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Estimates on the solution of an elliptic equation related to Brownian motion with drift

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1. Introduction.

In this paper we are concerned with studying the Dirichlet problem for an elliptic equation on a domain in $\mathbb{R}^3$. For simplicity we shall assume that the domain is a ball Ω_R of radius R. Thus

$$\Omega_R = \{x \in \mathbb{R}^3 : \ |x| < R\} . \tag{1.1}$$

The equation we are concerned with is given by

$$(-\Delta - \mathbf{b}(x) \cdot \nabla)\, u(x) = f(x) , \qquad x \in \Omega_R , \tag{1.2}$$

with zero Dirichlet boundary conditions,

$$u(x) = 0 , \qquad x \in \partial\Omega_R . \tag{1.3}$$

Here we shall think of the functions $\mathbf{b}(x), f(x)$ as defined on all of $\mathbb{R}^3$. Thus we shall assume that

$$\mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^3 , \qquad f : \mathbb{R}^3 \to \mathbb{R} , \tag{1.4}$$

are Lebesgue measurable functions. It is well known [5], [11] that the solution of (1.2)-(1.3) has -at least in the case of smooth functions $\mathbf{b}, f$- a representation as an expectation value with respect to Brownian motion with drift $\mathbf{b}$. Thus

$$u(x) = E_x \left[\int_0^\tau f(X_{\mathbf{b}}(t))\, dt \right] , \tag{1.5}$$

where E_x denotes the expectation is taken with respect to the drift process $X_{\mathbf{b}}(t)$ starting at $x \in \Omega_R$, and τ is the first hitting time on the boundary $\partial\Omega_R$.

Our main goal here is to prove existence and uniqueness of solutions to the boundary value problem (1.2)-(1.3) when the drift $\mathbf{b}$ is allowed to have singularities. To specify which kind of singularities $\mathbf{b}$ can have we define the Morrey spaces $M_p^q(\mathbb{R}^3)$ for $1 \le p \le q < \infty$. A measurable function $g : \mathbb{R}^3 \to \mathbb{C}$ is in $M_p^q(\mathbb{R}^3)$ if $|g|^p$ is locally integrable and there is a constant C such that

$$\int_Q |g|^p\, dx \le C^p\, |Q|^{1-p/q} , \tag{1.6}$$

for all cubes $Q \subset \mathbb{R}^3$. Here $|Q|$ denotes the volume of Q. The norm of g, $\|g\|_{q,p}$ is defined as

$$\|g\|_{q,p} = \inf\{C : \ (1.6) \text{ holds for } C \text{ and all cubes } Q\} . \tag{1.7}$$

It is easy to see that, with the definition (1.7) of norm, the space $M_p^q(\mathbb{R}^3)$ is a Banach space. Let $L^q(\mathbb{R}^3)$ be the standard L^q space on $\mathbb{R}^3$ with norm denoted by $\|\cdot\|_q$. Then one has the relationships for $1 \le r \le p \le q < \infty$,

$$\begin{gathered} L^q(\mathbb{R}^3) = M_q^q(\mathbb{R}^3) \subset M_p^q(\mathbb{R}^3) \subset M_r^q(\mathbb{R}^3) , \\ \|g\|_q = \|g\|_{q,q} \ge \|g\|_{q,p} \ge \|g\|_{q,r} . \end{gathered} \tag{1.8}$$

Our first theorem is a perturbation theory result.

Theorem 1.1. *Suppose* $1 < r < p \le q$ *and* $|\mathbf{b}| \in M_p^3$, $f \in M_r^q$ *for some* q, *with* $3/2 < q < 3$. *Then there exists an* $\varepsilon_0 > 0$ *depending only on* r, p, q *such that if* $\varepsilon \in \mathbb{C}$, $|\varepsilon| < \varepsilon_0/\|\mathbf{b}\|_{3,p}$, *then the boundary value problem*

$$(-\Delta - \varepsilon\, \mathbf{b}(x)\cdot\nabla)\, u_\varepsilon(x) = f(x) , \qquad x \in \Omega_R , \tag{1.9}$$

$$u_\varepsilon(x) = 0 , \qquad x \in \partial\Omega_R , \tag{1.10}$$

has a unique solution u_ε *in the following sense:*

a) u_ε *is uniformly Hölder continuous on* Ω_R *and satisfies the boundary condition* (1.10),

b) The distributional Lapacian Δu_ε *of* u_ε *on* Ω_R *is in* M_r^q *and the equation* (1.9) *holds for almost every* $x \in \Omega_R$,

c) $u_\varepsilon(x)$ *is an analytic function of* ε *in the disk* $|\varepsilon| < \varepsilon_0$ *for any fixed* $x \in \Omega_R$,

d) The L^∞ *norm of* u_ε *is bounded by*

$$\|u_\varepsilon\|_\infty \leq C\, R^{2-3/q}\, \|f\|_{q,r}\,, \tag{1.11}$$

where the constant C *depends only on* p, q, r.

REMARK. The restriction that f is L^q integrable for $q < 3$ is artificial since if $f \in L^{q_0}$ for some $q_0 \geq 3$ then $f \in L^q$ for all $q \leq q_0$. The $q < 3$ restriction is related to $b)$ and the value of ε_0.

Theorem 1.1 will be derived from a theorem on integral equations. Let T be an integral operator with measurable kernel $k_T : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$. Thus for measurable $f : \mathbb{R}^3 \to \mathbb{C}$ one defines Tf by

$$Tf(x) = \int_{\mathbb{R}^3} k_T(x,y)\, f(y)\, dy\,. \tag{1.12}$$

Theorem 1.2. *Suppose the kernel* k_T *of the integral operator* T *satisfies the inequality*

$$|k_T(x,y)| \leq \frac{|\mathbf{b}(x)|}{|x-y|^2}\,, \qquad x, y \in \mathbb{R}^3\,, \tag{1.13}$$

where $|\mathbf{b}| \in M_p^3$, $1 < p \leq 3$. *Then for any* r, q *which satisfy the inequalities*

$$1 < r < p\,, \qquad r \leq q < 3\,, \tag{1.14}$$

the operator T *is a bounded operator on the space* M_r^q. *The norm of* T *satisfies the inequality*

$$\|T\| \leq C\, \|\mathbf{b}\|_{3,p}\,, \tag{1.15}$$

where the constant C depends only on r, p, q.

Theorem 1.2 generalizes a result of Kerman and Sawyer [8] which proves the theorem in the case of L^q spaces, *i.e.* $r = q$. The more general Theorem 1.2 is necessary to prove Theorem 1.1 even if we assume $f \in L^q$. The Kerman-Sawyer theorem does apply to Theorem 1.1 if we assume $\mathbf{b} \in M_p^3$ with $p > 3/2$.

Next we turn to the non perturbative situation. It is easy to see -by considering the case of $|\mathbf{b}(x)| = \varepsilon/|x|$ with large ε- that (1.2) need not have a solution for $|\mathbf{b}| \in M_p^3$ if we make no restriction on the norm of $|\mathbf{b}|$. To obtain an appropriate non perturbative theorem we pursue an analogy with a problem which has already been studied in great detail. Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a measurable potential and consider the problem of estimating the number of bound states $N(V)$ of the Schrödinger operator $-\Delta + V$. It was shown independently by Cwickel-Lieb-Rosenbljum [10] that $N(V)$ satisfies the inequality

$$N(V) \le C \int_{\mathbb{R}^3} |V(x)|^{3/2}\, dx\,, \tag{1.16}$$

for some universal constant C. The best value for the constant C was obtained by Lieb [9] and is $C = .116$. This is to be contrasted with the lower bound on C, $C \ge .078$ obtained from semi-classical asymptotics. Hence the bound (1.16) with constant $C = .116$ is in some sense very sharp. However it may in fact be a bad estimate such as in the case $V(x) = -\varepsilon/|x|^2$ with ε small. In this situation the right hand side of (1.16) is infinity whereas in fact $N(V) = 0$.

In order to understand the cases where (1.16) gives bad estimates Fefferman and Phong [3] obtained new estimates on $N(V)$ which imply (1.16) and remain finite in the case $V(x) = -\varepsilon/|x|^2$ for small ε. The price one pays is that the constant C in (1.16) which follows from their estimates is far from optimal. The Fefferman-Phong estimate is as follows: Suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. Let $\varepsilon > 0$ be an arbitrary positive number. A cube Q is said to be minimal with respect to ε if

$$\begin{aligned} &\int_Q |V|^p\, dx \ge \varepsilon^p\, |Q|^{1-2p/3}\,, \\ &\int_{Q'} |V|^p\, dx < \varepsilon^p\, |Q'|^{1-2p/3}\,, \quad Q' \subset Q\,, \end{aligned} \tag{1.17}$$

for all dyadic subcubes $Q' \subset Q$. Here p is some fixed number, $1 < p \le 3/2$. Let $N_\varepsilon(V)$ be the number of minimal cubes in the dyadic decomposition. Then the Fefferman-Phong inequality is given by

$$N(V) \le C_\varepsilon N_\varepsilon(V), \tag{1.18}$$

where the constant C_ε is finite provided $\varepsilon > 0$ is sufficiently small. Since it is clear that

$$N_\varepsilon(V) \le \varepsilon^{-3/2} \int_{\mathbb{R}^3} |V(x)|^{3/2}\, dx, \tag{1.19}$$

the inequality (1.16) follows from (1.18).

The analogy between the drift problem (1.2)-(1.3) and the bound state problem for the Schrödinger operator is roughly in making the identification $-|\mathbf{b}|^2 = V$. It has been shown in a previous paper [1] that one can directly estimate the solution of the drift problem with $V = -|\mathbf{b}|^2$. However these estimates are not sharp. In fact there are important differences between the drift problem and the potential problem. For example there is no semi-classical asymptotic limit in which the inequality analogous to (1.16) becomes an identity. That said, our analysis will be close in spirit to the Fefferman-Phong analysis of the potential problem.

We consider the drift problem with non perturbative drift $\mathbf{b}$. Let p be a fixed number $1 < p < 3$ and $\varepsilon > 0$ be arbitrary. Suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. A cube Q is minimal with respect to ε if

$$\begin{aligned} &\int_Q |\mathbf{b}|^p\, dx \ge \varepsilon^p\, |Q|^{1-p/3}, \\ &\int_{Q'} |\mathbf{b}|^p\, dx < \varepsilon^p\, |Q'|^{1-p/3}, \quad Q' \subset Q, \end{aligned} \tag{1.20}$$

for all dyadic subcubes $Q' \subset Q$. Let $N_\varepsilon(\mathbf{b})$ be the number of minimal cubes in the dyadic decomposition. Then we have the following theorem:

Theorem 1.3. *Suppose $f \in M_r^q$, $1 < r \le q$, $r < p$, $p > 2$, $3/2 < q < 3$. Then there exists $\varepsilon > 0$ depending only on p, q, r such that if $N_\varepsilon(\mathbf{b}) < +\infty$ the boundary value problem (1.2)-(1.3) has a unique solution $u(x)$ in the following sense:*

a) u is uniformly Hölder continuous on Ω_R and satisfies the boundary condition (1.3),

b) The distributional Laplacian Δu of u is in M_r^q and the equation (1.2) *holds for almost every $x \in \Omega_R$.*

Our final theorem generalizes the estimate (1.11) on the L^∞-norm of the solution u of (1.2)-(1.3), to the nonperturbative situation. For $\mathbf{b} \in M_p^3$, $p > 1$, and n an integer define a function $a_n : \mathbb{R}^3 \to \mathbb{R}$ by

$$a_n(x) = \Big(2^{n(3-p)} \int_{|x-y|<2^{-n}} |\mathbf{b}|^p \, dy \Big)^{1/p} . \tag{1.21}$$

We then have the following

Theorem 1.4. *For $f \in M_r^q, \mathbf{b} \in M_p^3$ with $N_\varepsilon(\mathbf{b}) < +\infty$, let $u(x)$, $x \in \Omega_R$, be the solution of the Dirichlet problem* (1.2)-(1.3) *given by Theorem* 1.3. *Let n_0 be the integer which satisfies the inequality*

$$4R > 2^{-n_0} \geq 2R. \tag{1.22}$$

Then there exists γ, $0 < \gamma < 1$, depending only on $p > 2$ such that u satisfies the L^∞ estimate

$$\|u\|_\infty \leq C_1 R^{2-3/q} \|f\|_{q,r} \sum_{m=0}^{\infty} \gamma^m \sup_{x \in \Omega'_R} \exp \Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(x) \Big) . \tag{1.23}$$

The constant C_1 depends only on p, q, r and C_2 only on $p > 2$.

Theorem 1.4 will be proved in Section 6. We shall also show there that Theorem 1.4 implies the bound

$$\|u\|_\infty \leq C_1 R^{2-3/q} \|f\|_{q,r} \exp(C_2 N_\varepsilon(\mathbf{b})) , \tag{1.24}$$

provided ε is sufficiently small depending only on $p > 2$. Since $N_\varepsilon(\mathbf{b})$ satisfies the inequality

$$N_\varepsilon(\mathbf{b}) \leq \varepsilon^{-3} \|\mathbf{b}\|_3^3 , \tag{1.25}$$

the inequality (1.24) implies the bound

$$\|u\|_\infty \leq C_1 R^{2-3/q} \|f\|_{q,r} \exp(C_2 \|\mathbf{b}\|_3^3) . \tag{1.26}$$

Inequality (1.26) with $q = r = 3$, is already known [6]. This is the Alexandrov-Pucci estimate, which has been proved here using different ideas.

Our method is based on studying how the drift process $X_{\mathbf{b}}(t)$ differs from Brownian motion $X(t)$. The technical tool we use for this is the Cameron-Martin formula [10] which expresses expectations with respect to the drift process as Brownian motion expectations. Our main idea is that if $p > 2$ then the sets on which $|\mathbf{b}|$ is large have dimension strictly less than 1. Hence, by the nonrecurrence property of Brownian motion in dimension strictly larger than 2, most paths do not often visit sets where $|\mathbf{b}|$ is large.

While there is an extensive recent literature on elliptic equations with nonsmooth coefficients [2], [4], [7], there appears to be little studying the singular drift problem. The most recent paper we could find on the subject was the 1980 paper of Trudinger [12]. See also the book by Friedlin [5] for the relation between functional integration and partial differential equations.

2. A Theorem in Integral Equations.

Our goal in this section is to prove Theorem 1.2. For $x \in \mathbb{R}^3$ and $r > 0$, let $B(x,r)$ be the ball of radius r centered at x,

$$B(x,r) = \{y \in \mathbb{R}^3 : \ |x-y| < r\}. \tag{2.1}$$

We define operators S_n on locally integrable functions u on $\mathbb{R}^3$ for any integer $n \in \mathbb{Z}$ by

$$S_n u(x) = 2^{-n} \, |B(x, 2^{-n})|^{-1} \int_{B(x,2^{-n})} |u(y)| \, dy. \tag{2.2}$$

It is evident then from (1.3) that the operator T satisfies the inequality

$$|Tu(x)| \le C \sum_{n=-\infty}^{\infty} |\mathbf{b}(x)| S_n u(x), \tag{2.3}$$

for some universal constant C.

Let Q_0 be the cube centered at the origin with side of length 2^{-n_0}. We define an operator T_0 by

$$\begin{aligned} T_0 u(x) &= Tu(x), \qquad x \notin Q_0, \\ T_0 u(x) &= 0, \qquad x \in Q_0. \end{aligned} \tag{2.4}$$

Lemma 2.1. *Suppose the support of u is contained in the ball $B(0,2^{-n_0-2})$. Then for $1<r<p\le 3$, $r\le q\le 3$, there is a constant C depending only on p, q, r such that*

$$\|T_0 u\|_{q,r} \le C\,\|\mathbf{b}\|_{3,p}\,\|u\|_{q,r}\,. \tag{2.5}$$

PROOF. We need to show

$$\Big(\int_Q |T_0 u|^r\,dx\Big)^{1/r} \le C\,\|\mathbf{b}\|_{3,p}\,\|u\|_{q,r}\,|Q|^{1/r-1/q}\,, \tag{2.6}$$

for all cubes Q. First let us consider the case

$$|Q| \le 2^{-3n_0}\,. \tag{2.7}$$

We use the inequality

$$|T_0 u(x)| \le A\,|\mathbf{b}(x)|\,2^{2n_0}\,\|u\|_1\,, \qquad x\in\mathbb{R}^3\,, \tag{2.8}$$

where the constant A is universal. Hence the left hand side of (2.6) is bounded by

$$A\,2^{2n_0}\|u\|_1\Big(\int_Q |\mathbf{b}|^r\,dx\Big)^{1/r} \le A\,2^{2n_0}\|u\|_1\,\|\mathbf{b}\|_{3,p}\,|Q|^{1/r-1/3}\,, \tag{2.9}$$

on using Hölder's inequality. Next we use the fact that

$$\|u\|_1 \le \|u\|_{q,r}\,|Q_0|^{1-1/q}\,, \tag{2.10}$$

whence

$$\Big(\int_Q |T_0 u|^r\,dx\Big)^{1/r} \le A\,\|\mathbf{b}\|_{3,p}\,\|u\|_{q,r}\,|Q|^{1/r-1/q}\,, \tag{2.11}$$

in view of (2.7).

Next let us suppose that

$$2^{-3k} < |Q| \le 2^{-3(k-1)}\,, \qquad k\le n_0\,, \tag{2.12}$$

and the double of Q contains the origin. Then, by the property of the support of u, one has

$$\int_Q |T_0 u|^r\,dx \le A\,\|u\|_1^r \sum_{n=k-3}^{n_0} 2^{2nr}\int_{Q_n} |\mathbf{b}|^r\,dx\,, \tag{2.13}$$

where the cubes Q_n have side of length 2^{-n} and center at the origin. Using the fact that $\mathbf{b} \in M_p^3$, $p > r$, the inequality (2.13) yields

$$\begin{aligned} \int_Q |T_0 u|^r \, dx &\leq A \, \|u\|_1^r \sum_{n=k-3}^{n_0} 2^{2nr} \, \|\mathbf{b}\|_{3,p}^r \, |Q_n|^{1-r/3} \\ &= A \, \|u\|_1^r \, \|\mathbf{b}\|_{3,p}^r \sum_{n=k-3}^{n_0} 2^{3n(r-1)} \\ &\leq B \, \|u\|_1^r \, \|\mathbf{b}\|_{3,p}^r \, 2^{3n_0(r-1)} , \end{aligned} \tag{2.14}$$

for some constant B depending on $r > 1$.

We have then

$$\left(\int_Q |T_0 u|^r \, dx \right)^{1/r} \leq C \, \|u\|_1 \, \|\mathbf{b}\|_{3,p} \, |Q_0|^{1/r-1} . \tag{2.15}$$

Using (2.10) again we conclude that

$$\begin{aligned} \left(\int_Q |T_0 u|^r \, dx \right)^{1/r} &\leq C' \, \|\mathbf{b}\|_{3,p} \, \|u\|_{q,r} \, |Q_0|^{1/r-1/q} \\ &\leq C' \, \|\mathbf{b}\|_{3,p} \, \|u\|_{q,r} \, |Q|^{1/r-1/q} . \end{aligned} \tag{2.16}$$

Finally, if Q satisfies (2.12) and the double of Q does not contain the origin then the inequality (2.16) continues to hold.

Let K be an arbitrary cube in $\mathbb{R}^3$ with side of length 2^{-n_K} for some integer n_K. We associate with K an operator T_K on integrable functions $u : K \to \mathbb{C}$. To do this we decompose K into a dyadic decomposition of cubes Q_n with sides of length 2^{-n}, where $n \geq n_K$. For any cube $Q_n \subset K$ let u_{Q_n} be the average of $|u|$ on Q_n. Then for any $n \geq n_K$ we define the operator S_n by

$$S_n u(x) = 2^{-n} \, u_{Q_n} , \qquad x \in Q_n . \tag{2.17}$$

The operator T_K is then given by

$$T_K u(x) = \sum_{n=n_K}^{\infty} |\mathbf{b}(x)| \, S_n u(x) , \qquad x \in K . \tag{2.18}$$

We relate the operators T_K to the operator T by the following

Lemma 2.2. *For $z \in Q_0$ let $\tilde{Q}_0(z)$ be the cube centered at z with side of length 2^{2-n_0}. Let u be an arbitrary integrable function supported in the ball $B(0, 2^{-n_0-2})$ and Q an arbitrary cube. Then there is a universal constant C such that for any $r \geq 1$,*

$$\int_{Q\cap Q_0} |Tu(x)|^r \, dx \leq C \int_{Q_0} \frac{dz}{|Q_0|} \int_{Q\cap Q_0} |T_{\tilde{Q}_0(z)} u(x)|^r \, dx \,. \tag{2.19}$$

PROOF. This is a consequence of Jensen's inequality. In fact Jensen implies that

$$\begin{aligned} \int_{Q_0} \frac{dz}{|Q_0|} \int_{Q\cap Q_0} & |T_{\tilde{Q}_0(z)} u(x)|^r \, dx \\ & \geq \int_{Q\cap Q_0} \Big(\int_{Q_0} \frac{dz}{|Q_0|} T_{\tilde{Q}_0(z)} u(x) \Big)^r \, dx \,. \end{aligned} \tag{2.20}$$

Now one merely has to note that, because of the restriction on the support of u, one has

$$\int_{Q_0} \frac{dz}{|Q_0|} T_{\tilde{Q}_0(z)} u(x) \geq C \, |Tu(x)| \,, \qquad x \in Q_0 \,, \tag{2.21}$$

for some universal constant C.

The main work in this section will be concerned with bounding the operators T_K.

Theorem 2.3. *Suppose $1 < r < p \leq 3$, $1 < r \leq q < 3$. Then there is a constant C depending only on p, q, r such that*

$$\Big(\int_Q |T_K u|^r \, dx \Big)^{1/r} \leq C \, \|\mathbf{b}\|_{3,p} \, \|u\|_{q,r} \, |Q|^{1/r - 1/q} \,, \tag{2.22}$$

where Q is any dyadic subcube of the cube K.

PROOF OF THEOREM 1.2. We can assume without loss of generality that u has compact support where the support of u is contained in a ball $B(0, 2^{-n_0-2})$ for some integer n_0. Let Q be an arbitrary cube. Then by Lemma 2.1 one has

$$\begin{aligned} \Big(\int_Q |Tu|^r \, dx \Big)^{1/r} & \leq \Big(\int_{Q\cap Q_0} |Tu|^r \, dx \Big)^{1/r} + \Big(\int_{Q\setminus Q_0} |Tu|^r \, dx \Big)^{1/r} \\ & \leq \Big(\int_{Q\cap Q_0} |Tu|^r \, dx \Big)^{1/r} + C \, \|\mathbf{b}\|_{3,p} \, \|u\|_{q,r} \, |Q|^{1/r - 1/q} \,. \end{aligned} \tag{2.23}$$

Let K be one of the cubes $\tilde{Q}_0(z)$ from Lemma 2.2. Then it is clear that the set $Q \cap Q_0$ is contained in the union of at most eight dyadic subcubes Q' of $\tilde{Q}_0(z)$ with $|Q'| \le |Q|$. Hence one has

$$\begin{aligned}\left(\int_{Q\cap Q_0} |T_K u|^r\,dx\right)^{1/r} &\le \sum_{Q'} \left(\int_{Q'} |T_K u|^r\,dx\right)^{1/r} \\ &\le 8\,C\,\|\mathbf{b}\|_{3,p}\,\|u\|_{q,r}\,|Q|^{1/r-1/q}\,,\end{aligned} \tag{2.24}$$

by Theorem 2.1.

Now Lemma 2.2 implies

$$\left(\int_{Q\cap Q_0} |Tu|^r\,dx\right)^{1/r} \le C'\,\|\mathbf{b}\|_{3,p}\,\|u\|_{q,r}\,|Q|^{1/r-1/q}\,, \tag{2.25}$$

for some constant C' depending only on p, q, r. Theorem 1.2 follows now from (2.23), (2.25).

We begin the proof of Theorem 2.3. We shall assume without loss of generality that

$$\|\mathbf{b}\|_{3,p} \le 1\,. \tag{2.26}$$

Lemma 2.4. *Suppose $u : K \to \mathbb{C}$ is an integrable function and $Q' \subset K$ is a dyadic subcube of K such that for all dyadic $Q \subset Q'$ there is the inequality*

$$|Q|^{1/3+\varepsilon} u_Q \le |Q'|^{1/3+\varepsilon} u_{Q'}\,, \tag{2.27}$$

for sufficiently small $\varepsilon > 0$ depending only on r, p. Then the inequality (2.22) holds on Q'.

PROOF. Let N be the integer such that the length of Q' is $2^{-N}, N \ge n_K$. Then one has

$$\begin{aligned}\left(\int_{Q'} |Tu_K|^r\,dx\right)^{1/r} &\le \left(\int_{Q'} \Big(|\mathbf{b}(x)| \sum_{n=n_K}^{N-1} S_n u(x)\Big)^r dx\right)^{1/r} \\ &\quad + \left(\int_{Q'} \Big(|\mathbf{b}(x)| \sum_{n=N}^{\infty} S_n u(x)\Big)^r dx\right)^{1/r}.\end{aligned} \tag{2.28}$$

We estimate the first term on the right in (2.28). Since $q < 3$ one has

$$\sum_{n=n_K}^{N-1} S_n u(x) \le C\, \|u\|_{q,r}\, |Q'|^{1/3-1/q}\,. \tag{2.29}$$

Thus the first term is bounded by

$$\begin{aligned} C\, \|u\|_{q,r}\, |Q'|^{1/3-1/q} \Big(\int_{Q'} |\mathbf{b}(x)|^r \, dx \Big)^{1/r} \\ \le C\, \|u\|_{q,r}\, |Q'|^{1/3-1/q}\, |Q'|^{1/r-1/3} \end{aligned} \tag{2.30}$$

in view of (2.26). Hence the first term is bounded by

$$C\, \|u\|_{q,r} |Q'|^{1/r-1/q}\,, \tag{2.31}$$

which has the form of the right side of (2.22).

To bound the second term on the right in (2.28) we need to decompose $|\mathbf{b}|$. For m an integer let E_m be the set

$$E_m = \{x \in \mathbb{R}^3 :\ 2^{m-1} < |\mathbf{b}(x)| \le 2^m\}\,. \tag{2.32}$$

We write the sum of $S_n u(x)$ over n as

$$\begin{aligned} &\Big(\sum_{n=N}^{\infty} S_n u(x) \Big)^r \\ &= S_N u(x)^r + \sum_{k=N}^{\infty} \Big(\Big(\sum_{n=N}^{k+1} S_n u(x) \Big)^r - \Big(\sum_{n=N}^{k} S_n u(x) \Big)^r \Big) \\ &= S_N u(x)^r \\ &\quad + \sum_{k=N}^{\infty} r \int_0^1 \Big(\sum_{n=N}^{k} S_n u(x) + t\, S_{k+1} u(x) \Big)^{r-1} S_{k+1} u(x)\, dt\,. \end{aligned} \tag{2.33}$$

Now we use (2.27) to obtain the bound

$$\sum_{n=N}^{k} S_n u(x) + t\, S_{k+1}\, u(x) \le (k+2-N)\, 2^{3\varepsilon(k+1-N)}\, |Q'|^{1/3}\, u_{Q'}\,. \tag{2.34}$$

We have then the estimate

$$\begin{aligned}(2.35)\qquad &\Big(\sum_{n=N}^{\infty} S_n u(x)\Big)^r \leq \big(|Q'|^{1/3}\, u_{Q'}\big)^{r-1} \\ &\cdot \Big(S_N u(x) + \sum_{k=N+1}^{\infty} r\,(k+1-N)^{r-1}\, 2^{3\varepsilon(r-1)(k-N)}\, S_k u(x)\Big) .\end{aligned}$$

For m, k integers with $k \geq N$ let

$$(2.36)\qquad a_{m,k} = \sum_{Q_k \subset Q'} |E_m \cap Q_k|\, u_{Q_k} ,$$

where the Q_k are dyadic subcubes of Q' with side of length 2^{-k}. Then one has

$$\begin{aligned}&\int_{Q'} \Big(|\mathbf{b}(x)| \sum_{n=N}^{\infty} S_n u(x)\Big)^r dx \\ (2.37)\qquad &\leq \Big(|Q'|^{1/3}\, u_{Q'}\Big)^{r-1} \sum_{m=-\infty}^{\infty} 2^{mr} \Big(2^{-N}\, a_{m,N} \\ &\quad + \sum_{k=N+1}^{\infty} r\,(k+1-N)^{r-1}\, 2^{3\varepsilon(r-1)(k-N)}\, 2^{-k}\, a_{m,k}\Big) .\end{aligned}$$

There are two estimates on $a_{m,k}$ which we use. The first follows from (2.27). Thus

$$(2.38)\qquad a_{m,k} \leq |E_m \cap Q'|\, 2^{(1+3\varepsilon)(k-N)}\, u_{Q'} .$$

The second is obtained by observing that $|E_m \cap Q_k| \leq |Q_k|$, whence

$$(2.39)\qquad a_{m,k} \leq |Q'|\, u_{Q'} .$$

It follows that for any $\alpha, 0 < \alpha < 1$, the right side of (2.37) is bounded by

$$\begin{aligned}&\Big(|Q'|^{1/3} u_{Q'}\Big)^{r-1} \sum_{m=-\infty}^{\infty} 2^{mr} \Big(2^{-N} \Big(|Q'|\, u_{Q'}\Big)^{\alpha} \Big(|E_m \cap Q'|\, u_{Q'}\Big)^{1-\alpha} \\ &+ \sum_{k=N+1}^{\infty} r(k+1-N)^{r-1} 2^{3\varepsilon(r-1)(k-N)} 2^{-k} \Big(|Q'|\, u_{Q'}\Big)^{\alpha}\end{aligned}$$

$$
\begin{aligned}
&\cdot\Big(|E_m \cap Q'|\, 2^{(1+3\varepsilon)(k-N)}\, u_{Q'}\Big)^{1-\alpha}\Big) \\
(2.40)\qquad & \leq C_\alpha\, |Q'|^{r/3+\alpha}\, u_{Q'}^r \sum_{m=-\infty}^{\infty} 2^{mr}\, |E_m \cap Q'|^{1-\alpha},
\end{aligned}
$$

where the constant C_α depends only on $\alpha > 0$.

We bound the sum with respect to m on the right in (2.40) by writing

$$
\begin{aligned}
&\sum_{m=-\infty}^{\infty} 2^{mr}|E_m \cap Q'|^{1-\alpha} \\
(2.41)\qquad & \leq |Q'|^{1-\alpha} \sum_{m=-\infty}^{N-1} 2^{mr} + \sum_{m=N}^{\infty} 2^{mr}|E_m \cap Q'|^{1-\alpha} \\
& \leq C\,|Q'|^{1-\alpha-r/3} + \sum_{m=N}^{\infty} 2^{mr}|E_m \cap Q'|^{1-\alpha}.
\end{aligned}
$$

In view of (2.26) one has

$$
(2.42)\qquad 2^{mp}|E_m \cap Q'| \leq 2^p|Q'|^{1-p/3},
$$

and consequently it follows because $r < p$ that

$$
(2.43)\qquad \sum_{m=N}^{\infty} 2^{mr}|E_m \cap Q'|^{1-\alpha} \leq C_\alpha\, |Q'|^{1-\alpha-r/3},
$$

provided $\alpha > 0$ is sufficiently small. We conclude then that there is a constant C depending only on r, p, q such that

$$
\begin{aligned}
(2.44)\qquad \Big(\int_{Q'} \Big(|\mathbf{b}(x)| \sum_{n=N}^{\infty} S_n u(x)\Big)^r dx\Big)^{1/r} & \leq C\,|Q'|^{1/r} u_{Q'} \\
& \leq C\,\|u\|_{q,r}\,|Q'|^{1/r-1/q}.
\end{aligned}
$$

The inequality (2.44) combined with (2.31) proves the result.

Next we need to remove the restriction (2.27) on the growth of the averages of u on dyadic cubes. To do this we define a Calderón-Zygmund decomposition of Q'. We can assume without loss of generality that $u \in L^\infty(Q')$. Define a function N_1 on Q',

$$
(2.45)\qquad N_1 : Q' \to \{k \in \mathbb{Z} \cup \{\infty\} :\ k \geq N\},
$$

by

a) $N_1(x) = \infty$ if $|Q|^{1/3+\varepsilon} u_Q \le |Q'|^{1/3+\varepsilon} u_{Q'}$ for all dyadic subcubes Q of Q' such that $x \in Q$,

b) Otherwise $2^{-N_1(x)}$ is the length of the side of the largest dyadic cube Q, $x \in Q \subset Q'$, such that $|Q|^{1/3+\varepsilon} u_Q > |Q'|^{1/3+\varepsilon} u_{Q'}$.

We define the set G_1 to be

$$G_1 = \{x \in Q' : \ N_1(x) = \infty\}. \tag{2.46}$$

Since $u \in L^\infty(Q')$ there is a unique finite family $\mathcal{F}_1$ of disjoint dyadic subcubes of Q' such that

$$\bigcup_{Q \in \mathcal{F}_1} Q = Q' \backslash G_1 . \tag{2.47}$$

If $\mathcal{F}_1$ is nonempty then we define a function N_2 on Q' which is analogous to N_1. Thus

a) $N_2(x) = \infty$ if $x \in G_1$,

b) $N_2(x) = \infty$ if $x \in Q' \backslash G_1$ and $|Q|^{1/3+\varepsilon} u_Q \le |\overline{Q}|^{1/3+\varepsilon} u_{\overline{Q}}$ for all dyadic subcubes with $x \in Q \subset \overline{Q} \in \mathcal{F}_1$,

c) Otherwise $2^{-N_2(x)}$ is the length of the side of the largest dyadic cube Q, $x \in Q \subset \overline{Q} \in \mathcal{F}_1$, such that $|Q|^{1/3+\varepsilon} u_Q > |\overline{Q}|^{1/3+\varepsilon} u_{\overline{Q}}$.

Observe that $N_2(x)$ is defined uniquely for x not on the boundary of any cube $\overline{Q} \in \mathcal{F}_1$. Thus it is defined up to a set of measure 0. Furthermore, one has

$$N_2(x) \ge N_1(x) + 1, \qquad \text{a.e. } x \in Q'. \tag{2.48}$$

Now define G_2 to be the set

$$G_2 = \{x \in Q' \backslash G_1 : \ N_2(x) = \infty\}. \tag{2.49}$$

Then, as with N_1, there is a unique finite family $\mathcal{F}_2$ of disjoint dyadic subcubes of Q' with

$$\bigcup_{Q \in \mathcal{F}_2} Q = Q' \backslash G_1 \backslash G_2 . \tag{2.50}$$

One can continue this procedure inductively to construct a sequence of functions $N_j, j \geq 1$, on Q', a sequence of disjoint subsets $G_j, j \geq 1$, of Q', and a sequence of families $\mathcal{F}_j$ with the properties:

$$a) \quad \bigcup_{j=1}^{\infty} G_j = Q' ,$$

b) $\mathcal{F}_j$ is a finite collection of disjoint dyadic subcubes of Q' such that

$$\bigcup_{Q \in \mathcal{F}_k} Q = Q' \setminus \bigcup_{j=1}^{k} G_j ,$$

c) For any $Q \in \mathcal{F}_k$ let $\overline{Q} \in \mathcal{F}_{k-1}$ be the unique subcube containing Q. Then

$$|Q|^{1/3+\varepsilon} u_Q > |\overline{Q}|^{1/3+\varepsilon} u_{\overline{Q}} ,$$

$$d) \quad N_k(x) = \infty \text{ for } x \in \bigcup_{j=1}^{k} G_j .$$

Otherwise $N_k(x)$ is defined by $2^{-3N_k(x)} = |Q|$ where Q is the unique cube in $\mathcal{F}_k$ with $x \in Q$.

We have constructed families $\mathcal{F}_j, j \geq 1$, of dyadic subcubes of Q'. Let $\mathcal{F}_0 = \{Q'\}$. Then we have

Lemma 2.5. *Suppose $u \in L^\infty(Q')$. Then there is a constant C depending only on r, p, q such that*

$$(2.51) \qquad \int_{Q'} \Big(|\mathbf{b}(x)| \sum_{n=N}^{\infty} S_n u(x) \Big)^r dx \leq C \sum_{j=0}^{\infty} \sum_{Q \in \mathcal{F}_j} |Q| \, u_Q^r .$$

PROOF. Define a sequence a_j, $j \geq 0$, by

$$\begin{aligned} a_0 = & \int_{Q'} |\mathbf{b}(x)|^r \big(S_N u(x)\big)^r dx \\ & + \int_{Q'} |\mathbf{b}(x)|^r \sum_{k=N}^{N_1(x)-2} r \int_0^1 \Big(\sum_{n=N}^{k} S_n u(x) + t\, S_{k+1} u(x) \Big)^{r-1} \\ & \qquad \cdot S_{k+1} u(x) \, dt \, dx , \end{aligned}$$

(2.52)

$$a_j = \int_{Q'} |\mathbf{b}(x)|^r \sum_{k=N_j(x)-1}^{N_{j+1}(x)-2} r \int_0^1 \Big(\sum_{n=N}^{k} S_n u(x) + t\, S_{k+1} u(x) \Big)^{r-1} \cdot S_{k+1} u(x)\, dt\, dx\,, \qquad j \geq 1\,.$$

In view of (2.33) the left hand side of (2.51) is given by

$$\sum_{j=0}^{\infty} a_j\,. \tag{2.53}$$

It follows directly from the proof of Lemma 2.4 that there is a constant C such that

$$a_0 \leq C\, |Q'|\, u_{Q'}^r\,. \tag{2.54}$$

We wish to show that for $j \geq 1$, one has

$$a_j \leq C \sum_{Q \in \mathcal{F}_j} |Q|\, u_Q^r\,. \tag{2.55}$$

Evidently one has

$$a_j = \int_{N_j(x) < \infty} (\,\cdot\,)\, dx = \sum_{Q \in \mathcal{F}_j} \int_Q (\,\cdot\,)\, dx\,, \tag{2.56}$$

where $(\,\cdot\,)$ denotes the integrand in the formula for a_j. Let us fix a particular $\overline{Q} \in \mathcal{F}_j$ with side of length 2^{-M}, $M > N$, whence $N_j(x) = M$, $x \in \overline{Q}$. By definition of the families $\mathcal{F}_j$ it follows that

$$\begin{aligned} \sum_{n=N}^{M} S_n u(x) &\leq \sum_{n=N}^{M} 2^{-3\varepsilon(M-n)} |\overline{Q}|^{1/3}\, u_{\overline{Q}} \\ &\leq C\, |\overline{Q}|^{1/3} u_{\overline{Q}}\,, \qquad x \in \overline{Q}\,. \end{aligned} \tag{2.57}$$

On the other hand, for $M < k+1 \leq N_{j+1}(x) - 1$, one has

$$\begin{aligned} &\sum_{n=M+1}^{k} S_n u(x) + t\, S_{k+1} u(x) \\ &\qquad \leq (k+1-M)\, 2^{3\varepsilon(k+1-M)} |\overline{Q}|^{1/3}\, u_{\overline{Q}}\,, \end{aligned} \tag{2.58}$$

which is analogous to (2.34). Now one can proceed just as in Lemma 2.4 to obtain (2.55). The result follows then from (2.53) and (2.55).

Lemma 2.6. *There is a constant C depending only on r, p, q such that*

$$\sum_{j=0}^{\infty} \sum_{Q \in \mathcal{F}_j} |Q|\, u_Q^r \le C \int_{Q'} |u|^r \, dx \,. \tag{2.59}$$

PROOF. Since we can assume $\|u\|_\infty < \infty$ there exists an integer $t \ge 1$ such that $\mathcal{F}_t$ is empty. Thus

$$Q' = \bigcup_{j=1}^{t} G_j \,. \tag{2.60}$$

Let us consider a particular $Q \in \mathcal{F}_j$, $0 \le j \le t-1$. It is evident that

$$Q \subset \bigcup_{m=j+1}^{t} G_m \,. \tag{2.61}$$

We wish to estimate $|Q \cap G_m|$ for $m \ge j+1$. We have now

$$\begin{aligned} |Q|\, u_Q &= \int_Q |u| \, dx \\ &\ge \sum_{i=m}^{t} \int_{Q \cap G_i} |u| \, dx \\ &= \sum_{\overline{Q} \in \mathcal{F}_{m-1}, \overline{Q} \subset Q} |\overline{Q}|\, u_{\overline{Q}} \\ &\ge \sum_{\overline{Q} \in \mathcal{F}_{m-1}, \overline{Q} \subset Q} |\overline{Q}| \left(\frac{|Q|}{|\overline{Q}|} \right)^{1/3} u_Q \\ &\ge 2^{(m-j-1)}\, u_Q \sum_{\overline{Q} \in \mathcal{F}_{m-1}, \overline{Q} \subset Q} |\overline{Q}| \\ &= 2^{(m-j-1)}\, u_Q \, |Q \cap \bigcup_{i=m}^{t} G_i| \,. \end{aligned} \tag{2.62}$$

We conclude therefore that

$$\frac{|Q \cap G_m|}{|Q|} \leq 2^{-(m-j-1)}. \tag{2.63}$$

Next we consider

$$\begin{aligned} |Q|\, u_Q^r &= \frac{1}{|Q|^{r-1}} \Big(\int_Q |u|\, dx \Big)^r \\ &= \frac{1}{|Q|^{r-1}} \Big(\sum_{m=j+1}^{t} \int_{Q\cap G_m} |u|\, dx \Big)^r \\ &\leq \frac{1}{|Q|^{r-1}} \Big(\sum_{m=j+1}^{t} a_m^{r'} \Big)^{r/r'} \sum_{m=j+1}^{t} a_m^{-r} \Big(\int_{Q\cap G_m} |u|\, dx \Big)^r, \end{aligned} \tag{2.64}$$

by Hölder's inequality, where a_m is an arbitrary positive sequence and $1/r + 1/r' = 1$. We choose a_m to be given by

$$a_m = \bigg(\Big(\frac{3}{2}\Big)^{m-j-1} \frac{|Q \cap G_m|}{|Q|} \bigg)^{1/r'} \tag{2.65}$$

In view of (2.63) the inequality (2.64) yields

$$\begin{aligned} |Q|\, u_Q^r &\leq C \sum_{m=j+1}^{t} \Big(\frac{2}{3}\Big)^{(m-j-1)(r-1)} \frac{1}{|Q \cap G_m|^{r-1}} \\ &\qquad \cdot \Big(\int_{Q\cap G_m} |u|\, dx \Big)^r \\ &= C \sum_{m=j+1}^{t} \Big(\frac{2}{3}\Big)^{(m-j-1)(r-1)} |Q \cap G_m|\, u_{Q\cap G_m}\,. \end{aligned} \tag{2.66}$$

We conclude then that

$$\sum_{Q\in\mathcal{F}_j} |Q|\, u_Q^r \leq C \sum_{m=j+1}^{t} \Big(\frac{2}{3}\Big)^{(m-j-1)(r-1)} \int_{G_m} |u|^r\, dx \tag{2.67}$$

by Jensen's inequality. Now if we sum (2.67) with respect to j and use the fact that the sets G_j are disjoint we obtain the inequality (2.59).

Theorem 2.3 now follows immediately from the previous two lemmas and the estimate (2.31) in Lemma 2.4 on the first term on the right hand side of (2.28).

3. Perturbative existence and uniqueness.

We turn to the proof of Theorem 1.1. We first consider the problem of uniqueness of the solution to (1.9), (1.10). Let us write $g(x) = -\Delta u_\varepsilon(x), x \in \Omega_R$, the distributional Laplacian which is assumed to exist by b) of Theorem 1.1. Since $g \in M_r^q$ and a) of the same theorem it follows by Weyl's lemma that u_ε is given by the formula

$$u_\varepsilon(x) = \int_{\Omega_R} G_D(x,y)\, g(y)\, dy\,, \tag{3.1}$$

where G_D is the Green's function for the Dirichlet Laplacian on Ω_R. Thus

$$G_D(x,y) = \frac{1}{4\pi\, |x-y|} - \frac{1}{4\pi}\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\,, \tag{3.2}$$

where $\overline{y}$ is the conjugate of y in the sphere $\partial\Omega_R$. It follows easily from the representation (3.1) that the distributional gradient ∇u_ε exists as an integrable function on Ω_R and is given by the formula

$$\nabla u_\varepsilon(x) = \int_{\Omega_R} \nabla_x G_D(x,y)\, g(y)\, dy\,, \qquad x \in \Omega_R\,. \tag{3.3}$$

Now let T be the integral operator with kernel k_T given by

$$k_T(x,y) = \begin{cases} \mathbf{b}(x)\cdot\nabla_x G_D(x,y)\,, & x,y \in \Omega_R\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{3.4}$$

It is clear from (3.2) that $\nabla_x G_D$ satisfies the inequality

$$|\nabla_x G_D(x,y)| \le \frac{1}{2\pi\, |x-y|^2}\,. \tag{3.5}$$

Hence Theorem 1.2 applies to the operator T. In view of (3.1), (3.3) equation (1.9) is the same as

$$(I - \varepsilon T)\, g = f\,, \tag{3.6}$$

provided we extend f, g by zero outside Ω_R. By Theorem 1.2 there is an appropriate $\varepsilon_0 > 0$ such that $|\varepsilon| < \varepsilon_0/\|\mathbf{b}\|_{3,p}$ implies that εT as an operator on M_r^q has norm strictly less than 1. Since f, g are assumed to be in M_r^q equation (3.5) implies that g is given by

$$g = (I - \varepsilon T)^{-1} f \,.$$

Hence g is uniquely determined by f. Since (3.1) shows that u_ε is uniquely determined by g uniqueness of the solution follows.

To prove existence we define g by (3.7) and u_ε by (3.1). Thus $g \in M_r^q$ and $\|g\|_{q,r} \le C \|f\|_{q,r}$ for some constant C depending on ε_0. We shall show that the estimate (1.11) holds. In fact from (3.1) we have

$$\begin{aligned} |u_\varepsilon(x)| &\le \frac{1}{4\pi} \int_{\Omega_R} \frac{|g(y)|}{|x-y|}\, dy \\ &\le \frac{1}{2\pi} \sum_{n=n_0}^{\infty} 2^n \int_{Q_n} |g(y)|\, dy \,, \end{aligned} \tag{3.8}$$

where Q_n is the cube centered at x with side of length 2^{-n}, and n_0 is the unique integer satisfying

$$4R \le 2^{-n_0} < 8R \,. \tag{3.9}$$

Using the fact that $g \in M_r^q$ it follows that

$$\begin{aligned} |u_\varepsilon(x)| &\le \sum_{n=n_0}^{\infty} 2^n \, |Q_n|^{1-1/q} \, \|g\|_{q,r} \\ &\le C \, 2^{-2n_0 + 3n_0/q} \, \|f\|_{q,r} \,, \end{aligned} \tag{3.10}$$

since $q > 3/2$. The inequality (1.11) follows from (3.9), (3.10). We can generalize the above argument to show that u_ε is Hölder continuous. It is also clear from (3.2) that u_ε satisfies the boundary condition (1.10). Hence *a*) of Theorem 1.1 holds. To prove *b*) we use the fact that the distributional gradient of u_ε must be given by (3.3) and the distributional Laplacian of u_ε satisfies $-\Delta u_\varepsilon = g$. Thus we have

$$-\Delta u_\varepsilon(x) - \varepsilon \, \mathbf{b}(x) \cdot \nabla u_\varepsilon(x) = (I - \varepsilon T) \, g(x) \,, \tag{3.11}$$

for almost every $x \in \Omega_R$.

It follows now from the definition (3.7) of g that the right hand side of (3.11) is just the function $f(x)$. This concludes the proof of *b*).

Part c) follows by expanding (3.7) out in a Taylor series in ε. This completes the proof of Theorem 1.1.

Next we prove some results which are perturbative in nature but which will be needed to understand the nonperturbative problem. Let $g : \partial\Omega_R \to \mathbb{R}$ be a continuous function and $u(x)$, $|x| < R$ be the solution of the Dirichlet problem

$$\begin{cases} -\Delta u(x) = 0\,, & |x| < R\,, \\ u(x) = g(x)\,, & x \in \partial\Omega_R\,. \end{cases} \tag{3.12}$$

Then u is given by the Poisson formula

$$u(x) = Pg(x) = \frac{1}{4\pi R} \int\limits_{|z|=R} \frac{R^2 - |x|^2}{|x - z|^3}\, g(z)\, dz\,. \tag{3.13}$$

Now the solution of the Dirichlet problem

$$\begin{cases} (-\Delta - \mathbf{b}(x) \cdot \nabla)\, u(x) = 0\,, & |x| < R\,, \\ u(x) = g(x)\,, & x \in \partial\Omega_R\,, \end{cases} \tag{3.14}$$

is given formally by the expression

$$u = Pg + (-\Delta_D)^{-1}(I - T)^{-1}\mathbf{b} \cdot \nabla Pg\,, \tag{3.15}$$

where $(-\Delta_D)^{-1}$ is the inverse of the Dirichlet Laplacian and has kernel (3.2).

The formula (3.15) is not appropriate for drifts $\mathbf{b} \in M_p^3$. The reason is that even if g is Hölder continuous on $\partial\Omega_R$ the function $\nabla Pg(x)$ is not in general an L^∞ function for $|x| < R$. To get around this difficulty we average over the radius of the ball on which we solve the Poisson problem. Thus let us suppose we have a Hölder continuous $g \in C^\alpha(\Omega_R)$ for some α, $0 < \alpha \le 1$. We define $Kg(x)$ formally for $x \in \Omega_{R/2}$ by

$$Kg(x) = \frac{2}{R} \int_{R/2}^{R} u_\lambda(x)\, d\lambda\,, \tag{3.16}$$

where u_λ denotes the solution (3.15) of the Poisson problem on the ball of radius λ. More precisely let P_λ, T_λ, $(-\Delta_{D,\lambda})^{-1}$ be the operators in

(3.15) acting on the ball of radius λ. Denote the kernel of $(I-T_\lambda)^{-1}$ by $H_\lambda(x,y)$ and $(-\Delta_{D,\lambda})^{-1}$ by $G_{D,\lambda}(x,y)$. We can think of H_λ and $G_{D,\lambda}$ as being defined on $\mathbb{R}^3\times\mathbb{R}^3$ by simply extending the functions by zero outside $\Omega_\lambda\times\Omega_\lambda$. It is clear that for $|x|<\lambda$ one has

$$(3.17)\qquad \nabla P_\lambda g(x)=\frac{1}{4\pi\lambda}\int\limits_{|z|=\lambda}\nabla_x\Big(\frac{|z|^2-|x|^2}{|x-z|^3}\Big)\,(g(z)-g(x))\,dz$$

since the left hand side of (3.13) is constant for g a constant. It follows therefore that the formal definition (3.16) of K corresponds to

$$(3.18)\qquad \begin{aligned} Kg(x)&=\frac{2}{R}\int\limits_{R/2<|z|<R}\frac{1}{4\pi|z|}\,\frac{|z|^2-|x|^2}{|x-z|^3}\,g(z)\,dz\\ &+\frac{2}{R}\int_{\Omega_R}dw\int_{\Omega_R}dy\int\limits_{R/2<|z|<R}dz\;G_{D,|z|}(x,w)\,H_{|z|}(w,y)\\ &\cdot\mathbf{b}(y)\cdot\Big(\nabla_y\frac{|z|^2-|y|^2}{4\pi|z|\,|y-z|^3}\Big)\,(g(z)-g(y))\,. \end{aligned}$$

Proposition 3.1. *There exists a constant $\varepsilon_0>0$ depending only on $p>1$ such that if $\|\mathbf{b}\|_{3,p}<\varepsilon_0$ then $u(x)=Kg(x)$ defined by (3.18) on $\Omega_{R/2}$ exists and is Hölder continuous. Further, the distributional Laplacian Δu is in M^q_r for any $r<p$, $q<3$ and*

$$(3.19)\quad -\Delta u(x)-\mathbf{b}(x)\cdot\nabla u(x)=0\,,\qquad \textit{for almost every } x\in\Omega_{R/2}\,.$$

PROOF. Evidently the first integral on the right in (3.18) yields a Hölder continuous function. We shall show first that the second integral is uniformly bounded. Let T be the integral operator with kernel $|\mathbf{b}(x)|/2\pi|x-y|^2$. Then if ε_0 is sufficiently small the operator $(I-T)^{-1}$ exists in the sense of Theorem 1.2 with kernel $H(x,y)$ say. It is easy to see now that

$$(3.20)\qquad |H_{|z|}(w,y)|\le H(w,y)\,,\qquad w,y\in\mathbb{R}^3\,.$$

We also have

$$|G_{D,|z|}(x,w)|\le\frac{1}{4\pi\,|x-w|}\,,\qquad x,w\in\mathbb{R}^3\,.$$

Consequently the second integral is bounded in absolute value by

$$
\begin{aligned}
&\frac{2}{R}\int_{\Omega_R} dw \int_{\Omega_R} dy \int\limits_{R/2<|z|<R} dz \frac{1}{4\pi\,|x-w|} \\
&\qquad \cdot H(w,y)\,|\mathbf{b}(y)|\left|\nabla_y \frac{|z|^2-|y|^2}{4\pi\,|z|\,|z-y|^3}\right| |g(z)-g(y)| \\
&\qquad \le C\int_{\Omega_R} dw \int_{\Omega_R} dy\, \frac{1}{|x-w|} H(w,y)\,|\mathbf{b}(y)|\,,
\end{aligned} \tag{3.22}
$$

using the fact that g is Hölder continuous. Since $|\mathbf{b}| \in M_r^q$ for any $r < p, q < 3$, Theorem 1.2 implies that the last integral is uniformly bounded in x. The Hölder continuity of u follows similarly.

Let us define $h(x)$ for $x \in \Omega_R$ by

$$
\begin{aligned}
h(x) = \frac{2}{R}\int_{\Omega_R} dy \int\limits_{R/2<|z|<R} dz\; H_{|z|}(x,y) \\
\cdot\, \mathbf{b}(y)\cdot\left(\nabla_y \frac{|z|^2-|y|^2}{4\pi\,|z|\,|y-z|^3}\right)(g(z)-g(y))\,.
\end{aligned} \tag{3.23}
$$

By our previous argument it follows that $h \in M_r^q$ for any $r < p$, $q < 3$. We wish to show that the distributional Laplacian Δu on Ω_R is given by

$$
-\Delta u(x) = h(x)\,, \qquad x \in \Omega_R\,. \tag{3.24}
$$

We have

$$
h(x) = \int\limits_{R/2<|z|<R} h(x,z)\,dz \tag{3.25}
$$

and

$$
\begin{aligned}
u(x) &= \frac{2}{R}\int_{R/2}^{R} d\lambda\; P_\lambda g(x) \\
&\quad + \int_{\Omega_R} dw \int\limits_{R/2<|z|<R} dz\; G_{D,|z|}(x,w)\,h(w,z)\,.
\end{aligned} \tag{3.26}
$$

Let φ be a C^∞ function with compact support in $\Omega_{R/2}$. Then it follows from (3.26) that

$$\begin{aligned}
&\int -\Delta\varphi(x)\,u(x)\,dx \\
&= \int_{\Omega_R} dw \int_{R/2<|z|<R} dz \left(\int -\Delta\varphi(x)\,G_{D,|z|}(x,w)\,dx\right) h(w,z) \\
&= \int dw\,dz\;\varphi(w)\,h(w,z) \\
&= \int dw\;\varphi(w)\,h(w),
\end{aligned} \tag{3.27}$$

on application of Fubini's theorem. Hence we have (3.24). Similarly one has that the distributional gradient ∇u is given by

$$\begin{aligned}
\nabla u(x) &= \frac{2}{R}\int \frac{1}{4\pi\,|z|}\nabla_x\left(\frac{|z|^2-|x|^2}{|x-z|^3}\right) g(z)\,dz \\
&\quad + \frac{2}{R}\int_{\Omega_R} dw \int_{\Omega_R} dy \int dz\;\nabla_x G_{D,|z|}(x,w)\,H_{|z|}(w,y) \\
&\qquad\cdot \mathbf{b}(y)\cdot\left(\nabla_y \frac{|z|^2-|y|^2}{4\pi\,|z|\,|y-z|^3}\right)(g(z)-g(y)).
\end{aligned} \tag{3.28}$$

Let us define for $\delta > 0$, $y\in\Omega_R$, $z\in\Omega_R\backslash\Omega_{R/2}$, $f_{\delta,|z|}(y)$ by

$$f_{\delta,|z|}(y) = \mathbf{b}(y)\cdot\left(\nabla_y \frac{|z|^2-|y|^2}{4\pi\,|z|\,|y-z|^3+\delta}\right)(g(z)-g(y)). \tag{3.29}$$

Then we have for almost every $x\in\Omega_{R/2}$, the identity

$$\begin{aligned}
&\int_{\Omega_R} dw \int_{\Omega_R} dy\;\mathbf{b}(x)\cdot\nabla_x G_{D,|z|}(x,w)\,H_{|z|}(w,y)\,f_{\delta,|z|}(y) \\
&\qquad = -f_{\delta,|z|}(x) + \int_{\Omega_R} dy\;H_{|z|}(x,y)\,f_{\delta,|z|}(y).
\end{aligned} \tag{3.30}$$

This follows since the left hand side of (3.30) is the operator $T_{|z|}(I-T_{|z|})^{-1}$ applied to the function $f_{\delta,|z|}\in M_r^q$ for any $r<p$, $q<3$, and the right hand side is the operator $-I+(I-T_{|z|})^{-1}$ applied to the same function. It is clear then by dominated convergence that

$$\lim_{\delta\to 0}\frac{2}{R}\int_{\Omega_R} dy \int_{R/2<|z|<R} dz\;H_{|z|}(x,y)\,f_{\delta,|z|}(y) = h(x), \tag{3.31}$$

for almost every $x \in \Omega_{R/2}$. It follows trivially that

$$\lim_{\delta \to 0} \frac{2}{R} \int\limits_{R/2<|z|<R} dz\, f_{\delta,|z|}(x) = \mathbf{b}(x) \cdot \frac{2}{R} \int\limits_{R/2<|z|<R} \frac{1}{4\pi\,|z|}\, \nabla_x \Big(\frac{|z|^2 - |x|^2}{|x-z|^3} \Big)\, g(z)\, dz\,, \tag{3.32}$$

for any $x \in \Omega_{R/2}$. Again dominated convergence and (3.28) implies that

$$\begin{aligned} \lim_{\delta \to 0} \frac{2}{R} \int_{\Omega_R} dw \int_{\Omega_R} dy & \\ \int\limits_{R/2<|z|<R} & dz\, \mathbf{b}(x) \cdot \nabla_x G_{D,|z|}(x,w)\, H_{|z|}(w,y)\, f_{\delta,|z|}(y) \\ = \mathbf{b}(x) \cdot \nabla u(x) & \\ - \mathbf{b}(x) \cdot \frac{2}{R} & \int\limits_{R/2<|z|<R} \frac{1}{4\pi\,|z|}\, \nabla_x \Big(\frac{|z|^2 - |x|^2}{|x-z|^3} \Big)\, g(z)\, dz\,. \end{aligned} \tag{3.33}$$

It follows then from (3.30) to (3.33) that

$$\mathbf{b}(x) \cdot \nabla u(x) = h(x)\,, \qquad \text{for almost every } x \in \Omega_{R/2}\,. \tag{3.34}$$

Hence (3.24) and (3.34) implies (3.19).

Proposition 3.2. *Suppose u is a Hölder continuous function on the closure of Ω_R, the distributional Laplacian Δu is in M_r^q for some r, q, $1 < r < p$, $3/2 < q < 3$ and u satisfies the equation*

$$-\Delta u(x) - \mathbf{b}(x) \cdot \nabla u(x) = 0\,, \qquad \textit{for almost every } x \in \Omega_{R/2}\,. \tag{3.35}$$

Then

$$u(x) = K u(x)\,, \qquad \textit{for all } x \in \Omega_{R/2}\,, \tag{3.36}$$

provided $\|\mathbf{b}\|_{3,p} < \varepsilon_0$ where ε_0 depends only on r, p, q.

PROOF. Let us consider the problem

$$\begin{cases} -\Delta u(x) - \mathbf{b}(x) \cdot \nabla u(x) = f(x), & |x| < \lambda, \\ u(x) = g(x), & |x| = \lambda. \end{cases} \tag{3.37}$$

We shall assume that $f \in M_r^q$ and $\nabla P_\lambda g(x)$ is an L^∞ function for $|x| < \lambda$. Then it is easy to see that (3.37) has a unique solution u given by

$$\begin{aligned} u = P_\lambda g &+ (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1}\mathbf{b} \cdot \nabla P_\lambda g \\ &+ (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1} f. \end{aligned} \tag{3.38}$$

In fact if we put

$$v = u - P_\lambda g, \tag{3.39}$$

then

$$\begin{cases} -\Delta v(x) - \mathbf{b}(x) \cdot \nabla v(x) = \mathbf{b}(x) \cdot \nabla P_\lambda g(x) + f(x), & |x| < \lambda, \\ v(x) = 0, & |x| = \lambda. \end{cases} \tag{3.40}$$

Since we are assuming $\nabla P_\lambda g$ is L^∞, Theorem 1.1 implies (3.38).

Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a C^∞ function with support in the unit ball centered at the origin and with integral 1. Then for $\delta > 0$ the functions φ_δ,

$$\varphi_\delta(x) = \delta^{-3}\varphi(x/\delta), \qquad x \in \mathbb{R}^3, \tag{3.41}$$

are approximate Dirac δ functions. We consider functions $u_\delta = \varphi_\delta * u$ where u is the solution of (3.35) given in the statement of Proposition 3.2. Then u_δ is a C^∞ function in the ball $|x| < R - \delta$, and

$$-\Delta u_\delta(x) - \mathbf{b}(x) \cdot \nabla u_\delta(x) = f_\delta(x), \qquad |x| < R - \delta, \tag{3.42}$$

where

$$f_\delta(x) = \varphi_\delta * (\mathbf{b} \cdot \nabla u)(x) - \mathbf{b}(x) \cdot \nabla u_\delta(x). \tag{3.43}$$

Since $f \in M_r^q$ for some $r < p$, $q > 3/2$, it follows that if $\lambda < R - \delta$, then

$$\begin{aligned} u_\delta = P_\lambda u_\delta &+ (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1}\mathbf{b} \cdot \nabla P_\lambda u_\delta \\ &+ (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1} f_\delta. \end{aligned} \tag{3.44}$$

Let η satisfy $0 < \eta < R/2$ and consider $\delta < \eta$. Then if we integrate (3.44) with respect to λ over the interval $R/2 < \lambda < R-\eta$, we have for any x with $|x| < R/2$,

$$
\begin{aligned}
u_\delta(x) = {} & \frac{2}{R-2\eta} \int\limits_{R/2<|z|<R-\eta} \frac{1}{4\pi |z|} \frac{|z|^2 - |x|^2}{|x-z|^3} u_\delta(z)\, dz \\
& + \frac{2}{R-2\eta} \int_{\Omega_R} dw \int_{\Omega_R} dy \int\limits_{R/2<|z|<R-\eta} dz\, G_{D,|z|}(x,w)\, H_{|z|}(w,y) \\
& \qquad \cdot \mathbf{b}(y) \cdot \left(\nabla_y \frac{|z|^2 - |y|^2}{4\pi |z|\, |y-z|^3}\right) (u_\delta(z) - u_\delta(y)) \\
& + \frac{2}{R-2\eta} \int_{\Omega_R} dw \int_{\Omega_R} dy \\
& \qquad \int\limits_{R/2<|z|<R-\eta} dz\, G_{D,|z|}(x,w)\, H_{|z|}(w,y)\, f_\delta(y)\,.
\end{aligned} \tag{3.45}
$$

Since $\delta < \eta$ it follows that $f_\delta \in M_r^q(\Omega_{R-\eta})$.

We shall show that

$$\lim_{\delta \to 0} \|f_\delta\|_{q,r} = 0\,. \tag{3.46}$$

Let us put $h = -\Delta u$. Then by (3.35) we have that

$$\lim_{\delta \to 0} \|\varphi_\delta * (\mathbf{b} \cdot \nabla u) - h\|_{q,r} = 0\,. \tag{3.47}$$

Next we consider the limit of $\mathbf{b} \cdot \nabla u_\delta$. By Weyl's lemma we have that

$$u(x) = \int_{\Omega_R} G_{D,R}(x,y)\, h(y)\, dy + P_R u(x)\,, \tag{3.48}$$

for all $x \in \Omega_R$. Hence the distributional gradient of u is given by the formula

$$\nabla u(x) = \int_{\Omega_R} \nabla_x G_{D,R}(x,y)\, h(y)\, dy + \nabla_x P_R u(x)\,, \quad |x| < R\,. \tag{3.49}$$

Thus

$$\nabla u(x) = \int_{\Omega_R} \frac{1}{4\pi} \nabla_x \left(\frac{1}{|x-y|}\right) h(y)\, dy + \mathbf{w}(x)\,, \tag{3.50}$$

where $\mathbf{w}$ is a C^∞ function in $|x| < R$. Hence we have

$$\nabla u_\delta(x) = \int_{\mathbb{R}^3} \frac{1}{4\pi} \nabla_x\Big(\frac{1}{|x-y|}\Big)\, \varphi_\delta * h(y)\, dy + \varphi_\delta * \mathbf{w}(x)\,, \tag{3.51}$$

in $|x| < R-\delta$. Here we have extended h to all of $\mathbb{R}^3$ by setting h to zero outside Ω_R. Evidently $\varphi_\delta * \mathbf{w}(x)$ converges uniformly in $|x| < R - \eta$ as $\delta \to 0$ to $\mathbf{w}(x)$. Also $\varphi_\delta * h$ converges to h as $\delta \to 0$ in the space M_r^q. It follows then from the fact that the operator with kernel

$$\mathbf{b}(x) \cdot \frac{1}{4\pi} \nabla_x\Big(\frac{1}{|x-y|}\Big) \tag{3.52}$$

is bounded on the space M_r^q that

$$\lim_{\delta\to 0} \|\mathbf{b}\cdot\nabla u_\delta - \mathbf{b}\cdot\nabla u\|_{q,r} = 0\,.$$

The identity (3.46) follows from (3.47), (3.53).

Next we take the limit as $\delta \to 0$ in (3.45). In view of (3.46) the final integral on the right hand side vanishes in the limit. Since u is Hölder continuous the first 2 integrals converge to identical integrals with u_δ replaced by u. Now if we let $\eta \to 0$ we obtain the formula (3.36).

4. Nonperturbative uniqueness.

We shall prove the uniqueness part of Theorem 1.3 in this section. Throughout the section we shall assume that $\mathbf{b} \in M_p^3$ with $2 < p < 3$.

Assume for the moment that $\mathbf{b}$ is a C^∞ function. Then for any $\lambda > 0$, the solution of the Poisson problem

$$\begin{cases} -\Delta u(x) - \mathbf{b}(x)\cdot\nabla u(x) = 0\,, & |x| < \lambda\,, \\ u(x) = g(x)\,, & |x| = \lambda\,, \end{cases} \tag{4.1}$$

is given as an integral

$$u(x) = \int_{|y|=\lambda} \rho(x,y)\, g(y)\, dy\,. \tag{4.2}$$

The function $\rho(x, y)$ in (4.2) is defined and continuous on the set $\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| < |y|\}$. The solution (4.2) can be represented as an expectation value with respect to Brownian motion with drift $\mathbf{b}$. One has

$$u(x) = E_x[g(X_{\mathbf{b}}(\tau_\lambda)], \tag{4.3}$$

where τ_λ is the first hitting time of the drift process started at x on the sphere $|y| = \lambda$. Let $R < \lambda$. Then if we condition on the hitting distribution of the process on the sphere $|y| = R$, we have from (4.3),

$$\begin{aligned} u(0) &= E_0\left[E_{X_{\mathbf{b}}(\tau_R)}[g(X_{\mathbf{b}}(\tau_\lambda))]\right] \\ &= E_0[u(X_{\mathbf{b}}(\tau_R))] \\ &= \int\limits_{|z|=R} \rho(0, z)\, u(z)\, dz\,. \end{aligned} \tag{4.4}$$

We conclude then that

$$\rho(0, y) = \int\limits_{|z|=R} \rho(0, z)\, \rho(z, y)\, dz\,, \tag{4.5}$$

for any y with $|y| > R$.

The Cameron-Martin formula [10] enables one to write the probability measure for the drift process $X_{\mathbf{b}}(t)$ in terms of the Wiener measure for Brownian motion $X(t)$. In particular, the drift expectation (4.3) becomes a Brownian motion expectation given by

$$\begin{aligned} u(x) = E_x\Big[\exp\Big(\frac{1}{2}\int_0^\tau \mathbf{b}(X(t)) \cdot dX(t) \\ - \frac{1}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\, dt\Big)\, g(X(\tau_\lambda))\Big]\,. \end{aligned} \tag{4.6}$$

Since $g \equiv 1$ implies $u \equiv 1$ it follows from (4.6) that for any $\theta \in \mathbb{R}$ one has the identity

$$E_x\Big[\exp\Big(\frac{\theta}{2}\int_0^\tau \mathbf{b}(X(t)) \cdot dX(t) - \frac{\theta^2}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\, dt\Big)\Big] \equiv 1\,. \tag{4.7}$$

For any integer $n \in \mathbb{Z}$ let A_n be the spherical shell,

$$A_n = \{x \in \mathbb{R}^3 : \ 2^{-n} < |x| < 2^{-n+1/2}\}. \tag{4.8}$$

We define a measure μ on $\cup_{n=-\infty}^{\infty} A_n$ by

$$d\mu(x) = \frac{dx}{4\pi\,|x|^2\,(2^{-n+1/2} - 2^{-n})}, \qquad x \in A_n,\ n \in \mathbb{Z}. \tag{4.9}$$

Hence $\mu(A_n) = 1$ so A_n is a probability space with respect to μ. We define an integral operator T_n from functions on A_n to functions on A_{n-1} by

$$T_n f(x) = 4\pi\,|x|^2 \int_{A_n} \rho(y,x)\, f(y)\, \mu(y), \qquad x \in A_{n-1}. \tag{4.10}$$

Let $\rho_n : A_n \to \mathbb{R}$ be given by

$$\rho_n(x) = 4\pi\,|x|^2\,\rho(0,x), \qquad x \in A_n. \tag{4.11}$$

Then (4.5) imples that

$$T_n \rho_n = \rho_{n-1}, \qquad n \in \mathbb{Z}. \tag{4.12}$$

We write

$$T_n = P_n + Q_n, \tag{4.13}$$

where P_n is the same operator as T_n for the case $\mathbf{b} \equiv 0$. Hence the kernel $\rho(y,x)$ for P_n is just the Poisson kernel. It follows easily that

$$P_n(1) = 1, \tag{4.14}$$

where 1 denotes the function identically equal to 1.

Lemma 4.1. *Suppose $f \in L^r_\mu(A_n)$ for some r, $1 \le r \le \infty$ and satisfying*

$$\int_{A_n} f\, d\mu = 0. \tag{4.15}$$

Then there exists a universal constant $\gamma, 0 < \gamma < 1$, such that $P_n f \in L^r_\mu(A_{n-1})$ and there is the inequality

$$\|P_n f\|_r \leq \gamma \, \|f\|_r \,. \tag{4.16}$$

PROOF. First we prove (4.16) with $\gamma = 1$. We have

$$P_n f(x) = 4\pi \, |x|^2 \int_{A_n} \rho(y, x) \, f(y) \, d\mu(y) \,, \tag{4.17}$$

where ρ is the Poisson kernel. It follows then from (4.14) and Jensen's inequality that

$$|P_n f(x)|^r \leq 4\pi \, |x|^2 \int_{A_n} \rho(y, x) \, |f(y)|^r \, d\mu(y) \,. \tag{4.18}$$

Now (4.16) with $\gamma = 1$ follows on integrating (4.18) and using the fact that

$$\int_{A_{n-1}} 4\pi \, |x|^2 \, \rho(y, x) \, d\mu(x) = 1 \,. \tag{4.19}$$

To obtain $\gamma < 1$ we use (4.15). Observe that (4.15) implies

$$\int_{A_{n-1}} P_n f(x) \, d\mu(x) = 0 \,. \tag{4.20}$$

Since $P_n f(x)$ is a continuous function there exists $x_0 \in A_{n-1}$ with $P_n f(x_0) = 0$. Now let us write f as a sum of its positive and negative parts,

$$f = f_+ - f_- \,, \qquad P_n f = P_n f_+ - P_n f_- \,. \tag{4.21}$$

By the properties of the Poisson kernel there exist universal constants c_1, c_2 such that for $x \in A_{n-1}$,

$$P_n f_-(x) \geq c_1 \, P_n f_-(x_0) = \frac{c_1}{2} \, P_n |f|(x_0) \geq c_2 \, P_n |f|(x) \,, \tag{4.22}$$

where $0 < c_2 < 1$. Hence

$$P_n f(x) = P_n f^+(x) - P_n f^-(x) \leq (1 - c_2) \, P_n |f|(x) \,. \tag{4.23}$$

Since we can obtain a similar lower bound on $P_n f(x)$ we conclude that

$$\|P_n f\|_r \le (1-c_2)\,\|P_n|f|\,\|_r \le (1-c_2)\,\|f\|_r\,. \tag{4.24}$$

Thus we can take $\gamma = 1 - c_2 < 1$.

Lemma 4.2. *Let $1 < r < \infty$. Then for any $\delta > 0$ there exists $\varepsilon > 0$ depending only on r, p, δ such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies the inequality*

$$\|Q_n f\|_r \le \delta\,\|f\|_r\,, \qquad f \in L^r_\mu(A_n)\,,\ n \in \mathbb{Z}\,. \tag{4.25}$$

PROOF. Let r' be the conjugate to r, $1/r + 1/r' = 1$. We consider the adjoint Q_n^* of Q_n. We shall show that Q_n^* is a bounded operator from $L^{r'}_\mu(A_{n-1})$ to $L^{r'}_\mu(A_n)$ and satisfies

$$\|Q_n^* f\|_{r'} \le \delta\,\|f\|_{r'}\,. \tag{4.26}$$

This will imply (4.25).

We have from (4.10) that T_n^* is given by the formula

$$T_n^* f(x) = \frac{2^{n+1}}{\sqrt{2}-1}\int_{A_{n-1}} \rho(x,y)\,f(y)\,dy\,. \tag{4.27}$$

To obtain Q_n^* we need to subtract off from (4.27) the operator corresponding to $\mathbf{b} = 0$. This can easily be done from the formula (4.6). Comparing (4.2), (4.6), (4.27) we have

$$\begin{aligned} Q_n^* f(x) = {} & \frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \int_0^1 ds \\ & \cdot E_x\Big[\Big(\frac{1}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big) \\ & \cdot \exp\Big(\frac{s}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) \\ & \qquad - \frac{s}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\, f(X(\tau_\lambda))\Big]\,. \end{aligned} \tag{4.28}$$

Equation (4.28) can be written as

$$Q_n^* f(x) = \frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \int_0^1 ds$$

$$\begin{aligned}
(4.29) \qquad & \cdot E_x\Big[\Big(\frac{1}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big) \\
& \cdot \exp\Big(\frac{s}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - \frac{rs}{2}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big) \\
& \cdot \exp\Big(s\,(\frac{r}{2}-\frac{1}{4})\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\, f(X(\tau_\lambda))\Big]\,.
\end{aligned}$$

Now we apply the generalized Hölder inequality to (4.29). Let m be an integer satisfying $m > r$. Then, observing that

$$(4.30) \qquad \frac{1}{2m} + \frac{1}{2r} + \Big(\frac{1}{2r} - \frac{1}{2m}\Big) + \frac{1}{r'} = 1\,,$$

we have

$$\begin{aligned}
&|Q_n^* f(x)| \\
&\le \int_0^1 ds \Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\
&\quad \cdot E_x\Big[\Big(\frac{1}{2}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big]\Big)^{1/2m} \\
&\quad \cdot \Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\
&\quad \cdot E_x\Big[\exp\Big(rs\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) - r^2 s\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{1/2r} \\
(4.31)\quad & \\
&\quad \cdot \Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \\
&\quad \cdot E_x\Big[\exp\Big(\frac{mrs(2r-1)}{2(m-r)}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{(1/2r-1/2m)} \\
&\quad \cdot \Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda\, E_x\big[|f(X(\tau_\lambda))|^{r'}\big]\Big)^{1/r'}\,.
\end{aligned}$$

If we use (4.7) with $\theta = 2\,r$, we can conclude from (4.31) that for $x \in A_n$,

$$\begin{aligned}
&|Q_n^* f(x)|^{r'} \\
&\le C\,\|f\|_{r'}^{r'}\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda
\end{aligned}$$

$$(4,32)\quad \cdot E_x\Big[\Big(\frac{1}{2}\int_0^{\tau} \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^{\tau} |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big]\Big)^{r'/2m}$$
$$\cdot\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda$$
$$\cdot E_x\Big[\exp\Big(\frac{m\,r(2r-1)}{2(m-r)}\int_0^{\tau} |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{r'(1/2r-1/2m)}$$

where C is a universal constant. Observe now that

$$(4.33)\quad \begin{aligned} E_x\Big[\Big(\frac{1}{2}\int_0^{\tau} \mathbf{b}(X(t))\cdot dX(t) - \frac{1}{4}\int_0^{\tau} |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big] \\ \le E_x\Big[\Big(\int_0^{\tau} \mathbf{b}(X(t))\cdot dX(t)\Big)^{2m}\Big] \\ + \frac{1}{4^m} E_x\Big[\Big(\int_0^{\tau} |\mathbf{b}|^2(X(t))dt\Big)^{2m}\Big]. \end{aligned}$$

It follows from (4.7) that there exists a constant C_m depending only on m such that

$$(4.34)\quad \begin{aligned} E_x\Big[\Big(\int_0^{\tau} \mathbf{b}(X(t))\cdot dX(t)\Big)^{2m}\Big] \\ \le C_m\, E_x\Big[\Big(\int_0^{\tau} |\mathbf{b}|^2(X(t))\,dt\Big)^{m}\Big]. \end{aligned}$$

We can assume without loss of generality that $2m > r'$. Hence if we integrate (4.32) with respect to x over A_n and apply Hölder with exponents $2m/r'$ and $2m/(2m-r')$ we obtain the inequality

$$\|Q_n^* f\|_{r'}^{r'} \le C'\, \|f\|_{r'}^{r'}$$
$$\cdot\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda$$
$$\cdot \int_{A_n} d\mu(x)\, E_x\Big[\Big(\int_0^{\tau} |\mathbf{b}|^2(X(t))\,dt\Big)^{m}$$
$$+\Big(\int_0^{\tau} |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big]\Big)^{r'/2m}$$
$$\cdot\Big(\int_{A_n} d\mu(x)\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda$$

(4.35)

$$\cdot E_x\Big[\exp\Big(\frac{m\,r(2r-1)}{2(m-r)} \cdot \int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{\frac{2m}{2m-r'}\frac{r'(m-r)}{2rm}}\Big)^{\frac{2m-r'}{2m}},$$

for some constant C' depending only on m. It follows from Jensen's inequality that

$$\Big(\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \cdot E_x\Big[\exp\Big(\frac{m\,r(2r-1)}{2(m-r)}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\Big)^{\frac{2m}{2m-r'}\frac{r'(m-r)}{2rm}} \leq \frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda\, E_x\Big[\exp\Big(\alpha\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big] \tag{4.36}$$

where

$$\alpha = \frac{m\,r(2r-1)}{2(m-r)}\max\Big\{1, \frac{2m}{2m-r'}\frac{r'(m-r)}{2rm}\Big\}. \tag{4.37}$$

It follows from Theorem 1.1.*b*) of [1] that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ and ε is sufficiently small depending only on α, then

$$\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \cdot \int_{A_n} d\mu(x)\, E_x\Big[\exp\Big(\alpha\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big] \leq 2. \tag{4.38}$$

On the other hand by the same argument one has

$$\frac{2^{n+1}}{\sqrt{2}-1}\int_{2^{-n-1}}^{2^{-n-1/2}} d\lambda \cdot \int_{A_n} d\mu(x)\, E_x\Big[\Big(\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)^m + \Big(\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)^{2m}\Big] \leq C\,\varepsilon^{2m}, \tag{4.39}$$

where C depends only on m. We conclude therefore from (4.35), (4.38), (4.39) that

$$\|Q_n^* f\|_{r'} \le \delta \, \|f\|_{r'} \,, \tag{4.40}$$

provided ε is sufficiently small. The inequality (4.25) follows directly from this.

Lemma 4.3. *Let ρ_n be the density* (4.11), *and* $1 < r < \infty$. *Then for any $\delta > 0$ there exists $\varepsilon > 0$ depending only on r, p, δ such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies the inequality*

$$\|\rho_n - 1\|_r < \delta \,, \qquad \textit{for all } \; n \in \mathbb{Z} \,. \tag{4.41}$$

PROOF. From (4.12) we have

$$\begin{aligned} \rho_{n-1} - 1 &= T_n \rho_n - 1 \\ &= P_n \rho_n + Q_n \rho_n - 1 \\ &= P_n(\rho_n - 1) + Q_n(\rho_n - 1) + Q_n 1 \,. \end{aligned} \tag{4.42}$$

Hence by Lemmas 4.1 and 4.2 we have

$$\|\rho_{n-1} - 1\|_r \le \gamma \, \|\rho_n - 1\|_r + \delta' \, \|\rho_n - 1\|_r + \delta' \,, \tag{4.43}$$

where δ' can be chosen arbitrarily small depending on ε. Since $\gamma < 1$ we can therefore have $\gamma + \delta' < 1$. It follows then by induction from (4.43) that for any $M \ge n$ one has the inequality

$$\|\rho_n - 1\|_r \le (\gamma + \delta')^{M-n} \, \|\rho_M - 1\|_r + \frac{\delta'}{1 - \gamma - \delta'} \,. \tag{4.44}$$

Letting $M \to \infty$ and choosing δ' such that $\delta'/(1 - \gamma - \delta') < \delta$ yields the inequality (4.41).

Now let us return to the operator K on functions $g \in C^\alpha(\Omega_R)$ defined by (3.18).

Lemma 4.4. *Suppose $x_0 \in \Omega_R \backslash \Omega_{R/2}$ and $g(x) \ge g_0 > 0$ for $|x - x_0| < r_0$, $g(x) \ge 0, x \in \Omega_R \backslash \Omega_{R/2}$. Then there exists a positive constant $c(r_0/R)$ depending only on r_0/R such that*

$$K \, g(0) \ge c(r_0/R) \, g_0 \,, \tag{4.45}$$

provided $\|\mathbf{b}\|_{3,p}$ *is sufficiently small.*

PROOF. With $\rho(x,y)$ defined as in (4.2) we have the identity

$$
\begin{aligned}
K\,g(0) &= \frac{2}{R}\int_{R/2}^{R} d\lambda \int_{|z|=\lambda} \rho(0,z)\,g(z)\,dz \\
&= \frac{2}{R}\int_{\Omega_R\backslash\Omega_{R/2}} \rho(0,z)\,g(z)\,dz\,.
\end{aligned} \tag{4.46}
$$

Since $g(z) \geq 0$ for $z \in \Omega_R\backslash\Omega_{R/2}$ it follows that

$$
K\,g(0) \geq \frac{2}{R}\,g_0 \int_{\{z\in\Omega_R\backslash\Omega_{R/2}:|z-x_0|<r_0\}} \rho(0,z)\,dz\,. \tag{4.47}
$$

For $x, y \in \mathbb{R}^3$ with $x \neq y$ we define a function $\xi(x,y)$ as follows: For $\lambda > 0$ let O_λ be an arbitrary open subset of the sphere $\{z : |z| = \lambda\}$. Then

$$
\int_{y-x\in O_\lambda} \xi(x,y)\,dy = \begin{array}{l}\text{probability that the drift}\\ \text{process started at } x\\ \text{exits the sphere } |y-x|=\lambda\\ \text{through the set } x+O_\lambda\,.\end{array} \tag{4.48}
$$

It is clear that ξ and the previously defined function ρ are related by the equation

$$
\rho(0,y) = \xi(0,y)\,, \qquad y \in \mathbb{R}^3\backslash\{0\}\,. \tag{4.49}
$$

Let N be the integer $N = [4|x_0|/r_0]$, where $[\cdot]$ denotes integer part. For $j = 1, \ldots, N-1$, let $z_j \in \mathbb{R}^3$ be given by

$$
z_j = j\,\frac{r_0}{4}\,\frac{x_0}{|x_0|}\,. \tag{4.50}
$$

It is clear then from the definition of N that

$$
\frac{r_0}{4} \leq |z_{N-1} - x_0| \leq \frac{r_0}{2}\,, \qquad |z_{N-1}| \leq |x_0| - \frac{r_0}{4}\,. \tag{4.51}
$$

Next we define z_N by

$$z_N = x_0 + \frac{r_0}{4}\frac{x_0}{|x_0|}\,. \tag{4.52}$$

For $j = 1,\dots,N$, $\delta > 0$, let $B_{j,\delta}$ be the ball of radius $\delta\, r_0$ centered at z_j. We first choose $\delta < 1/8$. This ensures that the spheres $B_{j,\delta}$, $j = 1,\dots,N$ are disjoint. Let $K(\delta)$ be given by

$$K(\delta) = \sup\{|z-w|/r_0 :\ z \in B_{N-1,\delta}\,,\ w \in B_{N,\delta}\}\,. \tag{4.53}$$

It is clear that

$$K(\delta) \le \frac{3}{4} + 2\,\delta\,. \tag{4.54}$$

Now for arbitrary $z \in B_{N-1,\delta}$ let y satisfy

$$|y-z| = K(\delta)\, r_0\,, \qquad |x_0| - \frac{r_0}{8} \le |y| \le |x_0| + \frac{r_0}{8}\,. \tag{4.55}$$

Then we need to choose δ sufficiently small such that if y satisfies (4.55) then $|y-x_0| < r_0$. This is clearly possible provided δ is chosen to depend on the ratio $r_0/R < 1$. We then have the inequality

$$\begin{aligned}\int_{\{z\in\Omega_R\setminus\Omega_{R/2}:|z-x_0|<r_0\}} \rho(0,z)\,dz \ge \Big(\frac{1}{4r_0\delta}\Big)^N \frac{r_0}{8}\,\Big(\prod_{j=1}^N \int_{B_{j,\delta}} dy_j\Big)& \\ \cdot\,\xi(0,y_1)\,\xi(y_1,y_2)\cdots\xi(y_{N-1},y_N)&\,.\end{aligned} \tag{4.56}$$

The inequality (4.56) can be explained as follows: First constrain the integration on the left hand side to the surface of the sphere $|z| = |x_0| + \varepsilon$, where $-r_0/8 < \varepsilon < r_0/8$. Second, constrain the variables $y_j, j = 1,\dots,N$ to lie on surfaces $|y_1| = \varepsilon_1$, $|y_j - y_{j-1}| = \varepsilon_j$, $j = 2,\dots,N$, where

$$\begin{aligned} r_0\,(1/4-\delta) < \varepsilon_1 < r_0\,(1/4+\delta)\,,& \\ r_0\,(1/4-2\,\delta) < \varepsilon_j < r_0\,(1/4+2\,\delta)\,,\quad j = 2,\dots,N-1,& \\ (K(\delta)-4\,\delta)\, r_0 < \varepsilon_N < K(\delta)\, r_0\,.&\end{aligned} \tag{4.57}$$

Then we have the inequality

$$\int \rho(0,z)\,dz \ge \Big(\prod_{j=1}^N \int dy_j\Big)\xi(0,y_1)\,\xi(y_1,y_2)\cdots\xi(y_{N-1},y_N)\,. \tag{4.58}$$

This is true because the left hand side is the probability of the drift process starting at 0 exiting the sphere $|z| = |x_0| + \varepsilon$ where it intersects the ball $|z - x_0| < r_0$. The right hand side gives the probability of a set of paths which accomplish this. The second condition on δ following (4.55) guarantees that any path included on the right hand side exits through the intersection with the ball $|z - x_0| < r_0$. The inequality (4.56) is obtained from (4.58) by doing the radial integrations and observing the constraints (4.57) on the ε_j, $j = 1, \dots, N$.

The inequality (4.45) will follow if we can show that

$$\frac{1}{4r_0\delta} \int_{B_{j,\delta}} \xi(y_{j-1}, y_j)\, dy_j \geq \gamma > 0, \tag{4.59}$$

where γ depends only on $\|\mathbf{b}\|_{3,p}$. However, this is an immediate consequence of Lemma 4.3.

The previous lemmas enable us to prove a maximum principle for the solutions of the elliptic equation (1.2). This will then imply uniqueness of the solution as given in Theorem 1.3.

Theorem 4.5. *Suppose* $\mathbf{b}$ *satisfies the conditions of Theorem* 1.3, u *is a Hölder continuous function on* Ω_R *with distributional Laplacian* Δu *in* M_r^q *for some* r, q, $1 < r < p$, $3/2 < q < 3$ *and* u *satisfies the equation*

$$-\Delta u(x) - \mathbf{b}(x) \cdot \nabla u(x) = 0, \qquad \text{for almost every } x \in \Omega_R. \tag{4.60}$$

Then if u *has a maximum interior to* Ω_R *the function* u *is a constant.*

PROOF. Suppose u has a maximum at an interior point $x_0 \in \Omega_R$. By the conditions on $\mathbf{b}$ there exists an open ball $B(x_0, \delta)$ centered at x_0 with radius δ such that the closure is contained in Ω_R and $\|\mathbf{b}\|_{3,p} < \varepsilon$ when $\mathbf{b}$ is restricted to $B(x_0, \delta)$. We can therefore apply Proposition 3.2 to u on $B(x_0, \delta)$ to conclude that

$$u(x_0) = K_\delta u(x_0), \tag{4.61}$$

where K_δ is the operator (3.18) for the ball $B(x_0, \delta)$. It follows from (4.61) that

$$0 = K_\delta g(x_0), \tag{4.62}$$

where $g(x) = u(x_0) - u(x)$. Since $g(x) \geq 0$, $x \in B(x_0, \delta)$ it follows from Lemma 4.4 that $g(x) = 0$ for all $x, \delta/2 < |x - x_0| < \delta$. One can further deduce that $g(x) = 0$ for all $x \in B(x_0, \delta)$. The result then is a consequence of the connectedness of Ω_R.

5. Nonperturbative existence.

We shall complete the proof of Theorem 1.3 in this section. The basic input is that the boundary value problem (1.2)-(1.3) has a C^∞ solution $u(x)$ provided $\mathbf{b}$ and f are C^∞ functions. This is a well known result [6]. We then prove existence of the solution to (1.2)-(1.3) for nonsmooth $\mathbf{b}$ and f by smoothing $\mathbf{b}$ and f with approximate Dirac δ functions and taking limits.

Let Q_0 be the smallest cube concentric with Ω_R and containing it which has side of length $2^{-n_0}, n_0$ an integer. Suppose now $\mathbf{b} \in M_p^3, \varepsilon > 0$ and $N_\varepsilon(\mathbf{b}) < +\infty$. Then there exists a unique minimal integer $m_\varepsilon(\mathbf{b}) \geq n_0$ such that every dyadic subcube $Q \subset Q_0$ with side of length 2^{-m_ε} has the property

$$\int_{Q'} |\mathbf{b}|^p \, dx < \varepsilon^p \, |Q'|^{1-p/3}, \tag{5.1}$$

for all dyadic subcubes $Q' \subset Q$.

Our main theorem in this section is the following

Theorem 5.1. *Suppose* $\mathbf{b}$ *and* f *are* C^∞ *functions and* u *is the solution of the boundary value problem* (1.2)-(1.3). *Then there exists* $\varepsilon > 0$ *depending only on* p, q, r *such that*

$$\|u\|_\infty \leq C_1 \, R^{2-3/q} \, \|f\|_{q,r} \, \exp(C_2 \, m_\varepsilon(\mathbf{b})/n_0), \tag{5.2}$$

for some constants C_1, C_2 *depending only on* p, q, r.

Next we consider a possibly singular $\mathbf{b} \in M_p^3$ with $N_\varepsilon(\mathbf{b}) < +\infty$ for some $\varepsilon > 0$. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a nonnegative C^∞ function such that

$$\int_{\mathbb{R}^3} \varphi(x) dx = 1, \qquad \operatorname{supp} \varphi \subset \{x : \ |x| \leq 1\}. \tag{5.3}$$

For $\delta > 0$ let $\varphi_\delta(x) = \delta^{-3}\varphi(x/\delta)$, $x \in \mathbb{R}^3$, and put $\mathbf{b}_\delta = \varphi_\delta * \mathbf{b}$. Since $\mathbf{b} \in M_p^3$ it is clear that $\mathbf{b}_\delta$ is a C^∞ function. We choose δ to satisfy the inequality

$$\delta < \delta_0 = 2^{-m_\varepsilon - 1}\,. \tag{5.4}$$

Let $y \varepsilon \mathbb{R}^3$ be an arbitrary vector satisfying $|y| < \delta$ and $Q \subset Q_0$ be a dyadic subcube with side of length 2^{-m_ε}. Then for all dyadic subcubes Q' of Q we have

$$\int_{Q'} |\mathbf{b}(x+y)|^p\, dx \le 8\,\varepsilon^p\, |Q'|^{1-p/3} < (8\,\varepsilon)^p\, |Q'|^{1-p/3}\,, \tag{5.5}$$

since the translate of Q' by y intersects at most 8 dyadic cubes with side of length $2\delta_0$. We see from Jensen's inequality that

$$\int_{Q'} |\mathbf{b}_\delta|^p\, dx < (8\,\varepsilon)^p\, |Q'|^{1-p/3}\,, \tag{5.6}$$

for all dyadic subcubes Q' of the cube Q. It follows in particular that

$$m_{8\varepsilon}(\mathbf{b}_\delta) \le m_\varepsilon(\mathbf{b})\,, \tag{5.7}$$

provided δ satisfies (5.4). We shall need the following

Lemma 5.2. *Let $R > 0$, $g : \Omega_R \backslash \Omega_{R/2} \to \mathbb{R}$ an L^∞ function and $\mathbf{b}$ be a C^∞ drift. Let τ_λ be the first hitting time for the drift process started at $x, |x| < \lambda$, on the sphere $\{y : |y| = \lambda\}$. Define $v(x)$ for $|x| < R/4$ by*

$$v(x) = \frac{2}{R}\int_{R/2}^{R} E_x[g(X_{\mathbf{b}}(\tau_\lambda))]\, d\lambda\,. \tag{5.8}$$

Then there exists $\varepsilon > 0$ depending only on $p > 2$ such that, if $\|\mathbf{b}\|_{3,p} < \varepsilon$, the function $v(x)$ is Hölder continuous for $|x| \le R/4$. In particular $v(x)$ satisfies the inequalities

$$\|v\|_\infty \le \|g\|_\infty\,, \qquad |v(x) - v(y)| \le C\,\|g\|_\infty \left(\frac{|x-y|}{R}\right)^\alpha, \tag{5.9}$$

where C and $\alpha > 0$ depend only on p, ε.

PROOF. The first estimate in (5.9) is immediate from the definition (5.8). To obtain the second estimate we use the method employed in Section 4 to prove Lemmas 4.1 and 4.2. For x, y satisfying $|x|, |y| < R/4$ we choose $(x+y)/2$ as our origin and define regions A_n as in (4.8). Let n_1 be the smallest integer such that $|x-y| \geq 2^{-n_1-1}$ and n_0 be the smallest integer such that $R/8 \geq 2^{-n_0-1}$. If $n_1 \leq n_0$ then the second inequality of (5.9) follows from the first inequality. Hence we shall assume $n_1 \geq n_0 + 1$. For $n \leq n_1$ let $\rho_{x,n}$ be the density corresponding to (4.11) on the set A_n for the drift process starting at x. This can be constructed exactly as in Lemmas 4.1 and 4.2 by using spherical shells centered at x up to radius 2^{-n_1-2} and then making the next transformation to the spherical shell A_{n_1} centered at $(x+y)/2$. We conclude that for ε sufficiently small there is an inequality $\|\rho_{x,n_1}\|_r \leq C_r$ where the constant C_r depends only on $r > 1$. Since there is a similar inequality for ρ_{y,n_1} we conclude that

$$\|\rho_{x,n_1} - \rho_{y,n_1}\|_r \leq C_r \, , \tag{5.10}$$

for some suitable universal constant depending only on $r > 1$. Now by Lemmas 4.1 and 4.2 one has the inequality

$$\|\rho_{x,n} - \rho_{y,n}\|_r \leq \gamma^{n_1-n} C_r \, , \qquad n \geq n_1 \, , \tag{5.11}$$

where γ is a constant depending only on ε, r, $0 < \gamma < 1$. In particular (5.11) holds for $n = n_0$. Next we can use the method of Lemmas 4.1 and 4.2 to estimate the densities of the drift process starting at x and y on $\Omega_R \backslash \Omega_{R/2}$. If we denote these by ρ_x, ρ_y it easily follows from (5.11) that

$$\|\rho_x - \rho_y\|_r \leq \gamma^{n_1-n_0} C_r' \, . \tag{5.12}$$

Since

$$v(x) = \int_{\Omega_R \backslash \Omega_{R/2}} \rho_x(z)\, g(z)\, d\mu(z) \, , \tag{5.13}$$

where

$$d\mu(z) = \frac{2}{R} \frac{dz}{4\pi |z|^2} \, , \tag{5.14}$$

it easily follows from (5.12), (5.13) that the second inequality of (5.9) holds with α defined by

$$2^{-\alpha} = \gamma, \qquad \text{where } 1/2 < \gamma < 1. \tag{5.15}$$

This completes the proof.

PROOF OF THEOREM 1.3: EXISTENCE. We shall use Theorem 5.1 and Lemma 5.2 to construct a solution of the boundary value problem. Let ε_0 be chosen so that Lemma 5.2 and the perturbation Theorem 1.1 holds for $\|\mathbf{b}\|_{3,p} < \varepsilon_0$, while Theorem 5.1 holds for $\varepsilon = \varepsilon_0$. We restrict ε so that $\varepsilon < \varepsilon_0/64$. Now for δ satisfying (5.4) let u_δ be the solution of the boundary value problem (1.2)-(1.3) with drift $\mathbf{b}_\delta$ and observable $f_\delta = \varphi_\delta * f$, $f \in M_r^q$, $q > 3/2$. In view of (5.7) and Theorem 5.1 we have the inequality

$$\|u_\delta\|_\infty \leq C_1\, R^{2-3/q}\, \|f\|_{q,r} \exp(C_2\, m_\varepsilon(\mathbf{b})/n_0), \tag{5.16}$$

since $\|f_\delta\|_{q,r} \leq \|f\|_{q,r}$.

Now let x_0 be an arbitrary point in Ω_R and consider $\mathbf{b}_\delta$ restricted to the ball centered at x_0 with radius δ_0, $B(x_0, \delta_0)$. It follows from (5.6) that $\|\mathbf{b}_\delta\|_{3,p} < \varepsilon_0$. We consider x in the ball $B(x_0, \delta_0/4)$ and for $\delta_0/2 < \lambda < \delta_0$ let τ_λ be the hitting time for the drift process started at x on the boundary of the ball $B(x_0, \lambda)$. Then if τ is the hitting time on the sphere $\partial\Omega_R$ we have

$$\begin{aligned} u_\delta(x) &= E_x\Big[\int_0^\tau f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] \\ &= E_x\Big[\int_0^{\tau_\lambda} f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] + E_x\Big[\int_{\tau_\lambda}^{\tau} f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] \\ &= E_x\Big[\int_0^{\tau_\lambda} f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] + E_x\Big[u_\delta(X_{\mathbf{b}_\delta}(\tau_\lambda))\Big]. \end{aligned} \tag{5.17}$$

Integrating with respect to λ we have then for $|x - x_0| < \delta_0/4$ the representation

$$\begin{aligned} u_\delta(x) = \frac{2}{\delta_0}\int_{\delta_0/2}^{\delta_0} d\lambda\; E_x\Big[\int_0^{\tau_\lambda} f_\delta(X_{\mathbf{b}_\delta}(t))\,dt\Big] \\ + \frac{2}{\delta_0}\int_{\delta_0/2}^{\delta_0} d\lambda\; E_x\Big[u_\delta(X_{\mathbf{b}_\delta}(\tau_\lambda))\Big] \\ = w_\delta(x) + v_\delta(x). \end{aligned} \tag{5.18}$$

In view of Lemma 5.2 and (5.16) we have v_δ is Hölder continuous and

$$
\begin{aligned}
&|v_\delta(x) - v_\delta(y)| \\
&\qquad \leq C\left(\frac{|x-y|}{\delta_0}\right)^{\alpha} R^{2-3/q}\|f\|_{q,r}\exp(C_2\, m_\varepsilon(\mathbf{b})/n_0)\,.
\end{aligned}
\tag{5.19}
$$

It follows now from Theorem 1.1 that w_δ is Hölder continuous and

$$
|w_\delta(x) - w_\delta(y)| \leq C\left(\frac{|x-y|}{\delta_0}\right)^{\beta} \delta_0^{2-3/q}\,\|f\|_{q,r}\,, \tag{5.20}
$$

where the exponent β depends on $q > 3/2$. Hence the functions $u_\delta, \delta < \delta_0$, form an equicontinuous family, which by (5.16) is uniformly bounded. The Ascoli-Arzelà theorem implies then that there exists a sequence δ_n, $n \geq 1$, with $\lim_{n\to\infty}\delta_n = 0$ such that the u_{δ_n} converge uniformly to a limiting function u. The function u must necessarily be Hölder continuous in view of the uniform Hölder continuity of the functions u_δ.

We shall show that u is the solution to the boundary value problem (1.2)-(1.3) in the sense of Theorem 1.3. Evidently *a*) of Theorem 1.3 follows immediately from our preceding work. To prove *b*) we consider equation (5.18) again. Letting K_δ be the operator K of (3.18) adapted to the ball $B(x_0,\delta_0)$ with drift $\mathbf{b}_\delta$ and $T_{\delta,\lambda}$ be the integral operator on M_r^q with kernel (3.4) corresponding to the drift $\mathbf{b}_\delta$ and ball $B(x_0,\lambda)$, we can write (5.18) as

$$
u_\delta(x) = \frac{2}{\delta_0}\int_{\delta_0/2}^{\delta_0} d\lambda\,(-\Delta_{D,\lambda})^{-1}(I - T_{\delta,\lambda})^{-1} f_\delta(x) + K_\delta(u_\delta)\,. \tag{5.21}
$$

Now we take $\delta = \delta_n, n \geq 1$, in (5.21) and let $\delta \to 0$. Since $\mathbf{b}_\delta$ converges to $\mathbf{b}$ in M_p^3 and f_δ to f in M_r^q and u_δ is uniformly Hölder continuous as $\delta \to 0$ it follows that

$$
u(x) = \frac{2}{\delta_0}\int_{\delta_0/2}^{\delta_0} d\lambda\,(-\Delta_{D,\lambda})^{-1}(I - T_{0,\lambda})^{-1} f(x) + K_0(u)\,, \tag{5.22}
$$

where K_0 and $T_{0,\lambda}$ are the operators which correspond to the drift $\mathbf{b}$. It follows easily from (5.22) that the distributional Laplacian $\Delta u(x)$ for $|x - x_0| < \delta_0/4$ is given by

$$
-\Delta u(x) = \frac{2}{\delta_0}\int_{\delta_0/2}^{\delta_0} d\lambda\,(I - T_{0,\lambda})^{-1} f(x) - \Delta K_0(u)\,. \tag{5.23}
$$

Proposition 3.1 and Theorem 1.2 then imply that $\Delta u \in M_r^q$. Finally Proposition 3.1 and the perturbative existence argument at the beginning of Section 3 imply from (5.23) that

$$-\Delta u(x) - \mathbf{b}(x) \cdot \nabla u(x) = f(x), \qquad |x - x_0| < \delta_0/4, \tag{5.24}$$

where ∇u is the distributional gradient of u. The proof of Theorem 1.3 is complete.

We turn to the proof of Theorem 5.1. We shall pursue the same method we used in Section 4 to prove uniqueness. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^∞ observable and $\mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^3$ a C^∞ drift. For any $\eta > 0$ we define a function $\rho_{f,\eta}(x, y)$ on the set $\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| < |y|\}$ as follows: Let $\lambda > 0$ be arbitrary and τ_λ be the first hitting time for the drift process started at a point $x, |x| < \lambda$, on the sphere $|y| = \lambda$. Then for any continuous function g on the sphere $|y| = \lambda$, one has

$$\begin{aligned}\int_{|y|=\lambda} & \rho_{f,\eta}(x,y)\, g(y)\, dy \\ &= E_x\Big[g(X_{\mathbf{b}}(\tau_\lambda))\, \chi\Big(\eta\, \lambda^{2-3/q} \|f\|_{q,r} - \int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\, dt\Big)\Big],\end{aligned} \tag{5.25}$$

where χ is the Heaviside function, $\chi(s) = 1$, for $s \geq 0$, $\chi(s) = 0$, for $s < 0$. It is clear from our definition that $\rho_{f,\eta}$ is an increasing function of η and

$$\lim_{\eta \to \infty} \rho_{f,\eta}(x, y) = \rho(x, y), \tag{5.26}$$

where $\rho(x, y)$ is defined by (4.2). Let A_n be the region (4.8). For any integer $n \in \mathbb{Z}$ we can define $\rho_{f,\eta,n} : A_n \to \mathbb{R}$ in analogy to $\rho_n : A_n \to \mathbb{R}$ given by (4.11). Thus we define $\rho_{f,\eta,n}$ by

$$\rho_{f,\eta,n}(x) = 4\pi\, |x|^2\, \rho_{f,\eta}(0, x), \qquad x \in A_n. \tag{5.27}$$

Lemma 5.3. *There exists $\varepsilon > 0, C > 0$ depending only on r, p, q such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies the inequality*

$$\|\rho_n - \rho_{f,\eta,n}\|_1 < \frac{C}{\eta}. \tag{5.28}$$

PROOF. Since $\|\rho_n\|_1 = 1$, $\|\rho_{f,\eta,n}\|_1 \le 1$, the inequality (5.28) holds for small η. Therefore we may assume that η is large. Now we have

$$\|\rho_n - \rho_{f,\eta,n}\|_1 = \frac{2^n}{\sqrt{2}-1} \cdot \int_{2^{-n}}^{2^{-n+1/2}} d\lambda\, E_0\Big[\chi\Big(\int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\,dt - \eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big)\Big]. \tag{5.29}$$

From Theorem 1.1 and (1.11) we have that if ε is sufficiently small then

$$E_0\Big[\int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\,dt\Big] \le C_1\,\lambda^{2-3/q}\,\|f\|_{q,r} \tag{5.30}$$

for some constant C_1 depending only on r, p, q. Hence (5.29), (5.30) and Chebyshev's inequality implies that

$$\|\rho_n - \rho_{f,\eta,n}\|_1 \le \frac{2^n}{\sqrt{2}-1}\int_{2^{-n}}^{2^{-n+1/2}} d\lambda\,\frac{C_1}{\eta} = \frac{C_1}{\eta}\,. \tag{5.31}$$

The proof is complete.

To complete the proof of Theorem 5.1 we follow the argument of Lemma 4.4. Thus for $x, y \in \mathbb{R}^3$ we define a function $\xi_{f,\eta}(x,y)$ in analogy to the function $\xi(x,y)$ of Lemma 4.4. For $\lambda > 0$ and O_λ an arbitrary open subset of the sphere $\{z : |z| = \lambda\}$ we define

$$\int_{y-x\in O_\lambda} \xi_{f,\eta}(x,y)\,dy = \begin{array}{l}\text{probability that the drift}\\ \text{process started at } x\\ \text{exits the sphere } |y-x| = \lambda\\ \text{through the set } x + 0_\lambda, \text{ and}\end{array} \tag{5.32}$$
$$\int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\,dt < \eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\,.$$

Now let x_0 be an arbitrary point in Ω_R and τ be the time for the drift process starting at x_0 to hit $\partial 0\Omega_R$. We define points z_j, $j = 0, 1, 2, \ldots$ by

$$z_j = x_0 + j\,2^{-m_\varepsilon(\mathbf{b})}\mathbf{k}\,, \tag{5.33}$$

where $\mathbf{k} = (0,0,1)$ is the unit vector in $\mathbb{R}^3$ in the positive z direction, $m_\varepsilon(\mathbf{b})$ is as given in the statement of Theorem 5.1. Let $B_{j,\delta}$ be the ball of radius $\delta\, 2^{-m_\varepsilon(\mathbf{b})}$ centered at $z_j, j = 1,2,\dots$ We choose $\delta < 1/2$ so that the balls $B_{j,\delta}$ do not intersect. Then, in analogy to the inequality (4.56) we have

$$P_{x_0}\Big(\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt < N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big) \geq \Big(\frac{1}{2\lambda\delta}\Big)^N \Big(\prod_{j=1}^N \int_{B_{j,\delta}} dy_j\Big) \cdot \xi_{f,\eta}(x_0,y_1)\,\xi_{f,\eta}(y_1,y_2)\cdots\xi_{f,\eta}(y_{N-1},y_N)\,, \tag{5.34}$$

where

$$N = \frac{m_\varepsilon(\mathbf{b})}{n_0} + 1\,, \qquad \lambda = 2^{-m_\varepsilon(\mathbf{b})+1}\,. \tag{5.35}$$

Lemma 4.3 and Lemma 5.3 imply that

$$\frac{1}{2\lambda\delta}\int_{B_{j,\delta}} \xi_{f,\eta}(y_{j-1},y_j)\,dy_j \geq \gamma > 0\,. \tag{5.36}$$

where γ depends only on p,q,r,ε, provided η is sufficiently large. We conclude then from (5.34) that

$$P_{x_0}\Big(\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt < N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big) \geq \gamma^N\,, \tag{5.37}$$

whence it follows that

$$\sup_{x\in\Omega_R} P_x\Big(\int_0^{\tau_\lambda} |f|(X_{\mathbf{b}}(t))\,dt > N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big) < 1-\gamma^N\,, \tag{5.38}$$

where $0 < \gamma < 1$. The estimate (5.2) follows from (5.38) and the Markov property. In fact

$$\begin{aligned} |u(x)| &\leq E_x\Big[\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt\Big] \\ &\leq N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\sum_{k=0}^\infty (1-\gamma^N)^k \\ &= N\,\eta\,\gamma^{-N}\,\lambda^{2-3/q}\,\|f\|_{q,r}\,, \end{aligned} \tag{5.39}$$

since (5.38) implies by the Markov property that

$$(5.40)\quad \sup_{x\in\Omega_R} P_x\Big(\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt > k\,N\,\eta\,\lambda^{2-3/q}\,\|f\|_{q,r}\Big) < (1-\gamma^N)^k\,,$$

for $k = 1,2,\ldots$ It is finally easy to see that

$$(5.41)\qquad N\,\gamma^{-N}\,\lambda^{2-3/q} \le C_1\,R^{2-3/q}\,\exp(C_2 m_\varepsilon(\mathbf{b})/n_0)\,,$$

for some constants C_1, C_2 depending only on p, q, r.

6. L^∞-bounds.

We shall prove Theorem 1.4 here by refining the estimates already proved in [1]. It is clear we may assume $\mathbf{b}$ and f are C^∞ functions on Ω_R. Hence the drift process $X_{\mathbf{b}}(t)$ is defined and also the expectations of f we shall be considering.

Let Q_0 be a cube concentric with Ω_R having side of length 2^{-n_0}, where n_0 is defined by (1.22). We have the following

Lemma 6.1. *Suppose for some integer $m \ge 0$, the drift $\mathbf{b}$ satisfies the inequality*

$$(6.1)\qquad \int_Q |\mathbf{b}|^p\,dx < \varepsilon^p\,|Q|^{1-p/3}\,,$$

on all dyadic subcubes $Q \subset Q_0$ with side of length 2^{-n}, $n \ge m + n_0$. Let u be the solution of the Dirichlet problem (1.2)-(1.3). Then if ε is sufficiently small, depending only on $p > 2$, there exist constants C_1 depending only on p, q, r, and C_2 only on $p > 2$, such that

$$(6.2)\qquad \|u\|_\infty \le C_1\,R^{2-3/q}\,\|f\|_{q,r}\,\sup_{x\in\Omega_r}\exp\Big(C_2\sum_{j=0}^m a_{n_0+j}(x)\Big)\,.$$

PROOF. If $m = 0$, the perturbative Theorem 1.1 applies and the estimate (6.2) is just the same as (1.11). Therefore we may assume m is large. In that case we modify the proof of Theorem 1.4 of [1]. Consider the function $\xi(x)$, $x \in \Omega_R$, given by

$$(6.3)\qquad \xi(x) = E_x\Big[\exp\Big(-\frac{1}{\mu}\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt\Big)\Big]\,,$$

where τ is the time for the drift process starting at x to hit ∂Q_0. The parameter μ is given by the formula

$$\mu = C\, R^{2-3/q}\, \|f\|_{q,r}\,, \tag{6.4}$$

where the constant C is to be chosen large, depending only on p, q, r. Let U be the set

$$U = \{y:\ 2^{-m-n_0-1} \le |x-y| \le 2^{-m-n_0}\}\,. \tag{6.5}$$

We define a density $\rho : U \to \mathbb{R}$ by the relation

$$\begin{aligned} 2^{m+n_0+1} \int_{2^{-m-n_0-1}}^{2^{-m-n_0}} d\lambda\, E_x\Big[g\,(X(\tau_\lambda))\,\exp\Big(-\frac{1}{\mu}\int_0^{\tau_\lambda} |f|\,(X_{\mathbf{b}}(t))\,dt\Big)\Big] \\ = \int_U \rho(y)\, g(y)\, dy\,, \end{aligned} \tag{6.6}$$

for all continuous functions $g : U \to \mathbb{R}$. Here τ_λ denotes the hitting time for the drift process started at x on the sphere $\{y : |x-y| = \lambda\}$. From Sections 2 and 3 it is clear that

$$\int_{\rho(y)<2^{3(m+n_0)}} \rho(y)\, dy > \frac{1}{2}\,, \tag{6.7}$$

provided ε is sufficiently small depending on $p > 2$, and C in (6.4) is chosen sufficiently large depending on p, q, r.

It follows from (6.3) and (6.6) that

$$\xi(x) = \int_U dy\ \rho(y)\, E_y\Big[\exp\Big(-\frac{1}{\mu}\int_0^{\tau} |f|(X_{\mathbf{b}}(t))\,dt\Big)\Big]\,. \tag{6.8}$$

Now we apply the same argument as in Section 5 of [1] to conclude that

$$\xi(x) \ge \eta(x)^2\,, \tag{6.9}$$

where

$$\begin{aligned} \eta(x) = \int_U dy\ \rho(y)\, E_y\Big[\exp\Big(-\frac{1}{2\mu}\int_0^{\tau} |f|(X(t))\,dt \\ -\frac{1}{4}\int_0^{\tau} |\mathbf{b}|^2(X(t))\,dt\Big)\Big]\,, \end{aligned} \tag{6.10}$$

and $X(t)$ is standard Brownian motion. Applying the same argument as in the proof of Theorem 1.1.*a*) of [1] to (6.10) we obtain the inequality

$$\eta(x) \geq \frac{1}{2} \exp\Big(- C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big), \tag{6.11}$$

where $C_2 > 0$ is universal provided C in (6.4) is chosen sufficiently large depending on p, q, r.

Evidently (6.11) implies a lower bound on $\xi(x)$. The inequality (6.2) follows from this bound and Lemma 5.1 of [1].

Next we consider the probability of hitting a dyadic subcube Q_n of Q_0 with side of length 2^{-n}, $n > n_0$, before exiting Ω_R.

Lemma 6.2. *For $n \in \mathbb{Z}$, let Ω_n be the region*

$$\Omega_n = \{x \in \mathbb{R}^3 :\ 2^{-n-1} < |x| < 2^{-n+1}\}. \tag{6.12}$$

For $x \in \Omega_n$ let P_x be the probability that the drift process started at x exits Ω_n through the sphere $\{y : |y| = 2^{-n+1}\}$. Let δ be a number satisfying $0 < \delta < 2/3$. Then if $|x| = 2^{-n}$ there is a constant C depending only on $\delta < 2/3$ and $p > 2$ such that

$$P_x \geq \delta \exp(-C\, a_{n-1}(0)). \tag{6.13}$$

PROOF. Let χ be the function defined on the boundary $\partial\Omega_n$ of Ω_n by

$$\chi(z) = \begin{cases} 1, & \text{if } |z| = 2^{-n+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{6.14}$$

We define a function $\xi(x)$ analogous to (6.3) by

$$\xi(x) = E_x[\chi(X_{\mathbf{b}}(\tau))], \tag{6.15}$$

where τ is the first hitting time on $\partial\Omega_n$ for the drift process started at $x \in \Omega_n$. Hence $\xi(x)$ is the probability of exiting Ω_n though the outer sphere. We wish to generalize the inequality (6.9). Let $K > 0$ be some arbitrary constant to be specified later and put

$$\eta(x) = E_x\Big[\chi(X(\tau)) \exp\Big(- \frac{K}{4} \int_0^{\tau} |\mathbf{b}|^2(X(t))\, dt\Big)\Big], \tag{6.16}$$

where $X(t)$ denotes Brownian motion in (6.16) and τ is its first exit time from Ω_n. Then for any $s > 1, 1/s + 1/s' = 1$, we have by Hölder's inequality and the Cameron-Martin formula.

$$
\begin{aligned}
\eta(x) = E_x\Big[\chi(X(\tau) \exp\Big(\frac{1}{2s}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) \\
- \frac{1}{4s}\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big) \\
\cdot \exp\Big(-\frac{1}{2s}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) \\
- \frac{1}{4}\Big(K - \frac{1}{s}\Big)\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big] \\
\le \xi(x)^{1/s} E_x\Big[\exp\Big(\frac{-s'}{2s}\int_0^\tau \mathbf{b}(X(t))\cdot dX(t) \\
- \frac{s'}{4}\Big(K - \frac{1}{s}\Big)\int_0^\tau |\mathbf{b}|^2(X(t))\,dt\Big)\Big]^{1/s'} .
\end{aligned}
\tag{6.17}
$$

In view of (4.7) we have the inequality

$$\eta(x) \le \xi(x)^{1/s}, \tag{6.18}$$

provided

$$s'\Big(K - \frac{1}{s}\Big)\Big(\frac{s}{s'}\Big)^2 \ge 1 . \tag{6.19}$$

Hence if we choose K by the formula

$$K = \frac{1}{s-1}, \tag{6.20}$$

then the inequality (6.18) holds. Note that K diverges if we let s approach 1.

We can estimate $\eta(x)$ from below in exactly the same way Theorem 1.1.a) of [1] was proved. In fact we have the inequality

$$\eta(x) \ge E_x[\chi(X(\tau))] \exp\Big(-C K^{1/2} \sum_{m=n-2}^{\infty} a_m(x)\Big), \tag{6.21}$$

where the constant C is universal, provided x lies in the region

$$2^{-n-1/2} < |x| < 2^{-n+1/2} . \tag{6.22}$$

It is a simple matter to compute $E_x[\chi(X(\tau))]$. In fact we have

$$E_x[\chi(X(\tau))] = \frac{4}{3}\left(1 - \frac{2^{-n-1}}{|x|}\right). \tag{6.23}$$

Now consider an arbitrary point $x_0 \in \Omega_n$ with $|x_0| = 2^{-n}$. Choose $s > 1$ sufficiently small so that $\delta^{1/s} < 2/3$, and let B be the ball

$$B = \left\{ x : |x - x_0| < \min\left\{ 2^{-n}\left(\frac{2 - 3\,\delta^{1/s}}{4 - 3\,\delta^{1/s}}\right), 2^{-n}(1 - 2^{-1/2}) \right\} \right\}. \tag{6.24}$$

It is clear from (6.23) that B is contained in the region (6.22) and

$$E_x[\chi(X(\tau))] \geq \delta^{1/s}, \qquad x \in B. \tag{6.25}$$

Let $X(t)$ be an arbitrary continuous path with $X(0) = x_0$, $X(t) \in B$, $t < \tau$, and $X(\tau) \in \partial B$. We claim that there exists an $x = X(t)$ for some t, $0 \leq t \leq \tau$, such that

$$\sum_{m=n-2}^{\infty} a_m(x) \leq C_1\, a_{n-1}(0), \tag{6.26}$$

where the constant C_1 depends only on $\delta^{1/s} < 2/3$ and $p > 2$. Here we are taking $\mathbf{b}(x) = 0$ for $x \notin \Omega_n$ in our definition of $a_m(x)$. The inequality (6.13) clearly follows from (6.18), (6.21), (6.25), (6.26).

We are left to prove (6.26). Let $C_1 > 0$, β be constants with $0 < \beta < 1$ and consider the sets

$$S_m = \{x \in B : \ a_m(x) > C_1\, \beta^{m-n}\, a_{n-1}(0)\}, \qquad m \geq n-2. \tag{6.27}$$

We shall assume that

$$\{X(t) : \ 0 \leq t \leq \tau\} \subset \bigcup_{m=n-2}^{\infty} S_m. \tag{6.28}$$

Otherwise there exists a point x on the path $X(t)$ in the complement of all the sets S_m, $m \geq n-2$, in which case (6.26) clearly holds since $\beta < 1$. For each $x \in S_m$ let D_x be the open ball centered at x with radius 2^{-m}. From (6.28) it follows that the sets $\{D_x : x \in S_m,\, m \geq n-2\}$ form an open cover of the path $X(t)$, $0 \leq t \leq \tau$. By compactness of the path there exists a finite subcover $\Gamma = \{D_j : 1 \leq j \leq N\}$ for some integer

N. For each integer $m \geq n-2$, let Γ_m be the subset of Γ consisting of balls with radius 2^{-m}. Let D be an arbitrary ball and $\tilde{D}$ the ball concentric with D but with three times the radius. Then there exists a subset $\tilde{\Gamma}_m \subset \Gamma_m$ of disjoint balls such that

$$\bigcup_{D \in \Gamma_m} D \subset \bigcup_{D \in \tilde{\Gamma}_m} \tilde{D}. \tag{6.29}$$

Since the balls in $\tilde{\Gamma}_m$ are disjoint it follows from the definition (6.27) of S_m that the cardinality $|\tilde{\Gamma}_m|$ of the set $\tilde{\Gamma}_m$ satisfies the inequality

$$\begin{aligned} |\tilde{\Gamma}_m| \, (C_1 \, \beta^{m-n} \, a_{n-1}(0))^p &\leq 2^{m(3-p)} \int_{\Omega_n} |\mathbf{b}|^p \, dy \\ &\leq 2^{(3-p)(m+1-n)} \, a_{n-1}(0)^p \,, \end{aligned} \tag{6.30}$$

which implies the bound,

$$|\tilde{\Gamma}_m| \leq \frac{2^{3-p}}{C_1^p} \left(\frac{2^{3-p}}{\beta^p} \right)^{m-n}. \tag{6.31}$$

We choose β so that

$$\frac{2^{3-p}}{\beta^p} < 2\,. \tag{6.32}$$

This is possible since $2 < p < 3$. It is clear that for any point x on the path $X(t)$, $0 \leq t \leq \tau$, one must have the inequality

$$|x - x_0| \leq \sum_{m=n-2}^{\infty} 6 \cdot 2^{-m} \, |\tilde{\Gamma}_m| \leq A \, \frac{2^{-n}}{C_1^p} \,, \tag{6.33}$$

where A depends only on β satisfying (6.32). Since $X(\tau)$ lies on the boundary of the ball B in (6.24) the inequality (6.33) is violated for $x = X(\tau)$ provided C_1 is chosen sufficiently large. Hence we have a contradiction to our assumption (6.28). The proof is complete.

Lemma 6.3. *Let $S_0, S_1, \ldots, S_N$ be a set of concentric spheres with radii $r_0, r_1, \ldots, r_N$ satisfying $r_0 < r_1 < r_2 < \cdots < r_N$. For $j = 1, \ldots, N-1$ let $q_j(x,y)$ be non negative functions of $x \in S_j$, $y \in S_{j-1}$ satisfying*

$$0 < \int_{S_{j-1}} q_j(x,y) \, dy \leq q_j < 1\,, \qquad x \in S_j\,, \tag{6.34}$$

for some positive numbers $q_1, \dots, q_{N-1}$.

Suppose now the $q_j(x,y)$ *are probability density functions for a stochastic process* $Y(t)$ *with continuous paths in the following sense: For any open set* $0 \subset S_{j-1}$,

$$\text{Prob}\{Y \textit{ started at } x \in S_j \textit{ exits the region between } S_{j-1} \textit{ and } S_{j+1} \textit{ through } 0\} = \int_0 q_j(x,y)\,dy\,. \tag{6.35}$$

Let $x \in S_{N-m}$ *for some* m, $1 \le m \le N-1$, *and* P_x *be the probability that* Y *started at* x *exits the region between* S_0 *and* S_N *through* S_0. *Then there is the inequality*

$$P_x \cdot \le \frac{1 + \dfrac{p_{N-1}}{q_{N-1}} + \dfrac{p_{N-1}}{q_{N-1}}\dfrac{p_{N-2}}{q_{N-2}} + \cdots + \displaystyle\prod_{j=1}^{m-1} \dfrac{p_{N-j}}{q_{N-j}}}{1 + \dfrac{p_{N-1}}{q_{N-1}} + \dfrac{p_{N-1}}{q_{N-1}}\dfrac{p_{N-2}}{q_{N-2}} + \cdots + \displaystyle\prod_{j=1}^{N-1} \dfrac{p_{N-j}}{q_{N-j}}}\,, \tag{6.36}$$

where the p_j *are defined by* $p_j = 1 - q_j$, $j = 1, \dots, N-1$.

PROOF. Observe that the right hand side of (6.36) is just u_{N-m} where u_n, $n = 0, 1, \dots, N$ is the solution of the finite difference equation

$$\begin{aligned} &u_n = p_n\, u_{n+1} + q_n\, u_{n-1}\,, \qquad 1 \le n \le N-1\,, \\ &u_0 = 1\,, \qquad u_N = 0\,. \end{aligned} \tag{6.37}$$

Hence the lemma merely states that P_x is bounded by the probability for a random walk on the spheres S_j, $j = 0, \dots, N$, with transition probabilities determined by the upper bound q_j in (6.34), $j = 1, \dots, N-1$.

To prove the lemma we first consider the case when $N = 3$. For $x \in S_1 \cup S_2$ let $u(x)$ be the probability that Y started at x hits S_0 before hitting S_3. Then $u(x)$ must satisfy the equations

$$\begin{aligned} u(x) &= \int_{S_1} q_2(x,y)\, u(y)\, dy, \qquad x \in S_2\,, \\ u(x) &= \int_{S_2} p_1(x,y)\, u(y)\, dy + \int_{S_0} q_1(x,y)\, dy\,, \qquad x \in S_1\,, \end{aligned} \tag{6.38}$$

where $p_1(x,y)$ is the probability density for the process started at $x \in S_1$ of hitting S_2 before S_0. It follows from (6.38) that

$$u(x) = \int_{S_1}\int_{S_2} q_2(x,y)\,p_1(y,z)\,u(z)\,dz\,dy + \int_{S_1}\int_{S_0} q_2(x,y)\,q_1(y,z)\,dz\,dy\,, \quad x \in S_2. \tag{6.39}$$

Putting $u_2 = \sup_{x\in S_2} u(x)$ it follows from (6.39) that

$$u_2 \le \sup_{x\in S_2} A(x)\,, \tag{6.40}$$

where

$$A(x) = \frac{\int_{S_1}\int_{S_0} q_2(x,y)\,q_1(y,z)\,dz\,dy}{1 - \int_{S_1}\int_{S_2} q_2(x,y)\,p_1(y,z)\,dz\,dy}\,. \tag{6.41}$$

Using the fact that for any $y \in S_1$,

$$\int_{S_2} p_1(y,z)\,dz = 1 - \int_{S_0} q_1(y,z)\,dz\,, \tag{6.42}$$

we have that

$$\frac{1}{A(x)} = \frac{1 + \int_{S_1} q_2(x,y)\,dy\Big(\Big(\int_{S_1} q_2(x,y)\,dy\Big)^{-1} - 1\Big)}{\int_{S_1}\int_{S_0} q_2(x,y)\,q_1(y,z)\,dz\,dy}\,. \tag{6.43}$$

Using (6.34) for $j = 1$, we have from (6.43) that $A(x)$ satisfies the inequality

$$\frac{1}{A(x)} \ge 1 + \frac{1}{q_1}\left(\Big(\int_{S_1} q_2(x,y)\,dy\Big)^{-1} - 1\right). \tag{6.44}$$

Next, applying (6.34) for $j = 2$ to the right side of (6.44) yields

$$\frac{1}{A(x)} \ge 1 + \frac{1}{q_1}\Big(\frac{1}{q_2} - 1\Big) = 1 + \frac{p_2}{q_2} + \frac{p_2}{q_2}\,\frac{p_1}{q_1}\,. \tag{6.45}$$

Inequality (6.45) together with (6.40) implies (6.36) for $N = 3$, $m = 1$.

Next we consider the case $N = 3$, $m = 2$. From (6.38) we have

$$\begin{aligned} u(x) &= \int_{S_2}\int_{S_1} p_1(x,y)\, q_2(y,z)\, u(z)\, dz\, dy \\ &\quad + \int_{S_0} q_1(x,y)\, dy\,, \qquad x \in S_1\,. \end{aligned} \tag{6.46}$$

Setting $u_1 = \sup_{x\in S_1} u(x)$ we obtain from (6.46) the inequality

$$u_1 \le \sup_{x\in S_1} B(x)\,, \tag{6.47}$$

where

$$B(x) = \frac{\displaystyle\int_{S_0} q_1(x,y)\, dy}{1 - \displaystyle\int_{S_2}\int_{S_1} p_1(x,y)\, q_2(y,z)\, dz\, dy}\,. \tag{6.48}$$

Using (6.34) with $j = 2$ it follows that

$$B(x) \le \frac{\displaystyle\int_{S_0} q_1(x,y)\, dy}{1 - q_2 \displaystyle\int_{S_2} p_1(x,y)\, dy} = \frac{\displaystyle\int_{S_0} q_1(x,y)\, dy}{1 - q_2 + q_2 \displaystyle\int_{S_0} q_1(x,y)\, dy}\,. \tag{6.49}$$

Next, applying (6.34) with $j = 1$, yields the inequality

$$B(x) \le \frac{q_1}{1 - q_2 + q_2\, q_1} = \frac{1 + \dfrac{p_2}{q_2}}{1 + \dfrac{p_2}{q_2} + \dfrac{p_2}{q_2}\dfrac{p_1}{q_1}}\,. \tag{6.50}$$

Hence (6.36) for $N = 3$, $m = 2$ follows from (6.50) and (6.47).

The situation for $N \ge 4$ can be derived from the $N = 3$ case by induction. Suppose we already know that (6.36) holds for any sequence of less than $N + 1$ spheres. We consider the case of $N + 1$ spheres $S_0, S_1, \ldots, S_N$. Let k be an integer satisfying $2 \le k \le N - 1$, and consider the case of the four spheres S_0, S_{k-1}, S_k, S_N. Let Q_1 be an upper bound on the probability of Y starting at $x \in S_{k-1}$ of hitting S_0 before S_k. Similarly let Q_2 be an upper bound on the probability of Y starting at $x \in S_k$ of hitting S_{k-1} before S_N. Then by our result already obtained for the 4 sphere case, we have

$$P_x \le \left(1 + \frac{1}{Q_1}\left(\frac{1}{Q_2} - 1\right)\right)^{-1}, \quad x \in S_k . \tag{6.51}$$

By our inductive assumptions we have bounds on Q_1, Q_2, namely

$$\frac{1}{Q_1} \ge 1 + \frac{p_{k-1}}{q_{k-1}} + \frac{p_{k-1}}{q_{k-1}}\frac{p_{k-2}}{q_{k-2}} + \cdots + \prod_{j=1}^{k-1}\frac{p_{k-j}}{q_{k-j}},$$

$$\frac{1}{Q_2} - 1 \ge \frac{\prod_{j=1}^{N-k}\frac{p_{N-j}}{q_{N-j}}}{1 + \frac{p_{N-1}}{q_{N-1}} + \cdots + \prod_{j=1}^{N-k-1}\frac{p_{N-j}}{q_{N-j}}} . \tag{6.52}$$

Substituting the right hand side of (6.52) into (6.51) clearly implies the bound (6.36).

Finally we must deal with the case $k = 1$. Here the four spheres are S_0, S_1, S_2, S_N. Let Q_2 be an upper bound on the probability of Y started at $x \in S_2$ of hitting S_1 before S_N. Then from (6.47), (6.50) we have

$$P_x \le \frac{q_1}{1 - Q_2 + Q_2 q_1}, \qquad x \in S_1 . \tag{6.53}$$

By our induction assumption we have the bound

$$Q_2 \le \frac{1 + \frac{p_{N-1}}{q_{N-1}} + \cdots + \prod_{j=1}^{N-3}\frac{p_{N-j}}{q_{N-j}}}{1 + \frac{p_{N-1}}{q_{N-1}} + \cdots + \prod_{j=1}^{N-2}\frac{p_{N-j}}{q_{N-j}}} . \tag{6.54}$$

Hence we have

$$\begin{aligned}\frac{1}{q_1}(1 - Q_2) + Q_2 &= 1 - Q_2 + \frac{p_1}{q_1}(1 - Q_2) + Q_2 \\ &\ge \frac{1 + \frac{p_{N-1}}{q_{N-1}} + \cdots + \prod_{j=1}^{N-1}\frac{p_{N-j}}{q_{N-j}}}{1 + \frac{p_{N-1}}{q_{N-1}} + \cdots + \prod_{j=1}^{N-2}\frac{p_{N-j}}{q_{N-j}}} .\end{aligned} \tag{6.55}$$

Substituting (6.55) into (6.53) yields the estimate (6.36) for $x \in S_1$.

Lemma 6.4. *For any integer $m \geq 0$ and arbitrary $\varepsilon > 0$ let $U_m \subset Q_0$ be the union of all dyadic subcubes Q of Q_0 with side of length 2^{-n_0-m} such that*

$$\int_Q |\mathbf{b}|^p \, dx \geq \varepsilon^p \, |Q|^{1-p/3} \, . \tag{6.56}$$

For $x \in \Omega_R$ let $X_{\mathbf{b}}(t)$ be the process with drift $\mathbf{b}$ starting at x, where we set $\mathbf{b}$ to be identically zero outside Q_0. For $\lambda > 0$ let τ_λ be the first hitting time on the sphere of radius λ centered at x and $P_m(x)$ be the probability

$$\begin{aligned} &P_m(x) \\ &\quad = \mathrm{Prob}\,\{X_{\mathbf{b}}(t) \textit{ hits } U_m \textit{ in the time interval } \tau_{R/2} < t < \tau_R\} \, . \end{aligned} \tag{6.57}$$

Then there exists a constant $\gamma, 0 < \gamma < 1$, and constants C_1, C_2 depending only on $p > 2$ such that

$$P_m(x) \leq C_1 \, \varepsilon^{-p} \, \gamma^m \sup_{y \in Q_{n_0}} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(y)\Big) \, . \tag{6.58}$$

PROOF. We can assume m large since the right hand side will be larger than 1 if m is small. Let $Q \subset Q_0$ be a dyadic cube with side of length 2^{-n_0-m} and center x_0. Then for any path $X_{\mathbf{b}}(t)$ there exists a time t_0 with $0 \leq t_0 < \tau_{R/2}$ such that

$$|X_{\mathbf{b}}(t_0) - x_0| \geq \frac{R}{4} \, . \tag{6.59}$$

For $j = 0, 1, 2, \ldots$ let S_j be the sphere centered at x_0 with radius 2^{-n_0-m+j}. Then from (1.22) we see that $X_{\mathbf{b}}(t_0)$ lies in the region outside the sphere S_{m-4}. Also the ball of radius R centered at x lies inside the sphere S_{m+1}. Hence the probability P_Q of $X_{\mathbf{b}}(t)$ hitting Q in the time interval $\tau_{R/2} < t < \tau_R$ is less than the supremum of the probabilities of the drift process started at $x \in S_{m-4}$ of hitting S_0 before S_{m+1}.

Now we can use Lemmas 6.2 and 6.3 to estimate this last probability. From Lemma 6.2 we have that

$$q_j \leq 1 - \delta \, \exp\Big(- C \, a_{n_0+m-j-1}(x_0)\Big) \, . \tag{6.60}$$

Hence the inequality (6.36) yields

$$P_Q \le A\left(\frac{1}{\delta}-1\right)^m \exp\left(\frac{C}{1-\delta}\sum_{j=0}^{m} a_{n_0+j}(x_0)\right), \tag{6.61}$$

for some universal constant A. Here we have used the fact that $\mathbf{b} \equiv 0$ outside Q_{n_0} and that

$$\delta^{-1} e^{\xi} - 1 \le (\delta^{-1}-1)\exp\frac{\xi}{1-\delta}, \qquad \xi \ge 0. \tag{6.62}$$

Finally we estimate the number N_m of cubes $Q \subset U_m$. From (6.56) we have, if 0 is the center of Q_0, that

$$\varepsilon^p\, 2^{-(n_0+m)(3-p)}\, N_m \le 2^{-n_0(3-p)}\, a_{n_0}(0)^p, \tag{6.63}$$

whence

$$N_m \le \varepsilon^{-p}\, 2^{m(3-p)}\, a_{n_0}(0)^p. \tag{6.64}$$

Hence $P_m(x)$ is bounded by the product of the right side of (6.64) and the supremum over $x_0 \in Q_0$ of the right side of (6.61). Now using the fact that, $p > 2$ and δ can be chosen as close as we please to $2/3$, yields (6.58).

Lemma 6.5. *Let $f \in M^q_r$, $q > 3/2$, $1 < r \le q$. Then there exists γ, $0 < \gamma < 1$, depending only on $p > 2$ such that*

$$\begin{aligned} &\sup_{x\in\Omega_R} E_x\left[\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\right] \\ &\quad\le C_1\, R^{2-3/q}\, \|f\|_{q,r} \sum_{m=0}^{\infty} \gamma^m \sup_{x\in\Omega_R} \exp\left(C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\right). \end{aligned} \tag{6.65}$$

The constant C_1 depends only on p, q, r and C_2 only on $p > 2$.

PROOF. We write

$$\begin{aligned} &E_x\left[\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\right] \\ &\quad= \sum_{m=0}^{\infty} E_x\left[\chi_m(X_{\mathbf{b}}) \int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\right], \end{aligned} \tag{6.66}$$

where χ_m is the characteristic function of the set of paths which visit U_m between times $\tau_{R/2}$ and τ_R but do not visit any U_n with $n > m$. We use Schwarz's inequality to obtain

$$\begin{aligned} &E_x\Big[\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big] \\ &\leq \sum_{m=0}^{\infty} E_x\Big[\chi_m(X_{\mathbf{b}})\Big(\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big)^2\Big]^{1/2} P_m(x)^{1/2}. \end{aligned} \tag{6.67}$$

We have now that

$$\begin{aligned} &E_x\Big[\chi_m(X_{\mathbf{b}})\Big(\int_{\tau_{R/2}}^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big)^2\Big] \\ &\qquad\leq E_x\Big[\Big(\int_0^{\tau_R} |f|(X_{\mathbf{b}_m}(t))\,dt\Big)^2\Big], \end{aligned} \tag{6.68}$$

where $\mathbf{b}_m$ is equal to $\mathbf{b}$ on $Q_0 \setminus \cup_{j=m+1}^{\infty} U_j$ but zero otherwise. This is true because the characteristic function χ_m restricts to paths which do not visit $\cup_{j=m+1}^{\infty} U_j$. The drift $\mathbf{b}_m$ satisfies the conditions for Lemma 6.1 and hence if ε is sufficiently small there is the inequality

$$\begin{aligned} &\sup_{x\in\Omega_R} E_x\Big[\int_0^{\tau_R} |f|(X_{\mathbf{b}_m}(t))\,dt\Big] \\ &\qquad\leq C_1\, R^{2-3/q}\, \|f\|_{q,r} \sup_{x\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big). \end{aligned} \tag{6.69}$$

It follows from (6.69) using the Chebyshev inequality and the Markov property that

$$\begin{aligned} &\sup_{x\in\Omega_R} E_x\Big[\Big(\int_0^{\tau_R} |f|(X_{\mathbf{b}}(t))\,dt\Big)^2\Big]^{1/2} \\ &\qquad\leq 40\, C_1\, R^{2-3/q}\, \|f\|_{q,r} \sup_{x\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big). \end{aligned} \tag{6.70}$$

The inequality (6.65) follows now from (6.70), (6.67) and Lemma 6.4.

PROOF OF THEOREM 1.4. We bound the solution $u(x)$ of (1.2)-(1.3) by

$$|u(x)| \leq \sum_{k=1}^{\infty} E_x\Big[\int_{\tau_{R/2^k}}^{\tau_{R/2^{k-1}}} |f|(X_{\mathbf{b}}(t))\,dt\Big]. \tag{6.71}$$

From Lemma 6.5 we have the inequality

$$
\begin{aligned}
E_x\Big[\int_{\tau_{R/2^k}}^{\tau_{R/2^{k-1}}} & |f|(X_{\mathbf{b}}(t))\,dt\Big] \\
&\le C_1\,2^{-(2-3/q)(k-1)}\,R^{2-3/q}\,\|f\|_{q,r} \\
&\quad\cdot \sum_{m=k-1}^{\infty} \gamma^{m-(k-1)} \sup_{y\in\Omega_R} \exp\Big(C_2 \sum_{j=k-1}^{m} a_{n_0+j}(y)\Big) \qquad (6.72)\\
&\le C_1\,2^{-(2-3/q)(k-1)/2}\,R^{2-3/q} \\
&\quad\cdot \sum_{m=0}^{\infty} \gamma_1^m \sup_{y\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(y)\Big)\,,
\end{aligned}
$$

where $\gamma_1 = \max\{\gamma, 2^{-(2-3/q)/2}\}$.

Summing the right side of the last inequality with respect to k proves the theorem.

Our last result shows the inequality (1.24) follows from Theorem 1.4.

Proposition 6.6. *There exist universal constants C, c such that for any $\varepsilon > 0$, $x \in \mathbb{R}^3$, there is the inequality*

$$\sum_{n=-\infty}^{\infty} a_n(x)\,H(a_n(x)-\varepsilon) \le C\,N_{c\varepsilon}(\mathbf{b})\,, \qquad (6.73)$$

where $H(t)$ is the Heaviside function, $H(t) = 1$, if $t > 0$ and $H(t) = 0$, if $t \le 0$.

PROOF. For $n \in \mathbb{Z}$ let $c_n(x)$ be defined by

$$c_n(x) = \Big(2^{n(3-p)} \int_{2^{-n-1}<|x-y|<2^{-n}} |\mathbf{b}|^p\,dy\Big)^{1/p}. \qquad (6.74)$$

It is clear from (6.74), (1.21) that there is the identity

$$2^{-(3-p)}\,a_{n+1}(x)^p + c_n(x)^p = a_n(x)^p\,. \qquad (6.75)$$

Let α be an arbitrary positive number, $0 < \alpha < 1$, such that

$$\delta^{-p} = (1-\alpha^p)\,2^{3-p} > 1\,. \qquad (6.76)$$

If $c_n(x) > \alpha\, a_n(x)$ it is clear that

$$a_n(x)\, H(a_n(x) - \varepsilon) \le \alpha^{-1}\, c_n(x)\, H(c_n(x) - \alpha\, \varepsilon)\,. \tag{6.77}$$

On the other hand if $c_n(x) < \alpha a_n(x)$, then (6.75) implies that $a_n(x) < \delta\, a_{n+1}(x)$, and hence

$$a_n(x)\, H(a_n(x) - \varepsilon) \le \delta\, a_{n+1}(x)\, H(a_{n+1}(x) - \varepsilon)\,. \tag{6.78}$$

Putting (6.77), (6.78) together we conclude that for all values of $a_n(x)$ there is the inequality

$$\begin{aligned} a_n(x)\, H(a_n(x) - \varepsilon) \le \delta\, a_{n+1}(x)\, H(a_{n+1}(x) - \varepsilon) \\ + \alpha^{-1}\, c_n(x)\, H(c_n(x) - \alpha\, \varepsilon)\,. \end{aligned} \tag{6.79}$$

If we sum (6.79) over $n \in \mathbb{Z}$ we obtain the inequality

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} a_n(x)\, H(a_n(x) - \varepsilon) \\ &\qquad \le \frac{1}{\alpha(1-\delta)} \sum_{n=-\infty}^{\infty} c_n(x)\, H(c_n(x) - \alpha\, \varepsilon)\,. \end{aligned} \tag{6.80}$$

Now let us suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. For any $n \in \mathbb{Z}$, let S_n be the set,

$$\begin{aligned} S_n = \{Q : Q \cap \{y : 2^{-n-1} < |x - y| < 2^{-n}\} \\ \text{is not empty and } |Q| \le 2^{-3(n+3)}\}\,. \end{aligned} \tag{6.81}$$

For any $\varepsilon > 0$ let $N_{\varepsilon,n}(\mathbf{b})$ be the number of minimal cubes for $\mathbf{b}$ which are in S_n. It is clear from the definition (6.81) of S_n that

$$\sum_{n=-\infty}^{\infty} N_{\varepsilon,n}(\mathbf{b}) \le 2\, N_\varepsilon(\mathbf{b})\,. \tag{6.82}$$

Next let us suppose $c_n(x) > \alpha\, \varepsilon$. Then we have

$$(\alpha\, \varepsilon)^p < 2^{n(3-p)} \sum_{\substack{Q \in S_n \\ |Q| = 2^{-3(n+3)}}} \int_Q |\mathbf{b}|^p\, dy\,. \tag{6.83}$$

Since there are at most 2^{12} cubes $Q \in S_n$ with side of length 2^{-n-3} it follows from (6.83) that one of them must satisfy the inequality

$$\int_Q |\mathbf{b}|^p \, dy \geq (\alpha \, \varepsilon \, 2^{-12/p} \, 2^{3(3/p-1)})^p \, |Q|^{1-p/3} . \tag{6.84}$$

Hence if $c_n(x) > \alpha\varepsilon$ and c satisfies the inequality

$$c < \alpha \, 2^{-12/p} \, 2^{3(3/p-1)} , \tag{6.85}$$

then from (1.20) one must have $N_{c\,\varepsilon,n}(\mathbf{b}) \geq 1$.

Finally we use a result of Fefferman [3]. Let $\varepsilon > 0$ be arbitrary. Then there exist disjoint sets $E_1, E_2, \ldots, E_M$ with the properties:

a) $\displaystyle\bigcup_{Q \in S_n} Q = \bigcup_{j=1}^{M} E_j$.

b) Each E_j is a subset of a cube $Q_j \in S_n$.

c) $\displaystyle\int_{E_j} |\mathbf{b}|^p \, dy \leq C_1 \, \varepsilon^p \, |Q_j|^{1-p/3}$, for some universal constant C_1, $j = 1, \ldots, M$.

d) $M \leq C_2 \, (N_{\varepsilon,n}(\mathbf{b}) + 1)$, for some universal constant C_2.

We can bound $c_n(x)$ by using the Fefferman deconposition. Thus for any $c > 0$, we have

$$\begin{aligned} c_n(x)^p &\leq 2^{n(3-p)} \sum_{j=1}^{M} \int_{E_j} |\mathbf{b}|^p \, dy \\ &\leq 2^{n(3-p)} \sum_{j=1}^{M} C_1 \, (c\,\varepsilon)^p \, |Q_j|^{1-p/3} \\ &\leq C_1 \, (c\,\varepsilon)^p \, M \\ &\leq C_1 \, (c\,\varepsilon)^p \, C_2 \, (N_{c\,\varepsilon,n}(\mathbf{b}) + 1) . \end{aligned} \tag{6.86}$$

Now if we choose c to satisfy the inequality (6.85) we have that

$$c_n(x)^p \leq 2 \, C_1 \, (c\,\varepsilon)^p \, C_2 \, N_{c\,\varepsilon,n}(\mathbf{b}) , \tag{6.87}$$

provided $c_n(x) > \alpha\,\varepsilon$. The inequality (6.73) follows from (6.80), (6.82) and (6.87).

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Formulas for approximate solutions of the $\partial\bar\partial$-equation in a strictly pseudoconvex domain

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Abstract. Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. We construct approximative solution formulas for the equation $i\partial\bar\partial u = \theta$, θ being an exact (1,1)-form in D. We show that our formulas give simple proofs of known estimates and indicate further applications.

Introduction.

Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. The main result of this paper is a weighted approximate solution formula for the equation

$$(0.1) \qquad i\,\partial\bar\partial\, u = \theta\,,$$

where θ is an exact (1,1)-form (current) in D.

The equation (0.1) is of interest mainly because of its connection to divisors (zero sets) of holomorphic functions. Namely, to each such divisor there is associated a positive closed (1,1)-current and the solutions to (0.1) are precisely functions of the form $u = \log|f|$ (disregarding for the moment possible topological obstructions) where f is a holomorphic function that defines the divisor. Hence bounds on the solution u in (0.1) proves the existence of holomorphic functions in various function classes with given zero set θ. For results of this kind, see for instance [H], [S], [V2] and [B].

When $n = 1$, (0.1) is just Poisson's equation and can be solved by the Newton kernel $(1/2\pi)\log|\zeta - z|$ if θ has finite mass in D. However, if *e.g.* θ just satisfies the Blaschke condition

$$\int_D d(\zeta, \partial D)\, |\theta(\zeta)| < +\infty, \tag{0.2}$$

then one has to use a weighted solution kernel; the Green's functions in D, which is not explicitly given in general but nevertheless well understood, at least if ∂D has some regularity.

If $n > 1$, (0.1) is usually solved by following a two step method that goes back to Lelong. One first solves

$$i\, dw = \theta \tag{0.3}$$

so that $w_{0,1} = -\bar{w}_{1,0}$. (assuming θ real). For bidegree reasons $\bar{\partial} w_{0,1} = 0$ so one can solve

$$\bar{\partial} v = w_{0,1}\,. \tag{0.4}$$

Then $u = v + \bar{v}$ solves (0.1).

In 1975 it was proved independently by Henkin [H] and Skoda [S] that (0.1) admits a solution in $L^1(\partial D)$ if θ satisfies the Blaschke condition (0.2). This result had been conjectured for some time and the main problem for the solution was to get $L^1(\partial D)$-estimates for (0.4). This was solved by Henkin and Skoda by introducing solution formulas with weights that gave the desired estimate.

However, in other situations it is (0.3) that offers the greatest difficulties, as for instance in Varopoulos' result, [V2], that there is a solution u in BMO(∂D) if θ satisfies a certain Carleson condition.

Explicit solution formulas for the L^2_α-minimal solutions to (0.1) in the ball in $\mathbb{C}^n$ were obtained in [An1] and [An2] ($L^2_\alpha = L^2((1 - |\zeta|^2)^{\alpha-1} d\lambda)$). For appropriate choices of α these formulas admitted simple and natural proofs of the (known) estimates for (0.1) discussed above. Earlier Berndtsson [B] used an explicit formula for the ball in $\mathbb{C}^2$ to show that (0.1) has a negative solution if θ has finite mass. Recently this result has been generalized by Arlebrink [Ar] to the strictly pseudoconvex case.

Let ρ be a strictly plurisubharmonic C^4 defining function for D and put $L^p_\alpha = L^p((-\rho)^{\alpha-1}d\lambda)$ if $\alpha > 0$ and $L^p_0 = L^p(\partial D)$. In this paper we use ideas from [An1], [An2] and [Ar] to construct operators M_α, acting on (1,1)-forms θ in D, and P_α and F_α, acting on functions, such that

$$u = M_\alpha(i\,\partial\bar{\partial}u) + P_\alpha u + F_\alpha u\,, \tag{0.5}$$

where $P_\alpha u$ is pluriharmonic and $F_\alpha u$ is a weakly singular integral operator and hence somewhat smoothing; roughly $F^k_\alpha M_\alpha\theta$ is nicer than $M_\alpha\theta$ and F^m_α maps L^1_α into $C(\bar{D})$ if m is large enough. If θ is an exact (1,1) current and u_0 is an L^1_α-solution of $i\partial\bar{\partial}u = \theta$, then by (0.5), $u_1 = M\theta + F_\alpha u_0$ also solves (0.1). Repeating this argument, we get a new solution u_m,

$$u_m = M_\alpha\theta + F_\alpha M_\alpha\theta + F^2_\alpha M_\alpha\theta + \cdots + F^{m-1}_\alpha M_\alpha\theta + F^m_\alpha u_0\,. \tag{0.6}$$

Thus, given a starting solution $u_0 \in L^1_\alpha$, estimates of the solution u_m are reduced to estimates for the explicitly given $M_\alpha\theta$.

We also get a similar expansion of the L^2_α-minimal solution of (0.1) in terms of $M_\alpha\theta$. As a by-product we get an expression for the L^2_α-orthogonal pluriharmonic projection

$$\Pi_\alpha : L^2_\alpha \cap \mathcal{H} \longrightarrow D\,,$$

($\mathcal{H}$ denotes pluriharmonic functions), such that

$$\Pi_\alpha u = P_\alpha u + R^1_\alpha u + R^2_\alpha \Pi_\alpha u\,,$$

where P_α and R^j_α are explicit and R^j_α are regularizing (compact). In particular, when $\alpha = 0$, $P_0 u$ only depends on the boundary values of u, so if $u \in L^2(\partial D)$ has pluriharmonic extension U to D, then

$$U = P_0 u + K_0 u\,,$$

where K_0 is a compact operator on $L^2(\partial D)$. When $n = 1$, $P_0 u$ is the classical double layer potential of u, which provides an approximate solution to Dirichlet's problem.

In order to clarify our argument for (0.5), we conclude this paragraph with a sketch of the proof in a simple nontrivial case, namely when D is the unit disc Δ in $\mathbb{C}$, $\alpha = 1$ and $z \in \partial\Delta$.

Claim. *If u is smooth on $\bar{\Delta}$, and $z \in \partial\Delta$, then*

$$
\begin{aligned}
u(z) = &\int_\Delta \frac{(1-|\zeta|^2)^2}{|1-\bar{\zeta}z|^2} \frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}}\, d\lambda \\
&+ 2\,\mathrm{Re}\, \frac{1}{\pi}\int_\Delta \frac{u(\zeta)\, d\lambda}{(1-\bar{\zeta}z)^2} - \frac{1}{\pi}\int_\Delta u(\zeta)\, d\lambda\,.
\end{aligned}
\tag{0.7}
$$

The following argument for (0.7) is possibly not the most simple, but it follows the general scheme in Section 3. First we write ($w = \partial u/\partial\bar{\zeta}$)

$$
u(z) = \frac{1}{\pi}\int \frac{u(\zeta)\, d\lambda}{(1-\bar{\zeta}z)^2} + \frac{1}{\pi}\int \frac{(1-|\zeta|^2)\,\bar{z}\,w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} = Gu + Kw\,. \tag{0.8}
$$

Then we rewrite Kw as

$$
\begin{aligned}
Kw &= \frac{1}{\pi}\int \frac{(1-|\zeta|^2)(\bar{z}-\bar{\zeta})\,w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} + \frac{1}{\pi}\int \frac{(1-|\zeta|^2)\,\bar{\zeta}\,w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} \\
&= \frac{1}{\pi}\int \frac{(1-|\zeta|^2)\,\bar{z}\,w}{1-\zeta\bar{z}} + I\,.
\end{aligned}
$$

An integration by parts shows that

$$
\begin{aligned}
I &= -\frac{1}{2\pi}\int \frac{1}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} \frac{\partial}{\partial\zeta}(1-|\zeta|^2)^2 w \\
&= \frac{1}{2\pi}\int \frac{(1-|\zeta|^2)^2}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} \frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}} + \frac{1}{2\pi}\int \frac{(1-|\zeta|^2)^2\bar{z}\,w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})^2}\,.
\end{aligned}
\tag{0.9}
$$

Now we apply a trivial instance of the crucial Proposition 3.2, namely

$$
(1-|\zeta|^2)w\bar{z} = (1-\bar{\zeta}z)w\bar{z} + (1-\zeta\bar{z})\bar{\zeta}w\,,
$$

to the last term in (0.9) and get

$$
\begin{aligned}
I = &\frac{1}{2\pi}\int \frac{1-|\zeta|^2}{(1-\bar{\zeta}z)(1-\zeta\bar{z})} \frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}} + \frac{1}{2\pi}\int \frac{(1-|\zeta|^2)\,\bar{z}\,w}{(1-\zeta\bar{z})^2} \\
&+ \frac{1}{2\pi}\int \frac{(1-|\zeta|^2)\,\bar{\zeta}\,w}{(1-\bar{\zeta}z)(1-\zeta\bar{z})}\,,
\end{aligned}
$$

and since the last term here is $I/2$, we can solve for I and get

$$Kw = \frac{1}{\pi}\int \frac{(1-|\zeta|^2)\,\bar z\,w}{1-\zeta\bar z} + \frac{1}{\pi}\int \frac{(1-|\zeta|^2)\,\bar z\,w}{(1-\zeta\bar z)^2} + \frac{1}{\pi}\int \frac{(1-|\zeta|^2)^2}{(1-\bar\zeta z)(1-\zeta\bar z)}\,\frac{\partial^2 u}{\partial\zeta\partial\bar\zeta}\,.$$

Finally one can integrate by parts in the two integrals involving w, and then summing up one arrives at (0.7).

To prove (0.5) in general we need suitable integral formulas to replace G and K in (0.8). We describe them in Section 1. In Section 2 we state our main results and also point out some applications, as *e.g.* $L^p(\partial D)$-estimates for (0.1) and the BMO-estimate of Varopoulos. In Section 3 and Section 4 we construct our operators and show their relations, whereas some estimates are left to Section 5.

1. Some preliminaries.

Let $D = \{\rho < 0\}$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ where ρ is a C^3 strictly plurisubharmonic defining function. Suppose that $\phi(\zeta, z) : \bar D \times \bar D \to \mathbb{C}$ is C^1 and satisfies

$$2\,\mathrm{Re}\,\phi \geq -\rho(\zeta) - \rho(z) + \delta\,|\zeta - z|^2 \tag{1.1}$$

and

$$d_\zeta\phi\,|_{\zeta=z} = \partial\rho\,|_\zeta = d_\zeta\bar\phi\,|_{\zeta=z}\,, \qquad \zeta = z \in D\,. \tag{1.2}$$

Then $|\phi(\zeta,p)| \sim |\phi(p,\zeta)|$ if $p \in \partial D$ and $\zeta \in \bar D$, and for $p \in \partial D$,

$$B_t(p) = \{\zeta \in \partial D :\ |\phi(\zeta,p)| < t\}$$

and

$$Q_t(p) = \{\zeta \in D :\ |\phi(\zeta,p)| < t\}$$

are the Koranyi balls in ∂D and in D around $p \in \partial D$, see for example the discussion in [AnC]; indeed $B_t(p)$ ($Q_t(p)$) is $\sim \sqrt{t}$ in the complex tangential directions at p and t in the last one (ones). In particular, $|B_t(p)| \sim t^n$ and $|Q_t(p)| \sim t^{n+1}$. Also, if $\tilde\phi$ is another function that satisfies (1.1) and (1.2) then $|\tilde{\phi}| \sim |\phi|$. We recall the following well known estimates

$$\int_{\partial D} \frac{d\sigma(\zeta)}{|\phi(\zeta,z)|^{n+\alpha}} \lesssim \left(\frac{1}{-\rho(z)}\right)^{\alpha}, \quad \alpha > 0, \tag{1.3}$$

and

$$\int_{D} \frac{(-\rho(\zeta))^{\beta}\, d\lambda(\zeta)}{|\phi(\zeta,z)|^{n+1+\alpha+\beta}} \lesssim \left(\frac{1}{-\rho(z)}\right)^{\alpha}, \quad \alpha > 0, \quad \beta > -1. \tag{1.4}$$

There is a C^1-function $\phi(\zeta, z) : \bar{D} \times \bar{D} \to \mathbb{C}^n$ which is holomorphic in z for fixed $\zeta \in \bar{D}$ and such that

$$\phi(\zeta, z) = \langle q(\zeta, z), z - \zeta\rangle - \rho(\zeta)$$

satisfies (1.1) and (1.2), see [F]. If we put $s(\zeta, z) = -q(z, \zeta)$ and make the identifications $s \sim \Sigma s_j d\zeta_j$ and similarly $q \sim \Sigma q_j d\zeta_j$, we can define, for $\alpha > 0$,

$$\begin{aligned} H_\alpha u(z) &= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, n!}\left(\frac{i}{2\pi}\right)^n \\ &\quad \cdot \int_D \frac{(-\rho)^{\alpha-1} u[-\rho\bar\partial q - nq \wedge \bar\partial\rho] \wedge (\bar\partial q)^{n-1}}{\phi^{n+\alpha}}, \end{aligned} \tag{1.5}$$

for functions u, and

$$\begin{aligned} Q_\alpha w(z) &= \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)\,(n-1)!}\left(\frac{i}{2\pi}\right)^n \\ &\quad \cdot \int_D \frac{(-\rho)^{\alpha-1} s \wedge w \wedge [-\rho\bar\partial q - (n-1) q \wedge \bar\partial\rho] \wedge (\bar\partial q)^{n-2}}{\phi(\zeta,z)^{n+\alpha-1}\phi(z,\zeta)}, \end{aligned} \tag{1.6}$$

for (0,1)-forms w and $z \in \partial D$. Then, see [AnB, Example 1], $Q_\alpha w(z)$ is the boundary values of a function $Q_\alpha w(z)$ on D, such that

$$Q_\alpha \bar\partial u = u - H_\alpha u \tag{1.7}$$

and

$$\bar\partial Q_\alpha w = w \qquad \text{if } \bar\partial w = 0\,. \tag{1.8}$$

REMARK 1. If we let α tend to 0 in (1.5) and (1.6) we get

$$H_0u(z) = \left(\frac{i}{2\pi}\right)^n \int_{\partial D} \frac{uq \wedge (\bar\partial q)^{n-1}}{\phi^n} \tag{1.5'}$$

and

$$Q_0w(z) = \left(\frac{i}{2\pi}\right)^n \int_{\partial D} \frac{w \wedge s \wedge q \wedge (\bar\partial q)^{n-2}}{\phi(\zeta, z)^{n-1}\phi(z,\zeta)} \tag{1.6'}$$

and (1.7) and (1.8) still hold. We also notice that

$$H_\alpha : L^2_\alpha \to L^2_\alpha \cap O(D)$$

boundedly. For $\alpha > 0$ this follows by (1.4) and Shur's lemma, whereas for $\alpha = 0$, H_0 is a singular integral operator and the argument is more involved and uses Cotlar's lemma, see [KS].

We will use the solution operator Q_α above later on, but for our primary purposes we need analogues of (1.5)-(1.8) with a modification v of ϕ that we now construct.

To begin with we let $\eta_j = z_j - \zeta_j$ and put

$$-v(\zeta, z) = \rho + \Sigma\rho_j\eta_j + \frac{1}{2}\Sigma\rho_{jk}\eta_j\eta_k\ , \tag{1.9}$$

where $\rho = \rho(\zeta)$, $\rho_j = \partial\rho/\partial\zeta_j$ and so on. Clearly v satisfies (1.2) and since ρ is strictly plurisubharmonic it also satisfies (1.1) near the diagonal $\Delta \subset \bar D \times \bar D$. Let $\chi = \chi(|z - \zeta|)$ be a smooth function supported and identically 1 near Δ and put

$$-q_j(\zeta, z) = \chi\Big(\rho_j + \frac{1}{2}\sum_k \rho_{jk}\eta_k\Big) - (1 - \chi)\,\bar\eta_j\ .$$

Then we define v globally by

$$-v(\zeta, z) = \langle q(\zeta, z), z - \zeta\rangle + \rho(\zeta)\,. \tag{1.10}$$

This v coincides with v in (1.9) near Δ and (1.1) holds globally (with v instead of ϕ) on $\bar D \times \bar D$. The main reason for requiring that v be as in (1.9) near Δ is that

$$v(\zeta, z) = \overline{v(z, \zeta)} + O(|z - \zeta|^3)\,, \tag{1.11}$$

i.e. $v(\zeta, z)$ is almost conjugate symmetric. For further discussion of $v(\zeta, z)$, see Propositions 3.1 and 3.2 in Section 3. We also put $s(\zeta, z) = -q(z, \zeta)$ so that

$$-\langle s(\zeta, z), z-\zeta\rangle + \rho(z) = v(z, \zeta) = \overline{v(\zeta, z)} + O(|\eta|^3).$$

Again identifying s and q by (1,0)-forms, we can build the operators H_α and Q_α, *cf.* (1.5) and (1.6). Then (1.7) still holds, but since $H_\alpha u$ no longer has a holomorphic kernel, (1.8) cannot hold in general. However, since $q(\zeta, z)$ is holomorphic in z near Δ, we have, *cf.* Example 1 and the proof of Theorem 1 in [AnB],

$$\bar\partial Q_\alpha w = w + \mathcal{X}_\alpha w, \qquad \text{if } \bar\partial w = 0, \tag{1.8'}$$

where

$$\mathcal{X}_\alpha \bar\partial u = \int_D O((-\rho)^\alpha) \wedge \bar\partial u = \int_D O((-\rho)^{\alpha-1}) u$$

and $O((-\rho)^\alpha)$ denotes a smooth kernel which is $O((-\rho)^\alpha)$ and has bidegree (0,1) in z (for $\alpha = 0$ the last integral is over the boundary). Since clearly $\mathcal{X}_\alpha w$ is $\bar\partial$-closed if w is, we can apply any reasonable solution operator for $\bar\partial$, *e.g.* Q_α from (1.6), to $\mathcal{X}_\alpha w$ and then obtain new operators $\tilde{\mathcal{L}}_\alpha$, such that

$$\bar\partial \tilde{\mathcal{L}}_\alpha w = \bar\partial Q_\alpha w - w,$$

and $\mathcal{L}_\alpha$ such that

$$\bar\partial \mathcal{L}_\alpha u = \bar\partial Q_\alpha \bar\partial u - \bar\partial u.$$

Moreover, it follows from *e.g.* Section 5 that $\tilde{\mathcal{L}}_\alpha$ and $\mathcal{L}_\alpha$ have smooth kernels that are $O((-\rho)^\alpha)$ and $O((-\rho)^{\alpha-1})$, respectively. Finally, we put

$$K_\alpha w = Q_\alpha w - \tilde{\mathcal{L}}_\alpha w, \qquad G_\alpha u = H_\alpha u + \mathcal{L}_\alpha u. \tag{1.12}$$

Then

$$K_\alpha \bar\partial u = u - G_\alpha u \tag{1.13}$$

and

$$\bar\partial K_\alpha w = w \qquad \text{if } \bar\partial w = 0. \tag{1.14}$$

Moreover, G_α has a holomorphic kernel and maps $L^2_\alpha \to L^2_\alpha \cap O(D)$ boundedly, since H_α does. An important consequence of (1.11) (and (1.12)) that we need later on is that (letting small letters denote the corresponding kernels)

$$
\begin{aligned}
g_\alpha(\zeta,z) - \overline{g_\alpha(z,\zeta)} &\sim h_\alpha(\zeta,z) - \overline{h_\alpha(z,\zeta)} \\
&= (-\rho)^{\alpha-1}\, O\Big(\frac{|\eta|^3}{|v|^{n+\alpha+1}}\Big),
\end{aligned}
\tag{1.15}
$$

which makes it a weakly singular kernel and hence represents a (somewhat) smoothing operator (and a compact operator on L^2_α).

2. Main results.

With the notation introduced in Section 1 we can describe the boundary values of our solutions for $\partial\bar\partial$. However to describe them for $z \in D$ we also need the following notation. If $\alpha > -1$ and $(\zeta, z) \in \bar D \times \bar D$ we let

$$
h_{\alpha,j,k}(\zeta,z) = \frac{\alpha+1}{\pi}\int_{|\tau|<1} \frac{(1-|\tau|^2)^\alpha\, d\lambda(\tau)}{(1-a\bar\tau)^j\,(1-\bar a\tau)^k}\,, \tag{2.1}
$$

and if $\alpha = -1$,

$$
h_{-1,j,k}(\zeta,z) = \frac{1}{\pi}\int_{|\tau|=1} \frac{d\sigma(\tau)}{(1-a\bar\tau)^j\,(1-\bar a\tau)^k}\,, \tag{2.1$'$}
$$

where

$$
a(\zeta,z) = \frac{\sqrt{\rho(\zeta)\,\rho(z)}}{v(\zeta,z)}\,.
$$

Because of (1.1), $|a| \le 1$ with equality if and only if $\zeta = z$. Also $h_{\alpha,j,k} \equiv 1$ if z or ζ is on ∂D. Moreover, one easily verifies that $h_{\alpha,j,0} = h_{\alpha,0,k} \equiv 1$, $h_{\alpha,j,k} \sim (|v|^2/\sigma)^{j+k-\alpha-2}$ if $j+k > \alpha+2$, $h_{\alpha,j,k} \sim 1 + \log|v|^2/\sigma$ if $j+k = \alpha+2$ and $h_{\alpha,j,k} \sim 1$ if $j+k < \alpha+2$, where $\sigma = (1-|a|^2)|v|^2 = |v|^2 - \rho(\zeta)\rho(z)$, so that $\sigma \sim |z-\zeta|^2$ if $(\zeta,z) \in K \subset\subset D \times D$. More precisely, $h_{\alpha,j,k}$ can be expressed in terms of hypergeometric functions, see [An2], but we are only interested in its asymptotic behaviour. We also put $h_{\alpha,\ell} = h_{\alpha,\ell/2,\ell/2}$, so that $|h_{\alpha,j,k}| \le h_{\alpha,j+k}$ and $h_{\alpha+1,\ell+1} \sim h_{\alpha,\ell}$ if $l > \alpha+2$. Now we can state our main result (recall (1.12) for the definition of G_α).

Theorem 1. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and ρ a strictly plurisubharmonic C^4 defining function. For each integer $\alpha \geq 1$ we have operators P_α, F_α and M_α such that for smooth functions u,*

$$u = M_\alpha(i\,\partial\bar{\partial}u) + P_\alpha u + F_\alpha u\,. \tag{2.2}$$

Here

$$P_\alpha u = G_\alpha u + \bar{G}_\alpha u$$

is pluriharmonic,

$$|F_\alpha u| \lesssim \int_D \frac{(-\rho)^{\alpha-1}|u|}{|v|^{n+\alpha-1/2}}\, h_{\alpha-2,n+\alpha-1/2}\, d\lambda\,, \tag{2.3}$$

$$\begin{aligned} M_\alpha\theta = &\sum_{\substack{j+k=n+\alpha\\ j,k\geq 1}} c_{jk} \int_D \frac{(-\rho)^{\alpha+1}\theta\wedge(i\,\partial\bar{\partial}\rho)^{n-1}}{v^j\bar{v}^k}\, h_{\alpha,j,k} \\ &- \sum_{\substack{j+k=n+\alpha+1\\ j,k\geq 1}} c'_{jk} \\ &\int_D \frac{(-\rho)^{\alpha+1}\theta\wedge\Big(-\rho(z)i\partial\bar{\partial}\rho + i(n-1)\,\partial\bar{v}\wedge\bar{\partial}v\Big)\wedge(i\partial\bar{\partial}\rho)^{n-2}}{v^j\bar{v}^k}\, h_{\alpha,j,k} \\ &+ \sum_{\substack{j+k=n+\alpha\\ j,k\geq 1}} c''_{jk} \\ &\int_D \frac{(-\rho)^{\alpha+2}\theta\wedge\Big(-\rho(z)i\partial\bar{\partial}\rho + i(n-1)\partial\bar{v}\wedge\bar{\partial}v\Big)\wedge(i\partial\bar{\partial}\rho)^{n-2}}{v^{j+1}\bar{v}^{k+1}} \\ &\cdot h_{\alpha+1,j+1,k+1} + R_\alpha\theta\,, \end{aligned} \tag{2.4}$$

where

$$|R_\alpha\theta| \lesssim \int_D \frac{(-\rho)^\alpha(-\rho|\theta| + \sqrt{-\rho}\,(|\theta\wedge\partial\rho| + |\theta\wedge\bar{\partial}\rho|))}{|v|^{n+\alpha-1/2}}\, h_{\alpha-1,n+\alpha-1/2}\,. \tag{2.5}$$

In particular,

$$|M_\alpha\theta| \lesssim \int_D \frac{(-\rho)^\alpha}{|v|^{n+\alpha}}$$

$$\cdot\Big(-\rho|\theta| + \sqrt{-\rho}\,(|\theta\wedge\partial\rho| + |\theta\wedge\bar\partial\rho|) + |\theta\wedge\partial\rho\wedge\bar\partial\rho|\Big)\, h_{\alpha-1,n+\alpha}\,. \tag{2.6}$$

The exact values of the constants are

$$c_{j,k} = \frac{1}{(2\pi)^n}\,\frac{(n+\alpha)\,(n+\alpha-j-1)!\,(n+\alpha-k-1)!}{(n-1)!\,(\alpha+1)!\,(n+\alpha-j-k)!}\,,$$

$$c'_{j,k} = \frac{1}{(2\pi)^n}\,\frac{(n+\alpha-j)!\,(n+\alpha-k)!}{(n-1)!\,(\alpha+1)!\,(n+\alpha-j-k+1)!}$$

and

$$c''_{j,k} = \frac{1}{(2\pi)^n}\,\frac{j\,k\,(n+\alpha-j-1)!\,(n+\alpha-k-1)!}{(n-1)!\,(\alpha+2)!\,(n+\alpha-j-k)!}\,.$$

Notice that the kernel for M_α is $\sim |h_{\alpha-1,n+\alpha}| \sim |\zeta - z|^{-(2n-2)}$ if $n > 1$ (and $\sim \log|\zeta - z|$ for $n = 1$) when (ζ, z) is in the interior of $D \times D$.

REMARK 1. If D is the ball, then M_α is the solution operator M_α from [An1] and [An2], G_α is the L^2_α-Bergman projection, $F_\alpha u = -G_\alpha u(0)$ so that $P_\alpha + F_\alpha$ is the L^2_α-orthogonal pluriharmonic projection.

REMARK 2. For $\alpha = 0$, we have the same result as in Theorem 1 for $z \in \partial D$; *i.e.* everything holds for $\alpha = 0$ if $z \in \partial D$, and

$$|F_0 u(z)| \lesssim \int_{\partial D} \frac{|u(\zeta)|}{|v|^{n-1/2}} + \int_D \frac{|u(\zeta)|}{|v|^{n+1/2}}\,, \qquad z \in \partial D\,.$$

Our main application of Theorem 1 is to estimate solutions to the $\partial\bar\partial$-equation, and to this end we need the following technical result:

Proposition 2.1. *Under the conditions of Theorem* 2.1,

$$|F^k_\alpha u(z)| \lesssim \int_D \frac{(-\rho)^{\alpha-1}|u|}{|v|^{n+\alpha-k/2}}\, h_{\alpha-2,n+\alpha-k/2}\, d\lambda\,, \tag{2.7}$$

if $k < 2(n+\alpha) - 1$,

$$|F_\alpha^k M_\alpha \theta(z)| \lesssim \int_D \frac{(-\rho)^\alpha}{|v|^{n+\alpha-k/2}} \Big(-\rho|\theta| + \sqrt{-\rho}\,(|\theta\wedge\partial\rho| + |\theta\wedge\bar\partial\rho|)$$

$$+ |\theta\wedge\partial\rho\wedge\bar\partial\rho|\Big)\, h_{\alpha-1,n+\alpha-k/2}\, d\lambda\,, \tag{2.8}$$

if $k < 2(n+\alpha-1)$, *and if k is large enough* F_α^k *maps* L_α^1 *into* $C(\bar D)$ *boundedly.*

This proposition is proved in Section 5.

Now, suppose u_0 is a solution to $i\,\partial\bar\partial u = \theta$ in L_α^1. Then $u_1 = M_\alpha\theta + F_\alpha u_0$ is another solution, and by iteration so is

$$u = M_\alpha\theta + F_\alpha M_\alpha\theta + F_\alpha^2 M_\alpha\theta + \cdots + F_\alpha^m M_\alpha\theta + F_\alpha^m u_0\,.$$

From Proposition 2.1 we then get

Theorem 2. *If* $u_0 \in L_\alpha^1$ *solves* $i\,\partial\bar\partial u_0 = \theta$ *so does*

$$u = M_\alpha\theta + \mathcal{R}_\alpha\theta + T_\alpha u_0\,,$$

where $M_\alpha\theta$ *is as in Theorem* 1, $T_\alpha u_0 \in C(\bar D)$ *and*

$$|\mathcal{R}_\alpha\theta| \lesssim \int_D \frac{(-\rho)^\alpha}{|v|^{n+\alpha-1/2}} \cdot \Big(-\rho|\theta| + \sqrt{-\rho}\,(|\theta\wedge\partial\rho| + |\theta\wedge\bar\partial\rho|) + |\theta\wedge\partial\rho\wedge\bar\partial\rho|\Big) \cdot h_{\alpha-1,n+\alpha-1/2}\,.$$

In order to apply Theorem 2 to get various estimates of solutions to $i\,\partial\bar\partial u = \theta$ we need an a priori solution u_0 in some $L_\alpha^1(D)$. This is provided by

Theorem 3. *Assume that* θ *is a d-exact* $(1,1)$*-form (current) such that*

$$(-\rho)^\alpha\Big(-\rho|\theta| + \sqrt{-\rho}\,(|\theta\wedge\partial\rho| + |\theta\wedge\bar\partial\rho|) + |\theta\wedge\partial\rho\wedge\bar\partial\rho|\Big) \tag{2.9}$$

is a finite measure in D, $\alpha \geq 0$. Then there is a solution $u \in L^1_\alpha(D)$ to $i\,\partial\bar\partial u = \theta$.

If $\alpha = 0$ this is the Henkin-Skoda theorem, [H] and [S], and the case $\alpha > 0$ is due to Dautov and Henkin, [DH]. This theorem was the first outcome of solution formulas for the $\bar\partial$-equation with weights. With the modern technique the proof is rather simple and for the readers convenience we sketch it when α is an integer.

SKETCH OF PROOF. Assume first that $\alpha = 0$ and that θ is a d-exact real (1,1)-current such that

$$|||\theta||| = -\rho|\theta| + \sqrt{-\rho}\,(|\theta \wedge \partial\rho| + |\theta \wedge \bar\partial\rho|) + |\theta \wedge \partial\rho \wedge \bar\partial\rho|$$

is a finite measure (if θ is positive this is equivalent to $\int_D -\rho\,\mathrm{trace}\,\theta < +\infty$). First we look for a solution to $id\omega = \theta$ such that

$$\int_D |\omega| + \frac{1}{\sqrt{-\rho}}\,(|\partial\rho \wedge \omega| + |\bar\partial\rho \wedge \omega|) < +\infty\,. \tag{2.10}$$

If D is convex, one can use the simple homotopy for d obtained by contracting D to a point, *i.e.* $\omega = i\int_0^1 h^*\theta\,dt$, where $h(t,z) = tz$ (assuming $0 \in D$). For a general strictly pseudoconvex domain D one can piece together such local solutions to a global one by a cohomology argument. It is at this point the d-exactness of θ comes into play. If $\omega_{0,1}$ is the (0,1)-part of ω, then for bidegree reasons, $\bar\partial\omega_{0,1} = 0$. We may assume that ω is chosen so that $\bar\omega_{0,1} = -\omega_{1,0}$. We now apply the solution operator Q_α from (1.6) and put $v = Q_1\omega_{0,1}$. Then (1.3), (2.10) and Fubini's theorem immediately yield that

$$\int_{\partial D} |v|\,d\sigma \lesssim \int_D |\omega_{0,1}| + \frac{1}{\sqrt{-\rho}}\,|\omega_{0,1} \wedge \partial\rho| < \infty\,.$$

Putting $u = 2\,\mathrm{Re}\,v = v + \bar v$, we get a solution $u \in L^1(\partial D)$ to $i\partial\bar\partial u = \theta$.

It follows from the $\mathbb{R}^{2n}$-Riesz decomposition that also $u \in L^1(D)$; however we show this with another argument that also covers the case $\alpha \in \mathbb{Z}_+$. Let $\tilde D \subset \mathbb{C}^{n+\alpha}$ be the strictly pseudoconvex domain defined by $\tilde D = \{(z, z') \in \mathbb{C}^{n+\alpha} : \tilde\rho(z, z') = \rho(z) + |z'|^2 < 0\}$. Then (2.9) implies, *cf.* Section 5,

$$\int_{\tilde D} -\tilde\rho|\theta| + \sqrt{-\tilde\rho}\,(|\theta \wedge \partial\tilde\rho| + |\theta \wedge \bar\partial\tilde\rho|) + |\theta \wedge \partial\tilde\rho \wedge \bar\partial\tilde\rho| < +\infty\,.$$

Hence, by the case $\alpha = 0$ applied to $\theta(z, z') = \theta(z)$ in $\tilde{D}$, we obtain a solution $u \in L^1(\partial\tilde{D})$. We may assume that u is independent of z' and then, by Lemma 5.1, $\int_D (-\rho)^\alpha |u| \sim \int_{\partial\tilde{D}} |u| \, d\tilde{\sigma} < +\infty$.

Recall that a measure μ in D is a Carleson measure if $\mu(Q_t(p)) \lesssim t^n$ where $Q_t(p)$ are the Koranyi balls in D. As an application of Theorem 2 we can prove Varopoulos' theorem:

Theorem 4. *Assume that θ is d-exact and that $|||\theta|||$ is a Carleson measure. Then there is a solution $u \in \mathrm{BMO}\,(\partial D)$ to $i\,\partial\bar{\partial}u = \theta$.*

By Theorem 3, there is a solution $u_0 \in L^1(D)$ and hence $T_\alpha u_0 \in C(\bar{D})$. It is easy to see that $\mathcal{R}_\alpha\theta$ is bounded on ∂D, and a standard estimation of the integral defining $M_\alpha\theta$ shows that $M_\alpha\theta \in \mathrm{BMO}\,(\partial D)$. For the details of this argument see Section 6. Thus we have obtained a relatively simple proof of Varopoulos' theorem that avoids the delicate task of solving the Poincaré equation $id\omega = \theta$ with Carleson estimates, *cf.* Section 0.

By interpolation, Theorems 3 and 4 imply that there is a solution in $L^p(\partial D)$, $1 < p < \infty$, if $|||\theta||| \in W^{1-1/p}$. Here W^α are the interpolation spaces between the finite measures W^0 and the Carleson measures W^1 in D. This can also be seen by simple estimates of the integrals, using the following charctacterization of W^α, see [AmB],

$$\mu \in W^\alpha \quad \text{if and only if} \quad \mu = k\, d\tau\,,$$

where

$$\tau \in W^1 \quad \text{and} \quad k \in L^{1/(1-\alpha)}(d\tau)\,.$$

EXAMPLE 1. If ∂D has enough regularity then the operator T_α in Theorem 2 will map L^1_α into $C^k(\bar{D})$, and so our technique can be used also to study $C^{n+\alpha}$-regularity for the solution. However, we do not pursue these things in more detail in this paper.

We will go one step further and show that in fact $M_\alpha\theta$ is the principal term of the L^2_α-minimal solution $N_\alpha\theta$ to $i\,\partial\bar{\partial}u = \theta$, but to this end we first have to study the L^2_α-orthogonal projection

$$\Pi_\alpha : L^2_\alpha \to L^2_\alpha \cap \mathcal{H}\,,$$

where $\mathcal{H}$ denotes the space of pluriharmonic functions in D. First we note that, *cf.* (2.2),

$$u = (P_\alpha + F_\alpha)u \tag{2.11}$$

if u is pluriharmonic, $\alpha \geq 1$.

Since Π_α is pluriharmonic, $P_\alpha \Pi_\alpha = \Pi_\alpha - F_\alpha \Pi_\alpha$ and since $P_\alpha : L^2_\alpha \to L^2_\alpha \cap \mathcal{H}$, *cf.* Section 1, $\Pi_\alpha P_\alpha = P_\alpha$. Taking adjoints we get $P^*_\alpha = P^*_\alpha \Pi^*_\alpha$, and after subtracting $\Pi_\alpha - F_\alpha \Pi_\alpha - P^*_\alpha = (P_\alpha - P^*_\alpha)\Pi_\alpha$ and thus

$$\Pi_\alpha = P_\alpha + F_\alpha + A_\alpha(I - \Pi_\alpha), \tag{2.12}$$

if $A_\alpha = P^*_\alpha - P_\alpha - F_\alpha$. Note that, since $P_\alpha = G_\alpha + \bar{G}_\alpha$, (1.15) and (2.3) imply that A_α is weakly singular. By iteration we get

Theorem 5. *The L^2_α-orthogonal projection $\Pi_\alpha : L^2_\alpha \to L^2_\alpha \cap \mathcal{H}$ can be written*

$$\begin{aligned} \Pi_\alpha = P_\alpha + F_\alpha + A_\alpha(I - P_\alpha - F_\alpha) - A^2_\alpha(I - P_\alpha - F_\alpha) \\ + \cdots + (-1)^m A^{m-1}_\alpha(I - P_\alpha - F_\alpha) + (-1)^{m+1} A^m_\alpha(I - \Pi_\alpha). \end{aligned} \tag{2.13}$$

Since A^m_α maps L^2_α into $C(\bar{D})$ (or even into $C^k(\bar{D})$ if ∂D is sufficiently smooth) if m is large enough, *cf.* Proposition 2.1, $\Pi_\alpha u$ has the same regularity as $P_\alpha u$; *e.g.* if ∂D is C^∞ then P_α maps $C^\infty(\bar{D})$ into $C^\infty(\bar{D}) \cap \mathcal{H}$ and hence also Π_α does.

EXAMPLE 2. Formula (2.11) also holds for $\alpha = 0$, *cf.* Remark 2, for $z \in \partial D$. Then $P_0 u(z)$, $z \in \partial D$, has to be interpreted as the boundary values of the pluriharmonic function $P_0 u(z)$. Since $P_0 u$ only depends on the boundary values of u, we can take any operator V, *e.g.*, the $\mathbb{R}^{2n}$-Poisson integral, which represents a pluriharmonic function in terms of its boundary values, and then

$$U = P_0 u + F_0 V u \tag{2.14}$$

is a representation of the pluriharmonic function U in terms of its boundary values u, $P_0 u$ is pluriharmonic and one can show that $F_0 V : L^2(\partial D) \to L^2(\partial D)$ is compact. When $n = 1$,

$$P_0 u(z) = 2 \operatorname{Re} \frac{1}{2\pi i} \int_{\partial D} \frac{u(\zeta)\, d\zeta}{\zeta - z}$$

is the classical double layer potential of u, which provides an approximative solution of Dirichlet's problem.

Finally we state our result about the L^2_α-minimal solution $N_\alpha\theta$ to $i\,\partial\bar\partial u = \theta$.

Theorem 6. *If θ is exact in D, then the L^2_α-minimal solution $N_\alpha\theta$ of $i\,\partial\bar\partial u = \theta$ is given by*

$$
\begin{aligned}
N_\alpha\theta &= M_\alpha\theta - A_\alpha M_\alpha\theta + A_\alpha^2 M_\alpha\theta \\
&\quad + \cdots + (-1)^m A_\alpha^{m-1} M_\alpha\theta + (-1)^{m+1} A_\alpha^m N_\alpha\theta\,.
\end{aligned}
\tag{2.15}
$$

PROOF. This follows immediately from (2.2) and (2.13) once one has noticed that if u is any solution, then

$$N_\alpha\theta = u - \Pi_\alpha u\,.$$

3. Proof of Theorem 1 when $z \in \partial D$.

We first assume that $z \in \partial D$. The general case will then be obtained by applying the first result to a certain set $\tilde D \subset \mathbb{C}^{n+1}$ where $D = \mathbb{C}^n \cap \tilde D$. Our starting point is the relation ($w = \bar\partial u$)

$$u = G_\alpha u + K_\alpha w\,, \tag{3.1}$$

see (1.13). For convenience we recall that

$$
\begin{aligned}
G_\alpha u(z) &= \frac{\Gamma(n+\alpha)}{n!\,\Gamma(\alpha)}\left(\frac{i}{2\pi}\right)^n \\
&\quad \cdot \int_D \frac{(-\rho)^{\alpha-1} u[-\rho\bar\partial q - nq \wedge \bar\partial\rho] \wedge (\bar\partial q)^{n-1}}{v^{n+\alpha}(\zeta,z)} \\
&\quad + \text{nice operator}
\end{aligned}
\tag{3.2}
$$

and, for $z \in \partial D$,

$$
\begin{aligned}
K_\alpha w(z) &= \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)\,(n-1)!}\left(\frac{i}{2\pi}\right)^n \\
&\quad \cdot \int_D \frac{(-\rho)^{\alpha-1} s \wedge w \wedge [-\rho\bar\partial q - (n-1)q \wedge \bar\partial\rho] \wedge (\bar\partial q)^{n-2}}{v(\zeta,z)^{n+\alpha-1} v(z,\zeta)} \\
&\quad + \text{nice operator}\,.
\end{aligned}
\tag{3.3}
$$

The objective now is to generalize the argument given at the end of Section 0, *i.e.* rewrite $K_\alpha w$ in a appropriate manner to obtain (2.2), and to this end we need

Proposition 3.1. *If ρ is C^3 and v, s, q are defined as in Section* 1, *then*

$$\partial\bar v = -s + O(|\eta|) = q + O(|\eta|) = -\partial\rho + O(|\eta|) \tag{3.4}$$

$$s \wedge q = O(|\eta|)\,, \qquad \partial\rho \wedge \partial\bar v = O(|\eta|)\,, \tag{3.5}$$

$$\bar\partial q = \partial\bar\partial\rho + O(|\eta|)\,, \tag{3.6}$$

$$\partial\rho \wedge \partial\bar v = s \wedge q + O(|\eta|^2)\,, \tag{3.7}$$

$$v(z,\zeta) = \overline{v(\zeta,z)} + O(|\eta|^3)\,, \tag{3.8}$$

and

$$\partial v(\zeta,z) = O(|\eta|^2)\,. \tag{3.9}$$

and

Proposition 3.2. *If ρ is C^3, v is defined by* (1.9) *and $z \in \partial D$, then*

$$\begin{aligned}(n-1)&w \wedge d\bar v \wedge \bar\partial\rho \wedge \partial\rho \wedge (\partial\bar\partial\rho)^{n-2}\\ &= \Big(-(n-1)w \wedge d\bar v \wedge dv \wedge \partial\rho - \rho w \wedge d\bar v \wedge \partial\bar\partial\rho\\ &\quad - vw \wedge d\bar v \wedge \partial\bar\partial\rho + \bar v w \wedge \partial\rho \wedge \partial\bar\partial\rho\Big) \wedge (\partial\bar\partial\rho)^{n-2} + |w|\,O(|\eta|^3)\end{aligned}$$

and O is in $C^1(\bar D \times \partial D)$.

All differentials are with respect to ζ. Note that (3.8) means that $v(\zeta,z)$ is nearly conjugate symmetric (self-adjoint) and (3.9) means that it is nearly antiholomorphic in ζ.

The equations (3.4) to (3.6) follow quite easily from the definitions. Clearly (3.9) follows from (3.8), which is wellknown and used *e.g.* in [KS]. It can be verified by a straightforward computation but it also follows immediately from (3.11) below, which we anyway need in the proof of the much harder Proposition 3.2.

The equation (3.7) was first used in [Ar] and can be verified as follows:

PROOF OF (3.7). Since $\sum q_k\eta_k = -v(\zeta,z) - \rho(\zeta)$ and, by (3.8), $\sum s_k\eta_k = -v(\zeta,z) - \rho(z) + O(|\eta|^3)$, we have for small η, $\sum \eta_k\partial q_k - q = -\partial\rho$ and $\sum \eta_k\partial s_k - s = -\partial\bar{v} + O(|\eta|^2)$. Hence, since by (3.4) $s = -q + O(|\eta|)$,

$$\partial s \wedge \partial\bar{v} = q \wedge s + \sum \eta_k\, \partial(q_k + s_k) + O(|\eta|^2)\,.$$

Now

$$q_k = -\rho_k(\zeta) - \frac{1}{2}\sum_j \rho_{jk}(\zeta)\,\eta_j$$

and

$$s_k = \rho(z) - \frac{1}{2}\sum_j \rho_{jk}(z)\,\eta_j$$

so that $\partial(q_k + s_k) = O(|\eta|)$ and thus (3.7) follows.

PROOF OF PROPOSITION 3.2. By (3.9) and for bidegree reasons it is enough to prove that $A \wedge (\partial\bar{\partial}\rho)^{n-2} = O(|\eta|^3)$, where

$$A = (n-1)w \wedge \partial\bar{v} \wedge \bar{\partial}(\rho + v) \wedge \partial\rho + (\rho + v)w \wedge \partial\bar{v} \wedge \partial\bar{\partial}\rho - \bar{v}w \wedge \partial\rho \wedge \partial\bar{\partial}\rho\,.$$

To simplify the computation we want to choose suitable holomorphic coordinates. The definition (1.9) depends on the choice of coordinates but if v' is defined by (1.9) with respect to new holomorhpic coordinates z', we claim that

$$v = v' + O(|\eta|^3) \quad \text{and} \quad \bar{\partial}_\zeta v = \bar{\partial}_\zeta v' + O(|\eta|^3)\,. \tag{3.10}$$

Let as assume (3.10) for a moment and complete the proof of Proposition 3.2. Let $\zeta \in D$ be fixed. By (3.10) we may assume that v is defined with respect to holomorphic coordinates such that

$$\partial\bar{\partial}\rho(z) = \sum d\zeta_i \wedge d\bar{\zeta_i} + O(|\eta|^2)$$

and hence also $\rho_{ij\bar{k}}(\zeta) = \rho_{\bar{i}j\bar{k}}(\zeta) = 0$. By linearity we may assume that $w = d\bar{\zeta_1}$. Now $(\partial\bar{\partial}\rho)^{n-2}$ is a sum of terms $\wedge_i(d\zeta_i \wedge d\bar{\zeta_i})$ where i assumes all but two of the numbers $1,\ldots,n$, and only the differentials in A with respect to these to variables make any contribution to such a term.

Thus if we let $\gamma = d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_k \wedge d\bar{\zeta}_k \wedge (\partial\bar{\partial}\rho)^{n-2}$ (γ is independent of k), we have

$$\begin{aligned}
&w \wedge \partial\bar{v} \wedge \bar{\partial}(\rho + v) \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\
&\quad = \sum_{k=2}^{n} \bar{d}\bar{\zeta}_1 \wedge (\partial_{1k}\rho + \bar{\eta}_1 d\zeta_1 + \bar{\eta}_k d\zeta_k) \wedge \eta_k d\bar{\zeta}_k \wedge \partial_{1k}\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\
&\quad = \sum_{k=2}^{n} (-\rho_1\eta_k\bar{\eta}_k + \rho_k\bar{\eta}_1\eta_k)\,\gamma\,,
\end{aligned}$$

$$\begin{aligned}
&(\rho + v)w \wedge \partial\bar{v} \wedge \partial\bar{\partial}\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\
&\qquad\qquad = \sum_{k=2}^{n} (\rho + v)\, d\zeta_1 \wedge \partial_1\bar{v} \wedge d\zeta_k \wedge d\bar{\zeta}_k \wedge (\partial\bar{\partial}\rho)^{n-2} \\
&\qquad\qquad = (n-1)\,(\rho + v)\,(\rho_1 + \bar{\eta}_1)\,\gamma
\end{aligned}$$

and

$$\begin{aligned}
\bar{v} \wedge \partial\rho \wedge \partial\bar{\partial}\rho \wedge (\partial\bar{\partial}\rho)^{n-2} &= \sum_{k=2}^{n} \bar{v}\, d\bar{\zeta}_1 \wedge \partial_1\rho \wedge d\zeta_k \wedge d\bar{\zeta}_k \wedge (\partial\bar{\partial}\rho)^{n-2} \\
&= -(n-1)\,\bar{v}\,\rho_1\,\gamma\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
A \wedge (\partial\bar{\partial}\rho)^{n-2} &= (n-1)\Big(\rho_1(\rho - v - \bar{v} - \sum_{k=2}^{n} \eta_k\bar{\eta}_k) \\
&\qquad + (\rho + v)\bar{\eta}_1 + \sum_{k=2}^{n} \rho_k\bar{\eta}_1\eta_k\Big)\gamma + O(|\eta|^3) \\
&= (n-1)\rho_1(-\rho(z) + O(|\eta|^3)) + O(|\eta|^3) \\
&= O(|\eta|^3)
\end{aligned}$$

as $\rho(z) = 0$, and the proof is complete.

PROOF OF (3.10). We first assume that ρ is real analytic and let $u(\zeta, z)$ be the unique function that equals $-\rho$ for $\zeta = z$, is holomorphic in z and antiholomorphic in ζ. Then $\overline{u(\zeta, z)} = u(z, \zeta)$. Since v (and v') is nothing but the Taylor expansion of u up to second order, we have

$$v = u + O(|\eta|^3)\,. \tag{3.11}$$

Note also that $\bar{\partial}_\zeta v(\zeta, z) = \bar{\partial}_\zeta u(\zeta, z) + O(|\eta|^3)$. The same formulas hold for v' and since u and $\bar{\partial} u$ are invariantly defined, (3.10) now follows if ρ is real analytic. The general case can be obtained by approximation.

We now replace $v(z,\zeta) = \bar{v}(\zeta, z) + O(|\eta|^3)$ by $\bar{v}$ in (3.3), $\bar{\partial} q$ by $\partial\bar{\partial}\rho$ and $s \wedge w$ by $\partial\rho \wedge \partial\bar{v}$. Then we get

$$K_\alpha w = \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)(n-1)!}\Big(\frac{i}{2\pi}\Big)^n\Big(-\int_D \frac{(-\rho)^\alpha d\bar{v} \wedge w \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^{n+\alpha-1}\bar{v}}$$

$$+(n-1)\int_D \frac{(-\rho)^{\alpha-1}\partial\rho \wedge w \wedge d\bar{v} \wedge \bar{\partial}\rho \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^{n+\alpha-1}\bar{v}}\Big) + \tilde{F}_\alpha w\,, \tag{3.12}$$

where

$$\tilde{F}_\alpha w = \sum \int \frac{(-\rho)^{\alpha-1+\ell} w \wedge O(|\eta|^{2m+1})}{v^j \bar{v}^k} + \int \frac{(-\rho)^{\alpha-1+\ell} w \wedge \bar{\partial}\rho \wedge O(|\eta|^{2m})}{v^j \bar{v}^k}\,,$$

if $\ell, j, k, m \geq 0$, and $\ell + m - (j+k) \geq 1 - n - \alpha$, and $O(|\eta|^\beta)$ is in $C^1(\bar{D} \times \bar{D})$. To obtain (3.12) we have used Proposition 3.1 and that

$$\frac{1}{\bar{v} + O(|\eta|^3)} - \frac{1}{\bar{v}} = O\Big(\frac{|\eta|^3}{|v|^3}\Big).$$

We need the following auxiliary notation:

$$A_{\alpha,j,k} = \Big(\frac{i}{2\pi}\Big)^n \int_D \frac{(-\rho)^\alpha w \wedge d\bar{v} \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^j \bar{v}^k}\,,$$

$$B_{\alpha,j,k} = \Big(\frac{i}{2\pi}\Big)^n \int_D \frac{(-\rho)^\alpha w \wedge d\bar{v} \wedge \bar{\partial}\rho \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k}\,,$$

$$C_{\alpha,j,k} = \Big(\frac{i}{2\pi}\Big)^n \int_D \frac{(-\rho)^\alpha \partial w \wedge \partial\bar{v} \wedge \bar{\partial} v \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k}$$

and

$$D_{\alpha,j,k} = \Big(\frac{i}{2\pi}\Big)^n \int_D \frac{(-\rho)^\alpha \partial w \wedge (\partial\bar{\partial}\rho)^{n-1}}{v^j \bar{v}^k}\,.$$

Hence (3.12) can be written

$$K_\alpha w = \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)(n-1)!} A_{\alpha,n+\alpha-1,1} - \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)(n-2)!} B_{\alpha,1,n+\alpha-1} + \tilde{F}_\alpha w\,.$$

Let $R_\alpha = R_\alpha\theta$ denote any term that satisfies (2.5) (since $z \in \partial D$, $h_{\alpha,n+\alpha-1/2} = 1$). Then we have

Lemma 3.3. *If $j+k = n+\beta+1$ and $\beta \geq \alpha - 1$ then*

$$B_{\beta,j,k} = -\frac{1}{\beta+1} A_{\beta+1,j,k} + \frac{j}{(\beta+1)(\beta+2)} C_{\beta+2,j+1,k} + \tilde{F}_\alpha + R_\alpha\,. \tag{3.13}$$

Thus

$$K_\alpha w = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)(n-1)!} A_{\alpha,n+\alpha-1,1} + \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+2)(n-2)!} C_{\alpha+1,n+\alpha,1} + \tilde{F}_\alpha + R_\alpha\,. \tag{3.14}$$

By repeated use of

Lemma 3.4. *If $j+k = n+\alpha$ then*

$$\begin{aligned} A_{\alpha,j,k} = {} & \frac{k}{n+\alpha-k} A_{\alpha,j-1,k+1} + \frac{n+\alpha}{(n+\alpha-k)(\alpha+1)} D_{\alpha+1,j,k} \\ & + \frac{jk(n-1)}{(\alpha+1)(\alpha+2)(n+\alpha-k)} C_{\alpha+2,j+1,k+1} \\ & - \frac{k(n-1)}{(\alpha+1)(n+\alpha-k)} C_{\alpha+1,j,k+1} + \tilde{F}_\alpha + R_\alpha\,, \end{aligned}$$

and recalling that $\theta = i\,\partial w$ we obtain from (3.14) that ($z \in \partial D$)

$$K_\alpha w = M_\alpha + A_{\alpha,0,n+\alpha} + \tilde{F}_\alpha \tag{3.15}$$

and then by (3.1), (2.2) is proved since we have

Lemma 3.5.

$$A_{\alpha,0,n+\alpha} = \bar{G}_\alpha u + F_\alpha u \tag{3.16}$$

and

$$\tilde{F}_\alpha w = F_\alpha u\,. \tag{3.17}$$

As $-\rho \lesssim |v|$ and $|\eta| \lesssim \sqrt{|v|}$, (2.6) follows from (2.4) by (3.4), and Theorem 1 is completely proved for $z \in \partial D$.

The rest of this paragraph is devoted to the proofs of Lemmas 3.3 to 3.5. Lemmas 3.3 and 3.5 are obtained only by some elementary integrations by parts whereas the proof of Lemma 3.4 is somewhat involved and depends on the crucial Proposition 3.2.

PROOF OF LEMMA 3.3. Note that

$$B_{\beta,j,k} = \int_D \frac{d(-\rho)^{\beta+1} \wedge w \wedge d\bar{v} \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k}\,.$$

Thus an integration by parts yields

$$\begin{aligned} B_{\beta,j,k} = A_{\beta+1,j,k} &+ \int_D \frac{(-\rho)^{\beta+1} \wedge w \wedge d\bar{v} \wedge dv \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^{j+1}\bar{v}^k} \\ &+ \int_D \frac{(-\rho)^{\beta+1} \partial w \wedge d\bar{v} \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^j \bar{v}^k}\,. \end{aligned} \tag{3.18}$$

To handle the first integral, notice that

$$\begin{aligned} &(-\rho)^{\beta+1} w \wedge d\bar{v} \wedge dv \wedge \partial\rho \wedge (\partial\bar{\partial}\rho)^{n-2} \\ &\qquad = \frac{1}{\beta+2} d(-\rho)^{\beta+2} \wedge w \wedge d\bar{v} \wedge dv \wedge (\partial\bar{\partial}\rho)^{n-2} \\ &\qquad\quad + (-\rho^{\beta+1}) w \wedge \bar{\partial}\rho \wedge O(|\eta|^2)\,, \end{aligned}$$

and so after an integration by parts, the integral becomes

$$-\frac{1}{\beta+2} \int \frac{(-\rho)^{\beta+2} \wedge \partial w \wedge d\bar{v} \wedge dv \wedge (\partial\bar{\partial}\rho)^{n-2}}{v^{j+1}\bar{v}^k} + \tilde{F}_\alpha\,.$$

However, for bidegree reasons and (3.9),

$$\partial w \wedge d\bar v \wedge dv \wedge (\partial\bar\partial\rho)^{n-2} = \partial w \wedge \partial\bar v \wedge \bar\partial v \wedge (\partial\bar\partial\rho)^{n-2} + \partial w \wedge O(|\eta|^4)$$

and hence the first integral in (3.18) is

$$C_{\beta+2,j+1,k} + \tilde F_\alpha + R_\alpha .$$

Now consider the last integral in (3.18). Again, for bidegree reasons, $d\bar v \wedge \partial\rho$ can be replaced by $\bar\partial v \wedge d\rho$ and then an integration by parts yields

$$\frac{1}{\beta+2}\int_D (-\rho)^{\beta+2}\partial w \wedge d\frac{\partial\bar v}{v^j\bar v^k} \wedge (\partial\bar\partial\rho)^{n-2}$$

which is an R_α if $\beta \ge \alpha - 1$ and $j + k = n + \beta + 1$.

PROOF OF LEMMA 3.4. In this proof $\equiv$ means equality modulo terms R_α and $\tilde F_\alpha$. By Lemma 3.3,

$$A_{\alpha+1,j,k+1} \equiv -(\alpha+1)\,B_{\alpha,j,k+1} + \frac{j}{\alpha+2}\,C_{\alpha+2,j+1,k+1} . \tag{3.19}$$

Now we apply Proposition 3.2 to the B-term in (3.19) and get, using the same argument as when handling the first integral in (3.18) in the proof of Lemma 3.3,

$$\begin{aligned} A_{\alpha+1,j,k+1} \equiv &- C_{\alpha+1,j,k+1} + \frac{j}{\alpha+2}\,C_{\alpha+2,j+1,k+1} \\ &- \frac{\alpha+1}{n-1}\,A_{\alpha+1,j,k+1} + \frac{\alpha+1}{n-1}\,A_{\alpha,j-1,k+1} + \frac{\alpha+1}{n-1}\,A_{\alpha,j,k} . \end{aligned}$$

Solving for $A_{\alpha+1,j,k+1}$ yields

$$\begin{aligned} A_{\alpha+1,j,k+1} \equiv &-\frac{n-1}{n+\alpha}\,C_{\alpha+1,j,k+1} + \frac{j(n-1)}{(n+\alpha)(\alpha+2)}\,C_{\alpha+2,j+1,k+1} \\ &+ \frac{\alpha+1}{n+\alpha}\,A_{\alpha,j-1,k+1} + \frac{\alpha+1}{n+\alpha}\,A_{\alpha,j,k} . \end{aligned} \tag{3.20}$$

Note that

$$\begin{aligned} A_{\alpha,j,k} &\equiv -\int_D \frac{(-\rho)^\alpha w \wedge \partial\rho \wedge (\partial\bar\partial\rho)^{n-1}}{v^j\bar v^k} \\ &\equiv -\frac{1}{\alpha+1}\int_D \frac{d(-\rho)^{\alpha+1} \wedge w \wedge (\partial\bar\partial\rho)^{n-1}}{v^j\bar v^k} , \end{aligned}$$

and hence an integration by parts yields

$$A_{\alpha,j,k} \equiv \frac{k}{\alpha+1} A_{\alpha+1,j,k+1} + \frac{1}{\alpha+1} D_{\alpha+1,j,k} \,. \tag{3.21}$$

Combining (3.20) with (3.21) and solving for $A_{\alpha,j,k}$ finally one gets Lemma 3.4.

PROOF OF LEMMA 3.5. For bidegree reasons, $w \wedge d\bar{v} = \bar{\partial} u \wedge d\bar{v}$ can be replaced by $du \wedge \partial \bar{v}$ in the definition of $A_{\alpha,0,n+\alpha}$ and hence

$$\begin{aligned} A_{\alpha,0,n+\alpha} &= \Big(\frac{i}{2\pi}\Big)^n \int_D \frac{(-\rho)^\alpha du \wedge \partial\bar{v} \wedge (\partial\bar{\partial}\rho)^{n-1}}{\bar{v}^{n+\alpha}} \\ &= \Big(\frac{i}{2\pi}\Big)^n \alpha \int_D \frac{(-\rho)^{\alpha-1} u \wedge \partial\bar{v} \wedge \bar{\partial}\rho \wedge (\partial\bar{\partial}\rho)^{n-1}}{\bar{v}^{n+\alpha}} + F_\alpha u \\ &= \bar{G}_\alpha u + F_\alpha u \,, \end{aligned}$$

cf. (3.2) and Proposition 3.1. This proves (3.16), and (3.17) is obtained in the same manner.

A FINAL REMARK ABOUT THE CASE $\alpha = 0$. Anything we have done above works equally well for $\alpha = 0$ if only integrals as $\alpha \int_D (-\rho)^{\alpha-1} d\rho \wedge \gamma$ are interpreted as $\int_{\partial D} \gamma$. In particular,

$$|F_0 u| \lesssim \int_{\partial D} |u|\, O(\frac{1}{|v|^{n-1/2}}) + \int_D |u|\, O(\frac{1}{|v|^{n+1/2}}) \,.$$

4. Proof of Theorem 1 for $z \in D$.

Let ρ and D be as before and consider the strictly pseudoconvex domain

$$\tilde{D} = \{(\zeta, \zeta_{n+1}) \in \mathbb{C}^{n+1} : \rho(\zeta) + |\zeta_{n+1}|^2 < 0\}$$

and put $\tilde{\rho}(\tilde{\zeta}) = \rho(\zeta) + |\zeta_{n+1}|^2$, where $\tilde{\zeta} = (\zeta, \zeta_{n+1})$. Then $D = \tilde{D} \cap \{\zeta_{n+1} = 0\}$ so a function u in D can be considered as a function in $\tilde{D}$, not depending on the last variable. By Theorem 1 we now have operators $\tilde{P}_{\alpha-1} u$, $\tilde{F}_{\alpha-1} u$, $\tilde{M}_{\alpha-1} \partial\bar{\partial} u$ and so on so that (2.2) holds for $\tilde{z} \in \partial\tilde{D}$, $\alpha \geq 1$.

Proposition 4.1. *With the notation above, $\tilde P_{\alpha-1}u(z, z_{n+1})$ does not depend on the last variable and $P_\alpha u(z) = \tilde P_{\alpha-1}u(z, z_{n+1})$. For $z \in D$ we thus have*

$$u(z) = P_\alpha u(z) + \tilde F_{\alpha-1}u(z, \sqrt{-\rho(z)}) + \tilde M_{\alpha-1}(i\,\partial\bar\partial u)(z, \sqrt{-\rho(z)}) . \tag{4.1}$$

It is therefore natural to *define*

$$M_\alpha(i\,\partial\bar\partial u)(z) = \tilde M_{\alpha-1}(i\,\partial\bar\partial u)(z, \sqrt{-\rho(z)})$$

for $z \in D$, and $F_\alpha u(z)$ similarly, and then it remains to check that (2.3)-(2.6) hold.

PROOF OF PROPOSITION 4.1. To be precise, we first construct $\tilde P_{\alpha-1}$ in the following way: We let $\tilde v(\tilde\zeta, \tilde z) = v(\zeta, z) - \bar\zeta_{n+1}z_{n+1}$ and form the corresponding weighted formula $\tilde H_{\alpha-1}$, *cf.* (1.5), in $\tilde D$. Then $\tilde H_{\alpha-1}u(z, z_{n+1})$ will not depend on z_{n+1} if $u = u(z)$ does not. Then we modify it by a smooth kernel $\tilde{\mathcal L}_{\alpha-1}u$ as in (1.12). Since $\tilde H_{\alpha-1}$ is already holomorphic in the last variable, this can be done in such a way that $\tilde G_{\alpha-1}u$ and hence $\tilde P_{\alpha-1} = \tilde G'_{\alpha-1} + \tilde{\bar G}_{\alpha-1}$ is independent of z_{n+1} if u is. Thus the proposition follows from

$$\tilde H_{\alpha-1}\, u(z, z_{n+1}) = H_\alpha u(z) . \tag{4.2}$$

To see (4.2), we first note that (with obvious notation)

$$\tilde q(\tilde\zeta, \tilde z) = q(\zeta, z) + \bar\zeta_{n+1}\, d\zeta_{n+1}$$

and

$$\tilde H_{\alpha-1}\, u = \int_{\tilde D} \frac{(-\tilde\rho)^{\alpha-2}u(\zeta)(-\tilde\rho\bar\partial\tilde q - (n+1)\tilde q \wedge \bar\partial\tilde q) \wedge (\bar\partial\tilde q)^n}{(v(\zeta, z) - \bar\zeta_{n+1}z_{n+1})^{n+\alpha}} . \tag{4.3}$$

Now,

$$\begin{aligned}
&(-\tilde\rho\bar\partial\tilde q - (n+1)\tilde q \wedge \bar\partial\tilde\rho) \wedge (\bar\partial\tilde q)^n \\
&\quad = \Big((-(\rho + |\zeta_{n+1}|^2)\bar\partial q - (n+1)q \wedge \bar\partial\rho) \wedge n(\bar\partial q)^{n-1} \\
&\qquad + (-\rho + |\zeta_{n+1}|^2 - (n+1)|\zeta_{n+1}|^2)(\bar\partial q)^n\Big) \wedge d\zeta_{n+1} \wedge d\bar\zeta_{n+1} \\
&\quad = (n+1)(-\rho\bar\partial q - nq \wedge \bar\partial\rho) \wedge (\bar\partial q)^{n-1} \wedge d\zeta_{n+1} \wedge d\bar\zeta_{n+1} .
\end{aligned}$$

Thus

$$\tilde{H}_{\alpha-1}\, u(z) = (n+1)\int_D (-\rho\,\bar\partial q - nq \wedge \bar\partial\rho) \wedge (\bar\partial q)^{n-1} u$$
$$\int\limits_{|\zeta_{n+1}|<\sqrt{-\rho(\zeta)}} \frac{(-\rho(\zeta) - |\zeta_{n+1}|^2)^{\alpha-2} d\zeta_{n+1} \wedge d\bar\zeta_{n+1}}{(v - \bar\zeta_{n+1} z_{n+1})^{n+\alpha}}$$

and if we make the change of variables $\sqrt{-\rho(\zeta)}\tau = \zeta_{n+1}$ in the inner integral we get

$$\tilde{H}_{\alpha-1} u(z) = \int_D \frac{(-\rho)^{\alpha-1} u(-\rho\,\bar\partial q - nq \wedge \bar\partial q) \wedge (\bar\partial q)^{n-1}}{v^{n+\alpha}} h_{\alpha-2,n+\alpha,0}\ ,$$

cf. (2.1), with $a = \sqrt{-\rho(\zeta)}\sqrt{-\rho(z)}/v(\zeta, z)$ and hence (4.2) follows, since $h_{\alpha-2,n+\alpha,0} \equiv 1$.

