Revista Matemática Iberoamericana
Vol. 13, N.º 1, 1997

Rough maximal functions and rough singular integral operators applied to integrable radial functions

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Abstract. Let Ω be homogeneous of degree 0 in $\mathbb{R}^n$ and integrable on the unit sphere. A rough maximal operator is obtained by inserting a factor Ω in the definition of the ordinary maximal function. Rough singular integral operators are given by principal value kernels $\Omega(y)/|y|^n$, provided that the mean value of Ω vanishes. In an earlier paper, the authors showed that a two-dimensional rough maximal operator is of weak type (1,1) when restricted to radial functions. This result is now extended to arbitrary finite dimension, and to rough singular integrals.

1. Introduction.

Let $\Omega \geq 0$ be an integrable function on the unit sphere S^{n-1} in $\mathbb{R}^n$, and extend it to a function in $\mathbb{R}^n \backslash \{0\}$, homogeneous of degree 0. The rough maximal operator corresponding to Ω is defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} \Omega(y)\, |f(x-y)|\, dy\,, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)\,.$$

This operator is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, as seen by the method of rotations. It is, however, unknown whether it is of weak type (1,1).

Under (weak) additional assumptions on Ω, several authors have proved the weak type; see the authors' paper [S-S] for details. That paper contains a proof that M_Ω is of weak type (1,1) in the plane when restricted to radial functions f, for a general $\Omega \in L^1$. In fact, the same result is proved for $n = 2$ when M_Ω is replaced by the larger operator

$$M_\Omega^* f(x) = \int_{S^{n-1}} \Omega(\omega)\, M_\omega f(x)\, d\omega\,.$$

Here and below, $d\omega$ is the area measure on S^{n-1}. Further, M_ω is the one-dimensional maximal operator in the direction $\omega \in S^{n-1}$, defined by

$$M_\omega f(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x - t\,\omega)|\, dt\,.$$

As we pointed out in [S-S], M_Ω^* cannot be of weak type (1,1) on general functions even when Ω is the constant function 1. In this paper, we shall extend the above to $\mathbb{R}^n$, as follows.

Theorem 1. *The operator M_Ω^* is of weak type $(1,1)$ when restricted to radial functions in $\mathbb{R}^n$, for any nonnegative $\Omega \in L^1(S^{n-1})$ and any n. The same is true for M_Ω.*

Rough singular integral operators can be defined analogously. Now $\Omega \in L^1(S^{n-1})$ must have mean value 0. Let

$$T_\Omega f(x) = \text{p.v.} \int \frac{\Omega(y)}{|y|^n}\, f(x-y)\, dy = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\varepsilon<|y|<R} \frac{\Omega(y)}{|y|^n}\, f(x-y)\, dy\,,$$

whenever the limit exists. The L^p boundedness of such operators (which is easy when Ω is odd) was proved by Calderón and Zygmund [C-Z] assuming $\Omega \in L\log L(S^{n-1})$. There is a nice proof due to J. Duoandikoetxea and J. L. Rubio de Francia [D-RF] when $\Omega \in L^q(S^{n-1})$, $q > 1$. With the same condition on Ω, S. Hofmann [H] proved the weak type (1,1) in the plane. The same was proved for $\Omega \in L\log L(S^{n-1})$ by M. Christ and J. L. Rubio de Francia [Ch-RF]. In an unpublished work, they also extended the result to dimension at most 7. More recently, A. Seeger [Se] has proved it in any dimension, again under the hypothesis $\Omega \in L\log L(S^{n-1})$. We remark that the L^p inequality, $1 < p < \infty$, cannot hold without additional assumptions on Ω, since the Fourier multiplier corresponding to T_Ω need not be bounded (*cf.* [St, Chapter II]). In our result, we have no additional assumption on Ω, but apply the operator only to radial functions.

Theorem 2. *Let $\Omega \in L^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega \, d\omega = 0$. The operator $T_\Omega f$ is well defined for any radial function $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, in the sense that the principal value exists for almost every x. Moreover, when restricted to radial functions, T_Ω is of weak type $(1,1)$ and bounded on L^p, $1 < p < \infty$, and so is the maximal singular integral operator*

$$T^*_\Omega f(x) = \sup_{0<\varepsilon<R<\infty} \left| \int_{\varepsilon<|y|<R} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy \right| .$$

To prove the two-dimensional estimate for M^*_Ω in [S-S, Theorem 3], we applied Theorems 1 and 2 of [S-S]. These two results say that $y^{-1}G(y\,\cdot) * f(x) \in L^{1,\infty}(y\,dx\,dy,\ y>0)$ for any $f \in L^1(\mathbb{R})$ and suitable $G \in L^1(\mathbb{R})$. Our method to prove Theorem 1 in the present paper is similar. Implicit in our proof is a version of Theorem 2 of [S-S], where $\mathbb{R}$ is replaced by S^{n-1}. We point out that a version with $\mathbb{R}$ replaced by $\mathbb{R}^n$ also follows from the arguments below. However, we leave it to the interested reader to state it explicitly.

Theorem 1 of this paper is proved in Section 2. It is then one of the tools used to prove Theorem 2 in Section 3.

Finally, with respect to the notation in this paper, an integral $\int_a^b$ with $a > b$ should be interpreted as 0. Further, C denotes many different positive finite constants.

2. Proof of Theorem 1.

We write $x \in \mathbb{R}^n$ as $x = r\,\theta$ with $r \ge 0$ and $\theta \in S^{n-1}$, and denote as in [S-S] by $A(\omega,\theta) = A(\omega,x) \in [0,\pi)$ the angle between $\omega \in S^{n-1}$ and θ. Also let $s(\omega,\theta) = \max\{\sin A(\omega,\theta), A(\omega,\theta)/2\}$. With $0 \le g \in L^1(t^{n-1}\,dt)$ defined on $\mathbb{R}_+$, we follow [S-S] in defining

$$A_\omega g(x) = \frac{1}{r} \int_{rs(\omega,\theta)}^\infty g(t)\, \frac{t\,dt}{(t^2 - r^2 s(\omega,\theta)^2)^{1/2}} .$$

Consider the operator

$$Pg(x) = \int_{s(\omega,\theta)<\delta} \Omega(\omega)\, A_\omega g(x)\, d\omega ,$$

where $\delta > 0$ will be a small constant. The first part of the proof of [S-S, Theorem 3], which is carried out for each $n \geq 2$, now shows that we need only find an estimate

$$P : L^1([0,\infty),\, t^{n-1}\, dt) \longrightarrow L^{1,\infty}(S^{n-1} \times [0,\infty),\, r^{n-1}\, d\theta\, dr)\,.$$

Notice that

$$A_\omega g(x) \leq \frac{1}{r}\, G(r\, s(\omega,\theta))\,,$$

where

$$G(u) = \int_u^\infty g(t)\, \frac{t^{1/2}\, dt}{(t-u)^{1/2}}\,, \qquad u > 0\,.$$

Essentially as in [S-S, proof of Theorem 3], we majorize G by

$$G \leq C \sum_{\nu=0}^{\infty} 2^{-\nu/2} G_\nu + C\, h\,,$$

where

$$\begin{aligned} G_\nu(u) &= 2^\nu u^{1-n} \int_u^{2^{2^{-\nu}} u} g(t)\, t^{n-1}\, dt \\ &\leq C \sum_{k\in\mathbb{Z}} \int_{2^{-k2^{-\nu}}}^{2^{(2-k)2^{-\nu}}} g(t)\, t^{n-1}\, dt \\ &\qquad \cdot 2^\nu 2^{k2^{-\nu}(n-1)} \chi_{[2^{-k2^{-\nu}},\, 2^{(1-k)2^{-\nu}}]}(u) \end{aligned} \tag{2.1}$$

and

$$h(u) = \int_u^\infty g(t)\, dt\,.$$

This implies

$$\begin{aligned} Pg(x) &\leq C \sum_{\nu=0}^{\infty} 2^{-\nu/2} r^{-1} \int_{s(\omega,\theta)<\delta} \Omega(\omega)\, G_\nu(r\, s(\omega,\theta))\, d\omega \\ &\quad + C\, r^{-1} \int_{s(\omega,\theta)<\delta} \Omega(\omega)\, h(r\, s(\omega,\theta))\, d\omega \\ &= C \sum_{\nu=0}^{\infty} 2^{-\nu/2} P_\nu g(x) + C\, Qg(x)\,, \end{aligned} \tag{2.2}$$

the last equality defining P_ν and Q.

To extend the technique used to control P_ν and Q from [S-S], we need analogues of dyadic cubes in S^{n-1}. First, we divide S^{n-1} into a finite number of disjoint subsets E_s, $s = 1, \dots, s_0$, with piecewise smooth boundaries and of small diameters. In each E_s, we can then introduce coordinates simply by projecting E_s orthogonally onto a hyperplane of $\mathbb{R}^n$ tangent to E_s at some point of E_s. In this hyperplane, *i.e.* in $\mathbb{R}^{n-1}$, we introduce the ordinary hierarchy of dyadic cubes. Thus for each $j \in \mathbb{Z}$, we have a partition of $\mathbb{R}^{n-1}$ into cubes of side 2^{-j}. Some of these cubes have images in E_s under the inverse projection. These images will be denoted $(I_j^i)_i$ and called 2^{-j}-cubes. This is for $j \geq j_0$, some j_0. Suitably adapted near ∂E_s, all these sets will form a hierarchy of partitions of E_s and, hence, of S^{n-1}.

The conditional expectation at level j, $j \geq j_0$, of a function $f \in L^1(S^{n-1})$ is now defined by

$$E_j f(x) = |I_j^i|^{-1} \int_{I_j^i} f \,, \qquad x \in S^{n-1} \,,$$

where I_j^i is that 2^{-j}-cube in S^{n-1} which contains the given point x.

Now consider Q. The desired estimate

$$Q : L^1(t^{n-1}\, dt) \longrightarrow L^{1,\infty}(r^{n-1}\, d\theta\, dr) \,,$$

can be seen as a version of Theorems 1 and 4 of [S-S], where $\mathbb{R}$ and $\mathbb{R}^n$, respectively, are replaced by S^{n-1}. Instead of a convolution, we now have the integral defining Qg in (2.2). However, the proof technique carries over without problems. We can assume that the decreasing function h has the form $h = \sum a_k \chi_{[0,2^{-k-C}]}$. Also, it is enough to consider dyadic values of r (*cf.* the inequality (2.3) below). One can now easily relate Q to the conditional expectation, essentially as in [S-S]. The estimates needed for conditional expectation carry over. This takes care of Q.

To control the operator P, we must also estimate the P_ν. It is enough to prove that each P_ν maps $L^1(t^{n-1}\, dt)$ boundedly into $L^{1,\infty}(r^{n-1}\, d\theta\, dr)$, with a constant that grows only polynomially in ν. This will allow summing in $L^{1,\infty}$. As in the proof of Theorem 2 in [S-S], we let r take only the values $r = 2^{2\nu j}$, $j \in \mathbb{Z}$, and prove that

$$\sum_j 2^{2\nu j n} |\{\theta \in S^{n-1} \; : \; P_\nu g(2^{2\nu j}\theta) > \lambda\}| \leq C\,(1+\nu)^C \,\frac{1}{\lambda}\, \|g\|_{L^{1,\infty}(t^{n-1} dt)} \,. \tag{2.3}$$

Here $|\cdot|$ is the area measure of S^{n-1}. This will complete the proof. To verify (2.3), it is enough, as in [S-S, proof of Theorem 2], to sum in (2.1) only over those k of the form $k = \ell\, 2^{\nu+1}\nu + \kappa$, $\ell \in \mathbb{Z}$, for each $\kappa = 0, \dots, 2^{\nu+1}\nu - 1$. For simplicity, we shall consider only $\kappa = 0$. The level set in (2.3) will thus be replaced by the set of those $\theta \in S^{n-1}$ for which

$$
\begin{aligned}
(2.4) \qquad 2^{-2\nu j} \int_{s(\omega,\theta)<\delta} \Omega(\omega) \Big(\sum_{\ell} \int_{2^{-2\nu\ell}}^{2^{2^{1-\nu}-2\nu\ell}} & g(t)\, t^{n-1}\, dt \Big) \\
& \cdot 2^{\nu} 2^{2(n-1)\nu\ell} \chi_{R_{\ell+j}(\theta)}(\omega)\, d\omega > \lambda\,,
\end{aligned}
$$

where $R_m(\theta)$ is the ring

$$
R_m(\theta) = \{\omega \in S^{n-1} \,:\, 2^{-2\nu m} \le s(\omega,\theta) \le 2^{2^{-\nu}-2\nu m}\}\,.
$$

Because of the condition $s(\omega,\theta) < \delta$ in the integral in (2.4), we need only consider $m \ge m_0$ here, for some $m_0 > 0$. This means that the sum in (2.4) is taken over $\ell \ge m_0 - j$. Notice that the radius and the width of $R_m(\theta)$ are approximately $2^{-2\nu m}$ and $2^{-\nu-2\nu m}$, respectively.

Next, we let the point θ move within a $2^{-\nu(1+2m)}$-cube $I^i_{\nu(1+2m)}$ and form

$$
R^i_m = \bigcup_{\theta \in I^i_{\nu(1+2m)}} R_m(\theta)\,.
$$

This set is contained in a ring of width at most $C\,2^{-\nu(1+2m)}$. Clearly, R^i_m is covered by those $2^{-\nu(1+2m)}$-cubes intersecting it. Their number is at most $C\,2^{(n-2)\nu}$. Among these $2^{-\nu(1+2m)}$-cubes, we discard those which are not in the same E_s as $I^i_{\nu(1+2m)}$. Then we enumerate the remaining ones as $I^{\lambda(i,q)}_{\nu(1+2m)}$, $q = 1, \dots, q_0 = \mathrm{O}(2^{(n-2)\nu})$, in a coherent way as i varies. By this we mean that the direction from the midpoint of $I^i_{\nu(1+2m)}$ (which is the approximate centre of the ring-like set R^i_m) to the midpoint of $I^{\lambda(i,q)}_{\nu(1+2m)}$ should not vary too much with i, for a fixed q. It is enough if two such directions never form an angle greater than $\pi/4$, say, measured in the coordinate system of each E_s.

In (2.4), we shall now replace $R_{\ell+j}(\theta)$ by $I^{\lambda(i,q)}_{\nu(1+2\ell+2j)}$ when $\theta \in I^i_{\nu(1+2\ell+2j)}$, for a fixed q. More precisely, this means that the level set in (2.3) is replaced by the union of those $I^i_{\nu(1+2\ell+2j)}$ for which

$$2^{-2\nu j}\int \Omega(\omega)\Big(\sum_{\ell\geq m_0-j}\int_{2^{-2\nu\ell}}^{2^{2^{1-\nu}-2\nu\ell}} g(t)\,t^{n-1}\,dt\Big) \cdot 2^{\nu}2^{2(n-1)\nu\ell}\chi_{I^{\lambda(i,q)}_{\nu(1+2\ell+2j)}}(\omega)\,d\omega>\lambda\,.$$

This version of (2.3), call it (2.3'), implies the theorem, since we can sum in q by means of the adding-up lemma in $L^{1,\infty}$ as in [S-S].

The mean value of Ω in $I^{\lambda(i,q)}_{\nu(1+2\ell+2j)}$ can be seen as an S^{n-1} version of the translated conditional expectation from the proof of Theorem 2 of [S-S]. In fact, the arguments used in that proof now carry over and prove (2.3'). We leave the details to the reader. This ends the proof of Theorem 1.

3. Proof of Theorem 2.

We start with the L^1 case. Let

$$T^{\varepsilon,R}_{\Omega}f(x)=\int_{\varepsilon<|y|<R}\frac{\Omega(y)}{|y|^n}\,f(x-y)\,dy\,.$$

Notice that all the conclusions follow from the weak type estimate for the maximal operator T^*_Ω. Also, in the definition of $T^*_\Omega f(x)$, we need only take $R\geq 10\,|x|=10\,\rho$. This is because in the case $R<10\,\rho$, one has

$$\int_{\varepsilon<|y|<R}\frac{\Omega(y)}{|y|}\,f(x-y)\,dy=T^{\varepsilon,10\rho}_{\Omega}f(x)-T^{R,10\rho}_{\Omega}f(x)\,.$$

Together with $T^{\varepsilon,R}_{\Omega}$, we consider

$$\tilde{T}^{\varepsilon,R}_{\Omega}f(x)=\int_{\substack{|\,|y-x|-|x|\,|>\varepsilon\\ |y|<R}}\frac{\Omega(y)}{|y|^n}\,f(x-y)\,dy\,. \tag{3.1}$$

We shall estimate the difference between these two operators.

The notation $x=\rho\,\theta$, $y=r\,\omega$, $A=A(\theta,\omega)$ will be as in Section 2. A radial function $f\in L^1$ will be written $f(x)=g(|x|)$, with $g\in L^1(\mathbb{R}_+;\rho^{n-1}\,d\rho)$. The distance $t=|x-y|$ satisfies

$$t^2=\rho^2+r^2-2\,\rho\,r\cos A\,. \tag{3.2}$$

Hence,

$$r = \rho \cos A \pm \sqrt{t^2 - \rho^2 \sin^2 A}\,. \tag{3.3}$$

Proposition 3. *The operator*

$$\tilde{T}^*_\Omega f(x) = \sup_{\substack{\varepsilon>0 \\ R>10|x|}} |\tilde{T}^{\varepsilon,R}_\Omega f(x)|\,,$$

is of weak type $(1,1)$ *when restricted to radial functions.*

Proposition 4. *The operator*

$$D^*_\Omega f(x) = \sup_{\substack{\varepsilon>0 \\ R>10|x|}} |T^{\varepsilon,R}_\Omega f(x) - \tilde{T}^{\varepsilon,R}_\Omega f(x)|\,,$$

is of weak type $(1,1)$ *when restricted to radial functions.*

It is clear that the L^1 part of Theorem 2 follows from these two results.

PROOF OF PROPOSITION 3. In the integral defining $\tilde{T}^{\varepsilon,R}_\Omega f(x)$, we pass to polar coordinates, getting

$$\tilde{T}^{\varepsilon,R}_\Omega f(x) = \int_{S^{n-1}} \Omega(\omega)\, d\omega \int_{\substack{|\,|x-r\omega|-\rho\,|>\varepsilon \\ 0<r<R}} \frac{g(|x - r\,\omega|)}{r}\, dr\,.$$

Next, we shall transform the inner integral here, using $t = |x - r\,\omega|$ as a new variable of integration. One has $dr = t\,dt/(r - \rho\cos A)$. The correspondence between r and t is not quite one-to-one, and the sign in (3.3) must be chosen correctly. As seen geometrically, one obtains a sum of four integrals. Indeed,

$$\tilde{T}^{\varepsilon,R}_\Omega f(x) = \int_{A>\pi/2} \Omega(\omega)\, d\omega \int_{\rho+\varepsilon}^{R_1(\rho)} \frac{g(t)}{\rho\cos A + \sqrt{t^2 - \rho^2\sin^2 A}} \cdot \frac{t\,dt}{\sqrt{t^2 - \rho^2 \sin^2 A}}$$

$$
\begin{aligned}
&+\int_{A<\pi/2}\Omega(\omega)\,d\omega\int_{\rho+\varepsilon}^{R_2(\rho)}\frac{g(t)}{\rho\cos A+\sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad\qquad\cdot\frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&+\int_{A<\pi/2}\Omega(\omega)\,d\omega\int_{\rho\sin A}^{\rho-\varepsilon}\frac{g(t)}{\rho\cos A-\sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad\qquad\cdot\frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&+\int_{A<\pi/2}\Omega(\omega)\,d\omega\int_{\rho\sin A}^{\rho-\varepsilon}\frac{g(t)}{\rho\cos A+\sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad\qquad\cdot\frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&=I_1+I_2+I_3+I_4\,.
\end{aligned}
$$

Here $R_j(\rho)\in[R-\rho,R+\rho]$ for $j=1,2$.

The integrand is the same in I_1 and I_2, and one finds

$$
\begin{aligned}
I_1+I_2=\int_{S^{n-1}}\Omega(\omega)\,d\omega\int_{\rho+\varepsilon}^{R} g(t)\,\frac{\sqrt{t^2-\rho^2\sin^2 A}-\rho\cos A}{t^2-\rho^2}& \\
\cdot\frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}}+E\,.&
\end{aligned}
$$

Here the error E is due to the fact that $R_j(\rho)$ need not equal R, $j=1,2$. It follows that

$$
|E|\le\int_{R-\rho\le|y|\le R+\rho}\frac{|\Omega(y)|}{|y|^n}\,|f(x-y)|\,dy\le C\,M_\Omega f(x)\,,
$$

and Theorem 1 gives the weak type estimate for $\sup_{\varepsilon,R}|E|$. Thus we have

$$
\begin{aligned}
I_1+I_2&=\int_{\rho+\varepsilon}^{R} g(t)\,\frac{t}{\rho^2-t^2}\,dt\int_{S^{n-1}}\Omega(\omega)\,\frac{\rho\cos A}{\sqrt{t^2-\rho^2\sin^2 A}}\,d\omega+E \\
&=J_1+E\,,
\end{aligned}
$$

where we used the equality $\int\Omega(\omega)\,d\omega=0$. Moreover,

$$
\begin{aligned}
I_3+I_4&=\int_0^{\rho-\varepsilon} g(t)\,\frac{t}{\rho^2-t^2}\,dt\int_{\substack{A<\pi/2\\ \sin A<t/\rho}} 2\,\Omega(\omega)\,\frac{\rho\cos A}{\sqrt{t^2-\rho^2\sin^2 A}}\,d\omega \\
&=J_2\,.
\end{aligned}
$$

That part of J_1 obtained by integrating only over $t > 2\,\rho$ is easy to control, since its absolute value is at most

$$C \int_{2\rho}^{R} |g(t)|\, \frac{1}{t}\, \frac{\rho}{t}\, dt\, \|\Omega\|_1 \,.$$

It is then enough to observe that

$$\int_0^\infty \rho^{n-1}\, d\rho \int_{2\rho}^\infty |g(t)|\, \frac{\rho}{t^2}\, dt \le C \int_0^\infty |g(t)|\, t^{n-1}\, dt \,.$$

This takes care of the supremum in R.

That part of J_2 which corresponds to $t < \rho/2$ can also be easily handled. Indeed, it equals what one gets by restricting the integral defining $\tilde{T}_\Omega^\varepsilon f(x)$ to the region $|y - x| < \min\{\rho/2, \rho - \varepsilon\}$. Since $|y| \sim |x|$ in this region, we can dominate by $M_\Omega f(x)$ and apply Theorem 1 to get the desired weak type estimate.

The remaining integrals are thus

$$J_1' = \int_{\rho+\varepsilon}^{2\rho} g(t)\, \frac{t}{\rho^2 - t^2}\, dt \int_{S^{n-1}} \Omega(\omega)\, \frac{\rho \cos A}{\sqrt{t^2 - \rho^2 \sin^2 A}}\, d\omega$$

and

$$J_2' = \int_{\rho/2}^{\rho-\varepsilon} g(t)\, \frac{t}{\rho^2 - t^2}\, dt \int_{\substack{A<\pi/2 \\ \sin A < t/\rho}} 2\,\Omega(\omega)\, \frac{\rho \cos A}{\sqrt{t^2 - \rho^2 \sin^2 A}}\, d\omega \,.$$

Notice that the value for $t = \rho$ of the inner integral in J_1' is

$$a(\theta) = \int_{S^{n-1}} \Omega(\omega)\, \operatorname{sgn} \cos A\, d\omega \,.$$

The corresponding quantity for J_2' is

$$\int_{A<\pi/2} 2\,\Omega(\omega)\, d\omega = a(\theta)\,,$$

because of the vanishing mean value of Ω. Clearly a is a continuous function on S^{n-1}.

If we replace the inner integrals of J_1' and J_2' by $a(\theta)$, the resulting expressions will add up to

$$a(\theta)\int_{[\rho/2,2\rho]\setminus[\rho-\varepsilon,\rho+\varepsilon]} g(t)\,\frac{t}{\rho^2-t^2}\,dt\,.$$

This integral is a truncation of a smooth principal value singular integral on $\mathbb{R}_+$. By standard methods, it can be shown to define a weak type (1,1) operator for the measure $t^{n-1}\,dt$. So does the corresponding maximal singular integral, defined as the supremum in ε of the integral.

Since a is a bounded function, we also get a bounded operator from $L^1(t^{n-1}\,dt)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Thus, to prove Proposition 3, it only remains to estimate the difference operators arising when we subtract $a(\theta)$ from the inner integrals in J_1' and J_2'. For these operators, we shall actually derive strong type estimates.

For the case of J_1', we write

$$\begin{aligned}\Big|\frac{\rho\cos A}{\sqrt{t^2-\rho^2\sin^2 A}}-\operatorname{sgn}\cos A\Big| &= \Big|\rho\cos A\Big(\frac{1}{\sqrt{t^2-\rho^2\sin^2 A}}-\frac{1}{\rho\,|\cos A|}\Big)\Big|\\ &\le \rho\,|\cos A|\,\frac{\sqrt{t^2-\rho^2\sin^2 A}-\rho\,|\cos A|}{\rho\,|\cos A|\sqrt{t^2-\rho^2\sin^2 A}}\\ &\le \frac{t^2-\rho^2}{t^2-\rho^2\sin^2 A}\,,\end{aligned}$$

where we multiplied and divided by the conjugate quantity of the numerator, to get the last inequality. Our difference operator is thus controlled by

$$V_1 g(\rho,\theta)=\int_\rho^{2\rho}|g(t)|\,t\,dt\int_{S^{n-1}}|\Omega(\omega)|\,\frac{d\omega}{t^2-\rho^2\sin^2 A}\,.$$

One finds

$$\begin{aligned}&\int_{S^{n-1}} V_1 g(\rho,\theta)\,d\theta\\ &\qquad\le \int_\rho^{2\rho}|g(t)|\,t\,dt\int_{S^{n-1}}|\Omega(\omega)|\,d\omega\int_{S^{n-1}}\frac{d\theta}{t^2-\rho^2\sin^2 A}\,.\end{aligned}$$

Writing $s = t/\rho \in (1,2)$, we see that the innermost integral here is

$$\begin{aligned} C\rho^{-2} \int_0^{\pi/2} \frac{\sin^{n-2}\alpha\, d\alpha}{s^2 - \sin^2\alpha} &= C\rho^{-2} \int_0^1 \frac{u^{n-2}\, du}{\sqrt{1-u^2}\,(s^2-u^2)} \\ &\le C\rho^{-2} \int_0^1 \frac{du}{\sqrt{1-u}\,(s-u)} \\ &= C\rho^{-2} \int_0^1 \frac{dv}{\sqrt{v}\,(s-1+v)} \\ &= C\rho^{-2} \Big(\int_0^{s-1} + \int_{s-1}^1 \Big) \\ &\le \frac{C\rho^{-2}}{\sqrt{s-1}} \\ &= \frac{C\rho^{-3/2}}{\sqrt{t-\rho}} \; . \end{aligned}$$

This implies

$$\begin{aligned} &\int_0^\infty \rho^{n-1}\, d\rho \int_{S^{n-1}} V_1 g(\rho,\theta)\, d\theta \\ &\qquad \le C \int_0^\infty |g(t)|\, t\, dt \int_{t/2}^t \rho^{n-1-3/2} \frac{d\rho}{\sqrt{t-\rho}} \, \|\Omega\|_1 \\ &\qquad = C \int_0^\infty |g(t)|\, t^{n-1}\, dt \, \|\Omega\|_1 \; . \end{aligned}$$

Since $V_1 g$ does not depend on ε, this is the desired strong type (1,1) estimate.

To deal with the difference operator coming from J_2', we observe that, almost as in the case of J_1',

$$\begin{aligned} &\Big| \int_{\substack{A<\pi/2 \\ \sin A < t/\rho}} 2\,\Omega(\omega)\, \frac{\rho\cos A}{\sqrt{t^2 - \rho^2 \sin^2 A}}\, d\omega - \int_{A<\pi/2} 2\,\Omega(\omega)\, d\omega \Big| \\ &\qquad \le 2 \int_{\substack{A<\pi/2 \\ \sin A < t/\rho}} |\Omega(\omega)|\, \frac{\rho^2 - t^2}{\rho\cos A \sqrt{t^2 - \rho^2 \sin^2 A}}\, d\omega \\ &\qquad\quad + 2 \int_{\substack{A<\pi/2 \\ \sin A > t/\rho}} |\Omega(\omega)|\, d\omega \\ &\qquad = K_1 + K_2 \; . \end{aligned}$$

With $s = t/\rho \in (1/2, 1)$, we now get

$$\begin{aligned}&\int_{S^{n-1}} d\theta \int_{\rho/2}^{\rho-\varepsilon} |g(t)| \frac{t}{\rho^2 - t^2}\, dt\, K_1 \\ &\leq 2 \int_{\rho/2}^{\rho} |g(t)|\, t\, dt\, \rho^{-2} \int_{S^{n-1}} |\Omega(\omega)|\, d\omega \int_{\substack{A<\pi/2 \\ \sin A < s}} \frac{d\theta}{\cos A \sqrt{s^2 - \sin^2 A}}\ .\end{aligned}$$

Here the innermost integral is

$$\begin{aligned}C \int_0^{\arcsin s} \frac{\sin^{n-2}\alpha\, d\alpha}{\cos\alpha\, \sqrt{s^2 - \sin^2\alpha}} &= C \int_0^s \frac{u^{n-2}\, du}{(1-u^2)\sqrt{s^2 - u^2}} \\ &\leq C \int_0^s \frac{du}{(1-u)\sqrt{s-u}} \\ &= C \int_0^s \frac{du}{(1-s+u)\sqrt{u}} \\ &\leq \frac{C}{\sqrt{1-s}}\ .\end{aligned}$$

This implies

$$\begin{aligned}&\int_0^\infty \rho^{n-1}\, d\rho \int_{S^{n-1}} d\theta \int_{\rho/2}^{\rho} |g(t)| \frac{t}{\rho^2 - t^2}\, dt\, K_1 \\ &\qquad \leq C \int_0^\infty |g(t)|\, t\, dt \int_t^{2t} \rho^{n-1-2+1/2} \frac{d\rho}{\sqrt{\rho - t}}\, \|\Omega\|_1 \\ &\qquad \leq C \int_0^\infty |g(t)|\, t^{n-1}\, dt\, \|\Omega\|_1\ .\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_{S^{n-1}} d\theta \int_{\rho/2}^{\rho-\varepsilon} |g(t)| \frac{t}{\rho^2 - t^2}\, dt\, K_2 \\ &\qquad \leq 2 \int_{\rho/2}^{\rho} |g(t)| \frac{t}{\rho^2 - t^2}\, dt \int_{S^{n-1}} |\Omega(\omega)|\, d\omega \int_{\substack{A<\pi/2 \\ \sin A > s}} d\theta\, .\end{aligned}$$

Here the innermost integral is found to be $\mathrm{O}(\sqrt{1-s})$. Integrating the above against $\rho^{n-1}\, d\rho$, we get at most

$$C \int |g(t)|\, t^{n-1}\, dt\, \|\Omega\|_1\ ,$$

as before. This strong type estimate ends the proof of Proposition 3.

PROOF OF PROPOSITION 4. Observe that $T_\Omega^{\varepsilon,R}f(x) - \tilde{T}_\Omega^{\varepsilon,R}f(x)$ is independent of R. One has

$$D_\Omega^* f(x) \le \sup_{\varepsilon>0} \int_{\substack{|y|>\varepsilon \\ |\,|x-y|-|x|\,|<\varepsilon}} \frac{|\Omega(y)|}{|y|^n}\, |f(x-y)|\, dy\,. \tag{3.4}$$

We assume that $f, g \ge 0$. Notice that $r = \varepsilon$ is equivalent to $t = t_\varepsilon$, where

$$t_\varepsilon^2 = \rho^2 + \varepsilon^2 - 2\,\rho\,\varepsilon\cos A\,. \tag{3.5}$$

One can assume that $\varepsilon < \rho/2$, since otherwise the integral in (3.4) is taken over a region where $\varepsilon < |y| < C\varepsilon$. Then the rough maximal operator of Theorem 1 applies.

As in the preceding proof, we write the integral in (3.4) in polar coordinates and replace the integration in r by integration in t. Again, we divide the resulting integral into four parts, though not quite in the same way as before. For the supremum of each part, we shall derive a strong or weak type (1,1) estimate.

Part 1: $A > \pi/2$. Then $\cos A < 0$, and $t > \rho$. This part of the integral in (3.4) is dominated in absolute value by

$$\begin{aligned}
&\int_{A>\pi/2} |\Omega(\omega)|\, d\omega \int_{t_\varepsilon}^{\rho+\varepsilon} \frac{g(t)}{\rho\cos A + \sqrt{t^2-\rho^2\sin^2 A}}\, \frac{t\, dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad = \int_{A>\pi/2} |\Omega(\omega)|\, d\omega \int_{t_\varepsilon}^{\rho+\varepsilon} \frac{\rho\,|\cos A| + \sqrt{t^2-\rho^2\sin^2 A}}{t^2-\rho^2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \frac{g(t)\, t\, dt}{\sqrt{t^2-\rho^2\sin^2 A}} \\
&\qquad \le 2\int_{A>\pi/2} |\Omega(\omega)|\, d\omega \int_{t_\varepsilon}^{\rho+\varepsilon} \frac{g(t)\, t\, dt}{t^2-\rho^2}\,,
\end{aligned} \tag{3.6}$$

since here $\rho\,|\cos A| \le \sqrt{t^2-\rho^2\sin^2 A}$.

The last inner integral is no larger than

$$\int_{t_\varepsilon}^{\rho+\varepsilon} g(t)\, \frac{dt}{t-\rho} \le \int_{\rho}^{\rho+\varepsilon} g(t) \min\left\{\frac{1}{t-\rho}, \frac{1}{t_\varepsilon-\rho}\right\} dt\,.$$

Since the minimum here is decreasing in t for $\rho < t < \rho + \varepsilon$, it is well known that the right hand integral is dominated by the maximal function of g at ρ times

$$\int_\rho^{\rho+\varepsilon} \min\Big\{\frac{1}{t-\rho}, \frac{1}{t_\varepsilon - \rho}\Big\}\, dt = 1 + \log \frac{\varepsilon}{t_\varepsilon - \rho}\,.$$

Instead of the ordinary maximal function $Mg(\rho)$, we can here use

$$M_\ell g(\rho) = M(g\chi_{[\rho/2, 2\rho]})(\rho)\,,$$

since $\varepsilon < \rho/2$. Because of (3.5), we have

$$\log \frac{\varepsilon}{t_\varepsilon - \rho} = \log \frac{\varepsilon\,(t_\varepsilon + \rho)}{\varepsilon^2 + 2\,\rho\,\varepsilon\,|\cos A|} \le \log \frac{1}{|\cos A|}\,.$$

Altogether, the expressions in (3.6) are majorized by

$$2\, M_\ell g(\rho) \int_{S^{n-1}} |\Omega(\omega)| \Big(1 + \log \frac{1}{|\cos A|}\Big)\, d\omega\,.$$

Here the first factor is in $L^{1,\infty}(\rho^{n-1}\, d\rho)$ and the second in $L^1(S^{n-1})$ as a function of θ, as shown via Fubini's theorem. A product of this type belongs to $L^{1,\infty}(\rho^{n-1}\, d\rho\, d\theta)$. Since the product is independent of ε, this ends Part 1.

Part 2: $A < \pi/2$ and $r < (\rho \cos A)/2$. Since

$$r = \rho\cos A - \sqrt{t^2 - \rho^2 \sin^2 A}\,,$$

this implies

$$\sqrt{t^2 - \rho^2\sin^2 A} > \frac{1}{2}\,\rho\cos A\,. \tag{3.7}$$

We can assume that

$$\frac{1}{2}\,\rho\cos A > \varepsilon\,, \tag{3.8}$$

because otherwise we get nothing. The part of the integral in (3.4) we get is

$$\int_{A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho-\varepsilon}^{t_\varepsilon} \frac{g(t)}{\rho\cos A - \sqrt{t^2-\rho^2\sin^2 A}} \frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}}$$
$$= \int_{A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho-\varepsilon}^{t_\varepsilon} \frac{\rho\cos A + \sqrt{t^2-\rho^2\sin^2 A}}{\rho^2-t^2} \frac{g(t)\,t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}}$$
$$\leq C\int_{A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho-\varepsilon}^{t_\varepsilon} \frac{g(t)\,t\,dt}{\rho^2-t^2}\ ,$$

the last step because of (3.7).

We proceed as in Part 1. The logarithm to be estimated is now

$$\log\frac{\varepsilon}{\rho-t_\varepsilon} = \log\frac{\varepsilon\,(\rho+t_\varepsilon)}{2\,\rho\,\varepsilon\cos A-\varepsilon^2}\ .$$

By means of (3.8), we get rid of the ε^2 term in the denominator, and the logarithm is seen to be dominated by $\log(1/\cos A)$. The rest is like Part 1.

Part 3: $A<\pi/2$ and $(\rho\cos A)/2<r<2\,\rho\cos A$. This part of the integral in (3.4) is dominated by the rough maximal function $M^*_\Omega f(x)$. We apply Theorem 1.

Part 4: $A<\pi/2$ and $r>2\,\rho\cos A$. Notice that this inequality for r is equivalent to $t>\rho$. We can assume that $A>\pi/4$, because otherwise $\rho/C\leq r\leq C\rho$ for some C, and M^*_Ω will apply.

The integral we now get is

$$\int_{\pi/4<A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho}^{\rho+\varepsilon} \frac{g(t)}{\rho\cos A + \sqrt{t^2-\rho^2\sin^2 A}} \frac{t\,dt}{\sqrt{t^2-\rho^2\sin^2 A}}$$
$$\leq \int_{\pi/4<A<\pi/2} |\Omega(\omega)|\, d\omega \int_{\rho}^{2\rho} \frac{g(t)\,t\,dt}{t^2-\rho^2\sin^2 A}\ .$$

Notice that the last expression does not contain ε. Its integral with respect to $dx = C\rho^{n-1}\,d\rho\,d\theta$ is

$$C\int_0^\infty \rho^{n-1}\,d\rho\int_{S^{n-1}} d\theta\int_{\pi/4<A<\pi/2} |\Omega(\omega)|\,d\omega\int_\rho^{2\rho}\frac{g(t)\,t\,dt}{t^2-\rho^2\sin^2 A}$$
$$\le C\int_0^\infty g(t)\,t\,dt\int_{S^{n-1}}|\Omega(\omega)|\,d\omega\int_{t/2}^t \rho^{n-1}\,d\rho \cdot\int_{\pi/4<A<\pi/2}\frac{d\theta}{t^2-\rho^2+\rho^2\cos^2 A}\,.$$

The innermost integral here is

$$C\int_{\pi/4}^{\pi/2}\frac{\sin^{n-2}\alpha\,d\alpha}{t^2-\rho^2+\rho^2\cos^2\alpha}\le C\int_0^{1/\sqrt{2}}\frac{du}{t^2-\rho^2+\rho^2u^2}\le\frac{C}{\rho\sqrt{t^2-\rho^2}}\,.$$

It follows that the fourfold integral is no larger than

$$C\int g(t)\,t^{n-1}\,dt\,\|\Omega\|_1\,.$$

This ends Part 4 and the proof of Proposition 4.

For the L^p part of Theorem 2, it is clearly enough to prove versions of Propositions 3 and 4 with strong type (p,p) instead of weak type $(1,1)$. This requires only small modifications of the proofs just given. For instance, in the proof of Proposition 3 one obtains several strong type $(1,1)$ inequalities by integrating various expressions with respect to $\rho^{n-1}\,d\rho\,d\theta$. For the L^p inequality, one can instead estimate these expressions by quantities like

$$C\,M_\ell g(\rho)\int_{S^{n-1}}|\Omega(\omega)|\Big(1+\log\frac{1}{|\cos A|}\Big)\,d\omega\,,$$

which is in $L^p(\rho^{n-1}\,d\rho\,d\theta)$ if $g\in L^p(\rho^{n-1}\,d\rho\,d\theta)$. We leave the details of the rest of the L^p case to the reader.

This ends the proof of Theorem 2.

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Recibido: 21 de junio de 1.995
Revisado: 6 de noviembre de 1.995

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Both authors were partially supported by the Spanish D.G.I.C.Y.T., the first one during a sabbatical leave. They were also supported under the E.U. contract ERBCHRXCT 930083.

Revista Matemática Iberoamericana
Vol. 13, N.° 1, 1997

Elliptic gaussian random processes

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Abstract. We study the Gaussian random fields indexed by $\mathbb{R}^d$ whose covariance is defined in all generality as the parametrix of an elliptic pseudo-differential operator with minimal regularity asumption on the symbol. We construct new wavelet bases adapted to these operators; the decomposition of the field on this corresponding basis yields its iterated logarithm law and its uniform modulus of continuity. We also characterize the local scalings of the field in term of the properties of the principal symbol of the pseudodifferential operator. Similar results are obtained for the *Multi-Fractional Brownian Motion.*

Résumé. Nous étudions les processus aléatoires gaussiens X indexés par $\mathbb{R}^d$ tels qu'il existe un opérateur pseudo-différentiel A d'ordre donné admettant pour parametrix la covariance de X.

Nous construisons une base d'ondelettes adaptée à l'opérateur A. La décomposition du processus X sur cette base conduit à la loi du module de continuité uniforme et à la loi du logarithme itéré. Nous caractérisons aussi les propriétés d'échelle locales du processus au moyen de la partie principale du symbole. Nous étendons ces résultats au *Mouvement Brownien Multi-Fractionnaire* qui est associé à un opérateur d'ordre variable.

1. Introduction and statement of results.

1.1. Introduction.

Let $X(x)$ be a (centered real valued) Gaussian Random Process defined on $\mathbb{R}^d$, of covariance $R(x,y) = \mathbb{E}(X(x)\,X(y))$. Two isomorphic Hilbert spaces are associated with X: the space $\mathcal{H}$ defined by the closure of the random variables $Z = \sum \alpha_i\, X(t_i)$ for the scalar product $(Z \mid T) = \mathbb{E}(Z\,T)$ and the *Reproducing Kernel Hilbert Space* (R.K.H.S.) H composed of the functions which can be written

$$f_Z(t) = \mathbb{E}(X(t)\,Z)\,, \tag{1}$$

with $Z \in \mathcal{H}$; the scalar product in H is

$$(f_Z, f_Y)_H = \mathbb{E}(Z\,Y)\,.$$

By Riesz's representation theorem, we can define a self-adjoint positive operator $A : H \to H'$, the dual of H by

$$(f,g)_H = \langle A(f) \mid g \rangle_{(H',H)}\,, \tag{2}$$

where $\langle \cdot \mid \cdot \rangle_{(H',H)}$ means the (H',H) duality.

Of particular significance is the case where the norm in H is equivalent with the norm of one of the Sobolev spaces H^s or of the homogeneous spaces $\dot{H}^s$ (in this last case H is defined by additional conditions, for instance by vanishing conditions at the origin). We will call *Elliptic Gaussian Random Processes* the processes such that

$$C_1 \|f\|^2_{\dot{H}^s} \leq (A(f) \mid f)_{L^2} \leq C_2 \|f\|^2_{\dot{H}^s}\,,$$

which is an ellipticity asumption on the operator A (we borrow this terminology from Guyon [17] where it covers a similar idea). These norm estimates imply that the operator is everywhere of order $2s$. We will show later that the techniques we introduce allow also to study the Multifractional Brownian Motion, a case where the order of the operator is a function of x.

We specify the setting by requiring A to be a pseudodifferential operator, and we will make some limited regularity asumptions on its symbol $\sigma(x,\xi)$. Since theoretically all the information on X is contained in the operator A, we want to investigate in details the correspondence between the properties of X and A. Some points are classical; for instance X has the Markov field property if and only if A is differential; X has stationary increments if and only if the symbol $\sigma(x,\xi)$ does not depend on x (the norm in H is then shift-invariant).

In this work, we will mainly study two properties of the process:

1) local self-similarity,

2) regularity of the sample paths, looking for exact constants in the laws of local and uniform moduli of continuity.

Let us recall that a process X is said to be selfsimilar of order α at the origin if, for all $\rho > 0$,

$$\text{Law}\,\{\rho^{-\alpha}X(\rho\,x),\, x\in\mathbb{R}^d\} = \text{Law}\,\{X(x),\, x\in\mathbb{R}^d\}\,. \tag{3}$$

For instance, the *Fractional Brownian Motion of order* α is selfsimilar (of order α) at the origin. Dobrushin in [13] gives a complete characterization of selfsimilar gaussian fields with stationary increments; and it follows from [13] that the exact scaling law (3) can hold only for very specific processes. The renormalisation operators $R^{\alpha}_{x_0,r}$ are defined by

$$R^{\alpha}_{x_0,\rho}X(x) = \frac{1}{\rho^{\alpha}}\left(X(x_0+\rho\,x) - X(x_0)\right)$$

and, by definition, a process X is *locally asymptotically self-similar* (L.A.S.S.) of order $\alpha\in(0,1)$ at x_0 if $R^{\alpha}_{x_0,\rho}X$ has a non trivial limit in law when $\rho\to\infty$.

The case $\alpha = 1$ requires a different renormalisation formula, see (39), and the corresponding processes will be called *locally asymptotically critical processes* (L.A.C.).

Regularity properties for Gaussian processes have been considered in full generality in [16]. In the general case the uniform and local moduli of continuity are known only up to a multiplicative constant. For a large class of stationary increments processes Kono in [26] and Marcus in [29] obtain the exact constants in laws of the moduli, but these cases do not include in the elliptic setting the critical order $s - d/2\in\mathbb{N}$. One of our purposes is to solve completely this problem (Theorem 1.3) in the general elliptic pseudodifferential setting.

One of the main ideas behind the results we will describe is that the local properties of an elliptic gaussian process are contained in the principal part of the symbol associated with the operator A. Let us illustrate this idea on a very simple example.

Let W be the one dimensional Brownian motion issued from 0. The corresponding operator A_0 is the second derivative, whose symbol $\sigma_0(x,\xi)$ is $|\xi|^2$. The fact that σ_0 is an homogeneous function of ξ independent of x implies that W is selfsimilar of degree $1/2$. The Orstein-Uhlenbeck process V is the solution of the stochastic differential equation $dV = -q\, V\, dt + dW$. One easily checks that the R.K.H.S. of this process is exactly the Sobolev space H^1 so that the corresponding operator is $A = q^2\mathrm{Id} - \partial^2/\partial x^2$ of symbol $\sigma(x,\xi) = |\xi|^2 + q^2$. Since the modification of the symbol bears on low order terms only, for every bounded open subset U of $\mathbb{R}$, $\mathrm{Law}\,(V|_U) \equiv \mathrm{Law}\,(W|_U)$ (the two processes restricted to U are locally undistinguishable on one realization), see [32]. This has two more consequences:

1) The Orstein-Uhlenbeck process V will satisfy the following local scaling property at any point t

$$\lim_{\rho\to 0^+} \mathrm{Law}\Big\{\frac{V(t+\rho\, u) - V(t)}{\rho^{1/2}}\,,\ u \in \mathbb{R}\Big\} = \mathrm{Law}\, W\,,$$

and we observe that the symbol of the "asymptotic process" is the "principal part" of σ.

2) The uniform modulus of continuity and the iterated logarithm law (the local modulus of continuity) of V and W are the same.

1.2. The Model.

In this paper we consider triples (A, H_A, X_A) constituted by

• An elliptic symetric positive pseudodifferential operator A derived from a symbol $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by the usual formula

$$(Af)(x) = \frac{1}{(2\pi)^{d/2}} \int e^{ix\xi}\sigma(x,\xi)\, \hat{f}(\xi)\, d\xi\,, \tag{4}$$

where $\hat{f}$ is the Fourier transform of f. We will use the notation $A = \mathrm{O}p(\sigma)$.

• A Hilbert space H_A whose scalar product is given by the generalized Dirichlet form

$$\mathcal{A}(f,g) = \int A(f)(x)\, \bar{g}(x)\, dx\,,$$

defined at least for $f, g \in \mathcal{D}$, with $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$.

• A gaussian centered process X_A with H_A as Reproducing Kernel Hilbert Space (R.K.H.S.), see [32]. The covariance function r of X_A

$$r(x, y) = \mathbb{E}\,(X_A(x)\, X_A(y))\,,$$

is the kernel of A^{-1} (defined on appropriate spaces) and a parametrix of the operator A.

Let us now define a class of symbols and state some precise asumptions for the symbols σ we use.

Definition 1.1. *A symbol σ defined on $\mathbb{R}^d \times \mathbb{R}^d \backslash \{0\}$ belongs to S_n^m, $m \in \mathbb{R}$, $n \in \mathbb{N}$ if,*

1) *for any multi-index s with length $|s| = s_1 + \cdots + s_d \leq n$, exists C_s such that*

$$|\partial_\xi^s \sigma(x, \xi)| \leq C_s\,(1 + |\xi|)^{m-|s|}\,, \qquad \textit{for } \xi \neq 0\,, \tag{6}$$

2) *there exists $\varepsilon > \varepsilon' \geq 0$ such that*

$$|\sigma(x, \xi) - \sigma(y, \xi)| \leq C\,(1 + |\xi|)^{m+\varepsilon'} |x - y|^{\varepsilon}\,. \tag{7}$$

Note that these properties are not the minimal asumptions which imply continuity of the operator between Sobolev spaces (see [12] for such conditions). But they are the minimal regularity asumptions implying that the symbol behaves "locally as if it were constant in x at high frequencies", a fact we will need to characterize the local scaling properties of the process X.

HYPOTHESIS HA$\,(m, \gamma)$. Let $m \geq 0$ and $\gamma > 0$. $A = Op\,(\sigma)$ satisfies hypothesis HA$\,(m, \gamma)$ if

• $\sigma \in S^m_{[\gamma]+d+1}$,

• there exists $c > 0$ such that

$$c\,|\xi|^m \leq \sigma(x, \xi)\,.$$

• There exist C_1 and $C_2 > 0$ such that

$$C_1 \int |\xi|^m\, |\hat{f}(\xi)|^2\, d\xi \leq (A(f) \mid f)_{L^2} \leq C_2 \int (1 + |\xi|)^m\, |\hat{f}(\xi)|^2\, d\xi\,.$$

Remarks. a) The last inequalities can be rewritten as

$$C_1\|f\|^2_{\dot{H}^{m/2}} \le (A(f)\,|\,f)_{L^2} \le C_2\|f\|^2_{\dot{H}^{m/2}}\,.$$

b) The hypothesis HA (m,γ) is related to the existence of a dual process for X (see [31] or [23]).

Hypothesis HAS (m,γ). We say HAS (m,γ) is satisfied when there exists $c>0$, such that

$$c\,(1+|\xi|^m) \le \sigma(x,\xi)\,, \qquad \text{if } \xi \ne 0$$

and

$$(A(f)\,|\,f)_{L^2} \sim \|f\|^2_{H^{m/2}}\,.$$

The model under Hypothesis HAS (m,γ) when $m>d$. Let us suppose $m>d$ and HAS (m,γ) holds for the symbol σ. Let $H_A = \mathrm{cl}_{\mathcal{A}}(\mathcal{D})$, the closure of $\mathcal{D}=\mathcal{D}(R^d)$ with respect to the inner product $\mathcal{A}$. Then $H^{m/2}\equiv H_A \subset L^2 \subset H'_A$ and $A^{-1}: H'_A \to H_A$ can be written using the kernel theorem as

$$A^{-1}f(x) = \int r(x,y)\,f(y)\,dy\,,$$

with r a continuous kernel on $(R^d)^2$. As r is symetric and of positive type we know (see [32]) that there exists a centered gaussian process X_A with covariance function r. The triple (A,H_A,X_A) satisfies the conditions we ask for our model.

The model under Hypothesis HA (m,γ) when $m>d$. Let us now suppose only HA (m,γ) holds. As the operator A may be non-inversible, a definition of H_A requires more care. In [9] Bourdaud gives a dilation invariant realization of the homogeneous Sobolev space $\dot{H}^{m/2}$. Similar ideas will be used here. Let us start with the case that turns out to be the most important for us, for it leads to processes which are limits in law of local renormalisations.

Suppose that $m-d=2\,(l+\alpha)>0$ with l an integer and $\alpha\in(0,1)$. Let $\sigma(\xi)=|\xi|^m S^2(\xi/|\xi|)$, $A=Op\,(\sigma)$, where the function S is continuous on the unit sphere of R^d and takes only positive values. In this case we set $H=\mathrm{cl}_{\mathcal{A}}(\mathcal{D}_0)$ with

$$\mathcal{D}_0 = \left\{\psi\in\mathcal{D} \,:\, D^\beta\psi(0)=0,\ \text{if } |\beta| < \frac{m-d}{2}\right\}.$$

Let W be a gaussian white noise on R^d. The gaussian process X_x is defined by

$$X_x = \int \exp_l(i\,x\,\xi)\,\hat{\varphi}(\xi)\,\frac{d\overline{\hat{W}}(\xi)}{\sqrt{\sigma(\xi)}}\,, \tag{8}$$

where $\exp_l(y) = e^y - \sum_{0\le k\le l} y^k/k!$. For every $\varphi \in L^2$ let

$$f_\varphi(x) = \int \exp_l(i\,x\,\xi)\,\hat{\varphi}(\xi)\,\frac{d\xi}{\sqrt{\sigma(\xi)}}\,.$$

The following result shows that (A, H, X) fullfills our conditions.

Lemma 1.1. *The symbol σ satisfies* HA$\,(m,\gamma)$. *The Hilbert space H is the* R.K.H.S. *of the gaussian process X and we have*

$$H = \{f_\varphi \,:\, \varphi \in L^2\}\,, \tag{9}$$

$$\mathcal{A}(f_{\varphi_1}, f_{\varphi_2}) = (\varphi_1 \,|\, \varphi_2)_{L^2}\,. \tag{10}$$

As this lemma can be deduced from results of [13] we only sketch the proof. It is easy to check that $\{f_\varphi,\, \varphi \in L^2\}$ is a Hilbert space with $\mathcal{A}$ as inner product and that (10) holds. We can also notice that for $\psi \in \mathcal{D}_0$

$$\mathcal{A}(f_\varphi, \psi) = (\sqrt{\sigma}\,\hat{f}_\varphi \,|\, \sqrt{\sigma}\,\hat{\psi})_{L^2} = (\hat{\varphi} \,|\, \sqrt{\sigma}\,\hat{\psi})_{L^2}\,.$$

Therefore if $\mathcal{A}(f_\varphi, \psi) = 0$, for all $\psi \in \mathcal{D}_0$, we get $\hat{\varphi} = 0$, and then $f_\varphi = 0$. This shows that $\mathcal{D}_0$ is dense in H and thus (9) holds. It remains to prove that H is the R.K.H.S. of X, *i.e.* for all x we have

$$f_\varphi(x) = \mathcal{A}(K_x, f_\varphi)\,,$$

where $K_x(y) = E\,(X_x X_y)$ is the covariance function of the process X. Since $K_x(y) = f_{k_x}$, where

$$\hat{k}_x(\xi) = \frac{\exp_l(-i\,x\,\xi)}{|\xi|^{m/2} S(\xi/|\xi|)}\,,$$

thus

$$\mathcal{A}(K_x, f_\varphi) = \Big(\hat{\varphi}\,\Big|\,\frac{\exp_l(-i\,x\,\xi)}{|\xi|^{m/2} S(\xi/|\xi|)}\Big)_{L^2} = f_\varphi(x)\,.$$

REMARKS. When $S(\xi) \equiv 1$ and $l = 0$, X is the d-dimensional Fractional Brownian motion of order α.

If $\alpha = 1$, the integral (8) does not define a process any longer (to study this case, we would have to split the integral). This is coherent with the facts that in this case $\dot{H}^{l+1+d/2}$ has no dilation invariant realization (see [9]) and that a different normalization is required for Elliptic Gaussian Processes of critical order $2\,l + d$ ($l \in \mathbb{N}$) to have a local asymptotic scaling law.

Finally one should notice that other spaces H_A can be associated with a single operator $A = Op\,(\sigma)$. But the associated processes are locally the same. For example if Hypothesis HAS (m, γ) holds, define H_A (respectively $H_{A,0}$) equal to $\mathrm{cl}_{\mathcal{A}}(\mathcal{D})$ (respectively $\mathrm{cl}_{\mathcal{A}}(\mathcal{D}_0)$) and denote by X_A, $X_{A,0}$ the associated processes. For any open bounded subset $U \subset R^d \backslash \{0\}$ it is easy to see (think of the brownian motion and bridge) that the laws of the restricted processes are equivalent, that is

$$\text{Law}\,(X_{A,0}|_U) \equiv \text{Law}\,(X_A|_U)\,. \tag{11}$$

CONVENTION. From now on we suppose the triple (A, H_A, X_A) is given and satisfies the conditions of our model and, unless otherwise specified, Hypothesis HA (m, γ).

1.3. Outline of the method.

The method we will use in order to obtain the modulus of continuity and the local scaling laws of the elliptic processes is the following.

a) For an operator A satisfying HAS (m, γ) we will construct in Section 2 an orthonormal wavelet basis Φ_λ of H_A indexed by the dyadic cubes, and such that each Φ_λ is localized near the corresponding dyadic cube (precise localization estimates are stated in Theorem 1.1 of Section 1.4). Using the canonical isomorphism between $\mathcal{H}$ and H_A we get

$$X_A(x) = \sum_\lambda \xi_\lambda \Phi_\lambda\,, \tag{12}$$

where the ξ_λ are independant normalized centered Gaussian; In Section 4, the local properties of the process X_A will be deduced from this decomposition.

b) In the general case HA (m, γ) we will perform a modification of the symbol at low frequencies in order to obtain a new process for which the stronger asumption HAS (m, γ) holds, and such that the two processes have the same local properties. This will be true because low frequency modifications do not alter such properties as local regularity or asymptotic scaling. Let us state the modification and prove this result. Let g be a nonnegative function in $\mathcal{D}(\mathbb{R}^n)$ such that supp $(g) \subset B(0,2)$, and

$$g(\xi) = 1\,, \qquad \text{if } |\xi| \leq 1\,. \tag{13}$$

Let G be the operator of convolution with $\hat{g}$ and set

$$A_g = (\mathrm{Id} - G)\, A\, (\mathrm{Id} - G) + G\,. \tag{14}$$

Clearly, if A is selfadjoint positive, so is A_g. X_A, X_{A_g} will denote the associated gaussian elliptic processes.

Proposition 1.1. *The operator A_g satisfies* HAS (m, γ) *and for any bounded open subset U of $\mathbb{R}^d$ such that $0 \notin U$*

$$\mathrm{Law}\,(X_A|_U) \equiv \mathrm{Law}\,(X_{A_g}|_U)\,.$$

Proof of Proposition 1.1. The symbol σ_g of A_g is given by

$$\sigma_g(x,\xi) = g(\xi) + (1 - g(\xi))^2 \sigma(x,\xi) + r(x,\xi)\,,$$

with $r(x,\xi)$ a regularizing kernel. It is easy to check that σ_g fulfills the conditions of HA (m, γ),

$$C_1 \int |\xi|^{2s} |\hat{f}(\xi)|^2\, d\xi \leq (A(f) \mid f)_{L^2} \leq C_2 \int (1 + |\xi|^{2s}) |\hat{f}(\xi)|^2\, d\xi\,.$$

The conditions for HAS (m, γ) are satisfied because

$$\begin{aligned} (A_g(f) \mid f)_{L^2} &\sim \|(\mathrm{Id} - G) f\|^2_{\dot{H}^s} + (G(f) \mid f)_{L^2} \\ &\sim \int (1 - g(\xi))\, |\xi|^{2s} |\hat{f}(\xi)|^2\, d\xi + \int g(\xi)\, |\hat{f}(\xi)|^2\, d\xi \\ &\sim \int (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2\, d\xi\,. \end{aligned}$$

Using (11) we can assume that the processes are starting from 0, *i.e.* the related R.K.H.S. are the closure of $\mathcal{D}_0$ for $\mathcal{A}$ and $\mathcal{A}_g$. The local equivalence result follows from [32, Theorem 8.6] if we check that

i) $C_U(x,y) := A^{-1}|_{U\times U}(x,y) - A_g^{-1}|_{U\times U}(x,y) \in H_{A_g}^{\otimes 2}(U\times U)$,

ii) -1 is not an eigenvalue of $C_U : H_{A_g} \to H_{A_g}$.

Let us consider the operator B

$$B := A - A_g = G\,A\,G - A\,G - G\,A + G\,. \tag{15}$$

As the function g belongs to $\mathcal{D}(\mathbb{R}^d)$ we know that B is a regularizing operator;

$$A^{-1} - A_g^{-1} = A_g^{-1}((I + B\,A_g^{-1})^{-1} - I)$$

and

$$(I + B\,A_g^{-1})^{-1} = \sum_{n\geq 0}(-1)^n(B\,A_g^{-1})^n\,.$$

Now if we consider the restrictions to open bounded U which are small enough, the last series converges and the operator $A^{-1} - A_g^{-1}$ is of Hilbert-Schmidt type with a spectral radius less than 1, so that condition ii) is satisfied.

For the first condition, it is sufficient to show that

$$(-\Delta)_x^{m/4}(-\Delta)_y^{m/4}C(x,y) \in L^2_{\mathrm{loc}}(\mathbb{R}^d\otimes\mathbb{R}^d)\,.$$

But, as before

$$\begin{aligned}(-\Delta)_x^{m/4}&(-\Delta)_y^{m/4}C(x,y)\\ &= \sum_{n\geq 1}(-1)^n(-\Delta)_x^{m/4}A_g^{-1}(B\,A_g^{-1})^n(-\Delta)_x^{m/4}(x,y)\,,\end{aligned}$$

which converges in $L^2(U\times U)$ for U small enough, since A_g^{-1} is an operator of order $-m$ and B is regularizing.

Finally we obtain the equivalence of laws for every bounded open subset U of $\mathbb{R}^d\backslash\{0\}$, by decomposing U in a finite number of small enough open subsets.

1.4. Wavelets and pseudodifferential operators.

We will construct a wavelet basis associated with an operator A_g satisfying HAS (m, γ). We obtain this basis by applying ${A_g}^{-1/2}$ on the "Littlewood-Paley" orthonormal wavelet basis of L^2 defined by Lemarié and Meyer (see [30]). Let us recall some properties of this basis.

1.4.1. The "Littlewood-Paley" wavelet basis.

There exists ϕ and $\psi^{(l)}$, $l \in L := \{0,1\}^d \backslash \{(0,\dots,0)\})$ such that $\hat{\phi}$ is C^∞ and supported in the domain $|\xi| \leq 4\pi/3$; $\widehat{\psi^{(l)}}$ are C^∞ with support included in the domain $2\pi/3 \leq |\xi| \leq 8\pi/3$; the following translations and dilations of these functions

$$\begin{aligned}\phi_k(x) &= \phi(x-k)\,, \qquad k \in \mathbb{Z}^d\,,\\ \psi^{(l)}_{j,k}(x) &= 2^{dj/2}\psi^{(l)}(2^j x - k)\,, \qquad j \in \mathbb{N},\ k \in \mathbb{Z}^d,\ l \in L\,,\end{aligned}$$

are an orthonormal basis of $L^2(\mathbb{R}^d)$ (notice that the family $\{\psi^{(l)}_{j,k} : j \in \mathbb{Z},\, k \in \mathbb{Z}^d,\, l \in L\}$ is also an orthonormal basis of $L^2(\mathbb{R}^d)$).

In order to simplify the notations, let $\psi^{(0,\dots,0)}_{0,k} := \phi_k$, $k \in \mathbb{Z}^d$ and

$$\psi_\lambda := \psi^{(l)}_{j,k},\ \ \lambda = (j,k,l) \in \mathbb{Z} \times \mathbb{Z}^d \times L \cup \{0\} \times \mathbb{Z}^d \times \{(0,\dots,0)\}\,.$$

For a given $\lambda = (j,k,l)$, the integer j will be often referred to as j_λ, and called the scale of λ. By abuse, λ will often be identified with the dyadic point $\tilde{\lambda} = k\,2^{-j} + l\,2^{-j-1}$ and the corresponding dyadic cube $c_\lambda = \tilde{\lambda} + [0,1]^d\,2^{-j-1}$. Let Λ be the set of λ's such that $j \geq 0$, and $\underline{\Lambda}$ for the whole set ($j \in \mathbb{Z}$).

The correlation (or Gram) matrix of a $(\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d))$ continuous operator A is

$$M_A(\lambda, \lambda') = (A(\psi_\lambda) \,|\, \psi_{\lambda'})_{L^2}\,, \qquad \lambda,\, \lambda' \in \Lambda\,. \tag{16}$$

1.4.2. Wavelet orthonormal basis associated with a pseudo-differential operator.

Let us define

$$\Phi_\lambda = A_g^{-1/2}(\psi_\lambda)\,, \qquad \lambda \in \Lambda\,. \tag{17}$$

We can restate the norm equivalence of Proposition 1.1 as follows: $A_g^{-1/2}$ is of the form $D\overline{M}D$ with $\overline{M}$ bounded on l^2 and $D(\lambda,\lambda') = 2^{-jm/2}\delta_{\lambda,\lambda'}$. The important result that we will prove at the beginning of Part 2 is decay of the entries of $\overline{M}$: we will show that this matrix is "almost diagonal" (in a sense that will be made precise in Definition 2.1). This will easily imply that the Φ_λ have the following "wavelet-like" decay properties and have an "asymptotic behavior" for large j's.

Theorem 1.1. *Let $m,\gamma > 0$, suppose that Hypothesis* HA (m,γ) *holds and that A_g satisfies* HAS (m,γ). *The $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ defined by* (17) *form an orthonormal basis of H_{A_g} with the following smoothness and localization properties.*

If $|s| \le [m/2]$,

$$|\partial^s \Phi_\lambda(x)| \le \frac{C_\gamma\, 2^{j(d/2+|s|-m/2)}}{(1+2^j|x-\lambda|)^{d+\gamma}}\,. \tag{18}$$

If $|s| = [m/2]$,

$$|\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)| \le \frac{C_\gamma\, |x-y|^{m/2-|s|}\, 2^{jd/2}}{(1+2^j|x-\lambda|)^{d+\gamma}}\,. \tag{19}$$

If $[m/2] \le |s| \le [\gamma+m/2]$,

$$|\partial^s \Phi_\lambda(x)| \le \frac{C_\gamma\, 2^{j(d/2+|s|-m/2)}}{(1+2^j|x-\lambda|)^{d+\gamma+m/2-|s|}}\,. \tag{20}$$

If $|s| = [\gamma+m/2]$,

$$|\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)| \le \frac{C_\gamma\, |x-y|^{\gamma+m/2-|s|}\, 2^{jd/2}}{(1+2^j|x-\lambda|)^{d}}\,. \tag{21}$$

The following theorem which describes the asymptotic behaviour of the wavelets when $j \to +\infty$ shows that in this limit the wavelets are the same as wavelets associated with a Selfsimilar Gaussian Process; thus it will allow us to derive the local scaling properties of the process X.

Theorem 1.2. *The hypotheses are the same as in Theorem* 1.1. *Let* (g_λ) *be defined by its Fourier transform as follows*

$$\hat{g}_\lambda(\xi) = (\sigma(\lambda,\xi))^{-1/2}\hat{\psi}_\lambda(\xi)\,.$$

Then for all ε, *there exists* J *such that, for* $s \le m/2$ *and* $j \ge J$

$$|\partial^s g_\lambda(x) - \partial^s \Phi_\lambda(x)| \le \frac{\varepsilon\, 2^{(d/2+|s|-m/2)j}}{(1+2^j|x-\lambda|)^{d+\gamma}}\,.$$

1.4.3. REMARKS. Let us now give a few remarks concerning the kind of symbols we consider here and the wavelets we use. First we used nonnegative scales ($j \ge 0$) for the following reason. If we used all the $\psi^{(l)}_{j,k}$ even for negative and arbitrary large j (and no ϕ_k) we would not be able to decompose symbols that depend on x (and then in Part 4, to analyse stochastic processes that have nonstationary increments). In fact when the symbol depends on x and thus presents oscillation at (say) scale 2^{-j_0}, its action on a wavelet indexed by $-j \ll -j_0$ does not give a "vaguelette" at scale 2^{-j}; the function we obtain oscillates too much. Thus the matrix of A in a basis composed of all the $\psi^{(l)}_{j,k}$ (including negative and arbitrary large j) would not be "almost diagonal".

On the other hand, since we have to use the $\phi(x-k)$ we may not allow the symbol to vanish or to have a pole at 0; otherwise it would introduce a singularity at 0.

1.5. Regularity of the Elliptic Gaussian Processes.

In this part $m > d$, $(l,\alpha) \in \mathbb{N}\times]0,1]$ is defined by

$$\frac{m-d}{2} = l + \alpha\,. \tag{22}$$

Before giving the uniform modulus and the iterated logarithm law of the processes X_A, let us start with a "global" regularity result which is a straightforward consequence of the wavelet decomposition of X_{A_g}, under hypothesis HA (m,γ) for A. Let us recall that a function f belongs to the Besov space $B^s_{p,q}$ if

$$|f|_{s,p,q} := \|f\|_{L^p} + \sum_{|r|=[s]} |\partial^r f|_{\beta,p,q} < \infty\,,$$

where

$$\omega_p(g,t) = \sup_{|y|\leq t} \|(g(\,\cdot\,+y) - g(\,\cdot\,))\|_{L^p}\ ,$$

$$\beta = s-[s]\,,$$

$$|g|^q_{\beta,p,q} = \int_0^\infty \Big(\frac{\omega_p(g,t)}{t^\beta}\Big)^q\,\frac{dt}{t}\,,$$

with the usual modification when $q=\infty$. Let us also recall that Sobolev and Hölder spaces are given by $H^s = B^s_{2,2}$ and $C^s = B^s_{\infty,\infty}$.

1.5.2. Regularity of the process X_{A_g}.

Proposition 1.2. *If the symbol σ satisfies* HA (m,γ)*, then,*

i) *for each $\Phi \in H_{A_g}$, $\mathcal{A}_g(X_{A_g},\Phi)$ is a well defined random variables of law $\mathcal{N}(0,\|\Phi\|_{A_g})$;*

ii) *for each bounded open set $U\subset\mathbb{R}^d$,*

$$X_{A_g}(x) = \sum_{\lambda\in\Lambda} \Phi_\lambda(x)\,\mathcal{A}_g(X_{A_g},\Phi_\lambda)\,,$$

with uniform convergence of the serie and its derivatives up to order l on U.

iii) *The above series converges locally in $B^s_{p,q}$ when $s<l+\alpha$ $\mathbb{P}$ almost surely.*

In dimension $d=1$ and for the fractionnal brownian motion of order α, assertion ii) of this proposition is proved in [11]. Note that Besov spaces have also been used by D. Donoho and his collaborators (see [14]) as a particularly convenient setting for wavelet based methods in statistics.

1.5.2. Laws of uniform and local moduli of continuity.

Let us define $d_{i,s}(x,y)$ ($(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, $i = 1, 2$) by

$$d^2_{1,s}(x,y) = \mathbb{E}\,[(\partial^s X_A(x) - \partial^s X_A(y))^2]\,, \tag{23}$$

$$d^2_{2,s}(x,y) = \mathbb{E}\Big[\Big(\partial^s X_A(x) - 2\,\partial^s X_A\Big(\frac{x+y}{2}\Big) + \partial^s X_A(y)\Big)^2\Big]\,. \tag{24}$$

Recall that $m - d = 2\,(l + \alpha)$, $l \in \mathbb{N}$, $0 < \alpha \leq 1$. For any multi-index s of lenght $|s| = l$ we define,

1) when $\alpha = 1$,

$$c_{2,s}(y) = \limsup_{x \to y} \frac{d_{2,s}(x,y)}{|x-y|} \tag{25}$$

and

$$c_{1,s}(y) = \limsup_{x \to y} \frac{d_{1,s}(x,y)}{|x-y|\sqrt{\log\,(|x-y|^{-1})}}\,. \tag{26}$$

2) When $\alpha < 1$,

$$c_{1,s}(y) = \limsup_{x \to y} \frac{d_{1,s}(x,y)}{|x-y|^{\alpha}}\,. \tag{27}$$

Lemma 1.2. *Under the hypothesis* HA$\,(m,\gamma)$ *and if* $|s| = l$*, the functions* $c_{1,s}$*,* $c_{2,s}$ *belong to* $C^{(\varepsilon-\varepsilon')/2}(\mathbb{R}^d)$*.*

Let us now set, when D is a bounded open subset of $\mathbb{R}^d$,

$$c_{i,s,D} = \sup_{y \in D} c_{i,s}(y)\,.$$

We can express the main result of this paragraph, where we use the notation, for r small enough

$$L(r) = \log\Big(\frac{1}{r}\Big)\,, \qquad L_k(r) = \log \circ \cdots \circ \log\Big(\frac{1}{r}\Big)\,, \qquad k \text{ times}\,.$$

Theorem 1.3. *Under the hypothesis* HA (m,γ), *if* $m > d$, $s \in \mathbb{N}^d$, $|s| = l$,

i) *law of the uniform modulus*:

- *when* $\alpha < 1$,

$$\limsup_{x,y\in D,|x-y|\to 0} \frac{|\partial^s X_A(x) - \partial^s X_A(y)|}{|x-y|^\alpha \sqrt{L(|x-y|^{-1})}} = \sqrt{2\,d}\; c_{1,s,D}\ , \tag{28}$$

$\mathbb{P}$ *almost everywhere,*

- *when* $\alpha = 1$,

$$\begin{aligned}\limsup_{x,y\in D,|x-y|\to 0} \frac{\left|\partial^s X_A(x) - 2\,\partial^s X_A\left(\frac{x+y}{2}\right) + \partial^s X_A(y)\right|}{|x-y|\sqrt{L(|x-y|^{-1})}}& \\ = \sqrt{2\,d}\; c_{2,s,D}\ ,&\end{aligned} \tag{29}$$

$\mathbb{P}$ *almost everywhere,*

- *and*

$$\limsup_{x,y\in D,|x-y|\to 0} \frac{|\partial^s X_A(x) - \partial^s X_A(y)|}{|x-y|\, L(|x-y|^{-1})} = \sqrt{2\,d}\; c_{1,s,D}\ , \tag{30}$$

$\mathbb{P}$ *almost everywhere.*

ii) *Law of the iterated logarithm*:

- *when* $\alpha < 1$, *for all* $y \in \mathbb{R}^d$,

$$\limsup_{x\to y} \frac{|\partial^s X_A(x) - \partial^s X_A(y)|}{|x-y|^\alpha \sqrt{L_2(|x-y|^{-1})}} = \sqrt{2}\; c_{1,s}(y)\,, \tag{31}$$

$\mathbb{P}$ *almost everywhere,*

- *when* $\alpha = 1$, *for all* $y \in \mathbb{R}^d$,

$$\limsup_{x\to y} \frac{|\partial^s X_A(x) - \partial^s X_A(y)|}{|x-y|\sqrt{L(|x-y|^{-1})}\,\sqrt{L_3(|x-y|^{-1})}} = \sqrt{2}\; c_{1,s}(y)\,, \tag{32}$$

$\mathbb{P}$ *almost everywhere.*

Note that the law of the iterated logarithm may be used to identify the "principal part" of the symbol, when it exists. If we assume that

$$\sigma(x,\xi) = a_x\Big(\frac{\xi}{|\xi|}\Big)|\xi|^m + o\left(|\xi|^m\right), \tag{33}$$

we will obtain later, see (60), in the case $\alpha < 1$, the very explicit formula

$$c_{1,s}^2(y) = \sup_{|v|=1}\int_{\mathbb{R}^d} 4\,\frac{\sin^2(v\,\xi/2)}{|\xi|^{d+2\alpha}\,a_y(\xi/|\xi|)}\,d\xi\,, \tag{34}$$

(31) shows the precise relationship between the "principal part" of the symbol σ and the exact local modulus of continuity of the process X.

Theorem 1.3 is proved in Section 4. The main idea is first to get the results for the modified process X_{A_g}, using its wavelet decomposition (see Theorem 1.1) and then to transfer the regularity properties of X_{A_g} to X_A, as a consequence of Proposition 1.1.

1.5.3. Remarks. When comparing (26) with (27), or (28) with (30), or (31) with (32) it appears that the case $\alpha = 1$ is critical. In fact this goes back to formula (18) which for $\alpha = 1$, $|s| = l$ gives

$$\partial^s\Phi_\lambda(x) - \partial^s\Phi_\lambda(y) \simeq C_\gamma|x-y|\,, \qquad |x-y|\to 0\,,$$

with C_γ independent of the scale j of λ. Assuming Φ_λ is supported by the dyadic cube c_λ, with center λ and sidelenght 2^{-j}, we would have when $l=0$, $\alpha=1$, $|y-x|\le 2^{-n}$

$$\frac{|\partial^s X(x) - \partial^s X(y)|}{|x-y|} \simeq C \sum_{\lambda\in\Lambda\cap E,\, j_\lambda\le n} \mathbf{1}_{\{c_\lambda(x)\}}\,\xi_\lambda\,. \tag{35}$$

(30) and (35) will be obtained by studying large deviations for sums of normal random variable indexed by a tree in Section 3.

On the other hand, formula (18) implies that in the case $\alpha<1$ the uniform and local moduli can be studied with sums restricted to the scales j near $\log_2(|x-y|)$, so that the proofs of (28) and (31) are close to the proofs of [3].

1.5.4. Comparison with known results.

When the symbol σ is a function of ξ only, and $m > d$ the process X_A has stationary increments. Let us then define $\rho(h)$ near the origin by

$$\rho(|x-y|) = d_s(x,y)\,. \tag{36}$$

Kono in [26] assumes that ρ^2 is concave and increasing near 0. Formula (27) shows that, even in the stationary case, none of these two conditions needs to be satisfied. In dimension 1, Marcus [29, Theorem 3.8] obtains the modulus of continuity under wider asumptions than Kono, which however do not include the critical case $\alpha = 1$ that we consider.

The results of Lemma 1.2 imply the hypotheses of Theorem 2.10 in [15] which asserts the existence of a bounded random variable $K(\omega)$ such that if $\alpha < 1$

$$|\partial^s X_A(x) - \partial^s X_A(y)| \leq K(\omega)\,|x-y|^\alpha \log\left(\frac{1}{|x-y|}\right)^{1/2},$$

$\mathbb{P}$ almost surely, and when $\alpha = 1$

$$|\partial^s X_A(x) - \partial^s X_A(y)| \leq K(\omega)\,|x-y| \log\left(\frac{1}{|x-y|}\right),$$

$\mathbb{P}$ almost surely, which is clearly less accurate than Theorem 1.3 in the elliptic context. If $A = \prod_{i=1}^d(-\Delta + c_i^2)$, the results of Proposition 1.1 are proved in [8]. If A is differential with C^∞ coefficients, it is proved in [1] that

$$\mathbb{P}\,(X_A \in H_{\mathrm{loc}}^{l-d-\varepsilon}) = 1\,, \qquad \text{for all } \varepsilon > 0\,.$$

1.6. Local scaling properties of Elliptic Gaussian Processes.

In this paragraph, we suppose that the symbol σ satisfies HA (m,γ). We show that the process X_A satisfies some local scaling property when its symbol σ admits a "principal part" (which is positively homogeneous).

We will distinguish the two cases $\alpha < 1$ and $\alpha = 1$ (l and α are defined by (22)).

Definition 1.2. *Suppose that* $0 < \alpha < 1$. *The* E.G.P. X_A *is Asymptotically Self Similar* (L.A.S.S.) *of order* (α, l) *if*

$$\lim_{\rho \to 0^+} \mathrm{Law}\Big\{\frac{(-\Delta)^{l/2} X_A(x+\rho\, u) - (-\Delta)^{l/2} X_A(x)}{\rho^\alpha},\ u \in \mathbb{R}^d\Big\}, \tag{37}$$

exists for every x *and is not trivial.* X_A *is Weakly Asymptotically Self Similar* (W.L.A.S.S.) *of order* (α, l) *if for every* x*, we can find a sequence* $(\rho_n) \longrightarrow 0^+$ *such that*

$$\frac{(-\Delta)^{l/2} X_A(x+\rho_n u) - (-\Delta)^{l/2} X_A(x)}{\rho_n^\alpha}, \qquad u \in \mathbb{R}^d,$$

converges in law to a non trivial limit.

When $m = d + 2\,\alpha$, $0 < \alpha < 1$ $(l = 0)$ the following theorem characterizes the L.A.S.S. property. The general case is similar, after l differentiations.

Theorem 1.4. *If the symbol* σ *satisfies* HA $(d + 2\,\alpha, \gamma)$ *for* $0 < \alpha < 1$, *the following assertions are equivalent.*

i) X_A *is a* L.A.S.S. *of order* $(\alpha, 0)$.

ii) *For all* $x_0 \in \mathbb{R}^d$,

$$\lim_{\rho \to \infty} \frac{\sigma(x_0, \rho\,\xi)}{\rho^{d+2\alpha}} = \theta_{x_0}(\xi), \tag{38}$$

exists, θ *is an* $(d+2\,\alpha)$*-homogeneous non trivial symbol and* $\theta \in S^{d+2\,\alpha}_{\infty,0}$.

iii) *For all* $x_0 \in \mathbb{R}^d$,

$$\lim_{h \to 0^+} \mathbb{E}\Big[\frac{(X_A(x_0 + hu) - X_A(x_0))^2}{h^{2\alpha}}\Big] = c_{x_0}^2(u),$$

exists and the function c_{x_0} *is an* α*-homogeneous non trivial function.*

We now consider the critical case $\alpha = 1$.

Definition 1.3. *Suppose that* $\alpha = 1$. *The E.G.P.* X_A *belongs to the weakly Locally Critical* (W.L.C.) *class of order* l *if for every* x*, we can find a sequence* (ρ_n) *which goes to* 0^+ *such that*

$$\frac{(-\Delta)^{l/2} X_A(x+\rho_n u) - (-\Delta)^{l/2} X_A(x)}{\rho_n \sqrt{\log(1/\rho_n)}}, \qquad u \in \mathbb{R}^d,$$

converges in law to the process $\{(G,u),\, u\in\mathbb{R}^d\}$ *with* G *a gaussian random variable on* $\mathbb{R}^d$*. It belongs to the Locally Critical* (L.C.) *class of order* (α,l) *if for every* x *there exists* G *a gaussian random variable on* $\mathbb{R}^d$ *such that*

$$\lim_{\rho\to 0^+}\mathrm{Law}\Big\{\frac{(-\Delta)^{l/2}X_A(x+\rho\, u)-(-\Delta)^{l/2}X_A(x)}{\rho\sqrt{\log(1/\rho)}},\ u\in\mathbb{R}^d\Big\} = \mathrm{Law}\,\{(G,u),u\in\mathbb{R}^d\}\,. \tag{39}$$

for every x*.*

The following theorem gives a characterization of the Locally Critical class.

Theorem 1.5. *If the symbol* σ *satisfies* HA $(d+2,\gamma)$*, the following assertions are equivalent.*

i) X_A *is Locally Critical of order* 0*.*

ii) *For all* $x_0\in\mathbb{R}^d$*,*

$$\lim_{\rho\to\infty}\frac{\sigma(x_0,\rho\,\xi)}{\rho^{d+2}}=\theta_{x_0}(\xi)\,, \tag{40}$$

exists and θ_{x_0} *is an* $(d+2)$*-homogeneous and non trivial symbol which belongs to* $S^{d+2}_{\infty,0}$*.*

iii) *For all* $x_0\in\mathbb{R}^d$

$$\lim_{h\to 0^+}\mathbb{E}\Big[\frac{(X_A(x+hu)-X_A(x))^2}{h^2\log(1/h)}\Big]=c^2_{x_0}(u)\,,$$

exists and the function c_{x_0} *is an* 1*-homogeneous non trivial function.*

Condition (38) or (40) means that the symbol has the following asymptotic behavior

$$\sigma(x,\xi)=h(x)\,F\Big(\frac{\xi}{|\xi|}\Big)|\xi|^m+o\,(|\xi|^m)\,,$$

it excludes symbols which have some slow oscillations at high frequencies like

$$\sigma(x,\xi)=|\xi|^m(1+\sin^2(\log(1+|\xi|^2)))\,,$$

such symbols give rise to processes which belongs only to the above weak classes. More precisely, let us consider a symbol σ such that

$$\sigma(\xi) = \sigma_0(\xi)\, f(\xi)\,, \tag{41}$$

where the symbol σ_0 is supposed to satisfy one of the equivalent conditions of Theorem 1.4. Let X_0 be the gaussian process associated to σ_0, wich belongs to the L.A.S.S. class of order $(\alpha, 0)$.

Proposition 1.3. *Let f be an even and C^∞ function on $\mathbb{R}^d$ such that the operator of symbol $1/f$ is positive definite, $1/f$ belongs to L^1_{loc} and, for all $s \in \mathbb{N}^d$ there exists $C_s > 0$ such that*

$$|\partial^s f(\xi)| \leq C_s(1 + |\xi|)^{-|s|}\,.$$

The gaussian process associated with σ in (41) *belongs to the* W.L.A. S.S. *class of order* $(\alpha, 0)$. *Moreover X belongs to the* L.A.S.S. *class of order* $(\alpha, 0)$ *if and only if* $\lim_{|\xi|\to\infty} f(\xi)$ *exists and does not vanish.*

The proofs of the local scaling properties are given in Part 5.

1.7. Complements.

We now consider two interesting cases that do not fit strictly speaking in the framework of Elliptic Gaussian Random Fields (E.G.R.F.), but can nonetheless be studied by the methods introduced in this paper. First we will consider the *Generalized Gaussian Processes* where the order of A is less than $d/2$; in that case the corresponding process is no more a function but a distribution. The second one is a *Fractional Brownian motion of nonconstant order* that we will define in (42). It will not be an E.G.R.F. but we will see that there is also a wavelet basis "adapted" to this process so that the technique we developed will immediately yield its regularity and scaling properties.

1.7.1. Extension to Generalized Gaussian Processes.

For the sake of simplicity we will consider only two cases which are important in applications: the "$1/f$ noise" which is used in signal analysis (see [28]) and the "free field" which is used in quantum field theory (see [6] and [7]).

We consider in dimension 1 the operator $A = (-\Delta)^{1/2}$; the process X_A is no longer a random function, but a random distribution, *i.e.* a Generalized Gaussian Process (G.G.P.) which is called the $1/f$ noise (see [33]), because of its spectral function.

Let now $d \geq 2$, $q \in \mathbb{R}^+$ and $A = -\Delta + q^2$. The process associated with A is by definition the free field of mass q.

In both situations, let us define a "truncated process" as follows

$$X_n(x) = \sum_{\lambda \subset E, j_\lambda \leq n} \Phi_\lambda(x)\, \xi_\lambda \ ,$$

with $E = (0,1)^d$.

Theorem 1.6. *For every $d \in \mathbb{N}^*$ there exists $C_d > 0$ such that,*

1) *if $d = 1$ or 2,*

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in E} |X_n(x)| = C_d \ , \qquad \mathbb{P} \textit{ almost everywhere},$$

2) *if $d > 2$,*

$$\limsup_{n\to\infty} \sqrt{\frac{2^{-n(d-1)}}{n}} \sup_{x\in E} |X_n(x)| = C_d \ , \qquad \mathbb{P} \textit{ almost everywhere.}$$

This result, which will be proved in Section 5, shows the rate of divergence in the space of bounded functions of the processes X_n which are approximations of X in the distribution sense. If $d = 1$, we see that X_n diverges very slowly. This shows why the fact that X is not a function but a distribution is hard to detect on numerical simulations, see [33]. In field theory [8] the difficulties of the renormalization increase with d (If $d = 4$, $|X_n|_\infty$ diverges like $\sqrt{n}\ 2^{3n/2}$ which shows one of the reasons of the difficulty of $P\Phi_4$ theory). We must also mention the connected work [6] where the renormalisation of sums like

$$X_n(x) = \sum_{j_\lambda \leq n} \phi_\lambda(x)\, \xi_\lambda\, 2^{|\lambda| d/2} \ ,$$

is studied when the ξ_λ are Rademacher or Gaussian random variables ξ_λ, and $\phi_\lambda(x)$ is the indicatrix function of a dyadic cell λ.

1.7.2. Multifractional Brownian Motion.

Let us state the definition we adopt for the *Multifractional Brownian Motion* which extends (8).

Definition 1.4. *Let $a \in C^r(\mathbb{R}^d, (0,1))$ for some $r > \sup a(x)$ and W a white noise. The Multifractional Brownian motion of order $a(x)$ is defined by*

$$B_a(x) = \int \frac{e^{ix\xi} - 1}{|\xi|^{a(x)+d/2}} \, d\overline{\hat{W}}(\xi) \,. \tag{42}$$

The function $C : \mathbb{R}^d \to \mathbb{R}$ defined by

$$C^2(x) = \int \frac{1 - \cos^2(x\,\nu)}{|\nu|^{d+2a(x)}} \, d\nu \,,$$

belongs to $C^r(\mathbb{R}^d)$ and

$$\mathbb{E}\,(|B_a(x+h) - B_a(x)|^2) = C^2(x)\,|h|^{2a(x)} + \mathrm{o}(h)\,.$$

In order to obtain a wavelet decomposition of B_a, one uses the following decomposition of the white noise on the Fourier transforms of the Littlewood-Paley wavelet basis

$$dW(\xi) = \sum \eta_\lambda \, \hat{\psi}_\lambda(\xi) \, d\xi \,,$$

where the η_λ are i.i.d. standard gaussian; if

$$\omega_\lambda(x) = \int \frac{e^{ix\xi} - 1}{|\xi|^{a(x)+d/2}} \, \overline{\hat{\psi}}_\lambda(\xi) \, d\xi \,, \tag{43}$$

then

$$B_a(x) = \sum \eta_\lambda \, \omega_\lambda(x) \tag{44}$$

and the following result will be a consequence of "vaguelettes" decay estimates for the ω_λ. In this part we define $a_E = \inf_{x\in E} a(x)$, $C_E = \sup_{x\in a^{-1}(a_E)\cap\overline{E}} C(x)$, when E is a bounded open set.

Theorem 1.7. *Let E be an open bounded set. The Multifractional Brownian motion B_a satisfies the following law of the uniform modulus of continuity*

$$\limsup_{x,y\in E, |x-y|\to 0} \frac{|B_a(x) - B_a(y)|}{|x-y|^{a_E}\sqrt{\log(1/|x-y|)}} = C_E\sqrt{2\,d}\,, \tag{45}$$

$\mathbb{P}$ almost everywhere, and the law of the iterated logarithm, for all $y \in \mathbb{R}^d$

$$\limsup_{x\to y} \frac{|B_a(x) - B_a(y)|}{|x-y|^{a(y)}\sqrt{\log\log(1/|x-y|)}} = \sqrt{2}\;C(x)\,, \tag{46}$$

$\mathbb{P}$ almost everywhere. Furthermore, B_a is Asymptotically Self Similar of order $a(x_0)$ at x_0; i.e.

$$\lim_{\rho\to 0^+} \mathrm{Law}\Big\{\frac{B(x_0+\rho\, u) - B(x_0)}{\rho^{a(x_0)}}\,,\ u \in \mathbb{R}^d\Big\} = \mathrm{Law}\,\{B_{a(x_0)}\}\,. \tag{47}$$

The reader can check that the same analysis would work after introducing in (42) a directional dependancy $S(\xi)$.

2. Wavelets and Elliptic Operators.

In this part we will construct the wavelet basis of H_A and prove Theorems 1.1 and 1.2 under regularity hypotheses on the symbol σ. We will also prove the equivalence in law stated in Proposition 1.1. This opens the way to Theorem 1.3 (proved in Section 4) which gives the uniform and local moduli of continuity of the process.

2.1. Wavelet matrices of pseudodifferential elliptic operators.

The basic idea here is not to work directly on the operator itself but rather on the infinite matrix of its coefficients on a wavelet basis. We will show that the matrices of the pseudodifferential operators we consider and of their inverses are of the form DMD where D is a diagonal matrix in a wavelet basis whereas M and M^{-1} are "almost diagonal" in the following sense (see [30]).

Definition 2.1. 1) *A matrix $M(\lambda,\lambda')$ belongs to the algebra $\mathcal{M}^\gamma$ if*

$$|M(\lambda,\lambda')| \le C\,\omega_\gamma(\lambda,\lambda')\,,$$

where

$$\omega_\gamma(\lambda,\lambda') = \frac{2^{-(d/2+\gamma)|j-j'|}}{(1+|j-j'|^2)(1+2^{\inf(j,j')}|\lambda-\lambda'|)^{d+\gamma}}\,.$$

2) *A matrix $M(\lambda,\lambda')$ belongs to $\mathcal{M}^{\gamma,m}$ $(m\in\mathbb{R})$ if $M = D\overline{M}D$ with $\overline{M}\in\mathcal{M}^\gamma$ and $D(\lambda,\lambda') = 2^{jm/2}\,\delta_{\lambda,\lambda'}$.*

For operators, we have the corresponding classes.

Definition 2.2. *An operator A belongs to* $\mathrm{OP}\,(\mathcal{M}^{\gamma,m})$ *if its matrix $M_{\lambda,\lambda'} = \langle A(\psi_\lambda)\,|\,\psi_{\lambda'}\rangle$ in the "Littlewood-Paley" wavelet basis belongs to $\mathcal{M}^{\gamma,m}$.*

The following Proposition shows that the class of symbols considered here is related to the class of matrices just defined. Therefore let

$$\delta(m,\gamma) := \min\Big\{[\gamma]+1,\frac{m}{2}\Big\}\,. \tag{48}$$

Proposition 2.1. *If the symbol σ satisfies* $\mathrm{HA}\,(m,\gamma)$*, then $A\in$* $\mathrm{OP}\,(\mathcal{M}^{\delta,m})$*, for all $\delta<\delta(m,\gamma)$.*

The following theorem asserts a kind of symbolic calculus for the operators we consider.

Theorem 2.1. *If A satisfies* $\mathrm{HAS}\,(m,\gamma)$*, A^{-1} belongs to* $\mathrm{OP}\,(\mathcal{M}^{\delta,-m})$*, for all $\delta<\delta(m,\gamma)$.*

We now prove Proposition 2.1. Let $r=[\gamma]+d+1$, $\delta=\delta(m,\gamma)$. We know that σ belongs to S^m_r. Denote by $M_{\lambda,\lambda'}$ the entries of M_A, the matrix of A in $(\psi_\lambda)_{j_\lambda\ge 0}$. Since A is self-adjoint, we consider only the case $j'\ge j$;

$$\begin{aligned} M_{\lambda,\lambda'} &= \iint \sigma(x,\xi)\,e^{ix\xi}\hat{\psi}_\lambda(\xi)\,\psi_{\lambda'}(x)\,dx\,d\xi \\ &= 2^{d(j'-j)/2}\iint \sigma(x,\xi)\,e^{i(x-\lambda)\xi}\,\hat{\psi}^{(l)}\Big(\frac{\xi}{2^j}\Big)\psi^{(l')}(2^{j'}x-k')\,dx\,d\xi \end{aligned}$$

and thus

$$M_{\lambda,\lambda'} = 2^{-d(j'-j)/2} \iint \sigma\Big(\frac{x}{2^{j'}} + \lambda', 2^j\xi\Big) \cdot e^{i(x/2^{j'}+\lambda'-\lambda)2^j\xi}\, \hat\psi^{(l)}(\xi)\, \psi^{(l')}(x)\, dx\, d\xi\,. \tag{49}$$

Since the functions $\psi^{(l)}$ have fast decay, there exists a $K > 0$ such that, for all $l \in \{0,1\}^d$ and $\rho \geq 1/2$,

$$\int_{|x|>\rho/2d} |\psi^{(l)}(x)|\, dx \leq K\rho^{-r}\,. \tag{50}$$

We distinguish two more cases.

Case 1. $l' \neq 0$ and $2^j|\lambda - \lambda'| < 1/2$.

The function $F(\lambda, \lambda', x, \xi) = \sigma(x + \lambda', \xi)\, e^{i(x+\lambda'-\lambda)\xi}$ satisfies estimates (6) and (7). Thus, since $\psi^{(l')}$ has a vanishing integral

$$\begin{aligned} M_{\lambda,\lambda'} &= 2^{-d(j'-j)/2} \iint F(\lambda, \lambda', 2^{-j'}x, 2^j\xi)\, \hat\psi^{(l)}(\xi)\, \psi^{(l')}(x)\, dx\, d\xi \\ &= 2^{-d(j'-j)/2} \iint (F(\lambda, \lambda', 2^{-j'}x, 2^j\xi) - F(\lambda, \lambda', 0, 2^j\xi)) \\ &\qquad \cdot \hat\psi^{(l)}(\xi)\, \psi^{(l')}(x)\, dx\, d\xi\,, \end{aligned}$$

so that

$$\begin{aligned} |M_{\lambda,\lambda'}| &\leq C_0\, 2^{-d(j'-j)/2} \\ &\quad \cdot \iint |2^{-j'}x|^{\varepsilon}(1 + 2^j|\xi|)^{m+\varepsilon}|\hat\psi^{(l)}(\xi)|\, |\psi^{(l')}(x)|\, dx\, d\xi \\ &\leq C_1\, 2^{-(d/2+\varepsilon+m/2)(j'-j)} 2^{mj/2} 2^{mj'/2} \\ &\leq C_2\, \omega_{\delta(m,\gamma)}(\lambda, \lambda'). \end{aligned}$$

Case 2. $2^j|\lambda - \lambda'| \geq 1/2$.

Let

$$J := 2^{-d(j'-j)/2} \iint_{|x|>\rho/2d} \sigma\Big(\frac{x}{2^{j'}} + \lambda', 2^j\xi\Big) e^{i(x/2^{j'}+\lambda'-\lambda)2^j\xi} \cdot \hat\psi^{(l)}(\xi)\, \psi^{(l')}(x)\, dx\, d\xi\,.$$

Using (50) and hypothesis HA (m,γ), we have

$$
\begin{aligned}
|J| &\le C_3\, 2^{-d(j'-j)} \iint_{|x|>\rho/2d} (1+2^j|\xi|)^m |\hat{\psi}^{(l)}(\xi)|\,|\psi^{(l')}(x)|\,dx\,d\xi \\
&\le C_4\, 2^{-d(j'-j)/2+mj}\,\rho^{-r}
\end{aligned}
$$

if $\rho \ge 1/2$. With $\rho = 2^{j'}|\lambda'-\lambda|$ we obtain

$$
|J| \le C_4\, \frac{2^{-d(j'-j)/2+mj}}{2^{rj'}|\lambda'-\lambda|^r} = C_4\, 2^{m(j+j')/2}\, \frac{2^{-(d/2+m/2+r)(j'-j)}}{(2^j|\lambda'-\lambda|)^r}\,.
$$

The result will be achieved if we get a similar bound for

$$
\begin{aligned}
\tilde{M}_{\lambda,\lambda'} = 2^{-d(j'-j)/2} \iint_{|x|\le 2^{j'}|\lambda'-\lambda|/2d} & \sigma\Big(\frac{x}{2^{j'}}+\lambda', 2^j\xi\Big) e^{i(x/2^{j'}+\lambda'-\lambda)2^j\xi} \\
& \cdot \hat{\psi}^{(l)}(\xi)\,\psi^{(l')}(x)\,dx\,d\xi\,.
\end{aligned}
$$

In fact there exists a coordinate direction, say the k^{th} one, such that

$$
|\lambda'_k - \lambda_k| \ge \frac{1}{d}\,|\lambda'-\lambda|\,.
$$

Integrating by parts r times in the direction k, we get

$$
\begin{aligned}
\tilde{M}_{\lambda,\lambda'} &= 2^{-d(j'-j)/2} \\
&\quad \cdot \iint_{|x|\le 2^{j'}|\lambda'-\lambda|/2d} \frac{e^{i(x/2^{j'}+\lambda'-\lambda)2^j\xi}}{\Big(i\,2^j\Big(\frac{x_k}{2^{j'}}+\lambda'_k-\lambda_k\Big)\Big)^r} \\
&\qquad \cdot \partial^r_{\xi_k}\Big(\sigma\Big(\frac{x}{2^{j'}}+\lambda', 2^j\xi\Big)\hat{\psi}^{(l)}(\xi)\Big)\psi^{(l')}(x)\,dx\,d\xi\,.
\end{aligned}
$$

In the domain of integration, $|x_k/2^{j'}+\lambda'_k-\lambda_k| \ge |(\lambda-\lambda')/(2d)|$ so that

$$
\begin{aligned}
|\tilde{M}_{\lambda,\lambda'}| &\le C_5\, \frac{2^{-d(j'-j)/2}}{(2^j|\lambda-\lambda'|)^r} \\
&\quad \cdot \iint \Big|\partial^r_{\xi_k}\Big(\sigma\Big(\frac{x}{2^{j'}}+\lambda', 2^j\xi\Big)\hat{\psi}^{(l)}(\xi)\Big)\psi^{(l')}(x)\Big|\,dx\,d\xi \\
&\le C_6\, \frac{2^{-d(j'-j)/2}}{(1+2^j|\lambda-\lambda'|)^r}\, 2^{mj} \int |\psi_{l'}(x)|\,dx \\
&\le C_7\, 2^{mj/2} 2^{mj'/2}\, \frac{2^{-(d/2+m/2)(j'-j)}}{(2^j|\lambda-\lambda'|)^r}\,.
\end{aligned}
$$

Observe that if $2^j|\lambda-\lambda'| < 1/2$ and $l' = 0$ we have necessarily $j' = j = 0$ and thus $|k + l/2 - k'| < 1/2$, which gives $\lambda = \lambda'$; hence Proposition 2.1.

2.2. Construction of the wavelets.

In all this subsection we suppose that HAS$\,(m, \gamma)$ holds and we will construct the basis (Φ_λ) under this hypothesis. We define $\Phi_\lambda = A^{-1/2}(\psi_\lambda)$, where we can use for instance Kato's formula to define $A^{-1/2}$

$$A^{-1/2} = \frac{2}{\pi}\int_0^\infty \frac{dt}{t^2\mathrm{Id} + A}\,.$$

The fact that the (Φ_λ) form an orthonormal basis of H_A is just an algebraic computation since the definition of $A^{-1/2}$ is such that $A^{-1/2}$ is a positive selfadjoint operator satisfying $A^{-1/2} \circ A^{-1/2} = A^{-1}$.

Let us recall that a family of functions $(f_\lambda)_\Lambda$ is a system of δ-vaguelettes if and only if the matrix of the family in any wavelet basis (with regularity strictly larger than δ) belongs to $\mathcal{M}^\delta$, see [30].

Proposition 2.2. *We have* $\Phi_\lambda = 2^{-mj/2}\zeta_\lambda$ *where* ζ_λ *are* δ*-vaguelettes for every* $\delta < \delta(m, \gamma)$*.*

Let us sketch the proof of Proposition 2.2. Let $t \geq 0$. We define $H(t)$ as the completion of $\mathcal{D}(\mathbb{R}^d)$ for the norm

$$\|u\|^2_{H(t)} = \langle (t^2\mathrm{Id} + A)u \,|\, u\rangle\,.$$

Lemma 2.1. *Wavelets are an unconditional basis of* $H(t)$ *and the following norm equivalences* (*uniform in* t) *hold*

$$\|u\|^2_{H(t)} \sim \|u\|^2_{H^m} + t^2\|u\|^2_{L^2} \sim \sum_\lambda |U_\lambda|^2\theta^2_\lambda\,, \tag{51}$$

where U_λ *are the wavelet coefficients of* u *and* $\theta_\lambda = \sqrt{t^2 + 2^{mj}}$*.*

The first equivalence is nothing but the asumption $H1$ on A, and the second comes from the wavelet characterization of H^s (see [30]).

Proposition 2.3. *The following decomposition holds*

$$t^2\mathrm{Id} + M = D'\overline{N}D'\,,$$

where D' is diagonal on the L^2-orthonormal (Littlewood-Paley) wavelet basis, $D' = Diag(\theta_\lambda)$ and for all $\delta < \delta(m,\gamma)$, $\overline{N}$ and $\overline{N}^{-1}$ belongs to $\mathcal{M}^\delta$ (uniformly in t).

Let us admit this proposition for now and see why Proposition 2.2 is a consequence of Proposition 2.3. Using the definition of $A^{-1/2}$, we have the following estimate for the matrix coefficients of $A^{-1/2}$

$$|M^{-1/2}_{\lambda,\lambda'}| \leq \frac{2}{\pi}\int_0^\infty \frac{1}{\sqrt{t^2+2^{mj}}}\,|\omega(\lambda,\lambda',t)|\,\frac{1}{\sqrt{t^2+2^{mj'}}}\,dt\,,$$

where $\omega(\lambda,\lambda',t) = \overline{N}^{-1}_{\lambda,\lambda'}$. But for every $\delta < \delta(m,\gamma)$ the Proposition 2.3 gives

$$|\omega(\lambda,\lambda',t)| \leq\; C\,\omega_\delta(\lambda,\lambda')\,,$$

uniformly in t (see the definition of $\omega_\delta(\lambda,\lambda')$ in Definition 2.1). Thus

$$|M^{-1/2}_{\lambda,\lambda'}| \leq C\,\omega_\delta(\lambda,\lambda')\int_0^\infty \frac{1}{\sqrt{t^2+2^{mj}}\sqrt{t^2+2^{mj'}}}\,dt\,,$$

hence

$$|M^{-1/2}_{\lambda,\lambda'}| \leq C\,\omega_\delta(\lambda,\lambda')\,(1+|j-j'|)2^{-m\sup\{j,j'\}/2}\,, \tag{52}$$

and since $\Phi_\lambda = \sum_{\lambda'} M^{-1/2}_{\lambda,\lambda'}\psi_{\lambda'}$, we have $\Phi_\lambda = 2^{-mj/2}\sum_{\lambda'}\theta_{\lambda,\lambda'}\,\psi_{\lambda'}$ where $\theta_{\lambda,\lambda'}$ belongs to $\mathcal{M}^{\gamma'}$ for any $\gamma' < \delta(m,\gamma)$. Hence Proposition 2.2.

We will now prove Proposition 2.3. In the following, $t \geq 0$ will be fixed; the dependancy of the coefficients in t will often be forgotten, but all estimates will be uniform in t.

From Theorem 2.1 we have

$$M = DND\,, \tag{53}$$

where D is diagonal, $D = Diag(2^{mj/2})$, and $N \in \mathcal{M}^{\gamma'}$ for any $\gamma' < \delta(m,\gamma)$; then we get

$$t^2\mathrm{Id} + M = D'\overline{N}D'\,,$$

with

$$\overline{N}_{\lambda,\lambda'} = \frac{t^2\delta_{(\lambda,\lambda')} + 2^{m(j+j')/2}N_{\lambda,\lambda'}}{\sqrt{t^2+2^{mj}}\sqrt{t^2+2^{mj'}}} \ ,$$

where $\delta_{(\lambda,\lambda')}$ is the Kronecker symbol. As $N \in \mathcal{M}^{\gamma'}$, for all $\gamma' < \delta(m,\gamma)$ we obtain the same property for $\overline{N}$ and the first part of Proposition 2.3 is proved. We prove the second part after a study of invertibility of operators in the algebra $\mathcal{M}^\gamma$ performed in the next subsection. Basically, we will "freeze" the coefficients of the operator $t^2\mathrm{Id} + M$ at the center of the "numerical support" of the wavelets. The matrice of $t^2\mathrm{Id}+M$ in a wavelet basis will thus be approximated by another matrix that will be "invertible in $\mathcal{M}^\gamma$ for large j's". We will give a precise definition of these approximations of matrices, and this will lead to the "symbolic calculus" result stated in Theorem 2.1. In Subsection 2.4, these general results will be applied to the operator $t^2\mathrm{Id} + M$.

2.3. The "quasi-ideals" $\mathcal{I}^\gamma$.

Definition 2.3. *A matrix S belongs to $\mathcal{I}^\delta$ if $S \in \mathcal{M}^\delta$ and for all $\varepsilon > 0$ exists J such that $j \geq J$ or $j' \geq J$ implies*

$$|S_{\lambda,\lambda'}| \leq \varepsilon\, \omega_\delta(\lambda,\lambda')\,.$$

REMARK. Suppose that $M \in \mathcal{M}^\delta$ and $\delta' < \delta$. Then $M \in \mathcal{I}^{\delta'}$ if for all ε, $C > 0$, there exists J such that if j or $j' \geq J$,

(54) $|j-j'| \leq C$ and $|k\,2^{-j} - k'2^{-j'}| \leq \ C\,2^{-j}$ implies $|M_{\lambda,\lambda'}| \leq \varepsilon\,.$

In fact if j and j' are small, there is nothing to prove, and if either $|j-j'|$ or $|k\,2^{-j} - k'2^{-j'}|$ is large, the result holds because $\omega_\delta(\lambda,\lambda') \leq \varepsilon\,\omega_{\delta'}(\lambda,\lambda')$.

Lemma 2.2. *If $S \in \mathcal{I}^\delta$ and $M \in \mathcal{M}^\delta$ then for all $\delta' < \delta$, $SM \in \mathcal{I}^{\delta'}$ and $MS \in \mathcal{I}^{\delta'}$.*

Note that $\mathcal{I}^\delta$ is not an ideal in the algebra $\mathcal{M}^\delta$. The above lemma shows that it shares the same property as ideals if we are ready to admit an arbitrary small loss on the value of δ.

Let

$$\mathrm{dist}(\lambda,\lambda') = |j-j'| + |\tilde{\lambda} - \tilde{\lambda}'|\,,$$

which gives a distance on Λ. We can now sketch the proof of this lemma. We know that $SM \in \mathcal{M}^\delta$. If λ and λ' are distant ($|j-j'|$ large or $|k/2^j - k'/2^{j'}|$ large) then $\omega_\delta(\lambda,\lambda') \le \varepsilon\, \omega_{\delta'}(\lambda,\lambda')$ hence the result in that case.

Suppose now that λ and λ' are close; if j is small, we have nothing to prove. If j (and thus j') is large

$$\begin{aligned} |SM_{\lambda,\lambda'}| &= \Big|\sum_{\lambda''} S_{\lambda,\lambda''} M_{\lambda'',\lambda'}\Big| \\ &\le \sum_{\lambda''} \varepsilon\, \omega_\delta(\lambda,\lambda'')\, \omega_\delta(\lambda'',\lambda') \\ &\le C\,\varepsilon\, \omega_\delta(\lambda,\lambda')\,. \end{aligned}$$

Hence the lemma in this case. The proof for MS is the same.

The importance of $\mathcal{I}^\delta$ comes from the following Proposition which shows that $\mathcal{I}^\delta$ will play a role similar to compact perturbations of invertible operators.

Proposition 2.4. *Suppose that M and M^{-1} belong to $\mathcal{M}^\delta$ and that S belongs to $\mathcal{I}^\delta$. If $M+S$ is invertible on l^2 then $(M+S)^{-1} \in \mathcal{M}^\delta$ and for all $\delta' < \delta$, $(M+S)^{-1} - M^{-1} \in \mathcal{I}^{\delta'}$.*

PROOF OF PROPOSITION 2.4. The first step is to reduce the proposition to the case where $S_{\lambda,\lambda'} = 0$ if $j \ge J$ or $j' \ge J$. Let $\overline{S}$ the restriction of Q to indexes (λ,λ') such that $j \ge J$ or $j' \ge J$. The norm of $\overline{S}$ in $\mathcal{I}^\delta$ can be made arbitrarily small by choosing J large enough. The set of invertible elements in an algebra being open, $M+\overline{S}$ will be invertible if J is large enough, hence the reduction that we claimed. We suppose now $S_{\lambda,\lambda'} = 0$ if $j \ge J$ or $j' \ge J$. We have

$$M + S = M\,(\mathrm{Id} + M^{-1}S)$$

and

$$(\mathrm{Id} - M^{-1}S)^{-1} = \mathrm{Id} + M^{-1}S\,(\mathrm{Id} - M^{-1}S)^{-1}\,.$$

Let E be the restriction of M^{-1} to the indexes (λ,λ') such that $j \le J$ and $j' \le J$, and $E_{\lambda,\lambda'} = 0$ elsewhere. Then, one easily checks that Id$-ES$ is invertible, and that $S\,(\mathrm{Id} - M^{-1}S)^{-1} = Q\,(\mathrm{Id} - ES)^{-1}$.

The fact that Id$-ES$ belongs to $\mathcal{M}^\delta$ is equivalent to

$$|(\mathrm{Id} - ES)_{\lambda,\lambda'}| \le \frac{C}{|1+\mathrm{dist}(\lambda,\lambda')|^{d+\delta}}\,, \tag{55}$$

for all $j, j' \leq J$. The set of indexes we consider is a subset of $\mathbb{Z}^d \times [0, \ldots, J]$. If it were a subset of $\mathbb{Z}$, a symbolic calculus result (see [21]) would show that (55) and the l^2-inversibility of $(\mathrm{Id} - ES)$ imply that estimate (55) holds for the invert of $\mathrm{Id} - ES$, hence that $(\mathrm{Id} - ES)^{-1}) \in \mathcal{M}^\delta$ since all other non diagonal entries of this matrix vanish. Actually, one checks by inspection that theorem of [21] also holds in the d-dimensional case.

Thus $(M+S)^{-1} \in \mathcal{M}^\delta$. Actually

$$(M+S)^{-1} - M^{-1} = M^{-1} S\,(\mathrm{Id} - ES)^{-1} M^{-1}$$

and since $S \in \mathcal{I}^\delta$, Lemma 2.2 implies that $(M+S)^{-1} - M^{-1} \in \mathcal{I}^{\delta'}$ for all $\delta' < \delta$.

Corollary 2.1. *Suppose that $P \in \mathcal{M}^\delta$ and is selfadjoint positive and invertible on l^2. If there exists $Q \in \mathcal{M}^\delta$ such that $PQ - \mathrm{Id} \in \mathcal{I}^\delta$ then*

$$P^{-1} \in \mathcal{M}^\delta \qquad \textit{and} \qquad P^{-1} - Q \in \mathcal{I}^{\delta'}\,, \qquad \textit{for all } \delta' < \delta\,.$$

PROOF. Let $J > 0$ be given. By hypothesis $PQ = \mathrm{Id} + R + S$ where $R_{\lambda,\lambda'} = 0$ if $j \geq J$ or $j' \geq J$ and $\varepsilon = \|S\|_{\mathcal{M}^\gamma}$ can be choosen arbitrarily small if J is large enough. Let I_J be the operator

$$\begin{cases} (I_J)_{\lambda,\lambda'} = 1\,, & \text{if } \lambda = \lambda' \text{ and } j \leq J\,, \\ (I_J)_{\lambda,\lambda'} = 0\,, & \text{else}\,. \end{cases}$$

For $\theta > 0$ we consider $A\,(B + \theta I_J) = \mathrm{Id} + R + \theta P I_J + S$. First note that if θ is large enough $R + \mathrm{Id} + \theta A I^J$ is invertible on l^2 because, decomposing the matrices according to their action on $j \leq J$ and $j > J$, we can write

$$P = \begin{pmatrix} P_1 & P_3 \\ P_2 & P_4 \end{pmatrix}, \qquad R = \begin{pmatrix} R' & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$\mathrm{Id} + R + \theta P I_J = \begin{pmatrix} R' + \mathrm{Id} + \theta P_1 & 0 \\ \theta P_2 & \mathrm{Id} \end{pmatrix}.$$

Since P is selfadjoint positive invertible, P_1 has the same property, and

$$\|P_1\| \leq \|P\| \qquad \text{and} \qquad \|P_1^{-1}\|\,\|P^{-1}\|\,.$$

Thus choosing θ large enough, $\mathrm{Id}+R'+\theta P_1$ is invertible, and, using the same argument as in the proof of Proposition 2.4, its inverse belongs to $\mathcal{M}^\delta$.

The inverse of $\mathrm{Id}+R+\theta PI^J$ is

$$\begin{pmatrix} (\mathrm{Id}+R'+\theta P_1)^{-1} & 0 \\ -\theta P_2(\mathrm{Id}+R'+\theta P_1)^{-1} & \mathrm{Id} \end{pmatrix}.$$

For a fixed θ, the norms of $\mathrm{Id}+R+\theta PI_J$ and of its inverse are bounded indenpendantly of J. Choosing J large enough (which can be done independantly of the choice of θ), ε $(=\|S\|)$ can be arbitrarily small, and thus $\mathrm{Id}+R+\theta PI_J+S$ is invertible in l^2.

Applying Proposition 2.4, we see that $P(Q+\theta I_J)$ is invertible in $\mathcal{M}^\delta$. The same property holds for P because $P^{-1}=(Q+\theta I_J)\,(P(Q+\theta I_J))^{-1}$.

Furthermore $P^{-1}-(Q+\theta I_J)\in\mathcal{I}^{\delta'}$, for all $\delta'<\delta$ and since $I_J\in\mathcal{I}^\delta$, we see that $P^{-1}-Q\in\mathcal{I}^{\delta'}$, for all $\delta'<\delta$.

2.4. Application of the Quasi-ideals.

We first end the proof of Proposition 2.3.

Recall that $\overline{N}$ is the matrix

$$\overline{N}_{\lambda,\lambda'}=\frac{((t^2\mathrm{Id}+M)\psi_\lambda\,|\,\psi_{\lambda'})_{L^2}}{\theta_\lambda\theta_{\lambda'}}$$

and let

$$\begin{aligned} P_{\lambda,\lambda'} &= (\theta_\lambda\theta_{\lambda'})^{-1} < ((t^2+\sigma(\lambda,\xi))\hat\psi_\lambda\,|\,\hat\psi_{\lambda'})_{L^2}\,, \\ Q_{\lambda,\lambda'} &= \theta_\lambda\theta_{\lambda'}((t^2+\sigma(\lambda,\xi))^{-1}\hat\psi_\lambda\,|\,\hat\psi_{\lambda'})_{L^2}\,. \end{aligned}$$

Lemma 2.3. *Under hypothesis* HA (m,γ) *the matrices P and Q belong to $\mathcal{M}^{\gamma'}$ for $\gamma'<\delta(m,\gamma)$.*

PROOF. As before we suppose $0\le j\le j'$. If $|j-j'|\ge 2$, $P_{\lambda,\lambda'}=0$ because of the supports of the $\hat\psi_\lambda$. If $|j-j'|<2$ and $|\lambda-\lambda'|\le C\,2^{-j}$,

$$|P_{\lambda,\lambda'}|\le\int\frac{1+t^2+|\xi|^m}{1+t^2+2^{jm}}\,2^{-dj/2}2^{-dj'/2}\Big|\hat\psi\Big(\frac{\xi}{2^j}\Big)\Big|\,\Big|\hat\psi\Big(\frac{\xi}{2^{j'}}\Big)\Big|\,d\xi\le C\,.$$

If $|j-j'|<2$ and $2^j|\lambda-\lambda'|$ is large,

$$P_{\lambda,\lambda'}=\int \frac{t^2+\sigma(\tilde{\lambda},\xi)}{\theta_\lambda\theta_{\lambda'}}\,\hat{\psi}\Big(\frac{\xi}{2^j}\Big)\overline{\hat{\psi}}\Big(\frac{\xi}{2^{j'}}\Big)e^{i(\lambda-\lambda')\xi}\,2^{-dj/2}\,2^{-dj'/2}\,d\xi$$

and integrating by parts in a chosen direction as above,

$$\begin{aligned}P_{\lambda,\lambda'}=&\frac{1}{(\lambda_l-\lambda_l')^r\,2^{(j+j')d/2}\theta_\lambda\theta_{\lambda'}}\\&\cdot\sum_{p+q=r}\int \partial_{\xi_l}^p\sigma(\lambda,\xi)\,\partial_{\xi_l}^q\Big(\psi\Big(\frac{\xi}{2^j}\Big)\overline{\hat{\psi}}\Big(\frac{\xi}{2^{j'}}\Big)\Big)\,d\xi\,,\end{aligned}$$

so that

$$|P_{\lambda,\lambda'}|\le\frac{C}{|\lambda-\lambda'|^r\,\theta_\lambda\theta_{\lambda'}}\sum_{p+q=r}\int_{|\xi|\le 2^j 8\pi/3}(1+|\xi|)^{m-p}\,2^{-qj}\,d\xi$$

and finally

$$|P_{\lambda,\lambda'}|\le\frac{C}{(2^j|\lambda-\lambda'|)^r}\,.$$

Hence Lemma 2.3 for P. The proof for Q is similar.

Lemma 2.4. *The matrix $\overline{N}-P$ belongs to $\mathcal{I}^{\gamma'}$ and $PQ^*-\mathrm{Id}$ belongs to $\mathcal{I}^{\gamma'}$ for any $\gamma'<\delta(m,\gamma)$.*

PROOF. By symmetry we can suppose $j\le j'$;

$$(\overline{N}-P)_{\lambda,\lambda'}=\frac{1}{\theta_\lambda\theta_{\lambda'}}\int(\sigma(x,\xi)-\sigma(\lambda,\xi))\,\hat{\psi}_\lambda(\xi)\,e^{ix\xi}\,\overline{\psi_{\lambda'}(x)}\,dx\,d\xi\,.$$

Using the hypothesis HA (m,γ),

$$\begin{aligned}|(\overline{N}-P)_{\lambda,\lambda'}|&\le\frac{C}{\theta_\lambda\theta_{\lambda'}}\iint|x-\tilde{\lambda}|^\varepsilon(1+|\xi|)^{m+\varepsilon'}2^{-dj/2}\\&\qquad\cdot|\hat{\psi}(\xi 2^{-j})|\,2^{dj'/2}\,|\psi(2^{j'}x-k')|\,dx\,d\xi\\&\le\frac{C}{\theta_\lambda\theta_{\lambda'}}\,2^{-\varepsilon j'}2^{d(j-j')/2}\\&\quad\cdot\iint(1+2^j|\xi|)^{m+\varepsilon'}|\hat{\psi}(\xi)|\\&\qquad\cdot|x-2^{j'}(\lambda-\lambda')|\,2^\varepsilon\,|\psi(x)|\,dx\,d\xi\end{aligned}$$

and, because of (54),

$$|(\overline{N}-P)_{\lambda,\lambda'}| \leq \frac{C}{\theta_\lambda\theta_{\lambda'}}\,2^{-\varepsilon j'}2^{\varepsilon' j}(1+2^{mj}) \leq C_1\,2^{-(\varepsilon-\varepsilon')j}\,.$$

Thus $\overline{N}-P \in \mathcal{I}^{\gamma'}$.

Let us now prove the second result of Lemma 2.4. We have

$$\begin{aligned}(PQ^*)_{\lambda,\lambda'} &= \sum_{\lambda''} \frac{((t^2+\sigma(\lambda,\xi))\hat{\psi}_\lambda \,|\, \hat{\psi}_{\lambda''})_{L^2}}{\theta_\lambda\theta_{\lambda''}} \\ &\qquad\cdot \theta_{\lambda''}\theta_{\lambda'}\Big(\hat{\psi}_{\lambda''}\,\Big|\,\frac{1}{t^2+\sigma(\lambda',\xi)}\,\hat{\psi}_{\lambda'}\Big)_{L^2} \\ &= \sum_{\lambda''}\frac{\theta_{\lambda'}}{\theta_\lambda}\,((t^2+\sigma(\lambda,\xi))\,\hat{\psi}_\lambda \,|\, \hat{\psi}_{\lambda''})_{L^2} \\ &\qquad\cdot\Big(\hat{\psi}_{\lambda''}\,\Big|\,\frac{1}{t^2+\sigma(\lambda',\xi)}\,\hat{\psi}_{\lambda'}\Big)_{L^2} \\ &= \frac{\theta_{\lambda'}}{\theta_\lambda}\int\frac{t^2+\sigma(\lambda,\xi)}{t^2+\sigma(\lambda',\xi)}\,\hat{\psi}_\lambda(\xi)\,\overline{\hat{\psi}}_{\lambda''}(\xi)\,d\xi\,.\end{aligned}$$

If $|j-j'|\geq 2$, $(PQ^*)_{\lambda,\lambda'}=0$ because the supports of $\hat{\psi}_\lambda$ and $\hat{\psi}_{\lambda'}$ are disjoint. If $\lambda=\lambda'$,

$$(PQ^*)_{\lambda,\lambda'} = \int \hat{\psi}_\lambda(\xi)\,\overline{\hat{\psi}}_\lambda(\xi)\,d\xi = 1\,.$$

The remaining case is thus $|j-j'|<2$, $\lambda\neq\lambda'$. Since we can suppose that (54) holds, $\theta_{\lambda'}/\theta_\lambda$ is of the order of magnitude of 1, and we have to estimate

$$\begin{aligned}\chi_{\lambda,\lambda'} &= \int\frac{t^2+\sigma(\lambda,\xi)}{t^2+\sigma(\lambda',\xi)}\,\hat{\psi}_\lambda(\xi)\,\overline{\hat{\psi}}_{\lambda''}(\xi)\,d\xi \\ &= \int\frac{\sigma(\lambda,\xi)-\sigma(\lambda',\xi)}{t^2+\sigma(\lambda',\xi)}\,\hat{\psi}_\lambda(\xi)\,\overline{\hat{\psi}}_{\lambda''}(\xi)\,d\xi\,,\end{aligned}$$

(because of the orthogonality of the wavelets); but

$$\begin{aligned}|\chi_{\lambda,\lambda'}| &\leq C\int\frac{|\lambda-\lambda'|^\varepsilon(1+|\xi|)^{m+\varepsilon'}}{(1+|\xi|)^m}\,\Big|\hat{\psi}_\lambda\Big(\frac{\xi}{2^j}\Big)\Big|\,\Big|\hat{\psi}_\lambda\Big(\frac{\xi}{2^{j'}}\Big)\Big|\,d\xi \\ &\leq C\,|k-k'|^\varepsilon\,2^{j(\varepsilon'-\varepsilon)}\,.\end{aligned}$$

This proves Lemma 2.4.

END OF THE PROOF OF PROPOSITION 2.3. From Lemma 2.4, $\overline{N}Q^* - \mathrm{Id} \in \mathcal{I}^{\gamma'}$, for all $\gamma' < \delta(m,\gamma)$. From lemma 2.1, $\overline{N}$ is invertible on l^2. Using Corollary 2.1, $\overline{N}^{-1} \in \mathcal{M}^{\gamma'}$ and $\overline{N}^{-1} - Q^*$ belongs to $\mathcal{I}^{\gamma'}$, for all $\gamma' < \delta(m,\gamma)$, hence Proposition 2.3.

2.5. Properties of the Φ_λ.

Let us check that Theorems 1.1, 1.2 and Proposition 1.2 are a direct consequence of the results given in the previous section. We first prove Theorem 1.1 which gives the localization and regularity of the wavelets Φ_λ. Recall that

$$\Phi_\lambda = \sum_{\lambda'} M^{-1/2}_{\lambda,\lambda'} \psi_{\lambda'} \ ,$$
$$|M^{-1/2}_{\lambda,\lambda'}| \le C\,\omega_\gamma(\lambda,\lambda')\,2^{-m\sup\{j,j'\}/2}\,,$$

thus

$$|\partial^s \Phi_\lambda(x)| \le C \sum_{\lambda'} \omega_\gamma(\lambda,\lambda')\,2^{-m\sup\{j,j'\}/2}|\partial^s\psi_{\lambda'}(x)|\,,$$

where

$$\partial^s\psi_\lambda(x) = 2^{dj/2}(\partial^s\psi)(2^jx-k)\,.$$

If $|s| < m/2$, we have

$$|\partial^s\Phi_\lambda(x)| \le C\,2^{-(m/2-|\alpha|)j}\sum_{\lambda'}\tilde{\omega}_\gamma(\lambda,\lambda')\,|\partial^s\psi_{\lambda'}(x)|\,.$$

Since $\partial^s\psi_\lambda$ are vaguelettes and $\omega_\gamma \in \mathcal{M}^\gamma$, using standard calculations explicited in [30], we deduce (18) and (19).

If $|s| > m/2$,

$$|\partial^s\phi_\lambda(x)| \le C\,2^{-(m/2-|s|)j} \cdot \sum_{\lambda'} 2^{(m/2-|s|)j}\omega_\gamma(\lambda,\lambda')\,(1+|j-j'|)\,2^{(m/2-|s|)j'}\partial^s\psi_\lambda\ .$$

As the matrix $2^{(m/2-|s|)j}\omega_\gamma(\lambda,\lambda')$ belongs to $\mathcal{M}^{\gamma-|s|-m/2}$ (20) and (21) follow.

As regards Theorem 1.2 we deduce from Lemma 2.4

$$M_{\lambda,\lambda'}^{-1/2} - \langle 2^{mj/2} g_\lambda \mid \psi_{\lambda'} \rangle = \alpha_{\lambda,\lambda'} \ ,$$

where the matrix $(2^{m \sup(j,j')/2}\alpha_{\lambda,\lambda'})$ belongs to $\mathcal{I}^{\gamma'}$ for all $\gamma' < \delta(m,\gamma)$. The inequality of Theorem 1.2 is now straightforward.

As an application of the smoothness and decay properties of the wavelets, we now prove Proposition 1.2. We use the notation $\mathcal{A}(X_A, f)$ for the random variable associated to the function f by the isomorphism $H_A \to \mathcal{H}$, see (1).

On account of Theorem 1.1 the results i), ii) can be proven exactly as in [3]. For the third result, we can use the following wavelet criterium (see [30]) for Besov spaces: if $(\psi_\lambda)_{\lambda\in\Lambda}$ is a wavelet basis of $L^2(\mathbb{R}^d)$, the function $f = \sum_{\lambda\in\Lambda} \alpha_\lambda \psi_\lambda$ belongs to the Besov space $B^s_{p,q}$ if and only if the sequence $\{2^{j(d(1/2-1/p)+s)}(\sum_{j_\lambda=j} |\alpha_\lambda|^p)^{1/p}\}_j$ belongs to l^q.

As the functions $2^{mj/2}\Phi_\lambda$ define a Riesz basis of $L^2(\mathbb{R}^d)$, see Theorem 1.2, and satisfy wavelet localization properties, see Theorem 1.1, we have only to show that

$$\sum_{j\geq J} 2^{jq(s+d/2-d/p-m/2)} \Big(\sum_{\lambda\in U, j_\lambda = j} |\mathcal{A}(X_A, \Psi_\lambda)|^p \Big)^{q/p} < \infty \,,$$

with probability one. The domain U being bounded, the cardinal of $\{\lambda \in U,\ j_\lambda = j\}$ is of order 2^{jd} so that we get this inequality as consequence of the Borel-Cantelli Lemma when $s + d/2 - m/2 < 0$.

2.6. Equivalence in law of X_A and X_{A_g}.

PROOF OF PROPOSITION 1.1. Let g be the function defined in (13) and A_g be the operator defined in (14). The symbol σ_g of A_g is given by

$$\sigma_g(x,\xi) = g(\xi) + (1 - g(\xi))^2 \sigma(x,\xi) + r(x,\xi) \,,$$

with $r(x,\xi)$ a regularizing kernel. It is easy to check that σ_g fulfills the conditions of HA (m,γ). Moreover

$$C_1 \int |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \leq C_1 (A(f) \mid f)_{L^2} \int (1 + |\xi|^{2s}) \, |\hat{f}(\xi)|^2 \, d\xi \,,$$

hence the following equivalences

$$\begin{aligned}(A_g(f)\mid f)_{L^2} &\sim \|(\mathrm{Id}-G)f\|^2_{\dot H^s} + (G(f)\mid f)_{L^2}\\ &\sim \int (1-g(\xi))\,|\xi|^{2s}|\hat f(\xi)|^2\,d\xi + \int g(\xi)\,|\hat f(\xi)|^2\,d\xi\\ &\sim \int (1+|\xi|)^{2s}|\hat f(\xi)|^2\,d\xi\,.\end{aligned}$$

Using the notation $C(x,y)$ for the kernel of an operator C, we can write

$$\begin{aligned}X_{A_g}(x) &= \int_{\mathbb{R}^d} A_g^{-1/2}(x,y)\,dW(y)\,,\\ X_A(x) &= \int_{\mathbb{R}^d} A^{-1/2}(x,y)\,dW(y)\,,\end{aligned}$$

where $W(dy)$ denotes the brownian standard measure on $\mathbb{R}^d$.

In order to prove the equivalence of laws Law $(X_{A_g}|_U)$ = Law $(X_A|_U)$ for every bounded open subset U of $\mathbb{R}^d$ we apply Theorem 8.6 of [32]. Therefore, we will check that

i) $C_U(x,y) := A^{-1}|_{U\times U}(x,y) - A_g^{-1}|_{U\times U}(x,y) \in H_{A_g}^{\otimes 2}(U\times U)$,

ii) -1 is not an eigenvalue of $C_U : H_{A_g} \to H_{A_g}$.

Let us consider the operator B defined in (15). As the function g belongs to $\mathcal{D}(\mathbb{R}^d)$ we know that B is a regularizing operator

$$A^{-1} - A_g^{-1} = A_g^{-1}((I+BA_g^{-1})^{-1} - I)$$

and

$$((I+BA_g^{-1})^{-1} = \sum_{n\geq 0}(-1)^n (BA_g^{-1})^n\,.$$

Now if we consider the restrictions to open bounded U which are small enough, the last serie converges and the operator $A^{-1} - A_g^{-1}$ is of Hilbert-Schmidt type with a spectral radius less than 1, so that condition ii) is satisfied.

For the first condition, it is sufficient to show that

$$(-\Delta)_x^{m/4}(-\Delta)_y^{m/4}C(x,y) \in L^2_{\mathrm{loc}}(\mathbb{R}^d\otimes\mathbb{R}^d)\,.$$

But, as before

$$\begin{aligned}&(-\Delta)_x^{m/4}(-\Delta)_y^{m/4}C(x,y)\\ &\qquad = \sum_{n\geq 1}(-1)^n(-\Delta)_x^{m/4}A_g^{-1}(BA_g^{-1})^n(-\Delta)_x^{m/4}(x,y)\,,\end{aligned}$$

which converges in $L^2(U \times U)$ for U small enough, since A_g^{-1} is an operator of order $-m$ and B is regularizing.

Finally we obtain the equivalence of laws for every bounded open subset U of $\mathbb{R}^d$, by decomposing U in a finite number of small enough open subsets.

2.7. Quadratic variations.

In this paragraph, we will prove Lemma 1.2. For this purpose, we will study some quadratic variations related to wavelets.

For $y \in \mathbb{R}^d$ and $s \in \mathbb{N}^d$, $|s| = l$, we define

$$c_{1,s}^2(y) = \limsup_{x \to y} \frac{1}{|x-y|^\alpha \delta_\alpha(|x-y|)} \sqrt{\sum_\lambda |\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2}\ ,$$

where $\delta_\alpha(h) = 1$ if $\alpha < 1$ and $\delta_\alpha(h) = \sqrt{\log(1/h)}$ if $\alpha = 1$. If n is the integer defined by $2^{-n} \le |x-y| < 2^{1-n}$ we deduce from (18)

a) for $j \le n$,

$$\sum_{k,l} |\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2 \le C\, |x-y|^2\, 2^{2(1-\alpha)j}\ ,$$

b) for $j > n$,

$$\sum_{k,l} |\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2 \le C\, 2^{-2j\alpha}\ .$$

Summing up these inequalities for $j \ge 0$ yields

$$\sum_\lambda |\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2 \le C\, (u_{\alpha,n}|x-y|^2 + 2^{-2n\alpha})\ ,$$

with $u_{\alpha,n} = n$ (respectively $2^{2(1-\alpha)n}$) if $\alpha = 1$ (respectively < 1). As $2^{-n} \le |x-y| < 2^{1-n}$,

$$c_{1,s}^2(y) \le C < \infty\ , \qquad \text{for all } y \in \mathbb{R}^d\ .$$

Let us now show that $c_{1,s}^2$ is Hölderian of order $\varepsilon'' := \varepsilon - \varepsilon'$, where ε and ε' are defined in (7). Let us distinguish two cases.

Case 1. $\alpha < 1$.

For $\varepsilon > 0$ fixed, using as above the results of Theorem (1.1), we have

$$\sum_{j \le n - \log_2(1/\varepsilon)} \frac{|\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2}{|x-y|^{2\alpha}} \le C\,\varepsilon^{2(1-\alpha)}\,, \tag{56}$$

$$\sum_{j \ge n + \log_2(1/\varepsilon)} \frac{|\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2}{|x-y|^{2\alpha}} \le C\,\varepsilon^{2\alpha}\,, \tag{57}$$

$$\sum_{\lambda \in \Lambda_{y,n,\varepsilon}} \frac{|\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2}{|x-y|^{2\alpha}} \le C\,\varepsilon^{2(1-\alpha)}\,, \tag{58}$$

where in the last inequality

$$\Lambda_{y,n,\varepsilon} = \Big\{\lambda \in \Lambda \;:\; n - \log_2\Big(\frac{1}{\varepsilon}\Big) < j < n + \log_2\Big(\frac{1}{\varepsilon}\Big),\ |y-\lambda| > \frac{2^{-n}}{\varepsilon}\Big\}\,.$$

Let $\varepsilon = 1/n$; when n grows to ∞ the value of $c_{1,s}(y)$ is given by the sum restricted to $V_{y,n,\varepsilon} = \Lambda \backslash \Lambda_{y,n,\varepsilon}$. Define h_λ by its Fourier transform

$$\hat{h}_\lambda(\xi) = \frac{1}{\sqrt{\sigma(y,\xi)}}\,\hat{\psi}_\lambda(\xi)$$

and observe that the estimates (18), (19) hold for h_λ. Then inequalities (56), (57) and (58) hold with Φ_λ replaced by h_λ. Using Theorem 1.2, for n large enough

$$\sum_{\lambda \in V_{y,n,\varepsilon}} \big|\,|\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2 - |\partial^s g_\lambda(x) - \partial^s g_\lambda(y)|^2\big| \le C\,\varepsilon' |x-y|^{2\alpha}\,, \tag{59}$$

thus hypothesis HA (m,γ) implies that for $\lambda \in V_{y,n,\varepsilon}$ (59) holds for h_λ instead of g_λ. Thus

$$c_{1,s}^2(y) = \limsup_{x \to y} \sum_\lambda \frac{|\partial^s h_\lambda(x) - \partial^s h_\lambda(y)|^2}{|x-y|^{2\alpha}}\,.$$

Define the function H by

$$\hat{H}(\xi) = \frac{1}{\sqrt{\sigma(y,\xi)}}\,,$$

so that

$$|\partial^s h_\lambda(x) - \partial^s h_\lambda(y)|^2 = (\partial^s H(x - \cdot\,) - \partial^s H(y - \cdot\,) \mid \psi_\lambda)^2_{L^2} \,.$$

Since ψ_λ is an orthonormal basis of L^2,

$$\begin{aligned} c^2_{1,s}(y) &= \limsup_{x\to y} \frac{1}{|x-y|^{2\alpha}} \int \Big|(e^{ix\xi} - e^{iy\xi})\, \frac{(i\xi)^s}{\sqrt{\sigma(y,\xi)}}\Big|^2 \, d\xi \\ &= \limsup_{u\to 0} \frac{4}{|u|^{2\alpha}} \int \sin^2\Big(\frac{u\xi}{2}\Big) \frac{|\xi|^{2|s|}}{\sigma(y,\xi)}\, d\xi \,. \end{aligned} \tag{60}$$

We want to bound $I(z,u) - I(y,u)$ where

$$I(y,u) := \frac{1}{|u|^{2\alpha}} \int \sin^2\Big(\frac{u\xi}{2}\Big) \frac{|\xi|^{2|s|}}{\sigma(y,\xi)}\, d\xi \,.$$

Recalling that $|s| = l$, $2(l+\alpha) = m-d$, and using the change of variable $\zeta = |u|\xi$,

$$\begin{aligned} |I(z,u) - I(y,u)| \le \int & \frac{|\zeta/|u|\,|^m}{|\sigma(y,\zeta/|u|)\,\sigma(z,\zeta/|u|)|} \\ & \cdot |\sigma(y,\zeta/|u|) - \sigma(z,\zeta/|u|)|\, \frac{\sin^2(\zeta u/2|u|)}{|\zeta|^{d+2\alpha}}\, d\zeta \,. \end{aligned}$$

As

$$|\sigma(\,\cdot\,,\xi)| \ge C_1(1+|\xi|)^m \,, \tag{61}$$

$$|\sigma(y,\xi) - \sigma(z,\xi)| \le C_2(1+|\xi|)^m |y-z|^{\varepsilon''} \,, \tag{62}$$

we get, using $0 \le \sin^2(t) \le \min\{1, t^2\}$,

$$\begin{aligned} |I(z,u) - I(y,u)| &\le C_3\, |y-z|^{\varepsilon''} \Big(\int_0^1 r^{-2\alpha+1}\, dr + \int_1^\infty r^{-1-2\alpha}\, dr \Big) \\ &\le C_4\, |y-z|^{\varepsilon''} \end{aligned}$$

and thus the ε''-Hölder property for $c^2_{1,s}$.

Case 2. $\alpha = 1$.

The difference with the previous case is that

$$\sum_{k,l} \frac{|\partial^s \Phi_\lambda(x) - \partial^s \Phi_\lambda(y)|^2}{|x-y|^{2\alpha}}$$

no longer decreases (as j increases). We must replace the set $\Lambda_{y,n,\varepsilon}$ by

$$\tilde{\Lambda}_{y,n,\varepsilon} = \Big\{\lambda \in \Lambda \ : \ \sqrt{n} \le j < n + \log_2\Big(\frac{1}{\varepsilon}\Big),\ |y-\lambda| > \frac{2^{-n}}{\varepsilon}\Big\}$$

and define now $V_{y,n,\varepsilon} = \Lambda \backslash \tilde{\Lambda}_{y,n,\varepsilon}$. We can then proceed exactly as above and obtain

$$c_{1,s}^2(y) = \limsup_{x\to y} \sum_{\lambda} \frac{|\partial^s h_\lambda(x) - \partial^s h_\lambda(y)|^2}{|x-y|^2 \log\Big(\dfrac{1}{|x-y|}\Big)} = 4\,\tilde{I}(y,u)\,, \tag{63}$$

with

$$\tilde{I}(y,u) = \frac{1}{|u|^2\log(1/|u|)} \int \sin^2\Big(\frac{u\xi}{2}\Big)\frac{|\xi|^{2l}}{\sigma(y,\xi)}\,d\xi\,.$$

Using again (61), (62), we see that $c_{1,s}^2$ is Hölder of order ε''. Hence Lemma 1.2. Let us, still in the case $\alpha=1$, consider the expression

$$c_{2,s}^2(y) = \limsup_{x\to y} \frac{1}{|x-y|}\sqrt{\sum_{\lambda}\Big|\partial^s\Phi_\lambda(x) - 2\,\partial^s\Phi_\lambda\Big(\frac{x+y}{2}\Big) + \partial^s\Phi_\lambda(y)\Big|^2}\,,$$

with $y\in\mathbb{R}^d$, $s\in\mathbb{N}^d$, $|s|=l$. Using once again the bounds for $\partial^r\Phi_\lambda$ (with $|r|=l+2$) given in (18), we have for $n := [\log_2(|x-y|)]$

$$\sum_{k,l}\Big|\partial^s\Phi_\lambda(x) - 2\,\partial^s\Phi_\lambda\Big(\frac{x+y}{2}\Big) + \partial^s\Phi_\lambda(y)\Big|^2 \le C\,|x-y|^4\,2^{2j}\,,$$

if $j\le n$, and

$$\sum_{k,l}\Big|\partial^s\Phi_\lambda(x) - 2\,\partial^s\Phi_\lambda(\frac{x+y}{2}) + \partial^s\Phi_\lambda(y)\Big|^2 \le C\,2^{-2j}\,,$$

if $j>n$. Thus, after summation

$$c_{2,s}^2(y) \le C < \infty\,, \qquad \text{for all } y\in\mathbb{R}^d\,.$$

The required smoothness of $c_{2,s}$ follows as above.

3. Sums on the 2^d-adic tree.

The key idea to prove the law of uniform modulus in the critical case $\alpha = 1$ is to notice the relationship between the expression of the process X_A decomposed on the Φ_λ's and sums of normal random variable on the 2^d-adic tree.

As explained in Section 1, we have to study when $n = \log_2(|x - y|) \to \infty$, the following sums

$$\frac{|\partial^s X_A(x) - \partial^s X_A(y)|}{|x-y|} \simeq \text{Const.} \sum_{\lambda \in \Lambda \cap D, j_\lambda \leq n} \mathbf{1}_{c_\lambda(x)}\, \xi_\lambda \ ,$$

see (35). But the last sum is exactly the sum of Gaussian standard random variables on the paths of length n of a 2^d-adic tree. This will be performed after introducing some notations.

Let $\mathcal{T}$ be the 2^d-adic tree of root $*$ (each "father" has 2^d children). We denote by $\overline{L}$ the set $\{0,1\}^d$ and by L the set $\overline{L} \backslash \{(0,\dots,0)\}$. The elements of $\mathcal{T}$ can be coded in the following manner

$$t = t_0 t_1 t_2 \cdots t_j \ , \qquad \text{with } j \in \mathbb{N},\ t_0 = *,\ t_i \in \overline{L} \text{ for } i = 1,\dots,j\,.$$

The lenght j of t is denoted by $|t| := j$. For integers $0 \leq k \leq |t|$ we write

$$\underline{t}_k = t_0 t_1 \cdots t_k \ ,$$

so that the path from the root to t is

$$C_t^* = \{*, \dots, \underline{t}_k, \dots, t\}\,.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space on which is defined the i.i.d. family $\{\xi_t,\ t \in \mathcal{T}\}$ of Gaussian standard random variables. We set

$$S(t) = \sum_{s \in C_t^*} \xi_s \ ,$$

$$S_n^* = \max_{|t|=n} S(t)\,.$$

Proposition 3.1. *With the above notations the following limit holds*

$$\limsup_{n\to\infty} \frac{S_n^*}{n} = \sqrt{2\,d \log 2} \ ,$$

$\mathbb{P}$ *almost surely.*

The aim of this paragraph is to prove Proposition 3.1 and to give a corollary used in the proof of Theorem 1.3 in the critical case ($\alpha = 1$). A proof of Proposition 3.1 can be found in a recent work, see [10], where more general trees are considered. Our proof is very different and intends to show the production of asymptotic independent sums in the tree, so that we the study will be reduced to the i.i.d. case.

A few more notations will be needed.

The sub-tree of index j of $\mathcal{T}$ is defined by

$$\mathcal{T}_j = \{t \in \mathcal{T} \,:\, |t| \leq j\}\,.$$

The set of leaves of $\mathcal{T}_j$ is defined by

$$\wedge_j = \{t \in \mathcal{T} \,:\, |t| = j\}\,.$$

The tree is ordered by $u \succ t$ which means that $t \in C_u^*$. The cells $(q_j(t),\, t \in \mathcal{T}_j)$ are defined by

$$q_j(t) = \{s \in \wedge_j \,:\, s \succ t\}\,.$$

We now define the Haar basis of $l^2(\wedge_j)$.

For $l \in L$ and $\theta \in \overline{L}$, let

$$\varepsilon_l(\theta) = \prod_{i=1}^{d} \varepsilon_{l_i}(\theta_i)\,,$$

where $\varepsilon_0(0) = \varepsilon_0(1) = \varepsilon_1(0) = +1$, $\varepsilon_1(1) = -1$.

Let us now define the functions $\psi_{j,s}^l$ on $l^2(\wedge_j)$ by

$$\psi_{j,s}^l(t) = 2^{d(|s|-j)/2}\, \mathbf{1}_{\{t \in q_j(s) : t \neq s\}}\, \varepsilon_l(t_{|s|+1})\,.$$

If we add to the family $\{\psi_{j,s}^l\}_{s \in \delta_{j-1}, l \in L}$ the function ψ_ϕ which is identically 2^{dj} on $l^2(\wedge_j)$ and if we set $|\phi| = -1$, we obtain the following result whose proof is straightforward.

Lemma 3.1. *The family* $\{\psi_{j,t}^l\}_{-1 \leq t \leq j-1, l \in L}$ *is an orthonormal basis of* $l^2(\wedge_j)$.

3.1. Upper bound.

For $t, u \in \mathcal{T}$ let

$$\begin{aligned} tu &= *t_1 \cdots t_{|t|} u_1 \cdots u_{|u|} \,, \\ S_t(u) &= S(tu) - S(t) \,, \\ (S_t)^*_j &= \max_{|u|=j} S_t(u) \,, \end{aligned}$$

so that

$$S^*_j = \xi_* + \max_{|t|=1} \{S_t\}^*_{j-1} \,.$$

If $\mathbb{E}_{\xi_*}$ denotes the expectation with respect to the law of ξ_*, we get

$$\begin{aligned} \mathbb{P}\,(S^*_j < \theta) &= \mathbb{E}_{\xi_*}\Big(\prod_{|t|=1}^{d} \mathbb{P}\,(\xi_* + (S_t)^*_{j-1} < \theta)\Big) \\ &= \mathbb{E}_{\xi_*}(\mathbb{P}\,(\xi_* + S^*_{j-1} < \theta)^{2^d}) \\ &\geq \mathbb{P}\,(\xi_* + S^*_{j-1} < \theta)^{2^d} \qquad \text{(Jensen)} \end{aligned}$$

and by induction on j,

$$\mathbb{P}\,(S^*_j < \theta) \geq \mathbb{P}\,(\xi_* + \cdots + \xi_j < \theta)^{2^{jd}} \,.$$

Let η be a gaussian normal random variable and $\theta = \beta\,(j+1)$;

$$\begin{aligned} \mathbb{P}\,(S^*_j > \beta\,(j+1)) &\leq 1 - (1 - \mathbb{P}\,(\eta > \beta\sqrt{j+1}))^{2^{jd}} \\ &\leq 2^{jd}\,\mathbb{P}\,(\eta > \beta\sqrt{j+1}) \\ &\leq \frac{2^{jd}\, e^{-\beta^2 (j+1)}}{\sqrt{2 + \beta\,(j+1)}} \,, \end{aligned}$$

using a classical estimation on the gaussian tail.

Choosing $\beta > \sqrt{2\,d \log 2}$, we have

$$\sum_j \mathbb{P}\,(S^*_j > \beta\,(n+1)) < \infty \,,$$

and from the Borel-Cantelli Lemma we can conclude

$$\limsup_{j\to\infty} \frac{S^*_j}{j} \leq \sqrt{2\,d \log 2} \,, \tag{64}$$

$\mathbb{P}$ almost surely.

3.2. Lower bound.

Let $G_j(t,u) = \mathbb{E}[S(t)\,S(u)]$ be the covariance of S on $\wedge_j$. In the following lemma we give the spectral decomposition of G_j. We define

$$\alpha_s = \frac{2^{d(j-|s|)} - 1}{2^d - 1}\,.$$

Lemma 3.2. *For $t, u \in \wedge_j$ we have*

$$G_j(t,u) = \sum_{\substack{-1 \le |s| \le j-1 \\ l \in L}} \alpha_s\, \psi^l_{j,s}(t)\, \psi^l_{j,s}(u)\,. \tag{65}$$

This lemma is a direct consequence of the obvious formula

$$G_j(t,u) = \sum_{k=1}^{j} \delta_{\underline{t}_k, \underline{u}_k}\,,$$

where $\underline{t}_k = * t_1 t_2 \cdots t_k$, and δ is the Kroneker symbol.

Now we define the kernel $G_j^{-1/2}\,(t,u)$ by

$$G_j^{-1/2}(t,u) = \sum_{\substack{-1 \le |s| \le j-1 \\ l \in L}} \alpha_s^{-1/2}\, \psi^l_{j,s}(t)\, \psi^l_{j,s}(u)$$

and the random variables η_s by

$$\eta(s) := \sum_{t \in \wedge_j} G_j^{-1/2}(s,t)\, S(t)\,, \qquad s \in \wedge_j\,.$$

Lemma 3.3. *The family $\eta(s)$, $s \in \wedge_j$ is i.i.d. with common law $\mathcal{N}(0,1)$.*

The proof is immediate since in the gaussian centered case $\mathbb{E}\,(\xi\eta) = 0$ is equivalent to the independance of ξ and η.

Let us introduce some more notations. For $x > 0$ let $l(x) = [\log(x)/d\log 2] = [\log_{2^d}(x)]$, $\tilde{j} = j - l(j)$ and $\tilde{\eta}_j(s) = \sum_{t\in q_j(s)} \eta(t)/j$, the last expression being the arithmetic mean on $q_j(s)$ when $|s| = \tilde{j}$.

The upper bound will be obtained by proving

$$\limsup_{j\to\infty} \frac{1}{\tilde{j}} \max_{|s|=\tilde{j}} S(s) \geq \sqrt{2\,d\log 2} - \varepsilon\,, \qquad \text{for all } \varepsilon > 0\,. \tag{66}$$

First , we observe that

$$\tilde{\eta}_j(s) = \frac{1}{j} \sum_{t\in q_j(s)} \sum_{u\in\wedge_j} \sum_{\substack{|r|\leq j-1\\ l\in L}} \alpha_r^{-1/2}\, \psi_{j,r}^l(t)\, \psi_{j,r}^l(u)\, S(u)\,. \tag{67}$$

As $\sum_{t\in q_j(s)} \psi_{j,r}^l(t)$ is equal to $j\,\varepsilon_l(\underline{s}_{|r|+1})\, 2^{d(|r|-j)/2}$ or to 0 according to $(r \prec s,\ r\neq s)$ or not, the expression (67) can be simplified in

$$\tilde{\eta}_j(s) = \sum_{u\in\wedge_j} \sum_{\substack{r\prec s\\ l\in L}} \varepsilon_l(\underline{s}_{|r|+1})\, \alpha_r^{-1/2}\, 2^{d(|r|-j)/2}\, \psi_{j,r}^l(u)\, S(u)\,.$$

We consider now the decomposition $\tilde{\eta}_j(s) = \eta_j^{\varepsilon}(s) + \eta_j^0(s)$ with

$$\eta_j^{\varepsilon}(s) = \sum_{u\in q_j(\underline{s}_{j(\varepsilon)})} \sum_{\substack{l\in L\\ r\prec s}} \alpha_r^{-1/2}\, 2^{d(|r|-j)/2}\, \psi_{j,r}^l(u)\, S(u)\,,$$

$$\eta_j^0(s) = \sum_{u\in\wedge_j\setminus q_j(\underline{s}_{j(\varepsilon)})} \sum_{\substack{l\in L\\ r\prec s}} \varepsilon_l(\underline{s}_{|r|+1})\, \alpha_r^{-1/2}\, 2^{d(|r|-j)/2}\, \psi_{j,r}^l(u)\, S(u)$$

and $j(\varepsilon) = j - 2\,l\,(j/\varepsilon)$. Using the same cancellation property as above, the summation in η_j^0 can be restricted to $r \prec \underline{s}_{j(\varepsilon)}$. The following Lemma allows us to bound $|\eta_j^0|$.

Lemma 3.4. *For every $\varepsilon > 0$ there exists a random variable N and a constant C such that for all $s \in \wedge_{j(\varepsilon)}$*

$$\sum_{u\in\wedge_j} \sum_{\substack{l\in L\\ r\prec s}} |\alpha_r^{-1/2}\, 2^{d(|r|-j)/2}\, \psi_{j,r}^l(u)\, S(u)| \leq C\,\varepsilon\,, \quad \text{on } \{j \geq N\}\,, \tag{68}$$

$\mathbb{P}$ *almost surely.*

PROOF. Let $\theta > \sqrt{2\,d\log 2}$, and let N be some random variable such that $|S_j^*| \le \theta j$ on $\{N(\omega) < j\}$ which we determined during proof of the upper bound. From the inequalities $|\alpha_r^{-1/2}\psi_{j,r}^l(u)| \le 2^{d(|r|-j)}$ and $\operatorname{card}\{u \in \wedge_j \,:\, u \succ r\} \le 2^{d(j-|r|)}$, we get

$$\sum_{u\in\wedge_j}\sum_{\substack{l\in L\\ r\prec s}} |\alpha_r^{-1/2}\, 2^{d(|r|-j)/2}\, \psi_{j,r}^l(u)\, S(u)| \le \theta j \sum_{k=0}^{j(\varepsilon)} 2^{k-d/2} = C\,\varepsilon\,,$$

on $\{j > N(\omega)\}$. Hence Lemma 3.4.

Consider now the following decomposition

$$\eta_j^\varepsilon(s) = \sum_{\substack{v\succ \underline{s}_{j(\varepsilon)}\\ |v|=j-2l(j)}} B_v + C_v\ ,$$

where

$$B_v = \Big(\sum_{u\in q_j(v)}\sum_{l,r\prec s}\mu_{u,l,r}\Big)S(v)\,,$$

$$C_v = \sum_{u\in q_j(v)}\sum_{l,r\prec s}\mu_{u,l,r}S_v(u)\,,$$

and

$$\mu_{u,l,r} = \alpha_r^{-1/2}\,\varepsilon_l(\underline{s}_{|r|+1})\,2^{d(|r|-j)/2}\,\psi_{j,r}^l(u)\,.$$

Lemma 3.5.

$$\lim_{j\to\infty}\sum_{\substack{v\succ\underline{s}_{j(\varepsilon)}\\ |v|=j-2l(j)\\ v\neq\underline{s}_{j-2l(j)}}} B_v = 0\,,$$

$\mathbb{P}$ *almost surely.*

PROOF. The summation on r is in this case reduced to $r \prec \underline{s}_{(j-2l(j))}$. We have

$$\sum_{\substack{r\prec\underline{s}_{j-2l(j)}\\ l}} |\mu_{u,l,r}| \le \sum_{k=0}^{j-2l(j)} 2^{3(k-j)d/2} \le j^{-3}$$

and thus

(69)
$$\Big|\sum_{u\in q_j(v)}\ \sum_{\substack{l\\ r\prec \underline{s}_{j-2l(j)}}} \mu_{u,l,r}\Big| \le j^{-1}\,.$$

As we can bound the cardinal of $\{v \succ \underline{s}_{j(\varepsilon)},\ |v| = j-2\,l\,(j)\}$ by $2^{2dl(1/\varepsilon)}$, we get when $j\to\infty$

$$A := \sum_{\substack{v\succ \underline{s}_{j(\varepsilon)}\\ |v|=j-2l(j)\\ v\neq \underline{s}_{j-2l(j)}}}\ \sum_{\substack{u\in q_j(v)\\ r\prec \underline{s}_{j-2l(j)}\\ l}} \mu_{u,l,r}\, S(\underline{s}_{j(\varepsilon)}) \to 0\,,$$

$\mathbb{P}$ almost surely. We still have to study

$$R := \Big(\sum_{\substack{v\succ \underline{s}_{j(\varepsilon)}\\ |v|=j-2l(j)\\ v\neq \underline{s}_{j-2l(j)}}} B_v\Big) - A\,.$$

But

$$R = \Big(\sum_{\substack{u\in q_j(v)\\ r\prec \underline{s}_{j-2l(j)}\\ l}} \mu_{u,l,r}\Big)\Big(\sum_{\substack{v\succ \underline{s}_{j(\varepsilon)}\\ |v|=j-2l(j)\\ v\neq \underline{s}_{j-2l(j)}}} S_{\underline{s}_{j(\varepsilon)}}(v)\Big)$$

and using the independance of the random variable $S_{\underline{s}_{j(\varepsilon)}}(v)$ we have $\mathbb{E}(R^2) \le C\,j^{-2}$. The convergence we claimed is now clear.

Lemma 3.6. *The following limit holds*

$$\lim_{j\to\infty} \sum_{\substack{v\succ \underline{s}_{j(\varepsilon)}\\ |v|=j-2l(j)}} C_v = 0\,,$$

$\mathbb{P}$ *almost surely.*

PROOF. Using the definition of C_v we can write

$$\sum_{\substack{v\succ \underline{s}_{j(\varepsilon)}\\ |v|=j-2l(j)\\ v\neq \underline{s}_{j-2l(j)}}} C_v = Q_j + R_j\ ,$$

where

$$R_j = \sum_{u \in q_j(\underline{s}_{j-2l(j)})} \sum_{\substack{\underline{s}_{j-2l(j)} \prec r \prec s \\ l}} \mu_{u,l,r}\, S_{\underline{s}_{j-2l(j)}}(u)\,,$$

so that only $r \prec \underline{s}_{j-2l(j)}$ are involved in Q_j. We can proceed as in the preceding lemma to get $\lim_{j\to\infty} Q_j = 0$, $\mathbb{P}$ almost surely. Now we use the upper bound to obtain for j large

$$R_j \leq C\sqrt{l(j)}\Big(\sum_{k=j-2l(j)}^{j-l(j)} \operatorname{card}\{q_j(\underline{s}_k)\, 2^{3(k-j)d/2}\}\Big)$$

(recall $|s| = j - l(j)$ and $|\mu_{u,l,r}| \leq 2^{3(|r|-j)d/2}$) and then $R_j \leq C\sqrt{l(j)}/j$, hence the lemma.

It remains to estimate

$$B_j = \Big(\sum_{l,r} \sum_{u \in q_j(\underline{s}_{j-2Lj})} \mu_{u,l,r}\Big) S(\underline{s}_{j-2Lj})\,.$$

As the summation in r is reduced to $r \prec \underline{s}_{j-2Lj}$, we get $B_j = S(\underline{s}_{j-2Lj})$ $j\,(1+\varepsilon_j)/j$ where $\varepsilon_j \to 0$.

The previous Lemmas and estimations give us

$$\eta_j^{\varepsilon}(s) = \frac{S(\underline{s}_{j-2Lj})}{j}\,(1+T_j)\,,$$

with $\lim_{j\to\infty} T_j = 0$, $\mathbb{P}$ almost surely. and the lower bound is now a direct consequence of the following lemma

Lemma 3.7.

$$\limsup_{j\to\infty} \max_{|s|=j-Lj} |\tilde{\eta}_j(s)| = \sqrt{2\,d\log 2}\,.$$

PROOF. The random variables $\sqrt{j}\,\tilde{\eta}_j(s)$ are independant Gaussian centered and of variance 1; so that the lemma is a classical asymptotic result, see [32] for instance.

3.3. A corollary.

We can identify the 2^d-adic tree δ and D the set of dyadic points in $(0,1)^d$. Let c be a continuous function on $(0,1)^d$, and $(\xi_t)_{t\in Dy}$ an i.i.d. family of centered Gaussian random variables such that $\operatorname{var}(\xi_t) = c(t)$. We define the process Z_t by

$$Z_t = \sum_{s\in C_t^*} \xi_t \ .$$

Let $\overline{c} = \max_{t\in(0,1)^d} c(t)$.

Proposition 3.2.

$$\limsup_{j\to\infty} \frac{1}{j} \max_{|t|=j} |Z_t| = \overline{c}\,\sqrt{2\,d\log 2}\ , \tag{70}$$

$\mathbb{P}$ *almost surely.*

PROOF. The upper and lower bounds of the previous demonstration, $c\equiv 1$, can be adapted to the present case. We need only to change the constant of Lemma (3.7) which becomes $\overline{c}\,\sqrt{2\,d\log 2}$.

4. Regularity of Elliptic Gaussian Processes.

In this part we prove Theorems 1.3 and 1.7. Recall that here $m > d$ (then X_A is an ordinary Gaussian process), $l\in\mathbb{N}$ and $\alpha\in(0,1]$ are the numbers defined by $(m-d)/2 = l+\alpha$. Recall also that we can suppose that $\operatorname{HAS}(m,\gamma)$ hold.

We begin with the proof of Theorem 1.3. For the results of this section the process X_A is restricted to a bounded domain D. Without loss of generality, we suppose that $D=(0,1)^d$. We prove first the law of the uniform modulus with $l=0$, $\alpha=1$, then we study the case $l=0$, we prove the law of iterated logarithm (local modulus), when $\alpha=1$ and also when $\alpha<1$. Finally we explain how to get the results without restrictions on l.

As explained in Section 1, we will use the decomposition of X_A on a wavelet orthonormal basis of H_A. We introduce therefore a few more notations. If $\{\Phi_\lambda,\ \lambda\in\Lambda\}$ is the wavelet basis of the Hilbert space H_A, given by Theorem 1.1, for each $f\in H_A$,

$$f = \sum_{\lambda\in\Lambda} f_\lambda \Phi_\lambda\ , \qquad \text{with } f_\lambda := \mathcal{A}(f,\Phi_\lambda)\,.$$

If g denotes a strictly increasing function of $\mathbb{N}$ in $\mathbb{R}_+$ (which will be later chosen), we define the functions $\tilde{f}, f_n$, with $\lambda = (j,k,l)$, by

$$\tilde{f}(x) = \sum_{j\geq 0} \sum_{|x-\lambda|\leq 2^{-j} g(j)\sqrt{j}} \Phi_\lambda(x)\, f_\lambda\,, \tag{71}$$

$$f_n(x) = \sum_{0\leq j\leq n} \Phi_\lambda(x)\, f_\lambda\,, \tag{72}$$

and in addition,

$$\tilde{R}f = f - \tilde{f}\,, \qquad R_n f = f - f_n\,. \tag{73}$$

We need another operation which will perform averages. Recall that c_λ is the dyadic cell with center λ and side length 2^{-j}. For $f \in L^1_{\mathrm{loc}}(R^d)$, let

$$\overline{f}(\lambda) = 2^{jd} \int_{c_\lambda} f(x)\, dx\,. \tag{74}$$

4.1. Law of the uniform modulus when $l = 0$, $\alpha = 1$.

The main idea is to make reductions in order to be able to use Lemma 3.1 and its corollary. This is done with the help of the projectors defined above. Let $\tilde{X}_n$ be the process defined by (71), (72) and define

$$\Delta_n = \{(x,y) \in D \times D \,:\, 2^{n-1} < |x-y| \leq 2^{-n}\}\,. \tag{75}$$

Let us explain the reductions we plan to do.

First reduction. We will prove

$$\begin{aligned}\limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} &\frac{|\tilde{X}_n(x) - \tilde{X}_n(y)|}{|x-y|\sqrt{\log|x-y|^{-1}}} \\ &= \limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{|X(x) - X(y)|}{|x-y|\sqrt{\log|x-y|^{-1}}}\,,\end{aligned} \tag{76}$$

$\mathbb{P}$ almost surely. That is,

$$\limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{|R_n X(x) - R_n X(y)|}{|x-y|\sqrt{\log|x-y|^{-1}}} = 0\,, \tag{77}$$

$\mathbb{P}$ almost surely, and

$$\limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{|\tilde{R}X_n(x) - \tilde{R}X_n(y)|}{|x-y|\sqrt{\log|x-y|^{-1}}} = 0\,, \tag{78}$$

$\mathbb{P}$ almost surely.

Second reduction. In order to describe this second reduction we must first introduce some additional notations. For $n \in \mathbb{N}$ let us define $\overline{n} \in \mathbb{N}$ and the set $\overline{\Lambda}_n \subset \Lambda$ by

$$\overline{n} = \left[\log \frac{2^n}{g(n)\sqrt{n}}\right], \qquad \overline{\Lambda}_n = \{\lambda \in \Lambda \,:\, j_\lambda = \overline{n}\}\,. \tag{79}$$

Now if $\mu \in (0,1)$, the integers n_μ and m_μ and the set Λ_n^μ are defined by

$$n_\mu = [(1-\mu)\,n]\,, \qquad m_\mu = [\mu\, n]\,, \qquad \Lambda_n^\mu = \{\lambda \in \Lambda \,:\; j_\lambda = m_\mu\}\,.$$

Denote by $\overline{Q}_n$ (respectively Q_n^μ) the set of dyadic cells $\{\overline{c}_\lambda,\, j_\lambda = \overline{n}\}$ (respectively $\{c_n^\mu,\, j_\lambda = m_\mu\}$). In a c_n^μ-cell there are $2^{\overline{n}-m_\mu}\overline{c}_n$-cells.

Remark 5.1. Let n^0 be the integer defined by

$$\frac{\log(g(n^0)^2 n^0)}{n^0} < 2\,\mu < \frac{\log(g(n^0+1))^2(n^0+1)}{n^0+1}\,, \tag{80}$$

then $\overline{n} > m_\mu$, for all $n \geq n^0$.

When $(x,x') \in \Delta_n$, $(y,y') \in \Delta_n$ and $|x-y| > 2^{-m_\mu}$, the random variables $(\tilde{X}_n(x) - \tilde{X}_n(x'))$ and $(\tilde{X}_n(y) - \tilde{X}_n(y'))$ are conditionally independent knowing $\sigma\{\xi_\lambda,\, j_\lambda \leq n_0\}$. Now for every $\lambda \in \Lambda$, the neighbourhood $\vee(\lambda)$ of λ is defined by

$$\vee(\lambda) = \{\lambda' \in \Lambda \,:\, j_\lambda = j_{\lambda'} \text{ and } \partial c_{\lambda'} \cap \partial c_\lambda \neq \varnothing\}\,. \tag{81}$$

Then if $\overline{\tilde{X}}_n(\lambda)$ is defined as in (74), let $d_n(\lambda)$ be defined by

$$d_n(\lambda) = \max_{\lambda' \in \vee(\lambda)} \frac{|\overline{\tilde{X}}_n(\lambda) - \overline{\tilde{X}}_n(\lambda')|}{|\lambda - \lambda'|}\,. \tag{82}$$

Thanks to Remark 5.1, we are now in the situation of applying Proposition 3.2 and the second reduction consists in proving that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \max_{\lambda \in \overline{\Lambda}_n} d_n(\lambda) \\ &\qquad = \limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{|\tilde{X}_n(x) - \tilde{X}_n(y)|}{|x-y|\sqrt{\log|x-y|^{-1}}}\,, \end{aligned} \tag{83}$$

$\mathbb{P}$ almost surely, and

$$\limsup_{n\to\infty} \frac{1}{n} \max_{\lambda\in\overline{\Lambda}_n} \; d_n(\lambda) = \sqrt{2\,d}\, C_D \;, \tag{84}$$

$\mathbb{P}$ almost surely, with

$$C_D^2 = \limsup_{\substack{x,t\in D\\ |x-y|\to 0}} \frac{\mathbb{E}[(X(x)-X(y))^2]}{|x-y|^2 \log|x-y|^{-1}} \;.$$

Step 1. Proof of (84).

As we have seen, if $j_\lambda = m_\mu$ there is $2^{\overline{n}-m_\mu}$ cells of $\overline{Q}_n$ in each q_λ^μ cell. Let K_n^μ be the set $\{1,\ldots,2^{\overline{n}-m_\mu}\}$ and if $\lambda \in \Lambda_n^\mu$ let $i(\lambda)$ be the position in c_n^μ of the i^{rd} cell c_λ^μ of $\overline{Q}_n \cap c_n^\mu$ (with the abuse $\overline{Q}_n \cap c_n^\mu = \{c \in \overline{Q}_n \;:\; \text{exists } \tilde{c} \in c_n^\mu \text{ and } c \subset \tilde{c}\}$). Let

$$\sigma^2(\lambda,\lambda') = \mathbb{E}\Big[\frac{(\overline{\tilde{X}}_n(\lambda) - \overline{\tilde{X}}_n(\lambda'))^2}{|\lambda-\lambda'|^2 \log|\lambda-\lambda'|}\Big]$$

and

$$\sigma^2(\lambda) = \max_{\lambda'\in\vee(\lambda)} \sigma^2(\lambda,\lambda') \,.$$

On the other hand, let us define functions $\{\phi_k(\lambda,\lambda'),\, k=1,\ldots,n\}$ and random variables $\{\eta_k(\lambda),\, k=1,\ldots,n\}$ such that

$$\begin{aligned}\frac{\big(\overline{\tilde{X}}_n(\lambda) - \overline{\tilde{X}}_n(\lambda')\big)}{|\lambda-\lambda'|} &= \sum_{k\le n} \sum_{|x-r|\le 2^{-kd} g(k)\sqrt{k}} \Big(\frac{\overline{\Phi}_r(\lambda) - \overline{\Phi}_r(\lambda')}{|\lambda-\lambda'|}\Big)\xi_r \\ &:= \sum_{k\le n} \phi_k(\lambda,\lambda')\,\eta_k(\lambda)\,.\end{aligned}$$

Let n^0 be the integer of Remark 5.1. It is clear that $\{\eta_\lambda : j_\lambda > n_0\}$ is an i.i.d. family of Gaussian normal random variables. Furthermore,

$$\sum_{k=1}^{n} \phi_k^2(\lambda,\lambda') = \sigma^2(\lambda,\lambda')\,.$$

Hence, if μ is fixed and n is large enough we will be in the situation of Proposition 3.2. Therefore, for every sequence $\{i_n\}_{n\ge 0}$ such that $i_n \in K_n^\mu$,

$$\limsup_{n\to\infty} \frac{1}{n} \max_{|\nu|=n_\mu} d_n(i_n(\nu)) = \sqrt{(1-\mu)\,2\log d}\; C_D \;,$$

$\mathbb{P}$ almost surely. Let $\mu \to 0$, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \max_{j_\lambda = n} d_n(\lambda) \geq \sqrt{2\,d}\, C_D\ ,$$

$\mathbb{P}$ almost surely. Since the upper bound is easily deduced from the one of Proposition 3.2, we have proved (84).

Step 2. Proof of (83).

Here we must go from averages to pointwise values. Let

$$S(\lambda, \lambda'; x, y) = \frac{1}{n}\, d_n(\lambda, \lambda') - \frac{\tilde{X}_n(x) - \tilde{X}_n(y)}{|x-y|\sqrt{\log|x-y|^{-1}}}\ ,$$

for $x \in c_\lambda$, $(x,y) \in \Delta_n$. In order to prove (83) it is enough to show that,

$$\limsup_{n\to\infty} \max_{\substack{\lambda\in\Lambda_n \\ \lambda'\in\vee(\lambda) \\ x\in c_\lambda \\ (x,y)\in\Delta_n}} |S(\lambda, \lambda'; x, y)| = 0\,, \tag{85}$$

$\mathbb{P}$ almost surely, but

$$\begin{aligned}
&|S(\lambda, \lambda'; x, y)| \\
&\quad = \frac{1}{n}\Big|\sum_{k\leq n} \sum_{\substack{j_r = k \\ 2^k|r-\lambda|\leq g(k)\sqrt{k}}} \Big(2^{nd}\int_{c_\lambda} \Big(\frac{\Phi_r(x') - \Phi_r(x)}{2^{-n}}\Big)\, dx' \\
&\qquad\qquad - 2^{nd}\int_{c_\lambda} \Big(\frac{\Phi_r(y') - \Phi_r(y)}{2^{-n}}\Big)\, dy'\Big)\xi_r\Big|\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{c_\lambda} (\Phi_r(x') - \Phi_r(x))\, dx' - \int_{c_\lambda} (\Phi_r(y') - \Phi_r(y))\, dy' \\
&\qquad = C\Big(2^{nd}\int_{c_\lambda} D^2\Phi_r(\lambda, \lambda')\,(dx, dx') + \varepsilon\Big)\,,
\end{aligned}$$

for $C > 0$. Denote by A_r this quantity; using Theorem 1.1,

$$|A_r| \leq C\, 2^{-n}\, 2^{(d/2 - m/2 + 2)r}\, 2^{-2n} = C\, 2^{(d/2 - m/2 - 1)n}\ .$$

Thanks to a result proved in [3] we have

$$\sum_{2^k|\lambda-r|\le g(k)\sqrt{k}} |A_r| \le 4\,n\,,$$

if n is large enough. So

$$|S(\lambda,\lambda';x,y)| \le 2\,C\,\frac{2^{-n}}{n}\sum_{k\le n}\sqrt{k}\,2^k\,2^{(d/2-m/2-1)n}\,,$$

if n is large enough. Then (83) follows.

Step 3. Proof of (76).

It is sufficient to prove (77) and (78). Let us begin by (78). Taking into account that $|\xi_r| \le \sqrt{2}\,\sqrt{r}$ if $|r|$ is large enough, and that

$$\sum_{|l|\ge g(k)\sqrt{k}} \frac{1}{(1+|l|)^{d+1}} \simeq \frac{1}{g(k)\sqrt{k}}\,,$$

(78) becomes

$$\frac{|\tilde{R}X_n(s)-\tilde{R}X_n(y)|}{|x-y|\sqrt{\log|x-y|^{-1}}} \le \frac{1}{n}\sum_{1}^{n}\frac{1}{g(k)}\,,$$

but with a correct choice of function g we can deduce (78).

To prove (77) we use the same method as above. For $\beta>0$ using again Theorem 1.1, (77) becomes

$$\begin{aligned}\frac{|R_nX(x)-R_nX(y)|}{|x-y|\sqrt{\log|x-y|^{-1}}} &\le \frac{1}{n}\sum_{k=n}^{n+\beta\log n}\sqrt{k}+2\,\frac{2^n}{n}\sqrt{k}\,2^{-k}\\ &\le \frac{2}{n\,\beta}+\frac{3}{2}\,\beta\,\frac{\log n}{\sqrt{n}}\,,\end{aligned}$$

if n is large enough; in the first part of the proof of the upper bound we have used (2.2) with $|\alpha|=1$ and with $|\alpha|=0$ in the second one, where furthermore the inequality $|x-y|<|x|+|y|$ has been used, hence the factor 2.

Conclusion. The proof of the law of the uniform modulus (30) results then from (84), (83) and (76).

4.2. Law of the uniform modulus when $l = 0$, $0 < \alpha < 1$.

In the present case, the reductions we perform will lead to a Brownian motion-like situation (*cf.* Introduction) or more precisely [3]-like situation. We set $\ell_\alpha(r) = r^\alpha$.

First reduction. Let $\beta > 0$, let us introduce the integers $n_\beta^\pm = n \pm [\beta \log n]$. We have to prove

$$\limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{\left|X_{n_\beta^-}(x) - X_{n_\beta^-}(y)\right|}{\ell_\alpha(|x-y|)} = 0\,, \tag{86}$$

$\mathbb{P}$ almost surely, and

$$\limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{\left|R_{n_\beta^+}X(x) - R_{n_\beta^+}X(y)\right|}{\ell_\alpha(|x-y|)} = 0\,, \tag{87}$$

$\mathbb{P}$ almost surely. That is to say, low and high scales have no contribution to the result. Let

$$S_n(x,y) = \left(R_{n_\beta^-}X(x) - R_{n_\beta^-}X(y)\right) - \left(R_{n_\beta^+}X(x) - R_{n_\beta^+}X(y)\right),$$

corresponding to the terms of scale between n_β^- and n_β^+;

$$\limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{|S_n(x,y)|}{l_\alpha(|x-y|)} = \limsup_{(x,y)\in\Delta_n} \frac{|X(x)-X(y)|}{l_\alpha(|x-y|)}\;, \tag{88}$$

$\mathbb{P}$ almost surely, is a consequence of (86) and (87).

Second reduction. The second reduction will lead to a situation where the wavelets will be thought of as compactly supported. We have to show that

$$\limsup_{n\to} \max_{(x,y)\in\Delta_n} \frac{|S_n(x,y)|}{l_\alpha(|x-y|)} = \limsup_{(x,y)\in\Delta_n} \frac{|\tilde{S}_n(x,y)|}{l_\alpha(|x-y|)}\;,$$

$\mathbb{P}$ almost surely, where $\tilde{S}(x,y)$ is obtained by applying the operator (71) in each variable x and y.

Third reduction. The third reduction consists in defining a sequence of partitions $(P_n)_{n\geq 0}$ of the domain D such that if q and q' are two elements of P_n sufficiently far away then $\{\tilde{S}_n(x,y) : (x,y) \in q \times q\}$ and $\{\tilde{S}_n(x,y) : (x,y) \in q' \times q'\}$ become independent.

The integer $\overline{n}$ is, as before, equal to $[\log(2^n/g(n)\sqrt{n})]$. When $s > 0$ let $\overline{n}_s = \overline{n}/n^s$. Consider the subsets

$$\overline{\Lambda}_n = \{\lambda \in \Lambda \,:\, j_\lambda = \overline{n}\}\,, \qquad \overline{\Lambda}_n^s = \{\lambda \in \Lambda \,:\, j_\lambda = \overline{n}_s\}\,.$$

Looking for the cells $\overline{q}_\lambda$ and q_λ^s if $\lambda \in \overline{\Lambda}_n$ and $\lambda \in \overline{\Lambda}_n^s$ respectively, we define $\tilde{q}_\lambda^s$ as the cell with the same center as q_λ^s, with faces parallel to the axes and with side lengths $2^{-n}(\overline{n} - \overline{n}_s)$. In these conditions, for all $s > 0$, by construction we have independence between $\sigma\{\tilde{S}_n(x,y) : (x,y) \in \tilde{q}_\lambda^s \times \tilde{q}_\lambda^s\}$ and $\sigma\{\tilde{S}_n(x,y) : (x,y) \in \tilde{q}'_\lambda(x,y) \times \tilde{q}'_\lambda(x,y)\}$ for $\lambda \neq \lambda'$, $j_\lambda = j_{\lambda'} = \overline{n}_s$.

Now let $\vee(\lambda)$ as before the set of neighbours of λ. We have to show that

$$\limsup_{n\to\infty} \max_{\substack{j_\lambda = \overline{n} - s\log\overline{n} \\ \lambda' \in \vee(\lambda)}} \frac{|\overline{S}_n(\lambda,\lambda')|}{2^{-\alpha n}\sqrt{n\,d\log 2}} = \limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{|\tilde{S}_n(x,y)|}{l_\alpha(x,y)}\,, \tag{89}$$

$\mathbb{P}$ almost surely, and this is a consequence of

$$\limsup_{n\to\infty} \max_{\substack{j_\lambda = \overline{n} - s\log\overline{n} \\ (x,y)\in\Delta_n}} \left| \frac{\overline{S}_n(\lambda,\lambda')}{2^{-\alpha n}\sqrt{n\,d\log 2}} - \frac{\overline{S}_n(x,y)}{l_\alpha(|x-y|)} \right| = 0\,, \tag{90}$$

$\mathbb{P}$ almost surely. The proof of (90) is in every way analogous to the one of (83). Now the method of [3] can be directly used for showing

$$\limsup_{n\to\infty} \max_{j_\lambda = \overline{n} - s\log\overline{n}} \frac{|\overline{S}_n(\lambda,\lambda')|}{2^{-dn}\sqrt{n}} = \sqrt{d\log 2}\; C_D\,,$$

$\mathbb{P}$ almost surely, and also (86), (87). For the last results we use well known bounds for independant gaussian random variables and the inequalities (18), (19).

4.3. Law of uniform "Zygmund-class"-modulus when $l = 0$, $\alpha = 1$.

We can proceed as in the last paragraph, using (18), (19) to restrict the sum only to scales $j_\lambda \sim \log(|x-y|)$. Following the method of [3] the above reductions give the result.

4.4. Law of the iterated logarithm when $l = 0$, $\alpha = 1$.

Here we set $l_1^{(2)}(r) = |r|\sqrt{\log r^{-1} \log\log\log r^{-1}}$. To prove that for $y \in D$ we have

$$\limsup_{x\to y} \frac{|X(x)-X(y)|}{l_1^{(2)}(|x-y|)} = \sqrt{2}\, C(y)\,,$$

$\mathbb{P}$ almost surely. We consider reductions of the problem absolutely similar to the preceding ones. We will also use the well known result of Levy-Kinchin,

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n\log\log n}} \Big|\sum_{k=1}^{n} \xi_k\Big| = \sqrt{2}\,, \tag{91}$$

$\mathbb{P}$ almost surely, where $(\xi_k,\, k \in \mathbb{N})$ is an i.i.d. sequence of Gaussian normal random variables.

Hence, using the modulus $l_1^2(\rho)$ in place of $l_1(r) = |r|\log(r^{-1})$ it is possible to prove an inequality analogous to (77) and (78), therefore an equality similar to (76). In these conditions (83) will become

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{\sqrt{n\log\log n}} \Big(\max_{\lambda\in\overline{\Lambda}_k(y)} d_n(\lambda)\Big) \\ &\qquad = \limsup_{n\to\infty} \max_{(x,y)\in\Delta_n} \frac{|\tilde{X}_n(x)-\tilde{X}_n(y)|}{l_1^{(2)}(|x-y|)} \end{aligned} \tag{92}$$

where $\overline{\Lambda}_k(y) = \{\lambda \in \overline{\Lambda}_k \text{ and } 2^n d(\lambda, y) \le g(n)\sqrt{n}\}$.

Then using (91) the first member of (92) converges to $\sqrt{2}\,C(y)$ almost surely.

4.5. Law of the iterated logarithm when $l = 0$, $0 < \alpha < 1$.

To prove the law of the iterated logarithm we have only to show that

$$\limsup_{x \to y} \frac{|X(x) - X(y)|}{l_\alpha^{(2)}(|x-y|)} = \limsup_{n \to \infty} \frac{1}{\sqrt{\log n}} \max_{(x,y) \in \Delta_n} K \,, \tag{93}$$

$\mathbb{P}$ almost surely, with

$$K = 2^{\alpha n} \Big| \sum_{\substack{n - \beta \log\log n < k < n + \beta \log\log n \\ 2^k |y - r| < g(k)\sqrt{k}}} (\Phi_r(x) - \Phi_r(y))\, \xi_r \Big| \sqrt{2 \log 2}\, C_s(g) \,.$$

The same reductions as above show that we can use the proof of the same result given in [BJR] for the one dimensional case. Hence (93).

4.6. The laws of moduli when $l \neq 0$.

In order to end the proof of Theorem 1.3 we still have to consider the case $l \neq 0$.

Let s be a multi-index of length l. Let us set $Y(x) = \partial^s X(x)$. Thanks to Proposition 2.1,

$$Y(x) = \sum_{\lambda \in \Lambda} \partial^s \Phi_\lambda(x)\, \xi_\lambda :\simeq \sum_{\lambda \in \Lambda} \theta_\lambda\, \xi_\lambda \,.$$

Let $\tilde{A}$ be the elliptic operator defining the topology of the auto-reproducing Hilbert space H_y of Y. As θ_λ is an orthonormal basis of H_y, it follows that $\theta_\lambda = \tilde{A}^{-1/2} \psi_\lambda$; as $\partial^s \varphi_\lambda = \partial^s A^{-1/2} \psi_\lambda$, we get $\tilde{A}^{-1/2} = \partial^s A^{-1/2}$. So, the symbol σ_y of $\tilde{A}$ is of degree $m - 2l$; it satisfies Hypothesis HA$\,(m - 2l, \gamma)$. Therefore, performing the same calculus as above we obtain the theorem in the general case.

4.7. Approximation of elliptic gaussian generalized processes.

Let us now prove Theorem 1.6 which concerns generalized Gaussian processes. From the estimations of Theorem 1.1 we know there exists $r_d > 0$ such that

$$\text{i)} \qquad \lim_{h \to 0} \sum_{0 \le j_\lambda \le \log_2(h^{-1})} \frac{(\Phi_\lambda(x+h) - \Phi_\lambda(x))^2}{|h|^2 \log(|h|^{-1})} = r_d^2 \,, \tag{94}$$

when $d = 1$ or $d = 2$, and

$$\text{(95)} \qquad \text{ii)} \qquad \lim_{h \to 0} \sum_{0 \le j_\lambda \le \log_2(h^{-1})} \frac{(\Phi_\lambda(x+h) - \Phi_\lambda(x))^2}{|h|^2} = r_d^2\,,$$

if $d \geq 3$. Using these limits we can transpose the proofs we gave for the law of the uniform modulus to get the results of Theorem 1.6. Note that when $d = 1$, $d = 2$ then $\alpha = 1$ (critical case) so that $C_d = r_d \sqrt{2 \log 2}$. When $d \geq 3$, $\alpha = 1/2$ and $C_d = r_c \sqrt{2\, d \log 2}$.

4.8. Moduli of continuity for the multifractional Brownian motion.

We constructed a collection of wavelets ω_λ which, because of the decomposition (44), plays for the multifractional Brownian motion exactly the same role as the Φ_λ for Elliptic Processes. The proofs of regularity results for the multifractional Brownian motion are similar to [3] and we will just sketch them. We first prove "vaguelettes-type" localization estimates for the ω_λ defined in (43).

Proposition 4.1. *We assume the function a belongs to $C^r(\mathbb{R}^d, (0,1))$, $\sup a(x) \leq r$, $r > 0$, $(j,l) \neq (0,0)$ and $K \in \mathbb{N}$. Then there exists a constant C (which depends on K) such that*

$$\text{(96)} \qquad |\omega_\lambda(x)| \leq C\, 2^{-j\,a(x)} \Big(\frac{1}{(1 + |2^j x - k|)^K} + \frac{1}{(1+|k|)^K} \Big)$$

and

$$\begin{aligned} &|\omega_\lambda(x) - \omega_\lambda(y)| \\ \text{(97)} \qquad &\leq C\, 2^{-j a(x)} \Big(\frac{2^j |x-y| + j\, |a(x) - a(y)|}{(1 + |2^j x - k|)^K} + \frac{j\, |a(x) - a(y)|}{(1+|k|)^K} \Big)\,. \end{aligned}$$

PROOF. We want to bound

$$H = \int \frac{e^{ix\xi}}{|\xi|^{a(x)+d/2}}\, \hat{\psi}_\lambda(\xi)\, d\xi = 2^{-jd/2} \int \frac{e^{i(x-\lambda)\xi}}{|\xi|^{a(x)+d/2}}\, \hat{\psi}^{(l)}\Big(\frac{\xi}{2^j}\Big)\, d\xi\,.$$

Let us recall that the support of $\hat{\psi}$ is included in $\{\xi : 2\pi/3 \leq |\xi| \leq 8\pi/3\}$. Setting $\nu = \xi/2^j$ in the integral we get easily

$$|H| \leq c\, 2^{-j a(\lambda)}\,.$$

If this change of variable is made after K integrations by part in a direction where

$$|x-\lambda| \le d\,|x_p - \lambda_p|\,,$$

we get

$$H \le c\,2^{-j(K+a(x))}|x-\lambda|^{-K}\,.$$

From these two inequalities we deduce (96).

For the second result we write $\omega_\lambda(x) - \omega_\lambda(y) = R + S$, where

$$R = 2^{-jd/2}\int \frac{e^{i(x-\lambda)\xi}\,(e^{i(y-x)\xi}-1)}{|\xi|^{a(x)+d/2}}\,\hat{\psi}^{(l)}\Big(\frac{\xi}{2^j}\Big)\,d\xi$$

and

$$S = 2^{-jd/2}\int \frac{(e^{i(y)\xi}-1)\,e^{-i\xi\lambda}}{|\xi|^{a(x)+d/2}}\Big(\frac{1}{|\xi|^{a(y)-a(x)}}-1\Big)\hat{\psi}^{(l)}\Big(\frac{\xi}{2^j}\Big)\,d\xi\,.$$

To give a bound for R we use $|e^{ih\xi}-1| \le |h|\,|\xi|$ and proceed as in the proof of (96), so that

$$|R| \le c\,2^{-ja(x)}\,\frac{2^j\,|x-y|}{(1+|2^jx-k|)^K}\,.$$

Now we can split S as $S = S_1 + S_2$,

$$S_1 = 2^{-j(a(x)+d)}\int \frac{(e^{i(y)\xi}-1)\,e^{-i\xi\lambda}}{|\xi/2^j|^{a(x)+d/2}}\Big(\frac{1}{|\xi/2^j|^{a(y)-a(x)}-1}\Big)\hat{\psi}^{(l)}\Big(\frac{\xi}{2^j}\Big)\,d\xi\,,$$

$$S_2 = 2^{-jd}(2^{j(a(x)-a(y))}-1)\int \frac{(e^{i(y)\xi}-1)\,e^{-i\xi\lambda}}{|\xi/2^j|^{a(y)+d/2}}\,\hat{\psi}^{(l)}\Big(\frac{\xi}{2^j}\Big)\,d\xi\,.$$

With the same integrations by part, change of variable and using the inequality

$$r^{a(y)-a(x)} - 1 = O\big(|a(y)-a(x)|\log r\,\max\{r^{a(x)}, r^{a(y)}\}\big)\,,$$

we obtain

$$|S_i| \le c\,2^{-ja(x)}\,\frac{j\,|a(x)-a(y)|}{(1+|k|)^K}\,,$$

and then the last estimate of the Proposition holds.

Let us now prove the law of the uniform modulus. We use the decomposition (44) in order to estimate $(B_\alpha(x+h) - B_\alpha(x))$. Setting $n = [\log_2 |h|^{-1}]$ (so that $2^{-n-1} < |h| \leq 2^{-n}$), we separate the sum into four terms T_i which correspond to the cases

1) $j < n - \beta \log n$,

2) $j > n + \beta \log n$,

3) $n - \beta \log n \leq j \leq n + \beta \log n$, $j(a_E - a(\lambda)) \leq \delta \log j$,

4) $n - \beta \log n \leq j \leq n + \beta \log n$, $j(a_E - a(\lambda)) \geq \delta \log j$.

Using well known properties of an independant sequence of standard gaussian random variables as in [3], when $\beta(1-a_E) > 1/2$, we get from (97)

$$\lim_{h\to 0} |h|^{-a_E} T_1 = 0\,,$$

and from (96)

$$\lim_{h\to 0} |h|^{-a_E} T_2 = 0\,.$$

In the same way we deduce also from (97) that

$$\lim_{h\to 0} |h|^{-a_E} T_3 = 0\,.$$

The relevant contribution of the sum is given by T_4. Now, using the continuity of the function C_E and proceeding as in [3], we get

$$\limsup_{\substack{x,y\in E\\ |x-y|\to 0}} \frac{|B_a(x) - B_a(y)|}{|x-y|^{a_E}\sqrt{\log 1/|x-y|}} = C_E\sqrt{d}\,,$$

$\mathbb{P}$ almost surely. The proof of the law of the iterated logarithm follows exactly the corresponding proof for E.G.R.P. in the non-critical case.

The asymptotic self similarity of the Multifractional B.M. B_a is a straightforward application of the following Proposition. We define

$$\theta_\lambda = \int \frac{e^{ix\xi} - 1}{|\xi|^{a(\lambda)+d/2}}\, \hat{\psi}_\lambda(\xi)\, d\xi\,.$$

Proposition 4.2. *If the function a belongs to* $C^r(\mathbb{R}^d, (0,1))$, $(r > 0)$, *we have the following asymptotic behavior*

$$|\omega_\lambda(x) - \theta_\lambda(x)| \le c\, j\, |a(x) - a(\lambda)| \\ \cdot 2^{-j \min\{a(x), a(\lambda)\}} \Big(\frac{1}{(1 + |2^j x - k|)^K} + \frac{1}{(1+|k|)^K} \Big) .$$

The proof is along the lines of Proposition 4.1.

5. Scaling properties for Elliptic Gaussian processes.

In this part Theorems 1.4, 1.5 and Proposition 1.3 are proved. Recall that we want to study the local scaling properties for Elliptic Gaussian processes. They will be connected them with scaling properties of the associated symbols or wavelets. Consider a point x_0 in $\mathbb{R}^d$ which remains fixed for the whole paragraph. The whole-scale Littlewood-Paley basis (of L^2) is denoted by $\{\psi_\mu\}_{\mu \in \underline{\Lambda}}$, where $\underline{\Lambda} = \mathbb{Z} \times \mathbb{Z}^d \times L$.

Let s be a symbol on $\mathbb{R}^d$. We define when it makes sense the function g_λ^s by its Fourier transform

$$\hat{g_\lambda^s}(\xi) = \frac{\hat{\psi_\lambda}(\xi)}{\sqrt{s(x,\xi)}} \; .$$

5.1. Scaling properties for elliptic symbols.

We suppose here that the symbol σ fullfills hypotheses HA (m, γ). We consider only the case $m = d + 2\,\alpha$, $0 < \alpha \le 1$.

For $\rho > 0$ we set

$$\sigma^{x_0}(x, \xi) = \sigma(x_0 + x, \xi)\, , \qquad \sigma_\rho^{x_0}(x,\xi) = \rho^m \sigma^{x_0}\Big(\rho\, x, \frac{\xi}{\rho}\Big) .$$

Using the scaling properties of $\{\psi_\lambda\}_\lambda$, we have

$$g_\lambda^{\sigma_\rho^{x_0}}(x) = \rho^{-\alpha} g_{\lambda(\rho)}^{\sigma^{x_0}}(\rho\, x)\, , \tag{98}$$

if $\rho = 2^{-p}$, $\lambda(\rho) = 2^{-j-p}(k + l/2) = \rho\lambda$. The extension to ρ positive real is obvious.

Consequently, when $(\xi_\lambda)_{\lambda\in\underline{\Lambda}}$ is an i.i.d. standard gaussian family, we get

$$\begin{aligned}\rho^{-\alpha}\sum_{j_\lambda\geq 0}\big(g_\lambda^{\sigma^{x_0}}(x\,2^{-p})-g_\lambda^{\sigma^{x_0}}(0)\big)\xi_\lambda \\ =\sum_{j_\lambda\geq 0}\big(g_{\lambda(1/\rho)}^{\sigma_\rho^{x_0}}(x)-g_{\lambda(1/\rho)}^{\sigma_\rho^{x_0}}(0)\big)\xi_\lambda\,.\end{aligned}\tag{99}$$

This gives the following equality in law

$$\begin{aligned}\rho^{-\alpha}\sum_{j_\lambda\geq 0}\big(g_\lambda^{\sigma^{x_0}}(x\,2^{-p})-g_\lambda^{\sigma^{x_0}}(0)\big)\xi_\lambda \\ \overset{\text{(d)}}{=}\sum_{j_\mu\geq -p}\big(g_\mu^{\sigma_\rho^{x_0}}(x)-g_\mu^{\sigma_\rho^{x_0}}(0)\big)\xi_\mu\,.\end{aligned}\tag{100}$$

Lemma 5.1. *With the above notations, the convergence and the limit of $\sigma_\rho^{x_0}(x,\xi)$ when $\rho\to 0^+$ is independant of x. In case of convergence, the limit function θ satisfies, for all ξ and $r>0$,*

$$\theta(r\,\xi)=r^m\theta(\xi)\,,\tag{101}$$

and also, for all ξ,

$$c\,|\xi|^m\leq|\theta(\xi)|\leq C|\xi|^m\,,\tag{102}$$

where c, C are the ellipticity constants given by hypothesis HA$\,(m,\gamma)$ *for the symbol σ.*

PROOF. We know from HA$\,(m,\gamma)$ that

$$\rho^m\Big|\sigma\Big(x_0+\rho\,x,\frac{\xi}{\rho}\Big)-\sigma\Big(x_0,\frac{\xi}{\rho}\Big)\Big|\leq K\rho^m\Big(1+\frac{|\xi|}{\rho}\Big)^{m+\varepsilon'}|\rho\,x|^\varepsilon\,,$$

with $\varepsilon>\varepsilon'\geq 0$. This is bounded by $K(\rho+|\xi|)^{m+\varepsilon'}|x|^\varepsilon\rho^{\varepsilon-\varepsilon'}=o\,(\rho)$. The first assertion of the Lemma is now clear. The homogeneity property of the limit function is classical. And the last inequalities are deduced from

$$c\,|\xi|^m\leq\sigma(z,\xi)\leq C|\xi|^m\,,\qquad\text{if }|\xi|\geq R\,,$$

which are part of our hypothesis.

5.2. Local scaling for processes.

We define the scaling operators $R_{\alpha,\rho}$ when $\alpha < 1$, L_ρ when $\alpha = 1$ by

$$R_{\alpha,\rho}(f) = \frac{f(x_0 + \rho\,\cdot) - f_\lambda(x_0)}{\rho^\alpha}\,,$$
$$L_\rho(f) = \frac{1}{\sqrt{\log(1/\rho)}}\,\frac{f(x_0 + \rho\,\cdot) - f_\lambda(x_0)}{\rho}\,.$$

We suppose here that the symbol σ fullfills hypotheses HA (m,γ), $H1$. We consider $A = \mathrm{op}\,(\sigma)$ and X the gaussian process associated with. According to Proposition 1.2 we can write

$$X_x = \sum_{\lambda\in\Lambda} \xi_\lambda\,\Phi_\lambda(x)\,,$$

with ξ_λ i.i.d. standard gaussian. We complete the family with ξ_λ, $j_\lambda < 0$ keeping the i.i.d. property valid.

We say that the symbol σ satisfies hypothesis $H(x_0)$ when

$$\lim_{\rho\to 0^+} \sigma_\rho^{x_0}(0,\xi) = \theta(\xi)\,, \qquad \text{for almost every } \xi \in \mathbb{R}^d\,.$$

In this case we set

$$Y_x = \sum_{\lambda\in\underline{\Lambda}} \xi_\lambda\, g_\lambda^\theta(x)\,.$$

We can now state convergence in law ((d)-lim) and equality in law ($\overset{(\mathrm{d})}{=}$) for the locally scaled processes.

Lemma 5.2. *We suppose that the symbol σ of the* E.G.P. X *fullfills* HA (m,γ), $H(x_0)$ *and* HAS (m,γ). *If $\alpha < 1$,*

$$\text{(d)-}\lim_{\rho\to 0^+} R_{\alpha,\rho}X = Y\,, \tag{103}$$

and for all $\rho > 0$,

$$R_{\alpha,\rho}Y \overset{(\mathrm{d})}{=} Y\,. \tag{104}$$

If $\alpha = 1$, there exists a gaussian vector G on $\mathbb{R}^d$ such that

$$\text{(d)-}\lim_{\rho\to 0^+} L_\rho X = \text{(d)-}\lim_{\rho\to 0^+} L_\rho Y = (G\,|\,x)\,. \tag{105}$$

PROOF. *Case $\alpha < 1$.* Let us give first the idea of the proof. We approximate

$$R_{\alpha,\rho}X = \rho^{-\alpha}\sum_{j_\lambda\geq 0}(\Phi_\lambda(x_0+\rho\,\cdot) - \Phi_\lambda(x_0))\,\xi_\lambda$$

by

$$\rho^{-\alpha}\sum_{j_\lambda\geq 0}\left(g_\lambda^{\sigma^{x_0}}(\rho\,\cdot) - g_\lambda^{\sigma^{x_0}}(0)\right)\xi_\lambda\;.$$

But as far as the laws are concerned we know from (100) that the renormalization of the above process is equivalent to a shift on the scales. We obtain

$$R_{\alpha,\rho}X \overset{\text{(d)}}{\sim} \sum_{j_\mu\geq \log_2\rho}\left(g_\mu^{\sigma_\rho^{x_0}}(x) - g_\mu^{\sigma_\rho^{x_0}}(0)\right)\xi_\mu\;.$$

As $\lim_{\rho\to 0^+}\log_2\rho = -\infty$ and the symbol $\sigma_\rho^{x_0}$ converges to θ (by hypothesis) we get (103)

$$\text{(d)-}\lim_{\rho\to 0^+} R_{\alpha,\rho}X_x = \sum_{\lambda\in\underline{\Lambda}}\xi_\lambda\, g_\lambda^\theta(x)\,.$$

Now let us give the technical justifications for the three steps just described. For the first step we use the approximation of wavelets given by the Theorem 1.3, so that

$$\lim_{\rho\to 0^+}\rho^{-\alpha}\sum_{j_\lambda\geq 0}\left(g_\lambda^{\sigma^{x_0}}(\rho\,x) - g_\lambda^{\sigma^{x_0}}(0)\right)\xi_\lambda = 0\,,$$

uniformly on every bounded set, $\mathbb{P}$ almost surely.

In the second step we apply directly (100) so that

$$\sum_{j_\lambda\geq 0}\left(g_\lambda^{\sigma^{x_0}}(x\,2^{-p}) - g_\lambda^{\sigma^{x_0}}(0)\right)\xi_\lambda \overset{\text{(d)}}{=} \sum_{j_\mu\geq -p}\left(g_\mu^{\sigma_\rho^{x_0}}(x) - g_\mu^{\sigma_\rho^{x_0}}(0)\right)\xi_\mu\;.$$

For the last step we use the convergence of symbols given by hypothesis $H(x_0)$ and also Lemma 5.1. Then

$$\text{(d)- } \lim_{\rho\to 0^+} \sum_{j_\mu \geq -\log_2 \rho} \left(g_\mu^{\sigma_\rho^{x_0}}(x) - g_\mu^{\sigma_\rho^{x_0}}(0) \right) \xi_\mu = \sum_{j_\mu \in \mathbb{Z}} \left(g_\mu^{\theta}(x) - g_\mu^{\theta}(0) \right) \xi_\mu \ .$$

This gives the first result of the Lemma. The second one is another direct application of the scaling result (100).

Case $\alpha = 1$. The canonical basis of $\mathbb{R}^d$ is denoted by $(e_1, \dots, e_d)$. We know from our construction that $\partial_i \Phi_\mu(x) \sim \delta(l_i, 1)$, when $x \to \mu = (j,k,l)$, where δ denotes the Kronecker symbol, and at the same time

$$|\partial_i \Phi_\mu(x)| \leq K \, \frac{1}{(1 + |x-\mu|)^{d+\gamma}} \ .$$

Then, if $\rho \to 0^+$, using the proof that led to the uniform modulus result in the critical case $\alpha = 1$, we get

$$\frac{1}{\sqrt{\log(1/\rho)}} \sum_{\mu\in\Lambda} \frac{\Phi_\mu(x_0 + \rho\, x) - \Phi_\mu(x_0)}{\rho} \, \xi_\mu$$

$$\sim \sum_{i=1}^{d} \frac{(x \,|\, e_i)}{\sqrt{\log(1/\rho)}} \sum_{0\leq j \leq \log(1/\rho)} \xi_{\mu_j(x_0, i)} \ ,$$

where $\mu_j(x,i)$ is defined by $\mu_j(x,i) = (j,k,l)$ if and only if $l = \delta(\,\cdot\,, i)$ and x belong to a dyadic cube $q_{j,k}$. The end of the proof is now an application of the Central Limit Theorem.

As an immediate consequence we can now prove Theorem 1.4, 1.5.

5.3. Local scalings for X_A.

We first prove Theorems 1.4 and 1.5.

Lemma 5.2 gives ii) implies i) for both Theorem 1.4 and Theorem 1.5. As i) implies iii) is clear, we have only to prove iii) implies ii).

Let us consider the symbols

$$\underline{\theta}(\xi) := \liminf_{\rho\to 0^+} \sigma_\rho^{x_0}(x,\xi) \, , \qquad \overline{\theta}(\xi) := \limsup_{\rho\to 0^+} \sigma_\rho^{x_0}(x,\xi)$$

and recall that they satisfy (102) (see Lemma 5.1).

In the case $\alpha < 1$, we deduce from the result (60) and the hypothesis of convergence

$$\begin{aligned} 0 &< \int \frac{\sin^2(u\xi)}{\overline{\theta}(\xi)}\, d\xi \\ &= \lim_{\rho \to 0^+} \mathbb{E}\Big(\frac{X(x_0 + \rho\, u) - X(x_0)^2}{\rho^{2\alpha}} \Big) \\ &= \int \frac{\sin^2(u\,\xi)}{\underline{\theta}(\xi)}\, d\xi < \infty\,. \end{aligned}$$

This leads to the almost everywhere equality

$$\underline{\theta}(\xi) = \overline{\theta}(\xi)\,,$$

and then to existence of the limit stated in ii).

In the case $\alpha = 1$ the proof is the same, except that we use (63) instead of (60).

We now prove Proposition 1.3.

The fact that $\sigma = \sigma_0 f$ satisfies HA (m, γ) and that Proposition 1.1 can be applied is easy to check.

As we know that

$$\lim_{\rho \to \infty} \frac{\sigma_0(\rho\,\xi)}{\rho^{d+2\alpha}} = \theta_{x_0}(\xi)\,,$$

(see (38)) and also that the function f is bounded we obtain the existence of

$$\limsup_{\rho \to \infty} \frac{\sigma(\rho\,\xi)}{\rho^{d+2\alpha}}\ .$$

Then, with the same arguments as in the proof of Lemma 5.2, case $\alpha < 1$, we see that the process X_A belongs to the weak L.A.S.S. class.

Moreover the process X_A belongs to the L.A.S.S. class if and only if

$$\lim_{\rho \to \infty} \frac{\sigma(\rho\,\xi)}{\rho^{d+2\alpha}}\ ,$$

exists and this, within our hypotheses, is equivalent to the existence of $\lim_{\rho \to \infty} f(\xi)$.

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Recibido: 7 de septiembre de 1.995
Revisado: 30 de julio de 1.996

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Revista Matemática Iberoamericana
Vol. 13, N.° 1, 1997

Fonctions d'échelle interpolantes, polynômes de Bernstein et ondelettes non stationnaires

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Résumé. La théorie de la convergence des fonctions d'échelle (non-stationnaires) et l'approximation des filtres d'échelle interpolants à l'aide de polynômes de Bernstein, permettent la construction d'une fonction d'échelle interpolante non-stationnaire aux propriétés d'approximation remarquables.

Abstract. The theory of convergence for (non-stationary) scaling functions and the approximation of interpolating scaling filters by means of Bernstein polynomials, allow us to construct a non-stationary interpolating scaling function with interesting approximation properties.

0. Introduction.

Dans cet article, nous nous intéressons essentiellement à des distributions φ du type suivant: leur transformée de Fourier $\hat{\varphi}$, définie formellement par

$$\hat{\varphi}(\xi) = \langle \varphi, e^{ix\xi} \rangle = \int_{-\infty}^{+\infty} \varphi(x)\, e^{-ix\xi}\, dx\,, \tag{1}$$

est un produit infini

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_j\Big(\frac{\xi}{2^j}\Big)\,, \tag{2}$$

où les fonctions m_j vérifient pour deux constantes $C_0 \geq 1$ et $N \geq 0$ indépendantes de j

$$\begin{aligned} &m_j \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})\,, \qquad m_j(0) = 1\,,\\ &\|m_j\|_\infty \leq C_0\;, \qquad \Big\|\frac{d}{d\xi}\,m_j\Big\|_\infty \leq C_0\, j^N\,. \end{aligned} \tag{3}$$

Sous les hypothèses (3), il est facile de voir que le produit infini

$$\prod_{j=1}^{\infty} m_j\Big(\frac{\xi}{2^j}\Big)\,,$$

converge ponctuellement, uniformément sur tout compact et dans $\mathcal{S}'(\mathbb{R})$ vers une fonction continue $\hat{\varphi}$ à croissance lente. Pour le vérifier, il suffit d'écrire $j^N \leq C_N\, 2^{j/2}$ et donc, pour $|\xi| \geq \sqrt{2}$,

$$\begin{aligned} |\hat{\varphi}(\xi)| &\leq \prod_{j=1}^{\infty} \min\Big\{C_0, 1 + C_0 C_N \Big(\frac{|\xi|}{\sqrt{2}}\Big)^j\Big\}\\ &\leq |\xi|^{2\log C_0/\log 2} \prod_{j=0}^{+\infty} (1 + C_0 C_N 2^{-j/2})\,. \end{aligned}$$

En fait, il suffit de supposer que

$$\sum_0^\infty 2^{-j}\Big\|\frac{d}{d\xi}\,m_j\Big\|_\infty < +\infty\,,$$

puisque pour $|\xi|/2^j$ assez petit

$$\Big|\log\Big(\frac{m_j}{\xi/2^j}\Big)\Big| \leq C\,2^{-j}\Big\|\frac{d}{d\xi}\,m_j\Big\|_\infty |\xi|\,.$$

Les produits infinis du type (2) avec $m_j = m_0$ indépendants de j font depuis 1986 l'objet d'une étude intensive. Nous sommes alors dans le cadre de l'*analyse multi-résolution* de S. Mallat et Y. Meyer [20], [21], du moins lorsque φ est de carré intégrable et que la famille $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ est une base de Riesz d'un sous-espace fermé de $L^2(\mathbb{R})$. Les propriétés de la *fonction d'échelle* φ en fonction du *filtre d'échelle* m_0 ont été abondamment décrites, en particulier celles qui décrivent la régularité de φ (A. Cohen [4], A. Cohen et I. Daubechies [6], I. Daubechies et J. Lagarias [9], T. Eirola [12], L. Hervé [14], O. Rioul [23], L. Villemoes [30], par exemple ...). Ces résultats forment ce que nous appellerons dans la suite la "théorie classique" des ondelettes, dont nous rappellerons dans la Section 2 quelques traits fondamentaux sur lesquels nous baserons la suite de nos résultats. Le résultat principal (Proposition 2) indique, sous l'hypothèse que le filtre m_0 vérifie le *critère d'Albert Cohen,* l'existence d'un indice $\sigma_0 \in]-\infty, +\infty]$ tel que pour $s > \sigma_0$, $\varphi \notin H^s$ (espace de Sobolev) et pour $s < \sigma_0$, φ est à *décroissance rapide* dans H^s (*i.e.* pour tout $k \in \mathbb{N}$, $x^k\varphi \in H^s$); ce résultat donne de plus le calcul de σ_0 en fonction des propriétés spectrales de l'*opérateur de transition* T_2 associé à m_0, et défini par

$$T_2 f(\xi) = \left|m_0\Big(\frac{\xi}{2}\Big)\right|^2 f\Big(\frac{\xi}{2}\Big) + \left|m_0\Big(\frac{\xi}{2}+\pi\Big)\right|^2 f\Big(\frac{\xi}{2}+\pi\Big).$$

Une première série de résultats, présentés dans la Section 3, concerne le problème de l'approximation de la fonction d'échelle φ par des fonctions d'échelle plus simples φ_N (essentiellement, on demandera à φ_N d'être à support compact). La question étudiée est essentiellement la suivante: comment l'approximation du filtre m_0 par des filtres m_N se traduit-elle sur la qualité de l'approximation de φ par les fonctions φ_N associées? Par rapport à la "théorie classique", il s'agit donc essentiellement d'un théorème de dépendance continue par rapport aux paramètres, et il y aura donc besoin de peu d'innovation réelle pour obtenir ces résultats. (La Section 3 contient donc des résultats originaux, dont la démonstration est très courte et renvoie à la Section 2; la Section 2 contient une présentation de résultats classiques refondus pour être immédiatement opérationnels dans les sections 3 et 7).

Le design de filtres d'échelle nous imposera parfois d'introduire des filtres m_N dont la limite n'est pas C^∞ (par exemple, si m_N est le N-ième filtre de Daubechies [8], le filtre $|m_N(\xi)|^2$ converge vers la fonction 2π-périodique m_∞ valant 1 sur $]-\pi/2, \pi/2[$, $1/2$ en $\pm\pi/2$ et 0 sur $]\pi/2, \pi]$; le problème de m_N lui-même se complique encore de l'étude de sa phase [16]). Nous discuterons brièvement dans la Section 10 des fonctions d'échelle à décroissance lente, correspondant à des filtres peu réguliers.

L'étude des produits infinis (2) avec m_j dépendant de j est beaucoup plus récente, et a été théorisée sous le nom d'*analyse multi-résolution non-stationnaire* [10]. La principale motivation de cette étude est qu'elle permet l'obtention de fonctions d'échelle (non-stationnaires) C^∞ et à support compact, ce qui n'est pas possible dans le cas stationnaire. (Si φ est une fonction d'échelle de classe C^N, alors le diamètre de son support est au moins $N+2$). Les exemples usuels de telles fonctions d'échelle non-stationnaires sont la fonction de Rvachev $up(x-1)$ [26], [11] et la base de Berkolaïko et Novikov [2], [7] qui permettent une approximation spectrale des fonctions régulières.

