(Q_0)$, $g \in L_0^p(Q_0)$.

Recall that $E_{M+1}f = f_M$. Assume $M < N$ and let T^* denote the adjoint of T. Recall that $P_{b_N}g = P_{b_N}g_N + (g - g_N)$.

Notice that by the product and composition rules (6), $P_{b_N}g_N = E_{N+1}(P_b g)$. Also by (4) $\Pi_{b_N}^* f_M = \Pi_b^* f_M$, for all $M \leq N$. Then using these equalities and (10) we get that, for $M \leq N$,

$$\begin{aligned}
\int g\,f_M &= \int (I-\Pi_{b_N})P_{b_N}g\,f_M \\
&= \int P_{b_N}g\,(I-\Pi_{b_N}^*)f_M \\
&= \int (P_{b_N}g_N + g - g_N)\,(I-\Pi_b^*)f_M \\
&= \int E_{N+1}(P_b(g))\,(I-\Pi_b^*)f_M + \int (g-g_N)\,(I-\Pi_b^*)f_M\,.
\end{aligned}$$

Clearly the left hand side of the string of equalities converges to $\int gf$, for all $f \in L^q(Q_0)$, $g \in L_0^p(Q_0)$ when N and $M \to \infty$.

Fix M and let $N \to \infty$. Since Π_b^* is a bounded operator on $L^q(Q_0)$ and $\|g - g_N\|_p \to 0$ as $N \to \infty$, the second term in the last equality goes to zero by Hölder's inequality. Since P_b is, by hypothesis, bounded on $L^p(Q_0)$ we see that, as $N \to \infty$, the first term converges to

$$\int P_b g\,(I - \Pi_b^*) f_M = \int (I - \Pi_b) P_b g\, f_M \,,$$

by the dominated convergence theorem.

Finally let $M \to \infty$ and again by dominated convergence we see that

$$\int f\, g = \int (I - \Pi_b) P_b g\, f \,, \qquad \text{for all } f \in L^q(Q_0),\ g \in L_0^p(Q_0)\,.$$

Similarly for $P_b(I - \Pi_b)$. This ends the proof of the theorem.

2.3.3. Proof of the main formula.

Fix N, and let $b_N = \sum_{j=0}^{N} \Delta_j b$.

We want to prove that the operator P_{b_N} is the inverse of $I - \Pi_{b_N}$ on the finite dimensional subspace of functions of frequencies lower or equal to N and mean value zero on Q_0, namely

$$L_0^N = \{g \in L^1_{\mathrm{loc}}(Q_0) :\ g = \sum_{j=0}^{N} \Delta_j g,\ E_0 g = 0\}\,.$$

It is enough to prove one of the equalities in the lemma, the other is given automatically since we are dealing with a finite dimensional subspace.

We defined

$$P_b^m g = \sum_{k=0}^{m-1} \Delta_k g \prod_{j=k+1}^{m} (1 + \Delta_j b) + \Delta_m g\,.$$

It follows immediatly from the definition, that the operators P_b^m satisfy the following *recurrence equation*

$$P_b^m g = \Delta_m g + (1 + \Delta_m b) P_b^{m-1} g\,. \tag{11}$$

Let $g \in L_0^N(Q_0)$, hence $g = g_N = \sum_{k=0}^{N} \Delta_k g$. Clearly $P_b^N g = P_{b_N} g_N$. It is not hard to check that for $m \le N$, $P_{b_N}^m g_N = E_{m+1}(P_{b_N} g_N)$ and $\Delta_m b = \Delta_m b_N$. Substituting into (11) we get, for $m \le N$,

$$E_{m+1}(P_{b_N} g_N) = \Delta_m g + (1 + \Delta_m b_N) E_m(P_{b_N} g_N)\,, \tag{12}$$

which is the the same recurrence equation we obtained when we solved formally the equation $(I - \Pi_b)f = g$, in Section 2.2.

Recall that by definition of the paraproduct, $\Delta_k(\Pi_b f) = \Delta_k b\, E_k f$ (see (5)). Also recall that $\Delta_k = E_{k+1} - E_k$. With this in mind, equation (12) becomes

$$\Delta_m((I - \Pi_{b_N})P_{b_N} g_N) = \Delta_m g\,.$$

This is true for all positive integers $m \leq N$. Recall that $E_{N+1} = \sum_{m=0}^{N} \Delta_m$, hence we conclude that

$$E_{N+1}((I - \Pi_{b_N})P_{b_N} g_N) = E_{N+1} g\,.$$

We are almost done; just recall that by definition $E_{N+1}g = g_N$, and it is not hard to chek that $E_{N+1}((I - \Pi_{b_N})P_{b_N} g_N) = (I - \Pi_{b_N})P_{b_N} g_N$.

Hence we showed that

$$(I - \Pi_{b_N})P_{b_N} g_N = g_N\,.$$

This finishes the proof of the main formula.

Once the main formula is known, we can show some amusing identities.

Lemma 2. $P_{b_N} g_N = \sum_{j=0}^{N} \Pi_{b_N}^j g_N\,.$

PROOF. By the previous result, it is enough to show that the Neumann polynomial $\sum_{j=0}^{N} \Pi_{b_N}^j g_N$ is the inverse of $I - \Pi_{b_N}$ on the subspace $L_0^N(Q_0)$.

Now certainly

$$(I - \Pi_{b_N})\Big(\sum_{j=0}^{N} \Pi_{b_N}^j g_N\Big) = g_N - \Pi_{b_N}^{N+1} g_N\,.$$

We will be done if we show that $\Pi_{b_N}^{N+1} g_N = 0$. We will prove this by induction.

For $N = 0$ it is true that $\Pi_{b_0} g_0 = \Delta_0 b\, E_0 g = 0$, since $E_0 g = 0$.

Assume that $\Pi_{b_{N-1}}^N g_{N-1} = 0$. We want to show that it is true for $N + 1$. It is enough to show that, for all $k \leq N$,

$$\Pi_{b_N}^k g_N = \Pi_{b_{N-1}}^k g_{N-1} + E_N(\Pi_{b_{N-1}}^{k-1} g_{N-1})\Delta_N b\,. \tag{13}$$

Suppose for a moment that (13) is true, in particular for $k = N$ we get

$$\Pi_{b_N}^N g_N = \Pi_{b_{N-1}}^N g_{N-1} + E_N(\Pi_{b_{N-1}}^{N-1} g_{N-1})\Delta_N b\,.$$

But by the inductive hypothesis, $\Pi_{b_{N-1}}^N g_{N-1} = 0$, hence

$$\Pi_{b_N}^{N+1} g_N = \Pi_{b_N}(E_N(\Pi_{b_{N-1}}^{N-1} g_{N-1})\Delta_N b)\,.$$

Finally, just recall that by the product and composition rules of the expectation and difference operators (*cf.* Section 2.1.2),

$$\Pi_{b_N}(\Delta_N h) = \sum_{k=0}^{N} \Delta_k b\, E_k(\Delta_N h) = 0$$

and

$$E_N(\Pi_{b_{N-1}}^{N-1} g_{N-1})\Delta_N b = \Delta_N h\,,$$

for certain function h. Therefore $\Pi_{b_N}^{N+1} g_N = 0$, and the lemma is proved.

PROOF OF (13). We will show (13) by induction on $k \leq N$ for fixed N.

By the observations made above, the last term in the right hand side of (13) can be written as $\Delta_N(h_N^k)$, for some function h_N^k, and moreover $\Pi_{b_N}(\Delta_N(h_N^k)) = 0$. Hence when applying the paraproduct Π_{b_N} to (13) we get

$$\Pi_{b_N}^{k+1} g_N = \Pi_{b_N}(\Pi_{b_{N-1}}^k g_{N-1})\,.$$

Using the linear property on b of Π_b, we get that the right hand side is equal to

$$\Pi_{b_{N-1}}^{k+1} g_{N-1} + \Pi_{b_N - b_{N-1}}(\Pi_{b_{N-1}}^k g_{N-1})\,.$$

By the definition of the paraproduct and since $\Pi_{b_N - b_{N-1}} f = \Delta_N b\, E_N f$, then

$$\Pi_{b_N}^{k+1} g_N = \Pi_{b_{N-1}}^{k+1} g_{N-1} + \Delta_N b\, E_N(\Pi_{b_{N-1}}^k g_{N-1})\,.$$

This proves (13) for $k+1$ assuming that it is known for k. The only missing step is to check the equation for $k = 1$.

Notice that by the bilinear properties of the paraproduct, then

$$\Pi_{b_N} g_N = \Pi_{b_{N-1}} g_{N-1} + \Pi_{b_N}(g_N - g_{N-1}) + \Pi_{b_N - b_{N-1}}(g_{N-1})\,.$$

By previous observations $\Pi_{b_N - b_{N-1}}(g_{N-1}) = \Delta_N b\, E_N(g_{N-1})$, and $\Pi_{b_N}(g_N - g_{N-1}) = 0$, hence

$$\Pi_{b_N} g_N = \Pi_{b_{N-1}} g_{N-1} + \Delta_N b\, E_N(g_{N-1}),$$

which is exactly what we wanted to show, since $\Pi^0_{b_{N-1}}$ is the identity operator.

We can also show that

Lemma 3. $\Pi^N_{b_N} g_N = \Delta_0 g\, \Delta_1 b \cdots \Delta_N b$.

REMARK. From this lemma and property (7) we conclude that

$$\Pi^{N+1}_{b_N} g_N = \sum_{k=0}^{N} \Delta_k b\, E_k(\Delta_0\, g \Delta_1 b \cdots \Delta_N b) = 0.$$

PROOF. We know that equation (13) is valid for $k = N - 1$, *i.e.*

$$\Pi^{N-1}_{b_N} g_N = \Pi^{N-1}_{b_{N-1}} g_{N-1} + \Delta_N b\, E_N(\Pi^{N-2}_{b_{N-1}} g_{N-1}). \tag{14}$$

We will proceed once more by induction. For $N = 1$ it is true that $\Pi_{b_1} g_1 = \Delta_0 b\, E_0 g + \Delta_1 b\, E_1 g$, but $E_0 g = 0$ hence $\Delta_0 g = E_1 g$, and therefore $\Pi_{b_1} g_1 = \Delta_0 g\, \Delta_1 b$.

Assume that $\Pi^{N-1}_{b_{N-1}} g_{N-1} = \Delta_0 g\, \Delta_1 b \cdots \Delta_{N-1} b$. We want to show that the same holds for N. Recall that $\Pi_{b_N}(\Delta_N h) = 0$, hence, using (14) and the inductive hypothesis, we get that

$$\begin{aligned} \Pi^N_{b_N} g_N &= \Pi_{b_N}(\Delta_0\, g \Delta_1 b \cdots \Delta_{N-1} b) \\ &= \sum_{k=0}^{N} E_k(\Delta_0 g\, \Delta_1 b \cdots \Delta_{N-1} b) \Delta_k b, \end{aligned} \tag{15}$$

the last equality by the definition of the paraproduct. By property (7), we see that all the summands in (15) are cancelled except the last one, more precisely, $E_k(\Delta_0 g\, \Delta_1 b \cdots \Delta_{N-1} b) = 0$ for all $0 \le k < N$, and $E_N(\Delta_0 g\, \Delta_1 b \cdots \Delta_{N-1} b) = \Delta_0 g \Delta_1 b \cdots \Delta_{N-1} b$. Hence

$$\Pi^N_{b_N} g_N = \Delta_0 g\, \Delta_1 b \cdots \Delta_{N-1} b\, \Delta_k b.$$

This finishes the proof of the lemma.

3. Boundedness of P_b, or T_ω vs P_ω.

In Section 2.2 we introduced the operator P_b, associated to a given function $b \in \mathrm{BMO}$. Formally the operator is

$$P_b g = \sum_{k\in\mathbb{Z}} \Delta_k g \prod_{j=k+1}^{\infty} (1+\Delta_j b)\,. \tag{16}$$

We are multiplying the Haar decomposition at level j of g, $\Delta_j g$, by an infinite product. Suppose that for each j, the product is a function. Then P_b will be a particular case of the *model operator*

$$Tf(x) = \sum_{k\in\mathbb{Z}} \omega_k(x)\, \Delta_k f(x)\,, \tag{17}$$

where the ω_k's are a sequence of functions. For example, if $\omega_k = 1$ then T will be the identity operator.

Let us consider the following non trivial examples. Fix ω, a positive locally integrable function (a *weight*); consider

1) $\omega'_k(x) = \dfrac{\omega(x)}{E_k\omega(x)}$,

2) $\omega''_k(x) = \dfrac{\omega(x)}{E_{k+1}\omega(x)}$,

and call T_ω and P_ω the operators associated to the first and second examples respectively.

In Section 4 we will see that there is a very nice correspondence between some classes of weights ω and functions b in BMO. Under this correspondence

$$P_b = P_\omega\,.$$

Therefore it is enough to study P_ω.

Suppose T_ω is a bounded operator in $L^p(\mathbb{R}^n)$. It can be checked with no difficulty (computing the action of T_ω on Haar functions), that the weight ω must satisfy the so called *dyadic reverse Hölder p condition* (RH_p^d condition, see definition in the next section).

We will prove the following theorem,

Theorem 4 (Theorem I). *Given a dyadic doubling weight ω (defined in the next Section), for $1 < p < \infty$ and $1/p + 1/q = 1$, the following properties are equivalent,*

i) T_ω *is bounded in* $L^p(\mathbb{R}^n)$,

ii) P_ω *is bounded in* $L^p(\mathbb{R}^n)$,

iii) $\omega \in RH_p^d(\mathbb{R}^n)$,

iv) M_ω *is bounded in* $L^q(\mathbb{R}^n)$,

v) S_ω *is bounded in* $L^q(\mathbb{R}^n)$,

where M_ω *and* S_ω *are weighted maximal and square functions, namely,* $M_\omega f = \sup_j E_j^\omega |f|$, *and* $S_\omega f = (\sum_j |\Delta_j^\omega f|^2)^{1/2}$, *where* $E_j^\omega f = E_j f\omega / E_j\omega$, *and* $\Delta_j^\omega = E_{j+1}^\omega - E_j^\omega$.

This will be enough to prove Theorem 2, once the relation $\omega \leftrightarrow b$ is understood. Under this correspondence, a weight ω is dyadic doubling if and only if there exists a constant $0 < \varepsilon < 1$ such that $|\Delta_j b| \leq 1-\varepsilon$ for all $j \in \mathbb{Z}$. The condition b is of RH_p^d-type means that the corresponding weight $\omega \in RH_p^d$.

We could also give sufficient conditions for the model operator (17) to be bounded, when a larger class of weights is considered. That can be found in [P] for $n = 1$, $p = 2$.

The proof of the theorem reduces to the boundedness of some weighted dyadic square functions, which in turn are controlled by a weighted maximal function. These are well known objects. We will present a proof, identical to S. Buckley's proof of the boundedness of the standard dyadic square function on weighted L^2. Our proof will allow enough weights in the game so that not only we capture our L^2 result, but also we have enough room to use a standard *extrapolation* argument to produce the L^p results, following J. L. Rubio de Francia's philosophy that *"there is no L^p but weighted L^2"*.

Once the boundedness on L^p of the weighted square functions is established, we get our results using classical Littlewood-Paley theory.

First we will recall some definitions and standard results. Then we will prove Theorem 4, and finally we will clarify the relation between ω and b, and produce the examples of Theorem 3.

3.1. Preliminaries.

3.1.1. Weights.

Recall that throughout this paper, we are using "C" to indicate a constant that depends only on p and the dimension n. $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ indicates the set of all dyadic cubes in $\mathbb{R}^n$. For any $Q \in \mathcal{D}$, $\mathcal{D}(Q)$ is the collection of proper dyadic subcubes of Q, $\tilde{Q}$ is the *parent* of Q (the smallest dyadic cube properly containing Q). For any weight ω and set S, $\omega(S)$ denotes the integral of ω over S, $|S|$ denotes the Lebesgue measure of S, and $m_S\omega = \omega(S)/|S|$. Unless otherwise specified, $1 < p < \infty$, but p is otherwise arbitrary.

Definition. *We say that ω is a dyadic doubling weight, if there exists a constant $C > 0$ such that $\omega(\tilde{Q}) \leq C\,\omega(Q)$ for all dyadic cubes Q, $\tilde{Q}$ is the parent of Q.*

Definition. *We say that ω is an A_p^d weight if there exists $C > 0$ such that*

$$\left(\frac{1}{|Q|}\int_Q \omega\right)\left(\frac{1}{|Q|}\int_Q \omega^{-1/p-1}\right)^{p-1} \leq C\,, \qquad \text{for all } Q \in \mathcal{D}.$$

Definition. *We say that ω is an A_∞^d weight if there exists $C > 0$, such that for all $Q \in \mathcal{D}$, we have*

$$\frac{1}{|Q|}\int_Q \omega \leq C \exp\left(\frac{1}{|Q|}\int_Q \log\omega\right).$$

Definition. *We say that $\omega \in RH_p^d$ (dyadic reverse Hölder p) if there exists $C > 0$ such that*

$$\frac{1}{|Q|}\int_Q \omega^p(x)\,dx \leq C\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)^p, \qquad \text{for all } Q \in \mathcal{D}.$$

These are dyadic versions of the classical Muckenhoupt classes. For equivalent definitions of weight classes see [GC-RF] and [CF].

It is clear how to define weight classes with respect to a positive measure σ.

We will assume that all our weights are dyadic doubling, and we will consistently ommit the word dyadic. This is necessary for the theory of such weights to closely mirror the non-dyadic case, for example to get that $\omega \in RH_p^d$ implies that $\omega \in A_\infty^d$. S. Buckley studied a characterization of these classes of weights by summation conditions, [B]. In particular, given a doubling weight ω,

$$\text{(18)} \quad \omega \in A_\infty^d \text{ if and only if } \sum_{Q\in\mathcal{D}(Q_0)} |Q| \left(\frac{m_Q\omega - m_{\tilde{Q}}\omega}{m_{\tilde{Q}}\omega} \right)^2 \leq C\,|Q_0|\,.$$

This is a Carleson condition.

Definition. *We will say that a sequence of numbers, $\{a_Q\}_{Q\in\mathcal{D}}$, is a μ-Carleson sequence, for a given positive measure μ, if for all $Q_0 \in \mathcal{D}$,*

$$\sum_{Q\in\mathcal{D}(Q_0)} \mu(Q)\, a_Q^2 \leq \mu(Q_0)\,.$$

It is clear what is a doubling measure.

Definition. *Given a doubling measure σ, we will say that the weight ω is in $A_\infty^d(d\sigma)$, if the sequence $a_Q = (m_Q^\sigma\omega - m_{\tilde{Q}}^\sigma\omega)/m_{\tilde{Q}}^\sigma\omega$ is a σ-Carleson sequence, where $m_Q^\sigma\omega = \int_Q(\omega/\sigma(Q))\,d\sigma$.*

Definition. *Given a doubling measure σ, we will say that a weight ω is σ-doubling if the measure $d\mu = \omega d\sigma$ is doubling.*

We will say that a measure μ is in $A_\infty^d(d\sigma)$ if $d\mu = \omega d\sigma$, and ω is an $A_\infty^d(d\sigma)$ weight.

We will need the following lemmas

Lemma 4. *Given a doubling measure σ in $A_\infty^d(dx)$, and a doubling measure $\mu \in A_\infty^d(d\sigma)$, then any σ-Carleson sequence is also μ-Carleson.*

Lemma 5. *Given a doubling measure σ in $A_\infty^d(dx)$, and a doubling weight $\mu \in A_\infty^d(d\sigma)$. Then the sequence $(m_Q\mu - m_{\tilde{Q}}\mu)/m_{\tilde{Q}}\mu$ is a μ-Carleson sequence.*

We will prove these lemmas at the end. We will also need a weighted version of Carleson's lemma.

Lemma 6 (Weighted Carleson's lemma). *Given a doubling measure μ in $A^d_\infty(dx)$. Given a μ-Carleson sequence $\{a_Q\}_{Q\in\mathcal{D}}$, and a sequence of positive numbers $\{\lambda_Q\}_{Q\in\mathcal{D}}$ then,*

$$\sum_{Q\in\mathcal{D}} \mu(Q)\, a_Q^2\, \lambda_Q \leq C \int \lambda^*(x)\, d\mu(x)\,,$$

where $\lambda^*(x) = \sup_{x\in Q} \lambda_Q$.

For a proof see [M, p. 273], with the obvious changes.

In particular if $\lambda_Q = (m^\sigma_Q f)^2$ then we can bound the sum of the products by the $L^2(d\mu)$ norm of the weighted maximal function M_σ to be defined in the next section.

3.1.2. Littlewood-Paley theory.

We will use the notation $a \sim b$, for positive numbers a and b, whenever there exists a positive and finite constant C such that $C^{-1}b \leq a \leq Cb$; we will say, in that case, that a and b are *comparable*.

The expectation and difference operators E_j, Δ_j where introduced in Section 2.1.2.

The dyadic maximal and square functions are,

$$Mf(x) = \sup_j E_j|f|(x)\,,$$

and

$$Sf(x) = \Big(\sum_j |\Delta_j f(x)|^2\Big)^{1/2}\,,$$

respectively. It is well known that (see [Ch], [Ga] or [S])

Theorem 5 (Littlewood-Paley). $\|Sf\|_{L^p(dx)} \sim \|f\|_{L^p(dx)}$.

We will be interested in weighted versions of the square and maximal functions. Given a doubling measure σ, define the weighted expectation and difference operators by

$$E_j^\sigma f(x) = \frac{1}{\sigma(Q)} \int_Q f\, d\sigma, \qquad x \in Q \in \mathcal{D}_j \,,$$
$$\Delta_j^\sigma = E_{j+1}^\sigma - E_j^\sigma \,.$$

Define the corresponding maximal and square functions, M_σ and S_σ

$$M_\sigma f(x) = \sup_j E_j^\sigma |f|(x)\,,$$
$$S_\sigma f(x) = \Big(\sum_j |\Delta_j^\sigma f(x)|^2 \Big)^{1/2} .$$

It is certainly true, for doubling measures $\sigma \in A_\infty^d$, that the $L^p(d\sigma)$ norms of the weighted square function and the function are comparable, *i.e.* $\|S_\sigma f\|_{L^p(d\sigma)} \sim \|f\|_{L^p(d\sigma)}$, see [Ga].

The following classical results are known.

Theorem 6. (Coifman-Fefferman). *Given a doubling measure $\sigma \in A_\infty^d$, and a σ-doubling weight ω_0, then M_σ is bounded in $L^p(\omega_0 d\sigma)$ if and only if $\omega_0 \in A_p^d(d\sigma)$.*

For a proof see [CF] with the obvious changes. It is also true that

Theorem 7. *Given a doubling measure $\sigma \in A_\infty^d$, and a σ-doubling weight ω_0, then $\omega_0 \in A_p^d(d\sigma)$ if and only if S_σ is bounded in $L^p(\omega_0 d\sigma)$.*

We will present a proof of this theorem at the end for completeness. This is an extension of the results of S. Buckley, who proves it for $p = 2$ and $d\sigma = dx$ (see [B]). Extrapolation will then give the result for $1 < p < \infty$.

Theorem 8 (Rubio de Francia's extrapolation). *Given a doubling measure $\sigma \in A_\infty^d$, and a σ-doubling weight ω_0, if $\omega_0 \in A_2^d(d\sigma)$ implies that T is bounded in $L^2(\omega_0 d\sigma)$ then $\omega_0 \in A_p^d(d\sigma)$ implies that T is bounded in $L^p(\omega_0 d\sigma)$, for all $1 < p < \infty$.*

For a proof see [GC-RF].

Let us just mention the following tautology:

$$\omega_0 \in A_p^d(d\sigma) \text{ if and only if } \frac{1}{\omega_0} \in RH_q^d(\omega_0 d\sigma)\,, \quad \frac{1}{q}+\frac{1}{p}=1\,.$$

In particular, given ω a doubling weight, let $\omega_0 = 1/\omega$, $d\sigma = \omega\, dx$; then $\omega_0\, d\sigma = dx$. Denote by $S_\omega = S_\sigma$, then using the tautology, the theorem for the square function reads,

$$S_\omega \text{ is bounded in } L^p(dx) \text{ if and only if } \omega \in RH_q^d(dx)\,,$$

which is the equivalence between v) and iii) in Theorem I.

Similarly, denoting by $M_\omega = M_\sigma$, Theorem 6 for the maximal function reads,

$$M_\omega \text{ is bounded in } L^p(dx) \text{ if and only if } \omega \in RH_q^d(dx)\,,$$

which is the equivalence of iv) and iii) in Theorem I.

3.2. P_ω vs T_ω - Proof of Theorem I.

After the remarks at the end of previous section, only the equivalence of i), ii) and iii) are left in the proof of Theorem I.

Given a doubling weight ω, define at least formally the operators

$$P_\omega f(x) = \sum_j \frac{\omega(x)}{E_{j+1}\omega(x)}\, \Delta_j f(x)\,,$$

$$T_\omega f(x) = \sum_j \frac{\omega(x)}{E_j\omega(x)}\, \Delta_j f(x)\,.$$

As we pointed out before, if T_ω is bounded in $L^p(\mathbb{R}^n)$ then $\omega \in RH_p^d(\mathbb{R}^n)$. Computing formally the adjoints with respect to the ordinary pairing, we get that

$$P_\omega^* f = \sum_j \Delta_j \Big(\frac{\omega f}{E_{j+1}\omega}\Big)\,,$$

$$T_\omega^* f = \sum_j \Delta_j \Big(\frac{\omega f}{E_j\omega}\Big)\,.$$

Certainly if we can prove that these "adjoints" are well defined and bounded operators in $L^q(\mathbb{R}^n)$, then the operators T_ω and P_ω will be well defined themselves and bounded in $L^p(\mathbb{R}^n)$.

These suggest us to introduce the following cousins of the weighted square function S_σ, defined in the preliminaries. Define

$$S'_\sigma f(x) = \Big(\sum_j \Big|\Delta_j\Big(\frac{\sigma f}{E_{j+1}\sigma}\Big)(x)\Big|^2\Big)^{1/2},$$

$$S''_\sigma f(x) = \Big(\sum_j \Big|\Delta_j\Big(\frac{\sigma f}{E_j\sigma}\Big)(x)\Big|^2\Big)^{1/2}.$$

Proposition 1. *Given a doubling measure $\sigma \in A^d_\infty(dx)$, and a doubling measure $\mu \in A^d_2(d\sigma)$, then S'_σ and S''_σ are bounded in $L^2(d\mu)$.*

We will prove it at the end.

If that is the case, then by the Extrapolation Theorem 8, we will conclude that

Corollary 1. *Given a doubling measure $\sigma \in A^d_\infty(dx)$, and a doubling measure $\mu \in A^d_q(d\sigma)$, then S'_σ and S''_σ are bounded in $L^q(d\mu)$.*

In particular, let ω be a doubling weight, let $d\sigma = \omega dx$, $\omega_0 = 1/\omega$, and $d\mu = \omega_0 d\sigma = dx$, assume that $\omega \in RH^d_p(dx)$ (if and only if $1/\omega \in A^d_q(\omega dx)$); and let us denote $S'_\omega = S'_\sigma$, $S''_\omega = S''_\sigma$, then by Corollary 1,

$$\begin{aligned} \|S'_\omega f\|_{L^q(dx)} &\le C\,\|f\|_{L^q(dx)}\,, \\ \|S''_\omega f\|_{L^q(dx)} &\le C\,\|f\|_{L^q(dx)}\,. \end{aligned} \tag{19}$$

Notice that

$$\Delta_j(P^*_\omega f) = \Delta_j\Big(\frac{f\omega}{E_{j+1}\omega}\Big),$$

$$\Delta_j(T^*_\omega f) = \Delta_j\Big(\frac{f\omega}{E_j\omega}\Big).$$

Therefore, if S is the standard dyadic square function, $S(P^*_\omega f) = S'_\omega f$ and $S(T^*_\omega f) = S''_\omega f$. And by the Littlewood-Paley theory and (19), if $\omega \in RH^d_p(dx)$ then

$$\|P^*_\omega f\|_q \sim \|S(P^*_\omega f)\|_q = \|S'_\omega f\|_q \le C\,\|f\|_q\,.$$

Similarly $\|T^*_\omega f\|_q \le C\|f\|_q$.

This implies that P^*_ω and T^*_ω are bounded operators on $L^q(dx)$; so certainly P_ω and T_ω will be bounded on $L^p(dx)$, provided $\omega \in RH^d_p(dx)$.

We pointed out before that it is trivial to check that if T_ω is bounded in L^p then $\omega \in RH^d_p(dx)$. For P_ω the same holds; just recall that since ω is a doubling weight, in particular $E_{j+1}\omega \sim E_j\,\omega$, uniformly on j. It is also true that T_ω and S_ω are simultaneously bounded in L^p.

Therefore we almost proved Theorem I. After our previous remarks, the only missing step in the proof of this theorem is the proof of the Proposition 1.

PROOF OF PROPOSITION 1. First note that we can compute the $L^2(d\mu)$ norms of the square functions, since

$$
\begin{aligned}
\|(\sum_j |\Delta_j g|^2)^{1/2}\|^2_{L^2(d\mu)} &= \|(\sum_j |E_{j+1}g - E_j g|^2)^{1/2}\|^2_{L^2(d\mu)} \\
&= \sum_{Q\in\mathcal{D}} \mu(Q)\, |m_Q g - m_{\tilde{Q}} g|^2 ,
\end{aligned}
\tag{20}
$$

where $m_Q g = \int_Q g/|Q|$.

We will get

$$
\begin{aligned}
\|S_\sigma f\|^2_{L^2(d\mu)} &= \sum_{Q\in\mathcal{D}} \mu(Q)\Big(\frac{m_Q f\sigma}{m_Q \sigma} - \frac{m_{\tilde{Q}} f\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 ,\\
\|S'_\sigma f\|^2_{L^2(d\mu)} &= \sum_{Q\in\mathcal{D}} \mu(Q)\Big(\frac{m_Q f\sigma}{m_Q \sigma} - \frac{1}{2^n}\sum_{Q'\in S(\tilde{Q})} \frac{m_{Q'} f\sigma}{m_{Q'}\sigma}\Big)^2 ,\\
\|S''_\sigma f\|^2_{L^2(d\mu)} &= \sum_{Q\in\mathcal{D}} \mu(Q)\Big(\frac{m_Q f\sigma}{m_{\tilde{Q}} \sigma} - \frac{m_{\tilde{Q}} f\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 ,
\end{aligned}
$$

where $m_Q\sigma = \sigma(Q)/|Q|$, and $S(\tilde{Q})$ denotes the subset of dyadic cubes which are direct children of $\tilde{Q}$.