Next we compute $\tilde{M}_{\alpha-1}\theta(z, \sqrt{-\rho(z)})$ for $\theta = \theta(\zeta)$. For simplicity we just consider a typical term, namely

$$I = \int_{\tilde D} \frac{(-\tilde\rho)^\alpha \theta \wedge \partial\bar{\tilde v} \wedge \bar\partial \tilde v \wedge (\partial\bar\partial\tilde\rho)^{n-1}}{\tilde v^j \bar{\tilde v}^k}$$

with $\tilde z = (z, \sqrt{-\rho(z)})$. We first notice that $\partial\bar{\tilde v} = \partial\bar v - \bar z_{n+1} d\zeta_{n+1}$ and $\bar\partial\tilde v = \bar\partial v - z_{n+1} d\bar\zeta_{n+1}$ so that

$$\theta \wedge \partial\bar{\tilde v} \wedge \bar\partial\tilde v \wedge (\partial\bar\partial\tilde\rho)^{n-1}$$
$$= \theta \wedge (|z_{n+1}|^2 \partial\bar\partial\rho + (n-1)\partial\bar v \wedge \bar\partial v) \wedge (\partial\bar\partial\rho)^{n-2} \wedge d\zeta_{n+1} \wedge d\bar\zeta_{n+1}\ .$$

Noting that $|z_{n+1}|^2 = -\rho(z)$ and proceeding as in the proof of Proposition 4.1 above we get

$$I = \int_D \frac{(-\rho)^{\alpha+1}\theta \wedge (-\rho(z)\partial\bar\partial\rho + (n-1)\partial\bar v \wedge \bar\partial v) \wedge (\partial\bar\partial\rho)^{n-2}}{v^j \bar v^k} h_{\alpha,j,k}$$

with $a = \sqrt{-\rho(\zeta)}\sqrt{-\rho(z)}/v(\zeta, z)$. In the same way,

$$F_\alpha u(z) = \tilde{F}_{\alpha-1} u(z, \sqrt{-\rho(z)})$$

so that

$$|F_\alpha u(z)| \lesssim \int_{\tilde D} \frac{(-\tilde\rho)^{\alpha-2}|u|}{|\tilde v|^{n+\alpha-1/2}} \tag{4.4}$$

(if $\alpha = 1$,

$$F_1 u(z) \lesssim \int_{\partial\tilde D} |u| \frac{d\sigma(\tilde\zeta)}{|\tilde v|^{n+1/2}} + \int_{\tilde D} |u| \frac{d\lambda(\tilde\zeta)}{|\tilde v|^{n+3/2}} \;)$$

and proceeding as before we get

$$|F_\alpha u(z)| \lesssim \int_D \frac{(-\rho)^{\alpha-1}|u|}{|v|^{n+\alpha-1/2}} h_{\alpha-2,n+\alpha-1/2} \,.$$

This completes the proof of Theorem 1.

5. Proof of Proposition 2.1.

We recall from the last paragraph that if u is defined on D, then $F_\alpha u(z) = \tilde F_{\alpha-1} u(\tilde z)$ and $M_\alpha \theta(z) = \tilde M_{\alpha-1}\theta(\tilde z)$ where $\tilde z = (z, \sqrt{-\rho(z)}) \in \partial\tilde D$. By writing the operators this way we avoided the factors $h_{\alpha,j,k}$ in their integral representation. However, since $\tilde F_{\alpha-1}$ was defined as an integral over $\tilde D$, to compute compositions such as $\tilde F_{\alpha-1} \circ \tilde F_{\alpha-1}$, etc., we need to know $F_{\alpha-1}u(\tilde z)$ also for $\tilde z \in \tilde D$. We will avoid this difficulty by rewriting $\tilde F_{\alpha-1}u$ as an integral over the boundary of a domain $D_\alpha \subset \mathbb{C}^{n+\alpha}$.

REMARK. When $\alpha = 1$, $\tilde F_{\alpha-1}u$ consists of both a boundary integral and an integral over $\tilde D$. So in this case the argument is slightly different as we only need to rewrite this last integral as an integral over ∂D_2. We omit those details and assume that $\alpha > 1$ in the sequel.

Let $\zeta^\alpha = (\zeta_1, \dots, \zeta_n, \zeta_{n+1}, \dots, \zeta_{n+\alpha}) = (\zeta, \zeta') \in \mathbb{C}^{n+\alpha}$ where $\zeta = (\zeta_1, \dots, \zeta_n)$. We define the strictly pseudoconvex domain D_α by $D_\alpha = \{\zeta^\alpha \in \mathbb{C}^{n+\alpha} : \rho^\alpha(\zeta^\alpha) < 0\}$ where $\rho^\alpha(\zeta^\alpha) = \rho(\zeta) + \sum_{i=1}^{\alpha} |\zeta_{n+i}|^2$. We then have $v_\alpha(\zeta^\alpha, z^\alpha) = v(\zeta, z) + \bar\zeta_{n+1} z_{n+1} + \cdots + \bar\zeta_{n+\alpha} z_{n+\alpha}$. When $\alpha = 1$, we write $\tilde D$ for D_1 and $\tilde\zeta$ for ζ^1 (as in Section 4). Note that D_α is obtained from D by applying the map $D \mapsto \tilde D$, α times.

Lemma 5.1. *If u is defined on D, then*

a) $$\int_D (-\rho(\zeta))^\alpha u(\zeta)\, d\lambda(\zeta) = \frac{\alpha}{\pi} \int_{\tilde D} (-\tilde\rho(\tilde\zeta))^{\alpha-1} u(\zeta)\, d\lambda(\tilde\zeta)$$

and

b) $$\int_D u(\zeta)\, d\lambda(\zeta) \sim \int_{\partial\tilde D} u(\zeta)\, d\sigma(\tilde\zeta)\,.$$

PROOF. We have

$$\int_{\tilde D}(-\tilde\rho)^{\alpha-1}u\,d\lambda=\int_D u(\zeta)\,d\lambda(\zeta)\int\limits_{|\zeta_{n+1}|<\sqrt{-\rho(\zeta)}}(-\rho(\zeta)-|\zeta_{n+1}|^2)^{\alpha-1}\,d\lambda(\zeta_{n+1})$$

$$=\frac{\pi}{\alpha}\int_D(-\rho(\zeta))^\alpha u(\zeta)\,d\lambda(\zeta)\,.$$

We obtain $b)$ from $a)$ as

$$\int_{\partial\tilde D}\frac{u(\zeta)\,d\sigma(\tilde\zeta)}{|d\tilde\rho|}=\lim_{\alpha\to 0}\alpha\int_{\tilde D}(-\tilde\rho)^{\alpha-1}u(\zeta)\,d\lambda(\tilde\zeta)\,.$$

We also need

Lemma 5.2. *If D is a strictly pseudoconvex domain in $\mathbb{C}^n$, then for $z\in\bar D$, $w\in\partial D$*

$$\int_{\partial D}\frac{d\sigma(\zeta)}{|v(z,\zeta)|^a|v(\zeta,w)|^b}\lesssim\frac{1}{|v(z,w)|^{a+b-n}}$$

if $0<a,b<n$ and $a+b>n$. If $a+b<n$ the integral is bounded.

PROOF. We first observe that

$$\text{(5.1)}\qquad \mathrm{I}=\int\limits_{d(\zeta,z)<\delta}\frac{d\sigma(\zeta)}{d^a(\zeta,z)}\lesssim\delta^{n-a}\,,\qquad\text{if }\ 0<a<n\,,$$

and

$$\text{(5.2)}\qquad \mathrm{II}=\int\limits_{d(\zeta,z)>\delta}\frac{d\sigma(\zeta)}{d^a(\zeta,z)}\lesssim\delta^{n-a}\,,\qquad\text{if }\ a>n\,.$$

Here d is the pseudometric that defines the Koranyi balls in D, see Section 1 and [AnC]. Then $d(z,w)\sim|v(z,w)|$ if w and/or z is on the boundary.

We first prove (5.1) and (5.2) for $z\in\partial D$. Then

$$\mathrm{I}\sim\int\limits_{d(\zeta,z)<\delta}d\sigma(\zeta)\int\limits_{t>d(\zeta,z)}\frac{dt}{t^{a+1}}=\int_0^{+\infty}\frac{dt}{t^{a+1}}\int\limits_{d(\zeta,z)<\min\{t,\delta\}}d\sigma(\zeta)$$

$$\lesssim\int_{-\infty}^{\delta}\frac{t^n}{t^{a+1}}dt+\delta^n\int_\delta^{+\infty}\frac{dt}{t^{a+1}}\lesssim\delta^{n-a}\,.$$

Similarly,

$$\mathrm{II} \sim \int_\delta^{+\infty} \frac{dt}{t^{a+1}} \int_{\delta<d(\zeta,z)<t} d\sigma(\zeta) \lesssim \int_\delta^{+\infty} \frac{t^n}{t^{a+1}} \sim \delta^{n-a}.$$

If $z \in D$, let $z_0 \in \partial D$ satisfy $d(z) = d(z, z_0) = d(z, \partial D)$. Then $d(\zeta, z) \gtrsim d(\zeta, z_0)$ and (5.1) follows if we apply it to z_0. To prove (5.2) we consider two cases. If $d(z) \lesssim C\delta$, then if $d(\zeta, z) > \delta$, we have $d(\zeta, z) \sim d(\zeta, z_0)$, and we are done by the case $z_0 \in \partial D$. If $d(z) > C\delta$, then (1.3) (it is proved in the same way as (5.2) by observing that $d(\zeta, z) \sim -\rho(z) + d(\zeta, z_0)$) implies that $\mathrm{II} \lesssim (-\rho(z))^{n-a} \sim d(z)^{n-a} \lesssim \delta^{n-a}$.

Now choose c small enough (so that $cC \leq 1/2$, where C is the constant in the triangle inequality for d), let $\delta = c\,d(z, w)$ and put $B^c = \partial D \backslash (B_\delta(z) \cup B_\delta(w))$. Then

$$\int_{\partial D} \frac{d\sigma(\zeta)}{|v(z,\zeta)|^a\,|v(\zeta,w)|^b} \sim \int_{B_\delta(z)} + \int_{B_\delta(w)} + \int_{B^c} \frac{d\sigma(\zeta)}{d^a(z,\zeta)\,d^b(\zeta,w)}$$
$$= A + B + C.$$

The integrals A and B are estimated in the same way. Observe that if $\zeta \in B_\delta(z)$, then $d(z, w) \leq C\,(d(z,\zeta) + d(\zeta, w)) \leq d(z, w)/2 + C\,d(\zeta, w)$ and hence $d(\zeta, w) \gtrsim d(z, w)$. Thus by (5.1),

$$A \lesssim \frac{1}{d^b(z,w)} \int_{d(\zeta,w)<\delta} \frac{d\sigma(\zeta)}{d^a(\zeta,z)} \lesssim \frac{1}{d(z,w)^{a+b-n}}.$$

To estimate C, we note that if $\zeta \in B^c$ then $d(z,\zeta) \geq c\,d(z,w)$. Hence $d(\zeta,w) \lesssim d(\zeta,z) + d(z,w) \lesssim d(\zeta,z)$ and by symmetry we have $d(\zeta,w) \sim d(\zeta,z)$. This implies by (5.2)

$$C \leq \int_{d(\zeta,z)>\delta} \frac{d\sigma(\zeta)}{d^{a+b}(\zeta,z)} \leq \frac{1}{d^{a+b-n}(z,w)}$$

if $a + b > n$. If $a + b < n$ the integral is bounded.

PROOF OF PROPOSITION 2.1. First we claim that

$$\begin{aligned}(5.3)\qquad \int_D \frac{(-\rho)^{\alpha-1}|u(\zeta)|}{|v(\zeta,z)|^l}\, h_{\alpha-2,l}\, d\lambda(\zeta) &= \int_{\tilde D} \frac{(-\tilde\rho)^{\alpha-2}|u(\zeta)|}{|\tilde v(\tilde\zeta,\tilde z)|^l}\, d\lambda(\tilde\zeta) \\ &= \int_{\partial D_\alpha} \frac{|u(\zeta)|}{|v_\alpha(\zeta^\alpha, z^\alpha)|^l}\, d\sigma(\zeta^\alpha),\end{aligned}$$

if $|(z_{n+1},\dots,z_{n+\alpha})| = \sqrt{-\rho(z)}$. In fact, for the first equality *cf.* the proof of Proposition 4.1; the last equality is obtained for $z^\alpha = (z, \sqrt{-\rho(z)}, 0, \dots, 0)$ by repeated use of Lemma 5.1, and then the general case follows since the integral is rotation invariant with respect to $(\zeta_{n+1},\dots,\zeta_{n+\alpha})$.

By (5.3), (2.7) is equivalent to

$$|F_\alpha^k u(z)| \lesssim \int_{\partial D_\alpha} \frac{|u(\zeta)|}{|v_\alpha(z^\alpha,\zeta^\alpha)|^{n+\alpha-k/2}}\, d\sigma(\zeta^\alpha), \tag{5.4}$$

$1 \le k < 2(n+\alpha)$. We will prove (5.4) by induction. It is true for $k=1$ since then (2.7) is nothing but (2.3). Assuming (5.4) for k we have

$$\begin{aligned} |F_\alpha^{k+1}u(z)| &\lesssim \int_{\partial D_\alpha} \frac{|F_\alpha^k u(\zeta)|\, d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha,\zeta^\alpha)|^{n+\alpha-1/2}} \\ &\lesssim \int_{\partial D_\alpha} \frac{d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha,\zeta^\alpha)|^{n+\alpha-k/2}} \int_{\partial D_\alpha} \frac{|F_\alpha^k u(w)|\, d\sigma(w^\alpha)}{|v_\alpha(\zeta^\alpha,w^\alpha)|^{n+\alpha-1/2}}\,. \end{aligned}$$

By Fubini's theorem and Lemma 5.2 we now get (5.4) for $k+1$.

We also see that if $k > 2(n+\alpha)-1$, $F_\alpha^k u$ has a bounded kernel (when $k = 2(n+\alpha)-1$ it has a logarithmic singularity), and since the kernel of F_α is continuous (and much more) off the diagonal and this is preserved under composition, we have $F_\alpha^k u \in C(\bar{D})$.

To see (2.8) first note that, by (2.6) and the argument for (5.3),

$$|M_\alpha\theta(\zeta)| \lesssim \int_{D_\alpha} \frac{|||\theta|||_D\, d\lambda(w^\alpha)}{|v_\alpha(\zeta^\alpha,w^\alpha)|^{n+\alpha}}$$

if $|(\zeta_{n+1},\dots,\zeta_{n+\alpha})| = \sqrt{-\rho(\zeta)}$ (recall that $|||\theta||| = -\rho|\theta| + \sqrt{-\rho}(|\theta\wedge\partial\rho| + |\theta\wedge\bar\partial\rho|) + |\theta\wedge\partial\rho\wedge\bar\partial\rho|$). Hence, by (5.4),

$$\begin{aligned} |F_\alpha^k M_\alpha\theta(z)| &\lesssim \int_{\partial D_\alpha} \frac{|M_\alpha\theta(\zeta)|\, d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha,\zeta^\alpha)|^{n+\alpha-k/2}} \\ &\lesssim \int_{\partial D_\alpha} \frac{d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha,\zeta^\alpha)|^{n+\alpha-k/2}} \int_{D_\alpha} \frac{|||\theta|||_D\, d\lambda(w^\alpha)}{|v_\alpha(\zeta^\alpha,w^\alpha)|^{n+\alpha}} \\ &\sim \int_{D_\alpha} |||\theta|||_D\, d\lambda(w^\alpha) \\ &\quad\cdot \int_{\partial D_\alpha} \frac{d\sigma(\zeta^\alpha)}{|v_\alpha(z^\alpha,\zeta^\alpha)|^{n+\alpha-k/2}\,|v_\alpha(\zeta^\alpha,w^\alpha)|^{n+\alpha}} \end{aligned}$$

$$\begin{aligned} &\lesssim \int_{D_\alpha} \frac{|||\theta|||_D \, d\lambda(w^\alpha)}{|v_\alpha(\zeta^\alpha, w^\alpha)|^{n+\alpha-k/2}} \\ &\sim \int_D (-\rho)^\alpha \frac{|||\theta||| \, d\lambda(w)}{|v(z,w)|^{n+\alpha-k/2}} \, h_{\alpha-1,n+\alpha-k/2} \,, \end{aligned}$$

and the proof of Proposition 2.1 is complete.

6. Proof of Theorem 4.

In this section we assume that $\mu = -\rho|\theta| + \sqrt{-\rho}(|\theta\wedge\partial\rho| + |\theta\wedge\bar\partial\rho|) + |\theta \wedge \partial\rho \wedge \bar\partial\rho|$ is a Carleson measure and prove that $M_\alpha\theta \in \mathrm{BMO}\,(\partial D)$. It is easy to see that $R_\alpha\theta \in L^\infty$; in fact,

$$\begin{aligned} |R_\alpha\theta(z)| &\lesssim \int_D \frac{(-\rho)^\alpha}{|v(\zeta,z)|^{n+\alpha-1/2}} \, d\mu(\zeta) \\ &\sim \int_D d\mu(\zeta) \int_{d(\zeta,z)}^{+\infty} \frac{dt}{t^{n+1/2}} \\ &= \int_0^{+\infty} \frac{dt}{t^{n+1/2}} \int_{d(\zeta,z)<t} d\mu(\zeta) = \int_0^{+\infty} \frac{\mu(Q_t(\zeta))}{t^{n+1/2}} \, dt < +\infty \,, \end{aligned}$$

as $\mu(Q_t(\zeta)) \lesssim t^n$. The other terms in $M_\alpha\theta$ are estimated in the same fashion, so instead of giving detailed arguments for each of them we concentrate on a typical one. Our choice is

$$f(z) = \int_D \frac{(-\rho)^{\alpha+1}\theta \wedge \partial\bar v(\zeta,z) \wedge \bar\partial v(\zeta,z) \wedge (\partial\bar\partial\rho)^{n-2}}{v(\zeta,z)^{n+1}\,\bar v(\zeta,z)^\alpha} \,.$$

We want to estimate

$$M_h(p) = \frac{1}{|B_h(p)|} \int_{B_h(p)} |f(z) - f_h| \, d\sigma(z) \,,$$

where f_h is the mean value of f over $B_h(p)$. To this end we need

Lemma 6.1. *If μ is a Carleson measure then*

$$J = \int_{-\rho(\zeta)>h} \frac{d\mu(\zeta)}{d^{n+\alpha}(\zeta,p)} \lesssim \frac{1}{h^\alpha} \,.$$

PROOF. Let $\zeta = (y, x)$ where $y = -\rho(\zeta)$ and $x \in \partial D$. Put

$$E_{k,m} = \{\zeta : \; 2^{k-1}h \le y \le 2^k h, \; m\, 2^k h \le d(x,p) \le (m+1)\, 2^k h\}$$

if $k \ge 1$. When $k = 0$, we replace the lower bound for y by 0. Then $\{-\rho(\zeta) > h\} = \cup_{k\ge 1} \cup_{m \ge 0} E_{k,m}$. Since $|E_{k,m}| \approx (2^k h)^{n+1} m^{n-1}$, $E_{k,m}$ can be covered by $\lesssim m^{n-1}$ Koranyi balls $Q_{2^k h}(q_i)$, and hence

$$\mu(E_{k,m}) \le \sum \mu(Q_{2^k h}(q_i)) \lesssim m^{n-1} (2^k h)^n .$$

From this we obtain

$$\begin{aligned} J &\lesssim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2^k h (1+m))^{n+\alpha}} \mu(E_{k,m}) \\ &\lesssim \frac{1}{h^\alpha} \sum_{k=1}^{\infty} 2^{-k\alpha} \sum_{m=0}^{\infty} \frac{m^{n-1}}{(1+m)^{n+\alpha}} \lesssim \frac{1}{h^\alpha} , \end{aligned}$$

as desired.

We also need

$$|v(z,\zeta) - v(w,\zeta)| \lesssim (h\, d(\zeta, p))^{1/2} , \tag{6.1}$$

if $z, w \in B_h(p)$, $\zeta \notin B_{Ch}(p)$. This follows immediately if we write

$$\begin{aligned} v(z,\zeta) - v(w,\zeta) = v(z,w) &+ \langle q(z,w) - q(z,\zeta), z - w\rangle \\ &+ \langle q(z,\zeta) - q(w,\zeta), \zeta - w\rangle . \end{aligned}$$

Let us return yo the estimate of $M_h(p)$. We have

$$M_h(p) \lesssim \frac{1}{|B_h(p)|^2} \int_{z\in B_h(p)} \int_{w \in B_h(p)} |f(z) - f(w)|\, d\sigma(z)\, d\sigma(w) ,$$

and

$$\begin{aligned} f(z) - f(w) &= \Big(\int_{y>Ch} + \int_{y\le Ch} \Big) (-\rho)^{\alpha+1} \\ &\quad \cdot \left(\frac{\partial \bar v(z) \wedge \bar\partial v(z)}{v^{n+1}(z) \bar v^{\alpha}(z)} - \frac{\partial \bar v(w) \wedge \bar\partial v(w)}{v^{n+1}(w) \bar v^{\alpha}(w)} \right) \wedge \theta \wedge (\partial\bar\partial\rho)^{n-2} \\ &= m_\infty(z,w) + m_0(z,w) , \end{aligned}$$

where $y = -\rho(\zeta)$ and $v(z) = v(z,\zeta)$ for short. Consider first the part where $y > Ch$. Then

$$\begin{aligned}&\frac{\partial\bar v(z)\wedge\bar\partial v(z)}{v^{n+1}(z)\bar v^{\alpha}(z)} - \frac{\partial\bar v(w)\wedge\bar\partial v(w)}{v^{n+1}(w)\bar v^{\alpha}(w)}\\ &\qquad = \partial\bar v(z)\wedge\bar\partial v(z)\left(\frac{1}{v^{n+1}(z)\bar v^{\alpha}(z)} - \frac{1}{v^{n+1}(w)\bar v^{\alpha}(w)}\right)\\ &\qquad\quad + \frac{1}{v^{n+1}(z)\bar v^{\alpha}(z)}\left(\partial\bar v(z)\wedge\bar\partial v(z) - \partial\bar v(w)\wedge\bar\partial v(w)\right).\end{aligned}$$

By (6.1),

$$\begin{aligned}\left|\frac{1}{v^{n+1}(z)\bar v^{\alpha}(z)} - \frac{1}{v^{n+1}(w)\bar v^{\alpha}(w)}\right| &\lesssim \frac{|v(z)-v(w)|}{d^{n+\alpha+2}(\zeta,p)}\\ &\le \frac{\sqrt{h}}{d^{n+\alpha+3/2}(\zeta,p)}.\end{aligned}$$

Furthermore by (3.4),

$$\partial\bar v(z)\wedge\bar\partial v(z) = \partial\rho\wedge\bar\partial\rho + O(|\eta|)\,(\partial\rho + \bar\partial\rho) + O(|\eta|^2)$$

and

$$\begin{aligned}\partial\bar v(z)\wedge\bar\partial v(z) - \partial\bar v(w)\wedge\bar\partial v(w) &= \partial\bar v(z)\wedge(\bar\partial v(z) - \bar\partial v(w))\\ &\quad + (\partial\bar v(z) - \partial\bar v(w))\wedge\bar\partial v(w)\\ &= O(\sqrt{h})\,(\partial\rho + \bar\partial\rho + O(|\eta|)).\end{aligned}$$

Hence the integrand in m_∞ is bounded by $\sqrt{h}\,d(\zeta,p)^{-(n+1/2)}\,d\mu$. By Lemma 6.1 this implies

$$\begin{aligned}&\frac{1}{|B_h(p)|^2}\int_{B_h(p)}\int_{B_h(p)} m_\infty(z,w)\,d\sigma(z)\,d\sigma(w)\\ (6.2)\qquad &\lesssim \frac{1}{|B_h(p)|^2}\int_{B_h(p)} d\sigma(z)\int_{B_h(p)} d\sigma(w)\int_{y>Ch}\frac{\sqrt{h}}{d(\zeta,p)^{n+1/2}}\,d\mu\\ &\lesssim \frac{\sqrt{h}}{\sqrt{h}} = 1.\end{aligned}$$

The contribution from the part where $y < Ch$, is dominated by two terms of the form

$$\int_{y\leq Ch} \frac{1}{|B_h(p)|} \int_{z\in B_h(p)} (-\rho)^{\alpha+1} \frac{|\partial\bar{v}\wedge\bar{\partial}v\wedge\theta\wedge(\partial\bar{\partial}\rho)^{n-2}|}{|v^{n+\alpha+1}(\zeta,z)|}\,d\sigma(z)$$

$$\lesssim \sum_{m=0}^{\infty}\int_{E_{0,m}} \frac{1}{h^n}\int_{z\in B_h(p)} (-\rho)\frac{|\partial\bar{v}\wedge\bar{\partial}v\wedge\theta\wedge(\partial\bar{\partial}\rho)^{n-2}|}{d(\zeta,z)^{n+1}}\,.$$

Again we use $\partial\bar{v}(z)\wedge\bar{\partial}v(z) = \partial\rho\wedge\bar{\partial}\rho + O(|\eta|)(\partial\rho+\bar{\partial}\rho) + O(|\eta|^2)$ to obtain

$$\begin{aligned}
\mathrm{I} &= \frac{1}{|B_h(p)|^2}\int\int m_0(z,w)\,d\sigma(z)\,d\sigma(w)\\
&\lesssim \sum_{m=0}^{\infty}\frac{1}{h^n}\int_{E_{0,m}} d\mu \int_{z\in B_h(p)} \frac{-\rho}{d(\zeta,z)^{n+1}}\,d\sigma(z)\\
&\lesssim \sum_{m=0}^{\infty}\frac{1}{h^n}\int_{E_{0,m}} d\mu(y,x) \int_{z\in B_h(p)} \frac{y}{(y+d(x,z))^{n+1}}\,d\sigma\,.
\end{aligned}$$

But if $\zeta = (y,x)\in E_{0;m}$, then

$$\int_{z\in B_h(p)} \frac{y}{(y+d(x,z))^{n+1}}\,d\sigma \lesssim \frac{1}{(1+m)^{n+1}}\,,$$

and we obtain

$$\mathrm{I}\lesssim \sum_{m=0}^{\infty}\frac{1}{(1+m)^{n+1}}\,\mu(E_{0,m}) \lesssim \sum_{m=0}^{\infty}\frac{m^{n-1}}{(1+m)^{n+1}} \lesssim 1\,. \tag{6.3}$$

By (6.2) and (6.3) we get $M_h(p)\lesssim 1$, and the proof is complete.

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Hiperbolic singular integral operators

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Abstract. We define a class of integral operators which are singular relative to the hyperbolic metric on simply connected domains of the plane. We study the necessary and sufficient conditions for such operators to be bounded on L^2 of the upper half plane relative to the hyperbolic metric.

Introduction.

Let Ω be an open simply connected subset of $\mathbb{R}^2$. Let us denote by $\partial\Omega$ the boundary of Ω and by $\delta(x)$ the euclidean distance from x to $\partial\Omega$. Let $\rho(x,y)$ be the hyperbolic distance between x and y in Ω. And let $m(x,y) = \inf\{\mathrm{vol}_h(B) : B \text{ is a ball containing } x \text{ and } y\}$, where by $\mathrm{vol}_h(B)$ we denote the hyperbolic volume of B and B is the ball defined relative to the hyperbolic metric (when $\Omega = \mathbb{R}^2_+$ and B has hyperbolic radius r, $\mathrm{vol}_h(B)$ is like $\sinh^2(r/2)$).

We consider a class of operators given by kernels satisfying standard estimates -like the usual Calderón-Zygmund operators- but with respect to the hyperbolic metric. We study the necessary and sufficient conditions for such "hyperbolic singular integral operators" T to extend to a bounded operators on $L^2(\Omega, dx/\delta(x)^2)$.

In hyperbolic spaces, the volume of a ball grows exponentially as a function of its radius. We can not then have a doubling measure. Therefore, hyperbolic spaces are not examples of spaces of homogeneous type and we cannot view them within the same framework of general Calderón-Zygmund theory developed for these spaces (*cf.* for example [CW], [DJS])

The motivation for considering operators given by this kind of kernels arises when looking at the Green's function of the upper half plane in two-dimensions:

$$G(x,y) = \log \frac{|x-\bar{y}|}{|x-y|}, \qquad \text{for } x, y \in \mathbb{R}^2_+ .$$

It is well known that the Green's operator is not bounded on $L^2(\mathbb{R}^2_+, dx)$, where dx is Lebesgue measure. But if we consider

$$\tilde{G}(x,y) = \log \frac{|x-\bar{y}|}{|x-y|} \, \chi_{\{\rho(x,y)>1\}} ,$$

then the operator associated to $\tilde{G}(x,y)$ is bounded on $L^2(\mathbb{R}^2_+, dh)$, $dh = dx/\delta(x)^2 = dx/x_2^2$. This is a consequence of $|G(x,y)| \le c/m(x,y)$; that is the Green's function decays like the inverse of the volume of the smallest ball -relative to the hyperbolic metric- containing x and y.

The philosophy to deal with such "hyperbolic singular integral operators" would be the following. If the hyperbolic distance between x and y is larger than one, then the kernel would decrease "exponentially" and we have enough decay to handle the L^2-boundedness via Schur's Lemma. If the distance between x and y is less than one then these points lie "in the same" Whitney cube where euclidean distance and hyperbolic distance are comparable. We are then reduced to the euclidean case and the $T1$-Theorem of David and Journé applies (*cf.* [DJ]).

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1. Definitions, examples and statement of results.

Definition 1.1. *A hyperbolic standard kernel is a continuous function* $K : \Omega \times \Omega \setminus \Delta \longrightarrow \mathbb{C}$ *for which there exists a constant* $c > 0$ *such that*

1) $|K(x,y)| \le \dfrac{c}{m(x,y)}$, *for all* $(x,y) \in \Omega \times \Omega \setminus \Delta$.

2) *If* $(x,y) \in \Omega \times \Omega \setminus \Delta$ *are such that* $\rho(x,y) < 1$, *then*

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq c \, \frac{\delta(x)\,\delta(y)}{|x-y|^3}$$

($\Delta = \{(x,y) : x = y\}$ *and the gradients are taken in the distributional sense and assumed to be functions*).

Definition 1.2. *A hyperbolic singular integral operator* T *is an operator taking* $C_0^\eta(\Omega)$ *into* $L^1_{\text{loc}}(\Omega)$ *and associated to a hyperbolic standard kernel* K*: for every* $f \in C_0^\eta(\Omega)$,

$$Tf(x) = \int K(x,y)\, f(y)\, \frac{dy}{\delta(y)^2} \,,$$

for $x \notin \operatorname{supp} f$.

Notice that when $\rho(x,y) < 1$ we have that $m(x,y)$ is comparable to $\delta(x)\,\delta(y)/|x-y|^2$ and that $1/4 \leq \delta(x)/\delta(y) \leq 4$; therefore our hyperbolic singular integral operator coincides with a usual singular integral operator in euclidean geometry.

We refer the reader to [B] and [BP] for precise definitions and properties about hyperbolic geometry.

EXAMPLES. i) Let $\Omega = \mathbb{R}^2_+$, $(x_1, x_2) = x$, $y = (y_1, y_2)$ and consider the Riesz transforms

$$\Big(x_2 \, \frac{\partial}{\partial x_1}\Big)^2 G(x,y)\,; \qquad G(x,y) = \log \frac{|x-\bar{y}|}{|x-y|}\,.$$

where the derivatives are taken in the distributional sense.

Then $(x_2\, \partial/\partial x_1)^2 G(x,y)$ equals

$$K(x,y) = -x_2^2 \Big(\frac{(x_1-y_1)^2 - (x_2-y_2)^2}{|x-y|^4} - \frac{(x_1-y_1)^2 - (x_2+y_2)^2}{|x-\bar{y}|^4} \Big)\,.$$

$K(x,y)$ is a C^1-function away from the diagonal and it is easy to see that it satisfies 1) and 2).

ii) Take $\Omega = \mathbb{R}^2_+$ and for $0 < r < 1$, let $B(i,r)$ be the ball of hyperbolic radius r centered at i. Given $x, y \in \mathbb{R}^2_+$ there exists a

Möbius transformation γ such that $\rho(x,y)=\rho(\gamma x,\gamma y)=\rho(i,pi)$. Since $\rho(i,pi)=|\log\ p|$ we have that,

$$\mathrm{vol}_h B(x,\rho(x,y))=\mathrm{vol}_h B(i,|\log\ p|)\,.$$

Let k be a smooth function so that $|k(r)|\le c/r^2$ and $|k'(r)|\le c/r^3$ and define $k(r)=k(\rho(x,y))=k(|\log\ p|)$. Then let $K(x,y)=k(\rho(x,y))$, if $\rho(x,y)<1$ and $K(x,y)=0$ otherwise. Clearly K satisfies 1) and 2). And we have that,

$$Tf(x)=Tf(\gamma i)=\int_G k(\rho(\gamma i,\mu i))\,f(\mu i)\,dh(\mu)=k*_G f$$

if $\gamma i=x$ and $\mu i=y$; $\gamma,\mu\in G$, the group of Möbius transformations (*cf.* [CW2, Chapter 10]).

iii) Let $k(r)=(1/\sinh^2 r)^{1+i}$ and let $K(x,y)=k(\rho(x,y))$. Then it is clear that K satisfies 1) and 2).

Let $\Omega=\mathbb{R}^2_+$. In Section 2 we prove,

Theorem 2.1. *Let T be a hyperbolic singular integral operator. Then, T extends to a bounded operator in $L^2(\mathbb{R}^2_+,dx/x_2^2)$ if and only if, for any $0<\varepsilon<1$,*

1) $T(w)\in w\,\mathrm{BMO}(dh)$; $w(x)=x_2^{\varepsilon}$,
2) $T^*(w)\in w\,\mathrm{BMO}(dh)$; $w(x)=x_2^{\varepsilon}$,
3) *T satisfies the "local weak boundedness property"* (LWBP):

Let $\{Q_j\}$ be the Whitney decomposition of $\mathbb{R}^2_+$. Fix Q_j and let $d\omega_j$ be $dx/|Q_j|$. Then $\omega_j(Q)=|Q|/|Q_j|$ for $Q\subseteq Q_j$. Let $f,g\in C_0^\eta$ such that $\mathrm{supp}\,f,\mathrm{supp}\,g\subseteq Q$, $Q\subseteq Q_j$ and $|f(x)-f(y)|\le c\,|x-y|^\eta\,\omega_j(Q)^{-\eta/2}$; same condition also for g. Then,

$$|\langle Tf,g\rangle|=\left|\int Tf(x)\,g(x)\,d\omega_j\right|\le c\,\omega_j(Q)\,\|f\|_\infty\,\|g\|_\infty\,,$$

where c is a constant independent of j.

By $\mathrm{BMO}(dh)$ we mean, modulo constants the space of functions f such that

$$\sup_{\{Q:\mathrm{vol}_h(Q)\le 1\}}\frac{1}{\mathrm{vol}_h(Q)}\int_Q|f-(mh)_Q f|\,dh(x)\,,$$

where Q is a cube in $\mathbb{R}^2_+$ with sides parallel to the coordinate axis, $dh(x) = dx/x_2^2$ and

$$(mh)_Q f = \frac{1}{\mathrm{vol}_h(Q)} \int f \, dh .$$

For other domains different than $\mathbb{R}^2_+$ we can get a partial result for Ω a simply connected domain in $\mathbb{R}^2$ bounded by a Jordan curve that is a K-quasicircle with $K = 1 + \varepsilon$, $\varepsilon > 0$ very small and R an operator associated to a kernel $R(z, w)$ that is closely related to a hyperbolic standard kernel when $\rho^*(z, w) \geq 1$, ρ^* is the hyperbolic distance in Ω.

In Section 3 we prove,

Theorem 3.1. *Let Ω be a simply connected domain in $\mathbb{R}^2$ bounded by a Jordan curve that is a K-quasicircle with $K = 1 + \varepsilon$ and $\varepsilon > 0$ very small. Let ρ^* be the hyperbolic distance function in Ω and $\delta(z')$ the euclidean distance* $\operatorname{dist}\{z', \partial\Omega\}$. *Let $R(z', w')$ be a kernel defined on $\Omega \times \Omega$ such that $R(z', w') = 0$, if $\rho^*(z', w') < 1$ and $|R(z', w')| \leq c\, e^{-\rho^*(z',w')/K^2}$, if $\rho^*(z', w') \geq 1$.*

Then, if R is the operator associated to $R(z', w')$, we have that there exists $\eta = \eta(\varepsilon) > 0$ such that

$$R(\delta^\eta)(z') \leq c\, \delta^\eta(z') , \qquad dh \text{ almost everywhere,}$$

where $dh = dz'/\delta(z')^2$ is equivalent to the hyperbolic measure on Ω.

By Schur's Lemma it is an immediate consequence of Theorem 3.1 that R defines a bounded operator on $L^2(\Omega, dz'/\delta(z')^2)$.

Actually, if $\tilde{G}(z', w')$ is the Green's function on any simply connected domain of $\mathbb{R}^2$ (with non trivial boundary), then $|\tilde{G}(z', w')| \leq c\, e^{-\rho^*(z',w')}$ if $\rho^*(z', w') \geq 1$.

REMARK. We also have that

$$|\tilde{G}(z', w')| \leq c \left(1 + \log \frac{1}{\rho^*(z', w')^2}\right), \qquad \text{for } \rho^*(z', w') < 1 .$$

This estimate is enough to prove that $\tilde{G}$, the operator associated to the kernel $\tilde{G}(z', w')\, \chi_{\rho^*(z',w')<1}$, satisfies $\tilde{G}(\delta^\eta)(z') \leq \delta^\eta(z')$, dh almost everywhere, for $0 < \eta < 1$, Ω simply connected in $\mathbb{R}^2$.

The Green's operator on Δ, the unit disk, defines a bounded operator on $L^2(\Delta, dh)$. Therefore the Green's operator on Ω, any simply connected domain in the plane, defines a bounded operator on $L^2(\Omega, dh)$.

REMARK. Kernels $K(z', w')$ defined on $\Omega \times \Omega$, $\partial\Omega$ a K-quasicircle, $K = 1 + \varepsilon$, satisfying: $K(z', w') = 0$ if $\rho^*(z', w') < 1$ and

$$|K(z', w')| \leq c\, \frac{\min\{\delta(z')^2, \delta(w')^2\}}{|z' - w'|^2}\,, \qquad \text{if } \rho^*(z', w') > 1\,,$$

are also kernels of the kind described in Theorem 3.1:

Being $\partial\Omega$ a K-quasicircle, there exists $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $F(\infty) = \infty$, $f = F|_\Delta : \Delta \longrightarrow \Omega$ is univalent and F is a K^2-global quasiconformal map (recall that F^{-1} is also K^2-quasiconformal).

Also F satisfies: if $z, w, u \in \mathbb{R}^2$ then (*cf.* [A])

$$\min\left\{\frac{|z-w|}{|w-u|}^{1/K^2}, \frac{|z-w|}{|w-u|}^{K^2}\right\} \leq \left|\frac{F(z) - F(w)}{F(w) - F(u)}\right| \leq \max\left\{\frac{|z-w|}{|w-u|}^{1/K^2}, \frac{|z-w|}{|w-u|}^{K^2}\right\}.$$

Therefore,

$$\frac{\min\{\delta(z')^2, \delta(w')^2\}}{|z' - w'|^2} \leq e^{-\rho(h(z'), h(w'))/K^2}\,,$$

where $h = f^{-1}$ and ρ is the hyperbolic distance function in Δ.

Before proving Theorem 2.1 we wish to recall,

Schur's Lemma. *If $K(x, y)$ is a nonnegative kernel, if p and q are strictly positive measurable functions on X and Y respectively, and if α and β are positive numbers such that*

$$\int K(x, y)\, q(y)\, d\mu(y) \leq \alpha\, p(x)\,, \qquad \textit{for almost every } x\text{-}\, d\mu'\,,$$

$$\int K(x, y)\, p(x)\, d\mu'(x) \leq \beta\, q(y)\,, \qquad \textit{for almost every } y\text{-}\, d\mu\,,$$

then $K(x, y)$ is a bounded kernel and $\|K\|^2 \leq \alpha\,\beta$. That is, the operator associated to $K(x, y)$ maps $L^2(X, d\mu') \to L^2(Y, d\mu)$ continuously. $(X, d\mu')$ and $(Y d\mu)$ are measure spaces, μ' and μ are positive measures (*cf.* [HS]).

2. Proof of Theorem 2.1.

Write,

$$T(f) = \int_{\{\rho(x,y)<1\}} K(x,y)\, f(y)\, \frac{dy}{\delta(y)^2} + \int_{\{\rho(x,y)\geq 1\}} K(x,y)\, f(y)\, \frac{dy}{\delta(y)^2}$$
$$= T_1(f) + T_2(f)\,,$$

for $f \in C_0^\eta$ and $x \notin \operatorname{supp} f$. Then, the theorem will follow from:

A) $T_2 : x_2^\varepsilon L^\infty \longrightarrow x_2^\varepsilon L^\infty$. Recall that on $\mathbb{R}^2_+$, $\delta(x) = x_2$ if $x = (x_1, x_2)$.

B) If $T : L^2(dh) \longrightarrow L^2(dh)$, then $T_1 : x_2^\varepsilon L^\infty \longrightarrow x_2^\varepsilon \operatorname{BMO}(dh)$.

C) If T_1 has the LWBP on each Q_j and $T_1(1) \in \operatorname{BMO}(dh)$, $T_1^*(1) \in \operatorname{BMO}(dh)$, then T_1 is bounded on $L^2(dh)$.

PROOF OF A). We need to show that there exists a constant $c = c(\varepsilon) > 0$ such that $\|T_2(f)(\cdot)/\delta(\cdot)\|_\infty \leq c\,\|F\|_\infty$; $f = x_2^\varepsilon F$, $F \in L^\infty$.

$$|T_2(f)(x)| \leq \int_{\{\rho(x,y)>1\}} |K(x,y)|\, y_2^\varepsilon\, F(y)\, \frac{dy}{y_2^2}$$
$$\leq c\,\|F\|_\infty \int_{\{\rho(x,y)>1\}} \frac{1}{m(x,y)}\, y_2^\varepsilon\, \frac{dy}{y_2^2}$$
$$\leq c''\,\|F\|_\infty \int_{\{\rho(x,y)>1\}} \frac{x_2 y_2}{|x-\bar{y}|^2}\, y_2^\varepsilon\, \frac{dy}{y_2^2}\,,$$

since for $r = \rho(x,y) > 1$, $\sinh^2(r/2) \sim e^r$, and then,

$$\frac{1}{m(x,y)} \leq c' \frac{x_2\, y_2}{|x-\bar{y}|^2}\,.$$

By means of a (Möbius) transformation we get,

$$\leq c\,\|F\|_\infty\, x_2^\varepsilon \int \frac{x_2^2}{|x - \overline{x_2 w}|^2}\, w_2^{\varepsilon-1}\, dw$$
$$\leq c\,\|F\|_\infty\, x_2^\varepsilon \int_0^{+\infty} \frac{x_2^2}{w_2^{1-\varepsilon}(x_2 + x_2 w_2)^2} \int_{-\infty}^{+\infty} \frac{1}{1 + \left(\dfrac{x_1 - x_2 w_1}{x_2 + x_2 w_2}\right)^2}\, dw_1\, dw_2$$

$$\leq c\,\pi\,\|F\|_\infty\, x_2^\varepsilon \int_0^{+\infty} \frac{1}{w_2^{1-\varepsilon}(1+w_2)}\, dw_2$$
$$\leq c(\varepsilon)\,\pi\,\|F\|_\infty\, x_2^\varepsilon\,,$$

where
$$c(\varepsilon) = c\left(\frac{1}{\varepsilon} + \frac{1}{1-\varepsilon}\right).$$

PROOF OF B). Assume $T : L^2(dh) \longrightarrow L^2(dh)$. By A) and Schur's Lemma we have that $T_2 : L^2(dh) \longrightarrow L^2(dh)$ continuously, therefore we know that $T_1 : L^2(dh) \longrightarrow L^2(dh)$. We define the action of T_1 on $x_2^\varepsilon L^\infty$:

Let Q be a cube such that $\mathrm{vol}_h(Q) < 1$, then everything is like in the euclidean case ([DJ], [DJS]).

Let $f = f\,\chi_{\bar{Q}} + f(1-\chi_{\bar{Q}}) = f_1 + f_2$. $\bar{Q} = 2\,Q$ is the cube with the same center as Q and sidelength $2\,\ell(Q)$.

$$(T_1 f)_Q = \frac{T_1(f_1)(x)}{x_2^\varepsilon} + \left|\frac{T_1(f_2)(x)}{x_2^\varepsilon} - c_Q\right|,$$

where
$$c_Q = \int\limits_{\{y:\, \rho(x_Q,y)<1,\, y\notin\bar{Q}\}} \frac{K(x_0,y)}{x_2^\varepsilon}\, f_2(y)\, \frac{dy}{y_2^2}\,,$$

where x_Q is the center of Q.

$$(T_1 f)_Q(x) = \frac{T_1(f_1)(x)}{x_2^\varepsilon} + \int\limits_{\{\rho(x,y)<1\}\cap\bar{Q}^c} \left(\frac{K(x,y)}{x_2^\varepsilon} - \frac{K(x_Q,y)}{x_2^\varepsilon}\right) f_2(y)\, \frac{dy}{y_2^2}\,.$$

$(T_1 f)_Q$ is well defined up to constants (depending on Q).

Now, denote by $d_Q = d(Q,\mathbb{R})$, and let $f_1(x) = x_2^\varepsilon F(x)\chi_{\bar{Q}}(x) \in L^2(\mathbb{R}^2_+, dh)$. By the boundedness of T_1 and Jensen's inequality we have,

$$\frac{1}{\mathrm{vol}_h(Q)} \int_Q \frac{|T_1 f_1(x)|}{x_2^\varepsilon}\, \frac{dx}{x_2^2} \leq c\, \frac{d_Q^{-\varepsilon}}{\mathrm{vol}_h(Q)} \int_Q |T_1 f_1(x)|\, \frac{dx}{x_2^2}$$
$$\leq d_Q^{-\varepsilon} \left(\frac{1}{\mathrm{vol}_h(Q)} \int_Q |T_1 f_1(x)|^2\, \frac{dx}{x_2^2}\right)^{1/2}$$

$$\leq d_Q^{-\varepsilon}\left(\frac{1}{\mathrm{vol}_h(Q)}\int_Q |f_1(x)|^2\,\frac{dx}{x_2^2}\right)^{1/2}.$$

But $|f_1(x)| \leq \|F\|_\infty\, d_Q^\varepsilon$, therefore the last expression is less or equal than $c\,\|F\|_\infty$.

On the other hand,

$$\Big|\int_{\{\rho(x,y)<1\}\cap\bar{Q}^c}\left(\frac{K(x,y)}{x_2^\varepsilon}-\frac{K(x_Q,y)}{x_2^\varepsilon}\right)f_2(y)\,\frac{dy}{y_2^2}\Big|$$

$$\leq c\,\|F\|_\infty \int_{\{\rho(x,y)<1\}\cap\bar{Q}^c} |K(x,y)-K(x_Q,y)|\,\frac{dy}{y_2^2}$$

$$\leq c\,|Q|^{1/2}\,\|F\|_\infty \int_{\{y:\rho(x,y)<1\}\cap\bar{Q}^c} \frac{1}{|x_Q-y|^3}\,dy$$

$$\leq c'\,|Q|^{1/2}\,\|F\|_\infty \int_{|Q|^{1/2}}^{+\infty}\frac{1}{r^2}\,dr$$

$$\leq c\,\|F\|_\infty .$$

Therefore, $T_1 : x_2^\varepsilon L^\infty \longrightarrow x_2^\varepsilon\,\mathrm{BMO}(dh)$, which proves B). Similarly, we have that if $T^*\colon L^2(dh) \longrightarrow L^2(dh)$ then, $T^*\colon x_2^\varepsilon L^\infty \longrightarrow x_2^\varepsilon\,\mathrm{BMO}(dh)$.

It is easy, and left to the reader, to check that if $T\colon L^2(dh) \longrightarrow L^2(dh)$ then T has the LWBP on each Whitney Q_j.

Now we should prove the converse. Once more we remark that A) has already established, by Schur's Lemma, the $L^2(dh)$-boundedness of T_2.

PROOF OF C). If T_1 has the LWBP on each Q_j and $T_1(1) \in \mathrm{BMO}(dh)$, $T_1^*(1) \in \mathrm{BMO}(dh)$, then T_1 is bounded on $L^2(dh)$.

To see this we take the Whitney decomposition $\{Q_j\}$ for the upper half plane and we divide it into 9 subfamilies $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_9$ so that if f is a function supported in Q_{j_i} and g a function supported in Q_{k_i}, and $Q_{k_i} \in \mathcal{F}_i$, $Q_{j_i} \in \mathcal{F}_i$, $k_i \neq j_i$. Then, $\operatorname{supp} T_1(f)$ and $\operatorname{supp} T_1(g)$ have disjoint interiors.

For Q_k a Whitney cube denote by $N_k = Q_k \cup \{8 \text{ neighbors}\}$. A neighbor of Q_k is a Whitney cube having one side or one vertex in common with Q_k.

Now let f be in $C_0^\eta(\mathbb{R}^2_+)$ and let $\{Q_j\}$ be the Whitney decomposition for $\mathbb{R}^2_+$ and write

$$f = \sum_j f\,\chi_{Q_j} = \sum_{i=1}^{9} \sum_{j_i} f\,\chi_{Q_{j_i}} = \sum_{i=1}^{9} \sum_{j_i} f_{j_i}\,.$$

Then,

$$\|T_1(f)\|_2^2 \leq c(9) \sum_{i=1}^{9} \sum_{j_i} \|T_1(f_{j_i})\|_2^2\,.$$

We wish to conclude that $\|T_1(f_{j_i})\|_2^2 \leq c_0\,\|f_{j_i}\|_2^2$ where c_0 is independent of Q_j, any Whitney cube. Observe that $\|f\|_2^2 = \sum_{i=1}^{9} \sum_{j_i} \|f_{j_i}\|_2^2\,$.

To see this, let us write T_1 in the following way: for h such that $\operatorname{supp} h \subseteq Q_j$

$$\begin{aligned} T_1(h) &= L_j(h) + E_j(h)\,, \\ L_j(h)(x) &= \int_{\{\rho(x,y)<1\}} \chi_{Q_j}(x)\,K(x,y)\,h(y)\,\frac{dy}{y_2^2}\,, \\ E_j(h)(x) &= \int_{\{\rho(x,y)<1\}} \chi_{N_j \setminus Q_j}(x)\,K(x,y)\,h(y)\,\frac{dy}{y_2^2}\,. \end{aligned}$$

Then $L_j : L^2(Q_j, d\omega_j) \longrightarrow L^2(Q_j, d\omega_j)$ continuously. Indeed, $T_2(w) \in w\,\mathrm{BMO}$ and $T(w) \in w\,\mathrm{BMO}$, $w(x) = x_2^\varepsilon$. Then $T_1(w) \in w\,\mathrm{BMO}$, and

$$\frac{T_1(w)}{w} \in \mathrm{BMO}(dh) \qquad \text{implies} \qquad \frac{T_1(w)}{w} \in \mathrm{BMO}(Q_j, d\omega_j)\,.$$

Now, if $x \in Q_j$ then

$$w(x) = x_2^\varepsilon = d_j^\varepsilon\, b(x)^\varepsilon\,,$$

where $1/4 \leq b(x) \leq 4$ is independent of j and $d_j = d(Q_j, \mathbb{R})$. Therefore, on Q_j,

$$\frac{T_1(w)(x)}{w(x)} = \frac{T_1(b)(x)}{b(x)}\,,$$

and so

$$\frac{T_1(b)(x)}{b(x)} \in \mathrm{BMO}(Q_j, d\omega_j)\,.$$

Observe that $b(x)$ is a positive, Hölder continuous function on Q_j with constant $c\,d_j^{-\varepsilon}$.

Then by Stegenga [Sg] we can prove that if

$$\frac{T_1(b)(x)}{b(x)} \in \mathrm{BMO}(Q_j, d\omega_j) \qquad \text{then} \qquad T_1(b)(x) \in \mathrm{BMO}(Q_j, d\omega_j)$$

with constant depending on the BMO constant of $T_1(w)/w$, on $\|b\|_\infty$ and on

$$\alpha = \sup_{Q\subseteq Q_j} \frac{1}{\omega_j(Q)} \left(\log \frac{1}{\omega_j(Q)}\right) \int_Q |b(x) - m_Q b| \, d\omega_j \,,$$

$$m_Q b = \frac{1}{\omega_j(Q)} \int_Q b(x)\, d\omega_j \,.$$

It is easy to see now that since $|b(x) - b(x')| \leq c\,|x - x'|^\varepsilon / d_j^\varepsilon$, we have that $\alpha \leq 1$.

Then $T_1(b) \in \mathrm{BMO}(Q_j, d\omega_j)$ with constant independent of j.

On the other hand,

$$T_1(b) = T_1(b\,\chi_{Q_j}) + T_1(b\,(1 - \chi_{Q_j}))\,.$$

But $T_1(b(1-\chi_{Q_j}))$ is in $\mathrm{BMO}(Q_j, d\omega_j)$ with constant independent of j (this follows from Lemma 2.1 at the end of this section; in fact, $T_1(b(1-\chi_{Q_j}))(x) \sim \log(|Q_j|^{1/2}/d(x, \partial Q_j))$). Then $T_1(b\,\chi_{Q_j}) \in \mathrm{BMO}(Q_j, d\omega_j)$. In the same way $T_1^*(b\,\chi_{Q_j}) \in \mathrm{BMO}(Q_j, d\omega_j)$. Therefore,

i) $L_j(b\,\chi_{Q_j}) \in \mathrm{BMO}(Q_j, d\omega_j)$,

ii) $L_j^*(b\,\chi_{Q_j}) \in \mathrm{BMO}(Q_j, d\omega_j)$,

iii) L_j has the WBP on Q_j -relative to $d\omega_j$.

Therefore, by the $T1$-Theorem (*cf.* [DJ])

$$L_j : L^2(Q_j, \frac{dx}{x_2^2}) \longrightarrow L^2(Q_j, \frac{dx}{x_2^2})$$

with constant independent on j.

Now we concentrate on E_j. We wish to show that

$$E_j : L^2(N_j, \frac{dx}{x_2^2}) \longrightarrow L^2(N_j, \frac{dx}{x_2^2}),$$

$$E_j(f)(x) = \int \chi_{Q_j}(x) \frac{K(x,y)}{y_2^2} \chi_{N_j \setminus Q_j}(y) f(y)\, dy .$$

If we call

$$e_j(x,y) = \chi_{Q_j}(x) \frac{K(x,y)}{y_2^2} \chi_{N_j \setminus Q_j}(y)$$

then

$$|e_j(x,y)| \le \frac{c}{|x-y|^2}, \qquad \text{for } x \in Q_j,\ y \in N_j \setminus Q_j .$$

We will show that there exist two positive measurable functions $p(x)$ and $q(y)$ and two positive numbers α and β such that

$$(\mathrm{S}_1) \qquad \int |e_j(x,y)|\, q(y)\, dy \le \alpha\, p(x), \qquad \text{for almost every } x,$$

$$(\mathrm{S}_2) \qquad \int |e_j(x,y)|\, p(x)\, dx \le \beta\, q(y), \qquad \text{for almost every } y.$$

Let

$$q(y) = d(y, \partial Q_j)^{-1/2}, \qquad y \in N_j \setminus Q_j ,$$
$$p(x) = d(x, \partial Q_j)^{-1/2}, \qquad x \in Q_j .$$

We need to show that

$$(\mathrm{S}_1) \quad \chi_{Q_j}(x) \int_{N_j \setminus Q_j} |e_j(x,y)|\, d(y, \partial Q_j)^{-1/2}\, dy \le \alpha\, d(x, \partial Q_j)^{-1/2},$$

$$(\mathrm{S}_2) \quad \chi_{N_j \setminus Q_j}(x) \int_{Q_j} |e_j(x,y)|\, d(y, \partial Q_j)^{-1/2}\, dy \le \beta\, d(x, \partial Q_j)^{-1/2}.$$

First we observe that it is enough to prove (S_1) and (S_2) for $\overline{\overline{Q}}_j$ instead of $N_j \setminus Q_j$ where $\overline{\overline{Q}}_j$ is the cube with the same center as Q_j and 5 times its side length.

Next we observe that it is enough to show

$$(\mathrm{S}_1') \quad \chi_{B_j}(x) \int_{\overline{\overline{B}}_j \setminus B_j} |e_j(x,y)|\, d(y, \partial B_j)^{-1/2}\, dy \le \alpha\, d(x, \partial B_j)^{-1/2}$$

$$(\mathrm{S}_2') \quad \chi_{\overline{\overline{B}}_j \setminus B_j}(x) \int_{B_j} |e_j(x,y)|\, d(y, \partial B_j)^{-1/2}\, dy \le \beta\, d(x, \partial B_j)^{-1/2},$$

where B_j is a ball with same center as Q_j and radius comparable to $\ell(Q_j)$.

Indeed, given Q_j and B_j, there is a bilipschitz map $h : \mathbb{R}^2 \to \mathbb{R}^2$ sending B_j to Q_j. The interior of B_j goes to the interior of Q_j and there exists $M > 0$ such that

$$\frac{1}{M} \leq \frac{h(z) - h(w)}{z - w} \leq M .$$

Therefore if Jh is the jacobian of h, $|Jh(z)| \leq 2M^2$ almost everywhere. Moreover, there exists a bilipschitz map $h : \mathbb{R}^2 \to \mathbb{R}^2$ sending $\overline{\overline{B}}_j \setminus B_j$ to $\overline{\overline{Q}}_j \setminus Q_j$. First consider h_1 bilipschitz from $\mathbb{R}^2$ to $\mathbb{R}^2$ sending $\overline{\overline{B}}_j$ to $\overline{\overline{Q}}_j$ and the interior to the interior. Then we consider a neighborhood C_j of $h_1(B_j)$ at a distance proportional to $\partial\overline{\overline{B}}_j$ and ∂B_j and consider $h_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a bilipschitz map such that $h_2(z) = \tilde{h}_2(z)$ if $z \in C_j$ and $h_2(z) = z$ if $z \in \mathbb{R}^2 \setminus C_j$.

Here, $\tilde{h}_2$ is the map that sends C_j to another neighborhood C_j' and maps $h_1(B_j)$ -inside C_j- to Q_j -inside C_j'-. Also, $C_j \setminus h_1(B_j)$ is mapped to $C_j' \setminus Q_j$.

Finally, we take $h = h_2 \circ h_1$, $\quad h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\overline{\overline{B}}_j \setminus B_j$ to $\overline{\overline{Q}}_j \setminus Q_j$ and is bilipschitz with constant M' independent of j, (*cf.* [T], [JK]).

To prove (S_1') and (S_2') we can proceed in two different ways. One is elementary, the other more constructive. We choose to prove (S_1') in the elementary and (S_2') in the other way. We can assume B_j is centered at 0, that x lies in the real axis, and it is in the interior of B_j.

Let r be the radius of B_j. Then,

$$\begin{aligned}
&\int_{\overline{\overline{B}}_j \setminus B_j} |e_j(x,y)|\, d(y, \partial B_j)^{-1/2}\, dy \\
&\qquad \leq \int_{\overline{\overline{B}}_j \setminus B_j} \frac{1}{|x-y|^2}\, d(y, \partial B_j)^{-1/2}\, dy \\
&\qquad = \int_{-\pi}^{\pi} \int_r^{3r} \frac{\rho\,(\rho - r)^{-1/2}}{\rho^2 + |x|^2 - 2\rho |x| \cos\,\theta}\, d\rho\, d\theta \\
&\qquad = 2\pi \int_r^{3r} \frac{\rho}{(\rho - r)^{1/2}(\rho^2 - |x|^2)}\, d\rho \\
&\qquad \leq 2\pi \int_r^{3r} \frac{d\rho}{(\rho - r)^{1/2}(\rho - |x|)}
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \int_{r-|x|}^{3r+|x|} \frac{d\rho}{\rho(\rho+|x|-r)^{1/2}} \\
&\le \frac{\pi}{\sqrt{r-|x|}} \\
&= \pi\, d(x, \partial B_j)^{-1/2},
\end{aligned}
$$

where we have used

1) if $b^2 > c^2$,

$$
\int \frac{dx}{b + c\cos ax} = \frac{2}{a(b^2-c^2)^{1/2}} \arctan\Big(\frac{(b-c)\tan(ax/2)}{(b^2-c^2)^{1/2}}\Big) + c,
$$

2) if $b < 0$,

$$
\int \frac{dx}{x\,(ax+b)^{1/2}} = \frac{2}{\sqrt{-b}} \arctan \frac{(ax+b)^{1/2}}{\sqrt{-b}} + c\,.
$$

To prove (S_2') we can assume with no loss of generality that $B_j = \Delta$, the unit disk centered at 0. We wish to prove that there exists $\beta > 0$ such that

$$
\begin{aligned}
&\int_\Delta \frac{1}{|x-y|^2}\,(1-|y|^2)^{-1/2}\,dy \le \beta\,(|x|^2-1)^{-1/2}, \qquad \text{for } x \notin \Delta, \\
&\int_\Delta \frac{1}{|x-y|^2}\,(1-|y|^2)^{-1/2}\,dy \le \frac{1}{|a|}\int_\Delta \frac{1}{|x-y|^2}\,(1-|y|^2)^{-1/2}\,dy,
\end{aligned}
$$

where $\bar a = 1/x$ and $|a| < 1$. There is no loss of generality in assuming that x is real and that $|a| \ge 3/4$. So, it is enough to prove that

$$
\int_\Delta \frac{1}{|1-\bar a y|^2}\,(1-|y|^2)^{-1/2}\,dy \le c\,(1-|a|^2)^{-1/2}.
$$

To do so we look at the level lines for $1/|1-\bar a y|$, at the points $c_n = 2^{-n}/(1-|a|^2)$

$$
\begin{aligned}
C_n &= \Big\{ y \in \Delta : \ \frac{1}{|1-\bar a y|} = \frac{2^{-n}}{(1-|a|^2)} \Big\} \\
&= \Big\{ y \in \Delta : \ \Big|\frac{1}{\bar a} - y\Big|^2 = \frac{2^n(1-|a|^2)^2}{|a|^2} \Big\}.
\end{aligned}
$$

These are circles centered at $1/\bar{a}$ and radius $2^n(1-|a|^2)/|a|$. Observe that if $n = 0$, the radius is $(1-|a|^2)/|a| \sim 1/|\bar{a}| - 1$. This gives us the first circle whose intersection with Δ occurs in its boundary. As n increases we obtain a sequence of circles each time with double radius. Then

$$\int_\Delta \frac{1}{|1-\bar{a}y|^2}(1-|y|^2)^{-1/2}\,dy \leq \sum_{n=0}^{\infty}\int_{A_n}\frac{1}{|1-\bar{a}y|^2}(1-|y|^2)^{-1/2}\,dy\,,$$

where

$$A_n = \{y \in \Delta :\ 2^{n-1}(1-|a|^2) \leq (1-\bar{a}y) \leq 2^n(1-|a|^2)\}\,, \quad \Delta \subseteq \bigcup_{n=0}^{\infty} A_n\,.$$

But the last inequality is less than or equal to

$$c\sum_{n=0}^{M_0}\frac{2^{-2n}}{(1-|a|^2)^2}\int_{A_n}\frac{dy}{(1-|y|^2)^{1/2}} + \int\limits_{R_{M_0}}\frac{1}{|1-\bar{a}y|^2}(1-|y|^2)^{-1/2}\,dy\,,$$

where $R_{M_0} = \Delta \setminus \bigcup_{n=0}^{M_0} A_n$. Now, A_n is contained in a larger region T_n for $n \leq M_0$ where M_0 is the last integer n before C_n touches or includes 0 in A_{n+1}. In polar coordinates, T_n is defined by letting θ vary between 0 and $\ell(\Gamma_n)$ and r between $1-2^n(1-|a|^2)$ and 1. We call $\Gamma_n = \partial(\cup_{j=0}^n A_j)\cap\partial\Delta$.

If $y \in R_{M_0}$ then $|1-\bar{a}y| > 1/10$, and

$$\begin{aligned}\int_{R_{M_0}}\frac{1}{|1-\bar{a}y|^2}\frac{1}{(1-|y|^2)^{-1/2}}\,dy &\leq \frac{4}{3}10^2\int_\Delta\frac{1}{(1-|y|^2)^{-1/2}}\,dy\\ &\leq c_0\,(1-|a|^2)^{-1/2}\,.\end{aligned}$$

Now,

$$\begin{aligned}\int_{T_n}\frac{1}{(1-|y|^2)^{-1/2}}\,dy &= \int_0^{\ell(\Gamma_n)}d\theta\int_{1-2^n(1-|a|^2)}^1\frac{r}{(1-r^2)^{1/2}}\,dr\\ &\leq c\,\ell(\Gamma_n)\,2^{n/2}(1-|a|^2)^{1/2}\,.\end{aligned}$$

But circle is chord arc and so we have that $\ell(\Gamma_n) \sim |b_n - d_n| \leq c\,2^n(1-|a|^2)$, where c is a positive absolute constant and b_n, d_n are the points where C_n crosses $\partial\Delta$. Therefore,

$$\begin{aligned}c\sum_{n=0}^{M_0}\frac{2^{-2n}}{(1-|a|^2)^2}\,2^{3n/2}(1-|a|^2)^{3/2} &\leq c\,(1-|a|^2)^{-1/2}\sum_{n=0}^{\infty}2^{-n/2}\\ &\leq c\,(1-|a|^2)^{-1/2}\,.\end{aligned}$$

Therefore we have (S'_2) and with this we conclude the proof of Theorem 2.1.

Lemma 2.1. *Let S be a singular integral operator associated to a kernel $s(x,y)$ satisfying standard estimates. Then $S(1-\chi_{Q_j})(x)$ is in* $\mathrm{BMO}(Q_j)$ *with constant independent of the size of Q_j.*

PROOF. The proof of this uses the same sort of argument used to show that a Calderón-Zygmund operator maps L^∞ into BMO continuously. We refer the reader to [DJS], [N].

Preliminaries for Section 3.

We would like to recall some of the results necessary for the proof of Theorem 3.1. No proofs are shown but they can be found at the indicated references.

Given a simply connected domain Ω (with nontrivial boundary) the Riemann mapping Theorem tells us we can construct a univalent mapping -that is a conformal homeomorphism- f of the unit disk Δ onto Ω.

Lemma P.1. *Suppose Ω and Ω' are domains on $\bar{\mathbb{C}}$ and $f : \Omega' \longrightarrow \Omega$ is conformal. Then if $G(z,w)$ is a Green's function on Ω, $G'(z,w) \equiv G(f(z), f(w))$ is a Green's function for Ω'.*

Theorem K. (Weak form of Koebe 1/4 Theorem). *Suppose $f\colon \Delta \longrightarrow \mathbb{C}$ is univalent, $f(0) = 0$, and $f'(0) = 1$. Then there exists $c_0 > 0$ (independent of f) such that $D(0, c_0) \subset f(\Delta)$.*

Corollary K. *If $f : \Omega \longrightarrow \Omega'$ is conformal, then for all $z \in \Omega$,*

$$|f'(z)| \sim \frac{\operatorname{dist}\{f(z), \partial\Omega'\}}{\operatorname{dist}\{z, \partial\Omega\}} \,.$$

If f is a univalent function on Δ, f' never vanishes, so we can write $f' = e^{\varphi}$ for some holomorphic φ on Δ.

Theorem P.2. *There is a universal constant $c_0 > 0$ such that if f is univalent on Δ and $f' = e^{\varphi}$ then $|\varphi'(z)| \le c_0\,(1-|z|)^{-1}$ for all $z \in \Delta$.*

Theorem P.3. *There is a universal constant $\varepsilon_0 > 0$ such that if $f' = e^{\varphi}$ where φ is holomorphic and $|\varphi'(z)| \le \varepsilon(1-|z|)^{-1}$ for $\varepsilon \le \varepsilon_0$ and $z \in \Delta$, then $f : \Delta \longrightarrow f(\Delta)$ is a conformal map onto a Jordan domain bounded by a quasicircle (with constant $\sim 1 + c\varepsilon$).*

And conversely, let Ω be a simply connected domain in $\mathbb{R}^2$ bounded by a Jordan curve that is a K-quasicircle (∞ is fixed), $K = 1 + \varepsilon$. Let f be the Riemann map from Δ to Ω (f has a K^2-quasiconformal extension to $\mathbb{R}^2$, ∞ remains fixed) and let $f' = e^{\varphi}$. Then,

$$\sup_{\Delta} |\varphi'(z)|\,(1-|z|^2) \le c_0\,\varepsilon\,,$$

where $c_0 > 0$ is an absolute constant (cf. [P], [L]).

A quasicircle is a Jordan curve Γ in $\mathbb{R}^2$ that is the image of the unit circle $\mathbb{T}$ under a globally quasiconformal homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$. Conformal maps are mappings sending small circles to small circles. Quasiconformal mapping take small circles to ellipses of bounded eccentricity, (*cf.* [A], [LV], for an exposition in the subject).

Definition P.4. *For g defined on Δ we define*

$$\|g\|_{\mathcal{B}} = \sup_{\Delta} |g'(z)|\,(1-|z|^2)\,.$$

The set of holomorphic functions g on Δ with $\|g\|_{\mathcal{B}} < +\infty$ is called the Bloch class $\mathcal{B}$; $\|\cdot\|_{\mathcal{B}}$ is conformally invariant.

Note that on Δ, $1-|z| \sim 1-|z|^2$ so if f is univalent and $f' = e^{\varphi}$ then by Theorem P.2, φ is in the Bloch class.

Recall that on Δ the hyperbolic distance is defined by

$$\rho(z,w) = \log \frac{|1-z\bar{w}| + |z-w|}{|1-z\bar{w}| - |z-w|}\,.$$

Lemma P.5. *If φ is holomorphic on Δ then $\varphi \in \mathcal{B}$ if and only if there is an $A > 0$ such that $|\varphi(z) - \varphi(w)| \le A\,\rho(z,w)$ for all $z, w \in \Delta$. Moreover, if A_0 is the smallest such constant $A_0 \sim \|\varphi\|_{\mathcal{B}}$ (cf.* [P]).

3. Proof of Theorem 3.1.

Set $K = 1+\varepsilon$, then $K^2 \sim 1+3\varepsilon$, ε is very small and it will become clear at the end of the proof how small it should be.

Let $f : \Delta \to \Omega$ univalent. Let $h = f^{-1} : \Omega \to \Delta$ univalent, ρ^* the hyperbolic metric on Ω, $\rho^*(z', w') = \rho(h(z'), h(w')) = \rho(z, w)$, $z' = f(z)$, $w' = f(w)$, ρ the hyperbolic metric on Δ. We want to prove that if $\delta(w) = \operatorname{dist}\{w, \partial\Omega\}$ then there exists $\eta = \eta(\varepsilon)$, $0 < \eta < 1$ and $c > 0$ an absolute constant such that for $z' \in \Omega$,

$$\int R(z', w')\, \delta(w')^\eta \, \frac{dw'}{\delta(w')^2} \le c\, \delta(z')^\eta .$$

Recall that if $A \subseteq \Omega$, $\operatorname{vol}_h(A) = \int_A dw'/\delta(w')^2$,

$$\int R(z', w')\, \delta(w')^{\eta-2}\, dw' \le \int_{\{w' : \rho^*(z', w') > 1\}} \frac{(1-|h(z')|^2)^{1/K^2} (1-|h(w')|^2)^{1/K^2}}{|1 - h(z')\overline{h(w')}|^{2/K^2}}\, \delta(w')^{\eta-2}\, dw' ,$$

$$1 - |h(z')|^2 = d(h(z'), \partial\Delta) \qquad \text{and} \qquad f'(z) = \frac{1}{h(z')} \quad \text{if} \quad h(z') = z .$$

By Theorem K we have that

$$\begin{aligned} d(w', \partial\Omega) &\sim |f'(w)|\, d(w, \partial\Delta) \\ &\sim \frac{1}{|h'(w')|}\, (1 - |h(w')|^2) , \end{aligned}$$

h is conformal. Therefore, if $w = h(w')$

$$\frac{dw}{\delta(w)^2} \sim \frac{dw'}{\delta(w')^2} .$$

Then, the last inequality is

$$\begin{aligned} &\le c\, \delta(z')^{1/K^2} |h'(z')|^{1/K^2} \int_{\{w : \rho(w, z) > 1\}} \frac{(1-|w|^2)^{1/K^2+\eta}}{|1 - h(z')\bar{w}|^{2/K^2}}\, |f'(w)|^\eta \, \frac{dw}{\delta(w)^2} \\ &\le c\, (1-|z|^2)^{1/K^2} \int_{\{w : \rho(z, w) > 1\}} \frac{\delta(w)^{1/K^2+\eta}}{|1 - z\bar{w}|^{2/K^2}}\, |f'(w)|^\eta \, \frac{dw}{\delta(w)^2} . \end{aligned}$$

We know that $f' = e^{\varphi}$ for some φ in the Bloch class such that $\|\varphi\|_{\mathcal{B}} \leq c_0\,\varepsilon$; call $\delta = c_0\,\varepsilon$. Then we can make the last expression less than or equal to

$$c \int_{\Delta} e^{-\rho(z,w)/K^2}\, e^{\eta \operatorname{Re}\varphi(w)}\, \delta(w)^{\eta}\, \frac{dw}{\delta(w)^2}$$
$$= c\, e^{\eta u(z)} \int_{\Delta} e^{-\rho(z,w)/K^2}\, e^{\eta(u(w)-u(z))}\, \delta(w)^{\eta}\, \frac{dw}{\delta(w)^2}\,,$$

where $u = \operatorname{Re}\varphi$ and $|f'(w)|^{\eta} = e^{\eta u(w)}$. But,

$$\|\varphi\|_{\mathcal{B}} \leq \delta \qquad \text{implies} \qquad u(w) - u(z) \leq |\varphi(w) - \varphi(z)| \leq \delta\, \rho(w,z)\,.$$

Therefore,

$$\leq c\, e^{\eta u(z)} \int_{\Delta} e^{-(1/K^2 - \eta\delta)\,\rho(w,z)}\, \delta(w)^{\eta-2}\, dw$$
$$\leq c\, e^{\eta u(z)} \int_{\Delta} \frac{(1-|z|^2)^{1/K^2-\eta\delta}\,(1-|w|^2)^{-(1/K^2-\eta+\eta\delta)}}{(|1-z\bar{w}|^2)^{1/K^2-\eta\delta}}\, dw\,.$$

Let $1+\alpha = 1/K^2 - \eta\delta$, $r = -(-1/K^2 + 2 - \eta + \eta\delta)/2$. Then

$$= c\, e^{\eta u(z)}\,(1-|z|^2)^{1+\alpha} \int_{\Delta} \frac{(1-|w|^2)^{2r}}{|1-z\bar{w}|^{2(1+\alpha)}}\, dw\,.$$

Call $B(z,z)^{-r} = (1-|z|^2)^{2r}$. We know that for $r > -1/2$ and $\alpha - r > 0$ (*cf.* [CR]),

$$\int_{\Delta} \frac{B(w,w)^{-r}}{|1-z\bar{w}|^{2(1+\alpha)}}\, dw \leq \frac{1}{(1-|z|^2)^{2(\alpha-r)}}\,.$$

But,

$$r = \frac{1}{2}\Big(\frac{1}{K^2} - 2 - \eta\delta + \eta\Big) > -\frac{1}{2} \quad \text{if and only if} \quad \eta > \Big(1 - \frac{1}{K^2}\Big)\frac{1}{1-\delta}\,.$$

And,

$$\alpha - r > 0 \qquad \text{if and only if} \qquad \eta < \frac{1}{(\delta+1)K^2}\,.$$

Recall that $\delta = c_0\,\varepsilon$, and that $K^2 \sim 1 + 3\,\varepsilon$. Then, for ε sufficiently small, we have that

$$0 < \Big(1 - \frac{1}{K^2}\Big)\frac{1}{1-\delta} < \frac{1}{(1+\delta)K^2}\,.$$

Therefore, altogether, we have that

$$c\, e^{\eta u(z)} \frac{(1-|z|^2)^{1+\alpha}}{(1-|z|^2)^{2(\alpha-r)}} \leq c\,|f'(z)|^{\eta}\,(1-|z|^2)^{\eta} \leq c\, d(z',\partial\Omega)^{\eta} = c\,\delta(z')^{\eta}$$

for

$$\Big(1-\frac{1}{K^2}\Big)\frac{1}{1-\delta} < \eta < \frac{1}{(1+\delta)K^2}\,,$$

since $1+\alpha-2\alpha+2r=\eta$. This concludes the proof of Theorem 3.1.

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Fourier coefficients of Jacobi forms over Cayley numbers

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Abstract. In this paper, we shall compute explicitly the Fourier coefficients of the Eisenstein series

$$E_{k,m}(z,w) = \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k} \sum_{t \in \mathfrak{o}} \exp\Big\{2\pi i m\Big(\frac{az+b}{cz+d} N(t) + \sigma\big(t, \frac{w}{cz+d}\big) - \frac{cN(w)}{cz+d}\Big)\Big\}$$

which is a Jacobi form of weight k and index m defined on $\mathcal{H}_1 \times \mathcal{C}_{\mathbb{C}}$, the product of the upper half-plane and Cayley numbers over the complex field $\mathbb{C}$. The coefficient of $e^{2\pi i(nz+\sigma(t,w))}$ with $nm > N(t)$, has the form

$$-\frac{2(k-4)}{B_{k-4}} \prod_p S_p \, .$$

Here S_p is an elementary factor which depends only on $\nu_p(m)$, $\nu_p(t)$, $\nu_p(n)$ and $\nu_p(nm - N(t))$. Also $S_p = 1$ for almost all p. Indeed, one has $S_p = 1$ if $\nu_p(m) = \nu_p(nm - N(t)) = 0$. An explicit formula for S_p will be given in details. In particular, these Fourier coefficients are rational numbers.

1. Notation and Introduction.

As usual $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, and the field of rational numbers, real numbers, and complex numbers, respectively. $\mathcal{C}_f$ is the Cayley numbers over the field f and $\mathbf{o}$ is the ring of integral Cayley numbers in $\mathcal{C}_{\mathbb{R}}$. $\mathcal{C}_f$ is an eight-dimensional vector space over f with a basis $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ which is characterized by the follwing rules for multiplication ([1]):

1) $xe_0 = e_0x = x$, for all $x \in \mathcal{C}$,

2) $e_i^2 = -e_0, \quad i = 1, 2, \ldots, 7$,

3) $e_1e_2e_4 = e_2e_3e_5 = e_3e_4e_6 = e_4e_5e_7 = e_5e_6e_1 = e_7e_1e_3 = -e_0$.

Also $\mathbf{o}$ has a $\mathbb{Z}$-basis α_0, α_1, α_2, α_3, α_4, α_5, α_6, α_7 given by

$$\alpha_0 = e_0\ , \quad \alpha_1 = e_1\ , \quad \alpha_2 = e_2\ , \quad \alpha_3 = e_4\ ,$$
$$\alpha_4 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4)\,, \quad \alpha_5 = \frac{1}{2}(-e_0 - e_1 - e_4 + e_5)\,,$$
$$\alpha_6 = \frac{1}{2}(-e_0 + e_1 - e_2 + e_6)\,, \quad \alpha_7 = \frac{1}{2}(-e_0 + e_2 + e_4 + e_7)\,.$$

For $x = \sum_{j=0}^{7} x_j e_j\,,\ y = \sum_{j=0}^{7} y_j e_j\,;\ x_j, y_j \in f$, we define

$$N(x) = \sum_{j=0}^{7} x_j^2\,, \qquad \sigma(x,y) = 2\sum_{j=0}^{7} x_j\, y_j\ .$$

Let k, m be a pair of positive integers. A holomorphic function ψ on $\mathcal{H}_1 \times \mathcal{C}_{\mathbb{C}}$ is a Jacobi form of weight k and index m if it satisfies the following conditions

$$\text{(J.1)} \quad \psi\Big(\frac{az+b}{cz+d}, \frac{w}{cz+d}\Big) = (cz+d)^k \exp\big\{2\pi i m N(w)\,\frac{c}{cz+d}\big\}\,\psi(z,w)\,,$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL_2(\mathbb{Z})$.

$$\text{(J.2)} \quad \psi(z, w + \lambda z + \mu) = \exp\{-2\pi i m(zN(\lambda) + \sigma(\lambda, w))\}\,\psi(z,w)\,,$$

for all $\lambda, \mu \in \mathbf{o}$.

(J.3) ψ possesses a Fourier expansion of the form

$$\psi(z,w)=\sum_{n\geq 0}\sum_{\substack{t\in\mathbf{o}\\ nm\geq N(t)}}\alpha_\psi(n,t)\,e^{2\pi i(nz+\sigma(t,w))}\,.$$

For positive integers k,m with k even and $k\geq 10$, we let

$$E_{k,m}(z,w)=\frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-k}\sum_{t\in\mathbf{o}}\exp\Big\{2\pi im\Big(\frac{az+b}{cz+d}N(t)+\sigma\big(t,\frac{w}{cz+d}\big)-\frac{cN(w)}{cz+d}\Big)\Big\}$$

with $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$. Then a direct verification shows that $E_{k,m}$ satisfies (J.1) and (J.2). For the proof, see [5].

In this paper, we shall show that the Fourier coefficient $e_{k,m}(n,t)$ with $nm>N(t)$ in the Fourier expansion

$$E_{k,m}(z,w)=\sum_{n=0}^{\infty}\sum_{\substack{nm\geq N(t)\\ t\in\mathbf{o}}}e_{k,m}(n,t)\,e^{2\pi i(nz+\sigma(t,w))}$$

of $E_{k,m}(z,w)$ is a rational number of the form

$$-\frac{2(k-4)}{B_{k-4}}\prod_p S_p\,.$$

Let ν_p be the standard discrete valuation in $\mathbb{Q}_p$ with $\nu_p(p^j)=j$ for all $j\in\mathbb{Z}$. For $t=\sum_{j=0}^{7}t_j\alpha_j\in\mathbf{o}$, we set $\nu_p(t)=\min_{0\leq j\leq 7}\nu_p(t_j)$. For our convenience, we set $\triangle=mn-N(t)$ and $\triangle'=\triangle/m=n-N(t)/m$.

Theorem. *For positive integers m, k with k even and $k\geq 10$, the Fourier coefficient $e_{k,m}(n,t)=0$ if $nm<N(t)$. If $mn=N(t)$, then*

$$e_{k,m}(n,t)=\begin{cases}1, & \text{if } t=mt' \text{ and } n=mN(t') \text{ for some } t'\in\mathbf{o}\,,\\ 0, & \text{otherwise}.\end{cases}$$

If $mn>N(t)$, then

$$e_{k,m}(n,t)=-\frac{2(k-4)}{B_{k-4}}\prod_p S_p\,,$$

where S_p is given by

1) *If* $\nu_p(m)=0$, *then* $S_p=\sum_{j=0}^{\nu_p(\triangle)} p^{j(k-5)}$.

2) *If* $\nu_p(m)>\nu_p(t)$, *then*

$$S_p=p^{l_1}\frac{1-p^{8-k}}{1-p^{4-k}}\sum_{j=0}^{\alpha} p^{j(9-k)}+\begin{cases} p^{l_2}(1-p^{4-k})^{-1}, & \text{if } \nu_p(t)\le\nu_p(n),\\ 0, & \text{if } \nu_p(t)>\nu_p(n).\end{cases}$$

3) *If* $\nu_p(m)\le\nu_p(t)$, *then*

$$S_p=p^{l_1}\frac{1-p^{8-k}}{1-p^{4-k}}\sum_{j=0}^{\beta} p^{j(9-k)}+\begin{cases} p^{l_3}(1-p^{4-k})^{-1}, & \text{if } \nu_p(m)\le\nu_p(n),\\ 0, & \text{if } \nu_p(m)>\nu_p(n).\end{cases}$$

$$+\sum_{j=\gamma+1}^{\nu_p(\triangle')} p^{(k-5)(\nu_p(\triangle')-j)}-\begin{cases} p^{l_4}(1-p^{4-k})^{-1}, & \text{if } \nu_p(m)\le\nu_p(\triangle'),\\ 0, & \text{if } \nu_p(m)>\nu_p(\triangle').\end{cases}$$

In 2) *and* 3), *one has*

$$l_1=-(k-1)\,\nu_p(m)+(k-5)\,\nu_p(\triangle),$$
$$l_2=-(k-1)\,\nu_p(m)+(k-5)\,\nu_p(\triangle)+(9-k)\,\nu_p(t)+8-k,$$
$$l_3=(10-2k)\,\nu_p(m)+(k-5)\,\nu_p(\triangle)+8-k,$$
$$l_4=(10-2k)\,\nu_p(m)+(k-5)\,\nu_p(\triangle)+4-k.$$

Also

$$\alpha=\min\{\nu_p(t),\nu_p(n)\}, \qquad \beta=\min\{\nu_p(m),\nu_p(n)\}$$

and

$$\gamma=\min\{\nu_p(m),\nu(\triangle')\}.$$

REMARK. Note that in the above, (1) is a special case of (3) and S_p in (1) can be obtained from S_p given in (3) by setting $\nu_p(m)=0$.

In particular, we have

$$e_{k,1}(n,t)=\begin{cases} 1, & \text{if } n=N(t),\\ -\dfrac{2(k-4)}{B_{k-4}}\sigma_{k-5}(n-N(t)), & \text{if } n>N(t).\end{cases}$$

From this, we conclude that $E_{k,1}(z,w)$ is a product of $E_{k-4}(z)$, the normalized Eisenstein series of weight $k-4$, and

$$\theta(z,w)=\sum_{t\in\mathfrak{o}} e^{2\pi i(zN(t)+\sigma(t,w))}$$

which is a Jacobi form of weight 4 and index 1.

The Fourier coefficient $\tilde{e}_{k,m}(n,r)$ of the Eisenstein series

$$\tilde{E}_{k,m}(z,w) = \frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-k}\sum_{\lambda\in\mathbb{Z}}\exp\left\{2\pi i m\left(\frac{az+b}{cz+d}\lambda^2 + 2\lambda\frac{w}{cz+d} - \frac{cw^2}{cz+d}\right)\right\}$$

was given in [4] in terms of Cohen's function $H(s,N)$; *i.e.*

$$\tilde{e}_{k,1}(n,r) = \frac{H(k-1,4n-r^2)}{\zeta(3-2k)}\,.$$

If m is square free, then

$$\tilde{e}_{k,m}(n,r) = \frac{\sigma_{k-1}(m)^{-1}}{\zeta(3-2k)}\sum_{d|(n,m,r)} d^{k-1}H(k-1,\frac{4nm-r^2}{d^2})$$

by the relation $\tilde{E}_{k,1}\ |_{T(m)}\ (z,w) = \sigma_{k-1}(m)\,\tilde{E}_{k,m}(z,w)$. Here $T(m)$ is the Hecke operator on the space of Jacobi forms of weight k and index 1, $J_{k,1}$, defined by

$$\psi|_{T(m)}(z,w) = m^{k-1}\sum_{ad=m}\sum_{0\leq b<d} d^{-k}\,\psi(\frac{az+b}{d},\frac{mw}{d})\,.$$

However, we do not see any relation such as

$$E_{k,1}\ |_{T(m)}\ (z,w) = \sigma_{k-1}(m)E_{k,m}(z,w)$$

in the cases for Cayley numbers even if m is a prime number.

2. Fourier Coefficients of $E_{k,m}$.

From the formula for $E_{k,m}(z,w)$, we separate the sum over c and d into two sums $E^0_{k,m}(z,w)$ and $E^1_{k,m}(z,w)$ according to c is zero or not. If $c=0$, then $d=1$ or -1. We choose $a=d=1$ or -1, and $b=0$, so that

$$(1)\qquad E^0_{k,m}(z,w) = \sum_{t\in\mathfrak{o}} e^{2\pi i m(zN(t)+\sigma(t,w))}\,.$$

Obviously, $E^0_{k,m}$ is a linear combination of $e^{2\pi i(nz+\sigma(t,w))}$ with coefficient 1 or 0 according to $nm = N(t)$ with $t = mt'$, $n = mN(t')$ for some $t' \in \mathbf{o}$ or not. For those terms with $c \neq 0$, we can rewrite the sum as

$$E^1_{k,m}(z,w) = \frac{1}{2}\sum_{(c,d)=1} c^{-k} \sum_{t\in\mathbf{o}} \Big(z+\frac{d}{c}\Big)^{-k}$$

$$\cdot \exp\Big\{2\pi i m\Big(-\frac{N(w-t/c)}{z+d/c}+\frac{aN(t)}{c}\Big)\Big\}. \tag{2}$$

Note that the substitutions $d \mapsto d + cp$ and $t \mapsto t + c\lambda$ correspond to $z \mapsto z+p$ and $w \mapsto w+\lambda$ in $E^1_{k,m}$, respectively. Here p is an integer and λ is an integral Cayley number. Hence

$$E^1_{k,m}(z,w) = \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{(c,d)=1\\ d(\mathrm{mod}\ c)}} \sum_{t\in\mathbf{o}/c\mathbf{o}} \exp\Big(\frac{2\pi i m N(t)}{cd}\Big)$$

$$\cdot F_{k,m}\big(z+\frac{d}{c}, w-\frac{t}{c}\big) \tag{3}$$

with

$$F_{k,m}(z,w) = \sum_{p\in\mathbb{Z}}\sum_{\lambda\in\mathbf{o}} (z+p)^{-k} \exp\Big(-2\pi i m\,\frac{N(w+\lambda)}{z+p}\Big). \tag{4}$$

The function $F_{k,m}(z,w)$ is a periodic function in z and w, so it has Fourier expansion of the form

$$F_{k,m}(z,w) = \sum_{n\in\mathbb{Z}}\sum_{t\in\mathbf{o}} \gamma(n,t)\, e^{2\pi i(nz+\sigma(t,w))}.$$

In order to compute the Fourier coefficient $\gamma(n,t)$ of $F_{k,m}$, we need the following lemma which follows from the well known Poisson summation formula.

Lemma 1. *For any $h > 0$, we have*

$$\sum_{\lambda\in\mathbf{o}} \exp\{-2\pi h N(w+\lambda)\} = \frac{1}{h^4}\sum_{t\in\mathbf{o}} \exp\Big\{-2\pi\Big(\frac{N(t)}{h} + i\,\sigma(t,w)\Big)\Big\}.$$

Proposition 1. *Notation as above, then one has for $k \geq 10$,*

$$(5) \quad F_{k,m}(z,w) = \frac{\alpha_k}{m^{k-1}} \sum_{n\in\mathbb{N}} \sum_{\substack{t\in\mathfrak{o} \\ nm>N(t)}} (nm - N(t))^{k-5} e^{2\pi i(nz+\sigma(t,w))}$$

with

$$\alpha_k = \frac{(-2\pi i)^{k-4}}{(k-5)!} .$$

PROOF. By Lemma 1 and a standard argument (see [2, p. 226]), we get

$$\sum_{\lambda\in\mathfrak{o}} \exp\Big(- 2\pi m \frac{N(w+\lambda)}{z+p}\Big) = \frac{(z+p)^4}{m^4} \sum_{t\in\mathfrak{o}} \exp\Big(2\pi i \frac{N(t)\,(z+p)}{m + 2\pi i\,\sigma(t,w)}\Big) ,$$

for any $z \in \mathcal{H}_1$ and $p \in \mathbb{Z}$. It follows

$$F_{k,m}(z,w) = \frac{1}{m^4} \sum_{p\in\mathbb{Z}} \sum_{t\in\mathfrak{o}} (z+p)^{-k+4} \exp\Big(2\pi i \frac{N(t)\,(z+p)}{m}\Big)\, e^{2\pi i\,\sigma(t,w)} .$$

Note that the series

$$\sum_{p\in\mathbb{Z}} (z+p)^{-k+4} \exp\Big(2\pi i \frac{N(t)\,(z+p)}{m}\Big)$$

is a periodic function in $z = x + iy$. Let

$$\sum_{p\in\mathbb{Z}} (x+iy+p)^{-k+4} \exp\Big(2\pi i \frac{N(t)\,(x+iy+p)}{m}\Big) = \sum_{p\in\mathbb{Z}} c_n(y)\, e^{2\pi i n x} .$$

Then

$$\begin{aligned} c_n(y) &= \int_0^1 \sum_{p\in\mathbb{Z}} (x+iy+p)^{-k+4} \exp\Big\{2\pi i\Big(\frac{N(t)\,(x+iy+p)}{m} - nx\Big)\Big\}\, dx \\ &= \exp(-s\pi N(t)\, y/m) \int_{-\infty}^{+\infty} \frac{e^{2\pi i x(-n+N(t)/m)}}{(x+iy)^{k-4}}\, dx \\ &= \begin{cases} \alpha_k(\Delta')^{k-5}\, e^{-2\pi n y} , & \text{if } nm > N(t) , \\ 0 , & \text{if } nm \leq N(t) . \end{cases} \end{aligned}$$

Here $\triangle' = n - N(t)/m$. For detail of calculations, see [4, p. 19]. Hence

$$\sum_{p\in\mathbb{Z}} (z+p)^{-k+4} \exp(2\pi i N(t)\,(z+p)/m) = \frac{\alpha_k}{m^{k-1}} \sum_{\substack{n\in\mathbb{N}\\ nm>N(t)}} (nm - N(t))^{k-5}\, e^{2\pi i n z}$$

and our assertion for $F_{k,m}(z,w)$ follows.

In our next proposition, we shall express $e_{k,m}(n,t)$ as a Dirichlet series with an Euler product.

Proposition 2. *For $nm > N(t)$ and k even, $k \geq 10$, one has*

$$e_{k,m}(n,t) = \frac{\alpha_k}{m^{k-1}} (nm - N(t))^{k-5} \frac{1}{\zeta(k-8)} \sum_{a=1}^{\infty} T_a(Q)\, a^{-(k-1)}$$

with

$$T_a(Q) = \#\{\lambda \in \mathfrak{o}/a\mathfrak{o} :\ m\,N(\lambda) - \sigma(t,\lambda) + n \equiv 0 \,(\mathrm{mod}\ a)\}.$$

PROOF. We substitute $F_{k,m}(z,w)$ in Proposition 1 into (3), and get

$$E^1_{k,m}(z,w) = \sum_{n\in\mathbb{N}} \sum_{\substack{t\in\mathfrak{o}\\ nm>N(t)}} e_{k,m}(n,t)\, e^{2\pi i (nz+\sigma(t,w))} \tag{6}$$

with

$$e_{k,m}(n,t) = \frac{\alpha_k}{m^{k-1}} (nm - N(t))^{k-5} \cdot \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{(c,d)=1\\ d\leq c}} \sum_{t\in\mathfrak{o}/c\,\mathfrak{o}} \exp\left\{2\pi i \left(\frac{m\,N(\lambda)}{cd} + \sigma(t, -\frac{\lambda}{c}) + \frac{nd}{c}\right)\right\}. \tag{7}$$

Since $(c,d) = 1$, we can replace λ by $d\lambda$ in the third summation of $e_{k,m}(n,t)$. Hence

$$e_{k,m}(n,t) = \frac{\alpha_k}{m^{k-1}} (nm - N(t))^{k-5} \cdot \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{(c,d)=1\\ d (\mathrm{mod}\ c)}} \sum_{\lambda\in\mathfrak{o}/c\,\mathfrak{o}} \exp\left(2\pi i \frac{d}{c} (m\,N(\lambda) + \sigma(t,-\lambda) + n)\right). \tag{8}$$

Let $Q(\lambda) = m\,N(\lambda) - \sigma(t,\lambda) + n$. Use the well known formula

$$\sum_{\substack{(c,d)=1 \\ d(\mathrm{mod}\ c)}} e^{2\pi i d N/c} = \sum_{a|(c,N)} \mu(\frac{c}{a})\,a \tag{9}$$

with $\mu(a)$ the Möbius function. Hence

$$\begin{aligned} e_{k,m}(n,t) &= \frac{\alpha_k}{m^{k-1}}\,(nm - N(t))^{k-5} \sum_{c=1}^{\infty} c^{-k} \sum_{a|(c,Q(\lambda))} \mu(\frac{c}{a})\,a \sum_{\substack{\lambda\in \mathbf{o}/c\mathbf{o} \\ a|Q(\lambda)}} 1 \\ &= \frac{\alpha_k}{m^{k-1}}\,(nm - N(t))^{k-5} \sum_{c=1}^{\infty} c^{-k} \sum_{a|(c,Q(\lambda))} \mu(\frac{c}{a})\,a\,(\frac{c}{a})^8 \sum_{\substack{\lambda\in \mathbf{o}/c\mathbf{o} \\ a|Q(\lambda)}} 1\,. \end{aligned}$$

Let $c = ab$ and use the formula

$$\sum_{b=1}^{\infty} \mu(b)\,b^{-s} = \frac{1}{\zeta(s)}\,, \qquad \text{for } \operatorname{Re} s > 1\,,$$

to get

$$e_{k,m}(n,t) = \frac{\alpha_k}{m^{k-1}}\,(nm - N(t))^{k-5}\,\frac{1}{\zeta(k-8)} \sum_{a=1}^{\infty} T_a(Q)\,a^{-(k-1)}\,. \tag{10}$$

Here

$$T_a(Q) = \#\{\lambda \in \mathbf{o}/a\mathbf{o} :\ Q(\lambda) \equiv 0\,(\mathrm{mod}\ a)\}\,.$$

To obtain the explicit formula for $e_{k,m}(n,t)$ when $nm > N(t)$, we have to find the value of the Dirichlet series

$$\sum_{a=1}^{\infty} T_a(Q)\,a^{-s}$$

at $s = k-1$. Here

$$T_a(Q) = \#\{\lambda \in \mathbf{o}/a\mathbf{o} :\ Q(\lambda) = m\,N(\lambda) - \sigma(\lambda,t) + n \equiv 0\,(\mathrm{mod}\ a)\}\,.$$

By the multiplicativity of $T_a(Q)$, it suffices to consider the case $a = p^{\nu}$ ($\nu \in \mathbb{Z},\ \nu \geq 0$).

In the following consideration, we set $T_\nu(Q)=T_{p^\nu}(Q)$, $\omega_\nu=e^{2\pi i/p^\nu}$. We also set

$$Z(s)=\sum_{\nu=0}^{\infty}T_\nu(Q)\,p^{-\nu s}.$$

Proposition 3. *For any positive integer ν, we have for*

$$\lambda=\sum_{j=0}^{7}\lambda_j\alpha_j\,,\qquad t=\sum_{j=0}^{7}t_j\alpha_j\,,$$

that

$$(11)\quad T_\nu(Q)=p^{3\nu}\sum_{\tau=0}^{\nu}p^{4\tau}\sum_{\alpha'}{}^{\tau}\,\omega_{\nu-\tau}^{\alpha' n}\prod_{j=0}^{3}\Big(\sum_{m\lambda_j\equiv t_j\,(\mathrm{mod}\;p^{\nu-\tau})}\omega_{\nu-\tau}^{\alpha' t_{j+4}\lambda_j}\Big).$$

where α' ranges over all positive integers between 1 and $p^{\nu-\tau}$ with $(\alpha',p)=1$ in the summation $\sum^{\tau}$.

PROOF. By the p-adic version of Siegel's Babylonian reduction process, we can express $T_\nu(Q)$ as a Gaussian sum given by

$$T_\nu(Q)=p^{-\nu}\sum_{\alpha=1}^{p^\nu}\sum_{\lambda\in\mathfrak{o}/p^\nu\mathfrak{o}}\omega_\nu^{\alpha(mN(\lambda)-\sigma(\lambda,t)+n)}.$$

Over the p-adic integers, the quadratic form N is equivalent to the quadratic form with matrix $\left(\begin{smallmatrix}0&E\\E&0\end{smallmatrix}\right)$, where E is the 4×4 identity matrix. Thus

$$\begin{aligned}\sum_{\lambda\in\mathfrak{o}/p^\nu\mathfrak{o}}&\omega_\nu^{\alpha(mN(\lambda)-\sigma(\lambda,t)+n)}\\&=\omega_\nu^{\alpha n}\prod_{j=0}^{3}\Big(\sum_{\lambda_j=1}^{p^\nu}\sum_{\lambda_{j+4}=1}^{p^\nu}\omega_\nu^{\alpha(m\lambda_j\lambda_{j+4}-t_j\lambda_{j+4}-t_{j+4}\lambda_j)}\Big)\\&=\omega_\nu^{\alpha n}\prod_{j=0}^{3}\Big(\sum_{\lambda_j=1}^{p^\nu}\omega_\nu^{-at_{j+4}\lambda_j}\sum_{\lambda_{j+4}=1}^{p^\nu}\omega_\nu^{\alpha(m\lambda_j-t_j)\lambda_{j+4}}\Big).\end{aligned}$$

Note that

$$\sum_{\lambda_{j+4}=1}^{p^\nu}\omega_\nu^{\alpha(m\lambda_j-t_j)\lambda_{j+4}}=\begin{cases}p^\nu\,, & \text{if }\alpha(m\lambda_j-t_j)\equiv0\,(\mathrm{mod}\;p^\nu)\,,\\0\,, & \text{otherwise.}\end{cases}$$

On the other hand, we let $\alpha = \alpha' p^{\tau}$ with $(\alpha', p) = 1$ and get our assertion by an elementary calculation.

REMARK. For fixed $\nu \geq 1$ and $0 \leq \tau \leq \nu$, the product

$$(12) \qquad \prod_{j=0}^{3} \Big(\sum_{\lambda_j+1}^{p^{\nu}} \sum_{\lambda_{j+4}=1}^{p^{\nu}} \omega_{\nu-\tau}^{\alpha'(m\lambda_j \lambda_{j+4} - t_j \lambda_{j+4} - t_{j+4}\lambda_j)} \Big)$$

is zero unless the congruences

$$m\lambda_j \equiv t_j \pmod{p^{\nu-\tau}}, \qquad j = 0,1,2,3$$

have a solution. By the symmetry of t_j and $t_{j+4'}$ we conclude that the product in (12) is zero unless the congruences

$$m\lambda_j \equiv t_j \pmod{p^{\nu-\tau}}, \qquad \text{for } j = 0,1,2,3,4,5,6,7$$

have at least a solution.

3. Cases with $\nu_p(m) = 0$.

From Proposition 3 and its remark, we note that the evaluation of $T_\nu(Q)$ depends on solving the congruences

$$(13) \qquad m\lambda_j \equiv t_j \pmod{p^{\nu-\tau}}, \qquad j = 0,1,2,\ldots,7.$$

Obviously, the solvability of the congruences is wholy determined by $\nu_p(m)$, $\nu_p(t)$ and $\nu - \tau$.

In this Section, we shall investigate those cases with $(m,p) = 1$. Under such assumption, the congruences in (13) have always a unique solution.

Proposition 4. *If $(m,p) = 1$ and $\delta = \nu_p(n - N(t)/m)$, then one has for* $\operatorname{Re} s > 8$,

$$\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = \frac{1-p^{3-s}}{1-p^{7-s}} \sum_{j=0}^{\delta} p^{-(s-4)j}.$$

PROOF. Denote the solution of the congruences

$$m\lambda_j \equiv t_j \,(\mathrm{mod}\ p^{\nu-\tau}), \qquad j = 0,1,2,3$$

by $\lambda_j = t_j/m$, $j = 0,1,2,3$. Hence by Proposition 3, we have

$$T_\nu(Q) = p^{3\nu} \sum_{\tau=0}^{\nu} p^{4\tau} \sum_{\alpha'}{}^{\tau} \omega_{\nu-\tau}^{\alpha'(n-N(t)/m)} .$$

Apply (9) to the second summation; we get

$$T_\nu(Q) = p^{3\nu} \sum_{\tau=0}^{\nu} p^{4\tau} \sum_{j=0}^{\min\{\delta,\nu-\tau\}} \mu(p^{\nu-\tau-j}) p^j .$$

Note that $\mu(1) = 1$, $\mu(p) = -1$ and $\mu(p^l) = 0$ for $l \geq 2$. It follows

$$\begin{aligned} T_\nu(Q) &= p^{3\nu}\Big(\sum_{0 \leq \nu-\tau \leq \delta} p^{4\tau} p^{\nu-\tau} - \sum_{0 \leq \nu-\tau-1 \leq \delta} p^{4\tau} p^{\nu-\tau-1} \Big) \\ &= p^{4\nu}\Big(\sum_{0 \leq \nu-\tau \leq \delta} p^{3\tau} - \sum_{0 \leq \nu-\tau-1 \leq \delta} p^{3\tau-1} \Big) \end{aligned}$$

Now we shall prove by induction on δ that our assertion is true. In order to distinguish the cases for different δ, we let

$$Z_q(s) = \sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} , \qquad \delta = q .$$

When $\delta = 0$, then $\tau = \nu$ in the first summation and $\tau = \nu - 1$ in the second summation. Hence

$$Z_0(s) = \sum_{\nu=0}^{\infty} p^{7\nu} p^{-\nu s} - \sum_{\nu=1}^{\infty} p^{7\nu-4} p^{-\nu s} = \frac{1-p^{3-s}}{1-p^{7-s}} .$$

Suppose that for $\delta = q$ the assertion is true. Now

$$\begin{aligned} Z_{q+1}(s) - Z_q(s) &= \sum_{\nu=q+1}^{\infty} p^{3(\nu-q-1)} p^{-\nu(s-4)} - \sum_{\nu=q+2}^{\infty} p^{3(\nu-q-2)-1} p^{-\nu(s-4)} \\ &= p^{-(q+1)(s-4)} \frac{1-p^{3-s}}{1-p^{7-s}} . \end{aligned}$$

Thus the formula is also true for $\delta = q + 1$ and our proof is complete.

Corollary. *If* $n > N(t)$, *then*

$$e_{k,1}(n,t) = -\frac{2(k-4)}{B_{k-4}}\,\sigma_{k-5}(n - N(t)).$$

PROOF. From (10) and Proposition 4 we have

$$\begin{aligned} e_{k,1}(n,t) &= \alpha_k (n - N(t))^{k-5}\,\frac{1}{\zeta(k-8)}\,\frac{\zeta(k-8)}{\zeta(k-4)} \sum_{d \| [n-N(t)]} d^{-(k-5)} \\ &= \frac{\alpha_k}{\zeta(k-4)}\,\sigma_{k-5}(n - N(T)). \end{aligned}$$

But

$$\frac{\alpha_k}{\zeta(k-4)} = \frac{(-2\pi i)^{k-4}}{(k-5)!\,\zeta(k-4)} = -\frac{2(k-4)}{B_{k-4}},$$

hence our assertion follows.

Corollary. $E_{k,1}(z,w) = E_{k-4}(z)\,\theta(z,w)$ *with*

$$\theta(z,w) = \sum_{t \in \mathfrak{o}} e^{2\pi i (N(t) z + \sigma(t,w))}.$$

PROOF. Note that $e_{k,1}(n,t) = 0$ unless $n \geq N(t)$. Also $e_{k,1}(N(t),t) = 1$ by an observation. Then we have

$$\begin{aligned} E_{k,1}(z,w) &= \sum_{t \in \mathfrak{o}} e^{2\pi i (N(t) z + \sigma(t,w))} \\ &\quad - \frac{2(k-4)}{B_{k-4}} \sum_{n > N(t)} \sigma_{k-5}(n - N(t))\, e^{2\pi i (nz + \sigma(t,w))} \\ &= \sum_{t \in \mathfrak{o}} e^{2\pi i (N(t) z + \sigma(t,w))} \\ &\quad - \frac{2(k-4)}{B_{k-4}} \sum_{n=1}^{\infty} \sum_{t \in \mathfrak{o}} \sigma_{k-5}(n)\, e^{2\pi i (n + N(t)) z + \sigma(t,w)} \\ &= \Big(1 - \frac{2(k-4)}{B_{k-4}} \sum_{n=1}^{\infty} \sigma_{k-5}(n)\, e^{2\pi i n z}\Big)\, \theta(z,w) \\ &= E_{k-4}(z)\,\theta(z,w). \end{aligned}$$

Corollary.

$$\theta(z,w) = \sum_{t\in\mathfrak{o}} e^{2\pi i(N(t)z+\sigma(t,w))},$$

is a Jacobi form of weight 4 *and index* 1.

4. Cases with $0 \le \nu_p(t) < \nu_p(m)$.

For fixed $\nu \ge 1$ and $0 \le \tau \le \nu$. If $0 \le \nu_p(t) < \nu_p(m)$, then the congruences

$$m\lambda_j \equiv t_j \,(\mathrm{mod}\ p^{\nu-\tau}), \qquad j = 0,1,2,3$$

have solutions only if $\nu - \tau \le \nu_p(t)$. Moreover the number of solutions is $p^{4(\nu-\tau)}$.

Proposition 5. *Under the condition* $0 \le \nu_p(t) < \nu_p(m)$, *then one has for* $\mathrm{Re}\, s > 8$,

$$\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = \sum_{j=0}^{\alpha} p^{(8-s)j} + \begin{cases} p^{(8-s)\nu_p(t)+7-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(t) \le \nu_p(n), \\ 0, & \text{if } \nu_p(t) > \nu_p(n), \end{cases}$$

where $\alpha = \min\{\nu_p(n), \nu_p(t)\}$.

PROOF. Begin with (11) of Proposition 3 and the observation above, and get

$$\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = 1 + \sum_{\nu=1}^{\infty} p^{-\nu s} \sum_{\nu-\tau\le\nu_p(t)} p^{3\nu+4\tau} p^{4\nu-4\tau} \sum_{\alpha'}^{\tau} \omega_{\nu-\tau}^{\alpha' n}.$$

Apply (9) to the third summation, we get

$$\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = 1 + \sum_{\nu=1}^{\infty} p^{-\nu s} \sum_{\nu-\tau\le\nu_p(t)} \sum_{j=0}^{\min\{\nu-\tau,\nu_p(n)\}} \mu(p^{\nu-\tau-j}) p^j.$$

Denote the coefficient of $p^{(7-s)\nu}$ by A_ν. According to $\nu_p(t) \le \nu_p(n)$ or $\nu_p(t) > \nu_p(n)$, we have the following two cases.

Case I. $\nu_p(t) \le \nu_p(n)$. Then $\min\{\nu-\tau, \nu_p(n)\} = \nu - \tau$ since $\nu - \tau \le \nu_p(t) \le \nu_p(n)$. Therefore

$$A_\nu = \sum_{\nu-\tau\le\nu_p(t)} \sum_{0\le j\le \nu-\tau} \mu(p^{\nu-\tau-j})p^j = \begin{cases} p^\nu, & \text{if } \nu \le \nu_p(t), \\ p^{\nu_p(t)}, & \text{if } \nu > \nu_p(t). \end{cases}$$

Hence

$$\begin{aligned} \sum_{\nu=0}^{\infty} T_\nu(Q)p^{-\nu s} &= 1 + \sum_{\nu=1}^{\infty} p^{(7-s)\nu} A_\nu \\ &= \sum_{j=0}^{\nu_p(t)} p^{j(8-s)} + p^{(8-s)\nu_p(t)+7-s}(1-p^{7-s})^{-1}. \end{aligned}$$

Case II. $\nu_p(t) > \nu_p(n)$. Then

$$A_\nu = \sum_{\nu-\tau\le\nu_p(n)} \sum_{0\le j\le\nu-\tau} \mu(p^{\nu-\tau-j})p^j + \sum_{\nu-\tau>\nu_p(n)}^{\nu_p(t)} \sum_{0\le j\le\nu_p(n)} \mu(p^{\nu-\tau-j})p^j.$$

Note that the first sum in A_ν can be computed as in the case I. The second sum in A_ν is zero unless $\nu \ge \nu_p(n)+1$, $\nu-\tau = \nu_p(n)+1$ and $j = \nu_p(n)$. For such exceptional cases, the sum is $-p^{\nu_p(n)}$. Consequently, we have

$$A_\nu = \begin{cases} p^\nu, & \text{if } \nu \le \nu_p(t), \\ 0, & \text{if } \nu > \nu_p(t). \end{cases}$$

It follows

$$\sum_{\nu=0}^{\infty} T_\nu(Q)p^{-\nu s} = 1 + \sum_{\nu=1}^{\infty} p^{(7-s)\nu} A_\nu = \sum_{j=0}^{\nu_p(n)} p^{j(8-s)}.$$

This proves our assertions.

5. Cases with $\nu_p(m) \le \nu_p(t)$.

For fixed $\nu \ge 1$ and $0 \le \tau \le \nu$. If $\nu_p(m) \le \nu_p(t)$, then the congruences

$$m\lambda_j \equiv t_j \,(\mathrm{mod}\ p^{\nu-\tau}), \qquad j = 0,1,2,3$$

always have solutions. The number of solutions is $p^{4\nu_p(m)}$ if $\nu_p(m) < \nu - \tau$, and if $\nu_p(m) \geq \nu - \tau$, the number of solutions is $p^{4(\nu-\tau)}$.

Proposition 6. *Under the condition $\nu_p(m) \leq \nu_p(t)$, one has for* $\operatorname{Re} s > 8$

$$\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = \sum_{j=0}^{\beta} p^{(8-s)j} + \begin{cases} p^{(8-s)\nu_p(m)+7-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(m) \leq \nu_p(n), \\ 0, & \text{if } \nu_p(m) > \nu_p(n), \end{cases}$$
$$+ p^{4\nu_p(m)} \frac{1-p^{3-s}}{1-p^{7-s}} \sum_{j=\gamma+1}^{\nu_p(\triangle')} p^{(4-s)j} - \begin{cases} p^{(8-s)\nu_p(m)+7-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(m) \leq \nu_p(\triangle'), \\ 0, & \text{if } \nu_p(m) > \nu_p(\triangle'). \end{cases}$$

Here $\beta = \min\{\nu_p(m), \nu_p(n)\}$ *and* $\gamma = \min\{\nu_p(m), \nu_p(\triangle')\}$.

PROOF. We begin with (11) of Proposition 3, and separate the series into two subseries according to $\nu - \tau > \nu_p(m)$ or $\nu_p(m) \leq \nu - \tau$. Hence

$$\sum_{\nu=0}^{\infty} T_\nu(Q) p^{-\nu s} = 1 + \sum_{\nu=1}^{\infty} p^{(3-s)\nu} \sum_{\nu-\tau > \nu_p(m)} p^{4\tau+4\nu_p(m)} {\sum_{\alpha'}}^{\tau} \omega_{\nu-\tau}^{\alpha'\triangle'} + \sum_{\nu=1}^{\infty} \sum_{\nu-\tau \leq \nu_p(m)} p^{(7-s)\nu} {\sum_{\alpha'}}^{\tau} \omega_{\nu-\tau}^{\alpha' n},$$

where α' ranges over all positive integers between 1 and $p^{\nu-\tau}$ with $(\alpha', p) = 1$ in the summation $\sum^{\tau}$.

Let $Z_1(s)$ be the subseries corresponding to the summation $\nu - \tau > \nu_p(m)$ and $Z_2(s)$ be the remaining sum. By the computations in Proposition 5, we have

$$Z_2(s) = \sum_{j=0}^{\beta} p^{(8-s)j} + \begin{cases} p^{(8-s)v_p(m)+7-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(m) \leq \nu_p(n), \\ 0, & \text{if } \nu_p(m) > \nu_p(n)', \end{cases}$$

where $\beta = \min\{\nu_p(m), \nu_p(n)\}$.

Also we have

$$Z_1(s) = p^{4\nu_p(m)}\Big(\sum_{\nu=0}^{\infty} p^{(3-s)\nu} \sum_{0\le\tau\le\nu} p^{4\tau} \sum_{j=0}^{\min\{\nu-\tau,\nu_p(\Delta')\}} \mu(p^{\nu-\tau-j})p^j - \sum_{\nu=0}^{\infty} p^{(3-s)\nu} \sum_{\nu-\tau\le\nu_p(m)} \sum_{j=0}^{\min\{\nu-\tau,\nu_p(\Delta')\}} \mu(p^{\nu-\tau-j})p^j\Big)$$

$$= p^{4\nu_p(m)}\Big(\sum_{\nu=0}^{\infty} p^{(4-s)\nu}\Big(\sum_{0\le\tau\le\nu(\Delta')} p^{3\tau} - \sum_{0\le\tau-1\le\nu(\Delta')} p^{3\tau-1}\Big)\Big) - p^{4\nu_p(m)}\sum_{\nu=0}^{\infty} p^{(4-s)\nu}\Big(\sum_{0\le\tau\le\gamma} p^{3\tau} - \sum_{0\le\tau-1\le\gamma} p^{3\tau-1}\Big) - \begin{cases} p^{5\nu_p(m)} \sum_{\nu=\nu_p(m)+1}^{\infty} p^{(3-s)\nu} p^{4(\nu-\nu_p(m)-1)}, & \text{if } \nu_p(m)\le \nu_p(\Delta'), \\ 0, & \text{if } \nu_p(m)> \nu_p(\Delta'). \end{cases}$$

Here $\gamma = \min\{\nu_p(m), \nu_p(\Delta')\}$. Now by the computations of Proposition 4, we conclude that

$$Z_1(s) = p^{4\nu_p(m)}\frac{1-p^{3-s}}{1-p^{7-s}}\sum_{j=\gamma+1}^{\nu(\Delta')} p^{-j(s-4)} - \begin{cases} p^{(8-s)\nu_p(m)+3-s}(1-p^{7-s})^{-1}, & \text{if } \nu_p(m)\le \nu_p(\Delta'), \\ 0, & \text{if } \nu_p(m)> \nu_p(\Delta'). \end{cases}$$

Combine Proposition 2 and Propositions 4, 5, 6 with $s = k-1$ together. Also using the well known result

$$\frac{\alpha_k}{\zeta(k-4)} = \frac{(-2\pi i)^{k-4}}{(k-5)!\,\zeta(k-4)} = -\frac{2(k-4)}{B_{k-4}},$$

we get

$$e_{k,m}(n,t) = -\frac{2(k-4)}{B_{k-4}}\prod_p S_p,$$

where S_p is as we claimed in the Theorem.

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Trajectoires de groupes à 1-paramètre de quasi-isométries

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1. Introduction.

Un homéomorphisme $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ est une $(L\text{-})$*quasi-isométrie* si pour tout $x, y \in \mathbb{R}^2$

$$\frac{1}{L} \|x - y\| \leq \|g(x) - g(y)\| \leq L \|x - y\| .$$

Soit $G = \{g_t : t \in \mathbb{R}\}$ un groupe à 1-paramètre. Il est dit *quasi-isométrique* s'il existe $L \geq 1$ tel que tout élément de G est une L-quasi-isométrie.

Le point de départ de ce travail est l'étonnant exemple de P. Tukia [T2] d'un groupe quasi-isométrique du plan $\mathbb{R}^2$ n'étant pas quasi-isométriquement conjugable à un groupe d'isométries, *i.e.* il ne s'écrit pas sous la forme

$$G = f \circ \Phi \circ f^{-1} ,$$

où $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ est une quasi-isométrie et Φ est un groupe d'isométries. La raison pour laquelle ceci n'a pas lieu est simple. Le groupe G de Tukia a une trajectoire $\Gamma = \{g_t(0) : t \in \mathbb{R}\}$ le "snowflake" ou encore la courbe de Von Koch. Cette trajectoire Γ n'est pas rectifiable; elle a même une dimension de Hausdorff strictement plus grande que un.

Le fait qu'un groupe quasi-isométrique peut agir transitivement sur une courbe fractale est remarquable et la question se pose: quelles sont les trajectoires d'un groupe à 1-paramètre quasi-isométrique en général ? On les appelle dans la suite *quasi-isométrique cercle* (QI-*cercle*) ou, plus précisément, L-QI-*cercle* si les éléments du groupe sont L-quasi-isométriques.

D. Sullivan [Su1] et P. Tukia [T1] ont montré qu'en dimension deux tout groupe quasiconforme est quasiconformément conjugué à un groupe conforme. Un groupe quasi-isométrique à 1-paramètre est alors le conjugué quasiconforme d'un groupe d'isométries à 1-paramètre parabolique ou elliptique; *cf.* la classification des applications conformes, par exemple dans Greenberg [4]. Par conséquent, les QI-cercles sont des quasicercles, c'est à dire les images d'une droite (ou d'un cercle) par une application quasi-conforme de $\mathbb{R}^2$.

Avant d'énoncer des caractérisations des QI-cercles fixons les notations necessaires. On note par $D(p,r)$ le disque ouvert centré en p de rayon $r > 0$. Dans tout le texte, disque signifie toujours disque ouvert. Le bord du disque $D(0,1)$ est noté $\mathbb{S}^1$. Le symbole $\Gamma(p,q)$ désigne un sous-arc demi-ouvert d'une courbe Γ, joignant p à q et contenant le point p mais pas q. Quand Γ est une courbe de Jordan on prend pour $\Gamma(p,q)$ le sous-arc avec le plus petit diamètre (s'il existe; sinon on choisit librement un des deux sous-arcs). Si $E \subset \Gamma$ est un sous-ensemble, $N(r,E)$ est le plus petit nombre de disques de rayon r necessaires pour couvrir E.

Théorème 1.1. *Une courbe Γ est un L-QI-cercle si et seulement si une des propriétés suivantes est vérifiée.*

(I) *Il existe $M \geq 1$ et $h : \mathbb{R} \longrightarrow \Gamma$ ou $h : \mathbb{S}^1 \longrightarrow \Gamma$ un homéomorphisme tel que*

$$\|h(x+t) - h(x)\| \leq M \; \|h(y+s) - h(y)\| \;,$$

pour tous $x, y \in \mathbb{R}$ ou $\mathbb{S}^1$ et pour tout $0 < t \leq s$.

(II) *Il existe une mesure ω de Γ non-triviale et σ-finie, i.e. $0 < \omega(\Gamma(p,q)) < \infty$ pour tout $p, q \in \Gamma$ distincts, vérifiant pour une constante $A \geq 1$,*

$$\omega(\Gamma(p_1,p_2)) \leq A\,\omega(\Gamma(q_1,q_2))\,,$$

pour tous $p_i, q_i \in \Gamma$ avec $\|p_1 - p_2\| = \|q_1 - q_2\|$.

(III) Γ *vérifie la propriété géométrique suivante: il existe une constante $H \geq 1$ telle que, si $p_1, p_2, q_1, q_2 \in \Gamma$ sont deux paires de points telles que $\|p_1 - p_2\| = \|q_1 - q_2\|$, alors*

$$N(r, \Gamma(p_1, p_2)) \leq H\, N(r, \Gamma(q_1, q_2))\,, \qquad \textit{pour tout } r > 0\,.$$

En plus, toutes les constantes dépendent l'une de l'autre. Quand l'une d'entre elle est égale à un, les autres peuvent aussi être prises égales à un.

Le fait qu'un QI-cercle doit être un quasicercle se reflète dans ce théorème. Si on remplace par exemple dans l'inégalité de (I) y par x on a précisement la condition qui dit que h est une quasi-symétrie et donc que Γ est un quasicercle. On va aussi voir que (II) aussi bien que la propriété géométrique (III) implique la propriété des trois points de Ahlfors: une courbe Γ est un quasicercle s'il existe une constante $c \geq 1$ telle que $\operatorname{diam} \Gamma(p, q) \leq c\, \|p - q\|$.

Remarquons encore que si Γ vérifie (II) ou (III) avec constante 1, alors Γ est forcément une droite ou un cercle (*cf.* Proposition 3.1).

Dans [FM] K.J. Falconer et T.D. Marsh étudient les "quasi-self-similar" cercles. Grâce à leur étude on peut dire que ce sont des courbes de Jordan Γ paramétrisables par un homéomorphisme $h : \mathbb{S}^1 \longrightarrow \Gamma$ vérifiant

$$\text{(1.1)} \qquad \frac{\|x - y\|^\alpha}{c} \leq \|h(x) - h(y)\| \leq c\, \|x - y\|^\alpha\,,$$

pour tous $x, y \in \mathbb{S}^1$, où $c \geq 1$ et $1/\alpha = \operatorname{Hdim}(\Gamma) \in [1, 2[$. Ceci et le Théorème 1.1 impliquent que toutes ces courbes sont des exemples de QI-cercles. En particulier les ensembles de Julia des fonctions $f_\lambda(z) = z^2 + \lambda$, λ appartenant à l'intérieur de la cardioïde principale de l'ensemble de Mandelbrot [Su2], aussi bien que les ensembles limites de certains groupes kleiniens [Bo].

Les "quasi-self-similar" cercles ont des jolies propriétés fractales. Par exemple, toutes les différentes dimensions (Box, Hausdorff) d'une telle courbe coïncident et, à une quasi-isométrie près, ils sont définis uniquement en fonction de leur dimension [FM]. Dans le Paragraphe 5 nous étudions comment étendre ces résultats aux QI-cercles.

Je tiens à remercier Michel Zinsmeister pour des nombreuses discussions et pour m'avoir aidé à clarifier ce travail. Je remerci également

le referee. La condition (II) du Théorème 1.1 est basée sur une idée à lui.

2. Étude de la propriété géométrique.

Nous résumons dans ce paragraphe quelques résultats importants pour la suite.

L'astuce suivante va être utilisée à plusieures reprises: il est clair que le nombre de disques de rayon $r > 0$ necessaires pour pouvoir couvrir un disque $D(p, R)$, $R \geq r$, se majore par une constante $\nu = \nu(R/r)$ dépendant que du rapport des rayons. Une conséquence immédiate de ceci est

$$N(r, \Gamma(p,q)) \leq \nu\left(\frac{R}{r}\right) N(R, \Gamma(p,q))\,, \tag{2.2}$$

quand $r \leq R$ et $p, q \in \Gamma$. Voyons pourquoi (II) et (III) sont des conditions plus fortes que la propriété des trois points de Ahlfors:

Lemme 2.1. *Toute courbe Γ vérifiant la propriété géométrique* (III) *est un $2H$-quasicercle, i.e.*

$$\operatorname{diam}\Gamma(p,q) \leq 2H\,\|p-q\|\,, \qquad \textit{pour tous } p, q \in \Gamma\,.$$

Quand une courbe Γ vérifie (II) *elle est un $2A$-quasicercle.*

PREUVE. Supposons le contraire dans le cas où Γ vérifie (III): il existe $p, q \in \Gamma$ tels que $\operatorname{diam}\Gamma(p,q) > 2H\|p-q\|$. Soit $r = \|p-q\|$. Il existe $a, b \in \Gamma(p,q)$ avec $\|a-b\| = r$ et $\|a-x\| < r$ pour tout $x \in \Gamma(a,b)$. D'où $N(r, \Gamma(a,b)) = 1$. Comme $\operatorname{diam}\Gamma(p,q) > 2Hr$, le nombre $N(r, \Gamma(p,q))$ est strictement plus grand que H. On a alors une contradiction avec (III).

Considérons maintenant une courbe Γ vérifiant (II). Soient $p, q \in \Gamma$ et $a, b \in \overline{\Gamma(p,q)}$ tel que $\operatorname{diam}\Gamma(p,q) = \|a-b\|$. On choisi $m \in \mathbb{N}$ de sorte que $(m+1)\|p-q\| \geq \|a-b\| \geq m\,\|p-q\|$. Alors, par (II), $\omega(\Gamma(a,b)) \geq m\,\omega(\Gamma(p,q))/A$. Or, $\Gamma(a,b) \subset \Gamma(p,q)$, et donc $m \leq A$. D'où, $\operatorname{diam}\Gamma(p,q) \leq (m+1)\,\|p-q\| \leq 2A\,\|p-q\|$.

Les nombres $N(r,\gamma)$ sont définis dans l'introduction. Nous aurons besoin d'une quantité analogue: on appelle r-ensemble de $\Gamma(p,q)$ un ensemble de points $\{x_0 = p, x_1, \dots, x_n = q\} \subset \overline{\Gamma(p,q)}$ ordonné et maximal pour que $\|x_{i-1} - x_i\| = r$, $i = 1, 2, \dots, n-1$. On note $n(r, \Gamma(p,q)) = n$.

Lemme 2.2. *Pour un K-quasicercle Γ les nombres $N(r,\Gamma(p,q))$ et $n(r,\Gamma(p,q))$ sont équivalents:*

$$N(r,\Gamma(p,q)) \leq n(r,\Gamma(p,q)) \leq \nu(4K)\, N(r,\Gamma(p,q))\,.$$

PREUVE. Choisissons un ensemble $\{x_0,\dots,x_n\} \subset \overline{\gamma}$, $\gamma = \Gamma(p,q)$, maximal pour la condition suivante: on prend $x_0 = p$ et successivement $x_i \in \Gamma(x_{i-1},q)$, $i = 1,\dots,n-1$, de sorte que $\|x_{i-1} - x_i\| = r$ et $\|x - x_{i-1}\| < r$ pour tout $x \in \Gamma(x_{i-1},x_i)$. Enfin on prend comme dernier point $x_n = q$. Ce choix assure que les disques $D(x_i,r)$, $i = 0,1,\dots,n-1$ couvrent γ. D'où $N(r,\gamma) \leq n \leq n(r,\gamma)$.

Partons maintenant d'un r-ensemble $\{x_0,x_1,\dots,x_n\}$ de $\gamma = \Gamma(p,q)$. Quand $n = 1$ tout est clair. Soit alors $n \geq 2$. Dans $\gamma_i = \Gamma(x_{i-1},x_i)$, $i = 1,\dots,n-1$, il existe un point p_i tel que $\min\{\|p_i - x_{i-1}\|, \|p_i - x_i\|\} \geq r/2$. Γ étant un K-quasicercle, $\|x - p_i\| \geq r/2K$ pour tout $x \in \gamma\backslash\gamma_i$. Un disque de rayon $r/4K$ contenant un des points p_i n'a pas de point en commun avec $\gamma\backslash\gamma_i$. D'où et par (2.2) on a bien

$$n(r,\gamma) \leq N\big(\frac{r}{4K},\gamma\big) \leq \nu(4K)\, N(r,\gamma)\,.$$

Lemme 2.3. *Soit Γ une courbe vérifiant* (III). *Alors, il existe une constante $\eta(H) > 0$ dépendant que de H telle que pour tout $p,q \in \Gamma$ et $0 < r \leq R \leq \|p-q\|$ ainsi que $a,b \in \Gamma$ avec $\|a-b\| = R$ on a*

$$\begin{aligned}\eta(H)\, n(R,\Gamma(p,q))\, N(r,\Gamma(a,b)) &\leq N(r,\Gamma(p,q)) \\ &\leq H\, n(R,\Gamma(p,q))\, N(r,\Gamma(a,b))\,.\end{aligned}$$

PREUVE. Notons $\gamma = \Gamma(p,q)$, $\{x_0,\dots,x_n\}$ un R-ensemble de γ, $\gamma_i = \Gamma(x_{i-1},x_i)$ et soit $a,b \in \Gamma$ avec $\|a-b\| = R$. Quand $n = n(R,\gamma) = 1$ necessairement $\|p-q\| = \|a-b\|$ et le lemme suit directement de (III). Supposons alors $n \geq 2$. À cause de (III) on a

$$N(r,\gamma) \leq \sum_{i=1}^{n} N(r,\gamma_i) \leq H\, n\, N(r,\Gamma(a,b))\,.$$

Pour voir l'autre inégalité considérons d'abord le cas $r \leq R/4H$. Cette restriction fait qu'un disque $D(x,r)$ a une intersection non vide avec au plus deux des arcs γ_i qui doivent être des arcs voisins. Sinon il existe $u, v \in \Gamma(p,q)$ avec $\|u-v\| < 2r$ et tel que $\gamma_i \subset \Gamma(u,v)$ pour un $i \in \{2, \dots, n-1\}$. Or, Γ est un $2H$-quasicercle et donc

$$\|u-v\| \geq \frac{1}{2H} \operatorname{diam} \Gamma(u,v) \geq \frac{1}{2H} R \geq 2r .$$

Il est alors possible d'extraire d'un recouvrement minimal de $\Gamma(p,q)$ de disques de rayon r un recouvrement de $\bigcup_{1\leq 2k+1\leq n} \gamma_{2k+1}$ tel que chaque disque a une intersection non vide avec qu'un seul arc γ_{2k+1}. D'où, $N(r, \Gamma(p,q)) \geq \sum_{1\leq 2k+1 \leq n} N(r, \gamma_{2k+1})$. Le même raisonnement s'applique aux γ_i d'indice pair, ce qui implique

$$\begin{aligned} 2\,N(r,\Gamma(p,q)) &\geq \sum_{i=1}^{n} N(r,\gamma_i) \\ &\geq \frac{1}{H}\,(n-1)\,N(r,\Gamma(a,b)) \\ &\geq \frac{1}{2H}\,n(R,\gamma)\,N(r,\Gamma(a,b)) . \end{aligned}$$

Il reste à voir le cas $R \geq r > R/4H$. Par les lemmes précédents et avec (2.2) on voit que

$$N(r,\Gamma(a,b)) \leq N\big(\frac{R}{4H}, \Gamma(a,b)\big) \leq \nu(8H^2)\,N(2HR, \Gamma(a,b)) = \nu(8H^2)$$

et

$$n(R,\Gamma(p,q)) \leq \nu(8H)\,N(R,\Gamma(p,q)) \leq \nu(8H)\,N(r,\Gamma(p,q)) .$$

La constante $\eta(H) = \min\{1/4H,\ 1/\nu(8H)\nu(8H^2)\}$ vérifie alors l'inégalité de gauche dans les deux cas.

Lemme 2.4. *Soit Γ une courbe vérifiant* (III). *Alors, pour toute constante $\tilde{H} \geq 1$ il existe $d > 0$ dépendant que de $\tilde{H}$ et de la constante H de* (III) *telle que si $N(r,\Gamma(p_1,p_2)) \leq \tilde{H}\,N(r,\Gamma(q_1,q_2))$ pour un $r > 0$ avec $r \leq \|q_1 - q_2\|$ alors $\|p_1 - p_2\| \leq d\,\|q_1 - q_2\|$.*

PREUVE. Soit $\|p_1 - p_2\| > \|q_1 - q_2\| = R$. Par le Lemme 2.3 on a

$$N(r,\Gamma(p_1,p_2)) \geq \eta(H)\,n(R,\Gamma(p_1,p_2))\,N(r,\Gamma(q_1,q_2)) .$$

En appliquant l'hypothèse on en déduit $\tilde{H}/\eta(H) \geq n(R, \Gamma(p_1, p_2))$, ce qui montre $\|p_1 - p_2\| \leq d\,\|q_1 - q_2\|$ avec $d = \tilde{H}/\eta(H)$.

3. Démonstration du Théorème 1.1.

(I) *implique* Γ *est un QI-cercle.* Cette preuve est fortement basée sur un argument utilisé par Tukia dans [T2].

a) Le cas non compact; on suppose que Γ est homéomorphe à $\mathbb{R}$. On cherche un groupe quasi-isométrique à 1-paramètre $G = \{g_t : t \in \mathbb{R}\}$ pour lequel Γ est une trajectoire: $\Gamma = \{g_t(p) : t \in \mathbb{R}\}$ avec $p \in \Gamma$ un point quelconque.

Soit $h : \mathbb{R} \longrightarrow \Gamma$ une paramétrisation vérifiant (I) et notons par $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ l'extension $K = K(M)$-quasiconforme de Beurling-Ahlfors-Tukia [T3] de h. On considére le groupe

$$G = H \circ \{x \longrightarrow x + t :\ t \in \mathbb{R}\} \circ H^{-1} = H \circ T \circ H^{-1}\,.$$

Ce groupe a Γ comme trajectoire. Afin de montrer que G est un groupe quasi-isométrique on va d'abord établir l'inégalité

$$\frac{\operatorname{dist}\{H(u_1, v), \Gamma\}}{\operatorname{dist}\{H(u_2, v), \Gamma\}} \leq L_1\,, \tag{3.3}$$

pour tous $u_1, u_2 \in \mathbb{R}$ et $v \in \mathbb{R}\backslash\{0\}$, pour une constante $L_1 = L_1(M) \geq 1$.

Il existe $t_i \in \mathbb{R}$ avec $\operatorname{dist}\{H(u_i, v), \Gamma\} = \|H(u_i, v) - h(t_i)\|$, $i = 1, 2$. Soit $s_1 \in \mathbb{R}$ tel que

$$\|(u_1, v) - (s_1, 0)\| = \|(u_2, v) - (t_2, 0)\| = l$$

et utilisons le fait qu'une application quasiconforme de $\mathbb{R}^2$ est K'-quasisymétrique avec $K' = K'(K)$ est une constante dépendant que de K et donc que de M, [Ge2]. Alors,

$$\operatorname{dist}\{H(u_1, v), \Gamma\} \leq \|H(u_1, v) - h(s_1)\| \leq K'\,\|h(s_1 + l) - h(s_1)\|$$

et donc, avec (I),

$$\begin{aligned}\operatorname{dist}\{H(u_1, v), \Gamma\} &\leq K'M\,\|h(t_2 + l) - h(t_2)\| \\ &\leq K'^2M\,\|H(u_2, v) - h(t_2)\|\,.\end{aligned}$$

Par le choix de t_2, dist $\{H(u_1,v),\Gamma\} \le K'^2 M$ dist $\{H(u_2,v),\Gamma\}$ ce qui est précisément (3.3) avec $L_1 = K'^2 M$.

On conclut maintenant comme dans [T2] preuve du Lemme 1. Soient $U_1 = H(\{x_2 > 0\})$ et $U_2 = H(\{x_2 < 0\})$ les composantes connexes de $\mathbb{R}^2 \backslash \Gamma$. L'application H, étant l'extension de Beurling-Ahlfors-Tukia de h, est, restreinte à $\{x_2 > 0\}$ ou à $\{x_2 < 0\}$, difféomorphe et une quasi-isométrie hyperbolique. La constante de quasi-isométrie hyperbolique dépend aussi que de M [T3]. On définit sur U_j la métrique quasi-hyperbolique par $|dy|/\text{dist}\{y,\Gamma\}$. Cette métrique étant équivalente à la métrique hyperbolique, il existe $L_2 = L_2(M) \ge 1$ tel que

$$\frac{1}{L_2}\frac{|dy|}{\text{dist}\{y,\Gamma\}} \le \frac{|dg_t(y)|}{\text{dist}\{g_t(y),\Gamma\}} \le L_2 \frac{|dy|}{\text{dist}\{y,\Gamma\}},$$

pour tout $t \in \mathbb{R}$ et $y \in U_j$, $j = 1,2$. Or, y et $g_t(y)$ sont dans une même courbe $H(\{(x_1,v) : x_1 \in \mathbb{R}\})$ pour un $v \ne 0$. L'inégalité (3.3) implique donc

$$\frac{1}{L_1 L_2}|dy| \le |dg_t(y)| \le L_1 L_2 |dy|, \qquad \text{pour presque tout } \; y \in \mathbb{R}^2.$$

Les g_t étant quasiconformes on en déduit que le groupe G est bien un groupe quasi-isométrique (pour la métrique euclidienne) et que sa constante bilipschitzienne dépend que de M.

b) Le cas compact. On suppose maintenant qu'il existe $h : \mathbb{S}^1 \longrightarrow \Gamma$ vérifiant (I). Soit encore H l'extension de Beurling-Ahlfors-Tukia de h et $G = H \circ \mathcal{R} \circ H^{-1}$ où $\mathcal{R} = \{x \longmapsto Ux : U \text{ matrice orthogonale}\}$ le groupe des rotations euclidiennes fixant 0 et ∞. Ce groupe a Γ comme trajectoire. Pour montrer que G est un groupe quasi-isométrique on peut supposer, quitte à conjuguer G avec une translation, que 0 est un point fixe des éléments de ce groupe. Supposons pour l'instant aussi qu'il existe $p \in \Gamma$ avec $\|p\| = 1$. Ceci implique $\Gamma \subset A(K') = \{x \in \mathbb{R}^2 : 1/K' \le \|x\| \le K'\}$ puisque l'application H est K'-quasisymétrique avec K' dépendant que de M et on a $H(0) = 0$; le groupe G fixe 0.

Notons $d_{\mathbb{S}}$ la métrique sphérique (définie par $\|dx\|/(1 + \|x\|^2)$). L'involution $i(x) = x/\|x\|^2$ est une isométrie pour cette métrique. La métrique euclidienne et la métrique sphérique étant équivalentes dans $A(K')$, l'application $\overline{h} = i \circ h : \mathbb{S}^1 \longrightarrow \overline{\Gamma} = i(\Gamma)$ vérifie aussi l'inégalité de (I) pour une constante $\overline{M}$ dépendant que de M.

Soit C_1 la composante connexe de $\mathbb{R}^2 \backslash \Gamma$ contenant 0 et C_2 l'autre. On considère $G_1 = G|_{C_1}$ et $\overline{G}_2 = i \circ G|_{C_2} \circ i$. Comme dans le cas non compact on montre que G_1 et $\overline{G}_2$ sont quasi-isométriques -pour le groupe $\overline{G}_2$ on utilise que $\overline{h}$ vérifi (I) et que $\overline{H} = i \circ H \circ i$ est une extension quasiconforme de $\overline{h}$ ayant les propriétés de l'extension de Beurling-Ahlfors-Tukia.

Dans $C_1 \cup i(C_2) \subset D(0, K')$ la métrique euclidienne et la métrique sphérique sont équivalentes. G_1, $\overline{G}_2$ et donc aussi G sont alors des groupes quasi-isométriques pour la métrique sphérique. Or, un groupe de $\mathbb{R}^2$ dont tout élément fixe 0 est quasi-isométrique pour la métrique sphérique si et seulement s'il l'est pour la métrique euclidienne (*cf.* [T3], on peut le voir en utilisant l'expression infinitesimale de cette métrique). G est alors bien un groupe quasi-isométrique pour la métrique euclidienne et sa constante bilipschitzienne est contrôlé par M.

Quand aucun point $p \in \Gamma$ est à distance un de l'origine, le groupe $G^* = d_{1/r} \circ G \circ d_r$, avec $d_r(x) = rx$ et $r = \|p\|$ pour un $p \in \Gamma$ quelconque, est, par le précédent raisonnement, un groupe quasi-isométrique dont la constante bilipschitzienne dépend que de M. Il en est alors de même pour G puisque conjugaison d'un groupe quasi-isométrique par similitude ne change pas sa constante bilipschitzienne.

Γ *est un QI-cercle implique* (III). Il existe $G = \{g_t : t \in \mathbb{R}\}$ un groupe L-quasi-isométrique de $\mathbb{R}^2$ agissant transitivement sur Γ. La courbe Γ est un $K(L)$-quasicercle puisque G se conjugue quasiconformément en un groupe isométrique et on a un contrôle de la dilatation de l'application conjugant en fonction de L, voir [T1].

Soient $\gamma_1 = \Gamma(p_1, p_2)$ et $\gamma_2 = \Gamma(q_1, q_2)$ des sous-arcs de Γ avec $R = \|p_1 - p_2\| = \|q_1 - q_2\|$. On veut montrer $N(r, \gamma_2) \leq H\, N(r, \gamma_1)$ avec $H = H(L)$. Pour éviter un problème d'orientation on suppose que soit $g_t(\gamma_1) \subset \gamma_2$ soit $g_t(\gamma_1) \supset \gamma_2$ où $t \in \mathbb{R}$ tel que $g_t(p_1) = q_1$.

On prend $\{x_0 = q_1, x_1, \ldots, x_m\}$ ordonné et maximal de sorte que $x_i \in \gamma_2$ pour $i = 0, 1, \ldots, m-1$ et tel que $\|x_i - x_{i-1}\| = R/L$ ainsi que $\|x - x_{i-1}\| < R/L$ quand $x \in \Gamma(x_{i-1}, x_i)$. Le choix de ces points et le Lemme 2.2 impliquent $m \leq n(R/L, \gamma_2) \leq \nu(4K) N(R/L, \gamma_2)$. Γ est un K-quasicercle et donc $\gamma_2 \subset D(q_1, KR)$. Avec (2.2) on en déduit $m \leq \nu(4K)\, \nu(LK) = H_1$.

Pour conclure on majore les nombres $N(r, \Gamma(x_{i-1}, x_i))$ en fonction de $N(r, \gamma_1)$. Soit $t \in \mathbb{R}$ tel que $g_t(p_1) = x_{i-1}$. Par le choix des x_i et le fait que le groupe est L-quasi-isométrique $\Gamma(x_{i-1}, x_i) \subset g_t(\gamma_1)$. Prenons $D_j = D(y_j, r)$, $j = 1, \ldots, N$ un recouvrement minimal de γ_1. Or, $g_t(D_j) \subset D(g_t(y_j), Lr)$ et donc, encore avec (2.2), on a

$$\frac{1}{\nu(L)} N(r, \Gamma(x_{i-1}, x_i)) \leq N(Lr, \Gamma(x_{i-1}, x_i)) \leq N(r, \gamma_1) .$$

D'où

$$N(r,\gamma_2) \le \sum_{i=1}^{m} N(r,\Gamma(x_{i-1},x_i)) \le \nu(L)\,H_1\,N(r,\gamma_1)$$

ce qui termine la preuve. Remarquons que $H = H_1\,\nu(L)$ dépend que de L.

(III) *implique* (I). Par le Lemme 2.1 une courbe vérifiant (III) est un quasicercle. Elle est alors soit homéomorphe à $\mathbb{R}$ soit à $\mathbb{S}^1$.

a) Le cas non compact. Supposons d'abord que Γ est une courbe vérifiant (III) et homéomorphe à $\mathbb{R}$. On peut supposer que $0 \in \Gamma$. Notons Γ^+ l'une des composantes connexes de $\Gamma\setminus\{0\}$ et soit $\gamma = \Gamma(0,q) \subset \overline{\Gamma}^+$ choisi de sorte que $\|q\| = 1$.

Considérons un $m \in \mathbb{N}$ tel qu'il existe un $r_m > 0$ avec $N(r_m,\gamma) = m$. A ce m on associe une application continue $h_m : \mathbb{R} \longrightarrow \mathbb{R}^2$ de la façon suivante: soit $\{p_i \in \Gamma : i \in \mathbb{Z}\}$ un ensemble de points ordonnés avec $p_0 = 0$, $p_1 \in \Gamma^+$ et $\|p_i - p_{i-1}\| = r_m$ pour tout $i \in \mathbb{Z}$. L'application h_m est alors définie par $h(i/m) = p_i$ et par

$$h_m\Big|_{[(i-1)/m,\,i/m]} : \left[\frac{i-1}{m}, \frac{i}{m}\right] \longrightarrow [p_{i-1},p_i] \quad \text{affine.}$$

On montre que $\{h_m\}$ est equicontinue et que la limite d'une sous-suite convergente est bien un homéomorphisme de $\mathbb{R}$ sur Γ satisfaisant (I). La preuve de ceci est basée sur les inégalités suivantes: il existe $C_1(H)$ tel que

$$\left\|h_m\Big(\frac{i+k}{m}\Big) - h_m\Big(\frac{i}{m}\Big)\right\| \le C_1(H)\left\|h_m\Big(\frac{j+l}{m}\Big) - h_m\Big(\frac{j}{m}\Big)\right\|, \tag{3.4}$$

pour tous $i,j \in \mathbb{Z}$ et $0 < k \le l$, et il existe $C_2(H)$ tel que

$$\left\|h_M\Big(\frac{i}{M}\Big)\right\| \le C_2(H)\left\|h_m\Big(\frac{1}{m}\Big)\right\|, \tag{3.5}$$

quand $M \ge m$ et $i/M \le 1/m$.

Montrons (3.4). Γ étant un $2H$-quasicercle

$$\|p_{j+l} - p_j\| \ge \frac{\operatorname{diam}\Gamma(p_j,p_{j+l})}{2H} \ge \frac{r_m}{2H}\,.$$

L'inégalité (3.4) est alors une conséquence du Lemme 2.4 s'il existe $\tilde H$ dépendant que de H tel que

$$N\Big(\frac{r_m}{2H},\Gamma(p_i,p_{i+k})\Big) \le \tilde H\,N\Big(\frac{r_m}{2H},\Gamma(p_j,p_{j+l})\Big)\,. \tag{3.6}$$

Les disques $D(p_{i+\nu}, 2Hr_m)$, $\nu = 1, \dots, k$, couvrent $\Gamma(p_i, p_{i+k})$. De ceci et de (2.2) on tire

$$N\Big(\frac{r_m}{2H}, \Gamma(p_i, p_{i+k})\Big) \leq \nu(4H^2)\, N(2Hr_m, \Gamma(p_i, p_{i+k})) \leq \nu(4H^2)\, k\,.$$

D'autre part, $l \leq n(r_m, \Gamma(p_j, p_{j+l}))$ puisque $\{p_j, \dots, p_{j+l}\}$ est un r_m-ensemble de $\Gamma(p_j, p_{j+l})$ non nécessairement maximal. Par le Lemme 2.2 on a alors

$$l \leq \nu(8H)\, N(r_m, \Gamma(p_j, p_{j+l})) \leq \nu(8H)\, N(\frac{r_m}{2H}, \Gamma(p_j, p_{j+l}))\,.$$

Ces dernières estimations montrent bien que (3.6) a lieu avec $\tilde{H} = \nu(4H^2)\, \nu(8H)$.

De la même façon on montre (3.5): pour pouvoir appliquer le Lemme 2.4 on se convaint qu'il existe $\tilde{H}$ dépendant que de H tel que

$$N\Big(r_M, \Gamma\big(0, h_M(\frac{i}{M})\big)\Big) \leq \tilde{H}\, N\Big(r_M, \Gamma\big(0, h_m(\frac{1}{m})\big)\Big)\,, \tag{3.7}$$

quand $M \geq m$ et $i/M \leq 1/m$. On montre comme plus haut que

$$N\Big(r_M, \Gamma\big(0, h_M(\frac{i}{M})\big)\Big) \leq \nu(2H)\, N\Big(2Hr_M, \Gamma\big(0, h_M(\frac{i}{M})\big)\Big) \leq \nu(2H)\, i\,.$$

D'autre part, les Lemmes 2.2 et 2.3 font que

$$N(r_M, \gamma) \leq H\, \nu(8H)\, N(r_m, \gamma)\, N\Big(r_M, \Gamma\big(0, h_m(\frac{1}{m})\big)\Big)\,.$$

Comme $N(r_k, \gamma) = k$ on a alors

$$\frac{M}{m} \leq H\, \nu(8H)\, N\Big(r_M, \Gamma\big(0, h_m(\frac{1}{m})\big)\Big)\,.$$

Or, $i/M \leq 1/m$ et donc (3.7) est bien valable avec $\tilde{H} = H\, \nu(2H)\, \nu(8H)$.

En conséquence de (3.4) et (3.5) il est clair que $\{h_m\}$ est equicontinue. Par le Théorème d'Ascoli il est alors possible d'extraire une sous-suite de $\{h_m\}$ convergeant uniformément sur les compacts de $\mathbb{R}$. L'inégalité (3.4) implique que la limite h satisfait l'inégalité de (I) et que h est soit constante soit un homéomorphisme de $\mathbb{R}$ sur Γ. Afin de voir que h n'est pas une application constante il suffit d'établir que

$\|h(1)\| > 0$; h fixe 0. Comme $\{h_m(i/m) : i = 0, \dots, m\}$ est un r_m-ensemble de $\Gamma(0, h_m(1))$ on a

$$N(r_m, \gamma) = m \leq n(r_m, \Gamma(0, h_m(1))) \leq \nu(8H)\, N(r_m, \Gamma(0, h_m(1)))\,.$$

On peut alors appliquer le Lemme 2.4: il existe $d = d(H)$ tel que $\|h_m(1)\| \geq \|q\|/d = 1/d$. La fonction h est donc bien un homéomorphisme.

b) Le cas compact: Γ est homéomorphe à $\mathbb{S}^1$. Supposons que $1 \in \Gamma$. Soit $r > 0$ suffisament petit et $P = \{p_0 = 1, p_1, \dots, p_{m-1}\}$ ordonné r-espacé et maximal sur Γ tel que $r/2 \leq \|p_{m-1} - p_0\| \leq 3r/2$. A partir de ce choix on conclut comme dans le cas non compact.

(II) *si et seulement si* (I). Soit d'abord Γ une courbe vérifiant (II) et paramétrons cette courbe à l'aide de la mesure ω: comme Γ est un quasicercle (Lemme 2.1) c'est une courbe homéomorphe soit à $\mathbb{R}$ soit à $\mathbb{S}^1$. Considérons d'abord le cas non compact. Soit $p \in \Gamma$ et notons Γ^+ et Γ^- les composantes connexes de $\Gamma \backslash \{p\}$. On définit $h : \mathbb{R} \longrightarrow \Gamma$ par $h(x) = q \in \overline{\Gamma^+}$ le point avec $\omega(\Gamma(p, q)) = x$ quand $x \geq 0$ et par $h(x) = q \in \Gamma^-$ le point avec $\omega(\Gamma(p, q)) = -x$ quand $x < 0$. Quand Γ est homéomorphe à $\mathbb{S}^1$ on note λ la mesure de Lebesque du cercle normalisée de sorte que $\lambda(\mathbb{S}^1) = \omega(\Gamma)$. Pour définir $h : \mathbb{S}^1 \longrightarrow \Gamma$ on prend $x \in \mathbb{S}^1$, $p \in \Gamma$ et on munit les deux courbes d'une orientation. Alors, l'image de $y \in \mathbb{S}^1$ soit $h(y) = q \in \Gamma$ le point tel que les segments joignant dans le sens positif x à y, p à q respectivement, ont une même longueur (mesuré avec λ, ω respectivement).

Dans les deux cas on obtient ainsi un homéomorphisme $h : \mathbb{R}$ ou $\mathbb{S}^1 \longrightarrow \Gamma$. Montrons pour Γ non compact que cette paramétrisation vérifie l'inégalité de (I). Soit $h : \mathbb{R} \longrightarrow \Gamma$, $x, y \in \mathbb{R}$ et $s \geq t > 0$. On note $p_1 = h(x)$, $p_2 = h(x+t)$, $q_1 = h(y)$, $q_2 = h(y+s)$ et $R = \|q_2 - q_1\|$. Soit $\{a_0, a_1, \dots, a_n\}$ un R-ensemble de $\Gamma(p_1, p_2)$. Quand $n = 1$, forcément $\|p_1 - p_2\| \leq \|q_1 - q_2\|$. Sinon, les hypothèses sur ω et la définition de h impliquent

$$\begin{aligned} t = \omega(\Gamma(p_1, p_2)) &= \sum_{i=1}^{n} \omega(\Gamma(a_{i-1}, a_i)) \\ &\geq \frac{n-1}{A}\, \omega(\Gamma(q_1, q_2)) \geq \frac{n}{2A}\, s \geq \frac{n}{2A}\, t\,. \end{aligned}$$

On en déduit que $\|p_1 - p_2\| \leq nR \leq 2A\, \|q_1 - q_2\|$.

Pour la réciproque on pose $\omega(E) = |h^{-1}(E)| = \int_{h^{-1}(E)} dx$ où $E \subset \Gamma$ est un ensemble mesurable. C'est une mesure non triviale et (σ-)finie, *i.e.* $\omega(\Gamma(p,q)) = \omega(\Gamma(h(x+t), h(x))) = t$. Montrons qu'elle vérifie bien l'inégalité de (II). Soit $R = \|p_1 - p_2\| = \|q_1 - q_2\|$ avec $p_i, q_i \in \Gamma$ et soit $0 < t = \omega(\Gamma(p_1,p_2)) < \omega(\Gamma(q_1,q_2)) = s$. Notons $y_i = h^{-1}(q_i)$ et $a_j = h(y_1 + jt)$; on suppose que $y_1 < y_2$. (I) implique $\|a_i - a_{i-1}\| \geq \|p_2 - p_1\|/M = R/M$.

Soit $k \in \mathbb{N}$ le plus petit entier tel que $y_1 + kt \geq y_2$, *i.e.* tel que $kt \geq s$. Γ est un $K = K(M)$-quasicercle ce qui implique $\|a_i - a_j\| \geq R/(KM)$ quand $i \neq j$. Par conséquent tout disque de rayon $R/(2KM)$ contient au plus un des points a_i et donc

$$k \leq N\Big(\frac{R}{2KM}, \Gamma(q_1,q_2)\Big) \leq \nu(2K^2M)\, N(KR, \Gamma(q_1,q_2)) = \nu(2K^2M)$$

ce qui montre bien

$$\omega(\Gamma(q_1,q_2)) \leq \sum_{i=1}^{k} \omega(\Gamma(a_{i-1},a_i)) \leq \nu(2K^2M)\,\omega(\Gamma(p_1,p_2)).$$

3.1. *QI*-cercles avec constante un.

Pour terminer la démonstration du Théorème 1.1 il suffit de préciser ce qui se passe quand une des constantes est égale à un. Or, dans le cas où le groupe est 1-quasi-isométrique c'est un groupe d'isométries et donc Γ est soit une droite soit un cercle. Si la constante de (I) est un, l'homéomorphisme h est une 1-quasisymétrie et [McKV] implique que h est une transformation de Möbius. Quand Γ vérifi (II) avec constante $A = 1$ il est clair que l'homéomorphisme obtenu par paramétrisation comme dans la preuve (II) implique (I) est égalément une 1-quasisymétrie. Le cas restant se trouve dans la

Proposition 3.1. *Soit* Γ *une courbe vérifiant* (III) *avec constante égale à un, alors il s'agit soit d'une droite soit d'un cercle.*

La preuve de ce fait est basée sur le résultat suivant.

Lemme 3.2. *Soit* Γ *une courbe vérifiant* (III) *et soient* $\gamma_i = \Gamma(p_i, q_i)$, $i = 1, \ldots, k$, *des arcs disjoints. Alors*

$$\lim_{r\to 0} \frac{N(r,\gamma_1) + \cdots + N(r,\gamma_k)}{N(r, \gamma_1 \cup \cdots \cup \gamma_k)} = 1\,.$$

PREUVE. Il suffit de considérer deux arcs disjoints γ_1 et γ_2. Si $\overline{\gamma}_1 \cap \overline{\gamma}_2 = \varnothing$ tout est clair. Sinon, $\{p\} = \overline{\gamma}_1 \cap \overline{\gamma}_2$, on a toujours $N(r,\gamma_1)+N(r,\gamma_2) \geq N(r,\gamma_1 \cup \gamma_2)$.

Soit $\gamma_{i,r} = \gamma_i \backslash (\gamma_i \cap D(p, 10Hr))$ pour $i = 1,2$ et $r > 0$ petit par rapport à $\operatorname{diam} \gamma_i$, $i = 1,2$. Puisque Γ est un $2H$-quasicercle (Lemme 2.1) on a $\|x_1 - x_2\| \geq 10Hr/2H = 5r$ pour tout $x_i \in \gamma_{i,r}$, $i = 1,2$. Les disques de rayon r d'un recouvrement minimal de $\gamma_{1,r}$ sont alors disjoints des disques d'un tel recouvrement de $\gamma_{2,r}$. D'où

$$N(r,\gamma_{1,r}) + N(r,\gamma_{2,r}) \leq N(r,\gamma_1 \cup \gamma_2).$$

En réutilisant le fait que le disque $D(p, 10Hr)$ se couvre par au plus $\nu = \nu(10H)$ disques de rayon r, on en déduit

$$\begin{aligned} N(r,\gamma_1) + N(r,\gamma_2) &\leq N(r,\gamma_{1,r}) + N(r,\gamma_{2,r}) + 2\nu \\ &\leq N(r,\gamma_1 \cup \gamma_2) + 2\nu, \end{aligned}$$

ce qui achève la preuve.

PREUVE DE LA PROPOSITION 3.1. Il suffit de reprendre la preuve de (III) implique (I) et de voir que la constante $C_1(H)$ dans (3.4) peut être prise égale à un quand $H = 1$. Dans ce cas, on obtient encore une 1-quasisymétrie envoyant la droite ou le cercle sur Γ.

Notons $p_i = h_m(i/m)$ et supposons qu'on puisse avoir $\|p_{i+k} - p_i\| > \|p_{j+l} - p_j\|$ avec $k \leq l$. Dans ce cas il existe $x \in \Gamma(p_i, p_{i+k})$ avec $\|x - p_i\| = \|p_{j+l} - p_j\| = R$. Alors, par le Lemme 3.2 et par (III) avec $H = 1$

$$\begin{aligned} 1 \geq \frac{k}{l} &= \lim_{r\to 0} \frac{\sum_{\nu=1}^{k} N(r,\Gamma(p_{i+\nu-1}, p_{i+\nu}))}{\sum_{\nu=1}^{l} N(r,\Gamma(p_{j+\nu-1}, p_{j+\nu}))} \\ &= \lim_{r\to 0} \frac{N(r,\Gamma(p_i,x)) + N(r,\Gamma(x,p_{i+k}))}{N(r,\Gamma(p_j,p_{j+l}))}. \end{aligned}$$

Or, quand $\|p_{i+k} - x\| > R$ il est clair que $N(r,\Gamma(x,p_{i+k})) \geq N(r,\Gamma(p_i,x))$, $r < R$. Sinon on applique le Lemme 2.3,

$$n(\|p_{i+k} - x\|, \Gamma(p_i,x))\, N(r,\Gamma(x,p_{i+k})) \geq N(r,\Gamma(p_i,x)).$$

D'où, il existe $\varepsilon > 0$ tel que

$$1 \geq 1 + \lim_{r\to 0} \frac{N(r,\Gamma(x,p_{i+k}))}{N(r,\Gamma(p_j,p_{j+l}))} \geq 1 + \varepsilon \lim_{r\to 0} \frac{N(r,\Gamma(p_i,x))}{N(r,\Gamma(p_j,p_{j+l}))} = 1 + \varepsilon,$$

ce qui est impossible.

4. Les QI-cercles et la géométrie fractale.

Comme on l'a remarqué dans l'introduction, les "quasi-self-similar" cercles sont des QI-cercles. Quasi-auto-similarité veut dire que des petits morceaux d'une courbe s'agrandissent par une similitude en des arcs d'une taille standart, lesquels se plongent quasi-isométriquement dans la courbe elle même. Une conséquence de ceci est que les sous-arcs d'une même taille sont quasi-isométriques; *cf.* (1.1). Cette dernière propriété est aussi valable pour les QI-cercles. On peut même dire qu'elle les caractérise. Par contre, il n'y a plus de rapport entre des sous-arcs de différentes tailles.

Cet affaiblissement des propriétés fait que les résultats des "quasi-self-similar" cercles ne se transmettent pas directement à des QI-cercles. Néanmoins, on peut montrer que pour un QI-cercle la dimension de Hausdorff coïncide avec la "lower Box-dimension". En plus il est possible d'étendre le théorème de Falconer et Marsh [FM] à des QI-cercles.

Pour pouvoir faire ceci nous aurons besoin de mieux connaître le comportement de certaines mesures sur un QI-cercle Γ. On lui associe d'abord une fonction ρ, appelée *fonction de dimension* canonique, de la façon suivante: quand Γ est compact on pose $\rho(r) = 1/N(r,\Gamma)$, $0 < r \leq \operatorname{diam}\Gamma$. Sinon on choisit $\gamma = \Gamma(a,b)$ un sous-arc quelconque de Γ avec $\|a-b\| = 1$ et $\gamma_r = \Gamma(a,x) \supset \gamma$ avec $\|a-x\| = r$ ainsi que $\|a-y\| < r$ pour tout $y \in \Gamma(b,x)$. Alors, on définit maintenant $\rho(r) = 1/N(r,\gamma)$ pour $0 < r \leq 1$ et $\rho(r) = N(1,\gamma_r)$ pour $r > 1$.

Lemme 4.1. *Sur un QI-cercle Γ il existe une mesure ω et des constantes C_3 et C_4 telles que*

$$(4.8) \qquad \frac{1}{C_3} \leq \frac{\omega(\Gamma(p,q))}{\rho(\|p-q\|)} \leq C_3\,, \qquad \textit{pour tous}\ \ p,q \in \Gamma\,,\ p \neq q\,,$$

et

$$(4.9) \qquad \omega(D(x,r)) \leq C_4\,\rho(r)\,, \qquad \textit{pour tous}\ \ x \in \mathbb{R}^2\ \ \textit{et}\ \ r > 0\,.$$

PREUVE. Soit ω la mesure de (II) du Théorème 1.1 normalisée par $\omega(\Gamma) = 1$ quand Γ est compact et par $\omega(\gamma) = 1$ sinon, γ étant l'arc de la définition de ρ.

Soit Γ non compact et considérons d'abord le cas $r = \|p-q\| \leq 1$, $p,q \in \Gamma$. Notons $\{x_0,\ldots,x_n\}$ un r-ensemble de γ. Alors l'inégalité de

(II) implique

$$1=\omega(\gamma)=\sum_{i=1}^{n}\omega(\Gamma(x_{i-1},x_i))\le A\,n\,\omega(\Gamma(p,q))$$

et aussi $n\,\omega(\Gamma(p,q))/2\le A$. L'inégalité (4.8) en résulte immédiatement puisque les nombres n et $N(r,\gamma)$ sont équivalents (Lemme 2.2).

Quand $r=\|p-q\|>1$ on choisit $\{x_0,\dots,x_n\}$ un 1-ensemble de γ_r et on applique encore l'inégalité de (II):

$$\omega(\Gamma(p,q))\le A\,\omega(\gamma_r)=A\sum_{i=1}^{n}\omega(\Gamma(x_{i-1},x_i))\le A^2\,n\,\omega(\gamma)=A^2\,n$$

et $A^2\omega(\Gamma(p,q))\ge n/2$. Il suffit encore d'appliquer le Lemme 2.2 pour en déduire (4.8).

Quand Γ est compact on procède de la même façon.

De ces estimations on déduit (4.9). Effectivement, puisque Γ est un $2H$-quasicercle, avec H la constante de (III), l'ensemble $\Gamma\cap D(x,5Hr)$ contient un arc $\sigma=\Gamma(u,v)$, $u,v\in\partial D(x,5Hr)$, contenant l'ensemble $\Gamma\cap D(x,r)$. D'où $\omega(D(x,r))=\omega(\Gamma\cap D(x,r))\le\omega(\sigma)$ et par (4.8) ceci devient $\omega(D(x,r))\le C_3\,\rho(\|u-v\|)$. On en déduit (4.9) puisque $\|u-v\|\le 10\,Hr$.

La fonction ρ définie avant le lemme précédent est croissante et elle vérifie $\lim_{r\to 0}\rho(r)=0$. A une telle fonction on peut associer la mesure de Hausdorff suivante

$$(4.10)\quad m_\rho(E)=\lim_{r\to 0}\,\inf\left\{\sum_{i=1}^{\infty}\rho(r_i):\ E\subset\bigcup_{i=1}^{\infty}D(x_i,r_i)\text{ avec } r_i<r\right\},$$

pour $E\subset\Gamma$.

Lemme 4.2. *Soit Γ un QI-cercle, ρ sa fonction de dimension canonique et m_ρ la mesure de Hausdorff associée. Alors, il existe une constante C_5 telle que*

$$\frac{1}{C_5}\,\rho(\|p-q\|)\le m_\rho(\Gamma(p,q))\le C_5\,\rho(\|p-q\|)\,,\quad \textit{pour tous } p,q\in\Gamma\,.$$

PREUVE. L'inégalité de gauche est une conséquence du Lemme 4.1 et du lemme de Frostman. Effectivement, si $\Gamma(p,q) \subset \bigcup_{i=1}^{\infty} D(x_i, r_i)$ alors

$$\omega(\Gamma(p,q)) \le \sum_{i=1}^{\infty} \omega(D(x_i, r_i)) \le C_4 \sum_{i=1}^{\infty} \rho(r_i)$$

et donc $C_4\, C_3\, m_\rho(\Gamma(p,q)) \ge \rho(\|p-q\|)$.

Dans la preuve de l'autre inégalité on fait appel aux Lemmes 2.2 et 2.3. Quand $R = \|p-q\| > 1$ alors

$$\begin{aligned} m_\rho(\Gamma(p,q)) &\le \liminf_{r\to 0} \frac{N(r,\Gamma(p,q))}{N(r,\gamma)} \\ &\le H\, n(1,\Gamma(p,q)) \\ &\le H^2\, \nu(8H)\, \rho(\|p-q\|) \end{aligned}$$

et quand $R = \|p-q\| \le 1$ on a

$$m_\rho(\Gamma(p,q)) \le \liminf_{r\to 0} \frac{N(r,\Gamma(p,q))}{N(r,\gamma)} \le \frac{1}{\eta(H)}\, \frac{1}{n(R,\gamma)} \le \frac{\rho(\|p-q\|)}{\eta(H)} .$$

4.1. Les QI-cercles et les différentes dimensions.

Connaître le lien entre les différentes notions de dimensions a des avantages pratiques. Par exemple, la simple définition des "Box-dimensions" fait qu'elles s'évaluent facilement alors que l'estimation de la dimension de Hausdorff est souvent laborieuse et difficile.

Précisons ces termes pour un compact $K \subset \mathbb{R}^2$. Sa dimension de Hausdorff est $\mathrm{Hdim}(K) = \sup\{\delta > 0 :\ m_{r^\delta}(K) = +\infty\}$; m_{r^δ} est la mesure de Hausdorff de dimension δ. Les "Box-dimensions" sont données par

$$\underline{\mathrm{Bdim}}(K) = \liminf_{r\to 0} \frac{\log N(r,K)}{-\log r} ,$$

$$\overline{\mathrm{Bdim}}(K) = \limsup_{r\to 0} \frac{\log N(r,K)}{-\log r} .$$

On a toujours $\mathrm{Hdim}(K) \le \underline{\mathrm{Bdim}}(K) \le \overline{\mathrm{Bdim}}(K)$ et pour les "quasi-self-similar" cercles ces trois nombres coïncident.

Proposition 4.3. *Un QI-cercle Γ a toujours la dimension de Hausdorff égale à la "lower Box-dimension":* $\mathrm{Hdim}(\Gamma) = \underline{\mathrm{Bdim}}(\Gamma)$. *Par contre, il existe des exemples pour lesquels les différentes "Box-dimensions" sont distinctes:* $\underline{\mathrm{Bdim}}(\Gamma) < \overline{\mathrm{Bdim}}(\Gamma)$.

PREUVE DE $\mathrm{Hdim}(\Gamma) = \underline{\mathrm{Bdim}}(\Gamma)$. Il est suffisant de considérer $\gamma \subset \Gamma$ un arc quelconque puisque sur Γ agit un groupe quasi-isométrique transitif et les dimensions sont invariantes par quasi-isométrie. Prenons alors pour γ l'arc de la définition de la fonction de dimension canonique ρ quand Γ n'est pas compact et $\gamma = \Gamma$ sinon.

Pour tout

$$\beta < \delta = \underline{\mathrm{Bdim}}(\gamma) = \liminf_{r \to 0} \frac{\log \rho(r)}{\log r}$$

on a $\lim_{r \to 0} \rho(r)/r^\beta = 0$. Comme $m_\rho(\gamma) > 0$ (Lemme 4.2) il suit que $m_{r^\beta}(\gamma) = +\infty$.

Un exemple d'un QI-cercle ayant différents "upper" et "lower Box-dimension".

Utilisons la construction du "snowflake" et notons $T_{a,b}$ la transformation qui associe à un intervalle $I = [a,b]$ la première iterée du snowflake avec les extrémités a et b. Plus précisement, soient $a = (a_1, a_2)$, $b = (b_1, b_2)$ ainsi que

$$x_1 = a + \frac{b-a}{3}, \qquad x_2 = \frac{a+b}{2} + (b_2 - a_2, a_1 - b_1)\frac{1}{2\sqrt{3}}$$

et

$$x_3 = a + \frac{2(b-a)}{3}.$$

On associe à ces points quatre similitudes contractantes T_i , $i = 1, \dots, 4$, envoyant $\overrightarrow{ab}$ sur $\overrightarrow{ax_1}$, $\overrightarrow{x_1x_2}$, $\overrightarrow{x_2x_3}$, $\overrightarrow{x_3b}$, respectivement. Notons

$$T_{a,b}(K) = \bigcup_{i=1}^{4} T_i(K), \qquad K \text{ compact de } \mathbb{R}^2.$$

Soit $J_0 = [a,b]$ avec $\|a-b\| = 1$ et $J_1 = T_{a,b}(J_0)$. J_1 consiste en quatre côtés de longueur $1/3$, ce qui correspond à une dimension $\delta_1 = \log 4/\log 3$. Soit $0 < \varepsilon < (\delta_1 - 1)/4$. Il existe $k_1 \in \mathbb{N}$ tel que

$$4\,k_1 \left(\frac{1}{3k_1}\right)^\delta = 1, \qquad \text{avec } 1 \le \delta \le 1 + \varepsilon.$$

En fait, on saute k_1 itérations de $T_{a,b}$ pour ramener la dimension proche de un. Partageons alors J_1 en $4\,k_1$ $[a_i, b_i]$ de longueur $1/(3\,k_1)$: $J_1 = \cup_{i=1}^{4k_1}[a_i, b_i]$. Maintenant on va effectuer k_2 itérations de $T_{a_i b_i}$ sur les intervalles $[a_i, b_i]$. Soit donc

$$J_2 = \bigcup_{i=1}^{4k_1} T_{a_i b_i}^{k_2}\,([a_i, b_i])\ ,$$

où k_2 est choisi de sorte que

$$4^{k_2} 4\,k_1 \left(\frac{1}{3^{k_2}}\,\frac{1}{3k_1}\right)^{\delta} = 1\,,$$

avec cette fois ci $\delta_1 - \varepsilon < \delta \le \delta_1$. Les prochaines k_3 itérations de $T_{a_i b_i}$ on les saute pour avoir de nouveau une dimension proche de un et ainsi de suite.

Soit $\Gamma(a, b)$ la limite de ce procédé. L'exemple cherché est

$$\Gamma = \Gamma\big((0,0),(1,0)\big) \cup \Gamma\Big((0,0),(\frac{1}{2},\frac{\sqrt{3}}{2})\Big) \cup \Gamma\Big((\frac{1}{2},\frac{\sqrt{3}}{2}),(1,0)\Big)\,.$$

Clairement, Γ est un QI-cercle le critère géométrique (III) étant vérifié et on a

$$\underline{\mathrm{Bdim}}\,(\Gamma) < \overline{\mathrm{Bdim}}\,(\Gamma)\,.$$

4.2. Classification des QI-cercles par quasi-isométrie.

Si on veut établir un analogue du théorème de Falconer et Marsh [FM] pour les QI-cercles on est amené à prendre une notion plus forte que seulement la dimension, à cause de l'exemple du paragraphe précédent: ce sont les classes de fonctions de dimension.

On appelle une fonction $\rho : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ croissante et vérifiant $\lim_{r\to 0} \rho(r) = 0$ ainsi que $\lim_{r\to\infty} \rho(r) = +\infty$ *fonction de dimension.* Remarquons que pour les courbes compactes seul le comportement de ρ au voisinage de 0 est important. On se contente alors dans ce cas de définir la fonction de dimension au voisinage de 0. Deux fonctions de dimension ρ_1, ρ_2 sont équivalentes s'il existe $\alpha, \beta > 0$ tels que

$$\alpha \le \frac{\rho_1(r)}{\rho_2(r)} \le \beta\,, \qquad \text{pour tout } r > 0\,.$$

Au début de ce paragraphe nous avons associé à un QI-cercle Γ une fonction de dimension canonique ρ. L'ensemble de fonctions de dimensions équivalentes à ce ρ est appelé la classe de fonctions de dimension associée à Γ.

Théorème 4.4. *Soient Γ_1 et Γ_2 deux QI-cercles, tout deux compacts ou non. Il existe $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ une quasi-isométrie identifiant les courbes, $\Phi(\Gamma_1) = \Gamma_2$, si et seulement si Γ_1 et Γ_2 ont la même classe de fonctions de dimension.*

REMARQUE. Ce théorème contient le résultat de Falconer et Marsh [FM]: si Γ est un "quasi-self-similar" cercle, sa classe de fonctions de dimension est celle qui contient la fonction constante $\delta(r) = \dim \Gamma$. D'où, pour deux telles courbes Γ_1 et Γ_2 il existe une quasi-isométrie $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ si et seulement si les deux courbes ont la même dimension.

PREUVE. Montrons l'existence de la quasi-isométrie Φ sous l'hypothèse que les Γ_i ont la même classe de fonctions de dimension. On note ρ_1, ρ_2 les fonctions de dimension canoniques de Γ_1, Γ_2 respectivement et m_{ρ_i} les mesures de Hausdorff associées. Ces mesures permettent de paramétrer les courbes Γ_i; *cf.* la preuve (II) implique (I). Notons $h_i : \mathbb{R}$ ou $\mathbb{S}^1 \longrightarrow \Gamma_i$, $i = 1,2$, ces paramétrisations. On montre que $\varphi = h_2 \circ h_1^{-1} : \Gamma_1 \longrightarrow \Gamma_2$ est une quasi-isométrie.

Par le Lemme 4.2 il existe $C_5 \geq 1$ tel que

$$\frac{\rho_i\,(\|u-v\|)}{C_5} \leq m_{\rho_i}(\Gamma_i(u,v)) \leq C_5\,\rho_i(\|u-v\|)\,, \quad u,v \in \Gamma_i\,.$$

Si $p_1, p_2 \in \Gamma_1$ et si $q_i = \varphi(p_i) \in \Gamma_2$, $i = 1,2$, alors $m_{\rho_1}(\Gamma_1(p_1,p_2)) = m_{\rho_2}(\Gamma_2(q_1,q_2))$ et donc

$$\frac{1}{C_5^2} \leq \frac{\rho_1(\|p_1-p_2\|)}{\rho_2(\|q_1-q_2\|)} \leq C_5^2\,.$$

Par l'équivalence de ρ_1 et ρ_2 il existe $\alpha, \beta > 0$ telles que

$$\frac{\beta}{C_5^2} \leq \frac{\rho_1(\|p_1-p_2\|)}{\rho_1(\|q_1-q_2\|)} \leq \alpha\, C_5^2\,.$$

De cette condition il est facile à voir que φ est une quasi-isométrie. Il suffit d'expliciter $\rho_1(r)$ dans les différents cas et d'utiliser le Lemme 2.3.

On peut prolonger φ en une quasi-isométrie Φ du plan grâce à un résultat de Gehring [Ge1]. Une autre possibilité est de montrer, comme dans la preuve "(I) implique Γ est un QI-cercle", que $\Phi = H_2 \circ H_1^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$, avec H_i les extensions quasiconformes de Beurling-Ahlfors-Tukia des quasisymétries h_i, est une quasi-isométrie du plan.

Soit maintenant Φ une L-quasi-isométrie de $\mathbb{R}^2$ telle que $\Phi(\Gamma_1) = \Gamma_2$ et notons encore ρ_1, ρ_2 les fonctions de dimensions canoniques de Γ_1 , Γ_2 respectivement. On doit montrer qu'elles sont équivalentes.

Considérons d'abord le cas Γ_1 et Γ_2 compact. Si $D(x_1, r), \ldots, D(x_N, r)$ est un recouvrement minimal de Γ_1, alors les ensembles $E_i = \Phi(D(x_i, r))$ couvrent Γ_2 et $E_i \subset D(\Phi(x_i), Lr)$. D'où et avec (2.2) il est clair que $N(r, \Gamma_1) \geq N(Lr, \Gamma_2) \geq N(r, \Gamma_2)/\nu(L)$. Par symétrie du problème il en résulte $1/\nu(L) \leq \rho_1(r)/\rho_2(r) \leq \nu(L)$.

Quand les Γ_i ne sont pas compacts on doit montrer qu'il existe $C \geq 1$ tel que

$$\frac{1}{C} \leq \frac{N(r, \gamma_1)}{N(r, \gamma_2)} \leq C, \qquad \text{et} \qquad \frac{1}{C} \leq \frac{N(1, \gamma_{1,r})}{N(1, \gamma_{2,r})} \leq C$$

où les arcs viennent de la définition des ρ_i. Ce qui compte est que les extrémités des γ_i et des $\gamma_{i,r}$ sont à distance égale. D'où, la preuve de ceci est exactement le contenue de la preuve "QI-cercle implique (III)". Le rôle des g_t dans cette preuve prend ici $g_{2,t} \circ \Phi$ où $G_2 = \{g_{2,t} : t \in \mathbb{R}\}$ est le groupe quasi-isométrique de Γ_2 .

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Weyl sums and atomic energy oscillations

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*"...cuando hay vino beben vino
cuando no hay vino, agua fresca."*

A. Machado.

Abstract. We extend Van der Corput's method for exponential sums to study an oscillating term appearing in the quantum theory of large atoms. We obtain an interpretation in terms of classical dynamics and we produce sharp asymptotic upper and lower bounds for the oscillations.

1. Introduction.

The purpose of this paper is to study a certain sum that plays a crucial role in the asymptotic analysis of non-relativistic atomic energies. The sum is given by the expression

$$\Psi_Q(Z) = \sum_{l=1}^{l_{\rm TF}} \frac{2l+1}{\dfrac{1}{\pi}\displaystyle\int \Big(V_{\rm TF}^Z(r) - \frac{l(l+1)}{r^2}\Big)_+^{-1/2} dr} \cdot \mu\Big(\frac{1}{\pi}\int \Big(V_{\rm TF}^Z(r) - \frac{l(l+1)}{r^2}\Big)_+^{1/2} dr\Big),$$

where $\mu(x) = \text{dist}\{x, \mathbb{Z}\}^2 - 1/12$, V_{TF}^{Z} is the Thomas-Fermi potential with charge Z (see [Li]), which satisfies the perfect scaling condition

$$V_{\text{TF}}^{Z}(r) = Z^{4/3} V_{\text{TF}}^{1}\left(Z^{1/3}r\right) \tag{1.a}$$

and we have

$$V_{\text{TF}}^{1}(r) = \frac{y(ar)}{r}, \qquad a = \left(\frac{3\pi}{2}\right)^{2/3} \tag{1.b}$$

and y is the Thomas-Fermi function, solution of the Thomas-Fermi equation

$$\begin{cases} y''(r) = \dfrac{y^{3/2}(r)}{r^{1/2}}, \\ y(0) = 1, \\ \lim\limits_{r \to +\infty} y(r) = 0, \end{cases}$$

and l_{TF} is the greatest integer such that $V_{\text{TF}}^{Z}(r) - l(l+1)/r^2$ is positive somewhere. Here, and throughout this article, we set

$$(x)_+^{-1/2} = \begin{cases} x^{1/2}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

The role of the function $\Psi_Q(Z)$ in atomic physics is as follows:

Consider a non-relativistic atom, consisting of a nucleus of charge Z fixed at the origin, and N quantized electrons at positions $x_i \in \mathbb{R}^3$. The hamiltonian of such a system is given by

$$H_{Z,N} = \sum_{i=1}^{N}\left(-\Delta_{x_i} - \frac{Z}{|x_i|}\right) + \frac{1}{2}\sum_{i \neq j}\frac{1}{|x_i - x_j|}$$

acting on

$$\psi \in \mathcal{H} = \bigwedge_{i=1}^{N} L^2\left(\mathbb{R}^3 \otimes \mathbb{Z}_2\right).$$

We define the energy of such an atom as

$$E(Z) = \inf_{N \geq 0} E(Z, N), \qquad E(Z,N) = \inf_{\substack{\phi \in \mathcal{H} \\ \|\phi\| = 1}} \langle H_{Z,N}\phi, \phi\rangle.$$

The computation of $E(Z)$ can only be done explicitly for $Z = 1$, when it equals $-1/4$. For $Z = 2$ good upper and lower bounds are known, but the situation gets more and more complicated as Z grows. It was observed very early in the history of quantum mechanics, in 1927 (see [Th] and [Fe]), by Thomas and Fermi, that for Z large, $E(Z)$ must approximately equal $c_{\mathrm{TF}} Z^{7/3}$ for c_{TF} a well known explicit constant. This was made rigorous by Lieb and Simon in 1973 ([LS] and ([Li]), a very beautiful result which also holds for molecules.

Comparisons with numerical results showed that the Thomas-Fermi approximation was only good up to a term of size Z^2, and Scott ([Sc]) in 1950 was the first to realize that this Z^2 effect was due to electrons very near the nucleus, which behave as if they were in the exactly solvable model without electronic interaction. His argument was make rigorous in a series of papers by Hughes-Siedentop-Weikard ([Hu], [SW1], [SW2] and [SW3]) in 1985-89. This was proved to be true also for molecules by Ivrii-Sigal [IS].

A smaller effect, of size $Z^{5/3}$ was observed by Dirac, in 1930 ([Di]), which comes from a delicate analysis of electronic correlations. Additional effects were also found by Scott ([Sc]), corrected by March and Plaskett [MP], and then finally established by Schwinger ([Sch]), who argued that the asymptotic energy expansion should then contain the term $c_{DS} Z^{5/3}$, for c_{DS} an explicit constant. The proof of Schwinger's result was announced in [FS1], and is as follows:

$$E(Z) = c_{\mathrm{TF}}\, Z^{7/3} + \frac{1}{8}\, Z^2 + c_{DS}\, Z^{5/3} + O\Big(Z^{5/3-a}\Big)\,, \qquad a > 0\,.$$

Its complete proof appears in [FS2], [FS3], [FS4], [FS5], [FS6], [FS7] and [FS8].

It has been known for some time that nice asymptotics for atomic energies in powers of $Z^{1/3}$ will stop after the Dirac-Schwinger term. This can most easily be conjectured by looking at simpler, exactly solvable models such as the harmonic oscillator (see [Si]). Comparisons with numerical results also show that the next correction will be oscillatory in nature. We refer the reader to the book of Englert ([En]; see also [ES1] and [ES2]) for a physical discussion of the energy asymptotics up to including oscillatory terms. The exact form of the function Ψ_Q above originates from the proof of the Dirac-Schwinger's term in [FS1], where it is seen that

$$\text{(2)} \qquad E(Z) = c_{\mathrm{TF}}\, Z^{7/3} + \frac{1}{8}\, Z^2 + c_{DS}\, Z^{5/3} + \Psi_Q(Z) + O\Big(Z^{5/3-a}\Big)\,,$$

where $a > 0$, although the current estimates for a above do not yet guarantee that Ψ_Q really dominates over the O-term.

Note that in establishing (2) not only do we need estimates for the error terms with a large enough, but also we need lower bounds for the size of the function Ψ_Q, which are not completely obvious. It follows from our results in the present article that one would need $a > 1/6$ in order to show that Ψ_Q dominates over the error terms contained in the O-term.

From the abstract mathematical perspective, sums such as Ψ_Q are quite old, the best known going back to Gauss, which is related to estimating the number of integral lattice points inside a convex curve: most notably, a circle, which gave rise to the *circle* problem, and a hyperbola, which comes from the *divisor* problem, two of the most elusive problems in analytic number theory (see [GK] for a general description; [IM] and [Hx] for the latest results). It is worth noting the close similarity between our problem and the circle problem, which comes from a refined analysis of the number of bound states of quantum free particles in a box.

A step higher in sophistication, but still within the same realm of problems, is the Selberg trace formula, which, very loosely speaking, expresses spectral information about the laplacian on an abstract manifold in terms of the closed geodesics on that manifold, which can also be seen as the mathematical version of the Feymann Path integrals for abstract systems. We refer the reader to [G] and references thereof for a wealth of ideas in the theory of trace formulas, quantum chaos, classical mechanics, and all that.

An announcement of our results, which are described below, appeared in [CFS].

Our work is organized as follows: First, after making some trivial modifications to the well known stationary phase lemma (Section 1), we set out (in Section 2) to study sums of the type

$$S(\lambda) = \sum_{l=1}^{\lambda} f\left(\frac{l}{\lambda}\right) \mu\left(\lambda \phi\left(\frac{l}{\lambda}\right)\right),$$

where $\phi''(x) \geq c_0 > 0$, and μ is a periodic function of average 0. Examples of such sums are

1. If $f \equiv 1$, $\mu(x) = e^{2\pi ix}$, $\phi(x) = x^2$, we have the well-known Gauss sums modulo λ.

2. If $f \equiv 1$, $\mu(x) = x - [x] - 1/2$, then S represents the error term in the lattice point problem for a curve ϕ dilated by λ.

While the first item above is well understood, the second remains very hard. In our analysis, we will have to deal only with functions μ whose Fourier coefficients decrease rather rapidly ($\hat{\mu}(n) \sim |n|^{-3/2+\epsilon}$), and this allows a complete analysis of the sums via the usual method of Van der Corput (Poisson summation followed by stationary phase; see [GK}), since all expressions turn out to be absolutely convergent in this case. A little elementary number theory will be needed here to rule out the possibility of a small denominator problem, which gives rise to an error term whose size depends on whether a certain number is rational or irrational.

In Section 3, we apply the results of Section 2 to Ψ_Q, obtaining a new sum Ψ_0, a leading "dual" version of Ψ_Q, reminiscent of the Jacobi identity for the modular function. Sharp upper bounds for Ψ_Q are an easy consequence of this. However, obtaining the right regularity properties for the curve and amplitude involved in the formula for Ψ_Q turns out to be rather tedious.

In Section 4 we obtain *lower* bounds for Ψ_Q in the form of an Ω-result, by understanding how Ψ_0 behaves on average.

In Section 5 we use the *dual* expression Ψ_0 to give us a dynamical interpretation of the sum Ψ_Q as a sum of classical data extended over all closed trajectories of a classical hamiltonian. This result appears to have similarities also with a recent result of Bleher [B2].

Section 6 is devoted to side issues.

1. Stationary Phase Estimates.

We begin with a review of stationary phase. Consider $f \in C_0^\infty(\mathbb{R})$. Then, if $t > 0$,

$$\int_{-\infty}^{+\infty} e^{itx^2} f(x)\, dx = e^{\pi i/4} \sqrt{\frac{\pi}{t}} \int_{-\infty}^{+\infty} e^{-\pi^2 i\xi^2/t} \hat{f}(\xi)\, d\xi\, .$$

Using the identity

$$e^s = 1 + \int_0^1 e^{su}\, s\, du\,,$$

we deduce

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{itx^2} f(x)\, dx \\
&\quad = e^{\pi i/4}\sqrt{\frac{\pi}{t}}\left(f(0) + \int_{-\infty}^{+\infty} \hat{f}(\xi) \int_0^1 \frac{-\pi^2 i\,\xi^2}{t}\, e^{-\pi^2 iu\xi^2/t}\, du\, d\xi\right) \\
&\quad = e^{\pi i/4}\sqrt{\frac{\pi}{t}}\left(f(0) + \frac{i}{4t}\int_{-\infty}^{+\infty} \widehat{f''}(\xi) \int_0^1 e^{-\pi^2 iu\xi^2/t}\, du\, d\xi\right) \\
&\quad = e^{\pi i/4}\sqrt{\frac{\pi}{t}}\, f(0) + \frac{i}{4t}\int_{-\infty}^{+\infty} f''(x) \int_0^1 e^{itx^2/u}\, \frac{du}{u^{1/2}}\, dx \\
&\quad = e^{\pi i/4}\sqrt{\frac{\pi}{t}}\, f(0) + \frac{i}{4t^{3/2}}\int_{-\infty}^{+\infty} f''(x)\, g_t(x)\, dx\,,
\end{aligned}$$

for

$$g_t(x) = \int_0^1 e^{itx^2/u} \left(\frac{t}{u}\right)^{1/2} du\,.$$

Note that $g_t(x) = t^{1/2} g_1\left(t^{1/2}x\right)$, and

$$g_1(x) = i\, e^{ix^2/u}\, \frac{u^{3/2}}{x^2}\Big|_0^1 - \frac{3i}{2}\int_0^1 e^{ix^2/u}\, \frac{u^{1/2}}{x^2}\, du\,,$$

hence $|g_1(x)| \le 2/|x|^2$ and thus, g_1 is integrable. Furthermore

$$\|g_t\|_1 = \|g_1\|_1 = O(1)$$

and $|g_t(x)| \le 2\,|x|^{-2} t^{-1/2}$.

We also consider one-sided integrals of the form

$$\int_0^{+\infty} e^{itx^2} f(x)\, dx\,.$$

Define

$$f^+(x) = \begin{cases} f(x)\,, & \text{if } x \ge 0\,, \\ f(-x)\,, & \text{if } x \le 0\,, \end{cases}$$

and consider $f_\varepsilon = f^+ * \varphi_\varepsilon$, for a suitable approximation to the identity φ_ε. Using our previous identity, we obtain

$$\begin{aligned}\int_0^{+\infty} e^{itx^2} f(x)\,dx &= \frac{1}{2}\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} e^{itx^2} f_\varepsilon(x)\,dx\\ &= \frac{1}{2}\,e^{\pi i/4}\sqrt{\frac{\pi}{t}}\,f(0) + \frac{i}{8\,t^{3/2}}\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} f''_\varepsilon(x)\,g_t(x)\,dx\\ &= \frac{1}{2}\,e^{\pi i/4}\sqrt{\frac{\pi}{t}}\,f(0) + \frac{i}{4\,t^{3/2}}\int_0^{+\infty} f''(x)\,g_t(x)\,dx\,.\end{aligned}$$

The last step follows since g_t is integrable and both functions g_t and f^+ are even.

Definition. *Let ϕ such that*

$$|\phi^{(n)}(x)| \le C_n\,,\quad (0\le n\le 5)\,,\qquad c_0 = \inf|\phi''(x)| > 0\,,$$

for x in a certain interval which will be clear in our applications. We denote by

$$c_0^+ = \min\{1, c_0\}\,,\qquad B(\phi) = \left(\frac{1+\|\phi\|_{C^5}}{c_0^+}\right)^{54}\,.$$

Lemma 1 (Stationary Phase Lemma). *Let $f(x)\in C_0^2(\mathbb{R})$, such that*

$$|f'(x)| \le \begin{cases} N_1\,, & \text{if } |x|\le L\,,\\ N_2\,, & \text{if } |x|>L\,,\end{cases}\qquad (N_1\le N_2)$$

and

$$|f''(x)| \le \begin{cases} M_1\,, & \text{if } |x|\le L\,,\\ M_2\,, & \text{if } |x|>L\,,\end{cases}\qquad (M_1\le M_2)$$

and let $\phi(x)$ such that $\phi(0)=\phi'(0)=0$, $\left|\phi^{(n)}(x)\right|\le C_n$ for $0\le n\le 5$, and $\phi''(x)\ge c_0>0$ for all x in the support of f.

Then

$$\begin{aligned}&\left|\int_{-\infty}^{+\infty} e^{it\phi(x)} f(x)\,dx - \left(\frac{2\pi}{|t|\,\phi''(0)}\right)^{1/2} e^{\operatorname{sign}(t)\,\pi i/4} f(0)\right|\\ &\qquad\le A\,B(\phi)\,t^{-3/2}\left(\|f\|_\infty + N_1 + \frac{N_2}{t^{1/2}L} + M_1 + \frac{M_2}{t^{1/2}L}\right).\end{aligned}\tag{3.a}$$

Similarly,

$$\Big|\int_0^{+\infty} e^{it\phi(x)}\, f(x)\, dx - \Big(\frac{\pi}{2\,|t|\,\phi''(0)}\Big)^{1/2} e^{\operatorname{sign}(t)\,\pi i/4}\, f(0)\Big| \le A\,B(\phi)\,t^{-3/2}\Big(\|f\|_\infty + N_1 + \frac{N_2}{t^{1/2}L} + M_1 + \frac{M_2}{t^{1/2}L}\Big). \tag{3.b}$$

We also have the usual L-independent estimates

$$\Big|\int_{-\infty}^{+\infty} e^{it\phi(x)}\, f(x)\, dx - \Big(\frac{2\pi}{|t|\phi''(0)}\Big)^{1/2} e^{\operatorname{sign}(t)\,\pi i/4}\, f(0)\Big| \le A\,B(\phi)\,t^{-3/2}(\|f\|_\infty + \|f'\|_1 + \|f''\|_1) \tag{3.c}$$

and

$$\Big|\int_0^{+\infty} e^{it\phi(x)}\, f(x)\, dx - \Big(\frac{\pi}{2|t|\phi''(0)}\Big)^{1/2} e^{\operatorname{sign}(t)\,\pi i/4}\, f(0)\Big| \le A\,B(\phi)\,t^{-3/2}\,(\|f\|_\infty + \|f'\|_1 + \|f''\|_1), \tag{3.d}$$

where A is a universal constant, and $B(\phi)$ is as defined above for x in the support of f. Here, sign(t) *stands for the function which equals* 1 *if $t > 0$ and -1 if $t < 0$.*

PROOF. It will obviously be enough to consider the case $t > 0$. Consider the change of variables given by

$$u(x) = x\,\sqrt{\frac{\phi(x)}{x^2}}$$

and its inverse $z(u)$. We begin by obtaining regularity properties of u and z.

Let $k \ge 1$. In what follows, A_k will denote a collection of universal constants depending only on k. First, we consider $|x| \le 1$ and define

$$\phi_1(x) = x^{-2}\,\phi(x).$$

Since

$$\phi_1(x) = \int_0^1 \int_0^1 \phi''(s\,t\,x)\,s\,dt\,ds$$

we have

$$\|\phi_1\|_{C^k} \le A_k\,\|\phi\|_{C^{k+2}}.$$

Next, define

$$\phi_2(x) = \sqrt{\phi_1(x)}\,, \qquad |x| \le 1\,,$$

and note that

$$\frac{d^k\phi_2(x)}{dx^k} = \frac{\displaystyle\sum_{\substack{1\cdot i_1+\cdots+p\cdot i_p=k\\ i_j\ge 0}} c^{(k)}_{i_1,\dots,i_p}\,\phi_1'(x)^{i_1}\cdots\phi_1^{(p)}(x)^{i_p}}{\phi_1^{k-1/2}(x)}\,,$$

which can easily be checked by induction. As a result, we have

$$\|\phi_2\|_{C^k} \le A_k\,\frac{(1+\|\phi_1\|_{C^k})^k}{c_0^{k-1/2}} \le A_k\,\frac{(1+\|\phi\|_{C^{k+2}})^k}{c_0^{k-1/2}}\,,$$

where we have used the fact that $\phi_1(x) \ge c_0/2$. Therefore, since $u(x) = x\,\phi_2(x)$, we conclude that

$$\left|\frac{d^k u(x)}{dx^k}\right| \le A_k.\frac{(1+\|\phi\|_{C^{k+2}})^k}{c_0^{+k-1/2}}\,, \qquad \text{when } |x|\le 1\,.$$

When $|x| > 1$ we obviously have that

$$\frac{d^k u(x)}{dx^k} = \sum_{i=1}^{k}\phi^{1/2-i}(x)\sum_{\substack{1\cdot i_1+\cdots+p\cdot i_p=k\\ i_j\ge 0}} c^{i,k}_{i_1,\dots,i_p}\,\phi'(x)^{i_1}\cdots\phi^{(p)}(x)^{i_1}\,,$$

hence

$$|u^{(k)}(x)| \le A_k\,\frac{(1+\|\phi\|_{C^k})^k}{c_0^{+k-1/2}}\,, \qquad |x|>1\,,$$

so altogether we obtain

$$\|u\|_{C^k} \le A_k\,\frac{(1+\|\phi\|_{C^{k+2}})^k}{c_0^{+k-1/2}}\,.$$

Finally, since

$$z'(u(x))\,u'(x) = 1\,,$$

and, for $k\ge 2$,

$$\frac{d^k z}{du^k}(u(x))\,(u'(x))^k = \sum_{p=1}^{k-1} K_{k,p}\,\frac{d^p z}{du^p}(u(x)) \sum_{\substack{1\cdot i_1+\cdots+q\cdot i_q=k+1-p\\ i_j\ge 0}} c^{k,p}_{i_2,\dots,i_q}\,u'(x)^{i_1}\cdots u^{(q)}(x)^{i_q}\,,$$

and

$$|u'(x)| = \frac{1}{2}\Big|\frac{\phi'(x)}{\sqrt{\phi(x)}}\Big| \geq c\,, \qquad c = \frac{c_0}{\sqrt{C_2/2}}\,,$$

we obtain by induction that

$$\Big|\frac{d^k z(u)}{du^k}\Big| \leq A_k \Big(\frac{1+C_2}{c_0^+}\Big)^{k^2} (1+\|u\|_{C^k})^{k^2}\,.$$

With our previous estimate for $\|u\|_{C^k}$ we then conclude that

$$\Big|\frac{d^k z(u)}{du^k}\Big| \leq A_k \Big(\frac{1+\|\phi\|_{C^{k+2}}}{c_0^+}\Big)^{2k^3}\,.$$

Then

$$\int_{-\infty}^{+\infty} e^{it\phi(x)}\, f(x)\, dx = \int_{-\infty}^{+\infty} e^{itu^2}\, \tilde{f}(u)\, du\,,$$

for

$$\tilde{f}(u) = f(z(u))\, z'(u)\,.$$

Note that $\tilde{f}(0) = f(0)\,\sqrt{2/\phi''(0)}$. Since $z'(u) \leq c^{-1}$,

$$|f'(z(u))| \leq \begin{cases} N_1\,, & \text{if } |u| \leq c\,L\,,\\ N_2\,, & \text{otherwise}\,,\end{cases}$$

$$|f''(z(u))| \leq \begin{cases} M_1\,, & \text{if } |u| \leq c\,L\,,\\ M_2\,, & \text{otherwise}\,.\end{cases}$$

As a result, using stationary phase, we arrive at

$$\Big|\int_{-\infty}^{+\infty} e^{it\phi(x)}\, f(x)\, dx - \Big(\frac{2\pi}{t\phi''(0)}\Big)^{1/2} e^{\pi i/4}\, f(0)\Big| \leq t^{-3/2}\,(|I_1| + |I_2| + |I_3|)\,,$$

for

$$I_1 = \int g_t(u)\, f(z(u))\, z'''(u)\, du\,,$$

$$I_2 = \int g_t(u)\, f'(z(u))\, z'(u)\, z''(u)\, du\,,$$

$$I_3 = \int g_t(u)\, f''(z(u))\, z'(u)^3\, du\,.$$

Now,

$$|I_1| \le \|z\|_{C^3}\, \|f\|_\infty \int |g_t(u)|\, du \le A\left(\frac{1+\|\phi\|_{C^5}}{c_0^+}\right)^{54} \|f\|_\infty$$

Next,

$$\begin{aligned}|I_2| &\le A\left(\frac{C_0}{\sqrt{c_0}}\right)\left(\frac{1+\|\phi\|_{C^4}}{c_0^+}\right)^{16}\\ &\quad\cdot\Big(N_1 \int\limits_{|u|\le c\,L} |g_t(u)|\, du + 2\,N_2 \int\limits_{|u|\ge cL} t^{-1/2}u^{-2}\, du\Big)\\ &\le A\left(\frac{1+\|\phi\|_{C^5}}{c_0^+}\right)^{54}\left(N_1\, \|g_1\|_1 + \frac{2\,N_2}{t^{1/2}\,L}\right).\end{aligned}$$

Finally,

$$\begin{aligned}|I_3| &\le M_1\, c^{-3} \int\limits_{|u|\le c\,L} |g_t(x)|\, dx + 2\,M_2\, c^{-3} \int\limits_{|u|\ge c\,L} t^{-1/2}\, u^{-2}\, du\\ &\le M_1\, c^{-3}\, \|g_1\|_1 + \frac{2\,M_2}{c^3\, t^{1/2}\, L}\,,\end{aligned}$$

which proves the first claim in our lemma. The one-sided integral is estimated in the same way. The L-independent estimates are obtained in a similar manner, except that integrals I_2 and I_3 in this case are estimated directly by

$$|I_2| \le \frac{C_0}{\sqrt{c_0}}\left(\frac{1+\|\phi\|_{C^4}}{c_0^+}\right)^{16} \|f'\|_\infty\, \|g_t\|_1\,,$$

$$|I_3| \le c^{-3}\, \|f''\|_\infty\, \|g_t\|_1\,.$$

The one-sided estimate in this case is also analogous.

This lemma will be complemented with the following trivial results:

Lemma 2. *Let $f \in C_0^2(\mathbb{R})$, and ϕ such that $|\phi'(x)| \ge d$ for all x in the support of f. Then*

$$\Big|\int_{\mathbb{R}} e^{it\phi(x)} f(x)\, dx\Big| \le t^{-1}\left(\frac{\|f'\|_1}{d} + \frac{\|f\,\phi''\|_1}{d^2}\right), \tag{4.a}$$

$$\Big|\int_{\mathbb{R}} e^{it\phi(x)} f(x)\, dx\Big| \le t^{-1}\left(\frac{\|f\,\phi''\|_1}{d^2}\right) + 4\,t^{-2}\left(\frac{\|f''\|_1}{d^2} + \frac{\|f'\,\phi''\|_1}{d^3}\right), \tag{4.b}$$

$$\tag{4.c} \begin{aligned} \Big| \int_{\mathbb{R}} e^{it\phi(x)} f(x)\, dx \Big| \le 10\, t^{-2} \Big(& \frac{\|f''\|_1}{d^2} + \frac{\|f'\, \phi''\|_1}{d^3} \\ & + \frac{\|f\, \phi'''\|_1}{d^3} + \frac{\|f'\, (\phi'')^2\|_1}{d^4} \Big) . \end{aligned}$$

PROOF. Integration by parts yields

$$\tag{5} \begin{aligned} \int_{\mathbb{R}} e^{it\phi(x)} f(x)\, dx &= -\frac{1}{it} \int_{\mathbb{R}} e^{it\phi(x)} \frac{d}{dx}\Big(\frac{f(x)}{\phi'(x)}\Big)\, dx \\ &= \frac{1}{it} \int_{\mathbb{R}} e^{it\phi(x)} \frac{f'(x)}{\phi'(x)}\, dx \\ &\quad - \frac{1}{it} \int_{\mathbb{R}} e^{it\phi(x)} \frac{f(x)\, \phi''(x)}{\phi'(x)^2}\, dx . \end{aligned}$$

This yields (4.a). For (4.b) we perform another integration by parts to the first integral above, which equals

$$\frac{1}{t^2} \int_{\mathbb{R}} e^{it\phi(x)} \frac{f''(x)}{\phi'(x)^2}\, dx + \frac{2}{t^2} \int_{\mathbb{R}} e^{it\phi(x)} \frac{f'(x)\, \phi''(x)}{\phi'(x)^3}\, dx ,$$

which yields (4.b). For (4.c), we integrate by parts also the last integral in (5), which gives

$$-t^{-2} \int \Big(\frac{f'(x)\, \phi''(x)}{\phi'(x)^3} + \frac{f(x)\, \phi'''(x)}{\phi'(x)^3} - 3\, \frac{f(x)\, \phi''(x)^2}{\phi'(x)^4} \Big)\, e^{it\phi(x)}\, dx$$

as needed.

Lemma 3. *Let $f \in C_0^2((a,b))$ and ϕ such that $\phi''(x) \ge c_0 > 0$, and $\phi'(x) \ne 0$ for $x \in [a,b]$. Then,*

$$\Big| \int_a^b e^{it\phi(x)} f(x)\, dx \Big| \le t^{-1}\, |b-a| \Big(\frac{\|f''\|_\infty}{c_0} + \frac{\|f\|_\infty\, \|\phi''\|_\infty}{c_0^2} \Big) .$$

REMARK. The point in this result is that the estimate is independent of $\inf |\phi'(x)|$.

PROOF. f vanishes at a at order 2, which implies

$$|f(x)| \le \|f''\|_\infty\, |x-a|^2 , \qquad |f'(x)| \le \|f''\|_\infty\, |x-a| .$$

So,

$$\phi'(x) \geq c_0\,(x-a)\,.$$

The lemma follows trivially by integration by parts, since

$$\int_a^b e^{it\phi(x)}\, f(x)\,dx = \frac{i}{t}\int_a^b \left(\frac{f'(x)}{\phi'(x)} - \frac{f(x)\,\phi''(x)}{\phi'(x)^2}\right) e^{it\phi(x)}\,dx\,.$$

The following is a trivial variant of the usual Van der Corput lemmas.

Lemma 4. *Let f be differentiable in $[a,b]$, and ϕ such that $\phi''(x) \geq c_0 > 0$ for $x \in [a,b]$ Then,*

$$\left|\int_a^b e^{it\phi(x)}\, f(x)\,dx\right| \leq 8\,t^{-1/2}\,c_0^{-1/2}\,(1 + \|f\|_\infty + \|f'\|_1)\,.$$

PROOF. Let $R = t^{-1/2}\,c_0^{1/2}$, and consider

$$E_1 = \{x:\ |\phi'(x)| \geq R\}\,, \qquad E_2 = \{x:\ |\phi'(x)| < R\}\,.$$

It is obvious that E_1 has at most two components, and $|E_2| \leq R/c_0$. The contribution of the integral over E_2 is thus trivial. The integral over E_1, after integration by parts, equals

$$\left.\frac{f(x)}{it\,\phi'(x)}\right|_{\partial E_1} + i\,(I_1 - I_2)$$

for

$$I_1 = \int_{E_1} e^{it\phi(x)}\,\frac{f'(x)}{t\phi'(x)}\,dx\,, \qquad I_2 = \int_{E_1} e^{it\phi(x)}\,\frac{f(x)\phi''(x)}{t\,\phi'(x)^2}\,dx\,.$$

The boundary terms contribute with at most $4\,\|f\|_\infty/(t\,R)$, which is fine, and the I_i are trivially estimated by

$$|I_1| \leq \frac{\|f'\|_1}{t\,R}$$

and

$$|I_2| \leq \|f\|_\infty \int_{E_1} \frac{\phi''(x)}{t\,\phi'(x)^2}\,dx \leq \frac{4\,\|f\|_\infty}{t\,R}\,,$$

which gives us the bound in the claim of the lemma.

2. The heart of the matter.

In this Section we consider a function μ periodic with period 1, average 0, and Fourier coefficients satisfying

$$|\hat{\mu}(n)| \leq M\,|n|^{-\sigma}\,, \qquad \sigma \geq 1\,.$$

We also assume that

$$\sum_{n\neq 0} \sqrt{|n|}\;|\hat{\mu}(n)| < +\infty \tag{6}$$

Our estimates will depend on M in a trivial way, but since for the applications we will be satisfied with $M = 10$, we will not bother to keep track of the dependence on M. In fact, we will be mostly interested in $\hat{\mu}(n) = |n|^{-s}$, with $s = \sigma + it$ and $\sigma \geq 1$, and for the applications to the energy asymptotics we will be dealing with

$$\mu(x) = \operatorname{dist}\{x, \mathbb{Z}\}^2 - \frac{1}{12}\,, \qquad \hat{\mu}(n) = \frac{e^{-\pi i n}}{2\pi^2 n^2}\,.$$

However, our estimates will be independent of the value of the sum in (6), which could even be λ-dependent.

Consider also ϕ smooth, defined on $[a,b]$, and satisfying the crucial nondegeneracy condition $-\phi''(x) \geq c_0 > 0$: of course, the same argument would work if we assumed $\phi''(x) > c_0$, with only a few signs being flipped, but we choose this sign in our non-degeneracy condition because it is exactly the one satisfied by the function ϕ in our application to the sum $\Psi_Q(Z)$.

We also assume the bounds

$$\left|\phi^{(n)}(x)\right| \leq C_n\,, \qquad 0 \leq n \leq 5\,, \qquad \text{for } x \in [a,b]\,,$$

where $|b-a|$ is bounded by a universal constant, and define

$$S(\lambda) = \sum_{l \in (\mathbb{Z}+\gamma)\cap[a\lambda\,,\,b\lambda]} f\Big(\frac{l}{\lambda}\Big)\;\mu\Big(\lambda\,\phi(\frac{l}{\lambda})\Big)\,,$$

where γ is a real number.