La *fonction de Rvachev* $up(x)$ est encore mal connue du public (mathématique) occidental, les principales références étant en russe ou en ukrainien (dont le livre [27] paru en 1979). Nous en rappellerons les principales propriétés dans la Section 5. (Cette section expose les résultats de Rvachev. Nous avons préféré travailler dans $\mathbb{R}$ plutôt que dans $[-1,1]$).

La seconde série de résultats, que nous présentons dans la Section 7, concerne les *analyses multi-résolutions quasi-stationnaires.* L'idée est d'appliquer tout le mécanisme de l'analyse du comportement asymptotique des suites de filtres d'échelle développée dans la première série de résultats à la suite m_j qui intervient dans le produit (2). Nous verrons que si la suite m_j converge vers un filtre asymptote m_∞, la connaissance des propriétés de m_∞ et de sa fonction d'échelle φ_∞ simplifie grandement l'étude de la convergence du produit infini (2) et de la taille et de la régularité de la fonction d'échelle non-stationnaire ainsi définie. (Avec la Section 3, c'est le principal résultat du papier).

La troisième série de résultats (sections 7 et 8) concerne la construction effective des suites de filtres approximant un filtre donné. La construction est quasiment immédiate pour le cas des filtres associés à des *fonctions d'échelle interpolantes*, et fait intervenir l'approximation des fonctions par des *polynômes de Bernstein.* En particulier, nous sommes à même de construire une fonction d'échelle interpolante non-stationnaire, ou "ondelette de Kharkov", jouissant de propriétés d'approximation remarquables (*cf.* Théorème 4).

Le cas des fonctions d'échelle orthogonales est beaucoup plus complexe, à cause du problème de la phase. Nous essayerons ici de bien poser le problème, en réservant l'éventuelle solution à des travaux ultérieurs.

1. Décroissance et convergence rapides dans un espace local de distributions.

Nous avons regroupé dans cette section quelques lemmes techniques sur la convergence rapide (*cf.* Définition 2), qui nous seront utiles pour vérifier les qualités d'approximation dans les théorèmes des sections suivantes. Nous avons choisi de présenter ces lemmes dans un cadre axiomatique assez général (ce que nous appelons dans la Définition 1 un espace local de distributions), mais en réalité nous travaillerons avec des espaces simples comme L^2, L^∞, H^s ou $B_\infty^{\alpha,\infty}$ (c'est-à-dire l'espace de Hölder C^α si $\alpha > 0$ et $\alpha \notin \mathbb{N}$, l'espace de Zygmund C_*^α si $\alpha \in \mathbb{N}^*$).

Definition 1. *Un espace local de distribution* (*ou* E.L.D.) *est un espace de Banach* E, *continûment injecté dans* $\mathcal{D}'(\mathbb{R})$, *tel que*

i) *pour tout* $\varphi \in E$ *et tout* $\omega \in C_c^\infty(\mathbb{R})$, $\omega\,\varphi \in E$,

ii) *il existe* $C_0 \geq 0$ *et* $N_0 \in \mathbb{N}$ *tel que, pour tout* $\varphi \in E$ *et tout* $\omega \in C_c^\infty$ *avec* $\operatorname{supp}\omega \subset [-1,1]$,

$$\|\varphi\,\omega\|_E \leq C_0 \sum_{p=0}^{N_0} \left\| \frac{d^p}{dx^p}\,\omega \right\|_{L^\infty([-1,1])} \|\varphi\|_E\,, \tag{4}$$

iii) *pour tout* $\varphi \in E$ *et tout* $x_0 \in \mathbb{R}$, $\varphi(x-x_0) \in E$ *et il existe* $C_1 \geq 0$ *et* $N_1 \geq 0$ *tel que*

$$\|\varphi(x-x_0)\|_E \leq C_1(1+|x_0|)^{N_1}\|\varphi\|_E\,, \tag{5}$$

pour tout $\varphi \in E$ *et tout* $x_0 \in \mathbb{R}$.

Definition 2. *Soit* E *un* E.L.D.

i) φ *est à décroissance rapide dans* E *si pour tout entier* $k \in \mathbb{N}$, $x^k\varphi \in E$.

ii) *Une suite* $\{\varphi_n\}_{n\in\mathbb{N}}$ *est à convergence rapide dans* E *si les* φ_n *sont à décroissance rapide dans* E, *convergent dans* E *vers une distribution* φ *et pour tout entier* $k \in \mathbb{N}$, $x^k\varphi_n$ *converge dans* E *vers* $x^k\varphi$.

On dira de même qu'une série $\sum_{k\in\mathbb{Z}} \varphi_k$ est à convergence rapide si les sommes partielles $\{\sum_0^n \varphi_k\}_{n\in\mathbb{N}}$ et $\{\sum_{-n}^0 \varphi_k\}_{n\in\mathbb{N}}$ sont à convergence rapide.

Lemme A (Lemme des blocs). *Soit E un* E.L.D.

a) *Si* $\operatorname{supp}\varphi_k \subset [k-1, k+1]$ *pour tout* $k \in \mathbb{Z}$, *alors* $\sum_{\mathbb{Z}} \varphi_k$ *est à convergence rapide dans E si et seulement si la suite* $\{\|\varphi_k\|_E\}_{k\in\mathbb{Z}}$ *est à décroissance rapide (pour tout* $p \in \mathbb{N}$, $\{k^p\|\varphi_k\|_E\}_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$).

b) φ *est à décroissance rapide dans E si et seulement si φ se décompose en* $\sum_{\mathbb{Z}} \varphi_k$ *avec* $\operatorname{supp}\varphi_k \subset [k-1, k+1]$ *et* $\{\|\varphi_k\|_E\}_{k\in\mathbb{Z}}$ *à décroissance rapide.*

c) *pour qu'une suite* $\{\varphi_n\}_{n\in\mathbb{N}}$ *converge rapidement dans E, il suffit que* $\{\varphi_n\}_{n\in\mathbb{N}}$ *converge dans E et que pour tout* $p \in \mathbb{N}$, $\sup_k \|x^p\varphi_k\|_E < +\infty$.

Preuve. a) Le lemme est presque évident. Si $\{\varphi_n\}$ converge rapidement dans E, il est clair que $\sup_k \|x^p\varphi_k\|_E < +\infty$. En particulier, si $\sum \varphi_k$ converge rapidement, avec $\operatorname{supp}\varphi_k \subset [k-1, k+1]$, on a $\sup_k \|x^p\varphi_k\|_E < +\infty$. Mais si ω est C^∞, vaut 1 sur $[-1, 1]$ et a son support contenu dans $[-2, 2]$, on a, pour $|k| \geq 3$,

$$\varphi_k = \omega_k(x-k)\, x^p\varphi_k(x) \qquad \text{avec} \qquad \omega_k = (x+k)^{-p}\omega(x)\,.$$

(4) et (5) permettent de conclure que

$$\|\varphi_k\|_E \leq C_p\,(1+|k|)^{2N_1-p}\|x^p\varphi_k\|_E\ ,$$

et donc que $\{\|\varphi_k\|_E\}$ est à décroissance rapide. Inversement, si $\|\varphi_k\|_E$ est à décroissance rapide, on écrit $x^p\varphi_k = \tilde{\omega}_k(x-k)\varphi_k$ avec $\tilde{\omega}_k = (x+k)^p\omega(x)$; on a alors

$$\|x^p\varphi_k\|_E \leq C_p\,(1+|k|)^{2N_1+p}\|\varphi_k\|_E\ ,$$

de sorte que $\sum_{k\in\mathbb{Z}} \|x^p\varphi_k\|_E < +\infty$. La série $\sum x^p\varphi_k$ converge dans E, et converge vers $x^p\sum\varphi_k$ puisque E s'injecte dans $\mathcal{D}'$. Le point a) est donc prouvé.

b) est immédiat: si φ est à décroissance rapide, on prend $\gamma \in C_c^\infty$ avec $\operatorname{supp}\gamma \subset [-1, 1]$ et $\sum \gamma(x-k) = 1$ et on pose $\varphi_k = \varphi\,\gamma(x-k)$. Alors pour $|k| \geq 3$, on a

$$\|\varphi_k\|_E = \|\gamma(x-k)\,x^{-p}x^p\varphi\|_E \leq C_p\,(1+|k|)^{2N_1-p}\|x^p\varphi\|_E\ ,$$

et $\{\|\varphi_k\|_E\}$ est à décroissance rapide.

c) est tout aussi immédiat: si φ_n converge vers φ, $\gamma(x-k)\,\varphi_n$ converge vers $\gamma(x-k)\,\varphi$; or on a

$$\|\gamma(x-k)\,\varphi_n\|_E \leq C_p\,(1+|k|)^{2N_1-p}\|x^p\varphi_n\|_E\ .$$

Le point c) du Lemme A se généralise de la manière évident suivante:

Lemme B. *Soient E, E_1, E_2 trois* E.L.D. *tels que $E_1 \cap E_2 \subset E$ et, pour tout $f \in E_1 \cap E_2$*

$$\|f\|_E \leq C_2\|f\|_{E_1}^{\alpha}\|f\|_{E_2}^{1-\alpha}\ ,$$

pour un $\alpha \in]0,1[$. Alors si $\{\varphi_n\}_{n\in\mathbb{N}}$ converge dans E_1 et vérifie pour tout $p \in \mathbb{N}$, $\sup_n \|x^p\varphi_n\|_{E_2} < +\infty$, $\{\varphi_n\}$ converge rapidement dans E.

PREUVE. Même démonstration: $\{\gamma(x-k)\,\varphi_n\}_{n\in\mathbb{N}}$ converge dans E (puisque elle est de Cauchy dans E_1 et bornée dans E_2); de plus

$$\|\gamma(x-k)\,\varphi_n\|_{E_1} \leq C\,(1+|k|)^{2N_1}\|\varphi_n\|_{E_1}$$

et

$$\|\gamma(x-k)\,\varphi_n\|_{E_2} \leq C_p\,(1+|k|)^{2N_2-p}\|x^p\varphi_n\|_{E_2}\ .$$

Lemme C. *Soit E un* E.L.D. *et φ à décroissance rapide dans E.*

a) *Si $\varphi = \sum \varphi_k$ est une décomposition en blocs de φ (i.e.* $\operatorname{supp}\varphi_k \subset [k-1,k+1]$ *et la série converge rapidement dans E) alors si $b \in C^\infty(\mathbb{R})$ est à croissance lente ainsi que toutes ses dérivées (pour tout $p \in \mathbb{N}$, il existe $N \geq 0$, $(|x|+1)^{-N}d^pb/dx^p \in L^\infty$) la série $\sum_{k\in\mathbb{Z}}\langle\varphi_k, b\rangle$ converge vers une somme indépendante du choix des blocs φ_k et notée $\langle\varphi, b\rangle$.*

b) *La transformée de Fourier de φ, $\hat{\varphi}(\xi) = \langle\varphi, e^{ix\xi}\rangle$, est une fonction C^∞ à croissance lente ainsi que toutes ses dérivées.*

PREUVE. Remarquons d'abord que d'après (4) on a: si $\operatorname{supp}\omega \subset [-1,1]$ et $\varphi \in E$

$$|\langle\varphi,\omega\rangle| \leq C\,\|\varphi\|_E \sum_{p=0}^{N} \left\|\frac{d^p\omega}{dx^p}\right\|_{L^\infty(-1,1)}$$

en effet, on a pour $\psi \in E$ avec $\operatorname{supp}\psi \subset [-1,1]$, $|\langle \psi, 1\rangle| \le C\,\|\psi\|_E$ et on utilise ceci pour $\psi = \varphi\,\bar{\omega}$. On en conclut que

$$|\langle \varphi_k, b\rangle| \le C\,(1+|k|)^{M+2N_1}\|\varphi_k\|_E \sum_{p=0}^{N} \Big\|(1+|x|)^{-M}\,\frac{d^p b}{d^p x}\Big\|_\infty ,$$

de sorte que $\sum |\langle \varphi_k, b\rangle|$ converge. Le point a) est alors immédiat.

Pour le point b), il suffit de vérifier que $\hat\varphi$ est continue, puisque $(d/d\xi)^N\hat\varphi = \widehat{(-i\,x)^N\varphi}$ et que $x^N\varphi$ est encore à décroissance rapide dans E. Mais $\hat\varphi(\xi) = \sum_{k\in\mathbb{Z}}\hat\varphi_k(\xi)$ et on a

$$|\hat\varphi_k(\xi)| \le C\,(1+|k|)^{2N_1}\|\varphi_k\|_E(1+|\xi|)^N ,$$

de sorte que $\sum\hat\varphi_k$ converge uniformément sur tout compact.

Lemme D (Lemme de dérivation).

a) *Soient E_1 et E_2 deux* E.L.D. *tels que d/dx est continu de E_1 dans E_2. Alors si φ est à décroissance rapide dans E_1, $d\varphi/dx$ est à décroissance rapide dans E_2.*

b) *Le résultat reste vrai si l'on suppose seulement que d/dx est continu de $E_{1,0} = \{\varphi\in E_1 \,:\, \operatorname{supp}\varphi\subset[-1,1]\}$ dans E_2.*

c) *On a de plus*

$$\widehat{\frac{d\varphi}{dx}}(0) = \Big\langle \frac{d\varphi}{dx}, 1\Big\rangle = 0\,.$$

PREUVE. Il suffit de prouver b). On décompose φ en blocs $\varphi = \sum\varphi_k$. Alors

$$\frac{d\varphi}{dx} = \sum \frac{d\varphi_k}{dx}$$

dans $\mathcal{D}'$. De plus

$$\Big\|\frac{d\varphi_k}{dx}\Big\|_{E_2} \le C\,(1+|k|)^{N_1+N_2}\|\varphi_k\|_{E_1} ,$$

de sorte que $\|d\varphi_k/dx\|_{E_2}$ est à décroissance rapide.

Ce lemme a une réciproque.

Lemme E (Lemme de primitivation). *Soient* E_1 *et* E_2 *deux* E.L.D. *tels que:* $\varphi \to \int_{-\infty}^{x} \varphi(t)\, dt$ *est continu de* $E_{2,(0)} = \{\varphi \in E_2 : \operatorname{supp} \varphi \subset [-1,1]$ *et* $\langle \varphi, 1\rangle = 0\}$ *dans* E_1. *Alors si* $\varphi \in E_2$ *est à décroissance rapide,* φ *peut s'écrire* $\varphi = d\omega/dx$ *où* ω *est à décroissance rapide dans* E_1 *si et seulement si* $\langle \varphi, 1\rangle = 0$.

PREUVE. Si $\varphi = d\omega/dx$, on a $\hat{\varphi} = i\,\xi\,\hat{\omega}$ donc $\hat{\varphi}(0) = 0$. Pour le résultat inverse, il suffit de montrer qu'on peut écrire $\varphi = \sum \varphi_k$ une décomposition en blocs avec $\langle \varphi_k, 1\rangle = 0$ pour tout k. Si ω_k est le primitive à support compact de φ_k, on aura alors

$$\|\omega_k\|_{E_1} \leq C\,(1 + |k|)^{N_1+N_2} \|\varphi_k\|_{E_2}$$

et donc $\sum \omega_k$ sera une décomposition en blocs d'une distribution ω à décroissance rapide dans E_1.

On commence par décomposer φ en $\varphi = \sum \psi_k$, une décomposition en blocs sans condition sur ψ_k. On prend alors $\alpha \in E$, $\operatorname{supp} \alpha \subset [0,1]$ et $\langle \alpha, 1\rangle = 1$ (un tel α existe si $E \neq \{0\}$). On écrit

$$\psi_k = \psi_k - \langle \psi_k, 1\rangle\, \alpha\,(x-k) + \langle \psi_k, 1\rangle\, \alpha\,(x-k)\,.$$

La somme

$$\sum_{k\in\mathbb{Z}} \psi_k - \langle \psi_k, 1\rangle\, \alpha\,(x-k)\,,$$

est une décomposition par blocs avec des blocs de somme nulle. Nous sommes ramenés à traiter le cas de

$$\varphi = \sum_{k\in\mathbb{Z}} \varepsilon_k\, \alpha\,(x-k)\,,$$

avec $\{\varepsilon_k\}$ à décroissance rapide et $\sum \varepsilon_k = 0$. Il suffit d'écrire $\varepsilon_k = s_k - s_{k+1}$ et

$$\varphi = \sum_{k\in\mathbb{Z}} s_k\,(\alpha\,(x-k) - \alpha\,(x-k+1))\,,$$

on conclut en remarquant que $\{s_k\}$ est à décroissance rapide puisque

$$s_k = \sum_{k}^{+\infty} \varepsilon_p = -\sum_{-\infty}^{k-1} \varepsilon_p\ .$$

Lemme F (Lemme de primitivation discrète). *Soit* E *un* E.L.D. *et* φ *à décroissance rapide dans* E*. Alors les trois propriétés suivantes sont équivalentes:*

F1) *pour tout* $k \in \mathbb{Z}$, $\hat{\varphi}(2\,k\,\pi) = 0$,

F2) $\sum_{k\in\mathbb{Z}} \varphi(x-k)$ *converge dans* $\mathcal{D}'$ *vers* 0,

F3) *Il existe* $\omega \in E$ *à décroissance rapide tel que*

$$\varphi = \omega(x) - \omega(x-1)\,.$$

PREUVE. Il n'y a presque rien à démontrer. F3) implique F1) est évident, puisqu'alors $\hat{\varphi} = (1-e^{-i\xi})\,\hat{\omega}$. Pour F1) implique F2) et F2) implique F3), on commence par remarquer que si φ est à décroissance rapide dans E, alors $\sum \varphi(x-k)$ converge dans $\mathcal{D}'$ (et même dans $\mathcal{S}'$)

$$\sum \langle \varphi(x-k), \beta\rangle = \Big\langle \varphi, \sum \beta(x+k)\Big\rangle$$

et $\sum \beta(x+k)$ est C^∞ bornée ainsi que toutes ses dérivées.

Il reste à vérifier que

$$\sum \varphi(x-k) = \sum \hat{\varphi}(2\,k\,\pi)\,e^{ikx}\,,$$

c'est évident si le support de φ est compact. Dans le cas général, on décompose φ en blocs $\sum \varphi_p$; on a

$$\sum \varphi(x-k) = \sum \sum \varphi_p(x-k)\,,$$

au sens que pour tout $b \in \mathcal{S}$,

$$\sum \sum |\langle \varphi_p(x-k), b\rangle| < +\infty\,.$$

On a donc

$$\sum \varphi(x-k) = \sum_p \Big(\sum_k \hat{\varphi}_p(2\,k\,\pi)\,e^{ikx}\Big)$$

et comme

$$|\hat{\varphi}_p(2\,k\,\pi)| \le C_N\,\frac{(1+|k|)^N}{1+p^2}\,,$$

la convergence dans $\mathcal{D}'$ est immédiate.

Si $\sum \varphi(x-k) = 0$, on a $\hat{\varphi}(2\,k\,\pi) = 0$. Si $\hat{\varphi}(2\,k\,\pi) = 0$, on a $\sum \varphi(x-k) = 0$; on définit alors

$$\omega = \sum_{k=0}^{\infty} \varphi(x-k) = -\sum_{k=-\infty}^{-1} \varphi(x-k)\,.$$

On a

$$\|\gamma(x-p)\,\varphi(x-k)\|_E \leq C\,(1+|k|)^{N_1}\|\gamma(x-p+k)\,\varphi\|_E\;,$$

et on déduit tout de suite que ω est à décroissance rapide dans E.

2. La théorie classique des fonctions d'échelle.

La notion de fonction d'échelle régulière a été introduite en 1986 par Y. Meyer et S. Mallat [20], [21].

Definition 3. *Une fonction d'échelle régulière est une fonction $\varphi \in L^2(\mathbb{R})$ telle que*

i) *φ est à valeurs réelles,*

ii) *φ est à décroissance rapide dans L^2: pour tout $k \in \mathbb{Z}$, $x^k\varphi \in L^2$,*

iii) *φ engendre une base de Riesz $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ d'un sous-espace fermé V_0 de L^2,*

iv) *$\varphi(x/2) \in V_0$.*

L'étude des fonctions d'échelle se ramène à celle des *filtres d'échelle:* l'appartenance de $\varphi(x/2)$ à V_0 se réécrit en

$$\varphi\Big(\frac{x}{2}\Big) = \sum_{k\in\mathbb{Z}} a_k\,\varphi(x-k)\,, \tag{6}$$

où les a_k sont à valeurs réelles et dans $\ell^2(\mathbb{Z})$. Le filtre d'échelle associé à φ est la fonction 2π-périodique m_0 définie par

$$m_0(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}\,. \tag{7}$$

Quelques lemmes classiques permettent alors de se ramener à l'étude des propriétés de m_0.

Lemme 1. *Si φ et $x\,\varphi$ sont de carré intégrable, alors*

$$\sum_{k\in\mathbb{Z}} |\hat\varphi(\xi+2\,k\,\pi)|^2$$

converge uniformément sur $[-\pi,\pi]$.

PREUVE. $|\hat\varphi|^2 \in L^1$ donc

$$\sum_{k\in\mathbb{Z}} |\hat\varphi(\xi+2\,k\,\pi)|^2$$

converge presque partout, donc en au moins un point ξ_0. Il suffit alors d'écrire pour $\xi \in [\xi_0-\pi, \xi_0+\pi]$

$$\begin{aligned}
\Big|\Big(\sum_{|k|>K} |\hat\varphi(\xi+2\,k\,\pi)|^2\Big)^{1/2} &- \Big(\sum_{|k|>K} |\hat\varphi(\xi_0+2\,k\,\pi)|^2\Big)^{1/2}\Big| \\
&\le \Big(\sum_{|k|>K} |\hat\varphi(\xi+2\,k\,\pi)-\hat\varphi(\xi_0+2\,k\,\pi)|^2\Big)^{1/2} \\
&\le \Big(\sum_{|k|>K} |\xi-\xi_0| \int_{\xi_0+2k\pi}^{\xi+2k\pi} \Big|\frac{d\hat\varphi}{d\eta}(\eta)\Big|^2\,d\eta\Big)^{1/2} \\
&\le |\xi-\xi_0|^{1/2}\Big(\int_{|\eta-\xi_0|>2K\pi-\pi} \Big|\frac{d}{d\eta}\,\hat\varphi(\eta)\Big|^2\,d\eta\Big)^{1/2}.
\end{aligned}$$

Lemme 2.

i) *Si φ et $x\,\varphi$ sont de carré intégrable, alors $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ est une base de Riesz d'un sous-espace fermé de L^2 si et seulement si $\sum_{k\in\mathbb{Z}} |\hat\varphi(\xi+2\,k\,\pi)|^2$ ne s'annule pas sur $[-\pi,\pi]$.*

ii) *La fonction m_0, filtre d'échelle associé à une fonction d'échelle régulière φ, est C^∞ et vérifie $\hat\varphi(2\,\xi) = m_0(\xi)\,\hat\varphi(\xi)$.*

PREUVE. Le point i) est évident: si $\varphi \in L^2$, le fait que $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ soit une famille de Riesz est équivalent à ce que

$$\text{inf ess} \sum_{k\in\mathbb{Z}} |\hat\varphi(\xi+2\,k\,\pi)|^2 > 0$$

et que

$$\sup \text{ess} \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2 < +\infty\,,$$

si de plus $x\,\varphi \in L^2$, alors

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2$$

est continue. Pour ii), il suffit de remarquer que

$$m_0(\xi) = \frac{\displaystyle\sum_{k\in\mathbb{Z}} \hat{\varphi}(2\,\xi + 4\,k\,\pi)\,\hat{\varphi}(\xi + 2\,k\,\pi)}{\displaystyle\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2} = \frac{\displaystyle\sum_{k\in\mathbb{Z}} \Big\langle \varphi\Big(\frac{x}{2}\Big), \varphi(x-k)\Big\rangle e^{-ik\xi}}{\displaystyle\sum_{k\in\mathbb{Z}} \langle\varphi(x), \varphi(x-k)\rangle e^{-ik\xi}}$$

et que les coefficients $\langle\varphi(x/2), \varphi(x-k)\rangle$ et $\langle\varphi(x), \varphi(x-k)\rangle$ sont à décroissance rapide.

Lemme 3. *Si φ est une fonction d'échelle régulière, de filtre m_0, alors $m_0(0) = 1$, $\hat{\varphi}(0) \neq 0$ et*

$$\hat{\varphi}(\xi) = \hat{\varphi}(0) \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big)\,.$$

PREUVE. On remarque que $|m_0(0)| \leq 1$

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(4\,k\,\pi)|^2 = |m_0(0)|^2 \sum_{k\in\mathbb{Z}} |\hat{\varphi}(2\,k\,\pi)|^2\,.$$

On a alors

$$|\hat{\varphi}(\xi)| = \prod_{j=1}^{N} \Big|m_0\Big(\frac{\xi}{2^j}\Big)\Big|\,\Big|\hat{\varphi}\Big(\frac{\xi}{2^N}\Big)\Big| = |\hat{\varphi}(0)| \prod_{j=1}^{\infty} \Big|m_0\Big(\frac{\xi}{2^j}\Big)\Big|\,.$$

La convergence du produit infini est en effet immédiate: si $|m_0(0)| < 1$, il converge vers 0, si $|m_0(0)| = 1$, on a $|m_0(\xi)| = 1 + O(\xi/2^j)$. On en conclut que $\hat{\varphi}(0) \neq 0$, donc que $m_0(0) = 1$ et enfin que

$$\hat{\varphi}(\xi) = \hat{\varphi}(0) \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big)\,.$$

Lemme 4. *Si* $m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ *et* $m_0(0) = 1$, *alors la fonction*

$$\prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big)$$

est C^∞.

PREUVE. Comme

$$\hat\varphi(\xi) = \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big)$$

vérifie

$$\hat\varphi(\xi) = \prod_{j=1}^{N} m_0\Big(\frac{\xi}{2^j}\Big)\hat\varphi\Big(\frac{\xi}{2^N}\Big),$$

il suffit de le vérifier sur un voisinage assez petit de 0. Mais si ξ est assez petit, $\operatorname{Re} m_0(\xi/2^j) > 0$ pour tout j et on peut passer au logarithme

$$\hat\varphi(\xi) = e^{\sum_{j=1}^{\infty} \operatorname{Log} m_0(\xi/2^j)}.$$

Maintenant, si θ est définie sur $[-\varepsilon, \varepsilon]$ ($\varepsilon > 0$), C^∞ sur $[-\varepsilon,\varepsilon]$ et si $\theta(0) = 0$ alors $\sum_{j\geq 1}\theta(\xi/2^j)$ est C^∞ sur $[-\varepsilon,\varepsilon]$: la convergence des séries dérivées est immédiate et celle de la série de départ s'obtient par

$$\Big|\theta\Big(\frac{\xi}{2^j}\Big)\Big| \leq C\,\frac{|\xi|}{2^j}\,.$$

Les lemmes 3 et 4 ramènent donc l'étude des fonctions d'échelle régulières à celle des filtres d'échelle. Les filtres d'échelle ont été caractérisés par de nombreux travaux (nous utiliserons essentiellement [4], [6] et [14]). Une conséquence immédiate du Lemme 2 est que si φ est une fonction d'échelle régulière alors $\hat\varphi$ ne peut avoir de zéro 2π-périodique; cela se caractérise facilement sur le filtre m_0: c'est le critère d'Albert Cohen.

Definition 4. *Une fonction* $m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ *telle que* $m_0(0) = 1$ *satisfait le critère d'Albert Cohen s'il existe un compact* K *réunion finie d'intervalles fermés tel que* 0 *est un point intérieur à* K *et tel que*

$$\sum_{k\in\mathbb{Z}} \chi_K(\xi + 2\,k\,\pi) = 1\,, \qquad \textit{presque partout} \tag{9}$$

où

$$\chi_K(x) = \begin{cases} 1\,, & \text{si } x \in K\,, \\ 0\,, & \text{si } x \notin K\,. \end{cases}$$

Pour tous $\xi \in K$, $j \geq 1$,

$$m_0\Big(\frac{\xi}{2^j}\Big) \neq 0\,. \tag{10}$$

Un tel compact est appelé compact d'Albert Cohen associé à m_0.

Le rôle de ce critère est explicité par le lemme suivant

Lemme 5. *Soit* $m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ *telle que* $m_0(0) = 1$ *et soit*

$$\hat{\varphi} = \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big)\,.$$

i) $\hat{\varphi}$ *n'a pas de zéro* 2π*-périodique si et seulement si* m_0 *vérifie le critère d'Albert Cohen.*

ii) *Dans ce cas, si* K *est un compact d'Albert Cohen associé à* m_0, *on a* $\inf_{\eta\in K} |\hat{\varphi}(\eta)| > 0$. *De plus les fonctions*

$$\theta_N(\xi) = \chi_K\Big(\frac{\xi}{2^N}\Big) \prod_{j=1}^{N} m_0\Big(\frac{\xi}{2^j}\Big)$$

convergent ponctuellement vers $\hat{\varphi}$ *et sont dominées par* $\hat{\varphi}$

$$|\theta_N(\xi)| \leq \frac{1}{\inf\limits_{\eta\in K} |\hat{\varphi}(\eta)|}\, |\hat{\varphi}(\xi)|\,.$$

En particulier si $1 \leq p < +\infty$ *et* $w \in L^1_{\mathrm{loc}}$ *avec* $w(x) > 0$, *les trois assertions suivantes sont équivalentes:*

j) $\theta_N \to \hat{\varphi}$ *dans* $L^p(w\,dx)$ *quand* $N \to +\infty$,

jj) $\hat{\varphi} \in L^p(w\,dx)$,

jjj) $\sup_{N\geq 1} \|\theta_N\|_{L^p(w\,dx)} < +\infty$.

PREUVE. Ce lemme est évident. Si $\hat{\varphi}$ n'a pas de zéro 2π-périodique, on note $\mathcal{I}_0$ la collection des intervalles sur lesquels $\hat{\varphi}$ ne s'annule pas, et $\mathcal{I}$

la collection des intervalles I de la forme $I = I_0 + 2\,k\,\pi$, $I_0 \in \mathcal{I}_0$, $k \in \mathbb{Z}$; on a clairement $[-\pi, \pi] \subset \bigcup_{I \in \mathcal{I}} I$ et le compact K se construit à l'aide d'un sous-recouvrement fini de $[-\pi, \pi]$.

Inversement, si K est un compact d'Albert Cohen pour m_0, $\hat{\varphi}$ ne s'annule pas sur K (puisqu'aucun des termes $m_0(\xi/2^j)$ ne s'y annule) et y est continue; on a donc $\min_{\xi \in K} |\hat{\varphi}(\xi)| > 0$. Maintenant, si $\xi \in \mathbb{R}$, il existe nécessairement $k \in \mathbb{Z}$ tel que $\hat{\varphi}(\xi + 2\,k\,\pi) \in K$: en effet $\cup_{k\in\mathbb{Z}} K + 2\,k\,\pi$ est localement fermée, donc fermée, donc coïncide avec $\mathbb{R}$ tout entier (puisque $|\mathbb{R} \setminus \cup_{\mathbb{Z}}\, K + 2\,k\,\pi)| = 0$). Le reste du lemme est alors immédiat, puisque

$$\theta_N(\xi) = \begin{cases} \dfrac{\hat{\varphi}(\xi)}{\hat{\varphi}\Big(\dfrac{\xi}{2^N}\Big)}\,, & \text{si } \xi \in 2^N K\,, \\ 0\,, & \text{si } \xi \notin 2^N K\,. \end{cases}$$

Le résultat classique principal est alors le suivant [6], [14].

Proposition 1. *Soit* $m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ *telle que* $m_0(0) = 1$. *On note* $\hat{\varphi}$ *la fonction*

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big)$$

et T_2 *l'opérateur agissant sur les fonctions* 2π*-périodiques défini par*

$$T_2 f = \Big|m_0\Big(\frac{\xi}{2}\Big)\Big|^2 f\Big(\frac{\xi}{2}\Big) + \Big|m_0\Big(\frac{\xi}{2} + \pi\Big)\Big|^2 f\Big(\frac{\xi}{2} + \pi\Big)\,.$$

Alors les deux assertions suivantes sont équivalentes:

A1) $\hat{\varphi}$ *est la transformée de Fourier d'une fonction d'échelle régulière,*

A2) m_0 *satisfait les trois conditions suivantes:*

i) $m_0(\xi) = \overline{m_0(-\xi)}$,

ii) m_0 *satisfait le critère d'Albert Cohen,*

iii) $\sup_{N\in\mathbb{N}} \|T_2^N(1)\|_\infty < +\infty$.

De plus, lorsque A1) *ou* A2) *sont vérifiées, il existe* $\alpha > 0$ *tel que* $\varphi \in H^\alpha$ (*i.e. tel que* $|\xi|^\alpha \hat{\varphi} \in L^2$).

PREUVE. A1) implique A2) est évident: i) vient de ce que φ est à valeurs réelles, ii) de ce que $\hat{\varphi}$ n'a pas de zéro 2π-périodique, enfin iii) vient de ce que

$$\gamma(\xi) = \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2$$

ne s'annule pas et de ce que T_2 est un opérateur positif

$$1 \le \frac{1}{\inf\limits_{|\eta| \le \pi} \gamma(\eta)}\, \gamma(\xi)$$

et $T_2(\gamma) = \gamma$, de sorte que

$$T_2^N(1)(\xi) \le \frac{1}{\inf\limits_{|\eta| \le \pi} \gamma(\eta)}\, T_2^N(\gamma)(\xi) = \frac{\gamma(\xi)}{\inf\limits_{|\eta| \le \pi} \gamma(\eta)}\,.$$

A2) implique A1): On va commencer par montrer qu'il existe un $\rho_0 \in]0,1[$ tel que

$$\text{(11)} \qquad \sup_{\xi \in [-\pi,\pi]}\ \sup_{N \in \mathbb{N}} \rho_0^{-N} \sum_{2^N \pi \le |\xi + 2k\pi| < 2^{N+1}\pi} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2 < +\infty\,.$$

Cela implique en particulier que $\hat{\varphi}$ appartient à H^α dès que $2^\alpha \rho_0 < 1$ (*i.e.* pour tout $\alpha < \log(1/\rho_0)/\log 2$).

Pour cela, on note

$$I_N(\xi) = \sum_{2^N \pi \le |\xi + 2k\pi| < 2^{N+1}\pi} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2\,.$$

On a, puisque $\hat{\varphi}(\xi) = m_0(\xi/2)\,\hat{\varphi}(\xi/2)$,

$$\begin{aligned} I_N(\xi) &= \left|m_0\Big(\frac{\xi}{2}\Big)\right|^2 I_{N-1}\Big(\frac{\xi}{2}\Big) + \left|m_0\Big(\frac{\xi}{2} + \pi\Big)\right|^2 I_{N-1}\Big(\frac{\xi}{2} + \pi\Big) \\ &= T_2(I_{N-1})(\xi) \end{aligned}$$

et donc

$$I_N(\xi) = T_2^N(I_0(\xi))\,.$$

De même, on note

$$J_N(\xi) = \sum_{|\xi + 2k\pi| < 2^N \pi} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2$$

et on a $J_N = T_2^N(J_0)$, d'où

$$\|J_N(\xi)\|_\infty \leq \|J_0\|_\infty \|T_2^N(1)\|_\infty \,,$$

ce qui prouve

$$\sup_\xi \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2 < +\infty\,. \tag{12}$$

En particulier, $\hat{\varphi} \in L^2$ et les $\varphi(x-k)$, $k\in\mathbb{Z}$, engendrent une base de Riesz d'un sous-espace fermé V_0 de L^2 (car

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2 \geq \inf_{\eta\in K} |\hat{\varphi}(\eta)|^2 > 0\,,$$

pour K un compact d'Albert Cohen associé à m_0). De plus,

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(2\,k\,\pi)|^2 = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(2\,k\,\pi)|^2 + |m_0(\pi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\pi + 2\,k\,\pi)|^2\,,$$

ce qui prouve que $m_0(\pi) = 0$ (car $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\pi + 2\,k\,\pi)|^2 > 0$), et donc que $\hat{\varphi}(2\,k\,\pi) = 0$ pour $k \in \mathbb{Z}^*$. On en conclut que $I_0(\xi)$ vérifie, pour $|\xi| \leq \pi$,

$$I_0(\xi) = \sum_{\pi\leq|\xi+2k\pi|\leq 2\pi} |\hat{\varphi}(\xi + 2\,k\,\pi)|^2 \leq C\,|\xi| \leq C\,\pi \left|\sin\frac{\xi}{2}\right|,$$

de sorte que

$$I_N(\xi) \leq C\,\pi\,T_2^N\Big(\Big|\sin\frac{\xi}{2}\Big|\Big)\,.$$

Il reste à estimer

$$\Big\|T_2^N\Big(\Big|\sin\frac{\xi}{2}\Big|\Big)\Big\|_\infty\,.$$

On remarque d'abord que T_2 laisse invariant

$$E_0 = \Big\{f \in C^0(\mathbb{R}/2\pi\mathbb{Z}) \,:\, \frac{df}{d\xi} \in L^\infty \text{ et } f(0) = 0\Big\}\,.$$

De plus, si $f \in E_0$, on a

$$\|T_2^N(f)\|_\infty \leq \|T_2^N(1)\|_\infty \|f\|_\infty \leq C\,\|f\|_\infty$$

et

$$\begin{aligned}\Big\|\frac{d}{d\xi}\,T_2^N(f)\Big\|_\infty &\le \Big\|\frac{d}{d\xi}\,|m_0|^2\Big\|_\infty \|T_2^{N-1}f\|_\infty + \frac{1}{2}\,\Big\|T_2\Big(\Big|\frac{d}{d\xi}\,T_2^{N-1}f\Big|\Big)\Big\|_\infty \\ &\le \frac{1}{2^N}\Big\|T_2^N\Big(\Big|\frac{df}{d\xi}\Big|\Big)\Big\|_\infty \\ &\quad + \sum_{k=1}^{N}\Big\|\frac{d}{d\xi}\,|m_0|^2\Big\|_\infty \|T_2^{N-k}f\|_\infty \|T_2^{k-1}(1)\|_\infty\Big(\frac{1}{2}\Big)^{k-1} \\ &\le C\Big(\|f\|_\infty + \frac{1}{2^N}\Big\|\frac{df}{d\xi}\Big\|_\infty\Big)\,.\end{aligned}$$

En particulier, on obtient

$$\Big\|\frac{d}{d\xi}\,T_2^{N+M}(f)\Big\|_\infty \le C_0\Big(\|T_2^N(f)\|_\infty + \frac{1}{2^M}\Big(\|f\|_\infty + \Big\|\frac{df}{d\xi}\Big\|_\infty\Big)\Big)\,, \tag{13}$$

où C_0 ne dépend ni de f, ni de N, ni de M.

Par ailleurs, on obtient également que $\{T_2^N(|\sin(\xi/2)|)\}_{N\in\mathbb{N}}$ est bornée dans E_0, et le théorème d'Ascoli nous assure qu'il existe une sous-suite $\omega_{N_k} = T_2^{N_k}(|\sin(\xi/2)|)$ qui converge dans $C^0(\mathbb{R}/2\pi\mathbb{Z})$ vers une fonction ω. On considère alors K un compact d'Albert Cohen associé à m_0. La fonction

$$\theta_n(\xi) = \sqrt{\Big|\sin\frac{\xi}{2^{n+1}}\Big|}\;\chi_K\Big(\frac{\xi}{2^n}\Big)\prod_{j=1}^{n} m_0\Big(\frac{\xi}{2^j}\Big)\,,$$

converge ponctuellement vers $\sqrt{\sin(0)}\,\hat\varphi(0) = 0$ et se majore par $\hat\varphi(\xi)/\inf_{\eta\in K}|\hat\varphi(\eta)|$. Comme $\hat\varphi \in L^2$, le théorème de convergence dominée donne $\theta_n \to 0$ dans L^2; or

$$\int |\theta_n|^2\,d\xi = \int_{-\pi}^{\pi} T_2^n\Big(\Big|\sin\frac{\xi}{2}\Big|\Big)\,d\xi\,,$$

de sorte que $\int_{-\pi}^{\pi}\omega\,dx = 0$, et donc $\omega = 0$ (puisque $\omega \ge 0$).

On choisit dans (13) $N = N_k$ assez grand pour que

$$\Big\|T_2^N\Big(\Big|\sin\frac{\xi}{2}\Big|\Big)\Big\|_\infty$$

soit inférieur à $1/(4\pi C_0)$ et M assez grand pour que $3/(2\cdot 2^M)$ soit inférieur à $1/(4\pi C_0)$. On obtient alors, en posant $P = N + M$

$$\Big\|\frac{d}{d\xi}\,T_2^P\Big(\Big|\sin\frac{\xi}{2}\Big|\Big)\Big\|_\infty \le \frac{1}{2\pi}\,,$$

d'où, pour tout $\xi \in [-\pi, \pi]$,

$$T_2^P\Big(\Big|\sin\frac{\xi}{2}\Big|\Big) \le \frac{1}{2\pi}|\xi| \le \frac{1}{2}\Big|\sin\frac{\xi}{2}\Big|\,.$$

Maintenant, pour $Q \in \mathbb{N}$ quelconque, on obtient

$$\Big\|T_2^Q\Big(\Big|\sin\frac{\xi}{2}\Big|\Big)\Big\|_\infty \le \Big(\frac{1}{2}\Big)^{Q/P} \max_{0\le r<P} \Big\|T_2^r\Big(\Big|\sin\frac{\xi}{2}\Big|\Big)\Big\|_\infty\, 2^{r/P}\,.$$

Cela donne finalement: $|I_N(\xi)| \le C\rho_0^N$ avec $\rho_0 = (1/2)^{1/P}$. Nous avons donc démontré (11).

Il ne reste plus à vérifier que la décroissance rapide de φ dans L^2, ou encore que $(d/d\xi)^p\hat\varphi \in L^2$ pour tout $p \in \mathbb{N}$. On va en fait montrer quelque chose de plus fort: il existe $\rho_p \in]0,1[$ tel que

$$\sup_\xi \sup_{N\in\mathbb{N}} \rho_p^{-N} \sum_{2^N\pi\le|\xi+2k\pi|<2^{N+1}\pi} |\hat\varphi^{(p)}(\xi+2\,k\,\pi)|^2 < +\infty\,. \tag{14}$$

La démonstration de (14) se fait par récurrence sur p. Le cas $p=0$ a été traité avec l'inégalité (11). On va montrer que $\|I_{N,p}\|_\infty \le C_p\,\rho_p^N$ où

$$I_{N,p}(\xi) = \sum_{2^N\pi\le|\xi+2k\pi|<2^{N+1}\pi} |\hat\varphi^{(p)}(\xi+2\,k\,\pi)|^2\,.$$

Pour cela, il suffit d'écrire, pour $p \ge 1$,

$$\hat\varphi^{(p)}(\xi) = \sum_{k=0}^p C_p^k\,\frac{1}{2^p}\,m_0^{(p)}\Big(\frac{\xi}{2}\Big)\hat\varphi^{(p+k)}\Big(\frac{\xi}{2}\Big)\,.$$

On utilise l'inégalité

$$\Big(\sum_{k=0}^p|\alpha_k|\Big)^2 \le 2\,|\alpha_0|^2 + 2\Big(\sum_1^p|\alpha_k|^2\Big)^2 \le 2\,|\alpha_0|^2 + 2\,p\sum_1^p|\alpha_k|^2\,,$$

et on obtient

$$\begin{aligned} I_{N,p}(\xi) \le{}& \frac{1}{2^{2p-1}}\,T_2(I_{N-1,p}(\xi)) \\ &+ 2\,p\sum_{k=1}^p\frac{(C_k^p)^2}{4^p}\Big(\Big|m_0^{(k)}\Big(\frac{\xi}{2}\Big)\Big|^2 I_{N-1,p-k}\Big(\frac{\xi}{2}\Big) \\ &\qquad + \Big|m_0^{(k)}\Big(\frac{\xi}{2}+\pi\Big)\Big|^2 I_{N-1,p-k}\Big(\frac{\xi}{2}\Big)\Big) \\ \le{}& \frac{1}{2^{2p-1}}\,T_2(I_{N-1,p}(\xi)) + C_p'\,\sigma_p^{N-1}\,, \end{aligned}$$

avec $\sigma_p = \max_{0\leq q\leq p-1} \rho_q$ et

$$C'_p = \frac{p^2}{4^{p-1}}\left(C_p^{p/2}\right)^2 \max_{0\leq q\leq p-1}(C_q\|m_0^{(q)}\|_\infty)\,.$$

Comme T_2 est un opérateur positif, en itérant N fois on obtient

$$I_{N,p}(\xi) \leq \left(\frac{1}{2^{2p-1}}\right)^N T_2^N(I_{0,p}(\xi)) + \sum_{k=1}^N C'_p\,\sigma_p^{N-k}\left(\frac{1}{2^{2p-1}}\right)^{k-1} T_2^{k-1}(1)(\xi)\,.$$

Or on contrôle $\|T_2^k(1)\|_\infty$, et de même

$$\|T_2^N(I_{0,p}(\xi))\|_\infty \leq \|T_2^N(1)\|_\infty \|I_{0,p}\|_\infty\ ,$$

de sorte que

$$I_{N,p}(\xi) \leq C''_p\left(\frac{1}{(2^{2p-1})^N} + \sum_1^N \sigma_p^{N-k}\left(\frac{1}{2^{2p-1}}\right)^{k-1}\right),$$

d'où en choisissant ρ_p dans $]\max(\sigma_p, 1/2^{2p-1}), 1[$

$$I_{N,p}(\xi) \leq C''_p\,\rho_p^N\left(1 + \sum_1^N\left(\frac{\sigma_p}{\rho_p}\right)^{N-k}\right) \leq C'''_p\rho_p^N\ .$$

(14) est donc démontrée, ainsi que la Proposition 1.

La démonstration de la Proposition 1, qui donne l'appartenance de φ à H^α pour un $\alpha > 0$, peut être facilement adaptée pour l'étude de l'appartenance de φ à H^s, $s \in \mathbb{R}$. Parmi les multiples caractérisations, nous en choisissons une qui sera particulièrement bien adaptée à notre étude de la convergence des filtres.

Proposition 2. *Soit $m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ telle que $m_0(0) = 1$ et qui satisfait le critère d'Albert Cohen. Soit $\hat\varphi = \prod_{j=1}^\infty m_0(\xi/2^j)$ et T_2 l'opérateur*

$$T_2 f(\xi) = \left|m_0\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right) + \left|m_0\left(\frac{\xi}{2}+\pi\right)\right|^2 f\left(\frac{\xi}{2}+\pi\right).$$

Soit enfin $s \in \mathbb{R}$. Alors les assertions suivantes sont équivalentes:

B1) *Il existe* $\sigma > s$ *tel que* $\varphi \in H^\sigma$.

B2) *Il existe un entier* $N \geq 0$ *tel que* $N > s$, *un nombre* ρ *tel que* $\rho < 4^{N-s}$ *et un entier* $Q \geq 1$ *tel que:*

i) m_0 *se factorise en* $m_0(\xi) = ((1+e^{-i\xi})/2)^N \tilde{m}_0(\xi)$ *avec* $\tilde{m}_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$,

ii) *l'opérateur* $\tilde{T}_2$, *défini par*

$$\tilde{T}_2 f = \left|\tilde{m}_0\Big(\frac{\xi}{2}\Big)\right|^2 f\Big(\frac{\xi}{2}\Big) + \left|\tilde{m}_0\Big(\frac{\xi}{2}+\pi\Big)\right|^2 f\Big(\frac{\xi}{2}+\pi\Big)$$

vérifie

$$\|\tilde{T}_2^Q(1)\|_\infty \leq \rho^Q\,. \tag{15}$$

PREUVE. B2) implique B1) est relativement immédiat. On a

$$\begin{aligned}
&\int_{|\xi|\geq\pi} |\xi|^{2\sigma} |\hat{\varphi}(\xi)|^2\, d\xi \\
&\qquad \leq \sum_{k=0}^{\infty} \int_{2^k\pi\leq|\xi|<2^{k+1}\pi} (2^{k+1}\pi)^{2\sigma} \sup_{|\eta|\leq\pi/2} |\hat{\varphi}(\eta)|^2 \prod_{j=1}^{k+1} \left|m_0\Big(\frac{\xi}{2^j}\Big)\right|^2 d\xi \\
&\qquad \leq \sum_{k=0}^{\infty} (2^{k+1}\pi)^{2\sigma} \sup_{|\eta|\leq\pi} |\hat{\varphi}(\eta)|^2 \\
&\qquad\qquad \cdot \int_{|\xi|<2^{k+1}\pi} \left|\sin\frac{\xi}{2^{k+2}}\right|^{2N} 2^N \prod_{j=1}^{k+1} \left|m_0\Big(\frac{\xi}{2^j}\Big)\right|^2 d\xi \\
&\qquad = \sum_{k=0}^{\infty} (2^{k+1}\pi)^{2\sigma} 2^N \sup_{|\eta|\leq\pi} |\hat{\varphi}(\eta)|^2 \int_{-\pi}^{\pi} T_2^{k+1}\Big(\sin^{2N}\frac{\xi}{2}\Big)\, d\xi\,.
\end{aligned}$$

Mais on a

$$T_2\Big(\sin^{2N}\frac{\xi}{2} f\Big) = \frac{1}{2^{2N}} \sin^{2N}\frac{\xi}{2}\, \tilde{T}_2 f\,,$$

en effet,

$$\left|m_0\Big(\frac{\xi}{2}\Big)\right|^2 = \cos^{2N}\frac{\xi}{4} \left|\tilde{m}_0\Big(\frac{\xi}{2}\Big)\right|^2$$

et il suffit d'écrire

$$\begin{aligned}
\sin^{2N}\frac{\xi}{2} &= 2^{2N}\sin^{2N}\frac{\xi}{4}\cos^{2N}\frac{\xi}{4} \\
&= 2^{2N}\sin^{2N}\Big(\frac{\xi}{4}+\frac{\pi}{2}\Big)\cos^{2N}\Big(\frac{\xi}{4}+\frac{\pi}{2}\Big)\,.
\end{aligned}$$

On obtient donc

$$\int_{-\pi}^{\pi} T_2^{k+1}\Big(\sin^{2N}\frac{\xi}{2}\Big)\,d\xi = \Big(\frac{1}{2^{2N}}\Big)^{k+1}\int_{-\pi}^{\pi}\tilde{T}_2^{k+1}(1)\,d\xi$$

et donc

$$\int_{|\xi|\geq\pi}|\xi|^{2\sigma}|\hat{\varphi}(\xi)|^2\,d\xi$$

$$\leq 2^N\pi^{2\sigma}\sup_{|\eta|\leq\pi}|\hat{\varphi}(\eta)|^2\sum_{k=0}^{\infty}\Big(\frac{2^{2\sigma}}{2^{2N}}\Big)^{k+1}\|\tilde{T}_2^{k+1}(1)\|_{L^1(-\pi,\pi)}\,.$$

Cette série converge dès que $\|\tilde{T}_2^{k+1}(1)\|_{L^1(-\pi,\pi)} \leq C\tau^{k+1}$ avec $\tau < 2^{2(N-\sigma)}$. Or on a

$$\|\tilde{T}_2^{k+1}(1)\|_{L^1} \leq 2\pi\|\tilde{T}_2^{k+1}(1)\|_\infty \leq 2\pi\rho^{k+1}\max_{0\leq r<Q}\rho^{-r}\|\tilde{T}_2^{r}(1)\|_\infty$$

(cela provient de ce que $\|\tilde{T}_2^r(1)\|_\infty$ est la norme d'opérateur de $\tilde{T}_2^r$ sur $L^\infty(-\pi,\pi)$, et donc $\|\tilde{T}_2^{r+r'}(1)\|_\infty \leq \|\tilde{T}_2^r(1)\|_\infty\|\tilde{T}_2^{r'}(1)\|_\infty$). Si on a $\rho < 4^{N-\sigma}$, on a $\varphi \in H^\sigma$.

B1) implique B2). On commence par remarquer que si $\varphi \in H_\sigma$, pour un $\sigma > s$, alors φ est à décroissance rapide dans H^t pour tout $t < \sigma$ (et en particulier dans H^s). En fait, d'après le Lemme B, il suffit de le vérifier pour un seul $t_0 \in \mathbb{R}$. Pour $t \leq t_0$, cela sera alors immédiat; pour tout $t_0 < t < \sigma$, cela se fera par interpolation.

Puisque $|\hat{\varphi}(\xi)| \leq C\,(1+|\xi|)^L$ pour un $L \in \mathbb{N}$, on sait que la distribution

$$\hat{\varphi}_1(\xi) = \Big(\frac{1-e^{-i\xi}}{i\,\xi}\Big)^{L+1}\hat{\varphi}(\xi)$$

vérifie que $\sum_{k\in\mathbb{Z}}|\hat{\varphi}_1(\xi+2\,k\,\pi)|^2$ converge uniformément sur $[-\pi,\pi]$, vers une fonction γ_1 qui ne s'annule pas sur $[-\pi,\pi]$: si K est un compact d'Albert Cohen associé à m_0, on a

$$\inf_\xi\sum_{k\in\mathbb{Z}}|\hat{\varphi}_1(\xi+2\,k\,\pi)|^2 \geq \inf_{\eta\in K}\Big|\frac{1-e^{-i\eta}}{i\,\eta}\Big|^{2L+2}\inf_{\eta\in K}|\hat{\varphi}(\eta)|^2 > 0\,.$$

On a $\hat{\varphi}_1(\xi) = \prod_{j=1}^{\infty} m_1(\xi)$ avec $m_1 = ((1+e^{-i\xi})/2)^{L+1}m_0$. Si on définit $T_{1,2}$ par

$$T_{1,2}f = \Big|m_1\Big(\frac{\xi}{2}\Big)\Big|^2 f\Big(\frac{\xi}{2}\Big) + \Big|m_1\Big(\frac{\xi}{2}+\pi\Big)\Big|^2 f\Big(\frac{\xi}{2}+\pi\Big)\,,$$

on a nécessairement $\|T_{1,2}^N(1)\|_\infty \le \sup \gamma_1 / \inf \gamma_1$. La Proposition 1 montre alors que φ_1 est à décroissance rapide dans L^2, et que $(d/dx)^{L+1}\varphi_1$ est à décroissance rapide dans H^{-L-1}. En appliquant le Lemme F, du fait que $\widehat{(d/dx)^{L+1}\varphi_1} = (1-e^{-i\xi})^{L+1}\hat{\varphi}$, on obtient que φ elle-même est à décroissance rapide dans H^{-L-1}.

On remarque également qu'on peut supposer $s < 0$. En effet, si $s \ge 0$, φ est à décroissance rapide dans L^2. Cela implique que $m_0(\pi) = 0$, comme on l'a vu dans la démonstration de la Proposition 1. Alors m_0 se factorise en $m_0(\xi) = ((1+e^{-i\xi})/2)m_1(\xi)$ et $\hat{\varphi}$ en $\hat{\varphi}(\xi) = (1-e^{-i\xi})/(i\xi)\hat{\varphi}_1$. De plus φ est à décroissance rapide dans H^s si et seulement si $d\varphi/dx$ l'est dans H^{s-1} (Lemme E), c'est-à-dire si et seulement si φ_1 l'est dans H^{s-1} (Lemme F). Ainsi, si $N \le s < N+1$, on obtient que $m_0(\xi) = ((1+e^{-i\xi})/2)^{N+1}\tilde{m}_0(\xi)$ et que φ appartient à un H^σ avec $\sigma > s$ si et seulement si $\tilde{\varphi}$ (définie par sa transformée de Fourier $\prod_{j=1}^\infty \tilde{m}_0(\xi/2^j)$) appartient à un H^t avec $t > s - N$.

On suppose donc que $s < 0$ et que $\varphi \in H^\sigma$ pour un $\sigma \in]s, 0[$. On a alors, si K est un compact d'Albert Cohen associé à m_0

$$\begin{aligned}\int (1+|\xi|)^{2\sigma}|\hat{\varphi}(\xi)|^2\,d\xi &\ge \int (1+|\xi|)^{2\sigma}\chi_K\Big(\frac{\xi}{2^M}\Big)|\hat{\varphi}(\xi)|^2\,d\xi \\ &\ge \frac{1}{C}\,(2^M)^{2\sigma} \inf_{\eta\in K}|\hat{\varphi}(\eta)|^2 \\ &\quad \cdot \int \chi_K\Big(\frac{\xi}{2^M}\Big)\prod_{j=1}^M \Big|m_0\Big(\frac{\xi}{2^j}\Big)\Big|^2\,d\xi\,.\end{aligned}$$

D'où

$$\int_{-\pi}^{\pi} T_2^M(1)\,d\xi = \int \chi_K\Big(\frac{\xi}{2^M}\Big)\prod_{j=1}^M \Big|m_0\Big(\frac{\xi}{2^j}\Big)\Big|^2\,d\xi \le C\,(2^{2\sigma})^{-M}\,.$$

On obtient que $\|T_2^M(1)\|_{L^1(-\pi,\pi)} \le C\tau^M$, avec $\tau < 4^{-s}$. (Nous avons vu qu'inversement cette inégalité assure l'appartenance de φ à H^t pour $4^t\tau < 1$).

Nous avons obtenu une condition nécessaire et suffisante pour l'appartenance de φ à $\cap_{\sigma>s}H^\sigma$ lorsque $s < 0$, mais cette condition est malcommode puisqu'il s'agit de montrer que

$$\limsup_{M\to+\infty} \|T_2^M(1)\|_{L^1(-\pi,\pi)}^{1/M} < 4^{-s}\,.$$

Il s'agit maintenant de remplacer la norme L^1 par la norme L^∞ de sorte à estimer un rayon spectral ($\|T_2^M(1)\|_\infty$ est la norme d'opérateur de T_2^M agissant sur $L^\infty(-\pi,\pi)$). Dans la littérature, c'est souvent un jeu d'enfant: on n'étudie que le cas des polynômes trigonométriques; T_2 agit alors sur un espace de dimension fini, sur lequel les normes L^1 et L^∞ sont équivalentes. Mais nous traitons ici un cas un peu plus général ($m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$).

Pour contrôler la norme $\|T_2^M(1)\|_\infty$, il suffit de remarquer que si $\varphi \in H^\sigma$ pour un $\sigma > s$, alors φ est à décroissance rapide dans H^t pour tout $t < \sigma$ (et en particulier pour $t \in]s,\sigma[$); or ceci est équivalent à ce que φ_t, définie par $\hat\varphi_t = (1+|\xi^2|)^{t/2}\hat\varphi$, soit à décroissance rapide dans L^2 (il suffit de vérifier que

$$\Big(\frac{d}{d\xi}\Big)^k (1+|\xi|^2)^{t/2} = (1+|\xi|^2)^{t/2}\mu_{t,k}(\xi)$$

où $\mu_{t,k}$ est C^∞ bornée ainsi que toutes ses dérivées); le Lemme 1 donne donc

$$\sup_{\xi\in[-\pi,\pi]} \sum_{k\in\mathbb{Z}} (1+|\xi+2\,k\,\pi|^2)^t |\hat\varphi(\xi+2\,k\,\pi)|^2 < +\infty\,, \qquad \text{pour } t<\sigma\,. \tag{16}$$

Soit alors K un compact d'Albert Cohen associé à m_0 et $I_{M,t}$ la quantité

$$I_{M,t}(\xi) = \sum_{k\in\mathbb{Z}} \chi_K\Big(\frac{\xi+2\,k\,\pi}{2^M}\Big)(1+|\xi+2\,k\,\pi|^2)^t|\hat\varphi(\xi+2\,k\,\pi)|^2\,.$$

Si $t<0$, on a

$$\begin{aligned}
&\chi_K\Big(\frac{\xi+2\,k\,\pi}{2^M}\Big)(1+|\xi+2\,k\,\pi|^2)^t|\hat\varphi(\xi+2\,k\,\pi)|^2\\
&\geq \chi_K\Bigg(\frac{\frac{\xi}{2}+k\,\pi}{2^{M-1}}\Bigg)4^t\Big(1+\Big|\frac{\xi}{2}+k\,\pi\Big|^2\Big)^t\Big|m_0\Big(\frac{\xi}{2}+k\,\pi\Big)\Big|^2|\hat\varphi\Big(\frac{\xi}{2}+k\,\pi\Big)\Big|^2
\end{aligned}$$

et donc

$$\begin{aligned}
I_{M,t} &\geq 4^t\,T_2(I_{M-1,t})\\
&\geq 4^{Mt}\,T_2^M(I_{0,t})\\
&\geq C_t\,4^{Mt}\inf_K|\hat\varphi(\eta)|^2\,T_2^M(1)(\xi)\,,
\end{aligned}$$

de sorte que

$$\|T_2^M(1)\|_\infty \leq C'_t\, 4^{-Mt} \sup_\xi \sum_{k\in\mathbb{Z}} (1+|\xi+2\,k\,\pi|^2)^t |\hat{\varphi}(\xi+2\,k\,\pi)|^2\,.$$

Si on choisit r et t tels que $s < r < t < \sigma$, on a donc $\|T_2^M(1)\|_\infty \leq C''_t\, 4^{-Mt}$ pour tout M, et donc $\|T_2^M(1)\|_\infty \leq 4^{-Mr}$ pour M assez grand. Il suffit alors de poser $\rho = 4^{-r}$.

Le principal argument de la preuve des propositions 1 et 2 a été la positivité de l'opérateur T_2, et l'égalité corollaire de $\|T_2\|_{\mathcal{L}(L^\infty,L^\infty)}$ et $\|T_2(1)\|_\infty$. Les propositions 1 et 2 s'adaptent alors aisément pour des filtres à valeurs positives:

Proposition 1 bis. *Soit* $M_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ *telle que*

i) $M_0(0)=1$, $M_0(\xi)\geq 0$ *et* $M_0(\xi)=M_0(-\xi)$ *pour tout* ξ.

ii) M_0 *vérifie le critère d'Albert Cohen.*

On définit $\hat{\Phi}$ *par*

$$\hat{\Phi}(\xi) = \prod_{j=1}^{\infty} M_0\Big(\frac{\xi}{2^j}\Big)$$

et l'opérateur T_1 *par*

$$T_1 f(\xi) = M_0\Big(\frac{\xi}{2}\Big) f\Big(\frac{\xi}{2}\Big) + M_0\Big(\frac{\xi}{2}+\pi\Big) f\Big(\frac{\xi}{2}+\pi\Big)\,.$$

Alors les assertions suivantes sont équivalentes:

C1) Φ *est à décroissance rapide dans* L^∞ *et pour un* $\varepsilon > 0$, Φ *est* C^ε *(régularité höldérienne)*

C2) $\sup_{N\in\mathbb{N}} \|T_1^N(1)\|_\infty < +\infty$.

PREUVE. La démonstration de la Proposition 1 s'adapte presque immédiatement.

C1) implique C2). La seule chose à vérifier est que $\sum_{k\in\mathbb{Z}} \hat{\Phi}(\xi + 2\,k\,\pi)$ converge pour tout ξ vers la fonction C^∞, $\sum_{k\in\mathbb{Z}} \Phi(k)\, e^{-ik\xi}$. Pour cela, on considère la fonction $u_\xi(x) = \sum_{k\in\mathbb{Z}} \Phi(x+k)\, e^{-i(x+k)\xi}$.

C'est une fonction périodique, bornée et höldérienne: si ω est C^ε et à décroissance rapide dans L^∞, on a

$$\Big|\sum_{k\in\mathbb{Z}}\omega(x+k)\Big| \le \|\omega(x)\,(1+x^2)\|_\infty \sum_{k\in\mathbb{Z}} \frac{1}{1+(x+k)^2} \le C < +\infty\,,$$

$$\begin{aligned}\Big|\sum_{k\in\mathbb{Z}}&\omega(x+h+k)-\omega(x+k)\Big|\\ &\le \|\omega\|_{C^\varepsilon}\,|h|^\varepsilon\Big(\sum_{|x+k|\le K_0} 1\Big) + 2\,\|\omega(x)\,x^2\|_\infty \sum_{|x+k|>K_0}\frac{1}{|x+k|^2}\\ &\le C\Big(K_0|h|^\varepsilon+\frac{1}{K_0}\Big) \le C'|h|^{\varepsilon/2}\,,\end{aligned}$$

pour $|h|\le 1$ et $K_0 \approx |h|^{-\varepsilon/2}$.

Puisque u_ξ est höldérienne, on a

$$\sum_{k\in\mathbb{Z}}\Phi(x+k)\,e^{-i(x+k)\xi} = \sum_{k\in\mathbb{Z}}\hat{\Phi}(\xi+2\,k\,\pi)\,e^{+2ik\pi x}$$

pour tout x; en prenant $x=0$, on obtient la convergence souhaitée.

C2) implique C1). La démonstration est exactement similaire à celle de la Proposition 1. On obtient en particulier que pour tout $p\in\mathbb{N}$,

$$\sum_{k\in\mathbb{Z}}\Big|\frac{d^p}{d\xi^p}\,\hat{\Phi}(\xi+2\,k\,\pi)\Big| < +\infty\,,$$

ce qui entraîne $x^p\Phi\in C_0$.

Proposition 2 bis. *Soit $M_0\in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ telle que*

i) $M_0(0)=1$, $M_0(\xi)\ge 0$ *et* $M_0(\xi)=M_0(-\xi)$ *pour tout* $\xi\in\mathbb{R}$.

ii) M_0 *vérifie le critère d'Albert Cohen.*

iii) $\sup_{N\in\mathbb{N}}\|T_1^N(1)\|_\infty < +\infty$ *où*

$$T_1 f = M_0\Big(\frac{\xi}{2}\Big)f\Big(\frac{\xi}{2}\Big) + M_0\Big(\frac{\xi}{2}+\pi\Big)f\Big(\frac{\xi}{2}+\pi\Big)\,.$$

Soit $\hat{\Phi} = \prod_{j=1}^\infty M_0(\xi/2^j)$. *Alors les assertions suivantes sont équivalentes, pour* $\alpha\ge 0$.

D1) *Il existe* $\beta > \alpha$ *tel que* $|\xi|^\beta \hat{\Phi} \in L^1$ (*ce qui équivaut à* $\Phi \in B_1^{\beta,\infty}$ *puisque* $\hat{\Phi}$ *est positive ou nulle*).

D2) *En écrivant* $\alpha = 2N + \theta$, $N \in \mathbb{N}$, $0 \leq \theta < 2$, M_0 *vérifie les conditions suivantes:*

i) $M_0 = ((1+\cos\xi)/2)^{N+1} \tilde{M}_0$ *avec* $\tilde{M}_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$.

ii) *Il existe* $\rho \in [1, 2^{2-\theta}[$ *et* $Q \in \mathbb{N}^*$ *tels que*

$$\|\tilde{T}_1^Q(1)\|_\infty \leq \rho^Q ,$$

où

$$\tilde{T}_1 f(\xi) = \tilde{M}_0\Big(\frac{\xi}{2}\Big) f\Big(\frac{\xi}{2}\Big) + \tilde{M}_0\Big(\frac{\xi}{2}+\pi\Big) f\Big(\frac{\xi}{2}+\pi\Big) .$$

PREUVE. D2) implique D1). Comme pour la Proposition 2.

D1) implique D2). On commence par remarquer que si $\Phi \in B_1^{\beta,\infty}$, alors Φ est à décroissance rapide dans $B_1^{\gamma,\infty}$ pour tout $\gamma < \beta$ (on utilise l'inclusion $L^\infty \subset B_1^{\delta,\infty}$ pour tout $\delta < 0$ et le lemme B pour $B_1^{\delta,\infty}$ et $B_1^{\beta,\infty}$).

On montre alors que Φ_1, de transformée de Fourier $\prod_{j=1}^\infty \tilde{M}_0(\xi/2^j)$, est à décroissance rapide dans $B_1^{\gamma-2N-2,\infty}$ pour tout $\gamma < \beta$ (en particulier pour $\alpha < \gamma < \beta$). On obtient alors que

$$\int_{-\pi}^{\pi} \tilde{T}_1^M(1)\, d\xi \leq C\, 2^{(2N+2-\gamma)M} ,$$

pour tout M et le problème est à nouveau de passer de la norme L^1 à la norme L^∞. Pour cela, on note u_γ la fonction définie par

$$\hat{u}_\gamma = |\xi|^{2N+2}(1+|\xi|^2)^{(\gamma-2N-2)/2}\, \hat{\Phi} .$$

On a $(1+|\xi|)^{\beta-\gamma}\hat{u}_\gamma \in L^1$, de sorte que $u_\gamma \in B_1^{\beta-\gamma,\infty}$. De plus, Φ est à décroissance rapide dans L^∞, donc dans $B_1^{\delta,\infty}$ pour tout $\delta < 0$; on en conclut que u_γ est à décroissance rapide dans $B_1^{\delta-\gamma,\infty}$ (il suffit de vérifier que $(d^2/d\xi^2)^{N+1} u_\gamma$ est à décroissance rapide dans $B_1^{\delta-2N-2,\infty}$, puisque pour tout $\rho \in \mathbb{R}$ et $\sigma \in \mathbb{R}$, $(\mathrm{Id}-\Delta)^\rho$ envoie $B_1^{\sigma,\infty}$ dans $B_1^{\sigma-\rho,\infty}$ tandis que $[x, (\mathrm{Id}-\Delta)^\rho] = (\mathrm{Id}-\Delta)^\rho \circ S_\rho$ où S_ρ envoie $B_1^{\sigma,\infty}$ dans $B_1^{\sigma,\infty}$).

On en conclut que u_γ est à décroissance rapide dans $B_1^{\sigma,\infty}$ pour tout $\sigma < \beta - \gamma$, donc (puisque $\gamma < \beta$) dans L^∞. On sait alors que

$$\sum_{k\in\mathbb{Z}}(1 + |\xi + 2\,k\,\pi|)^{(\gamma-2N-2)/2}|\xi + 2\,k\,\pi|^{2N+2}\,\hat{\Phi}(\xi + 2\,k\,\pi)$$
$$= \sum_{k\in\mathbb{Z}} u_\gamma(k)\,e^{-ik\xi} \in C^\infty\,.$$

De plus, on a

$$\sum_{k\in\mathbb{Z}}(x-k)^p\,u_\gamma(x-k) = \sum_{k\in\mathbb{Z}}\Big(i\,\frac{\partial}{\partial\xi}\Big)^p\,\hat{u}_\gamma(2\,k\,\pi)\,e^{2ik\pi x}\,,$$

au sens des distributions (la première série convergeant uniformément sur tout compact); pour $0 \le p \le N+1$, on obtient

$$\sum_{k\in\mathbb{Z}}(x-k)^p u_\gamma(x-k) = 0\,.$$

D'où

$$\Big(\frac{d}{d\xi}\Big)^p \sum_{k\in\mathbb{Z}} u_\gamma(k)\,e^{-ik\xi}\Big|_{\xi=0} = \sum_{k\in\mathbb{Z}}(-i\,k)^p u_\gamma(k) = 0\,,$$

et donc

$$\sum_{k\in\mathbb{Z}} u_\gamma(k)\,e^{-ik\xi} = (1-\cos\xi)^{N+1}M(\xi)\,,\qquad \text{avec } M \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})\,.$$

Cela donne

$$\sum_{k\in\mathbb{Z}}(1 + |\xi + 2\,k\,\pi|)^{(\gamma-2N-2)/2)}\,\hat{\Phi}_1(\xi + 2\,k\,\pi) \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$$

(car $\hat{\Phi}(\xi) = (\xi^2/(2(1-\cos\xi))^{N+1}\hat{\Phi}_1(\xi))$ et la démonstration peut alors se faire.

Enfin, le dernier résultat classique caractérise les *fonctions d'échelle interpolantes* et les *fonctions d'échelle orthonormées*:

Proposition 3. *Soit* $m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ *telle que* $m_0(0) = 1$, $m_0(\xi) = \overline{m_0(-\xi)}$ *et* m_0 *satisfait le critère d'Albert Cohen, et soit*

$$\hat{\varphi} = \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big).$$

i) *Les deux assertions suivantes sont équivalentes:*

E1) *La famille* $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ *est orthonormée.*

E2) $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$, *pour tout* $\xi \in \mathbb{R}$.

ii) *Si* $m_0(\xi) \geq 0$ *pour tout* ξ, *alors les deux assertions suivantes sont équivalentes*:

F1) $\varphi \in C_0$, $\varphi(0) = 1$ *et pour* $k \in \mathbb{Z}$, $k \neq 0$, $\varphi(k) = 0$.

F2) $m_0(\xi) + m_0(\xi+\pi) = 1$, *pour tout* $\xi \in \mathbb{R}$.

PREUVE. Il suffit de démontrer l'équivalence entre F1) et F2). En effet, la famille $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ est orthonormée si et seulement si la fonction $\Phi(x) = \int \varphi(y)\,\varphi(y-x)\,dy$ vérifie $\Phi(0) = 1$ et $\Phi(k) = 0$; or on a $\hat{\Phi}(\xi) = \prod_{j=1}^{\infty} |m_0(\xi/2^j)|^2$.

Maintenant si $m_0(\xi) \geq 0$ et si φ est bornée, alors $\hat{\varphi}$ doit être intégrable de sorte que $\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi + 2\,k\,\pi)$ converge presque partout. De plus, au sens des distributions

$$\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi + 2\,k\,\pi) = \sum_{k\in\mathbb{Z}} \varphi(k)\, e^{-ik\xi}.$$

Si φ vérifie F1), on trouve $\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi + 2\,k\,\pi) = 1$ presque pour tout, ce qui entraîne F2) presque partout (et donc partout).

Inversement si m_0 vérifie F2), alors $T_1(1) = 1$ et donc $\hat{\varphi} \in L^1$ d'après la Proposition 1 bis. On sait alors que si K est un compact d'Albert Cohen,

$$\hat{\theta}_M = \prod_{j=1}^{M} m_0\Big(\frac{\xi}{2^j}\Big)\chi_K\Big(\frac{\xi}{2^M}\Big) \longrightarrow \hat{\varphi}$$

dans L^1, et donc $\theta_M \to \varphi$ dans L^∞. Mais il est facile de vérifier que $\theta(0) = 1$ et $\theta(k) = 0$ pour $k \in \mathbb{Z}^*$: cela se déduit par récurrence sur M à partir de $\sum_{k\in\mathbb{Z}} \chi_K(\xi + 2\,k\,\pi) = 1$ presque pour tout et donc $\theta_0(0) = 1$, $\theta_0(k) = 0$ pour $k \neq 0$.

Nous avons maintenant tous les outils de la théorie "classique" (légèrement remaniée pour les besoins de la cause) dont nous aurons besoin par la suite.

3. Dépendance de la fonction d'échelle par rapport à son filtre.

Nous pouvons maintenant facilement aborder le problème de la dépendance de la fonction d'échelle φ par rapport à son filtre m_0. C'est-à-dire que peut-on dire des fonctions d'échelle φ_n dont les filtres d'échelle m_n convergent dans $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ vers une fonction m_∞? En fait, la réponse est étonnamment simple. La convergence est simple à vérifier; pour la convergence des dérivées, il faut que celles-ci existent, et donc au moins que les filtres m_n s'annulent suffisamment en π (et en fait ce sera la seule restriction!).

Théoréme 1 (Premier Théorème d'approximation). *Soit $m_\infty \in C^\infty$ $(\mathbb{R}/2\pi\mathbb{Z})$ vérifiant le critère d'Albert Cohen et soit*

$$\{m_n\}_{n\in\mathbb{N}} \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})^{\mathbb{N}}$$

une suite de fonctions qui convergent vers m_∞ dans $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$. On suppose que pour tout $n \in \mathbb{N}$, $m_n(0) = 1$ et $m_n(\xi) = \overline{m_n(-\xi)}$. On définit φ_n $(n \in \mathbb{N} \cup \{\infty\})$ par

$$\hat{\varphi}_n = \prod_{j=1}^{\infty} m_n\left(\frac{\xi}{2^j}\right).$$

i) *Soit $\alpha \in \mathbb{R}$. Alors les assertions suivantes sont équivalentes:*

G1) *Pour un $\beta > \alpha$, φ_n est à décroissance rapide dans H^β pour n suffisamment grand et φ_n converge rapidement vers φ_∞ dans H^β.*

G2) *Pour un $\gamma > \alpha$, $\varphi_\infty \in H^\gamma$ et pour n suffisamment grand on a, pour $0 \leq p \leq N_\alpha$, $(d/d\xi)^p m_n(\pi) = 0$ où $N_\alpha = -1$ si $\alpha < 0$ et $N_\alpha = [\alpha]$ si $\alpha \geq 0$.*

ii) *De même si les m_N sont toutes à valeurs positives ou nulles, et si $\alpha \geq 0$ alors les assertions suivantes sont équivalentes:*

H1) *Pour un $\beta > \alpha$, φ_n est à décroissance rapide dans $B_1^{\beta,\infty}$ pour n suffisamment grand et φ_n converge rapidement vers φ_∞ dans $B_1^{\beta,\infty}$.*

H2) *Pour un* $\gamma > \alpha$, $\varphi_\infty \in B_1^{\gamma,\infty}$ *et pour* n *suffisamment grand, on a, pour* $0 \le p \le 2N_\alpha + 1$, $(d^p/d\xi^p)m_n(\pi) = 0$ *où* $N_\alpha = [\alpha/2]$.

REMARQUE. On peut bien sûr remplacer l'espace de Besov $B_1^{\beta,\infty}$ par l'espace plus naturel $B_\infty^{\beta,\infty}$ (puisque pour $\beta_1 < \beta_2$, $B_1^{\beta_2,\infty} \subset B_\infty^{\beta_2,\infty} \subset B_1^{\beta_1,\infty}$): si $\beta > 0$ et $\beta \notin \mathbb{N}$, $B_\infty^{\beta,\infty}$ est l'espace des fonctions de classe $C^{[\beta]}$, bornées ainsi que leurs dérivées jusqu'à l'ordre $[\beta]$ et telles que leur dérivée $[\beta]$-ième soit höldérienne d'exposant $\beta - [\beta]$.

PREUVE. On commence par remarquer qu'on peut supposer sans perte de généralité que les filtres $\{m_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ vérifient tous le critère d'Albert Cohen pour un même compact K. En effet, $\hat{\varphi}_n \to \hat{\varphi}_\infty$ uniformément sur tout compact: sur un voisinage V de 0, il suffit de passer au logarithme pour le voir; le cas général s'en déduit aisément en écrivant pour $\xi \in 2^L V$,

$$\varphi_n(\xi) = \hat{\varphi}_n\Big(\frac{\xi}{2^L}\Big)\prod_{j=1}^{L} m_n\Big(\frac{\xi}{2^j}\Big)\,.$$

Si K est un compact d'Albert Cohen associé à m_∞, on a $\inf_K |\hat{\varphi}_\infty(\xi)| > 0$; on obtient alors que $\inf_K |\hat{\varphi}_n(\xi)| > 0$ pour n assez grand.

Les implications G1) implique G2) et H1) implique H2) sont immédiates: d'après les propositions 2 et 2 bis, si $\varphi_n \in H^\beta$ pour un $\beta > \alpha$, alors m_n peut se factoriser en $((1+e^{-i\xi})/2)^{N_\alpha+1}\tilde{m}_n(\xi)$ et de même on a un résultat analogue si $\varphi_n \in B_1^{\beta,\infty}$.

G2) implique G1). On factorise m_n en $((1+e^{-i\xi})/2)^{N_\alpha+1}\tilde{m}_n(\xi)$ pour $n \ge n_0$ (où $n \ge n_0$ implique $(d^p/d\xi^p)m_n(\pi) = 0$ pour $0 \le p \le N_\alpha$). Alors $\tilde{m}_n \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ et converge dans C^∞ vers $\tilde{m}_\infty$. On pose

$$\tilde{\omega}_n = \prod_{j=1}^{\infty} \tilde{m}_n\Big(\frac{\xi}{2^j}\Big)$$

et

$$\tilde{T}_{2,n} f = \Big|\tilde{m}_n\Big(\frac{\xi}{2}\Big)\Big|^2 f\Big(\frac{\xi}{2}\Big) + \Big|\tilde{m}_n\Big(\frac{\xi}{2}+\pi\Big)\Big|^2 f\Big(\frac{\xi}{2}+\pi\Big)\,.$$

Puisque $\varphi_\infty \in H^\beta$, on sait qu'il existe $\gamma \in]\alpha,\beta[$ et $Q \in \mathbb{N}^*$ tel que

$$\|\tilde{T}_{2,\infty}^Q(1)\|_\infty \le 4^{(N_\alpha+1-\gamma)Q}\,.$$

Comme $\tilde{T}_{2,n}^Q(1) \to \tilde{T}_{2,\infty}^Q(1)$ dans $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$, on conclut que si $\delta \in]\alpha,\gamma[$ il existe n_1 tel que pour $n \ge n_1$

$$\|\tilde{T}_{2,n}^Q(1)\|_\infty \le 4^{(N_\alpha+1-\delta)Q}$$

et enfin

$$\sup_{n \geq n_1} \|\varphi_n\|_{H^\sigma} < +\infty\,, \qquad \text{pour tout } \sigma < \delta\,.$$

Par ailleurs, on vérifie facilement qu'il existe $B \in \mathbb{R}$ tel que

$$\sup_n \|(1+|\xi|)^B \hat{\varphi}_n\|_\infty < +\infty\,.$$

(Bien sûr, B est très négatif). La convergence ponctuelle de $\hat{\varphi}_n$ vers $\hat{\varphi}_\infty$ donne que φ_n converge vers φ_∞ dans H^σ pour tout $\sigma < B - 1/2$. On obtient donc que φ_n converge vers φ_∞ dans H^σ pour tout $\sigma < \delta$ (par interpolation entre H^{B-1} et $H^{(\sigma+\delta)/2}$).

Par ailleurs

$$\Big(\frac{1-e^{-i\xi}}{i\,\xi}\Big)^M \hat{\varphi}_n = \hat{\omega}_n\ ,$$

avec $M \in \mathbb{N}$, $M > -B + 1/2$, vérifie

$$\sum_{k\in\mathbb{Z}} |\hat{\omega}_n(\xi + 2\,k\,\pi)|^2 < +\infty\,,$$

pour $n \geq n_0$, et donc $((1+e^{-i\xi})/2)^M m_n$ dans L^2. De plus, $\|x^k \omega_n\|_2$ se majore à l'aide des quantités $\|\mu_n^{(p)}\|_\infty$ (où $\mu_n = ((1+e^{-i\xi})/2)^M m_n$) pour $0 \leq p \leq k$ et $\sup_{N\in\mathbb{N}} \|T_{2,n}^N(1)\|_\infty$ et $\rho_{0,n} < 1$ tel que pour un Q,

$$T_{2,n}^Q\Big(\Big|\sin\frac{\xi}{2}\Big|\Big) \leq \rho_{0,n}^Q \Big|\sin\frac{\xi}{2}\Big|\,,$$

oú

$$T_{2,n}(f) = \Big|\mu_n\Big(\frac{\xi}{2}\Big)\Big|^2 f\Big(\frac{\xi}{2}\Big) + \Big|\mu_n\Big(\frac{\xi}{2}+\pi\Big)\Big|^2 f\Big(\frac{\xi}{2}+\pi\Big)\,.$$

Or on a bien évidemment: $\sup_n \|\mu_n^{(p)}\|_\infty < +\infty$; par ailleurs, on a

$$\sup_{n\geq n_0}\sup_N \|T_{2,n}^N(1)\|_\infty \leq \sup_{n\geq n_0} \frac{\displaystyle\sup_{\xi\in[-\pi,\pi]} \sum_{k\in\mathbb{Z}} |\hat{\omega}_n(\xi+2\,k\,\pi)|^2}{\displaystyle\inf_{\xi\in K} |\hat{\omega}_n(\xi)|^2} < +\infty\,,$$

où K est un compact d'Albert Cohen commun à tous les m_n. Enfin, on écrit $\mu_n = ((1+e^{-i\xi})/2)\tilde{\mu}_n$ et

$$T_{2,n}\Big(\Big|\sin\frac{\xi}{2}\Big| f\Big) = \frac{1}{2}\Big|\sin\frac{\xi}{2}\Big| S_{2,n}(f)\,,$$

ou

$$S_{2,n}(f) = \left|\cos\frac{\xi}{2}\right|\left|\tilde{\mu}_n\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right) + \left|\sin\frac{\xi}{2}\right|\left|\tilde{\mu}_n\left(\frac{\xi}{2}+\pi\right)\right|^2 f\left(\frac{\xi}{2}+\pi\right).$$

On a

$$T^Q_{2,\infty}\left(\left|\sin\frac{\xi}{2}\right|\right) \le \lambda\left|\sin\frac{\xi}{2}\right|,$$

avec $\lambda < 1$ si et seulement si $S^Q_{2,\infty}(1) \le \lambda$ avec $\lambda < 2^Q$. Or $S^Q_{2,n}(1)$ converge uniformément vers $S^Q_{2,\infty}$. Si on choisit $\lambda' \in]\lambda, 2^Q[$, on trouve que $\|S^Q_{2,n}(1)\|_\infty \le \lambda'$ pour n assez grand, et donc

$$T^Q_{2,n}\left(\left|\sin\frac{\xi}{2}\right|\right) \le \rho_0^Q\left|\sin\frac{\xi}{2}\right|,$$

avec $\rho_0 = (2^{-Q}\lambda')^{1/Q}$. On en conclut que, pour tout $p \in \mathbb{N}$,

$$\sup_{n\ge n_1} \|x^p\omega_n\|_2 < +\infty\,,$$

et donc $\sup_{n\ge n_1} \|x^p\varphi_n\|_{H^{-M}} < +\infty$. Par interpolation, φ_n est à décroissance rapide et converge rapidement vers φ_∞ dans H^σ pour tout $\sigma < \delta$.

H2) implique H1). La démonstration est exactement similaire à G2) implique G1), mutatis mutandis.

Ce premier théorème d'approximation est donc une adaptation assez directe de la théorie "classique" des fonctions d'échelle. Mais sa méthodologie peut facilement s'étendre à des fonctions d'échelle non classiques: les fonctions d'échelle non stationnaires.

4. Analyse multi-résolution non-stationnaire.

Nous allons donner ici une définition très personnelle des analyses multi-résolutions non-stationnaires, et nous appellerons "analyse multi-résolution non-stationnaire normale" ce que d'autres auteurs appellent simplement "analyse multi-résolution non-stationnaire".

Definition 5. *Soit $\mu \in L^2_{\mathbb{R}}(\mathbb{R})$ une fonction à valeurs réelles et à support compact $\mu \neq 0$. L'analyse multi-résolution non stationnaire* *associée à μ est la famille des sous-espaces $\{V_j[\mu]\}_{j\in\mathbb{N}}$ de $L^2(\mathbb{R})$ définis par $f \in V_j[\mu]$ si et seulement si $f \in L^2$ et il existe $\{a_k\} \in \mathbb{C}^{\mathbb{Z}}$ tel que*

$$f = \sum_k a_k \mu\Big(x - \frac{k}{2^j}\Big) \qquad \textit{dans } \mathcal{D}' .$$

Bien sûr, $f \in V_j[\mu]$ si et seulement si $f(x/2^j) \in V_0[\mu(\,\cdot\,/2^j)]$. Dans un premier temps, nous commençons donc par décrire les propriétés de $V_0[\mu]$. Nous écrivons $L^2_{\text{comp}} = \{f \in L^2_{\mathbb{R}} : \operatorname{supp} f \text{ est compact}\}$.

Lemme 6.

i) *Il existe $\varphi_0 \in V_0[\mu] \cap L^2_{\text{comp}}$, $\omega_0 \in L^2_{\text{comp}}$ tel que*

$$\langle \varphi_0(x), \omega_0(x-k)\rangle = \delta_{k,0} , \qquad \textit{pour tout } k \in \mathbb{Z} .$$

ii) *Pour $f \in V_0[\mu]$,*

$$f(x) = \sum_k \langle f, \omega_0(x-k)\rangle \, \varphi_0(x-k) .$$

iii) *$V_0[\mu]$ est le plus petit sous-espace fermé de $L^2(\mathbb{R})$ contenant μ et ses translatées $\mu(x-k)$, $k \in \mathbb{Z}$, et la famille $\{\varphi_0(x-k)\}_{k\in\mathbb{Z}}$ est une base de Riesz de $V_0[\mu]$.*

iv) *Si $\tilde\varphi_0$, $\tilde\omega_0$ vérifient les mêmes propriétés que φ_0, ω_0 alors, il existe $\lambda \neq 0$, $k \in \mathbb{Z}$ telle que $\tilde\varphi_0(x) = \lambda\, \varphi_0(x-k)$.*

En fait φ_0 peut être définie comme la fonction de $V_0[\mu]$ de support minimum. Si $h \in L^2_{\text{comp}}$, on note $\langle h\rangle$ le nombre

$$\langle h\rangle = \sup \operatorname{supp} h - \inf \operatorname{supp} h .$$

On a alors

$$\langle \varphi_0\rangle = \min\{\langle\varphi\rangle \,:\, \varphi \in V_0[\mu] \cap L^2_{\text{comp}},\ \varphi \neq 0\} . \tag{17}$$

C'est évident: si $\varphi \in L^2_{\text{comp}}$, on applique le point ii) du Lemme 6 pour voir que φ est une *combinaison linéaire finie* des $\varphi_0(x-k)$: $\varphi(x) = \sum_{k_0}^{k_1} a_k \varphi_0(x-k)$, $a_{k_0} \neq 0$, $a_{k_1} \neq 0$ (si $\varphi \neq 0$); on a alors $\langle\varphi\rangle = \langle\varphi_0\rangle + k_1 - k_0$. En particulier, le point iv) est évident: $\langle\tilde\varphi_0\rangle = \langle\varphi_0\rangle$, et donc $k_1(\tilde\varphi_0) = k_0(\tilde\varphi_0)$ de sorte que $\tilde\varphi_0 = a_{k_0}\varphi_0(x-k_0)$.

PREUVE DU LEMME 6. Notons X_0 le plus petit sous-espace fermé de L^2 qui contienne toutes les fonctions $\mu(x-k)$, $k \in \mathbb{Z}$. Pour $h, g \in L^2_{\text{comp}}$, notons

$$P_{h,g}(\xi) = \sum_{k\in\mathbb{Z}} \langle h(x), g(x-k)\rangle \, e^{-ik\xi} \, .$$

Alors l'ensemble $I_g = \{P_{h,g} : h \in C_c^\infty\}$ est un idéal de l'anneau des polynômes trigonométriques: $P_{h_1,g} + P_{h_2,g} = P_{h_1+h_2,g}$ et $e^{-ik\xi} P_{h,g} = P_{h(x-k),g}$. Si $g \neq 0$, I_g a un unique générateur $P_g = 1 + \sum_{k\geq 1} \alpha_k e^{-ik\xi}$ (où la suite α_k est presque nulle). On note $d(g) = \deg P_g$. On peut alors choisir $\varphi_0 \in X_0 \cap L^2_{\text{comp}}$ et $\omega_0 \in C_c^\infty$ tels que

$$\begin{cases} d(\varphi_0) = \min\{d(\varphi)/\varphi \in L^2_{\text{comp}} \cap X_0 \; : \; \varphi \neq 0\} \, , \\ P_{\omega_0,\varphi_0} = P_{\varphi_0} \; . \end{cases}$$

Pour démontrer i), il suffit de montrer que $P_{\varphi_0} = 1$, ou encore que

$$P_{\varphi_0}(z) = \sum_{k\in\mathbb{Z}} \langle \omega_0(x), \varphi_0(x-k)\rangle \, z^k \, ,$$

ne s'annule jamais. Mais cela est (presque) évident. Si $P_{\varphi_0}(\bar{z}_0) = 0$, alors $P_{h,\varphi_0}(\bar{z}_0) = 0$ pour tout $h \in C_c^\infty$, de sorte que

$$\sum_{k\in\mathbb{Z}} z_0^k \varphi_0(x-k) = 0$$

dans $\mathcal{D}'$. On pose alors

$$\tilde{\varphi}_0 = \sum_{k\geq 0} z_0^k \varphi_0(x-k) = -\sum_{k<0} z_0^k \varphi_0(x-k) \, .$$

On a $\varphi(x) = \tilde{\varphi}_0(x) - z_0 \tilde{\varphi}_0(x-1)$, de sorte que

$$P_{\varphi_0} = P_{\omega_0,\varphi_0} = (1 - z_0 e^{ix\xi}) \, P_{\omega_0,\tilde{\varphi}_0} \; ,$$

ce qui contredit la définition de φ_0 si jamais $\tilde{\varphi}_0 \in X_0 \cap L^2_{\text{comp}}$. Mais bien évidemment $\tilde{\varphi}_0$ est à support compact (car le support de $\sum_{k\geq 0} z_0^k \varphi_0(x-k)$ est contenu dans $[\inf \operatorname{supp} \varphi_0, +\infty[$ et celui de $\sum_{k<0} z_0^k \varphi_0(x-k)$ dans $]-\infty, -1 + \sup \operatorname{supp} \varphi_0])$. Par ailleurs, si $|z_0| < 1$, $\sum_{k\geq 0} z_0^k \varphi_0(x-k)$ converge dans L^2 et $\tilde{\varphi}_0 \in X_0$; si $|z_0| > 1$, $\sum_{k<0} z_0^k \varphi_0(x-k) \in X_0$. Si $|z_0| = 1$, on a

$$\Big\| \sum_{k=0}^{N} z_0^k \varphi_0(x-k) \Big\|_2 = \|\tilde{\varphi}_0 - z_0^{N+1} \tilde{\varphi}_0(x-N-1)\|_2 \leq 2 \, \|\tilde{\varphi}_0\|_2 \; ,$$

($\tilde\varphi_0 \in L^2$, puisque $\tilde\varphi_0 \in L^2_{\rm loc}$ et $\tilde\varphi_0$ a un support compact); par ailleurs, $\tilde\varphi_0 - z_0^{N+1}\tilde\varphi_0(x-N-1)$ converge vers $\tilde\varphi_0$ dans $\mathcal{D}'$, et donc faiblement dans L^2, de sorte que $\tilde\varphi_0$ appartient à l'adhérence faible du convexe fermé X_0, donc à X_0. Le point i) est donc démontré. Pour le point ii), il suffit de remarquer que si $P_\mu = C\prod_1^K (X-z_k)$, alors

$$\hat\mu(\xi) = C' e^{iN\xi}\prod(e^{-i\xi}-z_k)\,\hat\varphi_0(\xi)\,.$$

En effet, la technique d'expulsion des zéros que nous avons utilisée entre φ_0 et $\tilde\varphi_0$ dans la preuve du point i) peut s'appliquer à $\hat\mu$ jusqu'à complète élimination. Nous obtenons alors une fonction φ_0 telle que $P_{\varphi_0} = 1$ et telle que μ s'exprime comme une combinaison linéaire finie des $\varphi_0(x-k)$. La famille $\varphi_0(x-k)$ est une base de Riesz de X_0 et pour cette fonction φ_0 au moins la propriété ii) est évidente. Il suffit alors de remarquer que si $\varphi \in X_0 \cap L^2_{\rm comp}$ alors $d(\varphi) > d(\varphi_0)$ sauf si $\varphi = \lambda\,\varphi_0(x-k)$, ce qui se vérifie en décomposant φ sur les $\varphi_0(x-k)$.

Le seul point qui reste à vérifier est $X_0 = V_0[\mu]$. On remarque d'abord que $X_0^\perp \cap L^2_{\rm comp}$ est dense dans $X_0^\perp$: si P_0 est le projecteur oblique de L^2 sur X_0

$$P_0 f = \sum_{k\in\mathbb{Z}}\langle f, \omega_0(x-k)\rangle\,\varphi_0(x-k)\,,$$

alors $X_0^\perp = (\mathrm{Id}-P_0^*)L^2$ et $X_0^\perp \cap L^2_{\rm comp} = (\mathrm{Id}-P_0^*)L^2_{\rm comp}$. Maintenant, si $f \in V_0[\mu]$, alors $f \in (X_0^\perp \cap L^2_{\rm comp})^\perp$, et donc $f \in X_0$. L'inclusion $X_0 \subset V_0[\mu]$ est évidente (il suffit de vérifier que $V_0[\mu]$ est fermé, ou encore que $V_0[\mu] = V_0[\varphi_0]$: mais si $T \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ et $P \in \mathbb{C}[X]$, il existe $S \in \mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ tel que $T = P(e^{-i\xi})S(\xi)$, et cela entraîne $V_0[\varphi_0] \subset V_0[\mu]$).

Nous comprenons maintenant pourquoi nous pouvons parler d'analyse multi-résolution:

Lemme 7.

i) $V_j[\mu]$ *est fermé dans* L^2 *et a une base de Riesz* $\{2^{j/2}\varphi_j(2^jx - k)\}_{k\in\mathbb{Z}}$ *telle que* $\varphi_j \in L^2_{\rm comp}$ *et telle qu'il existe* $\omega_j \in C_c^\infty$ *avec*

$$\langle 2^{j/2}\varphi_j(2^jx-k), 2^{j/2}\omega_j(2^jx-k)\rangle = \delta_{k,\ell}\,.$$

ii) $V_j[\mu] \subset V_{j+1}[\mu]$.

iii) $\bigcup_{j\ge 0} V_j[\mu]$ *est dense dans* L^2.

Preuve. i) est évident par renormalisation du Lemme 6. ii) est une évidence par définition de $V_j[\mu]$. Pour iii), on note $E = \overline{\bigcup_{j\geq 0} V_j[\mu]}$. Alors E est un sous-espace fermé de L^2, invariant par toute translation dyadique $k/2^j$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$, donc par toute translation réelle. Cela implique qu'il existe un ensemble mesurable $F \subset \mathbb{R}$ tel que: $f \in E$ si et seulement si $\{f \in L^2$ et $\operatorname{supp} \hat{f} \subset F\}$. Comme $\mu \in E$ et que $\hat{\mu}$ est une fonction analytique, on a $F = \mathbb{R}$ et $E = L^2$.

Ainsi, la fonction μ permet d'approximer dans L^2, par ses translatées dyadiques, toute fonction de carré intégrable. Un problème intéressant est alors de savoir à quelle vitesse cette approximation a lieu.

Definition 6. *Une fonction $\mu \in L^2_{\mathbb{R}} \cap L^2_{\mathrm{comp}}(\mathbb{R})$, $\mu \neq 0$, a une puissance d'approximation d'ordre N si: il existe $C \geq 0$, $j_N \in \mathbb{N}$, pour tout $f \in H^N$, $j \geq j_N$,*

$$\|f - \Pi_j f\|_2 \leq C\, 2^{-jN} \|f^{(N)}\|_2 \,,$$

où Π_j est le projecteur orthogonal de L^2 sur $V_j[\mu]$.

Par exemple, si μ est une fonction d'échelle associée à un filtre m_0 qui a un zéro d'ordre N en π, alors μ a une puissance d'approximation d'ordre N. Les fonctions d'échelle non-stationnaires ont été introduite pour permettre des puissances d'approximation d'ordre infini [11]. On parle alors d'*approximation spectrale.*

Remarquons également que si μ a une puissance d'approximation d'ordre N, $N \geq 1$, alors on a

$$\lim_{j\to+\infty} 2^{j(N-1)} \|f - \Pi_j f\|_2 = 0\,, \qquad \text{pour tout } f \in H^{N-1}\,.$$

En effet, si $f = g_A + h_A$, avec

$$\hat{g}_A(\xi) = \chi_{[-1,1]}\Big(\frac{\xi}{A}\Big)\hat{f}\,,$$

on a, en écrivant $g_A \in H^N$ et $h_A \in L^2$ avec

$$\|g_A^{(N)}\|_2 \leq A \|f^{(N-1)}\|_2$$

et

$$\|h_A\|_2 \leq A^{-N+1} \|f^{(N-1)}\|_2 \,,$$

$$2^{j(N-1)}\|f-\Pi_j f\|_2 \le C\,(2^{-j}A)\|f^{(N-1)}\|_2 \\ + 2\,(2^{j(N-1)}A^{-N+1})\|f^{(N-1)}\|_2\,,$$

d'où, si $A = 2^j$,

$$2^{j(N-1)}\|f-\Pi_j f\|_2 \le (C+2)\,\|f^{(N-1)}\|_2\,.$$

On réinjecte cette inégalité dans $f = g_A + h_A$, avec $g_A \in H^N$ et $h_A \in H^{N-1}$, $A = (\sqrt{2})^j$,

$$2^{j(N-1)}\|f-\Pi_j f\|_2 \le C\,(\sqrt{2})^{-j}\|f^{(N-1)}\|_2 \\ + (C+2)\Big(\int_{|\xi|>(\sqrt{2})^j} |f^{(N-1)}(\xi)|^2\,d\xi\Big)^{1/2}$$

et on a bien $2^{j(N-1)}\|f-\Pi_j f\|_2 \to 0$ quand $j \to +\infty$.

Dans une analyse multi-résolution non stationnaire $(V_j[\mu])$, nous avons obtenu pour chaque niveau $V_j[\mu]$ une fonction spéciale φ_j, et l'inclusion $V_j[\mu] \subset V_{j+1}[\mu]$ donne une *équation à deux échelles*

$$\varphi_j\Big(\frac{x}{2}\Big) = \sum_{k\in\mathbb{Z}} \Big\langle \varphi_j\Big(\frac{x}{2}\Big), \omega_{j+1}(x-k)\Big\rangle\,\varphi_{j+1}(x-k)\,,$$

qui se réécrit en

$$\hat{\varphi}_j(2\,\xi) = \frac{1}{2}\,P_{\varphi_j(x/2),\omega_{j+1}}(\xi)\,\hat{\varphi}_{j+1}(\xi)\,.$$

On écrit

$$m_{j+1}(\xi) = \frac{1}{2}\,P_{\varphi_j(x/2),\omega_{j+1}}(\xi)$$

et on a alors

$$\hat{\varphi}_0(\xi) = \prod_{j=1}^{M} m_j\Big(\frac{\xi}{2^j}\Big)\,\hat{\varphi}_M\Big(\frac{\xi}{2^M}\Big)\,, \tag{18}$$

formule qui est très proche de la formule qui relie la fonction d'échelle au filtre d'échelle. Mais ici, a priori, nous ne pouvons pas faire tendre M vers $+\infty$. Même si nous pouvions normaliser φ_M par $\hat{\varphi}_M(0) = 1$ (ce qui n'est pas toujours possible!), il n'y aurait aucune raison de conclure que

$$\lim_{M\to+\infty} \hat{\varphi}_M\Big(\frac{\xi}{2^M}\Big) = 1\,.$$

Lorsque cela est le cas, nous sommes dans le cas d'analyses multi-résolutions non-stationnaires "normales" (au sens où seuls les filtres m_j entrent en compte pour la détermination de φ_0).

Definition 7. *Une analyse multi-résolution non-stationnaire* $\{V_j[\mu]\}_{j\geq 0}$ *est dite normale si les fonctions d'échelle non-stationnaires* φ_j *(i.e. les fonctions* φ_j *telles que* $\varphi_j(2^j x) \in V_j[\mu]$ *et le support de* $\varphi_j(2^j x)$ *est minimum), peuvent être choisies de la forme*

$$\hat{\varphi}_j(\xi) = \prod_{\ell=1}^{\infty} m_{j+\ell}\Big(\frac{\xi}{2^\ell}\Big) \tag{19}$$

où m_ℓ *est un polynôme trigonométrique*

$$m_\ell = -\sum_{-\deg m_\ell}^{+\deg m_\ell} a_{k,\ell}\, e^{-ik\xi}\,,$$

à coefficients réels tel que $m_\ell(0) = 1$, $\deg m_\ell \leq C\ell^A$ *(où* $A \in \mathbb{N}$ *ne dépend pas de* ℓ*) et*

$$\sup_\ell \|m_\ell\|_\infty = B < +\infty\,.$$

Comme nous l'avons expliqué dans l'introduction, les produits partiels

$$\widehat{\theta_{j,N}}(\xi) = \prod_{\ell=1}^{N} m_{j+\ell}\Big(\frac{\xi}{2^\ell}\Big)$$

définissant $\hat{\varphi}_j$ admettent un majorant à croissance polynômiale, de sorte que $\widehat{\theta_{j,N}}$, qui converge vers $\hat{\varphi}_j$ dans $C^\infty(\mathbb{R})$ (*i.e.*, pour tout $p \in \mathbb{N}$, $(d/d\xi)^p\widehat{\theta_{j,N}}$ converge vers $(d/d\xi)^p\hat{\varphi}_j$ uniformément sur tout compact) converge vers $\hat{\varphi}_j$ également au sens des distributions tempérées. De plus $\operatorname{supp}\theta_{j,N} \subset [-M_j, M_j]$ avec

$$M_j = \sum_{\ell=1}^{\infty} \frac{\deg(m_{j+\ell})}{2^\ell} \leq C\sum_{\ell=1}^{\infty} \frac{(j+\ell)^A}{2^\ell}\,.$$

La convergence de $\theta_{j,N}$ vers φ_j a alors lieu avec des supports contenus dans un compact fixe (*i.e.*, fixe par rapport à N, pas par rapport à j).

Nous verrons dans la suite trois exemples de telles analyses multi-résolution non stationnaires: fonction de Rvachev (Section 5), base de Berkolaïko et Novikov (Section 6) et “ondelettes de Kharkov” (Section 8).

5. La fonction de Rvachev.

La fonction $up(x)$ a été introduite en 1971 par V. A. et V. L. Rvachev comme une fonction explicite C^∞, à support dans $[-1, 1]$ et dont les translatées par $1/2^j$, $\{up(x - k/2^j)\}_{k\in\mathbb{Z}}$, permettent de reconstruire tous les polynômes jusqu'au degré j

$$\mathbb{C}_j[X] \subset \Big\{ \sum_{k\in\mathbb{Z}} a_k \, up\Big(x - \frac{k}{2^j}\Big) \, : \, \{a_k\} \in \mathbb{C}^{\mathbb{Z}} \Big\} .$$

Nous translaterons cette fonction de 1 (c'est-à-dire que nous prendrons up à support dans $[0, 2]$) pour pouvoir appliquer le formalisme des analyses multirésolutions non stationnaires.

La fonction $up(x)$ peut être définie comme un exemple classique de fonction C^∞ à support compact par la formule

$$\widehat{up}(\xi) = \prod_{j=0}^{\infty} \hat{\chi}\Big(\frac{\xi}{2^j}\Big) , \tag{20}$$

où $\chi = \chi_{[0,1]}$ et donc

$$\hat{\chi}(\xi) = \frac{1 - e^{-i\xi}}{i\,\xi} .$$

De tels produits interviennent dans la théorie des classes de fonctions non quasi-analytiques (*cf.* le livre de Rudin [25, Chapter 19, p. 415], par exemple).

Il y a de nombreuses façons de justifier la convergence de (20). Par exemple,

$$\prod_{j=0}^{N} \hat{\chi}\Big(\frac{\xi}{2^j}\Big) = \hat{\mu}_N$$

est la transformée de Fourier de la fonction $\mu_N = \chi * 2\,\chi(2\,x) * \cdots * 2^N\chi(2^N x)$, qui est une fonction positive de masse totale 1 et à support contenu dans $[0,2]$. De plus, $\|\mu_N\|_\infty \leq 1$ (puisque

$$\|\mu_N\|_\infty = \|\chi * 2\,\mu_{N-1}(2\,x)\|_\infty \leq \|\chi\|_\infty \|\mu_{N-1}\|_1)\,,$$

$\|(d/dx)\mu_N\|_\infty \leq 2$ (puisque $d\mu_N/dx = 2\,\mu_{N-1}(2\,x) - 2\,\mu_{N-1}(2\,x - 2)$ et que $\mu_{N-1}(x)$ et $\mu_{N-1}(x-2)$ sont à support dans $[0,2]$ et $[2,4]$ respectivement) et enfin

$$\begin{aligned} &\|\mu_{N+1} - \mu_N\|_\infty \\ &\qquad = \sup_x \Big| \int (\mu_N(x) - \mu_N(x-y))\, 2^{N+1}\chi(2^{N+1}y)\, dy \Big| \leq 2^{-N}\,. \end{aligned}$$

μ_N converge donc uniformément vers une fonction up qui vérifie

$$\operatorname{supp} up \subset [0,2]\,, \qquad \int up\, dx = 1\,, \qquad \text{et } up = \chi * 2\, up(2\,x)\,.$$

En particulier, up est en quelque sorte sa propre régularisée et est dans C^∞.

La fonction up jouit de nombreuses propriétés remarquables, décrites dans [26] et [27]. Par exemple, il est clair que

$$\frac{d}{dx}\, up(x) = 2\, up(2\,x) - 2\, up(2\,x - 2) \tag{21}$$

et que $up(x) > 0$ sur $]0,2[$, $up(1) = 1$. (En effet, de (21) on tire que $0 = \inf \operatorname{supp} up$, et donc $up(x) > 0$ sur $]0,1]$ tandis que $up(1) = 2\int_0^1 up(2\,t)\, dt = \widehat{up}(0) = 1$). On en conclut alors pour $k \geq 1$

$$0 < x < 2 \qquad \text{et} \qquad \frac{d^k}{dx^k}\, up(x) = 0\,,$$

si et seulement si

$$x \in \Big\{ \frac{1}{2^{k-1}}, \frac{2}{2^{k-1}}, \ldots, \frac{2^k - 1}{2^{k-1}} \Big\}\,.$$

En fait c'est l'une des caractérisations de up

Proposition 4 (Caractérisation de up). *Chacun des problèmes* P1) *à* P5) *a pour seule et unique solution la fonction* up:

P1) $f \in \mathcal{E}'$, $\langle f, 1\rangle = 1$ *et* $f = \chi_{[0,1]} * 2\, f(2\, x)$

P2) $f \in \mathcal{E}'$, $\langle f, 1\rangle = 1$ *et* $df/dx = 2\, f(2\, x) - 2\, f(x-2)$

P3) $f \in C^\infty$, $\operatorname{supp} f \subset [0,2]$, $f(1) = 1$ *et pour tout* $k \geq 1$ *on a*:

$$0 < x < 2 \qquad \text{et} \qquad f^{(k)}(x) = 0\,,$$

si et seulement si

$$x \in \Big\{\frac{1}{2^{k-1}}, \frac{2}{2^{k-1}}, \dots, \frac{2^k-1}{2^{k-1}}\Big\}\,.$$

P4) $f \in C^\infty$, $\operatorname{supp} f \subset [0,2]$, $f(1) = 1$,

$$\lim_{n\to+\infty} 2^{-(n+1)(n+2)/2} \|f^{(n)}\|_\infty = 0\,,$$

et, pour tout $p \in \mathbb{N}$, *il existe* $\{a_k\} \in \mathbb{C}^{\mathbb{Z}}$, *tel que*

$$x^p = \sum_{k\in\mathbb{Z}} a_k f\Big(x - \frac{k}{2^p}\Big)\,.$$

P5) $f \in C^\infty$, $\operatorname{supp} f \subset [0,2]$, $f(1) = 1$,

$$\lim_{n\to+\infty} 2^{-(n+1)(n+2)/2} \|f^{(n)}\|_\infty = 0$$

et, pour tous $k \geq 1$, $\ell \in \{1, 2, \dots, 2^k - 1\}$,

$$f^{(k)}\Big(\frac{\ell}{2^{k-1}}\Big) = 0\,.$$

Quoique tous ces résultats aient été prouvés par V. A. et V. L. Rvachev [26], [27], nous en donnerons la preuve à titre d'introduction à la fonction *up*. Avant d'entamer cette preuve, nous donnons le lemme fondamental sur lequel repose une grande partie de la théorie de la fonction *up*.

Lemme 8. *a*) (Lemme d'encadrement) *Soit* $n \geq 0$ *et soit* A_n *l'ensemble des fonctions* f *qui vérifient:*

i) $f \in C^n$ *et* $\operatorname{supp} f \subset [0,2]$.

ii) $f^{(n)}$ *est strictement monotone sur chaque intervalle* $[k/2^n, (k+1)/2^n]$ $(0 \leq k < 2^{n+1})$.

iii) *Pour* $1 \leq p \leq n$ *et* $1 \leq k \leq 2^p - 1$,

$$f^{(p)}\Big(\frac{k}{2^{p-1}}\Big) = 0\,.$$

iv) $f(1) = 1$.

Alors si $f \in A_n$, *on a en tout point* x

$$\ell_n(x) \leq f(x) \leq h_n(x)\,, \tag{22}$$

où $\ell_0 = 0$, $h_0 = \chi_{[0,2]}$ *et* ℓ_{n+1} *et* h_{n+1} *sont définis par* $\ell_{n+1} = \int_0^x L_n(t)\,dt$ *et* $h_{n+1} = \int_0^x H_n(t)\,dt$, *où:*

- *pour* $0 \leq x \leq 1/2$, $L_n(t) = 2\,\ell_n(2\,t)$, $H_n(t) = 2\,h_n(2\,t)$,
- *pour* $1/2 \leq x \leq 1$, $L_n(t) = 2\,h_n(2\,t)$, $H_n(t) = 2\,\ell_n(2\,t)$,
- *pour* $1 \leq x \leq 3/2$, $L_n(t) = -2\,h_n(2\,t-2)$, $H_n(t) = -2\,\ell_n(2\,t-2)$,
- *pour* $3/2 \leq x \leq 2$, $L_n(t) = -2\,\ell_n(2\,t-2)$, $H_n(t) = -2\,h_n(2\,t-2)$.

En particulier, on a: si $f, g \in A_n$, *alors,*

$$\|f - g\|_\infty \leq \|h_n - \ell_n\|_\infty \leq \frac{2}{2^n}\,. \tag{23}$$

b) (Lemme d'extremalité) *Soit* σ_n *le spline de degré* n, *défini récursivement par*

$$\sigma_0 = \frac{1}{2}\,\chi_{[0,2]}$$

et

$$\sigma_{n+1} = \int_0^x (2\,\sigma_n(2\,t) - 2\,\sigma_n(2\,t-2))\,dt\,.$$

Alors $\sigma_{n+1} \in A_n$ *et on a, pour tout* $f \in A_n$,

$$\|f^{(n+1)}\|_\infty \geq \|\sigma_{n+1}^{(n+1)}\|_\infty = \frac{1}{2}\,2^{(n+1)(n+2)/2} \tag{24}$$

et l'égalité n'a lieu que pour $f = \sigma_{n+1}$.

c) (Lemme de Majoration) *Soit* $f \in C^n(\mathbb{R})$ *telle que* $f^{(n)}$ *soit lipschitzienne et telle que* $f(k) = 0$ *pour tout* $k \in \mathbb{Z}$ *et* $f^{(p)}(k/2^{p-1}) = 0$ *pour tout* $k \in \mathbb{Z}$ *et tout* $p \in \{1, \ldots, n\}$. *Alors on a*

$$\|f\|_\infty \leq 2^{1-n-((n+1)(n+2))/2}\|f^{(n+1)}\|_\infty\,. \tag{25}$$

PREUVE DU LEMME 8. Il est immédiat que si $f \in A_n$, alors on a

$$f'(x) = f'\Big(\frac{1}{2}\Big) f_1(2\,x) + f'\Big(\frac{3}{2}\Big) f_2(2\,x-2)\,,$$

où f_1 et f_2 sont dans A_{n-1}. L'idée est alors de démontrer (22) par récurrence. On commence par vérifier que si $f \in A_n$, alors pour $1 \leq p \leq n$, $\{f^{(p)}(x) = 0 \text{ et } 0 < x < 2\}$ si et seulement si

$$x \in \Big\{\frac{1}{2^{p-1}}, \frac{2}{2^{p-1}}, \dots, \frac{2^p-1}{2^{p-1}}\Big\}\,,$$

par hypothèse, puisque

$$f^{(n)}\Big(\frac{k}{2^{n-1}}\Big) = f^{(n)}\Big(\frac{k+1}{2^{n-1}}\Big) = 0$$

et que $f^{(n)}$ est strictement monotone sur $[2k/2^n, (2k+1)/2^n]$ et sur $[2k{+}1/2^n, (2k{+}2)/2^n]$, on voit que $f^{(n)}$ ne s'annule pas sur $]k/2^{n-1}, (k+1)/2^{n-1}[$ et donc que $f^{(n-1)}$ y est strictement monotone; ce qui prouve $A_n \subset A_{n-1}$ et par récurrence que les seuls zéros de $f^{(p)}$ sont les points $k/2^{p-1}$.

Pour démontrer (22), Rvachev introduit l'hypothèse de récurrence (H_n) suivante: "Si $f \in A_n$, $\alpha, \beta > 0$ et $x_0 \in [0,2]$ vérifient $\alpha\,\ell_n(x_0) \leq f(x_0) \leq \beta\,h_n(x_0)$ alors si $x_0 \leq 1$, on a $\alpha\,\ell_n(x) \leq f(x) \leq \beta\,h_n(x)$ pour tout $x \in [0, x_0]$, et si $x_0 \geq 1$, on a $\alpha\,\ell_n(x) \leq f(x) \leq \beta\,h_n(x)$ pour tout $x \in [x_0, 2]$".

H_0 est évident: puisque $\ell_0 = 0$, on a $f(x) \geq \alpha\,\ell_0(x)$ sur $[0,2]$: en effet f est croissante de 0 à 1 sur $[0,1]$ et décroissante de 1 à 0 sur $[1,2]$; de même si $f(x_0) \leq \beta\,h_0(x_0) = \beta$, on a nécessairement $f(x) \leq \beta$ sur $[0, x_0]$ si $x_0 \leq 1$, sur $[x_0, 2]$ si $x_0 \geq 1$.

Montrons que (H_n) implique (H_{n+1}). Comme 1 est centre de symétrie du problème (pour tout n on a $f(x) \in A_n$ si et seulement si $f(2-x) \in A_n$, $\ell_n(x) = \ell_n(2-x)$ et $h_n(x) = h_n(2-x)$), nous pouvons supposer $x_0 \leq 1$. On suppose donc $f \in A_{n+1}$, $x_0 \leq 1$ et $\alpha\,\ell_{n+1}(x_0) \leq f(x_0) \leq \beta\,h_{n+1}(x_0)$. Sur $[0,1]$, $f'(x) = f'(1/2)\,g(2\,x)$ avec $f'(1/2) > 0$ et $g \in A_n$. On considère $\varphi(x) = f(x) - \alpha\,\ell_{n+1}(x)$. Cette fonction vérifie $\varphi(x_0) \geq 0$ et $\varphi(0) = 0$. On note x_1 le point de $[0, x_0]$ où φ atteint son minimum. Si $x_1 = 0$ ou $x_1 = x_0$, alors $\varphi \geq 0$ sur $[0, x_0]$, ce qu'il faut démontrer. Si $0 < x_1 < 1/2$, alors $0 = \varphi'(x_1) = f'(1/2)\,g(2\,x_1) - 2\,\alpha\,\ell_n(2\,x_1)$. L'hypothèse (H_n) permet de conclure que $\varphi'(x) \geq 0$ sur $[0, x_1]$ et donc $\varphi(x_1) \geq \varphi(0) = 0$. De même, si $1/2 < x_1 < 1$, alors $0 = \varphi'(x_1) = f'(1/2)\, g(2\, x_1) - 2\, \alpha\, h_n(2\, x_1)$, de sorte que $\varphi'(x) \leq 0$ sur $[x_1, 1]$ et donc $\varphi(x_1) \geq \varphi(x_0) \geq 0$. Enfin si $x_1 = 1/2$, le même argument s'applique si $n+1 \geq 2$; si $n = 0$, ℓ_1 n'est pas dérivable en $1/2$; cependant $\ell_1(1/2) = 0$ et $f(1/2) > 0$, de sorte que $\varphi(1/2) > 0$. (H_{n+1}) est donc démontrée, le cas $f(x_0) \leq \beta\, h_{n+1}(x_0)$ se traitant de façon analogue.

(22) est alors immédiat. Pour $n = 0$, nous avons déjà vu que $0 \leq f \leq \chi_{[0,2]}$. Pour $n \geq 1$, on a $f(1) = \ell_n(1) = h_n(1)$ et (H_n) entraîne (22). Pour vérifier que $\ell_n(1) = h_n(1) = 1$, il suffit de remarquer que pour $t \in [0,1]$ on a $\ell_n(t) + h_n(1-t) = 1$. C'est vrai pour $n = 0$. Par ailleurs, on a

$$\begin{aligned} \ell_{n+1}(t) + h_{n+1}(1-t) &= \int_0^t 2\, \ell_n(2\, u)\, du + \int_0^{1/2} 2\, h_n(2\, u)\, du \\ &\quad + \int_{1/2}^{1-t} 2\, \ell_n(2\, u)\, du \\ &= \int_0^1 \ell_n(u)\, du + \int_0^{1/2} h_u(u)\, du \\ &= \int_0^1 (\ell_n(u) + h_n(1-u))\, du = 1\,, \qquad \text{si } t \leq \frac{1}{2}\,. \end{aligned}$$

En utilisant le fait que $\ell_n(2-u) = \ell_n(u)$ et donc

$$\int_{1/2}^{1-t} 2\, \ell_n(2\, u)\, du = \int_t^{1/2} 2\, \ell_n(2\, v)\, dv$$

et de même $\ell_{n+1}(t) + h_{n+1}(1-t) = 1$ si $t \geq 1/2$.

Pour démontrer a), il ne reste plus qu'à estimer $\|\ell_n - h_n\|_\infty$. En fait, on va montrer que pour $0 \leq x \leq 1$, $\ell_n(x) = h_n(x - 1/2^n)$. On commence par remarquer que c'est vrai pour $n = 0$ ou 1. Par ailleurs, $\ell_n(x) = 0$ pour $x \in [0, 1/2^n]$ et donc $h_n = 1$ pour $x \in [1 - 1/2^n, 1 + 1/2^n]$. On va montrer par récurrence que $\varphi_n(x) = \ell_n(x) - h_n(x - 1/2^n)$ est nulle sur $[0, 1)$. En effet, on a:

- si $0 \leq x \leq 1/2$,

$$\varphi'_{n+1}(x) = 2\, \ell_n(2\, x) - 2\, h_n\Big(2\, x - \frac{1}{2^n}\Big) = 2\, \varphi_n(2\, x) = 0\,,$$

- si $1/2 + 1/2^{n+1} \leq x \leq 1$,

$$\varphi'_{n+1}(x) = \begin{cases} 2\,h_n(2x) - 2\,\ell_n\Big(2\,x - \dfrac{1}{2^n}\Big)\,, \\ 2\,h_n(2-2\,x) - 2\,\ell_n\Big(2 - 2\,x + \dfrac{1}{2^n}\Big)\,, \end{cases}$$

$$= \begin{cases} -2\,\varphi_n\Big(2 - 2\,x + \dfrac{1}{2^n}\Big)\,, \\ 0\,, \end{cases}$$

- si $1/2 \leq x \leq 1/2 + 1/2^{n+1}$,

$$\varphi'_{n+1}(x) = 2\,h_n(2\,x) - 2\,h_n\Big(2\,x - \frac{1}{2^n}\Big) = 2 - 2 = 0\,,$$

de sorte que $\varphi_{n+1} = \varphi_{n+1}(0) = 0$.

On remarque alors que

$$\|h_n - \ell_n\|_\infty = \Big|h_n\Big(\frac{1}{2}\Big) - \ell_n\Big(\frac{1}{2}\Big)\Big| = \int_{1/2-1/2^n}^{1/2} 2\,h_{n-1}(2\,t)\,dt \leq \frac{2}{2^n}\,.$$

Le point a) est donc démontré.

Le point b) est quasiment immédiat. L'appartenance de σ_{n+1} à A_n est immédiate (il suffit d'écrire $\sigma_{n+1} = \chi_{[0,1]} * 2\,\sigma_n(2\,x)$ pour vérifier que pour tout $n \geq 0$, $\int \sigma_n\,dt = 1$ et donc pour tout $n \geq 1$, $\sigma_n(1) = 1$; les annulations de $\sigma^{(k)}_{n+1}$ sont alors immédiates par récurrence puisque $\sigma^{(k)}_{n+1} = 2^k \sigma^{(k-1)}_n(2\,t)$ sur $[0,1]$ et $= -2^k\sigma^{(k-1)}_n(2\,t-2)$ sur $[1,2]$). De même il est immédiat que sur $]k/2^n, (k+1)/2^n[$,

$$\sigma^{(n+1)}_{n+1} = \varepsilon_{k,n}\,2^{(n+1)(n+2)/2}$$

où $\varepsilon_{k,n}$ ne dépend que de k et n et appartient à $\{-1,1\}$. Le spline σ_{n+1} est donc un *spline parfait* (*i.e.* un spline de degré $n+1$ tel que

$$\Big|\frac{d^{n+1}}{dx^{n+1}}\,\sigma_{n+1}\Big| = c^{te}$$

presque partout sur son support). Il suffit alors de vérifier que si $\operatorname{supp} f \subset [0,2]$, $\|f^{(n+1)}\|_\infty \leq \|\sigma^{(n+1)}_{n+1}\|_\infty$ et $f^{(p)}(k/2^{p-1}) = 0$ pour $0 \leq k \leq 2^p$ alors $|f(x)| \leq \sigma_{n+1}(x)$ en tout point et $f(1) = \sigma_{n+1}(1) = 1$ si et seulement si $f = \sigma_{n+1}$. Cela est évident par récurrence. Si $n = 0$, on a $|f'| \leq 1$ presque pour tout et donc $|f(x) - f(x_0)| \leq |x - x_0|$; on prend $x_0 = 0$ pour $x \in [0,1]$ et $x_0 = 2$ pour $x \in [1,2]$ et on obtient $|f(x)| \leq (1 - |1-x|)_+ = \sigma_1(x)$; de plus on a

$$|f(1)| = \Big| \int_0^1 f'(t)\, dt \Big| \leq \int_0^1 1\, dt = 1\,,$$

avec égalité si et seulement si $f'(t) = \omega$ presque pour tout sur $[0,1]$, où $|\omega| = 1$ et ω ne dépend pas de t; on trouve donc $f(1) = 1$ si et seulement si $f' = 1$ sur $]0,1[$ et -1 sur $]1,2[$, et donc si et seulement si $f = \sigma_1$.

Maintenant, si on suppose le résultat démontré pour n, il est facile de le démontrer pour $n+1$. En effet, puisque

$$\|\sigma_{n+2}^{(n+2)}\|_\infty = 2^{n+2} \|\sigma_{n+1}^{(n+1)}\|_\infty\,,$$

on écrit $f'\chi_{[0,1]} = g(2\,x)$ et $f'\chi_{[1,2]} = h(2\,x - 2)$ et on peut appliquer la propriété au rang n à g et h:

$$\|g^{(n+1)}\|_\infty = 2^{-(n+1)} \|f^{(n+2)}\|_\infty \leq 2\, \|\sigma_{n+1}^{(n+1)}\|_\infty\,,$$

et donc $|f'(x)| \leq 2\, |\sigma_{n+1}(2\,x)|$ sur $[0,1]$ et $\leq 2\, |\sigma_{n+1}(2\,x-2)|$ sur $[1,2]$. En particulier,

$$|f(x)| \leq \int_0^x 2\, \sigma_{n+1}(2\,t)\, dt = \sigma_{n+2}(x)$$

si $0 \leq x \leq 1$, et

$$|f(x)| \leq \int_x^2 2\, \sigma_{n+1}(2\,t-2)\, dt = \sigma_{n+2}(x)$$

si $1 \leq x \leq 2$; de plus, l'égalité $f(1) = 1 = \sigma_{n+2}(1)$ n'est possible que si $f' = 2\,\sigma_{n+1}(2\,t)$ presque pour tout sur $[0,1]$ et $f' = 2\,\sigma_{n+1}(2\,t - 2)$ presque pour tout sur $[0,2]$, donc $f = \sigma_{n+2}$. Le point b) est donc démontré.

Le point c) est maintenant évident. Puisque $f^{(n)}$ est continue et que f a un zéro d'ordre $(n+1)$ aux points entiers, on peut écrire

$$f = \sum_{k=-\infty}^{+\infty} f_k(x - 2\,k)\,,$$

où f_k est à support dans $[0,2]$, $f_k^{(p)}(\ell/2^{p-1}) = 0$ pour $1 \leq p \leq n$, $1 \leq \ell \leq 2^n - 1$ et $f_k(1) = 0$, $f_k^{(n)}$ est lipschitzienne et $\|f_k^{(n+1)}\|_\infty \leq \|f^{(n+1)}\|_\infty$. La fonction

$$\tilde{f}_{k,\varepsilon} = \sigma_{n+1} + f_k \, \frac{\|\sigma_{n+1}^{(n+1)}\|_\infty}{\|f^{(n+1)}\|_\infty + \varepsilon}$$

(où $\varepsilon > 0$) appartient à A_n, de sorte que

$$\|\tilde{f}_{k,\varepsilon} - \sigma_{n+1}\|_\infty \leq \frac{2}{2^n} \, .$$

Faisant tendre ε vers 0, on obtient

$$\|f\|_\infty = \sup_k \|f_k\|_\infty \leq \frac{2}{2^n} \, 2^{-(n+1)(n+2)/2} \|f^{(n+1)}\|_\infty \, .$$

Le Lemme 8 est donc démontré.

PREUVE DE LA PROPOSITION 4. On vérifie facilement que up est solution des problèmes P1) à P5). En effet, on a

$$\frac{d}{dx} \, up = 2 \, up(2\,x) - 2 \, up(2\,x - 2)$$

et

$$\operatorname{supp} up(2\,x) \cap \operatorname{supp} up(2\,x - 2) = \{1\} \, .$$

On en conclut

$$\left\| \frac{d^N}{dx^N} \, up \right\|_\infty = 2 \left\| \frac{d^{N-1}}{dx^{N-1}} \, (up(2\,x)) \right\|_\infty = 2^N \left\| \frac{d^{N-1}}{dx^{N-1}} \, up \right\|_\infty ,$$

et donc

$$\left\| \frac{d^N}{dx^N} \, up \right\|_\infty = 2^{-(N+1)} \, 2^{(N+1)(N+2)/2} \, .$$

Il reste seulement à vérifier que tout polynôme s'écrit à l'aide de translatées de up. On va montrer plus précisément, pour $p \in \mathbb{N}$, pour $\ell \in \{0, \ldots, p\}$,

$$\sum_{k \in \mathbb{Z}} \left(\frac{k}{2^p} \right)^\ell up\Big(x - \frac{k}{2^p} \Big) = 2^p x^\ell \bmod \mathbb{C}_{\ell-1}[X] \, . \tag{26}$$

Pour $p=0$, (26) se réduit à $\sum_{k\in\mathbb{Z}} up(x-k)=1$, ou encore à $\widehat{up}(0)=1$ et $\widehat{up}(2k\pi)=0$ pour $k\in\mathbb{Z}^*$, ce qui est immédiat puisque

$$\widehat{up}(\xi)=\frac{1-e^{-i\xi}}{i\,\xi}\,\widehat{up}\Big(\frac{\xi}{2}\Big)\,.$$

Si (26) est vrai à l'ordre p, cela l'est encore à l'ordre $p+1$ pour $0\le \ell\le p$

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\Big(\frac{k}{2^{p+1}}\Big)^{\ell} up\Big(x-\frac{k}{2^{p+1}}\Big) &= \sum_{k\in\mathbb{Z}}\Big(\frac{k}{2^{p}}\Big)^{\ell} up\Big(x-\frac{k}{2^{p}}\Big)\\ &\quad+\sum_{k\in\mathbb{Z}}\Big(\frac{k}{2^{p}}\Big)^{\ell} up\Big(x-\frac{1}{2^{p+1}}-\frac{k}{2^{p}}\Big)\\ &\quad+\sum_{j=1}^{\ell}\Big(\frac{1}{2^{p+1}}\Big)^{j} C_\ell^j\\ &\qquad\cdot\sum_{k\in\mathbb{Z}}\Big(\frac{k}{2^{p}}\Big)^{\ell-j} up\Big(x-\frac{1}{2^{p+1}}-\frac{k}{2^{p}}\Big)\\ &=2^p x^\ell+2^p\Big(x-\frac{1}{2^{p+1}}\Big)^{\ell} \bmod \mathbb{C}_{\ell-1}[x]\\ &=2^{p+1}x^\ell \bmod \mathbb{C}_{\ell-1}[X]\,.\end{aligned}$$

Pour $\ell=p+1$, on utilise le fait que $\widehat{up}$ a un zéro d'ordre $p+2$ en $2^{p+2}k\pi$, $k\in\mathbb{Z}^*$, puisque

$$\widehat{up}(\xi)=\prod_{j=0}^{p+1}\frac{1-e^{-i\xi/2^j}}{i\,\xi/2^j}\,\widehat{up}\Big(\frac{\xi}{2^{p+2}}\Big)\,,$$

la formule sommatoire de Poisson donne alors que

$$\sum_{k\in\mathbb{Z}}\Big(x-\frac{k}{2^{p+1}}\Big)^{p+1} up\Big(x-\frac{k}{2^{p+1}}\Big)=C^{te}=\gamma_{p+1}\ ,$$

où γ_{p+1} est une constante ne dépendant pas de x. On obtient alors

$$\begin{aligned}0&=\sum_{j=0}^{p+1}C_{p+1}^j(-1)^j x^{p+1-j}\sum_{k\in\mathbb{Z}}\Big(\frac{k}{2^{p+1}}\Big)^{j} up\Big(x-\frac{k}{2^{p+1}}\Big) \bmod \mathbb{C}_p[X]\\ &=(-1)^{p+1}\sum_{k\in\mathbb{Z}}\Big(\frac{k}{2^{p+1}}\Big)^{p+1} up\Big(x-\frac{k}{2^{p+1}}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{p} C_{p+1}^{j}(-1)^{j} x^{p+1-j}\, 2^{p+1} x^{j} \bmod \mathbb{C}_p[X] \\
&= (-1)^{p+1} \sum_{k\in\mathbb{Z}} \Big(\frac{k}{2^{p+1}}\Big)^{p+1} up\Big(x-\frac{k}{2^{p+1}}\Big) \\
&\qquad\qquad - (-1)^{p+1} 2^{p+1} x^{p+1} \bmod \mathbb{C}_p[X]\,.
\end{aligned}
$$

(26) est donc prouvé.

up est donc bien solution des problèmes P1) à P5). Par ailleurs, les restrictions sur la taille des dérivées de f dans P4) ou P5) sont bien nécessaires pour ne pas avoir d'autre solution que up: pour P4), la fonction

$$f = up - \alpha \frac{d}{dx}\, up\,,$$

où $\alpha > 0$ est arbitraire, vérifie supp $f \subset [0,2]$, $\int_0^2 f\, dx = 1$ et

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \Big(\frac{k}{2^p}\Big)^{p} f\Big(x-\frac{k}{2^p}\Big) &= \sum_{k\in\mathbb{Z}} \Big(\frac{k}{2^p}\Big)^{p} up\Big(x-\frac{k}{2^p}\Big) \\
&\quad - \alpha \frac{d}{dx}\Big(\sum_{k\in\mathbb{Z}} \Big(\frac{k}{2^p}\Big)^{p} up\Big(x-\frac{k}{2^p}\Big)\Big) \\
&= 2^p x^p \bmod \mathbb{C}_{p-1}[X]\,.
\end{aligned}
$$

tandis que

$$\lim_{n\to+\infty} \frac{\|f^{(n)}\|_\infty}{2^{(n+1)(n+2)/2}} = \alpha\,.$$

Pour P5), la fonction $f = up(2\,x-1)$ vérifie supp $f \subset [0,2]$, $f(1) = 1$ et

$$f^{(k)}\Big(\frac{\ell}{2^{k-1}}\Big) = 2^k up^{(k)}\Big(\frac{2\ell}{2^{k-1}} - 1\Big) = 0\,,$$

tandis que

$$\|f^{(n)}\|_\infty = 2^n 2^{n(n+1)/2} = \frac{1}{2}\, 2^{(n+1)(n+2)/2}\,.$$

Il reste maintenant à vérifier l'unicité des solutions de P1) à P5).

• Pour P1), c'est évident: si $f = \chi * 2\, f(2\,x)$, alors

$$\hat f(\xi) = \hat\chi(\xi) \hat f\Big(\frac{\xi}{2}\Big) = \prod_{j=0}^{N} \hat\chi\Big(\frac{\xi}{2^j}\Big) \hat f\Big(\frac{\xi}{2^{N+1}}\Big)$$

et donc

$$\hat{f}(\xi) = \hat{f}(0) \prod_{j=0}^{\infty} \hat{\chi}\Big(\frac{\xi}{2^j}\Big) = \hat{f}(0)\, \widehat{up}(\xi)\,.$$

• Pour P2), on remarque que

$$\frac{d}{dx}(\chi * 2\, f(2\,x)) = 2\, f(2\,x) - 2\, f(2\,x-2)\,,$$

de sorte que pour $f \in \xi$ il est équivalent de dire que $f = \chi * 2\, f(2\,x)$ ou que

$$\frac{df}{dx} = 2\, f(2\,x) - 2\, f(2\,x-2)\,.$$

• Pour P3), on utilise le Lemme 8: si f vérifie les hypothèses de P3), alors $f \in A_n$ pour tout n. Comme $up \in A_n$, le lemme d'encadrement donne que

$$\|f - up\|_\infty \le \|h_n - \ell_n\|_\infty \le \frac{2}{2^n}\,.$$

Comme c'est vrai pour tout n, $f = up$.

• Pour P4), on commence par remarquer la chose suivante: si $\mu \in C_c^\infty$ et si ν est la fonction de support minimum dans $V_0[\mu]$, avec ω une fonction dans C_c^∞ vérifiant $\langle \omega(x-k), \nu\rangle = \delta_{k,0}$ pour $k \in \mathbb{Z}$, alors le fait que $f \in L^2_{\mathrm{loc}}$ s'écrive

$$f = \sum_{k\in\mathbb{Z}} \alpha_k\, \mu(x-k)\,, \qquad \text{avec } \{\alpha_k\} \in \mathbb{C}^{\mathbb{Z}}\,,$$

entraîne que

$$f = \sum_{k\in\mathbb{Z}} \langle f, \omega(x-k)\rangle\, \nu(x-k)\,.$$

En particulier, si $1 = \sum_{k\in\mathbb{Z}} \alpha_k\, \mu(x-k)$, on obtient que

$$1 = \langle 1, \omega\rangle \sum_{k\in\mathbb{Z}} \nu(x-k)\,,$$

et donc que $\sum_k (d\nu/dx)(x-k) = 0$; cela entraîne que

$$\frac{d\nu}{dx} = \alpha_1(x) - a_1(x-1)\,,$$

pour une fonction $\alpha_1 \in C_c^\infty$, et donc que

$$\hat{\mu}(\xi) = \frac{1 - e^{-i\xi}}{i\,\xi}\,\hat{\mu}_1(\xi)$$

pour une fonction $\mu_1 \in C_c^\infty$. On a alors $V_1[d\mu/dx] = V_1[\mu_1]$. A nouveau, si ν_1 est la fonction de support minimum de $V_1[d\mu/dx]$ et si $\omega_1 \in C_c^\infty$ vérifie

$$\Big\langle \omega_1\Big(x - \frac{k}{2}\Big), \nu_1 \Big\rangle = \delta_{k,0}\ ,$$

on a pour

$$f = \sum_{k\in\mathbb{Z}} \alpha_k\,\frac{d}{dx}\mu\Big(x - \frac{k}{2}\Big)\,, \qquad f = \sum \Big\langle f, \omega_1\Big(x - \frac{k}{2}\Big)\Big\rangle\,\nu_1\Big(x - \frac{k}{2}\Big)\,.$$

Ainsi, si en plus de $1 = \sum_{k\in\mathbb{Z}} \alpha_k\,\mu(x-k)$ on a

$$x = \sum_{k\in\mathbb{Z}} \beta_k\,\mu\Big(x - \frac{k}{2}\Big)\,,$$

on écrit

$$1 = \sum_{k\in\mathbb{Z}} \beta_k\,\frac{d}{dx}\mu\Big(x - \frac{k}{2}\Big)$$

et on obtient finalement

$$\frac{d}{dx}\,\nu_1 = \alpha_2(x) - \alpha_2\Big(x - \frac{1}{2}\Big)$$

et enfin

$$\hat{\mu}(\xi) = \frac{1 - e^{-i\xi}}{i\,\xi}\,\frac{1 - e^{-i\xi/2}}{i\,\xi/2}\,\hat{\mu}_2(\xi)\,.$$

Cet argument s'itère à l'infini et donne donc que si f vérifie les hypothèses de P4), on a pour tout $N \in \mathbb{N}$,

$$\hat{f}(\xi) = \prod_{j=0}^{N} \frac{1 - e^{-i\xi/2^j}}{(i\,\xi/2^j)}\,\hat{f}_{N+1}(\xi)\,,$$

avec $f_{N+1} \in C_c^\infty$. Le problème est alors de montrer que

$$\lim_{N\to+\infty} \hat{f}_{N+1}(\xi) = 1$$

ponctuellement. En notant $f_0 = f$, on a pour tout N,

$$f_{N+1} = \frac{1}{2^N} \sum_{k \geq 0} \frac{d}{dx} f_N\Big(x - \frac{k}{2^N}\Big) = -\frac{1}{2^N} \sum_{k<0} \frac{d}{dx} f_N\Big(x - \frac{k}{2^N}\Big) .$$

En particulier, on a: $\operatorname{supp} f_{N+1} \subset [\inf \operatorname{supp} f_N, \sup \operatorname{supp} f_N - 1/2^N] = [0, 1/2^N]$, de sorte qu'en fait on a

$$f_{N+1} = \chi_{[0,1/2^N]} \frac{1}{2^N} \frac{d}{dx} f_N = \chi_{[0,1/2^N]} \, 2^{-N(N+1)/2} \Big(\frac{d}{dx}\Big)^{N+1} f \, ,$$

et donc

$$\hat{f}_{N+1}(\xi) = 2^{-N(N+1)/2} \int_0^{1/2^N} f^{(N+1)}(x) \, e^{-ix\xi} \, dx \, .$$

Par ailleurs $\hat{f}_{N+1}(0) = 1$ et donc

$$\hat{f}_{N+1}(\xi) - 1 = 2^{-N(N+1)/2} \int_0^{1/2^N} f^{(N+1)}(x) \, (e^{-ix\xi} - 1) \, dx \, ,$$

d'où

$$\begin{aligned} |\hat{f}_{N+1}(\xi) - 1| &\leq 2^{-N(N+1)/2} \int_0^{1/2^N} \|f^{(N+1)}\|_\infty \frac{|\xi|}{2^N} \, dx \\ &\leq 2^{-N(N+1)/2} 2^{-N} \|f^{(N+1)}\|_\infty \frac{|\xi|}{2^N} \\ &= 2^{-(N+2)(N+3)/2} \|f^{(N+1)}\|_\infty \, 8 \, |\xi| \end{aligned}$$

et on a bien $\lim_{N \to +\infty} |\hat{f}_{N+1}(\xi) - 1| = 0$. Enfin P5) est immédiat, en appliquant le Lemme 8 (lemme de majoration) à $f - up$.

La Proposition 4 est donc démontrée.

Une propriété amusante de la fonction *up* est que ses valeurs aux points dyadiques sont rationnelles et aisément calculables. Ce qui en permet la tabulation et partant le calcul de toutes les quantités

$$\int_{k/2^j}^{(k+1)/2^j} x^\ell up(x) \, dx \, .$$

Lemme 9. *Pour tous* $j \in \mathbb{N}$, $k \in \{1, \dots, 2^{j+1} - 1\}$, $up(k/2^j) \in \mathbb{Q}$.

PREUVE. On remarque d'abord que tous les moments
$$m_\ell = \int_0^2 x^\ell up(x)\, dx$$
sont à valeurs rationnelles. En effet, on a
$$\begin{aligned}\int_0^2 x^\ell up(x)\, dx &= -\int_0^2 \frac{x^{\ell+1}}{\ell+1} \frac{d}{dx} up(x)\, dx \\ &= \int_0^2 (2\, up(2\,x-2) - 2\, up(2\,x)) \frac{x^{\ell+1}}{\ell+1}\, dx \\ &= \int_0^2 up(t) \Big(\Big(\frac{t+2}{2}\Big)^{\ell+1} - \Big(\frac{t}{2}\Big)^{\ell+1} \Big) \frac{dt}{\ell+1} \\ &= \frac{1}{2^\ell} \int_0^2 t^\ell up(t)\, dt \\ &\quad + \sum_{k=0}^{\ell-1} \frac{1}{\ell+1}\, C_{\ell+1}^k \, \frac{2^{\ell+1-k}}{2^{\ell+1}} \int_0^2 t^k up(t)\, dt\,.\end{aligned}$$
Partant de $\int_0^2 up(x)\, dx = 1$, on voit que $\int_0^2 t^\ell up(t)\, dt \in \mathbb{Q}$ par induction sur ℓ.

On calcule alors $up(k/2^j)$ par récurrence sur j à l'aide des formules
$$\sum_{k\in\mathbb{Z}} \Big(\frac{x-k}{2^j}\Big)^j up\Big(\frac{x-k}{2^j}\Big) = \int \Big(\frac{x}{2^j}\Big)^j up\Big(\frac{x}{2^j}\Big)\, dx = 2^j m_j\,.$$
Partant de $up(1) = 1$, on voit alors qu'on peut calculer les valeurs $up((2k-1)/2^{j+1})$ pour $1 \le k \le 2^{j+1}$ à l'aide des valeurs $up(k/2^j)$, $1 \le k \le 2^{j+1} - 1$, en résolvant le système de 2^{j+1} équations à 2^{j+1} inconnues:
$$(\mathrm{S}) \qquad \begin{cases} \text{pour } 0 \le \ell \le j, \text{ pour } 1 \le r \le 2^{j-\ell}, \\[2ex] \displaystyle\sum_{k=0}^{2^{\ell+1}-1} \Big(\frac{2r-1}{2^{j+1}} + \frac{k}{2^\ell}\Big)^\ell up\Big(\frac{2r-1}{2^{j+1}} + \frac{k}{2^\ell}\Big) = 2^\ell m_\ell\,, \\[4ex] \displaystyle\sum_{k=1}^{2^{j+1}} \Big(\frac{2k-1}{2^{j+1}}\Big)^{j+1} up\Big(\frac{2k-1}{2^{j+1}}\Big) \\ \displaystyle\qquad\qquad = 2^{j+1} m_{j+1} - \sum_{k=1}^{2^{j+1}-1} \Big(\frac{k}{2^j}\Big)^{j+1} up\Big(\frac{k}{2^j}\Big)\,. \end{cases}$$

Il reste à vérifier que le système est de Cramer. Mais les 2^j équations correspondant à $\ell = 0$ permettent d'exprimer $up((2r-1)/2^{j+1}+1)$ en fonction de $up((2r-1)/2^{j+1})$ $(1 \le r \le 2^j)$ et (S) devient alors pour ces valeurs $up((2r-1)/2^{j+1})$ $(1 \le r \le 2^j)$:

$$
(\mathrm{S}') \qquad \begin{cases}
\text{pour } 0 \le \ell \le j, \text{ pour } 1 \le r \le 2^{j-\ell}, \\[2ex]
\displaystyle\sum_{k=0}^{2^\ell-1} \Big(\Big(\frac{2r-1}{2^{j+1}} + \frac{k}{2^\ell} + 1\Big)^\ell - \Big(\frac{2r-1}{2^{j+1}} + \frac{k}{2^\ell}\Big)^\ell \Big) \\
\qquad\qquad \displaystyle\cdot\, up\Big(\frac{2r-1}{2^{j+1}} + \frac{k}{2^\ell}\Big) \\
\qquad\qquad\qquad\qquad \displaystyle = -2^\ell m_\ell + m_0 \sum_{k=0}^{2^\ell-1} \Big(\frac{2r-1}{2^{j+1}} + \frac{k}{2^\ell} + 1\Big)^\ell, \\[3ex]
\displaystyle\sum_{k=1}^{2^j} \Big(\Big(\frac{2k-1}{2^{j+1}} + 1\Big)^{j+1} - \Big(\frac{2k-1}{2^{j+1}}\Big)^{j+1}\Big)^{j+1} up\Big(\frac{2k-1}{2^{j+1}}\Big) \\
\qquad\qquad\qquad\qquad \displaystyle = -2^{j+1} m_{j+1} + \sum_{k=1}^{2^{j+1}-1} \Big(\frac{k}{2^j}\Big)^{j+1} up\Big(\frac{k}{2^j}\Big) \\
\qquad\qquad\qquad\qquad\quad \displaystyle + m_0 \sum_{k=1}^{2^j} \Big(\frac{2k-1}{2^{j+1}} + 1\Big)^{j+1}.
\end{cases}
$$

(S') se réécrit alors,

$$
(\mathrm{S}'') \qquad \begin{cases}
\text{pour } 0 \le \ell \le j-1, \text{ pour } 1 \le r \le 2^{j-1-\ell}, \\[2ex]
\displaystyle\sum_{k=0}^{2^{\ell+1}-1} \Big(\frac{2r-1}{2^j} + \frac{k}{2^\ell}\Big)^\ell up\Big(\frac{2r-1}{2^{j+1}} + \frac{k}{2^{\ell+1}}\Big) = c_{\ell,r}\ , \\[3ex]
\displaystyle\sum_{\ell=1}^{2^j} \Big(\frac{2k-1}{2^j}\Big)^j up\Big(\frac{2k-1}{2^{j+1}}\Big) = c_j\ .
\end{cases}
$$

où les $c_{\ell,r}$ et c_j sont des combinaisons linéaires des $up(k/2^j)$ à coefficients rationnels. On retrouve le système (S) au rang j (où les inconnues $up((2k-1)/2^j)$ sont remplacées par les inconnues $up(1/2\,(2k-1)/2^j))$ et par récurrence sur j, on obtient bien que (S) est un système de Cramer.

Nous pouvons maintenant présenter le résultat le plus frappant de Rvachev [26], [27]:

Proposition 5 (Série de Taylor généralisée). *Pour* $\rho \in]1,2[$, *on désigne par* K^ρ *l'espace de Banach*

$$K^\rho = \{f \in C^\infty(\mathbb{R}) \ : \ \sup_N \rho^{-N} 2^{-N(N+1)/2} \|f^{(N)}\|_\infty < +\infty\}\,.$$

Alors l'application

$$f \longrightarrow \{f(k)\}_{k\in\mathbb{Z}} \bigcup \Big\{\rho^{-N} 2^{-N(N+1)/2} f^{(N)}\Big(\frac{k}{2^{N-1}}\Big)\Big\}_{N\geq 1, k\in\mathbb{Z}}$$

est un isomorphisme de K^ρ *sur* $\ell^\infty(\mathbb{Z} \oplus \mathbb{N}^* \times \mathbb{Z})$*. Autrement dit toute fonction de* K^ρ *se décompose en*

$$f = \sum_{k\in\mathbb{Z}} f(k)\, \varphi_{0,k}(x) + \sum_{N=1}^{\infty} \sum_{k\in\mathbb{Z}} f^{(N)}\Big(\frac{k}{2^{N-1}}\Big) \varphi_{N,k}(x)\,,$$

où $\varphi_{N,k}$ *est l'unique solution de:* $\varphi_{N,k} \in K^\rho$, $\varphi_{N,k}(\ell) = \delta_{N,0}\delta_{k,\ell}$ *et*

$$\Big(\frac{d}{dx}\Big)^j \varphi_{N,k}\Big(\frac{\ell}{2^{j-1}}\Big) = \delta_{N,j}\, \delta_{\ell,k}\ ,$$

pour $j \geq 1$*. La convergence de la série a lieu uniformément sur tout compact de* $\mathbb{R}$ *pour* f *ainsi que pour toutes ses dérivées.*

PREUVE. Notons S_ρ l'application

$$f \longrightarrow \{f(k)\}_{k\in\mathbb{Z}} \bigcup \Big\{\rho^{-N} 2^{-N(N+1)/2} f^{(N)}\Big(\frac{k}{2^{N-1}}\Big)\Big\}_{N\geq 1, k\in\mathbb{Z}}\ .$$

Il est clair que S_ρ est injective de K^ρ dans $\ell^\infty(\mathbb{Z} \oplus \mathbb{N}^* \times \mathbb{Z})$. En effet, si $S_\rho f = 0$, le Lemme 8 (lemme de majoration) donne que pour tout n on a

$$\|f\|_\infty \leq 4\Big(\frac{\rho}{2}\Big)^{n+1} \rho^{-n-1} 2^{-(n+1)(n+2)/2} \|f^{(n+1)}\|_\infty\ .$$

Si $f \in K^\rho$, cela donne $\|f\|_\infty \leq C\,(\rho/2)^{n+1}$ où C ne dépend pas de n et donc $f = 0$.

Nous allons montrer maintenant l'existence de l'unique $\varphi_{N,k} \in K^\ell$ telle que

$$S_\rho(\varphi_{N,k}) = \{\delta_{N,0}\, \delta_{k,\ell}\}_{\ell\in\mathbb{Z}} \bigcup \{\rho^{-j}\, 2^{-j(j+1)/2} \delta_{N,j}\, \delta_{k,\ell}\}_{j\geq 1, \ell\in\mathbb{Z}}\ .$$

Nous montrerons de plus que $\varphi_{N,k} \in V_N[up]$ et que pour $N \geq 1$,

$$\operatorname{supp} \varphi_{N,k} \subset \Big[-E\Big(\frac{-k}{2^{N-1}}\Big) - 1, E\Big(\frac{k}{2^{N-1}}\Big) + 1 \Big] .$$

Nous allons construire $\varphi_{N,k}$ par récurrence sur N. Si $N = 0$, on connaît déjà la solution: $\varphi_{0,k} = up(x - 1 + k)$. Pour construire $\varphi_{N,k}$, une fois connues les $\varphi_{j,k}$ pour $j < N$, on part de l'égalité

$$\frac{d^N}{dx^N} \, up\Big(\frac{1}{2^N}\Big) = 2^{N(N+1)/2}$$

et on procède en trois étapes:

- en notant $I_{N,k}$ l'intervalle

$$I_{N,k} = \Big[-E\Big(\frac{-k}{2^{N-1}}\Big) - 1, E\Big(\frac{k}{2^{N-1}}\Big) + 1 \Big] ,$$

on définit $\omega_{N,k}$ par

$$\omega_{N,k} = \sum_{\ell/2^{N-1} \in I_{N,k} \text{ et } \ell \geq k} c_{N,k,\ell} \, up\Big(x - \frac{\ell}{2^{N-1}} + \frac{1}{2^N}\Big) ,$$

où les $c_{N,k,\ell}$ sont choisis de sorte que: pour $\ell/2^{N-1} \in I_{N,k}$ et $k \leq \ell$,

$$\frac{d^N}{dx^N} \, \omega_{N,k}\Big(\frac{\ell}{2^{N-1}}\Big) = \delta_{k,\ell} \, .$$

C'est–à-dire qu'on prend $c_{N,k,k} = 2^{-N(N+1)/2}$ et pour $\ell > k$,

$$c_{N,k,\ell} = -2^{-N(N+1)/2} \sum_{m=k}^{\ell-1} c_{N,k,m} \frac{d^N}{dx^N} \, up\Big(\frac{\ell - m}{2^{N-1}} + \frac{1}{2^N}\Big) .$$

Les $c_{N,k,\ell}$ sont donc bien définis et on a

$$\frac{d^N}{dx^N} \, \omega_{N,k}\Big(\frac{\ell}{2^{N-1}}\Big) = \delta_{k,\ell} \, ,$$

pour tous les points $\ell/2^{N-1} \in I_{N,k}$, tandis que pour tout $p > N$ et tout $\ell \in \mathbb{Z}$,

$$\frac{d^p}{dx^p} \, \omega_{N,k}\Big(\frac{\ell}{2^{p-1}}\Big) = 0 \, .$$

• On corrige alors $\omega_{N,k}$ en

$$\psi_{N,k} = \omega_{N,k} - \sum_{\ell\in I_{N,k}} \omega_{N,k}(\ell)\,\varphi_{0,\ell}(x) - \sum_{p=1}^{N-1} \sum_{\ell/2^{p-1}\in I_{N,k}} \frac{d^p}{dx^p}\,\omega_{N,k}\Big(\frac{\ell}{2^{p-1}}\Big)\varphi_{p,\ell}(x)\,.$$

On a alors $\psi_{N,k}(\ell) = 0$ pour $\ell \in I_{N,k}$ et

$$\frac{d^p}{dx^p}\,\omega_{N,k}\Big(\frac{\ell}{2^{p-1}}\Big) = \delta_{N,p}\,\delta_{k,\ell}\ ,$$

our $p \geq 1$ et $\ell/2^{p-1} \in I_{N,k}$. En particulier $\psi_{N,k}$ a un zéro d'ordre infini aux deux bornes de $I_{N,k}$.

• Il suffit alors de poser: $\varphi_{N,k} = \psi_{N,k}\,\chi_{I_{N,k}}$.

Bien sûr, on peut remplacer dans ce raisonnement $I_{N,k}$ par n'importe quel intervalle $I \supset I_{N,k}$ à bornes dans $\mathbb{Z}$. On obtient alors $\varphi_{N,k} = \omega_I\,\chi_I$ avec $\omega_I \in V_N[up]$; cela entraîne que $\varphi_{N,k} \in V_N[up]$. (En effet si ν_N est une fonction de $V_N[up]$ à support minimal et si $\nu_N^* \in C_c^\infty$ vérifie

$$\Big\langle \nu_N^*\Big(x - \frac{k}{2^N}\Big), \nu_N(x)\Big\rangle = \delta_{\ell,0}\ ,$$

on utilise le projecteur

$$P^N f = \sum_\ell \Big\langle f, \nu^*\Big(x - \frac{\ell}{2^N}\Big)\Big\rangle \nu\Big(x - \frac{\ell}{2^N}\Big)$$

sur $V_N[up]$. Son noyau est proprement supporté de sorte que $P_N\varphi_{N,k} = \varphi_{N,k}$ est immédiat).

Il reste à démontrer la convergence dans C^∞ (convergence uniforme sur tout compact de fonctions et de leurs dérivées) vers une fonction de K^ρ de

$$\sum_{N=1}^{\infty} \sum_{k\in\mathbb{Z}} c_{N,k}\,\varphi_{N,k}$$

lorsque $\{\rho^{-N} 2^{-N(N+1)/2} c_{N,k}\} \in \ell^\infty(\mathbb{N}^* \times \mathbb{Z})$. Pour cela, on va noter $K_{N,p}$ le nombre

$$K_{N,p} = \sup\Big\{ \Big\| \sum_{k\in\mathbb{Z}} c_{N,k}\,\frac{d^p}{dx^p}\,\varphi_{N,k}\Big\|_\infty\ :\ \sup_k |c_{N,k}| \leq 1\Big\}\,.$$

Comme $\varphi_{0,k} = up(x-1+k)$, on a (puisque $\operatorname{supp} up = [0,2]$)

$$K_{0,p} \leq 2 \left\| \frac{d^p}{dx^p}\, up \right\|_\infty = 2 \cdot 2^{p(p+1)/2}\,.$$

Nous allons alors estimer $K_{N,p}$ par récurrence sur N. On commence par remarquer que si $f \in K^\rho$ vérifie pour tout $p \geq p_0$ et tout $\ell \in \mathbb{Z}$,

$$\frac{d^p}{dx^p}\, f\Big(\frac{\ell}{2^{p-1}}\Big) = 0$$

alors

$$f = \sum_\ell f(\ell)\,\varphi_{0,\ell} + \sum_{p=1}^{p_0-1} \sum_\ell f^{(p)}\Big(\frac{f}{2^{p-1}}\Big)\varphi_{p,\ell}\,.$$

(En effet la convergence de

$$g = \sum_\ell f(\ell)\,\varphi_{0,2} + \sum_{p=1}^{p_0-1} \sum_\ell f^{(p)}\Big(\frac{\ell}{2^{p-1}}\Big)\varphi_{p,\ell}$$

est immédiate, la série étant localement finie. Par ailleurs, l'appartenance de g à K^ρ est immédiate: il suffit de remarquer que $\varphi_{p,\ell}(x+1) = \varphi_{p,\ell-2^{p-1}}(x)$, de sorte que les estimations de taille sur les dérivées de g sont immédiatement uniformes. Enfin $f = g$ puisque $S_\rho(f) = S_\rho(g)$). On remarque également que si $f \in K^\rho$ alors $(df/dx)(x/2) \in K^\rho$: si $g = (df/dx)(x/2)$ on a

$$\begin{aligned}\left\| \frac{d^N}{dx^N}\, g \right\|_\infty &= 2^{-N} \left\| \frac{d^{N+1}}{dx^{N+1}}\, f \right\|_\infty \\ &\leq 2^{-N}\, C\, \rho^{N+1} 2^{(N+1)(N+2)/2} \\ &= 2\, C\, \rho\rho^N\, 2^{N(N+1)/2}\,.\end{aligned}$$

On note alors $\varepsilon_N = \sum_{k\in\mathbb{Z}} c_{N,k}\,\varphi_{N,k}$. Si $N \geq 2$, on a

$$\begin{aligned}\frac{d\varepsilon_N}{dx} &= \frac{1}{2^N} \sum_{k\in\mathbb{Z}} c_{N,k}\, 2\, \varphi_{N-1,k}(2\,x) \\ &\quad + \sum_k \frac{1}{2}\, \frac{d\varepsilon_N}{dx}\Big(\frac{2k+1}{2}\Big) 2\, up(2\,x - 2\,k)\,.\end{aligned} \tag{27}$$

Cela nous donne

$$\Big\| \frac{d^N \varepsilon_N}{dx^N} \Big\|_\infty \le \Big\| \frac{d^{N-1}}{dx^{N-1}} \sum_{k\in\mathbb{Z}} c_{N,k}\, \varphi_{N-1,k} \Big\|_\infty + 2^{N(N+1)/2} \sup_k \Big| \frac{1}{2}\, \frac{d\varepsilon_N}{dx} \Big(\frac{2k+1}{2} \Big) \Big| \,.$$

Par ailleurs on a

$$\begin{aligned} \varepsilon_N(k_0+1) - \varepsilon_N(k_0) = 0 &= \int_{k_0}^{k_0+1} \frac{d\varepsilon_N}{dx}\, dx \\ &= \frac{1}{2}\, \frac{d\varepsilon_N}{dx} \Big(\frac{2k_0+1}{2} \Big) \\ &\quad + \int_{k_0}^{k_0+1} \frac{1}{2^N} \sum_{k\in\mathbb{Z}} c_{N,k}\, 2\, \varphi_{N-1,k}(2\,x)\, dx \,, \end{aligned}$$

ce qui donne

$$\Big| \frac{1}{2}\, \frac{d\varepsilon_N}{dx} \Big(\frac{2k_0+1}{2} \Big) \Big| \le \frac{2}{2^N} \Big\| \sum_{k\in\mathbb{Z}} c_{N,k}\, \varphi_{N-1,k} \Big\|_\infty \,.$$

Le Lemme 8 (lemme de majoration) permet alors d'écrire

$$\Big\| \sum_{k\in\mathbb{Z}} c_{N,k}\, \varphi_{N-1,k} \Big\|_\infty \le 2^{3-N-N(N-1)/2} \Big\| \frac{d^{N-1}}{dx^{N-1}} \sum_k c_{N,k}\, \varphi_{N-1,k} \Big\|_\infty \,.$$

Au total, on a donc obtenu

$$\begin{aligned} K_{N,N} &\le K_{N-1,N-1} \Big(1 + 2^{N(N+1)/2}\, \frac{2}{2^N}\, 2^{3-N-N(N-1)/2} \Big) \\ &= K_{N-1,N-1} \Big(1 + \frac{16}{2^N} \Big) \,. \end{aligned}$$

On obtient donc

$$K_{N,N} \le \prod_{j=2}^{N} \Big(1 + \frac{16}{2^j} \Big) K_{1,1} \le C_0 = \prod_{j=2}^{\infty} \Big(1 + \frac{16}{2^j} \Big) K_{1,1} \,.$$

Par ailleurs $K_{0,0} \leq 1$ et $K_{1,1}$ est bien fini puisque

$$K_{1,1} \leq \left\| \frac{d\varphi_{1,0}}{dx} \right\|_\infty + \left\| \frac{d\varphi_{1,1}}{dx} \right\|_\infty + \left\| \frac{d\varphi_{1,2}}{dx} \right\|_\infty .$$

On a donc montré

$$\sup_{N\in\mathbb{N}} K_{N,N} \leq \prod_{j=2}^{\infty} \Big(1 + \frac{16}{2^j}\Big) \cdot K_{1,1} = C_0 . \tag{28}$$

L'estimation de $K_{N,p}$ est alors facile:

- si $p < N$, on note

$$\Omega_{N,p} = 2^{N(N+1)/2}\, 2^{-p(p+1)/2}\, K_{N,p}$$

et en utilisant à nouveau (27) on obtient

$$K_{N,p} \leq 2^{p-N} K_{N-1,p-1} + 2^{p(p+1)/2}\, \frac{16}{2^N}\, 2^{-N(N+1)/2} K_{N-1,N-1}$$

et donc

$$\Omega_{N,p} \leq \Omega_{N-1,p-1} + \frac{16}{2^N}\, C_0 \leq \Omega_{N-p,0} + 16\, C_0 \sum_{N-p+1}^{N} \frac{1}{2^r} .$$

Par ailleurs, le Lemme 8 (lemme de majoration) donne que

$$K_{N-p,0} \leq K_{N-p,N-p}\, 4\, 2^{-(N-p)-(N-p)(N-p+1)/2}$$

et donc $\Omega_{N-p,0} \leq 4\, 2^{-(N-p)} C_0$. Au total

$$K_{N,p} \leq 2^{-N(N+1)/2}\, 2^{p(p+1)/2}\, 2^{-(N-p)}\, 5\, C_0 \,, \quad \text{pour } 0 \leq p \leq N . \tag{29}$$

- Si $p > N$, on remarque que

$$\frac{d^N}{dx^N}\Big(\sum c_{N,k}\, \varphi_{N,k}\Big) = \sum_{\ell\in\mathbb{Z}} \Big(\frac{d^N}{dx^N} \sum c_{N,k}\varphi_{N,k}\Big)\Big(\frac{\ell}{2^N}\Big) up(2^N x + 1 - \ell)$$

et donc (en utilisant que $up(x)$ et $up(x+2)$ sont à supports disjoints)

$$K_{N,p} \leq 2\, K_{N,N}\, 2^{-N(N+1)/2}\, 2^{p(p+1)/2} \,, \qquad \text{pour } p \geq N . \tag{30}$$

On conclut alors que si $\rho^{-N}2^{-N(N+1)/2}|c_{N,k}| \leq M$, on a

$$\begin{aligned}\Big\| \frac{d^p}{dx^p} \sum_N \sum_k c_{N,k}\, \varphi_{N,k} \Big\|_\infty &\leq \sum_{N=1}^{\infty} K_{N,p}\, M\, \rho^N 2^{N(N+1)/2} \\ &\leq M \sum_{N<p} 2\, C_0\, 2^{p(p+1)/2} \rho^N \\ &\quad + M \sum_{p\leq N} 5\, C_0\, 2^{p(p+1)/2} \rho^N 2^{-(N-p)} \\ &\leq C_0\, M\, \rho^p\, 2^{p(p+1)/2} \Big(2 \sum_{\ell \leq 0} \rho^{-\ell} + 5 \sum_{\ell \geq 0} \Big(\frac{\rho}{2}\Big)^\ell \Big).\end{aligned}$$

La Proposition 5 et donc démontrée.

Nous allons maintenant décrire l'analyse multi-résolution non-stationnaire engendrée par *up* (décrite dans [11] par N. Dyn et A. Ron).

Proposition 6. a) *Soit* up_k *la fonction de support minimum dans* $V_k[up]$ *normalisée par* $\inf \operatorname{supp} up_k = 0$ *et* $\widehat{up}_k(0) = 1$. *Alors on a*

$$\widehat{up}_k(2^k\xi) = \Big(\frac{1-e^{-i\xi}}{i\,\xi}\Big)^k \widehat{up}(\xi)\,, \tag{31}$$

$$\widehat{up}_k(2^k\xi) = \prod_{j=1}^{\infty} m_{j+k}\Big(\frac{\xi}{2^j}\Big) \quad \text{avec} \quad m_j(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^j. \tag{32}$$

En particulier, up engendre une analyse multirésolution non-stationnaire normale.

b) *Soit* Π_k *le projecteur orthogonal de* L^2 *sur* $V_k[up]$. *Alors pour tout* $N \geq 0$, *il existe une constante* $C_N > 0$ *telle que, pour tous* $k \geq N$, $f \in H^N$

$$\|f - \Pi_k f\|_2 \leq C_N\, 2^{-kN} \|f^{(N)}\|_2\,. \tag{33}$$

En particulier, la fonction up a un ordre d'approximation infini.

PREUVE. a) On a

$$\widehat{up}(\xi) = \prod_{j=0}^{\infty} \hat{\chi}\Big(\frac{\xi}{2^j}\Big)\,.$$

Par ailleurs, on sait que

$$\hat{\chi}(\xi) = \prod_{j=1}^{\infty} \Big(\frac{1 + e^{-i\xi/2^j}}{2} \Big) .$$

On obtient donc

$$\widehat{up}(\xi) = \prod_{j=0}^{\infty} \Big(\prod_{\ell=1}^{\infty} \frac{1 + e^{-i\xi/2^{j+\ell}}}{2} \Big) = \prod_{\ell=1}^{\infty} \Big(\frac{1 + e^{-i\xi/2^\ell}}{2} \Big)^{\ell} ,$$

Le regroupement des termes étant autorisé par le fait que si $|\xi| \leq \pi$,

$$\sum_{j=0}^{+\infty} \sum_{\ell=1}^{\infty} \Big| \mathrm{Log}\, \frac{1 + e^{-i\xi/2^{j+\ell}}}{2} \Big| < +\infty .$$

En particulier,

$$\widehat{up}(\xi) = \prod_{j=1}^{k} m_j \Big(\frac{\xi}{2^j} \Big) \hat{\gamma}_k(\xi)$$

avec

$$\hat{\gamma}_k = \prod_{j=k+1}^{\infty} m_j \Big(\frac{\xi}{2^j} \Big) .$$

On vérifie facilement que $\gamma_k \in V_k[up] \cap L^2_{\mathrm{comp}}$. (Il suffit de vérifier que si $\varphi \in L^2_{\mathrm{comp}}$ vérifie $\sum (-1)^k \varphi(x-k) = 0$, alors γ défini par

$$\hat{\gamma}(\xi) = \hat{\varphi}(\xi) \, \frac{2}{1 + e^{-i\xi}}$$

vérifie $\gamma \in V_0[\varphi] \cap L^2_{\mathrm{comp}}$: en fait on a $\varphi = (\gamma(x) + \gamma(x+1))/2$, ou encore

$$\gamma = 2 \sum_{0}^{\infty} \varphi(x+k) = -2 \sum_{-\infty}^{-1} \varphi(x+k) .$$

On a alors $\gamma \in L^2_{\mathrm{loc}}$, $\mathrm{supp}\, \gamma$ compact, donc $\gamma \in L^2_{\mathrm{comp}}$ et $\gamma \in V_0[\varphi]$). De plus γ_k est de support minimal dans $V_k[up]$: on peut utiliser le critère d'A. Ron [24] qui dit que ν est le support minimum dans $V_0[\mu]$ si la fonction entière $\hat{\nu}(z) = \int \nu(x) \, e^{-izx} \, dx$ n'a pas de zéro (complexe) 2π-périodique. Ce critère est évident: si ν n'est pas de support minimum,

$$\nu = \sum_{k_0}^{k_1} a_k \, \nu_0(x-k) \qquad \text{et} \qquad \hat{\nu}(z) = \Big(\sum_{k_0}^{k_1} a_k \, e^{-ikz} \Big) \hat{\nu}_0(z) ,$$

si ν par contre est de support minimum et si $\omega \in C_c^\infty$ vérifie $\langle \omega, \nu(x - k)\rangle = \delta_{k,0}$ alors

$$\sum_{k\in\mathbb{Z}} \hat{\nu}(z + 2k\pi)\, \hat{\omega}(z + 2k\pi)$$

converge uniformément sur tout compact de $\mathbb{C}$ vers la fonction 1.

Nous avons donc prouvé (32), et donc que $\{V_k[up]\}_{k\geq 0}$ est normale. (31) se déduit immédiatement de (32) en mettant $(1 + e^{-i\xi}/2)^k$ en facteur dans $m_{j+k}(\xi)$.

b) l'estimation (33) est relativement immédiate. On écrit $f = f_k + g_k$ où

$$\hat{f}_k = \chi_{[-\pi,\pi]}\Big(\frac{\xi}{2^k}\Big)\hat{f}\,.$$

On a

$$\|g_k - \Pi_k g_k\|_2 \leq 2\,\|g_k\|_2 \leq 2\,(2^k\pi)^{-N}\|f^{(N)}\|_2\,.$$

Par ailleurs, on a

$$\|\Pi_k f_k\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\Big|\displaystyle\sum_{\ell\in\mathbb{Z}} \hat{f}_k(\xi + \ell\, 2\pi\, 2^k)\, \widehat{up}_k(\xi + \ell\, 2\pi\, 2^k)\Big|^2}{\Big(\displaystyle\sum_{\ell\in\mathbb{Z}} |\widehat{up}_k(\xi + 2\pi\,\ell\, 2^k)|^2\Big)^2}\, |\widehat{up}_k(\xi)|^2\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\pi 2^k}^{\pi 2^k} |\hat{f}_k(\xi)|^2 \frac{|\widehat{up}_k(\xi)|^2}{\displaystyle\sum_{\ell\in\mathbb{Z}} |\widehat{up}_k(\xi + 2\pi\,\ell\, 2^k)|^2}\, d\xi\,,$$

et donc

$$\begin{aligned}\|f_k - \Pi_k f_k\|_2^2 &= \|f_k\|_2^2 - \|\Pi_k f_k\|_2^2 \\ &= \frac{1}{2\pi}\int_{-\pi 2^k}^{\pi 2^k} |\hat{f}_k(\xi)|^2 \frac{\displaystyle\sum_{\ell\neq 0} |\widehat{up}_k(\xi + 2\pi\,\ell\, 2^k)|^2}{\displaystyle\sum_{\ell\in\mathbb{Z}} |\widehat{up}_k(\xi + 2\pi\,\ell\, 2^k)|^2}\, d\xi\,.\end{aligned}$$

On écrit

$$\widehat{up}_k(\xi) = \Big(\frac{1 - e^{-i\xi/2^k}}{i\xi/2^k}\Big)^k \widehat{up}\Big(\frac{\xi}{2^k}\Big)\,,$$

d'où

$$\frac{\sum_{\ell\neq 0}|\widehat{up}_k(\xi+2\pi\,\ell\,2^k)|^2}{\sum_{\ell}|\widehat{up}_k(\xi+2\pi\,\ell\,2^k)|^2}\le\sum_{\ell\neq 0}\frac{|\widehat{up}_k(\xi+2\pi\,\ell\,2^k)|^2}{|\widehat{up}_k(\xi)|^2}$$

$$=\sum_{\ell\neq 0}\Big|\frac{\xi}{\xi+2\pi\,\ell\,2^k}\Big|^{2k}\frac{|\widehat{up}(\xi/2^k+2\pi\,\ell)|^2}{|\widehat{up}(\xi/2^k)|^2}\ .$$

Si $|\xi|\le\pi\,2^k$, alors

$$\frac{1}{|\widehat{up}(\xi/2^k)|}\le\sup_{|\eta|\le\pi}\frac{1}{|\widehat{up}(\eta)|}<+\infty$$

tandis que

$$\frac{|\xi|}{|\xi+2\pi\,\ell\,2^k|}\le\frac{|\xi|}{2\pi\,2^k-|\xi|}\le 1$$

et

$$\frac{|\xi|}{|\xi+2\pi\,\ell\,2^k|}\le\frac{|\xi|}{2\pi\,2^k-|\xi|}\le\frac{|\xi|}{\pi\,2^k}\ .$$

On obtient donc pour $k\ge N$,

$$\|f_k-\Pi_k f\|_2^2\le\frac{1}{2\pi}\int_{-\pi 2^k}^{\pi 2^k}|\hat f(\xi)|^2\Big(\frac{|\xi|}{\pi 2^k}\Big)^{2N}\cdot\frac{\sup_{|\eta|\le\pi}\sum_{\ell\neq 0}|\widehat{up}(\eta+2\pi\,\ell)|^2}{\inf_{|\eta|\le\pi}|\widehat{up}(\eta)|^2}\,d\xi$$

$$\le C_N\,2^{-2kN}\|f^{(N)}\|_2^2\ .$$

La Proposition 6 est donc démontrée.

En fait, non seulement l'approximation par up et ses translatées est spectrale, mais elle est de plus optimale. Plus précisément, Rvachev a étudié l'approximation des fonctions 2π-périodiques par des translatées (périodisées) de $up(x/\pi)$. Rappelons que si B est un espace de Banach et si A est une partie de B, le *diamètre de Kolmogorov* $d_n(A,B)$ est défini par

$$d_n(A,B)=\inf_{\substack{V_n\subset B\\ \dim V_n=n}}\ \sup_{a\in A}\ \inf_{v\in V_n}\|a-v\|_B\ .$$

C'est la mesure de la meilleure approximation possible de A par des vecteurs choisis dans un sous-espace de dimension n. On notera E_B (A, V) la quantité $\sup_{a\in A} \inf_{v\in V} \|a - v\|_B$.

Les 2π-périodisées de $up((x - 1/2^k)/\pi)$ forment un espace de dimension 2^{k+1}, noté Up_k. Rvachev montre alors [26]:

• si $B = L^2$ et $A_N = \{f \in H^N(\mathbb{R}/2\pi\mathbb{Z}) : \|f^{(N)}\|_2 \leq 1\}$, il existe $k_0(N)$ tel que pour $k \geq k_0(N)$, $E_{L^2}(A_N, Up_k) = d_{2^{k+1}}(A_N, L^2)$.

• Si $B = L^\infty$ et $\tilde{A}_N = \{f \in C^{N-1}(\mathbb{R}/2\pi\mathbb{Z}) : f^{(N)} \in L^\infty$ et $\|f^{(N)}\|_\infty \leq 1\}$, alors

$$\lim_{k\to+\infty} \frac{E_{L^\infty}(\tilde{A}_N, Up_k)}{d_{2^{k+1}}(\tilde{A}_N, L^\infty)} = 1 \,.$$

Comme le souligne Rvachev, les approximations optimales pour ces diamètres de Kolmogorov sont réalisées par les polynômes trigonométriques ou par les fonctions splines. Cependant les polynômes trigonométriques n'offrent pas de bonnes propriétés de localisation spatiale (puisque leur support est $\mathbb{R}/2\pi\mathbb{Z}$ tout entier) tandis que les splines ont un ordre d'approximation fini. La fonction de Rvachev combine les deux aspects (support compact, approximation spectrale) tout en étant optimale (pour l'approximation de H^N) ou asymptotiquement optimale (pour l'approximation de C^N).

On peut aussi étudier l'interpolation par des translatées de up. Plus précisément, Rvachev montre que $V_N[up]$ contient une interpolante Λ_N telle que

$$\delta_{k,0} = \begin{cases} \Lambda_N\Big(\dfrac{k}{2^N}\Big)\,, & \text{si } N \text{ est pair}\,, \\[2ex] \Lambda_N\Big(\dfrac{2k+1}{2^{N+1}}\Big)\,, & \text{si } N \text{ est impair}\,. \end{cases}$$

Il s'intéresse alors à l'opérateur d'interpolation

$$I_N f = \begin{cases} \displaystyle\sum_k f\Big(\frac{2k\pi}{2^N}\Big)\Lambda_N\Big(\frac{x}{\pi} - \frac{k}{2^N}\Big)\,, & \text{si } N \text{ est pair}\,, \\[2ex] \displaystyle\sum_k f\Big(\frac{2k+1}{2^{N+1}}\,2\pi\Big)\Lambda_N\Big(\frac{x}{\pi} - \frac{k}{2^N}\Big)\,, & \text{si } N \text{ est impair}\,. \end{cases}$$

Il montre que l'on a, si $f \in C^M(\mathbb{R}/2\pi\mathbb{Z})$,

$$\|f - I_N f\|_\infty \leq C_M \,\frac{1 + \log N}{2^{NM}}\, \|f^{(M)}\|_\infty$$

(alors que l'interpolation aux 2^{N+1} points $2k\pi/2^N$, $0 \leq k < 2^N$, par un polynôme trigonométrique ne donne une qualité d'approximation que de l'ordre de $N\,\|f^{(M)}\|_\infty/2^{NM}$ [31]; on passe donc d'une estimation d'erreur

$$\frac{2^{NM}\|f - I_N f\|_\infty}{\|f^{(M)}\|_\infty}$$

de l'ordre de $N = \mathrm{Log}_2\, 2^N$ à l'ordre de $\mathrm{Log}\, N = \mathrm{Log}\ \mathrm{Log}_2\, 2^N$).

6. La base de Berkolaïko et Novikov.

En 1992, V. Berkolaïko et I. Novikov ont introduit une modification de la fonction de Rvachev pour obtenir une base orthonormale de $L^2(\mathbb{R})$ composée d'ondelettes non stationnaires C^∞ à support compact [2]. Cette construction a été également introduite par A. Cohen et N. Dyn en 1993 [7], dans le cadre des travaux de N. Dyn sur les analyses multi-résolutions non stationnaires. L'idée est de remplacer dans la formule (32) le filtre

$$m_j(\xi) = \Big(\frac{1 + e^{-i\xi}}{2}\Big)^j$$

(qui est le fitre d'échelle du B-spline de degré $j - 1$) par le filtre de Daubechies

$$m_j(\xi) = \Big(\frac{1 + e^{-i\xi}}{2}\Big)^j \mu_j(\xi)$$

où $\mu_j = \sum_{k=0}^{j-1} \mu_{j,k}\, e^{-ik\xi}$ est tel que $|m_j(\xi)|^2 + |m_j(\xi + \pi)|^2 = 1$. En effet, ce filtre conduit à une fonction d'échelle qui a le même ordre d'approximation que le spline de degré $j - 1$ mais qui de plus engendre une base orthonormée d'ondelettes à support compact.

Proposition 7. *Soient $\{m_N\}_{N\geq 1}$ des filtres de Daubechies, avec*

$$m_N(\xi) = \Big(\frac{1 + e^{-i\xi}}{2}\Big)^N Q_N(e^{-i\xi})\,,$$

où $Q_N \in \mathbb{R}[X]$, $\deg Q_N = N - 1$, $Q_N(1) = 1$ et

$$|m_N(\xi)|^2 + |m_N(\xi + \pi)|^2 = 1\,.$$

On définit alors pour $N \geq 0$, Φ_N et Ψ_N par les formules

$$\hat{\Phi}_N(\xi) = \prod_{j=1}^{\infty} m_{N+j}(\xi)\,, \tag{34}$$

$$\hat{\Psi}_N(\xi) = e^{-i\xi/2}\, \overline{m}_{N+1}\Big(\frac{\xi}{2} + \pi\Big)\hat{\Phi}_{N+1}\Big(\frac{\xi}{2}\Big)\,. \tag{35}$$

Alors

i) Φ_N *et* Ψ_N *sont* C^∞ *à support compact,* $\operatorname{supp}\Phi_N \subset [0, 2N+3]$, $\operatorname{supp}\Psi_N \subset [-N, N+2]$.

ii) *La famille* $\{2^{N/2}\Phi_N(2^N x - k)\}_{k\in\mathbb{Z}}$ *est une base orthonormée de* $V_N[\Phi_0]$. *(En particulier* $\Phi_N(2^N x)$ *est de support minimal dans* $V_N[\Phi_0]$ *et* Φ_0 *engendre une analyse multi-résolution non-stationnaire normale).*

iii) *La famille* $\{2^{N/2}\Psi_N(2^N x - k)\}_{k\in\mathbb{Z}}$ *est une base orthonormée de* $V_{N+1}[\Phi_0] \cap V_N[\Phi_0]^\perp$. *En particulier, la famille* $\{\Phi_0(x-k)\}_{k\in\mathbb{Z}} \cup \{2^{N/2}\Psi_N(2^N x - k)\}_{N\geq 0, k\in\mathbb{Z}}$ *est une base orthonormée de* $L^2(\mathbb{R})$.

iv) Φ_0 *a un ordre d'approximation infini. Plus précisément, pour tout* $s \in \mathbb{R}$ *et tout* $f \in \mathcal{D}'(\mathbb{R})$, f *appartient à l'espace de Sobolev* $H^s(\mathbb{R})$ *si et seulement si* $N_s(f) < +\infty$ *où*

$$N_s(f) = \Big(\sum_{k\in\mathbb{Z}} |\langle f, \Phi_0(x-k)\rangle|^2 + \sum_{N=0}^{+\infty}\sum_{k\in\mathbb{Z}} 4^{Ns}|\langle f, 2^{N/2}\,\Psi_N(2^N x - k)\rangle|^2\Big)^{1/2}$$

et les normes $\|f\|_{H^s}$ *et* $N_s(f)$ *sont équivalentes.*

PREUVE. La convergence du produit infini (34) est immédiate. En effet, nous avons $\|m_N\|_\infty \leq 1$ et $\deg m_N = 2N - 1$, de sorte que l'inégalité de Bernstein nous donne $\|dm_N/d\xi\|_\infty \leq C\,N$, et donc

$$\Big|m_{N+j}\Big(\frac{\xi}{2^j}\Big) - 1\Big| \leq C\,\frac{N+j}{2^j}\,|\xi|\,,$$

comme

$$\sum_{j=1}^{\infty}(N+j)\,\frac{1}{2^j} = N + 2 < +\infty\,,$$

le produit converge. Par ailleurs, chaque produit fini est borné par 1; la convergence a donc lieu dans $\mathcal{S}'$. Par la transformation de Fourier inverse, on voit que Φ_N est un produit infini de convolution de sommes de masses de Dirac; le j-ième terme du produit de convolution a son support contenu dans $[0,(2N+2j-1)/2^j]$, de sorte que $\operatorname{supp}\Phi_N \subset [0,2N+3]$.

Par ailleurs, puisque $|m_N(\xi)|^2+|m_N(\xi+\pi)|^2=1$ pour tous N et ξ, les fonctions $\theta_{N,p}$ définies par

$$\hat{\theta}_{N,p}(\xi)=\chi_{[-\pi,\pi]}\Big(\frac{\xi}{2^p}\Big)\prod_{j=1}^{p} m_{N+j}\Big(\frac{\xi}{2^j}\Big) \tag{36}$$

engendrent des familles orthogonales $\{\theta_{N,p}(x-k)\}_{k\in\mathbb{Z}}$. Comme $\hat{\theta}_{N,p}\to\hat{\Phi}_N$ quand $p\to+\infty$ (la convergence étant ponctuelle), la preuve de la Proposition 7 se réduit aux estimations suivantes

$$|\hat{\theta}_{N,p}(\xi)|\le C_1|\xi|^{-\alpha|\mathrm{Log}\,\xi|}\,, \tag{37}$$

$$|\hat{\Psi}_N(\xi)|\le D_k|\xi|^k\,,\qquad \text{pour } k\in\mathbb{N},\ k\le N+1\,, \tag{38}$$

où C_1 et α sont des constantes positives ne dépendant ni de N ni de p, et où D_k ne dépend pas de N.

(37) est évident si $|\xi|\le 2^{30}\,2\pi$: on écrit $|\hat{\theta}_{N,p}(\xi)|\le 1$. Si $|\xi|\ge 2^{30}\,2\pi$, on note ℓ l'entier (≥ 30) tel que $2^{\ell}\,2\pi\le|\xi|<2^{\ell+1}\,2\pi$. Si $p\le\ell$, $\hat{\theta}_{N,p}(\xi)=0$. Si $p\ge\ell+1$, on a

$$|\hat{\theta}_{N,p}(\xi)|\le\prod_{j=1}^{\ell+1}\Big|m_{N+j}\Big(\frac{\xi}{2^j}\Big)\Big|=A_{N,\ell}(\xi)\,B_{N,\ell}(\xi)\,,$$

où

$$A_{N,\ell}(\xi)=\prod_{j=1}^{\ell+1}\Big|\frac{1+e^{-i\xi/2^j}}{2}\Big|^{N+j}\quad\text{et}\quad B_{N,\ell}(\xi)=\sum_{j=1}^{\ell+1}|Q_{N+j}(e^{-i\xi/2^j})|\,.$$

$A_{N,\ell}$ se majore facilement du fait que pour $|\xi|\le 2^{\ell+1}\pi$, $|\sin(\xi/2^{\ell+2})|\ge(2/\pi)(|\xi|/2^{\ell+2})$,

$$\begin{aligned}A_{N,\ell}(\xi)&=\prod_{j=1}^{\ell+1}\Big|\frac{1+e^{-i\xi/2^j}}{2}\Big|^{N}\prod_{k=0}^{\ell}\Big|\prod_{j=k+1}^{\ell+1}\frac{1+e^{-i\xi/2^j}}{2}\Big|\\&=\Big|\frac{\sin(\xi/2)}{2^{\ell+1}\,\sin(\xi/2^{\ell+2})}\Big|^{N}\prod_{k=0}^{\ell}\Big|\frac{\sin(\xi/2^{k+1})}{2^{\ell+1-k}\,\sin(\xi/2^{\ell+2})}\Big|\\&\le\Big(\frac{\pi}{|\xi|}\Big)^{N}\prod_{k=0}^{\ell}\frac{2^k\pi}{|\xi|}\\&\le 2^{-\ell N}\,2^{-\ell(\ell+1)/2}\,.\end{aligned}$$

Pour $B_{N,\ell}$, on part de la formule de Daubechies pour $|Q_N(e^{-i\xi})|$ [8]

$$|Q_N(e^{-i\xi})|^2 = \sum_{k=0}^{N-1} C_{N+k-1}^k \Big(\frac{1-\cos\xi}{2}\Big)^k .$$

Berkolaïko et Novikov ont remarqué que, puisque $C_{N+k-1}^k \le C_{N+k}^k$ si $k \le N-1$, on a $|Q_N(e^{-i\xi})| \le |Q_{N+1}(e^{-i\xi})|$ pour tous N et ξ. Par ailleurs, le contrôle de $Q_N(e^{-i\xi})\, Q_N(e^{-2i\xi})$ est classique

$$|Q_N(e^{-i\xi})\, Q_N(e^{-2i\xi})| \le |Q_N(-1)\, Q_N(e^{-2i\pi/3})| \le 2^{N-1/2}\, 3^{N/2} .$$

En effet,

$$A_N(X) = \sum_{k=0}^{N-1} C_{k+N-1}^k \Big(\frac{1-X}{2}\Big)^k$$

décroît sur $[-1,1]$; de plus on a toujours $\cos\xi \ge -1/2$ ou $\cos 2\,\xi \ge -1/2$, de sorte que

$$|Q_N(e^{-i\xi})\, Q_N(e^{-2i\xi})| \le |Q_N(-1)\, Q_N(e^{-2i\pi/3})| .$$

Par ailleurs, si $X<0$, on a

$$A_N(X) \le 2^{2N-2}\Big(\frac{1-X}{2}\Big)^{N-1}\Big(\frac{1-X}{-X}\Big)$$

il suffit d'écrire que $C_{N+k-1}^k \le C_{N+k}^{k+1}/2$ donc que

$$C_{N+k-1}^k \le 2^{k-N+1}\, C_{2N-2}^{N-1} \le 2^{2N-2}\, 2^{k-N+1} ,$$

cela donne $|Q_N(-1)| \le 2^{N-1/2}$ et $|Q_N(e^{-2i\pi/3})| \le 3^{N/2}$. Cette estimation sur $A_N(X)$ donne également lorsque $\cos\xi<0$,

$$|m_N(\xi)| \le \Big(\frac{1+\cos\xi}{2}\Big)^{N/2}(1-\cos\xi)^{N/2}\, 2^{N-1/2}\, \frac{1}{\sqrt{-\cos\xi}} = \frac{|\sin\xi|^N}{\sqrt{2\,|\cos\xi|}} .$$

L'estimation de $B_{N,\ell}$ est alors facile. En effet, si $\ell = 2\,q$, on écrit

$$\begin{aligned}
|B_{N,\ell}(\xi)| &\le \Big| Q_{N+1}(e^{-i\xi/2}) \prod_{r=1}^{q} Q_{2r+1+N}(e^{-i\xi/2^{2r}})\, Q_{2r+1+N}(e^{-i\xi/2^{2r+1}}) \Big| \\
&\le 2^{N+1/2} \prod_{r=1}^{q} 2^{N+2r+1/2}\, 3^{N/2+r+1/2} \\
&= (2\sqrt{3})^{N\ell/2+\ell(\ell+2)/4}\, 2^{N+1/2}\, 6^{\ell/4} ,
\end{aligned}$$

tandis que si $\ell = 2\,q - 1$ on écrit

$$\begin{aligned}|B_{N,\ell}(\xi)| &\leq \prod_{r=1}^{q} \left|Q_{N+2r}(e^{-i\xi/2^{2r-1}})\, Q_{N+2r}(e^{-i\xi/2^{2r}})\right| \\ &\leq \prod_{r=1}^{q} 2^{N+2r-1/2}\, 3^{N/2+r} \\ &= (2\sqrt{3})^{N(\ell+1)/2+(\ell+1)(\ell+3)/4}\, 2^{-(\ell+1)/2}\,.\end{aligned}$$

On obtient dans tous les cas

$$|B_{N,\ell}(\xi)| \leq (2\sqrt{3})^{N(\ell+2)/2}\, (2\sqrt{3})^{(\ell+2)(\ell+3)/4}$$

et donc

$$|A_{N,\ell}\, B_{N,\ell}| \leq 2^{-\ell N}(2\sqrt{3})^{N(\ell+2)/2}\, (2\sqrt{3})^{(\ell+2)(\ell+3)/4}\, 2^{-\ell(\ell+1)/2}\,.$$

Comme $\ell \geq 30$, $(\ell+2)/2 \leq 8\,\ell/15$ et donc

$$|A_{N,\ell}\, B_{N,\ell}| \leq \Big(\frac{(2\sqrt{3})^{8/15}}{2}\Big)^{\ell N} \Big(\frac{\sqrt{3}}{2}\Big)^{\ell^2/4} (3^{5/8}\, 2^{3/4})^{\ell}\, (2\sqrt{3})^{3/2}\,.$$

Comme $\gamma = (2\sqrt{3})^{8/15}/2$ est < 1 (car $\gamma^{30} = 3^8\, 2^{16}/2^{30} = 3^8/2^{14} = (81/128)^2$) et $\sqrt{3}/2 < 1$, on a en choisissant $\alpha < \log(\sqrt{3}/2)/4$ (et $\alpha > 0$)

$$|A_{N,\ell}\, B_{N,\ell}(\xi)| \leq C\, \gamma^{\ell N} e^{-\alpha \ell^2}\,,$$

où C ne dépend ni de N, ni de ℓ. Comme $\ell \geq \log(|\xi|/2\pi)/\log 2$, on a

$$|\hat{\theta}_N(\xi)| \leq C\Big(\frac{|\xi|}{2\pi}\Big)^{-N \mathrm{Log}\,(1/\gamma)/\mathrm{Log}\,2} \Big(\frac{|\xi|}{2\pi}\Big)^{-\alpha \mathrm{Log}\,(|\xi|/2\pi)/(\mathrm{Log}\,2)^2}\,,$$

(37) est donc démontré.

(38) est immédiat. En effet, si $|\xi| \geq 1$, on a $|\hat{\Psi}(\xi)| \leq 1$. Si $|\xi| \leq 1$, on a $|\xi/2 + \pi| \in [2\pi/3, 4\pi/3]$ (d'où $\cos(\xi+\pi) \leq -1/2$); on écrit alors

$$|\hat{\Psi}_N(\xi)| \leq \Big|m_{N+1}\Big(\frac{\xi}{2} + \pi\Big)\Big| \leq \frac{|\sin(\xi/2)|^{N+1}}{\sqrt{2\,\cos(\xi/2)}} \leq \Big(\frac{|\xi|}{2}\Big)^{N+1} \leq \Big(\frac{|\xi|}{2}\Big)^{k}.$$

La Proposition 7 est alors facile à établir. i) et ii) sont immédiats. Le point iii) est facile à vérifier: si $f, g \in V_N[\Phi_0]$,

$$\hat{f}(\xi) = F\Big(\frac{\xi}{2^N}\Big)\hat{\Phi}_N\Big(\frac{\xi}{2^N}\Big)$$

et
$$\hat{g}(\xi) = G\Big(\frac{\xi}{2^N}\Big)\hat{\Phi}_N\Big(\frac{\xi}{2^N}\Big)\,,$$
on a alors
$$\langle f, g\rangle = 2^N \int_{-\pi}^{\pi} F(\xi)\,\overline{G}(\xi)\,\frac{d\xi}{2\pi}\,.$$
On applique ceci à $f = \Psi_{N-1}$ et à $g = \Phi_{N-1}(x-k)$ ou $\Psi_{N-1}(x-k)$.

Enfin, la caractérisation de H^s est aisée à établir. En utilisant (37) et (38), on vérifie que pour tout $k \in \mathbb{N}$

$$\sup_{N\geq 0}\sup_{\xi}\sum_{p\in\mathbb{Z}} |\xi + 2\pi\, p|^k |\hat{\Psi}_N(\xi+2\pi\, p)|^2 < +\infty\,,$$
$$\sup_{N\geq k-1}\sup_{\xi}\sum_{p\in\mathbb{Z}} |\xi + 2\pi\, p|^{-k} |\hat{\Psi}_N(\xi+2\pi\, p)|^2 < +\infty\,.$$

Cela entraîne que, si Λ^s est l'opérateur $\widehat{\Lambda^s f} = |\xi|^s \hat{f}$, on a pour $s \in \mathbb{R}$ et $N \geq -s$
$$\|\Lambda^s Q_N f\|_2 \leq C_s\, 2^{Ns} \|Q_N f\|_2\,,$$
où
$$Q_N f = \sum_k 2^N \langle f, \Psi_N(2^N x - k)\rangle \Psi_N(2^N x - k)\,.$$
En particulier, on a
$$\begin{aligned}|\langle \Lambda^s Q_N f, \Lambda^s Q_{N'} f\rangle| &= |\langle \Lambda^{s\pm 1} Q_N f, \Lambda^{s\mp 1} Q_{N'} f\rangle| \\ &\leq 2^{Ns}\, 2^{N's}\, 2^{-|N-N'|} \|Q_N f\|_2 \|Q_{N'} f\|_2\end{aligned}$$
et cela entraîne
$$\|f\|_{H^s} \leq C\,\|N_s f\|_2 \simeq C(\|P_0 f\|_2^2 + \sum_0^\infty 4^{Ns}\|Q_N f\|_2^2)^{1/2}$$
(où P_0 est le projecteur orthogonal sur $V_0[\Phi_0]$). L'inégalité inverse s'obtient alors par dualité.

REMARQUE. On obtient facilement que
$$\lim_{N\to+\infty} |\hat{\Phi}_N(\xi)| = \begin{cases} 1\,, & \text{si } |\xi| < \pi\,, \\ \dfrac{1}{\sqrt{2}}\,, & \text{si } |\xi| = \pi\,, \\ 0\,, & \text{si } |\xi| > \pi\,. \end{cases}$$

En particulier, par le théorème d'Ascoli, cela entraîne que

$$\lim_{N\to+\infty} \inf_{x_0} \int |x - x_0|^2 \, |\Phi_N(x)|^2 \, dx = +\infty \,.$$

Ainsi, le support numérique de Φ_N tend vers l'infini. Cela entraîne des difficultés pour décrire les propriétés d'analyse fonctionnelle de la base de Berkolaïko et Novikov.

Dans les sections suivantes, nous allons décrire des analyses multi-résolutions non-stationnaires où les fonctions d'échelle non-stationnaires auront un bon comportement *global* en espace et en fréquence.

7. Analyse multi-résolution quasi-stationnaire.

Le manque de contrôle du comportement asymptotique des fonctions Φ_N de Berkolaïko et Novikov provient de ce que les filtres de Daubechies ont un mauvais comportement asymptotique lorsque $N \to +\infty$

$$\lim_{N\to+\infty} |m_N(\xi)| = \begin{cases} 1\,, & \text{si } |\xi| < \dfrac{\pi}{2}\,, \\ \dfrac{1}{\sqrt{2}}\,, & \text{si } |\xi| = \dfrac{\pi}{2}\,, \\ 0\,, & \text{si } \dfrac{\pi}{2} < |\xi| \le \pi\,. \end{cases}$$

La situation est différente lorsque les filtres convergent dans $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$

Definition. *Une analyse multi-résolution quasi-stationnaire est l'analyse multi-résolution non-stationnaire $\{V_k[\Phi_0]\}_{k\ge 0}$ associée à une fonction Φ_0 qui vérifie*

i) $\hat{\Phi}_0(\xi) = \prod_{j=1}^\infty m_j(\xi/2^j)$.

ii) m_j *est un polynôme trigonométrique à coefficients réels tel que*

$$\sum_{j=1}^\infty \frac{1}{2^j} \deg m_j < +\infty \,.$$

iii) m_j *converge dans* $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ *vers une fonction* m_∞ *qui est un filtre d'échelle associé à une fonction d'échelle régulière* φ_∞.

iv) *Pour* j *assez grand,* $m_j(\pi) = 0$ *et* $m_j(0) = 1$.

En fait, seules les propriétés de m_∞ déterminent les propriétés de l'analyse multi-résolution.

Théorème 2 (Deuxième théorème d'approximation). *Soient $\{m_j\}_{j\geq 1}$ des fonctions de $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ vérifiant les hypothèses suivantes:*

i) *m_j converge dans $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ vers le filtre d'échelle m_∞ d'une fonction d'échelle régulière φ_∞.*

ii) *Pour j assez grand, $m_j(0) = 1$ et $m_j(\pi) = 0$.*

Alors

a) *Les fonctions $\hat{\Phi}_j(\xi) = \prod_{\ell=1}^{\infty} m_{j+\ell}(\xi/2^\ell)$, $j \geq 0$, sont C^∞ et de carré intégrable. De plus leurs transformées de Fourier inverses Φ_j sont à décroissance rapide dans L^2 et convergent rapidement vers φ_∞ dans L^2.*

b) *De même si $\alpha > 0$, si $\varphi_\infty \in H^{\alpha+\varepsilon}$ pour un $\varepsilon > 0$ et si pour j assez grand on a $(\partial^p/\partial\xi^p)m_j(\pi) = 0$ pour $0 \leq p \leq [\alpha]$, les Φ_j sont à décroissance rapide dans H^α et convergent rapidement vers φ_∞ dans H^α.*

c) *Si tous les m_j sont à valeurs positives ou nulles, si $\alpha > 0$, si $\varphi_\infty \in B_\infty^{\alpha+\varepsilon,\infty}$ pour un $\varepsilon > 0$ et si pour j assez grand on a $(\partial^p/\partial\xi^p)m_j(\pi) = 0$ pour $0 \leq p \leq 2[\alpha/2] + 1$, alors les Φ_j sont à décroissance rapide dans $B_\infty^{\alpha,\infty}$ et convergent rapidement vers φ_∞ dans $B_\infty^{\alpha,\infty}$.*

d) *Tous les m_j vérifient $|m_j(\xi)|^2 + |m_j(\xi+\pi)|^2 = 1$ si et seulement si toutes les familles $\{\Phi_j(x-k)\}_{k\in\mathbb{Z}}$ sont des familles orthonormales, (i.e. $\langle\Phi_j(x-k), \Phi_j(x-\ell)\rangle = \delta_{k,\ell}$).*

e) *Si tous les m_j sont à valeurs positives ou nulles, alors tous les m_j vérifient $m_j(\xi)+m_j(\xi+\pi) = 1$ si et seulement si toutes les fonctions Φ_j sont interpolantes, (i.e. $\Phi_j(k) = \delta_{k,0}$).*

f) *Si tous les m_j sont des polynômes trigonométriques et si*

$$\sum_{j=1}^{\infty}\Big(\frac{\deg m_j}{2^j}\Big) < +\infty\,,$$

alors toutes les Φ_j sont à support compact et pour j assez grand $\{\Phi_j(2^j x - k)\}_{k\in\mathbb{Z}}$ est une base de Riesz de $V_j[\Phi_0]$.

REMARQUES. a) Parmi les propriétés qui ne peuvent pas se lire sur m_∞, on peut se demander dans f) si $\Phi_j(2\,x)$ est de support minimum dans $V_j[\Phi_0]$. C'est bien sûr le cas si Φ_j est interpolante (cas e)) ou orthogonale (cas d)).

b) Contrairement au cas de la fonction *up* [11], on a un contrôle de

$$\frac{\sup_{\|\lambda_k\|_2=1} \Big\| \sum_k \lambda_k \, \Phi_j(x-k) \Big\|_2}{\inf_{\|\lambda_k\|_2=1} \Big\| \sum_k \lambda_k \, \Phi_j(x-k) \Big\|_2},$$

uniforme en j (pour j assez grand): en effet, cette quantité est donnée par

$$\frac{\sup_\xi \Big(\sum_k |\hat{\Phi}_j(\xi+2k\pi)|^2 \Big)^{1/2}}{\inf_\xi \Big(\sum_k |\hat{\Phi}_j(\xi+2k\pi)|^2 \Big)^{1/2}}$$

et on montre facilement que $\sum_k |\hat{\Phi}_j(\xi+2k\pi)|^2$ converge uniformément vers $\sum_k |\hat{\varphi}_\infty(\xi+2k\pi)|^2$.

PREUVE. La preuve de a), b), c) est similaire à celle du Théorème 1, demandant juste un peu de précautions oratoires pour tenir compte de la non-stationarité.

On vérifie facilement que $\hat{\Phi}_j$ est C^∞. En effet, si V est un voisinage compact de $\xi_0 \in \mathbb{R}$, on a sur V, $\operatorname{Re} m_{j+\ell}(\xi/2^\ell) \geq 1/2$ pour ℓ assez grand; il suffit de remarquer que

$$|m_N(\xi) - m_N(0)| \leq \sup_k \Big\| \frac{d}{d\xi} \, m_k \Big\|_\infty |\xi| \, .$$

On a alors

$$\Big| \operatorname{Log} m_{j+\ell}\Big(\frac{\xi}{2^\ell}\Big) \Big| \leq 2 \sup_k \Big\| \frac{d}{d\xi} \, m_k \Big\|_\infty \frac{|\xi|}{2^\ell}$$

et pour $p \geq 1$

$$\Big| \frac{d^p}{d\xi^p} \operatorname{Log} m_{j+\ell}\Big(\frac{\xi}{2^\ell}\Big) \Big| \leq \Big(\frac{1}{2}\Big)^{\ell_p} 4^p p! \Big(\sum_{q=0}^{p} \sup_k \Big\| \frac{d^q}{d\xi^q} \, m_k \Big\|_\infty \Big)^p .$$

Cela assure que $\sum_{\ell \geq \ell_0} \operatorname{Log} m_{j+\ell}(\xi/2^\ell)$ est C^∞ sur $\overset{\circ}{V}$, et il en va de même pour

$$\hat{\Phi}_j = \prod_{1 \leq \ell < \ell_0} m_{j+\ell}\Big(\frac{\xi}{2^\ell}\Big) e^{\sum_{\ell \geq \ell_0} \operatorname{Log} m_{j+\ell}(\xi/2^\ell)} \, .$$

Cette démonstration permet également de voir qu'il existe C et $M \geq 0$ tels que pour tout j et tout ξ on ait: $|\hat{\Phi}_j(\xi)| \leq C\,(1+|\xi|)^M$. Par convergence dominée, cela entraîne la convergence de Φ_j vers φ_∞ dans H^s pour $s < -M-1/2$ et dans $B_\infty^{\delta,\infty}$ pour $\delta < -M-1$. Pour terminer la démonstration de a), b), c), il suffit de vérifier que dans le cas b) on a, pour $0 < \varepsilon' < \min\{\varepsilon, [\alpha]+1-\alpha\}$ pour tout $k \in \mathbb{N}$,

$$\sup_j \|x^k \Phi_j\|_{H^{\alpha+\varepsilon'}} < +\infty$$

et dans le cas c) on a, pour $0 < \varepsilon' < \min\{\varepsilon, 2[\alpha/2]+2-\alpha\}$ pour tout $k \in \mathbb{N}$,

$$\sup_j \|x^k \Phi_j\|_{B_\infty^{\alpha+\varepsilon',\infty}} < +\infty\,.$$

Bien évidemment, on peut se restreindre à $j \geq j_0$ et écrire:

- dans le cas b),

$$m_j(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^N \mu_j(\xi)$$

($j \geq j_0$ ou $j = +\infty$) où $N = [\alpha]+1$; on a $\mu_j \to \mu_\infty$ dans $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ et si

$$\hat{\Omega}_j = \prod_{k=1}^{\infty} \mu_{j+k}\Big(\frac{\xi}{2^\ell}\Big)\,,$$

on a

$$\hat{\Phi}_j = \Big(\frac{1-e^{-i\xi}}{i\,\xi}\Big)^N \hat{\Omega}_j$$

et le problème est alors de montrer que

$$\sup_j \|x^k \Omega_j\|_{H^{\alpha-N+\varepsilon'}} < +\infty\,,$$

- dans le cas c),

$$m_j(\xi) = \Big(\frac{1+\cos\xi}{2}\Big)^N \mu_j(\xi)\,,$$

où $N = 2[\alpha/2]+2$ et

$$\hat{\Phi}_j = \Big(\frac{2\,(1-\cos\xi)}{\xi^2}\Big)^N \hat{\Omega}_j$$

et le problème est de montrer que

$$\sup_k \|x^k \Omega_j\|_{B_\infty^{\alpha-2N+\varepsilon',\infty}} < +\infty\,.$$

Nous traitons le cas b), le cas c) se traitant de manière totalement analogue (en étudiant la norme $\|x^k\Omega_j\|_{B_1^{\alpha-2N+\varepsilon'',\infty}}$). Nous cherchons à estimer

$$I_{j,k} = \int_{|\xi|\geq\pi} |\xi|^{2\alpha+2\varepsilon'-2N} \Big|\frac{\partial^k}{\partial\xi^k}\,\hat\Omega_j(\xi)\Big|^2\,d\xi\,,$$

ou encore (puisque $\alpha+\varepsilon'-N<0$)

$$\begin{aligned}\tilde I_{j,k} &= \sum_{\ell=0}^{\infty}\int_{|\xi|\leq 2^\ell 2\pi} (1+|\xi|)^{2\alpha+2\varepsilon'-2N}\Big|\frac{\partial^k}{\partial\xi^k}\,\hat\Omega_j(\xi)\Big|^2\,d\xi\\ &\leq C\sum_{\ell=0}^{\infty}(2^\ell)^{2\alpha+2\varepsilon'-2N}\sup_\xi \sum_{|\xi+2\pi p|\leq 2^\ell 2\pi}\Big|\frac{\partial^k}{\partial\xi^k}\,\hat\Omega_j(\xi+2\pi\,p)\Big|^2\,.\end{aligned}$$

On note alors

$$M_{j,k,\ell}(\xi) = \sum_{|\xi+2\pi p|\leq 2^\ell 2\pi}\Big|\frac{\partial^k}{\partial\xi^k}\,\hat\Omega_j(\xi+2\pi\,p)\Big|^2\,.$$

Si $k=0$, on a

$$\begin{aligned}M_{j,0,\ell}(\xi) &= \Big|\mu_{j+1}\Big(\frac{\xi}{2}\Big)\Big|^2 M_{j+1,0,\ell-1}\Big(\frac{\xi}{2}\Big) + \Big|\mu_{j+1}\Big(\frac{\xi}{2}+\pi\Big)\Big|^2 M_{j+1,0,\ell-1}\Big(\frac{\xi}{2}\Big)\\ &= T_{j+1}\circ T_{j+2}\circ\cdots\circ T_{j+\ell}(M_{j+\ell,0,0})\,,\end{aligned}$$

où T_j est l'opérateur de transition associé à μ_j, *i.e.* T_j est défini par

$$T_j f = \Big|\mu_j\Big(\frac{\xi}{2}\Big)\Big|^2 f\Big(\frac{\xi}{2}\Big) + \Big|\mu_j\Big(\frac{\xi}{2}+\pi\Big)\Big|^2 f\Big(\frac{\xi}{2}+\pi\Big)\,.$$

Par positivité des opérateurs T_j et contrôle uniforme des $\hat\Omega_j$ sur $[-2\pi, 2\pi]$, on obtient

$$|M_{j,0,\ell}(\xi)| \leq C\,T_{j+1}\circ T_{j+2}\circ\cdots\circ T_{j+\ell}(1)(\xi)\,.$$

Par ailleurs, si ρ est choisi de façon à ce que $4^{N-\alpha-\varepsilon} < \rho < 4^{N-\alpha-\varepsilon'}$, on sait qu'il existe Q tel que $\|T_\infty^Q(1)\|_\infty \leq \rho^Q$; si maintenant ρ' est choisi de façon à ce que $\rho < \rho' < 4^{N-\alpha-\varepsilon'}$, on aura qu'il existe j_1 tel que si $j \geq j_1$ on a

$$\|T_{j+1} \circ T_{j+2} \circ \cdots \circ T_{j+Q}(1)\|_\infty \leq \rho'^Q$$

et cela entraîne, pour tous j et ℓ,

$$\|M_{j,0,\ell}\|_\infty \leq C\,\rho'^\ell$$

et donc $\tilde{I}_{j,0} \leq C \sum_0^\infty (\rho'^{4^{\alpha+\varepsilon'-N}})^\ell < +\infty$.