Let us first compare S_σ and S''_σ. Adding and subtracting $m_Q f\sigma / m_{\tilde{Q}}\sigma$ we get that

$$\|S_\sigma f\|^2_{L^2(d\mu)} \le C\, \|S''_\sigma f\|^2_{L^2(d\mu)}$$

$$
\begin{aligned}
(21) \qquad & + C \sum_{Q\in\mathcal{D}} \mu(Q)\Big(\frac{m_Q f\sigma}{m_Q\sigma}\Big)^2 \Big(\frac{m_Q\sigma - m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 \\
& = C\,\|S''_\sigma f\|^2_{L^2(d\mu)} + C\,W_1 \,.
\end{aligned}
$$

The last summand on the right hand side is bounded by a constant times $\|M_\sigma f\|^2_{L^2(d\mu)}$. This is a consequence of the weighted Carleson's lemma (*cf.* Section 3.1.1). Because by Buckley's summation condition (18), and the weight Lemma 4, the sequence $(m_Q\sigma - m_{\tilde{Q}}\sigma)/m_{\tilde{Q}}\sigma$ is a μ-Carleson sequence, since by hypothesis $\mu \in A_2^d(d\sigma) \subset A_\infty^d(d\sigma)$, and μ is doubling.

Similarly we conclude that

$$
\|S''_\sigma f\|^2_{L^2(d\mu)} \le C\,\|S_\sigma f\|^2_{L^2(d\mu)} + C\,\|M_\sigma f\|^2_{L^2(d\mu)} \,.
$$

Let us now compare S'_σ and S_σ. Adding and subtracting the correct terms, we get

$$
\begin{aligned}
\|S'_\sigma f\|^2_{L^2(d\mu)} \le{}& C\,\|S_\sigma f\|^2_{L^2(d\mu)} \\
& + C \sum_{Q\in\mathcal{D}} \mu(Q)\Big(\frac{m_{\tilde{Q}} f\sigma}{m_{\tilde{Q}}\sigma} - \frac{1}{2^n}\sum_{Q'\in S(\tilde{Q})} \frac{m_{Q'} f\sigma}{m_{Q'}\sigma}\Big)^2 .
\end{aligned}
$$

But

$$
m_{\tilde{Q}} f\sigma = \frac{1}{2^n} \sum_{Q'\in S(\tilde{Q})} m_{Q'} f\sigma \,.
$$

Therefore we can bound the last summand by

$$
S(f,\sigma) = \sum_{Q\in\mathcal{D}} \mu(Q)\,\Big(\frac{1}{2^n}\sum_{Q'\in S(\tilde{Q})} \frac{m_{Q'} f\sigma}{m_{Q'}\sigma}\,\frac{m_{Q'}\sigma - m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 .
$$

By hypothesis μ is doubling, *i.e.* $\mu(Q) \sim \mu(\tilde{Q})$, for all $Q \in S(\tilde{Q})$; and clearly

$$
S(f,\sigma) \le C \sum_{Q\in\mathcal{D}} \mu(\tilde{Q})\,\Big(\frac{m_Q\sigma - m_Q\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 \Big(\frac{m_Q f\sigma}{m_Q\sigma}\Big)^2 = W_1 \,.
$$

But W_1 is bounded by a constant multiple of $\|M_\sigma f\|^2_{L^2(d\mu)}$. We conclude that

$$
\|S'_\sigma f\|^2_{L^2(d\mu)} \le C\,\|S_\sigma f\|^2_{L^2(d\mu)} + C\,\|M_\sigma f\|^2_{L^2(d\mu)} \,.
$$

This finishes the proof of the proposition, since by hypothesis we know that M_σ is bounded on $L^2(d\mu)$ and so is then S_σ by Theorems 6 and 7.

Thus, Theorem I has been proved.

3.3. Proof of the weighted square function theorem d'après Buckley.

Let us denote the mean value of a function f on a cube Q with respect to the measure σ by $m^\sigma_Q f$. Hence, $m^\sigma_Q f = m_Q f\sigma / m_Q \sigma$.

With this notation the weighted square function S_σ becomes

$$S_\sigma f(x) = \Big(\sum_{x \in Q \in \mathcal{D}} |m^\sigma_Q f - m^\sigma_{\tilde{Q}} f|^2 \Big)^{1/2} .$$

Computing the $L^2(d\mu)$ norm (see (20)), we get

$$\|S_\sigma f\|^2_{L^2(d\mu)} = \sum_{Q \in \mathcal{D}} \mu(Q) \, (m^\sigma_Q f - m^\sigma_{\tilde{Q}} f)^2 .$$

We want to prove that if σ is a doubling weight in $A^d_\infty(dx)$, and $\omega_0 \in A^d_2(d\sigma)$ and is σ-doubling, then S_σ is a bounded operator in $L^2(d\mu)$, where $d\mu = \omega_0 d\sigma$.

We know that under those conditions the weighted maximal function M_σ is bounded in $L^2(d\mu)$ (see Theorem 6). Therefore it will be enough to control the square function with the maximal function. More precisely, we will show, exactly as in Buckley's proof for $d\sigma = dx$, $d\mu = \omega \, dx$, doubling $\omega \in A^d_2(dx)$ (see [B]), that

$$(22) \quad \|S_\sigma f\|^2_{L^2(d\mu)} \le C \, \|M_\sigma f\|^2_{L^2(d\mu)} + C \, \|M_\sigma f\|_{L^2(d\mu)} \|S_\sigma f\|_{L^2(d\mu)} .$$

With a bootstrapping argument, and Theorem 6 we get

$$\|S_\sigma f\|^2_{L^2(d\mu)} \le C \, \|f\|^2_{L^2(d\mu)} ,$$

provided that we can ensure that the $L^2(d\mu)$ norm of $S_\sigma f$ is finite.

We will show that (22) holds uniformly for a finite version of the square function. More precisely, fix $N > 0$ and Q_0 any cube. Denote by $\mathcal{D}^N(Q_0)$ those cubes $Q \in \mathcal{D}(Q_0)$ such that their side length $l(Q) \geq 2^{-N} l(Q_0)$. Define

$$S_\sigma^{N,Q_0} f(x) = \Big(\sum_{x \in Q \in \mathcal{D}^N(Q_0)} |m_Q^\sigma f - m_{\tilde{Q}}^\sigma f|^2 \Big)^{1/2} .$$

Clearly now $\|S_\sigma^{N,Q_0} f\|_{L^2(d\mu)} < \infty$. Now we are allowed to complete the bootstrapping argument to conclude that

$$\|S_\sigma^{N,Q_0} f\|^2_{L^2(d\mu)} \leq C \, \|f\|^2_{L^2(d\mu)} \,, \qquad \text{for all } N > 0, \text{ for all } Q_0 \,.$$

Finally letting N go to infinity, and then adding over the cubes in a given generation, $Q_0 \in \mathcal{D}_M$, and letting $M \to -\infty$, we will get the desired boundedness of S_σ.

Fix $N > 0$ and the base cube Q_0. To simplify the notation, let us drop the superscripts N and Q_0 from the operator and the dyadics. Since μ is a doubling measure, then

$$\|S_\sigma f\|^2_{L^2(d\mu)} \leq C \sum_{Q \in \mathcal{D}} \mu(\tilde{Q}) \, (m_Q^\sigma f - m_{\tilde{Q}}^\sigma f)^2 = W \,.$$

Add and subtract

$$a_Q = \frac{m_Q f\sigma}{m_{\tilde{Q}} \sigma} = \frac{m_Q \sigma}{m_{\tilde{Q}} \sigma} \, m_Q^\sigma f$$

then,

$$\begin{aligned} W &\leq \sum_{Q \in \mathcal{D}} \mu(\tilde{Q}) \, (m_Q^\sigma f)^2 \Big(\frac{m_Q \sigma - m_{\tilde{Q}} \sigma}{m_{\tilde{Q}} \sigma} \Big)^2 \\ &\quad + \sum_{Q \in \mathcal{D}} \mu(\tilde{Q}) \, (a_Q - m_{\tilde{Q}}^\sigma f)^2 \\ &\leq C \, W_1 + W_2 \,. \end{aligned}$$

We already showed in (21) that $W_1 \leq C \, \|M_\sigma f\|^2_{L^2(d\mu)}$. Recall the fact that given m numbers a_j, and denoting their mean value by $A = \sum_{j=1}^m a_j / m$ then, $\sum_{j=1}^m (a_j - A)^2 = C \sum_{j=1}^m (a_j^2 - A^2)$. In particular notice that $\sum_{Q \in S(\tilde{Q})} a_Q / 2^n = m_{\tilde{Q}}^\sigma f$. Let $m = 2^n$, $A = m_{\tilde{Q}}^\sigma f$, $a_j = a_Q$; hence

$$W_2 = C \sum_{Q \in \mathcal{D}} \mu(\tilde{Q}) \, (a_Q^2 - (m_{\tilde{Q}}^\sigma f)^2) \,.$$

Next, add and subtract $m_Q\mu\,(m^\sigma_Q f)^2/m_{\tilde{Q}}\mu$, then

$$\begin{aligned} W_2 &= C\sum_{Q\in\mathcal{D}} \mu(\tilde{Q})\,(m^\sigma_Q f)^2\Big(\Big(\frac{m_Q\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 - \frac{m_Q\mu}{m_{\tilde{Q}}\mu}\Big)\Big) \\ &\quad + C\sum_{Q\in\mathcal{D}} (2^n\mu(Q)\,(m^\sigma_Q f)^2 - \mu(\tilde{Q})\,(m^\sigma_{\tilde{Q}} f)^2) \\ &= W_3 + W_4 \,. \end{aligned}$$

The second summand is a telescoping sum, namely, $W_4 = \sum_k(b_k - b_{k-1})$, where $b_k = \sum_{Q\in\mathcal{D}_k(Q_0)} 2^n\mu(Q)(m^\sigma_Q f)^2$, and $\mathcal{D}_k(Q_0)$ is the k^{th} generation of the dyadic decomposition of the base cube Q_0. Clearly for all $k > 0$,

$$b_k \le C\int_{Q_0} |M_\sigma f(x)|^2\,d\mu(x) \le C\,\|M_\sigma f\|^2_{L^2(d\mu)}\,.$$

Hence $|W_4| \le \sup_{k>0}|b_k| \le C\,\|M_\sigma f\|^2_{L^2(d\mu)}$.

Observe that adding and subtracting 1, we get that,

$$\begin{aligned} W_3 &= C\sum_{Q\in\mathcal{D}} \mu(\tilde{Q})\,(m^\sigma_Q f)^2\Big(\Big(\frac{m_Q\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 - 1\Big) \\ &\quad + C\sum_{Q\in\mathcal{D}} \mu(\tilde{Q})\,(m^\sigma_Q f)^2\,\frac{m_{\tilde{Q}}\mu - m_Q\mu}{m_{\tilde{Q}}\mu} \\ &= W_5 + W_6 \,. \end{aligned}$$

Notice that

$$\Big(\frac{m_Q\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 - 1 = 2\,\frac{m_Q\sigma - m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\sigma} + \Big(\frac{m_Q\sigma - m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 .$$

Therefore,

$$\begin{aligned} W_5 &= \sum_{Q\in\mathcal{D}} \mu(\tilde{Q})\,(m^\sigma_Q f)^2\Big(\frac{m_Q\sigma - m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\sigma}\Big)^2 \\ &\quad + 2\sum_{Q\in\mathcal{D}} \mu(\tilde{Q})\,(m^\sigma_Q f)^2\Big(\frac{m_Q\sigma - m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\sigma}\Big) \\ &= W_1 + 2\,W_7 \,. \end{aligned}$$

We already showed that $W_1 \leq C\|M_\sigma f\|^2_{L^2(d\mu)}$.

By mean value properties,

$$\sum_{Q\in S(\tilde{Q})} \frac{m_Q\sigma - m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\sigma} = 0\,.$$

Hence,

$$W_7 \leq C \sum_{Q\in\mathcal{D}} \mu(\tilde{Q})\, \frac{m_{\tilde{Q}}\sigma - m_Q\sigma}{m_{\tilde{Q}}\sigma}\, ((m^\sigma_Q f)^2 - (m^\sigma_{\tilde{Q}} f)^2)\,.$$

Applying the Cauchy-Schwartz inequality, we conclude that

$$W_7 \leq C\, W_1^{1/2}\, \|S_\sigma f\|_{L^2(d\mu)} \leq C\, \|M_\sigma f\|_{L^2(d\mu)} \|S_\sigma f\|_{L^2(d\mu)}\,.$$

We are left with the term W_6. This time

$$\sum_{Q\in\mathcal{D}} \frac{m_{\tilde{Q}}\mu - m_Q\mu}{m_{\tilde{Q}}\mu} = 0$$

Hence,

$$\begin{aligned} W_6 &= C \sum_{\tilde{Q}\in\mathcal{D}} \mu(\tilde{Q}) \sum_{Q\in S(\tilde{Q})} \frac{m_{\tilde{Q}}\mu - m_Q\mu}{m_{\tilde{Q}}\mu}\, ((m^\sigma_Q f)^2 - (m^\sigma_{\tilde{Q}} f)^2) \\ &\leq C \Big(\sum_{Q\in\mathcal{D}} \mu(\tilde{Q}) \Big(\frac{m_{\tilde{Q}}\mu - m_Q\mu}{m_{\tilde{Q}}\mu}\Big)^2 (m^\sigma_Q f + m^\sigma_{\tilde{Q}} f)^2\Big)^{1/2} \|S_\sigma f\|_{L^2(d\mu)} \\ &\leq C\, W_8^{1/2}\, \|S_\sigma f\|_{L^2(d\mu)}\,. \end{aligned}$$

But W_8 will be bounded by the $L^2(d\mu)$ norm of the maximal function M_σ, if we can show that the sequence $(m_{\tilde{Q}}\mu - m_Q\mu)/m_{\tilde{Q}}\mu$ is a μ-Carleson sequence. If that is the case, we can apply Carleson's lemma as we did for W_1. But that is exactly the conclusion of Lemma 5.

Hence $W_6 \leq C\, \|M_\sigma f\|_{L^2(d\mu)} \|S_\sigma f\|_{L^2(d\mu)}$.

Putting together all the estimates we get inequality (22), with constants independent of the base cube Q_0 and N. This finishes the proof of the theorem.

3.4. Proof of the weight lemmas.

PROOF OF LEMMA 4. We are given a σ-Carleson sequence $\{b_Q\}_{Q\in\mathcal{D}}$; therefore

$$\sum_{Q\in\mathcal{D}(Q_0)} \sigma(Q)\, b_Q^2 \le C\,\sigma(Q_0)\,, \qquad \text{for all } Q_0 \in \mathcal{D}\,.$$

This is equivalent to saying that the function $b = \sum_j \Delta_j^\sigma b$ is in dyadic BMO$(d\sigma)$. Where the weighted difference operator is defined by $\Delta_j^\sigma b(x) = m_Q^\sigma b - m_{\tilde{Q}}^\sigma b = b_Q$, for all $x \in Q \in \mathcal{D}_{j+1}$.

For σ-doubling and $A_\infty^d(d\sigma)$ measures it is easy to deduce the John-Niremberg inequality for $\mathrm{BMO}_d(d\sigma)$ (see [G, Chapter VI]), and then use it to deduce the equivalent $L^p(d\sigma)$ characterizations of dyadic BMO$(d\sigma)$, namely

$$\Big(\frac{1}{\sigma(Q)}\int_Q |b - m_Q^\sigma b|^p\, d\sigma\Big)^{1/p} \le C\,\|b\|_{\mathrm{BMO}_d(d\sigma)}\,, \tag{23}$$

for all $Q \in \mathcal{D}$. Fix a cube $Q_0 \in \mathcal{D}$. Define the function

$$b^{Q_0}(x) = (b(x) - m_{Q_0}^\sigma b)\,\chi_{Q_0}(x)\,.$$

Clearly, $b_Q^{Q_0} = m_Q^\sigma b^{Q_0} - m_{\tilde{Q}}^\sigma b^{Q_0} = b_Q$ if $Q \in \mathcal{D}(Q_0)$, and zero otherwise.

Then $S_\sigma b^{Q_0} = (\sum_{x\in Q\in\mathcal{D}(Q_0)} b_Q^2)^{1/2}$; where S_σ is the weighted square function. Therefore

$$\int_{Q_0} S_\sigma^2 b^{Q_0}\, d\mu = \sum_{Q\in\mathcal{D}(Q_0)} \mu(Q)\, b_Q^2\,.$$

The right hand side is the sum we want to estimate, so it is enough to estimate the integral on the left hand side. It is certainly true by the standard Littlewood-Paley theory that S_σ is a bounded operator in $L^p(d\sigma)$ (not $L^p(d\mu)$!!), for $1 < p < \infty$. Hence, noting that $d\mu = \omega_0\, d\sigma$, and using Hölder's inequality we conclude that

$$\int_{Q_0} S_\sigma^2 b^{Q_0}\, d\mu \le \|S_\sigma^2 b^{Q_0}\|_{L^p(d\sigma,Q_0)}\, \|\omega_0\|_{L^q(d\sigma,Q_0)}\,.$$

Since by hypothesis $\omega_0 \in A^d_\infty(d\sigma)$, then there exists $1 < q < \infty$ such that $\omega_0 \in RH^d_q(d\sigma)$ (Gëhring's Theorem, see [Ge]), in particular it is true that for $1/q + 1/p = 1$,

$$\|\omega_0\|_{L^q(d\sigma,Q_0)} \le C\,\sigma(Q_0)^{-1/p}\mu(Q_0)\,.$$

And also by Littlewood-Paley theory for that particular p we can estimate,

$$\begin{aligned}\|S^2_\sigma b^{Q_0}\|_{L^p(d\sigma,Q_0)} &= \Big(\int_{Q_0} S^{2p}_\sigma b^{Q_0}\,d\sigma\Big)^{2/2p}\\ &= \|S_\sigma b^{Q_0}\|^2_{L^{2p}(d\sigma,Q_0)} \le C\,\|b^{Q_0}\|^2_{L^{2p}(d\sigma,Q_0)}\,.\end{aligned}$$

But by definition,

$$\begin{aligned}\|b^{Q_0}\|^2_{L^{2p}(d\sigma,Q_0)} &= C\,\Big(\int_{Q_0} |b - m^\sigma_{Q_0} b|^{2p}\,d\sigma\Big)^{1/p}\\ &\le C\,\|b\|^2_{\mathrm{BMO}(d\sigma)}\,\sigma^{1/p}(Q_0)\,,\end{aligned}$$

where the last inequality follows by remark (23).

Therefore, putting all these facts together, we get the desired inequality for all $Q_0 \in \mathcal{D}$, *i.e.*

$$\sum_{Q\in\mathcal{D}(Q_0)} \mu(Q)\,b_Q^2 \le C\,\mu(Q_0)\,.$$

Therefore the sequence $\{b_Q\}_{Q\in\mathcal{D}}$ is a μ-Carleson sequence, and the lemma is proved.

PROOF OF LEMMA 5. First observe that by hypothesis μ is a doubling measure, hence $m_Q\mu \sim m_{\tilde{Q}}\mu$. Next we want to show that

$$\sum_{Q\in\mathcal{D}(Q_0)} \mu(Q)\Big(\frac{m_Q\mu}{m_{\tilde{Q}}\mu} - 1\Big)^2 \le C\,\mu(Q_0)\,, \qquad \text{for all } Q_0 \in \mathcal{D}\,.$$

Adding and subtracting

$$\frac{m_Q\mu\, m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\mu\, m_Q\sigma}$$

inside the brackets, we can bound the left hand side by

$$\sum_{Q\in\mathcal{D}(Q_0)} \mu(Q)\Big(\frac{m_Q\mu\, m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\mu\, m_Q\sigma}-1\Big)^2+\sum_{Q\in\mathcal{D}(Q_0)} \mu(Q)\Big(\frac{m_Q\mu}{m_{\tilde{Q}}\mu}\Big)^2\Big(\frac{m_{\tilde{Q}}\sigma}{m_Q\sigma}-1\Big)^2.$$

By hypothesis the sequence

$$\frac{m^\sigma_Q\omega_0-m^\sigma_{\tilde{Q}}\omega_0}{m^\sigma_{\tilde{Q}}\omega_0}=\frac{m_Q\mu\, m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\mu\, m_Q\sigma}-1$$

is σ-Carleson, hence is μ-Carleson by Lemma 4. Therefore we can control the first summand by a constant multiple of $\mu(Q_0)$.

Similarly the sequence

$$\frac{m_Q\sigma-m_{\tilde{Q}}\sigma}{m_{\tilde{Q}}\sigma}\sim\frac{m_{\tilde{Q}}\sigma}{m_Q\sigma}-1$$

is dx-Carleson, hence is σ-Carleson by Lemma 4, and is also μ-Carleson, again by Lemma 4. Hence we can also bound the second summand by a constant multiple of $\mu(Q_0)$, since μ is doubling. The lemma is proved.

4. Correspondence $b\leftrightarrow\omega$, and examples.

In this section we will clarify the correspondence between dyadic doubling A^d_∞ weights and a subset of BMO_d. This has been borrowed from [KFP]. For much more about dyadic weight classes see [B].

Fix a cube Q_0, without loss of generality we can assume that $Q_0=[0,1]^n$. With the notation in Section 2.1.2, let

$$\omega_N(x)=\prod_{j=0}^{N}(1+\Delta_j b(x)),$$

where b is a locally integrable function on Q_0. Let $\omega=\prod_{j=0}^\infty(1+\Delta_j b)$, be the weak limit of the partial products $\{\omega_N\}$. The sequence $\{\omega_N\}$ is a (positive) dyadic martingale when $-1<\Delta_j b<1$, since $\int_{Q_0}(\omega_k-\omega_{k-1})(x)\,dx=0$ and $E(\omega:\mathcal{D}_k(Q_0))=\omega_k$, where $\mathcal{D}_k(Q_0)$ is the σ-field generated by all dyadic cubes contained in Q_0 of side length $2^{-k}|Q_0|$.

The necessary and sufficient condition for ω to be a dyadic doubling measure on Q_0 is that there exists $0 < \varepsilon < 1$ such that

$$|\Delta_j b(x)| \le 1 - \varepsilon\,, \qquad \text{for all } x \in Q_0,\ \text{for all } j \in \mathbb{N}\,.$$

(See [KFP] for a proof.)

We can now give another characterization of dyadic $A^d_\infty(Q_0)$, assuming doubling.

Theorem 9 (KFP). *Let b be a locally integrable function on Q_0 and $0 < \varepsilon < 1$, such that $|\Delta_j b(x)| \le 1 - \varepsilon$, for all $x \in Q_0$, and for all $j \in \mathbb{N}$. Then the product $\omega = \prod_{j=0}^{\infty}(1 + \Delta_j b)$ belongs to doubling $A^d_\infty(Q_0)$ if and only if there exists constant $C > 0$ such that for all $Q' \in \mathcal{D}(Q_0)$,*

$$\sum_{Q\in\mathcal{D}(Q')} |Q|\,(m_Q b - m_{\tilde{Q}} b)^2 \le C\,|Q'|\,. \tag{24}$$

For a proof see [KFP].

We have explained how to compute the weight ω given the function b. It is very easy to check, that given a doubling weight ω, $m_{Q_0}\omega = 1$, then,

$$\Delta_j b = \frac{\Delta_j \omega}{E_j\omega} = \frac{E_{j+1}\omega - E_j\omega}{E_j\omega}\,.$$

After this observation, we see that condition (24) is exactly what we called Buckley's summation condition (18).

Notice also, that the product is nothing more than a telescoping product, since $1 + \Delta_j b = E_{j+1}\omega/E_j\omega$; hence $\omega = \prod_{j=0}^{\infty} E_{j+1}\omega/E_j\omega$. And equality holds almost everywhere by Lebesgue's theorem, and since $E_0\omega = 1$.

The martingale condition $E(\omega : \mathcal{D}_k(Q_0)) = \omega_k$ is translated into,

$$E_k\omega = \prod_{j=0}^{k-1}(1 + \Delta_j b)\,.$$

Definition. *For a locally integrable function b on Q_0 as in the previous theorem, we will say that b is of A^d_∞-type on Q_0 if the doubling weight $\omega = \prod_{j=0}^{\infty}(1 + \Delta_j b)$ belongs to $A^d_\infty(Q_0)$.*

It is known (see [M, vol. 2]), that condition (24) on b means that $b \in \mathrm{BMO}_d(Q_0)$. Hence the set of functions b of A_∞^d-type is a subset of $\mathrm{BMO}_d(Q_0)$.

It is known, assuming doubling, that

$$A_\infty^d(Q_0) = \bigcup_{p>1} RH_p^d(Q_0) = \bigcup_{p>1} A_p^d(Q_0).$$

Definition. *For a locally integrable function b as in the previous theorem, we will say that b is of RH_p^d-type (respectively, A_p^d-type), on Q_0 if the associated doubling weight is in $RH_p^d(Q_0)$ (respectively, $A_p^d(Q_0)$).*

For the corresponding summation conditions on b, see [B].

Definition. *Let b be a locally integrable function on $\mathbb{R}^n$, such that $|\Delta_j b| \le 1 - \varepsilon$, for all $j \in \mathbb{Z}$ and some constant $0 < \varepsilon < 1$. We say that b is of A_∞^d-type (respectively, RH_p^d or A_p^d-type) if b is of A_∞^d-type (respectively, RH_p^d or A_p^d-type) on Q uniformly for every $Q \in \mathcal{D}$.*

Assume from now on that b is a locally integrable function and that there exists $0 < \varepsilon < 1$ such that for all $j \in \mathbb{Z}$, $|\Delta_j b| \le 1 - \varepsilon$. Let P_b be the operator defined formally for $g \in L_0^p(Q_0)$ by

$$P_b g = \sum_{k=0}^{\infty} \Delta_k g \prod_{k+1}^{\infty} (1 + \Delta_j b).$$

If b is of A_∞^d-type, then the weight $\omega = \prod_{j=0}^{\infty}(1+\Delta_j b)$ is a well defined doubling weight in $A_\infty^d(Q_0)$; moreover, $\omega / E_{k+1}\omega = \prod_{k+1}^{\infty}(1 + \Delta_j b)$, therefore,

$$P_b g = \sum_{k=0}^{\infty} \frac{\omega}{E_{k+1}\omega} \Delta_k g = P_\omega g ,$$

this last equality by definition of the operator P_ω, when restricted to a cube Q_0. The weight really depends on the base cube Q_0, but for the estimates we only need uniform bounds on these local versions of P_b. The bounds are uniform because they only depend on the $RH_p^d(Q_0)$ constants of the weights, and by definition of RH_p^d-type the local weights have uniform $RH_p^d(Q_0)$ constants.

We can then study P_ω instead of P_b, as we claimed.

Since the paraproduct $\Pi_b f$ is bilinear in b and f, then, $\lambda\Pi_b = \Pi_{\lambda b}$, $\lambda \in \mathbb{N}$. All the algebra (Section 2.2) is valid, and questions about the invertibility of $I - \lambda\Pi_b$ are reduced to questions about the weight ω_λ corresponding to the function λb.

Suppose that b is of RH_p^d-type; we showed that in that case $I - \Pi_b$ is invertible in L^p, with bounded inverse, and that is equivalent to $\omega \in RH_p^d$ and being doubling. By spectral theory (the resolvent is an open subset of $\mathbb{C}$), we know that there exist neighbourhoods of $\lambda = 0$ and $\lambda = 1$ such that $I - \lambda\Pi_b$ is invertible in L^p. Is this true for every $|\lambda| \leq 1$? This question can be translated into a question about weights.

Given b of RH_p^d-type, if $|\lambda| \leq 1$ then certainly λb is of A_∞^d-type and $|\Delta_j(\lambda b)| \leq 1 - \varepsilon$. We are saying that under this correspondence, multiplication by $|\lambda| \leq 1$ preserves doubling and A_∞^d weights. This is different from the standard correspondence $\tilde{\omega} = e^b$ (for example, consider $b = \log|x|$, and $\lambda = -1$, then $\tilde{\omega}_\lambda = 1/|x|$, which is not even locally integrable, not to say in A_∞^d.)

The question is if multiplication by $|\lambda| \leq 1$ preserves doubling RH_p^d weights.

For $-1 < \lambda < 0$ the answer is negative, and it is the content of Theorem III. We would like to know what happens for $0 < \lambda < 1$.

PROOF OF THEOREM III. We will produce examples stemming from the prototype function in BMO, namely, $C\log(1/|x|)$. It is enough to consider a dyadic version of this functions, in dimension $n = 1$. After being requested by electronic mail, S. Buckley produced the same sort of examples.

Fix $I_0 = [0,1]$. Let $I_k = (2^{-k}, 2^{-k+1}]$, $k \geq 1$. For $0 < \varepsilon < 2$, define the step function $b_\varepsilon(x) = k(1-\varepsilon)$, $x \in I_k$. Certainly $|\Delta_j b_\varepsilon| \leq |1-\varepsilon|$, and $b_\varepsilon \in \mathrm{BMO}_d(I_0)$. Hence $\omega_\varepsilon = \prod_{j=0}^\infty (1 + \Delta_j b_\varepsilon)$ is a doubling A_∞^d weight.

REMARK. This is not the case for the standard correspondence, $\tilde{\omega} = e^b$; for example, let $\varepsilon = 1 - \log 2$, then $\tilde{\omega}_\varepsilon$ will correspond to the dyadic version of $1/|x|$.