In our applications, we will be concerned with the following two situations: on the one hand, we will have functions f and ϕ independent of λ; this simplifies some estimates, but the amplitude function f does not vanish at the endpoint b, which gives origin to a certain diophantine analysis of the phase ϕ. On the other hand, we will have to deal with functions f and ϕ which depend on λ, which will force us to keep track of error terms in a careful way: furthermore, there is no obvious multiscale analysis in the problem and we thus have to analyze blow up manually. However, in this case the amplitude function is supported inside $[a,b]$ which avoids diophantine discussions.

We summarize both cases as follows.

Case I: $f \in C_0^\infty((a,b])$. In this case, we shall impose that the bounds satisfied by ϕ and f are universal, *i.e.*, independent of λ. The obvious singularity in the sum appearing around $l = b\lambda$ will give rise to a purely arithmetic behavior of the sum.

Case II: $f \in C_0^\infty((a,b))$. In this case, the functions ϕ and f will depend on λ in the sense that the bounds satisfied by ϕ will grow (slowly) as a function of λ. We will thus keep track carefully of the dependence of our error bounds in terms of the regularity assumptions of f and ϕ. The absence of singularities in this case will make the study of the sums purely analytical.

We wish to understand the behavior of $S(\lambda)$ for large λ in both cases.

Case I. As mentioned above, f and ϕ will satisfy universal bounds for its derivatives of the type

$$\|\phi'(x)\|_{C^5} \le C\,, \qquad -\phi''(x) \ge c_0\,, \qquad \|f\|_{C^2} \le C\,,$$

for constants C and c_0 independent of λ. As a consequence, we will not keep track of the dependence of constants on the regularity properties of either f or ϕ, and the constant C will be ubiquitously used to denote a universal constant depending on the regularity properties of f and ϕ as stated above. Another constant will play a role, though, which is $\phi'(b)$ in the case that it is a rational number p/q: in this case, some constants will depend on q, and this dependence will be made explicit.

Let $\varphi(x)$ supported on $(-\infty,b)$, identically equal to 1 on $[a,b-\lambda^{-1/2-\varepsilon}]$, for $\varepsilon = 1/20$, φ as smooth as possible. We denote by

$$I_\lambda = [b-\lambda^{-1/2-\varepsilon},b]$$

the set where φ' is supported.

It is clear that

$$S(\lambda) = \sum_{l\in\mathbb{Z}+\gamma} f\left(\frac{l}{\lambda}\right) \mu\left(\lambda\,\phi(\frac{l}{\lambda})\right) \varphi(\lambda^{-1}l) + O(\|f\|_\infty\,\lambda^{1/2-\varepsilon})$$

and that in the new sum above, only finitely many terms are non-zero. Moreover, $\mu\,(\lambda\,\phi(x/\lambda))\,(f\,\varphi)(\lambda^{-1}x)$ is a piecewise smooth function of compact support. We set

$$\varphi_f(x) = (f\,\varphi)(x)$$

which satisfies $\|\varphi_f\|_\infty \le \|f\|_\infty$, and

$$|\varphi_f'(x)| \le \begin{cases} C\,, & \text{if } x \notin I_\lambda\,, \\ \lambda^{1/2+\varepsilon}\,, & \text{if } x \in I_\lambda\,, \end{cases}$$

(7)

$$|\varphi_f''(x)| \le \begin{cases} C\,, & \text{if } x \notin I_\lambda\,, \\ \lambda^{1+2\varepsilon}\,, & \text{if } x \in I_\lambda\,, \end{cases}$$

The Poisson summation formula yields

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}+\gamma} \mu\left(\lambda\,\phi(\frac{l}{\lambda})\right)\,\varphi_f(\lambda^{-1}l) \\
&\quad= \sum_l e^{2\pi i l\gamma} \int_{-\infty}^{+\infty} \mu\left(\lambda\,\phi(\frac{x}{\lambda})\right)\,\varphi_f(\lambda^{-1}x)\,e^{-2\pi i x l}\,dx \\
&\quad= \sum_l e^{2\pi i l\gamma} \int_{\lambda a}^{\lambda b} \mu\left(\lambda\,\phi(\frac{x}{\lambda})\right)\,\varphi_f(\lambda^{-1}x)\,e^{-2\pi i x l}\,dx \\
&\quad= \sum_{\substack{l\in\mathbb{Z}\\ n\neq 0}} \hat{\mu}(n)\,e^{2\pi i l\gamma} \int_{\lambda a}^{\lambda b} e^{2\pi i(\lambda n\phi(x/\lambda)-xl)}\,\varphi_f(\lambda^{-1}x)\,dx \\
&\quad= \lambda \sum_{\substack{l\in\mathbb{Z}\\ n\neq 0}} \hat{\mu}(n)\,e^{2\pi i l\gamma} \int_a^b e^{2\pi i\lambda(n\phi(x)-xl)}\,\varphi_f(x)\,dx\,.
\end{aligned}$$

We will show below that the sum is absolutely convergent, due to the fast decrease of $\hat{\mu}$ assumed in (6), and the fast decrease of the integrals; therefore, the infinite sum can be taken in any order we like.

Define

$$I(n,l) = e^{2\pi i l\gamma} \int_a^b e^{2\pi i\lambda(n\phi(x)-xl)}\, \varphi_f(x)\, dx\,.$$

For integers n and l, define $x_{n,l}$ as the unique point (when it exists) satisfying $\phi'(x_{n,l}) = l/n$. Note that

$$c_0^{-1} \left|\frac{l}{n} - \frac{l'}{n'}\right| \geq |x_{n,l} - x_{n',l'}| \geq \|\phi''\|_\infty^{-1} \left|\frac{l}{n} - \frac{l'}{n'}\right|\,.$$

Define also

$$\theta(n,l) = n\,\phi(x_{n,l}) - l\,x_{n,l}$$

and

$$\sigma_\phi(n,l) = \lambda^{1/2} \frac{1}{|n\,\phi''(x_{n,l})|^{1/2}}\, e^{-\mathrm{sign}(n)\,\pi i/4 + 2\pi i(\lambda\,\theta(n,l)+\gamma\, l)}\,.$$

We write σ_ϕ to point out that σ depends only on ϕ: the amplitude f does not appear.

We begin with the following crude estimate, which is a trivial consequence of Lemma 4.

Lemma 5.

$$|I(n,l)| \leq C\,\lambda^{-1/2}\,|n|^{-1/2}\,.$$

This already implies that only the terms appearing for small n play a role in our sum.

Theorem 6. *With the previous notation, we have*

$$S(\lambda) = \sum_{\substack{n\neq 0\\ l\in\mathbb{Z}\\ x_{n,l}\in(a,b]}} \hat{\mu}(n)\,\varepsilon_{n,l}\, f(x_{n,l})\,\sigma(n,l) + A(\lambda)\,,$$

where $\varepsilon_{n,l} = 1$, *unless* $\phi'(b) = l/n$ *when* $\varepsilon_{n,l} = 1/2$, *and*

$$A(\lambda) = o(\lambda^{1/2})\,.$$

If $\phi'(b) = p/q$, *then we have*

$$A(\lambda) = O(C_q \lambda^{1/2-\gamma})\,, \qquad \gamma > 0\,.$$

If, however, $\phi'(b)$ is irrational, the o-term depends on the diophantine properties of $\phi'(b)$.

In any case, $|A(\lambda)| \leq C\sqrt{\lambda}$ for C which only depends on $\|f\|_{C^2}$ and $B(\phi)$.

PROOF. Note that there are three types of pairs (n,l): those for which $\phi'(x) = l/n$ for some $x = x_{n,l} \in [a,b)$, those such that $\phi'(x)$ never equals l/n for any $x \in [a,b)$, and those (if any) for which l/n equals $\phi'(b)$ ($\phi'(a)$ will play no role here since f vanishes to infinite order at a). We need to deal with these cases separately, and we thus write

$$S(\lambda) = S_1(\lambda) + S_2(\lambda) + S_3(\lambda),$$

where

$$S_1(\lambda) = \lambda \sum_{\phi'(a) \leq l/n < \phi'(b)} \hat{\mu}(n)\, I(n,l),$$

$$S_2(\lambda) = \lambda \sum_{l/n = \phi'(b)} \hat{\mu}(n)\, I(n,l),$$

$$S_3(\lambda) = \lambda \sum_{l/n \notin [\phi'(a), \phi'(b)]} \hat{\mu}(n)\, I(n,l).$$

Sum S_1: For every term in this sum, the integrand in $I(n,l)$ has a stationary point $x_{n,l}$. Our stationary phase analysis then shows, using (7), that

$$|\lambda\, I(n,l) - f(x_{n,l})\, \sigma_{n,l}| \leq \lambda\, E_{n,l}\,,$$

(8)
$$E_{n,l} = C\,(\lambda\,|n|)^{-3/2}\left(1 + \min\left\{\lambda^{1+2\varepsilon}\,,\, \frac{\lambda^{1/2+2\varepsilon}}{|n|^{1/2}\,\mathrm{dist}\,\{x_{n,l}\,,\, I_\lambda\}}\right\}\right),$$

where the minimum appears as the best of estimates (3.a) and (3.b) above.

The terms in S_1 will be grouped into three categories. First, those for which $x_{n,l}$ falls far from b, and second, those for which $x_{n,l}$ falls near b. Within the second class, we will have to consider separately those that appear only when n is large, and those with n small.

Fix n in the sum above. For each n, the number of terms in the sum in l is at most $C\,|n|$. And of those terms, the number of l for which

$x_{n,l}$ falls within d of I_λ is bounded by at most $1 + C\,|n|\,(d + \lambda^{-1/2-\varepsilon})$. If, say, $\phi'(b) = p/q$ is rational, and we take $d > \lambda^{-1/2}$, we have

$$d \geq |b - x_{n,l}| \geq C_2^{-1} \left| \frac{l}{n} - \frac{p}{q} \right| \geq \frac{C_2^{-1}}{|n\,q|} ,$$

which means that, if

$$|n| < \|\phi\|_{C^2}^{-1}\, q^{-1}\, d^{-1} ,$$

then there are no l such that $x_{n,l}$ falls within d if I_λ. Similarly, if $\phi'(b)$ is irrational, the number of such l is at most 1 for

$$|n| < \|\phi\|_{C^2}^{-1}\, d^{-1}$$

if $d > \lambda^{-1/2-\varepsilon}$. We denote this unique l (when it exists) by $l_0(n, d, \lambda)$, and we denote by $n_0(d, \lambda)$ the smallest $|n|$ for which n has such an l_0.

After all this, we choose

$$d = \lambda^{-7\varepsilon}, \qquad c^\sharp = \begin{cases} \|\phi\|_{C^2}^{-1} , & \text{if } \phi'(b) \text{ is irrational,} \\ \|\phi\|_{C^2}^{-1}\, q^{-1} , & \text{if } \phi'(b) = \dfrac{p}{q} , \end{cases}$$

and break up

$$\lambda^{-1}\, S_1(\lambda) = \sum_{\substack{|n| \leq c^\sharp d^{-1} \\ d(x_{n,l}, I_\lambda) \geq d}} \hat\mu(n)\, I(n,l) + \sum_{\substack{|n| > c^\sharp d^{-1} \\ \text{all } l}} \hat\mu(n)\, I(n,l) + \sum_{c^\sharp d^{-1} > |n| \geq n_0(d,\lambda)} \hat\mu(n)\, I\,(n, l_0(n,d,\lambda))$$

where it is understood that if $\phi'(b)$ is rational, the sum in the third term above is null, and the sum in n also in the third term is extended only to those n with a corresponding $l_0(n, d, \lambda)$.

For the first term we use (3.a) to obtain

$$E_{n,l} \leq C\,(\lambda\,|n|)^{-3/2} \left(1 + |n|^{-1/2}\, \lambda^{1/2+9\varepsilon}\right) ,$$

Since $|n| \leq c^\sharp\, \lambda^{7\varepsilon}$, we obtain (recall $\varepsilon = 1/20$),

$$E_{n,l} \leq C\,|n|^{-2}\, \lambda^{-1+9\varepsilon}$$

and, using again $\varepsilon = 1/20$, we obtain

$$\sum_{\substack{|n|\leq c^\sharp\, d^{-1}\\ d(x_{n,l},I_\lambda)\geq d}} \lambda\,\hat{\mu}(n)\,I(n,l)$$

$$= \sum_{\substack{|n|\leq c^\sharp\, d^{-1}\\ d(x_{n,l},I_\lambda)\geq d}} \hat{\mu}(n)\,f(x_{n,l})\,\sigma_{n,l} + O\Big(\lambda^{1/2-\varepsilon}\sum_{n\neq 0}\Big|\frac{\hat{\mu}(n)}{n}\Big|\Big).$$

For the second term in (9), we use the trivial estimate (3.b) in (8) to conclude that

$$E_{n,l} = O(\lambda^{-1/2+2\varepsilon}\,|n|^{-3/2})$$

and since $|n| \geq c^\sharp\,\lambda^{7\varepsilon}$,

$$E_{n,l} = O_q(\lambda^{-1/2-\varepsilon}\,|n|^{-15/14})$$

and we obtain

$$\sum_{|n|>c^\sharp\, d^{-1}} \lambda\,\hat{\mu}(n)\,I(n,l)$$

$$= \sum_{|n|>c^\sharp\, d^{-1}} \hat{\mu}(n)\,f(x_{n,l})\,\sigma_{n,l} + O_q\Big(\lambda^{1/2-\varepsilon}\sum_{n\neq 0}\frac{|\hat{\mu}(n)|}{|n|^{15/14}}\Big).$$

Note that, here, we could simply have used Lemma 4 to conclude that both $I(n,l)$ and $\sigma_{n,l}$ give a negligible contribution, but this would have required, either, to use the stronger assumption that $\sigma > 3/2$, or to obtain an error estimate which depends on the value of the sum (6).

Finally, for the third term, it is clear that

$$\lim_{\lambda\to\infty} n_0\left(\lambda^{-7\varepsilon},\lambda\right) = \infty\,,$$

which, using Lemma 5, implies that

$$\begin{aligned}\Big|\sum_{|n|\geq n_0(d,\lambda)} \hat{\mu}(n)\,I\,(n,l_0(n,d,\lambda))\Big| &\leq C\sum_{|n|\geq n_0(d,\lambda)} |\hat{\mu}(n)|\,(\lambda\,|n|)^{-1/2}\\ &\leq C\,\lambda^{-1/2}\,|n_0(d,\lambda)|^{-\sigma+1/2}\\ &= o(\lambda^{-1/2})\,.\end{aligned}$$

Similarly, observe that

$$\Big| \sum_{|n| \geq n_0(d,\lambda)} \hat{\mu}(n)\, f(x_{n,l})\, \sigma_{n,l_0(n,d,\lambda)} \Big| \leq C\,\lambda^{1/2}\, |n_0(d,\lambda)|^{-\sigma+1/2} = o(\lambda^{1/2}).$$

Therefore, we can conclude that

$$S_1(\lambda) = \sum_{l/n \in [\phi'(a),\phi'(b))} \hat{\mu}(n)\, f(x_{n,l})\, \sigma_{n,l} + o(\lambda^{1/2}).$$

Sum S_2: If $\phi'(b)$ is irrational, this sum is empty. We thus assume that $\phi'(b)$ is rational.

If we tried to proceed as we did for S_1, we find that $E_{n,l}$ is too big, and this has no remedy. This is so because we would be comparing $I(n,l)$ with the wrong thing: it is not $f(x_{n,l})\,\sigma_{n,l}$ what we should look at, but $f(x_{n,l})\,\sigma_{n,l}/2$ instead. We proceed as follows:

Say $\phi'(b) = l/n$. We have, by (3.d), that

$$\begin{aligned} e^{2\pi i l\gamma} \int_0^1 & e^{2\pi i\lambda(n\phi(x)-lx)}\, \varphi_f(x)\,dx \\ &= e^{2\pi i l\gamma} \int_0^1 e^{2\pi i\lambda(n\phi(x)-lx)}\, f(x)\,dx + O(\lambda^{-1/2-\varepsilon}) \\ &= \frac{1}{2\lambda}\, f(x_{n,l})\,\sigma_{n,l} + O((|n|\lambda)^{-3/2}) + O(\lambda^{-1/2-\varepsilon}), \end{aligned}$$

which yields

$$S_2(\lambda) = \frac{1}{2} \sum_{l/n=\phi'(b)} \hat{\mu}(n)\, f(x_{n,l})\, \sigma_{n,l} + O(\lambda^{1/2-\varepsilon}).$$

Sum S_3: As for S_1, we deal separately with those l and n for which $\phi'(x) - l/n$ is small or large, and for those for which it is small, we distinguish between small and large n.

When $|\phi'(x) - l/n| > d$ for all $x \in [a,b]$, we use (4.c) to obtain

$$I(n,l) = O\left(\frac{\lambda^{-3/2+\varepsilon}}{|n|^2\, d^2} + \frac{\lambda^{-2}}{|n|^2\, d^3} + \frac{\lambda^{-2}}{|n|^2\, d^4} \right),$$

which implies

$$\Big| \sum_{\{(n,l):\ |\phi'(x)-l/n|>d\}} \hat{\mu}(n)\, I(n,l) \Big| \le C\, \lambda^{-3/2} \sum_{\text{all } n\neq 0} \Big|\frac{\hat{\mu}(n)}{n}\Big| \left(\frac{\lambda^{\varepsilon}}{d} + \frac{1}{\lambda^{1/2}\, d^2} + \frac{1}{\lambda^{1/2}\, d^3} \right) .$$

If we now set $d = \lambda^{-7\varepsilon}$ we obtain

$$\sum_{\{(n,l):\ |\phi'(x)-l/n|>d\}} \hat{\mu}(n)\, I(n,l) = O(\lambda^{-1/2-\varepsilon}) .$$

When $0 < |\phi'(b) - l/n| \le d$, for a fixed n there are at most $1 + |n|\, d$ terms in the sum. And, as before, if $\phi'(b)$ is rational and $|n| < d^{-1}$, then there are no l, and if $\phi'(b)$ is irrational, there is at most one such l, which, if it really existed, we would denote by $l_0(n, d, \lambda)$; we denote by $n_0(d, \lambda)$ the first $|n|$ for which n has such an l. Therefore we break up the remaining part of S_3 given by $0 < |\phi'(b) - l/n| \le d$, into

$$\sum_{\substack{\{(n,l):\ |\phi'(x)-l/n|\le d\} \\ |n|>d^{-1}}} \hat{\mu}(n)\, I(n,l)$$

and

$$\sum_{|n|\ge n_0(d,\lambda)} \hat{\mu}(n)\, I\,(n, l_0(n, d, \lambda)) .$$

The first sum above is trivially controlled by (4.c), which implies

$$I(n,l) = O(\lambda^{-2+7/5}\, |n|^{-2}) ,$$

hence

$$\sum_{\substack{\{(n,l):\ |\phi'(x)-l/n|>d\} \\ |n|>d^{-1}}} \hat{\mu}(n)\, I(n,l) = O(\lambda^{-1/2-\varepsilon})$$

For the second term, note as before that $\lim_{\lambda\to\infty} n_0(d, \lambda) = \infty$, and therefore, using Lemma 4, we get

$$\begin{aligned} \Big| \sum_{|n|\ge n_0(d,\lambda)} \hat{\mu}(n)\, I\,(n, l_0(n, d, \lambda)) \Big| &\le C \sum_{|n|\ge n_0(d,\lambda)} |\hat{\mu}(n)|\, (\lambda\, |n|)^{-1/2} \\ &\le C\, \lambda^{-1/2}\, n_0^{-\sigma+1/2} \\ &= o(\lambda^{-1/2}) . \end{aligned}$$

All this implies that

$$S_3(\lambda) = o(\lambda^{1/2})$$

and the theorem follows.

Case II. In this case we will not need $\varphi(x)$ since f is compactly supported and smooth. In fact, $\mu\left(\lambda\,\phi(x/\lambda)\right) f(\lambda^{-1}x)$ is a piecewise smooth function of compact support, and by the Poisson summation formula, as before,

$$\sum_{l\in\mathbb{Z}+\gamma} \mu\left(\lambda\,\phi(\frac{l}{\lambda})\right) f(\lambda^{-1}l)$$

$$= \lambda \sum_{\substack{l\in\mathbb{Z}\\ n\neq 0}} \hat{\mu}(n)\, e^{2\pi i l\gamma} \int_a^b e^{2\pi i\lambda\,(n\phi(x)-xl)}\, f(x)\, dx\,.$$

Define, as before,

$$I(n,l) = e^{2\pi i l\gamma} \int_a^b e^{2\pi i\lambda\,(n\phi(x)-xl)}\, f(x)\, dx$$

and $x_{n,l}$ as the unique point (if it did exit) satisfying $\phi'(x_{n,l}) = l/n$. Also as before, we have

$$c_0^{-1}\left|\frac{l}{n}-\frac{l'}{n'}\right| \geq |x_{n,l}-x_{n',l'}| \geq \|\phi''\|_\infty^{-1}\left|\frac{l}{n}-\frac{l'}{n'}\right|.$$

Define also $\theta(n,l)$ and $\sigma(n,l)$ exactly as in Case I.

Theorem 7. *With the previous notation, we have*

$$S(\lambda) = \sum_{\substack{n\neq 0\\ l\in\mathbb{Z}}} \hat{\mu}(n)\,\sigma(n,l) + O(B(\phi)\,\|f\|_{C^2}\,(1+\|f'\|_\infty))\,.$$

PROOF. In this case now there are only two types of pairs (n,l): those for which $\phi'(x) = l/n$ for some $x = x_{n,l} \in [a,b]$, and those such that $\phi'(x)$ never equals l/n, and thus we write

$$S(\lambda) = S_1(\lambda) + S_2(\lambda)\,,$$

where

$$S_1(\lambda) = \lambda \sum_{l/n \in [\phi'(a), \phi'(b)]} \hat{\mu}(n) I(n,l),$$

$$S_2(\lambda) = \lambda \sum_{l/n \notin [\phi'(a), \phi'(b)]} \hat{\mu}(n) I(n,l).$$

Sum S_1: We proceed as in the previous section,

$$|\lambda I(n,l) - f(x_{n,l}) \sigma_{n,l}| \leq \lambda E_{n,l},$$

where, by (3.c), $E_{n,l}$ is given now by

$$E_{n,l} = C B(\phi) \|f\|_{C^2} (\lambda |n|)^{-3/2}.$$

For each n, the number of terms in the sum is at most $(3 + C_1)|n|$. Therefore, we can conclude that

$$S_1(\lambda) = \lambda \sum_{l/n \in [\phi'(a), \phi'(b)]} \hat{\mu}(n) f(x_{n,l}) \sigma_{n,l} + O(C_1^+ B(\phi) \|f\|_{C^2} \lambda^{-1/2})$$

for

$$C_1^+ = \max\{1, C_1\}.$$

Sum S_2: By Lemma 3, we have

$$|I(n,l)| \leq C \|f\|_{C^2} B(\phi) (\lambda |n|)^{-1},$$

which we use when $|l| \leq 2 C_1^+ |n|^{1-\delta}$, for $\delta > 0$, to obtain

$$\sum_{\{(n,l):\ |l| \leq 2 C_1^+ |n|^{1-\delta}\}} |\hat{\mu}(n) I(n,l)| \leq C \lambda^{-1} \|f\|_{C^2} B(\phi) C_1^+ \sum_{n \neq 0} \frac{|\hat{\mu}(n)|}{|n|^\delta}.$$

Outside of this range, we have

$$\left| \phi'(x) - \frac{l}{n} \right| \geq \frac{1}{2} \left| \frac{l}{n} \right|, \qquad \text{for all } x \in [a,b].$$

Therefore, (4.b) implies

$$|I(n,l)| \leq \frac{C \|f\|_{C^2} B(\phi)}{\lambda^2 |n|^2} \left(\frac{n^2}{l^2} + \frac{|n|^3}{|l|^3} + \frac{n^4}{l^4} \right).$$

Therefore,

$$\sum_{\{(n,l):\ |l|>2\,C_1^{+}|n|^{1-\delta}\}} |\hat{\mu}(n)\, I(n,l)| \le C\,\lambda^{-2}\,\|f\|_{C^2}\, B(\phi) \sum_{n\neq 0} \frac{|\hat{\mu}(n)|}{|n|^{1-4\delta}}\,.$$

3. Energy Asymptotics.

We plan to apply our previous estimates to the function

$$\Psi_C(Z) = 2\pi\, Z^{4/3} \sum_{l\in(\mathbb{Z}+1/2)\cap[1\,,\,a^{-1/2}Z^{1/3}\Omega_c]} \eta(l\,Z^{-1/3})\,\mu\big(Z^{1/3}\,\phi(l\,Z^{-1/3})\big)\,,$$

where

$$\mu(x) = \operatorname{dist}\{x,\mathbb{Z}\}^2 - \frac{1}{12}\,,$$

$$\phi(\Omega) = \frac{1}{\pi}\int \left(V^1_{\mathrm{TF}}(r) - \frac{\Omega^2}{r^2}\right)_+^{1/2} dr = \frac{a^{-1/2}}{\pi}\int \left(\frac{y(r)}{r} - \frac{a\,\Omega^2}{r^2}\right)_+^{1/2} dr\,,$$

$$\eta(\Omega) = \frac{\Omega}{P(\Omega)}\,,$$

$$P(\Omega) = \int \left(V^1_{\mathrm{TF}}(r) - \frac{\Omega^2}{r^2}\right)_+^{-1/2} dr = a^{-3/2}\int_{r_1(a^{1/2}\Omega)}^{r_2(a^{1/2}\Omega)} \left(\frac{y(r)}{r} - \frac{a\,\Omega^2}{r^2}\right)^{-1/2} dr\,.$$

Here, $r_i(\Omega)$ are the two points where $y(r)/r$ equals Ω^2/r^2 (see below) and Ω_c is the supremum of the Ω for which

$$\frac{y(r)}{r} - \frac{\Omega^2}{r^2}$$

is positive somewhere.

The crucial result we need is the non-vanishing of the second derivative of ϕ. This was proved in [FS8]. Because of its vital importance, we display it explicitly:

Theorem 8. *There exists a number* c_0 *such that*

$$-\phi''(\Omega) \geq c_0 > 0\,, \qquad \text{for all } \Omega \in (0, \Omega_c)\,.$$

We will first recall some known results (which appear, for example, in [FS8], [Hi] and [Hu]) which we will need here. After that, we will complement them with further properties of ϕ and P, some of which are taken from similar estimates appearing in [FS2-8].

Review of earlier results. If we set $u(r) = r\,y(r)$, then u has a unique maximum at $r = r_c$, where $r_c \sim 2.1$. We set $\Omega_c^2 = u(r_c)$. Then, u is increasing on $[0, r_c]$ and decreasing on $(r_c, +\infty)$. This is a crucial fact whose proof goes back to Sommerfeld, and can be found in [Hu].

Around 0, u satisfies the expansions

$$u(x) = \sum_{n=2}^{\infty} u_n\, x^{n/2}\,, \qquad u_2 = 1\,, \quad u_3 = 0\,, \quad u_4 \sim -1.588\,.$$

Rigorous numerical bounds for u_4 can be found in [FS8]. However, it is easy to see analytically that $u_4 < 0$. We also have

$$\sum_{n=2}^{\infty} |u_n|\, \rho_0^n < +\infty\,, \qquad \rho_0 > 0\,,$$

therefore, the function

$$f(z) = \sum_{n=2}^{\infty} u_n z^n$$

is analytic in a small neighborhood around 0.

Around infinity, we have the expansion

$$u(x) = \frac{144}{x^2} \sum_{n=0}^{\infty} b_n\, x^{-n\alpha/2}\,,$$

where

$$b_0 = 1\,, \quad b_1 \sim -13\,, \quad \alpha = \frac{\sqrt{73} - 7}{2} \sim 0.772\,.$$

Again, rigorous numerical bounds for b_1 are found in [FS8], and it can be seen analytically that $b_1 < 0$. We also have that

$$\sum_{n=0}^{\infty} |b_n|\, \rho_1^n < +\infty\,, \qquad \rho_1 > 0\,,$$

and, as a result, we have

$$u(x) = \frac{144}{x^2}\, f(x^{-\alpha/2}) \tag{10}$$

for a function f analytic in a neighborhood of 0.

Given any $\Omega \in (0, \Omega_c)$, there exist two numbers, $r_1(\Omega) \leq r_2(\Omega)$ where u equals Ω^2. We then have

Lemma 9. *The following formulas hold:*

$$\phi(\Omega) = \frac{a^{-1/2}}{\pi}\, F(a^{1/2}\,\Omega),$$

where

$$F(\Omega) = \int \left(u(x) - \Omega^2\right)_+^{1/2} \frac{dx}{x},$$

$$F'(\Omega) = -\Omega \int \left(u(x) - \Omega^2\right)_+^{-1/2} \frac{dx}{x},$$

$$F''(\Omega) = -\lim_{\delta \to 0} \left(\int_{r_1(\Omega)+\delta}^{r_2(\Omega)-\delta} \left(u(x) - \Omega^2\right)^{-3/2} y(x)\, dx + c(\Omega)\, \delta^{-1/2} \right),$$

where $c(\Omega)$ is uniquely specified by requiring the finiteness of the limit.

Moreover, if b is any number less than $r_2(\Omega)$, then

$$\frac{d^2}{d\Omega^2} \int_{r_1(\Omega)}^{b} \left(u(x) - \Omega^2\right)_+^{1/2} \frac{dx}{x}$$

equals

$$-\lim_{\delta \to 0} \left(\int_{r_1(\Omega)+\delta}^{b} \left(u(x) - \Omega^2\right)^{-3/2} y(x)\, dx + c_1(\Omega)\, \delta^{-1/2} \right)$$

again, for a constant c_1 that makes the limit finite. The corresponding symmetric case also holds.

Furthermore, F can be extended as an analytic function to a complex neighborhood of $(0, \Omega_c]$. However, 0 is an essential singularity of ϕ (or F), and, moreover,

$$\lim_{\Omega \to 0} \phi''(\Omega)\, \Omega^{\gamma} = \kappa, \qquad \gamma = \frac{9 - \sqrt{73}}{2} > 0,$$

where κ is a strictly negative real number.

A consequence of this which is of importance to us is that although ϕ and ϕ' remain bounded as we approach 0, the second derivative blows up slowly, and third will blow up much faster. In other words, ϕ does not satisfy sensible non-degenerate multiscale analysis bounds.

Further background results. Here we will obtain growth and regularity properties of the functions ϕ and P above. We define

$$g_\gamma(x) = \int_1^{x^{-2}} (t-1)^{-1/2}\, t^\gamma \, dt\,, \qquad \text{for } \ 0 < x \le \frac{1}{2}\,.$$

We begin listing several elementary results of calculus.

Lemma 10. *For $\gamma \in \mathbb{R}$, we have*

$$g_\gamma^{(k)}(x) \le C_k\,(|\gamma|+1)^{k-1}\, x^{-2\gamma-1-k}\,, \qquad \text{for } \ k \ge 1\,. \tag{11.a}$$

Furthermore, if $\gamma < -1/2$, then

$$\frac{1-2^{\gamma+1/2}}{|\gamma+1/2|} \le g_\gamma(x) \le 100 + \frac{100}{|\gamma+1/2|} \tag{11.b}$$

and if $\gamma > -1/2$, then

$$\frac{x^{-2\gamma-1}-1}{|\gamma+1/2|} \le g_\gamma(x) \le \Big(100 + \frac{100}{|\gamma+1/2|}\Big)\, x^{-2\gamma-1}\,, \tag{11.c}$$

where $0 < x \le 1/2$.

PROOF. Estimate (11.a) is completely trivial. For (11.b), we use

$$\int_1^2 t^{-1/2+\gamma}\, dt \le g_\gamma(x) \le 4 + \int_2^{+\infty} (t-1)^{-1/2+\gamma}\, dt\,.$$

For (11.c), we use the fact that $t-1 > t/2$ for $t > 2$ to write

$$\int_1^{x^{-2}} t^{\gamma-1/2}\, dt \le g_\gamma(x) \le 2^{|\gamma|+3} + 2\int_2^{x^{-2}} t^{\gamma-1/2}\, dt\,,$$

which implies (11.c) after using the fact that $2^{|\gamma|+1/2} \le 4\,x^{-2\gamma-1}$.

Lemma 11. *Define*

$$f(\Omega) = \left(\Omega_\varepsilon^2 - \Omega^2\right)^{-1/2} .$$

Then, $|f^{(k)}(\Omega)| \le C_k\, \Omega_\varepsilon^{-1-k}$ *for* $\Omega \le \Omega_\varepsilon/2$ *and* $k \ge 0$.

PROOF.

$$f(\Omega) = \Omega_\varepsilon^{-1}\, g\Big(\frac{\Omega}{\Omega_\varepsilon}\Big) , \qquad \text{for } g(x) = \left(1 - x^2\right)^{-1/2} . \tag{12}$$

Lemma 12. *Given* $\beta > 0$, $\delta > 0$, $\Omega_\varepsilon > 0$, τ, d *and* w_n *for* $n = 0, 1, 2, \ldots$ *let*

$$f(\Omega) = \sum_{n=0}^{\infty} w_n\, m_n(\Omega) , \qquad m_n(\Omega) = \Omega^{2\gamma_n + d}\, g_{\gamma_n}\Big(\frac{\Omega}{\Omega_\varepsilon}\Big) ,$$

where $\gamma_n = \tau + \beta n$ *and* $|\gamma_n + 1/2| > \delta$ *for all* $n \ge 0$.

Assume that $\sum w_n z^n$ *has a radius of convergence* ρ *and* $\Omega_\varepsilon^{2\beta} \le \rho/2$. *Then,*

$$\Big|\frac{d^k f}{d\Omega^k}(\Omega)\Big| \le \begin{cases} C\,\Omega^{2\tau+d-k} , & \text{if } \tau < -1/2 , \\ C\,\Omega^{d-1-k} , & \text{if } \tau > -1/2 , \end{cases} \qquad \text{when } \Omega \le \Omega_\varepsilon/2 ,$$

for a certain constant C *which depends on everything except* Ω.

PROOF. Let us consider first those n such that $\gamma_n > -1/2$. In this case, using Lemma 10 we obtain

$$\begin{aligned} \Big|\frac{d^k m_n}{d\Omega^k}(\Omega)\Big| &\le \sum_{l=0}^{k} C(k;\delta)\,(n+1)^{k-l}\, \Omega^{2\gamma_n + d - (k-l)}\, \Omega_\varepsilon^{-l} \Big|\frac{d^l g_{\gamma_n}(\Omega)}{d\Omega^l}\Big(\frac{\Omega}{\Omega_\varepsilon}\Big)\Big| \\ &\le C(k;\delta)\,(n+1)^k\, \Omega_\varepsilon^{2\gamma_n+1}\, \Omega^{d-1-k} . \end{aligned}$$

If, on the other hand, $\gamma_n < -1/2$, the $l = 0$ term above has to be estimated by

$$C(k;\delta)\,(n+1)^k\, \Omega^{2\gamma_n + d - k}$$

and we obtain

$$\begin{aligned}\left|\frac{d^k m_n}{d\Omega^k}(\Omega)\right| &\le C(k;\delta)\,(n+1)^k\,\left(\Omega_\varepsilon^{2\gamma_n+1}\,\Omega^{d-1-k}+\Omega^{2\gamma_n+d-k}\right)\\ &\le 2\,C(k;\delta)\,(n+1)^k\,\Omega^{2\gamma_n+d-k}\end{aligned}$$

because $\Omega \le \Omega_\varepsilon$. Since we can only have $\gamma_n < -1/2$ for finitely many n, we conclude that

$$\sum_{n=0}^{\infty} w_n\, m_n^{(k)}(\Omega) = \frac{d^k f}{d\Omega^k}(\Omega)\,,$$

where the sum converges absolutely, and we obtain the required estimate.

This ends our presentation of calculus results. In what follows we will develop the regularity bounds for ϕ first and then P.

Lemma 13. *For constants C_n and $c > 0$ we have*

$$|\phi(t)| \le C_0\,, \qquad |\phi'(t)| \le C_1\,, \qquad \left|\frac{d^n\phi}{d\Omega^n}\right| \le C_n\,\Omega^{-n+1+\alpha} \quad (n \ge 2)$$

and

$$-\phi''(\Omega) \ge c\,\Omega^{-1+\alpha}\,.$$

Proof. As in Lemma 9, in order not to bother with the presence of the constant a, we will prove this result for the function F instead.

The inequalities for ϕ and ϕ' are obvious. For the higher derivatives, the bounds outside a neighborhood of 0 are a direct consequence of the analytic extension of F to a complex neighborhood of $(0,\Omega_c]$, which is Corollary 1.3 in [FS8]. We are thus left with proving the bounds in an arbitrarily small neighborhood to the right of 0, given by $(0,\bar{\Omega}_\varepsilon)$, for a small universal number $\bar{\Omega}_\varepsilon$ to be picked up later in the proof.

Arguing as in formula (4.1.a.b.c) [FS8], using Lemma 9, we write

$$-F''(t) = I_1 + I_2 + I_3$$

for

$$I_1 = \int_a^b \left(u(r) - \Omega^2\right)^{-3/2} y(r)\, dr\,,$$
$$I_2 = \lim_{\delta \to 0} \left(\int_{r_1(\Omega)+\delta}^{a} \left(u(r) - \Omega^2\right)^{-3/2} y(r)\, dr - G_1(\Omega)\, \delta^{-1/2} \right)\,,$$
$$I_3 = \lim_{\delta \to 0} \left(\int_{b}^{r_2(\Omega)-\delta} \left(u(r) - \Omega^2\right)^{-3/2} y(r)\, dr - G_2(\Omega)\, \delta^{-1/2} \right)\,,$$

with G_i such that the limit is finite, and a and b any numbers such that $r_1(\Omega) < a < b < r_2(\Omega)$. In practice, we will take a and b such that

$$u(a) = u(b) = \Omega_\varepsilon^2$$

for Ω_ε a small number, and later, we will take $\bar{\Omega}_\varepsilon \ll \Omega_\varepsilon$.

First, $I_1(\Omega)$ is C^∞ in a neighborhood of 0, and therefore satisfies

$$\left|\frac{d^k I_1}{d\Omega^k}(\Omega)\right| \le C_k(\Omega_\varepsilon)\,, \qquad \text{for } \Omega \in (0, \Omega_\varepsilon)\,,$$

no matter which Ω_ε we will end up choosing.

For I_2, we write $I_2 = d\tilde{I}_2/d\Omega$, where, by Lemma 9,

$$\tilde{I}_2(\Omega) = \Omega \int_{r_1(\Omega)}^{a} \left(u(r) - \Omega^2\right)^{-1/2} \frac{dr}{r}\,.$$

Let $r(t)$ be the inverse of u near 0, $u(r(t)) = t$, and set $w(t) = r'(t)/r(t)$. Changing variables above we obtain,

$$\begin{aligned} \tilde{I}_2(\Omega) &= \Omega \int_{\Omega^2}^{\Omega_\varepsilon^2} \left(t - \Omega^2\right)^{-1/2} w(t)\, dt \\ &= \Omega^2 \int_1^{\Omega^{-2}\Omega_\varepsilon^2} (t-1)^{-1/2}\, w\left(t\,\Omega^2\right)\, dt \end{aligned}$$

which implies, after differentiation,

$$I_2(\Omega) = 2\,\Omega \int_1^{\Omega^{-2}\Omega_\varepsilon^2} (t-1)^{-1/2}\, h\left(t\,\Omega^2\right)\, dt - 2\left(\Omega_\varepsilon^2 - \Omega^2\right)^{-1/2} w(\Omega_\varepsilon^2)\, \Omega_\varepsilon^2$$

for

$$h(t) = t\, w'(t) + w(t) .$$

Next, we recall that $u(r) = r\, f(r^{1/2})$ for f analytic around 0 and $f(0)=1$. Therefore, $u^{1/2}(r) = r^{1/2}\, \tilde{f}(r^{1/2})$, for $\tilde{f}$ also analytic around 0, or $u^{1/2}(r) = \tilde{\tilde{f}}(r^{1/2})$, for $\tilde{\tilde{f}}$ also analytic around 0, $\tilde{\tilde{f}}(0) = 0$ and $\tilde{\tilde{f}}\,'(0) = 1$. Therefore, $\tilde{\tilde{f}}$ has an analytic inverse g, with $g(0) = 0$ and $g'(0) = 1$, and therefore, $r^{1/2} = u^{1/2}\, \tilde{g}(u^{1/2})$, for $\tilde{g}$ analytic and $\tilde{g}(0) = 1$. Squaring both sides, we obtain

$$r(t) = t\, \tilde{\tilde{g}}(t^{1/2})$$

for $\tilde{\tilde{g}}$ analytic around 0, $\tilde{\tilde{g}}(0) = 1$. As a consequence of this, we also have

$$w(t) = t^{-1}\, W(t^{1/2})$$

for W analytic around 0 and

$$h(t) = t^{-1}\, H(t^{1/2}) .$$

It was shown in [FS8] that $H(0) = H'(0) = 0$, and $H''(0) = -2\, y'(0)$, which implies that in fact

$$h(t) = f_h(t^{1/2}) , \qquad f_h(0) = -y'(0) .$$

Therefore,

$$f_h(z) = \sum_{n=0}^{\infty} h_n z^n , \qquad \sum_{n=0}^{\infty} |h_n|\, \rho_2^n < +\infty ,$$

for ρ_2 a small universal constant.

We break up $I_2(\Omega) = f_1(\Omega) + f_2(\Omega)$ for

$$f_1(\Omega) = 2\,\Omega \int_1^{\Omega^{-2}\Omega_\varepsilon^2} (t-1)^{-1/2} h\left(t\,\Omega^2\right)\, dt = \sum_{n=0}^{\infty} 2\, h_n\, \Omega^{1+n}\, g_{n/2}\Big(\frac{\Omega}{\Omega_\varepsilon}\Big) ,$$

$$f_2(\Omega) = -2\left(\Omega_\varepsilon^2 - \Omega^2\right)^{-1/2} w(\Omega_\varepsilon^2)\, \Omega_\varepsilon^2 .$$

Lemma 11 shows that $f_2^{(k)}(\Omega)$ is bounded for all $k \geq 0$ and $\Omega \leq \Omega_\varepsilon/2$ by a constant that may depend on Ω_ε. For f_2, we apply Lemma 12 with $d = 1$, $\tau = 0$ and $\beta = 1/2$ to obtain

$$\Big|\frac{d^k f_2}{d\Omega^k}(\Omega)\Big| \leq C(k, \Omega_\varepsilon)\, \Omega^{-k} .$$

We conclude the analysis of I_2 by observing that all the bounds we obtained are in agreement with the statement of the lemma.

We continue now with I_3. Denote by $r(t)$ the inverse function of $u(r)$, such that $u(r(t)) = t$. We proceed as in Section 4 in [FS8] to construct $w(t) = -r'(t)/r(t)$ and then set $h(t) = tw'(t) + w(t)$ which allows us to argue as before to obtain (equation (4.21.a) in [FS8])

$$I_3 = 2\Omega \int_1^{\Omega^{-2}\Omega_\varepsilon^2} (t-1)^{-1/2} h\left(t\,\Omega^2\right)\, dt - 2\left(\Omega_\varepsilon^2 - \Omega^2\right)^{-1/2} w(\Omega_\varepsilon^2)\,\Omega_\varepsilon^2 .$$

By (10) we have that $r^2u(r) = g(r^{-\alpha})$ for g analytic in a neighborhood of 0, with $g(0) = 1/144$. Therefore, setting $z = r^{-\alpha}$ and $u(r) = t$ we have $t = z^{2/\alpha}g(z)$, or $t^{\alpha/2} = \tilde{g}(z)$ for a new $\tilde{g}$ analytic in a small neighborhood of 0, with $\tilde{g}(0) = 0$ and $\tilde{g}'(0) \neq 0$. Thus, $\tilde{g}$ has an analytic inverse, f, with $f(0) = 0$, $f'(0) \neq 0$, and we have $z = f(t^{\alpha/2})$, or $z = t^{\alpha/2}\tilde{f}(t^{\alpha/2})$ for $\tilde{f}$ analytic around 0 and $\tilde{f}(0) \neq 0$. Therefore, $r(t) = z^{-1/\alpha} = t^{-1/2}v(t^{\alpha/2})$ for a new function v analytic around 0 which also satisfies $v(0) \neq 0$. Hence,

$$(13) \qquad r'(t) = t^{-3/2}\, v_p(t^{\alpha/2})\,, \qquad r''(t) = t^{-5/2} v_{pp}(t^{\alpha/2})\,,$$

for functions v_p and v_{pp} analytic in a small neighborhood around 0. Therefore,

$$h(t) = \frac{1}{4t}\, f_h(t^{\alpha/2})\,,$$

where, by (13), $f_h(z)$ is analytic in a small neighborhood around 0, $|z| < \rho_h$. It is observed in [FS8] that $f_h(0) = 0$ and $f_h'(0) > 0$ (Equation (4.20) in [FS8]). This allows us to put

$$I_3(\Omega) = f_1(\Omega) + f_2(\Omega)\,,$$

where

$$(14) \qquad \begin{aligned} f_1(\Omega) &= \sum_{n=1}^{\infty} 2\, h_n\, m_n(\Omega)\,, \\ m_n(\Omega) &= \Omega^{-1+n\alpha} \int_1^{\Omega^{-2}\Omega_\varepsilon^2} (t-1)^{-1/2} t^{-1+n\alpha/2}\, dt\,, \end{aligned}$$

and

$$f_2(\Omega) = -2\left(\Omega_\varepsilon^2 - \Omega^2\right)^{-1/2} w(\Omega_\varepsilon^2)\,\Omega_\varepsilon^2 .$$

If we make sure that

$$\Omega_\varepsilon^\alpha \le \frac{1}{2}\,\rho_h \tag{15}$$

we can invoke Lemma 12, with $\tau = -1+\alpha/2$, which is less than $-1/2$, $\beta = \alpha/2$ and $d = 1$ to obtain

$$\left|\frac{d^k f_1}{d\Omega^k}(\Omega)\right| \le C(\Omega_\varepsilon; k)\,\Omega^{-1+\alpha-k}\,.$$

This ends the proof of all upper bounds in the statement of the lemma. For the lower bound for $-\phi''$, we use the notation in (14) to write

$$f_1(\Omega) = 2\,h_1\,m_1(\Omega) + \tilde{f}_1(\Omega)\,, \qquad \tilde{f}_1(\Omega) = \sum_{n=2}^{\infty} 2\,h_n\,m_n(\Omega)\,.$$

Applying Lemma 10 to the first term above with $\gamma = -1+\alpha/2 < -1/2$, and Lemma 12 applied to $\tilde{f}_1(\Omega)$ with $\tau = -1+\alpha > -1/2$, we obtain

$$f_1(\Omega) \ge c\,h_1\Omega^{-1+\alpha}\,, \qquad |\tilde{f}_1(\Omega)| \le C(\Omega_\varepsilon)\,.$$

Since all other terms in the break-up of $-\phi''$ remain bounded as $\Omega \to 0$, we conclude that

$$-\phi''(\Omega) \ge c\,\Omega^{-1+\alpha}\,, \qquad \Omega \le \bar{\Omega}_\varepsilon\,,$$

for $\bar{\Omega}_\varepsilon \ll \Omega_\varepsilon$, as required.

We now turn our attention to P. In this case, rather than introducing a new function that allows us to do without the bothersome constant a, we will simply proceed as if a did not appear in the definition of P. This simplification clearly does not change the result, except of course, that the details of the proof will not contain the a dependence.

The following result is a trivial adaptation of Lemma 1.2 in [FS8] for P instead of ϕ.

Lemma 14. *We have $P \in C^\infty(0,\Omega_c)$. Furthermore, P admits an analytic extension to a neighborhood of Ω_c.*

PROOF. Let

$$H(\delta,\Omega)=\Omega\int_{r_1(\Omega)+\delta}^{r_2(\Omega)-\delta}\left(u(r)-\Omega^2\right)^{-1/2}r\,dr\,.$$

Consider the analytic change of variables given by

$$t(r)=\begin{cases}\left(\Omega_c^2-u(r)\right)^{1/2}, & \text{if } r\geq r_c\,,\\ -\left(\Omega_c^2-u(r)\right)^{1/2}, & \text{if } r\leq r_c\,.\end{cases}\tag{16}$$

Note that t is smooth and strictly increasing in the range $(0,+\infty)$. We can therefore consider its inverse, $r(t)$, and use it to rewrite

$$H(\delta,\Omega)=\Omega\int_{t_1(\delta,\Omega)}^{t_2(\delta,\Omega)}\left(D^2-t^2\right)^{-1/2}w(t)\,dt\,,$$

where

$$t_1=t\,(r_1+\delta)\,,\quad t_2=t\,(r_2-\delta)\,,\quad D^2=\Omega_c^2-\Omega^2\,,\quad w(t)=r'(t)\,r(t)\,.$$

Note that w is smooth on $(-\Omega_c,\Omega_c)$, and that

$$\begin{aligned} t_1&=-D\,(1+\tau_1(\delta))\,, & t_2&=D\,(1+\tau_2\,(\delta))\,,\\ c\,\delta&\leq|\tau_i|\leq C\,\delta & &\text{for } i=1,2\,,\end{aligned}\tag{17}$$

uniformly on compact subsets of $(-\Omega_c,\Omega_c)$, which implies that

$$H(\delta,\Omega)=\Omega\int_{D^{-1}t_1}^{D^{-1}t_2}\left(1-t^2\right)^{-1/2}w(tD)\,dt$$

converges as $\delta\to 0$ uniformly to the C^1 function

$$H(0,\Omega)=\Omega\int_{-1}^{1}\left(1-t^2\right)^{-1/2}w(tD)\,dt=P(\Omega)\,.\tag{18}$$

To show analyticity around Ω_c, note that $w(t)$ is analytic around 0; thus, it admits a convergent power series expansion given by

$$w(t)=\sum_{n=0}^{\infty}w_nt^n\,,\qquad |t|\leq\rho\,,$$

which implies

$$P(\Omega) = \Omega \sum_{n=0}^{\infty} w_n\, D^n \int_{-1}^{1} \left(1 - t^2\right)^{-1/2} t^n\, dt\,.$$

The integral corresponding to the odd terms in the sum is 0, which implies that in fact

$$P(\Omega) = \Omega \sum_{n=0}^{\infty} w_{2n}\, D^{2n} \int_{-1}^{1} \left(1 - t^2\right)^{-1/2} t^{2n}\, dt\,,$$

which defines an analytic function of Ω around Ω_c, since D^2 now is analytic in Ω.

Lemma 15. *For constants C_n and $c > 0$ we have*

$$\left|\frac{d^k P}{d\Omega^k}(\Omega)\right| \le C_k\, \Omega^{-3-k}\,, \qquad |P(\Omega)| \ge c\,\Omega^{-3}\,.$$

Proof. Lemma 14 establishes our inequalities outside an arbitrarily small neighborhood of 0. For a neighborhood to the right of 0 given by $(0,\bar{\Omega}_\varepsilon)$ we proceed as before, setting

$$f(\Omega) = I_1 + I_2 + I_3$$

for

$$I_1 = \int_a^b \left(u(r) - \Omega^2\right)^{-1/2} r\, dr\,,$$

$$I_2 = \int_{r_1(\Omega)}^{a} \left(u(r) - \Omega^2\right)^{-1/2} r\, dr\,,$$

$$I_3 = \int_b^{r_2(\Omega)} \left(u(r) - \Omega^2\right)^{-1/2} r\, dr\,,$$

with a and b any numbers such that $r_1(\Omega) < a < b < r_2(\Omega)$. We will take a and b such that $u(a) = u(b) = \Omega_\varepsilon^2$ for Ω_ε a small number to be picked later.

I_1 is C^∞ around 0, and thus satisfies all the required upper bounds.

For I_2, denote by $r(t)$ the inverse function of $u(r)$ around 0, such that $u(r(t)) = t$, $r(t) \leq r_c$, and set $w(t) = r'(t)\, r(t)$. By the same argument as before, we can see that

$$w(t) = t\, f_0(t^{1/2})$$

for

$$f_0(z) = \sum_{n=0}^{\infty} w_n z^n$$

analytic for $|z| \leq \rho_3$, ρ_3 a small universal number, and $f_0(0) \neq 0$.

Then,

$$\begin{aligned} I_2 &= \int_{\Omega^2}^{\Omega_\varepsilon^2} \left(t - \Omega^2\right)^{-1/2} w(t)\, dt \\ &= \Omega \int_1^{\Omega^{-2}\Omega_\varepsilon^2} (t-1)^{-1/2}\, w(t\,\Omega^2)\, dt \\ &= \sum_{n=0}^{\infty} w_n\, \Omega^{3+n}\, g_{1+n/2}\Big(\frac{\Omega}{\Omega_\varepsilon}\Big) . \end{aligned}$$

Thus, Lemma 12, for $\Omega_\varepsilon \leq \rho_3/2$, $\tau = 1$, $\beta = 1/2$, $d = 1$, yields

$$\Big|\frac{d^k I_2}{d\Omega^k}(\Omega)\Big| \leq C(k; \Omega_\varepsilon)\, \Omega^{-k} , \qquad \text{for } \ \Omega \leq \frac{1}{2}\, \Omega_\varepsilon .$$

For I_3, denote by $r(t)$ the inverse function of $u(r)$ around infinity, such that $u(r(t)) = t$, $r(t) \geq r_c$, and set $w(t) = -r'(t)\, r(t)$. By the same argument as before, we can see that

$$w(t) = t^{-2}\, f_1(t^{\alpha/2})$$

for

$$f_1(z) = \sum_{n=0}^{\infty} w_n\, z^n$$

analytic for $|z| \leq \rho_4$, ρ_4 a small universal number, and $f_1(0) \neq 0$.

Then,

$$I_3 = \int_{\Omega^2}^{\Omega_\varepsilon^2} \left(t - \Omega^2\right)^{-1/2} w(t)\, dt$$

$$
\begin{aligned}
&= \Omega \int_1^{\Omega^{-2}\Omega_\varepsilon^2} (t-1)^{-1/2}\, w(t\,\Omega^2)\, dt \\
&= \Omega^{-3} \sum_{n=0}^{\infty} w_n \Omega^{\alpha n} \int_1^{\Omega^{-2}\Omega_\varepsilon^2} (t-1)^{-1/2}\, t^{-2+\alpha n/2}\, dt \\
&= \sum_{n=0}^{\infty} w_n\, m_n(\Omega)\,,
\end{aligned}
$$

where

$$m_n(\Omega) = \Omega^{-3+\alpha n}\, g_{-2+\alpha n/2}\Big(\frac{\Omega}{\Omega_\varepsilon}\Big)\,.$$

Thus, Lemma 12, for $\Omega_\varepsilon \leq \rho_4/4$, $\tau = -2$, $\beta = \alpha/2$, $d = 1$, yields

$$\Big|\frac{d^k I_3}{d\Omega^k}(\Omega)\Big| \leq C(k;\Omega_\varepsilon)\,\Omega^{-3-k}\,, \qquad \text{for } \ \Omega \leq \frac{1}{2}\,\Omega_\varepsilon\,.$$

For the lower bound, we write

$$I_3(\Omega) = w_n\, m_0(\Omega) + \tilde{I}_3(\Omega)\,, \qquad \tilde{I}_3(\Omega) = \sum_{n=1}^{\infty} w_n\, m_n(\Omega)\,.$$

Lemma 10 applied to m_0 and Lemma 12 applied to $\tilde{I}_3$ with $\tau = -2 + \alpha/2 < -1/2$, $\beta = \alpha/2$, $d = 1$, yield that

$$|m_0(\Omega)| \geq c\,\Omega^{-3}\,, \quad |\tilde{I}_3(\Omega)| \leq C(\Omega_\varepsilon)\,\Omega^{-3+\alpha}\,, \qquad \text{for } \ \Omega \leq \frac{1}{2}\,\Omega_\varepsilon\,.$$

Therefore, for Ω small enough, we obtain

$$I_3(\Omega) \geq c\,\Omega^{-3}\,, \qquad \Omega \leq \bar{\Omega}_\varepsilon\,,$$

for a number $\bar{\Omega}_\varepsilon \ll \Omega_\varepsilon$. Since I_1 and I_2 remain bounded as $\Omega \to 0$, the lemma is proved.

Corollary 16. *For constants C_k and c_0 we have*

$$\eta(\Omega) \geq c_0\,\Omega^4\,, \qquad \Big|\frac{d^k \eta}{d\Omega^k}(\Omega)\Big| \leq C_k\,\Omega^{4-k}\,, \quad k \geq 0\,.$$

We apply now our growth estimates for ϕ and η to show that Ψ_C is very much like Ψ_Q in the introduction.

Lemma 17.

$$|\Psi_Q(Z) - \Psi_C(Z)| \le C\, Z^{4/3}\,.$$

PROOF. Set

$$\tilde{l} = \frac{l+1/2}{\lambda}\,, \qquad \lambda = Z^{1/3}\,,$$

and note that

$$l(l+1)\,\lambda^{-2} = \tilde{l}^2 - \frac{1}{4\lambda^2}\,.$$

We then use (1.a) and (1.b) to conclude that

$$\Psi_Q = 2\pi\, Z^{4/3} \sum_{l=1}^{l_{\mathrm{TF}}} \frac{\tilde{l}}{P\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big)}\, \mu\Big(Z^{1/3}\, \phi\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big)\Big)\,.$$

Define $l_{\max}$ as the largest l appearing in the sum defining Ψ_C. Note:

1. Each term appearing in the definition of either Ψ_Q above, or in Ψ_C, is bounded by a constant independent of Z.

2. The sum in Ψ_Q is taken over integers, while the sum in Ψ_Q is taken over half-integers.

3. The number of terms in either sum (l_{TF} for Ψ_Q and $l_{\max} - 1/2$ for Ψ_C) may differ slightly because in general, $l_{\mathrm{TF}} + 1/2 \ne l_{\max}$.

We show now that the number of terms in both sums differ by at most 1. $l_{\max}$ is the greatest element in $\mathbb{Z} + 1/2$ which is less than or equal to $a^{-1/2}\, Z^{1/3}\, \Omega_c$. Similarly, l_{TF} is the largest integer that satisfies

$$l_{\mathrm{TF}}\,(l_{\mathrm{TF}}+1) = \left(l_{\mathrm{TF}} + \frac{1}{2}\right)\sqrt{1 - \frac{1}{(2\,l_{\mathrm{TF}}+1)^2}} \le a^{-1/2}\, Z^{1/3}\, \Omega_c\,.$$

An immediate consequence of this is that

$$c\,Z^{1/3} \le l_{\mathrm{TF}} \le C\, Z^{1/3}\,, \qquad c\,Z^{1/3} \le l_{\max} \le C\, Z^{1/3}\,.$$

Then, for Z large enough, $l_{\mathrm{TF}} + 1/2 \le a^{-1/2}\, Z^{1/3}\, \Omega_c + 1$, which implies

$$l_{\mathrm{TF}} + \frac{1}{2} \le l_{\max} + 1\,. \tag{19}$$

On the other hand,

$$l_{\max} - l_{\mathrm{TF}} \leq a^{-1/2}\, Z^{1/3}\, \Omega_c - l_{\mathrm{TF}} \,.$$

Since we must have

$$l_{\mathrm{TF}} + \frac{3}{2} > \frac{a^{-1/2}\, Z^{1/3}\, \Omega_c}{\sqrt{1 - (2\, l_{\mathrm{TF}} + 3)^{-2}}} \,,$$

which implies

$$a^{-1/2}\, Z^{1/3}\, \Omega_c - l_{\mathrm{TF}} - \frac{1}{2} \leq \frac{3}{2} \,.$$

we conclude that $l_{\max} - 1/2 - l_{\mathrm{TF}} < 1$, or

(20) $$l_{\max} - \frac{1}{2} \leq l_{\mathrm{TF}} \,.$$

Thus, using (19),

(21) $$\left| l_{\max} - \frac{1}{2} - l_{\mathrm{TF}} \right| \leq 1 \,,$$

for Z large enough.

As a consequence of this, if we rewrite

$$\Psi_C(Z) = 2\pi\, Z^{4/3} \sum_{l=1}^{l_{\max}-1/2} \eta\Big(\big(l + \frac{1}{2}\big)\, Z^{-1/3}\Big) \cdot \mu\Big(Z^{1/3}\, \phi\Big(\big(l + \frac{1}{2}\big)\, Z^{-1/3}\Big)\Big) \,,$$

$$\tilde{\Psi}_Q(Z) = 2\pi\, Z^{4/3} \sum_{l=1}^{l_{\max}-1/2} \frac{\tilde{l}}{P\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big)}\, \mu\Big(Z^{1/3}\, \phi\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big)\Big) \,,$$

which makes sense by (20), and by noting that $l_{\max}$ in fact refers to a half-integer, we see that

$$\left| \Psi_Q(Z) - \tilde{\Psi}_Q(Z) \right| \leq C\, Z^{4/3} \,,$$

since, after all, the difference is at most one term of size at most $Z^{4/3}$. Next, we compare Ψ_C and $\tilde{\Psi}_Q$ term by term. To this end, we observe that, since $\tilde{l} \geq Z^{-1/3}$, we have

$$\left| \frac{1}{P(\tilde{l})} - \frac{1}{P\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big)} \right| \leq \frac{\left| P(\tilde{l}) - P\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big) \right|}{P(\tilde{l})\, P\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big)} \leq C\, \tilde{l}^6\, \tilde{x}^{-4}\, \tilde{l} \left(1 - \sqrt{1 - \frac{1}{4\, \tilde{l}^2\, \lambda^2}} \right)$$

(for $\tilde{x} \in [\sqrt{\tilde{l}^2 - 1/(4\lambda^2)}, \tilde{l}]$)

$$\leq C\, \tilde{l}\, \lambda^{-2},$$

which implies

$$\left| \eta\Big(\frac{l + 1/2}{\lambda}\Big) - \frac{\tilde{l}}{P\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big)} \right| \leq \tilde{l} \left| \frac{1}{P(\tilde{l})} - \frac{1}{P\Big(\tilde{l}^2 - \frac{1}{4\lambda^2}\Big)} \right| \leq C\, \tilde{l}^2\, \lambda^{-2}.$$

Similarly, since ϕ' is bounded,

$$\left| \phi(\tilde{l}) - \phi\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big) \right| \leq C\, \tilde{l} \Big(1 - \sqrt{1 - \frac{1}{4\, \tilde{l}^2\, \lambda^{-2}}} \Big) \leq C\, \tilde{l}^{-1}\, \lambda^{-2},$$

and since μ is Lipschitz, we conclude that

$$\left| \mu\Big(\lambda\, \phi\Big(\frac{l + 1/2}{\lambda}\Big)\Big) - \mu\Big(\lambda\, \phi\Big(\sqrt{\tilde{l}^2 - \frac{1}{4\lambda^2}}\Big)\Big) \right| \leq C\, \tilde{l}^{-1}\, \lambda^{-1}.$$

Therefore, the terms indexed by l in $\tilde{\Psi}_Q$ and Ψ_C differ by at most

$$C \left(\lambda^{-2}\, \tilde{l}^2 + \sup_{x \in [(\tilde{l}^2 - 1/(4\lambda^2))^{1/2}, \tilde{l}]} |\eta(x)|\, \lambda^{-1}\, \tilde{l}^{-1} \right),$$

which implies

$$\left| \Psi_Q(Z) - \tilde{\Psi}_C(Z) \right| \leq C\, Z^{4/3} \sum_{l=1}^{l_{\mathrm{TF}}} \left(Z^{-4/3}\, l^2 + Z^{-4/3}\, l^3 \right) \leq C\, Z^{4/3},$$

as required.

In what follows, we will denote either of the sums Ψ_C or Ψ_Q simply by Ψ, since we now know that both are the same modulo errors of order $Z^{4/3}$.

Define $\Omega_{n,l}$ such that $\phi'(\Omega_{n,l}) = l/n$, and

$$\theta(n,l) = n\,\phi(\Omega_{n,l}) - l\,\Omega_{n,l}\,.$$

Theorem 18.

$$\Psi(Z) = \Psi_0(Z) + o(Z^{3/2})\,,$$

where

$$\Psi_0(Z) = 2\pi\, Z^{3/2} \sum_{n,l} \frac{\eta(\Omega_{n,l})\,\hat{\mu}(n)}{|n\,\phi''(\Omega_{n,l})|^{1/2}}\; e^{2\pi i\, Z^{1/3}\,\theta(n,l) + \pi i(l - \mathrm{sign}(n)/4)}\,.$$

Also, Ψ_0 satisfies the bound

$$|\Psi_0(Z)| \le C\, Z^{3/2}$$

for a constant C.

Proof. Construct now a partition of unity given by $\{U_\nu, \theta_\nu\}$ for $\nu = 0, 1, \ldots$, such that

$$U_\nu = [\,a_\nu\,, b_\nu\,]\,, \qquad a_\nu = 2^{-\nu-2}\,a^{-1/2}\,\Omega_c\,, \quad b_\nu = 2^{-\nu}\,a^{-1/2}\,\Omega_c\,,$$

$$d_\nu \stackrel{\text{def}}{=} b_\nu - a_\nu\,, \qquad d_\nu \sim a_\nu \sim b_\nu\,,$$

$$\sum_\nu \theta(x) = \begin{cases} 1\,, & \text{if } 0 < x \le \Omega_c\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$

$$\theta_\nu \in C_0^\infty \quad \text{and} \quad \operatorname{supp}\theta_\nu \subset U_\nu\,, \qquad \text{for } \nu \ge 1\,,$$

$$\theta_0(x) \equiv 0\,, \qquad \text{for } x \notin [\,a_0\,, b_0\,]\,,$$

$$\left|\frac{d^k\theta_\nu}{dx^k}(x)\right| \le C_k\, d_\nu^{-k}\,,$$

for universal constants C_k independent of ν. Clearly, we have

$$\Psi_C(Z) = 2\pi\, Z^{4/3} \sum_{\nu=0}^{\infty} S_\nu(Z^{1/3})\,, \tag{22}$$

for

$$S_\nu = \sum_{l \in \mathbb{Z}+1/2} (\eta\,\theta_\nu)\Big(\frac{l}{\lambda}\Big)\,\mu\Big(\lambda\,\phi\big(\frac{l}{\lambda}\big)\Big), \qquad \lambda = Z^{1/3}.$$

Note that we have

$$\text{(23.a)} \qquad \begin{cases} |\phi(x)| \le C, \qquad |\phi'(x)| \le C, \\ c\,d_\nu^{-1+\alpha} \le |\phi''(x)| \le C\,d_\nu^{-1+\alpha}, \\ \Big|\dfrac{d^k\phi}{d\Omega^k}(x)\Big| \le C\,d_\nu^{1-k+\alpha} \qquad (k \ge 2), \\ \Big|\dfrac{d^k(\theta_\nu\,\eta)}{d\Omega^k}(x)\Big| \le C\,d_\nu^{4-k} \qquad (k \ge 0), \end{cases}$$

for $x \in U_\nu$, which implies that, in each S_ν, we have

$$\text{(23.b)} \qquad B(\phi) \le C\,a_{\nu_1}^{-400} \le C\,2^{400\,\nu}.$$

We consider

$$\nu_1 = \varepsilon_1\,|\log_2\lambda|, \qquad \varepsilon_1 = 10^{-3}.$$

For $\nu = 0$, we apply Theorem 6 to obtain

$$\text{(24)} \qquad S_0 = \sum_{\substack{n\neq 0\\ l\in\mathbb{Z}}} \varepsilon_{n,l}\,\hat{\mu}(n)\,\theta_0(\Omega_{n,l})\,\eta\,(\Omega_{n,l})\;\sigma(n,l) + o(Z^{1/6}).$$

For $0 < \nu \le \nu_1$, we use (23) and Theorem 7 to obtain

$$\text{(25)} \qquad S_\nu = \sum_{\substack{n\neq 0\\ l\in\mathbb{Z}}} \hat{\mu}(n)\;(\theta_\nu\,\eta)\,(\Omega_{n,l})\;\sigma(n,l) + O\big(2^{400\,\nu}\big).$$

If $\nu > \nu_1$, we argue directly as we did before Theorem 6 to obtain that

$$S_\nu = \lambda \sum_{\substack{n\neq 0\\ l\in\mathbb{Z}}} \hat{\mu}(n)\,I(n,l), \qquad I(n,l) = \int (\theta_\nu\,\eta)(x)\;e^{2\pi i\lambda(n\,\phi(x)-lx)}\,dx.$$

In order to analyze $I(n,l)$, we use Lemma 4 to obtain

$$|I(n,l)| \le C\,\big(\lambda\,|n|\big)^{-1/2}\,\big(d_\nu^{-1+\alpha}\big)^{-1/2}\,d_\nu^{3+\alpha},$$

which we will use when $|l| \le 4\,\|\phi'\|_\infty\,|n|$, to obtain

$$\left|\sum_{\{|l|\le 4\,\|\phi'\|_\infty\,|n|\}} \hat\mu(n)\,I(n,l)\right| \le C\,\lambda^{-1/2}\,d_\nu^{4+\alpha/2}\,. \tag{26.a}$$

When $|l| \ge 4\,\|\phi'\|_\infty\,|n|$, we have $|\phi'(x) - l/n| \ge |l|/(4|n|)$; thus, we apply (4.b) directly to $I(n,l)$ to obtain

$$|I(n,l)| \le (\lambda\,|n|)^{-1}\,\frac{d_\nu^{4+\alpha}}{\left(\dfrac{|l|}{4\,|n|}\right)^2} + 4\,(\lambda\,|n|)^{-2}\left(\frac{d_\nu^3}{\left(\dfrac{|l|}{4\,|n|}\right)^2} + \frac{d_\nu^{3+\alpha}}{\left(\dfrac{|l|}{4\,|n|}\right)^3}\right),$$

which implies that, for n fixed,

$$\left|\sum_{\{l:\ |l|\ge 4\,\|\phi'\|_\infty\,|n|\}} I(n,l)\right| \le C\left(\lambda^{-1}\,d_\nu^{4+\alpha} + \frac{\lambda^{-2}\,d_\nu^3}{|n|} + \frac{\lambda^{-2}\,d_\nu^{3+\alpha}}{|n|}\right),$$

which finally implies

$$\left|\sum_{|l|>4\,\|\phi'\|_\infty\,|n|} \hat\mu(n)\,I(n,l)\right| \le \lambda^{-1}\,d_\nu^3\,. \tag{26.b}$$

Putting (26.a) and (26.b) together, we obtain

$$|S_\nu| \le \lambda^{1/2}\,d_\nu^3\,,$$

which implies

$$\left|\sum_{\nu>\nu_1} S_\nu\right| \le \lambda^{1/2}\,2^{-3\,\nu_1} = \lambda^{1/2-3\,\varepsilon_1}\,. \tag{27}$$

Also,

$$\begin{aligned}\left|\sum_{\nu>\nu_1}\ \sum_{\{(n,l):\ \Omega_{n,l}\in U_\nu\}} \frac{\hat\mu(n)\,(\eta\,\theta_\nu)(\Omega_{n,l})}{|n\,\phi''(\Omega_{n,l})|^{1/2}}\,e^{2\pi i\lambda\,\theta(n,l)+\pi i(l-\mathrm{sign}(n)/4)}\right| \\ \le C \sup_{x\in\cup_{\nu=\nu_1+1}^{\infty} U_\nu} |\eta(x)| \\ = O(d_{\nu_1}^4) = O(\lambda^{-4\,\varepsilon_1})\,.\end{aligned}$$

and, putting (22), (24), (25) and (27) together, we obtain

$$\Psi(Z) = \Psi_0(Z) + o\Big(Z^{3/2}\Big) .$$

4. Lower Bounds.

Theorem 18 told us two things: that Ψ has a leading expression as a trigonometric sum, and that the size of this trigonometric sum, and therefore of Ψ, is *at most* of order $Z^{3/2}$. The question remains whether this bound is sharp or not. In related problems, such as the lattice point problem, sharp upper and lower bounds *on average* have been known for over fifty years (see [B1] and [HB] for recent developments). This, in our context, would translate into the statement that indeed $Z^{3/2}$ is best possible.

The aim of this section is to derive such estimates *on average* for the function Ψ_0. Classical ideas will work effortlessly after we show that a certain number is not 0. This number can be viewed as a certain (analytic, not necessarily arithmetic) L-function evaluated at the point $s = 2$. Thus, it is not surprising that such a condition appears if one thinks, for example, about the lattice point problem for parabola.

First, we will consider real values for Z, and then use this to study the case of interest, integer Z, as it relates to our original goal to understand the ground state energy of an atom of nuclear charge Z.

We begin by defining

$$a_{n,l} = \frac{\hat{\mu}(n)\,\eta(\Omega_{n,l})}{|n\,\phi''(x_{n,l})|^{1/2}}\; e^{\pi i(l-\mathrm{sign}(n)/4)} .$$

Let $\{\theta_\nu\}$ the set of all possible values of $\theta(n,l)$, selected such that $\theta_\nu \neq \theta_{\nu'}$ for $\nu \neq \nu'$. Define

$$L \stackrel{\mathrm{def}}{=} L_{\eta,\phi} = \sum_\nu \Big| \sum_{\theta(n,l)=\theta_\nu} a_{n,l} \Big|^2 ,$$

which Lemma 20 will prove to be non-zero; if, however, L were equal to 0, then it is easy to see that in fact we would have that

$$\Psi_0(Z) \equiv 0 , \qquad \text{all } Z .$$

In the meantime, we will simply assume $L \neq 0$.

Define also

$$C^* = 1 + \left(\frac{48 \, \|\eta\|_\infty \, \|\phi'\|_\infty}{c_0 \, L} \right)^{2/3}$$

$$c^\sharp = \inf \left\{ |\theta(n,l) - \theta(n',l')| : \ \theta(n,l) \neq \theta(n',l') \text{ and } |n|, \ |n'| \leq C^* \right\} .$$

Therefore, setting as usual $\lambda = Z^{1/3}$,

$$\begin{aligned} Z^{-3} \, |\Psi_0(Z)|^2 &= \sum_{n,\, n',\, l,\, l'} a_{n,l} \, \overline{a_{n',l'}} \, e^{2\pi i \lambda (\theta(n,l) - \theta(n',l'))} \\ &\geq \sum_{\substack{n,\, n',\, l,\, l' \\ \theta(n,l) = \theta(n',l')}} a_{n,l} \, \overline{a_{n',l'}} \\ &\qquad + \sum_{\theta(n,l) \neq \theta(n',l')} a_{n,l} \, \overline{a_{n',l'}} \, e^{2\pi i \lambda (\theta(n,l) - \theta(n',l'))} \, . \end{aligned}$$

The first term above is our L defined above. In the second term, we separate the terms for small $|n|$, $|n'|$, which we keep untouched, and the ones for which *either* n or n' are large, which we estimate using the fact that $|l| \leq \|\phi'\|_\infty \, |n|$ and $|l'| \leq \|\phi'\|_\infty \, |n'|$, to obtain

$$\begin{aligned} &\geq L - \sum_{\substack{|\theta(n,l) - \theta(n',l')| > 0 \\ |n|,\, |n'| \leq C^*}} a_{n,l} \, \overline{a_{n',l'}} \, e^{2\pi i \lambda (\theta(n,l) - \theta(n',l'))} \\ &\qquad - 2 \, c_0^{-1} \, \|\eta\|_\infty^2 \, \|\phi'\|_\infty^2 \sum_{\substack{|n| \geq C^* \\ \text{all } n'}} |n \, n'|^{-5/2} \\ &\geq L - \sum_{\substack{|\theta(n,l) - \theta(n',l')| > 0 \\ |n|,\, |n'| \leq C^*}} a_{n,l} \, \overline{a_{n',l'}} \, e^{2\pi i \lambda (\theta(n,l) - \theta(n',l'))} \\ &\qquad - 24 \, c_0^{-1} \, \|\eta\|_\infty^2 \, \|\phi'\|_\infty^2 \, (C^* - 1)^{-3/2} \, , \end{aligned}$$

which, by our choice of C^*, implies

$$Z^{-3} \, |\Psi_0(Z)|^2 \geq \frac{L}{2} - \sum_{\substack{|\theta(n,l) - \theta(n',l')| > 0 \\ |n|,\, |n'| \leq C^*}} a_{n,l} \, \overline{a_{n',l'}} \, e^{2\pi i \lambda (\theta(n,l) - \theta(n',l'))} \, .$$

Now we consider $Z_0 \geq 1$ and $Z \leq Z_0/2$ and prepare to integrate both sides from Z_0 to $Z_0 + Z$. For that, note that $dZ = 3\lambda^2\, d\lambda$ and, if we set $\lambda_0 = Z_0^{1/3}$, then

$$(Z_0 + Z)^{1/3} = \lambda_0 + \Lambda\,, \qquad \Lambda \leq Z/Z_0^{2/3}\,.$$

Also,

$$\left| \int_a^b \lambda^2\, e^{i\lambda\theta}\, d\lambda \right| \leq \frac{3\, b^2}{\theta}\,, \qquad a, b > 0\,.$$

Therefore,

$$\begin{aligned} \int_{Z_0}^{Z_0+Z} |\Psi_0(z)|^2\, \frac{dz}{z^3} &\geq \frac{Z}{2}\, L - 36\, Z_0^{2/3}\, c_0^{-1}\, \|\eta\|_\infty^2\, \|\phi'\|_\infty^2 \\ &\qquad \sum_{\substack{|\theta(n,l)-\theta(n',l')|>0 \\ |n|,\,|n'|\leq C^*}} \frac{|n\,n'|^{-5/2}}{2\pi\,|\theta(n,l) - \theta(n',l')|} \\ &\geq \frac{Z}{2}\, L - 36\, Z_0^{2/3}\, \frac{\|\eta\|_\infty^2\, \|\phi'\|_\infty^2}{c_0\, c^\sharp} \sum_{|n|,|n'|\leq C^*} |n\,n'|^{-5/2} \\ &\geq \frac{Z}{2}\, L - Z_0^{2/3}\, \frac{18000\, \|\eta\|_\infty^2\, \|\phi'\|_\infty^2}{c_0\, c^\sharp}\,. \end{aligned}$$

As a consequence, taking $Z/Z_0^{2/3}$ large depending on L, c_0, $c^\sharp$, $\|\eta\|_\infty$ and $\|\phi'\|_\infty$, but still Z not larger than $Z_0/2$, we have

$$\int_{Z_0}^{Z_0+Z} |\Psi_0(z)|^2\, \frac{dz}{z^3} \geq \frac{Z}{4}\, L\,.$$

Next, we turn to the non-vanishing of L. In preparation for the proof, set

$$\alpha_1 = \phi'(0)\,, \qquad \alpha_2 = \phi'(a^{-1/2}\,\Omega_c)$$

and note that $0 > \alpha_1 > \alpha_2$. Clearly, the (n, l) that enter in the sum for Ψ_0 are determined by the lattice points in $\mathbb{Z}^2$ which fall in the double-cone

$$\Gamma = \{(u, v) :\ \alpha_2\, |u| \leq -\,|v| < \alpha_1\, |u|\ ,\ u\, v < 0\}\,.$$

Define $x_{u,v}$, for $(u, v) \in \Gamma$, as the unique point satisfying

$$\phi'(x_{u,v}) = \frac{v}{n}\,.$$

Note that $x_{u,v}$ is strictly positive.

Then, we define

$$\theta(u,v) = u\,\phi(x_{u,v}) - v\,x_{u,v}\ , \qquad (u,v) \in \Gamma\,.$$

The following is a trivial fact

Lemma 19. *On* Γ *we have*

$$\nabla\theta(u,v) = (\phi(x_{u,v}), -x_{u,v})\ .$$

A consequence of this trivial fact is the result we mentioned above.

Lemma 20. $L \neq 0$.

PROOF. We will show that there is one θ_ν which only has one $(n,l) \in \Gamma$ such that $\theta(n,l) = \theta_\nu$. This clearly shows that L is not 0.

Let n_0 be the smallest positive integer such that $(n,l) \in \Gamma$ for some l. In our case, this l is negative. Choose the largest (negative) such l, which we denote by l_0. That is, l_0 is the largest negative integer that satisfies $l < \alpha_1\, n_0$. Then we claim that there is no other pair $(n',l') \in \Gamma$ such that $\theta(n_0,l_0) = \theta(n',l')$. Indeed, since $\theta(n,l)$ is strictly positive for $n > 0$, and strictly negative for $n < 0$, such n' would also have to be positive. It cannot equal n_0 because, since we should have $l' < l_0$, by the previous lemma we have

$$\theta(n_0,l') > \theta(n_0,l_0)\,.$$

We must then have $n' > n_0$, which also implies

$$l' < \alpha_1\, n' < \alpha_1\, n_0 \quad (< 0)$$

thus showing that $l' \leq l$. But this is also impossible because, also by the previous lemma, there exists a pair (ξ,η) on the segment joining (n_0,l_0) to (n',l'), such that

$$\theta(n',l') - \theta(n_0,l_0) = \phi(x_{\xi,\eta})(n'-n_0) - x_{\xi,\eta}(l'-l_0)$$

and this last expression is then strictly positive.

We summarize all this in the following lemma

Lemma 21. *There is a small constant κ_0 and a large constant K such that*

$$\int_{Z_0}^{Z_0+Z} z^{-3}\,|\Psi_0(z)|^2\,dz \geq \kappa_0\,Z\,,$$

whenever $Z \geq K\,Z_0^{2/3}$, and $Z_0 \geq K$.

PROOF. Our previous calculations show this result in the case that $Z \geq C\,Z_0^{2/3}$ but $Z \leq Z/2$, for a certain large constant C. For the general case, break up

$$\int_{Z_0}^{Z_0+Z} z^{-3}\,|\Psi_0(z)|^2\,dz = \sum_{n=0}^{N} \int_{Z_n}^{Z_{n+1}} z^{-3}\,|\Psi_0(z)|^2\,dz\,,$$

where

$$Z_{n+1} = 1.01\,Z_n \quad \text{when } n = 0,\ldots,N-1; \qquad Z_{N+1} = Z_0 + Z\,,$$

and N is chosen so that $1.01^{N+1}\,Z_0 \leq Z_0 + Z < 1.01^{N+2}\,Z_0$. Our previous calculations would apply to each of the integrals in the sum provided

$$Z_n - Z_{n-1} \geq C\,Z_{n-1}^{2/3}\,, \quad (n = 1,\ldots,N); \qquad Z + Z_0 \geq Z_N^{2/3}\,,$$

which amounts to requiring $Z_0 \geq 10^6$.

This mean value information can be used to obtain information about the oscillating behavior of Ψ_0, as follows

Let

$$I = [\,Z_0\,, Z_0 + \hat{C}\,Z_0^{2/3}\,] = \bigcup I_j$$

for

$$I_j = [\,Z_0 + \hat{c}\,j\,Z_0^{2/3}\,,\, Z_0 + \hat{c}\,(j+1)\,Z_0^{2/3}\,]\,,$$

where $\hat{C}$ is large depending on K and $\hat{c}$ is small depending on κ_0.

Denote also, for any function f,

$$m_j(f) = \inf_{x \in I_j} \left|x^{-3/2}\,f(x)\right|.$$

Corollary 22. *Given any $\varepsilon > 0$ small depending on k_0, any $\hat{C}$ and $\hat{c}$ as above, with the extra requirement that $\hat{c}$ is small depending on ε, and Z_0 also large enough, there exists a constant $0 < \alpha < 1$ such that $m_j(\Psi_0) > \varepsilon$ for at most $\alpha\hat{C}/\hat{c}$ of the I_j.*

PROOF. Put

$$Z^{-3/2}\,\Psi_0(Z) = F(Z) + E(Z)$$

such that $F(Z)$ contains only finitely many terms in the sum, and $|E(Z)|$ is always less than ε. In particular, we have that in order that $m_j(\Psi_0) < \varepsilon$ we must have $m_j(F) < 2\varepsilon$. It will therefore be enough to count how many of the $m_j(F)$ stay below 2ε to obtain the conclusion of the lemma.

Since both ϕ and ϕ' are bounded, we have the trivial bound

$$|F'(Z)| \le C_\varepsilon\, Z^{-2/3}\,,$$

for some constant C_ε which depends on ε. Thus, if $m_j(\Psi_0) \le \varepsilon$, we have $F(Z) \le 2\varepsilon + \hat{c}\,C_\varepsilon$ which implies

$$\int_{I_j} |F(z)|^2\, dz \le 16\,|I_j|\,\left(\varepsilon^2 + \hat{c}^2\, C_\varepsilon^2\right)\,.$$

Therefore, if we denote by

$$M = \text{number of } j \text{ such that } \ m_j(\Psi_0) < \varepsilon\,,$$

we use the trivial bound $|F(z)| \le C$ for all z, for a universal constant C, to get

$$\begin{aligned}\kappa_0\,\hat{C}\,Z_0^{2/3} &\le \int_I |F(z)|^2\ dz\\ &\le \left(\frac{\hat{C}}{\hat{c}} - M\right) C\,\hat{c}\,Z_0^{2/3} + 16\,M\,\hat{c}\,Z_0^{2/3}\ \left(\varepsilon^2 + \hat{c}^2\, C_\varepsilon^2\right)\,.\end{aligned}$$

This implies that

$$M \le \alpha\,\frac{\hat{C}}{\hat{c}}\,, \qquad \text{for } \alpha = \frac{C - \kappa_0}{C - 16(\varepsilon^2 + \hat{c}^2\, C_\varepsilon^2)}\,.$$

By adjusting $\hat{c}$ depending on ε it is easy to make $\alpha < 1$.

A consequence of this corollary is another which shows that the size of $\Psi_0(Z)$ is at most $c\,Z^{3/2}$, for a small constant c, even when we restrict our attention to integer values of Z.

Corollary 23.

$$\liminf_{\substack{Z\to\infty\\ Z=1,2,3,\dots}} \left|Z^{-2/3}\,\Psi_0(Z)\right| \neq 0\,.$$

PROOF. Apply the previous corollary to any $\varepsilon > 0$ small as required, and to any $\hat{C}$ large and $\hat{c}$ small also as needed, and then to infinitely Z_0 to conclude that there are infinitely many intervals of lengths going to infinity where $|x^{-3/2}\,\Psi_0(x)|$ is never smaller than ε.

5. The Classical Picture.

In this Section we identify all quantities appearing in the expression for Ψ_0 in terms of data coming from the classical dynamics of a particle in the field created by the Thomas-Fermi potential. We begin with a brief review of elementary classical mechanics, which can be found, among many other places, in [Ar].

Consider a particle with mass 1/2, in $\mathbb{R}^3$, moving in a negative radial potential $-V(r)$, which for us, will equal $-V^1_{\mathrm{TF}}(r)$. The motion is planar, and can be described by the distance to the origin $r(t)$ and the angle φ, which satisfy the relations

$$\dot{\varphi} = \frac{2M}{r^2}\,, \qquad \frac{1}{4}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r) = E$$

where M is the angular momentum, and E is the energy of the orbit. We begin assuming that the particle travels counter-clockwise in our frame of reference (r,φ). The motion takes place between radii $r_{\min}$ and $r_{\max}$ given by the two solutions of the equation

$$-V(r) + \frac{M^2}{r^2} = E\,.$$

This implies that all trajectories for a fixed energy and angular momentum can be obtained by rotation of a fixed one.

At energy 0, we denote by $t_{\min}$ and $t_{\max}$ the times at which the particle passes through $r_{\min}$ and $r_{\max}$ respectively. The angle of motion

φ and the distance to the origin r satisfy the equations

$$\frac{dr}{dt} = 2\sqrt{V(r) - \frac{M^2}{r^2}}\,, \qquad \frac{d\varphi}{dr} = \frac{M/r^2}{\sqrt{V(r) - \frac{M}{r^2}}}\,.$$

As a consequence, the particle going from $r_{\min}$ to $r_{\max}$ sweeps out an angle given by

$$M\int_{r_{\min}}^{r_{\max}} \left(V(r) - \frac{M}{r^2}\right)^{-1/2} \frac{dr}{r^2} = -\pi\,\phi'(M)\,,$$

and the trajectory is clearly symmetric with respect to either $r_{\min}$ or $r_{\max}$. Therefore, the angular momentum M will give rise to a closed orbit if and only if

$$-\phi'(M) = \frac{l}{n} \tag{28}$$

and in this case, n represents the number of times the particle oscillates between successive $r_{\min}$ (or $r_{\max}$) before closing, and l represents the winding number of the orbit around 0. Our initial assumption that the particle travels counter-clockwise means that $n, l \geq 0$. In our previous notation, we also have

$$M = \Omega_{n,-l}\,.$$

If $(l, n) = 1$ (where $(\cdot,\cdot)$ denotes greatest common divisor), the orbit is usually called *primitive*.

The period is given by

$$T(M) = 2n\int_{t_{\min}}^{t_{\max}} dt = n\int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{V(r) - \frac{M^2}{r^2}}} = n\,P(M)\,.$$

In order to find the action S along this closed trajectory,

$$S = 2n\int_{t_{\min}}^{t_{\max}} (\text{Kinetic Energy} + V)\;dt\,,$$

we note that, since we are at energy 0, Kinetic Energy $= V(r)$, which implies

$$\begin{aligned} S &= 4n \int_{r_{\min}}^{r_{\max}} \frac{V(r)}{2\sqrt{V(r) - \frac{M^2}{r^2}}}\, dr \\ &= 2n \left(\int_{r_{\min}}^{r_{\max}} \sqrt{V - \frac{M^2}{r^2}}\, dr + \int_{r_{\min}}^{r_{\max}} \frac{M^2/r^2}{\sqrt{V - \frac{M^2}{r^2}}}\, dr \right) \\ &= 2\pi\, (n\, \phi(M) + l\, M) \\ &= 2\pi\, (n\, \phi(\Omega_{n,-l}) + l\, \Omega_{n,-l}) \ . \end{aligned}$$

As a consequence, denoting by $S(M)$ the action along a closed counterclockwise trajectory at energy 0 with angular momentum M, we have

$$2\pi\, \theta(n, -l) = S\,(\Omega_{n,-l}) \ .$$

When the particle travels clockwise, we agree that S, T, n and l change sign, but we keep $M \geq 0$.

We have so far identified all terms in the definition of Ψ_0 except $n\, \phi''(\Omega_{n,l})$. For this one, consider a closed trajectory arising from angular momentum M, which gives rise to n oscillations between successive $r_{\max}(M)$; for ε small, consider a trajectory with angular momentum $M + \varepsilon$ which begins at $r_{\max}(M + \varepsilon)$ and denote by $2\pi\, \alpha_M(\varepsilon)$ the absolute value of the angle the particle forms between the initial position at $r_{\max}(M + \varepsilon)$ and the position after n oscillations also at $r_{\max}(M + \varepsilon)$, where we take α between 0 and 1/2. Then

$$D(M) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \varepsilon^{-1}\, \alpha_M(\varepsilon) = |n\, \phi''(M)|\, .$$

It is clear now that the nonvanishing of the second derivative of ϕ translates into the fact that closed trajectories are isolated modulo the trivial symmetry given by the rotation group.

The motion degenerates for the one circular orbit arising from $M_{\max} = a^{-1/2}\, \Omega_c$, the maximum angular momentum allowed in our system. In this case, we define the above classical variables simply in terms of ϕ' using the formulas we derived for the other trajectories.

Theorem 24.

$$\Psi_0(Z) = 2\pi Z^{3/2} \sum_{\substack{\text{closed trajectories}\\ \text{energy } 0}} \delta \frac{n\,\hat{\mu}(n)\,M}{T} \, |D(M)|^{-1/2} \cdot e^{i(Z^{1/3}S - \pi(l + \text{sign}(n)/4))},$$

where circular trajectories appear in the sum only when they have associated a finite number of oscillations n. We have $\delta = 1$ except for the circular trajectories, when $\delta = 1/2$. The sum is absolutely convergent.

REMARK. Note that the contribution of each trajectory depends on the particular frame of reference we take to compute n and l, but the sum is independent of it.

REMARK. One might think of the different values for δ as follows: non-circular trajectories contribute fully to eigenvalues, while the circular ones, being right at the outskirts of the classically allowed region, contribute half as eigenvalues, half as resonances.

6. Further Considerations.

In this section we will compute the derivatives of ϕ at the ends of our interval of interest $[0, a^{-1/2}\,\Omega_c]$. The derivative at $a^{-1/2}\,\Omega_c$ plays a role in the sense described on Section 2, since its rationality or irrationality translates into the appearance or absence of a certain contribution to Ψ or size $Z^{3/2}$. The derivative at 0 does not play such a role since the amplitude vanishes there.

Lemma 25.

$$-\frac{1}{\pi} F'(\Omega_c) = \left(1 - \frac{1}{2} r_c^{3/2} y_c^{1/2}\right)^{-1/2} = \frac{1}{\sqrt{1 - \frac{1}{2} r_c\,\Omega_c}} \sim 1.9376783\,.$$

PROOF. We use the change of variables $t(r)$ given by (16), and its inverse $r(t)$, to write

$$-F'(\Omega_c) = \Omega_c \int_{-1}^{1} (1 - t^2)^{-1/2}\, w(0)\, dt\,,$$

with

$$w(0) = \frac{r'(0)}{r_c} = \frac{1}{r_c\, t'(r_c)} = \frac{\sqrt{2}}{r_c\, |u''(r_c)|^{1/2}}\,.$$

Thus,

$$-\frac{1}{\pi}\, F'(\Omega_c) = \frac{\Omega_c}{r_c \left|\frac{1}{2}\, u''(r_c)\right|^{1/2}} = \left|\frac{2\, y(r_c)}{r_c\, u''(r_c)}\right|^{1/2},$$

since

$$\frac{1}{\pi}\int_{-1}^{1}(1-t^2)^{-1/2}\, dt = 1\,.$$

Manipulations using the identities

$$u(x) = x\, y(x), \qquad u'(x) = x\, y'(x) + y(x)\,,$$
$$u''(x) = x\, y''(x) + 2\, y'(x) = x^{1/2}\, y^{3/2}(x) + 2\, y'(x)$$

and

$$r_c\, y'(r_c) = -y(r_c)\,,$$

yield our result.

Lemma 26.

$$-\lim_{\Omega\to 0} F'(\Omega) = \frac{3\pi}{2}\,.$$

PROOF. Let $r_0(\Omega)$ and $r_1(\Omega)$ be the two solutions of $u(r) = \Omega^2$. We study first the asymptotics of r_0 and r_1.

For r_0, put $z = r_0^{1/2}$; then, for

$$f(z) = u(z^2) = z^2 - w\, z^4 + O(z^5)\ ,$$

we have that $f(z) = \Omega^2$. This implies that $z = \Omega + O(\Omega^2)$ and thus

$$r_0(\Omega) = \Omega^2 + O(\Omega^3)\ .$$

For r_1, since $u(r)$ decreases monotonically to 0, $r_1(\Omega) \to +\infty$. Since we have that $u(r) = 144\, r^{-2} + O(r^{-2-\alpha})$, for $\alpha > 0$, we get that

$$r_1(\Omega) = \frac{12}{\Omega} + o(\Omega^{-1})\ .$$

In order now to analyze $F'(\Omega)$, take ε be a small enough constant to be picked up later and rewrite

$$
\begin{aligned}
g(\Omega) &= \int_{r_0(\Omega)}^{\varepsilon} \left(u(r) - \Omega^2\right)^{-1/2} \frac{dr}{r} \\
&\quad + \int_{\varepsilon}^{r_1(\Omega)} \left(u(r) - \Omega^2\right)^{-1/2} \frac{dr}{r} \\
&= \int_{r_0(\Omega)}^{\varepsilon} \left(u'(r_0)\,(r - r_0)\right)^{-1/2} \frac{dr}{r} \\
&\quad + \int_{\varepsilon}^{r_1(\Omega)} \left(u(r) - \Omega^2\right)^{-1/2} \frac{dr}{r} + R_0(\Omega),
\end{aligned} \tag{29}
$$

for

$$R_0 = \int_{r_0}^{\varepsilon} \left(\left(u(r) - \Omega^2\right)^{-1/2} - \left(u'(r_0)\,(r - r_0)\right)^{-1/2}\right) \frac{dr}{r}\,.$$

We show first that $R_0 = O(1)$.

Fix Ω:

$$
\begin{aligned}
&\left(u(r) - \Omega^2\right)^{-1/2} - \left(u'(r_0)(r - r_0)\right)^{-1/2} \\
&= \sum_{n=1}^{\infty} c_n \left(u'(r_0)\,(r - r_0)\right)^{-n-1/2} \left(\left(u(r) - u(r_0)\right) - u'(r_0)\,(r - r_0)\right)^{n}.
\end{aligned}
$$

Throughout this analysis, c_n will denote a generic sequence of bounded constants.

Note that

$$
\begin{aligned}
|u(r) - u(r_0) - u'(r_0)\,(r - r_0)| &\le \frac{1}{2} \sup_{0 \le r \le \varepsilon} |u''(r)|\,(r - r_0)^2 \\
&\le \left(y'(0) + \frac{1}{2}\,\varepsilon^{1/2}\right)(r - r_0)^2 \\
&\le C_0\,(r - r_0)^2
\end{aligned}
$$

since $u''(r) = 2\,y'(r) + r\,y''(r)$, $|y'(r)| \le |y'(0)|$ and $y''(r) \le r^{-1/2}$. Therefore, the sum converges uniformly for $|r - r_0| < C_0/2$ and integrating with respect to dr/r on (r_0, ε), for $\varepsilon < C_0/2$, we obtain

$$\begin{aligned} R_0 &\leq \sum_{n=1}^{\infty} |c_n| \, |u'(r_0)|^{-n-1/2} \int_{r_0}^{\varepsilon} C_0^n \, |r - r_0|^{n-1/2} \, \frac{dr}{r} \\ &\leq \sum_{n=1}^{\infty} |c_n| \, |u'(r_0)|^{-n-1/2} \, C_0^n \, r_0^{n-1/2} \int_1^{\varepsilon/r_0} (y-1)^{n-1/2} \, \frac{dy}{y} \\ &\leq \sum_{n=1}^{\infty} |c_n| \, |u'(r_0)|^{-n-1/2} \, C_0^n \, r_0^{n-1/2} \, (y-1)^{n-1/2} \Big|_1^{r_0^{-1}\varepsilon} \\ &\leq \sum_{n=1}^{\infty} |c_n| \, |u'(r_0)|^{-n-1/2} \, C_0^n \, \varepsilon^{n-1/2} \, . \end{aligned}$$

For Ω small enough, we can make $|u'(r_0)| < 2$, and this will make the previous sum converge to $O(1)$ for ε small, thus proving that R_0 is bounded.

Recall now that $\Omega \, r_0(\Omega)^{-1/2} \to 1$, what implies

$$\begin{aligned} \Omega \int_{r_0}^{\varepsilon} (r - r_0)_+^{-1/2} \, \frac{dr}{r} &= \Omega \, r_0^{-1/2} \int_1^{r_0^{-1}\varepsilon} y^{-1} \, (y-1)^{-1/2} \, dy \\ &\to \int_1^{+\infty} y^{-1} \, (y-1)^{-1/2} \, dy = \pi \, , \end{aligned}$$

which, with the fact that $u'(r_0(\Omega)) \to 1$, implies that

$$\begin{aligned} \lim_{\Omega \to 1} \Omega & \int_{r_0}^{\varepsilon} \left(u(r) - \Omega^2\right)^{-1/2} \, \frac{dr}{r} \\ &= \lim_{\Omega \to 1} \Omega u'(r_0)^{-1/2} \int_{r_0(\Omega)}^{\varepsilon} (r - r_0(\Omega))^{-1/2} \, \frac{dr}{r} = \pi \, , \end{aligned}$$

which completes the analysis of the first integral in (29).

As for the other integral, we break it up into 5 pieces as follows

$$\begin{aligned} \int_{\varepsilon}^{r_1} & (u(r) - \Omega)^{-1/2} \, \frac{dr}{r} \\ &= \int_{\varepsilon}^{M} + \int_{M}^{r_1^{99/100}} + \int_{r_1^{99/100}}^{r_1/2} + \int_{r_1/2}^{r_1 - r_1^{2/3}} + \int_{r_1 - r_1^{2/3}}^{r_1} \\ &= I_1 + I_2 + I_3 + I_4 + I_5 \, . \end{aligned}$$

It is clear that $I_1 = O(1)$, so we do not worry about it.

For I_2, note that, if M is large enough so that u is decreasing from M on, we have on its domain that

$$|u(r) - \Omega^2| \geq \left|u\left(r_1^{99/100}\right) - u(r_1)\right| \geq C \left(r_1^{99/100}\right)^{-2} - c\,r_1^{-2} \geq c\,r_1^{-99/50}$$

and thus

$$|I_2| \leq C \int_M^{r_1^{99/100}} \left|r_1^{99/50}\right|^{1/2} \frac{dr}{r} \leq C\, r_1^{99/100} \log r_1 \,,$$

which does not contribute to the result.

Similarly, for I_5, note that

$$|u(r) - \Omega^2| \geq \Big(\inf_{|r-r_1| \leq r_1^{2/3}} |u'(r)| \Big) \, |r - r_1| \geq C\, r_1^{-3}\, |r - r_1| \,,$$

thus

$$\begin{aligned} |I_5| &\leq C \int_{r_1 - r_1^{2/3}}^{r_1} r_1^{3/2} (r_1 - r)^{-1/2} \frac{dr}{r} \\ &\leq C\, r_1^{1/2} \int_{r_1 - r_1^{2/3}}^{r_1} (r_1 - r)^{-1/2}\, dr \\ &\leq C\, r_1^{1/2}\, r_1^{1/3} \\ &\leq C\, r_1^{5/6} \,, \end{aligned}$$

and again, it does not contribute to the total outcome.

For I_4, note that

$$|u(r) - \Omega^2| \geq \Big(\inf_{r \sim r_1} |u'(r)| \Big)\, r_1^{2/3} \geq C\, r_1^{-3}\, r_1^{2/3} \,.$$

This implies two things:

First, since $|u(r) - 144\, r^{-2}| \leq C\, r^{-2-\alpha} \leq C\, r_1^{-2-\alpha}$, for $\alpha = (\sqrt{73} - 7)/2 > 1 - 2/3$, we have that $144\, r^{-2} - \Omega^2 > 0$ on this range, for Ω small enough.

And second,

$$\begin{aligned} &\left| \left(u(r) - \Omega^2\right)^{-1/2} - \left(\frac{144}{r^2} - \Omega^2\right)^{-1/2} \right| \\ &\qquad \leq \sum_{n=1}^{\infty} c_n\, |u(r) - \Omega^2|^{-n-1/2} \left(C\, r^{-2-\alpha}\right)^n \\ &\qquad \leq \sum_{n=1}^{\infty} c_n \left(C\, r_1^{3-2/3}\right)^{n+1/2} r^{-n(2+\alpha)} \,, \end{aligned}$$

and thus

$$\int_{r_1/2}^{r_1-r_1^{2/3}} \left|\left(u(r)-\Omega^2\right)^{-1/2} - \left(\frac{144}{r^2}-\Omega^2\right)^{-1/2}\right| \frac{dr}{r}$$

$$\leq \sum_{n=1}^{\infty} c_n \left(C\, r_1^{3-2/3}\right)^{n+1/2} r_1^{-n(2+\alpha)}$$

$$\leq C\, r_1^{(7/3)\,(3/2)-(2+\alpha)}$$

$$= o(r_1(\Omega))\ .$$

Finally, for I_3,

$$\int_{r_1^{99/100}}^{r_1/2} \left|\left(u(r)-\Omega^2\right)^{-1/2} - \left(\frac{144}{r^2}-\Omega^2\right)^{-1/2}\right| \frac{dr}{r}$$

$$\leq \int_{r_1^{99/100}}^{r_1/2} \sum_{n=1}^{\infty} c_n \left(u\left(\frac{r_1}{2}\right)-\Omega^2\right)^{-n-1/2} (c\,r)^{-n(2+\alpha)}\,\frac{dr}{r}$$

$$\leq \int_{r_1^{99/100}}^{r_1/2} \sum_{n=1}^{\infty} c_n \left|C\, r_1^{-2}\right|^{-n-1/2} r^{-n(2+\alpha)}\,\frac{dr}{r}$$

$$\leq \sum_{n=1}^{\infty} c_n \left|C\, r_1^{2}\right|^{n+1/2} r_1^{-99\,n(2+\alpha)/100}$$

$$\leq C\, r_1^{3}\, r_1^{-99\,(2+\alpha)/100}$$

$$= o(r_1(\Omega))\ .$$

Therefore, the second integral in (29) agrees modulo $o(\Omega^{-1})$ with

$$\int_{r_1^{99/100}}^{r_1-r_1^{2/3}} \left(\frac{144}{r^2}-\Omega^2\right)^{-1/2} \frac{dr}{r}$$

$$= \int_{r_1^{99/100}}^{r_1-r_1^{2/3}} \left(\frac{144}{r^2}\right)^{-1/2} \left(1-\frac{\Omega^2\,r^2}{144}\right)^{-1/2} \frac{dr}{r}$$

$$= \Omega^{-1} \int_{\Omega r_1^{99/100}/12}^{\Omega(r_1-r_1^{2/3})/12} (1-y^2)^{-1/2}\,dy$$

and therefore

$$\lim_{\Omega\to 0} \Omega \int_{\varepsilon}^{r_1(\Omega)} \left(u(r)-\Omega^2\right)^{-1/2} = \int_0^1 (1-y^2)^{-1/2}\,dy = \frac{\pi}{2}\ ,$$

which proves the lemma.

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A remark on gradients of harmonic functions

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Abstract. In any $C^{1,s}$ domain, there is nonzero harmonic function C^1 continuous up to the boundary such that the function and its gradient on the boundary vanish on a set of positive measure.

1. Introduction.

In this note, we will extend Bourgain and Wolff's result in [2] into the general $C^{1,s}$ domain of $\mathbb{R}^d$.

Theorem. *If D is a $C^{1,s}$ domain in $\mathbb{R}^d$ with $s > 0$ and $d \geq 3$, then there is a harmonic function $u : D \to \mathbb{R}$ which is C^1 up to the boundary and such that*

$$|\{Q \in \partial D : \ u(Q) = 0, \ \nabla u(Q) = 0\}| > 0 \, .$$

The idea for the proof of this theorem follows from the argument in [2]. We also need to use Aleksandrov-Kargaev function (see [1]) as our basic constructing factor. Since there is no reason to apply directly Alexandrov's result to the arbitrary domain, we have to work on the "almost flat" domain first and then get our final result by Kelvin transformation.

NOTATION. Let A and M be positive large numbers. Let Φ_M be a collection of $C^{1,s}$ domains in $\mathbb{R}^d$ which are of the form

$$\Omega = \{X = (x, x_d) \in \mathbb{R}^d : \ x_d > \varphi(x), \ x \in \mathbb{R}^{d-1}\}$$

such that φ is some $C^{1,s}$ function on $\mathbb{R}^{d-1}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = 0$ and

$$\|\nabla\varphi\|_\infty + \|\nabla\varphi\|_{C^s} \le e^{-M} \qquad \text{and} \qquad \nabla^2\varphi(x) = 0$$

when $|X| > 1$.

When $X = (x, x_d) \in \partial\Omega$, we denote by N_X the normal vector of $\partial\Omega$ at X, and n_x the normal vector of $\mathbb{R}^{d-1}$ at $(x, 0)$. By the assumption of Ω, we know that $|N_X - n_x| \le e^{-M}$ for any $X = (x, x_d) \in \partial\Omega$. We use notation $\nabla_T u$ to denote the tangent gradient of u on $\partial\Omega$. Finally, we usual use B to denote the ball in $\mathbb{R}^d$ and Q to denote the cube on $\mathbb{R}^{d-1}$. If Q is the cube on $\mathbb{R}^{d-1}$, then φQ denotes its image on $\partial\Omega$ by φ. C always denotes an absolute constant.

2. Several lemmas.

Lemma 1. *Suppose φ is a $C^{1,s}$ function on $\mathbb{R}^{d-1}$ and $\Omega = \{X = (x, x_d) \in \mathbb{R}^d : x_d > \varphi(x), x \in \mathbb{R}^{d-1}\}$ is a $C^{1,s}$ domain. Let $G(X, Y)$ be the Green's function of Ω. Then for any X, $Y \in \partial\Omega$, we have*

$$\left|\frac{d}{dN_X}\frac{d}{dN_Y} G(X, Y)\right| \le \frac{C}{|X - Y|^d} \,. \tag{1}$$

PROOF. When Ω is a bounded domain, the result is known (see [4], [6]) but we do not find a good reference for the proof. When Ω is unbounded, it is not true in general. So we would like to give a proof for such special case and one will see the proof still works for bounded domains with a tiny correction.

Claim. *Let $X \in \partial\Omega$ and $R > 0$. If u is a harmonic function in $\Omega \cap B(X, R)$, $|u| \le 1$ on $\Omega \cap \partial B(X, R)$ and $u = 0$ on $\partial\Omega \cap B(X, R)$, then $|\nabla u(Z)| \le C/R$ for all $Z \in \Omega \cap B(X, R/2)$.*

PROOF OF CLAIM. We may assume that $X = 0$. Let $D = \Omega \cap B(0,R)$ and $D_- = \Omega^c \cap B(0,R)$. Consider a map $\Phi : D_- \to D$ defined by $\Phi(z, z_d) = (z, 2\,\varphi(z) - z_d)$. Then the function $u \circ \Phi$ solves

$$\operatorname{div} A_0 \nabla (u \circ \Phi) = 0 \qquad \text{in } \; D_- \,,$$

where

$$A_0 = ((\Phi')^{\perp})^{-1} (\Phi')^{-1} = \begin{pmatrix} I & 2\,(\nabla\varphi)^{\perp} \\ 2\,\nabla\varphi & 1 + 4\,|\nabla\varphi|^2 \end{pmatrix}$$

is an elliptic matrix (see *e.g.* [3]). Let $\omega = u$ in D, $\omega = -u \circ \Phi$ in D_-. Then with $A = I$ in D and A_0 in D_-, ω solves $\operatorname{div} A\nabla\omega = 0$ in $B(0,R)$ in the weak sense, because the function ω is an odd 'reflection' of u. By the assumptions for the functions u and φ, we know $|\omega| \le 1$ and $\|A\|_{C^s} \le C\,e^{-M_1}$. If we define functions $v(Y) = \omega(RY)$ and $B(Y) = \min\{R^{-s}, 1\}\, A(RY)$ in $B(1)$, then v solves $\operatorname{div} B\nabla v = 0$ in $B(1)$ and B has uniform C^s bound. So by [4, Lemma 3.1], we have $|\nabla v(Y)| \le C$. Hence, $|\nabla \omega(Z)| \le C/R$. This proves the claim.

Now let us fix $X, Y \in \Omega$ and let $R = |X - Y|$. If we apply the claim to the function $R^{d-2} G(Z_1, Z_2)$ for $Z_2 \in B(Y, R/100)$ with $Z_1 \in B(X, R/100)$ fixed, then we have $|\nabla_{Z_2} G(Z_1, Z_2)| \le C/R^{d-1}$ for all $Z_1 \in B(X, R/100)$ and $Z_2 \in B(Y, R/100)$. Similarly, if we apply the claim to the function $R^{d-1} \nabla_{Z_2} G(Z_1, Z_2)$ for $Z_1 \in B(X, R/100)$ with $Z_2 \in B(Y, R/100)$ fixed, then we get $|\nabla_{Z_1} \nabla_{Z_2} G(Z_1, Z_2)| \le C/R^d$ for all $Z_1 \in B(X, R/100)$ and $Z_2 \in B(Y, R/100)$. Finally let Z_1 go to X and let Z_2 go to Y, to conclude.

Lemma 2. *Let Ω be as in Lemma 1 with $0 \in \partial\Omega$. Suppose u is a harmonic function in Ω which is $C^{1,s}$ up to the boundary. Assume the restriction function of u onto $\partial\Omega$ is supported in $\partial\Omega \cap B(0,1)$. Then for all $X \in \partial\Omega$ with $|X| \le 2$,*

$$\Big|\frac{d}{dN_X}\, u(X)\Big| \le C\, \|u\|_{C^{1,s}(\partial\Omega)} \,. \tag{2}$$

When $X \in \partial\Omega$ with $|X| \ge 2$, we have

$$\Big|\frac{d}{dN_X}\, u(X)\Big| \le C\, |X|^{-d}\, \|u\|_{C(\partial\Omega)} \,.$$

PROOF. The first result is a well known fact when Ω is either a bounded domain or the upper half space. Here we would like to give a short proof without using layer potential theory. Since $u(Y)$ has compact support on $\partial\Omega$, for $X \in \Omega$,

$$u(X) = \int_{\partial\Omega} \frac{d}{dN_Y} G(X,Y)\, u(Y)\, d\sigma(Y)\,.$$

If $X \in \partial\Omega$ with $|X| \geq 2$, then by (1) we have

$$\begin{aligned}\Big|\frac{d}{dN_X} u(X)\Big| &= \Big|\int_{\partial\Omega} \frac{d^2}{dN_X dZ_Y} G(X,Y)\, u(Y)\, d\sigma(Y)\Big| \\ &\leq C\, \|u\|_{C(\partial\Omega)} \int_{Y\in\partial\Omega,\ |Y|\leq 1} \frac{1}{|X-Y|^d}\, d\sigma(Y) \\ &\leq C\, |X|^{-d}\, \|u\|_{C(\partial\Omega)}\,.\end{aligned}$$

This proves (3). Now let us fix $X \in \partial\Omega$ with $|X| \leq 2$. Choose a function $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\phi(Z) = 1$ when $|Z| \leq 10$ and $\phi(Z) = 0$ when $|Z| \geq 20$. Let us write

$$u_\phi(Y) = u(Y) - u(X)\, \phi(Y-X) - \langle \nabla_T u(X), Y-X\rangle\, \phi(Y-X)\,,$$

which is supported in $\partial\Omega \cap \{|Y| \leq 15\}$ and bounded by

$$\|\nabla_T u\|_{C^s(\partial\Omega)}\, |Y-X|^{1+s}\,, \qquad \text{when } \ |Y-X| \leq 8\,.$$

So we have

$$\begin{aligned}u(Z) &= \int_{\partial\Omega} \frac{d}{dN_Y} G(Z,Y)\, u_\phi(Y)\, d\sigma(Y) \\ &\quad + u(X) \int_{\partial\Omega} \frac{d}{dN_Y} G(Z,Y)\, \phi(Y-X)\, d\sigma(Y) \\ &\quad + \int_{\partial\Omega} \frac{d}{dN_Y} G(Z,Y)\, \langle \nabla_T u(X), Y-X\rangle\, \phi(Y-X)\, d\sigma(Y) \\ &= u_1(Z) + u_2(Z) + u_3(Z)\,.\end{aligned}$$

For u_1,

$$\begin{aligned}\Big|\frac{d}{dN_X}\,u_1(X)\Big| &= \Big|\int_{\partial\Omega}\frac{d^2G(X,Y)}{dN_X dN_Y}\,u_\phi(Y)\,d\sigma(Y)\Big|\\ &\le C\int\limits_{X\in\partial\Omega,\,|X-Y|<8}\frac{|X-Y|^{1+s}}{|X-Y|^d}\,\|\nabla_T u\|_{C^s(\partial\Omega)}\,d\sigma(Y)\\ &\quad + C\int\limits_{X\in\partial\Omega,\,|X-Y|\ge 8}\frac{1}{|X-Y|^d}\,\|u\|_{C^1(\partial\Omega)}\\ &\le C\,\|u\|_{C^{1,s}(\partial\Omega)}\,.\end{aligned}$$

Notice that u_2 is a bounded harmonic function whose boundary value is 1 on $\partial\Omega\cap B(0,4)$. So apply the claim in Lemma 1 to the function $u(X)-u_2(\cdot)$, we have $|\nabla u_2(Z)|\le C\,\|u\|_\infty$ for $Z\in\Omega\cap B(0,2)$ so that

$$\Big|\frac{d}{dN_X}\,u_2(X)\Big|\le C\,\|u\|_{C^{1,s}(\partial\Omega)}\,.$$

Finally consider the harmonic function $u_3(Z)-\langle\nabla_T u(X), Z-X\rangle$ which is bounded in $\Omega\cap B(0,4)$ and whose boundary value is zero on $\partial\Omega\cap B(0,4)$. So again by using the claim in Lemma 1, we have

$$\Big|\frac{d}{dN_X}\,u_3(X)\Big|\le C\,\|u\|_{C^1(\partial\Omega)}\le C\,\|u\|_{C^{1,s}(\partial\Omega)}\,,$$

since $\langle\nabla_T u(X), Z-X\rangle$ is linear. This proves Lemma 2.

Now let a and ε be two positive numbers. Let

$$E^a_\varepsilon(X) = -a\,\frac{\varepsilon + x_d/a}{|X/a+\varepsilon\,e_d|^d}\,.$$

We denote by n_x the normal vector of $\mathbb{R}^{d-1}$ in $\mathbb{R}^d$ at $x\in\mathbb{R}^{d-1}$.

Lemma 3. *We have the following properties for $E^a_\varepsilon(X)$:*

$$\Big|\nabla^i E^a_\varepsilon(X)\Big|\le C\,a^{-i+1}\min\Big\{\varepsilon^{-d-i+1},\Big|\frac{X}{a}\Big|^{-d-i+1}\Big\},\tag{4}$$

for all $X \in \Omega$ *and* $i = 0, 1, 2$.

$$\begin{aligned}
(5) \qquad & \Big| \int_{\varphi Q(0,b)} \Big(\Big|1 + \frac{d}{dN} E_\varepsilon^a(X)\Big|^p - 1\Big)\, d\sigma(X) \\
& - \int_{Q(0,b)} \Big(\Big|1 + \frac{d}{dn} E_\varepsilon^a(x)\Big|^p - 1\Big)\, dx \Big| \\
& \qquad \le C\, e^{-pM}\, M^{d-1}\, a^{d-1}
\end{aligned}$$

if $\varepsilon \le e^{-M}$.

$$(6) \qquad \int_{\varphi Q(0,b)} \Big(\Big|1 + \frac{d}{dN} E_\varepsilon^a(X)\Big|^p - 1\Big)\, d\sigma(X) \le -\Big(\eta - \frac{C}{M}\Big)\, a^{d-1},$$

if $p > 0$ *is small but independent of* ε, M, a. *Here* $b = aM$ *in* (5) *and* (6), *and* η *is an absolute small positive number.*

PROOF. (4) follows directly from the calculation. After a change of variable, the left hand side of (5) is the integral over a subset $\{|x| < b\}$ of $\mathbb{R}^{d-1}$ of the following integrand

$$(|1+\langle \nabla E_\varepsilon^a(X), N_X\rangle|^p - 1)\,(1+|\nabla\varphi(x)|^2)^{1/2} - (|1+\langle \nabla E_\varepsilon^a(x), n_x\rangle|^p - 1)\,.$$

If we introduce a term $-|1+\langle \nabla E_\varepsilon^a(x), n_x\rangle|^p(1+|\nabla\varphi(x)|^2)^{1/2}$ and subtract it, then the integrand becomes

$$\begin{aligned}
& (|1+\langle \nabla E_\varepsilon^a(X), N_X\rangle|^p - |1+\langle \nabla E_\varepsilon^a(x), n_x\rangle|^p)\,(1+|\nabla\varphi(x)|^2)^{1/2} \\
& \quad + |1+\langle \nabla E_\varepsilon^a(x), n_x\rangle|^p\,(1+|\nabla\varphi(x)|^2)^{1/2} - 1) \\
& \quad - (1+|\nabla\varphi(x)|^2)^{1/2} - 1) \\
& = \mathrm{I} + \mathrm{II} + \mathrm{III}\,.
\end{aligned}$$

Up to multiplying a constant, I is bounded by

$$\begin{aligned}
|\langle \nabla E_\varepsilon^a(X), N_X\rangle - \langle \nabla E_\varepsilon^a(x), n_x\rangle|^p & \\
\le |\langle \nabla E_\varepsilon^a(X) - \nabla E_\varepsilon^a(x), n_x\rangle|^p &+ |\langle \nabla E_\varepsilon^a(X), N_X - n_x\rangle|^p \\
= \mathrm{I}_1 + \mathrm{I}_2\,. &
\end{aligned}$$

For I_1, since $X = (x, \varphi(x))$, by (4),

$$\mathrm{I}_1 = |\langle \nabla E_\varepsilon^a(X) - \nabla E_\varepsilon^a(x), n_x\rangle|$$

$$
\begin{aligned}
&\le a^{-2+1}\left|\frac{x}{a}\right|^{-d-2+1}|X-x|\\
&\le a^{-1}\left|\frac{x}{a}\right|^{-d-1}|\varphi(x)|\\
&\le e^{-M}\left|\frac{x}{a}\right|^{-d}.
\end{aligned}
$$

So

$$\int_{|x|\le b}|\mathrm{I}_1|\,dx \le M^{d(1-p)-1}\,e^{-pM}a^{d-1}\,.$$

Since $|N_X-n_x|\le C\,\|\nabla\varphi\|_\infty \le C\,e^{-M}$, again by (4),

$$\int_{|x|\le b}|\mathrm{I}_2|\,dx \le C\,e^{-pM}\int_{|x|\le b}\left|\frac{x}{a}\right|^{-dp}dx = C\,M^{d(1-p)-1}\,e^{-pM}\,a^{d-1}\,.$$

By a similar method, using (4) and the assumption of φ,

$$
\begin{aligned}
\int_{|x|\le b}|\mathrm{II}|\,dx &\le C\int_{|x|\le b}\Big(1+\left|\frac{x}{a}\right|^{-d}\Big)^p e^{-M}\,dx\\
&\le C\,a^{d-1}\,e^{-M}\,M^{d(1-p)-1}\,.
\end{aligned}
$$

And it is trivial to get

$$\int_{|x|\le b}|\mathrm{III}|\,dx \le C\,e^{-M}\int_{|x|\le b}dx \le C\,e^{-M}\,a^{d-1}\,M^{d-1}\,.$$

So combining those estimates of the integrations for $\mathrm{I}=\mathrm{I}_1+\mathrm{I}_2$, II and III, we have that the left hand side of (4) is bounded by

$$
\begin{aligned}
&C\,a^{d-1}\,(M^{d(1-p)-1}\,e^{-pM}+M^{d(1-p)-1}\,e^{-pM}+M^{d-1}\,e^{-M})\\
&\qquad\qquad \le C\,a^{d-1}\,M^{d-1}\,e^{-pM}\,.
\end{aligned}
$$

Let us now prove (6). From [1] or [2], we know that if $p>0$ is small enough then for all small ε,

$$\int_{\mathbb{R}^{d-1}}\Big(\Big|1+\frac{d}{dn}\,E^a_\varepsilon(x)\Big|^p-1\Big)dx \le -\eta\,a^{d-1}\,,$$

with a small absolute positive number η. From this integral, we easily get

$$\int_{|x|<b}\Big(\Big|1+\frac{d}{dn}\,E^a_\varepsilon(x)\Big|^p-1\Big)\,dx$$

$$
\begin{aligned}
&\le -\eta\, a^{d-1} + \int_{|x|>b} \Big|\Big|1 + \frac{d}{dn}\, E^a_\varepsilon(x)\Big|^p - 1\Big|\, dx \\
(7) \qquad &\le -\eta\, a^{d-1} + C \int_{|x|>b} \Big|\frac{x}{a}\Big|^{-d-1}\, dx \\
&\le -(\eta - C\, M^{-2})\, a^{d-1}
\end{aligned}
$$

by (4). So combine (7) and (5). We have that the left hand side of (6) is less than or equal to

$$-(\eta - C\, M^{-2})\, a^{d-1} + C\, e^{-pM}\, M^{d-1}\, a^{d-1} - \Big(\eta - \frac{C}{M}\Big)\, a^{d-1}$$

when M is large enough independently of a and ε.

We state our main lemma.

Lemma 4. *If $p > 0$ and $1/M > 0$ are small enough, then for any $\Omega \in \Phi_M$ and $\varepsilon > 0$ and any cube $Q = Q(0, l)$ on $\mathbb{R}^{d-1}$ with $l < 1$ there exists a harmonic function F^Q_ε which is C^1 up to $\bar\Omega$ with* $\operatorname{supp} F^Q_\varepsilon|_{\partial\Omega} \subset \varphi B(0, \varepsilon l)$ *such that*

$$\int_{\varphi Q} \Big|1 + \frac{d}{dN}\, F^Q_\varepsilon(X)\Big|^p\, d\sigma(X) \le e^{-2\beta p}\, |\varphi Q|\,, \tag{8}$$

$$\big|\nabla F^Q_\varepsilon(X)\big| \le C\, \min\Big\{\varepsilon^{-d},\ e^{-(d-1/2)M}\, \Big(\Big|\frac{X}{l}\Big|^{-d} + \Big|\frac{X}{l}\Big|^{-d+1/2}\Big)\Big\}, \tag{9}$$

if $X \in \partial\Omega$, where $\beta > 0$ is such that

$$e^{-2\beta p} = 1 - \frac{\eta}{2}\, e^{-(d-1)M}\,.$$

Proof. Let $a = e^{-M} l$ and $E^a_\varepsilon(X)$ be as above. Define

$$I_j = \{x \in \mathbb{R}^{d-1}\ :\ 2^{j-1} l \le |x| < 2^{j+1} l\}$$

and ρ_j a cut-off function in $\mathbb{R}^d$ with $\operatorname{supp} \rho_j \subset Q(0, 2^{j+1} l) \setminus Q(0, 2^{j-1} l)$ and $|\nabla^i \rho_j| \le 1/(2^j l)^i$, for $i = 0, 1, 2$ and for $j = 0, 1, \dots$ such that $\sum_j \rho_j(X) = 1$ for $|X| > 1$. Let $\rho = 1 - \sum \rho_j(X)$. Denote by F^Q_ε the harmonic extension into Ω of $\rho\, E^a_\varepsilon(X)|_{\partial\Omega}$ and $Q^a_{j,\varepsilon}$ the harmonic extension into Ω of $\rho_j\, E^a_\varepsilon(X)|_{\partial\Omega}$. Let $Q^a_\varepsilon = \sum_j Q^a_{j,\varepsilon}$. Then $E^a_\varepsilon = F^a_\varepsilon + Q^a_\varepsilon$.

An easy computation and (4) of Lemma 3 imply that

$$\begin{aligned} \|Q^a_{j,\varepsilon}\|_\infty &\le C\, a^d\, (2^j l)^{-d+1}\,, \\ \|\nabla_T Q^a_{j,\varepsilon}\|_\infty &\le C\, a^d\, (2^j l)^{-d}\,, \\ \|\nabla_T Q^a_{j,\varepsilon}\|_{C^s} &\le C\, a^d\, (2^j l)^{-d-s}\,. \end{aligned} \tag{10}$$

If we let $u(Y) = Q^a_{j,\varepsilon}(2^j l Y)$, then the restriction function of u onto $\partial\Omega$ is supported in $\partial\Omega \cap B(0,1)$ and $\|u\|_{C^{1,s}} \le C\, a^d\, (2^j l)^{-d+1}$ by the above estimates. So apply Lemma 2 to u. When $X \in \partial\Omega$ with $|X| \le 2^{j+2} l$, we get

$$\Big|\frac{d}{dN_X}\, Q^a_{j,\varepsilon}(X)\Big| \le C\, a^d\, (2^j l)^{-d}\,.$$

When $X \in \partial\Omega$ with $|X| > 2^{j+2} l$,

$$\Big|\frac{d}{dN_X}\, Q^a_{j,\varepsilon}(X)\Big| \le C\, a^d\, |X|^{-d} \le C\, a^d\, |X|^{-d+1/2}\, (2^j l)^{-1/2}\,.$$

Hence

$$\begin{aligned} \Big|\frac{d}{dN}\, Q^a_\varepsilon(X)\Big| &\le \Big(\sum_{|X|<2^{j+2}l} + \sum_{|X|>2^{j+2}l}\Big) \Big|\frac{d}{dN}\, Q^a_{j,\varepsilon}(X)\Big| \\ &\le C \sum_{|X|<2^{j+2}l} a^d\, (2^j l)^{-d} + C \sum_{|X|>2^{j+2}l} a^d\, (2^j l)^{-1/2} |X|^{-d+1/2} \\ &\le C\, a^d\, |X|^{-d} + C\, a^d\, |X|^{-d+1/2}\, l^{-1/2} \\ &\le C\Big(\Big|\frac{X}{a}\Big|^{-d} + \Big|\frac{X}{a}\Big|^{-d+1/2}\Big)\,, \end{aligned} \tag{11}$$

since $a = e^{-M} l$. We notice that if $|X| \le 4\, l$, the process of estimates above also give

$$\Big|\frac{d}{dN}\, Q^a_\varepsilon(X)\Big| \le C\, e^{-M}\,. \tag{12}$$

Now we estimate (7). Let $b = Ma$.

$$
\begin{aligned}
\int_{\varphi Q} \Big(\Big|1 + \frac{d}{dN} F_\varepsilon^Q(X)|^p - 1\Big)\, d\sigma(X) \\
&= \int_{\varphi Q(0,b)} \Big(\Big|1 + \frac{d}{dN}\,(E_\varepsilon^a(X) - Q_\varepsilon^a(X))\Big|^p - 1\Big)\, d\sigma(X) \\
&\quad + \int_{\varphi Q \setminus \varphi Q(0,b)} \Big(\Big|1 + \frac{d}{dN} F_\varepsilon^Q(X)\Big|^p - 1\Big)\, d\sigma(X) \\
&\le \int_{\varphi Q(0,b)} \Big(\Big|1 + \frac{d}{dN} E_\varepsilon^a(X)\Big|^p - 1\Big)\, d\sigma(X) \\
&\quad + \int_{\varphi Q(0,b)} \Big(\Big|1 + \frac{d}{dN} Q_\varepsilon^a(X)\Big|^p - 1\Big)\, d\sigma(X) \\
&\quad + \int_{\varphi Q \setminus \varphi Q(0,b)} \Big(\Big|1 + \frac{d}{dN} F_\varepsilon^Q(X)\Big|^p - 1\Big)\, d\sigma(X)
\end{aligned}
\tag{13}
$$

by triangle inequality. When $|x| > b$, the integrand in the last integral is bounded by $C\,|x/a|^{-d/2}$, by (4) and (11). So

$$
\int_{\varphi Q \setminus \varphi Q(0,b)} \Big(\Big|1 + \frac{d}{dN} F_\varepsilon^Q(X)\Big|^p - 1\Big)\, d\sigma(X) \le C\, a^{d-1}\, M^{-1/2}
\tag{14}
$$

Since $p < 1$, by Hölder inequality,

$$
\begin{aligned}
\int_{\varphi Q(0,b)} \Big|\frac{d}{dN} Q_\varepsilon^a(X)\Big|^p\, d\sigma \\
&\le \sum_j \int_{\varphi Q(0,b)} \Big|\frac{d}{dN} Q_{j,\varepsilon}^a(X)\Big|^p\, d\sigma \\
&\le C\, b^{(1-p/2)(d-1)} \sum_j \Big(\int_{\varphi Q(0,b)} \Big|\frac{d}{dN} Q_{j,\varepsilon}^a(X)\Big|^2\, d\sigma\Big)^{p/2}.
\end{aligned}
$$

Notice that $\varphi\Omega$ is a Lipschitz graph with uniform bound. Theorem 2.2.6 of [5] shows that

$$
\int_{\varphi\Omega} \Big|\frac{d}{dN} Q_{j,\varepsilon}^a\Big|^2\, d\sigma \le C \int_{\varphi\Omega} \Big|\nabla_T Q_{j,\varepsilon}^a\Big|^2\, d\sigma\,.
$$

So by (10),

$$
\begin{aligned}
\int_{\varphi Q(0,b)} \Big|\frac{d}{dN}\, Q^a_\varepsilon(X)\Big|^p\, d\sigma & \\
&\le C\, b^{(1-p/2)(d-1)} \sum_j \Big(\int_{\varphi\Omega} \Big|\nabla_T Q^a_{j,\varepsilon}\Big|^2\, d\sigma\Big)^{p/2} \\
&\le C\, b^{(1-p/2)(d-1)} \sum_j \Big(\int_{\varphi I_j} |a^d\,(2^j l)^{-d}|^2\, d\sigma\Big)^{p/2} \\
&\le C\, a^{dp}\, b^{(1-p/2)(d-1)} \sum_j (2^j l)^{-dp+p(d-1)/2} \\
&\le C\, a^{d-1}\,(al)^{p(d+1)/2}\, M^{(1-p/2)(d-1)} \\
&= C\, a^{d-1}\, e^{-pM(d+1)/2}\, M^{(1-p/2)(d-1)}\,.
\end{aligned}
\tag{15}
$$

Now apply (6) to the first term in the left hand side of (13) and combine (13), (14), and (15). The left hand side of (13) is less than or equal to

$$
\begin{aligned}
-\Big(\eta-\frac{C}{M}\Big)\, a^{d-1} + C\, a^{d-1}\, M^{-1/2} + C\, a^{d-1}\, e^{-pM(d+1)/2}\, M^{(1-p/2)(d-1)} & \\
\le -\frac{\eta}{2}\, a^{d-1} = -\frac{\eta}{2}\, e^{-M(d-1)}\, |\varphi Q|\,. &
\end{aligned}
$$

Hence

$$
\int_{\varphi Q} \Big|1+\frac{d}{dN}\, F^Q_\varepsilon(X)\Big|^p\, d\sigma \le \Big(1-\frac{\eta}{2}\, e^{-M(d-1)}\Big)\, |\varphi Q| = e^{-2\beta p}\, |\varphi Q|\,,
$$

with some $\beta > 0$.

Now we turn to prove (8). First we have

$$
\begin{aligned}
|\nabla F^a_\varepsilon(X)| &\le |\nabla Q^a_\varepsilon(X)| + |\nabla E^a_\varepsilon(X)| \\
&\le C\,\Big(\Big|\frac{X}{a}\Big|^{-d} + \Big|\frac{X}{a}\Big|^{-d+1/2}\Big) \\
&\le C\, e^{-M(d-1/2)}\,\Big(\Big|\frac{X}{l}\Big|^{-d} + \Big|\frac{X}{l}\Big|^{-d+1/2}\Big)
\end{aligned}
$$

by (4) and (11). In order to bound $|\nabla F^a_\varepsilon(X)|$ by $C\,\varepsilon^{-d}$, we notice that when $|X| < l\,\varepsilon < l$,

$$
\Big|\frac{d}{dN}\, Q^a_\varepsilon(X)\Big| \le C
$$

is bounded by (12) and of course is less than $C\varepsilon^{-d}$. The remaining part follows directly from calculation and (4).

Corollary 5. *Let p, M and ε be as in Lemma 4. Let $\Omega \in \Phi_{2M}$. For any $X_Q = (c_Q, \varphi(c_Q)) \in \partial\Omega$ and $s_0 > 0$, $Q \subset B(c_Q, s_0)$ is a cube in $\mathbb{R}^{d-1}$. For any function $I : \partial\Omega \to \mathbb{R}$ with $\operatorname{supp} I \subset \varphi\Omega$ such that*

$$|I(X_Q)^{-1}|\,\|I(\cdot) - I(X_Q)\|_\infty \le \frac{1}{8}\,\eta\, e^{-M(d-1)},$$

there exists a harmonic function F_ε^Q in Ω which is C^1 up to $\bar{\Omega}$ such that $\operatorname{supp} F_\varepsilon^Q \subset \varphi B(c_Q, \varepsilon\, l(Q))$ and

$$\int_{\varphi Q} \left| I(X) + I(X_Q)\,\frac{dF_\varepsilon^Q}{dN}(X)\right|^p d\sigma(X) \le e^{-2\beta p}\,|I(X_Q)|^p\,|\varphi Q|, \tag{16}$$

$$\begin{aligned} &|\nabla F_\varepsilon^Q(X)| \\ &\le C \min\left\{\varepsilon^{-d}, e^{-M(d-1/2)}\left(\left|\frac{X - X_Q}{l(Q)}\right|^{-d} + \left|\frac{X - X_Q}{l(Q)}\right|^{-d+1/2}\right)\right\}. \end{aligned} \tag{17}$$

PROOF. Under a new coordinate system such that X_Q is the new origin and the tangent space of $\partial\Omega$ at X_Q is the new $\mathbb{R}^{d-1}$, $\Omega \in \Phi_M$. Then this corollary follows directly from Lemma 4.

Let us first prove a weaker version of our theorem.

Theorem 1. *Let Ω be as in the Corollary 5. Then for any small number $s_0 > 0$, there is a harmonic function $u : \Omega \to \mathbb{R}$ which is C^1 up to $\bar{\Omega}$ and such that*

$$|\{X \in \partial\Omega :\ |X| \le s_0,\ u(X) = 0,\ \nabla u(X) = 0\}| > 0.$$

We will give the recursive construction. Let $Q_0 = Q_0(0, s_0) \subset \mathbb{R}^{d-1}$. Let $\{\delta_n\}_1^\infty$ be a sequence of small numbers such that $\delta_n^{-1} \in \mathbb{Z}$ which are chosen by induction later. Let $\{K_n\}$ and $\{\varepsilon_n\}$ be two sequences of numbers which are decided later with $K_n \nearrow +\infty$ and $\varepsilon \searrow 0$. a will be a large universal number also decided later.

Let $\Re_n$ be the collection of $\delta_n^{-(d-1)}$ cubes of side δ_n in $\mathbb{R}^{d-1}$ whose union is Q_0. Let $\Im_n$ be a subset of $\Re_n$ such that

$$\Big(\int_{\varphi V_n} \Big| \frac{du_n}{dN} \Big|^p \, d\sigma(X) \Big)^{1/p} \le A \, e^{-\beta n} \, , \tag{18}$$

where $V_n = \bigcup \{Q : Q \in \Im_n\}$ and u_n is the $C^{1,s}$ function in Ω defined by induction later. When $n = 1$, let $\delta_0 = s_0$ and $\Im_0 = \{Q_0\}$ and u_0 a harmonic function in Ω such that $u_0|_{\partial\Omega} \in C_0^{1,s}(\varphi Q_0/1000)$. Suppose we have $\Im_n$ and u_n. Let δ_{n+1} with $\delta_{n+1}/\delta_n \in \mathbb{Z}$ be such that δ_{n+1} small enough and decided in the following lemmas. Let $\Im_{n+1} \subset \Re_{n+1}$ be such that $Q \in \Im_{n+1}$ satisfies

$$Q' \in \Im_n \text{ for all } Q' \in \Re_n \text{ with } Q \subset Q' , \tag{19}$$

$$\Big(\frac{1}{|\varphi Q|} \int_{\varphi Q} \Big| \frac{du_n}{dN} \Big|^p \, d\sigma(X) \Big)^{1/p} \le K_{n+1} \, e^{-\beta n} \, . \tag{20}$$

Now let us define

$$u_{n+1}(X) = u_n(X) + \sum_{Q \in \Im_{n+1}} \frac{du_n}{dN}(X_Q) \, F^Q_{\varepsilon_{n+1}}(X) \, ,$$

where $F^Q_{\varepsilon_{n+1}}$ is as in Corollary 5.

Lemma 6. *For $X \in \partial\Omega$,*

$$\sum_{\substack{Q \in \Im_{n+1} \\ |x - c_Q| > \rho}} |\nabla F^Q_{\varepsilon_{n+1}}(X)| \le C \, e^{-M(d-1/2)} \Big(\frac{\delta_{n+1}}{\rho} + \Big(\frac{\delta_{n+1}}{\rho} \Big)^{1/2} \Big) , \tag{21}$$

if $\rho > C \, \delta_{n+1}$.

$$|\nabla u_{n+1}(X) - \nabla u_n(X)| \le C \, K_{n+1} \, e^{-\beta n} \, \varepsilon_{n+1}^{-d} \, , \tag{22}$$

if δ_{n+1} small enough.

PROOF. By (17) of Corollary 5, the left hand side of (21) is bounded by

$$\sum_{|x - c|_Q > \rho} e^{-M(d-1/2)} \Big(\Big| \frac{X - c_Q}{\delta_{n+1}} \Big|^{-d} + \Big| \frac{X - c_Q}{\delta_{n+1}} \Big|^{-d+1/2} \Big)$$

$$\le C \, e^{-M(d-1/2)} \Big(\frac{\delta_{n+1}}{\rho} + \Big(\frac{\delta_{n+1}}{\rho} \Big)^{1/2} \Big) ,$$

if $\rho > C\,\delta_{n+1}$. Let us prove (22) now. Since du_n/dN is continuous, after making δ_{n+1} small (20) will imply that

$$\Big|\frac{du_n}{dN}(X_{Q'})\Big| \le 2\,K_{n+1}\,e^{-\beta n}\,,$$

for all $Q' \in \Im_{n+1}$. For $X \in \partial\Omega$, there is at most one $Q \in \Im_{n+1}$ such that for any other $Q' \neq Q$, $Q' \in \Im_{n+1}$, $|x - c_{Q'}| > \rho$. Then the left hand side of (15) is bounded by

$$\sum_{\substack{Q'\in\Im_{n+1}\\ Q'\neq Q}} \Big|\frac{du_n}{dN}(X_{Q'})\Big|\,|\nabla F^{Q'}_{\varepsilon_{n+1}}(X)| + \Big|\frac{du_n}{dN}(X_Q)\Big|\,|\nabla F^{Q}_{\varepsilon_{n+1}}(X)|$$

$$\le 2\,K_{n+1}\,e^{-\beta n}\,(C + C\varepsilon_{n+1}^{-d})$$

$$\le C\,K_{n+1}\,e^{-\beta n}\,\varepsilon_{n+1}^{-d}\,,$$

where we used (21) and (17) in the first inequality.

The following lemma says our process in construction of u_n is possible, *i.e.* (18) is true.

Lemma 7. *There exists a large universal constant A such that*

$$\Big(\int_{\varphi V_{n+1}} \Big|\frac{du_{n+1}}{dN}\Big|^p\, d\sigma\Big)^{1/p} \le A\,e^{-\beta(n+1)}\,.$$

PROOF. As in [2], we first state a claim as follows. One may find the proof in [2].

Claim. *If δ_{n+1} is small enough, then for all $X \in \varphi Q$, $Q \in \Im_{n+1}$,*

$$\sum_{\substack{Q'\in\Im_{n+1}\\ Q'\neq Q}} \Big|\frac{du_n}{dN}(X_{Q'})\Big|\,|\nabla F^{Q'}_{\varepsilon_{n+1}}(X)|$$

$$\le C\,e^{-M(d-1/2)}\,\Big|\frac{du_n}{dN}(X)\Big| + e^{-M(d-1/2)}\,e^{-4\beta(n+1)}\,.$$

We would like to point out the idea. Divide the sum into two parts: $\sum_{\delta_{n+1}<|X_Q-X|<L\delta_{n+1}}$ and $\sum_{|X_Q-X|\ge L\delta_{n+1}}$. For the first term, use (21)

and notice that if δ_{n+1} is small then the numbers $du_n(X_{Q'})/dN$ are close to $du_n(X)/dN$ up to an error term. For the second term, when n fixed, $du_n(X_{Q'})/dN$ are bounded uniformly in $Q' \neq Q$. For the remaining part, apply (21) again and let L be big.

Now let us take

$$I = \Big\{ Q \in \Im_{n+1} : \ \Big| \frac{du_n}{dN}(X_Q) \Big| > e^{-4\beta(n+1)} \Big\}$$

and

$$\Pi = \Im_{n+1} \setminus I \, .$$

Let $X \in \varphi Q$ for some $Q \in \Im_{n+1}$. Denote

$$J(X) = \frac{du_n}{dN}(X) + \sum_{\substack{Q' \in \Im_{n+1} \\ Q' \neq Q}} \frac{du_n}{dN}(X_{Q'}) \, \frac{d}{dN} \, F^{Q'}_{\varepsilon_{n+1}}(X) \, .$$

Then

$$\frac{du_{n+1}}{dN}(X) = J(X) + \frac{du_n}{dN}(X_Q) \, \frac{d}{dN} \, F^{Q}_{\varepsilon_{n+1}}(X)$$

by the definition of u_{n+1}. If $Q \in I$, then after making δ_{n+1} small,

$$\begin{aligned} \Big| J(X) - \frac{du_n}{dN}(X_Q) \Big| &\le C \, e^{-M(d-1/2)} \, \Big| \frac{du_n}{dN}(X_Q) \Big| \\ &\le \frac{1}{8} \, \eta \, e^{-M(d-1)} \, \Big| \frac{du_n}{dN}(X_Q) \Big| \end{aligned}$$

by claim and by letting M large. So apply Corollary 5,

$$\begin{aligned} \int_{\varphi Q} \Big| \frac{du_{n+1}}{dN}(X) \Big|^p \, d\sigma(X) &\le e^{-2\beta p} \, |\varphi Q| \Big| \frac{du_n}{dN}(X_Q) \Big|^p \\ &\le e^{-3\beta p/2} \int_{\varphi Q} \Big| \frac{du_n}{dN} \Big|^p \, d\sigma(X) \, , \end{aligned} \tag{23}$$

if δ_{n+1} small again. When $Q \in \Pi$, write

$$J = \frac{du_n}{dN}(X_Q) + J_1 \, , \qquad \text{where} \ \ |J_1| \le C \, e^{-M(d-1/2)} \, e^{-4\beta(n+1)}$$

by claim. So apply Corollary 5 again,

$$
\begin{aligned}
&\int_{\varphi Q} \left|\frac{du_{n+1}}{dN}(X)\right|^p d\sigma(X) \\
&\qquad \le \left(e^{-2p\beta}\, e^{-4\beta(n+1)p} + C\, e^{-4\beta(n+1)p}\right) |\varphi Q| \\
&\qquad \le C\, e^{-4\beta(n+1)p}\, |\varphi Q|\,.
\end{aligned} \tag{24}
$$

Combine (23) and (24). By induction, we have

$$
\begin{aligned}
\int_{\varphi V_{n+1}} \left|\frac{du_{n+1}}{dN}\right|^p &\le e^{-3p\beta/2} \int_{\substack{\varphi Q \\ Q\in V_{n+1}\cap I}} \left|\frac{du_n}{dN}\right|^p + C\, e^{-4\beta(n+1)p} \sum_{Q\in V_{n+1}\cap \Pi} |\varphi Q| \\
&\le A^p \left(e^{-p\beta/2} + C\, A^{-p}\, e^{-3p(n+1)\beta}\right) e^{-p\beta(n+1)} \\
&\le A^p\, e^{p\beta(n+1)}\,,
\end{aligned}
$$

if A is large and independent of n.

Proof of Theorem 1. Choose K_n and ε_n such that

$$\sum \varepsilon_{n+1}^{-d} K_{n+1}\, e^{-\beta n} \le C_0 < +\infty$$

and

$$\sum K_{n+1}^{-p} + \varepsilon_{n+1}^{d-1} \le \frac{s_0^{d-1}}{1000\, C}\,.$$

Then the (21) of Lemma 6 implies that

$$\sum |\nabla u_{n+1} - \nabla u_n| \le C \sum K_{n+1}\, \varepsilon_{n+1}^{-d}\, e^{-\beta n} \le C\, C_0 < +\infty$$

i.e. $u = \sum u_n$ is C^1 up to Ω. On the other hand, by the definition of u and u_{n+1},

$$
\begin{aligned}
|\{X\in \varphi Q_0 :\ u(X)\ne 0\}| &\le \sum_n |\{X\in \varphi Q_0 :\ u_{n+1}(X)\ne u_n(X)\}| \\
&\le \sum_n \sum_{Q\in \Im_{n+1}} |\varphi(B(c_Q), \varepsilon_{n+1}\, l(Q))| \\
&\le \sum_n \delta_{n+1}^{-(d-1)}\, (\delta_{n+1}\, \varepsilon_{n+1})^{d-1} \\
&= C \sum \varepsilon_{n+1}^{d-1} \\
&\le \frac{s_0^{d-1}}{1000}\,.
\end{aligned}
$$

In order to estimate the gradient term, we notice that if a point $X \in \varphi(Q_0)$ is in some φQ_n's with infinite many n, then we know by (20) and the continuity of du/dN that $du(X)/dN = 0$. So again by (20) and (18),

$$\begin{aligned}\Big|\Big\{X \in \varphi Q_0 : \frac{du}{dN}(X) \neq 0\Big\}\Big| &\leq \sum_n |\varphi V_n \setminus \varphi V_{n+1}| \\ &= \sum_n \sum_{Q \in \varphi V_{n+1} \setminus \varphi V_n} |\varphi Q| \\ &\leq \sum_n K_{n+1}^{-p}\, e^{-\beta n p} \sum_{Q \in \varphi V_{n+1} \setminus \varphi V_n} \int_{\varphi Q} \Big|\frac{u_n}{dN}\Big|^p \\ &\leq \sum_n K_{n+1}^{-p}\, e^{-p\beta n} \int_{\varphi V_n} \Big|\frac{du_n}{dN}\Big|^p \\ &\leq \sum A^p\, K_{n+1}^{-p} \\ &\leq \frac{s_0^{d-1}}{1000}\,,\end{aligned}$$

by Lemma 7, this is because when $Q \in \Re_{n+1} \backslash \Im_{n+1}$, there exists $Q' \in \Im_n$ such that $Q' \supset Q$ but

$$\int_{\varphi Q} \Big|\frac{du_n}{dN}\Big|^p > K_{n+1}^{p}\, e^{-p\beta n}$$

by definition. And so

$$|\varphi Q| < K_{n+1}^{-p}\, e^{-p\beta n} \int_{\varphi Q} \Big|\frac{du_n}{dN}\Big|^p .$$

Finally we get

$$|\{X \in \varphi Q_0 : \; u(X) = 0,\, \nabla u(X) = 0\}| \geq \frac{1}{100}\, |\varphi Q_0| > 0\,.$$

3. Proof of Theorem.

Case 1: D is a bounded domain. Then we can assume $(0,0) \in \partial D$ and $\mathbb{R}^{d-1}$ is the tangent space of ∂D at $(0,0)$ and $D \subset \mathbb{R}^d_+$. So we may construct a domain $\Omega = \Omega_\varphi = \{(x, x_d) : x_d > \varphi(x)\}$ where $\varphi \in \Phi_{2M}$ and $D \subset \Omega$ and $\partial D \cap \partial\Omega \supset \varphi Q_0$ for some $Q_0 = Q_0(0, s_0)$ with $s_0 > 0$. Then apply Theorem 1 to Ω, we get a harmonic function U in Ω which is C^1 up to $\bar{\Omega}$ such that

$$|\{X \in \varphi Q_0 : \; U(X) = 0\,,\ \nabla U(X) = 0\}| > 0\,.$$

Now let $u = U|_D$, then this is the desired u.

Case 2: $D = \mathbb{R}^d \setminus B$ for some bounded $C^{1,s}$ domain B. We assume that $(0,0) \in B$. Let $T : X \to X/|X|^2$ be the Kelvin transformation and $\tilde{B} = \mathbb{R}^d|_{TB}$. Apply the result in Case 1 to $\tilde{B}$ and get a harmonic function U. Then $u(X) = U(TX)/|X|^{d-2}$ is the desired function for our domain D.

Case 3: D is a general $C^{1,s}$ domain. It is easy to find a domain $B \subset \bar{D}^c$ such that $\partial B \cap D$ contains some "ball" on ∂D. Then by the Case 2, there is a harmonic function U in $\mathbb{R}^d \setminus B$ which is C^1 up to ∂B and $|\{X \in \text{"ball"} : U = \nabla U = 0\}| > 0$. Then this $u(X) = U|_D$ is needed.

Some Remarks. 1. The theorems are true for C^1-Dini domains. The proof follows our argumens with minor corrections.

2. It is not hard to see that for every ε there exists a harmonic function which is C^1 up to the boundary and such that

$$|\partial D \setminus \{X \in \partial D : \; u(X) \neq 0 \text{ or } \nabla u(X) \neq 0\}| \leq \varepsilon\,.$$

3. We do not know if the theorem is true or not for Lipschitz domains. In fact, our method does not work even for C^1 domains (and even if we do not need the restriction $u = 0$).

4. We may also prove Lemma 4 by using the potential layer theory as in [2]. But again this method does not work even for C^1 domains.

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Uniqueness of positive solutions of nonlinear second order systems

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Abstract. In this paper we discuss the uniqueness of positive solutions of the nonlinear second order system $-u'' = g(v)$, $-v'' = f(u)$ in $(-R, R)$, $u(\pm R) = v(\pm R) = 0$ where f and g satisfy some appropriate conditions. Our result applies, in particular, to $g(v) = v$, $f(u) = u^p$, $p > 1$, or $f(u) = \lambda u + a_1 u^{p_1} + \cdots + a_k u^{p_k}$ with $p_j > 1$, $a_j > 0$ for $j = 1, \ldots, k$ and $0 \le \lambda < \mu_1^2$ where $\mu_1 = \pi^2/4R^2$.

Introduction.

In this paper we discuss the uniqueness of positive solutions $(u, v) \in (C^2[-R, R])^2$ of the nonlinear second order system with homogeneous Dirichlet data

$$(1.1) \qquad \begin{cases} -u''(t) = g(v(t)) \,, & -R < t < R \,, \\ -v''(t) = f(u(t)) \,, & -R < t < R \,, \\ u(\pm R) = v(\pm R) = 0 \,, \end{cases}$$

where $R > 0$ is fixed and the functions $f, g \in C^1(\mathbb{R})$ satisfy the following assumptions

$$(\mathrm{H}_1) \qquad 0 < g(v) \le v\, g'(v) \,, \qquad \text{for } \; v > 0 \,,$$

$$(\mathrm{H}_2) \qquad 0 < f(u) < u\, f'(u) \,, \qquad \text{for } \; u > 0 \,.$$

Of course $(u,v)>0$ means that $u>0$ and $v>0$ on $(-R,R)$.

It was proved by Troy [6] that u and v are symmetric about the origin and that $u'<0$ and $v'<0$ on $(0,R)$. It should be noted that in our situation the proof is considerably simpler. Moreover, by the Hopf boundary lemma [5, p. 4] here we also have $u'(R)<0$ and $v'(R)<0$. Therefore positive solutions of (1.1) can be treated as positive solutions of

$$\begin{cases} -u''(t)=g(v(t)), & 0\le t<R, \\ -v''(t)=f(u(t)), & 0\le t<R, \\ u(R)=v(R)=u'(0)=v'(0)=0. \end{cases} \tag{1.2}$$

The existence of positive solutions of nonlinear elliptic systems was examined by Clément, De Figueiredo and Mitidieri [1] in a bounded convex domain of $\mathbb{R}^n$ when $n\ge 2$ and by Peletier and Van Der Vorst [4] in a ball of $\mathbb{R}^n$ when $n\ge 4$. The question of the existence of positive solutions of problem (1.2) will be discussed in the last section of this paper.

Our main result is the following theorem.

Theorem 1.1. *Let $f,g\in C^1(\mathbb{R})$ satisfy* (H_1) *and* (H_2). *Let $(u,v)\in (C^2[-R,R])^2$ be a positive solution of problem* (1.1). *Then (u,v) is symmetric about the origin and is unique in the class of all positive solutions in $(C^2[-R,R])^2$.*

As a particular case of Theorem 1.1 we can state the following corollary concerning fourth order equations.

Corollary 1.1. *Let $f\in C^1(\mathbb{R})$ satisfy* (H_2). *Let $u\in C^4[-R,R]$ be a positive solution of*

$$\begin{cases} u^{(4)}(t)=f(u(t)), & -R<t<R, \\ u(\pm R)=u''(\pm R)=0. \end{cases} \tag{1.3}$$

Then u is symmetric about the origin and is unique in the class of all positive solutions in $C^4[-R,R]$.

In our proofs we shall make an intensive use of the one dimensional maximum principle and the related Hopf boundary lemma [5], which we recall:

Theorem A ([5, p. 2]). *Suppose $u \in C^2(a,b) \cap C[a,b]$ satisfies the differential inequality*

$$u'' + g(x)\, u' \geq 0\,, \qquad \text{for } a < x < b\,,$$

with g a bounded function. If $u \leq M$ in (a,b) and if the maximum M of u is attained at an interior point of (a,b), then $u \equiv M$.

Theorem B ([5, p. 4]). *Suppose $u \in C^2(a,b) \cap C^1[a,b]$ is a nonconstant function which satisfies the differential inequality $u'' + g(x)\, u' \geq 0$ in (a,b) and suppose g is bounded on every closed subinterval of (a,b). If the maximum of u occurs at $x = a$ and g is bounded below at $x = a$, then $u'(a) < 0$. If the maximum occurs at $x = b$ and g is bounded above at $x = b$, then $u'(b) > 0$.*

The outline of the paper is as follows. In Section 2 we introduce an initial value problem and we establish some preliminary results. Theorem 1.1 will be obtained as an immediate consequence of a crucial result that we state and prove in Section 3 (Theorem 3.1). Finally in Section 4 we prove an existence result and we give some examples to illustrate our theorem.

2. Preliminary results.

In order to prove our theorem we introduce the initial value problem

$$(2.1) \qquad \begin{cases} -u''(t) = g(v(t))\,, & t \geq 0\,, \\ -v''(t) = f(u(t))\,, & t \geq 0\,, \\ u(0) = \alpha\,,\ u'(0) = 0\,, & \\ v(0) = \beta\,,\ v'(0) = 0\,, & \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ are parameters. Throughout this section the functions $f, g \in C^1(\mathbb{R})$ are only assumed to be nondecreasing on $[0, +\infty)$ and such that $f(0) = g(0) = 0$, $f(s), g(s) > 0$ for $s > 0$ and $\lim_{s \to +\infty} g(s) = +\infty$.

Below we prove some propositions which will be needed to state and prove our crucial result: Theorem 3.1. In the following proposition we establish the local existence and uniqueness of solutions of problem (2.1).

Proposition 2.1. *For any* $\alpha > 0$, $\beta > 0$ *there exists* $T > 0$ *such that problem* (2.1) *on* $[0, T]$ *has a unique solution* $(u, v) \in (C^2[0, T])^2$.

PROOF. Let $\alpha > 0$ and $\beta > 0$ be given. Choose $T > 0$ such that

$$T^2 g(\beta) \le \alpha \qquad \text{and} \qquad T^2 f(\alpha) \le \beta$$

and consider the set of functions

$$Z = \Big\{(u, v) \in (C[0, T])^2 : \ \frac{\alpha}{2} \le u(t) \le \alpha \ \text{ and } \ \frac{\beta}{2} \le v(t) \le \beta \quad \text{for all } \ t \in [0, T]\Big\}.$$

Clearly, Z is a bounded closed convex subset of the Banach space $(C[0, T])^2$ endowed with the norm $\|(u, v)\| = \max\{\|u\|_\infty, \|v\|_\infty\}$. Define

$$L(u, v)(t) = \Big(\alpha - \int_0^t (t - s)\, g(v(s))\, ds\,,\ \beta - \int_0^t (t - s)\, f(u(s))\, ds\Big)$$

for $t \in [0, T]$ and $(u, v) \in Z$. It is easily verified that L is a compact operator mapping Z into itself, and so there exists $(u, v) \in Z$ such that $(u, v) = L(u, v)$ by the Schauder fixed point theorem. Clearly $(u, v) \in (C^2[0, T])^2$ and (u, v) is a solution of (2.1) on $[0, T]$. Since f and g are of class C^1 the uniqueness follows.

In view of Proposition 2.1, for any $\alpha, \beta > 0$ problem (2.1) has a unique local solution: let $[0, T_{\alpha,\beta})$ denote the maximum interval of existence of that solution ($T_{\alpha,\beta} = +\infty$, possibly). Define

$$P_{\alpha,\beta} = \{t \in (0, T_{\alpha,\beta}) : \ u(\alpha, \beta, s)\, v(\alpha, \beta, s) > 0\,, \ \text{for all } \ s \in [0, t]\}$$

where $(u(\alpha, \beta, \cdot\,), v(\alpha, \beta, \cdot\,))$ is the solution of problem (2.1) in $[0, T_{\alpha,\beta})$. Clearly $P_{\alpha,\beta} \ne \varnothing$.

Proposition 2.2. *For any* $\alpha, \beta > 0$ *we have*

$$t_{\alpha,\beta} = \sup P_{\alpha,\beta} < T_{\alpha,\beta}\ .$$

PROOF. If not, there exist $\alpha > 0$ and $\beta > 0$ such that $\sup P_{\alpha,\beta} = T_{\alpha,\beta}$. Suppose first that $T_{\alpha,\beta} < +\infty$. Noting $u = u(\alpha,\beta,\cdot)$ and $v = v(\alpha,\beta,\cdot)$ we have

$$0 \le u \le \alpha \quad \text{on} \quad [0, T_{\alpha,\beta}),$$
$$0 \le v \le \beta \quad \text{on} \quad [0, T_{\alpha,\beta}).$$

Since

$$u'(t) = -\int_0^t g(v(s))\,ds \quad \text{and} \quad v'(t) = -\int_0^t f(u(s))\,ds \tag{2.2}$$

for $t \in [0, T_{\alpha,\beta})$, we conclude that u, v, u' and v' are bounded on $[0, T_{\alpha,\beta})$ and we get a contradiction with the definition of $T_{\alpha,\beta}$. Now assume that $T_{\alpha,\beta} = +\infty$. Since $u'' < 0$ on $[0, +\infty)$ we deduce that

$$u'(t) \le u'(1) < 0, \qquad \text{for all} \ \ t \ge 1$$

from which we get

$$u(t) \le u(1) + u'(1)\,(t-1), \qquad \text{for all } t \ge 1.$$

Thus we can find $t \ge 1$ such that $u(t) < 0$ and we obtain a contradiction.

Proposition 2.3. *For any $\alpha > 0$ there exists a unique $\beta > 0$ such that $u(\alpha,\beta,t_{\alpha,\beta}) = v(\alpha,\beta,t_{\alpha,\beta}) = 0$.*

PROOF. We first prove the uniqueness. Let $\alpha > 0$ be fixed. Suppose that there exist $\beta > \gamma > 0$ such that $u(\alpha,\beta,t_{\alpha,\beta}) = v(\alpha,\beta,t_{\alpha,\beta}) = u(\alpha,\gamma,t_{\alpha,\gamma}) = v(\alpha,\gamma,t_{\alpha,\gamma}) = 0$. In order to simplify our notations, we denote $u(\alpha,\beta,t)$, $v(\alpha,\beta,t)$, $u(\alpha,\gamma,t)$ and $v(\alpha,\gamma,t)$ by $u(t)$, $v(t)$, $w(t)$ and $z(t)$. Define $b = \min\{t_{\alpha,\beta}, t_{\alpha,\gamma}\}$. Suppose that there exists $a \in (0,b]$ such that $v - z > 0$ on $[0,a)$ and $(v-z)(a) = 0$. Since $u'' - w'' = g(z) - g(v)$ and g is nondecreasing on $[0,+\infty)$, we deduce that $u'' - w'' \le 0$ on $[0,a]$. Using the fact that $(u-w)(0) = (u-w)'(0) = 0$, Theorems A and B imply that $u - w \le 0$ on $[0,a]$. Thus $v'' - z'' = f(w) - f(u) \ge 0$ on $[0,a]$ since f is nondecreasing on $[0,+\infty)$. We have $(v-z)(0) > 0$, $(v-z)'(0) = 0$ and $(v-z)(a) = 0$. Therefore Theorems A and B give a contradiction. Thus $v - z > 0$ on $[0,b]$. As before we show that $u - w \le 0$ on $[0,b]$. Since we have

$$(v-z)(b) = \begin{cases} v(t_{\alpha,\gamma}) > 0, & \text{if } t_{\alpha,\beta} > t_{\alpha,\gamma}, \\ 0, & \text{if } t_{\alpha,\beta} = t_{\alpha,\gamma}, \\ -z(t_{\alpha,\beta}) < 0, & \text{if } t_{\alpha,\beta} < t_{\alpha,\gamma}, \end{cases}$$

necessarily $b = t_{\alpha,\gamma} < t_{\alpha,\beta}$. Now $(u - w)(b) = u(t_{\alpha,\gamma}) > 0$ and we get a contradiction. The case $0 < \beta < \gamma$ can be handled in the same way.

Now we prove the existence. Suppose that there exists $\alpha > 0$ such that for any $\beta > 0$ $u(\alpha, \beta, t_{\alpha,\beta}) > 0$ or $v(\alpha, \beta, t_{\alpha,\beta}) > 0$. Since α is fixed we shall write u_β , v_β , t_β and T_β instead of $u(\alpha, \beta, \cdot)$, $v(\alpha, \beta, \cdot)$, $t_{\alpha,\beta}$ and $T_{\alpha,\beta}$. Define the following two sets

$$B = \{\beta > 0 :\ u_\beta(t_\beta) = 0 \ \text{ and } \ v_\beta(t_\beta) > 0\}\,,$$

$$C = \{\beta > 0 :\ u_\beta(t_\beta) > 0 \ \text{ and } \ v_\beta(t_\beta) = 0\}\,.$$

Then we have

$$(0, +\infty) = B \cup C\,. \tag{2.3}$$

The proof of the proposition is completed by using the next lemma which contradicts (2.3).

Lemma 2.1. $B = C = \varnothing$.

The proof follows readily from (2.3) and the next two lemmas.

Lemma 2.2.

i) *Suppose* $B \neq \varnothing$. *Then there exists* $m > 0$ *such that* $m \leq \inf B$.

ii) *Suppose* $C \neq \varnothing$. *Then there exists* $M > 0$ *such that* $M \geq \sup C$.

Lemma 2.3. B *and* C *are open.*

PROOF OF LEMMA 2.2. We have

$$u_\beta(t) = \alpha - \int_0^t (t - s)\, g(v_\beta(s))\, ds\,, \qquad 0 \leq t < T_\beta\,, \tag{2.4}$$

and

$$v_\beta(t) = \beta - \int_0^t (t - s)\, f(u_\beta(s))\, ds\,, \qquad 0 \leq t < T_\beta\,. \tag{2.5}$$

i) Let $\beta \in B$. (2.2) and (2.4) imply

$$t_\beta \geq \Big(\frac{2\alpha}{g(\beta)}\Big)^{1/2} \tag{2.6}$$

and from (2.5) we get

$$\beta > \int_0^{t_\beta} (t_\beta - s)\, f(u_\beta(s))\, ds\,. \tag{2.7}$$

Suppose that $\inf B = 0$ and let (β_j) be a sequence in B decreasing to zero. Then $t_{\beta_j} \to +\infty$ by (2.6). From (2.7) we deduce that

$$\beta_j \geq \int_0^1 (t_{\beta_j} - s)\, f(u_{\beta_j}(s))\, ds \tag{2.8}$$

for j large. Using (2.2) and (2.4) we have

$$u_{\beta_j}(t) \geq \alpha - \frac{g(\beta_j)}{2} \geq \frac{\alpha}{2} \tag{2.9}$$

for $t \in [0,1]$ and j large. From (2.8) and (2.9) we get

$$\beta_j \geq c$$

for j large where $c > 0$ is independent of j. This gives a contradiction.

ii) Suppose that $\sup C = +\infty$ and let (β_j) be a sequence in C increasing to $+\infty$. By virtue of (2.2) we have

$$0 < u_{\beta_j}(t) \leq \alpha\,, \qquad \text{for } \ t \in [0, t_{\beta_j}]\,. \tag{2.10}$$

(2.5) and (2.10) imply that $t_{\beta_j} \to +\infty$ as $j \to +\infty$. Then we can assume that $t_{\beta_j} \geq 1$ for all j and that

$$f(\alpha) \leq \beta_j\,, \qquad \text{for all } \ j\,. \tag{2.11}$$

(2.2), (2.5), (2.10) and (2.11) imply

$$\frac{\beta_j}{2} \leq v_{\beta_j}(t) \leq \beta_j\,, \qquad \text{for } \ t \in [0,1]\,,$$

and using (2.4) we deduce that $u_{\beta_j}(1) \leq \alpha - g(\beta_j/2)/2$. But then $u_{\beta_j}(1) < 0$ for j large contradicting (2.10).

The proof of Lemma 2.3 depends on the following two lemmas.

Lemma 2.4.

i) *Suppose that* $B \neq \varnothing$. *Then for any* $\beta \in B$ *we have* $u'_\beta < 0$ *on* $(0, t_\beta]$ *and* $v'_\beta < 0$ *on* $(0, t_\beta]$. *If in addition* $T_\beta < +\infty$, *then for any* $\gamma > \alpha$ (*respectively,* $\delta > \beta$) *there exists* $t \in (t_\beta, T_\beta)$ (*respectively,* $s \in (t_\beta, T_\beta)$) *such that* $|u_\beta(t)| = \gamma$ *and* $|u_\beta(r)| \leq \gamma$ *for* $r \in [0, t]$ (*respectively,* $|v_\beta(s)| = \delta$ *and* $|v_\beta(r)| \leq \delta$ *for* $r \in [0, s]$).

ii) *Suppose that* $C \neq \varnothing$. *Then for any* $\beta \in C$ *we have* $u'_\beta < 0$ *on* $(0, t_\beta]$ *and* $v'_\beta < 0$ *on* $(0, t_\beta]$. *If in addition* $T_\beta < +\infty$, *then for any* $\gamma > \alpha$ (*respectively,* $\delta > \beta$) *there exists* $t \in (t_\beta, T_\beta)$ (*respectively,* $s \in (t_\beta, T_\beta)$) *such that* $|u_\beta(t)| = \gamma$ *and* $|u_\beta(r)| \leq \gamma$ *for* $r \in [0, t]$ (*respectively,* $|v_\beta(s)| = \delta$ *and* $|v_\beta(r)| \leq \delta$ *for* $r \in [0, s]$).

Lemma 2.5. *Suppose that* $B \neq \varnothing$ *and* $C \neq \varnothing$. *Then for any* $\beta > 0$ *there exists* $\eta > 0$ *such that* $\min\{T_\beta, T_\gamma\} > \max\{t_\beta, t_\gamma\}$ *for any* $\gamma \in (\beta - \eta, \beta + \eta)$.

Proof of Lemma 2.4. The first part of i) is clear. Now assume that $T_\beta < +\infty$. If u_β (respectively, v_β) is bounded on $[0, T_\beta)$, then (2.5) (respectively, (2.4)) and (2.2) imply that v_β (respectively, u_β), u'_β and v'_β are also bounded on $[0, T_\beta)$ contradicting the definition of T_β. Thus u_β and v_β can not be bounded on $[0, T_\beta)$ and the last part of i) follows easily. ii) can be proved similarly.

Proof of Lemma 2.5. Let β be a fixed positive number. (2.3) implies that $\beta \in B \cup C$. Let $\gamma > 0$. In the same way $\gamma \in B \cup C$. From (2.4), (2.5) using Gronwall's inequality we obtain

$$\begin{aligned} &\max\left\{|u_\beta(t) - u_\gamma(t)|,\, |v_\beta(t) - v_\gamma(t)|\right\} \\ &\qquad \leq |\beta - \gamma| \left(1 + \int_0^t h(s)\, \exp\Big(\int_s^t r\, h(r)\, dr\Big)\, ds\right) \end{aligned} \tag{2.12}$$

for $t \in [0, \min\{T_\beta, T_\gamma\})$, where the function h is given by

$$\begin{aligned} h(t) = \max\Big\{ &\sup_{0 \leq \rho \leq 1} |f'(\rho\, u_\beta(t) + (1 - \rho)\, u_\gamma(t))|, \\ &\sup_{0 \leq \zeta \leq 1} |g'(\zeta\, v_\beta(t) + (1 - \zeta)\, v_\gamma(t))| \Big\} \end{aligned} \tag{2.13}$$

for $t \in [0, \min\{T_\beta, T_\gamma\})$. Suppose that $\max\{t_\beta, t_\gamma\} \geq \min\{T_\beta, T_\gamma\}$. Then, by Proposition 2.2, $\min\{T_\beta, T_\gamma\} < +\infty$. If $\min\{T_\beta, T_\gamma\} = T_\beta$, then necessarily $\max\{t_\beta, t_\gamma\} = t_\gamma$. By Lemma 2.4 there exists $t \in (t_\beta, T_\beta)$ such that $|u_\beta(t)| = 2\,\alpha$ and $|u_\beta(s)| \le 2\,\alpha$ for $s \in [0,t]$. Since $0 \le u_\gamma(s) \le \alpha$ and $0 \le v_\gamma(s) \le \gamma$ for $s \in [0,t]$ by Lemma 2.4, (2.12) and (2.13) imply

$$\alpha \le |u_\beta(t) - u_\gamma(t)| \le c\,|\beta - \gamma| \tag{2.14}$$

where $c > 0$ depends on α, β, γ and $t \in (t_\beta, T_\beta)$; clearly c is bounded with respect to γ when γ is in a bounded set. If $\min\{T_\beta, T_\gamma\} = T_\gamma$, then necessarily $\max\{t_\beta, t_\gamma\} = t_\beta$ and the proof is the same but now $t \in (t_\gamma, T_\gamma)$. Since in this case $T_\gamma \le t_\beta$ we can choose in (2.14) the same c as before. The lemma follows.

Proof of Lemma 2.3. 1) Suppose that B is not open. (2.3) implies that there exists $\beta \in B$ and a sequence $\{\beta_j\}$ in C such that $\beta_j \to \beta$ and $t_{\beta_j} \to T \in [0, +\infty]$. By Lemma 2.5 we can assume that $\min\{T_\beta, T_{\beta_j}\} > \max\{t_\beta, t_{\beta_j}\}$ for all j and so $T \le T_\beta$. We first show that $T < +\infty$. If not, we can assume that $t_{\beta_j} \ge t_\beta$ for all j by Proposition 2.2. Let $t \in [0, t_\beta]$. Using Lemma 2.4 we get

$$\int_0^t (t-s)\, g(v_{\beta_j}(s))\, ds \le \frac{g(\beta_j)\, t^2}{2}\,.$$

Choose $t \in (0, t_\beta]$ such that $g(\beta_j)\, t^2/2 \le \alpha/2$ for all j. Then using again Lemma 2.4 and the fact that

$$u_{\beta_j}(t) = \alpha - \int_0^t (t-s)\, g(v_{\beta_j}(s))\, ds$$

we obtain $u_{\beta_j}(s) \ge \alpha/2$ for $s \in [0,t]$ and for all j. Since

$$\begin{aligned}\beta_j &= \int_0^{t_{\beta_j}} (t_{\beta_j} - s)\, f(u_{\beta_j}(s))\, ds \\ &\ge \int_0^t (t_{\beta_j} - s)\, f(u_{\beta_j}(s))\, ds \\ &\ge f\Big(\frac{\alpha}{2}\Big)\, t\, \Big(t_{\beta_j} - \frac{t}{2}\Big)\end{aligned}$$

for all j we reach a contradiction. Now suppose that $T < T_\beta$. Then from (2.12), (2.13) and Lemma 2.4 we get

$$|v_\beta(t_{\beta_j}) - v_{\beta_j}(t_{\beta_j})| \le c\,|\beta_j - \beta|\,, \qquad \text{for all } j\,, \tag{2.15}$$

where c is a positive constant independent of j. Since $v_{\beta_j}(t_{\beta_j}) = 0$ for all j, (2.15) implies that $v_\beta(T) = 0$. Therefore $T > t_\beta$. We can assume that $t_{\beta_j} \geq (T + t_\beta)/2$ for all j. Let $t \in [t_\beta, T)$. Again we can assume that $t_{\beta_j} \geq t$ for all j. By Lemma 2.4 we have for all j

$$0 \leq u_{\beta_j}(s) \leq \alpha\,, \qquad \text{for} \;\; s \in [0,t]\,,$$

and

$$0 \leq v_{\beta_j}(s) \leq \beta_j\,, \qquad \text{for} \;\; s \in [0,t]\,.$$

Then (2.12) and (2.13) give for $s \in [0,t]$

$$|u_\beta(s) - u_{\beta_j}(s)| \leq c\,|\beta_j - \beta|\,, \qquad \text{for all } j\,, \tag{2.16}$$

where c is a positive constant independent of j. Let $s = t_\beta$ in (2.16), we get

$$u_{\beta_j}(t_\beta) \to 0 \qquad \text{when} \;\; j \to +\infty\,.$$

Since $u_{\beta_j}(t) \leq u_{\beta_j}(t_\beta)$ we obtain

$$u_{\beta_j}(t) \to 0 \qquad \text{when} \;\; j \to +\infty\,. \tag{2.17}$$

From (2.16) with $s = t$ and (2.17) we deduce that $u_\beta(t) = 0$. Since $t \in [t_\beta, T)$ is arbitrary we obtain a contradiction by using Lemma 2.4. Thus $T = T_\beta$. Then necessarily $T_\beta < +\infty$. By Lemma 2.4 we can find $s \in (t_\beta, T_\beta)$ such that $|v_\beta(s)| = 2\,\beta$ and $|v_\beta(r)| \leq 2\,\beta$ for $r \in [0,s]$. We can assume that $t_{\beta_j} \geq s$ and $\beta/2 < \beta_j < 3\beta/2$ for all j. Then from (2.12), (2.13) and Lemma 2.4 we obtain

$$\frac{\beta}{2} \leq |v_\beta(s) - v_{\beta_j}(s)| \leq c\,|\beta_j - \beta|\,, \qquad \text{for all } j\,,$$

where c is a positive constant independent of j. Clearly this is impossible.

2) Suppose that C is not open. (2.3) implies that there exists $\beta \in C$ and a sequence (β_j) in B such that $\beta_j \to \beta$ and $t_{\beta_j} \to T \in [0, +\infty]$. By Lemma 2.5 we can assume that $\min\{T_\beta, T_{\beta_j}\} > \max\{t_\beta, t_{\beta_j}\}$ for all j and so $T \leq T_\beta$. As in 1) we can show that $T < +\infty$. Now suppose that $T < T_\beta$. Then from (2.12), (2.13) and Lemma 2.4 we get

$$|u_\beta(t_{\beta_j}) - u_{\beta_j}(t_{\beta_j})| \leq c\,|\beta_j - \beta|\,, \qquad \text{for all } j\,, \tag{2.18}$$

where c is a positive constant independent of j. Since $u_{\beta_j}(t_{\beta_j}) = 0$ for all j, (2.18) implies that $u_\beta(T) = 0$. Therefore $T > t_\beta$. We can assume that $t_{\beta_j} \geq (T + t_\beta)/2$ for all j. Let $t \in [t_\beta, T)$. Again we can assume that $t_{\beta_j} \geq t$ for all j. By Lemma 2.4 we have for all j

$$0 \leq u_{\beta_j}(s) \leq \alpha\,, \qquad \text{for } \ s \in [0,t]\,,$$

and

$$0 \leq v_{\beta_j}(s) \leq \beta_j\,, \qquad \text{for } \ s \in [0,t]\,.$$

Then (2.12) and (2.13) give for $s \in [0,t]$

$$|v_\beta(s) - v_{\beta_j}(s)| \leq c\,|\beta_j - \beta|\,, \qquad \text{for all } j\,, \tag{2.19}$$

where c is a positive constant independent of j. Let $s = t_\beta$ in (2.19), we get

$$v_{\beta_j}(t_\beta) \to 0 \qquad \text{when } \ j \to +\infty\,.$$

Since $v_{\beta_j}(t) \leq v_{\beta_j}(t_\beta)$ we obtain

$$v_{\beta_j}(t) \to 0 \qquad \text{when } \ j \to +\infty\,. \tag{2.20}$$

From (2.19) with $s = t$ and (2.20) we deduce that $v_\beta(t) = 0$. Since $t \in [t_\beta, T)$ is arbitrary we obtain a contradiction by using Lemma 2.4. Thus $T = T_\beta$. Then necessarily $T_\beta < +\infty$. By Lemma 2.4 we can find $t \in (t_\beta, T_\beta)$ such that $|u_\beta(t)| = 2\,\alpha$ and $|u_\beta(r)| \leq 2\,\alpha$ for $r \in [0,t]$. We can assume that $t_{\beta_j} \geq t$ for all j. Then from (2.12), (2.13) and Lemma 2.4 we obtain

$$\alpha \leq |u_\beta(t) - u_{\beta_j}(t)| \leq c\,|\beta_j - \beta|\,, \qquad \text{for all } j\,,$$

where c is a positive constant independent of j. Clearly this is impossible. The proof of the lemma is complete.

Now we introduce

$$F(t) = \int_0^t f(s)\,ds \qquad \text{and} \qquad G(t) = \int_0^t g(s)\,ds\,.$$

The following lemma will be needed in the next section.

Lemma 2.6. *For any $\alpha > 0$, $\beta > 0$ we have*

$$u'(\alpha,\beta,t)\,v'(\alpha,\beta,t) + F(u(\alpha,\beta,t)) + G(v(\alpha,\beta,t)) = F(\alpha) + G(\beta) \tag{2.21}$$

for $t \in [0, T_{\alpha,\beta})$.

The proof is obvious.

3. Proof of Theorem 1.1.

We keep the notations introduced in Section 2. Clearly Theorem 1.1 is an immediate consequence of the following result.

Theorem 3.1. *Let* $f, g \in C^1(\mathbb{R})$ *satisfy* (H_1) *and* (H_2). *Then for any* $\alpha > 0$ *there exists a unique* $(\beta(\alpha), t(\alpha)) \in (0, +\infty) \times (0, +\infty)$ *such that* $u(\alpha, \beta(\alpha), t(\alpha)) = v(\alpha, \beta(\alpha), t(\alpha)) = 0$ *and* $u(\alpha, \beta(\alpha), t) > 0$, $v(\alpha, \beta(\alpha), t) > 0$ *for* $t \in [0, t(\alpha))$. *Moreover* $\beta(\alpha)$ *is a strictly increasing function of* α *and* $t(\alpha)$ *is a strictly decreasing function of* α.

Proof. Let $\alpha > 0$ be fixed. Since f and g verify the hypotheses used in Section 2 the existence and uniqueness of $(\beta(\alpha), t(\alpha))$ satisfying the conditions of the theorem are given by Proposition 2.3. Unfortunately the proof of the last part of the theorem is rather long. For $\alpha > 0$, $\beta > 0$ define

$$\varphi(\alpha, \beta, t) = \frac{\partial u}{\partial \alpha}(\alpha, \beta, t), \qquad \psi(\alpha, \beta, t) = \frac{\partial v}{\partial \alpha}(\alpha, \beta, t),$$

and

$$\rho(\alpha, \beta, t) = \frac{\partial u}{\partial \beta}(\alpha, \beta, t), \qquad \chi(\alpha, \beta, t) = \frac{\partial v}{\partial \beta}(\alpha, \beta, t),$$

for $t \in [0, T_{\alpha,\beta})$. Then φ, ψ, ρ and χ satisfy the linearized equations

$$(3.1) \qquad \begin{cases} -\varphi''(t) = g'(v(t))\,\psi(t), & 0 \le t < T_{\alpha,\beta}, \\ -\psi''(t) = f'(u(t))\,\varphi(t), & 0 \le t < T_{\alpha,\beta}, \\ \varphi(0) = 1, \ \psi(0) = \varphi'(0) = \psi'(0) = 0, \end{cases}$$

and

$$(3.2) \qquad \begin{cases} -\rho''(t) = g'(v(t))\,\chi(t), & 0 \le t < T_{\alpha,\beta}, \\ -\chi''(t) = f'(u(t))\,\rho(t), & 0 \le t < T_{\alpha,\beta}, \\ \chi(0) = 1, \ \rho(0) = \rho'(0) = \chi'(0) = 0. \end{cases}$$

We first prove the following lemma.

Lemma 3.1. *We have* $\varphi' > 0$ (*respectively,* $\chi' > 0$) *on* $(0, t_{\alpha,\beta}]$ *and* $\psi' < 0$ (*respectively,* $\rho' < 0$) *on* $(0, t_{\alpha,\beta}]$.

Proof. We have $\psi''(0) = -f'(\alpha) < 0$ (respectively, $\rho''(0) = -g'(\beta) < 0$). Then $\psi < 0$ (respectively, $\rho < 0$) in $(0, \eta]$ for some $\eta > 0$. Since the proof is the same in both cases we only prove that $\varphi' > 0$ and $\psi' < 0$ on $(0, t_{\alpha,\beta}]$. By what we have just seen we can define

$$t_0 = \sup\{t \in (0, t_{\alpha,\beta}] : \ \varphi\psi < 0 \ \text{ on } (0,t]\}\,.$$

Since

$$\varphi'(t) = -\int_0^t g'(v(s))\,\psi(s)\,ds$$

and

$$\psi'(t) = -\int_0^t f'(u(s))\,\varphi(s)\,ds$$

we deduce that $\varphi' > 0$ and $\psi' < 0$ on $(0, t_0]$. Therefore $\varphi(t_0)\psi(t_0) < 0$ and necessarily $t_0 = t_{\alpha,\beta}$.

Now let $D = \{(\alpha, \beta, t) : \ \alpha > 0, \ \beta > 0, \ \text{ and } \ t \in [0, T_{\alpha,\beta})\}$. It is well-known that D is open in $(0,+\infty) \times (0,+\infty) \times [0,+\infty)$. Consider the map $H : D \longrightarrow \mathbb{R}^2$ defined by

$$H(\alpha,\beta,t) = (u(\alpha,\beta,t), v(\alpha,\beta,t))\,.$$

Then $H \in C^1(D, \mathbb{R}^2)$ and

$$H(\alpha, \beta(\alpha), t(\alpha)) = 0\,, \qquad \text{for } \ \alpha > 0\,. \tag{3.3}$$

Since by Theorems A and B we have

$$u'(\alpha, \beta(\alpha), t) < 0 \qquad \text{and} \qquad v'(\alpha, \beta(\alpha), t) < 0 \tag{3.4}$$

for $t \in (0, t(\alpha)]$, using Lemma 3.1 we get

$$\det D_{(\beta,t)} H(\alpha, \beta(\alpha), t(\alpha)) = (\rho\, v' - \chi\, u')(\alpha, \beta(\alpha), t(\alpha)) > 0\,.$$

Therefore by the implicit function theorem $\alpha \to (\beta(\alpha), t(\alpha))$ is a C^1 map for $\alpha > 0$. Differentiating (3.3) with respect to α we get

$$\begin{aligned} &\varphi(\alpha, \beta(\alpha), t(\alpha)) + \rho(\alpha, \beta(\alpha), t(\alpha))\,\beta'(\alpha) \\ &\qquad\qquad + u'(\alpha, \beta(\alpha), t(\alpha))\,t'(\alpha) = 0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\psi(\alpha, \beta(\alpha), t(\alpha)) + \chi(\alpha, \beta(\alpha), t(\alpha))\,\beta'(\alpha) \\ &\qquad\qquad + v'(\alpha, \beta(\alpha), t(\alpha))\,t'(\alpha) = 0 \end{aligned} \tag{3.6}$$

for $\alpha > 0$. Since

$$\beta'(\alpha) = (\det D_{(\beta,t)} H(\alpha, \beta(\alpha), t(\alpha)))^{-1} (\psi\, u' - \varphi\, v')(\alpha, \beta(\alpha), t(\alpha))$$

we deduce from (3.4) and Lemma 3.1 that $\beta'(\alpha) > 0$. Define

$$X(\alpha, t) = \varphi(\alpha, \beta(\alpha), t) + \rho(\alpha, \beta(\alpha), t)\, \beta'(\alpha)$$

and

$$Y(\alpha, t) = \psi(\alpha, \beta(\alpha), t) + \chi(\alpha, \beta(\alpha), t)\, \beta'(\alpha)\,.$$

The proof of the theorem is completed by using (3.4), (3.5) and the next lemma.

Lemma 3.2. *There exists* $t_0 \in (0, t(\alpha))$ (*respectively,* $s_0 \in (0, t(\alpha))$) *such that* $X(\alpha, t) > 0$ *for* $t \in [0, t_0)$ (*respectively,* $Y(\alpha, t) > 0$ *for* $t \in [0, s_0)$) *and* $X(\alpha, t) < 0$ *for* $t \in (t_0, t(\alpha)]$ (*respectively,* $Y(\alpha, t) < 0$ *for* $t \in (s_0, t(\alpha)]$).

Proof. In order to simplify our notations, we denote $X(\alpha, t)$, $Y(\alpha, t)$, $u(\alpha, \beta(\alpha), t)$ and $v(\alpha, \beta(\alpha), t)$ by $X(t)$, $Y(t)$, $u(t)$ and $v(t)$. We have

$$\text{(3.7)} \qquad \begin{cases} -X''(t) = g'(v(t))\, Y(t)\,, & 0 \le t < T_{\alpha,\beta(\alpha)}\,, \\ -Y''(t) = f'(u(t))\, X(t)\,, & 0 \le t < T_{\alpha,\beta(\alpha)}\,, \\ X(0) = 1\,, X'(0) = 0\,, Y(0) = \beta'(\alpha) > 0\,, Y'(0) = 0\,. \end{cases}$$

Lemma 3.3. $X \ge 0$ *on* $[0, t(\alpha)]$ *if and only if* $Y \ge 0$ *on* $[0, t(\alpha)]\,.$

Proof. Suppose that $X \ge 0$ on $[0, t(\alpha)]$. From (3.4), (3.5) and (3.6) we get $Y(t(\alpha)) \ge 0$. Then Theorem A implies that $Y \ge 0$ on $[0, t(\alpha)]$. The converse can be proved in the same way.

Now suppose that $X \ge 0$ on $[0, t(\alpha)]$. By Lemma 3.3 we also have $Y \ge 0$ on $[0, t(\alpha)]$. Then using (H_1), (H_2) and (3.4) we obtain

$$\begin{aligned} 0 &< \int_0^{t(\alpha)} (f'(u)\, u - f(u))\, X = \int_0^{t(\alpha)} v''\, X - Y''\, u \\ &= (v'X)(t(\alpha)) + (u'Y)(t(\alpha)) + \int_0^{t(\alpha)} v\, X'' - u''\, Y \\ &= (v'X)(t(\alpha)) + (u'Y)(t(\alpha)) + \int_0^{t(\alpha)} (g(v) - g'(v)\, v)\, Y \le 0 \end{aligned}$$

and we reach a contradiction. In the same way Y can not remain nonnegative on $[0,t(\alpha)]$. Thus we can define t_0 (respectively, s_0) to be the first zero of X (respectively, Y) on $(0,t(\alpha))$. Moreover there exist $x \in (t_0, t(\alpha))$ and $y \in (s_0, t(\alpha))$ such that $X(x) < 0$ and $Y(y) < 0$. We shall prove that $X < 0$ on $(t_0, t(\alpha)]$ and $Y < 0$ on $(s_0, t(\alpha)]$ and this will complete the proof of Lemma 3.2. Suppose the contrary, then we have the following lemma.

Lemma 3.4. *There exist $s_1, t_1 \in (\max\{s_0, t_0\}, t(\alpha)]$ such that $X < 0$ on (t_0, t_1), $X(t_1) = 0$, $Y < 0$ on (s_0, s_1) and $Y(s_1) = 0$. Moreover if $t = \min\{s_1, t_1\}$, then we have $X'(t) > 0$ and $Y'(t) > 0$.*

Admitting the lemma for the moment, we show that we reach a contradiction. Differentiating (2.21) with respect to α and β respectively and taking the value at $(\alpha, \beta(\alpha), t)$ with $t \in [0, T_{\alpha,\beta(\alpha)})$ we get

$$\varphi'\, v' + u'\, \psi' + g(v)\, \psi + f(u)\, \varphi = f(\alpha)$$

and

$$\rho'\, v' + u'\, \chi' + g(v)\, \chi + f(u)\, \rho = g(\beta(\alpha))$$

for $t \in [0, T_{\alpha,\beta(\alpha)})$, from which we deduce

$$X'\, v' + Y'\, u' + g(v)\, Y + f(u)\, X = f(\alpha) + \beta'(\alpha)\, g(\beta(\alpha)) > 0 \tag{3.8}$$

for $t \in [0, T_{\alpha,\beta(\alpha)})$. Using (3.4), Lemma 3.4 and (3.8) for $t = \min\{s_1, t_1\}$ we see that the left hand side in (3.8) is negative and we get a contradiction.

In order to prove Lemma 3.4 we need

Lemma 3.5. *$X(t) < 0$ on $(t_0, t(\alpha)]$ if and only if $Y(t) < 0$ on $(s_0, t(\alpha)]$.*

PROOF. Suppose that $X(t) < 0$ on $(t_0, t(\alpha)]$. Then from (3.4), (3.5) and (3.6) we get $Y(t(\alpha)) < 0$. Suppose that $t_0 \le s_0$. Then Theorem A implies that $Y < 0$ on $(s_0, t(\alpha)]$. Now if $t_0 > s_0$, Theorems A and B imply that $Y' < 0$ on $(0, t_0]$. Thus $Y < 0$ on $(s_0, t_0]$. Then using Theorem A we get $Y < 0$ on $[t_0, t(\alpha)]$. The converse can be proved in the same way.

PROOF OF LEMMA 3.4. Recall that our assumption is that X can not remain negative on $(t_0, t(\alpha)]$ or that Y can not remain negative on $(s_0, t(\alpha)]$.

Case 1. $s_0 = t_0$. By Theorems A and B we have $X'(t_0) < 0$ and $Y'(t_0) < 0$. Our assumption and Lemma 3.5 imply that there exist $s_1, t_1 \in (t_0, t(\alpha)]$ such that $X < 0$ on $(t_0, t_1), X(t_1) = 0$, $Y < 0$ on (t_0, s_1) and $Y(s_1) = 0$. If $s_1 = t_1$ Theorems A and B imply that $X'(t_1) > 0$ and $Y'(t_1) > 0$. If $s_1 > t_1$ Theorems A and B imply that $X' > 0$ on $[t_1, s_1]$. Therefore $Y'' < 0$ on $(t_1, s_1]$. Since $Y(t_1) < 0$ and $Y(s_1) = 0$ Theorems A and B imply that $Y'(t_1) > 0$. In the same way if $s_1 < t_1$ we show that $X'(s_1) > 0$ and $Y'(s_1) > 0$.

Case 2. $s_0 < t_0$. By Theorems A and B we have $Y' < 0$ on $(0, t_0]$. Our assumption and Lemma 3.5 imply that there exists $s_1 \in (t_0, t(\alpha)]$ such that $Y < 0$ on (s_0, s_1) and $Y(s_1) = 0$. Let $d \in (t_0, s_1)$ be such that $Y(d) = \min_{t_0 \le s \le s_1} Y(s)$. Since $Y''(d) = -f'(u(d))X(d)$ we obtain $X(d) \le 0$. Then Theorems A and B imply that $X'(t_0) < 0$. By virtue of Lemma 3.5 there exists $t_1 \in (t_0, t(\alpha)]$ such that $X < 0$ on (t_0, t_1) and $X(t_1) = 0$. Then we conclude as in Case 1.

Case 3. $s_0 > t_0$. The proof is analogous to that given in Case 2.

The proof is complete.

4. An existence result and examples.

We begin this section with an existence result concerning positive solutions of problem (1.2).

The method we use to prove the existence of a positive solution of problem (1.2) consists of first obtaining a priori estimates on the positive solutions and then applying well-known properties of compact mapping taking a cone in a Banach space into itself (see [3]).

We denote by μ_1 the first eigenvalue of the operator $-d^2/dx^2$ on $(-R, R)$ with Dirichlet boundary conditions and φ_1 is the positive eigenfunction corresponding to μ_1 ($\mu_1 = \pi^2/4R^2$ and $\varphi_1(t) = C \cos(\pi t/2R)$ where $C > 0$ is a constant).

Theorem 4.1. *Let $f, g \in C(\mathbb{R})$ satisfy the following hypotheses*

$$(\mathrm{H}_3) \qquad f(s), g(s) \ge 0\,, \qquad \textit{for} \ \ s \ge 0\,,$$

$$(\mathrm{H}_4) \quad \liminf_{s \to +\infty} \frac{f(s)}{s} > a > 0\,, \ \liminf_{s \to +\infty} \frac{g(s)}{s} > b > 0 \ \ \textit{and} \ \ a\,b > \mu_1^2\,,$$

$$(\mathrm{H}_5) \quad \limsup_{s \to 0} \frac{f(s)}{s} < c\,, \quad \limsup_{s \to 0} \frac{g(s)}{s} < d \quad \textit{and} \quad c\,d < \mu_1^2\,.$$

Then problem (1.2) *possesses at least one positive solution* $(u,v) \in (C^2[0,R])^2$.

PROOF. We first prove that there exists $M > 0$ such that

$$\|u\|_\infty \le M \qquad \text{and} \qquad \|v\|_\infty \le M \tag{4.1}$$

for all positive solutions $(u,v) \in (C^2[0,R])^2$ of (1.2). By (H_4), there exist $K_j > 0$ for $j = 1, 2$ such that

$$f(s) \ge a\,s - K_1\,, \qquad \text{for } \ s \ge 0\,,$$

and

$$g(s) \ge b\,s - K_2 \qquad \text{for } \ s \ge 0\,.$$

Now let $(u,v) \in (C^2[0,R])^2$ be a positive solution of (1.2). Then, C denoting a generic positive constant, we have

$$\begin{aligned} \mu_1^2 \int_0^R \varphi_1\, u\, dt &= -\mu_1 \int_0^R u\, \varphi_1''\, dt = -\mu_1 \int_0^R \varphi_1\, u''\, dt \\ &= \mu_1 \int_0^R \varphi_1\, g(v)\, dt \ge b\,\mu_1 \int_0^R \varphi_1\, v\, dt - C \\ &= -b \int_0^R v\, \varphi_1''\, dt - C = -b \int_0^R \varphi_1\, v''\, dt - C \\ &= b \int_0^R \varphi_1\, f(u)\, dt - C \ge a\, b \int_0^R \varphi_1\, u\, dt - C\,. \end{aligned} \tag{4.2}$$

From (4.2) we deduce that

$$\begin{gathered} \int_0^R \varphi_1\, u\, dt \le C\,, \qquad \int_0^R \varphi_1\, v\, dt \le C\,, \\ \int_0^R \varphi_1\, f(u)\, dt \le C \quad \text{and} \quad \int_0^R \varphi_1\, g(v)\, dt \le C\,, \end{gathered} \tag{4.3}$$

where C is again a generic positive constant. Now we have

$$u(t) = \int_0^R G(t,s)\, g(v(s))\, ds \quad \text{and} \quad v(t) = \int_0^R G(t,s)\, f(u(s))\, ds \tag{4.4}$$

for $t \in [0,R]$, where $G(t,s)$ denotes the Green's function of the operator $-d^2/dx^2$ on $(-R,R)$ with Dirichlet boundary conditions. Since

$$G(t,s) = \begin{cases} R-t\,, & 0 \le s \le t \le R\,, \\ R-s\,, & 0 \le t \le s \le R\,, \end{cases}$$

we have

$$0 \le G(t,s) \le R-s\,, \qquad \text{for } \ 0 \le t,s \le R\,. \tag{4.5}$$

We also have

$$c_1\,(R-s) \le \varphi_1(s) \le c_2\,(R-s)\,, \qquad \text{for } \ s \in [0,R]\,, \tag{4.6}$$

for some positive constants c_j, $j = 1,2$. From (4.3)-(4.6) we easily get

$$u(t) \le C \quad \text{and} \quad v(t) \le C \quad \text{for} \quad t \in [0,R]\,,$$

where C is a positive constant and (4.1) is proved.

Now we can establish the existence of a positive solution of problem (1.2) by using Proposition 2.1 and Remark 2.1 of [3]. The arguments are by now well-known. However, in order that the paper be self contained, we provide details here (see [1], [2] or [4] for similar detailed proofs).

Let X denote the Banach space $(C[0,R])^2$ endowed with the norm $\|(u,v)\| = \max\{\|u\|_\infty, \|v\|_\infty\}$. Define the cone

$$C = \{(u,v) \in X : \ (u,v) \ge 0\}\,.$$

For $((u,v),x) \in C \times [0,+\infty)$ we define

$$F((u,v),x)(t) = (F_1((u,v),x)(t), F_2((u,v),x)(t))\,, \qquad \text{for } \ t \in [0,R]\,,$$

where

$$F_1((u,v),x)(t) = \int_0^R G(t,s)\, g(v(s)+x)\, ds\,,$$

$$F_2((u,v),x)(t) = \int_0^R G(t,s)\, f(u(s)+x)\, ds$$

and

$$\Phi(u,v) = F((u,v),0)\,.$$

By (H_3), F maps $C \times [0, +\infty)$ into C. Since G is continuous, it is well-known that F is compact. (H_3) and (H_5) imply that $f(0) = g(0) = 0$, hence $\Phi(0) = 0$. Now the following properties hold:

i) $(u, v) \neq \theta\,\Phi(u, v)$ for all $\theta \in [0, 1]$ and $(u, v) \in C$ such that $\|(u, v)\| = r$ for sufficiently small $r > 0$. Indeed by (H_5) we can choose $r > 0$ such that $f(s) \leq cs$ and $g(s) \leq ds$ for $0 \leq s \leq r$. Now suppose that there exist $\theta \in [0, 1]$ and $(u, v) \in C$ such that $(u, v) = \theta\,\Phi(u, v)$ with $\|(u, v)\| = r$. Then

$$\begin{cases} -u''(t) = \theta\, g(v(t))\,, & 0 \leq t < R\,, \\ -v''(t) = \theta\, f(u(t))\,, & 0 \leq t < R\,, \\ u(R) = v(R) = u'(0) = v'(0) = 0\,. \end{cases}$$

By Theorem A, $u, v > 0$ on $[0, R)$. We have

$$\begin{aligned} \mu_1^2 \int_0^R \varphi_1\, u\, dt &= -\mu_1 \int_0^R u\, \varphi_1''\, dt = -\mu_1 \int_0^R \varphi_1\, u''\, dt \\ &= \mu_1\, \theta \int_0^R \varphi_1\, g(v)\, dt \leq d\, \mu_1 \int_0^R \varphi_1\, v\, dt \\ &= -d \int_0^R v\, \varphi_1''\, dt = -d \int_0^R \varphi_1\, v''\, dt \\ &= d\, \theta \int_0^R \varphi_1\, f(u)\, dt \leq cd \int_0^R \varphi_1 u\, dt \end{aligned}$$

and we reach a contradiction because the integrals are nonzero.

ii) By (H_4), there exists $x_0 > 0$ such that

$$f(s + x) \geq a\,(s + x) \geq a\, s$$

and

$$g(s + x) \geq b\,(s + x) \geq b\, s\,, \qquad \text{for} \ \ s \geq 0\,,\ x \geq x_0 > 0\,.$$

Then using the same arguments as in the proof of (4.1) and Theorem A, we can show that $F((u, v), x) \neq (u, v)$ for all $(u, v) \in C$ and $x \geq x_0$.

iii) Now we note that the constant in (4.1) can be chosen independently of the parameter $x \in [0, x_0]$ for each fixed $x_0 \in (0, +\infty)$ if we consider positive solutions of (1.2) for the family of nonlinearities

$f_x(t) = f(t + x)$, $g_x(t) = g(t + x)$, $t \geq 0$. Thus we can find a constant $R > r$ such that $F((u, v), x) \neq (u, v)$ for all $x \in [0, x_0]$ and $(u, v) \in C$ with $\|(u, v)\| = R$.

Thus we may apply Proposition 2.1 and Remark 2.1 stated in [3] to conclude that Φ has a nontrivial fixed point $(u, v) \in C$. Theorem A and the properties of the Green's function imply that any nontrivial fixed point of Φ in C yields a positive solution of (1.2) in $(C^2[0, R])^2$. The proof of the theorem is complete.

REMARK. Note that, for the a priori estimates, condition (H_3) is not needed. We need it merely to insure that the maps Φ and F are cone-preserving.

We conclude this section with some examples to which our theorems apply.

a) We first consider problem (1.3) where $g(v) = v$. When $f(u) = \sum_{j=1}^{k} a_j u^{p_j}$ for $u > 0$ with $p_j > 1$ and $a_j > 0$ for $j = 1, \ldots, k$ and $k \geq 1$ or $f(u) = u^r/(1+u^s)$ for $u > 0$ with $r-1 > s > 0$, Theorem 4.1 implies the existence of a positive solution of (1.3) and Corollary 1.1 gives the uniqueness.

b) For problem (1.1) we can take f as in a) and g of the same type as f. Then the existence of a positive solution of (1.1) follows from Theorem 4.1 and the uniqueness is given by Theorem 1.1.

c) Take

$$f(u) = \lambda\, u + u^p \quad \text{and} \quad g(v) = \mu\, v + v^q\,, \qquad u, v > 0\,,$$

with $p, q > 1$, $\lambda, \mu > 0$ and $\lambda\, \mu < \mu_1^2$. By Theorem 4.1 there exists a positive solution of (1.1). Then Theorem 1.1 gives the uniqueness.

This is an example of a perturbed linear system. Consider the linear eigenvalue problem

$$\text{(4.7)} \qquad \begin{cases} -u'' = \lambda_2\, v\,, & \text{in } (-R, R)\,, \\ -v'' = \lambda_1\, u\,, & \text{in } (-R, R)\,, \\ u > 0\,,\ v > 0\,, & \text{in } (-R, R)\,, \\ u(\pm R) = v(\pm R) = 0\,. \end{cases}$$

The next lemma is a particular case of a result proved by Van Der Vorst [7] (see also [2] for an extension of this result).

Lemma 4.1. *Problem* (4.7) *has a solution if and only if*

$$\lambda_j > 0\,, \quad \textit{for} \;\; j = 1, 2\,, \quad \textit{and} \quad \lambda_1 \lambda_2 = \mu_1^2\,.$$

The solution is given by $u = c_1 \varphi_1$, $v = c_2 \varphi_1$ *where* $c_1 > 0$ *is an arbitrary constant and* $c_2 = c_1(\lambda_1/\lambda_2)^{1/2}$.

Clearly the above lemma shows that conditions (H_1) and (H_2) are sharp.

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L^p multipliers and their H^1-L^1 estimates on the Heisenberg group

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Abstract. We give a Hörmander-type sufficient condition on an operator-valued function M that implies the L^p-boundedness result for the operator T_M defined by $(T_M f)\hat{} = M\hat{f}$ on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$. Here "$\hat{}$" denotes the Fourier transform on $\mathbb{H}^n$ defined in terms of the Fock representations. We also show the H^1-L^1 boundedness of T_M, $\|T_M f\|_{L^1} \leq C\,\|f\|_{H^1}$, for $\mathbb{H}^n$ under the same hypotheses of L^p-boundedness.

1. Introduction.

Let $f \mapsto \hat{f}$ be the Fourier transform, $f \mapsto \check{f}$ the inverse Fourier transform, and m a bounded measurable function on $\mathbb{R}^n$. We say that m is a *multiplier* for $L^p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$, if $f \in L^2 \cap L^p$ implies $(m\hat{f})\check{}$ is in L^p and satisfies

$$\big\|(m\hat{f})\check{}\big\|_{L^p} \leq C_p\,\|f\|_{L^p}\,,$$

with C_p independent of f. The multiplier theorem was originally due to Hörmander [H] on $\mathbb{R}^n$:

Theorem (Multiplier theorem for $L^p(\mathbb{R}^n)$). *Let m be a function in $C^k(\mathbb{R}^n \setminus \{0\})$, $k > n/2$. Assume that $m \in L^\infty(\mathbb{R}^n)$ and*

$$\sup_{R>0} R^{2|\alpha|-n} \int_{R<|x|\le 2R} |D^\alpha m(x)|^2\, dx < +\infty$$

for all differential monomials D^α of order $|\alpha| \le k$. Then the multiplier operator mapping f into $(m\hat{f})\check{}$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

There are two general methods of proving multiplier theorems. The first one follows Hörmander's process [H] and works mostly on the Fourier transform side. The second one applies the theory developed by Coifman and Weiss [CW1] of constructing a well-behaved approximate identity and working mostly on the group. DeMichele and Mauceri [DMM] applied Coifman and Weiss' theory to extend the L^p multiplier theorem to the three-dimensional Heisenberg group $\mathbb{H}$. Here we follow the same machinery as in [DMM], and extend to the more general case of the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$.

Theorem 1 (Multiplier theorem for $L^p(\mathbb{H}^n)$). *Let M be an operator-valued function with each entry in $C^k(\mathbb{R}\setminus\{0\})$, $k \ge 4\,[(n+5)/4]$, where $[\,\cdot\,]$ denotes the greatest integer function. Also assume*

$$\sup_{\lambda\neq 0} \|M(\lambda)\| < +\infty\,,$$

$$\sup_{R>0} R^{\deg P-n-1} \int_{-\infty}^{+\infty} \left\|[\Delta_P M(\lambda)]\hat{\Pi}_R(\lambda)\right\|_{HS}^2 |\lambda|^n\, d\lambda < +\infty\,,$$

for every monomial P with $\deg P \le 4\,[(n+5)/4]$, where Δ_P is a difference-differential operator, $\hat{\Pi}_R(\lambda)$ a projection operator to a part of main diagonal (both operators will be defined in the next section). Then M is a multiplier of $L^p(\mathbb{H}^n)$, $1 < p < \infty$, and is of weak type $(1,1)$.

We also show the H^1-L^1 boundedness of T_M as follows.

Theorem 2. *Suppose M satisfies the same hypotheses as Theorem 1. Then T_M maps $H^1(\mathbb{H}^n)$ boundedly into $L^1(\mathbb{H}^n)$. Moreover, there exists a constant $C > 0$, independent of f, such that $\|T_M f\|_{L^1} \le C\,\|f\|_{H^1}$ for all $f \in H^1(\mathbb{H}^n)$.*

In Section 2 we review some basic tools of harmonic analysis on $\mathbb{H}^n$. In Section 3 we prove the $L^p(\mathbb{H}^n)$ multiplier theorem, and in Section 4 we show the H^1-L^1 estimate. Finally, we mention that C will be used to denote a constant which may vary from line to line.

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2. Preliminaries.

Most results in this section were given in [DMM] for the three-dimensional Heisenberg group $\mathbb{H}$. We extend those to $\mathbb{H}^n$ and give more detailed proofs here.

$\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{R}\times\mathbb{C}^n$ and multiplication defined by

$$(t,z)\,(t',z') = (t+t'+2\,\mathrm{Im}\,(z\cdot\bar{z}'), z+z')\,,$$

where $z\cdot\bar{z}' = \sum_{j=1}^{n} z_j\bar{z}'_j$. The Heisenberg Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on $\mathbb{H}^n$ is generated by

$$\begin{aligned} T &= \frac{\partial}{\partial t}\,,\\ Z_j &= \frac{\partial}{\partial z_j} + i\,\bar{z}_j\,\frac{\partial}{\partial t}\,,\\ \overline{Z}_j &= \frac{\partial}{\partial \bar{z}_j} - i\,z_j\,\frac{\partial}{\partial t}\,, \qquad j=1,2,\ldots,n\,, \end{aligned}$$

and the only non-zero commutations are

$$[Z_j,\overline{Z}_j] = -2\,i\,T\,, \qquad j=1,2,\ldots,n\,.$$

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure dV on $\mathbb{R}\times\mathbb{C}^n$. $\mathbb{H}^n$ is endowed with a family of dilations $\{\delta_r : r>0\}$ defined by $\delta_r(t,z) = (r^2t, rz)$. We say a function f on $\mathbb{H}^n$ is *homogeneous of degree* d if $f\circ\delta_r = r^d f$ for every $r>0$. Furthermore, we define the *homogeneous norm* of $(t,z)\in\mathbb{H}^n$, denoted by $|(t,z)|$, to be $(t^2+|z|^4)^{1/4}$. The norm is homogeneous of degree 1. For simplification of notation, sometimes we use $x = (t, z)$ to denote a point of $\mathbb{H}^n$, $rx = \delta_r(t, z) = (r^2 t, rz)$ a dilation of x, and $|x| = (t^2 + |z|^4)^{1/4}$ a homogeneous norm on $\mathbb{H}^n$. Guivarch [Gu] has shown that the triangle inequality $|xy| \leq |x| + |y|$, $x, y \in \mathbb{H}^n$, holds. Moreover, using polar coordinates, we have (*cf.* [FS, Corollary 1.16])

$$\int_{a \leq |x| \leq b} |x|^\alpha \, dx = \frac{C}{\alpha + 2n + 2} \left(b^{\alpha+2n+2} - a^{\alpha+2n+2} \right), \tag{1}$$

for $\alpha \neq -2n - 2$, $0 \leq a < b \leq +\infty$, where C is an absolute constant.

The convolution of two functions $f, g \in L^1(\mathbb{H}^n)$ is defined as usual,

$$(g * f)(x) = \int_{\mathbb{H}^n} g(xy^{-1}) \, f(y) \, dy = \int_{\mathbb{H}^n} g(y) \, f(y^{-1}x) \, dy \, .$$

Let $\mathcal{S}(\mathbb{H}^n)$ and $\mathcal{S}'(\mathbb{H}^n)$ denote the Schwartz space of rapidly decreasing smooth functions and space of tempered distributions, respectively.