On traite de même $M_{j,k,\ell}$: on a

$$\frac{\partial^k}{\partial \xi^k}\,\hat{\Omega}_j = \frac{1}{2^k} \sum_{\ell=0}^k C_k^\ell \,\frac{\partial^\ell}{\partial \xi^\ell}\,\hat{\Omega}_{j+1}\Big(\frac{\xi}{2}\Big)\,\frac{\partial^{k-\ell}}{\partial \xi^{k-\ell}}\,\hat{\mu}_{j+1}\Big(\frac{\xi}{2}\Big)\,,$$

d'où

$$\begin{aligned}
&|M_{j,k,\ell}(\xi)| \\
&\quad \leq \frac{1}{2^{2K-1}}\,T_{j+1}(M_{j+1,k,\ell-1})(\xi) \\
&\qquad + 2\,k \sum_{q=1}^k \frac{1}{4^k}\,(C_k^q)^2 \Big(\Big|m_{j+1}^{(q)}\Big(\frac{\xi}{2}\Big)\Big|^2 M_{j+1,k-q,\ell-1}\Big(\frac{\xi}{2}\Big) \\
&\qquad\qquad + \Big|m_{j+1}^{(q)}\Big(\frac{\xi}{2}+\pi\Big)\Big|^2 M_{j+1,k-q,\ell-1}\Big(\frac{\xi}{2}+\pi\Big)\Big)\,.
\end{aligned}$$

Par récurrence, on suppose montré que $|M_{N,q,L}(\xi)| \leq C\,\rho^L$ pour $q < k$ où C, ρ ne dépendent ni de N ni de L et $\max\{1, 4^{N-\alpha-\varepsilon}\} < \rho < 4^{N-\alpha}$. On a alors

$$|M_{j,k,\ell}(\xi)| \leq \frac{1}{2^{2k-1}}\,T_{j+1}(M_{j+1,k,\ell-1})(\xi) + C\,\rho^{\ell-1}\,,$$

ce qui donne

$$\begin{aligned}
M_{j,k,\ell}(\xi) \leq \Big(\frac{1}{2^{2k-1}}\Big)^\ell T_{j+1} \circ T_{j+2} \circ \cdots \circ T_{j+\ell}(M_{j+\ell,k,0}) \\
+ C \sum_{q=0}^{\ell-1} T_{j+1} \circ \cdots \circ T_{j+q}(1)\,\frac{\rho^{\ell-q-1}}{(2^{2k-1})^q}\,.
\end{aligned}$$

Or nous savons montrer que pour tous j et ℓ on a

$$\|T_{j+1} \circ T_{j+2} \circ \cdots \circ T_{j+\ell}(1)\|_\infty \leq C\,\rho^\ell$$

et par ailleurs $\sup_j \|M_{j,k,0}\|_\infty < +\infty$, de sorte que

$$\|M_{j,k,\ell}\|_\infty \leq C\Big(\frac{1}{2^{2k-1}}\Big)^\ell + \rho^{\ell-1}\sum_{q=0}^{\ell-1}\frac{1}{(2^{2k-1})^q} \leq C'\,\rho^\ell\,.$$

Nous avons donc prouvé

$$\sup_j \|x^k \Phi_j\|_{H^{\alpha+\varepsilon}} < +\infty\,.$$

Les points d) et e) sont relativement évidents. Démontrons par exemple d). Si Φ_j et Φ_{j+1} sont orthonormales (au sens que les familles $\{\Phi_j(x-k)\}_{k\in\mathbb{Z}}$ et $\{\Phi_{j+1}(x-k)\}_{k\in\mathbb{Z}}$ le sont), alors m_{j+1} vérifie

$$\begin{aligned}
1 &= \sum_{k\in\mathbb{Z}} |\hat{\Phi}_j(2\,\xi+2k\pi)|^2 \\
&= |m_{j+1}(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\Phi}_{j+1}(\xi+2k\pi)|^2 \\
&\quad + |m_{j+1}(\xi+\pi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\Phi}_{j+1}(\xi+\pi+2k\pi)|^2 \\
&= |m_{j+1}(\xi)|^2 + |m_{j+1}(\xi+\pi)|^2\,.
\end{aligned}$$

Inversement, supposons que $|m_j(\xi)|^2 + |m_j(\xi+\pi)|^2 = 1$ pour tout ξ; désignons par K un compact d'Albert Cohen associé à m_∞; pour tout p, la fonction $\theta_{N,p}$, définie par

$$\hat{\theta}_{N,p}(\xi) = \prod_{j=1}^{p} m_{N+j}\Big(\frac{\xi}{2^j}\Big)\chi_K\Big(\frac{\xi}{2^p}\Big)\,,$$

vérifie

$$\langle \theta_{N,p}(x), \theta_{N,p}(x-k)\rangle = \delta_{k,0}$$

par ailleurs, on a

$$|\hat{\theta}_{N,p}(\xi)| \leq |\hat{\Phi}_N(\xi)|\,\frac{1}{\inf\limits_{\eta\in K}|\hat{\Phi}_{N+p}(\eta)|}\,,$$

comme $\hat{\Phi}_{N+p}$ converge uniformément vers $\hat{\varphi}_\infty$ sur K et que $\inf_K |\hat{\varphi}_\infty| > 0$, on peut appliquer la convergence dominée: $\theta_{N,p} \to \Phi_N$ dans L^2 quand $p\to+\infty$, de sorte qu'on a bien

$$\langle \Phi_N(x), \Phi_N(x-k)\rangle = \delta_{k,0}\,.$$

Enfin, le point f) est évident, d'après la remarque qui suit le Théorème 2.

Les théorèmes 1 et 2 donnent deux résultats d'approximation d'une fonction d'échelle par d'autres fonctions d'échelle (stationnaires pour le Théorème 1, non-stationnaires pour le Théorème 2). La section suivante décrit de telles approximations.

8. Polynômes de Bernstein, fonctions d'échelle interpolantes et ondelettes de Kharkov.

Le point ii) de la Proposition 3 (qui caractérise les filtres des fonctions d'échelle interpolantes (à transformée de Fourier positive)) et le point e) du Théorème 2 (qui caractérise les filtres associés à une suite quasi-stationnaire de fonctions d'échelle interpolantes non-stationnaires) ont ramené l'étude de ces filtres à celle des fonctions m_0 vérifiant:

i) $m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$, $m_0(\xi) = \overline{m_0(-\xi)}$,

ii) $m_0(0) = 1$ et $m_0(\xi) \geq 0$ pour tout ξ,

iii) $m_0(\xi) + m_0(\xi + \pi) = 1$ pour tout ξ,

iv) m_0 vérifie le critère d'Albert Cohen.

Il est alors facile de vérifier que les conditions i) à iii) équivalent à $m_0(\xi) = F((1 + \cos\xi)/2)$ où

j) $F \in C^\infty([0, 1])$,

jj) $F(1) = 1$ et $F(t) \geq 0$ pour $t \in [0, 1]$,

jjj) $F(t) + F(1 - t) = 1$ pour tout $t \in [0, 1]$.

Le cas où m_0 est un polynôme trigonométrique est particulièrement simple: la condition jjj) s'écrit pour F, si $\deg m_0 \leq N$,

$$(39) \quad F(t) = \sum_{k=0}^{N} \varepsilon_{N,k}\, C_N^k\, t^k (1-t)^{N-k}\,, \qquad \text{avec } \varepsilon_{N,k} + \varepsilon_{N,N-k} = 1\,.$$

C'est-à-dire que F se représente particulièrement facilement dans la base des polynômes de Bernstein de degré N [1], [18], [29].

NOTATION. Nous noterons P_N la classe des polynômes F de degré $\leq N$ vérifiant $F(t)+F(1-t)=1$ (ou encore qui admettent une décomposition (39) dans la base des polynômes de Bernstein avec $\varepsilon_{N,k} = \varepsilon_{N,N-k}$). Nous noterons P_N^+ la classe des éléments de P_N pour lesquels on a de plus $\varepsilon_{N,0} = 0$ et pour $1 \leq k \leq N$, $0 \leq \varepsilon_{N,k} \leq 1$.

Remarquons que si $m_0(\xi) = F((1+\cos\xi)/2)$ avec $F \in P_N^+$ alors m_0 vérifie automatiquement les propriétés i) à iv). (Le critère d'Albert Cohen est automatiquement vérifié puisque dans ce cas m_0 ne s'annule qu'en π). La famille P_N^+ fournit donc une classe de filtres d'échelle associés à des fonctions d'échelle interpolantes à support compact (contenu dans $[-N, N]$).

De plus, si m_0 vérifie les propriétés i) à iv), on peut écrire grâce au lemme de Riesz $m_0(\xi) = |m_1(\xi)|^2$ où m_1 est un polynôme trigonométrique (à coefficients réels) associé à une fonction d'échelle orthogonale à support compact φ_1 (où "orthogonale" signifie que la famille $\{\varphi_1(x-k)\}_{k\in\mathbb{Z}}$ est orthonormale). Ainsi, P_N^+ est associée à une classe de filtres d'échelle orthogonaux.

Parmi les filtres d'échelle orthogonaux, les plus connus sont les filtres de Daubechies $m_1(\xi) = \mu_N(\xi)$, définis [8] par $\deg \mu_N \leq 2\,N+1$, $|\mu_N(\xi)|^2$ vérifie i) à iv) et μ_N se factorise par $((1+e^{-i\xi})/2)^{N+1}$. On a alors

$$|\mu_N(\xi)|^2 = F_N\Big(\frac{1+\cos\xi}{2}\Big)$$

où $F_N \in P_{2N+1}^+$: plus précisément on a

$$F_N(t) = \sum_{k=N+1}^{2N+1} C_{2N+1}^k\, t^k (1-t)^{2N+1-k}\,. \tag{40}$$

C'est-à-dire que

$$\varepsilon_{2N+1,k} = \begin{cases} 1\,, & \text{si } k > \dfrac{1}{2}\,(2\,N+1)\,, \\ 0\,, & \text{si } k < \dfrac{1}{2}\,(2\,N+1)\,. \end{cases}$$

F_N peut se caractériser de plusieurs façons. Nous venons de l'introduire comme le seul élément de P_{2N+1} qui a un zéro d'ordre au moins $N+1$ en 0. C'est aussi le seul polynôme de degré $2\,N+1$ qui vérifie $F(t) = O(t^{N+1})$ et $F(1+t) = 1 + O(t^{N+1})$; cette description correspond aux filtres maximalement plats de Herrmann [13].

Une autre caractérisation est liée à la régularité des fonctions d'échelle associées:

Proposition 8. *Pour F vérifiant les propriétés* j), jj), jjj) *et le critère d'Albert Cohen pour*

$$m_0(\xi) = F\Big(\frac{1+\cos\xi}{2}\Big)\,,$$

on note $\beta(F)$ le nombre $\beta(F) = \sup\{\alpha > 0 : |\xi|^\alpha\hat{\varphi} \in L^\infty\}$ où $\hat{\varphi}$ est donnée par

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big)\,.$$

Alors:

i) *$F \in P^+_{2N+1}$ vérifie $\beta(F) = \max_{G\in P^+_{2N+1}} \beta(G)$ si et seulement si $F = F_N$.*

ii) *Pour tout $\mu \in]1/4, 1/2[$ il existe $\lambda > 0$ et $N_0 > N$ tel que pour $N > N_0$ si $F \in P^+_N$ et vérifie $\varepsilon_{N,k} = 0$ pour $0 \le k \le \mu N$, on a $\beta(F) \ge \lambda N$.*

Preuve. i) est presque évident. On remplace $\beta(F)$ par

$$\beta_0(F) = \sup\Big\{\alpha > 0\,:\, \Big\{2^{k\alpha}\hat{\varphi}\Big(2^k\,\frac{2\pi}{3}\Big)\Big\}_{k\ge 0} \in \ell^\infty(\mathbb{N})\Big\}\,.$$

Il est clair que $\beta(F) \le \beta_0(F)$; par ailleurs

$$\Big|\hat{\varphi}\Big(2^k\,\frac{2\pi}{3}\Big)\Big| = \Big|\hat{\varphi}\Big(\frac{2\pi}{3}\Big)\Big|F\Big(\frac{1}{4}\Big)^k$$

de sorte que $\beta_0(F) = -\mathrm{Log}\,F(1/4)/\log 2$; enfin, d'après un résultat de Cohen et Conze ([5]), $\beta(F_N) = \beta_0(F_N)$.

i) se ramène donc à vérifier: pour $F \in P^+_{2N+1}$, $F \ne F_N$, on a $F(1/4) > F_N(1/4)$.

Mais ceci est évident,

$$\begin{aligned}F\Big(\frac14\Big)-F_N\Big(\frac14\Big)&=\sum_{k=0}^{N}\varepsilon_{2N+1,k}\,C_{2N+1}^k\\&\qquad\cdot\Big(\Big(\frac14\Big)^k\Big(\frac34\Big)^{2N+1-k}-\Big(\frac14\Big)^{2N+1-k}\Big(\frac34\Big)^k\Big)\\&=\sum_{k=0}^{N}\varepsilon_{2N+1,k}\,C_{2N+1}^k\Big(\frac14\Big)^k\Big(\frac34\Big)^{2N+1-k}\\&\qquad\cdot\Big(1-\Big(\frac13\Big)^{2N+1-2k}\Big).\end{aligned}$$

On a donc $F(1/4)-F_N(1/4)>0$ si l'un des $\varepsilon_{2N+1,k}$ $(0\le k\le N)$ est ≥ 0. Le point ii) est assez simple. On factorise dans $F(t)$ un facteur $t^{[\nu N]}$ où $\nu<\mu$

$$m_0(\xi)=\Big(\frac{1+\cos\xi}{2}\Big)^{[\nu N]}m_1(\xi)$$

et on obtient alors

$$\hat\varphi(\xi)=\Big(\frac{2\,(1-\cos\xi)}{\xi^2}\Big)^{[\nu N]}\prod_{j=1}^{\infty}m_1\Big(\frac{\xi}{2^j}\Big).$$

Le produit infini se contrôle aisément, comme nous l'avons fait déjà plusieurs fois, pour $|\xi|\ge\pi$,

$$\Big|\prod_{j=1}^{\infty}m_1\Big(\frac{\xi}{2^j}\Big)\Big|\le\sup_{|\eta|\le 2\pi}\Big|\prod_{j=1}^{\infty}m_1\Big(\frac{\eta}{2^j}\Big)\Big|\Big(\frac{|\xi|}{\pi}\Big)^{\log\|m_1\|_\infty/\log 2}$$

et donc on a

$$\beta(F)\ge 2\,[\nu N]-\frac{\log\|m_1\|_\infty}{\log 2}\,.$$

Il faut donc estimer $\|m_1\|_\infty=\sup_{0\le t\le 1}(F(t)/t^{[\nu N]})$. On écrit

$$\begin{aligned}\frac{F(t)}{t^{[\nu N]}}&=\sum_{k=[\mu N]+1}^{N}\varepsilon_{N,k}\,C_N^k\,t^{k-[\nu N]}\,(1-t)^{N-k}\\&\le\sum_{k=[\mu N]+1}^{N}C_N^k\,t^{k-[\nu N]}\,(1-t)^{N-k}\end{aligned}$$

$$\begin{aligned}
&\le \sum_{k=[\mu N]+1}^{N} \frac{C_N^{[\nu N]}}{C_k^{[\nu N]}}\, C_{N-[\nu N]}^{k-[\nu N]}\, t^{k-[\nu N]}\,(1-t)^{N-k} \\
&\le \sup_{1+[\mu N]\le k\le N} \frac{C_N^{[\nu N]}}{C_k^{[\nu N]}} \\
&= \frac{C_N^{[\nu N]}}{C_{1+[\mu N]}^{[\nu N]}}\,.
\end{aligned}$$

On utilise alors la formule de Stirling pour écrire pour $k \ge 1$

$$\frac{1}{C_0}\Big(\frac{k}{e}\Big)^k\sqrt{k} \le k! \le C_0\Big(\frac{k}{e}\Big)^k\sqrt{k}\,,$$

d'où

$$\begin{aligned}
\|m_1\|_\infty &\le C_1\, \frac{N^N}{(N-[\nu N])^{N-[\nu N]}}\, \frac{(1+[\mu N]-[\nu N])^{1+[\mu N]-[\nu N]}}{(1+[\mu N])^{1+[\mu N]}} \\
&\le C_2\, \frac{1}{\Big(1-\dfrac{[\nu N]}{N}\Big)^{N-[\nu N]}}\, \frac{\Big(\dfrac{[\mu N]-[\nu N]}{N}\Big)^{[\mu N][\nu N]}}{\Big(\dfrac{[\mu N]}{N}\Big)^{[\mu N]}} \\
&\le C_3\Big(\frac{(\mu-\nu)^{\mu-\nu}}{(1-\nu)^{1-\nu}\mu^\mu}\Big)^N
\end{aligned}$$

et donc

$$\beta(F) \ge \Big(2\,\nu - \frac{1}{\log 2}\,\log\frac{(\mu-\nu)^{\mu-\nu}}{(1-\nu)^{1-\nu}\mu^\mu}\Big)N + O(1)\,,$$

de sorte que ii) est prouvé si on peut choisir ν tel que

$$4^\nu > \frac{(\mu-\nu)^{\mu-\nu}}{(1-\nu)^{1-\nu}\mu^\mu}\,.$$

Cela est possible puisque quand $\nu \to 0^+$ on a

$$4^\nu = 1 + \nu\,\log 4 + O(\nu^2)\,,$$

$$\begin{aligned}
\frac{(\mu-\nu)^{\mu-\nu}}{(1-\nu)^{1-\nu}\mu^\mu} &= \frac{\Big(1-\dfrac{\nu}{\mu}\Big)^{\mu-\nu}}{(1-\nu)^{1-\nu}\mu^\nu} \\
&= \frac{(1-\nu+O(\nu^2))}{(1-\nu+O(\nu^2))\,(1+\nu\,\log\mu+O(\nu^2))} \\
&= 1-\nu\,\log\mu + O(\nu^2)\,,
\end{aligned}$$

Comme $4\,\mu > 1$, on a $\log 4 > -\log \mu$ et ii) est démontré.

REMARQUE. La démonstration du point ii) est basée sur une idée classique ([8]). Cependant si on factorisait tous les zéros ($\nu = \mu$) comme dans le cas classique, on ne pourrait pas conclure: il faudrait $4^\mu > 1/((1-\mu)^{1-\mu}\mu^\mu)$. Mais on a $\mu\,(1-\mu) < 1/4$ d'où $(4\,\mu\,(1-\mu))^\mu < 1$ tandis que $1/(1-\mu)^{1-2\mu} > 1$. Même dans le cas $\mu = 1/2$, qui correspond aux filtres de Daubechies, la factorisation totale ne permet pas de conclure $\liminf B(F_N)/N > 0$ mais seulement $\liminf \beta(F_N)/\log N > 0$. Pour obtenir $\liminf \beta(F_N)/N > 0$, il faut alors une étude plus fine de $\prod_{j=1}^{\infty} m_1(\xi/2^j)$.

Bien évidemment, les polynômes de Bernstein peuvent nous permettre d'approximer des fonctions. Nous obtenons alors le théorème suivant:

Théorème 3 (Approximation des fonctions d'échelle interpolantes). *Soit $m_\infty \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ une fonction qui vérifie*

i) $m_\infty(\xi) = m_\infty(-\xi) \geq 0$ *pour tout* ξ,

ii) $m_\infty(0) = 1$,

iii) $m_\infty(\xi) + m_\infty(\xi + \pi) = 1$ *pour tout* ξ,

iv) m_∞ *satisfait le critère d'Albert Cohen,*

et soit φ_∞ la fonction d'échelle interpolante régulière associée à m_∞.

Pour $N \geq 1$, on définit m_N, à l'aide de la fonction $F_\infty \in C^\infty([0,1])$ définie par

$$m_\infty(\xi) = F_\infty\Big(\frac{1+\cos\xi}{2}\Big),$$

par la formule suivante

$$m_N(\xi) = \sum_{k=0}^{N} F_\infty\Big(\frac{k}{N}\Big)\Big(\frac{1+\cos\xi}{2}\Big)^k\Big(\frac{1-\cos\xi}{2}\Big)^{N-k}. \tag{41}$$

Alors:

a) *Pour tout N, $m_N \in P_N^+$ et donc m_N définit une fonction d'échelle interpolante à support compact φ_N.*

b) $m_N \to m_\infty$ *dans* $C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ *quand* $N \to +\infty$.

c) *φ_N converge rapidement vers φ_∞ dans L^∞.*

d) *Si $\alpha < 1$ et si $\varphi_\infty \in B_\infty^{\alpha+\varepsilon,\infty}$ pour un $\varepsilon > 0$, alors il existe N_0 tel que pour $N > N_0$, $\varphi_N \in B_\infty^{\alpha,\infty}$ et φ_N converge rapidement vers φ_∞ dans $B_\infty^{\alpha,\infty}$.*

e) *Si m_∞ est identiquement nulle sur un voisinage de π, la conclusion* d) *est valable pour tout $\alpha > 0$.*

f) *De même les fonctions d'échelle non stationnaires Φ_N, définies par*

$$\hat{\Phi}_N(\xi) = \prod_{j=1}^{\infty} m_{N+j}\Big(\frac{\xi}{2^j}\Big),$$

sont à support compact, interpolantes et convergent rapidement vers φ_∞ dans L^∞, et dans $B_\infty^{\alpha,\infty}$ si $\alpha < 1$ et $\varphi_\infty \in B_\infty^{\alpha+\varepsilon,\infty}$ pour un $\varepsilon > 0$, et dans $B_\infty^{\alpha,\infty}$ si $\alpha > 0$, $\varphi_\infty \in B_\infty^{\alpha+\varepsilon,\infty}$ pour un $\varepsilon > 0$ et m_∞ est identiquement nulle sur un voisinage de π.

Ce Théorème 3 n'est bien sûr qu'un théorème fantôme, il s'agit d'une simple application des théorèmes 1 et 2 et de la théorie des approximations par les polynômes de Bernstein. Rappelons pour mémoire que si $F \in C^\infty([0,1])$ les polynômes de Bernstein

$$\sum_{k=0}^{N} F\Big(\frac{k}{N}\Big)\Big(\frac{1+x}{2}\Big)^k\Big(\frac{1-x}{2}\Big)^{N-k}$$

convergent vers F dans $C^\infty([0,1])$.

Cette construction d'interpolante non-stationnaire nous permet alors d'introduire l'*interpolante de Kharkov*, en hommage à la ville d'où proviennent et les polynômes de Bernstein [3] et la fonction de Rvachev. L'interpolante de Kharkov sera construite à partir d'une interpolante de Lemarié-Meyer [19], désignée également dans la littérature sous le nom d'interpolante de Littlewood-Paley-Meyer: on prend une fonction $F_\infty \in C^\infty([0,1])$ telle que:

i) $F_\infty(t) \geq 0$,

ii) $F_\infty(t) + F_\infty(1-t) = 1$,

iii) $F_\infty(t) = 0$ pour $|t| \leq 1/4$.

Un exemple de telle fonction peut être définie par

$$F_\infty(t) = \begin{cases} 1\,, & \text{si } t \geq \dfrac{3}{4}\,, \\ up\Big(2\,t - \dfrac{1}{2}\Big)\,, & \text{si } \dfrac{1}{4} \leq t \leq \dfrac{3}{4}\,, \\ 0\,, & \text{si } t \leq \dfrac{1}{4}\,. \end{cases}$$

ou encore $(d/dt)F_\infty(t) = 4\,up(4\,t-1)$. La fonction d'échelle φ_∞ associée à

$$m_\infty = F_\infty\Big(\frac{1+\cos\xi}{2}\Big)$$

vérifie alors $\operatorname{supp}\hat{\varphi}_\infty \subset [-4\pi/3, 4\pi/3]$ de sorte que φ_∞ appartient à la classe de Schwartz $\mathcal{S}(\mathbb{R})$. L'*interpolante de Kharkov* associée à F_∞ est la fonction Φ_0 définie par

$$\hat{\Phi}_0(\xi) = \prod_{N=1}^{\infty} m_N\Big(\frac{\xi}{2^N}\Big)\,,$$

où m_N est définie par (41).

Théorème 4 (Interpolante de Kharkov). *Soit F_∞ une interpolante de Lemarié-Meyer (de fonction d'échelle φ_∞), Φ_0 son interpolante de Kharkov et plus généralement $\{\Phi_N\}_{N\geq 0}$ les fonctions d'échelle interpolantes non-stationnaires associées*

$$\Big(\hat{\Phi}_N(\xi) = \prod_{j=1}^{\infty} m_{N+j}\Big(\frac{\xi}{2^j}\Big)\Big)\,.$$

On désigne par I_N l'opérateur d'interpolation

$$I_N(f) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k}{2^N}\Big)\Phi_N(2^N x - k) = \sum_{k\in\mathbb{Z}} 2^N\langle f, \delta(2^N x - k)\rangle\Phi_N(2^N x - k)$$

et par Ψ_N "l'ondelette de Kharkov" $\Psi_N = \Phi_{N+1}(2\,x-1)$. Alors

i) $\Phi_N \in C_c^\infty$ *pour tout N et $\Phi_N \to \varphi_\infty$ dans $\mathcal{S}(\mathbb{R})$ quand $N \to +\infty$.*

ii) I_N *et $I_{N+1} - I_N$ sont des projecteurs,* iii)

$$(I_{N+1}-I_N)f$$
$$=\sum_{k\in\mathbb{Z}}\Big(f\Big(\frac{2k+1}{2^{N+1}}\Big)-\sum_{\ell\in\mathbb{Z}}f\Big(\frac{\ell}{2^N}\Big)\Phi_N\Big(\frac{1}{2}+k-\ell\Big)\Big)\Psi_N(2^Nx-k)$$
$$=\sum_{k\in\mathbb{Z}}2^N\langle f,\theta_N(2^Nx-k)\rangle\Psi_N(2^Nx-k)\,,$$

où

$$\theta_N=\delta\Big(x-\frac{1}{2}\Big)-\sum_{\ell\in\mathbb{Z}}\Phi_N\Big(\frac{1}{2}+\ell\Big)\delta(x+\ell)\,.$$

iv) *Si $p,q\in[1,+\infty]$ et $s>1/p$ alors pour $f\in\mathcal{D}'(\mathbb{R})$ les trois assertions suivantes sont équivalentes:*

K1) $f\in B_q^{s,p}$,

K2) $I_0f\in L^p$ *et*

$$\|2^{Ns}\|(I_{N+1}-I_N)f\|_{L^p(\mathbb{R})}\|_{\ell^q(\mathbb{N})}<+\infty\,,$$

K3)
$$\sum_{k\in\mathbb{Z}}|f(k)|^p<+\infty\,,$$

et

$$\Big\|2^{N(s-1/p)}\Big\|f\Big(\frac{2k+1}{2^{N+1}}\Big)-\sum_{\ell}f\Big(\frac{\ell}{2^N}\Big)\Phi_N\Big(\frac{1}{2}+k-\ell\Big)\Big\|_{\ell^p(k)}\Big\|_{\ell^q(N)}<+\infty\,.$$

PREUVE. i) Le point i) est une conséquence directe du Théorème 2.

ii): le fait que I_N est un projecteur de $C(\mathbb{R})$ (fonctions continues sur $\mathbb{R}$) sur

$$\Big\{\sum_{k\in\mathbb{Z}}a_k\Phi_N(2^Nx-k)\ :\ \{a_k\}_{k\in\mathbb{Z}}\in\mathbb{C}^{\mathbb{Z}}\Big\}=V_N$$

est évident. Le fait que $I_{N+1}-I_N$ est un projecteur revient à ce que $I_{N+1}\circ I_N=I_N\circ I_{N+1}=I_N$; mais $I_{N+1}\circ I_N=I_N$ car $V_N\subset V_{N+1}$ (par construction de Φ_N) tandis que $I_N\circ I_{N+1}=I_N$ est évident puisque $I_{N+1}f$ et f coïncident sur $\mathbb{Z}/2^{N+1}$, donc sur $\mathbb{Z}/2^N$.

iii) est évident

$$(I_{N+1}-I_N)f=\sum_{k\in\mathbb{Z}}\Big(f\Big(\frac{k}{2^{N+1}}\Big)-I_Nf\Big(\frac{k}{2^{N+1}}\Big)\Big)\Phi_{N+1}(2^{N+1}x-k)\,.$$

Nous arrivons au point iv). Remarquons d'abord que l'équivalence entre K2) et K3) est immédiate: il suffit de vérifier qu'il existe une constante C_0 telle que pour tout N on ait

$$\frac{1}{C_0}\Big(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Big)^{1/p}\le\Big\|\sum_{k\in\mathbb{Z}}\lambda_k\Phi_N(x-k)\Big\|_p\le C_0\Big(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Big)^{1/p}\,. \tag{42}$$

Pour cela on note Ω_N la fonction duale de Φ_N

$$\hat{\Omega}_N(\xi)=\frac{\hat{\Phi}_N(\xi)}{\sum_{k\in\mathbb{Z}}|\hat{\Phi}_N(\xi+2k\pi)|^2}\,.$$

Il est immédiat que $\Omega_N\in\mathcal{S}$ et que $\Omega_N\to\omega_\infty$ dans $\mathcal{S}$ quand $N\to+\infty$, où

$$\hat{\omega}_\infty=\frac{\hat{\varphi}_\infty}{\sum_{k\in\mathbb{Z}}|\hat{\varphi}_\infty(\xi+2k\pi)|^2}\,.$$

On a alors

$$\Big\|\sum_{k\in\mathbb{Z}}\lambda_k\Phi_N(x-k)\Big\|_p\le\Big(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Big)^{1/p}\Big\|\sum_{k\in\mathbb{Z}}|\Phi_N(x-k)|\Big\|_\infty\,,$$

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Big)^{1/p}&=\sup_{\|\mu_k\|_{\ell^{p'}}\le 1}\Big|\Big\langle\sum_k\lambda_k\Phi_N(x-k),\sum_k\mu_k\Omega_N(x-k)\Big\rangle\Big|\\&\le\Big\|\sum_k\lambda_k\Phi_N(x-k)\Big\|_p\Big\|\sum_{k\in\mathbb{Z}}|\Omega_N(x-k)|\Big\|_\infty\,.\end{aligned}$$

(42) est alors immédiat.

K3) implique K1) est assez facile et plusieurs arguments peuvent s'appliquer. Par exemple, on prend une fonction d'échelle orthogonale phi de Lemarié-Meyer et psi son ondelette associée. On sait [19] que l'on a

$$\begin{aligned}&\Big\|\sum_{k\in\mathbb{Z}}\lambda_k\,\mathrm{phi}\,(x-k)+\sum_{N=0}^{\infty}\sum_{k\in\mathbb{Z}}\lambda_{N,k}\,\mathrm{psi}\,(2^Nx-k)\Big\|_{B^{s,p}_q}\\&\qquad\le C_{s,p,q}\Big(\|\lambda_k\|_p+\|2^{N(s-1/p)}\|\lambda_{N,k}\|_{\ell^p(k)}\|_{\ell^q(N)}\Big)\,,\end{aligned}$$

pour tous $s \in \mathbb{R}$ et $p,q \in [1,+\infty]$; par ailleurs l'opérateur T défini sur $\mathcal{S}$ par

$$\begin{cases} T(\mathrm{phi}\,(x-k)) = \Phi_0(x-k)\,, & k \in \mathbb{Z}\,, \\ T(\mathrm{psi}\,(2^N x-k)) = \Psi_N(2^N x-k)\,, & N \geq 0,\ k \in \mathbb{Z}\,, \end{cases}$$

est un opérateur pseudo-différentiel exotique appartenant à la classe $S_1^{0,1}$: son symbole est donné par

$$\begin{aligned}\sigma(x,\xi) &= \widehat{\mathrm{phi}}\,(\xi)\sum_{k\in\mathbb{Z}}\hat{\Phi}_0(\xi+2k\pi)\,e^{ikx} \\ &\quad + \sum_{N=0}^{\infty}\widehat{\mathrm{psi}}\Big(\frac{\xi}{2^N}\Big)\sum_{k\in\mathbb{Z}}\hat{\Psi}_N\Big(\frac{\xi}{2^N}+2k\pi\Big)e^{i2^N kx}\,.\end{aligned}$$

Or la classe $S_1^{0,1}$ opère continûment sur $B_q^{s,p}$ pour tout $s > 0$ [28] et donc K3) implique K1) est prouvé.

Il reste à prouver K1) implique K3). Par dualité, cela revient à vérifier que (en notant $p' = p/(p-1)$ et $q' = q/(q-1)$)

$$\begin{aligned}&\Big\|\sum_{\lambda_k\in\mathbb{Z}}\lambda_k\,\delta(x-k)+\sum_{N=0}^{\infty}\sum_{k\in\mathbb{Z}}\lambda_{N,k}\,\theta_N(2^N x-k)\Big\|_{B_{q'}^{-s,p'}} \\ &\qquad \leq C_{s,p,q}\Big(\|\lambda_k\|_{p'}+\|2^{N(-s-1/p')}\|\lambda_{N,k}\|_{\ell^{p'}(k)}\|_{\ell^{q'}(N)}\Big)\,.\end{aligned} \tag{43}$$

Pour cela, on va vérifier que si $\sigma < -1/p$ alors il existe $C_{\sigma,p,q} > 0$ tel que pour tout $N > -\sigma$ et toute suite $\{\lambda_{N,k}\}$ on ait

$$\Big\|\sum_{k\in\mathbb{Z}}\lambda_{N,k}\,\theta_N(2^N x-k)\Big\|_{B_{q'}^{\sigma,p'}} \leq C_{\sigma,p,q}\|\lambda_{N,k}\|_{\ell^{p'}}\,2^{N(-1/p'+\sigma)}\,. \tag{44}$$

Supposons (44) établi et choisissons $\alpha > 0$ tel que $-s+\alpha < -1/p$ et fixons $N_0 > s+\alpha$. On a alors

- pour $N < N_0$

$$\Big\|\sum_{k\in\mathbb{Z}}\lambda_{N,k}\,\theta_N(2^N x-k)\Big\|_{B_{q'}^{s,p'}} \leq C_N\|\lambda_{N,k}\|_{\ell^{p'}(k)}\,,$$

il suffit d'écrire $\|F\|_{B_{q'}^{-s,p'}} \leq \|F\|_{B_{p'}^{-s+\alpha,p'}}$ et de remarquer que la norme de $B_p^{s-\alpha,p}$ est localisable [22] de sorte que

$$\Big(\sum_{k\in\mathbb{Z}}|f(k)|^p\Big)^{1/p} \leq C_{s-\alpha,p}\|f\|_{B_p^{s,p}}\,.$$

• pour $N \geq N_0$: on a pour $\eta = \pm\alpha$

$$\Big\| \sum_{k\in\mathbb{Z}} \lambda_{N,k}\, \theta_N(2^N x - k) \Big\|_{B_{q'}^{-s+2,p'}} \leq C \|\lambda_{N,k}\|_{\ell^{p'}} 2^{N(-1/p'-s+\eta)}$$
$$= C\, 2^{N\eta} \varepsilon_N\,, \qquad \text{où } \varepsilon_N \in \ell^q\,.$$

Si

$$\omega = \sum_{N\geq N_0} \sum_{k\in\mathbb{Z}} \lambda_{N,k}\, \theta_N(2^N x - k) = \sum_{N\geq N_0} \omega_N\ ,$$

la décomposition de Littlewood-Paley $\omega = S_0\,\omega + \sum_{j\geq 0} \Delta_j\,\omega$ permet facilement de montrer que $\omega \in B_{q'}^{-s,p'}$

$$\|S_0\,\omega\|_{p'} \leq \sum_{N\geq N_0} \|S_0\,\omega_N\|_{p'} \leq C \sum_{N\geq N_0} 2^{-N\alpha} \varepsilon_N < +\infty\,,$$

tandis que

$$\begin{aligned} \|\Delta_j\,\omega\|_{p'} &\leq \sum_{N\geq N_0} \|\Delta_j\,\omega_N\|_{p'} \\ &\leq \sum_{N\geq N_0, N\leq j} \|\omega_N\|_{B_{q'}^{-s+\alpha,p'}} 2^{j(s-\alpha)} \\ &\qquad + \sum_{N\geq N_0, N\geq j} \|\omega_N\|_{B_{q'}^{-s-\alpha,p'}} 2^{j(s+\alpha)} \end{aligned}$$

de sorte que

$$2^{-js} \|\Delta_j\,\omega\|_{p'} \leq C \sum_N 2^{-\alpha|j-N|}\, \varepsilon_N$$

et donc $\{2^{-js}\|\Delta_j\,\omega\|_{p'}\} \in \ell^q(j)$ si $\{\varepsilon_N\} \in \ell^q(N)$.

Il reste à vérifier (44). C'est presque immédiat. En effet, si Ω est à décroissance rapide dans $B_{q'}^{\sigma+\varepsilon,p'}$, on a

$$\Big\| \sum_k \lambda_k\, \Omega(x-k) \Big\|_{B_{q'}^{\sigma,p'}} \leq \Big(\sum_k |\lambda_k|^{p'} \Big)^{1/p'} C(\Omega)$$

(ce quelque soit $\sigma \in \mathbb{R}$). En effet, c'est évident si $p' = q'$ par localisation; si $p' \neq q'$ on écrit $B_{q'}^{\sigma,p'} \subset B_{p'}^{\sigma+\varepsilon/2,p'} \subset B_{q'}^{\sigma+\varepsilon,p'}$. Si $\sigma > 0$, on en conclut que pour $A \geq 1$

$$\Big\| \sum_k \lambda_k\, \Omega(Ax-k) \Big\|_{B_{q'}^{\sigma,p'}} \leq C(\Omega) \Big(\sum_k |\lambda_k|^{p'} \Big)^{1/p'} A^{-1/p'+\sigma}\,.$$

Si $\sigma < 0$, cela reste vrai pourvu que $\int x^k \Omega \, dx = 0$ pour $0 \leq k \leq [-\sigma] = N_0$: dans ce cas on sait que $\Omega = (d/dx)^{N_0+1}\tilde{\Omega}$ où $\tilde{\Omega}$ est à décroissance rapide dans $B_{q'}^{\sigma+\varepsilon+N_0+1,p'}$. (44) est donc immédiat, car les θ_N vérifient que pour tout $p \in \mathbb{N}$, $\{x^p \theta_N\}$ est une famille bornée dans $B_{q'}^{\sigma+\varepsilon,p'}$ si $\sigma + \varepsilon < -1/p$ et de plus $\langle \theta_N, x^p \rangle = 0$ pour $0 \leq p \leq N$.

Le Théorème 4 est donc démontré.

REMARQUES. i) Si $0 < s \leq 1/p$ et si

$$\|2^{N(s-1/p)} \|\lambda_{N,k}\|_{\ell^p(k)}\|_{\ell^q(N)} < +\infty ,$$

on a démontré que

$$\sum_{N=0}^{\infty} \sum_{k \in \mathbb{Z}} \lambda_{N,k} \Psi_N(2^N x - k)$$

définissait un élément f de $B_q^{s,p}$. Cependant on ne peut écrire $\lambda_{N,k} = 2^N \langle f, \theta_N(2^N x - k)\rangle$ puisque $\theta_N \notin (B_q^{s,p})^*$. En particulier, on peut avoir une série d'ondelettes convergeant vers 0 dans $B_q^{s,p}$ avec des coefficients $\lambda_{N,k}$ non nuls.

ii) Le problème de l'interpolation des fonctions périodiques décrit à la fin de la section V est évident ici: si $f \in B_\infty^{r,\infty}(\mathbb{R}/2\pi\mathbb{Z})$ on a $\|f - I_N f\|_\infty \leq C/(2^N)^r$, ce qui est mieux que pour le système de Rvachev ou pour le système trigonométrique.

9. Le problème de la phase.

Dans la section précédente, nous avons décrit les fonctions d'échelle interpolantes et leur approximation par des fonctions d'échelle interpolantes (stationnaires ou non-stationnaires) à support compact.

Par intégration par parties [17], nous pouvons de même approximer des fonctions d'échelle en bi-orthogonalité avec des fonctions splines: en effet dire que la fonction φ est une fonction d'échelle régulière en dualité avec le B-spline N de degré k ($\hat{N} = ((1 - e^{-i\xi})/i\,\xi)^{k+1}$) revient à dire que la fonction Φ, définie par $(d/dx)^{k+1}\Phi = \Delta^{k+1}\varphi$ où $\Delta\varphi = \varphi(x+1) - \varphi(x)$, est une fonction d'échelle interpolante.

Il serait de même utile de savoir approximer les fonctions d'échelle orthogonales par des fonctions d'échelle orthogonales à support compact. Nous pourrions alors construire une base de Berkolaïko-Novikov sur le modèle de l'interpolante de Kharkov, et obtenir des fonctions d'échelle orthogonales non-stationnaires à support compact qui convergeraient dans $\mathcal{S}$, de sorte que la base d'ondelettes non stationnaires serait une "base universelle" (au moins dans les échelles $B_q^{s,p}$, $F_q^{s,p}, \dots$) à l'instar des bases de Lemarié-Meyer.

Nous butons alors sur un problème: nous n'avons pas de description directe des filtres d'échelle orthogonaux à réponse impulsionnelle finie. C'est-à-dire que pour construire un tel filtre m_0, on construit d'abord $|m_0(\xi)|^2$ (qui est un filtre d'échelle interpolant) puis on en prend une "racine carrée polynômiale" grâce au lemme de Riesz: on a alors $m_0(\xi) = |m_0(\xi)|\, e^{-i\omega(\xi)}$ où la phase ω dépend du choix des racines qu'on a conservées pour définir m_0. Comment alors contrôler cette phase pour que la convergence de $|m_0|^2$ entraine celle de m_0?

Ce problème reste extrêmement délicat à traiter. Il a motivé (à côté d'autres raisons) l'étude d'un cas particulier: les filtres de Daubechies $m_N(\xi)$. Pour lesquels on a

$$\lim_{N\to+\infty} |m_N(\xi)|^2 = \begin{cases} 1\,, & \text{si } |\xi| < \dfrac{\pi}{2}\,, \\ 0\,, & \text{si } \dfrac{\pi}{2} < |\xi| \le \pi,\ \xi = \pm\dfrac{\pi}{2}\,. \end{cases}$$

Dans ce cas, une étude de la phase peut être poussée assez loin [16], essentiellement parce que $|m_N(\xi)|^2$ est donné par

$$|m_N(\xi)|^2 = P_{2N+1}\Big(\frac{1+\cos\xi}{2}\Big)\,,$$

où P_{2N+1} est le polynôme de Bernstein associé à la fonction analytique par morceaux $\chi_{[1/2,1]}$. Mais nous ne disposons pas encore de résultats applicables à une interpolante de Lemarié-Meyer.

Pour avoir une "base universelle", on peut, au lieu de chercher à tout prix une base orthogonale, scinder $|m_0(\xi)|^2$ en un produit $m_1(\xi)\,\overline{m_2(\xi)}$ en imposant le convergence dans C^∞ de m_1 et de m_2. On obtiendrait alors une base bi-orthogonale d'ondelettes non-stationnaires qui serait une "base universelle" et dont les fonctions d'échelle non-stationnaires tendraient vers des fonctions d'échelle bi-orthogonales stationnaires.

10. Filtres peu réguliers.

Comme nous l'avons expliqué ci-dessus, la possibilité de décrire le comportement asymptotique de la phase des filtres de Daubechies tient à la simplicité de la limite de $|m_N(\xi)|^2$, c'est-à-dire de la fonction qui définit en retour les filtres de Daubechies comme des polynômes de Bernstein. Le point principal est que

$$\frac{d}{dx}\,\chi_{[1/2,+\infty[} = \delta\Big(x - \frac{1}{2}\Big)\,,$$

ce qui donne pour le polynôme

$$\sum_{k>N} C_{2N+1}^k x^k (1-x)^{2N+1-k} = P_{2N+1}(x)$$

la formule

$$\frac{d}{dx}\,P_{2N+1}(x) = (N+1)\,C_{2N+1}^{N+1} x^N (1-x)^N\,,$$

de sorte que P_{2N+1} s'écrit simplement

$$P_{2N+1}(x) = \int_0^x \frac{(2N+1)!}{(N!)^2}\,(t\,(1-t))^N\,dt\,.$$

Par ailleurs, les fonctions d'échelle φ_N héritent de la mauvaise localisation de fonction d'échelle φ_∞ associée à $m_0 = \chi_{[-\pi/2,\pi/2]}$ ($\varphi_\infty = (\sin \pi x)/(\pi x)$) de sorte que

$$\limsup_{N\to+\infty} \inf_{x_0} \int |x - x_0|^2\, |\varphi_N(x)|^2\, dx = +\infty\,.$$

On peut alors chercher à adoucir la limite de m_N tout en conservant des propriétés remarquables pour cette limite, afin d'être en mesure de contrôler la phase. Une idée prometteuse est alors de prendre pour $|m_\infty(\xi)|^2$ un spline par morceaux, c'est-à-dire un exemple élémentaire de fonction analytique par morceaux dont les dérivées se calculent aisément; un tel choix offre par ailleurs l'avantage de permettre des calculs explicites sur les filtres interpolants construits à l'aide de polynômes de Bernstein, donc d'échantillonnages de $|m_\infty(\xi)|^2$. Cependant, la convergence des filtres ne peut plus avoir lieu dans C^∞ et la convergence des fonctions d'échelle ne peut plus être rapide. Il faut alors reprendre toute la théorie ci-dessus développée pour l'adapter aux filtres peu réguliers et aux fonctions d'échelle peu décroissantes... A titre d'exemple, nous signalons que le lecteur intéressé trouvera dans [15] une démonstration du résultat suivant (qui généralise la Proposition 1).

Proposition 9. *Soit* $m_0 \in H^{1/2+\varepsilon}(\mathbb{R}/2\pi\mathbb{Z})$ *où* $\varepsilon > 0$ *telle que* $m_0(0) = 1$, $m_0(\xi) = \overline{m_0(-\xi)}$. *On note* $\hat{\varphi}$ *la fonction*

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big)$$

et T_2 *l'opérateur agissant sur les fonctions* 2π*-périodiques défini par*

$$T_2 f = \Big|m_0\Big(\frac{\xi}{2}\Big)\Big|^2 f\Big(\frac{\xi}{2}\Big) + \Big|m_0\Big(\frac{\xi}{2}+\pi\Big)\Big|^2 f\Big(\frac{\xi}{2}+\pi\Big).$$

Alors m_0 *satisfait le critère d'Albert Cohen et* $\sup_N \|T_2^N(1)\|_\infty < +\infty$ *si et seulement si* $(1+|x|)^{1/2+\varepsilon}\varphi \in L^2(\mathbb{R})$ *et la famille* $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ *est une base de Riesz d'un sous-espace fermé de* $L^2(\mathbb{R})$. *De plus pour tout* $\alpha > 1/2$, $|x|^\alpha\varphi \in L^2(\mathbb{R})$ *si et seulement si* $m_0 \in H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$.

Remerciements. Je tiens à remercier les nombreuses personnes sans qui ce travail n'aurait pu même être entamé: Djalil Kateb (de l'Université de Compiègne) pour nos travaux communs sur le problème de la phase des filtres d'échelle et nos longues discussions sur la fonction de Rvachev, Yves Meyer (de l'Université de Paris-Dauphine) pour avoir attiré mon attention sur la très belle thèse de Loïc Hervé, Igor Novikov (de l'Université de Voronezh) pour l'envoi de ses manuscrits sur la base de Berkolaïko et Novikov, Amos Ron (de l'Université de Wisconsin-Madison) pour l'envoi de nombreux tirés à part sur les analyses multi-résolution non stationnaires, Winifried Sickel (de l'Université de Iena) pour ses précieuses indications sur la théorie de l'approximation et la fonction de Rvachev, ainsi que Josette Dumas qui a tapé le manuscrit du présent texte.

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Recibido: 2 de octubre de 1.995
Revisado: 26 de abril de 1.996

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Revista Matemática Iberoamericana
Vol. 13, N.º 1, 1997

Checkerboards, Lipschitz functions and uniform rectifiability

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1. Introduction.

In his recent lecture at the International Congress [S], Stephen Semmes stated the following conjecture for which we provide a proof.

Theorem 1.1. *Suppose Ω is a bounded open set in $\mathbb{R}^n$ with $n > 2$, and suppose that $B(0,1) \subset \Omega$, $\mathcal{H}^{n-1}(\partial\Omega) = M < \infty$. Then there are $\varepsilon > 0$, $L < \infty$ (depending on n and M) and a Lipschitz graph Γ (with constant L) such that $\mathcal{H}^{n-1}(\Gamma \cap \partial\Omega) \geq \varepsilon$.*

Here $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure and $B(0,1)$ the unit ball in $\mathbb{R}^n$. By iterating our proof we obtain a slightly stronger result which allows us to cover most of the unit sphere S^{n-1}.

Theorem 1.2. *Same hypotheses. Given $\delta > 0$, there exist $\Gamma_1, \dots, \Gamma_N$, $N = N(\delta, M, n)$ so that each Γ_j is a $C(\delta, M, n)$ Lipschitz graph and*

$$\mathcal{H}^{n-1}\Big(\pi\big(\bigcup_{j=1}^{N} \Gamma_j \cup \partial\Omega\big)\Big) \geq \omega_n - \delta\,,$$

where π denotes the radial projection on S^{n-1} and ω_n is the area of S^{n-1}.

We remark that Theorems 1.1 and 1.2 are somewhat related to the results of [J]. David and Semmes have reported to us [DS2] that they also have proofs of the above theorems. The methods they use are, however, quite different from those we present. Whereas David and Semmes work directly on the domain, we prove a theorem that allows us to stitch together 2-dimensional slices (where the result is trivial). This result, which we call a Checkerboard Theorem, is perhaps the most interesting result of this paper.

Let $[0,1]^n$ be the unit cube in $\mathbb{R}^n$, and let $A, B \subset \mathbb{R}^n$ be Lebesgue measurable sets. We say that A is checkerboard connected through B if for any two points $x, y \in A$, there is a path from x to y which is a finite union of line segments, each line segment in one of the (n)-coordinate directions and having both endpoints in B. We define $d_{ch,B}(x,y)$, the checkerboard distance to be the infimum over the lengths of such paths. For example, if $A \subset [0,1]^2$ is any set and $B = [0,1]^2$, then for $x, y = (x_1, x_2), (y_1, y_2) \in A$ we have

$$d_{ch,B}(x,y) = |x_1 - y_1| + |x_2 - y_2| \,.$$

On the other hand if $A = B = [0,1/3]^2 \cup [2/3,1]^2$ then the points $(0,0), (1,1) \in A$ are not checkerboard connected through B.

Theorem 1.3. (The Checkerboard Theorem) *Given any $\delta > 0$ and any measurable set $B \subset [0,1]^n$ with $|B| = \varepsilon$, there exists a subset $A \subset B$ with*

$$|A| \geq (1-\delta)\, \varepsilon^n$$

and with A checkerboard connected through B. Furthermore, there exists a constant $C = C(\delta, \varepsilon, n) < \infty$ such that for all $x, y \in A$,

$$d_{ch,B}(x,y) \leq C\, |x-y| \,.$$

If $|B| = 1-\alpha$ with $\alpha \ll 1$, we can choose $|A| \geq 1 - c\,\alpha$ and $d_{ch,B}(x,y) \leq \sqrt{n}\, |x-y|$, for $x, y \in A$.

We remark that the final assertion of Theorem 1.3 provides an approach to a version of Almgren's Tilt Excess Theorem [A]. In that case one has $\varepsilon \sim 1$, $|B| = 1 - \alpha$, and one obtains a subset A with $|A| \geq 1 - C\alpha$ and a Lipschitz mapping F with Lipschitz constant $\leq C\sqrt{n}$. This will be explained more precisely in Section 7.

An examination of the constants in Theorem 1.3 will allow us to conclude the following

Corollary 1.4. *Suppose $B \subset [0,1]^n$, $|B| = \varepsilon$, and $F : B \to X$ (any metric space) satisfies a Lipschitz condition on any line parallel to the coordinate axes*

$$\rho(F(x_1, \dots, t, \dots, x_n), F(x_1, \dots, s, \dots, x_n)) \leq |t - s| \,,$$

for any two points on E differing only in one of the n coordinates. Then if $\delta \geq 0$ there exists $A \subset B$ with

$$|A| \geq (1 - \delta)\, \varepsilon^n \,,$$

and such that F is Lipschitz on A,

$$\rho(F(x), F(y)) \leq C(\delta)\, (n^{3/2} \varepsilon^{-n} + \sqrt{n}\, n^{2n} \varepsilon^{1-2n})\, |x - y| \,.$$

The outline of this paper is as follows. In Section 2, we recall a proof of Theorem 1.1 in $\mathbb{R}^2$ -this a known result included only for the sake of completeness. Section 3 is devoted to the proof of the Checkerboard Theorem. We then check constants to derive Corollary 1.4. This allows us to use the results of Section 2 to derive Theorem 1.1. In Section 4 we provide a counter-example for the checkerboard constant in Theorem 1.3 (equivalently for the Lipschitz constant of Corollary 1.4), showing it must be at least $(\log(1/\varepsilon))^{1/2} (\log\log(1/\varepsilon))^{-1/2}$.

In Section 5, we give another application of our methods by showing how to use two dimensional slices to obtain part of the "Structure Theorem" of geometric measure theory for sets of codimension 1. In Section 6, we discuss Almgren's Tilt Excess Theorem and the easy case of Theorem 1.3 (*i.e.* the case when $|B| \sim 1$).

2. A trivial result in $\mathbb{R}^2$.

Let D_1 denote the closed unit disk in $\mathbb{R}^2$. By a radial Lipschitz graph we shall mean a set in $\mathbb{R}^2$ given in polar coordinates by the equation $r = f(\theta)$ where f is a 2π-periodic Lipschitz function. We also define the map $\pi : \mathbb{R}^2 \backslash 0 \to S^1$ to be radial projection. In this section, we prove

Proposition 2.1. *Let $\alpha : S^1 \to \mathbb{R}^2$ be a closed curve in $\mathbb{R}^2 \backslash D_1$ with degree 1 about 0. Suppose $\mathcal{H}_1(\alpha(S^1)) < M$, for some $M < \infty$. Then for* *any* $\varepsilon > 0$ *there exists* Γ *a radial Lipschitz graph over* S^1 *with Lipschitz constant* $C(\varepsilon, M)$ *so that*

$$\mathcal{H}_1(\pi(\Gamma \cap \alpha(S^1))) > 2\pi - \varepsilon \, .$$

Proof. The idea of the proof is that first we prune $\alpha(S^1)$ into a graph and then we trim it to make it Lipschitz . The first observation is that $\nu = \pi(\alpha) : S^1 \longrightarrow S^1$ is a well defined continuous map with degree 1. We may lift it by the universal cover to

$$\tilde{\nu} : \mathbb{R} \longrightarrow \mathbb{R} ,$$

with $\tilde{\nu}(0) = 0$ and $\tilde{\nu}(2\pi) = 2\pi$. Furthermore, since the length of α is bounded, we have that $\tilde{\nu}'$ is a signed measure on $[0, 2\pi]$ with total measure 2π. Define $\rho_\varepsilon = \tilde{\nu}' - \varepsilon/(4\pi)$. This is a signed measure on $[0, 2\pi]$ with total measure $2\pi - \varepsilon/(4\pi) > 0$. We shall now modify α into a curve β in such a way that we change only pieces that give rise to sets of measure 0 under ρ_ε and replace them by line segments. Let $L = \sup |\tilde{\nu}'(I)|$ the sup taken over open intervals I in $[0, 2\pi]$ satisfying $\rho_\varepsilon(I) = 0$. We define $\beta_1 : [0, 2\pi] \longrightarrow \mathbb{R}^2$ to be equal to α except on an interval I with $\rho_\varepsilon(I) = 0$ and with $|\tilde{\nu}'(I)| > L/2$. Let x and y be the endpoints of I. Let $\beta_1(I)$ be the line segment between $\alpha(x)$ and $\alpha(y)$ parametrized so that $\pi(\beta_1)$ has constant speed. We define $\rho_{\varepsilon,1}$ as before, noting that $\rho_{\varepsilon,1}(I) = 0$. If the measure $\rho_{\varepsilon,1}$ is nonnegative then β_1 is a graph. Otherwise, $\rho_{\varepsilon,1}$ is a signed measure with total measure strictly less than that of ρ_ε so that we may proceed recursively, removing intervals with measure $\rho_{\varepsilon,j}$ measure 0 having large total measure and replacing their images with line segments. At last, we obtain β_∞ whose image is a graph, since associated to it is $\rho_{\varepsilon,\infty}$ which is nonnegative.

We define $\Gamma_0 = \beta_\infty(S^1)$. The next observation is that $\mu = \pi_* \mathcal{H}_1(\Gamma_0)$ is a well defined positive measure on S^1 and that,

$$\int_{S^1} \mu \le M \, .$$

Let $\mathcal{M}(\mu)$ be the Hardy-Littlewood maximal function of μ. We choose C so that $|\{M(\mu) > C/2\}| < \varepsilon/2$. The set $\{M(\mu) > C/2\}$ is open, hence a union of open intervals and we define the graph Γ by replacing the part of Γ_0 over each of these intervals by a line segment between the images of the endpoints. The result is the desired graph with Lipschitz constant C.

By the results of [J], one can in fact exhaust the image of α by a finite collection of Lipschitz graphs with universal Lipschitz constants and a garbage set with small Hausdorff content.

3. The checkerboard Theorem.

Proof of Theorem 1.3. Let $\mathcal{M}_j$ be the one dimensional Hardy-Littlewood maximal operator in the j-th coordinate direction. For any set A, let χ_A denote its characteristic function and let A^c denote $[-1,2]^n \backslash A$, *i.e.* the complement of A in the triple of the unit cube. Choose a small $\alpha > 0$ and define

$$B_1 = \{x \in B \ : \ \mathcal{M}_1(\chi_{B^c})(x) < 1 - \alpha\,\varepsilon\}\,,$$

and recursively for $j \leq n$,

$$B_j = \{x \in B_{j-1} \ : \ \mathcal{M}_j(\chi_{B^c_{j-1}})(x) < 1 - \alpha\,\varepsilon\}\,.$$

By choosing α sufficiently small, we may ensure that

$$|B_n| \geq \Big(1 - \frac{\delta}{2}\Big)^{1/n} \varepsilon\,.$$

(This follows easily from the Besicovitch covering Lemma, see [G, p. 39]) In fact, we may choose $\alpha \geq C(\delta)/n^2$. We shall now divide up B_n into its checkerboard connected components and choose one that suits our purposes. We shall do the same at each scale until we arrive at a set which satisfies a dyadic version of the theorem. A similar treatment as we have just given B will produce the desired set.

For any point in $x \in B_n$, we define its good set $\mathcal{G}(x)$ so that $y \in \mathcal{G}(x)$ provided $y \in B$ and there exist $z_j \in B_j$ when $1 \leq j \leq n-1$ so that $\pi_1(y) = \pi_1(z_1)$ and so that $\pi_j(z_{j-1}) = \pi_j(z_j)$ whenever $2 \leq j \leq n-1$, and so that $\pi_n(x) = \pi_n(z_{n-1})$. Here the π_j's are the $(n-1)$-dimensional projections into all but the j-th coordinate. In particular for any $y \in \mathcal{G}(x)$ we have that $d_{ch,B}(x,y) \leq 2\,n$. Further, by the definition of B_n, we have that $|\mathcal{G}(x)| \geq \alpha^n \varepsilon^n$. We define the neighborhod of x to be

$$N(x) = \{y \in B_n \ : \ \mathcal{G}(x) \cap \mathcal{G}(y) \neq \varnothing\}\,.$$

We cover B_n by neighborhoods $N(x_1), \dots, N(x_M)$ so that for $i \neq j$ we always have $x_j \notin N(x_i)$. Then we see that $M < 1/(\alpha^{n-1}\varepsilon^{n-1})$ since the $\pi_1(\mathcal{G}(x_i))$'s are disjoint and have total measure 1. Thus, in particular,

$$M \leq \frac{C(\delta)}{n^{2n-1}} .$$

The checkerboard connected components of B_n through B, call them $C_1, \dots, C_N$ with $N < M$ are just unions of disjoint subcollections of $N(x_1), \dots, N(x_M)$. We have that for any $x, y \in C_j$,

$$d_{ch,B}(x,y) < 4\, n\, M .$$

We pause for a brief lemma which we will use to estimate the size of one of the C_j's.

Lemma 3.2. *Let $A \subset \mathbb{R}^n$ be any measurable set of finite Lebesgue measure, then*

$$D_A = \frac{|A|^{n-1}}{\displaystyle\prod_{j=1}^{n} |\pi_j(A)|} \leq 1 .$$

We refer to D_A as the *checkerboard density* of A. The proof is simply to apply the 1-dimensional Hölder's inequality, n times to

$$\int \chi_{\pi_1(A)}(x_2, \dots, x_n) \chi_{\pi_2(A)}(x_1, x_3, \dots, x_n) \dots \chi_{\pi_n(A)}(x_1, \dots, x_{n-1}) .$$

It is of some interest to note that the above argument also gives a proof of the Sobolev imbedding Theorem (*cf.* [GT, p. 156, equation (7.27)]). That this link should exist is natural because both the Sobolev imbedding Theorem and Theorem 1.3 concern giving global properties of functions in terms of their behavior on one dimensional slices.

Now we proceed to estimate the size of a C_j. We observe first that

$$\sum_{j=1}^{N} |C_j| = |B_n| \geq \Big(1 - \frac{\delta}{2}\Big)^{1/n} \varepsilon\, |Q^n| .$$

On the other hand, since the C_j's are checkerboard disjoint, their $(n-1)$-dimensional projections are disjoint and hence we have for each $k \in \{1, \dots, n\}$,

$$\sum_{j=1}^{N} |\pi_k(C_j)| = |\pi_k(B_n)| .$$

Applying Hölder's inequality yields

$$\sum_{j=1}^{N}\prod_{k=1}^{n}|\pi_k(C_j)|^{1/n} \le \prod_{k=1}^{n}\Big(\sum_{j=1}^{N}|\pi_k(C_j)|\Big)^{1/n} \le \prod_{k=1}^{n}|\pi_k(B_n)|^{1/n}\,.$$

Hence, there is at least one j for which

$$\frac{|C_j|}{\prod_{k=1}^{n}|\pi_k(C_j)|^{1/n}} \ge \frac{|B_n|}{\prod_{k=1}^{n}|\pi_k(B)|^{1/n}}\,.$$

Taking the previous equation to the power n, we arrive at the main inequality

$$(\maltese)\qquad |C_j|\,D_{C_j} \ge |B_n|\,D_{B_n}\;.$$

Observe that in particular,

$$|B_n|\,D_{B_n} \ge \Big(\frac{|B_n|}{|Q^n|}\Big)^n|Q^n| \ge \Big(1-\frac{\delta}{2}\Big)\varepsilon^n\,.$$

Hence, since $D_{C_j}\le 1$, one has that $|C_j|\ge(3\,\varepsilon/4)^n|Q^n|$. But what is more, we have a procedure for taking any subset S of B_n in any cube Q' and finding a subset $\tilde S\subset S$ which is checkerboard connected through B with checkerboard diameter bounded by $8\,M\,l(Q')$, so that

$$|\tilde S|\,D_{\tilde S} \ge |S|\,D_S\;.$$

To see this, just dilate Q' into $[0,1]^n$ and follow the above argument. We require another lemma.

Lemma 3.3. *Let $t_1,\dots,t_{n-1},s\in(0,1)$, then we have*

$$\frac{s^n}{t_1t_2\cdots t_{n-1}}+\frac{(1-s)^n}{(1-t_1)(1-t_2)\cdots(1-t_{n-1})}\ge 1\,.$$

PROOF. It is clear that the minimum of

$$f(s)=\frac{s^n}{t_1t_2\cdots t_{n-1}}+\frac{(1-s)^n}{(1-t_1)(1-t_2)\cdots(1-t_{n-1})}\,,$$

lies on the interior of $(0,1)$. Setting $f'(s) = 0$ gives

$$s = \frac{(t_1 t_2 \cdots t_{n-1})^{1/(n-1)}}{(t_1 t_2 \cdots t_{n-1})^{1/(n-1)} + ((1-t_1)(1-t_2)\cdots(1-t_{n-1}))^{1/(n-1)}}\,.$$

substituting back into f, gives that for any s,

$$f(s) \geq \left(\frac{1}{(t_1 t_2 \cdots t_{n-1})^{1/(n-1)} + \Bigl(\prod_{j=1}^{n-1} (1-t_j) \Bigr)^{1/(n-1)}} \right)^{n-1}.$$

Now Jensen's inequality guarantees that $f(s) \geq 1$.

Now let A_0 be the set C_j and define f_0 to be the constant function D_{A_0} on A_0. Then the inequality (✠) may be rewritten as

$$\int_{A_0} f_0 \geq |B_n|\, D_{B_n}\ .$$

Then we obtain A_1 and f_1 as follows: We divide the cube $[0,1]^n$ into its dyadic children $Q_1, \dots, Q_{2^n}$ and the set A_0 into $A_{0,1}, \dots A_{0,2^n}$ with $A_{0,j} = A_0 \cap Q_j$. Then Lemma 3.3 implies that

$$\sum_{j=1}^{2^n} |A_{0,j}|\, D_{A_{0,j}} \geq \int_{A_0} f_0\ . \tag{1}$$

This is because when we chop a set C into C_l and C_r one the left and right sides of a hyperplane $x_j = c$ then $\pi_k(C_l)$ is disjoint from $\pi_k(C_r)$ for $k \neq j$. We chop A_0 once in each coordinate direction to obtain (1).

Now for each nonempty $A_{0,j}$ we find S_j, a checkerboard connected component of $A_{0,j}$ in Q_j. As before, it will have the properties that,

$$|S_j|\, D_{S_j} \geq |A_{0,j}|\, D_{A_{0,j}}\ ,$$

and for any $x, y \in S_j$,

$$d_{ch,B}(x,y) \leq 8\, M\, n\, l(Q_j)\,.$$

This last is true since $A_{0,j} \subset B_n \cap Q_j$ and any connected component of $B_n \cap Q_j$ which intersects $A_{0,j}$ is contained in $A_{0,j}$. Now we define

$\cup S_j = A_1 \subset A_0$ and we let f_1 be the function on A_1 which is constant on each S_j and equals D_{S_j}. We have shown that

$$\int_{A_1} f_1 \geq \int_{A_0} f_0 \,.$$

We proceed recursively producing A_j from A_{j-1} by letting the cubes at generation $j-1$ give birth, and letting f_j be the function which is constant on the intersection of A_j and cubes of the j-th generation and is equal there to the density of that intersection. Thus

$$\int_{A_j} f_j \geq \int_{A_{j-1}} f_{j-1} \,,$$

and we have found a decreasing sequence of sets A_j and a sequence of functions f_j supported on A_j and bounded by 1 so that for each j,

$$\int_{A_j} f_j \geq |B_n| \, D_{B_n} \,.$$

In particular, this implies that

$$|A_j| \geq |B_n| \, D_{B_n} \,,$$

and hence

$$|A_\infty| \geq |B_n| \, D_{B_n} \geq \Big(1 - \frac{\delta}{2}\Big) \varepsilon^n \,, \tag{2}$$

where $A_\infty = \cap A_n$. We have in addition that for any $x, y \in A_\infty$,

$$d_{ch,B}(x, y) \leq 8 \, M \, n \, d_d(x, y) \,, \tag{3}$$

where $d_d(x, y)$ is the dyadic distance between x and y, *i.e.* the side-length of the smallest dyadic cube containing both x and y. The equations (2) and (3) are almost the statement of the theorem but for the appearance of dyadic distance instead of Euclidean distance. We must trim A_∞ a little bit more in order to rectify this difficulty.

Now as we did to B, we define

$$A_{\infty,1} = \{x \in A_\infty \ : \ \mathcal{M}_1(\chi_{A_\infty^c})(x) < 1 - \alpha \, \varepsilon^n\} \,,$$

and recursively for $j \leq n$,

$$A_{\infty,j} = \{x \in A_{\infty,j-1} \;:\; \mathcal{M}_j(\chi_{A^c_{\infty,j-1}})(x) < 1 - \alpha\,\varepsilon^n\}\,.$$

By choosing α sufficiently small, we may arrange that $|A_{\infty,n}| \geq (\varepsilon/2)^n |Q^n|$. Let D be the set of points in $\mathbb{R}^n$ one of whose coordinates is a dyadic rational. The set D has measure 0. We claim that $A_{\infty,n} \backslash D$ is the desired set A.

We shall refer to a cube's face of codimension 1 as walls. Every cube has $2\,n$ walls which are naturally divided into n pairs of opposite walls. Each such pair corresponds canonically to a coordinate direction j, namely the direction for which the coordinate function x_j is constant on both faces in the pair. We say that two dyadic cubes Q_1 and Q_2 with the same sidelength are neighboring provided that the euclidean distance is 0. If this is the case and Q_1, Q_2 are distinct, then Q_1 and Q_2 share a face F of codimension k where $1 \leq k \leq n$ and k is an integer. Let $j_1 > j_2 > \cdots > j_k$ be the coordinate directions whose coordinate functions are constant on F. Then any two points $x \in Q_1$ and $y \in Q_2$ may be joined by a path which is piecewise linear with pieces in the coordinate directions $j_1, j_2, \ldots, j_k$ in that order and with corners in cubes $\tilde{Q}_r$ with $1 \leq r \leq k-1$ where each $\tilde{Q}_r$ neighbors Q_2. The cube $\tilde{Q}_r$ has a face in common with Q_2 of codimension $k-r$ which is associated to the directions $j_{r+1}, \ldots, j_k$.

For any x and y in A either $d_d(x,y) \leq C/(\alpha\,\varepsilon^n) d(x,y)$ or there is a scale l for which there are cubes Q_1 and Q_2 with $x \in Q_1$ and $y \in Q_2$ of sidelength $2^{-l} \leq 10\sqrt{n}/(\alpha\,\varepsilon^n)|x-y|$ which are neighbors with a common face F of codimension j associated to the directions and $j_1 > \cdots > j_k$ of the previous paragraph. These can be chosen so that the distance $d_{x,Q}$ from x to the boundary of Q_1 satisfies $d_{x,Q} \geq 2\,\alpha\,\varepsilon^n 2^{-l}$.

By the definition of $A_{\infty,n}$, we may find a point $x_1 \in A_{\infty,n-j_1} \cap \tilde{Q}_1$ which can be connected to x by a line in the direction j_1. We may proceed recursively choosing $x_2, \ldots, x_r, y_1$ with $x_l \in \tilde{Q}_l \cap A_{\infty,n-j_l}$ and $y_1 \in Q_2 \cap A_\infty$. But then

$$\begin{aligned} d_{ch,b}(x,y) &\leq (n + 8\,M\,n)\,2^{-l} \\ &\leq C(\delta)\,(n + 8\,n^{2(n-1)}\varepsilon^{n-1})\,n^{5/2}\sqrt{n}\;\varepsilon^{-n}|x-y| \\ &\leq C(\delta)\,(n^{3/2}\varepsilon^{-n} + \sqrt{n}\;n^{2n}\varepsilon^{2n-1})\,|x-y|\,. \end{aligned}$$

This proves Theorem 1.3 and Corollary 1.4

4. Proof of Theorems 1.1 and Theorem 1.2.

In this section, we apply the checkerboard Theorem and Proposition 2.1 to obtain a proof of Theorem 1.1. We observe that the proof of Theorem 1.2 is just a recursive iteration of the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Let $e_n = (0, 0, \ldots, 1)$ be the unit vector in the n-th coordinate direction and let π_n be the projection of $\mathbb{R}^n$ into the hyperplane perpendicular to e_n. Let Q be the cube in $\mathbb{R}^{n-1} \times \{0\}$ which is centered at the origin and which has sidelength $1/(2\sqrt{n-1})$. We will find a Lipschitz graph Γ having large intersection with S so that $\pi_n(\Gamma) \subset Q$.

For v any unit vector in $\mathbb{R}^{n-1}$ let $v^\perp$ denote the $(n-2)$-plane of vectors perpendicular to v in $\mathbb{R}^{n-1}$, let P_v denote the 2-plane in $\mathbb{R}^n$ spanned by v and e_n, and for $w \in v^\perp$ denote by $P_{v,w}$ the translate $P_v + w$.

Consider $S \cap P_{v,w}$ for $w \in Q$. For $\lambda > 0$, a real number, let

$$B_{\lambda,v} = \{w \in v^\perp : \ \mathcal{H}^1(S \cap P_{v,w}) > \lambda\} .$$

By the Slice Theorem ([Si, p. 156]) $|B_{\lambda,v}| \le M/\lambda$. Further for each $w \in Q$, we have that $P_{v,w} \cap S$ separates 0 from ∞ in $P_{v,w}$. If we have further that $w \notin B_{\lambda,v}$, then we can apply Proposition 2.1 to a subset of $P_{v,w} \cap S$. This is because any connected rectifiable set with finite length can be parametrized. (see [DS]) We parametrize a component of the boundary of the domain containing ∞ in $P_{v,w} \backslash S$ which separates 0 from ∞. We apply Proposition 2.1 to this subset to obtain a 1-dimension Lipschitz graph $\Gamma_{v,w,C}$ where C is the Lipschitz constant in Proposition 2.1. Recall that C depends only on the length of the curve and the choice of ε, which is at our disposal. Define

$$\Gamma_{v,C,\lambda} = \bigcup_{w \in Q \backslash B_\lambda} \Gamma_{v,w,C} .$$

By choosing C and λ sufficiently large but depending only on n and M, the measure of S, we may arrange that

$$|\pi_n(\Gamma_{v,C,\lambda}) \cap Q| \ge |Q| - \varepsilon ,$$

for whichever $\varepsilon > 0$ we wish.

We need a small Lemma.

Lemma 4.1. *Let X be a measure space of total measure M. Fix $\varepsilon > 0$ and $N \geq 2\,(M/\varepsilon)^2 + 1$. Let $A_1, \dots, A_N$ be subsets of X of measure $\geq \varepsilon$. Then there for ε sufficiently small there exist j, k with $|A_j \cap A_k| \geq c\,\varepsilon^3/M^2$ where $c \geq 0$ is a universal constant.*

Proof. Choose $N_0 \leq N = [(M/\varepsilon)^2 + 1]$ where $[x]$ denotes the greatest integer less than x. Let $\delta = c\,\varepsilon^2/M$ with c to be specified later. Suppose the Lemma is false. Then $|A_j \backslash A_1 \cup \dots \cup A_{j-1}| \geq (\varepsilon - (j-1))\delta$. Now

$$|X| = M \geq \sum_j |A_j \backslash A_1 \cup \dots \cup A_{j-1}| \geq \delta \sum_{j=1}^{N_0} j \geq c\,\delta\, N_0^2 \geq \frac{M^2}{\varepsilon}\,.$$

But for ε sufficiently small, this is a contradiction.

Hence we may find $v_1, \dots, v_{n-1}$ linearly independent vectors in $\mathbb{R}^{n-1}$ with uniformly large angle between any pair so that

$$\big|\pi_{n-1}\big(\bigcap \Gamma_{v_j,C,\lambda}\big) \cap Q\big| \geq \frac{1}{C\,M^{n-1}}\ ,$$

and so that the smallest covering of Q by parallelpipeds $P_1, P_2, \dots, P_K$ with edges in the directions v_j has K depending only on M and n. For some k then, we have that with

$$A_k = \pi_{n-1}\big(\bigcap \Gamma_{v_j,C,\lambda}\big) \cap P_k\ ,$$

then $|A_k| \geq 1/(10\,K\,M^N)$, where $N > 0$ depends only on n. To A_k, we apply the checkerboard theorem on P_k. This proves Theorem 1.1.

Proof of Theorem 1.2. Observe that in the proof of Theorem 1.1 above, we did not strongly use the fact that for some $\varepsilon > 0$ of our choosing, we have that $|\Gamma_{v,C,\lambda}| \geq 1 - \varepsilon$. If we had simply had that $|\Gamma_{v,C,\lambda}| \geq \delta/(10\,n^2)$ for instance, we would have found a universally large intersection with universally bounded Lipschitz constant where these depend only on δ and n. Also, we do not have to use the cube Q but may find a collection of $(n-1)$-cubes $Q_1, \dots, Q_{5n^2}$ so that the sectors over them cover the $(n-1)$-sphere, and define $\Gamma_{v,C,\lambda}$ for v in any of these cubes. Then we choose ε in Proposition 2.1 so that each $\Gamma_{v,C,\lambda}$ has projection onto Q_j with measure at least $|Q_j|(1 - \delta/2)$. Then we use the proof of Theorem 1.1 to extract Γ_1 above the cube Q_1. We now replace the $\Gamma_{v,C,\lambda}$'s by $\Gamma_{v,C,\lambda}\backslash\Gamma_1$ and we may continue recursively until for every cube Q_j there is a $v \in Q_j$ so that $\Gamma_{v,C,\lambda}\backslash\Gamma_1 \cup \Gamma_l$ has measure less than $\delta/(10\,n^2)$. At this point, $|\pi(\Gamma_1 \cup \cdots \cup \Gamma_l)| \geq \omega_n(1-\delta)$.

5. Counterexample.

Given two sets $E' \subset E \subset [0,1]^n$, we define the (checkerboard) connectivity of E' through E,

$$\gamma_E(E') = \sup_{x,y\in E'} \frac{d_{ch,E}(x,y)}{d(x,y)} \,,$$

where $d(x,y)$ denotes the euclidean distance between x and y.

Theorem 5.1. *For every $\varepsilon < 1/2$ there is a set $E \subset [0,1] \times [0,1]$, $|E| \geq \varepsilon/2$, with the following property: for every $c > 0$ there is $c' > 0$, depending only on c and not on E nor E' so that, if $E' \subset E$ and $|E'| \geq c\,\varepsilon^2$, then $\gamma_E(E') > c' \,(\log\,(1/\varepsilon))^{1/2}(\log\log\,(1/\varepsilon))^{-1/2}$.*

We pick $\varepsilon = 1/N^2$, N being a natural number. Divide the unit square into $1/\varepsilon$ disjoint subcubes,

$$Q_{n,m} = [n\,\sqrt{\varepsilon}, (n+1)\,\sqrt{\varepsilon}\,] \times [m\,\sqrt{\varepsilon}, (m+1)\,\sqrt{\varepsilon}\,]\,,$$

for $n, m = 0, 1, 2, \dots, 1/\sqrt{\varepsilon}-1$. To describe the set E we define $E \cap Q_{n,m}$

$$E \cap Q_{n,m} := \bigcup_{k=0}^{1/\varepsilon-1-nm} Q^k_{n,m} \,,$$

where

$$\begin{aligned} Q^k_{n,m} = {}&[n\,\varepsilon^{1/2} + (k+n\,m)\,\varepsilon^{3/2}, n\,\varepsilon^{1/2} + (k+n\,m+1)\,\varepsilon^{3/2}] \\ &\times [m\,\varepsilon^{1/2} + k\,\varepsilon^{3/2}, m\,\varepsilon^{1/2} + (k+1)\,\varepsilon^{3/2}]\,. \end{aligned}$$

It is easy to see that $|E| \geq \varepsilon/2$.

Let us denote $\gamma = \gamma_E(E')$. We can assume that $Q^i_{n,m} \subset E'$ whenever $Q^i_{n,m} \cap E' \neq \varnothing$.

Lemma 5.2.

a) *Assume* $Q^i_{n,m}$, $Q^j_{n+k,m+l} \subset E'$. *Then*

$$j - i = n\, l + O(\gamma^2 (l + k + 1)^2)\,.$$

b) *If* $\gamma \leq 1/(2\sqrt{\varepsilon})$, *then* $\#\{k : Q^k_{n,m} \cap E' \neq \varnothing\} \leq 1$ *for all* n, m.

Proof. The authors would recommend to the reader to sketch a picture of E.

a) We may restrict our attention only to paths which pass the centers of small cubes $Q^k_{n,m}$ thus reducing everything to essentially a problem in graph theory. Every such path is composed of elementary steps - *i.e.* vertical or horizontal lines from a large cube $Q_{n,m}$ to one of its neighbors. An elementary step has length $\sim \sqrt{\varepsilon}$ and connects $Q^k_{n,m}$ to one of $Q^k_{n+1,m}$, $Q^k_{n-1,m}$, $Q^{k+n}_{n,m-1}$, or $Q^{k-n}_{n,m+1}$.

Now, by assumption, $Q^i_{n,m}$ and $Q^j_{n+k,m+l}$ are connected by a path composed of M elementary steps with $M \leq \gamma(l+k+1)$. Then the cubes $Q^i_{n,m}$ and $Q^j_{n+k,m+l}$ are joined through a sequence $Q^{a(t)}_{b(t),c(t)}$ where a, b, c are integer valued functions and t runs from 1 to M. We always have $n - O(\gamma(l + k)) \leq b(t) \leq n + O(\gamma(l + k))$. Each upwards step increases $a(t)$ by $b(t)$ while each downwards one decreases it by $b(t)$. There must be l more upwards steps than downwards ones and at most M vertical steps. Thus $j - i = a(M) - a(0) = n\, l + O(\gamma^2(l + k + 1)^2)$.

b) Assume false. Then there are $Q^i_{n,m}$ and $Q^j_{n,m}$ $i \neq j$ joined by a path of consisting of less than or equal to $|i - j|\, \gamma\, \varepsilon$ elementary steps. This path must contain the same number of upward steps as downward steps. Thus by the argument above, one must have

$$|i - j| \leq |i - j|^2 \gamma^2 \varepsilon^2\,,$$

but this is a contradiction since $|i - j| \leq 1/\varepsilon$.

The following lemma tells us that given any cube Q the set E' has to skip a considerable part of Q. The iterated application of this lemma will give us the bound on the measure of E'.

Lemma 5.3. *Given any cube* Q *of sidelength* $D\sqrt{\varepsilon}$, *for some* $D \geq 9\,\gamma^2$, *there is a subcube* $Q' \subset Q$ *with sidelength* $D\sqrt{\varepsilon}/(9\,\gamma^2)$ *so that* $Q' \cap E' = \varnothing$.

PROOF OF THE LEMMA. Assume false. We divide Q into $M^2 = (9\,\gamma^2)^2$ squares of sidelength $D\sqrt{\varepsilon}/M$. Our assumption means that there is a point of E' in each of them.

Without loss of generality, we assume that $Q = [0, D\sqrt{\varepsilon}]^2$. (We can do this by simply renumbering the cubes.) Denote $\tilde{Q}_{u,v} = [u\,D\sqrt{\varepsilon}/M, (u+1)\,D\sqrt{\varepsilon}/M] \times [v\,D\sqrt{\varepsilon}/M, (v+1)\,D\sqrt{\varepsilon}/M]$ and $x_{u,v} \in \tilde{Q}_{u,v} \cap E'$. If $x_{u,v} \in Q^l_{n,m}$ denote also $l(x_{u,v}) = l$, $n(x_{u,v}) = n$, $m(x_{u,v}) = m$.

Then by the first part of Lemma 5.2

$$|l(x_{0,v}) - l(x_{0,v-1})| \le \frac{D^2}{M^2} + O\Big(\gamma^2 \frac{D^2}{M^2}\Big) = O\Big(\gamma^2 \frac{D^2}{M^2}\Big)$$

and

$$l(x_{u-1,0}) - l(x_{u,0}) = b_u u\,\frac{D}{M} + O\Big(\gamma^2 \frac{D^2}{M^2}\Big)\,,$$

where the sequence $b_u = m(x_{u,0}) - m(x_{u-1,0})$ satisfies $|\sum\limits_r^s b_u| \le D/M$ for all $r < s$. (The inequality is obvious, since the sum telescopes and for any u, one has $m(x_{u,0}) \le D/M$).

Also,

$$l(x_{u-1,M-1}) - l(x_{u,M-1}) = c_u u\,\frac{D}{M} + O\Big(\gamma^2 \frac{D^2}{M^2}\Big)\,,$$

where $|\sum\limits_r^s c_u| \le D/M$ for all $r < s$.

From all these inequalities, we can get an estimate of

$$|l(x_{M-1,0}) - l(x_{M-1,M-1})|$$

using the following

Fact 5.4. *Let $\{d_u\}$ be a sequence so that $|\sum\limits_r^s d_u| \le D/M$ for all $r < s$. Then*

$$\Big|\sum_{u=1}^{M} d_u u\Big| \le D\,.$$

PROOF OF FACT 5.4. One has simply

$$\Big|\sum_{u=1}^{M} u\,d_u\Big| = \Big|\sum_{l=1}^{M}\Big(\sum_{u=l}^{M} d_u\Big)\Big| \le \sum_{l=1}^{M}\frac{D}{M} = D\,.$$

Now we obtain,

$$|l(x_{M-1,0}) - l(x_{M-1,M-1})| \leq O\Big(\gamma^2\, \frac{D^2}{M}\Big)\,.$$

On the other hand, again by Lemma 5.2

$$l(x_{M-1,v-1}) - l(x_{M-1,v}) = [m(x_{M-1,v-1}) - m(x_{M-1,v})]D + O\Big(\gamma^2\, \frac{D^2}{M}\Big)\,.$$

Hence,

$$l(x_{M-1,0}) - l(x_{M-1,M-1}) = D^2 + O\Big(\gamma^2\, \frac{D^2}{M}\Big)\,.$$

Therefore $D^2 \leq O(\gamma^2 D^2/M)$, which is false if we take $M = c\,\gamma^2$, for a sufficiently big c.

REMARK 5.5. The argument of Lemma 5.3 actually proves that if $D \geq 18\,\gamma^2$, then for every $k \in \{9\,\gamma^2, 9\,\gamma^2 + 1, \dots, 18\,\gamma^2\}$ there is a cube $\tilde{Q}_{u,v}$ with $\max(u,v) = k$ having empty intersection with E'. Thus the measure of the union of those cubes is $|Q|/(36\,\gamma^2)$.

Now, we are ready to end the computations. Starting with the unit cube $Q_0 = [0,1] \times [0,1]$, we find a set A_1 which is a union of cubes of sidelength $1/(18\,\gamma^2)$ so that $E' \subset A_1$ and $|A_1| \leq 1 - 1/(18\,\gamma^2)$. We take a grid of cubes of sidelength $1/(18\,\gamma^2)$. Applying the remark again to each of them, we find $A_2 \subset A_1$, so that, $E' \subset A_2$, A_2 is a union of cubes of sidelength $1/(18\,\gamma^2)^2$ and $|A_2| \leq (1 - 1/(18\,\gamma^2))^2$. By induction, for any m so that $1/(18\,\gamma^2)^m \geq \sqrt{\varepsilon}$, we find a subset $E' \subset A_m$ union of cubes of sidelength $1/(18\,\gamma^2)^m$ with measure $|A_m| \leq (1 - 1/(18\,\gamma^2))^m$.

Moreover, the second part of Lemma 5.2 implies that $|E'| \leq \varepsilon^2 |A_m|$ for all such m. Hence,

$$|E'| \leq \varepsilon^2 \Big(1 - \frac{1}{18\,\gamma^2}\Big)^{\log(1/\varepsilon)/\log\gamma} < \varepsilon^2 e^{-d(\log(1/\varepsilon)/\log\gamma)1/\gamma^2} \leq c\,\varepsilon^2\,,$$

when $\gamma^2 \log\gamma \leq c' \log(1/\varepsilon)$.

6. The Structure Theorem.

The checkerboard theorem also provides a new approach to part of the "structure theorem" for n-dimensional ($n \in \mathbb{N}$) sets. To state that theorem in its "projection version" we need a definition. Here $G(m,n)$, $m \geq n$, denotes the Grassmannian manifold of all n-dimensional linear subspaces of $\mathbb{R}^m$, with its usual invariant measure.

DEFINITION. For every plane $P \in G(m,n)$ denote by π_P the orthogonal projection onto P. Given a set $E \subset \mathbb{R}^m$ we define the n-integral geometric measure of E as

$$\mathcal{U}_n(E) = \int_{G(m,n)} \mathcal{H}^n(\pi_P(E))\, dP\,.$$

Theorem 6.1 (The Structure Theorem). *Let $E \subset \mathbb{R}^m$, $\mathcal{H}^n(E) < \infty$. E has a decomposition into an n-rectifiable set A and an n-unrectifiable set $E \backslash A$ and $\mathcal{U}_n(E \backslash A) = 0$.*

The following theorem is the special case of the structure theorem which we will show.

Theorem 6.2. *Let $E \subset \mathbb{R}^{n+1}$, $0 < \mathcal{H}^n(E) < \infty$, so that $\mathcal{U}_n(E) > 0$. Then there is $P \in G(m,n)$ and a Lipschitz graph $\mathcal{L}_P$ onto P (i.e., there is a Lipschitz function $f_P : P \to \mathbb{R}^{m-n}$ whose graph is $\mathcal{L}_P$) such that $\mathcal{H}^n(E \cap \mathcal{L}_P) > 0$.*

Along with the density theorems, Theorem 6.1 is one of the central theorems in geometric measure theory. It was proven by Besicovitch [B] when $m = 2$, $n = 1$ and generalized by Federer [F]. The proof of Besicovitch's result can be found in any manual on that subject (see [Fa]). The checkerboard Theorem allows us to deduce the theorem for higher dimensions, when $m = n + 1$, from that case.

We will make use repeatedly of the following well known (and easy) fact

REMARK 6.3. Let E be a n-rectifiable set in $\mathbb{R}^{n+k}$. Then for almost every $P \in G(n+k,n)$ there is a Lipschitz graph $\mathcal{L}_P$ onto P so that $\mathcal{H}^n(E \cap \mathcal{L}_P) > 0$.

Proof of Theorem 6.2. The proof will be by induction on n. The case $n = 1$ is assumed to be known.

We will identify $G(l+1,l)$ and S^l in the standard way: $P \in G(l+1,l)$ is identified with its orthogonal vector $v \in S^l$. We will write $P = P_v$ and $\pi_{P_v} = \pi_v$.

Assume that E is a compact subset of $\mathbb{R}^{n+1}$, $\mathcal{H}^n(E) < \infty$ and $\mathcal{U}_n(E) > 0$. Given $\omega \in S^n$ we define the family of planes orthogonal to ω, $P_{\omega,t} = P_\omega + t\,\omega$. Denote $E_{\omega,t} = E \cap P_{\omega,t}$. Given any vector γ orthogonal to ω, (or, in other words $\omega \in P_\gamma$) we have

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\pi_\gamma(E_{\omega,t}))\, dt = \mathcal{H}^n(\pi_\gamma(E))\,.$$

Consider the equator of the unit sphere

$$S^{n-1} = \{x = (x_1, x_2, \dots, x_{n+1}) \in S^n \ : \ x_{n+1} = 0\}\,.$$

Given $\theta \in S^{n-1}$ we define the meridian through θ as

$$\mathcal{M}_\theta = \{x \in S^n \ : \ \pi(x) = t\,\theta,\, t > 0\}\,,$$

where π denotes the orthogonal projection onto the plane $\{x_n = 0\}$.

The assumption on the integral geometric measure of E implies that, for any $\theta \in S^{n-1}$,

$$\begin{aligned}\int_{\mathcal{M}_\theta} \int_{\mathbb{R}} & \mathcal{U}_{n-1}(E_{\omega,t})\, dt\, d\omega \\ &= \int_{\mathcal{M}_\theta} \int_{\mathbb{R}} \int_{\gamma \in \omega^\perp} \mathcal{H}^{n-1}(\pi_\gamma(E_{\omega,t}))\, d\mathcal{H}^{n-1}(\gamma)\, dt\, d\omega > 0\,.\end{aligned}$$

We now apply the $(n-1)$-dimensional theorem to $E_{\omega,t}$ whenever its integral geometric measure is positive. We obtain a set $C_\theta \subset \mathcal{M}_\theta$, $\mathcal{H}^1(C_\theta) > 0$, such that, for all $\omega \in C_\theta$ there is $B(\omega) \subset \{P \subset G(n+1, n),\, \omega \in P\} \approx \{v \in S^n : v \text{ is orthogonal to } \omega\}$, with $\mathcal{H}^{n-1}(B(\omega)) > 0$, and for every $P \in B(\omega)$ there is a graph $\mathcal{L}_{P,\omega}$ over P, Lipschitz in the direction of ω, $\mathcal{H}^n(\pi_P(E \cap \mathcal{L}_{P,\omega})) > 0$. In fact, for all $\omega \in C_\theta$, $\mathcal{H}^{n-1}(G(n+1,n) \cap \{\omega \in P\} \setminus B(\omega)) = 0$.

Let us denote

$$\tilde{B}(\theta) = \bigcup \{B(\omega) \ : \ \omega \in \mathcal{M}_\theta\}\,.$$

Then $\mathcal{H}^n(\tilde{B}(\theta)) > 0$. Therefore, using Fubini's theorem,

$$\begin{aligned}\int_{G(n+1,n)} \int_{S^{n-1}} & \chi_{\tilde{B}(\theta)}(P)\, d\mathcal{H}^{n-1}(\theta)\, d\mathcal{H}^n(P) \\ &= \int_{S^{n-1}} \int_{G(n+1,n)} \chi_{\tilde{B}(\theta)}(P)\, d\mathcal{H}^n(P)\, d\mathcal{H}^{n-1}(\theta) > 0\,.\end{aligned}$$

Hence, there is a plane P such that $\mathcal{H}^{n-1}(\{\theta : P \in \tilde{B}(\theta)\}) > 0$. By the definition of $\tilde{B}(\theta)$ this is equivalent to $\mathcal{H}^{n-1}(\{\omega : P \in B(\omega)\}) > 0$. (Notice that $P \cap S^n \sim S^{n-1}$).

Let us denote $D(P) = \{\omega \in P : P \in B(\omega)\}$, and $E_\omega = E \cap \mathcal{L}_{P,\omega}$, $\omega \in D(P)$. Since $\mathcal{H}^{n-1}(D(P)) > 0$, $\mathcal{H}^n(E) < \infty$ and $\mathcal{H}^n(\pi_P(E_\omega)) > 0$, for all $\omega \in D(P)$, then, we can find $\omega_1, \omega_2, \dots, \omega_n \in D(P)$ linearly independent so that, $\mathcal{H}^n(\pi_P(\cap E_{\omega_k})) > 0$. Now, we apply the checkerboard theorem and conclude that there is $\tilde{E} \subset \cap E_{\omega_k}$ of positive n-dimensional measure and contained in a Lipschitz graph $\mathcal{L}_P$ over P.

7. The tilt-excess Theorem.

In this section, we discuss a special case of the checkerboard Theorem which immediately implies a version of the Almgren tilt-excess Theorem. For our purposes, the tilt-excess Theorem (*cf.* [A]) is the following.

Theorem 7.1. *Let Ω be an open set and suppose the unit ball $B(0,1) \subset \Omega$. Suppose further that*

$$\mathcal{H}^{n-1}(\partial\Omega) \leq (1+\varepsilon)\, \mathcal{H}^{n-1}(S^{n-1})\,.$$

Then there exists a $C(n)\, \varepsilon^{1/3}$ Lipschitz graph Γ over S^{n-1} such that

$$\mathcal{H}^{n-1}(\Gamma \triangle \partial\Omega) \leq \varepsilon^{1/3}\,,$$

where $\triangle$ denotes symmetric difference.

Theorem 7.1 will follow from the following simple version of the checkerboard theorem.

Lemma 7.2. Let $A \subset Q^n$ the unit cube in $\mathbb{R}^n$ with $|A| = 1 - \varepsilon$. Then for sufficiently small ε (with small depending on n), there exists $B \subset A$ with $|B| \geq 1 - C^n \varepsilon$ (with C a universal constant) so that B is $\sqrt{n}$ checkerboard connected through A.

PROOF OF LEMMA 7.2. We define as in Section 3

$$A_1 = \Big\{ x \in A \ :\ \mathcal{M}_1(\chi_{B^c})(x) < \frac{1}{4} \Big\}\,,$$

and recursively for $j \leq n$,

$$A_j = \Big\{ x \in B_{j-1} \ : \ \mathcal{M}_j(\chi_{B_{j-1}^c})(x) < \frac{1}{4} \Big\} .$$

Then we have by induction $|A_n| \geq 1 - C^n \varepsilon$. We claim that A_n is the desired set B.

Let $x = (\tilde{x}, x_n)$ and $y = (\tilde{y}, y_n)$. Consider the sets in $[0,1]$,

$$S_1 = \{ t \in [0,1] \ : \ (\tilde{y}, x_n + t(y_n - x_n)) \in A_{n-1} \} ,$$

and

$$S_2 = \{ t \in [0,1] \ : \ (\tilde{x}, x_n + t(y_n - x_n)) \in A_{n-1} \} .$$

Then by the definition of A_n we have that $|S_1|, |S_2| \geq 3/4$. Hence, $|S_1 \cap S_2| \geq 1/2$. Thus there exists $s \in S_1 \cap S_2$. Thus letting $x^{n-1} = (\tilde{x}, s)$ and $y^{n-1} = (\tilde{y}, s)$, we have $x^{n-1}, y^{n-1} \in A_{n-1}$. Analogously we define x^{n-2} and y^{n-2} by replacing x and y by $\tilde{x}$ and $\tilde{y}$. We proceed recursively, always having $x^j, y^j \in A_j$. Then the path from x to y through $x^{n-1}, \dots, x^j, \dots, y^j, \dots, y^{n-1}$ has length at most $\sqrt{n}\, |x-y|$, and the lemma is proven.

PROOF OF THEOREM 7.1. By Lemma 7.2 and the argument used to prove Theorem 1.1, there exists Γ_0 a $\sqrt{n}$ Lipschitz graph over S^{n-1} such that

$$\mathcal{H}^{n-1}(\Gamma_0 \triangle \partial\Omega) \leq C(n)\, \varepsilon .$$

Now let $\mu = \pi^*(\mathcal{H}^{n-1}|_{\Gamma_0})$ where π is radial projection. Thus with $d\sigma$ defined as surface measure on S^n, we have that

$$d\mu = (1+g)\, d\sigma ,$$

with $||g||_{L^1(S^{n-1})} \leq C(n)\, \varepsilon$. (Notice that if $\Gamma_0 = \{ (\theta, n(\theta) F_0(\theta)) : \theta \in S^{n-1} \}$ where $n(\theta)$ is the unit normal vector to S^{n-1} at θ, then we have that g is on the order of $|\nabla F|^2$). Let $\mathcal{M}$ be the maximal operator on the sphere and let

$$B = \{ \mathcal{M} g > C(n)\, \varepsilon^{2/3} \} ,$$

so that $\mathcal{H}^{n-1}(S^{n-1} \cap A) < \varepsilon^{1/3}$. Then we can replace Γ_0 by Γ where, letting

$$\Gamma = \{ (\theta, n(\theta) F(\theta)) \ : \ \theta \in S^{n-1} \} ,$$

we have made F differ from F_0 only on B. The reader may easily verify that one way of doing this is by observing that B is open, and defining

$$F(x) = (\phi_{d(x)} * F_0)\,,$$

for x in B. Here $d(x)$ is the distance from x to the boundary of B. We have fixed some positive bump function ϕ supported in the unit ball, and $\phi_{d(x)}$ is a version of it scaled to have support in the ball of radius $d(x)$. Thus the theorem is proved.

The reader may find it an amusing exercise to verify that the exponent $1/3$ is sharp.

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Recibido: 19 de octubre de 1.995

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* The first author was supported by NSF grant DMS-92-13595.

† The second author was partially supported by a National Science Foundation Postdoctoral Fellowship.

‡ The third author was partially supported by a Spanish DGICYT grant, PB94-149.

Revista Matemática Iberoamericana
Vol. 13, N.º 1, 1997

On the two weights problem for the Hilbert transform

Nets Hawk Katz and Cristina Pereyra

0. Introduction.

In this paper, we prove sufficient conditions on pairs of weights (u, v) (scalar, matrix or operator valued) so that the Hilbert transform

$$Hf(x) = \text{p.v.} \int \frac{f(y)}{x-y} \, dy \,,$$

is bounded from $L^2(u)$ to $L^2(v)$. When $u = v$ are scalar, the classical results were given in [HMW] and [CF]. Earlier, [HS] gave a characterization of these weights by complex methods which has been generalized by [CS1] and [CS2] to the case of unequal weights. However these complex-analytic results give conditions which as stated by [CS2] "are not susceptible of being verified in practice". What follows shall be all in the category of real analysis. Matrix results for equal weights have recently been given in [TV]. For u and v scalar weights, a different sufficient condition from ours was given in [F]. More general conditions than ours for the scalar case have recently been given by [TVZ] using very different methods which do not seem to generalize to the operator valued case.

We shall consider only (u, v) so that $u^{-1}, v \in L^{1+\varepsilon}_{\text{loc}}$ are positive and u^{-1} and v are doubling. There will be an auxiliary Hilbert space $\mathcal{H}$, with scalar product denoted by $\langle \cdot \, , \cdot \rangle_{\mathcal{H}}$. The weights u and v shall be operator valued and we define for $\mathcal{H}$ valued functions f,

$$\|f\|^2_{L^2(u)} = \int \langle u(x) f(x), f(x) \rangle_{\mathcal{H}} \, dx \,.$$

Then we shall prove the following theorem:

Theorem. *If* (u,v) *as above satisfy conditions* a), b) *and* c) *then*

$$H : L^2(u) \longrightarrow L^2(v)\,.$$

For a full description of conditions a), b) and c) in the scalar and operator cases see Section 3. We briefly describe the conditions here.

Condition a) will state that for certain Haar multipliers M_u and M_v, the operators

$$u^{-1/2} M_u^{-1/2} \qquad \text{and} \qquad v^{1/2} M_v^{-1/2}$$

are bounded on $L^2(\mathbb{R},\mathcal{H})$. Operators of this form were first studied in [P1]. They were first used to study boundedness of the Hilbert transform in [TV]. In Section 1, we describe sufficient conditions for their boundedness in the scalar case. The weakness of these conditions, and their relation to the classical A_p conditions on weights make condition a) seem reasonable.

Condition b) is a sort of non-local A_2 condition for $(v^{1+\varepsilon}, u^{-(1+\varepsilon)})$. Condition c) is the boundedness of two weighted paraproducts. (In the operator case, part of condition c) is also a seemingly slightly stronger assumption -an inequality that in the scalar case automatically follows. We point out that in the setting of [TV], this inequality may be replaced by the reverse inequality to A_2, *i.e.* the inequality

$$\Big(\frac{1}{|I|}\int_I w^{-1}\Big)^{1/2}\Big(\frac{1}{|I|}\int_I w\Big)\Big(\frac{1}{|I|}\int_I w^{-1}\Big)^{1/2} \geq C\,,$$

which in the scalar case simply follows from Hölder's inequality. The inequality is also true in the operator valued case. For information on operator inequalities see [HP] and the references cited therein.)

In the matricial case when $u = v$, conditions a), b) and c) are equivalent to the classical Muckenhoupt A_2 condition.

Our theorem should be thought of as a sort of $T(1)$ theorem (see [D]) for two weights. In particular, condition c) should be seen as the analogue of requiring that $T(1)$ and $T^*(1)$ are in BMO. In this way, our proof differs from that of [TV] in the case that the weights are equal. We use only the standard kernel properties of the Hilbert transform H, namely the decay of matrix coefficients H_{IJ} when $3I \cap 3J = \varnothing$ and the general decay of Hh_I. Further, we prove our bounds using not the Senechkin-Vinogradov test (as in [TV]) but rather the two fundamental lemmas of linear algebra:

Lemma 0.1 (Cotlar). *Let T_j be operators on $\mathcal{H}$, a Hilbert space. Suppose that for any j,k, one has*

$$\|T_i\,T_j^*\|_{\mathcal{H}\to\mathcal{H}} \le a(|i-j|)\,,$$

and

$$\|T_j^*\,T_i\|_{\mathcal{H}\to\mathcal{H}} \le a(|i-j|)\,,$$

where $\sum_j a(j)^{1/2} \le C$ then

$$\Big\| \sum_{j=-N}^{j=N} T_j \Big\|_{\mathcal{H}\to\mathcal{H}} \le C\,,$$

with constant independent of N.

For a proof see [D]. Decomposition of an operator T into $\sum_j T_j$ with the T_j's satisfying the hypotheses of the Lemma is called Cotlarization. The other fundamental lemma of linear algebra (in the scalar case), is

Lemma 0.2 (Schur). *Let T be an operator on $L^2(X)$ with X a measure space and let $K(x,y)$, its scalar-valued kernel be positive. Suppose there are positive functions $w_1(x)$ and $w_2(x)$ with*

$$\int w_1(x)\,K(x,y)\,dx \le C_1 w_2(y)\,,$$

and

$$\int w_2(y)\,K(x,y)\,dy \le C_2 w_1(x)\,.$$

Then $\|T\|_{L^2(X)\to L^2(X)} \le (C_1C_2)^{1/2}$.

A proof may be found in [Da]. We state and prove a version in the operator case, (Lemma 3.1), which, while it is not deep, we have been unable to locate in the literature in this form.

Finally, we remark that the most important problem in the field of weighted norm inequalities for the Hilbert transform is to find the necessary and sufficient condition when $u = v$ in the case that $\mathcal{H}$ is not finite dimensional. It is conjectured that the condition is A_2. We do not know whether all A_2 weights satisfy our sufficient conditions, since the generalization of Gehring's theorem [G] is unclear. Also unclear is the correct definition for Carleson condition. We hope our paper inspires future work.

1. Carleson conditions and bounded operators.

We let $\mathcal{D}$ denote the set of dyadic intervals in the real line. We say that a sequence of real numbers $\{b_I\}$ indexed by $\mathcal{D}$ is a Carleson sequence provided that for any $I \in \mathcal{D}$, we have that

$$\sum_{J\in\mathcal{D}, J\subset I} b_J^2 \leq C\,|I|\,.$$

We recall the Carleson Lemma. (For a proof see [M, p. 261]).

Lemma 1.1 (Carleson). *Let λ_I be any sequence of real numbers. Define the function*

$$\lambda^*(x) = \sup_{x\in I} |\lambda_I|\,.$$

Then

$$\Big|\sum_I \lambda_I\, b_I^2\Big| \leq C \int \lambda^*(x)\, dx\,.$$

For any interval I, we define h_I to be the Haar function of I,

$$h_I(x) = \frac{1}{|I|^{1/2}}\,(\chi_{I^l} - \chi_{I^r})\,,$$

where I^l and I^r are the left and right children of I, the function χ_J for any interval J is the characteristic function of J, and $|I|$ denotes the length of I. The h_I's form an orthonormal basis of $L^2(\mathbb{R})$. To any sequence b_I, we associate an operator π_b, its paraproduct by

$$\pi_b f = \sum_{I\in\mathcal{D}} b_I\, h_I\, m_I(f)\,,$$

where $m_I(f) = \int_I f/|I|$ is the mean of f on I. (More commonly, π_b is referred to as the paraproduct with or by the BMO function $b =$

$\sum_{I\in\mathcal{D}} b_I h_I$. However throughout this paper the sequences $\{b_I\}$ occur far more naturally than the function b and we prefer to think of π_b as an operator associated to the sequence rather than as a modified product with the function).

Corollary 1.2. *The operator π_b is bounded on $L^2(\mathbb{R})$ if and only if b is a Carleson sequence.*

PROOF. For one direction, we simply compute that if b is a Carleson sequence,

$$\|\pi_b f\|^2_{L^2(\mathbb{R})} = \sum_I b_I^2\,(m_I f)^2 \le C\int (\mathcal{M}f)^2 \le C\,\|f\|^2_{L^2(\mathbb{R})}\,.$$

The first inequality follows from Lemma 1.1, where $\mathcal{M}f$ denotes the dyadic maximal function of f. The second inequality follows from the $L^2(\mathbb{R})$ boundedness of the dyadic maximal function, see [D]. On the other hand, if π_b is bounded then $\|\pi_b\,\chi_I\|^2_{L^2(\mathbb{R})} \le C\,|I|$. However

$$\|\pi_b\,\chi_I\|^2_{L^2(\mathbb{R})} \ge \sum_{J\subset I} b_J^2\,.$$

Hence b is a Carleson sequence.

Throughout this section, v shall be a weight -that is- a nonnegative L^1_{loc} function, and u_I and b_I shall be sequences indexed by intervals (all intervals in the remainder of this paper shall be dyadic). We shall concern ourselves with two kinds of operators

$$T_{v,u} f = v\sum_I \frac{\langle f, h_I\rangle}{u_I}\,h_I = v\,M_u^{-1} f \tag{1.1}$$

and

$$S_{v,u,b} f = M_u^{-1}\pi_b(vf)\,. \tag{1.2}$$

Here, obviously, M_u denotes the Haar multiplier with coefficients u_I, and $\langle\,\cdot\,,\cdot\,\rangle$ denotes the scalar product in $L^2(\mathbb{R})$. In this section, following [TV], we shall show that the L^2 boundedness of the operators in (1.1) and (1.2) is related. We shall give sufficient conditions, and we shall demonstrate their relationship with the now classical Coifman-Muckenhoupt conditions on weights (see [CF]).

It is clear that a necessary condition for $T_{v,u}$ to be bounded is that $m_I(v^2) \leq Cu_I^2$. Let $b_I = \langle v^2, h_I \rangle / m_I(v^2)$. Then we have:

Proposition 1.3. *If $m_I(v^2) \leq C\, u_I^2$ then $T_{v,u}$ is bounded on $L^2(\mathbb{R})$ if and only if $S_{v,u,b}$ is bounded on $L^2(\mathbb{R})$.*

PROOF. First observe that since $m_I(v^2) \leq C\, u_I^2$, we have that

$$S^*_{v,u,b}\, h_I = \frac{\langle v^2, h_I \rangle\, v\, \chi_I}{u_I\, m_I(v^2)\, |I|}$$

is a bounded set in $L^2(\mathbb{R})$. Hence, $g_I = T_{v,u} h_I - S^*_{v,u,b} h_I$ is a bounded set in $L^2(\mathbb{R})$. If g_I is also an orthogonal set in $L^2(\mathbb{R})$, then $T_{v,u} - S^*_{v,u,b}$ is a bounded operator on $L^2(\mathbb{R})$. Which would prove the proposition.

But in fact g_I is an orthogonal set. To see this observe that for each interval I, the function g_I is supported on the interval I and that restricted to each of the left and right halves of I, it is a constant multiple of v. Thus to show that g_I is an orthogonal set, it suffices to show that $g_I \perp v\, \chi_I$. But this is easy to verify since

$$\int g_I\, v\, \chi_I = \frac{1}{u_I} \Big(\int v^2 h_I - \frac{\langle v^2, h_I \rangle}{|I|\, m_I(v^2)} \int_I v^2 \Big) = 0\,,$$

which proves the proposition.

Next, we give a sufficient condition for $S_{v,u,b}$ to be bounded.

Proposition 1.4. *Suppose there exists $\delta > 0$ so that*

$$\Big(\frac{(m_I(v^{2+\delta}))^{1/(2+\delta)}}{u_I} \Big) b_I$$

is a Carleson sequence. Then the operator $S_{v,u,b}$ is bounded on $L^2(\mathbb{R})$.

PROOF. We have that

$$\|S_{v,u,b} f\|_{L^2}^2 = \sum_I \frac{b_I^2}{u_I^2}\, (m_I(v f))^2\,.$$

However by Hölder's inequality,

$$m_I(vf) \leq (m_I(v^{2+\delta}))^{1/(2+\delta)}(m_I(f^{(2+\delta)/(1+\delta)}))^{(1+\delta)/(2+\delta)}.$$

Now simply applying Carleson's lemma and the boundedness of the dyadic maximal function on $L^{(2+2\delta)/(2+\delta)}$ proves the proposition.

Corollary 1.5. *Suppose $w \in \mathrm{RH}_2$, that is there exists a constant C so that for any dyadic interval I, $m_I(w^2) \leq C(m_I w)^2$. Then $T_{w,m_I(w)}$ is bounded on $L^2(\mathbb{R})$.*

Proof. If $w \in \mathrm{RH}_2$ then $w^2 \in A_\infty$. Hence, $\langle w^2, h_I\rangle / m_I(w^2) = b_I$ is a Carleson sequence [FKP]. By Proposition 1.3, we need only show that $S_{w,m_I w,b}$ is bounded, but this follows immediately from the fact that for some $\delta \geq 0$, we have that $w \in \mathrm{RH}_{2+\delta}$ together with Proposition 1.4.

For other proofs, applications, and L^p versions of Corollary 1.5, see [P1], [P2], [KP].

Corollary 1.6. *Suppose that $w \in A_2$. Then the operators $T_{w^{1/2},(m_I w)^{1/2}}$ and $T_{w^{-1/2},(m_I w)^{-1/2}}$ are bounded on $L^2(\mathbb{R})$.*

Proof. By propositions 1.3 and 1.4, the operator $T_{w^{1/2},(m_I w)^{1/2}}$ is bounded for any $w \in A_\infty$. This follows from $\langle w, h_I\rangle / m_I(w)$ being a Carleson sequence, which occurs when $w \in A_\infty$, as well as the fact that $w \in \mathrm{RH}_{1+\varepsilon}$, for some $\varepsilon > 0$. Now since $w^{-1} \in A_\infty$, we have that $T_{w^{-1/2},(m_I(w^{-1}))^{1/2}}$ is bounded. But since $w \in A_2$, it is the case that $1/m_I(w) \geq m_I(w^{-1})$. This together with the boundedness of Haar multipliers with bounded coefficients proves the corollary.

For more information on the classical theory of Muckenhoupt weights, we refer the reader to [D].

We remark that Corollary 1.6 gives a trivial proof of the boundedness of Haar multipliers with bounded coefficients on $L^2(w)$ for any $w \in A_2$. The corollary says that $w^{1/2}M_w^{-1/2}$ and the adjoint of its inverse $w^{-1/2}M_w^{1/2}$ are bounded where M_w is the Haar multiplier with coefficients $m_I w$. Let L be a Haar multiplier with bounded coefficients. Then it is bounded on $L^2(w)$ if and only if $w^{1/2}Lw^{-1/2}$ is bounded on $L^2(\mathbb{R})$. By the boundedness of the operators from Corollary 1.6 and their adjoints, this is true if and only if $M_w^{1/2}LM_w^{-1/2}$ is bounded on

$L^2(\mathbb{R})$. But everything commutes and $M_w^{1/2} L M_w^{-1/2} = L$. Hence L is bounded on $L^2(w)$.

Similarly, one has a simple proof that π_b where b is a Carleson sequence is bounded on $L^2(w)$ when $w \in A_2$. We simply observe that it suffices to show that $M_w^{1/2} \pi_b w^{-1/2}$ is bounded on $L^2(\mathbb{R})$. Now we apply Proposition 1.4 using the fact that $w \in A_2$ implies $w \in A_{2-\delta}$.

The same ideas can be used to give simple sufficient conditions for L and π_b to satisfy two weight inequalities. For example, L takes $L^2(u) \longrightarrow L^2(v)$ provided there exist sequences c_1 and c_2 so that $T_{u^{-1/2},c_1}$ and $T_{v^{1/2},c_2}$ are bounded and (c_1, c_2) satisfy an A_2 condition, *i.e.* $1/(c_{1,I} c_{2,I}) \leq C$ for every dyadic I. Similarly, π_b is bounded from $L^2(u)$ to $L^2(v)$ provided $T_{v^{1/2},c_2}$ is bounded and there exists $\delta > 0$ so that $(m_I(u^{-1-\delta/2}))^{1/(2+\delta)} \leq C\, c_{2,I}$. The argument which proves this is the same as the proof of Proposition 1.4.

These ideas exactly form the basis for our two weights result for the Hilbert transform. Some pieces of the operator we will study will be treated like multipliers while others are treated like paraproducts. First, however, we discuss the relationship of the boundedness of $T_{v^{1/2},c_2}$ to $v \in A_\infty$.

As mentioned in the proof of Corollary 1.6, for any $w \in A_\infty$, we have that $T_{w^{1/2},(m_I(w))^{1/2}}$ is bounded. This followed from the fact that $w \in \mathrm{RH}^{1+\varepsilon}$, for some $\varepsilon > 0$. In what follows, define for any ε, $w_{I,\varepsilon} = (m_I(w^{1+\varepsilon}))^{1/(1+\varepsilon)}$. Fixing $\varepsilon > 0$, we may ask when $T_{w^{1/2},(w_{I,\varepsilon})^{1/2}}$ is bounded. Propositions 1.3 and 1.4 give as a sufficient condition that there exists a $\delta > 0$ so that

$$b_I = \frac{\langle w, h_I \rangle (w_{I,\delta})^{1/2}}{(m_I(w))(w_{I,\varepsilon})^{1/2}}\,,$$

is a Carleson sequence. By Hölder's inequality, it is certainly sufficient that $c_I = \langle w, h_I \rangle / w_{I,\mu}$ is a Carleson sequence provided that $0 < \mu < \varepsilon/(2+\varepsilon)$. We do not necessarily get the result when $\mu = \varepsilon/(2+\varepsilon)$ since weights not in A_∞ do not necessarily satisfy a reverse Hölder condition. When c_I is a Carleson sequence we say that $w \in A_{\infty+\mu}$. Certainly, if $w \in A_\infty$ then $w \in A_{\infty+\mu}$. A priori, one might believe that any weight in $A_{\infty+\mu}$ is in A_∞ or that all weights are in $A_{\infty+\mu}$. In fact, neither is the case. We thank Peter Jones for the following examples.

First consider $w(x) = |(\log x)^{-2}/x|$ on the interval $[0, 1/2]$. We have that $w(x)$ is not in A_∞ since it is not in $L^{1+\mu}$ for any μ. However, on every interval not containing 0, it satifies a reverse Hölder inequality with uniform estimates. Thus we have $w \in A_{\infty+\mu}$ since on those intervals I containing 0, we have $c_I = 0$ while on the others we simply apply [FKP]. In other words, to sum c_I^2 for I's contained in an interval of the form $[0, 2^{-j}]$, we need only sum it over intervals contained in intervals of the form $[2^{-k-1}, 2^{-k}]$ with $k > j$, apply [FKP] to each of these and sum the geometric series.

Next, we define a weight w_j with the parameter j an integer. The $A_{\infty+\mu}$ constant for any $\mu > 0$ will be unbounded as we vary the parameter j. We define $f_{j,[0,1]}$ to be the function defined on the interval $[0, 1]$ which takes on the value $2^j - \delta$ on $[0, 2^{-j}]$, is constant on the rest of $[0, 1]$ and has mean 1. For any interval I, we let $f_{j,I}$ be the same function rescaled to the interval I. Let $w_{j,0} = f_{j,[0,1]}$ We choose

$$\delta \ll 1/(2^j)^{(2^j)^{2^j}} ,$$

so that we may neglect it for what follows. Now we define $w_{j,1}$ by letting it equal $w_{j,0}$ in the interval $[0, 2^{-j}]$. Now in the interval $[k\, 2^{-j}, (k + 1)(2^{-j})]$ for $1 \leq k \leq 2^{j-1}$ we let $w_{j,1} = w_{j,0} f_{j,[k 2^{-j},(k+1)(2^{-j})]}$. We repeat the procedure 2^j times letting $w_j = w_{j,2^j}$. Now

$$r_j = 2^j \sum_{[0,2^{-j}] \subset J \subset [0,1]} \frac{\langle f_{j,[0,1]}, h_J\rangle^2}{f_J^2} ,$$

can readily be seen to be comparable to the $A_{\infty+\varepsilon}$ constant of w_j when

$$f_J = \Big(\frac{1}{|J|} \int_J (f_{j,[0,1]})^{1+\varepsilon}\Big)^{1/(1+\varepsilon)} .$$

But r_j is readily seen to be approximately $2^{j/(1+\varepsilon)}$.

2. A small section on operators.

The purpose of this section is just to discuss the generalizations of Jensen's inequality and Schur's lemma which we shall be using in the proof of the main theorem.

From this point on, $\mathcal{H}$ will be a Hilbert space. We will think of H, the Hilbert transform as acting on $L^2(\mathbb{R}, \mathcal{H})$, the space of square integrable Hilbert space valued functions. This space is the same as $L^2(\mathbb{R}) \otimes \mathcal{H}$. Naturally, we define the action of H by $H(f \otimes v) = (Hf \otimes v)$.

Our weights u and v shall be positive operator valued functions on $\mathcal{H}$. For any two self-adjoint operators A and B, we say that $A \le B$ when $B - A$ is positive; and for C a constant, $A \le C$ means that $(C\,\mathrm{Id} - A)$ is positive.

First, we state and prove the correct version of Schur's Lemma for operator valued kernels.

Lemma 2.1 (Schur). *Let X be a measure space. And let $K(x,y)$ be a $\mathcal{B}(\mathcal{H})$ valued function on $X \times X$. Suppose that $K(x,y) = A(x,y)\,B(x,y)$ where the multiplication is pointwise composition. Suppose further that*

$$\int A(x,y)\,A^*(x,y)\,dy \le C_1\ ,$$

and that

$$\int B^*(x,y)\,B(x,y)\,dx \le C_2\ .$$

Then $K(x,y)$ gives rise to a bounded operator on $L^2(\mathbb{R},\mathcal{H})$ with bound $C_1^{1/2} C_2^{1/2}$.

PROOF. We need only to bound

$$\int \langle f(x), K(x,y)\,g(y)\rangle_{\mathcal{H}}\,dx\,dy\,.$$

We observe

$$\begin{aligned}
&\int \langle f(x), A(x,y)\,B(x,y)\,g(y)\rangle_{\mathcal{H}}\,dx\,dy \\
&\qquad \le \int \langle A^*(x,y)\,f(x), B(x,y)\,g(y)\rangle_{\mathcal{H}}\,dx\,dy \\
&\qquad \le \Big(\int |A^*(x,y)\,f(x)|^2\,dx\,dy\Big)^{1/2}\Big(\int |B(x,y)\,g(y)|^2\,dx\,dy\Big)^{1/2},
\end{aligned}$$

here $|\cdot|$ denotes the norm in $\mathcal{H}$, *i.e.* $\|\cdot\|_{\mathcal{H}} = (\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})^{1/2}$.

We write the first integral as

$$\int \langle A(x,y)\,A^*(x,y)\,f(x), f(x)\rangle_{\mathcal{H}}\,dx\,dy\,,$$

and bound it by integrating first in y. We do the analogous thing for the second integral.

Further we need to state the operator version of Jensen's inequality.

Lemma 2.2. *Let $A(x)$ be a positive operator valued function on a measure space X. Let $d\mu(x)$ be a measure on X with total measure 1. Let $1 \le p \le \infty$. Then*

$$\Big(\int A(x)^p \, d\mu(x) \Big)^{1/p} \ge \int A(x) \, d\mu(x) \, .$$

For $1 \le p \le 2$, the only case in which we will use this, the result follows from [HP] and from the monotonicity of the function $f(t) = t^r$ when $0 \le r \le 1$, see [KR, Exercise 4.6.46]. All solutions are provided in [KR2]. Of course, we get immediately by scaling a version of Hölder's inequality:

Lemma 2.3. *Let $A(x)$ be a positive operator valued function and let $f(x)$ be a scalar, positive, integrable function. Then*

$$\int f(x) A(x) \, dx \le \Big(\int f(x) \Big)^{1/q} \Big(\int f(x) \, A(x)^p \, dx \Big)^{1/p} ,$$

whenever $1 < p < \infty$ and $1/p + 1/q = 1$.

Proof. Simply apply Lemma 2.2 to the measure

$$d\mu(x) = \frac{f(x) \, dx}{\displaystyle\int f(y) \, dy} \, .$$

Many norm estimates will be based on

Lemma 2.4. *Let T_1 and T_2 be positive operators with $T_1 \le T_2$. Let S be any fixed operator. Then*

$$\|T_1^{1/2} S\| \le \|T_2^{1/2} S\| \, .$$

Here $\| \cdot \|$ denotes the operator norm.

This is [KR, exercise 4.6.1].

3. The two weights problem.

In this section we will give a sufficient condition on pairs of doubling weights (u, v) ensuring that the operator $v^{1/2} H u^{-1/2}$ is bounded where H is the Hilbert transform. Here an operator valued weight v is said to be doubling if there exists a constant C so that for any dyadic interval I, whenever $\tilde{I}$ is its parent, one has

$$\int_{\tilde{I}} v \leq C \int_I v \,,$$

with the inequality in the sense of operators.

As always $\mathcal{D}$ shall denote the set of all dyadic intervals in the real line. The set of dyadic intervals of length 2^{-k} shall be $\mathcal{D}_k$. We shall divide the set of all ordered pairs of dyadic intervals into a union of 5 disjoint sets. Let

$$\begin{aligned} Z_1 &= \{(I,J) \,:\, |I| > |J|,\ 3I \cap 3J = \varnothing\}\,, \\ Z_2 &= \{(I,J) \,:\, |I| < |J|,\ 3I \cap 3J = \varnothing\}\,, \\ Z_3 &= \{(I,J) \,:\, |I| < |J|,\ 3I \cap 3J \neq \varnothing\}\,, \\ Z_4 &= \{(I,J) \,:\, |I| > |J|,\ 3I \cap 3J \neq \varnothing\}\,, \end{aligned}$$

and

$$Z_5 = \{(I,J) \,:\, |I| = |J|\}\,,$$

We let E_I denote the projection onto the Haar function on I, which we denote h_I. In other words, $E_I f = \langle f, h_I \rangle h_I$ for all $f \in L^2(\mathbb{R}, \mathcal{H})$, here $\langle f, h_I \rangle = \int f(x)\, h_I(x)\, dx \in \mathcal{H}$. We shall break up the Hilbert transform into corresponding pieces

$$H_\alpha = \sum_{(I,J) \in Z_\alpha} E_J \, H \, E_I \;.$$

Here α runs from 1 to 5.

We now state our conditions on the pair (u, v) and derive a few easy consequences.

First define

$$u_I = \Big(\frac{1}{|I|} \int u^{-(1+\varepsilon)} \Big)^{1/(1+\varepsilon)}$$

and

$$v_I = \Big(\frac{1}{|I|} \int v^{1+\varepsilon} \Big)^{1/(1+\varepsilon)} .$$

Here the number $\varepsilon > 0$ shall be fixed throughout. (By Lemma 2.2, it is clear that if we define the mean $m_I(u) = \int_I u/|I|$ then one has the operator inequalities $m_I(u^{-1}) \leq u_I$ and $m_I(v) \leq v_I$ which we shall use frequently).

We define the operators acting on $L^2(\mathbb{R}, \mathcal{H})$

$$T_u f = u^{-1/2} M_u^{-1/2} f = \sum_{I \in \mathcal{D}} u^{-1/2} u_I^{-1/2} \langle f, h_I \rangle \, h_I \ ,$$

and

$$T_v f = v^{1/2} M_v^{-1/2} f = \sum_{I \in \mathcal{D}} v^{1/2} v_I^{-1/2} \langle f, h_I \rangle \, h_I \ .$$

Here M_w^t denotes the Haar multiplier acting on $L^2(\mathbb{R}, \mathcal{H})$ with coefficient w_I^t, for w_I a given sequence of positive selfadjoint operators on $\mathcal{H}$.

We say that (u, v) satisfies condition a), provided that T_u and T_v are bounded operators on $L^2(\mathbb{R}, \mathcal{H})$.

We say that (u, v) satisfy condition b) provided that

$$\text{(3.1)} \quad u_I^{1/2} \Big(\Big(|I| \int_{(3I)^c} \frac{v^{1+\varepsilon}}{(x - y_I)^2} \Big)^{1/(1+\varepsilon)} + v_I + v_{I-|I|} + v_{I+|I|} \Big) u_I^{1/2} \leq C \, ,$$

and that

$$\text{(3.2)} \quad v_I^{1/2} \Big(\Big(|I| \int_{(3I)^c} \frac{u^{-(1+\varepsilon)}}{(x - y_I)^2} \Big)^{1/(1+\varepsilon)} + u_I + u_{I-|I|} + u_{I+|I|} \Big) v_I^{1/2} \leq C \, ,$$

where y_I denotes the center of I. We observe that for any A and B positive operators, writing $B^{1/2} A B^{1/2} \leq C$ with C a constant is the same as writing $\|A^{1/2} B^{1/2}\| \leq C^{1/2}$. We also point out that for any positive operator-valued function w, we have that

$$\int_{(3I)^c} \frac{w}{|x - y_I|^n} \leq \frac{1}{|I|^{n-2}} \int_{(3I)^c} \frac{w}{|x - y_I|^2} \ , \qquad \text{for } n > 2 \, .$$

This is not really Hölder's inequality but just the statement that on $(3I)^c$, we have

$$\frac{1}{|x - y_I|^{n-2}} \leq C \, \frac{1}{|I|^{n-2}} \ .$$

Finally, we come to condition c). If c_I is a sequence of bounded operators in $\mathcal{H}$ (not necessarilly selfadjoint this time) indexed by the dyadic intervals, we define the paraproduct

$$\pi_c f = \sum_I h_I \, c_I \, m_I(f) \, .$$

Let c_I^v be the operator on $\mathcal{H}$ given by $(m_I(v))^{-1}\langle v, H_3 h_I\rangle$ and c_I^u the analogous thing for u^{-1}, where we define

$$\langle v, H_3 h_I\rangle = \int v(x) \, (H_3 h_I)(x) \, dx \, .$$

Then we say that (u,v) satisfy condition c) provided that $M_u^{1/2}\pi_{c^v} v^{1/2}$ and $M_v^{1/2}\pi_{c^u} u^{-1/2}$ are bounded and that the following inequalities are satisfied for any dyadic $J \subset I$

$$\begin{aligned} (m_I(v))^{-1/2}\Big(\frac{1}{|I|}\int_{I^c} v\, H_3 h_J\Big) u_J \Big(\frac{1}{|I|}\int_{I^c} v\, H_3 h_J\Big)(m_I(v))^{-1/2} \\ \le C\Big(\frac{|J|^3}{(d_{IJ})^4}\Big) v_I^{-1/2} v_J \, v_I^{-1/2} \, . \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} (m_I(u^{-1}))^{-1/2}\Big(\frac{1}{|I|}\int_{I^c} u^{-1} H_3 h_J\Big) v_J & \\ \cdot \Big(\frac{1}{|I|}\int_{I^c} u^{-1} H_3 h_J\Big)(m_I(u^{-1}))^{-1/2} & \\ \le C\Big(\frac{|J|^3}{(d_{IJ})^4}\Big) u_I^{-1/2} u_J \, u_I^{-1/2} & \, . \end{aligned} \tag{3.4}$$

Here d_{IJ} denotes the distance from J to the boundary of I and the inequalities are in the sense of operators. (By contrast, we will define ρ_{IJ} to be the maximum of $|I|$, $|J|$, and the distance between I and J).

Let us observe a quick consequence of condition c). Let $h \in \mathcal{H}$ be a fixed vector. We apply the operator $M_u^{1/2}\pi_{c^v} v^{1/2}$ to the test function $v^{1/2} h \, \chi_I$. From the I-th summand of this we obtain the size estimate

$$\|u_I^{1/2} c_I^v (m_I(v))^{1/2}\|_{\mathcal{H}\to\mathcal{H}} \le C \, |I|^{1/2} \, . \tag{3.5}$$

There is of course an analogous inequality when the u's and v's are switched. In the case in which u and v are scalars, the inequality (3.5) (together with condition b)) implies the inequality (3.3).

We now give the proof of this implication. In the case where v and u are scalar, the definition of c_I^v together with (3.5) implies that

$$\int v\, H_3 h_I \leq C\, \frac{|I|^{1/2}(m_I(v))^{1/2}}{u_I^{1/2}}\,. \tag{3.6}$$

Now observe that $H_3 h_I$ is constant on intervals whose length is $|I|$. Hence, $|H_3 h_I| \leq C/|I|^{1/2}$ everywhere and is constant on I. Thus from (3.6),

$$\int_{I^c} v\, H_3 h_I \leq C\Big(|I|^{1/2} m_I(v) + \frac{|I|^{1/2}(m_I(v))^{1/2}}{u_I^{1/2}}\Big)\,. \tag{3.7}$$

But from condition b) and the fact that $m_I(v) \leq v_I$, one has that

$$(m_I(v))^{1/2} \leq C\, \frac{1}{u_I^{1/2}}\, (m_I(v) u_I)^{1/2} \leq C\, \frac{1}{u_I^{1/2}}\,.$$

So that (3.7) implies that

$$\int_{I^c} v H_3 h_I \leq C\, \frac{|I|^{1/2}(m_I(v))^{1/2}}{u_I^{1/2}}\,.$$

Now we observe that on I^c, the function $H_3 h_I$ is always positive. This is because $H h_I$ is positive on I^c and $H_3 h_I$ is given by the mean of $H h_I$ on a certain Whitney decomposition of I^c (we will say more about this in the proof of Theorem 3.1). In fact, we have on I^c that

$$H_3 h_I(x) \sim \frac{|I|^{3/2}}{\rho_{xI}^2}\,.$$

Where we define ρ_{xI} to be the maximum of $|I|$ and the distance from I to x. Thus one has

$$\int_{I^c} v H_3 h_J \leq C\, \frac{|J|^{3/2}|I|^{1/2}}{d_{IJ}^2} \int_{I^c} v H_3 h_I\,,$$

which immediately implies (3.3). In the last estimate we used the following facts,

(i)
$$\int \frac{dx}{\rho_{xI}^s} \sim \frac{1}{|I|^{s-1}}\,,$$

for any $1 < s$,

(ii)
$$\int_{I^c} \frac{dx}{\rho_{xJ}^2} \sim \frac{1}{d_{IJ}}\,,$$

for $J \subset I$.

This type of integral/series will appear repeatedly. Variations will be introduced as subtler sets/intervals are defined.

Theorem 3.1. *Suppose* (u,v) *satisfy conditions* a), b) *and* c). *Then the Hilbert transform* H *is bounded from* $L^2(u)$ *to* $L^2(v)$.

PROOF. Our goal is to show that the operator $v^{1/2}Hu^{-1/2}$ is bounded on $L^2(\mathbb{R})\otimes\mathcal{H}$. By condition a) which states that operators $v^{1/2}M_v^{-1/2}$ and $u^{-1/2}M_u^{-1/2}$ are bounded, it would suffice to show that $M_v^{1/2}HM_u^{1/2}$ is bounded. In fact, we will show that $M_v^{1/2}(H_1+H_2+H_5)M_u^{1/2}$, as well as $v^{1/2}H_3M_u^{1/2}$ and $M_v^{1/2}H_4u^{-1/2}$ are bounded. Then we shall write

$$\begin{aligned} v^{1/2}Hu^{-1/2} &= T_vM_v^{1/2}(H_1+H_2+H_5)M_u^{1/2}T_u^* \\ &\quad + (v^{1/2}H_3M_u^{1/2})T_u^* + T_v(M_v^{1/2}H_4u^{-1/2})\,, \end{aligned}$$

thereby proving the theorem.

By the symmetry between u and v, proving that $M_v^{1/2}H_1M_u^{1/2}$ and $v^{1/2}H_3M_u^{1/2}$ are bounded is the same as proving that $M_v^{1/2}H_2M_u^{1/2}$ and $M_v^{1/2}H_4u^{-1/2}$ are bounded. The proof bounding $M_v^{1/2}H_5M_u^{1/2}$ is also exactly the same as the proof that $M_v^{1/2}H_1M_u^{1/2}$ is bounded once one makes the trivial observation that for *any* two intervals I and J *of the same length*, one has

$$|H_{IJ}| \le C\,\frac{|I|^{3/2}|J|^{3/2}}{\rho_{IJ}^3}\,,$$

where ρ_{IJ} is the maximum of $|I|$, $|J|$, and the distance between I and J and where $H_{IJ} = \langle Hh_I, h_J\rangle$.

Thus we shall proceed to prove only that the operators $M_v^{1/2}H_1 \cdot M_u^{1/2}$ and $v^{1/2}H_3M_u^{1/2}$ are bounded on $L^2(\mathcal{H})$.

We begin with $M_v^{1/2}H_1M_u^{1/2}$. We shall denote its matrix coefficients by K_{IJ}. Each is a linear operator on $\mathcal{H}$. We have that for $|I| > |J|$ with $3I \cap 3J = \varnothing$,

$$K_{IJ} = u_I^{1/2} H_{IJ}\, v_J^{1/2}\,.$$

For these (I, J)'s, one has the classical estimate see [Da,TV],

$$|H_{IJ}| \le C\,\frac{|I|^{3/2}|J|^{3/2}}{\rho_{IJ}^3}\,.$$

Throughout this section whenever A is a real scalar or more generally a self-adjoint operator, we shall, by abuse of notation denote by $A^{1/2}$ some choice of normal square root for A always using the fact that $A^{1/2}(A^{1/2})^* = |A|$ where $|A|$ denotes the sum of the positive and negative parts of A.

We apply Lemma 2.1 to K_{IJ}. We let $A_{IJ} = u_I^{1/2}v_J^{1/2}H_{IJ}^{1/2}$. Hence, we let $B_{IJ} = H_{IJ}^{1/2}$. The desired estimate on $\sum_I B_{IJ}^*B_{IJ}$ is simply the corresponding estimate for the scalar Hilbert transform. We need only bound

$$\sum_{\substack{J:|J|<|I|\\ 3J\cap 3I=\varnothing}} u_I^{1/2} v_J |H_{IJ}| u_I^{1/2} \le C u_I^{1/2}\Big(\sum_{\substack{J:|J|<|I|\\ 3J\cap 3I=\varnothing}} \frac{v_J|J|^{3/2}|I|^{3/2}}{\rho_{IJ}^3}\Big) u_I^{1/2}\,,$$

where the last inequality is in the sense of positive operators. Suppose that $I \in \mathcal{D}_j$. Then we subdivide into a sum over the intervals $J \in \mathcal{D}_k$ and over all $k > j$. Hence,

$$\sum_{J\in\mathcal{D}_k} A_{IJ}A_{IJ}^* \le C\sum_{k>j} u_I^{1/2}\Big(\sum_{\substack{J:J\in\mathcal{D}_k\\ 3J\cap 3I=\varnothing}} \frac{v_J|J|^{3/2}|I|^{3/2}}{\rho_{IJ}^3}\Big) u_I^{1/2}\,. \tag{3.8}$$

Now we estimate

$$
\begin{aligned}
\sum_{\substack{J: J\in\mathcal{D}_k\\ 3J\cap 3I=\varnothing}} \frac{v_J|J|^{3/2}|I|^{3/2}}{\rho_{IJ}^3}
&\le \Big(\sum_{J\in\mathcal{D}_k} \frac{|I|^{3/2}|J|^{3/2}}{\rho_{IJ}^3}\Big)^{\varepsilon/(1+\varepsilon)} \\
&\quad\cdot\Big(\sum_{\substack{J\in\mathcal{D}_k\\ 3J\cap 3I=\varnothing}} \frac{|I|^{3/2}|J|^{3/2}v_J^{1+\varepsilon}}{\rho_{IJ}^3}\Big)^{1/(1+\varepsilon)} \\
&\le C\Big(\frac{|J|^{1/2}}{|I|^{1/2}}\Big)^{\varepsilon/(1+\varepsilon)}\Big(\sum_{\substack{J\in\mathcal{D}_k\\ 3J\cap 3I=\varnothing}} \frac{|J|^{3/2}|I|^{1/2}v_J^{1+\varepsilon}}{\rho_{IJ}^2}\Big)^{1/(1+\varepsilon)} \\
&\le C\,\frac{|J|^{1/2}}{|I|^{1/2}}\Big(|I|\int_{(3I)^c}\frac{v^{1+\varepsilon}}{(x-y_I)^2}\Big)^{1/(1+\varepsilon)}.
\end{aligned}
\tag{3.9}
$$

Now plugging (3.9) into (3.8) and using condition b) we conclude

$$
\sum_J A_{IJ}A_{IJ}^* \le \sum_{k>j}\frac{|J|^{1/2}}{|I|^{1/2}}\,.
$$

But this is a geometric sum, so that $M_v^{1/2}H_1M_u^{1/2}$ is bounded.

For the penultimate inequality in (3.9) we used the fact that

$$
\sum_{J\in D_k}\frac{1}{\rho_{IJ}^s} \sim \frac{|I|}{|J|\,|I|^s}\,, \tag{iii}
$$

for $1<s$, $|J|<|I|$; which we can compare to

$$
\sum_{I\in D_j}\frac{1}{\rho_{IJ}^s} \sim \frac{1}{|I|^s} \sim \sum_{I\in D_j}\frac{1}{\rho_{xI}^s}\,, \tag{iv}
$$

for $1<s$, $|J|<|I|$, and $x\in J$.

Now, we write $L = v^{1/2}(H_3-\pi_{cv}^*)M_u^{-1/2}$. If we can bound the operator L then we have proven the theorem by condition c). We shall apply Cotlar's lemma writing $L=\sum_j L_j$ with $L_j = L\Delta_j$, where $\Delta_j = \sum_{I\in\mathcal{D}_j} E_I$. We must bound $L_k^*L_j$, and it will be enough to consider only $k \leq j$ by symmetry considerations the case $k \geq j$ can be done using the half of hypothesis a), b), c) that are not used in what follows. Also we should not worry about bounding $L_k L_j^*$ because it can be seen that for $k \neq j$ one has $L_k L_j^* = 0$.

We write with $I \in \mathcal{D}_j$, $J \in \mathcal{D}_k$ and $z_1, z_2 \in \mathcal{H}$ that

$$\langle L(h_I \otimes z_1), L(h_J \otimes z_2) \rangle_{L^2(\mathbb{R},\mathcal{H})} = \sum_{\alpha=1}^{6} \langle z_1, L_{IJ}^{\alpha} z_2 \rangle_{\mathcal{H}} \ .$$

To define the decomposition L^α we now define a set of intervals B_{jk}. An interval J is contained in B_{jk} precisely if $J \in \mathcal{D}_k$ and there exists a $I \in \mathcal{D}_j$ so that the distance between J and ∂I is bounded by $|I|^{1/2}|J|^{1/2}$. Now we define

$$\langle z_1, L_{IJ}^1\, z_2 \rangle_{\mathcal{H}} = \langle v^{1/2} H_3 h_I\, u_I^{1/2} z_1, v^{1/2} H_3 h_J\, u_J^{1/2} z_2 \rangle_{L^2(\mathbb{R},\mathcal{H})} \ ,$$

when $J \in B_{jk}$ and 0 otherwise,

$$-\langle z_1, L_{IJ}^2\, z_2 \rangle_{\mathcal{H}} = \langle v^{1/2} H_3 h_I\, u_I^{1/2} z_1, v^{1/2} \pi_{c^v}^* M_u^{1/2} (h_J \otimes z_2) \rangle_{L^2(\mathbb{R},\mathcal{H})} \ ,$$

when $J \in B_{jk}$ and 0 otherwise,

$$-\langle z_1, L_{IJ}^3\, z_2 \rangle_{\mathcal{H}} = \langle v^{1/2} \pi_{c^v}^* M_u^{1/2} (h_I \otimes z_1), v^{1/2} H_3 h_J\, u_J^{1/2} z_2 \rangle_{L^2(\mathbb{R},\mathcal{H})} \ ,$$

when $J \in B_{jk}$ and 0 otherwise,

$$\langle z_1, L_{IJ}^4\, z_2 \rangle_{\mathcal{H}} = \langle v^{1/2} \pi_{c^v}^* M_u^{1/2} (h_I \otimes z_1), v^{1/2} \pi_{c^v}^* M_u^{1/2} (h_J \otimes z_2) \rangle_{L^2(\mathbb{R},\mathcal{H})} \ ,$$

when $J \in B_{jk}$ and 0 otherwise,

$$\langle z_1, L_{IJ}^5\, z_2 \rangle_{\mathcal{H}} = \langle v^{1/2} H_3 h_I u_I^{1/2} z_1, L(h_J \otimes z_2) \rangle_{L^2(\mathbb{R},\mathcal{H})} \ ,$$

when $J \notin B_{jk}$ and 0 otherwise and

$$-\langle z_1, L_{IJ}^6\, z_2 \rangle_{\mathcal{H}} = \langle v^{1/2} \pi_{c^v}^* M_u^{1/2} (h_I \otimes z_1), L(h_J \otimes z_2) \rangle_{L^2(\mathbb{R},\mathcal{H})} \ ,$$

when $J \notin B_{jk}$ and 0 otherwise.

It suffices to bound the operator-valued matrices L^α with exponential decay in $|j-k|$, and this is what we shall do.

The main point of the argument is as follows. By the definition of H_3, the Haar expansion of $H_3 h_I$ is the sum of all components

$\langle Hh_I, h_J\rangle h_J$ for J such that $3I \cap 3J \neq \varnothing$ and $|I| < |J|$, denote that collection of intervals by $Z_3(I)$, for each I. The dyadic intervals J with the property $3I \cap 3J = \varnothing$ which have $|I| \geq |J|$ and such that their parents $\tilde{J}$ belong to $Z_3(I)$ form a disjoint covering of $(5I)^c$ and we may define $J_I(x)$ for any point x in $(5I)^c$ to be the element of this covering containing x. For $x \in 5I$ we define $J_I(x)$ to be the dyadic interval of length $|I|$ containing x.

Then, by definition, we have that

$$(H_3 h_I)(x) = m_{J_I(x)}(Hh_I)\,.$$

Thus while Hh_I has logarithmic singularities, the function $H_3 h_I$ does not. In fact, we have the precise size estimates which we shall use from now on

$$|H_3 h_I(x)| \leq C\,\frac{|I|^{3/2}}{\rho_{xI}^2}\,. \tag{3.10}$$

Let us state some facts that will be used often in the proof. We let $S_{jk} = 2(\cup_{J\in B_{jk}} J)$. For $I \in \mathcal{D}_j$, $J \in \mathcal{D}_k$, and $j < k$,

$$\sum_{J\in B_{jk}} \frac{1}{\rho_{xJ}^2} \leq \frac{C}{|J|^2}\,, \tag{v}$$

for $x \in S_{jk}$,

$$\sum_{J\in B_{jk}} \frac{1}{\rho_{xJ}^2} \leq \frac{C}{|J|\,|I|}\,, \tag{vi}$$

for $x \in S_{jk}^c$,

$$\int_{S_{jk}} \frac{1}{\rho_{xI}^2} \leq C\,\frac{|J|^{1/2}}{|I|^{3/2}}\,. \tag{vii}$$

We begin with

$$L_{IJ}^1 = u_I^{1/2}\Big(\int v(H_3 h_I)\,(H_3 h_J)\Big)u_J^{1/2}\,.$$

We apply Lemma 2.1, letting

$$A_{IJ} = u_I^{1/2}\Big(\int v(H_3 h_I)\,(H_3 h_J)\Big)^{1/2}\,,$$

and

$$B_{IJ} = \Big(\int v(H_3 h_I)\,(H_3 h_J) \Big)^{1/2} u_J^{1/2}\,.$$

We let $S_{jk} = 2(\cup_{J\in B_{jk}} J)$ and shall estimate separately the integral on S_{jk} and on S_{jk}^c. We must estimate

$$\begin{aligned}\sum_{J\in B_{jk}} A_{IJ}A^*_{IJ} &= \sum_{J\in B_{jk}} u_I^{1/2} \Big| \int v(H_3h_I)(H_3h_J)\Big| u_I^{1/2}\\ &\le \sum_{J\in B_{jk}} u_I^{1/2}\Big(\int_{S_{jk}} \frac{v|I|^{3/2}|J|^{3/2}}{\rho_{xI}^2\,\rho_{xJ}^2} + \int_{S_{jk}^c} \frac{v|I|^{3/2}|J|^{3/2}}{\rho_{xI}^2\,\rho_{xJ}^2}\Big) u_I^{1/2}\,. \end{aligned}\tag{3.11}$$

Here again $|\cdot|$ denotes the sum of the positive and negative parts. Now, we estimate the integrals using the trivial bound $|J| \le \rho_{xJ}$, Hölder, (v) and (vii),

$$\begin{aligned}\sum_{J\in B_{jk}} \int_{S_{jk}} \frac{v|I|^{3/2}|J|^{3/2}}{\rho_{xI}^2\,\rho_{xJ}^2} &= \int_{S_{jk}} \frac{v|I|^{3/2}}{\rho_{xI}^2} \Big(\sum_{J\in B_{jk}} \frac{|J|^{3/2}}{\rho_{xJ}^2}\Big)\\ &\le \Big(\frac{|I|}{|J|}\Big)^{1/2} \Big(|I| \int_{S_{jk}} \frac{v}{\rho_{xI}^2}\Big)\\ &\le C\Big(\frac{|I|}{|J|}\Big)^{1/2} \Big(|I| \int_{S_{jk}} \frac{v^{1+\varepsilon}}{\rho_{xI}^2}\Big)^{1/(1+\varepsilon)}\\ &\qquad \cdot \Big(|I| \int_{S_{jk}} \frac{1}{\rho_{xI}^2}\Big)^{\varepsilon/(1+\varepsilon)}\\ &\le C\Big(\frac{|I|}{|J|}\Big)^{1/2} \Big(\frac{|S_{jk}\cap I|}{|I|}\Big)^{\varepsilon/(1+\varepsilon)}\\ &\qquad \cdot \Big(|I| \int \frac{v^{1+\varepsilon}}{\rho_{xI}^2}\Big)^{1/(1+\varepsilon)}\\ &\le C\Big(\frac{|I|}{|J|}\Big)^{1/2-\varepsilon/(2+2\varepsilon)} \Big(|I| \int \frac{v^{1+\varepsilon}}{\rho_{xI}^2}\Big)^{1/(1+\varepsilon)}\,. \end{aligned}\tag{3.12}$$

Here the factor of $(|S_{jk}\cap I|/|I|)$ comes from the fact that ρ_{xI} is almost constant on intervals of length I, also remember that $|S_{jk}\cap I| \sim (|I|\,|J|)^{1/2}$. However for $x\in S_{jk}^c$ we may use that $\rho_{xJ} \ge (|I|\,|J|)^{1/2}$, thus obtaining, using (vi), and Hölder,

$$\begin{aligned}
\sum_{J\in B_{jk}} \int_{S_{jk}^c} \frac{v|I|^{3/2}|J|^{3/2}}{\rho_{xI}^2\,\rho_{xJ}^2} &\le \Big(\frac{|J|}{|I|}\Big)^{1/2}\Big(|I|\int_{S_{jk}^c}\frac{v}{\rho_{xI}^2}\Big)\\
&\le C\Big(\frac{|J|}{|I|}\Big)^{1/2}\Big(|I|\int\frac{v}{\rho_{xI}^2}\Big)\\
&\le C\Big(\frac{|J|}{|I|}\Big)^{1/2}\Big(|I|\int\frac{1}{\rho_{xI}^2}\Big)^{\varepsilon/(1+\varepsilon)}\\
&\quad\cdot\Big(|I|\int\frac{v^{1+\varepsilon}}{\rho_{xI}^2}\Big)^{1/(1+\varepsilon)}\\
&\le C\Big(\frac{|J|}{|I|}\Big)^{1/2}\Big(|I|\int\frac{v^{1+\varepsilon}}{\rho_{xI}^2}\Big)^{1/(1+\varepsilon)}
\end{aligned} \tag{3.13}$$

Now we plug (3.12) and (3.13) into (3.11) using condition b) to obtain

$$\sum_{J\in B_{jk}} A_{IJ}A_{IJ}^* \le C\Big(\Big(\frac{|J|}{|I|}\Big)^{1/2}+\Big(\frac{|I|}{|J|}\Big)^{1/2-\varepsilon/(2+2\varepsilon)}\Big).$$

We compute directly

$$\sum_{I\in\mathcal{D}_j} B_{IJ}^*B_{IJ} \le \sum_{I\in\mathcal{D}_j} u_J^{1/2}\Big(\int\frac{v|I|^{3/2}|J|^{3/2}}{\rho_{xI}^2\,\rho_{xJ}^2}\Big)u_J^{1/2} \le C\,\frac{|J|^{1/2}}{|I|^{1/2}}\,.$$

Here the last inequality follows from summing inside the integral, and then applying Hölder, condition b), (i) and (iv) as in (3.12) and (3.13). But this provides the desired estimates on L_{IJ}^1 since the product of the estimates on $\sum_{J\in\mathcal{D}_k} A_{IJ}A_{IJ}^*$ and on $\sum_{I\in\mathcal{D}_j} B_{IJ}^*B_{IJ}$ decays exponentially in $k-j$.

We will use repeatedly the estimate deduced by Hölder and b) in (3.13)

$$u_I^{1/2}\Big(\int\frac{|I|\,v(x)}{\rho_{xI}^2}\Big)u_I^{1/2} \le C\,.$$

To bound L_{IJ}^2, naturally, we shall use condition c), recalling the estimate (3.5) namely

$$\|u_I^{1/2}c_I^v(m_I(v))^{1/2}\|_{\mathcal{H}\to\mathcal{H}} \le C\,|I|^{1/2}\,. \tag{3.5}$$

We abbreviate $D_I = u_I^{1/2} c_I^v (m_I(v))^{1/2}$. By definition, we have that for $J \in B_{jk}$, and $x_J \in J$,

$$L^2_{IJ} = u_I^{1/2}(H_3 h_I)(x_J)(m_J(v))^{1/2} D^*_J \,,$$

remember that

$$\pi^*_{c^v} M_u^{1/2}(h_J \otimes z) = \frac{\chi_J(x)}{|J|}\,(c^v_J)^* u_J^{1/2} z\,.$$

We apply Lemma 3.1, letting

$$A_{IJ} = u_I^{1/2}(m_J(v))^{1/2}\,\frac{|I|^{1/2}|J|^{1/2}}{\rho_{IJ}}$$

and letting

$$B_{IJ} = \frac{\rho_{IJ}}{|I|^{1/2}|J|^{1/2}}\,(H_3 h_I)(x_J) D^*_J \,.$$

Now we compute

$$\begin{aligned}
\sum_{J\in B_{jk}} A_{IJ}A^*_{IJ} &= u_I^{1/2}|I|\Big(\sum_{J\in B_{jk}} \frac{|J| m_J\, v}{\rho^2_{IJ}}\Big) u_I^{1/2} \\
&\le C\, u_I^{1/2}\Big(|I|\int_{S_{jk}} \frac{v}{\rho^2_{xI}}\Big) u_I^{1/2} \\
&\le C\Big(\frac{|J|}{|I|}\Big)^{\varepsilon/(2+2\varepsilon)} u_I^{1/2}\Big(|I|\int \frac{v^{1+\varepsilon}}{\rho^2_{xI}}\Big)^{1/(1+\varepsilon)} u_I^{1/2} \\
&\le C\Big(\frac{|J|}{|I|}\Big)^{\varepsilon/(2+2\varepsilon)} .
\end{aligned}$$

Here S_{jk} is as in (3.12), the penultimate inequality comes from the same application of Hölder as in (3.12), and the final inequality comes from condition b). On the other hand, by (3.5), we see that

$$\sum_I B^*_{IJ} B_{IJ} \le \sum_I \frac{|I|^2}{\rho^2_{IJ}} \le C\,.$$

Hence, we have obtained the desired estimate for L^2_{IJ}.

Next, we estimate L^3_{IJ}. We obtain immediately from the definition that

$$-L^3_{IJ} = \frac{1}{|I|}\, u_I^{1/2} c^v_I \Big(\int_I v(H_3 h_J)\Big) u_J^{1/2} \,.$$

As before, we shall use that from condition c), we have (3.5). We apply Lemma 3.1, letting

$$A_{IJ} = \frac{|J|^{1/4}}{|I|}\, u_I^{1/2} c_I^v \Big(\int_I v H_3 h_J \Big)^{1/2}, \tag{3.14}$$

and letting

$$B_{IJ} = |J|^{-1/4} \Big(\int_I v H_3 h_J \Big)^{1/2} u_J^{1/2}\,.$$

We compute

$$\sum_{J\in B_{jk}} A_{IJ} A_{IJ}^* \le \sum_{J\in B_{jk}} \frac{1}{|I|^2}\, u_I^{1/2} c_I^v \Big(\int_I |J|^2\, \frac{v}{\rho_{xJ}^2} \Big) (c_I^v)^* u_I^{1/2}, \tag{3.15}$$

by plugging in (3.14) into the sum and using the size estimates on $H_3 h_J$. Now, we estimate the integral in (3.15) by breaking up the interval I into $I\cap S_{jk}$ and $I\cap S_{jk}^c$ observing, by summing under the integral and using (v) and (vi), that

$$\sum_{J\in B_{jk}} \int_I |J|^2\, \frac{v}{\rho_{xJ}^2} \le C\Big(\int_{S_{jk}\cap I} v + \frac{|J|^{1/2}}{|I|^{1/2}} \int_I v \Big)\,. \tag{3.16}$$

The second piece in (3.16) is clearly bounded by $|J|^{1/2}|I|^{1/2} m_I(v)$. As for the first piece, we use doubling observing that $S\cap I$ is contained in the rightmost and leftmost dyadic subintervals of I having measure more that $2\,|J|^{1/2}|I|^{1/2}$. Recall doubling implies that if K is any dyadic interval and $\tilde K$ its parent then

$$\int_{\tilde K} v \le C \int_K v\,.$$

Now let K_b be K's twin sister. Since $\int_K v \ge \int_{\tilde K} v/C$ while $\int_{\tilde K} v = \int_K v + \int_{K_b} v$, one has that

$$\int_{K_b} v \le \Big(1 - \frac{1}{C}\Big) \int_{\tilde K} v\,.$$

Naturally, the same holds for K by applying the doubling condition on K_b. In fact if K' is any descendant of K after l generations, one has

$$\int_{K'} v \le \Big(1-\frac{1}{C}\Big)^l \int_K v \le \Big(\frac{|K'|}{|K|}\Big)^\delta \int_K v\,,$$

where $\delta \geq 0$ depends only on the doubling constant. Now since $S \cap I$ is contained in two descendants of I of length at most $4|J|^{1/2}|I|^{1/2}$, one has that

$$\int_{S\cap I} v \leq C\Big(\frac{|J|}{|I|}\Big)^{\delta/2} \int_I v\,. \tag{3.17}$$

Plugging our observations into (3.16) yields that

$$\sum_{J\in B_{jk}} \int_I |J|^2 \,\frac{v}{\rho_{xJ}^2} \leq C\,(|I|^{1-\delta/2}|J|^{\delta/2} + |I|^{1/2}|J|^{1/2}\Big) m_I(v)\,. \tag{3.18}$$

Now we plug (3.18) into (3.15), applying (3.5) and the fact that when $P_1 \leq P_2$ then $TP_1T^* \leq TP_2T^*$ for any P_1, P_2, and T to obtain that

$$\sum_{J\in B_{jk}} A_{IJ}A_{IJ}^* \leq C\Big(\Big(\frac{|J|}{|I|}\Big)^{\delta/2} + \Big(\frac{|J|}{|I|}\Big)^{1/2}\Big)\,.$$

Now,

$$\sum_{I\in\mathcal{D}_j} B_{IJ}^* B_{IJ} \leq \sum_{I\in\mathcal{D}_j} u_J^{1/2}\Big(\int_I \frac{|J|\,v}{\rho_{xJ}^2}\Big)u_J^{1/2} \leq u_J^{1/2}\Big(\int \frac{|J|\,v}{\rho_{xJ}^2}\Big)u_J^{1/2} \leq C\,.$$

Here we have used Hölder and condition b) as in (3.12) and (3.13). Thus, we have obtained the desired estimates on L_{IJ}^3.

We come now to L_{IJ}^4. By definition, when $J \in B_{jk}$ and $J \subset I$,

$$\begin{aligned} L_{IJ}^4 &= u_I^{1/2}c_I^v\Big(\int_J \frac{v}{|I|\,|J|}\Big)(c_J^v)^* u_J^{1/2} \\ &= \Big(\frac{1}{|I|}\Big) D_I (m_I(v))^{-1/2}(m_J(v))^{1/2} D_J^*\,, \end{aligned}$$

when $J \cap I = \varnothing$ then by support considerations $L_{IJ}^4 = 0$.

As usual we apply Lemma 2.1, though the sum over I will be over a set with only one element. We let

$$A_{IJ} = \frac{|J|^{1/2}}{|I|}\, D_I (m_I(v))^{-1/2}(m_J(v))^{1/2}\,,$$

and

$$B_{IJ} = \frac{1}{|J|^{1/2}}\, D_J^*\,.$$

We compute

$$\begin{aligned}\sum_{\substack{J\in B_{jk}\\ J\subset I}} A_{IJ}A^*_{IJ} &= \sum_{\substack{J\in B_{jk}\\ J\subset I}} \Big(\frac{1}{|I|}\Big) D_I \Big(\frac{|J|}{|I|}\Big)(m_I(v))^{-1/2}(m_J(v))(m_I(v))^{-1/2} D^*_I \\ &\le \Big(\frac{1}{|I|}\Big)^2 D_I(m_I(v))^{-1/2}\int_{S\cap I} v(m_I(v))^{-1/2} D^*_I \\ &\le C\Big(\frac{1}{|I|}\Big) D_I \Big(\frac{|J|}{|I|}\Big)^{\delta/2} D^*_I \\ &\le C\Big(\frac{|J|}{|I|}\Big)^{\delta/2}.\end{aligned}$$

Here the penultimate estimate is by (3.17) and the last one by (3.5). The bound on $B^*_{IJ}B_{IJ}$ independent of J is just (3.5). Hence, L^4_{IJ} satisfies the desired estimates.

Next we bound L^5_{IJ}. This time $J\notin B_{jk}$. We define

$$\hat{J} = \Big(\frac{|I|^{1/2}}{|J|^{1/2}}\Big) J$$

and we have that if $J\subset I$ then $\hat{J}\subset I$. Now the reason that we are Cotlarizing $L = v^{1/2}(H_3 - \pi^*_{c^v})M_u^{1/2}$ instead of just $v^{1/2}H_3M_u^{1/2}$ is precisely that it gives us the cancelation

$$\int v^{1/2} L h_I = 0\,,$$

for every interval I. We now simply use the fact that $H_3 h_I$ is constant on $\hat{J}$ while

$$\int_{\hat{J}} v^{1/2} L h_J = -\int_{\hat{J}^c} v^{1/2} L h_J$$

to write

$$\begin{aligned}L^5_{IJ} &= L^{5,1}_{IJ} - L^{5,2}_{IJ} \\ &= u_I^{1/2}\Big(\int_{(\hat{J})^c} v(H_3h_I)\,(H_3h_J)\Big)u_J^{1/2} \\ &\quad - u_I^{1/2}(H_3h_I)(x_J)\Big(\int_{(\hat{J})^c} v(H_3h_J)\Big)u_J^{1/2}\,.\end{aligned}$$

First, we bound $L^{5,1}_{IJ}$ by Lemma 2.1. We let

$$A_{IJ} = u_I^{1/2}\Big(\int_{(\hat{J})^c} v(H_3h_I)\,(H_3h_J)\Big)^{1/2},$$

and we let

$$B_{IJ} = \Big(\int_{(\hat{J})^c} v(H_3h_I)\,(H_3h_J)\Big)^{1/2} u_J^{1/2}\,.$$

We have

$$\begin{aligned}\sum_{J\notin B_{jk}} A_{IJ}A^*_{IJ} &\leq u_I^{1/2}\Big(\sum_{J\notin B_{jk}}\int_{(\hat{J})^c}\frac{v|I|^{3/2}|J|^{3/2}}{\rho^2_{xI}\,\rho^2_{xJ}}\Big)u_I^{1/2}\\ &\leq C\,u_I^{1/2}\Big(\int\frac{v|I|}{\rho^2_{xI}}\Big)u_I^{1/2}\\ &\leq C\,.\end{aligned}$$

Here, the penultimate inequality comes from the simple observation that for each x,

$$\text{(viii)}\qquad \sum_{J\in\mathcal{D}_k}\frac{1}{\rho^2_{xJ}}\,\chi_{\hat{J}^c}(x) \leq \frac{C}{|J|^{3/2}|I|^{1/2}}\,.$$

The inequality (viii) is obtained by majorizing the sum by

$$\frac{1}{|J|}\int_{|J|^{1/2}|I|^{1/2}}^{\infty}\frac{dx}{x^2}\,.$$

Furthermore,

$$\begin{aligned}\sum_{I\in\mathcal{D}_j} B^*_{IJ}B_{IJ} &\leq u_J^{1/2}\Big(\sum_{I\in\mathcal{D}_j}\int_{(\hat{J})^c}\frac{v|I|^{3/2}|J|^{3/2}}{\rho^2_{xI}\,\rho^2_{xJ}}\Big)u_J^{1/2}\\ &\leq C\Big(\frac{|J|^{1/2}}{|I|^{1/2}}\Big)\Big(u_J^{1/2}\int\frac{|J|v}{\rho^2_{xJ}}\Big)u_J^{1/2}\\ &\leq C\Big(\frac{|J|}{|I|}\Big)^{1/2},\end{aligned}$$

which is the desired estimate for $L^{5,1}_{IJ}$. Redefining A_{IJ} and B_{IJ}, we continue by decomposing $L^{5,2}_{IJ} = A_{IJ}B_{IJ}$. First, we let

$$K_{IJ} = \rho_{IJ}^{1/(1+\varepsilon)}(H_3h_I)(J)\,,$$

and we let

$$A_{IJ} = |I|^{1/(2(1+\varepsilon))} u_I^{1/2} \rho_{IJ}^{-1/(1+\varepsilon)} K_{IJ}^{1/2} \Big(\int_{(\hat{J})^c} v(H_3 h_J) \Big)^{1/2},$$

and

$$B_{IJ} = (|I|)^{-1/(2(1+\varepsilon))} K_{IJ}^{1/2} \Big(\int_{(\hat{J})^c} v(H_3 h_J) \Big)^{1/2} u_J^{1/2} .$$

We estimate

$$\begin{aligned} A_{IJ} A_{IJ}^* &\leq u_I^{1/2} |K_{IJ}|\, |J|^{3/2} \Big(|I| \int_{\hat{J}^c} \frac{v^{1+\varepsilon}}{\rho_{xJ}^2\, \rho_{IJ}^2} \Big)^{1/(1+\varepsilon)} \\ &\quad \cdot \Big(\int_{\hat{J}^c} \frac{1}{\rho_{xJ}^2} \Big)^{\varepsilon/(1+\varepsilon)} u_I^{1/2} \\ &\leq C\, |K_{IJ}|\, |J|^{3/2} u_I^{1/2} \Big(|I| \int_{\hat{J}^c} \frac{v^{1+\varepsilon}}{\rho_{xI}^2} \Big)^{1/(1+\varepsilon)} \\ &\quad \cdot u_I^{1/2} \Big(\frac{1}{|J|^{1/2} |I|^{1/2}} \Big)^{\varepsilon/(1+\varepsilon)} (|I|\,|J|)^{-1/(1+\varepsilon)} \\ &\leq C\, \frac{(|I|\,|J|)^{(1+2\varepsilon)/(2(1+\varepsilon))}}{\rho_{IJ}^{(1+2\varepsilon)/(1+\varepsilon)}} . \end{aligned}$$

Here we have used the fact that on $\hat{J}^c$, one has $\rho_{IJ}^2\, \rho_{xJ}^2 \geq |I|\,|J| \rho_{xI}^2$. We sum obtaining

$$\begin{aligned} \sum_{J \in \mathcal{D}_k} A_{IJ} A_{IJ}^* &\leq \sum_{J \in \mathcal{D}_k} \frac{|I|^{(1+2\varepsilon)/(2(1+\varepsilon))} |J|^{(1+2\varepsilon)/(2(1+\varepsilon))}}{\rho_{IJ}^{(1+2\varepsilon)/(1+\varepsilon)}} \\ &\leq \Big(\frac{|I|}{|J|} \Big)^{1/(2(1+\varepsilon))} . \end{aligned}$$

Meanwhile, we compute

$$\begin{aligned} B_{IJ}^* B_{IJ} &\leq (|I|\,|J|)^{-(2+\varepsilon)/(2(1+\varepsilon))} |J|^{3/2} |K_{IJ}|\, u_J^{1/2} \\ &\quad \cdot \Big(|J| \int_{\hat{J}^c} \frac{v^{1+\varepsilon}}{\rho_{xJ}^2} \Big)^{1/(1+\varepsilon)} u_J^{1/2} \\ &\leq C\, \frac{(|I|\,|J|)^{(1+2\varepsilon)/(2(1+\varepsilon))}}{\rho_{IJ}^{(1+2\varepsilon)/(1+\varepsilon)}} . \end{aligned}$$

We conclude that

$$\sum_{I \in \mathcal{D}_j} B^*_{IJ} B_{IJ} \leq \sum_{I \in \mathcal{D}_j} \frac{|I|^{(1+2\varepsilon)/(2(1+\varepsilon))} |J|^{(1+2\varepsilon)/(2(1+\varepsilon))}}{\rho_{IJ}^{(1+2\varepsilon)/(1+\varepsilon)}}$$
$$\leq \Big(\frac{|J|}{|I|}\Big)^{(1+2\varepsilon)/(2(1+\varepsilon))}.$$

which gives the desired estimate on $L^{5,2}_{IJ}$.

Finally, we come to L^6_{IJ}. We break up into $L^6_{IJ} = L^{6,1}_{IJ} + L^{6,2}_{IJ}$. Here, we let $L^{6,1}_{IJ} = L^6_{IJ}$ when $J \subset I$ and 0 otherwise. As before, we let $D_I = u_I^{1/2} c_I^v (m_I(v))^{1/2}$. We let

$$F_{IJ} = (m_I(v))^{-1/2} \Big(\frac{1}{|I|} \int_{I^c} vH_3 h_J\Big) u_J \Big(\frac{1}{|I|} \int_{I^c} vH_3 h_J\Big) (m_I(v))^{-1/2}.$$

We have $\|D_I\| \leq C|I|^{1/2}$ and we have

$$F_{IJ} \leq \frac{|J|^3}{d_{IJ}^4} \, u_I^{-1/2} u_J \, u_I^{-1/2}.$$

Recall d_{IJ} is the distance from J to the boundary of I. Here we are directly applying (3.3). Notice this is the only place where we use it. Since all J's we are considering are not in B_{jk}, we have that $d_{IJ} \geq |I|^{1/2}|J|^{1/2}$, but for most J, it is even bigger. We write by definition and cancellation,

$$-L^{6,1}_{IJ} = u_I^{1/2} c_I^v \Big(\frac{1}{|I|} \int_{I^c} vH_3 h_J\Big) u_J^{1/2}.$$

We let $A_{IJ} = L^{6,1}_{IJ}$ and $B_{IJ} = 1$. Then we have

$$\sum_{\substack{J : J \in \mathcal{D}_k \\ J \notin B_{jk}}} A_{IJ} A^*_{IJ} = \sum_J D_I F_{IJ} D^*_I$$
$$\leq D_I \Big(\sum_{J \in \mathcal{D}_k} \frac{|J|^3}{d_{IJ}^4} \, u_I^{-1/2} u_J \, u_I^{-1/2}\Big) D^*_I$$
$$\leq \frac{|J|^\delta}{|I|^{1+\delta}} \, D_I D^*_I$$
$$\leq C \Big(\frac{|J|}{|I|}\Big)^\delta,$$

for some $\delta > 0$ which is the desired estimate on $L^{6,1}_{IJ}$, provided we can show that

$$\sum_{\substack{J:J\in\mathcal{D}_k\\ J\notin B_{jk}}} \frac{|J|^3}{d_{IJ}^4}\, u_J \le \frac{|J|^\delta}{|I|^{1+\delta}}\, u_I \ ,$$

where the sum is over J's contained in I with $d_{IJ} \ge |I|^{1/2}|J|^{1/2}$. We let $\mathcal{Z}_1$ be the set of those J's with $d_{IJ} \le |I|^{3/4}|J|^{1/4}$ and $\mathcal{Z}_2$ be the set of those J's with $d_{IJ} \ge |I|^{3/4}|J|^{1/4}$. We estimate for $\mathcal{Z}_1$, noticing that $\operatorname{card}(\mathcal{Z}_1) \le (|I|/|J|)^{3/4}$,

$$\begin{aligned}
\sum_{J\in\mathcal{Z}_1} \frac{|J|^3}{d_{IJ}^4}\, u_J &\le \sum_{J\in\mathcal{Z}_1} \frac{|J|}{|I|^2}\, u_J \\
&\le \Big(\sum_{J\in\mathcal{Z}_1} \frac{|J|}{|I|^2}\, u_J^{1+\varepsilon}\Big)^{1/(1+\varepsilon)} \Big(\sum_{J\in\mathcal{Z}_1} \frac{|J|}{|I|^2}\Big)^{\varepsilon/(1+\varepsilon)} \\
&\le \Big(\frac{1}{|I|}\Big) u_I \Big(\frac{|J|}{|I|}\Big)^{\varepsilon/(4(1+\varepsilon))},
\end{aligned}$$

while for $\mathcal{Z}_2$,

$$\begin{aligned}
\sum_{J\in\mathcal{Z}_2} \frac{|J|^3}{d_{IJ}^4}\, u_J &\le \sum_{J\in\mathcal{Z}_2} \frac{|J|^2}{|I|^3}\, u_J \\
&\le \Big(\frac{|J|}{|I|}\Big)\Big(\sum_{J\in\mathcal{Z}_2} \frac{|J|}{|I|^2}\, u_J^{1+\varepsilon}\Big)^{1/(1+\varepsilon)} \Big(\sum_{J\in\mathcal{Z}_2} \frac{|J|}{|I|^2}\Big)^{\varepsilon/(1+\varepsilon)} \\
&\le u_I \Big(\frac{|J|}{|I|^2}\Big).
\end{aligned}$$

This leaves us to bound $L^{6,2}_{IJ}$.

By definition, for $I \cap J = \varnothing$,

$$L^{6,2}_{IJ} = D_I(m_I(v))^{-1/2}\Big(\frac{1}{|I|}\int_I vH_3h_J\Big)u_J^{1/2} \ ,$$

zero otherwise.

We break up

$$A_{IJ} = D_I(m_I(v))^{-1/2}\Big(\frac{1}{|I|}\int_I vH_3h_J\Big)^{1/2},$$

and

$$B_{IJ} = \Big(\frac{1}{|I|}\int_I vH_3h_J\Big)^{1/2} u_J^{1/2}\,,$$

and now we obtain bounds easily using the fact that $J \cap I = \varnothing$ and $d_{IJ} \geq (|I|\,|J|)^{1/2}$. We simply compute

$$\begin{aligned}\sum_{\substack{J:J\in\mathcal{D}_k\\ J\notin B_{jk}}} A_{IJ}A_{IJ}^* &\leq \Big(\sup_{x\in I}\sum_{J\in\mathcal{D}_k}\frac{|J|^{3/2}}{\rho_{xJ}^2}\Big) D_I(m_I(v))^{-1/2}\\ &\qquad\cdot\Big(\frac{1}{|I|}\int_I v\Big)(m_I(v))^{-1/2}D_I^*\\ &\leq C\,\frac{1}{|I|^{1/2}}\,D_ID_I^* \leq C\,|I|^{1/2}\,.\end{aligned}$$

Here we use the fact that

$$\text{(ix)}\qquad \sum_{\substack{J:J\in\mathcal{D}_k\\ J\notin B_{jk}\\ J\cap I=\varnothing}} \frac{|J|^{3/2}}{\rho_{xJ}^2} \leq \frac{C}{|I|^{1/2}}\,.$$

At the same time, seeing that the sum on I merely extends the support of the integral, we obtain

$$\sum_I B_{IJ}^*B_{IJ} \leq \frac{|J|^{1/2}}{|I|}\,u_J^{1/2}\Big(|J|\int\frac{v}{\rho_{xJ}^2}\Big)u_J^{1/2} \leq C\Big(\frac{|J|^{1/2}}{|I|}\Big)\,.$$

Multiplying these two estimates and obtaining decay, we prove Theorem 3.1.

It may be worth pointing out that if assumptions (3.3) and (3.4) seem unappealing, we can also obtain the same result by assuming a sort of "doubling at infinity" condition for v and u^{-1}. Thus it suffices, for example, to assume there is a $\delta > 0$ with

$$\begin{aligned}&\Big(\int_{I^c} vH_3h_J\Big)(m_J(v))^{-1}\Big(\int_{I^c} vH_3h_J\Big)\\ &\qquad\leq \Big(\frac{|J|}{d_{IJ}}\Big)^{\delta}\Big(\int_{J^c} vH_3h_J\Big)(m_J(v))^{-1}\Big(\int_{J^c} vH_3h_J\Big)\,.\end{aligned}$$

Then we simply use

$$\|F_{IJ}\| = \Big\| u_J^{1/2} \Big(\int_{I^c} vH_3h_J \Big) (m_J(v))^{-1} \Big(\int_{I^c} vH_3h_J \Big) u_J^{1/2} \Big\|,$$

together with the bound on the norm of D_J to obtain the same result. This doubling assumption may seem more natural to the reader than the assumption we make, until he realizes that it is not even automatic that this doubling assumption is true for $\delta = 0$.

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Recibido: 23 de octubre de 1.995

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* The first author was partially supported by the National Science Foundation Postdoctoral Fellowship.

Revista Matemática Iberoamericana
Vol. 13, N.º 2, 1997

The phase of the Daubechies filters

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Abstract. We give the first term of the asymptotic development for the phase of the N-th (minimum-phased) Daubechies filter as N goes to $+\infty$. We obtain this result through the description of the complex zeros of the associated polynomial of degree $2N+1$.

0. Introduction.

The *Daubechies filters* $m_N(\xi)$ are defined in the following way [2]:

i) $m_N(\xi)$ is a trigonometric polynomial of degree $2N+1$

$$m_N(\xi) = \sum_{k=0}^{2N+1} a_{N,k}\, e^{-ik\xi} \tag{1}$$

with *real-valued* coefficients $a_{N,k}$.

ii) $\sqrt{2}\, m_N(\xi)$ and $\sqrt{2}\, e^{-i\xi}\, \overline{m}_N(\xi+\pi)$ are *conjugate quadrature filters*

$$|m_N(\xi)|^2 + |m_N(\xi+\pi)|^2 = 1\,. \tag{2}$$

iii) $m_N(\xi)$ satisfies at 0 and π

$$m_N(0) = 1\,, \tag{3}$$

$$\frac{\partial^p}{\partial \xi^p} m_N(\pi) = 0\,, \qquad \text{for } p \in \{0, 1, \dots, N\}\,. \tag{4}$$

The importance of those filters is due to the following facts: the associated *wavelet* ψ_N defined by

$$\hat{\psi}_N(\xi) = e^{-i\xi/2}\,\overline{m}_N\Big(\frac{\xi}{2}+\pi\Big)\prod_{j=2}^{+\infty} m_N\Big(\frac{\xi}{2^j}\Big),$$

generates an orthonormal basis of $L^2(\mathbb{R})$ $\{2^{j/2}\psi_N(2^j x-k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ and satisfies the cancellation properties

$$\int x^p\psi_N(x)\,dx = 0\,, \qquad \text{for } p\in\{0,1,\cdots,N\}\,,$$

and has a support of minimal length among all orthonormal wavelets satisfying (6).

Conditions (1) to (4) don't define m_N in an unique way. As a matter of fact, there is exactly $2^{[(N+1)/2]}$ solutions m_N (where $[x]$ is the integer part of x). Indeed, conditions (1) to (4) determine only the *modulus* of m_N

$$|m_N(\xi)|^2 = Q_N(\cos\xi)\,, \tag{7}$$

$$Q_N(X) = \Big(\frac{1+X}{2}\Big)^{N+1}\sum_{k=0}^{N}\binom{N+k}{k}\Big(\frac{1-X}{2}\Big)^k. \tag{8}$$

We are going to check easily the following result on the roots of Q_N.

Proposition 1. *The roots of Q_N are $X=-1$ with multiplicity $N+1$ and N roots $X_{N,1},\cdots,X_{N,N}$ with multiplicity 1 such that*

i) *for $1\le k\le N$, $\operatorname{Re} X_{N,k}>0$ and $X_{N,N+1-k}=\overline{X_{N,k}}$,*

ii) *for $1\le k\le [N/2]$, $\operatorname{Im} X_{N,k}>0$,*

iii) *if N is odd, $X_{N,(N+1)/2}>1$.*

With help of Proposition 1, we may easily describe the solutions m_N of (1) to (4). Indeed, if $X_{N,k}=(z_{N,k}+1/z_{N,k})/2$ with $|z_{N,k}|>1$, then we have

$$m_N(\xi) = \prod_{k=1}^{[(N+1)/2]} S_{N,k}(\xi)\Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1}, \tag{9}$$

where, for $1 \le k \le [N/2]$,

$$
(10) \qquad \begin{aligned} S_{N,k}(\xi) &= \frac{(e^{-i\xi} - z_{N,k})\,(e^{-i\xi} - \overline{z}_{N,k})}{|1 - z_{N,k}|^2} \\ \text{or} \qquad S_{N,k}(\xi) &= \frac{(1 - z_{N,k}\, e^{-i\xi})\,(1 - \overline{z}_{N,k}\, e^{-i\xi})}{|1 - z_{N,k}|^2} \,. \end{aligned}
$$

If N is odd,

$$
(11) \qquad \begin{aligned} S_{N,(N+1)/2}(\xi) &= \frac{e^{-i\xi} - z_{N,(N+1)/2}}{1 - z_{N,(N+1)/2}} \\ \text{or} \qquad S_{N,(N+1)/2}(\xi) &= \frac{1 - z_{N,(N+1)/2}\, e^{-i\xi}}{1 - z_{N,(N+1)/2}} \,. \end{aligned}
$$

The case where all the roots of $M_N(z)$ (the polynomial such that $m_N(\xi) = M_N(e^{-i\xi})$) are outside the unit disk is the *minimum-phased Daubechies filter*

$$
(12) \qquad m_N(\xi) = \Big(\frac{1 + e^{-i\xi}}{2}\Big)^{N+1} \prod_{k=1}^{N} \frac{e^{-i\xi} - z_{N,k}}{1 - z_{N,k}} \,.
$$

The aim of this paper is to describe the phase of the Daubechies filters as N goes to $+\infty$. Indeed, the modulus of m_N is described by (7) and (8) and one easily checks that

$$
(13) \qquad \lim_{N \to +\infty} |m_N(\xi)| = \begin{cases} 1\,, & \text{if } |\xi| < \dfrac{\pi}{2}\,, \\ \dfrac{1}{\sqrt{2}}\,, & \text{if } |\xi| = \dfrac{\pi}{2}\,, \\ 0\,, & \text{if } \dfrac{\pi}{2} < |\xi| \le \pi\,. \end{cases}
$$

The phase of m_N, on the other hand, is much more delicate to study: it depends of course on the choice of the factors $S_{N,k}$ in (9), but even for the case of minimum-phased filters we are not aware of any previous results on the behaviour of the phase.

We are going to give an approximate value of $z_{N,k}$ which allows the determination of the phase of m_N. More precisely, if $Z_1, \ldots, Z_N$ are N complex numbers such that for $k \in \{1, \ldots, N\}$, $|Z_k| \ne 1$ and if

$$
\Pi(Z_1, \ldots, Z_N)(\xi) = \prod_{k=1}^{N} \frac{e^{-i\xi} - Z_k}{1 - Z_k} \,,
$$

we define *the phase* $\omega(Z_1, \ldots, Z_N)(\xi)$ as the C^∞ real-valued function such that $\omega(0) = 0$ and

$$\Pi(Z_1, \ldots, Z_N)(\xi) = \prod_{k=1}^{N} \left| \frac{e^{-i\xi} - Z_k}{1 - Z_k} \right| e^{-i\omega(Z_1,\ldots,Z_N)(\xi)} .$$

This function is easily computed as

$$\omega(Z_1, \ldots, Z_N)(\xi) = \operatorname{Im} \Big(\int_0^\xi \sum_{k=1}^{N} \frac{i\, e^{-is}}{e^{-is} - Z_k}\, ds \Big) . \tag{14}$$

Theorem 1. *Let $Q_N(X)$ be given by* (8)*, $X_{N,1}, \ldots, X_{N,N}$ be its roots which are not equal to -1 ordered by:*

- *for $1 \le k \le [(N+1)/2]$, $\operatorname{Im} X_{N,k} \ge 0$ and $X_{N,N+1-k} = \overline{X_{N,k}}$,*
- $|X_{N,1}| < |X_{N,2}| < \cdots < |X_{N,[(N+1)/2]}|$

and let $z_{N,k}$ be defined by $X_{N,k} = (z_{N,k} + 1/z_{N,k})/2$ and $|z_{N,k}| > 1$.

For $1 \le k \le N$, we approximate $z_{N,k}$ by $Z_{N,k}$ where:

i) *for $1 \le k \le [(N^{1/5})/\operatorname{Log} N]$, $Z_{N,k} = i - \overline{\gamma_k}/\sqrt{N}$, where $\gamma_1, \gamma_2, \ldots, \gamma_k, \ldots$ are the roots of* $\operatorname{erfc}(z) = 1 - (2/\sqrt{\pi}) \int_0^z e^{-s^2}\, ds$*, such that $\operatorname{Im} \gamma_k > 0$ and ordered by $|\gamma_1| < |\gamma_2| < \cdots < |\gamma_k| < \ldots$,*

ii) *for $[(N^{1/5})/\operatorname{Log} N] < k \le [(N+1)/2]$, $Z_{N,k} = \theta_{N,k} + \sqrt{\theta_{N,k}^2 - 1}$, where*

$$\operatorname{Im} \theta_{N,k} > 0 , \tag{15.a}$$

$$1 - \theta_{N,k}^2 = \Big(1 + \frac{1}{N} \operatorname{Log} \big(2\sqrt{2N\pi \sin \varphi_{N,k}}\big)\Big)\, e^{-2i\varphi_{N,k}} , \tag{15.b}$$

and

$$\varphi_{N,k} = \frac{8\,k - 1}{8N + 6}\, \pi , \tag{16}$$

iii) *for $[(N+1)/2] < k \le N$, $Z_{N,k} = \overline{Z}_{N,N+1-k}$.*

Then for any choice

$$m_N(\xi) = \Big(\frac{1 + e^{-i\xi}}{2}\Big)^{N+1} \Pi(z_{N,1}^{\varepsilon_1}, \ldots, z_{N,N}^{\varepsilon_N})(\xi)$$

of the Daubechies filter m_N *(where* $\varepsilon_k = \pm 1$ *and* $\varepsilon_{N+1-k} = \varepsilon_k$*), the approximation*

$$\tilde{m}_N(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1} \Pi(Z_{N,1}^{\varepsilon_1}, \dots, Z_{N,N}^{\varepsilon_N})(\xi)$$

satisfies

$$(17) \quad |\omega(z_{N,1}^{\varepsilon_1}, \dots, z_{N,N}^{\varepsilon_N})(\xi) - \omega(Z_{N,1}^{\varepsilon_1}, \dots, Z_{N,N}^{\varepsilon_N})(\xi)| \le C_0 \frac{(\operatorname{Log} N)^2}{N^{1/5}} \ ,$$

for all $\xi \in \mathbb{R}$*, where* C_0 *doesn't depend neither on* $N \ge 2$ *nor on* ξ *nor on the* ε_k*'s.*

Thus, due to Theorem 1, we may give the phase of m_N with an $o(1)$ precision! Of course, we need the knowledge of the roots of the complementary error function; these roots are described in [3] and our results give again the same estimates, as we shall see.

We may greatly simplify the approximating $Z_{N,k}$'s if we accept to get a greater error. For instance, we may characterize easily the minimum-phased filters with an $O(\sqrt{N})$ error:

Theorem 2. *Let*

$$m_N(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1} \Pi(z_{N,1}, \dots, z_{N,N})(\xi)$$

be the N*-th minimum-phased Daubechies filter. Then the phase*

$$\omega(z_{N,1}, \dots, z_{N,N})(\xi)$$

satisfies

$$(18) \quad |\omega(z_{N,1}, \dots, z_{N,N})(\xi) - N\omega(\xi)| \le C_0 \sqrt{N} \ , \qquad \textit{for all } \xi \in \mathbb{R},$$

where C_0 *doesn't depend on* ξ *nor on* N *and where*

$$(19) \quad \omega(\xi) = \frac{1}{2\pi}\left(\operatorname{Li}_2(-\sin\xi) - \operatorname{Li}_2(\sin\xi)\right) = \frac{-1}{\pi} \sum_{k=0}^{+\infty} \frac{(\sin\xi)^{2k+1}}{(2k+1)^2} \ .$$

The Li_2 function is the polylogarithm of order 2

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \int_0^z \frac{1}{u} \operatorname{Log} \frac{1}{1-u}\, du\,. \tag{20}$$

The function $(\mathrm{Li}_2(z) - \mathrm{Li}_2(-z))/2$ is known under the name of Legendre's χ_2 function.

Theorem 2 will be proved by approximating m_N by

$$\tilde{m}_N(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1} \pi(\tilde{Z}_{N,1}, \dots, \tilde{Z}_{N,N})(\xi)$$

with

$$\tilde{Z}_{N,k} = \sqrt{e^{-i\theta_{N,k}}} + \sqrt{1+e^{-i\theta_{N,k}}}\,, \qquad \theta_{N,k} = -\pi + \frac{16k-2}{8N+6}\,\pi\,,$$

Then $\omega(\tilde{Z}_{N,1}, \dots, \tilde{Z}_{N,N})/N$ is identified with a Riemann sum for the integral

$$\frac{1}{2\pi} \operatorname{Im} \int_{-\pi}^{\pi} \operatorname{Log} \frac{1}{\sqrt{e^{-i\theta}} + \sqrt{1+e^{-i\theta}} - e^{-i\xi}}\, d\theta = \omega(\xi)\,.$$

This approximating $\tilde{Z}_{N,k}$ is a simplified version of the approximating $Z_{N,k}$ of Theorem 1, obtained by neglecting the term

$$\frac{1}{N} \operatorname{Log} 2\sqrt{2N\pi \sin \varphi_{N,k}}\,.$$

We will be also able to give a description of a family of almost linear-phased Daubechies filters:

Theorem 3. *Let*

$$m_N(\xi) = \Big(\frac{1+e^{-i\xi}}{2}\Big)^{N+1} \pi(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi)$$

be the N-th Daubechies filter with $N = 4\,q$ and with the following choice of $\varepsilon_{N,k}$: for $1 \le p \le q$, $\varepsilon_{N,4p-3} = \varepsilon_{N,4p} = 1$ and $\varepsilon_{N,4p-2} = \varepsilon_{4p-1} = -1$ (so that $\varepsilon_{N,N+1-k} = \varepsilon_{N,k}$). Then the phase $\omega(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi)$ satisfies:

$$\Big|\omega(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi) - \frac{1}{2}\, N\xi\Big| \le C_0\,, \qquad \textit{for all } \xi \in \mathbb{R}, \tag{21}$$

where C_0 doesn't depend on ξ nor on N.

We are now going to prove Theorem 1 (and obtain theorems 2 and 3 as corollaries). Of course, it amounts to give a precise description of the roots $X_{N,k}$ of $Q_N(X)$. If we neglect the term $\operatorname{Log} 2\sqrt{2N\pi \sin\varphi_{N,k}}/N$ in $Z_{N,k}$, we obtain as a first approximation that the $z_{N,k}$ are close to the arc $\{|z-1| = \sqrt{2},\ \operatorname{Re} z \geq 0\}$ (which can be parameterized as $\{\sqrt{e^{-i\theta}} + \sqrt{1+e^{-i\theta}},\ -\pi \leq \theta \leq \pi\}$), or equivalently that the $X_{N,k}$ are close to the half-lemniscate $\{|1-X_{N,k}^2| = 1, \operatorname{Re} X_{N,k} \geq 0\}$. This will be obtained by representing $Q_N(X)$ as a Bernstein polynomial on $[-1,1]$ approximating the piecewise analytical function $\chi_{[0,1]}$

$$Q_N(X) = \sum_{k=N+1}^{2N+1} \binom{2N+1}{k} \Big(\frac{1+X}{2}\Big)^k \Big(\frac{1-X}{2}\Big)^{2N+1-k} \tag{22}$$

(a formula pointed by many authors [1], [6], [11]). In that form, $Q_N(X)$ corresponds to a Herrmann filter [4] and it is precisely the figure in Herrmann's paper representing the $z_{N,k}$'s for Q_{21} which lead us to conjecture the behaviour of the $z_{N,k}$'s.

A classical theorem of Kantorovitch [5], [7] on the behaviour of Bernstein polynomials of piecewise analytical functions ensures that $Q_N(X)$ converges to 0 uniformly on any compact subset of the interior of the half lemniscat $\{|1-x^2| < 1,\ \operatorname{Re} x < 0\}$ and to 1 uniformly on any compact subset of $\{|1-x^2| < 1,\ \operatorname{Re} x > 0\}$. We will use similar tools to study $Q_N(X)$ *outside* of the convergence subsets.

Near the critical point $X = 0$, the approximation by points on the lemniscat is no longer precise enough, and we will show that for the small roots $X_{N,k}$, $-\sqrt{N}X_{N,k}$ is to be approximated by a root of the complementary error function. Such an approximation occurs for instance in the study of the (spurious) zeros of the Taylor polynomials of the exponential function [12] and we will use quite similar tools to get our description. The main difference, however, is maybe that we are dealing with a divergent family of polynomials.

NOTATIONS. We will define as usually $\operatorname{Log} z$ and $\sqrt{z}$ as the reciprocal functions of

$$z = \operatorname{Log} w \in \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi\} \longmapsto w = e^z \in \{w \in \mathbb{C} : w \notin (-\infty, 0])\},$$

$$z = \sqrt{w} \in \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \longmapsto w = z^2 \in \{w \in \mathbb{C} : w \notin (-\infty, 0]\}.$$

The paper will be organized in the following way:

1. Q_N as a Bernstein polynomial and other preliminary results.
2. Small roots of Q_N: first estimates.
3. Big roots of Q_N: first estimates.
4. Big roots of Q_N: further estimates.
5. Small roots of Q_N: further estimates.
6. The phase of a general Daubechies filter.
7. Minimum-phased Daubechies filters.
8. Almost linear-phased Daubechies filters.

1. Q_N as a Bernstein polynomial and other preliminary results.

We begin by proving a first localization result:

Result 1. *For $N \geq 2$ and $t \neq -1$, if $Q_N(t) = 0$ then $\mid 1 - t \mid < 1$.*

Proof. This will be the only time where we use the Daubechies formula (8) for $Q_N(X)$. This formula gives that if $Q_N(t) = 0$ and $t \neq -1$, then

$$\sum_{k=0}^{N} \frac{1}{2^k} \binom{N+k}{k} (1-t)^k = 0\,. \tag{23}$$

If we define α_k as $\alpha_k = \binom{N+k}{k}/2^k$, $0 \leq k \leq N$, then we have obviously $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_{N-1} = \alpha_N$, and we may apply a very classical lemma of Eneström, Kakeya and Hurwirtz (quoted by G. Pólya and Szegö [10, Exercise III-22]):

Lemma 1. *If $0 < a_0 < a_1 < \cdots < a_{N-1} = a_N$ and if $\sum_{k=0}^{N} a_k s^k = 0$ then $|s| < 1$.*

Proof of the lemma. If $s \geq 0$ then $\sum_{k=0}^{N} a_k s^k > 0$; if $s \notin [0, +\infty)$, then

$$\Big| a_0 + \sum_{k=1}^{N} (a_k - a_{k-1}) s^k \Big| < a_0 + \sum_{k=1}^{N} (a_k - a_{k-1})\, |s|^k\,,$$

thus if $|s| \geq 1$ (so that $|s|^k \leq |s|^{N+1}$) and $s \notin [0, +\infty)$, we get

$$\Big| (1-s) \sum_{k=0}^{N} a_k s^k \Big| > |s|^{N+1} \Big(a_N - \sum_{k=1}^{N} (a_k - a_{k-1}) - a_0 \Big) = 0\,.$$

Thus, we have shown that the roots t of Q_N such that $t \neq -1$ are located in the open disk of radius 1 and of center 1, and that the associated values $1-t^2$ are located in the interior of a cardioid.

From now until the end, we will use formula (22) instead of formula (8) to represent Q_N. The main interest in the representation of Q_N as a Bernstein polynomial is that Q_N is easily differentiated: (22) gives

$$\frac{d}{dt}Q_N(t) = \frac{(2N+1)!}{4^N(N!)^2}\,\frac{1}{2}\,(1-t^2)^N\,. \tag{24}$$

This expression can be easily related to the expression of $Q_N(\cos\xi)$ given by Y. Meyer ([8])

$$\begin{aligned} Q_N(\cos\xi) &= \int_{-1}^{\cos\xi} \frac{(2N+1)!}{4^N(N!)^2}\,\frac{1}{2}\,(1-t^2)^N\,dt \\ &= \int_{\xi}^{\pi} \frac{(2N+1)!}{4^N(N!)^2}\,\frac{1}{2}\,(\sin\theta)^{2N+1}\,d\theta\,. \end{aligned}$$

We will use intensively formula (24) in the following. If t is small, we approximate $Q_N(t)$ by $Q_N(0) = 1/2$ and obtain

$$Q_N(t) = \frac{1}{2}\Big(1 + \frac{(2N+1)!}{4^N(N!)^2}\int_0^t (1-s^2)^N\,ds\Big)\,, \tag{25}$$

while for a bigger t (with $\operatorname{Re} t > 0$) we approximate $Q_N(t)$ by $Q_N(1) = 1$ and obtain

$$Q_N(t) = 1 - \frac{1}{2}\,\frac{(2N+1)!}{4^N(N!)^2}\int_t^1 (1-s^2)^N\,ds\,. \tag{26}$$

Stirling's formula $N! = (N/e)^N\sqrt{2\pi N}(1 + 1/(12N) + O\,(1/N^2))$ allows one to simplify formulas (25) and (26)

$$\frac{(2N+1)!}{4^N(N!)^2} = 2\sqrt{\frac{N}{\pi}}\,\Big(1 + O\,\Big(\frac{1}{N^2}\Big)\Big)\,. \tag{27}$$

Thus $Q_N(t) = 0$ may be rewritten as

$$1 + \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{N}t}\Big(1-\frac{s^2}{N}\Big)^N\,ds = 1 - 2\,\frac{\sqrt{N}}{\sqrt{\pi}}\,\frac{4^N(N!)^2}{(2N+1)!} = O\,(\frac{1}{N^2}) \tag{28}$$

or as

$$\sqrt{N}\int_t^1 (1-s^2)^N\,ds = 2\,\frac{4^N(N!)^2}{(2N+1)!} = \sqrt{\pi}+O\Big(\frac{1}{N^2}\Big)\,. \tag{29}$$

Formula (28) will be used for the small roots (sections 2 and 5) and formula (29) for the big roots (sections 3 and 4).

We mention a further application of (24) (which will not be used in the following): we may compute explicitly the generating series for $Q_N(t)$ when $\mathrm{Re}\,t<0$:

Proposition 2. *Assume that* $\mathrm{Re}\,t<0$ *and* $|(1-t^2)\,u|<1$. *Then*

$$\sum_{N=0}^{+\infty} Q_N(t)\,u^N = \frac{1}{2}\,\frac{1-t^2}{\sqrt{1-u\,(1-t^2)}\,(-t+\sqrt{1-u\,(1-t^2)})}\,. \tag{30}$$

Proof. We differentiate $\sum_{N=0}^{+\infty} Q_N(t)\,u^N$ with respect to t. Then (24) gives

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\sum_{N=0}^{+\infty} Q_N(t)\,u^N\Big) &= \sum_{N=0}^{+\infty}\frac{1}{2}\,\frac{(2N+1)!}{4^N N!}\,\frac{((1-t^2)\,u)^N}{N!}\\ &= \frac{1}{2}\,(1-u\,(1-t^2))^{-3/2}\,,\end{aligned}$$

hence

$$\sum_{N=0}^{+\infty} Q_N(t)\,u^N = \int_{-1}^t \frac{1}{2}\,\frac{ds}{(1-(1-s^2)\,u)^{3/2}}\,.$$

On the other hand, if we differentiate $t/(1-u\,(1-t^2))^{1/2}$, we get

$$\frac{\partial}{\partial t}\Big(\frac{t}{(1-u\,(1-t^2))^{1/2}}\Big) = \frac{1-u\,(1-t^2)-t^2u}{(1-u\,(1-t^2))^{3/2}} = \frac{1-u}{(1-u\,(1-t^2))^{3/2}}\,.$$

Thus we have

$$\begin{aligned}\sum_{N=0}^{+\infty} Q_N(t)\,u^N &= \frac{1}{2\,(1-u)}\Big(\frac{t}{(1-u\,(1-t^2))^{1/2}}+1\Big)\\ &= \frac{1}{2\,(1-u)}\,\frac{1-u\,(1-t^2)-t^2}{(1-u\,(1-t^2))^{1/2}((1-u\,(1-t^2))^{1/2}-t)}\\ &= \frac{1}{2}\,\frac{1-t^2}{(1-u\,(1-t^2))^{1/2}((1-u\,(1-t^2))^{1/2}-t)}\,.\end{aligned}$$

As a corollary, we get:

Result 2. *If $t \in \mathbb{C}$ is such that $|1-t^2| > 1$, then*

$$\limsup_{N \to +\infty} |Q_N(t)| = +\infty\,.$$

PROOF. If $\mathrm{Re}\, t < 0$, this is obvious by formula (30); the right-hand term of equality (30) has $1/|1-t^2|$ as its radius of convergence in u, so that

$$\limsup_{N \to +\infty} |Q_N(t)|^{1/N} = |1-t^2|\,.$$

If $\mathrm{Re}\, t > 0$, then $Q_N(t) = 1 - Q_N(-t)$ so that again

$$\limsup_{N \to +\infty} |Q_N(t)|^{1/N} = |1-t^2|\,.$$

If $\mathrm{Re}\, t = 0$ and $t \neq 0$, then

$$|Q_N(t)| \sim \frac{1}{2}\, 2\sqrt{\frac{N}{\pi}} \int_0^{|t|} (1+\rho^2)^N \, d\rho \longrightarrow +\infty\,, \qquad \text{as } N \longrightarrow +\infty\,.$$

A last (and direct) application of formula (24) is Proposition 1.

Result 3.

i) *If t is a root of $Q_N(t)$ and $t \neq -1$, then t has multiplicity 1.*

ii) *If N is even, $t = -1$ is the unique real root of Q_N.*

iii) *If N is odd, Q_N has only one other real root $x_{N,(N+1)/2} \neq -1$, and $x_{N,(N+1)/2} > 1$.*

PROOF. By (24), we know that the only roots of dQ_N/dt are 1 and -1, so i) is obvious. Moreover, if N is even, dQ_N/dt is non-negative on $\mathbb{R}$ and thus Q_N is increasing: -1 is the unique real root of Q_N. If N is odd, then Q_N decreases on $(-\infty, -1]$, vanishes at -1, increases between -1 and 1, and decreases again from the value 1 at $t = 1$ to the value $-\infty$ at $t = +\infty$: Q_N has another real root $x_{N,(N+1)/2} > 1$.

Results 1 and 3 imply obviously Proposition 1.

2. Small roots of Q_N: first estimates.

In this section, we are going to prove the following result:

Result 4. *Let $\varepsilon_0 \in (0,1/2)$ and $K = [\varepsilon_0 \mathrm{Log}\, N/(2\pi)]$. Then, if N is big enough, the number of roots t of $Q_N(t)$ such that $\mathrm{Im}\, t \geq 0$ and $|t| \leq \sqrt{2K\pi/N}$ is exactly K. Moreover, if we list those roots as $x_{N,1}, \dots, x_{N,K}$ with $|x_{N,k}| < |x_{N,k+1}|$ and fix $\varepsilon_1 \in (\varepsilon_0, 1/2)$, we have*

$$\Big| x_{N,k} + \frac{1}{\sqrt{N}}\, \overline{\gamma}_k \Big| \leq C(\varepsilon_0, \varepsilon_1)\, \frac{1}{\sqrt{N}\, N^{1-2\varepsilon_1}} \,, \tag{31}$$

where $\gamma_1, \dots, \gamma_K$ are the K first roots γ of $\mathrm{erfc}(\gamma) = 0$ *with* $\mathrm{Im}\, \gamma \geq 0$.

Proof. Assume that $|t| \leq \sqrt{\alpha_1 \,\mathrm{Log}\, N/N}$ for some fixed $\alpha_1 > 0$. Then, using formulas (25) and (27), we write

$$Q_N(t) = \Big(\frac{1}{2} + \eta_N\Big)\Big(1 + \eta'_N + \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{N} t} \Big(1 - \frac{s^2}{N}\Big)^N ds\Big) \,,$$

where η_N, η'_N are two constants (depending only on N) which are $O(1/N^2)$. Now, if $|u| \leq \sqrt{\alpha_1 \,\mathrm{Log}\, N}$, we have

$$\frac{|u^4|}{N} \leq \alpha_1^2\, \frac{(\mathrm{Log}\, N)^2}{N} = o(1) \,,$$

hence one may find $C_0 \geq 0$ so that for N big enough ($N \geq N_0$ where N_0 depends only on α_1)

$$\Big| \Big(1 - \frac{u^2}{N}\Big)^N - e^{-u^2} \Big| \leq C_0 \Big| e^{-u^2}\, \frac{u^4}{N} \Big| \leq C_0\, \alpha_1^2\, \frac{(\mathrm{Log}\, N)^2}{N^{1-\alpha_1}} \,.$$

Hence we get for fixed $\alpha_1 > 0$ and for $N \geq N_0(\alpha_1)$

$$\Big| \Big(\frac{1}{2} + \eta_N\Big)^{-1} Q_N(t) - \mathrm{erfc}(-\sqrt{N}\, t) \Big| \leq C_1\, \frac{(\mathrm{Log}\, N)^{5/2}}{N^{1-\alpha_1}} \,, \tag{32}$$

for $|t| \leq \sqrt{\alpha_1 \,\mathrm{Log}\, N/N}$, where C_1 depends only on α_1.

Now, assume that θ is such that $Q_N(\theta) = 0$ or $\mathrm{erfc}(-\sqrt{N}\,\theta) = 0$ and that $|\theta| \leq \sqrt{\alpha_1 \,\mathrm{Log}\, N/N}$; in every case we have

$$|\mathrm{erfc}(-\sqrt{N}\theta)| \leq C_1 \frac{(\mathrm{Log}\, N)^{5/2}}{N^{1-\alpha_1}} \,.$$

We are going to show that for δ_0 small enough, $\operatorname{erfc}(-\sqrt{N}\,\theta+z)$ is not too small on $|z|=\delta_0$. Indeed we have

$$\begin{aligned}|\operatorname{erfc}(-\sqrt{N}\,\theta+z)-\operatorname{erfc}(-\sqrt{N}\,\theta)| &= \frac{2}{\sqrt{\pi}}\Big|\int_0^z e^{-N\theta^2}e^{2\sqrt{N}\,\theta s}e^{-s^2}\,ds\Big| \\ &\geq \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\,|e^{-N\theta^2}|\,|z| \geq \frac{1}{\sqrt{\pi}}\,N^{-\alpha_1}|z|\,,\end{aligned}$$

provided that

$$|z|\leq \min\Big\{2\,\sqrt{\alpha_1\operatorname{Log}N},\,\frac{1}{8\,C_2\sqrt{\alpha_1\operatorname{Log}N}}\Big\}\,,$$

where $C_2=\max_{|w|\leq 1}|(e^w-1)/w|$.

Thus, if $|\theta|\leq\sqrt{\alpha_2\operatorname{Log}N/N}$, where $\alpha_2<\alpha_1<1/2$, and if N is big enough so that

$$\sqrt{\alpha_2\,\frac{\operatorname{Log}N}{N}}+\frac{1}{8\,C_2\sqrt{\alpha_1 N\operatorname{Log}N}}<\sqrt{\alpha_1\,\frac{\operatorname{Log}N}{N}}$$

and

$$C_1\sqrt{\pi}\,\frac{(\operatorname{Log}N)^{5/2}}{N^{1-2\alpha_1}}<\frac{1}{8\,C_2\sqrt{\alpha_1\operatorname{Log}N}}<2\sqrt{\alpha_1\operatorname{Log}N}\,,$$

we obtain that $Q_N(t)$ and $\operatorname{erfc}(-\sqrt{N}\,t)$ have the same number of zeros inside the open disk $D(\theta, C_1\sqrt{\pi}\,(\operatorname{Log}N)^{5/2}/N^{3/2-2\alpha_1})$ (by Rouché's theorem).

In order to conclude, we need some information on the zeros of $\operatorname{erfc}(z)$. A theorem by Fettis, Cuslin and Cramer ([3]) gives a development of γ_k

$$\begin{aligned}\gamma_k = e^{3i\pi/4}\Big(&\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi} \\ &-\frac{i}{2\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi}}\operatorname{Log}\Big(2\,\sqrt{\pi}\,\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi}\,\Big) \\ &+O\Big(\frac{(\operatorname{Log}k)^2}{k\,\sqrt{k}}\Big)\Big)\,.\end{aligned}\tag{33}$$

Thus if M_0 is a fixed number in $(-\pi/4, 3\pi/4)$, the number of roots γ of $\operatorname{erfc}(\gamma)=0$ such that $\operatorname{Im}\gamma\geq 0$ and $|\gamma|\leq\sqrt{2\,k\pi+M_0}$ is exactly k when k is large enough.

Now we may prove Result 4. Let $\varepsilon_0 < 1/2$ and $K=[\varepsilon_0 \operatorname{Log} N/(2\pi)]$. For each root t of $Q_N(s)$ such that $|\operatorname{Im} t| \geq 0$ and $|t| \leq \sqrt{2K\pi/N} \leq \sqrt{\varepsilon_0 \operatorname{Log} N/N}$ there is a root θ of $\operatorname{erfc}(-\sqrt{N}\, s)$ such that

$$|\theta - t| \leq C_1\sqrt{\pi}\, \frac{(\operatorname{Log} N)^{5/2}}{N^{3/2-2\varepsilon_1}}\ ,$$

(where $\varepsilon_0 < \varepsilon_1 < 1/2$ and $N \geq N_1(\varepsilon_1)$). Then we have

$$\begin{aligned}|\sqrt{N}\,\theta| &\leq \sqrt{2K\pi} + C_1\sqrt{\pi}\, \frac{(\operatorname{Log} N)^{5/2}}{N^{1-2\varepsilon_1}}\\ &\leq \sqrt{2K\pi} + \frac{\pi}{16\sqrt{2K\pi}}\\ &\leq \sqrt{\Big(2K+\frac{1}{8}\Big)\pi}\end{aligned}$$

provided that $N \geq N_2(\varepsilon_1)$. But we know that there are exactly $2K$ roots of $\operatorname{erfc}(-\sqrt{N}\, s)$ inside the disk $D(0, \sqrt{(2K+1/4)\pi}/\sqrt{N})$. Conversely, if θ is a root of $\operatorname{erfc}(-\sqrt{N}\, s)$ such that

$$|\theta| \leq \sqrt{\frac{2K\pi}{N}} - C_1\sqrt{\pi}\, \frac{(\operatorname{Log} N)^{5/2}}{N^{3/2} - 2\,\varepsilon_1} \leq \sqrt{\varepsilon_0\, \frac{\operatorname{Log} N}{N}}\ ,$$

there is a root t of $Q_N(s)$ such that

$$|\theta - t| \leq C_1\sqrt{\pi}\, \frac{(\operatorname{Log} N)^{5/2}}{N^{3/2-2\varepsilon_1}}\ ,$$

hence $|t| \leq \sqrt{2K\pi/N}$; moreover for $N \geq N_2(\varepsilon_1)$ we have

$$\sqrt{2K\pi} - C_1\sqrt{\pi}\, \frac{(\operatorname{Log} N)^{5/2}}{N^{1-2\varepsilon_1}} > \sqrt{2K\pi} - \frac{\pi}{16\sqrt{2K\pi}} > \sqrt{\Big(2K-\frac{1}{8}\Big)\pi}\ ,$$

so that we have again $2K$ roots of $\operatorname{erfc}(-\sqrt{N}\, s)$ such that

$$|\theta| \leq \sqrt{\frac{2K\pi}{N}} - C_1\sqrt{\pi}\, \frac{(\operatorname{Log} N)^{5/2}}{N^{1-2\varepsilon_1}}\ .$$

Finally, we conclude by noticing that (33) shows us that if $\operatorname{erfc}(-\sqrt{N}\theta_i) = 0$, $i = 1, 2$, $\theta_1 \neq \theta_2$ and $|\theta_i| \leq \sqrt{(2K+1/8)\pi/N}$ then $|\theta_1 - \theta_2| \geq$

$C_0/\sqrt{KN}$ and $|\operatorname{Im}\theta_i| \geq C_0\sqrt{K/N}$ for some positive C_0 which doesn't depend on K nor N; hence the balls

$$D\Big(\theta_i, C_1\sqrt{\pi}\,\frac{(\operatorname{Log} N)^{5/2}}{N^{3/2-2\varepsilon_1}}\Big)$$

are disjoint and don't meet the real axis (for N large enough). Thus (31) is proved, if we notice that

$$\frac{(\operatorname{Log} N)^{5/2}}{N^{1-2\varepsilon_1}} < \frac{1}{N^{1-2\varepsilon_1'}}$$

for $\varepsilon_1 < \varepsilon_1' < 1/2$ and N large enough.

3. Big roots of Q_N: first estimates.

In this section, we are going to devote our attention to formula (26). A straigthforward application of (26) is the following one:

Result 5. *For N large enough, if $t \neq -1$ and $Q_N(t)=0$, then $|1-t^2|>1$.*

PROOF. If $Q_N(t) = 0$, then we have $\sqrt{N}\int_t^1 (1-s^2)^N\,ds = \sqrt{\pi}\,(1+\eta_N)$ with $\eta_N = O\,(1/N^2)$. Now, since $\operatorname{Re} t > 0$ (due to Result 1), we may write

$$\begin{aligned}\int_t^1 (1-s^2)^N\,ds &= \int_0^{1-t^2} \omega^N\,\frac{d\omega}{2\sqrt{1-\omega}}\\ &= (1-t^2)^{N+1}\int_0^1 \lambda^N\,\frac{d\lambda}{2\sqrt{1-\lambda(1-t^2)}}\,.\end{aligned}$$

We write $\Omega = 1-t^2$. If $|\Omega| \leq 1$ then we will prove that

$$\inf_{\lambda\in[0,1]} |1-\lambda\Omega| \geq \frac{1}{2}\,|1-\Omega|\,.$$

This is obvious if $\operatorname{Re}\Omega \leq 0$: we have $|1-\lambda\Omega| \geq 1$ and $|1-\Omega| \leq 2$. If $\operatorname{Re}\Omega > 0$, $\Omega = \rho e^{i\varphi}$ $(0 < \rho \leq 1,\ \varphi \in (-\pi/2, \pi/2))$, we distinguish the case $\rho \leq \sin\varphi$ and $\rho > \sin\varphi$. If $\rho \leq \sin\varphi$, it is easily checked that $|1-\lambda\Omega| \geq |1-\Omega|$. If $\rho > \sin\varphi$, we have $|1-\lambda\Omega| \geq \sin\varphi$ and $|1-\Omega| \leq |1-e^{i\varphi}| = 2\,|\sin(\varphi/2)|$; hence

$$|1-\lambda\Omega| \geq \Big|\cos\frac{\varphi}{2}\Big|\,|1-\Omega| \geq \frac{\sqrt{2}}{2}\,|1-\Omega|\,.$$

Thus, we have for $\operatorname{Re} t > 0$ and $|1-t^2| \leq 1$

$$\Big|\int_t^1 (1-s^2)^N\,ds\Big| \leq \frac{|1-t^2|^{N+1}}{N+1}\,\frac{1}{|t|} \leq \frac{1}{\sqrt{N}}\Big(\frac{1}{\sqrt{N}\,|t|}\Big)\,.$$

If $|t\sqrt{N}| \geq 2/\sqrt{\pi}$, we get

$$\Big|\sqrt{N}\int_t^1 (1-s^2)^N\,ds\Big| \leq \frac{1}{2}\sqrt{\pi}\,,$$

and thus $Q_N(t) \neq 0$ (for N large enough so that $|\eta_N| < 1/2$). If $\sqrt{N}\,|t| \leq 2/\sqrt{\pi}$, then $t \sim -\overline{\gamma}/\sqrt{N}$ for a root γ of $\operatorname{erfc}(z)$ such that $|\gamma| \leq 2/\sqrt{\pi}$; but the roots of $\operatorname{erfc}(z)$ satisfy $\pi/2 < |\operatorname{Arg}\gamma| < 3\pi/4$ so that (for N large enough) $|\operatorname{Arg} t| > \pi/4$ and t cannot lie inside the lemniscate $|1-t^2| \leq 1$.

