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Cauchy problem for semilinear parabolic equations with initial data in $H_p^s(\mathbb{R}^n)$ spaces

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Abstract. We study local and global Cauchy problems for the Semilinear Parabolic Equations $\partial_t U - \Delta U = P(D)F(U)$ with initial data in fractional Sobolev spaces $H_p^s(\mathbb{R}^n)$. In most of the studies on this subject, the initial data $U_0(x)$ belongs to Lebesgue spaces $L^p(\mathbb{R}^n)$ or to supercritical fractional Sobolev spaces $H_p^s(\mathbb{R}^n)$ $(s > n/p)$. Our purpose is to study the intermediate cases (namely for $0 < s < n/p$). We give some mapping properties for functions with polynomial growth on subcritical $H_p^s(\mathbb{R}^n)$ spaces and we show how to use them to solve the local Cauchy problem for data with low regularity. We also give some results about the global Cauchy problem for small initial data.

1. Introduction and results.

1.1. The evolution equation.

We study the Cauchy problem for the Semilinear Parabolic Equation

$$(1) \qquad \begin{cases} \partial_t U - \Delta U = P(D)F(U)\,, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n\,, \\ U(0,x) = U_0(x)\,, \end{cases}$$

where $P(D)$ is a pseudodifferential operator of order $d \in [0,2[$ and where F is a nonlinear function which behaves like $|x|^\alpha$ or $x\,|x|^{\alpha-1}$ $(\alpha > 1)$. The most classical examples of such evolution equations are the semilinear heat equations

$$\partial_t u - \Delta u = a\,u\,|u|^{\alpha-1}\,,$$

the Burgers viscous equations

$$\partial_t u - \Delta u = a\,\partial_x(|u|^\alpha)$$

and the Navier-Stokes equation

$$\partial_t u - \Delta u = \mathcal{P}\,\nabla(u \otimes u)\,,$$

where $\mathcal{P}$ denotes the projector on the divergence free vector field (see [Ca] for instance).

We look for mild solutions of (1), *i.e.* for solutions of the integral equation

$$U(t,x) = e^{t\Delta}\,U_0 + \int_0^t e^{(t-\tau)\Delta} P(D)F(U(\tau))\,d\tau\,, \tag{2}$$

where $e^{t\Delta}$ is the heat kernel. As usual the fractional Sobolev spaces and their homogeneous versions are defined by

$$H_p^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) :\ \Lambda_s f \in L^p\}$$

and

$$\dot{H}_p^s = \{f \in \mathcal{S}'(\mathbb{R}^n) :\ \dot{\Lambda}_s f \in L^p\}\,,$$

where Λ_s and $\dot{\Lambda}_s$ are the operators with symbols $\Lambda_s(\xi) = (1+|\xi|^2)^{s/2}$ and $\dot{\Lambda}_s(\xi) = |\xi|^s$ (these spaces are sometimes also denoted $L^{p,s}(\mathbb{R}^n)$, see [Me]). In the sequel we will say that $H_p^s(\mathbb{R}^n)$ is supercritical if $s > n/p$, *i.e.* if the embedding $H_p^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ is verified and, on the contrary, we will say that $H_p^s(\mathbb{R}^n)$ is subcritical.

In the proofs of existence and uniqueness for (2), there always exists a tight connection between the regularity of the Cauchy's data U_0 and the properties of the nonlinear term $P(D)F(U)$. Thus, for $F(x) \approx |x|^\alpha$, Giga [Gi] proved existence and uniqueness for Equation (2) as long as U_0 belongs to an $L^p(\mathbb{R}^n)$ for p large enough. When U_0 belongs to supercritical $H_p^s(\mathbb{R}^n)$ spaces, Taylor [Ta] proved existence and uniqueness for (2) under the assumptions $F(0) = 0$ and $F \in C^{[s]+1}(\mathbb{R})$. One of our purpose is to study all the intermediate range of regularity, namely, to solve (2) for initial data in $H_p^s(\mathbb{R}^n)$ with s in $]0, n/p[$. About this problem, partial results have been found by Henry [He] who proved that, if $s < 2 - d$ and if F maps bounded sets from $H_p^s(\mathbb{R}^n)$ into bounded sets in $L^p(\mathbb{R}^n)$, then (2) is well posed. Let us remark that, in the examples considered by these authors, the action of F on the functional space of the initial data is well understood. This allows to obtain crucial estimates on the nonlinear terms to solve (2): in the first two cases $F : L^p \longrightarrow L^{p/\alpha}$ and $F : H_p^s(\mathbb{R}^n) \longrightarrow H_p^s(\mathbb{R}^n)$ is bounded and in the third one, the hypothesis on F implies some similar properties.

In this paper our goal is to improve Henry's results for the local Cauchy problem and Giga's results for the global Cauchy problem (for small initial data). We give the minimal regularity of U_0 (see Remark 3 after Theorem 1.3 about this), measured on the scale of $H_p^s(\mathbb{R}^n)$ spaces, which ensures both existence and uniqueness for (2). So, for a fixed p in $]1, +\infty[$, we are looking for the smallest exponent of regularity such that, for all U_0 in $H_p^s(\mathbb{R}^n)$ with s greater than this smallest exponent, existence and uniqueness occur. In such a framework one of the most important difficulty arises from the fact that the action of the nonlinear function F on subcritical $H_p^s(\mathbb{R}^n)$ spaces is badly understood. So, to solve (2) in subcritical $H_p^s(\mathbb{R}^n)$ spaces, we will need to prove some mapping properties on those spaces for functions with polynomial growth: this will be realized using harmonic analysis and paradifferential calculus techniques in Section 4.

As an example, let us consider the nonlinear heat equations

$$\partial_t U - \Delta U = a\, U\, |U|^{\alpha-1}\,, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n\,. \tag{3}$$

When $U(t,x)$ is a solution of (3) then, for each $\lambda > 0$, the functions U_λ defined by $U_\lambda(t,x) = \lambda^{(2-d)/(\alpha-1)} U(\lambda^2 t, \lambda x)$ are also solutions (here $d = 0$) and, one can check that U and U_λ have the same norm in $L^\infty(\mathbb{R}^+, \dot{H}_p^s)$ if and only if

$$s = s_c = \frac{n}{p} - \frac{2-d}{\alpha - 1}\,. \tag{4}$$

Without further assumptions on the nonlinear term, this scaling argument suggests that, for all data in $H_p^s(\mathbb{R}^n)$, there exists a unique solution of (3) as long as $s > s_c$. This also suggests that the "right spaces" for the study of global existence are the spaces $H_p^{s_c}(\mathbb{R}^n)$. For instance, we show (see Theorem 1.3) that, for all $U_0(x) \in H^1(\mathbb{R}^3)$, one can find a unique local solution of (3) as long as $\alpha \in]1,5]$ and, furthermore (see Theorem 1.5), this solution is global as long as $\|U_0\|_{H^1}$ is sufficiently small. This result improves Henry's results because, using his criterion, one can only prove existence and uniqueness in $H^1(\mathbb{R}^3)$ for $\alpha \in]1,3]$.

In fact, we will show that this scaling argument is true for Equation (2) even if $P(D)$ and F do not possess the exact homogeneity of Equation (3). For these reasons we will say that $H^s_p(\mathbb{R}^n)$ is supercritical (respectively critical) for (2) if $s > s_c$ (respectively if $s = s_c$).

To avoid technical problems we will always assume that

$$s \geq \frac{n}{p} - \frac{n}{\alpha} \tag{5}$$

and that

$$s \geq 0\,. \tag{6}$$

Indeed, according to the Sobolev embedding theorem, if $u \in C([0,T[, H^s_p)$ with s as in (5) and as in (6) then $u \in C([0,T[, L^{\tilde{p}})$ with $\tilde{p} \geq \alpha$. Hence, the term $F(u)$ in (2) is well defined in $\mathcal{D}'(]0,T[\times\mathbb{R}^n)$. On the contrary, if (5) or (6) is not satisfied, solutions in $C([0,T[, H^s_p)$ cannot be defined in a simple way: for instance, if $u \in C([0,T[, H^s_p)$ with $s < 0$, then $F(u)$ has no sense *a priori*. For the study of such cases, when (5) or (6) are not fulfilled, we refer to [Ri] where we show that (2) can sometimes be solved using some smoothing properties of the heat kernel.

1.2. Hypotheses on the nonlinear terms.

About the nonlinear terms $P(D)$ and $F(u)$ we will make the following assumptions.

H1) $P(D)$ is a pseudodifferential operator of degree $d \in [0,2[$ with constant coefficients (and so $P(D)$ is bounded from $H^{s+d}_p(\mathbb{R}^n)$ to $H^s_p(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ and for all $p \in]1,+\infty[$).

About F we will assume that

H2) there exists $\alpha > 1$ such that,

i) $s \leq n\,(\alpha - 1)/(p\,\alpha)$,

ii) $F : \mathbb{R} \longrightarrow \mathbb{R}$ verifies $|F(x) - F(y)| \leq C\,|x-y|(|x|^{\alpha-1} + |y|^{\alpha-1})$
or,

H3) there exists $\alpha > 1$ such that,

i) $n\,(\alpha-1)/(p\,\alpha) < s < \min\{(n/p+1)\,(\alpha-1)/\alpha, n/p\}$,

ii) $F : \mathbb{R} \longrightarrow \mathbb{R}$ is $[\alpha]$ time differentiable, $D^j F(0) = 0$ for $j = 0, \ldots, [\alpha]-1$, $D^{[\alpha]}F(0) = 0$ if $\alpha \notin \mathbb{N}$, and $|D^{[\alpha]}F(x) - D^{[\alpha]}F(y)| \leq C\,|x-y|^{\alpha-[\alpha]}$

or,

H4)

i) $n/p < s$,

ii) $F : \mathbb{R} \longrightarrow \mathbb{R}$ verifies $F(0) = 0$ and $F \in C^{[s]+1}(\mathbb{R})$.

Note that those assumptions on the nonlinear term F depend in a crucial way of the smoothness of the initial data $U_0(x)$. Indeed, when $U_0(x)$ belongs to a supercritical $H_p^s(\mathbb{R}^n)$ space then, since we look for a solution in $C([0,T], H_p^s)$, we look for a bounded solution of (2). Hence, in H4), we do not need any assumptions on the asymptotic behavior of F; we just need smoothness assumptions on F. On the contrary, when $U_0(x)$ belongs to a subcritical $H_p^s(\mathbb{R}^n)$ space, then $U_0(x)$ is possibily unbounded in a neighbourhood of some point x_0 and then we need assumptions on the behavior of F at infinity to "control" $F(U_0(x))$ near x_0.

Note also that, from the assumptions on F, we can easily deduce from H3.ii) the following properties for the intermediate derivatives of F.

Lemma 1.1. *If* H3.ii) *holds then there exists a constant C such that,*

$$|D^j F(x) - D^j F(y)| \leq C\,|x-y|\,(|x|^{\alpha-j-1} + |y|^{\alpha-j-1})\,, \tag{7}$$

for all $j = 0, \ldots, [\alpha]-1$,

$$|D^j F(x)| \leq C\,|x|^{\alpha-j}\,, \tag{8}$$

for all $j = 0, \ldots, [\alpha]$.

1.3. Statement of main results.

To solve (2) the main idea is to counterbalance the loss of smoothness coming from the nonlinear terms by the smoothing effects of the heat kernel. In the framework of $L^p(\mathbb{R}^n)$ spaces, according to H2) and Hölder's inequality, $F : L^p \longrightarrow L^{p/\alpha}$ is continuous. If H4) holds there is no loss of smoothness on the $H_p^s(\mathbb{R}^n)$ scale thanks to the following Theorem (see [Me] or [Ta]).

Theorem 1.1. *Let* $p \in \,]1, +\infty[$. *If* H4) *is fulfilled then, for all* $u \in H_p^s(\mathbb{R}^n)$, $F(u)$ *belongs to* $H_p^s(\mathbb{R}^n)$ *and furthermore*

$$\|F(u)\|_{H_p^s} \leq C\left(\|u\|_{L^\infty}\right)\|u\|_{H_p^s} \,.$$

On the other hand, in the case of subcritical $H_p^s(\mathbb{R}^n)$ spaces, there is no stability by composition with nonlinear functions. For instance, the $H_p^s(\mathbb{R}^n)$ spaces are algebras if and only if $s > n/p$. For $s \in [1+1/p, n/p[$ and $p \in \,]1, +\infty[$ one can also prove that the functional calculus is trivial in $H_p^s(\mathbb{R}^n)$ (see G. Bourdaud [Bo] for instance): if F maps $H_p^s(\mathbb{R}^n)$ into itself for s in this range then $f(x) = a\,x$.

To measure the loss of smoothness on the $H_p^s(\mathbb{R}^n)$ scale coming from the composition by F, we will prove the following Theorem in Section 4.

Theorem 1.2. *Let* $p \in \,]1, +\infty[$ *and* s *such that*

$$\max\left\{0, \frac{n}{p} - \frac{n}{\alpha}\right\} < s < \frac{n}{p}\,.$$

Let s_α *defined by*

$$s_\alpha = s - (\alpha - 1)\left(\frac{n}{p} - s\right). \tag{9}$$

If H2) *or* H3) *is fulfilled then, for all* $u \in H_p^s(\mathbb{R}^n)$, $F(u)$ *belongs to* $H_p^{s_\alpha}(\mathbb{R}^n)$ *and furthermore, there exists a constant* C *independent of* u *such that*

$$\|F(u)\|_{H_p^{s_\alpha}} \leq C\,\|u\|_{H_p^s}^\alpha\,.$$

Remarks.

1) Note that the condition $s \le n\,(\alpha-1)/(p\,\alpha)$ in H2.i) is equivalent to

$$s_\alpha \le 0\,. \tag{10}$$

In the same way, the conditions $n\,(\alpha-1)/(p\,\alpha) < s < (n/p+1)(\alpha-1)/\alpha$ in H3.i) are equivalent to

$$0 < s_\alpha < \alpha - 1\,. \tag{11}$$

2) The hypothesis $s > \max\{0, n/p - n/\alpha\}$ ensures that $F(u)$ is well defined as an element of $\mathcal{D}'$.

3) The restriction $s < (1+n/p)(\alpha-1)/\alpha$ in H3.i) (*i.e.* $s_\alpha < \alpha-1$) comes from the lack of smoothness of F at $x=0$. However, if F is C^∞ ($F(x) = x^m$ for instance), then in H3.i) we must only assume that

$$\frac{n\,(\alpha-1)}{p\,\alpha} < s < \frac{n}{p}$$

to obtain Theorem 1.3.

4) The value of s_α given by Theorem 1.2 is optimal. To see this we have just to consider the example of $u(x) = \psi(x)\,x^{-\beta}$ and $F(x) = |x|^\alpha$ where ψ is a cut of function near 0.

5) In order to solve nonlinear Schrödinger equations, T. Colin [Co] established a related result to Theorem 1.2 for the spaces $H^s_p(\mathbb{R}^n) \cap L^z(\mathbb{R}^n)$. Recently another proof of Theorem 1.2 has been found by T. Runst and W. Sickel in [RS]. First, using paraproduct techniques, they prove Theorem 1.2 in the special case of polynomial functions. Then, using a Taylor expantion of F and Poisson approximations of u, they prove Theorem 1.2 in the general seeting of H3). Our proof is in fact very different. First, we use different techniques (we only use paradifferential calculus) and, second, we do not need to distingue between the polynomial case and the general case.

Using the nonlinear estimates given by Theorem 1.2 and the fixed point Theorem, in Section 2 we prove the following result about the local Cauchy problem.

Theorem 1.3. *Let $p \in]1,+\infty[$. Assume that* (5) *and* (6) *holds, and that* H2) *or* H3) *or* H4) *is fulfilled.*

a) *For all initial data U_0 in $H_p^s(\mathbb{R}^n)$ with $s > s_c$ there exists a unique maximal solution $U(t,x)$ of* (2) *in $C([0,T_m[\,,H_p^s)$ with*

$$T_m \geq C\,\|U_0\|_{H_p^s}^{-\nu^{-1}}\,, \qquad \text{where } \nu = \frac{s-s_c}{2}\,,$$

and, if $T_m < +\infty$, then

$$\lim_{t\to T_m} \|U(t,\cdot)\|_{H_p^s} = +\infty\,.$$

b) *Furthermore the following smoothing effects occur:*

• *$U(t,x) - e^{t\Delta}U_0 \in C([0,T_m[\,,H_p^{s+\theta})$ for all $\theta < (\alpha-1)\,\nu$ if $s < n/p$ and for all $\theta < 2-d$ if $s > n/p$.*

• *If F is $C^\infty(\mathbb{R})$ then,*

$$U(t,x) \in C^\infty([\delta, T_m[\times \mathbb{R}^n)\,,$$

for all $\delta > 0$.

c) *Let us assume that $s < 2-d$. Let $U \in C([0,T_1[\,,H_p^s)$ and $V \in C([0,T_2[\,,H_p^s)$ be the maximal solutions for the respective initial data U_0 and V_0. Then,*

$$\|U-V\|_{C([0,T[,H_p^s)} \leq C(T)\,\|U_0 - V_0\|_{H_p^s}^{1/2}\,.$$

for all $T < \min\{T_1,T_2\}$.

REMARKS.

1) Let us consider Equation (3) with $U_0 \in H_p^s(\mathbb{R}^n)$. If $s < 2-d$ and if $\alpha < 1/(1-sp/n)$ then Henry's results [He] give existence and uniqueness of a solution in $C([0,T[\,,H_p^s)$. Theorem 1.3 improves this because one can consider larger values of α (see the example in Section 1.1) and because the condition $s < 2-d$ is not needed.

2) Because of (4) we see that $L^p(\mathbb{R}^n)$ is supercritical for (2) if and only if $p > p_c$ where p_c is defined as

$$p_c = \frac{n\,(\alpha-1)}{2-d}\,. \tag{12}$$

So, for $U_0 \in L^p(\mathbb{R}^n)$ with $p > p_c$ and $p \geq \alpha$ (to make sure that (6) and that (5) are fulfilled with $s = 0$), there exists a unique solution of (2) in $C([0,T[, L^p)$: this had ever been proved in [Gi]. However, when $U_0 \in H_p^s(\mathbb{R}^n)$ with $H_p^s(\mathbb{R}^n) \hookrightarrow L^{\tilde{p}}$ for supercritical $L^{\tilde{p}}(\mathbb{R}^n)$ space, Giga's results give existence and uniqueness only in $C([0,T[\,, L^{\tilde{p}})$ but nothing is said about the $H_p^s(\mathbb{R}^n)$ regularity of the solution. Theorem 1.3 answers precisely to this question.

3) If $U_0 \in L^p(\mathbb{R}^n)$ with $p < p_c$, phenomena of non-existence and non-uniqueness may occur (see [We1] and [HW]). Note also that non uniqueness could also occur in the space $H^s(\mathbb{R}^n)$ for subcritical value of s: see Tayachi [T] for the nonlinear heat equations and Dix [Di] for the nonlinear Burgers equations. Theorem 1.3 shows that this could occur only for subcritical $H_p^s(\mathbb{R}^n)$ spaces since it is sufficient to assume that U_0 belongs to $H_p^s(\mathbb{R}^n)$ with $s > s_c$ to ensure both existence and uniqueness. Thus, with no further assumptions than H2) or H3) on the nonlinear terms, our results are optimal. However, for some more specific nonlinear terms, one can sometimes prove that (2) is well posed in some subcritical spaces: for instance for the nonlinear heat equations with the "good" sign and for the Burgers viscous equation with nonlinear term in divergence form (see [EZ]).

4) We mentioned earlier that the restrictions (5) and (6) are only technical. Indeed, when $U_0(x) \in H_p^s(\mathbb{R}^n)$ with $0 \leq s_c < s \leq n/p - n/\alpha$, using $L^q([0,T[\,, L^z)$ estimates for the heat kernel we can always solve (2). Also, when $U_0(x) \in H_p^s(\mathbb{R}^n)$ with $s_c < s < 0$ we can sometimes solve (2): this allows us to solve (2) with measures or distributions as initial data: see [Ri].

In the critical case, we obtain existence of a solution but uniqueness occurs (*a priori*) only in a subspace of $C([0,T[\,, H_p^{s_c})$. However, in this case, we prove global existence for small initial data. We also prove some time decay estimates for those solutions in various $L^q(\mathbb{R}^n)$ norms.

For the study of the global Cauchy problem, we will assume that

H1') $P(D)$ is a pseudodifferential of order $d < 2$ with homogeneous symbol $P(\xi)$,

and

H5) $F(0) = 0$ and there exists $\alpha > 1$ such that,

$$|F(x) - F(y)| \leq C\,|x-y|\,(|x|^{\alpha-1} + |y|^{\alpha-1})\,.$$

First, let us recall a useful result about the Cauchy problem for small initial data in $L^{p_c}(\mathbb{R}^n)$ which has been proved by F. Weissler [We2] for the nonlinear heat equations, by T. Kato [Ka] for the Navier-Stokes equations and by Y. Giga [Gi] for the general problem (2).

Theorem 1.4. *Assume that* H1') *and that* H5) *are fulfilled. Assume furthermore that $p_c > 1$. Let $\gamma(q)$ defined as*

$$\gamma(q) = \frac{n}{2}\Big(\frac{1}{p_c} - \frac{1}{q}\Big). \tag{13}$$

Then, there exists an absolute constant A such that, for all $U_0 \in L^{p_c}(\mathbb{R}^n)$ with $\|U_0\|_{L^{p_c}} \le A$, there is a unique global solution $U(t,x)$ of (2) *such that*

$$t \longrightarrow t^{\gamma(q)} \|U(t,\cdot)\|_{L^q} \in BC([0,+\infty[)\,, \tag{14}$$

for all q and $\gamma(q)$ such that

$$p_c \le q < +\infty \qquad \text{and} \qquad 0 \le \gamma(q) < \alpha^{-1}\,, \tag{15}$$

and such that

$$\lim_{t\to 0^+} t^{\gamma(q)}\, \|U(t,\cdot)\|_{L^q} = 0\,, \tag{16}$$

for all q and $\gamma(q)$ such that

$$p_c < q < +\infty\,,\ \alpha < q \qquad \text{and} \qquad 0 < \gamma(q) < \alpha^{-1}\,. \tag{17}$$

REMARKS.

1) Generally, the assumption $p_c > 1$ is sharp. For the nonlinear heat equations (3) with $a > 0$ the blow-up for non-negative $C_0^2(\mathbb{R}^n)$ initial data has been proved when $p_c \le 1$ (see [Fu] and [We2]).

2) Note that uniqueness in $BC(\mathbb{R}^+, L^{p_c})$ occurs only on the subspace defined by (14)-(15) and (16)-(17): if $V(t,x)$ is a solution of (2) in $BC(\mathbb{R}^+, L^{p_c})$, we do not know if V satisfies (14)-(15) and (16)-(17) or not.

3) Note that, from Theorem 1.4, the asymptotic decay of $U(t,x)$ in $L^q(\mathbb{R}^n)$ norm is exactly the same as the asymptotic decay of $e^{t\Delta}\,U_0(x)$ as long as the decay rate $\gamma(q)$ satisfies $\gamma(q) < \alpha^{-1}$.

4) Note also that, since $p_c > 1$, there always exists a q_0 such that (17) holds: if $\alpha \le p_c$ this is obvious since $\gamma(p_c) = 0$ and if $p_c < \alpha$ one can check that for $q \in]\alpha, \alpha\, p_c[$ then $0 < \gamma(\alpha) < \gamma(q) < \gamma(\alpha\, p_c) = (2-d)/(2\,\alpha) \le \alpha$.

First, we will prove a slight improvement of the Giga's result,

Lemma 1.2. *Assume that* $\|U_0\|_{L^{p_c}} \le A$ *and let us consider* $U(t,x)$ *the Giga's solution of* (2). *Then,*

$$\|U(t,\cdot)\|_{L^q} \le C\, t^{-\gamma(q)}\, \|U_0\|_{L^{p_c}}\,, \qquad \text{for all } q \in [p_c, +\infty[\,. \tag{18}$$

REMARK. Note that, in the estimate (18), there is no any restrictions on the size of the decay rate $\gamma(q)$.

Then, using the Lemma 1.2, we will consider the case of initial data with arbitrarily high norm in subcritical $L^p(\mathbb{R}^n)$ spaces and small norm in the critical space $L^{p_c}(\mathbb{R}^n)$.

Proposition 1.1. *Let* $U_0 \in L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *with* $p \le p_c$ *and assume that* $\|U_0\|_{L^{p_c}} \le A$. *Let us consider* $U(t,x)$ *the global solution of* (2) *given by Theorem* 1.4. *Then,*

$$U(t,x) \in BC(\mathbb{R}^+, L^p) \cap BC(\mathbb{R}^+, L^{p_c})\,, \tag{19}$$

$$\|U(t,\cdot)\|_{L^r} \le C\, t^{-n/2(1/p-1/r)}\, \|U_0\|_{L^p}\,, \tag{20}$$

for all $r \ge p$ *and* $t > 0$.

REMARK. One more time we see that $U(t,x)$ decay in $L^r(\mathbb{R}^n)$ with the same rate than $e^{t\Delta}\, U_0(x)$ this, without any restriction on the decay rate. For the Navier-Stokes equations, such a result has ever been proved in [Ka] but only when $n\,(1/p - 1/r)/2 < 1/2$ in (20).

Using Proposition 1.1, in Section 3.3 we will prove the following result on the global Cauchy problem for initial data in $H^s_p(\mathbb{R}^n)$ spaces.

Theorem 1.5. *Assume that* H1') *and* H5) *hold. Assume that* $p_c > 1$ *and that* $p \in]p_c\, \alpha^{-1}, p_c]$. *Then,*

a) *There exists an absolute constant* A' *such that, for all* $U_0 \in H_p^{s_c}(\mathbb{R}^n)$ *with* $\|U_0\|_{H_p^{s_c}} \leq A'$, *there is a unique global solution* $U(t,x)$ *of* (2) *in* $C([0,+\infty[, H_p^{s_c})$ *which satisfies* (14)-(15) *and* (16)-(17). *Furthermore* $U(t,x)$ *satisfies the estimates* (19) *and* (20).

b) *Let* $U_0 \in H_p^s(\mathbb{R}^n)$ *with* $s > s_c$. *If* $\|U_0\|_{H_p^{s_c}} \leq A'$, *then the local solution of* (2) *given by Theorem* 1.3 *belongs to* $BC(\mathbb{R}^+, H_p^s)$ *and satisfies the estimates* (19) *and* (20).

REMARKS.

1) For data with an arbitrarily norm in $H_p^{s_c}(\mathbb{R}^n)$ one can also prove local existence and uniqueness in a subspace of $C([0,T[, H_p^{s_c})$ defined by a local version of (14)-(15) and (16)-(17).

2) There is no restriction on the size of $\|U_0\|_{H_p^s}$ in Part b) of Theorem 1.5: we just assume that $\|U_0\|_{H_p^{s_c}(\mathbb{R}^n)}$ is small enough (the only norm invariant by scaling).

3) For the Navier-Stokes equations, using Besov spaces of nonpositive order, one can also prove global existence under a weaker assumption than the natural assumption $\|U_0\|_{H_p^{s_c}} \leq A$ (for instance see [GM], [KM] or [Ca]).

In Section 2 we will study the local Cauchy problem under the assumptions of Theorem 1.3: we prove existence, uniqueness and continuous dependance with respect to the initial data; we also prove smoothing effects for the solution of (2). In Section 3 we study the global Cauchy problem for small initial data in the critical space $L^{p_c}(\mathbb{R}^n)$, for initial data in the space $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ for subcritical $L^p(\mathbb{R}^n)$ spaces and then, for initial data in the Sobolev spaces $H_p^s(\mathbb{R}^n)$: we will prove Lemma 1.2, Proposition 1.1 and Theorem 1.5. Next, in Section 4, we will prove the nonlinear estimate of Theorem 1.2 which is the key estimate to prove the Theorem 1.3.

2. The local Cauchy problem.

We first prove existence of a solution (Section 2.1) and then uniqueness (Section 2.2). In Section 2.3 we study smoothing effects for (2) and, in Section 2.4, we study continuous dependence of the solutions with respect to the initial data.

2.1. Existence.

First we assume that Theorem 1.2 holds and that U_0 belongs to subcritical $H^s_p(\mathbb{R}^n)$ spaces. In the sequel C will denote a non-negative constant which may be changed from one line to another. We also forget the time dependance of C since in this section we are only dealing with a local problem. To simplify the notations we define

$$L(u)(t,x) = \int_0^t e^{(t-\tau)\Delta} P(D) F(u(\tau))\, d\tau\,.$$

We introduce the exponent $\tilde{p}$ given by

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s}{n}\,, \tag{21}$$

and by (5), (6) and since $s > s_c$,

$$\tilde{p} \geq \alpha \qquad \text{and} \qquad \tilde{p} > p_c\,. \tag{22}$$

We define the spaces

$$Y = C([0,T[\,, H^s_p) \tag{23}$$

and

$$X = C([0,T[\,, L^{\tilde{p}})\,. \tag{24}$$

Hence, by the Sobolev embedding Theorem, $Y \hookrightarrow X$. Now, let us consider the sequence of functions

$$u^0 = e^{t\Delta}\, U_0(x)\,, \qquad u^{j+1} = u^0 + L(u^j)\,. \tag{25}$$

First we are going to prove that $\{u^j\}$ converges strongly in X to a limit U which verifies (2) (this proof follows closely Giga's proof but we detail it for the reader's convenience) and second, using the new estimates given by Theorem 1.2, we will show that U belongs also to Y. Let us recall the $(L^p - L^q)$ and $(H^{s+\theta}_p - H^s_p)$ estimates for the semigroup $e^{\tau\Delta}$ (see [Tr]).