It was observed by Stone, von Neumann, and Weyl in the early 1930's that the irreducible unitary representations of $\mathbb{H}^n$ split into two classes. The representations which are trivial on the center $\mathcal{Z} = \{(t, 0) : t \in \mathbb{R}\}$ of $\mathbb{H}^n$ are just the usual one-dimensional representations of $\mathbb{C}^n \cong \mathbb{H}^n/\mathcal{Z}$ lifted to $\mathbb{H}^n$. Since these representations form a set of measure zero in the decomposition of $L^2(\mathbb{H}^n)$, we will not consider them further. The representations which are nontrivial on $\mathcal{Z}$ are classified by a parameter $\lambda \in \mathbb{R}^*$ ($\equiv \mathbb{R} \setminus \{0\}$) and may be described as follows. For $\lambda > 0$, let $\mathcal{H}_\lambda$ be the Bargmann space

$$\mathcal{H}_\lambda = \Big\{ F \text{ holomorphic on } \mathbb{C}^n : \|F\|^2 = \left(\frac{2\lambda}{\pi} \right)^n \int_{\mathbb{C}^n} |F(\zeta)|^2 \, e^{-2\lambda|\zeta|^2} d\zeta < +\infty \Big\} .$$

Then $\mathcal{H}_\lambda$ is a Hilbert space and the monomials

$$F_{\alpha,\lambda}(\zeta) = \sqrt{\frac{(2\lambda)^{|\alpha|}}{\alpha!}} \, \zeta^\alpha, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \overline{\mathbb{N}}^n$$

($\overline{\mathbb{N}} \equiv \mathbb{N} \cup \{0\}$), form an orthonormal basis for $\mathcal{H}_\lambda$, where

$$\alpha! = (\alpha_1!) \, (\alpha_2!) \cdots (\alpha_n!) \, , \qquad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \, ,$$

and

$$\zeta^\alpha = \zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\cdots\zeta_n^{\alpha_n}\,.$$

For $\lambda \in \mathbb{R}^*$, the representation Π_λ acts on $\mathcal{H}_{|\lambda|}$ by

$$\Pi_\lambda(t,z)F(\zeta) = \begin{cases} e^{i\lambda t + 2\lambda(\zeta\cdot z - |z|^2/2)}\,F(\zeta - \bar{z})\,, & \text{for } \lambda > 0\,,\\ e^{i\lambda t - 2\lambda(\zeta\cdot\bar{z} - |z|^2/2)}\,F(\zeta - z)\,, & \text{for } \lambda < 0\,. \end{cases}$$

A straightforward calculation shows that $\Pi_\lambda(t,z)$ is unitary and its adjoint operator $\Pi_\lambda(t,z)^* = \Pi_\lambda(-t,-z)$. For $f \in L^1(\mathbb{H}^n)$, $\lambda \in \mathbb{R}^*$, set

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(t,z)\,\Pi_\lambda(t,z)\,dV$$

where the integral is defined in the weak sense, and the operator $\hat{f}(\lambda)$ is called the *Fourier transform* of f. It follows immediately from the definition that for $f, g \in L^1(\mathbb{H}^n)$

$$\begin{aligned}\Pi_\lambda(y)\,\hat{f}(\lambda)\,F(\zeta) &= \Pi_\lambda(y)\int f(x)\,\Pi_\lambda(x)\,F(\zeta)\,dx\\ &= \int f(x)\,\Pi_\lambda(yx)\,F(\zeta)\,dx\\ &= \int f(y^{-1}x)\,\Pi_\lambda(x)\,F(\zeta)\,dx\end{aligned}$$

and

$$\begin{aligned}\hat{g}(\lambda)\,\hat{f}(\lambda) &= \int g(y)\,\Pi_\lambda(y)\,dy\,\hat{f}(\lambda)\\ &= \iint g(y)\,f(y^{-1}x)\,\Pi_\lambda(x)\,dxdy\\ &= \int (g*f)(x)\,\Pi_\lambda(x)\,dx\\ &= (g*f)\hat{\ }(\lambda)\,.\end{aligned}$$

For $(\lambda, m, \alpha) \in \mathbb{R}^* \times \mathbb{Z}^n \times \overline{\mathbb{N}}^n$, we use the notations

$$\begin{aligned} m_i^+ &= \max\{m_i, 0\}\,, & m_i^- &= -\min\{m_i, 0\}\,,\\ m^+ &= (m_1^+, m_2^+, \ldots, m_n^+)\,, & m^- &= (m_1^-, m_2^-, \ldots, m_n^-)\,,\end{aligned}$$

and define the partial isometry $W_\alpha^m(\lambda)$ on $\mathcal{H}_{|\lambda|}$ by

$$W_\alpha^m(\lambda)\, F_{\beta,\lambda} = (-1)^{|m^+|}\, \delta_{\alpha+m^+,\beta}\, F_{\alpha+m^-,\lambda}\,, \qquad \text{for } \lambda > 0\,,$$

and for negative λ by

$$W_\alpha^m(\lambda) = [W_\alpha^m(-\lambda)]^*\,.$$

Thus $\{W_\alpha^m(\lambda) : m \in \mathbb{Z}^n, \alpha \in \overline{\mathbb{N}}^n\}$ is an orthonormal basis for the Hilbert-Schmidt operators on $\mathcal{H}_{|\lambda|}$, and W_α^m has the matrix expression $W_{\alpha_1}^{m_1} \otimes W_{\alpha_2}^{m_2} \otimes \cdots \otimes W_{\alpha_n}^{m_n}$ if we view $F_{\alpha,\lambda}(\zeta)$ as $F_{\alpha_1,\lambda}(\zeta_1) \otimes F_{\alpha_2,\lambda}(\zeta_2) \otimes \cdots \otimes F_{\alpha_n,\lambda}(\zeta_n)$, where $\otimes$ means tensor product, $W_{\alpha_i}^{m_i}$ corresponds to the infinite dimensional matrix whose $(\alpha_i, m_i+\alpha_i)$-entry is $(-1)^{m_i}$ and zero everywhere else for $m_i \geq 0$, $(\alpha_i - m_i, \alpha_i)$-entry is 1 and zero everywhere else for $m_i < 0$, and $F_{\alpha_i,\lambda}(\zeta_i)$ corresponds to the infinite dimensional vector whose α_i-th components is 1 and all other components are zero. (Note: all entries and components here are counted from the 0-th position.) For each i, the matrix form of $W_{\alpha_i}^{m_i}$ is

$$(-1)^{m_i}\begin{bmatrix} 0 & & & \\ & \ddots & \overset{m_i}{\longleftrightarrow} 1 & \\ & & \ddots & \\ & & & \\ \end{bmatrix}\begin{matrix}\leftarrow \text{0-th row} \\ \leftarrow \alpha_i\text{-th row} \\ \\ \\ \end{matrix} \qquad \text{for } m_i \geq 0,$$

$$\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \ddots & \\ & 1 \overset{-m_i}{\longleftrightarrow} & & \\ \end{bmatrix}\begin{matrix}\leftarrow \text{0-th row} \\ \\ \\ \leftarrow (\alpha_i - m_i)\text{-th row} \end{matrix} \qquad \text{for } m_i < 0.$$

Proposition [G]. *If* $f \in L^1 \cap L^2(\mathbb{H}^n)$ *is of the form*

$$f(t,z) = \sum_{m\in\mathbb{Z}^n} f_m(t, |z_1|, \ldots, |z_n|)\, e^{i(m_1\theta_1+\cdots+m_n\theta_n)}\,, \qquad z_j = |z_j|\, e^{i\theta_j}\,,$$

then

$$\hat{f}(\lambda) = \sum_{\substack{m\in\mathbb{Z}^n \\ \alpha\in\overline{\mathbb{N}}^n}} \mathcal{R}_f(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\,,$$

where

$$\mathcal{R}_f(\lambda, m, \alpha) = \int_{\mathbb{H}^n} f_m(t, |z_1|, \ldots, |z_n|)\, e^{i\lambda t} \cdot l_{\alpha_1}^{|m_1|}(2\,|\lambda|\,|z_1|^2) \cdots l_{\alpha_n}^{|m_n|}(2\,|\lambda|\,|z_n|^2)\, dV\,, \tag{2}$$

and $l_{\alpha_j}^{|m_j|}$ is the Laguerre function.

NOTE. For a poly-radial function $f(t,z) = f(t, |z_1|, \ldots, |z_n|)$, the summation $\sum_{m\in\mathbb{Z}^n}$ in the above proposition contains only the term with $m = 0$. Hence $\hat{f}(\lambda)$ is poly-diagonal.

Recall $(\delta_r f)\hat{\ }(\xi) = r^{-n}\, \hat{f}(r^{-1}\xi)$ on $\mathbb{R}^n$, where $\delta_r f(x) = f(rx)$. If we define

$$f_r(t,z) = r^{-(n+1)/2}\, f(r^{-1/2}t, r^{-1/4}z)\,, \qquad r > 0\,,$$

on $\mathbb{H}^n$, from identity (2) and a change of variables we have a similar relationship between Fourier coefficients $\mathcal{R}_{f_r}(\lambda, m, \alpha)$ and $\mathcal{R}_f(\lambda, m, \alpha)$ as follows:

$$\mathcal{R}_{f_r}(\lambda, m, \alpha) = \mathcal{R}_f(\sqrt{r}\lambda, m, \alpha)\,. \tag{3}$$

We also have $[-ix_j f(x)]\hat{\ }(\xi) = \partial_j \hat{f}(\xi)$ on $\mathbb{R}^n$. More generally if $P(x) = P(x_1, x_2, \ldots, x_n)$ is a polynomial on $\mathbb{R}^n$ and the differential operator $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ is defined as usual, then $[P(-ix)\, f(x)]\hat{\ }(\xi) = P(D)\hat{f}(\xi)$. Let P be a polynomial in $t, z_j, \bar{z}_j$ on $\mathbb{H}^n$. Define the difference-differential operator Δ_P acting on the Fourier transform of $f \in L^1 \cap L^2(\mathbb{H}^n)$ by

$$\Delta_P\Big(\sum_{m,\alpha} \mathcal{R}_f(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\Big) = \sum_{m,\alpha} \mathcal{R}_{Pf}(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\,.$$

Let $\{e_j : 1 \le j \le n\}$ be the standard basis of $\mathbb{Z}^n$. We have the following explicit expressions for Δ_t, Δ_{z_j}, and $\Delta_{\bar{z}_j}$.

$$\begin{aligned}\Delta_t \hat{f}(\lambda) = &-i \sum_{m,\alpha} \big(\frac{\partial}{\partial\lambda}\mathcal{R}_f(\lambda, m, \alpha)\big)\, W_\alpha^m(\lambda)\\ &- \frac{ni}{2\lambda}\sum_{m,\alpha} \mathcal{R}_f(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\end{aligned}$$

$$- \frac{i}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^{n} \sqrt{\alpha_j(\alpha_j + |m_j|)}\, \mathcal{R}_f(\lambda, m, \alpha - e_j)\, W_\alpha^m(\lambda)$$

$$+ \frac{i}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^{n} \sqrt{(\alpha_j + 1)(\alpha_j + |m_j| + 1)}$$

$$\cdot\, \mathcal{R}_f(\lambda, m, \alpha + e_j)\, W_\alpha^m(\lambda)\ ;$$

$$\Delta_{z_j} \hat{f}(\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + m_j}\, \mathcal{R}_f(\lambda, m - e_j, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + 1}\, \mathcal{R}_f(\lambda, m - e_j, \alpha + e_j)\, W_\alpha^m(\lambda)\,,$$

if $m_j \geq 1$;

$$\Delta_{z_j} \hat{f}(\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j - m_j + 1}\, \mathcal{R}_f(\lambda, m - e_j, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j}\, \mathcal{R}_f(\lambda, m - e_j, \alpha - e_j)\, W_\alpha^m(\lambda)\,,$$

if $m_j \leq 0$;

$$\Delta_{\bar{z}_j} \hat{f}(\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + m_j + 1}\, \mathcal{R}_f(\lambda, m + e_j, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j}\, \mathcal{R}_f(\lambda, m + e_j, \alpha - e_j)\, W_\alpha^m(\lambda)\,,$$

if $m_j \geq 0$;

$$\Delta_{\bar{z}_j} \hat{f}(\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j - m_j}\, \mathcal{R}_f(\lambda, m + e_j, \alpha)\, W_\alpha^m(\lambda)$$

$$- \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + 1}\, \mathcal{R}_f(\lambda, m + e_j, \alpha + e_j)\, W_\alpha^m(\lambda)\,,$$

if $m_j \leq -1$.

Using these formulas, we obtain similar results for polynomials and extend the operators Δ_P as formal difference-differential operators acting on operators which are of type

$$M(\lambda) = \sum_{m,\alpha} B(\lambda, m, \alpha)\, W_\alpha^m(\lambda)\,.$$

We establish the formula for Δ_{z_j}. The others are proved similarly by using the recurrence relations and differential properties of $\{l_\alpha^m\}$. We use identity (2) and write

$$\begin{aligned}\Delta_{z_j}\hat{f}(\lambda) &= \sum_{m,\alpha} \mathcal{R}_{z_j f}(\lambda, m, \alpha)\, W_\alpha^m(\lambda) \\ &= \sum_{m,\alpha} \int_{\mathbb{H}^n} |z_j|\, f_{m-e_j}\, e^{i\lambda t} l_{\alpha_1}^{|m_1|}(2\,|\lambda|\,|z_1|^2)\cdots l_{\alpha_j}^{|m_j|}(2\,|\lambda|\,|z_j|^2) \\ &\qquad\qquad \cdots l_{\alpha_n}^{|m_n|}(2\,|\lambda|\,|z_n|^2)\, dV\; W_\alpha^m(\lambda)\,.\end{aligned}$$

The recurrence relations for Laguerre functions tell us

$$\begin{aligned}&\sqrt{2|\lambda|}\;|z_j|\; l_{\alpha_j}^{|m_j|}(2\,|\lambda|\,|z_j|^2) \\ &\qquad = \sqrt{\alpha_j + m_j}\; l_{\alpha_j}^{m_j-1}(2\,|\lambda|\,|z_j|^2) - \sqrt{\alpha_j+1}\; l_{\alpha_j+1}^{m_j-1}(2\,|\lambda|\,|z_j|^2)\,,\end{aligned}$$

if $m_j \geq 1$, and

$$\begin{aligned}&\sqrt{2|\lambda|}\;|z_j|\; l_{\alpha_j}^{|m_j|}(2\,|\lambda|\,|z_j|^2) \\ &\qquad = \sqrt{\alpha_j - m_j + 1}\; l_{\alpha_j}^{-m_j+1}(2\,|\lambda|\,|z_j|^2) - \sqrt{\alpha_j}\; l_{\alpha_j-1}^{-m_j+1}(2\,|\lambda|\,|z_j|^2)\,,\end{aligned}$$

if $m_j \leq 0$.

Thus, we have

$$\begin{aligned}\Delta_{z_j}\hat{f}(\lambda) &= \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + m_j}\; \mathcal{R}_f(\lambda, m - e_j, \alpha)\, W_\alpha^m(\lambda) \\ &\quad - \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j + 1}\; \mathcal{R}_f(\lambda, m - e_j, \alpha + e_j)\, W_\alpha^m(\lambda)\,,\end{aligned}$$

for $m_j \geq 1$, and

$$\begin{aligned}\Delta_{z_j}\hat{f}(\lambda) &= \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j - m_j + 1}\; \mathcal{R}_f(\lambda, m - e_j, \alpha)\, W_\alpha^m(\lambda) \\ &\quad - \frac{1}{\sqrt{2|\lambda|}} \sum_{m,\alpha} \sqrt{\alpha_j}\; \mathcal{R}_f(\lambda, m - e_j, \alpha - e_j)\, W_\alpha^m(\lambda)\,,\end{aligned}$$

for $m_j \leq 0$.

This proves the formula for Δ_{z_j}. Similarly, applying the same techniques we can obtain the formulas for $\Delta_{\bar{z}_j}$ and Δ_t.

Denoting $\|A\|_{HS}^2 = \operatorname{tr}(A^*A)$, the square of the Hilbert-Schmidt norm of A, we have the following Plancherel formula

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{+\infty} \left\|\hat{f}(\lambda)\right\|_{HS}^2 |\lambda|^n \, d\lambda \,, \qquad f \in L^1 \cap L^2(\mathbb{H}^n)\,.$$

By this we can extend the Fourier transform as an isometry from $L^2(\mathbb{H}^n)$ onto the Hilbert space of the operator-valued functions $\lambda \mapsto A(\lambda)$, $\lambda \in \mathbb{R}^*$, satisfying

i) $A(\lambda)$ is a Hilbert-Schmidt operator on $\mathcal{H}_\lambda$, for almost every $\lambda \in \mathbb{R}^*$,

ii) $(A(\lambda)P, Q)$ is a measurable function of λ, for every polynomial P, Q on $\mathbb{C}^n$,

iii) $\|A\|_2^2 = \dfrac{2^{n-1}}{\pi^{n+1}} \displaystyle\int_{-\infty}^{+\infty} \|A(\lambda)\|_{HS}^2 \, |\lambda|^n \, d\lambda < +\infty\,.$

Definition. *A left invariant multiplier of $L^p(\mathbb{H}^n)$, $1 \le p \le \infty$, is an operator-valued function $M : \lambda \mapsto M(\lambda)$, $\lambda \in \mathbb{R}^*$, such that*

a) for every $\lambda \in \mathbb{R}^*$, $M(\lambda)$ is a bounded operator on $\mathcal{H}_\lambda$,

b) the operator T_M defined by

$$(T_M f)\hat{}\,(\lambda) = M(\lambda)\, \hat{f}(\lambda)\,, \qquad f \in L^1 \cap L^p(\mathbb{H}^n)\,,$$

extends to a bounded operator on $L^p(\mathbb{H}^n)$.

From the Plancherel formula iii) above it follows immediately that M is a left $L^2(\mathbb{H}^n)$ multiplier if and only if $\sup_{\lambda \neq 0} \|M(\lambda)\| < +\infty$. We also remark that everything we say for left multipliers may be translated for right multipliers similarly defined, because the group $\mathbb{H}^n$ is unimodular.

On $\mathbb{R}^n$ we have $(\partial_j f)\hat{}\,(\xi) = i\,\xi_j \hat{f}(\xi)$. On $\mathbb{H}^n$ we have the following analogues: for $\lambda > 0$,

$$\begin{aligned}
&(Z_j f)\hat{}\,(\lambda)\, F(\zeta) \\
&= -2\,\lambda\, \hat{f}(\lambda)\, \zeta_j\, F(\zeta) \\
&= -\sqrt{2\lambda}\, \hat{f}(\lambda) \left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \right\} F(\zeta)
\end{aligned}$$

and

$$\begin{aligned}&(\overline{Z}_j f)\hat{}(\lambda)\, F(\zeta)\\ &= \hat{f}(\lambda)\, \frac{\partial F}{\partial \zeta_j}(\zeta)\\ &= \sqrt{2\lambda}\, \hat{f}(\lambda)\, \left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes \begin{bmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \right\} F(\zeta)\,,\end{aligned}$$

where I_k, $k = 1, 2, 3, \ldots$, is the infinite dimensional identity matrix; for $\lambda < 0$, we switch the formulas for Z_j and $\overline{Z}_j$. For all $\lambda \in \mathbb{R} \setminus \{0\}$, $(Tf)\hat{}(\lambda) = -i\lambda \hat{f}(\lambda)$. Thus, for $\lambda > 0$, we can consider the corresponding multiplier of the differential operators Z_j, $\overline{Z}_j$, T to be

$$-\sqrt{2\lambda}\left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \right\},$$

$$\sqrt{2\lambda}\left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes \begin{bmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \right\},$$

and

$$-i\,\lambda \left\{ I_1 \otimes \cdots \otimes I_{j-1} \otimes I_j \otimes I_{j+1} \otimes \cdots \right\},$$

respectively. To prove these formulas we consider the case $n = 1$, $\lambda > 0$, and the formula for $\hat{Z}$ only; for all other cases the following proof can be easily carried over. By definition

$$(Zf)\hat{}(\lambda) = (\partial_z f)\hat{}(\lambda) + i\,(\bar{z}\,\partial_t f)\hat{}(\lambda)$$

and integration by parts yields

$$(\bar{z}\,\partial_t f)\hat{}(\lambda)\, F(\zeta) = -i\,\lambda\,(\bar{z} f)\hat{}(\lambda)\, F(\zeta)\,.$$

We have

$$(Zf)\hat{\ }(\lambda) = (\partial_z f)\hat{\ }(\lambda) + \lambda\,(\bar{z}f)\hat{\ }(\lambda)\,.$$

Furthermore,

$$\begin{aligned}\hat{f}(\lambda)\,\big(\zeta\,F(\zeta)\big) &= \int f\; e^{i\lambda t+2\lambda(\zeta\cdot z-|z|^2/2)}(\zeta-\bar{z})\,F(\zeta-\bar{z})\,dV\\ &= \zeta\,\hat{f}(\lambda)\,F(\zeta) - (\bar{z}f)\hat{\ }(\lambda)\,F(\zeta)\,,\end{aligned}$$

so

$$\begin{aligned}(\partial_z f)\hat{\ }(\lambda)\,F(\zeta) &= -\int f\;\partial_z(\Pi_\lambda F)\,dV\\ &= -2\,\lambda\,\zeta\,\hat{f}(\lambda)\,F(\zeta) + \lambda\,(\bar{z}f)\hat{\ }(\lambda)\,F(\zeta)\\ &= -2\,\lambda\,\hat{f}(\lambda)\,\big(\zeta\,F(\zeta)\big) - \lambda\,(\bar{z}f)\hat{\ }(\lambda)\,F(\zeta)\,.\end{aligned}$$

Hence,

$$(Zf)\hat{\ }(\lambda)\,F(\zeta) = -2\,\lambda\,\hat{f}(\lambda)\,\big(\zeta\,F(\zeta)\big)\,.$$

As for the matrix form, we recall that

$$F_{\alpha,\lambda}(\zeta) = \frac{(2\lambda)^{\alpha/2}}{\sqrt{\alpha!}}\,\zeta^\alpha$$

is an orthonormal basis for $\mathcal{H}_\lambda$, and write

$$F(\zeta) = \sum_{\alpha=0}^{\infty} a_\alpha\,F_{\alpha,\lambda}(\zeta) \equiv \begin{bmatrix} a_0\\ a_1\\ a_2\\ \vdots \end{bmatrix}, \qquad a_\alpha \in \mathbb{C}\,.$$

Then

$$\begin{aligned}\zeta\,F(\zeta) &= \sum_{\alpha=0}^{\infty} a_\alpha\,\zeta\,F_{\alpha,\lambda}(\zeta)\\ &= \frac{1}{\sqrt{2\lambda}}\sum_{\alpha=0}^{\infty}\sqrt{\alpha+1}\,a_\alpha\,F_{\alpha+1,\lambda}(\zeta)\\ &= \frac{1}{\sqrt{2\lambda}}\begin{bmatrix} 0\\ a_0\\ \sqrt{2}\,a_1\\ \sqrt{3}\,a_2\\ \vdots \end{bmatrix}\end{aligned}$$

$$= \frac{1}{\sqrt{2\lambda}} \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{bmatrix} F(\zeta)\,.$$

On $\mathbb{H}^n$ the sub-Laplacian is the differential operator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 = -\frac{1}{2}\sum_{j=1}^{n}\big(Z_j\overline{Z}_j + \overline{Z}_j Z_j\big)\,.$$

The above calculations can be used to compute $\hat{\mathcal{L}}_0$. We apply the matrix expressions of Z_j and $\overline{Z}_j$ to get

$$\begin{aligned}
\hat{\mathcal{L}}_0(\lambda) &= -\frac{1}{2}\sum_{j=1}^{n}\big\{\hat{\overline{Z}}_j(\lambda)\hat{Z}_j(\lambda) + \hat{Z}_j(\lambda)\hat{\overline{Z}}_j(\lambda)\big\} \\
&= \sum_{j=1}^{n}\Big\{I_1 \otimes \cdots \otimes I_{j-1} \otimes \\
&\qquad |\lambda| \left(\begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & \ddots \end{bmatrix} + \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{bmatrix} \right) \otimes I_{j+1} \otimes \cdots \Big\} \\
&= \sum_{j=1}^{n}\Big\{I_1 \otimes \cdots \otimes I_{j-1} \otimes |\lambda| \begin{bmatrix} 1 & & & & \\ & 3 & & & \\ & & 5 & & \\ & & & 7 & \\ & & & & \ddots \end{bmatrix} \otimes I_{j+1} \otimes \cdots \Big\} \\
&= \sum_{\alpha \in \overline{\mathbb{N}}^n} (2\,|\alpha| + n)\,|\lambda|\, W_\alpha^0(\lambda)\,.
\end{aligned}$$

We now introduce the partition of the identity $I = \sum_{k=-\infty}^{+\infty} \hat{\Pi}_{2^k R}$, $R > 0$, where $\hat{\Pi}_s$ is the spectral projection of $\mathcal{L}_0$ corresponding to the multiplier

$$\hat{\Pi}_s(\lambda) = \sum_{s<(2|\alpha|+n)\,|\lambda| \le \sqrt{2}s} W_\alpha^0(\lambda)\,.$$

Then the $L^p(\mathbb{H}^n)$ multiplier theorem can be stated in the following way.

Theorem 1 (Multiplier theorem for $L^p(\mathbb{H}^n)$). *Let M be an operator-valued function with each entry in $C^k(\mathbb{R}\setminus\{0\})$, $k \geq 4\,[(n+5)/4]$, where $[\,\cdot\,]$ denotes the greatest integer function. Also assume*

$$\sup_{\lambda\neq 0} \|M(\lambda)\| < +\infty\,, \tag{4}$$

$$\sup_{R>0}\; R^{\deg P-n-1}\int_{-\infty}^{+\infty} \left\|[\Delta_P M(\lambda)]\hat{\Pi}_R(\lambda)\right\|_{HS}^2 |\lambda|^n\, d\lambda < +\infty\,, \tag{5}$$

for every monomial P with $\deg P \leq 4\,[(n+5)/4]$. Then M is a multiplier of $L^p(\mathbb{H}^n)$, $1<p<\infty$, and is of weak type $(1,1)$.

3. Proof of the $L^p(\mathbb{H}^n)$ multiplier theorem.

We follow [DMM] fairly closely. According to [CW1, Theorem 3.1], to establish the multiplier theorem it suffices to construct a well-behaved approximate identity $\{\phi_r : r>0\}$ satisfying

$$\int_{\mathbb{H}^n} |T_M\psi_r(x)|\left(1+\left(\frac{\rho(x)}{r}\right)^{\varepsilon}\right)dx \leq C\,, \qquad 0<r<+\infty\,, \tag{6}$$

for some $\varepsilon>0$ and $C>0$, where

$$\psi_r = \phi_r - \phi_{r/2} \qquad \text{and} \qquad \rho(x) = |x|^4 = t^2 + \Big(\sum_{j=1}^{n} |z_j|^2\Big)^2 .$$

Note that [CW1, Theorem 3.1] adopts $|x|^{2n+2}$ rather than $|x|^4$. However, if we check their proof carefully, we find that the inequality (6) also implies the $L^p(\mathbb{H}^n)$ boundedness of T_M due to Lemma 3 and Lemma 4 below.

Since we assume $\sup_{\lambda\neq 0}\|M(\lambda)\| < +\infty$, by Plancherel formula, the homogeneity of $\phi_r(x) = r^{-(n+1)/2}\phi_1(r^{-1/4}x)$ (see Lemma 1 below), and changing variables, we have

$$\int_{\mathbb{H}^n} |T_M\psi_r(x)|^2\, dx \leq C\, r^{-(n+1)/2}\,, \qquad 0<r<+\infty\,. \tag{7}$$

If we can also obtain

$$\int_{\mathbb{H}^n} |T_M\psi_r(x)|^2\, \rho(x)^{2[(n+5)/4]}\, dx \leq C\, r^{2[(n+5)/4]-(n+1)/2}\,, \tag{8}$$

for $0 < r < +\infty$, then we claim both inequalities (7) and (8) imply (6), and hence the multiplier theorem for $L^p(\mathbb{H}^n)$ follows. To see this we choose $0 < \varepsilon < [(n+5)/4] - (n+1)/4$. Then by (8) and (1)

$$\begin{aligned}
&\int\limits_{\rho(x)>r} |T_{_M}\psi_r(x)|\,\rho(x)^{\varepsilon}\,dx \\
&\le \Big(\int\limits_{\rho(x)>r} |T_{_M}\psi_r(x)|^2\,\rho(x)^{2[(n+5)/4]}\,dx\Big)^{1/2}\Big(\int\limits_{\rho(x)>r} \rho(x)^{2\varepsilon-2[(n+5)/4]}\,dx\Big)^{1/2} \\
&\le C\,r^{[(n+5)/4]-(n+1)/4}\,r^{\varepsilon-[(n+5)/4]+(n+1)/4} \\
&= C\,r^{\varepsilon}\,.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int\limits_{\rho(x)>r} |T_{_M}\psi_r(x)|\,dx \\
&\le \Big(\int\limits_{\rho(x)>r} |T_{_M}\psi_r(x)|\,\Big(\frac{\rho(x)}{r}\Big)^{\varepsilon}\,dx\Big)^{1/2}\Big(\int\limits_{\rho(x)>r} |T_{_M}\psi_r(x)|\,\Big(\frac{r}{\rho(x)}\Big)^{\varepsilon}\,dx\Big)^{1/2} \\
&\le C\Big(\int\limits_{\rho(x)>r} |T_{_M}\psi_r(x)|\,dx\Big)^{1/2}.
\end{aligned}$$

Combining this and the previous inequality, we obtain

$$\int\limits_{\rho(x)>r} |T_{_M}\psi_r(x)|\,\Big(1+\Big(\frac{\rho(x)}{r}\Big)^{\varepsilon}\Big)\,dx \le C\,. \tag{9}$$

On the other hand, from (7) we have

$$\int\limits_{\rho(x)\le r} |T_{_M}\psi_r(x)|\,dx \le \Big(\int\limits_{\rho(x)\le r} dx\Big)^{1/2}\Big(\int\limits_{\rho(x)\le r} |T_{_M}\psi_r(x)|^2\,dx\Big)^{1/2} \le C\,.$$

Thus

$$\int\limits_{\rho(x)\le r} |T_{_M}\psi_r(x)|\,\Big(1+\Big(\frac{\rho(x)}{r}\Big)^{\varepsilon}\Big)\,dx \le 2\int\limits_{\rho(x)\le r} |T_{_M}\psi_r(x)|\,dx \le C\,. \tag{10}$$

Combining (9) and (10) establishes the claim. Therefore, we only have to prove the inequality (8) to establish the multiplier theorem.

Compare with the construction in [DMM]. The construction of the approximate identity is contained in the following.

Lemma 1. *Let $\phi_1 \in \mathcal{S}(\mathbb{H}^n)$ be the poly-radial function with Fourier coefficients*

$$\mathcal{R}_{\phi_1}(\lambda, 0, \alpha) = \exp\{-(2\,|\alpha| + n)^4\, \lambda^4\}\,, \qquad \lambda \in \mathbb{R}^*, \quad \alpha \in \overline{\mathbb{N}}^n\,.$$

Define

$$\phi_r(t,z) = r^{-(n+1)/2}\, \phi_1\big(r^{-1/2}t, r^{-1/4}z\big)\,, \qquad r > 0\,.$$

Then $\mathcal{R}_{\phi_r}(\lambda, 0, \alpha) = \exp\{-r^2\,(2\,|\alpha| + n)^4\,\lambda^4\}$ and satisfies, for some $\eta > 0$,

i) $\displaystyle\int_{\mathbb{H}^n} |\phi_r(t,z)|\, \Big(1 + \frac{\rho(t,z)}{r}\Big)^{\eta}\, dV \le C\,,$

ii) $\displaystyle\int_{\mathbb{H}^n} \phi_r(t,z)\, dV = 1\,,$

iii) $\phi_r * \phi_s = \phi_s * \phi_r\,,$

iv) $\displaystyle\int_{\mathbb{H}^n} \big|\phi_r\big((t,z)(t_0,z_0)^{-1}\big) - \phi_r(t,z)\big|\, dV \le C\,\Big(\frac{\rho(t_0,z_0)}{r}\Big)^{\eta}\,,$

v) $\phi_r(t,z) = \phi_r(-t,-z)\,.$

Proof. The identity (3) gives

$$\mathcal{R}_{\phi_r}(\lambda, 0, \alpha) = \mathcal{R}_{\phi_1}(\sqrt{r}\lambda, 0, \alpha) = \exp\{-r^2(2\,|\alpha| + n)^4\,\lambda^4\}\,.$$

By the homogeneity of ϕ_r and a change of variables, we have

$$\int_{\mathbb{H}^n} |\phi_r(t,z)|\, \Big(1 + \frac{\rho(t,z)}{r}\Big)^{\eta}\, dV = \int_{\mathbb{H}^n} |\phi_1(t,z)|\, (1 + \rho(t,z))^{\eta}\, dV < +\infty$$

for all $\eta > 0$, since $\phi_1 \in \mathcal{S}(\mathbb{H}^n)$. This proves i). Since $l^0_\alpha(0) = 1$ and

$$\begin{aligned}\exp\{(-r^2(2\,|\alpha| + n)^4\,\lambda^4\} &= \mathcal{R}_{\phi_r}(\lambda, 0, \alpha)\\ &= \int_{\mathbb{H}^n} \phi_r(t, |z_1|, \ldots, |z_n|)\, e^{i\lambda t}\\ &\qquad \cdot\, l^0_{\alpha_1}(2\,|\lambda|\,|z_1|^2) \cdots l^0_{\alpha_n}(2\,|\lambda|\,|z_n|^2)\, dV\,.\end{aligned}$$

Letting $\lambda \to 0$ and applying the Lebesgue dominated convergence theorem, we have ii). Properties iii) and v) follow from the facts that $\hat{\phi}_r(\lambda)$ is poly-diagonal and $\hat{\phi}_r(\lambda) = \hat{\phi}_r(-\lambda)$. To prove iv) it suffices to prove it for $r = 1$ by the homogeneity of ϕ_r and changing variables. Let $L \in \mathfrak{h}$ be the normalized generator of the one parameter subgroup through $(0, z_0)^{-1}$. Then the fundamental theorem of calculus gives

$$\begin{aligned}
\int_{\mathbb{H}^n} \big|\phi_1((t,z)(0,z_0)^{-1}) - \phi_1(t,z)\big|\, dV \\
&\le \int_{\mathbb{H}^n} \int_0^{|z_0|} \big|L\phi_1((t,z)\,\exp(sL))\big|\, ds\, dV \\
&= |z_0|\, \|L\phi_1\|_1 \\
&= \rho(0,z_0)^{1/4}\, \|L\phi_1\|_1\,,
\end{aligned}$$

which proves iv) for $t_0 = 0$. For the general case, we write

$$(t_0, z_0) = \big(t_0, (z_0)_1, \ldots, (z_0)_n\big) \in \mathbb{H}^n$$

and let

$$\begin{aligned}
h_{0j} &= \big(0, 0, \ldots, 0, (z_0)_j, 0, \ldots, 0\big)\,, \\
h_{1j} &= \big(0, 0, \ldots, 0, \frac{i}{2\sqrt{n}}\sqrt{t_0}, 0, \ldots, 0\big)\,, \\
h_{2j} &= \big(0, 0, \ldots, 0, \frac{1}{2\sqrt{n}}\sqrt{t_0}, 0, \ldots, 0\big)\,,
\end{aligned}$$

where each h_{kj}, $(k = 0, 1, 2)$, $(j = 1, 2, \ldots, 5n)$, has its only non-zero entry in the z_j-component. By a straightforward calculation we have

$$\begin{aligned}
(t_0, z_0) &= \prod_{j=1}^{n} \big(\frac{t_0}{n}, 0, \ldots, 0, (z_0)_j, 0, \ldots, 0\big) \\
&= \begin{cases} \prod\limits_{j=1}^{n} h_{0j} h_{1j} h_{2j} h_{1j}^{-1} h_{2j}^{-1}\,, & \text{if} \quad t_0 \ge 0\,, \\ \prod\limits_{j=1}^{n} h_{0j} h_{2j} h_{1j} h_{2j}^{-1} h_{1j}^{-1}\,, & \text{if} \quad t_0 < 0\,. \end{cases}
\end{aligned}$$

Thus we can express $\phi_1\big((t,z)(t_0,z_0)^{-1}\big) - \phi_1(t,z)$ as a sum of $5n$ differences

$$\begin{aligned}\phi_1\big((t,z)(t_0,z_0)^{-1}\big) - \phi_1(t,z) &\equiv \phi_1(xx_1x_2\cdots x_{5n}) - \phi_1(x)\\ &= \phi_1(xx_1x_2\cdots x_{5n}) - \phi_1(xx_1x_2\cdots x_{5n-1})\\ &\quad + \phi_1(xx_1x_2\cdots x_{5n-1}) - \phi_1(xx_1x_2\cdots x_{5n-2})\\ &\quad + \phi_1(xx_1x_2\cdots x_{5n-2}) - \phi_1(xx_1x_2\cdots x_{5n-3})\\ &\quad\ \vdots\\ &\quad + \phi_1(xx_1) - \phi_1(x)\end{aligned}$$

for which each x_j ($= h_{kj}$ or h_{kj}^{-1}, $k = 0,1,2$), $j = 1,2,\ldots,5n$, has t-component zero, and apply the result just established to complete the proof of iv).

Lemma 2. *For every homogeneous polynomial P in $\mathbb{H}^n$ with $1 \le \deg P \le 4\,[(n+5)/4]$, one has*

$$\begin{aligned}&\sup\Big\{\big|\mathcal{R}_{P\psi_r}(\lambda,m,\alpha)\big|^2 :\ m\in\mathbb{Z}^n,\ R<(2\,|\alpha|+n)\,|\lambda|\le\sqrt{2}\,R\Big\}\\ &\qquad\le C_{P}\, r^{(1-n)/2+2[(n+5)/4]}\,R^{1-n+4\,[(n+5)/4]-\deg P}\, f_{P}(rR^2)\,,\end{aligned}\tag{11}$$

for $0<r<+\infty$, where $f_P\in L^1(\mathbb{R}_+)$. Moreover,

$$\begin{aligned}&\big|\mathcal{R}_{\psi_r}(\lambda,0,\alpha)\big|\\ &\qquad\le\begin{cases} C_0\big(r\,(2\,|\alpha|+n)^2\,\lambda^2\big)^2, & \text{for}\ \ r\,(2\,|\alpha|+n)^2\,\lambda^2\le 1\,,\\ 1\,, & \text{for}\ \ r\,(2\,|\alpha|+n)^2\,\lambda^2> 1\,.\end{cases}\end{aligned}\tag{12}$$

Because the proof of Lemma 2 is messy, it will be postponed to the appendix. That will enable the reader to follow the paper without getting lost. We now let $k=[(n+5)/4]$, $\rho(t,z)^k=(t^2+|z|^4)^{[(n+5)/4]}$. By the Plancherel formula, the inequality (8) is equivalent to the following inequality:

$$\int_{-\infty}^{+\infty}\sum_{m,\alpha}\big|\mathcal{R}_{\rho^k T_M\psi_r}(\lambda,m,\alpha)\big|^2\,|\lambda|^n\,d\lambda\le C\,r^{2k-(n+1)/2}\,,\tag{13}$$

for $0<r<+\infty$. Assume we can express $\rho(t,z)^k$ as a linear combination of products of powers of z_j', $\bar z_j'$, z_j-z_j', $\bar z_j-\bar z_j'$, $\delta(t',z')$, $\bar\delta(t',z')$,

$\delta((t,z)(t',z')^{-1})$, $\bar{\delta}((t,z)(t',z')^{-1})$, and $\rho(t',z')$, where $\delta(t,z) = t + i|z|^2$ and each term is homogeneous of degree $4k$. (We use $\tilde{\sum}$ to denote this linear combination.) Then we have a Leibniz formula for the operator Δ_{ρ^k}

$$\Delta_{\rho^k}\left[M(\lambda)\,\hat{\psi}_r(\lambda)\right] = \sum_{\text{finite}} C_j\left[\Delta_{P_j}M(\lambda)\right]\hat{\psi}_r(\lambda) + \sum_{\substack{\text{finite}\\ \deg R_j \geq 1}} C'_j\left[\Delta_{Q_j}M(\lambda)\right]\left[\Delta_{R_j}\hat{\psi}_r(\lambda)\right] \tag{14}$$

for some homogeneous polynomials P_j, Q_j, R_j with $\deg P_j = \deg Q_j + \deg R_j = 4k$, and constants C_j, C'_j. To see this we check some of terms in $\tilde{\sum}$, for instance $[\rho(t',z')]^k$ and $\bar{z}'_j(z_j - z'_j)^{4k-3}\delta((t,z)(t',z')^{-1})$. Write $M(\lambda) = \hat{f}(\lambda)$ for some f and $\psi_r = g$. Then

$$\Delta_{\rho^k}[M(\lambda)\,\hat{\psi}_r(\lambda)] = \Delta_{\rho^k}[(f*g)\hat{\ }(\lambda)] = [\rho^k(f*g)]\hat{\ }(\lambda)$$

and

$$\begin{aligned}
&\rho^k(f*g)(t,z)\\
&\quad= \int_{\mathbb{H}^n} \rho(t,z)^k\, f((t,z)(t',z')^{-1})\, g(t',z')\, dV(t',z')\\
&\quad= \int_{\mathbb{H}^n} \left\{C_1\,\rho(t',z')^k + C_2\,\bar{z}'_j(z_j - z'_j)^{4k-3}\delta((t,z)(t',z')^{-1}) + \cdots\right\}\\
&\qquad\qquad \cdot f((t,z)(t',z')^{-1})\, g(t',z')\, dV(t',z')\\
&\quad\equiv C_1\,A_1(t,z) + C_2\,A_2(t,z) + \cdots,
\end{aligned}$$

where

$$\begin{aligned}
A_1(t,z) &= \int_{\mathbb{H}^n} \rho(t',z')^k\, f((t,z)(t',z')^{-1})\, g(t',z')\, dV(t',z')\\
&= (f*\rho^k g)(t,z)\,.
\end{aligned}$$

Thus

$$\hat{A}_1(\lambda) = \hat{f}(\lambda)\,(\rho^k g)\hat{\ }(\lambda) = M(\lambda)[\Delta_{\rho^k}\hat{\psi}_r(\lambda)]\,.$$

Also

$$\begin{aligned}
&A_2(t,z)\\
&= \int_{\mathbb{H}^n} \bar{z}'_j(z_j - z'_j)^{4k-3}\delta((t,z)(t',z')^{-1})\, f((t,z)(t',z')^{-1})\, g(t',z')\, dV(t',z')\\
&= (Qf * Rg)(t,z)\,,
\end{aligned}$$

where $Q(t,z) = (z_j)^{4k-3}\delta(t,z)$ and $R(t,z) = \bar{z}_j$. Thus,

$$\hat{A}_2(\lambda) = (Qf)\hat{\ }(\lambda)\,(Rg)\hat{\ }(\lambda) = [\Delta_{Q_j} M(\lambda)]\,[\Delta_{R_j}\hat{\psi}_r(\lambda)]\,,$$

and the same process can be carried over to the other terms of $\tilde{\sum}$. We now show that $\rho(t,z)^k$ can be expressed as the linear combination $\tilde{\sum}$. We note that

$$\begin{aligned}\rho(t,z) - \rho(t',z') = {} & (t-t')^2 + 2\,t'(t-t') \\ & + \Big(\sum |z_j|^2 - \sum |z_j'|^2\Big)^2 \\ & + 2\Big(\sum |z_j'|^2\Big)\Big(\sum |z_j|^2 - \sum |z_j'|^2\Big)\,.\end{aligned}$$

Since

$$\begin{aligned}\sum |z_j|^2 = {} & \sum |z_j'|^2 \\ & + \sum\big(|z_j - z_j'|^2 - 2|z_j'|^2 + (z_j - z_j')\bar{z}_j' + (\bar{z}_j - \bar{z}_j')z_j' + 2\,|z_j'|^2\big)\end{aligned}$$

is a linear combination of products of powers of z_j', $\bar{z}_j'$, $z_j - z_j'$, $\bar{z}_j - \bar{z}_j'$. Thus $\rho(t,z)$ is a linear combination of products of powers of t', $t - t'$, z_j', $\bar{z}_j'$, $z_j - z_j'$, $\bar{z}_j - \bar{z}_j'$, and $\rho(t',z')$ with homogeneous degree 4 in each term. Also

$$t' = \frac{1}{2}\big(\delta(t',z') + \bar{\delta}(t',z')\big)\,,$$

$$\begin{aligned}t - t' = {} & \frac{1}{2}\big(\delta((t,z)(t',z')^{-1}) + \bar{\delta}((t,z)(t',z')^{-1})\big) \\ & - i\sum_{j=1}^{n}\big((z_j - z_j')\bar{z}_j' - (\bar{z}_j - \bar{z}_j')\bar{z}_j'\big)\,.\end{aligned}$$

This gives $\rho(t,z)^k = \tilde{\sum}$ as a linear combination of products of powers of z_j', $\bar{z}_j'$, $z_j - z_j'$, $\bar{z}_j - \bar{z}_j'$, $\delta(t',z')$, $\bar{\delta}(t',z')$, $\delta((t,z)(t',z')^{-1})$, $\bar{\delta}((t,z)(t',z')^{-1})$ and $\rho(t',z')$, in which each term is homogeneous of degree $4k$.

By the Leibniz formula (14) we write

$$\int_{-\infty}^{+\infty} \sum_{m,\alpha} \big|\mathcal{R}_{\rho^k T_M \psi_r}(\lambda,m,\alpha)\big|^2\,|\lambda|^n\,d\lambda$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \left\| \Delta_{\rho^k} [M(\lambda) \hat{\psi}_r(\lambda)] \right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&\le C \Bigg(\sum_{\text{finite}} \int_{-\infty}^{+\infty} \left\| [\Delta_{P_j} M(\lambda)] \hat{\psi}_r(\lambda) \right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&\qquad + \sum_{\substack{\text{finite} \\ \deg R_j \ge 1}} \int_{-\infty}^{+\infty} \left\| [\Delta_{Q_j} M(\lambda)] \, [\Delta_{R_j} \hat{\psi}_r(\lambda)] \right\|_{HS}^2 |\lambda|^n \, d\lambda \Bigg) \qquad (15) \\
&\equiv C \Big(\sum_{\text{finite}} I_j + \sum_{\substack{\text{finite} \\ \deg R_j \ge 1}} J_j \Big) .
\end{aligned}
$$

Recall that $\{W_\alpha^m(\lambda) : m \in \mathbb{Z}^n, \ \alpha \in \overline{\mathbb{N}}^n\}$ is an orthonormal basis for the Hilbert-Schmidt operators on $\mathcal{H}_{|\lambda|}$, and

$$
\hat{\Pi}_s(\lambda) = \sum_{s < (2|\alpha|+n)|\lambda| \le \sqrt{2} s} W_\alpha^0(\lambda) .
$$

If P is a homogeneous polynomial with degree $4k$, then

$$
\begin{aligned}
\mathrm{I} &\equiv \int_{-\infty}^{+\infty} \left\| [\Delta_P M(\lambda)] \hat{\psi}_r(\lambda) \right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&= \int_{-\infty}^{+\infty} \left\| [\Delta_P M(\lambda)] \Big[\sum_{j \in \mathbb{Z}} \mathcal{R}_{\psi_r}(\lambda, 0, \alpha) \, \hat{\Pi}_{2^{j/2} r^{-1/2}}(\lambda) \Big] \right\|_{HS}^2 |\lambda|^n \, d\lambda \\
&= \sum_{j \in \mathbb{Z}} \int_{-\infty}^{+\infty} \left\| [\Delta_P M(\lambda)] \, \hat{\Pi}_{2^{j/2} r^{-1/2}}(\lambda) \right\|_{HS}^2 \left| \mathcal{R}_{\psi_r}(\lambda, 0, \alpha) \right|^2 |\lambda|^n \, d\lambda \\
&\equiv \sum_{j<0} + \sum_{j \ge 0} \\
&\equiv \mathrm{I}' + \mathrm{I}'' ,
\end{aligned}
$$

where the coefficients $\mathcal{R}_{\psi_r}(\lambda, 0, \alpha)$ satisfy

$$
2^{j/2} \, r^{-1/2} < (2\,|\alpha| + n)\,|\lambda| \le \sqrt{2}\; 2^{j/2} \, r^{-1/2} .
$$

For $j < 0$, we have $r^{1/2} (2\,|\alpha| + n)\,|\lambda| \le \sqrt{2}\; 2^{j/2} \le 1$, so

$$
\left| \mathcal{R}_{\psi_r}(\lambda, 0, \alpha) \right| \le C \left(r \, (2|\,\alpha| + n)^2 \, \lambda^2 \right)^2 \le C \, 2^{2j+2}
$$

by (12). For $j \geq 0$, we get $r^{1/2}(2\,|\alpha| + n)\,|\lambda| > 2^{j/2} \geq 1$, and hence from (12)

$$\left|\mathcal{R}_{\psi_r}(\lambda, 0, \alpha)\right| \leq 1\,.$$

The basic assumption (5) on the multiplier now implies

$$\begin{aligned}
\mathrm{I}' &\leq C \sum_{j<0} 2^{4j+4} \int_{-\infty}^{+\infty} \left\|[\Delta_P M(\lambda)]\,\hat{\Pi}_{2^{j/2}r^{-1/2}}(\lambda)\right\|_{HS}^2 |\lambda|^n \,d\lambda \\
&\leq C \sum_{j<0} 2^{4j+4} \left(2^{j/2}\,r^{-1/2}\right)^{1+n-4k} \\
&= C\,r^{2k-(n+1)/2}\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}'' &\leq \sum_{j\geq 0} \int_{-\infty}^{+\infty} \left\|[\Delta_P M(\lambda)]\,\hat{\Pi}_{2^{j/2}r^{-1/2}}(\lambda)\right\|_{HS}^2 |\lambda|^n \,d\lambda \\
&\leq C \sum_{j\geq 0} \left(2^{j/2}\,r^{-1/2}\right)^{1+n-4k} \\
&= C\,r^{2k-(n+1)/2}\,.
\end{aligned}$$

For n fixed, there are at most a finite number (depending only on n) of terms of the form I. This proves (13) for the first sum of (15). Next consider two homogeneous polynomials Q, R with $\deg Q + \deg R = 4k$, $\deg R \geq 1$, and

$$\begin{aligned}
\mathrm{J} &\equiv \int_{-\infty}^{+\infty} \left\|[\Delta_Q M(\lambda)]\,[\Delta_R \hat{\psi}_r(\lambda)]\right\|_{HS}^2 |\lambda|^n \,d\lambda \\
&= \int_{-\infty}^{+\infty} \sum_{\alpha\in\overline{\mathbb{N}}^n} \left\|[\Delta_Q M(\lambda)]\,W_\alpha^{m_R}(\lambda)\right\|_{HS}^2 \left|\mathcal{R}_{R\psi_r}(\lambda, m_R, \alpha)\right|^2 |\lambda|^n \,d\lambda
\end{aligned}$$

since $\hat{\psi}_r(\lambda)$ is a poly-diagonal matrix and each of $\{\Delta_t, \Delta_{z_j}, \Delta_{\bar{z}_j}\}_{j=1}^n$ maps a poly-diagonal matrix into a pseudo-poly-diagonal matrix (*i.e.* the one in which each factor has its only non-zero entries on one sub or super diagonal), $\Delta_R\hat{\psi}_r(\lambda)$ is pseudo-poly-diagonal and hence

$$\Delta_R \mathcal{R}_{\psi_r}(\lambda, m, \alpha) = 0 \qquad \text{except for some } m_R \in \mathbb{Z}^n\,.$$

Using (11), (5), and the orthonormality of $\{W_\alpha^m(\lambda)\}$, we have

$$\mathrm{J} = \sum_{j\in\mathbb{Z}} \int_{-\infty}^{+\infty} \left\|[\Delta_Q M(\lambda)]\,\hat{\Pi}_{2^{j/2}}(\lambda)\right\|_{HS}^2 \left|\mathcal{R}_{R\psi_r}(\lambda, m_R, \alpha)\right|^2 |\lambda|^n \,d\lambda$$

$$\le \sum_{j\in\mathbb{Z}} C_R\, r^{(1-n)/2+2k} \left(2^{j/2}\right)^{1-n+4k-\deg R} f_R(2^j r)$$
$$\cdot \int_{-\infty}^{+\infty} \left\| [\Delta_Q M(\lambda)]\, \hat{\Pi}_{2^{j/2}}(\lambda) \right\|_{HS}^2 |\lambda|^n \, d\lambda$$
$$\le \sum_{j\in\mathbb{Z}} C\, r^{(1-n)/2+2k} \left(2^{j/2}\right)^{1-n+4k-\deg R} f_R(2^j r) \left(2^{j/2}\right)^{1+n-\deg Q}$$
$$= C\, r^{(1-n)/2+2k-1} \sum_{j\in\mathbb{Z}} 2^j\, r\, f_R(2^j\, r)$$
$$\approx C\, r^{(1-n)/2+2k-1} \|f_R\|_1$$
$$= C\, r^{2k-(n+1)/2}\,.$$

There are only finitely many terms of the form J, so the inequality (13) for the second sum in (15) is proved. This establishes the multiplier theorem for $L^p(\mathbb{H}^n)$.

4. H^1-L^1 estimate.

In Theorem 1, the multiplier theorem is valid for L^p, $p > 1$. For $p = 1$ we only have a weak-type estimate, so in this section we are trying to extend to another sense of strong type $\|T_M f\|_{L^1} \le C\|f\|_{H^1}$. Here H^1 is the Hardy space on $\mathbb{H}^n$ defined either in terms of maximal functions or in terms of an atomic decomposition [FS]. When $p > 1$, L^p and H^p are essentially the same.

REMARK. We have in fact proved that $\|T_M f\|_{H^p} \le C\|f\|_{H^p}$ for $0 < p \le 1$. The proof is more complicated than the one here and requires the theory of molecules (*cf.* [TW]). The details of this proof will appear elsewhere.

Specifically, we define a $(1,2,0)$-*atom* as an L^2-function f having support in a ball $B_R = \{x \in \mathbb{H}^n : |x| \le R\}$ and satisfying

$$\|f\|_2 \le \left|B_R\right|^{-1/2} \qquad \text{and} \qquad \int_{\mathbb{H}^n} f(x)\, dx = 0\,.$$

It is obvious that $\|f\|_1 \le 1$ for any $(1,2,0)$-atom f.

Theorem (Atomic decomposition of H^1) [FS, Chapter 3]. *Any f in H^1 can be represented as a linear combination of $(1,2,0)$-atoms $f=\sum_{i=1}^{\infty}\lambda_i f_i$, $\lambda_i\in\mathbb{C}$, where the f_i are $(1,2,0)$-atoms and the sum converges in H^1. Moreover,*

$$\|f\|_{H^1}\approx \inf\Big\{\sum_{i=1}^{\infty}|\lambda_i| : \sum_{i=1}^{\infty}\lambda_i f_i \textit{ is a decomposition of } f \textit{ into } (1,2,0)\textit{-atoms}\Big\}.$$

Let $\{\phi_r : r>0\}$ be the approximate identity in Section 3. It is easy to check that $\{\phi_r*\phi_r : r>0\}$ is also an approximate identity and satisfies the same properties i)-v) of Lemma 1. Therefore,

$$\phi_{2^{-i}}*\phi_{2^{-i}}*f\to f \qquad \text{in } L^p$$

and

$$\begin{aligned} f&=\lim_{m\to\infty}\sum_{i=0}^{m}\big(\phi_{2^{-i-1}}*\phi_{2^{-i-1}}-\phi_{2^{-i}}*\phi_{2^{-i}}\big)*f+\phi_1*\phi_1*f\\ &=\lim_{m\to\infty}\sum_{i=0}^{m}-\psi_{2^{-i}}*\big(\phi_{2^{-i}}+\phi_{2^{-i-1}}\big)*f+\phi_1*\phi_1*f \qquad \text{in } L^p\,. \end{aligned}$$

Since we only concern the tail terms in the approach $\phi_{2^{-i}}*\phi_{2^{-i}}*f\to f$, we may assume $\phi_1\equiv 1$. Thus if $\int f(x)\,dx=0$, $\phi_1*\phi_1*f=0$.

In the proof of Theorem 1, we have shown

$$(6)\qquad \int_{\mathbb{H}^n}|T_M\psi_r(x)|\Big(1+\Big(\frac{\rho(x)}{r}\Big)^{\varepsilon}\Big)\,dx\le C\,, \qquad \text{for all } r>0\,.$$

Let η and ε be the constants in Lemma 1 and (6), respectively. Setting $\widetilde{\phi}_r=\phi_r+\phi_{r/2}$, $a_r=T_M\psi_r*\widetilde{\phi}_r$, $K_m=-\sum_{i=0}^{m}a_{2^{-i}}$, and $\delta=\min\{\eta,\varepsilon\}$, we now have the following two lemmas.

Lemma 3.

a) $$\int_{\mathbb{H}^n}|a_r(x)|\,\rho(x)^{\delta}\,dx\le C\,r^{\delta}\,,$$

b) $$\int_{\mathbb{H}^n}|a_r(xy^{-1})-a_r(x)|\,dx\le C\Big(\frac{\rho(y)}{r}\Big)^{\delta}.$$

PROOF. Lemma 1.i) and inequality (6) give the uniform boundedness of $\|\widetilde{\phi}_r\|_1$ and $\|T_{M}\psi_r\|_1$. Applying the triangle inequality, we have

$$\rho(x)^\delta \leq C_\delta \left(\rho(y)^\delta + \rho(y^{-1}x)^\delta\right)$$

and then

$$\begin{aligned}\int_{\mathbb{H}^n} |a_r(x)|\, \rho(x)^\delta\, dx &\leq \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} |T_{M}\psi_r(y)|\, |\widetilde{\phi}_r(y^{-1}x)|\, \rho(x)^\delta\, dxdy \\ &\leq C_\delta \left\|\widetilde{\phi}_r\right\|_1 \int_{\mathbb{H}^n} |T_{M}\psi_r(y)|\, \rho(y)^\delta\, dy \\ &\quad + C_\delta \|T_{M}\psi_r\|_1 \int_{\mathbb{H}^n} \left|\widetilde{\phi}_r(x)\right| \rho(x)^\delta\, dx \\ &\leq C\, r^\delta .\end{aligned}$$

The last inequality is given by Lemma 1.i) and (6) again. The inequality b) is an easy consequence of Lemma 1.iv).

Lemma 4. *Suppose M satisfies the same hypotheses as Theorem* 1. *Then there exist constants C_1 and C_2, independent of m and y, such that*

$$\int_{\rho(x)>C_1\rho(y)} |K_m(xy^{-1}) - K_m(x)|\, dx \leq C_2\,,$$

for all $y \in \mathbb{H}^n$ and for all $m \geq 0$.

PROOF. For $i \in \mathbb{Z}^+$, Lemma 3.a) shows

$$\int_{\rho(x)>\lambda} \left|a_{2^{-i}}(x)\right| dx \leq \frac{1}{\lambda^\delta} \int_{\rho(x)>\lambda} \left|a_{2^{-i}}(x)\right| \rho(x)^\delta\, dx \leq \frac{C}{(2^i\lambda)^\delta}\,.$$

Therefore, choosing $C_1 > 16$, we have

$$\begin{aligned}&\int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(xy^{-1}) - a_{2^{-i}}(x)\right| dx \\ &\qquad \leq \int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(xy^{-1})\right| dx + \int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(x)\right| dx\end{aligned}$$

$$\leq \int_{\rho(xy)>C_1\rho(y)} \left|a_{2^{-i}}(x)\right| dx + \int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(x)\right| dx$$

$$\leq \int_{\rho(x)>C_3\rho(y)} \left|a_{2^{-i}}(x)\right| dx + \int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(x)\right| dx$$

$$\leq \frac{C}{(2^i\,\rho(y))^\delta}$$

since $16\,(\rho(x)+\rho(y)) > \rho(xy) > C_1\,\rho(y)$ implies

$$\rho(x) > \frac{C_1 - 16}{16}\,\rho(y) \equiv C_3\,\rho(y)\,.$$

The above inequality and Lemma 3.b) get

$$\int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(xy^{-1}) - a_{2^{-i}}(x)\right| dx \leq C\,\min\left\{\left(2^i\,\rho(y)\right)^\delta, \frac{1}{(2^i\,\rho(y))^\delta}\right\}.$$

Taking the summation of these inequalities, we obtain

$$\int_{\rho(x)>C_1\rho(y)} |K_m(xy^{-1}) - K_m(x)|\,dx$$

$$\leq \sum_{i=0}^{m} \int_{\rho(x)>C_1\rho(y)} \left|a_{2^{-i}}(xy^{-1}) - a_{2^{-i}}(x)\right| dx$$

$$\leq C \sum_{i<-\log_2\rho(y)} \left(2^i\,\rho(y)\right)^\delta + C \sum_{i\geq -\log_2\rho(y)} \frac{1}{(2^i\,\rho(y))^\delta}$$

$$\leq \frac{C}{2^\delta - 1} + \frac{C}{1-2^{-\delta}}\,.$$

The proof is thus complete.

Now we are ready to prove the H^1-L^1 estimate of T_{M} .

Theorem 2. *Suppose M satisfies the same hypotheses as Theorem* 1. *Then T_{M} maps $H^1(\mathbb{H}^n)$ boundedly into $L^1(\mathbb{H}^n)$. Moreover, there exists a constant $C > 0$, independent of f, such that $\|T_{M} f\|_{L^1} \leq C\,\|f\|_{H^1}$ for all $f \in H^1(\mathbb{H}^n)$.*

PROOF. By the atomic decomposition of H^1, it suffices to show

$$\|T_M f\|_{L^1} \leq C\,, \qquad \text{for any } (1,2,0)\text{-atom } f\,.$$

Given a $(1,2,0)$-atom f with $\operatorname{supp} f \subseteq \{x \in \mathbb{H}^n : |x| \leq R\}$, then $\|f\|_2 \leq C\,R^{-n-1}$ and $\int f(x)\,dx = 0$. The L^2-boundedness of T_M implies

$$T_M f = \lim_{m\to\infty} \sum_{i=0}^{m} -T_M \psi_{2^{-i}} * \widetilde{\phi}_{2^{-i}} * f = \lim_{m\to\infty} K_m * f \qquad \text{in } L^2\,.$$

Thus there exists a subsequence $\{m_j\}$, such that

$$T_M f = \lim_{j\to\infty} K_{m_j} * f \qquad \text{almost everywhere.}$$

Let C_1, C_2 be the constants in Lemma 4. Then

$$\begin{aligned}
&\int\limits_{|x|>C_1^{1/4}R} |T_M f(x)|\,dx \\
&\qquad \leq \liminf_{j\to\infty} \int\limits_{|x|>C_1^{1/4}R} |K_{m_j} * f(x)|\,dx \\
&\qquad \leq \liminf_{j\to\infty} \int\limits_{|x|>C_1^{1/4}R} \left| \int\limits_{|y|\leq R} \big(K_{m_j}(xy^{-1}) - K_{m_j}(x)\big)\, f(y)\,dy \right| dx \\
&\qquad \leq \liminf_{j\to\infty} \int\limits_{|y|\leq R} |f(y)|\,dy \int\limits_{|x|>C_1^{1/4}|y|} \big|K_{m_j}(xy^{-1}) - K_{m_j}(x)\big|\,dx \\
&\qquad \leq C_2\,\|f\|_{L^1} \\
&\qquad \leq C_2\,.
\end{aligned}$$

On the other hand the Schwartz inequality gives

$$\int\limits_{|x|\leq C_1^{1/4}R} |T_M f(x)|\,dx \leq C\,R^{n+1}\,\|T_M f\|_2 \leq C\,R^{n+1}\,\|f\|_2 \leq C\,.$$

This completes the proof.

Appendix. The proof of Lemma 2.

The purpose of this appendix is to prove the lemma that occurs in Section 3.

Lemma 2. *For every homogeneous polynomial P in $\mathbb{H}^n$ with $1 \leq \deg P \leq 4\,[(n+5)/4]$, one has*

$$\begin{aligned} &\sup\Big\{\big|\mathcal{R}_{P\psi_r}(\lambda,m,\alpha)\big|^2 :\ m\in\mathbb{Z}^n,\ R<(2\,|\alpha|+n)\,|\lambda|\leq\sqrt{2}\,R\Big\} \\ &\qquad \leq C_{P}\, r^{(1-n)/2+2[(n+5)/4]}\, R^{1-n+4\,[(n+5)/4]-\deg P}\, f_{P}(rR^2)\,, \end{aligned} \tag{11}$$

for $0<r<+\infty$, where $f_{P}\in L^1(\mathbb{R}_+)$. Moreover,

$$\begin{aligned} &\big|\mathcal{R}_{\psi_r}(\lambda,0,\alpha)\big| \\ &\qquad \leq \begin{cases} C_0\big(r\,(2\,|\alpha|+n)^2\,\lambda^2\big)^2, & \text{for } r\,(2\,|\alpha|+n)^2\,\lambda^2\leq 1\,,\\ 1\,, & \text{for } r\,(2\,|\alpha|+n)^2\,\lambda^2> 1\,. \end{cases} \end{aligned} \tag{12}$$

PROOF. The mean value theorem gives $|e^{-x}-e^{-x/4}|\leq C_0\,x$ for $0\leq x\leq 1$. Then,

$$\begin{aligned} \big|\mathcal{R}_{\psi_r}(\lambda,0,\alpha)\big| &= \big|\mathcal{R}_{\phi_r}(\lambda,0,\alpha)-\mathcal{R}_{\phi_{r/2}}(\lambda,0,\alpha)\big| \\ &= \Big|e^{-r^2(2|\alpha|+n)^4\lambda^4}-e^{-r^2(2|\alpha|+n)^4\lambda^4/4}\Big| \\ &\leq C_0\,r^2\,(2\,|\alpha|+n)^4\,\lambda^4 \end{aligned}$$

for $0\leq r\,(2\,|\alpha|+n)^2\,\lambda^2\leq 1$. For the second estimate of (12), we note that $|e^{-x}-e^{-x/4}|\leq 1$ for $x>1$. Thus (12) is proved. As for (11) we let $\sigma\equiv(2\,|\alpha|+n)\,|\lambda|$ and claim

$$\begin{aligned} \big|\mathcal{R}_{P\phi_r}(\lambda,m,\alpha)\big| &\leq C_{P}\, e^{-r^2\sigma^4}\,\sigma^{-\deg P/2} \\ &\qquad\cdot\Big(\sum_{k=1}^{\deg P}(r\sigma^2)^{2k}+(r\sigma^2)^{2\deg P}\,e^{80\,r^2\sigma^4/81}\Big)\,, \end{aligned} \tag{16}$$

for $R<(2\,|\alpha|+n)\,|\lambda|\leq\sqrt{2}\,R$. Assuming the claim for a moment, we have

$$\begin{aligned} &\big|\mathcal{R}_{P\phi_r}(\lambda,m,\alpha)\big| \\ &\qquad\leq C_{P}\,(r\sigma^2)^{(1-n)/4+[(n+5)/4]}\,\sigma^{-\deg P/2}\,(r\sigma^2)^{2-(1-n)/4-[(n+5)/4]} \end{aligned}$$

$$\cdot \Big(e^{-r^2\sigma^4} \sum_{k=0}^{\deg P-1} (r\sigma^2)^{2k} + (r\sigma^2)^{2\deg P-2}\, e^{-r^2\sigma^4/81} \Big)$$
$$= C_P\, (r\sigma^2)^{(1-n)/4+[(n+5)/4]}\, \sigma^{-\deg P/2}\, g_P(r\sigma^2)\,,$$

where

$$g_P(x) = x^{2-(1-n)/4-[(n+5)/4]} \Big(e^{-x^2} \sum_{k=0}^{\deg P-1} x^{2k} + e^{-x^2/81}\, x^{2\deg P-2} \Big).$$

We note that $\sigma \approx R$ and the exponent $2-(1-n)/4-[(n+5)/4] \geq 1/2$ for $n \in \mathbb{N}$. Hence, we set $f_P \equiv g_P^2 \in L^1(\mathbb{R}_+)$.

To prove the claim, we need the following well-known summation formula

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} i^k = 0\,, \qquad \text{for } \ 0 \leq k \leq m-1\,,\ m \in \mathbb{N}. \tag{17}$$

The idea of the proof of (16) is quite simple, but calculation is messy. We use binomial expansion and apply (17) again and again to establish the inequality (16). We show detailedly the inequality only for $P(t,z) = z_1^a\, \bar z_1^b$ with $a \geq b$ and $P(t,z) = z_1^a\, \bar z_2^b$; the proof can be carried over to the other cases with minor modifications. It follows from (2) and the recurrence relations and differential properties of $\{l_\alpha^m\}$ that, for $a \geq b$,

$$\mathcal{R}_{z_1^a \bar z_1^b \phi_r}(\lambda, (a-b)e_1, \alpha)$$
$$= (2|\lambda|)^{-(a+b)/2} \sqrt{\frac{(\alpha_1+a-b)!}{\alpha_1!}}$$
$$\cdot \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\, \mathcal{R}_{\phi_r}(\lambda, 0, \alpha+(i-j)e_1)\,,$$

and $\mathcal{R}_{z_1^a \bar z_1^b \phi_r}(\lambda, m, \alpha) = 0$ for $m \neq (a-b)e_1$. Then

$$\big|\mathcal{R}_{z_1^a \bar z_1^b \phi_r}(\lambda, (a-b)e_1, \alpha)\big|$$
$$\leq C\, \frac{\sigma^{(a-b)/2}}{|\lambda|^a}$$
$$\cdot \Big| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\, e^{-r^2(\sigma+2(i-j)|\lambda|)^4} \Big|$$

$$
\begin{aligned}
&\le C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4}\,\Bigg|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\Bigg|\\
&\quad + C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4}\\
&\qquad\cdot\Bigg|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{k=1}^{a+b-1}\frac{A_{i,j}^k}{k!}\Bigg| \qquad (18)\\
&\quad + C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4}\\
&\qquad\cdot\Bigg|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{k=a+b}^{\infty}\frac{A_{i,j}^k}{k!}\Bigg|\\
&\equiv S_1+S_2+S_3\,,
\end{aligned}
$$

where

$$
\begin{aligned}
A_{i,j} &= -\,r^2\big(\{\sigma+2\,(i-j)\,|\lambda|\}^4-\sigma^4\big)\\
&= -\,8\,r^2\,|\lambda|\,(i-j)\\
&\qquad\cdot\big(\sigma^3+3\,\sigma^2\,|\lambda|\,(i-j)+4\,\sigma\,|\lambda|^2\,(i-j)^2+2\,|\lambda|^3\,(i-j)^3\big)\,.
\end{aligned}
$$

We immediately obtain $S_1=0$ since the summation $\sum_{j=0}^{b}(-1)^j\binom{b}{j}=0$. To estimate S_3, we have

$$
\begin{aligned}
S_3 &\le C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4}\\
&\qquad\cdot\Bigg|\sum_{\substack{0\le i\le a\\0\le j\le b\\ i\ge j}}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{k=a+b}^{\infty}\frac{A_{i,j}^k}{k!}\Bigg|\\
&\quad + C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4}\\
&\qquad\cdot\Bigg|\sum_{\substack{0\le i\le a\\0\le j\le b\\ i< j}}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{k=a+b}^{\infty}\frac{A_{i,j}^k}{k!}\Bigg|\\
&\le C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\;e^{-r^2\sigma^4}\sum_{\substack{0\le i\le a\\0\le j\le b\\ i\ge j}}\binom{a}{i}\binom{b}{j}\frac{\sigma^b}{|\lambda|^b}\,(r^2\,|\lambda|\,\sigma^3)^{a+b} \qquad (19)
\end{aligned}
$$

$$+ C \frac{\sigma^{(a-b)/2}}{|\lambda|^a} e^{-r^2\sigma^4} \cdot \sum_{\substack{0\le i\le a\\ 0\le j\le b\\ i<j}} \binom{a}{i}\binom{b}{j} \frac{\sigma^b}{|\lambda|^b} (r^2|\lambda|\sigma^3)^{a+b} e^{80 r^2\sigma^4/81}$$

$$\le C\sigma^{-(a+b)/2} e^{-r^2\sigma^4} (r^2\sigma^4)^{a+b} + C\sigma^{-(a+b)/2} e^{-r^2\sigma^4} (r^2\sigma^4)^{a+b} e^{80 r^2\sigma^4/81}$$

since we apply the property of alternating series and the following estimate to the last two inequalities

$$\begin{aligned}\sum_{k=a+b}^{\infty} \frac{A_{i,j}^k}{k!} &= C A_{i,j}^{a+b} \sum_{k=a+b}^{\infty} \frac{A_{i,j}^{k-(a+b)}}{k!}\\ &\le C A_{i,j}^{a+b} e^{A_{i,j}}\\ &\le C (r^2|\lambda|\sigma^3)^{a+b} e^{r^2(\sigma^4-\{\sigma-|2\lambda(i-j)|\}^4)}\\ &\le C (r^2|\lambda|\sigma^3)^{a+b} e^{80 r^2\sigma^4/81},\end{aligned}$$

for $i<j$ and $2|\alpha|+n \ne |2(i-j)|$.

For $i<j$ and $2|\alpha|+n = |2(i-j)|$, the term $e^{-r^2(\sigma+2(i-j)|\lambda|)^4}$ in (18) disappears. Thus

$$\begin{aligned}\left|\mathcal{R}_{z_1^a\bar z_1^b\phi_r}(\lambda,(a-b)e_1,\alpha)\right| &\le C\frac{\sigma^{(a-b)/2}}{|\lambda|^a} \cdot \left|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\right|\\ &= C\frac{\sigma^{(a-b)/2}}{|\lambda|^a} \cdot \left|\sum_{i=0}^{a}(-1)^i\binom{a}{i}\frac{(\alpha_1+i)!}{(\alpha_1+i-b)!}\sum_{j=0}^{b}(-1)^j\binom{b}{j}\right|\\ &= 0.\end{aligned}$$

Hence S_3 fits the format of (16). As to S_2, we write

$$\begin{aligned}
S_2 &= C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\,e^{-r^2\sigma^4} \\
&\quad\cdot\Big|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j}\Big(\sum_{1\le u+v\le b} c_{u,v}\,\alpha_1^u\,i^v\Big) \\
&\qquad\cdot\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\frac{\big(-8\,r^2|\lambda|\,(i-j)\big)^k c_{k,m}\,\sigma^{3k-m}|\lambda|^m(i-j)^m}{k!}\Big| \\
&= C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\,e^{-r^2\sigma^4} \\
&\quad\cdot\Big|\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\sum_{1\le u+v\le b}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\qquad\cdot c_{u,v}\,\alpha_1^u\,i^v(-8\,r^2)^k(i-j)^{k+m}c_{k,m}\,\sigma^{3k-m}|\lambda|^{k+m}\Big| \\
&= C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\,e^{-r^2\sigma^4} \\
&\quad\cdot\Big|\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\sum_{1\le u+v\le b}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\qquad\cdot c_{u,v}\,\alpha_1^u\,i^v(-8\,r^2)^k\sum_{l=0}^{k+m}c_{k,m,l}\,i^{k+m-l}(-j)^l\sigma^{3k-m}|\lambda|^{k+m}\Big| \\
&= C\,\frac{\sigma^{(a-b)/2}}{|\lambda|^a}\,e^{-r^2\sigma^4}\,|S_4+S_5|\,,
\end{aligned}$$

where $c_{u,v}$ and $c_{k,m}$ denote constants dependent on their indexes,

$$\begin{aligned}
S_4 = \sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a-1}\sum_{1\le u+v\le b}\sum_{l=0}^{k+m}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
\cdot\, c_{u,v}\,\alpha_1^u\,i^v(-8\,r^2)^k c_{k,m,l}\,i^{k+m-l}(-j)^l\sigma^{3k-m}\,|\lambda|^{k+m}\,,
\end{aligned}$$

and

$$S_5 = \sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\ge a}\sum_{1\le u+v\le b}\sum_{l=0}^{k+m}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j}$$

$$\cdot\, c_{u,v}\, \alpha_1^u\, i^v (-8\, r^2)^k c_{k,m,l}\, i^{k+m-l} (-j)^l \sigma^{3k-m}\, |\lambda|^{k+m}\,.$$

It follows from (17) that

$$\begin{aligned}
S_4 = & \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{1\le k+m\le a-1} \frac{1}{k!}\,(-8\, r^2)^k \sigma^{3k-m}\, |\lambda|^{k+m} \\
& \cdot \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{1\le u+v\le b} \Big(\sum_{l=0}^{b-1} + \sum_{l=b}^{k+m} \Big)\, c_{u,v,k,m,l}\, (-1)^{i+j} \\
& \qquad \cdot \binom{a}{i} \binom{b}{j} \alpha_1^u\, i^{k+m+v-l} (-j)^l \\
= & \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{1\le k+m\le a-1} \frac{1}{k!}\,(-8\, r^2)^k \sigma^{3k-m}\, |\lambda|^{k+m} \\
& \cdot \sum_{i=0}^{a} \sum_{1\le u+v\le b} \sum_{l=0}^{b-1} c_{u,v,k,m,l}\, (-1)^{i+l} \binom{a}{i} \alpha_1^u\, i^{k+m+v-l} \\
& \qquad \cdot \sum_{j=0}^{b} (-1)^j \binom{b}{j} j^l \\
& + \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{1\le k+m\le a-1} \frac{1}{k!}\,(-8\, r^2)^k \sigma^{3k-m}\, |\lambda|^{k+m} \\
& \cdot \sum_{j=0}^{b} \sum_{1\le u+v\le b} \sum_{l=b}^{k+m} c_{u,v,k,m,l}\, (-1)^{j+l} \binom{b}{j} \alpha_1^u\, j^l \\
& \qquad \cdot \sum_{i=0}^{a} (-1)^i \binom{a}{i} i^{k+m+v-l} \\
= & \; 0\,,
\end{aligned}$$

since $l \ge b$ implies $k+m+v-l \le k+m+b-l \le k+m \le a-1$. Thus we infer

$$S_2 = C\, \frac{\sigma^{(a-b)/2}}{|\lambda|^a}\, e^{-r^2\sigma^4}\, |S_5|\,.$$

Similarly we estimate S_5 as follows.

$$S_5 = \sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\geq a}\sum_{\substack{1\leq u+v\leq b\\ u\geq k+m-a+1}}\sum_{l=0}^{k+m}\frac{1}{k!}(-1)^{i+j}\binom{a}{i}\binom{b}{j}$$
$$\cdot c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k c_{k,m,l}\, i^{k+m-l}(-j)^l\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$+\sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\geq a}\sum_{\substack{1\leq u+v\leq b\\ u\leq k+m-a}}\sum_{l=0}^{k+m}\frac{1}{k!}(-1)^{i+j}\binom{a}{i}\binom{b}{j}$$
$$\cdot c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k c_{k,m,l}\, i^{k+m-l}(-j)^l\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$\equiv S_6 + S_7\ .$$

Using (17) again, we obtain

$$S_6 = \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\geq a}\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{\substack{1\leq u+v\leq b\\ u\geq k+m-a+1}}\Big(\sum_{l=0}^{b-1}+\sum_{l=b}^{k+m}\Big)\frac{1}{k!}(-1)^{i+j}\binom{a}{i}\binom{b}{j}$$
$$\cdot c_{u,v}\,\alpha_1^u\, i^v(-8\,r^2)^k c_{k,m,l}\, i^{k+m-l}(-j)^l\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$= \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\geq a}\frac{1}{k!}(-8\,r^2)^k\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$\cdot\sum_{i=0}^{a}\sum_{\substack{1\leq u+v\leq b\\ u\geq k+m-a+1}}\sum_{l=0}^{b-1}c_{u,v,k,m,l}\,(-1)^{i+l}\binom{a}{i}$$
$$\cdot\,\alpha_1^u\, i^{k+m+v-l}\sum_{j=0}^{b}(-1)^j\binom{b}{j}j^l$$
$$+\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\leq k+m\leq a-1}\frac{1}{k!}(-8\,r^2)^k\sigma^{3k-m}\,|\lambda|^{k+m}$$
$$\cdot\sum_{j=0}^{b}\sum_{\substack{1\leq u+v\leq b\\ u\geq k+m-a+1}}\sum_{l=b}^{k+m}c_{u,v,k,m,l}\,(-1)^{j+l}\binom{b}{j}$$

$$\cdot\, \alpha_1^u\, j^l \sum_{i=0}^{a} (-1)^i \binom{a}{i} i^{k+m+v-l}$$

$$= 0\,,$$

since $k+m+v-l \le b-l+a-1 \le a-1$ for $1 \le u+v \le b$, $u \ge k+m-a+1$, and $l \ge b$.

$$|S_7| = \Bigg| \sum_{i=0}^{a} \sum_{j=0}^{b} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\ge a} \sum_{\substack{1\le u+v\le b\\ u\le k+m-a}} \sum_{l=0}^{k+m} \frac{1}{k!}\, (-1)^{i+j} \binom{a}{i}\binom{b}{j}$$

$$\cdot\, c_{u,v}\, \alpha_1^u\, i^v (-8\, r^2)^k c_{k,m,l}\, i^{k+m-l} (-j)^l \sigma^{3k-m}\, |\lambda|^{k+m} \Bigg|$$

$$\le C_{a,b} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\ge a} |\alpha|^{k+m-a}\, r^{2k}\, \sigma^{3k-m}\, |\lambda|^{k+m}$$

$$= C_{a,b} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\ge a} \sigma^{k+m-a}\, r^{2k}\, \sigma^{3k-m}\, |\lambda|^{a}$$

$$\le C_{a,b} \sum_{k=1}^{a+b-1} r^{2k}\, |\lambda|^a\, \sigma^{4k-a}\,.$$

Hence,

$$S_2 \le C\, \frac{\sigma^{(a-b)/2}}{|\lambda|^a}\, e^{-r^2\sigma^4} \sum_{k=1}^{a+b-1} r^{2k}\, |\lambda|^a\, \sigma^{4k-a}$$

$$= C\, e^{-r^2\sigma^4}\, \sigma^{-(a+b)/2} \sum_{k=1}^{a+b-1} (r\, \sigma^2)^{2k}\,,$$

which combined with (18), (19), and $S_1 = 0$ proves (16) for $P(t,z) = z_1^a\, \bar z_1^b$, $a \ge b$. For $P(t,z) = z_1^a\, \bar z_2^b$, we use (2) and the recurrence relations and differential properties of Laguerre functions again to obtain

$$\mathcal{R}_{z_1^a \bar z_2^b \phi_r}(\lambda, ae_1 - be_2, \alpha) = (2\,|\lambda|)^{-(a+b)/2} \sqrt{\frac{(\alpha_1+a)!}{\alpha_1!}}\, \sqrt{\frac{(\alpha_2+b)!}{\alpha_2!}}$$

$$\cdot \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \mathcal{R}_{\phi_r}(\lambda, 0, \alpha + ie_1 + je_2)$$

and $\mathcal{R}_{z_1^a \bar{z}_2^b \phi_r}(\lambda, m, \alpha) = 0$ for $m \neq ae_1 - be_2$. Then

$$\begin{aligned}
&\left|\mathcal{R}_{z_1^a \bar{z}_2^b \phi_r}(\lambda, ae_1 - be_2, \alpha)\right| \\
&\quad \le C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} \left| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} e^{-r^2 (\sigma + 2(i+j)|\lambda|)^4} \right| \\
&\quad \le C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} e^{-r^2 \sigma^4} \left| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \right| \\
&\qquad + C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} e^{-r^2 \sigma^4} \left| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \sum_{k=1}^{a+b-1} \frac{B_{i,j}^k}{k!} \right| \\
&\qquad + C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} e^{-r^2 \sigma^4} \left| \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{a}{i} \binom{b}{j} \sum_{k=a+b}^{\infty} \frac{B_{i,j}^k}{k!} \right| \\
&\quad \equiv T_1 + T_2 + T_3 ,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
B_{i,j} &= -r^2 \big((\sigma + 2\,(i+j)|\,\lambda|)^4 - \sigma^4 \big) \\
&= -8\, r^2\, |\lambda|\, (i+j) \\
&\quad \cdot \big(\sigma^3 + 3\,\sigma^2\, |\lambda|\, (i+j) + 4\,\sigma\, |\lambda|^2\, (i+j)^2 + 2\, |\lambda|^3\, (i+j)^3 \big) .
\end{aligned}$$

We immediately obtain $T_1 = 0$ since the summation

$$\sum_{j=0}^{b} (-1)^j \binom{b}{j} = 0 .$$

To estimate T_3, we use $|\lambda| \le \sigma$ and the property of alternating series to get

$$\begin{aligned}
T_3 &\le C \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}} e^{-r^2 \sigma^4} \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{a}{i} \binom{b}{j} (r^2\, |\lambda|\, \sigma^3)^{a+b} \\
&\le C\, e^{-r^2 \sigma^4}\, \sigma^{-(a+b)/2}\, (r^2\, \sigma^4)^{a+b} .
\end{aligned} \tag{21}$$

As to T_2, we write

$$\begin{aligned}
T_2 &= C\,\frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\,e^{-r^2\sigma^4} \\
&\quad\cdot\Bigg|\sum_{i=0}^{a}\sum_{j=0}^{b}(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\qquad\cdot\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\frac{\big(-8\,r^2\,|\lambda|\,(i+j)\big)^k c_{k,m}\,\sigma^{3k-m}\,|\lambda|^m(i+j)^m}{k!}\Bigg| \\
&= C\,\frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\,e^{-r^2\sigma^4} \\
&\quad\cdot\Bigg|\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\qquad\cdot(-8\,r^2)^k\,(i+j)^{k+m}\,c_{k,m}\,\sigma^{3k-m}\,|\lambda|^{k+m}\Bigg| \\
&= C\,\frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\,e^{-r^2\sigma^4} \\
&\quad\cdot\Bigg|\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\qquad\cdot(-8\,r^2)^k\sum_{l=0}^{k+m}c_{k,m,l}\,i^{k+m-l}\,j^l\,\sigma^{3k-m}\,|\lambda|^{k+m}\Bigg| \\
&= C\,\frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\,e^{-r^2\sigma^4}\,|T_4+T_5|\,,
\end{aligned}$$

where $c_{k,m}$ and $c_{k,m,l}$ denote constants dependent on their indexes,

$$\begin{aligned}
T_4 = \sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a+b-1}\sum_{l=0}^{k+m}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j}& \\
\cdot(-8\,r^2)^k\,c_{k,m,l}\,i^{k+m-l}\,j^l\,\sigma^{3k-m}\,|\lambda|^{k+m}&\,,
\end{aligned}$$

and

$$T_5 = \sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\ge a+b}\sum_{l=0}^{k+m}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j}$$

$$\cdot\,(-8\,r^2)^k\,c_{k,m,l}\,i^{k+m-l}\,j^l\,\sigma^{3k-m}\,|\lambda|^{k+m}\,.$$

It follows from (17) that

$$\begin{aligned}
T_4 &= \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a+b-1} \frac{1}{k!}\,(-8\,r^2)^k\,\sigma^{3k-m}\,|\lambda|^{k+m} \\
&\qquad \cdot\sum_{i=0}^{a}\sum_{j=0}^{b}\left(\sum_{l=0}^{b-1}+\sum_{l=b}^{k+m}\right) c_{k,m,l}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j}\,i^{k+m-l}\,j^l \\
&= \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a+b-1} \frac{1}{k!}\,(-8\,r^2)^k\,\sigma^{3k-m}\,|\lambda|^{k+m} \\
&\qquad \cdot\sum_{i=0}^{a}\sum_{l=0}^{b-1} c_{k,m,l}\,(-1)^i\binom{a}{i}\,i^{k+m-l}\sum_{j=0}^{b}(-1)^j\binom{b}{j}\,j^l \\
&\quad + \underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{1\le k+m\le a+b-1} \frac{1}{k!}\,(-8\,r^2)^k\,\sigma^{3k-m}\,|\lambda|^{k+m} \\
&\qquad \cdot\sum_{j=0}^{b}\sum_{l=b}^{k+m} c_{k,m,l}\,(-1)^j\binom{b}{j}\,j^l\sum_{i=0}^{a}(-1)^i\binom{a}{i}\,i^{k+m-l} \\
&= 0\,,
\end{aligned}$$

since $l \ge b$ implies $k+m-l \le a+b-1-l \le a-1$. Thus

$$\begin{aligned}
T_2 &= C\,\frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\,e^{-r^2\sigma^4}\,|T_5| \\
&= C\,\frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\,e^{-r^2\sigma^4} \\
&\qquad \cdot\Bigg|\sum_{i=0}^{a}\sum_{j=0}^{b}\underbrace{\sum_{k=1}^{a+b-1}\sum_{m=0}^{3k}}_{k+m\ge a+b}\sum_{l=0}^{k+m}\frac{1}{k!}\,(-1)^{i+j}\binom{a}{i}\binom{b}{j} \\
&\qquad\quad \cdot\,(-8\,r^2)^k\,c_{k,m,l}\,i^{k+m-l}\,j^l\,\sigma^{3k-m}\,|\lambda|^{k+m}\Bigg|
\end{aligned}$$

$$\leq C_{a,b}\, \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\, e^{-r^2\sigma^4} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\geq a+b} r^{2k}\, \sigma^{3k-m}\, |\lambda|^{k+m}$$

$$\leq C_{a,b}\, \frac{\sigma^{(a+b)/2}}{|\lambda|^{a+b}}\, e^{-r^2\sigma^4} \underbrace{\sum_{k=1}^{a+b-1} \sum_{m=0}^{3k}}_{k+m\geq a+b} r^{2k}\, \sigma^{3k-m}\, |\lambda|^{a+b}\, \sigma^{k+m-a-b}$$

$$= C_{a,b}\, e^{-r^2\sigma^4}\, \sigma^{-(a+b)/2} \sum_{k=1}^{a+b-1} (r\,\sigma^2)^{2k}\,,$$

which combined with (20), (21), and $T_1 = 0$ proves (16) for

$$P(t,z) = z_1^a\, \bar{z}_2^b\,,$$

and ends the proof of Lemma 2.

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La "version ondelettes" du théorème du Jacobien

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Résumé. Nous définissons un "produit renormalisé" par ondelettes qui améliore, dans certains cadres fonctionnels, les propriétés du produit usuel de deux fonctions. Grâce à cette technique de renormalisation du produit nous obtenons une démonstration par ondelettes d'une version précisée du théorème du Jacobien. Finalement nous établissons le lien entre ce produit renormalisé par ondelettes et les paraproduits de J.M. Bony.

Introduction.

Certaines opérations algébriques bilinéaires, telles que le produit de deux distributions, deviennent impossibles dans des cadres fonctionnels inadéquats. Pour contourner cette difficulté, il convient, soit de modifier la définition de l'opération bilinéaire, soit d'ajuster le cadre fonctionnel. Nous définissons donc, à partir de la décomposition dans une base orthonormée d'ondelettes de deux fonctions $f, g \in L^2$, des opérateurs bilinéaires qui généralisent et améliorent le produit usuel, que nous appellerons opérateurs de "produit renormalisé". Dans ce texte, nous travaillerons dans des cadres fonctionnels du type $L^p \times L^q$, et nous ferons en sorte que, par exemple, le "produit renormalisé" de deux fonctions de L^2 fournisse une fonction de l'espace de Hardy $\mathcal{H}^1$. Remarquons cependant que "multiplier" une fonction $a \in \mathrm{BMO}$ par une fonction $f \in L^2$ nécessite une modification du produit usuel qui n'est pas du tout la même que celle que nous envisageons ici. Il n'y a donc pas de solution "universelle" au problème de généraliser la définition du produit.

Soulignons surtout le fait que d'autres techniques bien connues, en particulier la théorie de Littlewood-Paley, et les paraproduits de J. M. Bony permettant de "paramultiplier" deux distributions tempérées arbitraires, fournissent des opérateurs de produit renormalisé équivalents à ceux que nous obtenons à l'aide des ondelettes.

On désigne par $\mathcal{H}^1(\mathbb{R}^2)$ l'espace de Hardy dans la version définie par E. Stein et G. Weiss: f appartient à $\mathcal{H}^1(\mathbb{R}^2)$ si et seulement si f et les transformées de Riesz $R_1 f$ et $R_2 f$ appartiennent toutes trois à $L^1(\mathbb{R}^2)$. L'appartenance de f à cet espace $\mathcal{H}^1(\mathbb{R}^2)$ peut être caractérisée par une condition portant sur les modules des coefficients d'ondelettes de f, *cf.* [13].

Le théorème du Jacobien [5] est l'énoncé suivant:

Si $f(x,y)$ et $g(x,y)$ sont deux fonctions appartenant à $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ et si les quatre dérivées (prises au sens des distributions) $\partial f/\partial x$, $\partial f/\partial y$, $\partial g/\partial x$, $\partial g/\partial y$ appartiennent à $L^2(\mathbb{R}^2)$, alors le Jacobien

$$J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

appartient à l'espace de Hardy $\mathcal{H}^1(\mathbb{R}^2)$.

On observe bien évidemment que $(\partial f/\partial x)\,(\partial g/\partial y)$ appartient à $L^1(\mathbb{R}^2)$, et qu'il en est de même pour $(\partial g/\partial x)\,(\partial f/\partial y)$. C'est la différence entre ces deux termes qui introduit précisément la cancellation nécessaire pour passer de $L^1(\mathbb{R}^2)$ à $\mathcal{H}^1(\mathbb{R}^2)$.

Nous allons donner une nouvelle démonstration en même temps qu'une version précisée de ce résultat. Cette démonstration est basée sur la technique de renormalisation par ondelettes du produit de deux fonctions. Elle s'applique également à un cadre fonctionnel un peu plus général que $L^2 \times L^2$, et à des opérateurs bilinéaires plus généraux que le Jacobien, à savoir

$$B(f,g) = \sum_{i=1}^{K} A_i(f)\, B_i(g)\,,$$

où A_i et B_i sont des opérateurs d'intégrales singulières du type étudié par A. Calderón et A. Zygmund, liés par des conditions d'oscillation (les hypothèses seront précisées par la suite).

En appendice nous établissons l'équivalence, "modulo" un espace strictement contenu dans l'espace de Hardy $\mathcal{H}^1$, entre le produit renormalisé par ondelettes et le produit renormalisé par paraproduit.

Soulignons finalement que la méthode de renormalisation du produit à l'aide des paraproduits fournit une démonstration du lemme "Div-Curl" qui est "optimale " du point de vue de sa simplicité (voir [5]), même si toutes les preuves mènent à un résultat équivalent.

1. La renormalisation par ondelettes du produit usuel.

1.1. Les ondelettes utilisées.

Les ondelettes $(\psi_\lambda)_{\lambda\in\Lambda}$, dont nous rappelons ici la construction et les propriétés essentielles, constituent une base orthonormée remarquable de $L^2(\mathbb{R}^n)$.

On désigne par Λ l'ensemble des points $\lambda = (k+\varepsilon/2)2^{-j}$ où $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ et $\varepsilon \in E = \{0,1\}^n \setminus \{(0,0,\ldots,0)\}$.

La base orthonormée d'ondelettes $(\psi_\lambda)_{\lambda\in\Lambda}$ est engendrée à partir de $2^n - 1$ "ondelettes-mère", que nous notons $(\psi^\varepsilon)_{\varepsilon\in E}$ de la façon suivante

$$\psi_\lambda(x) = 2^{nj/2}\,\psi^\varepsilon(2^j x - k) \qquad \text{si} \quad \lambda = \Big(k + \frac{\varepsilon}{2}\Big)\,2^{-j}\,. \tag{1.1}$$

On demande à chacune de ces fonctions ψ^ε de satisfaire aux propriétés suivantes:

(1.2) *régularité*: ψ^ε de classe C^s pour un certain $s \geq 1$;

(1.3) *localisation*:

$$|\partial^\alpha \psi^\varepsilon(x)| \leq C_m\,(1+|x|)^{-m}$$

pour tout multi-indice α tel que $|\alpha| \leq s$, tout entier $m \geq 1$ et tout $x \in \mathbb{R}^n$;

(1.4) $(\psi_\lambda)_{\lambda\in\Lambda}$ est une base orthonormée de $L^2(\mathbb{R}^n)$.

Des propriétés précédentes découle alors celle de cancellation

$$\int_{\mathbb{R}^n} x^\beta\,\psi^\varepsilon(x)\,dx = 0\,, \qquad \text{si} \quad 0 \leq |\beta| \leq s\,. \tag{1.5}$$

La construction d'une base orthonormée d'ondelettes repose sur le concept d'analyse multirésolution introduit par S. Mallat et Y. Meyer ([11] et [12]). Rappelons tout d'abord la définition d'une analyse multirésolution dans le cas unidimensionnel.

Définition 1.1. *Une analyse multirésolution de* $L^2(\mathbb{R})$ *est une suite* $(V_j)_{j\in\mathbb{Z}}$ *de sous-espaces vectoriels fermés de* $L^2(\mathbb{R})$, *vérifiant les propriétés suivantes:*

$$V_j \subset V_{j+1} \tag{1.6}$$

$$f(x) \in V_j \quad \textit{si et seulement si} \quad f(2x) \in V_{j+1} \tag{1.7}$$

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\} \tag{1.8}$$

$$\bigcup_{j\in\mathbb{Z}} V_j \quad \textit{est dense dans} \quad L^2(\mathbb{R}) \tag{1.9}$$

$$\begin{array}{l}\textit{il existe une fonction } \varphi(x) \in V_0 \textit{ telle que} \\ \{\varphi(x-k)\}_{k\in\mathbb{Z}} \textit{ soit une base hilbertienne de } V_0\,.\end{array} \tag{1.10}$$

Considérons maintenant W_j, le supplémentaire orthogonal de V_j dans V_{j+1}. On a

$$L^2(\mathbb{R}) = \overset{\perp}{\bigoplus_{j\in\mathbb{Z}}} W_j \tag{1.11}$$

$$f(x) \in W_0 \quad \text{si et seulement si} \quad f(2^j x) \in W_j\,. \tag{1.12}$$

La construction d'une base orthonormée d'ondelettes de $L^2(\mathbb{R})$ revient finalement à celle d'une base de W_0 de la forme $(\psi(x-k))_{k\in\mathbb{Z}}$. La construction de cette fonction ψ se fait automatiquement à partir de φ (voir [13]).

Revenons maintenant au cas multidimensionel. On pose, pour $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in E$,

$$\psi^{\varepsilon}(x) = \varphi^{\varepsilon_1}(x_1) \cdots \varphi^{\varepsilon_n}(x_n)$$

où

$$\varphi^{\varepsilon_i} = \varphi\,, \qquad \text{si} \quad \varepsilon_i = 0\,,$$
$$\varphi^{\varepsilon_i} = \psi\,, \qquad \text{si} \quad \varepsilon_i = 1\,.$$

Alors, $\{2^{nj/2}\,\psi^\varepsilon(2^j x - k)\}_{j\in\mathbb{Z},\,k\in\mathbb{Z}^n,\,\varepsilon\in E}$ est une base orthonormée d'ondelettes de $L^2(\mathbb{R}^n)$.

Soulignons finalement que nous disposons de bases orthonormées d'ondelettes à support compact, réelles, de régularité s ($1 \le s < +\infty$) arbitraire, construites par I. Daubechies [6].

Note. Dans le but de simplifier les notations, nous supposerons par la suite que toutes les fonctions considérées sont à valeurs réelles.

1.2. La construction du produit renormalisé par ondelettes.

Notons P_j et Q_j les opérateurs de projection orthogonale de $L^2(\mathbb{R}^n)$ sur V_j et W_j respectivement. On a alors

$$(1.13) \qquad \lim_{j\to+\infty} \|P_j f - f\|_2 = 0\,, \qquad \text{pour tout } f \in L^2(\mathbb{R}^n)\,,$$

$$(1.14) \qquad \lim_{j\to-\infty} P_j f = 0 \qquad \text{pour tout } f \in L^2(\mathbb{R}^n)$$

$$(1.15) \qquad P_{j+1} = P_j + Q_j\,.$$

Les propriétés (1.13) et (1.14) nous permettent d'écrire le produit ponctuel fg de deux fonctions de $L^2(\mathbb{R}^n)$ sous la forme d'une série télescopique, convergente dans L^1:

$$fg = \sum_{j\in\mathbb{Z}} \left[(P_{j+1}f)(P_{j+1}g) - (P_j f)(P_j g)\right]\,.$$

Grâce à (1.15), on obtient

$$fg = \sum_{j\in\mathbb{Z}} (P_j f)(Q_j g) + \sum_{j\in\mathbb{Z}} (Q_j f)(P_j g) + \sum_{j\in\mathbb{Z}} (Q_j f)(Q_j g)\,.$$

Notons maintenant

$$\pi_1(f,g) = \sum_{j\in\mathbb{Z}} (P_j f)(Q_j g)\,,$$

$$\pi_2(f,g) = \sum_{j\in\mathbb{Z}} (Q_j f)(P_j g)\,,$$

$$S(f,g) = \sum_{\lambda} \langle f,\psi_\lambda\rangle\,\langle g,\psi_\lambda\rangle\,\psi_\lambda^2\,,$$

$$\pi_3(f,g) = \sum_{j\in\mathbb{Z}} (Q_j f)(Q_j g) - S(f,g)\,.$$

Ainsi, $S(f,g)$ regroupe tous les termes du produit dont l'intégrale n'est pas nulle. Notre but étant de définir un opérateur de produit renormalisé $P^\sharp$ qui soit borné de $L^2 \times L^2$ dans $\mathcal{H}^1$, nous adopterons la définition qui consiste tout simplement à ôter tous les termes dépourvus d'oscillations.

Définition 1.2. *Soient f et g deux fonctions de $L^2(\mathbb{R}^n)$. Nous appellerons produit renormalisé de f et g, associé à la base orthonormée d'ondelettes $(\psi_\lambda)_{\lambda\in\Lambda}$, et nous noterons $P^\sharp(f,g)$, la fonction de $L^1(\mathbb{R}^n)$ définie par*

$$P^\sharp(f,g) = fg - \sum_{\lambda\in\Lambda} \langle f,\psi_\lambda\rangle\,\langle g,\psi_\lambda\rangle |\psi_\lambda|^2\,. \tag{1.16}$$

Les propriétés particulières de la base orthonormée d'ondelettes entraînent alors la

Proposition 1.1. *L'opérateur $P^\sharp$ est continu de $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ dans l'espace de Hardy réel $\mathcal{H}^1(\mathbb{R}^n)$.*

Afin de simplifier les notations, supposons que les fonctions φ et ψ sont à support compact, et concentrons nous sur le cas de la dimension 1 (la démonstration que nous proposons se généralise immédiatement au cas multidimensionnel).

Dans ce contexte, les produits $\varphi_{jk}\ \psi_{j,\,k+m}$ s'annulent dès que l' entier m vérifie $|m| > C_0$, où la constante C_0 ne dépend que de la taille des supports des fonctions φ et ψ. Notons w_{jk}^m le produit $\varphi_{jk}\,\psi_{j,\,k+m}$. Pour chaque valeur de m fixée, la famille $(w_{jk}^m)_{j,\,k\in\mathbb{Z}}$, a essentiellement les mêmes propriétés que la base orthonormée d'ondelettes, la seule propriété manquante étant l'orthogonalité.

L'opérateur π_1 a donc pour expression

$$\pi_1(f,g) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{|m|\leq C_0} 2^{j/2}\langle f,\varphi_{jk}\rangle\,\langle g,\psi_{j,k+m}\rangle\, w_{jk}^m\,.$$

Rappellons maintenant la caractérisation par ondelettes de l'espace de Hardy $\mathcal{H}^1(\mathbb{R}^n)$:

$$\sum_{\lambda\in\Lambda}\alpha_\lambda\psi_\lambda \in \mathcal{H}^1(\mathbb{R}^n) \Longleftrightarrow \left(\sum_{\lambda\in\Lambda}|\alpha_\lambda|^2 2^{nj}\chi_\lambda(x)\right)^{1/2} \in L^1(\mathbb{R}^n)\,,$$

où χ_λ désigne la fonction indicatrice du cube dyadique associé à λ, qui est défini par la condition $2^jx-k\in[0,1[^n$.

Des calculs standard montrent que l'opérateur $U: L^2\to L^2$, défini par $U(\psi_{jk}) = w_{jk}$, est un opérateur de Calderón-Zygmund qui vérifie les conditions $U(1)=U^*(1)=0$, et est en conséquence borné dans $\mathcal{H}^1$ et dans BMO.

Pour obtenir la continuité $L^2\times L^2\to\mathcal{H}^1$, il suffit donc de montrer que, pour tout $m\in\mathbb{Z}, |m|\leq C_0$, la série

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{j/2}\langle f,\varphi_{jk}\rangle\,\langle g,\psi_{j,k+m}\rangle\,\psi_{jk}\in\mathcal{H}^1(\mathbb{R})\,,$$

ce qui revient à vérifier que

$$\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{2j}\,|\langle f,\varphi_{jk}\rangle\,\langle g,\psi_{j,k+m}\rangle|^2\,\chi_{jk}(x)\Big)^{1/2}\in L^1(\mathbb{R})\,.$$

Il suffit maintenant de montrer que la fonction

$$\Big(\sup_{\substack{j,k\\ x\in Q_{jk}}}|\langle f, 2^{j/2}\varphi_{jk}\rangle|\Big)\,\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^j\,|\langle g,\psi_{j,k+m}\rangle|^2\,\chi_{jk}(x)\Big)^{1/2}$$

appartient à $L^1(\mathbb{R})$.

Or, si nous désignons par f^* la fonction maximale de Hardy et Littlewood, nous avons, pour tout $x \in \mathbb{R}$,

$$\sup_{\substack{j,k \\ x \in Q_{jk}}} |\langle f, 2^{j/2} \varphi_{jk} \rangle| \leq C\, f^*(x)\,.$$

Pour conclure, il suffit alors d'appliquer l'inégalité de Hölder et de remarquer que

$$\|f^*\|_2 \left\| \Big(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^j \, |\langle g, \psi_{j,k+m} \rangle|^2 \, \chi_{jk}(x) \Big)^{1/2} \right\|_{L^2} \leq C \, \|f\|_2 \, \|g\|_2 \,.$$

La continuité de l'opérateur π_2 s'établit de manière identique.

Quant à $\pi_3(f,g)$, des calculs immédiats montrent en fait son appartenance à un espace strictement contenu dans $\mathcal{H}^1(\mathbb{R}^n)$, l'espace de Besov homogène $\dot{B}_1^{0,1}(\mathbb{R}^n)$, dont nous rappellerons la définition par la suite.

Proposition 1.2. *Si $f_q \rightharpoonup f$ pour la topologie faible $\sigma(L^2, L^2)$, et $g_q \rightharpoonup g$ pour la même topologie, alors la suite $P^\sharp(f_q, g_q)$ converge faiblement vers $P^\sharp(f,g)$ au sens de la topologie faible $\sigma(\mathcal{H}^1, \mathrm{VMO})$.*

Rappelons que VMO est la fermeture, pour la norme de BMO, de l'espace vectoriel des fonctions continues et nulles à l'infini. L'espace $\mathcal{H}^1$ est le dual de VMO.

Considérons l'exemple de l'opérateur π_1. Sous nos hypothèses, et grâce à la Proposition 1.1, la suite $(\pi_1(f_q, g_q))$ est bornée dans $\mathcal{H}^1$. Comme l'opérateur $U : \psi_{jk} \to w_{jk}$ est borné dans $\mathcal{H}^1$, il suffit maintenant de montrer que, pour chaque $|m| \leq C_0$, la suite

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{j/2} \, \langle f_q, \varphi_{jk} \rangle \, \langle g_q, \psi_{j,k+m} \rangle \, \psi_{jk}$$

converge au sens de la topologie faible $\sigma(\mathcal{H}^1, \mathrm{VMO})$ vers

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{j/2} \, \langle f, \varphi_{jk} \rangle \, \langle g, \psi_{j,k+m} \rangle \, \psi_{jk} \,.$$

Ceci découle immédiatement du lemme suivant

Lemme 1.1. *Soit* $(h_q)_{q\in\mathbb{N}}$ *une suite bornée dans* $\mathcal{H}^1(\mathbb{R})$*. Alors,* h_q *converge vers* h *au sens de la topologie* $\sigma(\mathcal{H}^1, \mathrm{VMO})$ *si et seulement si* $\langle h_q, \psi_{jk}\rangle \to \langle h, \psi_{jk}\rangle$ *pour tout* $j \in \mathbb{Z}$ *et tout* $k \in \mathbb{Z}$*.*

Cette équivalence est conséquence du fait que la base orthonormée d'ondelettes $(\psi_\lambda)_{\lambda\in\Lambda}$ constitue une famille totale dans l'espace de Banach VMO.

1.3. L'équivalence entre les opérateurs $P^\sharp$ et $\tilde{P}^\sharp$ associés à deux bases orthonormées d'ondelettes.

Une fois le produit renormalisé $P^\sharp$ construit, la question suivante se pose tout naturellement : cette renormalisation $P^\sharp(f, g)$ est-elle, en un certain sens, indépendante de la base d'ondelettes choisie pour la construction de $P^\sharp$?

Pour répondre à cette question, nous introduisons maintenant l'espace de Besov homogène $\dot{B}_1^{0,1}(\mathbb{R}^n)$ ([13],[15]).

Définition 1.3. *L'espace de Besov homogène* $\dot{B}_1^{0,1}(\mathbb{R}^n)$ *est le sous-espace de* $L^1(\mathbb{R}^n)$ *composé des fonctions* $f \in L^1(\mathbb{R}^n)$ *dont la série dans une base orthonormée d'ondelettes* $f = \sum_{\lambda\in\Lambda} \alpha_\lambda \psi_\lambda$ *vérifie*

$$\sum_{\lambda\in\Lambda} |\alpha_\lambda|\, 2^{-nj/2} < +\infty\,. \tag{1.17}$$

En d'autres termes, la fonction f appartient à $\dot{B}_1^{0,1}(\mathbb{R}^n)$ si et seulement si $\sum_{\lambda\in\Lambda} |\alpha_\lambda|\, \|\psi_\lambda\|_1 < +\infty$, c'est à dire si la série d'ondelettes de f converge normalement dans $L^1(\mathbb{R}^n)$.

Une autre caractérisation de $\dot{B}_1^{0,1}$ sera utile par la suite.

Définition 1.4. *Une famille* $\{m_{jk}\}_{j\in\mathbb{Z}\,,\,k\in\mathbb{Z}^n}$ *est une famille normalisée en norme* L^1 *de molécules s'il existe deux exposants* $\alpha \geq \beta > 0$*, et une constante* $C > 0$ *tels que*

$$|m_{jk}(x)| \leq C\, 2^{nj}\, (1 + |2^j x - k|)^{-n-\alpha}\,, \tag{1.18}$$

pour tout $x \in \mathbb{R}^n$,

$$|m_{jk}(x) - m_{jk}(y)| \le C\, 2^{j(n+\beta)}\, |x-y|^\beta \tag{1.19}$$

pour tout $(x,y) \in (\mathbb{R}^n)^2$, *et*

$$\int_{\mathbb{R}^n} m_{jk}(x)\, dx = 0\,. \tag{1.20}$$

Remarque. Toute base orthonormée d'ondelettes de régularité $s \ge 1$ constitue une famille normalisée en norme L^2 de molécules.

La deuxième caractérisation de l'espace de Besov homogène $\dot{B}_1^{0,1}(\mathbb{R}^n)$ découle du lemme suivant:

Lemme 1.2. *Soit* (m_{jk}) *une famille normalisée en norme* L^1 *de molécules et* (α_{jk}) *une suite dans* $\ell^1(\mathbb{Z} \times \mathbb{Z}^n)$*:* $\sum_j \sum_k |\alpha_{jk}| < +\infty$*. Alors* $\sum \alpha_{jk}\, m_{jk}$ *appartient à* $\dot{B}_1^{0,1}(\mathbb{R}^n)$*.*

Nous pouvons maintenant énoncer le

Théorème 1.1. *Soient* $(\psi_\lambda)_{\lambda\in\Lambda}$ *et* $(\tilde{\psi}_\lambda)_{\lambda\in\Lambda}$ *deux bases orthonormées d'ondelettes de même régularité* s*,* $P^\sharp$ *et* $\tilde{P}^\sharp$ *les opérateurs de produit renormalisé associés respectivement à chacune de ces bases. Alors l'opérateur* $\Delta = P^\sharp - \tilde{P}^\sharp$ *envoie continûment* $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *dans* $\dot{B}_1^{0,1}(\mathbb{R}^n)$*.*

Cet énoncé signifie que, modulo l'espace $\dot{B}_1^{0,1}$, qui est strictement contenu dans $\mathcal{H}^1$, toutes les renormalisations du produit par ondelettes d'une certaine régularité sont équivalentes.

Les idées essentielles dans la preuve de ce théorème sont les mêmes que celles qui nous permettront d'établir le théorème du Jacobien. C'est pourquoi nous donnerons ici les étapes principales de cette preuve.

Si

$$f = \sum_\lambda \alpha_\lambda \psi_\lambda = \sum_\lambda \tilde{\alpha}_\lambda \tilde{\psi}_\lambda$$

et

$$g = \sum_\lambda \beta_\lambda \psi_\lambda = \sum_\lambda \tilde{\beta}_\lambda \tilde{\psi}_\lambda\ ,$$

alors la différence entre les deux produits renormalisés $\Delta(f,g)$ s'écrit

$$\Delta(f,g) = P^\sharp(f,g) - \tilde{P}^\sharp(f,g) = \Delta_1(f,g) + \Delta_2(f,g)$$

où

$$\Delta_1(f,g) = \sum_\lambda \tilde{\alpha}_\lambda \tilde{\beta}_\lambda \big(\psi_\lambda^2 - \tilde{\psi}_\lambda^2\big)$$

et

$$\Delta_2(f,g) = \sum_\lambda \big(\alpha_\lambda \beta_\lambda - \tilde{\alpha}_\lambda \tilde{\beta}_\lambda\big) \psi_\lambda^2\,.$$

Il est facile de vérifier que le terme $\Delta_1(f,g)$ appartient à l'espace $\dot{B}_1^{0,1}(\mathbb{R}^n)$ car $(\psi_\lambda^2 - \tilde{\psi}_\lambda^2)_{\lambda\in\Lambda}$ constitue une famille normalisée en norme L^1 de molécules, et la suite $(\tilde{\alpha}_\lambda \tilde{\beta}_\lambda)_{\lambda\in\Lambda}$ appartient à $\ell^1(\Lambda)$. Observons, en effet, que

$$\int (\psi_\lambda^2 - \tilde{\psi}_\lambda^2) = \|\psi_\lambda\|_2^2 - \|\tilde{\psi}_\lambda\|_2^2 = 0\,.$$

La vérification des autres propriétés des molécules est immédiate.

Pour étudier le terme $\Delta_2(f,g)$, on remarque que les suites (α_λ), $(\tilde{\alpha}_\lambda)$ puis (β_λ), $(\tilde{\beta}_\lambda)$ sont liées par les équations

$$\text{(1.21)}\qquad \begin{cases} \tilde{\alpha}_\lambda = \displaystyle\sum_{\lambda'\in\Lambda} \mu(\lambda',\lambda)\,\alpha_{\lambda'} \\ \tilde{\beta}_\lambda = \displaystyle\sum_{\lambda'\in\Lambda} \mu(\lambda',\lambda)\,\beta_{\lambda'} \end{cases}$$

où $\mu(\lambda',\lambda) = \langle \tilde{\psi}_\lambda, \psi_{\lambda'}\rangle$.

La matrice de changement de base $M = (\mu(\lambda',\lambda))_{(\lambda',\lambda)\in\Lambda^2}$ est unitaire, et présente une certaine décroissance à partir de la diagonale, que nous détaillons par la suite.

Pour $\gamma > 0$, nous définissons les "poids" $p_\gamma(\lambda,\lambda')$ par

$$\text{(1.22)}\quad p_\gamma(\lambda',\lambda) = \frac{2^{-|j'-j|(\gamma+n/2)}}{1+(j-j')^2}\Big(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k\,2^{-j}-k'\,2^{-j'}|}\Big)^{n+\gamma}.$$

Ces $p_\gamma(\lambda',\lambda)$ ont la propriété essentielle suivante (voir [14])

Lemme 1.3. *Soit* $0 < \gamma_3 \le \gamma_2 \le \gamma_1$. *Il existe une constante* $c = c(n, \gamma_1, \gamma_2, \gamma_3) > 0$ *telle que*

$$\sum_{\lambda\in\Lambda} p_{\gamma_2}(\lambda', \lambda)\, p_{\gamma_1}(\lambda'', \lambda) \le c\, p_{\gamma_3}(\lambda', \lambda''),$$

pour tout $(\lambda', \lambda'') \in \Lambda^2$.

Les coefficients de la matrice M sont estimés par

$$|\mu(\lambda', \lambda)| \le c\, p_{\gamma_1}(\lambda', \lambda), \tag{1.23}$$

pour tout $(\lambda', \lambda) \in \Lambda^2$, où la valeur de γ_1 dépend de la régularité de la base orthonormée d'ondelettes (voir à ce sujet [14]).

Retournons maintenant à $\Delta_2(f, g)$ que nous réécrivons

$$\begin{aligned}\Delta_2(f,g) &= \sum_{\lambda\in\Lambda} \Big[\alpha_\lambda\beta_\lambda - \Big(\sum_{\lambda'} \alpha_{\lambda'}\, \mu(\lambda', \lambda)\Big)\Big(\sum_{\lambda''} \beta_{\lambda''}\, \mu(\lambda'', \lambda)\Big)\Big]\psi_\lambda^2 \\ &= C(f,g) - R(f,g)\end{aligned}$$

où

$$\begin{aligned}C(f,g) &= \sum_{\lambda\in\Lambda} \Big(\alpha_\lambda\beta_\lambda - \sum_{\lambda'} \alpha_{\lambda'}\beta_{\lambda'}\, \mu(\lambda', \lambda)^2\Big)\, \psi_\lambda^2 \\ &= \sum_{\lambda\in\Lambda} \alpha_\lambda\beta_\lambda \Big(\psi_\lambda^2 - \sum_{\lambda'} \mu(\lambda', \lambda)^2\, \psi_{\lambda'}^2\Big)\end{aligned}$$

et

$$R(f,g) = \sum_{\lambda} \sum_{\substack{(\lambda', \lambda'')\in\Lambda^2 \\ \lambda'\ne\lambda''}} \alpha_{\lambda'}\beta_{\lambda''}\, \mu(\lambda', \lambda)\, \mu(\lambda'', \lambda)\, \psi_\lambda^2 .$$

L'étude de $C(f, g)$ se ramène à celle des fonctions

$$m_\lambda = \psi_\lambda^2 - \sum_{\lambda'\in\Lambda} \mu(\lambda', \lambda)^2\, \psi_{\lambda'}^2 ,$$

et découle du

Lemme 1.4. *Soit*

$$m_\lambda = \psi_\lambda^2 - \sum_{\lambda'\in\Lambda} \mu(\lambda', \lambda)^2\, \psi_{\lambda'}^2 .$$

L'ensemble $(m_\lambda)_{\lambda\in\Lambda}$ constitue alors une famille normalisée en norme L^1 de molécules.

Dans la preuve du Lemme 1.4, les propriétés de la matrice de passage $M = (\mu(\lambda', \lambda''))$ jouent un rôle essentiel.

La série $\sum_{\lambda'} \mu(\lambda', \lambda)^2 \, \|\psi^2_{\lambda'}\|_1$ étant convergente, l'intégration terme à terme donne

$$\int m_\lambda = \int \psi^2_\lambda - \sum_{\lambda'} \mu(\lambda', \lambda)^2 \int \psi^2_{\lambda'} = 1 - \sum_{\lambda'} \mu(\lambda', \lambda)^2 = 0\,.$$

Les autres propriétés des molécules découlent de la localisation et de la régularité des ondelettes, ainsi que des propriétés particulières des poids $p_\gamma(\lambda', \lambda)$.

Considérons finalement la partie $R(f, g)$, que nous réécrivons

$$R(f,g) = \sum_{\lambda'} \sum_{\lambda''} \alpha_{\lambda'} \beta_{\lambda''} \, p_{\gamma_2}(\lambda', \lambda'') \sum_{\lambda\in\Lambda} \frac{\mu(\lambda', \lambda)\, \mu(\lambda'', \lambda)}{p_{\gamma_2}(\lambda', \lambda'')} \, \psi^2_\lambda\,,$$

où γ_2 vérifie $0 < \gamma_2 < \gamma_1$.

Posons

$$m_{\lambda', \lambda''} \;=\; \sum_{\lambda\in\Lambda} \frac{\mu(\lambda', \lambda)\, \mu(\lambda'', \lambda)}{p_{\gamma_2}(\lambda', \lambda'')} \, \psi^2_\lambda\,.$$

Alors, si le coefficient γ_2 est choisi convenablement, l'ensemble

$$(m_{\lambda', \lambda''})_{(\lambda', \lambda'')\in\Lambda^2,\, \lambda'\neq\lambda''}$$

constitue une famille normalisée en norme L^1 de molécules.

Reste à voir finalement que

$$\sum_{\lambda'} \sum_{\lambda''} |\alpha_{\lambda'}| \, |\beta_{\lambda''}| \, p_{\gamma_2}(\lambda', \lambda'') < +\infty\,.$$

L'application du lemme de Schur avec les coefficients $w(\lambda) = 2^{-nj/2}$ à la matrice $(p_{\gamma_2}(\lambda', \lambda''))_{(\lambda', \lambda'')\in\Lambda^2}$ montre que celle-ci définit un opérateur continu dans $\ell^2(\Lambda)$ (on pourra consulter [14] à ce sujet).

La série

$$\sum_{\lambda'\in\Lambda} \sum_{\lambda''\in\Lambda} |\alpha_{\lambda'}| \, |\beta_{\lambda''}| \, p_{\gamma_2}(\lambda'', \lambda')$$

est ainsi majorée par

$$c \Big(\sum_{\lambda'\in\Lambda} |\alpha_{\lambda'}|^2 \Big)^{1/2} \Big(\sum_{\lambda''\in\Lambda} |\beta_{\lambda''}|^2 \Big)^{1/2} = c\, \|f\|_2 \, \|g\|_2\,.$$

2. La "version ondelettes" du théorème du Jacobien.

Le théorème du Jacobien est l'énoncé suivant [5].

Théorème 2.1. *Si $f(x,y)$ et $g(x,y)$ sont deux fonctions appartenant à $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ et si les quatre dérivées (prises au sens des distributions) $\partial f/\partial x$, $\partial f/\partial y$, $\partial g/\partial x$, $\partial g/\partial y$ appartiennent à $L^2(\mathbb{R}^2)$, alors le Jacobien*

$$J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

appartient à l'espace de Hardy $\mathcal{H}^1(\mathbb{R}^2)$.

Notre énoncé précisé est le suivant

Théorème 2.2. *Soit $(\psi_\lambda)_{\lambda\in\Lambda}$ une base orthonormée d'ondelettes et $P^\sharp$ l'opérateur de produit renormalisé associé à cette base. Alors, sous les mêmes hypothèses que le Théorème 2.1, on a*

$$J(f,g) \in \mathcal{H}^1(\mathbb{R}^2)$$

et

$$J_B(f,g) = J(f,g) - P^\sharp\Big(\frac{\partial f}{\partial x}\,,\,\frac{\partial g}{\partial y}\Big) + P^\sharp\Big(\frac{\partial f}{\partial y}\,,\,\frac{\partial g}{\partial x}\Big) \in \dot{B}^{0,1}_1(\mathbb{R}^2)\,.$$

Notre méthode consiste à renormaliser chacun des produits

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} \qquad \text{et} \qquad \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\,.$$

Les termes $P^\sharp(\partial f/\partial x\,,\,\partial g/\partial y)$ et $P^\sharp(\partial f/\partial y\,,\,\partial g/\partial x)$ appartiennent de façon automatique à $\mathcal{H}^1(\mathbb{R}^2)$.

Considérons maintenant la somme des "mauvais" termes qui étaient individuellement dépourvus d'oscillations.

$$J_B(f,g) = \sum_{\lambda\in\Lambda}\Big(\langle\frac{\partial f}{\partial y}\,,\psi_\lambda\rangle\,\langle\frac{\partial g}{\partial x}\,,\psi_\lambda\rangle - \langle\frac{\partial f}{\partial x}\,,\psi_\lambda\rangle\,\langle\frac{\partial g}{\partial y}\,,\psi_\lambda\rangle\Big)\,\psi_\lambda^2\,.$$

Si

$$f = \sum_{\lambda \in \Lambda} \alpha_\lambda \psi_\lambda \ , \qquad g = \sum_{\lambda \in \Lambda} \beta_\lambda \psi_\lambda \ ,$$

l'appartenance à $L^2(\mathbb{R}^2)$ des dérivées partielles de f et g entraîne l'appartenance à $\ell^2(\Lambda)$ des suites $(2^j \alpha_\lambda)_{\lambda \in \Lambda}$ et $(2^j \beta_\lambda)_{\lambda \in \Lambda}$.

Posons maintenant

$$\text{(2.1)} \qquad \begin{cases} \mu_x(\lambda', \lambda) = \langle \dfrac{\partial \psi_{\lambda'}}{\partial x} \, , \psi_\lambda \rangle \, , \\ \mu_y(\lambda', \lambda) = \langle \dfrac{\partial \psi_{\lambda'}}{\partial y} \, , \psi_\lambda \rangle \, . \end{cases}$$

On peut alors écrire

$$J_B(f,g) = \sum_{\lambda \in \Lambda} \Bigg(\sum_{(\lambda', \lambda'') \in \Lambda^2} \alpha_{\lambda'} \beta_{\lambda''} \big(\mu_y(\lambda', \lambda) \, \mu_x(\lambda'', \lambda) \\ - \mu_x(\lambda', \lambda) \, \mu_y(\lambda'', \lambda) \big) \Bigg) \psi_\lambda^2 \ .$$

Remarquons maintenant que l'on a de façon évidente les estimations suivantes

$$|\mu_x(\lambda', \lambda)| \leq c \, 2^{j'} \, p_{\gamma_1}(\lambda', \lambda)$$

et

$$|\mu_y(\lambda', \lambda)| \leq c \, 2^{j'} \, p_{\gamma_1}(\lambda', \lambda)$$

où les $p_\gamma(\lambda', \lambda)$ sont définis par l'équation (1.22).

Posons maintenant, pour $0 < \gamma_2 < \gamma_1$

$$m_{\lambda' \lambda''} = \frac{2^{-j'} \, 2^{-j''}}{p_{\gamma_2}(\lambda', \lambda'')} \sum_{\lambda \in \Lambda} \big(\mu_y(\lambda', \lambda) \, \mu_x(\lambda'', \lambda) - \mu_x(\lambda', \lambda) \, \mu_y(\lambda'', \lambda) \big) \, \psi_\lambda^2 \ .$$

Nous pouvons alors réécrire $J_B(f,g)$ sous la forme suivante

$$J_B(f,g) = \sum_{(\lambda', \lambda'') \in \Lambda^2} 2^{j'} \alpha_{\lambda'} \, 2^{j''} \beta_{\lambda''} \, p_{\gamma_2}(\lambda', \lambda'') \, m_{\lambda' \lambda''} \ .$$

Les deux lemmes suivants nous permettent alors de conclure.

Lemme 2.1. *La suite* $\left(2^{j'+j''}\alpha_{\lambda'}\beta_{\lambda''}\,p_{\gamma_2}(\lambda',\lambda'')\right)_{(\lambda',\lambda'')\in\Lambda^2}$ *appartient à l'espace* $\ell^1(\Lambda^2)$ *.*

Lemme 2.2. *L'ensemble des* $(m_{\lambda',\lambda''})_{(\lambda',\lambda'')\in\Lambda^2}$ *décrit une famille normalisée en norme* L^1 *de molécules.*

Nous montrerons seulement ici pourquoi $m_{\lambda'\lambda''}$ est une fonction de $L^1(\mathbb{R}^2)$, d'intégrale nulle.

Calculons tout d'abord, pour $(\lambda',\lambda'')\in\Lambda^2$, la somme

$$\begin{aligned}\sum_{\lambda\in\Lambda}\int_{\mathbb{R}^2}\left(|\mu_y(\lambda',\lambda)\,\mu_x(\lambda'',\lambda)-\mu_x(\lambda',\lambda)\,\mu_y(\lambda'',\lambda)\,|\psi_\lambda^2(x)\right)\,dx\\ \le c\,2^{j'+j''}\sum_{\lambda\in\Lambda}p_{\gamma_1}(\lambda',\lambda)\,p_{\gamma_1}(\lambda'',\lambda)\\ \le c(n,\gamma_1)\,2^{j'+j''}\,p_{\gamma_1}(\lambda',\lambda'')<+\infty\,.\end{aligned}$$

Nous sommes donc en mesure d'intégrer terme à terme la série qui définit $m_{\lambda'\lambda''}$

$$\int m_{\lambda'\lambda''}=\frac{2^{-(j'+j'')}}{p_{\gamma_2}(\lambda',\lambda'')}\sum_{\lambda\in\Lambda}\left(\mu_y(\lambda',\lambda)\,\mu_x(\lambda'',\lambda)-\mu_x(\lambda',\lambda)\,\mu_y(\lambda'',\lambda)\right).$$

Or

$$\begin{aligned}&\sum_{\lambda\in\Lambda}\left(\mu_y(\lambda',\lambda)\,\mu_x(\lambda'',\lambda)-\mu_x(\lambda',\lambda)\,\mu_y(\lambda'',\lambda)\right)\\ &\quad=\sum_{\lambda\in\Lambda}\left(\langle\frac{\partial\psi_{\lambda'}}{\partial y},\psi_\lambda\rangle\,\langle\frac{\partial\psi_{\lambda''}}{\partial x},\psi_\lambda\rangle-\langle\frac{\partial\psi_{\lambda'}}{\partial x},\psi_\lambda\rangle\,\langle\frac{\partial\psi_{\lambda''}}{\partial y},\psi_\lambda\rangle\right)\end{aligned}$$

et cette dernière série vaut exactement

$$\langle\frac{\partial\psi_{\lambda'}}{\partial y},\frac{\partial\psi_{\lambda''}}{\partial x}\rangle-\langle\frac{\partial\psi_{\lambda'}}{\partial x},\frac{\partial\psi_{\lambda''}}{\partial y}\rangle=\int J(\psi_{\lambda'},\psi_{\lambda''})=0\,.$$

3. Généralisations.

Nous présenterons par la suite des généralisations possibles aux résultats ci-dessus.

3.1. Le cadre $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$.

Le même opérateur $P^\sharp$ défini en Section 1.2 s'étend en un opérateur continu de $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ dans l'espace de Hardy $\mathcal{H}^r(\mathbb{R}^n)$, où $1 < p < +\infty$, $1 < q < +\infty$, $1/p+1/q = 1/r$, à condition que $1 \geq r > n/(n+1)$.

Remarque. Cette restriction sur r est due au manque d'oscillations des produits $\varphi_{jk}\,\psi_{jl}$ qui apparaissent dans la construction de $P^\sharp$. Cette limitation disparaît si nous considérons des renormalisations du produit obtenues à partir de symboles bilinéaires que nous définissons dans [7] et [8].

La différence $P^\sharp - \tilde{P}^\sharp$ sera un opérateur borné de $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ dans l'espace de Lizorkin-Triebel homogène $\dot{F}_r^{0,1}(\mathbb{R}^n)$ avec les mêmes conditions sur p, q et r.

Nous ne donnons pas ici la définition ni les propriétés des espaces de Lizorkin-Triebel; le lecteur intéressé pourra consulter [9] à ce sujet. Remarquons tout simplement que, si $r = 1$, l'espace $\dot{F}_1^{0,1}$ coïncide avec l'espace de Besov homogène $\dot{B}_1^{0,1}$, et que, si $0 < r \leq 1$, on a les inclusions strictes $\dot{F}_r^{0,1} \subset \mathcal{H}^r$.

Le théorème du Jacobien se généralise alors immédiatement au cadre $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$: si $\nabla f \in L^p(\mathbb{R}^2)$, $\nabla g \in L^q(\mathbb{R}^2)$, alors $J(f,g) \in \mathcal{H}^r(\mathbb{R}^2)$ pour $1 < p < +\infty$, $1 < q < +\infty$, $1/r = 1/p + 1/q$, pourvu que $1 \geq r > 2/3$.

Signalons maintenant que cette restriction $r > n/(n+1)$ dans l'énoncé du théorème du Jacobien n'est plus une limitation technique due à notre méthode, mais une limitation qui découle du fait que $J(f,g)$ n'a que son intégrale nulle, alors que les moments d'ordre supérieur ne le sont pas.

Remarques. La définition du produit renormalisé que nous proposons ici dépend du cadre fonctionnel dans lequel nous nous situons. Si on voulait, par exemple, trouver un "produit renormalisé" dans le cadre $L^2 \times \mathrm{BMO}$, on ne devrait alors conserver que le terme $\sum_{j\in\mathbb{Z}}(P_jf)(Q_jg)$.

3.2. Application à d'autres opérateurs bilinéaires.

Nous obtenons également des résultats analogues pour des opérateurs plus généraux.

Nous établissons, à titre d'exemple, le résultat suivant

Théorème 3.1. *Soient $(A_i)_{i=1,K}$ et $(B_i)_{i=1,K}$ des opérateurs de Calderón-Zygmund qui vérifient les conditions*

$$A_i(1) = A_i^*(1) = B_i(1) = B_i^*(1) = 0 \qquad \text{pour tout } i = 1, K\,.$$

Supposons que pour tout couple (u, v) de fonctions de $L^2(\mathbb{R}^n)$ la fonction $\sum_{i=1}^K A_i(u)\, B_i(v)$ est d'intégrale nulle.

Alors l'opérateur bilinéaire B défini par

$$B(f, g) = \sum_{i=1}^{K} A_i(f)\, B_i(g)$$

est continu de $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ dans l'espace de Hardy $\mathcal{H}^r(\mathbb{R}^n)$, $1 < p < +\infty$, $1 < q < +\infty$, $1/r = 1/p + 1/q$, pourvu que $1 \geq r > n/(n+1)$.

De façon plus précise, l'opérateur B se décompose en $B = B_1 + B_2$, où B_1 est continu de $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ dans $\mathcal{H}^r(\mathbb{R}^n)$ et B_2 l'est de $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ dans $\dot{F}_r^{0,1}(\mathbb{R}^n)$.

REMARQUE. Dans le cas particulier des opérateurs de convolution de Calderón-Zygmund, ce résultat a déjà été obtenu par R. Coifman et L. Grafakos ([2], [10]).

Ces idées de la preuve sont essentiellement les mêmes que pour le théorème du Jacobien.

Si U est un opérateur de Calderón-Zygmund vérifiant $U(1) = U^*(1) = 0$, et si l'on pose $u(\lambda, \lambda') = \langle U(\psi_\lambda), \psi_{\lambda'} \rangle$ pour $(\psi_\lambda)_{\lambda \in \Lambda}$ base orthonormée d'ondelettes, la matrice $(u(\lambda', \lambda))_{(\lambda', \lambda) \in \Lambda^2}$ a une décroissance à partir de la diagonale donnée par l'estimation

$$|u(\lambda, \lambda')| \leq c\, p_\gamma(\lambda, \lambda')\,,$$

où p_γ est défini par (1.22) et $\gamma > 0$ dépend de la régularité de la base d'ondelettes.

Pour effectuer la décomposition de l'opérateur bilinéaire B en B_1+B_2, rappelons que, A_i et B_i étant des opérateurs bornés dans L^2 (ou bien L^p et L^q) on peut écrire

$$B(f,g)=\sum_{i=1}^{K} P^{\sharp}(A_i f, B_i g)+\sum_{i=1}^{K} S(A_i f, B_i g)$$

où

$$S(A_i f, B_i g)=\sum_{\lambda}\langle A_i f,\psi_\lambda\rangle\,\langle B_i g,\psi_\lambda\rangle\,\psi_\lambda^2\,.$$

Si on pose $B_1(f,g)=\sum_{i=1}^{K} P^{\sharp}(A_i f, B_i g)$, B_1 a les propriétés annoncées.

Etudions maintenant $B_2(f,g)=\sum_{i=1}^{K} S(A_i f, B_i g)$. Posons, pour $i=1,K$,

$$a_i(\lambda',\lambda)=\langle A_i\psi_{\lambda'},\psi_\lambda\rangle\,,$$
$$b_i(\lambda'',\lambda)=\langle B_i\psi_{\lambda''},\psi_\lambda\rangle\,.$$

Le terme $B_2(f,g)$ s'écrit alors, si $f=\sum_\lambda \alpha_\lambda\psi_\lambda$, $g=\sum_\lambda \beta_\lambda\psi_\lambda$

$$\begin{aligned}B_2(f,g)&=\sum_{i=1}^{K}\sum_{\lambda}\Big(\sum_{\lambda'}\alpha_{\lambda'}\,a_i(\lambda',\lambda)\Big)\Big(\sum_{\lambda''}\beta_{\lambda''}\,b_i(\lambda'',\lambda)\Big)\,\psi_\lambda^2\\&=\sum_{\lambda\in\Lambda}\ \sum_{(\lambda',\lambda'')\in\Lambda^2}\alpha_{\lambda'}\beta_{\lambda''}\Big(\sum_{i=1}^{K}a_i(\lambda',\lambda)\,b_i(\lambda'',\lambda)\Big)\,\psi_\lambda^2\,.\end{aligned}$$

L'appartenance de $B_2(f,g)$ à $\dot{B}_1^{0,1}(\mathbb{R}^n)$ (respectivement $\dot{F}_r^{0,1}(\mathbb{R}^n)$) découle alors d'une démonstration analogue à celle du Théorème 2.2.

Remarquons seulement que

$$\begin{aligned}\int B_2(f,g)&=\sum_{\lambda}\sum_{\lambda'}\sum_{\lambda''}\alpha_{\lambda'}\beta_{\lambda''}\sum_{i=1}^{K}a_i(\lambda',\lambda)\,b_i(\lambda'',\lambda)\\&=\sum_{\lambda'}\sum_{\lambda''}\alpha_{\lambda'}\beta_{\lambda''}\sum_{\lambda}\sum_{i=1}^{K}\langle A_i\psi_{\lambda'},\psi_\lambda\rangle\,\langle B_i\psi_{\lambda''},\psi_\lambda\rangle\\&=\sum_{\lambda'}\sum_{\lambda''}\alpha_{\lambda'}\beta_{\lambda''}\sum_{i=1}^{K}\int(A_i\psi_{\lambda'})\,(B_i\psi_{\lambda''})=0\end{aligned}$$

car $\int B(\psi_{\lambda'},\psi_{\lambda''})=0$ par hypothèse.

4. Appendice. Lien entre renormalisation par ondelettes et renormalisation par paraproduit.

Nous établirons ici le lien entre la renormalisation par ondelettes du produit, et d'autres méthodes de renormalisation. Dans le but d'alléger les notations, nous considérons par la suite le cas de la dimension 1.

Soit φ une fonction appartenant à la classe de Schwartz telle que

$$\text{(4.1)} \qquad \begin{cases} \hat{\varphi} \in \mathcal{D}(\mathbb{R}), \\ \hat{\varphi} \quad \text{est une fonction réelle et paire,} \\ \hat{\varphi}(\xi) \in [0,1], \quad \text{pour tout} \quad \xi \in \mathbb{R}, \\ \hat{\varphi}(\xi) = 1 \quad \text{si} \quad |\xi| \le \pi - \delta \quad (\delta < \pi/3), \\ \hat{\varphi}(\xi) = 0 \quad \text{si} \quad |\xi| > \pi + \delta, \\ \hat{\varphi}^2(\xi) + \hat{\varphi}^2(2\pi - \xi) = 1 \quad \text{si} \quad 0 \le \xi \le 2\pi. \end{cases}$$

Pour tout $j \in \mathbb{Z}$ on désigne par S_j l'opérateur de convolution avec $2^j\varphi(2^j x)$, et on pose $\Delta_j = S_{j+1} - S_j$.

Le paraproduit de J.M. Bony [1] est défini, pour f et g dans L^2, par

$$\text{(4.2)} \qquad \Pi(f,g) = \sum_{j\in\mathbb{Z}} S_{j-1}(f)\,\Delta_j(g).$$

Remarque. La définition du paraproduit dépend du choix de la fonction φ; mais deux choix différents pour cette fonction conduisent en fait à des paraproduits "équivalents".

Il est bien connu que le paraproduit est un opérateur continu de $L^2 \times L^2$ dans l'espace de Hardy $\mathcal{H}^1$.

Le produit de f et de g peut s'écrire, à l'aide du paraproduit,

$$fg = \Pi(f,g) + \Pi(g,f) + \sum_{j\in\mathbb{Z}} (\Delta_j f)\,(\Delta_j g).$$

La définition de renormalisation au moyen du paraproduit surgit alors tout naturellement; on pose

$$\text{(4.3)} \qquad \Pi^\sharp(f,g) = \Pi(f,g) + \Pi(g,f) = fg - \sum_{j\in\mathbb{Z}} (\Delta_j f)\,(\Delta_j g).$$

Nous obtenons le résultat suivant

Théorème 4.1. *Soit $(\psi_\lambda)_{\lambda\in\Lambda}$ une base orthonormée d'ondelettes. L'opérateur $P^\sharp - \Pi^\sharp$ envoie continûment $L^2 \times L^2$ dans l'espace de Besov homogène $\dot{B}_1^{0,1}$.*

L'invariance "modulo $\dot{B}_1^{0,1}$" de la renormalisation du produit par rapport au choix de la base orthonormée d'ondelettes nous permet de considérer le cas de l'analyse multi-résolution dite de Littlewood-Paley. Dans ce cas φ est la fonction définie par (4.1) et l'ondelette ψ est définie par

$$\hat{\psi}(\xi) = \big(\hat{\varphi}(\xi/2)^2 - \hat{\varphi}(\xi)^2\big)^{1/2}\, e^{-i\xi/2}\,. \tag{4.5}$$

Pour f et g dans $L^2(\mathbb{R})$, il s'agit de montrer que

$$\pi_1(f,g) + \pi_2(f,g) + \pi_3(f,g) - \Pi(f,g) - \Pi(g,f) \in \dot{B}_1^{0,1}(\mathbb{R})\,.$$

Or on sait déjà que $\pi_3(f,g) \in \dot{B}_1^{0,1}(\mathbb{R})$. Finalement, par symétrie, il suffit de montrer que

$$\sum_{j\in\mathbb{Z}} (P_j f)\,(Q_j g) - \sum_{j\in\mathbb{Z}} (S_{j-1} f)\,(\Delta_j g) \in \dot{B}_1^{0,1}(\mathbb{R})\,.$$

Cette preuve se décompose en trois étapes.

Lemme 4.1. *Soient f et g deux fonctions de $L^2(\mathbb{R})$. La différence*

$$\Delta_1(f,g) = \sum_{j\in\mathbb{Z}} (P_j f)\,(Q_j g) - \sum_{j\in\mathbb{Z}} (P_{j-1} f)\,(Q_j g)$$

appartient à l'espace de Besov homogène $\dot{B}_1^{0,1}(\mathbb{R})$.

La preuve de ce lemme est immédiate, car Δ_1 s'écrit

$$\Delta_1(f,g) = \sum_{j\in\mathbb{Z}} (Q_{j-1} f)\,(Q_j g)\,.$$

Lemme 4.2. *Soient f et g deux fonctions de $L^2(\mathbb{R})$. La différence*

$$\Delta_2(f,g) = \sum_{j\in\mathbb{Z}} (P_{j-1} f)\,(Q_j g) - \sum_{j\in\mathbb{Z}} (S_{j-1} f)\,(Q_j g)$$

appartient à l'espace de Besov homogène $\dot{B}_1^{0,1}(\mathbb{R})$.

La preuve repose sur une utilisation soigneuse de la structure particulière de l'analyse multirésolution de Littlewood-Paley. Remarquons tout d'abord que, si $\mathcal{F}$ désigne la transformée de Fourier, on a

$$\mathcal{F}(V_j) = \left\{m(\xi/2^j)\,\hat{\varphi}(\xi/2^j) :\ m \in L^2(0,2\pi)\,,\ m(\xi+2\pi) = m(\xi)\right\}.$$

Si, pour chaque $T > 0$, on pose

$$E_T = \{f \in L^2(\mathbb{R}) :\ \operatorname{supp} \hat{f} \subset [-T,T]\}\,,$$

alors on a

$$E_{(\pi-\delta)2^j} \subset V_j \subset E_{(\pi+\delta)2^j}\ .$$

Nous effectuons maintenant une décomposition de la projection $P_j f$, pour chaque valeur de j fixée,

$$P_j f = P_j f_1^j + P_j f_2^j + P_j f_3^j\ ,$$

où f_1^j, f_2^j et f_3^j sont définies par les conditions

$$\text{(4.6)}\qquad \begin{cases} \operatorname{supp} \hat{f}_1^j \subset \left\{\xi \in \mathbb{R} :\ |\xi| \le (\pi-\delta)\,2^j\right\}, \\ \operatorname{supp} \hat{f}_2^j \subset \left\{\xi \in \mathbb{R} :\ (\pi-\delta)\,2^j \le |\xi| \le (\pi+\delta)\,2^j\right\}, \\ \operatorname{supp} \hat{f}_3^j \subset \left\{\xi \in \mathbb{R} :\ |\xi| \ge (\pi+\delta)\,2^j\right\}, \\ f = f_1^j + f_2^j + f_3^j\ . \end{cases}$$

Dans ces conditions, on a $f_1^j \in E_{(\pi-\delta)2^j} \subset V_j$, d'où $P_j f_1^j = f_1^j$. La fonction f_3^j est orthogonale à V_j, d'où $P_j f_3^j = 0$. Quant à f_2^j, elle est, par construction, orthogonale à $E_{(\pi-\delta)2^j} \subset V_j$. En conséquence, sa projection $P_j f_2^j$ sera orthogonale à $E(\pi-\delta)2^j$, et le spectre de $P_j f_2^j$ contenu dans la couronne

$$\mathcal{C}_j = \left\{\xi \in \mathbb{R} :\ (\pi-j)\,2^j \le |\xi| \le (\pi+j)\,2^j\right\}.$$

Si on calcule maintenant $S_j f$ à l'aide de cette même décomposition de f, on obtient

$$S_j f_1^j = f_1^j\ , \qquad S_j f_3^j = 0\,, \qquad \operatorname{supp} \widehat{f_2^j} \subset \mathcal{C}_j\ .$$

Finalement, pour chaque valeur de j on a

$$(P_j - S_j) f = (P_j - S_j) f_2^j\ ,$$

et le spectre de $(P_j - S_j)f$ est contenu dans la couronne $\mathcal{C}_j$.

Quant au spectre de $Q_j g$, il est contenu, d'après (4.5) et (4.1) dans la couronne $\mathcal{D}_j = \{\xi \in \mathbb{R} :\ (\pi - \delta)\, 2^j \leq |\xi| \leq 2^{j+1}(\pi + \delta)\}$.

Ainsi, pour chaque valeur de j, le produit $(P_{j-1} - S_{j-1})(f)\, Q_j(g)$ a son spectre contenu dans la couronne

$$\left\{\xi \in \mathbb{R} :\ (\pi - 3\delta)\, 2^j \leq |\xi| \leq 5\,(\pi + \delta)\, 2^j\right\}.$$

Ceci permet d'appliquer un lemme de "presque orthogonalité" et d'estimer

$$\begin{aligned}
\Big\|\sum_{j\in\mathbb{Z}}(P_{j-1} - S_{j-1})(f)\, Q_j(g)\Big\|_{\dot{B}_1^{0,1}} & \\
&\leq c \sum_{j\in\mathbb{Z}} \big\|(P_{j-1} - S_{j-1})(f)\, Q_j(g)\big\|_{L^1(\mathbb{R})} \\
&\leq c \Big(\sum_{j\in\mathbb{Z}} \|(P_j - S_j)f\|_2^2\Big)^{1/2} \Big(\sum_{j\in\mathbb{Z}} \|Q_j g\|_2^2\Big)^{1/2} \\
&\leq c\, \|f\|_2\, \|g\|_2 .
\end{aligned}$$

Lemme 4.3. *Soient f et g deux fonctions de $L^2(\mathbb{R})$. La différence*

$$\Delta_3(f, g) = \sum_{j\in\mathbb{Z}} (S_{j-1}f)\,(Q_j g) - \sum_{j\in\mathbb{Z}} (S_{j-1}f)\,(\Delta_j g)$$

appartient à l'espace de Besov homogène $\dot{B}_1^{0,1}(\mathbb{R})$.

L'idée de départ est de réécrire

$$\Delta_3(f, g) = \sum_{j\in\mathbb{Z}} S_{j-1}(f)\,(P_{j+1} - P_j)(g) - \sum_{j\in\mathbb{Z}} S_{j-1}(f)\,(S_{j+1} - S_j)(g)\,,$$

et d'appliquer, à chacune des séries ci-dessus, la transformation d'Abel.

Finalement

$$\Delta_3(f, g) = \sum_{j\in\mathbb{Z}} (\Delta_{j-1}f)\,\big(S_{j+1}g - P_{j+1}g\big)\,.$$

La fin de la démonstration est identique à celle du Lemme 4.2.

Signalons finalement que nous avons developpé dans [7] et [8] la construction d'une renormalisation du produit associée à des symboles bilinéaires du type de ceux que R. Coifman et Y. Meyer ont considérés dans [3]; la renormalisation par paraproduit n'est qu'un cas particulier de cette renormalisation par symboles bilinéaires.

Modulo $\dot{B}_1^{0,1}(\dot{F}_r^{0,1})$, toutes les méthodes de renormalisation que nous définissons sont équivalentes.

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Localisation fréquentielle des paquets d'ondelettes

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Abstract. Orthonormal bases of wavelet packets constitute a powerful tool in signal compression. It has been proved by Coifman, Meyer and Wickerhauser that "many" wavelet packets w_n suffer a lack of frequency localization. Using the L^1-norm of the Fourier transform $\hat{w}_n$ as localization criterion, they showed that the average $2^{-j} \sum_{n=0}^{2^j-1} \|\hat{w}_n\|_{L^1}$ blows up as j goes to infinity. A natural problem is then to know which values of n create this blow-up in average. The present work gives an answer to this question, thanks to sharp estimates on $\|\hat{w}_n\|_{L^1}$ which depend on the dyadic expansion of n, for several types of filters. Let us point out that the value of $\|\hat{w}_n\|_{L^1}$ is a weak localization criterion, which can only lead to a lower estimate on the variance of $\hat{w}_n$.

Résumé.

Les bases orthonormées de paquets d'ondelettes sont un outil puissant en compression du signal. Coifman, Meyer et Wickerhauser ont prouvé que de "nombreux" paquets d'ondelettes w_n souffrent d'une mauvaise localisation fréquentielle. Utilisant la norme L^1 de la transformée de Fourier $\hat{w}_n$ comme critère de localisation, ils ont montré que la moyenne $2^{-j} \sum_{n=0}^{2^j-1} \|\hat{w}_n\|_{L^1}$ explose lorsque j tend vers l'infini. Il est alors naturel de se demander quelles valeurs de n sont à l'origine de cette explosion en moyenne. Le présent travail donne une réponse à cette question, grâce à des estimations précises sur $\|\hat{w}_n\|_{L^1}$ en fonction du développement dyadique de n, et pour plusieurs types de filtres.

Soulignons le fait que la valeur de $\|\hat{w}_n\|_{L^1}$ est un critère faible de localisation, qui ne peut conduire qu'à une estimation inférieure sur la variance de $\hat{w}_n$.

1. Introduction.

Les paquets d'ondelettes ont été introduits récemment en traitement et en compression du signal, où ils se sont révélés d'une grande efficacité (voir [2], [3]). On peut les considérer comme une réponse partielle au problème des bases d'atomes temps-fréquence (voir [7]).

Un atome temps-fréquence est une fonction du temps $f_R(t)$ associée à un rectangle R de surface 2π dans le plan temps-fréquence (t,ξ), et qui satisfait, outre la condition de normalisation $\|f_R\|_{L^2} = 1$, aux exigences de localisation suivantes

$$\int_{\mathbb{R}} (t-t_0)^2\,|f_R(t)|^2\,dt \le K^2\,h^2\,, \tag{1.1}$$

$$\int_{\mathbb{R}} (\xi-\omega_0)^2\,|\hat{f}_R(\xi)|^2\,d\xi \le \frac{2\pi K^2}{h^2}\,. \tag{1.2}$$

Ici, (t_0,ξ_0) est le centre du rectangle R, qui se définit comme la région du plan d'équations

$$t_0 - h \le t \le t_0 + h\,, \qquad \xi_0 - \frac{2\pi}{h} \le \xi \le \xi_0 + \frac{2\pi}{h}\,.$$

$\hat{f}$ désigne la transformée de Fourier de f, et K est une constante positive. Si l'on impose à f_R d'être à valeurs réelles, il faut modifier la condition (1.2) en remplaçant l'intégrale sur $\mathbb{R}$ par une intégrale sur $\mathbb{R}_+$.

Le but de l'analyse temps-fréquence est d'écrire un signal donné comme combinaison linéaire d'atomes temps-fréquence choisis de manière "optimale", suivant un critère qui dépend de l'utilisation de ce signal.

Une façon d'atteindre ce but est de disposer d'une bibliothèque de bases orthonormées d'atomes temps-fréquence. Cette bibliothèque doit être assez vaste pour s'adapter à la grande variété de signaux possibles, mais pas trop, car on veut également disposer d'un algorithme rapide pour sélectionner la base qui représente au mieux un signal.

Les bases de paquets d'ondelettes constituent une telle bibliothèque, à ceci près que ce ne sont pas toutes des bases d'atomes temps-fréquence. Coifman, Meyer et Wickerhauser [4] ont en effet montré que la bibliothèque contient de mauvaises "pages" (ou paquets d'ondelettes) dans certains de ses "livres" (ou bases). Cependant, tout n'est pas perdu en pratique. En effet, la bibliothèque est générée à l'aide de filtres conjugués en quadrature, et l'on explique dans [4] comment les choisir de façon à réduire les artefacts. Malheureusement, ce choix correspond à des filtres plus longs, d'où un temps de calcul augmenté.

Une autre solution est de n'utiliser, dans la bibliothèque, que les "bons livres". Comme le choix est plus restreint, on gagne en temps de calcul, mais on perd en pertinence de la représentation. Pour appliquer cette solution, il convient de disposer, pour des filtres donnés, d'un "guide" indiquant les qualités de localisation des différents paquets d'ondelettes de la bibliothèque.

Dans le présent travail, nous nous proposons de fournir de tels guides, pour une large classe de filtres conjugués en quadrature. Ces guides présentent une lacune: le critère de localisation fréquentielle utilisé est plus faible que (1.2), et ne peut conduire qu'à une estimation inférieure sur la "variance"

$$\int_{\mathbb{R}} (\xi - \omega_0)^2 \, |\hat{f}_R(\xi)|^2 \, d\xi \, .$$

2. Enoncé des résultats.

Dans tout ce qui suit, nous appellerons "couple de filtres conjugués en quadrature" la donnée de deux fonctions à valeurs complexes (m_0, m_1) ayant les propriétés suivantes

(2.1) m_0 et m_1 sont de classe C^∞, 2π-périodiques et vérifient $m_\varepsilon(-\xi) = \overline{m_\varepsilon(\xi)}$, $\xi \in \mathbb{R}$, $\varepsilon = 1, 2$.

(2.2) Pour tout ξ réel, la matrice $\begin{pmatrix} m_0(\xi) & m_1(\xi) \\ m_0(\xi+\pi) & m_1(\xi+\pi) \end{pmatrix}$ est unitaire.

(2.3) $m_0(0) = 1$ et $m_0(\xi) \neq 0$ pour tout $\xi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

En notant $M_\varepsilon(\xi) = |m_\varepsilon(\xi)|$, $\varepsilon = 1, 2$, il résulte de (2.1), (2.2) et (2.3), que M_ε est une fonction paire, 2π-périodique, Lipschitzienne, avec $M_1(\xi) = M_0(\xi + \pi)$ et $M_0(\xi)^2 + M_0(\xi + \pi)^2 = 1$.

La fonction somme $S(\xi) = M_0(\xi) + M_0(\xi + \pi)$ est donc paire, π-périodique, Lipschitzienne, à valeurs dans $[1, \sqrt{2}]$, avec $S(0) = 1$, $S(\pi/2) = \sqrt{2}$.

Les propriétés (2.1) et (2.3) entraînent (voir [1]) que le produit $\prod_{p=1}^{\infty} m_0(\xi/2^p)$ converge pour presque tout ξ réel, et qu'il existe une constante $\lambda > 0$ telle que

$$\text{(2.4)} \qquad \Big| \prod_{p=1}^{\infty} m_0(\xi/2^p) \Big| \geq \lambda\,, \qquad \text{pour tout } \xi \text{ dans } [-\pi, \pi]\,.$$

Ce produit est la transformée de Fourier $\hat{\varphi}(\xi)$ de la "fonction d'échelle" $\varphi(t)$.

Les paquets d'ondelettes sont des fonctions w_n, $n \geq 0$, définies par leur transformée de Fourier

$$\hat{w}_n(\xi) = m_{\varepsilon_1}(\xi/2) \cdots m_{\varepsilon_j}(\xi/2^j)\, \hat{\varphi}(\xi/2^j)\,,$$

pour $n = \sum_{q=1}^{j} \varepsilon_q \, 2^{q-1}$.

On a bien sûr $w_0 = \varphi$, et pour $2^{j-1} \leq n < 2^j$, w_n est combinaison linéaire de fonctions du type $\psi(2^{j-1}t - k)$, $k \in \mathbb{Z}$, où $\psi = w_1$ est l'ondelette associée au couple de filtres conjugués en quadrature.

Il résulte de (2.2) que les fonctions w_n sont normalisées dans $L^2(\mathbb{R})$ et deux à deux orthogonales. La bibliothèque de bases orthonormées de $L^2(\mathbb{R})$ associée à ces fonctions est indexée par les partitions de $\mathbb{R}_+$ en intervalles dyadiques, c'est-à-dire du type $[n/2^p\, ,\, (n+1)/2^p]$, $n \in \mathbb{N}$, $p \in \mathbb{Z}$.

La base ayant pour indice une telle partition $(I_\alpha)_{\alpha \in A}$, est la collection de fonctions

$$\Big(2^{-p_\alpha/2}\, w_{n_\alpha}(2^{-p_\alpha} t - k)\Big)_{\alpha \in A\,,\, k \in \mathbb{Z}} \quad , \quad \text{avec} \quad I_\alpha = \left[\frac{n_\alpha}{2^{p_\alpha}}\, ,\, \frac{n_\alpha + 1}{2^{p_\alpha}}\right] .$$

Si l'on remplace (m_0, m_1) par les filtres "parfaits"

$$\chi_0 = 1_{[-\pi/2\,,\,\pi/2]+2\pi\mathbb{Z}} \quad \text{et} \quad \chi_1 = 1_{[\pi/2\,,\,3\pi/2]+2\pi\mathbb{Z}}\,,$$

1_X étant la fonction caractéristique de X, on trouve

$$\hat{w}_n(\xi) = 1_{J\cup(-J)}(\xi)\,,$$

pour $n = \sum_{q\geq 1} \varepsilon_q\, 2^{q-1}$, $J = \pi[\mathcal{G}(n)\,,\, \mathcal{G}(n)+1]$, $\mathcal{G}(n) = \sum_{q\geq 1} \gamma_q\, 2^{q-1}$, $\gamma_q = |\varepsilon_{q+1} - \varepsilon_q|$. On appelle "code de Gray" la correspondance

$$(\varepsilon_q) \to (\gamma_q)\,.$$

Cette formule suggère que w_n est un atome temps-fréquence associé au rectangle R d'équations $-1 \leq t \leq 1$, $\pi\,\mathcal{G}(n) \leq \xi \leq \pi(\mathcal{G}(n)+1)$.

Dans le cas des filtres parfaits, la condition (1.1) ne peut être satisfaite par aucun des w_n . A l'opposé, si l'on travaille avec des filtres (m_0, m_1) qui sont des polynômes trigonométriques, alors $w_n(t)$ est à support inclus dans un compact indépendant de n, et (1.1) est vérifiée. Nous ne savons pas si (1.1) est encore vraie pour des filtres conjugués en quadrature plus généraux, sous les seules hypothèses (2.1), (2.2), (2.3).

Pour étudier la condition (1.2), on part de l'inégalité suivante (on rappelle que w_n est normalisée au sens L^2):

$$\int_{-\infty}^{+\infty} |\hat{w}_n(\xi)|\, d\xi \leq \pi\,(2+\sigma_n)^{1/2}\,, \tag{2.5}$$

avec

$$\sigma_n \;=\; \frac{1}{2\pi} \inf_{\omega_0 \geq 0} \int_0^{+\infty} |\xi - \omega_0|^2\, |\hat{w}_n(\xi)|^2\, d\xi\,. \tag{2.6}$$

D'après (2.5), une minoration sur $\|\hat{w}_n\|_{L^1(\mathbb{R})}$ donne une minoration sur la localisation de $\hat{w}_n$ au sens de (1.2). En revanche, une majoration de $\|\hat{w}_n\|_{L^1(\mathbb{R})}$ ne donne aucun renseignement sur σ_n . Cependant, dans tout ce qui va suivre, on étudiera uniquement $\|\hat{w}_n\|_{L^1(\mathbb{R})}$, qui est plus facile à manipuler que σ_n (cette démarche est la même que dans [4]).

Etant donné un couple (m_0, m_1) de filtres conjugués en quadrature, il est prouvé dans [4] qu'il existe une constante $\rho > 1$ telle que pour tout $j \geq 0$, on ait

$$\frac{1}{2^j} \sum_{n=0}^{2^j-1} \|\hat{w}_n\|_{L^1} \;\geq\; 2\pi\,\rho^j\,. \tag{2.7}$$

En d'autres termes, toutes les fonctions w_n ne sont pas des atomes temps-fréquence.

Dans l'hypothèse où $|m_0|$ vaut 1 sur $[-\pi/3\,,\,\pi/3]$, on trouve dans [4] l'estimation suivante:

$$\|\hat{w}_n\|_{L^1} \;\leq\; C\,n^{1/4}\,. \tag{2.8}$$

Si l'on recherche une croissance plus lente de $\|\hat{w}_n\|_{L^1}$, il est naturel de choisir m_0 proche du filtre "parfait" χ_0. Plus précisément (voir [4]), si $|m_0|$ vaut 1 sur l'intervalle $[-\pi/2+\delta\,,\,\pi/2-\delta]$ avec $0<\delta\leq\pi/6$, alors

$$\|\hat{w}_n\|_{L^1} \leq C(\delta)\,n^{\gamma(\delta)} \quad \text{et} \quad (C\to 2\pi\,,\ \gamma\to 0) \quad \text{lorsque} \quad \delta\to 0\,. \tag{2.9}$$

Remarquons que la minoration (2.7) n'interdit pas l'existence d'une suite $(n_p)_{p\geq 0}$ strictement croissante telle que $\|\hat{w}_{n_p}\|_{L^1}$ reste bornée. Les résultats que nous allons énoncer puis démontrer sont des encadrements de $\|\hat{w}_n\|_{L^1}$ en fonction du développement dyadique de n. Les formules obtenues dépendront d'hypothèses sur (m_0,m_1). Nous appellerons $(H_p)_{p\geq 1}$ les hypothèses donnant les majorations, et $(h_p)_{p\geq 1}$ celles qui donneront les minorations.

L'hypothèse (H_1) assure que $\|\hat{w}_0(\xi)\|_{L^1(\mathbb{R})}$ est finie. Elle est immédiatement vérifiée lorsque $M_0=1$ sur $[-\pi/3\,,\,\pi/3]$. Elle est vraie aussi pour les filtres d'Ingrid Daubechies m_0^N , $N\geq 2$ (voir [1]).

Hypothèse (H_1)**.** *On peut écrire*

$$m_0(\xi)=\Big(\frac{1+e^{i\xi}}{2}\Big)^N B(\xi)\,,$$

et si l'on pose

$$b_j=\frac{1}{j\,\log 2}\sup_{\xi\in\mathbb{R}}\sum_{k=1}^{j}\log\,|B(2^k\xi)|\,,$$

on a l'inégalité

$$b=\inf_{j\geq 1} b_j < N-1\,.$$

Si l'hypothèse (H_1) est vraie (voir [1]), alors pour $\varepsilon > 0$ assez petit, il existe une constante $K_\varepsilon > 0$ telle que pour tout ξ réel,

$$|\hat{\varphi}(\xi)| \leq \frac{K_\varepsilon}{(1+|\omega|)^{1+\varepsilon}} . \tag{2.10}$$

Nous démontrerons le résultat suivant au Section IV:

Théorème (H_1)**.** *Supposons l'hypothèse* (H_1) *vérifiée. Avec les notations*

$$C = \|\hat{\varphi}\|_{L^1(\mathbb{R})} , \qquad R = \sup_{\xi \in [0,\pi]} \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi - 2k\pi)| ,$$

$$N_1(n) = \sum_{q\geq 1} \varepsilon_q , \qquad n = \sum_{q\geq 1} \varepsilon_q \, 2^{q-1} ,$$

on a l'inégalité

$$\|\hat{w}_n\|_{L^1(\mathbb{R})} \leq C\, R^{N_1(n)} .$$

L'hypothèse (h_1) permettra d'obtenir une estimation inverse (on rappelle que $M_0 = |m_0|$, $S(\xi) = M_0(\xi) + M_0(\xi+\pi)$).

Hypothèse (h_1)**.** *Il existe* $\alpha > 1$ *tel que pour tout* $\xi \in [\pi/4\,,\ \pi/2]$, *on ait* $S(\xi) \geq \alpha$. *De plus,* $M_0(\xi) \geq 1/\sqrt{2}$ *sur* $[-\pi/2\,,\, \pi/2]$.

Cette estimation est la suivante:

Théorème (h_1)**.** *Si l'hypothèse* (h_1) *est vérifiée, en posant*

$$r = \frac{\alpha+1}{2} ,$$

on a l'inégalité

$$\|\hat{w}_n\|_{L^1(\mathbb{R})} \geq 2\pi\, r^{N_1(n)} ,$$

N_1 *étant défini comme dans le Théorème* (H_1) .

Les deux résultats qui précèdent peuvent s'appliquer aux filtres de longueur finie:

Corollaire 1. *Si* m_0 *est un filtre d'Ingrid Daubechies* m_0^N *avec* $N \geq 2$ *, il existe* $R > r > 1$ *et* $C > 0$ *tels que pour tout* $n \geq 0$ *, on ait*

$$2\pi\, r^{N_1(n)} \;\leq\; \|\hat{w}_n\|_{L^1(\mathbb{R})} \leq C\, R^{N_1(n)} ,$$

N_1 *étant défini comme dans le Théorème* (H_1) *.*

Remarque. Pour N grand, le Théorème (h_1) donne une valeur de r très proche de 1. En revanche, la valeur de R donnée par le Théorème (H_1) n'est jamais inférieure à $\sqrt{2}$, comme le montre le calcul suivant:

$$\begin{aligned} R &\geq \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(\pi - 2k\pi)\right| \\ &= M_1(\frac{\pi}{2}) \sum_{l\in\mathbb{Z}} \left|\hat{\varphi}(\frac{\pi}{2} - 2l\pi)\right| + M_0(\frac{\pi}{2}) \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(\frac{\pi}{2} - 2l\pi - \pi)\right| \\ &= \sqrt{2} \sum_{l\in\mathbb{Z}} \left|\hat{\varphi}(\frac{\pi}{2} - 2l\pi)\right| \\ &\geq \sqrt{2} \left|\sum_{l\in\mathbb{Z}} \hat{\varphi}(\frac{\pi}{2} - 2l\pi)\right| = \sqrt{2} \,. \end{aligned}$$

Le Corollaire 1 donne donc une estimation peu précise. Cependant, cette estimation est suffisante lorsqu'il s'agit de savoir si une sous-suite $\|\hat{w}_{n_p}\|_{L^1}$ est bornée ou non.

Les hypothèses (H_p) et (h_p) pour $p \geq 2$ concernent des filtres tels que $M_0 = 1$ sur $[-\pi/3\,,\,\pi/3]$.

Hypothèse (H_p)**.** *Pour* $\delta = \delta_p = \pi/(2(2^p - 1))$, M_0 *vaut* 1 *sur*

$$\Big[-\frac{\pi}{2} + \delta\,,\,\frac{\pi}{2} - \delta\Big]\,,$$

et il existe $\beta < 1$ *tel que*

$$M_0(\xi) \leq \beta\,\frac{(2^p-1)^2}{\pi}\,\Big(\xi - \frac{\pi}{2} + \delta\Big) \qquad sur\;\; \Big[\frac{\pi}{2} - \delta\,,\,\frac{\pi}{2} + \delta\Big]\,.$$

Hypothèse (h_p). M_0 *vaut* 1 *sur* $[-\pi/3\,,\,\pi/3]$, *et il existe* $\alpha > 1$ *tel que*

$$S(\xi) \geq \alpha \qquad pour\;\; \xi \in \Big[\frac{\pi}{2}\Big(1 - \frac{1}{2^p}\Big)\,,\,\frac{\pi}{2}\Big]\,.$$

De plus, $M_0(\xi) \geq 1/\sqrt{2}$ *sur* $[-\pi/2\,,\,\pi/2]$ *.*

Ces hypothèses donnent deux estimations inverses l'une de l'autre:

Théorème (H_p)**.** *Si l'hypothèse* (H_p) *est vérifiée, il existe deux constantes* $C_p > 2\pi$ *et* $R = R(\beta, p) > 1$ *telles que pour tout* $n \geq 0$*, on ait*

$$\|\hat{w}_n\|_{L^1(\mathbb{R})} \ \leq \ C_p \, R^{N_p(n)+1} \, .$$

Ici, $N_p(n)$ *est le nombre de fois qu'apparaît la suite*

$$\underbrace{0 \ldots 0}_{(p-1) \text{ fois } 0} \quad 1$$

dans le développement dyadique $(\varepsilon_1 \ldots \varepsilon_j \ldots)$ *de* $n = \sum_{q \geq 1} \varepsilon_q \, 2^{q-1}$*. De plus, on peut prendre*

$$C_p = 2\pi + 4\delta_p \, (\sqrt{2} - 1) \qquad \text{et} \qquad R(\beta, p) = 3 + \frac{2^{p+1}\beta}{1 - \beta} \, .$$

Théorème (h_p)**.** *Si l'hypothèse* (h_p) *est vérifiée, en posant*

$$r = \frac{\alpha + 1}{2} \, ,$$

on a l'inégalité

$$\|\hat{w}_n\|_{L^1(\mathbb{R})} \geq 2\pi \, r^{N_p(n)} \, ,$$

N_p *étant le même que dans le Théorème* (H_p) *.*

Remarquons que pour tout $p \geq 2$, (H_p) implique (H_{p-1}), (h_p) implique (h_{p+1}), et si (h_p) est vraie, alors (H_{p+1}) est fausse.

Enfin, si un couple de filtres conjugués en quadrature est tel que $M_0 = 1$ sur $[-\pi/3 \, , \, \pi/3]$ et que M_0 soit décroissante sur $[0 \, , \, \pi/2]$, alors il existe $p \geq 2$ tel que (h_p) et (H_{p-1}) soient simultanément satisfaites par ce couple, (H_p) pouvant être ou non vérifiée.

Tout ceci se voit immédiatement à partir des énoncés des hypothèses, et on a le résultat suivant:

Corollaire 2. *Supposons que* $M_0 = 1$ *sur* $[-\pi/3 \, , \, \pi/3]$ *avec* M_0 *décroissante sur* $[0 \, , \, \pi/2]$ *. Soit* p *le plus petit entier tel que*

$$M_0\Big(\frac{\pi}{2}\big(1 - \frac{1}{2^p}\big)\Big) < 1 \, .$$

Il existe $R > r > 1$ et $C > 0$ tels que pour tout $n \geq 0$, on ait

$$2\pi\, r^{N_p(n)} \leq \|\hat{w}_n\|_{L^1(\mathbb{R})} \leq C\, R^{N_{p-1}(n)}\,,$$

N_p étant défini comme dans le Théorème (H_p).

En pratique, les algorithmes de paquets d'ondelettes sont utilisés avec des filtres à réponse impulsionnelle finie. Pour de tels filtres, on n'a pas $M_0 = 1$ sur $[-\pi/3\,,\,\pi/3]$, et seul le Corollaire 1 s'applique. Cependant, pour des filtres assez longs, on peut avoir M_0 très proche de 1 sur $[-\pi/3\,,\,\pi/3]$, ce qui rend peu précis l'encadrement du Corollaire 1 (voir la remarque plus haut). Dans ce cas, le Corollaire 2 pourrait donner des indications sur le comportement de $\|\hat{w}_n\|_{L^1(\mathbb{R})}$. Nous n'étudierons pas cette question plus avant.

Par ailleurs, il existe un algorithme de décomposition adaptative en fonctions cosinus locales, imaginé par R.R. Coifman, et appelé algorithme de repliement multiple (voir [5]). Les bases générées par cet algorithme sont les transformées de Fourier des bases de paquets d'ondelettes, et les filtres (m_0, m_1) associés vérifient les hypothèses du Corollaire 2.

Signalons enfin que les estimations (2.7) et (2.9), obtenues par une méthode directe dans [4], peuvent facilement être déduites des Théorèmes (h_p) et (H_p).

3. Démonstration des Théorèmes (h_p).

On rappelle les notations $M_\varepsilon = |m_\varepsilon|$, $S = M_0 + M_1$.

Soit $n = \sum_{q\geq 1} \varepsilon_q\, 2^{q-1}$. On pose $W_n^0 = 1$, et pour $j \geq 1$,

$$W_n^j(\xi) = M_{\varepsilon_1}(\xi/2) \cdots M_{\varepsilon_j}(\xi/2^j)\,. \tag{3.1}$$

Posons également $n_j = \sum_{q=1}^{j} \varepsilon_q\, 2^{q-1}$. Si $j \geq \log_2(n)$, alors $n_j = n$, et d'après (2.4),

$$W_n^j(\xi) \leq \frac{|\hat{w}_n(\xi)|}{\lambda} \qquad \text{sur } [-2^j\pi, 2^j\pi]\,. \tag{3.2}$$

Donc, par le théorème de convergence dominée de Lebesgue,

$$\|\hat{w}_n(\xi)\|_{L^1(\mathbb{R})} = \lim_{j\to+\infty} \|W_n^j(\xi)\|_{L^1(-2^j\pi, 2^j\pi)}\,. \tag{3.3}$$

De plus, on dispose de la propriété de monotonie suivante:

Lemme 1. *Soit* (m_0, m_1) *un couple de filtres conjugués en quadrature, et soit* $(\varepsilon_1, \ldots, \varepsilon_j)$ *une suite dans* $\{0,1\}^j$*. On a*

$$2^j \left\| m_{\varepsilon_j}(\xi)\, m_{\varepsilon_{j-1}}(2\xi) \cdots m_{\varepsilon_1}(2^j\xi) \right\|_{L^1(0,2\pi)} \geq 2^{j-1} \left\| m_{\varepsilon_{j-1}}(\xi) \cdots m_{\varepsilon_1}(2^{j-1}\xi) \right\|_{L^1(0,2\pi)} .$$

Pour n fixé, la suite $\|W_n^j(\xi)\|_{L^1(-2^j\pi, 2^j\pi)}$ *est donc croissante, et*

$$\|\hat{w}_n(\xi)\|_{L^1(\mathbb{R})} = \sup_{j\geq 0} \|W_n^j(\xi)\|_{L^1(-2^j\pi,2^j\pi)} . \tag{3.4}$$

Démonstration. On pose $F(\xi) = 2^{j-1} \left| m_{\varepsilon_{j-1}}(\xi) \cdots m_{\varepsilon_1}(2^{j-1}\xi) \right|$. F est continue, 2π-périodique, paire et à valeurs positives ou nulles. On calcule

$$\begin{aligned} 2\int_0^{2\pi} \left| m_{\varepsilon_j}(\xi) \right| F(2\xi)\, d\xi &= 2\int_0^{2\pi} M_0(\xi)\, F(2\xi)\, d\xi \\ &= \int_0^{2\pi} S\Big(\frac{\xi}{2}\Big)\, F(\xi)\, d\xi , \end{aligned}$$

en rappelant que $S(x) = M_0(x) + M_0(x+\pi)$. Comme on a toujours $1 \leq S(x) \leq \sqrt{2}$, le Lemme 1 est démontré.

Nous allons démontrer les Théorèmes (h_p) en minorant les normes $\|W_n^j(\xi)\|_{L^1(-2^j\pi,2^j\pi)}$ à l'aide d'une récurrence sur j, pour n fixé. Il est clair que $\|W_n^0(\xi)\|_{L^1(-\pi,\pi)} = 2\pi$.

Supposons dans un premier temps que l'hypothèse (h_1) est satisfaite.

Si $\varepsilon_j = 0$, alors $N_1(n_j) = N_1(n_{j-1})$, et par le Lemme 1,

$$\|W_n^{j+1}(\xi)\|_{L^1(-2^{j+1}\pi,2^{j+1}\pi)} \geq \|W_n^j(\xi)\|_{L^1(-2^j\pi,2^j\pi)} .$$

Si $\varepsilon_j = 1$, alors $N_1(n_j) = N_1(n_{j-1}) + 1$. De plus, le quotient de $\|W_n^{j+1}(\xi)\|_{L^1(-2^{j+1}\pi,2^{j+1}\pi)}$ par $\|W_n^j(\xi)\|_{L^1(-2^j\pi,2^j\pi)}$ est égal au quotient de $2\int_{-\pi}^{\pi} M_{\varepsilon_{j+1}}(\xi)\, M_1(2\xi)\, f(4\xi)\, d\xi$ par $\int_{-\pi}^{\pi} M_1(\xi)\, f(2\xi)\, d\xi$, avec la notation $f(\xi) = W_n^{j-1}(\xi)$.

f est continue, 2π-périodique, paire et à valeurs positives ou nulles, et on trouve

$$\begin{aligned}
2\int_{-\pi}^{\pi} & M_{\varepsilon_{j+1}}(\xi)\, M_1(2\xi)\, f(4\xi)\, d\xi \\
&= \frac{1}{2}\int_{-\pi}^{\pi} f(\xi)\left[S\Big(\frac{\xi}{4}\Big)\, M_1\Big(\frac{\xi}{2}\Big) + S\Big(\frac{\xi}{4}+\frac{\pi}{2}\Big)\, M_1\Big(\frac{\xi}{2}+\pi\Big)\right] d\xi \\
&\geq \frac{1}{2}\int_{-\pi}^{\pi} f(\xi)\left(M_1\Big(\frac{\xi}{2}\Big) + \alpha\, M_1\Big(\frac{\xi}{2}+\pi\Big)\right) d\xi\,.
\end{aligned}$$

Or, d'après l'hypothèse (h_1), on a $M_1(\xi/2) \leq 1/\sqrt{2} \leq M_1(\xi/2+\pi)$ pour tout $|\xi| \leq \pi$. D'autre part, quels que soient $x \leq y$ et $a \leq b$, on a l'inégalité $xa + yb \geq (x+y)(a+b)/2$. Si l'on prend $x = 1$, $y = \alpha$, $a = M_1(\xi/2)$, $b = M_1(\xi/2+\pi)$, on en déduit

$$\begin{aligned}
2\int_{-\pi}^{\pi} M_{\varepsilon_{j+1}}(\xi)\, M_1(2\xi)\, f(4\xi)\, d\xi &\geq \frac{1+\alpha}{4}\int_{-\pi}^{\pi} f(\xi)\, S\Big(\frac{\xi}{2}\Big)\, d\xi \\
&= \frac{1+\alpha}{2}\int_{-\pi}^{\pi} M_1(\xi)\, f(2\xi)\, d\xi\,.
\end{aligned}$$

Si l'on pose $r = (1+\alpha)/2$, on vient de démontrer par récurrence sur j que

$$\|W_n^j(\xi)\|_{L^1(-2^j\pi,2^j\pi)} \geq 2\pi\, r^{N_1(n_{j-1})}\,. \tag{3.5}$$

En prenant $j > \log_2 n$, on déduit le Théorème (h_1) de (3.4) et (3.5).

Plus généralement, pour montrer le Théorème (h_p), il suffit de prouver que le quotient de l'intégrale

$$A = 2\int_{-\pi}^{\pi} f(2^{p+1}\xi)\, M_\varepsilon(\xi)\, M_1(2\xi)\prod_{q=1}^{p-1} M_0(2^{q+1}\xi)\, d\xi$$

par l'intégrale

$$B = \int_{-\pi}^{\pi} f(2^p\xi) M_1(\xi)\prod_{q=1}^{p-1} M_0(2^q\xi)\, d\xi$$

est supérieur ou égal à $r = (1+\alpha)/2$, pour toute fonction f continue, 2π-périodique, paire et à valeurs positives ou nulles, et tout $\varepsilon \in \{0,1\}$.

Pour cela, on commence par vérifier, par récurrence sur j, que le produit $M_0(\xi)\cdots M_0(2^{j-1}\xi)\, f(2^j\xi)$ vaut $M_0(2^{j-1}\xi)\, f(2^j\xi)$ pour ξ dans $[-\pi/(3\cdot 2^{j-2}),\ \pi/(3\cdot 2^{j-2})] + 2\pi\,\mathbb{Z}$ et vaut 0 partout ailleurs.

On en déduit que

$$M_1(\xi)\prod_{q=1}^{p-1} M_0(2^q\xi)\, f(2^p\xi) = M_0(2^{p-1}\xi)\, f(2^p\xi)$$

sur

$$\Big[\pi\Big(1-\frac{1}{3\cdot 2^{p-2}}\Big), \pi\Big(1+\frac{1}{3\cdot 2^{p-2}}\Big)\Big] + 2\pi\,\mathbb{Z}\,,$$

et vaut 0 partout ailleurs. Donc

$$\begin{aligned} A &= 4\int_{\pi(1-1/(3\cdot 2^{p-2}))/2}^{\pi(1+1/(3\cdot 2^{p-2}))/2} M_0(\xi)\, M_0(2^p\xi)\, f(2^{p+1}\xi)\, d\xi \\ &= 2\int_{\pi}^{\pi(1+1/(3\cdot 2^{p-2}))} S\big(\frac{\xi}{2}\big)\, M_0(2^{p-1}\xi)\, f(2^p\xi)\, d\xi\ . \end{aligned}$$

On découpe maintenant l'intervalle $I = [\pi,\ \pi(1+1/(3\cdot 2^{p-2}))]$ en trois parties

$$\begin{aligned} I_1 &= \Big[\pi,\ \pi\Big(1+\frac{1}{3\cdot 2^{p-1}}\Big)\Big]\,, \\ I_2 &= \Big[\pi\Big(1+\frac{1}{3\cdot 2^{p-1}}\Big),\ \pi\Big(1+\frac{1}{2^p}\Big)\Big] \\ I_3 &= \Big[\pi\Big(1+\frac{1}{2^p}\Big),\ \pi\Big(1+\frac{1}{3\cdot 2^{p-2}}\Big)\Big]\,. \end{aligned}$$

Sur $I_1 \cup I_2$, on a $S(\xi/2) \geq \alpha$, et I_3 est le symétrique de I_2 par rapport à $\pi(1+1/2^p)$. Donc

$$\int_{I_1} S\big(\frac{\xi}{2}\big)\, M_0(2^{p-1}\xi)\, f(2^p\xi)\, d\xi \geq \alpha \int_{I_1} M_0(2^{p-1}\xi)\, f(2^p\xi)\, d\xi\,,$$

et

$$\begin{aligned} &\int_{I_2\cup I_3} S\big(\frac{\xi}{2}\big)\, M_0(2^{p-1}\xi)\, f(2^p\xi)\, d\xi \\ &\qquad = \int_{I_2} f(2^p\xi)\Big(S\big(\frac{\xi}{2}\big) M_0(2^{p-1}\xi) \end{aligned}$$

$$\begin{aligned}
&\qquad + S\big(2\pi\,\big(1+\frac{1}{2^p}\big)-\frac{\xi}{2}\big)\,M_0(2^{p-1}\xi+\pi)\Big)\,d\xi \\
&\geq \int_{I_2}\big(\alpha\,M_0(2^{p-1}\xi)+1\,M_0(2^{p-1}\xi+\pi)\big)\,f(2^p\xi)\,d\xi \\
&\geq \frac{1+\alpha}{2}\int_{I_2} S(2^{p-1}\xi)\,f(2^p\xi)\,d\xi \\
&= \frac{1+\alpha}{2}\int_{I_1\cup I_2} M_0(2^{p-1}\xi)\,f(2^p\xi)\,d\xi\,.
\end{aligned}$$

Finalement, on trouve l'inégalité cherchée,

$$A \geq \frac{1+\alpha}{2}\int_{\pi}^{\pi(1+1/(3\cdot 2^{p-2}))} 2\,M_0(2^{p-1}\xi)\,f(2^p\xi)\,d\xi = r\,B\,.$$

On en déduit, par récurrence sur j, que

$$\|W_n^j(\xi)\|_{L^1(-2^j\pi,2^j\pi)} \geq 2\pi\,r^{N_p(n_{j-1})}\,. \tag{3.6}$$

En prenant $j > \log_2 n$, on déduit le Théorème (h_p) de (3.4) et (3.6).

4. Démonstration des Théorèmes (H_p).

On rappelle les notations

$$M_\varepsilon = |m_\varepsilon|\,,\qquad S = M_0+M_1\,,\qquad W_n^j(\xi) = M_{\varepsilon_1}(\xi/2)\cdots M_{\varepsilon_j}(\xi/2^j)\,.$$

D'après (3.3), il suffit, pour obtenir une majoration de $\|\hat{w}_n(\xi)\|_{L^1(\mathbb{R})}$, de majorer $\|W_n^j(\xi)\|_{L^1(-2^j\pi,2^j\pi)}$ indépendamment de j.

Si l'hypothèse $(H1)$ est satisfaite, en posant

$$s^\infty(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi-2k\pi)|\,,$$

il résulte de la formule (2.10) que $R = \sup_{\xi\in[0,\pi]} s^\infty(\xi) < +\infty$. On note

$$s^m(\xi) = \sum_{k=0}^{2^m-1} W_1^m(\xi+2k\pi)\,. \tag{4.1}$$

D'après (2.4) et la formule de convergence dominée de Lebesgue, s^m converge simplement vers s^∞ lorsque m tend vers l'infini. De plus, on a

$$s^{m+1}(\xi) = s^m(\xi)\, S(2^{-m-1}\xi) \geq s^m(\xi)\,.$$

Par conséquent,

$$s^m(\xi) \leq s^\infty(\xi) \leq R\,. \tag{4.2}$$

Par ailleurs, pour $m \geq j > \log_2 n$, on a l'identité

$$W^m_{n+2^j}(\xi) = W^j_n(\xi)\, W^{m-j}_1(\xi/2^j)\ . \tag{4.3}$$

On en tire

$$\begin{aligned} \|W^m_{n+2^j}\|_{L^1(-2^m\pi,2^m\pi)} &= \int_{-2^m\pi}^{2^m\pi} W^j_n(\xi)\, W^{m-j}_1(\xi/2^j)\, d\xi \\ &= \int_{-2^j\pi}^{2^j\pi} W^j_n(\xi)\, s^{m-j}(2^{-j}\xi)\, d\xi \\ &\leq R\, \|W^j_n(\xi)\|_{L^1(-2^j\pi,2^j\pi)}\ . \end{aligned} \tag{4.4}$$

Donc, pour $n = 2^{j_1} + \cdots + 2^{j_N}$, $j_1 < \cdots < j_N$, on a

$$\begin{aligned} \|W^m_n\|_{L^1(-2^m\pi,2^m\pi)} &\leq R^N\, \|W^{j_1}_0(\xi)\|_{L^1(-2^{j_1}\pi,2^{j_1}\pi)} \\ &\leq \|\hat{\varphi}(\xi)\|_{L^1(\mathbb{R})}\, R^N = C\, R^N\,. \end{aligned} \tag{4.5}$$

En faisant tendre m vers l'infini, on obtient le Théorème (H_1) comme conséquence de (3.3) et (4.5).

Supposons maintenant l'hypothèse (H_2) vérifiée.

Soit f une fonction mesurable, positive, paire, 2π-périodique. Pour avoir l'inégalité stricte

$$2\int_{-\pi}^{\pi} M_\varepsilon(\xi)\, f(2\xi)\, d\xi > \int_{-\pi}^{\pi} f(\xi)\, d\xi\,, \tag{4.6}$$

il faut que f ait une intégrale non nulle sur $[2\pi/3\,,\, 4\pi/3]$.

Donc, s'il existe g, elle aussi positive, paire et 2π-périodique, telle que $f(\xi) = M_{\varepsilon'}(\xi)\, g(2\xi)$, l'inégalité (4.6) n'est possible que pour $\varepsilon' = 1$.

Par conséquent, si l'on pose $h(\xi) = M_{\varepsilon_j}(\xi)\cdots M_{\varepsilon_1}(2^{j-1}\xi)$, on a

$$\int_{-\pi}^{\pi} 2^m\, M_0(\xi)\cdots M_0(2^m\xi)\, h(2^{m+1}\xi)\, d\xi = \int_{-\pi}^{\pi} M_0(\xi)\, h(2\xi)\, d\xi\,. \tag{4.7}$$

Si l'on admet l'existence d'une constante $R(\beta) > 1$ tel que pour tout $j \geq 1$, en posant

$$\Sigma_j(\xi) = \sum_{k=0}^{2^{j+1}-1} W_{2^j-1}^{j+1}(\xi + 2k\pi)\,,$$

on ait

$$\sup_{\xi\in[0,\pi]} \Sigma_j(\xi) \leq R(\beta)\,, \tag{4.8}$$

alors, par des calculs analogues à (4.3), (4.4) et (4.5), on peut majorer $\|W_n^m\|_{L^1(-2^m\pi,2^m\pi)}$ par $C\,R^{N_2(n)+1}$, avec

$$C = 2\int_{-\pi}^{\pi} M_0(\xi)\, d\xi\,, \qquad 2\pi < C < \frac{2\pi}{3}\,(2+\sqrt{2}) = C_2\,,$$

puis on peut passer à la limite $m \to \infty$ en utilisant (3.3).

Pour démontrer le Théorème $(H2)$, il reste donc à prouver que $R(\beta)$ existe.

Dans ce but, étudions la fonction

$$F_j(\eta) = W_{2^j-1}^{j+1}(2^{j+1}\eta) = M_0(\eta)M_1(2\eta)\cdots M_1(2^j\eta)\,.$$

Cette fonction est paire, 2π-périodique, positive, et le support de sa restriction à $[0,\pi]$ est inclus dans l'intervalle $[\pi/6\,,\,2\pi/3]$.

On appelle d_1 l'homothétie de rapport négatif qui transforme l'intervalle $[0,1]$ en l'intervalle $[\pi/6\,,\,\pi/3]$, et d_2 l'homothétie de rapport positif qui transforme $[0,1]$ en $[\pi/3\,,\,2\pi/3]$.

Soit μ la fonction périodique de période 4 telle que pour tout x dans l'intervalle $[0,4]$ on ait $\mu(x) = M_0(\pi(1+x)/3)$. D'après l'hypothèse (H_2), μ est continue sur $\mathbb{R}$, s'annule sur $[1,\,3]$, est symétrique par rapport à 2, et vérifie l'inégalité

$$0 \leq \mu(x) \leq \min\{1, 3\beta(1-x)\}\,, \qquad \text{pour tout } \ x \in [0,1]\,. \tag{4.9}$$

On pose $f_j(x) = \mu(x)\cdots\mu(4^jx)$ pour x dans $[0,1]$. Par récurrence sur j, on déduit aisément les égalités suivantes de l'hypothèse (H_2):

$$F_{2j} \circ d_2 = F_{(2j+1)} \circ d_2 = F_{(2j+1)} \circ d_1 = F_{(2j+2)} \circ d_1 = f_j \ . \tag{4.10}$$

$E(r)$ désignant la partie réelle d'un réel r, on note, pour $h > 1$ et $x \in [0, h/4^{j+1}]$,

$$\sigma_{j,h}(x) = \sum_{k=0}^{E(4^{j+1}/h)-1} f_j\Big(x + \frac{hk}{4^{j+1}}\Big) .$$

D'après (4.10), on a

$$\begin{cases} \sup\limits_{[0,\pi]} \Sigma_{2j} \le 2 \sup\limits_{[0,6/4^j]} \sigma_{j-1,6} + 2 \sup\limits_{[0,3/4^j]} \sigma_{j,12} \\ \sup\limits_{[0,\pi]} \Sigma_{2j+1} = 2 \sup\limits_{[0,3/4^j]} \sigma_{j,12} + 2 \sup\limits_{[0,6/4^{j+1}]} \sigma_{j,6} \end{cases} \tag{4.11}$$

Pour trouver $R(\beta)$, il faut donc étudier f_j. Dans ce but, on décompose x en base 4, $x = \sum_{q\ge 1} \varepsilon_q/4^q$, $0 \le \varepsilon_q \le 3$. Le support de f_j est inclus dans l'ensemble de Cantor des x de la forme $\sum_{q\ge1} 3\alpha_q/4^q$, $\alpha_q \in \{0,1\}$.

Nous allons majorer $f_j(x)$ pour x de cette forme, par un calcul similaire à [6, p. 102].

Avec la notation $\alpha_0 = 1$, on considère la suite $0 \le q_1 < q_2 < \cdots$ des indices tels que $\alpha_q \ne \alpha_{q+1}$.

Pour j fixé, on appelle I l'indice maximal tel que $q_I \le j$, et on pose $l_1 = q_2 - q_1, \ldots, l_I = (j+1) - q_I$. Prenons maintenant un entier i entre 0 et I.

Si $\alpha_{q_i} = 1$, alors $\alpha_q = 0$ pour tout $q_i < q \le q_{i+1}$, par conséquent $4^{q_i}x \in [3,\, 3 + 4^{-l_i}] \bmod (4)$.

Si $\alpha_{q_i} = 0$, alors $\alpha_q = 1$ pour tout $q_i < q \le q_{i+1}$, par conséquent $4^{q_i}x \in [3,\, 3 + 4^{-l_i}] \bmod (4)$.

Dans les deux cas, on trouve $\mu(4^{q_i}x) \le 3\beta 4^{-l_i}$, d'où

$$f_j(x) \le (3\beta)^I\, 4^{-(l_1+\cdots+l_I)} = (3\beta)^I\, 4^{q_1-(j+1)} . \tag{4.12}$$

Il en résulte l'existence d'une constante $\rho(\beta) > 1$ telle que pour tout $h > 1$, on ait

$$\sup_{0\le x\le hk/4^{j+1}} \sigma_{j,h}(x) \le \rho . \tag{4.13}$$

En effet, tout intervalle de type $[a/4^{j+1}\,,\,(a+1)/4^{j+1}]$ contient au plus un point du type $x + hk/4^{j+1}$, d'où

$$\begin{aligned}
\sigma_{j,h}(x) &\leq \sum_{a=0}^{4^{j+1}-1} \sup_{[a/4^{j+1}\,,\,(a+1)/4^{j+1}]} f_j \\
&\leq 1 + \sum_{q_1=0}^{j} 4^{q_1-j-1} \sum_{I=1}^{(j+1-q_1)} C_{j-q_1}^{I-1}\,(3\beta)^I \\
&= 1 + \sum_{q=0}^{j} 4^{q-j-1}\,3\beta\,(1+3\beta)^{j-q} \\
&= 1 + 3\beta \sum_{q=0}^{j} \Big(\frac{1+3\beta}{4}\Big)^q \\
&\leq \frac{1+3\beta}{1-\beta}\,.
\end{aligned}$$

Maintenant, en combinant (4.13) aux formules (4.11), on trouve $R(\beta)$ vérifiant (4.8), ce qui démontre le Théorème $(H2)$.

Pour démontrer les Théorèmes (H_p), $p \geq 3$, on procède de façon similaire.

Sous l'hypothèse (H_p), l'inégalité stricte (4.6) n'est possible que si f a une intégrale non nulle sur $[\pi - 2\delta_p, \pi + 2\delta_p]$.

Donc, si $f(\xi) = M_{\varepsilon_j}(\xi)\, M_{\varepsilon_{j-1}}(2\xi) \cdots M_{\varepsilon_1}(2^{j-1}\xi)\, g(2^j\xi)$ avec g positive, 2π-périodique et paire, f ne peut vérifier (4.6) que si la suite $\varepsilon_1 \cdots \varepsilon_j$ est constituée d'un premier bloc qui peut être ou bien

$$\big(1 \quad \underbrace{0\ldots0}_{(p-2)\text{ fois }0} \quad 1\big)\,,$$

ou bien

$$\big(\underbrace{0\ldots0}_{k\text{ fois }0}\ \ 1\big)\,,$$

avec $0 \leq k \leq p-1$, suivi d'un certain nombre de blocs de la forme

$$\big(1 \quad \underbrace{0\ldots0}_{(p-2)\text{ fois }0} \quad 1\big)\,.$$

Par conséquent, si l'on trouve un majorant $R(\beta,p)$ aux sommes de la forme

$$\Sigma_j^p(\xi) = \sum_{k=0}^{2^{jp+1}-1} W_{n(j,p)}^{jp+1}(\xi+2k\pi)\,,$$

où $\xi \in [0,\pi]$ et

$$n(j,p) = \sum_{q=0}^{j-1} 2^{qp}(1+2^{p-1})\,,$$

alors $\|W_n^m\|_{L^1(-2^m\pi,2^m\pi)}$ se laisse majorer par $C_p(R(\beta,p))^{N_p(n)+1}$, avec

$$C_p = 2\pi + 4\delta_p\,(\sqrt{2}-1) \geq 2\int_{-\pi}^{\pi} M_0(\xi)\,d\xi\,.$$

Etudions donc $F_j^p(\eta) = W_{n(j,p)}^{jp+1}(2^{jp+1}\eta)$, pour $j \geq 1$.

Le support de sa restriction à $[0,\pi]$ est inclus dans l'intervalle

$$I = \left[\frac{\pi}{2} - (1+2^{-(j-1)p})\delta_p\,,\ \frac{\pi}{2}+\delta_p\right] = I_1 \cup I_2\,,$$

avec

$$I_1 = \left[\frac{\pi}{2} - (1+2^{-(j-1)p})\delta_p\,,\ \frac{\pi}{2}-\delta_p\right]$$

et

$$I_2 = \left[\frac{\pi}{2} - \delta_p\,,\ \frac{\pi}{2}+\delta_p\right].$$

Soit D l'homothétie de rapport positif qui envoie l'intervalle $[0,1]$ sur l'intervalle I_2. Pour $x \in [0,1]$, on pose $f_j^p(x) = F_j^p \circ D(x)$. Etant donnés $h > 1$ et $x \in [0, h/2^{(j+1)p}]$, on note

$$\sigma_{j,h}^p(x) = \sum_{k=0}^{E(2^{p(j+1)}/h)-1} f_j^p\left(x + \frac{hk}{2^{p(j+1)}}\right).$$

L'intervalle I_1 peut contenir au plus un réel du type $2^{-(jp+1)}(\xi+2k\pi)$, pour ξ fixé et k variable. Par conséquent, on a l'inégalité

$$\text{(4.14)} \qquad \sup_{\xi\in[0,\pi]} \Sigma_j^p(\xi) \leq 1 + 2 \sup_{x\in[0,h(p)/2^{(j+1)p}]} \sigma_{j,h(p)}^p(x)$$

avec $h(p) = \pi/\delta_p = 2\,(2^p-1) > 1$.

D'après l'hypothèse (H_p), on a la formule

$$f_j^p(x) = \nu(x)\,\nu(2^p x)\cdots\nu(2^{jp}x) \tag{4.15}$$

pour tout $x \in [0,1]$, ν étant une fonction positive, continue, paire et de période 2^p, s'annulant sur $[1,\, 2^p - 1]$, et vérifiant, pour tout x dans $[0,1]$, l'inégalité

$$\nu(x) \le \min\big\{1, \beta\,(2^p-1)\,(1-x)\big\}. \tag{4.16}$$

On voit donc que la fonction f_j^p a son support inclus dans l'ensemble de Cantor des x de la forme $\sum_{q\ge 1}(2^p-1)\alpha_q/2^{qp}$, avec $\alpha_q \in \{0,1\}$, et on trouve une généralisation de (4.12):

$$f_j^p(x) \le \big[(2^p-1)\,\beta\big]^I\, 2^{p(q_1-j-1)}. \tag{4.17}$$

Puis on obtient, pour $h > 1$,

$$\begin{aligned}\sigma_{j,h}^p &\le 1 + (2^p-1)\,\beta \sum_{q=0}^{j} \Big(\frac{1+(2^p-1)\beta}{2^p}\Big)^q \\ &\le 1 + \frac{2^p\beta}{1-\beta}.\end{aligned} \tag{4.18}$$

En conclusion, d'après (4.14) et (4.18), on peut prendre

$$R(\beta, p) = 3 + \frac{2^{p+1}\beta}{1-\beta},$$

et le Théorème (H_p) est démontré.

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On the singularities of the inverse to a meromorphic function of finite order

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Abstract. Our main result implies the following theorem: Let f be a transcendental meromorphic function in the complex plane. If f has finite order ρ, then every asymptotic value of f, except at most 2ρ of them, is a limit point of critical values of f.

We give several applications of this theorem. For example we prove that if f is a transcendental meromorphic function then $f'f^n$ with $n \geq 1$ takes every finite non-zero value infinitely often. This proves a conjecture of Hayman. The proof makes use of the iteration theory of meromorphic functions.

Introduction and main results.

In this paper by meromorphic function we mean a transcendental meromorphic function in the complex plane $\mathbb{C}$, if the domain of definition is not explicitly specified. Let $f : \mathbb{C} \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a meromorphic function. The inverse function f^{-1} can be defined on a Riemann surface which is conformally equivalent to $\mathbb{C}$ via f^{-1}. In this paper we identify the Riemann surface of f^{-1} with $\mathbb{C}$. We want to study the singularities of f^{-1}. This can be done by adding to $\mathbb{C}$ some ideal points and defining neighborhoods of these points.

Let us start with precise definitions. Take $a \in \overline{\mathbb{C}}$ and denote by $D(r,a)$ the disk of radius $r > 0$ (in spherical metric) centered at a. For every $r > 0$ choose a component $U(r)$ of the preimage $f^{-1}(D(r,a))$ in such a way that $r_1 < r_2$ implies $U(r_1) \subset U(r_2)$. Note that the function $U : r \to U(r)$ is completely determined by its germ at 0. Two possibilities can occur:

a) $\bigcap_{r>0} U(r) = \{z\}$, $z \in \mathbb{C}$. Then $a = f(z)$. If $a \in \mathbb{C}$ and $f'(z) \neq 0$ or if $a = \infty$ and z is a simple pole of f, then z is called an *ordinary point.* If $a \in \mathbb{C}$ and $f'(z) = 0$ or if $a = \infty$ and z is a multiple pole of f, then z is called a *critical point* and a is called a *critical value.* We also say that the critical point z *lies over* a.

b) $\bigcap_{r>0} U(r) = \varnothing$. Then we say that our choice $r \to U(r)$ defines a (transcendental) *singularity* of f^{-1}. For simplicity we just call such U a singularity. We also say that the singularity U *lies over* a. For every $r > 0$ the open set $U(r) \subset \mathbb{C}$ is called a *neighborhood of the singularity* U. So if $z_k \in \mathbb{C}$, we say that $z_k \to U$ if for every $\varepsilon > 0$ there exists k_0 such that $z_k \in U(\varepsilon)$ for $k \geq k_0$.

If U is a singularity then a is an asymptotic value, which means that there exists a curve $\Gamma \subset \mathbb{C}$ tending to ∞ such that $f(z) \to a$ as $z \to \infty$, $z \in \Gamma$. Such Γ is called an asymptotic curve. To construct an asymptotic curve take a sequence $r_k \to 0$ and a sequence $z_k \in U(r_k)$ and connect z_k to z_{k+1} by a curve $\gamma_k \subset U(r_k)$, which is possible because the $U(r)$ are connected. Then $\Gamma = \cup \gamma_k$ is an asymptotic curve. In particular it follows that every neighborhood $U(r)$ of a singularity U is unbounded. If a is an asymptotic value of f, then there is at least one singularity over a. Indeed, let $\Gamma \subset \mathbb{C}$ be an asymptotic curve, on which $f(z) \to a$. Then for every $r > 0$ the "tail" of Γ where $f(z) \in D(r,a)$ belongs to $f^{-1}(D(r,a))$ and we define $U(r)$ as the component of $f^{-1}(D(r,a))$ which contains this tail.

Certainly there can be many different singularities as well as critical and ordinary points over the same point a. Remark that if f is a meromorphic function, and $D \subset \overline{\mathbb{C}}$ contains no critical values and no asymptotic values then $f : f^{-1}(D) \to D$ is a covering. This justifies the name "singularities of f^{-1}".

The connection between asymptotic values of f and singularities of f^{-1} was stated for the first time by A. Hurwitz [17]. The following classification of singularities is due to F. Iversen [18] (see also [21], [26]).

A singularity U over a is called *direct* if there exists $r > 0$ such that $f(z) \neq a$ for $z \in U(r)$. (Then this is also true for all smaller values of r.) The simplest case of a direct singularity is the so-called logarithmic branch point. We say that U is a *logarithmic branch point* (or *logarithmic singularity*) over a if $f : U(r) \to D(r, a)\backslash\{a\}$ is a universal covering for some $r > 0$. Thus if $f = \exp$ then the inverse function $f^{-1} = \log$ has two logarithmic branch points: one over 0 and one over ∞. The function arccos, inverse to cos, has two logarithmic singularities over ∞.

A singularity U over a is called *indirect* if it is not direct, *i.e.* for every $r > 0$ the function f takes the value a in $U(r)$. In this case evidently the function f takes the value a infinitely often in $U(r)$. A simple example of an indirect singularity is given by the inverse function of $f(z) = \sin z/z$. Note that in this example the asymptotic value 0 is a limit point of critical values. M. Heins [15, Theorem 5] proved that the set of direct singularities of a function inverse to a meromorphic function is always countable.

For a meromorphic function of finite order ρ the celebrated Denjoy-Carleman-Ahlfors Theorem states that the inverse function has at most $\max\{2\rho, 1\}$ direct singularities [21, p. 309]. This implies that an entire function of finite order ρ has at most 2ρ finite asymptotic values [21, p. 313]. On the other hand, there are meromorphic functions of any given order $\rho \geq 0$ such that every point in $\overline{\mathbb{C}}$ is an asymptotic value [7]. So in this case the number of indirect singularities is infinite.

In the simplest examples like $f(z) = \sin z/z$ the indirect singularities are limits of critical points. More complicated examples show that this is not the case in general. One such example is contained in the book of L. I. Volkovyskii [27, p. 70]. He constructs a meromorphic function f with no critical points such that the set of asymptotic values has the power of the continuum (it is actually a Cantor set on the unit circle). So the inverse function of this function has many indirect singularities because the set of direct ones is countable by the result of Heins mentioned above. See also [25], where a similar example is discussed.

Our main result is that in the case of finite order the nature of the singularities of f^{-1} is much simpler.

Theorem 1. *If f is a meromorphic function of finite order, then every indirect singularity of f^{-1} is a limit of critical points.*

We can easily derive from Theorem 1 a formally stronger version of this theorem.

Theorem 1'. *Let f be a meromorphic function of finite order. Then every indirect singularity of f^{-1} over $a \in \overline{\mathbb{C}}$ is a limit point of critical points z_k such that $f(z_k) \neq a$.*

Proof. Assume that f has an indirect singularity U over a such that for some $r > 0$ the set $V = U(r) \backslash f^{-1}(a)$ contains no critical points. As the number of direct singularities is finite we may assume that there are no direct singularities over $A = D(r, a) \backslash \{a\}$.

Let us show that

$$f : V \to A \tag{1}$$

has an asymptotic value $a' \in A$. If this is not the case then (1) is a covering. As the fundamental group of the annulus A is $\mathbb{Z}$ we conclude that the fundamental group of V is $\mathbb{Z}$ or trivial. In the first case V is a degenerate annulus and a cannot be an asymptotic value in $U(r)$. So the fundamental group of V is trivial, that is, (1) is a universal covering. Then $f : U(r) \to A$ is also a universal covering, which contradicts to our assumption that U is a neighborhood of an indirect singularity over a.

Thus there is an asymptotic value $a' \in A$ such that the corresponding (indirect) singularity U' has a neighborhood $U'(r') \subset V$. Now we apply Theorem 1 to U' to conclude that there are critical points $z_k \in U(r)$ such that $f(z_k) \neq a$. This proves Theorem $1'$.

Corollary 1. *If f is a meromorphic function of finite order and a is an asymptotic value of f, then a is a limit of critical values $a_k \neq a$ or all singularities of f^{-1} over a are logarithmic.*

Corollary 2. *If f is a meromorphic function of finite order ρ and E is the set of its critical values, then the number of asymptotic values of f is at most $2\rho + \operatorname{card} E'$, where E' stands for the derived set of E.*

Proof. Let a be an asymptotic value, $a \notin E'$. By Corollary 1 there is a logarithmic singularity over a. Let us show that the number of logarithmic singularities is at most 2ρ. For $\rho \geq 1/2$ this follows from the Denjoy-Carleman-Ahlfors Theorem quoted above. It remains to show that there are no logarithmic singularities if $\rho < 1/2$. Suppose that there is a logarithmic singularity over $a \in \overline{\mathbb{C}}$ and that $f : U(r) \to D(r,a)\backslash\{a\}$ is a universal covering. Then $U(r)$ is a simply-connected unbounded domain. Assume without loss of generality that $a = \infty$. Then there exists $R > 0$ such that $R < |f(z)| < +\infty$ for $z \in U(r)$ and $|f(z)| = R$ for $z \in \partial U(r)$. Define a function u by $u(z) = \log(|f(z)|/R)$ for $z \in U(r)$ and $u(z) = 0$ for $z \in \mathbb{C}\backslash U(r)$. It is easy to see that u is subharmonic in $\mathbb{C}$. Since u is bounded on $\partial U(r)$ we deduce from a classical theorem due to Wiman (see for example [14, Theorem 6.4]) that the order of u is at least $1/2$. But the order of f is greater or equal than the order of u.

Corollary 3. *If a meromorphic function of finite order ρ has only finitely many critical values, then it has at most 2ρ asymptotic values.*

Corollary 3 was conjectured by the second author in his talk on the A.M.S. meeting in Springfield, Missouri, in October 1991.

Theorem 1 and its corollaries may be useful in many questions involving meromorphic functions of finite order, in particular in the iteration theory of rational [1], [3], [24] and transcendental meromorphic [2] functions. The role of singularities in the iteration of transcendental functions is discussed in [2, Section 4.3]. The connection with rational functions is via Poincaré functions.

We will apply our result to the distribution of values of some differential polynomials. In [13, Problem 1.19] W. K. Hayman conjectured that if f is a nonconstant meromorphic function and $n \in \mathbb{N}$, then $f'f^n$ takes every finite non-zero value. Earlier he had proved this for $n \geq 3$. More precisely, he had shown that if f is transcendental, then $f'f^n$ takes every finite non-zero value infinitely often if f is meromorphic and $n \geq 3$ [11, Corollary to Theorem 9] or if f is entire and $n \geq 2$ [11, Theorem 10]. J. Clunie [5] proved this for the case that f is entire and $n = 1$. Later E. Mues [19, Satz 3] settled the case that f is meromorphic and $n = 2$ and W. Hennekemper [16] extended Clunie's result to functions which have few poles in some sense.

We prove here the last unsolved case ($n = 1$ for meromorphic functions). Our method gives also a unified proof of all results on Hayman's conjecture mentioned above.

Theorem 2. *If f is a transcendental meromorphic function and $m > l$ are positive integers then $(f^m)^{(l)}$ assumes every finite non-zero value infinitely often.*

Hayman's conjecture corresponds to the case $l = 1$ in this theorem. The example $f(z) = e^z$ shows that 0 and ∞ can actually be omitted. Actually only the case $m = 2$, $l = 1$ in Theorem 2 is new. Recently Y. F. Wang [28] proved the statement of Theorem 2 for all $m \geq 3$ and all $l \geq 0$. Applying Theorem 2 to $1/f$ instead of f with $m = 2$ and $l = 1$ we obtain the following result.

Corollary 4. *If f is a transcendental meromorphic function then $f' + f^3$ has infinitely many zeros.*

The corresponding result for $f' + f^n$, $n \geq 4$, can be found in the papers of Hayman and Mues cited above.

Theorem 2 will be deduced from the following result which may be of independent interest.

Theorem 3. *Let f be a meromorphic function of finite order. If f has infinitely many multiple zeros, then f' assumes every finite non-zero value infinitely often.*

The proof of Theorem 3 uses iteration theory of meromorphic functions. The deduction of Theorem 2 from Theorem 3 is based on a rescaling lemma of Zalcman and Pang (Lemma 4), which allows to reduce the matter to the case of finite order. On the other hand we will construct an example which shows that Theorem 3 fails for functions of infinite order.

As a second application of Theorem 1 we give a unified proof of the following results recently obtained by J. Clunie, J. Langley, J. Rossi, and the second author [6], [8].

Theorem 4. *Let f be a transcendental meromorphic function of order ρ.*

a) *If $\rho < 1$ then f' has infinitely many zeros.*

b) *If $\rho < 1/2$ then f'/f has infinitely many zeros.*

c) *If f is entire and $\rho < 1$ then f'/f has infinitely many zeros.*

Examples in [6] show that all bounds for ρ in this theorem are sharp.