We may now enter the core of our computations. We are going to give a precise description of $\int_t^1 (1-s^2)^N\,ds$. Integration by parts gives us

$$\begin{aligned}\int_t^1 (1-s^2)^N\,ds &= \frac{(1-t^2)^{N+1}}{2\,t\,(N+1)} - \int_t^1 \frac{(1-s^2)^{N+1}}{2\,s^2(N+1)}\,ds\\ &= \frac{(1-t^2)^{N+1}}{2\,t\,(N+1)} - \frac{(1-t^2)^{N+2}}{4\,(N+1)}\int_0^1 \frac{\lambda^{N+1}\,d\lambda}{(1-\lambda\,(1-t^2))^{3/2}}\,.\end{aligned}$$

We then define $\eta(t)$ as

$$\eta(t) = \frac{|t^2|}{\inf\limits_{\lambda\in[0,1]} |1-\lambda\,(1-t^2)|}\,. \tag{34}$$

We have

$$\int_t^1 (1-s^2)^N\,ds = \frac{(1-t^2)^{N+1}}{2\,t\,(N+1)}\Big(1 + \frac{(1-t^2)}{2\,(N+2)\,t^2}\,\mu_N(t)\Big)\,, \tag{35}$$

for $\operatorname{Re} t > 0$ with

$$|\mu_N(t)| \leq \eta(t)^{3/2}. \tag{36}$$

Of course, (35) is a good formula if $\mu_N(t)$ cannot explode. As a matter of fact, we will show that in the neighbourhood of the roots of $Q_N(s)$ we have $|\eta(t)| \leq C_0$ where C_0 doesn't depend on N nor t; but we are still far from being able to prove it! The only obvious estimations on η are the following ones: if $\operatorname{Re} t^2 \geq 1$, we have of course $|\eta(t)| = |t^2|$, while if $\operatorname{Re} t^2 < 1$ and $|1 - t^2| > 1$ we have

$$|\eta(t)| = \frac{|t^2|}{|\sin(\operatorname{Arg}(1 - t^2))|}.$$

With help of formula (35) and a careful estimate of $\eta(t)$ in (36), we are going to prove:

Result 6. *Let $\varphi_{N,k} = (8k-1)\pi/(8N+6)$. Then for N large enough, the roots $x_{N,1}, \dots, x_{N,N}$ of Q_N such that $x_{N,k} \neq -1$, ordered by*

- *for $1 \leq k \leq [(N+1)/2]$, $\operatorname{Re} x_{N,k} \geq 0$ and $x_{N,N+1-k} = \overline{x_{N,k}}$*
- *$|x_{N,1}| < |x_{N,2}| < \cdots < |x_{N,[(N+1)/2]}|$*

satisfy

$$\begin{aligned} &\Big| x_{N,k} - \sqrt{2\sin\varphi_{N,k}}\, e^{i(\pi/4 - \varphi_{N,k})/2} \\ &\quad - \frac{e^{i(3\pi/4 - 3\varphi_{N,k}/2)}}{2N\sqrt{2\sin\varphi_{N,k}}} \operatorname{Log}\big(2\sqrt{2N\pi\sin\varphi_{N,k}}\big)\Big| \\ &\qquad \leq C\,\frac{1}{\sqrt{N}} \max\left\{\frac{(1+\operatorname{Log} k)^2}{k^{3/2}}, \frac{(1+\operatorname{Log} N+1-k)^2}{(N+1-k)^{3/2}}\right\}, \end{aligned} \tag{37}$$

where C doesn't depend on k nor N.

PROOF. Since $\varphi_{N,N+1-k} = \pi - \varphi_{N,k}$, it is enough to prove (37), for $1 \leq k \leq [(N+1)/2]$, *i.e.* for the roots which lie in the upper half-plane. The proof is decomposed in the following steps: one first proves that $\operatorname{Arg}(1 - x_{N,k}^2)$ cannot be too small, so that we have a first control on $\mu_N(x_{N,k})$; then one gives through (35) a first estimate on $x_{N,k}$ and on the related error; this gives us a more precise information on $\operatorname{Arg}(1 - x_{N,k}^2)$ and thus we may conclude with our final estimate.

Step 1. We want to estimate $\operatorname{Arg}(1-x_{N,k}^2)$. We fix $\theta_0 \in (\pi/4, \pi/2)$ so that the sector $\{z : \pi/2 \leq |\operatorname{Arg} z| \leq \pi - \theta_0\}$ contains no zero of $\operatorname{erfc}(z)$ (remember that $\lim_{k\to+\infty} \operatorname{Arg}\gamma_k = 3\pi/4$). We now distinguish the cases $\operatorname{Arg} x_{N,k} \in [0, \theta_0]$ and $\operatorname{Arg} x_{N,k} \in]\theta_0, \pi/2[$. If $\operatorname{Re} 1 - x_{N,k}^2 \leq 0$, we know that $\eta(x_{N,k}) \leq |x_{N,k}|^2 \leq 4$. If $\operatorname{Re} 1 - x_{N,k}^2 > 0$ and $\operatorname{Arg} x_{N,k} \in [0, \pi/4]$, then we see that $|x_{N,k}|^2 \leq |\tan \operatorname{Arg}(1 - x_{N,k}^2)|$ (because $\omega = 1 - x_{N,k}^2$ satisfies $\operatorname{Re}\omega \in (0,1]$ and $|\omega| > 1$ so that $|\sin\operatorname{Arg}\omega| \leq |1 - \omega| \leq |\tan\operatorname{Arg}\omega|$); moreover we have $|x_{N,k}|^2 \leq 4$; thus if $|\tan(\operatorname{Arg}(1 - x_{N,k}^2)| \leq 4$, then we have

$$|\sin(\operatorname{Arg}(1-x_{N,k}^2))| = \frac{|\tan(\operatorname{Arg}(1-x_{N,k}^2)|}{\sqrt{1+\tan^2(\operatorname{Arg}(1-x_{N,k}^2))}} \geq \frac{|x_{N,k}|^2}{\sqrt{17}}$$

and $\eta(x_{N,k}) \leq \sqrt{17}$. On the other hand, if $|\tan(\operatorname{Arg}(1-x_{N,k}^2))| \geq 4$, then we have $|\operatorname{Arg}(1-x_{N,k}^2)| \in [\operatorname{Arg}\tan 4, \pi/2]$ and thus

$$|\sin(\operatorname{Arg}(1-x_{N,k}^2))| \geq \sin\operatorname{Arg}\tan 4 = \frac{4}{\sqrt{17}} \geq \frac{|x_{N,k}|^2}{\sqrt{17}}$$

and $\eta(x_{N,k}) \leq \sqrt{17}$ again.

If $\operatorname{Arg}(x_{N,k}) \in [\pi/4, \theta_0]$, we have

$$|\operatorname{Im}(1-x_{N,k}^2)| = |x_{N,k}^2|\,|\sin 2\operatorname{Arg} x_{N,k}|$$

so that

$$|\operatorname{Im}(1-x_{N,k}^2)| \geq |x_{N,k}|^2\,|\sin 2\,\theta_0|\,,$$

while

$$|\sin\operatorname{Arg}(1-x_{N,k}^2)| = \frac{|\operatorname{Im}(1-x_{N,k}^2)|}{|1-x_{N,k}^2|} \geq \frac{1}{3}\,|\operatorname{Im}(1-x_{N,k}^2)|\,,$$

so that

$$\eta(x_{N,k}) \leq \frac{3}{|\sin 2\,\theta_0|}\,.$$

The difficult case is when $\theta_0 \leq \operatorname{Arg} x_{N,k} \leq \pi/2$ (as a matter of fact, we will see in step 3 that this case never occurs when N is big enough!). For the moment, we will show that we have necessarily for such an $x_{N,k}$ (and provided N is large enough) the inequality

$$N\,|x_{N,k}|^4 \geq \frac{|\cos\theta_0|}{100\,C_0^2} = \varepsilon_1\,,$$

where C_0 is given by

$$C_0 = \max\Big\{ \sup_{|\sigma|\le 1/2} \Big|\frac{\sigma^2 + \operatorname{Log}(1-\sigma^2)}{\sigma^4}\Big|,\ \sup_{|\sigma|\le 1} \frac{|e^\sigma - 1|}{|\sigma|}\Big\}.$$

Indeed, let $A_0 > 0$ be large enough so that for $A \ge A_0$, $e^{3A^2\cos(2\theta_0)/4}(1 + A^2/2) < 1/100$ (remember that $\cos 2\theta_0 < 0$), $4/(A^2|\cos 2\theta_0|) < 1/100$ and $A\, e^{A^2\cos(2\theta_0)/4} < 1/100$. If $\sqrt{N}\,|x_{N,k}| \ge A_0$ and $N\,|x_{N,k}|^4 \le \varepsilon_1$, we write

$$Q_N(x_{N,k}) = \frac{1}{2} + \Big(1 + O\Big(\frac{1}{N^2}\Big)\Big)\sqrt{\frac{N}{\pi}}\int_0^{x_{N,k}} (1-s^2)^N\, ds$$

and thus

$$|Q_N(x_{N,k})| \ge \frac{1}{10}\sqrt{N}\,\Big|\int_0^{x_{N,k}} (1-s^2)^N\, ds\Big| - \frac{1}{2}\,.$$

We write

$$(1-s^2)^N = e^{-Ns^2}\, e^{N(s^2 - \operatorname{Log}(1-s^2))}\,,$$

since $|s| \le \sqrt{\varepsilon_1/N}/4$, we have $|s| \le 1/2$ for N large enough, thus

$$|N\,(s^2 - \operatorname{Log}(1-s^2)| \le C_0|N\,s^4| \le \frac{1}{100}\,,$$

thus

$$|e^{N(s^2-\operatorname{Log}(1-s^2))} - 1| \le C_0^2\,|N\,s^4|\,.$$

Thus, writing $x_{N,k} = \rho_{N,k}\, e^{i\theta_{N,k}}$, we get

$$\begin{aligned}
|Q_N(x_{N,k})| &\ge \frac{1}{10}\Big|\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\, ds\Big| \\
&\qquad - \frac{C_0^2}{10}\int_0^{\sqrt{N}\rho_{N,k}} e^{-s^2\cos 2\theta_{N,k}}\,\frac{s^4}{N}\, ds - \frac{1}{2} \\
&\ge \frac{1}{10}\Big|\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\, ds\Big| \\
&\qquad - \frac{C_0^2}{10}\,\frac{(\sqrt{N}\,\rho_{N,k})^3}{N\,|\cos 2\,\theta_{N,k}|}
\end{aligned}$$

$$\cdot \int_0^{\sqrt{N}\rho_{N,k}} e^{-s^2 \cos 2\,\theta_{N,k}} s\,|\cos 2\,\theta_{N,k}|\, ds - \frac{1}{2}$$

$$\geq \frac{1}{10}\Big|\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\, ds\Big|$$

$$- \frac{e^{-N\rho_{N,k}^2 \cos 2\theta_{N,k}}}{2\sqrt{N}\,\rho_{N,k}}\Big(\frac{C_0^2\,(\sqrt{N}\,\rho_{N,k})^4}{10\,|\cos 2\,\theta_0|}\Big) - \frac{1}{2}\ .$$

We have now to estimate $\int_0^{\sqrt{N}x_{N,k}} e^{-s^2} ds$. We write

$$\begin{aligned}
&\int_0^{\sqrt{N}x_{N,k}} e^{-s^2}\, ds \\
&= e^{i\theta_{N,k}}\Big(\int_0^{\sqrt{N}\rho_{N,k}/2} e^{-s^2 e^{2i\theta_{N,k}}}\, ds + \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} e^{-s^2 e^{2i\theta_{N,k}}}\, ds\Big) \\
&= e^{i\theta_{N,k}}(I_1 + I_2)\,.
\end{aligned}$$

We have $|I_1| \leq e^{-N\rho_{N,k}^2 \cos(2\theta_{N,k})/4}\rho_{N,k}\sqrt{N}/2$, while

$$\begin{aligned}
I_2 &= \Big[\frac{e^{-s^2 e^{2i\theta_{N,k}}}}{-2\,s\,e^{2i\theta_{N,k}}}\Big]_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} - \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} \frac{e^{-s^2 e^{2i\theta_{N,k}}}}{2\,s^2 e^{2i\theta_{N,k}}}\, ds \\
&= \frac{e^{-N\rho_{N,k}^2\, e^{2i\theta_{N,k}}}}{-2\sqrt{N}\,\rho_{N,k}\, e^{2i\theta_{N,k}}} - \frac{e^{-N\rho_{N,k}^2 e^{2i\theta_{N,k}/4}}}{-\sqrt{N}\,\rho_{N,k}\, e^{2i\theta_{N,k}}} - I_3\ .
\end{aligned}$$

We have

$$|I_3| \leq \frac{1}{4\Big(\frac{1}{2}\sqrt{N}\,\rho_{N,k}\Big)^3 |\cos 2\,\theta_{N,k}|}$$

$$\cdot \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} e^{-s^2 \cos 2\theta_{N,k}}\, 2\,s\,|\cos 2\,\theta_{N,k}|\, ds$$

$$\leq \frac{e^{-N\rho_{N,k}^2 \cos 2\theta_{N,k}}}{4\Big(\frac{1}{2}\sqrt{N}\,\rho_{N,k}\Big)^3 |\cos 2\,\theta_0|}\ .$$

Thus we get

$$
\begin{aligned}
&|Q_N(x_{N,k})| \\
&\quad \geq \frac{1}{10}\,\frac{e^{-N\rho_{N,k}^2\cos 2\theta_{N,k}}}{2\sqrt{N}\,\rho_{N,k}} \\
&\qquad \cdot \Big(1 - 2\,e^{3N\rho_{N,k}^2\cos 2\theta_{N,k}/4} - \frac{4}{N\,\rho_{N,k}^2|\cos 2\theta_0|} \\
&\qquad\quad - N\,\rho_{N,k}^2\,e^{3N\rho_{N,k}^2\cos 2\theta_{N,k}/4} - \frac{C_0^2\,\varepsilon_1}{|\cos 2\,\theta_0|} \\
&\qquad\quad - 10\,\sqrt{N}\,\rho_{N,k}\,e^{N\rho_{N,k}^2\cos 2\theta_{N,k}}\Big) \\
&\quad \geq \frac{1}{10}\,\frac{e^{-N\rho_{N,k}^2\cos 2\theta_{N,k}}}{2\sqrt{N}\,\rho_{N,k}}\Big(1 - \frac{2}{100} - \frac{1}{100} - \frac{1}{100} - \frac{1}{100} - \frac{10}{100}\Big) > 0\,,
\end{aligned}
$$

which contradicts $Q_N(x_{N,k}) = 0$. Up to now, we have proved that if $\arg x_{N,k} > \theta_0$ then either $\sqrt{N}\,|x_{N,k}| \leq A_0$ or $N\,|x_{N,k}|^4 \geq \varepsilon_1$. But if $|x_{N,k}| \leq A_0/\sqrt{N}$ and N is large enough, Result 4 ensures that $-\sqrt{N}\,x_{N,k}$ is close to a zero of $\operatorname{erfc}(z)$. This is not possible for N large enough since the distance between $\{z : \pi/2 \leq |\operatorname{Arg} z| \leq \pi - \theta_0\}$ and $\{z : \operatorname{erfc}(z) = 0\}$ is positive.

Thus we must have $N\,|x_{N,k}|^4 \geq \varepsilon_1$. Write again $x_{N,k} = \rho_{N,k}\,e^{i\theta_{N,k}}$; since $|x_{N,k} - 1| \leq 1$ by Result 1, we have $\rho_{N,k} \leq 2\cos\theta_{N,k}$; thus $2\cos\theta_{N,k} \geq (\varepsilon_1/N)^{1/4}$ and

$$
|\operatorname{Im} x_{N,k}^2| = |x_{N,k}^2|\,|\sin 2\,\theta_{N,k}| \geq \sin\theta_0\Big(\frac{\varepsilon_1}{N}\Big)^{1/4}|x_{N,k}|^2\,.
$$

We thus have proved

$$
\eta(x_{N,k}) = \frac{|x_{N,k}|^2|1 - x_{N,k}^2|}{|\operatorname{Im} x_{N,k}^2|} \leq \frac{3\,N^{1/4}}{(\sin\theta_0)\,\varepsilon_1^{1/4}} = C_1^{N^{1/4}}\,.
$$

We thus have proved

- if $\operatorname{Arg} x_{N,k} < \theta_0$,

$$
|\mu_N(x_{N,k})| \leq \eta(x_{N,k})^{3/2} \leq C_2\,,
$$

- if $\operatorname{Arg} x_{N,k} > \theta_0$,

$$\begin{aligned} |\mu_N(x_{N,k})| &\le \eta(x_{N,k})^{3/2} \\ &\le (C_1 N^{1/4})^{3/2} \\ &= C_1^{3/2} \, \frac{(N\,|x_{N,k}^2)^{3/4}}{(N\,(|x_{N,k}|^4)^{3/8}} \\ &\le \frac{C_1^{3/2}}{\varepsilon_1^{3/8}} \, (N\,|x_{N,k}|^2)^{3/4} \,. \end{aligned}$$

In any case, we have

$$|\mu_N(x_{N,k})| \le C\,(N\,|x_{N,k}|^2)^{3/4} \,. \tag{38}$$

(Remember that $\lim_{N\to+\infty} \inf_k N\,|x_{N,k}|^2 = |\gamma_1|^2 > 0$).

Step 2. We are now able to give an estimate for $x_{N,k}$. Let us consider a root $y \neq -1$ of Q_N such that $\operatorname{Im} y \ge 0$. We have

$$\int_y^1 (1-s^2)^N \, ds = 2\, \frac{4^N (N!)^2}{(2N+1)!} \,,$$

hence from (35) and (36),

$$\frac{(1-y^2)^{N+1}}{2\,(N+1)\sqrt{\pi}\, y} \Big(1 + O\Big(\frac{\eta(y)^{3/2}}{N\,|y|^2}\Big)\Big) = \sqrt{\frac{\pi}{N}}\, \Big(1 + O\Big(\frac{1}{N^2}\Big)\Big) \,, \tag{39}$$

(where $\alpha = O\,(\varepsilon(N,y))$ means that $|\alpha|/\varepsilon(N,y) \le C$ for a positive constant C which doesn't depend neither on N nor on y). Taking the $(N+1)$-th root of the modulus of both terms of equality (39), we get

$$\begin{aligned} |1-y^2| &= 1 + \frac{1}{N+1} \operatorname{Log}\Big(2\sqrt{N\pi}\, \frac{N+1}{N}\, |y|\Big) \\ &\quad + O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{1}{N^3}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2 |y|^2}\Big) \\ &= 1 + \frac{1}{N} \operatorname{Log}\,(2\sqrt{N\pi}\,|y|) + O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2 |y|^2}\Big) \,. \end{aligned}$$

Now, we write $1-y^2 = \rho\, e^{-i\varphi}$ ($\varphi \in [0,\pi]$, $\rho > 0$), so that $y = \sqrt{1-\rho\, e^{-i\varphi}}$. We have found

$$
\begin{aligned}
|1-\rho| &= O\Big(\frac{1}{N}\,\mathrm{Log}\,(\sqrt{N}\,|y|)\Big) + O\Big(\frac{(\mathrm{Log}\,N)^2}{N^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2\,|y|^2}\Big) \\
&= O\Big(\frac{1}{N}\,\mathrm{Log}\,(\sqrt{N}\,|y|)\Big)\,,
\end{aligned}
$$

(since $1/CN \le \mathrm{Log}\,(\sqrt{N}\,|y|)/N \le C\,\mathrm{Log}\,N/N$, while $\eta(y)^{3/2}/(N^2|y|^2) \le C/(N(N|y|^2)) \le C'/N$). Thus $1-\rho\, e^{-i\varphi} = 1-e^{-i\varphi} + (1-\rho)\, e^{-i\varphi}$ with

$$
\Big|\frac{(1-\rho)\, e^{-i\varphi}}{1-\rho\, e^{-i\varphi}}\Big| = O\Big(\frac{\mathrm{Log}\,(\sqrt{N}\,|y|)}{N|y|^2}\Big)
$$

and we find

$$
\begin{aligned}
y &= \sqrt{(1-e^{-i\varphi})\Big(1+O\Big(\frac{\mathrm{Log}\,\sqrt{N}\,|y|}{N|y|^2}\Big)\Big)} \\
&= \sqrt{2\sin\Big(\frac{\varphi}{2}\Big)}\; e^{i(\pi/4-\varphi/4)}\Big(1+O\Big(\frac{\mathrm{Log}\,\sqrt{N}\,|y|}{N|y|^2}\Big)\Big)\,.
\end{aligned}
$$

We insert this result in (39) and take the phase

$$
-(N+1)\,\varphi - \frac{\pi}{4} + \frac{\varphi}{4} + O\Big(\frac{\mathrm{Log}\,\sqrt{N}\,|y|}{N|y|^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N|y|^2}\Big) = -2k\pi
$$

or

$$
\varphi = \frac{8k-1}{4N+3}\,\pi + O\Big(\frac{\mathrm{Log}\,\sqrt{N}\,|y|}{N^2|y|^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2|y|^2}\Big)\,. \tag{40}
$$

If we assume $\sqrt{N}\,|y| \ge A_0$ where A_0 is big enough so that

$$
O\Big(\frac{\mathrm{Log}\,A_0}{NA_0^2}\Big) + O\Big(\frac{1}{NA_0^{1/2}}\Big)
$$

is less than $4\pi/(4N+3)$ (A_0 being chosen independently from N), we see that $0 \le \varphi \le \pi$ implies $0 \le k \le [(N+1)/2]$; moreover since

$$
|y| = \sqrt{2\sin\Big(\frac{\varphi}{2}\Big)}\,\Big(1+O\Big(\frac{\mathrm{Log}\,\sqrt{N}\,|y|}{N|y|^2}\Big)\Big)
$$

we must have

$$2\sin\Big(\frac{\varphi}{2}\Big) \geq \frac{A_0^2}{N} + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N^2|y|^2}\Big)\,.$$

We take $A_0^2 = \sqrt{2K_0\pi}$, where K_0 is big enough; we then see that we must have $k > K_0$.

If $\sqrt{N}\,|y| \leq \sqrt{2K_0\pi}$, we know that (provided N is big enough) $y \sim -\overline{\gamma}_k/\sqrt{N}$ for $k \in \{1, \dots, K_0\}$. We have moreover found candidates $y_{N,k}$ for the remaining roots $x_{N,k}$, $K_0 < k \leq [(N+1)/2]$, which are given by

$$(41)\qquad 1 - y_{N,k}^2 = \Big(1 + \frac{1}{N}\operatorname{Log} 2\,\sqrt{2N\pi \sin\varphi_{N,k}}\Big)e^{-2i\varphi_{N,k}}\,,$$

for $K_0 < k \leq [(N+1)/2]$ and $\varphi_{N,k} = (8k-1)\pi/(8N+6)$.

More precisely, we have shown that if $Q_N(y) = 0$, $\operatorname{Im} y \geq 0$, $y \neq -1$ and $\sqrt{N}\,|y| \geq \sqrt{2K_0\pi}$, then for some $k \in \{K_0+1, \dots, [(N+1)/2]\}$ we have

$$(42)\quad 1-y^2 = 1-y_{N,k}^2 + O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2|y|^2}\Big) + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N^2|y|^2}\Big)\,.$$

We are going now to prove that, provided that K_0 is fixed large enough (and provided thereafter that N is large enough), for each $y_{N,k}$ there is exactly one root y satisfying (42). Notice that $|y_{N,k}^2 - y_{N,k+1}^2| \geq C_0/N$ while

$$O\Big(\frac{(\operatorname{Log} N)^2}{N^2}\Big) + O\Big(\frac{\eta(y)^{3/2}}{N^2|y|^2}\Big) + O\Big(\frac{\operatorname{Log}\sqrt{N}\,|y|}{N^2|y|^2}\Big) \leq C\,\frac{1}{N}\Big(\frac{(\operatorname{Log} N)^2}{N} + \frac{1}{(\sqrt{N}\,|y|)^{1/2}}\Big)\,.$$

Indeed, let's write $s = \sqrt{y_{N,k}^2 - v}$ where $|v| = \eta_0/N$, η_0 small enough. We are going to estimate $Q_N(s)$. We know that

$$\int_s^1 (1-\sigma^2)^N\,d\sigma = \frac{(1-s^2)^{N+1}}{2\,s\,(N+1)}\,\Big(1 + O\Big(\frac{\eta(s)}{N|s|^2}\Big)\Big)\,,$$

where $\eta(s)$ is bounded independently of s provided that $|1-s| < 1$, $|1-s^2| > 1$ and $|\operatorname{Arg} s| < \theta_0$ (where $\theta_0 \in (\pi/4, \pi/2)$). Thus, we are going to estimate $|1-s|$, $|1-s^2|$ and $|\operatorname{Arg} s|$. We have obviously from (41)

$$y_{N,k}^2 = 1 - e^{-2i\varphi_{N,k}} + O\Big(\frac{\operatorname{Log} k}{N}\Big) = (1 - e^{-2i\varphi_{N,k}})\Big(1 + O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big)$$

and such an estimate holds as well for s^2. (We see also from (41) that

$$\begin{aligned} |1-s^2| &\geq 1 + \frac{1}{N} \operatorname{Log} 2\sqrt{2N\pi \sin \varphi_{N,k}} - \frac{\eta_0}{N} \\ &\geq 1 + \frac{1}{N} \operatorname{Log} 2\sqrt{4\pi K_0} - \frac{\eta_0}{N} \\ &> 1 \end{aligned}$$

provided η_0 is small enough). Thus we find that

$$\operatorname{Arg} s^2 = \frac{\pi}{2} - \varphi_{N,k} + O\Big(\frac{\operatorname{Log} k}{k}\Big) < 2\,\theta_0 \ ,$$

if K_0 is large enough (so that $O(\operatorname{Log} K_0/K_0) < 2\,\theta_0 - \pi/2$) and thus

$$\operatorname{Arg} s = \frac{\pi}{4} - \frac{1}{2}\,\varphi_{N,k} + O\Big(\frac{\operatorname{Log} k}{k}\Big) \in (-\theta_0, \theta_0)\,.$$

Moreover,

$$|s| = \sqrt{2\sin\varphi_{N,k}}\Big(1 + O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big)$$

and this latter estimate gives $|s| < 2\cos(\operatorname{Arg} s)$: if $\varphi_{N,k} > \varepsilon_0$ (where ε_0 is fixed small enough as we shall see below) and K_0 and N are large enough we have

$$\sqrt{2\sin\varphi_{N,k}}\,\Big(1 + O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big) \leq \sqrt{2}\,\Big(1 + C\,\frac{\operatorname{Log} K_0}{K_0}\Big) \leq \sqrt{2}\,\Big(1 + \frac{\varepsilon_0}{100}\Big)\,,$$

while

$$\begin{aligned} 2\cos\left(\operatorname{Arg} s\right) &\geq 2\cos\Big(\frac{\pi}{4} - C\,\frac{\operatorname{Log} K_0}{K_0}\Big) \\ &\geq 2\cos\Big(\frac{\pi}{4} - \frac{\varepsilon_0}{3}\Big) \\ &\geq \sqrt{2}\,\Big(1 + \frac{2\,\varepsilon_0}{3\pi} - \frac{\varepsilon_0^2}{2}\Big)\,. \end{aligned}$$

On the other hand, if $\varphi_{N,k} < \varepsilon_0$ we find

$$\sqrt{2\sin\varphi_{N,k}}\Big(1+O\Big(\frac{\operatorname{Log} k}{k}\Big)\Big) \le \sqrt{2\,\varepsilon_0}\,\sqrt{1+C\,\frac{\operatorname{Log} K_0}{K_0}} \le C'\sqrt{\varepsilon_0}\,,$$

while $2\cos(\operatorname{Arg} s) \ge 2\cos\theta_0$; thus if ε_0 is small enough to ensure $\varepsilon_0 < 4/(3\pi) - 1/50$ and $\varepsilon_0 < 4\cos^2\theta_0/C'^2$ we find $|s| < 2\cos(\operatorname{Arg} s)$. But this latter inequality is equivalent to $|1-s|<1$. Thus we found

$$Q_N(s) = 1 - \Big(1+O\Big(\frac{1}{N^2}\Big)\Big)\sqrt{\frac{N}{\pi}}\;\frac{(1-s^2)^{N+1}}{2\,s\,(N+1)}\Big(1+O\Big(\frac{1}{|Ns^2|}\Big)\Big)\,.$$

We have moreover:

$$\begin{aligned}(1-s^2)^{N+1} &= (1-y_{N,k}^2)^{N+1}\Big(1+\frac{v}{1-y_{N,k}^2}\Big)^{N+1}\\ &= (1-y_{N,k}^2)^{N+1}\Big(1+\frac{Nv}{1-y_{N,k}^2}+O\,(N^2v^2)\Big)\end{aligned}$$

$$s=\sqrt{y_{N,k}^2 - v} = y_{N,k}\Big(1-\frac{v}{2\,y_{N,k}^2}+O\Big(\frac{v^2}{y_{N,k}^4}\Big)\Big)\,.$$

This gives, since $|s|$ has $\sqrt{k/N}$ as order of magnitude

$$\begin{aligned}Q_N(s) = 1 - &\Big(1+O\Big(\frac{1}{k}\Big)\Big)\,\frac{(1-y_{N,k}^2)^{N+1}}{2\sqrt{N\pi}\,y_{N,k}}\\ &\cdot\Big(1+\frac{Nv}{1-y_{N,k}^2}+\frac{v}{2\,y_{N,k}^2}+O\,(N^2v^2)+O\Big(\frac{v^2}{y_{N,k}^4}\Big)\Big)\,.\end{aligned}$$

Moreover

$$|y_{N,k}| \ge 2\sqrt{\frac{8k-1}{8N+6}}\,\Big(1+O\Big(\frac{1}{k}\operatorname{Log} k\Big)\Big)$$

and

$$y_{N,k} = \sqrt{2\sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)}\;e^{i(\pi/4-(8k-1)\pi/(16N+12))}\Big(1+O\Big(\frac{1}{k}\operatorname{Log} k\Big)\Big)\,,$$

so that

$$\begin{aligned}
&\frac{(1-y_{N,k}^2)^{N+1}}{2\sqrt{N\pi}\,y_{N,k}}\\
&\quad=\frac{\left(1+\dfrac{1}{N}\operatorname{Log}2\sqrt{2N\pi\sin\Big(\dfrac{8k-1}{8N+6}\,\pi\Big)}\,\right)^N}{2\sqrt{2N\pi\sin\Big(\dfrac{8k-1}{8N+6}\,\pi\Big)}}\,\Big(1+O\Big(\frac{1}{k}\operatorname{Log}k\Big)\Big)\\
&\quad=\Big(1+O\Big(\frac{1}{N^2}\,(\operatorname{Log}k)^2\Big)\Big)^N\Big(1+O\Big(\frac{1}{k}\operatorname{Log}k\Big)\Big)
\end{aligned}$$

and finally

$$\begin{aligned}
Q_N(s)=1-&\Big(1+O\Big(\frac{1}{k}\,(\operatorname{Log}k)^2\Big)\Big)\\
&\cdot\Big(1+\frac{Nv}{1-y_{N,k}^2}+\frac{v}{2\,y_{N,k}^2}+O\,(N^2v^2)+O\Big(\frac{v^2}{y_{N,k}^4}\Big)\Big)\,.
\end{aligned}$$

Now, we write

$$R_{N,k}(s)=N\,\frac{v}{1-y_{N,k}^2}=N\,\frac{y_{N,k}^2-s^2}{1-y_{N,k}^2}\,.$$

Since $|v|=\eta_0/N$, we have

$$|R_{N,k}(s)|=\eta_0\Big(1+O\Big(\frac{\operatorname{Log}k}{N}\Big)\Big)\,,$$

while

$$|Q_N(s)-R_{N,k}(s)|=O\Big(\frac{(\operatorname{Log}k)^2}{k}\Big)+O\Big(\frac{\eta_0}{k}\Big)+O\,(\eta_0^2)\,.$$

We choose η_0 small enough to ensure that the $O\,(\eta_0^2)$ term is smaller than $\eta_0/2$ (independently of N and k), and then choose K_0 large enough to ensure that $O\,((\operatorname{Log}k)^2/k)+O\,(\eta_0/k)$ is smaller than $\eta_0/4$ for $k>K_0$. For this choice of K_0, we get

$$|Q_N(s)-R_{N,k}(s)|<\frac{3}{4}\,\eta_0<|R_{N,k}(s)|\,.$$

Thus, by Rouché's theorem, $Q_N(s)$ and $R_{N,k}(s)$ have the same number of roots inside the domain $\{|y^2_{N,k} - s^2| \le \eta_0/N,\ \mathrm{Re}\, s > 0\}$.

Step 3. We have thus found a number K_0 so that for N large enough we may list the roots $x_{N,1}, \dots, x_{N,[(N+1)/2]}$ of Q_N with $x_{N,k} \neq -1$, $\mathrm{Im}\, x_{N,k} \ge 0$, $|x_{N,k}| < |x_{N,k+1}|$ in the following way:

- for $k \le K_0$, $|x_{N,k}| < \sqrt{2K_0\pi/N}$ and $x_{N,k} \sim -\overline{\gamma}_k/\sqrt{N}$,
- for $k \ge K_0$,

$$|x^2_{N,k} - y^2_{N,k}| = O\Big(\frac{\eta(x_{N,k})^{3/2}}{N^2|x_{N,k}|^2}\Big) + O\Big(\frac{\mathrm{Log}\,(\sqrt{N}\,|x_{N,k}|)}{N^2|x_{N,k}|^2}\Big)\,,$$

where $y_{N,k}$ is given by (41).

Moreover, we have seen in step 2 that in that case we must have $\mathrm{Arg}\, x_{N,k} < \theta_0$, hence $\eta(x_{N,k})$ is bounded independently of N and k. Moreover $x_{N,k}$ is of order of magnituge $\sqrt{k/N}$, hence

$$|x^2_{N,k} - y^2_{N,k}| = O\Big(\frac{\mathrm{Log}\, k}{Nk}\Big)\,.$$

Thus we find

$$\begin{aligned} 1 - x^2_{N,k} = \Big(1 + \frac{1}{N}\,\mathrm{Log}\, 2\,\sqrt{2N\pi \sin\Big(\frac{8k-1}{8N+6}\pi\Big)}\,\Big) \\ \cdot e^{-2i\pi(8k-1)/(8N+6)} + O\Big(\frac{\mathrm{Log}\, k}{Nk}\Big) \end{aligned} \tag{43}$$

and thus

$$\begin{aligned} x^2_{N,k} = {}& \Big(1 - e^{-2i\pi(8k-1)/(8N+6)}\Big) \\ & \cdot \Big(1 - \frac{e^{-2i\pi(8k-1)/(8N+6)}}{N(1 - e^{-2i\pi(8k-1)/(8N+6)})}\,\mathrm{Log}\, 2\,\sqrt{2N\pi \sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)} \\ & \quad + O\Big(\frac{\mathrm{Log}\, k}{k^2}\Big)\Big) \end{aligned}$$

which gives

$$
\begin{aligned}
x_{N,k} &= e^{i(\pi/4-(8k-1)\pi/(16N+12))}\sqrt{2\sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)} \\
&\quad\cdot\left(1+\frac{e^{i(\pi/2-(8k-1)\pi/(8N+6))}}{4N\sin\Big(\dfrac{8k-1}{8N+6}\,\pi\Big)}\operatorname{Log}2\,\sqrt{2N\pi\sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)}\right. \\
&\qquad\left.+\,O\Big(\frac{\operatorname{Log}k}{k^2}\Big)\right),
\end{aligned}
\tag{44}
$$

which gives (37) for $k > K_0$. For $k \le K_0$, (37) says only that $x_{N,k}$ is $O\,(1/\sqrt{N})$, which we already known since $\sqrt{N}\,|x_{N,k}| \le \sqrt{2K_0\pi}$.

Thus we have proved Result 6.

A nice corollary of Result 6 is that we may recover formula (33) on the roots of $\operatorname{erfc}(z)$:

Corollary. *The k-th root γ_k of* $\operatorname{erfc}(z)$ *such that* $\operatorname{Im}\gamma_k > 0$ *is given by*

$$
\begin{aligned}
\gamma_k &= e^{3i\pi/4}\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi} \\
&\quad\cdot\left(1-\frac{i}{2\Big(2k-\dfrac{1}{4}\Big)\pi}\operatorname{Log}2\sqrt{\pi}\,\sqrt{\Big(2\,k-\frac{1}{4}\Big)\pi}+O\Big(\frac{(\operatorname{Log}k)^2}{k^2}\Big)\right).
\end{aligned}
\tag{45}
$$

PROOF. It is enough to use formula (37) for $x_{N,k}$ with $N, k \to +\infty$ and $k < \operatorname{Log}N/8$: we have

$$
x_{N,k} = -\frac{-\overline{\gamma}_k}{\sqrt{N}} + O\Big(\frac{1}{N}\Big) \qquad \text{and} \qquad \frac{k}{N} = O\Big(\frac{\operatorname{Log}N}{N}\Big),
$$

thus we find γ_k. The only thing to check is the exact number of roots γ such that $|\gamma| \le \sqrt{2K_0\pi}$ (since we used formula (33) to give it). But this is an old and classical result of Nevanlinna [9], and thus we may recover formula (33) from formula (37).

4. Big roots of Q_N: further estimates.

Though Result 6 is enough for the proof of theorems 1 to 3 (provided we improve result n° 4 for the smaller roots), we may give even more precise estimations for the roots $x_{N,k}$. For instance, we may integrate by parts one step further formula (35) and thus get an $O\,((\mathrm{Log}\,k)^3/Nk^2)$ error instead of $O\,(\mathrm{Log}\,k/Nk)$ for $1-x_{N,k}^2$.

More generally, how far can we compute $\int_t^1 (1-s^2)^N\,ds$? We have

$$\int_t^1 (1-s^2)^N\,ds = (1-t^2)^{N+1}\int_0^1 \lambda^N \frac{d\lambda}{2\sqrt{1-\lambda(1-t^2)}}\,.$$

If we write

$$1-\lambda(1-t^2) = t^2\Big(1+\frac{1-t^2}{t^2}\,(1-\lambda)\Big)\,,$$

we see that if $\mathrm{Re}\,t^2 > 1/2$ (so that $|1-t^2| < t^2$), we may develop $(\sqrt{1-\lambda(1-t^2)})^{-1}$ as a Taylor series in $(1-\lambda)$ and find (for $\mathrm{Re}\,t^2 > 1/2$)

$$\frac{1}{\sqrt{1-\lambda(1-t^2)}} = \frac{1}{t}\sum_{k=0}^{+\infty}(-1)^k\,\frac{2k!}{4^k(k!)^2}\Big(\frac{(1-\lambda)\,(1-t^2)}{t^2}\Big)^k,$$

which gives

$$(46)\quad \begin{cases} \text{for } \mathrm{Re}\,t>0 \text{ and } \mathrm{Re}\,t^2 > \dfrac{1}{2}\,, \\[2mm] \displaystyle\int_t^1 (1-s^2)^N\,ds \\[2mm] \displaystyle\qquad = \frac{(1-t^2)^{N+1}}{2\,t}\sum_{k=0}^{+\infty}(-1)^k\,\frac{(2k)!}{4^k(k!)^2}\,\frac{N!\,k!}{(N+k+1)!}\Big(\frac{1-t^2}{t^2}\Big)^k. \end{cases}$$

Unfortunately, we are mostly interested in small t's (remember that $x_{N,k} = O\,(\sqrt{k/N})$). (46) has to be replaced by an asymptotic formula (which is obtained by repeatedly integrating by parts)

$$(47)\quad \begin{cases} \text{for } \mathrm{Re}\,t>0 \text{ and } M\in\mathbb{N}, \\[2mm] \displaystyle\int_t^1 (1-s^2)^N\,ds \\[2mm] \displaystyle\qquad = \frac{(1-t^2)^{N+1}}{2\,t}\sum_{k=0}^{M}(-1)^k\,\frac{(2k)!}{4^k(k!)^2}\,\frac{N!\,k!}{(N+k+1)!}\Big(\frac{1-t^2}{t^2}\Big)^k \\[2mm] \qquad\quad +R_{M,N}(t)\,, \end{cases}$$

where the remainder

$$R_{M,N}(t) = (-1)^{M+1}(1-t^2)^{N+M+2}\,\frac{(2M+2)!}{4^{M+1}((M+1)!)^2}$$
$$\cdot\,\frac{N!\,(M+1)!}{(N+M+2)!}\int_0^1 \frac{\lambda^{N+M+1}\,d\lambda}{(1-\lambda(1-t^2))^{1/2+M+1}}$$

may be estimated by

$$\begin{aligned}|R_{M,N}(t)| \le \Big|\frac{(1-t^2)^{N+1}}{2\,t}\Big|\,\frac{(2M+2)!}{4^{M+1}((M+1)!)^2}\,\frac{(M+1)!\,N!}{(N+M+2)!}\\ \cdot\,\Big|\frac{1-t^2}{t^2}\Big|^{M+1}\eta(t)^{1/2+M+1}\,.\end{aligned}\tag{48}$$

$M=0$ gave Result 6. $M=1$ gives the following result:

Result 7. *Writing* $\varphi_{N,k} = (8k-1)\pi/(8N+6)$ *and*

$$\lambda_k = \operatorname{Log} 2\,\sqrt{2N\pi\sin\varphi_{N,k}}\;,$$

we have more precisely for all $k \in \{1,\dots,N\}$

$$\begin{aligned}1-x_{N,k}^2 = e^{-2i\varphi_{N,k}}&\\ \cdot\Big(1+\frac{1}{N}\,\lambda_k+\frac{1}{N^2}+\frac{\lambda_k}{N^2}+\frac{\lambda_k^2}{2N^2}+\frac{i\,e^{-i\varphi_{N,k}}}{4N^2\sin\varphi_{N,k}}\,(\lambda_k-1)\Big)&\\ +\,\varepsilon_{N,k}\;,&\end{aligned}\tag{49}$$

where

$$|\varepsilon_{N,k}| \le C\max\Big\{\frac{1+(\operatorname{Log} k)^3}{Nk^2},\ \frac{1+\operatorname{Log}(N+1-k)^3}{N(N+1-k)^2}\Big\}$$

and C *doesn't depend neither on* N *nor on* K.

PROOF. We assume $k \le [(N+1)/2]$. We write $1-x_{N,k}^2 = 1-y_{N,k}^2+v$ and the problem is to estimate v. We already know $v = O\left(\operatorname{Log} k/(Nk)\right)$. Furthermore, we know that

$$\int_{x_{N,k}}^1 (1-s^2)^N\,ds = \frac{2\,4^N(N!)^2}{(2N+1)!} = \sqrt{\frac{\pi}{N}}\,\Big(1+O\Big(\frac{1}{N^2}\Big)\Big)$$

and

$$\int_{x_{N,k}}^{1} (1-s^2)^N \, ds = \frac{(1-x_{N,k}^2)^{N+1}}{2\,(N+1)\,x_{N,k}} \Big(1 - \frac{1-x_{N,k}^2}{2\,(N+2)\,x_{N,k}^2} + O\Big(\frac{1}{N^2 x_{N,k}^4}\Big)\Big).$$

Now, write

$$\begin{aligned} \frac{1-x_{N,k}^2}{2\,(N+2)\,x_{N,k}^2} &= \frac{1-y_{N,k}^2}{2\,(N+2)\,y_{N,k}^2} + O\Big(\frac{v}{k}\Big) + O\Big(\frac{Nv}{k^2}\Big) \\ &= \frac{1-y_{N,k}^2}{2\,(N+2)\,y_{N,k}^2} + O\Big(\frac{\operatorname{Log} k}{k^3}\Big) \end{aligned}$$

and

$$\begin{aligned} \frac{1-y_{N,k}^2}{2\,(N+2)\,y_{N,k}^2} &= \frac{e^{-2i\varphi_{N,k}}}{2\,(N+2)\,y_{N,k}^2} + O\Big(\frac{\operatorname{Log} k}{Nk}\Big) \\ &= \frac{e^{-2i\varphi_{N,k}}}{2N(1-e^{-2i\varphi_{N,k}})} + O\Big(\frac{\operatorname{Log} k}{k^2}\Big), \end{aligned}$$

so that

$$1 - \frac{1-x_{N,k}^2}{2\,(N+2)\,x_{N,k}^2} + O\Big(\frac{1}{N^2 x_{N,k}^4}\Big) = 1 + \frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}} + O\Big(\frac{\operatorname{Log} k}{k^2}\Big).$$

We now turn our attention to $(1-x_{N,k}^2)^{N+1}/(2\,(N+1)\,x_{N,k})$. We have

$$\begin{aligned} &2\,(N+1)\,x_{N,k}\sqrt{\frac{\pi}{N}} \\ &\quad = 2\Big(1+\frac{1}{N}\Big)\sqrt{N\pi}\,\sqrt{y_{N,k}^2 - v} \\ &\quad = 2\Big(1+\frac{1}{N}\Big)\sqrt{N\pi}\,\sqrt{1-e^{-2i\varphi_{N,k}} - \frac{e^{-2i\varphi_{N,k}}}{N}\,\lambda_{N,k} + O\Big(\frac{\operatorname{Log} k}{Nk}\Big)} \\ &\quad = 2\Big(1+\frac{1}{N}\Big)\sqrt{N\pi}\,\sqrt{2\sin\varphi_{N,k}}\,e^{i(\pi/4-\varphi_{N,k}/2)} \\ &\qquad \cdot\Big(1+\frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}}\,\lambda_{N,k} + O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big)\Big) \end{aligned}$$

and

$$\begin{aligned}(1-x_{N,k}^2)^{N+1} &= (1-y_{N,k}^2)^{N+1}\Big(1+\frac{v}{1-y_{N,k}^2}\Big)^{N+1}\\ &= (1-y_{N,k}^2)^{N+1}\Big(1+\frac{(N+1)\,v}{1-y_{N,k}^2}+O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big)\Big)\\ &= (1-y_{N,k}^2)^{N+1}\Big(1+Nv\,e^{2i\varphi_{N,k}}+O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big)\Big).\end{aligned}$$

Finally we have

$$\begin{aligned}&\frac{(1-y_{N,k}^2)^{N+1}}{2\sqrt{2N\pi\sin\varphi_{N,k}}\,e^{i(\pi/4-\varphi_{N,k}/2)}}\\ &\qquad=\left(\frac{1+\dfrac{1}{N}\,\lambda_{N,k}}{1+\dfrac{1}{N+1}\,\lambda_{N,k}+\dfrac{1}{2\,(N+1)^2}\,\lambda_{N,k}^2+O\Big(\dfrac{(\operatorname{Log} k)^3}{N^3}\Big)}\right)^{N+1}\\ &\qquad=1-\frac{1}{N}\,\lambda_{N,k}-\frac{1}{2N}\,\lambda_{N,k}^2+O\Big(\frac{(\operatorname{Log} k)^3}{N^2}\Big).\end{aligned}$$

We have thus obtained

$$\begin{aligned}&\Big(1+\frac{1}{N}\Big)\Big(1+O\Big(\frac{1}{N^2}\Big)\Big)\\ &\qquad=\frac{(1-x_{N,k}^2)^{N+1}}{2\sqrt{N\pi}\,x_{N,k}}\Big(1-\frac{1-x_{N,k}^2}{2\,(N+2)\,x_{N,k}^2}+O\Big(\frac{1}{N^2x_{N,k}^4}\Big)\Big)\\ &\qquad=1-\frac{\lambda_{N,k}}{N}-\frac{1}{2N}\,\lambda_{N,k}^2-\frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}}\,\lambda_{N,k}+Nv\,e^{2i\varphi_{N,k}}\\ &\qquad\quad+\frac{i\,e^{-i\varphi_{N,k}}}{4N\sin\varphi_{N,k}}+O\Big(\frac{(\operatorname{Log} k)^3}{k^2}\Big)\end{aligned}$$

which gives the value of v with an $O\left((\operatorname{Log} k)^3/(Nk^2)\right)$ error.

As a corollary, we find a further development of γ_k, which is exactly the formula given in [3]:

Corollary. *If* $\mu_k = (2k - 1/4)\pi$*, then*

$$\gamma_k = e^{-3i\pi/4}\sqrt{\mu_k}\Big(1 - \frac{i}{2\,\mu_k}\operatorname{Log} 2\sqrt{\pi\mu_k} - \frac{1}{4\,\mu_k^2}\operatorname{Log} 2\sqrt{\pi\mu_k} + \frac{1}{4\,\mu_k^2} + \frac{1}{8\,\mu_k^2}(\operatorname{Log} 2\sqrt{\pi\mu_k})^2 + O\Big(\frac{(\operatorname{Log} k)^3}{k^3}\Big)\Big). \tag{50}$$

PROOF. From (31) and (49), we get

$$1 - \frac{\overline{\gamma}_k^2}{N} = \Big(1 - i\,\frac{\mu_k}{N}\Big)\Big(1 + \frac{1}{N}\operatorname{Log} 2\sqrt{\pi}\,\sqrt{\mu_k} + \frac{i}{2N\mu_k}(\operatorname{Log} 2\sqrt{\pi}\,\sqrt{\mu_k} - 1)\Big) + O\Big(\frac{(\operatorname{Log} k)^3}{Nk^2}\Big),$$

hence

$$\gamma_k^2 = -i\,\mu_k - \operatorname{Log} 2\sqrt{\pi\mu_k} + \frac{i}{2\,\mu_k}\operatorname{Log} 2\sqrt{\pi\mu_k} - \frac{i}{2\,\mu_k} + O\Big(\frac{(\operatorname{Log} k)^3}{k^2}\Big)$$

and

$$\gamma_k = \sqrt{-i\,\mu_k}\Big(1 - \frac{i}{2\,\mu_k}\operatorname{Log} 2\sqrt{\pi\mu_k} - \frac{1}{4\,\mu_k^2}\operatorname{Log} 2\sqrt{\pi\mu_k} + \frac{1}{4\,\mu_k^2} + \frac{1}{8\,\mu_k^2}(\operatorname{Log} 2\sqrt{\pi\mu_k})^2 + O\Big(\frac{(\operatorname{Log} k)^3}{k^3}\Big)\Big)$$

and the corollary is proved.

5. Small roots of Q_N: further estimates.

We are now able to give a much better estimate for the small roots of Q_N. Indeed, we used the rough estimate $|e^{-Nx_{N,k}^2}| \le e^{N|x_{N,k}^2|}$ which is far from being good since $x_{N,k}$ accumulates on the line $x = y$ for k big (and $k^2 = O(N)$), so that $e^{-Nx_{N,k}^2}$ is much smaller than $e^{N|x_{N,k}|^2}$: indeed if $k^2 = O(N)$ we find that

$$x_{N,k}^2 = -\frac{1}{N}\operatorname{Log} 2\sqrt{\pi\Big(2k - \frac{1}{4}\Big)\pi} + \frac{i}{N}\Big(2k - \frac{1}{4}\Big)\pi + O\Big(\frac{\operatorname{Log} k}{Nk}\Big),$$

hence

$$|e^{-Nx_{N,k}^2}| = e^{\mathrm{Log}\,2\,\sqrt{\pi(2k-1/4)\pi}}\; e^{O(\mathrm{Log}\,k/k)}$$
$$= 2\sqrt{\pi}\,\sqrt{\Big(2k-\frac{1}{4}\Big)\pi}\,\Big(1+O\Big(\frac{\mathrm{Log}\,k}{k}\Big)\Big)\,,$$

while

$$e^{N|x_{N,k}|^2} \geq e^{(2k-1/4)\pi}\Big(1+O\Big(\frac{\mathrm{Log}\,k}{k}\Big)\Big)\,.$$

Thus, we may improve Result 4 in an impressive manner: for a much bigger set of indexes k, $-\overline{\gamma}_k/\sqrt{N}$ provides a very precise approximation of $x_{N,k}$:

Result 8. *There exist $\eta_0 > 0$ and $C_0 > 0$ so that for N large enough and $k \leq \eta_0 N^{1/5}/(\mathrm{Log}\,N)^{2/5}$ we have*

$$\Big|x_{N,k}+\frac{\overline{\gamma}_k}{\sqrt{N}}\Big| \leq C_0\,\frac{1}{N\sqrt{N}}\,\Big(\frac{k^{5/2}}{1+\mathrm{Log}\,k}\Big)\,. \tag{51}$$

PROOF. We write

$$\tilde{Q}_N(t) = 4\sqrt{\frac{N}{\pi}}\,\frac{4^N(N!)^2}{(2N+1)!}\,Q_N(t) = 1+O\Big(\frac{1}{N^2}\Big)+2\sqrt{\frac{N}{\pi}}\int_0^t (1-s^2)^N\,ds$$

and approximate $(1-s^2)^N$ by e^{-Ns^2} (provided that Nt^4 remains bounded: $|Nt^4| \leq A_0$)

$$(1-s^2)^4 = e^{N\,\mathrm{Log}(1-s^2)} = e^{-Ns^2}(1+O\,(Ns^4))\,.$$

Thus

$$\tilde{Q}_N(t) = \mathrm{erfc}(-\sqrt{N}\,t)+O\Big(\frac{1}{N^2}\Big)+\sqrt{N}\int_0^t e^{-Ns^2}O\,(Ns^4)\,ds\,.$$

Let $\theta = \mathrm{Arg}\,t$ and assume $\theta \in (\pi/4, \pi/2)$. Then we have

$$\Big|\sqrt{N}\int_0^t e^{-Ns^2}O\,(Ns^4)\,ds\Big| \leq C\,N\,\sqrt{N}\,|t|^3\int_0^{|t|} e^{-N\lambda^2\cos 2\theta}\lambda\,d\lambda$$
$$\leq C\,\frac{|e^{-Nt^2}|\,\sqrt{N}\,|t|^3}{2\,|\cos 2\,\theta|}\,.$$

We have thus proved that for $|Nt^4| \le A_0$ and $\operatorname{Arg} t \in (\pi/4, \pi/2)$ we have

$$|\tilde{Q}_N(t) - \operatorname{erfc}(-\sqrt{N}\, t)| \le C\Big(\frac{1}{N^2} + \sqrt{N}\, |t|^3 \,\frac{|e^{-Nt^2}|}{2\,|\cos 2 \operatorname{Arg} t|}\Big)\,.$$

Now, we write $t = x_{N,k} + \delta$, $|\delta| \le \delta_0/N$. Remember that we have

$$|x_{N,k}| \approx \sqrt{\frac{\Big(2k - \dfrac{1}{4}\Big)\pi}{N}}$$

(hence we will look at $k \le \sqrt{A_0 N/(2\pi)}$) and

$$\begin{aligned}\operatorname{Arg} x_{N,k} &= \frac{\pi}{4} - \frac{1}{2}\,\varphi_{N,k} + \operatorname{Arg}\Big(1 + \frac{i\, e^{-i\varphi_{N,k}}}{4N \sin \varphi_{N,k}} \operatorname{Log}\,(2\sqrt{2N\pi \sin \varphi_{N,k}})\Big) \\ &\quad + O\Big(\frac{(\operatorname{Log} k)}{k^2}\Big) \\ &= \frac{\pi}{4} + \frac{\operatorname{Log}\Big(2\sqrt{\pi}\,\sqrt{\Big(2k - \dfrac{1}{4}\Big)\pi}\,\Big)}{2\Big(2k - \dfrac{1}{4}\Big)\pi} + O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{k}{N}\Big)\,,\end{aligned}$$

hence if $k \ge k_0$ where k_0 is large enough so that

$$O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{k}{N}\Big) = O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{1}{k}\Big)$$

is smaller than

$$\frac{1}{2}\,\frac{\operatorname{Log} 2\,\sqrt{\pi}\,\sqrt{\Big(2k - \dfrac{1}{4}\Big)\pi}}{2\Big(2k - \dfrac{1}{4}\Big)\pi}\,,$$

we find that $\operatorname{Arg} x_{N,k} \in (\pi/4, \pi/2)$. (This is also true for $k \le k_0$, if N is large enough, since $x_{N,k} \sim -\overline{\gamma}_k/\sqrt{N}$).

Moreover

$$\cos(2\operatorname{Arg} x_{N,k})$$

$$= -\sin\left(\frac{\operatorname{Log}\Big(2\sqrt{\pi}\sqrt{\Big(2k-\frac{1}{4}\Big)\pi}\Big)}{\Big(2k-\frac{1}{4}\Big)\pi} + O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{k}{N}\Big)\right)$$

$$= -\frac{\operatorname{Log}\Big(2\sqrt{\pi}\sqrt{\Big(2k-\frac{1}{4}\Big)\pi}\Big)}{\Big(2k-\frac{1}{4}\Big)\pi} + O\Big(\frac{(\operatorname{Log} k)^2}{k^2}\Big) + O\Big(\frac{k}{N}\Big),$$

hence $\cos(2\operatorname{Arg} x_{N,k})$ has order of magnitude $\operatorname{Log} k/k$. Thus we obtain for δ_0 small enough

- $t = x_{N,k}\Big(1 + O\Big(\frac{1}{\sqrt{Nk}}\Big)\Big)$,
- $e^{-Nt^2} = e^{-Nx_{N,k}^2}\left(1 + O\left(\sqrt{\frac{k}{N}}\right) + O\Big(\frac{1}{N}\Big)\right)$,
- $\operatorname{Arg} t = \operatorname{Arg} x_{N,k} + O\Big(\frac{1}{\sqrt{Nk}}\Big) = \operatorname{Arg} x_{N,k} + O\Big(\frac{1}{k\sqrt{k}}\Big)$,

thus we have

$$|\tilde{Q}_N(t) - \operatorname{erfc}(-\sqrt{N}\,t)| \le C\Big(\frac{1}{N^2} + \sqrt{N}\left(\frac{k}{N}\right)^{3/2}\frac{\sqrt{k}}{(\operatorname{Log} k)/k}\Big)$$

$$\le C'\,\frac{k^3}{N\operatorname{Log} k}\,.$$

On the other hand we have

$$|\operatorname{erfc}(-\sqrt{N}\,t) - \operatorname{erfc}(-\sqrt{N}\,x_{N,k})|$$

$$= \Big|2\sqrt{\frac{N}{\pi}}\int_{x_{N,k}}^{t} e^{-Ns^2}\,ds\Big|$$

$$= |e^{-Nx_{N,k}^2}|\,2\sqrt{\frac{N}{\pi}}\;\Big|\int_0^{\delta} e^{-2Nx_{N,k}s-Ns^2}\,ds\Big|.$$

We notice that

$$|2Nx_{N,k}\,s + Ns^2| \le 2\,|x_{N,k}|\,\delta_0 + \frac{\delta_0^2}{N} \le C\,\frac{\delta_0}{\sqrt{N}}\,,$$

so that if N is large enough,

$$|e^{-2Nx_{N,k}s-Ns^2}-1|\le\frac{1}{2}\,,$$

which gives

$$|\mathrm{erfc}(-\sqrt{N}\,t)-\mathrm{erfc}(-\sqrt{N}\,x_{N,k})|\ge 2\sqrt{\frac{N}{\pi}}\;|e^{-Nx_{N,k}^2}|\,\frac{1}{2}\,|\delta|\ge C\sqrt{Nk}\;|\delta|\,.$$

Thus

$$\text{(52)}\qquad\begin{cases}|\mathrm{erfc}(-\sqrt{N}\,t)|\ge C_1\sqrt{N}\,k\,\delta-C_2\,\dfrac{k^3}{N\,\mathrm{Log}\,k}\,,\\[2mm] |\mathrm{erfc}(-\sqrt{N}\,t)-\tilde{Q}_N(t)|\le C_2\,\dfrac{k^3}{\sqrt{N}\,\mathrm{Log}\,k}\,.\end{cases}$$

Now choose

$$\delta_{N,k}=\frac{3\,C_2}{C_1}\,\frac{k^{5/2}}{N^{3/2}\mathrm{Log}\,k}$$

(we have $\delta_{N,k}<\delta_0/N$ if $k^{5/2}/\mathrm{Log}\,k<\delta_0C_1\sqrt{N}/(3C_2)$); we obtain that

$$\sup_{|t-x_{N,k}|=\delta_{N,k}}|\mathrm{erfc}(-\sqrt{N}\,t)-\tilde{Q}_N(t)|\le\frac{1}{2}\inf_{|t-x_{N,k}|=\delta_{N,k}}|\,\mathrm{erfc}(-\sqrt{N}\,t)|\,,$$

hence by Rouché's theorem we find that $\tilde{Q}_N$ and $\mathrm{erfc}(-\sqrt{N}\,t)$ have the same number of roots in the disk $|t-x_{N,k}|<\delta_{N,k}$. Since

$$|x_{N,k}-x_{N,k+1}|\approx\sqrt{\frac{\pi}{2kN}}$$

and

$$\sqrt{kN}\,\delta_{N,k}=O\Big(\frac{k^3}{N\,\mathrm{Log}\,k}\Big)=O\Big(\frac{1}{N^{2/5}(\mathrm{Log}\,N)^{7/5}}\Big)=o\,(1)$$

(if $k\le CN^{1/5}/(\mathrm{Log}\,N)^{2/5}$), we find: for $k\le\eta_0N^{1/5}/(\mathrm{Log}\,N)^{2/5}$ (η_0 small enough)

$$|x_{N,k}+\frac{\overline{\gamma}_k}{\sqrt{N}}|\le C\,\frac{1}{N\sqrt{N}}\,\Big(\frac{k^{5/2}}{\mathrm{Log}\,k}\Big)\,.$$

Result 8 is proved.

Result 8 is enough for what we want to prove. But, of course, we may develop a bit further $(1-s^2)^N$ and get a better approximation for $x_{N,k}$:

Result 9. *For $k \le \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ we have more precisely*

$$x_{N,k} = -\frac{\overline{\gamma}_k}{\sqrt{N}} + \frac{1}{N\sqrt{N}} \Big(\frac{1}{2}\,\overline{\gamma}_k^3 + \frac{3}{8}\,\overline{\gamma}_k + O\,(\sqrt{\operatorname{Log} k}) \Big) .$$

PROOF. We write $\operatorname{Log}(1-s^2) = -s^2 - s^4/2 + O\,(s^6)$. Hence we have

$$(1-s^2)^N = e^{-Ns^2} \Big(1 - N\,\frac{s^4}{2} + O\,(Ns^6) + O\,(N^2 s^8) \Big) ,$$

provided that $|s| \le A_0/N^{1/4}$.

Thus we have for $|t| \le A_0/N^{1/4}$ and $\operatorname{Arg} t \in (\pi/4, \pi/2)$

$$\begin{aligned} &\Big| \tilde{Q}_N(t) - \operatorname{erfc}(-\sqrt{N}\,t) + 2\sqrt{\frac{N}{\pi}}\, N \int_0^t e^{-Ns^2} s^4\, ds \Big) \\ &\qquad \le C\Big(\frac{1}{N^2} + \sqrt{N}\, \Big| \frac{t^5 e^{-Nt^2}}{\cos(2\operatorname{Arg} t)} \Big| + N\sqrt{N}\, \Big| \frac{t^7 e^{-Nt^2}}{\cos(2\operatorname{Arg} t)} \Big| \Big) . \end{aligned}$$

Moreover we have

$$\begin{aligned} N \int_0^t e^{-Ns^2} s^4\, ds &= \Big[\frac{-e^{-Ns^2} s^3}{2} \Big]_0^t + \frac{3}{2} \int_0^t e^{-Ns^2} s^2\, ds \\ &= \frac{-e^{-Nt^2} t^3}{2} - \frac{3}{4N}\, e^{-Nt^2} t + \frac{3}{4N} \int_0^t e^{-Ns^2}\, ds . \end{aligned}$$

Now, we write $\eta = 1/\sqrt{2N|\cos(2\operatorname{Arg} t)|}$ (if $t \approx x_{N,k}$, we have $\eta \approx \sqrt{4k/(N \operatorname{Log} k)} < |t|$) and we write

$$\begin{aligned} \Big| \int_0^t e^{-Ns^2}\, ds \Big| &\le \int_0^\eta |e^{-Nt^2}|\, ds + \int_\eta^{|t|} e^{-Ns^2 \cos(2\operatorname{Arg} t)}\, \frac{s\, ds}{\eta} \\ &\le \eta |e^{-Nt^2}| + \frac{|e^{-Nt^2}|}{2N|\cos(2\operatorname{Arg} t)|\,\eta} \\ &= \frac{2\,|e^{-Nt^2}|}{\sqrt{2N|\cos(2\operatorname{Arg} t)|}} . \end{aligned}$$

Finally we get

$$\begin{aligned}\operatorname{erfc}(-\sqrt{N}\,x_{N,k}) = e^{-Nx_{N,k}^2}\sqrt{\frac{N}{\pi}}\,x_{N,k}^3 + e^{-Nx_{N,k}^2}\,\frac{3}{4\sqrt{N\pi}}\,x_{N,k} \\ + O\Big(\frac{1}{N^2}\Big) + O\Big(\frac{k^4}{N^2\log k}\Big) + O\Big(\frac{k^5}{N^2\log k}\Big) \\ + O\Big(\frac{\sqrt{\log k}}{N}\Big)\end{aligned}$$

and, assuming again $k < \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$,

$$\operatorname{erfc}(-\sqrt{N}\,x_{N,k}) = e^{-Nx_{N,k}^2}\sqrt{\frac{N}{\pi}}\,x_{N,k}^3\Big(1 + O\Big(\frac{1}{k}\Big)\Big)$$

On the other hand, we have $x_{N,k} = -\overline{\gamma}_k/\sqrt{N} + s$ with

$$s = O\Big(\frac{1}{N\sqrt{N}}\frac{k^{5/2}}{\operatorname{Log} k}\Big)$$

and we want a better estimate for s. We have

$$\sqrt{N}\,s\,\overline{\gamma}_k = O\Big(\frac{1}{N}\,\frac{k^3}{\operatorname{Log} k}\Big) = O\Big(\frac{1}{N^{2/5}}\Big)$$

and thus we may develop

$$\begin{aligned}\operatorname{erfc}(\overline{\gamma}_k - \sqrt{N}\,s) &= e^{-\overline{\gamma}_k^2}\,\frac{2}{\sqrt{\pi}}\int_0^{-\sqrt{N}\,s} e^{-2\overline{\gamma}_k u - u^2}\,du \\ &= -\frac{2}{\sqrt{\pi}}\,e^{-\overline{\gamma}_k^2}\,\sqrt{N}\,s\,(1 + O\,(\sqrt{N}\,s\,\overline{\gamma}_k) + O\,(Ns^2))\,.\end{aligned}$$

Hence we find

$$-\frac{2}{\sqrt{\pi}}\,e^{-\overline{\gamma}_k^2}\,\sqrt{N}\,s \sim \sqrt{\frac{N}{\pi}}\,x_{N,k}^3\,e^{-Nx_{N,k}^2}$$

and therefore

$$s \sim -\frac{1}{2}\,x_{N,k}^3 = O\Big(\frac{k^{3/2}}{N^{3/2}}\Big),$$

so that

$$
\begin{aligned}
&- e^{-\overline{\gamma}_k^2} \frac{2}{\sqrt{\pi}} \sqrt{N}\, s\Big(1 + O\Big(\frac{k^2}{N}\Big) + O\Big(\frac{k^3}{N^2}\Big)\Big) \\
&\qquad = \sqrt{\frac{N}{\pi}}\, x_{N,k}^3\, e^{-N x_{N,k}^2} + \frac{3}{4\sqrt{N\pi}}\, x_{N,k}\, e^{-N x_{N,k}^2} + O\Big(\frac{\sqrt{\operatorname{Log} k}}{N}\Big),
\end{aligned}
$$

so that (since $e^{-N x_{N,k}^2 + \overline{\gamma}_k^2} = 1 + O(\sqrt{N}\, s\, \overline{\gamma}_k) = 1 + O\left(k^2/N\right)$)

$$
\begin{aligned}
s &= -\frac{1}{2}\, x_{N,k}^3 - \frac{3}{8N}\, x_{N,k} + O\Big(\frac{\sqrt{\operatorname{Log} k}}{N\sqrt{N}}\Big) \\
&= \frac{1}{2}\, \frac{\overline{\gamma}_k^3}{N\sqrt{N}} + \frac{3\,\overline{\gamma}_k}{8N\sqrt{N}} + O\Big(\frac{\sqrt{\operatorname{Log} k}}{N\sqrt{N}}\Big)
\end{aligned}
$$

and Result 9 is proved.

6. The phase of a general Daubechies filter.

We have now almost achieved the proof of Theorem 1. Indeed, we have given estimates for $x_{N,k}$, hence for $z_{N,k}$, which is the solution of $x_{N,k} = (z_{N,k} + 1/z_{N,k})/2$ with $\operatorname{Re} z_{N,k} > 0$, hence which is given by $z_{N,k} = x_{N,k} + \sqrt{x_{N,k}^2 - 1}$. We thus have proved:

Proposition 3. *Let P_N be the N-th polynomial of I. Daubechies*

$$
P_N(z) = \Big(\frac{1+z}{2}\Big)^{2N+2} \sum_{k=0}^{N} (-1)^k \binom{N+k}{k} \Big(\frac{1-z}{2}\Big)^{2k} \tag{54}
$$

which is related to Q_N by

$$
e^{i(2N+1)\xi} P_N(e^{-i\xi}) = Q_N(\cos \xi) \tag{55}
$$

or equivalently

$$
P_N(z) = z^{2N+1} Q_N\Big(\frac{1}{2}\Big(z + \frac{1}{z}\Big)\Big). \tag{56}
$$

Then the roots of P_N are precisely given as the following ones:

- *$z = -1$ with multiplicity $2N + 2$,*

- *$2N$ roots with multiplicity 1 which can be decomposed into*

$$\Big\{z_{N,k}, \overline{z_{N,k}}, \frac{1}{z_{N,k}}, \frac{1}{\overline{z}_{N,k}}\Big\}_{1\leq k\leq[N/2]},$$

(together with $\{z_{N,(N+1)/2}, 1/z_{N,(N+1)/2}\}$ if N is odd), where $\operatorname{Im} z_{N,k} \geq 0$, $\operatorname{Re} z_{N,k} \geq 0$, $|z_{N,k}| > 1$, $\operatorname{Im} z_{N,k} > 0$ *for* $k < [(N+1)/2]$ *and* $\operatorname{Im} z_{N,(N+1)/2} = 0$.

Moreover we have, for N large enough:

- *if $k \leq \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ (where η_0 is fixed independently of N and is small enough)*

$$z_{N,k} = i - \frac{\overline{\gamma}_k}{\sqrt{N}} + O\Big(\frac{k}{N}\Big), \tag{57}$$

where γ_k is the k-th zero γ of $\operatorname{erfc}(z)$ *with* $\operatorname{Im}\gamma > 0$

- *for all k*

$$z_{N,k} = y_{N,k} + \sqrt{y_{N,k}^2 - 1} + O\Big(\frac{1+\operatorname{Log} k}{k\sqrt{Nk}}\Big), \tag{58}$$

where

$$y_{N,k} = \bigg(1 - e^{-2i(8k-1)\pi/(8N+6)} - \frac{1}{N}\, e^{-2i(8k-1)\pi/(8N+6)} \operatorname{Log} 2 \sqrt{2N\pi \sin\Big(\frac{8k-1}{8N+6}\,\pi\Big)}\bigg)^{1/2}.$$

PROOF. Just write $z_{N,k} = x_{N,k} + \sqrt{x_{N,k}^2 - 1}$ and apply results 6 and 8.

Of course, we could give better estimates using results 7 and 9, but we won't need them. We have easy estimates for $1/z_{N,k}$ as well since $1/z_{N,k} = x_{N,k} - \sqrt{x_{N,k}^2 - 1}$.

We are now going to use proposition 3 in the estimation of the phase of a Daubechies filter. We want to approximate for $\xi \in [-\pi, \pi]$, $1/(e^{-i\xi} - \lambda_{N,k})$ where

$$\lambda_{N,k} \in \Big\{z_{N,k}, \frac{1}{z_{N,k}}, \overline{z}_{N,k}, \frac{1}{\overline{z}_{N,k}}\Big\}.$$

A direct consequence of Proposition 3 is the following proposition:

Proposition 4. *Let* $\xi \in [-\pi,\pi]$ *and let* $z_{N,k}$, $1 \le k \le [(N+1)/2]$ *be the roots of* P_N *described in Proposition* 3. *Let* $\lambda_{N,k} \in \{z_{N,k}, 1/z_{N,k}, \overline{z}_{N,k}, 1/\overline{z}_{N,k}\}$. *Then*

i) *for* $1 \le k \le \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ *we have, writing* $\widetilde{z_{N,k}} = i - \overline{\gamma}_k/\sqrt{N}$,

$$\Big|\frac{1}{e^{-i\xi}-\lambda_{N,k}} - \frac{1}{e^{-i\xi}-\widetilde{\lambda_{N,k}}}\Big| \le C\,\frac{k}{N}\,\frac{1}{\dfrac{k}{N}+|\cos\xi|^2}\,, \tag{59}$$

where C *doesn't depend neither on* N *nor on* k *nor on* ξ (*and where* $\widetilde{\lambda_{N,k}} = \widetilde{z_{N,k}}$ *if* $\lambda_{N,k} = z_{N,k}$, $1/\widetilde{z_{N,k}}$, *if* $\lambda_{N,k} = 1/z_{N,k}$ *and so on* ...).

ii) *for* $k \ge k_0$ (k_0 *large enough independently of* N) *we have, writing* $\widehat{z_{N,k}} = y_{N,k} + \sqrt{y_{N,k}^2 - 1}$ *as in formula* (58),

$$\Big|\frac{1}{e^{-i\xi}-\lambda_{N,k}} - \frac{1}{e^{-i\xi}-\widehat{\lambda_{N,k}}}\Big| \le C\,\frac{\operatorname{Log} k}{k\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N}+|\cos\xi|^2}\,. \tag{60}$$

Proof. Of course, we may assume $\xi \in [0,\pi]$. If $\xi \in [\pi/2,\pi]$, the estimation is easy since $\operatorname{Re} e^{-i\xi} < 0$ and $\operatorname{Re}\lambda_{N,k} > 0$ (as well $\operatorname{Re}\widehat{\lambda_{N,k}}$ and $\operatorname{Re}\widetilde{\lambda_{N,k}}$). Thus,

$$|e^{-i\xi}-\lambda_{N,k}| \ge \operatorname{Re}(-e^{-i\xi}+\lambda_{N,k}) \ge C\sqrt{\frac{k}{N}} + |\cos\xi|$$

and the same for $|e^{-i\xi}-\widehat{\lambda_{N,k}}|$ and $|e^{-i\xi}-\widetilde{\lambda_{N,k}}|$. Of course, we must prove that $\min\{\operatorname{Re}\lambda_{N,k}, \operatorname{Re}\widehat{\lambda_{N,k}}, \operatorname{Re}\widetilde{\lambda_{N,k}}\} \ge C\sqrt{k/N}$. For $\operatorname{Re}\widetilde{\lambda_{N,k}}$, it is obvious, since

$$\operatorname{Re}\widetilde{\lambda_{N,k}} \ge \frac{-\operatorname{Re}\gamma_k}{\sqrt{N}\,\Big|i - \dfrac{\overline{\gamma}_k}{\sqrt{N}}\Big|^2} \approx \sqrt{\frac{k\pi}{N}}\,.$$

For $\operatorname{Re}\lambda_{N,k}$, if $k < \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$, we deduce that $\operatorname{Re}\lambda_{N,k} \ge C\sqrt{k/N}$ since

$$|\lambda_{N,k}-\widetilde{\lambda_{N,k}}| \le |z_{N,k}-\widetilde{z_{N,k}}| \le C\,\frac{k}{N} \le \sqrt{\frac{k}{N}}\,C'\,N^{-2/5}\,.$$

We thus turn our attention to $\operatorname{Re}\widehat{\lambda_{N,k}} \geq \operatorname{Re}\widehat{z_{N,k}}/|\widehat{z_{N,k}}|^2$ and $\operatorname{Re}\lambda_{N,k} \geq \operatorname{Re} z_{N,k}/|z_{N,k}|^2$ for large k's. We define $\mu_{N,k} = \sqrt{1-e^{-2i(8k-1)\pi/(8N+6)}}$ and $\xi_{N,k} = \mu_{N,k} + \sqrt{\mu_{N,k}^2 - 1}$. We have

$$\begin{aligned}\xi_{N,k} &= \sqrt{2\sin\Big(\frac{8k-1}{8N+6}\Big)}\, e^{i(\pi/4-(8k-1)\pi/(2(8N+6)))} \\ &\qquad + e^{i(\pi/2-(8k-1)\pi/(8N+6))} \\ &= 1+\sqrt{2}\, e^{i(\pi/4-(8k-1)\pi/(2(8N+6))+\arcsin\sqrt{2}\sin(\pi/4-(8k-1)\pi/2(8N+6)))}\end{aligned}$$

and thus we study $1+\sqrt{2}\, e^{i(\omega+\arcsin\sqrt{2}\sin\omega)}$ for $\omega \in [0,\pi/4]$. We have

$$\begin{aligned}&\operatorname{Re}\big(1+\sqrt{2}\, e^{i(\omega+\arcsin\sqrt{2}\sin\omega)}\big) \\ &\qquad = \sqrt{1-2\sin^2\omega}\,\big(\sqrt{1-2\sin^2\omega}+\sqrt{2\cos^2\omega}\,\big) \\ &\qquad = \sqrt{\cos 2\omega}\,\big(\sqrt{2\cos^2\omega}+\sqrt{1-2\sin^2\omega}\,\big) \geq \sqrt{\frac{4}{\pi}\Big(\frac{\pi}{4}-\omega\Big)}\,,\end{aligned}$$

which gives

$$\operatorname{Re}\xi_{N,k} \geq \sqrt{2\,\frac{8k-1}{8N+6}} \geq \sqrt{\frac{k}{N}}\,.$$

Now we have

$$|\widehat{z_{N,k}} - \xi_{N,k}| \leq C\sqrt{\frac{k}{N}}\,\frac{\operatorname{Log} k}{k}\,,$$

so that if k is large enough we have

$$\operatorname{Re}\widehat{z_{N,k}} \geq C'\sqrt{\frac{k}{N}}\,.$$

Moreover

$$|z_{N,k} - \widehat{z_{N,k}}| \leq C\sqrt{\frac{k}{N}}\,\frac{\operatorname{Log} k}{k^2}$$

and thus

$$\operatorname{Re} z_{N,k} \geq C''\sqrt{\frac{k}{N}}\,.$$

Finally, we control $|z_{N,k}|$ and $|\widehat{z_{N,k}}|$ by

$$|z_{N,k}| + |\widehat{z_{N,k}}| \leq 1+\sqrt{2}+O\Big(\sqrt{\frac{k}{N}}\,\frac{\operatorname{Log} k}{k}\Big) \leq C\,.$$

Thus we obtain

$$\operatorname{Re}\lambda_{N,k} \geq C\sqrt{\frac{k}{N}} \qquad \text{and} \qquad \operatorname{Re}\widehat{\lambda_{N,k}} \geq C\sqrt{\frac{k}{N}}\,.$$

We are going to prove that

$$|e^{-i\xi} - \lambda_{N,k}| \geq C\Big(\sqrt{\frac{k}{N}} + |\cos\xi|\Big)$$

and

$$|e^{-i\xi} - \widehat{\lambda_{N,k}}| \geq C\Big(\sqrt{\frac{k}{N}} + |\cos\xi|\Big)$$

holds for $\xi \in [0, \pi/2]$ as well. Notice that if $|\lambda_{N,k}| < 1$, we have

$$|\lambda_{N,k} - e^{-i\xi}| = \Big|\frac{1}{z_{N,k}}\Big|\,\Big|e^{-i\xi} - \frac{1}{\overline{\lambda_{N,k}}}\Big| \geq \frac{1}{C'}\Big|e^{-i\xi} - \frac{1}{\overline{\lambda_{N,k}}}\Big|$$

(and the same for $|e^{-i\xi} - \widehat{\lambda_{N,k}}|$) so that we may assume $|\lambda_{N,k}| > 1$. If $\lambda_{N,k} = z_{N,k}$, our equality is obvious: for $\xi_{N,k}$ we have either $\operatorname{Im}\xi_{N,k} \geq 1$ or $\operatorname{Re}\xi_{N,k} \geq 2$ and, since $\operatorname{Im} e^{-i\xi} < 0$, we find $|e^{-i\xi} - \xi_{N,k}| \geq 1$, hence (for k large), $|e^{-i\xi} - z_{N,k}| \geq 1/2$ and $|e^{-i\xi} - \widehat{z_{N,k}}| \geq 1/2$, while

$$\frac{1}{2} \geq \frac{1}{4}\Big(\sqrt{\frac{k}{N}} + |\cos\xi|\Big)\,.$$

Now if $\lambda_{N,k}$ is the conjugate of $z_{N,k}$ or $\widehat{z_{N,k}}$, we are going to show that

$$|e^{-i\xi} - \overline{\xi}_{N,k}| \geq C\Big(\sqrt{\frac{k}{N}} + |\cos\xi|\Big)\,,$$

which gives the control over $|e^{-i\xi} - \lambda_{N,k}|$ for large k's. Thus we are led to show that

$$\text{(61)} \qquad \begin{cases} \text{for } \xi \in \Big[0, \dfrac{\pi}{2}\Big] \text{ and } \omega \in \Big[0, \dfrac{\pi}{4}\Big]\,, \\[2ex] |e^{-i\xi} - 1 - \sqrt{2}\; e^{-i(\omega + \arcsin\sqrt{2}\,\sin\omega)}| \\[1ex] \qquad\qquad\qquad \geq C\Big(|\cos\xi| + \sqrt{\dfrac{\pi}{4} - \omega}\Big)\,. \end{cases}$$

We compute easily $\mu(\xi,\omega) = |e^{-i\xi} - 1 - \sqrt{2}\, e^{-i(\omega + \arcsin\sqrt{2}\sin\omega)}|^2$

$$\begin{aligned}
\mu(\xi,\omega) &= \big(\cos\xi - \sqrt{1-2\sin^2\omega}\,\big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\big)\big)^2 \\
&\qquad + \big(\sin\xi - \sqrt{2}\,\sin\omega\big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\big)\big)^2 \\
&= 1 + \big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\big)^2 \\
&\qquad - 2\big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\,\big) \\
&\qquad\quad \cdot \big(\cos\xi\sqrt{1-2\sin^2\omega} + \sin\xi\sqrt{2}\,\sin\omega\big) \\
&= \big(\sqrt{2}\,\cos\omega - 1 + \sqrt{1-2\sin^2\omega}\big)^2 \\
&\qquad + 2\big(\sqrt{2}\,\cos\omega + \sqrt{1-2\sin^2\omega}\big) \\
&\qquad\quad \cdot \big(1 - \cos\left(\xi - \arcsin\left(\sqrt{2}\,\sin\omega\right)\right)\big) \\
&\geq 1 - 2\sin^2\omega + 2\left(1 - \cos\left(\xi - \arcsin\left(\sqrt{2}\,\sin\omega\right)\right)\right).
\end{aligned}$$

We have

$$1 - 2\sin^2\omega = \cos 2\,\omega \geq \frac{2}{\pi}\Big(\frac{\pi}{2} - 2\,\omega\Big).$$

On the other hand, we have

$$\begin{aligned}
1 - \cos\left(\xi - \arcsin\sqrt{2}\,\sin\omega\right) &= 2\sin^2\Big(\frac{\xi}{2} - \frac{1}{2}\,\arcsin\sqrt{2}\,\sin\omega\Big) \\
&\geq \frac{2}{\pi^2}|\xi - \arcsin\sqrt{2}\,\sin\omega|^2.
\end{aligned}$$

Moreover we have

$$\frac{\pi}{2} - \arcsin\sqrt{2}\,\sin\omega = \arcsin\sqrt{\cos 2\,\omega} \leq \frac{\pi}{2}\,\sqrt{\cos 2\,\omega}\,,$$

hence we have (using $|a+b|^2 \geq a^2/3 - b^2/2$)

$$\begin{aligned}
\mu(\xi,\omega)^2 &\geq \cos 2\,\omega + \frac{4}{\pi^2}\Big|\xi - \frac{\pi}{2} + \frac{\pi}{2} - \arcsin\sqrt{2}\,\sin\omega\Big|^2 \\
&\geq \cos 2\,\omega + \frac{4}{3\pi^2}\Big|\xi - \frac{\pi}{2}\Big|^2 - \frac{2}{\pi^2}\Big|\frac{\pi}{2} - \arcsin\sqrt{2}\,\sin\omega\Big|^2 \\
&\geq \frac{1}{2}\,\cos 2\,\omega + \frac{4}{3\pi^2}\cos^2\xi \\
&\geq \frac{4}{3\pi^2}\Big(\cos^2\xi + \Big|\frac{\pi}{4} - \omega\Big|\Big)
\end{aligned}$$

and thus (61) is proved.

Proposition 4 is then obvious since

$$\Big|\frac{1}{e^{-i\xi}-\lambda_{N,k}}-\frac{1}{e^{-i\xi}-\tilde{\lambda}_{N,k}}\Big|=\frac{|\lambda_{N,k}-\tilde{\lambda}_{N,k}|}{|e^{-i\xi}-\lambda_{N,k}|\,|e^{-i\xi}-\tilde{\lambda}_{N,k}|}$$

and since we control each term due to (61) or to Proposition 3.

We may now obtain Theorem 1 as a corollary of Proposition 4:

Corollary. *With the same notation as in Proposition* 4, *if* $k_0\le k_N\le \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ *then*

$$\begin{aligned}\int_0^{2\pi}\Big|\sum_{k=1}^{[(N+1)/2]}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\lambda_{N,k}} &\\ -\sum_{k=1}^{k_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\tilde{\lambda}_{N,k}}&-\sum_{k_N+1}^{[(N+1)/2]}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\hat{\lambda}_{N,k}}\Big|\,d\xi \\ &\le C\Big(\frac{k_N^{3/2}}{\sqrt{N}}+\frac{\operatorname{Log} k_N}{k_N}\Big).\end{aligned}\tag{62}$$

PROOF. Using Proposition 4, and writing $I_N(\xi)$ for

$$I_N(\xi)=\sum_{k=1}^{N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\lambda_{N,k}}-\sum_{k=1}^{k_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\tilde{\lambda}_{N,k}}-\sum_{k_N+1}^{[(N+1)/2]}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\tilde{\lambda}_{N,k}}\,,$$

we get

$$I_N(\xi)\le\sum_{k=1}^{k_N}C\,\frac{k}{N}\,\frac{1}{\dfrac{k}{N}+|\cos\xi|^2}+\sum_{k_N+1}^{[(N+1)/2]}C\,\frac{\operatorname{Log} k}{k\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N}+|\cos\xi|^2}\,.$$

Thus we have to estimate

$$\begin{aligned}\int_0^{2\pi}\frac{d\xi}{k+N|\cos\xi|^2}&\le 4\int_0^{\arccos\sqrt{k/N}}\frac{d\xi}{N\cos^2\xi}+4\int_{\arccos\sqrt{k/N}}^{\pi/2}\frac{d\xi}{k}\\ &=\frac{4}{N}\tan\Big(\arccos\sqrt{\frac{k}{N}}\,\Big)+\frac{4}{k}\Big(\frac{\pi}{2}-\arccos\sqrt{\frac{k}{N}}\,\Big)\\ &\le\frac{4}{\sqrt{Nk}}+\frac{2\pi}{\sqrt{Nk}}\,,\end{aligned}$$

so that

$$\int_0^{2\pi} I_N(\xi)\,d\xi \leq C'\Big(\sum_{k=1}^{k_N}\sqrt{\frac{k}{N}}+\sum_{k_N+1}^{[(N+1)/2]}\frac{\operatorname{Log} k}{k^2}\Big) \leq C''\Big(\frac{k_N^{3/2}}{\sqrt{N}}+\frac{\operatorname{Log} k_N}{k_N}\Big)\,.$$

Now Theorem 1 is proved with $k_N = [N^{1/5}/\operatorname{Log} N]$. At least, we have proved it for $\xi \in [0, 2\pi]$. But $\omega(z_{N,1}^{\varepsilon_1}, \ldots, z_{N,N}^{\varepsilon_N}) - \omega(Z_{N,1}^{\varepsilon_1}, \ldots, Z_{N,N}^{\varepsilon_N})$ is 2π-periodical, since $\omega(Z_1, \ldots, Z_N)(\xi + 2\pi) - \omega(Z_1, \ldots, Z_N)(\xi) = 2i\pi M$ where M is the number of Z_k's which lie inside the open disk $|Z| < 1$.

7. Minimum-phased Daubechies filters.

This section is devoted to the proof of Theorem 2.

Result 10. *We have the following inequality*

$$(63)\quad \Big|\frac{d}{d\xi}\omega(z_{N,1}, \ldots, z_{N,N})(\xi) - \frac{N}{2\pi}\operatorname{Im}\int_{-\pi}^{\pi}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\xi(\omega)}\,d\omega\Big| \leq C\sqrt{N}\ ,$$

where $\xi(\omega) = \sqrt{e^{-i\omega}} + \sqrt{1 + e^{-i\omega}}$.

PROOF. We approximate $z_{N,k}$ by $Z_{N,k} = Z((8k-1)\pi/(8N+6))$, $(1 \leq k \leq N)$ where

$$Z(\omega) = \sqrt{2\sin\omega}\; e^{i(\pi/4-\omega/2)} + e^{i(\pi/2-\omega)}\ .$$

We have shown that for $k_0 \leq k \leq [(N+1)/2]$, ($k_0$ large enough) we have

$$\Big|\frac{1}{e^{-i\xi}-z_{N,k}} - \frac{1}{e^{-i\xi}-Z_{N,k}}\Big| \leq C\,\frac{\operatorname{Log} k}{\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N}+\cos^2\xi}$$

and

$$\Big|\frac{1}{e^{-i\xi}-\overline{z}_{N,k}} - \frac{1}{e^{-i\xi}-\overline{Z}_{N,k}}\Big| \leq C\,\frac{\operatorname{Log} k}{\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N}+\cos^2\xi}$$

(notice that $z_{N,N+1-k} = \overline{z}_{N,k}$ and $Z_{N,N+1-k} = \overline{Z}_{N,k}$). If $k < k_0$, we have to prove similarly

$$\Big|\frac{1}{e^{-i\xi}-z_{N,k}} - \frac{1}{e^{-i\xi}-Z_{N,k}}\Big| \leq C\,\frac{1}{\sqrt{N}}\,\frac{1}{\dfrac{1}{N}+\cos^2\xi}$$

and
$$\Big|\frac{1}{e^{-i\xi} - \overline{z}_{N,k}} - \frac{1}{e^{-i\xi} - \overline{Z}_{N,k}}\Big| \le C\,\frac{1}{\sqrt{N}}\,\frac{1}{\dfrac{1}{N} + \cos^2\xi}\,.$$

We have of course
$$|z_{N,k} - Z_{N,k}| \le |z_{N,k}| + |Z_{N,k}| \le \frac{C}{\sqrt{N}}\,,$$

so that we only have to check that
$$|e^{-i\xi} - Z_{N,k}| \ge \frac{1}{C}\Big(\frac{1}{\sqrt{N}} + |\cos\xi|\Big)$$

(which is an easy consequence of (61)) and that
$$|e^{-i\xi} - z_{N,k}| \ge \frac{1}{C}\Big(\frac{1}{\sqrt{N}} + |\cos\xi|\Big)\,.$$

If $|\xi + \pi/2| \ge 3\,|\gamma_{k_0}|/\sqrt{N}$ and $\xi \in [-2\pi, 0]$, we find
$$e^{-i\xi} - z_{N,k} = 2\,e^{-i(\xi/2+\pi/4)}\sin\Big(\frac{\xi}{2} + \frac{\pi}{4}\Big) - \frac{\overline{\gamma}_k}{\sqrt{N}} + O\Big(\frac{1}{N}\Big)\,,$$

hence
$$\begin{aligned}|e^{-i\xi} - z_{N,k}| &\ge \Big|\sin\Big(\frac{\xi}{2} + \frac{\pi}{4}\Big)\Big| - \frac{|\gamma_{k_0}|}{\sqrt{N}} + O\Big(\frac{1}{N}\Big)\\ &\ge \frac{1}{2}\Big|\sin\Big(\frac{\xi}{2} + \frac{\pi}{4}\Big)\Big|\\ &\ge \max\Big\{\frac{1}{4}\,|\cos\xi|, \frac{6}{\pi}\,\frac{|\gamma_{k_0}|}{\sqrt{N}}\Big\}\,.\end{aligned}$$

On the other hand, if $|\xi + \pi/2| \le 3\,|\gamma_{k_0}|/\sqrt{N}$, we have
$$e^{-i\xi} - z_{N,k} = -\Big(\frac{\xi}{2} + \frac{\pi}{4}\Big) - \frac{\overline{\gamma}_k}{\sqrt{N}} + O\Big(\frac{1}{N}\Big)\,,$$

hence
$$|e^{-i\xi} - z_{N,k}| \ge \frac{1}{2}\,\frac{\inf \operatorname{Im}\gamma_k}{\sqrt{N}} = \frac{c_0}{\sqrt{N}} \ge C_0\max\Big\{\frac{1}{\sqrt{N}}, \frac{1}{6\,|\gamma_{k_0}|}\,|\cos\xi|\Big\}\,.$$

Thus we have obtained

$$\Big|\frac{d}{d\xi}\,\omega(z_{N,1},\dots,z_{N,N})(\xi)-\sum_{k=1}^{N}\operatorname{Im}\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z_{N,k}}\Big|$$

$$\le C\sum_{k=1}^{N}\frac{(1+\operatorname{Log}k)}{\sqrt{Nk}}\,\frac{1}{\dfrac{k}{N}+\cos^2\xi}$$

$$\le C\sqrt{N}\sum_{1}^{\infty}\frac{1+\operatorname{Log}k}{k\sqrt{k}}\,.$$

Now we look at

$$S_N(\xi)=\operatorname{Im}\sum_{k=1}^{N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z_{N,k}}$$

as at a Riemann sum: we have

$$\frac{\pi}{N}\,S_N(\xi)\underset{N\to\infty}{\longrightarrow}\operatorname{Im}\int_0^{\pi}\frac{i\,e^{-i\xi}\,d\omega}{e^{-i\xi}-Z(\omega)}\,.$$

If $\xi\neq\pm\pi/2$, we have a proper Riemann integral; if $\xi=\pm\pi/2$, the integrand is unbounded at 0 ($\xi=-\pi/2$) or π ($\xi=\pi/2$); but for $\xi=-\pi/2$ we have $e^{-i\xi}-Z(\omega)=e^{i\pi/4}\sqrt{2\,\omega}+O\,(\omega)$ near $\omega=0$ and thus

$$\int_0^{\pi}\frac{1}{|i-Z(\omega)|}\,d\omega<+\infty\,.$$

It is easy to evaluate the distance between $\pi S_N/N$ and the integral. We have

$$\Big|\int_0^{7\pi/(8N+6)}\frac{d\omega}{e^{-i\xi}-Z(\omega)}\Big|\le C\int_0^{7\pi/(8N+6)}\frac{d\omega}{\sqrt{\omega}}\le C'\,\frac{1}{\sqrt{N}}\,,$$

$$\Big|\int_{(8N-1)\pi/(8N+6)}^{\pi}\frac{d\omega}{e^{-i\xi}-Z(\omega)}\Big|\le C\int_{(8N-1)\pi/(8N+6)}^{\pi}\frac{d\omega}{\sqrt{\pi-\omega}}$$

$$\le C'\,\frac{1}{\sqrt{N}}\,,$$

$$\frac{1}{N}\left|\frac{1}{e^{-i\xi}-Z\Big(\dfrac{8N-1}{8N+6}\,\pi\Big)}\right|\le C'\,\frac{\sqrt{N}}{N}\,,$$

and finally for $1 \le k < N$

$$\left| \int_{(8k-1)\pi/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{1}{e^{-i\xi} - Z(\omega)}\, d\omega - \frac{8\pi}{8N+6}\, \frac{1}{e^{-i\xi} - Z\Big(\dfrac{8k-1}{8N+6}\Big)\pi} \right|$$

$$\le C \int_{(8k-1)/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{\Big| Z(\omega) - Z\Big(\dfrac{8k-1}{8N+6}\Big)\pi \Big|}{|e^{-i\xi} - Z(\omega)|\, \Big| e^{-i\xi} - Z\Big(\dfrac{8k-1}{8N+6}\pi\Big) \Big|}\, d\omega$$

$$\le C' \int_{(8k-1)\pi/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{\dfrac{1}{\sqrt{Nk}}}{\sqrt{\dfrac{k}{N}}\, \sqrt{\dfrac{k}{N}}}\, d\omega$$

$$\le C'' \frac{1}{k^{3/2}\sqrt{N}}$$

and thus

$$\Big| \frac{\pi}{N}\, S_N(\xi) - \operatorname{Im} \int_0^\pi i\, e^{-i\xi}\, \frac{d\omega}{e^{-i\xi} - Z(\omega)} \Big| \le C\, \frac{1}{\sqrt{N}} \,.$$

Thus, Result 10 is proved since writing $-e^{-2i\omega} = e^{-i\sigma}$ gives

$$\int_0^\pi i\, e^{-i\xi}\, \frac{d\omega}{e^{-i\xi} - \sqrt{2\sin\omega}\; e^{i(\pi/4-\omega/2)} - e^{i(\pi/2-\omega)}}$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} i\, e^{-i\xi}\, \frac{d\sigma}{e^{-i\xi} - \sqrt{e^{-i\sigma}} - \sqrt{1+e^{-i\sigma}}} \,.$$

We will easily prove Theorem 2 if we know the value of $I(\xi) = \int_{-\pi}^{\pi} i\, e^{-i\xi}\, d\sigma/(e^{-i\xi} - \xi(\sigma))$:

Result 11. *Let* $\xi(\sigma) = \sqrt{e^{-i\xi}} + \sqrt{1+e^{-i\sigma}}$ *and* $\xi \in [-\pi, \pi]$. *Then*

$$\int_{-\pi}^{\pi} i\, e^{-i\xi} \frac{d\sigma}{e^{-i\xi} - \xi(\sigma)}$$

$$= \begin{cases} -\pi \tan\Big(\dfrac{\xi}{2}\Big) + i\, \dfrac{\cos\xi}{\sin\xi}\, \operatorname{Log}\Big(\dfrac{1-\sin\xi}{1+\sin\xi}\Big)\,, & \textit{if } |\xi| \le \dfrac{\pi}{2}\,, \\ -\pi \operatorname{cotan}\Big(\dfrac{\xi}{2}\Big) + i\, \dfrac{\cos\xi}{\sin\xi}\, \operatorname{Log}\Big(\dfrac{1-\sin\xi}{1+\sin\xi}\Big)\,, & \textit{if } |\xi| \ge \dfrac{\pi}{2}\,. \end{cases} \tag{64}$$

We find that $I(\xi)$ is continuous, which is obvious since by (61)

$$|e^{-i\xi} - \xi(\sigma)| \geq C\sqrt{\pi^2 - \sigma^2}\,,$$

so that we may apply Lebesgue's dominated convergence theorem.

PROOF. Since $\xi(\sigma) = \overline{\xi}(-\sigma)$, we find that

$$I(-\xi) = -\overline{\int_{-\pi}^{\pi} \frac{i\,e^{-i\xi}}{e^{-i\xi} - \overline{\xi}(\sigma)}\,d\sigma} = -\overline{I(\xi)}\,,$$

so that it is enough to compute $I(\xi)$ for $\xi \in [0, \pi]$.

Writing $e^{-i\sigma} = u$, we may write

$$I(\xi) = \int_{-1+i0}^{-1-i0} \frac{e^{-i\xi}}{\sqrt{u} + \sqrt{1+u} - e^{-i\xi}}\,\frac{du}{u}\,,$$

where u runs clockwise on the circle $|u| = 1$. The function

$$f(z) = \frac{e^{-i\xi}}{z\,(\sqrt{z} + \sqrt{1+z} - e^{-i\xi})}$$

is analytical on $\mathbb{C}\backslash(-\infty, 0]$ and may be extended continuously to $(-\infty, 0] + i\,0$ and $(-\infty, 0] - i\,0$ but at three points: $z = 0$ (both a pole and a branching point), $z = -1$ (a branching point) and if $\xi \in [0, \pi/2]$ at $-\sin^2\xi - i\,0 = z_\xi$. Thus we may write:

- for $\xi \in [\pi/2, \pi]$

$$\begin{aligned} I(\xi) = \lim_{\varepsilon\to 0} \int_{-1}^{-\varepsilon} & \frac{e^{-i\xi}}{\sqrt{u + i\,0} + \sqrt{1+u} - e^{-i\xi}}\,\frac{du}{u} \\ &+ \int_{-\varepsilon}^{-1} \frac{e^{-i\xi}}{\sqrt{u - i\,0} + \sqrt{1+u} - e^{-i\xi}}\,\frac{du}{u} \\ &+ \int_{-\varepsilon+i0}^{-\varepsilon-i0} \frac{e^{-i\xi}}{\sqrt{u} + \sqrt{1+u} - e^{-i\xi}}\,\frac{du}{u} \\ &= 2\,i \int_0^1 \frac{dt}{\cos\xi - \sqrt{1-t^2}} - 2\,i\,\pi\,\frac{e^{-i\xi}}{1 - e^{-i\xi}} \\ &= 2\,i \int_0^{\pi/2} \frac{\cos\alpha}{\cos\xi - \cos\alpha}\,d\alpha - \pi \operatorname{cotan}\Big(\frac{\xi}{2}\Big) + \pi\,i\,. \end{aligned}$$

• if $\xi \in (0,\pi/2)$ we have, writing $t_\varepsilon^+ = \sqrt{\sin^2\xi+\varepsilon}$ and $t_\varepsilon^- = \sqrt{\sin^2\xi-\varepsilon}$

$$I(\xi) = \lim_{\varepsilon\to 0} A_\varepsilon + B_\varepsilon + C_\varepsilon\ ,$$

where

$$\begin{aligned}
A_\varepsilon &= \int_{-1}^{-(t_\varepsilon^+)^2} + \int_{-(t_\varepsilon^-)^2}^{-\varepsilon} \frac{e^{-i\xi}}{\sqrt{u+i\,0}+\sqrt{1+u}-e^{-i\xi}}\,\frac{du}{u} \\
&\quad + \int_{-\varepsilon}^{-(t_\varepsilon^-)^2} + \int_{-(t_\varepsilon^+)^2}^{-1} \frac{e^{-i\xi}}{\sqrt{u-i\,0}+\sqrt{1+u}-e^{-i\xi}}\,\frac{du}{u} \\
&= 2\,i\int_{\sqrt{\varepsilon}}^{t_\varepsilon^-} + \int_{t_\varepsilon^+}^{1} \frac{dt}{\cos\xi-\sqrt{1-t^2}} \\
B_\varepsilon &= \int_{-\varepsilon+i\,0}^{-\varepsilon-i\,0} \frac{e^{-i\xi}}{\sqrt{u}+\sqrt{1+u}-e^{-i\xi}}\,\frac{du}{u} \\
&= -2\,i\,\pi\,\frac{e^{-i\xi}}{1-e^{-i\xi}} + O\left(\sqrt{\varepsilon}\right) \\
&= -\pi\operatorname{cotan}\left(\frac{\xi}{2}\right) + i\,\pi + O\left(\sqrt{\varepsilon}\right), \\
C_\varepsilon &= \int_{-(t_\varepsilon^+)^2}^{-(t_\varepsilon^-)^2} \frac{e^{-i\xi}}{\sqrt{u+i\,0}+\sqrt{1+u}-e^{-i\xi}}\,\frac{du}{u} \\
&\quad + \int_{z_\xi+\varepsilon}^{z_\xi-\varepsilon} \frac{e^{-i\xi}\,du}{\left(\sqrt{u}+\sqrt{1+u}-e^{-i\xi}\right)u} \\
&= -i\,\pi\,2\,i\operatorname{cotan}\xi + O(\varepsilon) \\
&= 2\pi\operatorname{cotan}\xi + O(\varepsilon)\,,
\end{aligned}$$

since the residue of

$$f(\xi) = \frac{e^{-i\xi}}{\sqrt{u}+\sqrt{1+u}-e^{-i\xi}}\,\frac{1}{u}$$

at $z_\xi = -\sin^2\xi - i\,0$ is equal to

$$\frac{e^{-i\xi}}{\dfrac{1}{2}\,\dfrac{1}{\sqrt{z_\xi}}+\dfrac{1}{2}\,\dfrac{1}{\sqrt{1+z_\xi}}}\,\frac{1}{z_\xi} = \frac{2\sqrt{z_\xi}\,\sqrt{1+z_\xi}}{z_\xi} = 2\,i\operatorname{cotan}\xi\,.$$

Hence we have

$$\begin{aligned} I(\xi) &= \pi\Big(2\operatorname{cotan}\xi - \operatorname{cotan}\Big(\frac{\xi}{2}\Big)\Big) \\ &\qquad + i\,\pi + 2\,i \lim_{\varepsilon\to 0} \int_0^{t_\varepsilon^-} + \int_{t_\varepsilon^+}^1 \frac{dt}{\cos\xi - \sqrt{1-t^2}} \\ &= -\pi\tan\left(\frac{\xi}{2}\right) + i\,\pi + 2\,i\lim_{\varepsilon\to 0}\int_0^{\alpha_\varepsilon^-} + \int_{\alpha_\varepsilon^+}^{\pi/2} \frac{\cos\alpha\, d\alpha}{\cos\xi - \cos\alpha}\,, \end{aligned}$$

where $\alpha_\varepsilon^- = \arcsin t_\varepsilon^-$ and $\alpha_\varepsilon^+ = \arcsin t_\varepsilon^+$.

Thus, for proving Result 11, we just have to estimate for $\xi \in (0,\pi)$, $\xi \neq \pi/2$

$$A(\xi) = \lim_{\varepsilon\to 0}\int_0^{\alpha_\varepsilon^-} + \int_{\alpha_\varepsilon^+}^{\pi/2} \frac{\cos\alpha\, d\alpha}{\cos\xi - \cos\alpha}$$

with $\alpha_\varepsilon^- = \arcsin\sqrt{\sin^2\xi - \varepsilon}$ and $\alpha_\varepsilon^+ = \arcsin\sqrt{\sin^2\xi + \varepsilon}$. We do the usual change of variable $\beta = \tan(\alpha/2)$. Then

$$A(\xi) = \lim_{\varepsilon\to 0}\int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \frac{2\,(1-\beta^2)}{(1+\beta^2)\,((1+\beta^2)\cos\xi - (1-\beta^2))}\, d\beta\,.$$

We write

$$\begin{aligned} (1+\beta^2)\cos\xi - (1-\beta^2) &= \beta^2(1+\cos\xi) - (1-\cos\xi) \\ &= 2\,\beta^2\cos^2\left(\frac{\xi}{2}\right) - 2\sin^2\left(\frac{\xi}{2}\right)\,, \end{aligned}$$

hence

$$\begin{aligned} A(\xi) &= \frac{1}{\cos^2\left(\frac{\xi}{2}\right)} \lim_{\varepsilon\to 0}\int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \frac{1-\beta^2}{(1+\beta^2)\Big(\beta^2 - \tan^2\left(\frac{\xi}{2}\right)\Big)}\, d\beta \\ &= \frac{1}{\cos^2\left(\frac{\xi}{2}\right)} \lim_{\varepsilon\to 0}\int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \Bigg(\frac{-2}{1+\tan^2\left(\frac{\xi}{2}\right)}\,\frac{1}{1+\beta^2} \\ &\qquad\qquad + \frac{1-\tan^2\Big(\frac{\xi}{2}\Big)}{1+\tan^2\Big(\frac{\xi}{2}\Big)}\,\frac{1}{\beta^2 - \tan^2\Big(\frac{\xi}{2}\Big)}\, d\beta\Bigg) \end{aligned}$$

$$= \lim_{\varepsilon\to 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \left(\frac{-2}{1+\beta^2} \right.$$

$$\left. + \frac{\cos\xi}{\sin\xi}\left(\frac{1}{\beta - \tan\left(\frac{\xi}{2}\right)} - \frac{1}{\beta + \tan\left(\frac{\xi}{2}\right)}\right)\right) d\beta$$

$$= \lim_{\varepsilon\to 0} -\frac{\pi}{2} + \frac{\cos\xi}{\sin\xi} \operatorname{Log}\left|\frac{1-\tan\left(\frac{\xi}{2}\right)}{1+\tan\left(\frac{\xi}{2}\right)}\right|$$

$$- \frac{\cos\xi}{\sin\xi} \operatorname{Log}\left|\frac{\beta_\varepsilon^+ - \tan\left(\frac{\xi}{2}\right)}{\beta_\varepsilon^+ + \tan\left(\frac{\xi}{2}\right)}\right| + \frac{\cos\xi}{\sin\xi} \operatorname{Log}\left|\frac{\beta_\varepsilon^- - \tan\left(\frac{\xi}{2}\right)}{\beta_\varepsilon^- + \tan\left(\frac{\xi}{2}\right)}\right|$$

$$= -\frac{\pi}{2} + \frac{\cos\xi}{2\sin\xi} \operatorname{Log}\left(\frac{1-\tan\left(\frac{\xi}{2}\right)}{1+\tan\left(\frac{\xi}{2}\right)}\right)^2$$

$$+ \frac{\cos\xi}{\sin\xi} \lim_{\varepsilon\to 0} \operatorname{Log}\left|\frac{\beta_\varepsilon^- - \tan\left(\frac{\xi}{2}\right)}{\beta_\varepsilon^+ - \tan\left(\frac{\xi}{2}\right)}\right|.$$

Now we have

$$\left(\frac{1-\tan\left(\frac{\xi}{2}\right)}{1+\tan\left(\frac{\xi}{2}\right)}\right)^2 = \frac{\cos^2\left(\frac{\xi}{2}\right) - 2\sin\left(\frac{\xi}{2}\right)\cos\left(\frac{\xi}{2}\right) + \sin^2\left(\frac{\xi}{2}\right)}{\cos^2\left(\frac{\xi}{2}\right) + 2\sin\left(\frac{\xi}{2}\right)\cos\left(\frac{\xi}{2}\right) + \sin^2\left(\frac{\xi}{2}\right)}$$

$$= \frac{1-\sin\xi}{1+\sin\xi},$$

while we have for $\xi \in (0, \pi/2)$

$$\beta_\varepsilon^- - \tan\left(\frac{\xi}{2}\right) \sim \frac{1}{2}\left(1+\tan^2\left(\frac{\xi}{2}\right)\right)(\alpha_\varepsilon^- - \xi)$$

$$\sim \frac{1}{2}\left(1+\tan^2\left(\frac{\xi}{2}\right)\right)\frac{\sqrt{\sin^2\xi - \varepsilon} - \sin\xi}{\cos\xi}$$

$$\sim \frac{-\varepsilon\left(1+\tan^2\left(\frac{\xi}{2}\right)\right)}{4\sin\xi\cos\xi}$$

and

$$\beta_\varepsilon^+ - \tan\Big(\frac{\xi}{2}\Big) \sim \frac{+\varepsilon\Big(1+\tan^2\frac{\xi}{2}\Big)\Big)}{4\sin\xi\cos\xi} \sim -\Big(\beta_\varepsilon^- - \tan\Big(\frac{\xi}{2}\Big)\Big).$$

Thus

$$A(\xi) = -\frac{\pi}{2} + \frac{\cos\xi}{2\sin\xi}\,\mathrm{Log}\,\frac{1-\sin\xi}{1+\sin\xi}$$

and Result 11 is proved.

Now, (63) gives

$$\Big|\frac{d}{d\xi}\,\omega(z_{N,1},\dots,z_{N,N})(\xi) - \frac{N}{2\pi}\,\frac{\cos\xi}{\sin\xi}\,\mathrm{Log}\frac{1-\sin\xi}{1+\sin\xi}\Big| \le C\sqrt{N}\,.$$

Integrating this for $\xi \in [-\pi,\pi]$ we get

$$\Big|\omega(z_{N,1},\dots,z_{N,N})(\xi) - \frac{N}{2\pi}\,(\mathrm{Li}_2(-\sin\xi) - \mathrm{Li}_2(\sin\xi))\Big| \le C\sqrt{N}\,.$$

Since both functions are 2π-periodical, this inequality can be extended to all $\xi \in \mathbb{R}$ and Theorem 2 is proved.

8. Almost linear-phased Daubechies filters.

In this section, we prove Theorem 3. The proof is very easy. Indeed, we want to estimate for $N = 4q$, $\omega(z_{N,1}^{\varepsilon_{N,1}},\dots,z_{N,N}^{\varepsilon_{N,N}})(\xi)$ with $\varepsilon_{N,k} = 1$ if $k = 0 \bmod 4$ or $k = 1 \bmod 4$, and $\varepsilon_{N,k} = -1$ otherwise.

We have (writing ω_N for $\omega(z_{N,1}^{\varepsilon_{N,1}},\dots,z_{N,N}^{\varepsilon_{N,N}})$, K_N for $\{k\in\mathbb{N} : 1 \le k \le N,\ \varepsilon_{N,k} = 1\}$ and $\tilde K_N$ for $\{k \in \mathbb{N} : 1 \le k \le N,\ \varepsilon_{N,k} = -1\}$)

$$\frac{d\omega_N}{d\xi} = \mathrm{Im}\sum_{k\in K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-z_{N,k}} + \sum_{k\in\tilde K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-\frac{1}{\overline{z}_{N,k}}}$$

(we have used that for $k\in\tilde K_N$, $N+1-k\in\tilde K_N$ and $z_{N,k} = \overline{z}_{N,N+1-k}$). Hence we have

$$\frac{d\omega_N}{d\xi} = \mathrm{Im}\Big(\sum_{k\in K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-z_{N,k}} - \sum_{k\in\tilde K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-z_{N,k}}\Big)$$

$$+\,\mathrm{Im}\Big(\sum_{k\in\tilde K_N}\frac{i\,e^{-i\xi}}{e^{-i\xi}-z_{N,k}} + \frac{i\,e^{-i\xi}}{e^{-i\xi}-\dfrac{1}{\overline{z}_{N,k}}}\Big).$$

But we have

$$\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z}+\frac{i\,e^{-i\xi}}{e^{-i\xi}-\dfrac{1}{\overline{\overline{Z}}}}=\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z}+\frac{i\,\overline{Z}}{\overline{Z}-e^{+i\xi}}$$

$$=\frac{i\,e^{-i\xi}(e^{i\xi}-\overline{Z})+i\,\overline{Z}\,(-e^{-i\xi}+Z)}{|e^{-i\xi}-Z|^2}$$

$$=\frac{i\,(1-2\,\overline{Z}\,e^{-i\xi}+|Z|^2)}{|Z-e^{-i\xi}|^2}$$

$$=i+\frac{i\,(Z\,e^{i\xi}-\overline{Z}\,e^{-i\xi})}{|Z-e^{-i\xi}|^2}\ ,$$

hence

$$\operatorname{Im}\left(\frac{i\,e^{-i\xi}}{e^{-i\xi}-Z}+\frac{i\,e^{-i\xi}}{e^{-i\xi}-\dfrac{1}{\overline{\overline{Z}}}}\right)=1\,.$$

Thus, we have obtained

$$\frac{d\omega_N}{d\xi}=\frac{N}{2}+\operatorname{Im}\sum_{k=1}^{q}i\,e^{-i\xi}\Big(\frac{1}{e^{-i\xi}-z_{N,4k-3}}-\frac{1}{e^{-i\xi}-z_{N,4k-2}}$$
$$-\frac{1}{e^{-i\xi}-z_{N,4k-1}}+\frac{1}{e^{-i\xi}-z_{N,4k}}\Big)\,.$$

Now we write, for $r\in\{1,2,3\}$

$$\frac{1}{e^{-i\xi}-z_{N,4k-r}}=\frac{1}{e^{-i\xi}-z_{N,4k}}+\frac{z_{N,4k-r}-z_{N,4k}}{(e^{-i\xi}-z_{N,4k})\,(e^{-i\xi}-z_{N,4k-r})}$$
$$=\frac{1}{e^{-i\xi}-z_{N,4k}}+\frac{z_{N,4k-r}-z_{N,4k}}{(e^{-i\xi}-z_{N,4k})^2}$$
$$+\frac{(z_{N,4k-r}-z_{N,4k})^2}{(e^{-i\xi}-z_{N,4k})^2(e^{-i\xi}-z_{N,4k-r})}\ .$$

We have, writing $\tilde{k}=\min\{k,q+1-k\}$

$$\left|\frac{(z_{N,4k-r}-z_{N,4k})^2}{(e^{-i\xi}-z_{N,4k})^2(e^{-i\xi}-z_{N,4k-r})}\right|\le C\,\frac{\dfrac{1}{N\tilde{k}}}{\left(\dfrac{\tilde{k}}{N}+\cos^2\xi\right)^{3/2}}\le\frac{C\,\dfrac{1}{\tilde{k}}\sqrt{\dfrac{1}{N\tilde{k}}}}{\dfrac{\tilde{k}}{N}+\cos^2\xi}$$

and

$$\int_{-\pi}^{\pi} \frac{d\xi}{\dfrac{\tilde{k}}{N} + \cos^2 \xi} \leq 4 \int_0^{\arccos \sqrt{\tilde{k}/N}} \frac{d\xi}{\cos^2 \xi} + \frac{4N}{\tilde{k}} \int_{\arccos \sqrt{\tilde{k}/N}}^{\pi/2} d\xi$$

$$= 4\sqrt{\frac{N}{\tilde{k}}} \sin\left(\arccos\sqrt{\frac{\tilde{k}}{N}}\right) + \frac{4N}{\tilde{k}} \arcsin\sqrt{\frac{\tilde{k}}{N}}$$

$$\leq 4\sqrt{\frac{N}{\tilde{k}}} + 2\pi\sqrt{\frac{N}{\tilde{k}}}\,,$$

so that

$$\int_{-\pi}^{\pi} \Big| \frac{d\omega_N}{d\xi} - \frac{N}{2} - \mathrm{Im} \sum_{k=1}^{q} \frac{z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}}{(e^{-i\xi} - z_{N,4k})^2} \Big|$$

$$\leq C \sum_{k=1}^{\infty} \frac{1}{k^2} = C' < +\infty$$

and

$$\int_{-\pi}^{\pi} \Big| \frac{d\omega_N}{d\xi} - \frac{N}{2} \Big| \, d\xi$$

$$\leq C' + C \sum_{k=1}^{q} \sqrt{\frac{N}{\tilde{k}}} \, |z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}| \,.$$

When $\tilde{k} \leq k_0$, we write

$$|z_{N,4k-r} - z_{N,4k+1-r}| = O\Big(\frac{1}{\sqrt{N\tilde{k}}}\Big)$$

and obtain

$$\sum_{\tilde{k} \leq k_0} \sqrt{\frac{N}{\tilde{k}}} \, |z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}| \leq C \operatorname{Log} k_0 \,.$$

When $\tilde{k} \geq k_0$, we may write as in formula (58)

$$z_{N,4k-r} = y_{N,4k-r} + \sqrt{y_{N,4k-r}^2 - 1} + O\Big(\frac{\operatorname{Log} \tilde{k}}{\tilde{k}\sqrt{N\tilde{k}}}\Big)$$

$$= \sqrt{\omega_{N,4k-r}} + \sqrt{\omega_{N,4k-r} + 1} + O\Big(\frac{\operatorname{Log} \tilde{k}}{\tilde{k}\sqrt{N\tilde{k}}}\Big)\,,$$

where

$$\omega_{N,\ell} = -e^{-2i\pi(8\ell-1)/(8N+6)} - \frac{1}{N}\, e^{-2i\pi(8\ell-1)/(8N+6)} \operatorname{Log}\Big(2\sqrt{2N\pi \sin\Big(\frac{8\ell-1}{8N+6}\,\pi\Big)}\,\Big).$$

We write

$$\sqrt{\alpha+\beta} = \sqrt{\alpha} + \frac{\beta}{\sqrt{\alpha}+\sqrt{\alpha+\beta}} = \sqrt{\alpha} + \frac{\beta}{2\sqrt{\alpha}} - \frac{\beta^2}{2\sqrt{\alpha}\,(\sqrt{\alpha}+\sqrt{\alpha+\beta})^2}\,.$$

Now, we have: $\omega_{N,\ell}$ is order of magnitude 1, $\omega_{N,\ell}+1$ is of order of magnitude $\min\{\sqrt{\ell/N}, \sqrt{(N+1-\ell)/N}\}$ and $\omega_{N,\ell+1}-\omega_{N,\ell}$ is of order of magnitude $1/N$. Thus, we may write

$$\sqrt{\omega_{N,4k-r}} = \sqrt{\omega_{N,4k}} + O\Big(\frac{1}{N}\Big)$$

$$\begin{aligned}\sqrt{1+\omega_{N,4k-r}} &= \sqrt{1+\omega_{N,4k}} + \frac{\omega_{N,4k-r}-\omega_{N,4k}}{2\sqrt{1+\omega_{N,4k}}} + O\Big(\frac{1}{\tilde{k}\sqrt{N\tilde{k}}}\Big)\\ &= \sqrt{1+\omega_{N,4k}} + \frac{e^{-2i\pi(32k-1)/(8N+6)}(1-e^{2i8r\pi/(8N+6)})}{2\sqrt{1+\omega_{N,4k}}}\\ &\quad + O\Big(\frac{\operatorname{Log}\tilde{k}}{N^2}\Big) + O\Big(\frac{1}{\tilde{k}\sqrt{N\tilde{k}}}\Big)\end{aligned}$$

and finally

$$\begin{aligned}&\frac{\sqrt{N}}{\tilde{k}}\,|z_{N,4k-3}-z_{N,4k-2}-z_{N,4k-1}+z_{N,4k}|\\ &\quad= \sqrt{\frac{N}{\tilde{k}}}\,\Big|\frac{e^{2i24\pi/(8N+6)}-e^{2i16\pi/(8N+6)}-e^{2i8\pi/(8N+6)}+1}{2\sqrt{1+\omega_{N,4k}}}\Big|\\ &\qquad + O\Big(\frac{\operatorname{Log}\tilde{k}}{\tilde{k}^2}\Big) + O\Big(\frac{1}{\sqrt{N\tilde{k}}}\Big) + O\Big(\frac{\operatorname{Log}\tilde{k}}{N\sqrt{N\tilde{k}}}\Big)\\ &\quad= O\Big(\frac{1}{N\tilde{k}}\Big) + O\Big(\frac{\operatorname{Log}\tilde{k}}{\tilde{k}^2}\Big) + O\Big(\frac{1}{\sqrt{N\tilde{k}}}\Big) + O\Big(\frac{\operatorname{Log}\tilde{k}}{N\sqrt{N\tilde{k}}}\Big)\,.\end{aligned}$$

We thus have proved Theorem 3, since

$$\sum_1^N \frac{1}{N\tilde{k}} \leq C\,\frac{\operatorname{Log} N}{N} = o\,(1)\,,$$

$$\sum_1^N \frac{1}{\sqrt{N\tilde{k}}} \leq C\,\frac{\sqrt{N}}{\sqrt{N}} = C < +\infty\,,$$

$$\sum_1^\infty \frac{\operatorname{Log} \tilde{k}}{\tilde{k}^2} < +\infty\,,$$

$$\sum_1^N \frac{\operatorname{Log} \tilde{k}}{N\sqrt{N\tilde{k}}} \leq C\,\frac{1}{N\sqrt{N}}\,\sqrt{N}\,\operatorname{Log} N = o\,(1)\,.$$

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Recibido: 20 de febrero de 1.995
Revisado: 10 de noviembre de 1.995

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Revista Matemática Iberoamericana
Vol. 13, N.º 2, 1997

L^p-bounds for spherical maximal operators on $\mathbb{Z}^n$

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Abstract. We prove analogue statements of the spherical maximal theorem of E. M. Stein, for the lattice points $\mathbb{Z}^n$. We decompose the discrete spherical "measures" as an integral of Gaussian kernels $s_{t,\varepsilon}(x) = e^{2\pi i |x|^2 (t+i\varepsilon)}$. By using Minkowski's integral inequality it is enough to prove L^p-bounds for the corresponding convolution operators. The proof is then based on L^2-estimates by analysing the Fourier transforms $\hat{s}_{t,\varepsilon}(\xi)$, which can be handled by making use of the "circle" method for exponential sums. As a corollary one obtains some regularity of the distribution of lattice points on small spherical caps.

1. Introduction.

Let us denote by σ_λ the characteristic function of the sphere of radius $\lambda^{1/2}$ in $\mathbb{Z}^n$, *i.e.*

$$\sigma_\lambda = \chi_{\{x\in\mathbb{Z}^n : |x|^2=\lambda\}} \qquad \text{and} \qquad S_\lambda = \sum_{x\in\mathbb{Z}^n} \sigma_\lambda(x)\,.$$

Let Λ be a fixed positive number and define the spherical maximal operator as

$$M_\Lambda f(x) = \sup_{\Lambda\le\lambda<2\Lambda} \Big| \Big(\frac{\sigma_\lambda}{S_\lambda} * f\Big)(x)\Big|\,. \tag{1}$$

It is proved that

Theorem 1. *If $n \geq 5$, $p > n/(n-2)$, $f \in L^p(\mathbb{Z}^n)$, then*

$$\|M_\Lambda f\|_{L^p(\mathbb{Z}^n)} \leq c_{n,p}\, \|f\|_{L^p(\mathbb{Z}^n)}\,, \tag{2}$$

where the constant $c_{n,p}$ is independent of Λ.

We generalize estimate (2) to the case of the k-spheres, which are defined by

$$\sigma_\lambda = \chi_{\{x \in \mathbb{Z}^n : |x|^2 = \lambda\}}$$

and it is proved.

Theorem 2. *Let $k \geq 2$, $K = 2^{k-1}$, then for $n > 4\,K\,k$, $p > n/(n-4Kk)$ we have*

$$\|M_{\Lambda,k} f\|_p \leq c_{n,k,p}\, \|f\|_p\,, \tag{3}$$

where the constant $c_{n,k,p}$ is independent of Λ.

It is well-known, that for $n \geq 5$, there exist constants $0 < c_n < C_n$ such that

$$c_n\, \lambda^{n/2-1} \leq S_\lambda \leq C_n\, \lambda^{n/2-1}\,. \tag{4}$$

We start with the decompositions

$$\sigma_\lambda(x) = \int_0^1 e^{2\pi i(|x|^2-\lambda)t}\, dt$$

and

$$e^{-2\pi\varepsilon\lambda}\, \sigma_\lambda(x) = \int_0^1 e^{2\pi i|x|^2(t+i\varepsilon)}\, e^{-2\pi i\lambda t}\, dt \tag{5}$$

and define the modified maximal operator as

$$\tilde{M}_{\lambda,\varepsilon} f(x) = \sup_{\Lambda \leq \lambda < 2\Lambda} |(e^{-2\pi\varepsilon\lambda}\lambda^{-n/2+1}\sigma_\lambda) * f(x)|\,. \tag{6}$$

From inequality (4) it follows, that if $\Lambda \leq \varepsilon^{-1}$ and $f \geq 0$, we have

$$M_\Lambda f(x) \leq c\, \tilde{M}_{\Lambda.\varepsilon} f(x)\,, \tag{8}$$

for every x, so it is enough to prove (2) for the modified maximal operator. Introducing the convolution operator

$$S_{t,\varepsilon} f = s_{t,\varepsilon} * f\,, \qquad \text{where} \qquad s_{t,\varepsilon}(x) = e^{2\pi i\,|x|^2(t+i\varepsilon)}\,.$$

Minkowski's integral inequality together with formulae (4) and (5) imply

$$\|\tilde{M}_{\Lambda,\varepsilon} f\|_p \le c_n\,\Lambda^{-n/2+1} \int_0^1 \|S_{t,\varepsilon} f\|_p\,dt\,. \tag{9}$$

In order to understand the integrand on the right side of inequality (9), we will apply the so-called "circle" method in the variable t (*cf.* [3]). First we decompose the interval $[0,1]$ into neighborhoods of rationals, whose denominator is smaller than a given number N as follows:

Let $N > 0$ be given and consider the set

$$H = \{p/q\,:\,1 \le q \le N,\ 0 < p \le q,\ (p,q) = 1\}\,,$$

and define the neighborhoods

$$V_{p,q} = \Big\{t \in [0,1]\,:\,\Big|t - \frac{p}{q}\Big| = \min_{r\in H} |t - r|\Big\}\,.$$

From the obvious inequalities:

if $p/q \ne p_1/q_1$ then

$$\Big|\frac{p}{q} - \frac{p_1}{q_1}\Big| \ge \frac{1}{2Nq} + \frac{1}{2Nq_1}\,. \tag{I}$$

For every $t \in [0,1]$ there exists $p/q \in H$ such that,

$$\Big|t - \frac{p}{q}\Big| \le \frac{1}{Nq}\,, \tag{II}$$

it follows

$$W^*_{p,q} \subseteq V_{p,q} \subseteq W_{p,q}\,,$$

where $W^*_{p,q} = \{t : |t - p/q| < 1/(2Nq)\}$ $W_{p,q} = \{t : |t - p/q| \le 1/(Nq)\}$.

The crucial point is that one can estimate the Fourier transform $\hat{s}_{t,\varepsilon}(\xi)$ separately in each neighborhood $V_{p,q}$ by using Poisson summation and the properties of Gaussian sums, as it is shown below.

2. Fourier transform estimates.

Lemma 1. *Let $t = p/q + \tau$, $t \in V_{p,q}$, then*

$$\|S_t f\|_{L^2} \le c_n\, q^{-n/2} \min\{\varepsilon^{-n/2}, \tau^{-n/2}\}\, \|f\|_2\,. \tag{10}$$

PROOF. The Fourier transform of the function $s_t = s_{t,\varepsilon}$ is defined by

$$\hat{s}_t(\xi) = \sum e^{2\pi i(|x|^2(t+i\varepsilon)+x\xi)}\,, \qquad \xi \in \Pi^n$$

and inequality (10) is equivalent to

$$\sup_\xi |\hat{s}_t(\xi)| \le c_n\, q^{-n/2} \min\{\varepsilon^{-n/2}, \tau^{-n/2}\}\,. \tag{11}$$

Since $\hat{s}_t(\xi)$ is the product of n one dimensional functions, *i.e.* $\hat{s}_t(\xi) = \Pi_j \hat{s}_t(\xi_j)$ it is enough to prove formula (11) in case when $n = 1$. By Poisson summation and substituting $x = rq + s$, we have

$$\begin{aligned}
\hat{s}_t(\xi) &= \sum_x e^{2\pi i x^2 p/q}\, s_\tau(x) e^{2\pi i x\xi} \\
&= \sum_{s=0}^{q-1} e^{2\pi i s^2 p/q} \sum_r s_\tau(rq+s)\, e^{2\pi i(rq+s)\xi} \\
&= \sum_{s=0}^{q-1} e^{2\pi i s^2 p/q} \sum_l \frac{1}{q}\, e^{-2\pi i s/q}\, \tilde{s}_\tau\Big(\frac{l}{q} - \xi\Big)\,,
\end{aligned}$$

where $\tilde{s}_\tau(\xi) = \int_{\mathbb{R}} s_\tau(x)\, e^{-2\pi i x\xi}\, dx$ is simply the Fourier transform of s_τ as function on $\mathbb{R}$, which has the simple form

$$\tilde{s}_\tau(\xi) = \int_{\mathbb{R}} e^{2\pi i(x^2(\tau+i\varepsilon)-x\xi)}\, d\xi = (\varepsilon - i\tau)^{-1/2}\, e^{-\xi^2/(\varepsilon-i\tau)}\,.$$

So we have the formula

$$\begin{aligned}
\hat{s}_t(\xi) &= (\varepsilon - i\tau)^{-1/2} \\
&\quad\cdot \sum_l \Big(\frac{1}{q} \sum_{s=0}^{q-1} e^{2\pi i(p/qs^2 - l/qs)}\Big) e^{-\pi(\xi-l/q)^2/(2(\varepsilon-i\tau))}\,.
\end{aligned} \tag{12}$$

In order to estimate this expression, first we note that because of the properties of Gaussian sums one has

$$\Big|\frac{1}{q}\sum_{s=0}^{q-1} e^{2\pi i(s^2p/q\,-\,sl/q)}\Big| \le \sqrt{2}\,q^{-1/2}\,.$$

Now we choose $\varepsilon = \Lambda^{-1}$, $N = [\Lambda^{1/2}]$ (where $[x]$ denotes the integer part of x), and since $t = p/q + \tau \in V_{p,q}$ we have $\tau \le 1/(Nq) \le \varepsilon^{1/2}q^{-1}$. It follows

$$\frac{\varepsilon}{q^2(\varepsilon^2+\tau^2)} \ge \frac{1}{2\,q^2\varepsilon} \ge \frac{1}{2}\,, \qquad \text{if } \tau \le \varepsilon$$

and

$$\frac{\varepsilon}{q^2(\varepsilon^2+\tau^2)} \ge \frac{\varepsilon}{2\,q^2\tau^2} = \frac{1}{2}\,(\varepsilon^{1/2}q^{-1}\tau^{-1})^2 \ge \frac{1}{2}\,, \qquad \text{if } \varepsilon \le \tau\,.$$

Now it is easy to estimate the right hand side of formula (12)

$$\begin{aligned}|\hat{s}_t(\xi)| &\le |\varepsilon - i\tau|^{-1/2}q^{-1/2}\sum_l e^{-\pi\varepsilon/(2(q\xi-l)^2(q^2(\varepsilon^2+\tau^2)))}\\ &\le c\,q^{1/2}(\varepsilon+\tau)^{-1/2}\Big(\sum_l e^{-\pi/4(q\xi-l)^l}\Big)\\ &\le c\,q^{-1/2}(\varepsilon+\tau)^{-1/2}\,,\end{aligned}$$

where the constant c is independent of q and ξ.

This proves inequality (10) and Lemma 1 follows.

PROOF OF THEOREM 1. It is easy to see that

$$\|S_t f\|_1 \le \|S_t\|_1\,\|f\|_1 \le c_n\,\varepsilon^{-n/2}\,\|f\|_1\,. \tag{13}$$

Let $1 < p \le 2$ and we choose the number α such that $1/p = \alpha/2 + (1-\alpha)$. Interpolating between estimates (10) and (13), we have

$$\|S_t f\|_p \le c_n\,q^{-n\alpha/2}\varepsilon^{-n/2}\min\Big\{1, \Big(\frac{\tau}{\varepsilon}\Big)^{-n\alpha/2}\Big\}\|f\|_p\,.$$

This implies

$$\begin{aligned}\int_{V_{p,q}} \|S_t f\|_p &\le c_n\,\Lambda^{-n/2+1}q^{-n\alpha/2}\\ &\qquad\cdot\Big(\int_0^\varepsilon \varepsilon^{-n/2}\,d\tau + \varepsilon^{-n/2}\int_\varepsilon^\infty \Big(\frac{\tau}{\varepsilon}\Big)^{-n\alpha/2} d\tau\Big)\\ &\le c_n\,q^{-n\alpha/2}(\varepsilon\,\Lambda)^{-n/2+1}\\ &\le c_n'\,q^{-n\alpha/2}\|f\|_p\,.\end{aligned}$$

It follows when $n > 4$, $\alpha > 4/n$ or equivalently $p > n/(n-2)$

$$\begin{aligned}\|\tilde{M}_{\Lambda,\varepsilon} f\|_p &\le c_n \Big(\sum_{p/q \in H} q^{-n\alpha/2} \Big) \|f\|_p \\ &\le c_n \|f\|_p \Big(\sum_{q=1}^{\infty} q^{-n\alpha/2+1} \Big) \\ &\le c_n \|f\|_p \,.\end{aligned}$$

This proves Theorem 1.

3. Estimates for k-spheres.

We now briefly describe how the L^2 estimate generalize to k-spheres. The extra complications arise are similar to those of the Waring problem. Indeed we refer to the analysis of Hardy-Littlewood in [3], where it was proved that for $n > 2^{k-1}k$

$$c_{n,k}\, \lambda^{n/k-1} \le S_{\lambda,k} \le C_{n,k}\, \lambda^{n/k-1}\,, \tag{14}$$

hence as for $k = 2$ one has

$$\|M_{\Lambda,k} f\|_p \le c_{n,k}\, \Lambda^{-n/k+1} \int_0^1 \|S_t f\|_p \, dt\,,$$

where the kernel of the operator S_t is $s_t(x) = e^{2\pi i (\sum_j |x_j|^k)(t+i\varepsilon)}$.

For $t = p/q + \tau$ Poisson summation yields

$$\hat{s}_t(\xi) = \sum_{l=-\infty}^{\infty} \Big(\frac{1}{q} \sum_{s=0}^{q-1} e^{2\pi i (s^k p/q - sl/q)} \Big) \tilde{s}_\tau \Big(\frac{l}{q} - \xi \Big)\,,$$

where $\tilde{s}_\tau(\eta) = \int_{\mathbb{R}} s_\tau(x)\, e^{-2\pi i x\eta}\, dx$ is the Fourier transform on $\mathbb{R}$.

The decomposition into neighborhoods of rationals for $k > 2$ looks as follows

$$H_{k,0} = \Big\{ \frac{p}{q} \,:\, q \le \Lambda^{1/k} \Big\}\,, \qquad H_{k,1} = \Big\{ \frac{p}{q} \,:\, \Lambda^{1/k} < q \le \Lambda^{1-1/k} \Big\}\,.$$

$V_{p,q}$ is called a major arc if $p/q \in H_{k,0}$ and a minor arc if $p/q \in H_{k,1}$.

The reasoning of Theorem 1 generalizes to the major arcs as is shown in

Lemma 2. *Let $p/q \in H_{k,0}$, $t \in V_{p,q}$. Then we have for $p > n/(n-k+1)$*

$$\Lambda^{-n/k+1} \int_{V_{p,q}} \|S_t f\|_p \, dt \leq c_{n,k,p} \, q^{-n\alpha/K} \|f\|_p \,, \tag{15}$$

where $\alpha = 2\,p/(p-1)$.

Proof. We make use of the following estimates which are proved in [3] using slightly different notations.

$$|\tilde{s}_\tau(\eta)| \leq c \, |\tau + i\,\varepsilon|^{-1/k} \tag{16}$$

holds uniformly in η. Let $\eta = l - q\,\xi$ then one has

$$\begin{aligned} \Big|\tilde{s}_\tau\Big(\xi - \frac{l}{q}\Big)\Big| &\leq c_k \, |\tau + i\,\varepsilon|^{-1/(2(k-1))} \, q^{(k-2)/(2(k-1))} \\ &\quad \cdot |\eta|^{-(k-2)/(2(k-1))} \, e^{-(c|\eta|^{k/(k-1)})} \,. \end{aligned} \tag{17}$$

From inequalities (16) and (17) it follows

$$\sum_{l=-\infty}^{\infty} \Big|\tilde{s}_\tau\Big(\xi - \frac{l}{q}\Big)\Big| \leq c_k \, (|\tau + i\,\varepsilon|^{-1/k} + |\tau + i\,\varepsilon|^{-1/(2(k-1))} \, q^{(k-2)/(2(k-1))}) \,,$$

where inequality (16) is used when $|l - q\,\xi|$ is minimal. Also one has the standard estimate for the Weyl sum

$$\Big| q^{-1} \sum_{s=0}^{q-1} e^{2\pi i (s^k p/q - sl/q)} \Big| \leq c \, q^{-1/K} \,,$$

which holds uniformly in l, when $K = 2^{k-1}$. Taking the n-th power of $\hat{s}_t$, we obtain (in n-dimension) on the major arcs

$$\begin{aligned} &\sup_\xi |\hat{s}_t(\xi)| \\ &\leq c_{n,k} \, q^{-n/K} (|\tau + i\,\varepsilon|^{-n/k} + |\tau + i\,\varepsilon|^{-n/(2(k-1))} q^{n(k-2)/(2(k-1))}) \,. \end{aligned}$$

Let $1/p = \alpha/2 + 1 - \alpha$ and using the trivial estimate

$$\|s_t\|_1 \leq \Big(\sum_{x \in Z} e^{-2\pi\varepsilon |x|^k} \Big)^n \leq c_n \, \varepsilon^{-n/k} \,,$$

we obtain by interpolation

$$\|S_t\|_{p\to p} \le c_{n,k,p}\, q^{-n\alpha/K}\, \varepsilon^{-n/k} \cdot \Big(\Big|i+\frac{\tau}{\varepsilon}\Big|^{-n/k} + \Big|i+\frac{\tau}{\varepsilon}\Big|^{-n/(2(k-1))} (\varepsilon^{1/k}q)^{n(k-2)/(2(k-1))}\Big)^{\alpha}.$$

Using the facts that on a major arc $\varepsilon^{1/k}q = \Lambda^{-1/k}q \le 1$ and the simple estimate

$$\int_{\mathbb{R}} \Big|i+\frac{\tau}{\varepsilon}\Big|^{-\beta} d\tau \le c_\beta\, \varepsilon\,, \qquad \text{for } \beta > 1\,.$$

Estimate (15) follows when $n\alpha/(2(k-1)) > 1$. This proves Lemma 2.

On the minor arcs one can give direct estimates for $\hat{s}_t(\xi)$ exploiting that the denominator q is large

Lemma 3. *Let $p/q \in H_{k,1}$, $t \in V_{p,q}$ then we have*

$$\sup_{\xi} \hat{s}_{t,\varepsilon}(\xi) \le c_{n,k}\, \Lambda^{n/k - n/(4kK)}\,. \tag{18}$$

PROOF. It is enough to prove (18) in one dimension. Let $L = \Lambda^{1/k+\delta}$, for some $\delta > 0$, then one has

$$\hat{s}_t(\xi) = \sum_{x=0}^{L} e^{2\pi i (x^k(t+i\varepsilon)+x\xi)} + O\Big(\sum_{x>L} e^{-\varepsilon x^k}\Big) \tag{19}$$

and

$$\sum_{x>L} e^{-\varepsilon x^k} \le e^{-\varepsilon \Lambda^{1+\delta}} \varepsilon^{-1/k} \le e^{-\Lambda^{\delta}} \Lambda^{1/k} = O(1)\,.$$

To estimate the maim term of formula (19) we use partial summation. Let us define the sums $s_{l,\xi} = \sum_{x=0}^{l} e^{2\pi i (x^k p/q + x\xi)}$, we have

$$\begin{aligned} &\sum_{l=0}^{L} (s_{l,\xi} - s_{l-1,\xi})\, e^{2\pi i l^k (\tau+i\varepsilon)} \\ &\qquad = \sum_{l=0}^{L} s_{l,\xi}\, \big(e^{2\pi i l^k (\tau+i\varepsilon)} - e^{2\pi i (l-1)^k (\tau+i\varepsilon)}\big)\,. \end{aligned} \tag{20}$$

Since on the minor arcs $\tau \le \Lambda^{1/k-1} q^{-1} \le \Lambda^{-1}$, it follows

$$|e^{2\pi i(\tau+i\varepsilon)l^k} - e^{2\pi i (\tau+i\varepsilon) l^{k-1}}| \le c_k\, \Lambda^{-1/k+k\delta}\,,$$

so the sum in the formula (19) is less than equal

$$|\hat{s}_t(\xi)| \leq c_{k,\delta}\big(\max_{l\leq L}|s_{l,\xi}|\big)\Lambda^{(k+1)\delta}\,.$$

Using the standard estimate for Weyl sums (*cf.* [2, Chapter 6]), one has

$$\Big|\sum_{x=0}^{l} e^{2\pi(x^k p/q + x\xi)}\Big| \leq c_{k,\delta}\,\Lambda^{(1/k)(1-1/(2K))+2k\delta} \tag{21}$$

holds uniformly in ξ and p, when $\Lambda^{1/k} \leq q \leq \Lambda^{1-1/k}$, $l \leq \Lambda^{1/k+\delta}$.

The above estimates imply for $\delta \leq \delta(k)$ the estimate

$$|\hat{s}_{t,\varepsilon}(\xi)| \leq c_{k,\delta}\,\Lambda^{1/k-1/(2kK)+4k\delta} \leq c_k\,\Lambda^{1/k-1/(4kK)} \leq c_{n,k} \tag{22}$$

holds uniformly in ξ , and Lemma 3 follows.

PROOF OF THEOREM 2. Interpolation between the trivial L^1, and the L^2 estimate (18), shows that for $1/p = 1-\alpha/2$ on a minor arc we have

$$\Lambda^{-n/k+1}\|S_t\|_{p\to p} \leq c_{n,k,p}\,\Lambda^{n/k-n\alpha/(4Kk)}\,.$$

Hence for $n > 4Kk$, $p > n/(n-4Kk)$ one has

$$\begin{aligned}\|M_{\Lambda,k}f\|_p &\leq c_{n,k,p}\Big(\sum_{p/q\in H_{0,k}} q^{-n\alpha/K} + \Lambda^{-n/k+1+n/k-n\alpha/(4kK)}\Big)\|f\|_p\\ &\leq c'_{n,k,p}\,\|f\|_p\ ,\end{aligned}$$

since $n\alpha/K > 2$ and $n\alpha/(4Kk) > 1$. This proves Theorem 2.

We would like to point out how these estimates are connected with the distribution of integer points on spherical caps. More precisely we define the maximal function

$$s^*_{\lambda,l} = \sup_{\lambda\leq\mu<2\lambda}|S^{n-1}_\mu \cap (x+D^n_l)|\,,$$

where $x + D^n_l = \{u \in \mathbb{Z}^n : |x-u| \leq l^{1/2}\}$. On the other hand the average number of integer points on a spherical cap of radius $l^{1/2}$ lying on the sphere of radius $\lambda^{1/2}$ is clearly

$$\tilde{S}_{\lambda,l} = l^{(n-1)/2}\lambda^{-1/2}\,.$$

Theorem 1. implies the following

Corollary 1. *Let $\varepsilon > 0$. If $l > \lambda^{1-\varepsilon/2}$ then*

$$\lambda^{-n/2}|\{x \in D^n_\lambda \,:\, s^*_{\lambda,l}(x) > \lambda^\varepsilon\, \tilde{s}_{\lambda,l}\}| < c_{n,\varepsilon}\, \lambda^{-\varepsilon n/(2(n-2))}\,. \tag{23}$$

PROOF. Let $\lambda > 0$, $f(x) = |x|^{-n+2}$. First we estimate $M_\lambda f(x)$ from below as follows

$$\begin{aligned}
&M_\lambda f(x)\\
&\quad \geq c_n\, \lambda^{-n/2+1} \sup_{\lambda\leq\mu<2\lambda} \sum_{l=1}^{\infty} l^{-n/2+1}|\{y \in \mathbb{Z}^n \,:\, |x-y|^2 = l,\ |y|^2 = \mu\}|\\
&\quad \geq c_n\, \lambda^{-n/2+1}\\
&\qquad \cdot \sup_{\lambda\leq\mu<2\lambda} \sum_{L \text{ dyadic}} L^{-n/2+1}|\{y \in \mathbb{Z}^n \,:\, L \leq |x-y|^2 < 2L,\ |y|^2 = \mu\}|\\
&\quad \geq c_n\, \lambda^{-n/2+1}\\
&\qquad \cdot \sup_{\lambda\leq\mu<2\lambda} \sum_{L \text{ dyadic}} L^{-n/2+1}|\{y \in Z^n \,:\, |x-y|^2 \leq L,\ |y|^2 = \mu\}|\,,
\end{aligned}$$

where the last inequality was obtained by partial summation. This immediately implies

$$M_\lambda f(x) \geq c_n\, \lambda^{-n/2+1}\, \frac{s^*_{\lambda,L}(x)}{\tilde{s}_{\lambda,L}}\, \frac{L^{1/2}}{\lambda^{1/2}}\,, \tag{24}$$

for every dyadic value $L = 2^l$, but it remains true for every integer l since the function $s^*_{\lambda,l}(x)$ is monotone increasing in l.

Choosing $p = n/(n-2) + \eta$, it follows for $l > \lambda^{1-\varepsilon/2}$

$$\begin{aligned}
\|M_\lambda f\|_p^p &\geq \lambda^{-\eta(n/2-1)}\lambda^{-n/2} \sum_{x\in D^n_\lambda} \Big(\frac{s^*_{\lambda,l}(x)}{\tilde{s}_{\lambda,l}}\, \frac{l^{1/2}}{\lambda^{1/2}}\Big)^{n/(n-2)+\eta}\\
&\geq \lambda^{3\varepsilon n/4(n-2)-\eta n/2}\lambda^{-n/2}|\{x \in D^n_\lambda \,:\, s^*_{\lambda,l} > \lambda^\varepsilon \tilde{s}_{\lambda,l}\}|\,.
\end{aligned}$$

Choosing η small enough estimate (23) follows immediately, since $f \in L^p(\mathbb{Z}^n)$ and the maximal operator M_λ is bounded in $L^p(\mathbb{Z}^n)$ by Theorem 1.

Acknowledgement. I am grateful to my thesis advisor E. M. Stein, who asked me about this problem and also suggested the generalization to the k-spheres, and whose encouragement made possible this research.

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Recibido: 2 de noviembre de 1.995
Revisado: 26 de abril de 1.996

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Revista Matemática Iberoamericana
Vol. 13, N.º 2, 1997

Hilbert transform, Toeplitz operators and Hankel operators, and invariant A_∞ weights

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Abstract. In this paper, several sufficient conditions for boundedness of the Hilbert transform between two weighted L^p-spaces are obtained. Invariant A_∞ weights are introduced. Several characterizations of invariant A_∞ weights are given. We also obtain some sufficient conditions for products of two Toeplitz operators or Hankel operators to be bounded on the Hardy space of the unit circle using Orlicz spaces and Lorentz spaces.

0. Introduction.

Let ∂D be the unit circle and dw denotes the Lebesgue measure on ∂D. For $p > 1$ and v a positive function on ∂D, $L^p(v)$ denotes the space of functions f on the unit circle such that

$$\int_{\partial D} |f(w)|^p \, v(w) \, dw < \infty \, .$$

We use L^p to denote $L^p(v)$ if $v = 1$. Let H^p be the subspace of L^p which those functions are analytic on the unit disk D. There is an orthogonal projection P from L^2 onto H^2. The Hilbert transform T is defined to be $T = -i\,P + i\,(I - P)$.

We are concerned with the problem of identifying those pair (v,u) of positive functions on ∂D for which the Hilbert transform T is bounded from $L^p(u)$ to $L^p(v)$, that is

$$\int_{\partial D} |Tf(w)|^p \, v(w)\, dw \le C \int_{\partial D} |f(w)|^p \, u(w)\, dw \,, \tag{1}$$

for all $f \in L^p(u)$.

This problem was first raised by Muckenhoupt and Wheeden in [16]. There is a very elegant theorem of Cotlar-Sadosky ([4], [5]) which gives a necessary and sufficient condition that (1) holds for a given constant C. The theorem of Cotlar and Sadosky generalizes the Helson-Szegö theorem in two weights case. On the other hand, it is a considerably interesting question in harmonic analysis to find explicit estimates of the norm of the Hilbert transform between two weighted spaces; see [2] for further references. So it seems interesting to find characterizations of the weight functions for (1) close in form to the following A_p condition [15]. It remains an open question ([6], [22], and [8]).

In case that $u = v$, Hunt, Muckenhoupt and Wheeden [12] have proved that (1) holds if and only if v satisfies a simpler condition

$$\sup_I \Big(\frac{1}{|I|}\int_I v(w)\, dw\Big)\Big(\frac{1}{|I|}\int_I v(w)^{-1/(p-1)}\, dw\Big)^{p-1} < \infty \,, \tag{A_p}$$

where the supremum is taken over all arcs I. Muckenhoupt [14] has shown that the A_p condition is a necessary and sufficient condition that the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(w)|\, dw \,,$$

satisfies the following inequality

$$\int_{\partial D} |Mf(w)|^p \, v(w)\, dw \le C \int_{\partial D} |f(w)|^p \, u(w)\, dw \,. \tag{2}$$

One may expect that the following condition is a necessary and sufficient condition that inequality (1) holds even $u \ne v$

$$\sup_I \Big(\frac{1}{|I|}\int_I v(w)\, dw\Big)\Big(\frac{1}{|I|}\int_I u(w)^{-1/(p-1)}\, dw\Big)^{p-1} < \infty \,. \tag{A'_p}$$

Simple examples [16] show that the A'_p condition is not sufficient for either (1) or (2) to hold. Sawyer [21] has shown that (2) holds if and only if

$$(S_p) \qquad \sup_I \Big(\int_I (M(\chi_I u^{-1/(1-p)})(w))^p \, v(w)\, dw \Big) \cdot \Big(\int_I u(w)^{-1/(p-1)}\, dw \Big)^{-1} < \infty\,,$$

where χ_I denotes the characteristic function of I. Sawyer [22] also showed that the S_p condition with additional conditions is sufficient for (1). Fujii [8] has obtained the following sufficient condition for (1):

There exist constants $0 < \alpha < 1$, β and $0 < C_0 < \infty$ such that, for every arc I and all measurable subsets E and F of I with $E \cap F = 0$ and $|F| \geq \alpha |I|$,

$$\begin{aligned} \Big(\int_E v(w)\, dw \Big) \Big(|I|^{-1} \int_{c(n,\alpha)I} u^{-1/(p-1)}(w)\, dw \Big)^p & \\ & \leq C_0 \Big(\frac{|E|}{|I|} \Big)^\beta \int_F u^{-1/(p-1)}\,(w)\, dw \\ & < \infty\,, \end{aligned}$$

where $c(n,\alpha)$ is a constant greater than 1 and $c(n,\alpha)I$ is the arc with the same center as I and expanded $c(n,\alpha)$ times.

Sawyer's condition involves the operator M, and it is interesting to obtain sufficient conditions close in form to the A'_p condition. In that direction, Neugebauer [17] has obtained the following sufficient condition for (2) for $r > 1$,

$$(3) \qquad \sup_I \Big(\frac{1}{|I|} \int_I v^r(x)\, dx \Big) \Big(\frac{1}{|I|} \int_I u^{-r/(p-1)}(x)\, dx \Big)^{p-1} < \infty\,.$$

Recently Pérez [19] has improved the condition (3) and obtained weaker sufficient conditions for (2) using the general maximal operator involving in Banach function spaces.

In this paper, using Banach function norms we define a maximal operator and a nontangential maximal operator. We shall show several sufficient conditions for (1). In particular, we prove that if (u, v) is a pair of weights such that for some $r > 1$,

$$\sup_{z\in D} v^r(z)\,(u^{-r/(p-1)}(z))^{p-1} < \infty\,, \tag{4}$$

then the Hilbert transform is bounded from $L^p(u)$ to $L^p(v)$. Here we follow the convention of identifying functions on the unit circle with their harmonic extensions, defined via Poisson's formula, into the unit disk D.

The condition (4) is analogous to the condition (3) and the following condition: for $0 < \alpha < 1$ and $r > 1$,

$$\begin{aligned}\sup_I |I|^{\alpha/n}\Big(\frac{1}{|I|}\int_I v^r(x)\,dx\Big)^{1/(pr)}&\\ \cdot\Big(\frac{1}{|I|}\int_I u^{(1-p')r}(x)\,dx\Big)^{1/(p'r)} &< \infty\,.\end{aligned} \tag{5}$$

Sawyer and Wheeden [23] have shown that the condition (5) is a sufficient condition for fractional integral operators

$$I_\alpha f(x) = \int |x-y|^{\alpha-1} f(y)\,dy$$

to be bounded from $L^p(u)$ to $L^p(v)$.

For $z \in D$, let $\phi_z(w)$ be the Möbius map

$$\phi_z(w) = \frac{z-w}{1-\overline{z}\,w}\,,$$

for $w \in \overline{D}$.

We can improve the condition (4) using the scale of Lorentz spaces or Orlicz spaces which are concrete examples of Banach function spaces. Let $P(z, x)$ be the Poisson kernel

$$P(z,x) = \frac{1-|z|^2}{|1-\overline{z}\,x|^2}\,.$$

We shall show that one of the following conditions is a sufficient condition for (1)

$$\begin{aligned}\sup_{z\in D}\Big(\sup_{t>0}\int_{v(x)>t} t^r P(z,x)\,dx\Big)&\\ \cdot\Big(\sup_{t>0}\Big(\int_{u^{-1}(x)>t} t^{r/(p-1)} P(z,x)\,dx\Big)^{p-1}\Big) &< \infty\,,\end{aligned} \tag{6}$$

for some $r > 1$, and

$$\sup_{z\in D} \|v^{1/p} \circ \phi_z\|_{L^\Phi} \, \|u^{-1} \circ \phi_z\|_{L^\Psi} < \infty\,, \tag{7}$$

for some Young functions such as

$$\Phi(t) = t^p \log^{p-1+\delta}(1+t)$$

and

$$\Psi(t) = t^{p'} \log^{p'-1+\delta}(1+t)\,,$$

or weaker ones

$$\Phi(t) = t^p \log^{p-1}(1+t)\,(\log\log(1+t))^{p-1+\delta}$$

and

$$\Psi(t) = t^{p'} \log^{p'-1}(1+t)\,(\log\log(1+t))^{p'-1+\delta}\,,$$

for some $\delta > 0$.

We will introduce invariant A_∞ weights and give several characterizations of invariant A_∞ weights. From these characterizations we can easily tell the difference between A_∞ and invariant A_∞ weights.

If we assume that both v and $u^{-1/(p-1)}$ are invariant A_∞ weights, we will show that the condition

$$\sup_{z\in D} v(z)\,(u^{-1/(p-1)}(z))^{p-1} < \infty\,, \tag{8}$$

is a necessary and sufficient condition for the Hilbert transform to be uniformly bounded from $L^p(u \circ \phi_z)$ to $L^p(v \circ \phi_z)$ for all $z \in D$.

Boundedness of the Hilbert transform between two weighted L^2 spaces is related to boundedness of products of two Toeplitz operators or Hankel operators on the Hardy space H^2. On products of Toeplitz operators Sarason [20] made the following conjecture:

Let f and g be outer functions in H^2. The product $T_f T_{\overline{g}}$ is bounded if and only if

$$\sup_{z\in D} |f|^2(z)\,|g|^2(z) < \infty\,. \tag{9}$$

On the product of Hankel operators it is natural to make the following conjecture [27]:

Let f and g be in L^2. Then the product $H_f^* H_g$ is bounded if and only if

$$\sup_{z \in D} \|H_f\, k_z\|_2\, \|H_g\, k_z\|_2 < \infty\,. \tag{10}$$

Let f_- denote $(1-P)\,f$ for $f \in L^2$. Then the condition (10) is equivalent to

$$\sup_{z \in D} \|f_- \circ \phi_z - f_-(z)\|_2\, \|g_- \circ \phi_z - g_-(z)\|_2 < \infty\,. \tag{11}$$

Treil [20] showed that if the product $T_f T_{\overline{g}}$ is bounded, then the condition (9) holds in Sarason's Conjecture. Conversely, it was shown [27] that the following condition implies that $T_f T_{\bar{g}}$ is bounded

$$\sup_{z \in D} |f|^{2r}(z)\, |g|^{2r}(z) < \infty\,, \tag{12}$$

for some $r > 1$. Also in [27] it was shown that the condition (10) is necessary for $H_f^* H_g$ to be bounded and that the following condition is sufficient

$$\sup_{z \in D} \|f_- \circ \phi_z - f_-(z)\|_{2r}\, \|g_- \circ \phi_z - g_-(z)\|_{2r} < \infty\,, \tag{13}$$

for some $r > 1$.

In this paper we will improve the above sufficient conditions for boundedness of the product of two either Toeplitz operators or Hankel operators using Orlicz spaces or Lorentz spaces. In particular, we will show that if for two outer functions f and g in H^2,

$$\sup_{z\ inD} \|f \circ \phi_z\|_{L^\Phi}\, \|g \circ \phi_z\|_{L^\Phi} < \infty\,, \tag{14}$$

then $T_f T_{\overline{g}}$ is bounded, and if for two functions f and g in L^2,

$$\sup_{z \in D} \|f_- \circ \phi_z - f_-(z)\|_{L^\Phi}\, \|g_- \circ \phi_z - g_-(z)\|_{L^\Phi} < \infty\,, \tag{15}$$

then $H_f^* H_g$ is bounded. Here the Young function Φ is $t^2 \log^{1+\delta}(1+t)$ or $t^2 \log(1+t)\,(\log\log(1+t))^{1+\delta}$ for some $\delta > 0$.

The letter C will denote a positive constant, possibly different on each occurrence.

1. Banach function spaces.

Recall some basic facts about the theory of Banach function spaces, Orlicz spaces and Lorentz spaces. We shall refer the reader to [1], [13] and [18] for a complete account. A Banach function space X over the unit circle is a subspace of the Lebesgue measurable functions with a Banach function norm. The most important property of the Banach function space is the generalized Hölder inequality

$$\int |f(x)\, g(x)|\, dx \le \|f\|_X \|g\|_{X'}\,, \tag{16}$$

where X' is the associate space to X.

Let us look at several concrete examples of Banach function spaces. A function $\Phi : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex and increasing satisfying $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. Each Young function Φ has associated a complementary Young function $\tilde{\Phi}$. The Orlicz space L^Φ consists of all Lebesgue measurable functions f such that

$$\int \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, dx < \infty\,,$$

for some $\lambda > 0$. The space L^Φ is a Banach function space with the Luxemburg norm defined by

$$\|f\|_\Phi = \inf\Big\{\lambda > 0 \,:\, \int \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, dx \le 1\Big\}\,.$$

Its associated space is $L^{\tilde{\Phi}}$. A Young function Φ is said to satisfy the Δ_2-condition if there exist $C > 0$ and $T \ge 0$ such that

$$\Phi(2\,t) \le C\,\Phi(t)\,,$$

for all $t \ge T$.

Let $1 < s,\ q \leq \infty$. The Lorentz space $L^{s,q}$ is the space of the Lebegue measurable functions f such that

$$\|f\|_{L^{s,q}} = \Big(q\int_0^\infty \big(t\,|\{x \in \partial D : |f(x)| > t\}|^{1/s}\big)^q \frac{dt}{t}\Big)^{1/q} < \infty\,,$$

if $q < \infty$, and

$$\|f\|_{L^{s,\infty}} = \sup_{0<t<\infty} t\,|\{x \in \partial D : |f(x)| > t\}|^{1/s} < \infty\,,$$

if $q = \infty$. The Lorentz space $L^{s,q}$ is a Banach function space with the associate space $L^{s',q'}$.

Let X be a Banach function space over ∂D with respect to the Lebesgue measure. Given a measurable function f and any interval I we define the X-average of f over I by

$$\|f\|_{X,I} = \|\tau_{|I|}(f\chi_I)\|_X\ ,$$

where τ_δ, with $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta\, x)$, χ_E is the characteristic function of E. We define a natural maximal operator $M_X f(x)$ associated to the space X by

$$M_X f(x) = \sup_{x\in I} \|f\|_{X,I}\ , \tag{17}$$

where the supremum is taken over all intervals containing x.

For any $x \in \partial D$, and a fixed $\alpha > 1$ let

$$\Gamma(x) = \Big\{z \in D : \frac{|x-z|}{1-|z|} \leq \alpha\Big\}\,.$$

We define a nontangential maximal operator $N_X f(x)$ associated to the space X by

$$N_X f(x) = \sup_{z\in\Gamma(x)} \|f\circ\phi_z\|_X\ . \tag{18}$$

Let $X = L^\Phi$ be the Orlicz space defined by a Young function Φ. Then the maximal operator M_X is defined in terms of the average

$$\|f\|_{X,I} = \inf\Big\{\lambda > 0 : \frac{1}{|I|}\int_I \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,dx \leq 1\Big\}\,.$$

If X is the Lorentz space $X = L^{s,q}$, then the maximal operator is

$$M_X f(x) = \sup_{x \in I} \frac{1}{|I|^{1/s}} \|f \chi_I\|_{L^{s,q}} .$$

Proposition 1. *Let X be either an Orlicz space or a Lorentz space. Let f be in X. Then there is a constant C such that*

$$N_X f(x) \leq C \, M_X f(x) , \tag{19}$$

for $x \in \partial D$.

PROOF. We shall consider only the case that X is the Orlicz space L^Φ with a Young function Φ. The same method will prove the theorem in the case that X is a Lorentz space.

We may assume $x = 1$. We will use polar coordinate re^{it} for points in the unit disk. Let us first consider the points $z = r$ on the real axis in the cone

$$\Gamma(1) = \Big\{ z \in D : \frac{|1 - z|}{1 - |z|} < \alpha \Big\} .$$

Then

$$f(r) = \int P(r, e^{it}) \, f(e^{it}) \, dt ,$$

and the kernel $P(r, e^{it})$ is a positive even function which is decreasing for positive t. That means $P(r, e^{it})$ is a convex combination of the box kernels $\chi_{(-h,h)}(t)/(2h)$. Take step functions $h_n(t)$, which are also nonnegative, even, and decreasing on $t > 0$, such that $h_n(t)$ increases with n to $P(r, e^{it})$. Then $h_n(t)$ has the form

$$\sum_{j=1}^{N} a_j \chi_{(-t_j, t_j)}(t)$$

with $a_j \geq 0$, and $\int h_n(t) \, dx = \sum_j 2 \, t_j a_j \leq 1$. Suppose that $\lambda > 0$ such that $\lambda > M_X f(1)$. Thus

$$\Big(\frac{1}{|I|} \Big) \int_I \Phi \Big(\frac{|f(e^{it})|}{\lambda} \Big) \, dt \leq 1 ,$$

for all intervals I containing 1. Then

$$\begin{aligned}\Big|\int h_n(t)\,\Phi\Big(\frac{|f(e^{it})|}{\lambda}\Big)\,dt\Big| &\le \sum_{j=1}^{N} 2\,t_j a_j\,\frac{1}{|(-t_j,t_j)|}\int_{(-t_j,t_j)}\Phi\Big(\frac{|f(e^{it})|}{\lambda}\Big)\,dt\\ &\le \sum_{j=1}^{n} 2\,t_j a_j\\ &\le 1\,.\end{aligned}$$

Then by monotone convergence

$$\int \Phi\Big(\frac{|f\circ\phi_r(e^{it})|}{\lambda}\Big)\,dt = \int \Phi\Big(\frac{|f(e^{it})|}{\lambda}\Big)P(r,e^{it})\,dt \le 1\,.$$

Hence $\|f\circ\phi_r\|_X \le \lambda$. Now fix $z\in\Gamma(1)$. Then $|1-re^{is}|\le\alpha(1-r)$, and $P(re^{is},e^{it})$ is majorized by a positive even function $\psi(t)$, which is decreasing on $t>0$, such that

$$\int \psi(t)\,dt \le A_\alpha\ ,$$

for some constant A_α. The function is $\psi(t)=\sup\{P(re^{is},e^{il}) : |l|>t\}$. Approximating $\psi(t)$ from below by step functions $h_n(t)$ just as before, we have

$$\int \psi(t)\,\Phi\Big(\frac{|f(e^{it})|}{\lambda}\Big)\,dt \le A_\alpha\ .$$

By convexity, we obtain

$$\begin{aligned}\int \Phi\Big(\frac{|f\circ\phi_z(e^{it})|}{A_\alpha\lambda}\Big)\,dt &= \int \Phi\Big(\frac{|f(e^{it})|}{A_\alpha\lambda}\Big)P(z,e^{it})\,dt\\ &\le \int \Phi\Big(\frac{|f(e^{it})|}{A_\alpha\lambda}\Big)\psi(t)\,dt\\ &\le 1\,.\end{aligned}$$

Thus by the definition of the Luxemburg norm, we have

$$\|f\circ\phi_z\|_X < A_\alpha\lambda\,.$$

Therefore

$$N_X f(1) \le A_\alpha\lambda\,.$$

So we conclude that

$$N_X f(1) \leq A_\alpha \, M_X f(1) \, .$$

2. Distribution function inequality.

In this section we will get two distribution function inequalities involving the Lusin area integral and the nontangential maximal operator N_X for Banach function spaces X.

For w a point in ∂D, we let Γ_w denote the angle with vertex w and opening $\pi/2$ which is bisected by the radius to w. The set of points z in Γ_w satisfying $|z-w| < \varepsilon$ will be denoted by $\Gamma_{w,\varepsilon}$. We fix the shape of our typical truncated cone $\Gamma_{w,\varepsilon}$. Whenever h is in L^1, we define the truncated Lusin area integral of h to be

$$A_\varepsilon(h)(w) = \Big(\int_{\Gamma_{w,\varepsilon}} |\nabla h(z)|^2 \, dA(z) \Big)^{1/2} . \tag{20}$$

Here $h(z)$ means the harmonic extension of h on D,

$$|\nabla h(z)|^2 = \Big| \frac{\partial h}{\partial z} \Big|^2 + \Big| \frac{\partial h}{\partial \overline{z}} \Big|^2 ,$$

and $dA(z)$ denotes the area measure on the unit disk. Then $A_\varepsilon(h)(w)^2$ represents the area (points counted with their multiplicity) of the image in the complex plane of the truncated cone $\Gamma_{w,\varepsilon}$ under the map $z \to h(z)$.

The Lebesgue measure of the subset E of ∂D will be denoted by $|E|$. For $z \in D$, we let I_z denote the closed subarc of ∂D with center $z/|z|$ and measure $\delta(z) = 1 - |z|$.

For a number $p > 1$, we use p' to denote the number so that $1/p + 1/p' = 1$. Let Q denote the operator $I - P$. As in [27], we have the following distribution function inequalities.

Theorem 1. *Let X and Y be two Banach function spaces. Let f and g be in X and Y respectively, and ϕ and ψ in X' and Y'. For $|z| > 1/2$ and $a > 0$ sufficiently large, there is a constant $C_a > 0$ such that*

$$\begin{aligned} |\{\lambda \in I_z \, : A_{2\delta(z)}(P(f\,\phi))(\lambda) \, A_{2\delta(z)}(P(g\,\psi))(\lambda) \\ < a \, \|f \circ \phi_z\|_X \|g \circ \phi_z\|_Y \\ \cdot \inf_{w \in I_z} N_{X'}(\phi)(w) \inf_{w \in I_z} N_{Y'}(\psi)(w)\}| \geq C_a |I_z| \end{aligned} \tag{21}$$

and

$$
\begin{aligned}
(22)\qquad &|\{\lambda \in I_z \, : A_{2\delta(z)}(Q(f\,\phi))(\lambda)\, A_{2\delta(z)}(Q(g\,\psi))(\lambda) \\
&\qquad < a\, \|f_- \circ \phi_z - f_-(z)\|_X\, \|g_- \circ \phi_z - g_-(z)\|_Y \\
&\qquad\quad \cdot \inf_{w\in I_z} N_{X'}(\phi)(w) \inf_{w\in I_z} N_{Y'}(\psi)(w)\}| \geq C_a |I_z| \,.
\end{aligned}
$$

Moreover, the constant C_a *can be chosen to satisfy* $C_a = 1 - 2a^{-1/2}C$ *for some positive constant* C.

PROOF. We will show only the first distribution function inequality. The same method will prove the second one.

For a fixed z in D, and $a > 0$ let $E(a)$ be the set of points in I_z where

$$A_{2\delta(z)}(P(f\,\phi))(\lambda) \leq a^{1/2}\, \|f \circ \phi_z\|_X \inf_{w\in I_z} N_{X'}\phi(w)$$

and $F(a)$ the set of points in I_z where

$$A_{2\delta(z)}(P(g\,\psi)(\lambda)) \leq a^{1/2}\, \|g \circ \phi_z\|_Y \inf_{w\in I_z} N_{Y'}\psi(w) \,.$$

We claim the following distribution function inequalities, *i.e.* for $a > 0$ sufficiently large

$$|E(a)| \geq K_a\, |I_z| \,, \tag{23}$$

and

$$|F(a)| \geq K_a\, |I_z| \,, \tag{24}$$

and $\lim_{a\to\infty} K_a = 1$.

First we show how Theorem 1 follows from those two distribution inequalities. If $w \in I_z$ is in $E(a) \cap F(a)$, then

$$A_{2\delta(z)}(P(f\,\phi))(\lambda) \leq a^{1/2}\, \|f \circ \phi_z\|_X \inf_{w\in I_z} N_{X'}\phi(w)$$

and

$$A_{2\delta(z)}(P(g\,\psi)(\lambda)) \leq a^{1/2}\, \|g \circ \phi_z\|_Y \inf_{w\in I_z} N_{Y'}\psi(w) \,.$$

Thus

$$
\begin{aligned}
&A_{2\delta(z)}(P(f\,\phi))(\lambda)\, A_{2\delta(z)}(P(g\,\psi)(\lambda) \\
&\qquad < a\, \|f \circ \phi_z\|_X\, \|g \circ \phi_z\|_Y \inf_{w\in I_z} N_{X'}(\phi)(w) \inf_{w\in I_z} N_{Y'}(\psi)(w) \,.
\end{aligned}
$$

So $E(a) \cap F(a)$ is a subset of

$$\{\lambda \in I_z : A_{2\delta(z)}(P(f\,\phi))(\lambda)\, A_{2\delta(z)}(P(g\,\psi))(\lambda) < a\,\|f \circ \phi_z\|_X\,\|g \circ \phi_z\|_Y \inf_{w \in I_z} N_{X'}(\phi)(w) \inf_{w \in I_z} N_{Y'}(\psi)(w)\}\,.$$

On the other hand, we have

$$|E(a) \cap F(a)| \geq |E(a)| + |F(a)| - |I_z|\,.$$

Since $\lim_{a \to \infty} K_a = 1$, we have

$$\begin{aligned} |\{\lambda \in I_z : A_{2\delta(z)}(P(f\,\phi))(\lambda)\, A_{2\delta(z)}(P(g\,\psi)(\lambda) \\ < a\,\|f \circ \phi_z\|_X\,\|g \circ \phi_z\|_Y \inf_{w \in I_z} N_{X'}(\phi)(w) \inf_{w \in I_z} N_{Y'}(\psi)(w)\}| \\ \geq |E(a) \cap F(a)| \geq |E(a)| + |F(a)| - |I_z| \geq (2K_a - 1)\,|I_z| \end{aligned}$$

if $C_a = 2K_a - 1$. This completes the proof of Theorem 1.

Now we turn to the proof of our claim. For simplicity we will present only the details of the proof of (23). Using the same method we can prove (24). The proof consists of three steps. Let χ_E denote the characteristic function of the subset E of ∂D. In order to prove (23) we write $P(f\,\phi)$ as $P(f\,\phi) = P(\phi_1) + (P\phi_2)$ where $\phi_1 = f(\chi_{2I_z}\phi)$ and $\phi_2 = f(\chi_{\partial D \setminus 2I_z}\phi)$.

Step 1. There is a constant $C > 0$ such that for all $t > 0$,

$$\begin{aligned} |\{\lambda \in I_z : A_{2\delta(z)}(P(\phi_1)) < t\}| \\ \geq \Big(1 - \frac{C\,\|f \circ \phi_z\|_X \inf_{w \in I_z} N_{X'}\phi(w)}{t}\Big)|I_z|\,. \end{aligned} \tag{25}$$

From the definition of the truncated Lusin area integral, we easily see that

$$A_{2\delta(z)}(P(\phi_1)) \leq A_{2\delta(z)}(\phi_1)\,.$$

Thus

$$\{\lambda \in I_z : A_{2\delta(z)}(\phi_1) < t\} \subset \{\lambda \in I_z : A_{2\delta(z)}(P(\phi_1)) < t\}\,.$$

To prove (25) we need only to show

$$\begin{aligned} |\{\lambda \in I_z : A_{2\delta(z)}(\phi_1) < t\}| \\ \geq \Big(1 - \frac{C\,\|f \circ \phi_z\|_X \inf_{w \in I_z} N_{X'}\phi(w)}{t}\Big)|I_z|\,. \end{aligned} \tag{26}$$

By the theorem of Marcinkiewicz and Zygmund, and the fact that the nontangential maximal function M is of weak type $(1,1)$ then the truncated Lusin area integral $A_\varepsilon f(w)$ is also of weak type $(1,1)$. So we have, for $t>0$

$$|\{\lambda \in I_z \,:\, A_{2\delta(z)}(\phi_1) \geq t\}| \leq \frac{C}{t}\int_{\partial D} |\phi_1|\, dw = \frac{C}{t}\int_{2I_z} |f(w)\,\phi(w)|\, dw\,.$$

Since an elementary estimate shows that for $w \in 2I_z$, $P(z,w) > C/|2I_z|$, it follows that

$$\begin{aligned}\int_{2I_z} |f(w)\,\phi(w)|\, dw &\leq C\,|I_z| \int |f(w)\,\phi(w)|\, P(z,w)\, dw \\ &= C\,|I_z| \int |f\circ\phi_z(w)\,\phi\circ\phi_z(w)|\, dw\,.\end{aligned}$$

By the generalized Hölder inequality we have

$$\int_{2I_z} |f(w)\,\phi(w)|\, dw \leq C\,|I_z|\, \|f\circ\phi_z\|_X \|\phi\circ\phi_z\|_{X'}\,.$$

Because for each $u \in I_z$, z is in $\Gamma(u)$, we have

$$|\{\lambda \in I_z \,:\, A_{2\delta(z)}(\phi_1) \geq t\}| \leq \frac{C\,|I_z|}{t}\, \|f\circ\phi_z\|_X \inf_{w\in I_z} N_{X'}\phi(w)\,.$$

Thus

$$|\{\lambda \in I_z \,:\, A_{2\delta(z)}(\phi_1) < t\}| \geq \Big(1 - \frac{C\,\|f\circ\phi_z\|_X \inf_{w\in I_z} N_{X'}\phi(w)}{t}\Big)|I_z|\,.$$

Step 2. On I_z,

$$A_{2\delta(z)}(P((\phi_2))(w) \leq C\,\|f\circ\phi_z\|_X \inf_{w\in I_z} N_{X'}(\phi)(w)\,, \tag{27}$$

for some $C>0$.

For ϕ_2, we shall use a pointwise estimate of the norm of gradient of $P(\phi_2)$. Since $P(\phi_2)$ is analytic in D, we have

$$\nabla(P(\phi_2)(w)) = \frac{1}{2\pi}\int \frac{\overline{\xi}\,\phi_2(\xi)}{(1-w\,\overline{\xi})^2}\, d\xi\,.$$

Thus

$$|\nabla P(\phi_2)(w)| \leq C \int \frac{|\phi_2(\xi)|}{|1-w\,\overline{\xi}|^2}\,d\xi \leq C \int_{\partial D\setminus 2I_z} \frac{|f(\xi)\,\phi(\xi)|}{|1-w\,\overline{\xi}|^2}\,d\xi\,.$$

On the other hand, there is a constant $C>0$ so that

$$\Big|\frac{1-(\xi,z)}{1-(\xi,w)}\Big| \geq C\,,$$

for all ξ in $\partial D\setminus 2I_z$ and w in $\Gamma_{u,2\delta(z)}$. Thus we obtain

$$|\nabla(P\phi_2)(w)| \leq C \int_{\partial D\setminus 2I_z} \frac{|f(\xi)\,\phi(\xi)|}{|1-\overline{z}\,\xi|^2}\,d\sigma(\xi)\,.$$

Applying the generalized Hölder inequality yields

$$|\nabla(P\phi_2)(u)| \leq \frac{C}{1-|z|^2}\,\|f\circ\phi_z\|_X\|\phi\circ\phi_z\|_{X'}\ .$$

Because z belongs to $\Gamma(u)$, for any $u \in I_z$, the last factor on the right is no larger than $CN_{X'}\phi(u)$, and the desired inequality is established.

Step 3. This step will complete the proof of the distribution function inequality (23) by combining last two steps. Since $P(f\phi) = P(\phi_1) + P(\phi_2)$, we have

$$A_{2\delta(z)}(P(f\phi))(w) \leq A_{2\delta(z)}((P\phi_1))(w) + A_{2\delta(z)}(P(\phi_2))(w)\,.$$

So for any $\lambda > 0$,

$$\bigcap_{i=1}^{2}\Big\{w\in I_z\,:\,A_{2\delta(z)}((P\phi_i)) \leq \frac{\lambda}{2}\Big\} \subset \{w\in I_z\,:\,A_{2\delta(z)}(P(f\phi)) \leq \lambda\}\,.$$

Let $E_i(a)$ be the subset of I_z such that

$$A_{2\delta(z)}((P\phi_i)) \leq a^{1/2}\,\|f\circ\phi_z\|_X \inf_{w\in I_z} N_{X'}\phi(w)$$

for $i \leq 2$.

Then we have

$$\bigcap_{i=1}^{2} E_i\Big(\frac{a}{2}\Big) \subset E(a)\,.$$

If in Step 1 we choose that $t = a^{1/2} \, \|f \circ \phi_z\|_X \inf_{w \in I_z} N_{X'} \phi(w)$, then we have

$$\Big| E_1 \Big(\frac{a}{2}\Big) \Big| \ge (1 - C \, a^{-1/2}) \, |I_z|$$

for a sufficiently large a.

By Step 2, for $a > 0$ sufficiently large we have

$$A_{2\delta(z)}(P(\phi_2))(u) < a^{1/2} \, \|f \circ \phi_z\|_X \inf_{w \in I_z} N_{X'} \phi(w)$$

everywhere on I_z, which implies $|E_2(a/2)| = |I_z|$. So

$$|E(a)| \ge (1 - a^{-1/2} \, C) \, |I_z| \, .$$

This completes the proof of (23) if we choose $K_a = 1 - a^{-1/2} \, C$.

3. Hilbert transform.

In this section we apply the distribution function inequality (21) in Theorem 1 to get a sufficient condition for the boundedness of the Hilbert transform on two weighted L^p. Let $\|T\|_p^z$ denote the norm of the Hilbert transform T from $L^p(u \circ \phi_z)$ to $L^p(v \circ \phi_z)$.

Given a Banach function space X, we will use X' to denote its associate space which is another Banach space.

Theorem 2. *Let u and v two positive functions on the unit circle, and $1 < p < \infty$. Suppose that X and Y are two Banach function spaces such that $N_{X'}$ maps $L^{p'}$ to $L^{p'}$ and $N_{Y'}$ maps L^p to L^p. Then there is a constant $C > 0$ such that*

$$(\|T\|_p^\lambda)^p \le C \Big(\sup_{z \in D} \|v^{1/p} \circ \phi_z\|_X \, \|u^{-1/p} \circ \phi_z\|_Y + \Big(\sup_{z \in D} v(z) \, (u^{-1/(p-1)}(z))^{p-1} \Big)^{1/p} \Big) \, , \tag{28}$$

for all $\lambda \in D$.

PROOF. Since $T = -i \, P + i \, (I - P)$, it suffices to show that there is a constant $C > 0$ such that

$$(\|P\|_p^\lambda)^p \le C \Big(\sup_{z \in D} \|v^{1/p} \circ \phi_z\|_X \, \|u^{-1/p} \circ \phi_z\|_Y + \Big(\sup_{z \in D} v(z) \, (u^{-1/(p-1)}(z))^{p-1} \Big)^{1/p} \Big) \, , \tag{29}$$

for all λ. As $(v \circ \phi_\lambda)(z) = v(\phi_\lambda(z))$ we see that

$$\sup_{z\in D} \|v^{1/p}\circ\phi_z\|_X \, \|u^{-1/p}\circ\phi_z\|_Y = \sup_{z\in D} \|v^{1/p}\circ\phi_\lambda\circ\phi_z\|_X \, \|u^{-1/p}\circ\phi_\lambda\circ\phi_z\|_Y \, .$$

So it suffices to show

$$\begin{aligned}(\|P\|_p^0)^p \le C\Big(&\sup_{z\in D} \|v^{1/p}\circ\phi_z\|_X \, \|u^{-1/p}\circ\phi_z\|_Y \\ &+ \big(\sup_{z\in D} v(z)\,(u^{-1/(p-1)}(z))^{p-1}\big)^{1/p}\Big) .\end{aligned}$$

This is equivalent to show

$$\begin{aligned}&\int |P(\phi)(w)|^p v(w)\,dw \\ &\le C\Big(\sup_{z\in D} \|v^{1/p}\circ\phi_z\|_X \, \|u^{-1/p}\circ\phi_z\|_Y \\ &\qquad + \big(\sup_{z\in D} v(z)\,(u^{-1/(p-1)}(z))^{p-1}\big)^{1/p}\Big) \int |\phi(w)|^p u(w)\,dw \, ,\end{aligned} \tag{30}$$

for all $\phi \in L^p(u)$. Let $\psi = \phi\, u^{1/p}$. Then the above inequality is equivalent to the following inequality

$$\begin{aligned}&\int |P(\psi\, u^{-1/p})(w)|^p v(w)\,dw \\ &\qquad \le C\Big(\sup_{z\in D} \|v^{1/p}\circ\phi_z\|_X \, \|u^{-1/p}\circ\phi_z\|_Y \\ &\qquad\qquad + \big(\sup_{z\in D} v(z)\,(u^{-1/(p-1)}(z))^{p-1}\big)^{1/p}\Big) \int |\psi(w)|^p \,dw \, ,\end{aligned}$$

for all $\psi \in L^p$. On the other hand,

$$\int |P(\psi\, u^{-1/p})(w)|^p v(w)\,dw = \int |v^{1/p}(w)\, P(\psi\, u^{-1/p})(w)|^p\,dw$$

and the dual space of L^p is $L^{p'}$. Then

$$\begin{aligned}&\int |P(\psi\, u^{-1/p})(w)|^p v(w)\,dw \\ &\qquad = \sup_{\substack{\|h\|_{p'}\le 1\\ h\in L^{p'}}} \Big| \int v^{1/p}(w)\, P(\psi\, u^{-1/p})(w)\, \overline{h(w)}\,dw \Big| .\end{aligned}$$

So we need only to estimate the pairing

$$\int v^{1/p}(w)\, P(\psi\, u^{-1/p})(w)\, \overline{h(w)}\, dw\,.$$

Now

$$\begin{aligned}\int v^{1/p}(w)\, P(\psi\, u^{-1/p})(w)\, \overline{h(w)}\, dw \\ = \int P(\psi\, u^{-1/p})(w)\, \overline{P(v^{1/p}\, h)(w)}\, dw\,.\end{aligned}$$

Using the Littlewood-Paley formula [10], we have

$$\begin{aligned}&\int P(\psi\, u^{-1/p})(w)\, \overline{P(v^{1/p}\, h)(w)}\, dw \\ &\qquad = P(\psi\, u^{-1/p})(0)\, \overline{P(v^{1/p}\, h)(0)} \\ &\qquad\quad + \iint \langle \nabla (P(u^{-1/p}\, \psi)(z), \nabla (P(v^{1/p}\, h)(z)\rangle \log \frac{1}{|z|}\, dA(z)\,.\end{aligned} \tag{31}$$

Define

$$\mathrm{Term}_I = \iint_{|z|\geq 1/2} \langle \nabla (P(u^{-1/p}\, \psi)(z), \nabla (P(v^{1/p}\, h)(z)\rangle \log \frac{1}{|z|}\, dA(z)\,.$$

and

$$\mathrm{Term}_{II} = \iint_{|z|< 1/2} \langle \nabla (P(u^{-1/p}\, \psi)(z), \nabla (P(v^{1/p}\, h)(z)\rangle \log \frac{1}{|z|}\, dA(z)\,.$$

It is easy to verify that

$$|P(\psi\, u^{-1/p})(0)\, \overline{P(v^{1/p}\, h)(0)}| \leq C\, (v(0)\, u^{-1/(p-1)}(0)^{p-1})^{1/p} \|\psi\|_p\, \|h\|_{p'}\ ,$$

and

$$\begin{aligned}&|\langle \nabla (P(u^{-1/p}\, \psi)(z), \nabla (P(v^{1/p}\, h)(z)\rangle| \\ &\qquad \leq C\, (v(0)\, u^{-1/(p-1)}(0)^{p-1})^{1/p} \|\psi\|_p\, \|h\|_{p'}\end{aligned}$$

for $|z| \leq 1/2$. So we need only an estimate of Term_I. We claim that there is a constant $C>0$ such that

$$|\mathrm{Term}_I| \leq C \sup_{z\in D} \|v^{1/p}\circ \phi_z\|_X\, \|u^{-1/p}\circ \phi_z\|_Y\, \|\psi\|_p\, \|h\|_{p'}\ .$$

So by (31), P is bounded from $L^p(u)$ to $L^p(v)$ and its norm is bounded by

$$C\Big(\sup_{z\in D}\|v^{1/p}\circ\phi_z\|_X\,\|u^{-1/p}\circ\phi_z\|_Y+\Big(\sup_{z\in D}v(z)\,(u^{-1/(p-1)}(z))^{p-1}\Big)^{1/p}\Big)\,.$$

Now we turn to the proof of the claim. Fix an $a>0$ for which the first distribution function inequality in Theorem 1 holds. For $w\in\partial D$, let $\rho(w)$ denote the maximum of those numbers ε for which

$$\begin{aligned}&A_\varepsilon(P(u^{-1/p}\,\psi))(w)\,A_\varepsilon(P(v^{1/p}\,h))(w)\\&\qquad\le a\sup_{z\in D}\|v^{1/p}\circ\phi_z\|_X\,\|u^{-1/p}\circ\phi_z\|_Y\,N_{X'}h(w)\,N_{Y'}\psi(w)\,.\end{aligned}$$

Thus

$$\begin{aligned}&\int_{\partial D}A_{\rho(w)}(P(u^{-1/p}\,\psi))(w)\,A_{\rho(w)}(P(v^{1/p}\,h))(w)\,dw\\&\qquad\le a\sup_{z\in D}\|v^{1/p}\circ\phi_z\|_X\,\|u^{-1/p}\circ\phi_z\|_Y\int N_{X'}h(w)\,N_{Y'}\psi(w)\,.\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}&\int_{\partial D}A_{\rho(w)}(P(u^{-1/p}\,\psi))(w)\,A_{\rho(w)}(P(v^{1/p}\,h))(w)\,dw\\&\qquad\le a\sup_{z\in D}\|v^{1/p}\circ\phi_z\|_X\,\|u^{-1/p}\circ\phi_z\|_Y\,\|N_{X'}h\|_{p'}\,\|N_{Y'}g\|_p\,.\end{aligned}$$

Since $N_{X'}$ maps $L^{p'}$ to $L^{p'}$ and $N_{Y'}$ maps L^p to L^p, we have

$$\begin{aligned}&\int_{\partial D}A_{\rho(w)}(P(u^{-1/p}\,\psi))(w)\,A_{\rho(w)}(P(v^{1/p}\,h))(w)\,dw\\&\qquad\le C\sup_{z\in D}\|v^{1/p}\circ\phi_z\|_X\,\|u^{-1/p}\circ\phi_z\|_Y\,\|\psi\|_p\,\|h\|_{p'}\,.\end{aligned}$$

On the other hand, let $\chi_w(z)$ denote the characteristic function of $\Gamma_{w,\rho(w)}$, we have

$$\begin{aligned}&\int_{\partial D}A_{\rho(w)}(P(u^{-1/p}\,\psi))(w)\,A_{\rho(w)}(P(v^{1/p}\,h))(w)\,dw\\&=\int_{\partial D}\Big(\int_{\Gamma_{w,\rho(w)}}|\nabla(P(u^{-1/p}\,\psi)(z)|^2\,dA(z)\Big)^{1/2}\\&\qquad\cdot\Big(\int_{\Gamma_{w,\rho(w)}}|\nabla(P(v^{1/p}\,h)(z)|^2\,dA(z)\Big)^{1/2}d\omega\\&\ge\int_{|z|>1/2}\int_{\partial D}\chi_w(z)\,|\nabla(P(u^{-1/p}\,\psi))(z)|\,|\nabla(P(v^{1/p}\,h))(z)|\,dw\,dA(z)\,.\end{aligned}$$

Now the distribution function inequality (21) tells us that $\rho(w) \geq 2(1-|z|)$ on a subset of I_z whose measure is at least $C_a(1-|z|)$. If $\rho(w) \geq 2(1-|z|)$, then z is in $\Gamma_{w,\rho(w)}$. Thus $\chi_w(z) = 1$ on a subset of I_z of measure at least $C_a(1-|z|)$. Combining this observation with previous inequality, we obtain

$$\begin{aligned}\int_{\partial D} & A_{\rho(w)}\,(P(u^{-1/p}\,\psi))(w)\,A_{\rho(w)}(P(v^{1/p}\,h))(w)\,dw \\ &\geq C_a \int_{|z|>1/2} |\nabla(P(u^{-1/p}\,\psi)(z)|\,|\nabla(P(v^{1/p}\,h)(z)|\,(1-|z|)\,dA(z) \\ &\geq C_a|\mathrm{Term}_I|\,.\end{aligned}$$

So

$$|\mathrm{Term}_I| \leq C \sup_{z\in D} \|v^{1/p}\circ\phi_z\|_X\,\|u^{-1/p}\circ\phi_z\|_Y\,\|\psi\|_p\,\|h\|_{p'}\,.$$

This completes the proof of the theorem.

Corollary 1. *Let X and Y be either Orlicz spaces or Lorentz spaces. Suppose that $M_{X'}$ maps $L^{p'}$ to $L^{p'}$ and $M_{Y'}$ maps L^p to L^p. Then the Hilbert transform T is uniformly bounded from $L^p(v\circ\phi_z)$ to $L^p(v\circ\phi_z)$ for all $z\in D$.*

Proof. Since X and Y are either Orlicz spaces or Lorentz spaces, by Proposition 1, the maximal operators $M_{X'}$ and $M_{Y'}$ dominate the nontangential maximal operators $N_{X'}$ and $N_{Y'}$ respectively. So $N_{X'}$ maps $L^{p'}$ to $L^{p'}$ and $N_{Y'}$ maps L^p to L^p. Also it was shown in [19] that if $M_{X'}$ maps $L^{p'}$ to $L^{p'}$, then

$$\|f\|_p \leq C\,\|f\|_X\,,$$

for any $f\in X$. Then

$$\sup_{z\in D} v(z)\,(u^{-1/(p-1)}(z))^{p-1} \leq C\big(\sup_{z\in D}\|v^{1/p}\circ\phi_z\|_X\,\|u^{-1/(p-1)}\circ\phi_z\|_Y\big)^p\,.$$

So the corollary follows immediately from Theorem 2.

A particular example is when $X = L^{pr}$ and $Y = L^{p'r}$, where $r>1$. In this case the associate spaces are $X' = L^{(pr)'}$ and $Y' = L^{(p'r)'}$ whose corresponding maximal operators are given by

$$M_{X'}f(x) = \sup_{x\in I}\Big(\frac{1}{|I|}\int_I |f(y)|^{(pr)'}\,dy\Big)^{1/(pr)'}$$

and

$$M_{Y'} f(x) = \sup_{x \in I} \Big(\frac{1}{|I|} \int_I |f(y)|^{(p'r)'} \, dy \Big)^{1/(p'r)'} ,$$

which are bounded on $L^{p'}$ and L^p, respectively. By Proposition 1, the maximal operator M_X dominates the nontangential maximal operator N_X if X is an Orlicz space. So we have the following corollary.

Corollary 2. *Let $1 < p < \infty$, and suppose that (u, v) is a pair of weights such that for some $r > 1$,*

$$\sup_{z \in D} v^r(z) \, (u^{-r/(p-1)}(z))^{p-1} < \infty .$$

Then the Hilbert transform is bounded from $L^p(u)$ to $L^p(v)$.

The condition in Corollary 2 is quite close to the necessary condition that the Hilbert transform is uniformly bounded from $L^p(u \circ \phi_\lambda)$ to $L^p(u \circ \phi_\lambda)$ for $\lambda \in D$.

Proposition 2. *Let u and v be two positive functions on the unit circle. Let $1 < p < \infty$. If the Hilbert transform is uniformly bounded from $L^p(u \circ \phi_\lambda)$ to $L^p(u \circ \phi_\lambda)$ for $\lambda \in D$, then*

$$\sup_{z \in D} v(z) \, (u^{-1/(p-1)}(z))^{p-1} < \infty .$$

Proof. For a fixed $\lambda \in D$, let P_0 be the operator

$$P - z \, P \, \overline{z}$$

from $L^p(u)$ to $L^p(v)$. It is easy to check that

$$P_0 f = \Big(\int f(x) \, e_0(x) \, dx \Big) e_0 ,$$

for $f \in L^p(u \circ \phi_\lambda)$ where $e_0(z) = 1$. Since the Hilbert transform is unformly bounded from $L^p(u \circ \phi_\lambda)$ to $L^p(v \circ \phi_\lambda)$, P is also uniformly bounded from $L^p(u \circ \phi_\lambda)$ to $L^p(v \circ \phi_\lambda)$. So there is a constant $C > 0$ such that

$$\|P_0\|_{L^p(u \circ \phi_\lambda) \to L^p(v \circ \phi_\lambda)} \le C ,$$

for all $\lambda \in D$. On the other hand, the norm of P_0 from $L^p(u\circ\phi_\lambda)$ to $L^p(v\circ\phi_\lambda)$ is $\|u^{-1/p}\circ\phi_\lambda\|_{p'}\,\|v\circ\phi_\lambda\|_1^{1/p}$. This completes the proof of the proposition.

We can improve Corollary 2 using the scale of Lorentz spaces: if $X = L^{pr,\infty}$, then $X' = L^{(pr)',1}$ and $M_{X'}$ is bounded on $L^{p'}$. By Proposition 1, the maximal operator M_X dominates the nontangential maximal operator N_X if X is a Lorentz space. Hence we have the following corollary.

Corollary 3. *Let $1 < p < \infty$, and $1 < r < \infty$. Suppose that (u,v) is a pair of weights such that*

$$\sup_{z\in D}\Big(\sup_{t>0}\Big(\int_{\{v(x)>t\}} t^r P(z,x)\,dx\Big)\Big) \cdot\Big(\sup_{t>0}\Big(\int_{\{u^{-1}(x)>t\}} t^{-r/(p-1)}P(z,x)\,dx\Big)^{p-1}\Big) < \infty\,.$$

Then the Hilbert transform is bounded from $L^p(u)$ to $L^p(v)$.

More interesting examples are provided by the theory of Orlicz spaces. We have the following theorem which improves Corollary 2. In particular, if Young functions Φ and Ψ are in the following forms

$$\Phi(t) = t^p \log^{p-1+\delta}(1+t)\,,$$

and

$$\Psi(t) = t^{p'} \log^{p'-1+\delta}(1+t)\,,$$

or weaker ones

$$\Phi(t) = t^p \log^{p-1}(1+t)\,(\log\log(1+t))^{p-1+\delta}$$

and

$$\Psi(t) = t^{p'} \log^{p'-1}(1+t)\,(\log\log(1+t))^{p'-1+\delta}\,,$$

then the correspoding Orlicz spaces satisfy conditions in the following theorem.

Theorem 3. *Let $1 < p < \infty$, and let $\Phi(t)$ and $\Psi(t)$ be Young functions satisfying Δ_2-condition such that*

$$\int_c^\infty \Big(\frac{t^p}{\Phi(t)}\Big)^{p'-1}\frac{dt}{t} < \infty \qquad \text{and} \qquad \int_c^\infty \Big(\frac{t^{p'}}{\Psi(t)}\Big)^{p-1}\frac{dt}{t} < \infty\,,$$

for some positive constant c. Let (u, v) be a pair of weights such that

$$\sup_{z\in D} \|v^{1/p} \circ \phi_z\|_{L^\Phi} \|u^{-1/p} \circ \phi_z\|_{L^\Psi} < \infty .$$

Then the Hilbert transform T is bounded from $L^p(u)$ to $L^p(v)$.

PROOF. Let $X = L^\Phi$ and $Y = L^\Psi$. To prove the theorem, by Theorem 2, we need to show that $N_{X'}$ maps $L^{p'}$ to $L^{p'}$ and $N_{Y'}$ maps L^p to L^p. By Proposition 1, we see that $M_{X'}$ and $M_{Y'}$ dominate $N_{X'}$ and $N_{Y'}$, respectively. On the other hand, it is shown in [19] that $M_{X'}$ maps L^p to L^p and $M_{Y'}$ maps $L^{p'}$ to $L^{p'}$. This completes the proof of the theorem.

4. Invariant A_∞ weights.

A_∞ weights are introduced in connection with several problems in harmonic analysis by Muckenhoupt [15] and Coifman-Fefferman [3]. First we introduce the A_∞ condition as [3] and [14]. A weight function v on the unit circle is in A_∞ if there are positive constants $C, \delta > 0$ so that given any arc I and any measurable subset $E \subset I$

$$\frac{\displaystyle\int_E v(w)\, dw}{\displaystyle\int_I v(w)\, dw} \le C\Big(\frac{|E|}{|I|}\Big)^\delta .$$

There are many characterizations of A_∞ weights [9]. A characterization of A_∞ similar to the A_p condition is found in [11]. That is, a weight function v is in A_∞ if and only if

$$\sup_I \frac{1}{|I|} \int_I v(w)\, dw \Big(\frac{1}{|I|} \exp\Big(\int_I \log v^{-1}(w)\, dw\Big)\Big) < \infty .$$

But the A_∞ condition is not Möbius invariant in the sense

$$\sup_{z\in D} \sup_I \frac{1}{|I|} \int_I v\circ\phi_z(w)\, dw \Big(\frac{1}{|I|} \exp\Big(\int_I \log v^{-1}\circ\phi_z(w)\, dw\Big)\Big) < \infty .$$

We define a weight v to be an invariant A_∞ weight if

$$\sup_{z\in D} v(z) \exp\left(-(\log v)(z)\right) < \infty . \tag{32}$$

The invariant A_∞ weights were first studied by Wolff in [26]. He [26] showed that a weight v is an invariant A_∞ weight if and only if for any $\alpha > 0$ there is a $\beta > 0$: for any subarc $I \subset \partial D$ and subset $E \subset I$, then

$$\int_E v(w)\, P(z,w)\, dw \le \beta \int_I v(w)\, P(z,w)\, dw$$

implies that

$$\int_E P(z,w)\, dw \le \alpha \int_I P(z,w)\, dw\,,$$

for all $z \in D$. So comparing an equivalent condition to A_∞ ([3, Lemma 5]) we see that invariant A_∞ weights have a certain invariant property.

If two weights are invariant A_∞ weights, first we will show a necessary and sufficient conditions for the Hilbert transform to be uniformly bounded between two weighted spaces.

Theorem 4. *Suppose that both v and $u^{-1/(p-1)}$ are invariant A_∞ weights. The invariant A_p condition*

$$\sup_{z\in D} v(z)\, (u^{-1/(p-1)}(z))^{p-1} < \infty$$

is a necessary and sufficient condition for the Hilbert transform to be uniformly bounded from $L^p(u\circ\phi_z)$ to $L^p(v\circ\phi_z)$ for all $z \in D$.

Proof. By Proposition 2, we see that the condition in Theorem 4 is a necessary condition that the Hilbert transform is uniformly bounded from $L^p(u\circ\phi_z)$ to $L^p(v\circ\phi_z)$ for all $z\in D$.

We need only to show that the condition in Theorem 4 is sufficient. As pointed out in [26], the invariant A_∞ weight condition is equivalent to that $v(w)\, P(z,w)\, dw$ is comparable to $P(z,w)\, dw$ in the sense of [3] uniformly over $z \in D$. So the equivalent conditions of [3, Lemma 5] are valid for the measures $v(w)\, P(z,w)\, dw$ and $P(z,w)\, dw$ uniformly over $z \in D$. So there are positive constants B and $r > 1$ such that

$$v^r(z) \le B\, (v(z))^r$$

for all $z \in D$.

Since both v and $u^{-1/(p-1)}$ are invariant A_∞ weights, we can choose a constant $r > 1$ so that

$$\begin{aligned} v^r(z) &\le B\, (v(z))^r\,,\\ u^{-r/(p-1)}(z) &\le B\, (u^{-1/(p-1)}(z))^r\,, \end{aligned}$$

for all $z \in D$. Thus

$$v^r(z)\,(u^{-r/(p-1)}(z))^{p-1} \le B^2(v(z))^r(u^{-1/(p-1)}(z))^r\,,$$

for all $z \in D$. If u and v satisfy

$$\sup_{z\in D} v(z)\,(u^{-1/(p-1)}(z))^{p-1} < \infty\,,$$

then they also satisfy

$$\sup_{z\in D} v^r(z)\,(u^{-r/(p-1)}(z))^{p-1} < \infty\,.$$

By Corollary 2, the Hilbert transform T is uniformly bounded from $L^p(u\circ\phi_z)$ to $L^p(v\circ\phi_z)$. This completes the proof of Theorem 4.

What is the difference between A_∞ weights and invariant A_∞ weights? The following theorem answers the question completely.

Let $f(z)$ and $g(z)$ be two nonnegative functions. We use $f(z)\cong g(z)$ to denote that there are two positive constants C_1 and C_2 such that

$$C_1 f(z) \le g(z) \le C_2 f(z)$$

for any z.

For each $z = re^{it}\in D$, recall I_z as in Section 2

$$I_z = \left\{e^{i\theta}\in\partial D : |\theta - t| \le \frac{1-r}{2}\right\}.$$

Theorem 5. *Let v be a positive function on the unit circle. Then v is an invariant A_∞ weight if and only if v is an A_∞ weight, and*

$$v(z)\cong\frac{1}{|I_z|}\int_{I_z} v(w)\,dw\,,$$

for all $z\in D$.

PROOF. Suppose that v is an invariant A_∞ weight. First we will prove that $\log v$ is in BMO. By the Jensen inequality, we have

$$\exp\Big(\int_D(\log v - (\log v)(z))P_z(e^{i\theta})\,d\theta\Big) \le \int_D \exp\left(\log v - (\log v)(z)\right)P_z(e^{i\theta})\,d\theta \le C\,.$$

Hence

$$\exp\Big(\int_D (\log v - (\log v)(z))^+ P_z(e^{i\theta})\,d\theta\Big) \\ \le \exp\Big(\int_D (\log v - (\log v)(z)) P_z(e^{i\theta})\,d\theta\Big) \le C\,,$$

and

$$\exp\Big(\int_D (\log v - (\log v)(z))^- P_z(e^{i\theta})\,d\theta\Big) \\ \le \exp\Big(\int_D (\log v - (\log v)(z)) P_z(e^{i\theta})\,d\theta\Big) \le C\,.$$

because

$$\int_D (\log v - (\log v)(z))^+ P_z(e^{i\theta})\,d\theta = \int_D (\log v - (\log v)(z))^- P_z(e^{i\theta})\,d\theta\,.$$

On the other hand,

$$\begin{aligned}\int_D |\log v - (\log v)(z)|\,P_z(e^{i\theta})\,d\theta &= \int_D (\log v - (\log v)(z))^+ P_z(e^{i\theta})\,d\theta \\ &\quad + \int_D (\log v - (\log v)(z))^- P_z(e^{i\theta})\,d\theta \\ &\le 2\log C\,.\end{aligned}$$

So $\log v$ is in BMO. For each interval I, there is a point z_I such that $I = I_{z_I}$. We use v_I to denote the average of v over I. Since $\log v$ is in BMO, we have

$$\begin{aligned}|(\log v)_I - (\log v)(z)| &\le \Big|\frac{1}{|I_z|}\int_{I_z} (\log v - (\log v)(z))\,d\theta\Big| \\ &\le \frac{1}{|I_z|}\int_{I_z} |\log v - (\log v)(z)|\,d\theta \\ &\le C\int_{I_z} |\log v - (\log v)(z)|\,P_z(e^{i\theta})\,d\theta \\ &\le C\,\|\log v\|_{\mathrm{BMO}}\,.\end{aligned}$$

Hence

$$v_I \le v(z_I) \le C\,e^{(\log v)(z) - (\log v)_I + (\log v)_I} \le C\,e^{(\log v)_I}\,.$$

So v is an A_∞ weight. In addition,

$$v(z) \leq C\, e^{(\log v)(z)-(\log v)_I+(\log v)_I} \leq C\, e^{(\log v)_I}\,.$$

By the Jensen inequality we have

$$v(z) \leq C\, v_I\,.$$

Conversely suppose that v is an A_∞ weight and satisfy

$$v(z) \cong \frac{1}{|I_z|}\int_{I_z} v(w)\, dw\,,$$

for all $z \in D$.

Since v is an A_∞ weight, $\log v$ is in BMO and

$$\frac{1}{|I_z|}\int_{I_z} v(e^{i\theta})\, d\theta \leq C \exp\Big(\frac{1}{|I_z|}\int_{I_z} \log v(e^{i\theta})\, d\theta\Big)\,,$$

for some $C > 0$. On the other hand,

$$\begin{aligned}\frac{1}{|I_z|}\int_{I_z} (\log v - (\log v)(z))\, d\theta &\leq \frac{1}{|I_z|}\int_{I_z} |\log v - (\log v)(z)|\, d\theta \\ &\leq C\int_{I_z} |\log v - (\log v)(z)| P_z(e^{i\theta})\, d\theta \\ &\leq C\,\|\log v\|_{\mathrm{BMO}}\,.\end{aligned}$$

Hence

$$\frac{1}{|I_z|}\int_{I_z} (\log v)\, d\theta \leq C\,\|\log v\|_{\mathrm{BMO}} + (\log v)(z)\,.$$

Thus

$$\begin{aligned}v(z) &\leq C\,\frac{1}{|I_z|}\int_{I_z} v(e^{i\theta}) d\theta \\ &\leq C \exp\Big(\frac{1}{|I_z|}\int_{I_z} (\log v)(e^{i\theta})\, d\theta\Big) \\ &\leq C\, e^{C_1\|\log v\|_{\mathrm{BMO}}+(\log v)(z)} \\ &\leq C\, e^{(\log v)(z)}\,.\end{aligned}$$

So $v(z)$ is an invariant A_∞ weight. This completes the proof of Theorem 5.

We remark that A_∞ condition is not comparable with

$$v(z) \cong \frac{1}{|I_z|}\int_{I_z} v(w)\,dw\,,$$

for all $z \in D$. There is an A_∞ weight v which $v(z)$ is not equivalent to $\int_{I_z} v(w)\,dw/|I_z|$. As in [26], for example $v(e^{i\theta}) = |\theta|$ is in A_∞. But it is not in invariant A_∞. So by Theorem 5, $v(z)$ is not equivalent to $\int_{I_z} v(w)\,dw/|I_z|$.

Also there is a function $v > 0$ such that $v(z)$ is equivalent to $\int_{I_z} v(w)\,dw/|I_z|$. But v is even not in A_∞, see examples in [24].

If v is in A_2, then $v(z)$ is equivalent to $\int_{I_z} v(w)\,dw/|I_z|$. Thus an A_2 weight is also an invariant A_∞ weight. So a weight function v is an A_2 weight if and only if both v and v^{-1} are invariant A_∞ weights. Therefore invariant A_∞ weights are somehow $1/2 - A_2$ weights. But there are invariant A_∞ weights which are not in A_2 [26].

Recently Fefferman-Kenig-Pipher [7] characterized A_∞ weights in terms of Carleson measures, which is very close to $\log v \in \mathrm{BMO}$. To state their result more precisely we consider weights v on the real line $\mathbb{R}$. Let $\Phi_t(x) = ct^{-1/2}e^{-|x|^2/t}$, fix a function v which verifies the doubling condition

$$\int_{|x-x_0|<2t} v(\theta)\,d\theta \le \rho \int_{|x-x_0|<t} v(\theta)\,d\theta\,,$$

for some $\rho > 1$. The heat extension of v will be defined by $v(x,t) = (v \star \Phi_t)(x)$, and $\nabla v(x,t)$ will denote the spatial gradient. Fefferman-Kenig-Pipher showed that $v \in A_\infty(R)$ if and only if for all $x_0 \in \mathbb{R}$ and $t > 0$,

$$\frac{1}{t}\int_0^{t^2}\int_{|x-x_0|<t} \frac{|\nabla v(x,s)|^2}{(v(x,s))^2}\,dx\,ds < C\,.$$

They [7] made a remark that the above result with the harmonic (Poisson) extension in place of the heat extension would not characterize $A_\infty(\mathbb{R})$, for the Poisson kernel need not have sufficiently rapid decay.

However, using harmonic extension of weights we will characterize invariant A_∞ weights in terms of Carleson measures.

Theorem 6. *Let v be a positive function on the unit circle. Then v is an invariant A_∞ weight if and only if $(|\nabla v(z)|^2/v(z)^2)(1-|z|^2)$ is a*

Carleson measure, i.e.,

$$\int_{S(I)} \Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\, dA(z) \leq C\,|I|\,,$$

for any subarcs I of the unit circle, where

$$S(I) = \{re^{it} \in D : 1-r < |I|,\ e^{it} \in I\}\,.$$

PROOF. Since $v(z)$ is harmonic on D, we have

$$-\Delta \log(v(z)) = \frac{|\nabla v(z)|^2}{(v(z))^2}\,.$$

By Green's formula, we obtain

$$\log{(v(z))} - (\log v)(z) = -\int_D \log\Big|\frac{1-\overline{z}\,w}{z-w}\Big|\Delta(\log{(v(w))})\, dA(w)\,.$$

So

$$\log{(v(z))} - (\log v)(z) = \int_D \log\Big|\frac{1-\overline{z}\,w}{z-w}\Big|\frac{|\nabla v(w)|^2}{(v(w))^2}\, dA(w)\,.$$

For $1/2 < |w| < 1$, it is easy to check that $\log|w|^{-1} \cong (1-|w|^2)$. Also since $v(w)$ is harmonic on D, $|\nabla v(w)|^2/(v(w))^2$ is bounded by a constant M for all w with $|w| \leq 1/2$. By the above equation, we have

$$\log{(v(z))} - (\log v)(z) \leq \int_{|w|>1/2} \Big(1-\Big|\frac{1-\overline{z}\,w}{z-w}\Big|^{-2}\Big)\frac{|\nabla v(w)|^2}{(v(w))^2}\, dA(w) + M$$

and

$$\log{(v(z))} - (\log v)(z) \geq \int_{|w|>1/2} \Big(1-\Big|\frac{1-\overline{z}\,w}{z-w}\Big|^{-2}\Big)\frac{|\nabla v(w)|^2}{(v(w))^2}\, dA(w)\,.$$

By [10], $(1-|w|^2)\,|\nabla v(w)|^2\, dA(w)/(v(w))^2$ is a Carleson measure if and only if

$$\sup_{z\in D}\int_D \frac{(1-|z|^2)}{|1-\overline{z}\,w|^2}\,\frac{|\nabla v(w)|^2}{|v(w)|^2}\,(1-|w|^2)\, dA(w) < \infty\,.$$

This is equivalent to

$$\log(v(z)) - (\log v)(z) \le C\,,$$

for all $z \in D$. This completes the proof of Theorem 6.

Another important property of A_∞ is that

$$A_\infty = \bigcup_{p>0} A_p\,.$$

We will show that invariant A_∞ weights have such property also.