Lemma 2.1.

a) *For all* $q \geq p$ *and* $\tau > 0$, *there exists* C *such that*

$$\|e^{\tau\Delta} f\|_{L^q} \leq C\, \tau^{-n/2(1/p-1/q)}\, \|f\|_{L^p}\ .$$

b) *For all* $\theta \geq 0$ *and* $\tau \in\,]0,T]$, *there exists* $C(T)$ *such that*

$$\|e^{\tau\Delta} f\|_{H^{s+\theta}_p} \leq C(T)\, \tau^{-\theta/2}\, \|f\|_{H^s_p}\ .$$

c) *For all* $\theta \geq 0$ *and* $\tau > 0$, *there exists* C *such that*

$$\|e^{\tau\Delta} f\|_{\dot H^{s+\theta}_p} \leq C\, \tau^{-\theta/2}\, \|f\|_{\dot H^s_p}\ .$$

By Part a) of Lemma 2.1

$$\|u^0\|_X \leq \|U_0\|_{L^{\tilde p}} \leq C\, \|U_0\|_{H^s_p}\ . \tag{26}$$

Let u and v in X then,

$$\|L(u)(t) - L(v)(t)\|_{L^{\tilde p}} \leq \int_0^t \left\|e^{(t-\tau)\Delta} P(D)\, (F(u)(\tau) - F(v)(\tau))\right\|_{L^{\tilde p}} d\tau\,.$$

Since we are working in the whole Euclidian space $\mathbb{R}^n$, the operators $e^{\tau\Delta}$ and $P(D)$ are some Fourier multipliers and so,

$$e^{(t-\tau)\Delta} P(D) = P(D)\, e^{(t-\tau)\Delta} = e^{\Delta(t-\tau)/2}\, P(D)\, e^{\Delta(t-\tau)/2}\,.$$

Furthermore by H1), $P(D) : H^d_p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is bounded and so, using Lemma 2.1,

$$\begin{aligned}
&\|e^{(t-\tau)\Delta}\, P(D)\, (F(u)(\tau) - F(v)(\tau))\|_{L^{\tilde p}} \\
&\qquad\qquad \leq C\, (t-\tau)^{-d/2}\, \|e^{\Delta(t-\tau)/2}\, (F(u)(\tau) - F(v)(\tau))\|_{L^{\tilde p}} \\
&\qquad\qquad \leq C\, (t-\tau)^{-\beta}\, \|F(u)(\tau) - F(v)(\tau)\|_{L^{\tilde p/\alpha}}\ ,
\end{aligned}$$

(note that the first part of (22) is needed) where, by (4),

$$\beta = \frac{d}{2} + \frac{n}{2}\,\frac{\alpha-1}{\tilde p} = 1 - \frac{(\alpha-1)\,(s-s_c)}{2} < 1\,. \tag{27}$$

Using this last estimate and Hölder's inequality, we obtain

$$\begin{aligned}&\|L(u)(t)-L(v)(t)\|_{L^{\tilde p}}\\&\qquad\le C\int_0^t (t-\tau)^{-\beta}\,\|u(\tau)-v(\tau)\|_{L^{\tilde p}}\big(\|u(\tau)\|_{L^{\tilde p}}^{\alpha-1}+\|v(\tau)\|_{L^{\tilde p}}^{\alpha-1}\big)\,d\tau\end{aligned}$$

and, since $\beta<1$,

$$\|L(u)-L(v)\|_X\le C\,T^{1-\beta}\,\|u-v\|_X\big(\|u\|_X^{\alpha-1}+\|v\|_X^{\alpha-1}\big)\,. \tag{28}$$

Furthermore $L(0)=0$ and from (26) and (28) we deduce that

$$\begin{cases}\|u^{j+1}\|_X\le\|U_0\|_{H_p^s}+C\,T^{1-\beta}\|u^j\|_X^{\alpha}\,,\\ \|u^{j+1}-u^j\|_X\le C\,T^{1-\beta}\|u^j-u^{j-1}\|_X\\ \qquad\qquad\qquad\cdot\big(\|u^j\|_X^{\alpha-1}+\|u^{j-1}\|_X^{\alpha-1}\big)\,.\end{cases} \tag{29}$$

Then, a standard fixed point argument shows that, for

$$T<\frac{C}{4}\,\|U_0\|_{H_p^s}^{-(\alpha-1)/(1-\beta)}\,, \tag{30}$$

the sequence $\{u^j\}$ converges strongly in X to a limit U which obviously solves (2) since $\tilde p\ge\alpha$ by (22).

Now, we must prove that this solution belongs also to Y. Let $u\in Y$, then,

$$\|L(u)(t)\|_{H_p^s}\le\int_0^t\|e^{(t-\tau)\Delta}P(D)F(u)(\tau)\|_{H_p^s}\,d\tau\,.$$

As previously

$$\begin{aligned}\|e^{(t-\tau)\Delta}P(D)F(u)(\tau)\|_{H_p^s}&\le C\,(t-\tau)^{-d/2}\,\|e^{\Delta(t-\tau)/2}F(u)(\tau)\|_{H_p^s}\\&\le C\,(t-\tau)^{-(d+s-s_\alpha)/2}\,\|F(u)(\tau)\|_{H_p^{s_\alpha}}\,.\end{aligned}$$

But now, using Theorem 1.2, we can bound the term $\|F(u)(\tau)\|_{H_p^{s_\alpha}}$ by $C\,\|u(\tau)\|_{H_p^s}^{\alpha}$ and furthermore, thanks to (27) and to (4), we obtain

$$\|e^{(t-\tau)\Delta}P(D)F(u)(\tau)\|_{H_p^s}\le C\,(t-\tau)^{-\beta}\,\|u(\tau)\|_{H_p^s}^{\alpha}\,.$$

This last inequality leads then to

$$\|L(u)(t)\|_{H^s_p} \le C \int_0^t (t-\tau)^{-\beta} \, \|u(\tau)\|^{\alpha}_{H^s_p} \, d\tau \le C\, T^{1-\beta} \, \|u\|^{\alpha}_Y$$

and so by (26),

$$\|u^{j+1}\|_Y \le \|U_0\|_{H^s_p} + C\, T^{1-\beta} \, \|u^j\|^{\alpha}_Y \,. \tag{31}$$

As previously, if T satisfies (30), thanks to (31) we see that the $\|u^j\|_Y$ remain bounded and so, we can always extract a subsequence $\{u^{j_k}\}$ which converges weakly-$\star$ to a limit $\tilde{U} \in Y$. Now the u^{j_k} converge to U and converge to $\tilde{U}$ in $\mathcal{D}'(]0,T[\times\mathbb{R}^n)$ and so U agrees with $\tilde{U}$. Thus we have proved the existence of a solution in $C([0,T[, H^s_p)$.

The estimate for T_m comes from (30) which gives

$$T_m \ge \frac{C}{8} \, \|U_0\|^{-2/(s-s_c)}_{H^s_p} \,.$$

If $T_m < +\infty$, this explicit lower bound obviously allows us to show the blow-up in $H^s_p(\mathbb{R}^n)$ norm (one can also prove the blow up in $L^{\tilde{p}}(\mathbb{R}^n)$ when it holds in $H^s_p(\mathbb{R}^n)$).

If $s > n/p$, using Theorem 1.1 instead of Theorem 1.2, the same proof gives existence under the hypothesis H4).

2.2 Uniqueness.

Let $U(t,x) \in Y$ and $V(t,x) \in Y$ be two solutions for the same initial data U_0 and let $T < \max\{T_m(V), T_m(U)\}$. Then, since U and V solve (2),

$$\|U - V\|_X = \|L(U) - L(V)\|_X$$

and so, by (28),

$$\|U - V\|_X \le 2\, T^{1-\beta} \, C \, M^{\alpha-1} \, \|U - V\|_X \,,$$

where

$$M = \sup_{[0,T]} \{\|U(t)\|_{L^{\tilde{p}}}, \|V(t)\|_{L^{\tilde{p}}}\} \,.$$

So, for T small enough,

$$\|U - V\|_X \le \frac{1}{2} \, \|U - V\|_X$$

and so $U = V$ on $[0,T]$. To conclude we just have to iterate this in order to prove that $T_m(U) = T_m(V)$ and that $U = V$ on $[0, T_m(U)[$.

2.3. Smoothing effects.

Let U be a solution of (2). Using Lemma 2.1 we easily see that

$$\|U(t,x) - e^{t\Delta}U_0\|_{H_p^{s+\theta}} \leq C \int_0^t (t-\tau)^{-\theta-\beta} \, \|U(\tau,\cdot)\|_{H_p^s}^\alpha \, d\tau$$

and so, for all $\theta < 1-\beta = (\alpha-1)\nu$,

$$\|U(t,x) - e^{t\Delta}U_0\|_{H_p^{s+\theta}} \leq C\, T^{1-\beta-\theta} \, \|U\|_Y^\alpha \,,$$

which gives the first part of Theorem 1.3.b). If $s > n/p$, the proof is the same using Theorem 1.1 instead of Theorem 1.2.

Now let us assume that $F \in C^\infty(\mathbb{R})$. For all $t > 0$, $e^{t\Delta}U_0$ is $C^\infty(\mathbb{R}^n)$ and so $U(\delta/2,\cdot) \in H_p^{s+\theta}(\mathbb{R}^n)$. Taking $\delta/2$ as initial time, we just have to repeat this argument to prove that $U(\delta/2+\delta/4,\cdot) \in H_p^{s+2\theta}(\mathbb{R}^n) \cdots$ finally, $U \in C([\delta, T[\,, C^\infty)$ for each $\delta > 0$. Thus we have proved the second part of Theorem 1.3.b).

2.4. Continuous dependence with respect to the data.

First we deal with continuity in X norm. Let U and V be two solutions of (2) for the respective initial data U_0 and V_0. Let

$$T < \min\{T_m(U_0), T_m(V_0)\}$$

and

$$M = \sup_{t\in[0,T]} \{\|U(t)\|_{H_p^s}, \|V(t)\|_{H_p^s}\}\,.$$

By (28),

$$\|U - V\|_X \leq \|U_0 - V_0\|_{L^{\tilde p}} + C\, T^{1-\beta} \, \|U-V\|_X \, 2\, M^{\alpha-1}\,.$$

Taking $T' \leq T$ such that $4\,C\,T'^{1-\beta}M^{\alpha-1} \leq 1$ then,

$$\|U - V\|_{C([0,T'[\,,L^{\tilde p})} \leq 2\, \|U_0 - V_0\|_{L^{\tilde p}}$$

and, if one can take $T' = T$, this ends the proof. On the contrary, solving (2) for the initial data $U(T')$ and $V(T')$, the uniqueness and the uniform bound for U and V in X norm allow us to iterate this last argument N times until $N\,T' \geq T$ and thus

$$\|U - V\|_X \leq C(T)\,\|U_0 - V_0\|_{L^{\tilde p}} \leq C(T)\,\|U_0 - V_0\|_{H_p^s}\ . \tag{32}$$

Now let us assume that $s_\alpha \leq 0$, *i.e.* that $s \leq n\,(\alpha - 1)/(p\,\alpha)$. Then,

$$\begin{aligned}
&\|U(t) - V(t)\|_{H_p^s} \\
&\qquad \leq \|U_0 - V_0\|_{H_p^s} + C \int_0^t (t-\tau)^{-\beta}\,\|F(U(\tau)) - F(V(\tau))\|_{H_p^{s_\alpha}}\,d\tau\ .
\end{aligned}$$

Since $s_\alpha \leq 0$ and $\alpha/\tilde p = 1/p - s_\alpha/n$, we can use the Sobolev embedding

$$L^{\tilde p/\alpha} \hookrightarrow H_p^{s_\alpha}\ ,$$

which leads to

$$\begin{aligned}
&\|U(t) - V(t)\|_{H_p^s} \\
&\qquad \leq \|U_0 - V_0\|_{H_p^s} + C \int_0^t (t-\tau)^{-\beta}\,\|F(U(\tau)) - F(V(\tau))\|_{L^{\tilde p/\alpha}}\,d\tau \\
&\qquad \leq \|U_0 - V_0\|_{H_p^s} + C\,T^{1-\beta}\,\|U - V\|_X \big(\|U\|_X^{\alpha-1} + \|V\|_X^{\alpha-1}\big)
\end{aligned}$$

and, according to (32) and to this last inequality, we obtain that

$$\|U - V\|_Y \leq C(T)\,\|U_0 - V_0\|_{H_p^s}\ . \tag{33}$$

To conclude we have to relax our assumption on s. Since U and V are solutions of (2),

$$\|U - V\|_Y \leq \|U_0 - V_0\|_{H_p^s} + \|L(U) - L(V)\|_Y\ .$$

First, let us recall the following interpolation inequality.

Lemma 2.2. *Let $p \in\,]1, +\infty[$, $\theta \in \mathbb{R}$ and $s \in \mathbb{R}$. Then, for all $f \in H_p^{s+\theta}(\mathbb{R}^n)$,*

$$\|f\|_{H_p^s}^2 \leq C\,\|f\|_{H_p^{s+\theta}}\,\|f\|_{H_p^{s-\theta}}\ .$$

For a proof see [Tr].

By Lemma 2.2 we see that

$$\begin{aligned}&\|L(U)(t) - L(V)(t)\|_{H_p^s}\\ &\le \big(\|L(U)(t))\|_{H_p^{s+\theta}} + \|L(V)(t)\|_{H_p^{s+\theta}}\big)^{1/2}\, \|L(U)(t) - L(V)(t)\|_{H_p^{s-\theta}}^{1/2}\end{aligned}$$

Now since $s < 2 - d$, one can choose $\theta < (\alpha - 1)\,(s - s_c)/2$ such that

$$s_c < s - \theta < \frac{\alpha - 1}{\alpha}\,\frac{n}{p}\,. \tag{34}$$

Using the smoothing effects, the first term of the left-hand side of the last inequality can be bounded by

$$C(T)\,(\|U\|_Y + \|V\|_Y)^{1/2} \le C'(T)\,M\,,$$

and, using (33), since $(s - \theta)$ satisfies (34), we bound the second term by

$$\begin{aligned}\|U_0 - V_0\|_{H_p^{s-\theta}}^{1/2} + \|U(t) - V(t)\|_{H_p^{s-\theta}}^{1/2} &\le C(T)\,\|U_0 - V_0\|_{H_p^{s-\theta}}^{1/2}\\ &\le C(T)\|U_0 - V_0\|_Y^{1/2}\,.\end{aligned}$$

Combining this two inequalities we obtain that

$$\|U - V\|_Y \le C(T)\,\|U_0 - V_0\|_Y^{1/2}$$

and the proof of Part c) is completed.

3. The global Cauchy problem.

In this section we study the global Cauchy problem for small initial data in $L^{p_c}(\mathbb{R}^n)$. First in Section 3.1 we study the case of initial data which belongs only to $L^{p_c}(\mathbb{R}^n)$ and we prove Lemma 1.2. In Section 3.2 we study the global Cauchy problem for initial data in $L^p(\mathbb{R}^n)\cap L^{p_c}(\mathbb{R}^n)$ when $L^p(\mathbb{R}^n)$ is subcritical for (2) and we prove the Proposition 1.1. Then, in Section 3.3, we consider initial data in $H_p^s(\mathbb{R}^n)$ space and we prove the Theorem 1.5.

3.1. Initial data in $L^{p_c}(\mathbb{R}^n)$.

Let us consider $U_0 \in L^{p_c}(\mathbb{R}^n)$. In [Gi] Giga proved that there exists a non-negative absolute constant A such that, if $\|U_0\|_{L^{p_c}} \leq A$, then there exists a unique global solution of (2) in $BC(\mathbb{R}^+, L^{p_c})$ which satisfies

$$t \longmapsto t^{\gamma(q)} \|U(t,\cdot)\|_{L^q} \in BC(\mathbb{R}^+),$$

for all q and $\gamma(q)$ such that

$$p_c \leq q < +\infty \qquad \text{and} \qquad 0 \leq \gamma(q) < \alpha^{-1},$$

and which satisfies

$$\lim_{t \to 0^+} t^{\gamma(q)} \|U(t,\cdot)\|_{L^q} = 0,$$

for all q and $\gamma(q)$ such that

$$p_c < q < +\infty, \ \alpha < q \qquad \text{and} \qquad 0 < \gamma(q) < \alpha^{-1}.$$

First we are going to prove that, for $p_c \leq q < +\infty$ and $0 \leq \gamma(q) < \alpha^{-1}$,

$$\|U(t,\cdot)\|_{L^q} \leq C\, t^{-\gamma(q)} \|U_0\|_{L^{p_c}}, \tag{35}$$

which is a little more precise than the estimate

$$\|U(t,\cdot)\|_{L^q} \leq C\, t^{-\gamma(q)}.$$

Second, we are going to relax the restriction $\gamma(q) < \alpha^{-1}$ in this estimate. Indeed, when $p_c \geq n\alpha/2$, the reader will check that the assumption $\gamma(q) < \alpha^{-1}$ is fulfilled for all $q \in [p_c, +\infty[$ and so, the asymptotic estimates (35) to. On the contrary, when $p_c < n\,\alpha/2$, on must assume that $q \in [p_c, (1/p_c - 2/(n\,\alpha))^{-1}[$ to be sure that $\gamma(q) < \alpha^{-1}$ holds. So, when $p_c < n\,\alpha/2$, the asymptotic estimates are proved only for q in the range $[p_c, (1/p_c - 2/(n\,\alpha))^{-1}[$ and we want to show that they holds for all exponent q in $[p_c, +\infty[$.

To prove Lemma 1.2 let us come back to the proof of Theorem 1.4 given in [Gi]. In the critical case (when $U_0 \in L^{p_c}(\mathbb{R}^n)$), to prove the existence of a solution for (2), one introduce, for $p_c < q < +\infty$, $\alpha < q$ and $0 < \gamma(q) < \alpha^{-1}$, the Banach spaces

$$X_q = \{f(t,x) : \ t \longmapsto t^{\gamma(q)} \|f(t,x)\|_{L^q} \in BC(\mathbb{R}^+)\}$$

and the space

$$Y = \{f(t,x):\ t \longmapsto \|f(t,x)\|_{L^{p_c}} \in BC(\mathbb{R}^+)\}\,.$$

Then, if we consider $\{u^j\}$ the sequence of functions defined by (25), we have the estimate (see [Gi])

$$\|u^{j+1}\|_{X_q} \le C_1\,\|u^0\|_{X_q} + C_2\,\|u^j\|^{\alpha}_{X_q}\,, \tag{36}$$

where,

$$\|f(t,x)\|_{X_q} = \sup_{t>0} t^{\gamma(q)}\,\|f(t,x)\|_{L^q}\,.$$

Then, when $\|U_0\|_{L^{p_c}} \le A$, using (36) and (16)-(17), one can prove that the $\{u^j\}$ converge in X_q to $U(t,x)$ the unique solution of (2) such that (16)-(17) is fulfilled (see [Gi] for a proof). Furthermore, to prove that $U(t,x)$ belongs also to $BC([0,+\infty[\,, L^{p_c}(\mathbb{R}^n))$, one can easily check that the nonlinear map $L : X_q \longrightarrow Y$ defined at the beginning of the Section 2.1 satisfies

$$\|L(U)\|_Y \le C\|U\|^{\alpha}_{X_q} \tag{37}$$

as soon as $p_c < q < +\infty$, $\alpha < q$ and as soon as $0 < \gamma(q) < \alpha^{-1}$.

Now let us come back to the proof of Lemma 1.2. By (36), it is obvious that the sequence $\{u^j\}$ stay in the ball $B(0, 2C_1\|u^0\|_{X_q})$ for the X_q topology as soon as

$$C_2\,(2\,C_1\,\|u^0\|_{X_q})^{\alpha} \le C_1\,\|u^0\|_{X_q}\,,$$

which holds for

$$\|u_0\|_{X_q} \le \Big(\frac{1}{2^{\alpha}\,C_1^{\alpha-1}\,C_2}\Big)^{1/(\alpha-1)}\,.$$

Now, by Lemma 2.1,

$$\|u^0\|_{X_q} \le C\,\|U_0\|_{L^{p_c}} \tag{38}$$

and so, for

$$\|U_0\|_{L^{p_c}} \le A = \frac{1}{C}\,\Big(\frac{1}{2^{\alpha}C_1^{\alpha-1}C_2}\Big)^{1/(\alpha-1)}\,,$$

there exists a global solution $U(t,x)$ of (2) which belongs to the ball

$$B(0, 2\,C_1\,\|u^0\|_{X_q}) \subset B(0, 2\,C_1\,C\,\|U_0\|_{L^{p_c}})\,,$$

for the X_q topology. Thus the proof of Lemma 1.2 is completed for the exponents q such that $p_c < q < +\infty$, $\alpha < q$ and $0 < \gamma(q) < \alpha^{-1}$. To conclude in the special case of $L^{p_c}(\mathbb{R}^n)$, we have just to use this last result and the estimate (37). Thus, if $p_c \geq \alpha$ the proof is over. On the contrary, if $p_c < \alpha$, we have just to interpolate the estimates in $L^q(\mathbb{R}^n)$ norm and in $L^{p_c}(\mathbb{R}^n)$ norm to end the proof.

Now we are going to prove that the asymptotic estimates

$$\|U(t,\cdot)\|_{L^q} \leq C\,t^{-\gamma(q)}\,\|U_0\|_{L^{p_c}}$$

holds also when $\gamma(q) \geq \alpha^{-1}$. First, for U_0 such that $\|U_0\|_{L^{p_c}} \leq A$, let us consider $U(t,x)$ the Giga's solution of (2) and let us consider q_0 an exponent such that $q_0 > p_c$ and such that $\gamma(q_0) < \alpha^{-1}$ (such a q_0 always exists since $p_c > 1$: see the Remark 4 after Theorem 1.4). Next let us consider the sequence $\{q_i\}$ defined by

$$\frac{n}{2}\Big(\frac{1}{q_i} - \frac{1}{q_{i+1}}\Big) = \delta < \alpha^{-1} \tag{39}$$

and note that $\{q_i\}$ is increasing and that there exists q_l such that $n/(2\,q_l) < \alpha^{-1}$.

Let us define

$$I(q_i, q_{i+1}) = \int_0^1 (1-s)^{-d/2 - n(\alpha-1)/(2q_{i+1})}\, s^{-\delta\alpha}\, ds\,.$$

Then, by (39), for all $i \geq 0$,

$$I(q_i, q_{i+1}) < +\infty\,.$$

Now we pick $t_0 > 0$ and we consider V the solution of

$$\begin{cases} V(t,x) = e^{t\Delta} V_0 + L(V)(t,x)\,, \\ V(0,x) = V_0(x) = U(t_0,x)\,. \end{cases} \tag{40}$$

First, by the previous result and since $0 < \gamma(q_0) < \alpha^{-1}$, it follows that

$$V_0 \in L^{p_c}(\mathbb{R}^n) \cap L^{q_0}(\mathbb{R}^n) \qquad \text{with} \qquad \|V_0\|_{L^{q_0}} \leq C\,t_0^{-\gamma_0}\,\|U_0\|_{L^{p_c}}\,. \tag{41}$$

Lemma 3.1. *Let $T = T(t_0)$ such that*

$$2^\alpha C^\alpha T^{(2-d)/2(1-p_c/q_o)} I(q_0, q_1) \|V_0\|_{L^{q_0}}^{\alpha-1} < 1 \tag{42}$$

then,

$$\|V(t)\|_{L^{q_1}} \leq 2\, C\, t^{-\delta} \|V_0\|_{L^{q_0}} \ , \tag{43}$$

for all $t \in]0, T[$.

Indeed, since $V_0(x) \in L^{q_0}(\mathbb{R}^n)$ with $q_0 > p_c$, using the proof of Theorem 1.3 we see that the sequence

$$v^0 = e^{t\Delta} V_0 \ , \qquad v^{j+1}(t,x) = v^0 + L(v^j)(t,x) \, ,$$

converges strongly to $V(t,x)$ in $C([0,T], L^{q_0})$. By Lemma 2.1, v^0 obviously satisfies (43) for all $t > 0$. Now, if v^j satisfies (43), then

$$\begin{aligned}
\|v^{j+1}(t)\|_{L^{q_1}} &\leq C\, \|V_0\|_{L^{q_0}}\, t^{-\delta} \\
&\qquad + C \int_0^t (t-\tau)^{-d/2 - n(\alpha-1)/(2q_1)} \|v^j(\tau)\|_{L^{q_1}}^\alpha \, d\tau \\
&\leq C\, \|V_0\|_{L^{q_0}}\, t^{-\delta} \\
&\qquad + C\, 2^\alpha C^\alpha \|V_0\|_{L^{q_0}}^\alpha \int_0^t (t-\tau)^{-d/2 - n(\alpha-1)/(2q_1)}\, \tau^{-\delta\alpha}\, d\tau
\end{aligned}$$

for all $t \in]0, T[$, and so,

$$\begin{aligned}
&\|v^{j+1}(t)\|_{L^{q_1}} \\
&\leq \frac{2\, C\, \|V_0\|_{L^{q_0}}}{t^{n/2(1/q_0 - 1/q_1)}} \left(\frac{1}{2} + 2^{\alpha-1} C^\alpha \|V_0\|_{L^{q_0}}^{\alpha-1} I_{q_0,q_1} T^{(2-d)/2(1-p_c/q_0)} \right),
\end{aligned}$$

for all $t \in]0, T[$. Hence, if T satisfies (42),

$$\|v^{j+1}(t,\cdot)\|_{L^{q_1}} \leq 2\, C\, t^{-\delta} \|V_0\|_{L^{q_0}} \ .$$

So, by induction, (43) holds for all j and thus the Lemma is proved.

Using the uniqueness result in the supercritical case and (40) we see that

$$V(t,x) = U(t + t_0, x)$$

and so, by Lemma 3.1,

$$\|U(t+t_0)\|_{L^{q_1}} \leq 2\,C\,t^{-\delta}\,\|U(t_0)\|_{L^{q_0}}\ ,$$

for each $t \in [0, T(t_0)[$ where $T(t_0)$ satisfies (42). Now, we claim that there exists an absolute constant A' such that, when $\|U_0\|_{L^{p_c}} \leq A'$, one can always take $T(t_0) = t_0/2$ in the previous inequality. Indeed by Lemma 3.1 we have only to make sure that

$$2^\alpha\, C^\alpha \Big(\frac{t_0}{2}\Big)^{(2-d)/2(1-p_c/q_0)}\,\|V_0\|^{\alpha-1}_{L^{q_0}}\, I(q_0,q_1) < 1\,,$$

which, combined with (41), leads to

$$2^\alpha\, C^\alpha \Big(\frac{t_0}{2}\Big)^{(2-d)/2(1-p_c/q_0)-n(\alpha-1)/2(1/p_c-1/q_0)}\,\|U_0\|^{\alpha-1}_{L^{p_c}}\, I(q_1,q_0) < 1$$

and, since

$$\frac{2-d}{2}\Big(1-\frac{p_c}{q_0}\Big) - \frac{n(\alpha-1)}{2}\Big(\frac{1}{p_c}-\frac{1}{q_0}\Big) = 0$$

it is sufficient to make sure that

$$2^\alpha\, C^\alpha\, \|U_0\|^{\alpha-1}_{L^{p_c}}\, I(q_1,q_0) < 1\,.$$

Thus, when U_0 is small enough in $L^{p_c}(\mathbb{R}^n)$, (42) holds for each $t_0 > 0$ and so

$$\Big\|U\Big(\frac{3\,t_0}{2}\Big)\Big\|_{L^{q_1}} \leq 2\,C\,t_0^{-\delta}\,t_0^{-n/2(1/p_c-1/q_0)}\,\|U_0\|_{L^{p_c}}$$

and, since t_0 is arbitrary,

$$\|U(t)\|_{L^{q_1}} \leq 2\,C\,t^{-n/2(1/p_c-1/q_1)}\,\|U_0\|_{L^{p_c}}\ ,$$

for all $t > 0$. Now, since $I(q_i, q_{i+1}) < +\infty$ and since we have prove the required estimate for q_1 defined by (39), we have just to iterate this proof to get the required estimate in L^{q_2} norm... Thus, for each q_i, the proof follows by induction. Now, if $q \in]q_i, q_{i+1}[$, we get the result by interpolation. Thus we have proved that $U(t,x)$, the global solution of (2), satisfies

$$\|U(t,x)\|_{L^q} \leq C\,t^{-\gamma(q)}\,\|U_0\|_{L^{p_c}}\ ,$$

for all $q \in [p_c, +\infty[$.