REMARK. First we proved Hayman's conjecture (Theorem 2 with $m = 2$ and $l = 1$) only for functions of finite order. A preprint with this result was widely circulated. It was then realized independently (and almost simultaneously) by H. H. Chen and M. L. Fang, by L. Zalcman, and by the second author of this paper how the infinite order case can be reduced to the finite order case (Step 2 in the proof of Theorem 2). We are grateful to L. Yang for telling us about H. H. Chen's and M. L. Fang's result and to Y. F. Wang for sending us a preprint of their work [4], to L. Zalcman for informing us about his work and to D. Drasin for bringing to our attention the papers of X. Pang [22], [23].

2. Lemmas.

The proofs of the following two lemmas use some ideas of A. Weitsman [29] (compare also [8, Proposition 2.1]).

Lemma 1. *Let $p > 3$ be an integer and g be a transcendental meromorphic function of order less than $p - 3$. Then there exists an integer $n_0 = n_0(g)$ and a sequence $R_n \in (2^{pn-2}, 2^{pn})$, $n \geq n_0$, such that the total length of the level curves $|g(z)| = R_n$ in $K_n = \{z : |z| \leq 2^n\}$ is at most $2^{pn/2}$.*

PROOF. We use the standard notations of Nevanlinna theory [8], [12], [21]. For $R > |g(0)| + 1$ (or $R > 0$ if $g(0) = \infty$) and $\theta \in [0, 2\pi]$ we have

$$n\left(2^n, \frac{1}{g - Re^{i\theta}}\right) \leq N\left(2^{n+2}, \frac{1}{g - Re^{i\theta}}\right) \leq T\left(2^{n+2}, g\right) + \log^+ R + C\,,$$

where C depends on g only. Thus

$$p_n(R) := \frac{1}{2\pi}\int_0^{2\pi} n\left(2^n, \frac{1}{g - Re^{i\theta}}\right) d\theta \leq T\left(2^{n+2}, g\right) + \log^+ R + C\,. \tag{2}$$

Let $l_n(R)$ be the total length of the level curves $|g(z)| = R$ in K_n. Put $\beta_n = 2^{pn}$ and $\alpha_n = 2^{pn-2}$. By the length-area principle [10, p. 18] we have

$$\int_{\alpha_n}^{\beta_n} \frac{l_n(R)^2 dR}{R p_n(R)} \leq 2\pi \,\mathrm{area}\,(K_n) = 2\pi^2\, 2^{2n}\,.$$

So there exists $R_n \in (\alpha_n, \beta_n)$ such that

$$l_n(R_n)^2 \le \frac{1}{\beta_n - \alpha_n} R_n\, p_n(R_n)\, 2\pi^2\, 2^{2n} \le 2^{pn}\,, \qquad n \ge n_0\ ,$$

in view of (2) and the estimate

$$T(2^{n+2}, g) \le 2^{(p-3)(n+2)}\,, \qquad n \ge n_0\ .$$

This proves the lemma.

Lemma 2. *Let $p > 3$ be an integer and f be a meromorphic function of order less than $p-3$. Given $\varepsilon > 0$ there exists $C > 0$ such that for every component B of the set $E = \{z : |f'(z)| < C^{-1}|z|^{-2p}\}$ we have*

$$\operatorname{diam} f(B) < \varepsilon\,. \tag{3}$$

Here $\operatorname{diam} S$ denotes the (Euclidean) diameter of a set $S \subset \mathbb{C}$.

PROOF. Apply Lemma 1 to the function $g = 1/f'$. Note that f and g have the same order because f and f' have the same order by a result of J. M. Whittaker [30]. Increase if necessary n_0 from Lemma 1 such that

$$\sum_{n=n_0}^{\infty} \frac{2^{np/2} + 2\pi\, 2^n}{R_n} < \frac{\varepsilon}{2} \tag{4}$$

and hence

$$\sum_{n=n_0}^{\infty} \frac{2^{n+1}}{R_n} < \frac{\varepsilon}{2}\ . \tag{5}$$

For $n \ge n_0$ we set

$$V_n = \{z :\ |z| < 2^n,\ |g(z)| > R_n\}$$

and

$$V = \bigcup_{n=n_0}^{\infty} V_n\ .$$

Note that the boundary ∂V consists of some arcs of the level curves $|g(z)| = R_n$ which are in K_n and some arcs of the circles $|z| = 2^n$ on which we have $R_n \le |g(z)| \le R_{n+1}$. Applying Lemma 1 and (4) we obtain

$$\int_{\partial V} |g(z)|^{-1}\,|dz| \le \sum_{n=n_0}^{\infty} \frac{2^{np/2} + 2\pi\,2^n}{R_n} < \frac{\varepsilon}{2}\,. \tag{6}$$

We may assume without loss of generality that there are no poles of g on $|z| = 2^{n_0}$. Choose $C > 1$ such that the set $E = \{z : |g(z)| > C\,|z|^{2p}\}$ does not meet the circle $|z| = 2^{n_0}$ and such that for all components B of this set contained in $\{z : |z| < 2^{n_0}\}$ the condition (3) is satisfied. Let us show that $E \cap \{z : |z| \ge 2^{n_0}\} \subset V$. If $z \in E$ and $|z| \ge 2^{n_0}$, we can find $n > n_0$ such that $2^{n-1} \le |z| < 2^n$. Then we have $|g(z)| > C\,|z|^{2p} \ge |z|^{2p} \ge 2^{2p(n-1)} \ge R_n$ so that $z \in V_n \subset V$.

Now let D be a component of V which contains a component B of E such that $B \subset \{z : |z| > 2^{n_0}\}$. If z_1 and z_2 are in B, connect them by the straight line segment L. If $L \subset D$ take $\gamma = L$. If $L \not\subset D$ consider a segment $[a, b] \subset L$ such that $(a, b) \subset \mathbb{C}\backslash D$ and $a, b \in \partial D$. Replace (a, b) by a bounded arc of ∂D connecting a and b. After performing this procedure on every segment of $L\backslash D$ we obtain a curve γ_1 connecting z_1 and z_2. Delete if necessary some parts of γ_1 to obtain a simple curve γ connecting z_1 and z_2. The part of γ in D consists of some segments of L. Denote by T_n the union of these segments which lie in $2^{n-1} \le |z| \le 2^n$. Then $|g(z)| \ge R_n$ for $z \in T_n$ and thus by (5) and (6)

$$\begin{aligned}
|f(z_1) - f(z_2)| &\le \int_\gamma |g(z)|^{-1}\,|dz| \\
&< \frac{\varepsilon}{2} + \sum_{n=n_0}^{\infty} \int_{T_n} |g(z)|^{-1}\,|dz| \\
&\le \frac{\varepsilon}{2} + \sum_{n=n_0}^{\infty} \frac{2^{n+1}}{R_n} \\
&< \varepsilon\,.
\end{aligned}$$

3. Proof of Theorem 1.

Let a be an asymptotic value and U be an indirect singularity over a such that $U(R_0)$ contains no critical points and $0 \notin U(R_0)$ for some $R_0 > 0$. Without loss of generality we may assume that $a = 0$. We are going to construct inductively the following objects:

- a sequence of asymptotic values a_n , $R_0/2 > |a_1| > |a_2| > \cdots$,
- a sequence of disjoint simply-connected domains $G_n \subset U(R_0/2)$ such that f is univalent in G_n and $D_n = f(G_n)$ is a disk, $0 \notin \overline{D}_n$,
- a sequence of asymptotic curves $\Gamma_n \subset G_n$ such that $f(\Gamma_n)$ is a straight line segment and $f(z) \to a_n$ as $z \to \infty$, $z \in \Gamma_n$.

Let us show how to construct a_n, G_n, and Γ_n assuming that a_k, G_k, and Γ_k are already constructed for $k < n$.

First choose a positive number $R_n < |a_{n-1}|$ (if $n = 1$ we take $R_1 < R_0/2$) such that $U(R_n) \cap G_k = \varnothing$ for $k < n$. This is possible because $0 \notin \overline{D}_k = \overline{f(G_k)}$. Then we take a point $z_n \in U(R_n)$ satisfying $f(z_n) = 0$. The existence of such a point follows from the definition of an indirect singularity. We have $f'(z_n) \neq 0$ by assumption. So there exists a branch φ of f^{-1} of the form

$$\varphi(w) = z_n + \sum_{m=1}^{\infty} c_m w^m .$$

Denote by r_n the radius of convergence of this series.

We claim that

$$0 < r_n < R_n . \tag{7}$$

To prove the right inequality, suppose that $r_n \geq R_n$. Then $A = \varphi(\{w : |w| < R_n\})$ is a component of $f^{-1}(\{w : |w| < R_n\})$, containing the point $z_n \in U(R_n)$. This implies that $A = U(R_n)$ because $U(R_n)$ is connected. Hence f is univalent in $U(R_n)$, which is a contradiction. This proves (7).

Let $a_n = r_n e^{is_n}$ be a singular point of φ. We have $|a_n| = r_n < R_n < |a_{n-1}| < \cdots < R_0/2$.

Consider the disk

$$D_n = \left\{ w : \left| w - \frac{2\,r_n}{3} e^{is_n} \right| < \frac{r_n}{3} \right\} .$$

Then φ is holomorphic on $\overline{D}_n \backslash \{a_n\}$ and $0 \notin \overline{D}_n$. Set $G_n = \varphi(D_n)$. Then G_n is a simply-connected domain in $\mathbb{C}$ bounded by one analytic curve tending to infinity in both directions. Indeed, if G_n is bounded, then $z^* = \varphi(a_n) \in \mathbb{C}$. If z^* is an ordinary point, then φ has no singular point at a_n. But $z^* \in U(R_n) \subset U(r_0)$ cannot be a critical point by assumption. Moreover, $G_n \subset U(R_n)$ so that in particular $G_n \cap G_k = \varnothing$ for $k < n$. Finally we consider the segment

$$L_n = \left\{ w = t\, e^{is_n} : \; \frac{2}{3}\, r_n \leq t < r_n \right\} \subset D_n$$

and put $\Gamma_n = \varphi(L_n)$. This completes our construction.

Now we want to estimate the rate of convergence $f(z) \to a_n$, $z \in \Gamma_n$. Let $q_n \in \partial G_n$, $x_n = |q_n|$. For $x > x_n$ we denote by $\theta_n(x)$ the angular measure of $\{\theta : xe^{i\theta} \in G_n\}$. Then

$$\sum_{n=1}^{\infty} \theta_n(x) \leq 2\pi \tag{8}$$

because the G_n are disjoint. Now, by the Ahlfors distortion Theorem [21, p. 98] applied to the conformal map $f : G_n \to D_n$, we have

$$\log \frac{1}{|f(z) - a_n|} \geq \pi \int_{x_n}^{|z|} \frac{dx}{x\, \theta_n(x)} - C_n\,, \qquad z \in \Gamma_n\,, \tag{9}$$

where the C_n are constants. We want to conclude from here that for all n with at most $4p + 2$ exceptions

$$\liminf_{z \to \infty,\, z \in \Gamma_n} \; |f(z) - a_n| |z|^{2p+1} = 0\,. \tag{10}$$

(Here $p > 3$ is a natural number such that the order of f is less than $p - 3$.) To prove (10) assume that $|f(z) - a_n| > c\,|z|^{-2p-1}$ for $K = 4p+3$ values of n and all large $|z|$, say for $n = 1, 2, \ldots, K$ and $|z| \geq x_0$, where $x_0 > \max\{x_n : 1 \leq n \leq K\}$. Then we have by (9)

$$\pi \int_{x_0}^{|z|} \frac{dx}{x\, \theta_n(x)} \leq (2p + 1) \log |z| + O(1)\,, \quad 1 \leq n \leq K\,. \tag{11}$$

Now using Schwarz's inequality and (11) we get

$$\left(\log \frac{|z|}{x_0} \right)^2 = \left(\int_{x_0}^{|z|} \frac{dx}{x} \right)^2 \leq$$

$$\le \int_{x_0}^{|z|} \frac{dx}{x\,\theta_n(x)} \int_{x_0}^{|z|} \frac{\theta_n(x)\,dx}{x}$$
$$\le \left(\frac{1}{\pi}\,(2p+1)\,\log|z| + O(1)\right) \int_{x_0}^{|z|} \frac{\theta_n(x)\,dx}{x}\,.$$

Adding these inequalities for $n = 1, 2, \ldots, K$ and using (8) we obtain

$$K\left(\log\frac{|z|}{x_0}\right)^2 \le \Big(2\,(2p+1)\,\log|z| + O(1)\Big)\,\log\frac{|z|}{x_0}$$

which is a contradiction because $K = 4p+3$. This proves that (10) is satisfied except possibly for $4p+2$ values of n. Dropping those a_n and Γ_n for which (10) is not satisfied and changing the enumeration of the remaining a_n and Γ_n we may assume that (10) is satisfied for all n.

Next we prove that for every n there exists a sequence $z_{n,j} \in \Gamma_n$, $z_{n,j} \to \infty$, such that

$$|f'(z_{n,j})| \le |z_{n,j}|^{-2p-1}\,. \tag{12}$$

Recall that f maps Γ_n monotonically onto a straight line segment. Thus

$$|f(z) - a_n| = \int_z^\infty |f'(z)|\,|dz|\,,$$

where the path of integration is Γ_n. If we assume contrary to (12) that $|f'(z)| > |z|^{-2p-1}$ for all $z \in \Gamma_n$ with sufficiently large moduli, then we obtain

$$|f(z) - a_n| \ge \int_z^\infty |z|^{-2p-1}\,|dz| \ge \frac{1}{2p}\,|z|^{-2p}$$

which contradicts (10). Hence (12) is true.

Recall that $R_0/2 > |a_1| > |a_2| > \cdots$ and put

$$\varepsilon = \frac{1}{4}\,\min\{|a_i - a_j| :\ 1 \le i < j \le 2p\}\,.$$

Then $\varepsilon < R_0/8$. Apply Lemma 2 using the value of ε just specified. Lemma 2 gives some value $C > 0$. For every n choose a point $z_n^* = z_{n,j(n)}$ using the relation (12) such that the following conditions are satisfied for $1 \le n \le 2p$:

$$|z_n^*| \ge C \tag{13}$$

and

$$|f(z_n^*) - a_n| < \varepsilon .$$

Then

$$|f(z_n^*) - f(z_k^*)| > 2\,\varepsilon , \qquad 1 \le n < k \le 2p , \tag{14}$$

and

$$|f(z_n^*)| + \varepsilon < \frac{3}{4}\, R_0 , \qquad 1 \le n \le 2p . \tag{15}$$

Using (12) and (13) we get

$$|f'(z_n^*)| < C^{-1}\, |z_n^*|^{-2p} , \qquad 1 \le n \le 2p . \tag{16}$$

Let B_n be the component of the set $\{z : |f'(z)| < C^{-1}|z|^{-2p}\}$ containing z_n^*. Applying Lemma 2 we conclude that

$$\operatorname{diam} f(B_n) < \varepsilon , \qquad 1 \le n \le 2p . \tag{17}$$

By (15) we have $f(B_n) \subset \{w : |w| < 3R_0/4\}$. But $U(R_0)$ is a component of $f^{-1}(\{w : |w| < R_0\})$ and $U(R_0)$ and B_n have a point z_n^* in common. So we conclude that

$$\overline{B}_n \subset U(R_0) , \qquad 1 \le n \le 2p . \tag{18}$$

Comparing (14) and (17) we conclude that the B_n are disjoint.

The function

$$u(z) = -\log|f'(z)| - 2p \log|z| - \log C$$

is subharmonic in $U(R_0)$ because $U(R_0)$ does not contain critical points of f by assumption. Also $0 \notin U(R_0)$ by assumption. Now the B_n are components of the set $\{z \in U(R_0) : u(z) > 0\}$ and we have $u(z) = 0$ for $z \in \partial B_n$ by (18).

Now a standard application of the subharmonic version of the Denjoy-Carleman-Ahlfors Theorem [14, Theorem 8.9] shows that the order of u is at least p. So the order of f' and hence f is at least p and we have a contradiction which proves the theorem.

4. Proof of Theorems 2 and 3.

Proof of Theorem 3. Let $c \in \mathbb{C}\backslash\{0\}$ and consider the function g defined by $g(z) = z - f(z)/c$. Then g has finite order because f has finite order.

We shall use some results from the iteration theory of meromorphic functions. By $g^{\circ n}$ we denote the n-th iterate of g. The largest open set where all $g^{\circ n}$ are defined and form a normal family is called the *Fatou set* of g and denoted by $F(g)$.

Let now ζ be a multiple zero of f. Then $g(\zeta) = \zeta$ and $g'(\zeta) = 1$. Classical results from iteration theory (see for example [1, Theorem 6.5.4]) now imply that there exists a component U of $F(g)$, a so-called *Leau domain*, such that $\zeta \in \partial U$ and $g^{\circ n} \to \zeta$ locally uniformly in U. Moreover, U contains a critical or asymptotic value of g, see for example [1, Theorem 9.3.2]. (In [1] as well as in [3], [24] only the case of rational functions is discussed, in which case only critical values need to be considered, but the proof extends to the transcendental case, if we also take asymptotic values into account.) Since f has infinitely many multiple zeros and since Leau domains related to distinct fixed points of g are disjoint, we deduce that the set of critical and asymptotic values of g is infinite. By Corollary 3 this is possible only if g has infinitely many critical values. In particular, g' has infinitely many zeros which implies that f' assumes the value c infinitely often. This completes the proof of Theorem 3.

For the proof of Theorem 2 we also need the following lemmas.

Lemma 3. *Let f be a transcendental meromorphic function. If f has only finitely many zeros, then $f^{(l)}$, $l \geq 1$, assumes every finite non-zero value infinitely often.*

Lemma 3 was proved by W. K. Hayman ([11, Theorem 3] or [12, Corollary to Theorem 3.5]).

Lemma 4. *Let F be a non normal family of meromorphic functions in the unit disk D, and $-1 < k < 1$. Then there exist sequences $f_n \in F$, $z_n \in D$ and $a_n > 0$ such that $|z_n| < r < 1$, $a_n \to 0$ and*

$$g_n(\zeta) = a_n^{-k} f_n(z_n + a_n\zeta) \to g(\zeta),$$

where g is a non-constant meromorphic function in the plane of order at most 2, normal type, and the convergence is uniform on compacta in $\mathbb{C}$ with respect to the spherical metric.

The case $k = 0$ in Lemma 4 was proved by L. Zalcman [31], [32], and the general case by X. Pang [22], [23].

Proof of Theorem 2.

Step 1. We first prove the theorem for the case when the order of f is finite. If f has finitely many zeros then the conclusion follows from Lemma 3. If f has infinitely many zeros then $h = (f^m)^{(l-1)}$ has infinitely many *multiple* zeros and we apply Theorem 3 to h.

Step 2. Now we reduce the general case to the case of finite order, using Lemma 4. We use the notation

$$f^{\#} = \frac{|f'|}{1 + |f|^2}$$

for the spherical derivative.

Suppose that there exists a transcendental meromorphic function f such that the equation $(f^m)^{(l)}(z) = a$ has a finite set of solutions for some $a \neq 0$. We may assume without loss of generality that $a = 1$.

Put $k = l/m$ and define a family F consisting of all functions

$$f_n(z) = 2^{-kn}\, f(2^n z)\,, \qquad 1/4 < |z| < 2\,, \quad n = 1, 2, \ldots$$

This family cannot be normal in $\{z : 1/4 < |z| < 2\}$. For otherwise we would have for some $M > 0$

$$M > f_n^{\#}(z) \geq 2^{(1-k)n}\, f^{\#}(2^n z) > f^{\#}(2^n z)\,, \qquad 1/2 < |z| < 1$$

from which follows that

$$\iint_{|x+iy|<r} (f^{\#}(x+iy))^2\, dxdy = O(r^2)\,, \qquad r \to +\infty\,,$$

so the order of f is finite which contradicts Step 1.

Now notice that $(f_n^m)^{(l)}(z) = (f^m)^{(l)}(2^n z)$, so $(h^m)^{(l)}(z) \neq 1$ for every $h \in F$.

Now we choose a disk in the annulus $\{z : 1/4 < |z| < 2\}$ such that F is not normal in this disk, apply Lemma 4 to F with $k = l/m$ and obtain a non-constant meromorphic function g of order at most 2 which also has the property $(g^m)^{(l)}(z) \neq 1$, $z \in \mathbb{C}$. This contradicts Step 1. So Theorem 2 is proved.

Here is another application of Theorem 3.

If P is a non-constant polynomial and if f is a transcendental meromorphic function of finite order, then $P(f)\,f'$ assumes every finite non-zero value infinitely often. This was proved by E. Mues [20, Satz 1] for the case that f is entire, but without the restriction on the order. To see this we choose a zero a of P, with the property that f has infinitely many a-points if such a zero exists. We define $Q(z) = \int_a^z P(t)\,dt$ and proceed as in the proof of Theorem 2, Step 1, with $h = Q(f)$.

Now we will show that Theorem 3 fails for functions of infinite order.

Example. Define

$$f(z) = z + a \int_0^z \exp(b \exp t - t)\,dt\,,$$

where a and b are complex numbers with the properties:

$$(19) \qquad 1 + a\,b = 0 \quad \text{and} \quad 1 + a\,\exp b = 0\,.$$

Such numbers are easy to find by taking any solution of $\exp(z) = z$ as b and putting $a = -1/b$. From the first condition (19) follows that $f(2\pi i) = 0$. (Use the substitution $w = \exp t$ and residues to evaluate the integral). So f has period $2\pi i$. From the second condition (19) follows that $f'(0) = 0$. By periodicity f has multiple zeros at the points $2\pi i k$. On the other hand f' omits the value 1.

5. Proof of Theorem 4.

We start with the following simple

Proposition *Let f be a meromorphic function with infinitely many zeros and no asymptotic values in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Then there are infinitely many critical points lying over $\mathbb{C}^*$.*

Proof. We have infinitely many branches of f^{-1} of the form

$$q_k(w) = \varphi_k(w^{1/p_k}) = z_k + \sum_{n=1}^{\infty} c_n\, w^{n/p_k}\,,$$

where p_k are some integers, z_k are the zeros of f and φ_k are univalent. If the radius of convergence r_k of the series φ_k is infinite then φ_k is linear as a univalent holomorphic function in $\mathbb{C}$. So all r_k are finite which means that the branches q_k have singularities in $\mathbb{C}^*$. These singularities are algebraic branch points and only finitely many q_k's may share one such singularity. Thus the total number of critical points of f over $\mathbb{C}^*$ is infinite. This completes the proof of the Proposition.

We will also use a theorem of F. Iversen [18], [21], which states that *if a transcendental meromorphic function takes some value $a \in \overline{\mathbb{C}}$ finitely many times then a is an asymptotic value.*

PROOF OF THEOREM 4. To prove a) assume that f' has finitely many zeros. Then all but a finite set of critical points lie over ∞. From Corollary 2 we conclude that there is at most one asymptotic value $a \in \overline{\mathbb{C}}$. If $a = \infty$ or there is no asymptotic value at all, then f has infinitely many zeros and we apply the Proposition to get a contradiction. If a is finite we may assume without loss of generality that $a = 0$. Then f has infinitely many poles by Iversen's theorem and we apply the Proposition to $1/f$.

To prove b) we assume that f'/f has finitely many zeros. This means that all except finitely many critical points lie over 0 and ∞. By Corollary 2 there are no asymptotic values. So we have infinitely many zeros by Iversen's theorem and the Proposition gives a contradiction.

To prove c) we assume that f'/f has finitely many zeros. Then all critical values lie over 0 and by Corollary 2 and Iversen's theorem the only asymptotic value is infinity. Again there are infinitely many zeros and the application of the Proposition finishes the proof.

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Multipliers of de Branges-Rovnyak spaces in H^2

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1. Introduction.

In 1966 de Branges and Rovnyak introduced a concept of complementation associated to a contraction between Hilbert spaces that generalizes the classical concept of orthogonal complement. When applied to Toeplitz operators on the Hardy space of the disc, H^2, this notion turned out to be the starting point of a beautiful subject, with many applications to function theory. The work has been in constant progress for the last few years. We study here the multipliers of some de Branges-Rovnyak spaces contained in H^2.

This introductory section is devoted mainly to general background on Hilbert spaces contained contractively in H^2; all its material can be found in [15], and especially in [13]. Also, at the end of the section we give an account of the main results obtained in this paper.

Let H, H_1 be Hilbert spaces, and $A : H_1 \to H$ be a contraction. We denote by $M(A)$ the space formed by the range of A with the Hilbert space structure that makes A a coisometry from H_1 onto $M(A)$. With this structure the inclusion of $M(A)$ in H is a contraction, so we say that $M(A)$ is contained contractively in H. The space $\mathcal{H}(A) = M[(1 - AA^*)^{1/2}]$ is called the complementary space of $M(A)$. The overlapping space $M(A) \cap \mathcal{H}(A)$ equals $A\,\mathcal{H}(A^*)$, and it is not difficult to prove that if $a \in H$, then $a \in \mathcal{H}(A)$ if and only if $A^*a \in \mathcal{H}(A^*)$. If A is a partial isometry (and only in this case), $M(A)$ and $\mathcal{H}(A)$ are closed subspaces of H, orthogonal complements of each other; otherwise the overlapping space $A\mathcal{H}(A^*)$ is always nontrivial.

Let b be an element of the unit ball $B(H^\infty)$ in H^∞, and let T_b and $T_{\overline{b}}$ be the Toeplitz operators associated to b and $\overline{b}$ acting on H^2. Since these operators are contractions, we can consider the spaces $\mathcal{H}(T_b)$ and $\mathcal{H}(T_{\overline{b}})$, which from now on will be denoted by $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$, respectively. Using a classical criterion of Douglas to factorize contractions, it is easy to show that $\mathcal{H}(\overline{b})$ is contained contractively in $\mathcal{H}(b)$ (see [15, II-2]). Now a simple calculation shows that if $f, g \in \mathcal{H}(b)$, then

$$\langle f, g\rangle_{\mathcal{H}(b)} = \langle f, g\rangle_{H^2} + \langle T_{\overline{b}}(f), T_{\overline{b}}(g)\rangle_{\mathcal{H}(\overline{b})}\,.$$

If $b = b_1 b_2$, with b_1 and b_2 in $B(H^\infty)$, then $\mathcal{H}(b) = \mathcal{H}(b_1) + b_1\mathcal{H}(b_2)$, where $\mathcal{H}(b_1)$ is contained contractively in $\mathcal{H}(b)$ and T_{b_1} implements a contraction from $\mathcal{H}(b_2)$ into $\mathcal{H}(b)$. Besides, this sum is direct (*i.e.* $\mathcal{H}(b_1) \cap b_1\mathcal{H}(b_2) = \{0\}$) if and only if $\mathcal{H}(b_1)$ is the orthogonal complement of $b_1\mathcal{H}(b_2)$ in $\mathcal{H}(b)$. In particular this holds if b_1 is an inner function, because since in this case T_{b_1} is an isometry, so that $(1 - T_{b_1}T_{\overline{b}_1})^{1/2}$ is the projection (in H^2) onto the orthogonal complement of $b_1 H^2$. Moreover, $\mathcal{H}(b_1)$ is an ordinary closed subspace of H^2.

For $\varphi \in H^\infty$, the Toeplitz operator $T_{\overline{\varphi}}$ is a bounded operator on $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ with norm (in both cases) not exceeding $\|\varphi\|_\infty$.

The spaces $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ can be represented in terms of Cauchy integrals. Let μ be a Borel finite positive measure on $\partial\mathbb{D}$, the boundary of the unit disc. For $f \in L^2(\mu)$, define the Cauchy transform of f respect to μ as

$$K_\mu(f)(z) = \int_{\partial\mathbb{D}} \frac{1}{1 - e^{-i\theta}z}\, f(e^{i\theta})\, d\mu(e^{i\theta})\,, \qquad z \in \mathbb{C} \setminus \partial\mathbb{D}.$$

It is an analytic function on $\mathbb{C} \setminus \partial\mathbb{D}$. We often (not always) use the restriction of this function to $\mathbb{D}$, its meaning being clear from the context. If the measure μ is given by a weight, $d\mu(e^{i\theta}) = g(e^{i\theta})\, d\theta/2\pi$ with $g \in L^1\,(= L^1(d\theta/2\pi))$, $g \geq 0$, we simply write K_g for K_μ. In particular, if $g \equiv 1$ we write K.

Let $b \in B(H^\infty)$. The real part of the function $(1+b(z))/(1-b(z))$ is $(1-|b(z)|^2)/|1-b(z)|^2 \geq 0$, so it can be represented by the Herglotz formula

$$(1) \quad \frac{1+b(z)}{1-b(z)} = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu_b(e^{i\theta}) + i\,\mathrm{Im}\left(\frac{1+b(0)}{1-b(0)}\right), \qquad z \in \mathbb{D},$$

where

$$d\mu_b(e^{i\theta}) = \frac{1-|b(e^{i\theta})|^2}{|1-b(e^{i\theta})|^2}\frac{d\theta}{2\pi} + d\mu_S(e^{i\theta}),$$

with μ_S a positive finite singular measure and

$$\sigma = \frac{1-|b|^2}{|1-b|^2} \in L^1.$$

First Clark [3] for b inner and then Sarason in general [17] proved that the operator given by $V_b(f)(z) = (1-b(z))K_{\mu_b}(f)(z)$ (for $f \in L^2(\mu_b)$ and $z \in \mathbb{D}$), establishes an isometry from $H^2(\mu_b)$ onto $\mathcal{H}(b)$, where $H^2(\mu_b)$ is the closure in $L^2(\mu_b)$ of the analytic polynomials (see [1] and [2] for vector valued versions). Also, in [13] it is proved that if $\rho(e^{i\theta}) = 1-|b(e^{i\theta})|^2$, then K_ρ is an isometry from $H^2(\rho)$ $(= H^2(\rho(e^{i\theta})\, d\theta/2\pi))$ onto $\mathcal{H}(\overline{b})$. For a given $b \in B(H^\infty)$, ρ, σ and μ_b will always denote the functions and measure associated to b as in the above paragraph.

At this point two very different cases appear in the study of the spaces $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$, according to whether b is or is not an extreme point of $B(H^\infty)$, or equivalently, according to whether ρ is not or is log-integrable on $\partial\mathbb{D}$ (see [11, p. 138]). The reason for this distinction is a famous theorem of Szegö ([11, p. 49]), which asserts that for a positive finite measure μ on $\partial\mathbb{D}$, $H^2(\mu) = L^2(\mu)$ if and only if the Radon-Nikodym derivative of μ with respect to the Lebesgue measure is not log-integrable. Thus, if b is extreme in $B(H^\infty)$ (and only in this case), $H^2(\rho) = L^2(\rho)$ and $H^2(\mu_b) = L^2(\mu_b)$. Notice that $\log\sigma = \log\rho - \log|1-b|^2$, where $\log|1-b|^2$ is integrable because $1-b \in H^1$ ([11, p. 51]).

A multiplier of $\mathcal{H}(b)$ (or of $\mathcal{H}(\overline{b})$) is a function $m \in H^\infty$ such that $\mathcal{H}(b)$ (respectively $\mathcal{H}(\overline{b})$) is invariant by T_m. If $f \in H^2$, then $f \in \mathcal{H}(\overline{b})$ if and only if $bf \in \mathcal{H}(b)$. This immediately implies that every multiplier of $\mathcal{H}(b)$ is also a multiplier of $\mathcal{H}(\overline{b})$. Also, for u an inner function, the decomposition $\mathcal{H}(ub) = u\mathcal{H}(b) + \mathcal{H}(u)$ together with the fact that $uH^2 \cap \mathcal{H}(u) = \{0\}$, implies that every multiplier of $\mathcal{H}(ub)$ is a multiplier of $\mathcal{H}(b)$. It is known that both inclusions of multipliers can be proper. D. Sarason [16] gave an example of a nonextreme outer function b for which the multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ are different. However, it is unknown if the multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ can be different when b is outer and extreme. If b is inner, $\mathcal{H}(\overline{b})$ is trivial, and it is easy to see that only the constant functions are multipliers of $\mathcal{H}(b)$; otherwise there are plenty of nonconstant multipliers (see [13]).

Information about multipliers for the nonextreme case can be found in [5], [12], [14] and [16]. The main source for the extreme case is the paper of Lotto and Sarason [13]. The latter case is the subject of this paper, so we assume from now on that b is an extreme point of $B(H^\infty)$ unless the contrary is stated. Also, we exclude the trivial case b inner.

Since in our case the backward shift S^* is an invertible operator on $\mathcal{H}(\overline{b})$ ([13, Theorem 3.6]), it is easy to prove that every multiplier of $\mathcal{H}(\overline{b})$ is in $\mathcal{H}(\overline{b}) + \mathbb{C}$. Since $\mathcal{H}(\overline{b})$ has no other constants except the zero function, the above space is a one-dimensional linear extension of $\mathcal{H}(\overline{b})$. If $f \in \mathcal{H}(\overline{b})+\mathbb{C}$, the Cauchy representation of $\mathcal{H}(\overline{b})$ shows that for $z \in \mathbb{D}$, $f(z) = K_\rho(q)(z) + c$ with $q \in L^2(\rho)$ and $c \in \mathbb{C}$. Now define the following conjugation in $\mathcal{H}(\overline{b}) + \mathbb{C}$, $f_*(z) = -K_\rho(\overline{q})(z) + K_\rho(\overline{q})(0) + \overline{c}$. A straightforward calculation shows that if we think of f as defined on $\mathbb{C} \setminus \partial\mathbb{D}$, then

$$f_*(z) = \overline{f(1/\overline{z})}\,.$$

Let us denote by $\mathcal{M}(b)$ and $\mathcal{M}(\overline{b})$ the algebras of multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ respectively. The above conjugation has the important property that if m belongs to any of these algebras, then m_* belongs to the same algebra. In particular, $m(z)$ and $\overline{m(1/\overline{z})}$ are in H^∞ (for $z \in \mathbb{D}$), which implies that $m(z) = K_\rho(q)(z) + c$ must be bounded for all $z \in \mathbb{C} \setminus \partial\mathbb{D}$. In other words, the algebras of multipliers are contained in the space

$$K^\infty(\rho) = \{m = K_\rho(q) + c :\ q \in L^2(\rho)\,,\ c \in \mathbb{C},\ \sup_{z\in\mathbb{C}\setminus\partial\mathbb{D}} |m(z)| < +\infty\}\,.$$

The space $K^\infty(\rho)$ is closed under multiplication, and if $f, g \in K^\infty(\rho)$ then $(fg)_* = f_* g_*$. Moreover, if $m = K_\rho(q) + c \in K^\infty(\rho)$, the norm $\|m\|_{K^\infty(\rho)} = \sup_{z\in\mathbb{C}\setminus\mathbb{D}} |m(z)| + \|q\|_{L^2(\rho)}$ makes $K^\infty(\rho)$ a $*$-Banach algebra. Summing up, we have the following string of inclusions

$$\mathcal{M}_\infty(b) \subset \mathcal{M}(ub) \subset \mathcal{M}(b) \subset \mathcal{M}(\overline{b}) \subset K^\infty(\rho)\,, \tag{2}$$

where u is an inner function and $\mathcal{M}_\infty(b) = \bigcap_{v \text{ inner}} \mathcal{M}(vb)$. If m belongs to any of these algebras, the spectrum of m in the respective algebra is the closure of $m(\mathbb{C}\backslash\partial\mathbb{D})$. Also, the operation $m \to m_*$ is a multiplicative conjugation in all the algebras (see [13]).

The paper is organized as follows. In Section 2 we give a characterization of the group $\Gamma = \{f \in K^\infty(\rho) : \; f_* = f^{-1}\}$ and we show that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are any of the algebras in (2), then $\mathcal{M}_1 = \mathcal{M}_2$ if and only if $\mathcal{M}_1 \cap \Gamma = \mathcal{M}_2 \cap \Gamma$. This observation will be fundamental in the sequel. In Section 3 we establish some known relations between multipliers and weighted norm inequalities. We study these relations in terms of our characterization of Γ. Section 4 answers a question by Lotto and Sarason by giving an example of $b \in B(H^\infty)$ extreme, such that $\mathcal{M}(\overline{b})$ does not coincide with $K^\infty(\rho)$. We obtain a complete characterization of $\mathcal{M}(\overline{b})$ for this example. In Section 5 it is proved that $\mathcal{M}_\infty(b)$ is dense in $\mathcal{M}(b)$ and $\mathcal{M}(\overline{b})$ with the respective strong operator topologies. Section 6 discusses the way in which the singular component of μ_b affects the algebras $\mathcal{M}(b)$ and $K^\infty(\rho)$. In Section 7 we introduce a partial isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(\overline{b})$, which is used to obtain a sufficient condition for a function $m \in K^\infty(\rho)$ to belong to $\mathcal{M}(b)$. It follows as a corollary that $\mathcal{H}(b)$ is imbedded in $L^2(\rho/|1-ub|^2)$ for every inner function u. Also, we show several characterizations of the equality $\mathcal{M}_\infty(b) = K^\infty(\rho)$. In particular, this turns out to be equivalent to $\mathcal{M}_\infty(b) = \mathcal{M}(b)$. In Section 8 we investigate how $\mathcal{H}(b)$, $\mathcal{H}(\overline{b})$ and their multipliers are affected if we replace b by $\tau \circ b$, where τ is an analytic authomorphism of the unit disk. In Section 9 we prove that the multipliers of $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ coincide when b is continuous up to the boundary of the disk. Finally, Section 10 contains some information about the interaction between the conjugation $*$ and the inner factors of functions in any of the algebras $\mathcal{M}(b)$, $\mathcal{M}(\overline{b})$ and $K^\infty(\rho)$.

I am indebted to D. Sarason for many valuable comments.

2. Some special functions in $K^\infty(\rho)$.

One of the main problems when studying the algebras of multipliers is the lack of examples, in particular, the difficulty to exhibit nonconstant elements of $K^\infty(\rho)$. The next theorem will allow us to construct functions $m = K_\rho(q) + c$ in $K^\infty(\rho)$, where $r = |q\rho|$ has a preestablished behaviour. We need two lemmas.

Lemma 2.1. *Let $f = K_\rho(q)+c \in \mathcal{H}(\overline{b})+\mathbb{C}$. Then the (inner) boundary function of $f(z) - \overline{f_*(z)}$ equals $q\rho$. Conversely, if f and g are analytic functions on $\mathbb{D}$ such that $f - \overline{g} = P * q\rho$, where P denotes the Poisson kernel and $q \in L^2(\rho)$, then in $\mathbb{D}$,*

$$f(z) = K_\rho(q)(z) + \overline{g(0)}$$

and

$$g(z) = \overline{K_\rho(q)(1/\overline{z})} + g(0)\,.$$

The lemma is just a particular case of Lemmas 10.1 and 10.2 in [13].

Lemma 2.2. *Let s be a real valued function in L^∞. Then*

$$2\,|s| \le |e^s - e^{-s}| \le 2\,e^{|s|}\,|s|\,.$$

Proof. Both inequalities follow from simple calculations with the Taylor series

$$\frac{|e^s - e^{-s}|}{|s|} = 2\sum_{n\ge 0} \frac{s^{2n}}{(2n+1)!}\,.$$

If f and g are functions defined almost everywhere in $\partial\mathbb{D}$, and f takes the the value zero whenever g does (except for a null set), the quotient f/g makes sense and it is finite almost everywhere with the convention $0/0 = 0$.

Theorem 2.3. *Let s be a real valued bounded function defined on ∂D such that $s^2/\rho \in L^1$. Then $m = e^{s+i\tilde{s}} \in K^\infty(\rho)$, where $\tilde{s}$ is any harmonic conjugate of s. Moreover, if $m = K_\rho(q) + c$ with $q \in L^2(\rho)$ and $c \in \mathbb{C}$, then*

1) $q\rho = (|m|^2 - 1)/\overline{m}\,.$

2) *If* $r = |q\rho|$, *then* $2\,|s| \le |r| \le 2\,e^{\|s\|_\infty}\,|s|\,.$

3) $m_* = m^{-1}\,.$

Conversely, every $m \in K^\infty(\rho)$ such that $m_* = m^{-1}$ is of the above form with $s = \log|m|$.

PROOF. The function $m = e^{s+i\tilde{s}}$ is invertible in H^∞. Hence, the bounded harmonic function $m-\overline{m}^{-1}$ is the Poisson integral of its (inner) boundary function $(|m|^2-1)/\overline{m}$. Write $q = (|m|^2-1)/\overline{m}\rho$. Since $|(|m|^2-1)/\overline{m}| = |e^s - e^{-s}|$, Lemma 2.2 asserts that

$$(3) \qquad 2\,\frac{|s|}{\rho} \le |q| \le C\,\frac{|s|}{\rho}\,, \qquad \text{with } C = 2\,e^{\|s\|_\infty}\,.$$

Therefore $|q|^2\rho \le C^2\,(s^2/\rho) \in L^1$ and consequently $q \in L^2(\rho)$. By Lemma 2.1, $m = K_\rho(q) + \overline{m^{-1}(0)}$ and $m_* = m^{-1}$.

On the other hand, if $m = K_\rho(q) + c$ is any element of $K^\infty(\rho)$ such that $m_* = m^{-1}$, then by Lemma 2.1 the boundary function of $m-\overline{m}_* = (|m|^2-1)/\overline{m}$ equals $q\rho$. Hence $q = (|m|^2-1)/\overline{m}\rho \in L^2(\rho)$. Since m is an invertible function of H^∞ then $m = e^{s+i\tilde{s}}$, where $s = \log|m| \in L^\infty$. A new application of Lemma 2.2 shows that the inequalities (3) hold for these q and s, thus $s^2/\rho \le (1/4)|q|^2\rho \in L^1$.

Definition. *Let $b \in B(H^\infty)$ and $\rho(e^{i\theta}) = 1 - |b(e^{i\theta})|^2$. If s is a real valued, essentially bounded function on $\partial\mathbb{D}$ such that $s^2/\rho \in L^1$, we will say that s is an admissible function for ρ, or simply, that s is admissible.*

Theorem 2.3 implies that for every s admissible there is $m = K_\rho(q) \in K^\infty(\rho)$, where $r = |q\rho|$ behaves like $|s|$. On the other hand, if $m = K_\rho(q) + c$ is any element of $K^\infty(\rho)$, then $r = |q\rho|$ is admissible.

We fix for the rest of the paper the notation E for the set where ρ does not vanish. That is,

$$E = \{e^{i\theta} \in \partial\mathbb{D} :\ \rho(e^{i\theta}) \ne 0\}\,.$$

In Theorem 13.3 of [13] it is proved that $m = K_\rho(q) + c \in K^\infty(\rho)$ is a multiplier of $\mathcal{H}(vb)$ for every inner function v if and only if $q^2\rho \in L^\infty$. If we write $r = |q\rho|$, this condition can be rewritten as $r^2/\rho \in L^\infty$. Since r is bounded, the above condition holds for all $m \in K^\infty(\rho)$ if $\chi_E/\rho \in L^\infty$ (where χ_E denotes the characteristic function of E). Theorem 2.3 immediately implies that the converse also holds, because if $\chi_E/\rho \notin L^\infty$ then there is an admissible function s such that $s^2/\rho \notin L^\infty$.

Theorem 2.3 gives a characterization of the functions in

$$\Gamma = \{f \in K^\infty(\rho) :\ f_* = f^{-1}\}\,.$$

Denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ two different algebras of the string of inclusions (2), with $\mathcal{M}_1 \subset \mathcal{M}_2$.

Proposition 2.4. *$\mathcal{M}_2 \subset \mathcal{M}_1$ if and only if $\mathcal{M}_2 \cap \Gamma \subset \mathcal{M}_1$.*

PROOF. Suppose that there is $m \in \mathcal{M}_2 \setminus \mathcal{M}_1$. Since $m = (m+m_*)/2 + i(m-m_*)/2i$, then $(m+m_*)/2$ or $(m-m_*)/2i$ is not in $\mathcal{M}_1$. Hence there is $f \in \mathcal{M}_2 \setminus \mathcal{M}_1$ such that $f = f_*$. Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$ be a number which does not belong to the spectrum of f. Then

$$\frac{f-\overline{\alpha}}{f-\alpha} = 1 + \frac{\alpha-\overline{\alpha}}{f-\alpha} \in (\mathcal{M}_2 \cap \Gamma) \setminus \mathcal{M}_1 .$$

For $m \in K^\infty(\rho)$ denote by $\operatorname{sp}(m)$ the spectrum of m. Let $\mathcal{M}$ be any of the Banach algebras $\mathcal{M}(b)$, $\mathcal{M}(\overline{b})$ or $K^\infty(\rho)$.

Lemma 2.5. *Let $m \in \mathcal{M}$ with $\operatorname{sp}(m) \cap \partial\mathbb{D} = \varnothing$. If f is a continuous function on $\partial\mathbb{D}$, then*

$$I_f(m) = \int_0^{2\pi} \frac{m_* - e^{-i\theta}}{m - e^{i\theta}}\, f(e^{i\theta})\, \frac{d\theta}{2\pi}$$

is in $\mathcal{M}$.

PROOF. The continuity of the map $\omega \to (m_* - \overline{\omega})/(m-\omega)$ for $\omega \notin \operatorname{sp}(m) = \overline{\operatorname{sp}(m_*)}$ assures that $I_f(m)$ is well defined (because $\operatorname{sp}(m) \cap \partial\mathbb{D} = \varnothing$), and that it is the limit (in norm) of

$$S_n = \sum_{k=0}^{n-1} \frac{1}{n}\, f(e^{2\pi i k/n})\, \frac{m_* - e^{-2\pi i k/n}}{m - e^{2\pi i k/n}} .$$

Proposition 2.6. *Let $m \in K^\infty(\rho)$ with $\|m\| < 1$. If $f_k(e^{i\theta}) = e^{ik\theta}$ (with k an integer) then*

$$I_{f_k}(m) = \begin{cases} m^{k-2}(1 - m\, m_*), & \text{if } k \geq 2, \\ -m_*, & \text{if } k = 1, \\ 0, & \text{if } k \leq 0. \end{cases}$$

PROOF. It is a straightforward calculation with the power series expansion (in $e^{i\theta}$) of $(m_* - e^{-i\theta})/(m - e^{i\theta})$.

Corollary 2.7. *Let $\mathcal{M}$ be as in the preceding lemma. Then the span of $\Gamma \cap \mathcal{M}$ is dense in $\mathcal{M}$.*

PROOF. The proof of Lemma 2.5 shows that if f is continuous on $\partial\mathbb{D}$ and $m \in \mathcal{M}$ is such that $\mathrm{sp}\,(m) \cap \partial\mathbb{D} = \varnothing$, then $I_f(m)$ is in the closure of span$(\Gamma \cap \mathcal{M})$. Given any $m \in \mathcal{M}$, take $m^1 = m_*/2\|m_*\|$, where the norm $\|m_*\|$ is taken in $K^\infty(\rho)$, and $f_1(e^{i\theta}) = e^{i\theta}$. By Proposition 2.6, $-I_{f_1}(m^1) = m^1_* = m/2\|m_*\|$. Hence, $m = -2\,\|m_*\|\,I_{f_1}(m^1)$ is in the closure of span$\,(\Gamma \cap \mathcal{M})$.

3. Weights and Multipliers.

In [13] some criteria are given for a function $m \in K^\infty(\rho)$ to belong to $\mathcal{M}(b)$ or $\mathcal{M}(\overline{b})$. Those criteria are the starting point of most of the sequel. The next theorem is a different formulation of Theorem 12.2 and Lemma 13.1 in [13].

Theorem 3.1. *Let $m = K_\rho(q) + c \in K^\infty(\rho)$. If $r = |q\rho|$, then*

1) *$m \in \mathcal{M}(\overline{b})$ if and only if $f \in L^2(r^2/\rho)$ for every $f \in \mathcal{H}(\overline{b})$.*

2) *$m \in \mathcal{M}(b)$ if and only if $f \in L^2(r^2/\rho)$ for every $f \in \mathcal{H}(b)$.*

3) *If $m \in \mathcal{M}(b)$ and u is an inner function, then $m \in \mathcal{M}(ub)$ if and only if $f \in L^2(r^2/\rho)$ for every $f \in \mathcal{H}(u)$.*

The advantage of this point of view for the present paper is that Theorem 3.1 is given in terms of the admissible function r. Theorems 3.1 and 2.3 immediately yield the fact that $\mathcal{M}(b)$ (or $\mathcal{M}(\overline{b})$) coincides with $K^\infty(\rho)$ if and only if for every admissible function r, $f \in L^2(r^2/\rho)$ for all $f \in \mathcal{H}(b)$ (respectively, for all $f \in \mathcal{H}(\overline{b})$).

By a standard argument involving the closed graph theorem, if any of the conditions of Theorem 3.1 holds, then it holds with continuity.

Let μ be a finite Borel measure on $\partial\mathbb{D}$ and $f \in L^1(\mu)$. Then, as a function on $\mathbb{D}$, $K_\mu(f)$ belongs to H^p for $0 < p < 1$; so it has a finite nontangential limit for almost every $e^{i\theta} \in \partial\mathbb{D}$ (see [8, pages 17 and 39]). Most of the time it will be convenient to think of $K_\mu(f)$ as its (inner) boundary function. Since $K_\rho : L^2(\rho) \to \mathcal{H}(\overline{b})$ is an onto isometry, then for $f = K_\rho(q)$,

$$\|f\|_{\mathcal{H}(\overline{b})} = \|q\|_{L^2(\rho)} = \|q\rho^{1/2}\|_{L^2} \ .$$

Thus every $f \in \mathcal{H}(\overline{b})$ can be written as $f = K_{\rho^{1/2}}(h)$ with $q\rho^{1/2} = h \in L^2$, $h = 0$ outside of E, and $\|f\|_{\mathcal{H}(\overline{b})} = \|h\|_{L^2} = \|h\|_{L^2(\chi_E)}$. Conversely, if $h \in L^2$ then $h\chi_E = q\rho^{1/2}$ with $q \in L^2(\rho)$ (take $q = h\chi_E/\rho^{1/2}$), and $\|h\chi_E\|_{L^2} = \|q\|_{L^2(\rho)}$. Then $K_{\rho^{1/2}} : L^2(\chi_E) \to \mathcal{H}(\overline{b})$ is an onto isometry. On the other hand, if $d\mu_b = \sigma\, d\theta/2\pi + d\mu_S$ is the measure associated to b by formula (1), then $K_{\mu_b} = K_\sigma + K_{\mu_S}$, and $V_b = (1-b)\, K_{\mu_b}$ is an onto isometry from $L^2(\mu_b)$ onto $\mathcal{H}(b)$. As before, we can replace the operator K_σ on $L^2(\sigma)$ by $K_{\sigma^{1/2}}$ on $L^2(\chi_E)$. We just obtained that $W_b = (1-b)\,(K_{\sigma^{1/2}} + K_{\mu_S})$ is an isometry from $L^2(\chi_E) \oplus L^2(\mu_S)$ onto $\mathcal{H}(b)$. With these facts in mind we can rewrite Theorem 3.1 once more.

Theorem 3.2. *Let $m = K_\rho(q) + c \in K^\infty(\rho)$. If $r = |q\rho|$, then*

1) *$m \in \mathcal{M}(\overline{b})$ if and only if $K_{\rho^{1/2}}$ maps $L^2(\chi_E)$ into $L^2(r^2/\rho)$.*

2) *$m \in \mathcal{M}(b)$ if and only if $K_{\sigma^{1/2}}$ maps $L^2(\chi_E)$ into $L^2(r^2/\sigma)$ and K_{μ_S} maps $L^2(\mu_S)$ into $L^2(r^2/\sigma)$.*

3) *If $m \in \mathcal{M}(b)$ then $m \in \mathcal{M}(ub)$ if and only if $(1-u)\, K_{\mu_u}$ maps $L^2(\mu_u)$ into $L^2(r^2/\rho)$, where μ_u is the measure associated to u in the representation* (1).

PROOF. 1) and 3) are immediate. By Theorem 3.1 and the above comment, $m \in \mathcal{M}(b)$ if and only if for every $q_1 \in L^2(\chi_E)$ and $q_2 \in L^2(\mu_S)$,

$$(1-b)\,(K_{\sigma^{1/2}}(q_1) + K_{\mu_S}(q_2)) \in L^2(r^2/\rho)\,.$$

Since $r^2/\sigma = r^2|1-b|^2/\rho$, this is equivalent to $K_{\sigma^{1/2}}(q_1) + K_{\mu_S}(q_2) \in L^2(r^2/\sigma)$, and clearly this is the same as 2).

Again, if any of the conditions of the theorem holds, it does with continuity. Then, the problem of establishing whether a given $m \in K^\infty(\rho)$ is a multiplier is transformed into a problem of weighted norm inequalities. It is not surprising then that Helson-Szegö weights play an important role in the theory. A Helson-Szegö weight is a function $\gamma = e^{\varphi+\tilde{\psi}}$, where φ and ψ are bounded real valued functions on $\partial\mathbb{D}$ and $\|\psi\|_\infty < \pi/2$. The relevance of these functions is that they are precisely the positive weights γ in L^1 such that the Cauchy transform is a bounded operator from $L^2(\gamma)$ into itself [10].

Theorem 3.3. *Let r be an admissible function. If there is a Helson-Szegö weight γ_r such that $r^2/\rho = \chi_E\,\gamma_r$, then $K_{\rho^{1/2}}$ is a bounded operator from $L^2(\chi_E)$ into $L^2(r^2/\rho)$. The statement also holds replacing ρ by σ everywhere.*

PROOF. Take $f \in L^2(\chi_E)$; then $f\rho^{1/2} \in L^2(\chi_E/\rho) \subset L^2(r^2/\rho)$, and since $f\rho^{1/2} = 0$ outside of E, then $f\rho^{1/2} \in L^2(\gamma_r)$. By the Helson-Szegö theorem $K_{\rho^{1/2}}(f) \in L^2(\gamma_r)$, hence $K_{\rho^{1/2}}(f) \in L^2(\gamma_r\chi_E) = L^2(r^2/\rho)$. The same argument works for σ.

Corollary 3.4. *Let $b \in B(H^\infty)$. If there is a Helson-Szegö weight γ such that $\chi_E/\rho = \chi_E\,\gamma$, then $K_{\rho^{1/2}}$ maps $L^2(\chi_E)$ into $L^2(r^2/\rho)$ for every admissible function r. If $d\mu_b = \sigma\,d\theta/2\pi$, the same holds replacing ρ by σ everywhere.*

PROOF. Since Helson-Szegö weights are in L^1, $\chi_E/\rho \in L^1$ (*i.e.* χ_E is admissible). By Theorem 3.3 $K_{\rho^{1/2}}$ maps $L^2(\chi_E)$ into $L^2(\chi_E/\rho)$, and since r is bounded, $L^2(\chi_E/\rho) \subset L^2(r^2/\rho)$.

The assertion for σ can be similarly deduced from Theorem 3.3 if we show that χ_E is admissible, that is, $\chi_E/\rho \in L^1$. So we assume that $\chi_E/\sigma = \chi_E\gamma$, with γ a Helson-Szegö weight. Clearly γ^{-1} is also a Helson-Szegö weight, thus $\sigma^2\gamma = \chi_E/\gamma \in L^1$, or what is the same, $\sigma \in L^2(\gamma)$. Then, by the Helson-Szegö theorem, $K(\sigma) \in L^2(\gamma) \subset L^2(\chi_E\gamma)$. Since $d\mu_b = \sigma\,d\theta/2\pi$, then by [15, III-7],

$$K(\sigma) = K_\sigma(1) = K_{\mu_b}(1) = (1-b)^{-1}(1-\overline{b(0)})^{-1}(1-\overline{b(0)}b)\,,$$

which implies that $(1-b)^{-1} \in L^2(\chi_E\gamma)$. Thus,

$$|1-b|^{-2}\chi_E\,\gamma = |1-b|^{-2}\,\chi_E/\sigma = \chi_E/\rho$$

is in L^1, as claimed.

The statement for σ in the above corollary already appears in [13, Theorem 14.1] with a different formulation and a similar (slightly different) proof.

4. An example.

It is asked in [13] if for b extreme, not an inner function, the algebras $\mathcal{M}(\bar{b})$ and $K^\infty(\rho)$ coincide. We give here an example for which those algebras do not coincide. We also obtain for this example a complete characterization of the multipliers of $\mathcal{H}(\bar{b})$ among the elements of $K^\infty(\rho)$.

When convenient, we identify a function $f(e^{i\theta})$ defined almost everywhere on $\partial\mathbb{D}$ with a function $f(\theta)$ defined for almost every $\theta \in (-\pi,\pi]$. Let β be a function in L^1 (of $\partial\mathbb{D}$). For $f \in L^1(\beta)$ define the Hilbert transform of f with weight β as

$$H_\beta(f)(\theta) = \lim_{\varepsilon \to 0} \int_{\varepsilon < |\varphi - \theta| < \pi} \frac{f(\varphi)}{\theta - \varphi}\, \beta(\varphi)\, d\varphi\,.$$

We write H if $\beta = 1$.

Proposition 4.1. *In Theorem* 3.2 *we can replace* $K_{\rho^{1/2}}$ *and* $K_{\sigma^{1/2}}$ *by* $H_{\rho^{1/2}}$ *and* $H_{\sigma^{1/2}}$, *respectively.*

PROOF. We prove the proposition for $K_{\sigma^{1/2}}$, the proof for $K_{\rho^{1/2}}$ is the same. Let $f \in L^2(\chi_E)$; then for $z \in \mathbb{D}$,

$$K_{\sigma^{1/2}}(f)(z) = \frac{1}{2}\left((P * f\sigma^{1/2})(z) + i(Q * f\sigma^{1/2})(z) + (P * f\sigma^{1/2})(0)\right),$$

where P is the Poisson kernel and Q is its harmonic conjugate. Since f and $\sigma^{1/2}$ are in L^2, $f\sigma^{1/2} \in L^1$; hence the boundary function of $(P * f\sigma^{1/2})(z)$ is $f\sigma^{1/2}$. The fact that $f \in L^2$ and $r \in L^\infty$ now implies that $f\sigma^{1/2} \in L^2(r^2/\sigma)$. Also $L^2(r^2/\sigma)$ contains the constants because $r^2/\sigma \in L^1$. That is, $K_{\sigma^{1/2}}(f) \in L^2(r^2/\sigma)$ if and only if the boundary function of $(Q * f\sigma^{1/2})(z)$ is in $L^2(r^2/\sigma)$. Let us denote this boundary function also by $Q * f\sigma^{1/2}$. A simple computation shows that

$$Q * f\sigma^{1/2} = \frac{1}{\pi}\, H_{\sigma^{1/2}}(f) + d * f\sigma^{1/2}\,,$$

where $d(\theta) = \operatorname{cotg}\theta/2 - 2/\theta$ is a bounded function, $|d(\theta)| \le 2/\pi$ (see [9, p. 105]). Hence $|d * f\sigma^{1/2}| \le C\, \|f\sigma^{1/2}\|_{L^1} < +\infty$, and then $d * f\sigma^{1/2}$ always belongs to $L^2(r^2/\sigma)$.

For $\theta \in (0, 2\pi]$ the function $(1 - e^{-1/\theta})^{1/2}$ is log-integrable, so that there is $b \in H^\infty$ such that $|b(e^{i\theta})| = (1 - e^{-1/\theta})^{1/2}$ almost everywhere with respect to $d\theta$. Furthermore, $\rho(\theta) = 1 - |b(e^{i\theta})|^2 = e^{-1/\theta}$ is not log-integrable; thus b is an extreme point of $B(H^\infty)$. We consider this b for the rest of the section. It will be convenient to think of ρ as defined on $(-\pi, \pi]$,

$$\rho(\theta) = \begin{cases} e^{-1/\theta}, & \text{if } 0 < \theta \le \pi, \\ e^{-1/(2\pi+\theta)}, & \text{if } -\pi < \theta \le 0. \end{cases}$$

Theorem 4.2. *For $m = K_\rho(q) + c \in K^\infty(\rho)$, put $r = |q\rho|$. If $m \in \mathcal{M}(\bar{b})$, there is a constant $C > 0$ such that*

$$\int_0^\varepsilon r^2(\theta)\, e^{1/\theta}\, d\theta \le C\,\varepsilon\,, \qquad \textit{for all } \varepsilon \in (0,\pi)\,.$$

PROOF. For $\theta \in (0, \pi)$, the function $r^2(\theta)\, e^{1/\theta} = r^2(\theta)/\rho(\theta) \in L^1$, from which it is immediate that the conclusion of the theorem is equivalent to

$$\sup \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon r^2(\theta)\, e^{1/\theta}\, d\theta < +\infty\,. \tag{4}$$

If (4) does not hold, there are γ, $0 < \gamma < 1$, and two sequences (α_k), $(\beta_k) \subset (0, \pi)$ such that $\alpha_k < \gamma\beta_k$ for all k, $\alpha_k \to 0$, $\beta_k \to 0$ and

$$\frac{1}{\beta_k - \alpha_k} \int_{\alpha_k}^{\beta_k} r^2(\theta)\, e^{1/\theta}\, d\theta \longrightarrow +\infty\,.$$

Taking suitable subsequences of (α_k) and (β_k) we can also assume that $\beta_{k+1} < \alpha_k$ for all k. Let (s_k) be a sequence in ℓ^1 (the space of absolutely summable sequences) such that $s_k > 0$ for all k, and

$$\sum_{k \ge 1} s_k \frac{1}{\beta_k - \alpha_k} \int_{\alpha_k}^{\beta_k} r^2(\theta)\, e^{1/\theta}\, d\theta = +\infty\,. \tag{5}$$

Take

$$d_k = \Big(\frac{s_k}{\beta_k - \alpha_k}\Big)^{1/2} \qquad \text{for } k \ge 1\,,$$

and consider the function

$$f(\theta) = \sum_{k\geq 1} d_k\, \chi_{(-\beta_k, -\alpha_k)}(\theta)\,.$$

Then $\rho^{-1/2} f \in L^2$, because

$$\begin{aligned}\int_{-\pi}^{\pi} \rho^{-1} |f|^2\, d\theta &\leq \sup_{-\pi<\theta<0} |\rho^{-1}(\theta)| \sum_{k\geq 1} \frac{s_k}{\beta_k - \alpha_k}\,(\beta_k - \alpha_k) \\ &= e^{1/\pi}\, \|(s_k)\|_{l^1} < +\infty\,.\end{aligned}$$

By Proposition 4.1, if we show that $H_{\rho^{1/2}}(\rho^{-1/2} f) = H(f)$ does not belong to $L^2(r^2/\rho)$, then m is not a multiplier of $\mathcal{H}(\overline{b})$.

A simple computation shows that the Hilbert transform of $\chi_{(-\beta_k - \alpha_k)}$ is $\log(|\theta + \beta_k|/|\theta + \alpha_k|)$, and this function is positive for $\theta > 0$. Thus, for $\theta > 0$ we have

$$H(f)(\theta) \geq d_k \log \frac{\theta + \beta_k}{\theta + \alpha_k}\,, \qquad \text{for all } k \geq 1\,. \tag{6}$$

In particular, (6) holds for $\alpha_k < \theta < \beta_k$. Besides, if $\alpha_k < \theta < \beta_k$, $(2\alpha_k)^{-1} > (\theta + \alpha_k)^{-1} > (\beta_k + \alpha_k)^{-1}$, and consequently

$$\begin{aligned}\frac{\theta + \beta_k}{\theta + \alpha_k} &= 1 + \frac{\beta_k - \alpha_k}{\theta + \alpha_k} > 1 + \frac{\beta_k - \alpha_k}{\beta_k + \alpha_k} = \frac{2\,\beta_k}{\beta_k + \alpha_k} \\ &= \frac{2}{1 + \alpha_k/\beta_k} > \frac{2}{1+\gamma} = c > 1\,.\end{aligned}$$

Therefore,

$$\log \frac{\theta + \beta_k}{\theta + \alpha_k} \geq \log c\,, \qquad \text{for all } \theta \in (\alpha_k, \beta_k)\,. \tag{7}$$

Now (6) and (7) yield

$$\begin{aligned}\int_{\alpha_k}^{\beta_k} |H(f)|^2\, r^2\, e^{1/\theta}\, d\theta &\geq \int_{\alpha_k}^{\beta_k} d_k^2\, \log^2\Big(\frac{\theta + \beta_k}{\theta + \alpha_k}\Big)\, r^2\, e^{1/\theta}\, d\theta \\ &\geq \frac{s_k}{\beta_k - \alpha_k}\, \log^2 c \int_{\alpha_k}^{\beta_k} r^2\, e^{1/\theta}\, d\theta\,.\end{aligned} \tag{8}$$

Then,

$$\int_0^\pi |H(f)|^2 r^2 e^{1/\theta}\, d\theta \geq \log^2 c \sum_{k\geq 1} \frac{s_k}{\beta_k - \alpha_k} \int_{\alpha_k}^{\beta_k} r^2\, e^{1/\theta}\, d\theta = +\infty$$

by (8) and (5). That is, $H(f) \notin L^2(r^2/\rho)$.

Theorem 4.3. *For $m = K_\rho(q) + c \in K^\infty(\rho)$, put $r = |q\rho|$. If for some constant $C > 0$,*

$$\int_0^\varepsilon r(\theta)^2\, e^{1/\theta}\, d\theta \leq C\,\varepsilon\,, \qquad \text{for } 0 < \varepsilon < \pi\,,$$

then m is a multiplier of $\mathcal{H}(\overline{b})$.

PROOF. By Proposition 4.1, we must show that $H_{\rho^{1/2}}(f) \in L^2(r^2/\rho)$ for every $f \in L^2$. For $f \in L^2$, the function $f\rho^{1/2}$ is in L^2, and the Hilbert transform maps L^2 into itself (see [9, III]), so that $H_{\rho^{1/2}}(f) \in L^2$. Besides, for $-\pi < \theta < 0$, $\rho^{-1}(\theta) = e^{1/(2\pi+\theta)}$ is bounded, and so is r^2/ρ. Thus $H_{\rho^{1/2}}(f)$ is square integrable with respect to the measure $r^2/\rho\, d\theta$ in $(-\pi, 0)$. So, we only have to show the square integrability in $(0, \pi)$. We can assume $f \geq 0$. Write $f = f_1 + f_2$, where $f_1 = f\chi_{(-\pi,0)}$ and $f_2 = f\chi_{(0,\pi)}$. For $0 < \theta < \pi$,

$$\begin{aligned} H_{\rho^{1/2}}(f_1)(\theta) &= \lim_{\varepsilon\to 0} \int\limits_{\varepsilon<|\varphi-\theta|<\pi} \frac{f_1(\varphi)\,\rho^{1/2}(\varphi)}{\theta - \varphi}\, d\varphi \\ &= \int_{-\pi}^0 \frac{f_1(\varphi)\,\rho^{1/2}(\varphi)}{\theta-\varphi}\, d\varphi\,. \end{aligned}$$

Since $f_1 \geq 0$, this equality shows that $H_{\rho^{1/2}}(f_1)(\theta)$ is decreasing for $0 < \theta < \pi$. Then for $\lambda > 0$, the set

$$E_\lambda = \{\theta \in (0, \pi) : \ |H_{\rho^{1/2}}(f_1)(\theta)| > \lambda\}$$

is some interval $(0, a_\lambda)$ with $0 \leq a_\lambda < \pi$ (the possibility $E_\lambda = \varnothing$ is covered by $a_\lambda = 0$). Denote by ν the measure on $(0, \pi)$ defined by $d\nu = r^2(\theta)\, e^{1/\theta}\, d\theta$. For a (Lebesgue) measurable set $F \subset \partial\mathbb{D}$ we write $|F|$ for its Lebesgue measure. By the hypothesis of the theorem,

$$\nu(E_\lambda) = \nu\big((0, a_\lambda)\big) = \int_0^{a_\lambda} r^2(\theta)\, e^{1/\theta}\, d\theta \leq C\, a_\lambda = C\,|E_\lambda|\,.$$

Hence,

$$\int_0^\pi |H_{\rho^{1/2}}(f_1)|^2\,d\nu = \int_0^\infty 2\,\lambda\,\nu(E_\lambda)\,d\lambda$$
$$\le C\int_0^\infty 2\,\lambda\,|E_\lambda|\,d\lambda = C\int_0^\pi |H_{\rho^{1/2}}(f_1)|^2\,d\theta\,,$$

and the last integral is finite because $f_1\rho^{1/2}\in L^2$. For f_2 and $\theta\in(0,\pi)$ we have

$$\begin{aligned}H_{\rho^{1/2}}(f_2)(\theta) &= H(f_2(\varphi)e^{-1/2\varphi})(\theta)\\ &= H[f_2(\varphi)\,(e^{-1/2\varphi}-e^{-1/2\theta})](\theta)+H(f_2(\varphi)\,e^{-1/2\theta})(\theta)\\ &= I_1(\theta)+I_2(\theta)\,.\end{aligned}$$

The function $I_2(\theta)$ is equal to $e^{-1/2\theta}H(f_2)(\theta)$, hence

$$\int_0^\pi |I_2(\theta)|^2\,r^2\,e^{1/\theta}\,d\theta = \int_0^\pi e^{-1/\theta}\,|H(f_2)(\theta)|^2\,r^2\,e^{1/\theta}\,d\theta$$
$$\le \|r\|^2_{L^\infty}\,\|H(f_2)\|^2_{L^2} < +\infty\,.$$

Finally,

$$I_1(\theta) = \lim_{\varepsilon\to 0}\int\limits_{\varepsilon<|\varphi-\theta|<\pi} f_2(\varphi)\,N(\varphi,\theta)\,d\theta\,,$$

where

$$N(\varphi,\theta) = \frac{e^{-1/2\varphi}-e^{-1/2\theta}}{\theta-\varphi}$$

can be continuously extended to $[0,\pi]\times[0,\pi]$, and therefore is bounded. Hence $|I_1(\theta)|\le C\,\|f_2\|_{L^1} < +\infty$, which implies that $I_1(\theta)$ is square integrable with respect to the (finite) measure $r^2/\rho\;d\theta = r^2\,e^{1/\theta}\,d\theta$ in $(0,\pi)$.

For our example, Theorems 4.2 and 4.3 give a complete characterization of the multipliers of $\mathcal{H}(\bar b)$ among the elements of $K^\infty(\rho)$. However, it is not clear at this point that there are elements in $K^\infty(\rho)$ which fail to satisfy the condition of the theorems. Theorem 2.3 will be the fundamental tool to construct such an element.

Corollary 4.4. *There are elements in $K^\infty(\rho)$ which are not multipliers of $\mathcal{H}(\bar b)$.*

PROOF. If s is an admissible function for ρ, then Theorem 2.3 asserts that $m = e^{s+i\tilde{s}}$ is in $K^\infty(\rho)$. Besides, if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon s^2(\theta)\, e^{1/\theta}\, d\theta = +\infty\,,$$

part 2) of Theorem 2.3 together with Theorem 4.2 immediately implies that $m \notin \mathcal{M}(\overline{b})$. A straightforward calculation shows that if $0 < \alpha < 1/2$, then

$$s(\theta) = \begin{cases} \dfrac{e^{-1/2\theta}}{\theta^\alpha}\,, & \text{if } 0 < \theta \le \pi\,, \\ 0\,, & \text{if } -\pi < \theta \le 0\,, \end{cases}$$

does the job.

If we take as b an outer function such that $|b(e^{i\theta})| = (1 - e^{-1/\theta})^{1/2}$ almost everywhere with respect to $d\theta$, then b is invertible in H^∞. Hence by [13, Theorem 7.1], $\mathcal{H}(b)$=$\mathcal{H}(\overline{b})$.

5. Strong operator topology.

Let f and q be measurable functions on $\partial\mathbb{D}$. Denote

$$J_q(f) = \|qf\|_{L^2(\rho)} = \Big(\frac{1}{2\pi} \int_0^{2\pi} |qf|^2\, \rho\, d\theta\Big)^{1/2}\,.$$

Notice that if $K_\rho(q) + c$ is in $\mathcal{M}(b)$ (or in $\mathcal{M}(\overline{b})$) then by Theorem 3.1, $J_q(f) < \infty$ for all $f \in \mathcal{H}(b)$ (respectively $f \in \mathcal{H}(\overline{b})$). Actually, the above conditions are equivalent.

Lemma 5.1. *Let $m = K_\rho(q) + c \in \mathcal{M}(\overline{b})$ and $f = K_\rho(g) \in \mathcal{H}(\overline{b})$. Then*

$$\|mf\|_{\mathcal{H}(\overline{b})} \le J_q(f) + J_{\overline{m}_*}(g)\,.$$

PROOF. By Lemma 2.1, $mf = K_\rho(h)$, where h is the boundary function of $mf - \overline{m}_* \overline{f}_*$. This boundary function is

$$mf - \overline{m}_* \overline{f}_* = (m - \overline{m}_*)\, f + \overline{m}_*\, (f - \overline{f}_*) = (qf + \overline{m}_* g)\, \rho\,.$$

Thus $\|mf\|_{\mathcal{H}(\overline{b})} = \|qf + \overline{m}_* g\|_{L^2(\rho)} \le J_q(f) + J_{\overline{m}_*}(g)\,.$

Lemma 5.2. *Let* $m = K_\rho(q) + c \in \mathcal{M}(b)$*,* $f \in \mathcal{H}(b)$ *and* g *be the function in* $L^2(\rho)$ *such that* $T_{\overline{b}} f = K_\rho(g)$*. Then*

$$\|mf\|_{\mathcal{H}(b)} \leq \|mf\|_{H^2} + 2\, J_q(T_{\overline{b}} f) + J_{\overline{m}_*}(g) + J_q(f)\,.$$

PROOF. The equality $\|mf\|^2_{\mathcal{H}(b)} = \|mf\|^2_{H^2} + \|T_{\overline{b}}(mf)\|^2_{\mathcal{H}(\overline{b})}$ implies

$$\|mf\|_{\mathcal{H}(b)} \leq \|mf\|_{H^2} + \|T_{\overline{b}}(mf)\|_{\mathcal{H}(\overline{b})}\,. \tag{9}$$

We have

$$T_{\overline{b}}(mf) = m\, T_{\overline{b}} f + P_+\{(\overline{b}f - P_+(\overline{b}f))\, m\}\,, \tag{10}$$

where P_+ is the orthogonal projection from L^2 onto H^2. The function $h = \overline{(1 - P_+)(\overline{b}f)}$ is in H^2_0, so Lemma 12.1 of [13] says that

$$P_+\{(\overline{b}f - P_+(\overline{b}f))\, m\} = P_+(\overline{h}(K_\rho(q) + c)) = K_\rho(\overline{h}q)\,.$$

Thus

$$\begin{aligned} \|P_+(\overline{h}m)\|_{\mathcal{H}(\overline{b})} &= \|\overline{h}q\|_{L^2(\rho)} \\ &= \|(\overline{b}f - P_+(\overline{b}f))\, q\|_{L^2(\rho)} \\ &\leq J_q(f) + J_q(T_{\overline{b}} f)\,. \end{aligned} \tag{11}$$

Besides, $m \in \mathcal{M}(\overline{b})$ (because $\mathcal{M}(b) \subset \mathcal{M}(\overline{b})$), so by Lemma 5.1,

$$\|m\, T_{\overline{b}} f\|_{\mathcal{H}(\overline{b})} \leq J_q(T_{\overline{b}} f) + J_{\overline{m}_*}(g)\,. \tag{12}$$

Therefore (9), (10), (11) and (12) yield the conclusion.

Theorem 5.3. $\mathcal{M}_\infty(b)$ *is dense in* $\mathcal{M}(b)$ *and* $\mathcal{M}(\overline{b})$ *with the respective strong operator topologies.*

PROOF. We prove the theorem for $\mathcal{M}(b)$; the same argument works for $\mathcal{M}(\overline{b})$. Let $\Gamma = \{m \in K^\infty(\rho) :\ m_* = m^{-1}\}$. By Corollary 2.7, $\operatorname{span}(\Gamma \cap \mathcal{M}(b))$ is dense in $\mathcal{M}(b)$ with the operator norm. So, it is enough to prove that every $m \in \Gamma \cap \mathcal{M}(b)$ can be approached (in the strong operator topology of $\mathcal{M}(b)$) by a sequence $(m_n) \subset \Gamma \cap \mathcal{M}_\infty(b)$.

By Theorem 2.3, $m = e^{s+i\tilde{s}}$, with s some admissible function. Consider

$$s_n(e^{i\theta}) = \begin{cases} s(e^{i\theta})\,, & \text{if } |s(e^{i\theta})| \le n\,\rho^{1/2}(e^{i\theta})\,, \\ n\,\rho^{1/2}(e^{i\theta})\,, & \text{if } |s(e^{i\theta})| > n\,\rho^{1/2}(e^{i\theta})\,. \end{cases}$$

Since $s_n^2/\rho \le n^2$, $m_n = e^{s_n+i\tilde{s}_n}$ is in $\mathcal{M}_\infty(b)$. Clearly $s_n \to s$ in L^2, so by the continuity in L^2 of the harmonic conjugation, also $\tilde{s}_n \to \tilde{s}$ in L^2. Taking a suitable subsequence, we can assume that $s_n(e^{i\theta}) \to s(e^{i\theta})$ and $\tilde{s}_n(e^{i\theta}) \to \tilde{s}(e^{i\theta})$ for almost every $e^{i\theta} \in \partial\mathbb{D}$.

By Theorem 2.3, $m = K_\rho(q) + c$ with $q = e^{i\tilde{s}}(e^s - e^{-s})/\rho$ and $c \in \mathbb{C}$; and $m_n = K_\rho(q_n) + c_n$ with $q_n = e^{i\tilde{s}_n}(e^{s_n} - e^{-s_n})/\rho$ and $c_n \in \mathbb{C}$. Hence, $m_n \to m$, $q_n \to q$ and $(m_n)_* = m_n^{-1} \to m^{-1} = m_*$ almost everywhere. Theorem 2.3 also shows that

$$|q_n| \le 2\, e^{\|s_n\|}\, \frac{|s_n|}{\rho} \le 2\, e^{\|s\|}\, \frac{|s|}{\rho} \le e^{\|s\|}\, |q|\,.$$

Thus $|q - q_n| \le C\,|q|$ for all $n \ge 1$, where $C > 0$. Since $m \in \mathcal{M}(b)$, then $hq \in L^2(\rho)$ for any $h \in \mathcal{H}(b)$. Hence, if $f \in \mathcal{H}(b)$ then $J_{q-q_n}(T_{\bar{b}} f)$ and $J_{q-q_n}(f)$ tend to zero when $n \to \infty$ by the dominated convergence theorem. Besides,

$$\max\{\|(m_n)_*\|_\infty\,,\ \|m_n\|_\infty\} \le e^{\|s\|_\infty}\,.$$

So, if $T_{\bar{b}} f = K_\rho(g)$, then $J_{m_* - (m_n)_*}(g)$ and $\|(m - m_n)f\|_{H^2}$ also tend to zero when $n \to \infty$ by the dominated convergence theorem. Thus, Lemma 5.2 shows that $\|(m - m_n)f\|_{\mathcal{H}(b)} \to 0$.

6. The singular component of the measure μ_b.

It is natural to ask how the singular component of the measure μ_b affects the algebras $\mathcal{M}(\bar{b})$, $\mathcal{M}(b)$ and $K^\infty(\rho)$. We address now this problem. Let b, b_1 be extreme points of $B(H^\infty)$, and u be an inner function such that $\mu_b = \mu_{b_1} + \mu_u$. Since u is inner, it is clear from the Herglotz representation (1) that μ_u is a singular measure. Conversely, every Borel positive finite singular measure is associated (via the Herglotz formula) to an inner function. Put $\rho_1 = 1 - |b_1|^2$, $\rho = 1 - |b|^2$ and σ for the Radon-Nikodym derivative of μ_b (and of μ_{b_1}) with respect to the normalized Lebesgue measure. In order to simplify notation, we assume without loss of generality that the respective additive imaginary constant for b_1, b and u in formula (1) is trivial.

Lemma 6.1. *Let $q \in L^2(\rho)$ and $q_1 \in L^2(\rho_1)$. Then $K_\rho(q) = K_{\rho_1}(q_1)$ if and only if $q\rho = q_1\rho_1$.*

PROOF. Suppose that $K(q\rho - q_1\rho_1) = 0$; then $q\rho - q_1\rho_1 \in \overline{H}_0^2$, so it must be trivial if it is not log-integrable. The equality

$$\frac{\rho_1}{|1-b_1|^2} = \sigma = \frac{\rho}{|1-b|^2}$$

implies that the sets $E = \{z \in \partial\mathbb{D} : \rho(z) \neq 0\}$ and $\{z \in \partial\mathbb{D} : \rho_1(z) \neq 0\}$ coincide almost everywhere. Then,

$$\begin{aligned} q\rho - q_1\rho_1 &= \Big(q\rho^{1/2}\Big(\frac{\rho}{\rho_1}\Big)^{1/2} - q_1\rho_1^{1/2}\Big)\,\rho_1^{1/2} \\ &= \Big(q\rho^{1/2}\Big|\frac{1-b}{1-b_1}\Big| - q_1\rho_1^{1/2}\Big)\,\rho_1^{1/2} \\ &= \big(q\rho^{1/2}|1-b| - q_1\rho_1^{1/2}|1-b_1|\big)\,\sigma^{1/2} = h\,\sigma^{1/2}\,, \end{aligned}$$

where the function h is in L^2. Thus, $\log|q\rho - q_1\rho_1| \leq \log^+|h| + (1/2)\log\sigma$ is not integrable and the lemma follows.

Lemma 6.2. *Let b, b_1 and u be as before. Then*

i) $\displaystyle 2\,\frac{1-b}{1-b_1} = 3 - b - 2\,\frac{1-b}{1-u}\,,$

ii) $\displaystyle 2\,\frac{1-b_1}{1-b} = 1 + b_1 + 2\,\frac{1-b_1}{1-u}\,.$

PROOF. Both formulas are straightforward calculations from the identity

$$\frac{1+b}{1-b} = \frac{1+b_1}{1-b_1} + \frac{1+u}{1-u}$$

given by the Herglotz representations associated to b, b_1 and u.

Theorem 6.3. *Let b, b_1 and u be as before.*

1) *Let $m = K_\rho(q) \in K^\infty(\rho)$, $|q\rho| = r$. Then*

$$m \in K^\infty(\rho_1) \quad \textit{if and only if} \quad (1-u)^{-1} \in L^2(r^2/\sigma)\,.$$

2) *Let $m_1 = K_{\rho_1}(q_1) \in K^\infty(\rho_1)$, $|q_1\rho_1| = r_1$. Then*

$$m_1 \in K^\infty(\rho) \quad \textit{if and only if} \quad (1-u)^{-1} \in L^2(r_1^2/\sigma)\,.$$

PROOF. 1) Let $m = K_\rho(q) \in K^\infty(\rho)$. If $m(z) = K_\rho(q)(z) = K_{\rho_1}(q_1)(z) + c$, with $q_1 \in L^2(\rho_1)$ and $c \in \mathbb{C}$, then letting $z \to \infty$ we obtain that $c = 0$. Hence $K_\rho(q) = K_{\rho_1}(q_1)$, and Lemma 6.1 says that this happens if and only if $r = |q\rho|$ is admissible for ρ_1 (so $q_1 = q\,\rho/\rho_1$). That is, if and only if $r^2/\rho_1 \in L^1$. Now

$$\frac{r^2}{\rho_1} = \frac{\rho}{\rho_1}\,\frac{r^2}{\rho} = \left|\frac{1-b}{1-b_1}\right|^2 \frac{r^2}{\rho} = \left|\frac{3-b}{2} - \frac{1-b}{1-u}\right|^2 \frac{r^2}{\rho}\,,$$

where the last equality follows from i) of Lemma 6.2. Since $(3-b)/2$ is bounded and r^2/ρ is in L^1, we have that $r^2/\rho_1 \in L^1$ if and only if

$$\left|\frac{1-b}{1-u}\right|^2 \frac{r^2}{\rho} \in L^1\,,$$

or, what is the same, if and only if $(1-u)^{-1} \in L^2(r^2/\sigma)$. Assertion 2) follows in the same way using formula ii) of Lemma 6.2.

Theorem 6.4. *Let b and b_1 be as before. Then $\mathcal{M}(b) \subset \mathcal{M}(b_1)$.*

PROOF. Let $m = K_\rho(q) + c \in \mathcal{M}(b)$. We will show first that m belongs to $K^\infty(\rho_1)$. The measure μ_{b_1} decomposes as $d\mu_{b_1} = \sigma\, d\theta/2\pi + d\mu_{S_1}$, where μ_{S_1} is the singular component of μ_{b_1}. On the other hand, μ_u can be decomposed as $d\mu_u = \alpha\, d\mu_{S_1} + d\mu_0$, where $\alpha \in L^1(\mu_{S_1})$, $\alpha \geq 0$ is the Radon-Nikodym derivative of μ_u with respect to μ_{S_1}, and μ_0 is singular with respect to μ_{S_1}. These decompositions together show that the measure $d\nu = (1+\alpha)\, d\mu_{S_1} + d\mu_0$ is the singular component of $d\mu_b$. Put $r = |q\rho|$; since $m \in \mathcal{M}(b)$, Theorem 3.2.2) asserts that $K_\nu(f) \in L^2(r^2/\sigma)$ for all $f \in L^2(\nu)$. Let χ be a function which takes the value 1 almost everywhere with respect to $d\mu_{S_1}$ and the value 0 almost everywhere with respect to $d\mu_0$, and consider $f = \alpha(1+\alpha)^{-1}\chi + 1 - \chi$. Since $\alpha \geq 0$ and μ_{S_1} and μ_0 are finite measures, then $f \in L^2(\nu)$ (f is bounded almost everywhere with respect to $d\nu$). Thus $K_\nu(f)$ is in $L^2(r^2/\sigma)$. But

$$\begin{aligned} K_\nu(f) &= K_{(1+\alpha)\mu_{S_1}}\,(\alpha(1+\alpha)^{-1}\chi) + K_{\mu_0}\,(1-\chi) \\ &= K_{\mu_{S_1}}(\alpha\chi) + K_{\mu_0}(1) \\ &= K_{\alpha\mu_{S_1}+\mu_0}(1) = K_{\mu_u}(1)\,. \end{aligned}$$

Hence $K_{\mu_u}(1) \in L^2(r^2/\sigma)$. It is well known [15, III-7] that

$$(1-u)\,K_{\mu_u}(1) = (1-\overline{u(0)})^{-1}\,(1-\overline{u(0)}u)\,. \tag{13}$$

Since $|(1-\overline{u(0)})^{-1}(1-\overline{u(0)}u)|$ is bounded from below by a positive constant, we obtain that $(1-u)^{-1} \in L^2(r^2/\sigma)$. Now Theorem 6.3.1) says that $m \in K^\infty(\rho_1)$.

The fact that $m \in \mathcal{M}(b)$ implies by Theorem 3.2.2), that $K_{\sigma^{1/2}}(f) \in L^2(r^2/\sigma)$ for all $f \in L^2(\chi_E)$. So, by the same theorem, in order to prove that $m \in \mathcal{M}(b_1)$ we must show that if $g \in L^2(\mu_{S_1})$ then $K_{\mu_{S_1}}(g) \in L^2(r^2/\sigma)$. Consider the function $g(1+\alpha)^{-1}\chi$. Since

$$\begin{aligned}|g\,(1+\alpha)^{-1}\chi|^2\,d\nu &= |g|^2\,|1+\alpha|^{-2}\,(1+\alpha)\,d\mu_{S_1}\\ &= |g|^2\,(1+\alpha)^{-1}\,d\mu_{S_1} \le |g|^2\,d\mu_{S_1}\,,\end{aligned}$$

then $g(1+\alpha)^{-1}\chi$ belongs to $L^2(\nu)$. Therefore, since m is a multiplier of $\mathcal{H}(b)$, Theorem 3.2.2) says that $K_\nu\,(g\,(1+\alpha)^{-1}\chi)$ is in $L^2(r^2/\sigma)$; but $K_\nu\,(g\,(1+\alpha)^{-1}\chi) = K_{(1+\alpha)\mu_{S_1}}(g\,(1+\alpha)^{-1}) = K_{\mu_{S_1}}(g)$, and the theorem follows.

Two particular cases are of special interest in Theorem 6.4, when μ_{b_1} is absolutely continuous, and when μ_u is singular with respect to the singular component of μ_{b_1} (*i.e.* $\alpha = 0$ in the proof of the theorem). If b_1 is a nonextreme point of $B(H^\infty)$ and μ_{b_1} is absolutely continuous, Theorem 6.4 was obtained by Davis and McCarthy [5].

Theorem 6.5. *Let b_1 be an extreme point of $B(H^\infty)$ and $\mu_S = \mu_1 + \cdots + \mu_n$ be a purely atomic measure, where each μ_j ($1 \le j \le n$) is an atom at the point $\omega_j = e^{i\varphi_j} \in \partial\mathbb{D}$ (with $\omega_j \ne \omega_k$ if $j \ne k$). Let $b \in B(H^\infty)$ such that $\mu_b = \mu_{b_1} + \mu_S$. If $m = K_{\rho_1}(q_1) + c$ (with $q \in L^2(\rho_1)$, $c \in \mathbb{C}$ and $r = |q\rho_1|$) is a multiplier of $\mathcal{H}(b_1)$, then the following conditions are equivalent.*

1) $m \in K^\infty(\rho)$.

2) $K_{\mu_S}(1) \in L^2(r^2/\sigma)$.

3) $K_{\mu_j}(1) \in L^2(r^2/\sigma)$ *for every* j.

4) $m \in \mathcal{M}(b)$.

5) $f_j(\theta) = (\theta - \varphi_j)^{-2}\, r^2(e^{i\theta})/\sigma(e^{i\theta}) \in L^1[d\theta, (\varphi_j - \pi, \varphi_j + \pi)]$ *for all* j.

PROOF. 1) if and only if 2) is in Theorem 6.3.2), using again that if u is the inner function associated to μ_S, then $(1-u)^{-1}$ behaves like $K_{\mu_S}(1)$ (formula (13)).

2) implies 3). Let $V \subset \partial\mathbb{D}$ be an open neighborhood of ω_1 such that the closure of V does not contain any of the ω_j, $2 \le j \le n$. Then $K_{\mu_1}(1)$ is continuous on $\partial\mathbb{D} \setminus V$ and therefore it is square integrable with respect to the measure $r^2/\sigma\, d\theta$ there. On the other hand,

$$K_{\mu_1}(1) = K_{\mu_S}(1) - \sum_{j=2}^{n} K_{\mu_j}(1)\,,$$

and since $\sum_{j=2}^{n} K_{\mu_j}(1)$ is continuous on V and by hypothesis $K_{\mu_S}(1) \in L^2(r^2/\sigma)$, then $K_{\mu_1}(1)$ is also square integrable with respect to $r^2/\sigma\, d\theta$ in V. Analogously, $K_{\mu_j}(1) \in L^2(r^2/\sigma)$ for all $2 \le j \le n$.

3) implies 4). Hypothesis 3) clearly implies that $K_{\mu_S}(f) \in L^2(r^2/\sigma)$ for all $f \in L^2(\mu_S)$. In particular 2) holds, and since 2) implies 1), $m \in K^\infty(\rho)$. Since $m \in \mathcal{M}(b_1)$, by Theorem 3.2.2) and the comments preceding it, $K_{\mu_{b_1}}(h) \in L^2(r^2/\sigma)$ for all $h \in L^2(\mu_{b_1})$. The decomposition $\mu_b = \mu_{b_1} + \mu_S$ now clearly implies that $K_{\mu_b}(f) \in L^2(r^2/\sigma)$ for all $f \in L^2(\mu_b)$. Hence by Theorem 3.2 again, $m \in \mathcal{M}(b)$.

Obviously 4) implies 1). To prove the equivalence between 3) and 5), write $\alpha_j = \|\mu_j\|$. Then $K_{\mu_j}(1)(e^{i\theta}) = \alpha_j(1 - \overline{\omega}_j e^{i\theta})^{-1}$. Therefore,

$$|K_{\mu_j}(1)(e^{i\theta})|^2 = |\alpha_j|^2 |e^{i\varphi_j} - e^{i\theta}|^{-2} = |\alpha_j|^2\, 2^{-1} \left(1 - \cos(\theta - \varphi_j)\right)^{-1}.$$

The equivalence now follows from the fact that $1 - \cos(\theta - \varphi_j)$ behaves like $(\theta - \varphi_j)^2$ when $|\theta - \varphi_j| < \pi$.

7. A partial isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(\overline{b})$.

If φ and f are measurable functions on $\partial\mathbb{D}$ such that $\varphi f \in L^2$, we define $T_\varphi(f) = P_+(\varphi f)$, where P_+ is the orthogonal projection from L^2 onto H^2. Hence T_φ is an operator defined on the space $\{f \text{ measurable: } \varphi f \in L^2\}$. If $\psi \in L^\infty$, M_ψ will denote the operator on L^2 of multilplication by ψ.

Lemma 7.1. *The operators $T_{1-\overline{b}} K_{\sigma^{1/2}}$ and $K_{\sigma^{1/2}} M_{1-\overline{b}}$ are contractions from $L^2(\chi_E)$ into L^2 and coincide.*

PROOF. Notice that since $(1-b)\, K_{\sigma^{1/2}}(f) \in \mathcal{H}(b) \subset H^2$ for $f \in L^2(\chi_E)$, then $(1 - \overline{b})\, K_{\sigma^{1/2}}(f) \in L^2$, so $T_{1-\overline{b}} K_{\sigma^{1/2}}$ is well defined on $L^2(\chi_E)$.

Let $f = (1-\overline{b})\, g$, with $g \in L^2(\chi_E)$, then

$$\begin{aligned} T_{1-\overline{b}}\, K_{\sigma^{1/2}}\big((1-\overline{b})\, g\big) &= T_{1-\overline{b}}\, K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\, g\Big) \\ &= K_{\rho^{1/2}}\Big(\frac{(1-\overline{b})^2}{|1-b|}\, g\Big) \\ &= K_{\sigma^{1/2}}\big((1-\overline{b})\, f\big) \\ &= K_{\sigma^{1/2}}\, M_{1-\overline{b}} f\,, \end{aligned}$$

where the second equality follows from [13, Corollary 3.5]. Hence both operators coincide on $(1-\overline{b})L^2(\chi_E)$. This is a dense subspace of $L^2(\chi_E)$, because if h is orthogonal to this subspace, then for all $g \in L^2(\chi_E)$,

$$0 = \langle h, (1-\overline{b})\, g\rangle = \langle (1-b)\, h, g\rangle\,,$$

which implies $(1-b)\, h\chi_E = 0$, so $h = 0$ almost everywhere with respect to $d\theta$ on E. Therefore, we only have to show that both operators are contractions. Let $f \in L^2(\chi_E)$; then

$$\begin{aligned} \|T_{1-\overline{b}}\, K_{\sigma^{1/2}}(f)\|_{L^2} &= \|P_+[(1-\overline{b})\, K_{\sigma^{1/2}}(f)]\|_{L^2} \\ &\le \|(1-\overline{b})\, K_{\sigma^{1/2}}(f)\|_{L^2} \\ &= \|(1-b)\, K_{\sigma^{1/2}}(f)\|_{H^2} \\ &\le \|(1-b)\, K_{\sigma^{1/2}}(f)\|_{\mathcal{H}(b)} \\ &= \|f\|_{L^2(\chi_E)}\,. \end{aligned}$$

Also,

$$\begin{aligned} \|K_{\sigma^{1/2}}\big((1-\overline{b})\, f\big)\|_{L^2} &= \Big\|K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\, f\Big)\Big\|_{L^2} \\ &= \Big\|K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\, f\Big)\Big\|_{H^2} \\ &\le \Big\|K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\, f\Big)\Big\|_{\mathcal{H}(\overline{b})} \\ &= \Big\|\frac{1-\overline{b}}{|1-b|}\, f\Big\|_{L^2(\chi_E)} = \|f\|_{L^2(\chi_E)}\,. \end{aligned}$$

The decomposition of the measure $\mu_b = \sigma\, d\theta/2\pi + d\mu_S$ induces an orthogonal decomposition $L^2(\mu_b) = L^2(\sigma) \oplus L^2(\mu_S)$, which according to our treatment we identify with $L^2(\chi_E) \oplus L^2(\mu_S)$ (via the onto isometry $(f, g) \mapsto (\sigma^{1/2} f, g)$). This decomposition translates into an orthogonal decomposition for $\mathcal{H}(b)$ as $\mathcal{H}(b) = \mathcal{H}(b)^\sigma \oplus \mathcal{H}(b)^S$, where

$$\mathcal{H}(b)^\sigma = (1-b)\, K_{\sigma^{1/2}}(L^2(\chi_E))$$

and

$$\mathcal{H}(b)^S = (1-b)\, K_{\mu_S}(L^2(\mu_S))\,.$$

Theorem 7.2. *$T_{(1-\overline{b})/(1-b)}$ is a partial isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(\overline{b})$ with initial space $\mathcal{H}(b)^\sigma$. Further, if $g \in L^2(\chi_E)$,*

$$T^*_{(1-\overline{b})/(1-b)}(K_{\rho^{1/2}}(g)) = (1-b)\, K_{\sigma^{1/2}}\Big(\frac{1-b}{|1-b|}\, g\Big)\,.$$

PROOF. First we show that $\mathcal{H}(b)^S$ is contained in the kernel of $T_{(1-\overline{b})/(1-b)}$. Denote by u the inner function associated to μ_S in (1). Let $f \in \mathcal{H}(b)^S$; then there is $g \in L^2(\mu_S)$ such that

$$f = (1-b)\, K_{\mu_S}(g) = \frac{1-b}{1-u}\,(1-u)\, K_{\mu_S}(g) \in \frac{1-b}{1-u}\, \mathcal{H}(u)\,.$$

Besides, $\|f\|_{\mathcal{H}(b)} = \|g\|_{L^2(\mu_S)} = \|(1-u)\, K_{\mu_S}(g)\|_{\mathcal{H}(u)}$. We can now begin with $g \in L^2(\mu_S)$, obtaining that

$$\mathcal{H}(b)^S = \frac{1-b}{1-u}\, \mathcal{H}(u)\,.$$

It is well known that the span of the functions

$$k^u_\omega(e^{i\theta}) = \frac{1-\overline{u(\omega)}\, u(e^{i\theta})}{1-\overline{\omega}\, e^{i\theta}}\,, \qquad \omega \in \mathbb{D}\,,$$

is dense in $\mathcal{H}(u)$. Thus the span of the functions $(1-b)\,(1-u)^{-1} k^u_\omega$ ($\omega \in \mathbb{D}$) is dense in $\mathcal{H}(b)^S$. Hence, it is enough to prove that these functions belong to the kernel of $T_{(1-\overline{b})/(1-b)}$. Let us denote by z the function $z(e^{i\theta}) = e^{i\theta}$. Then

$$T_{(1-\overline{b})/(1-b)}\Big(\frac{1-b}{1-u}\, k_\omega^u\Big) = P_+\Big(\frac{(1-\overline{b})\,(1-\overline{u(\omega)})\,u}{(1-u)\,(1-\overline{\omega}\,z)}\Big)$$
$$= P_+\Big(\frac{(1-\overline{b})\,(\overline{u}-\overline{u(\omega)})\,\overline{z}}{(\overline{u}-1)\,(\overline{z}-\overline{\omega})}\Big) = P_+(\overline{g})\,,$$

where

$$g = -\frac{(1-b)\,(u-u(\omega))\,z}{(1-u)\,(z-\omega)}\,.$$

In [15, III-11] it is proved that $(1-b)\,(1-u)^{-1}$ belongs to H^2; therefore $g \in H_0^2$ and consequently $P_+(\overline{g}) = 0$. Now let $f \in L^2(\chi_E)$. By Lemma 7.1,

$$T_{(1-\overline{b})/(1-b)}\big((1-b)\,K_{\sigma^{1/2}}(f)\big) = T_{1-\overline{b}}\,K_{\sigma^{1/2}}(f)$$
$$= K_{\sigma^{1/2}}\big((1-\overline{b})\,f\big) = K_{\rho^{1/2}}\Big(\frac{1-\overline{b}}{|1-b|}\,f\Big)\,,$$

and clearly

$$\Big\|\frac{1-\overline{b}}{|1-b|}\,f\Big\|_{L^2(\chi_E)} = \|f\|_{L^2(\chi_E)}\,.$$

That is, $T_{(1-\overline{b})/(1-b)}$ maps $\mathcal{H}(b)^\sigma$ isometrically into $\mathcal{H}(\overline{b})$. To see that this map is onto, let $g \in L^2(\chi_E)$ and take $f = (1-b)\,g/|1-b|$. By Lemma 7.1,

$$T_{(1-\overline{b})/(1-b)}\big((1-b)\,K_{\sigma^{1/2}}(f)\big) = K_{\sigma^{1/2}}\Big(\frac{|1-b|^2}{|1-b|}\,g\Big) = K_{\rho^{1/2}}(g)\,.$$

This also proves the formula for $T^*_{(1-\overline{b})/(1-b)}$.

Corollary 7.3. *The measure μ_b is absolutely continuous if and only if*

$$T_{(1-\overline{b})/(1-b)}\,(1-T_b\,T_{\overline{b}})^{1/2}$$

is one-to-one (from H^2 into H^2).

PROOF. By Theorem 7.2, μ_b is absolutely continuous if and only if $T_{(1-\overline{b})/(1-b)}|_{\mathcal{H}(b)}$ is one-to-one. Hence, the corollary will follow if we show that $(1-T_b T_{\overline{b}})^{1/2}$ is one-to-one. Since b is not an inner function, $\|T_{\overline{b}} f\|_{H^2} \le \|\overline{b} f\|_{L^2} < \|f\|_{H^2}$ unless $f=0$. Hence, $f \ne T_b T_{\overline{b}} f$ if $f \ne 0$.

Theorem 7.4. *K_{μ_b} maps $L^2(\mu_b)$ into $L^2(\rho)$.*

PROOF. Let $h \in L^2(\mu_b)$, and consider $f=(1-b)\,K_{\mu_b}(h) \in \mathcal{H}(b)$. Then $T_{\overline{b}} f$ is in $\mathcal{H}(\overline{b})$, and

$$\begin{aligned} T_{\overline{b}} f &= P_+\big((\overline{b}-1+1-|b|^2)\,K_{\mu_b}(h)\big) \\ &= -P_+\big((1-\overline{b})\,K_{\mu_b}(h)\big) + P_+\big(\rho\,K_{\mu_b}(h)\big) \\ &= -T_{(1-\overline{b})/(1-b)} f + K\big(\rho\,K_{\mu_b}(h)\big)\,. \end{aligned}$$

Notice that $\rho\,|K_{\mu_b}(h)| \le 2\,|(1-b)\,K_{\mu_b}(h)| \in L^2$. By Theorem 7.2 the first summand is in $\mathcal{H}(\overline{b})$, therefore $K(\rho\,K_{\mu_b}(h))$ belongs to $\mathcal{H}(\overline{b})$, too. Then there is $q \in L^2(\rho)$ such that $K(\rho\,K_{\mu_b}(h) - \rho\,q) = 0$, or equivalently, $\rho\,K_{\mu_b}(h) - \rho\,q \in \overline{H}^2_0$. Now,

$$\log|\rho\,K_{\mu_b}(h) - \rho\,q| \le \log^+|\rho^{1/2}\,K_{\mu_b}(h) - \rho^{1/2}\,q| + \frac{1}{2}\log\rho\,,$$

and since ρ is not log-integrable, $\rho\,K_{\mu_b}(h) - \rho\,q$ cannot be log-integrable if we prove that $\rho^{1/2}\,K_{\mu_b}(h) - \rho^{1/2}\,q$ is in L^1. The function $\rho^{1/2}\,q$ is in L^2. Besides

$$\rho^{1/2}\,|K_{\mu_b}(h)| = \frac{\rho^{1/2}}{|1-b|}\,|(1-b)\,K_{\mu_b}(h)| = \sigma^{1/2}\,|f|\,,$$

which is in L^1 because it is the product of two functions of L^2. Hence $K_{\mu_b}(h)(e^{i\theta}) = q(e^{i\theta})$ almost everywhere with respect to the measure $\rho(e^{i\theta})\,d\theta$, so $K_{\mu_b}(h) \in L^2(\rho)$.

A direct consequence of the above theorem is that $V_b = (1-b)\,K_{\mu_b}$ maps $L^2(\mu_b)$ into $L^2(\sigma)$, in other words $\mathcal{H}(b) \subset L^2(\sigma)$. Let us return to the multipliers.

Corollary 7.5. *Let $m = K_\rho(q) + c \in K^\infty(\rho)$, and put $r = |q\,\rho|$. A sufficient condition for m to be a multiplier of $\mathcal{H}(b)$ is that there exists a constant $C > 0$ such that $r^2/\sigma \le C\rho$ (or what is equivalent, $|q|\chi_E \le C^{1/2}|1-b|^{-1}\chi_E$, where $E = \{z \in \partial\mathbb{D} :\ \rho(z) \ne 0\}$).*

Proof. By Theorem 3.2, $m \in \mathcal{M}(b)$ if and only if $K_{\mu_b}(h) \in L^2(r^2/\sigma)$ for all $h \in L^2(\mu_b)$. By Theorem 7.4 this holds if $L^2(\rho) \subset L^2(r^2/\sigma)$, and this is clearly equivalent to $r^2/\sigma \le C\rho$ for some constant $C > 0$. Besides,

$$\frac{r^2}{\sigma} \le C\,\rho \ \text{ if and only if } \ |q|^2\,\rho^2 = r^2 \le C\,\rho\sigma = C\,\frac{\rho^2}{|1-b|^2}\,,$$

which is equivalent to

$$|q|^2\,\chi_E \le C\,\frac{\chi_E}{|1-b|^2}$$

and

$$|q|\,\chi_E \le C^{1/2}\,\frac{\chi_E}{|1-b|}\,.$$

Remark 7.6. If s is any bounded real valued function which satisfies $s^2/\sigma \le C\,\rho$ for some constant $C > 0$, then $s^2/\rho \le C\,\sigma \in L^1$, that is, s is admissible for ρ. Hence $m = e^{s+i\tilde{s}} \in K^\infty(\rho)$, and if $m = K_\rho(q) + c$, then $r = |q\,\rho|$ behaves like s. Therefore the corollary asserts that $m \in \mathcal{M}(b)$.

The unexpected condition for multipliers given by Corollary 7.5 is not always necessary. For instance, let b be an outer function such that $\rho(e^{i\theta}) = e^{-1/|\theta|}$ for $\theta \in [-\pi,\pi)$. Then b is continuous on $\partial\mathbb{D}$ because b is outer and $|b|$ is continuously differentiable on $\partial\mathbb{D}$. Moreover, $|b(1)| = 1$, so we can assume multiplying by $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ if need be, that $b(1) = -1$. The function $\rho^{1/2}$ is admissible, even more, $m = e^{\rho^{1/2}+\tilde{\rho}^{1/2}} \in \mathcal{M}_\infty(b)$ because $\left(\rho^{1/2}\right)^2/\rho = \chi_E$ is bounded (see Section 2). If $m = K_\rho(q) + c$, $r = |q\,\rho|$, and r satisfies the condition of Corollary 7.5, then also $\rho^{1/2}$ satisfies this condition, that is, $\rho/\sigma \le C\,\rho$. This is equivalent to $|1-b(z)|^2 \le C\,(1-|b(z)|^2)$ for all $z \in \partial\mathbb{D}$. And this inequality obviously does not hold for z close to 1.

Corollary 7.7. *Let b be an extreme point and u be an inner function. If $\sigma_{ub} = \rho/|1-ub|^2$, then $\mathcal{H}(b) \subset L^2(\sigma_{ub})$.*

Proof. If $s = \rho^{1/2}\,\sigma_{ub}^{1/2}$ then $s^2/\sigma_{ub} = \rho$, so by Remark 7.6, $m = e^{s+i\tilde{s}}$ belongs to $\mathcal{M}(ub)$. In particular, m is in $\mathcal{M}(b)$, thus

$$(1-b)\,K_{\mu_b}(f) \in L^2(s^2/\rho) = L^2(\sigma_{ub})\,, \qquad \text{for all } \ f \in L^2(\mu_b)\,.$$

That is, $\mathcal{H}(b) \subset L^2(\sigma_{ub})$.

The idea of the example in Remark 7.6 will be exploited more in the sequel. For expository reasons, it will be convenient to prove the next lemma in $\mathbb{C}_+ = \{z = x + iy \in \mathbb{C} : \ y > 0\}$. Of course, the result also holds in the disc (with obvious translation).

Lemma 7.8. *Let (α_k) be a sequence of real numbers such that $\alpha_k \neq \alpha_j$ if $k \neq j$ and $\lim \alpha_k = \alpha$, with $\alpha \neq \alpha_k$ for all k. Let (ω_k) be a sequence in $\partial\mathbb{D}$ and (ε_k) be a decreasing sequence of positive numbers that tends to zero. Then there exists an interpolating Blaschke product B, continuous on the closure of $\mathbb{C}_+$ except in $z = \alpha$, such that $|B(\alpha_k) - \omega_k| < \varepsilon_k$ for all k.*

PROOF. We can assume $\varepsilon_k < 1$ for all k. Take $d_1 = (1/4) \inf_{j\neq 1} |\alpha_1 - \alpha_j|$ and $r_1 = \varepsilon_1 d_1/2^2$. Consider the half circle $S_1 = \{z \in \mathbb{C}_+ : \ |z-\alpha_1| = r_1\}$. There is $z_1 \in S_1$ such that

$$\left|\operatorname{Arg}\left(\frac{\alpha_1 - z_1}{\alpha_1 - \overline{z}_1}\right) - \operatorname{Arg}\omega_1\right| < \frac{\varepsilon_1}{2} \, ,$$

where Arg is the argument taken in $[0, 2\pi)$. Hence, if $b_1(z) = (z - z_1)/(z-\overline{z}_1)$ then $|b_1(\alpha_1)-\omega_1| < \varepsilon_1/2$. If $x \in \mathbb{R}$ is such that $|x-\alpha_1| > d_1$, then $\operatorname{Arg}((x-z_1)/(x-\overline{z}_1))$ belongs to the union of the intervals $(0, a_1)$ and $(2\pi - a_1, 2\pi)$, where $a_1 = 2\arctan(r_1/d_1) \leq 2\,r_1/d_1 = \varepsilon_1/2$. We can repeat the process with α_2, taking $d_2 = (1/4)\inf_{j\neq 2} |\alpha_2 - \alpha_j|$, $r_2 = \varepsilon_2 d_2/2^3$ and $\overline{b_1(\alpha_2)}\,\omega_2$ instead of ω_2. So, we obtain a point $z_2 \in S_2 = \{z \in \mathbb{C}_+ : \ |z_2 - \alpha_2| = r_2\}$ such that if $b_2(z) = (z - z_2)/(z - \overline{z}_2)$, then

$$|b_2(\alpha_2) - \overline{b_1(\alpha_2)}\,\omega_2| < \frac{\varepsilon_2}{2} \, ,$$

and for $x \in \mathbb{R}$ with $|x - \alpha_2| > d_2$, $\operatorname{Arg} b_2(x) \in (0, a_2) \cup (2\pi - a_2, 2\pi)$, where $a_2 < 2\,r_2/d_2 = \varepsilon_2/2^2$. Consider the Blaschke product $B_2 = b_2\, b_1$. Then,

$$\begin{aligned} |B_2(\alpha_2) - \omega_2| &= |b_2(\alpha_2)\, b_1(\alpha_2) - \omega_2| \\ &= |b_2(\alpha_2) - \omega_2\, \overline{b_1(\alpha_2)}| < \frac{\varepsilon_2}{2} \end{aligned} \tag{1}$$

and

$$\begin{aligned} |B_2(\alpha_1) - \omega_1| &\leq |b_2(\alpha_1)\, b_1(\alpha_1) - b_2(\alpha_1)\,\omega_1| + |b_2(\alpha_1)\,\omega_1 - \omega_1| \\ &= |b_1(\alpha_1) - \omega_1| + |b_2(\alpha_1) - 1| \\ &< \frac{\varepsilon_1}{2} + a_2 < \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} \, , \end{aligned} \tag{2}$$

where (2) holds because $|\alpha_1 - \alpha_2| > d_2$. Repeating this process k times, where $d_k = (1/4)\inf_{j\neq k}|\alpha_k - \alpha_j|$, $r_k = \varepsilon_k d_k/2^{k+1}$ and ω_k is replaced by $\overline{B_{k-1}(\alpha_k)}\,\omega_k$, we obtain a point $z_k \in S_k = \{z \in \mathbb{C}_+ :\ |z - \alpha_k| = r_k\}$ such that if $B_k = b_k B_{k-1}$, then

$$(1') \qquad |B_k(\alpha_k) - \omega_k| < \frac{\varepsilon_k}{2}$$

and

$$(2') \qquad |B_k(\alpha_j) - \omega_j| < \frac{\varepsilon_j}{2^j} + \frac{\varepsilon_{j+1}}{2^{j+1}} + \cdots + \frac{\varepsilon_k}{2^k}\,, \qquad \text{for all } \ j < k\,.$$

For $j < k$ the fact that (ε_n) is a decreasing sequence implies

$$(14) \qquad |B_k(\alpha_j) - \omega_j| < \sum_{n=j}^{k} \frac{\varepsilon_n}{2^n} < \varepsilon_j \sum_{n=j}^{k} \frac{1}{2^n} < \varepsilon_j\ .$$

The sequence (B_k) obtained in this process is the sequence of partial products of $B(z) = \prod_{k=1}^{\infty}(z - z_k)/(z - \overline{z}_k)$, where the points z_k are as above. The usual factors used to make the arguments convergent are not required because $\{z_k :\ k \geq 1\}$ is bounded.

Simple estimations show that $|z_k - z_j|/|z_k - \overline{z}_j| > 1/3$ for $k \neq j$. Since $\operatorname{Im} z_k \leq r_k < C\,2^{-k}$ for some $C > 0$, it is clear that $B(z)$ is an interpolating Blaschke product (see [9, VII]). It is well known that the set of continuity on $\mathbb{C}_+$ of a Blaschke product coincides with the complement of the limit set of its zeros in $\mathbb{R}$. Then B is continuous on $\mathbb{C}_+ \setminus \{\alpha\}$ and by (14), $|B(\alpha_k) - \omega_k| < \varepsilon_k$ for all $k \geq 1$.

Theorem 7.9. *The following conditions are equivalent.*

1) $\mathcal{M}_\infty(b) = K^\infty(\rho)$.

2) $\mathcal{M}_\infty(b) = \mathcal{M}(b)$.

3) *There is a constant* $\delta > 0$ *such that* $\rho(e^{i\theta}) \geq \delta\,\chi_E(e^{i\theta})$ *almost everywhere with respect to* $d\theta$.

4) *For every inner function* u *there is a constant* $C = C(u) > 0$ *such that*

$$\frac{1 - |b(e^{i\theta})|^2}{|1 - u(e^{i\theta})\,b(e^{i\theta})|^2} \leq C$$

almost everywhere with respect to $d\theta$.

5) *For every inner function u there is a constant $\varepsilon = \varepsilon(u) > 0$ such that*

$$\varepsilon\, \chi_E(e^{i\theta}) \le \frac{1-|b(e^{i\theta})|^2}{|1-u(e^{i\theta})\, b(e^{i\theta})|^2}$$

almost everywhere with respect to $d\theta$.

6) *Condition* 4) *holds with C independent of u.*

7) *Condition* 5) *holds with ε independent of u.*

PROOF. The equivalence of 1) and 3) is in the comments following the definition of admissible function (Section 2). The string of inclusions (2) in Section 1 clearly shows that 1) implies 2).

2) implies 4). Take $s : \partial\mathbb{D}\to\mathbb{R}$ bounded such that $s^2/\rho \le C\,\sigma$ (where C is some positive constant). As we pointed out in Remark 7.6, s is admissible and $m = e^{s+i\tilde{s}} = K_\rho(q) + c$ belongs to $\mathcal{M}(b)$, where $r = |q\,\rho|$ behaves like s. Hypothesis 2) says that $m \in \mathcal{M}_\infty(b)$. This is equivalent to the boundedness of s^2/ρ (Section 2). So, $s^2/\rho \le C\,\sigma$ implies that s^2/ρ is bounded. Take $s = \rho^{1/2}\sigma^{1/2} = (1-|b|^2)\,|1-b|^{-1} \le 2$. Then $s^2/\rho = \rho\,\sigma/\rho = \sigma$, and consequently $s^2/\rho = \sigma$ must be bounded. We arrived to this conclusion only assuming $\mathcal{M}_\infty(b) = \mathcal{M}(b)$, and if this happens, then $\mathcal{M}_\infty(b) = \mathcal{M}(ub)$ for every inner function u. Besides, the characterization of $\mathcal{M}_\infty(b)$ given in Section 2 is not sensitive to the inner factor u, thus $\mathcal{M}_\infty(ub) = \mathcal{M}_\infty(b) = \mathcal{M}(ub)$. Therefore $\sigma_{ub} = (1-|b|^2)/|1-ub|^2$ must be bounded for every inner function u.

4) implies 3). If 3) does not hold, there is a positive decreasing sequence (ε_k) which tends to zero, such that the sets

$$T_k = \{z \in \partial\mathbb{D} :\ \varepsilon_k < \rho \le \varepsilon_{k-1}\}, \qquad k \ge 2,$$

all have positive measure. Then there are points $\omega_k \in \partial\mathbb{D}$ such that

$$E_k = \Big\{z \in T_k :\ \Big|\frac{\overline{b(z)}}{|b(z)|} - \omega_k\Big| < \varepsilon_k\Big\}$$

also have positive measure. For each $k \ge 2$ let α_k be a density point of E_k. By compactness we can extract a convergent subsequence of (α_k), we also denote this sequence by (α_k). Even more, we can assume that $\alpha_k \neq \alpha_j$ for $k \neq j$ and $\lim \alpha_k \neq \alpha_j$ for all j. By Lemma 7.8 there is an interpolating Blaschke product B continuous on $\{\alpha_k :\ k \ge 2\}$ such that

$$|B(\alpha_k) - \omega_k| < \varepsilon_k, \qquad \text{for all } k \ge 2.$$

Since α_k is a density point of E_k, any open arc-interval centered at α_k small enough satisfies $|E_k \cap I_k| > |I_k|/2$. Furthermore, by the continuity of B in α_k we can assume (shrinking I_k if necessary) that

$$|B(z) - \omega_k| < \varepsilon_k \,, \qquad \text{for all } z \in I_k \text{ and all } k \geq 2\,.$$

Hence, for almost every $z \in E_k \cap I_k$,

$$\begin{aligned} |B(z)\, b(z) - |b(z)|| &\leq |B(z)\, b(z) - \omega_k\, b(z)| + \left|\omega_k\, b(z) - \frac{\overline{b(z)}}{|b(z)|}\, b(z)\right| \\ &\leq |B(z) - \omega_k| + \left|\omega_k - \frac{\overline{b(z)}}{|b(z)|}\right| < 2\,\varepsilon_k\,. \end{aligned} \tag{15}$$

The first summand is smaller than ε_k because $z \in I_k$ and the second because $z \in E_k$. Then, for almost every $z \in E_k \cap I_k$,

$$\begin{aligned} |1 - B(z)\, b(z)| &\leq |1 - |b(z)|| + |\,|b(z)| - B(z)\, b(z)| \\ &< \rho(z) + 2\,\varepsilon_k < 3\,\rho(z)\,, \end{aligned}$$

because since $z \in E_k \cap I_k \subset T_k$ then $\varepsilon_k < \rho(z)$.

Hypothesis 4) says that there is a constant $C = C(B) > 0$ such that for almost every $z \in E$,

$$C^{-1}\,\rho(z) \leq |1 - B(z)\, b(z)|^2\,,$$

and since $T_k \subset E$ this equality holds in $E_k \cap I_k$. Therefore, for almost every $z \in E_k \cap I_k$,

$$C^{-1}\,\rho(z) \leq |1 - B(z)\, b(z)|^2 < 3^2\,\rho^2(z)\,.$$

Then $(9C)^{-1} \leq \rho$ in $E_k \cap I_k$, and since $E_k \cap I_k \subset T_k$, also $(9C)^{-1} \leq \rho \leq \varepsilon_{k-1}$, which contradicts the fact that (ε_k) tends to zero.

5) implies 3). We assume that 3) does not hold and retain the notations of the above proof. Consider the Blaschke product $-B$. For almost every $z \in E_k \cap I_k$,

$$\begin{aligned} |1 + B(z)\, b(z)| &\geq \left|\,\left|1 + b(z)\, \frac{\overline{b(z)}}{|b(z)|}\right| - \left|b(z)\, \frac{\overline{b(z)}}{|b(z)|} - b(z)\, B(z)\right|\,\right| \\ &= |1 + |b(z)| - |\,|b(z)| - b(z)\, B(z)|\,| \\ &> 1 + |b(z)| - 2\,\varepsilon_k > \frac{1}{2} \end{aligned} \tag{16}$$

if $\varepsilon_k < 1/4$ (*i.e.* for k big enough), by (15). By hypothesis there is $\varepsilon = \varepsilon(B) > 0$ such that

$$|1 + B(z)\, b(z)|^2 < \varepsilon^{-1}\, \rho(z)\,, \qquad \text{for almost every } z \in E\,.$$

In particular this holds for almost every $z \in E_k \cap I_k$, and since $\rho(z) \leq \varepsilon_{k-1}$ in this set, (16) implies

$$\frac{1}{4} \leq |1 + B(z)\, b(z)|^2 < \varepsilon^{-1}\, \rho(z) \leq \varepsilon^{-1}\, \varepsilon_{k-1}$$

for almost every $z \in E_k \cap I_k$. Again, this contradicts $\varepsilon_k \to 0$.

Clearly 6) implies 4) and 7) implies 5), so the theorem will follow if we show that 3) implies 6) and 7). If $\rho \geq \delta\, \chi_E$, then $|1 - ub|\, \chi_E \geq (\delta/2)\, \chi_E$ for every inner function u. Then,

$$\frac{\delta}{4}\, \chi_E \leq \frac{\rho}{4} \leq \frac{1 - |b|^2}{|1 - ub|^2} \leq 4\, \frac{\rho}{\delta^2} \leq \frac{4}{\delta^2}\, \chi_E\,.$$

8. Almost conformal invariance.

Lemma 8.1. *Let b be extreme and $\rho = 1 - |b|^2$. For $z_0 \in \mathbb{D}$ put $b_0 = (b - z_0)/(1 - \overline{z}_0 b)$, $\rho_0 = 1 - |b_0|^2$, $\sigma_{b_0} = \rho_0/|1 - b_0|^2$ and $\lambda = (1 + z_0)/(1 + \overline{z}_0)$. Then*

1) $\rho_0 = \rho\, \dfrac{1 - |z_0|^2}{|1 - \overline{z}_0 b|^2}\,,$

2) $1 - b_0 = (1 + z_0)\, \dfrac{1 - \overline{\lambda}\, b}{1 - \overline{z}_0\, b}\,,$

3) $\sigma_{b_0} = \dfrac{\rho_0}{|1 - b_0|^2} = \dfrac{1 - |z_0|^2}{|1 + z_0|^2}\, \dfrac{\rho}{|\lambda - b|^2}\,.$

PROOF. The above formulas follow from straightforward calculations with the following two identities (for $z \in \mathbb{C}$)

(i) $$1 - \Big|\frac{z - z_0}{1 - \overline{z}_0\, z}\Big|^2 = (1 - |z|^2)\, \frac{1 - |z_0|^2}{|1 - \overline{z}_0\, z|^2}\,,$$

(ii) $$1 - \frac{z - z_0}{1 - \overline{z}_0\, z} = \Big(\frac{1 + z_0}{1 + \overline{z}_0} - z\Big)\, \frac{1 + \overline{z}_0}{1 - \overline{z}_0\, z}\,.$$

Theorem 8.2. *Let b be extreme, $z_0 \in \mathbb{D}$ and $b_0 = (b - z_0)(1 - \overline{z}_0 b)^{-1}$. Then $\mathcal{H}(\overline{b}) = \mathcal{H}(\overline{b}_0)$ and $(1 - \overline{z}_0 b)\,\mathcal{H}(b_0) = \mathcal{H}(b)$.*

PROOF. The easy estimate

$$\frac{1 - |z_0|^2}{4} \le \frac{1 - |z_0|^2}{|1 - \overline{z}_0\, b|^2} \le \frac{1 + |z_0|}{1 - |z_0|}$$

together with Lemma 8.1.1) shows that b_0 is also an extreme point of $B(H^\infty)$, and that

$$E = \{e^{i\theta} \in \partial\mathbb{D} : \ \rho(e^{i\theta}) \ne 0\} = \{e^{i\theta} \in \partial\mathbb{D} : \ \rho_0(e^{i\theta}) \ne 0\}$$

almost everywhere. Also, if $f \in L^2(\chi_E)$, Lemma 8.1.1) implies

$$K_{\rho_0^{1/2}}(f) = K_{\rho^{1/2}}\left(f\, \frac{(1 - |z_0|^2)^{1/2}}{|1 - \overline{z}_0\, b|}\right),$$

and consequently $\mathcal{H}(\overline{b}) = \mathcal{H}(\overline{b}_0)$. Write $c = (1 - |z_0|^2)/|1 + \overline{z}_0|^2$ and $\lambda = (1 + z_0)/(1 + \overline{z}_0)$. By formula 3) of Lemma 8.1, for $z \in \mathbb{D}$,

$$\sigma_{b_0}(z) = c\, \frac{\rho(z)}{|1 - \overline{\lambda}\, b(z)|^2} = c\, \sigma_{\overline{\lambda} b}(z)\,.$$

Hence,

$$\operatorname{Re}\left(\frac{1 + b_0(z)}{1 - b_0(z)}\right) = \sigma_{b_0}(z) = c\, \sigma_{\overline{\lambda} b}(z) = c\, \operatorname{Re}\left(\frac{1 + \overline{\lambda}\, b(z)}{1 - \overline{\lambda}\, b(z)}\right).$$

Two analytic functions with the same real part must differ in an imaginary constant. Thus, there are $\gamma,\ \delta \in \mathbb{R}$ such that for $z \in \mathbb{D}$,

$$i\,\gamma = \frac{1 + b_0(z)}{1 - b_0(z)} - c\, \frac{1 + \overline{\lambda}\, b(z)}{1 - \overline{\lambda}\, b(z)} = \int_{\partial\mathbb{D}} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(e^{i\theta}) + i\,\delta\,,$$

where $d\mu(e^{i\theta}) = d\mu_{b_0}(e^{i\theta}) - c\, d\mu_{\overline{\lambda} b}(e^{i\theta})$. Since μ is a real measure, evaluating at $z = 0$ we obtain $\gamma = \delta$. The identity

$$\frac{e^{i\theta} + z}{e^{i\theta} - z} = 1 + 2 \sum_{n \ge 1} z^n\, e^{-in\theta}\,,$$

with uniform convergence of the series in $|z| \le r < 1$, now shows that $\int e^{-in\theta}\, d\mu(e^{i\theta}) = 0$ for all $n \ge 0$. Since μ is a real measure, taking complex conjugation we also obtain that $\int e^{in\theta}\, d\mu(e^{i\theta}) = 0$ for all $n \ge 1$. Then $\mu \equiv 0$ and therefore $\mu_{b_0} = c\,\mu_{\overline{\lambda}b}$. Thus, for $f \in L^2(\mu_{b_0}) = L^2(\mu_{\overline{\lambda}b})$,

$$\begin{aligned} V_{b_0}(f) &= (1-b_0)\, K_{\mu_{b_0}}(f) \\ &= (1+z_0)\frac{1-\overline{\lambda}\, b}{1-\overline{z}_0\, b}\, K_{c\mu_{\overline{\lambda}b}}(f) \\ &= \frac{1+z_0}{1-\overline{z}_0\, b}\, c\,(1-\overline{\lambda}\, b)\, K_{\mu_{\overline{\lambda}b}}(f) \\ &= \frac{1}{1-\overline{z}_0\, b}\,\frac{1-|z_0|^2}{1+\overline{z}_0}\, V_{\overline{\lambda}b}(f) \end{aligned}$$

by Lemma 8.1.2) and the equality of the measures. Thus

$$(1-\overline{z}_0\, b)\, V_{b_0}(f) = \frac{1-|z_0|^2}{(1+\overline{z}_0)}\, V_{\overline{\lambda}b}(f)\,,$$

which clearly implies that $(1-\overline{z}_0 b)\,\mathcal{H}(b_0) = \mathcal{H}(\overline{\lambda}b)$. Since $\mathcal{H}(\overline{\lambda}\, b) = \mathcal{H}(b)$, the theorem follows.

Corollary 8.3. *Let b be extreme, and denote by $\operatorname{sp}(b)$ the spectrum of b in H^∞. Then for $z_0 \ne 0$ the following conditions are equivalent.*

1) $z_0 \in \mathbb{D} \setminus \operatorname{sp}(b)\,.$
2) $(1-\overline{z}_0\, b)\,\mathcal{H}(\overline{b}) = \mathcal{H}(b)\,.$
3) $(1-\overline{z}_0\, b)^{-1} \in \mathcal{M}(b)\,.$

PROOF. 1) if and only if 2). $z_0 \in \mathbb{D} \setminus \operatorname{sp}(b)$ if and only if $b_0 = (b-z_0)/(1-\overline{z}_0 b)$ is invertible. Since b_0 is extreme, Theorem 7.1 of [13] says that b_0 is invertible if and only if $\mathcal{H}(b_0) = \mathcal{H}(\overline{b}_0)$. If this happens, Theorem 8.2 implies that $(1-\overline{z}_0 b)\,\mathcal{H}(\overline{b}) = \mathcal{H}(b)$. On the other hand, if this equality holds, then by Theorem 8.2,

$$(1-\overline{z}_0\, b)\,\mathcal{H}(\overline{b}) = \mathcal{H}(b) = (1-\overline{z}_0\, b)\,\mathcal{H}(b_0)\,.$$

Thus $\mathcal{H}(\overline{b}) = \mathcal{H}(b_0)$, and Theorem 8.2 again, shows that $\mathcal{H}(\overline{b}_0) = \mathcal{H}(b_0)\,.$

2) implies 3). $\mathcal{H}(b) \supset \mathcal{H}(\overline{b}) = (1-\overline{z}_0\, b)^{-1}\mathcal{H}(b)\,.$

3) implies 2). Let $f \in \mathcal{H}(b)$; then $(1-\overline{z}_0\, b)^{-1} f = g \in \mathcal{H}(b)$. Therefore

$$g - \overline{z}_0\, b\, g = (1-\overline{z}_0\, b)\, g = f\,.$$

Since $g \in \mathcal{H}(b)$, we have that $b\,g$ must be in $\mathcal{H}(b)$; but for a function $g \in H^2$ it is well known that $b\,g \in \mathcal{H}(b)$ if and only if $g \in \mathcal{H}(\bar{b})$ (see Section 1). Hence,

$$f = (1 - \overline{z}_0\, b)\, g \in (1 - \overline{z}_0\, b)\, \mathcal{H}(\bar{b})\,.$$

For $z_0 = 0$ condition 3) is trivial. The equivalence of 1) and 2) for this case is proved in Theorem 7.1 of [13]. More can be said now. Suppose that $z_0 \in \mathbb{D} \setminus \mathrm{sp}\,(b)$, then b_0 and b_0^{-1} are multipliers of $\mathcal{H}(b_0)$. Since by Theorem 8.2 $\mathcal{M}(b_0) = \mathcal{M}(b)$, we also have $b_0^{-1} \in \mathcal{M}(b)$. Besides, by Corollary 8.3 $(1 - \overline{z}_0\, b)^{-1} \in \mathcal{M}(b)$, then $b_0^{-1}(1 - \overline{z}_0\, b)^{-1} = (b - z_0)^{-1} \in \mathcal{M}(b)\,.$

Corollary 8.4. *Let b be extreme. If u is an inner function such that* $\mathrm{sp}\,(ub)$ *is not the whole closed disc, then $\mathcal{M}(ub) = \mathcal{M}(b) = \mathcal{M}(\bar{b})$.*

Proof. Since $\mathrm{sp}\,(ub)$ is compact, there must be some point $z_0 \neq 0$ such that $z_0 \in \mathbb{D} \setminus \mathrm{sp}\,(ub)$. By Corollary 8.3 $(1 - \overline{z}_0 u\, b)\mathcal{H}(\overline{ub}) = \mathcal{H}(ub)$; then clearly $\mathcal{M}(\overline{ub}) = \mathcal{M}(ub)$. The assertion now follows from Section 1, taking into account that $\mathcal{H}(\overline{ub}) = \mathcal{H}(\bar{b})\,.$

9. Continuity conditions.

Theorem 9.1. *Let $b \in B(H^\infty)$ with $d\mu_b = \sigma\, d\theta/2\pi + d\mu_S$. If $\varepsilon > 0$ and $0 < r < 1$, then*

$$\|\mu_S\| = \lim_{\varepsilon \to 0} \lim_{r \to 1} \int_{|1 - b(e^{i\theta})| < \varepsilon} \frac{1 - r^2\, |b(e^{i\theta})|^2}{|1 - r\, b(e^{i\theta})|^2}\, \frac{d\theta}{2\pi}\,.$$

Proof. Since the Poisson kernel

$$P_r(z) = \frac{1}{2\pi}\, \frac{1 - r^2\, |z|^2}{|1 - r\, z|^2}$$

is harmonic (for $z \in \mathbb{D}$ and $0 \le r \le 1$), then

$$P_r(b(z)) = \frac{1}{2\pi}\, \frac{1 - r^2\, |b(z)|^2}{|1 - r\, b(z)|^2}$$

is harmonic. Thus

$$\int_0^{2\pi} \frac{1 - r^2\,|b(e^{i\theta})|^2}{|1 - r\,b(e^{i\theta})|^2}\,\frac{d\theta}{2\pi} = \frac{1 - r^2\,|b(0)|^2}{|1 - r\,b(0)|^2}\,,$$

which tends to $(1 - |b(0)|^2)/|1 - b(0)|^2$ when $r \to 1$. By formula (1) of Section 1, this is the norm of μ_b. On the other hand, for $\varepsilon > 0$,

$$\lim_{r\to 1} \int_{|1-b(e^{i\theta})|\geq\varepsilon} P_r(b(e^{i\theta}))\,d\theta = \int_{|1-b(e^{i\theta})|\geq\varepsilon} P_1(b(e^{i\theta}))\,d\theta\,,$$

because the integrand converges uniformly in $|1 - b(e^{i\theta})| \geq \varepsilon$. Since $P_1 \circ b = \sigma/2\pi \in L^1$, the last integral tends to $\int_0^{2\pi} \sigma(e^{i\theta})\,d\theta/2\pi = \|\sigma\,d\theta/2\pi\|$ when ε tends to 0. Substracting, we obtain

$$\begin{aligned}
\|\mu_S\| &= \|\mu_b\| - \|\sigma\,d\theta/2\pi\| \\
&= \lim_{r\to 1} \int_0^{2\pi} P_r(b(e^{i\theta}))\,d\theta - \lim_{\varepsilon\to 0}\lim_{r\to 1} \int_{|1-b(e^{i\theta})|\geq\varepsilon} P_r(b(e^{i\theta}))\,d\theta \\
&= \lim_{\varepsilon\to 0}\lim_{r\to 1} \int_{|1-b(e^{i\theta})|<\varepsilon} P_r(b(e^{i\theta}))\,d\theta\,.
\end{aligned}$$

Corollary 9.2. *If $(1 - b)^{-1} \in L^2$, then μ_b is absolutely continuous.*

PROOF. Since $|1 - r\,b(e^{i\theta})| \geq |1 - b(e^{i\theta})|/2$ almost everywhere with respect to $d\theta$, then

$$\frac{1 - r^2\,|b(e^{i\theta})|^2}{|1 - r\,b(e^{i\theta})|^2} \leq \frac{4}{|1 - b(e^{i\theta})|^2} \in L^1\,.$$

Hence, by the dominated convergence theorem,

$$\lim_{r\to 1} \int_{|1-b(e^{i\theta})|<\varepsilon} P_r(b(e^{i\theta}))\,d\theta = \int_{|1-b(e^{i\theta})|<\varepsilon} \sigma(e^{i\theta})\,\frac{d\theta}{2\pi}\,,$$

and since $\sigma \in L^1$, the last integral tends to 0 when $\varepsilon \to 0$.

Notice that the above result also holds for b nonextreme. We keep assuming that b is not an inner function.

Theorem 9.3. *Let b be an extreme point of $B(H^\infty)$, continuous on $\partial\mathbb{D}$. Then $\mathcal{M}(b) = \mathcal{M}(\overline{b})$.*

PROOF. Factorize $b = ub_0$, where u is the inner factor of b and b_0 is its outer factor. Since b is continuous, b_0 is continuous (see [11 , p. 69]); and $\overline{u}\, b_0$ is also continuous. It is well known ([9, IV]) that for a function f continuous on $\partial\mathbb{D}$ there is a unique best approximation $g \in H^\infty$, and that $|f(e^{i\theta}) - g(e^{i\theta})| = \operatorname{dist}\{f, H^\infty\}$ for almost every $e^{i\theta} \in \partial\mathbb{D}$. Therefore, $\operatorname{dist}\{\overline{u}\, b_0, H^\infty\} < 1$, because otherwise since $\|\overline{u}\, b_0\| = 1$, the best approximation for $\overline{u} b_0$ in H^∞ must be the trivial function. So $|\overline{u}\, b_0| = 1$ almost everywhere, which is not the case. Thus, $\operatorname{dist}\{b_0, uH^\infty\} < 1$ and then Theorem 13.5 of [13] implies $\mathcal{M}(ub_0) = \mathcal{M}(b_0)$. Now it is clear from the equality $\mathcal{H}(\overline{ub_0}) = \mathcal{H}(\overline{b}_0)$ that we can assume $b = b_0$ outer.

Then b has square roots, and we will show that $\mathcal{M}(b) = \mathcal{M}(b^{2^n})$ for every integer n. We only have to prove that $\mathcal{M}(b) = \mathcal{M}(b^2)$. By Section 1, $\mathcal{H}(b^2) = \mathcal{H}(b) + b\,\mathcal{H}(b)$, thus $\mathcal{M}(b) \subset \mathcal{M}(b^2)$. Let $m \in \mathcal{M}(b^2)$ and $f \in \mathcal{H}(b)$. Then $bf \in \mathcal{H}(b^2)$ and therefore $m\, bf = g_1 + bg_2$ with $g_1,\ g_2 \in \mathcal{H}(b)$. Hence,

$$g_1 = b\,(mf - g_2) \in bH^2 \cap \mathcal{H}(b) = b\,\mathcal{H}(\overline{b}) \subset b\,\mathcal{H}(b)\,.$$

Thus $b\,mf = g_1 + b\,g_2 \in b\,\mathcal{H}(b)$, that is, $mf \in \mathcal{H}(b)$. Also, $\mathcal{H}(\overline{b}) = \mathcal{H}(\overline{b}^{2^n})$ for every integer n. As before, it is enough to take $n = 1$. This immediately follows from the inequalities

$$1 - |b|^2 \le 1 - |b^2|^2 \le 2\,(1 - |b^2|)$$

and the Cauchy transform representations of $\mathcal{H}(\overline{b})$ and $\mathcal{H}(\overline{b}^2)$.

It will therefore be enough to prove that there is an integer n such that $\mathcal{M}(b^{2^n}) = \mathcal{M}(\overline{b}^{2^n})$. Since the argument of b is continuous on the compact set $F = \{z :\ |z| \le 1,\ |b(z)| = 1\}$, there is some negative integer n such that the argument of $b^{2^n}(z)$ lives in $(-\pi/4, \pi/4)$ for $z \in F$. The continuity of b^{2^n} implies that for $\lambda \in \partial\mathbb{D}$ with $\operatorname{Re}\lambda < 0$,

$$|1 - \overline{\lambda}\, b^{2^n}(z)| \ge \delta > 0\,, \qquad \text{for all } z,\ |z| \le 1\,. \tag{17}$$

Therefore $|1 - \overline{\lambda}\, b^{2^n}(e^{i\theta})|^{-1} \in L^2$, and Corollary 9.2 implies that $\mu_{\overline{\lambda} b}$ is absolutely continuous, say $d\mu_{\overline{\lambda} b} = \sigma\, d\theta/2\pi$. Also, if $\rho = 1 - |b^{2^n}|^2$, condition (17) implies that the spaces $K_{\rho^{1/2}}(L^2(\chi_E))$ and $K_{\sigma^{1/2}}(L^2(\chi_E))$

coincide. Then,

$$\begin{aligned}(1-\overline{\lambda}\, b^{2^n})\,\mathcal{H}(\overline{b}^{2^n}) &= (1-\overline{\lambda}\, b^{2^n})\, K_{\rho^{1/2}}(L^2(\chi_E)) \\ &= (1-\overline{\lambda}\, b^{2^n})\, K_{\sigma^{1/2}}(L^2(\chi_E)) = \mathcal{H}(\overline{\lambda}\, b^{2^n}) = \mathcal{H}(b^{2^n})\,.\end{aligned}$$

Hence, $\mathcal{M}(b^{2^n}) = \mathcal{M}(\overline{b}^{2^n})$ and the theorem follows.

The argument to reduce the preceding theorem to the case in which b is an outer function is by D. Sarason (personal communication). My original proof of this fact was slightly more complicated.

The equality $\mathcal{M}(b^{2^n}) = \mathcal{M}(\overline{b}^{2^n})$ for n a suitable negative integer can be also proved using Corollary 8.4. Of course, Theorem 9.3 implies that the preceding algebras coincide for all integers n.

10. Inner factors in $\mathcal{H}(\overline{b}) + \mathbb{C}$.

Denote by $\mathcal{H}(\overline{b})_+$ the linear space $\mathcal{H}(\overline{b}) + \mathbb{C}$. The map $a \mapsto a_*$ defines a conjugation on $\mathcal{H}(\overline{b})_+$, where, for $a = K_\rho(q) + c \in \mathcal{H}(\overline{b})_+$, the function a_* is defined by $a_*(z) = -K_\rho(\overline{q})(z) + K_\rho(\overline{q})(0) + \overline{c} = \overline{a(1/\overline{z})}$ (see Section 1).

Theorem 10.1. *Let $a \in \mathcal{H}(\overline{b})_+$ and let u be an inner function. Then $ua \in \mathcal{H}(\overline{b})_+$ if and only if a_* is in uH^2. In this case, $(ua)_* = a_*/u$.*

PROOF. We can assume that u is not a constant function. If $a \in \mathcal{H}(\overline{b})_+$, then $a = K_\rho(q) + c$, with $q \in L^2(\rho)$ and $c \in \mathbb{C}$.

Sufficiency. The inner boundary function of $a - \overline{a}_*$ is $q\,\rho$, so the boundary function of $ua - u\overline{a}_*$ is $u\,q\,\rho$. By hypothesis a_*/u is in H^2, so $u(z)\,a(z) - (\overline{a_*(z)/u(z)})$ is harmonic, and since $\overline{u(z)}^{-1}$ and $u(z)$ have the same nontangential limit almost everywhere in $\partial\mathbb{D}$, the boundary function of $ua - (\overline{a_*/u})$ is also $u\,q\,\rho$. Hence, Lemma 2.1 gives

$$u(z)\,a(z) = K_\rho(uq)(z) + \overline{(a_*/u)(0)} \in \mathcal{H}(\overline{b})_+$$

and

$$a_*(z)/u(z) = \overline{K_\rho(uq)(1/\overline{z})} + (a_*/u)(0)\,.$$

Thus, $a_*(z)/u(z) = (ua)_*$.

Necessary condition. If $ua \in \mathcal{H}(\overline{b})_+$, then also $d = (ua)_* \in \mathcal{H}(\overline{b})_+$. Further, $d_* = ua \in uH^2$; so by the other implication of the theorem, $ud \in \mathcal{H}(\overline{b})_+$ and

$$(ud)_* = d_*/u = ua/u = a\,.$$

Hence, $a_* = ud \in u\mathcal{H}(\overline{b})_+ \subset uH^2$.

Corollary 10.2. *If m belongs to any of the algebras $\mathcal{M}(b)$, $\mathcal{M}(\overline{b})$ or $K^\infty(\rho)$, and u is an inner function, then $u\,m$ belongs to the same algebra as m if and only if $m_* \in uH^2$.*

PROOF. The necessary condition is immediate from the above theorem, since all the algebras are contained in $\mathcal{H}(\overline{b})_+$. For the other implication, the argument for $\mathcal{M}(b)$ and $\mathcal{M}(\overline{b})$ is the same. So, suppose that $m \in \mathcal{M}(b)$, $m_* \in uH^2$, and take $a \in \mathcal{H}(b)$. Since $m_* \in \mathcal{M}(b)$, then $m_*a \in \mathcal{H}(b)$. Thus, $(m_*/u)a = T_{\overline{u}}(m_*a) \in \mathcal{H}(b)$. That is, $m_*/u \in \mathcal{M}(b)$ and then $(m_*/u)_*$ also belongs to $\mathcal{M}(b)$. Besides, $(m_*/u)_* = um$ by Theorem 10.1.

If $m \in K^\infty(\rho) \subset \mathcal{H}(\overline{b})_+$ and $m_* \in uH^2$, then $m_* \in uH^2 \cap H^\infty = uH^\infty$. By Theorem 10.1, $u\,m \in \mathcal{H}(\overline{b})_+ \cap H^\infty$ and $(u\,m)_* = m_*/u \in \mathcal{H}(\overline{b})_+ \cap H^\infty$. Thus, $u\,m$ and $(u\,m)_*$ belong to H^∞, which means that $(u\,m)(z)$ is bounded for all $z \in \mathbb{C} \setminus \partial\mathbb{D}$. Consequently $u\,m \in K^\infty(\rho)$.

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On the Sphere Problem

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1. Introduction and Statement of Main Results.

One of the oldest problems in analytic number theory consists of counting points with integer coordinates in the d-dimensional ball. It is very easy to find a main term for the counting function, but the size of the error term is difficult to estimate. Namely the problem is to prove the approximate formula

$$\#\{\mathbf{x} \in \mathbb{Z}^d : \ \|\mathbf{x}\| \le R\} = \frac{\pi^{d/2}}{\Gamma\Big(\dfrac{d+2}{2}\Big)} \, R^d + O\big(R^\theta\big) \, ,$$

for any $R \ge 1$ with θ as small as possible. Let θ_d be the least number such that the above approximation is true with any $\theta > \theta_d$.

In dimension $d = 1$, $\theta_1 = 0$ follows trivially. The problem is also settled when $d \ge 4$. In this case, starting from a classical formula for the number of representations as sum of four squares and using elementary arguments, it can be proved $\theta_d = d - 2$ (see for instance [Fr]).

The evaluation of θ_d in the remaining cases $d = 2$, $d = 3$ is an outstanding problem in number theory and intractable by the methods of nowadays. The conjectures (supported by some mean results) are $\theta_2 = 1/2$, $\theta_3 = 1$.

The two dimensional problem is called "the circle problem" and it has a long history coming back to Gauss, who proved $\theta_2 \le 1$. In this century several authors gave some improvements of this result, very often creating new methods in the theory of exponential sums; the best result so far is $\theta_2 \le 46/73$ due to Huxley [Hu].

The three dimensional case, the so called "sphere problem", is also closely related with the work of Gauss about the average of the class number for negative discriminants (see Art. 302 of [Ga]). The literature about the sphere problem is not so wide as in the two dimensional case, although it seems more interesting because it has profound relations with others topics in number theory: class number, L-functions, etc. The best result until now was $\theta_3 \le 4/3$ due to Chen [Ch] and Vinogradov [Vi]. The purpose of this paper is to improve this bound ($\theta_3 \le 29/22$, see Theorem 1.1).

Before stating our main theorem we shall introduce some notation.

First of all we define $r_3(n)$, for a positive integer n, to be the number of representations as sum of three squares

$$r_3(n) = \#\{(n_1, n_2, n_3) \in \mathbb{Z}^3 :\ n_1^2 + n_2^2 + n_3^2 = n\}$$

and $R_3(n)$ to be the number of primitive representations

$$R_3(n) = \#\{(n_1, n_2, n_3) \in \mathbb{Z}^3 :\ \gcd(n_1, n_2, n_3) = 1,\ n_1^2 + n_2^2 + n_3^2 = n\}.$$

These two functions are related by the formula

$$r_3(n) = \sum_{d^2 \mid n} R_3\left(\frac{n}{d^2}\right). \tag{1.1}$$

There are also the following relations with other arithmetic quantities (see [Gr]):

$$R_3(n) = c_n\, h(-4n) = \frac{1}{\pi}\, c_n \sqrt{n}\, L(1, \chi_n), \qquad n > 1, \tag{1.2}$$

where $h(-4n)$ is the class number for the negative discriminant $-4n$, $L(s, \chi_n)$ is the L-function associated with the character

$$\chi_n(m) = (-4n/m)$$

and

$$c_n = \begin{cases} 0, & \text{if } n \equiv 0, 4, 7 \pmod 8, \\ 16, & \text{if } n \equiv 3 \pmod 8, \\ 24, & \text{if } n \equiv 1, 2, 5, 6 \pmod 8. \end{cases}$$

In order to write some formulas in a more symmetric way, we normalize $r_3(n)$ by

$$l(n) = r_3(n)\, n^{-1/2}.$$

The previous formulas show that $l(n)$ is very similar to $L(1,\chi_n)$. In fact our method is based in that we can consider a sum of $l(n)$ as a lattice point problem or as a sum of character sums. This duality allows us to employ, in different ranges, Poisson's summation or Burgess' inequality [Bu].

Let $S(R)$ be the number of lattice points in the sphere of radius R and $S(R,H)$ be the number of lattice points between the spheres of radius R and $R+H$, *i.e.*

$$S(R) = \sum_{n\le R^2} r_3(n) \qquad \text{and} \qquad S(R,H) = \sum_{R^2<n\le (R+H)^2} r_3(n)\,.$$

Our main results are the following:

Theorem 1.1. *For any $R\ge 1\ge H>0$ and $\varepsilon>0$ we have*

$$S(R) = \frac{4\pi}{3}\,R^3 + O\big(R^{29/22+\varepsilon}\big) \tag{1.3}$$

and

$$S(R,H) = 4\pi H R^2 + O\big(\big(H^{7/8}R^{15/8} + H^{2/3}R^{83/48} + R\big)R^{\varepsilon}\big)\,. \tag{1.4}$$

Remark. One should be able to establish an asymptotic formula for the mean value of $h(-n)$ with an error term as good as in Theorem 1.1. We intend to deal with this problem in another occasion.

2. A summation formula for $l(n)$.

In this section we shall establish

Lemma 2.1. *If $f\in C_0^3\big([0,\infty)\big)$ with $f''(0)=0$ then*

$$\sum_{n=1}^{\infty} l(n)\,f(\sqrt{n}) = 4\pi\int_0^{\infty} r\,f(r)\,dr - f'(0) + \sum_{n=1}^{\infty} l(n)\,\widetilde{f}(\sqrt{n})\,,$$

where $\widetilde{f}$ is the sine Fourier transform

$$\widetilde{f}(\xi) = 2\int_0^{\infty} f(x)\,\sin(2\pi x\xi)\,dx\,.$$

PROOF. By our hypothesis on f, the function

$$g(x,y,z)=\frac{f(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}$$

can be extended to a C_0^2 function in $\mathbb{R}^3$ and then by the classical Poisson summation formula we get

$$f'(0)+\sum_{n=1}^{\infty} l(n)\, f(\sqrt{n}) = \sum_{n_1,n_2,n_3} \widehat{g}(n_1,n_2,n_3)\,.$$

The Fourier transform of a radial function is radial, therefore we can suppose that the point (n_1,n_2,n_3) is on the z-axis, in which case by a change to spherical coordinates it follows easily that

$$\widehat{g}(n_1,n_2,n_3)=\frac{\widetilde{f}(\sqrt{n_1^2+n_2^2+n_3^2})}{\sqrt{n_1^2+n_2^2+n_3^2}}\,, \qquad \text{if } \; n_1^2+n_2^2+n_3^2\neq 0$$

and

$$\widehat{g}(0,0,0)=4\pi\int_0^\infty r\, f(r)\,dr\,,$$

hence the proof is complete.

3. Exponential sums over lattice points in spheres.

If we had chosen $f(x)=x$ for $0<x<R$ in Lemma 2.1 (actually one has to make some smoothing) then we could infer that

$$S(R)=\frac{4\pi}{3}\,R^3+O\big(R^{3/2+\varepsilon}\big)\,,$$

or equivalently $\theta_3\le 3/2$. This result was first proved by Landau [La], other better results were established by Walfisz [Wa1], Fomenko [Fo], Chen [Ch] and Vinogradov in several papers culminating in [Vi]. As we quoted in the introduction, the best exponent in the error term until now was $4/3+\varepsilon$ due to Chen and Vinogradov (in fact Vinogradov replaced R^ε by a logarithmic factor). These results require non-trivial estimates for exponential sums of the type

$$V_N(R)=\sum_{n\asymp N} r_3(n)\, e(R\sqrt{n})$$

where here and thereafter we write $n \asymp N$ to say that $c_1 N < n < c_2 N$ with some unspecified constants c_1, c_2, not necessarily the same ones in each ocurrence.

We prove in this section (by arguments similar to those in Chen [Ch] and Vinogradov [Vi])

Lemma 3.1. *For $R > 1$ we have*

$$V_N(R) \ll N^{5/4+\varepsilon} + N^{\varepsilon} \min\big\{R^{3/8}N^{15/16} + R^{1/8}N^{17/16}, \\ R^{7/24}N^{49/48} + R^{5/24}N^{53/48}\big\}.$$

Proof. We shall deduce that

$$V_N(R) \ll \Big|\sum_{a,b,c} e\big(R\sqrt{a^2+b^2+c^2}\big)\Big| \\ \ll N^{\varepsilon} \sum_{n \asymp N} \Big|\sum_{c \ll \sqrt{N}} e(\theta c)\, e\big(R\sqrt{n+c^2}\big)\Big|$$

for some $\theta \in \mathbb{R}$. To prove the above we select the smallest variable, say c, and apply [Gr-Ko, Lemma 7.3] for the sum over c in order to remove the involved summation conditions. From the variables a, b we create a new variable $n = a^2 + b^2$. Splitting the range of the inner sum into segments of lenght $N^{1/2-\varepsilon}$ by Cauchy's inequality,

$$V_N^2(R) \ll N^{1+\varepsilon} \sum_{c_1, c_2} \Big|\sum_{n \asymp N} e\Big(R\big(\sqrt{n+c_1^2} - \sqrt{n+c_2^2}\big)\Big)\Big|$$

where c_1, c_2 are restricted by $c_1, c_2 < N^{1/2}$ and $|c_1 - c_2| < N^{1/2-\varepsilon}$. Hence for a suitable $D < N^{1-\varepsilon}$ we get

$$V_N^2(R) \ll N^{5/2+\varepsilon} + N^{1+\varepsilon} \sum_{y \asymp D} \Big|\sum_{x \asymp N} e\big(f(x,y)\big)\Big|$$

where

$$f(x,y) = R\left(\sqrt{x} - \sqrt{x+y}\right).$$

If $D \ll N^{3/2}R^{-1}$ the innermost sum is $\ll N^{3/2}R^{-1}D^{-1}$ by [Gr-Ko, Theorem 2.1] of so it contributes $\ll N^{5/2+\varepsilon}$.

If $D \gg N^{3/2}R^{-1}$ we apply the B-process of the one-dimensional van der Corput's method (Poisson sumation and stationary phase), more precisely [Gr-Ko, Lemma 3.6] of with $F = RDN^{-1/2}$ getting

$$\begin{aligned} V_N^2(R) &\ll N^{5/2+\varepsilon} + R^{-1/2}\,D^{1/2}\,N^{9/4+\varepsilon} \\ &\quad + R^{-1/2}\,D^{-1/2}\,N^{9/4+\varepsilon}\sum_{y\asymp D}\Big|\sum_{x\asymp U} e\big(g(x,y)\big)\Big| \end{aligned} \tag{3.1}$$

where $U = RDN^{-3/2}$,

$$g(x,y) = f(\alpha(x,y),y) - x\,\alpha(x,y) \tag{3.2}$$

and $\alpha(x,y)$ is the implicit function defined by

$$f_x(\alpha(x,y),y) = x\,. \tag{3.3}$$

The middle term in (3.1) comes from the error term in [Gr-Ko, Lemma 3.6].

Next, by Cauchy's inequality and dividing into dyadic intervals, there exists $1 \le T < U$ such that

$$V_N^4(R) \ll N^{5+\varepsilon} + R^{-1}DN^{9/2+\varepsilon} + R^{-1}N^{9/2+\varepsilon}\,V_{UTD} \tag{3.4}$$

where

$$V_{UTD} \ll \sum_{x\asymp U}\sum_{z\asymp T}\Big|\sum_{y\asymp D} e\big(G(x,y,z)\big)\Big|$$

with

$$G(x,y,z) = g(x+z,y) - g(x,y)\,.$$

Now we apply two well known van der Corput's estimates (see [Gr-Ko, Theorem 2.2] and [Ti, Theorem 5.11] to obtain

$$V_{UTD} \ll U\,T\,\min\big\{D\,{\lambda_2}^{1/2} + {\lambda_2}^{-1/2}, D\,{\lambda_3}^{1/6} + D^{1/2}\,{\lambda_3}^{-1/6}\big\} \tag{3.5}$$

where $\lambda_2 \asymp |G_{yy}|$ and $\lambda_3 \asymp |G_{yyy}|$. Here, by the mean value theorem $\lambda_2 \asymp T|g_{xyy}|$ and $\lambda_3 \asymp T|g_{xyyy}|$. On the other hand, by (3.2) and (3.3) $g_x = -\alpha$, hence $\lambda_2 \asymp T|\alpha_{yy}|$ and $\lambda_3 \asymp T|\alpha_{yyy}|$.

It remains to estimate the partial derivatives α_{yy}, α_{yyy}. By the definition of f and (3.3) we have $\alpha^{-1/2} - (\alpha+y)^{-1/2} = 2\,xR^{-1}$. Differenciating with respect to y and eliminating $\alpha + y$, we get

$$\frac{1}{\alpha_y} = \Big(1 + \frac{y}{\alpha}\Big)^{3/2} - 1 \qquad \text{and} \qquad \frac{\alpha_y}{\alpha_y + 1} = (1-h)^3$$

with
$$h = \frac{2\,x\,\alpha^{1/2}}{R} \asymp \frac{D}{N} \ll N^{-\varepsilon}\,.$$
The first formula (recalling $y \asymp D$, $\alpha \asymp N$) implies $\alpha_y \asymp ND^{-1}$ and differentiating the second formula we obtain
$$-\alpha_{yy} = 6\,x^2\,R^{-2}\,h^{-1}\,(1-h)^2\,\alpha_y\,(\alpha_y+1)^2\,.$$
Using $\alpha_y = (1-h)^3/(1-(1-h)^3)$ we get
$$-\alpha_{yy} = 6\,x^2\,R^{-2}\,h^{-1}\,(1-h)^5\,(1-(1-h)^3)^{-3}\,.$$
Since $h \asymp DN^{-1}$ and $x \asymp U = R\,DN^{-3/2}$, this formula gives $|\alpha_{yy}| \asymp ND^{-2}$. Differentiating again
$$\alpha_{yyy} = \frac{h^2\,(2-h)}{3\,(1-h)^6}\,(3-3h+h^2)^2\,(5h^2-6h+6)\,\alpha_{yy}^2\,\alpha_y$$
whence $\alpha_{yyy} \asymp ND^{-3}$. From the above estimates for the partial derivatives we conclude that
$$\lambda_2 \asymp T\,ND^{-2} \qquad \text{and} \qquad \lambda_3 \asymp T\,ND^{-3}\,.$$
Substituting in (3.4) and (3.5) we have
$$\begin{aligned} V_N^4(R) &\ll N^{5+\varepsilon} + R^{-1}D\,N^{9/2+\varepsilon} \\ &\quad + N^{\varepsilon}R^{-1}U \min\big\{T^{3/2}N^5 + T^{1/2}D\,N^4, \\ &\qquad\qquad T^{7/6}D^{1/2}N^{14/3} + T^{5/6}D\,N^{13/3}\big\}\,. \end{aligned}$$
Finally, recalling that $U = R\,DN^{-3/2}$, $T < R\,DN^{-3/2}$ and $N^{3/2}R^{-1} \ll D < N$ the lemma follows.

4. A character sum estimate.

The objective of this section is to proof the following character sum estimate

Lemma 4.1. *For $1 < K < N^{1/2}$ and α_n , β_n arbitrary complex numbers*
$$\sum_{N<n\le N+K}\ \sum_{m\asymp M} \alpha_n\,\beta_m \left(\frac{n}{m}\right)$$
$$\ll \|\alpha\|\,\|\beta\|\,(K^{3/8}M^{1/2} + K^{1/2}M^{1/4}N^{3/64})\,(MN)^{\varepsilon}\,.$$

As a corollary it can be obtained the following L-function estimate

Corollary 4.2. *If* $1 < K < N^{1/2}$

$$\sum_{\substack{N<n\le N+K \\ n\equiv\nu\ (8)}} L(1,\chi_n) = \frac{3\,\zeta(2)}{28\,\zeta(3)}\,K + O\big(K^{7/8}N^{\varepsilon} + K^{2/3}N^{1/32+\varepsilon}\big)\,.$$

PROOF OF LEMMA 4.1. Dividing the range of summation into dyadic intervals, it is enough to estimate

$$S_{KM} = \sum_{k\asymp K}\sum_{m\asymp M} \alpha_{N+k}\,\beta_m\Big(\frac{N+k}{m}\Big)\,.$$

By Cauchy's inequality

$$S_{KM}^2 \ll \|\alpha\|^2 \sum_{k\asymp K}\Big|\sum_{m\asymp M}\beta_m\Big(\frac{N+k}{m}\Big)\Big|^2.$$

Again by Cauchy's inequality and interchanging the order of summation

$$\begin{aligned} S_{KM}^4 &\ll \|\alpha\|^4\,K\sum_{k\asymp K}\Big|\sum_{m\asymp M}\beta_m\Big(\frac{N+k}{m}\Big)\Big|^4 \\ &= \|\alpha\|^4\,K\sum_{\substack{m_1,m_2\\ m_3,m_4}}\beta_{m_1}\,\beta_{m_2}\,\overline{\beta}_{m_3}\,\overline{\beta}_{m_4}\sum_{k\asymp K}\Big(\frac{N+k}{m_1\,m_2\,m_3\,m_4}\Big)\,. \end{aligned}$$

Finally we apply Cauchy's inequality once more and put

$$h = m_1\,m_2\,m_3\,m_4$$

getting

$$\begin{aligned} S_{KM}^8 &\ll \|\alpha\|^8\,\|\beta\|^8\,K^2\,M^{\varepsilon}\sum_{h\asymp M^4}\Big|\sum_{k\asymp K}\Big(\frac{N+k}{h}\Big)\Big|^2 \\ &\ll \|\alpha\|^8\,\|\beta\|^8\,K^2\,M^{\varepsilon}\Big(KM^4 + \sum_{\substack{k_1,k_2\asymp K\\ k_1\neq k_2}}\sum_{h\asymp M^4}\Big(\frac{(N+k_1)\,(N+k_2)}{h}\Big)\Big)\,. \end{aligned}$$

Notice that $k_1 \neq k_2$ implies $(N+k_1)\,(N+k_2)$ is not a square because $k_1, k_2 < N^{1/2}$. Applying the Burgess bound [Bu] to the innermost character sum we get

$$S_{KM}^8 \ll \|\alpha\|^8\,\|\beta\|^8\,K^2\,\big(KM^4 + K^2(M^4)^{1/2}(N^2)^{3/16}\big)\,(MN)^{\varepsilon}$$

and this concludes the proof.

Proof of Corollary 4.2. By the Polya-Vinogradov inequality we truncate $L(1,\chi_n)$ to get

$$\sum_{\substack{N<n\le N+K\\ n\equiv\nu\ (8)}} L(1,\chi_n) = \sum_{\substack{N<n\le N+K\\ n\equiv\nu\ (8)}} \sum_{m<N} \frac{1}{m}\left(\frac{-4n}{m}\right) + O(N^\varepsilon)\,.$$

We shall consider separately the contributions of square and non-square m's.

The squares contribute

$$\sum_{\substack{N<n\le N+K\\ n\equiv\nu\ (8)}} \sum_{(k,4n)=1} \frac{1}{k^2} = \zeta(2) \sum_{\substack{N<n\le N+K\\ n\equiv\nu\ (8)}} \sum_{k|4n} \frac{\mu(k)}{k^2} = \frac{3\,\zeta(2)}{28\,\zeta(3)}\,K + O(1)\,.$$

Therefore

$$\sum_{\substack{N<n\le N+K\\ n\equiv\nu\ (8)}} L(1,\chi_n) = \frac{3\,\zeta(2)}{28\,\zeta(3)}\,K + W + O(N^\varepsilon)$$

where W denotes the contribution of the non-squares terms, *i.e.*

$$W = \sum_{\substack{N<n\le N+K\\ n\equiv\nu\ (8)}} \sum_{\substack{m<N\\ m\ne k^2}} \frac{1}{m}\left(\frac{-4n}{m}\right)\,.$$

Dividing the range of summation of W into dyadic intervals it is enough to estimate sums of the type

$$W_{KM} = M^{-1} \sideset{}{'}\sum_{k\le K} \sideset{}{''}\sum_{m\asymp M} \left(\frac{N+k}{m}\right)$$

for some $M\le N$ where the outer summation restricts k to a fixed arithmetic progression mod 8 and the inner summation restricts m to non-square odd integers belonging to a fixed arithmetic progression mod 4. Applying Polya-Vinogradov inequality for each character sum in k gives

$$W_{KM} \ll M^{1/2+\varepsilon}\,.$$

On the other hand by Lemma 4.1

$$W_{KM} \ll (K^{7/8} + KM^{-1/4}N^{3/64})\,N^\varepsilon\,.$$

From both estimates it follows that

$$W_{KM} \ll K^{7/8}N^{\varepsilon} + \min\left\{KM^{-1/4}N^{3/64}, M^{1/2}\right\}N^{\varepsilon}\,.$$

Choosing $M = K^{4/3}N^{1/16}$ the corollary is proved.

5. Proof of Theorem 1.1.

First we shall derive (1.4) from Corollary 4.2. By (1.1) and (1.2)

$$\begin{aligned} S(R,u) &= \frac{1}{\pi}\sum_{R<\sqrt{n}\le R+u}\sqrt{n}\sum_{d^2|n}\frac{c_{n/d^2}}{d}\,L(1,\chi_{n/d^2}) + O(1) \\ &= \frac{1}{\pi}\sum_{d\le R+u}\;\sum_{R/d<\sqrt{n}\le (R+u)/d} c_n\sqrt{n}\,L(1,\chi_n) + O(1)\,, \end{aligned}$$

where $O(1)$ is introduced to correct the contribution of terms with $d^2 = n$. This expression and summation by parts in Corollary 4.2 (or the trivial estimate if it does not apply) prove (1.4).

Now we proceed to prove (1.3). We shall not deal with $S(R)$ directly but first smooth by means of the following function

$$f(x) = \begin{cases} x\,, & \text{if } x\in[0,R]\,,\\ R\,(R+H-x)/H\,, & \text{if } x\in[R,R+H]\,,\\ 0\,, & \text{otherwise,}\end{cases}$$

more precisely we consider

$$S_f(R) = \sum_n l(n)\,f(\sqrt{n})\,.$$

This sum exceeds $S(R)$ by

$$S_f(R,H) = \sum_{R<\sqrt{n}\le R+H} l(n)\,f(\sqrt{n})\,,$$

so we have

$$S(R) = S_f(R) - S_f(R,H)\,. \tag{5.1}$$

The smoothed sum $S_f(R)$ will be treated by exponential sums from Section 3 and the short sum $S_f(R,H)$ will be treated by character sums from Section 4.

By Lemma 2.1 we have

$$S_f(R) = \frac{4\pi}{3}\, R^3 + 2\pi H R^2 + \frac{2\pi}{3}\, H^2 R - 1 + S_{\widetilde{f}}(R) \tag{5.2}$$

where

$$\widetilde{f}(\xi) = \frac{\sin(2\pi R\,\xi)}{2\pi^2\xi^2} - \frac{R}{H}\,\frac{\sin(\pi H \xi)}{\pi^2\xi^2}\,\cos(\pi\,(2R+H)\,\xi)\,.$$

Note that the application of Lemma 2.1 is justified, although the regularity conditions are not fulfilled, because the involved series converges uniformly on compacta by virtue of Lemma 3.1.

The first part of $\widetilde{f}$, namely $\sin(2\pi R\,\xi)/2\pi^2\xi^2$, contributes to $S_{\widetilde{f}}(R)$ at most $O\big(R^\varepsilon\big)$ by estimating the tail of the series using Lemma 3.1. Hence we have

$$S_{\widetilde{f}}(R) = -\frac{R}{\pi^2 H}\sum_{n=1}^{\infty}\frac{l(n)}{n}\,\sin(\pi H\sqrt{n})\,\cos(\pi\,(2R+H)\sqrt{n}) + O\big(R^\varepsilon\big)\,.$$

Dividing into dyadic intervals by partial summation we get

$$S_{\widetilde{f}}(R) \ll R\,H^{-1}\big(H^{1-\varepsilon}N_1^{-1}|V_{N_1}(R_1)| + N_2^{-3/2+\varepsilon}|V_{N_2}(R_2)|\big) + R^\varepsilon$$

for some $N_1 \le H^{-2} \le N_2$ and $R_1, R_2 = R + O\big(H\big)$. For simplicity we assume $R \le H^{-2}$. Now we estimate $V_{N_1}(R_1)$ and $V_{N_2}(R_2)$ by Lemma 3.1, if $N_1 \le R$ we get

$$N_1^{-1}V_{N_1}(R_1) \ll R^{5/16+\varepsilon},$$

if $R < N_1 \le H^{-2}$ we get

$$N_1^{-1}V_{N_1}(R_1) \ll \big(H^{-1/2} + R^{5/24}H^{-5/24}\big)\,H^{-\varepsilon}$$

and if $N_2 \ge H^{-2}$ we get

$$N_2^{-3/2+\varepsilon}V_{N_2}(R_2) \ll \big(H^{1/2} + R^{3/8}H^{9/8}\big)\,H^{-\varepsilon}.$$

Collecting the above estimates and substituting in (5.2) we obtain

$$S_f(R) = \frac{4\pi}{3} R^3 + 2\pi H R^2 + O\big((RH^{-1/2} + R^{9/8}H^{-1/8} + R^{21/16})\, H^{-\varepsilon}\big)\,. \tag{5.3}$$

Now we shall deal with $S_f(R,H)$. By Abel's Lemma and (1.4) we have

$$\begin{aligned} S_f(R,H) &= \frac{R\,(R+H)}{H} \int_0^H \frac{S(R,u)}{(R+u)^2}\, du \\ &= 2\pi H R^2 + O\big((H^{7/8}R^{15/8} + H^{2/3}R^{83/48} + R)\, R^{\varepsilon}\big)\,. \end{aligned} \tag{5.4}$$

Finally substituting (5.4) and (5.3) in (5.1) and choosing $H = R^{-7/11}$, (1.3) follows.

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Comportement asymptotique dans l'algorithme de transformée en ondelettes. Lien avec la régularité de l'ondelette.

Loïc Hervé

Résumé. Nous faisons l'étude du comportement asymptotique dans l'arbre de filtrage d'une transformée en ondelettes, en particulier en fonction de l'ordre de régularité de l'ondelette.

Abstract. We study the asymptotic performance for a Wavelets Transform, in particular as a function of the regularity order of the wavelet.

1. Introduction.

Analyses multirésolutions et transformées en ondelettes. On note $L^2(\mathbb{R})$ l'espace de Lebesgue usuel, et $\langle \cdot, \cdot \rangle$ son produit hilbertien. Rappelons qu'une analyse multirésolution ([10], [11]) est par définition une famille $(V_j)_{j\in\mathbb{Z}}$ de sous-espaces fermés de $L^2(\mathbb{R})$ tels que

a) $\displaystyle\bigcap_{j\in\mathbb{Z}} V_j = \{\vec{0}\}$ et $\displaystyle\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

b) $V_j \subset V_{j+1}$.

c) $f \in V_j$ est équivalent à $f(2^{-j}\cdot) \in V_0$.

d) Il existe une fonction $\phi \in V_0$, appelée *fonction d'échelle*, telle que la famille $\{\phi(\cdot + k) : k \in \mathbb{Z}\}$ forme une base de Riesz de V_0.

Pour tout $j \in \mathbb{Z}$, le système $\{\phi_{j,k} = 2^{j/2}\phi(2^j \cdot - k) : k \in \mathbb{Z}\}$ est une base de Riesz de V_j. Comme $V_0 \subset V_1$, il existe une suite $(h_n)_{n\in\mathbb{Z}}$ dans $\ell^2(\mathbb{Z})$ telle que

$$\phi(\cdot) = \sum_{n\in\mathbb{Z}} h_n\,\phi(2\cdot + n) \qquad \text{(équation d'échelle)}. \tag{1}$$

Les analyses multirésolutions fournissent un cadre pour l'étude des algorithmes d'analyse-synthèse appelés codages en sous-bande [5] ou encore transformées en ondelettes [10]. Par commodité pour le lecteur, nous donnons en appendice un bref descriptif de l'algorithme dans le cas simple orthogonal (ϕ engendre par translation entière une base orthonormée de V_0). Dans ce travail, nous nous intéressons uniquement à la partie "analyse" de la transformée qui, dans tous les cas (orthogonal et non orthogonal), utilise l'itération de l'opérateur

$$(T_0 x)(n) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h_k\, x(2n - k), \qquad n \in \mathbb{Z},\ x = (x(n))_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

Hypothèses et notations. La donnée principale d'une analyse multirésolution est la suite $(h_n)_{n\in\mathbb{Z}}$. On peut d'ailleurs caractériser les suites (assez générales) qui dérivent d'une analyse multirésolution [3], [1], [7]. En outre, la régularité de ϕ peut être calculée à l'aide de $(h_n)_{n\in\mathbb{Z}}$ (voir [4] pour un exposé des différentes méthodes). Dans ce travail, on considère une analyse multirésolution $(V_j)_{j\in\mathbb{Z}}$ telle que $(h_n)_{n\in\mathbb{Z}}$ soit à support fini. Pour fixer les idées, on supposera que $h_n = 0$ pour tout $n \notin [0, N]$, où $N \in \mathbb{N}^*$. D'autre part, on pose

$$H_0(\lambda) = \frac{1}{2} \sum_{k=0}^{N} h_k\, e^{2i\pi k\lambda}, \qquad \lambda \in [0,1].$$

On sait que $H_0(0) = 1$ et $H_0(1/2) = 0$, [11], [4]. Il existe donc $r \in \mathbb{N}^*$ et un polynôme trigonométrique v, avec $v(1/2) \neq 0$, tels que

$$H_0(\lambda) = 2^{-r}(1 + e^{2i\pi\lambda})^r\, v(\lambda). \tag{2}$$

Le choix du filtre H_0 est une question importante dans la pratique. Sans donner une liste exhaustive de tous les types de filtres et de leurs avantages, citons le cas orthogonal, décrit en appendice, qui fournit des formules d'analyse-synthèse très simples, le cas biorthogonal ([1], [4]) qui, tout en conservant l'avantage précédent, permet d'obtenir par exemple des filtres symétriques. D'autre part, on observe dans la pratique que la régularité de ϕ joue un rôle important dans la transformée en ondelettes, mais essentiellement quand l'ordre de régularité est ≤ 1, [12]. Au delà de la classe C^1, la régularité de ϕ ne semble pas fournir de gains substanciels en efficacité pour la transformée en ondelettes. Nous nous proposons de démontrer une propriété sur les itérées de T_0 qui corrobore l'observation précédente. Plus précisément,

L'objet de ce travail est de faire une étude précise de la convergence des itérées de T_0 (en particulier de la vitesse de convergence) en fonction de la régularité de ϕ, et plus exactement en fonction du coefficient

$$s_1 = s_1(\phi) = \sup\left\{s \in \mathbb{R} : \int_{-\infty}^{+\infty} |\lambda|^s \, |\hat{\phi}(\lambda)| \, d\lambda < +\infty\right\},$$

où $\hat{\phi}$ est la transformée de Fourier de ϕ. Rappelons que, si $s_1 > 0$, alors $\phi \in C^\alpha$ pour tout α tel que $0 < \alpha < s_1$. On a noté C^α l'ensemble des fonctions f telles que f est $[\alpha]$-fois dérivable et $f^{[\alpha]}$ est uniformément $(\alpha - [\alpha])$-hölderienne, où $[\alpha]$ est la partie entière de α. En outre on a

$$(3) \quad s_1 = \lim_{n\to+\infty} \left(r - \log_2 \frac{S_{n+1}}{S_n}\right), \qquad \text{où } S_n = \sum_{k=0}^{2^n-1} \left|v\left(\frac{k}{2^n}\right) \cdots v\left(\frac{k}{2}\right)\right|,$$

la vitesse de convergence étant exponentielle, voir [7], [8]. Notons que le but de ce travail est, en un certain sens, opposé à celui de [13], [14], où l'on fait l'étude de la régularité de ϕ à l'aide de l'analyse spectrale (essentiellement calcul du rayon spectral) d'opérateurs du type de T_0.

Les résultats sont présentés dans le Paragraphe 2, le principal s'énonçant de la manière suivante (Théorème 2.2): si $s_1 > 0$, alors dans un certain sous-espace de $\ell^2(\mathbb{Z})$ (assez gros), la suite d'opérateurs $(2^{j/2}\, T_0^j)_{j\geq 1}$ converge en norme avec une vitesse de convergence exponentielle, de l'ordre de 2^{-js_1} si $s_1 < 1$, et de l'ordre de 2^{-j} si $s_1 \geq 1$ (dans ce dernier cas, ϕ est au moins de classe C^1). Ces estimations montrent que la suite $\{T_0^j : j \geq 1\}$, qui est à la base de l'analyse dans la transformée en ondelettes, admet un comportement asymptotique étroitement lié à l'ordre de régularité de ϕ, mais uniquement quand ce dernier est ≤ 1. Pour $s_1 \geq 1$, nous préciserons les résultats précédents en donnant un développement asymptotique de $T_0^j x$, où x est une suite assez générale.

Les preuves, regroupées dans le Paragraphe 3, utilisent les opérateurs de transfert notés P_w définis, à partir d'une fonction w, 1-périodique, par

$$P_w f(\lambda) = w\big(\frac{\lambda}{2}\big)\, f\big(\frac{\lambda}{2}\big) + w\big(\frac{\lambda}{2} + \frac{1}{2}\big)\, f\big(\frac{\lambda}{2} + \frac{1}{2}\big),$$

où f est une fonction 1-périodique. Sous l'hypothèse $w \geq 0$, ces opérateurs ont fait l'objet de nombreux travaux (voir par exemple [3], [9]) et ont été beaucoup utilisés dans la théorie des ondelettes: caractérisation des filtres dérivant d'une analyse multirésolution, étude de la régularité de ϕ. La difficulté de ce travail réside dans le fait que les opérateurs P_w mis en jeu ici (par exemple P_{H_0}, P_v) ne sont pas nécessairement positifs. En revanche, la propriété de quasi-compacité ([6], [9]) est conservée. Pour résumer les techniques utilisées dans le Paragraphe 3, indiquons que l'étude spectrale de T_0 est ramenée à celle P_{H_0}, qui elle-même se déduit de celle de P_v. Enfin les propriétés spectrales de P_v sont comparées à celles de $P_{|v|}$ dont dépend le coefficient s_1 (en effet on a $S_n = P_{|v|}^n(1)$).

2. Etude des itérées de T_0.

• **Etude sur $\ell^2(\mathbb{Z})$** (Pour simplifier, on se place ici dans le cas orthogonal). On sait que, pour tout $f \in L^2(\mathbb{R})$,

$$\lim_{j \to +\infty} \|\Pi_{-j} f\|_{L^2(\mathbb{R})} = 0,$$

où l'on a noté Π_j la projection orthogonale sur V_j. Si en outre $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, on peut montrer que $\|\Pi_{-j} f\|_{L^2(\mathbb{R})} \leq C_\phi\, 2^{-j/2}\, \|f\|_{L^1(\mathbb{R})}$.

Soient $x = (x(n))_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ et $f(t) = \sum_{n \in \mathbb{Z}} x(n)\, \phi(t-n)$. Rappelons que

$$(4) \qquad \langle f, \phi_{-j,k} \rangle = (T_0^j x)(k), \qquad \text{pour tous } j \in \mathbb{N}^*,\ k \in \mathbb{Z}.$$

Comme $\{\phi_{j,k} = 2^{j/2}\phi(2^j \cdot - k) : k \in \mathbb{Z}\}$ est une base orthonormée de V_j, on en déduit que

$$\lim_{j \to +\infty} \|T_0^j x\|_{\ell^2(\mathbb{Z})} = 0,$$

et si f est intégrable sur $\mathbb{R}$, $\|2^{j/2}\, T_0^j x\|_{\ell^2(\mathbb{Z})} \leq C_\phi \, \|f\|_{L^1(\mathbb{R})}$.

A l'exception de la constante C_ϕ, ces dernières propriétés sont communes à toutes les analyses multirésolutions. Par conséquent, nous limiterons l'étude des itérées de T_0 à des sous-espaces propres de $\ell^2(\mathbb{Z})$ (largement représentatifs en pratique). Nous utiliserons pour cela les notations suivantes: pour $1 \leq p \leq +\infty$, on désigne par $(L^p(\mathbb{T}), \|\cdot\|_p)$ l'espace de Lebesgue usuel pour les fonctions 1-périodiques, à valeurs dans $\mathbb{C}$. On note $(E^0, \|\cdot\|_\infty)$ le sous-espace de $L^\infty(\mathbb{T})$ des fonctions continues, et $(E^1, \|\cdot\|)$ le sous-espace de E^0 des fonctions uniformément lipschitziennes, muni de la norme

$$\|X\| = \|X\|_\infty + \sup_{\lambda \neq \lambda'} \frac{|X(\lambda') - X(\lambda)|}{|\lambda' - \lambda|}\,.$$

Pour $X \in L^1(\mathbb{T})$, on note $J(X)$ la suite des coefficients de Fourier de X

$$J(X)(n) = \int_0^1 X(\lambda)\, e^{-2i\pi n\lambda}\, d\lambda\,, \qquad n \in \mathbb{Z}.$$

Pour tout espace de Banach $(B, \|\cdot\|)$ inclus dans $L^1(\mathbb{T})$, on définit l'espace de Banach $(J(B), \|\cdot\|_{J(B)})$ des suites $x = J(X)$, où $X \in B$, avec $\|x\|_{J(B)} = \|X\|_B$.

• **Etude sur $\mathcal{S} = J(L^\infty(\mathbb{T}))$.** Ce premier résultat met en évidence le lien entre le comportement des itérées de T_0 et le coefficient s_1.

Théorème 2.1. *Soit $\mathcal{S} = J(L^\infty(\mathbb{T}))$ muni de la norme $\|x\|_{\mathcal{S}} = \|X\|_\infty$, pour $x = J(X) \in \mathcal{S}$. L'opérateur T_0 est borné sur $\mathcal{S}$, et les trois propriétés suivantes sont équivalentes*

i) *La suite $\{2^{j/2}\, T_0^j : j \geq 1\}$ est uniformément bornée dans $\mathcal{S}$.*

ii) *$\hat{\phi}$ est intégrable sur $\mathbb{R}$.*

iii) $s_1 > 0$.

EXEMPLE. $H_0(\lambda) = 2^{-1}(1 + e^{2i\pi\lambda})$ (filtre de Haar). On a

$$\hat{\phi}(\lambda) = e^{i\pi\lambda}\, \frac{\sin \pi\lambda}{\pi\lambda} \notin L^1(\mathbb{R})\,.$$

Les itérées de T_0 ne sont donc pas uniformément bornées dans $\mathcal{S}$.

• **Etude sur $\mathcal{S}^1 = J(E^1)$ et $\mathcal{S}^0 = J(E^0)$.** On supposera désormais que $s_1 > 0$. On considère la matrice carrée suivante

$$A_0 = \big(\alpha_0(k,l)\big)_{k,\,l=0,\dots,N}\,, \qquad \text{où } \alpha_0(k,l) = h_{2k-\ell}\,,\ k,l = 0,\dots,N\,.$$

Les conditions $H_0(0) = 1$ et $H_0(1/2) = 0$, équivalentes à $\sum_n h_{2n} = \sum_n h_{2n+1} = 1$, assurent que la somme des coefficients sur chaque colonne de A_0 est égale à 1. D'où ${}^t\!A\vec{e} = \vec{e}$, avec $\vec{e} = {}^t[1,\dots,1]$. On en déduit que 1 est valeur propre de A_0.

Théorème 2.2. *Soit $\mu_0 = \sup\{|\mu| : \mu \in \operatorname{spect}(A_0),\ \mu \neq 1\}$. On a $\mu_0 < 1$, et il existe un unique vecteur $\gamma_0 = (\gamma_0(n))_{n=0}^N$ invariant par A_0 tel que $\sum_{n=0}^N \gamma_0(n) = 1$.*

Soit $\mathcal{S}^1 = J(E^1)$ muni de la norme $\|x\|_{\mathcal{S}^1} = \|X\|$, pour $x = J(X) \in \mathcal{S}^1$. L'opérateur T_0 est borné sur $\mathcal{S}^1$. Pour tout réel δ tel que $\delta > \max\{\mu_0, 1/2\}$, il existe une constante $C_\delta > 0$ telle que

$$(5) \qquad \|2^{j/2}\,T_0^j x - X(0)\,\gamma_0\|_{\mathcal{S}^1} \le C_\delta\,\delta^j\,\|x\|_{\mathcal{S}^1}\,,$$

pour tout $x = J(X) \in \mathcal{S}^1$, pour tout $j \in \mathbb{N}^*$.

Dans (5), le vecteur γ_0 a été identifié à un élément de $\ell^2(\mathbb{Z})$. En outre, du fait que E^1 est dense dans E^0, on obtient le

Corollaire 2.3. *Soit $\mathcal{S}^0 = J(E^0)$ muni de la norme $\|x\|_{\mathcal{S}^0} = \|X\|_\infty$, pour $x = J(X) \in \mathcal{S}^0$. L'opérateur T_0 est borné sur $\mathcal{S}^0$, et si $x = J(X) \in \mathcal{S}^0$, alors la suite $\{2^{j/2}\,T_0^j x : j \ge 1\}$ converge dans $\mathcal{S}^0$ vers $X(0)\gamma_0$.*

Rappelons que l'un des objectifs de ce travail est d'étudier le lien entre la vitesse de convergence des itérées de T_0 et la régularité de ϕ. La proposition suivante fournit une réponse en comparant les réels δ de (5) et le coefficient s_1 défini dans l'introduction.

Proposition 2.4. *Soit δ_0 la borne inférieure des réels δ tels que* (5) *soit satisfait. On sait que $\delta_0 \le \max\{\mu_0, 1/2\}$. Si en outre l'entier r dans* (2) *est tel que $r \ge 2$, alors $\delta_0 = \mu_0 \ge 1/2$, et*

$$(6) \qquad \frac{1}{2} \le \delta_0 \le \max\left\{2^{-s_1}, \frac{1}{2}\right\}.$$

REMARQUES.

a) La formule (6) assure que δ_0 décroit jusqu'à 1/2 quand s_1 croît dans $]0,1]$. En revanche, si $s_1 > 1$, on a $\delta_0 = 1/2$. Ceci est en accord avec les observations pratiques mentionnées dans l'introduction. Cependant, si $s_1 > 1$, il est possible de préciser l'estimation donnée par (5) (voir le développement asymptotique ci-dessous).

b) Pour $x = J(X)$, avec X suffisamment régulière, on obtient

$$\begin{aligned}\Big(\sum_n |x(n)|^2\Big)^{1/2} + 2\pi\Big(\sum_n |n\,x(n)|^2\Big)^{1/2} &\le \|X\|_\infty + \|X'\|_\infty \\ &= \|X\| = \|x\|_{S^1} \\ &\le \sum_n |x(n)| + 2\pi \sum_n |n\,x(n)|\,.\end{aligned}$$

c) La vitesse de convergence dans (5) dépend aussi de $\|x\|_{S^1}$. En pratique, x est une suite à support fini (éventuellement assez grand). On obtient, pour tout $j \ge 1$,

$$\begin{aligned}&\Big\|T_0^j x - 2^{-j/2}\Big(\sum_n x(n)\Big)\gamma_0\Big\|_{\ell^2(\mathbb{Z})} \\ &\qquad \le C_\delta\, 2^{-j/2}\, \delta^j \Big(\sum_n |x(n)| + 2\pi \sum_n |n\,x(n)|\Big)\,,\end{aligned}$$

et

$$\begin{aligned}&\Big(2\pi \sum_{n \notin \{0,\dots,N\}} |n\,(T_0^j x)(n)|^2\Big)^{1/2} \\ &\qquad \le C_\delta\, 2^{-j/2}\, \delta^j \Big(\sum_n |x(n)| + 2\pi \sum_n |n\,x(n)|\Big)\,.\end{aligned}$$

d) Le Théorème 2.2 se traduit également de la manière suivante: on suppose ici pour simplifier que $\{\phi(\cdot + k) : k \in \mathbb{Z}\}$ forme une base orthonormée de V_0. Soit $f \in V_0$. Sa transformée de Fourier $\hat{f}$ vérifie $\hat{f}(\lambda) = X(\lambda)\hat{\phi}(\lambda)$, où $X \in L^2(\mathbb{T})$. Supposons que $X \in E^1$. Rappelons qu'on a noté Π_j la projection orthogonale sur V_j. On obtient, grâce à (4), pour tout δ tel que $\delta > \max\{\mu_0, 1/2\}$,

$$\|2^{j/2}\,\Pi_{-j}(f) - X(0)\sum_{n=0}^{N} \gamma_0(n)\,\phi_{-j,n}\|_{L^2(\mathbb{R})} \le C_\delta\, \delta^j\, \|X\|\,,$$

pour tout $j \geq 1$.

• **Développement asymptotique de $T_0^j x$.** On note $M = [s_1]$, si $s_1 \notin \mathbb{N}$, et $M = [s_1] - 1$ sinon (on a $M < r \leq N$). Pour $m = 0, \dots, r$, on définit

$$H_m(\lambda) = 2^{r-m} (1 + e^{2i\pi\lambda})^{r-m} v(\lambda) = \frac{1}{2} \sum_{k=0}^{N-m} h_{m,k} e^{2i\pi k\lambda},$$

et on note $A_m = \big(\alpha_m(k,l)\big)_{k,l=0,\dots,N-m}$ la matrice carrée définie par

$$\alpha_m(k,l) = h_{m,2k-\ell}, \qquad k,l = 0, \dots, N-m.$$

Lemme 2.5. *Soit $\mu_M = \sup\{|\mu| : \mu \in \mathrm{spect}(A_M),\ \mu \neq 1\}$. On a $\mu_M < 1$. En outre, pour $m = 0, \dots, M$, il existe un unique vecteur $\gamma_m = (\gamma_m(n))_{n=0}^{N-m}$ invariant par A_m tel que $\sum_{n=0}^{N-m} \gamma_m(n) = 1$.*

On notera $\Gamma_m(\lambda) = \sum_{n=0}^{N-m} \gamma_m(n)\, e^{2i\pi n\lambda}$. Pour simplifier (voir Remarque 1) ci-dessous), on considère dans l'énoncé du théorème suivant une suite $x = (x(n))_{n\in\mathbb{Z}}$ à support fini, et on définit $X(\lambda) = \sum_n x(n)\, e^{2i\pi n\lambda}$. La fonction $X - X(0)\Gamma_0$ s'annule en 0, et est donc divisible par

$$S(\lambda) = \frac{e^{2i\pi\lambda} - 1}{2i} = e^{i\pi\lambda} \sin \pi\lambda.$$

On note α_m la suite des coefficients de Fourier de la fonction $S(\cdot)^m$

$$\alpha_m(n) = (2i)^{-m}(-1)^{m-n}\, C_m^n,$$

pour tout $n = 0, \dots, m$, et $\alpha_m(n) = 0$ sinon. Itérant le raisonnement précédent, on définit pour $m = 1, \dots, M-1$ le polynôme trigonométrique X_m par la formule de récurrence (où par convention $X_0 = X$)

$$S(\lambda)\, X_{m+1}(\lambda) = X_m(\lambda) - X_m(0)\, \Gamma_m(\lambda). \tag{7}$$

Théorème 2.6. *Pour tout réel δ tel que $\delta > \max\{\mu_M, 1/2\}$, il existe une constante $C_\delta > 0$, indépendante de x, telle que*

$$\|2^{j/2}\, T_0^j x - \sum_{m=0}^{M} 2^{-mj} X_m(0)\, (\alpha_m * \gamma_m)\|_{S^1} \leq C_\delta\, 2^{-Mj}\, \delta^j\, \|X_M\|. \tag{8}$$

En particulier, si x est de la forme $x=\alpha_M*y$ (i.e. $X(\lambda)=S(\lambda)^M\,Y(\lambda)$), alors

$$\|T_0^j(x)\|_{S^1} \leq C_\delta\, 2^{-j(M+1/2)}\, \delta^j\, \|y\|_{S^1}\ .$$

REMARQUES.

1) Si x est une suite de $\mathcal{S}^1$ telle que les formules (7) permettent de définir les fonctions X_m, avec $X_m \in E^1$, alors l'inégalité (8) est encore vérifiée. La preuve du Théorème 2.6 sera d'ailleurs donnée sous ces hypothèses.

2) Soit δ_M la borne inférieure des réels δ tels que l'inégalité (8) soit satisfaite pour tout $j \geq 0$, et tout $x \in \mathcal{S}^1$ satisfaisant à l'hypothèse de la remarque 1). On a $\delta_M \leq \max\{\mu_M, 1/2\}$. La démonstration du Théorème 2.6 montrera que, si $M \leq r-2$, alors $\delta_M = \max\{\mu_M, 1/2\}$ et $1/2 \leq \delta_M \leq 2^{-s_1+M}$ (voir Remarque b) à la fin du Paragraphe 3). Dans la plupart des exemples classiques d'analyses multirésolutions, les réels μ_M et 2^{-s_1+M} sont très peu différents.

3) La propriété (8) permet de préciser les inégalités dans les remarques c) et d) ci-dessus.

3. Démonstrations.

3.1. Passage à un opérateur quasi-compact.

Les espaces fonctionnels utilisés ci-dessous, et l'application J, ont été définis dans le Paragraphe 2. Pour $w \in E^1$, on notera P_w l'opérateur defini sur $L^\infty(\mathbb{T})$ par

$$P_w f(\lambda) = w\big(\frac{\lambda}{2}\big)\, f\big(\frac{\lambda}{2}\big) + w\big(\frac{\lambda}{2}+\frac{1}{2}\big)\, f\big(\frac{\lambda}{2}+\frac{1}{2}\big)\,.$$

Il est clair que P_w est un opérateur linéaire borné sur $L^\infty(\mathbb{T})$, E^0, et E^1. On note $u = |H_0|$, et

$$P_0 = P_{H_0}\,, \qquad P_u = P_{|H_0|}\,.$$

Lemme 3.1. *Soit $x \in \mathcal{S}$ ($x = J(X)$ avec $X \in L^\infty(\mathbb{T})$). Alors*

$$\sqrt{2}\, T_0 x = J(P_0 X)\,.$$

En particulier T_0 est un opérateur borné sur $\mathcal{S}$, $\mathcal{S}^0$, $\mathcal{S}^1$. En outre P_0 laisse invariant l'espace

$$\mathcal{T}_N = \text{vect}\{1, e^{2i\pi\lambda}, \dots, e^{2i\pi N\lambda}\},$$

et A_0 est la matrice de P_0 restreint à $\mathcal{T}_N$.

Dans la suite, l'étude de $\sqrt{2}\, T_0$ sur $\mathcal{S}$, $\mathcal{S}^0$, et $\mathcal{S}^1$ sera systématiquement remplacée par celle de P_0 respectivement sur $L^\infty(\mathbb{T})$, E^0, et E^1. Par commodité, pour chaque résultat du Paragraphe 2, nous donnerons la version fonctionnelle correspondante.

PREUVE DU LEMME 3.1. Toutes les propriétés du lemme découlent de la formule

$$\begin{aligned}\int_0^1 (P_0X)(\lambda)\, e^{-2i\pi n\lambda}\, d\lambda &= 2\int_0^1 X(\lambda)\, H_0(\lambda)\, e^{-2i\pi 2n\lambda}\, d\lambda \\ &= \sum_{k\in\mathbb{Z}} h_k \int_0^1 X(\lambda)\, e^{-2i\pi(2n-k)\lambda}\, d\lambda \\ &= \sum_{k\in\mathbb{Z}} h_k\, x(2n-k) = \sqrt{2}\,(T_0x)(n).\end{aligned}$$

Rappels sur la quasi-compacité. On désigne par $|\cdot|_\infty$ et $|\cdot|$ les normes d'opérateurs vis-à-vis respectivement de $L^\infty(\mathbb{T})$ et E^1. Pour un opérateur S borné sur un espace de Banach B, on note $\rho(S, B)$ son rayon spectral.

Soit w une fonction quelconque dans E^1. On montre aisément par récurrence que

$$P_w^j f(\lambda) = \sum_{k=0}^{2^j-1} w\Big(\frac{\lambda+k}{2}\Big)\cdots w\Big(\frac{\lambda+k}{2^j}\Big)\, f\Big(\frac{\lambda+k}{2^j}\Big), \tag{9}$$

pour tout $j \geq 1$.

Soit α une fonction réelle, mesurable, et 1-périodique, telle que $w(\lambda) = |w(\lambda)|\, e^{2i\pi\alpha(\lambda)}$, et soit

$$A(\lambda) = \exp\{-2i\pi[\alpha(2^{j-1}\lambda) + \alpha(2^{j-2}\lambda) + \cdots + \alpha(\lambda)]\}.$$

Il est clair que $P_{|w|}^j(1) = P_w^j(A)$. On en déduit les propriétés suivantes

$$\begin{aligned} \|P_{|w|}^j 1\|_\infty = |P_{|w|}^j|_\infty = |P_w^j|_\infty\,, \\ \rho(P_w, L^\infty(\mathbb{T})) = \rho(P_{|w|}, L^\infty(\mathbb{T})).\end{aligned} \tag{10}$$

De plus on montre, à partir de (9), que

$$\|P_w^j f\| \leq 2^{-j} \, \|P_{|w|}^j 1\|_\infty \, \|f\| + R_j \, \|f\|_\infty \, ,$$

pour tout $f \in E^1$, pour tout $j \geq 1$, où R_j est une constante positive. Cette dernière propriété est démontrée pour $w \geq 0$ dans [6], [9]; la preuve pour w de signe quelconque est identique. D'autre part, à l'aide de l'inégalité ci-dessus, on établit dans [6] que, si

$$\frac{1}{2} \, \rho(P_{|w|}, L^\infty(\mathbb{T})) < \rho(P_w, E^1) \, , \tag{11}$$

alors P_w est *quasi-compact sur* E^1, c'est-à-dire vérifie les propriétés suivantes

(A) *Soit* $\beta_w = \rho(P_{|w|}, L^\infty(\mathbb{T}))/2$. *L'ensemble* I *des valeurs spectrales* μ *de* P_w *sur* E^1, *telles que* $\beta_w \leq |\mu| \leq \rho(P_w, E^1)$ *est fini, et tout élément* $\mu \in I$ *est en fait une valeur propre d'indice fini* $\nu(\mu)$, *telle que* $\dim \ker(P_w - \mu\, I)^{\nu(\mu)} < +\infty$. *En outre on a*

$$E^1 = \Big(\bigoplus_{\mu \in I} \ker(P_w - \mu\, I)^{\nu(\mu)} \Big) \oplus \mathcal{F} \, , \tag{12}$$

où $\mathcal{F}$ *est un sous-espace fermé de* E^1, *stable par* P_w, *tel que* $\rho(P_{w|\mathcal{F}}, \mathcal{F}) < \beta_w$.

On rappelle que l'indice $\nu(\mu)$ d'une valeur propre μ est fini s'il existe un entier $n \geq 1$ tel que $\ker(P_w - \mu\, I)^n = \ker(P_w - \mu\, I)^{n+1}$, $\nu(\mu)$ étant le plus petit entier vérifiant cette dernière condition. Les propriétés de quasi-compacité ne suffiront pas pour prouver les résultats du Paragraphe 2. Nous utiliserons également des arguments de positivité. A cet effet, nous aurons besoin de la propriété suivante démontrée dans [6], [9].

(B) *Si* w *est à valeurs positives ou nulles, alors* P_w *est un opérateur positif* (*si* $f \geq 0$, *alors* $P_w f \geq 0$). *On a* $\rho(P_w, E^1) = \rho(P_w, L^\infty(\mathbb{T}))$. *Donc* P_w *est quasi-compact sur* E^1. *En outre,* $\rho(P_w, E^1)$ *est une valeur propre, d'indice maximal parmi les valeurs propres de module* $\rho(P_w, E^1)$, *et enfin il lui est associé une fonction propre à valeurs positives ou nulles.*

Le fait que la fonction d'échelle ϕ engendre par translations entières une base de Riesz de V_0 implique des conditions très précises sur les zéros de H_0. Sans entrer dans les détails, rappelons que ces conditions s'expriment par exemple en terme de compacts invariants (ou cycles périodiques) vis-à-vis de la transformation $\Delta(x) = 2x \pmod 1$ ([3], [8]). D'autre part, sous ces conditions, on démontre, dans [8] que, si $s_1 > 0$, alors $\rho(P_u, E^1) = 1$, et dans [9] que

(C) *l'espace des fonctions* 1*-périodiques continues* P_u*-invariantes est engendré par une fonction à valeurs strictement positives.*

3.2. Démonstration du Théorème 2.1.

En vertu du Lemme 3.1, l'énoncé du Théorème 2.1 est équivalent au suivant

i) *La suite* $\{P_0^j : j \geq 1\}$ *est uniformément bornée dans* $L^\infty(\mathbb{T})$.

ii) $\hat{\phi}$ *est intégrable sur* $\mathbb{R}$.

iii) $s_1 > 0$.

L'équivalence entre ii) et iii) est démontrée dans [8]. En outre on sait que $\rho(P_u, E^1)$ est une valeur propre pour P_u, d'indice fini qu'on notera ν (voir (B)), et on montre dans [8] que la condition ii) est équivalente à

iv) $\rho(P_u, E^1) = 1$ *et* $\nu = 1$.

Prouvons que i) est équivalent à iv).

i) $\Rightarrow$ iv). Comme 1 est valeur propre de A_0 (voir Paragraphe 2), on a $1 \leq \rho(A_0) \leq \rho(P_0, L^\infty(\mathbb{T})) \leq 1$, la dernière inégalité résultant de i). Donc $\rho(P_0, L^\infty(\mathbb{T})) = 1$. Finalement on a $\rho(P_u, E^1) = \rho(P_u, L^\infty(\mathbb{T})) = \rho(P_0, L^\infty(\mathbb{T})) = 1$ (la première égalité découlant de (B), la deuxième de (10)). En outre, du fait que $|P_0^j|_\infty = |P_u^j|_\infty$, la suite $\{P_u^j,\ j \geq 1\}$ est uniformément bornée dans $L^\infty(\mathbb{T})$. Comme $\rho(P_u, E^1)$ est une valeur propre de P_u sur E^1, son indice ν (qui est fini d'après (A)) est nécessairement égal à 1.

iv) $\Rightarrow$ i). Si $\rho(P_u, E^1) = 1$ et $\nu = 1$, l'indice de chaque valeur propre de module 1 pour P_u sur E^1 est égal à 1 d'après (B), d'où $\sup_{j \geq 1} |P_u^j| < +\infty$. On conclut en remarquant que $|P_0^j|_\infty = |P_u^j|_\infty = \|P_u^j 1\|_\infty \leq \|P_u^j 1\| \leq |P_u^j|$.

3.3. Démonstration du Théorème 2.2.

On a supposé pour le Théorème 2.2 que $s_1 > 0$. Rappelons que A_0, définie dans le Paragraphe 2, est la matrice de P_0 restreint à $\mathcal{T}_N$, et qu'on a noté μ_0 la plus grande valeur parmi les modules des valeurs propres de A_0 différentes de 1. En vertu du Lemme 3.1, le Théorème 2.2 est un corolaire du résultat suivant

Théorème 3.2. *On a $\mu_0 < 1$. L'espace des fonctions de E^1 P_0-invariantes est engendré par un polynôme trigonométrique $\Gamma_0 \in \mathcal{T}_N$ tel que $\Gamma_0(0) = 1$. Pour tout réel δ tel que $\delta > \max\{\mu_0, 1/2\}$, il existe une constante $C_\delta > 0$ telle que l'on ait, pour tout $X \in E^1$ et tout entier $j \geq 1$,*

$$\|P_0^j X - X(0)\,\Gamma_0\| \leq C_\delta\, \delta^j\, \|X\|\,. \tag{13}$$

DÉMONSTRATION DU THÉORÈME 3.2. D'après ce qui précède (*cf.* condition iv)), on a $\rho(P_u, E^1) = 1$. D'autre part, on sait que

$$\rho(P_0, L^\infty(\mathbb{T})) = \rho(P_u, L^\infty(\mathbb{T})) \quad \text{et} \quad \rho(P_u, L^\infty(\mathbb{T})) = \rho(P_u, E^1)$$

(*cf.* (10) et (B)). Donc $\rho(P_0, L^\infty(\mathbb{T})) = 1$. D'autre part, comme $\rho(P_0, E^1) \geq \rho(A_0) \geq 1$, P_0 est quasi-compact sur E^1 en vertu de (11). Donc P_0 admet au moins une valeur propre μ de module $\rho(P_0, E^1)$. On a $|\mu| \leq \rho(P_0, L^\infty(\mathbb{T})) = 1$, d'où $\rho(P_0, E^1) = 1$, et $\rho(A_0) = 1$. La suite de la preuve repose sur le

Lemme 3.3. a) *Soit μ une valeur propre de P_0 sur E^1 telle que $1/2 < |\mu| \leq 1$, et soit f une fonction propre associée à μ. Alors $f \in \mathcal{T}_N$. En particulier, toutes les valeurs propres μ de P_0 sur E^1 telles que $1/2 < |\mu| \leq 1$ sont valeurs propres de A_0.*

b) *1 est l'unique valeur propre de module 1 de P_0 sur E^1. En outre l'espace des fonctions de E^1 P_0-invariantes est engendré par un polynôme trigonométrique $\Gamma_0 \in \mathcal{T}_N$ tel que $\Gamma_0(0) = 1$.*

Ce lemme (que nous admettons pour le moment) prouve que $\mu_0 < 1$. Soit $X \in E^1$. Du fait que $H_0(0) = 1$ et $H_0(1/2) = 0$, on a $(P_0^j X)(0) = X(0)$ pour tout $j \geq 0$. D'autre part, d'après le Lemme 3.3 et (12), X s'écrit sous la forme: $X = a\Gamma_0 + g$, avec $a \in \mathbb{C}$, $g \in E^1$ telle que $\lim_{j\to+\infty} \|P_0^j g\| = 0$. D'où $a = X(0)$. Plus précisément, toujours d'après le Lemme 3.3 et (12) (remarquer également que $\beta_{H_0} = 1/2$, *cf.* (A)), pour tout $\delta > \max\{\mu_0, 1/2\}$, il existe une constante $D_\delta > 0$ telle que $\|P_0^j g\| \le D_\delta\, \delta^j\, \|g\| \le D_\delta\, (1 + \|\Gamma_0\|)\, \delta^j\, \|X\|$, ce qui prouve (13). Il reste à donner la

DÉMONSTRATION DU LEMME 3.3. a) Nous allons prouver que $f \in C^\infty$, puis que $f \in \mathcal{T}_N$.

• *f est de classe C^∞*. Nous reprenons ici un raisonnement présenté dans [3]. La fonction f étant lipschitzienne, elle est dérivable presque partout. Plus précisément f est la primitive d'une fonction mesurable bornée, qu'on notera f' pour simplifier. Dérivant l'équation $P_0 f = \mu f$, il vient que $(2\mu - P_0)f' = P_{H_0'} f = \xi_0$. Comme $\rho(P_0, L^\infty(\mathbb{T})) = 1$ et $|2\mu| > 1$, $(2\mu - P_0)$ est inversible sur $L^\infty(\mathbb{T})$, et

$$f' = \sum_{k=0}^{+\infty} (2\mu)^{-(k+1)}\, P_0^k \xi_0 \ .$$

Mais puisque $\xi_0 \in E^1$ et $\rho(P_0, E^1) = 1$, cette dernière série converge également dans E^1 vers une fonction $g \in E^1$, égale à f' presque partout. Il en résulte que f est la primitive d'une fonction lipschitzienne. Donc f est de classe C^1, et sa dérivée (au sens classique), g, est lipschitzienne.

On note désormais $f' = g$. Pour prouver que f est de classe C^p, pour tout $p \ge 1$, on itère la démonstration précédente. Par exemple, si $p = 2$, on part de l'équation $f' = (2\mu)^{-1}(P_0 f' + \xi_0)$: les fonctions f' et ξ_0 étant lipschtitziennes, elles s'écrivent comme primitives de fonctions mesurables bornées, qu'on notera respectivement f'' et ξ_0'. Par dérivation, on obtient, dans $L^\infty(\mathbb{T})$, $f'' = (4\mu)^{-1} P_0 f'' + \xi_1$, avec $\xi_1 \in E^1$. On conclut comme précédemment en considérant la série $\sum_{k=0}^{+\infty} (4\mu)^{-k} P_0^k \xi_1$.

• $f \in \mathcal{T}_N$. Soit $x = (x(n))_{n \in \mathbb{Z}}$ la suite des coefficients de Fourier de f. En vertu du Lemme 3.1, l'équation $P_0 f = \mu f$ est équivalente à $\sqrt{2}\, T_0 x = \mu x$, c'est-à-dire à

$$\mu\, x(n) = \sum_{k=0}^{N} h_k\, x(2n - k)\,, \qquad \text{pour tout } \ n \in \mathbb{Z}\,.$$

Soit $\ell \in \mathbb{N}^*$ quelconque, et $S_\ell = \sup_{n<0} |n^\ell\, x(n)|$ (on a $S_\ell < +\infty$ car f est de classe C^∞). Pour tout $n < 0$, on a

$$|\mu\, x(n)| \le \frac{C\, S_\ell}{2^\ell\, |n|^\ell}\ ,$$

avec $C = \sum_{k=0}^{N} |h_k|$. Donc $S_\ell \leq C\,|\mu|^{-1}2^{-\ell}S_\ell$. Pour ℓ assez grand, cette dernière inégalité n'est possible que si $S_\ell = 0$, c'est- à -dire $x(n) = 0$ pour tout $n < 0$.

Soit maintenant $R_\ell = \sup_{n \geq N+1} |n^\ell x(n)|$. Pour tout $n \geq N+1$, on a

$$|\mu\, x(n)| \leq \frac{C\,R_\ell}{|2n - N|^\ell} \leq \frac{C\,R_\ell}{2^\ell\,|n|^\ell \left|1 - \dfrac{N}{2N+2}\right|^\ell}.$$

D'où

$$R_\ell \leq \frac{C}{|\mu|}\, 2^{-\ell}\Big(\frac{2N+2}{N+2}\Big)^\ell R_\ell\,.$$

Pour ℓ assez grand, il vient que $R_\ell = 0$, c'est-à-dire $x(n) = 0$ pour tout $n \geq N+1$. Le a) du Lemme est démontré.

b) Soit $f \in E^1$ telle que $P_0 f = \mu f$, avec $|\mu| = 1, \mu \neq 1$ (on sait que $f \in \mathcal{T}_N$, mais cette propriété n'intervient pas ici). Démontrons que $f \equiv 0$: l'équation $P_0 f(\lambda) = \mu\, f(\lambda)$ appliquée avec $\lambda = 0$ donne $\mu\, f(0) = f(0)$, d'où $f(0) = 0$ (on a utilisé le fait que $H_0(0) = 1$ et $H_0(1/2) = 0$). En outre on a $|f| = |\mu f| = |P_0 f| \leq P_u(|f|)$. L'opérateur P_u étant positif, la suite $\{P_u^j(|f|)\ :\ j \geq 1\}$ est croissante. De la condition iv) et de la propriété sur l'indice des valeurs propres de module $\rho(P_u, E^1)$ pour P_u (*cf.* (B)), il vient que $\sup_{j\geq 1} |P_u^j| < +\infty$. Ainsi la suite de fonctions $\{P_u^j(|f|)\ :\ j \geq 1\}$ est relativement compacte (Théorème d'Ascoli): finalement elle converge vers une fonction $h \in E^1$ majorant $|f|$ et telle que $P_u h = h$. On a $(P_u^j(|f|))(0) = 0$ pout tout $j \geq 1$, d'où $h(0) = 0$. On déduit de la propriété (C) que h (et donc f) sont identiquement nulles, ce qui prouve la propriété annoncée, soit la première assertion du b).