Theorem 7. *Any weight function v satisfying invariant A_∞ already satisfies invariant A_p for some $p < \infty$.*

Proof. Since v is an invariant A_∞ weight, we have

$$\log(v(z)) - (\log v)(z) \le C\,. \tag{33}$$

By the Jensen inequality, we also have

$$\log(v(z)) - (\log v)(z) \ge 0\,. \tag{34}$$

On the other hand, by Theorem 5, we see that v is an A_∞ weight. It follows from [10] that $\log v$ is in BMO. By the theorem of John and Nirenberg [10], there exist positive constants C_1 and C_2 such that

$$|\{w \in \partial D : |\log v(w) - (\log(v))(0)| > \lambda\}| \le C_1\, e^{-C_2\lambda/\|\log v\|_{\mathrm{BMO}}}\,,$$

for $\lambda > 0$. Let $\phi_z(w)$ denote the Möbius map on the unit disk. Then $\|\log v \circ \phi_z\|_{\mathrm{BMO}} = \|\log v\|_{\mathrm{BMO}}$. Therefore

$$|\{w \in \partial D : |\log v \circ \phi_z(w) - \log(v \circ \phi_z)(0)| > \lambda\}| \le C_1\, e^{-C_2\lambda/\|\log v\|_{\mathrm{BMO}}}\,.$$

By (33) and (34), we have

$$\begin{aligned} |\{w \in \partial D : |\log v \circ \phi_z(w) - \log(v(z))| > \lambda\}| \\ \le C_1\, e^{-C_2\lambda/\|\log v\|_{\mathrm{BMO}}}\,. \end{aligned} \tag{35}$$

Let E be a subset of ∂D. For $z \in D$, let

$$\omega_z(E) = \int_E P(z,w)\, dw\,.$$

Then (35) is equivalent to

$$(36)\quad \omega_z\{w \in \partial D : |\log v(w) - (\log v)(z)| > \lambda\}| \le C_1 e^{-C_2\lambda/\|\log v\|_{\mathrm{BMO}}}\,.$$

Let $E_k = \{w \in \partial D : e^k v(z)^{-1} \le v(w)^{-1} \le e^{k+1} v(z)^{-1}\}$. Then

$$v^{-1/(p-1)}(z) \le v(z)^{-1/(p-1)} \Big(1 + \sum_{k=0}^{\infty} \omega(E_k) e^{(k+1)/(p-1)}\Big)\,.$$

By (36), we have

$$v(z)^{-1/(p-1)} \Big(1 + C_1 \sum_{k=0}^{\infty} e^{(k+1)/(p-1)-(C_2 k)/\|\log v\|_{\mathrm{BMO}}}\Big) \le C\, v(z)^{-1/(p-1)}$$

if $1/(p-1) < C_2/\|\log v\|_{\mathrm{BMO}}$. Therefore if $p > 1 + \|\log w\|_{\mathrm{BMO}}/C_2$, we have

$$v(z)\, (v^{-(1/(p-1))}(z))^{p-1} \le C\,,$$

for all $z \in D$. This completes the proof of the theorem.

Now we show another characterization of invariant A_∞ weights.

Theorem 8. *A weight function v is an invariant A_∞ if and only if there is a constant $C > 0$ such that*

$$\int_D P_s(z)\, v(z) \Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1 - |z|^2)\, dA(z) \le C\, v(s)\,,$$

for all $s \in D$ where $P_s(z) = (1 - |s|)/|1 - \overline{s}z|^2$ for $s, z \in D$, and

$$v(z) \cong \frac{1}{|I_z|} \int_{I_z} v(w)\, dw\,,$$

for all $z \in D$.

PROOF. Suppose that v is an invariant A_∞ weight. By Cauchy and Jensen inequalities

$$\exp\big((\log v)(z)\big) \le (v^{1/2}(z))^2 \le v(z)\,,$$

and since v is an invariant weight,

$$v(z) \exp\big(-(\log v)(z)\big) \le C\,.$$

Therefore

$$v(z) \le C\,(v^{1/2}(z))^2\,.$$

In addition by Theorem 6, $(|\nabla v(z)|^2/v(z)^2)(1-|z|^2)$ is a Carleson measure,

$$\begin{aligned}\int_D v\Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\,dA(z)\\ &\le \int_D ((v^{1/2})(z))^2\Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\,dA(z)\\ &\le C\int_{\partial D}(v^{1/2}(e^{i\theta}))^2\,d\theta\\ &= C\int_{\partial D} v(e^{i\theta})\,d\theta\\ &= C\,v(0)\,.\end{aligned}$$

In the above inequality replacing v by $v\circ\phi_z$ and making the change of variables give

$$\int_D P_s(z)\,v(z)\Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\,dA(z) \le C\,v(s)\,,$$

for all $s\in D$.

Suppose that a weight function v satisfies two conditions in Theorem 8. By Theorem 5, we need only to show that v is an A_∞ weight. By Green's formula, we have

$$\begin{aligned}v(z)\log(v(z)) - (v\log v)(z)\\ &= -\int_D \log\Big|\frac{1-\overline{z}\,w}{z-w}\Big|\,\Delta(v(w)\log(v(w)))\,dA(w)\,.\end{aligned}$$

It is easy to check that

$$\Delta(v(w)\log(v(w))) = v(w)\Big|\frac{|\nabla v(w)|^2}{v(w)^2}\Big|\,.$$

So

$$v(z)\log(v(z)) - (v\log v)(z) = -\int_D \log\Big|\frac{1-\overline{z}\,w}{z-w}\Big|\,v(w)\Big|\frac{|\nabla v(w)|^2}{v(w)^2}\Big|\,dA(w)\,.$$

By the first condition in Theorem 8, we have

$$\int P_z(w)\, v(w) \log\Big(\frac{v(w)}{v(z)}\Big)\, dw \le C\, v(z)\,.$$

Hence

$$\int P_z(w) \Big(\frac{v(w)}{v(z)}\Big)\Big(\log\Big(\frac{v(w)}{v(z)}\Big)\Big)\, dw \le C\,.$$

Now we write $\log x = \log^+ x - \log^- x$. It is easy to see that $x \log^- x \le C$. Then

$$\int P_z(w) \Big(\frac{v(w)}{v(z)}\Big)\Big(\log^+\Big(\frac{v(w)}{v(z)}\Big)\Big)\, dw \le C\,,$$

for all $z \in D$.

For $L > 10$, then we have

$$\int_{\{v > Lv(z)\}} P_z(w)\, v(w)\, dw \le \frac{C}{\log L}\, v(z)\,.$$

Hence

$$\int_{\{w \in I_z : v > Lv(z)\}} v(w)\, dw \le \frac{C}{\log L}\, v(z)\, |I_z|\,.$$

For any subset E of I_z with $|E| \le |I_z|/L^2$,

$$\begin{aligned}\int_E v(w)\, dw &= \int_{\{E : v(w) \le Lv(z)\}} v(w)\, dw + \int_{\{E : v(w) > Lv(z)\}} v(w)\, dw \\ &\le L\, v(z)\, |E| + \frac{C}{\log L}\, v(z)\, |I_z|\,.\end{aligned}$$

By the second condition in Theorem 8, we have

$$\begin{aligned}\int_E v(w)\, dw &\le \frac{1}{L}\int_{I_z} v(w)\, dw + \frac{C}{\log L}\int_{I_z} v(w)\, dw \\ &= \Big(\frac{1}{L} + \frac{C}{\log L}\Big)\int_{I_z} v(w)\, dw\,.\end{aligned}$$

If L is sufficiently large, there are two numbers $0 < \alpha = 1/L^2 < 1$ and $0 < \beta = 1/L + C/\log L < 1$ such that whenever $E \subset I_z$ and $|E| \le \alpha |I_z|$,

$$\int_E v(w)\, dw \le \beta\, |I_z|\,.$$

We have proved that v is an A_∞ weight by [3]. This completes the proof of Theorem 8.

In [25], it is shown that if a measure μ in D satisfies

$$\int_D P_s(z)\, v(z)^2\, d\mu(z) \le C\, v(s)\,,$$

then the following imbedding theorem

$$\int_D |(f\, v^{1/2})(z)|^2\, d\mu(z) \le K(C)\, \|f\|_2^2\,,$$

for all $f \in L^2$, holds. We will characterize invariant A_∞ weights by the so-called imbedding theorem.

Theorem 9. *A weight function v is an invariant A_∞ weight if and only if there is a constant $C > 0$ such that*

$$\int_D |(f\, v_s)(z)|^2 \frac{1}{v_s(z)} \Big(\frac{|\nabla v_s(z)|^2}{v_s(z)^2}\Big)(1-|z|^2)\, dA(z) \le C \int_{\partial D} |f(e^{i\theta})|^2\, v_s(e^{i\theta})\, d\theta\,,$$

for all $f \in L^2(v_s)$ and all $s \in D$, where $v_s = v \circ \phi_s$, and

$$v(z) \cong \frac{1}{|I_z|} \int_{I_z} v(w)\, dw\,,$$

for all $z \in D$.

Proof. Since the condition in v is an invariant A_∞, by Theorem 8, we have that there is a constant $C > 0$ such that

$$\int_D P_s(z)\, v(z) \Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\, dA(z) \le C\, v(s)\,,$$

for all $s \in D$. The above condition is invariant. So we need only to show that

$$\int_D |(f\, v)(z)|^2 \frac{1}{v(z)} \Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\, dA(z) \le C \int_{\partial D} |f(e^{i\theta})|^2\, v(e^{i\theta})\, d\theta\,,$$

for all $f \in L^2(v)$. Let $g = f v^{1/2}$ and

$$d\mu = \frac{\Big(\dfrac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\, dA(z)}{v}\,.$$

Then it is sufficient to show

$$\int_D |(g\, v^{1/2})(z)|^2\, d\mu(z) \le C \int_{\partial D} |g(e^{i\theta})|^2\, d\theta\,.$$

By Theorem 1.1 in [25], it is sufficient to show that

$$\int_D P_s(z)\, (v(z))^2\, d\mu(z) \le C\, v(s)\,.$$

This is equivalent to

$$\int_D P_s(z)\, v(z) \Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\, dA(z) \le C\, v(s)\,.$$

Conversely suppose that v satisfies the conditions in Theorem 9. If we let f be 1 in the first condition we have

$$\int_D |(v_s)(z)|^2\, \frac{1}{v_s(z)} \Big(\frac{|\nabla v_s(z)|^2}{v_s(z)^2}\Big)(1-|z|^2)\, dA(z) \le C \int_{\partial D} v_s(e^{i\theta})\, d\theta\,.$$

Making the change of variable implies

$$\int_D P_s(z)\, v(z) \Big(\frac{|\nabla v(z)|^2}{v(z)^2}\Big)(1-|z|^2)\, dA(z) \le C\, v(s)\,,$$

for all $s \in D$ where $P_s(z) = (1-|s|^2)/|1-\overline{s}z|^2$ for $s, z \in D$. By Theorem 8 we conclude that v is an invariant A_∞ weight. This completes the proof of Theorem 9.

5. Toeplitz operators and Hankel operators.

Let f be in L^2. The Toeplitz operator T_f and the Hankel operator H_f with symbol f are defined by $T_f p = P(fp)$, and $H_f p = (1-P)(fp)$, for all analytic polynomials p. Obviously they are densely defined on the Hardy space H^2. In this section we will show several sufficient conditions for the product of two Toeplitz operators or Hankel operators to be bounded.

Theorem 10. *Let X and Y be two Banach function spaces such that $N_{X'}$ and $N_{Y'}$ map L^2 to L^2. Suppose that (f,g) is a pair of functions in L^2 such that*

$$\sup_{z\in D} \|f_- \circ \phi_z - f_-(z)\|_X \, \|g_- \circ \phi_z - g_-(z)\|_Y < \infty \, .$$

Then the product $H_f^* H_g$ is bounded on the Hardy space H^2.

Proof. Let ϕ and ψ be in H^2. Then

$$\langle H_f^* H_g \psi, \phi\rangle = \langle H_g \psi, H_f \phi\rangle \, .$$

Using the Littlewood-Paley formula, we have

$$\langle H_f^* H_g \psi, \phi\rangle = \iint \langle \nabla (H_g\psi)(z), \nabla(H_f\phi)(z)\rangle \log \frac{1}{|z|} \, dA(z) \, .$$

Define

$$\mathrm{Term}_I = \iint_{|z|>1/2} \langle \nabla (H_g\psi)(z), \nabla(H_f\phi)(z)\rangle \log \frac{1}{|z|} \, dA(z)$$

and

$$\mathrm{Term}_{II} = \iint_{|z|<1/2} \langle \nabla (H_g\psi)(z), \nabla(H_f\phi)(z)\rangle \log \frac{1}{|z|} \, dA(z) \, .$$

It is easy to verify that there is a compact operator K on H^2 such that

$$\mathrm{Term}_{II} = \langle K\psi, \phi\rangle \, .$$

We claim that there is a constant $C>0$ such that

$$|\mathrm{Term}_I| \le C \sup_{z\in D} \|f_- \circ \phi_z - f_-(z)\|_X \, \|g_- \circ \phi_z - g_-(z)\|_Y \, \|\psi\|_2 \, \|\phi\|_2 \, .$$

So

$$\|H_f^* H_g\| \le \|K\| + C \sup_{z\in D} \|f_+ \circ \phi_z - f_+(z)\|_X \, \|g_- \circ \phi_z - g_-(z)\|_Y \, .$$

Now we turn to the proof of the claim. Fix an $a > 0$ for which the distribution function inequality (22) holds. For $w \in \partial D$, let $\rho(w)$ denote the maximum of those numbers ε for which

$$\begin{aligned} &A_\varepsilon(H_f\phi)(w)A_\varepsilon(H_g\psi)(w) \\ &\le a\|f_- \circ \phi_z - f_-(z)\|_X \, \|g_- \circ \phi_z - g_-(z)\|_Y \, N_{X'}(\phi)(w)\, N_{Y'}(\psi)(w)\,. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{\partial D} A_{\rho(w)}(H_f\phi)(w)\, A_{\rho(w)}(H_g\psi)(w)\, dw \\ &\qquad \le a \sup_{z\in D} \|f_- \circ \phi_z - f_-(z)\|_X \, \|g_- \circ \phi_z - g_-(z)\|_Y \\ &\qquad\quad \cdot \int_{\partial D} N_{X'}(\phi)(w)\, N_{Y'}(\psi)(w)\, dw \\ &\qquad \le a \sup_{z\in D} \|f_- \circ \phi_z - f_-(z)\|_X \, \|g_- \circ \phi_z - g_-(z)\|_Y \\ &\qquad\quad \cdot \|N_{X'}(\phi)\|_2 \, \|N_{Y'}(\psi)\|_2 \\ &\qquad \le a \sup_{z\in D} \|f_- \circ \phi_z - f_-(z)\|_X \, \|g_- \circ \phi_z - g_-(z)\|_Y \, \|\phi\|_2 \, \|\psi\|_2\,. \end{aligned}$$

The last inequality holds because $N_{X'}$ and $N_{Y'}$ are bounded on L^2.

On the other hand, letting $\chi_w(z)$ denote the characteristic function of $\Gamma_{w,\rho(w)}$, we have

$$\begin{aligned} &\int_{\partial D} A_{\rho(w)}(H_f\phi)(w)\, A_{\rho(w)}(H_g\psi)(w)\, dw \\ &\qquad = \int_{\partial D} \Big(\int_{\Gamma_{w,\rho(w)}} |\nabla(H_f\phi)(z)|^2\, dA(z)\Big)^{1/2} \\ &\qquad\qquad\quad \cdot \Big(\int_{\Gamma_{w,\rho(w)}} |\nabla(H_g\psi)(z)|^2\, dA(z)\Big)^{1/2} d\omega \\ &\qquad \ge \int_{|z|>1/2} \int_{\partial D} \chi_w(z)\, |\nabla(H_f\phi)(z)|\, |\nabla(H_g\psi)(z)|\, dw\, dA(z)\,. \end{aligned}$$

Now the distribution function inequality (22) tells us that $\rho(w) \ge 2(1 - |z|)$ on a subset of I_z whose measure is at least $C_a(1 - |z|)$. If $w \in I_z$ and $\rho(w) \ge 2(1 - |z|)$, then z in $\Gamma_{w,\rho(w)}$. Thus $\chi_w(z) = 1$ on a subset of I_z of measure at least $C_a(1-|z|)$. Combining this observation with the previous inequality, we obtain

$$\begin{aligned}\int_{\partial D} A_{\rho(w)}(H_f\phi)(w)\, A_{\rho(w)}(H_g\psi)(w)\, dw \\ &\geq C_a \int_{|z|>1/2} |\nabla(H_f\phi)(z)|\, |\nabla(H_g\psi)(z)|\, (1-|z|)\, dA(z) \\ &\geq C_a\, |\mathrm{Term}_I|\,.\end{aligned}$$

So

$$|\mathrm{Term}_I| \leq C \sup_{z\in D} \|f_-\circ\phi_z - f_-(z)\|_X\, \|g_-\circ\phi_z - g_-(z)\|_Y\, \|\phi\|_2\, \|\psi\|_2\,.$$

This completes the proof of the theorem.

Theorem 11. *Let X and Y be two Banach function spaces such that $N_{X'}$ and $N_{Y'}$ map L^2 to L^2. Suppose that (f,g) is a pair of outer functions in H^2 such that*

$$\sup_{z\in D} \|f\circ\phi_z\|_X\, \|g\circ\phi_z\|_Y < \infty\,.$$

Then the product $T_fT_{\overline{g}}$ is bounded on the Hardy space H^2.

PROOF. Let ϕ and ψ be two polynomials.

$$\langle T_fT_{\overline{g}}\,\psi, \phi\rangle\langle T_{\overline{g}}\,\psi, T_{\overline{f}}\,\phi\rangle\,.$$

Using the Littlewood-Paley formula, we have

$$\langle T_{\overline{g}}\,\psi, T_{\overline{f}}\,\phi\rangle = \int_D \langle \nabla(T_{\overline{g}}\,\psi)(z), \nabla(T_{\overline{g}}\,\phi)(z)\rangle \log\frac{1}{|z|}\, dA(z)\,.$$

Define

$$\mathrm{Term}_I = \iint_{|z|>1/2} \langle \nabla(T_{\overline{g}}\,\psi)(z), \nabla(T_{\overline{f}}\,\phi)(z)\rangle \log\frac{1}{|z|}\, dA(z)\,.$$

and

$$\mathrm{Term}_{II} = \iint_{|z|<1/2} \langle \nabla(T_{\overline{g}}\,\psi)(z), \nabla(T_{\overline{f}}\,\phi)(z)\rangle \log\frac{1}{|z|}\, dA(z)\,.$$

It is easy to verify that there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$|\mathrm{Term}_{II}| \le C_1 \, \|\psi\|_2 \, \|\phi\|_2$$

and

$$|\mathrm{Term}_{II}| \le C_2 \, \|\psi\|_2 \, \|\phi\|_2 \, .$$

We will show that there are the same estimates of Term_I as above ones.

Let X and Y be two Orlicz spaces. Fix an $a > 0$ for which the distribution function inequality (21) holds. For $w \in \partial D$, let $\rho(w)$ denote the maximum of those numbers ε for which

$$\begin{aligned} &A_\varepsilon(T_{\overline{f}}\,\phi)(w)\, A_\varepsilon(T_{\overline{g}}\,\psi)(w) \\ &\qquad\qquad \le a\, \|f \circ \phi_z\|_X \, \|g \circ \phi_z\|_Y \, N_{X'}(\phi)(w)\, N_{Y'}(\psi)(w)\, . \end{aligned}$$

Thus

$$\begin{aligned} &\int_{\partial D} A_{\rho(w)}(T_{\overline{f}}\,\phi)(w)\, A_{\rho(w)}(T_{\overline{g}}\,\psi)(w)\, dw \\ &\qquad \le a \sup_{z\in D} \|f \circ \phi_z\|_X \, \|g \circ \phi_z\|_Y \int_{\partial D} N_{X'}(\phi)(w)\, N_{Y'}(\psi)(w)\, dw\, . \end{aligned}$$

On the other hand, letting $\chi_w(z)$ denote the characteristic function of $\Gamma_{w,\rho(w)}$, we have

$$\begin{aligned} &\int_{\partial D} A_{\rho(w)}(T_{\overline{f}}\,\phi)(w)\, A_{\rho(w)}(T_{\overline{g}}\,\psi)(w)\, dw \\ &\quad = \int_{\partial D} \Big(\int_{\Gamma_{w,\rho(w)}} |\nabla (T_{\overline{f}}\,\phi)(z)|^2 \, dA(z) \Big)^{1/2} \\ &\qquad\qquad \cdot \Big(\int_{\Gamma_{w,\rho(w)}} |\nabla (T_{\overline{g}}\,\psi)(z)|^2 \, dA(z) \Big)^{1/2} d\omega \\ &\quad \ge \int_{|z|>1/2} \int_{\partial D} \chi_w(z) |\nabla (T_{\overline{f}}\,\phi)(z)|\, |\nabla (T_{\overline{g}}\,\psi)(z)|\, dw\, dA(z)\, . \end{aligned}$$

Now the distribution function inequality (21) tells us that $\rho(w) \ge 2(1-|z|)$ on a subset of I_z whose measure is at least $C_a(1-|z|)$. If $w \in I_z$ and $\rho(w) \ge 2(1-|z|)$, then z in $\Gamma_{w,\rho(w)}$. Thus $\chi_w(z) = 1$ on a subset of I_z of measure at least $C_a(1-|z|)$. Combining this observation with the previous inequality, we obtain

$$\begin{aligned}
\int_{\partial D} & A_{\rho(w)}(T_{\overline{f}}\,\phi)(w)\, A_{\rho(w)}(T_{\overline{g}}\,\psi)(w)\, dw \\
&\geq C_a \int_{|z|>1/2} |\nabla(T_{\overline{f}}\,\phi)(z)|\, |\nabla(T_{\overline{g}}\,\psi)(z)|\, (1-|z|)\, dA(z) \\
&\geq C_a\, |\mathrm{Term}_I|\,.
\end{aligned}$$

So

$$\begin{aligned}
|\mathrm{Term}_I| &\leq a \sup_{z\in D} \|f\circ\phi_z\|_X\, \|g\circ\phi_z\|_Y \int N_{X'}(\phi)(w)\, N_{Y'}(\psi)(w)\, dw \\
&\leq a \sup_{z\in D} \|f\circ\phi_z\|_X\, \|g\circ\phi_z\|_Y\, \|N_{X'}(\phi)\|_2\, \|N_{Y'}(\psi)\|_2\,.
\end{aligned}$$

Since $N_{X'}$ and $N_{Y'}$ are bounded on L^2, we have

$$|\mathrm{Term}_I| \leq a \sup_{z\in D} \|f\circ\phi_z\|_X\, \|g\circ\phi_z\|_Y\, \|\phi\|_2\, \|\psi\|_2\,.$$

This completes the proof of the theorem.

Acknowledgements. We thank the referee for useful comments.

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Recibido: 3 de noviembre de 1.995
Revisado: 30 de abril de 1.996

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This work is supported in part by the National Science Foundation.

Revista Matemática Iberoamericana
Vol. 13, N.º 2, 1997

Complete interpolating sequences for Paley-Wiener spaces and Muckenhoupt's (A_p) condition

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Abstract. We describe the complete interpolating sequences for the Paley-Wiener spaces L^p_π $(1 < p < \infty)$ in terms of Muckenhoupt's (A_p) condition. For $p = 2$, this description coincides with those given by Pavlov [9], Nikol'skii [8], and Minkin [7] of the unconditional bases of complex exponentials in $L^2(-\pi, \pi)$. While the techniques of these authors are linked to the Hilbert space geometry of L^2_π, our method of proof is based on turning the problem into one about boundedness of the Hilbert transform in certain weighted L^p spaces of functions and sequences.

1. Introduction.

In this paper we study interpolation in the Paley-Wiener spaces L^p_π $(1 < p < \infty)$, which consist of all entire functions of exponential type at most π whose restrictions to the real line are in L^p. The Paley-Wiener spaces are Banach spaces when endowed with the natural $L^p(\mathbb{R})$-norms. We want to describe those sequences $\Lambda = \{\lambda_k\}$, $\lambda_k = \xi_k + i\,\eta_k$, in the complex plane $\mathbb{C}$ for which the interpolation problem

$$f(\lambda_k) = a_k \;, \tag{1}$$

has a unique solution $f \in L^p_\pi$ for every sequence $\{a_k\}$ satisfying

$$\sum_k |a_k|^p e^{-p\pi|\eta_k|}\,(1+|\eta_k|) < \infty\,. \tag{2}$$

Such sequences Λ are termed *complete interpolating sequences for* L^p_π. A classical example of a complete interpolating sequence for L^p_π $(1 < p < \infty)$ is the sequence of integers $\mathbb{Z}$.

In the case $p = 2$ this problem is equivalent to that of describing all unconditional bases in $L^2(-\pi,\pi)$ of the form $\{\exp(i\,\lambda_k t)\}$. We refer to [4] for an account of this problem, including a detailed survey of its history. The unconditional basis problem was solved by Pavlov [9] under the additional restriction $\sup|\operatorname{Im}\lambda_k| < \infty$ and by Nikol'skii [8], assuming only $\inf \operatorname{Im}\lambda_k > -\infty$. Finally, Minkin [7] solved the problem without any a priori assumption on Λ.

The methods of [4], [7], [8], [9] are of a geometric nature and make crucial use of the Hilbert space structure of L^2_π. In this paper, we shall give a simpler proof, which works equally well for all p, $1 < p < \infty$. Incidentally, our method of proof shows that for $p = \infty$ or $0 < p \le 1$ there are no complete interpolating sequences. (See also [2], which "explains" this curious phenomenon). The core of our approach is a careful study of properties of the Hilbert transform in weighted spaces of functions and its discrete version in weighted spaces of sequences. More precisely, we turn our problem into one about boundedness of the discrete Hilbert transform in a weighted space, defined on a subsequence of Λ located in a horizontal strip, where the weight is expressed in terms of certain infinite products involving *all* the points of Λ.

As an application of our main theorem, we prove a counterpart of the well-known Kadets 1/4 theorem.

2. Preliminary observations and statement of the main result.

Suppose that Λ is a complete interpolating sequence for L^p_π. By a classical theorem of Plancherel and Pólya (see [6, Lecture 7, Theorem 4]),

$$\int_{-\infty}^{\infty} |f(x+i\,a)|^p\,dx \le e^{p\pi|a|}\|f\|^p_{L^p} \tag{3}$$

for every function $f \in L^p_\pi$ and each $a \in \mathbb{R}$, and so $\exp(i\,\pi\,z)\,f(z)$ belongs to the Hardy space H^p of $\mathbb{C}^+_a := \{z \in \mathbb{C} : \operatorname{Im} z > a\}$ for each

$a \in \mathbb{R}$. Hence the sequence $\Lambda \cap \mathbb{C}_a^+$ is H^p-interpolating in $\mathbb{C}_a^+$ (see [5, Chapter 9]). Similarly, $\Lambda \cap \mathbb{C}_a^-$ is H^p-interpolating in the half-plane $\mathbb{C}_a^- := \{z \in \mathbb{C} : \operatorname{Im} z < a\}$. So the sequences $\Lambda \cap \mathbb{C}_a^+$ and $\Lambda \cap \mathbb{C}_a^-$ satisfy the Carleson condition in the corresponding half-planes, *i.e.*,

$$\begin{aligned} &\inf_{\operatorname{Im}\lambda_j > a} \prod_{\substack{\operatorname{Im}\lambda_k > a\\ k\neq j}} \Big|\frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i\,2\,a}\Big| > 0\,,\\ &\inf_{\operatorname{Im}\lambda_j < a} \prod_{\substack{\operatorname{Im}\lambda_k < a\\ k\neq j}} \Big|\frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i\,2\,a}\Big| > 0\,. \end{aligned} \tag{4}$$

As a side remark, we mention that this condition may be expressed in different ways. For instance, by manipulating the Carleson condition in much the same way as in [1, p. 288-290] (we omit the details), we obtain the following equivalent condition

$$\sup_j \sum_{\substack{k\\ k\neq j}} \frac{(1+|\eta_j|)\,(1+|\eta_k|)}{|\lambda_j - \lambda_k|^2} < \infty\,. \tag{5}$$

Trivially, the inequalities in (4) imply that for each $a \in \mathbb{R}$

$$\inf_{\substack{\operatorname{Im}\lambda_j > a\\ \operatorname{Im}\lambda_k > a\\ k\neq j}} \Big|\frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i\,2\,a}\Big| > 0\,,$$

$$\inf_{\substack{\operatorname{Im}\lambda_j < a\\ \operatorname{Im}\lambda_k < a\\ k\neq j}} \Big|\frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda_k} - i\,2\,a}\Big| > 0\,.$$

Choosing respectively $a = -1$ and $a = 1$ in these two inequalities, we deduce that for some $\varepsilon > 0$ the disks

$$K(\lambda_k) := \{z : \ |z - \lambda_k| < 10\,\varepsilon\,(1 + |\eta_k|)\}$$

are pairwise disjoint. (We fix this value of ε until the end of the paper.)

Moreover, (4) implies that the measure

$$\mu_\Lambda^+ := \sum_{\eta_k > 0} \eta_k\,\delta_{\lambda_k}$$

(δ_λ is the unit point measure at λ) is a Carleson measure, *i.e.*,

$$\int_{\mathbb{C}^+} |f|^s \, d\mu_\Lambda^+ \le C \, \|f\|_{H^s}^s$$

for each function f in the Hardy space $H^s(\mathbb{C}^+)$, $s \ge 1$ (see [1, p. 63]). Similarly, Λ generates a Carleson measure in the lower half-plane as well as in each of the half-planes $\mathbb{C}_a^\pm$.

If Λ is a complete interpolating sequence for L_π^p, then

$$(6) \quad \|f\|_{L^p(\mathbb{R})} \le C \Big(\sum_k |f(\lambda_k)|^p \, e^{-p\pi|\eta_k|} \, (1 + |\eta_k|) \Big)^{1/p}, \qquad f \in L_\pi^p \, .$$

Indeed, since the interpolation problem (1) has a solution $f \in L_\pi^p$ whenever (2) holds, the operator

$$T : f \longmapsto \{ f(\lambda_k) \, e^{-\pi|\eta_k|} \, (1 + |\eta_k|)^{1/p} \}$$

is bounded from L_π^p onto l^p. By the uniqueness of the solution of the interpolation problem, we have $\ker T = \{0\}$, and it suffices to apply the Banach theorem on inverse operators.

Given $x \in \mathbb{R}$, $r > 0$, let $Q(x, r)$ be the square with center at x, side length $2\,r$, and sides parallel to the coordinate axes. We say that a sequence $\Lambda \subset \mathbb{C}$ is *relatively dense* if there exists $r_0 > 0$ such that

$$\Lambda \cap Q(x, r_0) \ne \varnothing \qquad \text{for each } x \in \mathbb{R} \, .$$

If Λ is a complete interpolating sequence for L_π^p, (6) forces Λ to be relatively dense: if this is not the case and there exist sequences $\{x_j\} \subset \mathbb{R}$ and $r_j \to \infty$ such that $Q(x_j, r_j) \cap \Lambda = \varnothing$, then, setting

$$f_j(z) = \frac{\sin \frac{\pi}{2} \, (z - x_j)}{z - x_j} \, ,$$

we find that

$$\sum_k |f_j(\lambda_k)|^p \, e^{-p\pi|\eta_k|} \, (1 + |\eta_k|) \longrightarrow 0 \, , \qquad j \longrightarrow \infty \, ,$$

while $\|f_j\|_{L^p}$ is independent of j.

Suppose that Λ is a complete interpolating sequence for L^p_π. Take $r > r_0$, where r_0 is as above, define

$$Q_j = Q(4\,r\,j, r)\,, \qquad j \in \mathbb{Z}\,,$$

and pick a sequence $\Gamma = \{\gamma_j\} \subset \Lambda$ such that $\gamma_j \in Q_j$. Let $\Sigma = \{\sigma_j\}$ be another sequence with $|\gamma_j - \sigma_j| = \varepsilon$. Suppose $w = \{w_j\}$ is a positive weight sequence. Associate with it the weighted space l^p_w consisting of all sequences $a = \{a_k\}$ satisfying

$$\|a\|^p_{w,p} := \sum_k |a_k|^p w_k < \infty\,.$$

We are interested in the boundedness of the *discrete Hilbert operator* $\mathcal{H}_{\Gamma,\Sigma}$ defined by the relation

$$\mathcal{H}_{\Gamma,\Sigma} : a = \{a_j\} \longmapsto \{(\mathcal{H}_{\Gamma,\Sigma} a)_j\}\,, \qquad (\mathcal{H}_{\Gamma,\Sigma} a)_j = \sum_k \frac{a_k}{\sigma_j - \gamma_k}\,,$$

on l^p_w. The following definitions are needed. We say that w satisfies the *discrete* (A_p) *condition* if

$$\sup_{\substack{k\in\mathbb{Z}\\ n>0}} \Big(\frac{1}{n}\sum_{j=k+1}^{k+n} w_j\Big)\Big(\frac{1}{n}\sum_{j=k+1}^{k+n} w_j^{-1/(p-1)}\Big)^{p-1} < \infty\,.$$

This condition is analogous to the classical continuous (A_p) condition for a positive weight $v(x) > 0$, $x \in \mathbb{R}$,

$$\sup_I \Big(\frac{1}{|I|}\int_I v\,dx\Big)\Big(\frac{1}{|I|}\int_I v^{-1/(p-1)}\,dx\Big)^{p-1} < \infty\,, \tag{7}$$

where I ranges over all intervals in $\mathbb{R}$ (see [3]). Recall that the latter condition is necessary and sufficient for boundedness of the classical Hilbert operator

$$\mathcal{H} : f \longmapsto (\mathcal{H}f)(t) = \frac{1}{i\,\pi}\int \frac{f(\tau)}{t-\tau}\,d\tau$$

on the weighted space of functions $L^p(\mathbb{R}; v)$ consisting of all functions f satisfying

$$\|f\|^p_{v,p} := \int |f(t)|^p\, v(t)\,dt < \infty\,.$$

We shall need the following lemma.

Lemma 1. *If $\mathcal{H}_{\Gamma,\Sigma}$ is bounded from l^p_w to l^p_w, then w satisfies the discrete (A_p) condition.*

PROOF. We adopt the proof for the continuous case (see [3]). Let $k \in \mathbb{Z}$ and $n > 0$ be given. For convenience, put $I_1 = \{k+1, k+2, \dots, k+n\}$, $I_2 = \{k+2n+1, k+2n+2, \dots, k+3n\}$. Suppose that a positive sequence a is supported on I_1. Then, for $j \in I_2$, we have

$$|(\mathcal{H}_{\Gamma,\Sigma}a)_j| \geq \sum_l a_l \frac{\mathrm{Re}\,(\sigma_j - \gamma_l)}{|\sigma_j - \gamma_l|^2} \geq \frac{C}{n} \sum_l a_l \ , \tag{8}$$

where C is independent of k and n. Putting $a_l = 1$, we get thus

$$\sum_{j \in I_2} w_j \leq C \sum_{l \in I_1} w_l \ ,$$

and by symmetry

$$\sum_{j \in I_1} w_j \asymp \sum_{l \in I_2} w_l \ . \tag{9}$$

Here and in what follows the sign $\asymp$ means that the ratio of the two sides lies between two positive constants. Now we put $a_l = w_l^\alpha$ for $l \in I_1$ and $a_l = 0$ otherwise, and get from (8) and the boundedness of $\mathcal{H}_{\Gamma,\Sigma}$

$$\Big(\sum_{j \in I_2} w_j\Big)\Big(\frac{1}{n}\sum_{l \in I_1} w_l^\alpha\Big)^p \leq C \sum_{m \in I_1} w_m^{1+\alpha p} \ .$$

Finally, we put $\alpha = -1/(p-1)$ and invoke (9), and the lemma is proved.

The converse of Lemma 1 is also true, but we will not need that fact. Note also that the boundedness of the operator $\mathcal{H}_{\Gamma,\Sigma}$ is independent of the choice of sequence Σ, provided the condition $|\gamma_j - \sigma_j| = \varepsilon$ holds.

Let Λ be a complete interpolating sequence for L^p_π. It may be that $0 \in \Lambda$, in which case we assume that $\lambda_0 = 0$. If the function $f_0 \in L^p_\pi$ solves the interpolation problem $f_0(\lambda_k) = \delta_{0,k}$, $k \in \mathbb{Z}$, then $f_0(\mu) \neq 0$ for $\mu \in \mathbb{C} \setminus \Lambda$, since otherwise the function $(z - \lambda_0)(z - \mu)^{-1} f_0(z)$ belongs to L^p_π and vanishes on Λ, contradicting the uniqueness of the solution of the interpolation problem (1). Since $f_0 \in L^p_\pi$, f_0 belongs to the Cartwright class $\mathcal{C}$ (see [6, Lecture 15]) and, in particular, the limit

$$S(z) = (z - \lambda_0) \lim_{R\to\infty} \prod_{|\lambda_k|<R, k\neq 0} \Big(1 - \frac{z}{\lambda_k}\Big) \tag{10}$$

exists and defines the *generating function* of the sequence Λ. Besides, the solution $f_k \in L^p_\pi$ of the interpolation problem $f_k(\lambda_n) = \delta_{k,n}$ has the form

$$f_k(z) = \frac{S(z)}{S'(\lambda_k)\,(z - \lambda_k)}\,. \tag{11}$$

We may now formulate our main theorem.

Theorem 1. $\Lambda = \{\lambda_k\}$, *where* $\lambda_k = \xi_k + i\eta_k$, *is a complete interpolating sequence for* L^p_π *if and only if the following three conditions hold.*

i) *The sequences* $\Lambda \cap \mathbb{C}^+$ *and* $\Lambda \cap \mathbb{C}^-$ *satisfy the Carleson condition in* $\mathbb{C}^+$ *and* $\mathbb{C}^-$ *respectively, i.e.* (4) *holds with* $a = 0$, *and also* $\inf_{k\neq j} |\lambda_k - \lambda_j| > 0$.

ii) *The limit* $S(z)$ *in* (10) *exists and represents an entire function of exponential type* π.

iii) *There exists a relatively dense subsequence* $\Gamma = \{\gamma_j\} \subset \Lambda$ *such that the sequence* $\{|S'(\gamma_j)|^p\}$ *satisfies the discrete* (A_p) *condition.*

Defining $F(x) = |S(x)|/\mathrm{dist}(x, \Lambda)$ $(x \in \mathbb{R})$, *we may replace statement* iii) *by the following*:

iii') $F^p(x)$ $(x \in \mathbb{R})$ *satisfies the* (*continuous*) (A_p) *condition.*

Note that that condition i) is equivalent to the statement that, for each $a \in \mathbb{R}$, the sequences $\Lambda \cap \mathbb{C}^{\pm}_a$ satisfy the Carleson condition (4). Another, more compact way of expressing i), is given by (5).

3. Proof of Theorem 1: necessity.

We have already proved the necessity of i) and ii), and also the existence of a relatively dense sequence $\Gamma = \{\gamma_j\} \subset \Lambda$. We prove now that iii) is necessary as well. Let ε be as above. Then, for every j, we can find a point σ_j with $|\sigma_j - \gamma_j| = \varepsilon$ and

$$|S(\sigma_j)| = \varepsilon\, |S'(\gamma_j)|\,.$$

This follows from the fact that $S(z)\,(z-\gamma_j)^{-1} \neq 0$ for $|z-\gamma_j| \leq \varepsilon$, hence

$$\min_{|z-\gamma_j|=\varepsilon} |S(z)\,(z-\gamma_j)^{-1}| \leq |S'(\gamma_j)| \leq \max_{|z-\gamma_j|=\varepsilon} |S(z)\,(z-\gamma_j)^{-1}|\,.$$

Set $\Sigma = \{\sigma_j\}$. The Plancherel-Pólya inequality (see [6, Lecture 20]) yields

$$\sum_j |f(\sigma_j)|^p \leq C\,\|f\|^p_{L^p}\,, \qquad f \in L^p_\pi\,. \tag{12}$$

Now let $a = \{a_j\}$ be a finite sequence. By (11), the unique solution of the interpolation problem $f(\gamma_j) = a_j$, $f(\lambda_k) = 0$, $\lambda_k \notin \Gamma$ has the form

$$f(z) = \sum_j \frac{a_j}{S'(\gamma_j)}\,\frac{S(z)}{(z-\gamma_j)}\,.$$

By (6) and (12), we have

$$\sum_j |f(\sigma_j)|^p \leq C \sum_j |a_j|^p\,.$$

Now, by our particular choice of the sequence Σ, we obtain iii) by observing that Lemma 1 applies with $w_j = |S'(\gamma_j)|^p$.

To prove that iii) implies iii'), we need the following lemma.

Lemma 2. *Suppose $x \in \mathbb{R}$ and $\operatorname{Re}\gamma_j \leq x \leq \operatorname{Re}\gamma_{j+1}$. Then there exists an $\alpha = \alpha(x) \in [0,1]$ such that*

$$|S'(\gamma_j)|^\alpha |S'(\gamma_{j+1})|^{1-\alpha} \asymp \frac{|S(x)|}{\operatorname{dist}(x,\Lambda)}\,,$$

uniformly with respect to $x \in \mathbb{R}$.

In fact, assuming this lemma to hold, we see that (7) with $v = F^p$ follows from iii) and the inequality $t^\alpha s^{1-\alpha} \leq t+s$, $t,s>0$, $\alpha \in [0,1]$.

PROOF OF LEMMA 2. We assume that $x \in [\operatorname{Re}\gamma_j, \operatorname{Re}\gamma_{j+1}]$ and, for simplicity, $x \notin \Lambda$. Set $\Lambda(x) = \{\lambda \in \Lambda : |\lambda - x| < 30\,r\}$. (Here r is the number used for constructing Γ.) For $\alpha \in [0,1]$ we have

$$\begin{aligned}
\rho &:= \frac{|S'(\gamma_j)|^\alpha |S'(\gamma_{j+1})|^{1-\alpha}}{|S(x)|\operatorname{dist}(x,\Lambda)^{-1}} \\
&= \Bigg(\frac{\Big|\dfrac{1}{\gamma_j} \displaystyle\prod_{\lambda_k \in \Lambda(x)\setminus\{\gamma_j\}} \Big(1 - \frac{\gamma_j}{\lambda_k}\Big)\Big|^\alpha}{\Big|\displaystyle\prod_{\lambda\in\Lambda(x)} \Big(1 - \frac{x}{\lambda}\Big)\Big|} \\
&\qquad \cdot \Big|\frac{1}{\gamma_{j+1}} \prod_{\lambda_k \in \Lambda(x)\setminus\{\gamma_{j+1}\}} \Big(1 - \frac{\gamma_{j+1}}{\lambda_k}\Big)\Big|^{1-\alpha} \operatorname{dist}(x,\Lambda)\Bigg) \\
&\quad \cdot \Big(\prod_{\lambda_k \in \Lambda\setminus\Lambda(x)} \frac{|\gamma_j - \lambda_k|^\alpha |\gamma_{j+1} - \lambda_k|^{1-\alpha}}{|x - \lambda_k|}\Big) \\
&= \Pi_1(x)\,\Pi_2(x)\,.
\end{aligned}$$

Writing

$$\begin{aligned}
\Pi_1(x) = {}& \frac{|\gamma_j - \gamma_{j+1}|}{\max\{|x-\gamma_j|, |x-\gamma_{j+1}|\}} \\
&\cdot \prod_{\lambda_k \in \Lambda(x)\setminus\{\gamma_j,\gamma_{j+1}\}} \frac{|\lambda_k - \gamma_j|^\alpha |\lambda_k - \gamma_{j+1}|^{1-\alpha}}{|x-\lambda_k|}\,,
\end{aligned}$$

we see that $\Pi_1(x) \asymp 1$ uniformly with respect to $\alpha \in [0,1]$.

To estimate $\Pi_2(x)$, we begin by writing

$$\gamma_j = x - x_j + i\,y_j\,, \qquad \gamma_{j+1} = x + x_{j+1} + i\,y_{j+1}\,.$$

The values x_j and x_{j+1} depend on x and also satisfy the inequalities $0 \le x_j,\ x_{j+1} \le 8\,r$. Recall also that $|y_j| < r$ for all j. We may then write

$$\begin{aligned}
\rho^2 \asymp \prod_{\lambda_k \notin \Lambda(x)} & \frac{((x - x_j - \xi_k)^2 + (y_j - \eta_k)^2)^\alpha}{(x-\xi_k)^2 + \eta_k^2} \\
&\cdot ((x + x_{j+1} - \xi_k)^2 + (y_{j+1} - \eta_k)^2)^{1-\alpha}
\end{aligned}$$

$$= \prod_{\lambda_k \notin \Lambda(x)} \Big(1 - \frac{2\,x_j\,(x-\xi_k) + 2\,y_j\,\eta_k + O\,(1)}{(x-\xi_k)^2 + \eta_k^2}\Big)^{\alpha}$$

$$\cdot \Big(1 + \frac{2\,x_{j+1}(x-\xi_k) - 2\,y_{j+1}\,\eta_k + O\,(1)}{(x-\xi_k)^2 + \eta_k^2}\Big)^{1-\alpha}.$$

Choosing $\alpha = \alpha(x)$ so that $\alpha\, x_j - (1-\alpha)\, x_{j+1} = 0$, *i.e.*,

$$\alpha = \frac{x_{j+1}}{(x_j + x_{j+1})}\,,$$

we find that

$$K_1 \exp\Big(c_1 \sum_{\lambda_k \notin \Lambda(x)} \frac{|\eta_k|}{(x-\xi_k)^2 + \eta_k^2}\Big) \le \rho^2$$

$$\le K_2 \exp\Big(c_2 \sum_{\lambda_k \notin \Lambda(x)} \frac{|\eta_k|}{(x-\xi_k)^2 + \eta_k^2}\Big)$$

for some c_1, c_2, K_1, K_2 independent of x. By Carleson's condition (5), the sum is uniformly bounded, and we are done.

4. Proof of Theorem 1: sufficiency.

We will now prove that i), ii), iii') imply that Λ is a complete interpolating sequence.

To begin with, note that

$$\int (F(x))^p \, \frac{dx}{1+|x|^p} < \infty \tag{13}$$

and

$$\int (F(x))^p \, dx = \infty\,. \tag{14}$$

The first relation follows from the fact that $\int (F(x))^p |\mathcal{H} f(x)|^p \, dx < \infty$ for each bounded finite function f; it suffices to take $f = \chi_{[0,1]}$. The second is a direct consequence of [3, Lemma 2]. (Alternatively, we may apply the operator $\mathcal{H}$ to an appropriate δ-sequence $\{\delta_n(x)\}$.)

First, we check that Λ is a uniqueness set. To this end, we need to estimate $|S(z)|$ from below.

Lemma 3. *Let as above $\varepsilon > 0$ be such that the disks*

$$K(\lambda_k) := \{z : |z - \lambda_k| < 10\,\varepsilon\,(1 + |\eta_k|)\}$$

are pairwise disjoint. Then

$$|S(z)| \geq C(1+|z|)^{-1/p}\, e^{\pi|\operatorname{Im} z|}, \qquad \textit{for } \operatorname{dist}(z, \Lambda) > \varepsilon\,(1+|\operatorname{Im} z|)\,. \tag{15}$$

Proof of Lemma 3. Put $\Lambda' = \Lambda \cap \{z : |\operatorname{Im} z| < \varepsilon\}$ and consider the auxiliary function

$$S_1(z) = S(z) \prod_{\lambda \in \Lambda'} \frac{z - \lambda + 2\,i\,\varepsilon}{z - \lambda}\,.$$

It is plain that

$$|S_1(z)| \asymp |S(z)|\,, \qquad |\operatorname{Im} z| > 3\,\varepsilon\,, \tag{16}$$

and, besides, iii') implies that $|S_1(x)|^p$ satisfies the (A_p) condition because $|S_1(x)| \asymp F(x)$.

The function $e^{i\pi z} S_1(z)/(z+i)$ belongs to H^p of the upper half-plane, as follows from (13), ii), and the Plancherel-Pólya theorem (3). Hence we have the following inner-outer factorization of S_1

$$S_1(z) = e^{-i\pi z} G(z)\, B_1(z)\,, \qquad \operatorname{Im} z > 0\,. \tag{17}$$

Here the Blaschke product B_1 corresponds to the Carleson sequence $(\Lambda \cap \mathbb{C}^+) \setminus \Lambda'$ and, in particular,

$$|B_1(z)| > c > 0\,, \qquad \text{for } \operatorname{dist}(z, \Lambda) > \varepsilon\,|\operatorname{Im} z|\,. \tag{18}$$

Moreover, G is an outer function and $|G(x)|^p$ satisfies the (A_p) condition. Therefore, $|G(x)|^{-q}$ is an (A_q) weight (here $1/p + 1/q = 1$), $G(x)^{-1}(1+|x|)^{-1} \in L^q(\mathbb{R})$, and thus

$$\frac{1}{(z+i)\,G(z)} = \frac{1}{2\pi i} \int \frac{1}{(t+i)\,G(t)}\, \frac{dt}{t - z}\,, \qquad \operatorname{Im} z > 0\,.$$

It follows that

$$\frac{1}{|G(z)|} \le C\,(1+|z|)^{1/p}\,. \tag{19}$$

Combining relations (16)-(19), we obtain (15) for $\operatorname{Im} z > 3\,\varepsilon$. The estimate for $\operatorname{Im} z < -3\,\varepsilon$ is similar, and to fill the gap $-3\,\varepsilon < \operatorname{Im} z < 3\,\varepsilon$, we may repeat the construction, taking another horizontal line instead of $\mathbb{R}$.

By (14) and the fact that $|S_1(x)|^p$ is an (A_p) weight, we have

$$\int |S_1(x)|^p\,dx = \infty\,.$$

Applying the Plancherel-Pólya theorem (3) to the function $f(z) = S_1(z + i\,a)$, we therefore obtain

$$\int |S_1(x + i\,a)|^p\,dx = \infty\,, \qquad a \in \mathbb{R}\,.$$

Hence, by (16),

$$\int |S(x+i)|^p\,dx = \infty\,.$$

By a second application of the Plancherel-Pólya theorem, we find that

$$\int |S(x)|^p\,dx = \infty\,. \tag{20}$$

We are now in position to prove the uniqueness. Indeed, if $f \in L^p_\pi$ and $f(\lambda) = 0$, $\lambda \in \Lambda$, then $\phi(z) = f(z)/S(z)$ is an entire function of exponential type 0. By (15) and the pointwise bound

$$|f(z)| \le C_p\,\|f\|_{L^p}(1 + |\operatorname{Im} z|)^{-1/p}\,e^{\pi|\operatorname{Im} z|}\,,$$

it follows that $|\phi(z)|$ is uniformly bounded for z satisfying $\operatorname{dist}(z,\Lambda) > \varepsilon(|\operatorname{Im} z| + 1)$. By the classical Phragmén-Lindelöf theorem, we get $\phi(z) \equiv C$, which is incompatible with (20), unless $C = 0$.

It remains only to check that we can actually solve the interpolation problem (1) for each sequence $a = \{a_k\}$ satisfying (2). It suffices to consider a finite sequence a and bound the norm of the solution by a constant times the left-hand side of (2). After doing so, we can apply a limit procedure. If a is a finite sequence, then, by (13), the unique solution of the interpolation problem (1) has the form

$$f(z) = \sum_k \frac{a_k}{S'(\lambda_k)} \frac{S(z)}{(z-\lambda_k)} \,. \tag{21}$$

We split the sum (21) into two parts, corresponding to points lying in $\mathbb{C}^+ \cup \mathbb{R}$ and in $\mathbb{C}^-$, respectively. We may estimate the norm of each sum separately, so let us assume that all the λ_k corresponding to $a_k \neq 0$ are in $\mathbb{C}^+ \cup \mathbb{R}$. Clearly, we may estimate the L^p integral along $\operatorname{Im}(z) = -1/2$. Let us, however, for conventional reasons, estimate it along $\mathbb{R}$ and assume all the points λ_k satisfy $\eta_k \geq 1/2$. Now let

$$B(z) = \prod_k \frac{z + \frac{i}{2} - \Big(\lambda_k + \frac{i}{2}\Big)}{z + \frac{i}{2} - \Big(\overline{\lambda_k} - \frac{i}{2}\Big)} \,.$$

Writing $S(z) = B(z)\, e^{-i\pi z} G(z)$, where G is an outer function in $\mathbb{C}^+$, we observe that iii') is equivalent to $|G(x)|^p$ satisfying the (A_p) condition. Since

$$S'(\lambda_k) = G(\lambda_k)\, \frac{e^{-i\pi\lambda_k}}{i\,(1+\eta_k)} \prod_{j\neq k} \frac{\lambda_k - \lambda_j}{\lambda_k - \overline{\lambda_j} + i} \,,$$

the Carleson condition (4) implies

$$|S'(\lambda_k)| \asymp |G(\lambda_k)|\, \frac{e^{\pi\eta_k}}{\eta_k} \,.$$

Thus it is enough to consider the L^p boundedness of

$$\tilde f(x) = \sum_k \frac{a_k\, \eta_k\, e^{-\pi\eta_k}}{G(\lambda_k)} \frac{G(x)}{x-\lambda_k} \,.$$

By duality,

$$\begin{aligned}
\|\tilde f\|_p &\asymp \sup_{\substack{\|h\|_q=1\\ h\in H^q}} \Big| \sum_k \frac{a_k\, \eta_k\, e^{-\pi\eta_k}}{G(\lambda_k)} \int_{\mathbb{R}} \frac{G(x)\, h(x)}{x-\lambda_k}\, dx \Big| \\
&\leq \sup_{\substack{\|h\|_q=1\\ h\in H^q}} \Big| \sum_k \frac{a_k\, \eta_k\, e^{-\pi\eta_k}}{G(\lambda_k)}\, (\mathcal{H}Gh)(\lambda_k) \Big| \\
&\leq \sup_{\substack{\|h\|_q=1\\ h\in H^q}} \Big(\sum_k |a_k|^p \eta_k\, e^{-p\pi\eta_k} \Big)^{1/p} \Big(\sum_k \Big| \frac{(\mathcal{H}Gh)(\lambda_k)}{G(\lambda_k)} \Big|^q \eta_k \Big)^{1/q} .
\end{aligned}$$

Since $|G(x)|^{-q}$ satisfies the (A_q) condition, G is an outer function in $\mathbb{C}^+$, and $h \in H^q, \|h\|_q \le 1$, we have $(\mathcal{H}Gh)(z)/G(z) \in H^q$, and $\|(\mathcal{H}Gh)(z)/G(z)\|_q \le C$. Since $\sum_k \eta_k \, \delta_{\lambda_k}$ is a Carleson measure, the last sum is uniformly bounded, and we get the desired conclusion.

The sum corresponding to points in $\mathbb{C}^-$ is treated similarly.

5. A stability result.

We will now show how Theorem 1 can be used to obtain a result similar to the Kadets 1/4 theorem. The same technique implies more sophisticated stability results for L^p_π, similar to the theorems of Avdonin and Katsnelson for L^2_π; see [4] for the latter results.

For $1 < p < \infty$ we denote by q the conjugate exponent, $1/p + 1/q = 1$, and put

$$p' = \max\{p, q\}\,.$$

We may now prove:

Theorem 2. *Suppose $\{\delta_k\}_{k\in\mathbb{Z}}$ is a sequence of real numbers, and put $\lambda_k = k + \delta_k$, $k \in \mathbb{Z}$. If $|\delta_k| \le d < 1/(2\,p')$ for every k, then $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for L^p_π. If merely $|\delta_k| < 1/(2\,p')$ for every k, then $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is not necessarily a complete interpolating sequence for L^p_π.*

Note that for $p = 2$ this is precisely the Kadets theorem (see [4]).

PROOF OF THEOREM 2. We prove first that the inequality

$$|\delta_k| < \frac{1}{2\,p'}$$

is not sufficient. If $\delta_0 = 1$ and otherwise $\delta_k = \operatorname{sgn}(k)\,\delta$, $-1 < \delta < 1$, standard estimates of infinite products yield

$$F(x) \asymp (1 + |x|)^{-2\delta}\,.$$

For $1 < p < 2$ we choose $\delta = -1/(2\,q)$. Then

$$\frac{1}{|x|}\int_0^x F^p\,dt\Big(\frac{1}{|x|}\int_0^x F^{-q}\,dt\Big)^{p-1} \ge C\,(\log(1+|x|))^{p-1}\,,$$

and the (A_p) condition fails. We obtain the same conclusion if $|\delta_k| < 1/(2\,q)$ and δ_k tends sufficiently fast to $\operatorname{sgn}(k)/(2\,q)$ as k tends to $\pm\infty$. If $2 < p < \infty$, we put $\delta = -1/(2\,p)$, and argue similarly.

With Λ as required in the theorem, define

$$\lambda_{\alpha,k} = k + \alpha\,\delta_k \qquad \text{and} \qquad \Lambda_\alpha = \{\lambda_{\alpha,k}\}\,,$$

where α is a real number. Suppose that $\delta < 1/2$ and $|\alpha|\,\delta < 1/2$, so that the distance between any two distinct points of Λ, and likewise the distance between any two distinct numbers of Λ_α, exceeds a certain positive number. Then estimates of infinite products show that

$$F_\alpha(x) \asymp (F(x))^\alpha\,, \tag{22}$$

where $F_\alpha(x) = |S_\alpha(x)|/\operatorname{dist}(x, \Lambda_\alpha)$ and S_α is the generating function of Λ_α.

Suppose first that $1 < p < 2$. If $d < 1/(2\,q)$, then $F^2_{q/2}$ satisfies the (A_2) condition, according to the classical $1/4$ theorem. By (22), it means that F^q satisfies the (A_2) condition, which implies, by Hölder's inequality, that F^p satisfies the (A_p) condition.

If $2 < p < \infty$, put $\alpha = p/2$ and argue similarly.

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Recibido: 9 de enero de 1.996
Revisado: 6 de diciembre de 1.996

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* Research supported by NATO linkage grant LG 930329.

† While at MSRI, Berkeley, during the fall 1995, the second author was supported in part by NSF grant DMS-9022140 and by the Research Council of Norway (NFR).

Revista Matemática Iberoamericana
Vol. 13, N.º 2, 1997

Statistic of the winding of geodesics on a Riemann surface with finite area and constant negative curvature

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Abstract. In this paper we show that the windings of geodesics around the cusps of a Riemann surface of finite area, behave asymptotically as independent Cauchy variables.

1. Introduction.

In this paper we show that the windings of geodesics around the cusps of a Riemann surface of finite area, behave asymptotically as independent Cauchy variables. Results of this type were originally given for Brownian paths. The original proof of [16] for the winding of planar Brownian motion around the origin was analytic. This theory was developed in many works including [1], [2], [14], [9] and [12] using excursion theory and geometric ideas. The idea that such a result might hold for geodesics is suggested by the central limit theorem of Ratner [13] and Sinaï, and the logarithm iterated law discovered by Sullivan [17]. Using coding theory a proof is given in [3] and [4] for modular surfaces. In the note [10], it was briefly shown that this result could be extended to arbitrary Riemann surfaces, by a simple argument that reduced the problem to the Brownian case. However, in these works, the contribution e_t^i of each cusp C_i was not identified. The asymptotic was actually obtained for linear combinations $\sum \lambda_i e_t^i$ under the condition that $\sum \lambda_i = 0$. We show that this condition is unnecessary, using the relation between the Brownian motion on the stable foliation and the geodesic flow which was obtained in [11]. It is reasonable to think that the constant curvature assumption could be relaxed as in [7], [8].

2. Presentation of the result.

Let M be a surface of constant negative curvature with finite area, represented as the quotient of the hyperbolic plane H, under the action of a Fuchsian group Γ.

The well known model of the hyperbolic plane, using the upper half-plane $\mathbb{C}^+$ with the metric $dl^2 = (dx^2 + dy^2)/y^2$ $(y > 0)$, can be transformed into the model of the open unit disc via a conformal map, the metric being then

$$dl^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}\,, \qquad x^2 + y^2 < 1\,.$$

In the representation of the disc, there exists a polygon (whose edges are geodesics) which is a fundamental domain for Γ. There comes out some invariants of the group, (independent from the choice of the system of generator) like its genus g and the multiplicity of the vertices of the polygon. M in our case, will be the union of a compact part and of n cusps $C_1, C_2, \ldots, C_n$, a cusp being the region of the polygon limited by two geodesics going at infinity to the same point of the boundary of the hyperbolic plane (though it is non compact, this region remains of finite area).

Let m be the normalized Liouville measure on the unit tangent bundle T^1M. Functions on T^1M can be viewed as random variables on the probability space $(T^1M, \mathcal{B}, m)$ ($\mathcal{B}$ denoting the Borel σ-field on T^1M).

We denote by θ_t the geodesic flow on T^1M, which preserves m and is known to be ergodic [5].

Let ω be a 1-form on M: we assume that $d\omega$ vanishes in a neighbourhood U_i of each cusp C_i. Let λ_i denote the residue of ω at C_i (which is the integral of ω along a loop around C_i, included in U_i, which doesn't depend on the loop as far as this form is locally closed).

If $\xi = (q, v)$, $q \in M$, $v \in T^1_q M$, set $\theta_t(\xi) = (q_t, v_t)$, and

$$e^i_t(\xi) = \int_0^t \langle \omega(q_s), v_s \rangle \, \mathbf{1}_{U_i}(q_s) \, ds \, .$$

(If λ_i does not vanish, e^i_t describes the winding of the geodesic in U_i).

We prove the following:

Theorem 1. *The joint distribution of* $(e^1_t/t, e^2_t/t, \ldots, e^n_t/t)$ *converges in law towards the product of* n *Cauchy distributions of parameter* $|\lambda_i|/|M|$ *where* $|M|$ *denotes the area of* M.

REMARKS. If $\hat{\omega}$ is another form, closed near the cusps, with the same residues, the theorem applied to $\omega - \hat{\omega}$ implies that $(\hat{e}^i_t - e^i_t)/t$ converges to 0 m almost surely.

If $d\omega = 0$ on M, $\sum \lambda_i$ vanishes. Since we assume only that $d\omega$ vanishes near the cusps, the residues can take arbitrary values. Therefore our theorem describes the winding of the geodesics around each cusp. This was not achieved in [4] and [10] where only the case of closed forms was treated.

Finally from the theorem we get the independence of the limit from the choice of the neighbourhoods.

If $\{\tilde{e}^i_t : 1 \leq i \leq n\}$ is defined using a different system of neighbourhoods $\{\tilde{U}_i : 1 \leq i \leq n\}$, $(\tilde{e}^i_t - e^i_t)/t$ converges to 0 m almost surely.

This comes from the lemma we shall use in the following:

Lemma 1. *If* ω *is a* 1*-form,* ϕ *is a* C^∞*-function of compact support in* M*, then*

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle \omega(q_s), v_s \rangle \, \phi(q_s) \, ds \longrightarrow 0 \, , \qquad \textit{almost surely}.$$

PROOF. This comes from the ergodic theorem, as far as

$$\int \omega(q, v) \, \phi(q) \, dm(q, v) = 0$$

because the transformation $\sigma : (q, v) \longmapsto (q, -v)$ changes the sign of the integrated function, and m is σ-invariant.

NOTATIONS. H will be represented by the complex upper half-plane $\{z = x + i\,y : y > 0\}$. We shall identify T^1H and $PSL_2(\mathbb{R})$ using the relations

$$q = \frac{a\,i + b}{c\,i + d} \qquad \text{and} \qquad v = \frac{i}{(c\,i + d)^2}\,.$$

Γ appears as a subgroup of $PSL_2(\mathbb{R})$. It is well known that T^1M can be identified with $\Gamma\backslash PSL_2(\mathbb{R})$, in such a way that $\theta_t(\xi)$ can be written $\xi\theta_t$, if we set

$$\theta_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Similarly the right actions of the 1-parameter subgroups

$$\theta_t^+ = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \theta_t^- = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

define the horocyclic flows on T^1M.

We can define the operators of derivation L_0, L_+ and L_- on C^1 functions of T^1M by

$$L_0 f(\xi) = \frac{d}{ds}\Big|_{s=0} f(\xi\theta_s)\,,$$

$$L_+ f(\xi) = \frac{d}{ds}\Big|_{s=0} f(\xi\theta_s^+)\,,$$

$$L_- f(\xi) = \frac{d}{ds}\Big|_{s=0} f(\xi\theta_s^-)\,.$$

For $\alpha > 0$ and $f \in L^2(m)$, we can also define a resolvent operator

$$R_\alpha f(\xi) = \int_0^\infty e^{-\alpha t} f(\xi\theta_t)\,dt\,.$$

We introduce the matrix T_z

$$T_z = \frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

and we recall the formulas $T_z T_{z'} = T_{x+yz'}$ and the decomposition of T_z in terms of the geodesic and horocyclic operators: $T_{x+iy} = \theta_x^+\,\theta_{\log y} = \theta_{\log y}\,\theta_{x/y}^+$. We deduce from there the commutation formulas

$$\theta_{-\log y}\,\theta_x^+\,\theta_{\log y} = \theta_{x/y}^+ \qquad \text{and} \qquad \theta_{-\log y}\,\theta_x^-\,\theta_{\log y} = \theta_{xy}^-\,.$$

From these commutation formulas we deduce the following equalities which will be useful especially in the proof of the convergence of some R_α-type integrals

$$L_+^k(\phi(\xi\theta_s)) = e^{-ks}(L_+^k\phi)\,(\xi\theta_s)\,,$$

$$L_-^k(\phi(\xi\theta_s)) = e^{ks}(L_-^k\phi)(\xi\theta_s)\,,$$

$$\text{for all } \phi \in C^\infty(T^1M) \text{ and } k\,.$$

The influence of the geodesic and horocyclic operators is described by the following formulas

$$T_z\theta_s^+ = T_{x+ys+iy} \qquad \text{and} \qquad T_z\theta_s = T_{x+iye^s}\ .$$

The foliation $\{\xi\, T_z, z \in\ H\}$, describes all the matrices we can obtain from ξ by the action of the geodesic and horocyclic flows.

Lastly, we shall denote the rotations of $PSL_2(\mathbb{R})$ by

$$K_t = \begin{pmatrix} \cos\left(\dfrac{t}{2}\right) & \sin\left(\dfrac{t}{2}\right) \\ -\sin\left(\dfrac{t}{2}\right) & \cos\left(\dfrac{t}{2}\right) \end{pmatrix}.$$

3. Reduction of the problem.

We shall denote by p the canonical projection of H on M and by π the canonical projection of T^1M on M.

Each cusp C_i is represented by a Γ-orbit on the boundary of H, *i.e.* the projective line $\mathbb{R} \cup \infty$. Picking up an element $\overline{C_i}$ in that orbit we can choose γ_i in $PSL_2(\mathbb{R})$ such that $\gamma_i^{-1}(\infty) = \overline{C_i}$. The subgroup of Γ which consists of the elements which fix $\overline{C_i}$, can be written $\{\gamma_i^{-1}\,\theta_{nX_i}^+\,\gamma_i,\, n \in \mathbb{Z}\}$ where X_i is a positive number independent of the choice of $\overline{C_i}$ and γ_i.

We define a fundamental domain $\mathcal{F}_i$ of Γ contained in $\{\gamma_i^{-1}z : 0 \leq x \leq X_i\}$, and containing $R_{h_i/4} = \{\gamma_i^{-1}z : 0 \leq x \leq X_i, y \geq h_i/4\}$ for some positive h_i. Choosing h_i large enough, we can take $U_i = p(R_{h_i/4})$ and assume the U_i's are disjoint. We shall denote $p(R_{h_i/3})$ by V_i and $p(R_{h_i})$ by W_i.

Lastly we denote $U = \cup_{i=1}^n U_i$, $V = \cup_{i=1}^n V_i$, and $W = \cup_{i=1}^n W_i$.

Let u be a C^∞ function on $\mathbb{R}^+$, such that $u=0$ on $[0,1/4]$, and $u=1$ on $[1/3,+\infty]$.

Let s_i denote the section of p relative to $\mathcal{F}_i$. There is a 1-form η on M that vanishes outside $U=\cup_{i=1}^n U_i$ and represented in U_i by

$$s_i^\star \gamma_i^\star\Big(\frac{\lambda_i}{X_i}\,dx\,u\Big(\frac{y}{h_i}\Big)\Big)$$

on U_i.

Inside W_i, $\omega-\eta$ is a closed form with 0-residue. Therefore (since W_i is isomorphic to a disc minus a point), it is exact. Let F_i be a smooth function on W_i such that $\omega-\eta=dF_i$ on W_i. F_i will be extended into a smooth function vanishing outside V_i. Then the 1-form $\omega_0=\omega-\eta-\sum_{i=1}^n dF_i$, vanishes on W.

Note that

$$\begin{aligned}\frac{e_t^i}{t}=\frac{1}{t}\int_0^t\langle\omega_0(q_s),v_s\rangle\,\mathbf{1}_{U_i}(q_s)\,ds\\ +\frac{1}{t}\sum_{j=1}^n(F_j(\xi\theta_t)-F_j(\xi))+\frac{1}{t}\int_0^t\langle\eta(q_s),v_s\rangle\,ds\,.\end{aligned}$$

Since θ_t preserves m, $F_i(\xi\theta_t)$ is a stationary process, so the middle term converges to 0 in probability (without any assumptions on the integrability of F).

The first term converges to 0 m *p.s.* by application of the ergodic theorem: indeed $\langle\omega_0(q),v\rangle\,\mathbf{1}_{U_i}(q)$ is an integrable function T^1M since it vanishes everywhere except on the compact set $({}^cW\cap U)\times S^1$. Moreover the mean value of this function is 0. Indeed, the transformation σ : $(q,v)\longmapsto(q,-v)$ changes the sign of the function, and m is σ-invariant.

Setting $\phi(\xi)=\langle\eta(q),v\rangle$, where $\xi=(q,v)$, the third term can be written

$$\frac{1}{t}\int_0^t\phi(\xi\theta_s)\,ds\,.$$

Since the residues λ_i are arbitrary, the theorem can be reduced to the

Proposition 1. *The law of*

$$\frac{1}{t}\int_0^t\phi(\xi\theta_s)\,ds$$

converges in law towards a Cauchy distribution of parameter

$$\sum_{i=1}^{n} \frac{|\lambda_i|}{|M|} \,.$$

4. Expression of ϕ.

We shall first introduce a fundamental domain for T^1M, as it was already for M:

$\hat{\mathcal{F}}_j = \{g \in PSL_2(\mathbb{R}) : g(i) \in \mathcal{F}_j\}$, is a fundamental domain for the left action of Γ on $PSL_2(\mathbb{R})$. It is possible to characterize any element ξ of T^1M by its representative $g_i(\xi)$ in $\hat{\mathcal{F}}_i$.

We can define the Iwasawa coordinates $z_i(\xi) = x_i(\xi) + i\, y_i(\xi)$ and $\theta_i(\xi)$ by the equation $\gamma_i\, g_i(\xi) = T_{z_i(\xi)} K_{\theta_i(\xi)}$.

Note that if

$$T_z K_\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,, \quad y = \frac{1}{c^2 + d^2} \quad \text{and} \quad \sin\theta = \frac{-2\, c\, d}{c^2 + d^2} \,.$$

It can be easily seen that $\theta_i(\xi)$ and $y_i(\xi)/h_i$ have a geometrical interpretation:

- y_i/h_i is the exponential of the distance from $\pi(\xi)$ to the boundary of W_i.

- θ_i is the angle between the geodesic going from $\pi(\xi)$ to C_i and the geodesic $\{\xi\theta_t, t \geq 0\}$.

We can deduce, from the definition of η the following expression of ϕ in the y_i, θ_i coordinates

$$\phi(\xi) = -\sum_{i=1}^{n} \frac{\lambda_i}{X_i}\, u\Big(\frac{y_i(\xi)}{h_i}\Big) y_i \sin\theta_i(\xi)\, \mathbf{1}_{U_i}(\pi(\xi)) \,.$$

for all $\xi \in T^1M$ (it is worth remarking that although ϕ is a function on T^1M, it depends only on 2 dimensions). It is useful to give the expression of the differential operators L_0 and L_+ in terms of y_i and θ_i:

Lemma 2. *Let F be a function on U_i, of the form $G(y_i(\xi), \theta_i(\xi))$. Then*

$$L_0 F(\xi)|_{U_i} = y_i \cos\theta_i \,\frac{\partial G}{\partial y_i} + \sin\theta_i \,\frac{\partial G}{\partial \theta_i}\,,$$

$$L_+ F(\xi)|_{U_i} = y_i \sin\theta_i \,\frac{\partial G}{\partial y_i} + (1-\cos\theta_i)\,\frac{\partial G}{\partial \theta_i}\,.$$

Let us finally introduce the function

$$\tilde{\phi}'(\xi) = \sum_{i=1}^{n} \frac{\lambda_i}{X_i}\, u\Big(\frac{y_i(\xi)}{h_i}\Big) y_i \cos\theta_i(\xi)\, \mathbf{1}_{U_i}(\pi(\xi))\,.$$

The interest of this function lies in the following:

Lemma 3. *Let ω'_ξ be the 1-form on H defined by the equation*

$$\omega'_\xi(z) = \tilde{\phi}'(\xi\, T_z)\,\frac{dx}{y} + \phi(\xi\, T_z)\,\frac{dy}{y}$$

and let j_ξ be the application from H to M which maps z onto $\pi(\xi T_z)$. Then we get

$$\omega'_\xi = j_\xi^\star \eta\,.$$

PROOF. The proof is just a matter of change of variables.

5. A differential form.

To follow the spirit of the proofs given in [10] and [11], we have to introduce closed forms. We first notice that since $\theta_s = T_{ie^s}$,

$$\int_0^t \phi(\xi\,\theta_s)\,ds = \int_1^{e^t} \phi(\xi\, T_{iy})\,\frac{dy}{y}\,.$$

We shall introduce a function $\tilde{\phi}$ such that

$$\omega^\xi = \phi(\xi\, T_z)\,\frac{dy}{y} + \tilde{\phi}(\xi\, T_z)\,\frac{dx}{y}$$

is a closed form on H, so that we will get

$$\int_0^t \phi(\xi\,\theta_s)\,ds = \int_i^{ie^t} \omega^\xi$$

(the second integral being independent of the path from i to $i\,e^t$).

$\tilde{\phi}(\xi)$ will be defined by the integral

$$-\int_0^\infty e^{-t} L_+\phi(\xi\,\theta_t)\,dt\,.$$

Its convergence will be proved using the following lemma:

Lemma 4. *Let χ be a locally bounded function on $\Gamma\backslash SL_2(\mathbb{R})$ such that for some positive constant P, $\chi(\xi)$ is bounded by*

$$Py_i\,(1-\cos\theta_i) = P\,\frac{2\,d_i^2}{(c_i^2+d_i^2)^2}$$

in V_i, for every i, where a_i, b_i, c_i, d_i denote the matrix coefficients of the matrix $\gamma_i\,g_i(\xi)$, and y_i and θ_i its Iwasawa coordinates. Then

$$\int_0^{+\infty} e^{-s}\chi(\xi\,\theta_s)\,ds$$

converges uniformly in ξ.

PROOF. As

$$\int_{t_0}^{+\infty} e^{-s}|\chi(\xi\,\theta_s)|\,dt = e^{-t_0}\int_0^{+\infty} e^{-s}|\chi(\xi\,\theta_{t_0+s})|\,ds\,,$$

for all $t_0\in\mathbb{R}$, it is enough to get an upper bound of $\int_0^{+\infty} e^{-t}|\chi|\,(\xi\,\theta_t)\,dt$, independent of ξ (the right integral being the value of this function for $\xi\,\theta_{t_0}$).

Outside V, $|\chi|$ is bounded so that the contribution of the part of the geodesic contained in ${}^cV$ is uniformly bounded.

Hence it is enough to show that

$$\sum_{i=1}^n \sum_{j\in\mathbb{N}} \int_{u_i^j}^{v_i^j} e^{-s}|\chi|(\xi\,\theta_s)\,ds$$

is uniformly bounded where the disjoint intervals $[u_i^j, v_i^j]$ are defined by recursion as follows: u_i^j denotes the first time after v_i^{j-1} (or 0 if $j=0$) where the geodesic enters W_i and v_i^j the next exit time of W_i.

We will in fact majorize the contribution of each interval of excursion $[u_i^j, v_i^j]$ by the contribution of an asymmetric excursion $[u_i^j, u_i^j + s_i^j]$ such that s_i^j is bounded below by a positive number and the geodesic between u_i^j and $u_i^j + s_i^j$ lies in V_i.

Let us denote by ξ_i^j the matrix $\gamma_i\, \xi_{u_i^j}$ and

$$\xi_i^j = \begin{pmatrix} a_i^j & b_i^j \\ c_i^j & d_i^j \end{pmatrix}.$$

We get $1/({c_i^j}^2 + {d_i^j}^2) = h_i$.

Let us show that $s_i^j = \log\left(2/({c_i^j}^2 h_i)\right)$ satisfies the required properties.

First

$$\frac{2}{{c_i^j}^2\, h_i} \geq \frac{2}{({c_i^j}^2 + {d_i^j}^2)\, h_i} = 2\,,$$

thus $s_i^j > \log 2$. Second

$${c_i^j}^2 {d_i^j}^2 h_i^2 \leq \frac{h_i^2({c_i^j}^2 + {d_i^j}^2)^2}{4} = \frac{1}{4} < 2\,,$$

so that

$$\frac{h_i}{3} < \frac{1}{{c_i^j}^2 e^{s_i^j} + {d_i^j}^2 e^{-s_i^j}} = \frac{h_i}{2 + \dfrac{{c_i^j}^2 {d_i^j}^2 h_i^2}{2}} < h_i\,.$$

All the conditions concerning s_i^j are hence satisfied.

We are going now to estimate the contribution of the j^{th} passage of the geodesic in the neighbourhood of C_i, by

$$\int_{u_i^j}^{u_i^j + s_i^j} e^{-s} |\chi|(\xi\, \theta_s)\, ds\,,$$

for which we are going to prove that it is the term of a convergent serie.

$$\int_{u_i^j}^{u_i^j + s_i^j} e^{-s} |\chi|(\xi\, \theta_s)\, ds = e^{-u_i^j} \int_0^{s_i^j} e^{-s} |\chi|(\xi_i^j\, \theta_s)\, ds\,,$$

with the above notation concerning the matrix ξ_i^j. We first notice that the minoration of s_i^j by $\log 2$, gives $u_i^j > (j-1)\log 2$, so that

$$e^{-u_i^j} < \frac{1}{2^{j-1}} \ .$$

Moreover,

$$\begin{aligned}
\Big| \int_0^{s_i^j} e^{-s} |\chi|(\xi_i^j\,\theta_s)\,ds \Big| &\le P \int_0^{\log(2/(hc_i^{j^2}))} e^{-s}\, \frac{c_i^{j^2} e^{3s}}{(d_i^{j^2} + c_i^{j^2} e^{2s})^2}\, ds \\
&= P \int_1^{2/(hc_i^{j^2})} \frac{c_i^{j^2} x}{(d_i^{j^2} + c_i^{j^2} x^2)^2}\, dx \\
&= P \Big[\frac{1}{(d_i^{j^2} + c_i^{j^2} x^2)} \Big]_1^{2/(hc_i^{j^2})} \\
&\le \frac{P}{2}\, \frac{1}{(d_i^j)^2 + (c_i^j)^2} \\
&= \frac{P\,h}{2} \ .
\end{aligned}$$

So the contribution of the j^{th} passage is less than $M_i h/2^j$, which is the term of a convergent serie. The lemma is proved.

Lemma 5. *The function $\tilde{\phi} = -R_1 L_+ \phi$ is continuous.*

PROOF. In each V_i we have an explicit formula for ϕ

$$\phi(\xi) = -\frac{\lambda_i}{X_i}\, y_i \sin\theta_i \ .$$

Lemma 1 yields

$$L_+\phi(\xi) = \frac{\lambda_i}{X_i}\, y_i\,(\cos\theta_i - 1)\,(2\cos\theta_i + 1)\,.$$

So $L_+\phi$ satifies the conditions of previous lemma, which ends the proof.

Lemma 6.

1) *$L_0\tilde{\phi}$ and $L_+\tilde{\phi}$ are well defined and continuous,*

2) ω^ξ *is a closed form with* C^1 *coefficients.*

PROOF. 1) For $L_0\tilde{\phi}$, we have to prove the uniform convergence in ξ of

$$\int_0^\infty e^{-s} L_0 L_+ \phi(\xi_i\,\theta_s)\,ds\,.$$

We just have to check the assumptions of Lemma 4. But using the formulas of Lemma 2, we get when $\pi(\xi) \in V_i$

$$L_0 L_+ \phi(\xi) = \frac{2\,\lambda_i}{X_i}\, y_i\,(1-\cos\theta_i)\,(1-4\cos\theta_i - 6\cos^2\theta_i)\,.$$

This function satisfies the assumptions of Lemma 4.

For $L_+\tilde{\phi}$, note that

$$L_+\Big(\int_0^T e^{-s} L_+\phi(\xi\,\theta_s)\,ds\Big) = \int_0^T e^{-s} L_+(L_+\phi(\xi\,\theta_s))\,ds\,.$$

To prove the uniform convergence in ξ of the last integral when T goes to ∞, we first note that

$$L_+(L_+\phi(\xi\,\theta_s)) = e^{-s} L_+^2\phi(\xi\,\theta_s)\,,$$

An easy calculation yields

$$L_+^2\phi(\xi) = \frac{6\lambda_i}{X_i}\, y_i \sin\theta_i \cos\theta_i(\cos\theta_i - 1)\,,$$

so we can conclude by Lemma 4.

2) It is easy to check [11] that

$$d\omega^\xi = (-L_+\phi + L_0\tilde{\phi} - \tilde{\phi})\,\frac{dx\wedge dy}{y^2}$$

and that the parenthesis vanishes by definition of $\tilde{\phi}$.

6. Geodesic flow and Brownian motion.

We are going to show the relation between the integral of ϕ along the flow between 0 and t, which is equal to $\int_i^{ie^t} \omega^\xi$, and the integral along the Brownian path on H starting at i.

Let us define the Brownian motion by the equations:

$$dx_t = \sqrt{2}\, y_t\, dW_t^{(1)}\,, \qquad x_0 = 0\,,$$

$$dy_t = \sqrt{2}\, y_t\, dW_t^{(2)}\,, \qquad y_0 = 1\,,$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are two real independent Brownian motions. The generator of the process so defined is

$$y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)$$

(the explanation of the choice of this normalization will appear in Lemma 10). We shall denote $z_t = x_t + i\, y_t$.

N.B. $\xi\, T_{z_t}$ is a Brownian motion on the leaf $\xi\, T_z$ (in the matricial sense).

The relation between both flows is given in the following lemma:

Theorem 2.

$$\lim_{t\to+\infty} \int m(d\xi) \exp\Big(\frac{i}{t}\int_i^{ie^t} \omega^\xi\Big) = \lim_{t\to+\infty} E\Big[\int m(d\xi) \exp\Big(-\frac{i}{t}\int_i^{z_{S_t}} \omega^\xi\Big)\Big]\,,$$

where S_t denotes the hitting time of the line of equation $y = e^{-t}$ by the Brownian motion on H starting at i.

PROOF. Using the invariance of the Liouville measure under the action of θ_t and performing first the change of variables $\xi\,\theta_t \to \xi$, and then change $s - t$ into s, the left hand side becomes

$$\int m(d\xi) \exp\Big(\frac{i}{t}\int_{-t}^{0} \phi(\xi\,\theta_s)\, ds\Big)\,.$$

With that remark we do not have to consider $\int_i^{ie^t} \omega^\xi$ anymore, but $-\int_i^{ie^{-t}} \omega^\xi$.

By the invariance of m under the right action of θ_u^+, we get

$$\int m(d\xi) \exp\Big(\frac{i}{t}\int_{-t}^0 \phi(\xi\,\theta_s)\,ds\Big) = \int m(d\xi) \exp\Big(\frac{i}{t}\int_{-t}^0 \phi(\xi\,\theta_u^+\theta_s)\,ds\Big),$$

for all $u \in \mathbb{R}$, thus

$$\int m(d\xi) \exp\Big(\frac{-i}{t}\int_0^{-t} \phi(\xi\,\theta_s)\,ds\Big) = \iint \nu_t(du)\, m(d\xi) \exp\Big(\frac{-i}{t}\int_0^{-t} \phi(\xi\,\theta_u^+\theta_s)\,ds\Big),$$

where $\nu_t(du)$ is any probability measure on $\mathbb{R}$.

But from Section 3, $\theta_u^+\theta_t = T_{u+ie^t}$, hence

$$\int_0^{-t} \phi(\xi\,\theta_u^+\theta_s)\,ds = \int_{u+i}^{u+ie^{-t}} \omega^\xi .$$

We have now to study

$$\begin{aligned}
&\iint \nu_t(du)\, m(d\xi) \exp\Big(\frac{-i}{t}\int_{u+i}^{u+ie^{-t}} \omega^\xi\Big) \\
&= \iint \nu_t(du)\, m(d\xi) \exp\Big(\frac{-i}{t}\int_i^{u+ie^{-t}} \omega^\xi\Big) \\
&\quad - \iint \nu_t(du)\, m(d\xi) \exp\Big(\frac{-i}{t}\int_i^{u+ie^{-t}} \omega^\xi\Big)\Big(1-\exp\Big(\frac{i}{t}\int_i^{u+i} \omega^\xi\Big)\Big).
\end{aligned}$$

We are now choosing for ν_t the Cauchy law with parameter $1-e^{-t}$, namely the hitting distribution of the line $y = e^{-t}$ by the Brownian motion. The last term vanishes as t goes to $+\infty$, by dominated convergence since

$$\frac{\nu_t(du)}{du} = \frac{1-e^{-t}}{(1-e^{-t})^2+u^2} \le \frac{1}{\frac{1}{4}+u^2}, \qquad \text{for } t \ge \log 2 .$$

With this choice of ν_t,

$$\iint \nu_t(du)\, m(d\xi) \exp\Big(\frac{-i}{t}\int_i^{u+ie^{-t}} \omega^\xi\Big) \\ = E\Big[\int m(d\xi) \exp\Big(\frac{-i}{t}\int_i^{z_{S_t}} \omega^\xi\Big)\Big]\,,$$

which can also be written,

$$E\Big[\int m(d\xi) \exp\Big(\frac{-i}{t}\int_0^{S_t} \langle \omega^\xi, \circ\, dz_s\rangle\Big)\Big]\,,$$

where $\circ$ denotes the Stratonovich integral as in [6]. It is indeed the stochastic integral for which the differential calculus coincides with the usual one; in other words, if

$$F(z) = \int_i^z \omega^\xi\,, \qquad F(z_t) = \int_0^t \langle \omega^\xi, \circ\, dz_s\rangle\,.$$

7. From Stratonovich to Itô.

By previous lemma, we have to study

$$\lim_{t\to+\infty} E\Big[\int m(d\xi) \exp\Big(\frac{-i}{t}\int_0^{S_t} \langle \omega^\xi, \circ\, dz_s\rangle\Big)\Big]\,.$$

The difficulty lies in the fact that ω^ξ is not a priori harmonic, and so the integral $\int_0^t \langle \omega^\xi, \circ\, dz_s\rangle$ is not a martingale, so that we cannot directly treat the problem using excursion theory as it was done in [10].

Let us examine the integral (we denote $\xi_s = \xi\, T_{z_s}$)

$$\begin{aligned}\int_0^t \langle \omega^\xi, \circ\, dz_s\rangle &= \int_0^t \Big(\frac{\phi(\xi\, T_{z_s})}{y_s} \circ dy_s + \frac{\tilde\phi(\xi\, T_{z_s})}{y_s} \circ dx_s\Big) \\ &= \int_0^t (\phi(\xi_s)\, dW_s^{(2)} + \tilde\phi(\xi_s)\, dW_s^{(1)}) \\ &\quad + \frac{1}{2}\int_0^t \Big(d\Big\langle \frac{\phi(\xi\, T_{z_s})}{y_s}, y_s\Big\rangle + d\Big\langle \frac{\tilde\phi(\xi\, T_{z_s})}{y_s}, x_s\Big\rangle\Big)\,.\end{aligned}$$

By Itô's formula,

$$d\Big\langle \frac{\phi(\xi\, T_{z_s})}{y_s}, y_s \Big\rangle = \frac{\partial}{\partial y}\Big(\frac{\phi(\xi\, T_z)}{y}\Big) d\langle y_s, y_s\rangle$$
$$= \Big(2\, y_s\, \frac{\partial \phi}{\partial y}(\xi\, T_{z_s}) - 2\,\phi(\xi\, T_{z_s})\Big)\, ds\,.$$

Similarly,

$$d\Big\langle \frac{\phi(\xi\, T_{z_s})}{y_s}, x_s \Big\rangle = \frac{\partial}{\partial x}\Big(\frac{\tilde\phi(\xi\, T_z)}{y}\Big) d\langle x_s, x_s\rangle = \Big(2\, y_s\, \frac{\partial \tilde\phi}{\partial x}(\xi\, T_{z_s})\Big)\, ds\,.$$

Thus

$$\int_0^t \langle \omega^\xi, \circ\, dz_s\rangle = \int_0^t \phi(\xi_s)\, dW_s^{(2)} + \tilde\phi(\xi_s)\, dW_s^{(1)}$$
$$+ \int_0^t \Big(y_s\, \frac{\partial \phi}{\partial y}(\xi\, T_{z_s}) + y_s\, \frac{\partial \tilde\phi}{\partial x}(\xi\, T_{z_s}) - \phi(\xi\, T_{z_s})\Big)\, ds\,,$$

which can also be written

$$\int_0^t \langle \omega^\xi, \circ\, dz_s\rangle = \int_0^t \phi(\xi_s)\, dW_s^{(2)} + \tilde\phi(\xi_s)\, dW_s^{(1)}$$
$$+ \int_0^t (L_0\phi + L_+\tilde\phi - \phi)(\xi\, T_{z_s})\, ds\,.$$

We notice that the last term describes the "lack of harmonicity" of the form ω^ξ. Indeed $(L_0\phi + L_+\tilde\phi - \phi = 0)$ as soon as ω^ξ is harmonic and we can then see that $\int_0^t \langle \omega^\xi, \circ\, dz_s\rangle$ is a martingale.

We show that the second term has no influence on the limit by the ergodic theorem, proving that $L_0\phi + L_+\tilde\phi - \phi$ is in $L^1(m)$, and that its mean value is equal to 0. For that purpose we shall prove two lemmas:

Lemma 7. *With the notations of Section 4, $L_+\tilde\phi' = -L_0\phi + \phi$.*

PROOF. By Lemma 2, we have just to check the following equality

$$\Big(y_i \sin\theta_i\, \frac{\partial}{\partial y_i} + (1-\cos\theta_i)\, \frac{\partial}{\partial \theta_i}\Big)\Big(u\Big(\frac{y_i}{h_i}\Big) y_i \cos\theta_i\Big)$$
$$= -\Big(y_i \cos\theta_i\, \frac{\partial}{\partial y_i} + \sin\theta_i\, \frac{\partial}{\partial \theta_i}\Big)\Big(- u\Big(\frac{y_i}{h_i}\Big) y_i \sin\theta_i\Big)$$
$$- u\Big(\frac{y_i}{h_i}\Big) y_i \sin\theta_i\ .$$

REMARK. $\tilde{\phi}'$ has the property to make coclosed the form

$$\tilde{\phi}'(\xi\, T_z)\,\frac{dx}{y} + \phi(\xi\, T_z)\,\frac{dx}{y}\;.$$

Lemma 8. *$f = L_0\phi + L_+\tilde{\phi} - \phi$, is m-integrable.*

PROOF. By lemmas 6 and 7, it is enough to prove that $L_+(\tilde{\phi} - \tilde{\phi}')$ is integrable on $\pi^{-1}(V_i)$.

Set for $\xi \in \pi^{-1}(V_i)$

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \gamma_i\, g_i(\xi)\,.$$

Note that $c_i^2 + d_i^2 \leq 3/h_i$. Then $\tilde{\phi}(\xi) = -R_1 L_+\phi(\xi)$ can be written

$$\begin{aligned}
\tilde{\phi}(\xi) &= -\int_0^{+\infty} e^{-s} L_+\phi(\xi\,\theta_s)\,ds \\
&= -\int_0^{\log(2/(c_i^2 h_i))} e^{-s} L_+\phi(\xi\,\theta_s)\,ds - \int_{\log(2/c_i^2 h_i))}^{+\infty} e^{-s} L_+\phi(\xi\,\theta_s)\,ds \\
&= -\frac{\lambda_i}{X_i}\int_0^{\log(2/(c_i^2 h_i))} \Big(\frac{2\,c_i^2\,e^{2s}}{(d_i^2 + c_i^2\,e^{2s})^2} - \frac{8\,c_i^2\,d_i^2\,e^{2s}}{(d_i^2 + c_i^2\,e^{2s})^3}\Big)\,ds \\
&\quad - \int_{\log(2/c_i^2 h_i))}^{+\infty} e^{-s} L_+\phi(\xi\,\theta_s)\,ds\,.
\end{aligned}$$

Since in matricial coordinates

$$L_+\phi(\xi) = \frac{\lambda_i}{X_i}\Big(\frac{2\,c_i^2}{(c_i^2 + d_i^2)^2} - \frac{8\,c_i^2\,d_i^2}{(c_i^2 + d_i^2)^3}\Big)\,,$$

$$\begin{aligned}
\tilde{\phi}(\xi) &= -\frac{\lambda_i}{X_i}\Big[\frac{d_i^2 - c_i^2\,x}{(d_i^2 + c_i^2\,x)^2}\Big]_1^{2/c_i^2 h_i} + \frac{h_i\,c_i^2}{2}\,\tilde{\phi}(\xi\,T_{2i/(c_i^2 h_i)}) \\
&= \tilde{\phi}'(\xi) + \frac{\lambda_i}{X_i}\,\frac{\dfrac{2}{h_i} - d_i^2}{\Big(\dfrac{4}{h_i^2} + d_i^2\Big)^2} + \frac{h_i\,c_i^2}{2}\,\tilde{\phi}(\xi\,T_{2i/(c_i^2 h_i)})\,.
\end{aligned}$$

It follows that $L_+(\tilde{\phi} - \tilde{\phi}')$ can be decomposed in the sum of two terms, which both appear to be bounded.

The first one is

$$L_+\left(\frac{\dfrac{2}{h_i} - d_i^2}{\left(\dfrac{4}{h_i^2} + d_i^2\right)^2}\right) = \left(a_i\,\frac{\partial}{\partial b_i} + c_i\,\frac{\partial}{\partial d_i}\right)\left(\frac{\dfrac{2}{h_i} - d_i^2}{\left(\dfrac{4}{h_i^2} + d_i^2\right)^2}\right)$$
$$= -\frac{\dfrac{8\,c_i\,d_i}{h_i}}{\left(\dfrac{4}{h_i^2} + d_i^2\right)^3} - \frac{2\,c_i\,d_i}{\left(\dfrac{4}{h_i^2} + d_i^2\right)^2}\,,$$

which is clearly bounded since $|c_i|$ and $|d_i|$ are bounded by $1/\sqrt{h_i}$.

The second one is $L_+(\psi)$ with

$$\psi(\xi) = \frac{h_i\,c_i^2}{2}\,\tilde{\phi}(\xi\,T_{2i/(c_i^2 h_i)})\,.$$

Note that for that z close to i,

$$\psi(\xi\,T_z) = \frac{h_i\,c_i^2\,y}{2}\,\tilde{\phi}(\xi\,T_z T_{2i/(c_i^2 y h_i)}) = \frac{h_i\,c_i^2\,y}{2}\,\tilde{\phi}(\xi\,T_{x+2i/(c_i^2 h_i)})$$

and therefore

$$L_+\psi(\xi\,T_z) = y\,\frac{\partial\psi(\xi\,T_z)}{\partial x}$$
$$= \frac{c_i^4\,h_i^2}{4}\,y^2\left(\frac{2}{h_i\,c_i^2}\,\frac{\partial}{\partial x}\tilde{\phi}(\xi\,T_{x+2i/(c_i^2 h_i)})\right)$$
$$= \frac{c_i^4\,h_i^2}{4}\,y^2\,(L_+\tilde{\phi})\,(\xi\,T_{x+2i/(c_i^2 h_i)})\,.$$

Hence

$$L_+\psi(\xi) = \frac{c_i^4\,h_i^2}{4}\,(L_+\tilde{\phi})\,(\xi\,T_{2i/(c_i^2 h_i)})\,,$$

$c_i^4\,h_i^2/4$ is clearly bounded, moreover as shown in the proof of Lemma 7, $\xi T_{2i/(c_i^2 h_i)}$ belongs to $V_i\backslash W_i$, which is relatively compact and $L_+\tilde{\phi}$ is continuous. The integrability of f on T^1M is now proven.

We can now state:

Lemma 9. *The integral of f on T^1M vanishes.*

Proof. From Lemma 7 it is enough to show that

$$\int_{T^1M} L_+(\tilde{\phi} - \tilde{\phi}')(\xi)\, m(d\xi) = 0\,.$$

Let g_n^0 a sequence of smooth positive functions on M, increasing towards 1 as n goes to ∞, and such that $\|\nabla g_n^0\|_\infty$ is less than some constant C for all n. Set $g_n = g_n^0 \circ \pi$. An integration by part yields

$$\int_{T^1M} g_n\, L_+(\tilde{\phi} - \tilde{\phi}')(\xi)\, m(d\xi) = \int_{T^1M} (\tilde{\phi}' - \tilde{\phi})\, L_+ g_n(\xi)\, m(d\xi)$$

and the result follows by dominated convergence, letting n increase to infinity.

Hence, we reduced our problem to the study of

$$\lim_{t\to+\infty} E\Big[\int m(d\xi) \exp\Big(\frac{-i\sqrt{2}}{t}\int_0^{S_t} \tilde{\phi}(\xi_s)\, dW_s^{(1)} + \phi(\xi_s)\, dW_s^{(2)}\Big)\Big]\,.$$

8. Calculation of the limit via excursion theory.

Lemma 10. *S_t/t converges almost surely towards 1 as $t \longrightarrow +\infty$.*

Proof. Since

$$y_t = \exp\big(\sqrt{2}\, W_t^{(1)} - t\big)\,, \qquad \text{for } t \geq 0\,,$$

we have

$$S_t - t = \sqrt{2}\, W_{S_t}^{(1)}\,.$$

So the graph of $t \longmapsto S_t$ is symmetric to the graph of $t \longmapsto t - \sqrt{2}\, W_t^{(1)}$, with respect to the first diagonal and

$$\frac{t - \sqrt{2}\, W_t^{(1)}}{t} \longrightarrow 1\,, \qquad \text{almost surely, as } t \to \infty\,.$$

Set

$$\frac{N_{t,1}}{t} = \frac{1}{t}\int_0^t \tilde{\phi}(\xi_s)\, \mathbf{1}_{\{\pi(\xi_s)\notin W\}}\, dW_s^{(1)} + \phi(\xi_s)\, \mathbf{1}_{\{\pi(\xi_s)\notin W\}}\, dW_s^{(2)}\,.$$

Lemma 11. *$N_{S_t,1}/t$ converges to 0 in L^2.*

PROOF. $N_{t,1}$ is a martingale with bracket

$$\int_0^t (\tilde\phi^2(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\notin W\}} + \phi^2(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\notin W\}})\,ds\,.$$

Since $\tilde\phi$ and ϕ are bounded on $\pi^{-1}(W^c)$, say by K, for all integer M,

$$E\,[N^2_{S_t\wedge M}] = E\Big[\int_0^{S_t\wedge M} (\tilde\phi^2(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\notin W\}} + \phi^2(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\notin W\}})\,ds\Big]$$

so that

$$E\,[N^2_{S_t\wedge M}] \le K^2 E\,[S_t\wedge M]\,.$$

But $S_t\wedge M + \log(y_{S_t\wedge M}) = 2\,W^{(1)}_{S_t\wedge M}$, and as far as $\log(y_{S_t\wedge M}) \ge -t$,

$$E\,[S_t\wedge M] \le t\,,$$

so that by Fatou's lemma, we get when M converges to ∞,

$$E\,[N^2_{S_t}] \le K^2 t$$

and we deduce the lemma.

Set now,

$$\frac{N_{S_t,2}}{t} = \frac{1}{t}\int_0^{S_t} \tilde\phi(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW^{(1)}_s + \phi(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW^{(2)}_s\,.$$

Lemma 12.

$$\frac{1}{t}\int_0^{S_t} (\tilde\phi(\xi_s) - \tilde\phi'(\xi_s))\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW^{(1)}_s$$

converges to 0 in L^2.

PROOF. The same proof as in the previous lemma yields the result, since $\tilde\phi - \tilde\phi'$ is easily seen to be bounded.

The averaged integral in the limit can therefore be replaced by

$$\frac{\sqrt{2}}{t}\int_0^{S_t} \tilde\phi'(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW^{(1)}_s + \phi(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW^{(2)}_s\,.$$

In order to use excursion theory, we have to get rid of "incomplete excursions" containing 0 and S_t. For that purpose we introduce T_ξ the first exit time of $\pi^{-1}(W)$ of the Brownian motion starting at ξ, and S_t^ξ its first exit time of $\pi^{-1}(W)$ after S_t. (*N.B.* T_ξ vanishes if $\pi(\xi) \notin W$ and $T_t^\xi = S_t$ when $\pi(\xi_{S_t}) \notin W$).

Note that under $m \otimes \mathbb{P}$, the distributions of

$$\int_0^{T^\xi} \tilde{\phi}'(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW_s^{(1)} + \phi(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW_s^{(2)}$$

and

$$\int_{S_t}^{S_t^\xi} \tilde{\phi}'(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW_s^{(1)} + \phi(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW_s^{(2)}$$

are independent of t (for the second integral, this follows from the T_z-invariance of m and the independence of ξ and S_t). Their quotients by t converge therefore to zero in probability. The averaged integral in the limit can finally be replaced by (Lemma 3),

$$\begin{aligned} H_t^\xi &= \frac{\sqrt{2}}{t}\int_{T^\xi}^{S_t^\xi} \tilde{\phi}'(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW_s^{(1)} + \phi(\xi_s)\,\mathbf{1}_{\{\pi(\xi_s)\in W\}}\,dW_s^{(2)} \\ &= \frac{1}{t}\int_{T^\xi}^{S_t^\xi} \sum_{i=1}^n \frac{\lambda_i}{X_i}\langle s_i^\star\,\gamma_i^\star(dx)\,\mathbf{1}_{\{z_s^\xi\in W_i\}}, \circ\, dz_s^\xi\rangle\,, \end{aligned}$$

where $z_s^\xi = \pi(\xi_s)$ is the Brownian motion on $\Gamma\backslash H$, starting from $\pi(\xi)$.

We now denote by E the expected value with respect to $m \otimes P$. Denote e_i^ξ the excursions of z_s^ξ in W_i, and $\hat{e}_i^\xi$ its lift into H, starting from the image in $\gamma_i\,\mathcal{F}_i$ of the starting point of e_i^ξ. Denote $a(\hat{e}_i^\xi)$ and $b(\hat{e}_i^\xi)$ the starting point and the endpoint of $\hat{e}_i^\xi$ in H and denote $[S(e_i^\xi), T(e_i^\xi)]$ the corresponding time interval.

With these notations,

$$H_t^\xi(\omega) = \frac{1}{t}\sum_{i=1}^n \frac{\lambda_i}{X_i}\Big(\sum_{\substack{e_i^\xi \\ 0<S(e_i^\xi)\le S_t}} b(\hat{e}_i^\xi) - a(\hat{e}_i^\xi)\Big)\,.$$

From excursion theory we get that

$$E\left[\exp\left(i\,H_t^\xi(\omega)\right)\right] = E\Big[\exp\Big(\sum_{i=1}^n \hat{E}_{h_i}\Big(\exp\Big(i\,\frac{\lambda_i}{X_i}\,\frac{X}{t}\Big) - 1\Big)\Big)L_{i,S_t}\Big)\Big]\,,$$

where $L_{i,t}$ is the value at time t of a local time on ∂V_i of z_s^ξ and $\hat{E}_{h_i}$ is the excursion law of the Brownian motion on H, above the line $y = h_i$. Its normalization depends on the choice of L_i, via the identity

$$E\Big[\frac{1}{t}\int_0^t \mathbf{1}_{\{z_s^\xi \in W_i\}}\,ds\Big] = E\Big[\frac{L_{i,t}}{t}\,\hat{E}_{h_i}(\zeta)\Big]$$

(ζ being the excursion lifetime).

X is the abscissa of the excursion endpoint.

By definition of $\hat{E}_{h_i}$,

$$E\Big[\exp\Big(\sum_{i=1}^n \hat{E}_{h_i}\Big(\exp\Big(i\,\frac{\lambda_i}{X_i}\,\frac{X}{t}\Big)-1\Big)\Big)L_{i,S_t}\Big)\Big]$$
$$= E\Big[\exp\Big(\sum_{i=1}^n \lim_{\varepsilon\to 0}\frac{1}{K\varepsilon}\,E_{x,h_i(1+\varepsilon)}\Big(\exp\Big(i\,\frac{\lambda_i}{tX_i}\,(x_{\tau_{h_i}}-x)-1\Big)\Big)L_{i,S_t}\Big)\Big],$$

where τ_{h_i} denotes the hitting time of the line $y = h_i$ by the Brownian motion on H starting from the point $(x, h_i(1+\varepsilon))$ and K is a normalization constant related to the normalization of L_i.

This last expression equals

$$E\Big[\exp\Big(\sum_{i=1}^n \lim_{\varepsilon\to 0}\frac{1}{K\varepsilon}\,E_{x,h_i(1+\varepsilon)}\Big(\Big(\exp\Big(-\frac{\lambda^2}{t^2X_i^2}\int_0^{\tau_{h_i}} y_s^2\,ds\Big)-1\Big)L_{i,S_t}\Big)\Big]$$
$$= E\Big[\exp\Big(\sum_{i=1}^n \lim_{\varepsilon\to 0}\frac{1}{K\varepsilon}\Big(\frac{\phi_i((1+\varepsilon)\,h_i)}{\phi_i(h_i)}-1\Big)L_{i,S_t}\Big)\Big]$$
$$= E\Big[\exp\Big(\sum_{i=1}^n \frac{h_i}{K}\,(\log\phi_i)'(h_i)L_{i,S_t}\Big)\Big],$$

where by the Feynman-Kac formula, ϕ_i solves the differential equation

$$y^2\phi_i'' - \frac{\lambda_i^2}{t^2X_i^2}\,y^2\phi_i = 0$$

with $\phi_i(h_i) = 1$ and ϕ_i bounded at $+\infty$. Therefore

$$\phi_i(y) = \exp\Big(-\frac{|\lambda_i|}{t\,X_i}\,(y-h_i)\Big)$$

and our expression takes the form

$$E\Big[\exp\Big(-\sum_{i=1}^{n}\frac{h_i|\lambda_i|}{K\,t\,X_i}\,L_{i,S_t}\Big)\Big]\,.$$

We now come back to the problem of normalizations. If $\hat{E}_{h_i}$ is normalized in such a way that $\hat{E}_{h_i}(\zeta)=1$, we have

$$E\Big[\frac{1}{t}\int_0^t \mathbf{1}_{\{z_s^\xi\in W_i\}}\,ds\Big]=\frac{E\,[L_{i,t}]}{t}\ ,$$

Since under $m\otimes P_\xi$, z_s is an ergodic process with invariant measure $dx\,dy/|M|y^2$,

$$\frac{E\,[L_{i,t}]}{t}=\frac{1}{|M|}\int_{V_i}\frac{dx\,dy}{y^2}=\frac{X_i}{|M|\,h_i}\ .$$

The ergodic theorem for additive functionals (*e.g.* [14]) yields the almost sure convergence of $L_{i,t}/t$ towards $X_i/(|M|\,h_i)$. As $S_t/t\longrightarrow 1$, $L_{i,S_t}/t$ converges also, almost surely, towards $X_i/(|M|\,h_i)$.

The expectation of the excursion lifetime equals

$$\lim_{\varepsilon\to 0}\frac{1}{K\varepsilon}\,E_{h_i(1+\varepsilon)}[\tau_{h_i}]=\lim_{\varepsilon\to 0}\lim_{\alpha\to 0}-\frac{1}{K\,\varepsilon\,\alpha}E_{h_i(1+\varepsilon)}[\exp\left(-\alpha\,\tau_{h_i}\right)-1]\,,$$

by monotone convergence (monotonicity in α follows from the convexity of the exponential).

The normalization of the excursion lifetime yields

$$1=\lim_{\varepsilon\to 0}\lim_{\alpha\to 0}-\frac{1}{K\,\varepsilon\,\alpha}(\psi_{i,\alpha}(h_i(1+\varepsilon))-1)\,,$$

where $\psi_{i,\alpha}$ is the solution of the differential equation

$$y^2\,\psi''_{i,\alpha}(y)-\alpha\,\psi_{i,\alpha}(y)=0$$

bounded at ∞ and such that $\psi_{i,\alpha}(h_i)=1$.

Hence $\psi_{i,\alpha}(y)=(y/h_i)^\mu$ where μ is the negative root of the equation $\mu\,(\mu-1)-\alpha=0$, namely

$$\mu=\frac{1}{2}\,(1-\sqrt{1+4\alpha})\,,$$

therefore

$$\lim_{\varepsilon\to 0}\lim_{\alpha\to 0}\frac{(1+\varepsilon)^{\mu}-1}{\varepsilon\,\alpha}=1 \qquad \text{and} \qquad K=\lim_{\alpha\to 0}-\frac{\mu}{\alpha}=1\,.$$

Finally

$$\lim_{t\to\infty} E\Big[\exp\Big(-\sum_{i=1}^{n}\frac{h_i\,|\lambda_i|}{K\,t\,X_i}\,L_{i,S_t}\Big)\Big]=\exp\Big(-\sum_{i=1}^{n}\frac{|\lambda_i|}{|M|}\Big)\,.$$

Hence

$$\lim_{t\to\infty} E\Big(\exp\Big(\sum_{i=1}^{n}\hat{E}_{h_i}\Big(\exp\Big(i\,\frac{\lambda_i}{X_i}\,\frac{X}{t}\Big)-1\Big)L_{i,S_t}\Big)\Big]=\exp\Big(-\sum_{i=1}^{n}\frac{|\lambda_i|}{|M|}\Big)\,,$$

and the average on T^1M of $\exp(iH_t^{\xi})$, converges towards

$$\exp\Big(-\sum\frac{|\lambda_i|}{|M|}\Big)\,,$$

which ends the proof of Theorem 1.

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Recibido: 20 de mayo de 1.996
Revisado: 21 de abril de 1.997

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Revista Matemática Iberoamericana
Vol. 13, N.º 2, 1997

A note on eigenvalues of ordinary differential operators

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In this follow-up on the work of [FS] an improved condition for the discrete eigenvalues of the operator $-d^2/dx^2 + V(x)$ is established for $V(x)$ satisfying certain hypotheses. The eigenvalue condition in [FS] establishes eigenvalues of this operator to within a small error. Through an observation due to C. Fefferman, the order of accuracy can be improved if a certain condition is true. This paper improves on the result obtained in [FS] by showing that this condition does indeed hold.

The theorem proven here relies on a version of WKB theory developed in [FS] and applies to operators with large slowly varying potentials. For example, it applies to potentials of the form $V(x) = \lambda^2 V_1(x)$ for fixed, smooth V_1, with $V'' > 0$, V having a local minimum, and $\lambda \gg 1$. The theorem applies to more general potentials as well.

Standard WKB theory yields the statement that all eigenvalues E of the differential operator $-d^2/dx^2 + V(x)$ satisfy

$$(1) \quad \int_{x_{\text{left}}}^{x_{\text{right}}} (E - V(x))^{1/2}\, dx = \pi\Big(k + \frac{1}{2}\Big) + O(\lambda^{-1}), \quad \text{for some } k \in \mathbb{Z},$$

where x_{left} and x_{right} are the two solutions of $E - V(x) = 0$.

[FS] shows that this condition for eigenvalues can be improved so that given $N > 0$, there exists $N' > 0$ and complex functions $h_l(E)$ defined in [FS] so that (1) becomes

$$\int_{x_{\text{left}}}^{x_{\text{right}}} (E - V(x))^{1/2}\, dx + \operatorname{Im}\log\Big(1 + \sum_{l=1}^{N'} h_l(E)\Big)$$

$$= \pi\Big(k + \frac{1}{2}\Big) + O\left(\Lambda^{-N}\right), \tag{2}$$

where Λ, which will be defined precisely in the theorem, plays a role analogous to λ. h_1 is explicitly given in [FS] and is purely imaginary. For the moment however, the critical property of h_l is that $h_l(E) = O(\Lambda^{-l})$, and the quantity $\sum h_l(E)$ is $O(\Lambda^{-1})$ in absolute value, and hence the Taylor series of log gives

$$\int_{x_{\text{left}}}^{x_{\text{right}}} (E - V(x))^{1/2}\, dx + i\, h_1(E) = \pi\Big(k + \frac{1}{2}\Big) + O\left(\Lambda^{-2}\right). \tag{3}$$

But if we were to carry out the same calculation to order Λ^{-3}, then since

$$\log\Big(1 + \sum_{l=1}^{N'} h_l(E)\Big) = h_1(E) + h_2(E) - \frac{1}{2}\, h_1^2(E) + O\left(\Lambda^{-3}\right), \tag{4}$$

we have

$$\begin{aligned} &\int_{x_{\text{left}}}^{x_{\text{right}}} (E - V(x))^{1/2}\, dx + \operatorname{Im}\left(h_1(E) + h_2(E)\right) \\ &\qquad = \pi\Big(k + \frac{1}{2}\Big) + O\left(\Lambda^{-3}\right). \end{aligned} \tag{5}$$

Note h_1^2 is real and therefore makes no contribution to the left-hand side of (5). Moreover, we shall show that h_k is purely imaginary whenever k is odd and real whenever k is even. This reduces the left-hand side of (5) to the simpler left-hand side of (3). This improves upon (3) since (5) holds to $O\left(\Lambda^{-3}\right)$ instead of $O\left(\Lambda^{-2}\right)$. Using the above fact we obtain an improved version of part of the WKB Eigenvalue Theorem. (*cf.* [FS, p. 239]). For the reader's convenience and for completeness we repeat the hypotheses here.

Theorem. *Suppose we are given positive functions $S(x)$ and $B(x)$ on I and a potential $V(x)$ supported on a possibly unbounded interval I_{BVP} with $I \subset I_{\text{BVP}}$. Furthermore, suppose we are given two real numbers $E_0 \leq E_\infty$, positive numbers $\varepsilon < 1/100$, $K > 1$ and $N > K\varepsilon^{-10}$. Define $N' = [\varepsilon N/500]$ and $N'' = 3\,\varepsilon\, N'/2 - K - 33$. And suppose we have the following hypotheses*:

Hyp0) *If* $x, y \in I$ *and* $|x - y| < c\,B(x)$, *then*

$$c < \frac{B(y)}{B(x)} < C \qquad \textit{and} \qquad c < \frac{S(y)}{S(x)} < C\,.$$

Hyp1) *For* $x \in I$ *and* $\alpha \geq 0$ *we have*

$$\left|\left(\frac{d}{dx}\right)^{\alpha} V(x)\right| \leq C_\alpha\, S(x) B^{\alpha}(x)\,.$$

Hyp2) *The equation* $V(x) = E_0$ *has two solutions* $x_{\text{left}} < x_{\text{right}}$ *in* I, *and they satisfy*

$$\operatorname{dist}(x_{\text{left}}, \partial I) > c\,B(x_{\text{left}})\,, \qquad \operatorname{dist}(x_{\text{right}}, \partial I) > c\,B(x_{\text{right}})\,.$$

Hyp3)

$$-V'(x) > c\,S(x_{\text{left}})B^{-1}(x_{\text{left}})\,, \qquad \textit{for } x \in [x_{\text{left}}, x_{\text{left}} + c_1 B(x_{\text{left}})]$$

and

$$V'(x) > c\,S(x_{\text{right}})B^{-1}(x_{\text{right}})\,, \qquad \textit{for } x \in [x_{\text{right}} - c_1 B(x_{\text{right}}), x_{\text{right}}]\,.$$

Hyp4)

$$c\,S(x) < E_0 - V(x) < CS(x)$$

for $x \in [x_{\text{left}} + c_1 B(x_{\text{left}}), x_{\text{right}} - c_1 B(x_{\text{right}})]$.

To state the remaining hypotheses, it is convenient to establish some notation. Set $\lambda(x) = S^{1/2}(x)B(x)$ *for* $x \in I$, *and set*

$$B_{\text{left}} = B(x_{\text{left}})\,, \qquad S_{\text{left}} = S(x_{\text{left}})\,, \qquad \lambda_{\text{left}} = \lambda(x_{\text{left}})\,.$$

$$B_{\text{right}} = B(x_{\text{right}})\,, \qquad S_{\text{right}} = S(x_{\text{right}})\,, \qquad \lambda_{\text{right}} = \lambda(x_{\text{right}})\,.$$

For $|E - E_0| < c\,S_{\text{left}}$, *let* $x_{\text{left}}(E)$ *be the solution of* $V(x) = E$ *nearest to* x_{left}, *and for* $|E - E_0| < cS_{\text{right}}$, *let* $x_{\text{right}}(E)$ *be the solution of* $V(x) = E$ *nearest to* x_{right}. *Define*

$$S_{\min} = \int_{x_{\text{left}} < x < x_{\text{right}}} S(x)\,dx$$

and

$$\Lambda = \int_{x_{\text{left}}}^{x_{\text{right}}} (S^{1/2}(x) B^2(x))^{-1}\, dx\,.$$

Our remaining hypotheses are as follows.

- *Assumptions on $V(x)$ in all of I_{BVP}*:

Hyp5) *If $|E - E_0| < c_2 S_{\min}$ and $E \le E_\infty$, then $V(x) > E$ for all $x \in I_{\text{BVP}} - [x_{\text{left}}(E), x_{\text{right}}(E)]$.*

Hyp6) *If $x \in I_{\text{BVP}}$ satisfies $x < x_{\text{left}} - \lambda^K_{\text{left}} B_{\text{left}}/2$ then $V(x) \ge E_\infty + 100/|x - x_{\text{left}}|^2$, and if $x \in I_{\text{BVP}}$ satisfies $x > x_{\text{right}} + \lambda^K_{\text{right}} B_{\text{right}}/2$, then $V(x) \ge E_\infty + 100/|x - x_{\text{right}}|^2$.*

- *Technical Assumptions*:

Hyp7) $\max_{x \in I} S(x) \le \lambda^K_{\text{left}} S_{\text{left}}$ *and* $\max_{x \in I} S(x) \le \lambda^K_{\text{right}} S_{\text{right}}$.

Hyp8)

$$\int_{x_{\text{left}}}^{x_{\text{right}}} \Big(\frac{dx}{S^{1/2}(x)}\Big) \le \Lambda^K \min\{S^{-1/2}_{\text{left}} B_{\text{left}}, S^{-1/2}_{\text{right}} B_{\text{right}}\}\,.$$

Hyp9)

$$\Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)\, B^4(x)}\Big)\Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}\Big) \le \Lambda^K\,.$$

- *WKB Condition*:

Hyp10) Λ *is bounded below by a positive constant depending only on ε, K and N, and on c, C, c_1, c_2, C_α in* Hyp0)-Hyp4)*.*

Then if E is an eigenvalue of $-d^2/dx^2 + V(x)$, we have that

$$\int_{x_{\text{left}}}^{x_{\text{right}}} (E - V(x))^{1/2}\, dx + i\, h_1(E) = \pi\Big(k + \frac{1}{2}\Big) + \phi_{\text{error}}(E)\,,$$

with $|\phi_{\text{error}}| \le C\Lambda^{-3}$ and

$$h_1(E) = \frac{i}{48} \lim_{\delta \to 0} \Big(\int_{x_{\text{left}}+\delta}^{x_{\text{right}}-\delta} V''(x)\,(E - V(x))^{-(3/2)}\, dx - q(E)\, \delta^{-1/2}\Big)$$

with $q(E)$ uniquely specified by demanding the finiteness of the limit.

PROOF. Let us say that a complex function f_l has the *alternating parity property on the index* l if it is real-valued for l even and purely imaginary for l odd. It suffices to show that h_l has the alternating parity property on the index l. Recall that the h_l's are inductively determined by

$$u_k^{\text{left}}(x,E) = \sum_{l=0}^{k} h_l(E)\, u_{k-l}^{\text{right}}(x,E)\,, \tag{6}$$

where u_k is the canonical solution of the transport equations

$$\begin{aligned} &u_0 \equiv 1\,,\\ 2\,i\,u_{k+1}' &+ \Big(\frac{5}{16}(p')^2 p^{-5/2} - \frac{1}{4}\,p'' p^{-3/2}\Big) u_k \\ &- \frac{1}{2}\,p'\,p^{-3/2}\,u_k' + p^{-1/2}\,u_k'' = 0\,, \qquad 0 \le k < N'\,. \end{aligned}$$

In particular, since $u_0^{\text{left}} = u_0^{\text{right}} = 1$,

$$h_2(E) = u_2^{\text{left}}(x,E) - h_1(E)\,u_1^{\text{right}}(x,E)\,.$$

Since h_1 is known to be purely imaginary, it suffices to show u_k^{left} and u_k^{right} each have the alternating parity property on the index k. Let us show u_k^{left} has the alternating parity property; the proof for u_k^{right} is totally analogous.

Lemma 10 of [FS] relate the canonical solution to the elementary solution of the transport equations in the following manner: if $u = (u_0(x), u_1(x), \ldots, u_{N''}(x))$ is the canonical solution of the transport equations, and if $\tilde{u} = (\tilde{u}_0, \ldots, \tilde{u}_{N''}(x))$ is the elementary solution, then

$$u_k(x) = \sum_{l=0}^{k} w_{k-l,0}\,\tilde{u}_l(x)\,,$$

where w_{kl} will be investigated in more detail below. Since the construction of the elementary solutions in [FS] makes it clear $\tilde{u}_l$ has the alternating parity property on the index l, we have reduced the problem to showing w_{kl} has the alternating parity property on the index k. Equivalently, letting $w_k(x) = \sum_{-3k \le l} w_{kl}\, x^{l/2}$, it suffices to show w_k has the alternating parity property on the index k.

Now all that is needed is to take account of the real and purely imaginary quantities that arise in the construction of w_k. (*cf.* [FS,

p. 155-162, 171]). We proceed as follows: $w_k(x)$ can be written in terms of $h_{kl}^{\#}, q_{kl}^{\#}$ and $\hat{h}_{kl}$ via the equation

$$\begin{aligned}\Big(1+\sum_{k=1}^{N}\lambda^{-k}w_k(x)\Big) &= \Big(\Big(1+\sum_{k=1}^{2N}\sum_{l=2-k}^{3N} h_{kl}^{\#}\,x^{l/2}\lambda^{-k}+O\,(\lambda^{-\varepsilon N/4}\Big)\\ &\quad\cdot\Big(1+\sum_{k=1}^{N}\sum_{l=-k}^{N} q_{kl}^{\#}\,x^{l}\lambda^{-2k}+O\,(\lambda^{-\varepsilon N/4})\Big)\\ &\quad\cdot\Big(1+\sum_{k=1}^{N}\sum_{l=-3k}^{N} \hat{h}_{kl}\,x^{l/2}\lambda^{-k}+O(\lambda^{-\varepsilon N/4})\Big)\Big)\,.\end{aligned}\tag{7}$$

To prove w_k has the alternating parity property on the index k, we will want to show both $h_{kl}^{\#}$ and $\hat{h}_{kl}$ have this property on the index k and $q_{kl}^{\#}$ is real. Let us first look at $h_{kl}^{\#}$. [FS] shows

$$\begin{aligned}&\exp\Big(\sum_{k=1}^{N}\sum_{l=-k}^{N} h_{kl}\,x^{l+3/2}\lambda^{-(2k-1)}\Big)\\ &\qquad= \Big(1+\sum_{k=1}^{2N}\sum_{l=2-k}^{3N} h_{kl}^{\#}\,x^{l/2}\lambda^{-k}+O\,(\lambda^{-\varepsilon N/4})\Big)\,,\end{aligned}\tag{8}$$

where the right-hand side is a high-order Taylor expansion with remainder. Let us consider more carefully how $h_{kl}^{\#}$ depends on h_{kl}. Note that

$$\begin{aligned}&\frac{2\,i}{3}\,\lambda(y_0(x))^{3/2}\sum_{k=1}^{N}\sum_{l=-k}^{N} f_{kl}^{\#\#}x^{l}\lambda^{-2k}\\ &\qquad=\sum_{k=1}^{N}\sum_{l=-k}^{N} h_{kl}\,x^{l+3/2}\lambda^{-(2k-1)}+O\,(\lambda^{-\varepsilon N/4})\,.\end{aligned}\tag{9}$$

Since $y_0(x)$ and $f_{kl}^{\#\#}$ are real, h_{kl} is purely imaginary since it depends only on these quantities multiplied by i. Now set

$$X=\sum_{k=1}^{N}\sum_{l=-k}^{N} h_{kl}\,x^{l+3/2}\lambda^{-(2k-1)}\,.$$

A sufficiently high power of X will be $O\,(\lambda^{-\varepsilon N/4})$, so the left-hand side of (8) has a Taylor expansion with remainder. Note that X^s is purely imaginary if and only if s is odd. Since X contains nothing but odd powers of λ, one finds upon collecting terms of the Taylor expansion with respect to λ that the coefficients are purely imaginary for all odd powers of λ, real for all even powers of λ. This says precisely that $h^{\#}_{kl}$ has the alternating parity property on the index k.

Now let us consider $q^{\#}_{kl}$. Quite simply, $q^{\#}_{kl}$ is real since all the other quantities in the following equation are real.

$$\begin{aligned}\Big(\frac{\partial y_N(x,\lambda)}{\partial x}\Big)^{-1/2}&(y_N(x,\lambda))^{-1/4}\\ &= (p(x))^{-1/4}\Big(1+\sum_{k=1}^{N}\sum_{l=-k}^{N} q^{\#}_{kl}\lambda^{-2k} + O\left(\lambda^{-\varepsilon N/4}\right)\Big)\,. \end{aligned}\tag{10}$$

Finally, let us consider $\hat{h}_{kl}$. We have that

$$\begin{aligned}\Big(1+\sum_{s=1}^{M} c_s\,\lambda^{-s}x^{-3s/2}&\Big(\sum_{k=0}^{N}\sum_{l=-k}^{N} h^{s}_{kl}\,x^{l}\lambda^{-2k} + O\left(\lambda^{-\varepsilon N/5}\right)\Big)\Big)\\ &= \Big(1+\sum_{k=1}^{N}\sum_{l=-3k}^{N} \hat{h}_{kl}\,x^{l/2}\lambda^{-k} + O\left(\lambda^{-\varepsilon N/6}\right)\Big)\,,\end{aligned}\tag{11}$$

where h^{s}_{kl} is real, and c_s has the alternating parity property on the index s. This is a consequence of the recurrence relation one finds upon substituting the asymptotic form of the Airey function

$$A(t) = \mathrm{Re}\Big(\frac{e^{\pm i\pi/4}e^{2it^{3/2}/3}}{t^{1/4}}\Big(1+\sum_{s=1}^{\infty} c_s\,t^{-(3/2)s}\Big)\Big)$$

into the Airey equation

$$\frac{d^2}{dy^2}A(y,\lambda) + \lambda^2 yA(y,\lambda) = 0\,.$$

Collecting the even and odd powers of λ on the left-hand side of (11) shows that $\hat{h}_{kl}$ has the alternating parity property on the index k.

Putting what we know about $h^{\#}_{kl}, q^{\#}_{kl}$ and $\hat{h}_{kl}$ into (7) reveals that coefficients of *even* powers of λ must involve products with *even* total numbers of $h^{\#}_{kl}$'s and $\hat{h}_{kl}$'s. Note also that the $q^{\#}_{kl}$ are always accompanied by *even* powers of λ. Therefore the coefficients of even powers of λ

on the left-hand side of (7) are real. On the other hand the coefficients of odd powers of λ are purely imaginary. Hence w_k has the alternating parity property on the index k.

Acknowledgement. The author wishes to thank Charles Fefferman for many helpful discussions.

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Recibido: 28 de mayo de 1.996

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Revista Matemática Iberoamericana
Vol. 13, N.º 2, 1997

The bilinear Hilbert transform is pointwise finite

Michael T. Lacey

To my father, H. E. Lacey, on the occasion of his sixtieth birthday

Abstract. Let $f \in L^\infty$ and $g \in L^2$ be supported on $[0,1]$. Then the principal value integral below exists in L^1.

$$\text{p. v.} \int f(x+y)\, g(x-y)\, \frac{dy}{y} \, .$$

1. Introduction.

The bilinear Hilbert transform is

$$H f g(x) = \int f(x+y)\, g(x-y)\, \frac{dy}{y} \, .$$

Two different rates of translation are incorporated into the integral, making it an extraordinarily subtle object. A beautiful conjecture, formulated first by A. Calderón in 1964,[1] is that H maps $L^2 \times L^\infty$ into L^2.

This paper gives some concrete indication that the conjecture could be true.

[1] This date was supplied to me by R. Coifman. Independently, C. Fefferman posed the same conjecture in the early 1970's. The motivation for both was the Cauchy integral on Lipschitz curves.

Theorem 1.1. *For functions f and g supported on $[0,1]$,*

$$\Big\| \int f(x+y)\, g(x-y)\, \frac{dy}{y} \Big\|_1 \leq C\|f\|_\infty\, \|g\|_2 \,.$$

In particular, the bilinear Hilbert transform is pointwise finite for $f \in L^\infty$ and $g \in L^2$.

There is only one natural antecedent to this conjecture of Calderón, namely Carleson's Theorem ([C]) on the pointwise convergence of Fourier series. And the starting point of this paper is C. Fefferman's proof ([F]) of that Theorem.

We then view one of the functions, say f, as a fixed bounded function. This allows us to use L^2 techniques to bound Hfg for $g \in L^2([-1,1])$. From this point, the large scale structure of our proof is borrowed from Fefferman. The operator is decomposed into pieces localized in space and frequency. These pieces have an intrinsic size, and they are grouped accordingly.

The proof depends upon combinatorial considerations to break up the small pieces of the decompostion into a relatively small number of orthogonal parts. A critical feature of the decomposition are certain Carleson measure estimates which arise from the decomposition of the function f.

The whole analysis is done solely on the function f, and only using the fact that f is bounded. The paper fully develops the line of reasoning, within these restrictions. Further progress on the conjectures can be made, by gaining a deeper understanding of the decompositions of the function f and their properties. As well, the structure of the function g must also be exploited, whereas in this paper g is essentially ignored. Both of these steps can be taken,[2] but will add significant complications to an already difficult paper. A discussion of this line of investigation will be postponed.

The proof occupies the rest of the paper, which is organized along the following lines. A strong working knowledge of Fefferman's proof [F] would certainly be an aid to the reader. Our approach uses especially, the combinatorial methods therein. But moreover, many of the details must be treated with more care in the present setting of the bilinear Hilbert transform.

[2] These words written in May of 1995 turned out to be prophetic. Together with C. Thiele [LT], a range of L^p bounds have been established for the bilinear Hilbert transform on the real line.

Section 2 gives the decomposition used in the proof, and a very crude outline of the proof. Section 3 treats some technical issues.

Section 4 initiates the real work. The first half of it takes up the most striking new element of the proof: the Carleson measure estimate of Lemma 4.4. It is the work-horse of the whole proof. And it's proof employs the full strength of the combinatorial ideas of [F]. Section 6 is exclusively devoted to a second Carleson measure estimate. Its proof is less difficult than the first, but the formulation is far from obvious.

The rest of Section 4, as well as Section 5 are devoted building up large units of bounded operators from the decomposition. The units of Section 5 are trees, which are in fact Calderón-Zygmund operators. Lemma 5.1 explains succinctly a fundamental difficulty. This difficulty is the sole subject of Section 6.

Concerning notation, a capital C will denote a constant, perhaps one that changes from line to line; in contrast, a lower case c will denote the center of a relevant interval; the indicator function of a set A will be denoted by $\mathbf{1}_A$, or when A has a complicated description, $\mathbf{1}[A]$; a similar notation will be used in summation

$$\sum_j [j \in A]\, f(j) = \sum_{j \in A} f(j)\,.$$

2. The decomposition.

This section will describe a decomposition of the bilinear singular integrals. Define the Fourier transform by

$$\mathcal{F}f(\xi) = \int e^{-2\pi i x \xi} f(x)\, dx\,,$$

and set $\langle f, g\rangle = \int f \overline{g}\, dx$.

We regard the bounded function f as fixed and supported on $[0,1]$, with g varying but supported on $[0,1]$. Notice that then the bilinear Hilbert transform is compactly supported, and we can restrict $1/y$ to $|y| < 2$. The proof is based upon a decomposition of $Hf\cdot$ into a relatively small number of components orthogonal in either space or frequency.

The place to start the decomposition is the kernel $1/y$. Consider a kernel $K(y) = \sum_{j\geq 1} K_j(y)$, where for each j,

$$\text{supp}\,(\mathcal{F}K_j) \subset \Big\{\xi \,:\, \Big(\frac{1}{3} - \frac{1}{100}\Big)3^j \leq \xi \leq \frac{1}{3}\,3^j\Big\}\,, \tag{2.1}$$

$$|K_j(y) - K_j(y')| \leq C\,\frac{|y-y'|}{|y|^2}\,, \qquad \text{if } 2\,|y-y'| \leq |y|\,, \tag{2.2}$$

as well as

$$|K_j(y)| \leq C\,3^j(1 \wedge (3^j y)^{-M})\,, \qquad \text{where } 30 < M < \infty\,. \tag{2.3}$$

The choice of M will be some large but fixed value. The negative values of j are irrelevant, as we are only concerned with $|y| < 2$.[3] We can write

$$\frac{1}{y} = J(y) + \sum_{v=0}^{V} K_+^v(y) + K_-^v(y)\,, \qquad 0 < |y| < 2\,,$$

where $\|J(y)\|_1 < \infty$. We take $V = \min\{v \,:\, (1+1/56)^v \geq 3\}$. For $0 \leq v < V$, the kernels

$$K_+^v\Big(\Big(1+\frac{1}{200}\Big)^v y\Big)$$

satifies (2.1), (2.2) and (2.3). The kernels

$$\overline{K_-^v\Big(\Big(1+\frac{1}{200}\Big)^v y\Big)}$$

satisfy the same conditions. We show how to treat

$$Tfg(x) = \int f(x+y)\,g(x-y)\,K(y)\,dy\,,$$

and the kernel J is trivial.

---

[3] The Fourier transform of K will be supported on $(0,\infty)$. The proof accomodates $\mathcal{F}K_j$ being supported on the positive and negative axes, but this slightly complicates some other combinatorial considerations.

Pairs. Fix triadic grids $\mathcal{G}$ of $\mathbb{R}$ and $\widehat{\mathcal{G}}$ of $\widehat{\mathbb{R}}$.[4] Call $p = [\omega, I\,]$ a *pair* if $\omega \in \widehat{\mathcal{G}}$, and $I \in \mathcal{G}$ with $|\omega| = |I|^{-1}$. We write $p = [\omega_p, I_p]$. Fix a function ϕ with L^2 norm 1, $\operatorname{supp}(\hat{\phi}) \subset [-1/100, 1/100]$,

$$|\phi(x)| \le C_M\,(1 \wedge |x|^{-M})\,, \tag{2.4}$$

where $30 < M < \infty$ is large, but fixed, and the collection of functions below is a tight frame in $L^2(\mathbb{R})$.

$$\{\phi^{m,n}(x) = e^{2\pi i m/200}\phi(x - 100\,n) : n, m \in \mathbb{Z}\}\,. \tag{2.5}$$

This last condition means that

$$\sum_{m,n} |\langle f, \phi^{m,n}\rangle|^2 = 2\,\|f\|_2^2\,,$$

for all square integrable f; and so

$$f(x) = \frac{1}{2}\sum_{m,n} \phi^{m,n}(x)\,\langle f, \phi^{m,n}\rangle\,, \tag{2.6}$$

at least in a L^2 sense. By considerations in [D, Section 3.4], this amounts to choosing ϕ so that

$$\sum_n \Big|\hat{\phi}\Big(\xi + \frac{n}{200}\Big)\Big|^2 \equiv \text{constant}\,.$$

All of these requirements can be met by choosing $\hat{\phi}$ to be symmetric, and for an increasing function $\nu(\xi)$ on $[0, 1/200]$ with $\nu(0) = 0$ and $\nu(1/200) = \pi/2$, setting

$$\hat{\phi}(\xi) = \lambda \begin{cases} \cos\nu(\xi)\,, & 0 \le \xi \le \dfrac{1}{200}\,, \\ \sin\nu\Big(\xi - \dfrac{1}{200}\Big)\,, & \dfrac{1}{200} \le \xi \le \dfrac{1}{100}\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

where λ is a normalizing constant. For any such ν, the function ϕ will satisfy (2.7); and for a smooth choice of ν, ϕ satisfies (2.4).

---

[4] That is, $\mathcal{G}$ is a union of intervals whose lengths are powers of 3. The set $\{I \in \mathcal{G} : |I| = 3^{-j}\}$ partitions $\mathbb{R}$, and the partitions are refining as $|I|$ decreases.

A further property of ϕ is that

$$\{\phi(x-200\,n) : n\in\mathbb{Z}\} \qquad \text{are orthonormal}. \tag{2.8}$$

This is due to (2.7).

For pairs $[\omega, I]$, set

$$\phi_{[\omega,I]}(x) = |I|^{-1/2}\, e^{2\pi i c(\omega)x} \phi\Big(\frac{x-c(I)}{|I|}\Big).$$

Here, $c(J)$ is the center of the interval J. We will also use the notation

$$\Phi(\omega) = c(\omega) + \frac{|\omega|}{27}\Big[-\frac{1}{2},\frac{1}{2}\Big], \tag{2.9}$$

to denote the support of the Fourier transform of $\phi_{[\omega,I]}$. Much more commonly, we will use the notation

$$k_\omega(y) = k_j(y)\,, \qquad \text{where } |\omega| = 3^j\,.$$

Let $f_{[\omega,I]}(x) = \phi_p(x)\langle f, \phi_p\rangle$, which forms our decomposition of f.

The smallest unit in the decomposition is

$$T_p\, g(x) = T_{[\omega,I]}\, g(x) = \int k_\omega(y)\, f_p(x+y)\, g(x-y)\, dy\,. \tag{2.10}$$

The proof will bound $T^0 g = \sum_{\text{all pairs}} T_p\, g$, which is certainly not the full singular integral. However, as will be explained in the next section, it is enough to bound T^0.

Here are some simple properties of the T_p. First, how big is T_p? Certainly, $|T_p g(x)| \le \|f_p\|_\infty |k_j| * |g|(x)$, hence

$$\|T_p\|_2 \le C\,\|f_p\|_\infty = C\,\frac{|\langle f, \phi_p\rangle|}{\sqrt{|I_p|}}\,. \tag{2.11}$$

We will use the notation

$$\operatorname{size}(p) = \frac{|\langle f, \phi_p\rangle|}{\sqrt{|I_p|}}\,.$$

The operators T_p has good space and frequency localization. The frequency localization has been made quite precise, as will be described by associating to every ω two sets $A(\omega)$ and $A^*(\omega)$. They are defined by

$$A(\omega) = c(\omega) + \frac{1}{3}\,|\omega| + \frac{|\omega|}{9}\Big[-\frac{1}{2},\frac{1}{2}\Big]\,, \tag{2.12}$$

and

$$A^*(\omega) = 2\,c(\omega) + \frac{1}{3}\,|\omega| + \frac{|\omega|}{9}\Big[-\frac{1}{2},\frac{1}{2}\Big]\,. \tag{2.13}$$

The point of these definitions is that

$$T_{[\omega,I\,]}\,g = T_{[\omega,I\,]}(\mathcal{F}^{-1}\mathbf{1}_{A(\omega)}\mathcal{F}g)\,,$$

and

$$T^*_{[\omega,I\,]}\,g = T^*_{[\omega,I\,]}(\mathcal{F}^{-1}\mathbf{1}_{A^*(\omega)}\mathcal{F}g)\,.$$

Or equivalently, by taking adjoints,

$$T_p\,g = \mathcal{F}^{-1}\mathbf{1}_{A^*(\omega)}\mathcal{F}\,T_p\,g \qquad \text{and} \qquad T^*_p g = \mathcal{F}^{-1}\mathbf{1}_{A(\omega)}\mathcal{F}\,T^*_p g\,. \tag{2.14}$$

To verify these equalities, write the "Fourier transform" of T_p as follows

$$\begin{aligned} T_{[\omega,I\,]}g(x) &= \mathcal{F}^{-1}_{\alpha}\Big(\int e^{-2\pi i\alpha y}k_\omega(y)\,f_p(x+y)\,dy\,\hat{g}(\alpha)\Big)(x)\\ &= \mathcal{F}^{-1}_{\alpha}\Big(\mathcal{F}^{-1}_{\beta}\big(\widehat{k_\omega}(\alpha-\beta)\,f_{[\omega,I\,]}(\beta)\big)(x)\,\hat{g}(\alpha)\Big)(x)\,. \end{aligned}$$

Then, the second half of (2.14) follows by looking back at the definitions of k_ω and $\phi_{[\omega,I\,]}$.

A similar calculation holds for T^*_p. Recalling (2.10), note that

$$\begin{aligned} T^*_p g(x) &= \int k_\omega(-y)\,f_p(x-2\,y)\,g(x-y)\,dy\\ &= \mathcal{F}^{-1}_{\alpha}\big(\mathcal{F}^{-1}_{\beta}\big(\widehat{k_\omega}(\alpha+2\,\beta)\,\hat{f}_p(-\beta)\big)(x)\,\hat{g}(\alpha)\big)(x)\,. \end{aligned}$$

From this, the other half of (2.14) follows.

Another simple property of the T_p's is that, up to modest changes in f and g, for any pair $p = [\omega, I\,]$, ω can be assumed to be centered at the origin. Let us carry out the calculation. For a pair $p = [\omega, I\,]$, and $c \in \hat{\mathbb{R}}$, (not necessarily the center of ω) notice that

$$\begin{aligned} f_{[\omega,I\,]}(x) &= \phi_{[\omega,I\,]}(x)\,\langle f, \phi_{[\omega,I\,]}\rangle \\ &= e^{2\pi ixc}\,\phi_{[\omega-c,I]}(x)\,\langle e^{-2\pi ic\cdot} f(\,\cdot\,), \phi_{[\omega-c,I]}(\,\cdot\,)\rangle \\ &= e^{2\pi ixc}\tilde{f}_{[\omega-c,I]}(x)\,, \end{aligned}$$

where $\tilde{f}(x) = e^{-2\pi icx} f(x)$. From this it follows that

$$\begin{aligned} T_{[\omega,I\,]}g(x) &= \int k_\omega(y)\, f_{[\omega,I\,]}(x+y)\, g(x-y)\, dy \\ &= \int k_\omega(y)\, e^{2\pi i(x+y)c}\tilde{f}_{[\omega-c,I]}(x+y)\, g(x-y)\, dy \qquad (2.15) \\ &= e^{4\pi ixc} \int k_\omega(y) \tilde{f}_{[\omega-c,I]}(x+y)\, (e^{-2\pi ic(x-y)} g(x-y))\, dy\,. \end{aligned}$$

It is convenient to shift ω's back to the origin because for such pairs p, f_p has derivative dominated by the scale of p. That is, $|d/dx\, f_p(x)| \le C/|I|$.

Partial Order for Pairs. The last topic for this section is a natural partial order for pairs. Write $p' = [\omega', I'] < p = [\omega, I\,]$ if $\omega \subset \omega'$ and $I' \subset I$. This partial order encodes the orthogonality properties of the T_p, in the following sense: if p and p' are not comparable then T_p and $T_{p'}$ are, roughly speaking, orthogonal.

C. Fefferman's approach focuses attention on three separate issues. The first is this: arbitrary collections of pairs must be controlled in terms of their maximal elements under the partial order '$<$'. The critical question concerns sets of pairs $\mathcal{P}$ which are mutually incomparable under $<$. How big is $\sum_{[\omega,I\,]\in\mathcal{P}} \mathbf{1}_I(x)$? The answer is contained in a Carleson measure estimate, the proof of which takes up the first half of Section 4 below.[5] The next issue concerns $T^{\mathcal{P}} = \sum_{p\in\mathcal{P}} T_p$, where $\mathcal{P}$ is a set of pairs which are incomparable under $<$. With the Carleson measure estimate in place, and it's implict orthogonality, one can check that

$$\|T^{\mathcal{P}}\|_2 \le C\,\sqrt{\sup_{p\in\mathcal{P}} \operatorname{size}(p)}\,.$$

[5] The corresponding estimate is required in [F] but is easily obtained.

The next important group of pairs are those which form a tree under the partial order $<$. They turn out to be Calderon-Zygmund operators, but now it can happen that

$$\|T^{\mathcal{P}}\|_2 \simeq 1\,,$$

regardless of how small the size of individual pairs in $\mathcal{P}$. This difficulty does not occur in [F]. Fortunately, it turns out that for trees, $\mathcal{P}$, the operator norm of $T^{\mathcal{P}}$ has an explicit form in terms of the decomposition of f. We take advantage of this in Section 6 to prove a second Carleson measure estimate. This lemma provides a way to control those trees which have large norm, even though their constituent parts are small.

Employing certain combinatorial tricks, one can then show that for an arbitrary collection of pairs $\mathcal{P}$,

$$\|T^{\mathcal{P}}\|_2 \simeq \sup\{\|T^{\mathcal{P}'}\|_2 : \mathcal{P}' \subset \mathcal{P}\,,\ \mathcal{P}' \text{ a tree}\}\,.$$

The concluding step in the proof is then to decompose the set of all pairs into collections $\mathcal{P}_n$ with $\|T^{\mathcal{P}_n}\|_2 \simeq 2^{-n}$. This estimate is then summed over n, completing the proof.

3. Technicalities.

This section serves as a catch-all; it includes all the steps that need to be explained, but would have hampered the flow of the previous section. The overall direction is to explain how to pass from $T^0 = \sum_{\text{all pairs}} T_p$ to the integral in Theorem 1.1. But first, T^0 will be further modified.

Central Intervals and Admissible Pairs. The modification of T^0 is needed to gain a certain improvement in triadic intervals.[6] Say that I is *central* if it is the middle third of a triadic interval. Phrased differently, I is central if both I and $3I$ are triadic. The "convenient property of central triadic intervals" is

$$I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_m\,,\ \text{all } I_m \text{ central implies } 3^{m-1}I_1 \subset I_m\,. \tag{3.1}$$

The proof is immediate.

[6] This notion doesn't enter into the proof until the very end.

Let us observe that for every $\omega \in \hat{\mathcal{G}}$, $A(\omega)$ is central in $\hat{\mathcal{G}}$. Indeed $A(\omega)$ is the middle third of $\hat{\mathcal{G}}$. But to acheive the same result for $A^*(\omega)$ we make a specific choice of grids. We take $\hat{\mathcal{G}} = \{3^j[n-1/2, n+1/2) : n, j \in \mathbb{Z}\}$. Then $A^*(\omega)$ is central in $\hat{\mathcal{G}}$, as is easily checked. More generally if $\hat{\mathcal{G}}^a = \{I + a : I \in \hat{\mathcal{G}}\}$ is the grid shifted by a, then for $\omega \in \hat{\mathcal{G}}^a$ the interval $A^*(\omega)$ is central in $\hat{\mathcal{G}}^{2a}$. These observations concerning $A^*(\omega)$ do not hold for an arbitary triadic grid, and the notion of centrality does not enter the proof until Section 6.

Call $p = [\omega, I] \in \hat{\mathcal{G}}^b \times \mathcal{G}^a$ *admissible* if ω is central in $\hat{\mathcal{G}}^b$, and I satisfies the following: $c(I) \in a + 200\,|I|\,\mathbb{Z}$, where a is fixed. This requirement on I, with (2.8), shows that the functions

$$\{\phi_{[\omega,I]} : [\omega, I] \text{ admissible; } \omega \text{ fixed}\} \text{ are orthogonal.} \tag{3.2}$$

This will be useful in Section 6.

The proof in the next four sections will bound $T^1 = \sum_{p \text{ admissible}} T_p$, and in the next section, "pair" will mean "admissible pair".

Lemma 3.3. *The bound* $\|T^1 g\|_1 \le C\,\|g\|_2$ *implies the same bound for the full singular integral.*

PROOF. The proof averages the bound for T^1 over space and frequency, with the central tool being the resolution of the identity below. For $a, b \in \mathbb{R}$, let

$$\Phi^{a,b} f(x) := e^{2\pi i a x} \phi(x-b)\, \langle f, e^{2\pi i a x}\phi(x-b)\rangle\,.$$

Then,

$$f(x) = C \iint (\Phi^{a,b} f)(x)\, da\, db\,. \tag{3.4}$$

One checks this by showing that for all $f, g \in L^2$,

$$C^{-2}\langle f, g\rangle = \iiint (\Phi^{a,b} f)(x)\, \overline{(\Phi^{a,b}) g(x)}\, da\, db\, dx\,.$$

(In the language of wavelets, $\Phi^{a,b} f$ is an example of a continuous windowed Fourier transform. See [D]). We apply this formula to f, which is bounded and supported on $[0,1]$.

The definition of T^1, pair, central and admissible pair all depend upon the choice of grids. Thus, if $\mathcal{G}$ is a grid, denote by $\mathcal{G}^a$ it's shift by a. Notice that the central intervals are just shifted in $\mathcal{G}^a$, and that they have density 1/3 at all scales. That is, amoung all the intervals $I \in \mathcal{G}$ with $|I| = 2^j$, every third one is central. Clearly, the admissible space intervals have density 1/200 at all scales. Then, let

$$f^{a,b,j}(x) = \sum_{[\omega,I]\in\hat{\mathcal{G}}^a\times\mathcal{G}^b} \big[p \text{ admissible}, |I| = 3^j\big]\phi_p(x)\,\langle f, \phi_p\rangle\,.$$

From Carleson's Theorem, one sees that this sum is convergent for almost every x. It follows from (3.4) that

$$f(x) = \lim_{M\to\infty} \frac{C}{M^2}\int_0^M\int_0^M f^{a,b,j}(x)\,da\,db\,.$$

And so, if $T^{a,b}$ is the operator T^1 formed over the grid $\hat{\mathcal{G}}^b \times \mathcal{G}^a$, we see that

$$\sum_j \int k_j(y) f(x+y)\,g(x-y)\,dy = \lim_{M\to\infty}\frac{C}{M^2}\int_0^M\int_0^M T^{a,b}g(x)\,da\,db\,.$$

The assumption is that we have appropriate norm bounds on $T^{a,b}$, independent of a and b. These same bounds then clearly apply to the averages above. In this way, we can pass to the full singular integrals.

4. No two pairs comparable.

The object of study herein are sets of pairs $\mathcal{P}$ for which no two distinct pairs in $\mathcal{P}$ are comparable with respect to "$<$" Call such a set of pairs a *thicket*. A convenient way to reformulate the not comparable condition is this. Two pairs $p = [\omega, I\,]$ and $p' = [\omega', I']$ are not comparable under '$<$' if and only if the two rectangles in space-frequency plane $I \times \omega$ and $I' \times \omega'$ are disjoint.

Here are the three facts this section is devoted to proving.

Lemma 4.1. *Let $0 < b < 1$. If $\mathcal{P}$ is a thicket and* size$(p) \geq b$ *for all $p \in \mathcal{P}$, then for any $\varepsilon > 0$,*

$$\Big\|\sum_{[\omega,I]\in\mathcal{P}} \mathbf{1}_I\Big\|_s \leq C_{s,\varepsilon}\, b^{-2-\varepsilon}\,, \qquad 1 < s < \infty\,.$$

In particular the function above is in every L^r class, for $r < \infty$.

Lemma 4.2. *If $\mathcal{P}$ is a thicket and* $\operatorname{size}(p) \le b$ *for all $p \in \mathcal{P}$ then for all $1 < r \le 2$,*

$$\|T^{\mathcal{P}} g\|_1 \le C_r\, b^{\delta(r)} \|g\|_r\,, \qquad \text{where } \delta(r) > 0\,. \tag{4.3}$$

Note that this is a $L^r \longrightarrow L^1$ estimate.

Both of these will follow from a study of f_p, which begins with the following Carleson measure lemma.

Lemma 4.4. *Let $0 < b < 1$. Let $\mathcal{P}$ be a thicket with* $\operatorname{size}(p) \ge b$ *for all $p \in \mathcal{P}$. Then for all intervals U, and all $\varepsilon > 0$,*

$$\sum_{[\omega, I] \in \mathcal{P}} [I \subset U]\, |\langle g, \phi_{[\omega, I]} \rangle|^2 \le C_\varepsilon\, b^{-\varepsilon} |U|\, \|g\|_\infty^2\,. \tag{4.5}$$

This Lemma, crucial to the whole line of reasoning of this paper, has an intricate and combinatorial proof. In fact, it already reflects the large scale structure of Fefferman's argument. The procedure is to identify nice subsets of thickets which satisfy (4.5), and decompose a thicket into a relatively small number of these nice subsets. In choosing our terminology, the thickets, spindly sets, shrubs and hedges of this section, we have chosen words, which in the American vernacular, pertain to the understory of a forest. This seems appropriate due to the close connection between thickets and the forests of Section 7.

The inequality is also probably in it's optimal form; at any rate, the purely L^2 estimate

$$\sum_{p \in \mathcal{P}} |\langle g, \phi_p \rangle|^2 \le C\, b^{-10{,}000}\, \|g\|_2^2$$

is false. Indeed, one can use the Fourier transform in this last inequality, and then it is quickly seen to be false for $\widehat{\phi_p}$ even if $\hat{g} = \mathbf{1}_{[-1,1]}$. Put another way, our Lemma 4.4 relies critically upon ϕ being compactly supported in the frequency variable.

Further, the proof given here will be applied to decompositions of other functions that arise later in this section. To this end, it is important to note that the only property of ϕ or rather ϕ_p used is this:

$$\omega \cap \omega' = \varnothing \qquad \text{implies} \qquad \operatorname{supp}(\widehat{\phi_p}) \cap \operatorname{supp}(\widehat{\phi_{p'}}) = \varnothing\,. \tag{4.6}$$

The other element of the proof is an estimate of the form

$$\sum_{[\omega,I]\in\mathcal{P}} [I\subset U]\,|I| \le C\,b^{-3}\,|U|\,, \qquad \text{for all intervals } U\,.$$

In the end, it is the desire to apply Lemma 4.4 to other "thickets" that justifies, even necessitates, our seemingly absurdly specific requirements on the decomposition given in Section 2. To emphasize this point, let us record here a second Lemma which we shall prove at the same time as the one above.

Lemma 4.7. *Let $\varepsilon > 0$. Let $\mathcal{P}$ be a thicket and in addition assume we have the estimate*

$$\sum_{[\omega,I]\in\mathcal{P}} [I\subset U]\,|I| \le C\,b^{-4}\,|U|\,, \qquad \textit{for all intervals } U\,. \tag{4.8}$$

Let $\{\varphi_p : p\in\mathcal{P}\}$ be functions satisfying

$$\omega_p\cap\omega_{p'}=\varnothing \qquad \textit{implies} \qquad \operatorname{supp}(\widehat{\varphi_p})\cap\operatorname{supp}(\widehat{\varphi_{p'}})=\varnothing\,, \tag{4.9}$$

$$|\varphi_p(x)|\le C_M\Big(1+\frac{\operatorname{dist}(x,I_p)}{|I_p|}\Big)^{-M}\,, \tag{4.10}$$

where M depends upon ε. Then for all intervals U and bounded functions g,

$$\sum_{p\in\mathcal{P}} [I_p\subset U]\,|\langle g,\varphi_p\rangle|^2 \le C_\varepsilon\, b^{-\varepsilon}\,|U|\,\|g\|_\infty^2\,. \tag{4.11}$$

In the Lemmas above and below, $0<\varepsilon<1$ is fixed. It's choice depends ultimately on the choice of $1<r<2$ in (4.3). The value of ε forces a certain rate of decay on ϕ. Namely in (2.4) and (2.3), M has to be chosen sufficiently large, but finite. The constants depending upon ε are independent of f and all pairs.

And, in the proof, it is convenient to assume these two conditions on pairs: for all pairs $p=[\omega,I]$ and $p'=[\omega',I']$

$$\omega=\omega' \text{ and } I\neq I' \quad \text{implies} \quad \operatorname{dist}(I,I')\ge 400\,b^{-\varepsilon}\,|I|\,. \tag{4.12}$$

$$\omega'\subsetneq\omega \qquad \text{implies} \quad |I|\le b^{1000}\,|I'|\,. \tag{4.13}$$

These conditions can be assumed by breaking up the set of all pairs into $O(b^{-2\varepsilon})$ disjoint sets. The effect of this will be to multiply our bound in (4.5) by (a trivial amount) of $b^{-2\varepsilon}$.

Spindly sets of pairs. Introduce a new partial order on pairs. Call $[\omega, I\,] \ll [\omega', I']$ if and only if $\omega \supset \omega'$ and

$$|c(I') - c(I)| \le \frac{1}{3}\, b^{-\varepsilon}\, |I'| \sum_{j\ge 0} [|I| < 3^{-j}|I'|]\, 3^{-j}\,. \tag{4.14}$$

One can check that this is indeed a partial order, namely if $p \ll p'$ and $p' \ll p''$ then $p \ll p''$.[7] (By (4.12), if $p \ll p'$, then $\omega' \subsetneq \omega$.) Call a set of pairs $\mathcal{P}$ *spindly* if no two pairs in $\mathcal{P}$ are comparable under $\ll$.

Lemma 4.15 *If $\mathcal{P}$ is spindly and* $\operatorname{size}(p) \ge b$ *for all $p \in \mathcal{P}$, then it satisfies* (4.5).

PROOF. Assume that $\mathcal{P}$ is finite, and $I \subset U$ for all $[\omega, I\,] \in \mathcal{P}$. Then let B denote the best constant in the inequality

$$\Big\| \sum_{p\in\mathcal{P}} \varepsilon_p\, \phi_p(x)\, \langle g, \phi_p\rangle \Big\|_2 \le B\, \|g\|_\infty\,, \tag{4.16}$$

g supported in $b^{-\varepsilon}U$, $\varepsilon_p \in \{\pm 1\}$. Our intention is to provide an estimate for B. Averaging over all choices of signs ε_p will give a square function inequality which is weaker than what is claimed in the Lemma, in that a restriction is place upon the support of g. We return to this point at the end of the proof.

We will have need of some trivial estimates below. Take g to be a function bounded by 1. Then $|\langle g, \phi_{[\omega,I]}\rangle| \le C\, \sqrt{|I|}\, \|g\|_\infty$. Also

$$\begin{aligned} &|\langle \phi_{[\omega,I\,]}, \phi_{[\omega',I']}\rangle| \\ &\quad\le \begin{cases} 0\,, & \text{if } \omega \cap \omega' = \varnothing\,, \\ C_\varepsilon \sqrt{\dfrac{|I'|}{|I|}} \Big(1 + \dfrac{\operatorname{dist}(I, I')}{|I|}\Big)^{-20/\varepsilon}\,, & \text{if } \omega \subset \omega'\,. \end{cases} \end{aligned} \tag{4.17}$$

[7] Roughly speaking, $p \ll p'$ if $\omega \supset \omega'$ and $(3b^{\varepsilon})^{-1} I \cap (3b^{\varepsilon})^{-1} I' \ne \varnothing$, and $\ll$ is the transitive hull of this pairwise relation. The trouble with the simplier condition is that it does not define a partial order. And we have a need yet for certain intricate combinatorial ideas, which depend upon a partial order, as in Lemma 4.28.

Write out the left hand side of (4.16) as

$$\Big\| \sum_{p\in\mathcal{P}} \varepsilon_p \, \phi_p(x) \, \langle g, \phi_p\rangle \Big\|_2^2 = \mathcal{D} + \mathcal{O} \,,$$

where $\mathcal{D}$ denotes the diagonal term

$$\mathcal{D} = \Big\langle \sum_{p\in\mathcal{P}} \phi_p(x) \, \langle \phi_p, \phi_p\rangle \, \langle g, \phi_p\rangle, g(x) \Big\rangle$$

and the off-diagonal term is, because of (4.9),

$$\mathcal{O} = \sum_{p=[\omega,I]\in\mathcal{P}} \varepsilon_p \, \langle \phi_p(x), g(x)\rangle \sum_{p'=[\omega',I']\in\mathcal{P}} [\omega \subsetneq \omega'] \, \varepsilon_{p'} \, \langle \phi_{p'}, \phi_p\rangle \, \langle g, \phi_{p'}\rangle \,.$$

The assumed inequality can be used on the diagonal term $\mathcal{D}$.

$$\mathcal{D} = \|g\|_2 \Big\| \sum_{p\in\mathcal{P}} \phi_p(x) \, \langle g, \phi_p\rangle \Big\|_2 \le B \, \sqrt{b^{-\varepsilon}|U|} \, \|g\|_\infty^2 \,.$$

For the off-diagonal term, fix a $p = [\omega, I] \in \mathcal{P}$. Then the sets $\{I' : [\omega', I'] \in \mathcal{P},\ \omega' \supset \omega,\ I' \neq I\}$ are pairwise disjoint, and do not intersect $b^{-\varepsilon} I$. Hence, by (4.17),

$$\begin{aligned}
\mathcal{S}_p &= \Big| \sum_{p'\in\mathcal{P}} [\omega \subsetneq \omega'] \, \varepsilon_{p'} \langle \phi_{p'}, \phi_p\rangle \, \langle g, \phi_{p'}\rangle \Big| \\
&\le C \, \|g\|_\infty \sum_{p'\in\mathcal{P}} [\omega \subsetneq \omega'] \, \frac{1}{\sqrt{|I|}} \Big(1 + \frac{\operatorname{dist}(I, I')}{|I|} \Big)^{-20/\varepsilon} |I'| \\
&\le C \, \|g\|_\infty \int_{(b^{-\varepsilon} I)^c} \frac{1}{\sqrt{|I|}} \Big(1 + \frac{\operatorname{dist}(I, x)}{|I|} \Big)^{-20/\varepsilon} \, dx \\
&\le C_\varepsilon \, b^{10} \, \|g\|_\infty \sqrt{|I|} \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{O} &\le \Big\| \sum_{p\in\mathcal{P}} |\phi_p(x)| \, \mathcal{S}_p \Big\|_1 \|g\|_\infty \\
&\le C_\varepsilon \, b^{10} \, \|g\|_\infty^2 \Big\| \sum_{p=[\omega,I]\in\mathcal{P}} |\phi_p(x)| \, \sqrt{|I|} \Big\|_1 \\
&\le C_\varepsilon \, b^{10} \, \|g\|_\infty^2 \sum_{[\omega,I]\in\mathcal{P}} |I| \,.
\end{aligned}$$

Putting these estimates together, we see that

$$B^2 \le B\sqrt{b^{-\varepsilon}|U|} + C_\varepsilon\, b^{10} \sum_{[\omega,I]\in\mathcal{P}} |I|\,. \tag{4.18}$$

If $B\sqrt{b^{-\varepsilon}|U|}$ is the larger of the two terms on the right, then $B \le 2\sqrt{b^{-\varepsilon}|U|}$ and we have proved (4.16). So we assume that this is not the case and we derive a contradiction by applying the inequality to f. Of course f is not supported on $b^{-\varepsilon}U$, nevertheless we have

$$\Delta_p^2 = |\langle f, \phi_p\rangle - \langle f\,\mathbf{1}_{b^{-\varepsilon}U}, \phi_p\rangle|^2 \le C_\varepsilon\, b^{10}\, |I|\,. \tag{4.19}$$

Note that this bound only depends upon the L^∞ bound on f. Hence

$$\begin{aligned}\frac{1}{4}\, b^2 \sum_{[\omega,I]\in\mathcal{P}} |I| &\le \sum_{p\in\mathcal{P}} |\langle f, \phi_p\rangle|^2\\ &\le 2 \sum_{[\omega,I]\in\mathcal{P}} |\langle f\,\mathbf{1}_{b^{-\varepsilon}U}, \phi_p\rangle|^2 + \Delta_p^2\\ &\le C_\varepsilon\, b^{10} \sum_{[\omega,I]\in\mathcal{P}} |I|\,.\end{aligned}$$

We therefore see a contradiction for small b, which is enough to prove (4.16), because we only assumed a lower bound on the size of $p \in \mathcal{P}$. To extend the square function inequality to all bounded functions, note that inequality (4.19) is valid for all such functions, and so shows that the condition that g be supported in $b^{-\varepsilon}U$ is superfluous, thereby establishing the Lemma.

We observe that the proof above contains the following Lemma.

Lemma 4.20. *Lemma* 4.7 *holds under the additional assumption that* $\mathcal{P}$ *is spindly.*

PROOF. We repeat the argument above, up the equation (4.18). Then we appeal to (4.8) to conclude that (4.16) holds, with best constant $B \le b^{-\varepsilon}\sqrt{|U|}$. That is the Lemma holds up the restriction on the support of g in (4.16). But this restriction is removed just as above.

Shrubs. Call $\mathcal{P}$ a *shrub* with top $p^t = [\omega^t, I^t]$ if $p \ll p^t$ for every $p \in \mathcal{P}$, but no two pairs in $\mathcal{P}$ are comparable with respect to $<$. Because the sets $\{I : [\omega, I\,] \in \mathcal{P}\}$ are disjoint and contained in $b^{-\varepsilon} I^t$, the inequality (4.5) is trivial. What is more to the point is the following decomposition, which is essentially a corollary to the Fefferman-Stein maximal inequalities. This Lemma depends only on the combinatorics of pairs.

Lemma 4.21. *Let $\mathcal{P}$ be a a shrub with top $p^t = [\omega^t, I^t]$. Then there is a set $\mathcal{P}' \subset \mathcal{P}$ and a set $F \subset b^{-\varepsilon} I^t$ for which the following three conditions hold.*

i) *$\mathcal{P}'$ can be written as a union of at most $O\,(b^{-4\varepsilon})$ spindly sets.*

ii) *For all $[\omega, I\,] \in \mathcal{P} \backslash \mathcal{P}'$, $b^{-\varepsilon} I \subset F$.*

iii) *And $|F| \leq C_\varepsilon\, b^{100}\, |I^t|$.*

PROOF. Notice that the sets $\{I : [\omega, I\,] \in \mathcal{P}\}$ are disjoint, for otherwise two pairs would be comparable under $<$. Set

$$F_0 = \Big\{ x : \sum_{[\omega, I\,] \in \mathcal{P}} (M\mathbf{1}_I)^\alpha(x) > b^{-(3-\alpha)\varepsilon} \Big\}, \qquad \alpha = 1 + \frac{\varepsilon}{2}\,.$$

By the Fefferman-Stein maximal inequalities

$$\Big\| \Big(\sum_{[\omega, I\,] \in \mathcal{P}} (M\mathbf{1}_I)^\alpha(x) \Big)^{1/\alpha} \Big\|_\beta \leq C_{\alpha,\beta}\, (b^{-\varepsilon} |I^t|)^{1/\beta}\,, \qquad 1 < \beta < \infty\,.$$

Using this with β large implies that $|F_0| \leq C_\varepsilon\, b^{100}\, |I^t|$.

Now, set $\mathcal{P}'_0 = \{[\omega, I\,] \in \mathcal{P} : b^{-\varepsilon} I \not\subset F_0\}$. Our claim is that

$$\Big\| \sum_{[\omega, I\,] \in \mathcal{P}'_0} \mathbf{1}[4b^{-\varepsilon} I](x) \Big\|_\infty \leq 64\, b^{-3\varepsilon}\,.$$

Consider an x and intervals $I_1, \ldots, I_J$ with $x \in 4b^{-\varepsilon} I_j$ for all $1 \leq j \leq J$, and $[\omega_j, I_j] \in \mathcal{P}'_0$. Suppose that $|I_1| \leq |I_j|$ for all j. Then, for all $y \in I_1$,

$$(M\mathbf{1}_{I_j}(y))^\alpha \geq 8^{-\alpha} b^{\varepsilon\alpha}\,, \qquad 1 \leq j \leq J\,,$$

so that

$$\sum_{j=1}^{J} (M\mathbf{1}_{I_j}(y))^\alpha \geq J\, 8^{-\alpha} b^{\varepsilon\alpha}\,, \qquad y \in I_1\,.$$

The right hand side above cannot be more than $64\, b^{-3\varepsilon}$, or we see that $[\omega_1, I_1] \notin \mathcal{P}'_0$. This gives an upper bound on J.

The set $\mathcal{P}_0'$ will have to have some more pairs deleted before it can be decomposed into spindly sets. To accomplish this, let

$$U_1 = \bigcup_{[\omega,I]\in\mathcal{P}_0'} b^{-\varepsilon} I\,.$$

Choose $\overline{p}_j = [\overline{\omega}_j, \overline{I}_j] \in \mathcal{P}_0'$ so that that $\{b^{-\varepsilon}\overline{I}_j : j \geq 1\}$ forms a minimal cover of the set U_1. Set $\mathcal{P}_1' = \{\overline{p}_j : j \geq 1\}$. Delete these pairs from $\mathcal{P}_0'$ and repeat this procedure. As a result, we have

$$\mathcal{P}_0' = \mathcal{P}_1' \cup \cdots \cup \mathcal{P}_J'\,, \qquad J \leq 64\, b^{-3\varepsilon}\,,$$

and for each $\mathcal{P}_j'$, the sets $\{b^{-\varepsilon}I : [\omega, I\,] \in \mathcal{P}_j'\}$ form a minimal cover of their union.

Last of all, we claim that for each $\mathcal{P}_j'$, $1 \leq j \leq J$, there is a set $F_j \subset b^{-\varepsilon} I^t$ and a $\mathcal{P}_j'' \subset \mathcal{P}_j'$ so that

i) $\mathcal{P}_j''$ is a union of $O\,(\log 1/b)$ spindly collections of pairs.

ii) For all $[\omega, I\,] \in \mathcal{P}_j' \backslash \mathcal{P}_j''$, $b^{-\varepsilon} I \subset F_j$.

iii) $|F_j| \leq C\, b^{200}\, |I^t|$.

These last three conditions in fact follow from the Vitali Covering Lemma: from $\{b^{-\varepsilon}I : [\omega, I\,] \in \mathcal{P}_j'\}$ select pairs $[\overline{\omega}_v, \overline{I}_v] \in \mathcal{P}_j'$ so that the intervals $b^{-\varepsilon}\overline{I}_v$ are disjoint in v and

$$\sum_v b^{-\varepsilon} |\overline{I}_v| \geq \frac{1}{2} \Big| \bigcup_{[\omega,I\,]\in\mathcal{P}_j'} b^{-\varepsilon} I \Big|\,.$$

The collection $\{[\overline{\omega}_v, \overline{I}_v] : v \geq 1\}$ is clearly spindly. Repeating this procedure $O\,(\log 1/b)$ times will prove these three conditions.

Last of all, we conclude the Lemma by taking $\mathcal{P}' = \bigcup_{j=1}^{J} \mathcal{P}_j''$, and $F = \bigcup_{j=0}^{J} F_j$.

In the first half of the proof just given we have made an observation which will be used below. Let us formulate it as a Lemma.

Lemma 4.22. *Let $\mathcal{I}$ be a collection of intervals. For a choice of $1 \leq \alpha < \infty$ set*

$$E = \Big\{x : \Big(\sum_{I\in\mathcal{I}} (M\mathbf{1}_I(x))^{\alpha}\Big)^{1/\alpha} > J\Big\}\,.$$

Then for all x

$$\sum_{I\in\mathcal{I}}[b^{-\varepsilon}I \not\subset E]\,\mathbf{1}_{b^{-\varepsilon}I}(x) \le 2^{\alpha}(b^{-\varepsilon\alpha}J)^{\alpha}\,.$$

A second Lemma of a general nature is also relevant at several parts of our argument. It shows that it suffices to prove the Carleson measure estimate, up to exceptional sets.

Lemma 4.23. *Let* $\{a_p : p\in\mathcal{P}\}$ *be non-negative numbers associated to pairs. Assume that for all intervals* U *there is an open set* $E\subset U$ *so that* $|E|\le|U|/2$ *and*

$$\sum_{p\in\mathcal{P}}[I_p\not\subset U]\,|a_p|\le C\,|U|\,.$$

Then, for all intervals U*,*

$$\sum_{p\in\mathcal{P}}|a_p|\le 2\,C\,|U|\,. \tag{4.24}$$

PROOF. Fix the interval U for which we want to prove (4.24). Set E_1 to be the set E of the Lemma, and let $\mathcal{P}_1'=\{p\in\mathcal{P}: I_p\not\subset U\}$. We apply the hypotheses of the Lemma to the components of the set E_1. Thus, let $U_{1,k}$ be the open components of E_1, and let $\mathcal{P}_{1,k}=\{p\in\mathcal{P}\backslash\mathcal{P}_1' : I_p\subset U_{1,k}\}$. Then for each k there is an open set $E_{1,k}\subset U_{1,k}$, with $|E_{1,k}|\le|U_{1,k}|/2$, so that setting $\mathcal{P}_{1,k}'=\{p\in\mathcal{P}: U_{1,k}\supset I_p, I_p\not\subset E_{1,k}\}$, we have

$$\sum_{p\in\mathcal{P}_{1,k}'}|a_p|\le C\,|U_{1,k}|\,.$$

But, $\sum_k|U_{1,k}|\le|E_1|\le 1/2|U|$. And so,

$$\sum_k|E_{1,k}|\le\frac12\sum_k|U_{1,k}|\le\frac12\,|E_1|\le\frac14\,|U|\,.$$

Set $\mathcal{P}_2'=\bigcup_k\mathcal{P}_{1,k}'$. We conclude that

$$\sum_{p\in\mathcal{P}_1'\cup\mathcal{P}_2'}|a_p|\le C\Big(1+\frac12\Big)|U|\,.$$

This argument can be continued inductively inside the components of $E_2 = \bigcup_k E_{1,k}$, thereby proving the Lemma.

Hedges. Call a set of pairs $\mathcal{P}$ a *hedge* if no two pairs in $\mathcal{P}$ are comparable under $<$ and $\mathcal{P}$ satisfies the following linearity or tree-like condition: for all $p, p', p'' \in \mathcal{P}$ with $p \ll p'$ and $p \ll p''$, either $p' \ll p''$ or $p'' \ll p'$.

Lemma 4.25. *If* $\mathcal{P}$ *is a hedge and* $\operatorname{size}(p) \geq b$ *for all* $p \in \mathcal{P}$*, then* (4.5) *holds.*

PROOF. Let $\overline{p}_k$ be the pairs in $\mathcal{P}$ maximal with respect to $\ll$. Set $\mathcal{P}_k = \{p \in \mathcal{P} : p \ll \overline{p}_k\}$. These sets are shrubs and are pairwise disjoint. Indeed, more is true: if $p \in \mathcal{P}_k$, $p' \in \mathcal{P}_{k'}$ and $p \ll p'$ then $\mathcal{P}_k = \mathcal{P}_{k'}$. The situation is this. $p \ll p'$ and $p \ll \overline{p}_k$. Hence from the definition of a hedge, $p' \ll \overline{p}_k$. But also $p' \ll \overline{p}_{k'}$, so that $\overline{p}_k$ and $\overline{p}_{k'}$ are comparable under $\ll$. Maximality then forces $\overline{p}_k = \overline{p}_{k'}$.

A corollary of this is that if $\mathcal{P}'_k \subset \mathcal{P}_k$ is spindly for all k then so is $\bigcup_k \mathcal{P}'_k$. But we know how to construct spindly sets from shrubs. And we can give a proof of (4.5) for a fixed interval U.

Assume that $\overline{I}_k \subset U$ for each k. Apply Lemma 4.21 to each $\mathcal{P}_k$. This gives us sets $\mathcal{P}'_k \subset \mathcal{P}_k$ and $F_k \subset b^{-\varepsilon}\overline{I}_k$ satisfying i)-iii) of that Lemma. It follows from Lemma 4.15 that $\cup\mathcal{P}'_k$ satisfies (4.5). Also from the fact that $\overline{p}_k$ is spindly and $\operatorname{size}(p) \geq b$, we see that (4.8) holds for $\{\overline{p}_k\}$. Hence

$$\sum_k |F_k| \leq C\, b^{100} \sum_k |\overline{I}_k| \leq C\, b^{90}\, |U| \,. \tag{4.26}$$

That is for b sufficently small, $C\, b^{90}$ will be no more than $1/2$, and then the assumptions of Lemma 4.23 are seen to hold, with $a_p = |\langle g, \phi_p \rangle|^2$. Therefore (4.5) holds. Again, it suffices to prove the Lemma for small b as only a lower bound on the size of pairs is assumed.

The next Lemma is a trivial adaptation of the previous proof.

Lemma 4.27. *Lemma* 4.7 *hold under the additional assumption that* $\mathcal{P}$ *is a hedge.*

Last of all, we want to decompose thickets into a small number of hedges. This Lemma depends upon the combinatorics of the pairs, as well as the Carleson measure estimate.

Lemma 4.28. *Let $\mathcal{P}$ be a thicket and assume that for some interval U, $I \subset U$ for all $[\omega, I] \in \mathcal{P}$. Assume that* (4.8) *holds for every hedge $\mathcal{P}' \subset \mathcal{P}$. Then there is a set $E \subset U$ and a set $\mathcal{P}' \subset \mathcal{P}$ so that*

i) *$\mathcal{P}'$ is a union of at most $O(\log 1/b)$ hedges.*

ii) *For all $[\omega, I] \in \mathcal{P} \backslash \mathcal{P}'$, $I \subset E$.*

iii) *And $|E| \leq C_\varepsilon \, b^{100} \, |U|$.*

This lemma, plus Lemma 4.23 and Lemma 4.25 will prove Lemma 4.4 and Lemma 4.7.

PROOF. Begin by letting $\overline{p}_k^1 = [\overline{\omega}_k^1, \overline{I}_k^1]$ be the maximal pairs in $\mathcal{P}$ with respect to $\ll$. The set $\{\overline{p}_k^1\}$ is spindly, and so in particular is a hedge. It satisfies (4.8) by assumption. That is,

$$\sum_k |\overline{I}_k^1| \leq C_\varepsilon \, b^{-4} \, |U| \, .$$

This allows us to delete some pairs from $\mathcal{P}$. Set

$$F_0 = \Big\{ x : \sum_k (M \mathbf{1}[\,\overline{I}_k^1\,](x))^2 > b^{-110} \Big\} .$$

It follows that $|F_0| \leq C_\varepsilon \, b^{100} \, |U|$; further let $\overline{p}_j = [\overline{\omega}_j, \overline{I}_j]$ be an enumeration of those $\overline{p}_k^1$ such that $b^{-\varepsilon} \overline{I}_k^1 \not\subset F_0$. As we have already seen in Lemma 4.22, we then have that

$$\sum_j \mathbf{1}[3 \, b^{-\varepsilon} \overline{I}_j](x) \leq 6 \, b^{-300} \, , \qquad \text{for all } x \in b^{-\varepsilon} U \, .$$

Take $\mathcal{P}^1 = \{ p \in \mathcal{P} : p \ll \text{some } \overline{p}_j \}$. Note that if $[\omega, I\,] \in \mathcal{P} \backslash \mathcal{P}^1$, then $[\omega, I\,] \ll$ some $\overline{p}_k^1$ with $b^{-\varepsilon} \overline{I}_k^1 \subset F_0$, and so $I \subset F_0$ as well.

A few more pairs must be deleted from $\mathcal{P}^1$ in order to gain a certain combinatorial advantage. We will proceed in an inductive fashion. Choose a pair $p_1 = [\omega_1, I_1] \in \mathcal{P}^1$ for which $|I_1|$ is maximal. Let

$$F_1 = \bigcup \Big\{ I : |I| \leq b^{1000} \, |I_1| \, , \\ \sum_{0 \leq j \leq 500 \log 1/b} 3^{-j} \leq \frac{|c(I) - c(I_1)|}{b^{-\varepsilon} |I_1|/3} \leq \sum_{j \geq 0} 3^{-j} \Big\} .$$

Note that $|F_1| \le b^{400}\,|I_1|$.[8] Choose a pair $p_2 = [\omega_2, I_2] \in \mathcal{P}^1\backslash\{p_1\}$, with $I_2 \not\subset F_1$ and $|I_2|$ maximal. Then define F_2 as above. It follows that $I_1 \not\subset F_2$. Continue this procedure until $\mathcal{P}^1$ is exhausted. Then the set E of ii) and iii) above is $E = F_0 \cup \bigcup_{j\ge 1} F_j$. Let $\mathcal{P}' = \{p_j : j \ge 1\}$. Observe that

$$|E| \le C_\varepsilon\, b^{100}\,|U| + b^{400} \sum_{[\omega, I]\in\mathcal{P}'} |I|\,. \tag{4.29}$$

Of course ii) holds. It remains to verify i) and iii). And here observe that the last inequality and i), together with (4.5) for hedges, trivially give iii). So it remains to check i).

The advantage gained in passing from $\mathcal{P}^1$ to $\mathcal{P}'$ is this: if $p = [\omega, I\,], p' = [\omega', I'] \in \mathcal{P}'$ and $p \ll p'$ then as follows from the removal of the sets F_j, $j \ge 1$,

$$|c(I) - c(I')| \le \frac{1}{3}\, b^{-\varepsilon}\,|I'| \sum_{j\le 500\log 1/b} 3^{-j}\,.$$

As a corollary, we see that $\mathcal{P}'$ satisfies the following good combinatorial condition: if $p_1 \ll p_2$, $p_3 \ll p_4$, all $p_i \in \mathcal{P}'$ then either $p_2 \ll p_3$ or $p_3 \ll p_2$.[9] To see this, let $p_i = [\omega_i, I_i]$. We then have $\omega_1 \supset \omega_2, \omega_3 \supset \omega_4$. Thus, we must have *e.g.* $\omega_2 \supset \omega_3$. Under the assumption of (4.12), $p_1 \ll p_2, p_3$, and $p_2 \ne p_3$ implies that we must have $\omega_3 \subsetneq \omega_2$. Then

$$\begin{aligned}|c(I_2) - c(I_3)| &\le |c(I_3) - c(I_1)| + |c(I_1) - c(I_2)|\\ &\le \frac{1}{3}\, b^{-\varepsilon}\,|I_3|\Big(\sum_{0\le j\le 500\log 1/b} 3^{-j} + \frac{|I_2|}{|I_3|}\sum_{0\le j\le 500\log 1/b} 3^{-j}\Big)\\ &\le \frac{1}{3}\, b^{-\varepsilon}\,|I_3| \sum_{0\le j} [\,3^{-j}|I_3| > |I_2|\,]\, 3^{-j}\,.\end{aligned}$$

The last line follows from the fact that $|I_2|/|I_3| \le b^{1000}$, (see (4.13)) and shows that $p_2 \ll p_3$.

We can now apply a combinatorial trick of Fefferman. Let $\mathcal{B}(p) = \sharp\{\overline{p}_k : p \ll \overline{p}_k\}$. For all $p \in \mathcal{P}'$, we have $\mathcal{B}(p) \le 6\, b^{-300}$, for if

$$[\omega, I\,] \ll \overline{p}_{k(1)}, \ldots, \overline{p}_{k(v)}\,, \qquad v = \mathcal{B}([\omega, I\,])\,,$$

---

[8] Here we are simply deleting a small neighborhood of the boundary of $(3b^\varepsilon)^{-1}I_1$. Recall that by (4.12), if $[\omega, I] \ll [\omega_1, I_1]$, then $|I| < b^{1000}|I_1|$.

[9] Note that for '$<$' this is trivial.

then for $x \in I$, $\sum_k \mathbf{1}[3\,b^{-\varepsilon}\overline{I}_k](x) \geq v$, but by construction this last sum can't be more that $6\,b^{-300}$. Furthermore, $\mathcal{B}(p)$ has the following combinatorial property: if $p \ll p'$, and $p \ll p''$, but p' and p'' not comparable under $\ll$, with all pairs in $\mathcal{P}'$, then $\mathcal{B}(p) \geq \mathcal{B}(p') + \mathcal{B}(p'')$.

Indeed, write $p' \ll \overline{p}_{k(1)}, \ldots, \overline{p}_{k(v)}$, and $p'' \ll \overline{p}_{j(1)}, \ldots, \overline{p}_{j(w)}$ where $v = \mathcal{B}(p')$ and $w = \mathcal{B}(p'')$. If some $\overline{p}_{k(\sigma)}$ equals some $\overline{p}_{j(\tau)}$ then one would have $p' \ll p''' = \overline{p}_{k(\sigma)} = p_{j(\tau)}$ and $p'' \ll p'''$. The situation is $p \ll p', p'' \ll p'''$ which by the good combinatorial property of $\mathcal{P}'$ forces p' and p'' to be comparable under $\ll$. This is a contradiction which forces the inequality $\mathcal{B}(p) \geq \mathcal{B}(p') + \mathcal{B}(p'')$.

But then the hedges are easy to define, simply take $\mathcal{H}_v = \{p \in \mathcal{P}' : 2^v \leq \mathcal{B}(p) < 2^{v+1}\}$ for $v \leq 120 \log 1/b$. The combinatorial property of $\mathcal{B}(p)$ show that each $\mathcal{H}_v$ is a hedge, finishing the proof.

We have completed the proof of the critical Carleson measure Lemma.

The next Lemma initiates the proof of the second estimate (Lemma 4.2) on the operator $T^{\mathcal{P}}$, but it's proof will also yield the first estimate (Lemma 4.1). We need an improvement of Lemma 4.4.

Lemma 4.30. *Let $\mathcal{P}$ be a thicket with $I_p \subset b^{-\varepsilon/2}[-1,1]$ for all $p \in \mathcal{P}$. One then has the inequality below for any $\varepsilon > 0$.*

$$\Big\| \Big(\sum_{p \in \mathcal{P}} |f_p(x)|^2 \Big)^{1/2} \Big\|_r \leq C_{\varepsilon,r}\, b^{-\varepsilon}\, \|f\|_\infty\,, \qquad 2 \leq r < \infty\,. \tag{4.31}$$

PROOF. To prove the Lemma, it suffices to establish that for all $\varepsilon > 0$,

$$\Big\| \Big(\sum_{p=[\omega,I] \in \mathcal{P}} \Big| \frac{\langle g, \phi_p \rangle}{\sqrt{|I|}}\, \mathbf{1}_I(x) \Big|^2 \Big)^{1/2} \Big\|_{\mathrm{BMO}} \leq C_\varepsilon\, b^{-\varepsilon}\, \|g\|_\infty\,. \tag{4.32}$$

For then properties of BMO give

$$\Big\| \Big(\sum_{p=[\omega,I] \in \mathcal{P}} \Big| \frac{\langle g, \phi_p \rangle}{\sqrt{|I|}}\, \mathbf{1}_I(x) \Big|^2 \Big)^{1/2} \Big\|_r \leq C_\varepsilon\, r\, b^{-\varepsilon}\, \|g\|_\infty\,,$$

which with the Fefferman-Stein maximal inequalities gives the Lemma.

But also, if $\mathcal{P}$ is a thicket and each pair p has size at least $b/2$, and so $b\,\mathbf{1}_I(x) \le C\,|f_p(x)|$, it follows from the decay of φ_p that $b^{-\varepsilon} I_p \subset [-1,1]$ for all $p \in \mathcal{P}$. Hence

$$\Big\| \sum_{[\omega,I]\in\mathcal{P}} \mathbf{1}_I(x) \Big\|_r \le C_\varepsilon\, r^2\, b^{-2-\varepsilon} ,$$

which is the conclusion of Lemma 4.1.

To check (4.32) it is enough to show that for all triadic intervals U,

$$\int_U \sum_{[\omega,I]\in\mathcal{P}} [I \subsetneq U] \Big| \langle g, \phi_{[\omega,I]} \rangle \frac{1}{\sqrt{|I|}}\, \mathbf{1}_I(x) \Big|^2 dx \le C_\varepsilon\, b^{-\varepsilon}\, |U|\, \|g\|_\infty^2 ,$$

where g is a function bounded by 1. But this is precisely (4.5) above.

To bound $T^{\mathcal{P}}$ as in (4.3) we will dualize and provide a proof of the estimate

$$\|T^{\mathcal{P}*} g\|_s \le C_s\, b^{\delta(s)}\, \|g\|_\infty , \tag{4.33}$$

where $2 \le s < \infty$ and $\delta(s) > 0$. We localize $T^{\mathcal{P}*}$ in the space variable.

Lemma 4.34. *Define, for $\varepsilon > 0$,*

$$T^*_{\sharp[\omega,I]} g(x) = \mathbf{1}[b^{-\varepsilon} I](x)\, T^*_{[\omega,I]}(\mathbf{1}[b^{-\varepsilon} I] g)(x) .$$

Let $\mathcal{P}$ be a thicket with size $(p) \ge b$ *for all $p \in \mathcal{P}$. And let g be a function bounded by* 1. *Then*

$$\Big\| \sum_{p\in\mathcal{P}} \big| T^*_p g(x) - T^*_{\sharp p} g(x) \big| \Big\|_r \le C_{r,\varepsilon}\, b^3 , \qquad 1 < r < \infty .$$

PROOF. We will do half of the proof, the other half being similar. Estimate ($p = [\omega, I\,]$)

$$\begin{aligned} &|T^*_p g(x) - \mathbf{1}[b^{-\varepsilon} I](x)\, T^*_p g(x)| \\ &\qquad \le \mathbf{1}[(b^{-\varepsilon} I)^c](x) \int |k_\omega(-y)\, f_p(x - 2\,y)\, g(x-y)|\, dy \\ &\qquad = \mathbf{1}[(b^{-\varepsilon} I)^c](x) \Big(\int_{|y|\le \operatorname{dist}(x,I)/3} + \int_{|y|\ge \operatorname{dist}(x,I)/3} \cdots\, dy \Big) \\ &\qquad = A_p(x) + B_p(x) . \end{aligned}$$

For the first term, use the decay of f_p away from I: from the definition, $|f_p(z)| \leq C_\varepsilon(\operatorname{dist}(z,I)/|I|)^{-6/\varepsilon}$, which implies that

$$\begin{aligned}
A_p(x) &\leq C\,\mathbf{1}[(b^{-\varepsilon}I)^c](x)\Big(\frac{\operatorname{dist}(x,I)}{|I|}\Big)^{-6/\varepsilon}|k_\omega| * |g|(x) \\
&\leq C\,b^6\,(M\mathbf{1}_I)^2(x)\,,
\end{aligned}$$

where M denotes the Hardy-Littlewood maximal function.

For the second term, use the decay of k_ω:

$$|k_\omega(y)| \leq C_\varepsilon\,|\omega|\,(|\omega|\,|y|)^{-6/\varepsilon}\,,$$

$$B_p(x) \leq C_\varepsilon\,\mathbf{1}[(b^{-\varepsilon}I)^c](x)\Big(\frac{\operatorname{dist}(x,I)}{|I|}\Big)^{-6/\varepsilon} \leq C_\varepsilon\,b^6\,(M\mathbf{1}_I)^2(x)\,.$$

Thus, using Lemma 4.1 and the Fefferman-Stein maximal inequalities we see that

$$\begin{aligned}
\Big\|\sum_{p\in\mathcal{P}} A_p(x)+B_p(x)\Big\|_r &\leq C_\varepsilon\,b^6\,\Big\|\sum_{[\omega,I]\in\mathcal{P}}|M\mathbf{1}_I(x)|^2\Big\|_r \\
&\leq C_\varepsilon\,b^6\Big\|\Big(\sum_{[\omega,I]\in\mathcal{P}}|M\mathbf{1}_I(x)|^2\Big)^{1/2}\Big\|_{2r}^2 \\
&\leq C_{r,\varepsilon}\,b^6\,\Big\|\sum_{[\omega,I]\in\mathcal{P}}\mathbf{1}_I(x)\Big\|_r \\
&\leq C_{r,\varepsilon}\,b^3\,.
\end{aligned}$$

This finishes the proof.

At this point, the top item on the agenda is a decomposition of g into functions analogous to f_p. But we have to abandon the luxury of reconstructing g after the fact, as is done for f in Lemma 3.3. This creates an extra problem, of an essentially technical nature.

Recall the definition of $A^*(\omega)$, (2.13). This interval is triadic in (a shift of) $\hat{\mathcal{G}}$. More can be said: $A^*(\omega)$ is central so that $3A^*(\omega)$ is triadic; $c(A^*(\omega)) = 2\,c(\omega)+|\omega|/6$; $|A^*(\omega)| = |\omega|/9$; and

$$\omega\cap\omega'=\varnothing \qquad \text{implies} \qquad 3A^*(\omega)\cap 3A^*(\omega')=\varnothing\,. \tag{4.35}$$

Indeed, if $\omega\cap\omega'=\varnothing$, then

$$|c(A^*(\omega))-c(A^*(\omega'))| = 2\,|c(\omega)-c(\omega')| \geq |\omega|+|\omega'|\,,$$

which demonstrates the assertion. These observations inform the definitions below.

To a pair $[\omega, I\,]$ associate the functions

$$\chi_{\delta,[\omega,I\,]}(x) \tag{4.36}$$
$$= |I|^{-1/2} \exp\Big(2\pi\, i\, x\Big(c(A^*(\omega)) + \delta\,\frac{|\omega|}{56}\Big)\Big)\phi\Big(\frac{x-c(I)}{|I|}\Big)$$

where $\delta = -56, -55, \dots, 55$. For each pair, the collection of functions

$$\big\{\chi_{\delta,[\omega,I+28n|I|]} : -56 \le \delta < 56,\ n \in \mathbb{Z},\ \omega \in \hat{\mathcal{G}}\big\}$$

is just a rescaling of the collection in (2.5) and so satisfies (2.6). (The constant A in that equation is irrelevant, and so we will take it to be 1). For a pair p, set

$$Q_{[\omega,I\,]} = \sum_{|\delta|\le 5}\ \sum_{|n|\le(300b^{\varepsilon})^{-1}} \chi_{\delta,[\omega,I+28n|I|]} \otimes \chi_{\delta,[\omega,I+28n|I|]}\,. \tag{4.37}$$

The sum over δ is restricted because if $|\delta| \ge 6$, then the support of $(\chi_{\delta,[\omega,J]})\widehat{\ }$ will not intersect $A^*(\omega)$. Therefore, one should have $T_p^* g \simeq T_p^* Q_p g$. This we will quantify in the next Lemma.

Lemma 4.38. *If $\mathcal{P}$ is a thicket then*

$$\Big\|\sum_{\mathcal{P}} |T_p^* g(x) - T_p^* Q_p g(x)|\Big\|_r \le C_r\, b^6\, \|g\|_\infty\,, \qquad r < \infty\,.$$

Proof. Since a dilate of (2.6) is in force, it follows from (2.14) that

$$T_p^* g = T_p^*\Big(\sum_{|\delta|\le 5}\ \sum_{n=-\infty}^{\infty} \chi_{\delta,[\omega,I+28n|I|]}\,\langle g, \chi_{\delta,[\omega,I+28n|I|]}\rangle\Big)\,.$$

The sum above is absolutely convergent. So we will give a pointwise estimate for

$$\langle g, \chi_{\delta,[\omega,J]}\rangle\, T_p^* \chi_{\delta,[\omega,J]}(x)\,, \qquad \text{where } \operatorname{dist}(I,J) \ge b^{-\varepsilon}|I|\,.$$

The δ is unimportant, and so will be dropped from notation.

Set $\Phi_I(x) = (1 \wedge (|x|/|I|)^{100/\varepsilon})$. We need to consider the integral

$$\begin{aligned}\Big|\int k_\omega(-y)\,\phi_p(x-2\,y)\,\chi_{[\omega,J]}(x-y)\,dy\Big| \\ &\le \frac{C_\varepsilon}{|I|}\int \Phi_I(y)\,\Phi_I(\mathrm{dist}(x-2\,y,I))\,\Phi_I(\mathrm{dist}(x-y,J))\,\frac{dy}{|I|} \\ &:= E(J,x)\,.\end{aligned}$$

Sum this over J to get

$$\begin{aligned}\sum_J [\,|I|=|J|,\ \mathrm{dist}(J,I)\ge b^{-\varepsilon}|I|\,]\,E(J,x) \\ &\le \frac{C_\varepsilon}{|I|}\int \Phi_I(y)\,\Phi_I(\mathrm{dist}(x-2\,y,I))\,\Phi_I(\mathrm{dist}(x-y,(b^{-\varepsilon}I)^c))\,\frac{dy}{|I|} \\ &\le \frac{C_\varepsilon}{|I|}\,b^{10}\,(M\mathbf{1}_I)^2(x)\,.\end{aligned}$$

Remembering that both f and g are bounded by 1, so that

$$|\langle \overline{f},\phi_p\rangle\,\langle \overline{g},\chi_{\delta,[\omega,J]}\rangle| \le C\,|I|\,,$$

it follows that

$$\Big\|\sum_{\mathcal{P}}|T_p^*(I-Q_p)\,g(x)|\Big\|_r \le C_\varepsilon\,b^{10}\,\Big\|\Big(\sum_{[\omega,I]\in\mathcal{P}}(M\mathbf{1}_I(x))^2\Big)^{1/2}\Big\|_{2r}^2 \le C\,b^6\,,$$

by Lemma 4.1.

With this last Lemma, we have associated with $\mathcal{P}$ a collection of pairs

$$\mathcal{P}^e = \{[\omega,J] : \text{for some } [\omega,I]\in\mathcal{P},\ \mathrm{dist}(J,I)\le(300\,b^\varepsilon)^{-1}|I|\}\,.$$

Our claim is that $\mathcal{P}^e$ essentially obeys Lemma 4.30.

Lemma 4.39. *If $\mathcal{P}$ is a thicket with* $\mathrm{size}(p)\ge b$ *for all $p\in\mathcal{P}$, then the following inequalities hold for all $\varepsilon>0$.*

$$\Big\|\Big(\sum_{|\delta|\le 5}\ \sum_{[\omega,J]\in\mathcal{P}^e}\Big|\frac{1}{\sqrt{|J|}}\,\mathbf{1}_J(x)\,\langle g,\chi_{\delta,[\omega,J]}\rangle\Big|^2\Big)^{1/2}\Big\|_r \le C_{\varepsilon,r}\,b^{-2\varepsilon}\,\|g\|_\infty\,,$$

where $r < \infty$.

PROOF. It suffices to prove the estimate above with the BMO norm replacing the L^r norm. This in turn follows from Lemma 4.7.

We see how to apply that Lemma to the pairs $\mathcal{P}^e$, and the functions $\{\chi_{\delta,p} : p \in \mathcal{P}^e\}$, where $|\delta| \le 5$ is fixed. The assumption (4.8) holds, because $\mathcal{P}$ is a thicket and we have proved Lemma 4.4. The condition (4.9) follows from the definition of the $\chi_{\delta,p}$. And (4.9) follows from (4.35).

The assumption of Lemma 4.7 that is not immediate is that $\mathcal{P}^e$ be a thicket. Indeed, it will not be so, in general. Yet if $\mathcal{P}$ is spindly, then $\mathcal{P}^e$ is a thicket. Therefore our Lemma follows from Lemma 4.7 if $\mathcal{P}$ is spindly. Continuing in this vein, we have Lemma 4.21. And therefore, by Lemma 4.23 our current Lemma holds under the more general assumption that $\mathcal{P}$ be a hedge. Finally, we have Lemma 4.28, so that our Lemma is seen to hold under the sole assumption that $\mathcal{P}$ is a thicket.

We can now give the final proof of this section.

PROOF OF LEMMA 4.2. We can assume that $b/2 \le \mathrm{size}(p) \le b$ for all $p \in \mathcal{P}$. For if not, we write $\mathcal{P} = \bigcup_n \mathcal{P}_n$, where $\mathcal{P}_n = \{p \in \mathcal{P} : 2^{-n-1} \le \mathrm{size}(p) \le 2^{-n} b\}$. For each n we will prove the estimate

$$\|T^{\mathcal{P}_n} g\| \le C_r\, (2^{-n} b)^{\delta(r)} \|g\|_r\,, \qquad 1 < r < 2\,.$$

This is summed over n to conclude the Lemma.

Recall that we wish to establish the inequalities (4.33). The interesting part is to establish the bound

$$\|T^{\mathcal{P}*} g\|_2 \le C_\varepsilon\, b^{1/2-\varepsilon}\, \|g\|_\infty\,, \qquad \varepsilon > 0\,. \tag{4.40}$$

Henceforth, g will be assumed to be bounded by 1.

Make the following definitions. For $p = [\omega, I\,] \in \mathcal{P}$, denote

$$\begin{aligned}
\mathcal{P}(p) &= \{[\omega', I'] \in \mathcal{P} : \omega \subset \omega',\ \mathrm{dist}(I', I) < b^{-\varepsilon}|I|\}\,,\\
\mathcal{P}^\flat(p) &= \{[\omega', I'] \in \mathcal{P} : \omega \subset \omega',\ \mathrm{dist}(I', I) \ge b^{-\varepsilon}|I|\}\,.
\end{aligned}$$

Expand

$$\begin{aligned}\|T^{\mathcal{P}*}g\|_2^2 &\le \sum_{p\in\mathcal{P}} \|T_p^*g\|_2^2 + \sum_{p\in\mathcal{P}} |\langle T_p^*g(x), T^{\mathcal{P}(p)*}g(x)\rangle| \\ &\quad + \sum_{p\in\mathcal{P}} |\langle T_p^*g(x), T^{\mathcal{P}^\flat(p)*}g(x)\rangle| \\ &= \mathcal{A}+\mathcal{B}+\mathcal{C}\,.\end{aligned}$$

This is justified since $\mathcal{F}T_p^*$ is supported in $A(\omega)\subset\omega$, hence if $\omega\cap\omega'=\varnothing$, $\langle T_p^*g, T_{p'}^*g\rangle = 0$.

For the diagonal term, by Lemma 4.38, we can write

$$\begin{aligned}\mathcal{A} &= \sum_{p\in\mathcal{P}} \|T_p^*g\|_2^2 \\ &\le C\,b^2 + \sum_{p\in\mathcal{P}} \|T_p^*Q_pg\|_2^2 \\ &\le C\,b^2 + C\,b^2\sum_{p\in\mathcal{P}} \|Q_pg\|_2^2 \\ &\le C\,b^2 + C\,b^{2-4\varepsilon}\sum_{p\in\mathcal{P}^e} \|\chi_p\langle g,\chi_p\rangle\|_2^2 \\ &\le C\,b^{2-10\varepsilon}\,,\end{aligned}$$

where in the last line we invoke Lemma 4.39. (And again, the $|\delta|<5$ is dropped from notation).

The diagonal estimate also enters into the second term, $\mathcal{B}$.

$$\begin{aligned}\mathcal{B} &= \sum_{p\in\mathcal{P}} |\langle T_p^*g(x), T^{\mathcal{P}(p)*}g(x)\rangle| \\ &\le C\,b\sum_{p\in\mathcal{P}} \Big\langle |T_p^*g(x)|, \sum_{[\omega',I']\in\mathcal{P}(p)} (M\mathbf{1}_{I'}(x))^2\Big\rangle \\ &\le C\,b\sum_{p\in\mathcal{P}} \|T_p^*g\|_2 \Big\| \sum_{[\omega',I']\in\mathcal{P}(p)} (M\mathbf{1}_{I'}(x))^2\Big\|_2\end{aligned}$$

But the last term on the right is no more than $C\sqrt{b^{-\varepsilon}|I|}$, since the sets $\{I' : [\omega',I']\in\mathcal{P}(p)\}$ are disjoint and contained in $b^{-\varepsilon}I$. We continue as follows.

$$\mathcal{B}\le C\,b^{1-2\varepsilon}\Big(\sum_{p\in\mathcal{P}} \|T_p^*g\|_2^2\Big)^{1/2}\Big(\sum_{[\omega,I]\in\mathcal{P}} b^{-\varepsilon}|I|\Big)^{1/2}\le C\,b^{1-15\varepsilon}\,.$$

The last term, $\mathcal{C}$, is the least interesting. All the intervals $\{I' : [\omega', I'] \in \mathcal{P}^\flat(p)\}$ are much smaller than I, and are pairwise disjoint. So if J is any interval of length $|I|$,

$$\begin{aligned}\int_J |T_p^* g(x)\, T^{\mathcal{P}^\flat(p)*} g(x)|\, dx &\le C_\varepsilon\, b\Big(1+\frac{\operatorname{dist}(I,J)}{|I|}\Big)^{-10/\varepsilon} \\ &\qquad \cdot \int_J \sum_{[\omega',I']\in\mathcal{P}^\flat(p)} (M\mathbf{1}_{I'})^2(x)\, dx \\ &\le C_\varepsilon\, b^{10}\Big(1+\frac{\operatorname{dist}(I,J)}{|I|}\Big)^{-5} |J|\,.\end{aligned}$$

It follows that

$$\mathcal{C} \le C_\varepsilon\, b^9 \sum_{[\omega,I]\in\mathcal{P}} |I| \le C\, b^6\,.$$

This finishes the proof of the L^2 estimate.

For the L^r bound, $r>2$, use Lemma 4.38 and Lemma 4.39 to see that

$$\begin{aligned}\Big\| \sum_{p\in\mathcal{P}} T_p^* g \Big\|_r &\le C\, b + \Big\| \sum_{p\in\mathcal{P}} T_p^*(Q_p g)(x)\Big\|_r \\ &\le C\, b + \Big\| \sum_{p\in\mathcal{P}} (|k_\omega| * |f_p|^2(x)\, |k_\omega| * |Q_p g|^2(x))^{1/2}\Big\|_r \\ &\le C_r\, b + \Big\| \Big(\sum_{p\in\mathcal{P}} |k_\omega| * |f_p|^2(x)\Big)^{1/2}\Big\|_r \qquad (4.41)\\ &\qquad \cdot \Big\| \Big(\sum_{p\in\mathcal{P}} |k_\omega| * |Q_p g|^2(x)\Big)^{1/2}\Big\|_r \\ &\le C_{\varepsilon,r}\, b^{-3\varepsilon}\,.\end{aligned}$$

Hence the Lemma follows by interpolating with the better L^2 bound.

5. Trees.

The emphasis in this section will be on sets of pairs $\mathcal{P}$ which are trees under the partial order on pairs. Call a set of pairs $\mathcal{P}$ a *tree with top* $p^t = [\omega^t, I^t]$ if all $p\in\mathcal{P}$ are less than p^t. (The top need not be in the tree, nor is the top unique.) It turns out that the $T^{\mathcal{P}}$ are familiar objects, namely Calderón-Zygmund operators. We need to estimate $\|T^{\mathcal{P}}\|_2$ and quantify the orthogonality between trees.

Lemma 5.1. *Let $\mathcal{P}$ be a tree then*

$$\|T^{\mathcal{P}}\|_r \leq C_r \,, \qquad 1 < r < \infty \,.$$

The norm estimate here is $O(1)$, which means that we will have to identify that part of a tree which contributes to the large norm estimate. This is the purpose of the next definitions.

Flavors of Trees. Call a tree $\mathcal{P}$ with top $p^t = [\omega^t, I^t]$ an *α-tree* if $c(\omega^t) \in A(\omega)$ for all $[\omega, I\,] \in \mathcal{P}$. Call $\mathcal{P}$ an *α^*-tree* if $2\,c(\omega^t) \in A^*(\omega)$ for all $[\omega, I\,] \in \mathcal{P}$. And call $\mathcal{P}$ a *β-tree* if $c(\omega^t) \notin A(\omega)$ and $2\,c(\omega^t) \notin A^*(\omega)$ for all $[\omega, I\,] \in \mathcal{P}$.

Of these three flavors of trees, β-trees are the easiest, since they are especially nice Calderón-Zygmund operators. The other flavors of trees are essentially the paraproducts, as we shall see.

PROOF OF LEMMA 5.1. For $\omega \supset \omega^t$, let $\mathcal{P}(\omega) = \{p \in \mathcal{P} : p = [\omega, I\,] \text{ for some } I\}$. This is useful since $\mathcal{P} = \bigcup_{\omega \supset \omega^t} \mathcal{P}(\omega)$. That is, the relevant frequency intervals form an increasing sequence. Let $b = \sup_{p \in \mathcal{P}} \operatorname{size}(p)$.

The top interval ω^t can be assumed to be centered at the origin, by the considerations in (2.15). This means that

$$|f_p(x) - f_p(y)| \leq C\, b\, \frac{|x-y|}{|I|} \,, \qquad \text{for all } p \in \mathcal{P} \,.$$

So, writing

$$T^{\mathcal{P}} g(x) = \int \sum_{p \in \mathcal{P}} k_\omega(x-y)\, f_p(2\,x - y)\, g(y)\, dy = \int K(x,y)\, g(y)\, dy \,,$$

one easily checks that $K(x,y)$ is a standard Calderón-Zygmund kernel. In particular, K satisfies the conditions

$$|K(x,y)| \leq \frac{C\,b}{|x-y|} \,, \tag{5.2}$$

and if $2\,|x-z| \le |x-y|$,

$$(5.3)\qquad |K(x,y)-K(z,y)| + |K(y,x)-K(y,z)| \le C\,b\,\frac{|x-z|}{|x-y|^2}\,.$$

It therefore suffices to establish the boundedness of $T^{\mathcal{P}}$ on L^2.

In case of a β-tree everything is easy; we see that $T^{\mathcal{P}}\mathbf{1} = T^{\mathcal{P}*}\mathbf{1} = 0$, so that the L^2 boundedness follows from Cotlar's Lemma. In particular, $\|T^{\mathcal{P}}\|_2 \le C\,b$.

The two remaining two cases are duals of one another, so we only consider the case of an α-tree. In that case $T^{\mathcal{P}}\mathbf{1} \ne 0$ but $T^{\mathcal{P}*}\mathbf{1} = 0$. Indeed, $T_p g$ acts on $\mathcal{F}^{-1}\mathbf{1}[A(\omega)]\mathcal{F}g$, by (2.14), and $0 \notin A^*(\omega)$ as $\mathcal{P}$ is an α-tree. Thus $T_p^*\mathbf{1} = 0$. On the other hand, $T_p\mathbf{1}$ need not be zero.

For pairs $p \in \mathcal{P}$, set

$$\frac{T_p\mathbf{1}(x)}{\langle f, \phi_p\rangle} = \alpha_p(x) = \int k_\omega(y)\,\phi_p(x+y)\,dy\,.$$

One easily checks that $\operatorname{supp}(\widehat{\alpha_{[\omega,I\,]}}) \subset A^*(\omega)$, which are disjoint lacunary intervals, as ω varies. One can also see that the inequality below holds.

$$(5.4)\qquad \Big\| \sum_p [\,p = [\omega, I\,] \text{ for some } I\,]\, a_p\, \alpha_p(x) \Big\|_2^2 \le C \sum_p |a_p|^2\,,$$

where ω is fixed. See for instance [D, equation (3.4.4)]. Then

$$T^{\mathcal{P}}\mathbf{1}(x) = \sum_{p\in\mathcal{P}} \langle f, \phi_p\rangle\, \alpha_p(x)\,,$$

and the BMO norm of this last term is easily seen to be

$$(5.5)\qquad \|T^{\mathcal{P}}\mathbf{1}\|_{\mathrm{BMO}} \le C_* \sup_U \Big(\frac{1}{|U|} \sum_{[\omega,I\,]\in\mathcal{P}} [\,I \subset U\,]\, |\langle f, \phi_p\rangle|^2 \Big)^{1/2}\,.$$

The supremum is finite as f is bounded. This estimate is in general, sharp. Namely, for all $\varepsilon > 0$, one can construct a bounded function f and an α-tree $\mathcal{P}$ so that $\operatorname{size}(p) < \varepsilon$ for all $p \in \mathcal{P}$, and $\|T^{\mathcal{P}}\mathbf{1}\|_{\mathrm{BMO}} \simeq 1$.

This lemma, and it's proof, demonstrate clearly a central reason why the bilinear Hilbert transform is so difficult to understand: regardless of the size of individual T_p's their sum can (and will) have large norm. And any estimate of the form $T^{\mathcal{P}} : L^2 \longrightarrow L^1$, say will also be $O(1)$. Thus the weaker inequalities of the previous section are of no use for the paraproducts.

Despite the bad features of α-trees, they do admit a certain control. Namely, there cannot be a great number of disjoint α-trees, all of a fixed norm. This point is made precise with a Carleson measure estimate in the next section.

These considerations suggest the following definition for the intrinsic size of an arbitrary collection of pairs $\mathcal{P}$. Set

$$\operatorname{size}(\mathcal{P}) = \sup\{\|T^{\mathcal{P}'}\|_2 : \text{ for all trees } \mathcal{P}' \subset \mathcal{P}\}. \tag{5.6}$$

We shall see, in Section 7, that this really is the appropriate notion for the size of $\mathcal{P}$.

The objective we take up in the remainder of this section is to provide a control for large unions of trees with small norm. And in the Lemmas, we will make no distinction on the flavor (α, α^*, or β) of the trees involved.

Normal Trees. Let $0 < b < 1$. (b is associated with the size of pairs and trees). Beginning with this definition, we will use a parameter $A > 3$ which won't play a role until the end of proof. Call a tree $\mathcal{P}$ with top $p^t = [\omega^t, I^t]$ *normal* if these two conditions are satisfied. For all $[\omega, I\,] \in \mathcal{P}$, one has $|I| \le (b/A)^{40000}\,|I^t|$; and $\operatorname{dist}(I, \partial I^t) \ge (b/A)^{400}\,|I^t|$.

For a normal tree we nearly have that

$$T^{\mathcal{P}} g \simeq \mathbf{1}[I^t]\, T^{\mathcal{P}}(g\mathbf{1}[I^t]) .$$

We make this precise, and due to some Fourier calculations yet to be done, the truncation in the space variable needs to be done in a smooth way. Thus let $\zeta(x)$ be a smooth function satisfying

$$\mathbf{1}_{[-1/4,1/4]}(x) \le \zeta(x) \le \mathbf{1}_{[-1/2,1/2]}(x)\,, \tag{5.7}$$

and $|\mathcal{F}\zeta(\xi)| \le C\,|\xi|^{-100}$. For an interval J let $\zeta_J(x) = \zeta((x-c(J))/|J|)$. Define for $p = [\omega, I\,]$ in a normal tree $\mathcal{P}$,

$$T_{\sharp p} g(x) = \zeta_{\mu(I)}(x)\, T_p(\zeta_{\mu(I)} g)(x)\,, \tag{5.8}$$

where $\mu(I)$ is $(b/A)^{100}\sqrt{|I^t|/|I|}\,I$. Normality guarantees that $\mu(I) \subset I^t$. Then let $T_\sharp^{\mathcal{P}} = \sum_{p\in\mathcal{P}} T_{\sharp p}$.

Lemma 5.9. *For a normal tree $\mathcal{P}$, we have the inequality*

$$|T_\sharp^{\mathcal{P}} g(x) - T^{\mathcal{P}} g(x)| \le C\Big(\frac{b}{A}\Big)^{1000} (M\mathbf{1}_{I^t})^2(x)\, Mg(x)\,. \tag{5.10}$$

The same inequality holds for $T_\sharp^{\mathcal{P}*}$. In particular,

$$\|T_\sharp^{\mathcal{P}}\|_r \le C_r \max\Big\{\mathrm{size}(\mathcal{P}), \Big(\frac{b}{A}\Big)^{100}\Big\}, \qquad 1 < r < \infty\,. \tag{5.11}$$

PROOF. For $j = 1, 2, \dots$ let

$$\mathcal{P}_j = \{[\omega, I\,] \in \mathcal{P} : |I| = 3^{-j}|I^t|\}\,.$$

(As $\mathcal{P}$ is normal, $\mathcal{P}_j$ is empty if $j \le 40000 \log A/b$.) This set is linearly ordered in the I coordinate. We can estimate for $[\omega, I\,] \in \mathcal{P}$

$$\begin{aligned}
&|T_{\sharp[\omega,I\,]} g(x) - T_{[\omega,I\,]} g(x)| \\
&\qquad \le \mathbf{1}\Big[\frac{\mu(I)}{8}\Big](x) \int_{x-y\notin\mu(I)/4} |k_\omega(y)\, f_p(x+y)\, g(x-y)|\, dy \\
&\qquad\quad + \mathbf{1}\Big[\Big(\frac{\mu(I)}{8}\Big)^c\Big](x) \int_{-\infty}^{\infty} |k_\omega(y)\, f_p(x+y)\, g(x-y)|\, dy \\
&\qquad \le C\, Mg(x)\Big(\Big(\frac{|\mu(I)|}{|I|}\Big)^{-20} \mathbf{1}\Big[\frac{\mu(I)}{8}\Big](x) \\
&\qquad\qquad\qquad + \mathbf{1}\Big[\Big(\frac{\mu(I)}{8}\Big)^c\Big](x)\Big(\frac{\mathrm{dist}(x, \mu(I)/16)}{|I|}\Big)^{-50}\Big)\,.
\end{aligned}$$

This follows on the one hand by the decay estimate for k_ω, and on the other by the decay of f_p.