3.2. Initial data in $L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

Let $p < p_c$. We consider now an initial data U_0 which belongs to $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$, we assume that $\|U_0\|_{L^{p_c}} \leq A$ and we denote by $U(t,x)$ the Giga's solution of (2) which belongs to $BC(\mathbb{R}^+, L^{p_c})$ and satisfies the estimates (14)-(15) and (16)-(17). Using the slight improvement about the decay of the $L^q(\mathbb{R}^n)$ norms (estimates (18) of Lemma 1.2) that we previously proved, we are first going to show that the Giga's solution belongs to $L^p(\mathbb{R}^n)$ for all t (step one), then we will prove that $U(t,x)$ belongs to $BC(\mathbb{R}^+,\ L^p(\mathbb{R}^n))$ (step two) and next, that $U(t,x)$ satisfies the asymptotic estimates (20) (step three).

Step one. Here we consider $U_0(x) \in L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ and we want to prove that,

$$\|U(t)\|_{L^p} \leq C(T)\,, \qquad \text{for all } T > 0 \text{ and } t \in [0,T]\,. \tag{44}$$

First let us assume that

$$\max\left\{1, \frac{p_c}{\alpha}\right\} \leq p < p_c\,. \tag{45}$$

Then, since $U(t,x)$ is a solution for (2), for all $T > 0$ and $t \in [0,T]$

$$\begin{aligned}
\|U(t)\|_{L^p} &\leq \|U_0\|_{L^p} + \|L(U)(t)\|_{L^p} \\
&\leq \|U_0\|_{L^p} + \int_0^t \|e^{(t-\tau)\Delta} P(D) F(U)(\tau)\|_{L^p}\, d\tau \\
&\leq \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2}\, \|F(U)(\tau)\|_{L^p}\, d\tau \\
&\leq \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2}\, \|U(\tau)\|_{L^{p\alpha}}^{\alpha}\, d\tau
\end{aligned}$$

Now, by (45), $p\,\alpha \geq p_c$ and so, using the estimates (18) of the Lemma 1.2, we obtain that

$$\begin{aligned}
\|U(t)\|_{L^p} &\leq \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2}\, \tau^{-\alpha\gamma(p\alpha)}\, \|U_0\|_{L^{p_c}}^{\alpha}\, d\tau \\
&\leq \|U_0\|_{L^p} + C(T)\, \|U_0\|_{L^{p_c}}^{\alpha}
\end{aligned}$$

since $d < 2$ and

$$0 < \gamma(p\,\alpha) < \gamma(p_c\,\alpha) = \frac{2-d}{2\,\alpha} \le \frac{1}{\alpha}\,,$$

for all $p_c/\alpha < p < p_c$.

Thus, the estimate (44) is proved for all p which verify (45) and, if $p_c < \alpha$, the proof is over.

Assume now that

$$\max\Big\{1, \frac{p_c}{\alpha^2}\Big\} \le p < \frac{p_c}{\alpha}\,. \tag{46}$$

First, if $U_0(x) \in L^p \cap L^{p_c}$, then $U_0(x)$ belongs to $L^q(\mathbb{R}^n)$ for all $q \in [p_c\,\alpha^{-1}, p_c]$ and then, by the previous result, $\|U(t)\|_{L^q} \le C(T, U_0)$ for all q in the range $[p_c\,\alpha^{-1}, p_c]$. Second, since $U(t,x)$ is a solution of (2)

$$\begin{aligned}
\|U(t)\|_{L^p} &\le \|U_0\|_{L^p} + \|L(U)(t)\|_{L^p}\\
&\le \|U_0\|_{L^p} + \int_0^t \|e^{(t-\tau)\Delta} P(D) F(U)(\tau)\|_{L^p}\, d\tau\\
&\le \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2} \|F(U)(\tau)\|_{L^p}\, d\tau\\
&\le \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2}\, \|U(\tau)\|^{\alpha}_{L^{p\alpha}}\, d\tau\,.
\end{aligned}$$

Next, we remark that $d < 2$ and that, by (46), $p\,\alpha$ belongs to the range $[p_c\,\alpha^{-1}, p_c[$. Hence, by the previous result, we can use the bound

$$\|U(t)\|_{L^{p\alpha}} \le C(T, U_0)\,,$$

which leads to

$$\|U(t)\|_{L^p} \le \|U_0\|_{L^p} + C(T, U_0)$$

and so, the estimate (44) holds for all p in the range $[\max\{1, p_c/\alpha^2\}, p_c]$. Next, for $p \in [p_c\,\alpha^{-n-1}, p_c\,\alpha^{-n}[$, the proof of (44) follows easily by induction.

Step two. In step one, we have proved that $U(t,x)$ the Giga's solution of (2) belongs to $L^p(\mathbb{R}^n)$ for all $t \ge 0$ when U_0 belongs to $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ and when U_0 is small enough in $L^{p_c}(\mathbb{R}^n)$. Now, we are going to prove that $U(t,x)$ belongs to $BC(\mathbb{R}^+, L^p)$. Let us consider $T > 0$ and t in $[0,T]$. First, since $U(t,x)$ is a mild solution of (1),

$$U(t,x) = e^{t\Delta}U_0(x) + L(U)(t,x)\,.$$

So, by Lemma 2.1,

$$\begin{aligned}\|U(t)\|_{L^p} &\le \|U_0\|_{L^p} + \|L(U)(t)\|_{L^p}\\ &\le \|U_0\|_{L^p} + \int_0^t \|e^{(t-\tau)\Delta}P(D)F(U)(\tau)\|_{L^p}\,d\tau\\ &\le \|U_0\|_{L^p} + C\int_0^t (t-\tau)^{-\xi(q)}\,\|F(U(\tau))\|_{L^q}\,d\tau\,,\end{aligned}$$

where q is any exponent in $[1,p[$ which will be fixed latter and where $\xi(q)$ is defined by

$$\xi(q) = \frac{d}{2} + \frac{n}{2}\Big(\frac{1}{q} - \frac{1}{p}\Big)\,, \tag{47}$$

Using Hölder's inequality we get

$$\|U(t)\|_{L^p} \le \|U_0\|_{L^p} + C\int_0^t (t-\tau)^{-\xi(q)}\,\|U(\tau)\|_{L^{qq_1}}\,\|U(\tau)\|^{\alpha-1}_{L^{qq_2(\alpha-1)}}\,d\tau\,,$$

where $1/q_1 + 1/q_2 = 1$ and furthermore, we choose q_1 such that $q\,q_1 = p$ to obtain

$$\|U(t)\|_{L^p} \le \|U_0\|_{L^p} + C\,\|U\|_{L^\infty([0,T],L^p)}\int_0^t (t-\tau)^{-\xi(q)}\|U(\tau)\|^{\alpha-1}_{L^{qq_2(\alpha-1)}}\,d\tau\,.$$

Now if we choose q such that $q \approx p$ with $q < p$ then, since $q\,q_1 = p$, $q_1 \approx 1$. Hence it follows that $z = q\,q_2\,(\alpha-1) \ge p_c$. Next, for $z = q\,q_2\,(\alpha-1) \ge p_c$, by Lemma 1.2, we can bound $U(t,x)$ in $L^{qq_2(\alpha-1)}(\mathbb{R}^n)$ norm by

$$\|U(t,x)\|_{L^{qq_2(\alpha-1)}} \le C\,t^{-\gamma(qq_2(\alpha-1))}\,\|U_0\|_{L^{p_c}}$$

and so,

$$\|U(t)\|_{L^p} \le C\,\|U_0\|_{L^p} + \|U\|_{L^\infty([0,T],L^p)}\,\|U_0\|^{\alpha-1}_{L^{p_c}}\int_0^t (t-\tau)^{-\xi(q)}\tau^{-\theta(q)}\,d\tau\,,$$

where

$$\theta(q) = (\alpha - 1)\,\gamma(q\,q_2\,(\alpha - 1)) = \frac{1}{2}\Big(2 - d - \frac{n}{q\,q_2}\Big)\,. \tag{48}$$

One can easily check that choosing $q \approx p$ then q_2 is large enough to makes sure that $0 < \xi(q) < 1$ and that $0 < \theta(q) < 1$ (since $d < 2$). Furthermore $\xi(q) + \theta(q) = 1$ and so,

$$\|U\|_{L^\infty([0,T],L^p)} \leq \|U_0\|_{L^p} + C\,\|U\|_{L^\infty([0,T],L^p)}\,\|U_0\|_{L^{p_c}}^{\alpha-1}\,. \tag{49}$$

Now, if $\|U_0\|_{L^{p_c}}$ is small enough then

$$1 - C\,\|U_0\|_{L^{p_c}}^{\alpha-1} \geq \frac{1}{2}$$

and then, by (49), and since $\|U\|_{L^\infty([0,T[,L^p)} < +\infty$ for all $T > 0$,

$$\|U\|_{L^\infty([0,T],L^p)} \leq \frac{\|U_0\|_{L^p}}{1 - C\,\|U_0\|_{L^{p_c}}^{\alpha-1}} \leq 2\,\|U_0\|_{L^p}\,.$$

To conclude, we have just to remark that the right side of this estimate do not depend of T. Thus, we have proved that $U(t,x)$ the mild solution of (1) belongs to $BC(\mathbb{R}^+, L^p(\mathbb{R}^n))$.

Step three. Now we have to prove the $L^r(\mathbb{R}^n)$ estimates (20) of Theorem 1.4. They hold obviously for the term $e^{t\Delta}U_0$ by Lemma 2.1, hence, we just deal with the nonlinear term $L(U)$. First let us suppose that

$$\delta(r) = \frac{n}{2}\Big(\frac{1}{p} - \frac{1}{r}\Big) < \frac{2-d}{2}\,. \tag{50}$$

Then,

$$\begin{aligned}
\|L(U)(t)\|_{L^r} &\leq \int_0^t \|e^{(t-\tau)\Delta}P(D)F(U(\tau))\|_{L^r}\,d\tau \\
&\leq C\int_0^t (t-\tau)^{-\delta(r)}\|e^{(t-\tau)\Delta}P(D)F(U(\tau))\|_{L^p}\,d\tau \\
&\leq C\int_0^t (t-\tau)^{-\delta(r)-\xi(q)}\,\|U(\tau)\|_{L^{qq_1}}\,\|U(\tau)\|_{L^{qq_2(\alpha-1)}}^{\alpha-1}\,d\tau\,,
\end{aligned}$$

where $q \in [1, p[$, $1/q_1 + 1/q_2 = 1$ and $\xi(q)$ is given by (47). Now, taking $q\, q_1 = p$ with $q \approx p$ then $q_1 \approx 1$ and $q\, q_2\, (\alpha - 1) \geq p_c$ and so, we using the estimates of Lemma 1.2 we obtain

$$\|L(U)(t)\|_{L^r} \leq C\big(\sup_{t\in\mathbb{R}^+} \|U(t)\|_{L^p}\big) \|U_0\|_{L^{p_c}}^{\alpha-1} \int_0^t (t-\tau)^{-\delta(r)-\xi(q)}\, \tau^{-\theta(q)}\, d\tau\,,$$

where $\theta(q)$ is given by (48). If (50) holds then one can choose q, q_1 and q_2 such that $\theta(q) < 1$, $\delta(r) + \xi(q) < 1$ and $\xi(q) + \theta(q) = 1$, and so,

$$\|L(U)(t)\|_{L^r} \leq C\, t^{-\delta(r)} \big(\sup_{t\in\mathbb{R}^+} \|U(t)\|_{L^p}\big) \|U_0\|_{L^{p_c}}^{\alpha-1}\,.$$

Then, since $\|U(t)\|_{L^p} \leq C\, \|U_0\|_{L^p}$ (by step two),

$$\|L(U)(t)\|_{L^r} \leq C\, t^{-\delta(r)}\, \|U_0\|_{L^{p_c}}^{\alpha-1}\, \|U_0\|_{L^p} \leq C\, t^{-\delta(r)}\, \|U_0\|_{L^p}\,,$$

which completes the proof.

Now, if (50) is not fulfilled, we build a sequence $\{r_i\}$ defined by

$$r_0 = p\,, \qquad \frac{n}{2}\Big(\frac{1}{r_i} - \frac{1}{r_{i+1}}\Big) = \delta < \max\Big\{\frac{(2-d)}{2}, \alpha^{-1}\Big\}\,.$$

And, if $p < r_1 < r_2 < p_c$, since $U(t,\cdot)$ is bounded in $L^p \cap L^{p_c}$, then $U(t,\cdot)$ is also bounded in L^r for all r in $[p, p_c]$ and for each $t \geq 0$.

Now let $t_0 > 0$ and let W be the solution of

$$\left\{\begin{array}{l} W(t,x) = e^{t\Delta} V_0 + L(W)(t,x)\,, \\ W(0,x) = W_0(x) = U(t_0,x)\,. \end{array}\right. \tag{51}$$

We have already proved that

$$W_0 \in L^{p_c}(\mathbb{R}^n) \cap L^{r_1}(\mathbb{R}^n) \quad \text{with} \quad \|W_0\|_{L^{r_1}} \leq C\, t_0^{-\delta(r_1)}\, \|U_0\|_{L^{p_c}} \tag{52}$$

and furthermore $W(t,\cdot)$ is bounded in $L^{r_1}(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$. So we just have to iterate the previous proof to estimate $W(t_0,x) = U(2\,t_0,x)$ in $L^{r_2}(\mathbb{R}^n)$ norm with respect to $W_0(x) = U(t_0,x)$ in $L^{r_1}(\mathbb{R}^n)$ norm to obtain the required estimate and we can do this until $r_i \leq p_c$.

Now let us denote by I the first index such that $r_I > p_c$. We have proved that

$$\left\{\begin{array}{l} \|U(t)\|_{L^{p_c}} \leq C\,(1+t)^{-\delta(p_c)}\,, \\ \|U(t)\|_{L^{r_I}} \leq C\, t^{-\delta(r_I)}\,. \end{array}\right. \tag{53}$$

and, to conclude, we just have to use the same proof as in Section 3.1 with the estimate (53) instead of the estimates (41). This end the proof of the Proposition.

3.3. Initial data in $H_p^s(\mathbb{R}^n)$.

We give now the proof of Theorem 1.5. Let us consider an initial data U_0 such that $\|U_0\|_{H_p^{s_c}} \leq A'$. Then, by the Sobolev embedding theorem, U_0 belongs to $L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and, if A' is small enough, then $\|U_0\|_{L^{p_c}} \leq A$. So, according to Theorem 1.4, there exists a unique global solution $U(t,x)$ of (2) and this solution satisfies (19) and (20). Hence, to prove that U belongs to $BC(\mathbb{R}^+, H_p^{s_c})$, we have only to check that U remains bounded in the homogeneous space $\dot{H}_p^{s_c}(\mathbb{R}^n)$ thanks to the following well know inequality

$$\|f\|_{H_p^s} \leq C\,(\|f\|_{L^p} + \|f\|_{\dot{H}_p^s})\,, \qquad \text{for all } s \geq 0\,.$$

Now since U is a solution of (2),

$$\begin{aligned}
\|U(t)\|_{\dot{H}_p^{s_c}} &\leq \|U_0\|_{\dot{H}_p^{s_c}} + \|L(U)(t)\|_{\dot{H}_p^{s_c}} \\
&\leq \|U_0\|_{H_p^{s_c}} + \int_0^t \|e^{(t-\tau)\Delta} P(D) F(U(\tau))\|_{\dot{H}_p^{s_c}}\, d\tau \\
&\leq \|U_0\|_{H_p^{s_c}} + C \int_0^t (t-\tau)^{-(s_c+d)/2}\, \|F(U(\tau))\|_{L^p}\, d\tau \qquad (54)\\
&\leq \|U_0\|_{H_p^{s_c}} + C \int_0^t (t-\tau)^{-\lambda(q)}\, \|F(U(\tau))\|_{L^{q/\alpha}}\, d\tau \\
&\leq \|U_0\|_{H_p^{s_c}} + C \int_0^t (t-\tau)^{-\lambda(q)}\, \|U(\tau)\|_{L^q}^{\alpha}\, d\tau\,,
\end{aligned}$$

where

$$\lambda(q) = \frac{s_c+d}{2} + \frac{n}{2}\Big(\frac{\alpha}{q} - \frac{1}{p}\Big)\,, \qquad q \in \,]p_c, p\,\alpha]\,. \tag{55}$$

and where, in the third inequality, we used the hypothesis of homogeneity on $P(D)$.

Since $p > p_c/\alpha$, one can check that $s_c < 2 - d$, and so taking $q \approx p\,\alpha$, one can always choose q such that $0 < \lambda(q) < 1$. Then, for this choice of q we obtain

$$\begin{aligned}\|U(t)\|_{\dot H_p^{s_c}} \le \|U_0\|_{H_p^{s_c}} \\ &+ C\big(\sup_{\mathbb{R}^+} t^{\gamma(q)}\,\|U(t)\|_{L^q}\big)^{\alpha} \int_0^t (t-\tau)^{-\lambda(q)}\,\tau^{-\alpha\gamma(q)}\,d\tau \\ &\le \|U_0\|_{H_p^{s_c}} + C\big(\sup_{\mathbb{R}^+} t^{\gamma(q)}\,\|U(t)\|_{L^q}\big)^{\alpha},\end{aligned}$$

since $\lambda(q) + \alpha\,\gamma(q) = 1$. But, by Lemma 1.2, we know that $t^{\gamma}\|U(t)\|_{L^q}$ remains bounded for all $t \ge 0$ and so U belongs to $BC(\mathbb{R}^+, \dot H_p^{s_c})$. Thus we have proved that U belongs to $BC(\mathbb{R}^+, H_p^{s_c})$.

Now, let $U_0 \in H_p^s(\mathbb{R}^n)$ such that $\|U_0\|_{H_p^{s_c}} \le A'$. Then, according to Part a) of Theorem 1.3 and to Part a) of Theorem 1.5, there exists a unique solution of (2) in $C([0,T[\,, H_p^s) \cap BC(\mathbb{R}^+, H_p^{s_c})$ and so, to prove Part b) of Theorem 1.5, we must show that blow up in $H_p^s(\mathbb{R}^n)$ norm cannot occur. But, like in Part b) of Theorem 1.3, one can easily show that smoothing effects occur namely that

$$\|U(t) - e^{t\Delta}U_0\|_{H_p^{s_c+\theta}} \le C\,\|U(t)\|_{H_p^{s_c}}\ ,$$

this, as long as $\lambda(q) + \theta < 1$, where $\lambda(q)$ is given by (55). Hence, if blow up holds in $H_p^{s_c+\theta}(\mathbb{R}^n)$ norm, it holds also in $H_p^{s_c}(\mathbb{R}^n)$ norm: this contradicts Part a) of Theorem 1.5. Now, since $s > s_c$ is arbitrary, we have just to iterate this proof to obtain the required result.

4. Composition on $H_p^s(\mathbb{R}^n)$ spaces.

4.1. Introduction.

In this section we prove the nonlinear estimate

$$\|F(u)\|_{H_p^{s_\alpha}} \le C\,\|u\|_{H_p^s}^{\alpha}$$

that we used in a crucial way in the proof of Theorem 1.3 (our result about local existence and uniqueness for Equation (2)). First we are going to consider the case H2) (*i.e.* when $s_\alpha \le 0$). Then, after recalling a few results about Littlewood-Paley analysis, we will prove Theorem 1.3 when H3) is fulfilled ($0 < s_\alpha < \alpha - 1$).

4.2. The case $s_\alpha \leq 0$.

Here, we suppose that $\max\{0, n/p - n/\alpha\} < s$ and that H2) is fulfilled, *i.e.* that $s_\alpha \leq 0$ and that $|F(x)| \leq C\,|x|^\alpha$. Now consider $u(x) \in H^s_p(\mathbb{R}^n)$. Then, by the Sobolev embedding Theorem ($s \geq 0$ and $p \in]1, +\infty[$) we have

$$H^s_p(\mathbb{R}^n) \hookrightarrow L^{(1/p - s/n)^{-1}}(\mathbb{R}^n)\,. \tag{56}$$

Now, since $s > n/p - n/\alpha$ we have $(1/p - s/n)^{-1} > \alpha$ and, on the other hand, we have $|F(x)| \leq C|x|^\alpha$. Thus, by (56)

$$\|F(u)\|_{L^{(\alpha/p-(s\alpha)/n)^{-1}}} \leq C\,\|u\|^\alpha_{L^{(1/p-s/n)^{-1}}} \leq C\,\|u\|^\alpha_{H^s_p}\,. \tag{57}$$

Next, to conclude, we remark that

$$\frac{1}{p} - \frac{s_\alpha}{n} = \frac{\alpha}{p} - \frac{s\,\alpha}{n}$$

and then, since $s_\alpha \leq 0$ and $(\alpha/p - (s\alpha)/n)^{-1} > 1$, we can use the Sobolev embedding

$$L^{(\alpha/p-(s\alpha)/n)^{-1}}(\mathbb{R}^n) \hookrightarrow H^{s_\alpha}_p(\mathbb{R}^n)\,, \tag{58}$$

which, with the estimate (57), gives

$$\|F(u)\|_{H^{s_\alpha}_p} \leq C\,\|u\|^\alpha_{H^s_p}$$

as we claim.

4.3. Littlewood-Paley analysis.

Let us first recall the Littlewood-Paley dyadic decomposition for a tempered distribution. Let φ_{-1} be a non-negative radial test function such that $\widehat{\varphi_{-1}}(\xi) = 1$ for $|\xi| \leq 3/4$ and such that $\widehat{\varphi_{-1}}(\xi) = 0$ for $|\xi| \geq 1$.

Let $\varphi_j(x) = 2^{nj}\,\varphi_{-1}(2^j x)$ and let us consider the partial sum operators S_j associated with the φ_j and defined by

$$S_j(f)(x) = \varphi_j \star f(x)\,. \tag{59}$$

Now define $\psi_{-1}(x) = \varphi_{-1}(x)$ and $\psi_j(x) = \varphi_j(x) - \varphi_{j-1}(x)$ and, in the same way as previously, consider the operators Δ_j defined by

$$\Delta_j(f)(x) = S_j(f)(x) - S_{j-1}(f)(x) = \psi_j \star f(x)\,. \tag{60}$$

Thus,

$$f = \lim_{j\to\infty} S_j(f) = \Delta_{-1}(f) + \sum_{j=0}^{\infty} \Delta_j(f)\,. \tag{61}$$

More precisely one can prove the following result (see [Tr]).

Proposition 4.1. *The convergence in* (61) *occurs in* $H_p^s(\mathbb{R}^n)$ *for all* p *in* $]1,\infty[$ *and for all* s *in* $\mathbb{R}$. *Furthermore, for all* f *in* $H_p^s(\mathbb{R}^n)$,

$$\|f\|_{H_p^s} \sim \|\Delta_{-1}(f)\|_{L^p} + \Big\| \Big(\sum_{j=0}^{\infty} 4^{js}\,|\Delta_j(f)|^2 \Big)^{1/2} \Big\|_{L^p}\,.$$

Now we give some classical Lemmas which will be of great use in the sequel.

Lemma 4.1 (Bernstein's inequalities)**.** *Let* $p \in [1,\infty]$.

a) *If* f *has its spectrum in the ball* $B(0,r)$ *then there exists a constant* C *independent of* f *and* r *such that*

$$\|\Lambda_s f\|_{L^p} \le C\,r^s\,\|f\|_{L^p}\,, \qquad \textit{for all } s > 0\,.$$

b) *If* f *has its spectrum in the ring* $C(0,Ar,Br) = \{\xi : A\,r \le |\xi| \le B\,r\}$ *then there exists some constants* C_1 *and* C_2 *independent of* f *and* r *such that*

$$C_1\,r^s\,\|f\|_{L^p} \le \|\Lambda_s f\|_{L^p} \le C_2\,r^s\,\|f\|_{L^p}\,, \qquad \textit{for all } s > 0\,.$$

For a proof see [AG]. The second Lemma describes the behavior of $S_j(u)$ and $\Delta_j(u)$ in $L^\infty(\mathbb{R}^n)$ norm when u belongs to $H^s_p(\mathbb{R}^n)$ spaces.

Lemma 4.2. *Let*

$$s_n = s - \frac{n}{p}\,. \tag{62}$$

Then,

a) *For all s in $\mathbb{R}$, $\|\Delta_k(u)\|_{L^\infty} \le C\,2^{-ks_n}\,\|u\|_{H^s_p}$.*

b) *If $s < n/p$ then, $\|S_k(u)\|_{L^\infty} \le C\,2^{-ks_n}\,\|u\|_{H^s_p}$.*

The proof is left to the reader (hint: use Bernstein's inequalities).

Lemma 4.3. *Let $\{f_k\}_{k=0}^\infty$ be a sequence of functions in $\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\operatorname{supp}(\hat{f}_k) \subset B(0, C\,2^k)\,.$$

Then there exists a constant C such that

$$\Big\|\Big(\sum_{j=0}^\infty\sum_{k=0}^\infty |\Delta_j(f_k)|^2\Big)^{1/2}\Big\|_{L^p} \le C\Big\|\Big(\sum_{k=0}^\infty |f_k|^2\Big)^{1/2}\Big\|_{L^p}\,.$$

For a proof see [Me].

4.4. The paracomposition formula.

To prove Theorem 1.2 we use the paracomposition technique (see [Me], [Ta], [AG], [Co], ...) which generalizes the paraproduct technique introduced by J. M. Bony. We rewrite $F(u)$ as the serie

$$\begin{aligned} F(u) = F(S_0(u)) &+ (F(S_1(u)) - F(S_0(u))) + \cdots \\ &+ (F(S_{k+1}(u)) - F(S_k(u))) + \cdots \end{aligned}$$

and since F is C^1 at least

$$F(u) = F(S_0(u)) + \sum_{k=0}^\infty \Delta_k(u)\,m_k(u)\,, \tag{63}$$

where

$$m_k(u) = \int_0^1 F'(S_k(u) + t\,\Delta_k(u))\,dt\,. \tag{64}$$

To relocate the $m_k(u)$ spectrums we introduce a second Littlewood-Paley's partition of unity

$$\widehat{\varphi_{-1}}\Big(\frac{\xi}{A\,2^k}\Big) + \sum_{p=0}^{\infty} \widehat{\psi}\Big(\frac{\xi}{A\,2^{k+p}}\Big) = 1$$

and so,

$$m_k(u) = m_{k,-1}(u) + \sum_{p=0}^{\infty} m_{k,p}(u)\,, \tag{65}$$

where

$$\begin{cases} m_{k,-1}(u) = \mathcal{F}^{-1}\Big(\hat{\varphi}_{-1}\Big(\dfrac{\xi}{A\,2^k}\Big)\Big) \star m_k(u)\,, \\ m_{k,p}(u) = \mathcal{F}^{-1}\Big(\hat{\psi}_{-1}\Big(\dfrac{\xi}{A\,2^k}\Big)\Big) \star m_k(u)\,. \end{cases} \tag{66}$$

So, by (63) and (66),

$$F(u) = F(S_0(u)) + \sum_{k=0}^{\infty} \Delta_k(u)\, m_{k,-1}(u) + \sum_{k,p=0}^{\infty} \Delta_k(u)\, m_{k,p}(u) \tag{67}$$

and we want to prove that each of those terms belongs to $H_p^{s_\alpha}(\mathbb{R}^n)$ where $s_\alpha > 0$ is given by (9).

For the term $F(S_0(u))$ we refer to [Co] (one uses bounds on the maximal function of $F(S_0(U))$ to get the proof).

4.4.1. The series $\sum_{k=0}^{\infty} \Delta_k(u)\, m_{k,-1}(u)$ belongs to $H_p^{s_\alpha}(\mathbb{R}^n)$.

We begin with the following Lemma.

Lemma 4.4. *Under* H3)*,*

$$\|m_{k,-1}(u)\|_{L^\infty} \le C\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1}\,, \qquad \textit{for all } k \in \mathbb{N}\,.$$

Lemma 4.4 follows from Lemma 4.2 since

$$\begin{aligned}\|m_{k,-1}(u)\|_{L^\infty} &= \|\mathcal{F}^{-1}(\hat\varphi_{-1}(\xi\, A^{-1}\, 2^{-k})) \star m_k(u)\|_{L^\infty}\\ &\le \|\mathcal{F}^{-1}(\hat\varphi_{-1}(\xi\, A^{-1}\, 2^{-k}))\|_{L^1}\, \|m_k(u)\|_{L^\infty}\\ &\le C\, \|m_k(u)\|_{L^\infty}\ .\end{aligned}$$

Now $|F'(x)| \le C|x|^{\alpha-1}$ and so

$$|m_k(u)| \le \int_0^1 C\, |S_k(u) + t\, \Delta_k(u)|^{\alpha-1}\, dt$$

and the estimates of Lemma 4.2 for $S_k(u)$ and $\Delta_k(u)$ in $L^\infty(\mathbb{R}^n)$ norm lead to the proof.