On sait que 1 est valeur propre de A_0, donc de P_0. Démontrons que deux fonctions quelconques f_1 et f_2 de E^1 P_0-invariantes sont nécessairement proportionnelles. On peut toujours trouver a et $b \in \mathbb{C}$ tels que la fonction $g = af_1 + bf_2$ soit nulle en 0. On raisonne alors comme précédemment (remplacer $|f|$ par $|g|$) pour prouver que $g \equiv 0$, ce qui montre bien que f_1 et f_2 sont colinéaires. Par conséquent, l'espace des fonctions de E^1 P_0-invariantes est engendré par une fonction Γ de E^1 qui, d'après ce qui précède, appartient en fait à $\mathcal{T}_N$. Il reste à prouver que $\Gamma(0) \neq 0$. Or, si on avait $\Gamma(0) = 0$, le raisonnement ci-dessus (remplacer $|f|$ par $|\Gamma|$) montrerait que $\Gamma \equiv 0$, ce qui est absurde.

DÉMONSTRATION DE LA PROPOSITION 2.4. Comme par hypothèse $r \geq 2$, $1/2$ est valeur propre de A_0 (cette propriété est bien connue [4];

nous y reviendrons à la fin de l'article). Donc $\mu_0 \geq 1/2$. On déduit de (12) ($\beta_{H_0} = 1/2$) et du Lemme 3.3 que $\delta_0 = \mu_0$. Il reste à démontrer (6): Si $\mu_0 = 1/2$, (6) est évident. Supposons que $\mu_0 > 1/2$. Soit f une fonction propre associée à une valeur propre μ de P_0 telle que $|\mu| = \mu_0$. On a nécessairement $f \in \mathcal{T}_N$ et $f(0) = 0$. Plus exactement, utilisant un développement limité dans l'équation $P_0 f = \mu f$, on prouve que f est de la forme: $f(\lambda) = 2^{-r}(1 - e^{2i\pi\lambda})^r g(\lambda)$, et un calcul simple montre que $P_v g = 2^r \mu g$, où v est le polynôme trigonométrique de la formule (2). On en déduit que $2^r \mu_0 \leq \rho(P_v, L^\infty(\mathbb{T})) = \rho(P_{|v|}, L^\infty(\mathbb{T}))$. Or on a $\rho(P_{|v|}, L^\infty(\mathbb{T})) = 2^{r-s_1}$ (voir [8]), d'où $\mu_0 \leq 2^{-s_1}$.

3.4. Démonstration du Théorème 2.6.

Pour simplifier les notations, donnons la preuve pour $M = 1$ (la généralisation pour $M \geq 2$ étant immédiate). Rappelons que

$$H_1(\lambda) = 2^{r-1}\,(1 + e^{2i\pi\lambda})^{r-1}\,v(\lambda) = \frac{1}{2}\sum_{k=0}^{N-1} h_{1,k}\,e^{2i\pi k\lambda}.$$

On a noté $A_1 = \big(\alpha_1(k,l)\big)_{k,l=0,\ldots,N-1}$ la matrice carrée définie par

$$\alpha_1(k,l) = h_{1,2k-\ell}\,, \qquad k,l = 0,\ldots,N-1\,,$$

et $\mu_1 = \sup\{|\mu| :\ \mu \in \operatorname{spect}(A_1),\ \mu \neq 1\}$. L'idée de la démonstration est de prouver que la suite $(h_{1,k})_k$ dérive d'une analyse multirésolution, et d'appliquer ensuite les résultats précédents au filtre H_1. A cet effet, on considère la fonction

$$\hat{\phi}_1(\lambda) = \prod_{k=1}^{+\infty} H_1\Big(\frac{\lambda}{2^k}\Big)\,, \qquad \lambda \in \mathbb{R}\,,$$

et

$$\tilde{s}_1 = s_1(\phi_1) = \sup\Big\{s \in \mathbb{R} :\ \int_{-\infty}^{+\infty} |\lambda|^s\,|\hat{\phi}_1(\lambda)|\,d\lambda < +\infty\Big\}.$$

En vertu de la propriété (3) appliquée à $\hat{\phi}_1$, et par définition de M, on a $\tilde{s}_1 = s_1 - 1 > 0$. Par conséquent, la transformée de Fourier inverse ϕ_1 de $\hat{\phi}_1$ est continue, à support compact, et satisfait à l'équation d'échelle

$$\phi_1(\cdot) = \sum_{n\in\mathbb{Z}} h_{1,n}\,\phi_1(2\cdot+n)\,.$$

En outre, $\{\phi_1(\cdot + k) : k \in \mathbb{Z}\}$ est un système de Riesz: en effet les translatées entières d'une fonction $g \in L^2(\mathbb{R})$ forment un système de Riesz s'il existe deux constantes a, b, avec $0 < a \leq b < +\infty$, telles que $a \leq \sum_{k\in\mathbb{Z}} |\hat{g}(\lambda + k)|^2 \leq b$ pour presque tout $\lambda \in [0,1]$ (*cf.* [11], [4]). Cette dernière condition étant satisfaite par hypothèse pour ϕ, l'est évidemment pour ϕ_1. Pour $j \in \mathbb{Z}$, on définit l'espace V_j^1 engendré par $\{2^{j/2}\phi_1(2^j \cdot -k) : k \in \mathbb{Z}\}$. Alors la famille $(V_j^1)_{j\in\mathbb{Z}}$ est une analyse multirésolution [4]. Ainsi la suite $(h_{1,k})_{k\in\mathbb{Z}}$ satisfait aux mêmes hypothèses que $(h_k)_{k\in\mathbb{Z}}$, et le Théorème 2.2 s'applique à l'opérateur T_1 défini sur $\ell^2(\mathbb{Z})$ par

$$(T_1 x)(n) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h_{1,k}\, x(2n - k), \qquad n \in \mathbb{Z},\ x \in \ell^2(\mathbb{Z}).$$

D'autre part, on démontre (comme pour le Lemme 3.1) que $T_1 \circ J = J \circ P_1$, où $P_1 = P_{H_1}$. Le Théorème 3.2, appliqué à P_1, s'énonce de la manière suivante:

On a $\mu_1 < 1$. L'espace des fonctions de E^1 P_1-invariantes est engendré par un polynôme trigonométrique $\Gamma_1 \in \mathcal{T}_{N-1}$ tel que $\Gamma_1(0) = 1$.

Pour tout réel δ tel que $\delta > \max\{\mu_1, 1/2\}$, il existe une constante $C_\delta > 0$ telle que l'on ait, pour tout $X \in E^1$ et tout entier $j \geq 1$,

$$\|P_1^j X - X(0)\,\Gamma_1\| \leq C_\delta\, \delta^j\, \|X\|\,. \tag{14}$$

Soit $X \in E^1$. On sait d'après le Théorème 3.2 que $\lim_{j\to+\infty} P_0^j X = X(0)\,\Gamma_0$ dans E^1. Notons que $X - X(0)\Gamma_0$ est nulle en 0. Supposons qu'il existe $X_1 \in E^1$ tel que

$$X(\lambda) - X(0)\,\Gamma_0(\lambda) = S(\lambda)\, X_1(\lambda)\,,$$

où $S(\lambda) = e^{i\pi\lambda} \sin \pi\lambda$ (la décomposition ci-dessus est presque toujours satisfaite, par exemple si X est un polynôme trigonométrique). L'énoncé du Théorème 2.6 (pour $M = 1$) est équivalent au suivant

Théorème 3.4. *Pour tout réel δ tel que $\delta > \max\{\mu_1, 1/2\}$, il existe une constante $C_\delta > 0$, indépendante de X, telle que*

$$\|P_0^j X - X(0)\,\Gamma_0 - 2^{-j} X_1(0)\, S \cdot \Gamma_1\| \leq C_\delta\, 2^{-j}\, \delta^j\, \|X_1\|\,. \tag{15}$$

PREUVE DU THÉORÈME 3.4. On a $P_0^j X - X(0)\,\Gamma_0 = P_0^j\,(X - X(0)\,\Gamma_0)$ $= P_0^j(SX_1)$. Or, grâce à la formule classique $\sin 2u = 2\sin u\cos u$, on établit facilement, par récurence sur j, l'égalité

$$P_0^j(SX_1)(\lambda) = 2^{-j}\,S(\lambda)(P_1^j X_1)(\lambda)\,,$$

d'où

$$\begin{aligned}\|P_0^j X - X(0)\,\Gamma_0 - 2^{-j} X_1(0)\,S\cdot\Gamma_1\| &= 2^{-j}\,\|S\cdot(P_1^j X_1 - X_1(0)\,\Gamma_1)\| \\ &\leq C\,2^{-j}\,\|P_1^j X_1 - X_1(0)\,\Gamma_1\|\,.\end{aligned}$$

On déduit (15) en appliquant (14) à X_1.

REMARQUES. a) Pour prouver que $1/2$ est valeur propre de A_0 quand $r \geq 2$, on peut procéder de la manière suivante: rappelons que $P_1\Gamma_1 = \Gamma_1$. Soit $Z(\lambda) = S(\lambda)\,\Gamma_1(\lambda)$. Utilisant la formule $\sin 2u = 2\sin u\cos u$, il vient que $P_0 Z = Z/2$.

b) On suppose toujous $M = 1$. La Proposition 2.4 s'applique à μ_1, à savoir: soit δ_1 la borne inférieure des réels δ tels que (14), et donc (15), soient satisfaites. On a $\delta_1 \leq \max\{\mu_1, 1/2\}$. Si en outre l'entier r dans (2) est tel que $r \geq 3$, alors $\delta_1 = \mu_1 \geq 1/2$, et $1/2 \leq \delta_1 \leq \max\{2^{-\bar{s}_1}, 1/2\} = \max\{2^{-s_1+1}, 1/2\}$.

APPENDICE: Rappels des formules d'analyse-synthèse dans la transformée en ondelettes.

Pour simplifier on suppose ici que $\{\phi(\cdot + k) \,:\, k \in \mathbb{Z}\}$ forme une base orthonormée de V_0, et pour $j \in \mathbb{Z}$, on note W_j le sous-espace de V_{j+1} orthogonal à V_j. Soit

$$\tilde{h}_n = (-1)^n\,\overline{h_{1-n}}\,, \qquad n \in \mathbb{Z}.$$

L'ondelette ψ est définie par

$$\psi(\cdot) = \sum_{n\in\mathbb{Z}} \tilde{h}_n\,\phi(2\cdot + n)\,. \tag{16}$$

La fonction ψ est dans W_0 et $\{\psi(\cdot - k) \,:\, k \in \mathbb{Z}\}$ forme une base orthonormée de W_0. Le système $\{2^{j/2}\psi(2^j\cdot - k) \,:\, k \in \mathbb{Z}\}$ est alors une base orthonormée de W_j pour tout $j \in \mathbb{Z}$, et $\{2^{j/2}\psi(2^j\cdot - k) \,:\, j, k \in \mathbb{Z}\}$

une base orthonormée de $L^2(\mathbb{R})$. Rappelons également que les fonctions ϕ et ψ sont à support compact et admettent la même régularité. On notera $g_{j,k}(\cdot) = 2^{j/2} g(2^j \cdot -k)$ pour $g = \phi$ et ψ. Pour $f \in V_0$, on considère les suites x, x_0, x_1 de $\ell^2(\mathbb{Z})$ définies par

$$\begin{aligned} x(n) &= \langle f, \phi(\cdot - n) \rangle, \qquad n \in \mathbb{Z}, \\ x_0(n) &= \langle f, \phi_{-1,n} \rangle, \qquad n \in \mathbb{Z}, \\ x_1(n) &= \langle f, \psi_{-1,n} \rangle, \qquad n \in \mathbb{Z}. \end{aligned}$$

Grâce à (1) et (16), on obtient les formules (dites d'analyse)

$$\begin{aligned} x_0(n) &= \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} h_k \, x(2n-k), \\ x_1(n) &= \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} \tilde{h}_k \, x(2n-k). \end{aligned}$$

On reconnait dans la première formule l'action de l'opérateur T_0. En retour, l'égalité

$$f = \sum_{n \in \mathbb{Z}} x(n) \, \phi_{0,n} = \sum_{n \in \mathbb{Z}} x_0(n) \, \phi_{-1,n} + \sum_{n \in \mathbb{Z}} x_1(n) \, \psi_{-1,n}$$

fournit la formule de synthèse (qu'on notera pour simplifier $x = \mathcal{S}(x_0, x_1)$)

$$x(k) = \frac{1}{\sqrt{2}} \left(\sum_{n \in \mathbb{Z}} h_{2n-k} \, x_0(n) + \sum_{n \in \mathbb{Z}} \tilde{h}_{2n-k} \, x_1(n) \right).$$

Si on oublie l'aspect fonctionnel dans la description ci-dessus, on remarque que, partant d'une suite x, on a construit par un procédé de filtrage-décimation deux suites x_0, x_1, qui grâce à la formule de synthèse, redonnent x. Soit $\tilde{T}_0$ l'opérateur défini sur $\ell^2(\mathbb{Z})$ par

$$(\tilde{T}_0 x)(n) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} \tilde{h}_k \, x(2n-k), \qquad n \in \mathbb{Z}, \ x \in \ell^2(\mathbb{Z}).$$

Soient $x \in \ell^2(\mathbb{Z})$ et $j \in \mathbb{N}^*$. L'algorithme se décompose de la manière suivante

- *Analyse.* On calcule les $(j+1)$ suites $x_1 = \tilde{T}_0 x$, $x_2 = \tilde{T}_0 T_0 x, \ldots,$ $x_j = \tilde{T}_0 T_0^{j-1} x$ et $y_j = T_0^j x$.

- *Synthèse.* Pour $m = j, j-1, \ldots, 1$, on itère la procédure $y_{m-1} = \mathcal{S}(y_m, x_m)$.

- On retrouve finalement $y_0 = x$.

Dans la pratique, la suite x analysée admet un support fini de longueur L (en général L est nettement supérieure à la taille N du filtre H_0). Les supports des suites $x_1, x_2, \ldots, x_j$ et y_j sont alors respectivement de longueur $L/2, L/4, \ldots, L/2^j$, et $L/2^j$ (noter que la somme est égale à L). On réduit ensuite la longueur de ces supports en effectuant une approximation $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_j, \tilde{y}_j$ de $x_1, x_2, \ldots, x_j, y_j$ (phase de compression des données). Enfin on applique l'algorithme de synthèse aux suites $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_j, \tilde{y}_j$, ce qui fournit une approximation de x.

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Analytic continuation of Dirichlet series

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1. Introduction

The questions considered in this paper arose from the study [KS] of I. Fredholm's (insufficient) proof that the gap series $\sum_0^\infty a^n \zeta^{n^2}$ (where $0 < |a| < 1$) is nowhere continuable across $\{|\zeta| = 1\}$. The interest of Fredholm's method ([F], [ML]) is not so much its efficacy in proving gap theorems (indeed, much more general results can be got by other means, *cf.* the Fabry gap theorem in [Di]) as in the connection it made between certain special gap series and partial differential equations. For a full discussion of this see [KS]; here we shall only outline the salient points to provide motivation for a study of some function-theoretic questions that arise naturally when one tries to extend Fredholm's method to other kinds of gaps. As our starting point we take a slightly more general gap series than that of Fredholm, namely

$$\varphi(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{n^2}, \tag{1.1}$$

where $\{a_n\}$ are complex and

$$0 < \overline{\lim_{n\to\infty}} |a_n|^{1/n} < 1. \tag{1.2}$$

Note that the radius of convergence is 1 (this would be so also under the weaker, and more natural condition where the right hand inequality in (1.2) is replaced by

$$\overline{\lim_{n\to\infty}}\, |a_n|^{1/n} < +\infty \tag{1.3}$$

but the method to be employed is simpler when (1.2) is assumed). Now,

$$u(z,w) = \sum_{n=0}^{\infty} a_n\, e^{nz+n^2 w} \tag{1.4}$$

is convergent to a holomorphic function for $(z,w) \in \mathbb{C} \times \mathcal{L}$, where

$$\mathcal{L} = \{w \in \mathbb{C} :\ \operatorname{Re} w < 0\}$$

and satisfies

$$\frac{\partial u}{\partial w} = \frac{\partial^2 u}{\partial z^2}\,. \tag{1.5}$$

For real z, w this is of course the "heat equation", with w as the time variable, but here we consider the variables as complex. The initial value problem for (1.5) with data on $\{w = w_0\}$ is characteristic so, as S. Kovalevskaya already explained in her Habilitationsschrift [Ko], even holomorphic data $z \mapsto u(z, w_0)$ does not in general suffice to guarantee a local holomorphic solution of (1.5). (Weierstrass expressed great surprise at this result, and admiration for his pupil's discovery; *cf.* especially his letter to P. du Bois-Reymond of 15 December 1874, reproduced in *Acta Math.* **39**). In fact, implicit in her reasoning is the following stronger statement: *a solution to* (1.5) *holomorphic in a bidisk* $D_z \times D_w$, *where*

$$D_z = \{z :\ |z - z_0| < R_1\}\,, \qquad D_w = \{w :\ |w - w_0| < R_2\} \tag{1.6}$$

extends holomorphically to $\mathbb{C} \times D_w$. (This can nowadays be deduced from general theorems, *cf.* [Ki] or [BS], also [H, Theorem 9.4.8]. See also [KS] for a simple proof).

Fredholm misunderstood Kovalevskaya's result, interpreting it to imply that if, for a solution u to (1.5) in the bidisk (1.6), the function $w \mapsto u(z_0, w)$ extends holomorphically across a boundary point w_1 of D_w, then $z \mapsto u(z, w_1)$ extends holomorphically to all of $\mathbb{C}$. This was the tool for Fredholm's attempt to prove the non-continuability of (1.1), and is (as shown in [KS]) incorrect. We emphasize that the error lies in attempting to draw conclusions from the behaviour of $w \mapsto u(z_0, w)$ *for just one value of* z_0.

To "save" Fredholm's idea one can first establish the following refinement of the above-mentioned result of Kovalevskaya. We precede it with a convenient definition.

Definition. *Let f be a function of one complex variable, holomorphic on a neighborhood of z_0. Then, for $k \in \mathbb{N}$ the k-fold symmetrization of f about z_0 is the function $t \mapsto F(t; z_0, k)$ where*

$$F(t; z_0, k) = \frac{1}{k} \sum_{j=0}^{k-1} f(z_0 + \omega^j t), \qquad \omega = e^{2\pi i/k}.$$

Note that F is holomorphic on a neighborhood of $t = 0$. The following is proven in [KS].

Theorem A. *If u is holomorphic on the bidisk* (1.6) *and satisfies* (1.5) *there, and $w \mapsto u(z_0, w)$ extends holomorphically to a neighborhood of a boundary point w_1 of D_w, then the 2-fold symmetrization of $z \mapsto u(z, w_1)$ about z_0 extends to $\mathbb{C}$ as an entire function of order at most* 2.

For later purposes note that if (1.5) is replaced by

$$\frac{\partial u}{\partial w} = \frac{\partial^k u}{\partial z^k}, \qquad k \geq 3 \tag{1.7}$$

the corresponding conclusion holds for the k-fold symmetrization of f about z_0.

Now we can apply Fredholm's idea correctly to show that φ in (1.1) is not continuable across any point $\zeta = e^{i v_0}, v_0 \in \mathbb{R}$. Indeed, if it were then, with u given by (1.4), $u(0, w)$ would extend from $\mathcal{L}$ to a neighborhood of its boundary point $w_0 = e^{iv_0}$ and so, by Theorem A, the 2-fold symmetrization about 0 of $\sum_{n=0}^{\infty} a_n e^{in^2 v_0} e^{nz}$ would extend as an entire function, that is

$$\sum_{n=0}^{\infty} a_n e^{in^2 v_0} \left(e^{nz} + e^{-nz}\right)$$

would extend from a neighborhood of $z = 0$ to the entire z-plane without singularities. But, because of assumption (1.2) this is a Laurent series in e^z with finite positive convergence radii. Since a Laurent series must have at least one singularity on each boundary circle of its annulus of convergence, we have a contradiction, and the noncontinuability of φ in (1.1) is proved.

Mittag-Leffler's exposition [ML] of Fredholm's idea ends with the suggestion that the method employed can be applied to more general situations. Let us see what happens when we try to apply the (corrected) Fredholm method to showing that $\sum_{n=0}^{\infty} a_n \zeta^{n^3}$ is not continuable across any point of $\partial\mathbb{D}$, where again we assume (1.2) (since the gaps are *bigger* one might expect the proof to be *easier*, but the strangeness of the method is that it does not work this way, as we will see). Introduce again the variable change $\zeta = e^w$ and look at

$$u(z,w) = \sum_{n=0}^{\infty} a_n \, e^{nz} \, e^{n^3 w}$$

which is holomorphic on $\mathbb{C} \times \mathcal{L}$ and satisfies

$$\frac{\partial u}{\partial w} = \frac{\partial^3 u}{\partial z^3} \, . \tag{1.8}$$

By the generalized form of Theorem A, if $u(0,w)$ were continuable across a point $w = i\, v_0$, $(v_0 \in \mathbb{R})$ of $\partial\mathcal{L}$, then the 3-fold symmetrization of $z \mapsto u(z, i\, v_0)$ about 0 would be entire, *i.e.*

$$\sum_{n=0}^{\infty} a_n \, e^{in^3 v_0} \left(e^{nz} + e^{\omega n z} + e^{\omega^2 n z} \right) , \tag{1.9}$$

where $\omega = e^{2\pi i/3}$, would be entire. But, could this happen? Now (1.9) is no longer a Laurent series in e^z, but a Dirichlet series of quite general type: $\sum c_m \, e^{\lambda_m z}$ with *complex* exponents $\{\lambda_m\}$ lying on *three rays* through 0. Even if (1.2) prevents the series from converging on the whole z-plane, there are no general theorems that rule out the analytic continuability of (1.9) to the whole plane. (Indeed, see [L] for discussion of phenomena which may occur).

It is fairly easy to show (see below, Section 4.2) that if we strengthen (1.2) to

$$0 < \overline{\lim_{n\to\infty}} \, |a_n|^{1/n} \le c$$

for a *sufficiently small* $c > 0$, then (1.9) cannot extend to all of $\mathbb{C}$, and thus, in this case, we do obtain the noncontinuability of $\sum_0^{\infty} a_n \, \zeta^{n^3}$.

But, perhaps surprisingly, the argument really fails essentially if only (1.2) is assumed. That is (and this is one of the main results of the present paper):

There exists complex $\{c_n\}_{n=0}^{\infty}$ *with*

$$0 < \overline{\lim}\, |c_n|^{1/n} = \delta < 1 \tag{1.10}$$

such that

$$\sum_{n=0}^{\infty} c_n \left(e^{nz} + e^{\omega nz} + e^{\omega^2 nz}\right), \tag{1.11}$$

where $\omega = e^{2\pi i/3}$ (note that we have absolute convergence on a neighborhood of $z = 0$) *extends without singularities to all of* $\mathbb{C}$. *Indeed, the sum of this series can vanish identically.*

An equivalent form of the last statement is obtained by evaluating the Taylor coefficients of (1.11) at $z = 0$:

There exist $\{c_n\}$ *satisfying* (1.10) *such that*

$$\sum_{n=0}^{\infty} c_n\, n^{3k} = 0\,, \qquad k = 0, 1, 2, \ldots$$

(*where* 0^0 *is interpreted as* 1) .

This formulation naturally leads to the consideration of the equations

$$\sum_{n=0}^{\infty} c_n\, n^{pk} = 0\,, \qquad k = 0, 1, 2, \ldots \tag{1.12}$$

We shall show that solutions satisfying (1.10) exist for each $p > 2$, but never for $p \leq 2$. Moreover, for $p > 2$ there is no solution if $\delta < \delta_p$ where δ_p is sufficiently small, and for p integral we shall find the best possible value of δ_p . In the course of this work, certain other questions which arise naturally will also be discussed.

The rest of the paper is organized as follows. Section 2 deals with cases where (1.12) (and some more general equation systems) admits only the solution $c_n = 0$. This is closely interwoven with known results concerning quasi-analytic functions. Section 3 contains our main result (Theorem 3.1) which shows the sharpness, in an important case, of the uniqueness theorem of Section 2; this example sheds light on the possibility of extending Fredholm's method to other kinds of gaps. In Section 4 it is shown that under certain conditions a function defined by a Dirichlet series of fairly general type cannot be analytically continued much beyond its domain of absolute convergence; this enables one to prove non-continuability of certain gap series by (a modification of) Fredholm's method. Section 5 contains a brief discussion of integral analogues of the problem treated in Sections 2 and 3; here fairly complete results are much easier to obtain.

2. A uniqueness problem for Dirichlet series.

Let us first consider a rather general situation, a Dirichlet series

$$\sum_{n=1}^{\infty} c_n\, e^{\lambda_n z}\,, \tag{2.1}$$

where $\{\lambda_n\}$ and $\{c_n\}$ are complex. We may of course assume the λ_n are pairwise distinct. From this point on various combinations of hypotheses could be made, some leading to uniqueness theorems and others not.

J. Wolff [W] constructed in 1921 examples that imply one can find $\{\lambda_n\}$ bounded and $\{c_n\}$ not all zero satisfying

$$\sum_{n=1}^{\infty} |c_n| < +\infty \tag{2.2}$$

and such that (2.1) (which then converges for all complex z) sums to 0 (however, Dirichlet series are not discussed in [W]). This is equivalent to finding a nontrivial solution $\{c_n\}$ satisfying (2.2) to the infinite system of linear equations

$$\sum_{n=1}^{\infty} c_n\, \lambda_n^k = 0\,, \qquad k = 0, 1, 2, \ldots \tag{2.3}$$

Wolff's result is not given in terms of (2.3) but rather as the solution of a then long-standing uniqueness question concerning series of the type

$$\sum_{n=1}^{\infty} \frac{c_n}{z - z_n}\,, \tag{2.4}$$

where $\{z_n\} \subset \mathbb{C}$. If (2.2) holds, (2.4) converges uniformly on compact subsets of $\mathbb{C} \setminus K$, where K denotes the closure of $\{z_n\}$, and various investigators (Borel, Carleman, Denjoy, Wolff, Beurling, ...) have studied conditions under which (a) the "apparent singularities" $\{z_n\}$ of the sum (2.4) really are singular points for the sum function (which is analytic on each component of $\mathbb{C} \setminus K$), and (b) in case there is more than one component, the sum functions corresponding to different components are analytic continuations of one another. (*e.g.* Borel showed that (a) and (b) may fail if only (2.2) is imposed while they hold if $\overline{\lim}\, |c_n|^{1/n} = 0$.) The uniqueness problem for (2.4) is of course subsumed under (a). Henceforth we will not mention interpretations of our results involving series (2.4), but refer the reader to [BSZ] for this connection.

A. Beurling showed [Be, pp. 209-210] that a series (2.1) can converge everywhere to zero with bounded $\{\lambda_n\}$ and non-zero $\{c_n\}$ that satisfy not merely (2.2) but much stronger conditions, *e.g.*

$$|c_n| \le \exp\left(- n/(\log n)^2 \right), \tag{2.5}$$

whereas this is not possible if

$$\overline{\lim}\, |c_n|^{1/n} < 1 .$$

Returning to Dirichlet series (2.1), we will in the remainder of this section be considering cases where $\lambda_n > 0$ and $\lambda_n \to \infty$. We begin with a basic uniqueness theorem. This is in principle known, as well as the corollaries we present; these results are scattered in the literature on quasi-analytic functions and Banach algebras. We need them to put in proper perspective the results of Section 3, and we include proofs for the reader's convenience.

Theorem 2.1. *Let* $0 < \lambda_1 < \lambda_2 < \dots$, *and*

$$\overline{\lim_{n\to\infty}}\, \frac{(\log n)^2}{\lambda_n} = 0 . \tag{2.6}$$

Suppose, for some $\varepsilon > 0$,

$$|c_n| \le e^{-\varepsilon\sqrt{\lambda_n}} . \tag{2.7}$$

If

$$\sum_{n=1}^{\infty} c_n\, \lambda_n^k = 0 , \qquad k = 0, 1, 2, \dots, \tag{2.8}$$

then all c_n *vanish.*

REMARK. This can be interpreted as a uniqueness theorem for series (2.1). Indeed, (2.6) and (2.7) imply that (2.1) as well as all its formally differentiated series converge absolutely for $\{z : \operatorname{Re} z \leq 0\}$, to some function f. Then (2.8) is the assertion that f and all its derivatives vanish at the boundary point 0 of this half-plane.

Before giving the proof, let us note some corollaries.

Corollary 1. ([Ca 2]). *If* $p > 0$, *and for some* $\varepsilon > 0$

$$|c_n| \leq \exp\big(-\varepsilon\, n^{p/2}\big) \tag{2.9}$$

then

$$\sum_{n=1}^{\infty} c_n\, n^{pk} = 0\,, \qquad k = 0, 1, 2, \ldots \tag{2.10}$$

implies $c_n = 0$ *for all* n *.*

This is just the case $\lambda_n = n^p$ of the theorem, and much of the rest of this paper is devoted to the question of *sharpness of the condition* (2.9). A few cases follow from well known results.

First of all, look at the case $p = 2$. The corollary says that if $\{c_n\}$ decay exponentially, and $\sum_1^\infty c_n\, n^{2k}$ all vanish then all c_n vanish. Here we certainly cannot weaken the hypothesis of exponential decay to, say

$$|c_n| \leq \exp\big(-a\, n^{\alpha}\big) \tag{2.11}$$

for some $\alpha < 1$ since, as is well known from the theory of quasi-analytic classes (*cf.* [M]), given $\alpha < 1$, there is a nontrivial function $\sum_{n=1}^\infty c_n \cos n\theta$, where $\{c_n\}$ satisfies (2.11), for which all derivatives vanish at $\theta = 0$, which is to say $\sum_1^\infty c_n\, n^{2k} = 0$ for $k = 0, 1, 2, \ldots$ See also [Ha, p. 27 ff.] for a pioneering discussion in this vein.

Next, examine the case $p = 1$. The corollary says that

$$|c_n| \leq \exp\big(-\varepsilon\, n^{1/2}\big) \tag{2.12}$$

and

$$\sum_1^{\infty} c_n\, n^{k} = 0\,, \qquad k = 0, 1, \ldots, \tag{2.13}$$

imply that all c_n vanish, or what is the same, (2.12) and the presence of an infinite order zero of $\sum_1^\infty c_n\, e^{in\theta}$ at some θ_0 imply all c_n vanish. This is due to Carleson [Ca2]. Here again, one cannot weaken hypothesis (2.12), say to

$$|c_n| \le \exp\big(-b\, n^\beta\big) \tag{2.14}$$

with $\beta < 1/2$. Indeed, it can be shown that if $\beta < 1/2$ the unique outer function F_σ in the unit disk satisfying

$$|F_\sigma(e^{i\theta})| = \exp\Big(-\Big|\sin\frac{\theta}{2}\Big|^{-\sigma}\Big), \qquad |\theta| \le \pi,$$

where $\sigma < 1$, has Taylor coefficients $\{c_n\}$ satisfying (2.14) if $\sigma = \sigma(\beta)$ is sufficiently close to 1. (Again, *cf.* [Ha, pp. 27 ff.] for closely related material.)

In the next section we shall discuss the sharpness of (2.9) in some other, more delicate cases. We may remark (as we will see in Section 5) that for the *integral analogue* of these problems matters are much simpler: different values of p are reducible to one another by a simple scaling argument (change of variables) but that is not possible with series. From a technical point of view, we stress that *examples to show the sharpness of* (2.9) *are the main concern of this paper.*

Corollary 2. ([Ca2]). *If* $f(z) = \sum_{n=1}^\infty c_n\, z^n$*, where* $\{c_n\}$ *satisfy* (2.12)*, and* f *has infinitely many zeroes in the open unit disk* $\mathbb{D}$*, then* $f \equiv 0$.

Proof. By Corollary 1 it is enough to show $f(e^{i\theta})$ has an infinite order zero at $\theta = \theta_0$, if f vanishes at a sequence $\{z_j\} \subset \mathbb{D}$ with $\lim z_j = e^{i\theta_0}$. This is a well-known fact; we include the simple proof. It is based on

Lemma. [TW, Prop. 4.5]. *If* f *is analytic in* $\mathbb{D}$ *and its Taylor coefficients* $\{a_n\}$ *satisfy*

$$|a_n| = O(n^{-k}), \qquad n \to \infty, \tag{2.15}$$

for every positive k (*or, what is the same,* $f \in C^\infty\big(\overline{\mathbb{D}}\big)$), *and* $f(\xi) = 0$ *for some* $\xi \in \partial\mathbb{D}$*, then* $f(z) = (z-\xi)g(z)$ *for some* g *analytic in* $\mathbb{D}$ *and in* $C^\infty(\overline{\mathbb{D}})$.

PROOF OF LEMMA. We may assume $\xi = 1$. Write $f = \sum_0^\infty a_n z^n$, $g = \sum_0^\infty b_n z^n$ where $g = (1-z)^{-1} f$ is analytic in $\mathbb{D}$. Then,

$$b_n = a_0 + a_1 + \cdots + a_n = -(a_{n+1} + a_{n+2} + \cdots)$$

since $\sum_0^\infty a_n = f(1) = 0$. Hence

$$|b_n| \le |a_{n+1}| + |a_{n+2}| + \cdots$$

so that, using (2.15), also $\{b_n\}$ satisfies the estimates (2.15), hence $g \in C^\infty(\overline{\mathbb{D}})$ and the lemma is proved.

DEDUCTION OF COROLLARY 2. If f vanishes at infinitely many points $\{z_j\}$ of $\mathbb{D}$ and $\xi \in \partial\mathbb{D}$ is a limit point of $\{z_j\}$ then $f(\xi) = 0$, so $f = (z-\xi)\, g(z)$ where $g \in C^\infty(\overline{\mathbb{D}})$. Now, $g(z_j) = 0$, so $g(\xi) = 0$ and hence $g = (z-\xi)\, h$ for some $h \in C^\infty(\overline{\mathbb{D}})$. Thus,

$$f(z) = (z-\xi)^2\, h(z)\,, \qquad h \in C^\infty(\overline{\mathbb{D}})\,.$$

Continuing in this fashion we see that for each m we have

$$f(z) = (z-\xi)^m\, f_m(z)$$

for a suitable $f_m \in C^\infty(\overline{\mathbb{D}})$. Thus, f has a zero of infinite order at ξ, which completes the proof of Corollary 2.

REMARK. It is not hard to show that there are non-trivial functions analytic in $\mathbb{D}$ whose Taylor coefficients satisfy (2.14), for any prescribed $\beta < 1/2$, with infinitely many zeroes in $\mathbb{D}$.

PROOF OF THEOREM 2.1. Note that (2.6) and (2.7) imply the absolute convergence of each of the series (2.8). Consider now the function

$$g(x) = \sum_{n=1}^\infty c_n \cos(\lambda_n^{1/2} x)\,, \qquad x \in \mathbb{R}\,. \tag{2.16}$$

In view of (2.7), g extends as an analytic function of $z = x + iy$ into a strip $\{z : \ |\mathrm{Im}\, z| < \delta\}$ for some $\delta > 0$. Then (2.8) expresses the fact that all even-order derivatives of g vanish at $z = 0$. Since g is an even function, $g \equiv 0$. Now, $g(x)$ is the Fourier-Stieltjes transform of the discrete measure which places masses $c_n/2$ at points $\pm\lambda_n^{1/2}$. By the uniqueness theorem for Fourier-Stieltjes transforms this measure vanishes, *i.e.* all c_n are zero. This concludes the proof.

REMARK. The hypothesis $|c_n| \leq e^{-\varepsilon\lambda_n^{1/2}}$ in Theorem 2.1 could be weakened. What is essential is that c_n are small enough so that

$$\sum c_n \cos(\lambda_n^{1/2} x)$$

falls into a quasi-analytic class on $\mathbb{R}$, in the sense of Denjoy-Carleman. One knows precisely what decay of $\{c_n\}$ is necessary for this, *cf.* [M]. We shall not however pursue this kind of generalization, which involves only well-known ideas.

Carleson [Ca2] obtains Corollary 1 in a somewhat different manner. He introduces

$$\varphi(s) = \sum_{n=1}^{\infty} c_n\, n^s \tag{2.17}$$

which is clearly an entire function of s under the hypothesis (2.9). It is easy to see (2.9) implies the estimate

$$\log|\varphi(\sigma + i\tau)| \leq \frac{2}{p}\, \sigma \log \sigma + O(\sigma) \tag{2.18}$$

for $\sigma > 0$. He now applies the following theorem, for which see [Ca1]:

If φ is analytic in the right half-plane and satisfies

$$|\varphi(\sigma + i\tau)| \leq C\, e^{m(\sigma)}\,, \tag{2.19}$$

where $m(\sigma)$ is convex on $\mathbb{R}^+$ and for some $p > 0$

$$\int_1^{\infty} \exp\big((-p/2)\, m(\sigma)/\sigma\big)\, d\sigma = \infty\,, \tag{2.20}$$

and

$$\varphi(pk) = 0\,, \qquad k = 0, 1, 2, \ldots, \tag{2.21}$$

then $\varphi \equiv 0$.

To obtain Corollary 1 from this one uses (2.18) to verify that (2.19) and (2.20) hold, and (2.21) is just (2.10); hence $\varphi \equiv 0$, which easily implies that all c_n vanish.

The theorem employed by Carleson is known to be sharp, but that does not imply the sharpness of Corollary 1 because a function satisfying an estimate (2.19) is not necessarily representable as a Dirichlet series (2.17).

Since the theorem is only stated, but not proved in [Ca1], we refer the reader to [Mal, pp. 184-185] for a proof.

3. An example of non-uniqueness and some of its ramifications.

Theorem 3.1. *For any $p > 2$, writing $\lambda_n = n^p$ $(n \ge 0)$, there exists a complex sequence $\{c_n\}$ satisfying*

$$\overline{\lim_{n\to\infty}} |c_n|^{1/n} = \delta_p = \exp\Big(- \pi \operatorname{ctg} \frac{\pi}{p}\Big) \tag{3.1}$$

such that

$$f(z) = \sum_{n=0}^{\infty} c_n \, e^{-\lambda_n z} \tag{3.2}$$

(*which converges for* Re $z \ge 0$, *and extends as a C^∞ function to the closed right half-plane*) *has an infinite-order zero at $z = 0$. In other terms,*

$$\sum_{n=0}^{\infty} c_n \, n^{pk} = 0 \,, \qquad k = 0, 1, 2, \dots \tag{3.3}$$

Moreover, for positive x

$$|f(x)| \le C \, \exp\big(- c\, x^{-1/p}\big) \,, \tag{3.4}$$

where C, c are positive constants.

For integral p, the constant on the right side of (3.1) *is sharp, in the sense that no such sequence $\{c_n\}$ exists with $0 < \overline{\lim}_{n\to\infty} |c_n|^{1/n} < \delta_p$.*

We postpone the proof, and discuss some consequences of the theorem. Let $p \geq 3$ be an integer, and let $\{c_n\}$ be as in the theorem. As in Section 1, form the p-fold symmetrization of the function

$$g(z) = \sum_{n=0}^{\infty} c_n \, e^{-nz} \tag{3.5}$$

about the origin. We may denote this by $F(z;p)$.

Since the p-fold symmetrization about 0 of e^z is $\sum_{k=0}^{\infty} z^{pk}/(pk)!\,$, one computes easily

$$\begin{aligned} F(z;p) &= \sum_{n=0}^{\infty} c_n \sum_{k=0}^{\infty} \frac{(-nz)^{pk}}{(p\,k)!} \\ &= \sum_{k=0}^{\infty} \frac{z^{pk}}{(p\,k)!} \sum_{n=0}^{\infty} c_n \, (-n)^{pk} = 0 \end{aligned}$$

in view of (3.3). We thus have

Corollary. *For any integer* $p \geq 3$ *there exists a Dirichlet series* (3.5) *whose coefficients satisfy* (3.1) (*and hence: g is analytic in a half-plane* $\{\operatorname{Re} z \geq -\delta\}$ *for some* $\delta > 0$) *whose p-fold symmetrization about the origin vanishes identically. In other terms,*

$$\sum_{k=0}^{p-1} \sum_{n=0}^{\infty} c_n \exp\big(-\omega^k nz\big) \equiv 0\,, \qquad \omega = e^{2\pi i/p}\,. \tag{3.6}$$

Remark. Note that (3.6) is a Dirichlet series of general type whose "exponents" are the set $\{-\omega^k n : \ 0 \leq k \leq p-1,\, n \in \mathbb{N}\}$ which is distributed along p rays through the origin. Condition (3.1) guarantees that this series converges absolutely on a neighborhood of $z = 0$, yet not in the whole plane. But the sum is an entire function (indeed, zero!). This behaviour is in stark contrast with the cases $p = 1$ (Taylor series in e^{-z}) and $p = 2$ (Laurent series in e^{-z}). Recalling our discussion of Fredholm's method in Section 1, we see that (1.9) *could* in fact be entire, subject to (1.2) ... so this method encounters an unforeseen difficulty when applied to a series with gaps like $\sum a_n \, \zeta^{n^3}$. (Thus, Mittag-Leffler's opinion that Fredholm's method could be generalized may be too optimistic; however, *some* gap series of type $\sum a_n \, \zeta^{n^3}$ can be exhibited by Fredholm's method *by requiring* $\overline{\lim}\, |a_n|^{1/n}$ *suitably small*, see the discussion following Theorem 4.1 below.)

PROOF OF THEOREM 3.1. The proof is based on a construction that has been used previously by Hirschman and Jenkins [HJ1], [HJ2], Anderson [A] and others for somewhat different purposes. Let

$$\varphi(w) = \prod_{n=1}^{\infty} \left(1 + \frac{w}{n^p}\right) . \tag{3.7}$$

Clearly φ is an entire function. By estimates given later, we will show it has order $1/p$, and moreover that

$$f(x) = (2\pi i)^{-1} \int_{\gamma} \varphi(w)^{-1} e^{xw}\, dw , \tag{3.8}$$

where $x \in \mathbb{R}$, and γ denotes the imaginary axis traversed from $-\infty$ to $+\infty$, is an absolutely convergent integral; and that translating γ parallel to itself (to a position that does not contain a point $-n^{-p}$ ($n \in \mathbb{N}$)) preserves convergence, and changes the integral only by the sum of residues of the poles passed over. Moving the contour leftwards to the position

$$\gamma_m = \{\operatorname{Re} w = -(\lambda_m \lambda_{m+1})^{1/2}\} , \tag{3.9}$$

where for convenience we denote

$$\lambda_m = m^{-p} \tag{3.10}$$

and letting $m \to \infty$ gives, formally,

$$f(x) = \sum_{n=1}^{\infty} \varphi'(-\lambda_n)^{-1}\, e^{-\lambda_n x} . \tag{3.11}$$

As we will show later, for $n > n_0$ we have

$$\log |\varphi'\,(-\lambda_n)\,| \sim (\pi \operatorname{ctg}(\pi/p) + o(1))\, n \tag{3.12}$$

as $n \to \infty$, and so

$$f(z) = \sum_{n=1}^{\infty} \varphi'(\lambda_n)^{-1} e^{-\lambda_n z} \tag{3.13}$$

converges uniformly for z on compact subsets of $\{\operatorname{Re} z > 0\}$. We shall show that this function f satisfies the requirements of the theorem. Thus, $c_n = \varphi'(\lambda_n)^{-1}$, and (3.12) implies (3.1).

We will first verify (3.4) which, since clearly f is C^∞ on the closed right half-plane, implies (3.3) (of course (3.4) is much stronger than (3.3)). Fix $x > 0$ in (3.8) and move γ to the right, to $\{w : \operatorname{Re} w = x^{-1}\}$. A crude estimate gives

$$|f(x)| \leq \frac{e}{2\pi} \int_{(1/x)-i\infty}^{(1/x)+i\infty} |\varphi(w)|^{-1} \, |dw| \tag{3.14}$$

and to get (3.4) from this we require a lower bound for $|\varphi(w)|$. We have for Re $w = u > 0$,

$$|\varphi(w)| = |1 + w| \, |1 + 2^{-p} w| \prod_{n=3}^{\infty} |1 + n^{-p} w| \tag{3.15}$$

and the infinite product is not less than

$$\prod_{n=3}^{\infty} \big(1 + n^{-p} u\big) \geq \prod_{3 \leq n \leq u^{1/p}} \big(n^{-p} u\big) \geq (N!)^{-p} \, u^{N-3} \,,$$

where N denotes the least integer $\geq u^{1/p}$. Simple estimates based on Stirling's formula show the last expression exceeds $\exp(p\, u^{1/p} - c \log u)$ for some positive c (henceforth $c, c_1, \ldots$ will designate positive constants whose precise value is of no concern). Hence, from (3.15),

$$|\varphi(u + iv)| \geq c_1 \, (1 + v^2) \, \exp\big((p/2) \, u^{1/p}\big)$$

and inserting this in (3.14) (with $u = 1/x$) gives (3.4).

We turn now to the estimate (3.12). From (3.7),

$$\varphi' \left(-\lambda_n\right) = n^p \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left(1 - \frac{n^p}{m^p}\right) . \tag{3.16}$$

Now,

$$\log \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left|1 - \frac{n^p}{m^p}\right| = \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \log \left|1 - \frac{n^p}{m^p}\right| = \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \psi\left(\frac{m}{n}\right) , \tag{3.17}$$

where

$$\psi(t) = \log|1 - t^{-p}|\,, \qquad t > 0\,. \tag{3.18}$$

Note that for $p > 1$ the improper Riemann integral of ψ over $(0, +\infty)$ exists, and since ψ is piecewise monotone (decreasing on $(0, 1)$, increasing on $(1, +\infty)$) it is easy to verify that the Riemann sums

$$n^{-1} \sum_{\substack{m=1 \\ n \neq n}}^{\infty} \psi\left(\frac{m}{n}\right)$$

converge to $A_p = \int_0^\infty \psi(t)\,dt$. Thus

$$\sum_{\substack{m=1 \\ m \neq n}}^{\infty} \psi\left(\frac{m}{n}\right) \sim A_p\, n\,, \qquad \text{as } \; n \to \infty\,,$$

so

$$\prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left|1 - \frac{n^p}{m^p}\right| = \exp\big((A_p + o(1))\, n\big)$$

which yields (3.12) if we verify that the improper Riemann integral

$$A_p = \int_0^\infty \log|1 - t^{-p}|\,dt \tag{3.19}$$

has the value $\pi \operatorname{ctg}(\pi/p)$. For this, see [GH, p. 84, formula 8a]. This proves (3.12).

To conclude the proof of the theorem, we now derive the estimates for φ that were needed to justify moving the contour of integration in (3.8). These are well known (*cf.* [Boa, p. 19]), but for the reader's convenience we present the details since some of the intermediate estimates will be required. We first study φ in $\mathbb{C} \setminus \overline{\Omega}_\beta$ where $\beta < \pi/2$ and

$$\Omega_\beta = \{z : \; |\pi - \arg z| < \beta\}\,. \tag{3.20}$$

In $\mathbb{C} \setminus \overline{\Omega}_\beta$, $\log\varphi$ has a single-valued analytic branch that is real on the positive real axis. In the following calculation, we work with this branch, and restrict z to $\mathbb{C} \setminus \overline{\Omega}_\beta$.

$$\log\varphi(z) = \sum_{n=1}^{\infty} \log\left(1 + \frac{z}{n^p}\right) = \int_{0+}^{\infty} \log(1 + t^{-p} z)\, d[t]$$

where $[\cdot]$ denotes the greatest integer function. Applying partial integration to the last integral gives

$$\begin{aligned}\log\varphi(z) &= p\,z\int_1^\infty \frac{[t]}{t}\,\frac{dt}{z+t^p}\\ &= p\,z\int_1^\infty \frac{dt}{z+t^p} + O\Big(|z|\int_1^\infty \frac{dt}{t\,|z+t^p|}\Big)\,.\end{aligned} \tag{3.21}$$

The first integral on the right can be evaluated by applying Cauchy's theorem. First, observe that

$$\int_1^\infty \frac{dt}{z+t^p} = \int_0^\infty \frac{dt}{z+t^p} + O\big(|z|^{-1}\big)\,, \qquad |z|\to\infty$$

and, writing $z = re^{i\theta}$, we move the line of integration in the right-hand integral to $\{\arg t = \theta/p\}$, so that $t = se^{i\theta/p}$, $s>0$. We get

$$\int_0^\infty \frac{dt}{z+t^p} = \Big(\exp\,i\big(\frac{1}{p}-1\big)\theta\Big)\int_0^\infty \frac{ds}{r+s^p}$$

which after some simplification becomes $C_p\,(re^{i\theta})^{-1+1/p}$, where C_p is a positive constant ($C_p = \int_0^\infty (1+u^p)^{-1}\,du$). Thus, the first term on the right of (3.21) is $C'\,z^{1/p} + O(1)$ for large $z\in\mathbb{C}\setminus\overline{\Omega}_\beta$. We will now show that the second term in (3.21) is of smaller order. This will establish:

$$\varphi(z) \sim \exp\big(C'_p\,z^{1/p}\big) \tag{3.22}$$

holds for large z outside each sector symmetric with respect to the negative real axis (where C'_p is a positive constant depending only on p). In particular, φ *is of order* $1/p$. Observe that (3.22) gives the rapid decrease of $|\varphi\,(x+iy)|^{-1}$ as $|y|\to\infty$ which was required for moving the line of integration since, from (3.22) (with $x+iy = z = re^{i\theta}$), we get

$$|\varphi(z)| \sim \exp\big(C'_p\,\cos(\theta/p)\,r^{1/p}\big)\,, \qquad z\in\mathbb{C}\setminus\overline{\Omega}_\beta \tag{3.23}$$

and, since $p>2$, $\cos(\theta/p)$ is positive for $|\theta|\le\pi$.

Now we estimate the O-term in (3.21). Consider separately the cases $x\ge 0$ and $x<0$.

For $x\ge 0$, $|z+t^p|^2 \ge |z|^2 + t^{2p}$, so

$$|z|\int_1^\infty \frac{dt}{t|z+t^p|} \le |z|\int_1^\infty \frac{dt}{t\,(t^{2p}+|z|^2)^{1/2}} \le C\,\log(1+|z|)\,,$$

while for $\operatorname{Re} z < 0$, $z \in \mathbb{C} \setminus \overline{\Omega}_\beta$ we have

$$|z + t^p| \geq c\,|z|\,, \qquad c = c(\beta)\,.$$

Hence

$$\int_1^\infty \frac{dt}{t\,|z + t^p|} = \Big(\int_1^T + \int_T^\infty\Big) \frac{dt}{t\,|z + t^p|}$$

(where $T = (2\,|z|)^{1/p}$), which is

$$\leq \int_1^T \frac{dt}{c\,|z|\,t} + \int_T^\infty \frac{dt}{t\,(1/2)\,t^p}$$

(since $|z + t^p| \geq t^p - |z| \geq (1/2)\,t^p$ for $t \geq T$),

$$= (c\,|z|)^{-1} \log T + O(T^{-p}) = O\big(|z|^{-1} \log |z|\big)$$

for large $|z|$. Hence, *the* O*-term in* (3.21) *is* $O(\log |z|)$ *for large* $|z|$ *outside* Ω_β, and (3.22) is completely proved.

To conclude the proof of our theorem we need only verify one last point: that the integral (3.8) tends to zero as γ is moved sufficiently far to the left, since that was assumed in the passage from (3.8) to (3.11). For this purpose we recall that, since φ is of order $1/p < 1/2$ there is a sequence $R_j \to \infty$ such that

$$\log m\,(R_j) > \cos(\pi/p)\, \log M(R_j)\,, \tag{3.24}$$

where $m(R), M(R)$ denote the minimum and maximum of $|\varphi(w)|$ on $\{|w| = R\}$, respectively (see [Boa, p. 40, Theorem. 3.1.6]). Thus, we may move γ leftwards in (3.8) through the sequence γ_j, where (for some fixed β, say $\beta = \pi/4$) γ_j consists of an arc of $\{|w| = R_j\}$ inside Ω_β, completed by vertical half-lines outside Ω_β. It follows at once from (3.23) and (3.24) that $\int_{\gamma_j} |\varphi(w)|^{-1}\,|dw| \to 0$ as $j \to \infty$. This completes the proof of Theorem 3.1, apart from the sharpness of the constant in (3.1) to which point we shall return in the next Section.

4. A result implying existence of singularities.

We saw, in the Corollary to Theorem 3.1, that if p is an integer, $p \geq 3$, there exist complex $\{c_n\}$ with

$$0 < \varlimsup_{n\to\infty} |c_n|^{1/n} = \delta < 1 \tag{4.1}$$

such that

$$g(z) = \sum_{n=0}^{\infty} c_n\, e^{-nz} \tag{4.2}$$

(the series converges absolutely to a function holomorphic on a neighborhood of 0) whose p-fold symmetrization about 0,

$$F(z;p) = \frac{1}{p}\sum_{k=0}^{p-1} g\left(\omega^k z\right), \qquad \omega = e^{2\pi i/p}, \tag{4.3}$$

vanishes identically (and hence, is analytically continuable to all of $\mathbb{C}$). We now show that for small enough δ in (4.1), this cannot happen:

Theorem 4.1. *Let $p \geq 3$ be an integer and suppose $\{c_n\}$ satisfy* (4.1) *with*

$$\delta < \delta_p = \exp\big(-\pi \operatorname{ctg}(\pi/p)\big).$$

Then $F(z;p)$ does not extend to all of $\mathbb{C}$ without singularities; in fact, it has a singularity in the disk centered at 0 of radius $(\pi^2+(\log(1/\delta))^2)^{1/2}$.

Observe that this implies the assertion in Theorem 3.1 concerning the sharpness of the constant. We do not know whether it is sharp also for non-integral p.

Before giving the proof, we observe a consequence of the theorem: for $\{c_n\}$ satisfying (4.1) with $\delta < \delta_p$ *the power series* $\sum_0^\infty c_n \zeta^{n^p}$ *is not continuable across any point of* $\partial\mathbb{D}$; this follows by the (modified) Fredholm argument we presented in Section 1. Of course, this argument has the blemish that the upper bound imposed on δ is purely fortuitous; one could remove it by combining the argument given with Hadamard's multiplication of singularities theorem and a few other things (see [KS] for details).

Proof of Theorem 4.1. Observe that (4.2) converges absolutely for $\operatorname{Re} z > -\sigma$, where

$$\sigma = \log \frac{1}{\delta} \tag{4.4}$$

and since g has period $2\pi i$, it must have a singularity at a point $z_0 = -\sigma + iy_0$ for some y_0 with $-\pi < y_0 \le \pi$. Let L denote the line segment joining 0 to z_0. It is clear that if $\{\omega^k z_0 : \ k = 1, 2, \ldots, p-1\}$ all lie in $\{\operatorname{Re} z > -\sigma\}$, the analytic continuation of $F(z;p)$ from 0 to z_0 along L is possible as far as z_0, and encounters a singularity at z_0, since each $g(\omega^k z)$ for $1 \le k \le p-1$ is analytic on a neighborhood of the closure of L. And it is geometrically obvious that this occurs if the angle subtended by the points $-\sigma \pm iy_0$ at 0 is less than $2\pi/p$. Since this angle can not exceed $2 \operatorname{arctg}(\pi/\sigma)$, we will have a singularity of $F(z;p)$ at z_0 if

$$2 \operatorname{arctg} \frac{\pi}{\sigma} < \frac{2\pi}{p} \,,$$

i.e. if $\sigma > \pi \operatorname{ctg}(\pi/p)$, and in view of (4.4) this completes the proof.

5. The integral analogue.

Corollary 1 to Theorem 2.1 has an integral analogue:

Let f be a complex-valued continuous function $[0, +\infty)$ and $p > 0$. If

$$|f(x)| \le C \, \exp(-c\, x^{p/2}) \tag{5.1}$$

for some positive constants C, c and

$$\int_0^\infty f(x)\, x^{pk}\, dx = 0 \,, \qquad k = 0, 1, 2, \ldots, \tag{5.2}$$

then $f \equiv 0$.

The proof is similar to that given in the discrete case, and may be left to the reader. As before, we are mainly interested in examples to show the sharpness of the condition (5.1), and shall prove:

Theorem 5.1. *Given any $p > 0$ and $0 < q < p/2$ there is a continuous f on $[0, \infty]$, $f \not\equiv 0$, satisfying* (5.2) *and*

$$|f(x)| \le C \exp(-x^q), \qquad x > 0. \tag{5.3}$$

Proof. As is well known, for $0 < b < 1$ there is a non-null entire function F of exponential type satisfying

$$|F(x)| \le e^{-|x|^b}, \qquad x \in \mathbb{R}. \tag{5.4}$$

The Fourier transform $\widehat{F}$ of F (which is infinitely differentiable) has compact support. Multiplying F by a suitable exponential $e^{i\lambda z}$ we can arrange that $\widehat{F}$ vanishes on a neighborhood of 0, and that the even part of F,

$$F_e(x) = \frac{F(x) + F(-x)}{2}$$

does not vanish identically; we assume this is done. Since all derivatives of $\widehat{F}$ vanish at 0,

$$\int_{-\infty}^{\infty} F(x)\, x^n \, dx = 0, \qquad n = 0, 1, \ldots, \tag{5.5}$$

hence

$$\int_0^{\infty} F_e(x)\, x^{2k} \, dx = 0, \qquad k = 0, 1, 2, \ldots \tag{5.6}$$

Changing variables in (5.6),

$$\int_0^{\infty} F_e(t^{p/2})\, t^{pk}\, t^{p/2-1} \, dt = 0, \qquad k = 0, 1, \ldots \tag{5.7}$$

Letting $f(t) = t^{p/2-1} F_e(t^{p/2})$ and observing (5.4), it is clear that f satisfies (5.3) if b is chosen greater than $2q/p$. This completes the proof.

Remarks. The idea to look at the integral analogue was suggested to us by D. J. Newman, who also provided an elegant proof of a weaker variant of Theorem 6.1, which we here sketch briefly. Starting from

$$\Gamma(np) = \int_0^{\infty} e^{-t}\, t^{np-1} dt,$$

where $p > 2$, rotate the line of integration to $\{\arg t = \pi/p\}$ giving

$$\Gamma(np) = (-1)^n \int_0^\infty \exp(-e^{i\pi u/p})\, u^{np-1}\, du$$

whence, taking imaginary parts

$$\int_0^\infty e^{-\cos \pi u/p} \frac{\sin(\sin(\pi u/p))}{u}\, u^{np}\, du = 0$$

holds for $n = 0, 1, 2, \ldots$ Thus, writing $a = \cos(\pi/p) > 0,\ b = (1-a^2)^{1/2}$ we see that, setting

$$f(u) = e^{-au}\left(\frac{\sin(\sin b\,u)}{u}\right), \tag{5.8}$$

$$\int_0^\infty f(u)\, u^{np}\, du = 0, \qquad \text{for} \quad n = 0, 1, \ldots \tag{5.9}$$

This gives f which is precisely the continuous analogue of the sequence $\{c_n\}$ we constructed in Theorem 3.1: it decays exponentially on $\mathbb{R}^+$ and the moments (5.9) vanish. But for fixed $p > 2$, this result is weaker than Theorem 5.1 (compare (5.8) with (5.3)). Moreover, the method we used to prove Theorem 5.1 can be made to yield more, since F could be chosen to satisfy not merely (5.4), but

$$|F(x)| \le \exp(-\varphi(|x|))$$

where φ is any sufficiently regular positive increasing function on $\mathbb{R}^+$ with

$$\int_0^\infty \frac{\varphi(t)}{1+t^2}\, dt < \infty.$$

Since these ideas are very well known, we do not pursue the details.

It would be interesting to extend Theorem 3.1 to the discrete analogue of Theorem 5.1, but we do not know how to do this.

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Ondelettes et espaces de Besov

Gérard Bourdaud

1. Introduction.

Soit $(\psi_\varepsilon)_{\varepsilon\in E}$ une famille finie d'ondelettes telle que l'ensemble des fonctions

$$x \mapsto 2^{jn/2}\, \psi_\varepsilon(2^j x - k)\,,$$

où $\varepsilon \in E, j \in \mathbb{Z}, k \in \mathbb{Z}^n$, constitue une base orthonormée de $L^2(\mathbb{R}^n)$. L'un des traits caractéristiques des bases d'ondelettes, c'est d'être des bases non seulement de $L^2(\mathbb{R}^n)$ mais encore de "tous" les espaces fonctionnels usuels. Ainsi, pour la distribution

$$f = \sum_{\varepsilon,j,k} c_{\varepsilon,j,k}\, 2^{j(n/p-s)}\, \psi_\varepsilon(2^j(\cdot) - k)\,,$$

on peut espérer l'équivalence de normes

$$(1) \qquad \|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^n)} \approx \sum_{\varepsilon}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n} |c_{\varepsilon,j,k}|^p\Big)^{q/p}\Big)^{1/q}\,,$$

où $\dot{B}_p^{s,q}(\mathbb{R}^n)$ désigne l'espace de Besov *homogène*. Si les fonctions ψ_ε appartiennent à la classe de Schwartz et ont tous leurs moments nuls, l'équivalence (1) est satisfaite quel que soit $s \in \mathbb{R}$. Par contre, si les ψ_ε sont des ondelettes à supports compacts, des ondelettes-splines, ou, plus généralement, des ondelettes r-régulières, au sens d'Yves Meyer [ME], l'équivalence (1) n'est satisfaite que pour $-r < s < r$. Il est dès lors naturel de rechercher les conditions de régularité minimales que doivent vérifier les ondelettes-mères ψ_ε de telle sorte qu'on ait l'équivalence (1) pour un s donné.

Nous nous proposons d'établir que ces conditions minimales sont l'appartenance de ψ_ε à certains espaces de Besov d'ordre $\pm s$. Il s'agira d'espaces de Besov construits à partir non plus de L^p mais d'un sous-espace $\mathcal{E}_p$ de L^p que nous mettrons en évidence.

Soyons plus précis: l'équivalence (1) comporte en fait deux volets: l'*analyse*, qui consiste à estimer les $c_{\varepsilon,j,k}$ à partir de la norme de f dans l'espace de Besov $\dot{B}_p^{s,q}(\mathbb{R}^n)$, la *synthèse* qui permet de retrouver la norme de f connaissant la norme "amalgamée" de la suite $(c_{\varepsilon,j,k})$. Nous verrons que l'appartenance de ψ_ε à $\dot{B}^{s,1}(\mathcal{E}_p)$ permet la synthèse, alors que l'analyse est possible si ψ_ε appartient à $\dot{B}^{-s,1}(\mathcal{E}_{p'})$.

Ces considérations s'appliquent au cas $1 \leq p \leq +\infty$ et $1 \leq q \leq +\infty$, mais nous verrons qu'elles s'adaptent dans une certaine mesure aux cas $0 < p < 1$ ou $0 < q < 1$.

Equipés d'un tel critère, nous retrouverons aisément le théorème d'Yves Meyer sur les ondelettes r-régulières et les résultats plus ou moins classiques sur les splines ([CI], [O], [S]); nous en déduirons également une condition portant sur la transformée de Fourier de ψ_ε, qui conviendra aux ondelettes construites suivant l'algorithme de Mallat [MA] et Daubechies [D].

L'orthogonalité joue un rôle secondaire dans nos résultats, qui s'appliquent aussi bien à la transformation de Frazier-Jawerth ([FJ1], [FJ2]) qu'aux bases bi-orthogonales [CDF].

2. Inégalités d'échantillonnage.

2.1. Généralités.

Le calcul des coefficients d'ondelettes $c_{\varepsilon,j,k}$ peut s'interpréter comme la succession de deux opérations: un *filtrage* -la convolution avec une ondelette analysatrice de résolution 2^{-j}- puis un *échantillonnage*: le calcul des valeurs aux points du réseau $2^{-j}\mathbb{Z}^n$. On ne perdra pas de généralité en supposant $j = 0$, ce qui conduit à étudier l'opérateur

$$S_\psi(f) = \big((\psi * f)(k)\big)_{k\in\mathbb{Z}^n} \,.$$

Supposons ψ localement intégrable. Dès que f est continue, à support compact, $\psi * f$ est une fonction continue, dont les valeurs ponctuelles sont bien définies; cela signifie que S_ψ est un opérateur linéaire défini sur $C_c(\mathbb{R}^n)$, à valeurs dans l'espace des suites indexées par $\mathbb{Z}^n$. De la même façon

$$T_\psi(c) = \sum_{k\in\mathbb{Z}^n} c_k \psi(\cdot - k)$$

est un opérateur linéaire défini sur l'ensemble des suites à supports finis, à valeurs dans $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Les relations

$$(2) \qquad \langle S_\psi(f), c\rangle = \langle f, T_{\tilde\psi}(c)\rangle = \langle \tilde f, T_\psi(\tilde c)\rangle$$

(où l'on a posé $\tilde f(x) = f(-x)$) sont vérifiées par toute fonction $f \in C_c(\mathbb{R}^n)$ et toute suite c à support fini.

Définition 1. *Pour $p \in\,]0,+\infty]$, on définit*

i) *$\mathcal{E}_p$ comme l'ensemble des fonctions $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ pour lesquelles il existe $C = C(\psi) > 0$ tel que*

$$\|T_\psi(c)\|_{L^p(\mathbb{R}^n)} \le C\,\|c\|_{\ell^p(\mathbb{Z}^n)}\,,$$

pour toute suite c à support fini,

ii) *$\mathcal{E}^*_p$ comme l'ensemble des fonctions $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ pour lesquelles il existe $C = C(\psi) > 0$ tel que*

$$\|S_\psi(f)\|_{\ell^p(\mathbb{Z}^n)} \le C\,\|f\|_{L^p(\mathbb{R}^n)}\,,$$

pour tout $f \in C_c(\mathbb{R}^n)$.

La "norme" de ψ dans $\mathcal{E}_p$ (respectivement $\mathcal{E}^*_p$) est la plus petite constante C qui intervienne dans i) (respectivement ii)) (nous utilisons les guillemets pour rappeler qu'il s'agit seulement d'une quasi-norme, quand $p < 1$). L'espace fonctionnel $\mathcal{E}^*_p$ est l'objet d'un travail de T. Tararykova [TA].

2.2. Etude de $\mathcal{E}_p$.

Une première approche consiste à utiliser au mieux l'inégalité de Young:

Proposition 1.

i) *Pour $0 < p \le 1$ on a $\mathcal{E}_p = L^p$, avec égalité des "normes".*

ii) *Soit* $1 \leq p \leq +\infty$; *pour que* ψ *appartienne à* $\mathcal{E}_p$, *il suffit que la fonction* $\mathbb{Z}^n$*-périodique*

$$x \mapsto \sum_{k \in \mathbb{Z}^n} |\psi(x-k)|$$

appartienne à $L^p(\mathbb{R}^n/\mathbb{Z}^n)$; *de plus la norme* $\|\psi\|_{\mathcal{E}_p}$ *est majorée par*

$$\Big(\int_{[0,1]^n} \Big(\sum_{k \in \mathbb{Z}^n} |\psi(x-k)|\Big)^p dx\Big)^{1/p}.$$

iii) *Pour que* ψ *appartienne à* $\mathcal{E}_\infty$, *il faut et il suffit que*

$$\sum_{k \in \mathbb{Z}^n} |\psi(\cdot - k)|$$

appartienne à L^∞; *de plus*

$$\|\psi\|_{\mathcal{E}_\infty} = \Big\|\sum_{k \in \mathbb{Z}^n} |\psi(\cdot - k)|\Big\|_\infty.$$

iv) *Il existe une fonction* ψ *telle que, pour tout* $p \in]1, +\infty[$, *on ait* $\psi \in \mathcal{E}_p$ *alors que*

$$\Big(\int_{[0,1]^n} \Big(\sum_{k \in \mathbb{Z}^n} |\psi(x-k)|\Big)^p dx\Big)^{1/p} = +\infty.$$

PREUVE. i) Le plongement de $\mathcal{E}_p \subset L^p$ étant clair, quel que soit $p > 0$, il suffit de prouver $L^p \subset \mathcal{E}_p$ pour tout $p \leq 1$; mais cela provient de l'inégalité

$$\Big\|\sum_{k \in \mathbb{Z}^n} c_k \psi(\cdot - k)\Big\|_p^p \leq \sum_{k \in \mathbb{Z}^n} |c_k|^p \|\psi(\cdot - k)\|_p^p = \|\psi\|_p^p \sum_{k \in \mathbb{Z}^n} |c_k|^p.$$

ii) Supposons $p \geq 1$; on a

$$\Big\|\sum_{k \in \mathbb{Z}^n} c_k \psi(\cdot - k)\Big\|_p = \Big(\int_{[0,1]^n} \sum_{m \in \mathbb{Z}^n} \Big|\sum_{k \in \mathbb{Z}^n} c_k \psi(x+m-k)\Big|^p dx\Big)^{1/p}.$$

On applique alors l'inégalité de Young $\ell^p * \ell^1 \subset \ell^p$ à x fixé; cela donne

$$\Big\| \sum_{k\in\mathbb{Z}^n} c_k \psi(\cdot - k) \Big\|_p \leq \Big(\int_{[0,1]^n} \Big(\sum_{k\in\mathbb{Z}^n} |c_k|^p \Big) \Big(\sum_{k\in\mathbb{Z}^n} |\psi(x-k)| \Big)^p dx \Big)^{1/p} .$$

iii) Supposons $\psi \in \mathcal{E}_\infty$ et, dans un premier temps, ψ *continue*; l'inégalité

$$\Big| \sum_{k\in\mathbb{Z}^n} c_k \psi(x-k) \Big| \leq C \sup_k |c_k|$$

est alors vraie *quel que soit* $x \in \mathbb{R}^n$. Fixons $x_0 \in \mathbb{R}^n$ et donnons-nous une partie finie Λ de $\mathbb{Z}^n$. On définit la suite $\{c_k\}$ par

$$c_k \, \psi(x_0 - k) = |\psi(x_0 - k)|$$

si $k \in \Lambda$ et $\psi(x_0 - k) \neq 0$, $c_k = 0$ sinon. On a alors $\sup |c_k| \leq 1$, ce qui donne

$$\sum_{k\in\Lambda} |\psi(x_0 - k)| \leq C .$$

Dans le cas général, où l'on a seulement $\psi \in L^\infty$, on considère une fonction $\theta \in C_c(\mathbb{R}^n)$, positive, d'intégrale 1 et on pose $\theta_j(x) = j^n \, \theta(jx)$ ($j \in \mathbb{N}^*$) puis $\psi_j = \theta_j * \psi$. On montre classiquement que ψ_j est une fonction continue telle que $\psi_j \to \psi$ (presque partout). On écrit ensuite

$$\begin{aligned} \Big| \sum_k c_k \, \psi_j(x-k) \Big| &\leq j^n \int \theta(jy) \Big| \sum_k c_k \, \psi(x-y-k) \Big| \, dy \\ &\leq \Big(\sup_k |c_k| \Big) \, \|\psi\|_{\mathcal{E}_\infty} . \end{aligned}$$

La continuité des ψ_j conduit à

$$\sum_k |\psi_j(x-k)| \leq \|\psi\|_{\mathcal{E}_\infty} ,$$

pour tout $x \in \mathbb{R}^n$; d'où l'on déduit aisément

$$\sum_k |\psi(x-k)| \leq \|\psi\|_{\mathcal{E}_\infty} ,$$

pour presque tout x.

iv) La fonction

$$\psi(x) = \frac{\sin \pi x}{x}$$

appartient à $\mathcal{E}_p$ pour $1 < p < +\infty$: c'est -comme l'observe Y. Meyer [ME, Chapitre I, Théorème 1]- une conséquence de la continuité ℓ^p de la transformation de Hilbert discrète; en revanche

$$\sum_{k\in\mathbb{Z}} |\psi(x-k)| = |\sin \pi x| \sum_{k\in\mathbb{Z}} \frac{1}{|x-k|}$$

est égal à $+\infty$ pour tout x non entier.

REMARQUE. La partie ii) de la proposition est un résultat de R. Jia et C. Micchelli [JM].

Proposition 2. *ψ appartient à $\mathcal{E}_2$ si et seulement si la fonction $2\pi\mathbb{Z}^n$-périodique*

$$\xi \mapsto \Big(\sum_{k\in\mathbb{Z}^n} |\widehat{\psi}(\xi + 2k\pi)|^2 \Big)^{1/2}$$

appartient à L^∞; de plus la norme $\|\psi\|_{\mathcal{E}_2}$ n'est autre que la norme L^∞ de cette fonction.

PREUVE. A la suite $\{c_k\}_{k\in\mathbb{Z}^n}$, à support fini, associons le polynôme trigonométrique

$$m(\xi) = \sum_{k\in\mathbb{Z}^n} c_k e^{-ik\cdot\xi} \ ;$$

il vient alors

$$\Big\| \sum_{k\in\mathbb{Z}^n} c_k \psi(\cdot - k) \Big\|_2^2 = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} |m(\xi)|^2 \Big(\sum_{k\in\mathbb{Z}^n} |\widehat{\psi}(\xi + 2k\pi)|^2 \Big) \, d\xi \, .$$

En prenant la borne supérieure pour toutes les suites $\{c_k\}$ telles que

$$\sum_{k\in\mathbb{Z}^n} |c_k|^2 = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} |m(\xi)|^2 \, d\xi \le 1 \, ,$$

on obtient précisément la norme L^∞ de la fonction

$$\xi \mapsto \sum_{k\in\mathbb{Z}^n} |\widehat{\psi}(\xi + 2k\pi)|^2 \, .$$

Le théorème de Riesz-Thorin nous fournit une troisième méthode pour estimer la norme dans $\mathcal{E}_p$:

Proposition 3. *Pour tout $p > 1$, on a $\mathcal{E}_1 \bigcap \mathcal{E}_\infty \subset \mathcal{E}_p$ et*

$$\|\psi\|_{\mathcal{E}_p} \leq \|\psi\|_{\mathcal{E}_1}^{1/p} \|\psi\|_{\mathcal{E}_\infty}^{1-1/p} ;$$

autrement dit

$$\|\psi\|_{\mathcal{E}_p} \leq \|\psi\|_1^{1/p} \Big\| \sum_{k\in\mathbb{Z}^n} |\psi(\,\cdot\,-k)| \Big\|_\infty^{1-1/p} .$$

Dès que $\psi \in \mathcal{E}_p$, l'opérateur T_ψ se prolonge par continuité à $\ell^p(\mathbb{Z}^n)$ (pour $p < +\infty$) et à $c_0(\mathbb{Z}^n)$ (pour $p = +\infty$); comme on peut s'y attendre, le prolongement continu de T_ψ est encore l'opérateur qui, à la suite $\{c_k\}$, associe

$$\sum_{k\in\mathbb{Z}^n} c_k\psi(\,\cdot\,-k)\,,$$

mais il convient de donner un sens précis à la somme infinie ci-dessus. Ce sera chose faite dans l'énoncé suivant, dont la preuve aisée est laissée au lecteur:

Proposition 4.

i) *Supposons $\psi \in \mathcal{E}_p$ ($p < +\infty$). Alors, pour toute suite $\{c_k\} \in \ell^p(\mathbb{Z}^n)$, la famille $\{c_k\psi(\,\cdot\,-k)\}_{k\in\mathbb{Z}^n}$ est sommable dans $L^p(\mathbb{R}^n)$; l'opérateur*

$$\{c_k\} \mapsto \sum_{k\in\mathbb{Z}^n} c_k\psi(\,\cdot\,-k)$$

ainsi défini n'est autre que le prolongement continu de T_ψ.

ii) *Supposons $\psi \in \mathcal{E}_\infty$. Alors, pour toute suite $\{c_k\} \in \ell^\infty(\mathbb{Z}^n)$, la série $\sum_{k\in\mathbb{Z}^n} c_k\,\psi(x-k)$ converge absolument pour presque tout $x \in \mathbb{R}^n$ et sa somme est une fonction essentiellement bornée; l'opérateur*

$$\{c_k\} \mapsto \sum_{k\in\mathbb{Z}^n} c_k\psi(\,\cdot\,-k)$$

ainsi défini est continu de $\ell^\infty(\mathbb{Z}^n)$ dans $L^\infty(\mathbb{R}^n)$.

2.3. Etude de $\mathcal{E}_p^*$.

Proposition 5.

i) *Pour tout* $p \in [1,+\infty]$, *on a* $\mathcal{E}_p^* = \mathcal{E}_{p'}$, *avec égalité des normes; de plus, si* $\psi \in \mathcal{E}_{p'}$, *alors, pour tout* $f \in L^p$, *la fonction* $\psi * f$ *est continue; l'opérateur qui à* $f \in L^p$ *associe la suite* $\{(f*\psi)(k)\}_{k\in\mathbb{Z}^n}$ *est continu de* $L^p(\mathbb{R}^n)$ *dans* $\ell^p(\mathbb{Z}^n)$ *et il prolonge l'opérateur* S_ψ.

ii) *Pour* $p < 1$, *on a* $\mathcal{E}_p^* = \{0\}$.

PREUVE. i) La première assertion est une conséquence immédiate de (2). Supposons $\psi \in \mathcal{E}_{p'}$; on a *a fortiori* $\psi \in L^{p'}$ et l'on sait que la convolution entre $\psi \in L^{p'}$ et $f \in L^p$ est une fonction continue. Supposons d'abord $p < +\infty$ et considérons une suite $\{f_j\}$ de fonctions continues à supports compacts telle que $f_j \to f$ dans L^p; alors la suite $S_\psi(f_j)$ converge dans ℓ^p vers une certaine suite $\{c_k\}_{k\in\mathbb{Z}^n}$. Pour un $k \in \mathbb{Z}^n$ fixé, on a $(f_j * \psi)(k) \to c_k$; l'inégalité de Hölder

$$\|(f-f_j)*\psi\|_\infty \le \|f-f_j\|_p \, \|\psi\|_{p'}$$

entraîne alors $c_k = (f*\psi)(k)$. Si $\psi \in \mathcal{E}_\infty^*$, autrement dit $\psi \in L^1$, on a aussitôt

$$|\psi * f(k)| \le \|\psi\|_1 \, \|f\|_\infty ,$$

pour tout $f \in L^\infty$; on obtient donc un opérateur linéaire continu de L^∞ dans ℓ^∞ qui prolonge évidemment S_ψ.

ii) Soit $p < 1$; supposons l'existence de $C > 0$ tel que

$$\Big(\sum_{k\in\mathbb{Z}^n} |f*\psi(k)|^p\Big)^{1/p} \le C\,\|f\|_p ,$$

quel que soit $f \in C_c(\mathbb{R}^n)$; en appliquant cette estimation à la fonction translatée $f(\cdot - a)$, on obtient *a fortiori*

$$|(f*\psi)(a)| \le C\,\|f\|_p .$$

cela montre que $f \mapsto (f*\psi)(a)$ est une forme linéaire continue sur $L^p(\mathbb{R}^n)$; le Théorème de Day entraîne alors $(f*\psi)(a) = 0$. Ceci étant vrai quels que soient $f \in C_c(\mathbb{R}^n)$ et $a \in \mathbb{R}^n$, il vient $\psi = 0$.

2.4. L'échantillonnage pour $0 < p < 1$.

On vient de voir qu'il n'y pas d'opérateur d'échantillonnage S_ψ défini sur $L^p (p < 1)$. Nous tournerons cette difficulté en échantillonnant exclusivement des fonctions entières de type exponentiel. On dispose alors des inégalités classiques de Plancherel-Polya.

Donnons nous un nombre $\gamma \in]0, \pi[$ et une fonction $\theta \in \mathcal{S}(\mathbb{R}^n)$ telle que $\widehat{\theta}(\xi) = 1$ sur le cube $[-\gamma, \gamma]^n$ et que le support de $\widehat{\theta}$ soit inclus dans $]-\pi, \pi[^n$. Les différentes constantes C dépendront exclusivement de p, γ, θ et de la dimension n. On a alors classiquement

$$f(x) = \sum_{k \in \mathbb{Z}^n} f(\frac{k}{R})\, \theta(Rx - k)\,, \tag{3}$$

pour toute fonction entière f telle que $\mathrm{supp}\,(\widehat{f}) \subset [-\gamma R, \gamma R]^n$.

Lemme 1. *Il existe des constantes positives C_1 et C_2 telles que, pour toute fonction f à spectre dans $[-\gamma, \gamma]^n$, on ait*

$$C_1 \, \|f\|_p \leq \Big(\sum_{k \in \mathbb{Z}^n} |f(k)|^p \Big)^{1/p} \leq C_2 \, \|f\|_p \,.$$

PREUVE. On part de l'identité (3), avec $R = 1$; en translatant f, il vient

$$f(k) = \sum_{l \in \mathbb{Z}^n} f(l + x)\, \theta(l - x - k)\,,$$

quels que soient $x \in [0,1]^n$ et $k \in \mathbb{Z}^n$; la décroissance rapide de θ conduit à l'inégalité

$$|\theta(k - x)|^p \leq C\,(1 + |k|)^{-n-1}\,,$$

uniformément en $x \in [0,1]^n$; d'où

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} |f(k)|^p &\leq C \sum_{l,k} \Big(\int_{[0,1]^n + l} |f(x)|^p \, dx \Big) \, (1 + |l - k|)^{-n-1} \\ &\leq C' \int_{\mathbb{R}^n} |f(x)|^p \, dx \,. \end{aligned}$$

Dans l'autre sens, on utilise directement (3):

$$|f(x)|^p \leq \sum_{k\in\mathbb{Z}^n} |f(k)|^p \, |\theta(x-k)|^p \,,$$

d'où

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx \leq \sum_{k\in\mathbb{Z}^n} |f(k)|^p \, \|\theta\|_p^p \,.$$

Lemme 2. *Il existe une constante $C > 0$ telle que, pour toutes fonctions f et g, à spectres dans $[-\gamma, \gamma]^n$, on ait*

$$\Big(\sum_{k\in\mathbb{Z}^n} |f*g(k)|^p\Big)^{1/p} \leq C \, \|f\|_p \, \|g\|_p \,.$$

PREUVE. On écrit

$$(f*g)(k) = \sum_{l,m} f(l)\, g(m)\, c_{k+m-l} \,,$$

où $c_k = \int \theta(k-z)\,\theta(z)\,dz$; il vient alors

$$\sum_{k\in\mathbb{Z}^n} |f*g(k)|^p \leq \Big(\sum_{k\in\mathbb{Z}^n} |f(k)|^p\Big)\Big(\sum_{k\in\mathbb{Z}^n} |g(k)|^p\Big)\Big(\sum_{k\in\mathbb{Z}^n} |c_k|^p\Big) \,.$$

On conclut en utilisant la décroissance rapide de la suite $\{c_k\}$ et le Lemme 1. Un simple changement d'échelle conduit alors au

Lemme 3. *Il existe une constante $C > 0$ telle que, pour tout $R > 0$ et toutes fonctions f et g, à spectres dans $[-\gamma R, \gamma R]^n$, on ait*

$$\Big(\sum_{k\in\mathbb{Z}^n} |f*g(\frac{k}{R})|^p\Big)^{1/p} \leq C \, R^{n(2/p-1)} \, \|f\|_p \, \|g\|_p \,.$$

On en arrive enfin au résultat principal de ce paragraphe:

Proposition 6. *Il existe une constante $C > 0$ telle que, pour tout $R > 0$ et toutes fonctions f et g, à spectres dans $[-\gamma R, \gamma R]^n$, on ait*

$$\Big(\sum_{k\in\mathbb{Z}^n} |f * g(k)|^p\Big)^{1/p} \leq C\, R^{n(2/p-1)} \sup\{R^{-n/p}, 1\}\, \|f\|_p\, \|g\|_p\,.$$

PREUVE. Admettons un instant l'inégalité

$$\sum_{k\in\mathbb{Z}^n} \frac{1}{(1+R\,|x-k|)^{n+1}} \leq C \sup\{R^{-n}, 1\}\,, \tag{4}$$

uniforme par rapport à $x \in \mathbb{R}^n$. En remplaçant x par l/R dans (4), il vient

$$\sum_{k\in\mathbb{Z}^n} \frac{1}{(1+|Rk-l|)^{n+1}} \leq C \sup\{R^{-n}, 1\}\,. \tag{5}$$

On écrit alors $(f * g)(k)$ suivant la formule (3) ce qui donne

$$\sum_{k\in\mathbb{Z}^n} |f * g(k)|^p \leq \sum_{l,k} \big|(f * g)(\frac{l}{R})\big|^p\, |\theta(Rk-l)|^p\,.$$

La décroissance rapide de θ, l'inégalité (5) et le Lemme 3 permettent alors de conclure.

Il nous reste à prouver (4). Le premier membre de cette inégalité est une fonction $u_R(x)$, $\mathbb{Z}^n$-périodique, bornée. Si $R \geq 1$, il suffit d'écrire

$$u_R(x) \leq u_1(x) \leq \|u_1\|_\infty\,.$$

Supposons $R < 1$ et $x \in [0,1]^n$. Pour une constante $\alpha > 0$ convenable ($\alpha = 2\sqrt{n}$ par exemple), on a

$$|k| \geq \alpha R^{-1} \qquad \text{implique} \qquad |x-k| \geq \frac{|k|}{2}\ ;$$

cela donne

$$\sum_{|k|\geq\alpha R^{-1}} \frac{1}{(1+R\,|x-k|)^{n+1}} \leq \frac{2^{n+1}}{R^{n+1}} \sum_{|k|\geq\alpha R^{-1}} |k|^{-n-1} \leq \frac{C}{R^n}$$

et

$$\sum_{|k|<\alpha R^{-1}} \frac{1}{(1+R\,|x-k|)^{n+1}} \leq \sum_{|k|<\alpha R^{-1}} 1 \leq \frac{C}{R^n}\,.$$

3. Les espaces de Besov construits sur $\mathcal{E}_p$.

Le cadre naturel des espaces de Besov homogènes est celui des distributions *modulo les polynômes*, que nous rappellerons d'abord brièvement.

Pour un entier $\nu \in \mathbb{N}$, on désigne par $\mathcal{S}_\nu$, le sous-espace de $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ constitué des fonctions f telles que

$$\int f(x)\, x^\alpha\, dx = 0\,,$$

quel que soit $\alpha \in \mathbb{N}^n$, $|\alpha| \leq \nu$; il est naturel de poser $\mathcal{S}_{-1} = \mathcal{S}$ et $\mathcal{S}_\infty = \cap_{\nu\in\mathbb{N}}\mathcal{S}_\nu$. Le dual topologique de $\mathcal{S}_\nu$, noté $\mathcal{S}'_\nu$, n'est autre que l'espace des distributions tempérées modulo les polynômes de degrés au plus ν (modulo *tous* les polynômes si $\nu = +\infty$).

La définition des espaces de Besov repose traditionnellement sur une partition dyadique de l'unité

$$\sum_{j\in\mathbb{Z}} \widehat{\lambda}(2^j\xi) = 1 \qquad \xi \neq 0\,,$$

où la fonction $\widehat{\lambda}$ est supposée de classe C^∞, positive, radiale, à support compact, disons dans la couronne $1 \leq |\xi| \leq 3$; on définit alors les opérateurs Δ_j par

$$\widehat{\Delta_j f}(\xi) = \widehat{\lambda}(\frac{\xi}{2^j})\widehat{f}(\xi)\,.$$

Δ_j est un opérateur linéaire continu de $\mathcal{S}$ dans $\mathcal{S}_\infty$ et de $\mathcal{S}'_\nu$ dans $\mathcal{S}'$. Pour tout $f \in \mathcal{S}'_\infty$, on a

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f\,,$$

la série convergeant dans $\mathcal{S}'_\infty$ -ce qui signifie que, quel que soit $g \in \mathcal{S}_\infty$, on a

$$\sum_{j\in\mathbb{Z}} |\langle \Delta_j f, g\rangle| < +\infty\,.$$

Soit E un sous-espace de $\mathcal{S}'_\nu$, muni d'une norme complète rendant continue l'injection canonique $E \hookrightarrow \mathcal{S}'_\nu$, invariant isométriquement par translations. Pour $s \in \mathbb{R}$ et $q \in]0,+\infty]$, l'espace de Besov $\dot{B}^{s,q}(E)$ ("construit" sur E) est l'ensemble des $f \in \mathcal{S}'_\infty$ telles que

$$\left\{2^{js}\,\|\Delta_j f\|_E\right\}_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

L'invariance de E par translations a pour conséquence que l'espace de Besov ainsi défini ne dépend pas du choix spécifique de la fonction λ; on montre en effet l'énoncé suivant:

Proposition 7. *Pour toute fonction $g \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$, il existe une constante $C > 0$ telle que*

$$\left\|\{2^{js}\,\|g(2^{-j}D)f\|_E\}_{j\in\mathbb{Z}}\right\|_{\ell^q} \leq C \left\|\{2^{js}\,\|\Delta_j f\|_E\}_{j\in\mathbb{Z}}\right\|_{\ell^q}.$$

J. Peetre [P, Chapitre 11] a montré que cette définition s'étend au cas $E = L^p$ $(0 < p < 1)$, bien que E ne soit alors ni un espace de Banach, ni même un sous-espace de $\mathcal{S}'_\infty$. L'espace de Besov homogène usuel est $\dot{B}^{s,q}(L^p(\mathbb{R}^n))$, noté encore $\dot{B}^{s,q}_p(\mathbb{R}^n)$.

Proposition 8. *Pour $s \in \mathbb{R}$, $p \in\ ]0,+\infty]$, $q \in\ ]0,+\infty]$, on définit l'entier $\nu = \nu(s,p,q,n)$ par*

$$\nu = \begin{cases} \sup\left\{[s - \frac{n}{p}], -1\right\}, & \text{si } q > 1 \text{ ou } s - \frac{n}{p} \notin \mathbb{N}, \\ s - \frac{n}{p} - 1, & \text{si } q \leq 1 \text{ et } s - \frac{n}{p} \in \mathbb{N}. \end{cases}$$

Alors, pour tout $f \in \dot{B}^{s,q}_p(\mathbb{R}^n)$, la série $\sum_{j\in\mathbb{Z}} \Delta_j f$ converge dans $\mathcal{S}'_\nu$, de sorte que $\dot{B}^{s,q}_p(\mathbb{R}^n)$ s'identifie à un sous-espace, invariant par translations et par dilatations, de $\mathcal{S}'_\nu$.

PREUVE. Il s'agit d'établir

$$\sum_{j\in\mathbb{Z}} |\langle \Delta_j f, g\rangle| < +\infty, \tag{6}$$

quel que soit $g \in \mathcal{S}_\nu$. A cet effet, on introduit une fonction $\mu \in \mathcal{S}$ telle que $\widehat{\mu}$ soit positive, radiale, portée par la couronne $1/2 \leq |\xi| \leq 4$ et que $\widehat{\mu}\widehat{\lambda} = \widehat{\lambda}$. La suite d'opérateurs $\{M_j\}_{j\in\mathbb{Z}}$ est définie par

$$\widehat{M_j f}(\xi) = \widehat{\mu}(\frac{\xi}{2^j})\, \widehat{f}(\xi) \ ;$$

on a alors $\Delta_j M_j = \Delta_j$, ce qui permet d'écrire

$$\begin{aligned} |\langle \Delta_j f, g\rangle| &= |\langle \Delta_j f, M_j g\rangle| \\ &\leq \|\Delta_j f\|_\infty \, \|M_j g\|_1 \\ &\leq C\, 2^{jn/p} \, \|\Delta_j f\|_p \, \|M_j g\|_1 \end{aligned}$$

-la dernière inégalité résulte classiquement du fait que le spectre de $\Delta_j f$ est inclus dans la boule $|\xi| \leq 3 \cdot 2^j$ (dans le cas $p < 1$, on se reportera au Paragraphe 2.4 ou à [P, Lemme 1, p. 234]).

Pour $j \geq 0$ et tout entier $N \geq 0$, on a

$$\text{(7)} \quad M_j g(x) = \int 2^{jn}\, \mu(2^j y) \Big(g(x-y) - \sum_{|\alpha|\leq N} \frac{1}{\alpha!} (-y)^\alpha \, g^{(\alpha)}(x) \Big)\, dy \ ;$$

l'hypothèse $g \in \mathcal{S}$ conduit alors à

$$\text{(8)} \qquad \|M_j g\|_1 \leq C_N \, 2^{-j(N+1)} \, ,$$

d'où

$$|\langle \Delta_j f, g\rangle| \leq C_N \, 2^{-j(N+1-n/p)} \, \|\Delta_j f\|_p$$

et il nous suffira de choisir N tel que $N + 1 + s - n/p > 0$.

Pour $j < 0$, les fonctions μ et g échangent leurs rôles:

$$\text{(9)} \quad \begin{aligned} M_j g(x) = 2^{jn} \int g(y) \Big(&\mu(2^j x - 2^j y) \\ &- \sum_{|\alpha|\leq \nu} \frac{1}{\alpha!} (-2^j y)^\alpha \, \mu^{(\alpha)}(2^j x) \Big)\, dy \, , \end{aligned}$$

égalité qui conduit à l'estimation

$$\text{(10)} \qquad \|M_j g\|_1 \leq C\, 2^{j(\nu+1)} \ ;$$

pour en finir avec (6), il suffit de noter qu'on a $\nu + 1 + n/p - s > 0$ (l'inégalité large étant suffisante pour $q \leq 1$).

REMARQUE. On peut montrer que l'entier ν est le plus petit possible tel que $\dot{B}_p^{s,q}(\mathbb{R}^n)$ se "réalise" comme un sous-espace invariant par translations et par dilatations de $\mathcal{S}'_\nu$ [B].

En conclusion de ce paragraphe, on note que l'espace de Besov $\dot{B}^{s,q}(\mathcal{E}_p)$ est défini quels que soient $s \in \mathbb{R}$, $p, q \in]0, +\infty]$: pour $p \geq 1$, il suffit de faire $E = \mathcal{E}_p$ dans la définition générale de $\dot{B}^{s,q}(E)$, pour $p < 1$, on a (Proposition 1) $\mathcal{E}_p = L^p$, de sorte que $\dot{B}^{s,q}(\mathcal{E}_p)$ est l'espace de Besov $\dot{B}_p^{s,q}(\mathbb{R}^n)$ introduit par J. Peetre [P, Chapitre 11]. Le plongement $\mathcal{E}_p \subset L^p$ entraîne que $\dot{B}^{s,q}(\mathcal{E}_p)$ s'identifie naturellement à un sous-espace de $\mathcal{S}'_\nu$, suivant la Proposition 8.

4. Condition de synthèse.

Théorème 1. *Soit $s \in \mathbb{R}$, $p, q \in]0, +\infty]$, $u = \inf\{p, q, 1\}$. Quel que soit $\psi \in \dot{B}^{s,u}(\mathcal{E}_p)$ et quelle que soit la suite $\{c_{j,k}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ telle que*

$$A = \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p\Big)^{q/p}\Big)^{1/q} < +\infty\,,$$

la série

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} c_{j,k} 2^{j(n/p-s)} \psi(2^j x - k)$$

converge dans $\mathcal{S}'_\nu$ (où $\nu = \nu(s, p, q, n)$ suivant la Proposition 8) et sa somme appartient à $\dot{B}_p^{s,q}(\mathbb{R}^n)$ avec une norme majorée par $A\,\|\psi\|_{\dot{B}^{s,u}(\mathcal{E}_p)}$.

PREUVE. Vérifions d'abord la convergence dans $\mathcal{S}'_\nu$. Il suffit, pour cela, de prendre $g \in \mathcal{S}_\nu$ et de vérifier que la famille de nombres complexes

$$\big\{c_{j,k}\, 2^{j(n/p-s)}\, \langle \psi(2^j(\cdot) - k), g\rangle\big\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$$

est sommable; ou encore de trouver une constante $C = C(\psi, g) > 0$ telle que, pour tout ensemble fini $F \subset \mathbb{Z} \times \mathbb{Z}^n$, la somme

$$\sum_{(j,k)\in F} c_{j,k}\, 2^{j(n/p-s)}\, \langle \psi(2^j(\cdot) - k), g\rangle \tag{11}$$

soit dominée par C. Ecrivons $F = \cup_{j\in\Lambda}(\{j\} \times \Lambda_j)$, où Λ est une partie finie de $\mathbb{Z}$ et chaque Λ_j une partie finie de $\mathbb{Z}^n$. D'après (6), on a

$$f = \sum_{m\in\mathbb{Z}} \Delta_m f$$

dans $\mathcal{S}'_\nu$, quel que soit $f \in \dot{B}^{s,u}(\mathcal{E}_p)$, ce qui permet de réécrire la somme (11) sous la forme

$$\sum_{m\in\mathbb{Z}} \sum_{j\in\Lambda} \sum_{k\in\Lambda_j} \langle c_{j,k}\, 2^{j(n/p-s)} \Delta_m(\psi(2^j(\cdot) - k)), M_m g\rangle \,.$$

Ensuite on se sert de la relation

$$\Delta_m \left(\psi(2^j(\cdot) - k)\right) = (\Delta_{m-j}\psi)(2^j(\cdot) - k) \;;$$

(11) se trouve alors dominé par

$$\sum_{m\in\mathbb{Z}} \sum_{j\in\Lambda} 2^{j(n/p-s-n)} \left|\left\langle \sum_{k\in\Lambda_j} c_{j,k}\, \Delta_{m-j}\, \psi(\cdot\, - k), M_m g((\cdot)\, 2^{-j})\right\rangle\right|$$
$$\leq \sum_{m\in\mathbb{Z}} \sum_{j\in\Lambda} 2^{j(n/p-s)} \left\| \sum_{k\in\Lambda_j} c_{j,k}\, \Delta_{m-j}\, \psi(\cdot\, - k)\right\|_\infty \|M_m g\|_1 \,.$$

Puisque $\Delta_{m-j}\psi(\cdot\, - k)$ est à spectre dans la couronne $2^{m-j} \leq |\xi| \leq 3\cdot 2^{m-j}$, on a

$$\left\| \sum_{k\in\Lambda_j} c_{j,k}\, \Delta_{m-j}\, \psi(\cdot\, - k)\right\|_\infty \leq C\, 2^{(m-j)n/p} \left\| \sum_{k\in\Lambda_j} c_{j,k}\, \Delta_{m-j}\, \psi(\cdot\, - k)\right\|_p ,$$

de sorte que (11) est dominé par

$$C \sum_{m\in\mathbb{Z}} \sum_{j\in\Lambda} \left(2^{(m-j)s}\, \|\Delta_{m-j}\psi\|_{\mathcal{E}_p}\right) \Big(\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p\Big)^{1/p} 2^{m(n/p-s)}\, \|M_m g\|_1 \,.$$

Cette dernière expression est de la forme

$$\sum_{m\in\mathbb{Z}} \sum_{j\in\Lambda} a_{m-j}\, b_j u_m \;,$$

où, par hypothèse, $\{a_j\} \in \ell^{\inf\{q,1\}}(\mathbb{Z})$ et $\{b_j\} \in \ell^q(\mathbb{Z})$. Pour conclure, il suffit de prouver que la suite de terme général

$$u_m = 2^{m(n/p-s)}\, \|M_m g\|_1$$

appartient à $\ell^{\sup\{q,1\}'}(\mathbb{Z})$, mais cela est une conséquence immédiate des estimations (8) et (10).

On va maintenant estimer la "norme" $\dot{B}_p^{s,q}(\mathbb{R}^n)$ de la distribution

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} c_{j,k}\, 2^{j(n/p-s)}\, \psi(2^j x - k)\,.$$

La convergence de la série dans $\mathcal{S}'_\nu$ permet d'écrire

$$\Delta_m f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} c_{j,k}\, 2^{j(n/p-s)}\, (\Delta_{m-j}\psi)(2^j(\,\cdot\,) - k)\,.$$

A partir de là, il est commode de traiter séparément les cas $p \geq 1$ et $p < 1$. Dans le premier cas, on a

$$2^{ms}\, \|\Delta_m f\|_p \leq \sum_{j\in\mathbb{Z}} 2^{(m-j)s}\, \|\Delta_{m-j}\psi\|_{\mathcal{E}_p} \Big(\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p \Big)^{1/p},$$

autrement dit, la suite $\{2^{ms}\, \|\Delta_m f\|_p)_{m\in\mathbb{Z}}$ est majorée par la convolution entre les suites $\{a_j\}$ et $\{b_j\}$. On conclut alors à l'aide de l'inclusion

$$\ell^q * \ell^{\inf\{q,1\}} \subset \ell^q\,.$$

Pour $p < 1$, on a

$$\begin{aligned} 2^{ms}\, \|\Delta_m f\|_p &\leq \Big(\sum_{j\in\mathbb{Z}} 2^{(m-j)sp}\, \|\Delta_{m-j}\psi\|_p^p \Big(\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p \Big) \Big)^{1/p} \\ &\leq \Big(\sum_{j\in\mathbb{Z}} b_j^p\, a_{m-j}^p \Big)^{1/p}. \end{aligned}$$

Si $q \geq p$, on utilise l'inégalité de Young $\ell^{q/p} * \ell^1 \subset \ell^{q/p}$, ce qui donne

$$\begin{aligned} \Big(\sum_{m\in\mathbb{Z}} (2^{ms}\, \|\Delta_m f\|_p)^q \Big)^{p/q} &\leq \Big(\sum_{m\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}} b_j^p\, a_{m-j}^p \Big)^{q/p} \Big)^{p/q} \\ &\leq \Big(\sum_{j\in\mathbb{Z}} b_j^q \Big)^{p/q} \Big(\sum_{j\in\mathbb{Z}} a_j^p \Big) \\ &\leq \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p \Big)^{q/p} \Big)^{p/q} \|\psi\|^p_{\dot{B}_p^{s,p}(\mathbb{R}^n)}\,. \end{aligned}$$

Si $q < p$, on a

$$\begin{aligned}\sum_{m\in\mathbb{Z}}(2^{ms}\,\|\Delta_m f\|_p)^q &\le \sum_{m\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} b_j^p\, a_{m-j}^p\Big)^{q/p}\\ &\le \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} b_j^q\, a_{m-j}^q\\ &= \sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n}|c_{j,k}|^p\Big)^{q/p}\,\|\psi\|^q_{\dot B_p^{s,q}(\mathbb{R}^n)}\,.\end{aligned}$$

5. Condition d'analyse.

Théorème 2. *Pour tous $s\in\mathbb{R}$, $p\in[1,+\infty]$, $q\in]0,+\infty]$ et $u=\inf\{q,1\}$, il existe $C=C(s,n,p,q)>0$ tel que, pour tout $\psi\in\dot B^{-s,u}(\mathcal{E}_{p'})$ et tout $f\in\dot B_p^{s,q}(\mathbb{R}^n)$, la suite des coefficients*

$$c_{j,k}=2^{j(s-n/p)}\langle 2^{jn}\,\psi(2^j(\,\cdot\,)-k),f\rangle \tag{12}$$

soit bien définie et qu'on ait

$$\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n}|c_{j,k}|^p\Big)^{q/p}\Big)^{1/q}\le C\,\|\psi\|_{\dot B^{-s,u}(\mathcal{E}_{p'})}\|f\|_{\dot B_p^{s,q}(\mathbb{R}^n)}\,.$$

PREUVE. L'hypothèse $\psi\in\dot B^{-s,u}(\mathcal{E}_{p'})$ entraîne *a fortiori*

$$\psi\in\dot B_{p'}^{-s,\sup\{q,1\}'}(\mathbb{R}^n)\,.$$

Or $\dot B_{p'}^{-s,\sup\{q,1\}'}(\mathbb{R}^n)$ n'est autre que le dual de $\dot B_p^{s,q}(\mathbb{R}^n)$, si $p<+\infty$ et $q<+\infty$, et, dans les cas $p=+\infty$ ou $q=+\infty$, c'est un prédual de $\dot B_p^{s,q}(\mathbb{R}^n)$; le crochet $\langle\psi,f\rangle$ est donc bien défini, pour tout $f\in\dot B_p^{s,q}(\mathbb{R}^n)$.

Pour estimer les $c_{j,k}$, on introduit

$$\theta(x)=\overline{\psi}(-x)\qquad\text{et}\qquad\theta_j(x)=2^{jn}\,\theta(2^jx)\ ;$$

alors

$$c_{j,k}=2^{j(s-n/p)}\,(\theta_j*f)(k\,2^{-j})$$

$$= \sum_{m\in\mathbb{Z}} 2^{j(s-n/p)} (\Delta_m\theta_j * M_m f)(k\,2^{-j})$$
$$= \sum_{m\in\mathbb{Z}} 2^{j(s-n/p)} (\Delta_{m-j}\theta_j * M_m f((\cdot)2^{-j}))(k)\,,$$

ce qui conduit, grâce à la Proposition 5, à

$$\Big(\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p\Big)^{1/p} \le \sum_{m\in\mathbb{Z}} 2^{js}\, \|\Delta_{m-j}\theta\|_{\mathcal{E}_{p'}}\, \|M_m f\|_p$$
$$\le \sum_{m\in\mathbb{Z}} (2^{(j-m)s}\, \|\Delta_{m-j}\theta\|_{\mathcal{E}_{p'}})\,(2^{ms}\, \|M_m f\|_p)\,.$$

Cette dernière expression n'est autre que la convolution entre les suites de termes généraux

$$a_j = 2^{-js}\, \|\Delta_j\theta\|_{\mathcal{E}_{p'}} \qquad \text{et} \qquad b_j = 2^{js}\, \|M_j f\|_p\,.$$

La Proposition 7 conduit à

$$\Big(\sum_{j\in\mathbb{Z}} |b_j|^q\Big)^{1/q} \le C\, \|f\|_{\dot{B}^{s,q}_p(\mathbb{R}^n)}$$

et, par hypothèse sur ψ, on a

$$\{a_j\}_{j\in\mathbb{Z}} \in \ell^{\inf\{q,1\}}(\mathbb{Z})\,.$$

L'inclusion $\ell^q * \ell^{\inf\{q,1\}} \subset \ell^q$ fournit alors l'inégalité annoncée.

Théorème 3. *Soit $s \in \mathbb{R}$, $0 < p < 1$, $q \in]0, +\infty]$ et $u = \inf\{q,p\}$. Soit ψ une fonction telle que la suite $\{a_j\}_{j\in\mathbb{Z}}$ définie par*

$$a_j = \begin{cases} 2^{j(2n/p-n-s)}\, \|\Delta_j\psi\|_p\,, & j \ge 0\,,\\ 2^{j(n/p-n-s)}\, \|\Delta_j\psi\|_p\,, & j < 0\,,\end{cases}$$

appartienne à $\ell^u(\mathbb{Z})$. Alors, pour tout $f \in \dot{B}^{s,q}_p(\mathbb{R}^n)$, la suite des coefficients $\{c_{j,k}\}$ définie par (12) vérifie l'estimation

$$\Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p\Big)^{q/p}\Big)^{1/q} \le C \Big(\sum_{j\in\mathbb{Z}} |a_j|^u\Big)^{1/u} \|f\|_{\dot{B}^{s,q}_p(\mathbb{R}^n)}\,.$$

PREUVE. On commence par s'assurer que $c_{j,k}$ est bien défini; autrement dit que la condition $\{a_j\} \in \ell^u$ entraîne que la fonction ψ appartient au dual de $\dot{B}_p^{s,q}(\mathbb{R}^n)$. J. Peetre [P] a montré que le dual de $\dot{B}_p^{s,q}(\mathbb{R}^n)$ contient l'espace de Besov $\dot{B}_\infty^{n(1/p-1)-s,\sup\{q,1\}'}(\mathbb{R}^n)$. Or on a

$$\|\Delta_j\psi\|_\infty \le C\, 2^{jn/p}\, \|\Delta_j\psi\|_p \;;$$

l'hypothèse $\{a_j\} \in \ell^u$ entraîne alors $\psi \in \dot{B}_\infty^{n(1/p-1)-s,u}(\mathbb{R}^n)$; on conclut à l'aide de l'inégalité $u \le \sup\{q,1\}'$.

Pour estimer $c_{j,k}$, on utilise la Proposition 6, ce qui donne

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p & \\
&\le \sum_{k,m} 2^{j(sp-n)}\, |(\Delta_{m-j}\theta * M_m f(2^{-j}(\,\cdot\,)))(k)|^p \\
&\le C\Big(\sum_{m\ge j} 2^{j(sp-n)}\, 2^{(m-j)n(2-p)}\, \|\Delta_{m-j}\theta\|_p^p\, \|M_m f(2^{-j}(\,\cdot\,))\|_p^p \\
&\qquad + \sum_{m<j} 2^{j(sp-n)}\, 2^{(m-j)n(1-p)}\, \|\Delta_{m-j}\theta\|_p^p\, \|M_m f(2^{-j}(\,\cdot\,))\|_p^p\Big) \\
&= C\Big(\sum_{m\ge j} 2^{(m-j)(n(2-p)-sp)}\, \|\Delta_{m-j}\theta\|_p^p\, 2^{msp}\, \|M_m f\|_p^p \\
&\qquad + \sum_{m<j} 2^{(m-j)(n(1-p)-sp)}\, \|\Delta_{m-j}\theta\|_p^p\, 2^{msp}\, \|M_m f\|_p^p\Big)\,.
\end{aligned}$$

En posant $b_j = 2^{js}\|M_j f\|_p$, il vient

$$\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p \le C \sum_{m\in\mathbb{Z}} a_{m-j}^p\, b_m^p\,.$$

Exactement comme dans la preuve du Théorème 2, on en tire

$$\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n} |c_{j,k}|^p\Big)^{q/p}\Big)^{1/q} \le C\Big(\sum_{j\in\mathbb{Z}} |a_j|^u\Big)^{1/u} \|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^n)}\,.$$

6. Le théorème d'Yves Meyer.

Une ondelette est r-régulière si elle est à décroissance rapide, ainsi que ses dérivées jusqu'à l'ordre r. Il est facile -comme le fait P. Auscher [A]- d'étendre cette définition aux r non entiers:

Définition 2. *Soit r un nombre positif non entier. On dit que ψ est r-régulière si, quel que soit l'entier N, la fonction*

$$x \mapsto (1+|x|^2)^N \, \psi(x)$$

appartient à l'espace de Hölder $B_\infty^{r,\infty}(\mathbb{R}^n)$.

Voici l'énoncé du théorème de Meyer, où le mot "base" signifie base inconditionnelle ou base inconditionnelle faible suivant que l'espace de Besov considéré est ou non séparable:

Théorème 4. *Soit $\{\psi_\varepsilon\}_{\varepsilon\in E}$ une famille finie de fonctions r-régulières telle que*

$$\{2^{jn/2}\,\psi_\varepsilon(2^j(\,\cdot\,)-k):\ \varepsilon\in E,\ k\in\mathbb{Z}^n,\ j\in\mathbb{Z}\} \tag{13}$$

soit une base orthonormée de $L^2(\mathbb{R}^n)$. On suppose de plus:

i) $|s|<r$, *si* $p\in[1,+\infty]$,

ii) $n(2/p-1)-r<s<r$, *si* $0<p<1$.

Alors la famille (13) *est une base de l'espace de Besov $\dot{B}_p^{s,q}(\mathbb{R}^n)$, quel que soit $q\in\,]0,+\infty]$. Plus précisément, il existe des constantes $C_j(r,s,p,q,n)>0$ $(j=1,2)$ telles que*

i) *pour toute suite $\{c_{\varepsilon,j,k}\}_{\varepsilon\in E,\,j\in\mathbb{Z},\,k\in\mathbb{Z}^n}$ de nombres complexes telle que le second membre de l'inégalité soit fini,*

$$\Big\|\sum_{\varepsilon,j,k} c_{\varepsilon,j,k}\,2^{j(n/p-s)}\,\psi_\varepsilon(2^j(\,\cdot\,)-k)\Big\|_{\dot{B}_p^{s,q}(\mathbb{R}^n)} \le C_2\sum_\varepsilon\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n}|c_{\varepsilon,j,k}|^p\Big)^{q/p}\Big)^{1/q},$$

ii) *toute distribution $f\in\dot{B}_p^{s,q}(\mathbb{R}^n)$ s'écrit d'une manière et d'une seule*

$$f=\sum_{\varepsilon,j,k} c_{\varepsilon,j,k}\,2^{j(n/p-s)}\psi_\varepsilon(2^j(\,\cdot\,)-k)\,,$$

avec

$$\sum_\varepsilon\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n}|c_{\varepsilon,j,k}|^p\Big)^{q/p}\Big)^{1/q}\le C_1\,\|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^n)}\,.$$

PREUVE. Désignons par m la partie entière de r. Le Théorème de Meyer sera une conséquence des théorèmes précédents et des inégalités

$$\|\Delta_j\psi\|_{\mathcal{E}_p} \leq C\, 2^{j(m+1-n(1/p-1)_+)}\,, \qquad j<0\,, \tag{14}$$

$$\|\Delta_j\psi\|_{\mathcal{E}_p} \leq C\, 2^{-jr}\,, \qquad j\geq 0\,, \tag{15}$$

où ψ est l'une quelconque des ondelettes ψ_ε.

PREUVE DE (15). Admettons un instant les estimations

$$|\Delta_j\psi(x)| \leq C_N\Big(\frac{2^{-jr}}{(1+|x|)^N} + \frac{2^{-jm}}{(1+|2^jx|)^N}\Big)\,, \tag{16}$$

pour tout $j \geq 0$ et n'importe quel $N>0$,

$$\sum_{k\in\mathbb{Z}^n} |\Delta_j\psi(x-k)| \leq C\,2^{-jr}\,, \qquad j\geq 0\,. \tag{17}$$

Commençons par prouver (15) pour $p\leq 1$. Il suffit d'élever (16) à la puissance p et de choisir $N > n/p$; il vient

$$\int |\Delta_j\psi(x)|^p\,dx \leq C\,(2^{-jrp} + 2^{-j(pm+n)})$$

et l'inégalité $m+n/p \geq r$ permet de conclure.

L'inégalité (17) s'écrit encore

$$\|\Delta_j\psi\|_{\mathcal{E}_\infty} \leq C\,2^{-jr}\;;$$

en interpolant entre les cas $p=1$ et $p=+\infty$, suivant la Proposition 3, on obtient (15) quel que soit $p\in\,]1,+\infty[$.

Il nous faut maintenant vérifier (16) et (17). Supposons d'abord r entier, autrement dit $r=m$. On estime $\Delta_j\psi$ à l'aide de l'identité (7), qui nous donne

$$\begin{aligned}|\Delta_j\psi(x)| \leq \sum_{|\alpha|=m} \frac{m}{\alpha!}\int_0^1 (1-t)^{m-1}\,dt \\ \cdot\Big(\int_{\mathbb{R}^n} |y|^m\,|\psi^{(\alpha)}(x-ty)|\,2^{jn}\,|\lambda(2^jy)|\,dy\Big)\,.\end{aligned} \tag{18}$$

La décroissance rapide de $\psi^{(\alpha)}$ conduit à

$$\begin{aligned}\int_{|y|\leq\frac{|x|}{2}} |y|^m\,|\psi^{(\alpha)}(x-ty)|\,2^{jn}\,|\lambda(2^jy)|\,dy \\ &\leq C_N\,(1+|x|)^{-N}\int |y|^m\,2^{jn}|\,\lambda(2^jy)|\,dy \\ &= C'_N\,(1+|x|)^{-N}\,2^{-jm}\ ;\end{aligned}$$

par ailleurs

$$\begin{aligned}\int_{|y|\geq\frac{|x|}{2}} |y|^m\,|\psi^{(\alpha)}(x-ty)|\,2^{jn}\,|\lambda(2^jy)|\,dy \\ &\leq 2^{-jm}\,\|\psi^{(\alpha)}\|_\infty \int_{|y|\geq 2^{j-1}|x|} |y|^m\,|\lambda(y)|\,dy \\ &\leq C_N\,(1+|2^jx|)^{-N}\,2^{-jm}\end{aligned}$$

-la dernière égalité résultant de la décroissance rapide de λ- et cela termine la preuve de (16). Pour obtenir (17), on utilise de nouveau (18), se ramenant ainsi à vérifier

$$\sum_{k\in\mathbb{Z}^n} |\psi^{(\alpha)}(x-k-ty)|\leq C$$

quels que soient x et y dans $\mathbb{R}^n$ et $t\in[0,1]$; or cette inégalité équivaut à $\psi^{(\alpha)}\in\mathcal{E}_\infty$, propriété qui découle aussitôt de la décroissance rapide de $\psi^{(\alpha)}$.

Passons au cas où r n'est pas entier. L'identité (7) conduit alors à

$$\begin{aligned}&|\Delta_j\psi(x)|\leq \sum_{|\alpha|=m}\frac{m}{\alpha!}\int_0^1(1-t)^{m-1}\,dt \\ (19)\qquad &\cdot\Big(\int_{\mathbb{R}^n}|y|^m\,|\psi^{(\alpha)}(x-ty)-\psi^{(\alpha)}(x)|\,2^{jn}\,|\lambda(2^jy)|\,dy\Big)\,.\end{aligned}$$

L'hypothèse de r-régularité se traduit par les estimations

$$(20)\qquad |\psi^{(\alpha)}(x+h)-\psi^{(\alpha)}(x)|\leq C_N\,(1+|x|)^{-N}\,|h|^{r-m}\,,$$

pour $|\alpha| = m$, $|h| \leq |x|/2$ et quel que soit $N > 0$; cela nous donne

$$\int_{|y|\leq \frac{|x|}{2}} |y|^m \, |\psi^{(\alpha)}(x-ty) - \psi^{(\alpha)}(x)| \, 2^{jn} \, |\lambda(2^j y)| \, dy$$

$$\leq C_N \, (1+|x|)^{-N} \, 2^{-jr} \int |y|^r \, |\lambda(y)| \, dy \,,$$

puis

$$\int_{|y|\geq \frac{|x|}{2}} |y|^m \, |\psi^{(\alpha)}(x-ty) - \psi^{(\alpha)}(x)| \, 2^{jn} \, |\lambda(2^j y)| \, dy$$

$$\leq 2 \, \|\psi^{(\alpha)}\|_\infty \, 2^{-jm} \int_{|y|\geq 2^{j-1}|x|} |y|^m \, |\lambda(y)| \, dy$$

$$\leq \; C_N \, 2^{-jm} \, (1+|2^j x|)^{-N} \,,$$

ce qui termine la preuve de (16). Pour obtenir (17), on combinera l'inégalité (19) avec

$$\text{(21)} \qquad \sum_{k\in\mathbb{Z}^n} |\psi^{(\alpha)}(x-k-y) - \psi^{(\alpha)}(x-k)| \leq C \, |y|^{r-m} \,,$$

estimation qu'il nous reste donc à établir. Notons d'abord que le premier membre de (21) est majoré par $2 \, \|\psi^{(\alpha)}\|_{\mathcal{E}_\infty}$, ce qui nous donne (21) dès qu'on a $|y| \geq 1$. Supposons maintenant $|y| < 1$: l'estimation (20) donne

$$\sum_{|x-k|\geq 2|y|} |\psi^{(\alpha)}(x-k-y) - \psi^{(\alpha)}(x-k)|$$

$$\leq C \, |y|^{r-m} \sum_{k\in\mathbb{Z}^n} \frac{1}{(1+|x-k|)^{n+1}} \leq C' \, |y|^{r-m} \,.$$

Pour estimer les termes restant, on suppose $x \in [0,1]^n$; alors $|x-k| < 2|y|$ entraîne $|k| \leq 2 + \sqrt{n}$, d'où

$$\sum_{|x-k|<2|y|} |\psi^{(\alpha)}(x-k-y) - \psi^{(\alpha)}(x-k)|$$

$$\leq (5 + 2\sqrt{n})^n \, \|\psi^{(\alpha)}\|_{\dot{B}^{r,\infty}_\infty(\mathbb{R}^n)} \, |y|^{r-m} \,.$$

Preuve de (14). L'inégalité (14) reflète le comportement de $\widehat{\psi}$ au voisinage de 0, autrement dit le degré d'oscillation de la fonction ψ. De ce fait on est conduit à utiliser le résultat suivant de Meyer ([ME, Chapitre III, Proposition 4]): sous les hypothèses du Théorème 4, on a

$$\int \psi_\varepsilon(x)\, x^\alpha\, dx = 0\,, \tag{22}$$

pour tout $\varepsilon \in E$ et $|\alpha| \leq m$. Nous allons voir que ces identités impliquent les estimations

$$|\Delta_j \psi(x)| \leq C_N\,(1 + |2^j x|)^{-N}\, 2^{j(n+m+1)} \tag{23}$$

(pour tout $j \leq 0$ et n'importe quel $N > 0$) et

$$\sum_{k \in \mathbb{Z}^n} |\Delta_j \psi(x-k)| \leq C\, 2^{j(m+1)}\,, \qquad j \leq 0\,. \tag{24}$$

On procède alors comme dans la preuve de (15): pour $p \leq 1$, l'inégalité (14) découle aussitôt de (23), (24) est précisément l'estimation $\mathcal{E}_\infty$ et le cas $1 < p < +\infty$ s'obtient en interpolant entre $p = 1$ et $p = +\infty$.

Pour terminer la preuve du Théorème 4, il nous reste à établir les estimations (23) et (24). Les oscillations de ψ nous permettent d'utiliser l'identité (9), qui donne

$$\begin{aligned} |\Delta_j \psi(x)| \leq 2^{j(n+m+1)} \sum_{|\alpha|=m+1} \frac{m+1}{\alpha!} \int_0^1 (1-t)^m\, dt \\ \cdot \Big(\int_{\mathbb{R}^n} |y|^{m+1}\, |\lambda^{(\alpha)}(2^j(x-ty))|\, |\psi(y)|\, dy \Big)\,. \end{aligned} \tag{25}$$

Pour $t \in [0,1]$, la décroissance rapide de $\lambda^{(\alpha)}$ conduit à

$$\begin{aligned} \int_{|y|<|x|/2} |\lambda^{(\alpha)}(2^j(x-ty))|\, |y|^{m+1}\, |\psi(y)|\, dy \\ \leq C_N\,(1+|2^j x|)^{-N} \int |y|^{m+1}\, |\psi(y)|\, dy\,, \end{aligned}$$

et la décroissance rapide de ψ à

$$\int_{|y|\geq|x|/2} |\lambda^{(\alpha)}(2^j(x-ty))|\, |y|^{m+1}\, |\psi(y)|\, dy \leq C_N\, \|\lambda^{(\alpha)}\|_\infty\, (1+|x|)^{-N}\,,$$

d'où l'inégalité (23). L'estimation (24) sera une conséquence de (25) et de

$$\sum_{k\in\mathbb{Z}^n} |\lambda^{(\alpha)}(2^j(x-ty-k))| \leq C\, 2^{-jn}\,,$$

(pour x et y dans $\mathbb{R}^n$, $t\in[0,1]$, $j\leq 0$), inégalité qui résulte elle-même de la décroissance rapide de $\lambda^{(\alpha)}$ et de (4).

7. Les bases de splines tensoriels.

Rappelons brièvement la construction des ondelettes splines dans $\mathbb{R}$, puis dans $\mathbb{R}^n$.

L'entier $m\geq 0$ étant donné, il existe une fonction $\psi_1\in L^2(\mathbb{R})$ ayant les propriétés suivantes:

(26) ψ_1 est de classe C^{m-1} (pas de condition si $m=0$),

(27) sur chaque intervalle $]k,k+1[$ $(k\in\mathbb{Z})$, ψ_1 coïncide avec un polynôme de degré m,

(28) il existe des nombres $C>0$, $\gamma>0$, tels que, pour tout $\alpha=0,\dots,m$ et presque tout $x\in\mathbb{R}$,

$$|\psi_1^{(\alpha)}(x)|\leq C\,e^{-\gamma|x|}\,,$$

(29) $\{2^{j/2}\,\psi_1(2^j(\,\cdot\,)-k):\ j\in\mathbb{Z},\ k\in\mathbb{Z}\}$ est une base orthonormée de $L^2(\mathbb{R})$.

Il s'agit dès lors d'une base d'ondelettes m-régulières, ce qui donne encore

$$\int \psi_1(x)\,x^\alpha\,dx = 0\,, \qquad \alpha=0,\dots,m. \tag{30}$$

Enfin l'ondelette ψ_1 est issue d'une fonction d'échelle ψ_0 possédant les propriétés (26), (27), (28).

En dimension $n>1$, on pose, pour tout $\varepsilon\in E=\{0,1\}^n\setminus\{0\}$,

$$\psi_\varepsilon(x)=\psi_{\varepsilon_1}(x_1)\cdots\psi_{\varepsilon_n}(x_n)\;;$$

alors $\{2^{jn/2}\,\psi_\varepsilon(2^j(\,\cdot\,)-k)\ :\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n,\varepsilon\in E\}$ est une base orthonormée de $L^2(\mathbb{R}^n)$, que nous appellerons désormais "la" base des *splines tensoriels de degré m.*

Le Théorème 4 s'applique évidemment à cette base: pour $p \geq 1$, c'est une base inconditionnelle de $\dot{B}_p^{s,q}(\mathbb{R}^n)$, dès que s appartient à l'intervalle $]-m,+m[$. Nous allons voir que cet intervalle peut être remplacé par $]-m-1/p', m+1/p[$; le résultat est d'autant plus intéressant qu'il est optimal:

Théorème 5. *La conclusion du Théorème 4 est vraie pour la base des splines tensoriels de degré m, pourvu que s vérifie les inégalités*

i) $-m-1/p' < s < m+1/p\,,\quad$ *pour* $p \geq 1\,,$

ii) $-m+n(2/p-1)-1/p < s < m+1\,,\quad$ *pour* $p < 1\,.$

Pour $p, q \in [1,+\infty[$, *ce résultat est optimal, en effet* ψ_ε *n'appartient ni à* $\dot{B}_p^{s,q}(\mathbb{R}^n)$, *pour* $s \geq m+1/p$, *ni au dual de* $\dot{B}_p^{s,q}(\mathbb{R}^n)$ *-pour* $s < -m-1/p'$ *ou pour* $s = -m-1/p'$ *et* $q > 1$.

PREUVE. 1) LE CAS $n = 1$. La fonction ψ_1 satisfait encore les inégalités (14) et (15) (avec $r = m$ et $n = 1$); le point nouveau, c'est l'estimation

$$\|\Delta_j \psi_1\|_{\mathcal{E}_p} \leq C\, 2^{-j(m+1/p)}\,, \qquad j \geq 0\,. \tag{31}$$

En combinant (31), (14) et les Théorèmes 1, 2 et 3, on obtiendra aisément la conclusion annoncée.

Pour obtenir (31), on commence par prouver

$$\|\Delta_j \psi_1\|_p \leq C\, 2^{-j(m+1/p)}\,, \qquad j \in \mathbb{Z}\,. \tag{32}$$

L'inégalité (32) signifie encore $\psi_1 \in \dot{B}_p^{m+1/p,\infty}$; il nous suffira donc de prouver

$$\psi_1^{(m+1)} \in \dot{B}_p^{1/p-1,\infty}\ ;$$

or on a $\psi_1^{(m+1)}(x) = \sum_{k\in\mathbb{Z}} a_k\, \delta(x-k)$, où la suite $\{a_k\}$ est à décroissance exponentielle, et il est facile de vérifier que la mesure de Dirac δ appartient à $\dot{B}_p^{1/p-1,\infty}$.

(32) implique (31), immédiatement pour $p \leq 1$, par interpolation avec le cas $p = +\infty$ pour $p > 1$.

2) LE CAS $n > 1$. La décomposition de Littlewood-Paley usuelle, définie à partir de la fonction radiale λ, est mal adaptée aux estimations de produits tensoriels. Aussi allons-nous substituer à la fonction λ une suite

$\lambda_1, \ldots, \lambda_n$ de fonctions qui seront elles-mêmes des produits tensoriels de fonctions d'une seule variable.

Voici les détails de la construction: les fonctions $u, v \in \mathcal{S}(\mathbb{R})$ vérifient:

i) $\operatorname{supp} \widehat{u} = \{\xi \in \mathbb{R} : 1/\sqrt{n} \leq |\xi| \leq 3\}$, et $\widehat{u}(\xi)$ ne s'annule pas pour $1/\sqrt{n} < |\xi| < 3$.

ii) $\operatorname{supp} \widehat{v} = \{\xi \in \mathbb{R} : |\xi| \leq 3\}$, et $\widehat{v}(\xi)$ ne s'annule pas pour $|\xi| < 3$.

On pose

$$\lambda_k(x) = u(x_k) \prod_{l \neq k} v(x_l)\,,$$

ce qui nous donne

i) $\operatorname{supp} \widehat{\lambda_k} \subset \{\xi \in \mathbb{R}^n : 1/\sqrt{n} \leq |\xi| \leq 3\sqrt{n}\}$,

ii) $\sum_{k=1}^n |\widehat{\lambda_k}(\xi)| > 0$, pour $1 < |\xi| < 3$.

Sous ces conditions, on montre classiquement (voir [P], par exemple) que la "norme" $\dot{B}_p^{s,q}(\mathbb{R}^n)$ est équivalente à

$$\sum_{k=1}^n \|\{2^{js} \|\Delta_{j,k} f\|_p\}_{j \in \mathbb{Z}}\|_{\ell^q}\,,$$

où les opérateurs $\Delta_{j,k}$ sont définis par

$$(\Delta_{j,k} f)\widehat{\ }(\xi) = \widehat{\lambda_k}(2^{-j}\xi) \widehat{f}(\xi)\,.$$

Nous allons vérifier successivement

$$\|\Delta_{j,k} \psi_\varepsilon\|_p \leq C\, 2^{-j(m+1/p)}\,, \qquad j \geq 0\,, \tag{33}$$

$$\|\Delta_{j,k} \psi_\varepsilon\|_p \leq C\, 2^{j(m+1-n(1/p-1)_+)}\,, \qquad j < 0\,, \tag{34}$$

$$\|\Delta_{j,k} \psi_\varepsilon\|_{\mathcal{E}_\infty} \leq C\, 2^{-jm}\,, \qquad j \geq 0\,, \tag{35}$$

$$\|\Delta_{j,k} \psi_\varepsilon\|_{\mathcal{E}_\infty} \leq C\, 2^{j(m+1)}\,, \qquad j < 0\,. \tag{36}$$

Il en résultera

$$\|\Delta_{j,k} \psi_\varepsilon\|_{\mathcal{E}_p} \leq C\, 2^{-j(m+1/p)}\,, \qquad j \geq 0\,,$$

$$\|\Delta_{j,k} \psi_\varepsilon\|_{\mathcal{E}_{p'}} \leq C\, 2^{j(m+1)}\,, \qquad j < 0,\ p \geq 1\,,$$

et la démonstration du Théorème 5 sera achevée.

Nous prouverons (33)-(36) pour $k = 1$, ce cas étant entièrement typique. Il sera commode de considérer les opérateurs 1-dimensionnels U_j, V_j tels que

$$\widehat{U_j f}(\xi) = \widehat{u}(2^{-j}\xi)\,\widehat{f}(\xi)\,, \qquad \widehat{V_j f}(\xi) = \widehat{v}(2^{-j}\xi)\,\widehat{f}(\xi)\,,$$

pour tout $\xi \in \mathbb{R}$. Il vient alors

$$\Delta_{j,1}\psi_\varepsilon(x) = U_j\psi_{\varepsilon_1}(x_1)\,V_j\psi_{\varepsilon_2}(x_2)\cdots V_j\psi_{\varepsilon_n}(x_n)\,,$$

d'où

$$\|\Delta_{j,1}\psi_\varepsilon\|_p = \|U_j\psi_{\varepsilon_1}\|_p\,\|V_j\psi_{\varepsilon_2}\|_p\cdots\|V_j\psi_{\varepsilon_n}\|_p\,.$$

Les estimations (33) et (34) résulteront alors de

$$\|U_j\psi_k\|_p \leq C\,2^{-j(m+1/p)}\,, \qquad j \geq 0,\ k = 0,1\,, \tag{37}$$

$$\|V_j\psi_k\|_p \leq C\,, \qquad j \geq 0,\ k = 0,1\,, \tag{38}$$

$$\|U_j\psi_1\|_p + \|V_j\psi_1\|_p \leq C\,2^{j(m+1-(1/p-1)_+)}\,, \qquad j < 0\,, \tag{39}$$

$$\|U_j\psi_0\|_p + \|V_j\psi_0\|_p \leq C\,2^{-j(1/p-1)_+}\,, \qquad j < 0\,. \tag{40}$$

(37) est essentiellement l'estimation (31) -la fonction d'échelle ψ_0 vérifie également (31), car cette estimation n'utilise que les propriétés (26), (27) et (28). Pour prouver (38), on imite la démonstration de (16): il vient

$$|V_j\psi_k(x)| \leq C_N\,(1+|x|)^{-N}\,,$$

d'où le résultat, en faisant $N > n/p$ (pour $p \geq 1$, (38) est aussi une conséquence immédiate de l'inégalité de Young: $\|V_j\psi_k\|_p \leq \|v\|_1\,\|\psi_k\|_p$).

L'inégalité (39) est une version unidimensionnelle de (14) -on notera que, la preuve de (14) n'utilisant aucune propriété d'oscillation de la fonction λ, rien n'empêche de remplacer l'opérateur U_j par l'opérateur V_j. (40) s'obtient en faisant $m = -1$ dans l'estimation (39), puisque la fonction d'échelle ψ_0 n'a pas *a priori* de moment nul.

Pour démontrer les estimations (35) et (36), on note l'inégalité

$$\|\Delta_{j,1}\psi_\varepsilon\|_{\mathcal{E}_\infty} \leq \|U_j\psi_{\varepsilon_1}\|_{\mathcal{E}_\infty}\,\|V_j\psi_{\varepsilon_2}\|_{\mathcal{E}_\infty}\cdots\|V_j\psi_{\varepsilon_n}\|_{\mathcal{E}_\infty}\ ;$$

on est alors ramené à prouver les estimations

(41) $$\|U_j\psi_k\|_{\mathcal{E}_\infty} \le C\,2^{-jm}\,, \qquad j \ge 0,\ k = 0,1\,,$$

(42) $$\|V_j\psi_k\|_{\mathcal{E}_\infty} \le C\,, \qquad j \ge 0,\ k = 0,1\,,$$

(43) $$\|U_j\psi_1\|_{\mathcal{E}_\infty} + \|V_j\psi_1\|_{\mathcal{E}_\infty} \le C\,2^{j(m+1)}\,, \qquad j < 0\,,$$

(44) $$\|U_j\psi_0\|_{\mathcal{E}_\infty} + \|V_j\psi_0\|_{\mathcal{E}_\infty} \le C\,, \qquad j < 0\,.$$

(41) et (43) sont des versions unidimensionnelles de (15) et (14), respectivement, alors que (42) et (44) sont des conséquences immédiates de $\psi_k \in \mathcal{E}_\infty$ $(k = 0,1)$.

3) OPTIMALITÉ. Nous allons prouver que ψ_ε n'appartient pas $\dot{B}_p^{s,q}$, pour $s > m + 1/p$ ou pour $s = m + 1/p$ et $q < +\infty$. Cela entraînera l'optimalité de la borne $m + 1/p$ et, par dualité, celle de la borne $-m - 1/p'$.

Le cas $\varepsilon = (1, 0, \dots, 0)$ est entièrement typique et nous montrerons, en fait, que la dérivée $\partial_1^{m+1}\psi_{(1,0,\dots,0)}$ n'appartient à $\dot{B}_p^{s,q}$ ni pour $s > 1/p - 1$, ni pour $s = 1/p - 1$ et $q < +\infty$.

Introduisons une fonction $\theta \in \mathcal{D}(\mathbb{R})$, portée par l'intervalle $[-1/2, 1/2]$, telle que
$$\int \theta(x)\, x^\alpha\, dx = 0\,,$$
pour $0 \le \alpha \le [s] + 1$, puis $\kappa \in \mathcal{D}(\mathbb{R})$ telle que $\int \kappa(x)\, dx = 1$; posons
$$\rho(x) = \theta(x_1)\,\kappa(x_2)\cdots\kappa(x_n)\,,$$
$\rho_j(x) = 2^{jn}\rho(2^j x)$, $\theta_j(x) = 2^j\,\theta(2^j x)$ et de même pour κ_j. On a $\rho \in \mathcal{D}(\mathbb{R}^n)$ et
$$\int \rho(x)\, x^\alpha\, dx = 0\,,$$
pour tout $\alpha \in \mathbb{N}^n$, $|\alpha| \le [s] + 1$, de sorte que $\partial_1^{m+1}\psi_{(1,0,\dots,0)} \in \dot{B}_p^{s,q}(\mathbb{R}^n)$ entraînerait

(45) $$\{2^{js}\,\|\rho_j * \partial_1^{m+1}\psi_{(1,0,\dots,0)}\|_p\}_{j\in\mathbb{Z}} \in \ell^q$$

(voir [P, Chapitre 8]). Or
$$\rho_j * \partial_1^{m+1}\psi_{(1,0,\dots,0)} = (\theta_j * \psi_1^{(m+1)}) \otimes (\kappa_j * \psi_0) \otimes \cdots \otimes (\kappa_j * \psi_0)\,,$$

ce qui donne

$$\|\rho_j * \partial_1^{m+1}\psi_{(1,0,\dots,0)}\|_p = \|\theta_j * \psi_1^{(m+1)}\|_p \, \|\kappa_j * \psi_0\|_p^{n-1} .$$

Par construction de ψ_1, on a

$$\psi_1^{(m+1)} = \sum_{k\in\mathbb{Z}} a_k \, \delta(x-k)$$

et, quitte à translater ψ_1, on peut supposer $a_0 \neq 0$; on a alors, pour $j \geq 0$,

$$\theta_j * \psi_1^{(m+1)} = a_0 \, \theta_j(x) ,$$

ce qui donne

$$\|\theta_j * \psi_1^{(m+1)}\|_p = 2^{j(1-1/p)} \, |a_0| \, \|\theta\|_p \; ;$$

en outre $\|\kappa_j * \psi_0\|_p$ tend vers $\|\psi_0\|_p$ quand $j \to +\infty$; finalement il existe une constante $c > 0$ telle que, pour j assez grand

$$\|\rho_j * \partial_1^{m+1}\psi_{(1,0,\dots,0)}\|_p \geq c \, 2^{j(1-1/p)} ,$$

mais cette estimation est incompatible avec (45).

8. Les ondelettes de Mallat et Daubechies.

Rappelons l'essentiel de la construction de ces ondelettes dans $L^2(\mathbb{R})$.

On part d'une fonction m, de classe C^∞, 2π-périodique, telle que $m(0) = 1$ et $|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1$, pour tout réel ξ, ce qui conduit à la factorisation

$$m(\xi) = A(\xi) \, \cos^N \frac{\xi}{2} , \tag{46}$$

où A est une fonction de classe C^∞, 2π-périodique, et N un entier positif. On définit alors la transformée de Fourier de la fonction d'échelle φ par le produit infini

$$\widehat{\varphi}(\xi) = \prod_{j\geq 1} m(2^{-j}\xi) \tag{47}$$

et l'ondelette ψ par la formule usuelle:

$$\widehat{\psi}(\xi) = e^{-i\xi/2}\,\overline{m}\Big(\frac{\xi}{2}+\pi\Big)\,\widehat{\varphi}\Big(\frac{\xi}{2}\Big). \tag{48}$$

Une ultime condition sur m -mise en évidence par A. Cohen [CO]- garantit que $\{2^{j/2}\,\psi(2^j(\cdot)-k)\ :\ j\in\mathbb{Z},\, k\in\mathbb{Z}\}$ est une base orthonormée de $L^2(\mathbb{R})$. Nous souhaitons savoir s'il s'agit également d'une base inconditionnelle de $\dot{B}_2^{s,q}(\mathbb{R})$.

Théorème 6. *Soit* $\{2^{j/2}\,\psi(2^j(\cdot)-k)\ :\ j\in\mathbb{Z},\, k\in\mathbb{Z}\}$ *une base orthonormée de* $L^2(\mathbb{R})$ *obtenue suivant les formules* (46), (47) *et* (48). *On suppose* $\gamma=\log_2\|A\|_\infty < N-1/2$. *Alors* $\{2^{j/2}\,\psi(2^j(\cdot)-k) : j\in\mathbb{Z},\, k\in\mathbb{Z}\}$ *est une base inconditionnelle de* $\dot{B}_2^{s,q}(\mathbb{R})$ *pour*

$$|s| < N-\gamma-\frac{1}{2}\,.$$

On a, plus précisément, l'équivalence de normes (1) *pour* $p=2$, $n=1$, $q\in\,]0,+\infty]$.

PREUVE. La factorisation (46) nous donne

$$\widehat{\psi}(\xi)=O(|\xi|^N)\,,\qquad |\xi|\to 0\,,$$

et le lemme de Daubechies ([D, Lemme 3.2])

$$\widehat{\psi}(\xi)=O(|\xi|^{\gamma-N})\,,\qquad |\xi|\to +\infty\,.$$

Nous allons voir que ces deux estimations conduisent à

$$\psi\in\dot{B}^{\pm s,\inf\{q,1\}}(\mathcal{E}_2)\,.$$

On utilisera la description de $\mathcal{E}_2$ fournie par la Proposition 2, ce qui nous amène à estimer les fonctions 2π-périodiques

$$A_j(\xi)=\sum_{k\in\mathbb{Z}}|\widehat{\Delta_j\psi}(\xi+2k\pi)|^2\,. \tag{49}$$

Le lemme suivant nous permet de conclure:

Lemme 4. *Soit $\alpha > 0$, $\beta > 1/2$ et ψ une fonction telle que*

$$\widehat{\psi}(\xi) = O(|\xi|^{\alpha}), \qquad |\xi| \to 0,$$

et

$$\widehat{\psi}(\xi) = O(|\xi|^{-\beta}), \qquad |\xi| \to +\infty.$$

Il existe alors une constante $C = C(\psi) > 0$ telle que les fonctions 2π-périodiques A_j définies suivant (49) *vérifient*

$$\|A_j\|_\infty \le C\, 4^{j\alpha}, \qquad j \le 0,$$

$$\|A_j\|_\infty \le C\, 2^{j(1-2\beta)}, \qquad j > 0.$$

PREUVE DU LEMME. Il nous suffit de considérer $\xi \in [0, 2\pi]$. On écrit d'abord

$$A_j(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + 2k\pi)|^2\, |\widehat{\lambda}(2^{-j}(\xi + 2k\pi))|^2 .$$

Supposons $j \le 0$; par hypothèse sur le support de $\widehat{\lambda}$, on a $\widehat{\lambda}(2^{-j}(\xi + 2k\pi)) = 0$ pour tout $k \neq 0$, d'où

$$A_j(\xi) = |\widehat{\psi}(\xi)\, \widehat{\lambda}(2^{-j}\xi)|^2 \le C\, 4^{j\alpha} .$$

Supposons $j \ge 4$; alors $\widehat{\lambda}(2^{-j}(\xi + 2k\pi))$ est non nul seulement si l'on a $k \ge 2^{j-2}/\pi$, auquel cas $\xi + 2k\pi$ est de l'ordre de grandeur de k; cela nous donne

$$A_j(\xi) \le C \sum_{k \ge 2^{j-2}/\pi} k^{-2\beta} \le C\, 2^{j(1-2\beta)} .$$

Nous laissons le lecteur se convaincre que A_1, A_2 et A_3 sont des fonctions bornées ...

9. Conclusion et remarques.

1) Nous avons choisi de considérer les espaces de Besov *homogènes* car ils conduisent aux caractérisations les plus simples et les plus naturelles en termes d'ondelettes. Que peut-on dire à propos des espaces de Besov *inhomogènes*? Le point de départ est la décomposition

$$f(x) = \sum_{k\in\mathbb{Z}^n} a_k\,\varphi(x-k) + \sum_{\varepsilon}\sum_{k\in\mathbb{Z}^n}\sum_{j=0}^{\infty} c_{\varepsilon,j,k}\,2^{j(n/p-s)}\,\psi_\varepsilon(2^j x - k)\,,$$

où φ est une fonction d'échelle et $\{\psi_\varepsilon\}$ la famille finie d'ondelettes associée à φ. Le résultat escompté est l'équivalence de normes

$$\|f\|_{B_p^{s,q}(\mathbb{R}^n)} \approx \Big(\sum_{k\in\mathbb{Z}^n}|a_k|^p\Big)^{1/p} + \sum_{\varepsilon}\Big(\sum_{j\geq 0}\Big(\sum_{k\in\mathbb{Z}^n}|c_{\varepsilon,j,k}|^p\Big)^{q/p}\Big)^{1/q}.$$

On peut l'obtenir de deux façons. La première consiste à reprendre pas -à -pas la méthode utilisée pour les espaces homogènes. La seconde -inapplicable au cas $s=0$- consiste à utiliser directement les résultats "homogènes", grâce aux relations

$$B_p^{s,q}(\mathbb{R}^n) = \dot{B}_p^{s,q}(\mathbb{R}^n)\cap L^p\,,\qquad s>0\,,$$

$$B_p^{s,q}(\mathbb{R}^n) = \dot{B}_p^{s,q}(\mathbb{R}^n) + L^p\,,\qquad s<0\,.$$

Le lecteur intéressé par cette question trouvera sans peine les bonnes conditions.

2) Nous avons souligné l'optimalité du Théorème 5. Il y a tout lieu de penser qu'il en est de même pour le Théorème de Meyer (Théorème 4); autrement dit, pour un réel positif r donné, on peut *conjecturer l'existence d'une base orthonormée d'ondelettes r-régulières qui ne soit une base de $\dot{B}_p^{s,q}(\mathbb{R}^n)$ ni pour $s\geq r$, ni pour $s\leq -r$.*

3) Nos résultats ne sont vraisemblablement pas optimaux dans le cas $p<1$. En ce qui concerne les splines de degré m, DeVore et Popov [DVP] obtiennent l'intervalle de validité $0<s<m+1$, alors que notre borne inférieure, $-m+n(2/p-1)-1/p$ n'est pas toujours négative. On ignore toutefois si les espaces de Besov de DeVore et Popov coïncident avec ceux de Peetre et Triebel pour $s\leq n(1/p-1)_+$ (voir [TR, Théorème 2.6.1]).

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Convex domains and unique continuation at the boundary

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Abstract. We show that a harmonic function which vanishes continuously on an open set of the boundary of a convex domain cannot have a normal derivative which vanishes on a subset of positive surface measure. We also prove a similar result for caloric functions vanishing on the lateral boundary of a convex cylinder.

1. Introduction.

Bourgain and Wolff [BW] have constructed a counterexample of a C^1-harmonic function u in $\mathbb{R}^d_+ = \{X \in \mathbb{R}^d : X_d \geq 0\}$, $d \geq 3$, for which u and its gradient vanish on a set of positive measure on $\partial\mathbb{R}^d_+$. On the other hand, it has been shown (see [F2]) that when D is a $C^{1,1}$ domain in $\mathbb{R}^d$, $d \geq 2$, and u is a non-constant harmonic function in D with $u = 0$ on an open set V contained in the boundary ∂D of D, then the Hausdorff measure of the set $\{Q \in V : \nabla u(Q) = 0\}$ is less or equal than $d-2$.

In general, the following conjecture still remains an open question: if u is a harmonic function on a connected Lipschitz domain D in $\mathbb{R}^d$ vanishing continuously on an open subset V of ∂D and whose normal derivative vanishes on a subset of V of positive measure, then u is identically zero on D.

When u is non-negative, we have from the comparison principle for harmonic funtions vanishing continuously on an open subset of ∂D, [D], that the normal derivative of u is pointwise comparable to the density of the harmonic measure with respect to surface measure $d\sigma$ on any compact subset K contained in V, and it is well known that the harmonic measure is mutually absolutely continuous with respect to $d\sigma$, [D]. Therefore, in this case the answer to the conjecture is positive.

Let D denote a Lipschitz domain in $\mathbb{R}^d$ and w be a non-negative function defined on ∂D. We recall that a nonnegative function w is a $B_2(d\sigma)$-weight provided that there is a constant C such that for all $Q \in \partial D$ and $r > 0$ the following holds

$$\Big(\frac{1}{\sigma(\Delta_r(Q))}\int_{\Delta_r(Q)} w^2\, d\sigma\Big)^{1/2} \le C\,\frac{1}{\sigma(\Delta_r(Q))}\int_{\Delta_r(Q)} w\, d\sigma\,,$$

where $\Delta_r(Q) = \partial D \cap B_r(Q)$.

In this note, we will prove the following regularity theorem.

Theorem 1. *Let D be a Lipschitz domain in $\mathbb{R}^d$, $d \ge 2$, $Q_0 \in \partial D$, and u be harmonic in D vanishing continuously on $\Delta_6(Q_0)$. Assume that there exists a constant M, possibly depending on u, such that for all $Q \in \Delta_3(Q_0)$ and $0 < r < 2$ the following doubling property holds,*

$$\int_{\Gamma_{2r}(Q)} u^2\, dX \le M \int_{\Gamma_r(Q)} u^2\, dX\,, \tag{1.1}$$

where $\Gamma_r(Q) = B_r(Q) \cap D$. Then, there exists a constant C depending on M, the Lipschitz character D and d, such that for all $Q \in \Delta_2(Q_0)$ and $0 < r < 1$

$$\Big(\frac{1}{\sigma(\Delta_r(Q))}\int_{\Delta_r(Q)} \Big|\frac{\partial u}{\partial N}\Big|^2 d\sigma\Big)^{1/2} \le C\,\frac{1}{\sigma(\Delta_r(Q))}\int_{\Delta_r(Q)} \Big|\frac{\partial u}{\partial N}\Big|\, d\sigma\,.$$

In particular, the absolute value of the normal derivative of u is a B_2-weight when restricted to $\Delta_2(Q_0)$.

In [F2] it is shown that the doubling property (1.1) holds for such a harmonic function when D is a $C^{1,1}$ domain, with M depending on the $C^{1,1}$ character and u. In this note we will show that the doubling property (1.1) also holds when the domain D is convex, obtaining the following theorem.

Theorem 2. *Let D, u and Q_0 be as in the previous theorem, and assume that either D is a $C^{1,1}$ or a convex domain. Then, the absolute value of the normal derivative of u on $\Delta_1(Q_0)$ is a $B_2(d\sigma)$-weight.*

It is well known that the above $B_2(d\sigma)$ condition implies that the set $\{Q \in \Delta_2(Q_0) : \partial u(Q)/\partial N = 0\}$ has zero surface measure unless $\partial u/\partial N = 0$ almost everywhere on $\Delta_2(Q_0)$. Therefore, if u is harmonic in D, vanishing continuously on an open subset V of ∂D, and $\{Q \in V : \partial u(Q)/\partial N = 0\}$ has positive surface measure, both u and $\partial u/\partial N$ must vanish identicaly on V. Extending u as zero outside of D, we obtain a new function which is harmonic in an open set Ω of $\mathbb{R}^d$ containing V and identically zero on $\Omega \setminus D$. It is well known that this implies that u must be identically zero in the connected component of its domain of definition containing V. Hence we obtain the following theorem.

Theorem 3. *Let D be a convex connected domain in $\mathbb{R}^d$, $d \geq 2$, and u be harmonic in D. Then, if u vanishes on an open subset V contained in ∂D and the set $\{Q \in V : \partial u(Q)/\partial N = 0\}$ has a positive surface measure, u must be identically zero on D.*

We want to remark that in [F2], the author claimed that his methods also applied to prove the above doubling property in the case of $C^{1,\alpha}$ domains, $0 < \alpha < 1$. But in a personal communication we learnt that his claim was incorrect.

This article is divided in two sections. In Section 2 we prove the theorems 1, 2 and 3, and in Section 3 we show that a similar result holds for solutions to the heat operator in convex cylinders.

2. Proofs of the main results.

To prove Theorem 1 we will need the following inequality.

Lemma 1. *There exists a constant C depending only on d, such that if $r > 0$, n is a positive integer, $0 < \beta < 1$, and $f \in C_0^\infty(B_{4r} \setminus B_{\beta r}(0))$, the following holds*

$$\int_{B_r} |f(X)|\, dX \leq C\, n\, \beta^{3-d-n}\, r^2 \int_{B_{2r}} |\Delta f(X)|\, dX$$

$$+ C\, 2^{-n} r^2 \int_{B_{4r}\setminus B_{2r}} |\Delta f(X)|\, dX\,.$$

PROOF. By rescaling we may assume that $r = 1$. Let $f \in C_0^\infty(B_r \setminus B_\beta)$ and n be an integer greater than 1. If $\Gamma(X, Y)$ denotes the fundamental solution for the Laplace operator on $\mathbb{R}^d$, we have

$$f(X) = \int \Gamma_n(X, Y)\, \Delta f(Y)\, dY\,, \qquad \text{for all } \; X \in \mathbb{R}^d\,,$$

where

$$\Gamma_n(X, Y) = \Gamma(X, Y) - \sum_{k=0}^{n-1} \sum_{|\alpha|=k} \frac{1}{\alpha!}\, D_X^\alpha \Gamma(0, Y)\, X^\alpha\,.$$

Since $|D_X^\alpha \Gamma(0, Y)| \leq C(d)^{|\alpha|}\, |Y|^{-(d-2+|\alpha|)}$, we have for $|X| \leq 1$ and $|Y| \geq 2$

$$|\Gamma_n(X, Y)| \leq C(d) \sum_{k \geq n} 2^{-k} \leq C(d)\, 2^{-n}\,,$$

and, for $|X| \leq 1$ and $\beta \leq |Y| \leq 2$,

$$|\Gamma_n(X, Y)| \leq |\Gamma(X, Y)| + C(d)\, n\, \beta^{3-d-n}\,.$$

From these estimates and the support properties of f we obtain

$$\int_{B_1} |f(X)|\, dX \leq C\, n\, \beta^{3-d-n} \int_{B_2} |\Delta f(X)|\, dX$$
$$+ C\, 2^{-n} \int_{B_4\setminus B_2} |\Delta f(X)|\, dX\,.$$

PROOF OF THEOREM 1. Let u and Q_0 be as in Theorem 1, $Q \in \Delta_3(Q_0)$ and $0 < r < 1$. Let β denote a vector field supported in $\Gamma_{2r}(Q)$ with $|\nabla\beta| \leq C\, r^{-1}$, $\beta \cdot N \geq C^{-1}$ on $\Delta_r(Q)$ for some constant C depending on the Lipschitz character of D and $\beta \cdot N \geq 0$ on $\Delta_{2r}(Q)$, where N denotes the exterior unit normal to D at points of ∂D.

Integrating the Rellich-Necas identity

$$\operatorname{div}(\beta \cdot |\nabla u|^2) = 2 \operatorname{div}((\beta \cdot \nabla u)\, \nabla u) + O(|\nabla \beta|\, |\nabla u|^2)$$

over $\Gamma_{2r}(Q)$, and since $\nabla u = \partial u/\partial N$ almost everywehre on $\Delta_5(Q_0)$, we obtain

$$\int_{\Delta_r(Q)} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma \le C\, r^{-1} \int_{\Gamma_{2r}(Q)} |\nabla u|^2\, dX\,,$$

and from Cacciopoli's inequality and the doubling property of u

$$\int_{\Delta_r(Q)} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma \le C\, r^{-3} \int_{\Gamma_{r/40}(Q)} u^2\, dX\,,$$

where C depends on the Lipschitz character of D, and M.

From standard estimates for subharmonic functions vanishing at the boundary [GT, Theorem 8.25] we can bound the L^2 averages of u by L^1 averages, obtaining

$$\left(\int_{\Delta_r(Q)} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma\right)^{1/2} \le C\, r^{-(3+d)/2} \int_{\Gamma_{r/30}(Q)} |u|\, dX\,.$$

We claim that for some constant C as above

$$\int_{\Gamma_{r/30}(Q)} |u|\, dX \le C\, r^2 \int_{\Delta_r(Q)} \left|\frac{\partial u}{\partial N}\right| d\sigma\,,$$

and assuming the claim, the theorem follows from the last two inequalities.

To prove the last claim, we may assume without loss of generality that $Q = 0$ and that near 0, ∂D coincides with $\{(x, y) :\ x \in \mathbb{R}^{d-1},\ y = \varphi(x)\}$ for some Lipschitz function φ with $\varphi(0) = 0$.

Let Z denote the point whose coordinates with respect to this coordinate system are $x = 0$ and $y = -r/2$. From the Lipschitz character of D we can find $0 < \beta < 1/8$ such that $B_{2\beta r}(Z)$ is contained in the complement of D. We extend u to be zero outside D and define $u_\varepsilon = u * \theta_\varepsilon$, where θ_ε is a regularization of the identity. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on $B_{2r}(Z)$ and whose support is contained in $B_{4r}(Z)$.

Setting $f_\varepsilon = \varphi u_\varepsilon$, we have that $f_\varepsilon \in C_0^\infty(B_{4r} \setminus B_{\beta r}(Z))$ for $\varepsilon > 0$ sufficiently small, and for X in $B_{4r}(Z)$

$$\begin{aligned} |\Delta f_\varepsilon(X)| &\le 2\, |\Delta\varphi|\, |(\nabla u) * \theta_\varepsilon| + |u_\varepsilon|\, |\Delta\varphi| \\ &\quad + \int_{\partial D} \left|\frac{\partial u}{\partial N}(Q)\right| \varphi(X)\, \theta_\varepsilon(X - Q)\, d\sigma(Q)\,. \end{aligned}$$

Applying to f_ε the translation to Z of the inequality in Lemma 1, and letting ε tend to zero, we obtain from the support properties of φ and standard estimates for harmonic functions the following estimate

$$\int_{\Gamma_{r/2}(Q)} |u|\, dX \le C(d,\beta,n)\, r^2 \int_{\Delta_{5r}(Q)} \left|\frac{\partial u}{\partial N}\right| d\sigma + C(d)\, 2^{-n} \int_{\Gamma_{5r}(Q)} |u|\, dX\,.$$

Using the doubling property of u, the second term above can be hidden on the left hand side of the inequality after choosing n large enough, getting

$$\int_{\Gamma_{r/2}(Q)} |u|\, dX \le C\, r^2 \int_{\Delta_{5r}(Q)} \left|\frac{\partial u}{\partial N}\right| d\sigma\,,$$

where C depends on d, the Lipschitz character of D and M; and this proves the claim.

PROOF OF THE DOUBLING PROPERTY. Assume now that D is convex and let u be as in the statement of Theorem 2. For $Q \in \Delta_3(Q_0)$ we define

$$H(r,Q) = \int_{\partial B_r(Q)\cap D} u^2\, d\sigma \qquad \text{and} \qquad D(r,Q) = \int_{\Gamma_r(Q)} |\nabla u|^2\, dX\,.$$

As Almgren, [A], we consider the frequency function

$$N(r,Q) = \frac{2\, r\, D(r,Q)}{H(r,Q)}\,.$$

We will prove that

$$r\,\frac{d}{dr}\,(\log(H(r,Q)\, r^{1-d})) = \frac{2\, r\, D(r,Q)}{H(r,Q)} = N(r,Q)\,,$$

and that the frequency function $N(\,\cdot\,,Q)$ is non-decreasing for $Q \in \Delta_3(Q_0)$. Therefore, if $0 < r < 2$, we have

$$r\,\frac{d}{dr}\,(\log(H(r,Q)\, r^{1-d})) \le N(2,Q)\,.$$

Standard arguments imply that the doubling constant M is bounded by $2^{d+\beta}$, where β is an upper bound of $N(2,Q)$ on $\Delta_3(Q_0)$.

To prove our claim, we may assume that $Q = 0$, $H(r,0) = H(r)$, $B_r = B_r(0)$, and $D(r,0) = D(r)$. Then,

$$\frac{d}{dr} H(r) = \frac{d-1}{r} H(r) + 2\,D(r)\,. \tag{2.1}$$

From the Rellich-Necas identity with vector field X, *i.e.*,

$$\operatorname{div}(X\,|\nabla u|^2) = 2\operatorname{div}((X\cdot\nabla u)\,\nabla u) + (d-2)\,|\nabla u|^2\,,$$

and the fact that the tangential derivative of u is zero on $\Delta_3(Q_0)$ we get

$$\frac{d}{dr} D(r) = 2\int_{\partial B_r \cap D}\Big(\frac{\partial u}{\partial N}\Big)^2 d\sigma + \frac{d-2}{r} D(r) + \frac{1}{r}\int_{\Delta_r}(Q\cdot N)\Big(\frac{\partial u}{\partial N}\Big)^2 d\sigma.$$

But in a convex domain with $0 \in \partial D$, $Q\cdot N$ is non-negative on ∂D. Hence,

$$\frac{d}{dr} D(r) \geq 2\int_{\partial B_r \cap D}\Big(\frac{\partial u}{\partial N}\Big)^2 d\sigma + \frac{d-2}{r} D(r)\,.$$

From the above inequality (2.1), and the quotient rule we obtain

$$\frac{d}{dr}\Big(\frac{r\,D(r)}{H(r)}\Big) \geq 2\,r\,H(r)^{-2}\Big(\int_{\partial B_r \cap D}\Big(\frac{\partial u}{\partial N}\Big)^2 d\sigma \int_{\partial B_r \cap D} u^2\, d\sigma - \Big(\int_{\partial B_r \cap D} u\,\frac{\partial u}{\partial N}\, d\sigma\Big)^2\Big),$$

and from Hölder's inequality we get

$$\frac{d}{dr}\Big(\frac{r\,D(r)}{H(r)}\Big) \geq 0\,,$$

as we wanted.

3. The parabolic case.

Here we show that a similar result holds for caloric functions vanishing continuously on the lateral boundary of a convex cylinder $D \times (0,\infty)$. The reader will observe in the next proof, that in general, the same result can be obtained when D is just a Lipschitz domain and the corresponding "unique continuation property" holds at points $(Q, 0)$ with $Q \in \partial D$, for harmonic functions defined on $W \cap D \times (0, \infty)$ and vanishing continuously on $W \cap \partial(D \times (0, \infty))$, where W is an open set in R^{d+1} containing the boundary point $(Q, 0)$.

Theorem 4. *Let D be a convex connected domain in $\mathbb{R}^d$, $d \geq 2$ and $u(X, t)$ satisfy*

$$\begin{cases} \Delta u - \partial_t u = 0\,, & \text{on } D \times (0, \infty)\,, \\ u(X, 0) = f(X)\,, & \\ u(Q, t) = 0\,, & \text{for } Q \in \partial D \text{ and } t > 0\,, \end{cases}$$

for some f in a suitable class. Assume that the set

$$E = \Big\{(Q, t) \in \partial D \times (0, \infty) : \frac{\partial u}{\partial N}(Q, t) = 0\Big\}$$

has positive surface measure on $\partial D \times (0, \infty)$. Then, u must be identically zero.

PROOF. Without loss of generality we may assume that (Q_0, τ), where $Q_0 \in \partial D$ and $\tau > 0$, is a density point of E, *i.e.*,

$$\lim_{r \to 0} \frac{m(E \cap (\Delta_r(Q_0) \times [\tau - r^2, \tau]))}{\sigma(\Delta_r(Q_0)) r^2} = 1\,, \tag{3.1}$$

where $dm = d\sigma\, dt$ on $\partial D \times (0, \infty)$.

We claim that this implies that $u(\,\cdot\,, \tau)$ vanishes to infinity order at Q_0:

$$\int_{\Gamma_r(Q_0)} |u(X, \tau)|\, dX = O(r^k)\,, \qquad \text{for all } k \geq 1 \text{ as } r \to 0\,.$$

Assuming this claim, we have

$$u(X, t) = \sum_{k \geq 1} a_k\, \varphi_k(X)\, e^{-\lambda_k t}\,,$$

where

$$f(X) = \sum_{k \geq 1} a_k\, \varphi_k(X)$$

and $\{\varphi_1, \varphi_2, \dots, \varphi_k, \dots\}$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \cdots$ denote respectively the eigenfunctions and eigenvalues for the Laplace operator on D.

Defining $v(X,y)$ for $X \in D$ and $y > 0$ as

$$v(X,y) = \sum_{k\geq 1} a_k \, \varphi_k(X) \, e^{-\lambda_k \tau} \, \frac{\sinh(\lambda_k^{1/2} y)}{\lambda_k^{1/2}}$$

we have that v is a harmonic function on $\Omega = D \times (0,\infty)$ with $v = 0$ on the bottom and sides of Ω. From the convexity of D, Ω is a convex domain in $\mathbb{R}^{d+1}$, and from our previous estimates for harmonic functions in convex domains vanishing on a boundary portion we have for $0 < r < 1$

$$\int_{\Gamma_r(Q_0,0)} |v(X,y)| \, dX \, dy \leq C \, r^2 \int_{\Delta_r(Q_0,0)} \Big| \frac{\partial v}{\partial N} \Big| \, d\gamma \,,$$

where $\Gamma_r(Q_0,0) = B_r(Q_0,0) \cap \Omega$, $\Delta_r(Q_0,0) = B_r(Q_0,0) \cap \partial\Omega$, $d\gamma$ denotes surface measure on $\partial\Omega$ and $C > 0$ is a constant depending on d, the Lipschitz character of D and v. From the doubling property of the absolute value of the normal derivative of v on $\partial\Omega$ and the fact that $\partial v(X,0)/\partial N = -\partial_y v(X,0) = -u(X,\tau)$ on the bottom of Ω, we obtain

$$\int_{\Delta_r(Q_0,0)} \Big| \frac{\partial v}{\partial N} \Big| \, d\gamma \leq C \int_{\Gamma_r(Q_0)} |u(X,\tau)| \, dX = O(r^k) \,, \quad \text{for all } \; k \geq 1 \,.$$

From the above inequalities we conclude that v vanishes to infinity order at $(Q_0, 0)$ and since v is doubling with respect to balls centered at $(Q_0,0)$, it can only happen when v is identically zero, which implies that u must be equal to zero.

Therefore, to finish our proof we must prove the above claim. It will follow from the analogue of the Lemma 1 in the parabolic case.

Lemma 2. *There exists a constant C depending on $d \geq 2$, such that if $r > 0$, $0 < \beta < 1$, $f \in C_0^\infty((B_{4r} \setminus B_{\beta r}) \times (-(4r)^2, 0])$, and n is an integer greater than 1, the following holds*

$$\begin{aligned}\int_{-r^2}^{0} \int_{B_r} |f| \, dX \, dt &\leq C \, n \, \beta^{-(d+2(n-1))} \, r^2 \int_{-(2r)^2}^{0} \int_{B_{2r}} |(\Delta - \partial_t) f| \, dX \, dt \\ &\quad + C \, 2^{-n} \, r^2 \int \int_{H_r} |(\Delta - \partial_t) f| \, dX \, dt \,,\end{aligned}$$

where $H_r = B_{4r} \setminus B_{2r} \times (-(4r)^2, 0] \cup B_{2r} \times (-(4r)^2, -(2r)^2]$.

PROOF. As usual we may assume that $r = 1$. Setting $\Gamma(X,t,Y,s) = (4\pi(t-s))^{-d/2} \exp(-|X-Y|^2/4(t-s))$ for $t > s$ and $\Gamma(X,t,Y,s) = 0$ for $t \le s$ we obtain using the fact that $D^\alpha \partial_t^j f(0,0) = 0$ for all d-tuples and $j \ge 0$, and a simple argument of integration by parts, that for $t \le 0$ and X in $\mathbb{R}^d$

$$f(X,t) = \int_{-\infty}^{0} \int \Gamma_n(X,t,Y,s)\,(\Delta - \partial_s)\, f(Y,s)\, dY\, ds\,,$$

where

$$\Gamma_n(X,t,Y,s) = \Gamma(X,t,Y,s) - \sum_{k=0}^{n-1} \sum_{|\alpha|+j=k} \frac{1}{\alpha! j!} D_X^\alpha \partial_t^j \Gamma(0,0,Y,s)\, X^\alpha t^j\,.$$

Interior estimates for caloric functions imply that for $s < 0$ we have

$$|D_X^\alpha \partial_t^j \Gamma(0,0,Y,s)| \le C(d)^{|\alpha|+2j}\, |s|^{-(|\alpha|+2j+d)/2}\,, \qquad \text{for } \ |Y|^2 \le |s|\,,$$

$$|D_X^\alpha \partial_t^j \Gamma(0,0,Y,s)| \le C(d)^{|\alpha|+2j}\, |Y|^{-(|\alpha|+2j+d)}\,, \qquad \text{for } \ |Y|^2 \ge |s|\,.$$

These estimates imply that for $|X| < 1$, $-1 < t < 0$ we have

$$|\Gamma_n(X,t,Y,s)| \le C(d)\,(\Gamma(X,t,Y,s) + n\,\beta^{-(d+2(n-1))})\,,$$

for $\beta < |Y| < 2$ and $-4 < s < 0$, and for $(Y,s) \in H_1$ we can estimate Γ_n using the generalized mean value theorem, obtaining

$$|\Gamma_n(X,t,Y,s)| \le C(d)\, 2^{-n}\,, \qquad \text{for } \ (Y,s) \in H_1\,.$$

The inequality follows from these estimates and the support properties of f.

Let now u be as in Theorem 4. Without loss of generality we may assume that $(Q_0, \tau) = (Q_0, 0)$, and that u is caloric for $X \in D$, $-2 < t < 0$. As before, for $0 < r < 1$ we let Z denote a point outside of D such that $|Z - Q_0| = r/2$, and β be a number with $0 < \beta < 1/8$, and such that $B_{2\beta r}(Z)$ does not intersect D.

We extend u to be zero outside the cylinder with base D and define $f_\varepsilon(x,t) = u_\varepsilon(x,t)\,\varphi(x,t)$, where $u_\varepsilon(X,t) = u(\,\cdot\,,t) * \theta_\varepsilon(X)$, and where $\varphi \in C_0^\infty(B_{4r}(Z) \times (-(4r)^2, 0])$ is such that $\varphi = 1$ on $B_{2r}(Z) \times (-(2r)^2, 0]$.

Applying to f_ε the translation of the inequality in Lemma 2 to $(Z, 0)$, observing that

$$|(\Delta - \partial_t) f_\varepsilon(X,t)| \le 2\,|\nabla\varphi|\,|(\nabla u) * \theta_\varepsilon| + |u_\varepsilon|\,|(\Delta - \partial_t)\varphi| + \int_{\partial D} \left|\frac{\partial u}{\partial N}(Q,t)\right| \varphi(X,t)\,\theta_\varepsilon(X-Q)\,d\sigma(Q)\,,$$

for X in $\mathbb{R}^d$ and $-1 < t < 0$, letting ε tend to zero, and using standard estimates for caloric functions we obtain

$$\int_{-r^2}^0 \int_{\Gamma_{r/2}(Q_0)} |u|\,dX\,dt \le C(d,n,\beta)\,r^2 \int_{-(5r)^2}^0 \int_{\Delta_{5r}(Q_0)} \left|\frac{\partial u}{\partial N}\right| d\sigma\,dt + C(d)\,2^{-n} \int_{-(5r)^2}^0 \int_{\Gamma_{5r}(Q_0)} |u|\,dX\,dt\,.$$

On the other hand, the first term of the right hand side of the last inequality can be bounded by

$$r^2\, m((\Delta_{5r}(Q_0) \times [-(5r)^2, 0]) \setminus E)^{1/2} \Big(\int_{-(5r)^2}^0 \int_{\Delta_{5r}(Q_0)} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma\,dt\Big)^{1/2}$$

If v denotes the solution to $\Delta v - \partial_t v = 0$ on $\Gamma_{6r}(Q_0) \times (-(6r)^2, 0]$ satisfying $v(Q,t) = 0$ for $Q \in B_{6r}(Q_0) \cap \partial D$ and $t \in (-(6r)^2, 0]$, and $v(Q,t) = 1$ on the remaining part of the parabolic boundary of $\Gamma_{6r}(Q_0) \times (-(6r)^2, 0]$, we have from [FS] that for some constant C depending on d, and the Lipschitz character of D,

$$\Big(\int_{-(5r)^2}^0 \int_{\Delta_{5r}(Q_0)} \left|\frac{\partial v}{\partial N}\right|^2 d\sigma\,dt\Big)^{1/2} \le C\,r^{(d+1)/2}\,.$$

On the other hand, from the parabolic maximum principle and standard estimates for caloric functions vanishing on the lateral boundary of a Lipschitz cylinder we have

$$|u(X,t)| \le C\,v(X,t)\,r^{-(d+2)} \int_{-(8r)^2}^0 \int_{\Gamma_{8r}(Q_0)} |u|\,dX\,dt\,,$$

for all (X,t) in $\Gamma_{6r}(Q_0) \times (-(6r)^2, 0]$.

Thus,

$$\Big(\int_{-(5r)^2}^{0}\int_{\Delta_{5r}(Q_0)}\Big|\frac{\partial u}{\partial N}\Big|^2\,d\sigma\,dt\Big)^{1/2} \le C\,r^{-(d+5)/2}\int_{-(8r)^2}^{0}\int_{\Gamma_{8r}(Q_0)}|u|\,dX\,dt\,,$$

where C depends only on the Lipschitz character of D, and d.

From the above chain of inequalities we have that for $0<r<1$

$$\begin{aligned}\int_{-r^2}^{0}\int_{\Gamma_{r/2}(Q_0)}|u|\,dX\,dt &\le \Big(C(d,n,\beta)\Big(\frac{m((\Delta_{5r}(Q_0)\times[-(5r)^2,0])\setminus E)}{\sigma(\Delta_{5r}(Q_0))(5r)^2}\Big)^{1/2}\\ &\quad + C(d)\,2^{-n}\Big)\int_{-(8r)^2}^{0}\int_{\Gamma_{8r}(Q_0)}|u|\,dX\,dt\,.\end{aligned}$$

Therefore, from (3.1) and choosing n large enough, we find that for all $\varepsilon>0$ there exists $r(\varepsilon)>0$, such that for $0<r<r(\varepsilon)$

$$\int_{-r^2}^{0}\int_{\Gamma_r(Q_0)}|u|\,dX\,dt \le \varepsilon\int_{-(12r)^2}^{0}\int_{\Gamma_{12r}(Q_0)}|u|\,dX\,dt\,.$$

This is well known to imply that

$$\int_{-r^2}^{0}\int_{\Gamma_r(Q_0)}|u|\,dX\,dt = O(r^k)\,,\qquad \text{for all } \ k\ge 1\,.$$

On the other hand, estimates for caloric functions vanishing on the lateral boundary give

$$\int_{\Gamma_r(Q_0)}|u(X,0)|\,dX \le C\,r^{-2}\int_{-(2r)^2}^{0}\int_{\Gamma_{2r}(Q_0)}|u|\,dX\,dt\,,$$

where C depends on d and the Lipschitz character of D, and this implies our claim.

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Two problems on doubling measures

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Doubling measures appear in relation to quasiconformal mappings of the unit disk of the complex plane onto itself. Each such map determines a homeomorphism of the unit circle on itself, and the problem arises, which mappings f can occur as boundary mappings? A famous theorem of Beurling and Ahlfors [2] states a necessary and sufficient condition: the Lebesgue measures $|f(I)|$ and $|f(J)|$ are comparable, $|f(I)| \simeq |f(J)|$, whenever I and J are adjacent arcs of equal length. Denoting by μ the measure on the unit circle such that $\mu(I) = |f(I)|$, this can be expressed by the inequality $\mu(2I) \leq c\,\mu(I)$, where $2I$ denotes an arc on the circle, concentric with I, of twice the length. The measure μ in the Beurling-Ahlfors theorem is the harmonic measure for a certain elliptic operator in divergence form, whence the problem of null-sets for doubling measures is closely related to that of null-sets for harmonic measure [3].

Certain estimations, such as those for singular integrals and maximal functions, which are classical in the case of Lebesgue measure, can be obtained for doubling measure in Euclidean space ([4], [5] and [6]). Doubling measures also appear in relation to inner functions in several complex variables [1].

The definition of doubling measure has meaning for any metric space (X, ρ), *i.e.* $\mu(B(x, 2r)) \leq c\,\mu(B(x, r))$, and it is natural to ask which compact metric spaces (X, ρ) carry non-trivial doubling measures. A necessary and sufficient condition was found by Vol'berg and Konyagin [8], and called finite uniform metric dimension: in each ball

$B(x, 2r)$ at most N points can be found with mutual distances at least r.

In view of the original interest in singular mappings and singular measures, mutually singular doubling measures on the same metric space are of interest. We prove that such measures exist provided (X, ρ) carries a doubling measure and is perfect. This answers a question stated in [8, p. 637].

A measure μ on $\mathbb{R}^1$, is called *dyadic-doubling* if $\mu(I) \leq c\,\mu(J)$ whenever I and J are adjacent dyadic intervals of the same length, whose union is also dyadic. These measures occur in the theory of weights and are completely characterized [7]. It is hardly surprising that the class of doubling measures and the class of dyadic doubling measures are different, but less trivial that the corresponding classes of null-sets (which we abbreviate as $\mathcal{N}$ and $\mathcal{N}_d$) are different. The class $\mathcal{N}$ is bilipschitz invariant. The class $\mathcal{N}_d$ lacks an invariance property of $\mathcal{N}$: we find a closed set E, not in $\mathcal{N}_d$, and a set T of full measure in $\mathbb{R}^1$, $|\mathbb{R}^1 \backslash T| = 0$, such that $t + E$ is in $\mathcal{N}_d$ for each t in T. A previous example [9] accomplished this with a set T of dimension 1. The class $\mathcal{N}_d$ is not invariant under multiplication by positive numbers t, but our example is not as strong as the one for addition.

1. Singular doubling measures on compact metric space.

Vol'berg and Konyagin proved [8] that *a compact metric space (X, ρ) carries a nontrivial doubling measure μ on X:*

$$(1.1) \qquad \mu(B(x, 2R)) \leq \Lambda\,\mu(B(x, R)), \qquad \text{for all } x \in X,\ R > 0,$$

where $\Lambda \geq 1$ and $B(x, R) = \{y \in X : \rho(x, y) < R\}$, if and only if it has finite uniform metric dimension. In particular *any compact set X in $\mathbb{R}^n$ carries a nontrivial doubling measure.*

They also raised the question: *on which compact metric spaces (X, ρ) are all doubling measures mutually absolutely continuous?* It follows from a well-known example of Beurling and Ahlfors [2] that this is not the case even for the unit circle. We prove the following.

Theorem 1. *Let (X, ρ) be a compact metric space and μ be a doubling measure on X having no atoms. Then there exists a doubling measure on X singular with respect to μ.*

We emphasize that a doubling measure on X satisfies the doubling condition on X only; that is, only balls with centers in X figure in the definition.

We say that (X,ρ) has *finite uniform metric dimension* if there exists a finite $N = N(X,\rho)$ such that for any $x \in X$ and $R > 0$, there are at most N points in $B(x, 2R)$ separated from one another by a distance at least R.

PROOF OF THEOREM 1. Let (X,ρ) satisfy all conditions in Theorem 1, and μ be a doubling measure on X with $\mu(X) = 1$. To construct a doubling measure on X singular with respect to μ, we invoke the idea of Riesz product on the measure space (X,μ). The functions w_k in the next lemma play the role of $1 + a_k \cos kx$ in the usual Riesz product.

Lemma 1. *There exist measurable functions w_k on X taking values $1/2$ and $3/2$ only, so that*

$$\mu(w_k = 1/2) = \mu(w_k = 3/2) = 1/2\,,$$

$$w_k \longrightarrow 1 \qquad \textit{weakly in } L^2(d\mu)\,, \tag{1.2}$$

and

$$w_k^{1/2} \longrightarrow \frac{1}{2}(\sqrt{1/2} + \sqrt{3/2}) \qquad \textit{weakly in } L^2(d\mu)\,. \tag{1.3}$$

PROOF. We observe that every measurable set E of measure $\mu(E) > 0$, can be divided into two subsets, each of measure $\mu(E)/2$. This is a general property of measure spaces with no atoms. Hence there is a measurable function w such that $\mu(w < t) = t$, $0 \le t \le 1$. Let $g(t)$ have period 1 on $[0, +\infty)$ with $g = 1/2$ on $[0, 1/2)$, $g = 3/2$ on $[1/2, 1]$. We set $w_k = g(2^k w)$.

To see that $w_k \to 1$ weakly in $L^2(d\mu)$, we observe that the functions w_k are independent. Therefore w_k tends weakly to its mean, as does $w_k^{1/2}$.

For $x \in X$ and $r > 0$, define

$$h_{x,r}(y) = \begin{cases} 1\,, & \text{if } \rho(x,y) \le r\,, \\ 0\,, & \text{if } \rho(x,y) \ge 3r/2\,, \\ 3 - 2\,\rho(x,y)\,r^{-1}\,, & \text{if } r < \rho(x,y) < 3r/2\,. \end{cases}$$

By the doubling property of μ, there exists $A > 1$ independent of x and r, so that

$$\int_X h_{x,2r}(y)\,d\mu(y) \le A \int_X h_{x,r}(y)\,d\mu(y)\,, \tag{1.4}$$

for all $x \in X$ and $r > 0$.

Let $\alpha = (\sqrt{1/2}+\sqrt{3/2})/2$, $\beta = 21\,\alpha/20$ and $\{w_k\}$ be the functions in Lemma 1. Note that $\beta < 1$.

We shall construct continuous functions $\{u_n\}_1^\infty$ and $\{v_n\}_1^\infty$ on X, so that the following inequalities are true:

$$\frac{1}{2} - \frac{1}{100\,(n+1)} \le u_n \le \frac{3}{2} + \frac{1}{100\,(n+1)}\,, \tag{1.5}$$

$$\int_X \prod_0^n u_i\, d\mu = 1\,, \tag{1.6}$$

$$\int_X \Big(\prod_0^n u_i\Big)^{1/2} d\mu \le \beta^n\,, \tag{1.7}$$

$$\int_X h_{x,2r} \prod_0^n u_i\, d\mu \le \Big(7 - \frac{1}{n+1}\Big) A \int_X h_{x,r} \prod_0^n u_i\, d\mu\,, \tag{1.8}$$

for all $x \in X$ and $r > 0$;

$$0 \le v_n \le 1\,, \tag{1.9}$$

$$\int_X v_n\, d\mu \le \beta^{n/2}\,, \tag{1.10}$$

and for all $0 \le j \le n$,

$$\int_X (1 - v_j) \prod_0^n u_i\, d\mu \le \Big(3 - \frac{1}{n+1}\Big)\beta^j\,. \tag{1.11}$$

Let $u_0 \equiv 1$ and $v_0 \equiv 0$ on X.

Assume that $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ have been chosen so that (1.5) to (1.11) are satisfied; we shall construct u_{n+1} and v_{n+1}.

Because of (1.2), (1.3), (1.7) and (1.11), for sufficiently large $k > k(u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_n)$,

$$\int_X \Big(\prod_0^n u_i\Big)^{1/2} w_k^{1/2}\, d\mu \le \Big(1 + \frac{1}{100\,(n+1)}\Big)\, \alpha\, \beta^n\,, \tag{1.12}$$

and

$$\int_X (1 - v_j)\Big(\prod_0^n u_i\Big) w_k\, d\mu \le \Big(3 - \frac{1}{\sqrt{(n+1)(n+2)}}\Big)\, \beta^j\,, \tag{1.13}$$

for all $0 \le j \le n$.

Because $u_i, 0 \le i \le n$, are uniformly continuous on X with values in $[1/4, 2]$, it follows from (1.4) that for all $x \in X$,

$$\int_X h_{x,2r} \prod_0^n u_i\, d\mu \le A\Big(1 + \frac{1}{n+1}\Big) \int_X h_{x,r} \prod_0^n u_i\, d\mu\,,$$

provided that $0 < r < r(u_0, u_1, \ldots, u_n)$. Since $1/2 \le w_k \le 3/2$,

$$\int_X h_{x,2r} \prod_0^n u_i\, w_k\, d\mu \le 3A\Big(1 + \frac{1}{n+1}\Big) \int_X h_{x,r} \prod_0^n u_i\, w_k\, d\mu \tag{1.14}$$

for all $x \in X$ and $0 < r < r(u_0, u_1, \ldots, u_n)$.

Now we can see by (1.2), the compactness of X and the continuity of $h_{x,r}(y)$ with respect to the variables x, y and r that

$$\lim_{k \to \infty} \int_X h_{x,r}(y) \prod_0^n u_i(y)\, w_k(y)\, d\mu(y) = \int_X h_{x,r}(y) \prod_0^n u_i(y)\, d\mu(y)$$

uniformly for $x \in X$, $r \ge r(u_0, \ldots, u_n)$. Moreover the integrals on the right have a positive lower bound for all x and $r \ge r(u_0, \ldots, u_n)$. We deduce from (1.8) and (1.14) that for sufficiently large k,

$$\begin{aligned} &\int_X h_{x,2r} \prod_0^n u_i\, w_k\, d\mu \\ &\qquad \le \Big(7 - \frac{1}{\sqrt{(n+1)(n+2)}}\Big) A \int_X h_{x,r} \prod_0^n u_i\, w_k\, d\mu\,, \end{aligned} \tag{1.15}$$

for all $x \in X$ and $r > 0$.

Now choose and fix one $w_{k(n)}$, so that (1.12), (1.13) and (1.15) are satisfied.

Denote by $d\nu_n = \prod_0^n u_i \, d\mu$. It follows from Lusin's theorem that there exists a continuous $\tilde{w}_{k(n)}$ on X taking values in $[1/2, 3/2]$ that agrees with $w_{k(n)}$ on X outside a set E_n of small ν_n measure. And let

$$u_{n+1} = \Big(\int_X \tilde{w}_{k(n)} \, d\nu_n \Big)^{-1} \tilde{w}_{k(n)} .$$

Clearly

$$\int \prod_0^{n+1} u_i \, d\mu = \int u_{n+1} \, d\nu_n = 1 ,$$

and

$$\frac{1}{2} - \frac{1}{100\,(n+2)} \le u_{n+1} \le \frac{3}{2} + \frac{1}{100\,(n+2)}$$

if $\nu_n(E_n)$ is sufficiently small. Moreover, $\nu_n(E_n)$ can be chosen small enough, so that (1.12), (1.13) and (1.15) remain true for slightly bigger constants when w_k is replaced by u_{n+1} :

$$\text{(1.16)} \qquad \int_X \Big(\prod_0^{n+1} u_i \Big)^{1/2} d\mu \le \beta^{n+1} ,$$

$$\int_X (1 - v_j) \prod_0^{n+1} u_i \, d\mu \le \Big(3 - \frac{1}{n+2}\Big) \beta^j , \qquad 0 \le j \le n ,$$

(the case $j = n+1$ shall be provided later) and

$$\int_X h_{x,2r} \prod_0^{n+1} u_i \, d\mu \le \Big(7 - \frac{1}{n+2}\Big) A \int_X h_{x,r} \prod_0^{n+1} u_i \, d\mu .$$

Finally, choose v_{n+1} continuous on $X, 0 \le v_{n+1} \le 1$, and

$$v_{n+1} = \begin{cases} 0, & \text{on the set where } \prod_0^{n+1} u_i \le \beta^{n+1} , \\ 1, & \text{on the set where } \prod_0^{n+1} u_i \ge 2\beta^{n+1} . \end{cases}$$

It follows from (1.16) that

$$\int_X v_{n+1}\,d\mu \le \mu\Big(\prod_0^{n+1} u_i \ge \beta^{n+1}\Big) \le \beta^{(n+1)/2},$$

and

$$\int_X (1-v_{n+1})\prod_0^{n+1} u_i\,d\mu \le \int_{\prod_0^{n+1} u_i \le 2\beta^{n+1}} \prod_0^{n+1} u_i\,d\mu \le 2\,\beta^{n+1}.$$

Hence u_{n+1} and v_{n+1} satisfy all properties (1.5) to (1.11).

Finally let ν be a w^* limit of $\prod_0^n u_i\,d\mu$, or of some subsequence.

In view of (1.10) and (1.11), $\int v_j\,d\mu \le \beta^{j/2}$ and $\int(1-v_j)\,d\nu \le 3\beta^j$ for all j. Thus $v_j \to 0$ almost everywhere with respect to $d\mu$ and $v_j \to 1$ almost everywhere with respect to $d\nu$. Therefore μ and ν are mutually singular.

From (1.8), it follows that for all $x \in X$ and $r > 0$,

$$\nu(B(x,2r)) \le \int h_{x,2r}\,d\nu \le 7A \int h_{x,r}\,d\nu \le 7A\,\nu\Big(B\Big(x,\frac{3}{2}r\Big)\Big).$$

Therefore ν is a doubling measure on X. This completes the proof of Theorem 1.

Let (X,ρ) be a compact metric space of finite uniform metric dimension. Let E_X be the set of accumulation points in X and F_X be the set of isolated points in X. Then $X = E_X \cup F_X$.

Lemma 2. *Let μ be any doubling measure on X. Then every point in F_X has positive μ-measure, and every point in E_X has zero μ-measure.*

PROOF. It is clear that every isolated point has positive μ-measure by the doubling condition. Let $x \in E_X$, and pick $\{x_n\} \subseteq X$ so that $0 < \rho(x,x_n) < \rho(x,x_{n-1})/10$. Then $\{B(x_n, 2\,\rho(x,x_n)/3)\}$ are mutually disjoint, and $x \in B(x_n, 4\,\rho(x,x_n)/3)$. Therefore

$$\begin{aligned}\mu(X) &\ge \sum_n \mu\Big(B\big(x_n, \frac{2}{3}\,\rho(x,x_n)\big)\Big)\\ &\ge c\sum_n \mu\Big(B\big(x_n, \frac{4}{3}\,\rho(x,x_n)\big)\Big) \ge c\sum_n \mu(\{x\}).\end{aligned}$$

Since $\mu(X) < \infty$, $\mu(\{x\})$ must be zero.

Therefore, for a doubling measure on (X, ρ) we may call $\mu|_{E_X}$ the *continuous part of* μ *and* $\mu|_{F_X}$ *the atomic part of* μ.

Corollary 1. *If* X *is a perfect set, then with respect to each doubling measure on* X *there exists a singular one.*

The following statement, which follows easily from the proof of Theorem 1, answers the question of Vol'berg and Konyagin.

Corollary 2. *All doubling measures on* (X, ρ) *are mutually absolutely continuous if and only if every doubling measure on* X *is purely atomic. A necessary topological condition is that* $\overline{F}_X = X$.

Although $\overline{F}_X = X$ is a necessary condition, it is far from being a sufficient condition for the mutual absolute continuity of all doubling measures on X; see the example below.

EXAMPLE. There are compact subsets X, Y and Z of $\mathbb{R}^1$, so that the sets of accumulation points E_X, E_Y and E_Z are all perfect sets, and the closures of isolated points $\overline{F}_X$, $\overline{F}_Y$ and $\overline{F}_Z$ equal to X, Y and Z respectively. However, all doubling measures on X are purely atomic; every doubling measure on Y contains a nontrivial continuous part; some doubling measures on Z are purely atomic and others have a nontrivial continuous part.

CONSTRUCTION OF X. Let E be the Cantor ternary set on $[0,1]$, F be the centers of all maximal intervals in $[0,1]\backslash E$, and $X = E \cup F$. Let $\{a_{n,j}\}_{j=1}^{2^{n-1}}$ be all points in F of distance $3^{-n}/2$ to E and $I_{n,j} = [a_{n,j} - 3^{-n+1}/2, a_{n,j} + 3^{-n+1}/2]$, thus $\{I_{n,j}\}_{j=1}^{2^{n-1}}$ forms a covering of E. Let μ be a doubling measure on X. Then there exists $c > 0$, so that

$$\sum_j \mu(\{a_{n,j}\}) \geq c \sum_j \mu(I_{n,j} \cap X) \geq c\, \mu(E),$$

for each $n \geq 1$. Since $\mu(X) < \infty$, $\mu(E)$ must be zero.

CONSTRUCTION OF Y. Let E be the Cantor ternary set on $[0,1]$, and $[0,1]\backslash E = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} I_{n,j}$, where $\{I_{n,j}\}_{j=1}^{2^{n-1}}$ are the maximal intervals in $[0,1]\backslash E$ of length exactly 3^{-n}, arranged in ascending order with respect to j. Given $0 < \beta_n < 1/4$, let $a_{n,j}$ and $b_{n,j}$ $(a_{n,j} > b_{n,j})$ be the two points in $I_{n,j}$ of distance $\beta_n 3^{-n}$ to E,

$$F = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} \{a_{n,j}, b_{n,j}\}$$

and $Y = E \cup F$.

Suppose that

$$\sum \left(log \frac{1}{\beta_n}\right)^{-1} < \infty. \tag{1.17}$$

Then every doubling measure μ on Y has a nontrivial continuous part.

Denote by $\{E_{n,j}\}_{j=1}^{2^n}$ the intervals in $[0,1]\backslash \bigcup_{k=1}^{n} \bigcup_{j=1}^{2^{k-1}} I_{k,j}$ in ascending order with respect to j. Note that $|E_{n,j}| = 3^{-n}$, and that

$$\bigcup_{j=1}^{2^n} E_{n,j} \cap F = \bigcup_{k=n+1}^{\infty} \bigcup_{j=1}^{2^{k-1}} \{a_{k,j}, b_{k,j}\}. \tag{1.18}$$

Moreover for each (n, j),

$$\begin{aligned} \operatorname{dist}(a_{n,j}, E) &= \operatorname{dist}(a_{n,j}, E_{n,2j}) \\ &= \operatorname{dist}(b_{n,j}, E) \\ &= \operatorname{dist}(b_{n,j}, E_{n,2j-1}) = \beta_n 3^{-n}. \end{aligned}$$

Suppose that μ is a doubling measure on Y which is purely atomic, *i.e.* $\mu(E) = 0$. Then there exists $c > 0$ depending only on μ, so that

$$\mu(E_{n,2j} \cap F) \geq c \left(\log \frac{1}{\beta_n}\right) \mu(\{a_{n,j}\}), \tag{1.19}$$

and

$$\mu(E_{n,2j-1} \cap F) \geq c \left(\log \frac{1}{\beta_n}\right) \mu(\{b_{n,j}\}). \tag{1.20}$$

In fact, fix (n, j) and write $E_{n,2j}$ as $[p/3^n, (p+1)/3^n]$ for some integer p and $a_{n,j}$ as $p/3^n - \beta_n/3^n$. Let $x_q = p/3^n + 1/3^{n+2q} \in E$, and $B_q = B(x_q, 2 \cdot 3^{-n-2q-1})$. Note that $\{B_q\}_1^{\infty}$ are mutually disjoint and

$B_q \subseteq E_{n,2j}$; moreover if $\beta_n 3^{-n} < 3^{-n-2q-1}$, then $a_{n,j} \in 2B_q$ and $2B_q \cap F = \{a_{n,j}\} \cup (2B_q \cap E_{n,2j})$, where $2B_q$ is the interval $B(x_q, 4 \cdot 3^{-n-2q-1})$. Therefore there exist $c' > 1$, so that for $1 \le q \le (1/3)\log 1/\beta_n$,

$$\mu(\{a_{n,j}\}) \le \mu(2B_q) \le c'\,\mu(B_q) = c'\,\mu(B_q \cap F).$$

Summing over q, $1 \le q \le (1/3)\log 1/\beta_n$, we obtain

$$\mu(\{a_{n,j}\}) \log\frac{1}{\beta_n} \le 3\,c'\,\mu(E_{n,2j} \cap F).$$

The proof of (1.20) is similar.

Denote by $m_n = \mu(\bigcup_{j=1}^{2^{n-1}} \{a_{n,j}, b_{n,j}\})$ and recall that $\sum_1^\infty m_n = \mu(Y) < \infty$. Summing over j's in (1.19) and (1.20), we deduce from (1.18) that $\sum_{n+1}^\infty m_k \ge c(\log 1/\beta_n)\,m_n$, for each $n \ge 1$. Denote $\sum_n^\infty m_k$ by r_n and $\log 1/\beta_n$ by N_n, we have $r_{n+1} \ge r_n(cN_n)/(1 + cN_n)$ for $n \ge 1$. Thus

$$r_{n+1} \ge \prod_{k=1}^{n} \frac{cN_k}{1 + cN_k}\,\mu(Y).$$

As $n \to \infty$, the left hand side approaches 0, and the right hand side has a positive limit under the hypothesis (1.17), which is impossible.

Therefore every doubling measure on Y must have a nontrivial continuous part.

The construction of Z uses Whitney modification of measures. Let E be a closed set on $\mathbb{R}^1$ and μ be a measure on $\mathbb{R}^1$. We call μ^E, a measure on $\mathbb{R}^1$, a *Whitney modification of* μ if $\mu^E \equiv \mu$ on E, and for some Whitney decomposition $\mathcal{W} = \{I\}$ of $\mathbb{R}^1 \setminus E$, $\mu^E(\{x_I\}) = \mu^E(I) = \mu(I)$ for every $I \in \mathcal{W}$ and x_I the center of I.

Recall that intervals in $\mathcal{W}$ have mutually disjoint interiors, $\cup_{\mathcal{W}} I = \mathbb{R}^1 \setminus E$ and $\operatorname{dist}(I, E)/4 \le |I| \le 4\operatorname{dist}(I, E)$ for each $I \in \mathcal{W}$. A measure μ is said to have the doubling property on a closed set S, if (1.1) is satisfied for all $x \in S$ and $R > 0$.

Lemma 3. *If μ is a doubling measure on $\mathbb{R}^1$, then any Whitney modification μ^E of μ has the doubling property on $E \cup F$, where F consists of the centers of intervals in $\mathcal{W}$, and $\mathcal{W}$ is the Whitney decomposition associated with μ^E.*

PROOF. For $x \in E$, let $I_x = \{x\}$, and for $x \in F$, let I_x be the interval in $\mathcal{W}$ centered at x. For any $x \in E \cup F$ and $R > 0$, we claim that

$$\mu^E(B(x,R)) \cong \mu(B(x, \operatorname{dist}(x,E)), \tag{1.21}$$

if $B(x,R) \cap (E \cup F) = \{x\}$, and

$$\mu^E(B(x,R)) \cong \mu(B(x,R)), \tag{1.22}$$

if $B(x,R) \cap (E \cup F)$ has at least two points.

By $c \cong d$ we mean c/d is bounded above and below by positive numbers depending only on the constant Λ in the doubling property of μ.

Let $a = \inf\{y \in E \cup F : \ y > x - R\}$ and $b = \sup\{y \in E \cup F : \ y < x + R\}$. Note that $a = x - R$ if $x - R \in E$, and $a \in F$ otherwise; and that $b = x + R$ if $x + R \in E$, and $b \in F$ otherwise.

If $a = b$, then $a = b = x$. In this case,

$$\mu^E(B(x,R)) = \mu^E(\{x\}) = \mu(I_x) \cong \mu(B(x, \operatorname{dist}(x,E)).$$

If $a \neq b$, then $b - a \geq \max\{\operatorname{dist}(a,E), \operatorname{dist}(b,E)\}/64$. Note that $x + R - b \leq 64 \operatorname{dist}(b,E)$ and $a - (x - R) \leq 64 \operatorname{dist}(a,E)$. Therefore $2R \geq b - a \geq 2^{-12}R$. In this case

$$\mu^E(B(x,R)) = \mu^E([a,b]) = \mu([a - |I_a|/2, b + |I_b|/2]) \cong \mu(B(x,R)).$$

Doubling property of μ^E on $E \cup F$ follows immediately from (1.21) and (1.22).

This property of the Whitney modification has a natural generalization to $\mathbb{R}^n$. For the converse, we raise the following question.

QUESTION. For which $(X, \mu), X$ perfect in $\mathbb{R}^n$ and μ doubling on X, is μ the restriction of a doubling measure in $\mathbb{R}^n$?

CONSTRUCTION OF Z. It follows from Lusin's theorem and an example of Beurling and Ahlfors [1] that there exist a nontrivial doubling measure μ on $\mathbb{R}^1$ and a perfect set $E \subseteq [0,1]$ of positive length so that $\mu(E) = 0$. Let $\mathcal{W}$ be any Whitney decomposition of $\mathbb{R}^1 \backslash E, F$ be the centers of intervals in $\mathcal{W}$ and $Z = E \cup (F \cap [-100, 100])$. Then the Whitney modification μ^E has the doubling property on Z (a modification of Lemma 3) and is purely atomic.

Let σ be the Lebesgue measure on $\mathbb{R}^1$ and σ^E be a Whitney modification. Then $\sigma^E(E) = \sigma(E) > 0$, and σ^E has the doubling property on Z.

QUESTION. Do there exist X compact in $\mathbb{R}^1$, a doubling measure μ on X, such that $\overline{F}_X = X$ and that $\mu|_{E_X}$ is also a nontrivial doubling measure on E_X? (Recall that E_X is the set of accumulation points and F_X is the set of isolated points.)

QUESTION. Given a compact set X on $\mathbb{R}^n$, and $\alpha > 0$, does there exist a measure μ doubling on X such that μ has full measure on a Borel set of Hausdorff dimensions less or equal than α?

We believe that the answer is positive when $n = 1$.

2. Null sets for dyadic doubling measures.

A measure μ on $\mathbb{R}^1$ is called a *doubling measure* if (1.1) holds for all $x \in \mathbb{R}^1$ and $R > 0$, equivalently, there exists $\lambda \geq 1$ so that $\mu(I) \leq \lambda\,\mu(J)$ for all neighboring intervals I and J of the same length. A measure μ on $\mathbb{R}^1$ is called a *dyadic doubling measure* if there exists $\lambda \geq 1$ so that $\mu(I) \leq \lambda\,\mu(J)$ whenever I and J are two dyadic neighboring intervals of same length and $I \cup J$ is also a dyadic interval. We shall refer to the constant λ above as $\lambda(\mu)$.

Denote by $\mathcal{D}$ the collection of all doubling measures on $\mathbb{R}^1$ and by $\mathcal{D}_d$ the collection of all dyadic doubling measures on $\mathbb{R}^1$. Denote by $\mathcal{N}$ the collection of null sets for $\mathcal{D}$, *i.e.*, $\mathcal{N} = \{E \subseteq \mathbb{R}^1 : \mu(E) = 0$ for all $\mu \in \mathcal{D}\}$, and $\mathcal{N}_d$ its dyadic counterpart $\{E \subseteq \mathbb{R}^1 : \mu(E) = 0$ for all $\mu \in \mathcal{D}_d\}$. Clearly $\mathcal{N}_d \subseteq \mathcal{N}$, and $\mathcal{N}$ is invariant under any bilipschitz mapping on $\mathbb{R}^1$. However $\mathcal{N}_d$ is not invariant under translation, or under multiplication.

Theorem 2. *There exist a perfect set $S \subseteq [0,1]$ which is in $\mathcal{N}\backslash\mathcal{N}_d$, and a set $T \subseteq \mathbb{R}^1$ of full measure (i.e., $\mathbb{R}^1\backslash T$ has zero length) such that $t + S \in \mathcal{N}_d$ for each $t \in T$.*

A weaker version of Theorem 2 was proved in [9] with $\dim T = 1$. The present proof has the same structure, but uses more refined estimations.

The analogue of Theorem 2 under multiplication is more difficult. We are only able to find perfect sets S and T with $\dim T = 0$, so that $S \in \mathcal{N}\backslash\mathcal{N}_d$ but $tS \in \mathcal{N}_d$ for each $t \in T$. We shall report this elsewhere.

The following lemmas from [9] are needed in our proof.

Lemma 4. *Let μ be a dyadic doubling measure on $\mathbb{R}^1$. Then there exists $c > 1$ depending on $\lambda(\mu)$ only, so that for any dyadic interval S and any subinterval T of S,*

$$\frac{1}{4}\left(\frac{|T|}{|S|}\right)^c \mu(S) \le \mu(T) \le 4\left(\frac{|T|}{|S|}\right)^{1/c} \mu(S).$$

Lemma 5. *Let S be any dyadic interval and μ and ν be two dyadic doubling measures on $\mathbb{R}^1$ satisfying $\mu(S) = \nu(S)$. Then the new measure $\omega \equiv \nu$ on S, $\omega \equiv \mu$ on $\mathbb{R}^1\backslash S$ is a dyadic doubling measure on $\mathbb{R}^1$ with $\lambda(\omega) \le \max\{\lambda(\mu), \lambda(\nu)\}$.*

Lemma 6. *Given a, ε, $\delta \in (0,1)$ with $\varepsilon + \delta^a < 1/16$, then there exists a measure $\tau \in \mathcal{D}_d$, with $\lambda(\tau) \le 10^{1/a}$, which satisfies $\tau([0,1]) = 1$, $\tau([0,\varepsilon]) = \varepsilon$ and $\tau([1-\delta, 1]) = \delta^a$.*

We shall use $\tau_{a,\varepsilon,\delta}$ to denote the restriction of this τ measure to the interval $[0,1]$.

PROOF OF THEOREM 2. Let $\alpha > 1$ and choose $\beta > \alpha$, $0 < a < \min\{1/5, \alpha\beta^{-1}\}$, $0 < c_m < 1/4$ and positive integers L_m $(m \ge 1)$ so that the following are true:

$$c_m^{-1} m^{\alpha-\beta} = o(1), \qquad \text{as } m \to \infty, \tag{2.1}$$

$$\sum m^{\beta a - \alpha} < \infty, \tag{2.2}$$

$$\sum (1 - m^{-\beta a})^{L_m} < \infty, \tag{2.3}$$

and

$$\sum (1 - 4c_m)^{L_m} = \infty. \tag{2.4}$$

For example, choose $\beta = 4\alpha$, $a = (\alpha - 1)/5\alpha$, $c_m = (4m^{2\alpha})^{-1}$ and $L_m = [m^{2\alpha}]$, with $[\cdot]$ the greatest integer function.

Let $K_1 = 0$, and $K_{m+1} = K_m + L_m$ for $m \geq 1$. Define n_k inductively by letting $n_0 = 10$ and

$$n_{k+1} = n_k + 10 + [\beta \log_2 m - \log_2 c_m] \tag{2.5}$$

when $K_m \leq k < K_{m+1}$.

Given $m \geq 1$, $1+K_m \leq k \leq K_{m+1}$ and integer j, denote by $L's, I's$ and $J's$ the dyadic intervals:

$$L_{k,j} = \left[\frac{j}{2^{n_k}}, \frac{j+1}{2^{n_k}}\right],$$

$$I_{k,j} = \left[\frac{j}{2^{n_k}}, \frac{j}{2^{n_k}} + \frac{1}{2^{n_k+5}\dot{m}^\alpha}\right]$$

and

$$J_{k,j} = \left[\frac{j+1}{2^{n_k}} - \frac{1}{2^{n_k+5}\dot{m}^\beta}, \frac{j+1}{2^{n_k}}\right],$$

where $\dot{m}^\alpha = 2^{[\alpha \log_2 m]}$, $\dot{m}^\beta = 2^{[\beta \log_2 m]}$ and $[\cdot]$ is the greatest integer function. Note that for $1 + K_m \leq k \leq K_{m+1}$,

$$|J_{k,j}|/|I_{k,j'}| = O(m^{\alpha-\beta}) \to 0, \qquad \text{as } m \to \infty \tag{2.6}$$

and

$$|J_{k,j}|/|L_{k+1,j'}| = 2^{n_{k+1}-n_k-[\beta \log_2 m]-5} \geq m^{2\alpha}.$$

To *construct* S, first we permanently remove from $S_1 \equiv [0,1]$ a group of mutually disjoint intervals $I_{k,j}$ of different sizes, with k ranging from $1+K_1$ to K_2, and collect some of the J intervals from the remaining part of S_1; call the union S_2. Next we permanently remove from S_2 a group of intervals $I_{k,j}$ with k ranging from $1 + K_2$ to K_3, and collect some of the J intervals from the remaining part of S_2; call the union S_3, etc. Finally let $S = \cap S_m$.

Let $S_1 = [0,1]$,

$$\mathcal{C}_1^I = \{I_{1,j} : \ J_{1,j-1} \cup I_{1,j} \subseteq S_1\},$$

$$\mathcal{C}_1^J = \{J_{1,j} : \ J_{1,j} \cup I_{1,j+1} \subseteq S_1\}.$$

For $1 = 1 + K_1 \leq k < K_2$, let

$$\begin{aligned}\mathcal{C}^I_{k+1} = \mathcal{C}^I_k \cup \{I_{k+1,j} :\ & J_{k+1,j-1} \cup I_{k+1,j} \text{ is contained in } S_1 ,\\ & \text{but neither } J_{k+1,j-1} \text{ nor } I_{k+1,j}\\ & \text{is contained in any interval in } \mathcal{C}^I_k \cup \mathcal{C}^J_k\} ,\end{aligned}$$

$$\begin{aligned}\mathcal{C}^J_{k+1} = \mathcal{C}^J_k \cup \{J_{k+1,j} :\ & J_{k+1,j} \cup I_{k+1,j+1} \text{ is contained in } S_1 ,\\ & \text{but neither } J_{k+1,j} \text{ nor } I_{k+1,j+1}\\ & \text{is contained in any interval in } \mathcal{C}^I_k \cup \mathcal{C}^J_k\} .\end{aligned}$$

Note that all intervals in $\mathcal{C}^I_{K_2} \cup \mathcal{C}^J_{K_2}$ have mutually disjoint interiors, and that intervals in $\mathcal{C}^I_{K_2}$ and those in $\mathcal{C}^J_{K_2}$ appear in pairs sharing common end points.

Let

$$S^I_2 = \text{union of all intervals in } \mathcal{C}^I_{K_2} ,$$

and

$$S_2 = \text{union of all intervals in } \mathcal{C}^J_{K_2} .$$

We keep the interior of S^I_2 in the complement of S permanently, and construct S^I_3 and S_3 as subsets of S_2. Let

$$\mathcal{C}^I_{1+K_2} = \{I_{1+K_2,j} :\ J_{1+K_2,j-1} \cup I_{1+K_2,j} \subseteq S_2\} ,$$

and

$$\mathcal{C}^J_{1+K_2} = \{J_{1+K_2,j} :\ J_{1+K_2,j} \cup I_{1+K_2,j+1} \subseteq S_2\} .$$

And define for $1 + K_2 \le k < K_3$,

$$\begin{aligned}\mathcal{C}^I_{k+1} = \mathcal{C}^I_k \cup \{I_{k+1,j} :\ & J_{k+1,j-1} \cup I_{k+1,j} \text{ is contained in } S_2 ,\\ & \text{but neither } J_{k+1,j-1} \text{ nor } I_{k+1,j}\\ & \text{is contained in any interval in } \mathcal{C}^I_k \cup \mathcal{C}^J_k\} ,\end{aligned}$$

$$\begin{aligned}\mathcal{C}^J_{k+1} = \mathcal{C}^J_k \cup \{J_{k+1,j} :\ & J_{k+1,j} \cup I_{k+1,j+1} \text{ is contained in } S_2 ,\\ & \text{but neither } J_{k+1,j} \text{ nor } I_{k+1,j+1}\\ & \text{is contained in any interval in } \mathcal{C}^I_k \cup \mathcal{C}^J_k\} ;\end{aligned}$$

and let

$$S^I_3 = \text{union of all intervals in } \mathcal{C}^I_{K_3} ,$$

and

$$S_3 = \text{union of all intervals in } \mathcal{C}^J_{K_3} .$$

We keep the interior of S_3^I in the complement of S permanently, and construct S_4^I and S_4 from S_3 as above. Continue this process to obtain $\mathcal{C}_{K_m}^I$, $\mathcal{C}_{K_m}^J$, S_m^I and S_m for all $m \geq 5$. Let

$$S = \bigcap_1^\infty S_m \, .$$

The rest of the proof is based on the simple fact that $J_{k,j}$ and $I_{k,j+1}$ are two adjacent intervals of very uneven sizes (2.6), whose common boundary point $(j+1)/2^{n_k}$ is a dyadic number.

To prove $S \in \mathcal{N}$, we note from (2.6) that for any $\nu \in \mathcal{D}$, and $1 + K_m \leq k \leq K_{m+1}$,

$$\nu(J_{k,j}) \leq m^{c(\alpha-\beta)} \, \nu(J_{k,j} \cup I_{k,j+1}) \, ,$$

for some $c > 0$ depending only on the doubling constant of ν. Summing over all $J_{k,j}$ in $\mathcal{C}_{K_m+1}^J$ we have $\nu(S) \leq \nu(S_{m+1}) \leq m^{c(\alpha-\beta)}\nu([0,1])$. Thus $\nu(S) = 0$. Alternatively, S is a porous set with large holes, therefore it is in $\mathcal{N}$, see [9].

To show $S \notin \mathcal{N}_d$, we apply scaled versions of Lemmas 5 and 6 repeatedly, to obtain a measure $\mu \in \mathcal{D}_d$ on $\mathbb{R}^1$, periodic with period 1, such that for $1 + K_m < k \leq K_{m+1}$ and all integers j

$$\mu(I_{k,j}) = (32\,\dot{m}^\alpha)^{-1} \, \mu(L_{k,j}) \tag{2.7}$$

and

$$\mu(J_{k,j}) = (32\,\dot{m}^\beta)^{-a} \, \mu(L_{k,j}) \, . \tag{2.8}$$

More precisely, μ is the weak limit of a subsequence of measures $\{\mu_{k_m}\}$ to be constructed as follows. Let μ_0 be the Lebesgue measure on $\mathbb{R}^1$. Assume that $\mu_{k_m} \in \mathcal{D}_d$, has been constructed with period 1. Then inductively for $1 + K_m \leq k \leq K_{m+1}$, let $f_{k,j}$ be the linear map that maps $L_{k,j}$ onto $[0,1]$, and define for $E \subseteq L_{k,j}$,

$$\mu_{k_{m+1}}(E) = \mu_{k_m}(L_{k,j}) \, \tau_{a,(32\dot{m}^\alpha)^{-1},(32\dot{m}^\beta)^{-1}}(f_{k,j}(E)) \, ,$$

where τ is the measure in Lemma 6. In view of Lemma 5, the measure $\mu_{k_{m+1}}$ is in $\mathcal{D}_d$ and satisfies (2.7) and (2.8) with μ replaced by $\mu_{k_{m+1}}$.