We can say more about b_ε; it corresponds to the dyadic version of the function $\alpha(\varepsilon)\log(1/|x|)$, where $\alpha(\varepsilon) = (1-\varepsilon)/\log 2$. (Hence the remark, for that ε, $\alpha(\varepsilon) = 1$.) The weight ω_ε corresponding to b_ε can be explicitely computed, $\omega_\varepsilon = C\,(2-\varepsilon)^k$, $x \in I_k$. Hence, ω_ε corresponds to the continuous weight $|x|^{\gamma(\varepsilon)}$, where $\gamma(\varepsilon) = -\log(2-\varepsilon)/\log 2$. Observe that $-1 < \gamma(\varepsilon) < \infty$ for $0 < \varepsilon < 2$. We captured the full range of prototype A^d_∞ weights,.namely $|x|^\gamma$, $\gamma > -1$. It is easy to check that a power $|x|^\gamma$ belongs to RH^d_p if and only if $\gamma > -1/p$, $1 < p < \infty$. Hence, solving for $0 < \varepsilon < 2$ the inequality $\gamma(\varepsilon) > -1/p$, we get that only for $2 - 2^{1/p} < \varepsilon < 2$, $\omega_\varepsilon \in RH^d_p$. Given $-1 < \lambda < 0$, there exists $1 < p < \infty$ such that $\lambda < 1 - 2^{1/p}$. Fix such p. Observe that $\lambda b_\varepsilon = b_{\varepsilon(\lambda)}$, where $\varepsilon(\lambda) = 1 - (1 - \varepsilon)\lambda$. If we could find $2 - 2^{1/p} < \varepsilon < 2$ such that $\varepsilon(\lambda) \le 2 - 2^{1/p}$ then we would be done, because $\omega_\varepsilon \in RH^d_p$ but $\omega_{\varepsilon(\lambda)}$ is not in RH^d_p. We can certainly find such ε; any $2^{1/p} < \varepsilon < 2$ will do the job. Hence we find function $b = b_\varepsilon$ of RH^d_p-type so that $\lambda b = b_{\varepsilon(\lambda)}$ is not. In particular, by Theorems 1 and 2, we conclude that $I - \lambda \Pi_b$ does not have a bounded inverse in L^p, hence $\lambda \Pi_b$ cannot be a contraction in L^p, *i.e.* $\|\Pi_b\|_{p,p} \ge 1/|\lambda|$.

This finishes the proof of Theorem III.

Remark. For these examples, multiplication by $0 < \lambda < 1$ preserves the RH^d_p-type condition on I_0.

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Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces

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1. Introduction.

In the Fourier theory of functions of one variable, it is common to extend a function and its Fourier transform holomorphically to domains in the complex plane $\mathbb{C}$, and to use the power of complex function theory. This depends on first extending the exponential function $e^{ix\xi}$ of the real variables x and ξ to a function $e^{iz\zeta}$ which depends holomorphically on both the complex variables z and ζ.

Our thesis is this. The natural analog in higher dimensions is to extend a function of m real variables monogenically to a function of $m+1$ real variables (with values in a complex Clifford algebra), and to extend its Fourier transform holomorphically to a function of m complex variables. This depends on first extending the exponential function $e^{i\langle \mathbf{x},\xi\rangle}$ of the real variables $\mathbf{x} \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^m$ to a function $e(x,\zeta)$ which depends monogenically on $x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1}$ and holomorphically on $\zeta = \xi + i\eta \in \mathbb{C}^m$.

We explore this thesis for functions Φ whose monogenic extensions are bounded by a constant multiple of $|x|^{-m}$ on a cone

$$C^{\circ}_{\mu+} = \{x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1} : \ x_L > -|\mathbf{x}| \tan\mu\} \ ,$$

on $C^{\circ}_{\mu-} = -C^{\circ}_{\mu+}$, or on $S^{\circ}_{\mu} = C^{\circ}_{\mu+} \cap C^{\circ}_{\mu-}$. The Fourier transforms b of these functions extend holomorphically to bounded functions on certain cones $S^{\circ}_{\nu}(\mathbb{C}^m)$ in $\mathbb{C}^m$ (for all $\nu < \mu$). Conversely, every bounded holomorphic function b on $S^{\circ}_{\mu}(\mathbb{C}^m)$ can be decomposed as $b = b_+ + b_-$, where $b_\pm$ are the Fourier transforms of functions f whose monogenic extensions are bounded by $c\,|x|^{-m}$ on $C^{\circ}_{\nu\pm}$ (for all $\nu < \mu$).

Such functions were studied in [LMcS], where it was shown that if Φ is a right-monogenic function which is bounded by $c\,|x|^{-m}$ on $C^{\circ}_{\mu+}$, then the singular convolution operator T_Φ defined by

$$(T_\Phi u)(x) = \lim_{\delta\to 0+} \int_\Sigma \Phi(x + \delta e_L - y)\, n(y)\, u(y)\, dS_y$$

is a bounded linear operator on $L_p(\Sigma)$ for $1 < p < \infty$. Here Σ is a Lipschitz surface consisting of all the points $x = \mathbf{x} + g(\mathbf{x})\, e_L \in \mathbb{R}^{m+1}$, where $\mathbf{x} \in \mathbb{R}^m$, and g is a real-valued Lipschitz function which satisfies $\|\nabla g\|_\infty \le \tan\omega$ for some $\omega < \mu$. We have embedded $\mathbb{R}^m$ in a complex Clifford algebra with at least m generators in the usual way, and identified the extra basis element e_L of $\mathbb{R}^{m+1}$ with either another generator such as e_{m+1}, or with the identity e_0. We use Clifford multiplication in the above integrand, in which $n(y)$ denotes the unit normal (which is defined at almost all $y \in \Sigma$).

So the Fourier transform b of Φ can be thought of as the Fourier multiplier corresponding to T_Φ. But we can also think of the mapping from b to $T_\Phi \in \mathcal{L}(L_p(\Sigma))$ as giving us a bounded H_∞ functional calculus of a differential operator

$$-i\,\mathbf{D}_\Sigma = \sum_{k=1}^{m} -i\, e_k\, D_{k,\Sigma}\ ,$$

and write

$$T_\Phi = b(-i\,\mathbf{D}_\Sigma) = b(-i\,D_{1,\Sigma}, -i\,D_{2,\Sigma}, \dots, -i\,D_{m,\Sigma})\,.$$

Such functional calculi are studied at length later in this paper. The operators $D_{k,\Sigma}$ are given by

$$D_{k,\Sigma}\, u = \frac{\partial U}{\partial x_k}\Big|_\Sigma$$

when u is the restriction to Σ of a function U which is left-monogenic on a neighbourhood of Σ.

Not surprisingly, $\mathbf{D}_\Sigma$ is the operator considered previously by Murray [M] and McIntosh [McI], when using Clifford analysis to prove the L_p-boundedness of the Cauchy singular integral C_Σ on Σ. For, if Σ is parametrized by $x = \mathbf{s} + g(\mathbf{s})\, e_L$, $\mathbf{s} \in \mathbb{R}^m$, then

$$(\mathbf{D}_\Sigma\, e_L u)(\mathbf{s} + g(\mathbf{s})\, e_L) = (e_L - \mathbf{D}g)^{-1}\mathbf{D}_s\, u(\mathbf{s} + g(\mathbf{s})\, e_L)\,, \quad u \in W_p^1(\Sigma)\,.$$

It is not easy to extend a given function monogenically from a domain in $\mathbb{R}^m$ to a domain in $\mathbb{R}^{m+1}$. In particular, it is not easy to tell whether a given function Φ defined on $\mathbb{R}^m \setminus \{0\}$ extends to a right-monogenic function which is bounded by $c|x|^{-m}$ on S_μ°, and hence whether the results of [LMcS] can be used to conclude that the singular convolution operator T_Φ is a bounded linear operator on $L_p(\Sigma)$. The use of Fourier theory helps. For example, consider the functions defined on $\mathbb{R}^m \setminus \{0\}$ by

$$\Phi_k(\mathbf{x}) = -\frac{2\, x_k}{\sigma_m\, |\mathbf{x}|^{m+1}}\,.$$

Their Fourier transforms are $r_k(\xi) = i\, \xi_k/|\xi|$, which extend holomorphically to bounded functions on the subsets $S_\mu^\circ(\mathbb{C}^m)$ of $\mathbb{C}^m$ (as we shall see). Therefore the functions Φ_k extend monogenically to functions satisfying the appropriate bounds on S_μ°, and the corresponding singular convolution operators $R_{k,\Sigma}$ are bounded on $L_p(\Sigma)$ for $1 < p < \infty$. These operators can be thought of as *Riesz transforms* on Σ. They satisfy $R_{j,\Sigma}R_{k,\Sigma} = R_{k,\Sigma}R_{j,\Sigma}$, $\sum e_k R_{k,\Sigma} = C_\Sigma$ and $\sum(R_{k,\Sigma})^2 = -I$.

As other examples, take $b_s(\xi) = |\xi|^{2is}$ for s real. Such b_s extend holomorphically to bounded functions on $S_\mu^\circ(\mathbb{C}^m)$, so their inverse Fourier transforms extend monogenically to functions satisfying the appropriate bounds, and the corresponding singular convolution operators are bounded on $L_p(\Sigma)$ for $1 < p < \infty$. These operators are purely imaginary powers of $\mathbf{D}_\Sigma^2$, which can be thought of as the negative of the Laplacian on Σ. See Sections 6 to 8.

In Section 7, we indicate the kind of application of these results that we have in mind by considering a boundary value problem for harmonic functions.

Let us briefly recall the main results from [McQ] concerning Fourier transforms of holomorphic functions defined on sectors in the complex plane. This material is generalized to higher dimensions in Sections 3 and 4.

For $0 < \mu \le \pi/2$, define the sectors

$$S^\circ_{\mu^+}(\mathbb{C}) = \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \mu\}, \qquad S^\circ_{\mu^-}(\mathbb{C}) = -S^\circ_{\mu^+}(\mathbb{C}),$$

and the cones

$$C^\circ_{\mu^+}(\mathbb{C}) = \{Z = X + iY \in \mathbb{C} :\ Z \neq 0,\ Y > |X| \tan \mu\},$$
$$C^\circ_{\mu^-}(\mathbb{C}) = -C^\circ_{\mu^+}(\mathbb{C}),$$

and also the double sector

$$S^\circ_\mu(\mathbb{C}) = S^\circ_{\mu^+}(\mathbb{C}) \cup S^\circ_{\mu^-}(\mathbb{C}) = C^\circ_{\mu^+}(\mathbb{C}) \cap C^\circ_{\mu^-}(\mathbb{C}).$$

Let $H_\infty(S^\circ_{\mu^+}(\mathbb{C}))$ be the Banach space of bounded complex-valued holomorphic functions defined on $S^\circ_{\mu^+}(\mathbb{C})$, and let $K(C^\circ_{\mu^+}(\mathbb{C}))$ be the Banach space of complex-valued holomorphic functions Φ for which $\sup\{|Z\,\Phi(Z)| :\ Z \in C^\circ_{\mu^+}(\mathbb{C})\} < +\infty$.

For every function $\Phi \in K(C^\circ_{\mu^+}(\mathbb{C}))$, there is a unique holomorphic function B defined on $S^\circ_{\mu^+}(\mathbb{C})$ which satisfies Parseval's identity

$$\frac{1}{2\pi}\int_0^{+\infty} B(\lambda)\,\hat{u}(-\lambda)\,d\lambda = \lim_{\alpha \to 0^+} \int_{\mathbb{R}} \Phi(X + i\alpha)\,u(X)\,dX$$
$$= \lim_{\varepsilon \to 0}\Big(\int_{|X| \ge \varepsilon} \Phi(X)\,u(X)\,dX + \Phi_1(\varepsilon)\,u(0)\Big),$$

for all u in the Schwartz space $\mathcal{S}(\mathbb{R})$, where $\Phi_1(\varepsilon) = \int_{\delta(\varepsilon)} \Phi(Z)\,dZ$, the integral being along a contour $\delta(\varepsilon)$ from $-\varepsilon$ to ε in $C^\circ_{\mu^+}$. Moreover, if $0 < \nu < \mu$, then $B \in H_\infty(S^\circ_{\nu^+}(\mathbb{C}))$, and

$$\|B\|_\infty \le c_{\nu,\mu} \sup\{|Z\,\Phi(Z)| :\ Z \in C^\circ_{\mu^+}(\mathbb{C})\}.$$

Conversely, for every function $B \in H_\infty(S^\circ_{\mu^+}(\mathbb{C}))$, there is a unique holomorphic function Φ defined on $C^\circ_{\mu^+}(\mathbb{C})$ which satisfies Parseval's identity. Moreover, if $0 < \nu < \mu$, then $\Phi \in K(C^\circ_{\nu^+}(\mathbb{C}))$, and

$$\sup\{|Z\,\Phi(Z)| :\ Z \in C^\circ_{\nu^+}(\mathbb{C})\} \le c_{\nu,\mu}\,\|B\|_\infty .$$

We write $B = \mathcal{F}(\Phi)$, and call B the *Fourier transform* of Φ, and we write $\Phi = \mathcal{G}(B)$, and call Φ the *inverse Fourier transform* of B. Also let $\Phi_1 = \mathcal{G}_1(B)$.

Similar results hold when $C^\circ_{\mu+}(\mathbb{C})$ is replaced by $C^\circ_{\mu-}(\mathbb{C})$ and $S^\circ_{\mu+}(\mathbb{C})$ is replaced by $S^\circ_{\mu-}(\mathbb{C})$, provided the limit in α is taken over negative α, and the contour $\delta(\varepsilon)$ from $-\varepsilon$ to ε is in $C^\circ_{\mu-}$.

Characterization of the inverse Fourier transform of functions in $H_\infty(S^\circ_\mu(\mathbb{C}))$ is slightly more complicated.

Define $\chi_{\mathrm{Re}>0} \in H_\infty(S^\circ_\mu(\mathbb{C}))$ by $\chi_{\mathrm{Re}>0}(\lambda) = 1$ when $\mathrm{Re}\,\lambda > 0$, and $\chi_{\mathrm{Re}>0}(\lambda) = 0$ when $\mathrm{Re}\,\lambda < 0$. Similarly define $\chi_{\mathrm{Re}<0}$.

Now consider $B \in H_\infty(S^\circ_\mu(\mathbb{C}))$. Write $B = B_+ + B_-$, where

$$B_+ = B\,\chi_{\mathrm{Re}>0} \in H_\infty(S^\circ_{\mu+}),$$

and

$$B_- = B\,\chi_{\mathrm{Re}<0} \in H_\infty(S^\circ_{\mu-}),$$

and let $\Phi = \mathcal{G}(B) = \mathcal{G}(B_+) + \mathcal{G}(B_-)$ and $\Phi_1(B) = \mathcal{G}_1(B) = \mathcal{G}_1(B_+) + \mathcal{G}_1(B_-)$. Then

i) Φ is a holomorphic function on $S^\circ_\mu(\mathbb{C})$,

ii) Φ_1 is a holomorphic function on $S^\circ_{\mu+}(\mathbb{C})$ which satisfies $\Phi_1'(Z) = \Phi(Z) + \Phi(-Z)$, and

iii) $$\frac{1}{2\pi}\int_0^{+\infty} B(\lambda)\,\hat{u}(-\lambda)\,d\lambda = \lim_{\varepsilon\to 0}\Big(\int_{|X|\geq\varepsilon} \Phi(X)\,u(X)\,dX + \Phi_1(\varepsilon)\,u(0)\Big),$$

for all $u \in \mathcal{S}(\mathbb{R})$. Moreover, if $0 < \nu < \mu$, then

$$\sup\{|Z\,\Phi(Z)| : \ Z \in S^\circ_\nu(\mathbb{C})\} \leq c_{\nu,\mu}\,\|B\|_\infty$$

and

$$\sup\{|\Phi_1(Z)| : \ Z \in S^\circ_\nu(\mathbb{C})\} \leq c_{\nu,\mu}\,\|B\|_\infty\,.$$

Conversely, given functions Φ and Φ_1 which satisfy i) and ii), such that $Z\,\Phi(Z)$ and $\Phi_1(Z)$ are bounded in Z, then there exists a unique holomorphic function B on $S^{\circ}_{\mu}(\mathbb{C})$ which satisfies Parseval's identity iii). Moreover, if $0 < \nu < \mu$, then, on $S^{\circ}_{\nu}(\mathbb{C})$,

$$\|B\|_{\infty} \leq c_{\nu,\mu} \sup\{|Z\,\Phi(Z)| : Z \in S^{\circ}_{\mu}(\mathbb{C})\} + \sup\{|\Phi_1(Z)| : Z \in S^{\circ}_{\mu^+}(\mathbb{C})\}.$$

In [McQ], this material is used to show that singular convolution operators $T_{(\Phi,\Phi_1)}$ are bounded linear operators on $L_p(\gamma)$ when $1 < p < \infty$, where γ is a Lipschitz curve in the complex plane $\mathbb{C}$. This is done by representing $T_{(\Phi,\Phi_1)}$ as $B(-iD_\gamma)$, where

$$D_\gamma = \frac{d}{dz}\Big|_\gamma,$$

and proving that $-iD_\gamma$ has a bounded H_∞ functional calculus in $L_p(\gamma)$. See also [McQ1].

We would like to take this opportunity to thank all those with whom we have discussed this material, in particular Raphy Coifman and John Ryan. The work was mainly done at Macquarie University, but parts were achieved at Yale University, the Mittag-Leffler Institute and Flinders University, and we thank them for their support. It may be worth noting that some of the early ideas concerning the role of the exponential functions $e(x,\zeta)$ in relating functional calculi of $\mathbf{D}_\Sigma\, e_L$ to singular convolution operators were developed while the second author was visiting Coifman at Yale in 1987.

2. Clifford analysis.

Throughout this paper m and M denote positive integers, L is equal to either 0 or $m+1$, and $M \geq \max\{m, L\}$.

The real 2^M-dimensional Clifford algebra $\mathbb{R}_{(M)}$ or the complex 2^M-dimensional Clifford algebra $\mathbb{C}_{(M)}$ have basis vectors e_S, where S is any subset of $\{1,2,\ldots,M\}$. Under the identifications $e_0 = e_\phi$ and $e_j = e_{\{j\}}$ for $1 \leq j \leq M$, the associative multiplication of basis vectors satisfies

$$\begin{gathered} e_0 = 1, \qquad e_j^2 = -e_0 = -1, \quad \text{for } 1 \leq j \leq M, \\ e_j\, e_k = -e_k\, e_j = e_{\{j,k\}}, \quad \text{for } 1 \leq j < k \leq M, \quad \text{and} \\ e_{j_1}\, e_{j_2} \cdots e_{j_s} = e_S, \quad \text{if } 1 \leq j_1 < j_2 < \cdots < j_s \leq M, \quad \text{and} \\ S = \{j_1, j_2, \ldots, j_s\}. \end{gathered}$$

The product of two elements $u = \sum_S u_S e_S$ and $v = \sum_T v_T e_T$ in $\mathbb{R}_{(M)}$ (or in $\mathbb{C}_{(M)}$) is $uv = \sum_{S,T} u_S v_T e_S e_T$, where $u_S,\, v_T \in \mathbb{R}$ (or $\mathbb{C}$). The term $u_\phi e_\phi$ is usually written as $u_0 e_0$ or just u_0, and is called the *scalar part* of u.

We embed the vector space $\mathbb{R}^m$ in the Clifford algebras $\mathbb{R}_{(M)}$ and $\mathbb{C}_{(M)}$ by identifying the standard basis vectors $e_1, e_2, \ldots, e_m$ of $\mathbb{R}^m$ with their counterparts in $\mathbb{R}_{(M)}$ or $\mathbb{C}_{(M)}$.

There are two common ways of embedding $\mathbb{R}^{m+1}$ in the Clifford algebras. Both ways are useful. We treat them together by denoting standard basis vectors of $\mathbb{R}^{m+1}$ by $e_1, e_2, \ldots, e_m, e_L$ and identifying e_L with either e_0 or e_{m+1}.

We use the euclidean norms $|u| = (\sum |u_S|^2)^{1/2}$ on $\mathbb{R}_{(M)}$ and on $\mathbb{C}_{(M)}$, and remark that $|uv| \le C\,|u|\,|v|$ for some constant C depending only on M. This constant can be taken as 1 if $u \in \mathbb{R}^{m+1}$, and as $\sqrt{2}$ if $u \in \mathbb{C}^{m+1}$.

We write an element $x \in \mathbb{R}^{m+1}$ as $x = \mathbf{x} + x_L e_L$ where $x \in \mathbb{R}^m$ and $x_L \in \mathbb{R}$, and its Clifford conjugate as $\bar{x} = -\mathbf{x} + x_L \bar{e}_L$ where $\bar{e}_L e_L = 1$. Then $\bar{x}\, x = x\, \bar{x} = \sum_{j=1}^m x_j^2 + x_L^2 = |x|^2$.

The Clifford algebras $\mathbb{R}_{(0)}, \mathbb{R}_{(1)}$ and $\mathbb{R}_{(2)}$ are the real numbers, complex numbers, and quaternions, respectively. An important property of these three algebras is that every non-zero element has an inverse. Although this is not true in general it is an important fact that every element $x = \mathbf{x} + x_L e_L$ of $\mathbb{R}^{m+1}$ does have an inverse x^{-1} in $\mathbb{R}_{(M)}$. Indeed $x^{-1} = |x|^{-2}\bar{x} \in \mathbb{R}^{m+1} \subset \mathbb{R}_{(M)}$.

For $\xi \in \mathbb{R}^m$, $\xi \neq \mathbf{0}$, define $\chi_\pm(x) = (1 \pm i\,\xi e_L |\xi|^{-1})/2$, so that $\chi_+(x) + \chi_-(x) = 1$. Using $(i\,\xi e_L)^2 = |\xi|^2$, we obtain

$$\chi_+(\xi)^2 = \chi_+(\xi)\,, \qquad \chi_-(\xi)^2 = \chi_-(\xi)\,,$$
$$\chi_+(\xi)\,\chi_-(\xi) = 0 = \chi_-(\xi)\,\chi_+(\xi)\,.$$

Further, $i\,\xi e_L = |\xi|\,\chi_+(\xi) - |\xi|\,\chi_-(\xi)$, and indeed, for any polynomial $P(\lambda) = \sum a_k \lambda_k$ in one variable with scalar coefficients, we have $P(i\xi e_L) = \sum a_k (i\xi e_L)^k = P(|\xi|)\,\chi_+(\xi) + P(-|\xi|)\,\chi_-(\xi)$. Therefore the polynomial p in m variables defined by $p(\xi) = P(i\xi e_L)$ satisfies $p(\mathbf{0}) = P(0)$ and

$$p(\xi) = P(i\xi e_L) = P(|\xi|)\,\chi_+(\xi) + P(-|\xi|)\,\chi_-(\xi)\,, \qquad \xi \neq \mathbf{0}\,.$$

It is natural to associate with every function B of one real variable a function b of m real variables defined at x by

$$b(\xi) = B(i\xi e_L) = B(|\xi|)\,\chi_+(\xi) + B(-|\xi|)\,\chi_-(\xi)$$

if $|\xi|$ and $-|\xi|$ are in the domain of B, and by $b(\mathbf{0}) = B(0)$ if 0 is in the domain of B.

Let us repeat this procedure for holomorphic functions of complex variables. First extend $|\xi|^2$ holomorphically to $\mathbb{C}^m$ by defining

$$|\zeta|_{\mathbb{C}}^2 = \sum_{j=1} \zeta_j^2 = |\xi|^2 - |\eta|^2 + 2i\langle \xi, \eta\rangle,$$

for $\zeta = \xi + i\eta \in \mathbb{C}^m$ (where $\xi, \eta \in \mathbb{R}^m$) and note that $(i\zeta e_L)^2 = |\zeta|_{\mathbb{C}}^2$. When $|\zeta|_{\mathbb{C}}^2 \neq 0$, take $\pm|\zeta|_{\mathbb{C}}$ as its two square roots, and define $\chi_\pm(\zeta) = (1 \pm i\,\zeta e_L |\zeta|_{\mathbb{C}}^{-1})/2$, so that, as before,

$$\chi_+(\zeta) + \chi_-(\zeta) = 1, \qquad \chi_+(\zeta)^2 = \chi_+(\zeta), \qquad \chi_-(\zeta)^2 = \chi_-(\zeta),$$
$$\chi_+(\zeta)\,\chi_-(\zeta) = 0 = \chi_-(\zeta)\,\chi_+(\zeta),$$

and $i\,\zeta e_L = |\zeta|_{\mathbb{C}}\,\chi_+(\zeta) - |\zeta|_{\mathbb{C}}\,\chi_-(\zeta)$.

Given any polynomial $P(\lambda) = \sum a_k \lambda_k$ in one variable with complex coefficients, the associated polynomial p in m variables defined by $p(\zeta) = P(i\zeta e_L) = \sum a_k (i\zeta e_L)^k$ satisfies

$$\begin{aligned} p(\zeta) &= P(i\zeta e_L) = P(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + P(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta) \\ &= \frac{1}{2}(P(|\zeta|_{\mathbb{C}}) + P(-|\zeta|_{\mathbb{C}})) + \frac{1}{2}\frac{(P(|\zeta|_{\mathbb{C}}) - P(-|\zeta|_{\mathbb{C}}))i\,\zeta e_L}{|\zeta|_{\mathbb{C}}}, \end{aligned}$$

if $|\zeta|_{\mathbb{C}}^2 \neq 0$, and

$$p(\zeta) = P(0) + P'(0)\,i\,\zeta e_L\,, \qquad \text{if } |\zeta|_{\mathbb{C}}^2 = 0\,.$$

It is natural to associate with every complex-valued holomorphic function B of one variable, defined on an open subset S of $\mathbb{C}$, a Clifford-valued holomorphic function b of m complex variables, defined, for all $\zeta \in \mathbb{C}^m$ such that $\{\pm|\zeta|_{\mathbb{C}}\} \subset S$, by

$$\begin{aligned} b(\zeta) &= B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta) \\ &= \frac{1}{2}(B(|\zeta|_{\mathbb{C}}) + B(-|\zeta|_{\mathbb{C}})) + \frac{1}{2}\frac{(B(|\zeta|_{\mathbb{C}}) - B(-|\zeta|_{\mathbb{C}}))i\,\zeta e_L}{|\zeta|_{\mathbb{C}}}, \end{aligned}$$

if $|\zeta|_{\mathbb{C}}^2 \neq 0$, and

$$b(\zeta) = B(0) + B'(0)\, i\,\zeta e_L\,, \qquad \text{if } |\zeta|_{\mathbb{C}}^2 = 0\,.$$

The reason we say that this is natural, is not only because b is the required polynomial when B is a polynomial, but also because the mapping from B to b is an algebra homomorphism. That is, if F is another holomorphic function defined on S, and $c_1, c_2 \in \mathbb{C}$, then $(c_1F + c_2B)(i\zeta e_L) = c_1F(i\zeta e_L) + c_2B(i\zeta e_L)$ and $(FB)(i\zeta e_L) = F(i\zeta e_L)B(i\zeta e_L)$.

Important examples, defined for each real t, are the holomorphic functions of $\lambda \in \mathbb{C}$ given by $E_t(\lambda) = e^{-t\lambda}$. The associated functions of m variables are given by

$$\begin{aligned} e(te_L, \zeta) = E_t(i\zeta e_L) &= e^{-t|\zeta|_{\mathbb{C}}}\chi_+(\zeta) + e^{t|\zeta|_{\mathbb{C}}}\chi_-(\zeta) \\ &= \cosh(t\,|\zeta|_{\mathbb{C}}) - \sinh(t\,|\zeta|_{\mathbb{C}})\,|\zeta|_{\mathbb{C}}^{-1} i\,\zeta e_L\,, \end{aligned}$$

if $|\zeta|_{\mathbb{C}}^2 \neq 0$, and

$$e(te_L, \zeta) = 1 - t\,i\,\zeta e_L\,, \qquad \text{if } |\zeta|_{\mathbb{C}}^2 = 0\,.$$

Then $e(te_L,\zeta)\,e(se_L,\zeta) = e((t+s)e_L,\zeta)$ and $e(te_L,-\zeta) = e(-te_L,\zeta)$. Also,

$$\frac{d}{dt}e(te_L,\zeta) = -i\,\zeta e_L\, e(te_L,\zeta) = -e(te_L,\zeta)\,i\,\zeta e_L\,.$$

Other important examples, defined for each complex α, are the functions $R_\alpha(\lambda) = (\lambda - \alpha)^{-1}$, $\lambda \neq \alpha$. Then

$$R_\alpha(i\,\zeta e_L) = (i\zeta e_L - \alpha)^{-1} = (i\zeta e_L + \alpha)\,(|\zeta|_{\mathbb{C}}^2 - \alpha^2)^{-1}\,, \qquad |\zeta|_{\mathbb{C}}^2 \neq \alpha^2\,.$$

(What we are really doing, is studying the spectral theory of the elements $i\zeta e_L$ of the complex algebra $\mathbb{C}_{(M)}$. The spectrum of $i\zeta e_L$ is $\{\pm|\zeta|_{\mathbb{C}}\}$, and the spectral decomposition of $i\zeta e_L$ is $i\zeta e_L = |\zeta|_{\mathbb{C}}\,\chi_+(\zeta) - |\zeta|_{\mathbb{C}}\,\chi_-(\zeta)$ when $|\zeta|_{\mathbb{C}}^2 \neq 0$, while $i\zeta e_L$ is nilpotent when $|\zeta|_{\mathbb{C}}^2 = 0$).

So far it has been unimportant which sign we assign to each square-root of $|\zeta|_{\mathbb{C}}^2$, though from now on we shall assume that $|\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0]$, and take $\operatorname{Re}|\zeta|_{\mathbb{C}} > 0$.

It is time to prove some estimates.

Theorem 2.1. *Let* $\zeta = \xi + i\eta \in \mathbb{C}^m$ *(where* $\xi, \eta \in \mathbb{R}^m$*), and assume that* $|\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0]$. *Let*

$$\theta = \tan^{-1}\left(\frac{|\eta|}{\operatorname{Re}|\zeta|_{\mathbb{C}}}\right) \in [0, \pi/2)\,.$$

Then

a) $0 < \operatorname{Re}|\zeta|_{\mathbb{C}} \le |\xi| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}}$,

b) $\operatorname{Re}|\zeta|_{\mathbb{C}} \le ||\zeta|_{\mathbb{C}}| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}} \le |\zeta| \le (1 + 2\tan^2\theta)^{1/2} \operatorname{Re}|\zeta|_{\mathbb{C}}$,

c) $-\theta \le \arg|\zeta|_{\mathbb{C}} \le \theta$, *and*

d) $|\chi_\pm(\zeta)| \le \dfrac{\sec\theta}{\sqrt{2}}$.