To prove that the series belongs to $H_p^{s_\alpha}(\mathbb{R}^n)$ by Proposition 4.1 it is then sufficient to show that the function

$$\sigma(x) = \Big(\sum_{j=0}^{\infty} 4^{js_\alpha} \Big|\Delta_j\Big(\sum_{k=0}^{\infty} \Delta_k(u) m_{k,-1}(u)\Big)\Big|^2\Big)^{1/2}$$

belongs to $L^p(\mathbb{R}^n)$. By construction the $m_{k,-1}(u)$ spectrums are in the balls $B(0, A\, 2^k)$ and the $\Delta_k(u)$ spectrums are in the rings $C(0, 2^{-1}A\, 2^k, 2\, A\, 2^k)$. Taking $A = 50$ (for instance) then the $m_{k,-1}(u)\Delta_k(u)$ spectrums are in some extended balls $B(0, A'2^k)$ and so, there exists an integer N such that $\Delta_j(m_{k,-1}(u)\Delta_k(u)) = 0$ for $j > k + N$ since the spectrums of φ_j and $\Delta_k(u)\, m_{k,-1}(u)$ are disjointed. So,

$$\begin{aligned}&\Big|\Delta_j\Big(\sum_{k=0}^{\infty} \Delta_k(u)\, m_{k,-1}(u)\Big)\Big|^2\\ &\qquad = \Big|\Delta_j\Big(\sum_{k=j+N}^{\infty} \Delta_k(u)\, m_{k,-1}(u)\Big)\Big|^2\\ &\qquad \le C\, 4^{-js_\alpha}\Big(\sum_{k=j+N}^{\infty} 4^{ks_\alpha}\, |\Delta_j(\Delta_k(u)\, m_{k,-1}(u))|^2\Big)\end{aligned}$$

by Cauchy-Schwartz inequality applied to the sequences

$$\{2^{-ks_\alpha}\, \mathbf{1}_{k\ge j+N}\} \qquad \text{and} \qquad \{2^{ks_\alpha}\Delta_k(u)\, m_{k,-1}(u)\, \mathbf{1}_{k\ge j+N}\}$$

(note that $s_\alpha > 0$ is needed). Then by definition of $\sigma(x)$ we get

$$\sigma(x) \leq C\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_j(2^{ks_\alpha}\Delta_k(u)\, m_{k,-1}(u))|^2\Big)^{1/2}$$

and, Lemma 4.3 applied to the sequence $\{2^{ks_\alpha}\Delta_k(u)\, m_{k,-1}(u)\}$ leads to

$$\|\sigma(x)\|_{L^p} \leq \Big\|\Big(\sum_{k=0}^{\infty} 4^{ks_\alpha}|\Delta_k(u)\, m_{k,-1}(u)|^2\Big)^{1/2}\Big\|_{L^p}.$$

Now, using Lemma 4.4

$$\begin{aligned}|\Delta_k(u)\, m_{k,-1}(u)|^2 &\leq \|m_{k,-1}(u)\|^2_{L^\infty}\, |\Delta_k(u)|^2\\ &\leq C\, 4^{-ks_n(\alpha-1)}\, \|u\|^{2(\alpha-1)}_{H^s_p}\, |\Delta_k(u)|^2\end{aligned}$$

and so,

$$\|\sigma(x)\|_{L^p} \leq C\, \|u\|^{\alpha-1}_{H^s_p}\Big\|\Big(\sum_{k=0}^{\infty} 4^{k(s_\alpha - s_n(\alpha-1))}\, |\Delta_k(u)|^2\Big)^{1/2}\Big\|_{L^p}.$$

But, $s = s_\alpha - s_n(\alpha-1)$ and so

$$\|\sigma(x)\|_{L^p} \leq C\, \|u\|^{\alpha-1}_{H^s_p}\Big\|\Big(\sum_{k=0}^{\infty} 4^{ks}\, |\Delta_k(u)|^2\Big)^{1/2}\Big\|_{L^p} \leq C\, \|u\|^{\alpha}_{H^s_p}.$$

Thus the series belongs to $H^{s_\alpha}_p(\mathbb{R}^n)$ and its norm is bounded by $C\,\|u\|^\alpha_{H^s_p}$.

4.4.2. The series $\sum_{k=0}^{\infty}\big(\sum_{p=0}^{\infty}\Delta_k(u)\, m_{k,p}(u)\big)$ belongs to $H^{s_\alpha}_p(\mathbb{R}^n)$.

For fixed $p \geq 0$ we define

$$l_p(x) = \sum_{k=0}^{\infty}\Delta_k(u)\, m_{k,p}(u).$$

Taking the constant A large enough one can check that the $\Delta_k(u)\, m_{k,p}(u)$ spectrums are in some rings $\{\xi : C_1\, 2^{p+k} \leq |\xi| \leq C_2\, 2^{p+k}\}$. So, there exists an integer K (which does not depend of p) such that those rings are K to K disjointed. So we can use the Littlewood-Paley analysis on the K partial sums $l_p^r(x)$ defined by

$$(68)\quad l_p^r(x) = \sum_{k=r \bmod(K)} \Delta_k(u)\, m_{k,p}(u) \qquad \text{with } r \in \{0,\dots,K-1\}$$

and, by Proposition 4.1, we know that for all r in $\{0,\dots,K-1\}$,

$$\|l_p^r\|_{H_p^{s_\alpha}} \le C \Big\| \Big(\sum_{k=r \bmod(K)} 4^{(k+p)s_\alpha}\, |\Delta_k(u)\, m_{k,p}(u)|^2 \Big)^{1/2} \Big\|_{L^p}.$$

Let us assume that the following Lemma holds.

Lemma 4.5. *Under* H3),

$$\|m_{k,p}(u)\|_{L^\infty} \le C\, 2^{-(\alpha-1)p}\, 2^{-k(\alpha-1)s_n}\, \|u\|_{H_p^s}^{\alpha-1}, \qquad \textit{for all } k \in \mathbb{N}.$$

Then by Lemma 4.5,

$$\begin{aligned}
\|l_p^r\|_{H_p^{s_\alpha}} &\le C\, 2^{p(s_\alpha-(\alpha-1))}\, \|u\|_{H_p^s}^{\alpha-1} \\
&\quad \cdot \Big\| \Big(\sum_{k=r \bmod(K)} 4^{k(s_\alpha - s_n(\alpha-1))}\, |\Delta_k(u)|^2 \Big)^{1/2} \Big\|_{L^p} \\
&\le C\, 2^{p(s_\alpha-(\alpha-1))}\, \|u\|_{H_p^s}^{\alpha-1} \Big\| \Big(\sum_{k=r \bmod(K)} 4^{ks}\, |\Delta_k(u)|^2 \Big)^{1/2} \Big\|_{L^p} \\
&\le C\, 2^{p(s_\alpha-(\alpha-1))}\, \|u\|_{H_p^s}^{\alpha}.
\end{aligned}$$

Thus, for $s_\alpha < \alpha - 1$, the K series $\{l_p^r\}_{p\in\mathbb{N}}$ are uniformly convergent in $H_p^{s_\alpha}(\mathbb{R}^n)$ and furthermore, for $r \in \{0,\dots,K-1\}$,

$$\sum_p \|l_p^r\|_{H_p^{s_\alpha}} \le C\, \|u\|_{H_p^s}^{\alpha},$$

which ends the proof of Theorem 1.2.

So, to conclude, we have just to prove Lemma 4.5. Let us define

$$(69)\qquad \theta = \alpha - 1 = N + \nu, \qquad \text{where } N = [\theta] \text{ and } \nu \in [0,1[\,.$$

and

$$P_k^t(x) = S_k(x) + t\,\Delta_k(x)\,. \tag{70}$$

By Lemma 4.1 applied with $p = \infty$,

$$\|m_{k,p}(u)\|_{L^\infty} \le C\,2^{-(k+p)\theta}\,\|m_k(u)\|_{C^\theta}\ , \tag{71}$$

where $C^\theta(\mathbb{R}^n)$ denotes the Hölder space of order θ endowed with the norm

$$\|h\|_{C^\theta} = \|h\|_{L^\infty} + \cdots + \|D^\theta h\|_{L^\infty}\ , \qquad \text{if } \theta \in \mathbb{N}, \tag{72}$$

and

$$\|h\|_{C^\theta} = \|h\|_{C^N} + \sup_{|x-y|<1} \frac{|D^N h(x) - D^N h(y)|}{|x-y|^\nu}\ , \qquad \text{if } \theta \notin \mathbb{N} \tag{73}$$

(for more details see [Tr] for instance).

So, by (71),

$$\begin{aligned}\|m_{k,p}(u)\|_{L^\infty} \le C\,2^{-(k+p)\theta}\Big(&\|m_k(u)\|_{L^\infty} + \cdots + \|D^N m_k(u)\|_{L^\infty} \\ &+ \sup_{|x-y|<1} \frac{|D^N m_k(u)(x) - D^N m_k(u)(y)|}{|x-y|^\nu}\Big)\,.\end{aligned} \tag{74}$$

The bound of $\|m_k(u)\|_{L^\infty}$ is easy to establish: we have just to argue as in the proof of Lemma 4.4 to get

$$\|m_k(u)\|_{L^\infty} \le C\,2^{-ks_n(\alpha-1)}\,\|u\|_{H_p^s}^{\alpha-1}\ . \tag{75}$$

Next we must bound $\|D^j m_k(u)\|_{L^\infty}$ for $j \in \{1,\dots,N\}$. Let γ be a multi-index such that $\gamma = \gamma_1 + \cdots + \gamma_n$ with total length $|\gamma| = |\gamma_1| + \cdots + |\gamma_n| = j$ then,

$$\partial^\gamma m_k(x) = \int_0^1 \sum_{q=1}^{j} \sum_{\gamma_1+\cdots+\gamma_q=\gamma} D^{q+1}F(P_k^t(x))\,\partial^{\gamma_1}P_k^t(x)\cdots\partial^{\gamma_q}P_k^t(x)\,dt\,,$$

where the second sum is taken on all the decompositions of $\gamma = \gamma_1 + \cdots + \gamma_q$. By Lemma 4.1 and Lemma 4.2,

$$\|\partial^{\gamma_i}P_k^t(x)\|_{L^\infty} \le C\,2^{|\gamma_i|k}\,\|P_k^t(x)\|_{L^\infty} \le C\,2^{|\gamma_i|k}\,2^{-ks_n}\,\|u\|_{H_p^s}$$

and so,

$$\begin{aligned}\|\partial^{\gamma_1} P_k^t(x) \cdots \partial^{\gamma_q} P_k^t(x)\|_{L^\infty} & \\ &\leq C\,(2^{|\gamma_1|k}\, 2^{-ks_n}\, \|u\|_{H_p^s}) \cdots (2^{|\gamma_q|k}\, 2^{-ks_n}\, \|u\|_{H_p^s}) \\ &\leq C\, 2^{k(|\gamma_1|+\cdots+|\gamma_q|)}\, 2^{-kqs_n}\, \|u\|_{H_p^s}^q\,.\end{aligned}$$

Furthermore by Lemma 1.1 and Lemma 4.2

$$\begin{aligned}\|D^{q+1}F(P_k^t)(x)\|_{L^\infty} &\leq C\,\|S_k(u) + t\,\Delta_k(u)\|_{L^\infty}^{\alpha-q-1} \\ &\leq C\, 2^{-ks_n(\alpha-1-q)}\, \|u\|_{H_p^s}^{\alpha-1-q}\,.\end{aligned}$$

And so, for fixed q,

$$\begin{aligned}&\Big\| \int_0^1 \sum_{\gamma_1+\cdots+\gamma_q=\gamma} D^{q+1}F(P_k^t(x))\, \partial^{\gamma_1} P_k^t(x) \cdots \partial^{\gamma_q} P_k^t(x)\, dt \Big\|_{L^\infty} \\ &\qquad \leq C\, 2^{-ks_n(\alpha-1-q)}\, \|u\|_{H_p^s}^{\alpha-1-q}\, 2^{k(|\gamma_1|+\cdots+|\gamma_q|)}\, 2^{-kqs_n}\, \|u\|_{H_p^s}^q \\ &\qquad \leq C\, 2^{-ks_n(\alpha-1)}\, 2^{jk}\, \|u\|_{H_p^s}^{(\alpha-1)}\,.\end{aligned}$$

Thus, for $j \in \{1, \ldots, N\}$,

$$\|D^j m_k(u)\|_{L^\infty} \leq C\, 2^{-ks_n(\alpha-1)}\, 2^{kj}\, \|u\|_{H_p^s}^{(\alpha-1)}\,. \tag{76}$$

To conclude we must estimate

$$\sup_{|x-y|<1} \frac{|D^{[N]} m_k(u)(x) - D^{[N]} m_k(u)(y)|}{|x-y|^\nu}\,.$$

Let γ be a multi-index of length N. Then,

$$\partial^\gamma m_k(x) = \int_0^1 \sum_{q=1}^N \sum_{\gamma_1+\cdots+\gamma_q=\gamma} D^{q+1}F(P_k^t(x))\, \partial^{\gamma_1} P_k^t(x) \cdots \partial^{\gamma_q} P_k^t(x)\, dt$$

and so, $\partial^\gamma m_k(x) - \partial^\gamma m_k(y) = I(x,y) + J(x,y)$ where

$$\begin{aligned}I(x,y) = \int_0^1 \sum_{q=1}^N \sum_{\gamma_1+\cdots+\gamma_q=\gamma} &(D^{q+1}F(P_k^t(x)) - D^{q+1}F(P_k^t(y))) \\ &\cdot \prod_{\gamma_i} \partial^{\gamma_i} P_k^t(x)\, dt\end{aligned}$$

and,

$$J(x,y) = \int_0^1 \sum_{q=1}^{N} \sum_{\gamma_1+\cdots+\gamma_q=\gamma} D^{q+1}F(P_k^t(y)) \cdot \Big(\prod_{\gamma_i} \partial^{\gamma_i} P_k^t(x) - \prod_{\gamma_i} \partial^{\gamma_i} P_k^t(y) \Big)\, dt\,.$$

We deal first with the term $I(x,y)$. For $q \in \{1,\dots,N\}$ and $\{\gamma_i\}_{i=1,\dots,q}$ a decomposition of γ we must estimate

$$I_k^q = \sup_{|x-y|<1} \left\{ \frac{1}{|x-y|^\nu} \Big| \int_0^1 \sum_{\gamma_1+\cdots+\gamma_q=\gamma} (D^qF'(P_k^t(x)) - D^qF'(P_k^t(y))) \cdot \prod_{\gamma_i} \partial^{\gamma_i} P_k^t(x)\, dt \Big| \right\}.$$

First suppose that $q \le N-1$. Then, by Lemmas 4.1 and 4.2

$$\Big\| \prod_{i=1}^{q} \partial^{\gamma_i} P_k^t(x) \Big\|_{L^\infty} \le C\, 2^{kN}\, 2^{-kqs_n}\, \|u\|_{H_p^s}^q\,. \tag{77}$$

Next we must bound

$$\sup_{|x-y|<1} \frac{|D^{q+1}F(P_k^t)(x) - D^{q+1}F(P_k^t)(y)|}{|x-y|^\nu}\,.$$

But, by Lemma 1.1, for $q \le N-1$,

$$|D^{q+1}F(x) - D^{q+1}F(y)| \le C\, |x-y|\, (|x|^{\alpha-q-2} + |y|^{\alpha-q-2})$$

and so

$$\begin{aligned} &|D^{q+1}F(P_k^t)(x) - D^{q+1}F(P_k^t)(y)| \\ &\qquad \le C\, |P_k^t(x) - P_k^t(y)|\, (|P_k^t(x)|^{\alpha-q-2} + |P_k^t(y)|^{\alpha-q-2})\,. \end{aligned}$$

But, by definition of the $C^\nu(\mathbb{R})$ norm,

$$\begin{aligned} \sup_{|x-y|<1} \frac{|P_k^t(x) - P_k^t(y)|}{|x-y|^\nu} &\le C\, \|P_k^t\|_{C^\nu} \\ &\le C\, 2^{k\nu}\, \|P_k^t\|_{L^\infty} \\ &\le C\, 2^{k\nu}\, 2^{-ks_n}\, \|u\|_{H_p^s} \end{aligned}$$

by Lemmas 4.1 and 4.2. Thus, for all $x, y \in \mathbb{R}^n$ with $|x-y|<1$,

$$|P_k^t(x) - P_k^t(y)| \le C\,|x-y|^\nu\, 2^{k\nu}\, 2^{-ks_n}\, \|u\|_{H_p^s}$$

and so

$$\sup_{|x-y|<1} \frac{|D^{q+1}F(P_k^t)(x) - D^{q+1}F(P_k^t)(y)|}{|x-y|^\nu} \le C\, 2^{k\nu}\, 2^{-ks_n}\, \|u\|_{H_p^s}\, \|P_k^t\|_{L^\infty}^{\alpha-q-2}$$

and so, by Lemma 4.2, for all q in $\{0,\dots,N-1\}$,

$$\sup_{|x-y|<1} \frac{|D^{q+1}F(P_k^t)(x) - D^{q+1}F(P_k^t)(y)|}{|x-y|^\nu} \le C\, 2^{k\nu}\, 2^{-ks_n(\alpha-q-1)}\, \|u\|_{H_p^s}^{\alpha-q-1}\,. \tag{78}$$

Then, from (69), (77) and (78), we deduce that for all q in $\{0,\dots,N-1\}$,

$$I_k^q \le C\, 2^{k(\alpha-1)}\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1}\,. \tag{79}$$

Now we deal with the terms I_k^N. By lemmas 4.1 and 4.2,

$$\Big\| \prod_{i=1}^N \partial^{\gamma_i} P_k^t(x) \Big\|_{L^\infty} \le C\, 2^{Nk}\, 2^{-kNs_n}\, \|u\|_{H_p^s}^N\,. \tag{80}$$

Now by H3)

$$|D^N F'(x) - D^N F'(y)| \le C\,|x-y|^\nu$$

and so

$$\begin{aligned}\sup_{t\in[0,1]} |D^N F'(P_k^t(x)) - D^N F'(P_k^t(y))| &\le C \sup_{t\in[0,1]} |P_k^t(x) - P_k^t(y)|^\nu \\ &\le C \sup_{t\in[0,1]} |x-y|^\nu\, \|\nabla P_t^k\|_{L^\infty}^\nu \\ &\le C\,|x-y|^\nu\, (2^k\, 2^{-ks_n}\, \|u\|_{H_p^s})^\nu\,,\end{aligned}$$

by Lemma 4.1 and Lemma 4.2. Combining these inequalities we get

$$\sup_{|x-y|<1} \frac{|P_k^t(x) - P_k^t(y)|}{|x-y|^\nu} \le C\,(2^k\, 2^{-ks_n}\, \|u\|_{H_p^s})^\nu\,. \tag{81}$$

Then, by (80) and (81)

$$I_k^N \leq C\,(2^k\,2^{-ks_n}\,\|u\|_{H_p^s})^{N+\nu}\,,$$

which leads to

$$I_k^N \leq C\,2^{-ks_n(\alpha-1)}\,2^{k(\alpha-1)}\,\|u\|_{H_p^s}^{\alpha-1}\,. \tag{82}$$

Now by (79) and (82),

$$\sup_{|x-y|<1}\frac{|I(x,y)|}{|x-y|^\nu} \leq C\,2^{-ks_n(\alpha-1)}\,2^{k(\alpha-1)}\,\|u\|_{H_p^s}^{\alpha-1}\,. \tag{83}$$

Now we deal with the term J. It can be rewritten as

$$\begin{aligned}J(x,y) = \sum_{q=1}^{N}\sum_{\gamma_1+\dots+\gamma_q=\gamma}\sum_{j=1}^{q}\int_0^1 D^{q+1}F(P_k^t(y))\,\partial^{\gamma_j}(P_k^t(x)-P_k^t(y))\\ \cdot\Big(\prod_{i>j}\partial^{\gamma_i}P_k^t(x)\Big)\Big(\prod_{i<j}\partial^{\gamma_i}P_k^t(y)\Big)\,dt\,.\end{aligned}$$

and we denote by $J_{q,\gamma_i,j}$ each term of the sum and we estimate them for all fixed triplet (q,γ_i,j). As previously, by Lemma 1.1

$$\|D^{q+1}F(P_k^t(y))\|_{L^\infty} \leq C\,2^{-ks_n(\alpha-q-1)}\,\|u\|_{H_p^s}^{\alpha-q-1}\,. \tag{84}$$

Now, let $i\neq j$, then by Lemmas 4.1 and 4.2,

$$\|\partial_x^{\gamma_i}P_k^t(x)\| \leq C\,2^{|\gamma_i|k}\,\|P_k^t(x)\|_{L^\infty} \leq C\,2^{|\gamma_i|k}\,2^{-ks_n}\,\|u\|_{H_p^s}$$

and so

$$\begin{aligned}&\Big\|\prod_{i>j}\partial^{\gamma_i}P_k^t(x)\prod_{i<j}\partial^{\gamma_i}P_k^t(y)\Big\|_{L^\infty}\\ &\qquad\leq C\,2^{k(\sum_{i\neq j}|\gamma_i|)}\,2^{-k(q-1)s_n}\,\|u\|_{H_p^s}^{q-1}\,.\end{aligned} \tag{85}$$

By definition of the $C^s(\mathbb{R})$ norm

$$\begin{aligned}\sup_{|x-y|<1}\frac{\partial^{\gamma_j}(P_k^t(x)-P_k^t(y))}{|x-y|^\nu} &\leq \|P_k^t\|_{C^{|\gamma_j|+\nu}}\\ &\leq C\,2^{k(|\gamma_j|+\nu)}\,\|P_k^t\|_{L^\infty}\\ &\leq C\,2^{k(|\gamma_j|+\nu)}\,2^{-ks_n}\,\|u\|_{H_p^s}\,.\end{aligned}$$

And so,

$$\sup_{|x-y|<1} \frac{\partial^{\gamma_j}(P_k^t(x) - P_k^t(y))}{|x-y|^\nu} \le C\, 2^{k(|\gamma_j|+\nu)}\, 2^{-ks_n}\, \|u\|_{H_p^s}\,. \tag{86}$$

Then by (84), (85) and (86) we get

$$\begin{aligned}\sup_{|x-y|<1} \frac{|J(x,y)|}{|x-y|^\nu} &\le C\, 2^{-ks_n(\alpha-q-1)}\, \|u\|_{H_p^s}^{\alpha-q-1}\, 2^{k(|\gamma_j|+\nu)}\, 2^{-ks_n}\, \|u\|_{H_p^s} \\ &\quad \cdot 2^{k(\sum_{i\ne j}|\gamma_i|)}\, 2^{-k(q-1)s_n}\, \|u\|_{H_p^s}^{q-1} \\ &\le C\, 2^{k(N+\nu)}\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1}\end{aligned}$$

and so, since $N+\nu = \alpha-1$,

$$\sup_{|x-y|<1} \frac{|J(x,y)|}{|x-y|^\nu} \le C\, 2^{k(\alpha-1)}\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1}\,. \tag{87}$$

Thus by (83) and (87)

$$\sup_{|x-y|<1} \left| \frac{D^{[N]}m_k(x) - D^{[N]}m_k(y)}{(x-y)^\nu} \right| \le 2^{-ks_n(\alpha-1)}\, 2^{k(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1}\,. \tag{88}$$

Now by (69), (71), (75), (76) and (88) we see that

$$\begin{aligned}&\|m_{k,p}\|_{L^\infty} \\ &\quad \le C\, 2^{-(k+p)\theta} \Big(\sum_{j=0}^{N} 2^{-ks_n(\alpha-1)}\, 2^{jk} + 2^{-ks_n(\alpha-1)}\, 2^{k(\alpha-1)} \Big) \|u\|_{H_p^s}^{\alpha-1} \\ &\quad \le C\, 2^{-p(\alpha-1)}\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1} \Big(\sum_{j=0}^{N} 2^{k(j-(\alpha-1))} + 1 \Big)\,.\end{aligned}$$

And for $j \in \{0,\dots,N\}$, $j-\alpha+1 \le 0$ from which we deduce that

$$\|m_{k,p}(u)\|_{L^\infty} \le C\, 2^{-ks_n(\alpha-1)}\, 2^{-p(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1}\,,$$

which ends the proof of Lemma 4.5.

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A proof of the smoothing properties of the positive part of Boltzmann's kernel

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Abstract. We give two direct proofs of Sobolev estimates for the positive part of Boltzmann's kernel. The first deals with a cross section with separated variables; no derivative is needed in this case. The second is concerned with a general cross section having one derivative in the angular variable.

Résumé. Nous donnons deux preuves directes des estimations de Sobolev pour la partie positive du noyau de Boltzmann. La première concerne les sections efficaces à variables séparées; aucune dérivée n'est nécessaire dans ce cas. La deuxième traite des sections efficaces générales ayant une dérivée dans la variable angulaire.

1. Introduction.

The Boltzmann quadratic kernel Q models binary collisions occurring in a rarefied monatomic gas (*cf.* [3], [4], [9]). It can be written under the form

$$Q(f)(v) = Q^+(f)(v) - f(v)\, Lf(v)\,, \tag{1.1}$$

where Lf is a linear convolution operator, and Q^+ is the positive part of Q, defined by

$$Q^+(f)(v) = \iint\limits_{\substack{v_*\in\mathbb{R}^N \\ \sigma\in S^{N-1}}} f\Big(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma\Big) f\Big(\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma\Big)$$

$$\cdot\, B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, d\sigma\, dv_*\ . \tag{1.2}$$

The nonnegative cross section B depends on the type of interaction between the particles of the gas.

In a gas in which particles interact with respect to forces proportional to r^{-s}, $s \geq 2$, the cross section B writes

$$B(x,u) = b(x)\,\beta(u)\,, \tag{1.3}$$

where

$$b(x) = x^{(s-5)/(s-1)}\,, \tag{1.4}$$

and β has a strong singularity near $u = 1$.

The classical assumption of angular cutoff of Grad [6] (that is $\beta \in L^1(]-1,1[)$) is used to remove this singularity. It will always be made in this paper. To get an idea of the properties of Q when this assumption is not made, we refer the reader for example to [5] or [8].

The properties of Q^+ (with the assumption of angular cutoff of Grad) have first been investigated by P.-L. Lions in [7]. In this work, it is proved that if B is a very smooth function with support avoiding certain points, then there exists $C_{N,B}$ such that

$$\|Q^+(f)\|_{\dot H^{(N-1)/2}(\mathbb{R}^N_v)} \leq C_{N,B}\, \|f\|_{L^1(\mathbb{R}^N_v)}\, \|f\|_{L^2(\mathbb{R}^N_v)} \tag{1.5}$$

for any $f \in L^1 \cap L^2(\mathbb{R}^N_v)$.

The proof of this estimate used the theory of Fourier integral operators. The very restricting conditions on B were not a serious inconvenience since in the application to the inhomogeneous Boltzmann equation, only the strong compactness in L^1 of $Q^+(f)$ was used, and not the estimate itself, so that these smoothness assumptions could be relaxed by suitable approximations of B. Notice that the use of the Fourier transform in the velocity variable in the context of the Boltzmann equation was already used by Bobylev in [2].

An extension of this work to the case of the relativistic Boltzmann kernel can be found in [1].

Then, another proof of (1.5) was given by Wennberg [10] with the help of the regularizing properties of the (generalized) Radon transform. The hypothesis on B were considerably diminished, so that for example forces in r^{-s} with angular cutoff and $s \geq 9$ were included. Considerations on related kernels (for example the relativistic kernel) can also be found in [11].

This work is intended to give yet another proof of (1.5)-like estimates, using only elementary properties of the Fourier transform. Moreover, we prove that the estimate holds for a large class of cross sections B, including all hard potentials with cutoff (that is when $s \geq 5$) and also soft potentials up to $s > 13/5$.

One of the drawbacks of our proof is that instead of having a L^1 norm times a L^2 norm in the right-hand side of (1.5), we only get a L^2 norm to the square.

In Section 2, we deal with the case when B is a tensor product (that is of the form (1.3)). Then, we present in Section 3 the case of general dependence for B with a reasonable smoothness assumption.