Sum this estimate over $p \in \mathcal{P}_j$.

$$\begin{aligned}
&\sum_{[\omega,I\,]\in\mathcal{P}_j} |T_{\sharp[\omega,I\,]} g(x) - T_p g(x)| \\
&\qquad\qquad \le C\Big(\frac{|\mu(I)|}{|I|}\Big)^{-10}\Big(1 + \frac{\mathrm{dist}(x, I^t)}{|I^t|}\Big)^{-2} Mg(x) \\
&\qquad\qquad \le C\Big(\frac{b}{A}\Big)^{1000} 3^{-5j}\Big(1 + \frac{\mathrm{dist}(x, I^t)}{|I^t|}\Big)^{-2} Mg(x)\,.
\end{aligned}$$

This is summed over j such that $3^{-j} \leq (b/A)^{400}$, and so yields the Lemma.

Separation of Trees. Two normal trees $\mathcal{P}$ with top $[\omega^t, I^t]$ and $\mathcal{P}'$ with top $[\omega^{t'}, I^{t'}]$ are *separated* if $I^t \cap I^{t'} = \varnothing$, or otherwise if

α) $p' = [\omega', I'] \in \mathcal{P}'$, and $I' \subset I^t$ implies $\operatorname{dist}(\omega', \omega^t) > (A/b)^{3000}|\omega'|$.

β) $p = [\omega, I\,] \in \mathcal{P}$, and $I \subset I^{t'}$ implies $\operatorname{dist}(\omega, \omega^{t'}) > (A/b)^{3000}|\omega|$.

γ) and finally, assuming $I^{t'} \subset I^t$, for all $[\omega, I\,] \in \mathcal{P}$, $|I| \leq (b/A)^{500}|I^{t'}|$ implies $\operatorname{dist}(I, \partial I^{t'}) \geq (b/A)^{400}|I^{t'}|$.

This next Lemma quantifies the essential orthogonality between trees.

Lemma 5.12. *For separated trees as above, we have*

$$\|T_\sharp^{\mathcal{P}} T_\sharp^{\mathcal{P}'*}\|\,,\ \|T_\sharp^{\mathcal{P}*} T_\sharp^{\mathcal{P}'}\|_2 \leq C\Big(\frac{b}{A}\Big)^{500}.$$

More importantly, we have the following local estimates on inner products. Assuming that $I^{t'} \subset I^t$,

$$|\langle T_\sharp^{\mathcal{P}} g, T_\sharp^{\mathcal{P}'} h\rangle| \leq C\Big(\frac{b}{A}\Big)^{500} \|Mg + |T_\sharp^{\mathcal{P}} g|\,\|_{L^2(I^{t'})} \|h\|_{L^2(I^{t'})}\,. \tag{5.13}$$

A similar inequality holds for $T_\sharp^{\mathcal{P}*}$ and $T_\sharp^{\mathcal{P}'*}$.

In the special case of $I^t = I^{t'}$, we have $T^{\mathcal{P}*}T^{\mathcal{P}'} = T^{\mathcal{P}}T^{\mathcal{P}'*} = 0$, which is obvious from the Fourier transform side, and so the norm estimate above is easy. The local inner product estimate, which is essential, contains more information and so the proof is not as easy. But the underlying idea is no different.

PROOF. The interesting case is $I^{t'} \subset I^t$. Let

$$\lambda' = \min\{|I'| : [\omega', I'] \in \mathcal{P}'\}\,,$$

and split $\mathcal{P}$ into the trees $\mathcal{P}^\sharp = \{[\omega, I\,] \in \mathcal{P} : |I| \geq \lambda'\}$ and $\mathcal{P}^\flat = \{[\omega, I\,] \in \mathcal{P} : |I| < \lambda'\}$. A technical point hinges on the fact that a pair $[\omega, I\,] \in \mathcal{P}^\flat$ could satisfy $I \cap I^{t'} = \varnothing$ and yet $\mu(I) \supset I^{t'}$. We treat the point by redefining $T_{\sharp p}$.

Recalling (5.8), $T_\sharp^{\mathcal{P}}$ is defined in terms of the intervals $\mu(I)$. Instead, define

$$\check{\mu}(I) = \Big(\frac{b}{A}\Big)^{100} \sqrt{\frac{|I^{t'}|}{|I|}}\, I\,, \tag{5.14}$$

and

$$\check{T}^{\mathcal{P}^\flat} g = \sum_{p\in\mathcal{P}^\flat} \zeta_{\check{\mu}(I)} T_p(\zeta_{\check{\mu}(I)} g)\,.$$

Then a trivial adaptation of Lemma 5.9 shows that

$$|T_\sharp^{\mathcal{P}^\flat} g(x) - \check{T}^{\mathcal{P}^\flat} g(x)| \le C\Big(\frac{b}{A}\Big)^{500} Mg(x)\,,$$

and so to prove the lemma it suffices to replace $T_\sharp^{\mathcal{P}^\flat}$ by $\check{T}^{\mathcal{P}^\flat}$. Now observe that

$$\langle \check{T}^{\mathcal{P}^\flat} g, T_\sharp^{\mathcal{P}'} g\rangle = \sum_{[\omega,I]\in\mathcal{P}} \sum_{[\omega',I']\in\mathcal{P}'} [\check{\mu}(I)\cap\mu(I')\ne\varnothing]\, \langle \check{T}_p g, T_{\sharp p'} h\rangle\,.$$

But now, for $[\omega, I\,] \in \mathcal{P}$ and $[\omega', I'] \in \mathcal{P}'$, part γ) of the definition of separated shows that if $\check{\mu}(I)\cap\mu(I')\ne\varnothing$, then $I\subset I^{t'}$. This is because $|I|<|I'|$, and so $|\check{\mu}(I)|<|\mu(I')|$. So

$$\begin{aligned}
\operatorname{dist}(I,I') &\le 2\,|\mu(I')| \\
&\le \Big(\frac{b}{A}\Big)^{100} \sqrt{|I^{t'}|\,|I'|} \\
&\le \Big(\frac{b}{A}\Big)^{20000} |I^{t'}| \\
&\le \operatorname{dist}(I', \partial I^{t'})\,,
\end{aligned}$$

by the definition of normality. But then $I\subset I^{t'}$.

With this done, we can assume that for all $[\omega, I\,]\in\mathcal{P}^\sharp$, $\mu(I)\cap I^{t'}\ne\varnothing$, and for all $[\omega, I\,]\in\mathcal{P}^\flat$, we have $I\subset I^{t'}$. Then, let

$$\lambda = \min\{|I'| : [\omega', I'] \in \mathcal{P}'\cup\mathcal{P}^\flat\}.$$

By an abuse of notation, we will write $T^{\mathcal{P}} = T_\sharp^{\mathcal{P}^\sharp} + \check{T}^{\mathcal{P}^\flat}$ for the purposes of this proof.

From the definition of separated it follows that for all $[\omega, I] \in \mathcal{P}$ and $[\omega', I'] \in \mathcal{P}'$

$$\operatorname{dist}(\omega, \omega') \geq \Big(\frac{A}{b}\Big)^{3000} \lambda^{-1} := \Big(\frac{A}{b}\Big)^{500} D\,.$$

Here $D = (A/b)^{2500}/\lambda \geq (A/b)^{40000}|I^{t'}|$. We shall see that the lemma is true because $\widehat{T^{\mathcal{P}}}$ and $\widehat{T_\sharp^{\mathcal{P}'}}$ live on disjoint sets.

Let c and c' denote respectively the centers of ω^t and $\omega^{t'}$. Recalling that $\widehat{T_p}$ is supported on $A^*(\omega)$ which is centered at $2\,c(\omega) + |\omega|/3$, let $\Phi(x)$ be a function with

$$\operatorname{supp}(\Phi) \subset \Big[-\frac{1}{D}, \frac{1}{D}\Big]\,, \tag{5.15}$$

$$|\check{\Phi}(\xi) - 1| \leq C\Big(\frac{|\xi - 2\,c'|}{D}\Big)^{-50}\,, \tag{5.16}$$

$$\|\Phi(x)\|_1 \leq C\,, \tag{5.17}$$

and

$$\check{\Phi}(2\,c) = 0\,. \tag{5.18}$$

Since $|c - c'|$ is so large, there is no problem accommodating this last condition.

Write

$$\mathcal{E}'h = T_\sharp^{\mathcal{P}'} h - \Phi * T_\sharp^{\mathcal{P}'} h\,.$$

Note that as D^{-1} is so small, the definition of normal and (5.15) imply that $\mathcal{E}'h$ is supported in $I^{t'}$. Further, we can write

$$\langle T^{\mathcal{P}} g, T_\sharp^{\mathcal{P}'} h\rangle = \langle T^{\mathcal{P}} g, \Phi * T_\sharp^{\mathcal{P}'} h\rangle + \langle T^{\mathcal{P}} g, \mathcal{E}'h\rangle = \mathcal{A} + \mathcal{B}\,.$$

Our first claim is that

$$|T^{\mathcal{P}*}(\Phi * H)(x)| \leq C\Big(\frac{b}{A}\Big)^{500} \chi * |H|(x)\,, \tag{5.19}$$

where $\chi(x) = D\,(1 + D|x|)^{-3/2}$. From this, it follows that

$$\begin{aligned}
|\mathcal{A}| &= |\langle g, T^{\mathcal{P}*}(\Phi * T_\sharp^{\mathcal{P}'} h)\rangle| \\
&\leq C\Big(\frac{b}{A}\Big)^{500} \langle |g|, \chi * |T_\sharp^{\mathcal{P}'} h|\rangle \\
&= C\Big(\frac{b}{A}\Big)^{500} \langle \chi * |g|, |T_\sharp^{\mathcal{P}'} h|\rangle \\
&\leq C\Big(\frac{b}{A}\Big)^{500} \|Mg\|_{L^2(I^{t'})}\, \|h\|_{L^2(I^{t'})}\,.
\end{aligned}$$

This is the principal estimate.

The second claim is that

$$\|\mathcal{E}'\|_2 \le C\Big(\frac{b}{A}\Big)^{500}, \tag{5.20}$$

hence, as $\operatorname{supp}\mathcal{E}'h \subset I^{t'}$, and $\mathcal{E}'h = \mathcal{E}'(h\mathbf{1}_{I^{t'}})$,

$$|\mathcal{B}| \le \langle |T^{\mathcal{P}}g|, |\mathcal{E}'h|\rangle \le C\Big(\frac{b}{A}\Big)^{500} \|T^{\mathcal{P}}g\|_{L^2(I^{t'})}\, \|h\|_{L^2(I^{t'})}\,.$$

The estimates on $\mathcal{A}$ and $\mathcal{B}$ prove the Lemma.

We turn to the proof of the two claims (5.19) and (5.20). There is no harm in assuming that ω^t is centered at the origin. See (2.15). For the proof of (5.19), write

$$T^{\mathcal{P}*}H(x) = \int K(x,y)\, H(y)\, dy\,.$$

As the cut-off function ζ in (5.7) is assumed to be smooth, $K(x,y)$ is a generalized Calderón-Zygmund kernel, and in particular satisfies the gradient condition (5.3) above, with b in that inequality replaced by 1. (Recall that we are not making any assumption about the size of pairs in the current Lemma). Then $g \longrightarrow T^{\mathcal{P}*}(\Phi * g)$ has kernel

$$\int K(x,z)\, \Phi(z-y)\, dz\,.$$

Note that $\int \Phi = 0$, due to (5.18), and our assumption that $c=0$; also the support of Φ is small, see (5.15). There are two estimates to be made. On the one hand, with the assumption $|I| > \lambda = (A/b)^{2500}/D$ for all $p \in \mathcal{P}$, we see that

$$\Big|\frac{d}{dz}K(x,z)\Big| \le C\,\lambda^{-2} = C\Big(\frac{b}{A}\Big)^{5000} D^2\,, \qquad \text{for all } x \text{ and } z\,.$$

Consequently using (5.15) and $\int \Phi\, dx = 0$,

$$\begin{aligned}\Big|\int K(x,z)\,\Phi(z-y)\,dz\Big| &= \Big|\int_{|z-y|\le 2/D} (K(x,z)-K(x,y))\,\Phi(z-y)\,dz\Big| \\ &\le \Big(\frac{b}{A}\Big)^{2500} D\,.\end{aligned}$$

On the other hand, if $|x-y|>4/D$ use (5.3) and (5.17) to see that

$$\begin{aligned}\Big|\int K(x,z)\,\Phi(z-y)\,dz\Big| &= \Big|\int (K(x,z)-K(x,y))\,\Phi(z-y)\Big|\,dz\\ &\leq C\,|x-y|^{-2}\int_{|z-y|\leq 2/D}|z-y|\,|\Phi(z-y)|\,dz\\ &\leq C\,D^{-1}\,|x-y|^{-2}\,.\end{aligned}$$

Notice that

$$D^{-1}|x|^{-2}\leq\Big(\frac{b}{A}\Big)D\,(1+D\,|x|)^{-3/2}\,,\qquad \text{if } D\,|x|\geq\Big(\frac{A}{b}\Big)^{1000}\,,$$

$$D\,\Big(\frac{b}{A}\Big)^{2000}=\min_{D|x|\leq(A/b)^{1000}}\Big(\frac{b}{a}\Big)^{500}D\,(1+|x|)^{3/2}\,,$$

so that (5.19) follows.

For the second claim (5.20), verify the dual inequality

$$\|T_\sharp^{\mathcal{P}'*}(h-\Phi*h)\|_2\leq C\Big(\frac{b}{A}\Big)^{500}\|h\|_2\,.$$

But we can estimate by (5.10), Lemma 5.1, (2.14) and (5.16) to see that

$$\begin{aligned}&\|T_\sharp^{\mathcal{P}'*}(h-\Phi*h)\|_2\\ &\qquad\leq C\Big(\frac{b}{A}\Big)^{1000}\|M(h-\Phi*h)\|_2+\|T^{\mathcal{P}'*}(h-\Phi*h)\|_2\\ &\qquad\leq C\Big(\frac{b}{A}\Big)^{1000}\|h\|_2+C\Big\|\mathbf{1}\Big[2\,c'-\frac{3}{\lambda},2\,c'+\frac{3}{\lambda}\Big]\,\mathcal{F}(h-\Phi*h)\Big\|_2\\ &\qquad\leq C\Big(\frac{b}{A}\Big)^{1000}\|h\|_2\,.\end{aligned}$$

The case of the adjoints being similar, we have completed the proof of the Lemma.

A point which will come up several times below is that the intervals I_p can range over the whole of the real line. (This was also an issue in the preceeding section). Here, let us note that if $\mathcal{P}$ is a tree with top t, then

$$\text{(5.21)}\qquad \operatorname{size}(\mathcal{P})\geq b\qquad\text{implies}\qquad I^t\subset b^{-\varepsilon}[-1,1]\,,$$

as easily follows from the decay of φ_p.

The emphasis of the next group of Lemmas is on forming large unions of trees.

Rows. Define a *row* to be a union of normal trees $\mathcal{P}_j$ with tops $p_j^t = [\omega_j^t, I_j^t]$ for which the sets $\{I_j^t\}$ are pairwise disjoint. Two rows $\mathcal{R} = \bigcup_j \mathcal{P}_j$ with tops $[\omega_j^t, I_j^t]$, and $\mathcal{R}' = \bigcup_j \mathcal{P}'_{j'}$ with tops $[\omega_j^{t\,\prime}, I_{j'}^{t\,\prime}]$ are *separated* if each $I_{j'}^{t\,\prime}$ is contained in some I_j^t, where $\mathcal{P}_j$ and $\mathcal{P}'_{j'}$ are separated.

Lemma 5.22. *Let $\mathcal{R}$ be a row with* $\operatorname{size}(\mathcal{R}) \le b$. *Then*

$$\|T_\sharp^{\mathcal{R}}\|_r \le C_r\, b\,, \qquad 1 < r < \infty\,. \tag{5.23}$$

If $\mathcal{R}$ and $\mathcal{R}'$ are separated rows then

$$\|T_\sharp^{\mathcal{R}} T_\sharp^{\mathcal{R}'*}\|_2, \|T_\sharp^{\mathcal{R}*} T_\sharp^{\mathcal{R}'}\|_2 \le C\Big(\frac{b}{A}\Big)^{500}. \tag{5.24}$$

PROOF. As the operators $T_\sharp^{\mathcal{R}}$ act coordinatewise on $\oplus_j L^r(I_j^t)$, the first assertion follows from Lemma 5.9.

And for the second assertion, again due to the coordinatewise action of the operators, it suffices to consider the case where $\mathcal{R}$ is in fact a tree $\mathcal{P}$ with top $[\omega^t, I^t]$, and each $I_{j'}^{t\,\prime}$ is contained in I^t. Then, using (5.13) and Cauchy-Schwartz,

$$\begin{aligned}
|\langle T_\sharp^{\mathcal{P}} g, T_\sharp^{\mathcal{R}'} h\rangle| &= \sum_{j'} |\langle T_\sharp^{\mathcal{P}} g, T_\sharp^{\mathcal{R}'_{j'}} h\rangle| \\
&\le C\Big(\frac{b}{A}\Big)^{500} \sum_{j'} \|Mg + |T_\sharp^{\mathcal{P}} g|\|_{L^2(I_{j'}^{t\,\prime})}\, \|h\|_{L^2(I_{j'}^{t\,\prime})} \\
&\le C\Big(\frac{b}{A}\Big)^{500} \|g\|_{L^2(I^t)}\, \|h\|_{L^2(I^t)}\,.
\end{aligned}$$

The last Lemma of this section provides an estimate for a large number of rows.

Lemma 5.25. *Let $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_N$ be pairwise separated rows where the number of rows N is at most $C\,(A/b)^{220}$. If* $\operatorname{size}(\mathcal{R}) \le b$, *then, for $1 < r < 2$,*

$$\|T^{\mathcal{R}} g\|_r \le C_r\, b\, \|g\|_2\,.$$

Proof. Cotlar's Lemma, and Lemma 5.22 provide the estimate

$$\Big\| \sum_{n=1}^{N} T_{\sharp}^{\mathcal{R}_n} g \Big\|_2 \le C\, b\, \|g\|_2 \,.$$

To finish the Lemma, we need to remove the sharp above. We do this by assuming that for each tree $\mathcal{T}$ in a row, we have $\operatorname{size}(\mathcal{T}) \ge b/2$, so that (5.21) is in force. Denote the top space intervals of the trees in the n-th row by $\{I_{n,k} : k \ge 1\}$. These intervals are disjoint in k. For a choice of $1 < r < 2$, let $1/s + 1/2 = 1/r$. We have by Lemma 5.9,

$$\begin{aligned}
\Big\| \sum_{n=1}^{N} |T_{\sharp}^{\mathcal{R}_n *} g - T^{\mathcal{R}_n *} g| \Big\|_r &\le C \Big(\frac{b}{A}\Big)^{1000} \Big\| \sum_{n=1}^{N} \sum_{k=1}^{\infty} (M \mathbf{1}[I_{n,k}])^2 \Big\|_s^2 \|Mg\|_2 \\
&\le C_r \Big(\frac{b}{A}\Big)^{1000} \Big\| \sum_{n=1}^{N} \sum_{k=1}^{\infty} \mathbf{1}[I_{n,k}] \Big\|_s^2 \|g\|_2 \\
&\le C_r \Big(\frac{b}{A}\Big)^{100} \|g\|_2 \,.
\end{aligned}$$

6. Orchards.

An important point left uncovered in the previous section is the behavior of the α and $\alpha *$ trees. These operators are in essence paraproducts, and can lead to operators of large norm, regardless of how small their constituent parts are. An adequate control of these objects requires a new Carleson measure estimate, which is the subject of this section.

Let $\mathcal{P}_i$, $i \ge 1$, be α-trees with tops $p_i^t = [\omega_i^t, I_i^t]$. Assume

i) For $i \ne i'$, $\mathcal{P}_i$ and $\mathcal{P}_{i'}$ are disjoint.

ii) For an absolute constant $c_1 = (12\, C_*)^{-1}$, with C_* as in (5.5),

$$c_1\, b^2\, |I_i^t| \le \sum_{p \in \mathcal{P}_i} |\langle f, \phi_p \rangle|^2 \,.$$

iii) For all $p \in \bigcup_i \mathcal{P}_i$, $\operatorname{size}(p) \le b^{30}$.

Such a collection of α-trees we will refer to as an *α-orchard.* An analogous collection of $\alpha*$-trees we will refer to as an *$\alpha*$-orchard.*[10]

Of these three conditions, the first is a modest regularity condition. The second is easily seen to be related to the norm of the tree operator $T^{\mathcal{P}_i}$. In particular, for a α-tree $\mathcal{P}$, let

$$\lambda^2 = \sup_U \frac{1}{|U|} \sum_{p=[\omega, I\,]\in\mathcal{P}} [\, I \subset U\,]\, |\langle f, \phi_p\rangle|^2 \,.$$

Then, our analysis from the previous section, and in particular (5.5), shows that

$$d_1 \lambda \le \|T^{\mathcal{P}}\|_2 \le D_1 \lambda \,.$$

Thus the trees in an orchard have a minimal size. The last condition iii) is in contrast to the analysis of say Section 4, in which a lower bound on pairs is imposed. Yet, in the context of α-trees, the interesting case is when the sizes of individual pairs are all quite small.

An essential fact about α-orchards is that the top intervals I_i^t obey a Carleson measure estimate much like the one established for sets of incomparable pairs.

Lemma 6.1. *There is a $b_0 > 0$ so that for all $0 < b < b_0$, and any α-orchard $\mathcal{O} = \bigcup_i \mathcal{P}_i$ as above, and all intervals U,*

$$\sum_i [\, I_i^t \subset U\,]\, |I_i^t| \le C\, b^{-2}\, |U| \,. \tag{6.2}$$

The rest of this section is taken up with the proof of this inequality. The general approach is to identify trees $\mathcal{P}_i' \subset \mathcal{P}_i$, which still satisfy ii) above, with the additional property that the functions $\{\phi_p : p \in \bigcup_i \mathcal{P}_i'\}$ form a basis in L^2. Applying this basis to our function $f(x)$ will complete the proof.

Recall the combinatorial structure of an α-tree $\mathcal{P}$ with top frequency interval ω^t. Each pair $[\omega, I\,] \in \mathcal{P}$ has $c(\omega^t) \in A(\omega)$. These intervals are central triadic intervals. From (2.12), we have $c(\omega) \simeq c(\omega^t) - |\omega|/3$. And so the sets $\Phi(\omega)$ of (2.9) form a lacunary sequence of intervals, from which one would have Littlewood-Paley estimates.

[10] Because condition iii) is an upper bound, any sufficently small value of c_2 is acceptable.

(See (6.3) below). Also, recalling (3.2), we have that the functions $\phi_{[\omega,I]}$ and $\phi_{[\omega,J]}$ are orthogonal, provided $I \neq J$.

The interval U in (6.2) can be fixed, and we assume that $I_i^t \subset U$ for all i. A formally weaker statement will imply (6.2). Namely, there is an open set $F \subset U$ so that

α) $|F| \le |U|/4$.

β) $\sum_i [\, I_i^t \not\subset F\,]\, |I_i^t| \le C\, b^{-2}\, |U|$.

We then appeal to Lemma 4.23 to prove (6.2). Yet, in proving this weaker statement, we can further impose the assumption

δ) $\| \sum_i [\, I_i^t \subset U\,]\, \mathbf{1}_{I_i^t}(x) \|_\infty \le b^{-22}$.

To be quite specific, the statement to be proved is this: given c_1 in ii), we can choose b_0 so that for every $0 < b < b_0$ and every orchard which satisfies δ) also satisfies α) (with $1/4$ replaced by $1/2$) and β) (with a possibly larger constant).

Let us now argue that there is no loss of generality in assuming δ). For this, we take an arbitrary orchard, $\mathcal{O}$ and select appropriate subcollection of the trees $\mathcal{P}_i$ which satisfy δ). The subcollection should be taken in this way. Denote by $I_{j(1,v)}^t$, for $v \ge 1$, the maximal elements from among $\{I_i^t : i \ge 1\}$. Remove the intervals $I_{j(1,v)}^t$ from the list, and and again take the maximal intervals $I_{j(2,v)}^t$, for $v \ge 1$. Repeat this procedure until the orchard is exhausted. The α-trees $\{\mathcal{P}_{j(k,v)} : 1 \le k \le K = b^{-22},\ v \ge 1\}$ satisfy δ). Assume α) and β) also hold. Set

$$E_0 = \Big\{ x : \sum_{k=1}^{\infty} \sum_{v=1}^{\infty} \mathbf{1}[\, I_{j(k,v)}^t\,](x) \ge (2\, b)^{-22} \Big\}\,,$$

that is, this set is the set on which δ) fails. We see that

$$\begin{aligned} |E| &\le \Big| \Big\{ x : \sum_{k=1}^{K} \sum_{v=1}^{\infty} \mathbf{1}[\, I_{j(k,v)}^t\,](x) = b^{-22} \Big\} \Big| \\ &\le |F| + b^{22} \sum_{k=1}^{K} \sum_{v=1}^{\infty} [\, I_{j(k,v)}^t \not\subset F\,]\, |I_{j(k,v)}^t| \\ &\le \Big(\frac{1}{4} + C\, b^{-20} \Big) |U|\,. \end{aligned}$$

This will be less than $|U|/2$, provided b is small enough. Thus, a slightly weaker form of α) holds. Taking $E = E_0 \cup F$, we see that

α') $|E| \leq |U|/2$

and that β) holds for the orchard $\mathcal{O}$ with the set F replaced by the set E. Again, by Lemma 4.23, this is enough to prove (6.2).

Assume δ). We turn our attention to the deletion of certain small sets of pairs. First we can assume each tree $\mathcal{P}_i$ is finite, without violating our condition ii). Second, for any α-tree $\mathcal{P}$, we have the fact that

$$\Big(\sum_{p\in\mathcal{P}} |\phi_p \otimes \phi_p|^2\Big)^{1/2} : L^\infty \longrightarrow \mathrm{BMO}\,. \tag{6.3}$$

By observing that the supports of the functions ϕ_p in frequency form a lacunary disjoint sequence, the operation above is seen to be an ordinary Littlewood-Paley square function, albeit conjugated by an exponential to account for the location of the tree in frequency, and then the bound above is immediate.

With this observation, we can delete pairs in $\mathcal{P}_i$ which fall close to the boundary of I_i^t. Specifically, set

$$\mathcal{P}_i^\partial = \{[\omega, I\,] \in \mathcal{P}_i : \mathrm{dist}(I, \partial I_i^t) \leq b^{100}\,|I_i^t|\}\,. \tag{6.4}$$

It follows that

$$\sum_{p\in\mathcal{P}_i^\partial} |\langle f, \phi_p\rangle|^2 \leq C\, b^{100}\,|I_i^t|\,.$$

Therefore, after removal of this set of pairs ii) holds with a slightly smaller constant.

The top of $\mathcal{P}_i$ must be removed. Namely, say that $p \in \mathcal{P}_i^t$ if there is no chain

$$p = p_1 \precneqq p_2 \precneqq \cdots \precneqq p_M\;,$$

with $M = 10{,}000\log(b/\mathrm{size}(p))$, and all $p_m \in \mathcal{P}_i$. Then it follows that

$$\sum_{p\in\mathcal{P}_i^t} |\langle f, \phi_p\rangle|^2 \leq b^5\,|I_i^t|\,,$$

due to the condition iii) above. Thus, for b sufficently small, these pairs can also be removed from $\mathcal{P}_i$ with only a minimal weakening of ii). Set $\mathcal{P}^t = \bigcup_i \mathcal{P}_i^t$.

The import of these last two conditions is that for $[\omega, I\,] \in \mathcal{P}_i\backslash(\mathcal{P}_i^\partial \cup \mathcal{P}_i^t)$,

$$|I| \leq (b\,\mathrm{size}(p))^{5000}\,|I_i^t|\,. \tag{6.5}$$

We arrive at a critical point, which centers on the relationship between distinct trees. Consider two top intervals I_i^t and $I_{i'}^t$ which intersect. Consider the following subtree of $\mathcal{P}_i$.

$$\begin{aligned} \mathcal{Q}_{i,i'} = \{[\omega, I\,] \in \mathcal{P}_i : I \subset I_{i'}^t, \\ \Phi(\omega') \subsetneq \Phi(\omega) \text{ for some } [\omega', I'] \in \mathcal{P}_{i'}\}\,. \end{aligned} \tag{6.6}$$

Recall that our objective is to identify a highly orthogonal set of functions ϕ_p. Since $\operatorname{supp}(\widehat{\phi_p}) \subset \Phi(\omega)$, the sets of pairs above are certainly a cause of concern.[11] But our claim is that $\mathcal{Q}_{i,i'}$ can only admit chains of bounded length in the partial order '$<$'. In particular, there are no ten pairs

$$p_1 \lneq p_2 \lneq \cdots \lneq p_{10}\ ,$$

with all $p_k = [\omega_k, I_k] \in \mathcal{Q}_{i,i'}$ for $1 \le k \le 10$. For assuming otherwise, the sets $A(\omega_k)$ are central, and $A(\omega_{10}) \subsetneq \cdots \subsetneq A(\omega_1)$. By the good property of centrality, (3.1),

$$\operatorname{dist}(A(\omega_{10}), \partial A(\omega_1)) \ge 3^9\,|A(\omega_{10})|\,.$$

Recall that $A(\omega_{10}) \subset \omega_{10}$, and $|A(\omega_{10})| = |\omega_{10}|/9$. Hence, $9\,\omega_{10} \subset A(\omega_1)$. Yet, $c(\omega_{i'}^t) \in p_{10} \in \mathcal{O}_{i,i'}$ and so for some $p' \in \mathcal{P}_{i'}$, $\Phi(\omega') \subsetneq \Phi(\omega_{10})$, and so $A(\omega') \subset \omega_{10} \subset A(\omega_1)$. This means that $p_1 \in \mathcal{P}_i$ is in the α-tree with top $[\omega_{i'}^t, I_{i'}^t]$. But recall that in the α-tree $\mathcal{P}_{i'}$, the intervals $\{\Phi(\omega') : [\omega', I'] \in \mathcal{P}_{i'}\}$ are lacunary. As a consequence, for any $[\omega', I'] \in \mathcal{P}_{i'}$, the intervals $\Phi(\omega')$ and $\Phi(\omega_1)$ are either equal or disjoint. This contradicts the assumption that $p_1 \in \mathcal{Q}_{i,i'}$.

Thus, this set of pairs cannot contain chains of length ten, as claimed. Using the upper bound on the size of pairs, iii), and the tree structure of $\mathcal{P}_i$, we see that

$$\sum_{p \in \mathcal{Q}_{i,i'}} |\langle f, \phi_p\rangle|^2 \le 10\, b^{30}\, |I_{i'}^t \cap I_i^t|\,.$$

Set $\mathcal{Q}_i = \bigcup_{i'} \mathcal{Q}_{i,i'}$. Using the assumption δ), we see that

$$\sum_{p \in \mathcal{Q}_i} |\langle f, \phi_p\rangle|^2 \le 10\, b^8\, |I_i^t|\,.$$

---

[11] Recall that $\phi_{[\omega, I\,]}$ and $\phi_{[\omega, J]}$ are orthogonal for $I \ne J$, due to (3.2). In $\mathcal{Q}_{i,i'}$, we need only concern ourselves with the case of ω' being a strict subset of ω.

This last estimate will be quite small. The pairs in $\mathcal{Q}_i$ can be deleted without affecting ii).

To summarize, we can without loss of generality assume that the sets $\mathcal{P}_i^\partial$, $\mathcal{P}_i^t$, and $\mathcal{Q}_i$ are empty for all i. For if they are not, we remove the pairs in these sets, and (a trivial weakening of) ii) above continues to hold. The essential advantage gained by these manipulations is as follows. For $p = [\omega, I\,] \in \mathcal{O}$, set

$$\mathcal{O}(p) = \{p' = [\omega', I'] \in \mathcal{O} : \Phi(\omega) \subsetneq \Phi(\omega')\}\,. \tag{6.7}$$

That is $\mathcal{O}(p)$ contains all pairs p' for which $|I'| < |I|$ and $\phi_{p'}$ is not orthogonal to ϕ_p. Then

$$\begin{aligned}&\{I' : [\omega', I'] \in \mathcal{O}(p)\} \text{ are pairwise disjoint}\\ &\text{and contained in } [(b\,\mathrm{size}(p))^{-500} I]^c\,.\end{aligned} \tag{6.8}$$

To see this claim, it is enough to verify that if $p' = [\omega', I'] \in \mathcal{O}$ is such that $\Phi(\omega) \subsetneq \Phi(\omega')$, then

$$\mathrm{dist}(I, I') \geq (b\,\mathrm{size}(p))^{-500}\,|I|\,. \tag{6.9}$$

This clearly proves the last half of (6.8). Yet it also proves disjointness, as is easily seen: consider $p_1 \neq p_2 \in \mathcal{O}(p)$, with $p_i = [\omega_i, I_i]$, for $i = 1, 2$. We want to show that $I_1 \cap I_2 = \varnothing$. The frequency intervals both contain $\Phi(\omega)$, and hence $\omega_1 \cap \omega_2$ is not empty. If $\omega_1 = \omega_2$, it is clear that I_1 and I_2 are disjoint. And otherwise, (6.9) shows that I_1 and I_2 are disjoint.

We establish (6.9). As $\Phi(\omega) \subsetneq \Phi(\omega')$, the α-tree structure dictates that the two pairs are in distinct trees: $p \in \mathcal{P}_i$, and $p' \in \mathcal{P}_{i'}$ with $i \neq i'$. The first case is $I' \not\subset I_i^t$. For then, it follows from the removal of the sets $\mathcal{P}_i^\partial$ and $\mathcal{P}_i^t$, that

$$\begin{aligned}\mathrm{dist}(I, I') &\geq \mathrm{dist}(I, \partial I_i^t)\\ &\geq b^{100}\,\frac{|I_i^t|}{|I|}\,|I|\\ &\geq (b\,\mathrm{size}(p))^{-1000}\,|I|\,.\end{aligned} \tag{6.10}$$

Recall (6.4) and (6.5). This is stronger than (6.9).

Thus we can assume that $I' \subset I_i^t$. But then, reversing primes in (6.6), $p' \in \mathcal{Q}_{i',i}$, a set of pairs that has been removed. The verification of (6.9) is complete.

The combinatorial work of the proof is complete. We have a set of pairs $\mathcal{O}$ which satisfy (6.8). This condition is a rather strong incomparability condition. Hence the final portion of the argument is a reprise of the techniques of Section 4, and in particular the proof of Lemma 4.4.

Our objective is to establish the following inequality. Fix an interval U, and assume $I_i^t \subset U$ for all i. For all bounded functions g supported on $2\,U$, and choices of signs $\{\varepsilon_p : p \in \mathcal{O}\}$,

$$\Big\| \sum_{p\in\mathcal{O}} \varepsilon_p\, \phi_p(x)\, \langle g, \phi_p\rangle \Big\|_2 \le C\sqrt{|U|}\;\|g\|_\infty^2\,. \tag{6.11}$$

To establish it, we can assume that $\mathcal{O}$ is finite, and therefore the inequality must hold with some finite constant on the right hand side. Let B denote the best constant in this assumed inequality; an upper bound on B can be given.

Set $S_p = \phi_p \otimes \phi_p$, then

$$\Big\| \sum_{p\in\mathcal{O}} \varepsilon_p\, \phi_p\, \langle g, \phi_p\rangle \Big\|_2^2 = \Big\| \sum_{p\in\mathcal{O}} \varepsilon_p\, S_p g \Big\|_2^2 = \mathcal{D} + \mathcal{O}\,, \tag{6.12}$$

where $\mathcal{D}$ and $\mathcal{O}$ are the diagonal and off-diagonal terms respectively.

In the diagonal, we can use the fact that S_p is a self-adjoint projection to write

$$\begin{aligned} \mathcal{D} &= \sum_{p\in\mathcal{O}} \langle S_p g, S_p g\rangle \\ &= \Big\langle \sum_{p\in\mathcal{O}} S_p g, g \Big\rangle \\ &\le \Big\| \sum_{p\in\mathcal{O}} S_p g \Big\|_2 \|g\|_2 \\ &\le B\sqrt{|U|}\;\|g\|_\infty^2\,. \end{aligned} \tag{6.13}$$

This estimate employs the assumed inequality (6.11), with best constant, together with the fact that g is supported on $2\,U$.

In the off-diagonal term, we have, recalling the notation $\mathcal{O}(p)$ of (6.7),

$$\begin{aligned} \mathcal{O} &\le 2 \sum_{p\in\mathcal{O}} \sum_{p'\in\mathcal{O}(p)} |\langle S_p g, S_{p'} g\rangle| \\ &\le 2 \sum_{p\in\mathcal{O}} |\langle g, \phi_p\rangle| \sum_{p'\in\mathcal{O}(p)} |\langle \phi_p, \phi_{p'}\rangle\, \langle g, \phi_{p'}\rangle| \end{aligned} \tag{6.14}$$

Denote the inner sum by $\mathcal{S}_p$. Recall the estimate

$$|\langle \phi_{[\omega,I]}, \phi_{[\omega',I']}\rangle| \le \begin{cases} 0\,, & \text{if } \omega \cap \omega' = \varnothing\,, \\ C_\varepsilon \sqrt{\dfrac{|I'|}{|I|}} \Big(1 + \dfrac{\operatorname{dist}(I,I')}{|I|}\Big)^{-20/\varepsilon}\,, & \text{if } \omega \subset \omega'\,. \end{cases}$$

We noted this in Section 4, and it is easy to verify. Use it in the estimate of $\mathcal{S}_p$.

$$\begin{aligned} \mathcal{S}_p &= \sum_{p' \in \mathcal{O}(p)} |\langle \phi_p, \phi_{p'}\rangle \langle g, \phi_{p'}\rangle| \\ &\le C\, \|g\|_\infty \sum_{[\omega',I'] \in \mathcal{O}(p)} \frac{1}{\sqrt{|I|}} \Big(1 + \frac{\operatorname{dist}(I,I')}{|I|}\Big)^{-20} |I'|\,. \end{aligned}$$

At this point, the essential ingredient from the first half of the proof enters in. Namely, using (6.8), we can continue the estimate of $\mathcal{S}_p$ as follows. The intervals $\{I' : [\omega', I'] \in \mathcal{O}(p)\}$ are pairwise disjoint and contained in the complement of $\tilde{I} = (b\,\operatorname{size}(p))^{-100} I$. Hence, the sum above can be dominated by

$$\begin{aligned} \mathcal{S}_p &\le C\, \|g\|_\infty \int_{\tilde{I}^c} \frac{1}{\sqrt{|I|}} \Big(1 + \frac{\operatorname{dist}(I,x)}{|I|}\Big)^{-20} dx \\ &\le C\, (b\,\operatorname{size}(p))^{100} \sqrt{|I|}\; \|g\|_\infty\,. \end{aligned}$$

Placing this estimate into that for the off-diagonal, (6.14), we get

$$\begin{aligned} \mathcal{O} &\le C\, \|g\|_\infty \sum_{p = [\omega, I] \in \mathcal{O}} (b\,\operatorname{size}(p))^{100} |\langle g, \phi_p\rangle| \sqrt{|I|} \\ &\le C\, \|g\|_\infty^2\, b^{400} \sum_{p \in \mathcal{O}} |\langle f, \phi_p\rangle|^2\,, \end{aligned}$$

with the last line following from iii) at the beginning of this section.

We collect estimates. Namely, the last display together with (6.12) and (6.13), to see that

$$B^2 \le C\Big(B\sqrt{|U|} + b^{400} \sum_{p \in \mathcal{O}} |\langle f, \phi_p\rangle|^2\Big)\,.$$

If the first term is the larger of the two on the right, then we see that the best constant B is no more than $C\sqrt{|U|}$, which is a perfectly adequate estimate.

If the second term is the larger of the two, a contradiction is seen. We can apply the inequality to $f\,\mathbf{1}_{2U}$. In particular,

$$\Big\| \sum_{p\in\mathcal{O}} \varepsilon_p\,\phi_p\,\langle f\,\mathbf{1}_{2U},\phi_p\rangle \Big\|_2 \le C\Big(b^{400}\sum_{p\in\mathcal{O}} |\langle f,\phi_p\rangle|^2\Big)^{1/2}, \tag{6.15}$$

since f is bounded by 1. It follows from the removal of the tops $\mathcal{P}_i^t$, and in particular (6.5), that

$$\begin{aligned} |\langle f,\phi_p\rangle - \langle f\,\mathbf{1}_{2U},\phi_p\rangle| &\le \|f\|_\infty\, \|\mathbf{1}[(2\,U)^c]\,\phi_p(x)\|_1 \\ &\le (b\,\mathrm{size}(p))^{10}\,\sqrt{|I|} \\ &\le \frac{1}{100}\,|\langle f,\phi_p\rangle|\,\sqrt{|I|}\,. \end{aligned} \tag{6.16}$$

Hence, we can average over choices of signs in (6.15), to see that

$$\sum_{p\in\mathcal{O}} |\langle f,\phi_p\rangle|^2 \le C\,b^{400} \sum_{p\in\mathcal{O}} |\langle f,\phi_p\rangle|^2\,,$$

which can only hold if $b \ge C'$, an absurdity.

The final touch in the proof our Lemma is short and sweet. We have established (6.11); apply the inequality to f, averaging over choices of signs. Noting (6.16), we see that

$$\sum_{p\in\mathcal{O}} |\langle f,\phi_p\rangle|^2 \le C\,|U|\,.$$

Yet, in the combinatorial half of the proof, we were careful to preserve the lower half of condition ii) at the beginning of this section, and thus, as the Lemma claims,

$$\sum_i [\, I_i^t \subset U\,]\,|I_i^t| \le C\,b^{-2}\,|U|\,.$$

7. Forests.

In this section, we combine the principal estimates of the previous three sections, and complete the proof of the bound for the bilinear Hilbert transform. We begin with a definition of a collection of pairs, a forest.

Call a set of pairs $\mathcal{F}$ a *forest* if

α) $\operatorname{size}(\mathcal{F}) \le b$. (See (5.6)).

β) If $p, p', p'' \in \mathcal{P}$, and $p < p', p < p''$ then p' and p'' are comparable, *e.g.* $p' < p''$.

γ) No point x is in more than $J = O\left((A/b)^{210}\right)$ intervals $I_1, \dots, I_J$, where the pairs $[\omega_j, I_j]$ are in $\mathcal{P}$, and mutually incomparable under $<$.

δ) $I_p \subset b^{-40}[-1,1]$ for all $p \in \mathcal{F}$.

The first condition is a natural restraint on the size of of the collection of pairs; the middle condition is a critical combinatorial condition imposing a tree-like structure on the forest; and the next to last condition is used to write a forest as a small number of rows. The last assumption will be satisfied by appealing to (5.21).

Lemma 7.1. *If $\mathcal{F}$ is a forest, then there is a set $E \subset (0,1)$ of measure at most $C(b/A)^{80}$ so that for all $1 < r < 2$, some $\delta > 0$,*

$$\|T^{\mathcal{P}} g\|_{L^r(E^c)} \le C_r\, b^{\delta} (\log A)\, \|g\|_2\,.$$

Two preparatory Lemmas are in order. First of all, our various estimates break down on small subsets, and the next Lemma justifies the deletion of these bad, thin sets.

Lemma 7.2. *Suppose that T is an operator on a finite measure space $(X, \mathcal{A}, \mu)$ so that for some $0 < b < 1$, and all $A > 10$ there is a set $F \subset X$ of measure at most $C\,(b/A)^{20}$ so that*

$$\|Tg\|_{L^r(X\setminus F)} \le C\, b\, A\, \|g\|_{L^2(X)}\,, \qquad r < 2\,.$$

Then,

$$\|Tg\|_r \le C\, \mu(X)^{1/r-1/2}\, b^{1/2}\, \|g\|_2\,.$$

PROOF. It suffices to assume that $\mu(X) = 1$, for the general case follows from this. Let $g \in L^2(X)$ have norm 1, and $\lambda > 0$. We have the estimate

$$\mu(\{Tg > \lambda\}) \leq |F| + \lambda^{-r} \int_{X \setminus F} |Tg|^r \, du \leq C\Big(\Big(\frac{b}{A}\Big)^{20} + (b\, A\, \lambda^{-1})^r\Big) .$$

Minimizing the estimate over A will prove the Lemma.

Deleting small subsets of $(0, 1)$ also requires us to delete sets of pairs which live on these sets. This is the subject of the next Lemma.

Lemma 7.3. *Let $\{I_j : j \geq 1\}$ be a collection of disjoint triadic intervals. Let $\mathcal{B}$ be a set of pairs with* size$(p) \leq b$*, and for all $[\omega, I\,] \in \mathcal{B}$, $I \subset I_j$ for some j. Set $E = \bigcup_j 2\, I_j$. Then for a choice of $\delta = \delta(r) > 0$,*

$$\|T^{\mathcal{B}} g\|_{L^r(E^c)} \leq C_r\, b^\delta\, \|g\|_2 \,, \qquad 1 < r < 2 \,.$$

PROOF. In the proof, we can in addition assume that size$(p) \geq b/2$, for then we can sum the estimate obtained for the sets $\mathcal{B}_n = \{p \in \mathcal{B} : 2^{-n} b \leq \text{size}(p) \leq 2^{-n+1} b\}$, for $n \geq 1$, to get the Lemma as stated.

Let $\overline{p}_k = [\overline{\omega}_k, \overline{I}_k]$ be the maximal pairs in $\mathcal{B}$. Remove the top from $\mathcal{B}$. Namely let $\mathcal{B}^t$ be those pairs in $p \in \mathcal{B}$ for which there is no chain

$$p = p_1 \lneq p_2 \lneq \cdots \lneq p_m \,,$$

with $m > M = 100\,(\log 1/b)$ and $p_1, p_2, \ldots, p_m \in \mathcal{B}$. Then $\mathcal{B}^t$ can be written as a union of $O\,(\log 1/b)$ sets which are not comparable under $<$. Hence, Lemma 4.2 implies

$$\|T^{\mathcal{B}^t} g\|_r \leq C_r\, b^\delta\, \|g\|_2 \,, \qquad 1 \leq r < 2 \,,$$

which is stronger than our conclusion.

Let $\mathcal{B}^\sharp = \mathcal{B} \setminus \mathcal{B}^t$, and set $\mathcal{P}_k = \{p \in \mathcal{B}^\sharp : p < \overline{p}_k\}$. This is a tree, with top interval $\overline{I}_k$ much larger than $|I|$ for all $[\omega, I\,] \in \mathcal{P}_k$. Thus, $T^{\mathcal{P}_k}$ will be quite small off of the set $E = \bigcup_j 2\, I_j$. In particular, choose j so that $\overline{I}_k \subset I_j$, which must exist. Then, one easily sees that

$$|T^{\mathcal{P}_k} g(x)| \leq C\, b^{100}\, (M \mathbf{1}_{\overline{I}_k}(x))^2\, Mg(x) \,, \qquad \text{if } x \notin 2\, I_j \,.$$

The explicit calculation is much in the spirit of Lemma 5.9. Of course $\mathcal{B}^\sharp = \bigcup_k \mathcal{P}_k$, hence for $r<2$, let $1/r = 1/2 + 1/s$, and write

$$\begin{aligned} \|T^{\mathcal{B}^\sharp} g\|_{L^r(E^c)} &\le C\, b^{100} \Big\| Mg \sum_k (M\mathbf{1}_{\overline{I}_k})^2 \Big\|_r \\ &\le C\, b^{100}\, \|Mg\|_2 \Big\| \sum_k (M\mathbf{1}_{\overline{I}_k})^2 \Big\|_s^2 \\ &\le C\, b^{100}\, \|g\|_2 \Big\| \sum_k \mathbf{1}_{\overline{I}_k} \Big\|_s . \end{aligned}$$

Here the lower bound on the size of pairs enters in. The $\overline{p}_k$, being maximal, are incomparable under $<$. Hence Lemma 4.1 applies to show that

$$\Big\| \sum_k \mathbf{1}_{\overline{I}_k} \Big\|_s \le C_s\, b^{-3} ,$$

which will finish the proof of the Lemma.

We turn to the proof of the bound for forests.

PROOF OF LEMMA 7.1. The first task is to rephrase the definition of a forest in terms of trees, which depends critically on the condition β). Let $\overline{p}_j = [\overline{\omega}_j, \overline{I}_j]$ be the maximal pairs in $\mathcal{F}$. Let $\mathcal{P}_j = \{p \in \mathcal{F} : p < [\overline{\omega}_j, \overline{I}_j]\}$. Each $\mathcal{P}_j$ is a tree and $\mathcal{F} = \bigcup_j \mathcal{P}_j$. Moreover, if $j \ne j'$, no two pairs $p \in \mathcal{P}_j$ and $p' \in \mathcal{P}_{j'}$ are comparable. For if not, assuming $p < p'$, then one has $p < \overline{p}_j$ as well as $p < \overline{p}_{j'}$. But these last two pairs being maximal, are incomparable, contradicting β).

The last condition in the definition of a forest, condition γ) implies that

$$\sum_j \mathbf{1}_{\overline{I}_j}(x) \le C\Big(\frac{A}{b}\Big)^{210} , \qquad \text{for all } x .$$

The next steps of the proof are made with the intent of extracting normal separated trees from the $\mathcal{P}_j$. The process starts by deleting the top and bottom from $\mathcal{F}$. First the top. Let $\mathcal{F}^t$ be the set of pairs $p \in \mathcal{F}$ for which there is no strictly ascending chain

$$p = p_1 \lneq p_2 \lneq \cdots \lneq p_m ,$$

with $m > M = 40000\,(\log A/b)$ and $p_1, p_2, \ldots, p_m \in \mathcal{F}$. Then $\mathcal{F}^t$ can be written as a union of M sets $\mathcal{F}^t_m$ for which no two pairs in any $\mathcal{F}^t_m$ are comparable. Thus, by Lemma 4.2.,

$$\|T^{\mathcal{F}^t} g\|_r \le C\, b^{\delta(r)} (\log A)\, \|g\|_2 , \qquad 1 < r < 2 . \tag{7.4}$$

This is more than what is claimed in the conclusion above.

Let $\mathcal{F}^0 = \mathcal{F} \backslash \mathcal{F}^t$, and now remove the bottom of $\mathcal{F}^0$. Let $\mathcal{F}^b$ be the set of $p \in \mathcal{F}^0$ for which there is no descending chain

$$p_1 \lneq p_2 \lneq \cdots \lneq p_m = p\,,$$

with $m > M = 40000 \log{(A/b)}$ and $p_1, p_2, \ldots, p_m \in \mathcal{F}^0$. As before,

$$\|T^{\mathcal{F}^b} g\|_r \leq C\, b^{\delta(r)}\, (\log A)\, \|g\|_r\ , \qquad 1 < r < 2\,.$$

Let $\mathcal{F}^1 = \mathcal{F} \backslash \big(\mathcal{F}^t \cup \mathcal{F}^b\big)$ and $\mathcal{P}^1_j = \mathcal{F}^1 \cap \mathcal{P}_j$.

The exceptional set enters in. Two sets are defined below to conform with the formulation of Lemma 7.3. Set

$$E_i = \bigcup_j \Big\{x : \operatorname{dist}(x, \partial \overline{I}_j) \leq 2\, i \Big(\frac{b}{A}\Big)^{400} |\overline{I}_j|\Big\} \qquad \text{for } i = 1, 2\,.$$

The set $E = E_2$ is the exceptional set of our Lemma. By part γ) and δ) of the definition of a forest, $|E| \leq C\,(b/A)^{90}$. Next, we delete some pairs. Let $\mathcal{B}$ denote those pairs in $\mathcal{F}$ for which $I \subset E_1$. It follows from Lemma 7.3 that

$$\|T^{\mathcal{B}} g\|_{L^r(E^c)} \leq C_r\, b^{\delta}\, \|g\|_2\ , \qquad 1 < r < 2\,,\ \delta > 0\,.$$

The exceptional set will play no other role in the proof.

We have identified all the pairs to remove from $\mathcal{F}$. Set $\mathcal{F}^\sharp = \mathcal{F} \backslash (\mathcal{F}^t \cup \mathcal{F}^b \cup \mathcal{B})$, and $\mathcal{P}^\sharp_j = \mathcal{P}_j \cap \mathcal{F}^\sharp$. The $\mathcal{P}^\sharp_j$ are much nicer trees. In particular,

i) $\mathcal{P}^\sharp_j$ is a normal tree.

ii) If $j \neq j'$ then $\mathcal{P}^\sharp_j$ and $\mathcal{P}^\sharp_{j'}$ are separated trees.

Here is the verification of the first claim. There are two conditions to check. The second, that $\operatorname{dist}(I, \partial \overline{I}_j) > (b/A)^{400}\, |\overline{I}_j|$ follows from the removal of the set $\mathcal{B}$. Finally, by the removal of the top, for each $[\omega, I\,] \in \mathcal{P}^\sharp_j$, there is a chain of triadic intervals

$$I = I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_M = \overline{I}_j\ , \qquad M > 40000 \log \frac{A}{b}\ .$$

And so $|I| \leq (b/A)^{40000}\, |\overline{I}_j|$. This shows normality.

The second claim will follow from the removal of the bottom: for $j \neq j'$, recall that the top of $\mathcal{P}_j^\sharp$ is $[\overline{\omega}_j, \overline{I}_j]$, and assume that $\overline{I}_j \cap \overline{I}_{j'} \neq \varnothing$. Let's check the condition $\alpha)$ in the definition of separated. For $p = [\omega, I] \in \mathcal{P}_j^\sharp$ with $I \subset \overline{I}_{j'}$, as $p \notin \mathcal{P}^b$, there is a descending chain $[\omega_1, I_1] = p_1 \lneq p_2 \lneq \cdots \lneq p_M = p$ where $M = 3000 \log A/b$, and all of the $p_m \in \mathcal{F}^0$. The situation is that $I_1 \subset I \subset \overline{I}_{j'}$, and $p_1 \in \mathcal{P}_j$. But p_1 and $[\overline{\omega}_{j'}, \overline{I}_{j'}]$ cannot be comparable, by the condition $\beta)$ in the definition of forest. That is, $\omega_1 \cap \overline{\omega}_{j'} = \varnothing$. But all of the ω_n's are central, so by the convenient property of central dyadic intervals, (3.1), ω sits well inside ω_1. In particular, $\operatorname{dist}(\omega, \omega_{j'}^0) \geq \operatorname{dist}(\omega, \partial\omega_1) \geq 3^M |\omega| \geq (A/b)^{3000} |\omega|$. This verifies $\alpha)$ in the definition of separated, with the proof of $\beta)$ following by symmetry. The last condition in the definition of separated follows from the removal of the pairs in $\mathcal{B}$. This finishes ii) above.

The next task is to show that $\mathcal{F}^\sharp$ can be written as a union of at most $J = (A/b)^{210}$ separated rows $\mathcal{R}_1, \ldots, \mathcal{R}_J$. To see this, let $\{I^i\}$ be the maximal intervals from the $\{\overline{I}_j\}$ with no repetitions. For each I^i, let $[\overline{\omega}_{j(i)}, I^i]$ be one of the maximal pairs $\overline{p}_{j(i)}$. Then $\mathcal{R} = \bigcup_i \mathcal{P}_{j(i)}^\sharp$. Delete the maximal pairs $[\overline{\omega}_{j(i)}, I^i]$ from the list of all maximal pairs, and repeat the procedure above. Condition $\gamma)$ guarantees that the procedure stops in at most $(A/b)^{210}$ steps. Separability of the rows $\mathcal{R}_j$ follows from the construction.

With this decomposition Lemma 5.25 concludes the proof is concluded by appealing to Lemma 5.25, Lemma 7.2 and part $\delta)$ of the definition of a forest.

A critical combinatorial trick will permit us to write much larger sets of pairs as a union of a small number of forests.

Lemma 7.5. *Let $\mathcal{P}$ be a set pairs with* $\operatorname{size}(\mathcal{P}) \leq b$. *Futher, letting $\overline{p}_k = [\overline{\omega}_k, \overline{I}_k]$ denote the maximal pairs in $\mathcal{P}$, assume that they obey a Carleson measure estimate*

$$\sum_k [\, \overline{I}_k \subset U \,] \, |\overline{I}_k| \leq C \, b^{-140} \, |U| \,. \tag{7.6}$$

for all intervals U. In addition assume that $I_k \subset b^{-40} \, [-1, 1]$ for all k. Then, for some $\delta > 0$,

$$\|T^{\mathcal{P}} g\|_1 \leq C \, b^\delta \, \|g\|_2 \,.$$

PROOF. The Carleson measure estimate implies that

$$\Big\| \sum_k \mathbf{1}_{\overline{I}_k} \Big\|_r \leq C_r\, b^{-180}\,, \qquad 1 < r < \infty\,.$$

Hence, the two sets

$$E_i = \Big\{ x : \sum_k \mathbf{1}[i\,\overline{I}_k](x) \geq \Big(\frac{A}{b}\Big)^{210} \Big\}\,, \qquad i = 1, 2\,,$$

have very small measure: $|E_2| \leq C\,(b/A)^{100}$. Use this set to delete some pairs. Set $\mathcal{B} = \{[\omega, I\,] \in \mathcal{P} : I \subset E_1\}$. By Lemma 7.3,

$$\|T^{\mathcal{B}}\|_{L^r(E_2^c)} \leq C_r\, b^{\delta}\, \|g\|_2\,, \qquad 1 < r < 2\,.$$

For the set $\mathcal{P}^\sharp = \mathcal{P} \backslash \mathcal{B}$, we will show that there is a set $F \subset (0,1)$ of measure at most $(b/A)^{20}$, so that

$$\|T^{\mathcal{P}^\sharp}\|_{L^r(F^c)} \leq C_r\, b^{\delta}\, (\log A)\, \|g\|_2\,, \qquad 1 < r < 2\,.$$

The estimate of this Lemma will then follow from Lemma 7.2.

We shall see that $\mathcal{P}^\sharp$ can be decomposed into $O\,(\log A/b)$ forests. Therefore, the estimate above follows from Lemma 7.1. The decomposition is accomplished by means of the following combinatorial trick, which has already been used in Section 4.

Let $B(p) =$ the number of k for which $p < \overline{p}_k$. Simply define

$$\mathcal{F}_m = \{p \in \mathcal{P} : 2^{m-1} \leq B(p) < 2^m\}\,.$$

By construction, this set is empty if $m > O\,(\log A/b)$.

It remains to verify that each $\mathcal{F}_m$ is a forest. The first condition in the definition is trivial, the next to last condition follows from the deletion of the set of pairs $\mathcal{B}$ and the last condition follows from the hypothesis of the Lemma. The middle condition β) in the definition of a forest must be checked. But it is a consequence of the following combinatorial property of $B(p)$: for $p, p', p'' \in \mathcal{P}$, with $p < p'$,$p < p''$ but p' and p'' not comparable implies that $B(p) \geq B(p') + B(p'')$. So if in addition p',$p'' \in \mathcal{F}_m$, then $p \notin \mathcal{F}_m$, proving that $\mathcal{F}_m$ is a forest. To see the super-additive property, write $p' < \overline{p}_{j(1)}, \overline{p}_{j(2)}, \dots, \overline{p}_{j(s)}$ where $B(p') = s$, and $p'' < \overline{p}_{k(1)}, \overline{p}_{k(2)}, \dots, \overline{p}_{k(t)}$ where $B(p'') = t$. Now, if some $\overline{p}_{j(u)}$ equals some $\overline{p}_{k(v)}$, the situation would be $p < p'\,, p'' <$

$p''' = \overline{p}_{j,u} = \overline{p}_{k,v}$. But it is a simple property of triadic intervals that the last condition forces p' and p'' to be comparable, a contradiction. Thus, $p < p' < \overline{p}_{j(1)}, \overline{p}_{j(2)}, \dots, \overline{p}_{j(s)}, \overline{p}_{k(1)}, \overline{p}_{k(2)}, \dots, \overline{p}_{k(t)}$ all pairs being distinct, which means that $B(p) \geq s+t$.

The previous Lemma, with it's reliance on the Carleson measure estimate, clearly implies the following two corollaries, which are stated for specificity. For orchards, the necessary Carleson measure estimate is Lemma 6.1. (Recall that this Lemma applies only for sufficently small b).

Corollary 7.7. *Let $\mathcal{O}$ be an α or an $\alpha*$-orchard, with* $\operatorname{size}(\mathcal{O}) \leq b$, *where $0 < b < b_0$. Asumme that $I_p \subset b^{-40}[-1,1]$ for all $p \in \mathcal{O}$. Then, for some $\delta > 0$,*

$$\|T^{\mathcal{O}} g\|_1 \leq C\, b^{\delta}\, \|g\|_2 \,.$$

And, assuming a lower bound on the size of pairs, the necessary Carleson measure estimate is Lemma 4.1.

Crollary 7.8. *Let $\mathcal{P}$ be a set of pairs with* $\operatorname{size}(\mathcal{P}) \leq b$, *and with* $\operatorname{size}(p) \geq b^{30}$ *for all $p \in \mathcal{P}$. Then,*

$$\|T^{\mathcal{P}} g\|_1 \leq C\, b^{\delta}\, \|g\|_2 \,.$$

The last Lemma describes the inductive procedure with which the set of all pairs can be broken up into sets to which the previous two corollaries can be applied.

Lemma 7.9. *Let $0 < b < b_0$. Let $\mathcal{P}$ be a set of pairs satisfying*

α) $\operatorname{size}(p) \leq b^{30}$ *for all $p \in \mathcal{P}$.*

β) $\operatorname{size}(\mathcal{P}) \leq b$.

Then $\mathcal{P} = \mathcal{P}^{\sharp} \cup \mathcal{P}^{\flat}$ where, for some $\delta > 0$,

$$\|T^{\mathcal{P}^{\flat}} g\|_1 \leq C\, b^{\delta}\, \|g\|_2 \,,$$

and $\mathcal{P}^{\sharp}$ satisfies α) and β) above, with b replaced by $b/2$.

PROOF. Consider those pairs $\overline{p} = [\overline{\omega}, \overline{I}]$ in $\mathcal{P}$ for which there is an α-tree $\mathcal{T}$ with top $\overline{p}$ so that

$$\sum_{p\in\mathcal{T}} |\langle f, \phi_p\rangle|^2 \geq c_1 b\, |\overline{I}|\,. \tag{7.10}$$

Here, c_1 is constant which appears in the definiton of an orchard. However, it follows from the hypothesis β) that the sum above cannot be more than an absolute constant times b. In addition, $\overline{I} \subset b^{-1}[-1,1]$, as follows from (5.21).

Let $\overline{p}_1 = [\overline{\omega}_1, \overline{I}_1]$ be such a pair, so that the interval $\overline{I}_1$ is maximal among all such pairs. Take $\mathcal{T}_1^\alpha$ to be all pairs $p = [\omega, I] \in \mathcal{P}$ so that $A(\omega)$ contains the center of $\overline{\omega}_1$, and $I \subset \overline{I}_1$. That is, $\mathcal{T}_1^\alpha$ is the largest α-tree in $\mathcal{P}$ with top $\overline{p}_1$. Repeat this procedure to the collection $\mathcal{P}\backslash\mathcal{T}_1^\alpha$ to get an α-tree $\mathcal{T}_2^\alpha$ with top $\overline{p}_2$. Continue this procedure indefinitely, thereby obtaining a sequence of α-trees $\mathcal{T}_j^\alpha$ with tops $\overline{p}_j$.

We claim that $\mathcal{O}^\alpha = \bigcup \mathcal{T}_j^\alpha$ is an α-orchard. There are three conditions to check, yet each of these follows immediately from the construction. Clearly, conditions ii) and iii) hold. And condition i) follows from maximality of the $\overline{I}_j$. Therefore, Corollary 7.7 applies, showing that

$$\|T^{\mathcal{O}^\alpha} g\|_1 \leq C\, b^\delta\, \|g\|_2\,, \qquad \delta > 0\,.$$

Remove the pairs $\mathcal{T}^\alpha$ from $\mathcal{P}$, and call the resulting set $\mathcal{P}^1$. By our choice of the constant c_1 in (7.10), which was made in the definition of an orchard, we see that for any α-tree $\mathcal{T}$ in $\mathcal{P}^1$,

$$\|T^{\mathcal{T}}\|_2 \leq \frac{b}{6}\,.$$

Continue by removing $\alpha*$-trees from $\mathcal{P}^1$ in exactly the same manner as the α-trees were removed. We get a set $\mathcal{T}^{\alpha*} \subset \mathcal{P}^1$ with

$$\|T^{\mathcal{T}^{\alpha*}} g\|_1 \leq C\, b^\delta\, \|g\|_2\,, \qquad \delta > 0\,.$$

Let $\mathcal{P}^2$ be the collection of pairs obtained by removing those pairs in $\mathcal{T}^{\alpha*}$ from $\mathcal{P}^1$. It follows that any α^*-tree in $\mathcal{P}^2$ has norm at most $b/6$. Now, since $\operatorname{size}(p) < b^{30}$, for all p, we see that any β-tree in $\mathcal{P}^2$ has very small norm. Hence, for any tree $\mathcal{T}$ in $\mathcal{P}^2$,

$$\|T^{\mathcal{T}}\|_2 \leq \frac{b}{2}\,,$$

which is seen by writing $\mathcal{T}$ as a union of an α-tree, a α^*-tree and a β-tree. That is, $\text{size}(\mathcal{P}^2) \le b/2$, and the set satisfies the condition β) of the Lemma with b replaced by $b/2$.

We turn our attention to the condition α). Set

$$\mathcal{P}^\flat = \left\{ p \in \mathcal{P}^2 : \left(\frac{b}{2}\right)^{30} \le \text{size}(p) \le b^{30} \right\},$$

and

$$\mathcal{P}^\sharp = \left\{ p \in \mathcal{P}^2 : \text{size}(p) \le \left(\frac{b}{2}\right)^{30} \right\}.$$

The second collection satisfies α) and β) with b replaced by $b/2$. And it remains to see that the first collection of pairs leads to an operator with small norm. Yet, with the lower bound on the size of pairs, we are in a position to apply Corollary 7.8 to $\mathcal{P}^\flat$, and so the proof of the Lemma is complete.

A brief argument will finish the proof of the boundedness of the bilinear Hilbert transform. Recall from Section 3 that set $\mathcal{P}$ to be the set of all (admissible) pairs, we need only prove

$$\|T^{\mathcal{P}} g\|_1 \le C\,\|g\|_2 \tag{7.11}$$

Let $\mathcal{P}^\flat = \{p : \text{size}(p) \ge b_0^{30}\}$, and $\mathcal{P}^\sharp$ be the complementary set of pairs. Now, $\text{size}(\mathcal{P}^\flat) \le C$, and so with the lower bound on the size of pairs, we can apply Corollary 7.8 to see that

$$\|T^{\mathcal{P}^\flat} g\|_1 \le C_0\,\|g\|_2\,.$$

Iteratively applying Lemma 7.9 to $\mathcal{P}^\sharp$, we can write this collection of pairs as a union of collections $\mathcal{P}_n$, with

$$\|T^{\mathcal{P}_n} g\|_1 \le C\,2^{-\delta n}\,\|g\|_2\,.$$

This estimate is summable in n, as $\delta > 0$, and so it proves (7.11), finishing the proof of the boundedness of the bilinear Hilbert transform.

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Recibido: 29 de octubre de 1.996

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* Research supported in part by an NSF Postdoctoral Fellowship, and NSF grants.

Revista Matemática Iberoamericana
Vol. 13, N.° 3, 1997

Diffusive limit for finite velocity Boltzmann kinetic models

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Abstract. We investigate, in the diffusive scaling, the limit to the macroscopic description of finite-velocity Boltzmann kinetic models, where the rate coefficient in front of the collision operator is assumed to be dependent of the mass density. It is shown that in the limit the flux vanishes, while the evolution of the mass density is governed by a nonlinear parabolic equation of porous medium type. In the last part of the paper we show that our method adapts to prove the so-called Rosseland approximation in radiative transfer theory.

1. Introduction.

In the kinetic theory of rarefied gases, two-velocity models of the Boltzmann equation are supposed to describe the evolution of the velocity distribution of a fictitious gas composed of two kinds of particles that move parallel to the x-axis with constant and equal speeds, either in the positive x-direction with a density u, or in the negative x-direction with a density v. The most general two-speed gas which is in local equilibrium when $u = v$ is described by the equations

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + c \, \dfrac{\partial u}{\partial x} = k(u, v, x) \, (v - u) \, , \\ \dfrac{\partial v}{\partial t} - c \, \dfrac{\partial v}{\partial x} = k(u, v, x) \, (u - v) \, , \qquad x \in \mathbb{R} \, , \ t \geq 0 \, , \end{cases}$$

where c is the modulus of the constant speed of the particles, and k is a nonnegative rate coefficient.

The most famous example of these models was proposed by Carleman's in the 1930's and appeared in print for the first time in 1957 in [Car]. In Carleman's model $k(u, v, x) = u + v$, so that the "collision" terms on the right-hand side of (1.1) describe binary interactions between particles. An interaction between two molecules of the former type results into two molecules of the latter type and vice versa. Clearly, Carleman equations have no meaningful physical interpretation; in particular, there is no conservation of momentum.

Choosing $k(u, v, x) = 1$, we obtain a linear system, known as Goldstein-Taylor model [Gol], [Tay]. The system represents the forward equation for the density of a molecule moving with constant speed along the x-axis, subject to spontaneous reversals of directions, at the jump times of a standard Poisson process of unit rate.

The macroscopic variables for these models are the mass density $\rho = u + v$, and the flux $j = c(u - v)$. It is interesting to remark that, since u and v can be expressed in terms of ρ and j, so that $k(u, v, x) = k(\rho, j, x)$, system (1.1) is equivalent to the following macroscopic equations for the mass density and the flux

$$\left\{ \begin{aligned} & \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0\,, \\ & \frac{\partial j}{\partial t} + c^2 \frac{\partial \rho}{\partial x} = -2\, k\,(\rho, j, x)\, j\,, \qquad x \in \mathbb{R},\ t \geq 0\,, \end{aligned} \right. \tag{1.2}$$

Basically, two different types of problems for the system (1.1) can be formulated. The first one is the initial or initial-boundary value problem. The second one is an asymptotic problem. Let us assume that the mean free path is not normalized to unity, but is left in the equation as a "small" parameter ε. More precisely, this means that in (1.1) we replace k by k/ε. The following question then naturally arises: what is the limiting form of system (1.2) as $\varepsilon \to 0$, and how do the initial data of the limiting equation match the initial data associated with (1.2)?

The limit $\varepsilon \longrightarrow 0$ corresponds to the transition from a kinetic description of the gas to that of a gas as a continuum, and we refer to the asymptotic problem as the hydrodynamic limit associated with the kinetic system (1.1).

Much is known for Carleman's equation in the scaling

$$\frac{\partial u}{\partial t} + \frac{1}{\varepsilon} \frac{\partial u}{\partial x} = \frac{1}{\varepsilon^2} \left(v^2 - u^2\right),$$

$$\frac{\partial v}{\partial t} - \frac{1}{\varepsilon}\frac{\partial v}{\partial x} = \frac{1}{\varepsilon^2}\,(u^2 - v^2)\,. \tag{1.3}$$

This asymptotic problem was first investigated by Kurtz [Kur]. By means of the theory of nonlinear semigroups, he proved that, for initial data $u_0(x) = v_0(x) \in L^1(\mathbb{R})$ the mass density $\rho_\varepsilon(x,t)$ converges in L^1_x for all $t \geq 0$ to $\rho(x,t)$ satisfying the nonlinear diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{4}\frac{\partial}{\partial x}\Big(\frac{1}{\rho}\frac{\partial \rho}{\partial x}\Big)\,, \qquad x \in \mathbb{R}\,,\ t \geq 0\,, \tag{1.4}$$

while $j_\varepsilon(x,t)$ converges to zero. In other words, (1.4) is the hydrodynamical limit of the Carleman's equation (1.3).

Subsequently, McKean [McK] generalized the preceding result, by removing the restriction that the initial flux has to be taken equal to zero.

Further results are due to Kaper, Leaf and Reich [KLR], who investigated the problem treated by Kurtz with ε-dependent initial data, and to Fitzgibbon [Fi1], [Fi2] who studied the problem in a bounded domain with specular reflecting boundary conditions. The method of proof of all the aforementioned papers relies mainly on the theory of nonlinear semigroups, and on the fact that these problems are L^1-accretive.

McKean's result [McK], has been recently extended by Toscani and Pulvirenti [PTo] to the system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial u}{\partial x} = \dfrac{1}{\varepsilon^2}\,\rho^\alpha\,(v - u)\,, \\[2mm] \dfrac{\partial v}{\partial t} - \dfrac{1}{\varepsilon}\dfrac{\partial v}{\partial x} = \dfrac{1}{\varepsilon^2}\,\rho^\alpha\,(u - v)\,, \end{cases} \tag{1.5}$$

with $0 \leq \alpha \leq 1$. The system (1.5) includes as particular cases both Carleman's equation ($\alpha = 1$) and the Golstein-Taylor model ($\alpha = 0$).

In the present paper, we will investigate in the diffusive limit the system

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \dfrac{\partial j_\varepsilon}{\partial x} = 0\,, \\[2mm] \varepsilon^2\,\dfrac{\partial j_\varepsilon}{\partial t} + \dfrac{\partial \rho_\varepsilon}{\partial x} = -2\,\rho_\varepsilon^\alpha\,j_\varepsilon\,, \qquad x \in \mathbb{R}\,,\ t \geq 0\,. \end{cases} \tag{1.6}$$

for any value of $\alpha < 1$. The case $\alpha < -1$ is of particular interest since we obtain in the limit the well-known porous media equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2\,(1+|\alpha|)}\,\frac{\partial^2}{\partial x^2}\,\rho^{1+|\alpha|}\,. \tag{1.7}$$

In the second part of the paper we will investigate the multidimensional and in particular the three dimensional version of system (1.1). In three dimensions of space, the molecules of the fictitious gas can move in directions parallel to one of the axes x_1, x_2, x_3 either in the positive direction or in the negative direction. Denoting by u_i (respectively u_{i+3}), $i = 1, 2, 3$ the densities of molecules moving in the positive (respectively negative) x_i-directions, the most general system which is in local equilibrium when $u_i = \rho/6$, $i = 1, 2, \ldots, 6$, where $\rho = \sum u_i$ is the mass density, takes the form

$$(1.8) \qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + c\,\dfrac{\partial u_i}{\partial x_i} = k(u_1, \ldots, u_6, x)\,(\rho - 6\,u_i)\,, \\[2ex] \dfrac{\partial u_{i+3}}{\partial t} - c\,\dfrac{\partial u_{i+3}}{\partial x_i} = k(u_1, \ldots, u_6, x)\,(\rho - 6\,u_{i+3})\,, \end{cases}$$

with $i = 1, 2, 3$, $x \in \mathbb{R}^3$, and $t \geq 0$. With a few modification, our one-dimensional analysis extends to the three-dimensional case when $k(u_1, \ldots, u_6, x) = \rho^\alpha$, $\alpha < 1$.

Other models (one or multidimensional) can be studied with our technique. Among them, let us mention the cases $k(u, v, x) = a(x)$, and $k(u, v, x) = \sum_{l=1}^{m} u^{m-l}\, v^l$, $m \in \mathbb{N}^+$.

The main object of the present investigation is to justify the passage from the mesoscopic description of kinetic theory to the macroscopic one of continuum theory. This passage is usually described by the asymptotic relations between solutions of the Boltzmann equation and solutions of Euler and Navier-Stokes equations. It is worthwhile mentioning that the target equations of continuum theory can be obtained directly from a microscopic description. In particular, the deduction of diffusion equations as a hydrodinamic limit of particle model is a well-studied subject. We quote here the paper by K. Oelschläger [Oel], in which the porous medium equation is obtained as a limit of a particle system that interact under the action of adequate potentials, as the number of particles tends to infinity. Depending on the scaling parameter applied, different versions of the porous medium equation in the limit dynamics are obtained. A different aspect of the limit dynamics for a Markov system of many particles, and the convergence to the porous media equation of the empirical density of the number of particles has been investigate by Inoue [Ino]. In this paper, the Kac-McKean propagation of chaos for the system is shown to hold.

In Section 2 we discuss the initial and the initial-boundary value problems associated with (1.1), and we will recover elementary a priori estimates for the solution. Here, the models (1.5) naturally separate in two subclasses, corresponding to $|\alpha| \leq 1$ and $\alpha < -1$ respectively. When $|\alpha| < 1$, at least in one dimension, the problem is shown to be L^1-accretive. Entropy bounds are discussed in Section 3, and the limit theorems in Section 4. When the model is accretive, given initial values of bounded variation, L^1-contraction and translational invariance imply total variation bounds on the solution, and one can pass to the limit for general L^1 initial conditions, in a rather straightforward (and standard) way. Let us briefly discuss the case $\alpha > 0$. The entropy bounds of Section 3, Theorem 3.1, imply that $\{j_\varepsilon\}$ is bounded in L^2, and thus by the second of equations (1.6) $\partial\rho_\varepsilon/\partial t$ is bounded in $L^2(0,T;H^{-1}_{\mathrm{loc}})$ for all $T > 0$. In view of the a priori bounds of Section 2, $\{\rho_\varepsilon\}$ is bounded in $L^\infty(\mathbb{R})$.

These bounds, combined with Proposition 2.3, imply that the family $\{\rho_\varepsilon\}$ is relatively compact in $C([0,T];L^1(\mathbb{R}))$ for all $T > 0$.

Hence $\rho_\varepsilon^\alpha\, j_\varepsilon \longrightarrow \rho^\alpha j$ in L^2-weak, $\varepsilon^2\, j_\varepsilon \longrightarrow 0$ strongly in L^2 and from the flux equation we deduce

$$\frac{\partial\rho}{\partial x} = -2\,\rho^\alpha j$$

at least in the sense of distributions (and in fact in L^2). Considering that $\rho \in L^\infty$, the above equality implies that we have

$$j = -\frac{1}{2\,(1-\alpha)}\,\frac{\partial\rho^{1-\alpha}}{\partial x}$$

in L^2. The case $-1 \leq \alpha < 0$ follows with similar arguments.

The case $\alpha \leq -1$ is more delicate, and the result is achieved by compensated compactness theory (see F. Murat [Mu1], [Mu2], and L. Tartar [Ta1], [Ta2]). In Section 5, we extend our analysis to the three-dimensional models (1.8). Finally, we state without proofs various extensions and variants of the results obtained below. In fact, our method of proof adapts to models with a continuous set of velocities. In particular we make contact (and propose more general proofs) with the so-called Rosseland approximation in radiative transfer theory (see C. Bardos, F. Golse, B. Perthame and R. Sentis [BGPS], and the references therein).

2. Basic a priori estimates and global existence.

In this section we discuss the initial and the initial-boundary value problems for system (1.1). Many arguments that follow are very elementary, and the proofs will be omitted. Besides, it has to be pointed out that the general a priori estimates we will use in the sequel, to our knowledge has never been used before.

For our purposes, as will be clear later on, we need to study (1.1) in a bounded interval $\Omega = (-a, a)$ with periodic boundary conditions. This limitation allows us to prove existence and uniqueness of a solution under weak conditions on the rate coefficient k.

Useful a priori estimates for the solution to system (1.1) follow by the structure of the "collision" term. Taking the sum of the two equations, and integrating over Ω, we obtain the mass conservation, namely $\int_\Omega \rho(x,t)\,dx$ is independent of $t \ge 0$. Let now $\varphi(r)$, $r \ge 0$ be a (regular) convex function. If we multiply the first equation of system (1.1) by $\varphi'(u)$ and the second by $\varphi'(v)$, after integrating over Ω we obtain

$$\begin{aligned} &\int_\Omega \frac{\partial \varphi(u)}{\partial t} + \int_\Omega \frac{\partial \varphi(v)}{\partial t} \\ &\qquad = - \int_\Omega k(u,v,x)\,(u-v)\,(\varphi'(u) - \varphi'(v))\,dx\,. \end{aligned} \tag{2.1}$$

Since $\varphi'(r)$ is non decreasing, the right-hand side of (2.1) is non positive. Thus we deduce, at least formally, that $\int_\Omega (\varphi(u) + \varphi(v))\,dx$ is monotone non increasing in $t \ge 0$, and

$$\begin{aligned} &\int_\Omega (\varphi(u(x,t)) + \varphi(v(x,t)))\,dx \\ &\qquad \le \int_\Omega (\varphi(u_0(x)) + \varphi(v_0(x)))\,dx\,. \end{aligned} \tag{2.2}$$

In particular, if the initial densities u_0, v_0 belong to $L^\infty(\Omega)$, taking $\varphi(r) = r^p$ for any $p \ge 1$, and letting p go to $+\infty$, we deduce the following bound

$$\max\{\|u(t)\|_\infty, \|v(t)\|_\infty\} \le \max\{\|u_0\|_\infty, \|v_0\|_\infty\}\,. \tag{2.3}$$

Similarly, we may assume that $u_0(x) \ge \delta$, $v_0(x) \ge \delta$ in Ω for some $\delta \ge 0$. Then, choosing $\varphi(r) = r^{-p}$, $(p > 0)$ in (2.2) above, and letting

p go to $+\infty$ we obtain

$$\max\left\{\left\|\frac{1}{u(t)}\right\|_\infty, \left\|\frac{1}{v(t)}\right\|_\infty\right\} \le \max\left\{\left\|\frac{1}{u_0}\right\|_\infty, \left\|\frac{1}{v_0}\right\|_\infty\right\},$$

or, equivalently, for all $t \ge 0$

$$\inf_{x\in\Omega}\{u(x,t), v(x,t)\} \ge \inf_{x\in\Omega}\{u_0(x), v_0(x)\}\,. \tag{2.4}$$

These formal a priori estimates are sufficient to yield the global existence of a unique solution of system (1.1) for a large class of rate coefficients. More precisely, we now need to specify our basic assumptions on $k(u,v,x)$.

Definition 2.1. 1) *$k(u,v,x)$ is an admissible rate coefficient of type* 1 *for system* (1.1) *if*

a) *$k(u,v,x) \le c_1(\mu) < \infty$, if $u,v \le \mu$ for any $\mu > 0$, and $x \in \Omega$.*

2) *$k(u,v,x)$ is an admissible rate coefficient of type* 2 *if $k(0,0,x) = \infty$ and*

b) *$k(u,v,x) \le c_2(\lambda) < \infty$, if $u,v \ge \lambda > 0$ for any $\lambda > 0$.*

A simple example is given by the rate coefficient

$$k(u,v,x) = (u+v)^\alpha\,.$$

k is of type 1 if $\alpha \ge 0$, and of type 2 if $\alpha < 0$.

We then have the following

Proposition 2.1. *Let $0 \le u_0(x), v_0(x) \in L^\infty(\Omega)$. Then, the initial-boundary value problem for the system* (1.1) *with a rate coefficient of type* 1 *has a unique solution $u(x,t), v(x,t) \in L^\infty(\Omega\times(0,T)) \cap C([0,T];L^p(\Omega))$ for all $T > 0, 1 \le p < \infty$. In addition, the solution satisfies the bound* (2.3)*.*

Proposition 2.2. *Let $0 \le u_0(x), v_0(x) \in L^\infty(\Omega)$ satisfy $u_0, v_0 \ge \delta$ on Ω for some $\delta > 0$. Then, the initial-boundary value problem for the system* (1.1) *with a rate coefficient type* 2 *has a unique solution bounded away from zero $u(x,t), v(x,t) \in L^\infty(\Omega\times(0,T)) \cap C([0,T];L^p(\Omega))$ for all $T > 0$, $1 \le p < \infty$. In addition, this solution satisfies the bounds* (2.3) *and* (2.4)*.*

A particular choice of k obviously allows to obtain additional results for the initial-boundary value problem (1.1). As precised in the introduction, we are interested in the fluid-dynamical limit of system (1.5), that corresponds to the choice $k(u,v,x) = (u+v)^\alpha = \rho^\alpha$, where $\alpha \le 1$ is a fixed constant. Since $(u+v)^\alpha$ is admissible, existence and uniqueness of a solution in L^∞ follows by Proposition 2.1 when α is positive, or by Proposition 2.2 when α is negative.

We are now going to use a few simple facts from the theory of dissipative operators. Let $f = (u,v)$, and let A_α be the operator defined by

$$A_\alpha f = (\rho^\alpha(v-u), \rho^\alpha(u-v)) .$$

Then we have

Lemma 2.1. *Let $0 \le \alpha \le 1$. Then, the operator A_α is dissipative from the domain*

$$D^+(A_\alpha) = \{(u,v) \in L^1(\Omega) \times L^1(\Omega),\ \|u\|_\infty, \|v\|_\infty < \infty\}$$

into $L^1(\Omega) \times L^1(\Omega)$.

If $-1 \le \alpha < 0$, and if $\delta > 0$, the operator A_α is dissipative from the domain

$$D^+_\delta(A_\alpha) = \{(u,v) \in L^1(\Omega) \times L^1(\Omega),\ u,v \ge \delta,\ \|u\|_\infty, \|v\|_\infty < \infty\}$$

into $L^1(\Omega) \times L^1(\Omega)$.

PROOF. Let us recall that a closed operator A from the domain $D(A) \subset L^+(X)$ into $L^1(X)$ is dissipative if, for any functions $f_1, f_2 \in D(A)$

$$\int_X (Af_1 - Af_2)\,\mathrm{sign}\,(f_1 - f_2)\,dx \le 0 .$$

In our case, $f = (u,v)$, so that

$$\begin{aligned}(A_\alpha f_1 - A_\alpha f_2)\,&\mathrm{sign}\,(f_1 - f_2)\\ &= ((u_1+v_1)^\alpha(v_1-u_1) - (u_2+v_2)^\alpha(v_2-u_2))\,\mathrm{sign}\,(u_1-u_2)\\ &\quad + ((u_1+v_1)^\alpha(u_1-v_1) - (u_2+v_2)^\alpha(u_2-v_2))\,\mathrm{sign}\,(v_1-v_2) .\end{aligned}$$

Then, the conclusion of the lemma follows by observing that, for $a \ge 0$, the function

$$y = (x+a)^\alpha(x-a) .$$

is monotone non decreasing for any fixed $\alpha \in [-1,1]$.

Let us now set

$$B_\alpha f = \Big(-\frac{\partial u}{\partial x} + \rho^\alpha (v-u),\ \frac{\partial v}{\partial x} + \rho^\alpha (u-v) \Big) = \frac{\partial}{\partial x}(-u,v) + A_\alpha f\,.$$

Then, the following lemma is immediate

Lemma 2.2. *Let $0 \le \alpha \le 1$. Then, the operator B_α is dissipative from the domain*

$$D^+(B_\alpha) = \{(u,v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega)\}$$

into $L^1(\Omega) \times L^1(\Omega)$.

If $-1 \le \alpha < 0$, and if $\delta > 0$, the operator B_α is dissipative from the domain

$$D^+_\delta(B_\alpha) = \{(u,v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega),\ u,v \ge \delta\}$$

into $L^1(\Omega) \times L^1(\Omega)$.

Remark 2.1. If $0 \le \alpha \le 1$, the existence theory in L^∞ can be extended to all of $\mathbb{R}$ without any difficulty. A further consequence of Proposition 2.1, combined with the a priori estimate (2.2) is that, if the initial data $u_0(x), v_0(x) \in L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ for some $p \ge 1$, the solution $u(x,t), v(x,t) \in L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ and

$$\Big(\int_{\mathbb{R}} (u(x,t)^p + v(x,t)^p)\,dx \Big)^{1/p}$$

is monotone non increasing for $t \ge 0$.

By Lemma 2.2, provided the initial values are in $D^+(B_\alpha)$, the solution of the system (1.5) can be written as

$$(u(\,\cdot,t), v(\,\cdot,t)) = e^{tB_\alpha}(u_0(\,\cdot\,), v_0(\,\cdot\,))$$

and, given $f_1 = (u_1,v_1)$, $f_2 = (u_2,v_2)$

$$\|e^{tB_\alpha} f_1 - e^{tB_\alpha} f_2\|_1 \le \|f_1 - f_2\|_1\,. \tag{2.5}$$

In particular, if the initial densities $(u_0(x), v_0(x))$ are of bounded variation, we see that the solution $(u(x,t), v(x,t))$ is of bounded variation, and

$$\max\left\{\left\|\frac{\partial u(x,t)}{\partial x}\right\|_1, \left\|\frac{\partial v(x,t)}{\partial x}\right\|_1\right\} \leq \max\left\{\left\|\frac{\partial u_0(x)}{\partial x}\right\|_1, \left\|\frac{\partial v_0(x)}{\partial x}\right\|_1\right\}. \tag{2.6}$$

Let us now consider the case $-1 \leq \alpha < 0$. By Proposition 2.2, given any $\delta > 0$, we have a unique global solution of system (1.5) in $L^\infty(\Omega)$. Furthermore, by Lemma 2.2, given initial data $f_1 = (u_1, v_1)$ and $f_2 = (u_2, v_2)$, with $f_1, f_2 \in D_\delta^+(B_\alpha)$ the solutions at any subsequent time $t > 0$ satisfy inequality (2.5).

As is well-known for accretive nonlinear semigroups, this allows to extend the semigroup to all L^1-data. In addition, if $u_0(x), v_0(x)$ have bounded variations, the solution $(u(x,t), v(x,t))$ has bounded variation, and inequality (2.6) holds. The previous arguments are summarized by the following

Proposition 2.3. *Let $0 \leq u_0(x)$, $v_0(x) \in L^1(\Omega)$. Then, provided $|\alpha| \leq 1$, the initial-boundary value problem* (1.5) *has a unique global solution $u(x,t), v(x,t) \in C([0,T]; L^1(\Omega))$ for all $T \geq 0$. In addition, if $u_0(x), v_0(x) \in BV(\Omega)$, then $u(x,t), v(x,t) \in L^\infty(0,\infty; BV(\Omega))$ and* (2.10) *holds. Furthermore, for any $p \geq 1$, if $u_0, v_0 \in L^p(\Omega)$, we have*

$$\Big(\int_{\mathbb{R}} (u(x,t_2)^p + v(x,t_2)^p)\, dx\Big)^{1/p} \leq \Big(\int_{\mathbb{R}} (u(x,t_1)^p + v(x,t_1)^p)\, dx\Big)^{1/p},$$

for $t_1 < t_2$. If $0 \leq \alpha \leq 1$, these results extend to $\Omega = \mathbb{R}$.

REMARK 2.2. We emphasize that neither BV-bounds, nor L^p-bounds for the system (1.5) depend on ε. This is not the case if we look for L^p-bounds on the derivatives.

Indeed we have

Proposition 2.4. *Let $0 \leq u_0(x)$, $v_0(x) \in D^+(B_\alpha)$, if $0 \leq \alpha \leq 1$, and $0 \leq u_0(x)$, $v_0(x) \in D_\delta^+(B_\alpha)$, for some $\delta > 0$, if $\alpha < 0$. Then, if $0 \leq u_0(x)$, $v_0(x) \in W^{m,p}$, $m \geq 0$, $1 \leq p \leq \infty$, $u(x,t), v(x,t) \in W^{m,p}$,* *and, for $t \le T$*

$$\int_\Omega \Big(\Big|\frac{\partial^m u(t)}{\partial x^m}\Big|^p + \Big|\frac{\partial^m v(t)}{\partial x^m}\Big|^p\Big)\,dx$$
$$\le c_{m,p}\,(\varepsilon, T, \delta, \|u\|_\infty, \|v\|_\infty) \int_\Omega \Big(\Big|\frac{\partial^m u_0}{\partial x^m}\Big|^p + \Big|\frac{\partial^m v_0}{\partial x^m}\Big|^p\Big)\,dx\,.$$

The case $\alpha = 0$ seems to be exceptional. Let us consider the system (1.1) with a constant rate $k = k_0$, that is let us consider the Goldstein-Taylor model

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\,\dfrac{\partial u}{\partial x} = k_0(v-u)\,, \\ \dfrac{\partial v}{\partial t} - c\,\dfrac{\partial v}{\partial x} = k_0(u-v)\,, \qquad x \in \mathbb{R},\ t \ge 0\,. \end{cases} \tag{2.7}$$

Easy computations show that

$$\frac{d}{dt}\int_{\mathbb{R}} \Big(\Big|\frac{\partial u}{\partial x}\Big|^p + \Big|\frac{\partial v}{\partial x}\Big|^p\Big)\,dx \le 0\,. \tag{2.8}$$

Hence, combining (2.8) with the result of Proposition 2.3 we conclude that, if the initial data $u_0, v_0 \in W^{1,p}(\mathbb{R})$, $1 \le p \le \infty$, the solution $u(x,t), v(x,t) \in W^{1,p}(\mathbb{R})$, and

$$\Big(\int_{\mathbb{R}} (u^p + v^p)\,dx\Big)^{1/p} + \Big(\int_{\mathbb{R}} \Big(\Big|\frac{\partial u}{\partial x}\Big|^p + \Big|\frac{\partial v}{\partial x}\Big|^p\Big)\,dx\Big)^{1/p}$$

is monotone non increasing in time.

Since the problem is linear, the same conclusion can be reached for higher order derivatives. So, we proved

Proposition 2.5. *Let $0 \le u_0(x)$, $v_0(x) \in W^{m,p}$, for $m \ge 0$, $1 \le p \le \infty$. Then the unique solution $u(x,t), v(x,t)$ to the initial value problem for the Goldstein-Taylor model* (2.14) *belongs to $W^{m,p}$ for all $t \ge 0$, and*

$$\sum_{k=0}^{m} \Big(\int_{\mathbb{R}} \Big|\frac{\partial^m u}{\partial x^m}\Big|^p + \Big|\frac{\partial v^m}{\partial x^m}\Big|^p\Big)\,dx\Big)^{1/p}$$

is monotone non increasing with time.

To end this section, let us recall that B_0 is in fact accretive in L^p for all $p \in [1, \infty]$.

3. Entropy bounds.

Having in mind the passage to the fluid dynamic limit, we discuss in this section further a priori bounds for the system (1.5), when $\alpha \le 1$. Let us introduce nonnegative functions $u_0(x), v_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}} (u_0(x)\,|\log u_0(x)| + v_0(x)\,|\log v_0(x)|)\,dx = M_1 < \infty \tag{3.1}$$

and, for $\omega(x) = (1+x^2)^{\beta/2}$, $0 < \beta < 1/2$,

$$\int_{\mathbb{R}} \omega(x)\,(u_0(x) + v_0(x))\,dx = M_2 < \infty\,. \tag{3.2}$$

The bounds (3.1) and (3.2) ensure a suitable decay at infinity of u_0 and v_0. In section 2 we showed that different values of α produce different results of existence, uniqueness and regularity of the solution to system (1.5). Nevertheless, since we wish to give a unified treatment of our system for any value of α, we first maintain our analysis as general as possible, postponing to the end of the section the remarks concerning the various possible extensions of the results for particular values of the parameter α.

To this end, given $\gamma > 0$, let us denote by Ω_ε the domain $[-1/\varepsilon^\gamma, 1/\varepsilon^\gamma]$. In addition, given $\mu > 0$, let

$$u_0^\varepsilon = \max\{u_0(x), \varepsilon^\mu\}\,, \qquad v_0^\varepsilon = \max\{v_0(x), \varepsilon^\mu\}\,. \tag{3.3}$$

By the results of Section 2, the initial boundary value problem for system (1.5), with periodic boundary conditions on Ω_ε, and initial values (3.3), has a unique global solution $u^\varepsilon(x,t), v^\varepsilon(x,t)$, for all $\alpha \le 1$.

Moreover, provided $\mu > \gamma\,(1+\beta)$, u_0^ε and v_0^ε satisfy bounds (3.1) and (3.2) with different but finite constants M_1^ε and M_1^ε. In fact we have

$$\begin{aligned}
&\Big|\int_{\Omega_\varepsilon} (u_0^\varepsilon(x)\,|\log u_0^\varepsilon(x)| + v_0^\varepsilon(x)\,|\log v_0^\varepsilon(x)|)\,dx \\
&\quad - \int_{\Omega_\varepsilon} (u_0(x)\,|\log u_0(x)| + v_0(x)\,|\log v_0(x)|)\,dx\Big| \\
&\qquad = \Big|\int_{\Omega_\varepsilon \cap \{u_0 < \varepsilon^\mu\}} (\varepsilon^\mu\,|\log \varepsilon^\mu| - u_0(x)\,|\log u_0(x)|)\,dx
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&+\int_{\Omega_\varepsilon\cap\{v_0<\varepsilon^\mu\}}(\varepsilon^\mu\,|\log\varepsilon^\mu|-v_0(x)\,|\log v_0(x)|)\,dx\Big|\\
&\le 4\,\varepsilon^\mu\,|\log\varepsilon^\mu|\,\varepsilon^{-\gamma}\\
&=4\,\mu\,\varepsilon^{\mu-\gamma}\,|\log\varepsilon|
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\int_{\Omega_\varepsilon}\omega(x)\,(u_0^\varepsilon(x)+v_0^\varepsilon(x))\,dx-\int_{\Omega_\varepsilon}\omega(x)\,(u_0(x)+v_0(x))\,dx\Big|&\\
\le 2\,\varepsilon^\mu\,\omega(\varepsilon^\gamma)\,2\,\varepsilon^{-\gamma}&\\
=4\,(1+\varepsilon^{-2\gamma})^{\beta/2}\varepsilon^{\mu-\gamma}\,.&
\end{aligned}
\tag{3.5}
$$

Let us choose $\varphi(r)=r\log r$, for $r\ge 0$. Then, by (2.1) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_\varepsilon}(u^\varepsilon(x,t)\log u^\varepsilon(x,t)+v^\varepsilon(x,t)\log v^\varepsilon(x,t))\,dx\\
&\quad=-\int_{\Omega_\varepsilon}(u^\varepsilon(x,t)+v^\varepsilon(x,t))^\alpha\,\frac{u^\varepsilon(x,t)-v^\varepsilon(x,t)}{\varepsilon^2}\,\log\frac{u^\varepsilon(x,t)}{v^\varepsilon(x,t)}\,dx\,.
\end{aligned}
\tag{3.6}
$$

On the other hand, if we multiply both equations (1.5) by ω, after integrating over Ω_ε, we get

$$
\frac{d}{dt}\int_{\Omega_\varepsilon}\omega(x)\,(u^\varepsilon(x,t)+v^\varepsilon(x,t))\,dx+\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\omega(x)\,\frac{\partial}{\partial x}\,(u^\varepsilon(x,t)-v^\varepsilon(x,t))\,dx=0\,.
$$

Integrating by parts, and making use of the periodicity, we deduce

$$
\frac{d}{dt}\int_{\Omega_\varepsilon}\omega(x)\,(u^\varepsilon(x,t)+v^\varepsilon(x,t))\,dx-\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\omega'(x)\,\frac{u^\varepsilon(x,t)-v^\varepsilon(x,t)}{\varepsilon}\,dx=0\,.
\tag{3.7}
$$

Then, taking the sum of (3.6) and (3.7) we conclude

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_\varepsilon}(u^\varepsilon(x,t)\log u^\varepsilon(x,t)+v^\varepsilon(x,t)\log v^\varepsilon(x,t)\\
&\qquad+\omega(x)\,(u^\varepsilon(x,t)+v^\varepsilon(x,t)))\,dx\\
&\quad+\int_{\Omega_\varepsilon}(u^\varepsilon(x,t)+v^\varepsilon(x,t))^\alpha\,\frac{u^\varepsilon(x,t)-v^\varepsilon(x,t)}{\varepsilon^2}\,\log\frac{u^\varepsilon(x,t)}{v^\varepsilon(x,t)}\,dx\\
&\qquad\qquad\le\int_{\Omega_\varepsilon}\left|\omega'(x)\,\frac{u^\varepsilon(x,t)-v^\varepsilon(x,t)}{\varepsilon}\right|dx\,.
\end{aligned}\tag{3.8}
$$

Let $\rho_\varepsilon = u^\varepsilon + v^\varepsilon$, and by $j_\varepsilon = (u^\varepsilon(x,t) - v^\varepsilon(x,t))/\varepsilon$. Then, for $0 < \theta < 1$, we obtain

$$
\begin{aligned}
\rho_\varepsilon^\alpha\,\frac{u^\varepsilon-v^\varepsilon}{\varepsilon^2}\,\log\frac{u^\varepsilon}{v^\varepsilon}&=\rho_\varepsilon^\alpha\,j_\varepsilon^2\,\frac{\log u^\varepsilon-\log v^\varepsilon}{u^\varepsilon-v^\varepsilon}\\
&=\rho_\varepsilon^\alpha\,j_\varepsilon^2\,\frac{1}{\theta\,u^\varepsilon+(1-\theta)\,v^\varepsilon}\\
&=\rho_\varepsilon^{\alpha-1}\,j_\varepsilon^2\,\frac{u^\varepsilon+v^\varepsilon}{\theta\,u^\varepsilon+(1-\theta)\,v^\varepsilon}\\
&\ge 2\,\rho_\varepsilon^{\alpha-1}\,j_\varepsilon^2\,.
\end{aligned}\tag{3.9}
$$

Now, by the a priori bound (2.3),

$$\rho_\varepsilon \le 2\max\{\|u_0\|_\infty, \|v_0\|_\infty\} = \nu(u_0, v_0)\,. \tag{3.10}$$

Hence, since $\alpha - 1 \le 0$,

$$\rho_\varepsilon^\alpha\,\frac{u^\varepsilon-v^\varepsilon}{\varepsilon^2}\,\log\frac{u^\varepsilon}{v^\varepsilon}\ge 2\,\nu^{\alpha-1}\,j_\varepsilon^2\,. \tag{3.11}$$

In all cases we obtained a bound from below in terms of $\nu(u_0, v_0)$, which depends only on the L^∞-norm of the initial values, and not on ε.

Let us now consider the case $-1 \le \alpha < 0$. Then, by (3.9) it follows

$$\rho_\varepsilon^\alpha\,\frac{u^\varepsilon-v^\varepsilon}{\varepsilon^2}\,\log\frac{u^\varepsilon}{v^\varepsilon}\ge 2\,\rho_\varepsilon^{\alpha-1}\,j_\varepsilon^2\ge 2\,\rho_\varepsilon^{2\alpha}\,j_\varepsilon^2\,\nu^{|\alpha|-1}\,. \tag{3.12}$$

We next observe that

$$
\begin{aligned}
\int_{\Omega_\varepsilon}\omega'\,|j_\varepsilon|\,dx&=\frac12\int_{\Omega_\varepsilon}\frac{\omega'}{\sqrt{\nu}}\,|2\sqrt{\nu}\,j_\varepsilon|\,dx\\
&\le\frac{\nu}{2}\int_{\Omega_\varepsilon}j_\varepsilon^2\,dx+\frac{1}{2\,\nu}\int_{\Omega_\varepsilon}(\omega')^2\,dx\,.
\end{aligned}\tag{3.13}
$$

By definition, since $0 < \beta < 1/2$, $\omega' \in L^2(\mathbb{R})$. Thus, by (3.8) we deduce

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_\varepsilon} (u^\varepsilon(x,t)\log u^\varepsilon(x,t) + v^\varepsilon(x,t)\log v^\varepsilon(x,t) \\
&\qquad + \omega(x)\,(u^\varepsilon(x,t) + v^\varepsilon(x,t)))\,dx + \frac{\nu}{2}\int_{\Omega_\varepsilon} j_\varepsilon^2(x,t)\,dx \\
&\qquad\qquad \le \frac{1}{2\,\nu}\int_{\Omega_\varepsilon} (\omega')^2(x)\,dx\,.
\end{aligned} \tag{3.14}
$$

In particular, for any $t \ge 0$

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} (u^\varepsilon(t)\log u^\varepsilon(t) + v^\varepsilon(t)\log v^\varepsilon(t) + \omega(u^\varepsilon(t) + v^\varepsilon(t)))\,dx \\
&\quad \le \frac{t}{2\,\nu}\int_{\Omega_\varepsilon} (\omega')^2(x)\,dx + \int_{\Omega_\varepsilon} \omega(u_0^\varepsilon(t) + v_0^\varepsilon(t))\,dx \\
&\qquad + \int_{\Omega_\varepsilon} (u_0^\varepsilon(t)\log u_0^\varepsilon(t) + v_0^\varepsilon(t)\log v_0^\varepsilon(t))\,dx \\
&\quad \le \frac{t}{2\,\nu}\int_{\mathbb{R}} (\omega')^2(x)\,dx + \int_{\mathbb{R}} \omega(u_0 + v_0)\,dx \\
&\qquad + \int_{\Omega_\varepsilon} (u_0\log u_0 + v_0\log v_0)\,dx + o\,(\varepsilon)\,,
\end{aligned} \tag{3.15}
$$

where the rest $o\,(\varepsilon)$ is given by the sum of the right-hand sides of (3.4) and (3.5). By (3.15), the monotonicity of

$$
\int_{\Omega_\varepsilon} (u^\varepsilon(t)\log u^\varepsilon(t) + v^\varepsilon(t)\log v^\varepsilon(t))\,dx
$$

implies

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \omega(u^\varepsilon(t) + v^\varepsilon(t))\,dx &\le \frac{t}{2\,\nu}\int_{\mathbb{R}} (\omega')^2(x)\,dx + \int_{\mathbb{R}} \omega(u_0 + v_0)\,dx \\
&\quad + \int_{\mathbb{R}} (u_0(x)\,|\log u_0(x)| + v_0(x)\,|\log v_0(x)|)\,dx \\
&\quad + \int_{\Omega_\varepsilon} (u^\varepsilon(t)\log^- u^\varepsilon(t) + v^\varepsilon(t)\log^- v^\varepsilon(t))\,dx \\
&\quad + o\,(\varepsilon)\,,
\end{aligned} \tag{3.16}
$$

where $\log^- r$ denotes the negative part of the logarithm.

By the classical inequality $z\log^- z \le y - z\log y$, $0 < z,\ y \le 1$, choosing $y = \exp(-\omega(x)/2)$, and $z = u_\varepsilon(x)$, we obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon} u^\varepsilon \log^- u^\varepsilon \,dx \le \int_{\mathbb{R}} \exp\Big(-\frac{\omega(x)}{2}\Big)\,dx + \frac{1}{2}\int_{\Omega_\varepsilon} \omega\, u^\varepsilon \,dx\,,\\
&\int_{\Omega_\varepsilon} v^\varepsilon \log^- v^\varepsilon \,dx \le \int_{\mathbb{R}} \exp\Big(-\frac{\omega(x)}{2}\Big)\,dx + \frac{1}{2}\int_{\Omega_\varepsilon} \omega\, v^\varepsilon \,dx\,.
\end{aligned}\tag{3.17}$$

Finally, for any $\varepsilon \le 1$, making use of inequalities (3.17) on the right-hand side of (3.16) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_\varepsilon} \omega(u^\varepsilon(t) + v^\varepsilon(t))\,dx\\
&\qquad \le \frac{t}{2\,\nu}\int_{\mathbb{R}} (\omega')^2(x)\,dx + 2\int_{\mathbb{R}} \exp\Big(-\frac{\omega(x)}{2}\Big)\,dx + \int_{\mathbb{R}} \omega(u_0 + v_0)\,dx\\
&\qquad\quad + \int_{\mathbb{R}} (u_0(x)|\log u_0(x)| + v_0(x)\,|\log v_0(x)|)\,dx + c\,.
\end{aligned}\tag{3.18}$$

In conclusion, for any $t \ge 0$ we obtained the bound

$$\int_{\Omega_\varepsilon} \omega(u^\varepsilon(t) + v^\varepsilon(t))\,dx \le c_1(t, u_0, v_0)\,,\tag{3.19}$$

where the constant c_1 does not depend on ε. By applying (3.19) into (3.17) we obtain an upper bound for the negative part of $u^\varepsilon(t)\log^- u^\varepsilon(t) + v^\varepsilon(t)\log^- v^\varepsilon(t)$ in terms of c_1. Hence, if the initial data satisfy conditions (3.1) and (3.2), for any $T > 0$ there exists a constant C_T, depending only on u_0 and v_0, such that, for all $t \le T$ and $\varepsilon > 0$,

$$\begin{aligned}
&\int_{\Omega_\varepsilon} (u^\varepsilon(t)\,|\log u^\varepsilon(t)|\\
&\qquad + v^\varepsilon(t)\,|\log v^\varepsilon(t)| + \omega(u^\varepsilon(t) + v^\varepsilon(t)))\,dx \le C_T\,.
\end{aligned}\tag{3.20}$$

Now, by (3.14) we argue that for any $\alpha \le 1$ and $T > 0$, j_ε is bounded in $L^2([0,T]\times\Omega_\varepsilon)$. In addition, in view of (3.9), if $\alpha \le 1$, $\rho_\varepsilon^{(\alpha-1)/2} j_\varepsilon$ is bounded in $L^2([0,T]\times\Omega_\varepsilon)$. Next, if $|\alpha| \le 1$, $\rho_\varepsilon^\alpha j_\varepsilon$ is bounded in $L^2([0,T]\times\Omega_\varepsilon)$. This follows by (3.12) when $\alpha < 0$, and by (3.11) and the L^∞-bound (3.10) when $\alpha \ge 0$.

By the Propositions 2.3 and 2.4, it follows that the previous bounds can be extended to all initial data satisfying (3.1) and (3.2) and to all of $\mathbb{R}$ when α is positive, to all initial data satisfying (3.1) and (3.2) and to any domain Ω_ε when $-1 \leq \alpha \leq 0$. In conclusion we proved

Theorem 3.1. *Let* $0 \leq u_0,\ v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *satisfy conditions* (3.1) *and* (3.2).

i) *Let* $0 \leq \alpha \leq 1$. *Then, for all* $T > 0$, *and* $\varepsilon > 0$ *there exist constants* $d_1 = d_1(u_0, v_0, T)$, $d_2 = d_2(u_0, v_0, T)$ *and* $d_3 = d_3(u_0, v_0, T)$ *such that, the unique solution* $u^\varepsilon(x,t), v^\varepsilon(x,t)$ *of the initial value problem for system* (1.5) *satisfies*

$$\int_0^T \int_{\mathbb{R}} \Big(\frac{u^\varepsilon(x,t) - v^\varepsilon(x,t)}{\varepsilon}\Big)^2 \,dx\,dt \leq d_1\ , \tag{3.21}$$

$$\int_0^T \int_{\mathbb{R}} \Big(\frac{u^\varepsilon(x,t) - v^\varepsilon(x,t)}{\varepsilon}\Big)^2 \cdot (u^\varepsilon(x,t) + v^\varepsilon(x,t))^{\alpha-1} \,dx\,dt \leq d_2\ , \tag{3.22}$$

$$\int_{\mathbb{R}} (u^\varepsilon(t)\,|\log u^\varepsilon(t)| + v^\varepsilon(t)\,|\log v^\varepsilon(t)| + \omega(u^\varepsilon(t) + v^\varepsilon(t)))\,dx \leq d_3\ . \tag{3.23}$$

ii) *Let* $-1 \leq \alpha < 0$. *Then, for all* $T > 0$, *and* $k \in \mathbb{N}^+$, *the unique solution* $u^\varepsilon(x,t), v^\varepsilon(x,t)$ *of the initial-boundary value problem for system* (1.5) *on the domain* $[-1/\varepsilon^k, 1/\varepsilon^k]$ *satisfies*

$$\int_0^T \int_{-1/\varepsilon^k}^{1/\varepsilon^k} \Big(\frac{u^\varepsilon(x,t) - v^\varepsilon(x,t)}{\varepsilon}\Big)^2 \,dx\,dt \leq d_1\ , \tag{3.24}$$

$$\int_0^T \int_{-1/\varepsilon^k}^{1/\varepsilon^k} \Big(\frac{u^\varepsilon(x,t) - v^\varepsilon(x,t)}{\varepsilon}\Big)^2 \cdot (u^\varepsilon(x,t) + v^\varepsilon(x,t))^{2\alpha} \,dx\,dt \leq d_2\ , \tag{3.25}$$

$$\int_{-1/\varepsilon^k}^{1/\varepsilon^k} (u^\varepsilon(t)|\log u^\varepsilon(t)| + v^\varepsilon(t)\,|\log v^\varepsilon(t)| + \omega(u^\varepsilon(t) + v^\varepsilon(t)))\,dx \leq d_3\ . \tag{3.26}$$

iii) *Let* $\alpha < -1$. *Given* $k \in \mathbb{N}^+$, *let* $\mu > k(1+\beta)$, *and let* u_0^ε, v_0^ε *be defined by* (3.3). *Then, for all* $T > 0$, *the unique solution of the initial-boundary value problem for system* (1.5) *on the domain* $[-1/\varepsilon^k, 1/\varepsilon^k]$, *with initial data* u_0^ε, v_0^ε, *satisfies the bounds* (3.24), (3.26) *and*

$$\int_0^T \int_{-1/\varepsilon^k}^{1/\varepsilon^k} \Big(\frac{u^\varepsilon(x,t) - v^\varepsilon(x,t)}{\varepsilon}\Big)^2 \cdot (u^\varepsilon(x,t) + v^\varepsilon(x,t))^{2\tau}\, dx\, dt \le d_2 \,, \tag{3.27}$$

for all $0 < \tau \le (|\alpha| + 1)/2$.

REMARK 3.1. Let $\alpha < 0$. Given $\varepsilon > 0$, the velocity of propagation of the hyperbolic system (1.5) is exactly $1/\varepsilon$. This means that, given any time $T > 0$, for $t \le T$ the solution $u^\varepsilon(x,t)$, $v^\varepsilon(x,t)$ on the interval $[-(1+T)/\varepsilon, (1+T)/\varepsilon]$ depends only on the initial values on the interval $[-(1+2T)/\varepsilon, (1+2T)/\varepsilon]$, provided the boundaries are located at $[-1/\varepsilon^k, 1/\varepsilon^k]$, for k large enough. In other words, for ε small enough, the presence of the boundaries does not affect the solution on the interval $[-(1+T)/\varepsilon, (1+T)/\varepsilon]$. This explain why the presence of the boundaries does not affect the solution in any bounded set in the limit procedure.

4. Limit theorems.

The macroscopic equations for the system (1.5) can be expressed in terms of the mass density $\rho_\varepsilon(x,t)$ and of the rescaled flux

$$j_\varepsilon(x,t) = \frac{u_\varepsilon(x,t) - v_\varepsilon(x,t)}{\varepsilon} \tag{4.1}$$

as follows

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \dfrac{\partial j_\varepsilon}{\partial x} = 0\,, \\ \varepsilon^2 \dfrac{\partial j_\varepsilon}{\partial t} + \dfrac{\partial \rho_\varepsilon}{\partial x} = -2\,\rho_\varepsilon^\alpha\, j_\varepsilon\,, \qquad x \in \mathbb{R},\ t \ge 0\,. \end{cases} \tag{4.2}$$

In this section, we study the limiting behaviour, as ε goes to zero, of the solutions $(\rho_\varepsilon, j_\varepsilon)$ to system (4.2). In our passage to the limit, we will consider various relatively compact sequences. In these cases, without risk of misunderstanding, when we say that the sequence converges to a limit, we mean that there exists a subsequence that converges to a limit.

In the rest of the section, we will consider initial values for the kinetic problem (1.5) satisfying the conditions of Theorem 3.1.

Let us first consider the case $0 \leq \alpha < 1$. In this case, by Proposition 2.3, if the initial values u_0, v_0 belong to $L^1 \cap L^\infty$, (2.9) holds. Hence we see that, for all $T > 0$

$$\lim_{h \to 0} \sup_{0 \leq t \leq T} \|\rho_\varepsilon(x+h,t) - \rho_\varepsilon(x,t)\|_1 = 0\,. \tag{4.3}$$

Moreover, by (3.23) we deduce that

$$\lim_{R \to \infty} \sup_{t \in [0,T]} \int_{|x|>R} \rho_\varepsilon(x,t)\, dx = 0\,. \tag{4.4}$$

In addition, by Theorem 3.1, $\{j_\varepsilon\}$ is bounded in L^2, and thus by (4.2) $\partial \rho_\varepsilon / \partial t$ is bounded in $L^2(0,T; H^{-1}_{\mathrm{loc}})$ for all $T > 0$. In view of the bound (2.3) $\{\rho_\varepsilon\}$ is bounded in $L^\infty(\mathbb{R})$.

These bounds, combined with (4.3) and (4.4), imply that the family $\{\rho_\varepsilon\}$ is relatively compact in $C([0,T]; L^1(\mathbb{R}))$ for all $T > 0$.

Hence $\rho_\varepsilon^\alpha j_\varepsilon \longrightarrow \rho^\alpha j$ in L^2-weak, $\varepsilon^2 j_\varepsilon \longrightarrow 0$ strongly in L^2 and from the flux equation (4.2) we deduce

$$\frac{\partial \rho}{\partial x} = -2\, \rho^\alpha j\,, \tag{4.5}$$

at least in the sense of distributions (and in fact in L^2). Considering that $\rho \in L^\infty$, (4.5) implies that we have, at least formally

$$j = -\frac{1}{2\,(1-\alpha)}\, \frac{\partial \rho^{1-\alpha}}{\partial x} \tag{4.6}$$

in L^2. This fact follows indeed from (4.5), since $\rho \in H^1_x$ and thus

$$\frac{\partial}{\partial x}\, \frac{(\varepsilon + \rho)^{1-\alpha}}{1-\alpha} = -2\, \frac{\rho^\alpha}{(\varepsilon+\rho)^\alpha}\, j\,,$$

which converges in L^2 to $-2\, j\, \mathbf{1}_{\{\rho>0\}}$. Therefore, $\rho^{1-\alpha} \in H^{-1}_x$, and (4.6) holds on the set $\{\rho > 0\}$. In addition, the entropy bound (3.22) shown in Section 3 implies at the limit that $\rho^{(\alpha-1)/2} j \in L^2(\mathbb{R} \times (0,T))$

for all $T > 0$. Thus, if $\alpha < 1$, $j = 0$, almost everywhere on $\rho = 0$, almost everywhere, and the proof of (4.6) is complete.

If we now replace (4.6) in the continuity equation, we recover that the limit density ρ satisfies the fast diffusion equation

$$\frac{\partial \rho}{\partial t} - \frac{1}{2\,(1-\alpha)} \frac{\partial^2 \rho^{1-\alpha}}{\partial x^2} = 0\,. \tag{4.7}$$

Since we assumed the initial values $u_0, v_0 \in L^1 \cap L^\infty$, so is the initial density $\rho_0(x) = \rho(x, t = 0)$. On the other hand, the fast diffusion equation (4.7) has a unique global solution in D', provided $\rho(0, x) \in L^1_{\rm loc}(\mathbb{R})$ (*cf.* M.A. Herrero and M. Pierre [HePi]). The uniqueness result guarantees the existence of a unique limit point for the whole family.

Hence, we obtained the result of [PTo] as a particular case ($\alpha > 0$) of the limit behaviour of the system (4.2).

Theorem 4.1. *Let $0 \le \alpha < 1$, and let $(\rho_\varepsilon, j_\varepsilon)$ be a sequence of solutions to the initial value problem for the system* (4.2)*, where the initial values u_0, v_0 satisfy the hypotheses of Proposition* 2.3 *and Theorem* 3.1*. Then, there exists $\rho \in L^1 \cap L^\infty$ such that $\rho_\varepsilon(x,t)$ converges to $\rho(x,t)$ strongly in $C([0,T]; L^1(\mathbb{R}))$ for all $T \ge 0$, while $\varepsilon j_\varepsilon$ converges to zero strongly in $L^2(\mathbb{R} \times [0,T])$. The limit density $\rho(x,t)$ is the (unique) weak solution to the Cauchy problem for the fast diffusion equation* (4.7)*, in $D'(\mathbb{R}) \times (0,\infty)$, with initial datum $\rho_0 = u_0 + v_0$.*

The proof for the case $-1 \le \alpha < 0$ is similar. By Theorem 3.1, we deduce that $n_\varepsilon = \rho_\varepsilon^\alpha j_\varepsilon$ converges to v in L^2-weak, while ρ_ε is relatively compact in $C([0,T]; L^1_{\rm loc}(\mathbb{R}))$ for all $T > 0$. In addition $\varepsilon^2 j_\varepsilon \to 0$ strongly in L^2, exactly as in the case $\alpha > 0$. Let us rewrite the second equation of system (4.2) in the form

$$\varepsilon^2 \frac{\partial j_\varepsilon}{\partial t} + \frac{\partial \rho_\varepsilon}{\partial x} = -2\, n_\varepsilon\ . \tag{4.8}$$

Passing to the limit in (4.8), we deduce

$$n = -\frac{1}{2} \frac{\partial \rho}{\partial x} \tag{4.9}$$

in D'. Now, from the continuity equation, considering that

$$j_\varepsilon = \rho_\varepsilon^{|\alpha|}\, n_\varepsilon \longrightarrow -\frac{\rho^{|\alpha|}}{2} \frac{\partial \rho}{\partial x}$$

in L^2-weak, we obtain that the limit density satisfies (in a weak sense) the slow diffusion equation

$$\frac{\partial \rho}{\partial t} - \frac{1}{2\,(1+|\alpha|)}\,\frac{\partial^2 \rho^{1+|\alpha|}}{\partial x^2} = 0\,. \tag{4.10}$$

We proved

Theorem 4.2. *Let $-1 \le \alpha < 0$, and let $(\rho_\varepsilon, j_\varepsilon)$ be a sequence of solutions to the initial-boundary value problem for the system* (4.2), *where the initial values u_0, v_0 satisfy the hypotheses of Proposition* 2.3 *and Theorem* 3.1*. Then, there exists $\rho \in L^1_{\text{loc}} \cap L^\infty$ such that $\rho_\varepsilon(x,t)$ converges to $\rho(x,t)$ in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$ for all $T \ge 0$, while $\varepsilon j_\varepsilon(x,t)$ converges to zero strongly in $L^2_{\text{loc}}(\mathbb{R}\times[0,T])$. The limit density $\rho(x,t)$ is the* (*unique*) *weak solution to the Cauchy problem for the slow diffusion equation* (4.10)*, in $D'((0,\infty)\times\mathbb{R})$, with initial datum $\rho_0 = u_0 + v_0$.*

Remark 4.1. An easy consequence of the previous results is that both $u_\varepsilon(x,t)$ and $v_\varepsilon(x,t)$, solutions to the initial (if $0 \le \alpha < 1$) (or initial-boundary (if $-1 \le \alpha < 0$)) value problem for the kinetic system (1.5) converge strongly to $\rho/2$, where ρ is the solution of the corresponding nonlinear diffusion equation.

We will now examine the case $\alpha < -1$. The main argument in our proof of the passage to the limit will be the "div-curl" lemma of compensated compactness theory (see F. Murat [Mu1], [Mu2] and L. Tartar [Ta1], [Ta2]).

Lemma 4.1. *Let A be an open set of $\mathbb{R}^n$, and v_ε and w_ε be two sequences such that*

$$\begin{aligned} v_\varepsilon &\longrightarrow v\,, \qquad \textit{in } [L^2(A)]^n\textit{-weak}\,,\\ w_\varepsilon &\longrightarrow w\,, \qquad \textit{in } [L^2(A)]^n\textit{-weak}\,, \end{aligned} \tag{4.11}$$

$$\operatorname{div} v_\varepsilon \textit{ is bounded in } L^2(A) \; (\textit{or compact in } H^{-1}(A))\,, \tag{4.12}$$

$$\operatorname{curl} w_\varepsilon \textit{ is bounded in } [L^2(A)]^n \; (\textit{or compact in } [H^{-1}(A)]^{n^2})\,.$$

Let $\langle\cdot\,,\cdot\rangle$ denote the inner product in $\mathbb{R}^n$, i.e. $\langle v, w\rangle = \sum_{i=1}^n v_i\,w_i$. Then

$$\langle v_\varepsilon, w_\varepsilon\rangle \longrightarrow \langle v, w\rangle \textit{ in } D'\,. \tag{4.13}$$

Let us take initial data satisfying part iii) of Theorem 3.1. Since, with these hypotheses, Proposition 2.2 holds, $\rho_\varepsilon \geq 2\,\varepsilon^\mu$, and, for given $n \in \mathbb{N}$, $n \geq 1$, the system (4.2) is equivalent to

$$\left\{ \begin{aligned} & \frac{\partial \rho_\varepsilon}{\partial t} + \frac{\partial j_\varepsilon}{\partial x} = 0\,, \\ & \varepsilon^2\,\rho_\varepsilon^n\,\frac{\partial j_\varepsilon}{\partial t} + \frac{1}{n+1}\,\frac{\partial \rho_\varepsilon^{n+1}}{\partial x} = -2\,\rho_\varepsilon^{\alpha+n} j_\varepsilon\;. \end{aligned} \right. \tag{4.14}$$

The second equation of the system (4.14) can be written as follows

$$\frac{\partial}{\partial t}\,(\varepsilon^2 \rho_\varepsilon^n j_\varepsilon) + \frac{1}{n+1}\,\frac{\partial \rho_\varepsilon^{n+1}}{\partial x} - n\,\varepsilon^2 j_\varepsilon\,\rho_\varepsilon^{n-1}\,\frac{\partial \rho_\varepsilon}{\partial t} = -2\,\rho_\varepsilon^{\alpha+n} j_\varepsilon\;. \tag{4.15}$$

Making use of the continuity equation, we deduce

$$\begin{aligned} -n\,\varepsilon^2 j_\varepsilon\,\rho_\varepsilon^{n-1}\,\frac{\partial \rho_\varepsilon}{\partial t} &= \frac{n}{2}\,\varepsilon^2 \rho_\varepsilon^{n-1}\,\frac{\partial j_\varepsilon^2}{\partial x} \\ &= \frac{n}{2}\,\varepsilon^2\,\frac{\partial}{\partial x}\,(\rho_\varepsilon^{n-1} j_\varepsilon^2) - \frac{n\,(n-1)}{2}\,\varepsilon^2 \rho_\varepsilon^{n-2} j_\varepsilon^2\,\frac{\partial \rho_\varepsilon}{\partial x}\;. \end{aligned} \tag{4.16}$$

Hence, choosing $n = 1$, we conclude that the system (4.14) is equivalent to

$$\left\{ \begin{aligned} & \frac{\partial \rho_\varepsilon}{\partial t} + \frac{\partial j_\varepsilon}{\partial x} = 0\,, \\ & \frac{\partial}{\partial t}\,(\varepsilon^2 \rho_\varepsilon\, j_\varepsilon) + \frac{\partial}{\partial x}\Big(\frac{\rho_\varepsilon^2}{2} + \frac{\varepsilon^2}{2}\,j_\varepsilon^2\Big) = -2\,\rho_\varepsilon^{\alpha+1}\, j_\varepsilon\;. \end{aligned} \right. \tag{4.17}$$

If $n > 1$, we substitute the result of (4.16) into (4.15) to obtain

$$\begin{aligned} \frac{\partial}{\partial t}\,(\varepsilon^2 \rho_\varepsilon^n\, j_\varepsilon) + \frac{\partial}{\partial x}\Big(\frac{\rho_\varepsilon^{n+1}}{n+1} + \varepsilon^2\,\frac{n}{2}\,\rho_\varepsilon^{n-1} j_\varepsilon^2\Big) - \varepsilon^2\,\frac{n\,(n-1)}{2}\,j_\varepsilon^2\,\rho_\varepsilon^{n-2}\,\frac{\partial \rho_\varepsilon}{\partial x} \\ = -2\,\rho_\varepsilon^{\alpha+n}\, j_\varepsilon\;. \end{aligned} \tag{4.18}$$

The term $\partial \rho_\varepsilon/\partial x$ in (4.18) can be evaluated by the second equation of the system (4.2), to give

$$\begin{aligned} -\varepsilon^2\,\frac{n\,(n-1)}{2}\,j_\varepsilon^2\,\rho_\varepsilon^{n-2}\,\frac{\partial \rho_\varepsilon}{\partial x} \\ = n\,(n-1)\,\varepsilon^2 j_\varepsilon^3 \rho_\varepsilon^{-2+\alpha} + \varepsilon^4\,\frac{n\,(n-1)}{6}\,\rho_\varepsilon^{n-2}\,\frac{\partial j_\varepsilon^3}{\partial t}\;. \end{aligned} \tag{4.19}$$

If we now take $n = 2$, we find that the second equation of system (4.2) can be written as follows

$$\frac{\partial}{\partial t}\Big(\varepsilon^2 \rho_\varepsilon^2 j_\varepsilon + \varepsilon^4 \frac{j_\varepsilon^3}{3}\Big) + \frac{\partial}{\partial x}\Big(\frac{\rho_\varepsilon^3}{3} + \varepsilon^2 \rho_\varepsilon j_\varepsilon^2\Big) = -2\,\rho_\varepsilon^{\alpha+2} j_\varepsilon - 2\,\varepsilon^2 j_\varepsilon^3 \,\rho_\varepsilon^{-2+\alpha}\,. \tag{4.20}$$

The general formula follows easily by induction. In particular, consider that, given $\alpha < -2$, there exists $n \in \mathbb{N}$ such that $\tau = |\alpha| - 2\,n \leq (|\alpha|+1)/2$. This implies, by Theorem 3.1, that $\rho_\varepsilon^{-\tau} j_\varepsilon$ is bounded in L^2. With this choice, the second equation of the system (4.14) can be written in equivalent form as

$$\frac{\partial}{\partial t}\Big(\varepsilon^2 \sum_{k=0}^{n} a_{n,k}\,\varepsilon^{2k} \rho_\varepsilon^{2(n-k)} j_\varepsilon^{2k+1}\Big) + \frac{\partial}{\partial x}\Big(\frac{\rho_\varepsilon^{2n+1}}{2n+1} + \sum_{k=1}^{n} b_{n,k}\,\varepsilon^{2k} \rho_\varepsilon^{2(n-k)+1} j_\varepsilon^{2k}\Big) = -2\Big(\rho_\varepsilon^{-\tau} j_\varepsilon + \sum_{k=1}^{n} c_{n,k}\,\varepsilon^{2k} \rho_\varepsilon^{-\tau-2k} j_\varepsilon^{2k+1}\Big)\,, \tag{4.21}$$

where $a_{n,k}, b_{n,k}$ and $c_{n,k}$ are suitable bounded constants that can be computed explicitly.

The previous argument allows us to prove the following

Proposition 4.1. *Let $0 \leq u_0, v_0$ satisfy the hypotheses of Theorem 3.1. Given $\alpha < -1$, let $n \in \mathbb{N}$ be such that $0 \leq \tau = |\alpha| - 2\,n \leq (|\alpha|+1)/2$. Then, if $\rho = w^* - \lim \rho_\varepsilon$ in L^∞,*

$$\begin{aligned} &\rho_\varepsilon^{2n+1} \longrightarrow \rho^{2n+1}\,, \qquad \text{in } D'\,, \\ &\rho_\varepsilon \longrightarrow \rho\,, \qquad \text{in } L^p_{\mathrm{loc}} \text{ for all } 1 \leq p < \infty\,. \end{aligned} \tag{4.22}$$

PROOF. Let us set $U_\varepsilon = (\rho_\varepsilon, j_\varepsilon)$. The continuity equation (4.2) becomes $\{\operatorname{div} U_\varepsilon\} = 0$. Given any region $A \Subset \mathbb{R} \times \mathbb{R}^+$, from equation (4.21) we see that the right-hand side is bounded in $L^2(A)$. In fact, by definition of j_ε, $|j_\varepsilon/\rho_\varepsilon| \leq 1/\varepsilon$. Thus

$$\int_A (\varepsilon^{2k} \rho_\varepsilon^{-\tau-2k} j_\varepsilon^{2k+1})^2\,dx\,dt \leq \int_A \rho_\varepsilon^{-2\tau} j_\varepsilon^2\,dx\,dt \tag{4.23}$$

and the above integral is bounded in view of (3.27). Moreover

$$\begin{aligned} \int_A |\varepsilon^{2k} \rho_\varepsilon^{-\tau-2k} j_\varepsilon^{2k+1}|\,dx\,dt &= \varepsilon^{2k} \int_A \rho_\varepsilon^{-\tau-1} j_\varepsilon^2 \Big|\frac{j_\varepsilon}{\rho_\varepsilon}\Big|^{2k-1} dx\,dt \\ &\leq \varepsilon \int_A \rho_\varepsilon^{-\tau-1} j_\varepsilon^2\,dx\,dt\,, \end{aligned} \tag{4.24}$$

which implies that $\varepsilon^{2k}\rho_\varepsilon^{-\tau-2k}j_\varepsilon^{2k+1}$ converges to 0 strongly in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Let us set $V_\varepsilon = (p_\varepsilon, q_\varepsilon)$, where

$$
\begin{aligned}
p_\varepsilon &= -\frac{\rho_\varepsilon^{2n+1}}{2\,n+1} - \sum_{k=1}^{n} b_{n,k}\,\varepsilon^{2k}\rho_\varepsilon^{2(n-k)+1} j_\varepsilon^{2k}\,, \\
q_\varepsilon &= \varepsilon^2 \sum_{k=0}^{n} a_{n,k}\,\varepsilon^{2k}\rho_\varepsilon^{2(n-k)} j_\varepsilon^{2k+1}\,.
\end{aligned} \tag{4.25}
$$

Then, equation (4.21) shows that $\{\mathrm{curl}\, V_\varepsilon\}$ is bounded in $L^2(A)$ for all A as before. Since V_ε is also bounded in $L^2(A)$, as can be easily checked from the definitions of p_ε and q_ε with the same argument leading to (4.23), we are in a position to apply the div-curl lemma 4.1, and deduce that the product $\langle U_\varepsilon, V_\varepsilon\rangle$ converges (along subsequences) in D' to $\langle U, V\rangle$, where

$$
\begin{aligned}
U &= w - \lim U_\varepsilon = (\rho, j)\,, \\
V &= w - \lim V_\varepsilon = (p, q)\,.
\end{aligned} \tag{4.26}
$$

By the same bounds we used in (4.24), we deduce that

$$
\sum_{k=1}^{n} b_{n,k}\,\varepsilon^{2k}\rho_\varepsilon^{2(n-k)+1} j_\varepsilon^{2k} \longrightarrow 0\,, \quad \sum_{k=1}^{n} \rho_\varepsilon\, b_{n,k}\,\varepsilon^{2k}\rho_\varepsilon^{2(n-k)+1} j_\varepsilon^{2k} \longrightarrow 0\,,
$$

$$
\varepsilon^2 \sum_{k=0}^{n} a_{n,k}\,\varepsilon^{2k}\rho_\varepsilon^{2(n-k)} j_\varepsilon^{2k+1} \longrightarrow 0\,, \quad \varepsilon^2 j_\varepsilon \sum_{k=0}^{n} a_{n,k}\,\varepsilon^{2k}\rho_\varepsilon^{2(n-k)} j_\varepsilon^{2k+1} \longrightarrow 0\,,
$$

strongly in $L^1(A)$. Hence we see that (along subsequences)

$$
p = w - \lim\Big(-\frac{\rho_\varepsilon^{2n+1}}{2\,n+1}\Big)\,, \qquad q = 0\,, \tag{4.27}
$$

while

$$
\langle U, V\rangle = -w - \lim\Big(\frac{\rho_\varepsilon^{2n+2}}{2\,n+1}\Big)\,. \tag{4.28}
$$

In other words, we have shown that

$$
w - \lim \rho_\varepsilon^{2n+2} = (w - \lim \rho_\varepsilon^{2n+1})\,\rho\,. \tag{4.29}
$$

As is well-known in such contexts, the conclusion of Proposition 4.1 automatically holds. One possible proof consists in recalling that, by convexity,

$$w - \lim \rho_\varepsilon^{2n+2} \geq (w - \lim \rho_\varepsilon^{2n+1})^{(2n+2)/(2n+1)} ,$$

so that

$$w - \lim \rho_\varepsilon^{2n+2} \geq \rho^{2n+2} . \tag{4.30}$$

Inequality (4.30), combined with (4.29) yields that $w - \lim \rho_\varepsilon^{2n+2} = \rho^{2n+2}$.

We may also use Minty's trick, as in [Lio], [MaMi] to conclude that $w - \lim \rho_\varepsilon^{2n+1} = \rho^{2n+1}$. In both cases, we deduce the strong convergence of ρ_ε to ρ using the strict convexity of $f^m(t)$, $t \in [0, \infty)$, for $m \geq 1$.

We are now able to handle the singular case.

Theorem 4.3. *Let $\alpha < -1$, and let $(\rho_\varepsilon, j_\varepsilon)$ be a sequence of solutions to the initial-boundary value problem for the system* (4.2)*, where the initial values $u_0^\varepsilon, v_0^\varepsilon$ for the kinetic system* (1.5) *satisfy the hypotheses of Theorem* 3.1*, part* iii)*. Then, there exists $\rho \in L^1_{\text{loc}} \cap L^\infty$ such that $\rho_\varepsilon(x,t)$ converges to $\rho(x,t)$ in $L^p_{\text{loc}}(\mathbb{R} \times (0,T])$ for all $p \in [1, +\infty)$ and all $T > 0$, while $\varepsilon j_\varepsilon(x,t)$ converges to zero strongly in $L^2_{\text{loc}}(\mathbb{R} \times [0,T])$. The limit density $\rho(x,t)$ is the weak solution to the Cauchy problem for the porous media equation in $D'(\mathbb{R} \times (0,T])$, with initial datum ρ_0 that is the weak limit of $\rho_\varepsilon(x, t=0) = u_0^\varepsilon + v_0^\varepsilon$.*

Proof. Let us rewrite the second equation of the system (4.2) in the equivalent form (4.21), where n has been chosen in such a way that τ satisfies the hypotheses of Theorem (3.1) (bound (3.27)), and at the same time $\tau > 1$. Let us remark that this is always possible in view of (4.17) and (4.20). Let

$$v_\varepsilon = -2\Big(\rho_\varepsilon^{-\tau} j_\varepsilon + \sum_{k=1}^{n} c_{n,k}\, \varepsilon^{2k} \rho_\varepsilon^{-\tau-2k} j_\varepsilon^{2k+1}\Big) . \tag{4.31}$$

By the proof of Proposition 4.1 we see that the sequence $\{v_\varepsilon\}$ has a weak limit in L^2, and that

$$v = w - \lim v_\varepsilon = w - \lim (-2\, \rho_\varepsilon^{-\tau} j_\varepsilon) . \tag{4.32}$$

Using the definitions (4.25), (4.21) becomes

$$\frac{\partial q_\varepsilon}{\partial t} - \frac{\partial p_\varepsilon}{\partial x} = v_\varepsilon \,. \tag{4.32}$$

We pass to the limit in the sense of distributions in (4.32), and recall that q_ε converges to zero in D. We find

$$v = \frac{\partial}{\partial x}\frac{\rho^{2n+1}}{2n+1} \tag{4.33}$$

in D'. However, since $v \in L^2$, $\partial \rho^{2n+1}/\partial x \in L^2$ as well. Let us write now the continuity equation as

$$\frac{\partial \rho_\varepsilon}{\partial t} - \frac{\partial}{\partial x}\Big(\frac{\rho_\varepsilon^\tau v_\varepsilon}{2} + \sum_{k=1}^{n} c_{n,k}\,\varepsilon^{2k}\rho_\varepsilon^{-2k} j_\varepsilon^{2k+1}\Big) = 0\,. \tag{4.34}$$

Since $\rho_\varepsilon \longrightarrow \rho$ in L^p_{loc} strongly for $p \in [1, +\infty)$, and ρ_ε is bounded in L^∞, we have

$$\rho_\varepsilon^\tau v_\varepsilon \longrightarrow \rho^\tau v = (\rho^{2n+1})^{\tau/(2n+1)}\frac{\partial}{\partial x}\frac{\rho^{2n+1}}{2n+1}\,\frac{\partial}{\partial x}\frac{\rho^{1+|\alpha|}}{1+|\alpha|}\,, \tag{4.35}$$

in L^2-weak. Consequently, as $\varepsilon \longrightarrow 0$, for all $\phi \in C^\infty$, such that $\operatorname{supp}\phi \subset \mathbb{R} \times \mathbb{R}^+$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\Big(\phi_t\,\rho - \phi_x\,\frac{\rho^{1+|\alpha|}}{1+|\alpha|}\Big)\,dx\,dt + \int_{-\infty}^{+\infty}\phi(x,0)\,\rho(x,0)\,dx\,. \tag{4.36}$$

The Cauchy problem for the porous media equation, with initial data ρ_0 satisfying the hypotheses of the theorem, is well-posed in the weak sense (4.36). In fact, existence, uniqueness and continuous dependence on the data for (4.34) is known (*cf.* Aronson [Aro]). The existence theorem guarantees a unique limit to the singular perturbation problem (4.2).

5. Extension to higher dimensions.

In this section, we shall discuss the three dimensional model (1.7). In fact, all the results can be adapted to an arbitrary number of dimensions, and we just choose to emphasize the three dimensional example.

Moreover, the largest part of the one-dimensional arguments can be adapted to higher dimensions, so we just sketch the main differences.

We shall study equations (1.8) in the box $\Omega = (-a, a)^3$ with periodic boundary conditions. Besides, we will limit our analysis to the case of a rate function k of the type ρ^α, with $\alpha < 1$. In analogy with the one-dimensional model, we will also write system (1.7) in the equivalent form

$$\begin{cases} \dfrac{\partial u^\pm}{\partial t} \pm \dfrac{1}{\varepsilon}\,\dfrac{\partial u^\pm}{\partial x} = \dfrac{1}{\varepsilon^2}\,\rho^\alpha\,(\rho - 6\,u^\pm)\,, \\[2ex] \dfrac{\partial v^\pm}{\partial t} \pm \dfrac{1}{\varepsilon}\,\dfrac{\partial v^\pm}{\partial y} = \dfrac{1}{\varepsilon^2}\,\rho^\alpha\,(\rho - 6\,v^\pm)\,, \\[2ex] \dfrac{\partial w^\pm}{\partial t} \pm \dfrac{1}{\varepsilon}\,\dfrac{\partial w^\pm}{\partial z} = \dfrac{1}{\varepsilon^2}\,\rho^\alpha\,(\rho - 6\,w^\pm)\,, \end{cases} \tag{5.1}$$

where $u = (u^+, u^-, v^+, v^-, w^+, w^-) = (u_1, u_4, u_2, u_5, u_3, u_6)$.

As in the one-dimensional case, given any convex function $\varphi(r)$, $r \geq 0$, we deduce the estimate

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{6} \frac{\partial \varphi(u_i)}{\partial t}\,dx \\ &\qquad = -\frac{1}{2\,\varepsilon^2}\int_\Omega \rho^\alpha \sum_{i\neq j} (\varphi'(u_i) - \varphi'(u_j))\,(u_i - u_j)\,dx \\ &\qquad \leq 0\,. \end{aligned} \tag{5.2}$$

In particular, if $\varphi(r) = r^p$, $p > 1$, we obtain

$$\int_\Omega \frac{\partial}{\partial t}\sum_{i=1}^{6} u_i^p\,dx = -\frac{p}{2\,\varepsilon^2}\int_\Omega \rho^\alpha \sum_{i\neq j}(u_i - u_j)\,(u_i^{p-1} - u_j^{p-1})\,dx\,. \tag{5.3}$$

The existence theory follows as in the one-dimensional case, and we obtain the corresponding of Propositions 2.1 and 2.2.

Proposition 5.1. *Let $0 \leq \alpha < 1$, and let $0 \leq u_{0,j}(x) \in L^\infty(\Omega)$, $j = 1, \ldots, 6$. Then, the initial-boundary value problem for system* (5.1) *has a unique solution $u(x,t)$, such that, for $j = 1, \ldots, 6$, $u_j(x,t) \in L^\infty(\Omega \times (0,T)) \cap C([0,T]; L^p(\Omega))$ for all $T > 0$ and $1 \leq p < \infty$. In addition, the solution satisfies the following bound*

$$\max_{j\leq 6} \|u_j(\cdot\,, t_2)\|_\infty \leq \max_{j\leq 6} \|u_j(\cdot\,, t_1)\|_\infty\,, \qquad \textit{if}\; t_1 \leq t_2\,. \tag{5.4}$$

Proposition 5.2. *Let $\alpha < 0$, and let $u_{0,j}(x) \in L^\infty(\Omega)$ satisfy the lower bound $u_{0,j} \geq \delta$ on Ω for some $\delta > 0$. Then, the initial-boundary value problem for the system* (5.1) *has a unique solution bounded away from zero $u(x,t)$ such that for $j = 1, \dots, 6$, $u_j(x,t) \in L^\infty(\Omega \times (0,T)) \cap C([0,T]; L^p(\Omega))$ for all $T > 0$ and $1 \leq p < \infty$. In addition, this solution satisfies the bounds* (5.4) *and*

$$\inf_{\substack{x\in\Omega \\ j\leq 6}} u_j(x,t_1) \leq \inf_{\substack{x\in\Omega \\ j\leq 6}} u_j(x,t_2)\,, \qquad \text{if } t_1 \leq t_2\,. \tag{5.5}$$

Given $u = (u_1, \dots, u_6)$, let us now introduce the operator B^α defined by components by

$$B_j^\alpha u = -\frac{\partial u_j}{\partial x_j} + \rho^\alpha\,(\rho - 6\,u_j)\,, \qquad i = 1, 2, 3\,,$$

$$B_j^\alpha v = \frac{\partial u_j}{\partial x_j} + \rho^\alpha\,(\rho - 6\,u_j)\,, \qquad i = 4, 5, 6\,.$$

Then, the following lemma is immediate

Lemma 5.1. *Let $0 \leq \alpha \leq 1/5$. Then, the operator B^α is dissipative from the domain*

$$D^+(B^\alpha) = \{u \in [W^{1,1}(\Omega)]^6\}$$

into $[L^1(\Omega)]^6$.

If $-1 \leq \alpha < 0$, and if $\delta > 0$, the operator B_α is dissipative from the domain

$$D_\delta^+(B^\alpha) = \{u \in [W^{1,1}(\Omega)]^6,\ u_j \geq \delta,\ j = 1, \dots, 6\}$$

into $[L^1(\Omega)]^6$.

Thus, we obtain the analogue of Proposition 2.3.

Proposition 5.3. *Let $0 \leq u_{0,j}(x) \in L^1(\Omega) \cap L^\infty(\Omega)$, $j = 1, \dots, 6$. Then, provided $-1 \leq \alpha \leq 1/5$, the initial-boundary value problem* (5.1) *has a unique global solution $u(x,t)$ such that $u_j(x,t) \in L^\infty(\Omega \times (0,T)) \cap$*

$C([0,T];L^1(\Omega)\cap L^\infty(\Omega))$ for all $T \geq 0$ and $j = 1,\dots,6$. Moreover, for all $t \geq 0$

$$\sum_{j=1}^{6} \|u_j(x+h,t) - u_j(x,t)\|_1 \leq \sum_{j=1}^{6} \|u_{0,j}(x+h) - u_{0,j}(x)\|_1 \,. \tag{5.6}$$

If $0 \leq \alpha \leq 1/5$, these results extend to $\Omega = \mathbb{R}^3$.

As in Section 3, we can obtain entropy bounds, that allow us to pass to the limit. Let us briefly outline the main differences. We introduce functions $0 \leq u_{0,j} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $j = 1,\dots,6$ such that

$$\int_{\mathbb{R}^3} \sum_{j=1}^{6} u_{0,j}(x)\, |\log u_{0,j}(x)|\, dx = M_1 < \infty \tag{5.7}$$

and, for $\omega(x) = (1+x^2)^\beta$, $0 < \beta < 1/8$,

$$\int_{\mathbb{R}^3} \omega(x) \sum_{j=1}^{6} u_{0,j}^2(x)\, dx = M_2 < \infty \,. \tag{5.8}$$

Given $\gamma > 0$, let us denote with Ω_ε, the domain $(-1/\varepsilon^\gamma, 1/\varepsilon^\gamma)^3$. In addition, given $\mu > 0$, let for $j = 1,\dots,6$

$$u_{0,j}^\varepsilon = \max\{u_0(x), \varepsilon^\mu\} \,. \tag{5.9}$$

By Proposition 5.2, if $\alpha < 0$, the initial boundary value problem for the system (5.1), with periodic boundary conditions on Ω_ε, and initial values (5.9), has a unique global solution $u^\varepsilon(x,t)$.

Moreover, provided $\mu > \gamma\,(3+\beta)$, u_0^ε satisfies bounds (5.7) and (5.8) with different but finite constants M_1^ε and M_2^ε.

The proof of the entropy bounds follows along the same lines of Section 3. We only remark that, in consequence of (5.8), one has to study the time evolution of

$$\int_{\Omega_\varepsilon} \Big(\sum_j u_j^\varepsilon(x,t) \log u_j^\varepsilon(x,t) + \omega(x) \sum_j (u_j^\varepsilon(x,t))^2 \Big)\, dx \,.$$

Since the integrand on the right-hand side of (5.3) is nonpositive for any convex function $\varphi(r)$, choosing $\varphi(r) = r^2$ we obtain

$$\frac{d}{dt} \int_{\Omega_\varepsilon} \omega(x) \sum_j (u_j^\varepsilon(x,t))^2\, dx \\ - \frac{1}{\varepsilon} \sum_{j=1}^{3} \int_{\Omega_\varepsilon} \omega(x) \, \frac{\partial}{\partial x_j} \left((u_j^\varepsilon)^2 - (u_{j+3}^\varepsilon)^2\right) dx \leq 0 \,.$$

Hence, integrating by parts and making use of the periodicity, we deduce the analogous of (3.7)

$$\frac{d}{dt}\int_{\Omega_\varepsilon}\omega(x)\sum_j (u_j^\varepsilon(x,t))^2\,dx \\ -\frac{1}{\varepsilon}\sum_{j=1}^{3}\int_{\Omega_\varepsilon}\frac{\partial}{\partial x_j}\,\omega(x)\,((u_j^\varepsilon)^2-(u_{j+3}^\varepsilon)^2)\,dx \le 0\,.$$

We next observe that $(\partial/\partial x_j)\,\omega \in L^4(\mathbb{R}^3)$, $j=1,2,3$. This implies the analogous of (3.14)

A further step consists in the identification of the limit Maxwellian.

Let us set $p=2$ in (5.3). Then, integrating over time we get

$$\frac{1}{\varepsilon^2}\int_0^T\int_{\Omega_\varepsilon}\rho_\varepsilon^\alpha\sum_{i\ne j}(u_i^\varepsilon-u_j^\varepsilon)^2\,dx\,dt \le \int_{\Omega_\varepsilon}\sum_i u_{0,i}^2\,dx\,. \tag{5.10}$$

Now, considering that the solution satisfies the bound (5.4), if $\alpha<0$,

$$\rho_\varepsilon^\alpha \ge \Big(6\max_{i\le 6}\|u_{0,i}\|_\infty\Big)^\alpha = \nu^\alpha \tag{5.11}$$

and by (5.10) we obtain

$$\frac{\nu^\alpha}{\varepsilon^2}\int_0^T\int_{\Omega_\varepsilon}\sum_{i\ne j}(u_i^\varepsilon-u_j^\varepsilon)^2\,dx\,dt \le \int_{\Omega_\varepsilon}\sum_i u_{0,i}^2\,dx\,. \tag{5.12}$$

Inequality (5.12) implies that, for $i\ne j$, $u_i^\varepsilon-u_j^\varepsilon \longrightarrow 0$ strongly in $L^2([0,T]\times\Omega_\varepsilon)$.

The same conclusion can be derived when $0<\alpha<1$. Since in this case (5.11) does not hold, we shall make use of a different argument. Let us write (5.3) by taking $p=2-\alpha$. We obtain

$$\frac{2-\alpha}{2\,\varepsilon^2}\int_0^T\int_{\mathbb{R}^3}\rho_\varepsilon^\alpha\sum_{i\ne j}(u_i^\varepsilon-u_j^\varepsilon)\,((u_i^\varepsilon)^{1-\alpha}-(u_j^\varepsilon)^{1-\alpha})\,dx\,dt \\ \le \int_{\mathbb{R}^3}\sum_{i=1}^{6}u_{0,i}^{2-\alpha}\,dx\,. \tag{5.13}$$

This implies

$$(5.14)\qquad (1-\alpha)\int_0^T\int_{\mathbb{R}^3}\sum_{i\neq j}(u_i^\varepsilon-u_j^\varepsilon)^2\,dx\,dt\le\frac{\varepsilon^2}{2-\alpha}\int_{\mathbb{R}^3}\sum_i u_{0,i}^{2-\alpha}\,dx$$

and also in this case, $u_i^\varepsilon-u_j^\varepsilon\to 0$ strongly in $L^2([0,T]\times\mathbb{R}^3)$ for $i\neq j$.

The rest of the proof leading to Theorem 3.1 follows along the same lines of the one-dimensional situation. So we have

Theorem 5.1. *Let $0\le u_{0,j}\in L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$, $j=1,\dots,6$ satisfy conditions* (5.7) *and* (5.8).

i) *Let $0\le\alpha<1$. Then, for all $T>0$, and $\varepsilon>0$ there exist constants $d_1=d_1(u_0,T)$, $d_2=d_2(u_0,T)$ and $d_3=d_3(u_0,T)$ such that, the unique solution $u^\varepsilon(x,t)$ to the initial value problem for system* (5.1) *satisfies*

$$(5.15)\qquad \int_0^T\int_{\mathbb{R}^3}\Big(\frac{u_i^\varepsilon(x,t)-u_{i+3}^\varepsilon(x,t)}{\varepsilon}\Big)^2\,dx\,dt\le d_1\,,$$

$$(5.16)\qquad \int_0^T\int_{\mathbb{R}^3}\Big(\frac{u_i^\varepsilon(x,t)-u_{i+3}^\varepsilon(x,t)}{\varepsilon}\Big)^2\rho_\varepsilon(x,t)^{\alpha-1}\,dx\,dt\le d_2\,,$$

where, in both cases, $i=1,2,3$, and

$$(5.17)\qquad \int_{\mathbb{R}^3}\Big(\sum_i u_i^\varepsilon(t)\,|\log u_i^\varepsilon(t)|+\omega\sum_i(u_i^\varepsilon(t))^2\Big)\,dx\le d_3\,,$$

$$(5.18)\qquad \lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^3}\sum_{i\neq j}(u_i^\varepsilon-u_j^\varepsilon)^2\,dx\,dt=0\,.$$

ii) *Let $-1\le\alpha<0$. Then, for all $T>0$, and $k\in\mathbb{N}^+$, the unique solution $u^\varepsilon(x,t)$ to the initial-boundary value problem for system* (5.1) *on the domain $\Omega_\varepsilon=(-1/\varepsilon^k,1/\varepsilon^k)^3$ satisfies the bounds* (5.15), (5.16), (5.17) *and the property* (5.18) *in $[0,T]\times\Omega_\varepsilon$.*

iii) *Let $\alpha<-1$. Given $k\in\mathbb{N}^+$, let $\mu>k\,(3+\beta)$, and let u_0^ε be defined by* (5.9). *Then, for all $T>0$, the unique solution to the initial-boundary value problem for system* (5.1) *on the domain $\Omega_\varepsilon=$*

$(-1/\varepsilon^k, 1/\varepsilon^k)^3$, *with initial data* u_0^ε, *satisfies the bounds* (5.10), (5.15), (5.17) *and the property* (5.18) *in* Ω_ε. *In addition we have*

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\frac{u_i^\varepsilon(x,t) - u_{i+3}^\varepsilon(x,t)}{\varepsilon}\Big)^2 \rho_\varepsilon(x,t)^{-2\tau}\, dx\, dt \le d_2 \,, \tag{5.19}$$

where $i = 1, 2, 3$, *for all* $0 < \tau \le (|\alpha| + 1)/2$.

Let us denote by $j_\varepsilon(x,t)$ the flux, *i.e.*

$$j_\varepsilon = \Big(\frac{u_\varepsilon^+ - u_\varepsilon^-}{\varepsilon}, \frac{v_\varepsilon^+ - v_\varepsilon^-}{\varepsilon}, \frac{w_\varepsilon^+ - w_\varepsilon^-}{\varepsilon}\Big)\,. \tag{5.20}$$

Then, the macroscopic equations for the system (5.1) can be expressed in the form

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \operatorname{div} j_\varepsilon = 0\,, \\ \varepsilon^2 \dfrac{\partial j_\varepsilon}{\partial t} + \dfrac{1}{3}\operatorname{grad}\rho_\varepsilon = -6\,\rho_\varepsilon^\alpha j_\varepsilon + E_\varepsilon\,, \end{cases} \tag{5.21}$$

where E_ε denotes the vector

$$\begin{aligned} E_\varepsilon = \Big(&\frac{\partial}{\partial x}\Big(\frac{1}{3}\,\rho_\varepsilon - (u_\varepsilon^+ - u_\varepsilon^-)\Big), \\ &\frac{\partial}{\partial y}\Big(\frac{1}{3}\,\rho_\varepsilon - (v_\varepsilon^+ - v_\varepsilon^-)\Big), \frac{\partial}{\partial z}\Big(\frac{1}{3}\,\rho_\varepsilon - (w_\varepsilon^+ - w_\varepsilon^-)\Big)\Big)\,. \end{aligned} \tag{5.22}$$

The presence of the vector E_ε is the main difference between the one-dimensional system (4.2) and the three-dimensional system (5.21). This is a consequence of the fact that in more than one-dimension the system of the macroscopic equations can not be expressed only in terms of the mean quantities. Let us rewrite

$$E_\varepsilon = \Big(\frac{\partial}{\partial x}\, e_\varepsilon^1, \frac{\partial}{\partial y}\, e_\varepsilon^2, \frac{\partial}{\partial z}\, e_\varepsilon^3\Big)\,,$$

with obvious meaning of $e_\varepsilon^1, e_\varepsilon^2$, and e_ε^3. We have

$$e_\varepsilon^1 = \frac{1}{3}\,(v_\varepsilon^+ - u_\varepsilon^-) + \frac{1}{3}\,(v_\varepsilon^- - u_\varepsilon^-) + \frac{1}{3}\,(w_\varepsilon^+ - u_\varepsilon^+) + \frac{1}{3}\,(w_\varepsilon^- - u_\varepsilon^-)\,, \tag{5.23}$$

so that, thanks to (5.18) we conclude that e_ε^1 converges to zero strongly in $L^2_{x,t}$. The same conclusion holds for the other components. We are in a position to obtain the corresponding of Theorems 4.1 and 4.2, provided $-1 \le \alpha \le 1/5$. In this range of α, by dissipativity, we have the compactness of $\rho_\varepsilon(x,t)$ in L^1_x or L^1_{loc} if $\alpha < 0$. Let us remark that when α is positive, $1/5 < \alpha < 1$ we can derive the same result of compactness as a consequence of the "div-curl" Lemma 4.1. Let us set $U_\varepsilon = (j_\varepsilon, \rho_\varepsilon)$ and $V_\varepsilon = (0, \rho_\varepsilon)$. Then, the first of equations (5.21) reads $\operatorname{div} U_\varepsilon = 0$, and this obviously implies that $\operatorname{div} U_\varepsilon$ is bounded in $L^2_{x,t}$. In view of (5.15) we deduce that $\varepsilon^2 j_\varepsilon$ converges to zero strongly in $L^2_{x,t}$, and by (5.16) follows that $\rho_\varepsilon^\alpha j_\varepsilon$ is bounded in $L^2_{x,t}$. But $e_\varepsilon^1, e_\varepsilon^2$, and e_ε^3 converge to zero strongly in $L^2_{x,t}$. Consequently, from the second equation of the system (5.21) we obtain that $\operatorname{curl} V_\varepsilon$ is compact in $H^{-1}_{x,t}$, and $U_\varepsilon V_\varepsilon = \rho_\varepsilon^2$ passes to the limit. At this point, we deduce as in Section 4 that $\rho_\varepsilon(x,t)$ converges to $\rho(x,t)$ in L^p_{loc}, $1 \le p < \infty$. Finally, thanks to the bound (5.17), we can show that

$$\lim_{R\to\infty} \sup_{0\le t\le T} \int_{|x|>R} \rho_\varepsilon(x,t)\,dx = 0 \tag{5.24}$$

and, since ρ_ε is bounded in L^∞, ρ_ε converges to ρ in $L^p(\mathbb{R})^3$. So we have

Theorem 5.2. *Let $0 \le \alpha < 1$, and let $(\rho_\varepsilon, j_\varepsilon)$ be a sequence of solutions to the initial value problem for the system* (5.21)*, where the initial values $u_{0,j}$, $j = 1, \ldots, 6$ satisfy the hypotheses of Proposition* 5.1 *and Theorem* 5.1*. Then, there exists $\rho \in L^1 \cap L^\infty$ such that $\rho_\varepsilon(x,t)$ converges to $\rho(x,t)$ strongly in $C([0,T]; L^p(\mathbb{R}^3))$ for all $T \ge 0$ and $1 \le p < \infty$, while $\varepsilon\, j_\varepsilon$ converges to zero strongly in $L^2(\mathbb{R}^3 \times [0,T])$. The limit density $\rho(x,t)$ is the* (*unique*) *weak solution to the Cauchy problem for the fast diffusion equation*

$$\frac{\partial \rho}{\partial t} - \frac{1}{18\,(1-\alpha)}\,\Delta \rho^{1-\alpha} = 0\,, \qquad \text{in } D'(\mathbb{R}^3 \times (0,\infty))\,,$$

with initial datum $\rho_0 = \sum_{j=1}^6 u_{0,j}$.

Theorem 5.3. *Let $-1 \le \alpha < 0$, and let $(\rho_\varepsilon, j_\varepsilon)$ be a sequence of solutions to the initial-boundary value problem for the system* (5.21)*, where the initial values $u_{0,j}$, $j = 1, \ldots, 6$ satisfy the hypotheses of Proposition* 5.3 *and Theorem* 5.1*. Then, there exists $\rho \in L^1_{\text{loc}} \cap L^\infty$ such that $\rho_\varepsilon(x,t)$*

converges to $\rho(x,t)$ *in* $C([0,T];L^p_{\mathrm{loc}}(\mathbb{R}^3))$ *for all* $T\geq 0$ *and* $p\in[1,\infty)$, *while* $\varepsilon\, j_\varepsilon(x,t)$ *converges to zero strongly in* $L^2_{\mathrm{loc}}(\mathbb{R}^3\times[0,T])$. *The limit density* $\rho(x,t)$ *is the (unique) weak solution to the Cauchy problem for the slow diffusion equation*

$$\frac{\partial\rho}{\partial t}-\frac{1}{18\,(1+|\alpha|)}\,\Delta\rho^{1+|\alpha|}=0\,,\qquad \textit{in } D'(\mathbb{R}^3\times(0,\infty))\,,$$

with initial datum $\rho_0=\sum_{j=1}^{6}u_{0,j}$.

REMARK 5.1. As in Section 4 we observe that the $u_j^\varepsilon(x,t), j=1,2,\ldots,6$, solutions to the initial (if $0\leq\alpha<1$) (or initial-boundary, if $-1\leq\alpha<0$) value problem for the kinetic system (5.1) converge strongly to $\rho/6$, where ρ is the solution of the corresponding nonlinear diffusion equation.

We will now examine the case $\alpha<-1$. Let $\rho_{1,\varepsilon}=u_\varepsilon^+ + u_\varepsilon^-$, $\rho_{2,\varepsilon}=v_\varepsilon^+ + v_\varepsilon^-$, $\rho_{3,\varepsilon}=w_\varepsilon^+ + w_\varepsilon^-$, and let $j_{i,\varepsilon}, i=1,2,3$ be the components of j_ε, given by (5.20). Then, summing and subtracting the first two equations (5.1), we obtain the system

$$\left\{\begin{aligned}&\frac{\partial\rho_{1,\varepsilon}}{\partial t}+\frac{\partial j_{1,\varepsilon}}{\partial x}=\frac{2}{\varepsilon^2}\,\rho_\varepsilon^\alpha\,(\rho_\varepsilon-3\,\rho_{1,\varepsilon})\,,\\ &\varepsilon^2\,\frac{\partial j_{1,\varepsilon}}{\partial t}+\frac{\partial\rho_{1,\varepsilon}}{\partial x}=-2\,\rho_\varepsilon^\alpha\, j_{1,\varepsilon}\,,\qquad x\in\mathbb{R},\ t\geq 0\,.\end{aligned}\right. \tag{5.25}$$

The analysis of Section 4, following Lemma 4.1 can easily be applied to system (5.25). With few differences, due to the presence of the term on the right side of the first equation, we will arrive to analogous conclusions. In particular, since ρ_ε is bounded in L^∞, and, in view of (5.19) $\rho_\varepsilon^{-(|\alpha|+1)/2}j_{1,\varepsilon}$ is bounded in $L^2_{x,t}$, $\rho_\varepsilon^\alpha\rho_{1,\varepsilon}\,j_{1,\varepsilon}$ is bounded in $L^2_{x,t}$ provided $\alpha\geq-3$. The same argument shows that $j_{1,\varepsilon}\,\rho_\varepsilon^\alpha\,(\rho_\varepsilon-3\,\rho_{1,\varepsilon})/\varepsilon^2$ is bounded in $L^2_{x,t}$ if $\alpha\geq-3$. Hence, $(\partial/\partial x)\rho_{1,\varepsilon}^2$ is compact in $H^{-1}_{x,t}$.

By identical computations, summing and subtracting the third and fourth (respectively the fifth and sixth) equation (5.1), we deduce that, if $\alpha\geq-3$, both $(\partial/\partial y)\,\rho_{2,\varepsilon}^2$ and $(\partial/\partial z)\,\rho_{3,\varepsilon}^2$ are compact in $H^{-1}_{x,t}$. Moreover, in view of (5.18), $\rho_{i,\varepsilon}^2-\rho_{j,\varepsilon}^2$ converges to zero in $L^2_{x,t}$. So we conclude that $\operatorname{grad}\rho_\varepsilon^2$ is compact in $H^{-1}_{x,t}$.

This result, coupled with the first equation of the system (5.21) enables us to handle, by the "div-curl" lemma, the passage to the limit for $-3\leq\alpha<-1$, exactly as in the one-dimensional case.

The extension to $\alpha < -3$ follows by the same strategy we adopted above, along the same lines of the one-dimensional proof, first multiplying the second equation of the system (5.25) by $\rho_{1,\varepsilon}^n$, $n > 1$, and then recovering the equivalent system in which the terms on the right-hand sides are bounded in $L^2_{x,t}$.

Finally we prove

Theorem 5.4. *Let $\alpha < -1$, and let $(\rho_\varepsilon, j_\varepsilon)$ be a sequence of solutions to the initial-boundary value problem for the system* (5.21)*, where the initial values $u^\varepsilon_{0,j}$, $j = 1, \ldots, 6$ for the kinetic system* (5.1) *satisfy the hypotheses of Theorem* 5.1*, part* iii)*. Then, there exists $\rho \in L^1_{\text{loc}} \cap L^\infty$ such that $\rho_\varepsilon(x,t)$ converges to $\rho(x,t)$ in $L^p_{\text{loc}}(\mathbb{R}^3 \times (0,T])$ for all $p \in [1, +\infty)$ and all $T > 0$, while $\varepsilon\, j_\varepsilon(x,t)$ converges to zero strongly in $L^2_{\text{loc}}(\mathbb{R} \times [0,T])$. The limit density $\rho(x,t)$ is the weak solution to the Cauchy problem for the porous media equation*

$$\frac{\partial \rho}{\partial t} - \frac{1}{18\,(1+|\alpha|)}\,\Delta \rho^{1+|\alpha|} = 0$$

in $D'(\mathbb{R}^3 \times (0,T])$, with initial datum ρ_0 that is the weak limit of $\rho_\varepsilon(x, t=0) = \sum_{j=1}^6 u^\varepsilon_{0,j}$.

6. Variants and extensions.

In this section, we present briefly a few variants and extensions of the previous problems and results. For most of these variants and extensions the proofs are straightforward adaptations of the proofs introduced above. First of all, we can allow the rate function k to depend on x, satisfying for instance

$$k_1(x)\,\rho^\alpha \le k(x,\rho) \le k_2(x)\,\rho^\beta\,, \qquad \text{for } 0 < \rho \le 1\,, \tag{6.1}$$

where $k_1, k_2 \in L^\infty(\mathbb{R})$, $\inf_{x\in\mathbb{R}} k_1 > 0$, and $\alpha, \beta < 1$.

Another variant consists in replacing the right-hand side in (1.1), $k(v-u)$ (respectively, $k(u-v)$), by $\varphi(v) - \varphi(u)$ (respectively, $\varphi(u) - \varphi(v)$), where φ is increasing.

Next, we can treat the system in which the velocities in the streaming terms are $1/\varepsilon + a$ and $1/\varepsilon + b$, where $a \neq b \in \mathbb{R}$. In this case we obtain at the limit the additional presence of a linear term.

Finally we can study the system

$$\left\{\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{\varepsilon}\frac{\partial u^m}{\partial x} &= \frac{1}{\varepsilon^2}\,\rho^\alpha\,(v-u)\,,\\ \frac{\partial v}{\partial t} - \frac{1}{\varepsilon}\frac{\partial v^m}{\partial x} &= \frac{1}{\varepsilon^2}\,\rho^\alpha\,(u-v)\,, \end{aligned}\right. \tag{6.2}$$

where $m > \alpha$, which yields in the limit

$$\frac{\partial \rho}{\partial t} - c\,\frac{\partial^2}{\partial x^2}\,\rho^{m-\alpha} = 0\,. \tag{6.3}$$

where $c = m\,(2^m(m-\alpha))^{-1}$. Of course, (6.2) is a weakly coupled system of one-dimensional hyperbolic scalar conservation laws, and u, v are entropy solutions of (6.2).

To end this section, we now wish to look at different models involving velocity sets which are not finite anymore. We begin with a model arising in radiative transfer theory, recently studied by C. Bardos, F. Golse, B. Perthame and R. Sentis [BGPS]. This equation describes the transport of photons in a starlike medium and is, mathematically, a nonlinear version of the transport of neutrons. We look for $u_\varepsilon = u_\varepsilon(x,\omega,t) \geq 0$, where $x \in \mathbb{R}^N$, $\omega \in S^{N-1}$ (the unit sphere of $\mathbb{R}^N$), $t \geq 0$, solution of the initial (or initial-boundary) value problem for the equation

$$\frac{\partial u}{\partial t} + \frac{1}{\varepsilon}\,\omega\cdot\operatorname{grad} u + \frac{1}{\varepsilon^2}\,k(\rho_\varepsilon)\,(u_\varepsilon - \rho_\varepsilon) = 0\,. \tag{6.4}$$

In (6.4) ρ_ε denotes the integral

$$\rho_\varepsilon(x,t) = \int_{S^{N-1}} u_\varepsilon(x,\omega,t)\,d\omega\,,$$

where $d\omega$ is the normalized Lebesgue measure on S^{N-1} ($\int_{S^{N-1}} d\omega = 1$). The nonnegative function k is continuous on $[0,\infty)$ and is supposed to satisfy

$$\nu_1\,s^\alpha \leq k(s) \leq \nu_2\,s^\beta\,, \qquad \text{for } s \in (0,1]\,, \tag{6.5}$$

with $|\alpha| < 1$, $|\beta| < 1$, $\nu_1, \nu_2 > 0$.

Our method of proof adapts to this model and yields at the limit

$$\frac{\partial \rho}{\partial t} - \Delta F(\rho) = 0\,, \tag{6.6}$$

where

$$F(\rho) = \frac{1}{N} \int_0^\rho \frac{ds}{k(s)} \,.$$

In radiative transfer theory, this limit is known as the "Rosseland approximation". Not only this convergence can be shown in a slightly, more general setting than in [BGPS], but our method of proof is completely different and simply relies upon entropy bounds and the "div-curl" lemma. In particular, our proof works if $k(\rho) = \rho^\alpha$ with $|\alpha| < 1$.

In conclusion, let us indicate that it is possible to interpret both the finite-velocity models and the above model (6.4) in a single setting. We briefly mention this remark: let V be a bounded set on $\mathbb{R}^N$, and let μ be a probability measure on V satisfying

$$\begin{aligned} &\int_V v_k \, d\mu = 0 \,, \qquad \text{for all } 1 \le k \le N \,, \\ &\int_V (v \cdot \xi)^2 \, d\mu > 0 \,, \qquad \text{for all } \xi \in S^{N-1} \,. \end{aligned} \tag{6.7}$$

We then look for $u_\varepsilon = u_\varepsilon(x, v, t) \ge 0$, solution on $\mathbb{R}^N \times V \times [0, \infty)$ of

$$\frac{\partial u}{\partial t} + \frac{1}{\varepsilon} \, v \cdot \operatorname{grad} u = \frac{1}{\varepsilon^2} \, \rho_\varepsilon^\alpha \, (\rho_\varepsilon - u_\varepsilon) \,, \tag{6.8}$$

where $|\alpha| < 1$ and

$$\rho_\varepsilon(x, t) = \int_V u_\varepsilon(x, v, t) \, d\mu \,.$$

The hydrodynamical limit for this equation is then

$$\frac{\partial \rho}{\partial t} - \sum_{i,j=1}^N a_{ij} \, \frac{\partial^2}{\partial x_i \partial x_j} \Big(\frac{\rho^{1-\alpha}}{1-\alpha} \Big) = 0 \,, \tag{6.9}$$

where

$$a_{ij} = \int_V v_i \, v_j \, d\mu \,.$$

Let us observe that the two (or six in three-dimensions) velocity model we were primarily interested in this paper corresponds to $V = \{+1, -1\}$, with $(1/2, 1/2)$ for the probability measure μ, while the Rosseland approximation corresponds to $V = S^{N-1}$ and $d\mu = d\omega / |S^{N-1}|$.

We do not wish to give a detailed proof of the above claims about Rosseland approximation or (6.8)-(6.9), since it is a straightforward adaptation of our method of proof. In particular, using the ideas of Section 4, one obtains all the a priori estimates concerning the monotonicity for convex functionals of the solution.

However, there is one point that we need to detail. Indeed, since V is no more finite (in general), we cannot estimate from below $\rho_\varepsilon/u_\varepsilon$, and thus we have to modify a little bit our use of the dissipation of entropy. More precisely, we obtain $L_t^\infty(L^1 \cap L^\infty)$ bounds on u_ε (and thus on ρ_ε) independently of ε, and from the monotonicity of the entropy we obtain for all $T > 0$

$$\int_0^T dt \int dx \int_V d\mu\, \rho_\varepsilon^\alpha \, \frac{\rho_\varepsilon - u_\varepsilon}{\varepsilon^2} \, \log \frac{\rho_\varepsilon}{u_\varepsilon} \le C \,, \tag{6.10}$$

where C denotes various positive constants independent of ε.

Hence, if we denote by j_ε the flux

$$j_\varepsilon(x,t) = \int_V v \, \frac{u_\varepsilon}{\varepsilon} \, d\mu = \int_V v \, \frac{u_\varepsilon - \rho_\varepsilon}{\varepsilon} \, d\mu \,,$$

we have

$$\begin{aligned} |j_\varepsilon|^2 &\le \Big(\int_V |v| \, \Big| \frac{\sqrt{u_\varepsilon} - \sqrt{\rho_\varepsilon}}{\varepsilon} \Big| \, |\sqrt{u_\varepsilon} + \sqrt{\rho_\varepsilon}| \, d\mu \Big)^2 \\ &\le C \int_V \frac{(\sqrt{u_\varepsilon} - \sqrt{\rho_\varepsilon})^2}{\varepsilon^2} \, d\mu \int_V (u_\varepsilon + \rho_\varepsilon) \, d\mu \\ &\le C \rho_\varepsilon \int_V \frac{\rho_\varepsilon - u_\varepsilon}{\varepsilon^2} \, \log \frac{\rho_\varepsilon}{u_\varepsilon} \, d\mu \,, \end{aligned}$$

where we used the classical inequality, valid for all $a, b \ge 0$

$$(\sqrt{a} - \sqrt{b})^2 \le C\,(a-b) \log \frac{a}{b} \;.$$

In particular, we deduce from (6.10) that $|j_\varepsilon|^2 \rho_\varepsilon^{\alpha-1}$ is bounded in L^1 and thus, since $|\alpha| < 1$, j_ε is bounded in L^2.

The convergence analysis follows along the same lines as in the preceding sections, writing the macroscopic equations for the equation (6.4)

$$\left\{ \begin{aligned} &\frac{\partial \rho_\varepsilon}{\partial t} + \operatorname{div} j_\varepsilon = 0 \,, \\ &\varepsilon^2 \, \frac{\partial j_\varepsilon}{\partial t} + \operatorname{div} \int_V v \otimes v u_\varepsilon \, d\mu = -\rho_\varepsilon^\alpha j_\varepsilon \;, \end{aligned} \right. \tag{6.11}$$

observing that $j_\varepsilon, \rho_\varepsilon^\alpha j_\varepsilon$ are bounded in L^2, and finally that

$$\int_V v \otimes v\, u_\varepsilon\, d\mu = A\, \rho_\varepsilon + \int_V v \otimes v (u_\varepsilon - \rho_\varepsilon)\, d\mu\,, \tag{6.12}$$

from which we deduce that $A \operatorname{grad} \rho_\varepsilon$ and thus $\operatorname{grad} \rho_\varepsilon$, since by construction the matrix A is positive definite, lie in a compact set of $H^{-1}_{x,t}$. In addition, the integral on the right-hand side converges converges to zero in $L^2_{x,t}$ as a consequence of the fact that j_ε is bounded in L^2, while V is bounded.

Several relevant differences between this approach and the approach by C. Bardos, F. Golse, B. Perthame and R. Sentis [BGPS] are worth emphasizing. First of all, the above proof seems a bit simpler and yields, on the technical side, apparently more general results. But in addition, and this is more important, the compactness phenomena are somewhat different, since velocity averaging (as in [BGPS]) would require the measure μ to satisfy the condition

$$\mu\{v \in V:\ v \cdot \xi = 0\} = 0\,, \qquad \text{for all } \xi \in S^{N-1}\,,$$

while we only need that the measure μ satisfies the second condition in (6.7), namely

$$\int_V (v \cdot \xi)^2\, d\mu > 0\,, \qquad \text{for all } \xi \in S^{N-1}\,.$$

Acknowledgment. This work was begun while the first author was visiting the University of Pavia for the opening lecture of the Ph.D. courses of the Faculty of Sciences. G. Toscani acknowledges the support of the National Council for Researches of Italy (CNR), Project "Kinetic Equations and Diffusion Problems".

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Recibido: 4 de marzo de 1.996
Revisado: 29 de enero de 1.997

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Revista Matemática Iberoamericana
Vol. 13, N.° 3, 1997

A generalization of a theorem by Kato on Navier-Stokes equations

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Abstract. We generalize a classical result of T. Kato on the existence of global solutions to the Navier-Stokes system in $C([0,\infty); L^3(\mathbb{R}^3))$. More precisely, we show that if the initial data are sufficiently oscillating, in a suitable Besov space, then Kato's solution exists globally. As a corollary to this result, we obtain a theorem on existence of self-similar solutions for the Navier-Stokes equations.

0. Introduction.

In the study of the Cauchy problem for the Navier-Stokes equations governing the time evolution of the velocity $v(t,x)$ and the pressure $p(t,x)$ of an incompressible viscous fluid filling all of $\mathbb{R}^3$

$$(0.1) \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = -(v \cdot \nabla) v - \nabla p \,, \\ \nabla \cdot v = 0 \,, \\ v(0) = v_0 \,, \end{cases}$$

there is considerable interest in finding global solutions $v(t,x)$ which are strongly continuous from the interval $[0,\infty)$ and take values in an abstract Banach space, whose norm is invariant under the transformation $f(\,\cdot\,) \longmapsto \lambda f(\lambda\,\cdot)$, for all $\lambda > 0$.

Following [1], we will call such a space a *limit* space for the study of the Navier-Stokes equations. A typical example is given by the Lebesgue space $L^3(\mathbb{R}^3)$ [2] ,[3], but one can also consider the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3)$ [4], [5], or the homogeneous Morrey-Campanato space $\dot{M}_2^3(\mathbb{R}^3)$ [6]-[9], or more sophisticated and somewhat esoteric examples as the Besov or Triebel-Lizorkin spaces [1].

The reason why these *limit* spaces arise naturally in the study of the Navier-Stokes equations is very simple. Suppose that $v(t,x)$ and $p(t,x)$ solve the system

$$\left\{ \begin{array}{l} \dfrac{\partial v}{\partial t} - \Delta v = -(v \cdot \nabla) v - \nabla p \\ \nabla \cdot v = 0 \end{array} \right. \tag{0.2}$$

then, the same holds true for $v_\lambda = \lambda\, v(\lambda^2 t, \lambda x)$ and $p_\lambda = \lambda^2 p(\lambda^2 t, \lambda x)$ for any positive λ.

In other words, these spaces have, as far as the space variable is concerned, the same scaling invariance as the Navier-Stokes equations, so that it appears very natural to solve (0.1) when the initial data v_0 (which depend on the x-variable only) belong to a Banach *limit* space.

In a previous article [10] (see also [1]), we gave a detailed analysis of the local well-posedness of the Cauchy problem (0.1) with initial data in an abstract Banach space. Although the algorithm we presented in [1], [10] allows one to recover – in a very abstract way – many previous known results on existence of local solutions for the system (0.1), it does not shed any light on the case of a *limit* Banach space.

In order to understand why and where the difficulties arise when dealing with such a *limit* space, let us briefly recall the standard procedure used in the study of the initial value problem (0.1).

The idea is the following. One first transforms the *classical* system (0.1) into the following *mild* integral equation

$$v(t) = S(t)\, v_0 + B(v,v)(t)\,, \tag{0.3}$$

where

$$B(v,u)(t) = -\int_0^t \mathbb{P} S(t-s)\, \nabla \cdot (v \otimes u)(s)\, ds \tag{0.4}$$

$\mathbb{P}$ and S being respectively the projection onto divergence free vector fields and the heat semigroup.

Then, it is customary to obtain the existence and uniqueness of a strongly continuous global ($T = \infty$) or local ($T < \infty$) solution $v(t,x) \in C([0,T);X)$ of (0.3), X being an abstract Banach space, by means of the standard contraction algorithm. Of course, the main difficulty in applying such an algorithm is to establish, *a priori*, the bicontinuity of the bilinear operator $B(v,u)(t)$ in $C([0,T);X) \times C([0,T);X) \longrightarrow C([0,T);X)$.

In the case of the Lebesgue space $L^p(\mathbb{R}^3)$, a straightforward application of Young inequality implies that

$$\sup_{0\le t<T} \|B(v,u)(t)\|_p$$

$$\le C\Big(\int_0^T t^{-1/2-3/(2p)}\Big) \sup_{0\le t<T}\|v(t)\|_p \sup_{0\le t<T}\|u(t)\|_p \tag{0.5}$$

thus showing that $B(v,u)(t)$ is bicontinuous in $C([0,T);L^p(\mathbb{R}^3))$ as long as $p > 3$.

On the other hand, it is not known whether or not the bilinear operator $B(v,u)$ is continuous in the *limit* $L^3(\mathbb{R}^3)$ setting. What is certainly true is that, even if the bilinear operator $B(v,u)(t)$ turns out *not* to be bicontinuous in such a *limit* space $L^3(\mathbb{R}^3)$, say in $C([0,T);L^3(\mathbb{R}^3))$, this would not *necessarily* imply a nonexistence theorem of mild solutions $v(t,x) \in C([0,T);L^3(\mathbb{R}^3))$ for the Navier-Stokes equations.

The problem of solving the integral equation (0.3) in the $L^3(\mathbb{R}^3)$-setting was first tackled in 1984 by T. Kato [2], who was able to circumvent the problem of the *possible* noncontinuity of $B(v,u)(t)$ in $C([0,T);L^3(\mathbb{R}^3))$ (*i.e.* the nonintegrability at the origin of the function t^{-1} appearing in (0.5) for $p = 3$).

Kato's masterstroke was to remark that, in order to obtain an *existence* theorem in $C([0,T);L^3(\mathbb{R}^3))$ for the Navier-Stokes equations, it is *sufficient* to show that the bilinear operator $B(v,u)(t)$ is continuous in a suitable subspace of $C([0,T);L^3(\mathbb{R}^3))$. This subspace K is made up of the functions $v(t,x) \in C([0,T);L^3(\mathbb{R}^3))$ such that, moreover,

$$t^{\alpha/2}v(t,x) \in C([0,T);L^q(\mathbb{R}^3)) \tag{0.6}$$

and

$$\lim_{t\to 0} t^{\alpha/2}\|v(t)\|_q = 0 \tag{0.7}$$

and normed by

$$\|\,|v|\,\|_K = \sup_{0\le t<T}\|v(t)\|_3 + \sup_{0\le t<T} t^{\alpha/2}\|v(t)\|_q \tag{0.8}$$

q being a fixed constant satisfying $3 < q \leq 6$ and $\alpha = \alpha(q) = 1 - 3/q$.

In other words, Kato's idea was to look for solutions in a space of vector-valued functions equipped with two norms: the first is the *natural* one, while the second controls the balance of the smoothing property of the heat semigroup $S(t)$ against the apparition of singularities by the quadratic term $B(v,v)(t)$.

Now, it is easy to observe that not only is the bilinear operator $B(v,u)(t)$ bicontinuous in this norm $\||\cdot|\|_K$, but also $S(t)v_0 \in K$ as long as $v_0 \in L^3(\mathbb{R}^3)$. All this is sufficient to deduce, by means of a standard contraction procedure, an *existence* theorem of global mild solutions for the Navier-Stokes equations in $C([0,\infty); L^3(\mathbb{R}^3))$ and small initial data in $L^3(\mathbb{R}^3)$, the *uniqueness* of the solution being guaranteed only in K, and not, in general, in the natural space $C([0,\infty); L^3(\mathbb{R}^3))$. More precisely, Kato's theorem (in a somewhat simplified version [1]) reads as follows [2]

Kato's Theorem. *Let q be fixed, $3 < q \leq 6$, and $\alpha = \alpha(q) = 1 - 3/q$, then there exists an absolute constant $\delta > 0$, such that if v_0 belongs to $L^3(\mathbb{R}^3)$, $\|v_0\|_3 < \delta$ and $\nabla \cdot v_0 = 0$ (in the distributional sense), then there exists a global mild solution of the Navier-Stokes equations in $C([0,\infty); L^3(\mathbb{R}^3))$. Moreover, this solution is the only one such that*

$$v(t,x) \in C([0,\infty); L^3(\mathbb{R}^3)) \,, \tag{0.9}$$

$$t^{\alpha/2} v(t,x) \in C([0,\infty); L^q(\mathbb{R}^3)) \tag{0.10}$$

and

$$\lim_{t\to 0} t^{\alpha/2} \|v(t)\|_q = 0 \,. \tag{0.11}$$

The aim of this paper is to prove that Kato's result holds true under a much weaker condition on the initial data. In order to make it clear, let us introduce (Definition 1.3) the Besov space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$, q and $\alpha(q)$ being chosen as before. It is quite easy to prove that

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3) \,, \tag{0.12}$$

but that these two spaces are different, for $|x|^{-1} \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ and $|x|^{-1} \notin L^3(\mathbb{R}^3)$ (Lemma 1.2).

Now, if we observe that for any tempered distribution $v_0 \in \mathcal{S}'(\mathbb{R}^3)$ the $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}}$ norm is equivalent to $\sup_{t\geq 0} t^{\alpha/2}\|S(t)v_0\|_q$ (Lemma 1.1), then Kato's global solution exists and satisfies (0.9)-(0.11) under the weaker conditions: $v_0 \in L^3(\mathbb{R}^3)$, $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$ and $\nabla \cdot v_0 = 0$ in the sense of distributions (Theorem 1.1).

Before delving into the details of this weak formulation of Kato's theorem, let us comment on the condition $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$.

To fix the ideas, let us suppose that $v_0(x)$ is an arbitrary $L^3(\mathbb{R}^3)$ function and let $w_k(x)$ be a sequence of functions such that $\|w_k\|_\infty \leq c$ (uniformly in k) and that $w_k \rightharpoonup 0$ in the sense of distributions as k goes to infinity.

In other words, let us suppose that the functions w_k are uniformly bounded and are more and more oscillating as k increases. Under these hypotheses, it is easy to prove that $w_k(x)\, v(x)$ tends to 0 in the strong topology of $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ (Lemma 2.1) in spite of the fact that if $|w_k(x)| = 1$ (almost everywhere in x, for any k), then $\|w_k v_0\|_3 = \|v_0\|_3$. A typical example of a sequence $w_k(x)$ fulfilling all these conditions is given by the exponential function, say $w_k(x) = \exp(i\, x \cdot k)$ (here, k is a vector and we let $|k|$ go to ∞).

The importance of the weaker condition, $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$ instead of $\|v_0\|_3 < \delta$, is now clear and can be formulated as follows: in order to prove the existence of Kato's global solution, all we need is *sufficiently oscillating* initial data. Of course, we have to pay attention to the divergence condition on the initial data for, in general, the divergence operator $\nabla\cdot$ does not commutate with the functions $w_k(x)$. Nevertheless, this does hold true asymptotically as k goes to infinity (Lemma 2.2), which is exactly the situation we have to deal with.

Another remarkable property of the Besov spaces $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ is, as we pointed out before, that they contain among their elements homogeneous functions of degree -1, such that *e.g.* $|x|^{-1}$. This is a crucial point when looking for solutions to the Navier-Stokes equations which satisfy the scaling property

$$v(t,x) = v_\lambda(t,x) = \lambda\, v(\lambda^2 t, \lambda x)\,, \qquad \text{for all } \lambda > 0\,. \tag{0.13}$$

In fact, whenever they exist, these particular solutions $v(t,x)$, which are usually called *self-similar solutions*, are such that their initial value $v(0,x)$ is a homogeneous function of degree -1. We will show in this paper how to obtain, by using the above mentioned weak formulation of Kato's theorem, an existence theorem of self-similar solutions $v(t,x)$

with initial data v_0 homogeneous of degree -1, divergence-free, and sufficiently small in the Besov space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ (Theorem 3.2).

The plan of the paper is the following. Section 1 contains the basic definitions and the proof of the main theorem. Section is devoted to illustrating that $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$ is satisfied for sufficiently oscillating initial data v_0. Finally, sections 3 and 4 deal with the existence of self-similar solutions for the Navier-Stokes equations in $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$.

1. The proof of the main Theorem.

We study the Cauchy problem for the Navier-Stokes equations governing the time evolution of the velocity

$$v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$$

and the pressure $p(t,x)$ of an incompressible fluid filling all of $\mathbb{R}^3$

$$\left\{\begin{array}{l} \dfrac{\partial v}{\partial t} - \Delta v = -(v\cdot\nabla)v - \nabla p\,, \\ \nabla\cdot v = 0\,, \\ v(0) = v_0\,. \end{array}\right. \tag{1.1}$$

We will focus our attention on the existence of global solutions to (1.1) in $C([0,\infty); L^3(\mathbb{R}^3))$, the space of continuous functions $v(t,x)$ of $t\in[0,\infty)$ with values in the Banach space $L^3(\mathbb{R}^3)$ of vector distributions.

Here and in the following, we say that a vector $a = (a_1,a_2,a_3)$ belongs to a function space X if $a_j \in X$ holds for every $j = 1,2$ and 3, and we put $\|a\| = \max_{1\le j\le 3}\|a_j\|$.

Before stating the main hypotheses concerning the initial data v_0 under which the system (1.1) will be solved in $C([0,\infty); L^3(\mathbb{R}^3))$, let us recall some definitions, which will be useful in the sequel.

1.1. The operator $\mathbb{P}$.

We let $\partial_j = -i\,\partial/\partial x_j$, $(i^2 = -1)$ and we indicate by $R_j = \partial_j(-\Delta)^{-1/2}$, for $j = 1,2$ and 3, the Riesz transformation.

For an arbitrary vector field $v(x) = (v_1(x), v_2(x), v_3(x))$ on $\mathbb{R}^3$, we set

$$z(x) = \sum_{j=1}^{3}(R_j v_j)(x) \tag{1.2}$$

and finally we define the operator $\mathbb{P}$ by

$$(\mathbb{P}v)_k(x) = v_k(x) - (R_k z)(x)\,, \qquad 1 \le k \le 3\,. \tag{1.3}$$

$\mathbb{P}$ is a pseudo-differential operator of degree zero and is an orthogonal projection onto the kernel of the divergence operator.

Making use of this projection operator $\mathbb{P}$ and the heat semigroup $S(t) = \exp(t\Delta)$, it is now a straightforward procedure to reduce the *classical* partial differential system (1.1) into the *mild* integral equation

$$v(t) = S(t)\, v_0 - \int \mathbb{P} S(t-s)\, \nabla \cdot (v \otimes v)(s)\, ds \tag{1.4}$$

Accordingly, a solution of the equation (1.4) will be called a *mild* solution of the Navier-Stokes equations. It is not difficult to see that a mild solution of the Navier-Stokes (1.4) is actually a classical solution of the system (1.1) (and conversely). It would be inappropriate and beyond the scope of this paper to present a proof of this equivalence here. Fore more details on the subject, we refer the reader to [3], [4].

1.2. The Littlewood-Paley decomposition.

Let us choose a real rotation invariant function φ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ whose Fourier transform is such that

$$0 \le \hat{\varphi}(\xi) \le 1\,, \qquad \begin{cases} \hat{\varphi}(\xi) = 1\,, & \text{if } |\xi| \le \dfrac{3}{4}\,, \\ \hat{\varphi}(\xi) = 0\,, & \text{if } |\xi| \ge \dfrac{3}{2}\,, \end{cases} \tag{1.5}$$

and let

$$\psi(x) = 8\, \varphi(2\, x) - \varphi(x)\,, \tag{1.6}$$

$$\varphi_j(x) = 2^{3j} \varphi(2^j x)\,, \qquad j \in \mathbb{Z}\,, \tag{1.7}$$

$$\psi_j(x) = 2^{3j} \psi(2^j x)\,, \qquad j \in \mathbb{Z}\,. \tag{1.8}$$

We denote by S_j and Δ_j, respectively, the convolution operators with φ_j and ψ_j. Finally, the set $\{S_j, \Delta_j\}_{j \in \mathbb{Z}}$ (actually *a set*) is the Littlewood-Paley decomposition of the unity, say

$$I = S_0 + \sum_{j \ge 0} \Delta_j = \sum_{j \in \mathbb{Z}} \Delta_j\,. \tag{1.9}$$

It is worthwhile to recall that only the first of the two series, say $f = S_0 f + \sum_{j\geq 0} \Delta_j f$, applies without any restriction on the tempered distribution f. On the other hand, the identity $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ is to be understood *modulo* polynomials (see [11] for a complete and general discussion on the matter).

1.3. The Besov spaces $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$.

Let q be fixed in $1 \leq q \leq \infty$ and $\alpha \in \mathbb{R}$. A tempered distribution $v \in \mathcal{S}'(\mathbb{R}^3)$ belongs to the Besov space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ if and only if the following norm

$$\|v\|_{\dot{B}_q^{-\alpha,\infty}} = \sup_{j\in\mathbb{Z}} 2^{-j\alpha} \|\Delta_j v\|_q \tag{1.10}$$

is finite. Here, Δ_j is, for any $j \in \mathbb{Z}$, the convolution operator with the function ψ_j given in a Littlewood-Paley decomposition of unity.

The following lemma will provide a different characterization of the Besov space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ in terms of the heat semigroup and will be one of the staple ingredients of the proof of Theorem 1.1.

Lemma 1.1. *Let q be fixed in $1 \leq q \leq \infty$ and $\alpha > 0$. For any tempered distribution $v \in \mathcal{S}'(\mathbb{R}^3)$, the following four norms*

$$\sup_{j\in\mathbb{Z}} 2^{-j\alpha} \|\Delta_j v\|_q \ , \tag{1.11}$$

$$\sup_{j\in\mathbb{Z}} 2^{-j\alpha} \|S_j v\|_q \ , \tag{1.12}$$

$$\sup_{t\geq 0} t^{\alpha/2} \|S(t)\, v\|_q \ , \tag{1.13}$$

$$\sup_{t\geq 0} \|S(t)\, v\|_{\dot{B}_q^{-\alpha,\infty}} \ , \tag{1.14}$$

are equivalent, and will be referred to in the sequel by $\|v\|_{\dot{B}_q^{-\alpha,\infty}}$.

The first equivalence (1.11) if and only if (1.12) is easy to prove if we recall that $\Delta_j = S_{j+1} - S_j$ for all $j \in \mathbb{Z}$, that $S_{j+1} = \sum_{k\leq j} \Delta_k$ and that $\alpha > 0$. Of course, when passing from (1.11) to (1.12) the *proviso* stated after (1.9), on the equivalence modulo the polynomials, is still required.

Let us examine the equivalence (1.12) if and only if (1.13). In order to see this, it is sufficient to observe that for $t = 4^{-j}$ the convolution operator $S(t)$ *essentially* reduces to the operator S_j. Of course, this is the case because the Fourier transform of $S(t)$ is given by $\exp(-t\,|\xi|^2)$ which has *essentially* the same properties stated in (1.5). The equivalence (1.12) if and only if (1.13) can now be shown using the same techniques as in the proof of the independence of the particular choice of φ in the Littlewood-Paley decomposition (see [12] for a proof).

The next step is the equivalence (1.13) if and only if (1.14). Here the proof is evident and left to the reader.

Lemma 1.2. *Let q_1 and q_2 be two fixed constants in $3 \le q_1 \le q_2 \le \infty$ and put $\alpha_1 = 1 - 3/q_1$ and $\alpha_2 = 1 - 3/q_2$. We have the following chain of continuous imbeddings*

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{q_1}^{-\alpha_1,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}_{q_2}^{-\alpha_2,\infty}(\mathbb{R}^3)\,. \tag{1.15}$$

This result is a consequence of the Bernstein's inequalities [13] which allow us to deduce, for any $j \in \mathbb{Z}$, $3 \le q_1 \le q_2 \le \infty$ and any tempered distribution v, the following chain of inequalities

$$2^{-j\alpha_2}\|\Delta_j v\|_{q_2} \le c\,2^{-j\alpha_1}\|\Delta_j v\|_{q_1} \le c\,\|\Delta_j v\|_3 \le c\,\|v\|_3\,, \tag{1.16}$$

which finally implies (1.15).

At this point, we would like to remark that the above inclusions are strict ones. For example, if we consider the function $|x|^{-1}$, here $x = (x_1, x_2, x_3)$, then $|x|^{-1} \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ $(q > 3, \alpha = 1 - 3/q)$, in spite of the fact that $|x|^{-1} \notin L^3(\mathbb{R}^3)$.

In Section 4 we will give a complete characterization, *i.e.* a necessary and sufficient condition, of which homogeneous functions of degree -1 belong to the space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ $(q > 3,\ \alpha = 1 - 3/q)$.

Here we limit ourselves to the case of the function $|x|^{-1}$, $x = (x_1, x_2, x_3)$. More generally, we want to show that if the restriction to the unit sphere S^2 of a tempered distribution function v, homogeneous of degree -1, belongs to $L^\infty(S^2)$, then v belongs to $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ $(q > 3,$ $\alpha = 1 - 3/q)$. In fact, thanks to the homogeneity of v, we have

$$\begin{aligned} \|v\|_{\dot{B}_q^{-\alpha,\infty}} &= \sup_{j\in\mathbb{Z}} 2^{-j\alpha}\|S_j v\|_q \\ &= \sup_{j\in\mathbb{Z}} 2^{-j(\alpha-1+3/q)}\|S_0\|_q \\ &= \|S_0\|_q\,. \end{aligned} \tag{1.17}$$

Now, if $v|_{S^2} \in L^\infty(S^2)$, we have

$$|S_0 v| \leq \frac{C}{1+|x|}\,, \tag{1.18}$$

which finally gives $v \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ as long as $q > 3$. This concludes our remark.

We are now in a position to generalize Kato's result. To this end let us first introduce the Banach space G which is made up by functions $v(t,x)$ satisfying

$$v(t,x) \in C([0,\infty); L^3(\mathbb{R}^3))\,, \tag{1.19}$$

$$t^{\alpha/2} v(t,x) \in C([0,\infty); L^q(\mathbb{R}^3)) \tag{1.20}$$

and

$$\lim_{t \to 0} t^{\alpha/2} \|v(t)\|_q = 0 \tag{1.21}$$

and normed by

$$\|v\|_G = \sup_{t>0} \|v(t)\|_{\dot{B}_q^{-\alpha,\infty}} + \sup_{t>0} t^{\alpha/2} \|v(t)\|_q\,. \tag{1.22}$$

This definition makes sense because, as we noticed in Lemma 1.2, the Lebesgue space $L^3(\mathbb{R}^3)$ is continuously imbedded in $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$.

The theorem that we will prove here is the following.

Theorem 1.1. *Let q be fixed in $3 < q \leq 6$ and $\alpha = \alpha(q) = 1 - 3/q$, then there exists an absolute constant $\delta > 0$ such that if v_0 belongs to $L^3(\mathbb{R}^3)$, $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$, and $\nabla \cdot v_0 = 0$ (in the sense of distributions), then there exists a global mild solution of the Navier-Stokes equations in $C([0,\infty); L^3(\mathbb{R}^3))$. Moreover, this solution is the only one such that*

$$v(t,x) \in C([0,\infty); L^3(\mathbb{R}^3))\,, \tag{1.23}$$

$$t^{\alpha/2} v(t,x) \in C([0,\infty); L^q(\mathbb{R}^3)) \tag{1.24}$$

and

$$\lim_{t \to 0} t^{\alpha/2} \|v(t)\|_q = 0\,. \tag{1.25}$$

The proof of this Theorem 1.1 can be obtained easily by virtue of the following three lemmata.

Lemma 1.3. *If* $v_0 \in L^3(\mathbb{R}^3)$, *then* $S(t)\, v_0 \in G$.

First, by Lemma 1.1, the G norm of $S(t)\, v_0$ is equivalent to the $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ norm of v_0. Next, as v_0 belongs to $L^3(\mathbb{R}^3)$ and as $L^3(\mathbb{R}^3)$ is a *separable* space, a straightforward application of the Banach-Steinhaus theorem shows that (1.19)-(1.21) hold with $v(t)$ replaced by $S(t)\, v_0$. This concludes the proof of the lemma.

Lemma 1.4. *The bilinear operator* $B(v,u)(t)$ *defined by*

$$B(v,u)(t) = -\int_0^t \mathbb{P}S(t-s)\, \nabla \cdot (v \otimes u)(s)\, ds \tag{1.26}$$

is bicontinuous in $G \times G \longrightarrow G$.

For the sake of simplicity, we will prove the lemma in its scalar version. More precisely, following [1], we will consider the scalar version of the bilinear operator $B(v,u)(t)$, given by

$$B(f,g)(t) = -\int_0^t (t-s)^{-2}\, \Theta\Big(\frac{\cdot}{\sqrt{t-s}}\Big) * (fg)(s)\, ds\,, \tag{1.27}$$

where $f = f(t,x)$ and $g = g(t,x)$ are two *scalar* fields in G and $\Theta = \Theta(x)$ is an analytic function of x which is $O\left(|x|^{-4}\right)$ at infinity.

First of all, Young inequality (here the condition $q \leq 6$ appears) gives

$$\|B(f,g)(t)\|_3 \leq \int_0^t (t-s)^{-2} \Big\|\theta\Big(\frac{\cdot}{\sqrt{t-s}}\Big)\Big\|_m \|f(s)\, g(s)\|_{q/2}\, ds\,, \tag{1.28}$$

where

$$\frac{1}{3} = \frac{1}{m} + \frac{2}{q} - 1 \tag{1.29}$$

thus showing that

$$\begin{aligned} \|B(f,g)(t)\|_3 &\leq \Big(\int_0^t (t-s)^{-2+3/(2m)} s^{-\alpha}\, ds\Big) \|\Theta\|_m \\ &\quad \cdot \big(\sup_{t>0} t^{\alpha/2} \|f(t)\|_q\big) \big(\sup_{t>0} t^{\alpha/2} \|g(t)\|_q\big) \\ &= c\, \big(\sup_{t>0} t^{\alpha/2} \|f(t)\|_q\big) \big(\sup_{t>0} t^{\alpha/2} \|g(t)\|_q\big)\,. \end{aligned} \tag{1.30}$$

The next step is to evaluate the second term of the G-norm. To this end, we use again Young's inequality and find

$$\|B(f,g)(t)\|_q \le \int_0^t (t-s)^{-2}\Big\|\theta\Big(\frac{\cdot}{\sqrt{t-s}}\Big)\Big\|_n \|f(s)\,g(s)\|_{q/2}\,ds\,, \tag{1.31}$$

where

$$\frac{1}{q} = \frac{1}{n} + \frac{2}{q} - 1 \tag{1.32}$$

this gives the desired result

$$\begin{aligned} \|B(f,g)(t)\|_q &\le \Big(\int_0^t (t-s)^{-2+3/(2n)} s^{-\alpha}\,ds\Big)\|\Theta\|_n \\ &\quad \cdot \big(\sup_{t>0} t^{\alpha/2}\|f(t)\|_q\big)\big(\sup_{t>0} t^{\alpha/2}\|g(t)\|_q\big) \\ &= c\,t^{-\alpha/2}\big(\sup_{t>0} t^{\alpha/2}\|f(t)\|_q\big)\big(\sup_{t>0} t^{\alpha/2}\|g(t)\|_q\big)\,. \end{aligned} \tag{1.33}$$

Let us now check the validity of condition (1.21) for the bilinear term $B(v,u)(t)$. Actually, we will prove a more precise statement.

In fact, not only is

$$\lim_{t\to 0} t^{\alpha/2}\|B(f,g)(t)\|_q = 0\,, \tag{1.34}$$

whenever

$$\lim_{t\to 0} t^{\alpha/2}\|f(t)\|_q = \lim_{t\to 0} t^{\alpha/2}\|g(t)\|_q = 0\,, \tag{1.35}$$

but also, if the latter condition is fulfilled, we have

$$\lim_{t\to 0} \|B(f,g)(t)\|_3 = 0\,. \tag{1.36}$$

In particular, this last property is very important in the proof of Theorem 1.1, because it guarantees that any solution $v(t,x) \in G$ of the integral equation (1.4) with data v_0 in $L^3(\mathbb{R}^3)$ tends to v_0 in the strong topology of $L^3(\mathbb{R}^3)$ and is unique in G.

Let us now verify condition (1.36). This is trivial because, if

$$s^{\alpha/2}\|f(s)\|_q \le \varepsilon \qquad \text{and} \qquad s^{\alpha/2}\|f(s)\|_q \le \varepsilon\,, \tag{1.37}$$

for $0 \le s < h$, then an argument analogous to the one used in (1.28)-(1.30) shows that (for $0 \le s < h$)

$$\|B(f,g)(t)\|_3 \le \varepsilon \tag{1.38}$$

which is nothing more than the (ε, h) definition of (1.36). The proof of (1.34) is essentially the same and does not present any difficulties. The Lemma 1.4 follows.

Let us now recall without proof, a classical result.

Lemma 1.5. *Let X be an abstract Banach space and $B : X \times X \longrightarrow X$ a bilinear operator, $\|\cdot\|$ being the X-norm, such that for any $x_1 \in X$ and $x_2 \in X$, we have*

$$\|B(x_1, x_2)\| \le \eta \,\|x_1\|\,\|x_2\|\,, \tag{1.39}$$

then for any $y \in X$ such that

$$4\,\eta\,\|y\| < 1\,, \tag{1.40}$$

the equation

$$x = y + B(x,x) \tag{1.41}$$

has a solution x in X. Moreover, this solution x is the only one such that

$$\|x\| \le \frac{1 - \sqrt{1 - 4\,\eta\,\|y\|}}{2\,\eta}\,. \tag{1.42}$$

The proof of Theorem 1.1 now easily follows if we take into account all the previous lemmata.

2. A remarkable property.

In order to appreciate the above-mentioned weak formulation of Kato's theorem, in this Section, we shall devote ourselves to illustrating that the condition $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$ is satisfied in the particular case of a sufficiently oscillating function v_0.

A typical situation will be given by the following example. Let v_0 be an arbitrary (not identically vanishing) function belonging to $L^3(\mathbb{R}^3)$. If we multiply v_0 by an exponential, say the function $w_k = \exp(i\,x\cdot k)$, we obtain, for any $k \in \mathbb{R}^3$, a function $w_k v_0$ such that (Lemma 2.1)

$$\lim_{|k|\to\infty} \|w_k v_0\|_{\dot{B}_q^{-\alpha,\infty}} = 0 \tag{2.1}$$

in spite of the fact that

$$\lim_{|k|\to\infty} \|w_k v_0\|_3 = \|v_0\|_3\ . \tag{2.2}$$

In other words, the smallness condition $\|w_k v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$, is verified as long as we choose a sufficiently high frequency k. At this point, it is tempting to consider $w_k v_0$ as the new initial data of the problem and affirm that Kato's solution exists globally in time, provided we consider a sufficiently oscillating data; but one can argue that $w_k v_0$ is no longer a divergence-free function.

Nevertheless, this is true *asymptotically*, for $|k| \to \infty$, which is exactly the situation we are dealing with. More precisely, it turns out that (Lemma 2.2)

$$\lim_{|k|\to\infty} \|\nabla\cdot(w_k v_0) - w_k \nabla\cdot v_0\|_3 = 0\ . \tag{2.3}$$

Lemma 2.1. *Let v be an arbitrary function in $L^3(\mathbb{R}^3)$ and let $w_k(x)$, $k\in\mathbb{N}$ be a sequence of functions such that $\|w_k\|_\infty \le C$ and $w_k \rightharpoonup 0$ (as $k\to\infty$) in the distributional sense. Then, the products $w_k v$ tend to 0 in the strong topology of $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ $(\alpha = 1 - 3/q > 0)$.*

In order to prove the lemma we will make use of a density argument. To this end let us introduce the following decomposition of the function v,

$$v = h + g\,, \tag{2.4}$$

where $h \in L^3(\mathbb{R}^3)$ and

$$\|h\|_3 \le \varepsilon \tag{2.5}$$

and $g \in C_0^\infty(\mathbb{R}^3)$. The next step is to recall the continuous imbedding (Lemma 1.2).

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3) \tag{2.6}$$

to infer the following inequality ($k \geq 0$)

$$\|w_k h\|_{\dot{B}_q^{-\alpha,\infty}} \leq c\,\|w_k h\|_3 \leq c\,\varepsilon\,. \tag{2.7}$$

On the other hand, Young's inequality gives ($j \in \mathbb{Z}$)

$$\|S_j(w_k g)\|_q \leq \|2^{3j}\varphi(2^j\cdot)\|_r\,\|w_k g\|_p\,, \tag{2.8}$$

where

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1\,. \tag{2.9}$$

This implies

$$\begin{aligned} 2^{-\alpha j}\|S_j(w_k g)\|_q &\leq C\,2^{-j(1-3/q)}2^{-j(1-3/r)}\|g\|_p \\ &= C\,2^{-j(1-3/p)}\|g\|_p\,, \end{aligned} \tag{2.10}$$

so that, for any $k \geq 0$, any $j \geq j_1 > 0$ and any $j \leq j_0 < 0$, we have

$$2^{-\alpha j}\|S_j(w_k g)\|_q \leq C\,\varepsilon \tag{2.11}$$

(in fact, if $j \geq j_1$ we let $p = q > 3$ and if $j \leq j_0$ we let $1 \leq p < 3$).

We are now left with the terms $S_j(w_k g)$ for $j_0 < j < j_1$. Making use of the hypothesis $m_k \rightharpoonup 0$ together with the Lebesgue dominated convergence theorem, we finally find, for any $k \geq k_0$ and $j_0 < j < j_1$,

$$2^{-\alpha j}\|S_j(w_k g)\|_q \leq C\,\varepsilon\,, \tag{2.12}$$

which concludes the proof of the Lemma.

Lemma 2.2. *Let $m(\xi) \in C^\infty(\mathbb{R}^3\backslash\{0\})$ be a homogeneous function of degree 0 and let M be the convolution operator associated to the multiplier $m(\xi)$. If now we consider $|\xi_0| = 1$, $v \in L^p(\mathbb{R}^3)$ and $1 < p < \infty$, then*

$$\begin{aligned} \lim_{\lambda\to\infty}\ \sup_{|\xi_0|=1}\ &\|M(\exp(i\,\lambda\,\xi_0\cdot x)\,v(x)) \\ &- \exp(i\,\lambda\,\xi_0\cdot x)\,m(\xi_0)\,v(x)\|_p = 0\,. \end{aligned} \tag{2.13}$$

In the case we are interested in, this Lemma will be used for $p = 3$ and with M replaced by the projection operator $\mathbb{P}$ onto the divergence free vector fields and $m(\xi)$ replaced by a 3×3 matrix whose entries are homogeneous symbols of degree 0.

In order to prove the Lemma in its general form, we remark that the symbol of the operator $\exp(-i\,\lambda\,\xi_0 \cdot x) M(\exp(i\,\lambda\,\xi_0 \cdot x)\, v) - m(\xi_0)\, v(x)$ is given by $m(\xi + \lambda\,\xi_0) - m(\lambda\,\xi_0)$, this by virtue of the homogeneity of m.

Equation (2.13) will now be proved by means of a density argument. In fact, it is sufficient to limit ourselves to functions $v \in \mathcal{V} \subset L^p(\mathbb{R}^3)$, where $\mathcal{V}$ is the dense subspace of $L^p(\mathbb{R}^3)$ defined by $v \in \mathcal{S}(\mathbb{R}^3)$ and the Fourier transform $\hat{v}$ of v has compact support.

Now, we put

$$v_\lambda = \exp(-i\,\lambda\,\xi_0 \cdot x)\, M(\exp(i\,\lambda\,\xi_0 \cdot x)\, v) - m(\lambda\,\xi_0)\, v\,, \tag{2.14}$$

then the Fourier transform of v_λ is given by

$$\hat{v_\lambda}(\xi) = (m(\xi + \lambda\,\xi_0) - m(\lambda\,\xi_0))\,\hat{v}(\xi)\,. \tag{2.15}$$

Finally, $\hat{v}$ has compact support, say in $|\xi| \leq R$, and then

$$m(\xi + \lambda\,\xi_0) - m(\lambda\,\xi_0) = r_\lambda(\xi)\,, \tag{2.16}$$

where, on $|\xi| \leq R$, $r_\lambda(\xi) \to 0$, together with all its derivates, in the L^∞ norm. We thus have $v_\lambda \to 0$ in $\mathcal{S}(\mathbb{R}^3)$ when $\lambda \to \infty$. *A fortiori*, $\|v_\lambda\|_p \to 0$ when $\lambda \to \infty$, and the Lemma is proved.