PROOF. The simplest to prove is

$$||\zeta|_{\mathbb{C}}|^2 = ||\zeta|_{\mathbb{C}}^2| = \left((|\xi|^2 - |\eta|^2)^2 + 4\langle\xi,\eta\rangle^2\right)^{1/2} \le |\xi|^2 + |\eta|^2 = |\zeta|^2\,,$$

so that

$$\text{e)} \qquad \operatorname{Re}|\zeta|_{\mathbb{C}} \le ||\zeta|_{\mathbb{C}}| \le |\zeta|\,.$$

On taking real parts in the identity

$$-(\xi + i\eta)^2 = -\zeta^2 = |\zeta|_{\mathbb{C}}^2 = (\operatorname{Re}|\zeta|_{\mathbb{C}} + i\operatorname{Im}|\zeta|_{\mathbb{C}})^2$$

we obtain

$$(\#) \qquad |\xi|^2 - |\eta|^2 = (\operatorname{Re}|\zeta|_{\mathbb{C}})^2 - (\operatorname{Im}|\zeta|_{\mathbb{C}})^2$$

or

$$2\,|\xi|^2 - |\zeta|^2 = 2\,(\operatorname{Re}|\zeta|_{\mathbb{C}})^2 - ||\zeta|_{\mathbb{C}}|^2\,,$$

so that, by *e*), we obtain $\operatorname{Re}|\zeta|_{\mathbb{C}} \le |\xi|$. Also, from (#), we have $|\xi|^2 \le |\eta|^2 + (\operatorname{Re}|\zeta|_{\mathbb{C}})^2 = (\tan^2\theta + 1)(\operatorname{Re}|\zeta|_{\mathbb{C}})^2$, leading to $|\xi| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}}$. Thus we have proved *a*).

Another consequence of (#) is

$$2\sec^2\theta(\operatorname{Re}|\zeta|_{\mathbb{C}})^2 = 2\big((\operatorname{Re}|\zeta|_{\mathbb{C}})^2 + |\eta|^2\big) = |\zeta|^2 + ||\zeta|_{\mathbb{C}}|^2\,,$$

which, by *e*), gives $||\zeta|_{\mathbb{C}}| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}} \le |\zeta|$. A further application of (#) gives

$$|\zeta|^2 = 2\,|\eta|^2 + (\operatorname{Re}|\zeta|_{\mathbb{C}})^2 - (\operatorname{Im}|\zeta|_{\mathbb{C}})^2 \le (1 + 2\tan^2\theta)\,(\operatorname{Re}|\zeta|_{\mathbb{C}})^2\,,$$

so that we have proved *b*).

Part *c*) is an immediate consequence of the inequality $||\zeta|_{\mathbb{C}}| \le \sec\theta \operatorname{Re}|\zeta|_{\mathbb{C}}$, while *d*) follows from $|\zeta| \le (1 + 2\tan^2\theta)^{1/2}||\zeta|_{\mathbb{C}}|$.

On defining

$$S^{\circ}_{\mu}(\mathbb{C}^m) = \{\zeta = \xi + i\eta \in \mathbb{C}^m : \ |\zeta|^2_{\mathbb{C}} \notin (-\infty, 0] \text{ and } |\eta| < \mathrm{Re}(|\zeta|_{\mathbb{C}}) \tan\mu\},$$

we see, from part *c*) of the above theorem, that whenever $\zeta \in S^{\circ}_{\mu}(\mathbb{C}^m)$, then $|\zeta|_{\mathbb{C}} \in S^{\circ}_{\mu+}(\mathbb{C})$ and $-|\zeta|_{\mathbb{C}} \in S^{\circ}_{\mu-}(\mathbb{C})$ (these sectors are defined in Section 1). So, for every holomorphic function B defined on $S^{\circ}_{\mu}(\mathbb{C}) = S^{\circ}_{\mu+}(\mathbb{C}) \cup S^{\circ}_{\mu-}(\mathbb{C})$, the associated holomorphic function b of m variables given by

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\, \chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\, \chi_-(\zeta)$$

is defined on $S^{\circ}_{\mu}(\mathbb{C}^m)$. Moreover, by part *d*), if B is bounded, then

$$\|b\|_{\infty} \leq \sqrt{2}\, \sec\mu\, \|B\|_{\infty} .$$

On letting $H_{\infty}(S^{\circ}_{\mu}(\mathbb{C}^m)) = H_{\infty}(S^{\circ}_{\mu}(\mathbb{C}^m), \mathbb{C}_{(M)})$, the Banach space of bounded Clifford-valued holomorphic functions defined on $S^{\circ}_{\mu}(\mathbb{C}^m)$, we obtain the following result

Theorem 2.2. *The mapping $B \mapsto b$ defined above is a one-one bounded algebra homomorphism from $H_{\infty}(S^{\circ}_{\mu}(\mathbb{C}))$ to $H_{\infty}(S^{\circ}_{\mu}(\mathbb{C}^m))$.*

PROOF. All that remains to be proved is that the mapping is one-one. Actually we can do better and recover B from b by means of the formula

$$B(\lambda) = \frac{2}{\sigma_{m-1}} \int\limits_{|\xi|=1} b(\lambda\xi)\, \chi_{\pm}(\xi)\, d\xi , \qquad \lambda \in S^{\circ}_{\mu\pm}(\mathbb{C})$$

where σ_{m-1} is the volume of the unit $(m-1)$-sphere in $\mathbb{R}^m$. (Beware that, when $m = 1$, functions in $H_{\infty}(S^{\circ}_{\mu}(\mathbb{C}))$ are complex-valued, while functions in $H_{\infty}(S^{\circ}_{\mu}(\mathbb{C}^1))$ take their values in $\mathbb{C}_{(M)}$.)

It may seem strange to use the inequality $|\eta| < \mathrm{Re}(|\zeta|_{\mathbb{C}}) \tan\mu$ in the definition of $S^{\circ}_{\mu}(\mathbb{C}^m)$, rather than a simpler one such as $|\eta| < |\xi| \tan\mu$. (These inequalities are the same when $m = 1$.) However, when it comes to characterizing the Fourier transforms of monogenic functions on cones $C^{\circ}_{\mu\pm}$ and S°_{μ} in $\mathbb{C}^{m+1}$, then, as we shall see in Section 4, there is no choice but to use the sets $S^{\circ}_{\mu}(\mathbb{C}^m)$ as defined above.

So far we have been considering Clifford-valued holomorphic functions of m complex variables. What is usually called *Clifford analysis* is the study of monogenic functions of $m+1$ real variables. In Section 4, we shall relate these two concepts via the Fourier transform. This involves extensive use of the exponential function

$$\begin{aligned} e(x,\zeta) &= e(\mathbf{x}+x_Le_L,\zeta) \\ &= e^{i\langle \mathbf{x},\zeta\rangle}e(x_Le_L,\zeta) \\ &= e^{i\langle \mathbf{x},\zeta\rangle}(e^{-x_L|\zeta|_{\mathbb{C}}}\chi_+(\zeta)+e^{x_L|\zeta|_{\mathbb{C}}}\chi_-(\zeta)), \end{aligned}$$

which is a holomorphic function of $\zeta \in \mathbb{C}^m$ for each $x = \mathbf{x}+x_Le_L \in \mathbb{R}^{m+1}$, and is a left-monogenic function of $x \in \mathbb{R}^{m+1}$ for each $\zeta \in \mathbb{C}^m$. It satisfies $e(x,\zeta)\,e(y,\zeta) = e(x+y,\zeta)$ and $e(x,-\zeta) = e(-x,\zeta)$. Of course, when $\mathbf{x} \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^m$, then $e(\mathbf{x},\xi) = e^{i\langle \mathbf{x},\xi\rangle}$, the usual exponential function in Fourier theory.

Also, $e(x,\zeta)\,\overline{e_L}$ is a right-monogenic function of $x \in \mathbb{R}^{m+1}$ for each $\zeta \in \mathbb{C}^m$.

We remark that

$$e(x,\zeta) = \exp i(\langle \mathbf{x},\zeta\rangle - x_L\zeta e_L) = \sum_{k=0} \frac{1}{k!}\big(i(\langle \mathbf{x},\zeta\rangle - x_L\zeta e_L)\big)^k .$$

Let us briefly review some facts about Clifford analysis. The differential operator

$$D = \mathbf{D} + \frac{\partial}{\partial x_L}\,e_L\,, \qquad \text{where } \mathbf{D} = \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\,e_k\,,$$

acts on C^1-functions $f = \sum f_S e_S$ of $m+1$ real variables to give

$$Df = \sum_{k=1}^{m}\frac{\partial f_S}{\partial x_k}\,e_k e_S + \frac{\partial f_S}{\partial x_L}\,e_L e_S$$

and also

$$fD = \sum_{k=1}^{m}\frac{\partial f_S}{\partial x_k}\,e_S e_k + \frac{\partial f_S}{\partial x_L}\,e_S e_L\,.$$

A C^1-function defined on an open subset of $\mathbb{R}^{m+1}$ with values in $\mathbb{R}_{(M)}$ or $\mathbb{C}_{(M)}$ is called *left-monogenic* if $Df = 0$ and *right-monogenic* if $fD = 0$.

We remark that each component of every left-monogenic function is harmonic, as is each component of every right-monogenic function.

The function $e(x,\zeta)$ is a left-monogenic function of x (for fixed ζ) because

$$\begin{aligned}\frac{\partial}{\partial x_L}\, e_L e(x,\zeta) &= -e_L i\,\zeta e_L e(x,\zeta)\\ &= -e_L i\,\overline{e_L}\zeta e(x,\zeta)\\ &= -i\,\zeta e(x,\zeta) = -\mathbf{D}\, e(x,\zeta)\,.\end{aligned}$$

Similar reasoning shows that $e(x,\zeta)\overline{e_L}$ is right-monogenic in x.

Define the function k on $\mathbb{R}^{m+1}\setminus\{0\}$ by

$$k(x) = \frac{1}{\sigma_m}\,\frac{\bar{x}}{|x|^{m+1}}\,, \qquad \text{for } x \neq 0\,,$$

(where σ_m is the volume of the unit m-sphere in $\mathbb{R}^{m+1}$).

The Cauchy kernels $k(y-x)$ are left- and right-monogenic functions of both x and y (when $x \neq y$).

Let us state Cauchy's theorem and the Cauchy integral formula.

Theorem 2.3. *Let Ω be a bounded open subset of $\mathbb{R}^{m+1}$ with Lipschitz boundary $\partial\Omega$ and exterior unit normal $n(y)$ defined for almost all $y \in \partial\Omega$. Suppose f is left-monogenic and g is right-monogenic on a neighbourhood of $\Omega^{\rm cl} = \Omega \cup \partial\Omega$. Then*

i) $$\int_\Sigma g(y)\,n(y)\,f(y)\,dS_y = 0\,,$$

ii) $$\int_{\partial\Omega} g(y)\,n(y)\,k(y-x)\,dS_y = \begin{cases} g(x)\,, & x \in \Omega\,,\\ 0\;, & x \notin \Omega^{\rm cl}\,,\end{cases}$$

iii) $$\int_{\partial\Omega} k(y-x)\,n(y)\,f(y)\,dS_y = \begin{cases} f(x)\,, & x \in \Omega\,,\\ 0\;, & x \notin \Omega^{\rm cl}\,.\end{cases}$$

Part i) is a direct consequence of Gauss' divergence theorem, while parts ii) and iii) follow from i) in the usual way, together with the easily verified identity

$$\int\limits_{|y-x|=r} n(y)\,k(y-x)\,dS_y = \int\limits_{|y-x|=r} k(y-x)\,n(y)\,dS_y = 1\,, \quad r > 0\,.$$

These reduce to theorems known by Cauchy when $m = 1$, in which case $\mathbb{R}^{1+1} = \mathbb{R}_{(1)} \cong \mathbb{C}$. It appears that, for $m = M = 2$, A. C. Dixon [D] published the first such results in 1904.

Further information about monogenic functions can be found in the books [BDS], [GM], and in the papers of F. Sommen, J. Ryan and others. See [S1] in particular, where Sommen introduces the exponential function $e(x, \xi)$. The paper [PS] contains some related material.

We remark that parts i) and iii) of Theorem 2.1 remain valid when f is a left-monogenic function taking its values in a finite-dimensional left Clifford module $\mathcal{X}$. That is, $\mathcal{X}$ is a finite-dimensional real (or complex) linear space together with a representation of $\mathbb{R}_{(M)}$ (or $\mathbb{C}_{(M)}$) as linear operators on $\mathcal{X}$. If $u \in \mathbb{R}_{(M)}$ (or $\mathbb{C}_{(M)}$) and $v \in \mathcal{X}$, we denote the action of u on v by uv.

We consider $\mathcal{X}$ together with a norm $\|\cdot\|$ and note that there exists a constant C such that $\|uv\| \leq C\,|u|\,\|v\|$ for all $u \in \mathbb{R}_{(M)}$ (or $\mathbb{C}_{(M)}$) and $v \in \mathcal{X}$. (We do not equip $\mathcal{X}$ with an inner-product, and in particular, do not require that the basis vectors e_j are represented by skew-adjoint operators).

Part I: Monogenic extensions of functions and holomorphic extensions of their Fourier transforms.

3. Monogenic functions on cones.

We present a mildly generalized version of results presented in [LMcS]. Here we consider monogenic functions defined on cones in $\mathbb{R}^{m+1}$ which are unions of half-spaces, whereas in [LMcS] we only considered cones which are rotationally symmetric about the x_0 axis. Allowing e_L to be e_{m+1} rather than e_0 causes no problems. (One reason for generalizing in this way, is to incorporate the application to boundary value problems which is presented in Section 7.)

We start by specifying some sets of unit vectors in $\mathbb{R}^{m+1}_{+} = \{x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1} : x_L > 0\}$. The metric $\angle(n, y) = \cos^{-1}\langle n, y\rangle$ is used on these unit vectors.

Let N be a compact set of unit vectors in $\mathbb{R}^{m+1}_{+}$ which contains e_L, and let $\mu_N = \sup\{\angle(n, e_L) : n \in N\}$. Then $0 \leq \mu_N < \pi/2$. For

$0 < \mu \leq \pi/2 - \mu_N$, define the open neighbourhoods N_μ of N in the unit sphere by $N_\mu = \{y \in \mathbb{R}^{m+1}_+ : \ |y| = 1, \ \angle(y,n) < \mu$ for some $n \in N\}$. (In Section 6, N is the set of unit vectors normal to a surface).

For each unit vector n, let C^+_n be the open half-space $C^+_n = \{x \in \mathbb{R}^{m+1} : \ \langle x, n\rangle > 0\}$, and define open cones in $\mathbb{R}^{m+1}$ as follows. Let $C^+_{N_\mu} = \cup\{C^+_n : \ n \in N_\mu\}$, $C^-_{N_\mu} = -C^+_{N_\mu}$ and $S_{N_\mu} = C^+_{N_\mu} \cap C^-_{N_\mu}$.

We remark that in [LMcS] we considered the case when N is rotationally symmetric, namely $N = \{n = \mathbf{n} + n_L e_L \in \mathbb{R}^{m+1}_+ : |n| = 1, \ n_L \geq |n| \operatorname{cotan} w\}$ for some $\omega \in [0, \pi/2)$. Then $\mu_N = \omega$. In this case we use the symbols

$$C^\circ_{\mu^+} = C^+_{N_{\mu-\omega}} = \{x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1} : \ x_L > -|x| \tan\mu\},$$
$$C^\circ_{\mu^-} = -C^\circ_{\mu^+}, \qquad S^\circ_\mu = C^\circ_{\mu^+} \cap C^\circ_{\mu^-},$$

consistent with [LMcS].

Define the Banach space $K(C^+_{N_\mu})$ to be the space of right-monogenic functions Φ from $C^+_{N_\mu}$ to $\mathbb{C}_{(M)}$ for which

$$\|\Phi\|_{K(C^+_{N_\mu})} = \frac{1}{2}\sigma_m \sup\{|x|^m |\Phi(x)| : \ x \in C^+_{N_\mu}\} < +\infty .$$

Similarly define $K(C^-_{N_\mu})$.

Also define the Banach space $K(S_{N_\mu})$ to be the space of pairs $(\Phi, \underline{\Phi})$ of functions with Φ right-monogenic from S_{N_μ} to $\mathbb{C}_{(M)}$ and with $\underline{\Phi}$ continuous on $(0, +\infty)\, e_L$, such that

$$\underline{\Phi}(R\, e_L) - \underline{\Phi}(r\, e_L) = \int_{r \leq |\mathbf{x}| \leq R} \Phi(\mathbf{x})\, d\mathbf{x}\, e_L ,$$

and

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} = \frac{1}{2}\sigma_m \sup\{|x|^m |\Phi(x)| : \ x \in S_{N_\mu}\}$$
$$+ \sup\{|\underline{\Phi}(r\, e_L)| : \ r > 0\} < +\infty .$$

Note that $\underline{\Phi}$ is determined by Φ up to an additive constant, and that

$$\underline{\Phi}'(r\, e_L) = \int_{|\mathbf{x}| = r} \Phi(\mathbf{x})\, d\mathbf{x}\, e_L .$$

Moreover, $\underline{\Phi}$ has a unique continuous extension to the cone

$$T_{N_\mu} = \{y = \mathbf{y} + y_L e_L \in \mathbb{R}^{m+1}_+ : \ y^\perp \subset S_{N_\mu}\},$$

which satisfies

$$\underline{\Phi}(y) - \underline{\Phi}(z) = \int_{A(y,z)} f(x)\, n(x)\, dS_x ,$$

where $A(y,z)$ is a smooth oriented m-manifold in S_{N_μ} joining the $(m-1)$-sphere $S_y = \{x \in \mathbb{R}^{m+1} : \ \langle x, y\rangle = 0$ and $|x| = |y|\}$ to the $(m-1)$-sphere S_z, in which case $|\underline{\Phi}(y)| \leq \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})}$ for all $y \in T_{N_\mu}$.

When N is rotationally symmetric, namely

$$N = \{n = \mathbf{n} + n_L e_L \in \mathbb{R}^{m+1}_+ : \ |n| = 1\,,\ n_L \geq |\mathbf{n}| \operatorname{cotan}\omega\},$$

we use the symbol

$$T^\circ_\mu = T_{N_{\mu-\omega}} = \{y = \mathbf{y} + y_L e_L \in \mathbb{R}^{m+1} : \ y_L > |x| \operatorname{cotan}\mu\},$$

consistent with [LMcS].

Let us state the relationship between these spaces. Here $H_{y^\pm}$ denote the hemispheres $H_{y^\pm} = \{x \in \mathbb{R}^{m+1} : \ \pm\langle x, y\rangle \geq 0$ and $|x| = |y|\}$ with boundaries S_y.

Theorem 3.1. i) *Given $\Phi_\pm \in K(C^\pm_{N_\mu})$, define the functions $\underline{\Phi_\pm}$ on T_{N_μ} by*

$$\underline{\Phi_\pm}(y) = \pm \int_{H_{y^\pm}} \Phi_\pm(x)\, n(x)\, dS_x\,, \qquad y \in T_{N_\mu}\,,$$

where $n(x) = x/|x|$ is normal to the hemisphere $H_{y^\pm}$. Then

$$(\Phi, \underline{\Phi}) = (\Phi_+ + \Phi_-, \underline{\Phi_+} + \underline{\Phi_-}) \in K(S_{N_\mu})$$

and

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} \leq \|\Phi_+\|_{K(C^+_{N_\mu})} + \|\Phi_-\|_{K(C^-_{N_\mu})}\,.$$

ii) *Conversely, given* $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$, *there exist unique functions* $\Phi_\pm \in K(C^\pm_{N_\mu})$ *which satisfy* $\Phi = \Phi_+ + \Phi_-$ *and* $\underline{\Phi} = \underline{\Phi_+} + \underline{\Phi_-}$. *These functions are given by the formulae*

$$\Phi_\pm(x) = \pm \lim_{\varepsilon \to 0} \Big(\int_{\substack{\langle y,n\rangle = 0 \\ |y| \geq \varepsilon}} \Phi(y)\, n\; k(x-y)\, dS_y + \underline{\Phi}(\varepsilon\, e_L)\, k(x) \Big) ,$$

with $x \in C^\pm_n \subset C^\pm_{N_\mu}$, *for all* $n \in N_\mu$, *where*

$$k(x) = \frac{1}{\sigma_m} \frac{\bar{x}}{|x|^{m+1}} ,$$

and they satisfy the estimates

$$\|\Phi_\pm\|_{K(C^\pm_{N_\mu})} \leq c\, \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} ,$$

where c *depends only on* μ_N, μ *(and the dimension* m*).*

PROOF. i) To see that

$$\underline{\Phi_\pm}(y) - \underline{\Phi_\pm}(z) = \int_{A(y,z)} \Phi_\pm(x)\, n(x)\, dS_x ,$$

apply Cauchy's theorem to the right-monogenic functions $\Phi_\pm$. The bound is straightforward.

ii) This is a slight generalization of results proved in [LMcS].

In other words, there is a natural isomorphism

$$K(S_{N_\mu}) \simeq K(C^+_{N_\mu}) \oplus K(C^-_{N_\mu}) .$$

We also need the closed linear subspaces $M(C^\pm_{N_\mu})$ of $K(C^\pm_{N_\mu})$ which consist of those functions $\Phi \in K(C^\pm_{N_\mu})$ which are left-monogenic as well as right-monogenic. The subspace $M(S_{N_\mu})$ of $K(S_{N_\mu})$ for which

$$M(S_{N_\mu}) \simeq M(C^+_{N_\mu}) \oplus M(C^-_{N_\mu})$$

is then

$$M(S_{N_\mu}) = \{(\Phi, \underline{\Phi}) \in K(S_{N_\mu}) :\ \Phi \text{ is left-monogenic and } (*) \text{ holds}\} ,$$

where

$$(*) \qquad \int_{|\mathbf{y}|=r} \langle \mathbf{y},\mathbf{x}\rangle\, r^{-1} \left(e_L \Phi(\mathbf{y})\mathbf{y} - \mathbf{y}\Phi(\mathbf{y})e_L\right) dS_y + \mathbf{x}\,\underline{\Phi}(re_L) - e_L\,\underline{\Phi}(re_L)\,\mathbf{x}\,e_L = 0\,,$$

when $r > 0$. It is not difficult to see i) that the value of the integral is independent of r, and ii) that it equals 0 when $\Phi \in M(C^{\pm}_{N_\mu})$. The difficult part is to show that when $(\Phi, \underline{\Phi}) \in M(S_{N_\mu})$, then the functions $\Phi_\pm$ defined in Theorem 3.1.ii are left-monogenic. See [LMcS]. Section 7 of [LMcS] contains further information about $(*)$.

Let us now consider convolutions. Given $\Phi \in K(C^+_{N_\mu})$, $\Psi \in M(C^+_{N_\mu})$ and $x \in C^+_n \subset C^+_{N_\mu}$, define $(\Phi * \Psi)(x)$ by

$$\begin{aligned}(\Phi * \Psi)(x) &= \int_{\langle y,n\rangle=\delta} \Phi(x-y)\,n\,\Psi(y)\,dS_y \\ &= \lim_{\varepsilon\to 0}\Big(\int_{\substack{\langle y,n\rangle=0\\ |y|\geq\varepsilon}} \Phi(x-y)\,n\,\Psi(y)\,dS_y + \underline{\Phi}(\varepsilon e_L)\,\Psi(x)\Big)\,,\end{aligned}$$

where $0 < \delta < \langle x, n\rangle$. It follows from Cauchy's theorem and the hypotheses of Φ being right-monogenic and Ψ being left-monogenic, that the integral is independent of the precise surface chosen. On the other hand, it is a consequence of Ψ being right-monogenic, that $\Phi * \Psi$ is right-monogenic, and indeed that

$$\|\Phi * \Psi\|_{K(C^+_{N_\nu})} \leq c_{\nu,\mu}\, \|\Phi\|_{K(C^+_{N_\mu})}\, \|\Psi\|_{K(C^+_{N_\mu})}\,,$$

for all $\nu < \mu$, as is shown in [LMcS].

If moreover $\Psi_1 \in M(C^+_{N_\mu})$, then $\Psi * \Psi_1$ is left- as well as right-monogenic, and $\Phi * (\Psi * \Psi_1) = (\Phi * \Psi) * \Psi_1$.

Corresponding results hold for functions defined on $C^-_{N_\mu}$.

If $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$ and $(\Psi, \underline{\Psi}) \in M(S_{N_\mu})$, define

$$(\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi}) = (\Phi_+ * \Psi_+ + \Phi_- * \Psi_-, \underline{\Phi_+ * \Psi_+} + \underline{\Phi_- * \Psi_-})\,.$$

It then follows from the above material that

$$\|(\Phi,\underline{\Phi})*(\Psi,\underline{\Psi})\|_{K(S^\circ_{\nu+})} \le C_{\nu,\mu}\,\|(\Phi,\underline{\Phi})\|_{K(S^\circ_{\mu+})}\,\|(\Psi,\underline{\Psi})\|_{K(S^\circ_{\mu+})}\,,$$

for all $\nu < \mu$.

Let K_N^+ be the linear space of functions Φ on $\mathbb{R}^m\setminus\{0\}$ which extend monogenically to $\Phi \in K(C^+_{N_\mu})$ for some $\mu > 0$. Similarly define K_N^-, K_N, M_N^+, M_N^- and M_N, so that $K_N \simeq K_N^+ \oplus K_N^-$ and $M_N \simeq M_N^+ \oplus M_N^-$, while M_N^+, M_N^- and M_N are all convolution algebras. (We do not introduce topologies on these spaces, so that $\oplus$ is merely the direct sum of linear spaces.) We remark that the only functions Φ which belong to both K_N^+ and K_N^- are those of the form $\Phi(\mathbf{x}) = c\,k(\mathbf{x})$ for some $c \in \mathbb{C}_{(M)}$, where

$$k(\mathbf{x}) = \frac{1}{\sigma_m}\frac{-\mathbf{x}}{|\mathbf{x}|^{m+1}}\,, \qquad \text{for } \mathbf{x} \in \mathbb{R}^m \setminus \{0\}\,,$$

with monogenic extension

$$k(x) = \frac{1}{\sigma_m}\frac{\bar{x}}{|x|^{m+1}}\,.$$

(See Section 12 of [BDS].) The embedding of K_N^+ into K_N defined above takes $c\,k \in K_N^+$ to $(c\,k, c/2) \in K_N$, while the embedding of K_N^- into K_N takes $c\,k \in K_N^-$ to $(c\,k, -c/2) \in K_N$.

4. Fourier transforms.

Our aim in this section is to identify the Fourier transforms $\mathcal{F}_\pm(\Phi)$ of the functions $\Phi \in K_N^\pm$, and to define the Fourier transforms $\mathcal{F}(\Phi,\underline{\Phi})$ of $(\Phi,\underline{\Phi})$. The transforms turn out to be bounded holomorphic functions defined on cones in $\mathbb{C}^m$. We also show that $\mathcal{F}_+$, $\mathcal{F}_-$ and $\mathcal{F}$ are algebra homomorphisms from the convolution algebras M_N^+, N_N^- and M_N to algebras of holomorphic functions.

We first associate with every unit vector $n = \mathbf{n} + n_L e_L \in \mathbb{R}^{m+1}$ satisfying $n_L > 0$, a real m-dimensional surface $n(\mathbb{C}^m)$ in $\mathbb{C}^m$, defined as follows.

$$\begin{aligned} n(\mathbb{C}^m) &= \{\zeta = \xi + i\eta \in \mathbb{C}^m : \ \xi \ne 0 \text{ and } n_L\eta = (n_L^2|\xi|^2 + \langle x,\mathbf{n}\rangle^2)^{1/2}\mathbf{n}\} \\ &= \{\zeta = \xi + i\eta \in \mathbb{C}^m : \ |\zeta|^2_{\mathbb{C}} \notin (-\infty,0] \text{ and } n_L\eta = \mathrm{Re}\,(|\zeta|_{\mathbb{C}})\mathbf{n}\} \\ &= \{\zeta = \xi + i\eta \in \mathbb{C}^m : \ |\zeta|^2_{\mathbb{C}} \notin (-\infty,0] \text{ and} \\ &\qquad\qquad \eta + \mathrm{Re}\,(|\zeta|_{\mathbb{C}})e_L = \kappa\, n \text{ for some } \kappa > 0\}\,, \end{aligned}$$

where

$$|\zeta|_{\mathbb{C}}^2 = \sum_{j=1}^{m} \zeta_j^2 = |\xi|^2 - |\eta|^2 + 2i\langle x, \eta\rangle .$$

The surfaces associated with distinct unit vectors are disjoint, with, in particular, $e_L(\mathbb{C}^m) = \mathbb{R}^m \setminus \{0\}$.

On these surfaces $|\xi|$, $|\zeta|$, $\mathrm{Re}\,(|\zeta|_{\mathbb{C}})$ and $||\zeta|_{\mathbb{C}}|$ are all equivalent. Indeed, by Theorem 2.1,

$$\mathrm{Re}\,|\zeta|_{\mathbb{C}} \leq |\xi| \leq (n_L)^{-1}\,\mathrm{Re}\,|\zeta|_{\mathbb{C}} ,$$

and

$$\mathrm{Re}\,|\zeta|_{\mathbb{C}} \leq ||\zeta|_{\mathbb{C}}| \leq (n_L)^{-1}\,\mathrm{Re}\,|\zeta|_{\mathbb{C}} \leq |\zeta| \leq (n_L)^{-1}\,(1+|\mathbf{n}|^2)^{1/2}\mathrm{Re}\,|\zeta|_{\mathbb{C}} ,$$

for all $\zeta \in n(\mathbb{C}^m)$. Further, the parametrization $\xi \mapsto \zeta = \xi + i\eta$ used in the first definition of $n(\mathbb{C}^m)$ is smooth, with

$$\left|\det\left(\frac{\partial \zeta_j}{\partial \xi_k}\right)\right| \leq \frac{1}{n_L} , \qquad \xi \neq 0 .$$

In proving this, we can assume, without loss of generality, that $n = n_1 e_1 + n_L e_L$, so that

$$\zeta = \xi + i\frac{n_1}{n_L}\left(|\xi|^2 n_L^2 + \xi_1^2 n_1^2\right)^{1/2} e_1 .$$

Then, if $j \geq 2$, $\partial\zeta_j/\partial\xi_k = \delta_{jk}$, and

$$\frac{\partial \zeta_1}{\partial \xi_k} = \delta_{1k} + \frac{i\,n_1\xi_k(n_L^2 + \delta_{1k}n_1^2)}{n_L(|\xi|^2 n_L^2 + \xi_1^2 n_1^2)^{1/2}} .$$

Therefore

$$\left|\frac{\partial \zeta_1}{\partial \xi_1}\right| \leq \frac{1}{n_L} \qquad \text{and} \qquad \left|\frac{\partial \zeta_1}{\partial \xi_k}\right| \leq n_1 , \qquad \text{when } k \geq 2 .$$

The estimate for the Jacobian follows.