The following notations will be used throughout the paper. For any $p \geq 1$, $q \geq 0$, $L^p_q(\mathbb{R}^N)$ is the weighted space embedded with the norm

$$\|f\|_{L^p_q(\mathbb{R}^N)} = \Big(\int_{v\in\mathbb{R}^N} |f(v)|^p \, (1+|v|)^{pq} \, dv \Big)^{1/p}, \tag{1.6}$$

and if $0 < s < N/2$, $\dot{H}^s(\mathbb{R}^N)$ is the homogeneous Sobolev space of functions f of $L^{2N/(N-2s)}(\mathbb{R}^N)$ such that

$$\widehat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \qquad \text{and} \qquad |\xi|^s \widehat{f}(\xi) \in L^2(\mathbb{R}^N_\xi)\,.$$

Its norm is given by

$$\|f\|_{\dot{H}^s(\mathbb{R}^N)} = \Big(\int_{\xi\in\mathbb{R}^N} |\widehat{f}(\xi)|^2 \, |\xi|^{2s} \, d\xi \Big)^{1/2}. \tag{1.7}$$

We shall use the two following formulas to compute some integrals on the sphere S^{N-1} ($N \geq 2$). The first deals with functions which only depend on one component: for any function β defined on $]-1,1[$,

$$\int_{S^{N-1}} \beta(\omega_N)\, d\omega = \frac{2\pi^{(N-1)/2}}{\Gamma\Big(\dfrac{N-1}{2}\Big)} \int_{-1}^{1} \beta(u)\, (1-u^2)^{(N-3)/2}\, du\,. \tag{1.8}$$

The second is concerned with the change of variables $\sigma = 2\,(\xi\cdot\omega)\,\omega - \xi$, for a fixed $\xi \in S^{N-1}$. We have for any function φ defined on S^{N-1}

$$\int_{S^{N-1}} \varphi(\sigma)\, d\sigma = \int_{S^{N-1}} \varphi(2\,(\xi\cdot\omega)\,\omega - \xi)\,|2\,\xi\cdot\omega|^{N-2}\, d\omega\,. \tag{1.9}$$

Finally, constants will be denoted by C, or C_N when they depend on the dimension N.

2. The case of separated variables.

We investigate here the properties of Q^+ when

$$B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big) = b\,(|v-v_*|)\,\beta\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\,, \tag{2.1}$$

where b and β are Borel functions defined on $]0,\infty[$ and $]-1,1[$ respectively. We consider the multidimensional case $N \geq 2$. Let us state the main result of this section.

Theorem 2.1. *Assume that there exists $K \geq 0$, $\alpha \geq 0$ such that*

$$|b(x)| \leq K(1+x)^\alpha\,, \qquad \text{for all } x > 0\,, \tag{2.2}$$

and that

$$\beta \in L^2(\,]-1,1[\,, (1-u^2)^{(N-3)/2}\, du)\,. \tag{2.3}$$

Then for any $f \in L^2_{1+\alpha}(\mathbb{R}^N)$, $Q^+(f) \in \dot H^{(N-1)/2}(\mathbb{R}^N)$ and

$$\begin{aligned} &\|Q^+(f)\|_{\dot H^{(N-1)/2}(\mathbb{R}^N)} \\ &\qquad \leq C_N K\,\|\beta\|_{L^2(\,]-1,1[\,,(1-u^2)^{(N-3)/2}\,du)}\|f\|^2_{L^2_{1+\alpha}(\mathbb{R}^N)}\,. \end{aligned} \tag{2.4}$$

In order to prove Theorem 2.1, let us define the operator $\widetilde{Q}^+$ for functions of two variables $F(v_1,v_2)$, $v_1, v_2 \in \mathbb{R}^N$ by

$$\begin{aligned} \widetilde{Q}^+(F)(v) = \iint\limits_{\substack{v_*\in\mathbb{R}^N \\ \sigma\in S^{N-1}}} & F\Big(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma, \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma\Big) \\ & \cdot\beta\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, d\sigma\, dv_*\,. \end{aligned} \tag{2.5}$$

Proposition 2.2. *For the linear operator* (2.5), *we have*

i) *If* $\beta \in L^1(\,]-1,1[\,,(1-u^2)^{(N-3)/2}\,du)$, *then for any* $F \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, $\widetilde{Q}^+(F) \in L^1(\mathbb{R}^N)$ *and*

$$\begin{aligned}&\|\widetilde{Q}^+(F)\|_{L^1(\mathbb{R}^N)}\\ &\qquad \le \frac{2\pi^{(N-1)/2}}{\Gamma\Big(\dfrac{N-1}{2}\Big)}\,\|\beta\|_{L^1(]-1,1[,(1-u^2)^{(N-3)/2}\,du)}\,\|F\|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}\,.\end{aligned} \tag{2.6}$$

Moreover, (2.6) *is an equality if* β *and* F *are nonnegative.*

ii) *If* $\beta \in L^2(\,]-1,1[\,,(1-u^2)^{(N-3)/2}\,du)$, *then for any* $F \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ *such that* $(v_2-v_1)F \in L^2(\mathbb{R}^N\times\mathbb{R}^N)$, *the integral* (2.5) *is absolutely convergent for almost every* v, $\widetilde{Q}^+(F) \in \dot{H}^{(N-1)/2}(\mathbb{R}^N)$ *and*

$$\begin{aligned}&\|\widetilde{Q}^+(F)\|_{\dot{H}^{(N-1)/2}(\mathbb{R}^N)}\\ &\qquad \le C_N\|\beta\|_{L^2(]-1,1[,(1-u^2)^{(N-3)/2}\,du)}\,\|F\|_{L^2}^{1/2}\,\|(v_2-v_1)F\|_{L^2}^{1/2}\,.\end{aligned} \tag{2.7}$$

Let us postpone the proof of Proposition 2.2 and deduce Theorem 2.1.

Proof of Theorem 2.1. Let us define

$$F(v_1,v_2) = f(v_1)\,f(v_2)\,b(|v_2-v_1|)\,. \tag{2.8}$$

Then, definitions (1.2), (1.3) and (2.5) yield $Q^+(f) = \widetilde{Q}^+(F)$. Now, by (2.2) we have

$$\begin{aligned}|F(v_1,v_2)| &\le |f(v_1)|\,|f(v_2)|\,K\,(1+|v_2-v_1|)^\alpha\\ &\le K\,|f(v_1)|\,|f(v_2)|\,(1+|v_1|+|v_2|)^\alpha\\ &\le K\,|(1+|v_1|)^\alpha f(v_1)|\,|(1+|v_2|)^\alpha f(v_2)|\,.\end{aligned} \tag{2.9}$$

Therefore,

$$\|F\|_{L^1} \le K\,\|f\|_{L^1_\alpha}^2\,, \qquad \|F\|_{L^2} \le K\,\|f\|_{L^2_\alpha}^2\,, \tag{2.10}$$

and since

$$\begin{aligned}|(v_2 - v_1)F(v_1, v_2)| &\le |v_1|\,|F(v_1, v_2)| + |v_2|\,|F(v_1, v_2)| \\ &\le K|(1 + |v_1|)^{1+\alpha} f(v_1)|\,|(1 + |v_2|)^{\alpha} f(v_2)| \\ &\quad + K|(1 + |v_1|)^{\alpha} f(v_1)|\,|(1 + |v_2|)^{1+\alpha} f(v_2)|\,,\end{aligned}$$

we have also

$$\|(v_2 - v_1)F\|_{L^2} \le 2K\|f\|_{L^2_\alpha}\|f\|_{L^2_{1+\alpha}}\,. \tag{2.11}$$

Therefore, we can apply Proposition 2.2.ii), and we get $Q^+(f) = \widetilde{Q}^+(F) \in \dot{H}^{(N-1)/2}$,

$$\|Q^+(f)\|_{\dot{H}^{(N-1)/2}} \le C_N\,\|\beta\|_{L^2}\,K\|f\|_{L^2_\alpha}^{3/2}\,\|f\|_{L^2_{1+\alpha}}^{1/2}\,. \tag{2.12}$$

Finally, (2.4) follows since

$$\|f\|_{L^2_\alpha} \le \|f\|_{L^2_{1+\alpha}}\,.$$

PROOF OF PROPOSITION 2.2. Estimate i) is easy with (1.8), and we only prove ii). Let us first assume that $F \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. We perform the change of variables

$$\sigma = 2\Big(\frac{v - v_*}{|v - v_*|}\cdot\omega\Big)\,\omega - \frac{v - v_*}{|v - v_*|}\,. \tag{2.13}$$

According to (1.9),

$$\begin{aligned}\widetilde{Q}^+(F)(v) = \iint\limits_{\substack{v_*\in\mathbb{R}^N\\ \omega\in S^{N-1}}} & F(v - (v - v_*)\cdot\omega\,\omega, v_* + (v - v_*)\cdot\omega\,\omega) \\ & \cdot\beta\Big(2\Big(\frac{v-v_*}{|v-v_*|}\cdot\omega\Big)^2 - 1\Big)\Big|2\,\frac{v-v_*}{|v-v_*|}\cdot\omega\Big|^{N-2}\,dv_*\,d\omega\,.\end{aligned} \tag{2.14}$$

Since by i) $\widetilde{Q}^+(F) \in L^1$, we can compute its Fourier transform, which is given by

$$\begin{aligned}
&\widehat{\widetilde{Q}^+(F)}(\xi) \\
&= \iiint\limits_{\substack{v,v_*\in\mathbb{R}^N\\ \omega\in S^{N-1}}} e^{-i\xi\cdot v} F(v-(v-v_*)\cdot\omega\,\omega, v_*+(v-v_*)\cdot\omega\,\omega) \\
&\qquad\qquad \cdot\beta\Big(2\Big(\frac{v-v_*}{|v-v_*|}\cdot\omega\Big)^2-1\Big)\Big|2\,\frac{v-v_*}{|v-v_*|}\cdot\omega\Big|^{N-2}\,dv\,dv_*\,d\omega \\
&= \iiint\limits_{\substack{v_1,v_2\in\mathbb{R}^N\\ \omega\in S^{N-1}}} e^{-i\xi\cdot(v_1-(v_1-v_2)\cdot\omega\,\omega)} F(v_1,v_2) \\
&\qquad\qquad \cdot\beta\Big(2\Big(\frac{v_1-v_2}{|v_1-v_2|}\cdot\omega\Big)^2-1\Big)\Big|2\,\frac{v_1-v_2}{|v_1-v_2|}\cdot\omega\Big|^{N-2}\,dv_1\,dv_2\,d\omega\,,
\end{aligned}$$

by the usual pre-post collisional change of variables. Next we perform a change of variables in ω, given by an orthogonal hyperplane symmetry which exchanges the unitary vectors

$$\frac{v_1-v_2}{|v_1-v_2|} \qquad \text{and} \qquad \frac{\xi}{|\xi|}\,.$$

We obtain

$$\begin{aligned}
\widehat{\widetilde{Q}^+(F)}(\xi) &= \iiint\limits_{\substack{v_1,v_2\in\mathbb{R}^N\\ \omega\in S^{N-1}}} e^{-i(\xi\cdot v_1-(v_1-v_2)\cdot\omega\,\xi\cdot\omega)} F(v_1,v_2) \\
&\qquad\qquad \cdot\beta\Big(2\Big(\frac{\xi}{|\xi|}\cdot\omega\Big)^2-1\Big)\Big|2\,\frac{\xi}{|\xi|}\cdot\omega\Big|^{N-2}\,dv_1\,dv_2\,d\omega \\
&= \int\limits_{\omega\in S^{N-1}} \widehat{F}(\xi-\xi\cdot\omega\,\omega, \xi\cdot\omega\,\omega) \\
&\qquad\qquad \cdot\beta\Big(2\Big(\frac{\xi}{|\xi|}\cdot\omega\Big)^2-1\Big)\Big|2\,\frac{\xi}{|\xi|}\cdot\omega\Big|^{N-2}\,d\omega\,,
\end{aligned}$$

with $\widehat{F}$ the Fourier transform of F in both variables. Finally, we make the change of variables

$$\sigma = 2\Big(\frac{\xi}{|\xi|}\cdot\omega\Big)\,\omega - \frac{\xi}{|\xi|}\,,$$

and get according to (1.9)

$$\widehat{\widetilde{Q}^{+}(F)}(\xi) = \int\limits_{\sigma\in S^{N-1}} \widehat{F}\Big(\frac{\xi-|\xi|\sigma}{2}, \frac{\xi+|\xi|\sigma}{2}\Big)\beta\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\,d\sigma\,. \tag{2.15}$$

Now, in order to estimate (2.15) we assume that $F \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, so that $\widehat{F}$ is smooth. This assumption can easily be relaxed by cutoff and convolution of F to get (2.7) in the general case.

We have by Cauchy-Schwarz's inequality

$$\begin{aligned} |\widehat{\widetilde{Q}^{+}(F)}(\xi)|^2 \le &\int\limits_{\sigma\in S^{N-1}} \Big|\widehat{F}\Big(\frac{\xi-|\xi|\,\sigma}{2}, \frac{\xi+|\xi|\,\sigma}{2}\Big)\Big|^2\,d\sigma \\ &\cdot \int\limits_{\sigma\in S^{N-1}} \Big|\beta\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big|^2\,d\sigma\,, \end{aligned} \tag{2.16}$$

and the last integral can be computed using (1.8),

$$\begin{aligned} &\int\limits_{\sigma\in S^{N-1}} \Big|\beta\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big|^2\,d\sigma \\ &\qquad = \frac{2\pi^{(N-1)/2}}{\Gamma\Big(\dfrac{N-1}{2}\Big)}\int_{-1}^{1}|\beta(u)|^2\,(1-u^2)^{(N-3)/2}\,du\,. \end{aligned} \tag{2.17}$$

Then,

$$\begin{aligned} &\int\limits_{\sigma\in S^{N-1}} \Big|\widehat{F}\Big(\frac{\xi-|\xi|\,\sigma}{2}, \frac{\xi+|\xi|\,\sigma}{2}\Big)\Big|^2\,d\sigma \\ &\quad = \int\limits_{\sigma\in S^{N-1}}\int_{r=|\xi|}^{\infty} -\frac{\partial}{\partial r}\Big|\widehat{F}\Big(\frac{\xi-r\,\sigma}{2}, \frac{\xi+r\,\sigma}{2}\Big)\Big|^2\,d\sigma\,dr \\ &\quad \le \int\limits_{\sigma\in S^{N-1}}\int_{r=|\xi|}^{\infty} \Big|\widehat{F}\Big(\frac{\xi-r\,\sigma}{2}, \frac{\xi+r\,\sigma}{2}\Big)\Big| \\ &\qquad\qquad \cdot\Big|(\nabla_2\widehat{F}-\nabla_1\widehat{F})\Big(\frac{\xi-r\,\sigma}{2}, \frac{\xi+r\,\sigma}{2}\Big)\Big|\,d\sigma\,dr \\ &\quad = \int_{|\eta|>|\xi|}\Big|\widehat{F}\Big(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\Big)\Big| \\ &\qquad\qquad \cdot\Big|(\nabla_2\widehat{F}-\nabla_1\widehat{F})\Big(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\Big)\Big|\,\frac{d\eta}{|\eta|^{N-1}}\,, \end{aligned}$$

where $\nabla_1 \widehat{F}$ and $\nabla_2 \widehat{F}$ denote the gradients of $\widehat{F}$ with respect to the first and second variables. Therefore,

$$\begin{aligned}
&\int\limits_{\xi\in\mathbb{R}^N} |\xi|^{N-1}\, d\xi \int\limits_{\sigma\in S^{N-1}} \Big|\widehat{F}\Big(\frac{\xi-|\xi|\,\sigma}{2}, \frac{\xi+|\xi|\,\sigma}{2}\Big)\Big|^2\, d\sigma \\
&\quad\le \iint\limits_{\xi,\eta\in\mathbb{R}^N} \Big|\widehat{F}\Big(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\Big)\Big|\, \Big|(\nabla_2\widehat{F}-\nabla_1\widehat{F})\Big(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\Big)\Big|\, d\xi\, d\eta \\
&\quad= 2^N \|\, |\widehat{F}|\, |\nabla_2\widehat{F}-\nabla_1\widehat{F}|\, \|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} \\
&\quad\le 2^N \|\widehat{F}\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}\, \|\nabla_2\widehat{F}-\nabla_1\widehat{F}\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} \\
&\quad= 2^N (2\pi)^{2N} \|F\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}\, \|(v_2-v_1)F\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}\ ,
\end{aligned}$$

and together with (2.16)-(2.17), we obtain (2.7).

REMARK 2.1. A slightly weaker version of Theorem 2.1 is still true when one deals with (not too) soft potentials (with the angular cutoff of Grad).

Namely, for a cross section satisfying assumption (2.1) with

$$\beta \in L^2(\,]-1,1[\,,(1-u^2)^{(N-3)/2}\,du)$$

and

$$b(x) = x^{-\alpha}\,, \qquad 0<\alpha<\frac{N}{2}\ , \tag{2.18}$$

for any $f \in L^{2N/(N-\alpha)}_1(\mathbb{R}^N)$, we have that

$$Q^+(f) \in \dot{H}^{(N-1)/2}(\mathbb{R}^N)$$

with

$$\begin{aligned}
&\|Q^+(f)\|_{\dot{H}^{(N-1)/2}(\mathbb{R}^N)} \\
&\qquad\le C_{N,\alpha} \|\beta\|_{L^2(]-1,1[,(1-u^2)^{(N-3)/2}\,du)} \|f\|^2_{L_1^{2N/(N-\alpha)}}\ .
\end{aligned} \tag{2.19}$$

Actually, defining

$$F(v_1,v_2) = f(v_1)f(v_2)\,|v_2-v_1|^{-\alpha}\,,$$

we have

$$\begin{aligned}\|F\|^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} &= \iint |f(v_1)|^2\,|f(v_2)|^2\,|v_2-v_1|^{-2\alpha}\,dv_1\,dv_2 \\ &\le \|\,|f|^2\,\|_{L^{r'}}\,\|\,|f|^2 * |v|^{-2\alpha}\,\|_{L^r}\ .\end{aligned}$$

We choose $r = N/\alpha$, so that

$$\|\,|f|^2 * |v|^{-2\alpha}\,\|_{L^r} \le C_{N,\alpha}\,\|\,|f|^2\,\|_{L^{N/(N-\alpha)}}$$

and we obtain

$$\|F\|^2_{L^2} \le C_{N,\alpha}\,\|\,|f|^2\,\|^2_{L^{N/(N-\alpha)}}\ .$$

Therefore,

$$\|F\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} \le C_{N,\alpha}\,\|f\|^2_{L^{2N/(N-\alpha)}}$$

and similarly

$$\|(v_2-v_1)F\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} \le C_{N,\alpha}\,\|vf\|_{L^{2N/(N-\alpha)}}\,\|f\|_{L^{2N/(N-\alpha)}}\ .$$

We conclude by applying Proposition 2.2.ii).

3. The general case.

We now concentrate on the case when B is not a tensor product. The estimate is not as straightforward as in Section 2, and we have to make a regularity assumption on B. Moreover, we only treat here the three-dimensional case.

Theorem 3.1. *Let B be a continuous function from $]0,\infty[\times[-1,1]$ to $\mathbb{R}$, admitting a continuous derivative in the second variable. We assume that*

$$(3.1)\qquad |B(x,u)| + \left|\frac{\partial B}{\partial u}(x,u)\right| \le K(1+x)\,,$$

for all $x>0$ and $u\in[-1,1]$. Then, for any $\varepsilon>0$, there exists a constant C_ε only depending on ε such that for any

$$f \in L^1_1(\mathbb{R}^3)\cap L^2_{(3+\varepsilon)/2}(\mathbb{R}^3)\,,$$

$Q^+(f) \in \dot{H}^1(\mathbb{R}^3)$ *with*

$$\|Q^+(f)\|_{\dot{H}^1(\mathbb{R}^3)} \leq C_\varepsilon K \|f\|^2_{L^2_{(3+\varepsilon)/2}} . \tag{3.2}$$

PROOF. Since

$$|B(x,u)| \leq K\,(1+x)\,,$$

the result of Proposition 2.2.i) ensures that the integral (1.2) defining $Q^+(f)$ is absolutely convergent for almost every v, and that $Q^+(f) \in L^1(\mathbb{R}^3)$,

$$\|Q^+(f)\|_{L^1} \leq 4\pi K \|f\|^2_{L^1_1} . \tag{3.3}$$

Therefore, we can compute the Fourier transform of $Q^+(f)$,

$$\begin{aligned}
\widehat{Q^+(f)}(\xi) = & \iiint\limits_{\substack{v,v_*\in\mathbb{R}^3\\ \sigma\in S^2}} e^{-iv\cdot\xi} f\Big(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma\Big) f\Big(\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma\Big) \\
& \qquad \cdot B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, d\sigma\, dv\, dv_* \\
= & \iiint\limits_{\substack{v,v_*\in\mathbb{R}^3\\ \sigma\in S^2}} e^{-i\xi\cdot(v+v_*-|v-v_*|\sigma)/2} f(v) f(v_*) \qquad\qquad (3.4) \\
& \qquad \cdot B\Big(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\, d\sigma\, dv\, dv_*\ ,
\end{aligned}$$

according to the pre-post collisional change of variables. Thus we obtain

$$\widehat{Q^+(f)}(\xi) = \iint_{v,v_*\in\mathbb{R}^3} e^{-i\xi\cdot(v+v_*)/2} f(v) f(v_*)\, D(v-v_*,\xi)\, dv\, dv_*\ , \tag{3.5}$$

where for any $w, \xi \in \mathbb{R}^3 \setminus \{0\}$

$$\begin{aligned}
& D(w,\xi) \\
& \quad = \int\limits_{\sigma\in S^2} e^{i|w|\sigma\cdot\xi/2} B\Big(|w|, \frac{w}{|w|}\cdot\sigma\Big)\, d\sigma \\
& (3.6) \\
& \quad = \int_{s=-1}^{+1} e^{i|w||\xi|s/2} \\
& \qquad \cdot \int_{\varphi=0}^{2\pi} B\Big(|w|, s\,\frac{\xi}{|\xi|}\cdot\frac{w}{|w|} + \sqrt{1-s^2}\,\sqrt{1-\Big(\frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\Big)^2}\,\cos\varphi\Big)\, d\varphi\, ds\,,
\end{aligned}$$

with spherical coordinates and

$$s = \sigma \cdot \frac{\xi}{|\xi|} \, . \tag{3.7}$$

Integrating by parts, we get

$$\begin{aligned} D(w,\xi) = &- \int_{s=-1}^{+1} \frac{2\, e^{i|w||\xi|s/2}}{i\,|w|\,|\xi|} \\ &\cdot \int_{\varphi=0}^{2\pi} \Big(\frac{\xi}{|\xi|}\cdot\frac{w}{|w|} - \frac{s}{\sqrt{1-s^2}}\,\sqrt{1-\Big(\frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\Big)^2}\,\cos\varphi \Big) \\ &\cdot \frac{\partial B}{\partial u}\Big(|w|, s\,\frac{\xi}{|\xi|}\cdot\frac{w}{|w|} \\ &\qquad + \sqrt{1-s^2}\,\sqrt{1-\Big(\frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\Big)^2}\,\cos\varphi\Big)\, d\varphi\, ds \\ &+ \frac{2\, e^{i|w||\xi|/2}}{i\,|w|\,|\xi|}\, 2\pi B\Big(|w|, \frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\Big) \\ &- \frac{2\, e^{-i|w||\xi|/2}}{i\,|w|\,|\xi|}\, 2\pi B\Big(|w|, -\frac{\xi}{|\xi|}\cdot\frac{w}{|w|}\Big) \end{aligned} \tag{3.8}$$

and therefore

$$\begin{aligned} |D(w,\xi)| &\le \frac{4\pi}{|w|\,|\xi|}\, K(1+|w|) \int_{-1}^{+1} \Big(1 + \frac{|s|}{\sqrt{1-s^2}}\Big)\, ds \\ &\quad + \frac{8\pi}{|w|\,|\xi|}\, K(1+|w|) \\ &\le \frac{24\pi}{|\xi|}\, K\Big(1 + \frac{1}{|w|}\Big) \, . \end{aligned} \tag{3.9}$$

Coming back to (3.5) and using the variables

$$z = \frac{v+v_*}{2} \, , \qquad w = v - v_* \, , \tag{3.10}$$

we get

$$\widehat{Q^+(f)}(\xi) = \int\limits_{w\in\mathbb{R}^3} W(f)\,(w,\xi)\, D(w,\xi)\, dw \, , \tag{3.11}$$

where

$$W(f)(w,\xi) = \int\limits_{z\in\mathbb{R}^3} e^{-iz\cdot\xi} f\Big(z+\frac{w}{2}\Big) f\Big(z-\frac{w}{2}\Big)\,dz \tag{3.12}$$

is a Wigner-type transform of f. Then, according to Cauchy-Schwarz's inequality, we get for any $\varepsilon > 0$

$$\begin{aligned} |\widehat{Q^+(f)}(\xi)|^2 \le & \int\limits_{w\in\mathbb{R}^3} |W(f)(w,\xi)|^2 (1+|w|)^{3+\varepsilon}\,dw \\ & \cdot \int\limits_{w\in\mathbb{R}^3} |D(w,\xi)|^2 \frac{dw}{(1+|w|)^{3+\varepsilon}} \\ & \le C_\varepsilon \frac{K^2}{|\xi|^2} \int\limits_{w\in\mathbb{R}^3} |W(f)(w,\xi)|^2(1+|w|)^{3+\varepsilon}\,dw\,. \end{aligned} \tag{3.13}$$

Finally, using Plancherel's identity, we obtain

$$\begin{aligned} & \int\limits_{\xi\in\mathbb{R}^3} |\xi|^2\,|\widehat{Q^+(f)}(\xi)|^2\,d\xi \\ & \le C_\varepsilon K^2 \int\limits_{w\in\mathbb{R}^3} \Big(\int\limits_{\xi\in\mathbb{R}^3} |W(f)(w,\xi)|^2\,d\xi\Big)(1+|w|)^{3+\varepsilon}\,dw \\ & = C_\varepsilon\, K^2 (2\pi)^3 \\ & \quad \cdot \int\limits_{w\in\mathbb{R}^3} \Big(\int\limits_{z\in\mathbb{R}^3} \Big|f\Big(z+\frac{w}{2}\Big) f\Big(z-\frac{w}{2}\Big)\Big|^2\,dz\Big)(1+|w|)^{3+\varepsilon}\,dw \\ & = C_\varepsilon\, K^2(2\pi)^3 \iint\limits_{v,v_*\in\mathbb{R}^3} |f(v)f(v_*)|^2\,(1+|v-v_*|)^{3+\varepsilon}\,dv\,dv_* \\ & \le C_\varepsilon\, K^2(2\pi)^3\, \|f\|^4_{L^2_{(3+\varepsilon)/2}}\,, \end{aligned} \tag{3.14}$$

by the same estimate as in (2.9) and the proof is complete.

REMARK 3.1. As in Section 2, one could here also treat singular B (in the first variable) if one allowed to replace the weighted L^2 norms of f in (3.2) by suitable (weighted) L^p norms, with $p > 2$.

REMARK 3.2. As in [10], one could deduce from Theorems 2.1 and 3.1 regularity properties for the homogeneous Boltzmann equation. Notice that such properties give also counterexamples. For example one can prove that if f is the solution of the homogeneous Boltzmann equation and if $f(0)$ is not smooth (the exact smoothness considered here depends on the properties of B), then for any $t > 0$, $f(t)$ will also not be smooth.

This behavior is completely opposite to that of the Boltzmann equation without angular cutoff (*cf.* [5], [8]).

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Moyenne de localisation fréquentielle des paquets d'ondelettes

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Résumé. En utilisant le théorème de Ruelle d'opérateur de transfert, nous démontrons que la moyenne $2^{-k}\sum_{n=0}^{2^k-1}\|\hat{w}_n\|_{L^1}$ de la localisation fréquentielle pour les paquets d'ondelettes admet un équivalent de la forme $c\,\rho^k$ ($c > 0$, $1 < \rho < \sqrt{2}$). Cela améliore une inégalité antérieurement obtenue par Coifman, Meyer et Wickerhauser. Des estimations numériques de ρ sont obtenues pour des filtres de Daubechies.

1. Enoncé.

On considère un couple de filtres miroirs conjugués en quadrature (m_0, m_1), c'est-à-dire deux fonctions satisfaisant aux conditions suivantes:

C1) m_0 et m_1 sont d'une classe lipschitzienne, 2π-périodiques;

C2) $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$, $m_1(\xi) = e^{-i\xi}\,\overline{m_0(\xi+\pi)}$;

C3) $m_0(0) = 1$, $m_0(\xi) \neq 0$ sur un compact K tel que

$$\bigcup_{j\in\mathbb{Z}}(K + \pi j) = \mathbb{R}.$$

Le produit suivant est alors bien défini

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\Big(\frac{\xi}{2^j}\Big) .$$

C'est la transformée de Fourier d'une fonction d'échelle φ. Les paquets d'ondelettes w_n $(n \geq 0)$ sont définis par leurs transformées de Fourier comme suit

$$\hat{w}_n(\xi) = \Big(\prod_{j=1}^{k} m_{\epsilon_j}\Big(\frac{\xi}{2^j}\Big)\Big)\varphi\Big(\frac{\xi}{2^k}\Big) , \qquad \text{si } \ n = \sum_{j=1}^{k} \varepsilon_j \, 2^{j-1} .$$

Ainsi on a $w_0 = \varphi$; $w_1 = \psi$, l'ondelette associée au couple (m_0, m_1) de filtres miroirs en quadrature [M1].