We note from the construction that

$$\mu_{k_2}([0,1] \backslash (S_2^I \cup S_2)) \leq \Big(1 - \frac{1}{2}\,(32^{-1} + 32^{-a})\Big)^{K_2 - K_1} .$$

The occurrence of 1/2 above is due to the fact that each $J \in \mathcal{C}_2^J$ and its companion I interval are not contained in the same L interval, but rather in two adjacent L intervals. Therefore

$$\begin{aligned}\mu_{k_2}(S_2) &\geq \mu_{k_2}\left(S_2^I \cup S_2\right)\left(1-32^{a-1}\right)\\ &\geq \left(1-\left(1-\frac{1}{2}\left(32^{-1}+32^{-a}\right)\right)^{K_2-K_1}\right)\left(1-32^{a-1}\right)\\ &\geq \left(1-\left(1-\frac{1}{2}32^{-a}\right)^{K_2-K_1}\right)\left(1-32^{a-1}\right).\end{aligned}$$

From the construction of S,

$$\mu(S) \geq \prod_{m=1}^{\infty}\left(1-\left(1-\frac{1}{2}32^{-a}\,m^{-\beta a}\right)^{K_{m+1}-K_m}\right)\left(1-32^{a-1}\,m^{\beta a-\alpha}\right),$$

which is positive in view of (2.2) and (2.3). Therefore $S \notin \mathcal{N}_d$.

For $x \in \mathbb{R}^1$, denote by $\|x\|$ the distance from x to the nearest integer. Let T be the set of t's such that there are infinitely many m's so that

$$\| t\, 2^{5+n_k} \dot{m}^\alpha \| > c_m\,, \qquad \text{for every } k,\ 1+K_m \leq k \leq K_{m+1}\,. \tag{2.9}$$

Denote points t in $[0,1]$ by their binary expansion $\sum_{n=1}^{\infty} t_n\, 2^{-n}$ with $t_n = 1$ or 0. Then $\| t\, 2^{5+n_k} \dot{m}^\alpha \| > c_m$ provided that not all t_n equal 0 for those n in the interval $(5+n_k+[\alpha\log_2 m], 7+n_k+[\alpha\log_2 m]-\log_2 c_m)$, and not all t_n equal 1 for the same range of $n's$. In view of (2.5), $n_{k+1} > n_k + [\alpha\log_2 m] - \log_2 c_m + 7$; thus for $m \geq 1$,

$$|\{t \in [0,1] :\ (2.9) \text{ holds}\}| \geq (1-4\,c_m)^{K_{m+1}-K_m}\,.$$

Since $[0,1]\backslash T = \bigcup\limits_{M\geq 10}^{\infty}$ (2.9) fails for every $m \geq M$,

$$|[0,1]\backslash T| \leq \sum_{M=10}^{\infty}\prod_{m\geq M}\left(1-(1-4\,c_m)^{K_{m+1}-K_m}\right) = 0\,,$$

because of (2.4). Similar argument show that $|\mathbb{R}^1\backslash T| = 0$.

Given $t \in T$, assume that for a certain m,

$$\| t\, 2^{n_k+5} \dot{m}^\alpha \| > c_m\,, \qquad \text{for every } k,\ 1+K_m \leq k \leq K_{m+1}\,;$$

then for each integer j,

$$\frac{p}{2^{n_k+5}\dot{m}^\alpha} + \frac{c_m}{2^{n_k+5}\dot{m}^\alpha} \le t + \frac{j+1}{2^{n_k}} \le \frac{p+1}{2^{n_k+5}\dot{m}^\alpha} - \frac{c_m}{2^{n_k+5}\dot{m}^\alpha} \tag{2.10}$$

for some integer p. Note that $t + (j+1)/2^{n_k}$ is the common boundary point for intervals $t + J_{k,j}$ and $t + I_{k,j+1}$, and that in view of (2.10),

$$t + J_{k,j} \subseteq \left[\frac{p}{2^{n_k+5}\dot{m}^\alpha}, t + \frac{j+1}{2^{n_k}}\right],$$

$$t + I_{k,j+1} \supseteq \left[t + \frac{j+1}{2^{n_k}}, \frac{p+1}{2^{n_k+5}\dot{m}^\alpha}\right] \equiv I'_{k,j+1}\,.$$

Suppose ν is in $\mathcal{D}_d$. Because

$$\left[\frac{p}{2^{n_k+5}\dot{m}^\alpha}, \frac{p+1}{2^{n_k+5}\dot{m}^\alpha}\right]$$

is a dyadic interval, it follows from Lemma 4 that

$$\frac{\nu(t + J_{k,j})}{\nu(t + I_{k,j+1})} \le \left(\frac{|J_{k,j}|}{|I'_{k,j+1}|}\right)^c \le \left(\frac{\dot{m}^{\alpha-\beta}}{c_m}\right)^c$$

for some $c > 0$ depending on ν only. Summing over all $J_{k,j}$ in $\mathcal{C}^J_{K_m}$, we have

$$\nu(t + S) \le (m^{\alpha-\beta}\, c_m^{-1})^c\, \nu([0,1])\,.$$

Because t is in T, m can be made arbitrarily large. Therefore $\nu(t+S) = 0$ by (2.1). This proves that $t + S \in \mathcal{N}_d$ for every $t \in T$.

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Pluriharmonic interpolation and hulls of C^1 curves in the unit sphere

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1. Introduction.

Let Γ be a simple closed C^1 curve lying in the unit sphere $b\mathbb{B}^n$, $\mathbb{B}^n$ being the unit ball in $\mathbb{C}^n$. By Stolzenberg's theorem [12], either Γ is polynomially convex or $\widehat{\Gamma} \setminus \Gamma$ is a 1-dimensional analytic subvariety of $\mathbb{B}^n$. Čirka [6] and Forstneric [7] showed that if Γ is C^2 and not polynomially convex, then $\widehat{\Gamma} \setminus \Gamma$ is smooth near Γ (*i.e.*, singularities do not accumulate at Γ) and Γ is transverse at each of its points-in the sense that the tangents to Γ never lie in the complex tangent space to $b\mathbb{B}^n$. In particular, if Γ is C^2 and has at least one complex tangent, then Γ is polynomially convex. This is no longer true when Γ is only C^1: Rosay [9] constructed a C^1 Jordan curve in $b\mathbb{B}^2$, bounding an analytic disk in $\mathbb{B}^2$, and having a complex tangent at a single point. Motivated by this, the first author proved in [1] that a rectifiable curve Γ is polynomially convex if the set of points of Γ where its tangent (these exist almost everywhere) is complex-tangential has positive linear measure.

On the other hand, Berndtsson and Bruna [4] showed that, when Γ is of class C^3 (in fact, $C^{2+\epsilon}$ was enough), the functions in $C(\Gamma)$ which can be interpolated by functions pluriharmonic on $\mathbb{B}^n$ and continuous on $\overline{\mathbb{B}^n}$ form a closed subspace of $C(\Gamma)$ of finite codimension. When Γ is polynomially convex, this codimension is zero. When Γ is not polynomially convex, then by Forstneric's result, $\widehat{\Gamma} \setminus \Gamma$ is smooth near

Γ and the Berndtsson-Bruna theorem is related with the solvability of the Dirichlet problem in $\widehat{\Gamma} \setminus \Gamma$. In fact, Shcherbina [11] later used this approach to characterize the codimension, for $n = 2$ and for C^∞ curves, in terms of the topology of Γ.

Here we shall study the C^1 case of both types of problems-hulls and pluriharmonic interpolation. Our first result (Theorem 2.1) states that if Γ is not polynomially convex, then $\widehat{\Gamma} \setminus \Gamma$ is still nice near Γ. In fact, close to $Q \in \Gamma$, $\widehat{\Gamma} \setminus \Gamma$ is the graph, over its projection on a suitable complex line, of a holomorphic function -the complex line being the normal to the sphere in the case when the tangent to Γ is transverse and being a complex tangent line to the sphere at Q otherwise. From this local parametrization we deduce in Section 3 our second result: If Γ is not polynomially convex and $T(s)$ is the so-called index of transversality of Γ (*i.e.*, $i\,T(s)$ is the complex normal component of the unit tangent to Γ) then $T(s)$ is greater or equal than 0 (after a possible change of orientation of Γ) and

$$\int \frac{ds}{T(s)^p} < \infty,$$

for all $p > 0$. This captures both Forstneric's result (because, in the C^2 case, $T(s) = O(|s - s_0|)$ close to a complex-tangential point $\gamma(s_0)$) and the C^1 version of the Theorem in [1] for rectifiable curves.

Finally, in Section 4, we prove, under the hypotheses that T has constant sign and that $\int T^{-1} ds$ converges, that the Berndtsson-Bruna result on pluriharmonic interpolation carries over to C^1 curves; in particular, this holds for all non-polynomially convex C^1 curves.

To simplify the exposition, we assume in the rest of the paper that $n = 2$. It is routinely checked that all proofs generalize to $n > 2$ with straightforward modifications.

2. The local structure of the hull.

Let Γ be a simple closed curve of class C^1 lying on the unit sphere $S = b\,\mathbb{B}^2$, with arc-length parametrization $\gamma(s)$. We assume that Γ is not polynomially convex. By Stolzenberg's theorem (see [12] or [13, Theorem 30.1], [15, Chapter 13]), $V = \widehat{\Gamma} \setminus \Gamma$ is a one-dimensional analytic variety. We will prove here:

Theorem 2.1. *For each point $Q \in \Gamma$ there is a neighbourhood N and a complex line L through Q such that if π is the projection on L one* *has:*

a) π *is one-to-one from* $\overline{V} \cap N$ *onto a domain* $\overline{U} \subset L$ *of class* C^1, *and* π *maps* $\Gamma \cap N$ *onto an arc* $\tau \subset bU$.

b) There is an holomorphic function f *in* U *of class* $C^1(\overline{U})$ *such that* $\overline{V} \cap N$ *is the graph of* f *over* $\overline{U}$.

Thus V is locally a graph in the neighbourhood of Γ. In the proof of Theorem 2.1 we need the following "general principle" (see [12], [2], [3], [5] for the original argument; see also [15, Theorem 10.7], and [13, Lemmas 30.7 and 30.9]):

Lemma 2.2. *Let* $X \subset \mathbb{C}^2$ *be compact and* p *a polynomial. Let* Ω_∞ *be the unbounded component of* $\mathbb{C} \setminus p(X)$. *Suppose that there is an open Jordan arc* σ, *open in* $p(X)$, *such that*

a) $\sigma \subset b\,\Omega_\infty \cap b\,\Omega$, *where* Ω *is a bounded component of* $\mathbb{C} \setminus p(X)$.

b) $p^{-1}(\lambda) \cap X$ *contains exactly one point for all* $\lambda \in \sigma$.

Then, either $p^{-1}(\Omega) \cap \widehat{X}$ *is empty or* $p^{-1}(\Omega) \cap \widehat{X}$ *is single sheeted. In the later case, there exists* $\phi \in H^\infty(\Omega, \mathbb{C}^2)$ *such that*

$$p^{-1}(\Omega) \cap \widehat{X} = \{\phi(\lambda) : \ \lambda \in \Omega\}.$$

Moreover, there are no points of $\widehat{X} \setminus X$ *over* σ, *and* ϕ *has a continuous extension to* σ.

In case p is a coordinate function, say $p(z) = z_1$, then $\phi(\lambda) = (\lambda, f(\lambda))$, so over Ω, $\widehat{X} \setminus X$ is the graph of $f \in H^\infty(\Omega)$.

PROOF OF THEOREM 2.1. We shall distinguish two cases:

Case A: Γ *is transverse at* Q, *i.e. the tangent to* Γ *at* Q *has a non-zero complex normal component.* We can assume without loss of generality that $Q = (1,0) = \gamma(0)$ and transversality means that $\gamma_1'(0)$ is a non-zero (pure imaginary) number.

Then $\gamma_1(s)$ determines s for $|s|$ small enough, say $|s| < \varepsilon$. Since Γ is simple, $\gamma_1(s) \neq 1$ for $|s| \geq \varepsilon$. Hence, shrinking ε if needed we see that the points of

$$\sigma \stackrel{\text{def}}{=} \{\gamma_1(s) : \ |s| < \varepsilon\}$$

are covered only once by z_1 on Γ. We also assume that ε is small enough so that σ is a C^1-curve (because $\gamma_1'(0) \neq 0$). We apply Lemma 2.2 with $X = \Gamma$ (note that $\sigma \subset b\,\Omega_\infty$ because $1 \in b\,\Omega_\infty$), $p(z) = z_1$ and therefore over Ω, the bounded component of $z_1(\Gamma)$ having σ in the boundary, V is the graph of some holomorphic function f. On σ

$$f(\gamma_1(s)) = \gamma_2(s)\,, \qquad |s| < \varepsilon\,.$$

Thus f is of class C^1 on σ. Now we take as U a C^1 domain in the z_1-plane contained in Ω and such that $b\,U \cap b\,\Omega \stackrel{\text{def}}{=} \tau \subset \sigma$.

At this point we need:

Lemma 2.3. *Let U be a C^1-domain in the complex plane, let f be holomorphic in U, continuous on $\overline{U}$. Let $\tau \subset b\,U$ be an arc on which $f|_{bU}$ is of class C^1. Then f' extends continuously to the points of τ.*

PROOF. Let $g : \overline{\Delta} \to \overline{U}$ be the Riemann mapping function from the unit disk $\overline{\Delta}$ to $\overline{U}$. Let $I \subset \mathbb{T} = b\,\Delta$ the arc mapped onto τ. Let $\tau' \subset \tau$ be a closed subarc of τ and $I' \subset I$ its corresponding arc in $\mathbb{T}$. It is well-known ([8, Theorem 10.1]) that $\arg g'$ has a continuous extension to $\overline{\Delta}$, hence $\log g \in$ VMOA, and so g' and $1/g'$ are in $L^p(\mathbb{T})$ for all $p > 0$. Let $h = f \circ g$, which is in the disc algebra. The hypothesis implies that h is absolutely continuous in I with derivative

$$h' = (f' \circ g)\, g'$$

at almost all points of I, and h' is in $L^p_{\text{loc}}(I)$ because f' is continuous on τ. Assume without loss of generality that $I = \mathbb{T} \cap D(1,r)$, $I' = \mathbb{T} \cap \overline{D}(1,r')$. Let χ be a C^∞ function supported in $D(1,r)$ equal to 1 on $D(1,r')$. We consider

$$H(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\chi(\zeta)\, h(\zeta)}{\zeta - z}\, d\zeta\,, \qquad z \in \Delta\,.$$

Note that

$$H(z) = \chi(z)\, h(z) - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\chi(z) - \chi(\zeta)}{\zeta - z}\, h(\zeta)\, d\zeta$$

and that this last integral defines a smooth function on $\overline{\Delta}$. Therefore H is in the disk algebra and on $\mathbb{T}$ it is an absolutely continuous function with derivative in $L^p(\mathbb{T})$. This implies that H' is in the Hardy class H^p, $p > 0$, or which is the same, the non-tangential maximal function over the Stolz angle $S(\theta)$

$$(H')^*(\theta) = \sup\{|H'(z)| : z \in S(\theta)\}$$

belongs to $L^p(\mathbb{T})$, $p > 0$. From this it follows that

$$(h')^*(\theta) = \sup\{|h'(z)| : z \in S(\theta)\}$$

is in $L^p(I')$ for all $p > 0$ and hence is in $L^p_{\mathrm{loc}}(I)$ for all $p > 0$.

Since also $(g')^{-1}$ has non-tangential maximal function in $L^p(\mathbb{T})$ for all $p > 0$, we conclude that $(f' \circ g)^* \in L^1_{\mathrm{loc}}(I)$. We will show now that $f' \circ g$ extends continuously to all points of I, which obviously implies the lemma. Fix a closed subarc $J \subset I$ and let D be a C^∞-domain in Δ such that $J \subset bD \cap \mathbb{T} \subset I$. Then $f' \circ g$ has non-tangential maximal function (with respect to D) in $L^1(bD)$ and therefore belongs to $H^1(D)$. Now, $f' \circ g$ is continuous in $bD \cap \mathbb{T}$ and so $f' \circ g|_D$ extends continuously to the closure of D (here we use the fact that D being a C^∞ domain the holomorphic function theory of D is analogous to the one of Δ). By the choice of D, it then follows that $f' \circ g$ extends continuously to all points of J.

Shrinking the domain U a bit we conclude the proof of Theorem 2.1 in the case A.

Case B. *Γ is complex-tangential at Q, i.e. the tangent to Γ at Q points in the complex-tangential direction.* We can assume, without loss of generality, that $Q = (1, 0) = \gamma(0)$ and that $\gamma_2'(0) = 1$, $\gamma_1'(0) = 0$. It follows immediately that there is $\varepsilon > 0$ such that

$$s \mapsto |\gamma_2(s)|$$

is strictly increasing in $(0, \varepsilon)$ and strictly decreasing in $(-\varepsilon, 0)$.

Since $|\gamma_1(s)|^2 + |\gamma_2(s)|^2 = 1$,

$$s \mapsto |\gamma_1(s)|$$

is strictly decreasing in $(0, \varepsilon)$ and strictly increasing in $(-\varepsilon, 0)$. Let us define

$$\sigma_+ = \{\gamma_1(s) : 0 < s < \varepsilon\},$$

$$\sigma_- = \{\gamma_1(s) : \ -\varepsilon < s < 0\},$$
$$\sigma = \{\gamma_1(s) : \ -\varepsilon < s < \varepsilon\}.$$

As before, since Γ is simple, and shrinking ε if needed we may assume that $\gamma_1(s) \notin \sigma$ for $|s| \geq \varepsilon$.

We know that both σ_+, σ_- meet each circle $|\lambda| = \rho$ at most once. We claim that σ_+, σ_- do not intersect. This is seen as in [9]: suppose that σ_+ and σ_- meet at $\lambda_0 = \gamma_1(a) = \gamma_1(b)$, with $0 < a < \varepsilon$, $-\varepsilon < b < 0$. Let $\nu = \gamma_1(b, a)$. Let R be a smooth simply connected domain in the z_1 plane separating ν from $z_1(\Gamma) \setminus \nu$ and containing $z_1(\Gamma) \setminus \nu$. The domain R admits a peaking function $H(\lambda)$ for the point λ_0. The function equal to H on $\overline{R}$ and to 1 in the domain bounded by ν can be uniformly approximated, by Mergelyan's theorem, by a sequence of polynomials $p_n(\lambda)$. This shows that the arc $\Gamma_1 = \gamma([b, a]) \subset \Gamma$ is a peak set for the algebra $P(\Gamma)$. Analogously, $\Gamma_2 = \Gamma \setminus \Gamma_1$ is also a peak set for $P(\Gamma)$. But Γ_1, Γ_2 are smooth arcs and so $P(\Gamma_1) = C(\Gamma_1)$, $P(\Gamma_2) = C(\Gamma_2)$. By general theory of uniform algebras (in fact an easy duality argument works), it follows that $P(\Gamma) = C(\Gamma)$ and Γ would be polynomially convex.

Therefore, σ_+ and σ_- do not meet, which means that γ_1 is one to one in $(-\varepsilon, \varepsilon)$ and Lemma 2.2 applies again as before. The main difference here with respect the situation in case A is that here the curve σ is in general not smooth at 1. This is why the z_1 projection does not work in this case and we shall look now to the z_2 projection.

Let g be the holomorphic function on Ω given by Lemma 2.2 so that $\lambda \mapsto (\lambda, g(\lambda))$ parametrizes V over Ω. Note that on σ

$$g(\gamma_1(s)) = \gamma_2(s), \qquad |s| < \varepsilon,$$

defines a curve τ in the z_2-plane which we can assume smooth because $\gamma_2'(0) = 1$.

We denote, for small δ,

$$\Omega_\delta = \{\lambda \in \Omega : \ |\lambda| > 1 - \delta\}.$$

The function g extends continuously to $\overline{\Omega}_\delta$, and

$$V_\delta = \{(\lambda, g(\lambda)) : \ \lambda \in \overline{\Omega}_\delta\}$$

is a neighbourhood of $Q = (1, 0)$ in $\widehat{\Gamma}$. The boundary $b\Omega_\delta$ consists of

$$C_\delta = \{\lambda \in \Omega : \ |\lambda| = 1 - \delta\}$$

and two arcs σ_+^δ, σ_-^δ included respectively in σ_+, σ_-. We denote $\sigma^\delta = \sigma_+^\delta \cup \sigma_-^\delta \cup \{1\} = \gamma_1(I_\delta)$ and $\tau^\delta = \gamma_2(I_\delta) = g(\sigma^\delta)$, a smooth subarc of τ.

We claim that for small enough δ, z_2 does not vanish at V_δ except at Q, *i.e.* g does not have zeros in Ω_δ. To see this, choose first δ such that g does not vanish on $\overline{C_\delta}$. Ω_δ is a simply-connected rectifiable domain and $g(b\,\Omega_\delta)$ is a closed piecewise smooth curve containing the arc $\tau^\delta = g(\sigma^\delta)$. Since $g(C_\delta)$ does not pass through 0 in the neighbourhood of 0 there are two components of $\mathbb{C}\setminus g(b\Omega_\delta)$, which we call R_+^δ and R_-^δ. Let m_+, m_- be the number of preimages (counting multiplicities) in Ω_δ of points of R_+^δ, R_-^δ, respectively. Let $N = \max\{m_+, m_-\}$.

If λ is a zero of g in Ω_δ, as g is an open mapping, the image of a neighbourhood of λ is a neighbourhood of 0 and hence meets both R_+^δ and R_-^δ. Therefore there are at most N zeros of g in Ω_δ, and g has no zeros in Ω_δ for small enough δ.

We will show next that $m_+ - m_-$ is either $+1$ or -1. We have

$$m_+ = \frac{1}{2\pi}\,\Delta_{b\Omega_\delta}\arg(g(\lambda)-a)\,,\qquad a\in R_+^\delta\ ,$$

$$m_- = \frac{1}{2\pi}\,\Delta_{b\Omega_\delta}\arg(g(\lambda)-b)\,,\qquad b\in R_-^\delta\ ,$$

or

$$\begin{aligned}2\pi m_+ &= \Delta_{g(b\Omega_\delta)}\ \arg(\zeta-a)\\ &= \Delta_{g(C_\delta)}\ \arg(\zeta-a)+\Delta_{\tau^\delta}\ \arg(\zeta-a)\,,\end{aligned}$$

$$\begin{aligned}2\pi m_- &= \Delta_{g(b\Omega_\delta)}\ \arg(\zeta-b)\\ &= \Delta_{g(C_\delta)}\ \arg(\zeta-b)+\Delta_{\tau^\delta}\ \arg(\zeta-b)\,.\end{aligned}$$

Recall that τ^δ is a smooth curve. Now substract both equations and make $a, b \to 0$ to get

$$2\pi(m_+-m_-) = \lim_{a\to 0}\Delta_{\tau^\delta}\arg(\zeta-a) - \lim_{b\to 0}\Delta_{\tau^\delta}\arg(\zeta-b) = \pm 2\pi\,.$$

If instead we add the equations we get

$$2\pi(m_+ + m_-) = 2\,\Delta_{g(C_\delta)}\arg\zeta + \lim_{a\to 0}\Delta_{\tau^\delta}\arg(\zeta-a) + \lim_{b\to 0}\Delta_{\tau^\delta}\arg(\zeta-b)\,.$$

Since τ^δ is smooth the limits

$$\lim_{\delta\to 0}\lim_{a\to 0}\Delta_{\tau^\delta}\arg(\zeta-a)\,,\qquad \lim_{\delta\to 0}\lim_{b\to 0}\Delta_{\tau^\delta}\arg(\zeta-b)\,,$$

are π, $-\pi$ or $-\pi, \pi$ respectively. Hence we obtain

$$\lim_{\delta \to 0} \Delta_{C_\delta} \arg g(\lambda) = \lim_{\delta \to 0} \Delta_{g(C_\delta)} \arg \zeta = \pi \, (m_+ + m_-) \, .$$

Let $m = m_+ + m_-$, an odd integer; m is positive, because g is not constant. Since g does not vanish in Ω_δ we can consider $h = g^{1/m}$. Then

$$\lim_{\delta \to 0} \Delta_{C_\delta} \arg h(\lambda) = \pi \, .$$

As the argument shows, this holds for all arcs in Ω_δ joining σ_+^δ and σ_-^δ.

Next we will see that for small enough δ, h is a one-to-one map from Ω_δ to a domain R_δ, which is smooth in the neighbourhood of $0 \in b R_\delta$. Recall that $g(\gamma_1(s)) = \gamma_2(s)$, $\sigma^\delta = \gamma_1(I_\delta)$ and $\gamma_2(0) = 1$. Without loss of generality we can assume that $h(\gamma_1(s))$, $s > 0$, is the principal determination of $\gamma_2(s)^{1/m}$, so that $h(\sigma_+^\delta)$ is a C^1 arc having limiting tangent $(1, 0)$ at 0, as it easily seen using polar coordinates. In the same way, $h(\sigma_-^\delta)$ is a C^1 arc having as tangent at 0 the opposite of some m-root of (-1). Since π is the variation of the argument, and m is odd, this root must be of course -1 and hence $h(\sigma^\delta)$ is a smooth arc. The fact that h is one-to-one follows then from the argument principle.

Let $f : R_\delta \to \Omega_\delta$ be the inverse map of h, $h(\lambda) = \zeta$, $g(\lambda) = \zeta^m$. We thus get the parametrization

$$\begin{aligned} R_\delta &\longrightarrow V_\delta \\ \zeta &\mapsto (f(\zeta), \zeta^m) \, , \end{aligned}$$

$f(0) = 1$, and R_δ is smooth near 0. Also, $f \in C(\overline{R}_\delta)$.

The final step is to show that m must, in fact, be 1. Suppose that $m \geq 3$. Let $F : \Delta \to R_\delta$ be the Riemann mapping function, $F(1) = 0$ from the unit disk to R_δ. Then we have a parametrization

$$\begin{aligned} \overline{\Delta} &\longrightarrow V_\delta \\ z &\mapsto (f(F(z)), F(z)^m) = (G(z), F(z)^m) \, . \end{aligned}$$

Shrinking R_δ we may suppose that R_δ is a C^1-domain so the mapping F satisfies, as said in the proof of Lemma 2.3, that $F' \in H^p$ for all p. In particular, F satisfies a Lipschitz condition of order β for all $\beta < 1$ and G is then in the disk algebra. Let $\alpha \subset b\,\Delta$ be the arc parametrizing $V_\delta \cap \Gamma$, $1 \in \alpha$.

We will show that $G(e^{it})$ has a non-zero derivative μ at $t = 0$. Once this is seen, since

$$|F(e^{it})| = O(|t|^{\beta})$$

for all $\beta < 1$, taking $\beta > 1/m$ we see that $F^m(e^{it})$ has zero-derivative and then $(\mu, 0)$ is tangent to Γ at $Q = (1, 0)$, in contradiction with the assumption that Γ is complex-tangential at Q.

Let $G = IK$ be the inner-outer factorization of G,

$$K(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{e^{it}+z}{e^{it}-z} \log|G(e^{it})|\,dt\right).$$

Since $G(1) = 1$, the inner part I has an analytic continuation near 1; using that $|I(re^{it})|^2 \leq 1$ and $|I(e^{it})|^2 = 1$ it is immediate to obtain that $I'(1)\,\overline{I(1)} \geq 0$. We want to show now that the formal rule

$$K'(1) = -K(1)\,\frac{1}{4\pi}\int_{-\pi}^{+\pi} \frac{1}{\sin^2\dfrac{t}{2}} \log|G(e^{it})|\,dt$$

(which makes sense because $\log|G(e^{it})| \simeq 1 - |G(e^{it})| \simeq |F(e^{it})|^{2m} = O(t^2)$ near $t = 0$) obtained by differentiating K under the integral sign is fully justified. This will give

$$K'(1) = k\,K(1)$$

with $k > 0$, because $\log|G| \leq 0$ and then

$$\begin{aligned}\mu = G'(1) &= I'(1)\,K(1) + I(1)\,K'(1)\\ &= I'(1)\,\overline{I(1)} + K'(1)\,\overline{(K(1))} = k + I'(1)\,\overline{I(1)} > 0\,.\end{aligned}$$

It remains thus to show that the formal rule above holds true. For this it is enough to show that the part of the integral over α satisfies the rule. Now on α, $|G(e^{it})|^2 = 1 - |F(e^{it})|^{2m}$. Writing

$$u(t) = \frac{1}{2}\log(1 - |F(e^{it})|^{2m})$$

and

$$Hu(z) = \frac{1}{2\pi}\int_{\alpha} \frac{e^{it}+z}{e^{it}-z}\,u(t)\,dt$$

we shall show that $Hu(z)$ has an unrestricted derivative at 1, *i.e.*

$$\frac{Hu(z) - Hu(1)}{z-1} - \frac{1}{2\pi}\int_\alpha \frac{2e^{it}}{(e^{it}-1)^2}\,u(t)\,dt \underset{z\to 1}{\longrightarrow} 0\,.$$

The last expression can be written

$$\frac{1}{\pi}\int_\alpha \frac{u(t)}{e^{it}-1}\Big(\frac{e^{it}}{e^{it}-z} - \frac{e^{it}}{e^{it}-1}\Big)\,dt = C(v)(z) - C(v)(1)\,,$$

where $v(t) = u(t)/(e^{it}-1)$ and $C(v)$ denotes the Cauchy integral of v over the arc α. Now

$$v'(t) = \frac{u'(t)}{e^{it}-1} - \frac{i\,e^{it}\,u(t)}{(e^{it}-1)^2}\,.$$

But $u' = m\,|F^{2(m-1)}|(\mathrm{Re}\,\overline{F}F')(1-|F|^{2m})^{-1}$, so that $v' \in L^p$ for all p. Thus v satisfies a Lipschitz condition of order β for all β, hence so does $C(v)$ and we are done.

In conclusion we have proved that $m = 1$. This means, with the notations used before, that for small δ, g is a one-to-one map from Ω_δ to $R_\delta = g(\Omega_\delta)$ with inverse f. We now take as U a C^1 domain included in R_δ so that $bU \cap bR_\delta = \tau^\delta$. On τ^δ

$$f(\gamma_2(s)) = \gamma_1(s)$$

and hence f is C^1 on τ^δ. With Lemma 2.3 we conclude as before.

This completes the proof of Theorem 2.1.

3. Analytical properties of the curve Γ.

Theorem 2.1 has several consequences regarding the curve Γ itself. Here we will draw one of them, to be used in the next section. If $\gamma(s)$ is the arc-length parametrization of Γ, as mentioned in the introduction

$$\gamma(s)\,\overline{\gamma'(s)} = i\,T(s)$$

with T a real-valued continuous function (we use the notation $a\,\overline{b}$ for $\sum_j a_j\,\overline{b_j}$ if $a = (a_1,\dots,a_n)$, $b = (b_1,\dots,b_n)$). T can be said to measure the transversality of Γ. We prove:

Theorem 3.1. *If Γ is a C^1 simple closed curve which is not polynomially convex, then T has constant sign and*

$$\int \frac{ds}{|T(s)|^p} < +\infty\,, \qquad \textit{for all } p\,.$$

Corollary 3.2. *If* $\int \frac{ds}{T(s)} = +\infty$, *$\Gamma$ is polynomially convex.*

The Corollary implies Forstneric' result [7] according to which a C^2 curve which is complex-tangential at one point is polynomially convex and should be compared as well with Alexander's in [1], stating that a rectifiable curve whose set of complex tangencies has positive length is also polynomially convex.

PROOF OF THEOREM 3.1. Of course it is enough to prove the result locally around a complex-tangent point $Q \in \Gamma$. Let's consider the parametrization described in Theorem 2.1. Without loss of generality we assume $= (1,0)$, that U is a C^1-domain in the z_2-plane, f is holomorphic in U, of class C^1 up to $\overline{U}$ and

$$\begin{aligned} \overline{U} &\longrightarrow \overline{V} \cap N \\ \lambda &\mapsto (f(\lambda), \lambda) \end{aligned}$$

is the parametrization. Let $\tau \subset bU$ be the arc parametrizing $\Gamma \cap N$.

As before, let $F : \overline{\Delta} \to \overline{U}$ be the Riemann mapping function from the unit disk to U, $F(1) = 0$, and the parametrization

$$\begin{aligned} \overline{\Delta} &\overset{\phi}{\longrightarrow} \overline{V} \cap N \\ z &\mapsto (G(z), F(z))\,. \end{aligned}$$

Let $\alpha \subset \mathbb{T}$ be the arc mapped onto τ by F. We know that $F' \in H^p$ for all p. In particular $F(e^{it})$ is absolutely continuous and G is in the disk algebra. If $e^{it_0} \in \alpha$ is a point where $F(e^{it})$ is differentiable, since f is C^1 on τ, $G(e^{it})$ is also differentiable at t_0. Hence $G(e^{it})$ is differentiable almost everywhere on α and moreover $d(G(e^{it}))/dt$ is in $L^p(\alpha)$. By [14, Theorem IV.5], the non-tangential limit at such point

$$\lim_{\substack{z \to e^{it_0} \\ \text{n.t.}}} G'(z) \overset{\text{def}}{=} G'(e^{it_0})$$

exists and equals $-i\,e^{-it_0}\,d(G(e^{it}))/dt\,|_{t=t_0}$. It obviously follows that this limit also equals the non-tangential limit

$$\lim_{\substack{z\to e^{it_0}\\ \text{n.t.}}} \frac{G(z)-G(e^{it_0})}{z-e^{it_0}}\,.$$

Therefore at almost all points of α the tangential and radial derivatives of F and G exist and belong to $L^p(\alpha)$ for all p. In particular,

$$ds = (|G'|^2+|F'|^2)^{1/2}\,dt$$

and T is given almost everywhere on $\phi(\alpha)$ by

$$\begin{aligned} i\,T &= \frac{1}{(|G'|^2+|F'|^2)^{1/2}}\,\overline{\langle i\,e^{it}\,G', i\,e^{it}\,F'\rangle}\,\langle G,F\rangle \\ &= -\frac{i\,e^{-it}}{(|G'|^2+|F'|^2)^{1/2}}\,(\overline{G'}\,G+\overline{F'}\,F)\,. \end{aligned}$$

We claim that there is a constant $c>0$ such that

$$e^{it}(G'\,\overline{G}+F'\,\overline{F})\geq c\,, \qquad \text{almost everywhere on } \alpha\,.$$

Lemma 3.3. *Let $\phi:\overline{\Delta}\to\overline{\mathbb{B}^2}$, $\phi=\langle G,F\rangle$ be an analytic disk such that the tangential and radial derivatives of F, G exist almost everywhere on an arc $\alpha\subset\mathbb{T}$, with $\phi(\alpha)\subset S$. Let $a=\phi(0)\in\mathbb{B}^2$. Then*

$$e^{it}\,(G'\,\overline{G}+F'\,\overline{F})\geq\frac{1-|a|}{1+|a|}\,, \qquad \textit{almost everywhere on } \alpha\,.$$

PROOF. Since $|F|^2+|G|^2=1$ on α and $|F(re^{it})|^2+|G(re^{it})|^2\leq 1$, at one point where everything makes sense, one has

$$\begin{aligned} 0 &= \frac{d}{dt}(|F(e^{it})|^2+|G(e^{it})|^2) \\ &= 2\,\mathrm{Re}\; i\,e^{it}\left(F'(e^{it})\,\overline{F(e^{it})}+G'(e^{it})\,\overline{G(e^{it})}\right)\,, \end{aligned}$$

$$0\leq \left.\frac{d}{dr}\right|_{r=1}\left(|F(re^{it})|^2+|G(re^{it})|^2\right)=2\,\mathrm{Re}\;e^{it}\,(F'\,\overline{F}+G'\,\overline{G})\,.$$

Therefore $e^{it}(F'\overline{F}+G'\overline{G})$ is real and non-negative.

Assume now that $a=0$. Then we can write $F(z)=z\,F_0(z)$, $G(z)=z\,G_0(z)$ and apply the above to $\langle G_0,F_0\rangle$ because $|G_0|^2+|F_0|^2=1$ on α. Then

$$e^{it}(G'\overline{G}+F'\overline{F})=1+e^{it}(G_0'\overline{G_0}+F_0'\overline{F}_0)\geq 1$$

and the result is proved when $a=0$.

Assume now $a\neq 0$. We can choose complex orthonormal coordinates such that $a=(\lambda,0)$. Let $\varphi_a=\langle\varphi_1,\varphi_2\rangle$ be the automorphism of B with

$$\varphi_1=\frac{(s-1)\,z_1+\lambda-s\,z_1}{1-\overline{\lambda}\,z_1}\,,\qquad \varphi_2=-\frac{s\,z_2}{1-\overline{\lambda}\,z_1}$$

where $s^2=1-|\lambda|^2$, so that $\varphi_a(a)=0$, $\varphi_a^{-1}=\varphi_a$ (see [10, Chapter 2]).

If $\varphi_a\circ\phi=\phi_0=\langle G_0,F_0\rangle$, then $G=\varphi_1\langle G_0,F_0\rangle$, $F=\varphi_2\langle G_0,F_0\rangle$. Therefore, with $D_i=\partial/\partial z_i$

$$\begin{aligned}G'&=(D_1\varphi_1)\,G_0'+(D_2\varphi_1)\,F_0'\,,\\ F'&=(D_1\varphi_2)\,G_0'+(D_2\varphi_2)\,F_0'\,,\end{aligned}$$

and

$$\begin{aligned}G'\overline{G}+F'\overline{F}=&\left((D_1\varphi_1)\overline{\varphi_1}+(D_1\varphi_2)\overline{\varphi_2}\right)G_0'\\ &+\left((D_2\varphi_1)\overline{\varphi_1}+(D_2\varphi_2)\overline{\varphi}_2\right)F_0'\,.\end{aligned}$$

A computation shows that the brackets at (z_1,z_2) equal

$$\frac{s^2\,\overline{z_1}}{|1-\overline{\lambda}\,z_1|^2}\,,\qquad \frac{s^2\,\overline{z_2}}{|1-\overline{\lambda}\,z_1|^2}\,,$$

respectively. Hence,

$$e^{it}(G'\overline{F}+F'\overline{F})=\frac{s^2}{|1-\overline{a}\,\phi_0|^2}\,e^{it}(G_0'\,\overline{G}_0+F_0'\,\overline{F_0})\,.$$

Since $\phi_0(0)=0$, $e^{it}(G_0'\overline{G_0}+F_0'\overline{F_0})\geq 1$ almost everywhere on α and the lemma is proved.

Note that the lemma also gives a proof of the fact that an analytic disk $\phi : \overline{\Delta} \to \overline{\mathbb{B}^2}$ with $\phi(\mathbb{T}) \subset S$ passing through $a \in \mathbb{B}^2$ must have a boundary of length $\geq 2\pi(1-|a|)/(1+|a|)$.

This already shows that T has constant sign. Finally

$$\int_{\phi(\alpha)} \frac{ds}{|T(s)|^p} = \int_\alpha \frac{(|G'|^2 + |F'|^2)^{(p+1)/2}}{|\overline{G'}\,G + \overline{F'}\,F|^p}\,dt$$
$$\leq C \int_\alpha (|G'|^2 + |F'|^2)^{(p+1)/2}\,dt < +\infty\,,$$

which ends the proof of Theorem 3.1.

4. Pluriharmonic interpolation from Γ.

In this section we assume that Γ is a simple closed C^1-curve on $S = b\,\mathbb{B}^2$, with arc-length parametrization $\gamma(s)$, such that its index of transversality defined by

$$i\,T(s) = \overline{\gamma'(s)}\,\gamma(s)$$

satisfies

$$T(s) \geq 0 \qquad \text{and} \qquad \int \frac{ds}{T(s)} < +\infty\,.$$

We may say that Γ is *close to transverse*. As seen in the previous section, this is the case if Γ is not polynomially convex, but we don't assume this here. Our purpose is to prove

Theorem 4.1. *With the assumptions above, the space* PHC *of pluriharmonic functions in* $\mathbb{B}^2$, *continuous up to* $b\,\mathbb{B}^2$ *has a closed trace of finite codimension in* $C(\Gamma)$. *In particular, if* Γ *is polynomially convex, any continuous function on* Γ *can be interpolated by a pluriharmonic function in* PHC.

This was proved in [4] for C^3 curves without any other assumption.

PROOF. The scheme of the proof is the same as that in [4], but each of the steps needs substantial modifications due to the lack of extra smoothness. Let $E \subset \Gamma$ be the set of complex-tangential points of Γ

and let $C_0(\Gamma)$ be the space of continuous functions in Γ vanishing on E. The first step is to construct an operator

$$K : C_0(\Gamma) \longrightarrow \mathrm{PHC}$$

such that

$$(K\varphi)(\gamma(s)) = \int L(t,s)\,\varphi(t)\,dt + \varphi(s)\,,$$

where the integral operator of the right-hand side is compact. The second step consists in showing that E is an interpolation set for the ball algebra, so that by a general result in [13, Theorem. 22.2], there is a linear continuous operator

$$I : C(E) \longrightarrow A(B)$$

such that $I\psi|_E = \psi$. Then the operator

$$P : C(\Gamma) \longrightarrow C(\Gamma)$$

defined by

$$P\varphi = K(\varphi - I\varphi) + I\varphi$$

satisfies

$$(P\varphi)(\gamma(s)) = \varphi(s) + \int L(t,s)\,(\varphi(t) - I\varphi(t))\,dt\,.$$

Now, Range P consists of boundary values of pluriharmonic functions in PHC. Moreover, by Fredholm theory, Range P is closed and of finite codimension. Then, a functional analysis argument ends the proof of the theorem (see [4, Section 6]).

To start with, let

$$K(t,z) = \frac{1}{\pi}\,\mathrm{Im}\,\frac{\gamma(t)\overline{\gamma'(t)}}{1 - z\overline{\gamma(t)}}\,, \qquad K\varphi(z) = \int K(t,z)\,\varphi(t)\,dt\,,$$

$$L(t,x) = K(t,\gamma(x)) = \frac{1}{\pi}\,T(t)\,\mathrm{Re}\,\frac{1}{1 - \gamma(x)\,\overline{\gamma(t)}}\,.$$

Note that $K(t,z)$ is positive and that

$$\mathrm{Re}\,(1 - \gamma(x)\,\overline{\gamma(t)}) = \frac{1}{2}\,|\gamma(x) - \gamma(t)|^2 \simeq |t - x|^2\,,$$

$$\mathrm{Im}\,(1 - \gamma(x)\,\overline{\gamma(t)}) = \mathrm{Im}\int_x^t \gamma'(s)\,\overline{\gamma(t)}\,ds = \int_x^t T(s)\,ds + O(|x - t|^2)\,.$$

Hence

$$\left|1-\gamma(x)\overline{\gamma(t)}\right| \simeq (t-x)^2 + \left|\int_t^x T(s)ds\right| .$$

Lemma 4.2. *With the assumption of Theorem* 4.1,

a) $L(t,x)$ *satisfies*

$$\sup_x \int\limits_{|x-t|\leq\delta} L(t,x)\,dt \to 0\,, \qquad \text{as } \delta\to 0\,.$$

b) $\int K(t,z)\,dt \leq C\,,$ *for all* $z\in\mathbb{B}^2$.

PROOF.

$$|L(t,x)| \simeq T(t)\frac{|t-x|^2}{\left(|t-x|^2+\left|\int_t^x T(\xi)\,d\xi\right|\right)^2} \leq T(t)\frac{|t-x|^2}{\left|\int_t^x T(\xi)\,d\xi\right|^2}\,.$$

Let $\phi(t)=\int_0^t T(\xi)\,d\xi$, ϕ is stricty increasing; let $\psi=\phi^{-1}$, we make the change of variables $u=\phi(t)$. If $v=\phi(x)$, since $T(t)\,dt=du$.

$$\int\limits_{|t-x|\leq\delta} T(t)\frac{|t-x|^2}{\left|\int_t^x T(\xi)\,d\xi\right|^2}\,dt \leq \int\limits_{|u-v|\leq\varepsilon(\delta)} \frac{|\psi(u)-\psi(v)|^2}{|u-v|^2}\,du$$

with $\varepsilon(\delta)\to 0$ as $\delta\to 0$. Now we apply Hardy's inequality, or rather its proof:

$$\begin{aligned}|\psi(u)-\psi(u)|^2 &= \left|\int_v^u \psi'(\xi)\,d\xi\right|^2\\ &\leq \left(\int_v^u |\psi'(\xi)|\,|\xi-v|^{1/2}\,|\xi-v|^{-1/2}\,d\xi\right)^2\\ &\leq 2\,|u-v|^{1/2}\int_v^u |\psi'(\xi)|^2\,|\xi-v|^{1/2}\,d\xi\end{aligned}$$

by Holder's inequality with the measure $|\xi - v|^{-1/2}\, d\xi$. Then,

$$\begin{aligned}
&\int\limits_{|u-v|\le\varepsilon(\delta)} \frac{|\psi(u)-\psi(v)|^2}{|u-v|^2}\, du \\
&\qquad \le 2 \int\limits_{|u-v|\le\varepsilon(\delta)} |u-v|^{-3/2} \left| \int_v^u |\psi'(\xi)|^2\, |\xi - v|^{1/2}\, d\xi \right| du \\
&\qquad \le 2 \int\limits_{|\xi-v|\le\varepsilon(\delta)} |\psi'(\xi)|^2\, |\xi - v|^{1/2} \Bigg(\int\limits_{|u-v|\ge|\xi-v|} |u-v|^{-3/2}\, du \Bigg) d\xi \\
&\qquad \le K \int\limits_{|\xi-v|\le\varepsilon(\delta)} |\psi'(\xi)|^2\, d\xi\,.
\end{aligned}$$

Hence for a) it is enough to have $\int |\psi'(\xi)|^2\, d\xi < +\infty$. But changing variables again, $u = \psi(\xi)$, $du = \psi'(\xi)\, d\xi$, $\psi'(\xi) = 1/T(u)$ this is

$$\int \frac{du}{T(u)} < +\infty\,.$$

This proves part a) of Lemma 4.2. For b), let for fixed z, $s = s(z)$ be such that

$$|1 - \overline{\gamma(s)}\, z| = \min\{\, |1 - \overline{\gamma(t)}\, z| : \text{ all } t\}\,.$$

Then

$$|1 - \overline{\gamma(t)}\, z| \simeq |1 - \overline{\gamma(s)}\, z| + |1 - \overline{\gamma(t)}\, \gamma(s)|\,.$$

The inequality $\lesssim$ is immediate because $|1 - \overline{a}b|^{1/2}$ satisfies a triangle inequality. On the other hand,

$$\begin{aligned}
|1 - \overline{\gamma(t)}\, \gamma(s)| + |1 - \overline{\gamma(s)}\, z| &\lesssim |1 - \overline{\gamma(t)}\, z| + 2|1 - \overline{\gamma(s)}\, z| \\
&\lesssim 3|1 - \overline{\gamma(t)}\, z|
\end{aligned}$$

by the choice of s. Hence

$$|1 - \overline{\gamma(t)}\, z| \simeq |1 - \overline{\gamma(s)}\, z| + |t - s|^2 + \left| \int_s^t T(\xi)\, d\xi \right|\,,$$

$$K(t,z) = \frac{1}{\pi}\, T(t)\, \mathrm{Re}\, \frac{1}{1 - \overline{\gamma(t)}\, z}$$

$$= \frac{1}{\pi} T(t) \frac{\mathrm{Re}\,(1 - \overline{\gamma(t)}\, z)}{|1 - \overline{\gamma(t)}\, z|^2}$$

$$\simeq T(t) \frac{\mathrm{Re}\,(1 - \overline{\gamma(t)}\, z)}{\left(|1 - \overline{\gamma(s)}\, z| + |t-s|^2 + \left|\int_s^t T(\xi)\, d\xi\right|\right)^2}.$$

Next,

$$\begin{aligned} 2\,\mathrm{Re}\,(1 - \overline{\gamma(t)}\, z) &= 1 - \overline{\gamma(t)}\, z + 1 - \gamma(t)\, \overline{z} \\ &= |\gamma(t) - z|^2 + 1 - |z|^2 \\ &\lesssim |\gamma(t) - \gamma(s)|^2 + |z - \gamma(s)|^2 + 1 - |z|^2 \\ &\lesssim |t-s|^2 + |z - \gamma(s)|^2 + 1 - |z|^2 . \end{aligned}$$

Write $r = r(z) = |z - \gamma(s)|^2 + 1 - |z|^2$, $R = |1 - \overline{\gamma(s)}\, z|$. Then

$$\begin{aligned} K(t,z) &\lesssim T(t) \frac{|t-s|^2}{\left(|t-s|^2 + \int_0^t T(\xi)\, d\xi\right)^2} + T(t) \frac{r(z)}{\left(R + \int_s^t T(\xi)\, d\xi\right)^2} \\ &= K_1(t,s) + K_2(t,z) . \end{aligned}$$

In proving *a*) we have already seen that $\int K_1(t,s)\, dt = O(1)$.

Next, with l equal to the length of γ, assuming $s = 0$, and with the change of variables $u = \int_0^t T(\xi)\, d\xi$,

$$\int_0^l \frac{T(t)\, dt}{\left(R + \int_0^t T(\xi)\, d\xi\right)^2} = \int_0^M \frac{du}{(R+u)^2} \leq \int_0^{+\infty} \frac{du}{(R+u)^2} = \frac{1}{R} .$$

But $r \lesssim R$. Hence $\int K_2(t,z)\, dt \leq C$, for all $z \in \mathbb{B}^2$, and part *b*) is also proved.

Lemma 4.3. *If φ vanishes whenever T vanishes, then*

$$\lim_{z \to \gamma(x)} K\varphi(z) = \varphi(x) + \int \varphi(t)\, L(t,x)\, dt .$$

PROOF.

$$\begin{aligned}
&\left| \int K(t,z)\,\varphi(t)\,dt - \varphi(x) - \int \varphi(t)\,L(t,x)\,dt \right| \\
&\quad \le \int\limits_{|x-t|\le\delta} K(t,z)\,|\varphi(t)-\varphi(x)|\,dt + |\varphi(x)| \left| \int\limits_{|x-t|\le\delta} K(t,z)\,dt - 1 \right| \\
&\qquad + \int\limits_{|x-t|\ge\delta} |\varphi(t)|\,|K(t,z)-L(t,x)|\,dt + \int\limits_{|x-t|\le\delta} |\varphi(t)|\,|L(t,x)|\,dt \\
&\quad \overset{\text{def}}{=} T_1 + T_2 + T_3 + T_4 \,.
\end{aligned}$$

Let $w(\delta)$ be the modulus of continuity of φ. Then,

$$T_1 \le w(\delta) \int K(t,z)\,dt \le C\,w(\delta)\,,$$

$$T_4 \le \|\varphi\|_\infty \int\limits_{|x-t|\le\delta} L(t,x)\,dt\,.$$

We break the integral in T_2, accordingly to

$$\begin{aligned}
K(t,z) &= \frac{1}{\pi}\,\mathrm{Im}\,\frac{z\,\overline{\gamma'(t)}}{1 - z\,\overline{\gamma(t)}} + \frac{O(|z-\gamma(t)|)}{|1 - z\,\overline{\gamma(t)}|} \\
&= -\frac{1}{\pi}\frac{d}{dt}\,\mathrm{Im}\,\log(1 - z\,\overline{\gamma(t)}) + O(|1 - z\,\overline{\gamma(t)}|^{-1/2})\,.
\end{aligned}$$

Note that

$$|1 - z\,\overline{\gamma(t)}| \gtrsim \int_s^t T(\xi)\,d\xi\,.$$

If z is close to $\gamma(x)$, with $T(x) \neq 0$, and δ is small, so that T is bounded below by some constant $C(x)$ between s and t, one has $|1 - z\,\overline{\gamma(t)}| \ge C(x)\,|s-t|$. Hence

$$\int\limits_{|x-t|\le\delta} |1 - z\,\overline{\gamma(t)}|^{-1/2} \le C(x)\,\delta^{1/2}\,,$$

for δ small and z close to $\gamma(x)$. All these gives, taking into account that $\lim_{z\to\gamma(x)} T_3 = 0$ by dominated convergence ($K(t,z)$ is singular at

$z = \gamma(t)$ only), and Lemma 4.2,

$$\lim_{z\to\gamma(x)} \sup \left| K\varphi(z) - (\varphi(x) + \int \varphi(t)\, L(t,x)\, dt) \right|$$
$$\leq C\, w(\delta) + C\,\delta^{1/2} + |\varphi(x)| \Big(\frac{1}{\pi} \Big(\arg\big(1 - \gamma(x)\,\overline{\gamma(x-\delta)}\big) - \arg\big(1 - \gamma(x)\,\overline{\gamma(x+\delta)}\big)\Big) - 1 \Big)$$
$$+ \|\varphi\|_\infty \int_{|x-t|\leq\delta} L(t,x)\, dt\,.$$

Since

$$1 - \gamma(x)\,\overline{\gamma}(x+\delta) = -\,i\,T(x)\,\delta + o(\delta)\,,$$
$$1 - \gamma(x)\,\overline{\gamma}(x-\delta) = i\,T(x)\,\delta + o(\delta)\,,$$

we obtain Lemma 4.3 by making $\delta \to 0$. When $T(x) = 0$ this argument does not control the term T_2, but it vanishes because $\varphi(x) = 0$.

Lemmas 4.2 and 4.3 complete the first step of the proof. Indeed, obviously $K\varphi$ is pluriharmonic and $K\varphi$ has a continuous extension to $\overline{\mathbb{B}^2} \setminus \Gamma$. Lemma 4.3 shows that $K\varphi \in$ PHC if $\varphi \in C_0(\Gamma)$. Finally, the operator $\varphi \mapsto \psi$ where

$$\psi(x) = \int \varphi(t)\, L(t,x)\, dt$$

is compact in $C(\Gamma)$. For this, we must show, to prove equicontinuity, that

$$\int |L(t,x) - L(t,y)|\, dt$$

is small for $|x - y|$ small, and this follows from part *a*) of Lemma 4.2 and the continuity of L off the diagonal.

It only remains to prove that E is an interpolation set for the ball algebra. This is a well-known result that can be proved for instance applying the Davie-Øksendal theorem ([10, Theorem. 10.4.3]): it is enough to see that for each ε there are Koranyi balls $V(\xi_1,\delta_1),\ldots,V(\xi_m,\delta_m)$ where

$$V(\xi,\delta) = \{z \in S :\ |1 - \xi\,\overline{z}| < \delta\}$$

such that $\sum_{i=1}^{m} \delta_i < \varepsilon$ and $E \subset V(\xi_1, \delta_1) \cup \cdots \cup V(\xi_m, \delta_m)$.

Given ε, let $\gamma(s_1), \ldots, \gamma(s_m) \in E$ be such that

$$E \subset \gamma\Big(\bigcup_{i=1}^{m} (s_i - \varepsilon, s_i + \varepsilon)\Big)$$

with $m = O(1/\varepsilon)$. By the mean value theorem

$$\gamma(s_i - \varepsilon, s_i + \varepsilon) \subset V(\gamma(s_i), \delta_i)$$

with

$$\delta_i \simeq \varepsilon\, w(\varepsilon)\,,$$

where w is the modulus of continuity of γ'. Hence

$$\sum_{i=1}^{m} \delta_i \leq m\, \varepsilon\, w(\varepsilon) = O(w(\varepsilon))$$

can be made arbitrarily small.

This ends the proof of Theorem 4.1.

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A criterion of Petrowsky's kind for a degenerate quasilinear parabolic equation

Peter Lindqvist

Abstract. The celebrated criterion of Petrowsky for the regularity of the latest boundary point, originally formulated for the heat equation, is extended to the so-called p-parabolic equation. A barrier is constructed by the aid of the Barenblatt solution.

Little is known about the "Dirichlet boundary value problem" of genuinely nonlinear parabolic partial differential equations in arbitrary domains in space-time. Equations akin to the p-parabolic equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

are notoriously difficult to study in domains that are not space-time cylinders, that is, not of the form $G \times (0,T)$, $G \subset \mathbb{R}^n$. The aim of this note is to exhibit some interesting domains in $\mathbb{R}^n \times (-\infty, 0)$ for which the origin $(0,0)$ is a regular boundary point with the normal parallel to the time axis. This is the result of my efforts to extend the celebrated criterion of Petrowsky to a nonlinear situation.

In 1933 Petrowsky obtained a sharp criterion for the regularity of "the latest moment" in connexion with the heat equation, *cf.* [P]. For example, for the "one-dimensional" equation $u_t = u_{xx}$ the origin is a regular boundary point of the domain defined by

$$-\frac{x^2}{4t} < \log|\log(-t)|, \qquad -T < t < 0, \tag{1}$$

while the origin is not a regular boundary point of any domain defined by

$$-\frac{x^2}{4t} < (1+\varepsilon)\log|\log(-t)|, \qquad -T < t < 0, \tag{2}$$

if $\varepsilon > 0$. If continuous boundary values are prescribed on the Euclidean boundary of the domain (1) in the (x,t)-plane, then there is a solution to the heat equation taking these boundary values, in particular, at the origin. Notice that the boundary values are prescribed, as it were, for an elliptic problem, no special attention being paid to the parabolic boundary.

The boundary behaviour is a delicate question, indeed. A boundary point can be regular for the equation $u_t = \Delta u$ and, at the same time, irregular for the equation $2u_t = \Delta u$. Such a domain can be constructed with Petrowsky's criterion. A necessary and sufficient geometric condition for the regularity of an *arbitrary* boundary point, the so-called parabolic Wiener criterion, was proved in 1980 by Evans and Gariepy, *cf.* [EG]. The generalizations of the Wiener criterion to non-linear parabolic equations have not been completely successful, *cf.* [G] and [Z]. They do not include equations like the p-parabolic and the porous medium equation.

The objective of our study is the p-parabolic equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\,\nabla u), \qquad 2 \le p < \infty. \tag{3}$$

The singular case $1 < p < 2$ would require modifications in the calculations to come and, for simplicity, we take $p > 2$. This is a prototype for a vast class of equations of the type $u_t = \operatorname{div} \mathbb{A}(x,t,\nabla u)$. The p-parabolic equation is also of interest for non-Newtonian fluids, *cf.* [B][1].

[1] NOTE ADDED IN NOVEMBER 1994. It has come to my attention that the p-parabolic equation has a strong application. Its solution represents the temperature in the atmosphere after the explosion of a hydrogen bomb, and the finite speed of propagation is essential.

In general, the equation ought to be interpreted in the weak sense. We refer the reader to the book [D]. The gradient ∇u of a solution is known to be Hölder continuous, but, in general, the time derivative u_t is merely a distribution. See [C], [Y], and [KV], for example.

Our result is the following.

Theorem. *Let $p > 2$. For the p-parabolic equation the origin is a regular boundary point of the domain*

$$(4) \qquad \frac{|x|^{p/(p-1)}}{(-t)^{p/\lambda(p-1)}} < K(-t)^{n(p-2)/\lambda}\,|\ln(-t)|^{\alpha(p-2)}, \quad -T < t < 0,$$

K and α denoting arbitrarily large constants, $\lambda = n(p-2) + p$, and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

REMARKS: 1°) The origin is *a fortiori* a regular boundary point of any subdomain of (4), if it is a boundary point at all.

2°) By *the exterior sphere condition* all the other boundary points of (4) are regular. It is the origin that is crucial.

3°) The geometric situation is interesting, because the tangent plane at $(0,0)$ is perpendicular to the time axis.

To understand the strange quantity in (4) we mention the *Barenblatt solution*

$$\mathcal{B}_p(x,t) = t^{-n/\lambda}\left(C - \frac{p-2}{p}\,\lambda^{-1/(p-1)}\left(\frac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)}$$

defined when $t > 0$ and $x \in \mathbb{R}^n$. Here $p > 2$ and $\lambda = n(p-2) + p$. The positive constant C is usually determined so that $\int \mathcal{B}_p(x,t)\,dx = 1$, when $t > 0$, *i.e.*, $\mathcal{B}_p(x, 0+) = \delta(x)$, the Dirac measure. See [B], [D, Section V.4, Equation (4.7), p. 125], or [KV]. When $p \to 2+$, the normalized Barenblatt solution approaches the ordinary heat kernel

$$(4\pi t)^{-n/2}\,e^{-|x|^2/4t},$$

obtained by Weierstrass. The key point in deriving our theorem is to construct a barrier (a supersolution of a specific kind) by the aid of the Barenblatt solution. This approach counts for our difficulties in obtaining the asymptotically right formulas, as $p \to 2+$. There are too many quantities in the calculations blowing up, as $p \to 2+$, to lead to Petrowsky's inequality (1).

Condition (4) is rather good, when $p > 2$. *We conjecture that the origin is an irregular boundary point of the domain*

$$\frac{|x|^{p/(p-1)}}{(-t)^{p/\lambda(p-1)}} < K(-t)^{n(p-2)/\lambda - \varepsilon}, \qquad -T < t < 0,$$

if $\varepsilon > 0$. It would be interesting to know the truth in this matter.[2]

It seems to be well-known that a boundary point is regular if and only if there exists a barrier at this point. Especially, a boundary point satisfying the exterior sphere condition (the earliest moment of the sphere being excluded as a tangent point) has a barrier and hence it is regular. Thus our theorem means that, given continuous boundary values on the boundary of the domain defined by (4), there exists a unique p-parabolic function taking the prescribed values in the classical sense. For all this we refer the reader to [KL].

Let Ω be a domain in $\mathbb{R}^n \times \mathbb{R}$ having the Euclidean boundary $\partial\Omega$ and $(0,0) \in \partial\Omega$. To be on the safe side, we remind the reader that a function $u : \Omega \longrightarrow \mathbb{R}$ satisfying the conditions

i) $u_t \geq \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$ in Ω,

ii) $u > 0$ in Ω and $\liminf_{\zeta \to \xi} u(\zeta) > 0$ for all $\xi \in \partial\Omega$, $\xi \neq (0,0)$,

and

iii) $\lim_{\zeta \to (0,0)} u(\zeta) = 0$,

will do as a barrier at the origin (with respect to the domain Ω), see [KL]. Our barrier will be so smooth that (i) is satisfied in the classical sense. It will be constructed as a function of the form

$$(5)\quad \begin{aligned} u(x,t) = f(t)\left(C + \frac{p-2}{p}\,\lambda^{-1/(p-1)}\left(\frac{|x|}{(-t)^{1/\lambda}}\right)^{p/(p-1)}\right)^{(p-1)/(p-2)} \\ + \phi(t) \end{aligned}$$

when $x \in \mathbb{R}^n$ and $t < 0$. As in the Barenblatt solution $\lambda = n(p-2) + p$ and C is any positive constant. We will later choose

$$(6)\quad f(t) = -\varepsilon\,|\ln(-t)|^{\alpha}, \qquad \phi(t) = -C^{(p-1)/(p-2)}\,f(t) + \rho(t),$$

[2] It is not too difficult to show that, if the right-hand member of the inequality is replaced by $K(-t)^{\beta}$ where $\beta = n(p-2)/\lambda(p-1)$, then the origin is irregular, indeed. Here $p > 2$.

where $\varepsilon > 0$ and $\rho(t) > 0$.

We shall select $\rho(t)$ so that u is a supersolution in the domain where $u > 0$ and this domain is to contain the domain (4). We do not care about what happens when $u \le 0$. Notice that u is positive precisely when

$$\left(C + \frac{p-2}{p}\,\lambda^{-1/(p-1)}\left(\frac{|x|}{(-t)^{1/\lambda}}\right)^{p/(p-1)}\right)^{(p-1)/(p-2)} < C^{(p-1)/(p-2)} + \frac{\rho(t)}{-f(t)}\,. \tag{7}$$

This inequality is at our disposal in the proof of i).

Observe that

$$u(x,t) \le f(t)\,C^{(p-1)/(p-2)} - C^{(p-1)/(p-2)}\,f(t) + \rho(t) = \rho(t)\,.$$

Thus iii) is valid, if $\rho(t) \to 0$ as $t \to 0-$. This requirement restricts the choice of $\rho(t)$ in a decisive way.

Our aim is to show that u *is a supersolution in the domain defined by* (7), as required in i). Using the abbreviation

$$Q = C + \frac{p-1}{p}\,\lambda^{-1/(p-1)}\left(\frac{|x|}{(-t)^{1/\lambda}}\right)^{p/(p-1)} \tag{8}$$

we have

$$\nabla Q = \frac{p-2}{p-1}\,\lambda^{-1/(p-1)}\,\frac{|x|^{(2-p)/(p-1)}x}{(-t)^{p/\lambda(p-1)}},$$

$$\frac{\partial Q}{\partial t} = \frac{p-2}{p-1}\,\lambda^{-p/(p-1)}\left(\frac{|x|}{(-t)^{1/\lambda}}\right)^{p/(p-1)}\frac{1}{-t},$$

since $\nabla|x|^q = q\,|x|^{q-2}\,x$. Recall that

$$u(x,t) = f(t)\,Q^{(p-1)/(p-2)} + \phi(t)\,. \tag{9}$$

Thus

$$\nabla u = f(t)\,Q^{1/(p-2)}\,\lambda^{-1/(p-1)}\,\frac{|x|^{(2-p)/(p-1)}x}{(-t)^{p/\lambda(p-1)}}\,,$$

$$|\nabla u|^{p-2}\,\nabla u = |f(t)|^{p-2}\,f(t)\,\lambda^{-1}Q^{(p-1)/(p-2)}\,\frac{x}{(-t)^{p/\lambda}}\,.$$

We obtain

$$
\begin{aligned}
&\operatorname{div}(|\nabla u|^{p-2}\nabla u)\\
(10)\qquad &= |f(t)|^{p-2} f(t)\,\lambda^{-1}\, Q^{(p-1)/(p-2)} \frac{n}{(-t)^{p/\lambda}}\\
&\quad + |f(t)|^{p-2} f(t)\,\lambda^{-p/(p-1)}\, Q^{1/(p-2)} \Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)} \frac{1}{(-t)^{p/\lambda}}
\end{aligned}
$$

after some arithmetic. The last term in (10) can be written as

$$\frac{p\,|f(t)|^{p-2} f(t)}{\lambda\,(p-2)\,(-t)^{p/\lambda}}\, Q^{1/(p-2)}(Q-C)\,.$$

Further we have

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = f'(t)\,Q^{(p-1)/(p-2)}\\
(11)\qquad &\quad + \phi'(t) + f(t)\,\lambda^{-p/(p-1)}\, Q^{1/(p-2)}\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)} \frac{1}{-t}\,,
\end{aligned}
$$

where the last term can be written as

$$\frac{p\,f(t)}{\lambda\,(p-2)(-t)}\, Q^{1/(p-2)}\,(Q-C)\,.$$

Combining equations (10) and (11) we finally arrive at the expression

$$
\begin{aligned}
&u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u)\\
(12)\qquad &= \phi'(t) - \frac{pC}{\lambda\,(p-2)}\Big(\frac{1}{-t} - \frac{|f(t)|^{p-2}}{(-t)^{p/\lambda}}\Big) f(t)\,Q^{1/(p-2)}\\
&\quad + \Big(f'(t) + \frac{p\,f(t)}{\lambda\,(p-2)(-t)} - \frac{|f(t)|^{p-2} f(t)}{(p-2)\,(-t)^{p/\lambda}}\Big)\, Q^{(p-1)/(p-2)}\,,
\end{aligned}
$$

where we have used the identity

$$\frac{1}{p-2} = \frac{n}{\lambda} + \frac{p}{\lambda\,(p-2)}\,.$$

For $f(t) = -\varepsilon\,|\ln(-t)|^{\alpha}$ we have

$$
\begin{aligned}
&f'(t) + \frac{p\,f(t)}{\lambda\,(p-2)(-t)} - \frac{|f(t)|^{p-2} f(t)}{(p-2)\,(-t)^{p/\lambda}}\\
&\qquad = \frac{-\varepsilon\,\alpha\,|\ln(-t)|^{\alpha-1}}{-t} - \frac{\varepsilon\,p\,|\ln(-t)|^{\alpha}}{\lambda\,(p-2)(-t)} + \frac{\varepsilon^{p-1}\,|\ln(-t)|^{\alpha(p-1)}}{(p-2)\,(-t)^{p/\lambda}}
\end{aligned}
$$

when $-1 < t < 0$. This expression is certainly negative, if

$$\frac{\varepsilon\, p\, |\ln(-t)|^{\alpha}}{\lambda\, (p-2)\, (-t)} \geq \frac{\varepsilon^{p-1}\, |\ln(-t)|^{\alpha(p-1)}}{(p-2)\, (-t)^{p/\lambda}}$$

when $-1 < t < 0$. This yields the condition

$$\frac{p}{\lambda} = \left(\frac{\alpha\, \lambda}{n\, e}\right)^{\alpha(p-2)} \varepsilon^{p-2} \tag{13}$$

for the largest possible ε. Let us fix ε this way. Then

$$-\frac{p\, C}{\lambda\, (p-2)} \left(\frac{1}{-t} - \frac{|f(t)|^{p-2}}{(-t)^{p/\lambda}}\right) f(t) > 0$$

in (12). Thus we have, using (7),

$$\begin{aligned}
&u_t - \operatorname{div}(|\nabla u|^{p-2}\, \nabla u) \\
&\geq \phi'(t) - \frac{p\, C}{\lambda\, (p-2)} \left(\frac{1}{-t} - \frac{|f(t)|^{p-2}}{(-t)^{p/\lambda}}\right) f(t)\, C^{1/(p-2)} \\
&\quad + \left(f'(t) + \frac{p\, f(t)}{\lambda\, (p-2)\, (-t)} - \frac{|f(t)|^{p-2}\, f(t)}{(p-2)\, (-t)^{p/\lambda}}\right) \\
&\qquad \cdot \left(C^{(p-1)/(p-2)} + \frac{\rho(t)}{-f(t)}\right) \\
&= \rho'(t) - \frac{n\, C^{(p-1)/(p-2)}\, |f(t)|^{p-2}\, f(t)}{\lambda\, (-t)^{p/\lambda}} \\
&\quad + \rho(t) \left(\frac{f'(t)}{-f(t)} - \frac{p}{\lambda\, (p-2)\, (-t)} + \frac{|f(t)|^{p-2}}{(p-2)\, (-t)^{p/\lambda}}\right),
\end{aligned}$$

where we have used that

$$\phi'(t) = -C^{(p-1)/(p-2)}\, f'(t) + \rho'(t).$$

Substituting the expression for $f(t)$ we obtain

$$\begin{aligned}
&u_t - \operatorname{div}(|\nabla u|^{p-2}\, \nabla u) \\
&\qquad \geq \rho'(t) + \frac{n\, C^{(p-1)/(p-2)}\, \varepsilon^{p-1}\, |\ln(-t)|^{\alpha(p-1)}}{\lambda (-t)^{p/\lambda}} \\
&\qquad\quad + \rho(t) \left(\frac{-\alpha}{(-t)\, |\ln(-t)|} - \frac{p}{\lambda\, (p-2)\, (-t)} \right. \\
&\qquad\qquad \left. + \frac{\varepsilon^{p-2}\, |\ln(-t)|^{\alpha(p-2)}}{(p-2)\, (-t)^{p/\lambda}}\right),
\end{aligned} \tag{14}$$

when $-1 < t < 0$ and $u > 0$.

Let us choose

$$\rho(t) = A(-t)^{1-p/\lambda}\,|\ln(-t)|^{\alpha(p-1)}.$$

Notice that

$$1 - \frac{p}{\lambda} = \frac{n\,(p-2)}{\lambda} > 0$$

so that $\rho(t) \to 0$, as $t \to 0-$. Inserting $\rho(t)$ into (14) we obtain

$$\begin{aligned} &u_t - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) \\ &\geq \Big(n\,C^{(p-1)/(p-2)}\,\varepsilon^{p-1} - A\,n\,(p-2) - \frac{A\,p}{p-2}\Big)\,\frac{|\ln(-t)|^{\alpha(p-1)}}{\lambda(-t)^{p/\lambda}} \\ &\quad + A\,\alpha\,(p-2)\,|\ln(-t)|^{\alpha(p-1)-1}\,(-t)^{-p/\lambda} \\ &\quad + \frac{A\,\varepsilon^{p-2}\,|\ln(-t)|^{\alpha(2p-3)}\,(-t)^{1-2p/\lambda}}{p-2}. \end{aligned}$$

It is plain that $u_t \geq \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$, if A does not exceed the value

$$A = \frac{n\,(p-2)\,C^{(p-1)/(p-2)}\,\varepsilon^{p-1}}{n\,(p-2)^2 + p} \tag{15}$$

and (7) holds. Using (13) and (15), we can now write (7) in the form

$$\begin{aligned} &\Big(1 + \frac{p-1}{p\,C}\,\lambda^{-1/(p-1)}\,\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)}\Big)^{(p-1)/(p-2)} \\ &\qquad < 1 + \frac{n\,p\,(p-2)}{\lambda\,(n\,(p-2)^2+p)}\,\Big(\frac{n\,e}{\alpha\,\lambda}\Big)^{\alpha(p-2)} \\ &\qquad\quad \cdot (-t)^{n(p-2)/\lambda}\,|\ln(-t)|^{\alpha(p-2)}. \end{aligned} \tag{16}$$

Here the constant C is at our disposal.

In order to conclude the proof we have only to observe that (4) implies (16). Indeed, suppose that (4) holds for $-T < t < 0$, where $T < 1$. The right-hand member of (4) is less than $K(\alpha\lambda/ne)^{\alpha(p-2)}$. Hence

$$\begin{aligned} &\Big(1 + \frac{p-2}{p\,C}\,\lambda^{-1/(p-1)}\,\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{p/(p-1)}\Big)^{(p-1)/(p-2)} \\ &\qquad < 1 + \frac{p-1}{p\,C}\,\lambda^{-1/(p-1)}\,\Big(\frac{|x|}{(-t)^{1/\lambda}}\Big)^{(p-1)/(p-2)} \\ &\qquad\quad \cdot \Big(1 + \frac{p-2}{p\,C}\,\lambda^{-1/(p-1)}\,K\Big(\frac{\alpha\,\lambda}{n\,e}\Big)^{\alpha(p-2)}\Big)^{1/(p-2)} \end{aligned}$$

$$= 1 + \frac{B}{C}\left(\frac{|x|}{(-t)^{1/\lambda}}\right)^{p/(p-1)}$$
$$\leq 1 + \frac{B}{C} K(-t)^{n(p-2)/\lambda}\,|\ln(-t)|^{\alpha(p-2)}$$

where B denotes a constant. It is plain that this estimate implies (16), if C is large enough.

Enlarging C further, if necessary, we have ii) valid. This concludes our proof.

REMARK. Forgetting the logarithm we can write (4) as

$$|x| = O((-t)^{\kappa}),$$

where $\kappa = \kappa(n,p)$. Now $\kappa \to 1/2$ as $p \to 2+$ and $\kappa \to n/(n+1)$ as $p \to \infty$. Moreover $\kappa < 1/2$ in the range $2 < p < 2n/(n-1)$. If $n = 1$, then $\kappa < 1/2$ for all $p > 2$. The smaller κ is, the better the condition (4). The smallest value of κ occurs, when

$$p - 2 = \frac{2n(\sqrt{2}-1)+2}{n^2+2n-1}.$$

For this rather strange value of p the exponent κ is slightly less than $1/2$ ($\kappa = \sqrt{2}-1$, when $n = 1$ and $\kappa = 1/2 - (3-2\sqrt{2})/4$, when $n = 2$).

The result has a natural extension to domains in the (x,t)-plane bounded by two Hölder continuous curves and two characteristic lines:

$$s_1(t) < x < s_2(t), \qquad t_1 < t < t_2 .$$

Suppose that the curves $x = s_1(t)$ and $x = s_2(t)$ are Hölder continuous with the aforementioned exponent κ, when $t_1 \leq t \leq t_2$. That is

$$|s_j(t+\tau) - s_j(t)| \leq K\,|\tau|^{\kappa}, \qquad j = 1,2 .$$

Then *the boundary points lying on the curves are regular.* Some auxiliary constructions are needed to deduce this from the Theorem. We will not pursue the matter any further.

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Bilipschitz extensions from smooth manifolds

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Abstract. We prove that every compact C^1-submanifold of $\mathbb{R}^n$, with or without boundary, has the bilipschitz extension property in $\mathbb{R}^n$.

1. Introduction.

Let $X \subset \mathbb{R}^n$ and let $f : X \to \mathbb{R}^n$ be a map. We say that f is *L-bilipschitz* (abbreviated L-BL) if $L \geq 1$ and if

$$\frac{|x-y|}{L} \leq |f(x) - f(y)| \leq L\,|x-y|\,,$$

for all $x, y \in X$. Thus 1-BL maps preserve distances, and we call them *isometries*. Every isometry $f : X \to \mathbb{R}^n$ is the restriction of a unique affine isometry $g : \operatorname{aff}(X) \to \mathbb{R}^n$; we let $\operatorname{aff}(X)$ denote the affine subspace of $\mathbb{R}^n$ generated by a nonempty subset X of $\mathbb{R}^n$. Hence every isometry f has an extension to an isometry of $\mathbb{R}^n$.

In general, an L-BL map $f : X \to \mathbb{R}^n$ need not have an extension to a bilipschitz map of $\mathbb{R}^n$, even if X is a very simple set. For example, X may be the unit circle and $f : X \to \mathbb{R}^3$ a homeomorphism onto a knotted curve. The situation changes, however, if L is required to be close to 1. The following concept was introduced in [V, p. 239]:

A set $X \subset \mathbb{R}^n$ is said to have the *bilipschitz extension property* (abbreviated BLEP) in $\mathbb{R}^n$ if there is $L_0 > 1$ such that if $1 \leq L \leq L_0$, then every L-BL map $f : X \to \mathbb{R}^n$ has an L_1-BL extension $g : \mathbb{R}^n \to \mathbb{R}^n$, where $L_1 = L_1(L, X, n) \to 1$ as $L \to 1$.

In [V, 5.17] it was proved that every compact $(n-1)$-dimensional C^1-submanifold X of $\mathbb{R}^n$ has the BLEP in $\mathbb{R}^n$. In the present paper we are going to prove the same result for all compact p-dimensional C^1-submanifolds X of $\mathbb{R}^n$, $1 \le p \le n$, with or without boundary. The result is given as Theorem 3.14. For $p = 1$, a proof was recently given in [HP]. We shall modify the method of [HP] to cover the technically more challenging remaining dimensions as well. Our proof is based on the BLEP of compact polyhedra, which was established in [PV].

1.1. NOTATIONS. We let $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of real numbers, integers, and positive integers, respectively. If $1 \le p \le n-1$, we identify $\mathbb{R}^p$ with the subset $\{x : x_{p+1} = \cdots = x_n = 0\}$ of $\mathbb{R}^n$. The distance between two sets $A, B \subset \mathbb{R}^n$ is written as $\operatorname{dist}(A, B)$ with the convention that $\operatorname{dist}(A, B) = \infty$ if A or B is empty. The diameter of A is $\operatorname{diam}(A)$ with $\operatorname{diam}(\varnothing) = 0$. For $r > 0$ we set

$$B^n(A, r) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) < r\},$$
$$\bar{B}^n(A, r) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \le r\},$$

where we have simplified the notation by writing x for $\{x\}$.

If f and g are two functions defined in a set A and with values in $\mathbb{R}^n$, we set

$$|f - g|_A = \sup_{x \in A} |fx - gx|.$$

We often omit parentheses writing fx instead of $f(x)$. A map $f : A \to \mathbb{R}^n$ $(A \subset \mathbb{R}^n)$ is called a *similarity* if there is $\lambda > 0$ such that

$$|fx - fy| = \lambda\, |x - y|$$

for all $x, y \in A$. The *similarity class* of a set $A \subset \mathbb{R}^n$ consists of all the images fA of A under similarities $f : A \to \mathbb{R}^n$.

We use the notation $\mathcal{P}(X)$ for the set of subsets of a set X. The cardinality of X will be denoted by $\# X$. The symbol id is used to denote various inclusion maps.

Let $Q \subset \mathbb{R}^n$ be a closed (or open) p-cube, $1 \le p \le n$, and let $t > 0$. We use the notation $Q(t)$ to denote the closed (or open) p-cube of $\mathbb{R}^n$ with the same center as Q, with side length t times that of Q, and with edges parallel to those of Q. The interior of Q is written as $\mathring{Q}$; it is the topological interior of Q in aff(Q).

2. Preparations.

We begin with a purely set-theoretic lemma. Perhaps surprisingly, it has an important role in the sequel.

2.1. Lemma. *Let* $\varphi_j :]1, a_j] \to]1, \infty[$ $(j \in \mathbb{N}, a_j > 1)$ *be a sequence of functions satisfying* $\lim_{t\to 1} \varphi_j(t) = 1$ *for all* $j \in \mathbb{N}$. *Then there exists a function* $m :]1, a_1] \to \mathbb{N}$ *with the following properties*

1) $a_{m(t)} \geq t\,,$ *for all* $t \in]1, a_1]\,,$
2) $m(t) \to \infty$ *as* $t \to 1\,,$
3) $\varphi_{m(t)}(t) \to 1$ *as* $t \to 1\,.$

PROOF. Set $b_1 = a_1$ and construct inductively a sequence $b_1 > b_2 > b_3 > \cdots$ of numbers $b_j > 1$ such that for $j \geq 2$ we have $b_j \leq a_j$, $b_j \leq 1 + 1/j$, and $\varphi_j]1, b_j] \subset]1, 1 + 1/j]$. Define $m :]1, a_1] \to \mathbb{N}$ by setting

$$m(t) = \max\{j \in \mathbb{N} : \ b_j \geq t\}$$

for $t \in]1, a_1]$. Since $a_{m(t)} \geq b_{m(t)} \geq t$ for all $t \in]1, a_1]$, (1) holds. Since $m(t) = j$ for $t \in]b_{j+1}, b_j]$, (2) holds. Since $\varphi_{m(t)}(t) \leq 1 + 1/m(t)$ for $t \in]1, b_2]$, (3) follows now from (2).

We now introduce the relative BLEP, which can be considered a generalization of the ordinary BLEP. In Theorem 2.4 we derive a useful property of the relative BLEP. Our notation in Definition 2.2 and remarks 2.3 is chosen to suit the application in Theorem 2.4.

2.2. Definition. *Let* $X \subset \mathbb{R}^n$, *let* $K_0 > 1$, *and let* $A :]1, K_0] \to \mathcal{P}(X)$ *be a function. We say that* X *has the bilipschitz extension property relative to* A *in* $\mathbb{R}^n$ *(abbreviated* BLEP *rel* A*) if there is* $K' \in]1, K_0]$ *such that if* $1 < L \leq K'$, *then every* L-BL *map* $f : X \to \mathbb{R}^n$ *with* $f|_{A(L)} = \mathrm{id}$ *has a* K_1-BL *extension* $g : \mathbb{R}^n \to \mathbb{R}^n$, *where* $K_1 = K_1(L, X, A, n) \to 1$ *as* $L \to 1$.

2.3. REMARKS 1). Let $A, B :]1, K_0] \to \mathcal{P}(X)$ be two functions as in Definition 2.2. If there is $L_0 \in]1, K_0]$ such that $A(L) \subset B(L)$ for

$L \in]1, L_0]$, we write $A \subset B$. If $A \subset B$ and if X has the BLEP rel A, then X has the BLEP rel B.

2). If $C \subset X$, we write BLEP rel C for BLEP rel A, where $A :]1, K_0] \to \mathcal{P}(X)$ is the constant function with value $A(L) = C$. The ordinary BLEP defined in the introduction is then the same concept as BLEP rel $\varnothing$.

3). Trivially, X always has the BLEP rel X.

4). To simplify the notation, we usually write $K_1 = K_1(L)$ without explicitly mentioning the data X, A, n, on which K_1 also depends.

2.4. Theorem. *Let $X \subset \mathbb{R}^n$, let $K_0 > 1$, and let $A, B :]1, K_0] \to \mathcal{P}(X)$ be two functions. Suppose also that X has the* BLEP rel A. *Then X has the* BLEP rel B *if and only if the following condition holds*

(*) *There exists $L_0 \in]1, K_0]$ such that if $1 < L \leq L_0$ and if $f : X \to \mathbb{R}^n$ is an L-*BL *mapping satisfying $f|_{B(L)} = \mathrm{id}$, then the map $f|_{A(L)} : A(L) \to \mathbb{R}^n$ has an L_1-*BL *extension $g_A : \mathbb{R}^n \to \mathbb{R}^n$, where $L_1 = L_1(L) = L_1(L, X, A, B, n) \to 1$ as $L \to 1$.*

PROOF. It is obvious that if X has the BLEP rel B, then (*) is true. To prove the converse, assume (*) and let $K' \in]1, K_0]$ be the number and $K_1 :]1, K'] \to]1, \infty[$ the function given by Definition 2.2 for the BLEP of X rel A. We must find the corresponding objects $K'' = K'_B$ and $K_2 = K_1^B$ for the BLEP of X rel B. By choosing K' small enough, we may assume that $K' \leq L_0$.

Choose $K'' > 1$ such that $L L_1(L) \leq K'$ for all $L \in]1, K'']$. Let $1 < L \leq K''$ and let $f : X \to \mathbb{R}^n$ be an L-BL mapping satisfying $f|_{B(L)} = \mathrm{id}$. Then (*) implies that the map $f|_{A(L)} : A(L) \to \mathbb{R}^n$ has an $L_1(L)$-BL extension $g_A : \mathbb{R}^n \to \mathbb{R}^n$. The map $f_A = g_A^{-1} \circ f : X \to \mathbb{R}^n$ is $L L_1(L)$-BL and satisfies $f_A|_{A(L)} = \mathrm{id}$. Since $L L_1(L) \leq K'$, we may apply the BLEP of X rel A to find an L_2-BL extension $g_1 : \mathbb{R}^n \to \mathbb{R}^n$ of f_A with $L_2 = K_1(L L_1(L))$. Now, the map $g = g_A \circ g_1 : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $g|_X = f$ and is K_2-BL with $K_2 = L_1 L_2$ satisfying $K_2 \to 1$ as $L \to 1$. It follows that X has the BLEP rel B.

2.5. Flatness.

Let $\Delta = v_0 \cdots v_p \subset \mathbb{R}^n$ be a p-simplex with vertices $v_0, \ldots, v_p$,

$p \geq 1$. As in [V, 2.6], we define the *flatness* $\rho(\Delta)$ of Δ by

$$\rho(\Delta) = \frac{\operatorname{diam}(\Delta)}{b(\Delta)},$$

where $b(\Delta)$ is the smallest height of Δ. Explicitly, we have

$$b(\Delta) = \min_{0 \leq i \leq p} \operatorname{dist}(v_i, \operatorname{aff}(\Delta_i)),$$

where Δ_i is the $(p-1)$-face of Δ opposite to v_i. The simplex $\Delta = v_0 \cdots v_p$ is called a *corner* if there is $i \in \{0, \ldots, p\}$ such that the vectors $v_j - v_i$ $(0 \leq j \leq p,\ j \neq i)$ are mutually orthogonal and of equal length $|v_j - v_i| = t$. The number $t > 0$ is called the *size* of the corner Δ. The flatness of a corner Δ is

$$\rho(\Delta) = \begin{cases} 1, & p = 1, \\ \sqrt{2p}, & p \geq 2. \end{cases}$$

Since $\rho(\Delta)$ is a continuous function of $(v_0, \ldots, v_p)$, we can choose an integer $m_n \geq 2$ with the following property: If $1 \leq p \leq n-1$, if $\Delta_0 = u_0 \cdots u_p \subset \mathbb{R}^n$ is a p-corner with size t, and if $v_0, \ldots, v_p$ are points in $\mathbb{R}^n$ with $|u_j - v_j| \leq t/m_n$ for all $j \in \{0, \ldots, p\}$, then $\Delta = v_0 \cdots v_p$ is a p-simplex with $\rho(\Delta) \leq 2p$.

3. The main result.

3.1. Basic assumptions.

Let $1 \leq p \leq n$ and let X be a compact p-dimensional C^1-submanifold of $\mathbb{R}^n$. The purpose of this paper is to prove that X has the BLEP in $\mathbb{R}^n$. In Section 3 we give a detailed exposition of the case where X has no boundary. The modifications needed to cover the case of manifolds with boundary will be briefly discussed in Section 4.

We begin by giving our assumptions on X more explicitly. Thus, in Section 3, $1 \leq p \leq n-1$ and X is a compact subset of $\mathbb{R}^n$ such that for every point y of X there is an open set U of $\mathbb{R}^p$ and an embedding $f : U \to \mathbb{R}^n$ satisfying the following conditions 1)-3):

1) $y \in fU \subset X$ and fU is open in X,

2) f is continuously differentiable in U,

3) for $x \in U$, the linear map $f'(x) : \mathbb{R}^p \to \mathbb{R}^n$ is injective.

As usual, such a map $f : U \to \mathbb{R}^n$ is called a *chart* of X at y.

3.2. The cube family $\mathcal{K}$.

Let $\delta > 0$ and let $\mathcal{J} = \mathcal{J}(\delta)$ be the family of all closed n-cubes $Q \subset \mathbb{R}^n$ with side length δ and with vertices in $\delta\mathbb{Z}^n$. Let $m_n \in \mathbb{N}$ be as in 2.5, and define

$$N_0 = N_0(n) = 2\,n\,(m_n + 1)\,, \qquad N = N(n) = (N_0 + 1)^n\,,$$
$$W = \{0, 1, \ldots, N_0\}^n\,.$$

For $w \in W$ we set

$$\mathcal{J}_w = [0, \delta]^n + \delta w + (N_0 + 1)\,\delta\mathbb{Z}^n\,.$$

Then the N subfamilies $\mathcal{J}_w$, $w \in W$, of $\mathcal{J}$ are disjoint, and

$$\mathcal{J} = \bigcup_{w \in W} \mathcal{J}_w\,.$$

Moreover, we have

$$\operatorname{dist}(Q, R) \geq N_0\,\delta$$

whenever $Q, R \in \mathcal{J}_w$, $Q \neq R$, $w \in W$. Choose an arbitrary enumeration

$$W = \{w(1), \ldots, w(N)\}$$

of W, set $\mathcal{J}_i = \mathcal{J}_{w(i)}$ for $i \in \{1, \ldots, N\}$, and note that

$$\mathcal{J} = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_N\,.$$

Set

$$\mathcal{K} = \mathcal{K}(\delta) = \{Q \in \mathcal{J} :\ Q \cap X \neq \varnothing\}\,, \qquad \mathcal{K}_i = \mathcal{K}_i(\delta) = \mathcal{K}(\delta) \cap \mathcal{J}_i\,,$$

and observe that $\mathcal{K}$ is the disjoint union

$$\mathcal{K} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N\,.$$

3.3. The numbers $\delta(L)$ and the sets $D(L)$.

Let $j \in \mathbb{N}$ and choose $\delta = 1/j$ in Paragraph 3.2. For every $Q \in \mathcal{K}$, choose a point $x_Q \in Q \cap X$. Since X is compact, the set

$$D_j = \{x_Q : \ Q \in \mathcal{K}\}$$

is finite. By *e.g.* [P, 2.4], D_j has the BLEP in $\mathbb{R}^n$. Hence there are $L_0^j > 1$ and a function $L_1^j :]1, L_0^j] \to]1, \infty[$ such that $L_1^j(L) \to 1$ as $L \to 1$ and such that if $1 < L \le L_0^j$, then every L-BL map $f : D_j \to \mathbb{R}^n$ has an $L_1^j(L)$-BL extension $g : \mathbb{R}^n \to \mathbb{R}^n$.

We set $K_0 = L_0^1$. Applying Lemma 2.1 with $a_j = L_0^j$, $\varphi_j = L_1^j$, we get a function $m :]1, K_0] \to \mathbb{N}$ with the following properties:

1) $L_0^{m(L)} \ge L$, for all $L \in]1, K_0]$,

2) $m(L) \to \infty$ as $L \to 1$,

3) $L_1^{m(L)}(L) \to 1$ as $L \to 1$.

If $1 < L \le K_0$, we define

$$\delta(L) = \frac{1}{m(L)}, \qquad L_2(L) = L_1^{m(L)}(L), \qquad D(L) = D_{m(L)}.$$

Then $\delta(L) \to 0$ and $L_2(L) \to 1$ as $L \to 1$, and $D :]1, K_0] \to \mathcal{P}(X)$ is a function. Moreover, if $1 < L \le K_0$, then every L-BL map $f : D(L) \to \mathbb{R}^n$ has an $L_2(L)$-BL extension $g : \mathbb{R}^n \to \mathbb{R}^n$. This fact, together with Theorem 2.4 (with $A = D$, $B = \varnothing$) and Remark 2.3.2, immediately implies the next lemma, which reduces our task to that of proving the BLEP of X rel D.

3.4. Lemma. *If X has the* BLEP rel D *in $\mathbb{R}^n$, then X has the* BLEP *in $\mathbb{R}^n$.*

Since X and n are fixed, we mostly do not indicate the dependence of various quantities on them in our notation. In many considerations we may also think of $L \in]1, K_0]$ as being fixed, at least temporarily. Then we simplify the notation by dropping the parameter L out of it. For example, from now on we write

$$(3.5) \quad \delta = \delta(L), \qquad D = D(L), \qquad \mathcal{K} = \mathcal{K}(\delta(L)) = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N,$$

whenever $1 < L \le K_0$.

3.6. Constructions.

Let $L \in]1, K_0]$, and let δ, D, and $\mathcal{K}$ be as in (3.5). Then $D = \{x_Q \colon Q \in \mathcal{K}\}$, where $x_Q \in Q \cap X$ is the point chosen in Paragraph 3.3 with $\delta = 1/m(L)$.

If $Q \in \mathcal{K}$, we let T_Q be the tangent plane of X at x_Q. Explicitly, if $f : U \to \mathbb{R}^n$ is a chart of X at x_Q as in Paragraph 3.1 and if $x_Q = f(x)$, then

$$T_Q = x_Q + \operatorname{im} f'(x) = x_Q + T_Q^0 ,$$

where $T_Q^0 = \operatorname{im} f'(x)$ is a p-dimensional linear subspace of $\mathbb{R}^n$, which can be shown to be independent of the chart f.

For each $Q \in \mathcal{K}$ we choose a closed n-cube Q^* of $\mathbb{R}^n$ with center x_Q, with side length

$$\lambda = \lambda(L) = 2\,(m_n + 1)\sqrt{n}\,\delta(L) ,$$

and such that Q^* has a p-face parallel to T_Q. Figure 1 illustrates the situation with $n = 2$, $p = 1$. The maps π_Q and ψ_Q will be defined below.

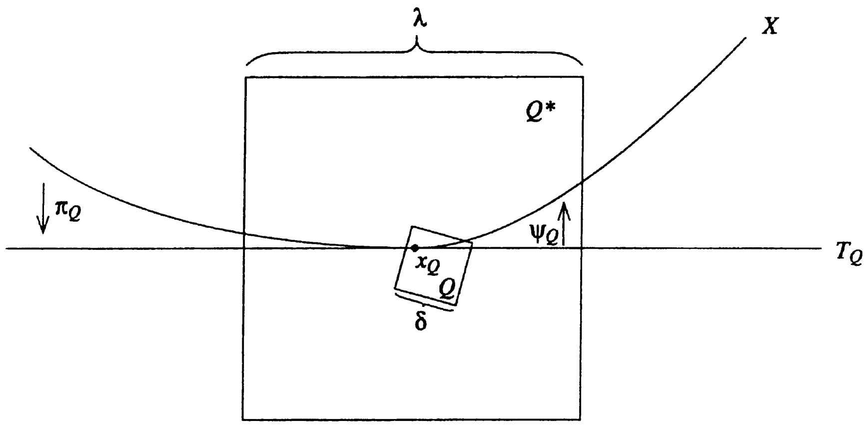

Figure 1

For $Q \in \mathcal{K}$ we let $\pi_Q \colon \mathbb{R}^n \to T_Q$ be the orthogonal projection. We omit the elementary but long proof of the following lemma. The result is geometrically obvious, because X is a compact C^1-manifold without boundary and because $\delta \to 0$ as $L \to 1$.

3.7. Lemma. *There exists a number* $K_1 \in]1, K_0]$ *such that if* $1 < L \leq K_1$, *then for every* $Q \in \mathcal{K}$ *the map*

$$\pi_Q^* : X \cap Q^* \longrightarrow T_Q \cap Q^*$$

defined by π_Q *is a homeomorphism with inverse*

$$\psi_Q = (\pi_Q^*)^{-1} : T_Q \cap Q^* \longrightarrow X \cap Q^*$$

satisfying the following conditions

1) *If* $x, y \in T_Q \cap Q^*$, *then we have*

$$|x - y| \leq |\psi_Q x - \psi_Q y| \leq M\,|x - y|\,,$$

where $M = M(L) \in [1,2]$ *and* $M(L) \to 1$ *as* $L \to 1$. *In particular,* ψ_Q *is* M-BL.

2) $|\psi_Q - \mathrm{id}\,|_{T_Q \cap Q^*} \leq \varepsilon_1\,\delta$, *where* $\varepsilon_1 = \varepsilon_1(L) \in [0, 1/2]$ *and* $\varepsilon_1(L) \to 0$ *as* $L \to 1$.

3.8. The maps φ_Q.

Let $1 < L \leq K_1$. Next we extend the maps $\psi_Q : T_Q \cap Q^* \to X \cap Q^*$ obtained from Lemma 3.7 to homeomorphisms $\varphi_Q : Q^* \to Q^*$ as follows:

Let $Q \in \mathcal{K}$, let $y \in T_Q \cap Q^*$, and set

$$R_y = \pi_Q^{-1}(y) \cap Q^*\,, \qquad B_y = \bar{B}^n(y, \delta) \cap R_y\,.$$

Then R_y is an $(n-p)$-cube, and B_y is an $(n-p)$-ball with center y. Let S_y be the boundary $(n-p-1)$-sphere of B_y. Since $|y - \psi_Q y| \leq \varepsilon_1\,\delta \leq \delta/2$ by Lemma 3.7.2), we can represent B_y as a cone in two ways: $B_y = yS_y = \psi_Q(y)S_y$. Let $\varphi_Q^y : B_y \to B_y$ be the $\psi_Q(y)$-cone of the identity map of S_y with vertex y, *i.e.*, φ_Q^y maps each segment $[y, z]$, $z \in S_y$, affinely onto the segment $[\psi_Q y, z]$. By the proof of [P, 2.3] we deduce that φ_Q^y is M_1-BL with

$$M_1 = M_1(L) = \frac{1}{1 - \varepsilon_1(L)}$$

satisfying $M_1 \to 1$ as $L \to 1$. We extend φ_Q^y to a map $\varphi_Q^y : R_y \to R_y$ by letting $\varphi_Q^y = \text{id}$ in $R_y \setminus B_y$. The desired map $\varphi_Q : Q^* \to Q^*$ can now be defined by letting φ_Q agree with φ_Q^y in R_y for all $y \in T_Q \cap Q^*$. Then φ_Q is a homeomorphism and $\varphi_Q|_{T_Q \cap Q^*} = \psi_Q$. Moreover, by the construction of φ_Q the following assertions are clearly true with M_1 and ε_1 as above:

1) φ_Q is M_1-BL,

2) $|\varphi_Q - \text{id}\,|_{Q^*} \le |\psi_Q - \text{id}\,|_{T_Q \cap Q^*} \le \varepsilon_1\, \delta$.

The choice of $N_0 = 2n(m_n + 1)$ in Paragraph 3.2 was made to guarantee that the interiors $\mathring{Q}^*$ of the cubes Q^*, $Q \in \mathcal{K}_i$, are disjoint. In the next lemma we verify this. We also derive an estimate for the cardinality of the set $\mathring{Q}^* \cap D$, $Q \in \mathcal{K}$.

3.9. Lemma. *Let $i \in \{1, \dots, N\}$, let $1 < L \le K_0$, and let $Q, R \in \mathcal{K}_i$, $Q \ne R$. Then we have*

1) $\mathring{Q}^* \cap \mathring{R}^* = \varnothing$,

2) $\#(\mathring{Q}^* \cap D) \le N$.

PROOF. To prove 1), observe that $\mathring{Q}^* \subset B^n(x_Q, N_0\delta/2)$ and $\mathring{R}^* \subset B^n(x_R, N_0\delta/2)$, because $\text{diam}\,(Q^*) = \lambda\sqrt{n} = N_0\,\delta = \text{diam}\,(R^*)$. Since $|x_Q - x_R| \ge \text{dist}\,(Q, R) \ge N_0\,\delta$ by Paragraph 3.2, we get $\mathring{Q}^* \cap \mathring{R}^* = \varnothing$.

For 2), note that $\mathring{Q}^* \subset B^n(x_Q, N_0\delta/2) \subset \mathring{Q}(N_0+1)$, where $\mathring{Q}(N_0+1)$ is the interior of $Q(N_0+1)$ (*cf.* 1.1). If $x = x_S \in \mathring{Q}^* \cap D$, $S \in \mathcal{K}$, then obviously $S \subset Q(N_0+1)$. Since

$$\#\{S \in \mathcal{J} : \; S \subset Q(N_0+1)\} = (N_0+1)^n = N$$

and since $\mathcal{K} \subset \mathcal{J}$, we get 2).

3.10. The basic polyhedra Z.

In [PV] it was proved that every compact polyhedron $Z \subset \mathbb{R}^n$ has the BLEP in $\mathbb{R}^n$. We are going to apply this result to some basic polyhedra, which belong to a finite number of similarity classes. Here we choose a set of representatives Z for these classes.

If $k \in \mathbb{N}$ and $t > 0$, we set $I^k(t) = [-t,t]^k$. We let $N_1 = N_1(n)$ be the unique integer satisfying

$$\frac{2}{3}(m_n+1)N\sqrt{n} \le N_1 < \frac{2}{3}(m_n+1)N\sqrt{n}+1\,,$$

where $N = (N_0+1)^n$ is as above. We divide $I^p(3)$ into $(6N_1)^p$ closed p-cubes R with side length $1/N_1$. Let $\mathcal{R}$ be the family of these p-cubes R, and let $\mathcal{Y}$ be the family of all polyhedra Y satisfying the conditions

1) $Y = \bigcup \mathcal{R}_1$ for some $\mathcal{R}_1 \subset \mathcal{R}$,
2) $I^p(2) \subset Y$.

Since $I^p(3) \subset I^n(3)$ by the identification of 1.1, we can now define a finite family $\mathcal{F}$ of compact polyhedra $Z \subset I^n(3)$ by setting

$$\mathcal{F} = \{Z : \ Z = Y \cup \partial I^n(3)\,,\ Y \in \mathcal{Y}\}\,.$$

3.11. The sets E_i.

Let $1 < L \le K_1$. For $Q \in \mathcal{K}$ we set

$$P_Q = (Q^* \cap T_Q)(2/3)\;,$$

i.e., P_Q is the closed p-cube with center x_Q, side length $\frac{2}{3}\lambda$, and edges parallel to those of $Q^* \cap T_Q$. If $1 \le i \le N$, we define a subset $E_i = E_i(L)$ of X by setting

$$E_i = \bigcup \{\psi_Q P_Q : \ Q \in \mathcal{K}_i\}\,,$$

where $\psi_Q : T_Q \cap Q^* \to X \cap Q^*$ is the homeomorphism of Lemma 3.7. We also set

$$q = q(L) = \frac{\delta(L)\sqrt{n}}{N}\,.$$

The next lemma is the decisive tool in our proof of the BLEP of X.

3.12. Lemma. *Let $A, B :]1, K_1] \to \mathcal{P}(X)$ be two functions satisfying the following conditions:*

1) $A(L) \subset B(L)$, *for all* $L \in]1, K_1]$,

2) $\operatorname{dist}(A(L), X \setminus B(L)) \geq q(L)$, *for all* $L \in]1, K_1]$.

Let $1 \leq i \leq N$, *and suppose that* X *has the* BLEP rel $A \cup D \cup E_i$ *in* $\mathbb{R}^n$. *Then* X *has the* BLEP rel $B \cup D$ *in* $\mathbb{R}^n$.

PROOF. To begin with, the reader should be informed that we allow the case where $A(L) = \varnothing$ or $B(L) = \varnothing$. In fact, we shall apply Lemma 3.12 with $A = B = \varnothing$ in the proof of Theorem 3.14.

Applying Theorem 2.4 with the substitution $K_0 \mapsto K_1$, $A(L) \mapsto A(L) \cup D(L) \cup E_i(L)$, $B(L) \mapsto B(L) \cup D(L)$, we first observe that it suffices to prove the following statement:

(3.13) *There exists* $L_0 \in]1, K_1]$ *such that if* $1 < L \leq L_0$ *and if* $f : X \to \mathbb{R}^n$ *is an* L-BL *mapping satisfying* $f|_{B(L)\cup D(L)} = \mathrm{id}$, *then* $f|_{A(L)\cup D(L)\cup E_i(L)}$ *has an* L_1-BL *extension* $F : \mathbb{R}^n \to \mathbb{R}^n$, *where* $L_1 = L_1(L) \to 1$ *as* $L \to 1$.

Let $1 < L \leq K_1$, and let $f : X \to \mathbb{R}^n$ be an L-BL map satisfying $f|_{B(L)\cup D(L)} = \mathrm{id}$. To be able to construct the desired L_1-BL extension F of $f|_{A(L)\cup D(L)\cup E_i(L)}$ we shall introduce new restrictions on L of the type $L \leq K_j = K_j(X, n) > 1$ ($j \geq 2$) whenever need arises. The proof below will imply (3.13) with $L_0 = \min_j K_j$. In it we use the notation $\varepsilon_j = \varepsilon_j(L)$, $j \geq 2$, for positive functions depending only on (X, n) and satisfying $\varepsilon_j(L) \to 0$ as $L \to 1$.

Fix $Q \in \mathcal{K}_i$, and let $\psi_Q = (\pi_Q^*)^{-1} : T_Q \cap Q^* \to X \cap Q^*$ be as in Lemma 3.7. Define a map $f_Q : T_Q \cap Q^* \to \mathbb{R}^n$ by setting

$$f_Q x = f\psi_Q x = f\varphi_Q x$$

for all $x \in T_Q \cap Q^*$. We prove that f_Q has the following properties:

a) f_Q is LM-BL with $M = M(L)$ as in Lemma 3.7.1),

b) $f_Q(x) = \psi_Q(x)$ for all $x \in \pi_Q(D \cap Q^*)$,

c) $|f_Q - \mathrm{id}|_{T_Q \cap Q^*} \leq \varepsilon_2(L)\,\delta$.

Since a) and b) are obvious, we only need to verify c). We first construct a not too flat p-simplex Δ such that the set Δ^0 of vertices of Δ is contained in $\pi_Q(D \cap Q^*)$.

Choose an orthonormal basis $(v_1, \ldots, v_p)$ of T_Q^0 (see Paragraph 3.6), and set

$$t = m_n \sqrt{n}\,\delta\,, \qquad z_j = x_Q + t\,v_j$$

for $1 \le j \le p$. Since $|z_j - x_Q| = m_n\sqrt{n}\,\delta < \lambda/2$, we have $z_j \in T_Q \cap Q^*$, $\psi_Q z_j \in X \cap Q^*$. Hence we can choose cubes $R_j \in \mathcal{K}$ so that $\psi_Q z_j \in R_j$, $1 \le j \le p$. We set $y_j = \pi_Q x_{R_j}$, $\Delta_0 = x_Q z_1 \cdots z_p$. Then Δ_0 is a p-corner with size t, see Paragraph 2.5. Since π_Q decreases distances, we have

$$|y_j - z_j| \le |x_{R_j} - \psi_Q z_j| \le \operatorname{diam}(R_j) = \sqrt{n}\,\delta = t/m_n\ ,$$

for all $j \in \{1, \dots, p\}$. By the definition of m_n in Paragraph 2.5, this implies that $\Delta = x_Q y_1 \cdots y_p$ is a p-simplex with $\rho(\Delta) \le 2p$. Moreover, we have $\Delta^0 \subset Q^*$, because

$$|y_j - x_Q| \le |y_j - z_j| + |z_j - x_Q| \le \sqrt{n}\,\delta + m_n \sqrt{n}\,\delta = \lambda/2$$

for all j. Since $|x_{R_j} - \psi_Q z_j| \le \sqrt{n}\,\delta$ and since $\varepsilon_1(L) \le 1/2$ in Lemma 3.7.2), it easily follows that $x_{R_j} \in Q^*$ for all j. Hence $\Delta^0 \subset \pi_Q(D \cap Q^*)$, and Δ is the desired p-simplex.

Next we observe that $f_Q|_{\Delta^0} = \psi_Q|_{\Delta^0}$ by (b). Applying this and Lemma 3.7.2) we get

$$|f_Q - \operatorname{id}|_{\Delta^0} \le |\psi_Q - \operatorname{id}|_{T_Q \cap Q^*} \le \varepsilon_1(L)\,\delta\,.$$

By the approximation theorem [V, 3.1] there exists an isometry $h : T_Q \to \mathbb{R}^n$ such that we have

$$|f_Q - h|_{T_Q \cap Q^*} \le \varepsilon_3(L) \operatorname{diam}(T_Q \cap Q^*) < \varepsilon_4(L)\,\delta\,,$$

where $\varepsilon_4(L) = 2n\,(1 + m_n)\,\varepsilon_3(L)$. Then h satisfies

$$|h - \operatorname{id}|_{\Delta^0} \le |h - f_Q|_{\Delta^0} + |f_Q - \operatorname{id}|_{\Delta^0} \le \varepsilon_5(L)\,\delta$$

with $\varepsilon_5(L) = \varepsilon_4(L) + \varepsilon_1(L)$. From [V, 2.11] it follows that for all $x \in T_Q$ we have

$$|hx - x| \le \varepsilon_5(L)\,\delta\, \left(1 + \operatorname{diam}(\Delta)^{-1} M_0\, |x - x_Q|\right)\ ,$$

where

$$M_0 = 4 + 6\,\rho(\Delta)\,p\,(1 + \rho(\Delta))^{p-1} \le M'$$

with $M' = 4 + 12\,p^2\,(1 + 2p)^{p-1}$. Since $m_n \ge 2$, we have

$$\begin{aligned} \operatorname{diam}(\Delta) &\ge |y_1 - x_Q| \ge |z_1 - x_Q| - |z_1 - y_1| \\ &\ge t - \frac{t}{m_n} \ge \frac{t}{2} = \frac{m_n\,\sqrt{n}\,\delta}{2}\,. \end{aligned}$$

Applying this estimate and the fact that $Q^* \subset \bar{B}^n(x_Q, \lambda\sqrt{n}/2)$ we get

$$|h - \mathrm{id}\,|_{T_Q \cap Q^*} \le \varepsilon_5(L)\,\delta\,\left(1 + 2\,M'\,\sqrt{n}\,\frac{m_n+1}{m_n}\right) \le \varepsilon_6(L)\,\delta\,,$$

where $\varepsilon_6(L) = (1 + 3\,M'\,\sqrt{n})\,\varepsilon_5(L)$. We now get the desired estimate

$$|f_Q - \mathrm{id}\,|_{T_Q \cap Q^*} \le |f_Q - h|_{T_Q \cap Q^*} + |h - \mathrm{id}\,|_{T_Q \cap Q^*} \le \varepsilon_2(L)\,\delta$$

with $\varepsilon_2(L) = \varepsilon_4(L) + \varepsilon_6(L)$. Hence c) is true.

Writing $A = A(L)$, $B = B(L)$ we set

$$A_Q = \pi_Q(Q^* \cap A)\,, \qquad B_Q = \pi_Q(Q^* \cap B)\,, \qquad D_Q = \pi_Q(Q^* \cap D)\,.$$

Since π_Q^* is M-BL with $M \le 2$ as in Lemma 3.7, the assumption 2) in Lemma 3.12 implies that we have

d) $$\mathrm{dist}\,(A_Q, T_Q \cap Q^* \setminus B_Q) \ge q/2\,.$$

Let N_1 be the integer defined in Paragraph 3.10. We divide the p-cube $T_Q \cap Q^*$ into $(6N_1)^p$ closed p-cubes R with side length $\lambda/6N_1$. Let $\mathcal{L}$ be the family of all these p-cubes R. If $R \in \mathcal{L}$, then by paragraphs 3.6, 3.10 and the definition of q before Lemma 3.12 we get the estimate

$$\mathrm{diam}\,(R) = \frac{\lambda\,\sqrt{p}}{6\,N_1} < \frac{(m_n+1)\,n\,\delta}{3\,N_1} \le \frac{q}{2}\,.$$

Applying this together with d) we see that the implication

e) $$R \cap A_Q \ne \varnothing \qquad \text{implies} \qquad R \subset B_Q$$

is true for all $R \in \mathcal{L}$. We set

$$\mathcal{L}_A = \{R \in \mathcal{L} :\ R \cap A_Q \ne \varnothing\}\,, \qquad \mathcal{L}_D = \{R \in \mathcal{L} :\ R \cap D_Q \ne \varnothing\}\,.$$

We divide the set $\mathring{Q}^* \cap T_Q \setminus \mathring{P}_Q$ into N_1 disjoint sets

$$H_j = \mathring{P}_Q\left(1 + \frac{j}{2\,N_1}\right) \setminus \mathring{P}_Q\left(1 + \frac{j-1}{2\,N_1}\right)\,, \qquad 1 \le j \le N_1\,,$$

as illustrated in Figure 2. For notation, see 1.1.

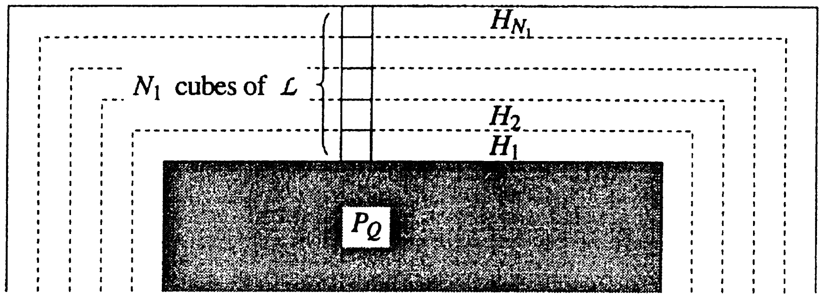

Figure 2

By Lemma 3.9.2) and Paragraph 3.10 we have

$$\#(\mathring{Q}^* \cap D) \leq N < N_1 .$$

Since the sets $H_j \subset \mathring{Q}^* \cap T_Q = \pi_Q(\mathring{Q}^* \cap X)$, $1 \leq j \leq N_1$, are disjoint, we can choose $j_0 \in \{1, \ldots, N_1\}$ such that

$$H_{j_0} \cap D_Q = \varnothing .$$

We define

$$\mathcal{L}_1 = \{R \in \mathcal{L}_D : \ \mathring{R} \subset P_Q \cup H_1 \cup \cdots \cup H_{j_0 - 1}\} ,$$
$$\mathcal{L}_2 = \{R \in \mathcal{L}_D : \ \mathring{R} \subset H_{j_0+1} \cup \cdots \cup H_{N_1}\} .$$

Then $\mathcal{L}_1$ consists of the cubes $R \in \mathcal{L}_D$ inside H_{j_0} and $\mathcal{L}_2$ of those outside H_{j_0} in Figure 2. We set

$$Y_A = \bigcup \mathcal{L}_A , \qquad Y_1 = \bigcup \mathcal{L}_1 , \qquad Y_2 = \bigcup \mathcal{L}_2 .$$

Then Y_1 and Y_2 satisfy the conditions

$$D_Q \subset Y_1 \cup Y_2 \cup (T_Q \cap \partial Q^*) , \qquad \operatorname{dist}(Y_1, Y_2) \geq \frac{\lambda}{6 N_1} ,$$
$$\operatorname{dist}(Y_1, \partial Q^*) \geq \frac{\lambda}{6 N_1} , \qquad \operatorname{dist}(Y_2, P_Q) \geq \frac{\lambda}{6 N_1} .$$

We define two polyhedra Y and Z_Q by setting

$$Y = Y_A \cup Y_1 \cup Y_2 \cup P_Q , \qquad Z_Q = Y \cup \partial Q^* .$$

Obviously Z_Q is similar to some member Z_0 of $\mathcal{F}$, *cf.* Paragraph 3.10. Since $\mathcal{F}$ is finite and since every $Z \in \mathcal{F}$ has the BLEP in $\mathbb{R}^n$ by [PV, 1.1], there exists $L_0^* > 1$ and a function $L_1^* :]1, L_0^*] \to]1, \infty[$ satisfying $L_1^*(K) \to 1$ as $K \to 1$ and such that if $Z \in \mathcal{F}$ and $1 < K \leq L_0^*$, then every K-BL map $g : Z \to \mathbb{R}^n$ has an $L_1^*(K)$-BL extension $G : \mathbb{R}^n \to \mathbb{R}^n$. Since Z_Q and Z_0 are similar, the same is true for all K-BL maps $g : Z_Q \to \mathbb{R}^n$, $1 < K \leq L_0^*$.

We next show that after restricting L we can define a function $g_Q : Z_Q \to \mathbb{R}^n$ by setting

$$\text{f)} \qquad g_Q x = \begin{cases} \varphi_Q^{-1} f \psi_Q x = \varphi_Q^{-1} f_Q x\,, & \text{if } x \in P_Q \cup Y_A \cup Y_1\,, \\ x\,, & \text{if } x \in Y_2 \cup \partial Q^*\,. \end{cases}$$

If $R \in \mathcal{L}_A$, then $R \subset B_Q$ by e). Hence $\psi_Q R \subset Q^* \cap B$, and we get $(\varphi_Q^{-1} \circ f_Q)|_R = \text{id}$, because $f|_B = \text{id}$ by the assumption of (3.13). It follows that $\varphi_Q^{-1} \circ f_Q = \text{id}$ in the set Y_A, which contains the intersection of $P_Q \cup Y_A \cup Y_1$ and $Y_2 \cup \partial Q^*$. Since φ_Q^{-1} is defined in Q^* only, we must yet verify that if L is chosen small enough, then $f_Q(P_Q \cup Y_1) \subset Q^*$.

Let $x \in P_Q \cup Y_1$. Since $\operatorname{dist}(x, \partial Q^*) \geq \lambda/6\,N_1$, the desired condition $f_Q x \in Q^*$ would follow if we had $|f_Q x - x| < \lambda/6N_1$. To arrange this we let $L \leq K_2$, where $K_2 \in]1, K_1]$ is such that the function ε_2 of c) satisfies for all $K \in]1, K_2]$ the estimate

$$\text{g)} \qquad \varepsilon_2(K) < \mu\,, \qquad \mu = \mu(n) = \frac{(m_n + 1)\sqrt{n}}{3\,N_1}\,.$$

By c) and Paragraph 3.6 we then indeed have

$$|f_Q x - x| \leq \varepsilon_2(L)\,\delta < \mu\,\delta = \frac{\lambda}{6\,N_1}\,.$$

Hence we get $f_Q(P_Q \cup Y_1) \subset Q^*$, if $L \leq K_2$.

From now on we always assume $L \leq K_2$. Then g_Q is well defined by f). Next we show that after restricting L once more, g_Q is M_2-BL, where $M_2 = M_2(L) \to 1$ as $L \to 1$. For this, let $x, y \in Z_Q$. We must derive suitable estimates for the number

$$\alpha = \frac{|g_Q x - g_Q y|}{|x - y|}\,.$$

If $\{x, y\} \subset P_Q \cup Y_A \cup Y_1$, then $1/M_1\,L\,M \leq \alpha \leq M_1\,L\,M$ by f), 3.8.1) and a). If $\{x, y\} \subset Y_A \cup Y_2 \cup \partial Q^*$, then $\alpha = 1$ by f) and the above observation that $g_Q|_{Y_A} = \text{id}$.

It remains to consider the case $x \in P_Q \cup Y_1$, $y \in Y_2 \cup \partial Q^*$. Then we have

$$|x-y| \geq \frac{\lambda}{6\,N_1} = \mu\,\delta$$

with μ as in g). Hence by 3.8.2) and c) we get

$$\begin{aligned}|g_Q x - g_Q y| &= |\varphi_Q^{-1} f_Q x - y| \\ &\leq |\varphi_Q^{-1} f_Q x - f_Q x| + |f_Q x - x| + |x-y| \\ &\leq \varepsilon_1(L)\,\delta + \varepsilon_2(L)\,\delta + |x-y| \\ &\leq (1+\varepsilon_7(L))\,|x-y|\,,\end{aligned}$$

where $\varepsilon_7(L) = (\varepsilon_1(L)+\varepsilon_2(L))/\mu$. Similarly we get

$$|g_Q x - g_Q y| \geq (1-\varepsilon_7(L))\,|x-y|\,.$$

Hence $1-\varepsilon_7(L) \leq \alpha \leq 1+\varepsilon_7(L)$.

Let $K_3 \in\,]1, K_2]$ be such that $\varepsilon_7(K) < 1$ for all $K \in\,]1, K_3]$. From now on we assume that $L \leq K_3$. By the above estimates, g_Q is then M_2-BL, where

$$M_2 = M_2(L) = \max\left\{L\,M_1(L)\,M(L),\ \frac{1}{1-\varepsilon_7(L)}\right\}$$

satisfies $M_2 \to 1$ as $L \to 1$.

Choose $K_4 \in\,]1, K_3]$ so that $M_2(K) \leq L_0^*$ for all $K \in\,]1, K_4]$. From now on we assume that $L \leq K_4$. Then g_Q has an L_3-BL extension

$$G_Q : \mathbb{R}^n \to \mathbb{R}^n\,,$$

where $L_3 = L_3(L) = L_1^*(M_2(L)) \to 1$ as $L \to 1$. Obviously, we have $G_Q|_{\partial Q^*} = \mathrm{id}$ and $G_Q Q^* = Q^*$.

We define a homeomorphism $F_Q : Q^* \to Q^*$ by letting $F_Q = \varphi_Q \circ G_Q \circ \varphi_Q^{-1}$. We prove that F_Q has the following properties:

h) F_Q is L_4-BL with $L_4 = L_4(L) = L_3(L)\,M_1(L)^2$,

i) $F_Q|_{\partial Q^*} = \mathrm{id}$,

j) $F_Q x = fx$, for all $x \in Q^* \cap (A \cup D \cup E_i)$.

By 3.8.1), h) is obvious. Since $\varphi_Q \partial Q^* = \partial Q^*$ and since $G_Q|_{\partial Q^*} =$ id, we get i). To prove j), let $x \in Q^* \cap (A \cup D \cup E_i)$ and set $y =$

$\varphi_Q^{-1}(x) = \pi_Q x$. Then we have $G_Q y = g_Q y$. If $x \in A$, then $g_Q y = y$, because $A_Q \subset Y_A$ and because $g_Q|_{Y_A} = \mathrm{id}$. Since $f|_A = \mathrm{id}$ by (3.13) and Lemma 3.12.1), we get

$$F_Q x = \varphi_Q y = x = fx$$

as desired. If $x \in D$, then $fx = x$, and by f) we have $g_Q y = y$. Hence we get $F_Q x = x = fx$. If $x \in E_i$, then $y \in P_Q$ by Paragraph 3.11 and Lemma 3.9.1). Hence f) implies that

$$g_Q y = \varphi_Q^{-1} f \psi_Q y = \varphi_Q^{-1} f x\,,$$

and we obtain j) in this last case as well:

$$F_Q x = \varphi_Q g_Q y = fx\,.$$

Letting $Q \in \mathcal{K}_i$ vary, we get a family of maps $F_Q : Q^* \to Q^*$, $Q \in \mathcal{K}_i$, as above. By Lemma 3.9.1) and by i) these maps can be glued together into a homeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$Fx = \begin{cases} F_Q x\,, & \text{if } x \in Q^* \text{ with } Q \in \mathcal{K}_i \\ x\,, & \text{if } x \in \mathbb{R}^n \setminus \bigcup\{Q^* : Q \in \mathcal{K}_i\}\,. \end{cases}$$

By h) and j) it is easy to see that F is L_4-BL and satisfies $Fx = fx$ for all $x \in A \cup D \cup E_i$. Hence (3.13) is true with $L_1 = L_4$, and Lemma 3.12 is proved.

We are now ready to prove our main theorem.

3.14. Theorem. *Let $1 \le p \le n$ and let X be a compact p-dimensional C^1-submanifold of $\mathbb{R}^n$ with or without boundary. Then X has the* BLEP *in $\mathbb{R}^n$.*

PROOF. As before in Section 3, we assume that $1 \le p \le n-1$ and that X has no boundary. Manifolds with boundary will be considered in Section 4.

Let $1 < L \le K_1$, where $K_1 \in]1, K_0]$ is as in Lemma 3.7. We shall freely use the definitions and results of 3.3-3.11. The parameter L will often be dropped out of the notation as *e.g.* in (3.5).

Letting N be as in Paragraph 3.2, we set

$$r_j = \sqrt{n}\,\delta + j\,q = (1 + j/N)\sqrt{n}\,\delta\,, \qquad 1 \le j \le N\,,$$
$$B_{ij} = \bigcup\{\psi_Q(\bar{B}^n(x_Q, r_j) \cap T_Q) :\ Q \in \mathcal{K}_i\}\,, \qquad 1 \le i, j \le N\,.$$

Note that we have $B_{ij} \subset E_i$, because $\bar{B}^n(x_Q, r_j) \cap T_Q \subset P_Q$ by the definitions of P_Q and Q^* in paragraphs 3.11 and 3.6 and by the fact that

$$\frac{1}{3}\,\lambda = \frac{2}{3}\,(m_n + 1)\sqrt{n}\,\delta \geq 2\sqrt{n}\,\delta \geq r_j\,.$$

We define N sets $B_i = B_i(L) \subset X$, $1 \leq i \leq N$, by setting

$$\begin{aligned} B_1 &= B_{1N}\,, \\ B_2 &= B_{1,N-1} \cup B_{2N}\,, \\ &\vdots \\ B_i &= B_{1,N-i+1} \cup B_{2,N-i+2} \cup \cdots \cup B_{iN} \\ &\vdots \\ B_N &= B_{11} \cup B_{22} \cup \cdots \cup B_{NN}\,. \end{aligned}$$

Let $Q \in \mathcal{K}$. Since $Q \subset \bar{B}^n(x_Q, \sqrt{n}\,\delta)$, since π_Q decreases distances, and since $\pi_Q x_Q = x_Q$, we have

$$\pi_Q(X \cap Q) \subset T_Q \cap \bar{B}^n(x_Q, \sqrt{n}\,\delta) \subset T_Q \cap Q^*\,.$$

Applying the map ψ_Q we get the inclusion

$$X \cap Q \subset \psi_Q(\bar{B}^n(x_Q, \sqrt{n}\,\delta) \cap T_Q)\,.$$

Since this holds for all $Q \in \mathcal{K} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N$ and since $r_j \geq \sqrt{n}\,\delta$ for all $j \in \{1, \ldots, N\}$, it follows that $X \subset B_N$. Hence X trivially has the BLEP rel $B_N \cup D$, see Remark 2.3.3.

We continue by induction. Suppose that $1 \leq i \leq N-1$ and that X has the BLEP rel $(B_{i+1} \cup D)$. With the aid of Lemma 3.12 we want to prove that X has the BLEP rel $(B_i \cup D)$.

We define a set $A_i = A_i(L) \subset X$ by

$$A_i = B_{1,N-i} \cup B_{2,N-i+1} \cup \cdots \cup B_{i,N-1}\,.$$

Obviously, we have $A_i \subset B_i$. Moreover, we get

$$\operatorname{dist}(A_i, X \setminus B_i) \geq q\,,$$

because ψ_Q increases distances by Lemma 3.7.1). Observe that the set

$$B_{i+1} = B_{1,N-i} \cup B_{2,N-i+1} \cup \cdots \cup B_{i,N-1} \cup B_{i+1,N}$$

satisfies $B_{i+1} \subset A_i \cup E_{i+1}$, because $B_{i+1,N} \subset E_{i+1}$, as noted above. Hence $B_{i+1} \cup D \subset A_i \cup D \cup E_{i+1}$, and X has the BLEP rel $(A_i \cup D \cup E_{i+1})$ by the inductive hypothesis and Remark 2.3.1). Applying Lemma 3.12 with the substitution $A \mapsto A_i$, $B \mapsto B_i$, $i \mapsto i+1$, we deduce that X indeed has the BLEP rel $(B_i \cup D)$.

By induction, we see that X has the BLEP rel $(B_1 \cup D)$. Since $B_1 = B_{1N} \subset E_1$, X also has the BLEP rel $(E_1 \cup D)$. Applying Lemma 3.12 again, now with the substitution $A \mapsto \varnothing$, $B \mapsto \varnothing$, $i \mapsto 1$, we see that X has the BLEP rel D in $\mathbb{R}^n$. By Lemma 3.4, this implies that X has the BLEP in $\mathbb{R}^n$.

3.15. REMARK. An analysis of our method in the proof of Theorem 3.14 reveals that the actual extension of an L-BL map $f : X \to \mathbb{R}^n$ with $L-1$ small enough can be done in $N+1$ steps. The first step begins by extending $f|_D$ to an $L_2(L)$-BL map $g : \mathbb{R}^n \to \mathbb{R}^n$ as in Paragraph 3.3. Replacing f by $g^{-1} \circ f$ a normalization $f|_D = \text{id}$ is then obtained. In the second step, the restriction $f|_{B_1 \cup D}$ (or even $f|_{E_1 \cup D}$) of this normalized map f is then extended and the stronger normalization $f|_{B_1 \cup D} = \text{id}$ is seen to be possible. The remaining $N-1$ steps correspond to the inductive steps above in reverse order. We proved that the BLEP of X rel $B_{i+1} \cup D$ implies the BLEP of X rel $B_i \cup D$, $1 \le i \le N-1$. This corresponds to extending $f|_{B_{i+1} \cup D}$ (or even $f|_{A_i \cup D \cup E_{i+1}}$), where $f : X \to \mathbb{R}^n$ is normalized by $f|_{B_i \cup D} = \text{id}$, and using the extended map for a new normalization $f|_{B_{i+1} \cup D} = \text{id}$. Since $B_N \cup D = X$, we finally see that f can be normalized by $f|_X = \text{id}$. This implies that f indeed has an extension.

3.16. The case $p = 1$.

If X is one-dimensional, the proof of Theorem 3.14 can be essentially simplified. By [P, 2.5] X can be assumed to be connected. Then X is a C^1 arc or a C^1 Jordan curve. Assuming that it is a Jordan curve, we present an outline of the method used in [HP] for the extension of an L-BL map $f : X \to \mathbb{R}^n$ with $L-1$ small enough.

The sets D_j of Paragraph 3.3 with j odd are not needed. For j even we simply let D_j consist of j points, which divide X into j subarcs of equal length. Then the number $m = m(L) \in 2\mathbb{N}$ and the set $D = D(L) = D_{m(L)}$ are obtained as in Paragraph 3.3; for this we have to replace $\mathbb{N}$ by $2\mathbb{N}$ in Lemma 2.1. The set D so defined consists of m points $a_1, \dots, a_m, a_{m+1} = a_1$, which divide X into m equally long subarcs C_j joining a_j to a_{j+1}, $1 \le j \le m$. As in Remark 3.15 we obtain the normalization $f|_D = \mathrm{id}$ by making use of an extension of $f|_D$. However, in the rest of the proof only the two steps described below are necessary. Moreover, only two rather simple similarity types of compact polyhedra are needed. The following constructions are possible if $L-1$ is small enough.

Let J_j denote the line segment joining a_j to a_{j+1}, $1 \le j \le m$. Choose closed n-cubes Q_j, $1 \le j \le m$, of $\mathbb{R}^n$ in such a way that a_j and a_{j+1} are the centers of two opposite $(n-1)$-faces of Q_j. Define the polyhedra

$$X_j = \partial Q_j\,(5/4) \bigcup J_j\,, \qquad Y_j = \partial Q_j \cup J_j\,,$$

and let $\psi_j : J_j \to C_j$ be the inverse of the orthogonal projection $C_j \to J_j$. Consider the maps $F_j : X_j \to \mathbb{R}^n$ and $\Psi_j : X_j \to \mathbb{R}^n$ ($1 \le j \le m$, j odd) defined by letting F_j coincide with $(f|_{C_j}) \circ \psi_j$ and Ψ_j with ψ_j in J_j and letting $F_j = \mathrm{id} = \Psi_j$ in $\partial Q_j\,(5/4)$. Extending these maps with the aid of the BLEP of X_j and using the extensions glued together we are able to complete the first step by obtaining the normalization

$$(*) \qquad f|_{\cup\{C_j:\ j \text{ odd}\}} = \mathrm{id}\,.$$

In the second step we then apply the same argument for the subarcs C_j of X with $j \in \{1, \dots, m\}$ even. From the normalization $(*)$ it here follows that we can use the polyhedra Y_j (j even) in the same role as the polyhedra X_j had above for j odd. Hence this step actually leads to the normalization $f|_X = \mathrm{id}$, showing that f can be extended.

4. Manifolds with boundary.

In this section we give an outline of the proof of Theorem 3.14 in the case where the compact p-dimensional C^1-manifold $X \subset \mathbb{R}^n$ has boundary. The case $p = n$ follows easily from the BLEP of ∂X, which was proved in Section 3 and in [V, 5.17]. Suppose that $p \le n-1$.

As in Section 3 we consider a number $\delta > 0$ and the cube families $\mathcal{J}$ and $\mathcal{K}$. However, the numbers N_0 and N are larger. We now let N_0 and N be the integers satisfying the conditions

$$6\,(m_n + 3)\,n^{3/2} < N_0 \le 6\,(m_n + 3)\,n^{3/2} + 1\,, \qquad N = (N_0 + 1)^n\,.$$

In Paragraph 3.3 the points $x_Q \in Q \cap X$, $Q \in \mathcal{K}$, are chosen so that $x_Q \in Q \cap \partial X$ whenever Q meets the boundary ∂X of X. As in Paragraph

3.3, we apply Lemma 2.1 to find the number $K_0 > 1$ and for $L \in]1, K_0]$ the numbers $\delta(L) > 0$, $L_2(L) > 1$, and the finite set $D(L) \subset X$. Then Lemma 3.4 holds verbatim.

Let Q^* be as in Paragraph 3.6. We define the new cube families

$$\mathcal{K}^1 = \{Q \in \mathcal{K} : \ Q^* \cap \partial X = \varnothing\},$$
$$\mathcal{K}^2 = \{Q \in \mathcal{K} : \ Q \cap \partial X \neq \varnothing\},$$
$$\mathcal{K}^0 = \mathcal{K}^1 \cup \mathcal{K}^2 .$$

In the sequel, only the cubes of $\mathcal{K}^0$ will be used.

If $Q \in \mathcal{K}^2$, we let T_Q and T'_Q denote the tangent planes of X and ∂X at x_Q, respectively. Let H_Q be the closed half plane of T_Q with $\partial H_Q = T'_Q$ such that H_Q and $\pi_Q X$ are in a natural sense on the same side of T'_Q near x_Q.

We set

$$\alpha = (m_n + 3)\, n\, \delta .$$

If $Q \in \mathcal{K}^2$, we replace Q^* by the larger cube $\tilde{Q}$ with center x_Q, side 6α, and having p-dimensional and $(p-1)$-dimensional faces parallel to T_Q and T'_Q, respectively.

For $Q \in \mathcal{K}^1$, the homeomorphism $\varphi_Q : Q^* \to Q^*$ and the p-cube P_Q are defined as in paragraphs 3.8 and 3.11. For $Q \in \mathcal{K}^2$, the corresponding homeomorphism $\varphi_Q : \tilde{Q} \to \tilde{Q}$ is defined in two steps. First, we define a homeomorphism $\varphi'_Q : \tilde{Q} \to \tilde{Q}$ such that $\varphi'_Q(\tilde{Q} \cap H_Q) = \pi_Q(\tilde{Q} \cap X)$. Next, we extend the map $\psi_Q : \pi_Q(\tilde{Q} \cap X) \to \tilde{Q} \cap X$ to a homeomorphism $\varphi''_Q : \tilde{Q} \to \tilde{Q}$. These maps are chosen in such a way that the homeomorphism $\varphi_Q = \varphi''_Q \circ \varphi'_Q : \tilde{Q} \to \tilde{Q}$ has the properties 1) and 2) of Paragraph 3.8. The sets P_Q, $Q \in \mathcal{K}^2$, are defined by setting

$$P_Q = \frac{2}{3}\,(H_Q \cap \tilde{Q} - x_Q) + x_Q .$$

The number N_1 of 3.10 is replaced by the larger number $N_1 = 2N\,(m_n + 3)\, n$. We divide $\mathcal{K}$ into disjoint subfamilies $\mathcal{K}_1, \ldots, \mathcal{K}_N$ as in (3.5), and we set

$$\mathcal{K}_i^0 = \mathcal{K}^0 \cap \mathcal{K}_i , \qquad E_i = \bigcup \{\varphi_Q P_Q : \ Q \in \mathcal{K}_i^0\}.$$

Then Lemma 3.12 holds verbatim with $q = \delta\sqrt{n}/N$. Its proof requires some modifications when considering the cubes Q of $\mathcal{K}^2$. For example, we define the maps

$$f_Q : \tilde{Q} \cap H_Q \to \mathbb{R}^n , \qquad Q \in \mathcal{K}^2 ,$$

by $f_Q x = f \varphi_Q x$. To prove that f_Q is close to the identity mapping in $\tilde{Q} \cap H_Q$ we again need a basis $(v_1, \dots, v_p)$ of $T_Q - x_Q$. This basis is now chosen so that $v_1 + \dots + v_p$ is a normal vector of T'_Q in T_Q, pointing to H_Q.

In the final proof of the BLEP of X, we define the sets B_{ij}, $1 \leq i, j \leq N$, as follows: If $Q \in \mathcal{K}^1$, we set

$$r_j = \sqrt{n}\, \delta + j\, q\,, \qquad U(Q,j) = \bar{B}^n(x_Q, r_j) \cap T_Q\,.$$

If $Q \in \mathcal{K}^2$, we write

$$s_j = \alpha + 2j\, q\,, \qquad U(Q,j) = \bar{B}^n(x_Q, s_j) \cap H_Q\,.$$

Then we define

$$B_{ij} = \bigcup \{\varphi_Q U(Q,j) :\ Q \in \mathcal{K}_i^0\}\,.$$

The rest of the proof remains essentially unchanged.

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Générateurs infinitésimaux et propiétés géométriques pour certaines équations complètement non linéaires

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Introduction.

Nous nous proposons, dans ce travail, d'étudier certaines propriétés géométriques telles que diverses symétries et diverses concavités radiales, directionnelles etc ..., pour des équations complètement non linéaires, par exemple, du type

$$(*) \qquad \begin{cases} F(x,u,\nabla u,\Delta u) = 0 & \text{dans } \Omega\,, \\ u|_{\partial\Omega} = 0\,, \qquad u > 0 & \text{dans } \Omega\,. \end{cases}$$

Nous nous attacherons en premier lieu à mettre en évidence des hypothèses aussi simples que possibles, applicables à des situations concrètes pour des solutions classiques. Dans l'étude de certains problèmes de contrôle optimal, les données ne sont pas suffisamment régulières pour permettre l'obtention de solutions classiques. Aussi notre deuxième souci sera d'adapter notre méthode à des problèmes où interviennent des données discontinues. Notre exemple modèle sera le suivant: étudier les points critiques de la solution de

$$\begin{cases} -\Delta u = f(x) & \text{dans } \Omega\,, \\ \quad u|_{\partial\Omega} & \text{à préciser,} \end{cases}$$

où $f(x) \in \{\alpha, \beta\}$ presque pour tout $x \in \Omega$. Ce type de question à données discontinues ne semble pas avoir été abordé. Notre méthode, comme beaucoup d'autres méthodes, nécessite l'utilisation du principe du maximum [1], [2], combinée à une idée qui nous semble nouvelle dans ce cadre: l'utilisation de générateurs infinitésimaux, de leurs itérés et de leurs commutateurs avec les opérateurs différentiels intervenant dans l'équation $(*)$.

Le sujet a connu des développements importants depuis les travaux de [3] et [4] utilisant une méthode efficace: la méthode des plans mobiles [24] doublée du principe du maximum; rappelons que dans [3] et [4] on trouve des résultats de symétries [4], de symétries radiales [3], [4], et de localisation des points critiques [4]. Pour des équations moins générales que $(*)$, l'étude de la concavité -ou quasiconcavité- a également connu un développement important depuis le "principe du maximum de concavité" de N. Korevaar [5], [6], appliqué aux surfaces capillaires de $\mathbb{R}^{n+1}$. Par une approche différente, Caffarelli et Spruck [7] ont également abordé diverses questions de convexité dont un résultat classique de [8] concernant la première fonction propre de Δ. Kenmington [9] et Kawhol [10] ont obtenu divers résultats sur le sujet en utilisant le principe de N. Korevaar. D'autres méthodes intéressantes ont été introduites par divers auteurs. Citons à titre d'exemples les résultats de [11], [12], [13], [14]. Un point commun à la plupart de ces méthodes est l'utilisation de solutions classiques. Enfin signalons que la liste de travaux cités ci-dessus ne prétend pas être exhaustive, et que nous renvoyons à [10] pour une liste plus complète.

I. Notations et position du problème.

On se donne un ouvert Ω borné (non nécessaire) régulier dont les propriétés géométriques seront précisées plus loin, et une fonction $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \ni (x, r, t, s) \longrightarrow F(x, r, t, s) \in \mathbb{R}$ de classe C^1 ou C^2, selon le besoin, par rapport à tous ses arguments $x = (x_1, \ldots, x_n)$, r, $t = (t_1, t_2, \ldots, t_n)$ et s. On suppose que les dérivées de F vérifient les hypothèses suivantes:

$$(1.1) \qquad \frac{\partial F}{\partial s} \geq \gamma_0 > 0\,, \qquad F(x, r, t, 0) \geq 0\,, \qquad \Big|\frac{\partial F}{\partial t}\Big| \leq K\,.$$

Dans toute la suite, pour ne pas alourdir le texte, nous ne considèrerons que l'opérateur Δ ou son itéré Δ^2: il est possible d'utiliser des opérateurs du type

$$\sum_i \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}$$

en s'entourant de quelques précautions.

Nous dirons que Ω est fortement étoilé par rapport à x_0 si et seulement si Ω est étoilé par rapport à x_0 et $(\vec{x} - \vec{x}_0) \cdot \vec{\nu} > 0$ pour tout $x \in \partial\Omega$, $\vec{\nu}$ étant la normale en x à $\partial\Omega$, orientée extérieurement.

1) Objet de ce travail: Etant donnée une solution classique u des équations

$$\begin{cases} F(x, u, \nabla u, \Delta u) = 0\,, \\ u|_{\partial\Omega} = 0\,, \qquad u > 0 \quad \text{dans } \Omega\,. \end{cases} \tag{1.2}$$

posées dans un ouvert Ω possédant certaines symétries ou autres propriétés géométriques, il s'agit d'étudier l'incidence de cette géométrie sur celle de u: propriété d'être étoilé, symétrique par rapport à un hyperplan, radialement concave, concave dans une direction etc ... Bien entendu nous nous fixons pour but de dégager des hypothèses aussi simples que possible sur F. Pour mener cette étude nous aurons besoin de la notion suivante:

2) Générateur infinitésimal d'une propriété géométrique: On se donne une famille d'opérateurs $S(t) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continus tels que

$$\begin{cases} S(0) = I = \text{identité}\,, \\ S(t)\,\Omega \subset \Omega\,, \quad \text{pour tout } t \neq 0\,, \end{cases} \tag{P}$$

où t appartient à une partie adéquate I de $\mathbb{R}$.

Nous dirons que Ω possède la propriété géométrique (P). On se donne une fonction dérivable sur Ω et pour tout $x \in \Omega$ on pose

$$Gu = \lim_{t \to 0} \frac{u(S(t)x) - u(x)}{t}\;; \tag{1.3}$$

il est clair que G est un opérateur linéaire ne dépendant que de S.

Définition. *On appelle générateur infinitésimal de la propriété* (P) *de* Ω *(ou plus simplement de* Ω*) l'opérateur* G *défini par* (1.3).

Si $S(\lambda) = (s_{ij}(\lambda))$ est une famille de matrices telles que $S(0) = I$ et $S(\lambda)\,\Omega \subset \Omega$, alors

$$G \;=\; {}^T x \cdot {}^T \frac{dS}{d\lambda}(0) \cdot \nabla$$

où

$${}^T x = (x_1, \ldots, x_n) \qquad \text{et} \qquad {}^T\nabla = \Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big).$$

Donnons quelques exemples de générateurs:

i) dérivation radiale: Ω est étoilé, $G = x \cdot \nabla = \sum_i x_i\, \partial/\partial x_i$,

ii) dérivation perpendiculairement à un hyperplan (π) qui est orthogonal à un vecteur de base e_{i_0}: $G = x_{i_0}\, \partial/\partial x_{i_0}$,

iii) dérivation dans une direction d: Ω est convexe,

$$G = d \cdot \nabla = \sum_i d_i \frac{\partial}{\partial x_i} ,$$

iv) la dérivation angulaire: pour $n = 2$, Ω étant un secteur circulaire,

$$G = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} .$$

Deux questions naturelles motivant notre travail se posent:

1) que doit vérifier F pour que la solution u de (1.2) vérifie

$$u(S(t)x) - u(x) \geq 0 , \qquad \text{pour tout } x \in \Omega ,\ \text{pour tout } t \in I \ ?$$

2) peut-on répondre à la même question pour avoir

$$u(S^2(t)x) - 2u(S(t)x) + u(x) \geq 0 , \quad \text{pour tout } x \in \Omega ,\ \text{pour tout } t \in I \ ?$$

Ceci fait apparaître l'idée d'itérer les opérateurs definis précédemment *i.e.* d'utiliser les itérés de générateurs et de donner leur signification. Cette idée est nouvelle.

3) Quelques itérés de générateurs et propriétés géométriques: Pour la convexité radiale, $G^2 = (x \cdot \nabla)^2$; pour la convexité perpendiculairement à un hyperplan (π), $G^2 = \left(x_{i_0}\, \partial/\partial x_{i_0}\right)^2$; le générateur de la convexité dans une direction d est

$$G^2 = (d \cdot \nabla)^2 \;=\; \sum_{ij} d_i d_j \, \frac{\partial^2}{\partial x_i \, \partial x_j}\,, \qquad \text{etc} \ldots$$

4) Commutateurs d'opérateurs: Précisons ici que les commutateurs de ces générateurs et des opérateurs différentiels intervenant dans (1.2) jouent un rôle crucial: de la facilité de leur calcul et de leur simplicité dépend la finalité de la réponse. Dans le cas de l'opérateur Δ on a

$$(1.4) \qquad \begin{cases} \Delta(x \cdot \nabla) - x \cdot \nabla(\Delta) = 2\,\Delta\,, \\ (x \cdot \nabla)^2 \Delta - \Delta(x \cdot \nabla)^2 = 4\,\Delta - 4\,\Delta(x \cdot \nabla)\,, \\ \Delta(d \cdot \nabla)^2 - (d \cdot \nabla)^2 \Delta = 0\,. \end{cases}$$

C'est grâce à ces relations que l'on pourra montrer que les fonctions auxiliaires $v = x \cdot \nabla u$, $w_1 = (x \cdot \nabla)^2 u$ et $w_2 = (d \cdot \nabla)^2 u$ satisfont des équations, exploitables, pour lesquelles a lieu le principe du maximum. On rencontre, dans la littérature, d'autres fonctions auxiliaires associées à la géométrie de Ω, comme par exemple dans [2] avec $v = |\nabla u|^2$ et dans [6] avec à la place de w_2 la fonction C définie sur $\Omega \times \Omega$ par

$$C(x,y) = u\Big(\frac{x+y}{2}\Big) - \frac{1}{2}\, u(x) - \frac{1}{2}\, u(y)\,.$$

Il nous semble que ces deux types de fonctions associées à la géométrie de Ω ne soient pas adaptées aux équations du type $(*)$:

i) par exemple, partant de

$$\begin{cases} -\Delta u = f(x,u)\,, & \text{dans } \Omega\,, \\ \quad u|_{\partial\Omega = 0}\,, & \end{cases}$$

l'équation satisfaite par $v = |\nabla v|^2$ est

$$-\Delta(|\nabla u|^2) = -2 \sum_{i,j} \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2 + \frac{\partial f}{\partial u}(x,u)\,|\nabla u|^2 + \sum_i \frac{\partial u}{\partial x_i}\,\frac{\partial f}{\partial x_i}(x,u)\,.$$

La présence, dans cette équation, du terme

$$\sum_i \frac{\partial u}{\partial x_i}\,\frac{\partial f}{\partial x_i}(x,u)$$

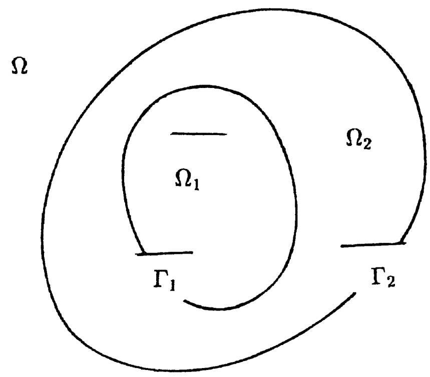
ne permet pas de l'exploiter par application du principe du maximum.

ii) La méthode utilisée dans [6], [9], [10], utilisant $C(x,y)$ ne permet pas de traiter des équations du type $(*)$ du fait que $C(x,y)$ est une fonction de $2 \times n$ variables définie sur $\Omega \times \Omega \subset (\mathbb{R}^n)^2$ (*cf.* preuve dans [10]). L'idée que nous présentons ici a l'avantage d'être systématique pour établir des équations convenables pour l'application du principe du maximum.

II. Résultats pour équations complètement non linéaires à données régulières.

A. Localisation des points critiques.

Nous nous proposons de mettre en œuvre cette idée générale dans un cadre simple: considérons par exemple $x \cdot \nabla$ sur Ω un ouvert régulier tel que $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ où Ω_1 et Ω_2 sont fortement étoilés par rapport à un même point, par exemple, 0. On dira que Ω est "un 2-connexe fortement étoilé". Dans Ω,



on suppose que les équations

$$(2.1) \qquad \begin{cases} F(x,u,\nabla u,\Delta u) = 0\,, \quad u > 0\,, \quad \text{dans } \Omega \\ u|_{\Gamma_1} = c_1\,, \quad u|_{\Gamma_2} = c_2\,, \quad c_1 > c_2 \geq 0\,. \end{cases}$$

possèdent une solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, et que F satisfait l'inégalité

$$(2.2) \qquad x \cdot \frac{\partial F}{\partial x} - 2\frac{\partial F}{\partial s}\, s - \frac{\partial F}{\partial t} \cdot t \leq 0\,, \qquad \text{pour tout } (x,r,t,s)\,.$$

Pour pouvoir appliquer le principe du maximum nous supposons que l'on a

$$-m \leq \frac{\partial F}{\partial r} \leq M(\Omega)\,, \tag{2.2.1}$$

où m est une constante positive et $M(r)$ une constante positive convenable, *cf.* [1].

REMARQUE 2.1. L'opérateur $x \cdot \nabla$ associé à Ω étoilé intervient naturellement dans les identités de type Pohozaev [16].

Nous avons alors le résultat suivant:

Théorème 2.1. *On suppose F de classe C^1 vérifiant* (1.1) *et* (2.2)*; et on suppose que* (2.1) *possède une solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfaisant: il existe, dans Ω, deux voisinages $\mathcal{V}(\Gamma_1)$ et $\mathcal{V}(\Gamma_2)$ tels que*

$$\begin{cases} c_1 = \sup\{u(x) : \ x \in \mathcal{V}(\Gamma_1)\}\,, \\ c_2 = \inf\{u(x) : \ x \in \mathcal{V}(\Gamma_2)\}\,. \end{cases} \tag{2.3}$$

Alors $c_1 = \sup\{u(x) : \ x \in \overline{\Omega}\}$, $c_2 = \inf\{u(x) : \ x \in \overline{\Omega}\}$ et les ensembles de niveau $E(t) = \{x \in \Omega : \ u(x) > t\}$ sont "des 2-connexes fortement étoilés", de plus u ne possède pas de point critique dans Ω.

DÉMONSTRATION. L'utilisation de la première relation (1.4) est essentielle. La fonction $v(x) = x \cdot \nabla u(x)$ vérifie au sens des distributions l'équation suivante

$$-\frac{\partial F}{\partial s}\Delta v - \frac{\partial F}{\partial t} \cdot \frac{\partial v}{\partial x} - \frac{\partial F}{\partial r} v = x \cdot \frac{\partial F}{\partial x} - 2\frac{\partial F}{\partial s}\Delta u - \frac{\partial F}{\partial t} \cdot \nabla u \tag{2.4}$$

où

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial s}(x, u, \nabla u, \Delta u)\,, \qquad \frac{\partial F}{\partial r} = \frac{\partial F}{\partial r}(x, u, \nabla u, \Delta u)\,,$$

$$\frac{\partial F}{\partial t} \cdot \frac{\partial v}{\partial x} = \sum_i \frac{\partial F}{\partial t_i}(x, u, \nabla u, \Delta u)\, \frac{\partial v}{\partial x_i}$$

$$x \cdot \frac{\partial F}{\partial x} = \sum_i x_i \frac{\partial F}{\partial x_i}(x, u, \nabla u, \Delta u)\,.$$

D'après (1.1) u vérifie une équation du type

$$\begin{cases} -\Delta u = g\,, \quad g \geq 0\,, & \text{dans } \Omega\,, \\ u|_{\Gamma_i} = c_i\,, \quad i = 1,2\,. \end{cases}$$

L'hypothèse (2.3) et le principe du maximum de Hopf entraînent que l'on a $v|\Gamma = x \cdot \nabla u|_\Gamma < 0$. Comme l'équation (2.4) obéit au principe du maximum classique grâce aux hypothèses (2.2) et (1.1), il s'ensuit que l'on a $v(x) = x \cdot \nabla u(x) < 0$ dans $\overline{\Omega}$; ce qui prouve que u ne possède pas de point critique dans Ω et que pour tout t, $c_2 < t < c_1$ l'ensemble $\{x \in \Omega:\ u(x) > t\}$ est un 2-connexe.

REMARQUE 2.2. Nous montrerons plus loin (*cf.* Propositions 3.2 et 3.3) que l'hypothèse (2.3) est nécessaire *i.e.* que le résultat ne peut être vrai pour toutes constantes c_1 et c_2 telles que $c_2 < c_1$.

Application: équation du 4ème ordre et système. Supposons qu'il existe une fonction u régulière solution de l'équation

$$\begin{cases} \Delta^2 u = f(x, \Delta u, u) & \text{dans } \Omega = \Omega_2 \setminus \overline{\Omega}_1\,, \\ u|_{\Gamma_1} = 0\,, \quad \Delta u|_{\Gamma_1} = 0\,, \\ u|_{\Gamma_2} = -1\,, \quad \Delta u|_{\Gamma_2} = 1\,, \end{cases}$$

où $f(x,t,r)$ est une fonction régulière, négative.

Posons

$$v = x \cdot \nabla u\,, \quad -\Delta u = \omega\,, \quad s = 2\omega + x \cdot \nabla \omega\,.$$

Un calcul élémentaire montre que ces fonctions vérifient les deux systèmes suivants

$$(S_1) \qquad \begin{cases} -\Delta u = \omega\,, \\ u|_{\Gamma_1} = 0\,, \quad u|_{\Gamma_2} = -1\,, \\ -\Delta \omega = f(x, -\omega, u)\,, \\ \omega|_{\Gamma_1} = 0\,, \quad \omega|_{\Gamma_2} = -1\,, \end{cases}$$

$$(S_2) \qquad \begin{cases} -\Delta s = 4\,f + x \cdot \dfrac{\partial f}{\partial x} - 2\,\dfrac{\partial f}{\partial t}\,\omega + \dfrac{\partial f}{\partial r} \cdot v - \dfrac{\partial f}{\partial t}\,s\,, \\ -\Delta v = s\,, \end{cases}$$

où

$$x \cdot \frac{\partial f}{\partial x} = \sum_i x_i \frac{\partial f}{\partial x_i}(x, -\omega, u), \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}(x, -\omega, u),$$
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial r}(x, -\omega, u).$$

Le principe du maximum de Hopf appliqué à (S_1) entraîne que l'on a

$$\left.\frac{\partial u}{\partial \eta_2}\right|_{\Gamma_2} < 0, \qquad \left.\frac{\partial u}{\partial \eta_1}\right|_{\Gamma_1} > 0,$$

où $\partial/\partial\eta_i$ représente la dérivée dans la direction η_i, normale extérieure à Γ_i. Comme Ω_1 et Ω_2 sont fortement étoilés par rapport à 0, la fonction $v(x) = x \cdot \nabla u$ possède une trace négative sur $\Gamma_1 \cup \Gamma_2$. De même la fonction $x \cdot \nabla\omega$ possède une trace négative sur $\Gamma_1 \cup \Gamma_2$. Ainsi la trace de $s = 2\omega + x \cdot \nabla\omega$ est négative puisque ω est négatif par le principe du maximum classique.

Pour conclure positivement nous faisons les hypothèses suivantes sur la fonction f:

$$\begin{cases} 4\,f(x, -t, r) + x \cdot \dfrac{\partial f}{\partial x}(x, -t, r) - 2\,t\,\dfrac{\partial f}{\partial t}(x, -t, r) \leq 0, \\ \dfrac{\partial f}{\partial r}(x, -t, r) \geq 0, \end{cases}$$

pour tout (x, t, r). Nous sommes alors dans les conditions d'application du résultat de [19] concernant le principe du maximum des systèmes *i.e.* que l'on a

$$s < 0, \qquad v < 0, \qquad \text{dans } \Omega.$$

Ainsi u ne possède pas de points critiques dans $\Omega = \Omega_2 \setminus \overline{\Omega}_1$.

B. Concavité radiale, concavité perpendiculairement à un hyperplan, concavité, symétrie radiale partielle.

Dans cette section nous travaillons sous les hypothèses (1.1) et (2.2.1).

1) *Concavité radiale:* considérons le générateur $(x \cdot \nabla)^2$ associé à la propriété d'être étoilé de Ω. On pose $v = (x \cdot \nabla)^2 u$; pour simplifier

la présentation, on particularise l'équation (*) par

$$(2.4.1) \qquad \begin{cases} -\Delta u = f(x,u)\,, \quad u > 0 \qquad \text{dans } \Omega\,, \\ u|_{\partial\Omega} = 0\,, \end{cases}$$

i.e. on prend $F(x,r,t,s) = s + f(x,r)$.

REMARQUE 2.2.1. Lorsque Ω est une boule de centre 0, pour toute fonction radiale u définie sur Ω, on a

$$(x\cdot\nabla)^2 u = (r\,\frac{d}{dr})^2 u = r\,\frac{d}{dr}(r\,\frac{du}{dr})\,.$$

Si on suppose que $u \in C^4(\Omega)$, la fonction $v = (x\cdot\nabla)^2 u$ vérifie l'équation suivante, établie à l'aide de (1.4):

$$-\Delta v = A(x,u)\,(x\cdot\nabla u)^2 + B(x,u)\,(x\cdot\nabla u) + C(x,u)\,,$$

où

$$A(x,r) = \frac{\partial^2 f}{\partial r^2}(x,r)\,,$$

$$B(x,r) = \frac{\partial^2 f}{\partial r^2}(x,r) + x\cdot\frac{\partial^2 f}{\partial x\partial r}(x,r) + 5\,\frac{\partial f}{\partial r}(x,r)\,,$$

$$C(x,r) = 5\,x\,\frac{\partial f}{\partial x}(x,r) + {}^t x\cdot\frac{\partial^2 f}{\partial x^2}(x,r)\cdot x + 4\,f(x,r)\,.$$

On suppose que f vérifie les hypothèses suivantes:

$$(2.4.2) \qquad \begin{array}{l} A(x,r) \le 0\,, \qquad C(x,r) \le 0\,, \\ B^2(x,r) \le 4\,A(x,r)\cdot C(x,r)\,, \qquad \text{pour tout } (x,r)\,. \end{array}$$

Théorème 2.1.1. *On suppose que u solution de* (2.4.1) *est C^4, que f vérifie* (2.4.2) *et Ω fortement étoilé par rapport à 0. Alors la fonction concavité radiale $v = (x\cdot\nabla)^2 u$ ne peut atteindre son maximum à l'intérieur de Ω.*

La preuve de ce résultat est semblable à celle du Théorème 2.1. Signalons que, à notre connaissance, cette notion de concavité partielle ne semble pas avoir été abordée auparavant.

2) *Concavité:* On se donne Ω convexe. Ici la situation est plus complexe: on travaille avec une famille de générateurs de concavité *i.e.* on considère

$$(\alpha \times \nabla)^2 = \sum_{i,j} \alpha_i \, \alpha_j \, \frac{\partial^2}{\partial x_i \, \partial x_j} \,,$$

pour toute direction α de dérivation. On suppose que l'équation

$$\begin{cases} F(x, u, \nabla u, \Delta u) = 0 & \text{dans } \Omega\,, \\ u|_{\partial\Omega} = 0\,, \quad u > 0\,, \end{cases} \tag{2.5}$$

possède une solution u.

Définition 2.1. *Pour toute direction α on pose*

$$\omega(\alpha, u) = \omega_\alpha = \omega = (\alpha \times \nabla)^2 u \,.$$

Nous appellerons $\omega(\alpha, u)$ fonction de concavité (ou convexité suivant le cas).

Nous allons chercher, à partir de (2.5), l'équation satisfaite par la famille de fonctions ω_α. Vu le commutateur correspondant (*cf.* (1.4)) les résultats sont, en définitive, simples.

On pose $G(x) = F(x, u(x), \nabla u(x), \Delta u(x))$. Nous utiliserons la notation classique suivante

$$\frac{\partial G(x)}{\partial x_i} = \frac{d}{dx_i} \left[F(x, u(x), \nabla u(x), \Delta u(x))\right]$$

ce qui donne

$$(\alpha \times \nabla)^2 \, G(x) = (\alpha \times D)^2 \, F(x, u(x), \nabla u(x), \Delta u(x))$$

où

$$\alpha \times D = \sum_i \alpha_i \, \frac{d}{dx_i} \,.$$

Pour simplifier, nous écrirons

$$(\alpha \times \nabla)^2 G = (\alpha \times D)^2 F \,.$$

Proposition 2.1.

$$(\alpha \times D)F = \alpha \cdot \frac{\partial F}{\partial x} + \frac{\partial F}{\partial r}(\alpha \times \nabla u) + \frac{\partial F}{\partial t} \cdot \nabla(\alpha \times \nabla u) + \frac{\partial F}{\partial s} \Delta(\alpha \times \nabla u)$$

où

$$\alpha \cdot \frac{\partial F}{\partial x} = \sum_i \alpha_i \frac{\partial F}{\partial x_i}, \qquad \frac{\partial F}{\partial t} \cdot \nabla = \sum_j \frac{\partial F}{\partial t_j} \frac{\partial}{\partial x_j}.$$

La preuve est élémentaire.

Proposition 2.2. *L'équation dérivée de* (1)*, vérifiée par* $\omega(\alpha, u) = \omega$ *est*

$$-\frac{\partial F}{\partial s} \Delta\omega - \frac{\partial F}{\partial t} \cdot \nabla\omega - \frac{\partial F}{\partial r}\omega = (\alpha, a, b, c)^t \cdot \operatorname{Hess} F_{(x,u,\nabla u,\Delta u)} \cdot (\alpha, a, b, c)$$

où $\operatorname{Hess} F_{(x,u,\nabla u,\Delta u)}$ *est la matrice Hessienne de* $F(x, r, t, s)$ *évaluée au point* $(x, u(x), \nabla u(x), \Delta u(x))$ *et*

$$a = \alpha \cdot \nabla u, \quad b = \nabla(\alpha \cdot \nabla u), \quad c = \Delta(\alpha \cdot \nabla u).$$

DÉMONSTRATION. $(\alpha \times D)^2 F = (\alpha \times D)(\alpha \times D)F$. On applique la proposition précédente aux différents termes de $(\alpha \times D)F$.

Théorème 2.2. *On suppose que u solution de* (2.5) *appartient à* $C^4(\overline{\Omega})$ *et* $\operatorname{Hess} H_{(x,r,t,s)}$ *est négative. Alors pour toute direction* α *la fonction* $\omega(\alpha, u)$ *ne peut atteindre son maximum à l'intérieur de* Ω.

La preuve de ce résultat découle d'une version du principe du maximum classique donnée dans [1].

REMARQUES 2.3. 1) On peut remplacer l'opérateur Δ par tout opérateur $A = \sum a_{ij}\, \partial^2/\partial x_i \partial x_j$ elliptique uniformément, à coefficients constants.

2) Si u est concave au voisinage du bord $\partial\Omega$ alors le théorème précédent dit que u est concave sur Ω.

REMARQUE 2.4. En réalité nous avons

$$(\alpha, a, b, c)^t \cdot \operatorname{Hess} F_{(x,u,\nabla u,\Delta u)} \cdot (\alpha, a, b, c) = Q(x,u)(\alpha,\alpha)\,,$$

où $Q(x,u)(\cdot,\cdot)$ est une forme quadratique dépendant de x et u. La condition optimale de validité du théorème est donc

$$Q(x,u)(\alpha,\alpha) \le 0\,, \qquad \text{pour tout } \alpha \in \mathbb{R}^n\,;$$

elle dépend a priori de la solution u. La condition du Théorème 2.2 retenue ci-dessus est suffisante.

Comparaison avec les résultats existants. Prenons l'exemple suivant

$$F(x,r,t,s) = s + f(x,r)\ ,$$

$$\text{(2.6)} \qquad \begin{cases} -\Delta u = f(x,u) & \text{dans } \Omega\,, \qquad f(x,r) \le 0\,, \\ u|_{\partial\Omega} = 0\,, \quad u < 0\,. \end{cases}$$

L'équation dérivée de (2.6) est

$$\begin{aligned} -\Delta\omega - \frac{\partial f}{\partial r}\,\omega &= {}^t\alpha \cdot \frac{\partial^2 f}{\partial x^2} \cdot \alpha + 2\,\alpha \cdot \frac{\partial^2 f}{\partial x\,\partial r} \cdot (\alpha \cdot \nabla u) \\ &\quad + \frac{\partial^2 f}{\partial r^2}\,(\alpha \cdot \nabla u)^2 \\ &= Q(x,u)(\alpha,\alpha)\,. \end{aligned}$$

Posons pour tout $i,j \in \{1,2,\ldots,n\}$

$$q_{ij}(x,u) = \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x,u) + 2\,\frac{\partial^2 f}{\partial x_i\,\partial r}(x,u)\,\frac{\partial u}{\partial x_j} + \frac{\partial^2 f}{\partial r^2}(x,u)\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\ .$$

Si la solution u de (2.6) vérifie la condition

$$\text{(2.7)} \qquad \begin{cases} \text{la forme quadratique } Q = (q_{ij}(x,u)) \text{ est positive,} \\ \text{non identiquement nulle,} \end{cases}$$

le théorème précédent est valide *i.e.* la fonction de convexité $\omega(\alpha, u) = (\alpha \cdot \nabla)^2 u$ ne peut atteindre son minimum à l'intérieur de Ω.

Si on pose

$$R = \|\nabla u\|_{L^\infty(\Omega)} = \sup_{x\in\overline{\Omega}} \|\nabla u(x)\|\,,$$

une condition suffisante assurant (2.7) est

$$\text{(2.8)} \qquad \begin{cases} \text{pour tout } \ (x,r,t) \in \overline{\Omega} \times \mathbb{R}^- \times B(0,R)\,, \\ \text{la forme quadratique } Q = (q_{ij}(x,r,t))_{ij} \text{ est positive} \end{cases}$$

avec

$$q_{ij}(x,r,t) = \frac{\partial^2 f}{\partial x_i\, \partial x_j}(x,r) + 2\,\frac{\partial^2 f}{\partial x_i\, \partial r}(x,r)\, t_j + \frac{\partial^2 f}{\partial r^2}(x,r)\, t_i\, t_j\ .$$

Le premier résultat de ce type est dû à N. Korevaar ([5], [6]) qui a introduit un principe de concavité en définissant sur $\Omega \times \Omega$ la fonction de concavité suivante

$$C(x,y) = u\Big(\frac{x+y}{2}\Big) - \frac{1}{2}\,u(x) - \frac{1}{2}\,u(y)\ .$$

Cette idée a ensuite été utilisée par divers auteurs, *cf.* [10], [9] par exemple.

Cette fonction ne peut atteindre un maximum positif dans $\Omega \times \Omega$ si $-f(x,r)$ est strictement croissante par rapport à r et si

$$\text{(2.9)} \qquad -\frac{1}{f(x,r)} \quad \text{est convexe en } \ (x,r)\,.$$

Notre résultat ne nécessite pas la stricte croissance par rapport à r. Ce qui autorise, par exemple, des équations de la forme $f(x, \nabla u) = -\Delta u$, exclues dans [9]. La comparaison de (2.8) et (2.9) semble difficile. Cependant il convient de remarquer que si $-f(x,r)$ est concave alors (2.8) et (2.9) sont vérifiées simultanément. Enfin la méthode utilisée ne semble pas s'adapter aux équations complètement non linéaires, *cf.* la preuve dans [10, p. 116].

REMARQUE 2.5. Pour les équations du type (2.6) le résultat le plus général semble appartenir à [9]; la preuve de ce résultat a été améliorée par [10]; et c'est à cette dernière que l'on se réfère.

REMARQUE 2.6. Lorsque $f(x, r)$ n'est pas strictement monotone par rapport à r, dans [9] A. U. Kennington s'y ramène en faisant le changement de fonction puissance $v = u^\alpha$, $\alpha > 0$. En opérant de la sorte la méthode de Korevaar-Kennington-Kawohl perd de l'information dans le cas des solutions régulières comme le montrent les exemples qui suivent.

EXEMPLE 1. $\Omega =$ Boule unité de $\mathbb{R}^2$.

$$\begin{cases} -\Delta u = -16\,k(x^2+y^2)+1\,, \\ u|_{\partial\Omega} = 0\,, \quad 0 < k < \dfrac{1}{32}\,. \end{cases}$$

La solution $u(x, y) = -k\,(1-(x^2+y^2)) + (1-(x^2+y^2))/2$ est concave. La méthode citée ci-dessus que nous désignerons par commodité par méthode K donne que $\sqrt{u}$ est concave.

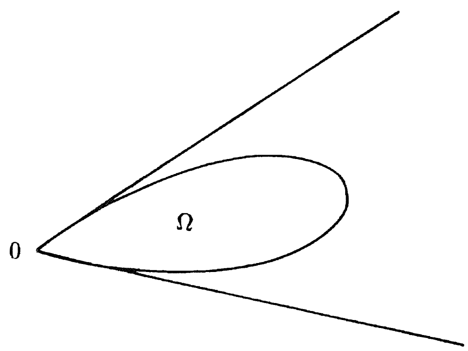
EXEMPLE 2. $\Omega =$ Ellipse de grand axe porté par l'axe $\overrightarrow{Ox}$ et d'excentricité 1/2.

$$\begin{cases} -\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{4}\,\dfrac{\partial^2 u}{\partial y^2}\right) = 1\,, \\ u|_{\partial\Omega} = 0\,. \end{cases}$$

La méthode K entraîne que $(u)^\alpha$ est concave avec $\alpha = 1/2$, puissance optimale -pour la méthode- alors que la solution $u(x, y) = (1 - x^2 - 4y^2)/2$ est concave.

Par contre si Ω n'est pas régulier, par exemple si Ω présente un point anguleux, le résultat obtenu dans [9] est optimal:

$$\begin{cases} -\Delta u = 1\,, \\ u|_{\partial\Omega} = 0\,. \end{cases}$$



La solution u n'est pas C^2 au voisinage de 0, elle n'est pas concave et $(u)^\alpha$ est concave, où $\alpha = 1/2$ est une puissance optimale. Ainsi la méthode K perd de l'information dans le cas régulier quand f n'est pas strictement monotone par rapport à u.

REMARQUE 2.7. De la même façon on peut étudier la concavité dans une direction d. Nous ne le ferons pas afin d'éviter de se répéter.

3. Groupe de symétrie: une remarque sur la symétrie radiale partielle:

On se donne un ouvert Ω de $\mathbb{R}^n$ possédant un groupe de symétrie:

$$\text{pour tout } x \in \Omega\,, \qquad y = R\cdot x \in \Omega\,,$$

pour toute transformation orthogonale R appartenant à un groupe G (fini ou pas).

Soit une fonction $F(x,r,t,s)$ de $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ dans $\mathbb{R}$ telle que

$$\begin{cases} \text{pour tous } x \in \Omega\,,\ (r,t,s) \in \mathbb{R}^{n+2}\,,\ R \in G\,, \\ F(R\cdot x, r, R\cdot t, s) = F(x,r,t,s)\,. \end{cases} \tag{2.10}$$

Soit u une solution de l'équation

$$\begin{cases} F(x,u,\nabla u, \Delta u) = 0\,, \\ u|_{\partial\Omega} = 0 \quad \text{(Dirichlet)} \quad \text{ou} \quad \dfrac{\partial u}{\partial \eta}\Big|_{\partial\Omega} = 0 \quad \text{(Neumann).} \end{cases} \tag{2.11}$$

Théorème 2.3. *Supposons que u est solution unique de* (2.11)*; alors u vérifie $u(R\cdot x) = u(x)$, pour tous $x \in \Omega$, $R \in G$, i.e. u possède le même groupe de symétrie que Ω.*

DÉMONSTRATION. On montre aisément que

$$\nabla(u(R\cdot x)) = {}^t R\,\nabla u(R\cdot x)\,, \qquad \text{pour tout } x \in \Omega\,,$$

où ${}^tR$ est la transposée de $R = (R)^{-1}$. Cette relation entraîne que la condition de Neumann est invariante par R,

$$\Delta[u(R\cdot x)] = (\Delta u)(R\cdot x)\,, \qquad \text{pour tout } x \in \Omega\,.$$

Comme l'équation (2.11) s'écrit aussi

$$F\big(R\cdot x\,,\ u(R\cdot x)\,,\ \nabla u(R\cdot x)\,,\ (\Delta u)(R\cdot x)\big) = 0\,,$$

puisque $R\,\Omega = \Omega$, on obtient

$$F\big(R\cdot x\,,\ u(R\cdot x)\,,\ R\cdot\nabla(u(R\cdot x))\,,\ \Delta[u(R\cdot x)]\big) = 0\,,$$

soit en utilisant l'hypothèse (2.10)

$$F\big(x\,,\ u(R\cdot x)\,,\ \nabla(u(R\cdot x))\,,\ \Delta[u(R\cdot x)]\big) = 0\,,$$

i.e. $v(x) = u(R\cdot x)$ est aussi une solution de (2.11); l'unicité entraîne que

$$u(R\cdot x) = u(x)\,, \qquad \text{pour tous } x \in \Omega\,,\ R \in G\,.$$

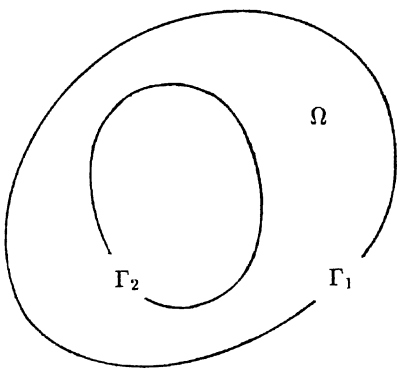
III. Problème semi-linéaire à donnée peu régulière.

A. f dépend de u.

Considérons le problème

$$\text{(3.0)} \qquad \begin{cases} -\Delta u = f(x,u) & \text{dans} \quad \Omega = \Omega_1 \setminus \overline{\Omega}_2\,, \\ u|_{\Gamma_1} = c_1 = 0\,, \quad u|_{\Gamma_2} = c_2\,, & c_2 > 0\,, \end{cases}$$

où Ω_1 et Ω_2 sont deux ouverts bornés réguliers connexes tels que $\overline{\Omega}_2 \subset \Omega_1$ et $\Gamma_i = \partial\Omega \cap \partial\Omega_i$



On suppose que $f(x,t)$ est de Carathéodory, positive, bornée:

$$\text{(3.1)} \qquad 0 \le f(x,t) \le M\,,$$

presque pour tout $x \in \Omega$, pour tout $t \in \mathbb{R}$. Il est connu [26] qu'il existe α, $0 < \alpha < 1$ tel que la solution u de (3.0) soit dans $C^{1,\alpha}(\overline{\Omega})$.

On suppose que cette solution u vérifie

$$2\,f(x,u) + \sum_i x_i \frac{d}{dx_i}\,[f(x,u)] \leq a \tag{3.2}$$

au sens de $H^{-1}(\Omega)$, où $a \in L^2(\Omega)$, est négative, non nulle;

$$\sup\left\{u(x) :\ x \in \overline{\Omega}\right\} = c_2\,. \tag{3.2.1}$$

REMARQUE 3.1. Lorsque f est indépendant de t (3.2) se réduit à

$$2\,f(x) + \sum_i x_i \frac{\partial f}{\partial x_i} \leq a \qquad \text{dans } H^{-1}(\Omega)\,.$$

Un exemple intéressant d'une telle fonction sera donné plus tard.

Théorème 3.1. *On suppose que Ω est un 2-connexe fortement étoilé par rapport à un point, $0 \in \Omega_2$, par exemple. Alors si u et f satisfont* (3.1) *à* (3.2.1), *les ensembles de niveau de u sont 2-connexes. De façon plus précise on a*

$$\inf\left\{\|\nabla u(x)\| :\ x \in \overline{\Omega}\right\} > 0\,. \tag{3.3}$$

REMARQUE 3.2. Par le principe du maximum de Hopf (*cf.* [1]) on a

$$x \cdot \nabla u|_\Gamma < 0\,, \qquad \Gamma = \Gamma_1 \cup \Gamma_2\,, \tag{3.3.1}$$

en utilisant (3.2.1).

DÉMONSTRATION. Nous allons précéder par régularisation. On se donne $f_n(x,t)$ une suite de fonctions régulières telles que

$$0 \leq f_n(x,t) \leq M+1\,, \qquad f_n(x,t) \to f(x,t)\,,$$

presque pour tout $x \in \Omega$, pour tout $t \in \mathbb{R}$, et on considère les deux problèmes suivants:

$$\begin{cases} -\Delta u_n = f_n(x,u)\,, \\ u_n|_{\Gamma_1} = c_1\,, \quad u_n|_{\Gamma_2} = c_2\,; \end{cases} \tag{3.4}$$

$$\begin{cases} -\Delta v_n = 2\,f_n(x,u) + \sum_i x_i \dfrac{d}{dx_i}\,[f_n(x,u)]\,, \\ v_n|_\Gamma = x \cdot \nabla u_n|_\Gamma\,, \qquad \text{où} \quad \Gamma = \Gamma_1 \cup \Gamma_2\,. \end{cases} \tag{3.5}$$

La fonction u_n étant régulière, par un calcul élémentaire de $\Delta(x \cdot \nabla u_n)$, en utilisant (1.4) on voit à partir de (3.4) que la fonction $x \cdot \nabla u_n$ est également solution de (3.5). L'unicité du problème de Dirichlet (3.5) entraîne que

$$v_n = x \cdot \nabla u_n\,. \tag{3.6}$$

Il s'agit maintenant de passer à la limite dans (3.4), (3.5) et (3.6). Il est aisé de voir que, à une sous-suite près, le problème (3.4) tend vers le problème

$$\begin{cases} -\,\Delta u = f(x,u)\,, \\ u|_{\Gamma_1} = c_1\,, \quad u|_{\Gamma_2} = c_2\,, \end{cases} \tag{3.9}$$

grâce à l'unicité du problème de Dirichlet. D'autre part la distribution

$$h_n = 2\,f_n(x,u) + \sum_i x_i \frac{d}{dx_i}\,[f_n(x,u)]$$

appartient à un borné de $H^{-1}(\Omega)$. Par conséquent le problème (3.5) possède une unique solution v_n bornée dans $H^1(\Omega)$ si on montre que $v_n|_\Gamma$ est borné dans $L^\infty(\Gamma)$; ce que nous ferons plus loin. Par conséquent à une sous-suite près $v_n = x \cdot \nabla u_n$ tend dans $H^1(\Omega)$ fort vers $v = x \cdot \nabla u$ solution de

$$\begin{cases} -\,\Delta v = 2\,f(x,u) + \sum_i x_i \dfrac{d}{dx_i}\,[f(x,u)]\,, \\ v|_\Gamma = x \cdot \nabla u|_\Gamma < 0\,. \end{cases} \tag{3.10}$$

Considérons alors la solution ω du problème suivant:

$$\begin{cases} -\,\Delta \omega = a \qquad \text{dans } \Omega\,, \\ \omega|_\Gamma = v|_\Gamma < 0\,; \end{cases} \tag{3.11}$$

elle satisfait, grâce au principe du maximum fort [22],

$$\omega < 0 \qquad \text{dans } \Omega . \tag{3.12}$$

Enfin, grâce à (3.2) on peut appliquer le principe de comparaison [22] dans $H_0^1(\Omega)$ *i.e.* on a

$$v - \omega \leq 0 \tag{3.13}$$

au sens de $H_0^1(\Omega)$, [22]; (3.12) et (3.13) entraînent que

$$x \cdot \nabla u(x) < 0 \qquad \text{dans } \overline{\Omega} . \tag{3.14}$$

Ainsi la preuve du théorème sera achevée si on montre que la trace de v_n sur Γ est bornée par exemple dans $L^\infty(\Gamma)$. Pour cela nous aurons besoin d'un résultat d'estimations établi dans [26, Théorème 14.1, p. 337]: puisque Ω est régulier, il satisfait la condition de la sphère extérieure en tout point de $\partial\Omega$; soit $r_0 > 0$ le rayon de cette sphère; la condition (14.9) de [26] est trivialement satisfaite dans le cas de l'équation (3.4). Ainsi nous obtenons l'estimation escomptée

$$\|\nabla u_n(x)\| \leq C , \qquad \text{pour tout } \ x \in \Gamma ,$$

où C est une constante ne dépendant que de r_0, M et c_2 .

B. f est indépendant de u.

Ici nous avons besoin d'un résultat préliminaire de trace.

1. Un résultat de trace. Considérons l'espace fonctionnel

$$E = \left\{ g \in L^2(\Omega) : \ x \cdot \nabla g(x) \in L^2(\Omega) \right\} ,$$

où $x \cdot \nabla g = \sum_i x_i \, \partial g / \partial x_i$ s'entend au sens des distributions. Muni de sa structure naturelle E est un espace de Hilbert. Nous avons le résultat suivant.

Lemme 3.1. *Supposons que Ω soit un ouvert borné régulier de $\mathbb{R}^n$, fortement étoilé. Alors pour tout g de E, on peut définir une trace sur le bord $\Gamma = \partial\Omega$, au sens de $H^{-1/2}(\Gamma)$.*

La preuve de ce résultat nécessite deux étapes.

1ÈRE ÉTAPE: UN RÉSULTAT DE DENSITÉ. Ici Ω n'est pas nécessairement borné.

Lemme 3.2. *$\mathcal{D}(\overline{\Omega})$ est dense dans E.*

Pour la démonstration de ce résultat de densité nous utiliserons le résultat suivant dû à Friedrichs [28].

Lemme 3.3. *Soit $\rho \in \mathcal{D}(\mathbb{R}^m)$, $\rho \geq 0$ tel que $\int_{\mathbb{R}^m} \rho\, dx = 1$; soit $v \in L^2(\mathbb{R}^m)$ à support compact, et $b \in C^1$ dans un voisinage du support de v. Alors on a, pour tout $k = 1, \ldots, m$,*

$$b \frac{\partial}{\partial x_k} (v * \rho_n) - \left(b \frac{\partial v}{\partial x_k}\right) * \rho_n$$

tend vers 0 dans $L^2(\mathbb{R}^m)$ fort quand $n \to +\infty$, avec

$$w * \rho_n(x) = \int_{\mathbb{R}^m} w\left(x - \frac{1}{n} y\right) \rho(y)\, dy$$

pour tout w dans $L^2(\mathbb{R}^m)$.

DÉMONSTRATION DU LEMME 3.2. Soit g appartenant à E; le Lemme 3.3 entraîne que

$$x \cdot \nabla(g * \rho_n) - (x \cdot \nabla g) * \rho_n \to 0$$

dans $L^2(\Omega)$ fort; or par définition de E

$$(x \cdot \nabla g) * \rho_n \ \to \ x \cdot \nabla g$$

dans $L^2(\Omega)$ fort; par conséquent

$$x \cdot \nabla(g * \rho_n) \to x \cdot \nabla g$$

dans $L^2(\Omega)$ fort. Ainsi la suite $g_n = g * \rho_n$ appartient à $\mathcal{D}(\overline{\Omega})$ et tend vers g dans E pour la topologie de ce dernier.

REMARQUE 3.3. Quand Ω n'est pas $\mathbb{R}^n$ le produit de convolution peut être défini comme dans [27].

2ÈME ÉTAPE: DÉMONSTRATION DU LEMME 3.1. Pour tout $i \in \{1, 2, \ldots, n\}$, posons

$$p_i(x) = x_i\, f(x)\,.$$

Il est clair que par définition $p = (p_1, \ldots, p_n) \in (L^2(\Omega))^n$. De plus par définition de E, $\operatorname{div} p \in L^2(\Omega)$.

Par conséquent, d'après [27], pour tout $R > 0$, $\vec{p}\cdot\vec{\nu} = f(x)\cdot(\vec{x}\cdot\vec{\nu}(x))$ possède sur $\partial\Omega \cap B(0, R)$, une trace dans $H^{-1/2}(\Gamma)$, où $\vec{\nu}$ représente la normale unitaire à $\partial\Omega$, orientée extérieurement. De plus on a

$$\|\vec{p}\cdot\vec{\nu}\|_{H^{-1/2}(\Gamma)} \leq C\left(\|x\, f\|_{L^2(\Omega)} + \|\operatorname{div}(x\, f)\|_{L^2(\Omega)}\right).$$

Avant de conclure il nous faut construire un relèvement de $\vec{p}\cdot\vec{\nu}$. Pour cela considérons le problème aux limites suivant:

$$\begin{cases} -\Delta w = 0\,, \\ w|_{\partial\Omega} = a(x)\,, \end{cases}$$

où $a(x) = \vec{x}\cdot\vec{\nu}(x)$. Comme Ω est régulier, étoilé et borné, $a(x)$ est $C^\infty(\partial\Omega)\cap L^\infty(\partial\Omega)$ telle que

$$a_1 \geq a(x) \geq a_0 > 0 \qquad \text{sur } \partial\Omega\,.$$

Des résultats classiques de régularité des problèmes elliptiques [26], [28] on déduit que $w \in C^\infty(\overline{\Omega})$. Posons alors

$$h(x) = \frac{f(x)}{w(x)}\,.$$

Il est clair que $h \in E$ et que d'après ce qui précède on peut définir $h\cdot(\vec{x}\cdot\vec{\nu})$ sur $\Gamma = \partial\Omega$ *i.e.*

$$\begin{aligned}(\vec{x}\cdot\vec{\nu})\, h|_{\partial\Omega} &= (\vec{x}\cdot\vec{\nu})\left(\frac{f}{w}\right)\Big|_{\partial\Omega} \\ &= \left(\frac{f(x)}{w(x)}\right) a(x)|_{\partial\Omega} = \vec{p}\cdot\vec{\nu}|_{\partial\Omega} \in H^{-1/2}(\Gamma)\end{aligned}$$

avec $p = x\,(f/w)$, et d'après [27]

$$\|(\vec{x}\cdot\vec{\nu})\, h\|_{H^{-1/2}(\Gamma)} \leq C\left(\left\|x\,\frac{f}{w}\right\|_{L^2(\Omega)} + \left\|\operatorname{div}\left(x\,\frac{f}{w}\right)\right\|_{L^2(\Omega)}\right).$$

Comme par le principe du maximum on a

$$\inf \left\{ w(x) : \ x \in \overline{\Omega} \right\} \geq a_0 > 0 ,$$

il s'ensuit, moyennant quelques estimations élémentaires, que

$$\|(\vec{x}\cdot\vec{\nu})\, h\|_{H^{-1/2}(\Gamma)} \leq C \left(\|x\, f\|_{L^2(\Omega)} + \|x \cdot \nabla f\|_{L^2(\Omega)} \right)$$

i.e.

$$\|(\vec{x}\cdot\vec{\nu})\, h\|_{H^{-1/2}(\Gamma)} \leq C \, \|f\|_E \, .$$

Considérons alors l'application linéaire suivante définie sur $\mathcal{D}(\overline{\Omega})$ pour

$$\begin{aligned} \tilde{\gamma}_0 : \mathcal{D}(\overline{\Omega}) &\longrightarrow H^{-1/2}(\Gamma) \\ f &\longrightarrow \tilde{\gamma}_0 f = \left(\frac{f}{w} \right) \cdot (\vec{x}\cdot\vec{\nu})|_{\partial\Omega} \, . \end{aligned}$$

Il est clair que pour tout $f \in \mathcal{D}(\overline{\Omega})$ on a $\tilde{\gamma}_0 f = f|_{\partial\Omega}$ (au sens de la restriction) et que $\tilde{\gamma}_0$ et continue. Le Lemme 3.2 et le Théorème de Hahn-Banach entraînent que $\tilde{\gamma}_0$ se prolonge de façon unique à E en une application linéaire continue telle que (*cf.* [20])

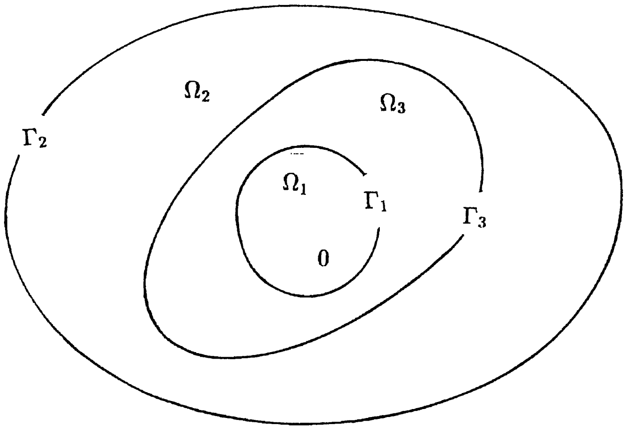
$$\|\tilde{\gamma}_0 f\|_{H^{-1/2}(\Gamma)} \leq C \, \|f\|_E \, , \qquad \text{pour tout } f \in E \, .$$

On écrira, pour simplifier les notations,

$$\tilde{\gamma}_0 f = f|_{\partial\Omega} \, .$$

2. Un résultat de connexité adapté aux problèmes de contrôle.

Considérons trois ouverts Ω_i, $(i = 1, 2, 3)$, possédant les propriétés suivantes: ils sont fortement étoilés par rapport à 0, réguliers tels que $\Omega_2 \supset \overline{\Omega}_3$, $\Omega_3 \supset \overline{\Omega}_1$. On pose



$$D = \Omega_3 \setminus \overline{\Omega}_1 , \qquad \Omega = \Omega_2 \setminus \overline{\Omega}_1 .$$
$$\Gamma_i = \partial\Omega_i , \qquad i = 1,2,3 .$$

On se donne une fonction f appartenant à $L^\infty(\Omega)$ telle que

(3.15) $$f(x) = \begin{cases} f_1(x), & \text{si } x \in D \text{ presque partout,} \\ f_2(x), & \text{si } x \in \Omega \setminus \overline{D} \text{ presque partout,} \end{cases}$$

(3.16) $$T_1 = \sum_i x_i \frac{\partial f_1}{\partial x_i} \geq 0 \qquad \text{dans } \mathcal{D}'(D),$$

(3.17) $$T_2 = \sum_i x_i \frac{\partial f_2}{\partial x_i} \geq 0 \qquad \text{dans } \mathcal{D}'(\Omega \setminus \overline{D}) ,$$

(3.18) $$T_1 \in L^2(D), \quad T_2 \in L^2(\Omega \setminus \overline{D}).$$

(3.19) $$f_1|_{\partial D} \leq f_2|_{\partial D} ,$$

au sens suivant des traces

$$\langle f_2 - f_1, \varphi\rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)} \geq 0, \qquad \text{pour tout } \varphi \in \mathcal{D}(\overline{\Omega}) .$$

Proposition 3.1. *On suppose les hypothèses* (3.15) *à* (3.19). *Alors la distribution* $T = \sum_i x_i\, (\partial f/\partial x_i)(x)$ *appartient à* $H^{-1}(\Omega)$ *et vérifie*

$$T = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} \geq 0 \qquad \textit{dans } \mathcal{D}'(\Omega) .$$

DÉMONSTRATION. Rappelons que d'après le Lemme 3.1 les traces de f_1 et f_2 sur ∂D et $\partial(\Omega \setminus \overline{D})$ respectivement sont bien définies.

On peut donc, après régularisation, faire des intégrations par parties et passer à la limite. Posons

$$p_i = x_i f, \qquad \text{pour tout } i = 1, 2, \ldots, n;$$

on a

$$p = (p_1, \ldots, p_n) = x f = \begin{cases} x f_1 & \text{sur } D, \\ x f_2 & \text{sur } \Omega \setminus \overline{D}. \end{cases}$$

Nous avons au sens des distributions

$$\operatorname{div} p = n f + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}. \tag{3.20}$$

Soit p_k une suite régulière telle que l'on ait [27]

$$\begin{aligned} p_k &\to p && \text{dans } (L^2(D))^n \text{ fort}, \\ \operatorname{div} p_k &\to \operatorname{div} p && \text{dans } L^2(D). \end{aligned}$$

Pour tout $\varphi \geq 0$ appartenant à $\mathcal{D}(\Omega)$, nous avons

$$\begin{aligned} \int_D p_k \cdot \nabla\varphi \, dx = &-\langle p_k \cdot \nu_1, \varphi \rangle_{H^{-1/2}(\Gamma_3) \times H^{1/2}(\Gamma_3)} \\ &+ \int_D \operatorname{div} p_k \, \varphi \, dx. \end{aligned} \tag{3.21}$$

On passe à la limite dans (3.21) en utilisant le résultat de continuité [27] suivant:

$$p_k \cdot \nu_1|_{\Gamma_3} \to p \cdot \nu_1|_{\Gamma_3} = f_1(x \cdot \nu_1)|_{\Gamma_3}$$

dans $H^{-1/2}(\Gamma_3)$, *i.e.* que l'on obtient

$$\begin{aligned} \int_D p \cdot \nabla\varphi \, dx = &-\langle f_1(x \cdot \nu_1), \varphi \rangle_{H^{-1/2}(\Gamma_3) \times H^{1/2}(\Gamma_3)} \\ &+ \int_D \operatorname{div} p \, \varphi \, dx. \end{aligned} \tag{3.22}$$

De même nous obtenons

$$\begin{aligned} -\int_{\Omega \setminus \overline{D}} p \cdot \nabla\varphi \, dx = &-\langle f_2(x \cdot \nu_2), \varphi \rangle_{H_{1/2}(\Gamma_3) \times H^{1/2}(\Gamma_3)} \\ &+ \int_{\Omega \setminus \overline{D}} \operatorname{div} p \, \varphi \, dx. \end{aligned} \tag{3.23}$$

Ainsi pour tout $\varphi \geq 0$, dans $\mathcal{D}(\Omega)$ on a

$$\begin{aligned}\langle \operatorname{div} p, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} &= -\langle p, \nabla\varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} \\ &= -\int_D p\cdot\nabla\varphi\, dx - \int_{\Omega\setminus\overline{D}} p\cdot\nabla\varphi\, dx \\ &= \langle (f_2-f_1)(x\cdot\nu), \varphi\rangle_{H^{-1/2}(\Gamma_3)\times H^{1/2}(\Gamma_3)} \\ &\quad + \int_D \operatorname{div} p\ \varphi\, dx + \int_{\Omega\setminus\overline{D}} \operatorname{div} p\ \varphi\, dx\end{aligned}$$

où $\nu_2 = -\nu_1 = -\nu$, ν étant la normale unitaire à ∂D orientée extérieurement. Or par (3.18) nous avons

$$\begin{aligned}&\operatorname{div} p = nf_1 + T_1 \in L^2(D) \qquad \text{dans } \mathcal{D}'(D)\,,\\ &\operatorname{div} p = nf_2 + T_2 \in L^2(\Omega\setminus\overline{D}) \qquad \text{dans } \mathcal{D}'(\Omega\setminus\overline{D})\,,\end{aligned}$$

d'où

$$\begin{aligned}\langle \operatorname{div} p, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} &= \int_{\partial D} (f_2-f_1)(x\cdot\nu)\,\varphi\, d\sigma + \int_\Omega nf\,\varphi\, dx \\ &\quad + \int_D T_1\varphi\, dx + \int_{\Omega\setminus\overline{D}} T_2\varphi\, dx\,;\end{aligned} \tag{3.24}$$

mais on a

$$\int_{\partial D} (f_2-f_1)(x\cdot\nu)\,\varphi\, d\sigma \geq 0$$

par (3.19) et puisque D est fortement 2-étoilé. Ainsi (3.17) et (3.24) entraînent que l'on a

$$\langle \operatorname{div} p, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} \geq n\int_\Omega f\,\varphi\, dx\,, \tag{3.25}$$

pour tout $\varphi \geq 0$, $\varphi \in \mathcal{D}(\Omega)$; la conclusion vient de (3.25) et (3.20).

REMARQUE 3.3.1. Si dans (3.16), (3.17) et (3.19) les inégalités sont inversées la proposition précédente donne

$$T = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} \leq 0 \qquad \text{dans } \mathcal{D}'(\Omega)\,.$$

Théorème 3.2. *Soit* $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ *un 2-connexe fortement étoilé par rapport à 0, régulier. Soit* f *une fonction positive, dans* $L^\infty(\Omega)$, *définie par* (3.15) *et telle que* $-f_1$ *et* $-f_2$ *vérifient* (3.16), (3.17) *et* (3.19). *On suppose de plus que l'on a*

$$2 f_1 + \sum x_i \frac{\partial f_1}{\partial x_i} \leq 0 \qquad \textit{dans } H^{-1}(D),$$

$$2 f_2 + \sum x_i \frac{\partial f_2}{\partial x_i} \leq 0 \qquad \textit{dans } H^{-1}(\Omega \setminus \overline{D}).$$

Alors si la solution u *de*

$$\begin{cases} -\Delta u = f & \textit{dans } \Omega = \Omega_2 \setminus \overline{\Omega}_1 , \\ u|_{\Gamma_2} = c_2 , \quad u|_{\Gamma_1} = c_1 , \end{cases}$$

satisfait

$$c_2 = \inf\{u(x) : \ x \in \overline{\Omega}\} , \qquad c_1 = \sup\{u(x) : \ x \in \overline{\Omega}\} , \tag{3.26}$$

les ensembles de niveaux de u *sont des 2-connexes fortement étoilés. De façon plus précise on a l'estimation*

$$\inf\{\|\nabla u(x)\| : \ x \in \overline{\Omega}\} > 0 .$$

DÉMONSTRATION. Il suffit de voir que l'on peut appliquer le Théorème 3.1. En effet (3.1) et (3.2) sont satisfaits *i.e.* on a

$$0 \leq f(x) \leq M = \operatorname{sup\,ess} f ,$$

puisque $f \in L^\infty(\Omega)$, et on a

$$2 f + \sum_i x_i \frac{\partial f}{\partial x_i} \leq 0 \qquad \text{dans } H^{-1}(\Omega)$$

par application de la Proposition 3.1.

3. Exemples.

EXEMPLE 1: intervenant en contrôle de domaines [18].

Etant données deux constantes α_1 et α_2 telles que $0 \leq \alpha_1 < \alpha_2$, on pose

$$\begin{aligned} f_1(x) &= \alpha_1\,, && \text{sur } D\,,\\ f_2(x) &= \alpha_2\,, && \text{sur } \Omega\setminus\overline{D}\,,\\ f(x) &= \begin{cases} \alpha_1\,, & \text{si } x\in D\,,\\ \alpha_2\,, & \text{si } x\in\Omega\setminus\overline{D}\,. \end{cases} \end{aligned}$$

La Proposition 3.1 montre que l'on a

$$\Big\langle \sum_i x_i \frac{\partial f}{\partial x_i}\,,\,\varphi \Big\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = (\alpha_2-\alpha_1)\int_{\Gamma_3} \varphi\,\vec{x}\cdot\vec{\nu}\;d\sigma \geq 0\,,$$

pour tout $\varphi \geq 0$, dans $\mathcal{D}(\Omega)$ puisque

$$(f_2-f_1)|_{\Gamma_3} = \alpha_2 - \alpha_1 > 0\,.$$

Si dans (3.26) on inverse le rôle de c_1 et c_2 *i.e.* si $c_1 = \inf u$ et $c_2 = \sup u$ alors on aura

$$2f + x\cdot\frac{\partial f}{\partial x} \geq 0 \qquad \text{dans} \quad H^{-1}(\Omega)$$

et le Théorème 3.2 est applicable *i.e.* que l'on a $x\cdot\nabla u(x) > 0$, pour tout $x\in\Omega$. Donc u n'a pas de points critiques dans Ω.

EXEMPLE 2: Ω est défini comme précédemment.

Cet exemple montre réellement la généralité du Théorème 3.2. Dans ce cas f dépend de x et de u à travers les ensembles de niveau de u. On se donne deux fonctions $f_1(x,t)$, $f_2(x,t)$ positives, de Caratheodory sur $\Omega\times\mathbb{R}^+$ telles que l'on ait

$$M \geq f_1(x,t) > f_2(x,t) > 0\,, \tag{3.27}$$

presque pour tout $x\in\Omega$, pour tout $t\in\mathbb{R}^+$. On suppose que $\partial f_1/\partial t$ et $\partial f_2/\partial t$ sont de Caratheodory, la dérivation étant prise au sens des distributions.

On définit la fonction f par

$$(3.28)\qquad f(x,t) = \begin{cases} f_1(x,t), & \text{pour tous } x \in \Omega,\ t > t_0 , \\ f_2(x,t), & \text{pour tous } x \in \Omega,\ t < t_0 . \end{cases}$$

Nous allons montrer d'abord que l'équation

$$(3.29)\qquad \begin{cases} -\Delta u = f(x,u) & \text{dans} \quad \Omega = \Omega_2 \setminus \overline{\Omega}_1 , \\ u|_\Gamma = 0, \quad u > 0, \end{cases}$$

possède une solution. La difficulté est liée à la discontinuité de f en t pour donner un sens au terme $f(x,u(x))$.

Proposition 3.1.1. *L'équation* (3.29) *possède une solution* u.

DÉMONSTRATION. On procède par régularisation. Soit f_n régulière telle que

$$\begin{cases} M \geq f_n(x,t) \geq (f_2) * \rho_n(x,t), & \text{pour tous } x,\ t, \\ f_n(x,t) \longrightarrow f(x,t), & \text{presque pour tout } x, \text{ pour tout } t \neq t_0 , \end{cases}$$

où ρ_n est un noyau régularisant; et soit u_n une solution de

$$(3.30)\qquad \begin{cases} -\Delta u_n = f_n(x,u_n) & \text{dans } \Omega, \\ u_n|_\Gamma = 0 , \quad u_n > 0; \end{cases}$$

elle vérifie les estimations suivantes:

$$\|u_n\|_{H_0^1(\Omega)} \leq C , \qquad \|u_n\|_{L^\infty(\Omega)} \leq C .$$

A une sous-suite près nous avons les convergences suivantes:

$u_n \rightharpoonup u$ dans $H_0^1(\Omega)$ faible,
$u_n \rightarrow u$ dans $L^p(\Omega)$ fort, pour tout $p < +\infty$,
$u_n(x) \rightarrow u(x)$ dans Ω presque partout,
$f_n(x,u_n(x)) \rightharpoonup g(x)$ dans $L^p(\Omega)$ faible, pour tout $p < +\infty$,
$(f_2) * \rho_n(x,u_n) \rightarrow f_2(x,u)$ dans $L^p(\Omega)$ fort, pour tout $p < +\infty$,

car

$$f_2 * \rho_n(x,\cdot) \rightarrow f_2(x,\cdot), \qquad \text{presque pour tout } x ,$$

uniformément sur tout compact de $\mathbb{R}$. La fonction u vérifie ainsi

$$(3.31)\qquad \begin{cases} -\Delta u = g(x)\,, & 0 < f_2(x,u(x)) \le g(x) \le M\,, \\ u|_\Gamma = 0\,, \quad u > 0\,, & u \in C^{1,\alpha}(\overline{\Omega})\,, \quad 0 < \alpha < 1\,. \end{cases}$$

on voit aisément que $E = \{x \in \Omega :\ u(x) = t_0\}$ est de mesure nulle. Il suffit donc de montrer que

$$g(x) = f(x,u(x))\,, \qquad \text{presque pour tout } x \in \Omega\,.$$

Pour cela on pose

$$F_n = \{x \in \Omega :\ u_n(x) = t_0\}\,,$$

$$F = \bigcup_n F_n\,, \qquad \tilde{\Omega} = \Omega \setminus (F \cup E)\,.$$

Cet ensemble F est de mesure nulle car il est facile de voir que

$$|F_n| = 0\,, \qquad \text{pour tout } n \in \mathbb{N}\,.$$

Par le théorème d'Egorof on a: *pour tout $\varepsilon > 0$ donné, il existe un mesurable $G \subset \tilde{\Omega}$ tel que $|\tilde{\Omega} \setminus G| \le \varepsilon$, $u_n \to u$ uniformément sur G; ce qui entraîne que $f_n(x,u_n(x)) \to f(x,u(x))$, pour tout $x \in G$.*

Pour tout $\varphi \in \mathcal{D}(\Omega)$ tel que $\|\varphi\|_{L^2(\Omega)} = 1$ on a

$$\begin{aligned} \int_\Omega \nabla u_n \cdot \nabla \varphi\, dx &= \int_\Omega f_n(x,u_n(x))\, \varphi\, dx \\ &= \int_{\tilde{\Omega}} f_n(x,u_n(x))\, \varphi\, dx \\ &= \int_G f_n(x,u_n(x))\, \varphi\, dx + \int_{\tilde{\Omega}\setminus G} f_n(x,u_n(x))\, \varphi\, dx\,; \end{aligned}$$

soit par passage à la limite quand n tend vers l'infini

$$\int_\Omega \nabla u \cdot \nabla \varphi\, dx = \int_G f(x,u(x))\, \varphi\, dx + \int_{\tilde{\Omega}\setminus G} g(x)\, \varphi\, dx\,,$$

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla \varphi\, dx - \int_{\tilde{\Omega}} f(x,u(x))\, \varphi\, dx & \\ = -\int_{\tilde{\Omega}\setminus G} f(x,u)\, \varphi\, dx &+ \int_{\tilde{\Omega}\setminus G} g\, \varphi\, dx\,. \end{aligned}$$

L'inégalité de Cauchy-Schwartz, appliquée au second membre, donne

$$\left|\int_\Omega \nabla u \cdot \nabla \varphi \, dx - \int_{\tilde{\Omega}} f(x,u)\, \varphi \, dx\right| \leq 2M\sqrt{\varepsilon}\,,$$

$$\left|\int_\Omega \nabla u \cdot \nabla \varphi \, dx - \int_{\tilde{\Omega}} \chi_\Omega f(x,u(x))\, \varphi \, dx\right| \leq 2M\sqrt{\varepsilon}\,,$$

pour tout $\varepsilon > 0$. Donc

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \chi_{\tilde{\Omega}}\, f(x,u(x))\, \varphi \, dx \tag{3.31.1}$$

pour tout $\varphi \in \mathcal{D}(\Omega)$ tel que $\|\varphi\|_{L^2(\Omega)} = 1$, ce qui signifie que l'on a (3.31.1) pour tout $\varphi \in \mathcal{D}(\Omega)$ par homogénéité. Ainsi u vérifie

$$-\Delta u = \chi_{\tilde{\Omega}}\, f(x,u(x)) \qquad \text{dans } \Omega\,,$$

i.e.

$$\chi_{\tilde{\Omega}}\, f(x,u(x)) = g(x)\,, \qquad \text{presque pour tout } x \in \Omega\,,$$

ou encore

$$f(x,u(x)) = g(x)\,, \qquad \text{presque pour tout } x \in \Omega\,,$$

car $|\tilde{\Omega}| = |\Omega|$. Fin de la preuve de la Proposition 3.1.1.

Pour traiter l'Exemple 2 nous avons besoin des hypothèses suivantes:

L'ouvert $D = \{x \in \Omega : \ u(x) > t_0\}$ est un 2-connexe fortement étoilé par rapport, par exemple, à 0. Les fonctions suivantes sont de Caratheodory et satisfont

$$0 \leq -\sum_i x_i \frac{\partial f_j}{\partial x_i}(x,t) \leq a\,|t| + b(x)\,, \tag{3.32}$$

pour tout $j = 1, 2$, où a est une constante positive et b une fonction de $L^2(\Omega)$. La fonction suivante est de Caratheodory et vérifie

$$0 \geq \frac{\partial f_j}{\partial t}(x,t) \in L^\infty(\Omega \times \mathbb{R}^+)\,, \qquad j = 1, 2\,. \tag{3.33}$$

On pose

$$g_j(x) = f_j(x, u(x)), \qquad j = 1, 2,$$

$$g(x) = f(x, u(x)), \qquad \text{presque pour tout } x,$$

$$k(x,t) = \begin{cases} \dfrac{\partial f_1}{\partial t}(x,t), & \text{presque pour tout } x, \text{pour tout } t < t_0, \\ \dfrac{\partial f_2}{\partial t}(x,t) & \text{presque pour tout } x, \text{pour tout } t > t_0. \end{cases}$$

Il est clair que les hypothèses (3.32) et (3.33) entraînent que g_j et $x \cdot \nabla g_j(x)$ appartiennent à $L^2(\Omega)$ pour $j = 1, 2$. Le Lemme 3.1 permet alors de définir les traces $g_j|_{\partial D}$, $j = 1, 2$. Ce qui rend licite l'hypothèse

$$(f_2(x,u) - f_1(x,u))\big|_{\partial D} \leq 0 \tag{3.34}$$

au sens de $H^{-1/2}(\partial D)$.

En procédant comme dans la Proposition 3.1 on obtient: si $p = x \cdot g$

$$\begin{aligned} \langle \operatorname{div} p, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} &= \langle n f + \sum_i x_i \frac{\partial g}{\partial x_i}, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} \\ &= \int_\Omega n f \varphi + \int_\Omega k(x,u)\varphi \\ &\quad + \int_D \sum_i x_i \frac{\partial f_1}{\partial x_i}(x,u)\varphi \\ &\quad + \int_{\Omega\setminus\overline{D}} \sum_i x_i \frac{\partial f_2}{\partial x_i}(x,u)\varphi \\ &\quad + \langle (f_2 - f_1)\cdot(x\cdot\nu), \varphi\rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)}, \end{aligned}$$

soit

$$\begin{aligned} \langle \sum_i x_i \frac{\partial g}{\partial x_i}, \varphi\rangle_{\mathcal{D}'\times\mathcal{D}} &= \int_\Omega k(x,u)\varphi + \int_\Omega T(x)\varphi\, dx \\ &\quad + \langle (f_2 - f_1)\cdot(x\cdot\nu), \varphi\rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)} \end{aligned} \tag{3.35}$$

avec

$$T(x) = \begin{cases} \displaystyle\sum_i x_i \frac{\partial f_1}{\partial x_i}(x,u), & \text{si } x \in D, \\ \displaystyle\sum_i x_i \frac{\partial f_2}{\partial x_i}(x,u), & \text{si } x \in \Omega\setminus\overline{D}. \end{cases}$$

En procédant par régularisation et en utilisant l'unicité du problème de Dirichlet on montre que la fonction $v = x \cdot \nabla u$ vérifie au sens des distributions

$$\text{(3.36)} \qquad \begin{cases} -\Delta v = 2\, f(x,u) + \sum_i x_i \dfrac{\partial g}{\partial x_i}(x)\,, \\ v|_{\partial\Omega} < 0\,. \end{cases}$$

Compte tenu de l'hypothèse (3.34), l'équation (3.36) entraîne que l'on a

$$\langle -\Delta v - k(x,u)\, v\,,\, \varphi\rangle \leq \int_\Omega T\varphi + 2\int_\Omega f(x,u)\,\varphi\; dx\,,$$

pour tout $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, *i.e.*

$$-\Delta v - k(x,u)\, v \leq T + 2\, f(x,u)\,.$$

Considérons alors la solution ω de

$$\text{(3.37)} \qquad \begin{cases} -\Delta\omega - k(x,u)\omega = T + 2\, f(x,u)\,, \\ \omega|_{\partial\Omega} = v|_{\partial\Omega}\,. \end{cases}$$

Si on suppose que l'on a

$$2\, f(x,u) + T(x) \leq 0\,, \qquad \text{presque pour tout } x \in \Omega\,,\ \not\equiv 0\,,$$

le principe du maximum ([22], [26]) donne

$$v(x) \leq \omega(x) < 0\,.$$

Et la conclusion s'impose comme dans l'Exemple 1.