For the open sets N_μ of unit vectors defined in Section 3, we associate the open cones $N_\mu(\mathbb{C}^m)$ in $\mathbb{C}^m$ given by

$$\begin{aligned} N_\mu(\mathbb{C}^m) &= \bigcup\{n(\mathbb{C}^m) :\ n \in N_\mu\} \\ &= \{\zeta = \xi + i\eta \in \mathbb{C}^m :\ |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ and} \\ &\qquad\qquad \eta + \mathrm{Re}\,(|\zeta|_{\mathbb{C}})\,e_L = \kappa n \\ &\qquad\qquad \text{for some } \kappa > 0 \text{ and } n \in N_\mu\} . \end{aligned}$$

Since $N_\mu(\mathbb{C}^m) \subset S^\circ_{\mu_N+\mu}(\mathbb{C}^m)$, the estimates in Theorem 2.1 all hold with $\theta = \mu_N + \mu$.

When N is rotationally symmetric, namely

$$N = \{n = \mathbf{n} + n_L e_L \in \mathbb{R}^{m+1}_+ : \ |n| = 1, \ n_L \geq |\mathbf{n}| \cot w\},$$

we have $S^\circ_\mu(\mathbb{C}^m) = N_{\mu-w}(\mathbb{C}^m)$. Again we allow functions to take their values in the complex Clifford algebra $\mathbb{C}_{(M)}$, so for example $H_\infty(N_\mu(\mathbb{C}^m))$ denotes the Banach space of all bounded holomorphic functions from $N_\mu(\mathbb{C}^m)$ to $\mathbb{C}_{(M)}$ under the norm $\|b\|_\infty = \sup\{|b(\zeta)| : \ \zeta \in N_\mu(\mathbb{C}^m)\}$.

Crucial to this paper are the exponential functions

$$e(x,\zeta) = e_+(x,\zeta) + e_-(x,\zeta),$$

where

$$e_+(x,z) = e^{i\langle \mathbf{x},\zeta\rangle}\, e^{-x_L|\zeta|_\mathbb{C}}\, \chi_+(\zeta)$$

and

$$e_-(x,\zeta) = e^{i\langle \mathbf{x},\zeta\rangle}\, e^{x_L|\zeta|_\mathbb{C}}\, \chi_-(\zeta).$$

They are entire left-monogenic functions of $x \in \mathbb{R}^{m+1}$ (for fixed ζ), and holomorphic functions of $\zeta \in N_\mu(\mathbb{C}^m)$ (for fixed x), which satisfy the bounds

$$\begin{aligned} |e_+(x,\zeta)| &= e^{-\langle \mathbf{x},\eta\rangle - x_L \operatorname{Re}|\zeta|_\mathbb{C}}\, |\chi_+(\zeta)| \\ &\leq \frac{\sec(\mu_N+\mu)}{\sqrt{2}}\, e^{-\langle x,n\rangle \operatorname{Re}|\zeta|_\mathbb{C}/n_L}, \qquad \zeta \in n(\mathbb{C}^m), \end{aligned}$$

and

$$\begin{aligned} |e_-(x,\zeta)| &= e^{-\langle \mathbf{x},\eta\rangle + x_L \operatorname{Re}|\zeta|_\mathbb{C}}\, |\chi_-(\zeta)| \\ &\leq \frac{\sec(\mu_N+\mu)}{\sqrt{2}}\, e^{\langle x,n\rangle \operatorname{Re}|\zeta|_\mathbb{C}/n_L}, \qquad \zeta \in \bar{n}(\mathbb{C}^m). \end{aligned}$$

Let

$$H^\pm_\infty(N_\mu(\mathbb{C}^m)) = \{b \in H_\infty(N_\mu(\mathbb{C}^m)) : \ b\chi_\pm = b\}.$$

Then every function $b \in H_\infty(N_\mu(\mathbb{C}^m))$ can be uniquely decomposed as $b = b_+ + b_-$ where $b_\pm = b\chi_\pm \in H^\pm_\infty(N_\mu(\mathbb{C}^m))$ These are closed linear subspaces of $H_\infty(N_\mu(\mathbb{C}^m))$, and indeed

$$H_\infty(N_\mu(\mathbb{C}^m)) = H^+_\infty(N_\mu(\mathbb{C}^m)) \oplus H^-_\infty(N_\mu(\mathbb{C}^m))$$

because

$$\|b\chi_\pm\|_\infty \leq \sqrt{2}\,\|b\|_\infty\,\|\chi_\pm\|_\infty \leq \sec(\mu_N + \mu)\,\|b\|_\infty\,,$$

for all $b \in H_\infty(N_\mu(\mathbb{C}^m))$.

We also introduce the subalgebras

$$\mathcal{A}(N_\mu(\mathbb{C}^m)) = \{b \in H_\infty(N_\mu(\mathbb{C}^m)) :\ \zeta e_L b(\zeta) = b(\zeta)\zeta e_L \text{ for all } \zeta\}\,.$$

Define $\mathcal{A}^\pm(N_\mu(\mathbb{C}^m))$ similarly, and note that if $b \in \mathcal{A}(N_\mu(\mathbb{C}^m))$ then $b_\pm = b\chi_\pm \in \mathcal{A}^\pm(N_\mu(\mathbb{C}^m))$, so that

$$\mathcal{A}(N_\mu(\mathbb{C}^m)) = \mathcal{A}^+(N_\mu(\mathbb{C}^m)) \oplus \mathcal{A}^-(N_\mu(\mathbb{C}^m))\,.$$

Particular functions b belonging to $\mathcal{A}(N_\mu(\mathbb{C}^m))$ are those of the form

$$b(\zeta) = B(i\,\zeta e_L) = B(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)$$

defined in Section 2, where $B \in H_\infty(S^\circ_{\mu_N+\mu}(\mathbb{C}))$. Also in $\mathcal{A}(N_\mu(\mathbb{C}^m))$ are all scalar-valued holomorphic functions in $H_\infty(N_\mu(\mathbb{C}^m))$, the simplest being $r_k(\zeta) = i\,\zeta_k/|\zeta|_{\mathbb{C}}\,,\ k = 1, 2, \ldots, m\,.$

Let H_N^+ be the algebra of all those functions b on $\mathbb{R}^m \setminus \{0\}$ which extend holomorphically to $b \in H_\infty^+(N_\mu(\mathbb{C}^m))$ for some $\mu > 0$. The algebra H_N^- however is defined to be the algebra of all those functions b on $\mathbb{R}^m \setminus \{0\}$ which extend holomorphically to $b \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$ for some $\mu > 0$, where $\overline{N} = \{\overline{n} \in \mathbb{R}^{m+1} :\ n \in N\}$. Then $H_N^+ \cap H_N^- = \{0\}$.

Define H_N by $H_N = H_N^+ + H_N^-$. Then $H_N = H_N^+ \oplus H_N^-$. Let $\mathcal{A}_N^+$, $\mathcal{A}_N^-$ and $\mathcal{A}_N$ be the subspaces of H_N^+, H_N^- and H_N consisting of all those functions which satisfy $\xi e_L b(\xi) = b(\xi)\xi e_L$ for all $\xi \neq \mathbf{0}$. Then $\mathcal{A}_N = \mathcal{A}_N^+ \oplus \mathcal{A}_N^-\,.$

We need to ensure that these holomorphic extensions are unique, which we can do by assuming that N is connected. In fact we shall make the stronger assumption that the compact sets N of unit vectors in $\mathbb{R}_+^{m+1}$ are starlike about e_L (in the sense that, whenever $n \in N$ and $0 \leq \tau \leq 1$, then $(\tau n + (1-\tau)e_L)/|\tau n + (1-t)e_L| \in N$). In this case, the open sets N_μ are also starlike about e_L in the same sense, and $N_\mu(\mathbb{C}^m)$ are connected open subsets of $\mathbb{C}^m$.

Theorem 4.1. *Let N be a compact set of unit vectors in $\mathbb{R}^{m+1}_+$ which is starlike about e_L. For every $\Phi \in K_N^+$, there exists a unique function $b \in H_N^+$ which satisfies Parseval's identity*

$$(\mathrm{P}) \qquad \begin{aligned} &(2\pi)^{-m} \int_{\mathbb{R}^m} b(\xi)\, \hat{u}(-\xi)\, d\xi \\ &\qquad = \lim_{\alpha \to 0+} \int_{\mathbb{R}^m} \Phi(\mathbf{x} + \alpha\, e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x} \\ &\qquad = \lim_{\varepsilon \to 0} \Big(\int_{|\mathbf{x}| \geq \varepsilon} \Phi(\mathbf{x})\, e_L\, u(\mathbf{x})\, d\mathbf{x} + \underline{\Phi}(\varepsilon\, e_L)\, u(\mathbf{0}) \Big), \end{aligned}$$

for all u in the Schwartz space $\mathcal{S}(\mathbb{R}^m)$, where $\underline{\Phi}$ is defined in Theorem 3.1. So b is the distribution Fourier transform of Φe_L, and we write $b = \mathcal{F}_+(\Phi) e_L$. We also call Φ the inverse Fourier transform of $b\,\overline{e_L}$, and write $\Phi = \mathcal{G}_+(b\,\overline{e_L})$.

The Fourier transform $\mathcal{F}_+$ is a linear transformation with the following properties.

i) *$\mathcal{F}_+$ is a one-one map of K_N^+ onto H_N^+. That is, for every $b \in H_N^+$ there exists a unique function $\Phi \in K_N^+$ such that $b = \mathcal{F}_+(\Phi) e_L$.*

ii) *If $0 < \nu < \mu \leq \pi/2 - \mu_N$ and $\Phi \in K(C_{N_\mu}^+)$ then $b \in H_\infty^+(N_\nu(\mathbb{C}^m))$ and $\|b\|_\infty \leq c_\nu \|\Phi\|_{K(C_{N_\mu}^+)}$ for some constant c_ν which depends on ν (as well as on μ_N and μ).*

iii) *If $0 < \nu < \mu \leq \pi/2 - \mu_N$ and $b \in H_\infty^+(N_\mu(\mathbb{C}^m))$ then $\Phi \in K(C_{N_\nu}^+)$ and $\|\Phi\|_{K(C_{N_\nu}^+)} \leq c_\nu \|b\|_\infty$ for some constant c_ν which depends on ν (as well as on μ_N and μ).*

iv) *$\Phi \in M_N^+$ if and only if $b \in \mathcal{A}_N^+$.*

v) *If $\Phi \in K_N^+$, $\Psi \in M_N^+$, $b = \mathcal{F}_+(\Phi) e_L$ and $f = \mathcal{F}_+(\Psi) e_L$, then*

$$bf = \mathcal{F}_+(\Phi * \Psi) e_L\ .$$

vi) *The mapping $\Phi \mapsto b$ is an algebra homomorphism from the convolution algebra M_N^+ onto the function algebra $\mathcal{A}_N^+$.*

vii) *Let p be a polynomial in m variables with values in $\mathbb{C}_{(M)}$, and let*

$$\Psi = p\Big(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_m}\Big)\, \Phi\,.$$

Then $\Psi \in K_N^+$ *if and only if* $p\,b \in H_N^+$, *in which case* $\mathcal{F}_+(\Psi)e_L = p\,b$.

viii) *If* $0 < \nu < \mu \le \pi/2 - \mu_N$, $s > -m$, *and* b *extends holomorphically to a bounded function which satisfies* $|b(\zeta)| \le c\,|\zeta|^s$ *for some* c_s *and all* $\zeta \in N_\mu(\mathbb{C}^m)$, *then there exists* $c_{s,\nu}$ *such that* $|\Phi(x)| \le c_{s,\nu}\,|x|^{-m-s}$ *for all* $x \in C_{N_\nu}^+$.

Hence $|\underline{\Phi}(y)| \le c_{s,\nu}\,|y|^{-s}$ *for all* $y \in T_{N_\mu}$, *so that, in particular, when* $-m < s < 0$, *we have* $\lim_{y\to 0}\underline{\Phi}(y) = 0$.

PROOF. In the estimates which follow, constants c may depend on μ_N, μ and the dimension m, and may vary from line to line. Dependence on ν will be specified by using c_ν.

Let $\Phi \in K(C_{N_\mu}^+)$. It is easy to see that either form of Parseval's identity uniquely determines b on $\mathbb{R}^m$ and therefore on $N_\mu(\mathbb{C}^m)$ (because it is a connected open set).

To construct b, we proceed as follows. For $\alpha > 0$, define $\Phi_\alpha(x) = \Phi(x + \alpha e_L)$, $x + \alpha e_L \in C_{N_\mu}^+$, in which case

$$\begin{aligned}\|\Phi_\alpha\|_{K(C_{N_\mu}^+)} &= \frac{1}{2}\sigma_m \sup\{|x|^m|\Phi(x+\alpha e_L)| : x \in C_{N_\mu}^+\}\\ &\le \sup\{|y|^m|\Phi(y)| : y \in C_{N_\mu}^+ + \alpha e_L\} \le \|\Phi\|_{K(C_{N_\mu}^+)}\,.\end{aligned}$$

For $\zeta \in n(\mathbb{C}^m) \subset N_\nu(\mathbb{C}^m) \subset N_\mu(\mathbb{C}^m)$, $\nu < \mu$, define

$$b_\alpha(\zeta) = \int_\sigma \Phi_\alpha(x)\,n(x)\,e_+(-x,\zeta)\,dS_x\ ,$$

where σ is the surface defined by

$$\sigma = \{x \in \mathbb{R}^{m+1} : \langle x, n\rangle = -|x|\,\sin(\mu-\nu)\}\,.$$

Note that the integrand is continuous and exponentially decreasing at infinity. (As usual, $n(x)$ denotes the normal to σ with $n_L(x) > 0$). Indeed, for $x \in \sigma$,

$$\begin{aligned}|e_+(-x,\zeta)| &\le \frac{\sec(\mu_N+\mu)}{\sqrt{2}}\,e^{\langle x,n\rangle \mathrm{Re}\,|\zeta|_{\mathbb{C}}/n_L}\\ &\le \frac{\sec(\mu_N+\mu)}{\sqrt{2}}\,e^{-|x|\,|\xi|\sin\theta}\,,\end{aligned}$$

where $\theta = \mu - \nu$.

Because of this fact, and Cauchy's theorem for monogenic functions (noting that Φ_α is right-monogenic and $e_+(-x,\zeta)$ is left-monogenic in x), we see that the definition of $b_\alpha(\zeta)$ does not depend on the precise surface σ chosen. So $b_\alpha(\zeta)$ depends holomorphically on $\zeta \in N_\mu(\mathbb{C}^m)$. Moreover

$$\begin{aligned} b_\alpha(\zeta)\, e^{\alpha|\zeta|_{\mathbb{C}}} &= \int_\sigma \Phi(x+\alpha e_L)\, n(x)\, e_+(-(x+\alpha e_L),\zeta)\, dS_x \\ &= \int_\sigma \Phi(x+\beta e_L)\, n(x)\, e_+(-(x+\beta e_L),\zeta)\, dS_x \\ &= b_\beta(\zeta)\, e^{\beta|\zeta|_{\mathbb{C}}}\,, \end{aligned}$$

for all $\alpha, \beta > 0$, so it makes sense to define b as the holomorphic function on $N_\mu(\mathbb{C}^m)$ which satisfies

$$b(z) = b_\alpha(\zeta)\, e^{\alpha|\zeta|_{\mathbb{C}}}\,, \qquad \text{for all } \alpha > 0\,.$$

We shall prove in a moment that

$$(\#) \qquad |b_\alpha(\zeta)| \le c_\nu\, \|\Phi\|_{K(C^+_{N_\mu})}\,, \qquad \text{for all } z \in N_\mu(\mathbb{C}^m)\,,$$

(where c_ν is independent of α) and

$$(\#\#) \qquad (2\pi)^{-m} \int_{\mathbb{R}^m} b_\alpha(\xi)\, \hat{u}(-\xi)\, d\xi = \int_{\mathbb{R}^m} \Phi(\mathbf{x}+\alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}\,.$$

The first version of Parseval's identity (P) follows as a consequence, as does the estimate in ii).

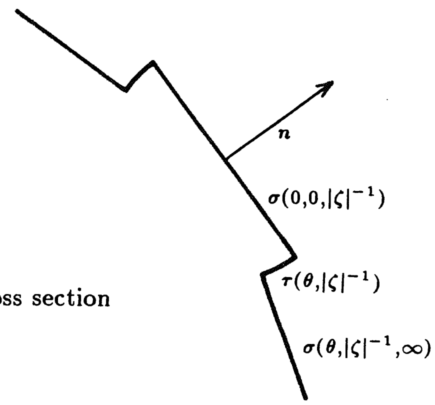
Let us prove (#). With $\zeta \in n(\mathbb{C}^m) \subset N_\nu(\mathbb{C}^m) \subset N_\mu(\mathbb{C}^m)$ and $\theta = \mu - \nu$ as before, apply Cauchy's theorem to change the surface of integration, so that

$$b_\alpha(\zeta) = \Big(\int_{\sigma(0,0,|\zeta|^{-1})} + \int_{\tau(\theta,|\zeta|^{-1})} + \int_{\sigma(\theta,|\zeta|^{-1},\infty)} \Big)\, \Phi_\alpha(x)\, n(x)\, e_+(-x,\zeta)\, dS_x\,,$$

where

$$\sigma(\theta,r,R) = \{x \in \mathbb{R}^{m+1} : \langle x,n\rangle = |x|\sin\theta,\ r \le |x| \le R\}\,,$$

$$\tau(\theta,R) = \{x \in \mathbb{R}^{m+1} : |x| = R,\ 0 \ge \langle x,n\rangle \ge -R\sin\theta\}\,.$$



Cross section

We need some estimates.

$$\Big|\int_{\sigma(0,0,R)} \Phi_\alpha(x)\, n\, e_+(-x,\zeta)\, dS_x\Big|$$

$$\le c\Big|\int_{\sigma(0,0,R)} \Phi_\alpha(x)\, n\, e(-x,\zeta)\, dS_x\Big|$$

$$\le c\Big|\int_{\sigma(0,0,R)} \Phi_\alpha(x)\, n\, \big(e(-x,\zeta)-1\big)\, dS_x\Big|$$

a)
$$+\, c\Big|\int_{\substack{\langle x,n\rangle \ge 0\\ |x|=R}} \Phi_\alpha(x)\, n\, dS_x\Big|$$

$$\le c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\Big(\sup\{|\nabla_y e(-y,\zeta)| :\ y\in\sigma(0,0,R)\} \cdot \int_{\sigma(0,0,R)} |x|^{-m}|x|\, dS_x + 1\Big)$$

$$\le c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\,(R\,|\zeta|+1)$$

$$\le c\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\,, \qquad \text{provided } R\le|\zeta|^{-1}\,,$$

b)

$$
\begin{aligned}
&\Big| \int\limits_{\tau(\theta,R)} \Phi_\alpha(x)\, n(x)\, e_+(-x,\zeta)\, dS_x \Big| \\
&\qquad \le c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})}\, R^{-m} \int\limits_{\tau(\theta,R)} e^{\langle x,n\rangle \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}\, dS_x \\
&\qquad = c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int\limits_{\tau(\theta,1)} e^{\langle x,n\rangle R \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}\, dS_x \\
&\qquad = c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int_{-\theta}^{0} e^{R \operatorname{Re}|\zeta|_{\mathbb{C}} \sin\Phi/n_L}\, d\Phi \\
&\qquad \le \frac{c}{R\,|\zeta|} \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \\
&\qquad \le c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})}\,, \qquad\qquad \text{provided } R \ge |\zeta|^{-1}\,,
\end{aligned}
$$

c)

$$
\begin{aligned}
&\Big| \int\limits_{\sigma(\theta,R,\infty)} \Phi_\alpha(x)\, n(x)\, e_+(-x,\zeta)\, dS_x \Big| \\
&\qquad \le c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int\limits_{\sigma(\theta,R,\infty)} |x|^{-m} e^{\langle x,n\rangle \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}\, dS_x \\
&\qquad = c\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int_{R}^{\infty} s^{-1} e^{-s\sin\theta \operatorname{Re}|\zeta|_{\mathbb{C}}/n_L}\, ds \\
&\qquad \le \frac{c_\nu}{R\,|\zeta|} \|\Phi_\alpha\|_{K(C^+_{N_\mu})} \\
&\qquad \le c_\nu\, \|\Phi_\alpha\|_{K(C^+_{N_\mu})}\,, \qquad\qquad \text{provided } R \ge |\zeta|^{-1}\,.
\end{aligned}
$$

On using the above three estimates with $R = |\zeta|^{-1}$, together with the preceding representation of b_α, we find that we have proved (#).

Now we shall prove (##). If we define $b_{\alpha,N}(\xi)$ for $\xi \in \mathbb{R}^m$ by

$$b_{\alpha,N}(\xi) = \int\limits_{|\mathbf{x}|\le N} \Phi_\alpha(\mathbf{x})\, e_L\, e^{i\langle \mathbf{x},\xi\rangle}\, d\mathbf{x}\,,$$

then the usual Parseval's identity gives

$$(2\pi)^{-m} \int_{\mathbb{R}^m} b_{\alpha,N}(\xi)\, \hat{u}(-\xi)\, d\xi = \int\limits_{|\mathbf{x}|\le N} \Phi(\mathbf{x} + \alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}\,,$$

for all $u \in \mathcal{S}(\mathbb{R}^m)$. We shall prove that

(∗) $|b_{\alpha,N}(\xi)| \leq c\,\|\Phi\|_{K(C^+_{N_\mu})}$ for all $\xi \in \mathbb{R}^m$ and $N > 0$,

(∗∗) for each $\xi \in \mathbb{R}^m$, $b_{\alpha,N}(\xi)\,\chi_+(\xi) \to b_\alpha(\xi)$, as $N \to \infty$, and

(∗ ∗ ∗) for each $\xi \in \mathbb{R}^m$, $b_{\alpha,N}(\xi)\,\chi_-(\xi) \to 0$, as $N \to \infty$.

Then (##) follows from these results and the Lebesgue dominated convergence theorem.

In proving (∗) and (∗∗), we use the estimates *a*), *b*) and *c*) above with $n = e_L$ in the definitions of σ, $\sigma(\theta, r, R)$ and $\tau(\theta, R)$.

First we prove (∗) when $|\xi|^{-1} \leq N$. Choose $0 < \theta < \mu$, and apply Cauchy's theorem to write

$$b_{\alpha,N}(\xi)\,\chi_+(\xi) = \Bigg(\int\limits_{\sigma(0,0,|\xi|^{-1})} + \int\limits_{\tau(\theta,|\xi|^{-1})} + \int\limits_{\sigma(\theta,|\xi|^{-1},N)} - \int\limits_{\tau(\theta,N)} \Bigg)\, \Phi_\alpha(x)\, n(x)\, e_+(-x,\xi)\, dS_x \ ,$$

so the uniform boundedness of $b_{\alpha,N}(\xi)\,\chi_+(\xi)$ in ξ and N follows from *a*), *b*) and *c*). On the other hand,

$$b_{\alpha,N}(\xi)\,\chi_-(\xi) = \int\limits_{H_{N+}} \Phi_\alpha(x)\, n(x)\, e_-(-x,\xi)\, dS_x \ ,$$

so, on using similar reasoning to the proof of *b*),

$$|b_{\alpha,N}(\xi)\,\chi_-(\xi)| \leq \frac{c}{N\,|\xi|}\, \|\Phi\|_{K(C^+_{N_\mu})} \leq c\,\|\Phi\|_{K(C^+_{N_\mu})} \ .$$

To prove (∗) when $|\xi|^{-1} \geq N$, only *a*) is needed.

To prove (∗∗), fix $\xi \in \mathbb{R}^m$, $\xi \neq \mathbf{0}$, and apply Cauchy's theorem to write

$$b_\alpha(\xi) - b_{\alpha,N}(\xi)\,\chi_+(\xi) = \Bigg(\int\limits_{\tau(\theta,N)} + \int\limits_{\sigma(\theta,N,\infty)} \Bigg)\, \Phi_\alpha(x)\, n(x)\, e_+(-x,\xi)\, dS_x \ ,$$

so, by b) and c),

$$|b_\alpha(\xi) - b_{\alpha,N}(\xi)\,\chi_+(\xi)| \le \frac{c}{N\,|\xi|}\,\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \to 0\,, \qquad \text{as } N \to \infty\,.$$

Moreover, $(***)$ follows from the estimate a few lines above.

As noted previously, the first version of Parseval's identity (P) follows. Our next task is to prove the second version of (P). Let $\varepsilon > 0$. Then

$$\begin{aligned}
(2\pi)^{-m} \int_{\mathbb{R}^m} b(\xi)\,\hat{u}(-\xi)\,d\xi
&= \lim_{\alpha\to 0+} \Big(\int\limits_{|\mathbf{x}|\ge\varepsilon} \Phi(\mathbf{x}+\alpha\,e_L)\,e_L\,u(\mathbf{x})\,d\mathbf{x} \\
&\qquad + \int\limits_{|\mathbf{x}|\le\varepsilon} \Phi(\mathbf{x}+\alpha\,e_L)\,e_L\,u(\mathbf{0})\,d\mathbf{x} \\
&\qquad + \int\limits_{|\mathbf{x}|\le\varepsilon} \Phi(\mathbf{x}+\alpha\,e_L)\,e_L\,(u(\mathbf{x})-u(\mathbf{0}))\,d\mathbf{x}\Big) \\
&= \int\limits_{|\mathbf{x}|\ge\varepsilon} \Phi(\mathbf{x})\,e_L\,u(\mathbf{x})\,d\mathbf{x} + \underline{\Phi}(\varepsilon)\,u(\mathbf{0}) \\
&\quad + \lim_{\alpha\to 0+} \Big(\int\limits_{|\mathbf{x}|\le\varepsilon} \Phi(\mathbf{x}+\alpha\,e_L)\,e_L\,(u(\mathbf{x})-u(\mathbf{0}))\,d\mathbf{x}\Big)\,,
\end{aligned}$$

(with Cauchy's theorem being used to evaluate the second integral). Now

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}}\ \lim_{\alpha\to 0+} \Big(\int\limits_{|\mathbf{x}|\le\varepsilon} |\Phi(\mathbf{x}+\alpha\,e_L)\,e_L(u(\mathbf{x})-u(\mathbf{0}))|\,d\mathbf{x}\Big) \\
&\quad\le \overline{\lim_{\varepsilon\to 0}}\ \lim_{\alpha\to 0+} \Big(C \int\limits_{|\mathbf{x}|\le\varepsilon} |\mathbf{x}+\alpha\,e_L|^{-m}\,|u(\mathbf{x})-u(\mathbf{0})|\,d\mathbf{x}\Big) \\
&\quad\le \overline{\lim_{\varepsilon\to 0}}\ \lim_{\alpha\to 0+} \Big(C \int\limits_{|\mathbf{x}|\le\varepsilon} |\mathbf{x}|^{-m}|u(\mathbf{x})-u(\mathbf{0})|\,d\mathbf{x}\Big) = 0
\end{aligned}$$

(as $u \in \mathcal{S}(\mathbb{R}^m)$), so

$$(2\pi)^{-m}\int_{\mathbb{R}^m} b(\xi)\,\hat{u}(-\xi)\,d\xi = \lim_{\varepsilon\to 0}\Big(\int_{|\mathbf{x}|\geq\varepsilon} \Phi(\mathbf{x})\,e_L\,u(\mathbf{x})\,d\mathbf{x} + \underline{\Phi}(\varepsilon)\,u(\mathbf{0})\Big)\,,$$

as required.

This completes the proof of the introductory statement in the theorem, together with the estimates in ii).

PROOF OF i) AND iii). It is easily verified that $\mathcal{F}_+$ is one-one. We prove it maps onto H_N^+ by constructing the inverse Fourier transform $\mathcal{G}_+$.

Consider functions $b \in H_\infty^+(N_\mu(\mathbb{C}^m))$. For $n \in N_\mu$, and $x = \mathbf{x} + x_L e_L \in C_n^+ \subset C_{N_\mu}^+$, define

$$\begin{aligned}\Phi_n(x) &= (2\pi)^{-m}\int_{n(\mathbb{C}^m)} b(\zeta)\,e(x,\zeta)\,d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_m\,\overline{e_L}\\ &= (2\pi)^{-m}\int_{n(\mathbb{C}^m)} b(\zeta)\,e_+(x,\zeta)\,d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_m\,\overline{e_L}\,.\end{aligned}$$

On the surface $n(\mathbb{C}^m)$, the integrand is exponentially decreasing at infinity. Indeed, when $\zeta \in n(\mathbb{C}^m)$, then

$$|e^{i\langle\mathbf{x},\zeta\rangle}\,e^{-x_L|\zeta|_{\mathbb{C}}}| \leq c\,e^{-\langle x,n\rangle\,\mathrm{Re}\,|\zeta|_{\mathbb{C}}/n_L}$$

and $\langle x,n\rangle > 0$. Moreover, $e(x,\zeta)\overline{e_L}$ is right-monogenic, so Φ_n is a right-monogenic function on C_n^+ which satisfies

$$|\Phi_n(x)| \leq \frac{c}{\langle x,n\rangle^m}\,\|b\|_\infty\,,\qquad x\in C_n^+\,,$$

where c depends only on μ_N and μ.

Moreover the integrand depends holomorphically on the single complex variable $z = \langle\zeta,\mathbf{n}\rangle$ (on writing $\zeta = z\mathbf{n} + \zeta'$ where $\langle\zeta',\mathbf{n}\rangle = 0$, and holding ζ' constant). So, by the starlike nature of N_μ, and Cauchy's theorem in the z-plane, we find that $\Phi_n(x) = \Phi_{e_L}(x)$ for all $x \in C_n^+$ with $x_L > 0$. Hence there is a unique right-monogenic function Φ on $C_{N_\mu}^+$ which coincides with each of the functions $\Phi_n(x)$ on C_n^+. We call

Φ the *inverse Fourier transform* of $b\,\overline{e_L}$, and write $\Phi = \mathcal{G}_+(b\,\overline{e_L})$. The above estimates for Φ_n imply that $\Phi \in K(C^+_{N_\nu})$ for all $\nu < \mu$, and

$$\|\Phi\|_{K(C^+_{N_\nu})} \leq c_\nu \, \|b\|_\infty \,.$$

We remark that, in the particular case when $x_L = 0$ and $|b(\zeta)| \leq c(1 + |\zeta|^{m+1})^{-1}$ for all $\zeta \in N_\mu(\mathbb{C}^m)$, then, by Cauchy's theorem, we can change the surface of integration to conclude that

$$\mathcal{G}_+(b\,\overline{e_L})(\mathbf{x}) = \Phi(\mathbf{x}) = (2\pi)^{-m} \int_{\mathbb{R}^m} b(\xi)\, e^{i\langle \mathbf{x},\xi\rangle}\, d\xi\, \overline{e_L} = \check{b}(\mathbf{x})\, \overline{e_L}\,,$$

which is the usual inverse Fourier transform of $b\,\overline{e_L}$.