Pour décrire la localisation fréquentielle de w_n, on fait appel à la variance de $\hat{w}_n$ définie par

$$\sigma_n = \inf_{\omega \in \mathbb{R}} \int_{\mathbb{R}} |\xi - \omega|^2 \, |\hat{w}_n(\xi)|^2 \, d\xi .$$

L'estimation de σ_n s'avère un peu délicate. Cependant on obtient facilement la minoration suivante pour σ_n

$$\|\hat{w}_n\|_{L^1(\mathbb{R})} \leq C\,(1 + \sigma_n)^{1/2} ,$$

où C est une constante indépendante de n. Il est prouvé dans [CMW] qu'il existe une constante $r > 1$ telle que pour tout $k \geq 1$ on ait

$$\frac{1}{2^k} \sum_{n=0}^{2^k - 1} \|\hat{w}_n\|_{L^1(\mathbb{R})} \geq 2\pi \, r^k .$$

Nous nous proposons de démontrer dans cette note le théorème suivant.

Théorème. *Soit* (m_0, m_1) *un couple de filtres miroirs conjugués en quadrature. Supposons que*

$$\Phi(\xi) = \sum_{j \in \mathbb{Z}} |\hat{\varphi}(\xi + 2\pi j)|$$

est continue. Alors il existe deux constantes $1 < \rho < \sqrt{2}$ *et* $c > 0$ *telles qu'on ait l'équivalence*

$$\frac{1}{2^k} \sum_{n=0}^{2^k-1} \|\hat{w}_n\|_{L^1(\mathbb{R})} \sim c\, \rho^k.$$

Les paquets d'ondelettes ont été introduits en traitement et en compression du signal par Coifman, Meyer et Wickerhauser [CMW]. L'idée remonte à l'analyse en temps-fréquence de signal, qui consiste à décomposer les signaux compliqués en signaux élémentaires caractérisés par leurs localisations en temps et en fréquence. On dit q'un signal $f_R(t)$ est localisé dans un rectangle $R = [t_0 - h, t_0 + h] \times [\omega_0 - \pi/h, \omega_0 + \pi/h]$ dans le plan temps-fréquence si

$$\int_{-\infty}^{+\infty} (t - t_0)^2 \, |f_R(t)|^2 \, dt \le K^2 h^2 \,,$$

$$\int_{-\infty}^{+\infty} (\omega - \omega_0)^2 \, |\hat{f}_R(\omega)|^2 \, d\omega \le \frac{2\pi K^2}{h^2} \,.$$

Un tel signal, dit atome temps-fréquence, est considéré comme élémentaire. On souhaite aussi que ces atomes soient orthogonaux. L'exemple le plus célèbre d'atomes temps-fréquence est celui de Gabor qui réalise la meilleure constante K. Mais les ondelettes de Gabor ne sont pas orthogonales, même pas presque orthogonales [S1]. Dans la littérature, il y avait aussi les ondelettes de Malvar. Celles-ci sont orthogonales par construction. Mais la localisation n'est pas satisfaite. D'ailleurs, les ondelettes de Malvar ne sont pas obtenues par translation, changement d'échelles et modulation d'une fonction fixée une fois pour toutes. C'est pour quoi on a introduit les paquets d'ondelettes [M2]. Elles sont orthogonales et obtenues à partir d'une seule fonction par translation, changement d'échelle et modulation. Mais comme le montrent l'inégalité de Coifman-Meyer-Wickerhauser et le théorème énoncé ci dessus, certain paquets d'ondelettes ont une mauvaise localisation temps-fréquence. Signalons que des estimations de la localisation pour un paquet individuel sont obtenues dans [S1].

2. Preuve.

La preuve du théorème se décompose en une série de lemmes. Pour simplifier, notons

$$M_\varepsilon = |m_\varepsilon|\,, \qquad \varepsilon = 0, 1\,, \qquad S = M_0 + M_1\ .$$

Lemme 1. *Soient $q > 0$ un réel, $h(\xi) \geq 0$ une fonction homogène d'ordre τ. Si $n = \sum_{j=1}^{k} \varepsilon_j\, 2^{j-1}$, on a*

$$\|h\hat{w}_n^q\|_{L^1(\mathbb{R})} = 2^{k(1+\tau)} \int_0^{2\pi} \Big(\prod_{j=1}^{k} M_{\varepsilon_j}(2^{j-1}\xi)\Big)^q \Phi_{h,q}(\xi)\, d\xi\,,$$

où

$$\Phi_{h,q}(\xi) = \sum_{j\in\mathbb{Z}} h(\xi + 2\pi j)\, |\hat{\varphi}(\xi + 2\pi j)|^q\,.$$

Preuve. Par le théorème de convergence monotone, on a

$$\|h\hat{w}_n^q\|_{L^1(\mathbb{R})} = \lim_{J\to\infty} \Big\| h(\xi)\Big(\prod_{j=1}^{J} M_{\varepsilon_j}\Big(\frac{\xi}{2^j}\Big)\Big)^q \hat{\varphi}^q\Big(\frac{\xi}{2^J}\Big)\Big\|_{L^1([-2\pi 2^J, 2\pi 2^J])}\ .$$

Or, si $J \geq k$, l'intégrale à droite est égale à

$$\begin{aligned}
&\int_{-2\pi 2^J}^{2\pi 2^J} h(\xi)\Big(\prod_{j=1}^{k} M_{\varepsilon_j}\Big(\frac{\xi}{2^j}\Big)\Big)^q \Big|\hat{\varphi}\Big(\frac{\xi}{2^k}\Big)\Big|^q d\xi \\
&\qquad = 2^{k(1+\tau)} \int_{-2\pi 2^{J-k}}^{2\pi 2^{J-k}} \Big(\prod_{j=1}^{k} M_{\varepsilon_j}(2^{j-1}y)\Big)^q h(y)\, |\hat{\varphi}(y)|^q\, dy \\
&\qquad = 2^{k(1+\tau)} \int_0^{2\pi} \Big(\prod_{j=1}^{k} M_{\varepsilon_j}(2^{j-1}y)\Big)^q \\
&\qquad\qquad \cdot \sum_{j=-2^{J-k}}^{2^{j-k}-1} h(y + 2\pi j)\, |\hat{\varphi}(y + 2\pi j)|^q\, dy\,.
\end{aligned}$$

Pour obtenir la première égalité on a fait le changement de variable $\xi = 2^k y$ et utilisé l'homogénéité de h. Pour obtenir la seconde on a utilisé la périodicité de $M_\varepsilon(2^j y)$. Afin de terminer la preuve il suffit d'appliquer encore une fois le théorème de convergence monotone.

Le Lemme 1 est énoncé sous une forme un peu plus générale que ce dont on a besoin. En fait, le choix $q = 1$ et $h = 1$ sera suffisant pour déduire le lemme suivant.

Lemme 2. *Pour tout $k \geq 1$, on a*

$$\frac{1}{2^k} \sum_{n=0}^{2^k-1} \|\hat{w}_n\|_{L^1(\mathbb{R})} = \int_0^{2\pi} \prod_{j=1}^{k} S(2^{j-1}\xi)\, \Phi(\xi)\, d\xi\,.$$

Preuve. Il suffit de remarquer que

$$\sum_{\varepsilon_1,\ldots,\varepsilon_k} \prod_{j=1}^{k} M_{\varepsilon_j}(2^{j-1}\xi) = \prod_{j=1}^{k} S(2^{j-1}\xi)\,.$$

Le Lemme 3 va établir une relation entre l'intégrale à droite dans le Lemme 2 et un opérateur de transfert, qui est défini par

$$L_S f(\xi) = S\Big(\frac{\xi}{2}\Big) f\Big(\frac{\xi}{2}\Big) + S\Big(\frac{\xi}{2}+\pi\Big) f\Big(\frac{\xi}{2}+\pi\Big)\,.$$

Lemme 3. *Pour toute fonction Φ définie sur $[0, 2\pi]$, on a*

$$2^k \int_0^{2\pi} \prod_{j=1}^{k} S(2^{j-1}\xi)\, \Phi(\xi)\, d\xi = \int_0^{2\pi} L_S^k \Phi(\xi)\, d\xi\,.$$

Preuve. On prouve l'égalité pour $n = 1$. Le cas général se demontre par récurrence.

$$\begin{aligned}
\int_0^{2\pi} L_S\Phi(\xi)\, d\xi &= \int_0^{2\pi} S\Big(\frac{\xi}{2}\Big)\Phi\Big(\frac{\xi}{2}\Big)\, d\xi + \int_0^{2\pi} S\Big(\frac{\xi}{2}+\pi\Big)\Phi\Big(\frac{\xi}{2}+\pi\Big)\, d\xi \\
&= 2\int_0^{\pi} S(y)\, \Phi(y)\, dy + 2\int_\pi^{2\pi} S(z)\, \Phi(z)\, dz \\
&= 2\int_0^{2\pi} S(x)\, \Phi(x)\, dx\,.
\end{aligned}$$

Pour la seconde égalité, on a fait les changements de variables $\xi/2 = y$ et $\xi/2 + \pi = z$.

PREUVE DU THÉORÈME. L'opérateur de transfert $L_S : C(\mathbb{T}) \to C(\mathbb{T})$ est bien défini et borné, où $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. On note λ son rayon spectral. Comme S est strictement positive et lipschitzienne, d'après le théorème d'opérateur de transfert dû à Ruelle [R] (voir aussi [B], [F] ,[W]), il existe une mesure de probabilité ν et une fonction continue $\ell > 0$ telles que, pour toute fonction continue f, $\lambda^{-k} L_S^k f$ converge uniformément vers $\langle f, \nu \rangle\, \ell$ où $\langle f, \nu \rangle$ désigne l'intégrale de f par rapport à ν. De plus, on sait que ν est diffuse, singulière et de support $[0, 2\pi]$. Par conséquent,

$$\lambda^{-k} \int_0^{2\pi} L_S^k \Phi(\xi)\, d\xi \longrightarrow \langle \Phi, \nu \rangle \, \langle \ell, \nu \rangle = c > 0\,,$$

d'où l'équivalence dans le théorème avec $\rho = \lambda/2$. Il reste à expliquer que $1 < \rho < \sqrt{2}$, ou bien $\log 2 < \log \lambda < 3 \log 2/2$. Or, $\log \lambda$ est la pression de $\log S$ sous la transformation $T(x) = 2x \pmod{2\pi}$. Rappelons le principe variationnel [B], [R], [W]

$$\log \lambda = \sup \Big(H(\mu) + \int_0^{2\pi} \log S(\xi)\, d\mu(\xi) \Big)\,,$$

où le sup est pris sur l'ensemble des mesures T-invariantes; il est atteint uniquement en $\mu_S = \ell\, \nu$, $H(\mu)$ désignant l'entropie de μ. Alors, comme $S(\xi) \leq \sqrt{2}$, on a

$$\log \lambda = h_{\mu_S} + \int_0^{2\pi} \log S(\xi)\, d\mu_S(\xi) \leq h_{\mu_S} + \frac{1}{2} \log 2 < \log 2 + \frac{1}{2} \log 2\,.$$

Pour la dernière inégalité on a utilisé le fait que l'entropie maximale de T est égale à $\log 2$ et que la mesure de Lebesgue est l'unique mesure d'entropie maximale. Notons μ_0 la mesure de Lebesgue. Comme $S(\xi) \geq 1$, on a

$$\log \lambda > h_{\mu_0} + \int_0^{2\pi} \log S(\xi)\, d\mu_0(\xi) \geq h_{\mu_0} = \log 2\,.$$

3. Filtres de Daubechies.

On considère les filtres de Daubechies de la forme suivante [D1], [D2], qui dépendent d'un paramètre entier $N \geq 1$ [D1], [D2]. Au lieu d'écrire m_0, on écrit $m_{0,N}$, qui est défini par

$$m_{0,N}(\xi) = \Big(\frac{1}{2}\,(1+e^{i\xi})\Big)^N Q_N(\xi)\,,$$

où Q_N est un polynôme tel que

$$|Q_N(\xi)|^2 = \sum_{k=0}^{N-1} \binom{N-1+k}{k} \sin^{2k}\Big(\frac{\xi}{2}\Big)\,.$$

Pour $N \geq 1$, notons ρ_N le ρ correspondant dans le théorème. Numériquement, on a

$$\begin{aligned} \rho_2 &= 1,20947\,, & \rho_3 &= 1,17270\,,\\ \rho_4 &= 1,14924\,, & \rho_5 &= 1,13283\,,\\ \rho_6 &= 1,12062\,, & \rho_7 &= 1,11114\,,\\ \rho_8 &= 1,10354\,, & \rho_9 &= 1,09730\,. \end{aligned}$$

La méthode de calcul numérique se trouve dans [FL]. Il s'agit d'approcher la fonction S par une fonction en escalier dont le rayon spectral de l'opérateur associé est celui d'une matrice.

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Asymptotic behavior of global solutions to the Navier-Stokes equations in $\mathbb{R}^3$

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Abstract. We construct global solutions to the Navier-Stokes equations with initial data small in a Besov space. Under additional assumptions, we show that they behave asymptotically like self-similar solutions.

0. Introduction.

When studying global solutions to an evolution problem, it is natural to study their asymptotic behavior, as it is usually a simpler way to describe the long term behavior than the solution itself. Global solution of the non-linear heat equation have been showed to be asymptotically close to self-similar solutions [7]. Under certain conditions, we will show how to obtain similar results for the incompressible Navier-Stokes system.

We recall the equations

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \otimes u) - \nabla p \,, \\ \nabla \cdot u = 0 \,, \\ u(x,0) = u_0(x) \,, \qquad x \in \mathbb{R}^3 \,, \ t \geq 0 \,. \end{cases}$$

As we are in the whole space, if $u(x,t)$ is a solution of (1), then for all $\lambda > 0$, $u_\lambda(x,t) = \lambda\, u(\lambda\, x, \lambda^2\, t)$ is also a solution.

We now note that studying the asymptotic behavior of $u(x,t)$ for large time is equivalent to studying the asymptotic behavior of $u_\lambda(x,t)$ for large λ with fixed time. Actually, we shall show that, as t goes to ∞, the natural space scale is $\sqrt{t}$ as in the heat equation. If we replace x by $x/\sqrt{t}$ and let $t \longrightarrow \infty$, we obtain the same result as if we let $\lambda \longrightarrow \infty$ in $u_\lambda(x,t)$. This new point of view is interesting for the following heuristic reason: we expect that the limit $v(x,t)$ of $u_\lambda(x,t)$ will also be a solution of (1). Furthermore, one might assume that $v(x,t)$ is the solution with initial data $v_0(x) = \lim_{\lambda\to\infty} \lambda\, u(\lambda\, x, 0)$. Of course, the limiting solution is invariant under the scaling, so

$$v(x,t) = \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big)\,,$$

and $v_0(x)$ is an homogeneous function of degree -1.

Such self-similar solutions have been studied previously (see [4], [2]), and we shall see in the present work how to make rigorous the previous heuristic approach.

Let us define the projection operator $\mathbb{P}$ onto the divergence free vector fields

$$\mathbb{P}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \begin{pmatrix} R_1\sigma \\ R_2\sigma \\ R_3\sigma \end{pmatrix}\,, \tag{2}$$

where R_j is the Riesz transform of symbol

$$\sigma_{R_j}(\xi) = \frac{\xi_i}{|\xi|}\,, \tag{3}$$

and where

$$\sigma = R_1\, u_1 + R_2\, u_2 + R_3\, u_3\ . \tag{4}$$

Therefore $\mathbb{P}$ is a pseudo-differential operator of order 0.

We transform the system (1) into an integral equation, where $S(t) = e^{t\Delta}$ denotes the heat kernel,

$$u(x,t) = S(t)\, u_0(x) - \int_0^t \mathbb{P}\, S(t-s)\, \nabla\cdot(u\otimes u)(x,s)\, ds\,. \tag{5}$$

This equation can be solved by a classical fixed point method (see [1], [5], [6]). Following the method of [1], we remark that the bilinear term in the previous equation can be reduced to a scalar operator

$$B(f,g)=\int_0^t \frac{1}{(t-s)^2}\,G\Big(\frac{\cdot}{\sqrt{t-s}}\Big)*(fg)\,ds\,, \tag{6}$$

where G is analytic, such that

$$|G(x)|\leq \frac{C}{1+|x|^4}\,, \tag{7}$$

$$|\nabla G(x)|\leq \frac{C}{1+|x|^4}\,. \tag{8}$$

This comes easily from the study of the symbol of B, as we have an exact expression under the integral. The matrix of this pseudo-differential operator has components like

$$-\frac{\xi_j\,\xi_k\,\xi_l}{|\xi|^2}\,e^{-t|\xi|^2} \tag{9}$$

off the diagonal, with an additional term $\xi_j\,e^{-t|\xi|^2}$ on it. The function G is then the inverse Fourier transform of any of these functions at $t=1$. The only thing we will need is that $G\in L^1\cap L^\infty$.

This paper is organized as follows. In a first part, we will define the functional setting which is well-suited for our study, then study global existence in this setting, and lastly the behavior of attracting solutions for large time, if they exist. Then in a second part, we will try to state a partial converse to the Theorem 3, that is a condition on the initial data in order to obtain a convergence to a self-similar solution for large time. The third part will be devoted to a better understanding of this condition, and will include reformulations of the condition and examples.

1. Global existence in Besov spaces.

A well suited functional space to study (1) is L^3 ([5]), as $\|u_\lambda\|_{L^3}=\|u\|_{L^3}$. But homogeneous functions of degree -1 are not in L^3, and we easily see that the weak limit of $u_{0,\lambda}$ is 0. We therefore have to enlarge this functional space to include homogeneous functions of degree -1. We have chosen the homogeneous Besov spaces $\dot{B}_p^{-(1-3/p),\infty}$. We will see later they arise naturally in our problem. Let us recall their definition ([9], [10]).

Definition 1. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\phi} \equiv 1$ in $B(0,1)$ and $\widehat{\phi} \equiv 0$ in $B(0,2)^c$, $\phi_j(x) = 2^{nj}\,\phi(2^j\,x)$, $S_j = \phi_j * \cdot$, $\Delta_j = S_{j+1} - S_j$. Let f be in $\mathcal{S}'(\mathbb{R}^n)$.*

- *If $s < n/p$, or if $s = n/p$ and $q = 1$, f belongs to $\dot{B}_p^{s,q}$ if and only if the following two conditions are satisfied*
  - *The partial sum*
  $$\sum_{-m}^{m} \Delta_j(f)$$
  converge to f for the topology $\sigma(\mathcal{S}', \mathcal{S})$.
  - *The sequence $\varepsilon_j = 2^{js}\|\Delta_j(f)\|_{L^p}$ belongs to ℓ^q.*
- *If $s > n/p$, or $s = n/p$ and $q > 1$, let us denote $m = E(s - n/p)$. Then $\dot{B}_p^{s,q}$ is the space of distributions f, modulo polynomials of degree less than $m+1$, such that*
  - *We have $f = \sum_{-\infty}^{\infty} \Delta_j(f)$ for the quotient topology.*
  - *The sequence $\varepsilon_j = 2^{js}\,\|\Delta_j(f)\|_{L^p}$ belongs to ℓ^q.*

We remark that nothing in this definition restricts s from being negative. In fact, we will use $s = -(1 - 3/p)$ which is indeed negative as $p > 3$. In the particular case where $s < 0$, it is worth noting that we can replace the condition $\varepsilon_j = 2^{js}\,\|\Delta_j(f)\|_{L^p} \in \ell^q$ by the equivalent condition $\tilde{\varepsilon}_j = 2^{js}\,\|S_j(f)\|_{L^p} \in \ell^q$. This second condition implies easily the first one, and conversely, we remark that $\tilde{\varepsilon}_j$ can be seen as a convolution between ε_j and $\eta_j = 2^{sj} \in \ell^1$. We shall obtain the following theorem which extends the results of [1].

Theorem 1. *There exists a positive function $\eta(q)$, $q > 3$ such that if $u_0 \in B_p^{-(1-3/p),\infty}$, $\nabla \cdot u_0 = 0$, $p \geq 3$, satisfies*

$$\|u_0\|_{B_q^{-(1-3/q),\infty}} < \eta(q)\,, \tag{10}$$

for a fixed $q > p$, then there exists a unique solution of (1) *such that*

$$u \in C_w([0+\infty), \dot{B}_p^{-(1-3/p),\infty})\,, \tag{11}$$

where C_w *denotes the weakly continuous functions, and, if* $p \leq 6$ *and* $u = S(t)\,u_0 + w(x,t)$*, then*

$$w \in L^\infty([0+\infty), L^3(\mathbb{R}^3)) \tag{12}$$

and

$$\|w\|_{L^3} < \gamma(q)\,, \tag{13}$$

where $\gamma(q)$ *depends only of* $\eta(q)$*.*

We remark that the restriction $p \leq 6$ in order to obtain (12) is merely due to the linear part: the equivalent of (12) actually holds for $p > 6$ if one considers higher order terms, if u is written as a sum of multilinear operators of u. For the sake of simplicity, we restrict ourselves to the first term, which yields this restriction.

We will prove the Theorem 1, using a fixed point argument via the following abstract lemma (Picard's theorem in a Banach space).

Lemma 1. *Let* $\mathcal{E}$ *be a Banach space,* B *a continuous bilinear application,* $x, y \in \mathcal{E}$

$$\|B(x,y)\|_{\mathcal{E}} \leq \gamma\, \|x\|_{\mathcal{E}}\, \|y\|_{\mathcal{E}}\,. \tag{14}$$

Then, if $4\,\gamma\,\|x_0\|_{\mathcal{E}} < 1$*, the sequence defined by*

$$x_{n+1} = x_0 + B(x_n, x_n)$$

converges to $x \in \mathcal{E}$ *such that*

$$x = x_0 + B(x,x) \qquad \textit{and} \qquad \|x\|_{\mathcal{E}} < \frac{1}{2\,\gamma}\,. \tag{15}$$

Let us define the space

$$F_q = \{f(x,t) : \sup_{t>0} \|f(x,t)\|_{L^q} < +\infty\}\,. \tag{16}$$

The following characterization will be very useful.

Proposition 1. *Take* $\alpha > 0$*,* $\gamma \geq 1$*,* $f \in \mathcal{S}(\mathbb{R}^n)$*, then*

$$\|f\| = \sup_{t>0} t^{\alpha/2}\, \|S(t) f\|_{L^\gamma} \tag{17}$$

is a norm in $\dot{B}_{\gamma}^{-\alpha,\infty}$ *equivalent to the usual dyadic one.*

Therefore, using the Sobolev inclusion

$$\dot{B}_p^{3/p-1,\infty} \hookrightarrow \dot{B}_q^{3/q-1,\infty}\,,$$

for $p \le q$, we see that $u_0 \in \dot{B}_q^{3/q-1,\infty}$, so that

$$\sqrt{t}\,\left(S(t)\,u_0\right)(\sqrt{t}\,x) \in F_q\,.$$

Then, in order to apply Lemma 1 to F_q, we are left to prove that if

$$D_t f = \sqrt{t}\, f(\sqrt{t}\,x, t)\,,$$

then $D_t B(D_t^{-1}\cdot, D_t^{-1}\cdot)$ is bicontinuous on F_q. Take $\tilde{f} = D_t f$ and $\tilde{g} = D_t g$ in F_q. We denote $M = \tilde{f}\,\tilde{g} \in F_{q/2}$. We observe that the bilinear operator (renormalized with D_t) can be written as follows

$$\widetilde{B}(\tilde{f},\tilde{g}) = \int_0^1 \frac{1}{(1-\lambda)^2}\, G\Big(\frac{x}{\sqrt{1-\lambda}}\Big) * M\Big(\frac{x}{\sqrt{\lambda}}, \lambda\, t\Big)\,\frac{d\lambda}{\lambda}\,.$$

Then, by Hölder and Young inequalities, we obtain

$$\|\widetilde{B}(\tilde{f},\tilde{g})\|_{F_q} \le \int_0^1 \frac{C\,d\lambda}{(1-\lambda)^{1/2+3/(2q)}\lambda^{1-3/q}}\, \|\tilde{f}\|_{F_q}\, \|\tilde{g}\|_{F_q}\,, \tag{18}$$

which gives us $\eta(q)$. Proceeding the same way, if $p \le 6$ gives

$$\|\widetilde{B}(\tilde{f},\tilde{g})\|_{F_3} \le \int_0^1 \frac{C\,d\lambda}{(1-\lambda)^{3/q}\lambda^{1-3/q}}\, \|\tilde{f}\|_{F_q}\, \|\tilde{g}\|_{F_q}\,. \tag{19}$$

This proves (12) and (13). We have now to prove the weak convergence when $t \to 0$. Clearly $S(t)\,u_0 \underset{t\to 0}{\rightharpoonup} u_0$ by a duality argument. As for the bilinear term, if $\phi \in C_0^\infty(\mathbb{R}^3)$ and if we denote by $Q(\theta)$ the convolution operator with $G(\,\cdot\,/\sqrt{\theta})/\theta^2$,

$$\langle Q(t-s) f g(s), \phi\rangle = \langle f g(s), S(t-s)\,\widetilde{Q}\,\phi\rangle,$$

where $\widetilde{Q}$ is defined by

$$\widehat{\widetilde{Q}\phi}(\xi) = \frac{\xi_j\,\xi_k\,\xi_\ell}{|\xi|^2}\,\widehat{\phi}(\xi)\,,$$

so that $\widetilde{Q}\phi \in L^1$, like the function G defined previously. Therefore $S(t-s)\,\widetilde{Q}\phi$ is (uniformly in $t-s$) in L^γ, with $1/\gamma + 2/q = 1$. Thus

$$\begin{aligned}
&\Big|\Big\langle \int_0^t Q(t-s)\, fg(s)\, ds, \phi \Big\rangle\Big| \\
&\qquad \le C \int_0^t \|fg(s)\|_{L^{q/2}}\, ds && (20)\\
&\qquad \le C \int_0^t \frac{ds}{s^{1-3/q}}\, \|\tilde f\|_{F_q}\, \|\tilde g\|_{F_q} && (21)\\
&\qquad \le C\, t^{3/q} \longrightarrow 0\ . && (22)
\end{aligned}$$

The uniqueness part of the theorem follows from the construction part, so we have proved the Theorem 1, in the case where $p = q$, with q for which (10) is verified. We next remark that the solution u actually satisfies

$$\sqrt{t}\, u(\sqrt{t}\, x, t) \in F_{q'}\ , \qquad \text{for all } q' \ge p,\ q > 3\,, \tag{23}$$

and that moreover the bilinear term w satisfies

$$\sqrt{t}\, w(\sqrt{t}\, x, t) \in F_{q'}\ , \qquad \text{for } \frac{p}{2} < q' \le p\,. \tag{24}$$

(23) is of course true for the linear part. Then, the bilinear term is in $F_{p/2}$ and in F_q for the particular q we have fixed. And by interpolation between $F_{p/2}$ and F_q it is in all $F_{q'}$ with $p/2 < q' < q$. We are left to prove (23) for the bilinear term when $q' > q$. An easy modification of (18) takes care of this situation

$$\|\widetilde{B}(\tilde f, \tilde g)\|_{F_{q'}} \le \int_0^1 \frac{C\, d\lambda}{(1-\lambda)^{1/2+3/q-3/(2q')}\, \lambda^{1-3/q}}\, \|\tilde f\|_{F_q}\, \|\tilde g\|_{F_q} \tag{25}$$

and if $q > 6$ we get all the $q' > q$. Otherwise, we have to proceed in several steps to reach a value $q' > 6$. Note that the great amount of flexibility provided by inequalities of type (18), (25) allows us to obtain this result in many different ways. In particular, we could establish the bicontinuity of the renormalized operator from $F_q \times F'_q$ to F'_q and carry along the fixed point iterations all the properties we want, provided the different continuity constants verify inequalities in the correct way, which happens to be the case. By the way, we remark that initial data in the the space $L^{3,\infty}$ are included. In fact, we have the following embedding,

Theorem 2.
$$L^{3,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}_p^{-(1-3/p),\infty}\,,$$
for all $p > 3$.

In order to prove this, we will make use of the following characterization of weak Lebesgue spaces

$$f \in L^{3,\infty} \qquad \text{if and only if} \qquad \int_E |f(x)|\,dx \leq C\,|E|^{2/3}\,,$$

for all Borel sets E. In particular, if $\varphi \in \mathcal{S}$ then $\varphi * f \in L^\infty$, and therefore is in L^p, for all $p > 3$. In fact $\varphi * f \in L^{3,\infty}$, and all bounded functions in $L^{3,\infty}$ are also in L^p, as the following estimate shows

$$\sum_{j\geq 0} 2^{-jp}\,|\{x:\ 2^{-j} \leq |g| \leq 2^{-j+1}|\} \leq C \sum_{j\geq 0} 2^{j(3-p)} < +\infty\,.$$

Thus,

$$\begin{aligned} S_j(f) &= 2^{3j}\int \varphi(2^j x - 2^j y)\,f(y)\,dy \\ &= \int \varphi(2^j x - y)\,f(2^{-j}y)\,dy \\ &= 2^j \int \varphi(2^j x - y)\,2^{-j} f(2^{-j}y)\,dy \\ &= 2^j\,h(2^j x)\,. \end{aligned}$$

Also, as h and f have the same norm in $L^{3,\infty}$, we obtain

$$\|S_j(f)\|_{L^p} \leq 2^{1-3/p}\,\|f\|_{L^{3,\infty}}\,,$$

which achieves the proof.