3. Self-similar solutions.

As we pointed out in the Introduction, a remarkable property of the Navier-Stokes equations

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = -(v \cdot \nabla)\, v - \nabla p\,, \\ \nabla \cdot v = 0\,, \end{cases} \tag{3.1}$$

is that they are invariant under the scaling (v, p) implies (v_λ, p_λ) for all $\lambda > 0$, where, respectively,

$$v_\lambda(t, x) = \lambda\, v(\lambda^2 t, \lambda x) \tag{3.2}$$

and

$$p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)\,. \tag{3.3}$$

In other words, suppose that (v,p) is a solution of the system (3.1), then the same holds true for (v_λ, p_λ) for any $\lambda > 0$.

An interesting question arises naturally. Are there solutions $v(t,x)$ of the Navier-Stokes equations which satisfy the scaling invariance

$$v(t,x) = v_\lambda(t,x)\,, \qquad \text{for all } \lambda > 0\,? \tag{3.4}$$

Whenever they exist, these particular solutions are called *self-similar* solutions to the Navier-Stokes equations and are, by definition, global in time.

Self-similar solutions are important because they describe the large time behavior of general global solutions to (3.1). A heuristic argument for this property is the following. Suppose that $v(t,x)$ is a global solution to (3.1), then $v_\lambda(t,x) = \lambda\, v(\lambda^2 t, \lambda x)$ is, for any $\lambda > 0$, a solution to (3.1). If, moreover, we suppose that $\lim_{\lambda\to\infty} v_\lambda(t,x) = u(t,x)$ exists (in a certain sense), then $u(t,x)$ is again a solution of the system (3.1) and, by taking $t = 1$ and $\lambda = \sqrt{t}$, we have $\lim_{t\to\infty} \sqrt{t}\, v(t, \sqrt{t}\, x) = u(1,x)$.

Here is another remarkable property of self-similar solutions. Suppose that $v(t,x)$ is a self-similar solution, then the value taken by $v(t,x)$ at $t = 0$, say the function $v(0,x)$, is necessary homogeneous of degree -1 and has divergence zero (in the sense of distributions). The explicit aim of this Section is to check whether or not the converse is true. More precisely, given a homogeneous function v_0 of degree -1, whose divergence is zero (in the sense of distributions), does the Cauchy problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = -(v\cdot\nabla)v - \nabla p\,, \\ \nabla\cdot v = 0\,, \\ v(0) = v_0\,, \end{cases} \tag{3.5}$$

associated to the system (3.1) admit a self-similar solution $v(t,x)$?

As we will discover in due course, the answer to this problem is positive, provided we choose a suitable functional setting for the problem.

It is worthwhile to recall that the above question is in general far from being trivial in $\mathbb{R}^3$ because, as is well-known, a uniqueness result for the Navier-Stokes equation is still lacking in dimension $n \geq 3$.

To clarify, suppose to the contrary that we dispose of such a uniqueness theorem for the solutions of the system (3.5) and suppose that, moreover, for v_0 homogeneous of degree -1, $\nabla \cdot v_0 = 0$, we were able to prove that *the* solution of the initial value problem (3.5) actually exists. Under these hypothesis, such a solution would be automatically self-similar.

On the other hand, the fact that v_0 should be homogeneous of degree -1 excludes practically all the classical spaces (*e.g.* Lebesgue and Sobolev) for which *at least* an existence theorem for the Navier-Stokes equations is available. As a matter of fact, if we forget for a while the divergence condition, the simplest example of a homogeneous function of degree -1 is given by $|x|^{-1}$, $x = (x_1, x_2, x_3)$, which does not belong to any Lebesgue nor Sobolev spaces.

The problem of finding self-similar solutions for the Navier-Stokes equations was tackled as early as 1933 by J. Leray in his pioneering Ph. D. dissertation [14]. More precisely, Leray was interested in the apparition of possible singularities for solutions $v(t,x)$ of the particular form

$$v(t,x) = \lambda(t)\, V(\lambda(t)\, x) \tag{3.6}$$

$\lambda(t)$ being a positive scalar function and $V(x)$ being a divergence-free vector field.

It is clear that if $v(t,x)$ is a self-similar solution of the Navier-Stokes equations, in other words, if $v(t,x) = \lambda\, v(\lambda^2 t, \lambda x)$, for all $\lambda > 0$, then by taking $\lambda = \sqrt{t}$ we find, in Leray's notation,

$$v(t,x) = \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big)\,, \tag{3.7}$$

where $V(x) = v(1,x)$ is an arbitrary divergence-free vector field.

After Leray's work, this particular representation, equation (3.7), was utilized by several authors with the aim of finding self-similar solutions by a direct approach, say *via* the elliptic equation (in the unknown functions V and Q)

$$\begin{cases} -\dfrac{1}{2}\, V - \dfrac{1}{2}\, (x \cdot \nabla)\, V - \Delta V = -(V \cdot \nabla)\, V - \nabla Q\,, \\ \nabla \cdot V = 0\,, \end{cases} \tag{3.8}$$

obtained by substituting (3.7) into (3.1).

Unfortunately, this strategy turned out to be unsuccessful (see, for a detailed discussion [15]) and the system (3.8) too difficult to solve.

The problem of finding self-similar solutions was still completely open when, in 1989, Y. Giga and T. Miyakawa [16] showed that in a suitable Morrey-Campanato space, self-similar solutions to the Navier-Stokes equations written in terms of the vorticity $\omega(t,x) = \operatorname{curl} v(t,x)$ exist as long as the initial data $\omega_0(x)$ are homogeneous of degree -2 and small enough.

In this Section, we tackle and solve the problem in a somewhat different way. More specifically, we will not deal with the elliptic equation (3.8), nor with the Navier-Stokes equations with vorticity as an unknown function but, in the previous notations, we look for self-similar solutions $v(t,x)$ of the mild equation

$$v(t) = S(t)\, v_0 + B(v,v)(t)\,, \tag{3.9}$$

where

$$B(v,u)(t) = -\int_0^t \mathbb{P}S(t-s)\; \nabla\cdot (v\otimes u)(s)\, ds\,. \tag{3.10}$$

The idea pursued here is to establish first a general existence and *uniqueness* theorem for mild solutions $v(t,x) \in C([0,\infty);X)$ of (3.9), X being an abstract Banach space containing homogeneous functions of degree -1, and then obtain the existence of self-similar solutions as a corollary.

In Section 1, we showed that the Besov spaces $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ ($3 < q < \infty, \alpha = 1 - 3/q$) have the remarkable property of allowing among their elements homogeneous functions of degree -1. Moreover, these spaces arise in a natural generalization of Kato's theorem (Theorem 1.1).

The starting point of this Part 3 is to restate this theorem in the full Besov setting, *i.e.* to remove the condition $v_0 \in L^3(\mathbb{R}^3)$. This will be a crucial step in the proof of the existence of self-similar solutions, because, for instance, $|x|^{-1} \notin L^3(\mathbb{R}^3)$. Some technical difficulties will appear when passing from $L^3(\mathbb{R}^3)$ to $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$, because the latter Banach space is not separable. This implies, among other things, that the property (1.21) (which plays an important role in the uniqueness part of Theorem 1.1) is no longer verified when $v(t)$ is replaced by $S(t)v_0$, $v_0 \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$. An easy way to see this is to consider, for example,

$$v_0(x) = \left(0, \frac{-x_3}{|x|^2}, \frac{x_2}{|x|^2}\right) \tag{3.11}$$

and remark that

$$\lim_{t\to 0} t^{\alpha/2}\|S(t)\,v_0\|_q = \|S(1)\,v_0\|_q \neq 0\,. \tag{3.12}$$

Another important limitation imposed when dealing with a non-separable Banach space X is that the heat semigroup is no longer a $\mathcal{C}_0$-semigroup. This means that $S(t)\,v_0$ is no longer a strongly continuous function from $[0,\infty)$ into X. A way to circumvent this difficulty is to replace the space $C([0,\infty);X)$ by the space $C_*([0,\infty);X)$ whose elements $v(t,x)$ are bounded flows in X, *viz.* $v(t,x)\in L^\infty([0,\infty);X)$ and are continuous in the weak sense of distributions.

With this modification in mind, and recalling that the standard fixed point algorithm (Lemma 1.5) gives the uniqueness of the solution in a small neighborhood of the origin, we will obtain the following result.

Theorem 3.1. *Let q be fixed in $3<q\le 6$ and $\alpha=\alpha(q)=1-3/q$, then there exists an absolute constant $\delta>0$ such that if v_0 belongs to $\dot B_q^{-\alpha,\infty}(\mathbb{R}^3)$, $\|v_0\|_{\dot B_q^{-\alpha,\infty}}<\delta$ and $\nabla\cdot v_0=0$ (in the sense of distributions), then there exists a global mild solution of the Navier-Stokes equations such that*

$$v(t,x)\in C_*((0,\infty);\dot B_q^{-\alpha,\infty}(\mathbb{R}^3))\,, \tag{3.13}$$

$$t^{\alpha/2}v(t,x)\in C_*((0,\infty);L^q(\mathbb{R}^3)) \tag{3.14}$$

and, if $3<q\le 4$,

$$v(t,x)-S(t)\,v_0\in C_*((0,\infty);\dot H^{1/2}(\mathbb{R}^3)) \tag{3.15}$$

and, if $4<q\le 6$,

$$v(t,x)-S(t)\,v_0\in C_*((0,\infty);L^3(\mathbb{R}^3))\,. \tag{3.16}$$

Moreover, there exists only one solution $v(t,x)$ verifying (3.13)-(3.14) *and such that*

$$\sup_{t\ge 0}\|v(t)\|_{\dot B_q^{-\alpha,\infty}}+\sup_{t\ge 0} t^{\alpha/2}\|v(t)\|_q\le R\,, \tag{3.17}$$

where $R=R(\|v_0\|_{\dot B_q^{-\alpha,\infty}})$ is a given constant.

The proof of Theorem 3.1 is essentially the same of that presented for Theorem 1.1. Here, equation (3.16) follows from an argument similar to the one given in (1.30).

The only point which merits clarification is the regularity property (3.15). To see this, let us recall the simplified bilinear scalar operator

$$B(f,g)(t) = -\int_0^t (t-s)^{-2}\,\Theta\Big(\frac{\cdot}{\sqrt{t-s}}\Big) * (fg)(s)\,ds\,. \tag{3.18}$$

Next, in order to evaluate the $\dot{H}^{1/2}(\mathbb{R}^3)$-norm of $B(f,g)(t)$, let us consider the operator $\Lambda^{1/2}$ (whose symbol is $|\xi|^{1/2}$), where $\Lambda = (-\Delta)^{1/2}$ is the usual Calderón operator.

We find

$$\Lambda^{1/2}B(f,g)(t) = c\int_0^t (t-s)^{-9/4}\,\Theta_1\Big(\frac{\cdot}{\sqrt{t-s}}\Big) * (fg)(s)\,ds\,, \tag{3.19}$$

where $\Theta_1 = \Lambda^{\frac{1}{2}}\Theta \in L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Finally, if $3 < q \le 4$, we obtain

$$\begin{aligned} \|\Lambda^{1/2}B(f,g)(t)\|_2 \le c\Big(\int_0^t (t-s)^{-9/4+3/(2m)} s^{-\alpha}\,ds\Big)\|\Theta_1\|_m \\ \cdot\Big(\sup_{t\ge 0} t^{\alpha/2}\|f(t)\|_q\Big)\Big(\sup_{t\ge 0} t^{\alpha/2}\|g(t)\|_q\Big)\,, \end{aligned} \tag{3.20}$$

where

$$\frac{1}{2} = \frac{1}{m} + \frac{2}{q} - 1 \tag{3.21}$$

and the estimate (3.15) follows.

As announced before, it is now elementary to obtain, as a particular case of Theorem 3.1, the following existence and uniqueness result of self-similar solutions for the Navier-Stokes equations. Here, the crucial point is that condition (3.17) is invariant under the transformation $v(t,x)$ implies $v_\lambda(t,x)$ for all $\lambda > 0$.

Theorem 3.2. *Let q be fixed in $3 < q \le 6$ and $\alpha = \alpha(q) = 1 - 3/q$, then there exists an absolute constant $\delta > 0$ such that if v_0 belongs to $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$, $\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$, $\nabla\cdot v_0 = 0$ (in the sense of distributions) and $v_0(x) = \lambda v_0(\lambda x)$ for all $\lambda > 0$, then there exists a global mild solution of the Navier-Stokes equations which is written in the form*

$$v(t,x) = \frac{1}{\sqrt{t}}\,V\Big(\frac{x}{\sqrt{t}}\Big)\,, \tag{3.22}$$

where $V \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ *is such that*

$$V(x) = S(1)\, v_0 + W(x)\,, \tag{3.23}$$

with $W \in \dot{H}^{1/2}(\mathbb{R}^3)$, *if* $3 < q \leq 4$, *and* $W \in L^3(\mathbb{R}^3)$, *if* $4 < q \leq 6$. *The initial value* v_0 *is taken by* $v(t,x)$ *at least in the weak sense of distributions. Finally, there is only one solution* $v(t,x)$ *such that* $V \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ *and*

$$\|V\|_{\dot{B}_q^{-\alpha,\infty}} + \|V\|_q \leq R\,, \tag{3.24}$$

where $R = R(\|v_0\|_{\dot{B}_q^{-\alpha,\infty}})$ *is a given constant.*

4. An equivalence theorem for homogeneous functions on Besov spaces.

Starting from the remark that a homogeneous function of degree -1 is known in all $\mathbb{R}^3$ by its restriction on the unit sphere S^2, in this Section we present an equivalence theorem for homogeneous functions of degree -1 which belong to the Besov space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$, with $1 \leq q \leq \infty$ and $\alpha = 1 - 3/q$. In Section 1 (Lemma 1.2), we showed that it is *sufficient* for a homogeneous function f of degree -1 to have an $L^\infty(S^2)$ restriction to the unit sphere S^2, to ensure that $f \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$, $q > 3, \alpha = 1 - 3/q$.

This remark is enough to guarantee that both Theorems 3.1 and 3.2 do not admit only the trivial data $v_0 = 0$ as initial condition. In fact if we consider

$$v_0 = \left(0, \frac{-x_3}{|x|^2}, \frac{x_2}{|x|^2}\right) \tag{4.1}$$

then v_0 is divergence free and belongs to $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ for $q > 3$ and $\alpha = 1 - 3/q$.

In this Section we want to prove the following more general and accurate result.

Theorem 4.1. *Let* $1 \leq q \leq \infty$ *and* $\alpha = 1 - 3/q$ *be fixed. Then, for any homogeneous distribution* f *of degree* -1 *the following three conditions are equivalent*

i) f *belongs to the homogeneous Besov space* $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$,

ii) *the restriction of f to a certain neighborhood Ω of the unit sphere S^2 belongs to the non-homogeneous space $B_q^{-\alpha,q}(\Omega)$,*

iii) *the restriction of f to the unit sphere S^2 belongs to the nonhomogeneous Besov space $B_q^{-\alpha,q}(S^2)$.*

Before delving into the details of the proof of this theorem, let us recall some simple properties of the homogeneous Besov spaces $B_q^{s,p}(\mathbb{R}^3)$, for $(s,p,q) \in \mathbb{R} \times [1,\infty] \times [1,\infty]$.

First of all $B_q^{s,p}(\mathbb{R}^3)$ is a *module* in the ring $C_0^\infty(\mathbb{R}^3)$. This means that the elements λ of $C_0^\infty(\mathbb{R}^3)$ define endomorphisms λf ($\lambda \in C_0^\infty(\mathbb{R}^3)$, $f \in B_q^{s,p}(\mathbb{R}^3)$) of $B_q^{s,p}(\mathbb{R}^3)$, as in the case of a vector space. The second remark is that it is now possible to define a *local space* $B_q^{s,p}$. The localization (in a neighborhood of x_0) is obtained by multiplying by a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ equal to one in a neighborhood of x_0.

The second group of observations concerns the possibility of extending a function $f(x) \in B_q^{s,p}(\mathbb{R}^2)$ into a function $\tilde{f}(x,y)$ that we can suppose to be either independent of y and equal to $f(x)$ if $y=0$, or equal to $f(x)\varphi(y)$, where φ is the function appearing in a Littlewood-Paley decomposition and is such that $\varphi(0) \neq 0$. These two points of view are equivalent in a neighborhood of the $\mathbb{R}^2 \times \{0\}$, because we can multiply by $1/\varphi(y)$ in a neighborhood of $\mathbb{R}^2 \times \{0\}$.

Let us start with a proof of Theorem 4.1 in a model global case as given by the following Lemma.

Lemma 4.1. *Let $g \in S(\mathbb{R}^3)$ be a function whose Fourier transform $\hat{g}$ has compact support. To fix ideas, let $g(0) \neq 0$. Then, for any function f (defined on $\mathbb{R}^2$) and any set of index $(s,p,q) \in \mathbb{R} \times [1,\infty] \times [1,\infty]$, the norm of $f(x)$ in the nonhomogeneous Besov space $B_q^{s,p}(\mathbb{R}^2)$ is equivalent to that of $f(x)\,g(y)$ in $B_q^{s,p}(\mathbb{R}^3)$*

Here is a simple proof of this result. First of all, let us consider a Littlewood-Paley decomposition in $\mathbb{R}$ associated to a function φ such that the support of $\hat{g}$ is included in the set of points $y \in \mathbb{R}$ such that $\hat{\varphi}(y) = 1$. Then, we use a Littlewood-Paley decomposition in $\mathbb{R}^3$ associated to the product structure. In other words,

$$
\begin{aligned}
I &= S_0 \otimes S_0 + \sum_0^\infty (S_{j+1} \otimes S_{j+1} - S_j \otimes S_j) \\
&= S_0 \otimes S_0 + \sum_0^\infty (S_j \otimes \Delta_j + \Delta_j \otimes S_j + \Delta_j \otimes \Delta_j)\,.
\end{aligned}
\tag{4.2}
$$

But $S_j(g) = g$ for $j \geq 0$ and $\Delta_j g = 0$ for $j \geq 0$. This implies that $(\Delta_j \otimes S_j)(fg) = (\Delta_j f)(S_j g) = (\Delta_j f)g$ and $(S_j \otimes \Delta_j)(fg) = (S_j f)(\Delta_j g) = 0$ for $j \geq 0$. Finally,

$$\begin{aligned}\Big(\sum_0^\infty 2^{pjs}\|\Delta_j(fg)\|_q^p\Big)^{1/p} &= \Big(\sum_0^\infty 2^{pjs}\|\Delta_j f\|_q^p\,\|g\|_q^p\Big)^{1/p} \\ &= \|g\|_q\Big(\sum_0^\infty 2^{pjs}\|\Delta_j f\|_q^p\Big)^{1/p},\end{aligned} \tag{4.3}$$

which concludes the proof of the Lemma.

Before proving Theorem 4.1 in all its generality, we recall here an other interesting result

Lemma 4.2. *Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. For any tempered distribution $f \in D'(S^2)$, let $\tilde{f}$ designate the distribution defined in the open set $\Omega = \{x \in \mathbb{R}^3 : 1/2 < |x| < 3/2\}$ by*

$$\tilde{f}(x) = f\Big(\frac{x}{|x|}\Big)\chi(|x|)\,, \tag{4.4}$$

where $\chi \in C_0^\infty[1/2, 3/2]$ and $\chi = 1$ in a neighborhood of 1. With these notations, for any $(s,p,q) \in \mathbb{R} \times [1,\infty] \times [1,\infty]$ we have

$$f \in B_q^{s,p}(S^2) \qquad \text{if and only if} \qquad \tilde{f} \in B_q^{s,p}(\Omega)\,. \tag{4.5}$$

The proof of this result is trivial, because S^2 is compact, so that one can first argue locally in a neighborhood of the point $x_0 \in S^2$ and conclude in a standard way. The details are left to the reader.

We are now in a position to prove Theorem 4.1. The equivalence ii) if and only if iii) is a consequence of Lemma 4.2. In order to prove that i) if and onlt if ii) we use the well-known wavelet characterization of the homogeneous Besov spaces [13]. More precisely, let $\psi(x) = \psi_\varepsilon(x)$, $x \in \mathbb{R}^3$ designate the mother wavelets ($\varepsilon = 1, 2, \ldots, 7$) and let us consider a Daubechies compactly supported orthonormal base of $\mathbb{R}^3$. Let us put

$$c(j,k) = \int_{\mathbb{R}^3} 2^{3j}\,\overline{\psi}(2^j x - k)\,f(x)\,dx \tag{4.6}$$

the wavelet coefficients of a function f, the normalization being intended in the $L^1(\mathbb{R}^3)$ sense. Then, the following characterization holds [13]

Lemma 4.3. *$f \in \dot{B}_q^{s,p}(\mathbb{R}^3)$ if and only if*

$$\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}^3} |c(j,k)|^q \, 2^{sjq} \, 2^{-3j} \Big)^{p/q} < \infty \,. \tag{4.7}$$

We will use this lemma with $s = -\alpha = 3/q - 1$, $p = \infty$ and f homogeneous of degree -1. Now, taking into account this last property, we can write

$$c(j,k) = 2^j c(0,k) =: 2^j c(k) \tag{4.8}$$

so that condition (4.7) becomes

$$f \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3) \qquad \text{if and only if} \qquad c(k) \in l^q(\mathbb{Z}^3)\,. \tag{4.9}$$

Finally, we consider the neighborhood $\Omega = \{x \in \mathbb{R}^3 : 1/2 < |x| < 3/2\}$ of the unit sphere S^2 and evaluate the $B_q^{-\alpha,q}(\Omega)$ norm of an arbitrary function f by using the wavelets whose support is included in Ω. This means $1/2 < |k\,2^{-j}| < 3/2$ and $j \geq j_0$. The last step in the proof of Theorem 4.1 is given by the following well-known result [17]

Lemma 4.4 *Let Ω be an open bounded set in $\mathbb{R}^3$. Let us suppose that f belongs to $B_q^{s,p}(\Omega)$, in other words, let us suppose that f is the restriction to Ω of a function in $B_q^{s,p}(\mathbb{R}^3)$. Then we have*

$$\sum_{j} \Big(\sum_{k} |c(j,k)|^q \, 2^{sjq} \, 2^{-3j} \Big)^{p/q} < \infty \,, \tag{4.10}$$

where the sum over j and k is restricted to the values for which the support $S(j,k)$ of $\psi_{j,k} = 2^{3j}\,\psi(2^j x - k)$ is included in Ω. If, conversely, this condition is verified, then for any $\Omega_\varepsilon \subset \Omega$, with $d(\Omega_\varepsilon, \Omega^c) \geq \varepsilon$, f belongs to $B_q^{s,p}(\Omega_\varepsilon)$.

Let us conclude the proof of Theorem 4.1. If f is homogeneous of degree -1, then $f \in B_q^{-\alpha,q}(\Omega)$ means

$$\sum_{j\geq j_0} \; \sum_{1/2\leq|k\,2^{-j}|\leq 3/2} |c(k)|^q = \sum_{k\in\mathbb{Z}^3} |c(k)|^q < \infty \,, \tag{4.11}$$

that is $c(k) \in l^q(\mathbb{Z}^3)$, which, in turn, is equivalent to $f \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$. Here the crucial point is that the dyadic coronas $1/2 \leq |k\,2^{-j}| \leq 3/2$ cover exactly all of $\mathbb{Z}^3\backslash\{0\}$. The proof of Theorem 4.1 is now completed.

Before ending, let us remark, as announced, (see also [1]) that if a function f homogeneous of degree -1 is such that its restriction to the unit sphere S^2 is bounded, then

$$\sum_0^\infty 2^{(3-q)j} \|\Delta_j f\|^q_{L^q(S^2)} \leq c \sum_0^\infty 2^{(3-q)j} = c \tag{4.12}$$

as long as $q > 3$, which implies that $f \in \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$. More generally, according to a celebrated result of J. E. Littlewood and R. Paley [18], the *sufficient* condition

$$f|_{S^2} \in L^q(S^2)\,, \tag{4.13}$$

would give (for $q > 3$) the same result, for

$$\sum_0^\infty 2^{(3-q)j} \|\Delta_j f\|^q_{L^q(S^2)} \leq c \sum_0^\infty \|\Delta_j f\|^q_{L^q(S^2)} \leq c\,. \tag{4.14}$$

Acknowledgements. This article is based on the author's Ph.D thesis. The author expresses his gratitude to his advisor Y. Meyer for his constant support and generous help. The kind hospitality received at the Department of Mathematics of Washington University in St. Louis, where this work was written under a post-doctoral fellowship, is also gratefully acknowledged.

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Recibido: 8 de abril de 1.996

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Revista Matemática Iberoamericana
Vol. 13, N.º 3, 1997

Some generalized Coxeter groups and their orbifolds

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Abstract. In this note we construct examples of geometric 3-orbifolds with (orbifold) fundamental group isomorphic to a ($\mathbb{Z}$-extension of a) generalized Coxeter group. Some of these orbifolds have either euclidean, spherical or hyperbolic structure. As an application, we obtain an alternative proof of theorem 1 of Hagelberg, Maclaughlan and Rosenberg in [5]. We also obtain a similar result for generalized Coxeter groups.

1. Introduction.

A group with presentation

$$\langle a_1, \ldots, a_n : a_i^{u_i} = (a_{i+1} a_i^{-1})^{v_i} \rangle \,,$$

where u_i, v_i are integers greater or equal to two, is called a generalized Coxeter group. In the particular case $u_i = v_i = 2$, the above is a Coxeter group with Coxeter diagram (see [1, p. 110]) as shown in figure 1.

A group having presentation

$$\Gamma = \langle x_1, \ldots, x_n, t : x_i^{k_i} = (x_{i+1} x_i^{-1})^{l_i} = (x_i t)^{m_i} = 1 \rangle \,,$$

where k_i, l_i, m_i are integers greater or equal to 2, is called a $\mathbb{Z}$-extension of a generalized Coxeter group. The reason for the name is the following:

if H denotes the smallest normal subgroup of Γ containing the generator t, then the quotient group $J = \Gamma/H$ has presentation

$$J = \langle \tilde{x}_1, \ldots, \tilde{x}_n : \tilde{x}_i^{\widehat{k_i}} = (\tilde{x}_{i+1}\tilde{x}_i^{-1})^{l_i} = 1 \rangle \,,$$

where $\widehat{k_i} = \mathrm{mcd}\,(k_i, m_i)$, that is, J turns out to be a generalized Coxeter group.

Figure 1.

Triangle and generalized tetrahedron groups are some examples of generalized Coxeter groups.

The main problem of three-orbifolds is their classification. As observed by W. Thurston in his project, geometry and topology are very well related, there are exactly eight geometries and essentially the conjecture is that all 3-orbifolds can be obtained by gluing a finite number of geometric orbifolds (that is, orbifolds of the form X/G, where X is one of the eight geometries and G is a discrete group of isometries of X). The main geometry is given by the hyperbolic one.

To understand this classification problem is good to have examples of geometric orbifolds, which are the parts to be glued to form the more general ones. For it, triangle and Coxeter groups have been of great interest (see, for instance, in Coxeter-Moser [2]).

Another (hyperbolic) orbifolds, with generalized triangle groups as (orbifold) fundamental groups, were studied in [7], [4], [5] and [6], where results concerning their discreteness and arithmeticity were obtained.

Generalized Coxeter groups were studied from the group theorical perspective in [12] and [13].

The main idea of this paper is roughly speaking the following. We first describe some graphs in the 3-sphere (perhaps with some deleted vertices) together a data given by some integers. Secondly, we try to figure out which kind of geometric orbifolds can be obtained with these objects. For it, we construct certain planes intersecting in a very special fashion and we consider the reflections on these planes. We apply Poincare's polygon theorem to obtain that the group generated by those reflection is discrete and has the desired presentation (*e.g.* generators and relations).

More technically, we construct explicit embeddings of some ($\mathbb{Z}$-extension of) generalized Coxeter groups as group of isometries of some 3-dimensional geometry. The orbifolds obtained are then geometric 3-orbifolds whose (orbifold) fundamental group is a ($\mathbb{Z}$-extension of a) generalized Coxeter group.

The presented geometries in these constructions are $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^3$, where S^2, $\mathbb{H}^2$ and $\mathbb{H}^3$ denote the 2-sphere, the hyperbolic plane and the hyperbolic three space, respectively. Some of these orbifolds were obtained in the paper of Dunbar [3].

This paper is organized as follows. We start in Section 2 with some basics definitions and the description of a particular type of 3-orbifolds denoted by $\mathcal{O}(n, K, L, M)$. The essential property of these orbifolds is the fact that their (orbifold) fundamental groups are $\mathbb{Z}$-extension of generalized Coxeter groups.

In Section 3 we produce geometries of type $\mathcal{Z} \times \mathbb{R}$, for the orbifolds $\mathcal{O}(n, K, L, K)$ with $K = (2, \ldots, 2)$, where $\mathcal{Z} \in \{\mathbb{H}^2, \mathbb{R}^2, S^2\}$. More precisely, we obtain the following

Theorem 1. *For $n \geq 3$, $K = (2, \ldots, 2)$ and $L = (l_1, \ldots, l_n)$, let $\kappa = n - 2 - \sum_{i=1}^{n} 1/l_i$. Then the orbifold $\mathcal{O}(n, K, L, K)$ has geometry:*

a) $\mathbb{H}^2 \times \mathbb{R}$, *if $\kappa > 0$,*

b) $S^2 \times \mathbb{R}$, *if $\kappa < 0$,*

c) $\mathbb{R}^3$, *if $\kappa = 0$.*

We also obtain some generalized triangle groups, yielding part i) of Theorem 1 in [5]. More precisely, we prove that the group $\langle x, r : x^2 = r^n = (r^{-1} x\, r\, x)^l = 1 \rangle$ can be embedded as group of isometries of $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$ and $\mathbb{R}^3$ (respectively) if $(l-1)\, n - 2\, l$ is bigger, less and equal to zero (respectively).

In Section 4 we construct generalized Coxeter groups, represented as $\mathbb{Z}_2$-extensions of Fuchsian groups of the first kind, that uniformize orbifolds $\mathcal{O}(n, K, L, K)$ with $K = (2, \ldots, 2)$, $n \geq 3$ and $n - 2 > \sum_{i=1}^{n} 1/l_i$. We also construct the generalized triangle group $\langle x, r : x^2 = r^n = (r^{-1} x\, r\, x)^l = 1\rangle$ as a $\mathbb{Z}$-extension of a Fuchsian group of the first kind for $(l-1)\, n - 2\, l > 0$.

In Section 5 we generalize the construction of Section 4 to obtain $\mathbb{Z}_2$-extensions of Fuchsian groups (maybe of the second kind).

In Section 6 we produce hyperbolic structures for the orbifolds $\mathcal{O}(n, K, L, K)$, where $n \geq 5$, $K = (3, \ldots, 3)$ and $L = (2, \ldots, 2)$; more precisely, we prove the following.

Theorem 2. *If $n \geq 5$, $K = (3, \ldots, 3)$ and $L = (2, \ldots, 2)$, then the orbifold $\mathcal{O}(n, K, L, K)$ has a hyperbolic structure.*

We also show that for $n \geq 5$ the generalized triangle group $\langle x, r : x^3 = r^n = (r^{-1} x^{-1} r x)^2 = 1\rangle$, can be realized as a group of hyperbolic isometries of $\mathbb{H}^3$.

In Section 7 we discuss the excluded cases $n \in \{3, 4\}$ of Theorem 2.

In Section 8 we generalize the constructions of the sections 3-6. In this way, we can obtain a many generalized Coxeter groups as group of isometries of the hyperbolic three space. In some of these cases, one may proceed as in Section 6 to produce $\mathbb{Z}$-extensions of generalized Coxeter groups as groups of isometries of $\mathbb{H}^3$.

In Section 9 we use the construction done in Section 8 to obtain a simple alternative proof of [5, Theorem 1]. We obtain a similar result for generalized Coxeter groups. Namely, we have the following.

Theorem 3. *Let n, k, l be integers with $n \geq 2$, $l \geq 2$ and $k \geq 2$ and $(n, k, l) \neq (3, 3, 2)$. There is a discrete and faithful representation of the generalized Coxeter group*

$$G = \langle x_1, \ldots, x_n : x_i^k = (x_{i+1} x\, i^{-1})^l = 1\rangle\,,$$

as group of isometries in one of the geometries $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $\mathbb{H}^3$. The geometry is $\mathbb{H}^3$ except when

$$(n, k, l) \in \{(3, 2, 2), (3, 2, 3), (3, 3, 2), (4, 2, 2)\}\,.$$

Part of this paper was written while the second author was visiting the Université Paul Sabatier.

2. Some basic definitions.

A n-orbifold $\mathcal{O}$ consists of a Hausdorff space X (the underlying topological space of the orbifold) and a collection

$$\{(U_\alpha, V_\alpha, G_\alpha, f_\alpha : U_\alpha \longrightarrow V_\alpha/G_\alpha),\ \alpha \in \mathcal{A}\},$$

satisfying the following properties:

1) The collection $\{U_\alpha : \alpha \in \mathcal{A}\}$ is an open covering of X.

2) G_α is a finite group of homeomorphisms of the open subset $V_\alpha \subset \mathbb{R}^n$.

3) The map $f_\alpha : U_\alpha \longrightarrow V_\alpha/G_\alpha$ is a homeomorphism.

4) If $U_\alpha \cap U_\beta \neq \varnothing$, and if $\pi_s : V_s \longrightarrow V_s/G_s$ is the natural (branched) covering induced by the action of G_s on V_s, then the map

$$f_\beta \circ f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \longrightarrow f_\beta(U_\alpha \cap U_\beta)$$

can be lifted to a homeomorphism

$$h_{\alpha,\beta} : \pi_\alpha^{-1}(f_\alpha(U_\alpha \cap U_\beta)) \longrightarrow \pi_\beta^{-1}(f_\beta(U_\alpha \cap U_\beta)).$$

Figure 2.

Some $\mathbb{Z}$-extensions of generalized Coxeter groups appear as (applying the Wirtinger algorithm and Haeflieger and Quach [8]) orbifold fundamental groups for orbifolds $\mathcal{O}(n, K, L, M)$ having the three sphere S^3 as underlying topological space and the planar graph $\pi_n(K, L, M)$ with the branching indices as shown in figure 2 as branching locus, where $K = (k_1, \ldots, k_n)$, $L = (l_1, \ldots, l_n)$ and $M = (m_1, \ldots, m_n)$ are n-tuples of integers greater or equal to two. It may happen that one or both vertices of the graph are not there, in which case the underlying topological space is either $\mathbb{R}^3$ or $\mathbb{R}^3$ minus a point. Generators X_i and T for the (orbifold) homotopy group of $\mathcal{O}(n, K, L, M)$ are given by (homotopy classes of) the loops shown in figure 3.

Figure 3.

We remark that there are some necessary restrictions on the n-tuples K, L, M in order to have an orbifold with the vertices included as described above. These restrictions are

$$\{k_{i-1}, l_i, k_i\}, \{m_{i-1}, l_i, m_i\} \in \{\{2,2,r\}, \{2,3,3\}, \{2,3,4\}, \{2,3,5\}\},$$

where $r \geq 2$. The above is consequence of the fact that the stabilizer of a point (to belong to the region of discontinuity of a discrete group) is a finite group which is either cyclic, dihedral or the group of isometries of a platonic solid.

A geometry is a simply-connected complete Riemannian manifold (M, ρ) with transitive isometry group G, having a co-finite volume subgroup and such that the stabilizer at a point is compact.

In dimension three, Thurston has shown that there are exactly eight geometries: $\mathbb{H}^3$, S^3, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $sl(2,\mathbb{R})$, Nil-geometry and Sol-geometry ([10], [1] and [11]).

3. First construction: splitting geometries.

We construct geometric structures for some of the above orbifolds $\mathcal{O}(n,K,L,K)$, where $K = (2,2,\ldots,2)$ and $n \geq 3$. The geometries are given by $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ and $\mathbb{H} \times \mathbb{R}$. In this way, the corresponding $\mathbb{Z}$-extension generalized Coxeter groups can be realized as a group of isometries in the respective geometry. The case $n = 2$ (where necessarily, $l_1 = l_2$) and some cases for $n = 3$ were considered by Dunbar in [3]. The above is summarized in the following.

Theorem 1. *For $n \geq 3$, $K = (2,\ldots,2)$ and $L = (l_1,\ldots,l_n)$, let $\kappa = n - 2 - \sum_{i=1}^{n} 1/l_i$. Then the orbifold $\mathcal{O}(n,K,L,K)$ has geometry:*

a) $\mathbb{H}^2 \times \mathbb{R}$, *if $\kappa > 0$,*

b) $S^2 \times \mathbb{R}$, *if $\kappa < 0$,*

c) $\mathbb{R}^3$, *if $\kappa = 0$.*

PROOF. Set $\mathcal{Z}$ equal to S^2, $\mathbb{R}^2$ and $\mathbb{H}^2$ for κ negative, positive and zero, respectively. We consider a geodesic polygon $P \subset \mathcal{Z}$ of n-sides and internal angles equals to (in cyclic order) $\pi/l_1,\ldots,\pi/l_n$, respectively. We label the sides of P as $e_1,\ldots,e_n$, in such a way that e_i connects the vertices with angles π/l_{i-1} and π/l_i. Let σ_i be reflection on the geodesic line containing the side e_i. We define the following isometries of $X \times \mathbb{R}$

$$x_i(p,q) = (\sigma_i(p),q) \qquad \text{and} \qquad t(p,q) = (p,q+1)\,,$$

where $(p,q) \in \mathcal{Z} \times \mathbb{R}$.

The group Γ generated by $x_1,\ldots,x_n$ and t is a discrete group acting on $\mathcal{Z}\times\mathbb{R}$ with fundamental domain given by $P\times[-1/2,1/2]$ (the transformations $x_1,\ldots,x_n$ generate a discrete group acting on $X \times \{q\}$ with fundamental polygon $P \times \{q\}$ for all $q \in \mathbb{R}$, and t is just an orthogonal translation to X. In fact, this is just Poincare's Polyhedron theorem on $X \times \mathbb{R}$).

It is evident that the above group is in fact a generalized Coxeter group and that the quotient $(\mathcal{Z} \times \mathbb{R})/\Gamma$ is an orbifold with S^3 as underlying space and as branching set the planar graph $\pi_n(K, L, M)$, where $K = M$ is the n-tuple formed by 2 at each component and $L = (l_1, \dots, l_n)$.

REMARK 1. Suppose that in the above construction we have the particular situation $l_i = l$, for some fixed $l \geq 2$. In this case, we may consider the polygon P to be invariant under an isometry $\tilde{r}$ of $\mathcal{Z}$ of order n. Set $r(p, q) = (\tilde{r}(p), q)$. We have that r is an isometry of $X \times \mathbb{R}$ of order n, keeping invariant $P \times [-1/2, 1/2]$. If we set $x = x_1$, then the group $\tilde{\Gamma}$ generated by Γ and r has the following presentation

$$\tilde{\Gamma} = \langle x, t, r : \; x^2 = r^n = (r\,t)^2 = (r^{-1} x\, r\, x)^l = r\, t\, r^{-1} t^{-1} = I \rangle \,.$$

The geometric orbifold uniformized by $\tilde{\Gamma}$ is shown in figure 4a. The group G generated by x and r has presentation

$$G = \langle x, r; x^2 = r^n = (r^{-1} x\, r\, x)^l = 1 \rangle \,,$$

(a generalized triangle group) and a fundamental domain given by the infinite volume cylinder $P^* \times \mathbb{R}$, where P^* is the triangle bounded e_1 and the two (geodesic) lines connecting the fixed point of $\tilde{r}$ with both ends of e_1. It follows that the geometric orbifold uniformized by G has infinite covolume, underlying topological space S^3 minus two points and branching locus has shown in figure 4b. In particular, if $\mathcal{Z}$ is either S^2 or $\mathbb{R}^2$, then we obtain (with the exception of one case) the cases in [5, Theorem 1.i)].

Figure 4.

4. Second construction: uniformizations by $\mathbb{Z}_2$-extensions of Fuchsian groups.

If in Theorem 1 the geometry is $\mathbb{H}^2 \times \mathbb{R}$ (that is, if $n \geq 3$ and $n-2-\sum_i^n 1/l_i > 0$) we can uniformize the orbifolds $\mathcal{O}(n,K,L,K)$, where $K=(2,\ldots,2)$ and $L=(l_1,\ldots,l_n)$, by the manifold $M=S^3-\widehat{\mathbb{R}}$, where $\widehat{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$. We can construct J so that $\mathcal{O}(n,K,L,K)=M/J$ and J is a group of Möbius transformations, in fact a $\mathbb{Z}_2-$extension of a Fuchsian group. The group J is constructed as follows: in the hyperbolic three space $\mathbb{H}^3=\{(z,t)\in\mathbb{C}\times\mathbb{R},\ t>0\}$ we consider the model of hyperbolic two space $\mathbb{H}^2_*=\{(z,t)\in\mathbb{H}^3,\ \mathrm{Im}\,(z)=0\}$.

On $\mathbb{H}^2_*$ we consider a geodesic polygon P of n sides and consecutive angles $\pi/l_1,\ldots,\pi/l_n$.

Let e_i be the side of P determined by the vertices with angles π/l_{i-1} and π/l_i.

Set x_i the isometry of $\mathbb{H}^3$ of order two having fixed point axes containing e_i.

Set D to be the common region determined by the geodesic planes $F_1,\ldots,F_n$, orthogonal to $\mathbb{H}^2_*$, such that $F_i\cap\mathbb{H}^2_*$ contains e_i.

As a consequence of Poincare's Polyhedron theorem [9], the group J, generated by $x_1,\ldots,x_n$, is a discrete group with D as a fundamental polyhedron.

The group J also acts as group of isometries of $\mathbb{H}^2_*$ (the transformations x_i are reflections) with P as compact fundamental domain for its action. It follows that the limit set of J is the boundary γ of $\mathbb{H}^2_*$ and, in particular, the group J is either a Fuchsian group or a $\mathbb{Z}_2$-extension of a Fuchsian group of the first kind. Since each generator x_i permutes both discs on $\widehat{\mathbb{C}}$ (the boundary of $\mathbb{H}^3$) bounded by γ, we have that J is a $\mathbb{Z}_2$-extension of a Fuchsian group of the first kind.

It is not hard to see that J is a generalized Coxeter group (with the generators $x_1,\ldots,x_n$) and that $(S^3-\widehat{\mathbb{R}})/J$ is the orbifold $\mathcal{O}(n,K,L,K)$.

REMARK 2. As before, by considering $l_i=l>n/(n-2)$, we may assume the geodesic polygon to be invariant under a rotation $\tilde{R}$ of order n. By performing the same construction as in Remark 1, we obtain an uniformization of the orbifold shown in figure 4a by a group G with presentation $(x=x_1)$

$$G=\langle x,r:\ x^2=r^n=(x\,r\,x\,r^{-1})^l=I\rangle\,,$$

that is, a generalized triangle group (compare to [5, Theorem 1]).

5. Generalization of the second construction.

The above construction can be generalized as follows. Let P be a hyperbolic polygon, maybe with an infinite number of sides, in the hyperbolic plane $\mathbb{H}^2_* \subset \mathbb{H}^3$.

We assume each internal angle $\theta_i = \pi/l_i$, where l_i is an integer greater or equal to two.

For each side e_j of P, we consider the isometry x_j of order two, with fix point set the geodesic containing e_j. Set G the group generated by all the transformations x_j.

A geodesic polyhedron Q is defined by the common region bounded by the geodesic planes F_i, orthogonal to $\mathbb{H}^2_*$, such that $F_i \cap \mathbb{H}^2_*$ contains e_i.

Poincare's Polyhedron theorem [9] applied to Q and the side pairings x_j asserts that G is a discrete group, Q is a fundamental polyhedron for the action of G, and a minimal set of relations for G given by $(x_{j-1}\, x_j)^{l_j} = I$ (if the sides e_{j-1} and e_j meet in $\mathbb{H}^2_*$ at the angle π/l_j).

The group G is again a $\mathbb{Z}_2$-extension of a Fuchsian group, and is of the first kind if and only if the polygon P has no sides in the boundary $\widehat{\mathbb{C}}$. In the case that G is not of the first kind, the orbifold $(S^3 - \Lambda)/G$, where Λ is the limit set of G, has universal covering $S^3 - \Lambda$. It has (orbifold) fundamental group isomorphic to G.

6. Third construction: hyperbolic uniformization.

One may think that the type of orbifolds we are considering cannot have a hyperbolic structure, that is, there is not a discrete group Γ of isometries of $\mathbb{H}^3$ which is a $\mathbb{Z}$-extension of a generalized Coxeter group.

The following theorem asserts that this is not the case, that is, we may have hyperbolic structures on some of our orbifolds.

Theorem 2. *If $n \geq 5$, $K = M = (3, \ldots, 3)$ and $L = (2, \ldots, 2)$, then the orbifold $\mathcal{O}(n, K, L, M)$ has a hyperbolic structure.*

PROOF. Set $n \geq 5$, $K = M = (3, \ldots, 3)$, $L = (2, \ldots, 2)$ and choose as α the angle between two faces of the regular Euclidean tetrahedron (that is, $\alpha \in (0, \pi)$ with $\cos \alpha = 1/3$).

We continue to consider the hyperbolic plane in the three space $\mathbb{H}^2_* = \{(z,t) \in \mathbb{H}^3, \operatorname{Im}(z) = 0\}$.

Let r be an isometry of $\mathbb{H}^3$ of order n, keeping invariant the plane $\mathbb{H}^2_*$.

We consider a hyperbolic polygon of n sides $P \subset \mathbb{H}$ which is invariant under the action of r. We label the sides of P consecutively in counterclockwise order by $e_1, \ldots, e_n$, respectively.

We may assume all internal angles to be equal to α. This is consequence of the fact that $\alpha < \pi/2$.

Choose a hyperbolic isometry x_1 of order three having as its fixed points set the geodesic line containing e_1. Since there are two possible choices for x_1, we take the one for which $x_1(P)$ consists of points (z, t) with $\mathrm{Im}\,(z) > 0$. Set $x_{i+1} = r x_i r^{-1}$.

The choice of the angle α ensures that both x_i and x_{i+1} generate the alternating group A_4. Moreover, every relation of A_4 is consequence of the relations $x_i^3 = x_{i+1}^3 = (x_{i+1} x_i^{-1})^2 = I$.

Consider the totally geodesic half-plane F_1^+ containing the side e_1, making an angle of $\pi/3$ with $\mathbb{H}^2_*$ and such that $F_1^- = x_1(F_1^+)$ also makes an angle of $\pi/3$ with $\mathbb{H}^2_*$ (see figure 5). The half-planes $F_i^* = r^{i-1}(F_1^*)$, for $* \in \{+, -\}$ and $i = 1, \ldots, n$, determine a hyperbolic polyhedron P^3 with $2\,n$ sides.

Figure 5.

We have that x_i is equal to $\sigma \circ \sigma_i$, where σ_i is reflection on F_i^+ and σ is reflection on $\mathbb{H}^2_*$.

The transformations $x_1, \ldots, x_n$, pair the sides by the rule $x_i(F_i^+) = F_i^-$. It is easy to check that the sides F_i^* and F_{i+1}^* meet at a right angle: it is a consequence of the fact that the transformation $x_{i+1}\, x_i^{-1}$

has order two.

Poincare's Polyhedron theorem [9] asserts that the group G generated by these transformations is a Kleinian group with P^3 as fundamental domain.

Figure 6.

Figure 7.

The intersection of P^3 with the boundary $\widehat{\mathbb{C}}$ of $\mathbb{H}^3$ consists of two right hyperbolic polygons P_1 and P_2, each one of n sides, such that each side of P_1 is paired by the transformations x_i to one side of P_2 (see figure 6). Each of these two polygons belongs to some component of the region of discontinuity of G which must be a round disc. The (circle) boundaries of these two components are disjoint, but both components are equivalent under G. It follows that G has no invariant component, each component is a round disc and they are all equivalent under the action of G. We also have that $(\mathbb{H}^3 \cup \Omega(G))/G$ is topologically the unit three ball with the boundary and branching set the planar graph shown in figure 7.

We may assume that the polygon P_1 is the one formed by the faces $F_1^+, \dots, F_n^+$. The (circle) boundary of the component of the region of discontinuity of G containing such a polygon determines a geodesic plane W on $\mathbb{H}^3$. The plane W necessarily cuts P^3 orthogonally. Let j be the reflection on W and set $t = j\sigma j\sigma$.

Set Q to be the hyperbolic polyhedron determined by the sides of P^3 and the two geodesic planes $j(\mathbb{H}^2_*)$ and $\sigma(j(\mathbb{H}^2_*))$, respectively. Side pairings of Q are given by $x_1, \dots, x_n$ and t.

The conditions for Poincare's polyhedron theorem [9] are satisfied in this case. In particular, we obtain that the group Γ generated by these transformations is a discrete group, Q is a fundamental polyhedron, and G is a $\mathbb{Z}$-extension of a generalized Coxeter group. Moreover, the orbifold $\mathbb{H}^3/\Gamma$ is an orbifold with underlying topological space S^3 and branching set the planar graph $\pi_n(K, L, M)$ with K, L, M as before.

Figure 8.

Remark 3. The group $\tilde{\Gamma}$ generated by Γ and the elliptic transformation r is again a hyperbolic group with the presentation

$$\tilde{\Gamma} = \langle x, t, r : \; x^3 = r^n = (x\,t)^3 = (r^{-1}x^{-1}r\,x)^2 = t^{-1}r^{-1}t\,r = I \rangle\,,$$

where $x = x_1$. It uniformizes the orbifold shown in figure 8a. The group G generated by x and r has the presentation

$$G = \langle x, r : \; x^3 = r^n = (r^{-1} x^{-1} r \, x)^2 = 1 \rangle \,,$$

that is, a generalized triangle group. It uniformizes a hyperbolic orbifold with S^3 minus a point as underlying topological space, and branch locus as shown in figure 8b.

7. The excluded cases $n = 3$ and $n = 4$ in theorem.

7.1. The case $n = 4$.

Let us consider the orbifold $\mathcal{O}(4, K, L, K)$, where $K = (3, 3, 3, 3)$, $L = (2, 2, 2, 2)$. For this, we proceed to construct in the hyperbolic plane $\mathbb{H}^2_* \subset \mathbb{H}^3$ a hyperbolic quadrilateral P, invariant under the rotation r of order 4 with fixed points axis orthogonal to $\mathbb{H}^2_*$.

In the same way as in Section 6, we obtain a hyperbolic polyhedron P^3 with 8 sides, labeled F_i^+ and F_i^- and side pairings x_i for $i = 1, ..., 4$. In this case, we have that the sides F_1^q, F_2^q, F_3^q and F_4^q, with $q \in \{+, -\}$, meet at a point $p_q \in \widehat{\mathbb{C}} \cap A(r)$, where $A(r)$ is the set of fixed points of r.

We choose W to be a horosphere centered at p_+ and disjoint from $\mathbb{H}^2_*$, and denote by j and σ the reflections (anticonformal involutions of S^3) across W and $\mathbb{H}^2_*$, respectively.

Figure 9.

Set $t = j\sigma j\sigma$, and consider Q to be the region bounded by P^3 and the spheres $j(\mathbb{H}^2_*)$ and $\sigma(j(\mathbb{H}^2_*))$. Denote by $\tilde{P}^3$ the image of the polyhedron P^3 under the reflection on the sphere $\widehat{\mathbb{C}}$.

The group Γ generated by $x_1, \ldots, x_4$ and t is a Kleinian group acting on S^3 for which the boundary sphere of $\mathbb{H}^3$ is contained in its limit set.

This group has exactly one invariant component $\Delta \subset \mathbb{H}^3$ which is simply connected.

We have $Q \subset \Delta$ is a fundamental domain for the action of Γ on Δ. The orbifold Δ/Γ is $\mathcal{O}(n, K, L, K)$. The other components are all equivalent to the complement of the closure of $\mathbb{H}^3$ in S^3, denoted by $\mathbb{H}^3_-$. The stabilizer of $\mathbb{H}^3_-$ is the group G generated by $x_1, \ldots, x_4$. The orbifold $\mathbb{H}^3_-/G$ is shown in figure 9.

Remark 4.

1) The group Γ is the analogous to a regular B-group in dimension 2 (a Kleinian group with a simply connected invariant component [9]).

2) The group $\tilde{\Gamma}$ generated by r and Γ uniformizes the orbifolds shown in figure 10.

Figure 10.

3) The group G generated by $x = x_1$ and r has presentation

$$G = \langle x, r : \; x^3 = r^4 = (r^{-1}x^{-1}r\,x)^2 = 1\rangle\,,$$

that is, G is a generalized triangle group. It uniformizes the hyperbolic orbifold of figure 10b.

7.2. The Case $n = 3$.

Let us consider now the orbifold $\mathcal{O}(3, K, L, K)$, where $K = (3, 3, 3)$, $L = (2, 2, 2)$. In this case the orbifold fundamental group has the representation

$$\Gamma = \langle x_1, x_2, x_3, t;\, x_i^3 = (x_{i+1}\, x_i^{-1})^2 = (x_i\, t)^3 = 1 \rangle \,.$$

In this case, the subgroup G generated by the transformations x_1, x_2 and x_3 is a finite group of order 60. Moreover, it has the representation

$$G = \langle x_1, x_2, x_3 :\ x_i^3 = (x_{i+1}\, x_i)^2 = 1 \rangle \,.$$

It follows that G has no elements of order 5. In particular, we cannot have an embedding of the group G (so neither of Γ) as group of isometries of S^3, $\mathbb{H}^3$, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $sl(2, \mathbb{R})$.

Theorem 1 (parts (2.a) and (3)) in [3] asserts that our orbifold cannot have neither Nil nor Sol geometry and, in particular, the group Γ cannot be embedded as group of isometries of any of these two geometries.

8. Generalization of the Constructions.

In this section, we generalize the previous constructions. These generalizations give embeddings of generalized Coxeter groups as group of isometries of the hyperbolic three space. In many of the cases, they also produce embeddings of $\mathbb{Z}$-extensions of generalized Coxeter groups as isometries of $\mathbb{H}^3$.

We use the unit three ball in $\mathbb{R}^3$ as model of the hyperbolic three space, that is, $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$. We need the following basic fact:

Lemma. *Let F_1, F_2 and F_3 be three hyperbolic planes in $\mathbb{H}^3$. We assume that the three planes intersect in a point $q \in \mathbb{H}^3 \cup \widehat{\mathbb{C}}$. Denote by l_i the intersection between the planes F_3 and F_i, for $i = 1, 2$. For $q \in \widehat{\mathbb{C}}$, set $\alpha = 0$, otherwise set $\alpha \in (0, \pi)$ to be the angle between l_1 and l_2 (we use counterclockwise orientation in the plane determined by those two lines). If $\beta_i \in (0, \pi/2)$ is the angle between F_i and F_3, and $\theta \in [\alpha, \pi]$ is the angle between F_1 and F_2, then*

$$\cos\theta = -\cos\beta_1 \cos\beta_2 + \cos\alpha \sin\beta_1 \sin\beta_2 \ .$$

Figure 11 shows the situation described in the above lemma.

Figure 11.

Proof. If $q \in \widehat{\mathbb{C}}$, then we may assume $q = \infty$. In this case, the planes are orthogonal euclidean planes in $\mathbb{R}^3$ to the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, and the three planes F_1, F_2 and F_3 determine a euclidean triangle of internal angles β_1, β_2 and θ and, in particular, the above formula holds trivially.

Let us assume now that q is inside the hyperbolic three space. In this case, we may assume the model of $\mathbb{H}^3$ given by the unit three ball in $\mathbb{R}^3$ with center in the origin $q = 0$. We also may assume that

a) $l_1 = \{(x, y, z) : x = z = 0\}$, $l_2 = \{(x, y, z) : z = 0, \tan\alpha = -x/y\}$,
b) $F_3 = \{(x, y, z) : z = 0\}$,
c) $F_1^{\perp} = \langle w_1 \rangle$, where

$$w_1 = (\cos(\pi/2 - \beta_1), 0, \sin(\pi/2 - \beta_1)),$$

d) $F_2^{\perp} = \langle w_2 \rangle$, where

$$w_2 = (\cos\alpha\cos(\pi/2 - \beta_2), -\sin\alpha\cos(\pi/2 - \beta_2), -\sin(\pi/2 - \beta_2)).$$

Now the equality $w_1 \cdot w_2 = \cos\theta$ gives the desired equality.

Now we proceed to the construction. For this, let us consider two n-tuples $K = (k_1, \ldots, k_n)$ and $L = (l_1, \ldots, l_n)$ of integers bigger or equal to two, such that

$$\text{(i)} \qquad \frac{1}{l_i} + \frac{1}{k_i} + \frac{1}{k_{i+1}} \geq 1\,.$$

We determine angles $\alpha_i \in [0, \pi/l_i]$ by the equation

$$\text{(ii)} \qquad \cos\alpha_i = \frac{\cos\left(\dfrac{\pi}{l_i}\right) + \cos\left(\dfrac{\pi}{k_i}\right)\cos\left(\dfrac{\pi}{k_{i+1}}\right)}{\sin\left(\dfrac{\pi}{k_i}\right)\sin\left(\dfrac{\pi}{k_{i+1}}\right)}\,.$$

The last equation gives some restrictions on l_i, k_i and k_{i+1}. We assume that

$$\text{(iii)} \qquad (n-2)\,\pi > \sum_{i=1}^{n} \alpha_i\,.$$

On the hyperbolic plane $\mathbb{H}^2 = \{(x,y,z) \in \mathbb{H}^3 : z = 0\}$, we draw an n-sided hyperbolic polygon P, where the sides are labeled cyclically as e_1, $e_2, \ldots, e_n$. We assume that the vertex v_i, determined by e_i and e_{i+1}, is contained inside $\mathbb{H}^3$ or on its boundary according as

$$\frac{1}{l_i} + \frac{1}{k_i} + \frac{1}{k_{i+1}}$$

is bigger than one or equal to one. The (internal) angle of the vertex v_i is α_i.

The existence of such a polygon is guaranteed by the inequality (iii). For each edge e_i, we consider a hyperbolic plane F_i that contains e_i. Let F_i^+ be the part of the above plane contained in the half-space $\{z \geq 0\}$. We assume that the angle between F_i^+ and P is exactly π/k_i.

Set σ and σ_i to be the reflection through $\mathbb{H}^2$ and F_i, respectively. Let K be the group generated by the reflections $\sigma, \sigma_1, \ldots, \sigma_n$. Since the angle between the planes F_i^+ and F_{i+1}^+ is π/l_i (this is a consequence of the above lemma and the definition of α_i), and the polyhedron P^3 determined by the faces P, $F_1^+, \ldots, F_n^+$, has finite number of sides, Poincare's polyhedron theorem applies to this case to obtain that:

a) K is a discrete group of isometries.

b) P^3 is a fundamental domain for K.

c) The generators for K are $\sigma, \sigma_1, \dots, \sigma_n$.

d) A set of maximal relations on these generators are: $\sigma^2 = \sigma_i^2 = (\sigma\,\sigma_i)^{k_i} = (\sigma_{i+1}\,\sigma_1)^{l_i} = 1$.

If we set $x_i = \sigma\,\sigma_i$, then we have that the group J generated by $x_1, \dots, x_n$ has the presentation

$$J = \langle x_1, \dots, x_n : \; x_i^{k_i} = (x_{i+1}\,x_i^{-1})^{l_i} = 1 \rangle \,,$$

that is, J is a generalized Coxeter group. A fundamental polyhedron for J is given by $Q = P^3 \cup \sigma(P^3)$, and the index of J in K is two.

In the particular cases that Q intersects the boundary of $\mathbb{H}^3$ in two polygons (both symmetric by the reflection σ), they will have internal angles equal to $\pi/l_1, \dots, \pi/l_n$. If it happens that $(n-2)\,\pi > \sum_{i=1}^{n} \pi/l_i$, and the group J is not Fuchsian group, then we may proceed as in construction of Section 6 to obtain a $\mathbb{Z}$-extension of a generalized Coxeter group.

9. An application.

In this section we proceed to obtain an alternative and easy proof of [5, Theorem 1], and an equivalent result for generalized Coxeter groups.

9.1. First construction.

We continue using the notation of Section 8, and assume $n \geq 3$, $l_i = l \geq 2$ and $k_i = k \geq 2$. In this situation the restrictions to the construction are the following:

i) $1/l + 2/k \geq 1$.

ii) The equation

$$\cos\alpha = \frac{\cos\left(\dfrac{\pi}{l}\right) + \cos^2\left(\dfrac{\pi}{k}\right)}{\sin^2\left(\dfrac{\pi}{k}\right)}$$

has a solution in the interval $[0, \pi/l]$.

iii) $(n-2)\,\pi/n > \alpha$.

In this case, the restriction ii) is a consequence of i), and restriction iii) is equivalent to:

iii.1) If $n = 3$, then

$$2\cos\left(\frac{\pi}{l}\right) > 1 - 3\cos^2\left(\frac{\pi}{k}\right).$$

iii.2) If $n = 4$, then

$$\sqrt{2}\,\cos\left(\frac{\pi}{l}\right) > 1 - (1+\sqrt{2})\cos^2\left(\frac{\pi}{k}\right).$$

iii.3) If $n \geq 5$, then there is no restriction.

In particular, the only exceptional cases for (n, k, l) are those with $1/l + 2/k < 1$ and $(3, 2, 2)$, $(3, 3, 2)$, $(3, 2, 3)$ and $(4, 2, 2)$.

We may construct the polygon P (as done in Section 8) with n sides, all internal angles equal to α, and invariant under an isometry R of order n. The group G generated by $x = x_1$ (constructed as in last section) and r is a group of isometries of $\mathbb{H}^3$ with presentation

$$G = \langle x, r : \; x^k = r^n = (r^{-1}x^{-1}r\,x)^l = 1\rangle\,.$$

Set M^+ and M^- the geodesic planes determined by the following properties:

1) M^+ and M^- are orthogonal to $\mathbb{H}^2$.

2) $M^+ \cap \mathbb{H}^2$ and $M^- \cap \mathbb{H}^2$ are geodesics through the vertices of P determined by the side e_1.

3) The angle between M^+ and M^- is $2\pi/n$.

Figure 12.

The hyperbolic polygon determined by the faces F_1^+, F_1^- (as in last section) and the two hyperbolic planes M^+, M^- is a fundamental polyhedron for G. The orbifold uniformized by G is shown in figure 12, where one or the two vertices may be deleted. The only cases for which the above orbifolds have finite volume are given when $n \in \{3, 4\}$ (these two are the only cases when such a polyhedron does not meet the boundary with non-empty interior).

9.2. Second construction.

Now we consider the case (n, k, l) such that

$$\frac{1}{l} + \frac{2}{k} < 1\,.$$

Let us consider the restrictions

i) If $n = 3$, then $l \geq 4$.

ii) If $n = 4$, then $l \geq 3$.

iii) If $n \geq 5$, then $l \geq 2$.

Draw in the complex plane a ray L_1, through zero, making an angle π/n with the positive real axis. Draw a circle $\mathcal{C}_1$ with center 1 and radius

$$R = \frac{\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{2\,l}\right)}\,.$$

This implies that the angle between L_1 and $\mathcal{C}_1$ is exactly $\pi/(2\,l)$. Choose $\lambda \in (\rho, (1+R)/(1-R))$, where $\rho > 1$ is a solution to the quadratic equation

$$\rho^2(1-R^2) - 2\,\rho\left(1 + R^2\cos\left(\frac{\pi}{l}\right)\right) + 1 - R^2 = 0\,.$$

Set $\mathcal{C}_\lambda$ the circle with center λ and radius λR. The angle between the positive real axis and $\mathcal{C}_\lambda$ is also $\pi/(2\,l)$ (this is consequence of the fact that $\mathcal{C}_\lambda = A(\mathcal{C}_1)$, where $A(z) = \lambda z$).

The angle between the circles $\mathcal{C}_1$ and $\mathcal{C}_\lambda$ varies continuously between 0 and $(l-1)\pi/l$ for λ varying continuously between in $(\rho, (1+R)/(1-R))$.

The restriction $1/l+2/k<1$ means that for any $k \geq 2$ we can find $\lambda \in (\rho, (1+R)/(1-R))$ such that the angle between the two circles is exactly $\pi/(2\,k)$. Moreover, the intersection of the two circles occurs in the interior of the sector

$$\left\{ z \in \mathbb{C} : \ \frac{-\pi}{n} < \operatorname{Arg}(z) < \frac{\pi}{n} \right\}.$$

We can find a transformation x in $\mathrm{PSL}(2,\mathbb{C})$ of order k whose fixed points are the two intersection points of both circles and mapping $\mathcal{C}_1$ onto $\mathcal{C}_\lambda$. The group G generated by x and r (as in the first construction) has presentation

$$G = \langle x, r : \ x^k = r^n = (r^{-1} x^{-1} r\, x)^l = 1 \rangle$$

Figure 13.

and is a Kleinian group. $\mathbb{H}^3/G$ is the hyperbolic orbifold with underlying topological space S^3 minus two points and branch locus as shown in figure 13 (it has infinite volume since the group G has non-empty region of discontinuity on the the Riemann sphere).

Using the fact that we can permute the roles of k and n in our construction, we have that the only exceptions to this construction is the one we have stated, that is, $1/l + 2/k < 1$.

9.3. Summary.

Both constructions give a faithful representation of the generalized triangle group

$$G = \langle x, r : \ x^k = r^n = (r^{-1}x^{-1}r\,x)^l = 1 \rangle\,,$$

as a discrete group in $\mathrm{PSL}(2,\mathbb{C})$, with the exceptions of the following triples $(n, k, l) \in \{(3,2,2),(3,2,3),(3,3,2),(4,2,2)\}$ (this is [5, theorem 1]). Three of these cases (that is, with the exception of $(3,3,2)$) can be represented inside the isometry group of $S^2 \times \mathbb{R}$ (see remark of Section 3). The case $(3,3,2)$ is in part discussed in the last part of Section 7.

Our constructions also permit us to say that we obtain finite covolume representations only in the first construction for $n \in \{3,4\}$.

The group J generated by the transformations $x_1, \ldots, x_n$, where $x_1 = x$ and $x_{i+1} = r\,x_i\,r^{-1}$, has presentation

$$G = \langle x_1, \ldots, x_n : \ x_i^k = (x_{i+1}\,x_i^{-1})^l = 1 \rangle\,,$$

that is, it is a generalized Coxeter group. Since a generalized Coxeter group

$$G = \langle x_1, x_2 : \ x_i^k = (x_{i+1}\,x_i^{-1})^l = 1 \rangle\,,$$

is in fact a triangle group, we have the following

Theorem 3. *For each triple of integers (n,k,l) with $n \geq 2$, $l \geq 2$ and $k \geq 2$ such that (n,k,l) is different from $(3,3,2)$, there is a discrete and faithful representation of the generalized Coxeter group*

$$G = \langle x_1, \ldots, x_n : \ x_i^k = (x_{i+1}\,x_i^{-1})^l = 1 \rangle\,,$$

as group of isometries in one of the geometries $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $\mathbb{H}^3$. Moreover, the geometry is $\mathbb{H}^3$ if and only if (n,k,l) is different from $(3,2,2)$, $(3,2,3)$, $(3,3,2)$ and $(4,2,2)$.

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Recibido: 5 de junio de 1.996
Revisado: 6 de enero de 1.997

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* The second author was supported by Projects FONDECYT 8970007 and UTFSM 971223

Revista Matemática Iberoamericana
Vol. 13, N.º 3, 1997

Estimates on the solution of an elliptic equation related to Brownian motion with drift (II)

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1. Introduction.

In this paper we continue the study of the Dirichlet problem for an elliptic equation on a domain in $\mathbb{R}^3$ which was begun in [5]. For $R > 0$ let Ω_R be the ball of radius R centered at the origin with boundary $\partial\Omega_R$. The Dirichlet problem we are concerned with is the following

$$(-\Delta - \mathbf{b}(x) \cdot \nabla)\, u(x) = f(x)\,, \qquad x \in \Omega_R\,, \tag{1.1}$$

with zero boundary conditions

$$u(x) = 0\,, \qquad x \in \partial\Omega_R\,. \tag{1.2}$$

Since we shall be obtaining estimates on the solution of (1.1), (1.2) in terms of R we shall think of the functions $\mathbf{b}(x)$, $f(x)$ as defined on all of $\mathbb{R}^3$. Thus we assume

$$\mathbf{b} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \qquad f : \mathbb{R}^3 \longrightarrow \mathbb{R}\,,$$

are Lebesgue measurable functions.

For $1 \le r \le q < \infty$ let M_r^q be the Morrey space on $\mathbb{R}^3$ defined as follows: a function $g : \mathbb{R}^3 \longrightarrow \mathbb{C}$ is in M_r^q if $|g|^r$ is locally integrable and there is a constant C such that

$$\int_Q |g|^r \, dx \le C^r \, |Q|^{1-r/q} \,, \tag{1.3}$$

for all cubes $Q \subset \mathbb{R}^3$. Here $|Q|$ denotes the volume of Q. The norm of g, $\|g\|_{q,r}$ is defined as

$$\|g\|_{q,r} = \inf \{C : \ (1.3) \text{ holds for } C \text{ and all cubes } Q \subset \mathbb{R}^3\}\,.$$

In our previous paper we proved that the problem (1.1), (1.2) has a unique solution if $|\mathbf{b}| \in M_p^3$, $p > 1$, and $\|\mathbf{b}\|_{3,p}$ is sufficiently small. This is a perturbative result. We also had a nonperturbative theorem. This theorem stated that if $\mathbf{b}$ is locally in M_p^3 with the local Morrey norm being small then (1.1), (1.2) has a unique solution. The proof of the nonperturbative theorem required $p > 2$. In fact the estimates diverge as p approaches 2. Our goal in this paper is to obtain nonperturbative theorems which are valid for $p > 1$.

To state our first nonperturbative theorem we need a quantity introduced by Fefferman [9]: suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. A cube Q is said to be minimal with respect to ε if

$$\int_Q |\mathbf{b}|^p \, dx \ge \varepsilon^p \, |Q|^{1-p/3} \,,$$

$$\int_{Q'} |\mathbf{b}|^p \, dx < \varepsilon^p \, |Q'|^{1-p/3} \,, \qquad Q' \subset Q \,,$$

for all proper dyadic subcubes Q' of Q. Then $N_\varepsilon(\mathbf{b})$ is the number of minimal cubes in the dyadic decomposition.

Theorem 1.1. *Suppose* $f \in M_r^q$, $1 < r \le q$, $r < p$, $p > 1$, $3/2 < q < 3$. *Then there exists* $\varepsilon > 0$ *depending only on* p, q, r *such that if* $N_\varepsilon(\mathbf{b}) < \infty$ *the boundary value problem* (1.1), (1.2) *has a unique solution* $u(x)$ *in the following sense*:

a) *u is uniformly Holder continuous on Ω_R and satisfies the boundary condition* (1.2).

b) *The distributional Laplacian Δu of u is in M_r^q and the equation* (1.1) *holds for almost every* $x \in \Omega_R$.

REMARK 1.1. The restriction $q < 3$ is required by b) while $q > 3/2$ is required by a). Thus if f is in L^q for any $q > 3/2$ the solution has property a).

Next we turn to the problem of obtaining good L^∞ estimates on the solution $u(x)$ given in Theorem 1.1. For $1 < p < 3$ and n an integer define a function $a_{n,p} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$a_{n,p}(x) = \Big(2^{n(3-p)} \int_{|x-y|<2^{-n}} |\mathbf{b}|^p(y)\, dy\Big)^{1/p} . \tag{1.4}$$

In [5] the following (Theorem 1.4) is proved:

Theorem 1.2. *Let n_0 be the integer which satisfies the inequality*

$$4R > 2^{-n_0} \geq 2R . \tag{1.5}$$

Then there exists γ, $0 < \gamma < 1$, depending only on $p > 2$ such that u satisfies the L^∞ estimate

$$\|u\|_\infty \leq C_1\, R^{2-3/q}\, \|f\|_{q,r} \sum_{m=0}^{\infty} \gamma^m \sup_{x \in \Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j,p}(x)\Big) . \tag{1.6}$$

The constant C_1 depends only on p, q, r and C_2 only on $p > 2$.

It is easy to see that the inequality (1.6) becomes stronger as p decreases. We shall show in Section 3 that Theorem 1.2 does not hold for $1 < p < 2$. We will accomplish this by constructing a counterexample to (1.6) for $f \equiv 1$ and any $p < 2$. This is somewhat surprising since (1.6) does hold for $1 < p < 2$ if the drift is spherically symmetric. In that case one can obtain an explicit formula for the solution of (1.1), (1.2). The counterexample constructed in Section 3 has a drift which is far from being spherically symmetric. In fact it is concentrated on a set with dimension 1. By the recurrence property of Brownian motion the process hits this set with high probability. Once inside the set, the drift pulls the Brownian particle towards the center of the ball Ω_R.

We wish to obtain a theorem which generalizes Theorem 1.2 to the case $1 < p < 2$. Let $s > 2$ be a parameter, and suppose we have a dyadic decomposition of $\mathbb{R}^3$ into cubes Q with $|Q| = 2^{-3m}$, m an integer. For m, n integers with $m \geq n$, and $x \in \mathbb{R}^3$ let

$$N_m(x) = \text{number of dyadic cubes } Q \text{ with } |Q| = 2^{-3m} ,$$

such that Q is contained in the ball centered at x with radius 2^{-n} and

$$\int_Q |\mathbf{b}|^p \, dx \geq \varepsilon^p \, |Q|^{1-p/3} \, ,$$

where $\varepsilon > 0$ is a given parameter. We define the function $a_{\varepsilon,n,s,p}(x)$ by

$$a_{\varepsilon,n,s,p}(x) = \Big(\frac{\sup_{m \geq n} N_m(x)}{2^{(m-n)(3-s)}} \Big)^{1/s} \, . \tag{1.7}$$

We may compare the functions $a_{n,p}$ and $a_{\varepsilon,n,s,p}$ defined by (1.4), (1.7) respectively. In fact by definition of $N_m(x)$ we have that

$$\varepsilon^p \, |Q_m|^{1-p/3} \, N_m(x) \leq \int_{|x-y|<2^{-n}} |\mathbf{b}|^p(y) \, dy = a_{n,p}(x)^p \, 2^{-n(3-p)} \, ,$$

whence

$$N_m(x) \leq \varepsilon^{-p} \, a_{n,p}(x)^p \, 2^{(m-n)(3-p)} \, ,$$

and so

$$\frac{N_m(x)}{2^{(m-n)(3-s)}} \leq \varepsilon^{-p} \, a_{n,p}(x)^p \, 2^{(m-n)(s-p)} \, .$$

We conclude that

$$a_{\varepsilon,n,p,p}(x) \leq \varepsilon^{-1} \, a_{n,p}(x) \, , \qquad x \in \mathbb{R}^3 \, . \tag{1.8}$$

Theorem 1.3. *Let n_0 be the integer satisfying* (1.5) *and suppose $2 < s \leq 3$, $1 < p \leq 3$. Then there exists ε, γ with $\varepsilon > 0$, $0 < \gamma < 1$, depending only on s, p such that the solution u of* (1.1), (1.2) *satisfies an inequality*

$$\|u\|_\infty \leq C_1 R^{2-3/q} \, \|f\|_{q,r} \sum_{m=0}^{\infty} \gamma^m \sup_{x \in \Omega_R} \exp \Big(C_2 \sum_{j=0}^{m} a_{\varepsilon,n_0+j,s,p}(x) \Big) \, .$$

The constant C_1 depends only on p, q, r, s and C_2 only on $s > 2$ and p, $1 < p \leq 3$.

It follows from (1.8) that Theorem 1.3 implies Theorem 1.2. We shall show in Section 8 that Theorem 1.3 implies that for $1 < p \leq 3$, there exists $\varepsilon > 0$ and constants C_1, C_2 depending only on p, q, r such that

$$\|u\|_\infty \leq C_1 R^{2-3/q} \|f\|_{q,r} \exp\left(C_2 N_\varepsilon(\mathbf{b})\right). \tag{1.9}$$

Theorem 1.1 will be proved in Section 2. It will be sufficient to give a proof of Lemma 4.2 of [5] which is valid for $1 < p \leq 3$. The remainder of the argument of the proof of Theorem 1.1 is then exactly as for [5, Theorem 1.3]. In the new proof of Lemma 4.2 we will introduce the notion of a weighted Morrey space. This notion will play a key role in sections 4, 5, 6, 7, 8 where we prove Theorem 1.3.

The main problem we need to solve to prove Theorem 1.3 is to estimate the exit probability from a spherical shell of Brownian motion with drift. Thus let us consider a particle started at $x \in \mathbb{R}^3$ with $|x| = R$ and let P be the probability that the particle exits the shell $\{y : R/2 < |y| < 2R\}$ through the outer sphere. For Brownian motion one can explicitly compute $P = 2/3$. For the case of Brownian motion with drift $\mathbf{b}$ we need to obtain a lower bound on P in terms of $\mathbf{b}$. In Section 4 we analyze this problem in the case when $\mathbf{b}$ is perturbative, that is when $\|\mathbf{b}\|_{3,p} \ll 1$. When $\mathbf{b}$ is not perturbative we estimate P by first defining a length scale $\lambda \ll R$ in terms of $\mathbf{b}$. Then we construct paths from x, $|x| = R$, to the outer sphere $\{|y| = 2R\}$ which are linear on scales larger than λ but diffusive on scales less than λ. Thus the paths of the drift process are confined to a cylinder of radius λ. The drift is propagated perturbatively on a length scale λ and ballistically on larger scales.

In order to propagate the drift perturbatively on the length scale λ we must limit the number of nonperturbative cubes on scales smaller than λ to have dimension less than 1. The requirement that the constant s in Theorem 1.3 exceeds 2 ensures that this holds on average. The analysis of this situation is in two parts. In sections 5, 6 we analyze the case when the number of nonperturbative cubes on a scale smaller than λ actually has dimension less than 1. Then in Section 7 we use an induction argument to show that we may relax this requirement to having dimension less than 1 on average.

Once we have a lower bound on the probability P of exiting from a spherical shell, Theorem 1.3 follows almost exactly as in the proof of [5, Theorem 1.4]. This is accomplished in Section 8.

The main task of this paper was to replace the use of the Cameron-Martin formula in [5]. The reason is that the Cameron-Martin formula involves integrals of $|\mathbf{b}|^2$ and hence cannot be used to estimate the solution of (1.1), (1.2) in terms of integrals of $|\mathbf{b}|^p$ with $p < 2$. In [5] we obtained a lower bound on the exit probability P from a spherical shell by combining the Cameron-Martin formula with [4, Theorem 1.1.a)]. In Appendix A we give a new proof of [4, Theorem 1.1.a)] which brings out the relationship between the methods employed in this paper and in [5]. We show that Theorem 1.1.a) is a consequence of the fact that Brownian motion confined to a long cylinder of radius λ behaves ballistically on length scales larger than λ. The proof of the ballistic behavior of Brownian motion depends on estimating accurately the Dirichlet Green's function for the heat equation on a disc of radius λ at large time. Estimates of this type are already known [2], [8] for operators in divergence form with L^∞ coefficients. It therefore seems reasonable that one might be able to generalize the results of Appendix A to the situation considered in [2], [8].

In the subsequent work we need to do more than simply estimate the exit probability from a spherical shell. We need to keep careful track of fluctuations of densities. The simplest example of this is as follows: Suppose we have a density ρ_1 on a sphere $|x| = R_1$ and that density is propagated by the drift process to a density ρ_2 on a sphere $|x| = R_2$, $R_1 < R_2$. In the case of Brownian motion the fluctuation of ρ_2 is smaller than ρ_1. Thus if $\|\cdot\|_q$ denotes the L^q norm, normalized so that $\|\mathbf{1}\|_q = 1$ we have that if $\|\rho_1 - \operatorname{Av}\rho_1\|_q \le \delta \operatorname{Av}\rho_1$ then $\|\rho_2 - \operatorname{Av}\rho_2\|_q \le \delta \operatorname{Av}\rho_2$, where $\operatorname{Av}\rho_1$, $\operatorname{Av}\rho_2$ denotes the average value, and δ is arbitrary. We shall show in Section 4 that for a perturbative drift this still holds provided $(R_2 - R_1) \sim R_1$. If $(R_2 - R_1) \ll R_1$ it may not hold. We investigate this question further in [3].

There is now an extensive literature on elliptic equations with nonsmooth coefficients. Within it there are roughly speaking two currents of thought. On the one hand there is the approach dominated by techniques from harmonic analysis as exemplified in [11], [12]. On the other hand there is the approach where functional integration and probability is at the fore as in [6], [7]. In the present paper the former approach dominates whereas in the previous paper [5] the latter approach was more prominent. See also [13] for results related to those of this paper.

2. Proof of Theorem 1.1.

Our goal in this section is to give a proof of [5, Lemma 4.2] which is valid for $p > 1$. Theorem 1.1 will follow from this and the proof of [5, Theorem 1.3].

First we need a generalization of [5, Theorem 1.2]. Let Ω_R be a ball in $\mathbb{R}^3$ with radius R and boundary $\partial\Omega_R$. For an arbitrary cube $Q \subset \mathbb{R}^3$ define $d(Q)$ by

$$d(Q) = \sup\{d(x, \partial\Omega_R) \,:\, x \in Q\}\,.$$

We define the Morrey space $M_r^q(\Omega_R)$ where $1 \le r \le q < \infty$ as follows: a measurable function $g : \Omega_R \to \mathbb{C}$ is in $M_r^q(\Omega_R)$ if $(R - |x|)^r \, |g(x)|^r$ is integrable on Ω_R and there is a constant $C > 0$ such that

$$R^{-r} \int_{Q\cap\Omega_R} (R - |x|)^r \, |g(x)|^r \, dx \le C^r \, |Q|^{1-r/q}\,, \tag{2.1}$$

for all cubes $Q \subset \mathbb{R}^3$. The norm of g, $\|g\|_{q,r,R}$ is defined as

$$\|g\|_{q,r,R} = \inf\{C : \ (2.1) \text{ holds for all cubes } Q\}\,.$$

Let χ_R be the characteristic function of the set Ω_R. Evidently g is in $M_r^q(\Omega_R)$ if and only if the function $(1 - |x|/R)\, \chi_R(x)\, g(x)$ is in the Morrey space M_r^q of [5].

Let T be an integral operator on functions with domain Ω_R which has kernel $k_T : \Omega_R \times \Omega_R \longrightarrow \mathbb{C}$. Thus for measurable $g : \Omega_R \longrightarrow \mathbb{C}$ one defines Tg by

$$Tg(x) = \int_{\Omega_R} k_T(x,y)\, g(y)\, dy\,, \qquad x \in \Omega_R\,.$$

Proposition 2.1. *Suppose the kernel k_T of the integral operator T satisfies the inequality*

$$|k_T(x,y)| \le \frac{|\mathbf{b}(x)|}{|x-y|^2} \min\Big\{1, \frac{R-|y|}{|x-y|}\Big\}\,, \qquad x, y \in \Omega_R\,,$$

where $|\mathbf{b}| \in M_p^3$, $1 < p \le 3$. Then for any r, q which satisfy the inequalities

$$1 < r < p\,, \ \ r \le q < 3\,,$$

the operator T is a bounded operator on the space $M_r^q(\Omega_R)$. The norm of T satisfies the inequality

$$\|T\| \le C\, \|\mathbf{b}\|_{3,p}\,,$$

where the constant C depends only on r, p, q.

REMARK. Observe that [5, Theorem 1.2] follows from Proposition 2.1 by letting $R \longrightarrow \infty$.

The proof of Proposition 2.1 follows the same lines as the proof of [5, Theorem 1.2]. Define an integer n_0 by

$$2^{-n_0-1} < 8\,R \leq 2^{-n_0}\,.$$

Let $Q_0(\xi)$ be a cube with side of length 2^{-n_0} and centered at ξ. It is clear that if $\xi \in \Omega_R$ then $\Omega_R \subset Q_0(\xi)$. Let K be one of the cubes $Q_0(\xi)$ with $\xi \in \Omega_R$. We define an operator T_K on functions $u : \Omega_R \longrightarrow \mathbb{C}$ which have the property that $(R - |x|)\, u(x)$ is integrable. To do this we decompose K into a dyadic decomposition of cubes Q_n with sides of length 2^{-n}, $n \geq n_0$. For any dyadic cube $Q \subset K$ with volume $|Q|$ let u_Q be defined by

$$u_Q = R^{-1}\, |Q|^{-1} \int_{\Omega_R \cap Q} (R - |x|)\, |u(x)|\, dx\,.$$

If Q is a distance of order R from $\partial\Omega_R$ then u_Q is comparable to the average of $|u|$ on Q. Otherwise u_Q can be much smaller than the average. For $n \geq n_0$ define the operator S_n by

$$S_n u(x) = 2^{-n} \Big(\frac{R}{d(Q_n)}\Big) u_{Q_n}\,, \qquad x \in Q_n\,.$$

The operator T_K is then given by

$$T_K u(x) = \sum_{n=n_0}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,, \qquad x \in \Omega_R\,.$$

It follows now by Jensen's inequality that there is a universal constant C such that for any $r \geq 1$ and cube Q there is the inequality

$$\begin{aligned} &\int_{Q \cap \Omega_R} (R - |x|)^r\, |T u(x)|^r\, dx \\ &\qquad \leq \frac{C^r}{|\Omega_R|} \int_{\Omega_R} d\xi \int_{Q \cap \Omega_R} (R - |x|)^r\, |T_{Q_0(\xi)} u(x)|^r\, dx\,. \end{aligned} \tag{2.2}$$

Hence it is sufficient to prove Proposition 2.1 with the operator T replaced by T_K where $K = Q_0(\xi)$ and $\xi \in \Omega_R$ is arbitrary.

The following lemma generalizes [5, Lemma 2.4]. It is proved in an exactly similar fashion.

Lemma 2.1. *Let $Q' \subset K$ be an arbitrary dyadic subcube of K with side of length $2^{-n_{Q'}}$. Suppose r, p satisfy the inequality $1 \le r < p$. Then there are constants ε, $C > 0$ depending only on r and p such that*

$$|Q|^{1/3+\varepsilon}\, u_Q \le |Q'|^{1/3+\varepsilon}\, u_{Q'}\ ,$$

for all dyadic subcubes Q of Q' implies the inequality

$$\int_{Q'} (R-|x|)^r \Big(\sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|\, S_n u(x) \Big)^r \, dx \le C^r \, \|\mathbf{b}\|_{3,p}^r \, |Q'| R^r \, u_{Q'}^r \ .$$

If we replace the function $u(x)$ by the function $(R-|x|)\, u(x)$ in the argument of [5] and use the previous lemma we conclude:

Corollary 2.1. *For any dyadic subcube $Q' \subset K$ one has*

$$\int_{Q'} (R-|x|)^r \Big(\sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|\, S_n u(x) \Big)^r \, dx$$
$$\le C^r \, \|\mathbf{b}\|_{3,p}^r \int_{Q'} (R-|x|)^r \, |u(x)|^r \, dx \, ,$$

for some constant C depending only on r and p.

Proposition 2.1 for T_K follows now from the last corollary in the same way as the corresponding theorem in [5] from [5, Lemma 2.4].

Next let $g \in L^q(\partial\Omega_R)$, $1 \le q < \infty$. We define a function $Bg(x)$ for $x \in \Omega_R$ by

$$Bg(x) = |\mathbf{b}(x)| \int_{|y|=R} \frac{|g(y)|}{|x-y|^3} \, dy \, , \qquad |x| < R \, . \tag{2.3}$$

Lemma 2.2. *Suppose $\mathbf{b} \in M_p^3$ with $1 < p < 3/2$, and r, q are numbers which satisfy the inequalities*

$$1 < r < p \, , \qquad \frac{1}{r} > \frac{1}{p} + \frac{2}{q} \, . \tag{2.4}$$

Then B is a bounded operator from $L^q(\partial\Omega_R)$ to $M_r^{q_1}(\Omega_R)$ where

$$\frac{1}{q_1} = \frac{1}{3} + \frac{2}{3\,q}\,. \tag{2.5}$$

Furthermore the norm of B satisfies an inequality

$$\|B\| \le CR^{-1}\,\|\mathbf{b}\|_{3,p} \text{ and } C \text{ is a universal constant.} \tag{2.6}$$

PROOF. From (2.1) we need to estimate the integral

$$R^{-r}\int_{Q\cap\Omega_R}(R-|x|)^r\,|Bg(x)|^r\,dx \tag{2.7}$$

on an arbitrary cube Q. From Holder's inequality this integral is bounded by

$$R^{-r}\Big(\int_{Q\cap\Omega_R}|\mathbf{b}(x)|^p\,dx\Big)^{r/p} \cdot\Big(\int_{Q\cap\Omega_R}(R-|x|)^{rp'}\Big(\int_{|y|=R}\frac{|g(y)|}{|x-y|^3}\,dy\Big)^{rp'}dx\Big)^{1/p'},$$

where $r/p+1/p'=1$.

Again from Holder we have

$$\int_{|y|=R}\frac{|g(y)|}{|x-y|^3}\,dy \le \|g\|_q\Big(\int_{|y|=R}\frac{dy}{|x-y|^{3q'}}\Big)^{1/q'},$$

where $1/q+1/q'=1$. Using the fact that

$$\int_{|y|=R}\frac{dy}{|x-y|^{3q'}} \le \frac{C^{q'}}{(R-|x|)^{3q'-2}},$$

for some universal constant C, we conclude that (2.7) is bounded by

$$R^{-r}\Big(\int_{Q\cap\Omega_R}|\mathbf{b}(x)|^p\,dx\Big)^{r/p} \cdot C^r\,\|g\|_q^r\Big(\int_{Q\cap\Omega_R}(R-|x|)^{-2rp'/q}\,dx\Big)^{1/p'}. \tag{2.8}$$

The inequality (2.4) implies that $2\,r\,p'/q < 1$.

Hence if we use the fact that $\mathbf{b} \in M^3_p$ then (2.8) implies that (2.7) is bounded by

$$C^r R^{-r}\,\|\mathbf{b}\|^r_{3,p}\,\|g\|^r_q\,|Q|^{r/p-r/3}\,d(Q)^{-2r/q}\,|Q|^{1/p'} \le C^r R^{-r}\,\|\mathbf{b}\|^r_{3,p}\,\|g\|^r_q\,|Q|^{1-r/q_1}\,,$$

on using the fact that $d(Q) \ge |Q|^{1/3}$.

Hence $Bg \in M^{q_1}_r(\Omega_R)$ and its norm satisfies the inequality (2.6).

Suppose $G_D(x,y), x, y \in \Omega_R$ is the Dirichlet kernel, whence

$$G_D(x,y) = \frac{1}{4\pi\,|x-y|} - \frac{1}{4\pi}\Big(\frac{R}{|y|}\Big)\frac{1}{|x-\overline{y}|}\,,$$

where $\bar{y}$ is the conjugate of y in the sphere $\partial\Omega_R$. Let $g \in M^q_1(\Omega_R)$, $1 \le q < \infty$ and define Hg by

$$Hg(x) = \int_{\Omega_R} G_D(x,y)\,g(y)\,dy\,, \qquad x \in \Omega_{R/2}\,.$$

Lemma 2.3. *Suppose $m > 1$ satisfies the inequality*

$$\frac{2}{3} + \frac{1}{m\,q} > \frac{1}{q}\,.$$

Then H is a bounded operator from $M^q_1(\Omega_R)$ to $L^m(\Omega_{R/2})$ and the norm of H, $\|H\|$, satisfies an inequality

$$\|H\| \le C_{q,m}\,R^{2+3/m-3/q}\,, \tag{2.9}$$

where $C_{q,m}$ is a constant depending only on q and m.

PROOF. We write $Hg = H_1 g + H_2 g$, where

$$H_1 g(x) = \int_{\Omega_{3R/4}} G_D(x,y)\,g(y)\,dy\,.$$

Since we are restricting x to the region $|x| < R/2$, there is a universal constant C such that

$$\|H_2 g\|_\infty \le \frac{C}{R^2}\int_{\Omega_R}(R-|y|)\,|g(y)|\,dy \le \frac{C}{R}\,|\Omega_R|^{1-1/q}\,\|g\|_{q,1,R}\,.$$

It follows that $H_2 g$ is in $L^m(\Omega_{R/2})$ for any $m \geq 1$ and

$$\|H_2\| \leq C\, R^{2+3/m-3/q}\,, \qquad \text{for some universal constant } C\,. \tag{2.10}$$

Next we bound $H_1 g$ by using the method of proof for the John-Nirenberg inequality [10]. For any α, $0 < \alpha < 1$, we have the inequality

$$\begin{aligned}|H_1 g(x)| &\leq \frac{1}{4\pi}\int_{\Omega_{3R/4}} \frac{|g(y)|}{|x-y|}\, dy \\ &\leq \frac{1}{4\pi}\Big(\int \frac{|g(y)|}{|x-y|^{\alpha m'}}\, dy\Big)^{1/m'} \Big(\int \frac{|g(y)|}{|x-y|^{(1-\alpha)m}}\, dy\Big)^{1/m}\,,\end{aligned} \tag{2.11}$$

where $1/m + 1/m' = 1$. Now

$$\begin{aligned}\int_{\Omega_{3R/4}} \frac{|g(y)|}{|x-y|^{\alpha m'}}\, dy &= \frac{1}{\alpha m'}\int_0^\infty \frac{d\rho}{\rho^{\alpha m'+1}} \int_{\Omega_{3R/4}\cap\{y:|x-y|<\rho\}} |g(y)|\, dy \\ &\leq \frac{1}{\alpha\, m'}\int_0^{2R} \frac{d\rho}{\rho^{\alpha m'+1}}\, \|g\|_{q,1,R}\, \rho^{3-3/q} \\ &\quad + \frac{1}{\alpha\, m'}\int_{2R}^\infty \frac{d\rho}{\rho^{\alpha m'+1}}\, \|g\|_{q,1,R}\, (2R)^{3-3/q} \\ &\leq C\, \|g\|_{q,1,R}\, R^{3-3/q-\alpha m'}\,,\end{aligned} \tag{2.12}$$

for some constant C provided

$$3 - \frac{3}{q} - \alpha\, m' > 0\,. \tag{2.13}$$

On the other hand

$$\begin{aligned}\int_{\Omega_{R/2}} dx \int_{\Omega_{3R/2}} \frac{|g(y)|\, dy}{|x-y|^{(1-\alpha)m}} &\leq C\, R^{3-(1-\alpha)m} \int_{\Omega_{3R/2}} |g(y)|\, dy \\ &\leq C\, R^{3-(1-\alpha)m}\, \|g\|_{q,1,R}\, R^{3-3/q}\,,\end{aligned} \tag{2.14}$$

for some constant C depending on $\alpha\, m$, provided

$$(1-\alpha)\, m < 3. \tag{2.15}$$

It is possible to choose an α, $0 < \alpha < 1$, satisfying both (2.13) and (2.15) provided m and q satisfy the inequality (2.9). Choosing such an α yields the inequality

$$\int_{\Omega_{R/2}} |H_1 g(x)|^m \, dx \le C_{q,m}^m \, R^{2m+3-3m/q} \, \|g\|_{q,1,R}^m \,, \tag{2.16}$$

upon using (2.11)-(2.14). Here the constant $C_{q,m}$ depends on q and m. Taking the m-th root of (2.16) and combining with (2.10) yields the result.

PROOF OF [5, LEMMA 4.2] FOR $p > 1$. We shall freely use the notation of [5]. Let us suppose that p and q satisfy the inequalities

$$\frac{1}{p} + \frac{2}{q} < 1 \,, \qquad 1 < p < \frac{3}{2} \,. \tag{2.17}$$

It will be sufficient for us to show that for any $\delta > 0$ there exists $\varepsilon > 0$ depending only on p, q such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies that the operator Q_n^* is a bounded operator from $L_\mu^q(A_{n-1})$ to $L_\mu^q(A_n)$ and satisfies the inequality

$$\|Q_n^* f\|_{q,\mu} \le \delta \, \|f\|_{q,\mu} \,, \tag{2.18}$$

where $\|\cdot\|_{q,\mu}$ is the norm in the space L_μ^q. To do this observe that $Q_n^* f(x)$ is given by the formula

$$Q_n^* f(x) = \frac{2^{n-1}}{(\sqrt{2}-1)} \int_{2^{-n+1}}^{2^{-n+3/2}} (-\Delta_{D,\lambda})^{-1} (I - T_\lambda)^{-1} \, \mathbf{b} \cdot \nabla P_\lambda f(x) \, d\lambda \,,$$

where $2^{-n} < |x| < 2^{-n+1/2}$. This follows from (3.38) of [5].

Now let us assume for the moment that λ is fixed and f is in $L^q(\partial\Omega_\lambda)$ with norm $\|f\|_{q,\partial\Omega_\lambda}$. It is easy to see from the explicit formula for the Poisson kernel that

$$|\mathbf{b}(x) \cdot \nabla P_\lambda f(x)| \le C B f(x) \,, \qquad x \in \Omega_\lambda \,,$$

where C is a universal constant and B is the operator defined by (2.3). In view of (2.17) we can choose $r > 1$ such that (2.4) holds. Hence by Lemma 2.2, $\mathbf{b} \cdot \nabla P_\lambda f$ is in the space $M_r^{q_1}(\Omega_R)$ where q_1 is determined from (2.5). It is easy to verify that the operator T_λ has kernel which satisfies the conditions for Proposition 2.1. Hence the function $(I - T_\lambda)^{-1}\,\mathbf{b}\cdot\nabla P_\lambda f$ is also in the space $M_r^{q_1}(\Omega_R)$ provided $\|\mathbf{b}\|_{3,p} < \varepsilon$ and ε is sufficiently small. Now Lemma 2.3 tells us that the function

$$g_\lambda(x) = (-\Delta_{D,\lambda})^{-1}(I - T_\lambda)^{-1}\,\mathbf{b}\cdot\nabla P_\lambda f(x)$$

is in the space $L^m(\Omega_{2^{-n+1/2}})$ provided m satisfies the inequality

$$\frac{2}{3} + \frac{1}{m\,q_1} > \frac{1}{q_1}\,, \tag{2.19}$$

with q_1 given by (2.5). Furthermore, the norm of g_λ satisfies an inequality

$$\|g_\lambda\|_m \le C_{p,q,m}\,\varepsilon\,2^{-n(1+3/m-3/q_1)}\,\|f\|_{q,\partial\Omega_\lambda}\,, \tag{2.20}$$

where the constant $C_{p,q,m}$ depends only on p, q, m. It is clear that the inequality (2.19) implies that $1/m > (2-q)/(2+q)$. Taking $m = q$, we have from (2.20) the inequality

$$\|g_\lambda\|_q \le C_{p,q}\,\varepsilon\,2^{-n/q}\,\|f\|_{q,\partial\Omega_\lambda}\,.$$

The triangle inequality now yields

$$\begin{aligned} &\left\| \frac{2^{n-1}}{(\sqrt{2}-1)} \int_{2^{-n+1}}^{2^{-n+3/2}} g_\lambda\,d\lambda \right\|_q \\ &\qquad \le \frac{2^{n-1}}{(\sqrt{2}-1)} \int_{2^{-n+1}}^{2^{-n+3/2}} \|g_\lambda\|_q\,d\lambda \\ &\qquad \le \frac{2^{n-1}}{(\sqrt{2}-1)} \int_{2^{-n+1}}^{2^{-n+3/2}} C_{p,q}\,\varepsilon\,2^{-n/q}\,\|f\|_{q,\partial\Omega_\lambda}\,d\lambda\,. \end{aligned} \tag{2.21}$$

From Jensen's inequality we see that

$$\frac{2^{n-1}}{(\sqrt{2}-1)} \int_{2^{-n+1}}^{2^{-n+3/2}} \|f\|_{q,\partial\Omega_\lambda}\,d\lambda \le C^{1/q}\,2^{-2n/q}\,\|f\|_{q,\mu}\,, \tag{2.22}$$

where C is a universal constant. Putting (2.21), (2.22) together gives us the inequality (2.18) with δ proportional to ε.

3. A counterexample.

Let r_0, R_0 be two numbers which satisfy $0 < r_0 < R_0 < \infty$, and let v be the solution of the two dimensional boundary value problem,

$$\tag{3.1} \begin{cases} \Delta v(x) = 0\,, & r_0 < |x| < R_0\,, \\ v(x) = 1\,, & |x| = r_0\,, \\ v(x) = 0\,, & |x| = R_0\,. \end{cases}$$

The function v is explicitly given by the formula,

$$\tag{3.2} v(x) = \frac{\log\Big(\dfrac{R_0}{|x|}\Big)}{\log\Big(\dfrac{R_0}{r_0}\Big)}\,.$$

For $a \in \mathbb{R}^2$ and $r > 0$, let $D(a,r)$ be the disc centered at a with radius r and let $\overline{D}(a,r)$ denote the closure of $D(a,r)$. We can extend v to $\mathbb{R}^2\backslash\overline{D}(0,r_0)$ by setting v to be zero for $|x| \geq R_0$. In that case v is a subharmonic function, and so in the distributional sense one has

$$\tag{3.3} \Delta v(x) \geq 0\,, \qquad |x| > r_0\,.$$

Lemma 3.1. *Let $U \subset \mathbb{R}^2$ be a domain and suppose $a_j \in U$, $j = 1,\dots,k$. Let $r_0 > 0$ be such that all the sets $\overline{D}(a_j,r_0)$ are disjoint and contained in U, $j = 1,\dots,k$. Let W be the domain*

$$W = U\backslash \bigcup_{j=1}^{k} \overline{D}(a_j,r_0)\,.$$

Let $u(x)$ be the solution of the equation

$$\begin{cases} \Delta u(x) = 0\,, & x \in W\,, \\ u(x) = 0\,, & x \in \partial U\,, \\ u(x) = 1\,, & x \in \partial D(a_j,r_0)\,,\ j = 1,\dots,k\,. \end{cases}$$

For $R_0 > r_0$, suppose S_0 is a subset of $\{1,\dots,k\}$ with the property that

$$\overline{D}(a_j,R_0) \subset U\,, \qquad j \in S_0\,.$$

For $x \in W$ define a function $\overline{u}(x)$ by

$$\overline{u}(x) = \frac{\sum_{j \in S_0} v(x - a_j)}{\sup\Big\{ \sum_{j \in S_0} v(a_i - a_j + \delta) : |\delta| = r_0 \ , \ 1 \le i \le k \Big\}} \ ,$$

where v is given by (3.2). *Then there is the inequality*

$$u(x) \ge \overline{u}(x) \, , \qquad x \in W \, . \tag{3.4}$$

PROOF. From (3.3) it follows that $\overline{u}$ is subharmonic on W. By definition of S_0 one has $\overline{u}(x) = 0$, $x \in \partial U$. Furthermore one has $\overline{u}(x) \le 1$ if $x \in \partial D(a_j, r_0)$, $j = 1, \ldots, k$. The maximum principle now yields the inequality (3.4).

Next let $\mathbb{Z}^2_\lambda$ be the lattice

$$\mathbb{Z}^2_\lambda = \{\lambda(n, m) : \ n, m \text{ integers}\} \, .$$

For $2\, r_0 < \lambda < R$ let $W_{\lambda,R}$ be the set

$$W_{\lambda,R} = D(0,R) \backslash \cup \{\overline{D}(a, r_0) : \ a \in \mathbb{Z}^2_\lambda, \ \overline{D}(a, r_0) \subset D(0,R)\} \, . \tag{3.5}$$

Consider the function $u(x)$ which is the solution of the boundary value problem

$$\begin{cases} \Delta u(x) = 0 \, , & x \in W_{\lambda,R} \, , \\ u(x) = 0 \, , & |x| = R \, , \\ u(x) = 1 \, , & x \in \partial D(a, r_0) \, , \ a \in \mathbb{Z}^2_\lambda \, , \ \overline{D}(a, r_0) \subset D(a, R) \, . \end{cases} \tag{3.6}$$

Evidently $u(x)$ is the probability that Brownian motion started at $x \in D(0, R)$ hits one of the discs radius r_0, centered at $a \in \mathbb{Z}^2_\lambda$, before exiting the region $D(0, R)$. Let us consider the quantity $\inf \{u(x) : |x| \le R/2\}$. If λ, r_0 are fixed and R becomes large we should expect this quantity to converge to 1 since a Brownian path is unlikely to avoid all the discs centered at points in $\mathbb{Z}^2_\lambda$ over large distances. The following lemma gives an estimate which verifies this intuition:

Lemma 3.2. *Suppose* $8\,r_0 < R$ *and* $u(x)$ *is the solution of* (3.6). *Then there is a universal constant* $c > 0$ *such that*

$$\inf_{|x|\le R/2} u(x) > 1 - \frac{c\,\lambda}{R}\,, \tag{3.7}$$

provided λ *lies in the region,*

$$2\,r_0 < \lambda < \frac{R}{\log\Big(\dfrac{R}{r_0}\Big)}\,. \tag{3.8}$$

PROOF. Let us take $R_0 = R/4$ in (3.2). Then by Lemma 3.1 we have that

$$u(x) \ge \sum_{a\in\mathbb{Z}^2_\lambda} \frac{v(x-a)}{\sup\Big\{\displaystyle\sum_{a\in\mathbb{Z}^2_\lambda} v(\delta-a) : |\delta| = r_0\Big\}}\,, \tag{3.9}$$

provided $|x| \le R/2$.

We have now that

$$\begin{aligned}
\sum_{a\in\mathbb{Z}^2_\lambda} v(\delta-a) &\sim \sum_{\substack{n\in\mathbb{Z}^2\\ 0<|n|<R/(4\lambda)}} \frac{\log\Big(\dfrac{R}{4\,\lambda\,|n|}\Big)}{\log\Big(\dfrac{R}{4\,r_0}\Big)} \\
&\sim \frac{1}{\log\Big(\dfrac{R}{4\,r_0}\Big)} \int_{|x|<R/(4\lambda)} \log\Big(\frac{R}{4\,\lambda\,|x|}\Big)\,dx \\
&= \frac{\dfrac{\pi}{2}\Big(\dfrac{R}{4\,\lambda}\Big)^2}{\log\Big(\dfrac{R}{4\,r_0}\Big)}\,.
\end{aligned}$$

By virtue of (3.8) we can conclude then that

$$\sum_{a\in\mathbb{Z}^2_\lambda} v(\delta-a) \ge c\,\frac{\pi}{2}\,\frac{\Big(\dfrac{R}{4\,\lambda}\Big)^2}{\log\Big(\dfrac{R}{4\,r_0}\Big)}\,, \tag{3.10}$$

for some universal constant c.

We estimate the numerator of (3.9) by Taylor expansion. Let $b \in \mathbb{Z}_\lambda^2$ be the nearest lattice point to x and $y = x - b$. Thus $|y| < \lambda/\sqrt{2}$. Hence we have

$$\begin{aligned} \sum_{a\in\mathbb{Z}_\lambda^2} v(x-a) &= \sum_{a\in\mathbb{Z}_\lambda^2} v(y-a) \\ &= \sum_{a\in\mathbb{Z}_\lambda^2} v(\delta-a) + \sum_{a\in\mathbb{Z}_\lambda^2} \left(v(y-a) - v(\delta-a)\right), \end{aligned} \tag{3.11}$$

where $|\delta| = r_0$. By Taylor's theorem we have

$$\sum_{a\in\mathbb{Z}_\lambda^2} \left(v(y-a) - v(\delta-a)\right) = \sum_{a\in\mathbb{Z}_\lambda^2} \int_0^1 (y-\delta)\cdot\nabla v\left(\delta - a + t\,(y-\delta)\right) dt\,.$$

Now if we use the inequality

$$|\nabla v(x)| \le \frac{\log\left(\dfrac{R}{4\,r_0}\right)}{|x|}\,,$$

we conclude from (3.10), (3.11) that

$$1 - u(x) \le \frac{C}{\left(\dfrac{R}{4\,\lambda}\right)^2} \sum_{\substack{n\in\mathbb{Z}^2\\ 0<|n|<R/(4\lambda)}} \frac{1}{|n|} \tag{3.12}$$

where C is a universal constant. The inequality (3.7) follows now by observing that the sum in (3.12) is of order λ/R.

Next we wish to obtain a three dimensional generalization of Lemma 3.2. First we consider a generalization of the boundary value problem (3.1).

Let $v(x)$ be the solution of the problem

$$\begin{cases} \Delta v(x) = \eta\, v(x)\,, & r_0 < |x| < R_0\,, \\ v(x) = 1\,, & |x| = r_0\,, \\ v(x) = 0\,, & |x| = R_0\,. \end{cases} \tag{3.13}$$

The function $v(x)$ is a Brownian motion expectation value. In fact let $X(t)$ be Brownian motion started at a point x and τ be the exit time from the region $\{y : r_0 < |y| < R_0\}$. Let χ be the characteristic function,

$$\chi(z) = \begin{cases} 1\,, & \text{if } |z| = r_0\,, \\ 0\,, & \text{if } |z| = R_0\,. \end{cases}$$

Then we have

$$v(x) = E_x[e^{-\eta\tau}\chi(X(\tau))]\,. \tag{3.14}$$

It is well known that the solution of (3.13) exists provided the parameter η is larger than the largest eigenvalue of the Dirichlet Laplacian. For $\eta = 0$ the solution of (3.13) is given by (3.2). For $\eta \neq 0$ we have the following:

Lemma 3.3. *Let I_0 be the modified Bessel function of the first kind defined by the infinite series,*

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}.$$

Suppose η satisfies the condition

$$I_0(\sqrt{\eta}\,t) \neq 0\,, \qquad r_0 \leq t \leq R_0\,. \tag{3.15}$$

Then the solution v of (3.13) *is given by*

$$v(x) = \frac{I_0(\sqrt{\eta}\,r)\displaystyle\int_r^{R_0} \frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)}}{I_0(\sqrt{\eta}\,r_0)\displaystyle\int_{r_0}^{R_0} \frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)}}\,, \tag{3.16}$$

where $r = |x|$.

PROOF. The problem (3.13) is rotation invariant. Hence $v(x)$ is just a function of $r = |x|$, $v(x) = v(r)$, and satisfies the equation

$$\begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\left(r\,\dfrac{dv}{dr}\right) = \eta\,v\,, \\ v(r_0) = 1\,, \\ v(R_0) = 0\,. \end{cases} \tag{3.17}$$

This is a Bessel equation of order zero. It is easy to see that $v(r) = I_0(\sqrt{\eta}\ r)$ is a solution of the equation (3.17), but not the boundary condition. A second solution can be found by the method of reduction of order provided (3.15) holds. It is given by

$$v(r) = I_0(\sqrt{\eta}\ r) \int_r^{R_0} \frac{dt}{t\, I_0^2(\sqrt{\eta}\, t)}\ . \tag{3.18}$$

It follows from (3.18) that the function (3.16) satisfies (3.17).

We consider a region in $\mathbb{R}^3$ which has a two dimensional structure. For $0 < 4\,r_0 < R_0$ consider the two cylinders

$$\begin{aligned} S_1 &= \{x = (x_1, x_2, x_3) :\ -R_0 < x_3 < R_0,\ x_1^2 + x_2^2 < R_0^2\}\,, \\ S_2 &= \left\{x = (x_1, x_2, x_3) :\ \frac{-R_0}{2} < x_3 < \frac{R_0}{2},\ x_1^2 + x_2^2 < r_0^2\right\}. \end{aligned} \tag{3.19}$$

The region $\mathcal{D}$ we wish to consider is given by $\mathcal{D} = S_1 \backslash \overline{S}_2$. The boundary $\partial\mathcal{D}$ of $\mathcal{D}$ is evidently the union of ∂S_1 and ∂S_2. We consider the problem

$$\begin{cases} \Delta v(x) = 0\,, & x \in \mathcal{D}\,, \\ v(x) = 1\,, & x \in \partial S_2\,, \\ v(x) = 0\,, & x \in \partial S_1\ . \end{cases} \tag{3.20}$$

Lemma 3.4. *Suppose $x = (x_1, x_2, x_3) \in \mathcal{D}$, and $r^2 = x_1^2 + x_2^2$. Then there is a universal constant $c > 0$ such that*

$$v(x) \ge c\, \frac{\log\left(\dfrac{R_0}{r}\right)}{\log\left(\dfrac{R_0}{r_0}\right)}\ , \tag{3.21}$$

provided $|x_3| < R_0/4$.

PROOF. Consider the two dimensional Brownian motion started at (x_1, x_2) and consider all paths which hit the circle $r = r_0$ before hitting the circle $r = R_0$. Let τ be the hitting time for such paths and suppose $\rho(r, t)$ is the density for τ. Then the function

$$\int_0^\infty e^{-\eta t}\, \rho(r, t)\, dt \tag{3.22}$$

is the solution to the problem (3.13). This follows from the representation (3.14). Next let $X_3(t)$ be Brownian motion started at $x_3 \in \mathbb{R}$ and let τ_3 be the first exit time from the interval $[-R_0, R_0]$. We define $w(x_3, t)$ by

$$w(x_3,t) = P_{x_3}\Big[-\frac{R_0}{2} < X_3(t \wedge \tau_3) < \frac{R_0}{2}\Big]. \tag{3.23}$$

Evidently $w(x_3, t)$ satisfies the heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x_3^2}, \qquad -R_0 < x_3 < R_0\,,\ t > 0\,, \tag{3.24}$$

with the boundary conditions

$$w(R_0, t) = w(-R_0, t) = 0\,, \qquad t > 0\,, \tag{3.25}$$

and the initial conditions

$$w(x_3, 0) = \begin{cases} 1\,, & -\dfrac{R_0}{2} < x_3 < \dfrac{R_0}{2}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.26}$$

It is clear from the definition (3.23) that the solution v of (3.20) has the representation

$$v(x_1, x_2, x_3) = v(r, x_3) = \int_0^\infty \rho(r,t)\, w(x_3,t)\, dt\,. \tag{3.27}$$

Observe that for any $\alpha > 0$ there is a constant $\gamma_\alpha > 0$ depending only on α, such that

$$w(x_3, t) \ge \gamma_\alpha > 0\,, \qquad |x_3| < \frac{R_0}{4}\,,\ 0 < t < \alpha R_0^2\,.$$

Hence, provided $|x_3| < R_0/4$ there is the inequality

$$v(r, x_3) \ge \gamma_\alpha \int_0^{\alpha R_0^2} \rho(r,t)\, dt\,. \tag{3.28}$$

Now from Lemma 3.3 we see there exists an $\varepsilon > 0$ independent of R_0 such that

$$\int_0^\infty \exp\Big(\frac{\varepsilon\, t}{R_0^2}\Big)\rho(r,t)\, dt \le C_\varepsilon \frac{\log\Big(\dfrac{R_0}{r}\Big)}{\log\Big(\dfrac{R_0}{r_0}\Big)}\,,$$

for some constant $C_\varepsilon > 0$ depending only on ε. Thus for any $\alpha > 0$ one has the inequality

$$\int_{\alpha R_0^2}^{\infty} \rho(r,t)\, dt \le \exp\left(-\varepsilon\alpha\right) C_\varepsilon \, \frac{\log\left(\dfrac{R_0}{r}\right)}{\log\left(\dfrac{R_0}{r_0}\right)} \, .$$

Choosing α such that $\exp\left(-\varepsilon\, \alpha\right) C_\varepsilon < 1/2$, we conclude that

$$\int_{0}^{\alpha R_0^2} \rho(r,t)\, dt \ge \frac{1}{2}\, \frac{\log\left(\dfrac{R_0}{r}\right)}{\log\left(\dfrac{R_0}{r_0}\right)} \, . \tag{3.29}$$

The inequality (3.21) follows now from (3.28), (3.29).

Lemma 3.5. *Let $v(x) = v(x_1, x_2, x_3) = v(r, x_3)$ be the solution of (3.20). Then there is a universal constant $C > 0$ such that*

$$\Big|\frac{\partial v(r,0)}{\partial r}\Big| \le \frac{C}{r} \log\left(\frac{R_0}{r_0}\right) . \tag{3.30}$$

PROOF. The eigenfunction expansion for the solution to the problem (3.24), (3.25), (3.26) is given by

$$w(x_3,t) = \frac{1}{R_0} \sum_{m=1}^{\infty} \exp\Big(-\frac{\pi^2 m^2}{4R_0^2}\, t\Big) \sin\Big(\frac{\pi\, m}{2R_0}\,(x_3 + R_0)\Big)$$
$$\cdot \int_{-R_0/2}^{R_0/2} \sin\Big(\frac{\pi\, m}{2R_0}\,(\zeta + R_0)\Big)\, d\zeta \, .$$

Hence from (3.27) we have

$$v(r,0) = \sum_{m=1}^{\infty} \frac{2}{\pi\, m}\,(1 - (-1)^m)\, u\Big(r, \frac{\pi^2 m^2}{4R_0^2}\Big) \sin\Big(\frac{\pi\, m}{4}\Big) ,$$

where $u(r,\eta)$ is the function given by (3.22). Consequently

$$\frac{\partial v(r,0)}{\partial r} = -\sum_{m=1}^{\infty} a_m(r) \sin\Big(\frac{\pi\, m}{4}\Big) , \tag{3.31}$$

where

$$a_m(r) = \frac{-2}{\pi\, m}\,(1-(-1)^m)\,\frac{\partial u}{\partial r}\Big(r,\,\frac{\pi^2 m^2}{4R_0^2}\Big) \geq 0\,. \tag{3.32}$$

The inequality (3.32) follows from the maximum principle applied to the equation (3.13) which $u(r,\eta)$ satisfies. We shall prove in the appendix that

$$\frac{\partial}{\partial r}\,\frac{\partial}{\partial \eta}\Big(\frac{u(r,\eta)}{\sqrt{\eta}}\Big) > 0\,, \qquad r_0 < r < R_0\ ,\ \eta > 0\,. \tag{3.33}$$

It follows then from (3.33) that $a_m(r)$ is a decreasing function of odd integers m. Hence by the alternating series theorem applied to (3.31) we conclude that

$$\Big|\frac{\partial v(r,0)}{\partial r}\Big| \leq a_1(r) + a_3(r)\,.$$

Next we use Lemma 3.3 to estimate $a_1(r)$, $a_3(r)$. From (3.16) we see that

$$\frac{\partial u}{\partial r}\,(r,\eta) = -\frac{\Big(\dfrac{I_0(\sqrt{\eta}\,r)}{r} - \sqrt{\eta}\,I_0'(\sqrt{\eta}\,r)\displaystyle\int_r^{R_0}\frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)}\Big)}{I_0(\sqrt{\eta}\,r_0)\displaystyle\int_{r_0}^{R_0}\frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)}}\,.$$

Thus we have

$$\Big|\frac{\partial u}{\partial r}\Big(r,\,\frac{\pi^2}{4R_0^2}\Big)\Big| \leq \frac{C}{r}\,\frac{1+\Big(\dfrac{r}{R_0}\Big)^2\log\Big(\dfrac{R_0}{r}\Big)}{\log\Big(\dfrac{R_0}{r_0}\Big)}\,,$$

for some universal constant $C>0$. In view of the fact that $z^2\log(1/z) \leq 1/e$, for $0<z<1$, we conclude that

$$a_1(r) \leq \frac{C}{r\log\Big(\dfrac{R_0}{r_0}\Big)}\,,$$

for some universal constant C. Since a similar inequality holds for $a_3(r)$ the inequality (3.30) follows.

We wish to obtain a three dimensional analogue of Lemma 3.2. For $a = (a_1, a_2) \in \mathbb{R}^2$ let $S_2(a)$ be the cylinder S_2 of (3.19) centered at the point $(a_1, a_2, 0) \in \mathbb{R}^3$. Then for $2\,r_0 < \lambda < L$ we define the set $W_{\lambda,L}$ to be

$$W_{\lambda,L} = S_1 \backslash \cup \{S_2(a) : \ a \in \mathbb{Z}^2_\lambda, \ \overline{S_2(a)} \subset S_1\}\,, \tag{3.34}$$

where we take $R_0 = L$ in (3.19). Thus $W_{\lambda,L}$ is a three dimensional analogue of the set (3.5). Consider the Dirichlet problem corresponding to (3.6),

$$\begin{cases} \Delta u(x) = 0\,, & x \in W_{\lambda,L}\ , \\ u(x) = 0\,, & x \in \partial S_1\ , \\ u(x) = 1\,, & x \in \partial S_2(a)\,, \ \overline{S_2(a)} \subset S_1\ , \ a \in \mathbb{Z}^2_\lambda\ . \end{cases} \tag{3.35}$$

The following lemma generalizes Lemma 3.2.

Lemma 3.6. *Suppose* $8\,r_0 < L$ *and* $u(x)$ *is the solution of* (3.35). *Then there is a universal constant* $c > 0$ *such that*

$$\inf_{|x| \le L/4} u(x) > 1 - \frac{c\,\lambda}{L}\ , \tag{3.36}$$

provided λ *lies in the region*

$$2\,r_0 < \lambda < \frac{L}{\log\left(\dfrac{L}{r_0}\right)}\ .$$

PROOF. First consider $x = (x_1, x_2, 0)$. Let $v(x)$ be the solution of the problem (3.20) with $R_0 = L/4$. Then

$$u(x) \ge \frac{\displaystyle\sum_{a \in \mathbb{Z}^2_\lambda} v(x-a)}{\sup\left\{\displaystyle\sum_{a \in \mathbb{Z}^2_\lambda} v(y-a) : \ y \in \partial S_2\right\}}\ . \tag{3.37}$$

From Lemma 3.4 it follows that

$$\sup\left\{\sum_{a \in \mathbb{Z}^2_\lambda} v(y-a) : \ y \in \partial S_2\right\} \ge c\,\frac{\left(\dfrac{L}{4\,\lambda}\right)^2}{\log\left(\dfrac{L}{4\,r_0}\right)}\ ,$$

for some universal constant $c > 0$. Now we can obtain a lower bound on the numerator in (3.37) by the same argument as in Lemma 3.2, using Lemma 3.5. Hence (3.36) follows for x of the form $x = (x_1, x_2, 0)$, $|x| < L/4$. Finally it is easy to extend these considerations to the case $x_3 \neq 0$, $|x| < L/4$, by observing that $u(x)$ is bounded below by the solution for cylinders centered on the x_3 constant plane of length $L/2$. This last situation is just the $x_3 = 0$ case again.

Next let $2 < r_0 < R_0$ and $\mathcal{D} \subset \mathbb{R}^3$ be the cylinder

$$\mathcal{D} = \{x = (x_1, x_3, x_3) : \ -R_0 < x_3 < R_0,\ r^2 = x_1^2 + x_2^2 < R_0^2\}\,.$$

We define a drift $\mathbf{b} : \mathcal{D} \longrightarrow \mathbb{R}^3$ as follows

$$\begin{aligned} &\mathbf{b}(x_1, x_2, x_3) = 0\,, \qquad r_0 < r < R_0\ ,\ -R_0 < x_3 < R_0\ , \\ &\mathbf{b}(x_1, x_2, x_3) = -\Big(\frac{x_1}{r}, \frac{x_2}{r}, 1\Big)\,, \qquad r < r_0\ ,\ -R_0 < x_3 < R_0\ . \end{aligned} \tag{3.38}$$

For $x \in \mathcal{D}$ let $P_x(\mathcal{D})$ be the probability that the Brownian process with drift $\mathbf{b}$, started at x, exits $\partial\mathcal{D}$ through the bottom of the cylinder, $\partial\mathcal{D} \cap \{x : x_3 = -R_0\}$. We wish to obtain a lower bound for $P_x(\mathcal{D})$ when $r = r_0$. To obtain this we consider an auxiliary region $\mathcal{D}'$ defined by

$$\mathcal{D}' = \Big\{x :\ r < 1\,,\ -R_0 < x_3 < \frac{R_0}{2}\Big\}\,.$$

Let $Q_x(\mathcal{D}')$ be the probability of exiting the region $\mathcal{D}\backslash\mathcal{D}'$ through the bottom of the cylinder $\partial\mathcal{D} \cap \{x_3 = -R_0\}$ or through $\partial\mathcal{D}'$. Then it is clear that for $x \in \mathcal{D}\backslash\mathcal{D}'$,

$$P_x(\mathcal{D}) \geq Q_x(\mathcal{D}') \inf\{P_y(\mathcal{D}) :\ y \in \partial\mathcal{D}'\}\,. \tag{3.39}$$

We shall estimate both quantities on the right in (3.39).

Lemma 3.7. *Let $\mathbf{b}'$ be a drift on $\mathcal{D}$ which is the same as $\mathbf{b}$ except the x_3 component is always zero. Let $Q_x'(\mathcal{D}')$ be the probability corresponding to $\mathbf{b}'$. Then*

$$Q_x(\mathcal{D}') \geq Q_x'(\mathcal{D}')\,. \tag{3.40}$$

PROOF. Let $u(x) = Q_x(\mathcal{D}')$, $x \in \mathcal{D}\backslash\mathcal{D}'$. Then u is the solution of the Dirichlet problem

$$\begin{cases} -\Delta u(x) - \mathbf{b}(x)\cdot\nabla u(x) = 0\,, \qquad x \in \mathcal{D}\backslash\mathcal{D}'\,, \\ u(x) = 1\,, \qquad x \in (\partial\mathcal{D}\cap\{x_3 = -R_0\})\cup\partial\mathcal{D}'\,, \\ u(x) = 0\,, \qquad x \in \partial\mathcal{D}\cap\{x_3 > -R_0\}\,. \end{cases} \tag{3.41}$$

Similarly if $v(x) = Q'_x(\mathcal{D}')$ then v satisfies the equation

$$-\Delta v(x) - \mathbf{b}'(x)\cdot\nabla v(x) = 0\,, \qquad x \in \mathcal{D}\backslash\mathcal{D}'\,,$$

with the same boundary conditions as in (3.41). We shall show later that

$$\frac{\partial v(x)}{\partial x_3} \le 0\,, \qquad x \in \mathcal{D}\backslash\mathcal{D}'\,. \tag{3.42}$$

Thus we have

$$-\Delta(u-v) - \mathbf{b}(x)\cdot\nabla(u-v) = (\mathbf{b}(x)-\mathbf{b}'(x))\cdot\nabla v(x) \ge 0\,, \qquad x \in \mathcal{D}\backslash\mathcal{D}'\,,$$

in view of (3.42). Since $u-v$ has zero boundary conditions on $\partial\mathcal{D}\cup\partial\mathcal{D}'$ it follows by the maximum principle that

$$u(x) \ge v(x)\,, \qquad x \in \mathcal{D}\backslash\mathcal{D}'\,.$$

This is exactly the inequality (3.40).

To prove (3.42) we use a representation for the function $v(x)$ which is analogous to (3.27). Consider two dimensional Brownian motion with drift $\mathbf{b}(x_1, x_2)$ defined by

$$\mathbf{b}(x_1,x_2) = \begin{cases} 0\,, & r > r_0\,, \\ -\left(\dfrac{x_1}{r}, \dfrac{x_2}{r}\right), & r < r_0\,. \end{cases}$$

Suppose the motion starts at (x_1, x_2) and consider only paths which hit the circle $r = 1$ before the circle $r = R_0$. Let τ_1 be the hitting time for such paths and $\rho_1(r,t)$ be the density for τ_1. Similarly let τ_2 be the hitting time for paths which first hit the circle $r = R_0$ and $\rho_2(r,t)$ the density for τ_2.

Next let $X_3(t)$ be Brownian motion started at $x_3 \in \mathbb{R}$ and τ_3 be the first exit time from the interval $[-R_0, R_0]$. Let $w(x_3, t)$ be given by

$$w(x_3,t) = P_{x_3}\Big(-R_0 < X_3(t\wedge\tau_3) < \frac{R_0}{2}\Big),$$

$$h(x_3,t) = P_{x_3}(\tau_3 < t,\ X_3(\tau_3) = -R_0)\,.$$

Then we have the representation,

$$\begin{aligned} v(x_1,x_2,x_3) &= \int_0^\infty \rho_1(r,t)\, w(x_3,t)\, dt \\ &\quad + \int_0^\infty (\rho_1(r,t)+\rho_2(r,t))\, h(x_3,t)\, dt\,. \end{aligned} \tag{3.43}$$

The function $w(x_3,t)$ satisfies the heat equation (3.24) with boundary condition (3.25) and initial condition given by

$$w(x_3,0) = \begin{cases} 1\,, & -R_0 < x_3 < \dfrac{R_0}{2}\,, \\ 0\,, & \dfrac{R_0}{2} < x_3 < R_0\,, \end{cases} \tag{3.44}$$

The function $h(x_3,t)$ satisfies the heat equation (3.24) with boundary conditions

$$h(-R_0,t) = 1\,,\ h(R_0,t) = 0\,, \qquad t > 0\,. \tag{3.45}$$

and initial conditions given by

$$h(x_3,0) = 0\,, \qquad -R_0 < x_3 < R_0\,. \tag{3.46}$$

Lemma 3.8. *The function $h(x_3,t)$ is a decreasing function of x_3 in the interval $[-R_0, R_0]$.*

PROOF. By the maximum principle one has

$$0 \le h(x_3,t) \le 1\,, \qquad -R_0 < x_3 < R_0\,.$$

Hence if we put $u(x_3,t) = \partial h(x_3,t)/\partial x_3$, then $u(x_3,t)$ satisfies the heat equation with initial and boundary conditions satisfying

$$u(x_3,0) = 0\,, \qquad -R_0 < x_3 < R_0\,,$$

$$u(-R_0,t) \le 0\,,\ u(R_0,t) \le 0\,, \qquad t > 0\,.$$

Again by the maximum principle for the heat equation it follows that

$$u(x_3, t) \le 0 \,, \qquad -R_0 < x_3 < R_0 \ , \ t > 0 \,.$$

Hence $h(x_3, t)$ is a decreasing function of x_3.

Lemma 3.9. *The function $w(x_3, t) + h(x_3, t)$ is a decreasing function of x_3 in the interval $[-R_0, R_0]$.*

PROOF. Putting $u(x_3, t) = w(x_3, t) + h(x_3, t)$, it is easy to see from (3.44), (3.45), (3.46) that u satisfies the heat equation with boundary and initial conditions given by

$$u(x_3, 0) = \begin{cases} 1 \,, & -R_0 < x_3 < \dfrac{R_0}{2} \,, \\ 0 \,, & \dfrac{R_0}{2} < x_3 < R_0 \,. \end{cases}$$

$$u(-R_0, t) = 1 \,, \ u(R_0, t) = 0 \,, \qquad t > 0 \,.$$

It follows again by the maximum principle for the heat equation that

$$0 \le u(x_3, t) \le 1 \,, \qquad -R_0 < x_3 < R_0 \ , \ t > 0 \,.$$

Now we apply the same argument as in Lemma 3.8 to complete the proof.

The inequality (3.42) follows easily now from (3.43) and Lemmas 3.8, 3.9.

Next we wish to estimate $Q'_x(\mathcal{D}')$. In view of the fact that the drift $\mathbf{b}'$ does not depend on x_3 this is easier to estimate than $Q_x(\mathcal{D}')$. Let us consider the function

$$u(r, \eta) = \int_0^\infty e^{-\eta t} \rho_1(r, t) \, dt \,.$$

Then $u(r, \eta)$ satisfies the equation

$$\begin{cases} \dfrac{d^2 u}{dr^2} + \left(b'(r) + \dfrac{1}{r} \right) \dfrac{du}{dr} = \eta \, u \,, & 1 < r < R_0 \,, \\ u(1, \eta) = 1 \,, \\ u(R_0, \eta) = 0 \,. \end{cases} \tag{3.47}$$

Here $b'(r)$ is given by the magnitude of $\mathbf{b}'$,

$$b'(r) = \begin{cases} 0, & r > r_0 \,, \\ -1\,, & r < r_0 \,. \end{cases}$$

Lemma 3.10 *Suppose* $2 < r_0 < R_0$, *and* $0 < \eta R_0 \le 1$. *Then there is a universal constant* C *such that*

$$u(r_0,\eta) \ge 1 - C\,\frac{\log r_0}{\log R_0}\,. \tag{3.48}$$

PROOF. By the maximum principle the solution of (3.47) is bounded below by the solution of the zero drift problem. Thus from Lemma 3.3 we have the inequality

$$\begin{aligned} u(r_0,\eta) &\ge \frac{I_0(\sqrt{\eta}\,r_0)\displaystyle\int_{r_0}^{R_0}\frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)}}{I_0(\sqrt{\eta})\displaystyle\int_{1}^{R_0}\frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)}} \\ &= \frac{I_0(\sqrt{\eta}\,r_0)}{I_0(\sqrt{\eta})}\left(1-\frac{\displaystyle\int_{1}^{r_0}\frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)}}{\displaystyle\int_{1}^{R_0}\frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)}}\right). \end{aligned}$$

Evidently one has

$$\frac{I_0(\sqrt{\eta}\,r_0)}{I_0(\sqrt{\eta})} \ge 1\,, \tag{3.49}$$

$$\int_1^{r_0}\frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)} \le \log r_0\,. \tag{3.50}$$

If we use the fact that there is a universal constant $C > 0$ such that $I_0(\sqrt{\eta}\,t) \le C$ for $0 \le t \le R_0^{1/2}$ then it is clear that

$$\int_1^{R_0}\frac{dt}{t\,I_0^2(\sqrt{\eta}\,t)} \ge C_1 \log R_0\,, \tag{3.51}$$

for some universal constant C_1. The inequality (3.48) follows now from (3.49), (3.50), (3.51).

Lemma 3.11. *Suppose $x = (x_1, x_2, x_3) \in \mathcal{D}\backslash\mathcal{D}'$ with $x_3 \le R_0/4$ and $r_0 = (x_1^2 + x_2^2)^{1/2}$. Then if $2 < r_0 < R_0$ there is a universal constant C such that*

$$Q'_x(\mathcal{D}') \ge 1 - C\,\frac{\log r_0}{\log R_0}\,. \tag{3.52}$$

PROOF. First we show that

$$\int_0^{R_0^{3/2}} \rho_1(r_0, t)\,dt \ge 1 - C_1\,\frac{\log r_0}{\log R_0}\,, \tag{3.53}$$

for some universal constant C_1. To see this observe from Lemma 3.10 that

$$\int_0^\infty e^{-t/R_0}\rho_1(r_0, t)\,dt \ge 1 - C_2\,\frac{\log r_0}{\log R_0}\,.$$

Thus

$$\int_0^{R_0^{3/2}} \rho_1(r, t)\,dt + e^{-R_0^{1/2}} \int_{R_0^{3/2}}^\infty \rho_1(r, t)\,dt \ge 1 - C_2\,\frac{\log r_0}{\log R_0}\,. \tag{3.54}$$

Now, if we use the fact that

$$\int_0^\infty \rho_1(r, t)\,dt \le 1\,,$$

we conclude from (3.54) that

$$\begin{aligned}(1 - e^{-R_0^{1/2}}) \int_0^{R_0^{3/2}} & \rho_1(r, t)\,dt \\ & \ge 1 - e^{-R_0^{1/2}} - C_2\,\frac{\log r_0}{\log R_0}\,.\end{aligned} \tag{3.55}$$

The inequality (3.55) clearly implies (3.53).

The result (3.52) follows now from the representation (3.43) by observing from the reflection principle that

$$\begin{aligned}P_{x_3}\Big[X_3(t) < \frac{R_0}{2}\,,\ 0 < t < R_0^{3/2}\Big] \\ = 1 - 2\int_{R_0/2 - x_3}^\infty \frac{1}{\big(4\pi R_0^{3/2}\big)^{1/2}} \exp\Big(-\frac{z^2}{4R_0^{3/2}}\Big)\,dz\,.\end{aligned}$$

Next we wish to obtain a lower bound on $P_y(\mathcal{D})$ for $y \in \partial\mathcal{D}'$. We shall show that if r_0 is of order $\log R_0$ then this bound is close to 1.

Lemma 3.12. *Let $X_3(t)$ be one dimensional Brownian motion started at $x_3 \in \mathbb{R}$ with constant drift $b(x_3) = -1$, $x_3 \in \mathbb{R}$. Let τ_3 be the exit time from the interval $[-R_0, R_0]$, where $R_0 > 1$. Then there is a universal constant $C > 0$ such that for x_3 in the interval $|x_3| \le R_0/2$ there is the inequality*

$$P_{x_3}(\tau_3 < R_0^2\,,\ X_3(\tau_3) = -R_0) \ge 1 - \frac{C}{R_0^{1/2}}\,. \tag{3.56}$$

PROOF. For $\eta > 0$ let $u(x_3, \eta)$ be defined by

$$u(x_3,\eta) = E_{x_3}[e^{-\eta\tau_3}\chi(X_3(\tau_3))]\,, \tag{3.57}$$

where

$$\chi(z) = \begin{cases} 1\,, & \text{if } z < 0\,, \\ 0\,, & \text{if } z \ge 0\,. \end{cases}$$

Then $u(x_3,\eta)$ satisfies the equation

$$\begin{cases} \dfrac{d^2u}{dx_3^2} - \dfrac{du}{dx_3} = \eta\, u\,, & -R_0 < x_3 < R_0\,, \\ u(-R_0,\eta) = 1\,, \\ u(R_0,\eta) = 0\,. \end{cases} \tag{3.58}$$

The equation (3.58) can be solved explicitly to yield

$$u(x_3,\eta) = \frac{e^{(x_3+R_0)/2}\sinh\left(\dfrac{(1+4\eta)^{1/2}(R_0-x_3)}{2}\right)}{\sinh\left((1+4\eta)^{1/2}R_0\right)}\,. \tag{3.59}$$

Next we take $\eta = 1/R_0^{3/2}$. Then it is clear from (3.59) that

$$u(x_3,\eta) \ge 1 - \frac{C_1}{R_0^{1/2}}\,, \qquad -\frac{R_0}{2} < x_3 < \frac{R_0}{2}\,. \tag{3.60}$$

Arguing as before we can see from (3.57) that

$$\begin{aligned} P_{x_3}(\tau_3 < R_0^2,\ X_3(\tau_3) = -R_0) \\ \ge (1 - e^{-\eta R_0^2})^{-1}\left(u(x_3,\eta) - e^{-\eta R_0^2}\right). \end{aligned} \tag{3.61}$$

The inequality (3.56) follows now from (3.60), (3.61).

Lemma 3.13. *Let $X(t)$ be two dimensional Brownian motion started at $x = (x_1, x_2) \in \mathbb{R}^2$ with drift* $\mathbf{b}$ *defined by*

$$\mathbf{b}(y_1, y_2) = -\frac{(y_1, y_2)}{(y_1^2 + y_2^2)^{1/2}} \,.$$

Suppose (x_1, x_2) lies on the unit circle and τ is the first hitting time on the circle radius $r_0 > 1$. Then for $R_0 > 2$ there exists a universal constant $C > 0$ such that if $r_0 = C \log R_0$ then there is the inequality

$$P(\tau > R_0^2) \ge 1 - \frac{C}{R_0} \,. \tag{3.62}$$

PROOF. Let us put

$$u(x) = E_x[e^{-\eta\tau}] \,, \qquad \eta > 0 \,,$$

and let $r = (x_1^2 + x_2^2)^{1/2}$. Then $u(x) = u(r)$ satisfies a boundary value problem,

$$\begin{cases} \dfrac{d^2u}{dr^2} + \Big(\dfrac{1}{r} - 1\Big)\dfrac{du}{dr} = \eta\, u \,, \qquad 0 < r < r_0 \,, \\ u(r_0) = 1 \,. \end{cases}$$

Let $v(r)$ be the solution of the boundary value problem

$$\begin{cases} \dfrac{d^2v}{dr^2} - \dfrac{1}{2}\,\dfrac{dv}{dr} = \eta\, v \,, \qquad 2 < r < r_0 \,, \\ v(r_0) = 1 \,, \\ v'(2) = 0 \,. \end{cases} \tag{3.63}$$

In view of the fact that $u'(2) \ge 0$ it follows from the maximum principle that

$$u(r) \le v(r) \,, \qquad 2 < r < r_0 \,.$$

Now we have

$$P(\tau < R_0^2) \le e\, u\Big(1, \frac{1}{R_0^2}\Big) \le e\, u\Big(2, \frac{1}{R_0^2}\Big) \le e\, v\Big(2, \frac{1}{R_0^2}\Big) \,. \tag{3.64}$$

We can estimate the last expression in (3.64) since the solution of (3.63) can be explicitly computed. It is given by

$$\begin{aligned} v(r,\eta) = {} & \frac{(\alpha-1)\exp\Big(\dfrac{(\alpha+1)\,(r-2)}{4}\Big)}{A} \\ & + \frac{(\alpha+1)\exp\Big(-\dfrac{(\alpha-1)(r-2)}{4}\Big)}{A} \end{aligned} \tag{3.65}$$

with

$$A = (\alpha-1)\exp\Big(\frac{(\alpha+1)(r_0-2)}{4}\Big) + (\alpha+1)\exp\Big(-\frac{(\alpha-1)(r_0-2)}{4}\Big),$$

where α is related to η by

$$\alpha = (1+16\,\eta)^{1/2}\,. \tag{3.66}$$

It is easy to see from (3.65), (3.66) that

$$v\Big(2,\frac{1}{R_0^2}\Big) \le \frac{2\alpha}{\alpha-1}\ \exp\Big(-\frac{(\alpha+1)\,(r_0-2)}{4}\Big) \le \frac{C}{R_0} \tag{3.67}$$

if $r_0 = C\log R_0$ and C is sufficiently large. The inequality (3.62) follows now from (3.64) and (3.67).

Corollary 3.1. *There exists a universal constant $C>0$ such that if $r_0 = C\log R_0$ then for $y\in\partial\mathcal{D}'$ there is the inequality*

$$P_y(\mathcal{D}) \ge 1 - \frac{C}{R_0^{1/2}}\,.$$

PROOF. From Lemmas 3.12 and 3.13 there is the inequality

$$P_y(\mathcal{D}) \ge \Big(1-\frac{C}{R_0^{1/2}}\Big)\Big(1-\frac{C}{R_0}\Big)\,.$$

Thus we are estimating the probability by restricting to paths which remain in the cylinder $r<r_0$ until they exit. For paths which remain in the cylinder, the components of the Brownian motion in the x_3 and (x_1,x_2) directions are independent.

Corollary 3.2. *Suppose $x \in \mathcal{D}\backslash\mathcal{D}'$ with $x_3 \le R_0/4$, $r_0 = (x_1^2 + x_2^2)^{1/2}$. Then there is a universal constant $C > 0$ such that for $r_0 = C \log R_0$, there is the inequality*

$$P_x(\mathcal{D}) \ge 1 - \frac{C}{(\log R_0)^{1/2}}\ . \tag{3.68}$$

PROOF. The inequality (3.68) follows from (3.39) and Lemmas 3.7, 3.11 and Corollary 3.1.

Lemma 3.14. *Suppose $R \ge 2$. Then there is a drift $\mathbf{b} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with the following properties*:

a) $\operatorname{supp}(\mathbf{b}) \subset \{x :\ 7R/8 < |x| < 9R/8\}$.

b) $\mathbf{b}(x) \cdot x \le 0$, $x \in \mathbb{R}^3$.

c) $\|\mathbf{b}\|_\infty \le 1$,

$$\int_{\mathbb{R}^3} |\mathbf{b}|\, dx \le CR\,(\log R)^4\,,$$

for some universal constant $C > 0$.

d) *For $x \in \mathbb{R}^3$ satisfying $|x| = R$ let P_x be the probability that the drift process exits the region $\{y : R/2 < |y| < 2R\}$ through the outer boundary $\{y : |y| = 2R\}$. Then there is a universal constant $C > 0$ such that*

$$P_x \le \min\left\{\frac{2}{3}, \frac{C}{(\log R)^{1/2}}\right\}. \tag{3.69}$$

PROOF. Let $a \in \mathbb{R}^3$ satisfy $|a| = R$, and $W_{\lambda,L}(a)$ denote the set $W_{\lambda,L}$ of (3.34) rotated and translated such that the origin corresponds to a and the (x_1, x_2) plane to the tangent plane to the sphere $\{x : |x| = R\}$ at the point a. We furthermore choose λ, L by

$$L = \alpha\, R\,, \qquad \lambda = \frac{L}{2\log L}\ , \tag{3.70}$$

where α satisfying $0 < \alpha < 1$ will be chosen independently of R.

We define a drift $\mathbf{b}_a(x), x \in \mathbb{R}^3$ as follows: Suppose S_2 is one of the cylindrical holes in $W_{\lambda,L}(a)$. Thus S_2 has radius r_0 and height L. Let (x_1, x_2, x_3) be orthogonal coordinates with x_3 in direction a and origin at the center of the circle formed by the intersection of S_2 with the tangent plane to the sphere $|y| = R$ at a. We define $\mathbf{b}_a(x)$ for $x \in S_2$ by (3.38). We similarly define $\mathbf{b}_a(x)$ for x in any cylindrical hole S_2 of $W_{\lambda,L}(a)$. Otherwise we set $\mathbf{b}_a(x) = 0$.

Next we choose a finite number of points $a_1, \ldots, a_N$ on $\{x : |x| = R\}$ with the properties: 1) For any $x \in \{y : |y| = R\}$ there is an $a_i, 1 \le i \le N$, such that $|x - a_i| < L/4$. 2) None of the holes S_2 in the cylinders $W_{\lambda,L}(a_i)$, $1 \le i \le N$, intersect.

Finally we choose $r_0 = \Gamma \log R_0, R_0 = L$, so that Corollary 3.2 holds and define the drift $\mathbf{b}$ by $\mathbf{b} = \sum_{i=1}^N \mathbf{b}_{a_i}$. It is easy to see now that the parameters α, Γ, N can be chosen in a universal way so that 1), 2), a), b), c) hold. It remains then to verify d).

To prove d) let x be such that $|x| = R$ and a_i satisfy $|x - a_i| < L/4$. Let Q_x be the probability of hitting one of the cylinders where $\mathbf{b} \ne 0$ before exiting the region $\{y : R(1-\varepsilon) < |x| < R(1+\varepsilon)\}$. Then by Lemma 3.6 and (3.70) there is a constant C_ε depending on ε such that

$$Q_x \ge 1 - \frac{C_\varepsilon}{\log R}\ . \tag{3.71}$$

Next, for $y \in \{z : R(1-\varepsilon) < |z| < R(1+\varepsilon), \mathbf{b}(z) \ne 0\}$, let H_y be the probability that the drift process exits the set $\{z : R(1-2\varepsilon) < |z| < R(1+2\varepsilon)\}$ through the inner boundary $\{z : |z| = R(1-2\varepsilon)\}$. Then by Corollary 3.2, ε can be chosen sufficiently small such that

$$H_y \ge 1 - \frac{C}{(\log R)^{1/2}}\ , \tag{3.72}$$

where the constant C depends only on α, Γ. Finally, for y satisfying $|y| = R(1-2\varepsilon)$ let K_y be the probability that the drift process exits the set $\{z : R/2 < |z| < R\}$ through the outer boundary $\{z : |z| = R\}$. In view of b) and the maximum principle this probability is less than the corresponding Brownian motion probabilty. Hence one has

$$K_y \le \frac{1-4\varepsilon}{1-2\varepsilon} < 1\ . \tag{3.73}$$

We use (3.71), (3.72), (3.73) to estimate P_x from above. In fact one clearly has

$$P_x \le (1 - Q_x) + Q_x(1 - \inf_y H_y) + Q_x \sup_y H_y \sup_y K_y \sup_y P_y\ . \tag{3.74}$$

The inequality (3.69) follows now from (3.74) and the previous inequalities since ε can be chosen in a universal way with $\varepsilon > 0$.

We use Lemma 3.14 to construct a drift on $\mathbb{R}^3$. In fact let $\mathbf{b}_n$ be the drift constructed in Lemma 3.14 with $R = 2^{-n}$, $n = -1, -2, \dots$ Then we put

$$\mathbf{b} = \sum_{n=-\infty}^{-1} \mathbf{b}_n \,. \tag{3.75}$$

Observe that $\operatorname{supp}(\mathbf{b}_n)$ do not overlap for different n. Hence from b), d) of Lemma 3.14 we have the inequality

$$\begin{aligned} p_{-n} = \sup_{|x|=2^{-n}} P\,(&\text{drift process started at } x \text{ with drift given by} \\ &(5.75) \text{ exits the region } 2^{-n-1} < |y| < 2^{-n+1} \\ &\text{through the outer boundary}) \\ \le \min\left\{\frac{2}{3}, \frac{C}{|n|^{1/2}}\right\}&, \end{aligned} \tag{3.76}$$

for some universal constant $C > 0$, $-n = 1, 2, \dots$

Lemma 3.15. *Let* $\mathbf{b}$ *be the drift given in* (3.75) *and suppose* $a_{n,p}$ *is defined by* (1.4) *and* n_0, R *related by* (1.5). *Then for any constants* γ, $C_2 > 0$, $0 < \gamma < 1$, *there is the inequality,*

$$\sum_{m=0}^{\infty} \gamma^m \sup_{x \in \Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j,p}(x)\Big) \le K R^{\alpha}\,, \tag{3.77}$$

for some constants K, α *depending only on* γ, C_2 *and* p *satisfying* $1 \le p < 2$.

PROOF. From c) of Lemma 3.14 we see that

$$\sum_{j=0}^{\infty} a_{n_0+j,p}(0) \le C\,, \tag{3.78}$$

where C is a universal constant. This follows because $p < 2$. On the other hand it is easy to see that if x satisfies $2^{-n-1} < |x| < 2^{-n+1}$ then

$$\sum_{j=0}^{\infty} a_{n_0+j,p}(x) \le C\,|n|\,, \tag{3.79}$$

for some universal constant C. Hence from (3.78), (3.79), we have

$$\sup_{x\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j,p}(x)\Big) \le \exp\left(C_2\, C\, |n_0|\right) = R^{\beta}\,,$$

for some β depending only on C_2C. Hence (3.77) follows.

Our final goal now is to use the inequality (3.76) to prove that the expected time to exit Ω_R, starting at the origin, exceeds R^{α} for any α, provided R is sufficiently large. In view of Lemma 3.15 this will show that there is no inequality (1.6) for $p < 2$.

Lemma 3.16. *Let $S_0, S_1, \dots, S_M$ be a set of concentric spheres with radii $r_0, r_1, \dots, r_M$ satisfying $r_0 < r_1 < \cdots < r_M$. Let $Y(t)$ be a stochastic process with continuous paths which is Brownian motion in the set $\{x : |x| \le r_1\}$. Consider every path of $Y(t)$ as being a random walk on the spheres $S_0, S_1, \dots, S_M$. For $x \in S_0$ let N_x be the number of times this random walk, started at x, hits S_0 before hitting S_M. Let τ_x be the amount of time taken for the process started at x to reach the sphere S_M. Then, if $2\, r_0 < r_1$, there is an inequality*

$$E\left[\tau_x\right] \ge C r_0^2\, E\left[N_x\right], \tag{3.80}$$

where C is a universal constant.

Proof. For $z \in S_1$ let $p(z)$ be the probability of the process started at z hitting S_M before S_0. For $n = 1, 2, \dots,$ and $x \in S_1$, $y \in S_0$ let $q_n(x, y)$ be the probability density for the process started at x and hitting S_0 n times without hitting S_M. Thus if $O \subset S_0$ is an open set,

$$\begin{aligned} &P\,(Y \text{ with } Y(0) = x \text{ hits } S_0\ n \text{ times without hitting } S_M \text{ and} \\ &\qquad \text{that on the } n\text{-th hit it lands in the set } O) = \int_O q_n(x, y)\, dy\,. \end{aligned}$$

For $x \in S_0$ let T_x be the first hitting time on S_1 for the process Y started at x. In view of our assumptions T_x is purely a Brownian motion variable. Then we have the identities

$$\begin{aligned} &P\left(N_x = 1\right) = E\left[p(Y(T_x))\right], \\ &P\left(N_x = m+1\right) = E\Big[\int_{S_0} q_m\left(Y(T_x), y\right) p(Y(T_y))\, dy\Big]\,, \end{aligned} \tag{3.81}$$

with $m = 1, 2, \ldots$ Clearly we also have the relation

$$q_m(x, y) = E\Big[\int_{S_0} q_n(x, z)\, q_{m-n}\,(Y(T_z), y)\, dz\Big], \tag{3.82}$$

with $n = 1, \ldots, m-1$. We shall use the functions p, q_m and the variables T_y to obtain a lower bound on $E\,[\tau_x]$. We do this by bounding $E\,[\tau_x]$ below by the amount of time the path spends in jumping from S_0 to S_1. Thus

$$\begin{aligned}
E\,[\tau_x] \geq{} & E\,[T_x\, p(Y(T_x))] \\
& + \sum_{m=1}^{\infty} \Big(E\Big[T_x \int_{S_0} q_m(Y(T_x), y)\, p(Y(T_y))\, dy\Big] \\
& \qquad + \sum_{n=1}^{m-1} E\Big[\int_{S_0}\int_{S_0} dy\, dz\, q_n(Y(T_x), y) \\
& \qquad\qquad \cdot T_y\, q_{m-n}\,(Y(T_y), z)\, p(Y(T_z))\Big] \\
& \qquad + E\Big[\int_{S_0} q_m(Y(T_x), y)\, T_y\, p(Y(T_y))\, dy\Big]\Big).
\end{aligned} \tag{3.83}$$

Since T_y is purely a Brownian motion variable and $2\,r_0 < r_1$, there is a universal constant $C > 0$ such that

$$E\,[T_y \mid Y(T_y)] \geq C\, r_0^2\,, \qquad y \in S_1\,. \tag{3.84}$$

Substituting (3.84) into (3.83) and using the identities (3.81), (3.82) yields the inequality (3.80).

Lemma 3.17. *Let $S_0, S_1, \ldots, S_M$ be a set of concentric spheres with radii $r_0, r_1, \ldots, r_M$ satisfying $r_0 < r_1 < \cdots < r_M$. For $j = 1, \ldots, M-1$ let $p_j(x, y)$ be nonnegative functions of $x \in S_j$, $y \in S_{j+1}$ satisfying*

$$0 < \int_{S_{j+1}} p_j(x, y)\, dy \leq p_j < 1\,, \qquad x \in S_j\,,$$

for some positive numbers $p_1, \ldots, p_{M-1}$. Suppose now that the $p_j(x, y)$, $j = 1, \ldots, M-1$, are probability density functions for a stochastic process $Y(t)$ with continuous paths in the following sense: for any open set $O \subset S_{j+1}$,

$$\begin{aligned}
& P\,(Y \text{ started at } x \in S_j \text{ exits the region} \\
& \quad \text{between } S_{j-1} \text{ and } S_{j+1} \text{ through } O) = \int_O p_j(x, y)\, dy\,.
\end{aligned}$$

Let $x \in S_0$ and N_x be the number of times the process hits S_0 before hitting S_M when viewed as a random walk on the spheres $S_0, \ldots, S_M$. Then

$$E\,[N_x] \geq 1 + \sum_{j=1}^{M-1} \prod_{i=1}^{j} \frac{q_i}{p_i}\,, \tag{3.85}$$

where $q_i = 1 - p_i$, $i = 1, \ldots, M-1$.

PROOF. We shall first prove (3.85) in the case $M = 2$. Thus if we put $u(x) = E\,[N_x]$ it follows that

$$u(x) = \begin{cases} \displaystyle\int_{S_0} q_1(x,y)\, u(y)\, dy\,, & x \in S_1\,, \\ \displaystyle\int_{S_1} p_0(x,y)\, u(y)\, dy + 1\,, & x \in S_0\,, \end{cases} \tag{3.86}$$

where

$$\begin{aligned} &P\,(Y \text{ started at } x \in S_0 \text{ exits the region inside} \\ &\qquad S_1 \text{ through the open set } O \subset S_1) = \int_O p_0(x,y)\, dy\,, \end{aligned} \tag{3.87}$$

$$\begin{aligned} &P\,(Y \text{ started at } x \in S_1 \text{ exits the region between } S_0 \\ &\qquad \text{and } S_2 \text{ through the open set } O \subset S_0) = \int_O q_1(x,y)\, dy\,. \end{aligned} \tag{3.88}$$

Evidently from the definitions (3.87), (3.88) one has

$$\int_{S_1} p_0(x,y)\, dy = 1\,, \qquad x \in S_0\,,$$

$$\int_{S_2} p_1(x,y)\, dy + \int_{S_0} q_1(x,y)\, dy = 1\,, \qquad x \in S_1\,.$$

From (3.86) we have

$$u(x) = \int_{S_1} p_0(x,y) \int_{S_0} q_1(y,z)\, u(z)\, dz\, dy + 1\,, \qquad x \in S_0\,. \tag{3.89}$$

Hence if we put $u_0 = \inf\{u(x) : x \in S_0\}$ then

$$\begin{aligned} u(x) &\geq u_0 \int_{S_1} p_0(x,y) \int_{S_0} q_1(y,z)\, dz\, dy + 1 \\ &= u_0 \int_{S_1} p_0(x,y) \Big(1 - \int_{S_2} p_1(y,z)\, dz\Big)\, dy + 1 \\ &\geq u_0 \int_{S_1} p_0(x,y)\,(1-p_1)\, dy + 1 = u_0\,(1-p_1) + 1\,, \qquad x \in S_0\,. \end{aligned}$$

Taking the infimum on the left in (3.89) we conclude

$$u_0 \geq \frac{1}{p_1}\,. \tag{3.90}$$

This last inequality is just (3.85) for $M = 2$.

To generalize this for $M > 2$ let $P_1(x,y)$ be defined by

$$\begin{aligned} &P\,(Y \text{ started at } x \in S_1 \text{ exits the region between } S_0 \\ &\qquad \text{and } S_M \text{ through the open set } O \subset S_M) = \int_O P_1(x,y)\, dy\,. \end{aligned}$$

From [5, Lemma 6.3] it follows that

$$\int_{S_M} P_1(x,y)\, dy \leq P_1\,, \qquad x \in S_1\,, \tag{3.91}$$

where

$$P_1 = \frac{1}{1 + \displaystyle\sum_{j=1}^{M-1} \prod_{i=1}^{j} \frac{q_i}{p_i}}\,. \tag{3.92}$$

Hence (3.85) follows from (3.90), (3.91), (3.92).

We use Lemmas 3.16 and 3.17 to obtain a lower bound on $u(0)$ where u is the solution of (1.1), (1.2) with $f \equiv 1$ and drift given by (3.75). Let $S_j, j = 0,1,2,\ldots$ be spheres centered at the origin with radius 2^j. Then the probabilities $p_j, j = 1,\ldots,M-1$, of Lemma 3.17 satisfy by (3.76) the inequality

$$p_j \leq \min\left\{\frac{2}{3}, \frac{C}{\sqrt{j}}\right\}, \qquad j = 1,2,\ldots$$

Consequently, if $R = 2^{-n_0}$ one has from Lemmas 3.16 and 3.17 the inequality

$$u(0) \geq C\Big(1 + \sum_{j=1}^{|n_0|-1} \prod_{i=1}^{j} \frac{q_i}{p_i}\Big) \geq C \exp\big(C_1\,|n_0| \log |n_0|\big)\,,$$

where C, C_1 are universal constants. Thus one has an inequality

$$u(0) \geq C R^{\alpha \log\log R}\,,$$

for some $C, \alpha > 0$. In view of Lemma 3.15 the inequality (1.6) does not hold for R sufficiently large.

4. Perturbative estimates on the exit probabilities from a spherical shell.

In this section we shall be interested in the drift process with perturbative drift $\mathbf{b}$. For $R_1 < R < R_2$ let U_{R_1,R_2} be the spherical shell

$$U_{R_1,R_2} = \{x \in \mathbb{R}^3 :\ R_1 < |x| < R_2\}\,.$$

Now suppose we start the process off on the sphere $\{x : |x| = R\}$ with density $f(x)$, $|x| = R$. Some of the paths of the process exit the shell U_{R_1,R_2} through the boundary $\{|x| = R_2\}$ and the others through $\{|x| = R_1\}$. Hence the density f induces densities f_1 on $\{|x| = R_1\}$ and f_2 on $\{|x| = R_2\}$. We shall be interested in comparing f_1, f_2 and f. To do this we shall need to define norms of these functions. Let $\rho > 0$ and g a measurable function on the sphere $\{|x| = \rho\}$. For $1 \leq q < \infty$ we define the L^q norm of g by

$$\|g\|_q = \Big(\frac{1}{4\pi\rho^2}\int_{|x|=\rho} |g(x)|^q\,dx\Big)^{1/q}\,.$$

Thus $\|\mathbf{1}\|_q = 1$. For an L^1 function g we define $\mathrm{Av}\,g$ by

$$\mathrm{Av}\,g = \frac{1}{4\pi\rho^2}\int_{|x|=\rho} g(x)\,dx\,.$$

It is clear that the functions f_1, f_2, f satisfy

$$\mathrm{Av}\,f_1 + \mathrm{Av}\,f_2 = \mathrm{Av}\,f\,.$$

We wish to obtain an expression for f_2 in terms of f . Let $g(x)$ be a function defined on the sphere $\{|x| = R_2\}$ and $u(x) = Pg(x)$ be defined for $x \in U_{R_1,R_2}$ as the solution of the boundary value problem

$$
\begin{cases} \Delta u(x) = 0\,, & R_1 < |x| < R_2\ , \\ u(x) = g(x)\,, & |x| = R_2\ , \\ u(x) = 0\,, & |x| = R_1\ . \end{cases} \tag{4.1}
$$

For $x, y \in U_{R_1,R_2}$ let $G_D(x,y)$ be the Dirichlet Green's function and k_T the kernel

$$
k_T(x,y) = \mathbf{b}(x) \cdot \nabla_x G_D(x,y)\,, \qquad x, y \in U_{R_1,R_2}\ . \tag{4.2}
$$

Suppose $g \in L^q(\{|x| = R_2\})$. Then we define the operator Q by

$$
Qg(x) = \int_{U_{R_1,R_2}} G_D(x,y)\,(I-T)^{-1}\,\mathbf{b}\cdot\nabla Pg(y)\,dy\,, \qquad |x| = R\,, \tag{4.3}
$$

where T is the operator induced by the kernel k_T. The expression (4.3) is purely formal. It takes functions with domain $\{|x| = R_2\}$ to functions with domain $\{|x| = R\}$. Similarly, the operator P defined above takes functions on the sphere $|x| = R_2$ to functions on the sphere $|x| = R$. Hence the formal adjoints P^* and Q^* of P and Q take functions on $|x| = R$ to functions on $|x| = R_2$. We have now the relation

$$
f_2 = P^*f + Q^*f\,.
$$

Our major goal here will be to show that the operator Q^* is dominated by the operator P^*. We shall prove this by showing that Q is dominated by P. To do this we shall need various estimates on the Green's function $G_D(x,y)$ and its derivatives. Observe that the Green's function for the shell U_{R_1,R_2} can be obtained from the Green's function for a sphere by the method of images. The estimates we need on $G_D(x,y)$ can easily be derived from this image representation. First we shall consider the simplest of cases $R_1 = 0$, $R_2 = 2R$. We obtain an improvement on Lemma 2.2.

Lemma 4.1. *Suppose $R_1 = 0$, $R_2 = 2R$. Let r, p, q satisfy the inequalities $1 < r < p \le 3$, $q > r$,*

$$
\frac{1}{q} < \frac{\frac{1}{r} - \frac{1}{p}}{1 - \frac{1}{p}}\ . \tag{4.4}
$$

Then if $g \in L^q(\{|x| = R_2\})$ *the function* $\mathbf{b} \cdot \nabla P g$ *is in the Morrey space* $M_r^{q_1}(\{|x| < R_2\})$, *where*

$$\frac{1}{q_1} = \frac{1}{3} + \frac{2}{3\,q} \tag{4.5}$$

and

$$\|\mathbf{b} \cdot \nabla P g\|_{q_1, r} \le C\, R^{2/q-1}\, \|\mathbf{b}\|_{3,p}\ \|g\|_q\,. \tag{4.6}$$

Proof. The idea of the proof here is to use the Harnack inequality. Thus it follows from Harnack that if g is a nonnegative function then there is a universal constant C such that

$$(R_2 - |x|)\, |\nabla P g(x)| \le C P g(x)\,.$$

Hence for any cube Q one has

$$\begin{aligned}\frac{1}{R_2^r} \int_{Q \cap \{|x| < R_2\}} &(R_2 - |x|)^r\, |\mathbf{b}(x)|^r\, |\nabla P g(x)|^r\, dx \\ &\le \frac{C^r}{R_2^r} \int_{Q \cap \{|x| < R_2\}} |\mathbf{b}(x)|^r\, |P g(x)|^r\, dx \\ &= \frac{C^r}{R_2^r} \int_{Q \cap \{|x| < R_2\}} |\mathbf{b}(x)|^{r(1-\alpha)}\, |\mathbf{b}(x)|^{r\alpha}\, |P g(x)|^r\, dx \\ &\le \frac{C^r}{R_2^r} \Big(\int_Q |\mathbf{b}(x)|^{r(1-\alpha)/(1-r/q)}\, dx \Big)^{1-r/q} \\ &\quad \cdot \Big(\int_{Q \cap \{|x| < R_2\}} |\mathbf{b}(x)|^{q\alpha} |P g(x)|^q\, dx \Big)^{r/q}.\end{aligned} \tag{4.7}$$

Since $P(1) = 1$ it follows by Jensen that

$$(P g(x))^q \le P g^q(x)\,.$$

Thus

$$\begin{aligned}\int_{Q \cap \{|x| < R_2\}} &|\mathbf{b}(x)|^{q\alpha}\, |P g(x)|^q\, dx \\ &\le \Big(\sup_{|x| = R_2} C_Q(x) \Big) \int_{|x| = R_2} |g(x)|^q\, dx\,,\end{aligned} \tag{4.8}$$

where

$$C_Q(x) = \int_{Q\cap\{|y|<R_2\}} |\mathbf{b}(y)|^{q\alpha} P\delta_x(y)\, dy\,,$$

and δ_x is the Dirac δ function concentrated at x, $|x| = R_2$. We suppose now that $\alpha > 0$ is chosen so that $q\,\alpha < 1$. Then we have

$$C_Q(x) \le C \int_Q \frac{|\mathbf{b}(y)|^{q\alpha}}{|y-x|^2}\, dy \le C \sum_{n=n_1}^{\infty} 2^{2n} \int_{Q_n} |\mathbf{b}(y)|^{q\alpha}\, dy\,,$$

where the Q_n are cubes with side 2^{-n} and n_1 is chosen so that $|Q| \sim 2^{-3n_1}$. Using the fact that $\mathbf{b} \in M^3_p$ we conclude that

$$C_Q(x) \le C \sum_{n=n_1}^{\infty} 2^{2n}\, |Q_n|^{1-q\alpha/3}\, \|\mathbf{b}\|^{q\alpha}_{3,p} \le C\, |Q|^{(1-q\alpha)/3}\, \|\mathbf{b}\|^{q\alpha}_{3,p}\,,$$

for some universal constant C. Hence from (4.7) and (4.8) we conclude that

$$\begin{aligned} &\frac{1}{R_2^r} \int_{Q\cap\{|x|<R_2\}} (R_2 - |x|)^r\, |\mathbf{b}(x)|^r\, |\nabla P g(x)|^r\, dx \\ &\qquad \le \frac{C^r}{R_2^r}\, |Q|^{1-r/q-r(1-\alpha)/3}\, |Q|^{(1-q\alpha)r/3q}\, \|\mathbf{b}\|^r_{3,p}\, R_2^{2r/q}\, \|g\|^r_q \qquad (4.9)\\ &\qquad = C^r R_2^{r(2/q-1)}\, |Q|^{1-r/q_1}\, \|\mathbf{b}\|^r_{3,p}\, \|g\|^r_q\,, \end{aligned}$$

where q_1 is given by (4.5) and α must satisfy the inequality

$$\frac{r\,(1-\alpha)}{1-\dfrac{r}{q}} \le\; p\,. \qquad (4.10)$$

The inequality (4.10) taken together with the condition $q\,\alpha < 1$ implies (4.4). The inequality (4.6) is an immediate consequence of (4.9).

REMARK. Observe that (4.5) is the same as (2.5) but (4.4) is an improvement on (2.4).

Proposition 4.1. *For $1 < q < \infty$ the operator Q defined by (4.3) is a bounded operator from $L^q(\{|x| = R_2\})$ to $L^q(\{|x| = R\})$ provided $\|\mathbf{b}\|_{3,p} < \varepsilon$ for sufficiently small ε depending on p and q. Furthermore* *the norm of Q, $\|Q\|$ satisfies an inequality $\|Q\| \le C\varepsilon$, where C is a universal constant.*

PROOF. We have by Lemma 4.1 and Proposition 2.1 that if ε is sufficiently small then

$$Qg(x) = \int_{|y|<R_2} G_D(x,y)\, h(y)\, dy\,, \qquad |x| = R\,,$$

where h is in the Morrey space $M_r^{q_1}(\{|x| < R_2\})$ and

$$\|h\|_{q_1,r} \le C R_2^{2/q-1} \, \|\mathbf{b}\|_{3,p} \, \|g\|_q \,, \tag{4.11}$$

for some universal constant C. Arguing as in Lemma 2.3 we see that if $m \ge 1$ satisfies the inequality

$$\frac{2}{3} + \frac{1}{q_1 m} > \frac{1}{q_1} + \frac{1}{3\,m} \,, \tag{4.12}$$

then

$$Qg \in L^m(\{|x| = R\})$$

and

$$\|Qg\|_m \le C R_2^{2-3/q_1} \, \|h\|_{q_1,r} \,, \tag{4.13}$$

for some constant C. This inequality (4.12) holds provided m satisfies the inequality

$$\frac{1}{m} > \frac{2-q}{2} \,. \tag{4.14}$$

It is easy to see that (4.14) holds with $m = q$ for all $q > 1$. The result now follows from (4.11) and (4.13) by observing that $2/q - 1 = -(2 - 3/q_1)$.

Corollary 4.1. *Suppose $R_1 = 0$, $R_2 = 2R$. Then for any p, $1 < p \le 3$ and $q > 1$ the following holds: there exists $\varepsilon, \delta > 0$ depending only on p, q such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ and $\|f - \mathrm{Av}\, f\|_q \le \delta |\mathrm{Av}\, f|$ then*

$$\|f_2 - \mathrm{Av}\, f_2\|_q \le \delta |\mathrm{Av}\, f_2| \,.$$

Proof. By Proposition 4.1 the operator Q^* is a bounded operator from $L^q(\{|x| = R\})$ to $L^q(\{|x| = R_2\})$ and $\|Q^*\| \le C\varepsilon$. We combine this with the fact that there exists γ, $0 < \gamma < 1$, such that

$$\|P^*(f - \mathrm{Av}\, f)\|_q \le \gamma \|f - \mathrm{Av}\, f\|_q \,. \tag{4.15}$$

The inequality (4.15) follows by the same argument as in [5, Lemma 4.1]. It is clear that

$$\mathrm{Av}\, f = P^*(\mathrm{Av}\, f) = \mathrm{Av} P^* f = \mathrm{Av}\, f_2 \,.$$

Thus

$$\begin{aligned}
\|f_2 - \mathrm{Av}\, f_2\|_q &= \|Q^* f - \mathrm{Av}\, Q^* f + P^*(f - \mathrm{Av}\, f)\|_q \\
&\le 2\, C\varepsilon\, \|f\|_q + \gamma\, \|f - \mathrm{Av}\, f\|_q \\
&\le 2\, C\varepsilon\, (1+\delta)\, |\mathrm{Av}\, f| + \gamma\, \delta\, |\mathrm{Av}\, f| \\
&\le \delta\, |\mathrm{Av}\, f| \\
&= \delta\, |\mathrm{Av}\, f_2| \,,
\end{aligned}$$

if ε is chosen so that

$$2\, C\varepsilon\, \frac{1+\delta}{\delta} + \gamma \le 1 \,.$$

The proof is complete.

Next we state an obvious generalization of Corollary 4.1.

Corollary 4.2. *Suppose $R_1 = R/2$, $R_2 = 2R$. Then for any p, $1 < p \le 3$ and $q > 1$ the following holds: there exist positive constants $c_1, c_2, \varepsilon, \delta$ depending only on p, q such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ and $\|f - \mathrm{Av}\, f\|_2 \le \delta\, |\mathrm{Av}\, f|$ then*

$$\begin{aligned}
&|\mathrm{Av}\, f_1| \ge c_1\, |\mathrm{Av}\, f| \qquad \textit{and} \qquad \|f_1 - \mathrm{Av}\, f_1\|_q \le \delta\, |\mathrm{Av}\, f_1| \,, \\
&|\mathrm{Av}\, f_2| \ge c_2\, |\mathrm{Av}\, f| \qquad \textit{and} \qquad \|f_2 - \mathrm{Av}\, f_2\|_q \le \delta\, |\mathrm{Av}\, f_2| \,.
\end{aligned}$$

PROOF. We shall just show that $|\mathrm{Av}\, f_2| \geq |\mathrm{Av}\, f|$. Observe that

$$\begin{aligned}\mathrm{Av}\,(P^* f) &= P^*(\mathrm{Av}\, f)\\ &= (\mathrm{Av}\, f)\, P^*(1)\\ &= (\mathrm{Av}\, f)\, P\,[\text{Brownian motion started at } x \text{ with } |x| = R\\ &\qquad\qquad \text{exits } U_{R_1,R_2} \text{ through the boundary } |y| = R_2]\\ &= \frac{\dfrac{1}{R_1} - \dfrac{1}{R}}{\dfrac{1}{R_1} - \dfrac{1}{R_2}}\,\mathrm{Av}\, f\\ &= \frac{2}{3}\,\mathrm{Av}\, f\,.\end{aligned}$$

Hence

$$\begin{aligned}|\mathrm{Av}\, f_2| &= |\mathrm{Av}\,(P^* f) + \mathrm{Av}\,(Q^* f)|\\ &\geq \frac{2\,|\mathrm{Av}\, f|}{3} - C\,\varepsilon\, \|f\|_q\\ &\geq \frac{2\,|\mathrm{Av}\, f|}{3} - C\,\varepsilon\,(1+\delta)\,|\mathrm{Av}\ f|\\ &\geq c_2\,|\mathrm{Av}\, f|\,.\end{aligned}$$

The proof is complete.

In Corollary 4.2 the distances $R - R_1$ and $R_2 - R$ are commensurable. Now we wish to consider the situation when $R - R_1$ is much smaller than $R_2 - R$.

Lemma 4.2. *Suppose $R/2 < R_1 < R < R_2 = 2R$. Then if $\mathbf{b} \equiv 0$ there exists a universal constant $c_2 > 0$ and a constant γ, $0 < \gamma < 1$ such that*

$$|\mathrm{Av}\, f_2| \geq c_2\,|\mathrm{Av}\, f|\,\frac{R - R_1}{R}\,, \tag{4.16}$$

$$\|f_2 - \mathrm{Av}\, f_2\|_q \leq \gamma\,|Av\ f_2|\,\frac{\|f - \mathrm{Av}\, f\|_q}{|\mathrm{Av}\, f|}\,, \tag{4.17}$$

for any q, $1 \leq q \leq \infty$.

PROOF. Since we are in the Brownian motion case we have $f_2 = P^* f$. The inequality (4.16) follows by the argument in Corollary 4.2. To get the inequality (4.17) we let $k > 0$ be such that $k\,P(1) = 1$. Since $f_2 = P^* f$ it follows that

$$\langle 1, f_2\rangle = \langle 1, P^* f\rangle = \langle P1, f\rangle\,,$$

and so we have

$$\operatorname{Av} f_2 = \frac{1}{k} \operatorname{Av} f\,. \tag{4.18}$$

Using Jensen's inequality and the fact that $P = P^*$ we have that for any q, $1 \le q \le \infty$ there is the inequality

$$\|k\,Pg\|_q \le \|g\|_q\,.$$

The inequality (4.17) will follow if we can show a version of the Harnack inequality, namely

$$C\,Pg(x_0) \ge Pg(x) \ge c\,Pg(x_0)\,, \qquad |x| = |x_0| = R\,, \tag{4.19}$$

for universal constants C, $c > 0$ and nonnegative functions g. In fact we need only repeat the argument of [5, Lemma 4.1] for the operator $k\,P$ and use (4.18).

To see (4.19) we write

$$Pg(x) = E_x[g(X(\tau))] = \int_{|y|=3R/2} \rho_x(y)\, E_y[g(X(\tau))]\, dy\,.$$

Here τ is the exit time from the shell U_{R_1,R_2} for Brownian motion. The density $\rho_x(y)$ is the density for paths started at x, $|x| = R$, which hit the sphere $|y| = 3R/2$ before hitting the sphere $|y| = R_1$. Thus

$$\int_{|y|=3R/2} \rho_x(y)\, dy = \frac{\dfrac{1}{R_1} - \dfrac{1}{R}}{\dfrac{1}{R_1} - \dfrac{2}{3R}}\,.$$

Now by the standard Harnack inequality applied to the shell U_{R_1,R_2} there exist universal constants C_1, $c_1 > 0$ such that

$$C_1 Pg(y_0) \ge Pg(y) \ge c_1 Pg(y_0)\,, \qquad |y| = |y_0| = \frac{3R}{2}\,.$$

Hence we have

$$\begin{aligned}
Pg(x) &= \int_{|y|=3R/2} \rho_x(y)\, Pg(y)\, dy \\
&\le \int_{|y|=3R/2} \rho_x(y)\, C_1\, Pg(y_0)\, dy \\
&= \int_{|y|=3R/2} \rho_{x_0}(y)\, C_1\, Pg(y_0)\, dy \\
&\le \int_{|y|=3R/2} \rho_{x_0}(y)\, C_1^2\, Pg(y)\, dy \\
&= C_1^2\, Pg(x_0)\,.
\end{aligned}$$

Similarly we obtain a lower bound $Pg(x) \ge c_1^2 Pg(x_0)$. Thus (4.19) follows with $C = C_1^2$, $c = c_1^2$.

Next we wish to generalize Lemma 4.2 to the case of nontrivial drift $\mathbf{b}$. To do this we shall need to generalize further the notion of a Morrey space. For Q a dyadic cube intersecting the spherical shell U_{R_1,R_2} let $d(Q)$ be defined by

$$d(Q) = \sup\{d(x, |y| = R_2) : x \in Q\}\,.$$

Observe that $d(Q)$ is *not* the maximum distance from points in Q to the boundary of U_{R_1,R_2}, only to the part of the boundary consisting of the sphere $|y| = R_2$. We define the Morrey space $M^q_{r,s}(U_{R_1,R_2})$ where $1 \le r \le q < \infty$ and $s > 0$ by the following: a measurable function $g : U_{R_1,R_2} \longrightarrow \mathbb{C}$ is in $M^q_{r,s}(U_{R_1,R_2})$ if $(R_2 - |x|)^r\, |g(x)|^r$ is integrable on U_{R_1,R_2} and there is a constant $C > 0$ such that

$$\frac{1}{R_2^r} \int_{Q \cap U_{R_1,R_2}} (R_2 - |x|)^r\, |g(x)|^r\, dx \le C^r\, |Q|^{1-r/q} \Big(\frac{R_2}{d(Q)}\Big)^{sr}\,, \tag{4.20}$$

for all cubes $Q \subset \mathbb{R}^3$. The norm of g, $\|g\|_{q,r,s}$ is defined as

$$\|g\|_{q,r,s} = \inf\{C : \text{ (4.20) holds for all cubes } Q\}\,.$$

Lemma 4.3. *Suppose $R/2 < R_1 < R < R_2 = 2R$. Let r, p, q satisfy the inequalities $1 < r < p \le 3$, $q > r$ and (4.4). Then if $g \in L^q(\{|x| = R_2\})$*

the function $\mathbf{b}\cdot\nabla Pg$ *is in the Morrey space* $M^3_{r,s}(U_{R_1,R_2})$ *where* $s = 2/q$ *and*

$$\|\mathbf{b}\cdot\nabla Pg\|_{3,r,s} \le CR_2^{-1}\,\|\mathbf{b}\|_{3,p}\,\|g\|_q\ .$$

Proof. This follows immediately from the argument of Lemma 4.1. The only modification is in estimating the function $C_Q(x)$. It is clear that if $|x| = R_2$ then

$$C_Q(x) \le C\,\frac{|Q|^{1-q\alpha/3}\,\|\mathbf{b}\|^{q\alpha}_{3,p}}{d(Q)^2}\ , \tag{4.21}$$

for some universal constant C. Observe that we also have an inequality

$$|\nabla Pg(x)| \le CR_2^{-1}\,\|g\|_q\ , \tag{4.22}$$

provided $R_1 < |x| < 3R/2$. This follows since $Pg(x) = 0$ for $|x| = R_1$. To get the inequality (4.20) we divide the cubes Q into two types, those with $d(Q) < R/2$ and those with $d(Q) \ge R/2$. For the first type we use the estimate (4.21) and the corresponding estimate in Lemma 4.1 to obtain (4.20) with $s = 2/q$. For the second category we use (4.22) and the fact that $\mathbf{b}$ is in M^3_p.

Lemma 4.4. *Suppose* $R/2 < R_1 < R < R_2 = 2R$. *Then the operator* T *with kernel* k_T *given by* (4.2) *is a bounded operator on the Morrey space* $M^q_{r,s}(U_{R_1,R_2})$ *provided* $1 < r < p$ *and* $1 < q < 3$, $s > 0$. *Furthermore, the norm of* T *is bounded as* $\|T\| \le C\,\|\mathbf{b}\|_{3,p}$ *where the constant* C *depends only on* r, s, q.

Proof. This follows from Corollary 2.1 and the fact that

$$\sum_{n=-\infty}^{n_{Q'}} |\mathbf{b}(x)|\,S_n u(x) \le |\mathbf{b}(x)| \sum_{n=-\infty}^{n_{Q'}} 2^{-n} u_{Q_n}\,\frac{R}{d(Q_n)}\ ,$$

where the Q_n are an increasing sequence of dyadic cubes containing the point x. We have now from (4.20) that

$$u_{Q_n} \le C\,|Q_n|^{-1/q}\Big(\frac{R_2}{d(Q_n)}\Big)^s\,\|u\|_{q,r,s}\ .$$

Hence,

$$\sum_{n=-\infty}^{n_{Q'}} 2^{-n}\,u_{Q_n}\,\frac{R}{d(Q_n)} \le C\,|Q'|^{1/3-1/q}\Big(\frac{R}{d(Q')}\Big)^{s+1}\|u\|_{q,r,s}\ ,$$

for some universal constant C, since $q<3$. Here we have used the fact that $d(Q_n)\ge d(Q')$ since $Q_n\supset Q'$. Thus

$$\begin{aligned}
&\frac{1}{R_2^r}\int_{Q'\cap U_{R_1,R_2}}(R_2-|x|)^r\Big(\sum_{n=-\infty}^{n_{Q'}}|\mathbf{b}(x)|\,S_nu(x)\Big)^r\,dx\\
&\qquad\le C^r\Big(\int_{Q'}|\mathbf{b}(x)|^r\,dx\Big)\|u\|_{q,r,s}^r\Big(\frac{R}{d(Q')}\Big)^{sr}|Q'|^{r/3-r/q}\\
&\qquad\le C^r\,\|\mathbf{b}\|_{3,p}^r\,|Q'|^{1-r/3}\,\|u\|_{q,r,s}^r\Big(\frac{R}{d(Q')}\Big)^{sr}|Q'|^{r/3-r/q}\\
&\qquad= C^r\,\|\mathbf{b}\|_{3,p}^r\,\|u\|_{q,r,s}^r\Big(\frac{R}{d(Q')}\Big)^{sr}|Q'|^{1-r/q}.
\end{aligned}$$

Proposition 4.2. *Suppose $R/2<R_1<R<R_2=2R$. For $1<q<\infty$ the operator Q defined by (4.3) is a bounded operator from $L^q(\{|x|=R_2\})$ to $L^q(\{|x|=R\})$ provided $\|\mathbf{b}\|_{3,p}<\varepsilon$ for sufficiently small ε depending on p and q. Furthermore, the norm of Q, $\|Q\|$ satisfies an inequality $\|Q\|\le C\varepsilon\,(R-R_1)/R$, where C is a universal constant.*

Proof. From Lemma 4.3 and Lemma 4.4 we have

$$Qg(x)=\int_{U_{R_1,R_2}}G_D(x,y)\,h(y)\,dy\,,\qquad |x|=R,$$

where h is in the Morrey space $M^{q_1}_{r,s}(U_{R_1,R_2})$ for any $1<r<p$, $r\le q_1<3$, provided (4.4) is satisfied and

$$\frac{1}{q_1}=\frac{1}{3}-\frac{s}{3}+\frac{2}{3\,q}\,, \tag{4.23}$$

with $0\le s<2/q$. The norm of h satisfies an inequality

$$\|h\|_{q_1,r,s}\le CR_2^{2/q-s-1}\,\|\mathbf{b}\|_{3,p}\,\|g\|_q\,. \tag{4.24}$$

We write $Qg(x)=g_1(x)+g_2(x)$, where

$$g_1(x)=\int_{U_{R_1,R_2}\cap\{|y|<3R/2\}}G_D(x,y)\,h(y)\,dy\,.$$

It follows that for $|x| = R$, there is an inequality

$$
\begin{aligned}
|g_2(x)| &\le \frac{C\,(R-R_1)}{R^3} \int_{U_{R_1,R_2}\cap\{|y|>3R/2\}} (R_2 - |y|)\,|h(y)|\,dy \\
&\le \frac{C\,(R-R_1)}{R^2}\, R^{3-3/q_1}\, \|h\|_{q_1,r,s} \qquad (4.25)\\
&= \frac{C\,(R-R_1)}{R}\, \|\mathbf{b}\|_{3,p}\, \|g\|_q\,.
\end{aligned}
$$

Next observe that

$$
|g_1(x)| \le C\,(R-R_1) \int_{U_{R_1,R_2}\cap\{|y|<3R/2\}} \frac{|h(y)|}{|x-y|^2}\,dy\,. \qquad (4.26)
$$

We estimate the integral in a similar way to Lemma 2.3. Thus

$$
\begin{aligned}
\int \frac{|h(y)|}{|x-y|^2}\,dy &= \int \frac{|h(y)|^{r/q}}{|x-y|^{2\alpha/q}}\, \frac{|h(y)|^{1-r/q}}{|x-y|^{2-2\alpha/q}}\,dy \\
&\le \Big(\int \frac{|h(y)|^r}{|x-y|^{2\alpha}}\,dy\Big)^{1/q} \Big(\int \frac{|h(y)|^{q'(1-r/q)}}{|x-y|^{(2-2\alpha/q)q'}}\,dy\Big)^{1/q'},
\end{aligned}
$$

where $1/q + 1/q' = 1$. We have used here the fact that $r < q$ which is a consequence of (4.4). We can estimate

$$
\int \frac{|h(y)|^{q'(1-r/q)}}{|x-y|^{(2-2\alpha/q)q'}}\,dy \le C \sum_{n=n_0}^{\infty} 2^{n(2-2\alpha/q)q'} \int_{Q_n} |h(y)|^{q'(1-r/q)}\,dy\,,
$$

where Q_n is the cube centered at x with side of length 2^{-n} and $2^{-n_0} \sim R$. In view of the fact that $q'(1-r/q) < r$ we have

$$
\int_{Q_n} |h(y)|^{q'(1-r/q)}\,dy \le \|h\|_{q_1,r,s}^{q'(1-r/q)}\, |Q_n|^{1-q'(1-r/q)/q_1}\,.
$$

Hence, provided α, $0 < \alpha < 1$, satisfies the inequality

$$
\Big(2 - \frac{2\,\alpha}{q}\Big) q' - 3 + \frac{3\,q'\Big(1 - \dfrac{r}{q}\Big)}{q_1} < 0\,, \qquad (4.27)
$$

we have the inequality

$$
\Big(\int \frac{|h(y)|^{q'(1-r/q)}}{|x-y|^{(2-2\alpha/q)q'}}\,dy\Big)^{1/q'} \le C\, \|h\|_{q_1,r,s}^{1-r/q}\, R^{1-(3-2\alpha)/q-3(1-r/q)/q_1}\,.
$$

There exists α, $0 < \alpha < 1$ satisfying (3.27) provided

$$\frac{1}{q_1} < \frac{1 - \dfrac{1}{q}}{3\Big(1 - \dfrac{r}{q}\Big)} \,.$$

Observe that since $r > 1$ the number on the right hand side of the last equation exceeds $1/3$. Since q_1 satisfies (4.23) and we can choose s as close as we please to $2/q$ the number q_1 may be chosen so that $1/q_1$ is less than any number larger than $1/3$. Hence we can find an α with $0 < \alpha < 1$ such that (4.27) holds. Then

$$\begin{aligned}
&\int_{|x|=R} \Big(\int \frac{|h(y)|}{|x-y|^2}\, dy \Big)^q \, dx \\
&\qquad \le C^q \, \|h\|_{q_1,r,s}^{q-r} \, R^{q-(3-2\alpha)-3(q-r)/q_1} \int_{|x|=R} \int \frac{|h(y)|^r}{|x-y|^{2\alpha}} \, dy \, dx \\
&\qquad \le C^q \, \|h\|_{q_1,r,s}^{q-r} \, R^{q-(3-2\alpha)-3(q-r)/q_1} R^{2-2\alpha} \, \|h\|_{q_1,r,s}^{r} \, R^{3-3r/q_1} \\
&\qquad = C^q \, \|h\|_{q_1,r,s}^{q} \, R^{qs} \\
&\qquad \le C^q R^{2-q} \, \|\mathbf{b}\|_{3,p}^q \, \|g\|_q^q \,,
\end{aligned}$$

for some universal constant C by (4.24). Hence by (4.26) we have

$$\|g_1\|_q \le C \, \frac{R - R_1}{R} \, \|\mathbf{b}\|_{3,p} \, \|g\|_q \,. \tag{4.28}$$

Putting (4.25) and (4.28) together we conclude that

$$\|Qg\|_q \le C \, \frac{R - R_1}{R} \, \|\mathbf{b}\|_{3,p} \, \|g\|_q \,,$$

and hence the result follows.

Next we put Lemma 4.2 and Proposition 4.2 together to obtain an analogue of Corollary 4.2 for the case when $R - R_1$ can be much smaller than $R_2 - R$.

Corollary 4.3. *Suppose $R/2 < R_1 < R < R_2 = 2R$. Then for any p, $1 < p \le 3$ and $q > 1$ the following holds: there exist positive constants c, ε, δ depending only on p, q such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ and*

$$\|f - \mathrm{Av}\, f\|_q \le \delta \, |\mathrm{Av}\, f| \,, \tag{4.29}$$

then

$$|\mathrm{Av}\, f_2| \geq C\, \frac{R-R_1}{R}\, |\mathrm{Av}\, f| \tag{4.30}$$

and

$$\|f_2 - \mathrm{Av}\, f_2\|_q \leq \delta\, |\mathrm{Av}\, f_2|\,. \tag{4.31}$$

PROOF. We have

$$|\mathrm{Av}\, f_2| = |\mathrm{Av}\,(P^* f) + \mathrm{Av}\,(Q^* f)| \geq |\mathrm{Av}\,(P^* f)| - C\,\varepsilon\, \frac{R-R_1}{R}\, \|f\|_q\,,$$

by Proposition 4.2. Now from the assumption (4.29) we conclude that

$$|\mathrm{Av}\, f_2| \geq |\mathrm{Av}\,(P^* f)| - \frac{C\,\varepsilon\,(R-R_1)}{R}\,(1+\delta)\,|\mathrm{Av}\, f|\,. \tag{4.32}$$

The inequality (4.30) follows now from (4.32) and (4.16) of Lemma 4.2, provided we choose ε sufficiently small. To get (4.31) observe that

$$\begin{aligned}\frac{\|f_2 - \mathrm{Av}\, f_2\|_q}{|\mathrm{Av}\, f_2|} &\leq \frac{\|P^* f - \mathrm{Av}\,(P^* f)\|_q}{|\mathrm{Av}\, f_2|} + \frac{\|Q^* f - \mathrm{Av}\,(Q^* f)\|_q}{|\mathrm{Av}\, f_2|} \\ &\leq \gamma\,\delta\, \frac{|\mathrm{Av}\,(P^* f)|}{|\mathrm{Av}\, f_2|} + \frac{2\,C\,\varepsilon\,(R-R_1)}{R}\,(1+\delta)\,\frac{|\mathrm{Av}\, f|}{|\mathrm{Av}\, f_2|}\,,\end{aligned}$$

where we have used (4.17) of Lemma 4.2 and Proposition 4.2 together with (4.29). Now from (4.32) and (4.16) it is clear that for sufficiently small ε we have

$$\frac{|\mathrm{Av}\,(P^* f)|}{|\mathrm{Av}\, f_2|} < \frac{1}{2} + \frac{1}{2\,\gamma}\,,$$

since $\gamma < 1$. Similarly we see that for sufficiently small ε there is the inequality

$$\frac{2\,C\,\varepsilon\,(R-R_1)}{R}\,(1+\delta)\,\frac{|\mathrm{Av}\, f|}{|\mathrm{Av}\, f_2|} \leq \Big(\frac{1}{2} - \frac{\gamma}{2}\Big)\delta\,.$$

Putting the last three inequalities together we conclude that (4.31) holds.

Observe that, in contrast to Corollary 4.2, we cannot expect the inequality $\|f_1 - \operatorname{Av} f_1\|_q \le \delta |\operatorname{Av} f_1|$ to hold in the situation of Corollary 4.3. The reason is that if $R - R_1$ is small then Brownian motion has a very small smoothing effect on a smooth density f. Thus the fluctuation of $P^* f$ decreases by a small amount proportional to $(R - R_1)/R$. On the other hand the perturbative part $Q^* f$ can generate high frequency modes with norm strictly larger than $(R - R_1)/R$ and hence the relative fluctuation of f_1 can be larger than that of f. We study this situation further in [3].

5. Perturbative estimates on the exit probabilities from a spherical shell with holes.

Consider a set $\mathcal{S} \subset \mathbb{R}^3$ which is a union of disjoint cubes. In this section we shall prove theorems analogous to the theorems of Section 4 for the drift process restricted to paths which do not intersect the set $\mathcal{S}$. To do this we associate with $\mathcal{S}$ a potential function $V_{\mathcal{S}}$ from which we can estimate the probability of hitting the set $\mathcal{S}$.

First we consider the case of Brownian motion $\mathbf{b} \equiv 0$. For each cube Q in $\mathcal{S}$ let $\tilde{Q}$ be the cube concentric with Q but double the size. We define a function $V_Q : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$V_Q(x) = \begin{cases} \dfrac{1}{|Q|^{2/3}}\,, & x \in \tilde{Q}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{5.1}$$

The potential $V_{\mathcal{S}}$ is then defined as

$$V_{\mathcal{S}} = \sum_{Q \subset \mathcal{S}} V_Q\,. \tag{5.2}$$

Now let $X(t)$, $t > 0$, be Brownian motion started at a point $x \in \mathbb{R}^3$. If X hits a cube $Q \subset \mathcal{S}$ then it will spend time of order $|Q|^{2/3}$ in the cube. Thus $\int_0^\infty V_Q(X(t))\,dt$ is of order 1 on paths $X(t)$ which hit Q. Hence we expect that the probability of Brownian motion hitting $\mathcal{S}$ can be estimated by the expectation of $\int_0^\infty V_{\mathcal{S}}(X(t))\,dt$ This is in fact the case.

Proposition 5.1. *Let $X(t)$ be Brownian motion in $\mathbb{R}^3$. Then there is a universal constant $C > 0$ such that*

$$P_x(X \text{ hits } \mathcal{S}) \le C\, E_x\Big[\int_0^\infty V_{\mathcal{S}}(X(t))\,dt\Big]\,.$$

PROOF. Putting $u(x) = P_x(X \text{ hits } \mathcal{S})$, $x \in \mathbb{R}^3 \backslash \mathcal{S}$ it is well known that $u(x)$ is the solution to the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0\,, & x \in \mathbb{R}^3 \backslash \mathcal{S}\,, \\ u(x) = 1\,, & x \in \partial\mathcal{S}\,. \end{cases}$$

On the other hand the function

$$w(x) = E_x\Big[\int_0^\infty V_{\mathcal{S}}(X(t))\,dt\Big] = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{V_{\mathcal{S}}(y)}{|x-y|}\,dy$$

satisfies

$$\Delta w(x) = 0\,, \qquad x \in \mathbb{R}^3 \backslash \mathcal{S}\,.$$

Suppose x is close to a boundary point of $\mathcal{S}$. Then this point is part of a cube Q. Thus

$$\lim_{x \to \partial\mathcal{S}} w(x) \geq \lim_{x \to \partial Q} \frac{1}{4\pi}\int_Q \frac{V_Q(y)}{|x-y|}\,dy \geq c > 0\,,$$

where c is a univeral constant. Consequently we have

$$u(x) \leq \frac{w(x)}{c}\,, \qquad x \in \partial\mathcal{S}\,.$$

Hence by the maximum principle we have the inequality

$$u(x) \leq \frac{w(x)}{c}\,, \qquad x \in \mathbb{R}^3 \backslash \mathcal{S}\,,$$

which proves the result.

We shall use the argument of Proposition 5.1 to prove an analogue of Corollary 4.1.

Proposition 5.2. *Suppose* $R_1 = 0$, $R_2 = 2R$, $\mathbf{b} \equiv 0$. *Let* f *be a density on the sphere* $|x| = R$ *and* f_2 *the density induced on* $|x| = R_2$ *by* f *propagated along Brownian paths which do not intersect* $\mathcal{S}$. *Then for any* q, $1 < q < \infty$, *there exists* $\delta, \eta > 0$ *depending only on* q *such that if*

$$\|f - \mathrm{Av}\, f\|_q \leq \delta\, |\mathrm{Av}\, f|$$

and

$$\mathrm{Av}_{|x|=R}\Big(E_x\Big[\int_0^{\tau_{R_2}} V_{\mathcal{S}}(X(t))\,dt\Big]\Big) < \eta\,,$$

then

$$\|f_2 - \mathrm{Av}\, f_2\|_q \le \delta |\mathrm{Av}\, f_2| \qquad \textit{and} \qquad |\mathrm{Av}\, f_2| \ge \frac{|\mathrm{Av}\, f|}{2}\,.$$

Proof. We consider the operator from functions g on $|x| = R_2$ to functions on $|x| = R$ given by

$$Ag(x) = E_x[g\,(X(\tau_{R_2}))\,,\ X(t) \in \mathcal{S}\,,\ \text{some } t,\ 0 < t < \tau_{R_2}]\,.$$

Then for any r, r', $1 < r < \infty, 1/r + 1/r' = 1$, we have

$$|Ag(x)|^r \le P_x(X(t) \in \mathcal{S},\ \text{some } t,\ 0 < t < \tau_{R_2})^{r/r'} E_x[|g(X(\tau_{R_2}))|^r]\,,$$

by Holder's inequality. Now by the property of the Poisson kernel we have that

$$E_x[|g(X(\tau_{R_2}))|^r] \le C\,\|g\|_r^r\,,$$

for some universal constant C. Hence if $r \ge r'$, we have

$$\|Ag\|_r^r \le C\,\|g\|_r^r\, \mathrm{Av}_{|x|=R}\, P_x(X(t) \in \mathcal{S},\ \text{some } t,\ 0 < t < \tau_{R_2})\,.$$

If $r < r'$ we have by Jensen,

$$\|Ag\|_r^r \le C\,\|g\|_r^r\, (\mathrm{Av}_{|x|=R}\, P_x(X(t) \in \mathcal{S},\ \text{some } t,\ 0 < t < \tau_{R_2}))^{r/r'}\,.$$

Now by the argument of Proposition 5.1 we conclude that

$$\|Ag\|_r \le C\,\|g\|_r\, \eta^{\min\{1/r, 1/r'\}}\,,$$

for some universal constant C. Thus the adjoint A^* of A is a bounded operator from $L^q(\{|x| = R\})$ to $L^q(\{|x| = R_2\})$ with norm $\|A\|$ bounded as

$$\|A\| \le C\eta^{\min\{1/q, 1/q'\}}\,,$$

for some constant C. Observe next that the densities f and f_2 are related by the equation

$$f_2 = P^* f - A^* f\,,$$

where P is the integral operator with Poisson kernel as in Section 4. Hence we have

$$\mathrm{Av}\, f_2 = \mathrm{Av}\, P^* f - \mathrm{Av}(A^* f) = \mathrm{Av}\, f - \mathrm{Av}\,(A^* f)\,.$$

Now since $q > 1$ it follows that

$$\begin{aligned}|\mathrm{Av}\,(A^* f)| &\le \|A^* f\|_q \\ &\le C\,\eta^{\min\{1/q,1/q'\}}\|f\|_q \\ &\le C\,\eta^{\min\{1/q,1/q'\}}\,(1+\delta)\,|Av\,f|\,.\end{aligned}$$

Thus by choosing η sufficiently small we have $|\mathrm{Av}\, f_2| \ge |\mathrm{Av}\, f|/2$.

Next observe that there exists γ, $0 < \gamma < 1$ such that

$$\|P^* f - \mathrm{Av}\,(P^* f)\|_q \le \gamma\,\|f - \mathrm{Av}\, f\|_q\,.$$

Hence

$$\begin{aligned}\|f_2 - \mathrm{Av}\, f_2\|_q &\le \|P^* f - \mathrm{Av}\,(P^*\, f)\|_q + \|A^* f - \mathrm{Av}\,(A^* f)\|_q \\ &\le \gamma\,\delta\,|\mathrm{Av}\, f| + 2\,C\,\eta^{\min\{1/q,1/q'\}}\,(1+\delta)\,|\mathrm{Av}\, f|\,.\end{aligned}$$

It is clear by choosing $\eta > 0$ sufficiently small that the right hand side of the last inequality is less than $\delta\,|\mathrm{Av}\, f_2|$. The result is complete.

REMARK 5.1. Observe that in Proposition 5.2 we have used the fact that if Q is a cube in $\mathcal{S}$ then $V_{\mathcal{S}}(x) \ge |Q|^{-2/3}$ on the double of Q, $\tilde{Q}$. The reason is that if Q has a small intersection with U_{R_1,R_2} then $\tilde{Q} \cap U_{R_1,R_2}$ has volume of order $|Q|$. Hence a Brownian path which hits Q makes an order 1 contribution to $\int_0^{\tau_{R_2}} V_{\mathcal{S}}(X(t))\,dt$.

Next we wish to generalize Propositions 5.1, 5.2 to the case of nontrivial drift. First we estimate the probability that the drift process visits a cube Q.

Proposition 5.3. *Let Q_m be a cube with side of length 2^{-m}, m an integer, and $P_x(Q_m)$ the probability that the process with drift $\mathbf{b}$ started at x visits Q_m before exiting to ∞. Then for any $\alpha < 1$ there exists $\varepsilon > 0$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then*

$$P_x(Q_m) \le \frac{C}{(2^m d(x, Q_m) + 1)^\alpha}\,, \tag{5.3}$$

for some universal constant C. Here $d(x, Q_m)$ is the distance from the point x to the cube Q_m.

PROOF. First we consider the solution of a boundary value problem on the shell U_{R_1,R_2} with $R_1 = R/2$ and $R_2 = 2R$. Thus we wish to estimate the solution of

$$\begin{cases} \Delta w(x) + \mathbf{b}(x) \cdot \nabla w(x) = 0\,, & x \in U_{R_1,R_2}\,, \\ w(x) = 0\,, & |x| = R_1\,, \\ w(x) = 1\,, & |x| = R_2\,. \end{cases}$$

Let w_0 be the solution when $\mathbf{b} \equiv 0$. Then, in the notation of Section 4, $w_0 = P\,1$. It is easy to see that w_0 is given by the formula

$$w_0(x) = \frac{4}{3}\Big(1 - \frac{R}{2\,|x|}\Big)\,.$$

We shall show that $\varepsilon > 0$ can be chosen so that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then there exists a universal constant $C > 0$ such that

$$|w(x) - w_0(x)| \le C\,\|\mathbf{b}\|_{3,p}\,, \qquad x \in U_{R_1,R_2}\,. \tag{5.4}$$

In fact we have

$$w(x) = w_0(x) + Q\,\mathbf{1}(x)\,, \qquad x \in U_{R_1,R_2}\,, \tag{5.5}$$

where Q is the operator (4.3). It is easy to see that if $1 < r < p$, $r \le q < 3$, the function $\mathbf{b} \cdot \nabla w_0$ is in the Morrey space M_r^q and

$$\|\mathbf{b} \cdot \nabla w_0\|_{q,r} \le C R^{3/q-2}\,\|\mathbf{b}\|_{3,p}\,,$$

for some universal constant $C > 0$. It follows then from [5, Theorem 1.2] that for ε sufficiently small

$$Q\,\mathbf{1}(x) = \int_{U_{R_1,R_2}} G_D(x,y)\,g(y)\,dy\,,$$

where $g \in M_r^q$ and

$$\|g\|_{q,r} \le C R^{3/q-2}\,\|b\|_{3,p}\,,$$

and C is a universal constant. If we take $q > 3/2$ then the inequality of (5.4) follows by standard argument.

To prove the inequality (5.3) let S_k, $k = 0, 1, 2, \ldots$ be spheres concentric with Q_m and with radius $2^k\, 2^{-m}$. Thus S_0 contains Q_m. From (5.4) we can choose ε sufficiently small such that w satisfies

$$\inf_{|x|=R} w(x) \geq \frac{2^\alpha}{(1+2^\alpha)}\,, \qquad 0 < R < \infty\,. \tag{5.6}$$

The inequality (5.3) follows immediately now from (5.6) and [5, Lemma 6.3].

Remark 5.2. Observe that in the Brownian motion case one can take $\alpha = 1$ in (5.3) but for the case of nontrivial $\mathbf{b}$ one must have $\alpha < 1$. This fact will determine our selection of the function $V_{\mathcal{S}}$ in the case of nontrivial $\mathbf{b}$.

The proof of Proposition 5.3 does not generalize to the situations we are interested in. We shall therefore give a different, more complicated proof of the Proposition which does generalize. Let us consider the region Ω_R external to the ball of radius $R > 0$ centered at the origin. The Dirichlet Green's function for this region is given by

$$G_D(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{R}{|y|}\,\frac{1}{|x-\overline{y}|}\right), \qquad |x|, |y| > R\,, \tag{5.7}$$

where $\overline{y}$ is the reflection of y in the boundary of Ω_R. We estimate G_D and its gradient $\nabla_x G_D$:

Lemma 5.1. a) *There is the inequality*

$$0 \leq G_D(x,y) \leq \frac{1}{4\pi\,|x-y|}\,, \qquad |x|, |y| > R\,.$$

b) $|\nabla_x G_D(x,y)| \leq k_1(x,y) + k_2(x,y)$, *where*

$$|k_1(x,y)| \leq \frac{C}{|x-y|^2}\,, \qquad |x|, |y| > R\,, \tag{5.8}$$

$$\begin{cases} |k_2(x,y)| \leq \dfrac{C}{|x|\,|y|}\,, & |y| > 3|x|\,,\ |x| > R\,, \\ |k_2(x,y)| = 0\,, & \text{otherwise}\,, \end{cases} \tag{5.9}$$

and C is a universal constant.

PROOF. Since a) follows easily from the maximum principle we shall just consider b). We have now

$$\nabla_x G_D(x,y) = \frac{-1}{4\pi}\Big(\frac{x-y}{|x-y|^3} - \frac{R}{|y|}\,\frac{x-\overline{y}}{|x-\overline{y}|^3}\Big)\,.$$

Since $G_D(x,y) \geq 0$ it follows that

$$|\nabla_x G_D(x,y)| \leq \frac{1}{4\pi}\Big(\frac{1}{|x-y|^2} + \frac{1}{|x-y|\,|x-\overline{y}|}\Big)\,.$$

We consider first the case $|y| > 3\,|x|$. It is easy to see that $|x-y| \geq 2\,|y|/3$ and

$$|x-\overline{y}| \geq |x| - |\overline{y}| \geq |x| - \frac{R}{3} \geq \frac{|x|}{2}\,.$$

Hence

$$\frac{1}{|x-y|\,|x-\overline{y}|} \leq \frac{3}{|x|\,|y|}\,.$$

Next consider the situation $R < |y| < 3|x|$. Suppose that $|x| > 2R$. Then

$$|x-\overline{y}| \geq |x| - R \geq \frac{|x|}{2} \geq \frac{|x|+|y|}{8} \geq \frac{|x-y|}{8}\,.$$

In the case $R < |x|$, $|y| < 2R$ it is clear that there exists a universal constant C_1 with $|x-\overline{y}| \geq C_1\,|x-y|$. We conclude then that in this situation one has

$$|\nabla_x G_D(x,y)| \leq \frac{C}{|x-y|^2}\,,$$

for some universal constant $C > 0$. The proof is complete.

Next we define Morrey spaces for the region Ω_R in a similar way to (4.20). Thus for $1 < r \leq q < \infty$ and $s > 0$ we say $g : \Omega_R \longrightarrow \mathbb{C}$ is in the Morrey space $M^q_{r,s}(\Omega_R)$ if

$$\int_{Q\cap\Omega_R} |g(x)|^r\,dx \leq C^r\,|Q|^{1-r/q}\Big(\frac{R}{d(Q)}\Big)^{rs}, \tag{5.10}$$

for all cubes Q and constant C. Here $d(Q)$ is defined by

$$d(Q) = \sup\,\{|y| : y \in Q\cap\Omega_R\}\,.$$

Evidently one has $d(Q) \geq R$. The norm of g, $\|g\|_{q,r,s}$ is then the infimum of all C such that (5.10) holds.

Lemma 5.2. *Let T_1 be the integral operator on functions with domain Ω_R which has kernel $|\mathbf{b}(x)|\, k_1(x,y)$ where $\mathbf{b} \in M_p^3$, $1 < p \le 3$ and k_1 satisfies* (5.8). *Then for $1 < r < p$, $r \le q < 3$, $s > 0$, T_1 is a bounded operator on $M_{r,s}^q(\Omega_R)$ and the norm of T_1, $\|T_1\|$ satisfies an inequality $\|T_1\| \le C\|\mathbf{b}\|_{3,p}$ where C depends only on r, p, q, s.*

PROOF. Same as for Lemma 4.4.

Lemma 5.3. *Let T_2 be the integral operator on functions with domain Ω_R which has kernel $|\mathbf{b}(x)|\, k_2(x,y)$ where $\mathbf{b} \in M_p^3$, $1 < p \le 3$ and k_2 satisfies* (5.9). *Then for $1 \le r \le p$, $r \le q$, $s \ge 0$ and $2 < 3/q + s < 3/r$, T_2 is a bounded operator on $M_{r,s}^q(\Omega_R)$ and the norm of T_2, $\|T_2\|$ satisfies an inequality $\|T_2\| \le C\|\mathbf{b}\|_{3,p}$, where C depends only on r, p, q, s.*

PROOF. For $n = 0, \pm 1, \dots$ let Q_n be the cube centered at the origin with side of length 2^{-n}. If $u : \Omega_R \longrightarrow \mathbb{C}$ is a locally integrable function we denote by u_{Q_n} the average value of $|u|$ on Q_n, whence

$$u_{Q_n} = |Q_n|^{-1} \int_{\Omega_R \cap Q_n} |u(x)|\, dx\,.$$

Hence we have

$$|T_2 u(x)| \le \frac{C\,|\mathbf{b}(x)|}{|x|} \sum_{|x| < 2^{-n}} 2^{-2n}\, u_{Q_n}\ ,$$

for some universal constant C. Hence for $2^{-m} > R$, we have

$$\begin{aligned}\int_{Q_m \cap \Omega_R} |T_2 u(x)|^r\, dx &\le C^r \sum_{k=m}^{\infty} \int_{Q_k} \Big(|\mathbf{b}(x)|\, 2^k \sum_{n=-\infty}^{k} 2^{-2n}\, u_{Q_n} \Big)^r dx \\ &\le C^r\, \|\mathbf{b}\|_{3,p}^r \sum_{k=m}^{\infty} 2^{k(2r-3)} \Big(\sum_{n=-\infty}^{k} 2^{-2n}\, u_{Q_n} \Big)^r .\end{aligned}$$

Observe next that

$$\begin{aligned}\sum_{n=-\infty}^{k} 2^{-2n}\, u_{Q_n} &\le \|u\|_{q,r,s} \sum_{n=-\infty}^{k} 2^{n(-2+3/q+s)} R^s \\ &\le C_1\, R^s\, \|u\|_{q,r,s}\, 2^{k(-2+3/q+s)}\,,\end{aligned}$$

since $2 < 3/q + s$. Thus we have

$$\int_{Q_m} |T_2 u(x)|^r \, dx \le C^r \, C_1^r \, R^{sr} \sum_{k=m}^{\infty} \|\mathbf{b}\|_{3,p}^r \, \|u\|_{q,r,s}^r \, 2^{k(3r/q+sr-3)}$$
$$\le C_2^r \, R^{sr} \, \|\mathbf{b}\|_{3,p}^r \, \|u\|_{q,r,s}^r \, 2^{m(3r/q+sr-3)} \,,$$

since $3/q + s < 3/r$. Consequently, we have

$$\int_{Q_m} |T_2 u(x)|^r \, dx \le C_2^r \, \|\mathbf{b}\|_{3,p}^r \, \|u\|_{q,r,s}^r \, |Q_m|^{1-r/q} \Big(\frac{R}{d(Q_m)} \Big)^{sr} .$$

We have shown therefore that (5.10) holds for cubes centered at the origin. It is easy now to generalize the previous argument to all cubes.