Let us show that b and $\Phi = \mathcal{G}_+(b\,\overline{e_L})$ satisfy Parseval's identity (P), from which we conclude that $\mathcal{G}_+$ really is the inverse of the Fourier transform $\mathcal{F}_+$, and complete our proof of i) and iii).

Let $b_\alpha(\zeta) = b(\zeta)\, e^{-\alpha|\zeta|_{\mathbb{C}}}$ for $\alpha > 0$. Then, for $\mathbf{x} \in \mathbb{R}^m$,

$$\Phi(\mathbf{x} + \alpha e_L) = \mathcal{G}_+(b\,\overline{e_L})(\mathbf{x} + \alpha e_L) = \mathcal{G}_+(b_\alpha\, \overline{e_L})(\mathbf{x}) = (b_\alpha)^\vee(\mathbf{x})\, \overline{e_L}$$

(by the above remark). Apply the usual Parseval's identity to obtain

$$(2\pi)^{-m} \int_{\mathbb{R}^m} b_\alpha(\xi)\, \hat{u}(-\xi)\, d\xi = \int_{\mathbb{R}^m} \Phi(\mathbf{x} + \alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}\,,$$

and hence

$$(2\pi)^{-m} \int_{\mathbb{R}^m} b(\xi)\, \hat{u}(-\xi)\, d\xi = \lim_{\alpha \to 0+} \int_{\mathbb{R}^m} \Phi(\mathbf{x} + \alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}\,,$$

for all $u \in \mathcal{S}(\mathbb{R}^m)$, as required.

PROOF OF iv). Choose $\Phi \in K(C^+_{N_\mu})$. Then Φ is left- (as well as right-monogenic) if and only if

$$\mathbf{D}\, e_L \Phi(x) = (\Phi e_L)\, \mathbf{D}(x)\,, \qquad \text{for all } x \in C^+_{N_\mu}\,.$$

(Both sides equal $-\dfrac{\partial \Phi}{\partial x_L}(x)$).

Let $b\,\overline{e_L} = \mathcal{F}_+(\Phi)$, define b_α as above, and use twice the version of Parseval's identity involving b_α, to see that, for all $u \in \mathcal{S}(\mathbb{R}^m)$,

$$(2\pi)^{-m} \int_{\mathbb{R}^m} \xi\, e_L\, b_\alpha(\xi)\, \hat{u}(-\xi)\, d\xi = -i \int_{\mathbb{R}^m} (\mathbf{D}\, e_L \Phi)(\mathbf{x} + \alpha e_L)\, e_L\, u(\mathbf{x})\, d\mathbf{x}$$

and

$$(2\pi)^{-m}\int_{\mathbb{R}^m} b_\alpha(\xi)\,\xi\, e_L\,\hat{u}(-\xi)\,d\xi = -i\int_{\mathbb{R}^m}(\Phi e_L\mathbf{D})(\mathbf{x}+\alpha e_L)\,e_L\,u(\mathbf{x})\,d\mathbf{x}\,.$$

So $\Phi \in M(C^+_{N_\mu})$ if and only if $\mathbf{D}\, e_L\Phi(x) = (\Phi e_L)\mathbf{D}(x)$ for all $x \in C^+_{N_\mu}$, which holds if and only if

$$\mathbf{D}\, e_L\Phi(\mathbf{x}+\alpha e_L) = (\Phi e_L)\mathbf{D}(\mathbf{x}+\alpha e_L)\,, \qquad \text{for all } x \in \mathbb{R}^m \setminus \{0\}$$

(by the right-monogenicity of the functions on both sides of this equation). By the above identities, this is true if and only if $\xi e_L b_\alpha(\xi) = b_\alpha(\xi)\xi e_L$. And this equation is satisfied if and only if $\zeta e_L b(\zeta) = b(\zeta)\zeta e_L$ for all $z \in N_\mu(\mathbb{C}^m)$. This proves iv).

The remaining parts can be proved in a similar way, with the estimates in viii) requiring a modification of the proof of iii).

Theorem 4.2. *The statement of Theorem* 4.1 *remains valid when the following changes are made.*

Replace $C^+_{N_\mu}$, $N_\mu(\mathbb{C}^m)$, K^+_N, M^+_N, $H^+_\infty(N_\mu(\mathbb{C}^m))$, H^+_N, $\mathcal{A}^+_N$ *and* $\mathcal{F}_+$ *by* $C^-_{N_\mu}$, $\overline{N}_\mu(\mathbb{C}^m)$, K^-_N, M^-_N, $H^-_\infty(\overline{N}_\mu(\mathbb{C}^m))$, H^-_N, $\mathcal{A}^-_N$ *and* $\mathcal{F}_-$ *respectively, and take the limit in* α *over negative* α.

Denote the inverse of $\mathcal{F}_-$ by $\mathcal{G}_- : H^-_N \longrightarrow K^-_N$, and call $\mathcal{F}_-$ the *Fourier transform* and $\mathcal{G}_-$ the *inverse Fourier transform.*

On combining Theorems 4.1 and 4.2, and using Theorem 3.1, the following result is obtained.

Theorem 4.3. *Let N be a compact set of unit vectors in $\mathbb{R}^{m+1}_+$ which is starlike about e_L. For every $(\Phi,\underline{\Phi}) \in K_N$, there exists a unique function $b \in H_N$ which satisfies Parseval's identity*

$$\text{(P)}\qquad \begin{aligned}&(2\pi)^{-m}\int_{\mathbb{R}^m} b(\xi)\,\hat{u}(-\xi)\,d\xi\\ &\qquad = \lim_{\varepsilon\to 0}\Big(\int_{|\mathbf{x}|\ge\varepsilon}\Phi(\mathbf{x})\,e_L\,u(\mathbf{x})\,d\mathbf{x} + \underline{\Phi}(\varepsilon\, e_L)\,u(\mathbf{0})\Big)\,,\end{aligned}$$

for all u in the Schwartz space $\mathcal{S}(\mathbb{R}^m)$. So $b\overline{e_L}$ is the distribution Fourier transform of $(\Phi,\underline{\Phi})$, and we write $b = \mathcal{F}(\Phi,\underline{\Phi})\,e_L$.

The Fourier transform $\mathcal{F}$ is a linear transformation with the following properties.

i) *$\mathcal{F}$ is a one-one map of K_N onto H_N. That is, for every $b \in H_N$ there exists a unique $(\Phi, \underline{\Phi}) \in K_N$ such that $b = \mathcal{F}(\Phi, \underline{\Phi})\, e_L$. Actually, if $b = b_+ + b_-$ with $b_\pm = b\,\chi_\pm \in H_N^\pm$, then $(\Phi, \underline{\Phi}) = (\Phi_+, \underline{\Phi_+}) + (\Phi_-, \underline{\Phi_-})$ where $\Phi_\pm = \mathcal{G}_\pm(b_\pm \overline{e_L}) \in K_N^\pm$.*

We write $(\Phi, \underline{\Phi}) = \mathcal{G}(b\overline{e_L})$, and call $\mathcal{G}$ the inverse Fourier transform.

ii) *If $0 < \nu < \mu \le \pi/2 - \mu_N$ and $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$ then $b_+ \in H_\infty^+(N_\nu(\mathbb{C}^m))$, $b_- \in H_\infty^-(\overline{N}_\nu(\mathbb{C}^m))$ and $\|b_\pm\|_\infty \le c_\nu\, \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})}$ for some constant c_ν which depends on ν (as well as on μ_N and μ).*

iii) *If $0 < \nu < \mu \le \pi/2 - \mu_N$ and $b_+ \in H_\infty^+(N_\mu(\mathbb{C}^m))$ and $b_- \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$, then $(\Phi, \underline{\Phi}) \in K(S_{N_\nu})$ and*

$$|(\Phi, \underline{\Phi})\|_{K(S_{N_\nu})} \le c_\nu\, (\|b_+\|_\infty + \|b_-\|_\infty)$$

for some constant c_ν which depends on ν (as well as on μ_N and μ).

iv) *$(\Phi, \underline{\Phi}) \in M_N$ if and only if $b \in \mathcal{A}_N$.*

v) *If $(\Phi, \underline{\Phi}) \in K_N$ and $(\Psi, \underline{\Psi}) \in M_N$ and $b = \mathcal{F}(\Phi, \underline{\Phi})\, e_L$ and $f = \mathcal{F}(\Psi, \underline{\Psi})\, e_L$, then*

$$bf = \mathcal{F}((\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi}))\, e_L\,.$$

vi) *The mapping $(\Phi, \underline{\Phi}) \mapsto b$ is an algebra homomorphism from the convolution algebra M_N onto the function algebra $\mathcal{A}_N$.*

vii) *If $(\Phi, \underline{\Phi}), (\Psi, \underline{\Psi}) \in K_N$, $b = \mathcal{F}(\Phi, \underline{\Phi})\, e_L$, $f = \mathcal{F}(\Psi, \underline{\Psi})\, e_L$, and if $f = p\, b$ where p is a polynomial in m variables with values in $\mathbb{C}_{(M)}$, then*

$$\Psi = p\Big(-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_m}\Big)\Phi\,.$$

viii) *If $0 < \nu < \mu \le \pi/2 - \mu_N$, $s > -m$, and b_+ (and b_-) extend holomorphically to bounded functions which satisfy $|b_\pm(\zeta)| \le c_s\, |\zeta|^s$ for some c_s and all $\zeta \in N_\mu(\mathbb{C}^m)$ (respectively, $\zeta \in \overline{N}_\mu(\mathbb{C}^m)$), then there exists $c_{s,\nu}$ such that $|\Phi(x)| \le c_{s,\nu}\, |x|^{-m-s}$ for all $x \in C_{N_\nu}^+$ and $|\underline{\Phi}(y)| \le c_{s,\nu}\, |y|^{-s}$ for all $y \in T_{N_\mu}$.*

Hence, in particular, when $-m < s < 0$, we have $\lim_{y \to 0} \underline{\Phi}(y) = 0$.

This result is a little nicer when $N = \overline{N}$. For then,

$$b_+ \in H_\infty^+(N_\mu(\mathbb{C}^m)) \qquad \text{and} \qquad b_- \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$$

if and only if
$$b \in H_\infty(N_\mu(\mathbb{C}^m)).$$

One application of the results of this section is to the investigation of monogenic extensions of functions defined on $\mathbb{R}^m \setminus \{0\}$. For example, consider
$$k_j(\mathbf{x}) = \frac{-x_j}{\sigma_m |\mathbf{x}|^{m+1}}, \qquad j = 1, 2, \ldots, m.$$
Our knowledge that the monogenic extension of $k(\mathbf{x}) = \sum_{j=1}^m k_j(\mathbf{x})\, e_j$ is
$$k(x) = \frac{\bar{x}}{|x|^{m+1}}, \qquad x \in \mathbb{R}^{m+1} \setminus \{0\},$$
does not help, because the individual components of k are not monogenic.

But we do know that the Fourier transform of $(2k_j, 0)$ is $r_j(\xi) = i\,\xi_j |\xi|^{-1}$, in the sense that $\Phi = 2k_j$, $\underline{\Phi} = 0$ and $b(\xi) = r_j(\xi)\, e_L$ satisfy Parseval's identity (P) in Theorem 4.3. We also know that r_j has a holomorphic extension which belongs to $H_\infty(S^\circ_\mu(\mathbb{C}^m))$ for all $\mu < \pi/2$, namely $r_j(\zeta) = i\,\zeta_j |\zeta|_{\mathbb{C}}^{-1}$. Hence $(2k_{(j)}, 0) = \mathcal{G}(r_j) \in K(S^\circ_\mu)$.

Therefore k_j has the right-monogenic extension $k_{(j)}$ to S°_μ for all $\mu < \pi/2$, and this extension satisfies $|k_{(j)}(x)| \le c_\mu / |x|^m$ for all $x \in S^\circ_\mu$.

We remark that, although the results in Section 3 were derived in [LMcS] without recourse to the Fourier theory just developed, many of these results were discovered, at least informally, by using Fourier transforms. In particular, the decomposition $K(S_{N_\mu}) \simeq K(C^+_{N_\mu}) \oplus K(C^-_{N_\mu})$ given in Theorem 3.1 can be obtained in this way, at least when $N = \overline{N}$, since Theorem 4.3 can be proved without making use of Theorem 3.1.

5. Connection with holomorphic functions of one variable.

Let $B \in H_\infty(S^\circ_\mu(\mathbb{C}))$, where $0 < \mu < \pi/2$. In Section 2 we saw that it is natural to associate with B the function $b \in H_\infty(S^\circ_\mu(\mathbb{C}^m))$, defined by $b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)$. Actually $b \in \mathcal{A}(S^\circ_\mu(\mathbb{C}^m)) = \{b \in H_\infty(S^\circ_\mu(\mathbb{C}^m)) : \zeta e_L b(\zeta) = b(\zeta)\zeta e_L \text{ for all } \zeta\}$, and the mapping $B \mapsto b$ is a one-one bounded algebra homomorphism from $H_\infty(S^\circ_\mu(\mathbb{C}))$ to $\mathcal{A}(S^\circ_\mu(\mathbb{C}^m))$.

Let us recall the symbols we are using for subsets of $\mathbb{C}$, $\mathbb{R}^{m+1}$ and $\mathbb{C}^m$:

$$C^\circ_{\mu^+}(\mathbb{C}) = \{Z = X + iY \in \mathbb{C} : \ Z \neq 0, \ Y > -|X| \tan \mu\},$$

$$C^\circ_{\mu^-}(\mathbb{C}) = -C^\circ_{\mu^+}(\mathbb{C}),$$

$$S^\circ_{\mu^+}(\mathbb{C}) = \{\lambda \in \mathbb{C} : \ \lambda \neq 0, \ |\arg(\lambda)| < \mu\}, \qquad S^\circ_{\mu^-}(\mathbb{C}) = -S^\circ_{\mu^+}(\mathbb{C}),$$

$$S^\circ_\mu(\mathbb{C}) = S^\circ_{\mu^+}(\mathbb{C}) \cup S^\circ_{\mu^-}(\mathbb{C}) = C^\circ_{\mu^+}(\mathbb{C}) \cap C^\circ_{\mu^-}(\mathbb{C}),$$

$$C^\circ_{\mu^+} = \{x = \mathbf{x} + x_L e_L \in \mathbb{R}^{m+1} : \ x_L > -|\mathbf{x}| \tan \mu\},$$

$$C^\circ_{\mu^-} = -C^\circ_{\mu^+}, \qquad S^\circ_\mu = C^\circ_{\mu^+} \cap C^\circ_{\mu^-},$$

$$T^\circ_\mu = \{y = \mathbf{y} + y_L e_L \in \mathbb{R}^{m+1} : \ y_L > |\mathbf{y}| \cot \mu\},$$

$$S^\circ_\mu(\mathbb{C}^m) = \{\zeta = \xi + i\eta \in \mathbb{C}^m : |\zeta|^2_{\mathbb{C}} \notin (-\infty, 0] \text{ and } |\eta| < \operatorname{Re}(|\zeta|_{\mathbb{C}}) \tan \mu\}.$$

Let us find the inverse Fourier transform of b in terms of the inverse Fourier transform of B.

We do this first for $B \in H_\infty(S^\circ_{\mu^+}(\mathbb{C}))$. In this case the inverse Fourier transform $\Phi = \mathcal{G}(B)$ of B is a complex-valued holomorphic function defined on $C^\circ_{\mu^+}(\mathbb{C})$. See Section 1. In particular

$$\Phi(Z) = \frac{1}{2\pi} \int_0^{+\infty} B(r)\, e^{irZ}\, dr, \qquad \text{when } \operatorname{Im}(Z) > 0.$$

The associated function b satisfies $b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\, \chi_+(\zeta)$, and therefore $b \in H^+_\infty(S^\circ_\mu(\mathbb{C}^m))$. Let $\Phi = \mathcal{G}_+(b\overline{e_L})$. Thus, when $x_L > 0$,

$$\begin{aligned}
\Phi(x) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} B(|\xi|)\, e_+(x,\xi)\, d\xi\, \overline{e_L} \\
&= \frac{1}{2(2\pi)^m} \int_{\mathbb{R}^m} \left(\overline{e_L} + \frac{i\xi}{|\xi|}\right) B(|\xi|)\, e^{-x_L|\xi|}\, e^{i\langle \mathbf{x},\xi\rangle}\, d\xi \\
&= \frac{1}{2(2\pi)^m} \int_{S^{m-1}} (\overline{e_L} + i\tau) \\
&\qquad\qquad \cdot \int_0^{+\infty} B(r)\, e^{-x_L r}\, e^{i\langle \mathbf{x},\tau\rangle r}\, r^{m-1}\, dr\, dS_\tau
\end{aligned}$$

$$
\begin{aligned}
(+) \qquad &= \frac{1}{2(2\pi i)^{m-1}} \int_{S^{m-1}} (\overline{e_L} + i\tau)\, \mathbf{\Phi}^{(m-1)}(\langle \mathbf{x}, \tau\rangle + ix_L)\, dS_\tau \\
&= \frac{1}{2(2\pi i)^{m-1}} \int_{S^{m-1}} (\overline{e_L} + i\langle \mathbf{x}, \tau\rangle \mathbf{x}|\mathbf{x}|^{-2}) \\
&\qquad\qquad \cdot \mathbf{\Phi}^{(m-1)}(\langle \mathbf{x}, \tau\rangle + ix_L)\, dS_\tau \\
&= \frac{\sigma_{m-2}}{2(2\pi i)^{m-1}} \int_{-1}^{1} (1-t^2)^{(m-3)/2} \Big(\overline{e_L} + \frac{it\,\mathbf{x}}{|\mathbf{x}|}\Big) \\
&\qquad\qquad \cdot \mathbf{\Phi}^{(m-1)}(|\mathbf{x}|t + ix_L)\, dt,
\end{aligned}
$$

where $\mathbf{\Phi}^{(m-1)}$ is the $(m-1)$-st derivative of $\mathbf{\Phi}$. As we know, Φ extends to a right- and left-monogenic function on $C^{\circ}_{\mu+}$ which belongs to $M(C^{\circ}_{\nu+})$ for all $\nu < \mu$.

When $B \in H_\infty(S^{\circ}_{\mu^-}(\mathbb{C}))$, $\mathbf{\Phi} = \mathcal{G}(B)$ and $b(\zeta) = B(i\zeta e_L) = B(-|\zeta|_{\mathbb{C}})\, \chi_-(\zeta)$, then $b \in H^-_\infty(S^{\circ}_{\mu}(\mathbb{C}^m))$, so we can form $\Phi = \mathcal{G}_-(b\,\overline{e_L})$. We see that when $x_L < 0$,

$$
\begin{aligned}
\Phi(x) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} B(-|\xi|)\, e_-(x,\xi)\, d\xi\, \overline{e_L} \\
&= \frac{1}{2(2\pi)^m} \int_{\mathbb{R}^m} \Big(\overline{e_L} - \frac{i\xi}{|\xi|}\Big) B(-|\xi|)\, e^{-x_L|\xi|}\, e^{i\langle \mathbf{x},\xi\rangle}\, d\xi \\
&= \frac{1}{2(2\pi)^m} \int_{S^{m-1}} (\overline{e_L} - i\tau) \int_0^{+\infty} B(-r)\, e^{x_L r}\, e^{i\langle \mathbf{x},\tau\rangle r}\, r^{m-1}\, dr\, dS_\tau \\
&= \frac{(-1)^{m-1}}{2(2\pi)^m} \int_{S^{m-1}} (\overline{e_L} + i\tau) \int_{-\infty}^{0} B(r)\, e^{-x_L r}\, e^{i\langle \mathbf{x},\tau\rangle r}\, r^{m-1}\, dr\, dS_\tau \\
&= \frac{1}{2(-2\pi i)^{m-1}} \int_{S^{m-1}} (\overline{e_L} + i\tau)\, \mathbf{\Phi}^{(m-1)}(\langle \mathbf{x},\tau\rangle + ix_L)\, dS_\tau \\
&= \frac{\sigma_{m-2}}{2(-2\pi i)^{m-1}} \int_{-1}^{1} (1-t^2)^{(m-3)/2} \Big(\overline{e_L} + \frac{it\mathbf{x}}{|\mathbf{x}|}\Big) \\
&\qquad\qquad \cdot \mathbf{\Phi}^{(m-1)}(|\mathbf{x}|t + ix_L)\, dt\,.
\end{aligned}
$$

When $B \in H_\infty(S^{\circ}_{\mu}(\mathbb{C}))$, write $B = B_+ + B_-$, where $B_+ = B\,\chi_{\mathrm{Re}>0} \in H_\infty(S^{\circ}_{\mu^+}(\mathbb{C}))$, and $B_- = B\,\chi_{\mathrm{Re}<0} \in H_\infty(S^{\circ}_{\mu^-})(\mathbb{C}))$. Then $b = b_+ + b_-$, where $b_\pm$ is associated with $B_\pm$. We can use this decomposition to relate the inverse Fourier transform $\mathcal{G}(b\,\overline{e_L}) = (\Phi, \underline{\Phi})$ of $b\,\overline{e_L}$ to the inverse Fourier transform $\mathcal{G}(B) = (\mathbf{\Phi}, \mathbf{\Phi}_1)$ of B.

In the following examples,

$$k(x) = \frac{1}{\sigma_m} \frac{\bar{x}}{|x|^{m+1}}$$

as usual.

$(\Phi(Z), \Phi_1(Z))$	$B(\lambda)$	$b(\zeta)$
$(0,1)$	1	1
$\left(\frac{i}{2\pi Z}, \frac{1}{2}\right)$	$\chi_{\mathrm{Re}>0}(\lambda)$	$\chi_+(\zeta)$
$\left(\frac{-i}{2\pi Z}, \frac{1}{2}\right)$	$\chi_{\mathrm{Re}<0}(\lambda)$	$\chi_-(\zeta)$
$\left(\frac{i}{\pi Z}, 0\right)$	$\mathrm{sgn}\,(\lambda)$	$\frac{i\zeta e_L}{\lvert\zeta\rvert_{\mathbb{C}}}$
$\frac{i}{2\pi}\Big(\frac{1}{Z+it}$, $-i\pi + \log\left(\frac{Z+it}{Z-it}\right)\Big)$	$\chi_{\mathrm{Re}>0}(\lambda)\, e^{-t\lambda}$ $(t>0)$	$\chi_+(\zeta)\, e^{-t\lvert\zeta\rvert_{\mathbb{C}}}$ $= \chi_+(\zeta)\, e^{-t(i\zeta e_L)}$
$\frac{t}{2\pi}\left(\frac{-1}{(Z+it)^2}, \frac{2Z}{Z^2+t^2}\right)$	$\chi_{\mathrm{Re}>0}(\lambda) t\lambda e^{-t\lambda}$ $(t>0)$	$\chi_+(\zeta)\, t\,\lvert\zeta\rvert_{\mathbb{C}} e^{-t\lvert\zeta\rvert_{\mathbb{C}}}$ $= i\chi_+(\zeta) t e_L \zeta\, e^{-t(i\zeta e_L)}$
$\Gamma(1+is)\big(\frac{i}{2\pi} e^{-\pi s/2} Z^{-1-is}$, $(\pi s)^{-1}\sinh(\pi s/2) Z^{-is}\big)$	$\chi_{\mathrm{Re}>0}(\lambda)\, \lambda^{is}$ $(s \in \mathbb{R})$	$\chi_+(\zeta)\, \lvert\zeta\rvert_{\mathbb{C}}^{is}$ $= \chi_+(\zeta)\,(i\zeta e_L)^{is}$
$(i\chi_{\mathrm{Re}>0}(Z)\, e^{i\alpha Z}$, $\alpha^{-1}(e^{i\alpha Z} - 1))$	$(\lambda - \alpha)^{-1}$ $(\mathrm{Im}\,\alpha > 0)$	$(i\zeta e_L - \alpha)^{-1}$
$(-i\chi_{\mathrm{Re}<0}(Z)\, e^{i\alpha Z}$, $\alpha^{-1}(e^{i\alpha Z} - 1))$	$(\lambda - \alpha)^{-1}$ $(\mathrm{Im}\,\alpha < 0)$	$(i\zeta e_L - \alpha)^{-1}$

$(\Phi(Z), \Phi_1(Z))$	$(\Phi(x), \underline{\Phi}(y))$
$(0,1)$	$(0,1)$
$\left(\frac{i}{2\pi Z}, \frac{1}{2}\right)$	$\left(k(x), \frac{1}{2}\right)$
$\left(\frac{-i}{2\pi Z}, \frac{1}{2}\right)$	$\left(-k(x), \frac{1}{2}\right)$
$\left(\frac{i}{\pi Z}, 0\right)$	$(2\,k(x), 0)$
$\frac{i}{2\pi}\left(\frac{1}{Z+it}, -i\pi + \log\left(\frac{Z+it}{Z-it}\right)\right)$ $(t>0)$	$(k(x+te_L), \underline{\Phi}(y))$ $\lim_{y\to 0} \underline{\Phi}(y) = 0$
$\frac{t}{2\pi}\left(\frac{-1}{(Z+it)^2}, \frac{2Z}{Z^2+t^2}\right)$ $(t>0)$	$\left(-t\frac{\partial k}{\partial t}(x+te_L), \underline{\Phi}(y)\right)$ $\lim_{y\to 0} \underline{\Phi}(y) = 0$
$\Gamma(1+is)\Big(\frac{i}{2\pi} e^{-\pi s/2} Z^{-1-is}$ $(\pi s)^{-1}\sinh(\pi s/2) Z^{-is}\Big)\Big)$ $(s \in \mathbb{R})$	$\Big(\frac{-1}{\Gamma(1-is)}$ $\int_0^{+\infty} t^{-is}\frac{\partial k}{\partial t}(x+te_L)\,dt, \underline{\Phi}_s(y)\Big)$ (see below)

The function $\underline{\Phi}_s$ (in the last row) has the form

$$\underline{\Phi}_s(r\,n) = \frac{r^{-is}}{\Gamma(1-is)} \int_0^{+\infty} t^{is-1} F(m, n_L, \tau)\, d\tau\, \overline{e_L}\, n\,,$$

where $r > 0$, $|n| = 1$, and F is a real-valued function which satisfies

$$|F(m, n_L, t)| \le c(m, n_L) \frac{t^m}{(1+t)^{m+1}}\,.$$

In particular, when $n = e_L$, then

$$\underline{\Phi}_s(r\,e_L) = \frac{\sigma_{m-1} r^{-is}}{\Gamma(1-is)} \int_0^{+\infty} \frac{t^{m+is-1}}{(1+t^2)^{(m+1)/2}}\, dt\,, \qquad r > 0\,.$$

(To prove this, first show that the function $\underline{\Phi}$ in the preceding row has the form $\underline{\Phi}(r\,n) = F(m, n_L, r/t)\,\overline{e_L}\,n$.)

The functions Φ_1 and $\underline{\Phi}$ are really only of interest near zero, and indeed do not enter into Parseval's identity or the convolution formulae when they tend to nought at zero. It is shown in [McQ] that if $|B(\lambda)| \leq c_s\,|\lambda|^s$ for all $\lambda \in S^\circ_\mu(\mathbb{C})$ and some $s < 0$, then $\Phi_1(Z) \to 0$ as $Z \to 0$ ($Z \in S_{\nu^+}(\mathbb{C})$, $\nu < \mu$). It also follows that $|b(\zeta)| \leq c_s\,|\zeta|^s$ for all $\zeta \in S^\circ_\mu(\mathbb{C})$, and hence from Theorem 4.3.viii) that $\underline{\Phi}(y) \to 0$ as $y \to 0$ ($y \in T^\circ_\nu$, $\nu < \mu$). So there is really no need to find Φ_1 and $\underline{\Phi}$ when $|B(\lambda)| \leq c_s\,|\lambda|^s$, $s < 0$.

It is important to realize however, that Φ_1 and $\underline{\Phi}$ do not always have limits at zero, in which case they are needed in Parseval's identity and in the definitions of the convolution operators presented in the next section. Principal-value integrals do not suffice. For example, Parseval's identity (P) connecting the function $\chi_+(\xi)\,|\xi|^{is}_{\mathbb{C}}$ with its inverse Fourier transform involves the function $\underline{\Phi}_s$ given above.

Let us turn our attention to the function $B = B_+ = B\,\chi_{\mathrm{Re}>0}$, and substitute the corresponding values of Φ and $\underline{\Phi}$ into the formula (+). We obtain (on using the fact that $(\overline{e_L}+i\tau)(a+ib)^k = (\overline{e_L}+i\tau)(a-be_L\tau)^k$ whenever $\tau \in S^{m-1}$ and $a, b \in \mathbb{R}$)

$$\frac{\bar{x}}{\sigma_m|x|^{m+1}} = \frac{1}{2(2\pi i)^{m-1}} \int_{S^{m-1}} (\overline{e_L} + i\tau)\frac{i}{2\pi}\,\frac{(-1)^{m-1}(m-1)!}{(\langle \mathbf{x}, \tau\rangle + ix_L)^m}\,dS_\tau$$

$$= \frac{(m-1)!}{2}\left(\frac{i}{2\pi}\right)^m \int_{S^{m-1}} (\overline{e_L} + i\tau)\,(\langle \mathbf{x}, \tau\rangle - x_L e_L \tau)^{-m}\,dS_\tau\,,$$

where $x_L > 0$, which, on taking the real part of the right hand side, is the plane wave decomposition of the Cauchy kernel presented by Sommen in [S] (at least in the case $L = 0$).