Now that we have solutions in the proper functional setting, we can study the asymptotic behavior of these solutions. We begin with a definition:

Definition 2. *We say that $u(x,t)$ "converges in L^p norm" to a function $V(x)$ if and only if one of the two equivalent conditions is satisfied*:

1) *For all compact intervals $[a,b] \subset (0+\infty)$*

$$u_\lambda(x,t) \overset{L^p(dx)}{\longrightarrow} \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big)\,, \qquad \text{as } \lambda \to \infty\,,$$

uniformly for $t \in [a,b]$

2) $\sqrt{t}\, u(\sqrt{t}\, x, t) \overset{L^p(dx)}{\longrightarrow} V(x)$, *as* $t \longrightarrow \infty$.

Then we will show the following

Theorem 3. *Let us take $3 < p < +\infty$. Let $u(x,t)$ be a solution of* (1) *such that*

$$\sup_{t>0} \|\sqrt{t}\, u(\sqrt{t}\, x, t)\|_{L^p} < +\infty \tag{28}$$

and

$$u(x,t) \text{ converges weakly to } u_0(x) \text{ when } t \longrightarrow 0\,. \tag{29}$$

If

$$u \text{ "converges in } L^p \text{ norm" to } V\,, \tag{30}$$

then the initial data $u_0(x)$ belongs to $B_p^{-(1-3/p),\infty}$, $V(x/\sqrt{t})/\sqrt{t}$ is a self-similar solution of (1), *and*

$$S(t)\, u_0 \text{ "converges in } L^p \text{ norm" to } v_1(x)\,, \tag{31}$$

where $v_1(x) = S(1)\, v_0$, and v_0 is the initial data of the self-similar solution.

Note that we did not make any smallness assumption on the initial data. In other respects, when $u_0 \in B_p^{-(1-3/p),\infty}$, the condition (31) implies that

$$\lambda\, u_0(\lambda\, x) \text{ converges weakly to } v_0 \text{ when } \lambda \longrightarrow 0\,, \tag{32}$$

but this is not equivalent, and we postpone the discussion on that matter to Section 3. We recall that the integral equation is

$$\begin{aligned}\sqrt{t}\, u(\sqrt{t}\, x, t) \\ &= \sqrt{t}\, (S(t)\, u_0)(\sqrt{t}\, x) - \int_0^t \mathbb{P} D_t(S(t-s)\, \nabla \cdot u \otimes u(s))\, ds\,.\end{aligned}$$

Let us denote $U(t) = \sqrt{t}\, u(\sqrt{t}\, x, t)$. Then we have

$$U(t) = \sqrt{t}\, (S(t)\, u_0)(\sqrt{t}\, x) - \widetilde{B}(U, U)(t)\,,$$

where we still use the usual notation for the bilinear operator. By hypothesis

$$M = U \otimes U \overset{L^{p/2}}{\longrightarrow} N = V \otimes V\,.$$

We consider the difference

$$\Delta_t(x) = \int_0^1 \frac{1}{(1-\lambda)^2}\, G\Big(\frac{x}{\sqrt{1-\lambda}}\Big) * \Big(M\Big(\frac{x}{\sqrt{\lambda}}, \lambda\, t\Big) - N\Big(\frac{x}{\sqrt{\lambda}}\Big)\Big) \frac{d\lambda}{\lambda}\,, \tag{34}$$

and we want to estimate the L^p-norm. Let

$$\omega(t) = \|M(x,t) - N(x)\|_{L^{p/2}}\,,$$

so

$$\Big\|M\Big(\frac{x}{\sqrt{t}}, \lambda t\Big) - N\Big(\frac{x}{\sqrt{\lambda}}\Big)\Big\|_{p/2} = \lambda^{3/p}\, \omega(\lambda\, t)\,,$$

and therefore,

$$\|\Delta_t(x)\|_{L^p} \le C \int_0^1 \frac{\omega(\lambda\, t)\, d\lambda}{(1-\lambda)^{1/2+3/(2p)}\, \lambda^{1-3/p}}\,. \tag{35}$$

We know that $\omega(t)$ is bounded, and

$$(1-\lambda)^{-1/2-3/(2p)}\, \lambda^{3/p-1} \in L^1(0,1)\,,$$

when $p > 3$, so we can apply the Lebesgue theorem and obtain

$$\lim_{t \longrightarrow \infty} \|\Delta_t(x)\|_{L^p} = 0\,.$$

Therefore, the bilinear term becomes

$$\frac{1}{\sqrt{t}}\, W\Big(\frac{x}{\sqrt{t}}\Big) + o(1)\,,$$

with

$$W(x) = \int_0^1 \frac{1}{(1-\lambda)^2}\, G\Big(\frac{x}{\sqrt{1-\lambda}}\Big) * N\Big(\frac{x}{\sqrt{\lambda}}\Big)\, \frac{d\lambda}{\lambda}\,.$$

The equation (33) can be written as

$$V(x) = t^{1/2}\,(S(t)\,u_0)(\sqrt{t}\,x) - W(x) + o(1)\,. \tag{36}$$

We see that the Fourier transform of $\sqrt{t}\,(S(t)\,u_0)(\sqrt{t}\,x)$ is

$$\frac{1}{t}\, e^{-|\xi|^2}\, \hat{u}_0\Big(\frac{\xi}{\sqrt{t}}\Big)\,,$$

which converges in $\mathcal{F}L^p$ to a distribution. Therefore, $\hat{u}_0(\xi/\sqrt{t})/t$ converges weakly to $\tilde{v}_0(\xi)$. On other hand, by means of (35) and (36),

$$\big\|\sqrt{t}\,(S(t)\,u_0)(\sqrt{t}\,x)\big\|_{L^p} \le C < +\infty\,, \qquad \text{for all } t > 0\,. \tag{37}$$

Hence,

$$\sup_{t>0} t^{1/2-3/(2p)}\, \|S(t)\,u_0\|_{L^p} \le C\,, \tag{38}$$

which is equivalent to $u_0 \in B_p^{3/p-1,\infty}$. Then for all λ, $u_{0,\lambda} \in B_p^{3/p-1,\infty}$, and

$$\|u_{0,\lambda}\|_{\dot{B}_p^{3/p-1,\infty}} = \|u_0\|_{\dot{B}_p^{3/p-1,\infty}}\,,$$

so that we can extract a subsequence which converges to v_0 in the space of tempered distribution, actually the convergence is in the sense of the topology $\sigma(\dot{B}_p^{3/p-1,\infty}, \dot{B}_p^{1-3/p,1})$. Then because the limit is unique, we have that $\hat{v}_0 = \tilde{v}_0$, and the whole sequence converges weakly to v_0, and moreover $v_0(x)$ belongs to $\dot{B}_p^{3/p-1,\infty}$. We remark that v_0 is necessarily homogeneous of degree -1. Let us prove that V is actually a solution of (1) where u_0 has been replaced by v_0. The set $(u_\lambda)_\lambda$ satisfies the estimates (28) and (38) uniformly in λ and indeed, for fixed $t > 0$,

$$\lambda\, u(\lambda\, x, \lambda^2\, t) \underset{\lambda\to+\infty}{\overset{L^p}{\longrightarrow}} \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big)\,.$$

Therefore, if we pass to the limit in the equation (5) which is satisfied by u_λ, we obtain

$$\frac{1}{\sqrt{t}} V\Big(\frac{x}{\sqrt{t}}\Big) = S(t)\, v_0 - \int_0^t \mathbb{P}S(t-s)\nabla \cdot V(s) \otimes V(s)\, ds\,. \tag{39}$$

We see that

$$\lim_{t\downarrow 0} \frac{1}{\sqrt{t}} V\Big(\frac{x}{\sqrt{t}}\Big) = v_0$$

weakly, which can be obtained in the same way as in the proof of Theorem 1.

2. Initial data and asymptotic convergence.

Theorem 3 was the easy part of the study. In some sense, if we have a convergence to a function, then this function must be a self-similar solution whose initial data is obtained in a natural way from the initial data, namely the weak limit of the rescaled initial data. It would be nice if the existence of such a weak limit was enough to ensure convergence toward a self-similar solution. Unfortunately, it is untrue, and this is the purpose of Proposition 4 to explain why. Nevertheless, we can obtain a necessary and sufficient condition in order to obtain this converse to the Theorem 3. We have seen in the first theorem that it is useful to see the solution $u(x,t)$ as the sum of two terms $u(x,t) = S(t)\, u_0 + w(x,t)$, the heat term which gives a tendency, and the bilinear term which is some sort of fluctuation, more regular than the linear term. We will do the same for the self-similar solution, so that

$$v(x,t) = \frac{1}{\sqrt{t}} V\Big(\frac{x}{\sqrt{t}}\Big) = S(t)\, v_0 + \frac{1}{\sqrt{t}} W\Big(\frac{x}{\sqrt{t}}\Big)\,.$$

Theorem 4. *Let u_0 be in $\dot{B}_p^{3/p-1,\infty}$, $\nabla \cdot u_0 = 0$, $3 \le p < +\infty$, such that for some $q > p$,*

$$\|u_0\|_{\dot{B}_q^{3/q-1,\infty}} < \eta(q)\,.$$

Moreover, suppose that there exists r, $r \ge p$ and $r > 3$, such that

$$S(t)\, u_0 \text{ “converges in } L^r \text{ norm” to } v_1(x)\,. \tag{40}$$

Then $\lambda\, u_0(\lambda x)$ converges weakly to a function v_0 such that $v_1 = S(1)\, v_0$. Further, if $u(x,t)$ is the solution of (1) with initial data u_0, $V(x/\sqrt{t})/\sqrt{t}$ is the solution with initial data v_0,

$$\lim_{t\to\infty} t^{1/2-3/(2\tilde{q})} \Big\| u(x,t) - \frac{1}{\sqrt{t}}\, V\Big(\frac{x}{\sqrt{t}}\Big)\Big\|_{L^{\tilde{q}}} = 0\,, \tag{41}$$

for all $\tilde{q} \geq p$, $\tilde{q} > 3$ and, if $p \leq 6$

$$\lim_{t\to\infty} \Big\| w(x,t) - \frac{1}{\sqrt{t}} W\Big(\frac{x}{\sqrt{t}}\Big)\Big\|_{L^3} = 0\,. \tag{42}$$

We remark first that the case $u_0 \in L^3$ leads to $v_1 = 0$, so that $v_0 = V = 0$. In this case, (41) and (42) become the usual estimates (see [5]). Therefore, we shall assume that $r > 3$. We easily see that the convergence (40) is in $L^{\tilde{q}}$, $\tilde{q} > p$ (and even $\tilde{q} = p$ if $p > 3$). In fact $\sqrt{t}\ (S(t)\, u_0)(\sqrt{t}\ x)$ is bounded for the norm $\|\cdot\|_{L^{\tilde{q}}}$, for all $\tilde{q} > p$, as $\dot{B}_p^{3/p-1,\infty} \hookrightarrow \dot{B}_{\tilde{q}}^{3/\tilde{q}-1,\infty}$. Therefore, we conclude by interpolation between L^p and L^r norms or between L^r and L^∞.

We obtained

Lemma 2. *Let $f \in \dot{B}_p^{3/p-1,\infty}$, $p > 3$, such that for some $r \geq p$,*

$$\lim_{t\longrightarrow\infty} t^{1/2-3/(2r)}\, \|S(t)\, f\|_{L^r} = 0\,.$$

Then, for all $\tilde{q} \geq p$

$$\lim_{t\to\infty} t^{1/2-3/(2\tilde{q})}\, \|S(t)\, f\|_{L^{\tilde{q}}} = 0\,. \tag{43}$$

From the proof of the Theorem 3, we already know that v_0, which is the weak limit of $u_{0,\lambda}$, belongs to the same Besov spaces as u_0. Therefore,

$$\|v_0\|_{\dot{B}_q^{3/q-1,\infty}} = \|u_0\|_{\dot{B}_q^{3/q-1,\infty}} < \eta(q)\,.$$

Furthermore we obtain the solutions $u(x,t)$ and $V(x/\sqrt{t})/\sqrt{t}$ by applying the Theorem 1, which used a fixed point argument. If we denote by $u^{(n)}$, respectively $V^{(n)}$, the successive approximations of u, respectively V, we remark that

$$u^{(1)}(x,t) = S(t)\, u_0\ ,$$

respectively
$$\frac{1}{\sqrt{t}}\,V^{(1)}\Big(\frac{x}{\sqrt{t}}\Big)=\frac{1}{\sqrt{t}}\,(S(1)\,v_0)\Big(\frac{x}{\sqrt{t}}\Big)\,.$$
If we recall that

$$u^{(n+1)}(x,t)=S(t)\,u_0-\int_0^t \mathbb{P}S(t-s)\,\nabla\cdot\big(u^{(n)}\otimes u^{(n)}\big)(s)\,ds\,,$$

we see from (40) that for $r=q$, we just have to prove, for a fixed n, that

$$\sqrt{t}\,u^{(n)}(\sqrt{t}\,x,t)\xrightarrow{L^q}V^{(n)}(x)\,. \tag{44}$$

This can be done using the estimates obtained in the proof of Theorem 3. Recall that we obtained a estimation on $S(t)\,u_0$ using an estimation on u and the equation. Here, the same technique applies, but we know an estimation on $S(t)\,u_0$ and u_n and deduce the estimation u_{n+1} using the equation. Then, by means of an estimates like (42) and (45) and the dominated convergence theorem, we obtain

$$\sqrt{t}\;B\big(u^{(n)},u^{(x)}\big)(\sqrt{t}\,x,t)\xrightarrow{L^3}B\big(V^{(n)},V^{(n)}\big)\,. \tag{45}$$

Therefore, splitting

$$u(x,t)-\frac{1}{\sqrt{t}}\,V\Big(\frac{x}{\sqrt{t}}\Big)=\big(u-u^{(n)}\big)+\big(V^{(n)}-V\big)+\big(u^{(n)}-V^{(n)}\big)\,,$$

we conclude with an $\varepsilon/3$ argument to obtain (41) using (44) for the fixed q we have chosen. We obtain the same result for all $\tilde{q}$ by interpolation between various L^σ norms, as in Lemma 2. We obtain (42) using (45) in the same way.

3. Understanding the condition on the initial data.

We might ask about the meaning of condition (40) and the relationship with the remark we made previously. Let us first introduce an equivalent definition of our Besov spaces $B_p^{-(1-3/p),\infty}$, $p>3$.

Proposition 2. *Let* $\{\psi_{\varepsilon_j}\}_\varepsilon$ *be a set of* 7 *wavelets such that the set* $\{\psi_\varepsilon(2^j x - k)\}_{\varepsilon; j,k\in\mathbb{Z}}$ *is an orthogonal basis of* $L^2(\mathbb{R}^3)$. *Then if*

$$f(x) = \sum_{\varepsilon,j,k} \alpha_\varepsilon(j,k)\, 2^j\, \psi_\varepsilon(2^j x - k)\,,$$

$f \in \dot{B}_p^{-(1-3/p),\infty}$ *is equivalent to*

$$\sup_j \Big(\sum_k |\alpha_\varepsilon(j,k)|^p \Big)^{1/p} < +\infty\,.$$

Then we have

Proposition 3. *The following two conditions are equivalent,*

If $\alpha_\varepsilon(j,k)$ *are the wavelets coefficients of* f *under the previous normalization, and*

$$f \in B_p^{-(1-3/p),\infty}\,, \qquad 3 < p < +\infty\,, \tag{46}$$

1) f *satisfies*

$$\lambda\, f(\lambda\, x) \text{ converges weakly to } 0\,, \tag{47}$$

and

$$\lim_{j\to-\infty} \Big(\sum_k |\alpha_\varepsilon(j,k)|^p \Big)^{1/p} = 0\,. \tag{48}$$

2) *The function* f *satisfies*

$$\lim_{t\to+\infty} t^{1/2-3/(2p)}\, \|S(t)f\|_{L^p} = 0\,. \tag{49}$$

Using the previous propositions, we will later prove the promised Proposition 4, which explains why the condition (40) is necessary and sufficient in order to obtain Theorem 4. It is in fact deeply linked to the nature of the functional space we are using, rather than to the equation itself. On the other hand, no other pathological examples are known to the author other than those constructed in the proof of this proposition. On simple practical examples, where we start with a rather regular initial data, the condition will be fulfilled. Let us give an example, where we forget about the divergence free vectors and deal with a scalar function for sake of simplicity. Take

$$u_0(x) = \frac{\varepsilon}{1+|x|} \ ,$$

then, by rescaling it converges weakly to

$$v_0(x) = \frac{\varepsilon}{|x|} \ .$$

We put an ε in order to comply with the smallness assumption. Then the condition (40) is verified, because the difference $\delta = u_0 - v_0$ belongs to L^3 outside of the unit ball, so that the solution of the heat equation with initial data $\delta(x)$ has its $L^3(\mathbb{R}^3 \backslash B(0,1))$ norm going to zero as time goes to infinity, and by Sobolev's embedding we get (40). In other words, what matters is the behavior of the initial data for low frequencies.

Proposition 4. *There exists a function $f \in B_3^{0,\infty}(\mathbb{R}^3)$, such that $\lambda f(\lambda x)$ converges weakly to* 0 *when $\lambda \to +\infty$, but such that, if $p > 3$*

$$\lim_{\lambda\to\infty} \|S(1)\,(\lambda\, f(\lambda\, x))\|_{L^p} \neq 0 \, .$$

We will now prove Proposition 3. Proposition 2 is nothing else than the usual characterization of Besov spaces with wavelets coefficients ([8]). We only changed the normalization. We restrict ourselves to Littlewood-Paley wavelets, as defined in [8], because they are closely related to Littlewood-Paley decomposition. But the same results hold for any wavelets basis, provided it has sufficient regularity. Let us recall a few useful properties of these particular wavelets basis, as they will be used later. The so-called scaling function of the wavelet basis is a function $\phi \in \mathcal{S}$, such that $\hat{\phi}(\xi) = 1$ if $-2\pi/3 < \xi < 2\pi/3$, $\hat{\phi}(\xi) = 0$ if $4\pi/3 < \xi$, $\hat{\phi}(\xi)$ is even, positive and such that $\hat{\phi}^2(\xi) + \hat{\phi}^2(2\pi - \xi) = 1$ if $0 < \xi < 2\pi$. Then the equivalent of operator S_j in the Littlewood-Paley analysis is an operator E_j, defined as follow:

Definition 3. *The operator E_j is a sum of three terms,*

$$E_j = \Sigma_j + M_j \Delta_j^- + M_j^{-1} \Delta_j^+ \ ,$$

where the three terms Σ_j, Δ_j^- and Δ_j^+ are the Fourier multipliers by $\hat\phi^2(2^{-j}\xi)$, $\hat\phi(2^{-j}\xi)\,\hat\phi(2^{-j}\,(2\pi+\xi))$, and $\hat\phi(2^{-j}\xi)\,\hat\phi(2^{-j}(2\pi-\xi))$. M_- is the multiplication by $\exp(2\pi\, i\, 2^j x)$*. We then define $D_j = E_{j+1} - E_j$, which is very close to the usual Δ_j from Definition* 1*.*

We see that (49) can be written as

$$\lim_{\lambda\to\infty} \|S(1)\,(\lambda\, f(\lambda\, x))\|_{L^p} = 0\,. \tag{50}$$

Then, if $\phi \in \mathcal{S}$ and supp $\hat\phi$ is compact,

$$\lim_{\lambda\to\infty} \|\phi * (\lambda\, f(\lambda\, x))\|_{L^p} = 0\,. \tag{51}$$

We remark then that

$$\Big(\sum_{\varepsilon,k} |\alpha_{j,k,\varepsilon}|^p\Big)^{1/p} = \|D_0(\lambda\, f(\lambda\, x))\|_{L^p}\ , \tag{52}$$

with $\lambda = 2^{-j}$, and D_0 defined as in [8, p. 45]. Then we know from (3) that D_0 is a sum of operators like $M\Delta$, where M is a multiplication by an imaginary exponential, and Δ is a convolution by a function whose Fourier transform is compactly supported. We deduce our result by using (51). Conversely, if we suppose that (46) is true, we first prove that for ϕ as defined above,

$$\lim_{\lambda\to\infty} \|\phi * f_\lambda\|_{L^p} = 0\,.$$

Doing a rescaling and taking λ of the order of 2^N, we are left to prove that

$$\lim_{N\to\infty} 2^{N(1-3/p)} \Big\| \sum_{j<-N} \sum_{\varepsilon,k} \alpha_{f,k,\varepsilon}\, 2^j\, \psi_\varepsilon(2^j x - k)\Big\|_{L^p} = 0\,.$$

The sum on $j < -N$ being the convolution with ϕ, if we assume the support of $\hat\phi$ to be contained in the unit ball. However, for a fixed j

$$\Big\| \sum_{\varepsilon,k} \alpha_{\varepsilon,j,k}\, 2^j\, \psi_\varepsilon(2^j x - k)\Big\|_{L^p} \le C\, 2^{j(1-3/p)} \Big(\sum_{\varepsilon,k} |\alpha_{j,k,\varepsilon}|^p\Big)^{1/p}\,.$$

Then, by means of (43)

$$\Big\| \sum_{k,\varepsilon} \alpha_{\varepsilon,j,k}\, 2^j\, \psi_\varepsilon\,(2^j x - k)\Big\|_{L^p} \leq 2^{j(1-3/p)} \varepsilon_j \ ,$$

with $\lim_{j\to -\infty} \varepsilon_j = 0$.

Then

$$2^{(1-3/p)N} \sum_{j\leq -N} \varepsilon_j\, 2^{(1-3/p)j} \longrightarrow 0\,,$$

as it it a convolution between ℓ^∞ and ℓ^1. Equation (49) follows by splitting $S(1)$ into a sum of dyadic blocks.

Let us go back to Proposition 4. It helps to understand why (31) is a necessary and sufficient condition, unlike (32). In fact, let us forget for a while the proposition and suppose only (32); in the opinion of the author, the following gives a good heuristic of the situation, and could be made rigorous except that in our case, and unlike [3], it doesn't produce any useful results. With the help of the Theorem 1, we can construct a set $(u_\lambda)_\lambda$ of solutions of (1) with initial data $u_{0,\lambda}$. All the estimates do not change by rescaling, which means they are independent of λ. Therefore, we can extract a subsequence which converges in $C([t_1, t_2], \times B(0,R))$, where $t_1 > 0$, for exactly the same reasons as in [3]: by bootstrap we obtain $u_\lambda \in C([t_1,t_2], W^{1,\infty})$, with a bound independent of λ, and then we know that $W^{1,p}(B(0,R)) \hookrightarrow C(B(0,R))$. We also obtain easily that $v(x,t)$ is actually the (self-similar) solution of (1) with an initial condition v_0, which is the weak limit of $(u_{0,\lambda})_\lambda$. But to prove (41), we just have to prove

$$\lim_{\lambda\to\infty} \|u_\lambda(x,1) - v(x,1)\|_{L^q} = 0\,. \tag{53}$$

This last sentence is true if we replace L^q by $L^q(B(0,R))$, and in order to prove (53), we should prove something like

$$\lim_{R\to\infty} \|\chi_R u_\lambda(x,1)\|_{L^q} = 0\,,$$

uniformly with regards to λ, where $\chi_R(x) = \chi(x/R)$ has value zero on $B(0,1)$, and one outside $B(0,2)$. Let us deal with the linear part: suppose that $u_0 \in L^3$, $\|\chi_R u_{0,\lambda}\|_{L^3} \leq \|u_0\|_{L^3(|x|>\lambda R)}$, we obtain easily

$$\lim_{R\to\infty} \|\chi_R S(1)\, u_{0,\lambda}\|_{L^q} = 0\,, \qquad \text{uniformly in } \lambda \geq 1\,.$$

We conclude with such a proof for the two dimensional case, as in ([3]). However, if $u_0 \in \dot{B}_3^{0,\infty}$ but $u_0 \notin L^3$, then $\|\chi_R u_0\|_{\dot{B}_3^{0,\infty}} \longrightarrow 0$ is not always true when $R \longrightarrow \infty$. For instance, if we take $f = 1/|x|$, then

$$\|\chi_R f\|_{\dot{B}_3^{0,\infty}} = \|\chi f\|_{\dot{B}_3^{0,\infty}} = \text{constant}\,.$$

We could hope to have a property like

$$\lim_{R\to\infty} \|\chi_R S(1)\, u_{0,\lambda}\|_{L^q} = 0\,,$$

uniformly if $\lambda \geq 1$. In fact, it is not possible, as we will see.

Proposition 5. *There exists $f \in \dot{B}_3^{0,\infty}$ such that for all R, there exists $\lambda \geq 1$ such that*

$$\|\chi_R S(1)\, f_\lambda\|_{L^4} = 1\,.$$

Here, we have chosen $p = 3$, $q = 4$, but we could have chosen any other values.

We remark that, if λ is fixed, $S(1)f_\lambda \in L^4$ and

$$\lim_{R\to\infty} \|\chi_R S(1) f_\lambda\|_{L^4} = 0\,.$$

We will need the following lemma:

Lemma 3. *If $f \in L^4, g \in L^1$, then*

$$\Big(\int_{|x|>R} |f * g|^4\, dx\Big)^{1/4} \leq \|g\|_{L^1}\Big(\int_{|x|>R/2} |f|^4\, dx\Big)^{1/4} + \|f\|_{L^4}\int_{|x|>R/2} |g|\, dx\,.$$

Therefore, in order to prove that $\|\chi_R S(1) f_\lambda\|_{L^4}$ is large enough, we just need to find a function $g \in L^1$ such that $\|\chi_R (g * S(1) f_\lambda)\|_{L^4}$ is large. Let $\phi \in \mathcal{S}$ be a function such that supp $\hat{\phi} \subset \{9/10 \leq |\xi| \leq 10/9\}$,

and

$$f(x) = \sum_0^\infty 2^{-j}\,\phi(2^{-j}x - x_j)\,, \qquad \text{where } |x_j| \longrightarrow \infty\,,$$

$$\begin{aligned} 2^m f(2^m x) &= \sum_0^{m-1} 2^{m-j}\phi(2^{m-j}x - x_j) + \phi(x - x_m) \\ &\quad + \sum_{m+1}^\infty 2^{m-j}\phi(2^{m-j}x - x_j) \\ &= u_m(x) + \phi(x - x_m) + v_m(x)\,. \end{aligned}$$

We observe that the frequencies of u_m are in $\{|\xi| \geq 9/5\}$ and the ones of v_m in $\{|\xi| \leq 5/9 \leq 9/10\}$. Thus there exists $g \in \mathcal{S}$ such that

$$\operatorname{supp}\hat{g} \subset \left\{\frac{10}{18} \leq |\xi| \leq \frac{18}{10}\right\}$$

and

$$\hat{g}(\xi) = e^{|\xi|^2}\,, \qquad \text{for } \frac{9}{10} \leq |\xi| \leq \frac{10}{9}\,.$$

We take $\lambda = 2^m$, $g * S(1) f_\lambda = \phi(x - x_m)$, and

$$\lim_{m\to\infty} \int_{|x|>R} |\phi(x - x_m)|^4\,dx = \|\phi\|_{L^4}^4\,.$$

We can go further in our study of f.

If $2^m < \lambda < 2^{m+1}$, we split f as

$$\lambda\, f(\lambda\, x) = u_m(x) + v_m(x)\,,$$

where u_m is the part of frequencies $2^{-j}\,\lambda$ with $|m - j| < N$ and v_m the one where $|m - j| \geq N$.

Then, we take a test function ψ such that $0 \notin \operatorname{supp}\hat{\psi}$, and N such that $\operatorname{supp}\hat{\psi} \subset [2^{-N}, 2^N]$. Then $\int \lambda\, f(\lambda\, x)\,\psi(x)\,dx$ contains only terms with $|j - m| \leq N$, which are in finite number and go to 0 when $|x_j| \longrightarrow \infty$.

We have proved the following proposition:

Proposition 6. *There exists* $f \in \dot{B}_3^{0,\infty}$ *such that* $f_\lambda \rightharpoonup 0$ *for the weak topology* $\sigma(\dot{B}_3^{0,\infty}, \dot{B}_{3/2}^{0,1})$*, but nevertheless,* $\|\chi_R S(1) f_\lambda\|_{L^4}$ *does not go to* 0 *when* $R \longrightarrow \infty$*, uniformly in* $\lambda \geq 1$*.*

The reader should consult [10] to see why the test functions ψ we used are dense into $\dot{B}_{3/2}^{0,1}$.

We have now to link the Proposition 6 and the condition (31).