PROOF OF PROPOSITION 5.3. Evidently $P_x(Q_m)$ is bounded above by the probability that the drift process started at x hits the ball concentric with Q_m of radius $R = 2^{-m}$. For Brownian motion this probability is given by $w_0(x)$, where

$$\begin{cases} \Delta w_0(x) = 0 \,, & |x| > R \,, \\ w_0(x) = 1 \,, & |x| = R \,. \end{cases}$$

Thus $w_0(x) = R/|x|$, $|x| > R$. For the drift process it is given by $w(x)$, where

$$w(x) = w_0(x) + \int_{\Omega_R} G_D(x,y) \, (I - T)^{-1} \, \mathbf{b} \cdot \nabla w_0(y) \, dy \,. \tag{5.11}$$

Here G_D is the Green's function (5.7) and T is the integral operator with kernel $\mathbf{b}(x) \cdot \nabla_x G_D(x,y)$. We wish to show that the function $\mathbf{b} \cdot \nabla w_0$ is in an appropriate Morrey space $M^q_{r,s}(\Omega_R)$. Evidently one has $|\mathbf{b}(x) \cdot \nabla w_0(x)| \le R \, |\mathbf{b}(x)|/|x|^2$. Now for the cube Q_n with side of length $2^{-n} > R$ centered at the origin one has

$$\begin{aligned} \int_{Q_n} \Big(\frac{R \, |\mathbf{b}(x)|}{|x|^2} \Big)^r \, dx &\le C \sum_{j=n}^{m} \|\mathbf{b}\|_{3,p}^r \, R^r \, 2^{-j(3-3r)} \\ &\le C_1^r \, \|\mathbf{b}\|_{3,p}^r \, R^{3-2r} \,, \end{aligned} \tag{5.12}$$

for some constant C_1, provided $1 < r \le p$. On the other hand if Q is a cube such that $d(Q) \gg |Q|^{1/3}$ then we have

$$\begin{aligned}\int_Q \Big(\frac{R\,|\mathbf{b}(x)|}{|x|^2}\Big)^r dx &\le \frac{R^r}{d(Q)^{2r}}\, \|\mathbf{b}\|_{3,p}^r\, |Q|^{1-r/3} \\ &\le R^{r(3/q-2)}\Big(\frac{R}{d(Q)}\Big)^{r(3-3/q)} \|\mathbf{b}\|_{3,p}^r\, |Q|^{1-r/q} \\ &\le R^{r(3/q-2)}\Big(\frac{R}{d(Q)}\Big)^{rs} \|\mathbf{b}\|_{3,p}^r\, |Q|^{1-r/q}\,,\end{aligned}$$

for any r, s, q with $1 \le r \le p$, $q \le 3$, $s \le 3 - 3/q$. Combining this last inequality with (5.12), we see that if r, s, q satisfy the inequalities

$$1 < r \le p\,, \qquad r \le q \le 3\,, \qquad s \le 3\Big(\frac{1}{r} - \frac{1}{q}\Big)\,, \tag{5.13}$$

then $\mathbf{b} \cdot \nabla w_0$ is in $M^q_{r,s}(\Omega_R)$ and

$$\|\mathbf{b} \cdot \nabla w_0\|_{q,r,s} \le C R^{3/q-2}\, \|\mathbf{b}\|_{3,p}\,,$$

for some constant C depending only on q, r, s.

Observe next that for any s, $0 < s < 1$, it is possible to find r, q such that $3/2 < q < 3$ as well as the inequalities (5.13) and the conditions of lemmas 5.2, 5.3 hold. Hence the function

$$g(x) = (I - T)^{-1}\, \mathbf{b} \cdot \nabla w_0$$

is also in $M^q_{r,s}(\Omega_R)$ for sufficiently small ε and has norm which satisfies

$$\|g\|_{q,r,s} \le C R^{3/q-2}\, \|\mathbf{b}\|_{3,p}\,,$$

for a constant C depending only on q, r, s. Now let us suppose that $|x| > 2R$. Then from (5.11) we have

$$\begin{aligned}|w(x) - w_0(x)| &\le \int_{\Omega_R} G_D(x,y)\, |g(y)|\, dy \\ &\le \int_{|x-y|<|x|/2} dy + \int_{|y|<|x|/2} dy \\ &\quad + \int_{\{|x-y|>|x|/2,\, |y|>|x|/2\}} dy \\ &= I_1 + I_2 + I_3\,.\end{aligned} \tag{5.14}$$

If we take now $2^{-n_0} \sim |x|$ for a suitable integer n_0 we have

$$\begin{aligned} I_1 &\le C \sum_{k=n_0}^{\infty} 2^k \int_{|x-y|<2^{-k}} |g(y)|\, dy \\ &\le C \sum_{k=n_0}^{\infty} 2^k\, R^{3/q-2}\, \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^s\, 2^{-3k(1-1/q)} \\ &\le C\, 2^{n_0(3/q-2)}\, R^{3/q-2}\, \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^s \\ &= C\, \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^{s+3/q-2}, \end{aligned}$$

since $q > 3/2$.

On the other hand we have

$$\begin{aligned} I_2 &\le C\, 2^{n_0} \sum_{k=n_0}^{m} \int_{2^{-k}<|y|<2^{-k+1}} |g(y)|\, dy \\ &\le C\, 2^{n_0} \sum_{k=n_0}^{m} R^{3/q-2} \Big(\frac{R}{2^{-k}}\Big)^s \|\mathbf{b}\|_{3,p}\, 2^{-3k(1-1/q)} \\ &\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^{s+3/q-2}, \end{aligned}$$

since $s + 3/q > 3$. Finally we have

$$\begin{aligned} I_3 &\le C \sum_{k=-\infty}^{n_0} 2^k \int_{2^{-k}<|x-y|<2^{-k+1}} |g(y)|\, dy \\ &\le C \sum_{k=-\infty}^{n_0} 2^k R^{3/q-2} \Big(\frac{R}{2^{-k}}\Big)^s \|\mathbf{b}\|_{3,p}\, 2^{-3k(1-1/q)} \\ &\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{R}{2^{-n_0}}\Big)^{s+3/q-2}, \end{aligned}$$

provided $s + 3/q - 2 > 0$. Now it is easy to see that we can choose s, q, r appropriately to make $s + 3/q - 2$ as close to 1 as we please. The inequality (5.3) easily follows from this.

Next we consider a cube Q_m with side of length 2^{-m} which is contained in the ball U_{0,R_2} of radius R_2. For $x \in U_{0,R_2}$ let $P_x(Q_m)$ be given now by

$$\begin{aligned} P_x(Q_m) = {} & \text{probability that the drift process started at } x \\ & \text{hits } Q_m \text{ before hitting the boundary of } U_{0,R_2} \, . \end{aligned}$$

It is easy to estimate this probability in the case of Brownian motion $\mathbf{b} \equiv 0$. In fact by the argument of Proposition 5.1 it is bounded by

$$P_x(Q_m) \leq C \int_{Q_m} 2^{2m} \, G_D(x,y) \, dy \, , \tag{5.15}$$

where G_D is the Dirichlet Green's kernel on U_{0,R_2} and C is a universal constant. Since G_D is given explicitly it is easy to estimate the right hand side of (5.15). Let $d(Q_m)$ be defined by

$$d(Q_m) = \sup \left\{ d(y, \partial U_{0,R_2}) : y \in Q_m \right\} .$$

Then we see from (5.15) that

$$P_x(Q_m) \leq \frac{C}{2^m d(x,Q_m) + 1} \, \min \left\{ 1, \frac{2^m \, d(Q_m)}{2^m \, d(x,Q_m) + 1} \right\} , \tag{5.16}$$

where C is a universal constant. In view of Proposition 5.3 it would seem that one could generalize (5.16) to the case of nontrivial $\mathbf{b}$ by

$$P_x(Q_m) \leq \frac{C_\alpha}{(2^m \, d(x,Q_m) + 1)^\alpha} \, \min \left\{ 1, \frac{2^m \, d(Q_m)}{2^m \, d(x,Q_m) + 1} \right\} , \tag{5.17}$$

where $0 < \alpha < 1$ and the constant C_α depends on α. We shall prove the inequality (5.17) following the same lines as the second proof of Proposition 5.3.

Let $B_a(\rho)$ be the ball of radius ρ centered at the point a. Suppose $a \in U_{0,R_2}$, the ball of radius R_2 centered at the origin and the distance from a to $\partial U_{0,R_2}$ is larger than 3ρ. Let w_0 be the solution of the Dirichlet problem

$$\begin{cases} \Delta w_0(x) = 0 \, , & x \in U_{0,R_2} \backslash B_a(\rho) \, , \\ w_0(x) = 1 \, , & x \in \partial B_a(\rho) \, , \\ w_0(x) = 0 \, , & x \in \partial U_{0,R_2} \, . \end{cases} \tag{5.18}$$

Lemma 5.4. *There is a universal constant $C > 0$ such that for $x \in U_{0,R_2} \backslash B_a(\rho)$*

$$|\nabla w_0(x)| \leq \frac{C\rho}{|x-a|^2} \min\Big\{1, \frac{d(a, \partial U_{0,R_2})}{|x-a|}\Big\}. \tag{5.19}$$

PROOF. Let $G_D(x,y)$, $x, y \in U_{0,R_2}$ be the Dirichlet Green's function for the ball. Then just as in Proposition 5.1, there exists a universal constant C such that

$$w_0(x) \leq C\rho^{-2} \int_{B_a(\rho)} G_D(x,y)\, dy\,, \qquad x \in U_{0,R_2} \backslash B_a(\rho)\,.$$

It is easy to estimate $w_0(x)$ from the last inequality since we have an explicit formula for G_D. Thus there is a universal constant $C > 0$ such that

$$w_0(x) \leq \frac{C\rho}{|x-a|} \min\Big\{1, \frac{d(a, \partial U_{0,R_2})}{|x-a|}\Big\}. \tag{5.20}$$

We obtain the estimate (5.19) from (5.20) and the Harnack principle. First let us consider the case where $\rho < |x-a| < 3\rho/2$. Now the function w_0 can be extended in a harmonic way inside the ball $B_a(\rho)$ by using the Kelvin transform [1]. Hence w_0 is harmonic in the region $\rho/2 < |x-a| < 7\rho/4$ and $\|w_0\|_\infty \leq C$ for some universal constant C. It follows then from the Harnack principle that

$$|\nabla w_0(x)| \leq \frac{C}{\rho}\,, \qquad \rho < |x-a| < \frac{3\rho}{2}\,,$$

for a suitable universal constant $C > 0$.

Next we consider the situation where

$$\frac{3\rho}{2} < |x-a| < \frac{d(a, \partial U_{0,R_2})}{2}\,.$$

Then w_0 is harmonic in the ball $|y-x| \leq |x-a|/4$. In fact we have

$$|x-a| \leq |x-y| + |y-a| \leq \frac{|x-a|}{4} + |y-a|\,,$$

whence $|y-a| \geq 3\,|x-a|/4 \geq 9\rho/8 > \rho$. On the other hand

$$|y-a| \leq |x-y| + |x-a| \leq \frac{5\,|x-a|}{4} \leq \frac{5\,d(a, \partial U_{0,R_2})}{8} < d(a, \partial U_{0,R_2})\,.$$

It follows easily now from (5.20) and the inequality $|y-a| \geq 3\,|x-a|/4$ that $|\nabla w_0(x)| \leq C\rho/|x-a|^2$ for some universal constant C.

Finally we consider the situation $|x-a| > d(a, \partial U_{0,R_2})/2$. Using the Kelvin transformation the function w_0 can be extended in a harmonic way to the entire ball $|x-y| \leq |x-a|/4$. Now, using Harnack and the estimate (5.20) we conclude that there is a universal constant C such that

$$|\nabla w_0(x)| \leq C\rho\, \frac{d(a, \partial U_{0,R_2})}{|x-a|^3}\,.$$

All cases of the inequality (5.19) are now covered.

Let $G_{D,1}(x,y)$ be the Dirichlet Green's function for the domain $U_{0,R_2} \backslash B_a(\rho)$. We wish to prove an analogue of Lemma 5.1.

Lemma 5.5. a) *Let $d = d(a, \partial U_{0,R_2})$. Then there is a universal constant C such that*

$$0 \leq G_{D,1}(x,y) \leq \frac{C}{|x-y|}\,\min\Big\{1, \frac{|y-a|+d}{|x-y|}\Big\}. \tag{5.21}$$

b) $|\nabla_x G_{D,1}(x,y)| \leq k_1(x,y) + k_2(x,y)$, *where*

$$|k_1(x,y)| \leq \frac{C}{|x-y|^2}\,\min\Big\{1, \frac{|y-a|+d}{|x-y|}\Big\}, \tag{5.22}$$

$$|k_2(x,y)| \leq \frac{C}{|x-a|\,|y-a|}\,\min\Big\{1, \frac{|x-a|+d}{|y-a|}\Big\}, \tag{5.23.1}$$

if $|y-a| > 3\,|x-a|$ and

$$|k_2(x,y)| = 0\,, \tag{5.23.2}$$

otherwise.

PROOF. a) Let $G_D(x,y)$ be the Dirichlet Green's function for the ball U_{0,R_2}. Then we have the inequality

$$0 \leq G_D(x,y) \leq \frac{C}{|x-y|}\,\min\Big\{1, \frac{d(y, \partial U_{0,R_2})}{|x-y|}\Big\},$$

for some universal constant C. The inequality (5.21) follows now from the fact that

$$0 \leq G_{D,1}(x,y) \leq G_D(x,y)\,, \qquad d(y, \partial U_{0,R_2}) \leq |y-a| + d\,.$$

b) Consider first the situation $|y-a| < 3\,|x-a|$. Then we have

$$|x-y| \leq |x-a| + |y-a| \leq 4\,|x-a|\,.$$

Consider next the ball $B_x(|x-y|/8)$ centered at x with radius $|x-y|/8$. For $z \in B_x(|x-y|/8)$ we have

$$|z-y| \geq |x-y| - |z-x| \geq \frac{7\,|x-y|}{8}$$

and

$$|z-a| \geq |x-a| - |x-z| \geq |x-a| - \frac{|x-y|}{8} \geq \frac{|x-a|}{2}\,.$$

Hence if $|x-a| > 2\rho$ the ball $B_x(|x-y|/8)$ does not intersect $B_a(\rho)$. Furthermore, the function $u(z) = G_{D,1}(z,y)$ can be extended in a harmonic way by the Kelvin transform to the entire ball $B_x(|x-y|/8)$. From (5.21) it follows that the L^∞ norm of u, $\|u\|_\infty$, on this ball satisfies

$$\|u\|_\infty \leq \frac{C}{|x-y|}\ \min\Big\{1, \frac{|y-a|+d}{|x-y|}\Big\}\,.$$

The inequality (5.22) follows now from this last inequality by the Harnack principle. To deal with the situation $|x-a| \leq 2\rho$ observe that the inequality (5.22) is just the same as $k_1(x,y) \leq C/|x-y|^2$.

We get this last inequality by exactly the same argument as before, extending the harmonic function $G_{D,1}(z,y)$ into the ball $B_a(\rho)$ as necessary.

Finally we consider the case $|y-a| > 3\,|x-a|$. As in Lemma 5.1 it follows that $|y-x| > 2\,|y-a|/3$. For $z \in B_x(|x-a|/4)$ we have

$$|y-z| \geq |y-x| - |z-x| \geq \frac{2\,|y-a|}{3} - \frac{|x-a|}{4} \geq \frac{7\,|y-a|}{12}\,.$$

Furthermore, $|z-a| \leq 5\,|x-a|/4$. Now consider again the function $u(z) = G_{D,1}(z,y)$ which can be continued in a harmonic way to the entire ball $B_x(|x-a|/4)$. By the symmetry of $G_{D,1}$ it follows from (5.21) that

$$0 \leq u(z) \leq \frac{C}{|z-y|}\ \min\Big\{1, \frac{|z-a|+d}{|z-y|}\Big\}\,.$$

The inequality (5.23) follows now from this last inequality and the Harnack principle.

Next let Ω_ρ be the domain

$$\Omega_\rho = \{x \in \mathbb{R}^3 : |x-a| > \rho\}\,.$$

We define Morrey spaces on Ω_ρ which generalize (5.10). For $1 < r \leq q < \infty$ and $s > 0$ we say that $g : \Omega_\rho \longrightarrow \mathbb{C}$ is in the weighted Morrey space $M^q_{r,s}(\Omega_\rho)$ with weight w if

$$\int_{Q\cap\Omega_\rho} w(x)^r\, |g(x)|^r\, dx \leq C^r |Q|^{1-r/q} \Big(\frac{\rho}{d(Q)}\Big)^{rs}\,, \tag{5.24}$$

for all cubes Q and constant C. Here $d(Q) = \sup\{|x-a| : x \in Q\cap\Omega_\rho\}$. The norm of g, $\|g\|_{q,r,s}$ is then the infimum of all C such that (5.24) holds.

Lemma 5.6. *Let T_1 be the integral operator on functions with domain Ω_ρ which has kernel $|\mathbf{b}(x)|\, k_1(x,y)$ where $\mathbf{b} \in M^3_p$, $1 < p \leq 3$ and k_1 satisfies* (5.22)*. Then for $1 < r < p$, $r \leq q < 3$, $s > 0$, T_1 is a bounded operator on the weighted Morrey space $M^q_{r,s}(\Omega_\rho)$ with weight w given by*

$$w(x) = \frac{1}{\min\Big\{1, \dfrac{d}{|x-a|}\Big\}}\,, \qquad x \in \Omega_\rho\,. \tag{5.25}$$

The norm $\|T_1\|$ of T_1 satisfies an inequality $\|T_1\| \leq C\,\|\mathbf{b}\|_{3,p}$, where C depends only on r, p, q, s.

Proof. We proceed in a similar way to the proof of Proposition 2.1. Consider a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. For $u : \Omega_\rho \longrightarrow \mathbb{C}$ we define u_Q by

$$u_Q = \frac{d(Q)}{d}\, |Q|^{-1} \int_{\Omega_\rho\cap Q} |u(x)|\, dx\,, \qquad |Q| < d^3\,,$$

$$u_Q = |Q|^{-1} \int_{\Omega_\rho\cap Q} w(x)\, |u(x)|\, dx\,, \qquad |Q| > d^3\,.$$

Let $n \in \mathbb{Z}$ and $S_n u(x)$ be given by

$$S_n u(x) = 2^{-n} \Big(\frac{d}{d(Q_n)}\Big) u_{Q_n}\,, \qquad x \in Q_n\,,$$

where Q_n is the unique dyadic cube with side of length 2^{-n} containing x. The operator S on functions $u : \Omega_\rho \longrightarrow \mathbb{C}$ is then defined as

$$Su(x) = \sum_{n=-\infty}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,, \qquad x \in \Omega_\rho\,. \tag{5.26}$$

Now we can think of the dyadic decomposition as being centered at some point $\xi \in \mathbb{R}^3$. The operator S of (5.26) should therefore be more accurately written as S_ξ. Then, in analogy to (2.2) we have

$$\int_{Q\cap\Omega_\rho} w(x)^r\, |T_1 u(x)|^r\, dx \le \frac{C^r}{|\Lambda|} \int_\Lambda d\xi \int_{Q\cap\Omega_\rho} w(x)^r\, |S_\xi u(x)|^r\, dx\,,$$

where Λ is a sufficiently large cube and C is a universal constant. This follows from the inequality (5.22). We can therefore restrict ourselves to showing that S_ξ is a bounded operator on the weighted Morrey space for an arbitrary ξ. Let n_0 be the smallest integer n such that $2^{-n} < d$. Then we may write $S_\xi = A + B$ where

$$Au(x) = \sum_{n=n_0}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,, \qquad x \in \Omega_\rho\,.$$

Suppose Q_m is a dyadic cube with side of length 2^{-m} where $m \ge n_0$. Then $\sup w / \inf w$ is bounded above by a universal constant on Q_m. We write $Au(x) = A_1 u(x) + A_2 u(x)$, for $x \in Q_m$ where

$$A_1 u(x) = \sum_{n=m}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,.$$

Then we have

$$\begin{aligned}
\int_{Q_m\cap\Omega_\rho} w(x)^r\, |A_1 u(x)|^r\, dx &\le (\sup w)^r \int_{Q_m\cap\Omega_\rho} |A_1 u(x)|^r\, dx \\
&\le (\sup w)^r\, C_1^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q_m\cap\Omega_\rho} |u(x)|^r\, dx \qquad (5.27)\\
&\le C_2^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q_m\cap\Omega_\rho} w(x)^r\, |u(x)|^r\, dx\,,
\end{aligned}$$

where C_1 and C_2 are constants depending only on $r < p$. Here we are using the boundedness of the operator A_1 as given in [5, Theorem 1.2].

Since $\sup w/\inf w$ is bounded above on the dyadic cube Q_{n_0} with side of length 2^{-n_0} which contains Q_m we have

$$\begin{aligned}
|A_2u(x)| &\le |\mathbf{b}(x)| \sum_{n=n_0}^{m} 2^{2n} \int_{Q_n\cap\Omega_\rho} |u(y)|\,dy \\
&\le \frac{C\,|\mathbf{b}(x)|}{\sup w} \sum_{n=n_0}^{m} 2^{2n} \int_{Q_n\cap\Omega_\rho} w(y)\,|u(y)|\,dy \\
&\le \frac{C\,|\mathbf{b}(x)|}{\sup w} \sum_{n=n_0}^{m} 2^{n(3/r-1)} \Big(\int_{Q_n\cap\Omega_\rho} w(y)^r\,|u(y)|^r\,dy\Big)^{1/r} \\
&\le \frac{C\,|\mathbf{b}(x)|}{\sup w} \sum_{n=n_0}^{m} 2^{n(3/r-1)}\, C\, 2^{-3n(1/r-1/q)} \Big(\frac{\rho}{d(Q_n)}\Big)^s \\
&\le \frac{C_1\,|\mathbf{b}(x)|}{\sup w}\, |Q_m|^{1/3-1/q} \Big(\frac{\rho}{d(Q_m)}\Big)^s .
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_{Q_m\cap\Omega_\rho} w(x)^r\,|A_2u(x)|^r\,dx \\
&\qquad \le C_2^r\,|Q_m|^{r/3-r/q}\Big(\frac{\rho}{d(Q_m)}\Big)^{rs} \int_{Q_m} |\mathbf{b}(x)|^r\,dx \\
&\qquad \le C_3^r\,\|\mathbf{b}\|_{3,p}^r\,|Q_m|^{1-r/q}\Big(\frac{\rho}{d(Q_m)}\Big)^{rs} .
\end{aligned} \tag{5.28}$$

If we put this last inequality together with (5.27) we conclude that

$$\int_{Q_m\cap\Omega_\rho} w(x)^r\,|Au(x)|^r\,dx \le C_4^r\,\|\mathbf{b}\|_{3,p}^r\,|Q_m|^{1-r/q}\Big(\frac{\rho}{d(Q_m)}\Big)^{rs} .$$

Suppose next that $m < n_0$. Then we have

$$\begin{aligned}
&\int_{Q_m\cap\Omega_\rho} w(x)^r\,|Au(x)|^r\,dx \\
&\qquad = \sum_{Q_{n_0}\subset Q_m} \int_{Q_{n_0}} w(x)^r\,|Au(x)|^r\,dx \\
&\qquad \underset{\text{by (5.27)}}{\le} \sum_{Q_{n_0}\subset Q_m} C^r\,\|\mathbf{b}\|_{3,p}^r \int_{Q_{n_0}} w(x)^r\,|u(x)|^r\,dx
\end{aligned}$$

$$= C^r \, \|\mathbf{b}\|_{3,p}^r \int_{Q_m} w(x)^r \, |u(x)|^r \, dx \, .$$

We conclude therefore that if $m > n_0$ then the inequality (5.28) holds. Therefore the operator A is bounded on the weighted Morrey space and $\|A\| \le C \, \|\mathbf{b}\|_{3,p}$ for some constant C depending only on r, p, q, s.

Next we turn to the operator B. To bound it we follow the same strategy as in Lemma 2.1 and Corollary 2.1. Observe that $Bu(x)$ is constant for $x \in Q_{n_0}$ where Q_{n_0} is an arbitrary dyadic cube with side of length 2^{-n_0}. We can bound $Bu(x)$ by

$$|Bu(x)| \le |\mathbf{b}(x)| \sum_{n=-\infty}^{n_0-1} 2^{2n} \, \frac{d}{d(Q_n)} \int_{\Omega_\rho \cap Q_n} w(y) \, |u(y)| \, dy \, ,$$

where the Q_n are the unique dyadic cubes with side of length 2^{-n} containing Q_{n_0}. Hence we have

$$\begin{aligned}
|Bu(x)| &\le |\mathbf{b}(x)| \sum_{n=-\infty}^{n_0-1} 2^{2n} \, \frac{d}{d(Q_n)} \, |Q_n|^{1-1/r} \Big(\int_{\Omega_\rho \cap Q_n} w(y)^r \, |u(y)|^r \, dy \Big)^{1/r} \\
&\le |\mathbf{b}(x)| \sum_{n=-\infty}^{n_0-1} 2^{n(3/r-1)} \, \frac{d}{d(Q_n)} \, C \, |Q_n|^{1/r-1/q} \Big(\frac{\rho}{d(Q_n)} \Big)^s \\
&= C \, |\mathbf{b}(x)| \sum_{n=-\infty}^{n_0-1} 2^{n(3/q-1)} \, \frac{d}{d(Q_n)} \Big(\frac{\rho}{d(Q_n)} \Big)^s \\
&\le C_1 \, |\mathbf{b}(x)| \, 2^{n_0(3/q-1)} \, \frac{d}{d(Q_{n_0})} \Big(\frac{\rho}{d(Q_{n_0})} \Big)^s .
\end{aligned}$$

Let Q_m be a dyadic cube with $m > n_0$. Then if $Q_m \subset Q_{n_0}$ we have

$$\begin{aligned}
\int_{Q_m \cap \Omega_\rho} w(x)^r \, |Bu(x)|^r \, dx \le \max \Big\{ 1, \frac{d(Q_m)}{d} \Big\}^r & \\
\cdot \Big(C_1 \, 2^{n_0(3/q-1)} \, \frac{d}{d(Q_{n_0})} \Big(\frac{\rho}{d(Q_{n_0})} \Big)^s \Big)^r & \\
\cdot \int_{Q_m} |\mathbf{b}(x)|^r \, dx & \\
\le C_1^r \, \|\mathbf{b}\|_{3,p}^r \, |Q_m|^{1-r/q} \Big(\frac{\rho}{d(Q_m)} \Big)^{rs} , &
\end{aligned}$$

since

$$d(Q_{n_0}) \geq d(Q_m)\,, \qquad |Q_{n_0}| \geq |Q_m|\,.$$

Next we consider dyadic cubes Q_m with $m \leq n_0$. Putting $Q' = Q_m$, one can easily verify the analogue of Lemma 2.1. Thus there are constants $\varepsilon, C > 0$, depending only on r and p such that

$$|Q|^{1/3+\varepsilon}\, u_Q \leq |Q'|^{1/3+\varepsilon}\, u_{Q'}\ ,$$

for all dyadic subcubes Q of Q' with $|Q| \geq 2^{-3n_0}$ implies the inequality

$$\int_{Q'\cap\Omega_\rho} w(x)^r \Big(\sum_{n=n_{Q'}}^{n_0} |\mathbf{b}(x)|\, S_n u(x) \Big)^r dx \leq C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'|\, u_{Q'}^r\ .$$

Now the analogue of Corollary 2.1 yields

$$\int_{Q'\cap\Omega_\rho} w(x)^r \Big(\sum_{n=n_{Q'}}^{n_0} |\mathbf{b}(x)|\, S_n u(x) \Big)^r dx \\ \leq C^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q'\cap\Omega_\rho} w(x)^r\, |u(x)|^r\, dx\,,$$

for some constant C depending only on r, p. We conclude therefore that

$$\int_{Q'\cap\Omega_\rho} w(x)^r \Big(\sum_{n=n_{Q'}}^{n_0} |\mathbf{b}(x)|\, S_n u(x) \Big)^r dx \\ \leq C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'|^{1-r/q} \Big(\frac{\rho}{d(Q')} \Big)^{rs}\,,$$

by virtue of the fact that u is in the weighted Morrey space. Finally we see just as in Lemma 4.4 that

$$\int_{Q'\cap\Omega_\rho} w(x)^r \Big(\sum_{n=-\infty}^{n_{Q'}-1} |\mathbf{b}(x)|\, S_n u(x) \Big)^r dx \\ \leq C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'|^{1-r/q} \Big(\frac{\rho}{d(Q')} \Big)^{rs}\,.$$

Hence the operator B is bounded on the weighted Morrey space. Since the operator A is also bounded it follows that T_1 is bounded.

Lemma 5.7. *Let T_2 be the integral operator on functions with domain Ω_ρ which has kernel $|\mathbf{b}(x)|\, k_2(x,y)$ where $\mathbf{b} \in M_p^3$ and k_2 satisfies* (5.23)*. Then for $1 \le r \le p$, $r \le q$, $s \ge 0$ and $2 < 3/q + s < 3/r$, T_2 is a bounded operator on the weighted Morrey space $M_{r,s}^q(\Omega_\rho)$ with weight w given by* (5.25)*. The norm $\|T_2\|$ of T_2 satisfies an inequality $\|T_2\| \le C\, \|\mathbf{b}\|_{3,p}$ where C depends only on r, p, q, s.*

Proof. We follow the same lines as the proof of Lemma 5.3. Thus for $n = 0, \pm 1, \ldots$, let Q_n be the cube centered at a with side of length 2^{-n} and assume that the integer n_0 satisfies $2^{-n_0} \sim d$. Then if $|x-a| < d$ we have the inequality

$$|T_2 u(x)| \le \frac{C\, |\mathbf{b}(x)|}{|x-a|} \sum_{|x-a|<2^{-n}<d} 2^{-2n}\, u_{Q_n} + \frac{C\, |\mathbf{b}(x)|\, d}{|x-a|} \sum_{n=-\infty}^{n_0} 2^{-n}\, u_{Q_n}\ ,$$

where C is a constant and u_{Q_n} is an average of u on Q_n given by

$$u_{Q_n} = |Q_n|^{-1} \int_{Q_n \cap \{|x-a| > 2^{-n-2}\}} |u(x)|\, dx\, .$$

Thus if $m > n_0$ we have

$$\begin{aligned}
\int_{Q_m \cap \Omega_\rho} & w(x)^r\, |T_2 u(x)|^r\, dx \\
& \le \int_{Q_m \cap \Omega_\rho} |T_2 u(x)|^r\, dx \\
& \le C^r \sum_{k=m}^{\infty} \int_{Q_k} \Big(|\mathbf{b}(x)|\, 2^k \sum_{n=n_0}^{k} 2^{-2n}\, u_{Q_n} \Big)^r dx \\
& \quad + C^r \sum_{k=m}^{\infty} \int_{Q_k} \Big(|\mathbf{b}(x)|\, 2^k \sum_{n=-\infty}^{n_0} d\, 2^{-n}\, u_{Q_n} \Big)^r dx\, .
\end{aligned} \tag{5.29}$$

Arguing as in Lemma 5.3 we see that

$$\begin{aligned}
\sum_{k=m}^{\infty} \int_{Q_k} & \Big(|\mathbf{b}(x)|\, 2^k \sum_{n=n_0}^{k} 2^{-2n}\, u_{Q_n} \Big)^r dx \\
& \le C_1^r\, \rho^{sr}\, \|\mathbf{b}\|_{3,p}^r\, \|u\|_{q,r,s}^r\, 2^{m(3r/q + sr - 3)}\, ,
\end{aligned}$$

since we are assuming $3/q+s<3/r$. To bound the second term on the right in (5.29) we estimate

$$\begin{aligned}\sum_{n=-\infty}^{n_0} d\,2^{-n}\,u_{Q_n} &\le \|u\|_{q,r,s}\sum_{n=-\infty}^{n_0} d^2\,2^{n(3/q+s)}\,\rho^s\\ &\le C_1\,\rho^s\,\|u\|_{q,r,s}\,d^2\,2^{n_0(3/q+s)}\\ &= C_1\,\rho^s\,\|u\|_{q,r,s}\,d^{2-3/q-s}\,,\end{aligned}$$

since $0<3/q+s$.

Hence

$$\begin{aligned}&\sum_{k=m}^{\infty}\int_{Q_k}\Big(|\mathbf{b}(x)|\,2^k\sum_{n=-\infty}^{n_0} d\,2^{-n}\,u_{Q_n}\Big)^r dx\\ &\qquad\le C_1^r\,\rho^{sr}\,\|u\|_{q,r,s}^r\,d^{(2-3/q-s)r}\sum_{k=m}^{\infty}\|\mathbf{b}\|_{3,p}^r\,2^{k(2r-3)}\\ &\qquad\le C_1^r\,\rho^{sr}\,\|\mathbf{b}\|_{3,p}^r\,\|u\|_{q,r,s}^r\,2^{m(3r/q+sr-3)}\,,\end{aligned}$$

since $2<3/q+s$. We conclude then that if $m>n_0$ there is the estimate

$$\begin{aligned}&\int_{Q_m\cap\Omega_\rho} w(x)^r\,|T_2u(x)|^r\,dx\\ &\qquad\le C_2^r\,\|\mathbf{b}\|_{3,p}^r\,\|u\|_{q,r,s}^r\,|Q_m|^{1-r/q}\Big(\frac{\rho}{d(Q_m)}\Big)^{sr}.\end{aligned}\tag{5.30}$$

Next we consider the case $m\le n_0$. Observe that if $|x-a|>d$ then

$$|T_2u(x)|\le C\,|\mathbf{b}(x)|\sum_{|x-a|<2^{-n}} 2^{-n}\,u_{Q_n}\ .$$

Hence we have for $k\le n_0$

$$\begin{aligned}&\int_{Q_k\cap\{|x-a|>2^{-k-2}\}} w(x)^r\,|T_2u(x)|^r\,dx\\ &\qquad\le C^r\Big(\sum_{n=-\infty}^{k+2} d^{-1}\,2^{-n-k}\,u_{Q_n}\Big)^r\int_{Q_k}|\mathbf{b}(x)|^r\,dx.\end{aligned}$$

We have now

$$\sum_{n=-\infty}^{k+2} d^{-1}\, 2^{-n-k}\, u_{Q_n} \le \sum_{n=-\infty}^{k+2} 2^{-k}\, \|u\|_{q,r,s}\, |Q_n|^{-1/q} \Big(\frac{\rho}{d(Q_n)}\Big)^s$$
$$\le C\, \rho^s\, \|u\|_{q,r,s}\, 2^{k(-1+3/q+s)}\,.$$

Combining the last two inequalities we conclude

$$\int_{Q_k \cap \{|x-a|>2^{-k-2}\}} w(x)^r\, |T_2 u(x)|^r\, dx$$
$$\le C^r\, \rho^{sr}\, \|\mathbf{b}\|_{3,p}^r\, \|u\|_{q,r,s}^r\, 2^{k(3r/q+sr-3)}\,.$$

Now by summing this last inequality over k, $m \le k \le n_0$ and using the fact that (5.30) holds with $m = n_0$ we conclude that (5.30) continues to hold for $m < n_0$.

We have shown that $g = T_2 u$ satisfies the inequality (5.24) provided $d(Q) \sim |Q|^{1/3}$. The inequality (5.24) for cubes Q with $d(Q) > |Q|^{1/3}$ follows by similar argument.

Proposition 5.4. *Let Q_m be a cube with side of length 2^{-m}, m an integer, which is contained in the ball U_{0,R_2} of radius R_2. For $x \in U_{0,R_2}$ let $P_x(Q_m)$ be the probability that the drift process started at x hits Q_m before hitting the boundary of U_{0,R_2}. Then for any $\alpha < 1$ there exists $\varepsilon > 0$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then the inequality* (5.17) *holds where the constant C_α depends only on α.*

PROOF. We follow the same argument as the second proof of Proposition 5.3. We can choose a point $a \in Q_m$ such that the ball $B_a(\rho)$ of radius $\rho \sim 2^{-m}$ centered at a is a distance larger than 3ρ from $\partial U_{0,R_2}$. Let $v(x)$ be the probability of the drift process started at $x \in U_{0,R_2}$ of hitting $B_a(\rho)$ before $\partial U_{0,R_2}$. Then we have

$$v(x) = w_0(x) + \int_\Omega G_{D,1}(x,y)\, (I-T)^{-1}\, \mathbf{b}\cdot\nabla w_0(y)\, dy\,,$$

where $\Omega = U_{0,R_2}\backslash B_a(\rho)$ and $G_{D,1}$ is the Dirichlet Green's function on Ω. The function w_0 is given by (5.18) and T is the integral operator with kernel $\mathbf{b}(x)\cdot\nabla_x G_{D,1}(x,y)$, $x, y \in \Omega$.

We wish to show that $\mathbf{b}\cdot\nabla w_0$ is in a weighted Morrey space $M^q_{r,s}$ with weight given by (5.25), where $d = d(Q_m)$. It is an immediate consequence of Lemma 5.4 that this is so provided r, q, s satisfy (5.13) and that

$$\|\mathbf{b}\cdot\nabla w_0\|_{q,r,s} \le C\rho^{3/q-2}\,\|\mathbf{b}\|_{3,p}\ .$$

Now $T = T_1 + T_2$ where T_1 and T_2 satisfy the conditions of lemmas 5.6, 5.7 respectively. Since the conditions in these lemmas on r, p, q, s are exactly the same as in lemmas 5.2, 5.3, we have that

$$|v(x) - w_0(x)| \le \int_\Omega G_{D,1}(x,y)\,|g(y)|\,dy\,, \tag{5.31}$$

where g is in the weighted Morrey space $M^q_{r,s}$,

$$\|g\|_{q,r,s} \le C\rho^{3/q-2}\,\|\mathbf{b}\|_{3,p}\ ,$$

and $3/2 < q < 3$, $0 < s < 1$, as well as the inequalities (5.13) hold.

We need then to estimate the integral on the right in (5.31). If $d(x, Q_m) \le d(Q_m)$ then the inequality (5.17) is the same as (5.3). Hence we may argue directly as in the second proof of Proposition 5.3. The estimates on the integrals I_1, I_2, I_3 in (5.14) are exactly as previously, since the weight function for our Morrey space is always greater than 1. Hence we may consider the situation when $d(x, Q_m) > d(Q_m)$. We write

$$\begin{aligned}\int_\Omega G_{D,1}(x,y)\,|g(y)|\,dy &= \int_{|x-y|<|x-a|/2} + \int_{|y-a|<|x-a|/2} \\ &\quad + \int_{\{|x-y|>|x-a|/2,\ |y-a|>|x-a|/2\}} \\ &= I_1 + I_2 + I_3\ .\end{aligned}$$

Then from Lemma 5.5 we have if $|x-a| \sim 2^{-n_1}$.

$$\begin{aligned}I_1 &\le C\sum_{k=n_1}^{\infty} 2^k \int_{|x-y|<2^{-k}} |g(y)|\,dy \\ &\le C\sum_{k=n_1}^{\infty} 2^k\,\rho^{3/q-2}\,\|\mathbf{b}\|_{3,p}\left(\frac{\rho}{2^{-n_1}}\right)^s 2^{-3k(1-1/q)}\frac{d}{2^{-n_1}} \\ &\le C\,\|\mathbf{b}\|_{3,p}\left(\frac{\rho}{2^{-n_1}}\right)^{s+3/q-2}\frac{d}{2^{-n_1}}\ ,\end{aligned}$$

as in the estimate of I_1 in the proof of Proposition 5.3. Similarly we can estimate I_3 as

$$\begin{aligned} I_3 &\le C \sum_{k=-\infty}^{n_1} 2^k \int_{2^{-k}<|y-a|<2^{-k+1}} |g(y)|\, dy \\ &\le C \sum_{k=-\infty}^{n_1} 2^{2k}\, d \int_{2^{-k}<|y-a|<2^{-k+1}} w(y)\, |g(y)|\, dy \\ &\le C \sum_{k=-\infty}^{n_1} 2^{2k}\, d\, \rho^{3/q-2} \Big(\frac{\rho}{2^{-k}}\Big)^s \|\mathbf{b}\|_{3,p}\, 2^{-3k(1-1/q)} \\ &\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2} \frac{d}{2^{-n_1}}\ , \end{aligned}$$

provided $s+3/q-1>0$.

Next we write I_2 as a sum,

$$I_2 = \int_{|y-a|<d} + \int_{d<|y-a|<|x-a|/2} = I_4 + I_5\ .$$

We can estimate I_4 from Lemma 5.5 as

$$\begin{aligned} I_4 &\le C\, \frac{d}{2^{-2n_1}} \int_{|y-a|<d} |g(y)|\, dy \\ &\le C\, \frac{d}{2^{-2n_1}}\, \rho^{3/q-2}\, \|\mathbf{b}\|_{3,p}\, d^{3-3/q} \Big(\frac{\rho}{d}\Big)^s \\ &= C\, \|\mathbf{b}\|_{3,p} \Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2} \frac{d}{2^{-n_1}} \Big(\frac{d}{2^{-n_1}}\Big)^{3-s-3/q} \\ &\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2} \frac{d}{2^{-n_1}}\ , \end{aligned}$$

since $d<2^{-n_1}$.

Finally, from Lemma 5.5 we have

$$\begin{aligned} I_5 &\le C \sum_{k=n_1}^{\infty} \frac{d}{2^{-2n_1}} \int_{2^{-k}<|y-a|<2^{-k+1}} w(y)\, |g(y)|\, dy \\ &\le C \sum_{k=n_1}^{\infty} \frac{d}{2^{-2n_1}}\, \rho^{3/q-2} \Big(\frac{\rho}{2^{-k}}\Big)^s \|\mathbf{b}\|_{3,p}\, 2^{-3k(1-1/q)} \\ &\le C\, \|\mathbf{b}\|_{3,p} \Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2} \frac{d}{2^{-n_1}}\ , \end{aligned}$$

since $s + 3/q < 3$.

We conclude therefore that there is a constant C such that

$$\int_\Omega G_{D,1}(x,y)\,|g(y)|\,dy \le C\,\|\mathbf{b}\|_{3,p}\Big(\frac{\rho}{2^{-n_1}}\Big)^{s+3/q-2}\frac{d}{2^{-n_1}}\,.$$

The result follows now from this last inequality just as in the proof of Proposition 5.3.

We can use Proposition 5.4 to generalize Proposition 5.1 to the case of nontrivial drift $\mathbf{b}$. First we need to modify the definition of $V_{\mathcal{S}}$ in (5.1), (5.2). For any Q such that $Q \cap U_{0,R_2} \neq \varnothing$ we define a potential function $V_{Q,\eta} : U_{0,R_2} \longrightarrow \mathbb{R}$ which depends on a parameter $\eta > 0$ by

$$V_{Q,\eta}(x) = \begin{cases} |Q|^{-2/3}\Big(\dfrac{R_2}{|Q|^{1/3}}\Big)^{\eta}, & x \in \tilde{Q}\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

With this new definition of $V_{Q,\eta}$ the potential $V_{\mathcal{S},\eta}$ is defined exactly as in (5.2). Thus

$$V_{\mathcal{S},\eta} = \sum_{Q \subset \mathcal{S}} V_{Q,\eta}\,.$$

Proposition 5.5. *Let $X(t)$ be Brownian motion in $\mathbb{R}^3$ and $X_{\mathbf{b}}(t)$ be the drift process with drift $\mathbf{b}$. Suppose $\mathcal{S}$ is a union of cubes with sides of length $\le R_2$. Then for any $\eta > 0$ there exists $\varepsilon > 0$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then*

$$P_x(X_{\mathbf{b}} \text{ hits } \mathcal{S} \text{ before exiting } U_{0,R_2}) \le C E_x\Big[\int_0^\tau V_{\mathcal{S},\eta}(X(t))\,dt\Big]\,,$$

where $|x| \le R_2/2$. Here τ is the first exit time out of the region U_{0,R_2} and C is a constant depending only on η, ε.

PROOF. It is sufficient for us to assume that $\mathcal{S}$ consists of a single cube Q with side $\le R_2$ which intersects U_{0,R_2}. In that case $\tilde{Q} \cap U_{0,R_2}$ contains a cube Q_m with side of length 2^{-m} which has the same order of magnitude as the length of Q. In view of Proposition 5.4 it will be sufficient for us to show that

$$E_x\Big[\int_0^\tau \chi_{Q_m}(X(t))\,dt\Big]$$

$$\ge c_\eta\Big(\frac{2^{-m}}{R_2}\Big)^{\eta}\frac{2^{-2m}}{(2^m d(x,Q_m)+1)^{\alpha}}\,\min\Big\{1, \frac{2^m d(Q_m)}{2^m d(x,Q_m)+1}\Big\}\,, \tag{5.32}$$

for some α, $0 < \alpha < 1$ and constant c_η depending only on η. Now the left hand side of the above inequality is just

$$\int_{Q_m} G_D(x,y)\, dy\,,$$

where G_D is the Dirichlet Green's function on the ball U_{0,R_2}. It is easy to see from the explicit formula for G_D that if $|x| \le R_2/2$ then

$$\int_{Q_m} G_D(x,y)\, dy \ge \frac{c\, 2^{-2m}}{2^m d(x,Q_m)+1} \min\Big\{1, \frac{2^m d(Q_m)}{2^m d(x,Q_m)+1}\Big\}\,,$$

for some universal constant $c > 0$. Thus the inequality (5.32) holds provided $\alpha \ge 1-\eta$.

Next we generalize Proposition 5.2 to the case of nontrivial $\mathbf{b}$.

Proposition 5.6. *Suppose $R_1 = 0$, $R_2 = 2R$, and suppose $\mathcal{S}$ consists of cubes of length $\le R_2$ Let f be a density on the sphere $|x| = R$ and f_2 the density on $|x| = R_2$ by f propagated by the process with drift $\mathbf{b}$ along paths which do not intersect $\mathcal{S}$. Let $\eta > 0$, $1 < q < \infty$, $1 < p \le 3$. Then there exist $\varepsilon, \delta, \xi > 0$ depending only on η, p, q such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$, $\|f - \mathrm{Av}\, f\|_q \le \delta\, |\mathrm{Av}\, f|$ and*

$$\mathrm{Av}_{|x|=R}\, E_x\Big[\int_0^{\tau_{R_2}} V_{\mathcal{S},\eta}(X(t))\, dt\Big] < \xi\,,$$

then

$$\|f_2 - \mathrm{Av}\, f_2\|_q \le \delta\, |\mathrm{Av}\, f_2| \qquad \text{and} \qquad |\mathrm{Av}\, f_2| \ge \frac{|\mathrm{Av}\, f|}{2}\,.$$

PROOF. We proceed as in Proposition 5.2. Letting q' satisfy $1/q + 1/q' = 1$, we need to show that the operator A defined by

$$Ag(x) = E_x[g(X_{\mathbf{b}}(\tau_{R_2}));\ X_{\mathbf{b}}(t) \in \mathcal{S},\ \text{some } t,\ 0 < t < \tau_{R_2}]$$

which maps functions on $|x| = R_2$ to functions on $|x| = R$ satisfies an inequality

$$\|Ag\|_{q'} \le \gamma(\xi)\, \|g\|_{q'}\,,$$

where $\gamma(\xi) \longrightarrow 0$ for $\xi \longrightarrow 0$. To prove this let $1 < r < q'$. Then it is sufficient to show that

$$E_x[|g(X_{\mathbf{b}}(\tau_{R_2}))|^r] \le C\, \|g\|_{q'}^r\,, \qquad |x| = R\,, \tag{5.33}$$

for some constant C depending only on r, q, p, ε. Now we can write

$$E_x[|g(X_{\mathbf{b}}(\tau_{R_2}))|^r] = \langle \rho_x, |g|^r \rangle ,$$

where ρ_x is the density of the drift process started at x, $|x| = R$, on the sphere $|y| = R_2$. Arguing similarly to the proof of [5, Lemma 4.3] and using Corollary 4.1 we see that for any s, $1 < s < \infty$, we can choose $\varepsilon > 0$ sufficiently small so that ρ_x is s integrable on $|y| = R_2$ and $\|\rho_x\|_s \leq C$ where C is a universal constant. Now we obtain the inequality (5.33) by choosing s to satisfy $1/s + r/q' = 1$ and applying Holder's inequality.

6. Auxiliary perturbative estimates.

In this section we shall prove a perturbative theorem which will be needed in the induction argument of Section 7. The theorem is similar in spirit to the results of sections 5 and 6 and our proof will depend on these. Let Ω_R be the ball of radius R in $\mathbb{R}^3$ centered at the origin and suppose a_1, a_2 are points which satisfy $|a_1| = |a_2| = R/2$, $|a_1 - a_2| = R$. Thus a_1 and a_2 lie on a diameter of Ω_R at a distance $R/2$ from the center. Let B_{r_1} be a ball of radius $r_1 \geq 10R$ such that $a_1 \in \partial B_{r_1}$ and the outward normal to ∂B_{r_1} at a_1 makes an angle less than $\pi/100$ with the vector $a_2 - a_1$. Similarly, let B_{r_2} be a ball of radius $r_2 \geq 10R$ such that $a_2 \in \partial B_{r_2}$ and the outward normal to ∂B_{r_2} at a_2 makes an angle less that $\pi/100$ with the vector $a_2 - a_1$. We shall be interested in the surfaces $D_1 = B(a_1, R/4) \cap \partial B_{r_1}$ and $D_2 = B(a_2, R/4) \cap \partial B_{r_2}$.

Next suppose we have a vector field $\mathbf{b} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and a dyadic decomposition of $\mathbb{R}^3$ into cubes Q. For n_0 an integer and $\varepsilon > 0$ let $\mathcal{S}$ be the set of all dyadic cubes Q_n with side of length 2^{-n}, $n \geq n_0$, such that

$$\int_{Q_n} |\mathbf{b}(x)|^p \, dx \geq \varepsilon^p \, |Q_n|^{1-p/3} . \tag{6.1}$$

For $Q \in \mathcal{S}$ and $\eta > 0$ define $V_{Q,\eta} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$V_{Q,\eta}(x) = \begin{cases} |Q|^{-2/3} \Big(\dfrac{2^{-n_0}}{|Q|^{1/3}} \Big)^{\eta} , & x \in \tilde{Q} , \\ 0 , & \text{otherwise} , \end{cases}$$

where $\tilde{Q}$ is the double of Q. The potential V_η is then given by

$$V_\eta = \sum_{Q \in \mathcal{S}} V_{Q,\eta} \, . \tag{6.2}$$

Observe that the potential V_η defined here is a particular case of the potential $V_{\mathcal{S},\eta}$ of Section 5. Suppose ρ_1 is a density on the surface D_1. Then $\mathrm{Av}_{D_1} \rho_1$ is the average of ρ_1 on D_1 and $\|\rho_1\|_{D_1,q}$, $1 < q < \infty$, is the L^q norm of ρ_1 normalized so that $\|\mathbf{1}\|_{D_1,q} = 1$. The theorem we wish to prove is as follows:

Theorem 6.1. *Let $R = 2^{-n}$, $n \geq n_0$, and ρ_1 be a density on D_1. Suppose $f \in M_r^t(\mathbb{R}^3)$ with $1 < r \leq t$, $r < p$, $3/2 < t < 3$. Let ρ_2 be the density induced on D_2 by the paths of the drift process $X_{\mathbf{b}}(t)$ which start on D_1, avoid the cubes $Q \in \mathcal{S}$ with $|Q| \leq 2^{-3n}$, exit the region $\Omega_R \cap B_{r_2}$ through D_2, and satisfy the inequality*

$$\int_0^\tau |f| \, (X_{\mathbf{b}}(t)) \, dt \leq C_1 \, R^{2-3/t} \, \|f\|_{t,r} \, ,$$

where C_1 is a constant. Let $0 < \eta' < \eta$ and suppose

$$\frac{1}{R} \int_{\Omega_R} V_\eta(x) \, dx \leq \xi \, 2^{\eta'(n-n_0)} \, , \tag{6.3}$$

where $\xi > 0$ is a constant. Then there exists a constant $\alpha > 1$ depending only on η' such that if $1 < q < \infty$ and C_1 is sufficiently large, ξ sufficiently small, one can find constants C_2, c_2 such that

$$\|\rho_1\|_{D_1,q} \leq C_2 \, \alpha^{n-n_0} \, \mathrm{Av}_{D_1} \rho_1 \tag{6.4}$$

implies that

$$\|\rho_2\|_{D_2,q} \leq C_2 \, \mathrm{Av}_{D_2} \rho_2 \, , \qquad \mathrm{Av}_{D_2} \rho_2 \geq c_2 \, \mathrm{Av}_{D_1} \rho_1 \, .$$

The constants ξ, C_1, C_2, c_2 can be chosen independent of R.

Remark. Theorem 6.1 is rather like the results we have already proven. In fact, if we take $C_1 = \infty$, $\xi = 0$, we are in the situation studied in Section 4. The case $C_1 = \infty$, $\xi > 0$, $n = n_0$, is the situation studied in Section 5. Observe that since the regions D_1, D_2 are not spheres the results of sections 4 and 5 do not immediately yield a proof of Theorem 6.1 in the above mentioned cases.

We shall prove Theorem 6.1 in a series of steps starting from the simplest situation. We first consider the case of Brownian motion where $\mathbf{b} \equiv 0$.

Lemma 6.1. *Let ρ_1 be a density on D_1 with $Av_{D_1}\rho_1 < \infty$ and ρ_2 the density induced on D_2 by Brownian paths started on D_1 which exit $\Omega_R \cap B_{r_2}$ through D_2. Then there exists universal constants c_2, C_2 such that for $1 \le q \le \infty$,*

$$\|\rho_2\|_{D_2,q} \le C_2 \operatorname{Av}_{D_2}\rho_2 \ , \qquad Av_{D_2}\rho_2 \ge c_2 \operatorname{Av}_{D_1}\rho_1 \ .$$

PROOF. Suppose g is a function defined on D_2 and let $u(x) = Pg(x)$, $x \in \Omega_R \cap B_{r_2}$ be given by the solution of the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0 \,, & x \in \Omega_R \cap B_{r_2} \ , \\ u(x) = g(x) \,, & x \in D_2 \ , \\ u(x) = 0 \,, & x \in \partial(\Omega_r \cap B_{r_2}) \backslash D_2 \ . \end{cases} \tag{6.5}$$

Thus P defines a mapping of functions on D_2 to functions on D_1. Let P^* be the adjoint of P defined by

$$\langle f, Pg \rangle_{D_1} = \langle P^* f, g \rangle_{D_2} \ ,$$

where $\langle \cdot\,,\cdot \rangle_{D_1}$, $\langle \cdot\,,\cdot \rangle_{D_2}$ are the standard inner products on $L^2(D_1)$ and $L^2(D_2)$ normalized so that $\|\mathbf{1}\|_{D_1,2} = \|\mathbf{1}\|_{D_2,2} = 1$. Then ρ_1 and ρ_2 are related by the equation $\rho_2 = P^*\rho_1$. We have therefore that

$$\operatorname{Av}_{D_2}\rho_2 = \langle \rho_2, 1 \rangle_{D_2} = \langle P^*\rho_1, 1 \rangle_{D_2} = \langle \rho_1, P1 \rangle_{D_1} \ .$$

Thus to show that $\operatorname{Av}_{D_2}\rho_2 \ge c_2 \operatorname{Av}_{D_1}\rho_1$ it is sufficient to prove that $P\mathbf{1}(x) \ge c_2 > 0$ for all $x \in D_1$. Hence we need to prove that there is a universal constant $c_2 > 0$ such that

$$P_x(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2) \ge c_2 \ , \qquad x \in D_1 \ . \tag{6.6}$$

To see this let $B_1, B_2, \ldots, B_N$ be balls with radius $\sim R$ having the following properties:

a) $B_i \subset \Omega_R \cap B_{r_2}$, $1 \leq i \leq N-1$, B_N is centered at a_2, B_1 is centered at x.

b) $|\partial B_i \cap B_{i+1}| \sim R^2$, $1 \leq i \leq N-1$.

Now for $1 \leq i \leq N$ let S_i be the sets

$$S_i = \{y \in \partial B_i : \; y \in B_{i+1}, \; d(y, \partial B_{i+1}) > c\, R\}\,, \qquad 1 \leq i \leq N-1\,,$$

$$S_N = \partial B_N \cap (\mathbb{R}^3 \backslash \Omega_R \cap B_{r_2})\,.$$

It is clear from a), b) that we may choose $c > 0$ such that $|S_i| \sim R^2$, $1 \leq i \leq N$. Next define $p_0, \ldots, p_{N-1}$ by

$$p_0 = P\,(BM \text{ started at } x \text{ exits } B_1 \text{ through } S_1)\,,$$

$$p_i = \inf_{y \in S_i} P\,(BM \text{ started at } y \in S_i \text{ exits } B_{i+1} \text{ through } S_{i+1})\,,$$

with $1 \leq i \leq N-1$. It is clear from the Poisson formula that there is a constant $c > 0$ such that $p_i \geq c$, $0 \leq i \leq N-1$. Hence we have

$$P_x(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2) \geq p_0\, p_1 \cdots p_{N-1} \geq c^N\,.$$

Since we can choose N to be an absolute constant the inequality (6.6) follows.

Next, to show that $\|\rho_2\|_{D_2,q} \leq C_2 \operatorname{Av}_{D_2} \rho_2$, we can prove that

$$|\langle \rho_2, f \rangle_{D_2}| \leq C_2 \operatorname{Av}_{D_2} \rho_2\, \|f\|_{D_2,q'}\,,$$

where $1/q + 1/q' = 1$. Since $\langle \rho_2, f \rangle_{D_2} = \langle \rho_1, Pf \rangle_{D_1}$ and we have already proved that $\operatorname{Av}_{D_2} \rho_2 \geq c_2 \operatorname{Av}_{D_1} \rho_1$, it is sufficient to show that

$$\|Pf\|_{D_1,\infty} \leq C\, \|f\|_{D_2,q'}\,, \tag{6.7}$$

for some universal constant C. We can prove this last inequality by observing that $|Pf(x)| \leq \overline{P}\,|f|(x)$, where $\overline{P}$ is the Poisson kernel for the ball B_{r_2}.

Lemma 6.2. *Let ρ_1 be a density on D_1 with $\operatorname{Av}_{D_1} \rho_1 < \infty$ and ρ_2 the density induced on D_2 by Brownian paths $X(t)$ started on D_1 which exit $\Omega_R \cap B_{r_2}$ through D_2 and satisfy*

$$\int_0^\tau |f|(X(t))\, dt \leq C_1\, R^{2-3/t}\, \|f\|_{t,r}\,, \qquad 1 \leq r \leq t\,,\; t > \frac{3}{2}\,.$$

Then there exist universal constants c_2, C_2 *such that for* $1 < q < \infty$ *and sufficiently large* C_1*, depending only on* r, t*, one has the inequalities*

$$\|\rho_2\|_{D_2,q} \le C_2 \operatorname{Av}_{D_2}\rho_2 \ , \qquad \operatorname{Av}_{D_2}\rho_2 \ge c_2 \operatorname{Av}_{D_1}\rho_1 \ .$$

PROOF. Suppose g is a function defined on D_2 and extend g to $\partial(\Omega_R \cap B_{r_2})$ by setting g to be zero on the rest of the boundary. Then $Pg(x)$ is defined for $x \in \Omega_R \cap B_{r_2}$ by

$$Pg(x) = E_x\Big[g(X(\tau))H\Big(C_1\, R^{2-3/t}\,\|f\|_{t,r} - \int_0^\tau |f|(X(t))\,dt\Big)\Big] ,$$

where H is the Heaviside function $H(z) = 1$, $z > 0$, $H(z) = 0$, $z \le 0$. Then just as in Lemma 6.1 we have $\rho_2 = P^*\rho_1$. It is furthermore clear that the inequality (6.7) continues to hold. Hence we need only prove that $\operatorname{Av}_{D_2}\rho_2 \ge c_2 \operatorname{Av}_{D_1}\rho_1$. This follows if we can show that

$$\begin{aligned} &P_x\Big(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2 \\ &\quad \text{and} \int_0^\tau |f|\,(X(t))\,dt \le C_1\, R^{2-3/t}\|f\|_{t,r}\Big) \ge c_2 \ , \quad x \in D_1 \ . \end{aligned} \tag{6.8}$$

Evidently from the Chebyshev inequality the left hand side of the previous inequality is bounded below by

$$P_x(X(t) \text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2) - \frac{E_x\Big[\int_0^\tau |f|\,(X(t))\,dt\Big]}{C_1\, R^{2-3/t}\,\|f\|_{t,r}} \ .$$

If we use now the fact that

$$E_x\Big[\int_0^\tau |f|(X(t))\,dt\Big] \le \frac{1}{4\pi}\int_{\Omega_R} \frac{|f(y)|}{|x-y|}\,dy \le K R^{2-3/t}\,\|f\|_{t,r} \ ,$$

for some constant K depending on t, then it is clear that (6.8) holds and hence the result.

Lemma 6.3. *Let* $\mathcal{S}$ *be a set of dyadic cubes and suppose* V_η *is defined by* (6.2)*. Let* ρ_1 *be a density on* D_1 *and* ρ_2 *the density induced on* D_2 *as in Theorem* 6.1*. Then if* $\mathbf{b} \equiv 0$ *the conclusion of Theorem* 6.1 *holds.*

PROOF. As in Lemma 6.2 we may confine ourselves to proving that $\mathrm{Av}_{D_2}\rho_2 \geq c_2 \,\mathrm{Av}_{D_1}\rho_1$. Thus we need to show

$$\begin{aligned}\int_{D_1} \rho_1(x)\, P_x\Big(X(t) &\text{ exits } \Omega_R \cap B_{r_2} \text{ through } D_2, \text{ avoids} \\ &\text{cubes } Q \in \mathcal{S} \text{ with } |Q| \leq 2^{-3n} \text{ and} \\ &\int_0^\tau |f|(X(t))\,dt \leq C_1\, R^{2-3/t}\|f\|_{t,r}\Big)\, d\mu(x) \\ &\geq c_2\,\mathrm{Av}_{D_1}\,\rho_1\ ,\end{aligned}$$

where μ is the surface measure on D_1 normalized so that $\mu(D_1) = 1$. From Lemmas 6.1, 6.2 it will be sufficient to show that

$$\begin{aligned}\int_{D_1} \rho_1(x)\, P_x\Big(X(t) \text{ hits } \bigcup_{\substack{Q\in\mathcal{S}\\ |Q|\leq 2^{-3n}}} Q \text{ before exiting } \Omega_R\cap B_{r_2}\Big)\, d\mu(x) \\ \leq \gamma\,\mathrm{Av}_{D_1}\rho_1\ ,\end{aligned} \tag{6.9}$$

where γ is a number which can be chosen arbitrarily small depending on ξ. Let Q_m be a cube in $\mathcal{S}$ with side of length 2^{-m}, $m \geq n$. In view of the inequality (6.3) m must satisfy the inequality

$$2^{(1-\eta)(m-n)} > \xi^{-1}\, 2^{(\eta-\eta')(n-n_0)}\ , \tag{6.10}$$

whence $m - n$ is larger than a constant times $n - n_0$ plus a constant which may be made arbitrarily large depending on ξ. Let $d(x, Q_m)$ be defined by $d(x, Q_m) = 2^{-m}$ if $x \in Q_m$, $d(x, Q_m) =$ distance from x to the center of Q_m if $x \notin Q_m$. Then as in Section 5 we have

$$\begin{aligned}\int_{D_1} \rho_1(x)\, P_x(X(t) &\text{ hits } Q_m \text{ before exiting } \Omega_R \cap B_{r_2})\, d\mu(x) \\ &\leq C \int_{D_1} \frac{2^{-m}\,\rho_1(x)}{d(x,Q_m)}\, d\mu(x) \\ &\leq C\Big(\int_{D_1} \frac{2^{-mq'}}{d(x,Q_m)^{q'}}\, d\mu(x)\Big)^{1/q'} \|\rho_1\|_{D_1,q'}\ ,\end{aligned}$$

where $1/q + 1/q' = 1$. We have now that

$$\int_{D_1} \frac{2^{-mq'}}{d(x,Q_m)^{q'}}\, d\mu(x) \leq C\, 2^{-(m-n)}\ , \qquad 1 \leq q' \leq \infty\ ,$$

for some constant C. Hence by the assumption (6.4) we conclude that there is a constant C such that

$$\int_{D_1} \rho_1(x)\, P_x(X(t) \text{ hits } Q_m \text{ before exiting } \Omega_R \cap B_{r_2})\, d\mu(x)$$
$$\leq C\, 2^{-(m-n)/q'}\, \alpha^{n-n_0}\, \mathrm{Av}_{D_1}\, \rho_1\ .$$

It is clear now from (6.10) that if α satisfied the inequality

$$1 < \alpha < 2^{(\eta-\eta')/q'(1-\eta)}\,, \tag{6.11}$$

then for any $\gamma > 0$, ξ can be chosen sufficiently small so that

$$\int_{D_1} \rho_1(x)\, P_x(X(t) \text{ hits } Q_m \text{ before exiting } \Omega_R \cap B_{r_2})\, d\mu(x)$$
$$\leq \gamma\, \mathrm{Av}_{D_1}\, \rho_1\ .$$

Suppose now that m satisfies (6.10) and N_m is the number of cubes Q_m in $\mathcal{S}$ with side of length 2^{-m}. Then from (6.3) it follows that

$$N_m \leq \xi\, 2^{(1-\eta)(m-n)-(\eta-\eta')(n-n_0)}\,.$$

Let g_m be the function defined by

$$g_m(x) = P_x\Big(X(t) \text{ hits } \bigcup_{Q_m\in\mathcal{S}} Q_m \text{ before exiting } \Omega_R \cap B_{r_2}\Big)\,.$$

Then we have

$$\begin{aligned}\|g_m\|_{D_1,1} &\leq \sum_{Q_m\in\mathcal{S}} C \int_{D_1} \frac{2^{-m}\, d\mu(x)}{d(x, Q_m)} \\ &\leq C N_m\, 2^{-(m-n)} \leq C\, \xi\, 2^{-\eta(m-n)-(\eta-\eta')(n-n_0)}\,.\end{aligned}$$

Now, using the obvious fact that $g_m(x) \leq 1$, we have that

$$\begin{aligned}\int_{D_1} \rho_1(x)\, g_m(x)\, d\mu(x) &\leq \|g_m\|_{D_1,q'}\, \|\rho_1\|_{D_1,q} \\ &\leq \|g_m\|_{D_1,1}^{1/q'}\, \|\rho_1\|_{D_1,q} \\ &\leq C\, \xi^{1/q'}\, 2^{-\eta(m-n)/q'-(\eta-\eta')(n-n_0)/q'} \\ &\quad \cdot \alpha^{n-n_0}\, \mathrm{Av}_{D_1} \rho_1\ .\end{aligned}$$

Letting m_0 be the minimum integer m such that (6.10) holds we conclude that

$$\begin{aligned}\int_{D_1} \rho_1(x)\, P_x\Big(X(t) \text{ hits } \bigcup_{\substack{Q\in\mathcal{S}\\ |Q|\le 2^{-3n}}} Q \text{ before exiting } \Omega_R\cap B_{r_2}\Big)\, d\mu(x) \\ \le \sum_{m=m_0}^{\infty} C\,\xi^{1/q'}\, 2^{-\eta(m-n)/q'-(\eta-\eta')(n-n_0)/q'}\,\alpha^{n-n_0}\,\mathrm{Av}_{D_1}\rho_1 \\ \le C'\,\xi^{1/q'}\, 2^{-\eta(m_0-n)/q'-(\eta-\eta')(n-n_0)/q'}\,\alpha^{n-n_0}\,\mathrm{Av}_{D_1}\rho_1\,.\end{aligned}$$

If we use the inequality (6.11) we have that

$$2^{-\eta(m_0-n)/q'-(\eta-\eta')(n-n_0)/q'}\,\alpha^{n-n_0}\,,$$

$$(2^{-(1-\eta)(m_0-n)+(\eta-\eta')(n-n_0)})^{\eta/q'(1-\eta)} < \xi^{\eta/q'(1-\eta)}\,,$$

from the definition of m_0 and (6.10). The inequality (6.9) immediately follows from this.

Next we wish to consider the case of nontrivial drift $\mathbf{b}$ with $\xi = 0$.

Lemma 6.4. *Let ρ_1 be a density on D_1 with $\mathrm{Av}_{D_1}\rho_1 < \infty$ and ρ_3 the density induced on the sphere $\partial B(a_1, R/3)$ by paths of the drift process $X_{\mathbf{b}}(t)$ started on D_1. Suppose that $\mathbf{b} \in M_p^3$ and $\|\mathbf{b}\|_{3,p} < \varepsilon$. Then for any q, $1 < q < \infty$, and sufficiently small ε, depending only on p, q one has, with $D_3 = \partial B(a, R/3)$,*

$$\mathrm{Av}_{D_1}\rho_1 = \mathrm{Av}_{D_3}\rho_3\,, \qquad \|\rho_3\|_{D_3,q} \le C_3\,\mathrm{Av}_{D_3}\rho_3\,,$$

where C_3 depends only on p, q, ε.

PROOF. For $y \in D_1$ let δ_y be the Dirac δ function concentrated at y. Then it follows from Corollary 4.1 that if h_y is the density induced on D_3 by δ_y then $\|h_y\|_{D_3,q} \le C_3$, for some constant C_3, provided ε is sufficiently small. Since

$$\rho_1 = \int_{D_1} \rho(y)\,\delta_y\,d\mu(y)\,,$$

it follows that

$$\begin{aligned}\|\rho_3\|_{D_3,q} &= \Big\| \int_{D_1} \rho(y)\, h_y \, d\mu(y) \Big\|_{D_3,q} \\ &\le \int_{D_1} \rho(y)\, \|h_y\|_{D_3,q}\, d\mu(y) \\ &\le C_3 \, \mathrm{Av}_{D_1} \rho_1 \ .\end{aligned}$$

The fact that $\mathrm{Av}_{D_1}\rho_1 = \mathrm{Av}_{D_3}\rho_3$ follows simply from the observation that $u(x) \equiv 1$ is a solution of the equation $\Delta u(x) + \mathbf{b}(x) \cdot \nabla u(x) = 0$.

Next let $G_D(x, y)$ be the Dirichlet kernel for $-\Delta$ on the domain $\Omega_R \cap B_{r_2}$. As in Section 4 we shall be concerned with the integral operator T on functions with domain $\Omega_R \cap B_{r_2}$ which has kernel k_T given by

$$k_T(x,y) = \mathbf{b}(x) \cdot \nabla_x G_D(x,y)\,, \qquad x, y \in \Omega_R \cap B_{r_2} \ . \tag{6.12}$$

Lemma 6.5. *There is a universal constant C such that*

$$|\nabla_x G_D(x,y)| \le \frac{C}{|x-y|^2}\,, \qquad x, y \in \Omega_R \cap B_{r_2} \ . \tag{6.13}$$

PROOF. Let $u(x) = G_D(x, y)$. We shall show that there is a universal constant C such that

$$u(x) \le \frac{C}{|x-y|} \ \min\Big\{1, d\Big(x, \frac{\partial(\Omega_R \cap B_{r_2})}{|x-y|}\Big)\Big\}\,, \tag{6.14}$$

where $x \in \Omega_R \cap B_{r_2} \backslash \{y\}$. The estimate (6.13) follows from (6.14) by using the fact that u is harmonic in $\Omega_R \cap B_{r_2} - \{y\}$ and the Poisson formula. One can easily prove (6.14) by constructing a barrier function. Thus let us suppose that $d(x, \partial(\Omega_R \cap B_{r_2})) < |x-y|/4$ and that x_0 is the nearest point on $\partial(\Omega_R \cap B_{r_2})$ to x. Let x_1 be the point $x_1 = x_0 + c\,|x-y|\,(x_0 - x)/|x_0 - x|$ where $c > 0$. Let $U_x = \{z : |z - x_1| > c\,|x-y|\}$. Then it is clear that we may choose $c < 1/8$ in a universal way so that $\Omega_R \cap B_{r_2} \subset U_x$ and $|z - x_1| > |x-y|/4$ if $|z-y| = |x-y|/4$. Let $v(z)$ be the function

$$v(z) = 1 - \frac{|x_0 - x_1|}{|z - x_1|}\,, \qquad z \in U_x \ .$$

Thus v is harmonic in the region $\Omega_R \cap B_{r_2} \backslash B(y, |x-y|/4)$ and satisfies the boundary conditions

$$v(z) \geq 0\,, \qquad z \in \partial(\Omega_R \cap B_{r_2})\,,$$

$$v(z) \geq 1 - 4\,c > \frac{1}{2}\,, \qquad z \in \partial B\Big(y, \frac{1}{4}\,|x-y|\Big)\,.$$

On the other hand $u(z)$ is also harmonic in the region

$$\Omega_R \cap B_{r_2} \backslash B\Big(y, \frac{|x-y|}{4}\Big)$$

and satisfies the boundary conditions

$$u(z) = 0\,, \qquad z \in \partial(\Omega_R \cap B_{r_2})\,,$$

$$u(z) \leq \frac{C}{|x-y|}\,, \qquad z \in \partial B\Big(y, \frac{1}{4}\,|x-y|\Big)\,,$$

where C is a universal constant. It follows then by the maximum principle that

$$u(z) \leq 2\,C\,\frac{v(z)}{|x-y|}\,, \qquad z \in \Omega_R \cap B_{r_2} \backslash B\Big(y, \frac{1}{4}\,|x-y|\Big)\,. \tag{6.15}$$

Observing now that

$$\begin{aligned}
v(x) &= 1 - \frac{|x_0 - x_1|}{|x - x_1|} \\
&= 1 - \frac{|x_0 - x_1|}{|x - x_0| + |x_0 - x_1|} \\
&= \frac{|x - x_0|}{|x - x_0| + |x_0 - x_1|} \\
&\leq \frac{|x - x_0|}{|x_0 - x_1|} \\
&= \frac{|x - x_0|}{c\,|x-y|}\,,
\end{aligned}$$

the inequality (6.14) follows from (6.15) on setting $z = x$.

The estimates in Lemma 6.5 can be improved when y is close to $\partial(\Omega_R \cap B_{r_2})$ but a distance from $\partial\Omega_R \cap \partial B_{r_2}$ In fact one can see this by using the Kelvin transform just as in the proof of Lemma 5.5. In particular we have an estimate similar to Proposition 2.1 for $\nabla_x G_D(x,y)$ when y is close to D_2.

Lemma 6.6. *There exist universal constants, c, C such that if $d(y, D_2) < cR$ then*

$$|\nabla_x G_D(x,y)| \le \frac{C}{|x-y|^2} \min\Big\{1, \frac{d(y,\partial B_{r_2})}{|x-y|}\Big\}, \qquad x, y \in \Omega_R \cap B_{r_2}\,.$$

PROOF. Suppose $d(y,D_2) < cR$. Then if c is sufficiently small one can choose γ, $0<\gamma<1/2$, in a universal way such that the harmonic function $u(z) = \nabla_x G_D(x,z)$ extends to the entire ball $B(y, \gamma|x-y|)$. This follows by using the Kelvin transform. Furthermore, by Lemma 6.5 there is a universal constant C such that

$$\sup_{z\in B(y,\gamma|x-y|)} |u(z)| \le \frac{C}{|x-y|^2}\,. \tag{6.16}$$

Let y_0 be the closest point on ∂B_{r_2} to y and suppose that $|y-y_0| < \gamma\,|x-y|/2$. Then from the Poisson integral formula and (6.16) one has that $|\nabla u(z)| \le C_1/|x-y|^3$ for all z on the line segment joining y to y_0, where C_1 is a constant. Since $u(y_0)=0$ if follows from this that $|u(y)| \le C_1\, d(y,\partial B_{r_2})/|x-y|^3$. The result easily follows.

We use Lemmas 6.5 and 6.6 to show that the operator T with kernel k_T given by (6.12) is a bounded operator on a weighted Morrey space. Let $\lambda > 0$ be a parameter and define the weight function w_λ on $\Omega_R \cap B_{r_2}$ by

$$w_\lambda(y) = \begin{cases} \dfrac{d(y,\partial B_{r_2})}{R}\,, & \text{if } d(y,D_2) \le \lambda R\,, \\ 1\,, & \text{if } d(y,D_2) \ge 2\lambda R\,, \end{cases}$$

and

$$w_\lambda(y) = \Big(2 - \frac{d(y,D_2)}{\lambda R}\Big)\frac{d(y,\partial B_{r_2})}{R} + \Big(\frac{d(y,D_2)}{\lambda R} - 1\Big)\,,$$

if $\lambda R < d(y,D_2) < 2\,\lambda\, R$.

Lemma 6.7. *Let Q be an arbitrary cube which intersects $\Omega_R \cap B_{r_2}$ and suppose*

$$d(Q) = \sup\{d(x, \partial B_{r_2}) : \ x \in Q\}\,.$$

Then there exists a constant C_λ depending only on λ such that

$$\frac{d(y,\partial B_{r_2})}{d(Q)} \le \frac{C_\lambda\, \omega_\lambda(y)}{\|\omega_\lambda\|_{\infty,Q}}\,, \qquad y \in Q \cap \Omega_R \cap B_{r_2}\,,$$

where $\|\omega_\lambda\|_{\infty,Q}$ denotes the L^∞ norm of ω_λ on Q.

PROOF. Suppose $|Q|^{1/3} \ge cR$ for some constant $c > 0$. Hence there are constants C_1, C_2, C_3 such that $d(Q) \ge C_1 R$, $\|\omega_\lambda\|_{\infty,Q} \le C_2$, $w_\lambda(y) \ge C_3\, d(y, \partial B_{r_2})/R$. The inequality (6.17) clearly follows from this and so we may assume from here on that $|Q|^{1/3} \le cR$ where $c > 0$ is an arbitrarily small universal constant.

Next suppose that for all $y \in Q$ one has $d(y, D_2) \le \lambda R$. In view of the definition of $w_\lambda(y)$ for $d(y, D_2) \le \lambda R$ the inequality (6.17) immediately follows. Similarly (6.17) follows if for all $y \in Q$ one has $d(y, D_2) \ge 2\,\lambda\, R$. Hence we may assume that there exists $y \in Q$ such that $\lambda R < d(y, D_2) < 2\,\lambda\, R$. We put $\gamma = d(y, D_2)/\lambda R - 1$, whence $0 \le \gamma \le 1$. Let $\delta = |Q|^{1/3}/\lambda R$. Then if $|Q|^{1/3} \le cR$ and $c > 0$ is small we have $0 < \delta < 1$. One has the inequalities

$$\|w_\lambda\|_{\infty,Q} \le (1-\gamma+\delta)\,\frac{d(Q)}{R} + \gamma + \delta\,,$$

$$w_\lambda(y) \ge (1-\gamma-\delta)\,\frac{d(y,\partial B_{r_2})}{R} + \gamma - \delta\,.$$

Suppose now that $2\,\delta < \gamma < 1 - 2\,\delta$. Then

$$\frac{w_\lambda(y)}{\|w_\lambda\|_{\infty,Q}} \ge \frac{\dfrac{(1-\gamma)\, d(y,\partial B_{r_2})}{2R} + \dfrac{\gamma}{2}}{\dfrac{3\,(1-\gamma)\, d(Q)}{2R} + \dfrac{3\,\gamma}{2}} \ge \frac{1}{3}\,\frac{d(y,\partial B_{r_2})}{d(Q)}\,,$$

since $d(y, \partial B_{r_2}) \le d(Q)$. Next suppose $0 < \gamma < 2\,\delta$. Since $\delta \le C\, d(Q)/R$ for some constant $C > 0$ we have that

$$\|w_\lambda\|_{\infty,Q} \le (1+3C)\,\frac{d(Q)}{R}\,.$$

On the other hand one also has $w_\lambda(y) \geq d(y, \partial B_{r_2})/(2R)$ if δ is sufficiently small. Hence (6.17) holds again. Finally, for $1-2\,\delta < \gamma < 1$ one has $w_\lambda(y) \geq 1/2$ for sufficiently small δ and hence (6.17) holds in this case also.

For $1 \leq r \leq q < \infty$ we define the weighted Morrey space $M^q_{r,w_\lambda}(\Omega_R \cap B_{r_2})$ as follows: a measurable function $g : \Omega_R \cap B_{r_2} \longrightarrow \mathbb{C}$ is in $M^q_{r,w_\lambda}(\Omega_R \cap B_{r_2})$ if $w_\lambda(y)^r\,|g(y)|^r$ is integrable on $\Omega_R \cap B_{r_2}$ and there is a constant C such that

$$\int_{Q\cap\Omega_R\cap B_{r_2}} w_\lambda(y)^r\,|g(y)|^r\,dy \leq C^r\,|Q|^{1-r/q}\,, \tag{6.18}$$

for all cubes $Q \subset \mathbb{R}^3$. The norm of g, $\|g\|_{q,r,w_\lambda}$ is defined as

$$\|g\|_{q,r,w_\lambda} = \inf\{C : \text{ (6.18) holds for all cubes } Q\}\,.$$

Lemma 6.8. *Suppose* $\mathbf{b} \in M^3_p$, $1 < p \leq 3$, *and* r, q *satisfy* $1 < r < p$, $r \leq q < 3$. *Then there exists a universal constant* $\lambda > 0$ *such that the operator* T *with kernel* k_T *given by* (6.12) *is a bounded operator on the space* $M^q_{r,w_\lambda}(\Omega_R \cap B_{r_2})$. *The norm of* T *satisfies the inequality* $\|T\| \leq C\,\|\mathbf{b}\|_{3,p}$, *where the constant* C *depends only on* r, p, q.

PROOF. We follow the same lines as the proof of Proposition 2.1. Define an integer n_0 by $2^{-n_0-1} < 8R \leq 2^{-n_0}$ and let $Q_0(\xi)$ be the cube centered at ξ with side of length 2^{-n_0}. It is clear that for $\xi \in \Omega_R \cap B_{r_2}$ then $\Omega_R \cap B_{r_2} \subset Q_0(\xi)$. We define an operator T_K on functions $u : \Omega_R \cap B_{r_2} \longrightarrow \mathbb{C}$ which have the property that $w_\lambda(x)\,u(x)$ is integrable. To do this we decompose K into a dyadic decomposition of cubes Q_n with sides of length 2^{-n}, $n \geq n_0$. For any dyadic cube $Q \subset K$ with volume $|Q|$ let u_Q be defined by

$$u_Q = |Q|^{-1} \int_{Q\cap\Omega_R\cap B_{r_2}} w_\lambda(x)\,u(x)\,dx\,.$$

For $n \geq n_0$ define the operator S_n by

$$S_n u(x) = 2^{-n}\,\frac{u_{Q_n}}{\|w_\lambda\|_{\infty,Q_n}}\,, \qquad x \in Q_n\,.$$

The operator T_K is then given by

$$T_K u(x) = \sum_{n=n_0}^{\infty} |\mathbf{b}(x)|\, S_n u(x)\,, \qquad x \in \Omega_R \cap B_{r_2}\ .$$

It follows now from Lemmas 6.5, 6.6, 6.7 and Jensen's inequality that one can choose λ in a universal way such that for every cube Q

$$\begin{aligned}\int_{Q\cap\Omega_R\cap B_{r_2}} & w_\lambda(x)^r\, |Tu(x)|^r\, dx \\ &\le \frac{C^r}{|\Omega_R \cap B_{r_2}|} \int_{\Omega_R\cap B_{r_2}} d\xi \int_{Q\cap\Omega_R\cap B_{r_2}} w_\lambda(x)^r\, |T_{Q_0(\xi)} u(x)|^r\, dx\,,\end{aligned}$$

for some universal constant C. Hence it is sufficient to prove the result of the lemma for the operator T_K.

Next we have the analogue of Lemma 2.1. Thus let $Q' \subset K$ be an arbitrary dyadic subcube of K with side of length $2^{-n_{Q'}}$. Suppose r, p satisfy the inequality $1 \le r < p$. Then there are constants ε, $C > 0$ depending only on r, p such that $|Q|^{1/3+\varepsilon}\, u_Q \le |Q'|^{1/3+\varepsilon}\, u_{Q'}$ for all dyadic subcubes Q of Q' implies the inequality

$$\int_{Q'} w_\lambda(x)^r \Big(\sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|\, S_n u(x) \Big)^r dx \le C^r\, \|\mathbf{b}\|_{3,p}^r\, |Q'| u_{Q'}^r\ .$$

The analogue of Corollary 2.1 follows from this last inequality. Thus we have for any dyadic subcube $Q' \subset K$,

$$\int_{Q'} w_\lambda(x)^r \Big(\sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|\, S_n u(x) \Big)^r dx \le C^r\, \|\mathbf{b}\|_{3,p}^r \int_{Q'} w_\lambda(x)^r\, |u(x)|^r\, dx\,.$$

To complete the proof of the lemma we need to show that for any dyadic subcube $Q' \subset K$ one has

$$\int_{Q'} w_\lambda(x)^r \Big(\sum_{n=n_0}^{n_{Q'}-1} |\mathbf{b}(x)|\, S_n u(x) \Big)^r dx \le C^r\, \|\mathbf{b}\|_{3,p}^r\, \|u\|_{q,r,w_\lambda}^r\, |Q'|^{1-r/q}\,,$$

for some constant C. This inequality is clear.

Next we prove the analogue of Lemma 4.1.

Lemma 6.9. *Suppose g is a function defined on D_2 and let $Pg(x)$, $x \in \Omega_R \cap B_{r_2}$, be the function given by the solution of the Dirichlet problem* (6.5)*. Let r, p, q, q_1 be as in Lemma* 4.1*. Then if $g \in L^q(D_2)$ the function $\mathbf{b} \cdot \nabla Pg$ is in the Morrey space $M^{q_1}_{r,w_\lambda}(\Omega_R \cap B_{r_2})$ for some universal $\lambda > 0$ and*

$$\|\mathbf{b} \cdot \nabla Pg\|_{q_1,r,w_\lambda} \le CR^{2/q-1} \|\mathbf{b}\|_{3,p} \, \|g\|_{D_2,q} \, .$$

PROOF. The inequality will follow just as in Lemma 4.1 if we can show that

$$(6.19) \quad w_\lambda(x) \, |\nabla Pg(x)| \le CR^{-1}(P|g|(x) + \|g\|_{D_2,1}) \, , \quad x \in \Omega_R \cap B_{r_2} \, ,$$

where C is a constant depending only on λ. To prove (6.19) first consider the case where $d(x, D_2) < \lambda R$. Since the Harnack principle implies that

$$d(x, \partial B_{r_2}) \, |\nabla Pg(x)| \le CP \, |g|(x) \, ,$$

the inequality follows. Next suppose $d(x, D_2) > \gamma R$. If $d(x, \partial(\Omega_R \cap B_{r_2})) > cR$ for an arbitrary constant $c > 0$ then the Harnack principle again implies that

$$w_\lambda(x) \, |\nabla Pg(x)| \le C_1 \, |\nabla Pg(x)| \le C_2 \, R^{-1} \, \|g\|_{D_2,1} \, ,$$

where C_2 depends on c. Hence we may assume that $d(x, D_2) > \lambda R$ and $d(x, \partial(\Omega_R \cap B_{r_2})) < cR$ where $c > 0$ can be arbitrarily small. We proceed now as in the argument of Lemma 6.5. Thus let x_0 be the nearest point on $\partial(\Omega_R \cap B_{r_2})$ to x and $x_1 = x_0 + \gamma R \, (x_0 - x)/|x_0 - x|$, where γ is to be chosen depending on λ, c. Let $U_x = \{z : |z - x_1| > \gamma R\}$. Then it is clear that we may choose γ sufficiently small so that $\Omega_R \cap B_{r_2} \subset U_x$ and $|z - x_1| > 3\gamma R$ if $z \in D_2$. Next let $v(z) = 1 - |x_0 - x_1|/|z - x_1|$, $z \in U_x$ and W be the region $W = \{z \in \Omega_R \cap B_{r_2} : |z - x_1| < 2\gamma R\}$. Evidently the functions $P|g|(z)$ and $v(z)$ are harmonic in W and there is a constant C depending on γ such that

$$P|g|(z) \le C \, \|g\|_{D_{2,1}} \, v(z) \, , \qquad z \in \partial W \, .$$

Hence by the maximum principle this last inequality holds for all $z \in W$. For c sufficiently small $x \in W$ and hence there is a constant C such that

$$P|g|(x) \le C \, \|g\|_{D_2,1} \, \frac{d(x, \partial(\Omega_R \cap B_{r_2}))}{R} \, .$$

Using the Harnack principle we immediately conclude that

$$|\nabla Pg(x)| \le C\,\frac{\|g\|_{D_2,1}}{R}\ ,$$

for some constant C. Hence (6.19) holds in all cases.

Lemma 6.10. *Let ρ_3 be a density on $D_3 = \partial B(a_1, R/3)$ which satisfies $\|\rho_3\|_{D_3,q} \le C_3 \operatorname{Av}_{D_3}\rho_3$. Let ρ_2 be the density induced on D_2 by the paths of the drift process $X_{\mathbf{b}}(t)$ which start on D_3 with density ρ_3 and exit the region $\Omega_R \cap B_{r_2}$ through D_2. Then if $\mathbf{b} \in M^3_p$, $\|\mathbf{b}\|_{3,p} < \varepsilon$ and ε is sufficiently small there are constants C_2, c_2 such that*

$$\|\rho_2\|_{D_2,q} \le C_2 \operatorname{Av}_{D_2}\rho_2\ , \qquad Av_{D_2}\rho_2 \ge c_2 \operatorname{Av}_{D_3}\rho_3\ .$$

Proof. Let $g \in L^q(D_2)$. We consider the operator Q analogous to (3.3), defined by

$$Qg(x) = \int_{\Omega_R\cap B_{r_2}} G_D(x,y)\,(I-T)^{-1}\,\mathbf{b}\cdot\nabla Pg(y)\,dy\,, \qquad x \in D_3\ .$$

Then $\rho_2 = P^*\rho_3 + Q^*\rho_3$. We shall show just as in Proposition 4.1 that for any q, $1 < q < \infty$, and ε sufficiently small Q is a bounded operator from $L^q(D_2)$ to $L^q(D_3)$ and $\|Q\| \le C\,\|\mathbf{b}\|_{3,p}$. The result follows from this by the same argument as in Section 4.

To prove that Q is bounded we use Lemmas 6.8, 6.9. Thus if ε is sufficiently small the function

$$h(y) = (I-T)^{-1}\,\mathbf{b}\cdot\nabla Pg(y)\,, \qquad y \in \Omega_R \cap B_{r_2}\ ,$$

is in the weighted Morrey space $M^{q_1}_{r,w_\lambda}(\Omega_R \cap B_{r_2})$ where q_1 is given by (3.5) and $\lambda > 0$ is universal. Furthermore by Lemma 6.9 there is the bound

$$\|h\|_{q_1,r,w_\lambda} \le CR^{2/q-1}\|\mathbf{b}\|_{3,p}\,\|g\|_{D_2,q}\ .$$

For $\gamma > 0$ let $W_1 = \{y \in \Omega_R \cap B_{r_2} : d(y,\partial B_{r_2}) > \gamma R\}$ and $W_2 = \Omega_R \cap B_{r_2}\backslash W_1$. It is clear that for γ and λ sufficiently small there is a constant C such that

$$G_D(x,y) \le C\,\frac{w_\lambda(y)}{|x-y|}\ , \qquad y \in W_1\ ,\ x \in D_3\ ,$$

$$G_D(x,y) \le C\,\frac{w_\lambda(y)}{R}\ , \qquad y \in W_2\ ,\ x \in D_3\ .$$

Hence we have

$$|Qg(x)| \le C \int_{W_1} \frac{w_\lambda(y)\,|h(y)|}{|x-y|}\,dy + \frac{C}{R}\int_{W_2} w_\lambda(y)\,|h(y)|\,dy\,, \qquad x \in D_3\ .$$

Now we argue exactly as in Proposition 4.1 to see that $\|Q\| \le C\,\|\mathbf{b}\|_{3,p}$.

PROOF OF THEOREM 6.1. If $\xi = 0$ and $C_1 = \infty$ the result is a consequence of Lemmas 6.4 and 6.10. Hence it is sufficient for us to prove that for $\xi > 0$ small and $C_1 < \infty$ large then $\mathrm{Av}_{D_2}\rho_2 \ge c\,\mathrm{Av}_{D_1}\rho_1$ for some constant $c > 0$. For $C_1 < \infty$ we argue as in Lemma 6.2 using [5, Theorem 1.1]. For $\xi > 0$ we argue as in Lemma 6.3 and use Proposition 5.3.

7. Nonperturbative estimates on the exit probabilities from a spherical shell.

In this section we shall generalize Corollary 4.2 to the nonperturbative case. The main tool we use to do this is the following nonperturbative version of Theorem 6.1:

Theorem 7.1. *Let $R = 2^{-n}$, n an integer, $n \ge n_0$, and ρ_1 be a density on D_1. Suppose $f \in M_r^t(\mathbb{R}^3)$ with $1 < r \le t$, $r < p$, $3/2 < t < 3$. Let $\overline{\rho}_2$ be the density induced on D_2 by the paths of the drift process $X_{\mathbf{b}}(t)$ which start on D_1, exit the region $\Omega_R \cap B_{r_2}$ through D_2, and satisfy the inequality*

$$\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt \le C_1\,R^{2-3/t}\,\|f\|_{t,r}\ ,$$

where C_1 is a constant. Then for $\eta > 0$, $1 < q < \infty$ and C_1 sufficiently large there exist constants $\alpha > 1$, β, C_2, $c_2 > 0$ such that

$$\|\rho_1\|_{D_1,q} \le C_2\,\alpha^{n-n_0}\,\mathrm{Av}_{D_1}\rho_1$$

implies that there is a function ρ_2 on D_2 such that $\overline{\rho}_2(x) \ge \rho_2(x) \ge 0$, $x \in D_2$, and

$$\|\rho_2\|_{D_2,q} \le C_2\,\mathrm{Av}_{D_2}\rho_2\ ,$$

$$\mathrm{Av}_{D_2}\rho_2 \ge c_2\,\mathrm{Av}_{D_1}\rho_1 \exp\Big(-\frac{\beta}{R}\int_{\Omega_R} V_\eta(x)\,dx\Big)\,.$$

REMARK. Theorem 6.1 implies Theorem 7.1 when (6.3) holds by taking $\beta > 0$. We can prove Theorem 7.1 under the assumption that $\mathbf{b} \in L^\infty$ since none of the constants depend on $\mathbf{b}$. In that case when $R = 2^{-n}$ and n is sufficiently large we are in the perturbative case and the theorem follows again from Theorem 6.1.

We shall prove Theorem 7.1 by induction. In particular we will prove that if m is an integer, $m \geq n_0$ and if Theorem 7.1 holds for $R = 2^{-n}$, $n > m$, then it also holds for $R = 2^{-m}$. The key fact in reducing the $R = 2^{-m}$ case to the case $R = 2^{-n}$, $n > m$, is the following:

Lemma 7.1. *For $x \in D_1$, $z \in D_2$, let $\Gamma_{x,z,k}$ be the cylinder whose axis is the line joining x to z and with radius 2^{-k}. Let $V : \Omega_R \longrightarrow \mathbb{R}$ be a nonnegative potential. Then there is a universal constant C such that*

$$\int_{D_1} d\mu(x) \int_{D_2} d\mu(z) \int_{\Gamma_{x,z,k} \cap \Omega_R} V(y)\, dy \leq C\Big(\frac{2^{-2k}}{R^2}\Big) \int_{\Omega_R} V(y)\, dy\,,$$

where $d\mu$ denotes the normalized euclidean measures on D_1, D_2.

PROOF. Let $\chi_{x,z,k}$ be the characteristic function of the set $\Gamma_{x,z,k} \cap \Omega_R$. For any $y \in \Omega_R$ either $|y - a_1| \geq R/2$ or $|y - a_2| \geq R/2$. Suppose $|y - a_1| \geq R/2$. Then there is a universal constant C such that

$$\int_{D_2} \chi_{x,z,k}(y)\, d\mu(z) \leq C\Big(\frac{2^{-2k}}{R^2}\Big)\,, \qquad x \in D_1\ .$$

Similarly if $|y - a_2| \geq R/2$ we have

$$\int_{D_1} \chi_{x,z,k}(y)\, d\mu(x) \leq C\Big(\frac{2^{-2k}}{R^2}\Big)\,, \qquad z \in D_2\ .$$

The lemma follows easily from these last two inequalities.

Lemma 7.2. *For $x \in D_1$ and $\delta > 0$ let $D_x = \{y \in D_1 : |y - x| < \delta\}$. Suppose $\gamma, q > 1$ and $\|f\|_{D_1,q} \leq \mathcal{K}\, \mathrm{Av}_{D_1} f$. Let G be the set*

$$G = \{x \in D_1 :\ d(x, \partial D_1) > 2\,\delta,\ \|f\|_{D_x,q} \leq \mathcal{K}\,\gamma\, \mathrm{Av}_{D_x} f\}\,.$$

Then there is a universal constant C such that

$$\int_G \mathrm{Av}_{D_x} f\, d\mu(x) \geq \Big(1 - \frac{1}{\gamma} - C\Big(\frac{\delta}{R}\Big)^{1/q'} \mathcal{K}\Big) \mathrm{Av}_{D_1} f\ .$$

PROOF. We have

$$\begin{aligned}
\mathrm{Av}_{D_1} f &= \frac{1}{|D_1|} \int_{D_1} f(y)\, dy \\
&= \frac{1}{|D_1|} \int_{D_1} f(y) \frac{1}{|D_1 \cap B(y,\delta)|}\, dy \int_{D_1} \chi_{D_x}(y)\, dx\,,
\end{aligned}$$

where χ_{D_x} is the characteristic function of D_x. Letting $H_i = \{x \in D_1 : d(x, \partial D_1) > i\,\delta\}$, $i = 1, 2, \ldots$, we can rewrite this last expression as

$$\begin{aligned}
\mathrm{Av}_{D_1} f = \frac{1}{|D_1|} &\int_{D_1 \setminus H_1} f(y) \frac{1}{|D_1 \cap B(y,\delta)|}\, dy \int_{D_1} \chi_{D_x}(y)\, dx \\
&+ \frac{1}{|D_1|} \int_{H_1} f(y) \frac{dy}{|D_x|} \int_{D_1} \chi_{D_x}(y)\, dx\,.
\end{aligned} \tag{7.1}$$

Next observe that

$$\begin{aligned}
\frac{1}{|D_1|} &\int_{H_1} f(y) \frac{dy}{|D_x|} \int_{D_1} \chi_{D_x}(y)\, dx \\
&= \frac{1}{|D_1|} \int_{H_1} f(y) \frac{dy}{|D_x|} \int_{D_1 \setminus H_2} \chi_{D_x}(y)\, dx \\
&\quad + \frac{1}{|D_1|} \int_{H_2} \mathrm{Av}_{D_x} f\, dx\,.
\end{aligned} \tag{7.2}$$

We can bound the first term in (7.1) as

$$\begin{aligned}
\frac{1}{|D_1|} \int_{D_1 \setminus H_1} f(y)\, dy &\le \frac{|D_1 \setminus H_1|^{1/q'}}{|D_1|^{1/q'}} \Big(\frac{1}{|D_1|} \int_{D_1} f(y)^q\, dy \Big)^{1/q} \\
&\le \Big(\frac{|D_1 \setminus H_1|}{|D_1|} \Big)^{1/q'} \|f\|_{D_1, q} \\
&\le \frac{1}{2}\, C \Big(\frac{\delta}{R} \Big)^{1/q'} \mathcal{K}\, \mathrm{Av}_{D_1} f\,,
\end{aligned}$$

for some universal constant C. Similarly we can bound the first term in (7.2) by

$$\frac{1}{|D_1|} \int_{D_1 \setminus H_3} f(y)\, dy \le \frac{1}{2}\, C \Big(\frac{\delta}{R} \Big)^{1/q'} \mathcal{K}\, \mathrm{Av}\, f\,.$$

We conclude from these last two inequalities that

$$\frac{1}{|D_1|}\int_{H_2} \mathrm{Av}_{D_x} f\,dx \geq \Big(1 - C\Big(\frac{\delta}{R}\Big)^{1/q'}\mathcal{K}\Big)\mathrm{Av}\,f\,.$$

Next observe that

$$\begin{aligned}\frac{1}{|D_1|}\int_{H_2\backslash G} \mathrm{Av}_{D_x} f\,dx &\leq \frac{1}{|D_1|\mathcal{K}\,\gamma}\int_{H_2\backslash G}\|f\|_{D_x,q}\,dx\\ &= \frac{1}{|D_1|\mathcal{K}\,\gamma}\int_{H_2\backslash G}\Big(\frac{1}{|D_x|}\int_{D_x} f(y)^q\,dy\Big)^{1/q}dx\\ &\leq \frac{|H_2\backslash G|^{1/q'}}{|D_1|\mathcal{K}\,\gamma}\Big(\int_{H_2\backslash G}\frac{dx}{|D_x|}\int_{D_x} f(y)^q\,dy\Big)^{1/q}\\ &\leq \Big(\frac{|H_2\backslash G|}{|D_1|}\Big)^{1/q'}\frac{1}{\mathcal{K}\,\gamma}\Big(\frac{1}{|D_1|}\int_{D_1} f(y)^q\,dy\Big)^{1/q}\\ &= \Big(\frac{|H_2\backslash G|}{|D_1|}\Big)^{1/q'}\frac{1}{\mathcal{K}\,\gamma}\,\|f\|_{D_1,q}\\ &\leq \frac{1}{\gamma}\,\mathrm{Av}_{D_1} f\,.\end{aligned}$$

The lemma follows from this last inequality and (7.3).

Let us assume now that Theorem 7.1 holds for $R = 2^{-n}$ with $n > m$, $m \geq n_0$, and consider the case $R = 2^{-m}$. If (6.3) holds the theorem is correct so we shall assume that (6.3) is violated. Put $k_0 = m$ and define an integer $k_1 > k_0$ by

$$2^{k_1-k_0} \sim 2^{\lambda_0}\Big(\frac{1}{R}\int_{\Omega_R} V_\eta(y)\,dy\Big)^{1/3}, \tag{7.4}$$

where $\lambda_0 \geq 0$ is a fixed integer to be chosen later. Since we are assuming that

$$\frac{1}{R}\int_{\Omega_R} V_\eta(y)\,dy \geq \xi\,2^{\eta'(m-n_0)} \tag{7.5}$$

and $m \geq n_0$ we should choose λ_0 to satisfy $2^{\lambda_0}\xi^{1/3} \geq 2$ to ensure $k_1 > k_0$.

Proposition 7.1. *Suppose that Theorem* 7.1 *holds for* $n > m \geq n_0$ *and that for every* $z \in D_2$ *the following inequality holds*

$$\frac{1}{2^{-k_1}} \int_{B(z,2^{-k_1})} V_\eta(y)\, dy \leq \xi\, 2^{\eta'(k_1-n_0)} . \tag{7.6}$$

Then Theorem 7.1 *holds for* $R = 2^{-m}$.

PROOF. From Lemma 7.1 we have that

$$\begin{aligned}\int_{D_1} d\mu(x) \int_{D_2} d\mu(z)\, \frac{1}{2^{-k_1}} \int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\, dy \\ \leq C\, 2^{-(k_1-k_0)}\, \frac{1}{R} \int_{\Omega_R} V_\eta(y)\, dy \\ \leq 2^{-\lambda_0}\, 2^{2(k_1-k_0-\lambda_0)} .\end{aligned} \tag{7.7}$$

Next for $x \in D_1$ and f a function on D_1 let $Av_{x,k_1} f$ be the average of f on the set $D_1 \cap B(x, 2^{-k_1-4})$ and $\|f\|_{x,k_1,q}$ be the corresponding L^q norm normalized so that $\|\mathbf{1}\|_{x,k_1,q} = 1$. Let $\tilde{D}_1$ be the set of $x \in D_1$ which satisfy the following properties:

a) $d(x, \partial D_1) > 2^{-k_1}$,

b) $\|\rho_1\|_{x,k_1,q} \leq C_2\, \alpha^{k_1+4-n_0}\, \mathrm{Av}_{x,k_1} \rho_1$,

c) $\displaystyle\int_{D_2} \frac{d\mu(z)}{2^{-k_1}} \int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\, dy \leq 2^{-\lambda_0/2}\, 2^{5\,(k_1-k_0-\lambda_0)/2}$.

In view of Lemma 7.2 we have that

$$\begin{aligned}\int_{\tilde{D}_1} \mathrm{Av}_{x,k_1} \rho_1\, d\mu(x) \\ \geq \Big(1 - \frac{1}{\alpha^{k_1+4-k_0}} - C\, 2^{-(k_1-k_0)/q'}\, \alpha^{k_0-n_0}\, C_2 \\ - C\, (2^{-\lambda_0/2} 2^{-(k_1-k_0-\lambda_0)/2})^{1/q'}\, \alpha^{k_0-n_0}\, C_2\Big) \mathrm{Av}_{D_1} \rho_1 .\end{aligned} \tag{7.8}$$

Observe that the last term in the previous expression is a consequence of the restriction c). In fact, in view of (7.7) one has

$$\frac{\mathrm{meas}\, \{x \in D_1 :\ \text{c) is violated}\}}{|D_1|} \leq 2^{-\lambda_0/2}\, 2^{-(k_1-k_0-\lambda_0)/2} .$$

Now from (7.4), (7.5) it follows that

$$2^{-(k_1-k_0)/q'}\alpha^{k_0-n_0} \le 2^{-\lambda_0/q'}\,\xi^{-\,1/3}\,2^{-\eta'(k_0-n_0)/3}\,\alpha^{k_0-n_0}\,.$$

Hence if we choose $\alpha < 2^{\eta'/3}$ and λ_0 sufficiently large depending on $\xi,\ C_2$ we can have

$$\int_{\tilde{D}_1} \mathrm{Av}_{x,k_1}\rho_1\,d\mu(x) \ge \frac{1}{2}\,\mathrm{Av}_{D_1}\rho_1\,. \tag{7.9}$$

For $x \in \tilde{D}_1$ we define a subset $\tilde{D}_2 \subset D_2$ as the set of $z \in D_2$ which satisfy

d) $d(z,\partial D_2) > 2^{-k_1}$,

e) $\displaystyle\frac{1}{2^{-k_1}}\int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\,dy \le 2^{-\lambda_0/4}\,2^{11(k_1-k_0-\lambda_0)/4}\,.$

From c), e) and the Chebyshev inequality we have that

$$\frac{|\tilde{D}_2|}{|D_2|} > 1 - 2^{-\lambda_0/4}\,2^{-(k_1-k_0-\lambda_0)/4} - C\,2^{-(k_1-k_0)}\,, \tag{7.10}$$

for some universal constant C. Evidently the set $\tilde{D}_2$ depends on $x \in \tilde{D}_1$.

Let $x \in \tilde{D}_1$, $z \in \tilde{D}_2$. Then we can use the induction hypothesis to propagate the density ρ_1 restricted to $D_1 \cap B(x, 2^{-k_1-4})$ through the cylinder Γ_{x,z,k_1}. To implement it we choose points $x_0, x_1, \ldots, x_N$ with the property that $x_0 = x$, $x_N = z$, $|x_i - x_{i+1}| = 2^{-k_1-2}$, $0 \le i \le N-1$, such that the balls centered at $(x_i + x_{i+1})/2$ with radius $2^{-(k_1+2)}$ are contained in Γ_{x,z,k_1}. Finally we insist that $N \le C\,2^{k_1-k_0}$ for some universal constant C.

Consider the ball B_0 centered at $(x_0+x_1)/2$ with radius $2^{-(k_1+2)}$. Letting $D_x = D_1 \cap B(x, 2^{-k_1-4})$ then from b) and the induction hypothesis ρ_1 restricted to D_x can be propagated to a density $\rho_1^{(1)}$ on $D_{x_1} = \partial B_r \cap B(x_1, 2^{-k_1-4})$ where B_r is a ball of radius $r \ge 10\,2^{-k_1-2}$ such that $x_1 \in \partial B_r$. Furthermore $\rho_1^{(1)}$ satisfies the conditions

$$\|\rho_1^{(1)}\|_{D_{x_1},q} \le C_2\,\mathrm{Av}_{D_{x_1}}\rho_1^{(1)}\,,$$

$$\mathrm{Av}_{D_{x_1}}\rho_1^{(1)} \ge c_2\,\mathrm{Av}_{D_x}\rho_1 \exp\Big(\frac{-\beta}{2^{-k_1-2}}\int_{B_0} V_\eta(x)\,dx\Big)\,.$$

In view of the above inequalities and the induction assumption we may propagate $\rho_1^{(1)}$ to a density $\rho_1^{(2)}$ on $D_{x_2} = \partial B_r \cap B(x_2, 2^{-k_1-4})$ and continue to do this until we obtain a density $\rho_1^{(N-1)}$ on $D_{x_{N-1}} = \partial B_r \cap B(x_{N-1}, 2^{-k_1-4})$ with the properties

$$\|\rho_1^{(N-1)}\|_{D_{x_{N-1}},q} \le C_2 \operatorname{Av}_{D_{x_{N-1}}} \rho_1^{(N-1)} , \tag{7.11}$$

$$\begin{aligned} &\operatorname{Av}_{D_{x_{N-1}}} \rho_1^{(N-1)} \\ &\quad \ge c_2^{N-1} \exp\Big(\frac{-\beta}{2^{-k_1-1}} \int_{\Gamma_{x,z,k_1}} V_\eta(y)\, dy\Big) \operatorname{Av}_{D_x} \rho_1 . \end{aligned} \tag{7.12}$$

In the inequalities (7.11), (7.12) the constants C_2, c_2, β are from Theorem 7.1. They are therefore part of the induction hypothesis. To ensure that these constants continue to hold on the next level up we use the assumption (7.6). Hence in propagating $\rho_1^{(N-1)}$ to $\rho_1^{(N)}$ we may use the perturbative Theorem 6.1. Let us denote the constants C_2, c_2 in Theorem 6.1 by $C_{2,\text{perturb}}$ and $c_{2,\text{perturb}}$ to distinguish them from the corresponding constants C_2, c_2 in Theorem 7.1. It is clear that by choosing λ_0 large enough we have

$$C_2 \le C_{2,\text{perturb}}\, \alpha^{k_1+2-n_0} . \tag{7.13}$$

Hence by Theorem 6.1 $\rho_1^{(N-1)}$ propagates to a density $\rho_1^{(N)}$ on $D_{x_N} = D_2 \cap B(z, 2^{-k_1-4})$ which has the properties

$$\|\rho_1^{(N)}\|_{D_{x_N},q} \le C_{2,\text{perturb}} \operatorname{Av}_{D_{x_N}} \rho_1^{(N)} , \tag{7.14}$$

$$\begin{aligned} &\operatorname{Av}_{D_{x_N}} \rho_1^{(N)} \ge c_{2,\text{perturb}}\, c_2^{N-1} \\ &\quad \cdot \exp\Big(\frac{-\beta}{2^{-k_1-1}} \int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\, dy\Big) \operatorname{Av}_{D_x} \rho_1 . \end{aligned} \tag{7.15}$$

Evidently we can assume $c_2 < 1$ and $c_2 < c_{2,\text{perturb}}$. Hence the inequality (7.15) yields

$$\begin{aligned} \operatorname{Av}_{D_{x_N}} \rho_1^{(N)} &\ge \exp\Big(- N \log\Big(\frac{1}{c_2}\Big) - \frac{\beta}{2^{-k_1-1}} \int_{\Gamma_{x,z,k_1}\cap\Omega_R} V_\eta(y)\, dy\Big) \\ &\quad \cdot \operatorname{Av}_{D_x} \rho_1 \\ &\ge \exp\Big(- C\, 2^{k_1-k_0} \log\Big(\frac{1}{c_2}\Big) - 2\beta 2^{-\lambda_0/4}\, 2^{11(k_1-k_0-\lambda_0)/4}\Big) \\ &\quad \cdot \operatorname{Av}_{D_x} \rho_1 , \end{aligned}$$

upon using e) and the fact that $N \le C\,2^{k_1-k_0}$. Observe now that

$$C\,2^{k_1-k_0}\log\Big(\frac{1}{c_2}\Big) + 2\,\beta\,2^{-\lambda_0/4}\,2^{11(k_1-k_0-\lambda_0)/4}$$
$$= \beta\,2^{3(k_1-k_0-\lambda_0)} \cdot \left(\frac{C\,2^{\lambda_0}\log\Big(\frac{1}{c_2}\Big)}{\beta}\,2^{-2(k_1-k_0-\lambda_0)} + 2^{1-\lambda_0/4}\,2^{-(k_1-k_0-\lambda_0)/4}\right).$$

In view of the assumption (7.5) we can choose λ_0 dependent only on ξ such that

$$2^{1-\lambda_0/4}\,2^{-(k_1-k_0-\lambda_0)/4} < \frac{1}{4}\,.$$

With this choice of λ_0 and arbitrary c_2, $0 < c_2 < 1$ we can choose $\beta > 0$ such that

$$\beta^{-1}\,C\,2^{\lambda_0}\log\Big(\frac{1}{c_2}\Big)2^{-2(k_1-k_0-\lambda_0)} < \frac{1}{4}\,.$$

Hence it follows that

$$\operatorname{Av}_{D_{x_N}}\rho_1^{(N)} \ge \exp\Big(-\frac{\beta}{2R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\operatorname{Av}_{D_x}\rho_1\,. \tag{7.16}$$

We wish to define the density ρ_2 on D_2. For $x \in \tilde{D}_1$ let

$$\gamma(k_1) = |D_1 \cap B(x, 2^{-k_1-4})|\,.$$

Evidently $\gamma(k_1)$ is independent of x and $\gamma(k_1) \sim 2^{-2k_1}$. Also

$$\rho_1(y) \ge \int_{D_1} \gamma(k_1)^{-1}\,\rho_1(y)\,\chi_{D_x}(y)\,dx\,, \qquad y \in D_1\,, \tag{7.17}$$

where χ_{D_x} is the characteristic function of $D_x = D_1 \cap B(x, 2^{-k_1-4})$. For $x \in \tilde{D}_1$, $z \in \tilde{D}_2$ let $\rho_1^{x,z}$ be the density $\rho_1^{(N)}$ defined above. Thus

$$\rho_1^{x,z}(y) = \begin{cases} \rho_1^{(N)}(y)\,, & y \in D_{x_N}\,, \\ 0\,, & y \in D_2 \backslash D_{x_N}\,. \end{cases}$$

It follows then from (7.17) that the density $\overline{\rho}_2$ induced on D_2 by ρ_1 as in Theorem 7.1 satisfies

$$\overline{\rho}_2(y) \ge \int_{\tilde{D}_1} \frac{dx}{\gamma(k_1)\,|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}(y)\,dz\,, \qquad y \in D_2\,. \tag{7.18}$$

From (7.16) and the above we have that

$$\mathrm{Av}_{D_2}\overline{\rho}_2 \geq \exp\Big(\frac{-\beta}{2R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\frac{1}{|D_2|}\int_{\tilde{D}_2}\mathrm{Av}_{D_x}\rho_1\,dx\,.$$

Now if we use the inequality (7.9) we conclude that

$$\mathrm{Av}_{D_2}\overline{\rho}_2 \geq c_2\exp\Big(\frac{-\beta}{R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\mathrm{Av}_{D_1}\rho_1\ ,$$

provided c_2 is sufficiently small. This last inequality is consistent with the lower bound on $\mathrm{Av}_{D_2}\rho_2$ in Theorem 7.1.

It seems reasonable from the previous argument that we shall define ρ_2 by the right hand side of (7.18). We need to be more subtle than this in order to keep control of $\|\rho_2\|_{D_2,q}$ as required by Theorem 7.1. We accomplish this by insisting that the integral of $\rho_1^{x,z}$ is independent of $z \in \tilde{D}_2$. In view of (7.16) we may insist that

$$\mathrm{Av}_{D_z}\rho_1^{x,z} = \exp\Big(-\frac{\beta}{2R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\mathrm{Av}_{D_x}\rho_1\ ,\qquad z\in\tilde{D}_2\ . \tag{7.19}$$

Then the density ρ_2 is defined like the right hand side of (7.18) by

$$\rho_2 = \int_{\tilde{D}_1}\frac{dx}{\gamma(k_1)\,|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}\,dz\,. \tag{7.20}$$

Evidently $\mathrm{Av}_{D_2}\rho_2$ satisfies the lower bound of Theorem 7.1. To estimate $\|\rho_2\|_{D_2,q}$ we use the Minkowski inequality. Thus

$$\|\rho_2\|_{D_2,q} \leq \int_{\tilde{D}_1}\frac{dx}{\gamma(k_1)}\Big\|\frac{1}{|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}\,dz\Big\|_{D_2,q}\,. \tag{7.21}$$

Now we have

$$\Big\|\frac{1}{|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}\,dz\Big\|_{D_2,q}^q = \frac{1}{|D_2|}\int_{D_2}\Big(\frac{1}{|\tilde{D}_2|}\int_{\tilde{D}_2}\rho_1^{x,z}(y)\,dz\Big)^q\,dy\,.$$

Observe next that $\rho_1^{x,z}(y) = 0$ if $|z-y| > 2^{-k_1}$, whence

$$\Big(\int_{\tilde{D}_2}\rho_1^{x,z}(y)\,dz\Big)^q \leq C^q\,2^{-2k_1(q-1)}\int_{\tilde{D}_2}\rho_1^{x,z}(y)^q\,dz\,,$$

for some universal constant C. Hence

$$\begin{aligned}\Big\| \frac{1}{|\tilde{D}_2|}\int_{\tilde{D}_2} \rho_1^{x,z}\,dz\Big\|_{D_2,q}^q &\le \frac{C^q}{|D_2|}\int_{D_2} \frac{2^{-2k_1(q-1)}}{|\tilde{D}_2|^q}\,dy \int_{\tilde{D}_2} \rho_1^{x,z}(y)^q\,dz \\ &= \frac{C^q\,2^{-2k_1(q-1)}}{|D_2|\,|\tilde{D}_2|^q}\int_{\tilde{D}_2} dz\int_{D_2}\rho_1^{x,z}(y)^q\,dy \\ &\le \frac{C^q\,2^{-2k_1(q-1)}}{|D_2|\,|\tilde{D}_2|^q} \\ &\quad\cdot\int_{\tilde{D}_2} 2^{-2k_1}\,C^q_{2,\mathrm{perturb}}\Big(\mathrm{Av}_{D_z}\rho_1^{x,z}\Big)^q\,dz\,. \end{aligned} \tag{7.22}$$

If we use now this last inequality together with (7.19) and (7.21) we can conclude that

$$\|\rho_2\|_{D_2,q} \le C\,C_{2,\mathrm{perturb}} \exp\Big(-\frac{\beta}{2R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\frac{\int_{\tilde{D}_1}\mathrm{Av}_{D_x}\rho_1\,dx}{(|D_2|\,|\tilde{D}_2|^{q-1})^{1/q}}\,.$$

It follows from (7.10) that we can choose λ_0 sufficiently large depending only on ξ so that $(|D_2|\,|\tilde{D}_2|^{q-1})^{1/q} \ge |D_2|/2$. In view of (7.19) we have that

$$\mathrm{Av}_{D_2}\rho_2 \ge \exp\Big(-\frac{\beta}{2R}\int_{\Omega_R} V_\eta(y)\,dy\Big)\frac{1}{|D_2|}\int_{\tilde{D}_1}\mathrm{Av}_{D_x}\rho_1\,dx\,.$$

We conclude therefore that

$$\|\rho_2\|_{D_2,q} \le C\;C_{2,\mathrm{perturb}}\,\mathrm{Av}_{D_2}\rho_2\,, \tag{7.23}$$

where C is a universal constant. Theorem 7.1 follows then if we have $C\,C_{2,\mathrm{perturb}} \le C_2$. This inequality is consistent with the inequality (7.13) provided we choose λ_0 large enough.

REMARK 7.2. The assumption (7.6) is only used in concluding (7.23). If we did not assume (7.6) then the constant in (7.23) would be $C\,C_2$ and we obviously cannot conclude that $C\,C_2 \le C_2$ if $C > 1$.

PROOF OF THEOREM 7.1. The idea is to extend the argument of Proposition 7.1 until a perturbative situation holds at the boundary of D_2. This will require introduction of further cylindrical decompositions until we are in a situation where (7.6) holds. We begin as in Proposition 7.1 by defining $k_0 = m$, k_1 by (7.4) and assume that (7.5) holds. To simplify notation we shall refer to the set D_2 from here on in this proof as E_1, and the density ρ_1 on D_1 as ρ.

The set $\tilde{D}_1$ is defined exactly as in Proposition 7.1 by a), b), c) following (7.7). For $x_1 \in \tilde{D}_1$ we define a subset $\tilde{E}_1 \subset E_1$ which depends on x_1 similarly to the set $\tilde{D}_2$ of Proposition 7.1. Thus we define it by the conditions d), e) following (7.9) but we also impose the requirement (7.6). Thus $z_1 \in \tilde{E}_1$ if

$$\mathrm{d}^{(1)})\ d(z_1, \partial E_1) > 2^{-k_1}\,,$$

$$\mathrm{e}^{(1)})\ \frac{1}{2^{-k_1}} \int_{\Gamma_{x_1,z_1,k_1} \cap \Omega_R} V_\eta(y)\, dy \le 2^{-\lambda_0/4}\, 2^{11(k_1-k_0-\lambda_0)/4}\,,$$

$$\mathrm{f}^{(1)})\ \frac{1}{2^{-k_1}} \int_{B(z_1,2^{-k_1})} V_\eta(y)\, dy \le \xi\, 2^{\eta'(k_1-n_0)}\,.$$

The set $\tilde{F}_1 \subset E_1$ is defined as the set of $z_1 \in E_1$ for which $\mathrm{d}^{(1)}$), $\mathrm{e}^{(1)}$) above hold but not $\mathrm{f}^{(1)}$). The inequality (7.10) yields therefore the inequality

$$\frac{|\tilde{E}_1 \cup \tilde{F}_1|}{|E_1|} \ge 1 - 2^{-\lambda_0/4}\, 2^{-(k_1-k_0-\lambda_0)/4} - C\, 2^{-(k_1-k_0)}\,. \tag{7.24}$$

Evidently if λ_0 is sufficiently large depending on ξ the right hand side of the above inequality is strictly positive.

Now for $x_1 \in \tilde{D}_1$, $z_1 \in \tilde{E}_1$ we can as in Proposition 7.1 propagate the density ρ restricted to $D_1 \cap B(x_1, 2^{-k_1-4})$ to a density ρ_{x_1,z_1} on $E_1 \cap B(z_1, 2^{-k_1-4})$ whose average value and fluctuation we can control exactly as in Proposition 7.1. Next suppose $z_1 \in \tilde{F}_1$. Then we may use the induction hypothesis to propagate ρ restricted to $E_1 \cap B(z_1, 2^{-k_1-4})$ to a density ρ_{x_1,z_1} which is concentrated on a set $D_2 = \partial B_r \cap B(\overline{x}, 2^{-k_1-4})$ and $\overline{x}$ has the property that $B(\overline{x}, 2^{-k_1-4}) \subset B(z_1, 2^{-k_1-1}) \cap \Omega_R$ but has no intersection with $B(z_1, 2^{-k_1-2})$. The density ρ_{x_1,z_1} on D_2 corresponds to $\rho_1^{(N-1)}$ in Proposition 7.1 and can be controlled by the inequalities (7.11) and (7.12).

For $z_1 \in \tilde{F}_1$ we define k_2 by

$$2^{k_2-k_1} \sim 2^{\lambda_0} \Big(\frac{1}{2^{-k_1}} \int_{B(z_1,2^{-k_1})} V_\eta(y)\, dy \Big)^{1/3}.$$

Thus k_2 has the same relationship to k_1 as k_1 has to k_0, but now it depends on the variable $z_1 \in \tilde{F}_1$. Let $E_2 = B(z_1, 2^{-k_1-4}) \cap E_1$ and define $\tilde{D}_2$ in analogy with $\tilde{D}_1$. Thus $\tilde{D}_2 \subset D_2$ and $x_2 \in \tilde{D}_2$ if

a$^{(2)}$) $d(x_2, \partial D_2) > 2^{-k_2}$,

b$^{(2)}$) $\|\rho_{x_1,z_1}\|_{x_2,k_2,q} \le C_2\, \alpha^{k_2+4-n_0}\, \mathrm{Av}_{x_2,k_2}\rho_{x_1,z_1}$,

c$^{(2)}$)

$$\frac{1}{|E_2|}\int_{E_2} \frac{dz_2}{2^{-k_2}} \int_{\Gamma_{x_2,z_2,k_2}\cap B(z_1,2^{-k_1})} V_\eta(y)\, dy \le 2^{-\lambda_0/2}\, 2^{5(k_2-k_1-\lambda_0)/2} .$$

By (7.11) we have that $\|\rho_{x_1,z_1}\|_{D_2,q} \le C_2\, Av_{D_2}\rho_{x_1,z_1}$. In analogy to the derivation of (7.8) we have that

$$\frac{1}{|D_2|}\int_{\tilde{D}_2} \mathrm{Av}_{x_2,k_2}\rho_{x_1,z_1}\, dx_2 \ge \mathrm{Av}_{D_2}\rho_{x_1,z_1} \cdot \Big(1 - \frac{1}{\alpha^{k_2+4-n_0}} - C\, 2^{-(k_2-k_1)/q'}\, C_2 - C\, 2^{-(k_2-k_1)/2q'}\, C_2\Big) . \tag{7.25}$$

For $x_2 \in \tilde{D}_2$ we define a subset $\tilde{E}_2 \subset E_2$ in analogy to E_1. Thus $z_2 \in \tilde{E}_2$ if

d$^{(2)}$) $d(z_2, \partial E_2) > 2^{-k_2}$,

e$^{(2)}$) $\displaystyle\frac{1}{2^{-k_2}}\int_{\Gamma_{x_2,z_2,k_2}\cap B(z_1,2^{-k_1})} V_\eta(y)\, dy \le 2^{-\lambda_0/4}\, 2^{11(k_2-k_1-\lambda_0)/4}$,

f$^{(2)}$) $\displaystyle\frac{1}{2^{-k_2}}\int_{B(z_2,2^{-k_2})} V_\eta(y)\, dy \le \xi\, 2^{\eta'(k_2-n_0)}$.

The subset $\tilde{F}_2 \subset E_2$ is the set of $z_2 \in E_2$ for which d$^{(2)}$) and e$^{(2)}$) hold but not f$^{(2)}$). In analogy with (7.24) we have the inequality

$$\frac{|\tilde{E}_2 \cup \tilde{F}_2|}{|E_2|} \ge 1 - 2^{-\lambda_0/4}\, 2^{-(k_2-k_1-\lambda_0)/4} - C\, 2^{-(k_2-k_1)} .$$

For $x_2 \in \tilde{D}_2$, $z_2 \in \tilde{E}_2$ we use Proposition 7.1 to propagate the density ρ_{x_1,z_1} restricted to $D_2 \cap B(x_2, 2^{-k_2-4})$ to a density on $E_2 \cap B(z_2, 2^{-k_2-4})$

whose average value and fluctuation we can control. This density is denoted by ρ_{x_1,z_1,x_2,z_2}. Just as previously if $z_2 \in \tilde{F}_2$ we use the induction hypothesis to propagate ρ_{x_1,z_1} restricted to $D_2 \cap B(x_2, 2^{-k_2-4})$ to a density ρ_{x_1,z_1,x_2,z_2} concentrated on a set $D_3 = B(\overline{x}, 2^{-k_2-4}) \cap \partial B_r$. The point $\overline{x}$ is to be chosen similarly to before. Thus we require that $B(\overline{x}, 2^{-k_2-4})$ is contained in $B(z_2, 2^{-k_2-1}) \cap \Omega_R$ but has no intersection with $B(z_2, 2^{-k_2-2})$.

Evidently we may continue this process by induction. Thus we obtain densities $\rho_{x_1,z_1}, \rho_{x_1,z_1,x_2,z_2}, \dots, \rho_{x_1,z_1,x_2,z_2,\dots,x_r,z_r,\dots}$, where ρ_{x_1,z_1} is defined for $x_1 \in \tilde{D}_1 \subset D_1$, $z_1 \in \tilde{E}_1(x_1) \subset E_1$. The function ρ_{x_1,z_1,x_2,z_2} is defined for $x_1 \in \tilde{D}_1 \subset D_1$, $z_1 \in \tilde{F}_1(x_1) \subset E_1, x_2 \in \tilde{D}_2(x_1, z_1) \subset D_2(x_1, z_1)$, $z_2 \in \tilde{E}_2(x_1, z_1, x_2) \subset E_2(z_1)$. Here we have shown the dependence of the sets $\tilde{E}_1$, $\tilde{E}_2$ etc. on the variables x_1, x_2, z_1, z_2. More generally the density $\rho_{x_1,z_1,\dots x_r,z_r}$ is defined for $x_1 \in \tilde{D}_1 \subset D_1$, $z_1 \in \tilde{F}_1(x_1) \subset E_1, \dots,$

$$x_{r-1} \in \tilde{D}_{r-1}(x_1, z_1, \dots, x_{r-2}, z_{r-2}) \subset D_{r-1}(x_1, z_1, \dots, x_{r-2}, z_{r-2}),$$

$$z_{r-1} \in \tilde{F}_{r-1}(x_1, z_1, \dots, x_{r-2}, z_{r-2}, x_{r-1}) \subset E_{r-1}(z_{r-2}),$$

$$x_r \in \tilde{D}_r(x_1, z_1, \dots, x_{r-1}, z_{r-1}) \subset D_r(x_1, z_1, \dots, x_{r-1}, z_{r-1}),$$

$$z_r \in \tilde{E}_r(x_1, z_1, \dots, x_{r-1}, z_{r-1}, x_r) \subset E_r(z_{r-1}).$$

Letting $\overline{\rho}$ be the density ρ propagated to E_1, it is clear by analogy with (7.18) that we have

$$\begin{aligned}
\overline{\rho} \geq & \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{E}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|} \rho_{x_1,z_1} \\
& + \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{F}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|} \int_{\tilde{D}_2} \frac{dx_2}{\gamma(k_2)} \\
& \qquad \cdot \int_{\tilde{E}_2} \frac{dz_2}{|\tilde{E}_2 \cup \tilde{F}_2|} \rho_{x_1,z_1,x_2,z_2} \\
& + \cdots + \int_{\tilde{D}_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde{F}_1} \frac{dz_1}{|\tilde{E}_1 \cup \tilde{F}_1|} \cdots \int_{\tilde{D}_r} \frac{dx_r}{\gamma(k_r)} \\
& \qquad \cdot \int_{\tilde{E}_r} \frac{dz_r}{|\tilde{E}_r \cup \tilde{F}_r|} \rho_{x_1,z_1,\dots,x_r,z_r} \\
& + \cdots,
\end{aligned} \tag{7.26}$$

where $\gamma(k) \sim 2^{-2k}$. Observe that the previous sum is finite. In fact k_r is defined by

$$2^{k_r - k_{r-1}} \sim 2^{\lambda_0} \Big(\frac{1}{2^{-k_{r-1}}} \int_{B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy \Big)^{1/3}, \tag{7.27}$$

and one also has the inequality

$$\frac{1}{2^{-k_{r-1}}} \int_{B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy \geq \xi\, 2^{\eta'(k_{r-1} - n_0)}\,.$$

Since we may assume $\mathbf{b} \in L^\infty(\mathbb{R}^3)$ this last inequality cannot hold for arbitrarily large k_{r-1}, whence r is bounded since $k_r \geq k_{r-1} + 1$. The last two inequalities imply that

$$2^{k_r - k_{r-1}} \geq 2^{\lambda_0} (\xi\, 2^{\eta'(k_{r-1} - n_0)})^{1/3},$$

and hence

$$k_r - n_0 \geq \Big(1 + \frac{\eta'}{3}\Big)(k_{r-1} - n_0) + 1\,, \tag{7.28}$$

if we choose λ_0 to satisfy $2^{\lambda_0} \xi^{1/3} > 2$. Thus $k_r - n_0$ is increasing exponentially fast as a function of r. Next we shall show that the difference $k_r - k_{r-1}$ actually *decreases*. To see this observe that

$$\begin{aligned}
&\frac{1}{2^{-k_r}} \int_{B(z_r, 2^{-k_r})} V_\eta(y)\, dy \\
&\qquad \leq \frac{1}{2^{-k_r}} \int_{\Gamma_{x_r, z_r, k_r} \cap B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy \\
&\qquad \leq 2^{-\lambda_0/4} \Big(\frac{1}{2^{-k_{r-1}}} \int_{B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy \Big)^{11/12} \\
&\qquad \leq 2^{-\lambda_0/4} (\xi\, 2^{\eta'(k_{r-1} - n_0)})^{-1/12} \frac{1}{2^{-k_{r-1}}} \int_{B(z_{r-1}, 2^{-k_{r-1}})} V_\eta(y)\, dy\,.
\end{aligned}$$

Here we have used the definition (7.27) of k_r and the condition e$^{(r)}$) corresponding to e$^{(2)}$). Hence from (7.27) we have the inequality

$$2^{3(k_{r+1} - k_r - \lambda_0)} \leq 2^{-\lambda_0/4} (\xi\, 2^{\eta'(k_{r-1} - n_0)})^{-1/12}\, 2^{3(k_r - k_{r-1} - \lambda_0)}\,.$$

It follows that we may choose λ_0 large enough depending only on ξ such that

$$2^{k_r-k_{r-1}} \le \frac{1}{2^r}\, 2^{k_1-k_0}. \tag{7.29}$$

We shall use (7.26), (7.28), (7.29) to get a lower bound on $Av_{E_1}\overline{\rho}$. Suppose $x_1 \in \tilde{D}_1$, $z_1 \in \tilde{F}_1$, $x_2 \in \tilde{D}_2$, $z_2 \in \tilde{F}_2 \cdots x_{r-1} \in \tilde{D}_{r-1}$, $z_{r-1} \in \tilde{F}_{r-1}$, $x_r \in \tilde{D}_r$. Then in analogy to (7.12) we have from the induction hypothesis that if $z_r \in \tilde{F}_r$,

$$\begin{aligned}
\int_{D_{r+1}} &\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy \\
&\ge \int_{D_r \cap B(x_r,2^{-k_r-4})} \rho_{x_1,z_1,\ldots x_{r-1},z_{r-1}}(y)\,dy \\
&\quad \cdot \exp\Big(-C\,2^{k_r-k_{r-1}}\log\Big(\frac{1}{c_2}\Big) \\
&\qquad - \frac{\beta}{2^{-k_r}}\int_{\Gamma_{x_r,z_r,k_r}\cap B(z_{r-1},2^{-k_{r-1}})} V_\eta(y)\,dy\Big).
\end{aligned}$$

Using now the condition e$^{(r)}$) corresponding to e$^{(1)}$) we conclude

$$\begin{aligned}
\int_{D_{r+1}} &\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy \\
&\ge \int_{D_r \cap B(x_r,2^{-k_r-4})} \rho_{x_1,z_1,\ldots x_{r-1},z_{r-1}}(y)\,dy \\
&\quad \cdot \exp\Big(-C\,2^{k_r-k_{r-1}}\log\Big(\frac{1}{c_2}\Big) - \beta\,2^{-\lambda_0/4}\,2^{11(k_r-k_{r-1}-\lambda_0)/4}\Big).
\end{aligned} \tag{7.30}$$

Similarly if $z_r \in \tilde{E}_r$ then

$$\begin{aligned}
\int_{E_1} &\rho_{x_1,z_1,\cdots,x_r,z_r}(y)\,dy \\
&\ge \int_{D_r \cap B(x_r,2^{-k_r-4})} \rho_{x_1,z_1,\ldots x_{r-1},z_{r-1}}(y)\,dy \\
&\quad \cdot \exp\Big(-C\,2^{k_r-k_{r-1}}\log\Big(\frac{1}{c_2}\Big) - \beta\,2^{-\lambda_0/4}\,2^{11(k_r-k_{r-1}-\lambda_0)/4}\Big).
\end{aligned}$$

Consequently we have that

$$\begin{aligned}\int_{\tilde{D}_r}\frac{dx_r}{\gamma(k_r)}\Big(\int_{\tilde{F}_r}\frac{dz_r}{|\tilde{E}_r\cup\tilde{F}_r|}\int_{\tilde{D}_{r+1}}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\\ +\int_{\tilde{E}_r}\frac{dz_r}{|\tilde{E}_r\cup\tilde{F}_r|}\int_{E_1}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\Big)\\ \geq\int_{\tilde{D}_r}\frac{dx_r}{\gamma(k_r)}\int_{D_r\cap B(x_r,2^{-k_r-4})}\rho_{x_1,z_1,\ldots,x_{r-1},z_{r-1}}(y)\,dy\\ \cdot\exp\Big(-C\,2^{k_r-k_{r-1}}\log\Big(\frac{1}{c_2}\Big)-\beta\,2^{-\lambda_0/4}\,2^{11(k_r-k_{r-1}-\lambda_0)/4}\Big).\end{aligned}$$

Next observe that in analogy to (7.25) we have

$$\begin{aligned}\int_{\tilde{D}_r}\frac{dx_r}{\gamma(k_r)}\int_{D_r\cap B(x_r,2^{-k_r-4})}\rho_{x_1,z_1,\ldots x_{r-1},z_{r-1}}(y)\,dy\\ \geq\int_{D_r}\rho_{x_1,z_1,\ldots,x_{r-1},z_{r-1}}(y)\,dy\\ \cdot\Big(1-\frac{1}{\alpha^{k_r+4-n_0}}-C\,2^{-(k_r-k_{r-1})/q'}C_2-C\,2^{-(k_r-k_{r-1})/2q'}C_2\Big).\end{aligned}$$

It is clear from (7.28) that there is a constant $a>1$ such that

$$\frac{1}{\alpha^{k_r+4-n_0}}+C\,2^{-(k_r-k_{r-1})/q'}C_2+C\,2^{-(k_r-k_{r-1})/2q'}C_2<\frac{1}{a^r}\,, \tag{7.31}$$

$r=1,2,\ldots.$ From the last three inequalities and (7.29) we conclude that

$$\begin{aligned}\int_{\tilde{D}_r}\frac{dx_r}{\gamma(k_r)}\Big(\int_{\tilde{F}_r}\frac{dz_r}{|\tilde{E}_r\cup\tilde{F}_r|}\int_{D_{r+1}}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\\ +\int_{\tilde{E}_r}\frac{dz_r}{|\tilde{E}_r\cup\tilde{F}_r|}\int_{E_1}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\Big)\\ \geq\Big(1-\frac{1}{a^r}\Big)\int_{D_r}\rho_{x_1,z_1,\ldots,x_{r-1},z_{r-1}}(y)\,dy\\ \cdot\exp\Big(-\frac{C}{2^r}\,2^{k_1-k_0}\log\Big(\frac{1}{c_2}\Big)-\beta\,2^{-\lambda_0/4}\,\frac{C^{11/4}}{2^{11r/4}}\,2^{11(k_1-k_0-\lambda_0)/4}\Big),\end{aligned}$$

for some constant C depending only on ξ. We may apply the previous inequality inductively to (7.26) to obtain

$$\int_{E_1} \overline{\rho}(y)\,dy \geq \prod_{r=1}^{\infty}\Big(1-\frac{1}{a^r}\Big)\int_{D_1}\rho(y)dy$$
$$\cdot \exp\Big(- C\, 2^{k_1-k_0}\log\Big(\frac{1}{c_2}\Big)\sum_{r=1}^{\infty}\frac{1}{2^r}$$
$$- \beta\, 2^{-\lambda_0/4}\, C^{11/4}\, 2^{11(k_1-k_0-\lambda_0)/4}\sum_{r=1}^{\infty}\frac{1}{2^{11r/4}}\Big).$$

Now we argue exactly as in Proposition 7.1 to verify that $Av_{E_1}\overline{\rho}$ is bounded below as the induction hypothesis requires.

Just as in Proposition 7.1 we cannot define ρ_2 by the right hand side of (7.26) since we cannot then control the fluctuation of ρ_2 in terms of its average value. We proceed as in Proposition 7.1 by generalizing (7.19). Thus we prescribe the averages of the densities $\rho_{x_1,z_1}, \rho_{x_1,z_1,x_2,z_2,\dots}$. First we modify (7.19) by insisting that

$$\begin{aligned} &\int_{E_1}\rho_{x_1,z_1}(y)\,dy = e^{-\eta_1}\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\,, \quad z_1\in\tilde{E}_1\,,\\ &\int_{D_2}\rho_{x_1,z_1}(y)\,dy = e^{-\eta_1}\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\,, \quad z_1\in\tilde{F}_1\,, \end{aligned} \tag{7.32}$$

where η_1 is a constant which satisfies

$$\eta_1 \geq C\, 2^{k_1-k_0}\log\Big(\frac{1}{c_2}\Big) + \beta\, 2^{-\lambda_0/4}\, 2^{11(k_1-k_0-\lambda_0)/4}\,.$$

In view of (7.30) this is clearly possible. More generally, let $x_1\in\tilde{D}_1$, $z_1\in\tilde{F}_1$, $x_2\in\tilde{D}_2$, $z_2\in\tilde{F}_2,\dots,x_r\in\tilde{D}_r$ Then from (7.30) we can insist that

$$\begin{aligned} &\int_{E_1}\rho_{x_1,z_1,\dots,x_r,z_r}(y)\,dy\\ &\qquad = e^{-\eta_r}\int_{D_r\cap B(x_r,2^{-k_r-4})}\rho_{x_1,z_1,\dots,x_{r-1},z_{r-1}}(y)\,dy\,, \quad z_r\in\tilde{E}_r\,,\\ &\int_{D_{r+1}}\rho_{x_1,z_1,\dots,x_r,z_r}(y)\,dy\\ &\qquad = e^{-\eta_r}\int_{D_r\cap B(x_r,2^{-k_r-4})}\rho_{x_1,z_1,\dots,x_{r-1},z_{r-1}}(y)\,dy\,, \quad z_r\in\tilde{F}_r\,, \end{aligned} \tag{7.33}$$

where η_r is a constant satisfying the inequality

$$\eta_r \geq \frac{C}{2^r}\, 2^{k_1-k_0} \log\Big(\frac{1}{c_2}\Big) + \beta\, 2^{-\lambda_0/4}\, \frac{C^{11/4}}{2^{11r/4}}\, 2^{11(k_1-k_0-\lambda_0)/4} ,$$

for some constant C depending only on ξ. Now we define ρ_2 by altering the right hand side of (7.26) to

$$\begin{aligned}\rho_2 = &\int_{\tilde D_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde E_1} \frac{dz_1}{|\tilde E_1 \cup \tilde F_1|}\, \rho_{x_1,z_1} \exp\Big(-\sum_{j=2}^{\infty} \eta_j\Big) \\ &+\cdots+ \int_{\tilde D_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde F_1} \frac{dz_1}{|\tilde E_1 \cup \tilde F_1|} \cdots \int_{\tilde D_r} \frac{dx_r}{\gamma(k_r)} \\ &\qquad \cdot \int_{\tilde E_r} \frac{dz_r}{|\tilde E_r \cup \tilde F_r|}\, \rho_{x_1,z_1,\dots,x_r,z_r} \exp\Big(-\sum_{j=r+1}^{\infty} \eta_j\Big) \\ &+\cdots .\end{aligned}$$

We consider the problem of estimating $\|\rho_2\|_{E_1,q}$ by writing ρ_2 as

$$\rho_2 = \int_{\tilde D_1} \frac{dx_1}{\gamma(k_1)} \int_{\tilde E_1 \cup \tilde F_1} \frac{dz_1}{|\tilde E_1 \cup \tilde F_1|}\, \psi_{x_1,z_1} ,$$

in analogy with the representation (7.20) of proposition 7.1. We argue now as in Proposition 7.1, using the Minkowski inequality to obtain the bound

$$\|\rho_2\|_{E_1,q} \leq \int_{\tilde D_1} \frac{dx_1}{\gamma(k_1)} \Big\| \int_{\tilde E_1 \cup \tilde F_1} \frac{dz_1}{|\tilde E_1 \cup \tilde F_1|}\, \psi_{x_1,z_1} \Big\|_{E_1,q} . \tag{7.34}$$

Since $\psi_{x_1,z_1}(y) = 0$ if $|z_1 - y| > 2^{-k_1}$ we have just as in (7.22) the inequality

$$\begin{aligned}&\Big\| \int_{\tilde E_1 \cup \tilde F_1} \frac{dz_1}{|\tilde E_1 \cup \tilde F_1|}\, \psi_{x_1,z_1} \Big\|_{E_1,q} \\ &\qquad\qquad \leq \frac{C\, 2^{-2k_1/q'}}{|\tilde E_1 \cup \tilde F_1|} \Big(\int_{\tilde E_1 \cup \tilde F_1} \|\psi_{x_1,z_1}\|_q^q \, dz_1 \Big)^{1/q} ,\end{aligned} \tag{7.35}$$

where $1/q + 1/q' = 1$, C is a universal constant and $\|\psi_{x_1,z_1}\|_q$ is the *unnormalized* L^q norm on E_1. Now

$$\psi_{x_1,z_1} = \exp\Big(-\sum_{j=2}^{\infty} \eta_j\Big) \rho_{x_1,z_1} , \qquad z_1 \in \tilde E_1 ,$$

whence

$$
\begin{aligned}
\|\psi_{x_1,z_1}\|_q^q &= \exp\Big(-q\sum_{j=2}^{\infty}\eta_j\Big)\|\rho_{x_1,z_1}\|_q^q \\
&\le \exp\Big(-q\sum_{j=2}^{\infty}\eta_j\Big)2^{2k_1(q-1)}\,C^q\,C_{2,\text{perturb}}^q\Big(\int_{E_1}\rho_{x_1,z_1}(y)\,dy\Big)^q \\
&= \exp\Big(-q\sum_{j=1}^{\infty}\eta_j\Big)2^{2k_1(q-1)}\,C^q\,C_{2,\text{perturb}}^q \\
&\quad\cdot\Big(\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\Big)^q,
\end{aligned}
$$

by (7.32) where C is a universal constant. Let us assume now that for $z_1\in\tilde F_1$ there is a universal constant C such that

$$
\begin{aligned}
\|\psi_{x_1,z_1}\|_q^q &\le \exp\Big(-q\sum_{j=2}^{\infty}\eta_j\Big)2^{2k_1(q-1)}\,C^q\,C_{2,\text{perturb}}^q \\
&\quad\cdot\Big(\int_{D_2}\rho_{x_1,z_1}(y)\,dy\Big)^q \\
&= \exp\Big(-q\sum_{j=1}^{\infty}\eta_j\Big)2^{2k_1(q-1)}\,C^q\,C_{2,\text{perturb}}^q \\
&\quad\cdot\Big(\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\Big)^q.
\end{aligned}
\tag{7.36}
$$

Then we have that

$$
\begin{aligned}
&\Big(\int_{\tilde E_1\cup\tilde F_1}\|\psi_{x_1,z_1}\|_q^q\,dz_1\Big)^{1/q} \\
&\qquad\le |\tilde E_1\cup\tilde F_1|^{1/q}\exp\Big(-\sum_{j=1}^{\infty}\eta_j\Big) \\
&\qquad\quad\cdot 2^{2k_1/q'}\,C\,C_{2,\text{perturb}}\int_{D_1\cap B(x_1,2^{-k_1-4})}\rho(y)\,dy\,.
\end{aligned}
$$

It follows now from (7.34), that

$$
\|\rho_2\|_q\le\int_{\tilde D_1}\frac{1}{|\tilde E_1\cup\tilde F_1|^{1/q'}}\exp\Big(-\sum_{j=1}^{\infty}\eta_j\Big)\frac{dx_1}{\gamma(k_1)}
$$

$$\begin{aligned}(7.37)\qquad &\cdot C\, C_{2,\mathrm{perturb}} \int_{D_1\cap B(x_1,2^{-k_1-4})} \rho(y)\,dy \\ &\le \frac{2\,C\,C_{2,\mathrm{perturb}}}{|D_1|^{1/q'}} \exp\Big(-\sum_{j=1}^{\infty}\eta_j\Big)\int_{D_1}\rho(y)\,dy\,,\end{aligned}$$

by making the ratio of $|D_1|$ to $|\tilde{E}_1\cup\tilde{F}_1|$ close to unity. This can be arranged in view of $\mathrm{d}^{(1)}$), $\mathrm{e}^{(1)}$) by choosing λ_0 large. It follows from this last inequality that

$$\|\rho_2\|_{E_1,q} \le C\,C_{2,\mathrm{perturb}} \exp\Big(-\sum_{j=1}^{\infty}\eta_j\Big)\mathrm{Av}_{D_1}\rho\,, \tag{7.38}$$

where C is a universal constant.

We shall show now that the inequality (7.38) holds in general. To do this we shall prove by induction that (7.36) holds. Thus for $x_1\in\tilde{D}_1$, $z_1\in\tilde{F}_1,\dots,x_r\in\tilde{D}_r$, $z_r\in\tilde{E}_r$,

$$\psi_{x_1,z_1,\dots,x_r,z_r} = \exp\Big(-\sum_{j=r+1}^{\infty}\eta_j\Big)\rho_{x_1,z_1,\dots,x_r,z_r}\,.$$

For $x_1\in\tilde{D}_1$, $z_1\in\tilde{F}_1,\dots,x_r\in\tilde{D}_r$, $z_r\in\tilde{F}_r$,

$$\begin{aligned}(7.39)\qquad \psi_{x_1,z_1,\dots,x_r,z_r} = &\int_{\tilde{D}_{r+1}} \frac{dx_{r+1}}{\gamma(k_{r+1})} \\ &\cdot\int_{\tilde{E}_{r+1}\cup\tilde{F}_{r+1}} \frac{dz_{r+1}}{|\tilde{E}_{r+1}\cup\tilde{F}_{r+1}|}\,\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\,.\end{aligned}$$

We now make the inductive assumption that for $z_{r+1}\in\tilde{F}_{r+1}$,

$$\begin{aligned}&\|\psi_{x_1,z_1,\dots x_{r+1},z_{r+1}}\|_q^q \le \exp\Big(-q\sum_{j=r+2}^{\infty}\eta_j\Big)\gamma(k_{r+1})^{-(q-1)}\,C^q\,C^q_{2,\mathrm{perturb}} \\ (7.40)\qquad &\cdot\Big(\int_{D_{r+2}}\rho_{x_1,z_1,\dots,x_{r+1},z_{r+1}}(y)\,dy\Big)^q\Big(1+\frac{1}{a^{r+1}}\Big)\,,\end{aligned}$$

where $a>1$ is some number to be specified and C is universal. To verify the assumption for $\psi_{x_1,z_1,\dots,x_r,z_r}$, $z_r\in\tilde{F}_r$, we argue as before using (7.39). Thus

$$
\begin{aligned}
\|\psi_{x_1,z_1,\dots,x_r,z_r}\|_q^q \le \Big(\int_{\tilde D_{r+1}} \frac{dx_{r+1}}{\gamma(k_{r+1})} \\
\cdot \Big\| \int_{\tilde E_{r+1}\cup \tilde F_{r+1}} \frac{dz_{r+1}}{|\tilde E_{r+1}\cup \tilde F_{r+1}|}\, \psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}} \Big\|_q \Big)^q \\
\le \Big(\int_{\tilde D_{r+1}} \frac{dx_{r+1}}{\gamma(k_{r+1})^{1/q}\, |\tilde E_{r+1}\cup \tilde F_{r+1}|} \\
\cdot \Big(\int_{\tilde E_{r+1}\cup \tilde F_{r+1}} \|\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\|_q^q \, dz_{r+1} \Big)^{1/q} \Big)^q .
\end{aligned}
$$

Now for $z_{r+1} \in \tilde E_{r+1}$ we have

$$
\begin{aligned}
\|\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\|_q^q &\le \exp\Big(-q \sum_{j=r+2}^{\infty} \eta_j \Big) \|\rho_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\|_q^q \\
&\le \exp\Big(-q \sum_{j=r+2}^{\infty} \eta_j \Big) \gamma(k_{r+1})^{-(q-1)}\, C_{2,\text{perturb}}^q \\
&\quad \cdot \Big(\int_{E_1} \rho_{x_1,z_1,\dots,x_{r+1},z_{r+1}}(y)\, dy \Big)^q \\
&= \exp\Big(-q \sum_{j=r+1}^{\infty} \eta_j \Big) \gamma(k_{r+1})^{-(q-1)}\, C_{2,\text{perturb}}^q \\
&\quad \cdot \Big(\int_{D_{r+1}\cap B(x_{r+1},2^{-k_{r+1}-4})} \rho_{x_1,z_1,\dots,x_r,z_r}(y)\, dy \Big)^q ,
\end{aligned}
$$

by (7.33) and Theorem 6.1. From the induction assumption (7.40) and (7.33) we have that if $z_{r+1} \in \tilde F_{r+1}$ then

$$
\begin{aligned}
\|\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}\|_q^q &\le \exp\Big(-q \sum_{j=r+1}^{\infty} \eta_j \Big) \gamma(k_{r+1})^{-(q-1)}\, C^q\, C_{2,\text{perturb}}^q \\
&\quad \cdot \Big(\int_{D_{r+1}\cap B(x_{r+1},2^{-k_{r+1}-4})} \rho_{x_1,z_1,\dots,x_r,z_r}(y)\, dy \Big)^q \\
&\quad \cdot \Big(1 + \frac{1}{a^{r+1}} \Big) .
\end{aligned}
$$

It follows then from these last three inequalities that

$$\|\psi_{x_1,z_1,\ldots,x_r,z_r}\|_q^q \le \exp\Big(-q\sum_{j=r+1}^{\infty}\eta_j\Big)C^q\,C_{2,\mathrm{perturb}}^q\Big(1+\frac{1}{a^{r+1}}\Big)$$
$$\cdot\Big(\int_{\tilde D_{r+1}}\frac{dx_{r+1}}{\gamma(k_{r+1})\,|\tilde E_{r+1}\cup\tilde F_{r+1}|^{1/q'}}$$
$$\cdot\int_{D_{r+1}\cap B(x_{r+1},2^{-k_{r+1}-4})}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\Big)^q .$$

We have now that

$$\int_{\tilde D_{r+1}}\frac{dx_{r+1}}{\gamma(k_{r+1})\,|\tilde E_{r+1}\cup\tilde F_{r+1}|^{1/q'}}$$
$$\cdot\int_{D_{r+1}\cap B(x_{r+1},2^{-k_{r+1}-4})}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy$$
$$\le C_r\,\gamma(k_r)^{-1/q'}\int_{D_{r+1}}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\,,$$

where we need

$$C_r \ge \Big(\frac{\gamma(k_r)}{|\tilde E_{r+1}\cup\tilde F_{r+1}|}\Big)^{1/q'}\,,\qquad x_{r+1}\in\tilde D_{r+1}\,.$$

From d$^{(r+1)}$), e$^{(r+1)}$) it is clear we can choose $a>1$ so that

$$\Big(\frac{|E_{r+1}|}{|\tilde E_{r+1}\cup\tilde F_{r+1}|}\Big)^{q-1} \le \frac{1+\dfrac{1}{a^r}}{1+\dfrac{1}{a^{r+1}}} = C_r^q\,.$$

We conclude that

$$\|\psi_{x_1,z_1,\ldots,x_r,z_r}\|_q^q \le \exp\Big(-q\sum_{j=r+1}^{\infty}\eta_j\Big)\gamma(k_r)^{-(q-1)}C^q\,C_{2,\mathrm{perturb}}^q$$
$$\cdot\Big(\int_{D_{r+1}}\rho_{x_1,z_1,\ldots,x_r,z_r}(y)\,dy\Big)^q\Big(1+\frac{1}{a^r}\Big)\,,$$

Thus we have verified the induction hypothesis (7.40) at the next level down. Setting $r=1$ in this last inequality yields (7.36).

The proof of the theorem will be complete if we can show that

$$\mathrm{Av}_{E_1}\rho_2 \geq c \exp\Big(-\sum_{j=1}^{\infty}\eta_j\Big)\mathrm{Av}_{D_1}\rho\,, \tag{7.41}$$

where the constant c is independent of the constant C_2 in the statement of Theorem 7.1. We can prove this in an exactly similar way to the proof of (7.37). Our induction assumption here is that for $z_{r+1} \in \tilde{F}_{r+1}$,

$$\begin{aligned}\int_{E_1} &\psi_{x_1,z_1,\dots,x_{r+1},z_{r+1}}(y)\,dy \\ &\geq \exp\Big(-\sum_{j=r+2}^{\infty}\eta_j\Big)\Big(1-\frac{1}{a^{r+1}}\Big)\int_{D_{r+1}}\rho_{x_1,z_1,\dots,x_{r+1},z_{r+1}}(y)\,dy\,.\end{aligned}$$

If we use now (7.39), (7.33) and (7.31) we can verify the induction hypothesis on the next level down. The fluctuation bound on ρ_2 in Theorem 7.1 follows from (7.37) and (7.41) if we choose the constant C_2 in the induction assumption sufficiently large.

We will use Theorem 7.1 to obtain estimates on the exit probability from a spherical shell. To do this we use the function $a_{\varepsilon,n,s,p}(x)$ defined in (1.7) in terms of the number of nonperturbative cubes inside the sphere of radius 2^{-n} centered at x. Let us suppose now that $a_{\varepsilon,n-1,s,p}(0) \gg 1$ so that we are in the nonperturbative situation and $a_{\varepsilon,n-1,s,p}(0) \sim 2^{n_0-n}$, $n_0 > n$. If we define V_η by (6.2) then we have:

Lemma 7.3. *Suppose $0 < \eta < s-2$. Then there is a constant C_η depending only on η such that*

$$\frac{1}{2^{-n}}\int_{B(0,2^{-n})} V_\eta(x)\,dx \leq C_\eta\, a_{\varepsilon,n-1,s,p}(0)^2\,.$$

PROOF. Clearly there is a universal constant C such that

$$\int_{B(0,2^{-n})} V_\eta(x)\,dx \leq \sum_{m=n_0}^{\infty} 2^{\eta(m-n_0)}\,2^{-m}N_m\,,$$

where N_m is the number of dyadic cubes with side of length 2^{-m} contained in the ball $B(0,2^{-n+1})$ such that (6.1) holds. In view of the fact that

$$N_m \leq 2^{(m-n)(3-s)}\,a_{\varepsilon,n-1,s,p}(0)^s \sim 2^{(m-n)(3-s)}\,2^{(n_0-n)s}\,,$$

it follows that

$$\int_{B(0,2^{-n})} V_\eta(x)\,dx \le 2^{-n}\,2^{2(n_0-n)} \sum_{m=n_0}^{\infty} 2^{(m-n_0)(\eta+2-s)}$$
$$\le C_\eta\,2^{-n}\,a_{\varepsilon,n-1,s,p}(0)^s,$$

since $\eta < s-2$.

Theorem 7.2. *Let f be a density on the sphere $|x| = 2^{-n}$. Suppose the drift process started on $|x| = 2^{-n}$ with density f induces a density $\overline{f}_2$ on the sphere $|x| = 2^{-n+1}$ when it exits the spherical shell $\{2^{-n-1} < |y| < 2^{-n+1}\}$. Then there exist constants C_1, C_2 such that if $1 < q < \infty$ and $\|f\|_q \le C_1\,Av\,f$ there is a density f_2 on $|x| = 2^{-n+1}$ with $0 \le f_2 \le \overline{f}_2$ such that $\|f_2\|_q \le C_1\,\mathrm{Av}\,f_2$ and*

$$\mathrm{Av}\,f_2 \ge \exp\left(-C_2\,a_{\varepsilon,n-2,s,p}(0)\right)\mathrm{Av}\,f\,,$$

provided $a_{\varepsilon,n-2,s,p}(0) \ge 1$. The L^q norm here is normalized so that $\|\mathbf{1}\|_q = 1$.

PROOF. For $|x| = 2^{-n}$, $|z| = 2^{-n+1}$ we consider cylinders $\Gamma_{x,z,k}$ with k defined by

$$2^{k-n} \sim 2^{\lambda_0}\,a_{\varepsilon,n-2,s,p}(0)\,.$$

Defining n_0 by $a_{\varepsilon,n-2,s,p}(0) \sim 2^{n_0-n}$, it follows that $k = \lambda_0 + n_0$. Letting $D_1 = \{|x| = 2^{-n}\}$, $E_1 = \{|x| = 2^{-n+1}\}$ it follows from Lemma 7.1, 7.3 that

$$\frac{1}{|D_1|}\int_{D_1}\frac{dx}{|E_1|}\int_{E_1}\frac{dz}{2^{-k}}\int_{\Gamma_{x,z,k}\cap B(0,2^{-n+2})} V_\eta(y)\,dy$$
$$\le C\,2^{n-k}\Big(\frac{1}{2^{-n}}\int_{B(0,2^{-n+2})} V_\eta(y)\,dy\Big)$$
$$\le C\,2^{n-k}\,2^{2(n_0-n)} = C\,2^{-2\lambda_0}\,2^{k-n}\,.$$

We follow now the lines of the proof of Proposition 7.1 and use Theorem 7.1 to propagate the drift process through the cylinder. Let D_1' be the set of $x \in D_1$ such that

$$\frac{1}{|E_1|}\int_{E_1}\frac{dz}{2^{-k}}\int_{\Gamma_{x,z,k}\cap B(0,2^{-n+2})} V_\eta(y)\,dy \le 2^{-\lambda_0/2}\,2^{(k-n-\lambda_0)}\,. \tag{7.42}$$

It follows by Chebyshev from the last two inequalities that

$$\frac{|D_1'|}{|D_1|} > 1 - C\,2^{-\lambda_0/2}\,.$$

Next we have by the argument of Lemma 7.2 that

$$\int_{D_1\backslash D_1'} \frac{dx}{2^{-2k}} \int_{D_1\cap B(x,2^{-k-4})} f(y)\,dy \le (C\,2^{-\lambda_0/2})^{1/q'}\,C_1\,\mathrm{Av}_{D_1} f\,.$$

For $x \in D_1$ let $\|f\|_{x,k}$ be the L^q norm on $D_1 \cap B(x, 2^{-k-4})$ normalized so that $\|\mathbf{1}\|_{x,k} = 1$. Let D_1'' be the set of $x \in D_1$ such that

$$\|f\|_{x,k} \le C_{2,\mathrm{thm}}\,\alpha^{k-n_0}\,\mathrm{Av}_{x,k} f\,,$$

where $C_{2,\mathrm{thm}}$ denotes the constant C_2 of Theorem 7.1 and α is the same constant as in the statement of the theorem. We choose λ_0 sufficiently large so that $2\,C_1 < C_{2,\mathrm{thm}}\alpha^{k-n_0}$. Since $\lambda_0 = k - n_0$ this is certainly possible. Setting $\tilde{D}_1 = D_1' \cap D_1''$, we conclude on taking $\gamma = 1/2$ in Lemma 7.2 that

$$\int_{\tilde{D}_1} \frac{dx}{2^{-2k}} \int_{D_1\cap B(x,2^{-k-4})} f(y)\,dy \ge \frac{1}{4}\,\mathrm{Av}_{D_1} f\,,$$

provided λ_0 is sufficiently large.

Next for $x \in \tilde{D}_1$ let $\tilde{E}_1$ be the set of $z \in E_1$ such that

$$\frac{1}{2^{-k}} \int_{\Gamma_{x,z,k}\cap B(0,2^{-n+2})} V_\eta(y)\,dy \le 2^{-\lambda_0/4}\,2^{(k-n-\lambda_0)}\,.$$

It follows from (7.42) that

$$\frac{|\tilde{E}_1|}{|E_1|} > 1 - 2^{-\lambda_0/4}\,.$$

Now we use Theorem 7.1 to propagate the density f restricted to $D_1 \cap B(x, 2^{-k-4})$ through the cylinder $\Gamma_{x,z,k}$ to $E_1 \cap B(z, 2^{-k-4})$. Let $f_{x,z}$ be this propagated density. In view of (7.43) we can arrange for this density to satisfy

$$\int_{E_1\cap B(z,2^{-k-4})} f_{x,z}(y)\,dy = e^{-\eta} \int_{D_1\cap B(x,2^{-k-4})} f(y)\,dy\,,$$

where $\eta = C\,2^{k-n}$ for some constant C. Theorem 7.1 also yields an estimate on the fluctuation of $f_{x,z}$. Thus

$$\|f_{x,z}\|_q \leq C_{2,\mathrm{thm}}\,\mathrm{Av}\,f_{x,z}\ .$$

The propagated density f_2 is defined now by

$$f_2 = \int_{\tilde{D}_1} \frac{dx}{2^{-2k}} \int_{\tilde{E}_1} f_{x,z}\,dz\,.$$

We can argue now exactly as in Proposition 7.1 to conclude that

$$\mathrm{Av}_{E_1} f_2 \geq \exp\left(-C\,2^{k-n}\right)\mathrm{Av}_{D_1} f\,,$$

$$\|f_2\|_q \leq 2\,C_{2,\mathrm{thm}}\,\mathrm{Av}_{E_1} f_2\ .$$

The result follows by taking $C_1 = 2\,C_{2,\mathrm{thm}}$.

8. Proof of Theorem 1.3.

Here we follow closely the argument of [5, Section 6]. In fact we shall repeat the entire argument of [5, Section 6] with the function $a_{n,p}(x)$ given in (1.4) replaced by the function $a_{\varepsilon,n,s,p}(x)$, $s > 2$ defined in (1.7). Our first lemma is identical to [5, Lemma 6.1]. In the following we shall denote the function $a_{\varepsilon,n,s,p}$ simply by a_n.

Lemma 8.1. *Let Q_0 be a cube containing Ω_R with side of length $2^{-n_0} \sim R$. Suppose for some integer $m \geq 0$, the drift $\mathbf{b}$ satisfies the inequality*

$$\int_Q |\mathbf{b}|^p\,dx \leq \varepsilon^p\,|Q|^{1-p/3}\,, \tag{8.1}$$

on all dyadic subcubes $Q \subset Q_0$ with side of length 2^{-n}, $n \geq m + n_0$. Let u be the solution of the Dirichlet problem (1.1), (1.2)*. Then if ε is sufficiently small, depending on $p > 1$, $s > 2$, there exist constants C_1 depending only on p, q, r and C_2 depending only on $p > 1$, $s > 2$, such that*

$$\|u\|_\infty \leq C_1 R^{2-3/q}\,\|f\|_{q,r} \sup_{x\in\Omega_R} \exp\Big(C_2 \sum_{j=0}^{m} a_{n_0+j}(x)\Big)\,.$$

PROOF. We consider the function $\xi(x)$, $x \in \Omega_R$, given by

$$\xi(x) = E_x\Big[\exp\Big(-\frac{1}{\mu}\int_0^\tau |f|(X_{\mathbf{b}}(t))\,dt\Big)\Big],$$

where τ is the time for the drift process $X_{\mathbf{b}}(t)$ to exit Q_0. By (8.1) the ball $B(x, 2^{-m})$ is perturbative for the drift $\mathbf{b}$. We need now an obvious generalization of Theorem 7.2. Thus let ρ_n be a density on the sphere $|x-y| = 2^{-n}$ and $\overline{\rho}_{n-1}$ be the density induced on the sphere $|x-y| = 2^{-n+1}$ by paths of the drift process which satisfy

$$\int_0^{\tau_{n-1}} |f|(X_{\mathbf{b}}(t))\,dt \le C\,2^{-n(2-3/q-\delta)}\,2^{-n_0\delta}\,\|f\|_{q,r}\,, \tag{8.2}$$

where τ_{n-1} is the first hitting time on $|y-x| = 2^{-n+1}$, C is a positive constant and $0 < \delta < 2 - 3/q$. Suppose now that $a_{n-2}(x) \ge \eta > 0$. It follows from Section 7 that for any t, $1 < t < \infty$, C can be chosen sufficiently large so that $\|\rho_n\|_t \le C\mathrm{Av}\,\rho_n$ implies that $\overline{\rho}_{n-1} \ge \rho_{n-1} \ge 0$, $\|\rho_{n-1}\|_t \le C\mathrm{Av}\,\rho_{n-1}$ and

$$\mathrm{Av}\,\rho_{n-1} \ge \mathrm{Av}\,\rho_n \exp\left(-K_\eta a_{n-2}(x)\right), \tag{8.3}$$

where K_η is a constant depending on η. If $a_{n-2}(x) \le \eta$ and η is sufficiently small then we are in the perturbative situation described in Section 5. Now ρ_{n-1} is the density induced on $|x-y| = 2^{-n+1}$ by paths which avoid nonperturbative cubes and such that (8.2) holds. By examining the proofs of Proposition 5.2, Proposition 5.6 and Lemma 7.3 we see that on choosing q sufficiently close to 2, depending on $s > 2$, one has

$$\mathrm{Av}\,\rho_{n-1} \ge (1 - \nu\,2^{-(n-n_0)\delta})\,(1 - C_1\,a_{n-2}(x))\,\mathrm{Av}\,\rho_n\,, \tag{8.4}$$

where the constant $\nu > 0$ can be made arbitrarily small and C_1 is independent of η. It follows from (8.2), (8.3), (8.4) that

$$\xi(x) \ge \frac{3}{4}\exp\Big(-\frac{1}{\mu}\sum_{n=n_0}^{\infty} C\,2^{-n(2-3/q-\delta)}\,2^{-n_0\delta}\,\|f\|_{q,r}\Big)$$
$$\cdot\exp\Big(-K\sum_{j=0}^{m} a_{n_0+j}(x)\Big).$$

If we choose now $\mu \sim 2^{-n_0(2-3/q)}\,\|f\|_{q,r}$ we can conclude that

$$\xi(x) \geq \frac{1}{2}\exp\Big(-K\sum_{j=0}^{m} a_{n_0+j}(x)\Big)\,.$$

The result follows from this last inequality and [4, Lemma 5.1].

Next we consider the analogue of [5, Lemma 6.2].

Lemma 8.2. *For $n \in \mathbb{Z}$ let Ω_n be the spherical shell $\Omega_n = \{x \in \mathbb{R}^3 : 2^{-n-1} < |x| < 2^{-n+1}\}$. For $x \in \Omega_n$ let P_x be the probability that the drift process started at x exits Ω_n through the sphere $|y| = 2^{-n+1}$. Let δ be a number satisfying $0 < \delta < 2/3$. Then if $|x| = 2^{-n}$ there is a constant C depending only on $\delta < 2/3$, $p > 1$, $s > 2$ and ε such that*

$$P_x \geq \delta \exp\left(-C\, a_{n-2}(0)\right).$$

PROOF. Observe that if $\mathbf{b} \equiv 0$ then

$$P_x = \frac{4}{3}\Big(1 - \frac{2^{-n-1}}{|x|}\Big)\,.$$

Hence if $|x| = 2^{-n}$ then $P_x = 2/3$. It follows that for fixed x_0 with $|x_0| = 2^{-n}$ then

$$P_x \geq \frac{1}{2}\Big(\delta + \frac{2}{3}\Big)\,, \tag{8.5}$$

for x in the set

$$B = \Big\{x : |x - x_0| < \frac{2^{-n}(2 - 3\,\delta)}{3\,(2-\delta)}\Big\}\,. \tag{8.6}$$

Consider next the case when $\mathbf{b} \not\equiv 0$, and let us first assume that we are in the perturbative case so that $a_{n-2}(0) < \eta$ and η is small. For $x \in \mathbb{R}^3$, m, k integers with $k \geq m$ let $N_{m,k}(x)$ be the number of dyadic cubes with side of length 2^{-k} contained in the ball $\{y : |x-y| < 2^{-n}\}$ which satisfy (6.1). Then from the definition of $a_{n-2}(0)$ we have that

$$N_{n-2,m}(0) \leq \eta^s\, 2^{(m+2-n)(3-s)}\,, \qquad m \geq n-2\,. \tag{8.7}$$

Let $X(t)$ be an arbitrary continuous path with $X(0) = x_0$, $X(t) \in B$, $t < \tau$, and $X(\tau) \in \partial B$. Let s' satisfy $2 < s' < s$, We claim that there are constants C_1, β depending only on s, s', such that $C_1 > 0$, $0 < \beta < 1$, and a point $x = X(t)$ for some t, $0 \le t \le \tau$, satisfying

$$N_{m,k}(x) \le C_1\, \eta^s \beta^{m-n}\, 2^{(k-m)(3-s')}\,, \qquad k \ge m \ge n\,. \tag{8.8.}$$

To prove this inequality we assume its negation and obtain a contradiction. Thus for each x on the path X there exists integers $m(x)$, $k(x)$ such that (8.8) is violated when $m = m(x)$, $k = k(x)$. Now the balls $B(x, 2^{-m(x)})$ form an open cover for the compact set X. Hence there exists a finite subcover $\Gamma = \{D_j : 1 \le j \le N\}$ for some integer N. For each integer $m \ge n$, let Γ_m be the subset of Γ consisting of balls with radius 2^{-m}. Let D be an arbitrary ball and $\tilde{D}$ the ball concentric with D but with three times the radius. Then there exists a subset $\tilde{\Gamma}_m \subset \Gamma_m$ of disjoint balls such that

$$\bigcup_{D\in\Gamma_m} D \subset \bigcup_{D\in\tilde{\Gamma}_m} \tilde{D}\,.$$

For $k \ge m$ let $\tilde{\Gamma}_{m,k}$ be the subset of $\tilde{\Gamma}_m$ consisting of balls $D = B(x, 2^{-m})$ such that $k(x) = k$. Since the balls in $\Gamma_{m,k}$ are disjoint it follows from (8.8), (8.7) that

$$|\tilde{\Gamma}_{m,k}|\, C_1\, \eta^s \beta^{m-n}\, 2^{(k-m)(3-s')} \le \eta^s\, 2^{(k+2-n)(3-s)}\,,$$

whence

$$|\tilde{\Gamma}_m| \le \sum_{k=m}^{\infty} |\tilde{\Gamma}_{m,k}| \le C\, C_1^{-1} \beta^{-(m-n)}\, 2^{(m-n)(3-s)}\,,$$

for some constant C depending on $s' < s$. We choose $\beta < 1$ now so that

$$\frac{2^{3-s}}{\beta} < 2\,.$$

This is possible since $s > 2$. It is clear that for any point x on the path $X(t)$, $0 \le t \le \tau$, one must have the inequality

$$|x - x_0| \le 6 \sum_{m=n}^{\infty} 2^{-m}\, |\tilde{\Gamma}_m| \le A\, \frac{2^{-n}}{C_1}\,,$$

for some constant A depending on s, s'. Since $X(\tau)$ lies on the boundary of the ball B in (8.6) this last inequality is violated for $x = X(\tau)$ provided C_1 is chosen sufficiently large. Hence we have a contradiction.

We may therefore assume that an $x = X(t)$ exists such that (8.8) holds. Now from Section 5 it follows that the Brownian particle at x can be propagated to the sphere of radius 2^{-n} centered at x with a loss of density which can be made arbitrarily small as $\eta \longrightarrow 0$. The density on the sphere of radius 2^{-n} is approximately uniform. Again from Section 5 the probability of exiting the outer sphere $\{|y| = 2^{-n+1}\}$ starting from $\partial B(x, 2^{-n})$ with approximately uniform density can be made arbitrarily close to the probability for Brownian motion $\mathbf{b} \equiv 0$ by choosing η sufficiently small. In view of (8.5) the result of the lemma follows if $a_{n-2}(0) < \eta$ and η is sufficiently small.

Next we turn to the nonperturbative case. We can assume now that there exists $\eta > 0$ and $a_{n-2}(0) \geq \eta$. Let n_1 be the unique integer such that

$$2^{n_1+1} > \frac{a_{n-2}(0)}{\eta} \geq 2^{n_1} \,.$$

Hence, analogously to (8.7) we have

$$N_{n-2,m}(0) \leq \eta^s \, 2^s \, 2^{sn_1} \, 2^{(m+2-n)(3-s)} \,, \qquad m \geq n-2 \,. \tag{8.9}$$

We shall show, in analogy to (8.8), that there exists $x = X(t)$ for some t, $0 \leq t \leq \tau$, satisfying

$$N_{m,k}(x) \leq C_1 \, \eta^s \, 2^{s'n_1} \beta^{m-n} \, 2^{(k-m)(3-s')} \,, \tag{8.10}$$

with $k \geq m + n_1$, $m \geq n$. To see this we argue exactly as in the perturbative case. Thus from (8.10), (8.9) the cardinality of the set $\tilde{\Gamma}_{m,k}$ satisfies

$$|\tilde{\Gamma}_{m,k}| \, C_1 \, \eta^s \, 2^{s'n_1} \beta^{m-n} 2^{(k-m)(3-s')} \leq \eta^s \, 2^s \, 2^{sn_1} \, 2^{(k+2-n)(3-s)} \,,$$

whence

$$|\tilde{\Gamma}_m| \leq \sum_{k=m+n_1}^{\infty} |\tilde{\Gamma}_{m,k}| \leq C \, C_1^{-1} \beta^{-(m-n)} \, 2^{(m-n)(3-s)} \,,$$

for some constant C depending on $s' < s$.

Now for x which satisfies (8.10) we see from the argument of Lemma 7.3 and Theorem 7.2 that the Brownian particle at x can be propagated to the sphere of radius 2^{-n} centered at x with a decrease in density by a factor

$$\exp\Big(-A\sum_{m=n}^{\infty}\beta^{m-n}\,2^{n_1}\Big)\,,$$

for some constant A. Now this density on the sphere of radius 2^{-n} centered at x can be propagated to the outer sphere $\{|y| = 2^{-n+1}\}$ with a further decrease in density by at most a factor

$$\exp\left(-A'\,2^{n_1}\right),$$

for some constant A'. Hence the total decrease in density from x_0 to the outer sphere is by a factor

$$\exp\left(-A''\,2^{n_1}\right),$$

for some constant A''.

The proof of the theorem follows now exactly as in [5, Section 6]. To prove (1.9) we need to prove the analogue of [5, Proposition 6.6].

Proposition 8.1. *Suppose $\eta > 0$. Then there exists a constant C depending on η, ρ and a universal constant c such that*

$$\sum_{n=-\infty}^{\infty} a_n(x)\,H\big(a_n(x)-\eta\big) \le C\,N_{c\,\varepsilon}(\mathbf{b})\,,$$

where $H(t)$ is the Heaviside function,

$$H(t) = \begin{cases} 1\,, & \text{if } t > 0\,,\\ 0\,, & \text{if } t < 0\,.\end{cases}$$

PROOF. We have

$$\sum_{n=-\infty}^{\infty} a_n(x)\,H(a_n(x)-\eta) \le \frac{1}{\eta^{s-1}}\sum_{n=-\infty}^{\infty} a_n(x)^s\,.$$

Letting $N_{m,n}(x)$ be the number of non perturbative dyadic cubes with side of length 2^{-m}, $m \ge n$, contained in the ball $|x-y| < 2^{-n}$ we have from the definition of $a_n(x)$ that

$$\sum_{n=-\infty}^{\infty} a_n(x)^s \le \sum_{n=-\infty}^{\infty}\sum_{m=n}^{\infty}\frac{N_{m,n}(x)}{2^{(m-n)(3-s)}} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{m}\frac{N_{m,n}(x)}{2^{(m-n)(3-s)}}\,.$$

Let N_m be the number of nonperturbative cubes with side of length 2^{-m} in $\mathbb{R}^3$. Then it is clear there is a constant C such that

$$\sum_{n=-\infty}^{m} \frac{N_{m,n}(x)}{2^{(m-n)(3-s)}} \leq C\, N_m \,.$$

It follows that

$$\sum_{n=-\infty}^{\infty} a_n(x)^s \leq C \sum_{m=-\infty}^{\infty} N_m = N\,,$$

where N is the total number of nonperturbative cubes in $\mathbb{R}^3$. Finally, we use the result of Fefferman [9] that $N \leq C\, N_{c\,\varepsilon}(\mathbf{b})$ for suitable universal constants C, c.

Appendix A. Brownian motion confined to a cylinder.

In this section we give a new proof of [4, Theorem 1.1.a)]. To do this we shall use a result concerning Brownian motion confined to a cylinder. For $\lambda > 0$ let D_λ be the disc of radius λ in $\mathbb{R}^2$,

$$D_\lambda = \{x = (x_1, x_2) : r^2 = x_1^2 + x_2^2 < \lambda^2\}\,.$$

Then for $m > 0$ the set

$$D_\lambda \times (-m\lambda, m\lambda) = \{x = (x_1, x_2, x_3) : (x_1, x_2) \in D_\lambda,\ x_3 \in (-m\lambda, m\lambda)\}$$

is a cylinder in $\mathbb{R}^3$. We are interested in studying Brownian motion confined to the cylinder when $m \gg 1$. In particular let $X(t)$ be Brownian motion in $\mathbb{R}^3$ started at the origin and τ be the first exit time from the cylinder. We shall consider Brownian motion under the constraint that $|X(\tau)_3| = m\lambda$. Thus the paths must exit the cylinder through one of the discs $D_\lambda \times \{m\lambda\}$ or $D_\lambda \times \{-m\lambda\}$. For $m \gg 1$ this is an unlikely event. Hence Brownian motion under this constraint behaves very differently to the standard Brownian motion. In fact it appears to behave ballistically on length scales much larger than λ. As a consequence one has

$$\text{(A.1)} \qquad E\,[\tau : |X(\tau)_3| = m\lambda] \sim m\lambda^2\,, \qquad m \gg 1\,.$$

One of our main goals here will be to prove (A.1). We shall need something more general to prove [4, Theorem 1.1.a)]. In fact we have the following:

Theorem A.1. *Let V be a nonnegative potential on $D_\lambda \times (-m\lambda, m\lambda)$. Then if $m \geq 1$ there is a universal constant C such that if $X(0)$ is uniformly distributed on a crosssection $D_\lambda \times \{\xi\}$,*

$$E\Big[\int_0^\tau V(X(t))\,dt : |X(\tau)_3| = m\lambda\Big] \leq \frac{C}{\lambda}\int_{D_\lambda \times (-m\lambda, m\lambda)} V(y)\,dy\,.$$

We recall [4, Theorem 1.1.a)]. Thus let V be a nonnegative potential on the ball Ω_R and for $n = 0, \pm 1, \pm 2, \ldots, x \in \Omega_R$ let $a_n(x)$ be the functions

$$a_n(x) = 2^n \int_{|x-y|<2^{-n}} V(y)\,dy\,.$$

[4, Theorem 1.1.a)] is then given by:

Theorem A.2. *For $x \in \Omega_R$ and Brownian motion $X(t)$ started at x, let τ be the first exit time from Ω_R. Then there is a universal constant $C > 0$ such that*

$$E_x\Big[\exp\Big(-\int_0^\tau V(X(t))\,dt\Big)\Big] \geq \exp\Big(-C\sum_{n=n_0}^\infty \min\big\{a_n(x), a_n(x)^{1/2}\big\}\Big)\,,$$

$x \in \Omega_R$, where n_0 is the unique integer which satisfies the inequality $2R < 2^{-n_0} \leq 4R$.

PROOF. We define a subset $\mathcal{S}$ of Brownian paths started at x. For $n \geq n_0$ define m_n, λ_n by

$$m_n = a_n(x)^{1/2}\,, \qquad m_n\lambda_n = 2^{-n}\,.$$

For a Brownian path $X(t)$ started at x let τ_n be the first hitting time on the sphere $|x-y| = 2^{-n}$. Thus $\tau_{n+1} < \tau_n$. The set $\mathcal{S}$ is then all Brownian paths started at x which for $\tau_{n+1} < t < \tau_n$ are contained within the cylinder centered at $X(\tau_{n+1})$ with axis given by the vector $X(\tau_{n+1}) - x$ and with radius λ_n, $n \geq n_0$. It is clear that there is a universal constant $c > 0$ such that if $m_n < c$ there is no constraint on the Brownian path for $\tau_{n+1} < t < \tau_n$.

We have now from Jensen's inequality that

$$
\begin{aligned}
&E_x\Big[\exp\Big(-\int_0^{\tau} V(X(t))\,dt\Big)\Big] \\
&\qquad\qquad \geq E_x\Big[\exp\Big(-\int_0^{\tau} V(X(t))\,dt\Big),\,\mathcal{S}\Big] \\
&\qquad\qquad \geq P(\mathcal{S})\exp\Big(-E\Big[\int_0^{\tau} V(X(t))\,dt:\mathcal{S}\Big]\Big)\,.
\end{aligned} \tag{A.2}
$$

It is obvious from the proof of Theorem A.1 that

$$
P(\mathcal{S}) \geq \exp\Big(-C\sum_{n=n_0}^{\infty} m_n\,H(m_n-c)\Big)\,, \tag{A.3}
$$

where C, c are universal constants and H is the Heaviside function, $H(t)=1$, $t>0$, $H(t)=0$, $t<0$. We can write now

$$
E\Big[\int_0^{\tau} V(X(t))\,dt:\mathcal{S}\Big] = \sum_{n=n_0}^{\infty} E\Big[\int_{\tau_{n+1}}^{\tau_n} V(X(t))\,dt:\mathcal{S}\Big]\,.
$$

By symmetry $X(\tau_{n+1})$ is uniformly distributed on the sphere $|y-x| = 2^{-n-1}$. Hence if $m_n < c$ one has

$$
\begin{aligned}
&E\Big[\int_{\tau_{n+1}}^{\tau_n} V(X(t))\,dt:\mathcal{S}\Big] \\
&\qquad \leq \frac{1}{4\pi\,2^{-2n-2}}\int_{|x-y|=2^{-n-1}}\frac{dy}{4\pi}\int_{|x-z|<2^{-n}}\frac{V(z)\,dz}{|y-z|} \\
&\qquad \leq K a_n(x)\,,
\end{aligned} \tag{A.4}
$$

for some universal constant K. For $m_n > c$ we use Theorem A.1 . Thus

$$
\begin{aligned}
&E\Big[\int_{\tau_{n+1}}^{\tau_n} V(X(t))\,dt:\mathcal{S}\Big] \\
&\qquad \leq \frac{1}{4\pi\,2^{-2n-2}}\int_{|x-y|=2^{-n-1}}\frac{C\,dy}{\lambda_n}\int_{\Gamma_{y,\lambda_n}\cap\{|x-z|<2^{-n}\}} V(z)\,dz\,,
\end{aligned}
$$

where Γ_{y,λ_n} is the cylinder centered at y with axis $y-x$ and radius λ_n. Arguing now as in Lemma 7.1 we have that

$$
\begin{aligned}
&\frac{1}{4\pi\,2^{-2n-2}}\int_{|x-y|=2^{-n-1}} dy\int_{\Gamma_{y,\lambda_n}\cap\{|x-z|<2^{-n}\}} V(z)\,dz \\
&\qquad\qquad \leq C\,(\lambda_n 2^n)^2\int_{\{|x-z|<2^{-n}\}} V(y)\,dy\,,
\end{aligned}
$$

for some universal constant C. Hence by the previous two inequalities we have

$$E\Big[\int_{\tau_{n+1}}^{\tau_n} V(X(t))\,dt : \mathcal{S}\Big] \le C\lambda_n\, 2^n\, a_n(x) = Ca_n(x)^{1/2}\,, \tag{A.5}$$

for some universal constant C. The result follows now from (A.2), (A.3), (A.4), (A.5).

REMARK. It is possible to prove Theorem A.2 after the fashion of the proof of Proposition 7.1, avoiding the use of the Jensen inequality in (A.2) and Theorem A.1. This would on a technical level be a simpler proof. Our main purpose here is to draw a comparison between the proof of Theorem A.2 above and the proof in [4]. In the latter proof Jensen's inequality was combined with restricting to Brownian paths under a time constraint. In the former, Jensen is combined with restricting to Brownian paths under a topological constraint. Thus in some sense time constraints on Brownian paths are equivalent to topological constraints.

Next we turn our attention to the proof of Theorem A.1. First we shall prove (A.1). In order to do this we need to examine the behavior of 2-dimensional Brownian motion on D_λ at large time.

Lemma A.1. *For $x, y \in D_\lambda$, $t > 0$, let $G_D(x,y,t)$ be the Green's function for the heat equation D_λ with Dirichlet boundary conditions. Then there is a universal constant $C > 0$ such that*

$$\int_{|y|<\lambda} G_D(x,y,t)\,dy \le C\int_{|y|<\lambda/2} G_D(x,y,t)\,dy\,, \tag{A.6}$$

for all $x \in D_\lambda$, $t \ge \lambda^2$.

PROOF. It follows easily from the semi-group property of G_D that it will be sufficient to prove (A.6) when $t = \lambda^2$. Evidently one has

$$\int_{|y|<\lambda} G_D(x,y,\lambda^2)\,dy = P_x(\tau_\lambda > \lambda^2)\,, \tag{A.7}$$

where τ_λ is the first exit time from D_λ of 2-dimensional Brownian motion $Y(t)$ started at $x \in D_\lambda$. By the Chebyshev inequality we have that

$$P_x(\tau_\lambda > \lambda^2) \le \lambda^{-2}\,E_x[\tau_\lambda] = \lambda^{-2}\,u(x)\,, \tag{A.8}$$

where $u(x)$ satisfies the equation

$$\begin{cases} -\Delta u(x) = 1\,, & |x| < \lambda\,, \\ u(x) = 0\,, & |x| = \lambda\,. \end{cases}$$

It is easy to see that the solution of this equation is given by

$$u(x) = \frac{1}{4}\,(\lambda^2 - r^2)\,, \qquad r = |x|\,. \tag{A.9}$$

Hence (A.7), (A.8), (A.9) yield an upper bound on the left hand side of (A.6).

Next we look for a lower bound on the right hand side of (A.6). Let α satisfy $0 < \alpha < 1$. We shall show that there is a positive constant C_α depending only on α such that

$$\int_{|y|<\lambda/2} G_D(x, y, \lambda^2)\, dy \geq C_\alpha\,, \qquad |x| \leq \alpha\lambda\,. \tag{A.10}$$

To see this let $G(z, w, t)$ be the heat kernel in $\mathbb{R}^2$,

$$G(z, w, t) = \frac{1}{4\pi t} \exp\Big(-\frac{|z-w|^2}{4\,t}\Big)\,.$$

Then for $|z|$, $|w| < \varepsilon\lambda$, $\varepsilon > 0$ there is a density $\rho(t, z')$, $0 < t < \varepsilon\lambda^2$, $|z'| = \lambda$ such that

$$G(z, w, \varepsilon\lambda^2) = G_D(z, w, \varepsilon\lambda^2) + \int_{|z'|=\lambda} dz' \int_0^{\varepsilon\lambda^2} \rho(t, z')\, G(z', w, t)\, dt\,.$$

The density $\rho(t, z')$ evidently satisfies the inequality

$$\int_{|z'|=\lambda} dz' \int_0^{\varepsilon\lambda^2} \rho(t, z')\, dt \leq 1\,.$$

Suppose now that $|z|$, $|w| \leq \alpha\lambda$, $|z - w| < (1-\alpha)\lambda/2$. Then it is clear that for ε sufficiently small, depending only on α one has

$$G(z', w, t) \leq \frac{1}{2}\, G(z, w, \varepsilon\lambda^2)\,, \qquad |z'| = \lambda\,,\ 0 < t < \varepsilon\lambda^2\,.$$

Hence from the last three inequalities we have that

$$G_D(z,w,\varepsilon\lambda^2) \geq \frac{1}{2}\, G(z,w,\varepsilon\lambda^2)\,, \quad |z|\,,\ |w| \leq \alpha\lambda\,,\ |z-w| < \frac{(1-\alpha)\,\lambda}{2}\,,$$

provided $\varepsilon > 0$ is sufficiently small. The inequality (A.10) follows from this last inequality by constructing paths from x to $|y| < \lambda/2$ in time steps of length $\varepsilon\lambda^2$ and using the semi-group property of G_D.

In view of the fact that the left hand side of (A.6) is bounded above by 1, the inequality (A.6) follows for $t = \lambda^2$ and all x satisfying $|x| \leq \alpha\lambda$, $\alpha < 1$, from (A.10). Our main problem then is to deal with the case $|x| \longrightarrow \lambda$ since the right hand side of (A.10) converges to zero as $\alpha \longrightarrow 1$. Let U_α be the set

$$U_\alpha = \{y \in D_\lambda : \lambda\alpha < |y| < \lambda\}\,.$$

Then for $x \in U_\alpha$ we have

$$\begin{aligned}
&\int_{|y|<\lambda/2} G_D(x,y,\lambda^2)\,dy \\
&\qquad = P_x\Big[Y(t) \text{ exits } U_\alpha \text{ through the boundary } \{y : |y| = \lambda\alpha\}\,, \\
&\qquad\qquad |Y(t)| < \lambda\,,\ 0 < t < \lambda^2\,,\ |Y(\lambda^2)| \leq \frac{\lambda}{2}\Big] \\
&\qquad \geq P_x\Big[Y(t) \text{ exits } U_\alpha \text{ through the boundary } \{|y| = \alpha\lambda\} \\
&\qquad\qquad \text{in time } < \frac{\lambda^2}{2}\Big] \\
&\qquad\qquad \cdot \inf_{\substack{|y|=\lambda\alpha \\ 0<s<\lambda^2/2}} P_y\Big[|Y(t)| < \lambda\,,\ 0 < t < \lambda^2 - s\,,\ |Y(\lambda^2 - s)| \leq \frac{\lambda}{2}\Big]\,.
\end{aligned} \tag{A.11}$$

It is clear from what we have just done that

$$\inf_{\substack{|y|=\lambda\alpha \\ 0<s<\lambda^2/2}} P_y\Big[|Y(t)| < \lambda\,,\ 0 < t < \lambda^2 - s\,,\ |Y(\lambda^2 - s)| \leq \frac{\lambda}{2}\Big] \geq c_\alpha > 0\,,$$

where c_α is a constant depending only on $\alpha < 1$. Thus we are left to estimate the first probability in the final expression of (A.11).

We do this by using the inequality

$$
\begin{aligned}
&P_x\Big[Y(t) \text{ exits } U_\alpha \text{ through the boundary } \{|y| = \alpha\lambda\} \text{ in time } < \frac{\lambda^2}{2}\Big] \\
&\geq P_x[Y(t) \text{ exits } U_\alpha \text{ through the boundary } \{|y| = \alpha\lambda\}] - \frac{2}{\lambda^2}\, E_x[\tau]\,,
\end{aligned}
$$

where τ is the first exit time from U_α. If we put $w(r) = E_x[\tau]$, $|x| = r$, then w satisfies the boundary value problem

$$
\begin{cases}
\dfrac{-d^2 w}{dr^2} - \dfrac{1}{r}\,\dfrac{dw}{dr} = 1\,, & \alpha\lambda < r < \lambda\,, \\
w(\alpha\lambda) = w(\lambda) = 0\,.
\end{cases}
$$

The solution of this boundary value problem is given by

$$
w(r) = \frac{1}{4}\,(\lambda^2 - r^2) - \frac{1}{4}\,\lambda^2\,(1-\alpha^2)\,\frac{\log\left(\dfrac{\lambda}{r}\right)}{\log\left(\dfrac{1}{\alpha}\right)}\,.
$$

If we put $v(r)$ to be the probability that Brownian motion started at x, $|x| = r$, exits U_α through the boundary $\{y : |y| = \alpha\lambda\}$ then v satisfies the boundary value problem,

$$
\begin{cases}
\dfrac{-d^2 v}{dr^2} - \dfrac{1}{r}\,\dfrac{dv}{dr} = 0\,, & \alpha\lambda < r < \lambda\,, \\
v(\alpha\lambda) = 1\,, & v(\lambda) = 0\,.
\end{cases}
$$

The solution of this last boundary value problem is given by the formula

$$
v(r) = \frac{\log\left(\dfrac{\lambda}{r}\right)}{\log\left(\dfrac{1}{\alpha}\right)}\,.
$$

Now consider the expression,

$$
v(r) - \frac{2}{\lambda^2}\,w(r) = \frac{1}{2}\,(3-\alpha^2)\,\frac{\log\left(\dfrac{\lambda}{r}\right)}{\log\left(\dfrac{1}{\alpha}\right)} - \frac{1}{2}\Big(1 - \frac{r^2}{\lambda^2}\Big)\,.
$$

It is clear that if α is sufficiently close to 1 there is a constant $k_\alpha > 0$ such that

$$v(r) - \frac{2}{\lambda^2}\, w(r) \ge \frac{k_\alpha}{4}\Big(1 - \frac{r^2}{\lambda^2}\Big) = k_\alpha\, \frac{u(r)}{\lambda^2}\,.$$

Thus by (A.7), (A.8), (A.9) we conclude that (A.6) holds with $t = \lambda^2$ if $|x| > \alpha\lambda$ with constant $C = (k_\alpha c_\alpha)^{-1}$ provided α is sufficiently close to 1.

We have proved therefore that (A.6) holds for all $x \in D_\lambda$ and $t = \lambda^2$. The result follows.

Lemma A.2. *Let $\kappa_0 > 0$ be the minimum eigenvalue of $-\Delta$ on the unit disc with Dirichlet boundary conditions. Let $G_D(x, y, t)$ be the Dirichlet Green's function for the heat equation on D_λ. Then for any $\alpha, 0 < \alpha < 1$, there exist positive constants c_α, C_α depending only on α such that*

$$c_\alpha \exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big) \le \int_{D_\lambda} G_D(x, y, t)\, dy \le C_\alpha \exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big), \tag{A.12}$$

with $t > 0$, provided $|x| \le \alpha\lambda$.

PROOF. Let $\varphi_0(x)$ be the eigenfunction on the unit disc corresponding to the eigenvalue κ_0 of $-\Delta$. Then $\varphi_0(x)$ is a positive C^∞ function for $|x| < 1$, and continuous on $|x| \le 1$ with $\varphi_0(x) = 0$, $|x| = 1$. By scaling we have that

$$\int_{D_\lambda} G_D(x, y, t)\, \varphi_0\Big(\frac{y}{\lambda}\Big)\, dy = \exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big)\varphi_0\Big(\frac{x}{\lambda}\Big)\,. \tag{A.13}$$

Hence it follows that

$$\int_{D_\lambda} G_D(x, y, t)\, dy \ge \exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big)\frac{\varphi_0\Big(\dfrac{x}{\lambda}\Big)}{\|\varphi_0\|_\infty}\,.$$

Now the first inequality in (A.12) follows from this last inequality by taking

$$c_\alpha = \inf\Big\{\frac{\varphi_0(z)}{\|\varphi_0\|_\infty} : |z| < \alpha\Big\} > 0\,.$$

We use Lemma A.1 to prove the upper bound in (A.12). Thus from (A.13) we have

$$\begin{aligned}\exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big)\varphi_0\Big(\frac{x}{\lambda}\Big) &\ge \inf\Big\{\varphi_0\Big(\frac{y}{\lambda}\Big) : |y| \le \frac{\lambda}{2}\Big\}\int_{|y|<\lambda/2} G_D(x, y, t)\, dy \\ &\ge c\int_{|y|<\lambda} G_D(x, y, t)\, dy\,,\end{aligned}$$

for some universal constant $c > 0$ provided $t \geq \lambda^2$. The upper bound in (A.12) clearly follows from this last inequality provided $t \geq \lambda^2$. The inequality for $t \leq \lambda^2$ is trivial by choosing C_α to satisfy $C_\alpha \exp(-\kappa_0) \geq 1$.

REMARK. The inequality (A.12) has already been proved in a much more general context [2], [8], for divergence form operators in domains with Lipschitz boundary.

Next we wish to prove the formula (A.1).

Proposition A.1. *Let τ be the time taken for 3-dimensional Brownian motion $X(t) = (X_1(t), X_2(t), X_3(t))$ started at the origin to exit the cylinder $D_\lambda \times (-m\lambda, m\lambda)$. There are universal constants $C, c > 0$ such that*

$$c\, m\lambda^2 \leq E\,[\tau : |X_3(\tau)| = m\lambda] \leq C m\lambda^2\,,$$

provided $m \geq 1$.

Consider one-dimensional Brownian motion $X_3(t)$ starting at the origin and let τ_1 be the first hitting time on the boundary of the interval $[-m\lambda, m\lambda]$, and $\rho(t)$, $t \geq 0$, be the probability density for τ_1. Next consider 2-dimensional Brownian motion starting at the origin and let τ_2 be the first hitting time on the boundary of D_λ. Then

$$E\,[\tau : |X_3(\tau)| = m\lambda] = \frac{\displaystyle\int_0^\infty P(\tau_2 > t)\, t\, \rho(t)\, dt}{\displaystyle\int_0^\infty P(\tau_2 > t)\, \rho(t)\, dt}\,. \tag{A.14}$$

Now from Lemma A.2 it follows that there are universal constants $C, c > 0$ such that

$$c\, I_m \leq E\,[\tau : |X_3(\tau)| = m\lambda] \leq C I_m\,, \tag{A.15}$$

where

$$I_m = \frac{\displaystyle\int_0^\infty \exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big) t\, \rho(t)\, dt}{\displaystyle\int_0^\infty \exp\Big(-\frac{\kappa_0\, t}{\lambda^2}\Big) \rho(t)\, dt}\,.$$

We have now that for $\eta > 0$,

$$\int_0^\infty e^{-\eta t}\rho(t)\,dt = E_0\left[\exp\left(-\eta\tau_1\right)\right] = \frac{1}{\cosh\left(\sqrt{\eta}\,m\lambda\right)}\ .$$

Differentiating this last expression with respect to η we obtain

$$\int_0^\infty e^{-\eta t}\,t\,\rho(t)\,dt = \frac{m\lambda\sinh\left(\sqrt{\eta}\,m\lambda\right)}{2\sqrt{\eta}\,\cosh^2\left(\sqrt{\eta}\,m\lambda\right)}\ .$$

If we take now $\eta = \kappa_0/\lambda^2$ and we use the last two formulas it follows that

$$I_m = \frac{m\lambda^2}{2\sqrt{\kappa_0}}\tanh\left(m\sqrt{\kappa_0}\right).$$

The result follows from this last formula and (A.15).

Proof of Theorem A.1. Let $G_m(\xi,\zeta,t)$ be the Dirichlet Green's function of the interval $[-m\lambda, m\lambda]$. Then if G_D is the Green's function for D_λ as in Lemma A.2 it follows that the Dirichlet Green's function for the cylinder $D_\lambda \times (-m\lambda, m\lambda)$ is given by

$$G_D(x,y,t)\,G_m(\xi,\zeta,t)\,, \qquad x,y\in D_\lambda\,,\ -m\lambda<\xi\,,\ \zeta<m\lambda\,,\ t>0\,.$$

For $x\in D_\lambda, \xi\in(-m\lambda,m\lambda)$ let $u(x,\xi)$ be the probability that Brownian motion started at (x,ξ) exits $D_\lambda\times(-m\lambda,m\lambda)$ through $D_\lambda\times\{m\lambda\}$ or $D_\lambda\times\{-m\lambda\}$. If we define $w(\xi)$ by

$$w(\xi) = \frac{1}{|D_\lambda|}\int_{D_\lambda} u(x,\xi)\,dx\,, \qquad \xi\in(-m\lambda,m\lambda)\,,$$

it follows from the argument of Proposition A.1 that there are positive universal constants $C, c > 0$ such that

$$\text{(A.16)} \qquad c\,E_\xi\Big[\exp\Big(-\frac{\kappa_0\tau_1}{\lambda^2}\Big)\Big] \le w(\xi) \le C E_\xi\Big[\exp\Big(-\frac{\kappa_0\tau_1}{\lambda^2}\Big)\Big]\,,$$

where τ_1 is the first exit time of Brownian motion started at ξ from the interval $(-m\lambda, m\lambda)$. Furthermore there is the identity

$$\text{(A.17)} \qquad E_\xi\Big[\exp\Big(-\frac{\kappa_0\tau_1}{\lambda^2}\Big)\Big] = \frac{\cosh\Big(\frac{\xi\sqrt{\kappa_0}}{\lambda}\Big)}{\cosh m\sqrt{\kappa_0}}\ .$$

We have now that if $X(t)$ denotes Brownian motion started uniformly on the cross section $D_\lambda \times \{\xi\}$, then

$$E\Big[\int_0^\tau V(X(t))\,dt : |X(\tau)_3| = m\lambda\Big] = \frac{1}{|D_\lambda|}\int_{D_\lambda} dx \int_0^\infty dt \int_{-m\lambda}^{m\lambda}\int_{D_\lambda} G_D(x,y,t)\,G_m(\xi,\zeta,t) \cdot u(y,\zeta)\,V(y,\zeta)\,\frac{dy\,d\zeta}{w(\xi)}\,. \tag{A.18}$$

It follows from Lemma A.2 that

$$\begin{aligned}\int_{D_\lambda} dx \int_0^\infty G_D(x,y,t)\,G_m(\xi,\zeta,t)\,dt &\le C\int_0^\infty \exp\Big(-\frac{\kappa_0\,t}{\lambda^2}\Big)G_m(\xi,\zeta,t)\,dt\\ &= CG(\xi,\zeta)\,,\end{aligned} \tag{A.19}$$

where G is the Green's function which satisfies

$$\begin{cases}\Big(-\dfrac{d^2}{d\xi^2}+\dfrac{\kappa_0}{\lambda^2}\Big)G(\xi,\zeta) = \delta\,(\xi-\zeta)\,, & -m\lambda < \xi < m\lambda\,,\\ G(\xi,\zeta) = 0\,, & |\xi| = m\lambda\,.\end{cases}$$

We can solve this boundary value problem to obtain the explicit formula

$$G(\xi,\zeta) = \lambda\sinh\Big(\frac{\sqrt{\kappa_0}}{\lambda}\,(m\lambda-\xi)\Big)\,\frac{\sinh\Big(\dfrac{\sqrt{\kappa_0}}{\lambda}\,(m\lambda+\zeta)\Big)}{\sqrt{\kappa_0}\,\sinh\,(2\,m\sqrt{\kappa_0})}\,,$$

if $\xi > \zeta$, and

$$G(\xi,\zeta) = \lambda\sinh\Big(\frac{\sqrt{\kappa_0}}{\lambda}\,(m\lambda+\xi)\Big)\,\frac{\sinh\Big(\dfrac{\sqrt{\kappa_0}}{\lambda}\,(m\lambda-\zeta)\Big)}{\sqrt{\kappa_0}\,\sinh\,(2\,m\sqrt{\kappa_0})}\,,$$

if $\xi < \zeta$. It follows now from this last identity and (A.16), (A.17) that there is a universal constant C such that

$$G(\xi,\zeta)\,w(\zeta) \le C\,\lambda\,w(\xi)\,, \qquad -m\lambda < \xi\,,\ \zeta < m\lambda\,.$$

Hence from this last inequality and (A.18), (A.19) we have that

$$E\Big[\int_0^\tau V(X(t))\,dt : |X(\tau)_3| = m\lambda\Big] \le \frac{C}{\lambda}\int_{-m\lambda}^{m\lambda}\int_{D_\lambda}\frac{u(y,\zeta)}{w(\zeta)}\,V(y,\zeta)\,dy\,d\zeta\,,$$

for some universal constant C.

It is obvious from Lemma A.2 that there is a universal constant C such that $u(y,\zeta) \le C\,w(\zeta)$, $y \in D_\lambda$, $\zeta \in (-m\lambda, m\lambda)$. The result follows from this and the last inequality.

Appendix B. A differential inequality.

Here we prove the inequality (5.33). Consider the solution $u(r,\eta)$ to the Sturm-Liouville problem,

$$\text{(B.1)} \qquad \begin{cases} \rho(r)\,\dfrac{d^2u}{dr^2} + \rho'(r)\,\dfrac{du}{dr} = \eta\,\rho(r)\,u\,, & 1 < r < R\,, \\ u(1) = 1\,, & \\ u(R) = 0\,, & \end{cases}$$

where $\rho(r) > 0$, $\rho'(r) > 0$, $1 \le r \le R$. We shall show that

$$\text{(B.2)} \qquad \frac{\partial}{\partial r}\frac{\partial}{\partial \eta}\Big(\frac{u(r,\eta)}{\sqrt{\eta}}\Big) > 0\,, \qquad 1 < r < R\,,\ \eta > 0\,.$$

This implies the inequality (5.33) on taking $\rho(r) = r$. The inequality (B.2) is sharp in the sense that the power of η, *i.e.* $\eta^{1/2}$ in the denominator cannot be improved. To see this consider for $\alpha > 0$, the function

$$w_\alpha(r) = \frac{\partial}{\partial \eta}\Big(\frac{u}{\eta^\alpha}\Big) = \frac{1}{\eta^\alpha}\Big(\frac{\partial u}{\partial \eta} - \alpha\,\frac{u}{\eta}\Big)\,.$$

Now it follows easily from the maximum principle that the function u decreases as a function of r. The function $w_0(r) = \partial u/\partial \eta$ satisfies the boundary conditions $w_0(1) = w_0(R) = 0$. It follows from the maximum principle again that $w_0(r) < 0$, $1 < r < R$. Hence there exists a minimum $\alpha_0 > 0$ such that

$$\frac{dw_\alpha}{dr} \ge 0\,, \qquad 1 < r < R\,,\ \alpha \ge \alpha_0\,.$$

We can explicitly compute α_0 in the exactly solvable case when $\rho \equiv 1$. Thus for $\rho \equiv 1$, we have

$$u(r,\eta) = \frac{\sinh\sqrt{\eta}\,(R-r)}{\sinh\sqrt{\eta}\,(R-1)}\,,$$

whence

$$\frac{1}{\sqrt{\eta}}\,\frac{\partial u}{\partial r} = \frac{-\cosh\sqrt{\eta}\,(R-r)}{\sinh\sqrt{\eta}\,(R-1)}\ ,$$

$$\frac{\partial}{\partial\eta}\Big(\frac{1}{\sqrt{\eta}}\,\frac{\partial u}{\partial r}\Big) = \frac{1}{2\sqrt{\eta}}\,\frac{(r-1)+(R-r)\cosh\sqrt{\eta}\,(r-1)}{\sinh^2\sqrt{\eta}(R-1)}\ .$$

Hence if $0<\alpha<1/2$ we have

$$\begin{aligned}\frac{dw_\alpha}{dr} = \frac{1}{2\,\eta^\alpha}\Big(&\frac{(r-1)\cosh\sqrt{\eta}\ (R-1)\cosh\sqrt{\eta}\ (R-r)}{\sinh^2\sqrt{\eta}\ (R-1)}\\ &+\frac{(R-r)\cosh\sqrt{\eta}\,(r-1)}{\sinh^2\sqrt{\eta}\ (R-1)}\Big)\\ &-\frac{\Big(\dfrac{1}{2}-\alpha\Big)}{\eta^{\alpha+1/2}}\,\frac{\cosh\sqrt{\eta}\ (R-r)}{\sinh\sqrt{\eta}\ (R-1)}\ .\end{aligned}$$

It is clear from this last identity that by choosing $r=1$ and R large we will have $dw_\alpha/dr<0$ at $r=1$ for any $\alpha<1/2$. Thus $\alpha_0=1/2$ is optimal in this case.

To prove (B.2) we first construct the Dirichlet Green's function for (B.1). Let $v(r)$ be the solution of the equation (B.1) with the boundary conditions $v(1)=0$, $v(R)=1$. Then the Dirichlet Green's function $G(r,r')$, $1<r$, $r'<R$, can be written as

$$G(r,r') = \begin{cases} c(r')\,u(r)\,v(r')\,, & 1<r'<r\,,\\ c(r')\,u(r')\,v(r)\,, & r<r'<R\,.\end{cases} \tag{B.3}$$

The constant $c(r')$ is determined by the jump discontinuity of $\partial G/\partial r$ at $r=r'$,

$$\lim_{r\to r'+}\frac{\partial G}{\partial r}(r,r') - \lim_{r\to r'-}\frac{\partial G}{\partial r}(r,r') = \frac{1}{\rho(r')}\ .$$

Thus if $W(r')$ denotes the Wronskian,

$$W(r') = u'(r')\,v(r') - u(r')\,v'(r')\,,\qquad 1<r'<R\,,$$

we have that

$$c(r') = \frac{1}{\rho(r')\,W(r')}\ .$$

Now using the fact that

$$W(r') = -\frac{\rho(1)\, v'(1)}{\rho(r')}\,, \qquad 1 < r' < R\,,$$

we conclude that

$$c(r') = -\frac{1}{\rho(1)\, v'(1)}\,, \qquad 1 < r' < R\,. \tag{B.4}$$

It is clear that the function $w(r) = w_{1/2}(r)$ satisfies the equation

$$\begin{cases} \rho(r)\, \dfrac{d^2 w}{dr^2} + \rho'(r)\, \dfrac{dw}{dr} = \eta\, \rho(r)\, w + \dfrac{\rho\, u}{\sqrt{\eta}}\,, \\ w(1) = -\dfrac{1}{2\,\eta^{3/2}}\,, \\ w(R) = 0\,. \end{cases}$$

Hence w has the representation

$$w(r) = \frac{-u(r)}{2\,\eta^{3/2}} + \int_0^R G(r, r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr'\,.$$

Using the formulas (B.3), (B.4) we have then

$$\begin{aligned} w(r) = \frac{-u(r)}{2\,\eta^{3/2}} &- \frac{1}{\rho(1)\, v'(1)} \int_0^r u(r)\, v(r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr' \\ &- \frac{1}{\rho(1)\, v'(1)} \int_r^R v(r)\, u(r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr'\,. \end{aligned}$$

On differentiating the above identity we have then

$$\begin{aligned} \frac{dw}{dr} = \frac{-u'(r)}{2\,\eta^{3/2}} &- \frac{u'(r)}{\rho(1)\, v'(1)} \int_0^r v(r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr' \\ &- \frac{v'(r)}{\rho(1)\, v'(1)} \int_r^R u(r')\, \frac{\rho(r')\, u(r')}{\sqrt{\eta}}\, dr'\,. \end{aligned}$$

It follows that $dw/dr \geq 0$, $1 < r < R$, if we can show that

$$2\,\eta\, v'(r) \int_r^R \rho(r')\, u(r')^2\, dr' \leq -\rho(1)\, v'(1)\, u'(r)\,, \qquad 1 < r < R\,. \tag{B.5}$$

We have now that

$$
\begin{aligned}
\eta \int_r^R \rho(r')\, u(r')^2\, dr' &= \int_r^R u\, \frac{d}{dr'}\Big(\rho(r')\, \frac{du}{dr'}\Big)\, dr' \\
&= -\int_r^R \rho(r')\, u'(r')^2\, dr' - \rho(r)\, u(r)\, u'(r)\,.
\end{aligned}
\tag{B.6}
$$

Next we show that

$$
\eta\, u(r')^2 \le u'(r')^2\,, \qquad 1 < r' < R\,. \tag{B.7}
$$

To see that (B.7) holds observe that

$$
\begin{aligned}
\frac{d}{dr'}\,(\eta\, u(r')^2 - u'(r')^2) &= 2\, u'(r')\, (\eta u(r') - u''(r')) \\
&= 2\, u'(r')\, \frac{\rho'(r')}{\rho(r')}\, u'(r') \ge 0\,, \qquad 1 < r' < R\,.
\end{aligned}
$$

We conclude from this that

$$
\eta\, u(r')^2 - u'(r')^2 \le \eta\, u(R)^2 - u'(R)^2 = -u'(R)^2 \le 0\,,
$$

whence (B.7) follows.

From (B.6), (B.7) it follows that

$$
2\,\eta \int_r^R \rho(r')\, u(r')^2\, dr' \le -\rho(r)\, u(r)\, u'(r)\,.
$$

Hence the left hand side of (B.5) is bounded above by

$$
-v'(r)\, \rho(r)\, u(r)\, u'(r)\,.
$$

Thus (B.5) holds if we can show

$$
\rho(1)\, v'(1) \ge v'(r)\, \rho(r)\, u(r)\,, \qquad 1 < r < R\,,
$$

since $u'(r) < 0$, $1 < r < R$. This last inequality follows from the fact that

$$
\begin{aligned}
v'(r)\, \rho(r)\, u(r) &= \rho(r)\, (u'(r)\, v(r) - W(r)) \\
&= \rho(r)\Big(u'(r)\, v(r) + \rho(1)\, \frac{v'(1)}{\rho(r)}\Big) \\
&\le \rho(1)\, v'(1)\,,
\end{aligned}
$$

since $u'(r) < 0$, $v(r) > 0$, $1 < r < R$.

Acknowledgement. The proof of the differential inequality in Appendix B was given to us by Paul Federbush. It is a pleasure to thank him for his elegant proof and for many other helpful conversations on the contents of this paper.

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Recibido: 26 de marzo de 1.996
Revisado: 11 de septiembre de 1.996

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* Research partially supported by U.S. National Science Foundation grants DMS 9100455 and DMS 9403399.

† Research supported by NFR-Norges Forskringsråd (Norwegian Research Council) grant 100222/40.