For the function $B = \chi_{\mathrm{Re}<0}$ we obtain

$$\frac{\bar{x}}{\sigma_m|x|^{m+1}} = \frac{-(m-1)!}{2}\left(\frac{-i}{2\pi}\right)^m \int_{S^{m-1}} (\overline{e_L}+i\tau)\,(\langle \mathbf{x}, \tau\rangle - x_L e_L \tau)^{-m}\,dS_\tau\,,$$

where $x_L < 0$, which also agrees with Sommen's formula. See also Ryan [R].

Part II: Convolution singular integrals on surfaces, and functional calculi.*

6. Convolution singular integrals on Lipschitz surfaces.

Let Σ denote the Lipschitz surface consisting of points $x = \mathbf{x} + g(\mathbf{x})\, e_L \in \mathbb{R}^{m+1}$, where $\mathbf{x} \in \mathbb{R}^m$, and g is a real-valued Lipschitz function which satisfies

$$\|\nabla g\|_\infty = \sup_{x\in\mathbb{R}^m} \Big(\sum_{j=1}^{m} \Big|\frac{\partial g}{\partial x_j}\Big|^2\Big)^{1/2} \leq \tan\omega < +\infty\,, \qquad \text{where } 0 \leq \omega < \frac{\pi}{2}\,.$$

A unit normal vector $n(x) \in \mathbb{R}^{m+1}_+$ is defined at almost all $x \in \Sigma$. Choose N to be a compact set of unit vectors in $\mathbb{R}^{m+1}_+$ which is starlike about e_L (as defined in Section 4), has $\mu_N \leq \omega$, and contains $n(x)$ for almost all $x \in \Sigma$.

Let $\mathcal{X}$ be a finite-dimensional left module over $\mathbb{C}_{(M)}$ (see Section 2). If $1 \leq p < \infty$ then $L_p(\Sigma)$ is the space of equivalence classes of functions $u : \Sigma \longrightarrow \mathcal{X}$ which are measurable with respect to $dS_{\mathbf{x}} = \sqrt{1 + |\nabla g(\mathbf{x})|^2}\, d\mathbf{x}$, and for which

$$\|u\|_p = \Big(\int_\Sigma |u(\mathbf{x})|^p dS_{\mathbf{x}}\Big)^{1/p} < +\infty\,.$$

Let us fix Σ, N and $\mathcal{X}$ for the remainder of this paper, and suppose that $1 < p < \infty$. As usual, $\mathcal{L}(L_p(\Sigma))$ denotes the Banach algebra of bounded linear operators on $L_p(\Sigma)$.

The following theorems are mild generalizations of the main results in [LMcS].

Theorem 6.1. *Suppose that $1 < p < \infty$.*

i) *Given $\Phi \in K_N^+$ or K_N^-, there exists $T_\Phi \in \mathcal{L}(L_p(\Sigma))$, defined for all $u \in L_p(\Sigma)$ and almost all $x \in \Sigma$, by*

$$\begin{aligned}(T_\Phi u)(x) &= \lim_{\delta\to 0+} \int_\Sigma \Phi(x \pm \delta e_L - y)\, n(y)\, u(y)\, dS_y \\ &= \lim_{\varepsilon\to 0} \Big(\int_{\substack{|x-y|\geq\varepsilon \\ y\in\Sigma}} \Phi(x-y)\, n(y)\, u(y)\, dS_y + \underline{\Phi}(\varepsilon\, n(x))\, u(x)\Big)\,.\end{aligned}$$

* Section 6 depends on Section 3, Section 7 depends on Section 4, Section 8 depends on Section 5.

Moreover, if $\Phi \in K(C^{\pm}_{N_\mu})$ *for* $0 < \mu \le \pi/2 - \omega$*, then*

$$\|T_\Phi u\|_p \le C_{\omega,\mu,p} \, \|\Phi\|_{K(C^{+}_{N_\mu})} \, \|u\|_p \,,$$

for some constants $C_{\omega,\mu,p}$ *which depend only on* ω*,* μ *and* p*.*

ii) *Given* $(\Phi, \underline{\Phi}) \in K_N$*, there exists* $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))$*, defined for all* $u \in L_p(\Sigma)$ *and almost all* $x \in \Sigma$*, by*

$$(T_{(\Phi,\underline{\Phi})}u)(x) = \lim_{\varepsilon \to 0} \Big(\int\limits_{\substack{|x-y|\ge\varepsilon \\ y\in\Sigma}} \Phi(x-y)\, n(y)\, u(y)\, dS_y + \underline{\Phi}(\varepsilon\, n(x))\, u(x) \Big) \,.$$

Moreover, if $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$ *for* $0 < \mu \le \pi/2 - \omega$*, then*

$$\|T_{(\Phi,\underline{\Phi})}u\|_p \le C_{\omega,\mu,p} \, \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} \, \|u\|_p \,,$$

for some constants $C_{\omega,\mu,p}$ *which depend only on* ω*,* μ *and* p*.*

We remark that part ii) follows directly from part i), together with Theorem 3.1, and that $T_{(\Phi,\underline{\Phi})} = T_{\Phi_+} + T_{\Phi_-}$ with Φ_+ and Φ_- the functions specified there.

Recall that the spaces K_N^+, K_N^- and K_N are not convolution algebras, but that the subspaces M_N^+, M_N^- and M_N are.

Theorem 6.2. *The mappings from* $\Phi \in M_N^{\pm}$ *to* $T_\Phi \in \mathcal{L}(L_p(\Sigma))$ *and from* $(\Phi, \underline{\Phi}) \in M_N$ *to* $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))$ *are algebra homomorphisms.*

Let $k(x) = \bar{x}/(\sigma_m |x|^{m+1})$, $x \ne 0$. Then k belongs to both M_N^+ and M_N^-, so let us write it as k_+ when considered in M_N^+ (with $\underline{k_+} = 1/2$), and as k_- when considered in M_N^- (with $\underline{k_-} = -1/2$). Also $(2k, 0) = (k_+, 1/2) + (k_-, -1/2) \in M_N$. The corresponding bounded linear operators on $L_p(\Sigma)$ are the Cauchy singular integral operators $C_\Sigma = T_{(2k,0)}$, $P_+ = T_{k_+}$ and $P_- = -T_{k_-}$. By Theorem 6.1 we know that they are defined for all $u \in L_p(\Sigma)$ and almost all $x \in \Sigma$ by

$$(P_\pm u)(x) = \pm \lim_{\delta \to 0+} \int_\Sigma k(x \pm \delta\, e_L - y)\, n(y)\, u(y)\, dS_y$$

and

$$(C_\Sigma u)(x) = 2 \lim_{\varepsilon \to 0} \int\limits_{\substack{|x-y|\ge\varepsilon \\ y\in\Sigma}} k(x-y)\, n(y)\, u(y)\, dS_y \,.$$

It is the inquiry into the boundedness of C_Σ that really started this study.

As noted in [LMcS], the following properties are immediate consequences of Theorems 6.1 and 6.2.

Theorem 6.3. *Let $\Phi_\pm \in M_N^\pm$. The Cauchy singular integral operators P_+, P_- and C_Σ are bounded linear operators on $L_p(\Sigma)$ which satisfy the following identities.*

(0) $P_+ + P_- = I\,, \qquad P_+ - P_- = C_\Sigma \qquad$ (*the Plemelj formulae*);

(1) $$\begin{aligned} P_+ T_{\Phi_+} &= T_{\Phi_+} P_+ = T_{\Phi_+}\,, & P_- T_{\Phi_+} &= T_{\Phi_+} P_- = 0\,,\\ P_- T_{\Phi_-} &= T_{\Phi_-} P_- = T_{\Phi_-}\,, & P_+ T_{\Phi_-} &= T_{\Phi_-} P_+ = 0\,; \end{aligned}$$

(2) $$P_+^2 = P_+\,, \quad P_-^2 = P_-\,, \quad P_+ P_- = P_- P_+ = 0\,, \quad C_\Sigma^2 = I\,;$$

(3) $$T_{\Phi_+} T_{\Phi_-} = T_{\Phi_-} T_{\Phi_+} = 0\,.$$

Define the *Hardy spaces* $L_p^\pm(\Sigma)$ to be the images of the projections $P_\pm$, so that $L_p(\Sigma) = L_p^+(\Sigma) \oplus L_p^-(\Sigma)$. The operators T_{Φ_+} map $L_p(\Sigma)$ into $L_p^+(\Sigma)$ and are zero on $L_p^-(\Sigma)$, while the operators T_{Φ_-} map $L_p(\Sigma)$ into $L_p^-(\Sigma)$ and are zero on $L_p^+(\Sigma)$. So alternatively we could define $T_{\Phi_\pm} \in \mathcal{L}(L_p^\pm(\Sigma))$, in which case $T_{(\Phi,\underline{\Phi})} = T_{\Phi_+} \oplus T_{\Phi_-}$, where $(\Phi, \underline{\Phi})$ is related to Φ_+ and Φ_- as in Theorem 3.1.

Let us make one observation which depends on Section 4. We used Fourier theory at the end of that section to show that $(2k_j, 0) \in K_N$, where $k_j(x) = -x_j/(\sigma_m |x|^{m+1})$, $x \in \mathbb{R}^m \setminus \{0\}$, $j = 1, 2, \ldots, m$. So the operators $R_{j,\Sigma} = T_{2k_j}$ are bounded linear operators on $L_p(\Sigma)$. The question as to whether these operators, which can be thought of as Riesz transforms on Σ, are L_p-bounded, was actually one of the motivations for developing the Fourier theory of this paper. (The boundedness of these operators is not a direct consequence of the boundedness of the Cauchy operator $C_\Sigma = \sum e_j R_{j,\Sigma}$, because $R_{j,\Sigma}$ is not merely the j-th component of C_Σ).

Theorem 6.4. *The Riesz transforms $R_{j,\Sigma}$ are bounded linear operators on $L_p(\Sigma)$ which satisfy $R_{j,\Sigma} R_{k,\Sigma} = R_{k,\Sigma} R_{j,\Sigma}$, $\sum e_j R_{j,\Sigma} = C_\Sigma$ and $\sum (R_{j,\Sigma})^2 = -I$.*

Here are some further consequences of Theorems 6.1 and 6.2. When $\Phi \in K_N^+$, and $\delta > 0$, then $\Phi_\delta \in K_N^+$ is defined by $\Phi_\delta(x) = \Phi(x + \delta e_L)$. In particular, $k_\delta \in M_N^+$, where $k_\delta(x) = k_{+\delta}(x) = k_+(x + \delta e_L)$.

If p is a polynomial in m variables with values in $C_{(M)}$, then $p(-i\,\mathbf{D})\,k_\delta \in K_N^+$, where

$$p(-i\,\mathbf{D})k_\delta(x) = p\Big(-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_m}\Big)\,k_+(x + \delta\, e_L)\,.$$

Theorem 6.5. *Let $\alpha > 0$ and $\delta > 0$.*

i) *If $\Phi \in K_N^+$, then $\Phi * k_\delta = \Phi_\delta \in K_N^+$, and $T_\Phi\, T_{k_\delta} = T_{\Phi_\delta}$.*

ii) *If $\Phi \in M_N^+$, then $k_\delta * \Phi = \Phi_\delta \in M_N^+$, and $T_{k_\delta}\, T_\Phi = T_{\Phi_\delta}$.*

iii) *$k_\alpha * k_\delta = k_{\alpha+\delta} \in M_N^+$, and $T_{k_\alpha}\, T_{k_\delta} = T_{k_{\alpha+\delta}}$.*

Suppose that p and q are two polynomials, with p satisfying $p(\xi)\xi e_L = \xi e_L p(\xi)$. Then $p(-i\,\mathbf{D})k_\delta \in M_N^+$, and

iv) *$k_\alpha * p(-i\,\mathbf{D})k_\delta = p(-i\,\mathbf{D})k_{\alpha+\delta} \in M_N^+$, and*

$$T_{k_\alpha}\, T_{p(-i\,\mathbf{D})k_\delta} = T_{p(-i\,\mathbf{D})k_{\alpha+\delta}}\,, \qquad \textit{and}$$

v) *$q(-i\,\mathbf{D})k_\alpha * p(-i\,\mathbf{D})k_\delta = (qp)(-i\,\mathbf{D})k_{\alpha+\delta} \in K_N^+$, and*

$$T_{q(-i\,\mathbf{D})k_\alpha}\, T_{p(-i\,\mathbf{D})k_\delta} = T_{(qp)(-i\,\mathbf{D})k_{\alpha+\delta}}\,.$$

Let Ω_+ be the open subset of $\mathbb{R}^{m+1}$ above Σ. That is, $\Omega_+ = \{X \in \mathbb{R}^{m+1} : X = x + \delta e_L,\ x \in \Sigma,\ \delta > 0\}$. For $u \in L_p(\Sigma)$, define $\mathcal{C}_\Sigma^+ u$ to be the left-monogenic function on Ω_+ given by

$$(\mathcal{C}_\Sigma^+ u)(X) = \int_\Sigma k(X-y)\, n(y)\, u(y)\, dS_y\,, \qquad X \in \Omega_+\,.$$

Then $(\mathcal{C}_\Sigma^+ u)(x + \delta e_L) = T_{k_\delta} u(x) \to P_+ u(x)$ as $\delta \to 0^+$ for almost all $x \in \Sigma$. This limit also exists in the L_p sense [LMcS]. That is, $\|T_{k_\delta} u - P_+ u\|_p \to 0$ as $\delta \to 0^+$.

Let us consider functions $u \in L_p^+(\Sigma)$, in which case $\|T_{k_\delta} u - u\|_p \to 0$ as $\delta \to 0^+$.

We could differentiate $(\mathcal{C}_\Sigma^+ u)(X)$ before taking the limit as X approaches Σ, though the limit need not always exist. More generally, given any polynomial p in m variables with values in $\mathbb{C}_{(M)}$, we could form

$$p(-i\,\mathbf{D})(\mathcal{C}_\Sigma^+ u)(X) = p\Big(-i\frac{\partial}{\partial X_1}, -i\frac{\partial}{\partial X_2}, \dots, -i\frac{\partial}{\partial X_m}\Big)(\mathcal{C}_\Sigma^+ u)(X),$$

though again the limit as X approaches Σ need not always exist. But let us define $p(-i\,\mathbf{D}_\Sigma)u(x)$ to be the limit of $p(-i\,\mathbf{D})(\mathcal{C}_\Sigma^+ u)(x+\delta e_L) = T_{p(-i\,\mathbf{D})k_\delta}u(x)$ as $\delta \to 0^+$, when it does exist in $L_p(\Sigma)$.

To be precise, define $p(-i\,\mathbf{D}_\Sigma)$ to be the linear transformation, from its domain $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma)) \subset L_p^+(\Sigma)$, into $L_p(\Sigma)$, given by

$$\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma)) = \{u \in L_p^+(\Sigma) : \ T_{p(-i\,\mathbf{D})k_\delta}u \to w \in L_p(\Sigma)\}$$

and $p(-i\,\mathbf{D}_\Sigma)u = w$.

If u is itself of the form $u = T_{k_\alpha}v$ for some $v \in L_p^+(\Sigma)$, then u is the restriction of the left monogenic function U to Σ, where

$$U(X) = (\mathcal{C}_\Sigma^+ v)(X+\alpha e_L), \qquad X+\alpha e_L \in \Omega_+ .$$

Such functions u belong to $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ and

$$p(-i\,\mathbf{D}_\Sigma)u = (p(-i\,\mathbf{D})U)|_\Sigma .$$

Consider, in particular, the functions $q_k(x) = i\xi_k$, $k = 1, 2, \dots, m$, and $q(\xi) = i\,\xi e_L = \sum_{k=1}^m i\xi_k e_k e_L$. Using them, we define the operators $D_{k,\Sigma} = q_k(-i\,\mathbf{D}_\Sigma)$, and $\mathbf{D}_\Sigma e_L = q(-i\,\mathbf{D}_\Sigma)$, so that, for functions u of the type specified in the preceding paragraph,

$$\mathbf{D}_\Sigma e_L u = (\mathbf{D} e_L U)|_\Sigma$$

and

$$D_{k,\Sigma}u = \frac{\partial U}{\partial X_k}, \qquad k = 1, 2, \dots, m.$$

It may be interesting to write these functions out in terms of the parameter $\mathbf{s}$, when Σ is parametrized by $x = \mathbf{s} + g(\mathbf{s})e_L$. We obtain

$$D_{k,\Sigma}u(\mathbf{s}+g(\mathbf{s})e_L) = \Big(\frac{\partial}{\partial s_k} + \frac{\partial g}{\partial s_k}(e_L - \mathbf{D}g)^{-1}\mathbf{D}_\mathbf{s}\Big)\, u(\mathbf{s}+g(\mathbf{s})e_L)$$

and

$$\begin{aligned}\mathbf{D}_\Sigma e_L u(\mathbf{s}+g(\mathbf{s})e_L) &= \sum_{k=1}^m e_k e_L D_{k,\Sigma} u(\mathbf{s}+g(\mathbf{s})e_L)\\ &= (e_L - \mathbf{D}g)^{-1}\mathbf{D}_\mathbf{s} u(\mathbf{s}+g(\mathbf{s})e_L),\end{aligned}$$

for all functions u such that $u = T_{k_\alpha} v$ for some $v \in L_p^+(\Sigma)$. In Theorem 8.2 we shall see that this expression for $\mathbf{D}_\Sigma$ is valid for every function u in its domain.

It follows from the next theorem that these are closed linear operators in $L_p^+(\Sigma)$. In the following two sections, we shall explore ways in which the convolution operators of Theorem 6.1 can be represented as bounded holomorphic functions of $(D_{k,\Sigma})$ and $\mathbf{D}_\Sigma$. We are still supposing that $1 < p < \infty$.

Theorem 6.6. *Let p be a polynomial in m variables with values in $\mathbb{C}_{(M)}$. Then $p(-i\,\mathbf{D}_\Sigma)$ is a linear transformation from $L_p^+(\Sigma)$ to $L_p(\Sigma)$ with domain $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ dense in $L_p^+(\Sigma)$.*

If $p(\xi)\xi e_L = \xi e_L p(\xi)$, then

$$p(-i\,\mathbf{D}_\Sigma)u \in L_p^+(\Sigma)\,, \qquad \text{for all } u \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))\,,$$

and indeed $p(-i\,\mathbf{D}_\Sigma)$ is a closed linear operator in $L_p^+(\Sigma)$.

Suppose that p and q are two polynomials, with p satisfying $p(\xi)\xi e_L = \xi e_L p(\xi)$. Let $u \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$. Then $p(-i\,\mathbf{D}_\Sigma)u \in \mathcal{D}^+(q(-i\,\mathbf{D}_\Sigma))$ if and only if $u \in \mathcal{D}^+((qp)(-i\,\mathbf{D}_\Sigma))$, in which case $q(-i\,\mathbf{D}_\Sigma)p(-i\,\mathbf{D}_\Sigma)u = (qp)(-i\,\mathbf{D}_\Sigma)u$.

PROOF. The domain $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ is dense in $L_p^+(\Sigma)$, because every function $u \in L_p^+(\Sigma)$ is the limit of $T_{k_\alpha}u \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ as $\alpha \to 0$.

For the remainder of this proof, suppose that $p(\xi)\xi e_L = \xi e_L p(\xi)$. Let $u \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$. We saw in Theorem 6.5 that $p(-i\mathbf{D})k_\delta \in M_N^+$ when $\delta > 0$, and that $T_{k\alpha}T_{p(-i\,\mathbf{D})k_\delta}u = T_{p(-i\,\mathbf{D})k_{\alpha+\delta}}u$ when $\alpha > 0$ also. On letting δ tend to 0, we obtain $T_{k_\alpha}p(-i\,\mathbf{D}_\Sigma)u = T_{p(-i\,\mathbf{D})k_\alpha}u$. On letting α to 0, we conclude that $p(-i\,\mathbf{D}_\Sigma)u = P_+p(-i\,\mathbf{D}_\Sigma)u \in L_p^+(\Sigma)$.

To prove that $p(-i\,\mathbf{D}_\Sigma)$ is a closed operator in $L_p^+(\Sigma)$, choose a sequence (v_n) in $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ such that $v_n \to v \in L_p^+(\Sigma)$ and $p(-i\,\mathbf{D}_\Sigma)v_n \to w \in L_p^+(\Sigma)$. We need to show that $v \in \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ and $p(-i\,\mathbf{D}_\Sigma)v = w$. For each $\alpha > 0$, $T_{k_\alpha}p(-i\,\mathbf{D}_\Sigma)v_n \to T_{k_\alpha}w$, and $T_{k_\alpha}p(-i\,\mathbf{D}_\Sigma)v_n = T_{p(-i\,\mathbf{D})k_\alpha}v_n \to T_{p(-i\,\mathbf{D})k_\alpha}v$, so that $T_{p(-i\,\mathbf{D})k_\alpha}v = T_{k_\alpha}w$. Therefore $T_{p(-i\,\mathbf{D})k_\alpha}v = T_{k_\alpha}w \to w$ as $\alpha \to 0$. We conclude that $v \in \mathcal{D}(p(-i\,\mathbf{D}_\Sigma))$ and that $p(-i\,\mathbf{D}_\Sigma)v = w$ as required.

From Theorem 6.5, we also have that

$$T_{q(-i\,\mathbf{D})k_\alpha}T_{p(-i\,\mathbf{D})k_\delta}u = T_{(qp)(-i\,\mathbf{D})k_{\alpha+\delta}}u\,,$$

and so, letting δ tend to 0, we obtain

$$T_{q(-i\,\mathbf{D})k_\alpha} p(-i\,\mathbf{D}_\Sigma)u = T_{(qp)(-i\,\mathbf{D})k_\alpha} u\,.$$

On letting α tend to 0, we conclude that $p(-i\,\mathbf{D}_\Sigma)u \in \mathcal{D}^+(q(-i\,\mathbf{D}_\Sigma))$ if and only if $u \in \mathcal{D}^+((qp)(-i\,\mathbf{D}_\Sigma))$, in which case $q(-i\,\mathbf{D}_\Sigma)p(-i\,\mathbf{D}_\Sigma)u = (qp)(-i\,\mathbf{D}_\Sigma)u$.

In a similar way, we can define the linear transformation $p(-i\,\mathbf{D}_\Sigma)$ from its domain $\mathcal{D}^-(p(-i\,\mathbf{D}_\Sigma)) \subset L_p^-(\Sigma)$ into $L_p(\Sigma)$.

Finally, we define the linear operator $p(-i\,\mathbf{D}_\Sigma)$ in $L_p(\Sigma)$ with dense domain

$$\begin{aligned}\mathcal{D}(p(-i\,\mathbf{D}_\Sigma)) &= \mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma)) \oplus \mathcal{D}^-(p(-i\,\mathbf{D}_\Sigma)) \\ &\subset L_p^+(\Sigma) \oplus L_p^-(\Sigma) = L_p(\Sigma)\end{aligned}$$

by $p(-i\,\mathbf{D}_\Sigma)u = p(-i\,\mathbf{D}_\Sigma)P_+u + p(-i\,\mathbf{D}_\Sigma)P_-u$.

Theorem 6.7. *The statement of Theorem 6.6 remains valid when $L_p^+(\Sigma)$ is replaced by $L_p(\Sigma)$ and $\mathcal{D}^+(p(-i\,\mathbf{D}_\Sigma))$ is replaced by $\mathcal{D}(p(-i\,\mathbf{D}_\Sigma))$.*

Suppose that U is a left-monogenic function on the strip $\Sigma + (-t,t)e_L$, and that the functions u_α defined by $u_\alpha(x) = U(x + \alpha e_L)$, $x \in \Sigma$, are uniformly bounded in $L_p(\Sigma)$, $\alpha \in (-t,t)$. Let $u = u_0 = U|_\Sigma$. Then, for each polynomial p,

$$p(-i\,\mathbf{D}_\Sigma)u = (p(-i\,\mathbf{D})U)\big|_\Sigma\,.$$

This follows from the remark following the definition of $p(-i\,\mathbf{D}_\Sigma)$ in $L_p^+(\Sigma)$, and the fact that $P_+u = T_{k_\alpha}P_+u_{-\alpha}$, together with a similar result for P_-u.

In particular, for such left-monogenic functions U,

$$\mathbf{D}_\Sigma\, e_L u = (\mathbf{D} e_L U)\big|_\Sigma \qquad \text{and} \qquad D_{k,\Sigma}u = \frac{\partial U}{\partial X_k}\Big|_\Sigma\,,$$

$k = 1, 2, \ldots, m$, when $u = U|_\Sigma$.

7. H_∞ functional calculi for functions of m variables.

Let $(\Phi,\underline{\Phi}) \in K(S_{N_\mu})$. We can think of $b = \mathcal{F}(\Phi,\underline{\Phi})e_L$ as the Fourier multiplier corresponding to the bounded linear operator $T_{(\Phi,\underline{\Phi})}$. But we can also think of the mapping from $b \in H_N$ to $T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))$ as giving us a bounded H_∞ functional calculus of $-i\,\mathbf{D}_\Sigma = \sum_{k=1}^m -i\,e_k D_{k,\Sigma}$, and write

$$T_{(\Phi,\underline{\Phi})} = b(-i\,\mathbf{D}_\Sigma) = b(-iD_{1,\Sigma}, -iD_{2,\Sigma}, \ldots, -iD_{m,\Sigma})\,.$$

In order to see that this is a natural thing to do, let us introduce a larger algebra than H_N, namely $\mathcal{P}_N$, consisting of all functions b from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{C}_{(M)}$ such that $b_+ = b\,\chi_+$ extends holomorphically to $N_\mu(\mathbb{C}^m)$ for some $\mu > 0$, and $b_- = b\,\chi_-$ extends holomorphically to $\overline{N_\mu}(\mathbb{C}^m)$, and the extensions satisfy $|b_\pm(\zeta)| \le c\,(1+|\zeta|^s)$ for some s and $c \ge 0$.

For such $b \in \mathcal{P}_N$, the functions $b_{+\delta}$ and $b_{-\delta}$ belong to H_N^+ and H_N^- respectively, where $b_{+\delta}(\zeta) = b_+(\zeta)\,e^{-\delta|\zeta|_{\mathbb{C}}}$ and $b_{-\delta}(\zeta) = b_-(\zeta)\,e^{-\delta|\zeta|_{\mathbb{C}}}$ for $\delta > 0$. Therefore $\Phi_{\pm\delta} = \mathcal{G}_\pm(b_{\pm\delta}\overline{e_L}) \in K_N^+$. Define $b(-i\,\mathbf{D}_\Sigma)$ to be the linear operator in $L_p(\Sigma)$ with domain

$$\mathcal{D}(b(-i\,\mathbf{D}_\Sigma)) = \{u \in L_p(\Sigma) : T_{\Phi_{\pm\delta}}u \to w_\pm \in L_p(\Sigma) \text{ as } \delta \to 0\}$$

by

$$b(-i\,\mathbf{D}_\Sigma)u = w_+ + w_-\ .$$

The fact that this terminology is reasonable follows from the following facts.

Theorem 7.1. *Suppose that $1 < p < \infty$. Let $b \in \mathcal{P}_N$.*

i) *If $b \in H_N$, then $b(-i\,\mathbf{D}_\Sigma) = T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))$, where $(\Phi,\underline{\Phi})e_L = \mathcal{G}(b)$. In particular, $1(-i\,\mathbf{D}_\Sigma) = I$, $\chi_\pm(-i\,\mathbf{D}_\Sigma) = P_\pm$, $(r_j e_L)(-i\,\mathbf{D}_\Sigma) = R_{j,\Sigma}$, and $r(-i\,\mathbf{D}_\Sigma) = C_\Sigma = \sum e_j R_{j,\Sigma}$ where $r(\xi) = i\,\xi|\xi|^{-1}e_L$.*

ii) *If $b_+ = b\,\chi_+ \in H_\infty^+(N_\mu(\mathbb{C}^m))$ and $b_- = b\,\chi_- \in H_\infty^-(\overline{N_\mu}(\mathbb{C}^m))$ with $0 < \mu \le \pi/2 - \omega$, then*

$$\|b(-i\,\mathbf{D}_\Sigma)u\|_p \le C_{\omega,\mu,p}\left(\|b_+\|_\infty + \|b_-\|_\infty\right)\|u\|_p\ ,$$

for some constants $C_{\omega,\mu,p}$ which depend only on ω, μ, p (and the dimension m).

iii) *If b is a polynomial in m variables, then the definition of $b(-i\,\mathbf{D}_\Sigma)$ coincides with that given in Section 6.*

iv) *The domain $\mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$ of $b(-i\,\mathbf{D}_\Sigma)$ is dense in $L_p(\Sigma)$.*

v) *If $b(\xi)\xi e_L = \xi e_L b(\xi)$ for all $\xi \in \mathbb{R}^m \setminus \{0\}$, then $b(-i\,\mathbf{D}_\Sigma)$ is a closed linear operator in $L_p(\Sigma)$.*

vi) *If $u \in \mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$, $f \in \mathcal{P}_N$ and $c \in \mathbb{C}_{(M)}$, then*

$$u \in \mathcal{D}(f(-i\,\mathbf{D}_\Sigma)) \quad \textit{if and only if} \quad u \in \mathcal{D}((c\,b+f)(-i\,\mathbf{D}_\Sigma)),$$

in which case $c\,b(-i\,\mathbf{D}_\Sigma)u + f(-i\,\mathbf{D}_\Sigma)u = (c\,b+f)(-i\,\mathbf{D}_\Sigma)u$.

vii) *If $b(\xi)\xi e_L = \xi e_L b(\xi)$ for all $\xi \in \mathbb{R}^m \setminus \{0\}$, $u \in \mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$ and $f \in \mathcal{P}_N$, then*

$$b(-i\,\mathbf{D}_\Sigma)u \in \mathcal{D}(f(-i\,\mathbf{D}_\Sigma)) \quad \textit{if and only if} \quad u \in \mathcal{D}((fb)(-i\,\mathbf{D}_\Sigma)),$$

in which case $f(-i\,\mathbf{D}_\Sigma)b(-i\,\mathbf{D}_\Sigma)u = (fb)(-i\,\mathbf{D}_\Sigma)u$.

PROOF. When $b \in H_N$, let $b_+ = b\chi_+$ and $\Phi_+ = \mathcal{G}_+(b_+\overline{e_L})$, so that $\Phi_{+\delta}(x) = \mathcal{G}_+(b_{+\delta}\overline{e_L})(x) = \Phi_+(x+\delta e_L)$. Therefore, for all $u \in L_p(\Sigma)$, $T_{\Phi_{+\delta}}u \to T_{\Phi_+}u$ in $L_p(\Sigma)$ as $\delta \to 0$. Similarly $T_{\Phi_{-\delta}}u \to T_{\Phi_-}u$. So $u \in \mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$, and $b(-i\,\mathbf{D}_\Sigma)u = T_{\Phi_+}u + T_{\Phi_-}u = T_{(\Phi,\underline{\Phi})}$.