REMARQUE 3.4. Un exemple de fonction positive satisfaisant l'inégalité

$$2\, f(x) + x \cdot \frac{\partial f}{\partial x}(x) \leq 0\,.$$

Supposons Ω_1 et Ω_2 convexes tels que $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Soit $g(x)$ concave sur Ω_2, régulière, positive telle que

$$g(0) = \sup\{g(x) :\ x \in \overline{\Omega}_2\}\,.$$

Dans ce cas on a

$$x \cdot \frac{\partial g}{\partial x}(x) \leq 0\,, \qquad \text{pour tout } x \in \Omega_2\,.$$

Posons

$$f(x) = \frac{g(x)}{x^2}\,, \qquad \text{pour tout } x \in \Omega_2 \setminus \overline{\Omega}_1\,.$$

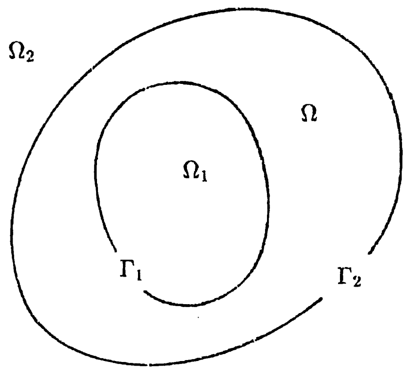
Un calcul élémentaire montre que

$$2\,f(x) + x \cdot \frac{\partial f}{\partial x}(x) \leq 0\,, \qquad \text{pour tout } x \in \Omega_2 \setminus \overline{\Omega}_1\,.$$

4. Nécessité de l'hypothèse (3.26).

Le résultat du Théorème 3.2 n'a pas lieu quelles que soient les constantes c_1 et c_2, $0 < c_2 < c_1$: en réalité ces constantes sont liées au second membre f. C'est le but de ce qui suit. Pour cela on considère deux ouverts connexes Ω_1 et Ω_2 bornés réguliers tels que $\Omega_2 \supset \overline{\Omega}_1$. On pose

$$\Omega = \Omega_2 \setminus \overline{\Omega}_1\,, \qquad \Gamma_i = \partial\Omega_i\,.$$



Soit g une fonction positive appartenant à $L^\infty(\Omega)$:

(3.38) $$0 \leq g(x) \leq M \neq 0\,, \qquad \text{avec} \quad g \not\equiv M\,.$$

(*cf.* Remarque 3.4).

Considérons le problème de Dirichlet suivant:

$$(3.39)\qquad \begin{cases} -\Delta r_c = g & \text{dans } \Omega\,, \\ r_c|_{\Gamma_2} = 0\,, \quad r_c|_{\Gamma_1} = c\,, & \text{constante} > 0\,. \end{cases}$$

On a

Proposition 3.2. *Il existe $c_0 = c_0(g) > 0$ tel que pour tout c, $0 \leq c \leq c_0$ la fonction r_c possède un point critique au moins à l'intérieur de Ω i.e. que r_c possède des ensembles de niveau qui ne sont pas des 2-connexes.*

REMARQUE 3.5. Il est connu que si $M = 0$ et Ω_i convexe ($i = 1, 2$), alors pour tout $c > 0$ la fonction r_c n'a pas de points critiques dans $\overline{\Omega}$.

De plus les lignes de niveau de r_c sont des courbes régulières convexes (*cf.* [14]).

DÉMONSTRATION DE LA PROPOSITION. On considère le problème

$$(3.40)\qquad \begin{cases} -\Delta\omega = g & \text{dans } \Omega\,, \\ \omega|_{\Gamma_1} = \omega|_{\Gamma_2} = 0\,, \end{cases}$$

et on choisit $0 < c \leq \|\omega\|_\infty = \sup\{\omega(x) : x \in \overline{\Omega}\}$. Il existe $x_0 \in \Omega$ tel que $\omega(x_0) = \|\omega\|_\infty$; par le principe du maximum classique on a

$$\|r_c\|_\infty \geq r_c(x_0) > \omega(x_0) = \|\omega\|_\infty \geq c = \sup\{r_c(x) : x \in \Gamma_1\}\ ;$$

par conséquent il existe $x_1 \in \Omega$ tel que

$$r_c(x_1) = \|r_c\|_\infty > c \qquad \text{et} \qquad c_0(g) = \|\omega\|_\infty\,.$$

Proposition 3.3. *Pour Ω défini ci-dessus, il existe un réel $c_1 = c_1(M) > 0$ tel que pour tout g satisfaisant (3.38) et tout $c > c_1$ la solution $r_c = r_{c,g}$ de (3.39) vérifie les estimations suivantes:*

$$\sup\{r_c(x) : x \in \overline{\Omega}\} = c\,,$$

$$r_c(x) < c\,, \qquad \textit{pour tout } x \in \Omega\,.$$

DÉMONSTRATION. Considérons les solutions ω_0, ω_1 et s_c respectivement des problèmes

$$(3.41) \qquad \begin{cases} -\Delta\omega_0 = 0 & \text{dans } \Omega\,, \\ \omega_0|_{\Gamma_2} = 0\,, & \omega_0|_{\Gamma_1} = 1\,, \end{cases}$$

$$(3.42) \qquad \begin{cases} -\Delta\omega_1 = 1 & \text{dans } \Omega\,, \\ \omega_1|_{\Gamma_2} = 0\,, & \omega_1|_{\Gamma_1} = 0\,, \end{cases}$$

$$(3.43) \qquad \begin{cases} -\Delta s_c = M & \text{dans } \Omega\,, \\ s_c|_{\Gamma_2} = 0\,, & s_c|_{\Gamma_1} = c\,. \end{cases}$$

Il est clair que l'on a

$$s_c = M\,\omega_1 + c\omega_0$$

et

$$\|\nabla s_c(x)\| = \|M\,\nabla\omega_1(x) + c\,\nabla\omega_0(x)\|\,.$$

Posons

$$\mathcal{L}(c) = \inf\{\|\nabla s_c(x)\| : \ x \in \overline{\Omega}\}\,,$$

et

$$E = \{c \geq 0 : \ \mathcal{L}(c) = 0\}\,.$$

D'après la Proposition 3.2, E contient l'intervalle $[0, \|s_0\|_\infty]$ où s_0 est la solution de (3.43) correspondant à $c = 0$, *i.e.* $E \neq \varnothing$. On définit alors $c_1(M)$ par

$$c_1(M) = \sup\{c \geq 0 : \ c \in E\}\,,$$

$$c_1(M) \geq \|s_0\|_\infty > 0\,.$$

Par définition de $c_1(M)$ tout $c > c_1(M)$ on a

$$(3.44) \qquad \mathcal{L}(c) = \inf\{\|\nabla s_c(x)\| : \ x \in \overline{\Omega}\} > 0\,.$$

D'autre part ($c > c_1(M)$) par le principe du maximum la solution r_c de (3.39) vérifie

$$r_c(x) < s_c(x)\,, \qquad \text{pour tout } x \in \Omega\,,$$

donc

$$r_c(x) < s_c(x) \leq \sup\{s_c(x) : \ x \in \overline{\Omega}\}\,,$$

pour tout $x \in \Omega$, *i.e.* d'après (3.44),

$$r_c(x) < \sup\{s_c(x) : x \in \overline{\Omega}\} = c = \sup\{r_c(x) : x \in \overline{\Omega}\},$$

pour tout $x \in \Omega$.

QUESTIONS. Que se passe-t-il pour s_c solution de (3.43) quand $c = c_1(M)$? Par exemple est-il exact que l'on a

$$\|\nabla s_c(x)\| > 0, \qquad \text{pour tout } x \in \Omega$$

et que $\|\nabla s_c(x)\|$ ne s'annule que sur Γ_1 ?

REMARQUE 3.6. Les Propositions 3.2 et 3.3 montrent que les Théorèmes 3.1 et 3.2 ne peuvent avoir lieu pour tout c_1 et c_2 tels que $c_2 < c_1$; ce qui justifie l'hypothèse (3.26).

5. Cas d'un ouvert fortement étoilé.

La difficulté, dans ce cas, est due au fait que l'hypothèse

$$\text{(3.44.1)} \qquad \begin{cases} 2\,f(x) + x \cdot \dfrac{\partial f}{\partial x} \leq 0 & \text{dans } \Omega, \\ f(x) \geq 0 & \text{dans } \Omega. \end{cases}$$

n'est jamais vérifiée au voisinage de 0 quand f est positive: on s'en convainc facilement en considérant une fonction positive concave atteignant son maximum en $x = 0$. La première inégalité (3.44.1) est satisfaite si f est négative au voisinage de 0. Mais dans ce cas la solution u correspondante n'est pas nécessairement positive.

i) *Un principe du maximum.*

On se donne une fonction $f : \Omega \to \mathbb{R}$ de signe non constant dans Ω. Et on se pose la question suivante: à quelles conditions sur f la solution u du problème

$$\text{(3.45)} \qquad \begin{cases} -\Delta u = f & \text{dans } \Omega, \\ u|_\Gamma = 0, \end{cases}$$

soit positive. Pour cela on introduit les notations suivantes:

$$\alpha(x,r) = \inf\{f(y) : \ y \in B(x,r)\},$$

pour tous $x \in \Omega$, $r > 0$ tel que $B(x,r) \subset \Omega$,

$$\beta(x,r,R) = \inf\{f(y) : \ y \in B(x,R) \setminus B(x,r)\},$$

pour tout $R > r$ tel que $B(x,R) \subset \Omega \subset \mathbb{R}^{N+1}$,

$$E = \{x \in \Omega : \ f(x) < 0\}, \qquad F = \{x \in \Omega : \ f(x) \geq 0\}.$$

Théorème 3.3. *On suppose qu'il existe $x \in \Omega$, $r_0 > 0$ et $R > r_0$ tels que l'on ait*

$$B(x,r_0) \supset E \tag{3.46}$$

et

$$1 - \frac{\alpha(x,r_0)}{\beta(x,r_0,R)} \leq \begin{cases} \left(\dfrac{R}{r_0}\right)^{N+1} \dfrac{(N-1)}{(N+1)\left(\dfrac{R}{r_0}\right)^{N-1} - 2}, & si\ N > 1, \\ \left(\dfrac{R}{r_0}\right)^{2} \dfrac{1}{1 + 2\log\left(\dfrac{R}{r_0}\right)}, & si\ N = 1. \end{cases} \tag{3.47}$$

Alors la solution u de (3.45) *est strictement positive dans Ω.*

La preuve de ce résultat nécessite deux résultats préliminaires.

On se donne un ouvert θ tel que $\theta \subset \overline{\Omega}$; et on considère le problème

$$\begin{cases} -\Delta v = g & \text{dans } \theta, \quad g \in L^2(\theta) \text{ par exemple,} \\ v|_{\partial\theta} = 0, \quad v \in H_0^1(\theta). \end{cases}$$

On pose

$$\tilde{g}(x) = \begin{cases} g(x) & \text{dans } \theta, \\ 0 & \text{dans } \Omega \setminus \overline{\theta}, \end{cases} \qquad \tilde{v}(x) = \begin{cases} v(x) & \text{dans } \theta, \\ 0 & \text{dans } \Omega \setminus \overline{\theta}, \end{cases}$$

il est clair que $\tilde{v} \in H_0^1(\Omega)$ (*cf.* [22] par exemple).

Lemme 3.1 (de prolongement). *Si on suppose que*

$$\frac{\partial v}{\partial \eta}\Big|_{\partial\theta} < 0$$

alors $\tilde{v}$ *vérifie dans* $H^{-1}(\Omega)$ *l'inégalité suivante*

$$-\Delta\tilde{v} \le \tilde{g}\,.$$

DÉMONSTRATION. C'est une simple application de la formule de Green: il est clair que $\Delta\tilde{v} \in H^{-1}(\Omega)$; et pour toute fonction φ dans $\mathcal{D}(\Omega)$, $\varphi \ge 0$, on a

$$\begin{aligned}\langle -\Delta\tilde{v}, \varphi\rangle_{H^{-1}\times H_0^1} &= \langle -\Delta\tilde{v}, \varphi\rangle_{\mathcal{D}'\times\mathcal{D}} = \langle -\Delta\varphi, \tilde{v}\rangle_{\mathcal{D}'\times\mathcal{D}} \\ &= \int_\Omega -\Delta\varphi\ \tilde{v}\ dx = \int_\theta \nabla\varphi\cdot\nabla v\ dx - \int_{\partial\theta}\frac{\partial\varphi}{\partial\eta}\, v\ d\sigma \\ &= \int_\theta \nabla\varphi\cdot\nabla v\ dx = -\int_\theta \Delta v\ \varphi\ dx + \int_{\partial\theta}\frac{\partial v}{\partial\eta}\,\varphi\ d\sigma\,,\end{aligned}$$

car

$$\nabla\tilde{v}(x) = \begin{cases}\nabla v(x) & \text{dans } \theta\,,\\ 0 & \text{dans } \Omega\setminus\overline{\theta}\,,\end{cases}$$

$$\begin{aligned}-\langle\Delta\tilde{v}, \varphi\rangle_{H^{-1}\times H_0^1} &= \int_\theta g\,\varphi\ dx + \int_{\partial\theta}\frac{\partial v}{\partial\eta}\,\varphi\ d\sigma \\ &= \int_\Omega \tilde{g}\,\varphi\ dx + \int_{\partial\theta}\frac{\partial v}{\partial\eta}\,\varphi\ d\sigma \\ &\le \int_\Omega \tilde{g}\,\varphi\ dx\,.\end{aligned}$$

Et on conclut par densité de $\mathcal{D}(\Omega)$ dans $H_0^1(\Omega)$.

Soit r_0 et R tels que $0 < r_0 < R$, et $g : B(0,R) \to \mathbb{R}$ définie par

$$g(x) = \begin{cases}\alpha < 0 & \text{sur} \quad B(0, r_0)\,,\\ \beta > 0 & \text{sur} \quad B(0,R)\setminus B(0,r_0)\,,\end{cases}$$

où α et β sont deux constantes.

On considère la solution v de

$$\text{(3.48)} \qquad \begin{cases} -\Delta v = g & \text{dans } B(0,R)\,, \\ v|_{\partial B} = 0\,. \end{cases}$$

Lemme 3.2 (de fonction radiale positive). *Si α, β, r_0 et R vérifie la relation*

$$1 - \frac{\alpha}{\beta} \le \begin{cases} \left(\dfrac{R}{r_0}\right)^{N+1} \dfrac{(N-1)}{(N+1)\left(\dfrac{R}{r_0}\right)^{N-1} - 2}\,, & si\ N > 1\,, \\ \left(\dfrac{R}{r_0}\right)^2 \dfrac{1}{1 + 2\log\left(\dfrac{R}{r_0}\right)}\,, & si\ N = 1\,, \end{cases}$$

alors (3.48) *possède une solution radiale positive.*

DÉMONSTRATION. La solution de (3.48) se calcule explicitement puisqu'elle vérifie

$$\text{(3.49)} \qquad \begin{cases} -\dfrac{1}{r^N}\dfrac{d}{dr}\left(r^N \dfrac{dv}{dr}\right) = g(r) & \text{dans } B(0,R)\,, \\ v(R) = 0\,, \quad v'(0) = 0\,, \quad N \ge 1\,, \end{cases}$$

$$v(r) = \int_r^R \frac{1}{\sigma^N}\left(\int_0^\sigma s^N g(s)\,ds\right) d\sigma\,,$$

$$v'(r) = -\frac{1}{r^N}\int_0^r s^N g(s)\,ds\,.$$

Par définition de g, on a $v'(r) \ge 0$, pour tout $r \in [0, r_0]$; *i.e.* si on impose la condition $v(0) \ge 0$, on aura $v(r) \ge 0$ sur $[0, r_0]$. De plus

$$\text{(3.50)} \qquad v''(r) = -\frac{(\beta - \alpha)N}{N+1}\left(\frac{r_0}{r}\right)^{N+1} - \frac{\beta}{N+1} \le 0\,,$$

pour tout $r \in [r_0, R]$, joint à la condition aux limites $v(R) = 0$, entraîne que l'on a $v(r) \ge 0$, pour tout $r \in [r_0, R]$.

Ainsi la seule condition à imposer pour que (3.48) ait une solution $v(r)$ positive est $v(0) \ge 0$, qui, pour $N \ne 1$, se traduit par

$$v(0) = \frac{\beta R^2}{2(N+1)} + \frac{\alpha - \beta}{2(N+1)} r_0^2 - \frac{(\beta - \alpha) r_0^{N+1}}{N^2 - 1}\left(\frac{1}{r_0^{N-1}} - \frac{1}{R^{N-1}}\right) \ge 0$$

qui est une condition équivalente à

$$1-\frac{\alpha}{\beta}\leq\left(\frac{R}{r_0}\right)^{N+1}\frac{(N-1)}{(N+1)\left(\frac{R}{r_0}\right)^{N-1}-2}\,.$$

Dans le cas $N=1$, le condition ci-dessus devient

$$v(0)=\frac{\alpha\,r_0^2}{4}+\frac{\beta}{4}(R^2-r_0^2)+\frac{(\alpha-\beta)\,r_0^2}{2}\,\log\frac{R}{r_0}\geq 0$$

ou encore

$$1-\frac{\alpha}{\beta}\leq\left(\frac{R}{r_0}\right)^{2}\frac{1}{1+2\,\log\dfrac{R}{r_0}}\,.$$

DÉMONSTRATION DU THÉORÈME 3.3. Soit v la solution radiale du problème

$$\begin{cases}-\Delta v=g & \text{dans } B(0,R)\,,\\ v|_{\partial B(0,R)}=0\,,\end{cases}$$

où l'on a supposé que le centre x est à l'origine, quitte à effectuer une translation, et où

$$g(r)=\begin{cases}\alpha(0,r_0) & \text{sur } B(0,r_0)\,,\\ \beta(0,r_0,R) & \text{sur } B(0,R)\setminus B(0,r_0)\,.\end{cases}$$

Par le Lemme 3.2 la fonction $v(r)$ est positive. De plus g est positive au voisinage du bord de $B(0,R)$, donc par le principe du maximum de Hopf on a

$$\frac{\partial v}{\partial\eta}\Big|_{\partial B(0,R)}<0\,.$$

On prolonge v par 0 à tout Ω: soit $\tilde v$ ce prolongement. D'après le Lemme 3.1 $\tilde v$ vérifie

$$\begin{cases}-\Delta\tilde v\leq\tilde g\,,\\ \tilde v|_{\partial\Omega}=0\,,\end{cases}\tag{3.51}$$

où $\tilde g$ est le prolongement de g par 0 à tout Ω. Puisque $f-\tilde g\geq 0$, dans Ω, le principe du maximum faible (*cf.* [26]) entraîne que l'on a

$$u(x)\geq\tilde v(x)\geq 0\qquad\text{dans }\Omega\,,$$

i.e. $u(x) > 0$ dans $B(0,R)$.

Il reste à montrer que $u(x) > 0$ dans Ω. On considère alors l'équation (3.45) restreinte à $\Omega \setminus \overline{B(0,R-\varepsilon)}$ où $\varepsilon > 0$ assez petit vérifie $R-\varepsilon > r_0$

$$\begin{cases} -\Delta u = f \geq 0 & \text{dans } \Omega \setminus \overline{B(0,R-\varepsilon)}, \\ u|_{\partial\Omega} = 0, \quad u|_{\partial B(0,R-\varepsilon)} \geq v|_{\partial B(0,R-\varepsilon)} > 0; \end{cases}$$

donc le principe du maximum classique (*cf.* [26]) entraîne que

$$u(x) > 0 \qquad \text{dans } \Omega \setminus \overline{B(0,R-\varepsilon)}.$$

REMARQUE 3.7. Ce théorème possède un certain "caractère local": en effet ce principe du maximum a encore lieu si

$$E = \bigcup_i E_i, \qquad i \in I \subseteq \mathbb{N},$$

et s'il existe $r_0^i > 0$ et $R_i > r_0^i$ satisfaisant (3.46) et (3.47) pour tout $i \in I$.

ii) *Localisation des points critiques.*

Etant donnés deux ouverts Ω et D convexes tels que $0 \in D$ et $\overline{D} \subset \Omega$, on considère une fonction $F(x)$ de Ω dans $\mathbb{R}^+$ satisfaisant les hypothèses suivantes:

$$F(x) = \begin{cases} \gamma, & \text{presque pour tout } x \in D, \quad \text{une const.} > 0, \\ f(x), & \text{pour tout } x \in \Omega \setminus \overline{D}, \quad f(x) \geq 0, \quad \text{pour tout } x, \end{cases}$$

où f est une fonction définie sur Ω qui vérifie les hypothèses suivantes (pour simplifier la présentation on choisit f régulière)

$$f|_{\partial D} \leq \gamma. \tag{3.51}$$

Pour tout $x \in \Omega$, tout $y \in D$, on pose

$$\begin{aligned} g(x,y) = &-\Big(2\,f(x+y)\,\chi_{\Omega\setminus\overline{D}}(x+y) + x\,\frac{\partial f}{\partial x}(x+y)\,\chi_{\Omega\setminus\overline{D}}(x+y)\Big) \\ &+ 2\gamma\,\chi_D(x+y), \end{aligned} \tag{3.52}$$

$$\alpha(y,r) = \inf\{g(x,y) :\ x \in B(-y,r)\}\,,$$

$$E(y) = \{x :\ g(x,y) \leq 0\}\,,$$

$$\beta(y,r,R) = \inf\big\{g(x,y) :\ x \in B(-y,R) \setminus \overline{B(-y,r)}\big\}\,,$$

où $r > 0$ et $R > 0$ sont tels que $\overline{D} \subset B(0,r) \subset B(0,R)$. Enfin on suppose que la solution u de

$$\begin{cases} -\Delta u = F & \text{sur } \Omega\,, \\ u|_\Gamma = 0\,, \end{cases} \tag{3.53}$$

possède la propriété suivante:

$$\begin{cases} \text{il existe } y_0 \in D \text{ point de maximum local pour } u \text{ tel que} \\ \qquad 2\,f(x+y_0) + x \cdot \dfrac{\partial f}{\partial x}(x+y_0) \leq 0\,, \\ \text{pour tout } x \text{ tel que } x + y_0 \in \Omega \backslash \overline{D}\,. \end{cases} \tag{3.54}$$

Il est clair que

$$E(y_0) = D - y_0 \qquad \text{et} \qquad \alpha(y_0,r) = -2\gamma\,.$$

Nous avons le résultat suivant:

Théorème 3.4. *On suppose que les hypothèses* (3.51), (3.54) *ont lieu et qu'il existe* r_0 *et* R *tels que* $\alpha(y_0,r_0)$, $\beta(y_0,r_0,R)$ *et* $E(y_0)$ *vérifient les inégalités* (3.47) *et* (3.46) *respectivement. Alors* y_0 *est le seul point critique de* u *dans* Ω. *De plus les ensembles de niveau de* u *sont strictement étoilés par rapport à* y_0.

DÉMONSTRATION. On se ramène au Théorème 3.2. Pour tout $r > 0$ suffisamment petit, il existe $t_r > 0$ tel que l'ouvert $\{x \in \Omega :\ u(x) > t_r\}$ possède une composante connexe Ω_r convexe, contenant y_0 et incluse dans $B(y_0,r)$, la boule de centre y_0 et de rayon r. Ceci est toujours possible car y_0 est un point de maximum local de u.

L'ouvert $\Omega \setminus \overline{\Omega}_r$ est donc un 2-connexe fortement étoilé par rapport à y_0 et on a

$$\begin{cases} -\Delta u = F(x) & \text{dans } \Omega \setminus \overline{\Omega}_r\,, \\ u|_{\partial\Omega} = 0\,, \quad u|_{\partial\Omega_r} = t_r > 0\,. \end{cases}$$

La fonction $v(x) = x \cdot \nabla u(x+y_0)$ pour tout $x \in \Omega - y_0 = \tilde{\Omega}$ vérifie

$$\begin{cases} -\Delta v(x) = 2\gamma\, \chi_D(x+y_0) + 2\, f(x+y_0)\, \chi_{\Omega\setminus\overline{D}}(x+y_0) \\ \qquad\qquad + \mathcal{T} + x \cdot \dfrac{\partial f}{\partial x}(x+y_0)\, \chi_{\Omega\setminus\overline{D}}(x+y_0)\,, \\ \\ v|_{\partial\tilde{\Omega}} = 0\,, \quad v|_{\partial\tilde{\Omega}_r} = t_r > 0\,, \quad x \in (\Omega \setminus \overline{\Omega}_r) - y_0\,, \end{cases}$$

où $\tilde{\Omega}_r = \Omega_r - y_0$, $\mathcal{T}$ est un élément de $H^{-1}(\tilde{\Omega})$ obtenu par application de la Proposition 3.1 et défini par

$$\langle \mathcal{T}, \varphi \rangle_{H^{-1}\times H_0^1} = \int_{\partial D} (f(y_0 + \cdot\,) - \gamma)\,(x \cdot \nu)\, \varphi \, d\sigma \leq 0\,,$$

pour tout $\varphi \geq 0$, $\varphi \in H_0^1(\tilde{\Omega})$. Par conséquent v satisfait l'inégalité

$$\begin{aligned} -\Delta v(x) \leq{}& 2\gamma\, \chi_D(x + y_0) + 2\, f(x + y_0)\, \chi_{\Omega\setminus\overline{D}}(x + y_0) \\ &+ x \cdot \frac{\partial f}{\partial x}(x + y_0)\, \chi_{\Omega\setminus\overline{D}}(x + y_0) = -g(x, y_0)\,, \end{aligned}$$

dans $H^{-1}(\tilde{\Omega})$, et les conditions aux limites

$$v|_{\partial\tilde{\Omega}} < 0\,, \qquad v|_{\partial\tilde{\Omega}_r} < 0\,,$$

grâce au principe du maximum de Hopf.

Si on considère ω solution dans $(\Omega \setminus \overline{\Omega}_r) - y_0$ de

$$\begin{cases} -\Delta \omega = 2\gamma\, \chi_D(x + y_0) + 2\, f(x + y_0)\, \chi_{\Omega\setminus\overline{D}}(x + y_0) \\ \qquad\qquad + x \cdot \dfrac{\partial f}{\partial x}(x + y_0)\, \chi_{\Omega\setminus\overline{D}}(x + y_0) = -g(x, y_0)\,, \\ \\ \omega|_{\partial\tilde{\Omega}} = v|_{\partial\tilde{\Omega}} < 0\,, \quad \omega|_{\partial\tilde{\Omega}_r} = v|_{\partial\tilde{\Omega}_r} < 0\,, \end{cases}$$

on peut montrer, par une adaptation de la preuve du Théorème 3.3, que l'on a $\omega(x) < 0$ dans $(\Omega \setminus \overline{\Omega}_r) - y_0$; ce qui entraîne que $v(x) < 0$ dans $(\Omega \setminus \overline{\Omega}_r) - y_0$, pour tout $r > 0$, par application du principe du maximum classique [26]; *i.e.* $|\nabla u(x)| > 0$ pour tout $x \in \Omega$, $x \neq y_0$.

iii) *localisation des points de maximum global.*

On suppose Ω convexe borné régulier et on considère l'ensemble $\mathcal{E}$ des fonctions positives, défini par

$$\mathcal{E} = \{f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : \ 0 \le a \le f(t) \le b\},$$

où a et b sont deux constantes. Pour tout f dans $\mathcal{E}$ considérons la solution $u = u_f$ de l'équation

$$\text{(3.55)} \qquad \begin{cases} -\Delta u = f & \text{dans } \Omega, \\ u|_\Gamma = 0. \end{cases}$$

Nous avons le résultat suivant

Proposition 3.4. *Il existe dans Ω un voisinage $\mathcal{V}$ de $\partial\Omega$ ne dépendant que de Ω, a et b tel que pour tout f appartenant à $\mathcal{E}$, la solution $u = u_f$ de* (3.55) *ne possède pas de points de maximum global dans $\mathcal{V}$. De plus le voisinage $\mathcal{V}$ est une "couronne convexe", donnée explicitement par:*

$$\mathcal{V} = \{x \in \Omega : \ u_b(x) < \sup\{u_a(x) : \ x \in \Omega\}\}.$$

DÉMONSTRATION. Par le principe du maximum on a

$$\text{(3.54.1)} \qquad u_a(x) \le u(x) = u_f(x) \le u_b(x), \qquad \text{pour tout } x \in \overline{\Omega}.$$

Comme les ensembles de niveaux $\{x \in \Omega : \ u_b(x) > t\}$ de u_b sont convexes (*cf.* [14]) il s'ensuit que $\mathcal{V} = \{x \in \Omega : \ u_b(x) < \|u_a\|_{L^\infty}\}$ est une "couronne convexe" ne dépendant que de a et b. On se propose alors de montrer que pour tout f dans $\mathcal{E}$, u_f n'a pas de point de maximum global dans $\mathcal{V}$.

D'après Stampacchia [25] il existe $K = K(\Omega, \beta)$ tel que l'on ait

$$\text{(3.55.1)} \qquad \|\nabla u_f\|_{L^\infty(\Omega)} \le K, \qquad \text{pour tout } f \in \mathcal{E}.$$

Soit y_0 quelconque appartenant à $\mathcal{V}$; on a

$$\text{(3.56)} \qquad u_b(y_0) < \|u_a\|_\infty = \|u_a\|_{L^\infty};$$

soit $r_0 > 0$ satisfaisant

$$\frac{\|u_a\|_\infty - u_b(y_0)}{r_0} > K \tag{3.57}$$

et tel que la boule $B(y_0, r_0)$ soit incluse dans $\mathcal{V}$.

Nous pouvons affirmer que $B(y_0, r_0)$ ne contient aucun point de maximum global de u_f. En effet supposons qu'il existe $\overline{x} \in B(y_0, r_0)$ tel que

$$u_f(\overline{x}) = u(\overline{x}) = \|u\|_{L^\infty(\Omega)} .$$

Nous avons, grâce à (3.55)

$$K \geq \frac{u(\overline{x}) - u(y_0)}{|\overline{x} - y_0|} \geq \frac{u(\overline{x}) - u(y_0)}{r_0}$$

qui donne, compte tenu de (3.54.1) et (3.57)

$$K \geq \frac{u(\overline{x}) - u(y_0)}{r_0} > \frac{\|a_a\|_\infty - u_b(y_0)}{r_0} > K\,, \tag{3.58}$$

ceci constitue une contradiction.

REMARQUE 3.8. Il serait intéressant de savoir si l'hypothèse (3.5.4) est nécessaire.

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An improved bound for Kakeya type maximal functions

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The purpose of this paper is to improve the known results (specifically [1]) concerning L^p boundedness of maximal functions formed using $1 \times \delta \times \cdots \times \delta$ tubes. We briefly recall the problem: let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then for $0 < \delta < 1$ one defines (we follow [1] in notation and in the definition of f_δ^*)

$$f_\delta^* : \mathbb{P}^{d-1} \to \mathbb{R}, \qquad f_\delta^*(e) = \sup_T \frac{1}{|T|} \int_T |f|,$$

where $\mathbb{P}^{d-1}$ is projective space and T runs through all cylinders with length 1, cross section radius δ and axis in the e direction. Also

$$f_\delta^{**} : \mathbb{R}^d \to \mathbb{R}, \qquad f_\delta^{**}(x) = \sup_T \frac{1}{|T|} \int_T |f|,$$

where T runs through cylinders containing x with length 1 and cross section radius δ. A conjecture (some years old for f_δ^{**}, formulated in [1] for f_δ^*) is that for any $\varepsilon > 0$

$$\|M_\delta f\|_d \leq C_\varepsilon\, \delta^{-\varepsilon} \|f\|_d\,,$$

where M_δ denotes either f_δ^* or f_δ^{**}. Interpolating this conjectured bound with the trivial $L^1 \to L^\infty$ bound, one obtains the equivalent conjecture

$$\|M_\delta f\|_q \leq C_\varepsilon\, \delta^{-(d/p-1+\varepsilon)} \|f\|_p\,, \qquad 1 \leq p \leq d,\ q = (d-1)p'.$$

In either case M_δ takes functions supported in a disc of radius 1 to functions whose support has measure bounded by a fixed constant , so for fixed p the range of q must be an interval $[1, q_0(p)]$ and the conjecture

$$(1) \qquad \|M_\delta f\|_q \leq C_\varepsilon\, \delta^{-(d/p-1+\varepsilon)}\, \|f\|_p\,, \qquad 1 \leq p \leq d\,,\ q \leq (d-1)p'$$

is again equivalent. Our purpose is to extend the range of p and q for which (1) is known to hold. We assume throughout the paper that $d \geq 3$. When $d = 2$, Theorem 1 below is well-known; see [6], and [1] for the case of f_δ^*.

Results like (1) with $p = q = 2$ go back to A. Cordoba's work in the mid-1970's, *e.g.* [6]. When $p = (d+1)/2$, $q = (d-1)p' = d+1$, (1) follows from S. Drury's result [7] and a somewhat stronger result is proved for f_δ^{**} in Christ-Duoandikoetxea-Rubio de Francia [8]. The exponent $p = (d+1)/2$ plays a natural role, and getting beyond it was accomplished only recently by Bourgain [1] who proved (1) with $p = p(d) \in ((d+1)/2, (d+2)/2)$ given by the recursion

$$p(2) = 2\,, \qquad p(d) = \frac{(d+2)p(d-1) - d}{2p(d-1) - 1}$$

for $d \geq 3$, and $q = p$. Our result is the following further improvement.

Theorem 1. *(1) holds for $M_\delta f = f_\delta^*$ or f_δ^{**}, $p = (d+2)/2$, $q = (d-1)p'$.*

Thus we improve the L^p exponent from *e.g.* 7/3 to 5/2 in three dimensions; our argument also gives the correct value of q. An immediate corollary (*cf.* [1]) is that any Besicovitch or Nikodym set in $\mathbb{R}^d$ has Hausdorff dimension at least $(d+2)/2$.

We wanted to avoid giving separate arguments for f_δ^* and f_δ^{**} and will therefore base the proof of Theorem 1 on certain axioms which are satisfied by both of them. In order to do this we first have to make a couple of (well-known) reductions in the problem. To begin with, it clearly suffices to prove Theorem 1 for functions f which are supported in a fixed compact set. Next, instead of f_δ^{**} it is more convenient to work with a certain variant. If $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ then define

$$f_\delta^{***} : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$$

via

$$f_\delta^{***}(x) = \sup_T \frac{1}{|T|} \int_T |f| \,,$$

where T runs through all cylinders with length 1 and cross section radius δ whose axis makes an angle less than $\pi/100$ with the d-th coordinate direction. We then have:

Any estimate of the form $\|f_\delta^{***}\|_q \leq A(\delta)\,\|f\|_p$ with $q \geq p$, valid for all functions f with fixed compact support, implies a corresponding estimate $\|f_\delta^{**}\|_q \leq C\,A(\delta)\,\|f\|_p$ for functions f with fixed compact support.

This is proved as follows: the assumption means that if

$$M_\delta f(x) = \sup_T \frac{1}{|T|} \int_T |f|$$

with T running through $1 \times \delta$ cylinders containing x whose axis makes an angle less than $\pi/100$ with e_d, then

$$\int_{\mathbb{R}^{d-1}} (M_d f(\alpha, 0))^q \, d\alpha \lesssim \|f\|_p^q \,.$$

Clearly 0 here could be replaced by t for any t. Integrating dt over a suitable compact set we obtain $\|M_\delta f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_p$. It remains only to remove the restriction that the axis of T make an angle less or equal than $\pi/100$ with e_d. However this is easily done by using finitely many different choices of coordinates, so the proof is complete. It follows that in order to prove Theorem 1 it suffices to prove

Theorem 1b. (1) *holds for $M_\delta f = f_\delta^*$ or f_δ^{***}, $p = (d+2)/2$, $q = (d-1)\,p'$, assuming f is supported in the unit disc.*

Next, let $M(d,1)$ be all lines in $\mathbb{R}^d$. We mean all lines here and not all lines through the origin, *i.e.* $M(d,1)$ is a $(2d-2)$-dimensional manifold. We can map $M(d,1)$ onto $\mathbb{P}^{d-1}$ via $\ell \to e_\ell$ where e_ℓ is the line through the origin parallel to ℓ. It is convenient to fix a Riemannian metric on $M(d,1)$ and let $\operatorname{dist}(\ell_1, \ell_2)$, $\ell_1, \ell_2 \in M(d,1)$ be the associated distance function. Since we will work locally we do not care what the metric is and just note the following. Let D be a disc in $\mathbb{R}^d$, $\tilde{D}$ the concentric disc with $\operatorname{radius}(\tilde{D}) = 100\operatorname{radius}(D)$. If ℓ_1 and ℓ_2 are lines which intersect D then dist (ℓ_1, ℓ_2) is comparable to the Hausdorff distance between $\ell_1 \cap \tilde{D}$ and $\ell_2 \cap \tilde{D}$ and therefore also satisfies

$$\text{(2)} \qquad \operatorname{dist}(\ell_1, \ell_2) \approx \theta(\ell_1, \ell_2) + d_{\min}(\ell_1, \ell_2),$$

where $\theta(\ell_1, \ell_2) \in [0, \pi/2]$ is the angle between e_{ℓ_1} and e_{ℓ_2} and

$$d_{\min}(\ell_1, \ell_2) = \inf\{|x - y| : \ x \in \ell_1 \cap \tilde{D}, \ y \in \ell_2 \cap \tilde{D}\}$$

(constants depend on D).

Now let $(\mathcal{A}, d)$ be a metric space (necessarily bounded, by (4) below) with a measure μ satisfying

$$\text{(3)} \qquad \mu(D(\alpha, \delta)) \approx \delta^m, \qquad \alpha \in \mathcal{A}, \quad \delta \leq \operatorname{diam} \mathcal{A},$$

for a certain $m \in \mathbb{R}^+$. Here $D(\alpha, \delta)$ is the δ-disc centered at α, and we will also use the notation $|E|$ for $\mu(E)$. Suppose that for each $\alpha \in \mathcal{A}$ a subset $F_\alpha \subset M(d, 1)$ is given, with $\overline{\cup_\alpha F_\alpha}$ compact and with

$$\text{(4)} \qquad d(\alpha, \beta) \lesssim \inf_{\substack{\ell \in F_\alpha \\ m \in F_\beta}} \operatorname{dist}(\ell, m), \qquad \text{for all } \alpha, \beta \in \mathcal{A}.$$

If $f : \mathbb{R}^d \to \mathbb{R}$ then we define $M_\delta f : \mathcal{A} \to \mathbb{R}$ by

$$\text{(5)} \qquad M_\delta f(\alpha) = \sup_{\ell \in F_\alpha} \sup_{a \in \ell} \frac{1}{|T_\ell^\delta(a)|} \int_{T_\ell^\delta(a)} |f|,$$

where $T_\ell^\delta(a)$ is the cylinder with length 1, radius δ, axis ℓ and center a.

REMARK. This setup arises as follows. Suppose m is an integer, U is an open subset of $M(d, 1)$ with compact closure, M is an m-dimensional manifold (with a Riemannian metric) and F a smooth map from U into M which can be extended smoothly to a neighborhood of $\overline{U}$. Assume that the range $F(U)$ contains a certain open subset $\mathcal{A}$ of M with smooth boundary. For $\alpha \in \mathcal{A}$ define $F_\alpha = F^{-1}(\alpha)$. It is easy to see that (3) and (4) will then hold.

In practice, m is always less or equal than $d - 1$. Examples with $m = d - 1$ are

I) $U =$ all lines intersecting $D(0, 1)$, $M = \mathcal{A} = \mathbb{P}^{d-1}$, $F(\ell) = e_\ell$. Then $M_\delta f = f_\delta^*$.

II) U = all lines intersecting $D(0,1)$ and making an angle less than $\pi/100$ with the x_d-axis, $M = \mathbb{R}^{d-1}$, $F(\ell) = \ell \cap \mathbb{R}^{d-1}$, $\mathcal{A} = F(U)$, where of course we are identifying $\mathbb{R}^{d-1}$ with the points in $\mathbb{R}^d$ with last coordinate zero. Then $M_\delta f = f_\delta^{***}$.

The natural analogue of the conjecture (1) in the general case where m is not necessarily $d-1$ is that

$$(6)\quad \|M_\delta f\|_q \leq C_\varepsilon\, \delta^{-(d/p-1+\varepsilon)}\, \|f\|_p\,, \quad 1 \leq p \leq m+1\,,\ 1 \leq q \leq mp'\,.$$

If $p \leq (m+2)/2$ then (6) is true in the general context (3), (4), (5). This is implicit both in [1] and in [7], [8]; we will give an argument based on [1] in Section 2 below. In many cases no improvement is possible as we will explain in Section 5. However, in examples I), II) the following additional property $(*)$ is satisfied. Here if Π is a 2-plane in $\mathbb{R}^d$ we let $M(\Pi,1)$ be the lines contained in Π, and $\operatorname{dist}(\ell, M(\Pi,1)) = \inf_{m\in M(\Pi,1)} \operatorname{dist}(\ell, m)$. A δ-separated subset of $\mathcal{A}$ means of course a subset $\{\alpha_j\}$ such that $j \neq k$ implies $d(\alpha_j, \alpha_k) \geq \delta$.

Property $(*)$. *If $\ell_0 \in \cup_\alpha F_\alpha$ and Π is a 2-plane containing ℓ_0, and if $\sigma \geq \delta$, and if $\{\alpha_j\}_{j=1}^N$ is a δ-separated subset of $\mathcal{A}$ and for each j there is $\ell_j \in F_{\alpha_j}$ with $\operatorname{dist}(\ell_j, M(\Pi,1)) < \delta$ and $\operatorname{dist}(\ell,\ell_0) < \sigma$, then $N \leq C\sigma/\delta$.*

REMARK (intended as motivation): When M_δ arises from a foliation as discussed in the preceding Remark, property $(*)$ is roughly the statement that there is no 2-plane Π such that each line contained in Π belongs to a different fiber F_α. More precisely, one should require the infinitesimal version of this condition -see Section 5.

To verify property $(*)$ in examples I), II) it suffices to show that the set $\{\alpha_j\}$ in question is contained in the intersection of a $C\sigma$-disc with a $C\delta$-neighborhood of a curve. It is clearly contained in a $C\sigma$-disc centered at $F(\ell_0)$, so it suffices to show that it is contained in a $C\delta$-neighborhood of a curve. This in turn will follow if the set

$$\gamma \stackrel{\text{def}}{=} \{\alpha :\ F^{-1}\alpha \cap M(\Pi,1) \neq \varnothing\}$$

is a curve. However, in cases I) and II) γ is respectively the great circle on the sphere (mod ± 1) obtained by intersecting the sphere with the translate of Π to the origin, and the intersection of Π with $\mathbb{R}^{d-1}$; note that in case II) Π and $\mathbb{R}^{d-1}$ intersect transversally since Π contains ℓ_0 which makes an angle less or equal than $\pi/100$ with the x_d axis.

Our proof of Theorem 1 works in the abstract context (3), (4), (5) provided property $(*)$ is satisfied. Namely, we will prove the following result.

Theorem 1c. *Assume* $2 \leq m \leq d-1$, (3), (4) *and property* $(*)$. *Then the estimate* (6) *is valid for the maximal function defined by* (5) *provided* $p \leq (m+3)/2$.

As indicated above, this result includes Theorems 1 and 1b. We had trouble deciding whether to use the axiomatic setup but eventually decided to do so since it gives a simultaneous proof for f_δ^* and f_δ^{**} and complicates the arguments only in technical respects. We also hope it may be of some interest. However, when $m = d-1$ (which would seem to be the main case) property $(*)$ and also the conclusion of Theorem 1c hold essentially only in examples I) and II); see Proposition 5.1.[1]

We finish this introduction by making some standard remarks about the definition (3), (4), (5). If $E \subset \mathcal{A}$ then a maximal δ-separated subset of E is of course a subset $\{\alpha_k\} \subset E$ which is δ-separated and is maximal with respect to this property. If $\{\alpha_k\}$ is a maximal δ-separated subset of E, then the discs $D(\alpha_k, \delta)$ cover E by the maximality property, and furthermore the concentric $\delta/2$-discs are disjoint. It follows that the cardinality of a maximal δ-separated subset of E is greater or equal than $C^{-1}\,|E|/\delta^m$ for a certain constant C, and also that if $\delta < \sigma$ then any δ-separated set $\{\alpha_k\}_{k=1}^M$ has a σ-separated subset $\{\alpha_{k_j}\}_{j=1}^{\overline{M}}$ with $\overline{M} \geq C^{-1}(\delta/\sigma)^m M$. Finally, if $\{\alpha_k\}$ is any δ-separated subset (maximal or not) and if A is any constant then there is a constant C_A such that no point $\alpha \in \mathcal{A}$ belongs to more than C_A discs of the form $D(\alpha_k, A\delta)$. This "bounded overlap" follows from the doubling property of μ (a consequence of (3)) in a standard way using disjointness of the discs $D(\alpha_k, \delta/2)$.

2. Preliminaries.

This section is expository. Its purpose is to give convenient forms of some known results. We first introduce some more notation. If $\rho \geq$

[1] We only consider straight lines in this paper.

$\delta > 0$ and $\ell \in M(d,1)$, $a \in \ell$, then $T_\ell^{\rho\delta}(a)$ will mean the cylinder with length ρ, cross section radius δ, axis contained in ℓ, and center a. $\tilde{T}_\ell^{\rho\delta}(a)$ will mean the same as $T_\ell^{(100\rho)(100\delta)}(a)$. We will usually take $\rho = 1$, and we abbreviate $T_\ell^{1\delta}(a)$ and $\tilde{T}_\ell^{1\delta}(a)$ by $T_\ell^\delta(a)$ and $\tilde{T}_\ell^\delta(a)$ respectively. We will also drop the a argument when no confusion will result. Thus T_ℓ^δ means "$T_\ell^\delta(a)$ for some $a \in \ell$", etc. Finally if Π is a 2-plane and $\delta > 0$ then

$$\Pi^\delta \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : \ |x - y| < \delta\}.$$

The first result we want to recall is a simple geometrical fact and was already used *e.g.* in [6].

(7) For any $T_{\ell_1}^\delta$ and $T_{\ell_2}^\delta$, $T_{\ell_1}^\delta \cap T_{\ell_2}^\delta$ is contained in a $T_{\ell_1}^{(C\delta/\theta(\ell_1,\ell_2))\delta}$

In particular,

$$\text{For any } T_{\ell_1}^\delta \text{ and } T_{\ell_2}^\delta, \ |T_{\ell_1}^\delta \cap T_{\ell_2}^\delta| \lesssim \frac{\delta^d}{\theta(\ell_1,\ell_2) + \delta}. \tag{8}$$

The second is a version of an argument of Bourgain [1, p. 153-154]. We formulate it as a lemma:

Lemma 2.1. *Suppose $\{T_{\ell_j}^\beta\}_{j=1}^M$ are tubes and E a set. Assume that $\varepsilon > \beta$, that*

$$T_{\ell_j}^\beta \cap T_{\ell_k}^\beta \neq \varnothing \quad \textit{implies} \quad \theta(\ell_j, \ell_k) \geq C^{-1}\varepsilon \tag{9}$$

and that for $a \in \mathbb{R}^d$, $1 \leq j \leq M$,

$$\left|T_{\ell_j}^\beta \cap E \cap (\mathbb{R}^d \backslash D(a, \frac{\beta}{\varepsilon}))\right| \geq \frac{\lambda}{2}\,|T_{\ell_j}^\beta|. \tag{10}$$

Then $|E| \gtrsim \lambda\,\beta^{d-1}\sqrt{M}$, where the constant depends on C.

PROOF. ([1]) We have $\sum_j |T_{\ell_j}^\beta \cap E| \gtrsim \lambda\,\beta^{d-1}M$, so there must be a point $a \in E$ which belongs to $\gtrsim (\lambda\,\beta^{d-1}M/|E|)\,T_{\ell_j}^\beta$'s. The sets $\tau_j \stackrel{\text{def}}{=} T_{\ell_j}^\beta \cap (\mathbb{R}^d \backslash D(a, \beta/\varepsilon))$, where $T_{\ell_j}^\beta$ runs over this set of tubes, have bounded overlap (*i.e.* no point belongs to more than a fixed finite number of them) for the following reason. First, it is easy to see using the hypothesis (9) that for any constant A and any given j there are at least C_A tubes $T^\beta_{\ell_k}$ containing a and with $\theta(\ell_j,\ell_k) \leq A\,\varepsilon$. Thus it suffices to show that if $\theta(\ell_j,\ell_k) \geq A\,\varepsilon$ for a large fixed constant A, then $\tau_j \cap \tau_k = \varnothing$, *i.e.* $T^\beta_{\ell_j} \cap T^\beta_{\ell_k} \subset D(a,\beta/\varepsilon)$. On the other hand by (7), $\theta(\ell_j,\ell_k) \geq A\,\varepsilon$ implies $\operatorname{diam}(T^\beta_{\ell_j} \cap T^\beta_{\ell_k}) \leq \beta/\varepsilon$ for large A. Since the point a belongs to each $T^\beta_{\ell_j}$ the bounded overlap follows. By assumption

$$|\tau_j \cap E| \geq \frac{\lambda}{2}\,|T^\beta_{\ell_j}| \approx \lambda\,\beta^{d-1}\,, \qquad \text{for each } j\,.$$

Hence

$$\frac{\lambda\,\beta^{d-1}\,M}{|E|}\,\lambda\,\beta^{d-1} \lesssim |E|$$

and the lemma follows.

REMARK. In [1], this lemma is used together with additional combinatorial arguments to obtain the result we mentioned in the introduction, that the conjecture (1) is true for certain $p > (d+1)/2$ (and $q = p$). If one reads [1] carefully then one sees that the $p = (d+1)/2$ case follows directly from Lemma 2.1. The argument proves the following[2]:

Proposition 2.1. *Assume* (3), (4). *Then the estimate* (6) *is valid for the maximal function defined by* (5) *provided* $p \leq (m+2)/2$.

PROOF. ([1]) It suffices to prove the following restricted weak type estimate at the endpoint $p = (m+2)/2$, $q = m\,p' = m+2$,

$$|\{\alpha \in \mathcal{A} :\ M_\delta f(\alpha) \geq \lambda\}| \lesssim \left(\frac{|E|}{\delta^{d-(m+2)/2}\,\lambda^{(m+2)/2}}\right)^2, \tag{11}$$

where E denotes a set and f its characteristic function. To prove (11), let $\varepsilon = A\,\delta/\lambda$ for a large fixed constant A, and choose a maximal ε-separated subset $\{\alpha_j\}_{j=1}^M$ of the set $\{\alpha \in \mathcal{A} :\ M_\delta f(\alpha) \geq \lambda\}$ and, for each j, a tube $T^\delta_{\ell_j}$ with $\ell_j \in F_{\alpha_j}$ and $|T^\delta_{\ell_j} \cap E| \geq \lambda\,|T^\delta_{\ell_j}|$. Then $M \gtrsim \varepsilon^{-m}\,|\{\alpha :\ M_\delta f(\alpha) \geq \lambda\}|$. The choice of ε implies that $|T^\delta_{\ell_j} \cap D(a,\delta/\varepsilon)| \leq \lambda\,|T^\delta_{\ell_j}|/2$, in particular (10) holds (with β replaced by δ).

[2] Actually the argument proves a slightly sharper result: the $\delta^{-\epsilon}$ factor is only needed at the endpoint $p=(m+2)/2$. Proposition 2.1 also follows from the results in [7] or [4] (at least if $m=d-1$), in fact the argument based on [4] gives a still sharper result where the $\delta^{-\epsilon}$ factor is not needed at all.

Also (9) holds for a certain constant C by (4) and (2). We therefore obtain $M \lesssim (|E|/\lambda\delta^{d-1})^2$, *i.e.*

$$|\{\alpha : M_\delta f(\alpha) \geq \lambda\} \lesssim \Big(\frac{|E|}{\delta^{d-(m+2)/2}\,\lambda^{(m+2)/2}}\Big)^2,$$

as claimed.

In order to prove Theorem 1c we will combine this type of argument with the L^2 argument of Cordoba and we now state a convenient form of the latter.

Lemma 2.2. *Assume $E \subset \mathbb{R}^d$ is a set, Π is a 2-plane in $\mathbb{R}^d$, and $\{T^\delta_{\ell_j}\}_{j=1}^M$ are tubes which are contained in $\Pi^{C_0\delta}$. Assume the following conditions:*

i) $|E \cap T^\delta_{\ell_j}| \geq \lambda\,|T^\delta_{\ell_j}|$, *for all* j.

ii) $\operatorname{card}(\{j : T^\delta_{\ell_j} \subset \tilde{T^\sigma_{\ell_k}}\}) \leq C_0\,\sigma/\delta$, *for all* $\sigma \in (\delta,1)$ *and all* k.

Then

$$M\,\delta^{d-1} \leq C\,\frac{|E \cap \Pi^{C_0\delta}|}{\lambda^2}\,\log\frac{1}{\delta}, \tag{12}$$

where C depends on C_0.

PROOF. ([6]) We may assume that $E \subset \Pi^{C_0\delta}$, and then we have

$$\begin{aligned}
M\,\lambda\,\delta^{d-1} &\leq \sum_j |E \cap T^\delta_{\ell_j}| \\
&= \int_E \sum \chi_{T^\delta_{\ell_j}} \\
&\leq |E|^{1/2}\Big\|\sum \chi_{T^\delta_{\ell_j}}\Big\|_2 \\
&= |E|^{1/2}\Big(\sum_i\sum_j |T^\delta_{\ell_i} \cap T^\delta_{\ell_j}|\Big)^{1/2} \\
&\leq |E|^{1/2}\Big(C\,M\,\delta^{d-1} + \sum_i\sum_{j\neq i} |T_{\ell_i} \cap T_{\ell_j}|\Big)^{1/2} \\
&\leq |E|^{1/2}\Big(C\,M\,\delta^{d-1}
\end{aligned}$$

$$+\sum_i \sum_{k=0}^{\log_2 \pi/\delta} \frac{\delta^{d-1}}{2^k} \operatorname{card}(\{j : T^\delta_{\ell_i} \cap T_{\ell_j} \neq \varnothing, \; 2^k\,\delta \le \theta(\ell_i,\ell_j) \le 2^{k+1}\,\delta\}) \Big)^{1/2}$$

$$\le C\,|E|^{1/2} \Big(M\,\delta^{d-1} + \sum_i \sum_{k=0}^{\log_2 \pi/\delta} \delta^{d-1}\Big)^{1/2},$$

where the last line follows from the assumption ii). So

$$M\,\lambda\,\delta^{d-1} \le C\,|E|^{1/2} \Big(M \log\frac{1}{\delta}\,\delta^{d-1}\Big)^{1/2}$$

which is equivalent to (12).

3. Main argument.

This section will be the proof of the following Lemma 3.1.

Lemma 3.1. *Assume $d \ge 3$, $2 \le m \le d-1$. Suppose that a set $E \subset \mathbb{R}^d$ and tubes $\{T^{\rho\delta}_{\ell_j}\}_{j=1}^M$, $\delta \le \rho/100$ are given and that with a sufficiently large constant C_0 and some constant C,*

i) $|E \cap T^{\rho\delta}_{\ell_j}| \ge \lambda\,|T^{\rho\delta}_{\ell_j}|$, *for all j.*

ii) *If $x \in \mathbb{R}^d$ then, for all j,*

$$\Big|E \cap T^{\rho\delta}_{\ell_j} \cap D\big(x, \big(\log\tfrac{\rho}{\delta}\big)^{-\nu}\rho\big)\Big| \le C_0^{-1}\,\frac{\lambda}{\log\frac{\rho}{\delta}}\,|\,T^{\rho\delta}_{\ell_j}|,$$

iii) *For any k and any $\sigma \in (\delta,\rho)$ the set*

$$\{j : \tilde T^{\rho\sigma}_{\ell_j} \cap \tilde T^{\rho\sigma}_{\ell_k} \neq \varnothing,\ \theta(\ell_j,\ell_k) \le \sigma/\rho\}$$

has cardinality less or equal than $C\,(\sigma/\delta)^m$.

iv) *For any k, any 2-plane Π with $\ell_k \subset \Pi$ and any $\sigma \in (\delta,\rho)$, the set*

$$\{j : T^{\rho\delta}_{\ell_j} \subset \tilde T^{\rho\sigma}_{\ell_k} \cap \Pi^{C_0\delta}\}$$

has cardinality less or equal than $C\,\sigma/\delta$.

Then for large ν

$$(13)\qquad \begin{aligned}&\rho^{-d}\,|E|\\ &\quad\geq C_\nu^{-1}\,\lambda^2\,(\rho^{-1}\delta)^{d-(m+3)/2}\,\Big(M\big(\frac{\delta}{\rho}\big)^m\Big)^{(1+1/m)/2}\,\Big(\log\frac{\rho}{\delta}\Big)^{-d\nu}.\end{aligned}$$

REMARKS. 1) Here $\tilde{T}^{\rho\sigma}_{\ell_k}$ is taken concentric with $T^{\rho\delta}_{\ell_k}$, *i.e.* if $T^{\rho\delta}_{\ell_k} = T^{\rho\delta}_{\ell_k}(a)$ then $\tilde{T}^{\rho\sigma}_{\ell_k} = T^{(100\rho)(100\sigma)}_{\ell_k}(a)$.

2) In the proof we may assume $\rho = 1$ since we can reduce to this case by scaling. We will in fact assume $\rho = 1$ throughout this section and will denote $T^{\rho\delta}_{\ell_j}$ by $T^{\delta}_{\ell_j}$, etc, as mentioned at the beginning of Section 2. We may then also assume that E is contained in a fixed compact set, say, the unit disc. It then follows by iii) with $\sigma \approx 1$ that $M \lesssim \delta^{-m}$.

3) This remark is intended as motivation. An immediate corollary of Lemma 3.1 is that any Besicovitch set in $\mathbb{R}^d$ has Minkowski dimension greater or equal than $(d+2)/2$, *i.e.* the following statement: *suppose* E *is a compact subset of* $\mathbb{R}^d$ *which contains a unit line segment in every direction. Let* $E_\delta = \{x :\ \mathrm{dist}\,(x,E) < \delta\}$. *Then*

$$(14)\qquad |E_\delta| \geq C_\nu^{-1}\,\delta^{(d-2)/2}\,\Big(\log\frac{1}{\delta}\Big)^{-d\nu}$$

for large ν. To prove this statement set $m = d-1$, $\rho = 1$. Let $\{e_j\}_{j=1}^{M}$ be a maximal δ-separated subset of $\mathbb{P}^{d-1}$. For each j there is a line in the e_j direction with a unit segment on it belonging to E, and therefore a tube $T^{\delta}_{\ell_j}$ with $e_{\ell_j} = e_j$ which is contained in E_δ. Thus i) of Lemma 3.1 holds with E replaced by E_δ and $\lambda = 1$, and then ii) holds tautologically if $\nu > 1$ and δ is small. Also iii) holds since the directions of the ℓ_j must belong to a $C\sigma$-disc in $\mathbb{P}^{d-1}$, and iv) holds since the directions of the ℓ_j must also belong to a $C\delta$-neighborhood of the great circle determined by Π. We conclude that (13) holds with E replaced by E_δ, and $\rho = 1$, $\lambda = 1$, $m = d-1$. Since $M \approx \delta^{-(d-1)}$ we obtain (14). Theorem 1 is a more refined result and requires an additional argument which we will give in Section 4.[3]

We start the proof by observing that we can make the following additional assumption:

$$(15)\qquad \text{If } T^{\delta}_{\ell_j}\cap T^{\delta}_{\ell_k} \neq \varnothing \text{ then } \theta(\ell_j,\ell_k) \geq \delta\,.$$

---

[3] Actually, for the statement about the dimension of Besicovitch sets we only needed Lemma 3.1 in the case λ=1. In this case the proof can be simplified.

Namely, if we let $\{T^\delta_{\ell_{j_i}}\}_{i=1}^{\overline{M}}$ be a subset of $\{T^\delta_{\ell_j}\}$ which satisfies (15) and is maximal with respect to this property, then every tube $T^\delta_{\ell_j}$ must satisfy $T^\delta_{\ell_j} \cap T^\delta_{\ell_{j_i}} \neq \varnothing$ and $\theta(\ell_j, \ell_{j_i}) \leq \delta$ for some i. It follows by iii) with $\sigma = \delta$ that $\overline{M} \geq A^{-1}M$. We could replace $\{T^\delta_{\ell_j}\}$ by $\{T^\delta_{\ell_{j_i}}\}$, so we may assume that $\{T^\delta_{\ell_j}\}$ satisfies (15).

We now fix a number N and consider the following possibilities.

I. (low multiplicity) There are at least $M/2$ values of j such that

$$|\{x \in T^\delta_{\ell_j} \cap E : \ \mathrm{card}(\{i : \ x \in T^\delta_{\ell_i}\}) \leq N\}| \geq \frac{\lambda}{2}\, |T^\delta_{\ell_j}|\,.$$

II_σ. (high multiplicity at angle σ) There are at least $C_1^{-1} M$ $(\log 1/\delta)^{-1}$ values of j such that

$$\begin{aligned} \Big|\Big\{x \in T^\delta_{\ell_j} \cap E :\ \mathrm{card}(\{i : x \in T^\delta_{\ell_i} \text{ and } \sigma \leq \theta(\ell_i, \ell_j) \leq 2\sigma\}) \\ \geq \Big(C_1 \log \frac{1}{\delta}\Big)^{-1} N\Big\}\Big| \\ \geq \Big(C_1 \log \frac{1}{\delta}\Big)^{-1} \lambda\, |T^\delta_{\ell_j}|\,. \end{aligned} \tag{16}$$

Lemma 3.2. *There is a number N for which we have both* I, *and also* II_σ *for some* $\sigma \in [\delta, \pi]$.

PROOF. Take the smallest $N \in \mathbb{Z}^+$ for which I holds. Then there are $M/2$ values of j for which

$$|\{x \in T^\delta_{\ell_j} \cap E : \ \mathrm{card}(\{i : \ x \in T^\delta_{\ell_i}\}) \geq N\}| \geq \frac{\lambda}{2}\, |T^\delta_{\ell_j}|\,. \tag{17}$$

For any j as in (17) and any $x \in T^\delta_{\ell_j}$ with $\mathrm{card}(\{i : \ x \in T^\delta_{\ell_i}\}) \geq N$, (15) implies there is some $k \in \{1, \dots, \log_2 \pi/\delta\}$ such that $\mathrm{card}(\{i : \ x \in T^\delta_{\ell_i},\ \theta(\ell_i, \ell_j) \in [2^{k-1}\delta, 2^k\delta]\}) \geq (\log_2 \pi/\delta)^{-1} N$. Thus for any j as in (17) there is some $k \in \{1, \dots, \log_2 \pi/\delta\}$ such that

$$\begin{aligned} \Big|\Big\{x \in T^\delta_{\ell_j} \cap E :\ \mathrm{card}(\{i : \ x \in T^\delta_{\ell_i},\ \theta(\ell_i, \ell_j) \in [2^{k-1}\delta, 2^k\delta]\}) \\ \geq \Big(\log_2 \frac{\pi}{\delta}\Big)^{-1} N\Big\}\Big| \\ \geq \Big(\log_2 \frac{\pi}{\delta}\Big)^{-1} \frac{\lambda}{2}\, |T^\delta_{\ell_j}|\,. \end{aligned}$$

It follows that II_σ holds for some $\sigma = 2^k\delta$.

The logic will now be as follows: we make separate estimates in the cases I and II_σ and then apply them both with N given by Lemma 3.2 to obtain (13). The estimate in case I is very simple:

Lemma 3.3. *If* i) *of Lemma* 2.1 *and* I *hold then* $|E| \gtrsim \lambda M \delta^{d-1}/N$.

PROOF. Let $\tilde{E} = \{x \in E : \operatorname{card}(\{i : x \in T^\delta_{\ell_i}\}) \le N\}$. Then $|T^\delta_{\ell_j} \cap \tilde{E}| \ge \lambda |T^\delta_{\ell_j}|/2$ for $M/2$ values of j, and

$$\begin{aligned} |E| &\ge |\tilde{E}| \ge |\tilde{E} \cap (\cup_j T^\delta_{\ell_j})| \\ &\ge N^{-1} \sum_j |\tilde{E} \cap T^\delta_{\ell_j}| \ge \frac{\lambda}{2N} \sum_j |T^\delta_{\ell_j}| \gtrsim \frac{\lambda M \delta^{d-1}}{N} \end{aligned}$$

and the lemma is proved.

The idea behind the estimate in case II_σ is as follows. We are evidently in a situation where many tubes intersect some given tube T. Each one of these tubes is then contained in a $C_0\delta$-neighborhood of some 2-plane containing the axis of T. The latter sets have nice intersection properties *i.e.* the same intersection properties as tubes through the origin in $\mathbb{R}^{d-1}$. This allows us to use Lemma 2.2 separately for each 2-plane and then sum the resulting estimates.

We carry this out as follows:

Lemma 3.4. *Assume* i), ii) *and* iv) *of Lemma* 3.1, *and suppose that* $T^\delta_{\ell_j}$ *is a tube for which* (16) *holds. Then*

$$|E \cap \tilde{T}^\sigma_{\ell_j}| \gtrsim \lambda^3 \sigma \delta^{d-2} N \left(\log \frac{1}{\delta}\right)^{-(d-2)\nu-3} \tag{18}$$

PROOF. Let $\mathcal{F}$ be the set of all tubes $T^\delta_{\ell_i}$ such that $T^\delta_{\ell_i} \cap T^\delta_{\ell_j} \cap E \neq \varnothing$ and $\sigma \le \theta(\ell_i, \ell_j) \le 2\sigma$ and let $|\mathcal{F}|$ be the cardinality of $\mathcal{F}$. If $T^\delta_{\ell_i} \in \mathcal{F}$ and $x \in \mathbb{R}^d$ then for a suitable constant C_2 the set

$$\left\{x \in T^\delta_{\ell_i} : \operatorname{dist}(x, \ell_j) \le C_2^{-1} \sigma \left(\log \frac{1}{\delta}\right)^{-\nu}\right\}$$

is contained in $T^\delta_{\ell_i} \cap \tilde{T}^{\sigma(\log 1/\delta)^{-\nu}}_{\ell_j}$ and therefore by (7) is contained in a disc of radius $C\,(\log 1/\delta)^{-\nu}$. Consequently by ii)

$$(19)\qquad \Big|\Big\{x \in T^\delta_{\ell_i} \cap E:\ \operatorname{dist}(x,\ell_j) \geq C_2^{-1}\,\sigma\Big(\log\frac{1}{\delta}\Big)^{-\nu}\Big\}\Big| \geq \frac{\lambda}{2}\,|T^\delta_{\ell_i}|\,,$$

provided C_0 has been chosen large enough. We choose 2-planes Π_k containing ℓ_j so that

A) any tube $T^\delta_{\ell_i} \in \mathcal{F}$ is contained in $\Pi_k^{C_0\delta}$ for some k,

B) any point x with $\operatorname{dist}(\ell_j, x) \geq C_2^{-1}\sigma(\log 1/\delta)^{-\nu}$ belongs to at most $C\,(\log 1/\delta)^{(d-2)\nu}\Pi_k^{C_0\delta}$'s.

For this, it suffices to choose a maximal δ/σ-separated set $\{\alpha_k\}$ in the unit sphere (modulo ±1) in the $d-1$-dimensional space $e_j^\perp \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d:\ \langle x, e_j\rangle = 0\}$, where $e_j = e_{\ell_j}$ is the direction of e_j and consider the planes through ℓ_j spanned by the α_k directions. It is easy to see that A) and B) will then hold; we will now prove this. For A), we let $e_i = e_{\ell_i} \in \mathbb{P}^{d-1}$ be the direction of ℓ_i and write $e_i = ae_j + be$ with $e \perp e_j$. Then $|b|$ is clearly the sine of the angle between e_i and e_j and is therefore $\lesssim \sigma$. Now choose k with $|e \pm \alpha_k| \leq \delta/\sigma$ (this is possible by maximality of $\{\alpha_k\}$). We assume for notational purposes that $|e - \alpha_k| \leq \delta/\sigma$. Since $T^\delta_{\ell_i} \cap T^\delta_{\ell_j} \neq \varnothing$ there are $x_1 \in \ell_j$ and $x_2 \in \ell_i$ with $|x_1 - x_2| \lesssim \delta$. Given $x \in T^\delta_{\ell_i}$, choose $y \in \ell_i$ with $|x - y| \lesssim \delta$. Then we have

$$x = x_1 + (y - x_2) + (x - y) + (x_2 - x_1)\,.$$

Here $y - x_2 = te_i$ with $t \lesssim 1$. So we may further write

$$x = (x_1 + t\,a\,e_j + t\,b\,\alpha_k) + t\,b\,(e - \alpha_k) + (x - y) + (x_2 - x_1)\,.$$

The term $x_1 + tae_j + tb\alpha_k$ belongs to Π_k, and the remaining terms are all $\lesssim \delta$ in absolute value, so we have proved A).

For B), we may clearly assume ℓ_j passes through the origin. We fix x as in B) and let $x_\perp$ be the orthogonal projection of x on $e_j^\perp$. Thus $x_\perp = te$ with $e \in e_j^\perp$, $|e| = 1$, $t \geq C_2^{-1}\sigma(\log 1/\delta)^{-\nu}$. If $x \in \Pi_k^{C_0\delta}$ then it follows that $\operatorname{dist}(x_\perp, \operatorname{span}\{\alpha_k\}) \leq C_0\delta$, so $|e \pm \alpha_k| \lesssim t^{-1}\delta \lesssim \delta(\log 1/\delta)^\nu/\sigma$. Since the α_k are δ/σ- separated there are $\lesssim (\log 1/\delta)^{(d-2)\nu}$ such values of k, so B) is proved.

For each k, let $\mathcal{F}_k$ be those tubes $T^\delta_{\ell_i} \in \mathcal{F}$ which are contained in $\Pi_k^{C_0\delta}$, and let $|\mathcal{F}_k|$ be the cardinality of $\mathcal{F}_k$. Note that by A) above, $\mathcal{F} = \cup_k \mathcal{F}_k$. Also, since $\theta(\ell_i, \ell_j) \le 2\sigma$, $T^\delta_{\ell_i}$ is contained in $\tilde{T}^\sigma_{\ell_j}$. We will apply Lemma 2.2 with E replaced by $E \cap \tilde{T}^\sigma_{\ell_j} \cap \{x : \operatorname{dist}(x, \ell_j) \ge C_2^{-1}\sigma(\log 1/\delta)^{-\nu}\}$, and with the 2-plane Π_k. Assumption i) of Lemma 2.2 holds (with $\lambda/2$ instead of λ) because of (19), and assumption ii) of Lemma 2.2 holds because of our current assumption iv). We conclude that

$$|\mathcal{F}_k|\,\delta^{d-1} \le C\,\frac{|E\cap \tilde{T}^\sigma_{\ell_j} \cap \Pi_k^{C_0\delta} \cap \{x : \operatorname{dist}(x,\ell_j) \ge C_2^{-1}\sigma(\log 1/\delta)^{-\nu}\}|}{\lambda^2}\,\log\frac{1}{\delta}\,,$$

for suitable C. It follows on summing over k that

$$\begin{aligned} |\mathcal{F}|\,\delta^{d-1} &\le \frac{C}{\lambda^2}\,\log\frac{1}{\delta} \\ &\quad\cdot \sum_k \left|E\cap \tilde{T}^\sigma_{\ell_j} \cap \Pi_k^{C_0\delta} \cap \left\{x : \operatorname{dist}(x,\ell_j) \ge C_2^{-1}\,\sigma\left(\log\frac{1}{\delta}\right)^{-\nu}\right\}\right| \\ &\lesssim \frac{C}{\lambda^2}\,|E\cap \tilde{T}^\sigma_{\ell_j}|\left(\log\frac{1}{\delta}\right)^{(d-2)\nu}\log\frac{1}{\delta} \end{aligned} \tag{20}$$

by B). On the other hand, since (16) holds we know that

$$\sum_{T^\delta_{\ell_i}\in\mathcal{F}} \chi_{T^\delta_{\ell_i}}(x) \ge \left(C_1\,\log\frac{1}{\delta}\right)^{-1} N\,,$$

for all x in a subset of $T^\delta_{\ell_j}$ with measure at least $\lambda(C_1\log 1/\delta)^{-1}|T^\delta_{\ell_j}|$. Accordingly

$$\begin{aligned} \left(C_1\,\log\frac{1}{\delta}\right)^{-1}\lambda\,|T^\delta_{\ell_j}| &\le N^{-1}C_1\,\log\frac{1}{\delta}\int_{T^\delta_{\ell_j}} \sum_{T^\delta_{\ell_i}\in\mathcal{F}} \chi_{T^\delta_{\ell_i}}(x)\,dx \\ &= N^{-1}C_1\,\log\frac{1}{\delta}\sum_{T^\delta_{\ell_i}\in\mathcal{F}} |T^\delta_{\ell_i}\cap T^\delta_{\ell_j}| \\ &\lesssim N^{-1}\,\frac{\delta^d}{\sigma}\,|\mathcal{F}|\,\log\frac{1}{\delta}\,, \end{aligned}$$

where the last line follows from (8). Thus $|\mathcal{F}| \gtrsim \sigma(\log 1/\delta)^{-2}N/\delta$. This inequality and (20) imply (18).

Lemma 3.5. *With the same assumptions as in Lemma* 3.4 *we also have*

$$|E \cap (\mathbb{R}^d \backslash D(a, (\log 1/\delta)^{-\nu})) \cap \tilde{T}^{\sigma}_{\ell_j}| \gtrsim \lambda^3 \sigma\, \delta^{d-2} N \left(\log \frac{1}{\delta}\right)^{-(d-2)\nu - 3},$$

for any $a \in \mathbb{R}^d$.

PROOF. We need only to observe that the hypotheses of Lemma 3.4 still hold (with λ replaced by $\lambda/2$) if E is replaced by

$$E \cap (\mathbb{R}^d \backslash D(a, (\log 1/\delta)^{-\nu})),$$

provided C_0 has been chosen large enough. The main point is that (16) holds; this follows from the corresponding statement for E since $|T^{\delta}_{\ell_j} \cap E \cap D(a, (\log 1/\delta)^{-\nu})|$ is small by assumption ii).

Lemma 3.6. *Assume the hypotheses of Lemma* 3.1, *and also* II_σ *for some* σ. *Then*

$$|E| \gtrsim \lambda^3 N\, \delta^{d-2} (M\, \delta^m)^{1/m} \left(\log \frac{1}{\delta}\right)^{-2d\nu}. \tag{21}$$

PROOF. If $\sigma \geq (M\delta^m)^{1/m} (\log 1/\delta)^{-\nu}$ this follows directly from Lemma 3.4, so we assume $\sigma \leq (M\delta^m)^{1/m} (\log 1/\delta)^{-\nu} \lesssim (\log 1/\delta)^{-\nu}$. By hypothesis there are $(C_1 \log 1/\delta)^{-1} M$ ℓ_j's for which (16) holds. Choose a subset $\{\ell_{j_k}\}_{k=1}^{\overline{M}}$ which is maximal with respect to the following property:

$$\tilde{T}^{\sigma}_{\ell_{j_k}} \cap \tilde{T}^{\sigma}_{\ell_{j_i}} \neq \varnothing \quad \text{implies} \quad \theta(\ell_{j_k}, \ell_{j_i}) \geq \sigma \left(\log \frac{1}{\delta}\right)^{\nu}.$$

Then

$$\overline{M} \gtrsim M \left(\log \frac{1}{\delta}\right)^{-1} \left(\frac{\delta}{\sigma \left(\log \frac{1}{\delta}\right)^{\nu}}\right)^{m};$$

this follows from the maximality using assumption iii) with σ replaced by $\sigma(\log 1/\delta)^{\nu}$, as in the argument after formula (15). We claim that Lemma 2.1 is applicable to the tubes $\tilde{T}^{\sigma}_{\ell_{j_k}}$ with β, λ, and ε there equal to σ,

$$C^{-1} \lambda^3 \frac{\delta^{d-2}}{\sigma^{d-2}} N \left(\log \frac{1}{\delta}\right)^{-(d-2)\nu - 3} \qquad \text{and} \qquad \sigma \left(\log \frac{1}{\delta}\right)^{\nu},$$

respectively. Namely, hypothesis (9) follows by construction, and (10) follows from Lemma 3.5. We conclude that

$$|E| \gtrsim \lambda^3 \sigma\, \delta^{d-2} N \left(\log\frac{1}{\delta}\right)^{-(d-2)\nu-3} \sqrt{M\left(\log 1/\delta\right)^{-1}\left(\frac{\delta}{\sigma\,(\log 1/\delta)^{\nu}}\right)^m}$$

$$\gtrsim \lambda^3 \sigma\, N\, \delta^{d-2} \left(\log\frac{1}{\delta}\right)^{-2d\nu} \sqrt{M\frac{\delta^m}{\sigma^m}}\,,$$

for large ν. The lemma now follows since $\sigma\,(M\,\delta^m/\sigma^m)^{1/2}$ is a decreasing function of σ when $m \geq 2$, hence minorized by its value at $(M\,\delta^m)^{1/m}$.

COMPLETION OF PROOF OF LEMMA 3.1. We need only choose N by Lemma 3.2 and then take the geometric mean of (21) and the estimate in Lemma 3.3.

4. Completion of the proof.

In order to prove Theorem 1c it suffices of course to prove the corresponding restricted weak type estimate at the endpoint, *i.e.* the estimate

$$|\{\alpha : M_\delta \chi_E(\alpha) \geq \lambda\}| \leq C_\varepsilon \left(\delta^{-\varepsilon}\frac{|E|}{\delta^{d-p}\lambda^p}\right)^{q/p},$$

where $p = (m+3)/2$ and $q = m\,p'$. We may also assume E is contained in the unit disc D. Furthermore (consider a maximal δ-separated subset) it suffices to prove the discrete analogue, *i.e.* that if $\{\alpha_j\}_{j=1}^M$ are δ-separated and if $\ell_j \in F_{\alpha_j}$ and $|E \cap T_{\ell_j}^{\delta}| \geq \lambda |T_{\ell_j}^{\delta}|$ then

$$M\,\delta^m \leq C_\varepsilon \left(\delta^{-\varepsilon}\frac{|E|}{\delta^{d-p}\lambda^p}\right)^{q/p}. \tag{22}$$

We will actually prove that if $\delta \leq \rho \leq 1$ and $\{\alpha_j\}_{j=1}^M$ are δ/ρ-separated and $\ell_j \in F_{\alpha_j}$ and if $|E \cap T_{\ell_j}^{\rho\delta}| \geq \lambda\, |T_{\ell_j}^{\rho\delta}|$, then

$$\rho^{-d}|E| \geq C_\varepsilon^{-1}\left(M\left(\frac{\delta}{\rho}\right)^m\right)^{p/q}\left(\frac{\delta}{\rho}\right)^{d-p}\lambda^p\left(\frac{\delta}{\rho}\right)^{\varepsilon}. \tag{23}$$

The case $\rho = 1$ gives (22).

In order to prove (23) we note first that any $T^{\rho\delta}_{\ell_j}$'s as there will automatically satisfy iii) and iv) of Lemma 3.1. Namely, to prove iii) fix k and σ. Let

$$J = \{j : \tilde{T}^{\rho\sigma}_{\ell_j} \cap \tilde{T}^{\rho\sigma}_{\ell_k} \neq \varnothing,\ \theta(\ell_j, \ell_k) \leq \sigma/\rho\}.$$

If $j \in J$ then $\operatorname{dist}(\ell_j, \ell_k) \lesssim \sigma/\rho$ by (2), so $d(\alpha_j, \alpha_k) \lesssim \sigma/\rho$ by (4). Thus the $\{\alpha_j\}_{j\in J}$ form a δ/ρ-separated subset of a $C\,\sigma/\rho$-disc, so $\operatorname{card} J \lesssim (\delta/\sigma)^m$, *i.e.* iii) holds.

The argument for iv) is similar. Fix k, a 2-plane Π and σ and let

$$J = \{j : T^{\rho\delta}_{\ell_j} \subset \tilde{T}^{\rho\sigma}_{\ell_k} \cap \Pi^{C_0\delta}\}.$$

If $j \in J$ then by assumption there is a segment of length ρ on ℓ_j which is contained in $\tilde{D}$ and is at distance less or equal than $C_0\,\delta$ from Π. Consequently by similar triangles, $\ell_j \cap \tilde{D}$ is at distance $\lesssim \delta/\rho$ from Π. We conclude that if $j \in J$ then $\operatorname{dist}(\ell_j, M(\Pi, 1)) \lesssim \delta/\rho$. We also know that $\operatorname{dist}(\ell_j, \ell_k) \lesssim \sigma/\rho$. Since the $\{\alpha_j\}_{j\in J}$ are δ/ρ-separated, property $(*)$ implies that $\operatorname{card} J \lesssim \sigma/\delta$, *i.e.* iv) holds.

We now fix ε, and will prove (23) for an appropriate constant C_ε by induction on ρ. The choice of C_ε requires some care. We first let B be a constant with the following property: *if $\delta < \varepsilon < 1$, then any δ-separated subset Y of $\mathcal{A}$ has an ε-separated subset Z with* $\operatorname{card}(Z) \geq B^{-1}(\delta/\varepsilon)^m \operatorname{card}(Y)$ (see the remarks at the end of the introduction). Next we fix ν large enough that $\nu\varepsilon > p$. (23) is trivial when $\rho \leq A\,\delta$ for any fixed constant A, provided C_ε is large enough.[4] We take A to satisfy $A \geq 100$, $3(\log A)^{-\nu} < 1$ and $(\log A)^{\nu\varepsilon - p} \geq (2B)^{p/q}\, C_0^p\, 3^\varepsilon$ (C_0 is the constant in Lemma 3.1), and determine C_ε by the following requirements: (23) should hold when $\rho \leq 3A\delta$, and $C_\varepsilon \geq 2^{(1+1/m)/2} C_\nu \sup_{t>A} t^{-\varepsilon}(\log t)^\nu$ where C_ν is the constant in (13).

In proving (23) we may suppose that $\rho \geq A\delta$ and (by the second requirement on A) that (23) has already been proved for parameters $\overline{\rho} \leq 3\rho(\log \delta/\rho)^{-\nu}$. We consider two cases: 1) There are at least $M/2$ values of j for which ii) of Lemma 3.1 holds; 2) There are at least $M/2$ values of j for which ii) fails.

In case 1) we simply apply (13) after deleting those ℓ_j for which

[4] In fact, if $\rho \leq A\delta$ then the δ/ρ-separation property implies an upper bound on M so that (23) follows from the obvious inequality $|E| \gtrsim \delta^d \lambda$.

ii) fails. Thus

$$\rho^{-d}|E| \geq C_\nu^{-1}\lambda^2(\rho^{-1}\delta)^{d-(m+3)/2}\Big(\frac{M}{2}\big(\frac{\delta}{\rho}\big)^m\Big)^{(1+1/m)/2}\Big(\log\frac{\delta}{\rho}\Big)^{-d\nu}$$
$$\geq C_\varepsilon^{-1}\lambda^2(\rho^{-1}\delta)^{d-(m+3)/2}\Big(M\big(\frac{\delta}{\rho}\big)^m\Big)^{(1+1/m)/2}\Big(\frac{\delta}{\rho}\Big)^\varepsilon,$$

where the last inequality holds by the second requirement on C_ε. Note that $(1+1/m)/2$ is identical with p/q. Also $p \geq 2$ and we may of course assume that $\lambda \leq 1$, so we may replace λ^2 by λ^p. Thus we obtain (23).

In case 2) we define $\overline{\rho} = 3\rho(\log\delta/\rho)^{-\nu}$, $\overline{\lambda} = \lambda(\log\delta/\rho)^{\nu-1}/3C_0$. We also drop the values of j for which ii) holds and choose a maximal $\delta/\overline{\rho}$-separated subset of the remaining $\{\alpha_j\}$; this sequence will still be denoted $\{\alpha_j\}$, and has cardinality greater or equal than

$$\overline{M} \stackrel{\text{def}}{=} (2B)^{-1}(\overline{\rho}/\rho)^m M.$$

We claim that for each j there is a tube $T_{\ell_j}^{\overline{\rho}\delta}$ such that $|E \cap T_{\ell_j}^{\overline{\rho}\delta}| \geq \overline{\lambda}\,|T_{\ell_j}^{\overline{\rho}\delta}|$. Namely, by the assumption that ii) fails there is a disc $D(x,\overline{\rho}/3)$ with $|E\cap T_{\ell_j}^{\rho\delta}\cap D(x,\overline{\rho}/3)| \geq C_0^{-1}\lambda\,|T_{\ell_j}^{\rho\delta}|/\log(\rho/\delta)$. It is easy to see that $T_{\ell_j}^{\rho\delta} \cap D(x,\overline{\rho}/3)$ is contained in a tube of the form $T_{\ell_j}^{\overline{\rho}\delta}$, and with this $T_{\ell_j}^{\overline{\rho}\delta}$ we have

$$|T_{\ell_j}^{\overline{\rho}\delta} \cap E| \geq C_0^{-1}\,\frac{\lambda}{\log\dfrac{\delta}{\rho}}\,|T_{\ell_j}^{\rho\delta}| = \overline{\lambda}\,|T_{\ell_j}^{\overline{\rho}\delta}|,$$

proving the claim. By the inductive assumption

$$\overline{\rho}^{-d}|E| \geq C_\varepsilon^{-1}\Big(\overline{M}\big(\frac{\delta}{\overline{\rho}}\big)^m\Big)^{p/q}\Big(\frac{\delta}{\overline{\rho}}\Big)^{d-p}\overline{\lambda}^p\Big(\frac{\delta}{\overline{\rho}}\Big)^\varepsilon.$$

A calculation shows this is equivalent with

$$\rho^{-d}|E| \geq (2B)^{-p/q}C_0^{-p}\,3^{-\varepsilon}\Big(\log\frac{\delta}{\rho}\Big)^{\nu\varepsilon-p}C_\varepsilon^{-1}$$
$$\cdot\Big(M\big(\frac{\delta}{\rho}\big)^m\Big)^{p/q}\Big(\frac{\delta}{\rho}\Big)^{d-p}\lambda^p\Big(\frac{\delta}{\rho}\Big)^\varepsilon.$$

Since $\rho/\delta \geq A$, the choice of A implies that the factor

$$(2B)^{-p/q}C_0^{-p}3^{-\varepsilon}(\log\delta/\rho)^{\nu\varepsilon-p}$$

appearing here is greater or equal than 1 and can be dropped. So we obtain (23).

5. Further remarks.

This section will consist of negative results, primarily the following.

Proposition 5.1. *Assume $d \geq 3$, let U be an open subset of $M(d,1)$, $\mathcal{A}$ a $d-1$-manifold, $F : U \to \mathcal{A}$ a submersion. Assume that there is $\alpha \in \mathcal{A}$ such that the following hold: $F^{-1}\alpha$ is connected, there is no vector $e \in \mathbb{P}^{d-1}$ such that $e_\ell = e$ for all $\ell \in F^{-1}\alpha$, and there is no point $p \in \mathbb{R}^d$ such that $p \in \ell$ for all $\ell \in F^{-1}\alpha$. Then property $(*)$ fails, and furthermore estimate (1) fails for the maximal function defined by (5) and the subsequent remark if $(d-3)/q < (p-2)/p$.*

REMARKS. 1) The relationship $(d-3)/q < (p-2)/p$ is satisfied if $p > (d+1)/2$ and $q = (d-1)p'$, so this shows in particular that (1) cannot hold for the full range of q for any $p > (d+1)/2$. If $d = 3$, then (1) cannot hold for any q if $p > 2$.

2) Proposition 5.1 and its proof are related to the counterexamples in Bourgain [2].

3) If $\mathcal{A}$ is an m-manifold with $m > d-1$ then property $(*)$ always fails. This follows from an abbreviated version of the proof below. When $m < d-1$ there are various examples where $(*)$ holds although it still fails generically.

The proof is easy, but it is convenient to split it into several lemmas. If $(s,t) \in \mathbb{P}^1$ then we define $Y_{st} \subset \mathbb{R}^n \times \mathbb{R}^n$ by $Y_{st} = \{(sx, tx) : x \in \mathbb{R}^n\}$, and if $x \in \mathbb{R}^n$ then we define $E_x \subset \mathbb{R}^n \times \mathbb{R}^n$ by $E_x = \operatorname{span}\{(x,0),(0,x)\}$.

Lemma 5.1. *Suppose Y is an n-dimensional subspace of $\mathbb{R}^n \times \mathbb{R}^n$, $n \geq 2$. Then either*

i) *$Y = Y_{st}$ for some $(s,t) \in \mathbb{P}^1$, or else*

ii) *There is $x \in \mathbb{R}^n \backslash \{0\}$ such that $E_x \cap Y = \{0\}$.*

PROOF. Let P_1 and P_2 be the projections of $Y \subset \mathbb{R}^n \times \mathbb{R}^n$ on $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ respectively. If ii) fails then it is clear that every $x \in \mathbb{R}^n$ is in the range of either P_1 or P_2. By linearity it follows that either P_1 or P_2 is onto, and therefore also 1-1. We may assume then that P_1 is 1-1 and onto. Since ii) fails and P_1 is 1-1 it follows that for every $x \in \mathbb{R}^n$ there is $\tau(x) \in \mathbb{R}$ such that $(x, \tau(x)\,x) \in Y$. The map $x \to (x, \tau(x)\,x)$ is a right inverse for P_1, hence linear, which implies τ is constant, *i.e.* $Y = Y_{1t}$ for some t.

From Lemma 5.1 we obtain a similar statement for submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$ which we now formulate. If M is a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ and $y \in M$ then $T_y M$ is the tangent space to M at y which we may of course identify with a subspace of $\mathbb{R}^n \times \mathbb{R}^n$.

Lemma 5.2. *Suppose Ω is an open set in $\mathbb{R}^n \times \mathbb{R}^n$, $n \geq 2$, and M is a (connected) n-dimensional submanifold of Ω. Then either*

i) *There are $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ and $(s, t) \in \mathbb{R} \times \mathbb{R}$, $(s, t) \neq (0, 0)$ such that M is the affine space $\{(a + sx, b + tx) : x \in \mathbb{R}^n\}$, or else*

ii) *There are $y \in M$ and $v \in \mathbb{R}^n$ such that $T_y M \cap E_v = \{0\}$.*

PROOF. We assume that ii) does not hold, and will show that then i) holds. Since ii) fails we know by Lemma 5.1 that for each $y \in M$ there is $(s(y), t(y)) \in \mathbb{P}^1$ such that $T_y M = Y_{s(y)t(y)}$. The map $y \to (s(y), t(y))$ is continuous from M into $\mathbb{P}^1$ so since the question is local and is symmetric with respect to $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ we may assume it does not take the value $(0, 1)$. Then locally M is a graph over $\mathbb{R}^n \times \{0\}$, $M = \{(x, F(x)) : x \in U\}$, where $U \subset \mathbb{R}^n$, and furthermore $DF(x) = \tau(x) \cdot \text{identity}$, where $\tau(x) = t(y)/s(y)$, $y = (x, F(x))$. It is well known (and easy to prove using equality of mixed second partials) that this property of F implies τ is constant, *i.e.* F has the form $F(x) = b + \tau x$ with $\tau \in \mathbb{R}$ and $b \in \mathbb{R}^n$. Thus i) holds, locally in a neighborhood of every point of M. The set where i) holds for a given a, b, s, t is then open-closed so a connectedness argument completes the proof.

Lemma 5.3. *Assume $\Omega \subset M(d, 1)$ is open and M is a connected $(d-1)$-dimensional submanifold of Ω. Then either*

i) *there is $e \in \mathbb{P}^{d-1}$ such that $e_\ell = e$ for all $\ell \in M$, or*

ii) *there is $p \in \mathbb{R}^d$ such that $p \in \ell$ for all $\ell \in M$, or*

iii) *there are $\ell \in M$ and a 2-plane Π with $\ell \subset \Pi$ such that $T_\ell M(\Pi, 1) \cap T_\ell M = \{0\}$.*

PROOF. We let ℓ_0 be the x_d axis, and may assume that $\ell_0 \in M$. It suffices to prove that either i), ii) or iii) holds in some neighborhood of ℓ_0. Introduce local coordinates near ℓ_0 on $M(d, 1)$ as follows: any line close to ℓ_0 is uniquely $\{(x, 0) + t\,(y, 1) : t \in \mathbb{R}\}$, with $x \in \mathbb{R}^{d-1}$, $y \in$

$\mathbb{R}^{d-1}$, and our coordinate system is defined by

$$\ell \to (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}.$$

We will apply Lemma 5.2 in these coordinates with $n = d-1$. First we note the following: if $\ell \in M(d,1)$ is a line close to ℓ_0 with coordinates (x,y) and if $v \in \mathbb{R}^{d-1}$, then v determines a 2-plane Π_v containing ℓ, namely the 2-plane $\Pi_v = \{(x + t\,v + s\,y, s) : s,t \in \mathbb{R} \subset \mathbb{R}^{d-1} \times \mathbb{R} \approx \mathbb{R}^d$. It is clear that $M(\Pi_v, 1)$ consists of all lines with coordinates $(x + t\,v, y + s\,v)$, where s,t are arbitrary real numbers, and therefore the tangent space to $M(\Pi_v, 1)$ at ℓ coincides (in our local coordinates) with the space E_v. We therefore conclude by Lemma 5.2 that either iii) of Lemma 5.3 holds, or else there are $s,t \in \mathbb{R}$, $(s,t) \neq (0,0)$, and $a \in \mathbb{R}^{d-1}$, $b \in \mathbb{R}^{d-1}$ such that M is all lines with coordinates $(a + s\,x, b + t\,x)$, $x \in \mathbb{R}^{d-1}$. If the latter possibility and if $t = 0$, then i) of Lemma 5.3 holds with $e = (b,1) \in \mathbb{P}^{d-1}$. On the other hand if $t \neq 0$ then ii) of Lemma 5.3 holds with $p = (a - sb/t, -s/t) \in \mathbb{R}^{d-1} \times \mathbb{R} \approx \mathbb{R}^d$.

PROOF OF PROPOSITION 5.1. We know by Lemma 5.3 that there are $\ell \in F^{-1}\alpha \cap U$ and a 2-plane $\Pi \supset \ell$ with $T_\ell(F^{-1}\alpha) \cap T_\ell M(\Pi, 1) = \{0\}$. It follows that the restriction of F to $M(\Pi,1)$ is an immersion in a neighborhood of ℓ, and therefore F maps a δ-neighborhood of $M(\Pi,1)$ onto a set S_δ containing a $C^{-1}\delta$-neighborhood of a surface S. This shows in the first place that property $(*)$ fails, let $\{\alpha_j\}$ be a maximal δ-separated subset of S. Next let $K \subset \mathbb{R}^d$ be a sufficiently large fixed compact set and let χ_δ be the characteristic function of the intersection of K with a δ-neighborhood of Π. Then $\|\chi_\delta\|_p \approx \delta^{(d-2)/p}$. On the other hand, $M_\delta \chi_\delta(\alpha) \geq C^{-1}\alpha$ if $\alpha \in S_\delta$. So $\|M_\delta \chi_\delta\|_q \gtrsim \delta^{(d-3)/q}$, whence (1) fails if $(d-3)/q < (d-2)/p - (d/p - 1)$, *i.e.* if $(d-3)/q < (p-2)/p$.

FINAL REMARKS. 1) The results in this paper can of course be applied to oscillatory integrals. We have nothing serious to say in this connection and will just record what follows by plugging Theorem 1 into the numerology in sections 4 and 5 of [3]. Namely, the Bochner-Riesz means are bounded on L^p for the optimal range of parameters provided that

$$p \in \Big(\frac{2\,(d^2+3d+3)}{d^2+5d+7}, \frac{2\,(d^2+3d+3)}{d^2+d-1}\Big),$$

and furthermore the adjoint of the restriction operator maps L^p to L^p if $p > 2(d^2+3d+3)/(d^2+d-1)$.

2) Proposition 5.1 suggests (as does [2]) that it should not be possible to prove Theorem 1 via space-time estimates for the x-ray transform, *i.e.* estimates for the x-ray transform from $L^p(\mathbb{R}^d)$ functions with fixed compact support to $W^{q\alpha}_{\text{loc}}(M(d,1))$ where $W^{q\alpha}$ is the Sobolev space with α derivatives in L^q, since such estimates do not distinguish situations where property $(*)$ is satisfied. Compare [8], where such estimates are used in the context of the circular maximal function, as well as [5]. We now sketch a concrete proof of this fact when $p = q$.

Proposition. *If the x-ray transform R maps*

$$L^q_{\text{comp}}(\mathbb{R}^d) \quad \text{to} \quad W^{q\alpha}_{\text{loc}}(M(d,1))$$

then $\alpha \le 1/q$.

We are concerned with the case $q \ge 2$ here. Conversely, if $q \ge 2$ and $\alpha \le 1/q$ then it is known that R maps L^q_{comp} to $W^{q\alpha}_{\text{loc}}$. This follows *e.g.* from formula (4.36) of [1] (if $q = 2$, which suffices since the $q = \infty$ case is trivial), or alternatively as pointed out in [10] from corresponding results [9] for general Fourier integral operators. As indicated by the proposition no improvement is possible.

We now sketch the proof of the proposition. Let $E = \{(\overline{x}, \overline{\overline{x}}) \in \mathbb{R}^{d-1} \times \mathbb{R} : |\overline{x}| \le 1, |\overline{\overline{x}}| \le \delta\}$, where δ is small. Let X be the set of lines in $M(d,1)$ which make an angle less or equal than δ with $\mathbb{R}^{d-1} \times \{0\}$ and intersect $D(0,1/2) \times \{0\} \subset \mathbb{R}^{d-1} \times \mathbb{R}$, and let Y be the set of lines obtained by translating the lines in X by $100\,\delta$ in the direction of the positive x_d axis. Then it is clear that $R\chi_E \ge$ constant > 0 on X, whereas $R\chi_E = 0$ on Y. In suitable local coordinates Y is a translate of X by an amount $\approx \delta$. This implies a lower bound: const $\delta^{-\alpha} |X|^{1/q}$ for the $W^{q,\alpha}$ Sobolev norm of $R\chi_E$. Since $|X| \approx \delta^2$ and $\|\chi_E\|_q = |E|^{1/q} \approx \delta^{1/q}$ we obtain the proposition.

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Følner Sequences in Polycyclic Groups

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Abstract. The isoperimetric inequality

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{\text{constant}}{\log|\Omega|}$$

for finite subsets Ω in a finitely generated group Γ with exponential growth is optimal if Γ is polycyclic.

1. Introduction and statements.

Let Γ be an infinite group generated by a finite set $S = S^{-1}$. If $\gamma \in \Gamma$ we denote by $\|\gamma\|_S$ the smallest number $k \in \mathbb{N}$ such that there exist $s_1, \dots, s_k \in S$ with $\gamma = s_1 \cdots s_k$. The distance between $\gamma, \gamma' \in \Gamma$ is defined as

$$d_S(\gamma, \gamma') = \|\gamma^{-1}\gamma'\|_S \,.$$

This distance on Γ, called the *word metric associated to* S, is left-invariant. We denote by $B(n) = \{\gamma \in \Gamma : \ \|\gamma\|_S \leq n\}$ the ball of radius n in Γ with center the identity. If $\Omega \subset \Gamma$ is a finite subset we denote by $|\Omega|$ its cardinal. Its boundary (relative to S) is defined by

$$\partial\Omega = \{\gamma \in \Gamma : \ \text{there exists } s \in S \text{ such that } \gamma s \notin \Omega\}\,.$$

Let $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{N}$ be the "inverse growth function of Γ"

$$\Phi(\lambda) = \min\{n \in \mathbb{N} : \ |B(n)| > \lambda\}\,.$$

A fundamental relation between the isoperimetric properties of the group and its growth is expressed in the following result (see [CSC93] and [Var91]).

Theorem 1.1 (Coulhon, Saloff-Coste). *Any finite non-empty subset* $\Omega \subset \Gamma$ *satisfies*

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{1}{4\,|S|\,\Phi(|\Omega|)}\,.$$

If Γ has polynomial growth of degree d this implies the existence of a constant $c > 0$, such that

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{c}{|\Omega|^{1/d}}\,, \qquad \text{for all } \Omega \subset \Gamma\,. \tag{1}$$

This result is due to Varopoulos [Var86]. Up to the changing of the value of c (which depends anyway on the choice of a generating set for Γ) this inequality is optimal (see [Gro93, 5. Eb]).

If Γ has exponential growth, Theorem 1.1 implies the existence of a constant $c > 0$, such that

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{c}{\log|\Omega|}\,, \qquad \text{for all } \Omega \subset \Gamma\,. \tag{2}$$

A finitely generated group Γ is non-amenable if and only if there is a constant $c > 0$ such that

$$\frac{|\partial\Omega|}{|\Omega|} \geq c\,, \qquad \text{for all } \Omega \subset \Gamma\,.$$

Hence the inequality of Theorem 1.1 is not optimal in this case.

The aim of this paper is to show that the inequality of Theorem 1.1 is optimal for polycyclic groups. A polycyclic group is solvable hence according to Milnor and Wolf (see [Mil68] and [Wol68]) its growth is either polynomial or exponential. Therefore, in view of (1) and (2) it is sufficient to prove the following statement.

Theorem 1.2. *Let* Γ *be an infinite polycyclic group. Let* $S = S^{-1}$ *be a finite generating set for* Γ. *There is a constant* $C > 1$ *and a family* Ω_n, $n \in \mathbb{N}$, *of finite subsets of* Γ *such that* $|\Omega_n| < |\Omega_{n+1}|$ *and*

$$\frac{C}{\log|\Omega_n|} \geq \frac{|\partial\Omega_n|}{|\Omega_n|}\,.$$

An inequality of the type

$$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{c}{\log|\Omega|}(\log|\Omega|)^{\epsilon}, \qquad \text{for all } \Omega \subset \Gamma,$$

for some $\epsilon > 0$ was known to be impossible (see [CSC93, 2.2]). This is an immediate consequence of Theorem 1.2.

2. Preliminaries to the proof.

Definition 2.1. *Two metric spaces X, Y are quasi-isometric if there exist a constant $\lambda > 1$ and applications $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that*

a) $d(f(x), f(x')) \leq \lambda\, d(x, x') + \lambda$, *for all* $x, x' \in X$.

b) $d(g(y), g(y')) \leq \lambda\, d(y, y') + \lambda$, *for all* $y, y' \in Y$.

c) $d(g \circ f(x), x) \leq \lambda$, *for all* $x \in X$.

d) $d(f \circ g(y), y) \leq \lambda$, *for all* $y \in Y$.

f is a quasi-isometry, g (which is also a quasi-isometry) is a quasi-inverse of f and λ is a quasi-isometry constant for f.

EXAMPLE 2.1. Let Γ be a group generated by a finite set $S = S^{-1}$. Let $H \subset \Gamma$ be a finite index subgroup. Let $T = T^{-1}$ be a finite generating set for H. Then H and Γ with the word metrics associated to T and S are quasi-isometric.

Let X be a metric space and let $R > 0$. Let $\Omega \subset X$. Let

$$V_R(\Omega) = \{x \in X : \text{ there exists } x' \in \Omega \text{ such that } d(x, x') \leq R\}$$

be the R-neighborhood of Ω. If $x \in X$ we denote by

$$d(x, \Omega) = \inf_{x' \in \Omega} d(x, x')$$

the distance between x and Ω.

Proposition 2.1. *Let X and Y be two finitely generated groups with word metrics. Let $f : X \longrightarrow Y$ be a quasi-isometry. Then they are constants $C > 1$ and $R > 0$ such that, for all finite subsets $\Omega \subset X$,*

$$|\partial V_R(f(\Omega))| \leq C\,|\partial\Omega|$$

and

$$|f(\Omega)| \leq |\Omega| \leq C\,|f(\Omega)|.$$

PROOF. We prove the first inequality. Let g be a quasi-inverse of f and let λ be a constant of quasi-isometry. We can assume $\lambda \in \mathbb{N}$. We choose $R = \lambda + 1$. We want to define an application

$$h : \partial V_R(f(\Omega)) \longrightarrow \partial\Omega$$

which is "almost injective". First, we notice that if $y \in \partial V_R(f(\Omega))$ then $g(y) \notin \Omega$. This is because if $g(y) \in \Omega$ then

$$d(y, f(\Omega)) \leq d(y, f \circ g(y)) \leq \lambda < R$$

and this contradicts $y \in \partial V_R(f(\Omega))$. We choose $x \in \Omega$ such that

$$d(g(y), x) = d(g(y), \Omega).$$

As $g(y) \notin \Omega$ it follows that $x \in \partial\Omega$. We put $h(y) = x$. Now we check that there is a constant $C > 1$ such that, if $x \in \partial\Omega$ then $|h^{-1}(x)| \leq C$. Let $y \in h^{-1}(x)$. Then

$$\begin{aligned} d(g(y), x) = d(g(y), \Omega) &\leq \lambda\, d(f \circ g(y), f(\Omega)) + \lambda \\ &\leq \lambda\,(\lambda + d(y, f(\Omega))) + \lambda \leq \lambda^2 + \lambda\, R + \lambda = M. \end{aligned}$$

Hence

$$d(y, f(x)) \leq d(f \circ g(y), f(x)) + \lambda \leq \lambda\, M + 2\,\lambda.$$

We choose

$$C = |B(\lambda\, M + 2\,\lambda)|.$$

This proves the first inequality of the proposition. The others are obvious.

Lemma 2.1. *Let N be a group generated by a finite set $B = B^{-1}$. Assume that a group G with finite generating set $A = A^{-1}$ acts on N by automorphisms. Then there is an integer $q > 1$, such that, for all $w \in G$,*

$$\|w(x)\|_B \leq q^{\|w\|_A}\, \|x\|_B\,, \qquad \textit{for all } x \in N.$$

ROOF. Let $q = \sup_{a \in A,\, b \in B} \|a(b)\|_B$. If $x \in N$ and $a \in A$, then

$$\|a(x)\|_B \leq q\, \|x\|_B\,.$$

·onclude by induction on $\|w\|_A$.

Lemma 2.2. *Let $F(a,b)$ be the free group on two letters. Let $k \in \mathbb{N}$. Using the notation $[b,a] = b\,a\,b^{-1}\,a^{-1}$ we have in $F(a,b)$ that*

$$b^k\,a = \Big(\prod_{j=1}^{k} b^{k-j}\,[b,a]\,b^{j-k}\Big)\,a\,b^k\,. \tag{3}$$

PROOF. Let $x_j = b^{k-j}\,[b,a]\,b^{j-k}$. We obtain the equality

$$b^{k+1-j}\,a = x_j\,b^{k-j}\,a\,b \tag{4}$$

by induction on j (where $1 \le j \le k$). We deduce the equality of the lemma by successively applying (4).

REMARK 2.1. If $k \in \mathbb{Z}^*$ the equality (3) generalizes to

$$b^k\,a = \Big(\prod_{j=1}^{|k|} b^{\epsilon(k)(|k|-j)}\,[b^{\epsilon(k)},a]\,b^{\epsilon(k)(j-|k|)}\Big)\,a\,b^k \tag{5}$$

where $\epsilon(k) = \pm 1$ is the sign of k.

Lemma 2.3. *Let $0 \to N \to \Gamma \to \mathbb{Z}^r \to 0$ be an exact sequence of groups where N is finitely generated. Let $B = B^{-1}$ be a finite generating set for N. Let $a_1,\dots,a_r$ be elements of Γ which project respectively on the canonical basis vectors $e_1,\dots,e_r$ of $\mathbb{Z}^r$. Then there is an integer $q > 1$, such that for each r-uple $K = (k_1,\dots,k_r) \in \mathbb{Z}^r$ and for each integer ν where $1 \le \nu \le r$, there exists a corresponding $x \in N$ with the following properties:*

a) $a_1^{k_1}\cdots a_\nu^{k_\nu}\cdots a_r^{k_r}\,a_\nu^{\pm 1} = x\,a_1^{k_1}\cdots a_\nu^{k_\nu\pm 1}\cdots a_r^{k_r}$.

b) $\|x\|_B \le q^{|K|}$.

(Where $|K| = \sum_{i=1}^{r} |k_i|$.*)*

PROOF. We assume the exponent of a_ν is positive (the other case is analogous). We assume that $K \ne 0$ (if $K = 0$ we choose $x = e$ and there is nothing to show). If $\nu = r$ then $x = e$. If $\nu < r$ we define for each $1 \le i \le r$

$$A_i = a_1^{k_1}\cdots a_i^{k_i}$$

and, if $k_i \neq 0$,

$$X_i = \prod_{j=1}^{|k_i|} x_{i,j}$$

where

$$x_{i,j} = a_i^{\epsilon(k_i)(|k_i|-j)} [a_i^{\epsilon(k_i)}, a_\nu] a_i^{\epsilon(k_i)(j-|k_i|)} .$$

If $k_i = 0$ we put $X_i = e$. For $2 \leq i \leq r$ the equality

$$A_i a_\nu = A_{i-1} X_i a_\nu a_i^{k_i} \tag{6}$$

follows from (5). We obtain the equality

$$a_1^{k_1} \cdots a_r^{k_r} a_\nu = \Big(\prod_{i=1}^{r-\nu} A_{r-i} X_i A_{r-i}^{-1} \Big) a_1^{k_1} \cdots a_\nu^{k_\nu+1} \cdots a_r^{k_r}$$

by successively applying (6) and by putting in terms of the form $A_{r-i}^{-1} A_{r-i}$;

$$x = \prod_{i=1}^{r-\nu} A_{r-i} X_i A_{r-i}^{-1}$$

belongs to N because it belongs to the derived group of Γ. Now we want an upper bound for $\|x\|_B$. According to Lemma 2.1 there is a constant $q > 1$ such that

$$\|x_{i,j}\| \leq q^{|K|} \sup_{1 \leq i \leq r} \|[a_i^{\epsilon(k_i)}, a_\nu]\|_B .$$

Hence, up to increasing q we obtain

$$\|x_{i,j}\| \leq q^{|K|} .$$

Using this last inequality and Lemma 2.1 again we have

$$\|x\|_B \leq r \sup_{1 \leq i \leq r-1} \|A_{r-i} X_i A_{r-i}^{-1}\|_B \leq r\, q^{|K|} \|X_i\|_B \leq r\, q^{|K|} |K|\, q^{|K|} .$$

By increasing q again we obtain the wanted inequality.

3. Proof of Theorem 1.2.

Let Γ be an infinite polycyclic group. According to Example 2.1 and Proposition 2.1 it is sufficient to prove the theorem for a finite index subgroup of Γ. According to a theorem of Mal'cev (see for example [Rob82, 15.1.6]) a polycyclic group has a finite index subgroup with nilpotent derived group. In order to avoid torsion elements in the abelianisation we again consider a finite index subgroup. Therefore it is sufficient to prove the theorem for polycyclic groups Γ of the form

$$0 \longrightarrow N \longrightarrow \Gamma \longrightarrow \mathbb{Z}^r \longrightarrow 0$$

where the sequence is exact and where N is nilpotent. As N is a subgroup in a polycyclic group it is polycyclic and hence finitely generated. Let $B = B^{-1}$ be a finite generating set for N. Choose elements $a_1, \dots, a_r \in \Gamma$ which project respectively on the canonical basis vectors $e_1, \dots, e_r$ of $\mathbb{Z}^r$. The set

$$S = B \cup \{a_1^{\pm 1}, \dots, a_r^{\pm 1}\}$$

generates Γ. Any $\gamma \in \Gamma$ can be written in a unique way

$$\gamma = x\, a_1^{k_1} \cdots a_r^{k_r}\,,$$

where $x \in N$ and $(k_1, \dots, k_r) \in \mathbb{Z}^r$. Let q_1 be the constant of Lemma 2.1 (with $G = \mathbb{Z}^r$) and let q_2 be the constant of Lemma 2.3. Let $q = \max\{q_1, q_2\}$. For each $n \in \mathbb{N}$ we define

$$\Omega_n = \{x\, a_1^{k_1} \cdots a_r^{k_r} \,:\, \|x\|_B \le q^{2n}\,,\ |K| \le n\}\,,$$
$$\omega_n = \{x\, a_1^{k_1} \cdots a_r^{k_r} \,:\, \|x\|_B \le q^{2n} - q^n\,,\ |K| \le n-1\}\,.$$

We want to show that

$$\partial\Omega_n \cap \omega_n = \varnothing\,. \tag{7}$$

That is, if $\gamma \in \omega_n$ and $s \in S$ then $\gamma s \in \Omega_n$.

a) Assume $s \in B$. If

$$x\, a_1^{k_1} \cdots a_r^{k_r} \in \omega_n$$

then

$$x\,a_1^{k_1}\cdots a_r^{k_r}\,s = x\,a_1^{k_1}\cdots a_r^{k_r}\,s\,a_r^{-k_r}\cdots a_1^{-k_1}\,a_1^{k_1}\cdots a_r^{k_r}$$

and according to Lemma 2.1

$$\begin{aligned}\|x\,a_1^{k_1}\cdots a_r^{k_r}\,s\,a_r^{-k_r}\cdots a_1^{-k_1}\|_B &\leq \|x\|_B + q^{|K|}\\ &\leq q^{2n} - q^n + q^{n-1} \leq q^{2n}\,.\end{aligned}$$

b) If $s \in \{a_1^{\pm 1}, \ldots, a_r^{\pm 1}\}$, we can assume $s = a_\nu$ where $1 \leq \nu \leq r$. Let

$$x\,a_1^{k_1}\ldots a_r^{k_r} \in \omega_n\,.$$

According to Lemma 2.3 there exists $x' \in N$, such that

$$x\,a_1^{k_1}\cdots a_\nu^{k_\nu}\cdots a_r^{k_r}\,a_\nu = x\,x'\,a_1^{k_1}\cdots a_\nu^{k_\nu+1}\cdots a_r^{k_r}$$

and such that

$$\|x'\|_B \leq q^{|K|}\,.$$

We have

$$\|x\,x'\|_B \leq \|x\|_B + \|x'\|_B \leq q^{2n} - q^n + q^{|K|} \leq q^{2n} - q^n + q^{n-1} \leq q^{2n}\,.$$

Let $B(n) \subset N$ be the ball of radius n with respect to B. Let d be the degree of the growth of N. According to Grunewald (see [Gri90, 7.2]) we have

$$|B(n)| = \alpha\,n^d + O(n^{d-1/2})\,,$$

where $\alpha > 0$ is a constant. We define

$$f(n) = \frac{|B(q^{2n} - q^n)|}{|B(q^{2n})|}\,.$$

Hence

$$f(n) = \frac{\alpha + \dfrac{O(1)}{q^n}}{\alpha + \dfrac{O(1)}{q^n}}\,.$$

Let $P(n)$ be the number of elements in $\mathbb{Z}^r$ of word norm less or equal than n with respect to the canonical generating set. The function $P(n)$ is polynomial of degree r. According to (7) we have

$$\frac{|\partial\Omega_n|}{|\Omega_n|} \leq \frac{|\Omega_n| - |\omega_n|}{|\Omega_n|}\,.$$

This last term is equal to

$$\begin{aligned}1 - f(n)\,\frac{P(n-1)}{P(n)} &= (1 - f(n)) + f(n)\,\frac{P(n) - P(n-1)}{P(n)} \\ &= \frac{O(1)}{q^n} + O(\frac{1}{n}) = O(\frac{1}{n}).\end{aligned}$$

Hence there is a constant $C_1 > 0$ such that

$$\frac{|\partial\Omega_n|}{|\Omega_n|} \leq \frac{C_1}{n}, \qquad \text{for all } n \in \mathbb{N}.$$

On the other hand, there exists a constant $C_2 > 0$ such that

$$\log|\Omega_n| \leq C_2\, n, \qquad \text{for all } n \in \mathbb{N}.$$

Eventually, $C = C_1 C_2$ is the constant we were looking for.

4. Remarks and questions.

a) Theorem 1.1 generalizes a result of Varopoulos (see [VCSC82, VI.3.1]) which shows that a group with superpolynomial growth has infinite isoperimetric dimension. As a solvable group is amenable, a solvable Lie group (with any left-invariant metric) containing a lattice with exponential growth (the group *Sol* for example [Thu82]) has infinite isoperimetric dimension but is not open at infinity (see [GLP81, Chapter 6]).

b) Theorem 1.1 combined with the Milnor-Wolf theorem on the growth of solvable groups [Mil68], [Wol68], shows that a finitely generated solvable group with finite isoperimetric dimension contains a finite index nilpotent subgroup (see [GLP81, 6.29]).

c) The isoperimetric profile of a finitely generated group (with a given generating set) is defined as (the asymptotic behaviour of) the function

$$I_\Gamma(n) = \inf_{|\Omega|=n} |\partial\Omega|$$

(see [Gro93, 5.E]). If Γ has exponential growth, Theorem 1.1 implies the existence of a constant $c > 0$ such that

$$I_\Gamma(n) \geq c\,\frac{n}{\log n}, \qquad \text{for all } n \in \mathbb{N}.$$

If moreover, the group Γ is polycyclic, it follows from the proof of Theorem 1.2 that there exist constants $p, q > 1$ and $C > 1$ such that for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$\frac{p^n}{C} \leq m \leq C\, q^n\,, \qquad I_\Gamma(m) \leq C \frac{m}{\log m}\,.$$

Can we replace p^n and q^n by n?

c) Theorem 1.2 is true for the solvable non-polycyclic group

$$\langle a, b \mid aba^{-1} = b^2 \rangle\,.$$

Is Theorem 1.2 true for finitely generated solvable groups?

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Variétés riemanniennes isométriques à l'infini

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1. Introduction.

Dans cet article, nous nous intéresserons à certaines propriétés des variétés riemanniennes non compactes qui ne dépendent que de leur géométrie à l'infini; pour cela, nous utiliserons un procédé de discrétisation qui associe un graphe (pondéré) à une variété. Nous reprenons ainsi les idées et les méthodes de [3], [17], [20], [21], [22], [31], [34] (voir aussi [1]), en leur incorporant des développements plus récents [11], [28], [29], qui font bien apparaître le rôle joué dans ces questions par les inégalités de Poincaré.

Soit (M, d, μ) un espace métrique mesuré; notons $V(x, r)$ le volume de la boule dans M de centre x et de rayon r, autrement dit $\mu(\{y \in M : \ d(y,x) \leq r\})$. Nous supposerons toujours $V(x,r) < +\infty$ pour $r < +\infty$.

Nous dirons que (M, d, μ) vérifie une condition de doublement du volume à rayon fixé si pour tout $r > 0$, il existe C_r tel que

$$(DV)_{\mathrm{loc}} \qquad V(x, 2r) \leq C_r V(x, r), \qquad \text{pour tout } x \in M,$$

Notons que $(DV)_{\mathrm{loc}}$ est une condition locale en r mais uniforme en $x \in M$.

Soient (M_i, d_i, μ_i), $i = 1, 2$, deux espaces métriques mesurés non compacts vérifiant $(DV)_{\mathrm{loc}}$. Posons

$$V_i(y, r) = \mu_i(\{z \in M_i : \ d_i(y, z) \leq r\}) .$$

Nous dirons qu'ils sont isométriques à l'infini s'il existe une application Φ de M_1 dans M_2 telle que

i) il existe $\varepsilon > 0$ tel que $[\Phi(M_1)]_\varepsilon = M_2$, où $[A]_\varepsilon$ désigne le voisinage d'ordre ε de A.

ii) il exite $a \geq 1$, $b \geq 0$ tels que $a^{-1}\, d_1(x,y) - b \leq d_2(\Phi(x), \Phi(y)) \leq a\, d_1(x,y) + b$, pour tout $x, y \in M_1$.

iii) il existe $C > 0$ tel que $C^{-1}\, V_1(x,1) \leq V_2(\Phi(x),1) \leq C\, V_1(x,1)$, pour tout $x \in M_1$.

L'existence d'une application vérifiant i) et ii) définit une relation d'équivalence entre espaces métriques; cette notion a été considérée par Kanai [20]. L'existence d'une application vérifiant i), ii) et iii) définit une relation d'équivalence entre espaces métriques mesurés vérifiant l'hypothèse $(DV)_{\text{loc}}$; cette relation apparaît dans [20].

Dans ce qui suit, nous considérerons essentiellement deux types d'espaces métriques mesurés: des variétés riemanniennes, munies de la distance et de la mesure riemannienne, et des graphes dénombrables, localement finis, connexes, munis de leur distance naturelle, et pondérés, c'est-à-dire munis de la mesure associée à une fonction positive fixée. Sur les graphes, nous ferons l'hypothèse que les sommets ont un nombre de voisins borné. Sur les variétés, nous ferons des hypothèses très faibles de géométrie locale.

Soit M une variété riemannienne connexe et complète, munie de sa mesure riemannienne que nous noterons tantôt μ tantôt dx. Pour $x \in M$ et $r > 0$, $B(x,r)$ désignera la boule riemannienne de centre x et de rayon r, et $V(x,r)$ son volume. Pour ψ une fonction sur M, $\psi_r(x)$ désignera sa moyenne sur $B(x,r)$, c'est-à-dire

$$\psi_r(x) = \frac{1}{V(x,r)} \int_{B(x,r)} \psi(y)\, dy\, .$$

Nous dirons que M vérifie l'inégalité de Poincaré sur les boules à rayon fixé, en abrégé vérifie $(P)_{\text{loc}}$, si pour tout $\sigma \geq 1$ et pour tout $r > 0$ il existe $C_{\sigma,r}$ tels que, pour toute fonction $\psi \in C_0^\infty(M)$ et tout $x \in M$,

$$(P)_{\text{loc}} \quad \Big(\int_{B(x,r)} |\psi(y) - \psi_r(x)|^\sigma\, dy \Big)^{1/\sigma} \leq C_{\sigma,r} \Big(\int_{B(x,2r)} |\nabla \psi(y)|^\sigma\, dy \Big)^{1/\sigma} .$$

Dans le second membre de $(P)_{\text{loc}}$ on peut toujours, sous l'hypothèse $(DV)_{\text{loc}}$, remplacer $B(x,2r)$ par $B(x,(1+\alpha)r)$, pourvu que $\alpha > 0$: les familles d'inégalités obtenues pour différents $\alpha \geq 0$ sont deux à deux équivalentes, sous l'hypothèse $(DV)_{\text{loc}}$. Si besoin est, on notera $(P_\sigma)_{loc}$ la version à σ fixé de l'inégalité ci-dessus.

Notons que les variétés à courbure de Ricci minorée vérifient des propriétés plus fortes que $(P)_{\text{loc}}$ et $(DV)_{\text{loc}}$ (voir $(P)_0$ et $(DV)_0$ au Paragraphe 8 ci-dessous); les conditions que nous venons de considérer n'imposent, elles, aucune contrainte vraiment locale sur M. Nous n'imposerons pas non plus $\sup_{x\in M} V(x,r) < +\infty$, pour r fixé, contrairement à ce qui a lieu dans les variétés à courbure de Ricci minorée.

Nous commencerons par montrer que certaines inégalités de type Poincaré ou Sobolev se transmettent entre graphes pondérés isométriques à l'infini. Puis nous discrétiserons les variétés vérifiant $(P)_{\text{loc}}$ et $(DV)_{\text{loc}}$, c'est-à-dire qu'à une telle variété nous associerons un graphe pondéré qui lui soit isométrique à l'infini, et nous vérifierons que cette opération préserve à nouveau certaines inégalités analytiques. Nous obtiendrons ainsi la stabilité par isométrie à l'infini de plusieurs propriétés des variétés vérifiant $(P)_{\text{loc}}$ et $(DV)_{\text{loc}}$. Ceci généralise les résultats de [20], [21], [22] car nous ne faisons pas d'hypothèse sur le rayon d'injectivité de M (en particulier, pour $r > 0$ fixé, $V(x,r)$ n'est pas supposé uniformément minoré, sauf lorsque nous considérons des inégalités de Sobolev qui l'imposent), ainsi que ceux de [28], [29], car la notion d'isométrie à l'infini est plus générale que celle de quasi-isométrie.

En couplant les considérations précédentes et les méthodes de [29], nous montrerons que les variétés à courbure de Ricci minorée qui sont isométriques à l'infini soit à une variété à courbure de Ricci positive ou nulle, soit à un groupe de Lie à croissance polynômiale du volume, ne possèdent pas de fonctions harmoniques positives non triviales. Ceci généralise dans deux directions le résultat de [20] qui traite le cas des variétés isométriques à l'infini à $\mathbb{R}^n$ et de dimension inférieure ou égale à n.

Les résultats de cet article valent dans un cadre plus général que celui des variétés riemanniennes: il suffit de considérer comme dans [29] des variétés munies d'un opérateur du second ordre localement sous-elliptique et de la distance qu'il induit; on peut ainsi traiter, par exemple, des opérateurs sous-elliptiques sur les groupes de Lie à croissance polynômiale du volume (voir Paragraphe 9).

2. Croissance du volume.

Ce paragraphe généralise [20, Paragraphe 3]. Commençons par un lemme technique dont la preuve est évidente et qui nous sera utile dans toute la suite.

Lemme 2.1. *Si l'espace métrique mesuré (M,d,μ) vérifie $(DV)_{\mathrm{loc}}$, alors, pour tous r_1, r_2 tels que $0 < r_1 < r_2$, il existe C_{r_1,r_2} tel que pour tout $x \in M$,*

$$V(x,r_2) \leq C_{r_1,r_2} V(x,r_1).$$

En particulier, pour tout $r \leq R$, pour tous $x, y \in M$ tels que $d(x,y) \leq R$, on a

$$V(x,r) \leq C_{r,2R} V(y,r).$$

Soient (M_1,d_1,μ_1) et (M_2,d_2,μ_2) deux espaces métriques mesurés vérifiant $(DV)_{\mathrm{loc}}$, et Φ une isométrie à l'infini de M_1 dans M_2. Soit $\varepsilon > 0$ tel que $[\Phi(M_1)]_\varepsilon = M_2$; à $z \in M_2$, associons un élément x de M_1 tel que $d_2(z,\Phi(x)) \leq \varepsilon$, et notons-le $\Phi^{-1}(z)$; Φ^{-1} définit une isométrie à l'infini de M_2 dans M_1.

Proposition 2.2. *Soient (M_1,d_1,μ_1) et (M_2,d_2,μ_2) deux espaces métriques mesurés vérifiant $(DV)_{\mathrm{loc}}$, et Φ une isométrie à l'infini de M_1 dans M_2. Alors, il existe $C > 0$ tel que*

$$C^{-1} V_1(x, C^{-1}r) \leq V_2(\Phi(x), r) \leq C\, V_1(x, C\, r),$$

pour tous $x \in M_1$, $r \geq 1$.

PREUVE. Soit $R \geq 1$ tel que $aR - b = R' > 0$ (nous reprenons, pour les propriétés de Φ, les notations de l'introduction). Soit $r \geq R$, $x \in M_1$, et $(x_i)_{i=1}^k$ une partie maximale R-séparée de $B(x,r)$. On a $B(x,r) \subset \cup_{i=1}^k B(x_i,R)$, et donc $V_1(x,r) \leq \sum_{i=1}^k V_1(x_i,R)$. De plus, d'après le Lemme 2.1,

$$V_1(x_i,R) \leq C_{1,R} V_1(x_i,1) \leq C_{1,R}\, C\, V_2(\Phi(x_i),1)$$

(la constante C provient de la propriété iii) de l'isométrie à l'infini). Comme les boules $B(x_i,R/2)$ sont deux à deux disjointes, il en est de même des boules $B(\Phi(x_i), R'/2)$. Mais, à nouveau d'après le Lemme 2.1, $V_2(\Phi(x_i), 1) \le C_{1,R'}\, V_2(\Phi(x_i), R'/2)$, d'où

$$V_1(x,r) \le \sum_{i=1}^{k} C_{1,R}\, C\, C_{1,R'}\, V_2(\Phi(x_i), R'/2)\,.$$

Enfin, comme $\Phi(x_i) \in B(\Phi(x), ar+b)$, $B(\Phi(x_i), R'/2) \subset B(\Phi(x), ar + b + R'/2)$, d'où

$$V_1(x,r) \le C'\, V_2(\Phi(x), ar + b + R'/2)\,.$$

On en déduit facilement l'inégalité de gauche, en passant de $r \ge R$ à $r \ge 1$ par l'intermédiaire de la condition $(DV)_{\text{loc}}$. Le même raisonnement appliqué à M_2, M_1 et Φ^{-1} montre que

$$V_2(\Phi(x), r) \le C\, V_1(\Phi^{-1} \circ \Phi(x), C\,r)\,, \qquad x \in M_1\,.$$

L'inégalité de droite s'en déduit en utilisant le fait que $d_1(\Phi^{-1} \circ \Phi(x), x)$ est borné.

Nous dirons que (M, d, μ) vérifie la propriété de doublement du volume à l'infini si pour tout $r_0 > 0$, il existe C_{r_0} tel que

$$(DV)_\infty \qquad V(x, 2r) \le C_{r_0}\, V(x,r)\,, \qquad \text{pour tous } x \in M\,,\ r > r_0\,.$$

On déduit de ce qui précède

Proposition 2.3 *Soient (M_1, d_1, μ_1) et (M_2, d_2, μ_2) deux espaces métriques mesurés vérifiant $(DV)_{\text{loc}}$ et isométriques à l'infini. Alors, si (M_1, d_1, μ_1) vérifie $(DV)_\infty$, il en est de même de (M_2, d_2, μ_2).*

3. Analyse sur les graphes pondérés.

Soit X un graphe localement uniformément fini: le nombre de voisins de chacun de ses sommets est borné par N. Soit d la distance naturellement associée au graphe X: $d(x,y)$ est le plus petit nombre de pas nécessaire pour relier x à y; nous noterons $x \sim y$ si x et y sont voisins dans X, c'est-à-dire si $d(x,y) = 1$, $x \simeq y$ si x et y sont voisins ou égaux. Soit m une fonction strictement positive fixée sur X. Nous supposerons

$$C_m = \sup_{\substack{x,y \\ x\sim y}} \frac{m(x)}{m(y)} < +\infty .$$

Nous dirons que (X, m) est un graphe pondéré. Si l'on pose

$$V(x,n) = \sum_{B(x,n)} m(y) ,$$

on a

$$m(x) \leq V(x,n) \leq m(x)\, C^n\, N^n , \qquad \text{pour tous } \ x \in X, \ n \in \mathbb{N}^* .$$

Autrement dit, si on considère m comme une mesure discrète sur X, l'espace métrique mesuré (X, d, m) vérifie $(DV)_{\text{loc}}$. La constante qui intervient dans $(DV)_{\text{loc}}$ est contrôlée par N et C_m.

Posons, pour $E \subset X$,

$$\|f\|_{p,E} = \Big(\sum_E |f(x)|^p\, m(x) \Big)^{1/p} .$$

Nous noterons $\|\cdot\|_{p,X}$, ou $\|\cdot\|_p$ s'il n'y a pas d'ambiguïté, la norme ℓ^p par rapport à la mesure m sur X. Nous désignerons par

$$\delta f(x) = \Big(\sum_{y\sim x} |f(y) - f(x)|^2 \Big)^{1/2}$$

la longueur du gradient discret en x de la fonction f.

Dans la suite on notera simplement X le graphe pondéré (X, m) et l'espace métrique mesuré correspondant.

A. Inégalités de Sobolev sur les graphes pondérés.

Soit

$$S_{p,q} = S_{p,q}(X) = \inf \Big\{ \frac{\|\delta f\|_p}{\|f\|_q} : \ f \in c_0(X), \ f \neq 0 \Big\} .$$

Nous dirons que X vérifie l'inégalité de Sobolev $(S_{p,q})$ si $S_{p,q}(X) > 0$; dans ce cas, on a

$$(S_{p,q}) \qquad S_{p,q}\, \|f\|_q \leq \|\delta f\|_p , \qquad \text{pour tout } \ f \in c_0(X) .$$

Ceci ne peut avoir lieu que si $q \geq p$. En appliquant $(S_{p,q})$ aux fonctions de Dirac, on voit que le cas $q > p$ ne peut se présenter que si $\inf_{x \in X} m(x) > 0$. On notera que

$$S_{p,p}(X) \leq N\, C_m \,.$$

Les inclusions entre espaces ℓ^p montrent facilement que $S_{p',q'}(X) \geq S_{p,q}(X)$, si $p' \leq p$ et $q' \geq q$, mais on a aussi:

Proposition 3.1. *Sur un graphe pondéré, l'inégalité $(S_{p,q})$ entraîne $(S_{p',q'})$ avec $1/p' - 1/q' = 1/p - 1/q$ pour $p' \geq p$, et $(S_{p',q'})$ avec $1/p' - 1/q' = 1 - p/q$ pour $1 \leq p' < p$. Dans le premier cas, on a*

$$S_{p',q'} \geq c\, S_{p,q} \,,$$

et dans le second,

$$S_{p',q'} \geq c'\, S_{p,q}^p \,,$$

où c, c' ne dépendent que de p, p', q, de la constante C_m, et de la borne N sur le nombre de voisins dans X.

REMARQUE. Les implications précédentes ne peuvent être améliorées comme le montrent les cas de $\mathbb{Z}^n$ (où $S_{p,q}(\mathbb{Z}^n) > 0$ si et seulement si $1/p - 1/q \geq 1/n$) et des graphes construits dans [12] qui vérifient $(S_{2,q})$ mais pas $(S_{1,q/2+\varepsilon})$.

PREUVE. La proposition est classique dans le cas non pondéré. Nous en donnons la preuve dans le cas général pour la commodité du lecteur. La première assertion s'obtient en appliquant $(S_{p,q})$ à $f^{q'/q}$, avec $f \geq 0$. Plus précisément,

$$\begin{aligned}
\|\delta(f^{q'/q})\|_p &= \Big(\sum_x \Big(\sum_{y\sim x} |f^{q'/q}(y) - f^{q'/q}(x)|^2\Big)^{p/2} m(x)\Big)^{1/p} \\
&\leq \frac{q'}{q}\Big(\sum_x \Big(\sum_{y\sim x} |f(y) - f(x)|^2 \\
&\qquad\qquad \cdot (\max\{f(y), f(x)\})^{2(q'/q-1)}\Big)^{p/2} m(x)\Big)^{1/p} \\
&\leq \frac{q'}{q}\Big(\sum_x \Big(\sum_{y\sim x} |f(y) - f(x)|^2\Big)^{p/2} g(x)^{p(q'/q-1)}\, m(x)\Big)^{1/p},
\end{aligned}$$

où $g(x) = \max\{f(y) : \; y \simeq x\}$. L'inégalité de Hölder donne alors

$$\|\delta(f^{q'/q})\|_p \leq \frac{q'}{q} \Big(\sum_x \Big(\sum_{y \sim x} |f(y) - f(x)|^2 \Big)^{p'/2} m(x) \Big)^{1/p'} \cdot \Big(\sum_x g(x)^{q'} m(x) \Big)^{1/q - 1/q'} .$$

Maintenant

$$\begin{aligned} \sum_x g(x)^{q'} m(x) &\leq \sum_x \Big(\sum_{y \simeq x} f(y)^{q'} \Big) m(x) \\ &\leq C_m \sum_x \Big(\sum_{y \simeq x} f(y)^{q'} m(y) \Big) \\ &\leq (N+1)\, C_m \sum_y f(y)^{q'} m(y) . \end{aligned}$$

On obtient donc

$$\|\delta(f^{q'/q})\|_p \leq \frac{q'}{q} (N+1)^{1/q - 1/q'} C_m^{q'/q - 1} \|\delta f\|_{p'} \, \|f\|_{q'}^{q'/q - 1} .$$

Comme

$$S_{p,q} \|f\|_{q'}^{q'/q} = S_{p,q} \|f^{q'/q}\|_q \leq \|\delta(f^{q'/q})\|_p ,$$

pour tout $f \in c_0(X)$, $f \geq 0$, on a

$$S_{p,q} \|f\|_{q'} \leq \frac{q'}{q} (N+1)^{1/q - 1/q'} C_m^{q'/q - 1} \|\delta f\|_{p'} ,$$

autrement dit, en utilisant le fait que $\delta\, |f| \leq \delta f$,

$$S_{p,q} \leq \frac{q'}{q} (N+1)^{1/q - 1/q'} C_m^{q'/q - 1} S_{p',q'} .$$

Pour obtenir la seconde assertion, on applique $(S_{p,q})$ à la fonction caractéristique d'un sous-ensemble fini Ω de X:

$$S_{p,q} \|1_\Omega\|_q \leq \|\delta 1_\Omega\|_p .$$

Mais $\|1_\Omega\|_q = m(\Omega)^{1/q}$, et

$$\delta 1_\Omega(x) = \Big(\sum_{y \sim x} |1_\Omega(y) - 1_\Omega(x)|^2 \Big)^{1/2} \begin{cases} = 0 , & \text{si } x \in (\Omega \setminus \partial\Omega) \cup (\Omega^c \setminus \partial\Omega^c) , \\ \leq \sqrt{N} , & \text{si } x \in \partial\Omega \cup \partial\Omega^c , \end{cases}$$

où, pour $A \subset X$, on note ∂A l'ensemble des points de A qui ont un voisin dans A^c. Donc

$$\begin{aligned}\|\delta 1_\Omega\|_p &= \Big(\sum_x \Big(\sum_{y\sim x} |f(y)-f(x)|^2\Big)^{p/2} m(x)\Big)^{1/p} \\ &\le N^{1/2}\, m(\partial\Omega \cup \partial\Omega^c)^{1/p}\,.\end{aligned}$$

Par ailleurs $m(\partial\Omega^c) = \sum_{x\in\partial\Omega^c} m(x) \le C_m \sum_{x\in\partial\Omega^c} m(y_x)$, où y_x est un point de $\partial\Omega$ voisin de x; chaque y_x ne pouvant intervenir plus de N fois dans la sommation, cela donne $m(\partial\Omega^c) \le C_m\, N\, m(\partial\Omega)$. Finalement,

$$m(\Omega)^{p/q} S_{p,q}^p \le N^{p/2}\,(1 + C_m\, N)\, m(\partial\Omega)\,,$$

soit, par la formule de coaire discrète ([8, preuve du Théorème 5], ou [33]),

$$\|f\|_{q/p}\, S_{p,q}^p \le C_{p,q}\, N^{p/2}\,(1 + C_m\, N)\, \|\delta f\|_1\ ,$$

pour tout $f \in c_0(X)$, soit

$$S_{1,q/p} \ge C_{p,q}^{-1}\, N^{-p/2}\,(1 + C_m\, N)^{-1}\, S_{p,q}^p\ .$$

Ici, $C_{p,q}$ est la constante amenée par la formule de coaire. On applique alors la première assertion pour conclure.

Corollaire 3.2. *Soit p, $1 \le p < +\infty$. Sur un graphe pondéré, l'inégalité $(S_{p,p})$ entraîne $(S_{p',p'})$, pour tout p', $1 \le p' < +\infty$, et, si $p' \le p$,*

$$c\, S_{p',p'} \le S_{p,p} \le C\, S_{p',p'}^{1/p}\,,$$

où c, C ne dépendent que de p, p', de la constante C_m, et de N.

Le cas $p = 1$, $p' = 2$, dans le cas non pondéré, figure dans [22, Proposition 4.2].

B. Marches aléatoires.

Considérons sur X le noyau markovien

$$p(x,y) = \begin{cases} \dfrac{m^{1/2}(y)}{\sum_{z\sim x} m^{1/2}(z)}\,, & \text{si } y \sim x\,,\\[2ex] 0\,, & \text{sinon,}\end{cases}$$

et P l'opérateur associé:

$$Pf(x) = \sum_y p(x,y)\, f(y)\,.$$

Le noyau p est réversible par rapport à la mesure

$$\pi(x) = m^{1/2}(x)\Big(\sum_{z\sim x} m^{1/2}(z)\Big)\,,$$

c'est-à-dire que

$$\pi(x)\, p(x,y) = \pi(y)\, p(y,x) = \begin{cases} m^{1/2}(x)\, m^{1/2}(y)\,, & \text{si } y\sim x\,, \\ 0\,, & \text{sinon}, \end{cases}$$

ou encore que P est auto-adjoint sur $\ell^2(\pi)$. Notons que $\pi(x)$ est de l'ordre de $m(x)$. Il en résulte que l'expression

$$\langle (I-P)f, f\rangle_{\ell^2(\pi)} = \sum_{x,y} |f(y)-f(x)|^2\, p(x,y)\, \pi(x)$$

est uniformément comparable à

$$\sum_x \Big(\sum_{y\sim x} |f(y)-f(x)|^2\Big) m(x) = \|\delta f\|_2\,.$$

REMARQUE. D'autres choix de p conduisent bien sûr au même résultat: il suffit de considérer une quantité $q(x,y)$ égale à 0 si $x \not\sim y$, et si $x\sim y$, à une expression symétrique et positive de $m(x)$ et $m(y)$, de l'ordre de $m(x)$ si l'on fait formellement $x=y$ (par exemple $m(x)+m(y)$, $\sup\{m(x), m(y)\}$ ou comme ci-dessus $m^{1/2}(x)\, m^{1/2}(y)$). On pose alors

$$\pi(x) = \sum_{z\sim x} q(x,z)\,, \qquad p(x,y) = \frac{q(x,y)}{\pi(x)}\,.$$

Définissons les noyaux itérés p_k, $k = 0,1,2,\ldots$, en posant

$$p_k(x,y) = \sum_z p_{k-1}(x,z)\, p(z,y)\,, \qquad p_0(x,y) = \delta_x(y)\,,$$

et

$$G(x,y) = \sum_0^\infty p_k(x,y)\,.$$

Rappelons que la chaîne de Markov associée à p est transiente si et seulement si $G(x,x) < +\infty$ pour un (et donc pour tous) $x \in X$. Nous dirons alors que X est transient et sinon que X est récurrent. Nous utiliserons le critère suivant (voir [32], [1]):

Théorème 3.3. *Le graphe pondéré X est transient si et seulement si il existe $x_0 \in X$ et $C = C(x_0)$ tels que*

$$f(x_0) \leq C\,\|\delta f\|_2\,, \qquad \textit{pour tout } f \in c_0(X)\,.$$

De plus, si X est transient, cette inégalité vaut pour tout $x_0 \in X$.

Nous dirons que X vérifie l'inégalité de Nash de dimension $\nu > 0$ si

$$N_\nu = N_\nu(X) = \inf\left\{\frac{\|\delta f\|_2\,\|f\|_1^{2/\nu}}{\|f\|_2^{1+2/\nu}} : f \in c_0(X)\,,\ f \neq 0\right\} > 0\,.$$

Dans ce cas, on a

$$(N_\nu) \qquad N_\nu\,\|f\|_2^{1+2/\nu} \leq \|\delta f\|_2\,\|f\|_1^{2/\nu}\,, \qquad \text{pour tout } f \in c_0(X)\,.$$

Notons que, comme pour l'inégalité $(S_{p,q})$ avec $p < q$, (N_ν) implique $\inf_X m > 0$.

Les inégalités de Nash et de Sobolev sont liées à la décroissance des noyaux itérés p_k comme l'indique le théorème suivant (voir [33], [6], [13]).

Théorème 3.4. *Pour $\nu > 0$, l'inégalité (N_ν) équivaut à*

$$\textit{il existe } C > 0 \textit{ tel que } p_k(x,x) \leq C\,m(x)\,k^{-\nu/2}\,, \textit{ pour tout } x \in X\,.$$

De plus, si $\nu > 2$ et $q = 2\nu/(\nu-2)$, ces propriétés sont aussi équivalentes à $(S_{2,q})$.

4. Graphes pondérés isométriques à l'infini.

La proposition suivante est l'adaptation de [22, Proposition 2.1] au cas des graphes pondérés. Nous n'en répéterons pas la preuve.

Proposition 4.1. *Soient* (X_1, m_1) *et* (X_2, m_2) *deux graphes pondérés isométriques à l'infini. Alors, si* (X_1, m_1) *vérifie l'inégalité* $(S_{p,q})$ *ou l'inégalité* (N_ν), *il en est de même de* (X_2, m_2).

Nous dirons qu'un graphe pondéré (X, m) vérifie l'inégalité de Poincaré à l'échelle (P) s'il existe une constante $C \geq 1$ et, pour tout $\sigma \geq 1$, une constante C_σ telles que, pour tout $x \in X$, $n \in \mathbb{N}^*$ et toute fonction f,

$$(P)\quad \Big(\sum_{y\in B(x,n)} |f(y) - f_n(x)|^\sigma\, m(y)\Big)^{1/\sigma} \leq C_\sigma\, n \Big(\sum_{y\in B(x,Cn)} |\delta f(y)|^\sigma\, m(y)\Big)^{1/\sigma},$$

où

$$f_n(x) = \frac{1}{V(x,n)} \sum_{y\in B(x,n)} f(y)\, m(y).$$

On distinguera la version L^σ dans cette famille d'inégalités en la notant (P_σ). L'inégalité (P) est satisfaite, par exemple, pour tous les graphes de Cayley des groupes finiment engendrés à croissance polynômiale du volume (la preuve est identique à celle donnée dans [35] pour les groupes de Lie). Elle est aussi satisfaite sur les espaces homogènes de ces groupes (voir [25], [30]).

Proposition 4.2. *Soient* (X_1, m_1) *et* (X_2, m_2) *deux graphes pondérés isométriques à l'infini. Alors, si* (X_1, m_1) *vérifie l'inégalité de Poincaré à l'échelle, il en est de même de* (X_2, m_2).

PREUVE. Pour alléger les notations, nous n'écrirons la preuve que pour $\sigma = 1$. Soit $\Phi : X_1 \to X_2$ une isométrie à l'infini, et $k \in \mathbb{N}^*$ tel que $[\Phi(X_1)]_k = X_2$. Si f est une fonction à support fini sur X_2, considérons sa moyenne f_k sur les boules de rayon k. Comme X_1 vérifie (P), on a, pour tout $x \in X_1$, et pour tout $n \in \mathbb{N}^*$,

$$(4.1)\quad \begin{aligned} &\sum_{y\in B(x,n)} |(f_k \circ \Phi)(y) - (f_k\circ\Phi)_n(x)|\, m_1(y) \\ &\qquad \leq C n \sum_{y\in B(x,C'n)} \delta(f_k\circ\Phi)(y)\, m_1(y). \end{aligned}$$

Commençons par traiter le second membre de cette inégalité. On a

$$(4.2)\qquad \sum_{y\in B(x,Cn)} \delta(f_k\circ\Phi)(y)\,m_1(y) \le C_1 \sum_{z\in B(\Phi(x),C_1'n)} \delta f_k(z)\,m_2(z)\,,$$

puisque pour C_1' assez grand, $B(\Phi(x),C_1'n)$ contient $\Phi(B(x,C'n))$ et que $m_2(\Phi(y))\approx m_1(y)$.

Ecrivons,

$$\begin{aligned}
|\delta f_k(z)|^2 &= \sum_{y\sim z} |f_k(z)-f_k(y)|^2\\
&= \sum_{y\sim z}\left|\frac{1}{V(z,k)}\sum_{t\in B(z,k)} f(t)\,m_2(t) - \frac{1}{V(y,k)}\sum_{s\in B(y,k)} f(s)\,m_2(s)\right|^2\\
&\le 2\sum_{y\sim z}\Big(\frac{1}{V(z,k)}\sum_{t\in B(z,k)} |f(t)-f(z)|^2\,m_2(t)\\
&\qquad\qquad + \frac{1}{V(y,k)}\sum_{s\in B(y,k)} |f(s)-f(z)|^2\,m_2(s)\Big)\\
&\le \frac{2\,C_2}{V(z,k)}\sum_{t\in B(z,k)} |f(t)-f(z)|^2\,m_2(t)\\
&\quad + \frac{C_3}{V(z,k)}\sum_{s\in B(z,k+1)} |f(s)-f(z)|^2\,m_2(s)\,.
\end{aligned}$$

Dans la dernière inégalité, C_2 désigne une borne sur le nombre de voisins dans X_2; pour le deuxième terme, on utilise le fait que $B(y,k)\subset B(z,k+1)$ et que $V(z,k)\le C_3V(y,k)$ (voir le Lemme 2.1). Comme $m_2(t)\approx V(z,k)$, pour tout $z\in X_2$, et pour tout $t\in B(z,k+1)$, on obtient

$$|\delta f_k(z)|^2 \le C_4 \sum_{t\in B(z,k+1)} |f(t)-f(z)|^2\,.$$

De plus,

$$|f(t)-f(z)|^2 \le (k+1)\sum_{i=1}^{j-1} |f(t_i)-f(t_{i+1})|^2 \le (k+1)\sum_{y\in B(z,k+1)} |\delta f(y)|^2\,,$$

où $t=t_1,\dots,t_i,\dots,t_j=z$ est un chemin minimisant de t à z. On en déduit

$$|\delta f_k(z)|^2 \le C_4\,(k+1)\,C_2^{k+1}\sum_{y\in B(z,k+1)} |\delta f(y)|^2\,,$$

soit

$$\delta f_k(z) \le C_5 \Big(\sum_{y \in B(z,k+1)} |\delta f(y)|^2 \Big)^{1/2} \le C_6 \sum_{y \in B(z,k+1)} \delta f(y) .$$

Finalement,

$$\sum_{z \in B(\Phi(x), C_1' n)} \delta f_k(z)\, m_2(z) \le C_7 \sum_{y \in B(\Phi(x), C_2' n)} \delta f(y)\, m_2(y) . \tag{4.3}$$

En mettant bout à bout (4.1),(4.2) et (4.3), on obtient

$$\begin{aligned} &\sum_{y \in B(x,n)} |(f_k \circ \Phi)(y) - (f_k \circ \Phi)_n(x)|\, m_1(y) \\ &\qquad \le C_8\, n \sum_{y \in B(\Phi(x), C_2' n)} \delta f(y)\, m_2(y) . \end{aligned} \tag{4.4}$$

Supposons maintenant que nous ayons démontré

$$\begin{aligned} &\sum_{z \in B(\Phi(x),n)} |f(z) - (f_k \circ \Phi)_n(x)|\, m_2(z) \\ &\qquad \le \sum_{y \in B(x, C_3' n)} |(f_k \circ \Phi)(y) - (f_k \circ \Phi)_n(x)|\, m_1(y) \\ &\qquad\quad + C_9 \sum_{z \in B(\Phi(x), C_3' n)} \delta f(z)\, m_2(z) . \end{aligned} \tag{4.5}$$

Alors, (4.4) et (4.5) donnent ensemble

$$\sum_{z \in B(\Phi(x),n)} |f(z) - (f_k \circ \Phi)_n(x)|\, m_2(z) \le C_{10}'\, n \sum_{z \in B(\Phi(x), C_4' n)} \delta f(z)\, m_2(z) .$$

On en déduit que X_2 possède la propriété (P), car

$$\sum_{z \in B(\Phi(x),n)} |f(z) - f_n(z)|\, m_2(z) \le 2 \inf_{\alpha} \sum_{z \in B(\Phi(x),n)} |f(z) - \alpha|\, m_2(z) .$$

Il nous reste donc à montrer (4.5). Ecrivons pour cela

$$\sum_{z \in B(\Phi(x),n)} |f(z) - (f_k \circ \Phi)_n(x)|\, m_2(z) \le \mathrm{I} + \mathrm{II} ,$$

où

$$\mathrm{I} = \sum_{z \in B(\Phi(x),n)} |f(z) - f_k \circ \Phi \circ \Phi^{-1}(z)|\, m_2(z)$$

et

$$\mathrm{II} = \sum_{z \in B(\Phi(x),n)} |f \circ \Phi \circ \Phi^{-1}(z) - (f_k \circ \Phi)_n(x)|\, m_2(z)\,.$$

On a, d'une part

$$\begin{aligned}\mathrm{II} &\leq C_{11} \sum_{z \in B(\Phi(x),n)} |f_k \circ \Phi \circ \Phi^{-1}(z) - (f_k \circ \Phi)_n(x)|\, m_1(\Phi^{-1}(z)) \\ &\leq C_{11} \sum_{y \in B(x, C_5' n)} |f_k \circ \Phi(y) - (f_k \circ \Phi)_n(x)|\, m_1(y)\,.\end{aligned}$$

En effet, pour C_5' assez grand, $\Phi^{-1}(B(\Phi(x),n)) \subset B(x, C_5' n)$, puisque Φ^{-1} est une quasi-isométrie et que $d_2(x, \Phi \circ \Phi^{-1}(x)) \leq k$.

D'autre part, si on pose $\Phi \circ \Phi^{-1}(z) = \bar{z}$, alors $d_2(z, \bar{z}) \leq k$ et

$$\begin{aligned}\mathrm{I} &= \sum_{z \in B(\Phi(x),n)} |f(z) - f_k(\bar{z})|\, m_2(z) \\ &\leq \sum_{z \in B(\Phi(x),n)} \Big(\frac{1}{V(\bar{z},k)} \sum_{t \in B(\bar{z},k)} |f(z) - f(t)|\, m_2(t) \Big)\, m_2(z)\,.\end{aligned}$$

En découpant comme nous l'avons fait précédemment $|f(z) - f(t)|$ suivant un chemin minimisant, on voit que

$$\mathrm{I} \leq C_{12} \sum_{z \in B(\Phi(x),n)} \Big(\sum_{y \in B(z,2k)} |\delta f(y)|^2 \Big)^{1/2} m_2(z)\,.$$

On en déduit facilement

$$\mathrm{I} \leq C_{13} \sum_{z \in B(\Phi(x), C_6' n)} \delta f(z)\, m_2(z)\,,$$

ce qui achève de démontrer (4.5) et la Proposition 4.2.

REMARQUE. Pour chaque $\sigma \geq 1$, on peut transférer d'un graphe pondéré à un autre la version L^σ de l'inégalité de Poincaré à l'échelle que nous avons notée (P_σ). On peut aussi considérer des inégalités analogues, mais où la dépendance en n n'est pas linéaire; si l'on part d'une dépendance en $K(n)$, K croissante, on obtient une dépendance en $CK(Cn)$ dans l'inégalité transférée.

5. Analyse à l'infini sur les variétés.

Ce paragraphe rappelle quelques résultats maintenant classiques et généralise [11]. Soit M une variété riemannienne connexe et complète munie de sa mesure canonique. Notons $\nabla\psi$ le gradient de la fonction ψ et

$$\|\psi\|_{p,E} = \Big(\int_E |\psi(x)|^p \, dx\Big)^{1/p}$$

pour $E \subset M$. Nous écrirons $\|\cdot\|_{p,M} = \|\cdot\|_p$ s'il n'y a pas d'ambiguïté.

Soit $p_t(x,y) > 0$ le noyau de la chaleur associé à l'opérateur de Laplace–Beltrami Δ sur M. Considérons la fonction de Green

$$G(x,y) = \int_0^\infty p_t(x,y)\, dt\,.$$

Nous dirons que M est transiente si $G(x,y) < +\infty$ en dehors de la diagonale et que M est récurrente sinon. Rappelons l'une des caractérisations classiques (voir [1], par exemple) de la transience.

Théorème 5.1. *Une variété M est transiente si et seulement si il existe un ouvert non vide U et une constante $C = C(U)$ tels que*

$$\int_U \psi(x)\, dx \le C\, \|\nabla\psi\|_2\,, \qquad \textit{pour tout } \psi \in C_0^\infty(M)\,.$$

De plus, si M est transiente, l'inégalité ci-dessus est satisfaite pour tout ouvert relativement compacte de M.

Soient $1 \le p \le q < +\infty$. Nous dirons que la variété M vérifie l'inégalité de Sobolev $(S_{p,q})$ si

$$S_{p,q} = S_{p,q}(M) = \inf\Big\{\frac{\|\nabla\psi\|_p}{\|\psi\|_q} \ :\ \psi \in C_0^\infty(M)\,,\ \psi \ne 0\Big\} > 0\,.$$

Dans ce cas, on a

$$(S_{p,q}) \qquad S_{p,q}\, \|\psi\|_q \le \|\nabla\psi\|_p\,, \qquad \text{pour tout } \psi \in C_0^\infty(M)\,.$$

De même, nous dirons que M vérifie l'inégalité de Nash de dimension ν notée (N_ν) si

$$N_\nu = N_\nu(M) = \inf\Big\{\frac{\|\nabla\psi\|_2\, \|\psi\|_1^{2/\nu}}{\|\psi\|_2^{1+2/\nu}} \ :\ \psi \in C_0^\infty(M)\,,\ \psi \ne 0\Big\} > 0\,.$$

Rappelons que l'inégalité (N_ν) est équivalente à

$$\text{il existe } C \text{ tel que } p_t(x,x) \leq C\, t^{-\nu/2}\,, \text{ pour tout } t > 0\,.$$

De plus, si $\nu > 2$, ces propriétés sont équivalentes à $(S_{2,q})$ avec $q = 2\nu/(\nu-2)$; voir [37] où ces équivalences sont discutées en détail.

Le but de ce paragraphe est de localiser à l'infini ce type d'équivalences sur les variétés vérifiant $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$. Supposons donc que M est une variété riemannienne connexe complète vérifiant $(P)_{\text{loc}}$ et $(DV)_{\text{loc}}$. Notons ρ la distance riemannienne sur M. Fixons $\varepsilon > 0$ ainsi qu'une partie X ε-séparée (pour tous $x, y \in X$, $\rho(x,y) \geq \varepsilon$) maximale de M. Nous utiliserons sans cesse le fait que les boules $B(x,\varepsilon)$, $x \in X$, recouvrent M, et que les boules $B(x,\varepsilon/2)$, $x \in X$, sont deux à deux disjointes. De plus, il découle facilement de l'hypothèse $(DV)_{\text{loc}}$ que, pour chaque $i \in \mathbb{N}^*$ fixé, tout point $z \in M$ appartient à un nombre uniformément borné de boules $B(x, i\varepsilon)$, $x \in X$.

Munissons X d'une structure de graphe en décidant que deux points distincts x, y de X sont voisins si $\rho(x,y) \leq 2\varepsilon$; nous noterons alors $x \sim y$. Il est facile de voir que, M étant connexe, X l'est aussi: on peut relier deux points arbitraires x et y de X par un chemin, c'est-à-dire qu'il existe une suite $x_0 = x, \ldots, x_i, x_{i+1}, \ldots, x_n = y$ telle que $x_i \sim x_{i+1}$, pour tout $i = 0, \ldots, n-1$. Il est clair que X est uniformément localement fini. Nous noterons N une borne sur le nombre de voisins de tout sommet de X.

Fixons $\{\theta_x\}_{x\in X}$ une partition de l'unité C^∞ sur M telle que $\theta_x \geq 1/N$ sur $\bar{B}(x,\varepsilon/2)$, $\theta_x \equiv 0$ sur $B(x, 3\varepsilon/2)^c$, et vérifiant $\|\nabla\theta_x\|_\infty \leq C$, pour tout $x \in X$. La construction d'une telle partition de l'unité est détaillée dans [21, p. 235].

Finalement, considérons l'opérateur S de $C_0^\infty(M)$ dans lui-même défini par

$$S\psi = \sum_{x\in X} \psi_\varepsilon(x)\, \theta_x,$$

où, suivant la notation de l'introduction

$$\psi_\varepsilon(x) = \frac{1}{V(x,\varepsilon)} \int_{B(x,\varepsilon)} \psi(y)\, dy\,.$$

Lemme 5.2. *Supposons que M vérifie $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$. Alors S opère sur L^p, $1 \leq p \leq +\infty$. Autrement dit,*

$$\|S\psi\|_p \leq C_1\, \|\psi\|_p\,, \qquad \textit{pour tout } \psi \in C_0^\infty(M)\,.$$

De plus,

$$\|\psi - S\psi\|_p \le C_2 \|\nabla\psi\|_p\,, \qquad \textit{pour tout } \psi \in C_0^\infty(M)\,. \tag{5.1}$$

Enfin, S est régularisant, c'est-à-dire opère de L^1 dans L^∞, si et seulement si

$$\inf_{x\in X} V(x,\varepsilon) > 0\,.$$

PREUVE. Seule la preuve de (5.1) mérite d'être donnée. Ecrivons

$$\begin{aligned}
\|\psi - S\psi\|_p^p &= \int \Big|\psi - \sum_{x\in X} \psi_\varepsilon(x)\,\theta_x\Big|^p \\
&= \int \Big|\sum_{x\in X} (\psi(y) - \psi_\varepsilon(x))\,\theta_x(y)\Big|^p\, dy \\
&\le N \sum_{z\in X} \int_{B(z,\varepsilon)} \Big(\sum_{x\sim z} |\psi(y) - \psi_\varepsilon(x)|\Big)^p\, dy \\
&\le N^p \sum_{z\in X} \sum_{x\sim z} \int_{B(z,\varepsilon)} |\psi(y) - \psi_\varepsilon(x)|^p\, dy \\
&\le 2^{p-1} N^p \sum_{z\in X} \Big(N \int_{B(z,\varepsilon)} |\psi(y) - \psi_\varepsilon(z)|^p\, dy \\
&\qquad + \sum_{x\sim z} |\psi_\varepsilon(z) - \psi_\varepsilon(x)|^p\, V(z,\varepsilon)\Big)\,.
\end{aligned}$$

Maintenant, d'après $(P)_{\text{loc}}$,

$$\int_{B(z,\varepsilon)} |\psi - \psi_\varepsilon(z)|^p \le C_{p,\varepsilon} \int_{B(z,2\varepsilon)} |\nabla\psi|^p\,.$$

Pour le second terme, nous utiliserons le résultat suivant qui découle de l'inégalité de Jensen et de l'inégalité de Poincaré $(P)_{\text{loc}}$, appliquée deux fois.

Lemme 5.3. *Soit M une variété vérifiant $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$. Etant donnés $\varepsilon > 0$ et $1 \le p < \infty$, il existe $C = C(\varepsilon, p)$ tel que, pour tous $x, y \in M$ tels que $\rho(x,y) \le 2\,\varepsilon$, on ait*

$$|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^p\, V(x,\varepsilon) \le C \int_{B(x,6\varepsilon)} |\nabla\psi(\xi)|^p\, d\xi\,.$$

Ceci termine la preuve de (5.1).

REMARQUE. Pour obtenir (5.1) nous n'avons utilisé les propriétés $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$ que pour un nombre fini de valeurs du paramètre r. Plus précisément, $\varepsilon > 0$ étant fixé, nous avons utilisé $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$ pour $r = \varepsilon/2, \varepsilon, 2\varepsilon, \ldots, 8\varepsilon$. Dans toute la suite, nous dirons qu'une constante C ne dépend que de $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$ à l'échelle ε si cette constante ne dépend que de $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$ avec $r \in \{\varepsilon/i\,,\ i\varepsilon :\ i = 1, \ldots, 16\}$. Par exemple, la constante C_2 dans (5.1) ne dépend que de $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$ à l'échelle ε.

Nous définissons maintenant la version à l'infini de $S_{p,q}(M)$ en posant

$$S_{p,q}^{\infty} = S_{p,q}^{\infty}(M) = \inf\left\{\frac{\|\nabla\psi\|_p}{\|S\psi\|_q}\ :\ \psi \in C_0^{\infty}(M)\,,\ \psi \neq 0\right\}.$$

Nous dirons que M vérifie $(S_{p,q}^{\infty})$ si $S_{p,q}^{\infty}(M) > 0$. Posons aussi

$$N_{\nu}^{\infty} = N_{\nu}^{\infty}(M) = \inf\left\{\frac{\|\nabla\psi\|_2\,\|\psi\|_1^{2/\nu}}{\|S\psi\|_2^{1+2/\nu}}\ :\ \psi \in C_0^{\infty}(M)\,,\ \psi \neq 0\right\}.$$

Nous dirons que M vérifie (N_{ν}^{∞}) si $N_{\nu}^{\infty}(M) > 0$.

Comme S opère sur L^q, $S_{p,q}$ (respectivement (N_{ν})) entraîne $S_{p,q}^{\infty}$ (respectivement (N_{ν}^{∞})) et l'inégalité (5.1) implique

Proposition 5.4. *Sur une variété M vérifiant $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$, l'inégalité $(S_{p,p})$ équivaut à $(S_{p,p}^{\infty})$. De plus, il existe c, C ne dépendant que de $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$ à l'échelle ε tels que*

$$c\, S_{p,p}^{\infty}(M) \leq S_{p,p}(M) \leq C\, S_{p,p}^{\infty}(M)\,.$$

REMARQUES: 1) On peut introduire la version locale de $S_{p,q}(M)$ en posant

$$S_{p,q}^{\text{loc}}(M) = \inf\left\{\frac{\|\nabla\psi\|_p}{\|\psi - S\psi\|_q}\ :\ \psi \in C_0^{\infty}(M)\,,\ \psi \neq 0\right\}.$$

On peut introduire de même une inégalité de Nash locale. Il est clair que $S_{p,q}(M) > 0$ si et seulement si $S_{p,q}^{\infty}(M) > 0$ et $S_{p,q}^{\text{loc}}(M) > 0$.

L'inégalité (5.1) signifie que l'on a toujours $S^{\text{loc}}_{p,p}(M) > 0$. Notons aussi que $S_{p,p}(M) \leq C$ où C ne dépend que des constantes apparaissant dans $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$.

2) Les inégalités $(S_{p,q})$, $(S^{\text{loc}}_{p,q})$ et $(S^{\infty}_{p,q})$ ne peuvent avoir lieu que si $p \leq q$. Ces inégalités ne peuvent avoir lieu avec $p < q$ que si $\inf_{x\in M} V(x,r)$ est strictement positif pour tout $r > 0$. Il suffit pour le voir d'appliquer l'une d'entre elles à la fonction $(r - d(x,\cdot))_+$. En faisant tendre r vers 0, on montre que, sur une variété riemannienne de dimension topologique n, $(S_{p,q})$ ou sa version locale ne peuvent avoir lieu que si $1/p - 1/q \leq 1/n$. Rappelons aussi que les variétés de dimension n à courbure de Ricci minorée vérifient $S^{\text{loc}}_{p,q}(M) > 0$ pour un ou tout couple (p,q) tels que $p < n$ et $0 < 1/p - 1/q \leq 1/n$ si et seulement si $\inf_M V(x,1) > 0$, voir [36]. Ces commentaires valent aussi pour les différentes versions (globale, locale, à l'infini) de l'inégalité de Nash si l'on remplace $1/p - 1/q$ par $1/\nu$.

3) D'après la remarque précédente et le Lemme 5.2, la validité de l'une des inégalités avec $p < q$ implique que l'opérateur S est régularisant. Il en va de même pour (N_ν) et (N^{∞}_{ν}).

Terminons ce paragraphe par la localisation à l'infini annoncée.

Théorème 5.5. *Soit $\nu > 0$. Si M vérifie $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$, les deux conditions suivantes sont équivalentes:*

1) *Il existe $t_0 > 0$ et C_0 tels que $\sup_x p_{t_0}(x,x) \leq C_0$ et $N^{\infty}_{\nu}(M) > 0$.*

2) *Il existe $r_0 > 0$ tel que $\inf_x V(x,r_0) > 0$, et $\sup_x p_t(x,x) = O(t^{-\nu/2})$ quand $t \to +\infty$.*

De plus, si $\nu > 2$ et $q = 2\nu/(\nu - 2)$, ces conditions sont aussi équivalentes à:

3) *Il existe $t_0 > 0$ et C_0 tels que $\sup_x p_{t_0}(x,x) \leq C_0$ et $S^{\infty}_{2,q}(M) > 0$.*

PREUVE. Notons $P_t = e^{-t\Delta}$ le semi-groupe de la chaleur sur M dont le noyau est p_t. Pour montrer que 1) implique 2), il suffit de montrer que 1) implique

$$(5.2) \quad \|P_{t_0}\psi\|_2^{1+2/\nu} \leq \|\nabla P_{t_0}\psi\|_2 \, \|\psi\|_1^{2/\nu}, \qquad \text{pour tout } \psi \in C_0^\infty(M)$$

puis d'utiliser l'argument de Nash (voir [26], [6], [37]). Mais, si on applique (N^{∞}_{ν}) à $P_{t_0}\psi$, on obtient

$$\|SP_{t_0}\psi\|_2^{1+2/\nu} \leq \|\nabla P_{t_0}\psi\|_2 \, \|P_{t_0}\psi\|_1^{2/\nu}, \qquad \text{pour tout } \psi \in C_0^\infty(M).$$

Comme le Lemme 5.2 donne

$$\|(I-S)P_{t_0}\psi\|_2 \leq \|\nabla P_{t_0}\psi\|_2$$

et que $\sup_x p_{t_0}(x,x) \leq C_0$ implique

$$\|(I-S)P_{t_0}\psi\|_2 \leq \|\psi\|_1 \ ,$$

on obtient facilement (5.2). La preuve de 2) implique 1) s'obtient à partir des arguments développés en détail dans [6] ou [37] en y ajoutant des manipulations similaires à celles présentées ci-dessus. En ce qui concerne 3), voir [11].

REMARQUE. Seule la version L^2 $(P_2)_{loc}$ de l'inégalité $(P)_{\text{loc}}$ a été utilisée dans la preuve du Théorème 5.5.

Corollaire 5.6. *Soit $\nu > 0$. Si M est à courbure de Ricci minorée (ou, plus généralement, si M vérifie les conditions $(DV)_0$ et $(P_2)_0$ introduites ci-dessous au Paragraphe 8), les deux conditions suivantes sont équivalentes:*

1) *M vérifie (N_ν^∞).*

2) *Pour tout $t_0 > 0$ il existe C_0 tel que $\sup_x p_t(x,x) \leq C_0\, t^{-\nu/2}$ pour tout $t \geq t_0$.*

De plus, si $\nu > 2$ et $q = 2\nu/(\nu-2)$, ces conditions sont aussi équivalentes à:

3) *M vérifie $(S_{2,q}^\infty)$.*

PREUVE. Sous les hypothèses de ce corollaire, pour tout $t_0 > 0$ il existe $c_0 > 0$, et C_0 tels que

$$c_0\, V(x,\sqrt{t})^{-1} \leq p_t(x,x) \leq C_0\, V(x,\sqrt{t})^{-1}\ , \quad \text{pour tout } t\,,\ 0 < t \leq t_0\ .$$

(voir Paragraphe 8) ce qui permet de conclure facilement.

REMARQUE. D'un point de vue plus géométrique, on notera que pour $q \geq 1$, l'inégalité $S_{1,q}$ équivaut, via la formule de coaire, à l'inégalité isopérimétrique $|A|^{1/q} \leq C\,|\partial A|$, où A décrit les sous-ensembles compacts de M à bord régulier; $S_{1,q}^\infty$ équivaut à la même inégalité, mais où A décrit les sous-ensembles compacts de M à bord régulier contenant une boule de rayon ε. Voir [8].

6. Discrétisation d'une variété riemannienne.

Nous reprenons les notations du paragraphe précédent. Autrement dit, M est une variété riemannienne connexe complète vérifiant $(P)_{\mathrm{loc}}$ et $(DV)_{\mathrm{loc}}$ et X est une partie ε-séparée maximale de M, pour un $\varepsilon > 0$ fixé. Rappelons que nous avons muni X d'une structure de graphe connexe tel que le nombre de voisins d'un point $x \in X$ est uniformément majoré (par N): deux points sont voisins dans X si $\rho(x,y) \leq 2\varepsilon$.

D'après [20, p.397], il existe $a \geq 1$ et $b \geq 0$ tels que pour tous $x, y \in X$,

$$\frac{1}{2\varepsilon}\,\rho(x,y) \leq d(x,y) \leq a\,\rho(x,y) + b\,,$$

où ρ est la distance riemannienne sur M et d est la distance naturellement associée au graphe X. Comme par construction le ε-voisinage X_ε est égal à M, l'inclusion de X dans M réalise une isométrie à l'infini de (X,d,m), où $m(x) = V(x,\varepsilon)$ (on identifie la fonction et la mesure), dans (M,ρ,μ). Remarquons que, d'après le Lemme 2.1, il existe une constante que nous noterons C_m, contrôlée par la constante de $(DV)_{\mathrm{loc}}$ sur M, à l'échelle ε, telle que

$$\sup_{\substack{x,y\\ x\sim y}} \frac{m(x)}{m(y)} \leq C_m\,.$$

(X,d,m) est donc un graphe pondéré au sens du Paragraphe 3. Dans la suite, on notera simplement M pour (M,ρ,μ) et X pour (X,d,m).

Le but de ce paragraphe est de montrer qu'un bon nombre de propriétés se transmettent entre une variété et son discrétisé. On généralisera ainsi des considérations issues de [1], [3], [11], [20], [21], [22], [31], [34].

Soit ψ une fonction sur M; nous lui associons une fonction $\tilde{\psi}$ sur X par

$$\tilde{\psi}(x) = \psi_\varepsilon(x) = \frac{1}{V(x,\varepsilon)} \int_{B(x,\varepsilon)} \psi(\xi)\,d\xi, \qquad x \in X\,.$$

L'inégalité de Jensen et les propriétés des boules $B(x,\varepsilon)$, $x \in X$, fournissent

Lemme 6.1. *Il existe C, C' tels que pour tout $x \in X$, tout entier n, tout $1 \leq p \leq +\infty$ et toute fonction $\psi \in C_0^\infty(M)$, on ait*

$$\|\tilde{\psi}\|_{p,B(x,n)} \leq C\,\|\psi\|_{p,B(x,C'n)}\,.$$

En particulier,

$$\|\tilde{\psi}\|_{p,X} \leq C\,\|\psi\|_{p,M}\,.$$

Inversement, soit f une fonction sur X; nous lui associons une fonction $\hat{f}$ sur M par

$$\hat{f}(y) = \sum_{x\in X} f(x)\,\theta_x(y)\,, \qquad y \in M\,,$$

où θ_x est la partition de l'unité déjà utilisée au Paragraphe 5.

Lemme 6.2. *Il existe C, C' tels que pour tout $z \in M$, tout $r > 0$, tout $1 \leq p \leq +\infty$ et toute fonction $f \in c_0(X)$, on ait*

$$\|\hat{f}\|_{p,B(z,r)} \leq C\,\|f\|_{p,B(\bar{z},\lceil C'r\rceil)}\,,$$

où $\bar{z}$ est un point de X tel que $\rho(z,\bar{z}) \leq \varepsilon$. En particulier,

$$\|\hat{f}\|_{p,M} \leq C\,\|f\|_{p,X}\,.$$

De plus, si $f \geq 0$, pour tout $x \in X$ et tout entier n, on a

$$\|f\|_{p,B(x,n)} \leq C\,\|\hat{f}\|_{p,B(x,C'n)}$$

et donc

$$\|f\|_{p,X} \leq C\,\|\hat{f}\|_{p,M}\,.$$

Rappelons que nous avons défini au Paragraphe 5 l'opérateur $S\psi = \sum_{x\in X}\tilde{\psi}(x)\,\theta_x$. Avec les notations ci-dessus, on a $S\psi = (\tilde{\psi})\hat{}$. Les lemmes 6.1 et 6.2 ont donc pour corollaire

Lemme 6.3. *Il existe C, C' tels que pour tout $z \in M$, $r > 0$ et toute fonction $\psi \in C_0^\infty(M)$,*

$$\|S\psi\|_{p,B(z,r)} \leq C\,\|\tilde{\psi}\|_{p,B(\bar{z},\lceil C'r\rceil)}$$

où $\bar{z} \in X$ est tel que $\rho(z,\bar{z}) \leq \varepsilon$. De plus, si $\psi \geq 0$, pour tout $x \in X$ et tout entier n, on a

$$\|\tilde{\psi}\|_{p,B(x,n)} \leq C\,\|S\psi\|_{p,B(x,C'n)}\,.$$

De même, si $f \in c_0(X)$ et $f \geq 0$,

$$\|f\|_{p,B(x,n)} \leq C \, \|S\hat{f}\|_{p,B(x,C'n)} \, .$$

Les constantes qui interviennent dans les trois lemmes ci-dessus et que nous avons uniformément notées C, ne dépendent que de $(DV)_{\mathrm{loc}}$, à l'échelle ε. Les constantes notées C' ne dépendent elles que de ε.

Les gradients discret et riemannien peuvent aussi être comparés.

Lemme 6.4. *Pour chaque $p \geq 1$, il existe des constantes C, C' telles que pour tout $x \in X$, tout entier n et toute fonction $\psi \in C_0^\infty(M)$, on ait*

$$\|\delta\tilde{\psi}\|_{p,B(x,n)} \leq C \, \|\nabla\psi\|_{p,B(x,C'n)} \, .$$

En particulier,

$$\|\delta\tilde{\psi}\|_{p,X} \leq C \, \|\nabla\psi\|_{p,M} \, .$$

De même, pour tout $z \in M$, $\bar{z} \in X$ tel que $\rho(z,\bar{z}) \leq \varepsilon$, tout $r > 0$ et toute fonction $f \in C_0(X)$,

$$\|\nabla\hat{f}\|_{p,B(z,r)} \leq C \, \|\delta f\|_{p,B(\bar{z},\lceil C'r\rceil)}$$

et donc

$$\|\nabla\hat{f}\|_{p,M} \leq C \, \|\delta f\|_{p,X} \, .$$

PREUVE. Les deux premières assertions découlent du Lemme 5.3. Pour les deux dernières, notons que, comme $\sum_{y\in X} \nabla\theta_y = 0$, on a

$$\nabla\hat{f} = \sum_{y\in X} (f(y) - f(x)) \, \nabla\theta_y \, , \qquad \text{pour tout } x \in X \, ,$$

et donc, pour tout $x \in X$ et tout $z \in B(x,\varepsilon)$,

$$\begin{aligned} |\nabla\hat{f}(z)| &\leq C_1 \sup\{|f(y) - f(x)| : \, d(y,x) \leq 2\} \\ &\leq C_2 \sum_{d(z,x)\leq 2} \delta f(z) \, . \end{aligned}$$

Il en résulte que

$$\int_{B(z,r)} |\nabla \hat{f}(\xi)|^p \, d\xi \le \sum_{x \in B(\bar{z}, \lceil C' r \rceil)} \int_{B(x,\varepsilon)} |\nabla \hat{f}(\xi)|^p \, d\xi$$
$$\le C_3 \sum_{x \in B(\bar{z}, \lceil C' r \rceil)} \sum_{d(z,x) \le 2} |\delta f(z)|^p \, m(x) ,$$

ce qui prouve l'inégalité annoncée. Les constantes C, C' du Lemme 6.4 ne dépendent que de ε et de $(DV)_{\text{loc}}$ et de $(P)_{\text{loc}}$, à l'échelle ε (en fait, pour les deux dernières inégalités, $(P)_{\text{loc}}$ n'intervient pas).

De ces lemmes techniques, on peut tirer immédiatement quelques résultats.

Proposition 6.5. *Soit $1 \le p \le q < +\infty$. Si X est un discrétisé de M, l'inégalité $(S^{\infty}_{p,q})$ sur M équivaut à $(S_{p,q})$ sur X. On a plus précisément*

$$c\, S_{p,q}(X) \le S^{\infty}_{p,q}(M) \le C\, S_{p,q}(X) ,$$

où c, C ne dépendent que de $(DV)_{\text{loc}}$ et de $(P)_{\text{loc}}$ sur M à l'échelle ε. Le même résultat vaut pour $N^{\infty}_{\nu}(M)$ et $N_{\nu}(X)$ avec $\nu > 0$.

Corollaire 6.6. *Si X est un discrétisé de M, $(S_{p,p})$ sur M équivaut à $(S_{p,p})$ sur X. Le rapport de $S_{p,p}(M)$ et de $S_{p,p}(X)$ est compris entre des bornes ne dépendant que des constantes de $(DV)_{\text{loc}}$ et de $(P)_{\text{loc}}$ sur M à l'échelle ε.*

Avec le Corollaire 3.2, on en déduit

Corollaire 6.7. *Sur M, les inégalités $(S_{p,p})$ sont deux à deux équivalentes pour $1 \le p < \infty$. Si $p' \le p$,*

$$C_1\, S_{p',p'}(M) \le S_{p,p}(M) \le C_2\, (S_{p',p'}(M))^{1/p} ,$$

où C_1 et C_2 ne dépendent que de p et p', et des constantes de $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$ à l'échelle ε.

CAS PARTICULIER: Prenons $p = 2$, $p' = 1$. D'après la formule de co-aire, $S_{1,1}(M) = h(M)$, avec

$$h(M) = \inf_{\Omega} \frac{|\partial \Omega|}{|\Omega|} ,$$

où Ω parcourt les sous-ensembles compacts à bord régulier de M ($h(M)$ est la constante de Cheeger). Comme $(S_{2,2}(M))^2 = \lambda_1$, où

$$\lambda_1 = \inf_{f \in C_0^\infty(M)} \frac{\langle -\Delta f, f \rangle}{\|f\|_2^2},$$

on obtient une forme plus faible mais plus générale de l'inégalité de Cheeger-Buser:

$$C_3\, h^2 \leq \lambda_1 \leq C_4\, h,$$

où C_3 et C_4 ne dépend que des constantes de $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$; voir [5].

Plus généralement, les propositions 3.1 et 6.5 donnent le

Corollaire 6.8. *Sur M, l'inégalité $(S^\infty_{p,q})$ entraîne $(S^\infty_{p',q'})$ avec $1/p' - 1/q' \geq 1/p - 1/q$ pour $p' \geq p$, et $(S^\infty_{p',q'})$ avec $1/p' - 1/q' \geq 1 - p/q$ pour $1 \leq p' \leq p$; dans le premier cas, on a*

$$S^\infty_{p',q'}(M) \geq c\, S^\infty_{p,q}(M),$$

et dans le second,

$$S^\infty_{p',q'}(M) \geq c\, (S^\infty_{p,q}(M))^p,$$

où c ne dépend que de p, p', q, q', et de $(DV)_{\text{loc}}$ et de $(P)_{\text{loc}}$ à léchelle ε.

On trouvera dans [12, Proposition 2], une esquisse de la deuxième partie de cet énoncé.

REMARQUE. Les implications précédentes ne peuvent être améliorées. Dans le cas $p' \geq p$, il suffit pour le voir de prendre $M = \mathbb{R}^n$, où $(S^\infty_{p,q})$ a lieu si et seulement si $1/p - 1/q \geq 1/n$. On construit d'autre part dans [12], pour tout $\varepsilon > 0$, et pour q arbitrairement proche de 1, une variété à géométrie bornée qui vérifie $(S^\infty_{2,q})$ mais pas $(S^\infty_{1,q/2+\varepsilon})$.

Concernant la transience, le Lemme 6.4 et les théorèmes 3.3 et 5.1 fournissent

Proposition 6.9. *Soit M une variété vérifiant $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$. Soit X un discrétisé de M. Alors, M est transiente si et seulement si X est transient.*

Nous passons maintenant aux inégalités de Poincaré. Nous dirons qu'une variété riemannienne M vérifie l'inégalité de Poincaré $(P)_\infty$ s'il existe C et si pour tout r_0 et tout $\sigma \geq 1$, il existe C_{σ,r_0} tels que pour tout $x \in M$, tout $r \geq r_0$ et tout $\psi \in C_0^\infty(M)$, on ait

$$(P)_\infty \quad \Big(\int\limits_{B(x,r)} |\psi(y) - \psi_r(x)|^\sigma \, dy \Big)^{1/\sigma} \leq C_{\sigma,r_0}\, r \Big(\int\limits_{B(x,Cr)} |\nabla\psi(y)|^\sigma \, dy \Big)^{1/\sigma}.$$

Les variétés riemanniennes à courbure de Ricci positive ou nulle vérifient $(P)_\infty$ (voir [5]). Il en est de même des groupes de Lie à croissance polynômiale du volume (voir [35]).

Proposition 6.10. *Soit M une variété satisfaisant $(P)_{\text{loc}}$ et $(DV)_{\text{loc}}$. Elle vérifie $(P)_\infty$ si et seulement si son discrétisé X vérifie (P).*

PREUVE. Nous n'écrirons la preuve que pour $\sigma = 1$. Les autres cas se traitent de la même façon Supposons que X vérifie (P). Soit $\psi \in C_0^\infty(M)$, $x \in M$, $r \geq \varepsilon$ et $\alpha \in \mathbb{R}$; on a

$$\begin{aligned}\int_{B(x,r)} |\psi(y) - \alpha| \, dy &\leq \int_{B(x,r)} \sum_{z \in X \cap B(x,r+\varepsilon)} |\psi(y) - \alpha| \, 1_{B(z,\varepsilon)}(y) \, dy \\ &\leq \sum_{z \in X \cap B(x,r+\varepsilon)} \int_{B(z,\varepsilon)} |\psi(y) - \tilde{\psi}(z)| \, dy \\ &\quad + \sum_{z \in X \cap B(x,r+\varepsilon)} m(z) \, |\tilde{\psi}(z) - \alpha| \, .\end{aligned}$$

Le premier terme est dominé, d'après $(P)_{\text{loc}}$, par

$$C_1 \sum_{z \in X \cap B(x,r+\varepsilon)} \int_{B(z,C_1'\varepsilon)} |\nabla\psi(y)| \, dy \, ,$$

donc, puisque $r \geq \varepsilon$, par

$$C_2 \int_{B(x,C_2'r)} |\nabla\psi(y)| \, dy \, .$$

Soit $x_0 \in X$ tel que $\rho(x,x_0) \leq \varepsilon$, et $n \in \mathbb{N}^*$ tel que $n \approx r$ et $X \cap B_M(x,r+\varepsilon) \subset B_X(x_0,n) \subset B_M(x,C''r)$; si l'on choisit $\alpha = \psi_n(x_0)$, le second terme est dominé, d'après (P), par

$$C_3 \, n \sum_{y \in B(x_0,C_3'n)} |\delta\tilde{\psi}(y)| \, .$$

Le Lemme 6.4 permet de majorer cette dernière quantité par

$$C_4\, r \int_{B(x,C_4' r)} |\nabla \psi(y)|\, dy\,.$$

En observant que

$$\int_{B(x,r)} |\psi(y) - \psi_r(x)|\, dy \leq 2 \inf_{\alpha\in\mathbb{R}} \int_{B(x,r)} |\psi(y) - \alpha|\, dy\,,$$

on obtient

$$\int_{B(x,r)} |\psi(y) - \psi_r(x)|\, dy \leq C_5\, r \int_{B(x,C_5' r)} |\nabla \psi(y)|\, dy\,,$$

pour $r \geq \varepsilon$. Jointe à la condition $(P)_{\mathrm{loc}}$, cette propriété donne $(P)_\infty$.

Supposons maintenant que M vérifie $(P)_\infty$. Soit f une fonction à support fini sur X, $x \in X$, $n \in \mathbb{N}^*$, et $\alpha \in \mathbb{R}$. Ecrivons

$$\sum_{y\in B(x,n)} |f(y) - \alpha|\, m(y) \leq C\Big(\sum_{y\in B(x,n)} \int_{B(y,\varepsilon/2)} |f(y) - \hat{f}(z)|\, dz + \sum_{y\in B(x,n)} \int_{B(y,\varepsilon/2)} |\hat{f}(z) - \alpha|\, dz\Big).$$

D'une part, on a

$$\mathrm{I} = \sum_{y\in B(x,n)} \int_{B(y,\varepsilon/2)} |\hat{f}(z) - \alpha|\, dz \leq \int_{B(x,C' n)} |\hat{f}(z) - \alpha|\, dz\,.$$

On choisit alors $\alpha = (\hat{f})_n(x)$, et on applique $(P)_\infty$. Il en découle

$$\mathrm{I} \leq C_1\, n \int_{B(x,C_1' n)} |\nabla \hat{f}(z)|\, dz\,.$$

Le Lemme 6.4 montre alors que

$$\int_{B(x,C_1' n)} |\nabla \hat{f}(z)|\, dz \leq C_2 \sum_{y\in B(x,C_2' n)} \delta f(y)\,.$$

Attention, dans ce calcul on a utilisé la convention que les boules $B(x,r)$ se rapportent à X quand elles apparaissent sous des sommes et à M quand elles apparaissent sous des intégrales.

D'autre part, comme $\theta_t(z) = 0$ dès que $d(z,t) \geq 3\varepsilon/2$, on a

$$\begin{aligned}\int_{B(y,\varepsilon/2)} |f(y) - \hat{f}(z)|\,dz &= \int_{B(y,\varepsilon/2)} |f(y) - \sum_{t\sim y} f(t)\,\theta_t(z)|\,dz \\ &= \int_{B(y,\varepsilon/2)} |\sum_{t\sim y}(f(y) - f(t))\,\theta_t(z)|\,dz \\ &\leq C_3 \sum_{t\sim y} |f(y) - f(t)|\,m(y) \\ &\leq C_4\,\delta f(y)\,m(y)\,.\end{aligned}$$

Il en résulte que

$$\mathrm{II} = \sum_{y\in B(x,n)} \int_{B(y,\varepsilon/2)} |f(y) - \hat{f}(z)|\,dz \leq C \sum_{y\in B(x,n)} \delta f(y)\,m(y)\,.$$

Ceci termine la démonstration puisque

$$\sum_{y\in B(x,n)} |f(y) - f_n(x)|\,m(y) \leq 2 \inf_{\alpha\in\mathbb{R}} \sum_{y\in B(x,n)} |f(y) - \alpha|\,m(y)\,.$$

REMARQUE. L'énoncé précédent vaut en fait séparément pour chacune des versions L^σ des inégalités de Poincaré à l'infini sur M que nous noterons $(P_\sigma)_\infty$: pour chaque $\sigma \geq 1$, sous l'hypothèse que M vérifie $(DV)_{\mathrm{loc}}$ et $(P_\sigma)_{loc}$, l'inégalité $(P_\sigma)_\infty$ sur M équivaut à (P_σ) sur X. L'énoncé s'étend aussi à d'autres types de dépendance en $r \geq 1$ et $n \in \mathbb{N}^*$.

7. Variétés et graphes isométriques à l'infini.

En mettant bout à bout le passage d'une variété à un discrétisé et le transfert entre graphes pondérés isométriques à l'infini, on peut ainsi résumer l'ensemble des paragraphes précédents:

Théorème 7.1. *Soit M_1, M_2 deux variétés vérifiant $(DV)_{\mathrm{loc}}$ et $(P)_{\mathrm{loc}}$ et isométriques à l'infini.*

1) *Soit $1 \leq p \leq q \leq +\infty$ et $\nu > 0$. Les conditions $S^\infty_{p,q}(M_i) > 0$, $i = 1,2$ sont équivalentes. Il en va de même des conditions $S_{p,p}(M_i) > 0$, $i = 1,2$, comme de $N_\nu(M_i) > 0$, $i = 1,2$.*

2) *M_1 et M_2 sont simultanément récurrentes ou transientes.*

3) *Pour chaque $\sigma \geq 1$, M_1 vérifie la condition $(P_\sigma)_\infty$ si et seulement si M_2 vérifie $(P_\sigma)_\infty$.*

4) *Soient $p_t^i(x,y)$, $i = 1,2$, les noyaux de la chaleur sur M_1, M_2. Supposons que pour chaque $r > 0$ et $i = 1,2$ on ait $\inf_{M_i} V(x,r) > 0$, et que pour $i = 1,2$ il existe $t_i > 0$ et C_i tels que $\sup_{M_i} p_{t_i}^i(x,x) \leq C_i$. Alors,*

$$\sup_{M_1} p_t^1(x,x) = O(t^{-\nu/2}), \qquad \textit{quand } t \to +\infty,$$

si et seulement si

$$\sup_{M_2} p_t^2(x,x) = O(t^{-\nu/2}), \qquad \textit{quand } t \to +\infty.$$

Théorème 7.2. *Soit M une variété vérifiant $(DV)_{\text{loc}}$ et $(P)_{\text{loc}}$ et soit (X,m) un graphe pondéré isométrique à l'infini à M (par exemple son discrétisé).*

1) *Soit $1 \leq p \leq q \leq +\infty$ et $\nu > 0$. Les conditions $S_{p,q}^\infty(M) > 0$ et $S_{p,q}(X) > 0$ sont équivalentes. Il en va de même des conditions $S_{p,p}(M) > 0$ et $S_{p,p}(X) > 0$ comme de $N_\nu(M) > 0$ et $N_\nu(X) > 0$.*

2) *M et X sont simultanément récurrents ou transients.*

3) *Pour chaque $\sigma \geq 1$, M vérifie la condition $(P_\sigma)_\infty$ si et seulement si X vérifie (P_σ).*

4) *Supposons que $\inf_X m > 0$ et qu'il existe $t_0 > 0$ et C_0 tels que $\sup_M p_{t_0}^M(x,x) \leq C_0$. Alors, les noyaux itérés p_k^X, $k = 1,2,\ldots$, sur X satisfont*

$$\sup_X p_k^X(x,x) = O(k^{-\nu/2})$$

si et seulement si le noyau de la chaleur p_t^M sur M satisfait

$$\sup_M p_t^M(x,x) = O(t^{-\nu/2}), \qquad \textit{quand } t \to +\infty.$$

Rappelons pour mémoire le cas typique d'application du Théorème 7.2 où M est une variété sur laquelle agit sans point fixe un groupe d'isométries Γ, finiment engendré, et tel que le quotient M/Γ soit compact. Le graphe de Cayley de Γ est alors isométrique à l'infini à M. En particulier, le Théorème 7.2 contient et généralise les résultats suivants:

1. Le revêtement universel $\tilde{N}$ d'une variété compacte N est transient si et seulement si le groupe fondamental $\pi_1(N)$ est transient (voir [31, [32]).

2. Le bas du spectre de $\tilde{N}$ est strictement positif si et seulement si $\pi_1(N)$ est non-moyennable (voir [3], [31]).

REMARQUE. Dans [4], Brooks étudie par discrétisation certaines propriétés de l'ensemble des revêtements compacts d'une variété compacte. Par exemple, si N est une variété compacte dont le π_1 a la propriété T de Kazhdan, alors la première valeur propre non nulle de tous les revêtements compacts de N est uniformément minorée. Pour le voir, Brooks utilise le fait que la première valeur propre non nulle (pour le laplacien discret) de tous les quotients finis d'un groupe finiment engendré ayant la propriété de Kazhdan est uniformément minorée. Le Théorème 7.2, 1, ci-dessus permet de retrouver ces résultats de Brooks par transport des inégalités de Poincaré. L'idée de base est, comme dans [4], la discrétisation mais notre approche est un peu différente.

8. Inégalités de Harnack et propriété de Liouville.

Soit M une variété riemannienne. Nous dirons que M a la propriété de Liouville (respectivement de Liouville forte) si elle n'admet pas d'autres fonctions harmoniques bornées (respectivement positives) que les constantes.

Nous dirons que M vérifie l'inégalité de Harnack elliptique s'il existe une constante $C > 0$ telle que pour tout $x \in M$, et tout $r > 0$, toute solution u positive de $\Delta u = 0$ dans $B(x,r)$ satisfait

$$\sup_{B(x,r/2)} u \leq C \inf_{B(x,r/2)} u .$$

Nous dirons que M vérifie l'inégalité de Harnack parabolique s'il existe une constante $C > 0$ telle que pour tout $x \in M$, et tout $r > 0$, toute solution u positive de $(\partial_t + \Delta)u = 0$ dans $Q(x,r) =]0, r^2[\times B(x,r)$ satisfait

$$\sup_{Q_-(x,r)} u \leq C \inf_{Q_+(x,r)} u ,$$

où

$$Q_-(x,r) =](r/4)^2, (r/2)^2[\times B(x, r/2)$$

et

$$Q_+(x,r) =](3r/4)^2, r^2[\times B(x,r/2).$$

Nous dirons enfin que M vérifie l'inégalité de Harnack parabolique locale si, pour tout $r_0 > 0$, il existe $C(r_0) > 0$ tel que pour tout $x \in M$, et tout r, $0 < r \leq r_0$, toute solution u positive de $(\partial_t + \Delta)u = 0$ dans $Q(x,r)$ satisfait

$$\sup_{Q_-(x,r)} u \leq C(r_0) \inf_{Q_+(x,r)} u .$$

Bien entendu, l'inégalité de Harnack parabolique entraîne l'inégalité de Harnack elliptique, qui entraîne à son tour la propriété de Liouville forte. Clairement, les inégalités de Harnack sont beaucoup plus précises que les propriétés de Liouville. A. Grigor'yan remarque dans [16, p. 340] que la Proposition 6 de [9] montre qu'une variété peut vérifier la version elliptique de l'inégalité de Harnack sans vérifier sa version parabolique. Dans le cadre des variétés à courbure de Ricci positive ou nulle, des versions précises de l' inégalité de Harnack parabolique sont fournies par P. Li et S-T. Yau dans [23].

T. Lyons a construit dans [24] (voir aussi [2]) deux variétés quasi-isométriques, à courbure de Ricci minorée, dont l'une a la propriété de Liouville forte et l'autre n'a pas la propriété de Liouville. Nous allons voir que, au contraire, l'inégalité de Harnack parabolique est stable par isométrie à l'infini.

Nous utiliserons pour cela le résultat principal de [29], qui donne une caractérisation de l'inégalité de Harnack parabolique en termes géométriques. Pour formuler ce résultat, nous devons introduire des formes de l'inégalité de Poincaré à l'échelle et de la propriété de doublement du volume plus fortes que celles que nous avons considérées jusqu'à présent. Nous dirons que M vérifie (DV) s'il existe C tel que

$$(DV) \qquad V(x,2r) \leq C\, V(x,r), \qquad \text{pour tous } x \in M,\ r > 0 .$$

Pour chaque $\sigma \geq 1$, nous dirons que M vérifie (P_σ) s'il existe C_σ tel que pour tout $x \in M$, $r > 0$, et toute fonction $\psi \in C_0^\infty(M)$ on ait

$$(P_\sigma) \quad \Big(\int_{B(x,r)} |\psi(y) - \psi_r(x)|^\sigma \, dy \Big)^{1/\sigma} \leq C_\sigma\, r \Big(\int_{B(x,2r)} |\nabla\psi(y)|^\sigma \, dy \Big)^{1/\sigma} .$$

Nous aurons aussi besoin d'une version plus locale de ces conditions. Nous dirons que M vérifie $(DV)_0$ si, pour tout $r_0 > 0$, il existe $C(r_0)$ tel que

$$(DV)_0 \quad V(x,2r) \leq C(r_0)\, V(x,r), \quad \text{pour tout } x \in M,\ 0 < r \leq r_0 .$$

Pour chaque $\sigma \geq 1$, nous dirons que M vérifie $(P_\sigma)_0$ si, pour tout $r_0 > 0$, il existe $C_\sigma(r_0)$ tel que pour tout $x \in M$, tout $0 < r \leq r_0$, et toute fonction $\psi \in C_0^\infty(M)$ on ait

$$(P_\sigma)_0 \quad \Big(\int_{B(x,r)} |\psi(y) - \psi_r(x)|^\sigma \, dy \Big)^{1/\sigma} \leq C_\sigma(r_0)\, r \Big(\int_{B(x,2r)} |\nabla\psi(y)|^\sigma \, dy \Big)^{1/\sigma}.$$

Nous dirons que M vérifie $(P)_0$ si elle vérifie $(P_\sigma)_0$ pour tout $\sigma \geq 1$. Avec ces notations, le résultat principal de [29] s'énonce:

Théorème 8.1. *Une variété riemannienne vérifie l'inégalité de Harnack parabolique si et seulement si elle vérifie l'inégalité de Poincaré à l'échelle* (P_2) *et la propriété de doublement du volume* (DV). *Elle vérifie l'inégalité de Harnack parabolique locale si et seulement si* $(P_2)_0$ *et* $(DV)_0$ *sont satisfaites.*

L'inégalité de Harnack parabolique est un outil puissant qui a de nombreuses applications. Quelques-unes sont rappelées dans [29]. Elle permet, par exemple, d'obtenir des bornes gaussiennes, inférieure et supérieure, pour le noyau de la chaleur. En particulier, l'inégalité de Harnack locale permet de montrer que pour tout t_0 il existe c_0, C_0 tels que

$$c_0\, V(x, \sqrt{t})^{-1} \leq p_t(x,x) \leq C_0\, V(x,\sqrt{t})^{-1}, \qquad x \in M,\ 0 < t \leq t_0 .$$

A l'aide du Théorème 7.1, on obtient donc

Proposition 8.2. *Soient* M_1 *et* M_2 *deux variétés riemanniennes isométriques à l'infini vérifiant* $(P_2)_0$ *et* $(DV)_0$. *Alors, pour tout* $\nu > 0$, *on a*

$$\sup_{M_1} p_t^1(x,x) = O(t^{-\nu/2}), \qquad \textit{quand } t \to \infty ,$$

si et seulement si

$$\sup_{M_2} p_t^2(x,x) = O(t^{-\nu/2}), \qquad \textit{quand } t \to \infty .$$

De même, si M *vérifie* $(P_2)_0$ *et* $(DV)_0$ *et est isométrique à l'infini au graphe pondéré* X, *alors*

$$\sup_{M} p_t^M(x,x) = O(t^{-\nu/2}), \qquad \textit{quand } t \to \infty ,$$

si et seulement si

$$\sup_X p_k^X(x,x) = O(k^{-\nu/2}), \qquad \textit{quand } k \to \infty .$$

Comme la propriété de doublement du volume (Proposition 2.3) et la propriété $(P_2)_\infty$ (théorèmes 7.1 et 7.2 ou, plus précisément, propositions 4.2 et 6.10 et les remarques qui les suivent) sont conservées par isométrie à l'infini, le Théorème 8.1 fournit:

Théorème 8.3. 1) *Soient M_1 et M_2 deux variétés riemanniennes isométriques à l'infini vérifiant $(DV)_0$ et $(P_2)_0$. Alors, si M_1 vérifie l'inégalité de Harnack parabolique, il en est de même de M_2.*

2) *Si M est une variété vérifiant $(DV)_0$ et $(P)_0$ et isométrique à l'infini à un graphe pondéré (X, m), alors M vérifie l'inégalité de Harnack parabolique si et seulement si X vérifie (DV) et (P_2).*

Corollaire 8.4. *Soit M une variété riemannienne à courbure de Ricci minorée. Supposons que M est isométrique à l'infini soit à une variété à courbure de Ricci positive ou nulle, soit à un groupe de Lie à croissance polynômiale du volume, soit encore à un groupe finiment engendré à croissance polynômiale du volume. Alors M vérifie l'inégalité de Harnack parabolique; en particulier, M a la propriété de Liouville forte.*

Cet énoncé généralise le Théorème 5.1 de [21], qui traite le cas où M est isométrique à l'infini à $\mathbb{R}^n$, avec $n \geq \dim M$, et les résultats de [16], [27], [28], [29], qui donnent certains cas des énoncés ci-dessus si l'on remplace isométrique à l'infini par quasi-isométrique.

REMARQUE: On peut vouloir essayer de décrire d'une façon ou d'une autre, à isométrie à l'infini près, les classes de variétés et de graphes vérifiant (DV) et (P). Nous nous contenterons des observations suivantes:

1. En utilisant les résultats de [30], on montre facilement qu'une variété M (ou un graphe) connexe qui vérifie (DV), (P) et $V(x,r) \geq c\,r^{1+\varepsilon}$, $r \geq 1$ ne peut posséder qu'une seule composante connexe à l'infini (*i.e.* $M \setminus B(x_0, r)$ est connexe pour tout r assez grand).

2. Tous les graphes de dimension 1 (*i.e.* tels que $C^{-1} \leq V(x,r)/r \leq C$) satisfont (DV) et (P). On en construit facilement une infinité

deux à deux non isométriques à l'infini. Il suffit par exemple de considérer la famille de graphes $(X_n)_{n\in\mathbb{N}^*}$, où X_n est constitué de n copies de $\mathbb{Z}$ qui se croisent en un point. Cet exemple montre que la Remarque 1 ne s'applique pas en dimension un. Il existe aussi une infinité de graphes de dimension un ne possédant qu'une composante connexe à l'infini et deux à deux non isométriques à l'infini. Il suffit de considérer deux copies de $\mathbb{N}$ et d'identifier par exemple les points a^i, $i \in \mathbb{N}$, où a est un entier donné. Les graphes associés à des entiers distincts ne sont pas isométriques à l'infini.

9. Opérateurs isométriques à l'infini.

La technique de discrétisation que nous avons décrite jusqu'ici pour les variétés riemanniennes peut en fait être mise en œuvre dès que l'on sait donner un sens à $(DV)_{\mathrm{loc}}$ et $(PL)_{loc}$.

Considérons une variété connexe M munie d'un opérateur du second ordre $\mathcal{L}$, à coefficients réels et sans terme constant. Supposons qu'il existe une mesure positive et C^∞ μ sur M telle que $\mathcal{L}$ soit auto-adjoint sur $L^2(M,\mu)$. Supposons enfin que $\mathcal{L}$ est localement sous-elliptique. On peut alors définir la longueur du gradient associé à $\mathcal{L}$ en posant

$$|\nabla\psi|^2 = |\nabla_{\mathcal{L}}\psi|^2 = \frac{1}{2}\left(-\mathcal{L}\psi^2 + 2\,\psi\mathcal{L}\,\psi\right).$$

Une métrique ρ est canoniquement associée à $\mathcal{L}$; on a en fait

$$\rho(x,y) = \sup\{\psi(x) - \psi(y) : \ \psi \in C_0^\infty(M),\ |\nabla f| \leq 1\}.$$

Nous disposons donc des notions nécessaires pour écrire les diverses inégalités de Poincaré à l'échelle et les diverses conditions de doublement du volume. Nous les noterons comme précédemment.

EXEMPLE. Sur $\mathbb{R}^2$, considérons l'opérateur de Grushin $\mathcal{L} = -(\partial_x^2 + x^2\partial_y^2)$. C'est un des opérateurs sous-elliptiques les plus simples (voir par exemple [19]). On peut montrer que $\mathcal{L}$ satisfait (DV) et (P). Ceci résulte de ce que $\mathcal{L}$ peut être obtenu par passage au quotient à partir du sous-laplacien sur le groupe de Heisenberg. De plus, pour $z=(x,y)$, $V(z,r) \simeq |x|\,r^2 + r^3$, ce qui montre que (DV) n'implique pas, en général, que $V(z,1)$ soit majoré ou minoré. Pour tout ceci, voir [25].

Soient maintenant deux opérateurs $\mathcal{L}_1, \mathcal{L}_2$ sur M comme ci-dessus, μ_1, μ_2 les mesures et ρ_1, ρ_2 les distances associées. Nous dirons que $\mathcal{L}_1$ et $\mathcal{L}_2$ sont isométriques à l'infini si (M, ρ_1, μ_1) et (M, ρ_2, μ_2) sont isométriques à l'infini au sens du Paragraphe 1.

Proposition 9.1. *Supposons que $\mathcal{L}_1$, $\mathcal{L}_2$ vérifient $(DV)_{\mathrm{loc}}$ et $(P)_{\mathrm{loc}}$ et qu'ils sont isométriques à l'infini.*

1) *Les opérateurs $\mathcal{L}_1$ et $\mathcal{L}_2$ vérifient simultanément chacune des propriétés $(S^{\infty}_{p,q})$, (N^{∞}_{ν}) ou $(S_{p,p})$ pour un ou pour tout $1 \leq p < +\infty$.*

2) *Soient p^1_t et p^2_t les noyaux de la chaleur respectivement associés à $\mathcal{L}_1$ et $\mathcal{L}_2$, et soit $\nu > 0$. Supposons que $\inf_M V_i(x,r) > 0$, pour tout $r > 0$ et $i = 1, 2$. Supposons de plus qu'il existe $t_i > 0$ et C_i tels que $\sup_M p^i_{t_i}(x,x) \leq C_i$ pour $i = 1, 2$. Alors*

$$\sup_M p^1_t(x,y) = O(t^{-\nu/2}), \qquad \textit{quand } t \to +\infty,$$

si et seulement si

$$\sup_M p^2_t(x,y) = O(t^{-\nu/2}), \qquad \textit{quand } t \to +\infty.$$

En particulier, cette équivalence vaut quand $\mathcal{L}_1$, $\mathcal{L}_2$ vérifient $(DV)_0$ et $(P)_0$ (ou même $(P_2)_0$).

EXEMPLE. Soit G un groupe de Lie muni d'une mesure de Haar invariante à droite. Considérons les opérateurs $\mathcal{L}$ sur G de la forme

$$\mathcal{L}\psi = -\sum_{i=1}^{k} X_i^2 \psi,$$

où les X_i sont des champs de vecteurs invariants à gauche qui engendrent algébriquement l'algèbre de Lie de G. Les propriétés $(DV)_0$ et $(P)_0$ sont alors vérifiées, voir [37], [35]. La première assertion de la Proposition 9.1 montre que G est non-moyennable si et seulement si pour un ou pour tout p, $1 \leq p < +\infty$, et un ou tout $\mathcal{L}$ comme ci-dessus, la propriété $(S_{p,p})$ est satisfaite. Ici on peut prendre comme définition de la non-moyennabilité le fait que le spectre de $\mathcal{L}_0 = -\sum_1^m Z_i^2$ sur $L^2(G)$ (mesure de Haar invariante à droite) ne contient pas zéro, où $(Z_i)_1^m$ est une famille fixée de champs de vecteurs invariants à gauche qui forme une base de l'algèbre de Lie de G. On peut aussi partir de la propriété de Folner et la discrétiser.

Théorème 9.2. *Supposons que $\mathcal{L}_1$, $\mathcal{L}_2$ vérifient $(DV)_0$ et $(P)_0$ et qu'ils sont isométriques à l'infini. Alors $\mathcal{L}_1$ et $\mathcal{L}_2$ vérifient simultanément (DV), (P_σ) ou l'inégalité de Harnack parabolique.*

Pour terminer, décrivons quelques situations où l'on peut appliquer la Proposition 9.1 et le Théorème 9.2. Sur $\mathbb{R}^n$, nous dirons qu'un opérateur $\mathcal{L}$ comme ci-dessus est uniformément sous-elliptique s'il existe C_1 tel que

$$|\nabla_{\mathcal{L}}\psi| \leq C_1\, |\nabla\psi|\,, \qquad \text{pour tout } \psi \in C_0^\infty(\mathbb{R}^n)\,, \tag{9.1}$$

et s'il existe $0 < \alpha \leq 1$ et C_2 tels que

$$\|\Delta^{\alpha/2}\psi\|_2 \leq C_2\,(\|\mathcal{L}^{1/2}\psi\|_2 + \|\psi\|_2)\,, \quad \text{pour tout } \psi \in C_0^\infty(\mathbb{R}^n)\,, \tag{9.2}$$

où $\nabla\psi = \left(\sum |\partial_i\psi|^2\right)^{1/2}$ et $\Delta = -\sum_{i=1}^n \partial_i^2$ est le laplacien usuel. La première inégalité n'est qu'une façon de dire que les coefficients de $\mathcal{L}$ sont bornés.

Il résulte des travaux de Fefferman, Phong, et Sanchez-Calle [14], [15], que, si $\mathcal{L}$ est uniformément sous-elliptique, il vérifie $(DV)_0$, et que la distance $\rho_{\mathcal{L}} = \rho$ satisfait

$$\sup_{|x-y|\leq 1} \rho(x,y) \leq C \quad \text{et} \quad \sup_{\rho(x,y)\leq 1} |x-y| \leq C$$

où $|x|$ est la norme euclidienne de $x \in \mathbb{R}^n$. On en déduit facilement que Δ et $\mathcal{L}$ sont isométriques à l'infini. De plus, Jerison et Sanchez-Calle montrent dans [19] que $\mathcal{L}$ satisfait $(P)_0$. On peut donc appliquer le Théorème 9.2 qui montre que $\mathcal{L}$ vérifie l'inégalité de Harnack parabolique. On peut en déduire que $\mathcal{L}$ a la propriété de Liouville forte et obtenir des estimations précises du noyau de la chaleur associé à $\mathcal{L}$. On retrouve ainsi certains des résultats de [23].

Maintenant, on peut facilement généraliser ces considérations en remplaçant $\mathbb{R}^n$ et sa structure euclidienne par une variété riemannienne M. La définition d'un opérateur sous-elliptique (par rapport à la structure riemannienne fixée sur M) s'obtient en copiant celle donnée ci-dessus dans le cas de $\mathbb{R}^n$: si ∇ et Δ sont le gradient et le laplacien riemanniens sur M, un opérateur $\mathcal{L}$ comme ci-dessus est uniformément sous-elliptique (par rapport à Δ) s'il vérifie (9.1) et (9.2) pour toute fonction $\psi \in C_0^\infty(M)$.

Dans ce cadre, il nous faut supposer que la structure locale de M est uniformément comparable à celle de $\mathbb{R}^n$ en un sens suffisamment fort pour pouvoir transplanter les résultats de Fefferman, Phong, Sanchez-Calle et Jerison. C'est certainement le cas si on prend pour M un groupe de Lie muni d'une métrique riemannienne invariante à gauche (l'opérateur $\mathcal{L}$ n'est lui pas nécessairement invariant à gauche). On peut dans ce cas appliquer la Proposition 9.1 et le Théorème 9.2. En particulier, les opérateurs uniformément sous-elliptiques sur les groupes de Lie à croissance polynômiale du volume vérifient l'inégalité de Harnack parabolique et ont la propriété de Liouville forte (voir aussi [27]). On peut étudier de même les opérateurs uniformément elliptiques sur les revêtements de variétés compactes.

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