The estimate in ii) is a consequence of Theorems 4.3.iii) and 6.1.ii).

To prove iii), use the identities $\mathcal{F}_\pm(p(-i\,\mathbf{D}_\Sigma)k_{\pm\delta})e_L = p_{\pm\delta}$, which are consequences of Theorem 4.1.vii) and Theorem 4.2.

The proof of the remaining parts mimics that of Theorem 6.6.

Let us indicate the kind of application of these results that we have in mind. Details will appear elsewhere.

Consider the following boundary value problem for harmonic functions.

$$\begin{cases} \Delta U(X) = \displaystyle\sum_{k=1}^{m} \frac{\partial^2 U}{\partial X_k^2}(X) + \frac{\partial^2 U}{\partial x_L^2}(X) = 0\,, & X \in \Omega_+\,, \\ \Big(\displaystyle\sum_{k=1}^{m} \beta_k \frac{\partial U}{\partial X_k} + \beta_L \frac{\partial U}{\partial X_L}\Big)\Big|_\Sigma = w \in L_p(\Sigma, \mathbb{C})\,, & \end{cases}$$

where $\beta_k, \beta_L \in \mathbb{C}$ and $2 \le p < \infty$.

In the special case when $\beta_L = 1$ and $\beta_k = 0$, $k = 1, 2, \ldots, m$, the solution to this problem is given by

$$U(X) = U(\mathbf{X} + X_L\, e_L) = -\int_{X_L}^{+\infty} (\mathcal{C}^+_{\Sigma_0} v)(\mathbf{X} + t\, e_L)\, dt\,,$$

where $v = (P_{+0})^{-1} w \in L_p(\Sigma)$. Here $\mathcal{C}^+_{\Sigma_0}$ denotes the scalar part of the Cauchy integral $\mathcal{C}^+_\Sigma$, namely

$$(\mathcal{C}^+_{\Sigma_0} v)(X) = \int_\Sigma \langle \overline{k(X-y)}, n(y)\rangle\, v(y)\, dS_y\,, \qquad X \in \Omega_+\,,$$

which is the double-layer potential operator on Σ, and $P_{+0} = \frac{1}{2}(I + C_{\Sigma_0})$, where C_{Σ_0} is the singular double-layer potential operator on Σ. The invertibility of P_{+0} in $L_p(\Sigma, \mathbb{C})$ was proved by Verchota [V].

In the general case of complex β_k and β_L, we assume that, for some $\kappa > 0$,

$$(\#) \qquad |\langle \beta, n + it\rangle| \geq \kappa\,, \quad \text{for all } n \in N \text{ and } t \in \mathbb{R}^{m+1} \text{ such that } |t| = 1 \text{ and } \langle n, t\rangle = 0\,,$$

where $\beta = \sum \beta_k e_k + \beta_L e_L$. (This is the weakest condition on β under which we can expect to solve the boundary value problem, because, if Σ is smooth in a neighbourhood of a point $x \in \Sigma$, then the covering condition of Agmon, Douglis, Nirenberg for this problem is, that there does not exist a unit tangent vector t to Σ at x satisfying $\langle \beta, n(x) + it\rangle = 0$, where $n(x)$ is the unit normal to Σ at x.)

It can be shown that (#) implies

$$(\#\#) \qquad |\langle \beta, |\zeta|_{\mathbb{C}}\, e_L - i\zeta\rangle| \geq \kappa |\, |\zeta|_{\mathbb{C}}|\,, \qquad \text{for all } \zeta \in N(\mathbb{C}^m)\,,$$

and hence that the holomorphic function b defined by

$$b(\zeta) = \frac{|\zeta|_{\mathbb{C}}}{\langle \beta, |\zeta|_{\mathbb{C}}\, e_L - i\zeta\rangle}$$

is bounded by κ^{-1} on $N(\mathbb{C}^m)$, and indeed is bounded by $2\kappa^{-1}$ on $N_\mu(\mathbb{C}^m)$ for μ small enough (to derive (##) from (#), choose $\zeta \in N(\mathbb{C}^m)$, meaning that there exists $n \in N$ and $c > 0$ such that $\eta +$

$\mathrm{Re}(|\zeta|_{\mathbb{C}})e_L = cn$. Apply (#) with this choice of n, and with $t = c^{-1}(-\xi + \mathrm{Im}(|\zeta|_{\mathbb{C}})e_L))$.

Therefore $b(-i\,\mathbf{D}_\Sigma)$ is a bounded linear operator on $L_p(\Sigma, \mathbb{C}_{(M)})$.

On noting the identity,

$$\Big(\sum_{k=1}^{m} \beta_k\zeta_k - \beta_L\zeta e_L\Big)\, b(\zeta)\,\chi_+(\zeta) = -\zeta e_L \chi_+(\zeta)\,,$$

it is straightforward to verify that the solution to our boundary value problem is

$$U(X) = U(\mathbf{X} + X_L\,e_L) = -\int_{X_L}^{+\infty} (\mathcal{C}_\Sigma^+ b(-i\,\mathbf{D}_\Sigma)v)_0(\mathbf{X} + t\,e_L)\,dt\,,$$

where $X \in \Omega_+$ and $v = (P_{+0})^{-1}w \in L_p(\Sigma, \mathbb{C})$.

Further, $(\mathcal{C}_\Sigma^+ b(-i\,\mathbf{D}_\Sigma)v)_0(x + \delta e_L) = (T_{\Phi_\delta}v)_0(x)$ when $x \in \Sigma$ and $\delta > 0$, where $\Phi = \mathcal{G}_+(b\,\chi_+\,\overline{e_L}) \in M_N^+$. So the integrand can be expressed as

$$(\mathcal{C}_\Sigma^+ b(-i\,\mathbf{D}_\Sigma)v)_0(X) = \int_\Sigma \langle\overline{\Phi(X-y)}, n(y)\rangle\, v(y)\, dS_y\,,$$

where $X \in \Omega_+$.

We stress that the Fourier theory developed in Section 4 has been used to show that the assumption (#) implies that $\Phi \in M_N^+$, and hence that $T_\Phi \in \mathcal{L}(L_p(\Sigma, \mathbb{C}_{(M)}))$.

In our other papers involving H_∞ functional calculi, frequent use is made of a Convergence Lemma of the following nature. In particular it can be used to show that other reasonable definitions of $b(-i\,\mathbf{D}_\Sigma)$ lead to the same operator as ours. We are still supposing that $1 < p < \infty$.

Convergence Lemma. *Suppose* $0 < \mu \le \pi/2 - \omega$. *Let*

$$b_{(\alpha)} = b_{(\alpha)+} + b_{(\alpha)-}\,,$$

where $b_{(\alpha)+}$ *is a uniformly bounded net of functions in* $H_\infty^+(N_\mu(\mathbb{C}^m))$ *which converges to a function* $b_+ \in H_\infty^+(N_\mu(\mathbb{C}^m))$ *uniformly on every set of the form* $\{\zeta \in N_\mu(\mathbb{C}^m) : 0 < \delta \le |\zeta| \le \Delta < \infty\}$, *and where* $b_{(\alpha)-}$ *is a uniformly bounded net of functions in* $H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$ *which converges in a similar way to a function* $b_- \in H_\infty^-(\overline{N}_\mu(\mathbb{C}^m))$. *Let* $b = b_+ + b_-$. *Then* $b_{(\alpha)}(-i\,\mathbf{D}_\Sigma)u$ *converges to* $b(-i\,\mathbf{D}_\Sigma)u$ *for every* $u \in L_p(\Sigma)$, *and consequently* $\|b(-i\,\mathbf{D}_\Sigma)\| \le \sup_\alpha \|b_{(\alpha)}(-i\,\mathbf{D}_\Sigma)\|$.

PROOF. It is actually quite straightforward to use the definitions to show that $\Phi_{(\alpha)\pm} = \mathcal{G}_\pm(b_{(\alpha)\pm}\overline{e_L})$ converge in a similar way to $\Phi_\pm = \mathcal{G}_\pm(b_\pm\overline{e_L})$ and then that, for each $u \in L_p(\Sigma)$, $b_{(\alpha)}(-i\,\mathbf{D}_\Sigma)u = T_{\Phi_{(\alpha)+}}u + T_{\Phi_{(\alpha)-}}u$ converges to $T_{\Phi_+}u + T_{\Phi_-}u = b(-i\,\mathbf{D}_\Sigma)u$.

Here is a small corollary. Let us state it for functions defined on sets of the form $S^\circ_\mu(\mathbb{C}^m)$, rather than on the more general sets $N_\mu(\mathbb{C}^m)$ and $\overline{N}_\mu(\mathbb{C}^m)$).

Theorem 7.2. *Let b be a holomorphic function which satisfies $|b(\zeta)| \le c(1+|\zeta|^d)$ on $S^\circ_\mu(\mathbb{C}^m)$ for some $\mu \in (\omega, \pi/2)$, and d and $c \ge 0$. Assume that $b(\xi)\xi e_L = \xi e_L b(\xi)$ for all $\xi \in \mathbb{R}^m$, and suppose that $b(\zeta)$ has an inverse $b(\zeta)^{-1} \in \mathbb{C}_{(M)}$ for all $\zeta \in S^\circ_\mu(\mathbb{C}^m)$, and that there exists $s \ge 0$ such that*

$$|b(\zeta)^{-1}| \le c(|\zeta|^s + |\zeta|^{-s}), \qquad \zeta \in S^\circ_\mu(\mathbb{C}^m).$$

Then, the operator $b(-i\,\mathbf{D}_\Sigma)$ is one-one, and has dense range $\mathcal{R}(b(-i\,\mathbf{D}_\Sigma))$ in $L_p(\Sigma)$.

PROOF. Let $F_{(n)}(\lambda) = (n\lambda)^s(i+n\lambda)^{-s}(\chi_{\mathrm{Re}>0}(\lambda)e^{-\lambda/n} + \chi_{\mathrm{Re}<0}(\lambda)e^{\lambda/n})$, $\lambda \in S^\circ_\mu(\mathbb{C})$, $n = 1, 2, 3, \ldots$ Then the sequence $(F_{(n)})$ is uniformly bounded and converges uniformly to 1 on every set of the form $\{\lambda \in S^\circ_\mu(\mathbb{C}) : 0 < \delta \le |\lambda| \le \Delta < +\infty\}$. For each n, define $f_{(n)} \in H_\infty(S^\circ_\mu(\mathbb{C}^m))$ by

$$f_{(n)}(\zeta) = F_{(n)}(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + F_{(n)}(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)$$

as in Section 5. Then the sequence $(f_{(n)})$ is uniformly bounded and converges uniformly to 1 on every set of the form $\{z \in S^\circ_\mu(\mathbb{C}^m) : 0 < \delta \le |\zeta| \le \Delta < +\infty\}$. Let

$$g_{(n)} = f_{(n)}b^{-1} \in H_\infty(S^\circ_\mu(\mathbb{C}^m)) \quad \text{and} \quad h_{(n)} = b^{-1}f_{(n)} \in H_\infty(S^\circ_\mu(\mathbb{C}^m)),$$

so that $f_{(n)} = g_{(n)}b = b\,h_{(n)}$.

Suppose that $u \in \mathcal{D}(b(-i\,\mathbf{D}_\Sigma))$, and that $b(-i\,\mathbf{D}_\Sigma)u = 0$. By Theorem 7.1.vii), $f_{(n)}(-i\,\mathbf{D}_\Sigma)u = g_{(n)}(-i\,\mathbf{D}_\Sigma)b(-i\,\mathbf{D}_\Sigma)u = 0$, and, by the Convergence Lemma, $f_{(n)}(-i\,\mathbf{D}_\Sigma)u$ tends to u. So $u = 0$. We conclude that $b(-i\,\mathbf{D}_\Sigma)$ is a one-one operator.

Let $w \in L_p(\Sigma)$. Then $f_{(n)}(-i\,\mathbf{D}_\Sigma)w = b(-i\,\mathbf{D}_\Sigma)h_{(n)}(-i\,\mathbf{D}_\Sigma)w \in \mathcal{R}(b(-i\,\mathbf{D}_\Sigma))$, and

$$\lim_{n\to\infty} f_{(n)}(-i\,\mathbf{D}_\Sigma)w = w.$$

We conclude that $\mathcal{R}(b(-i\,\mathbf{D}_\Sigma))$ is dense in $L_p(\Sigma)$.

8. H_∞ functional calculi for functions of one variable.

Let us turn our attention to functions b which are associated with holomorphic functions of one variable. Recall that, for every holomorphic function B defined on $S^\circ_\mu(\mathbb{C})$, where $\omega < \mu \le \pi/2$, there is an associated function b defined on $S^\circ_\mu(\mathbb{C}^m)$ by

$$b(\zeta) = B(i\,\zeta e_L) = B(|\zeta|_{\mathbb{C}})\,\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\,\chi_-(\zeta)\,.$$

So it is natural to define the operator $B(\mathbf{D}_\Sigma e_L)$ by $B(\mathbf{D}_\Sigma e_L) = b(-i\,\mathbf{D}_\Sigma)$ whenever $b(-i\,\mathbf{D}_\Sigma)$ is itself defined.

It is a consequence of Theorems 2.2 and 7.1 that the mapping from $H_\infty(S^\circ_\mu(\mathbb{C}))$ to $\mathcal{L}(L_p(\Sigma))$ given by $B \mapsto B(\mathbf{D}_\Sigma e_L)$ is a bounded algebra homomorphism.

We remark that the condition $b(\zeta)\zeta e_L = \zeta e_L b(\zeta)$ which we often use, is automatically satisfied by functions b of the type $b(\zeta) = B(i\zeta e_L)$.

Let H_ω be the linear space of functions B on $\mathbb{R} \setminus \{0\}$ which have holomorphic extensions $B \in H_\infty(S^\circ_\mu(\mathbb{C}))$ for some $\mu > \omega$, and let $\mathcal{P}_\omega$ be the linear space of functions B on $\mathbb{R}\setminus\{0\}$ which extend holomorphically to $S^\circ_\mu(\mathbb{C})$ for some $\mu > \omega$ and satisfy $|B(\zeta)| \le c(1+|\zeta|^s)$ on $S^\circ_\mu(\mathbb{C})$ for some s and $c \ge 0$.

Theorem 8.1. *Suppose that* $1 < p < \infty$. *Let* $B \in \mathcal{P}_\omega$.

i) *The operator* $B(\mathbf{D}_\Sigma e_L)$ *is a closed linear operator in* $L_p(\Sigma)$ *with domain* $\mathcal{D}(B(\mathbf{D}_\Sigma e_L))$ *dense in* $L_p(\Sigma)$.

ii) *If* $B \in H_\omega$, *then*

$$B(\mathbf{D}_\Sigma e_L) = T_{(\Phi,\underline{\Phi})} \in \mathcal{L}(L_p(\Sigma))\,,$$

where $\mathcal{F}(\Phi,\underline{\Phi})e_L = b$ *and* $b(\xi) = B(i\,\xi e_L)$. *In particular,* $1(\mathbf{D}_\Sigma e_L) = I$, $\chi_{\mathrm{Re}>0}(\mathbf{D}_\Sigma e_L) = P_+$, $\chi_{\mathrm{Re}<0}(\mathbf{D}_\Sigma e_L) = P_-$ *and* $\operatorname{sgn}(\mathbf{D}_\Sigma e_L) = C_\Sigma$.

iii) *If* $B \in H_\infty(S^\circ_\mu(\mathbb{C}))$ *with* $\omega < \mu \le \pi/2$, *then*

$$\|B(\mathbf{D}_\Sigma e_L)u\|_p \le C_{\omega,\mu,p}\,\|B\|_\infty\,\|u\|_p\,, \qquad u \in L_p(\Sigma)\,,$$

for some constants $C_{\omega,\mu,p}$ *which depend only on* ω, μ, p *(and the dimension* m*).*

iv) *If* $u \in \mathcal{D}(B(\mathbf{D}_\Sigma e_L))$, $F \in \mathcal{P}_\omega$ *and* $c \in \mathbb{C}$, *then* $u \in \mathcal{D}(F(\mathbf{D}_\Sigma e_L))$ *if and only if* $u \in \mathcal{D}((cB+F)(\mathbf{D}_\Sigma e_L))$, *in which case*

$$c\,B(\mathbf{D}_\Sigma e_L)u + F(\mathbf{D}_\Sigma e_L)u = (c\,B + F)(\mathbf{D}_\Sigma e_L)u\,.$$

v) *If* $u \in \mathcal{D}(B(\mathbf{D}_\Sigma e_L))$ *and* $F \in \mathcal{P}_\omega$, *then* $B(\mathbf{D}_\Sigma e_L)u \in \mathcal{D}(F(\mathbf{D}_\Sigma e_L))$ *if and only if* $u \in \mathcal{D}((FB)(\mathbf{D}_\Sigma e_L))$, *in which case*

$$F(\mathbf{D}_\Sigma e_L)B(\mathbf{D}_\Sigma e_L)u = (FB)(\mathbf{D}_\Sigma e_L)u\,.$$

vi) *The complex spectrum* $\sigma(B(\mathbf{D}_\Sigma e_L))$ *is a subset of*

$$\bigcap\{(B(S^\circ_\mu(\mathbb{C}))^{\mathrm{cl}} :\ \mu > \omega\}\,.$$

Indeed

$$\|(B(\mathbf{D}_\Sigma e_L) - \alpha I)^{-1}u\|_p \le C_{\omega,\mu,p}\,\frac{\|u\|_p}{\operatorname{dist}\{\alpha, B(S^\circ_\mu(\mathbb{C}))\}}\,,$$

for all $u \in L_p(\Sigma)$.

vii) *Suppose that there exists* $\mu \in (\omega, \pi/2)$, $s \ge 0$ *and* $c > 0$ *such that*

$$|B(\lambda)| \ge c\,|\lambda|^s(1 + |\lambda|^{2s})^{-1}\,, \qquad \lambda \in S^\circ_\mu(\mathbb{C})\,.$$

Then, the operator $B(\mathbf{D}_\Sigma e_L)$ *is one-one, and has dense range* $\mathcal{R}(B(\mathbf{D}_\Sigma e_L))$ *in* $L_p(\Sigma)$.

PROOF. The first five parts are immediate corollaries of Theorem 7.1.

To prove vi), let α be a complex number such that

$$d = \operatorname{dist}\{\alpha, B(S^\circ_\mu(\mathbb{C}))\} > 0\,, \qquad \text{for some } \mu > \omega\,.$$

Then $F = (B - \alpha)^{-1} \in H_\infty(S^\circ_\mu(\mathbb{C}))$ and $\|F\|_\infty \le d^{-1}$, so, by ii) and iii), $F(\mathbf{D}_\Sigma e_L) \in \mathcal{L}(L_p(\Sigma))$ and $\|F(\mathbf{D}_\Sigma e_L)u\|_p \le C_{\omega,\mu,p}\, d^{-1}\|u\|_p$ for all $u \in L_p(\Sigma)$.

Therefore, by iv) and v), $(B(\mathbf{D}_\Sigma e_L) - \alpha I)F(\mathbf{D}_\Sigma e_L)u = u$ for all $u \in L_p(\Sigma)$, and $F(\mathbf{D}_\Sigma e_L)(B(\mathbf{D}_\Sigma e_L) - \alpha I)u = u$ for all $u \in D(B(\mathbf{D}_\Sigma e_L))$. Hence $(B(\mathbf{D}_\Sigma e_L) - \alpha I)^{-1} = F(\mathbf{D}_\Sigma e_L)$. The result follows.

Part vii) is a consequence of Theorem 7.2.

The closed linear operator $\mathbf{D}_\Sigma e_L$ is defined on $L_p(\Sigma)$ by $\mathbf{D}_\Sigma e_L = B(\mathbf{D}_\Sigma e_L)$ when $B(\lambda) = \lambda$. It is a consequence of part vi) above, that its spectrum $\sigma(\mathbf{D}_\Sigma e_L)$ is a subset of $S_\omega(\mathbb{C}) = S_{\omega+}(\mathbb{C}) \cup S_{\omega-}(\mathbb{C})$ where $S_{\omega\pm}(\mathbb{C}) = \{\lambda \in \mathbb{C} : \lambda = 0 \text{ or } |\arg(\pm\lambda)| \le \omega\}$. Further, for all $\mu > \omega$, there exists $c_{\omega,\mu,p}$ such that

$$\|(\mathbf{D}_\Sigma e_L - \alpha I)^{-1}u\|_p \le c_{\omega,\mu,p}\,|\alpha|^{-1}\|u\|_p\,,$$

for all $\alpha \notin S_\mu(\mathbb{C})$ and all $u \in L_p(\Sigma)$. That is, $\mathbf{D}_\Sigma e_L$ is of type ω in $L_p(\Sigma)$ (provided "type ω" is defined using the double sector $S_\omega(\mathbb{C})$ as in [McQ]). Indeed, on applying part vii) as well, we find that $\mathbf{D}_\Sigma e_L$ is a one-one operator of type ω in $L_p(\Sigma)$ with dense domain $\mathcal{D}(\mathbf{D}_\Sigma e_L)$ and dense range $\mathcal{R}(\mathbf{D}_\Sigma e_L)$ in $L_p(\Sigma)$.

We see that the restrictions of $\mathbf{D}_\Sigma e_L$ to $L_p^\pm(\Sigma)$ are closed linear operators in $L_p^\pm(\Sigma)$ with spectra in $S_{\omega\pm}(\mathbb{C})$, and indeed that $\mp\mathbf{D}_\Sigma e_L$ are the infinitesimal generators of the holomorphic C_0-semigroups $u \mapsto T_{k_{\pm\alpha}}u$, $\alpha > 0$, in $L_p^\pm(\Sigma)$.

The next theorem states that resolvents and polynomials of $\mathbf{D}_\Sigma e_L$ are equal to their counterparts $B(\mathbf{D}_\Sigma e_L)$. Thus it is reasonable to say that the mapping $B \mapsto B(\mathbf{D}_\Sigma e_L)$ is a functional calculus of the single operator $\mathbf{D}_\Sigma e_L$, as well as to say that the mapping $b \mapsto b(-i\mathbf{D}_\Sigma) = b(-iD_{1,\Sigma}, -iD_{2,\Sigma}, \dots, -iD_{m,\Sigma})$ defined in Section 7 is a functional calculus of the m commuting operators $-iD_{k,\Sigma}$, $k = 1, 2, \dots, m$. It also states that $\mathbf{D}_\Sigma e_L$ is, not surprisingly, precisely the operator considered previously by Murray [M] and McIntosh [McI] (when $L = 0$). See also [GM].

Theorem 8.2. *Suppose that* $1 < p < \infty$.

i) *If* $\alpha \notin S_\omega(\mathbb{C})$, *define* $R_\alpha(\lambda) = (\lambda - \alpha)^{-1}$, *in which case* $R_\alpha(i\zeta e_L) = (i\zeta e_L - \alpha)^{-1}$. *Then* $R_\alpha(\mathbf{D}_\Sigma e_L) = (\mathbf{D}_\Sigma e_L - \alpha I)^{-1} \in \mathcal{L}(L_p(\Sigma))$.

ii) *For* k *a positive integer, define* $S_k(\lambda) = \lambda^k$, *in which case* $S_k(i\zeta e_L) = (i\zeta e_L)^k$. *Then* $\mathcal{D}(S_k(\mathbf{D}_\Sigma e_L)) = \mathcal{D}((\mathbf{D}_\Sigma e_L)^k)$ *and* $S_k(\mathbf{D}_\Sigma e_L)u = (\mathbf{D}_\Sigma e_L)^k u$ *for all* $u \in \mathcal{D}((\mathbf{D}_\Sigma e_L)^k)$.

iii) *Given a complex valued polynomial* $P(\lambda) = \sum_{k=0}^{k=d} a_k \lambda^k$ *of one variable with* $a_d \neq 0$, *define*

$$P(\mathbf{D}_\Sigma e_L)u = \sum a_k (\mathbf{D}_\Sigma e_L)^k u\,, \quad u \in \mathcal{D}(P(\mathbf{D}_\Sigma e_L)) = \mathcal{D}((\mathbf{D}_\Sigma e_L)^d)\,.$$

Then $\mathcal{D}(P(\mathbf{D}_\Sigma e_L)) = \mathcal{D}((\mathbf{D}_\Sigma e_L)^d)$, *and* $P(\mathbf{D}_\Sigma e_L)u = P(\mathbf{D}_\Sigma e_L)u$ *for all* $u \in \mathcal{D}(\mathbf{D}_\Sigma e_L)$.

iv) *If* Σ *is parametrized by* $x = \mathbf{s} + g(\mathbf{s})\, e_L$, $s \in \mathbb{R}^m$, *then*

$$\mathcal{D}(\mathbf{D}_\Sigma e_L) = W_p^1(\Sigma) = \Big\{ u \in L_p(\Sigma) : \frac{\partial}{\partial s_j} u(\mathbf{s} + g(\mathbf{s})\, e_L) \in L_p(\mathbb{R}^m, ds), \quad j = 1, 2, \dots, m \Big\}$$

and

$$(\mathbf{D}_\Sigma e_L u)(\mathbf{s}+g(\mathbf{s})\,e_L) = (e_L - \mathbf{D}g)^{-1}\mathbf{D}_s u(\mathbf{s}+g(\mathbf{s})\,e_L)\,, \quad u \in W^1_p(\Sigma)\,.$$

PROOF. The proofs of parts i) to iii) require repeated use of parts iv) and v) of Theorem 8.1. (*cf.* the proof of Theorem 8.1.vi).

To prove iv), let, for the moment, $\mathbf{A}_\Sigma$ be the closed linear operator with domain $W^1_p(\Sigma)$ in $L_p(\Sigma)$ defined by

$$(\mathbf{A}_\Sigma u)(\mathbf{s}+g(\mathbf{s})\,e_L) = (e_L - \mathbf{D}g)^{-1}\mathbf{D}_s u(\mathbf{s}+g(\mathbf{s})\,e_L)\,,$$

for all $u \in W^1_p(\Sigma)$, and note that $\mathbf{A}_\Sigma - iI$ is one-one [McI] (actually, one can see directly that $\mathbf{A}_\Sigma$ is of type ω).

Given $u \in \mathcal{D}(\mathbf{D}_\Sigma e_L)$, write $u = u_+ + u_-$ where $u_\pm = P_\pm u$, and, for $\delta > 0$, let $u_{+\delta} = T_{k+\delta} u_+$. We saw in Section 6 that $u_{+\delta} \in \mathcal{D}(\mathbf{D}_\Sigma e_L)$, $u_{+\delta} \to u_+$ and $\mathbf{D}_\Sigma e_L u_{+\delta} \to \mathbf{D}_\Sigma e_L u_+$ as $\delta \to 0$. Also $u_{+\delta} \in W^1_p(\Sigma)$, and we saw in Section 6 that $\mathbf{D}_\Sigma e_L u_{+\delta} = \mathbf{A}_\Sigma u_{+\delta}$. The fact that the operator $\mathbf{A}_\Sigma$ is closed implies that $u_+ \in \mathcal{D}(\mathbf{A}_\Sigma)$ and that $\mathbf{D}_\Sigma e_L u_+ = \mathbf{A}_\Sigma u_+$. On treating u_- in a similar way, we find that $u \in \mathcal{D}(\mathbf{A}_\Sigma)$ and that $\mathbf{D}_\Sigma e_L u = \mathbf{A}_\Sigma u$. Using the facts that $(\mathbf{A}_\Sigma - iI)$ is one-one and that $(\mathbf{D}_\Sigma e_L - iI)$ maps onto $L_p(\Sigma)$, we see that $\mathcal{D}(\mathbf{A}_\Sigma)$ can be no larger than $\mathcal{D}(\mathbf{D}_\Sigma e_L)$, and thus conclude the proof.

It is also true that, for $B \in H_\omega$, and indeed for $B \in \mathcal{P}_\omega$, the operator $B(\mathbf{D}_\Sigma e_L)$ coincides with that obtained using the definitions of holomorphic functional calculi in [Mc], [McQ], [McY], [CDMcY]. This is derived from Theorem 8.2 by using the Convergence Lemma of Section 7, and the Convergence Lemmas of those papers. We shall not go into details, but just wish to draw attention to the fact that the boundedness of the algebra homomorphism $B \mapsto B(\mathbf{D}_\Sigma e_L)$ is equivalent to the fact that $\mathbf{D}_\Sigma$ satisfies square function estimates in $L_p(\Sigma)$.

One particular consequence in the case $p = 2$ is the square function estimate,

$$\|u\|_2 \le C\left(\int_0^{+\infty} \|\Psi_+(t\mathbf{D}_\Sigma e_L)u\|_2 \,\frac{dt}{t}\right)^{1/2}, \qquad u \in L_2^+(\Sigma)\,,$$

where $\Psi_+(\lambda) = \chi_{\mathrm{Re}>0}(\lambda)\lambda e^{\lambda}$, or in other words, letting $U = \mathcal{C}_\Sigma^+ u$ denote the left-monogenic extension of u to Ω_+,

$$\begin{aligned}\|u\|_2 &\leq C\Big(\iint_{\Omega_+} |(\mathbf{D}U)(X)|^2 \operatorname{dist}\{X,\Sigma\}\, dX\Big)^{1/2} \\ &\leq C\Big(\iint_{\Omega_+} \Big(\sum_{k=1}^{m}\Big|\frac{\partial U}{\partial X_k}(X)\Big|^2 + \Big|\frac{\partial U}{\partial X_L}(X)\Big|^2\Big) \operatorname{dist}\{X,\Sigma\}\, dX\Big)^{1/2},\end{aligned}$$

where $u \in L_2^+(\Sigma)$. But this is the estimate proved in Theorem 4.1 of [LMcS], on which all our bounds are based. Now that we have traversed a full circle in [LMcS] and the current paper, it is time to stop.

ADDED IN PROOF. The reader may be interested in the lectures of Alan McIntosh on Clifford algebras, singular integrals, and harmonic functions on Lipschitz domains. These will appear as part of the Proceedings of the Conference on Clifford Algebras in Analysis held in Fayetteville, Arkansas, 1993, to be published by CRC Press.

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