Proposition 7. *Let*

$$f \in \dot{B}_3^{0,\infty}, \qquad f_\lambda(x) = \lambda f(\lambda x).$$

The two following properties are equivalent:

1) *The function* f *satisfies*

$$\lim_{t \to \infty} t^{1/8} \|S(t) f\|_{L^4} = 0. \tag{55}$$

2) $f_\lambda \rightharpoonup 0$ *for the topology* $\sigma(\dot{B}_3^{0,\infty}, \dot{B}_{3/2}^{0,1})$*, and*

$$\Big(\int_{|x|>R} |S(1) f_\lambda|^4 \Big)^{1/4} \leq \varepsilon_R, \tag{56}$$

with $\lim_{R\to\infty} \varepsilon_R = 0$ *independently of* $\lambda \geq 1$*.*

Let us prove that the first condition implies the second one. The weak convergence has already been proved. Knowing that (55) is equivalent to

$$\lim_{\lambda \to \infty} \|S(1) f_\lambda\|_{L^4} = 0,$$

this proves

$$\Big(\int_{|x|\geq R} |S(1) f_\lambda|^4 \, dx \Big)^{1/4} \leq \varepsilon,$$

for $\lambda \geq \lambda_0$. It remains the case where $\lambda \in [1, \lambda_0)$. As

$$S(1) f_\lambda(x) = \lambda \, (S(\lambda^2) f) \, (\lambda \, x),$$

we remark that the functions $S(\lambda^2) f$ are in a compact set of L^4. Then there exists R_ε such that, for $\lambda \in [1, \lambda_0)$ and $R > R_\varepsilon$,

$$\Big(\int_{|x|>R} |S(1)f_\lambda|^4\,dx\Big)^{1/4} \le \varepsilon\,,$$

the converse statement can be easily proved. In fact, if

$$f_\lambda \rightharpoonup 0\,,$$

we obtain that

$$S(1)f_\lambda(x) \longrightarrow 0$$

uniformly on any compact set. We can therefore estimate

$$\|S(1)f_\lambda\|_{L^4}$$

by splitting for $|x| \le R$ and $|x| > R$, which ends the proof.

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Subnormal operators of finite type I. Xia's model and real algebraic curves in $\mathbb{C}^2$

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Abstract. Xia proves in [9] that a pure subnormal operator S is completely determined by its self-commutator $C = S^*S - SS^*$, restricted to the closure M of its range and the operator $\Lambda = (S^*|M)^*$. In [9], [10], [11], he constructs a model for S that involves these two operators and the so-called mosaic, which is a projection-valued function, analytic outside the spectrum of the minimal normal extension of S. He finds all pure subnormals S with $\operatorname{rank} C = 2$. We will give a complete description of pairs of matrices (C, Λ) that correspond to some S for the case of the self-commutator C of arbitrary finite rank. It is given in terms of a topological property of a certain algebraic curve, associated with C and Λ. We also give a new explicit formula for Xia's mosaic.

0. Introduction.

One of the modern approaches to the spectral theory of a nonselfadjoint operator consists in constructing its functional model. The most developed theory of this kind is the Sz.-Nagy-Foiaş theory of Hilbert space contractions. Recently, several attempts have been made to construct functional models for other classes of operators. This paper concerns some questions that arise in connection with Xia's analytic model of subnormal operators.

A (bounded) linear operator S acting on a (complex) Hilbert space H is called *subnormal* if there exists a larger Hilbert space K, $K \supset H$, and a normal operator $N\colon K \to K$ such that $NH \subset H$ and $S = N|H$. The operator S is called pure if it has no nonzero reducing subspace on which it is normal. We will always assume S to be pure and the normal extension N of S to be minimal; the latter means that there is no subspace K', $H \subset K' \subsetneqq K$, such that $NK' \subset K'$ and $N|K'$ is normal.

This class of operators has been much investigated; we refer to [1] for a background.

It is known that for a subnormal operator S, if we put

$$C \stackrel{\text{def}}{=} S^*S - SS^*\,, \qquad M \stackrel{\text{def}}{=} \operatorname{clos}\operatorname{Range} C\,,$$

then $S^*M \subset M$. In [9], [10], [11], Xia constructs and studies an analytic model of a subnormal operator. He defines two functional model spaces that consist, respectively, of analytic and antianalytic M-valued functions on $\mathbb{C} \setminus \sigma(N)$ and gives formulas for the trancription of S and S^* in each of these two models (here $\sigma(N)$ is the spectrum of N).

One of the consequences of Xia's results is that if we put

$$\Lambda = \big(S^*|M\big)^*\,,$$

then the pair (C, Λ) of operators on M completely determines a pure subnormal operator S. If M is one-dimensional, then C, Λ are, essentially, complex numbers, and the spectrum of S is the closed disk with center in Λ and radius $C^{1/2}$. Therefore, by analogy, Λ and $C^{1/2}$ can be called *the matrix center* and *the matrix radius* of S.

The following question arises: which pairs (C, Λ) can appear in this way? The main result of this paper is a complete answer to this question in the case $\dim M < \infty$. It is given in terms of the algebraic curve

$$\Delta = \big\{(z, w) :\ \det\big(C - (w - \Lambda^*)\,(z - \Lambda)\big) = 0\big\}$$

in $\mathbb{C}P^2$. A crucial topological condition is that Δ has to be separated, that is, that $\Delta \cap \{(z, \bar z) : z \in \mathbb{C}\}$ divides each of the (nondegenerate) irreducible components of Δ into two connected components (see Theorem 1 below).

One of the main objects in Xia's model is Xia's mosaic

$$\mu(z) = P_M(N - SP_H)\,(N-z)^{-1}|M\,, \qquad z \in \mathbb{C} \setminus \sigma(N)\,,$$

here N is the minimal normal extension of S and P_W is the orthogonal projection onto a subspace W. We will give an explicit formula for $\mu(z)$ in terms of C, Λ and the curve Δ.

Sections 1-3 are devoted to preliminaries. Main results are formulated in Section 4; in Section 5, proofs are given. Section 6 collects some additional facts and examples. In the subsequent publication [12], we are going to continue the analysis of Xia's model.

The form of the main result resembles some results in the theory of commuting nonselfadjoint operators by Livšic, Vinnikov and others (see [5]). The connection between this theory and the topic of the present paper may exist, but does not seem to be obvious.

1. Xia's results.

We reproduce only those results by Xia that will be necessary for our exposition.

Let S be a pure subnormal operator, and define M, C, Λ as above. We will write $C = C(S)$, $\Lambda = \Lambda(S)$. Let us say that S is *of finite type* if $\dim M < \infty$. Denote by $\mathcal{L}(M)$ the space of bounded linear operators on M. Following Xia [9], define a $\mathcal{L}(M)$-valued measure $e(\,\cdot\,)$ by

$$e(\,\cdot\,) = P_M E(\,\cdot\,)\, P_M\,, \tag{1.1}$$

where $E(\cdot)$ is the spectral measure of N. Xia shows that $\sigma(N)$ is contained in the set

$$\gamma = \{u \in \mathbb{C} :\ \det\bigl(C - (\bar{u} - \Lambda^*)\,(u - \Lambda)\bigr) = 0\} \tag{1.2}$$

and that

$$\bigl(C - (\overline{u} - \Lambda^*)\,(u - \Lambda)\bigr)\, de(u) \equiv 0\,. \tag{1.3}$$

He also proves that the values of the function

$$\mu(z) = \int \frac{u-\Lambda}{u-z}\, de(u) = P_M(N - SP_H)\,(N-z)^{-1}|M\,, \quad z \in \mathbb{C}\setminus\gamma \tag{1.4}$$

are parallel projections on M (that is, $\mu(z)^2 \equiv \mu(z)$). We will call this function Xia's mosaic of S. Xia proves that

$$\big[\mu(z),\, C(z-\Lambda)^{-1}+\Lambda^*\big]=0\,, \qquad z\in\mathbb{C}\setminus\gamma\,, \tag{1.5}$$

where $[A,B]=AB-BA$.

For any non-negative $\mathcal{L}(M)$-valued measure e, we put $\mathcal{L}^2(e)$ to be the space of all measurable M-valued functions f satisfying

$$\|f\|^2 \overset{\text{def}}{=} \int_\gamma \big(de(u)f(u), f(u)\big) < \infty\,,$$

factorized by the linear manifold $\{f : \|f\|^2 = 0\}$. It is easy to see that $\mathcal{L}^2(e)$ is a Hilbert space.

The following result is part of [9, Theorems 1 and 2].

Theorem A (Xia [9]). *Let $C,\Lambda\in\mathcal{L}(M)$ and $C>0$. Suppose that there exists an $\mathcal{L}(M)$-valued positive measure e on a compact subset γ of $\mathbb{C}$ such that*

$$\int \frac{u-\Lambda}{u-z}\, de(u) = 0\,, \tag{1.6}$$

for z in the unbounded component of $\mathbb{C}\setminus\gamma$ and (1.3) holds. Let $\mathcal{D}$ be the set of all $z\in\mathbb{C}\setminus\gamma$ for which (1.6) holds, and $\mathcal{H}$ the closure in $\mathcal{L}^2(e)$ of all linear combinations of functions $(\lambda-(\,\cdot\,))^{-1}m$, $\lambda\in\mathcal{D}$, $m\in M$. Then the operator

$$(\widetilde{S}f)(u) = u\,f(u)\,, \qquad f\in\mathcal{H}\,, \tag{1.7}$$

is pure subnormal,

$$(\widetilde{N}f)(u) = u\,f(u)\,, \qquad f\in\mathcal{L}^2(e)\,, \tag{1.8}$$

is its minimal normal extension, $C=C(\widetilde{S})$, $\Lambda=\Lambda(\widetilde{S})$, and $e(\,\cdot\,)$ is connected with $\widetilde{N}$ in the same way as in formula (1.1). We imbed M into $\mathcal{L}^2(e)$ via the formula $c\longmapsto[c]$, where $[c](z)\equiv c$.

Conversely, if S is a subnormal operator of finite type and $C=C(S)$, $\Lambda=\Lambda(S)$, then the measure $e(\,\cdot\,)$, given by (1.1), enjoys the above properties, and (1.7), (1.8) define operators, unitarily equivalent to S and N, respectively.

The statements $C = C(S)$, $\Lambda = \Lambda(S)$ are not stated explicitly in [9, Theorem 2], but they follow at once from (1.7) and [9, formula (42)].

Theorem A gives a criterion for existence of a pure subnormal S with given matrices $C = C(S)$ and $\Lambda = \Lambda(S)$ in terms of the existence of a $\mathcal{L}(M)$-valued measure with certain properties. Our aim is to give a more explicit criterion.

2. The discriminant curve and its geometry.

Let M be a finite-dimensional Hilbert space and $C > 0$ and Λ operators on M. We associate with C, Λ the polynomial

$$\tau(z,w) = \det\big(C - (w - \Lambda^*)\,(z - \Lambda)\big) \tag{2.1}$$

and the algebraic curve

$$\Delta = \big\{(z,w) \in \mathbb{C}^2 :\ \tau(z,w) = 0\big\}\,,$$

which will be called *the discriminant curve* of S. As usual, we pass to homogeneous coordinates $(\zeta, \omega, \vartheta)$ in the complex projective plane $\mathbb{C}P^2$ by putting $z = \zeta\,\vartheta^{-1}$, $w = \omega\,\vartheta^{-1}$ and consider Δ as an algebraic curve in $\mathbb{C}P^2$, defined by the homogeneous polynomial equation $\vartheta^{2\dim M}\tau(\zeta\,\vartheta^{-1}, \omega\,\vartheta^{-1}) = 0$. Since

$$\tau(\overline{w}, \overline{z}) = \overline{\tau(z,w)}\,, \tag{2.2}$$

Δ possesses an antianalytic involution given by

$$\delta = (z,w) \longmapsto \delta^* \overset{\text{def}}{=} (\overline{w}, \overline{z})\,.$$

If we substitute $z = x+iy$, $w = x-iy$, then τ becomes a real polynomial in variables x, y. In this sense, Δ is a real algebraic curve. In terms of the coordinates (x,y) in $\mathbb{C}^2$, the map $\delta \longmapsto \delta^*$ is the usual complex conjugation $(x,y) \longmapsto (\overline{x}, \overline{y})$, that is, the reflexion with respect to the linear submanifold $\mathbb{R}^2 = \{x = \overline{x},\ y = \overline{y}\} = \{w = \overline{z}\}$ of real points of $\mathbb{C}^2$. In what follows, only the coordinates (z,w) will be used.

We observe that

a) $\delta \in \Delta$, $w(\delta) = \infty$ implies $z(\delta) \in \sigma(\Lambda)$;

b) $\delta \in \Delta$, $z(\delta) = \infty$ implies $w(\delta) \in \sigma(\Lambda^*)$.

For instance, to prove a), it suffices to rewrite the equation $\tau(z,w) = 0$ as

$$\det\bigl(Cw^{-1} - (1 - w^{-1}\Lambda^*)\,(z - \Lambda)\bigr) = 0$$

and to put here $w^{-1} = 0$.

Let

$$\tau(z,w) = \prod_{j=1}^{T} \tau_j(z,w)^{\alpha_j} \tag{2.3}$$

be the decomposition of τ into irreducible factors [3]; associated is a decomposition

$$\Delta = \bigcup_{j=1}^{T} \Delta_j\ , \tag{2.4}$$

where $\Delta_j = \bigl\{(z,w) : \tau_j(z,w) = 0\bigr\}$. We will call algebraic curves Δ_j the components of Δ.

A component $\widehat{\Delta}_k$ will be called *degenerate* if it has the form $z \equiv \text{const}$ or $w \equiv \text{const}$ and *nondegenerate* in the opposite case. Let $\widehat{\Delta}_{\text{deg}}$ be the union of degenerate components of $\widehat{\Delta}$ and $\widehat{\Delta}_{\text{ndeg}}$ the union of nondegenerate components.

Consider the following example. Let S be the shift operator

$$Sf(\,\cdot\,) = (\,\cdot\,)\, f(\,\cdot\,)\,,$$

acting of the Hardy space H^2, equipped with the modified norm $\|f\|_1^2 = \|f\|_{H^2}^2 + a\,|f(0)|^2$, where $a > 0$. It is easy to see that S is simple subnormal and that its discriminant surface is

$$\{z\,w = 1\} \cup \{z = 0\} \cup \{w = 0\}\,.$$

This shows that degenerate surfaces really can appear.

We put

$$\sigma_C(\Lambda) = \Bigl\{z \in \mathbb{C} :\ \det(C - (w - \Lambda^*)\,(z - \Lambda)) = 0\,,\quad \text{for all } w \in \mathbb{C}\Bigr\}\ ,$$

so that the degenerate components in the decomposition (2.4) are exactly the surfaces $z \equiv \lambda$ and $w \equiv \overline{\lambda}$, $\lambda \in \sigma_C(\Lambda)$. It is immediate that $\sigma_C(\Lambda) \subset \sigma(\Lambda)$ and $\sigma_C(\Lambda) \subset \gamma$.

A point δ of Δ will be called *regular* if it belongs to only one Δ_j and either

$$\frac{\partial \tau_j}{\partial z}(\delta) \neq 0 \qquad \text{or} \qquad \frac{\partial \tau_j}{\partial w}(\delta) \neq 0\,,$$

and *singular* in all other cases. The set Δ_s of singular points of Δ is finite. Put $\Delta_0 = \Delta \backslash \Delta_s$; then the sets $\Delta_j \cap \Delta_0$ are pairwise disjoint. *The blow-up* $\widehat{\Delta}$ of Δ can be defined as a unique abstract compact Riemann surface that consists of exactly T connected components $\widehat{\Delta}_j$, where each $\widehat{\Delta}_j$ is compact and is obtained by adding a finite number of points to $\Delta_j \cap \Delta_0$. There is a natural projection of $\widehat{\Delta}$ onto Δ which is identical on Δ_0. If $\delta \in \widehat{\Delta}$ and (z, w) is its image on Δ, we will write $\delta \sim (z, w)$. We refer to [3] for the background on the blow-up.

The functions $\delta \longmapsto z(\delta)$, $\delta \longmapsto w(\delta)$ extend to meromorphic functions on $\widehat{\Delta}$. The conjugation $\delta \longmapsto \delta^*$ also extends to $\widehat{\Delta}$.

The function

$$\eta = -\frac{dz}{dw}\,, \tag{2.5}$$

defined initially on regular points $\delta = (z, w) \in \widehat{\Delta}_{\mathrm{ndeg}}$, can be continued to a meromorphic function on $\widehat{\Delta}$. This function will play an important role in the sequel.

It is easy to check, using a) and b), that

$$\begin{aligned} z(\delta) \longrightarrow \infty \qquad &\text{implies} \qquad \eta(\delta) \longrightarrow \infty\,, \\ w(\delta) \longrightarrow \infty \qquad &\text{implies} \qquad \eta(\delta) \longrightarrow 0\,. \end{aligned} \tag{2.6}$$

Since both z-projection and w-projection of each nondegenerate component Δ_j is the whole sphere $\widehat{\mathbb{C}}$, it follows that η is non-constant on each nondegenerate component of Δ.

By (2.2), η has the following symmetry property

$$\eta(\delta^*) = (\overline{\eta}(\delta))^{-1}\,. \tag{2.7}$$

Put

$$\widehat{\Delta}_+ = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} :\ |\eta(\delta)| < 1\}\,, \qquad \widehat{\Delta}_- = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} :\ |\eta(\delta)| > 1\}\,,$$

then

$$\partial\widehat{\Delta}_+ = \partial\widehat{\Delta}_- = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} :\ |\eta(\delta)| = 1\}\,.$$

Let

$$\widehat{\Delta}_{\mathbb{R}} = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} : \ \delta = \delta^*\}$$

be the set of real points of $\widehat{\Delta}$, then $\widehat{\Delta}_{\mathbb{R}} \cap \Delta_0 = \Delta_0 \cap \{(z, \overline{z}) : z \in \mathbb{C}\}$. By (2.7),

$$\widehat{\Delta}_{\mathbb{R}} \subset \partial \widehat{\Delta}_+ \, .$$

Definition. *The algebraic curve Δ is called separated if for any non-degenerate component $\widehat{\Delta}_k$ of $\widehat{\Delta}$, $\widehat{\Delta}_{\mathbb{R}} \cap \widehat{\Delta}_k$ separates $\widehat{\Delta}_k$ into at least two connected components.*

Let Δ be separated. Then the set $\widehat{\Delta}_{\mathbb{R}} \cap \widehat{\Delta}_k$ is infinite for each nondegenerate $\widehat{\Delta}_k$ (and contains a continuous curve). In particular, it contains points of Δ_0. It follows that $\big(\widehat{\Delta}_k\big)^* = \widehat{\Delta}_k$ for each nondegenerate component $\widehat{\Delta}_k$. The conjugation transforms degenerate components $z = \text{const}$ into the components $w = \text{const}$, and vice versa. The general theory of Riemann surfaces with antianalytic convolution (see [6]) says that for each nondegenerate $\widehat{\Delta}_k$, $\widehat{\Delta}_{\mathbb{R}} \cap \widehat{\Delta}_k$ separates $\widehat{\Delta}_k$ into exactly two connected components.

Proposition 1. *Δ is separated if and only if $\widehat{\Delta}_{\mathbb{R}} = \partial \widehat{\Delta}_+$.*

PROOF. Clearly, $\partial \widehat{\Delta}_+ \cap \widehat{\Delta}_k$ separates $\widehat{\Delta}_k$ into at least two connected components for all nondegenerate $\widehat{\Delta}_k$; this proves the "if" part.

To prove the converse, suppose that Δ is separated, but $\widehat{\Delta}_{\mathbb{R}} \subsetneqq \partial \widehat{\Delta}_+$. The set $\partial \widehat{\Delta}_+$ has no isolated points. Since both $\partial \widehat{\Delta}_+$ and $\widehat{\Delta}_{\mathbb{R}}$ are closed, $\partial \widehat{\Delta}_+ \setminus \widehat{\Delta}_{\mathbb{R}}$ contains an arc, say, α. Then α is contained in a nondegenerate component $\widehat{\Delta}_k$. Therefore $\widehat{\Delta}_k \setminus \widehat{\Delta}_{\mathbb{R}}$ can be obtained from the connected set $(\widehat{\Delta}_k \setminus \partial \widehat{\Delta}_+) \cup \alpha$ by adding part of its boundary. Hence $\widehat{\Delta}_k \setminus \widehat{\Delta}_{\mathbb{R}}$ is connected, a contradiction.

Suppose Δ is separated. Put

$$\gamma_c = z\,(\partial \widehat{\Delta}_+)\,, \tag{2.8}$$

it is a finite union of piecewise analytic curves. We have that $\gamma_c \subset \gamma$ (see (1.2)), and $\gamma \setminus \gamma_c$ is a finite set.

3. The projection-valued function Q.

Let A be a square matrix. Then $\varphi(A)$ is defined by means of the Riesz-Dunford calculus for any function φ, analytic in a neighbourhood of $\sigma(A)$. It is easy to see that

$$\varphi(A|R) = \varphi(A)|R\,, \tag{3.1}$$

for any invariant subspace R of A.

For $\lambda \in \sigma(A)$, we put

$$\Pi_\lambda(A) = \chi_\lambda(A)\,,$$

where χ_λ is a locally constant function on a neighbourhood of $\sigma(A)$ such that $\chi_\lambda \equiv 1$ in a neighbourhood of λ and $\chi_\lambda \equiv 0$ in a neighbourhood of $\sigma(A) \setminus \{\lambda\}$. We put $\Pi_\lambda(A) = 0$ if $\lambda \notin \sigma(A)$. The operator $\Pi_\lambda(A)$ is a parallel projection; it is called the Riesz projection corresponding to the eigenvalue λ of A.

Let

$$\widehat{\Delta}' = \widehat{\Delta}_{\mathrm{ndeg}} \cup \bigcup_{w_0 \in \sigma_C(\Lambda)} \{(z, w) :\ w \equiv \overline{w}_0\}$$

be the algebraic curve obtained from $\widehat{\Delta}$ by excluding from it the "vertical" degenerate components $z \equiv z_0$. For $z \notin \sigma(\Lambda)$, a point (z, w) is in Δ if and only if w belongs to $\sigma(C(z-\Lambda)^{-1} + \Lambda^*)$. Therefore for any $\delta = (z, w) \in \Delta_0 \setminus z^{-1}(\sigma(\Lambda))$,

$$Q(\delta) \stackrel{\mathrm{def}}{=} \Pi_w(C\,(z-\Lambda)^{-1} + \Lambda^*) \tag{3.2}$$

is a non-zero parallel projection in M. The function Q is a projection-valued meromorphic function on $\widehat{\Delta}'$. The well-known properties of the functional calculus imply that

i) $Q(\delta_1)\,Q(\delta_2) = 0$ if $\delta_1,\ \delta_2 \in \Delta_0,\ z(\delta_1) = z(\delta_2) \notin \sigma(\Lambda),\ \delta_1 \neq \delta_2$,

ii) $\sum_{z(\delta)=z_0} Q(\delta) = I$ for any z_0 such that $z^{-1}(z_0) \subset \Delta_0$.

It follows from (1.5) that

$$\big[Q\big((z, w)\big),\ \mu(z)\big] = 0\,, \qquad \text{for } (z, w) \in \Delta_0 \setminus z^{-1}\big(\sigma(\Lambda)\big)\,. \tag{3.3}$$

4. Main results.

Theorem 1. *Let M be a finite-dimensional Hilbert space and C, Λ operators on M with $C > 0$. Define Δ, $\Delta_\pm$, Q as above, and put*

$$\mu(z) = \sum_{w:(z,w)\in\Delta_+} Q((z,w)), \qquad z \in \mathbb{C} \setminus \big(\sigma(\Lambda) \cup \gamma \cup z(\Delta_s)\big). \tag{4.1}$$

Then there exists a subnormal operator S satisfying $C = C(S)$ and $\Lambda = \Lambda(S)$ if and only if the following conditions hold:

i) *Δ is separated.*

ii) *There exists a positive $\mathcal{L}(M)$-valued measure $de(\,\cdot\,)$ such that*

$$(\Lambda - z)^{-1}(1 - \mu(z)) = \int \frac{de(u)}{u - z}, \qquad z \in \mathbb{C} \setminus \big(\sigma(\Lambda) \cup \gamma \cup z(\Delta_s)\big) \tag{4.2}$$

and

$$\big(C - (\bar{u} - \Lambda^*)\,(u - \Lambda)\big) de(u) \equiv 0\,. \tag{4.3}$$

If i), ii) *hold, then the measure $de(\,\cdot\,)$ is connected with the operator S by the formula* (1.1), *and μ is Xia's mosaic of S.*

It follows, in particular, that (4.1) expresses the mosaic of any subnormal operator S of finite type in terms of matrices $C = C(S)$, $\Lambda = \Lambda(S)$.

By (1.4), the set of singularities of the function $(\Lambda - z)^{-1}(1 - \mu(z))$ is contained in the set $\sigma(\Lambda) \cup \gamma$, which has zero area. By the Hartogs-Rosenthal theorem (see [4]), $e(\cdot)$ is uniquely determined by (4.2), whenever it exists.

The next Theorem 2 is a more detailed version of Theorem 1. Before formulating it, we need to introduce a few more notions.

Definition. *The pair (C, Λ) will be called non-exceptional if there exists a finite subset Z of $\mathbb{C}$ such that for $z \in \mathbb{C} \setminus Z$, all Jordan blocks of the matrix $C(z-\Lambda)^{-1} + \Lambda^*$ corresponding to eigenvalues w with $\overline{w} \notin \sigma_C(\Lambda)$ are simple.*

In fact, the author does not know whether exceptional pairs (C, Λ) exist.

Suppose that Δ is separated, that is, $\partial\widehat{\Delta}_+ = \widehat{\Delta}_{\mathbb{R}}$. Let γ_{cns} be the set of all nonsingular points of the curve γ_c (see (2.8)), then $\gamma_c \setminus \gamma_{\mathrm{cns}}$ is finite. If $(z,\overline{z}) \in \Delta$ and $\tau'_z(z,\overline{z}) \neq 0$, $\tau'_w(z,\overline{z}) \neq 0$, then $(z,\overline{z}) \in \Delta_0$ and $z \in \gamma_{\mathrm{cns}}$.

Let us orient the curve γ_c according to the positive orientation of $\partial\widehat{\Delta}_+$ as a boundary of $\widehat{\Delta}_+$. There is a continuous function $\xi : \gamma_{\mathrm{cns}} \longrightarrow \mathbb{C}$ with $|\xi| \equiv 1$ such that $dz = i\,\xi(z)\,|dz|$ on γ_{cns}. Then $\eta((z,\overline{z})) \equiv \xi(z)^2$, $z \in \gamma_{\mathrm{cns}}$.

Theorem 2. *In the above Theorem* 1, *conditions* i)-ii) *can be replaced by the following conditions.*

i') Δ *is separated.*

ii') *The pair* (C,Λ) *is non-exceptional.*

iii') *The matrix-valued measure* $\xi(z)\,(z-\Lambda)^{-1}Q((z,\overline{z}))\,|dz|\,\big|_{\gamma_c}$ *is positive and finite.*

iv') *There exists a finite subset* R *of* $\mathbb{C}$ *such that a representation*

$$(\Lambda - z)^{-1}(1-\mu(z)) = \frac{1}{2\pi}\int_{\gamma_c} \frac{(u-\Lambda)^{-1}Q((u,\overline{u}))}{u-z}\,\xi(u)\,|du| + \sum_{\zeta\in R}\frac{A_\zeta}{\zeta - z}\,, \tag{4.4}$$

holds for some non-negative matrices A_ζ, $\zeta \in R$.

v') $\big(C - (\bar{\zeta} - \Lambda^*)\,(\zeta - \Lambda)\big)A_\zeta = 0$ *for all* $\zeta \in R$.

If i')-v') *hold, the measure* $de(\,\cdot\,)$ *that corresponds to the* (*unique*) *subnormal operator* S *such that* $C = C(S)$, $\Lambda = \Lambda(S)$ *is given by*

$$de(u) = \frac{1}{2\pi}\,(u-\Lambda)^{-1}Q((u,\overline{u}))\,\xi(u)\,|du|\,\big|_{\gamma_c} + \sum_{\zeta\in R} A_\zeta\,\delta_\zeta(u)\,, \tag{4.5}$$

where δ_ζ *is the delta-measure concentrated in* ζ.

In fact, the difference between the left-hand side and the integral in the right-hand side in (4.4) is always a rational matrix function. So iv') is only a restriction on the form of this function.

We remark that if i'), ii') hold, then, by [10, formula (57)], the matrix $\xi(u)\,(u-\Lambda)^{-1}Q((u,\overline{u}))$ is self-adjoint for $u \in \gamma_c$. It seems that Xia uses ii') implicitly in some of his arguments. In Section 6 below, we give an example of a non-exceptional pair (C, Λ) such that this matrix fails to be positive on certain arcs of γ_c. It would be interesting to know whether i') and ii') imply iii').

Theorem 2 and the above Theorem A by Xia permit one to construct the operator S from matrices $C = C(S)$, $\Lambda = \Lambda(S)$ (whenever it is possible). In [12], we will discuss this construction in detail.

5. Proofs of Theorems 1 and 2.

Lemma 1. *Let S be a subnormal operator of finite type, and put $C = C(S)$, $\Lambda = \Lambda(S)$. Let Δ be the discriminant surface of S and μ its mosaic. Let U be an open connected set contained in $\widehat{\Delta} \setminus \widehat{\Delta}_{\mathbb{R}}$, and let $\delta_0, \delta_1 \in U$, with $z(\delta_0), z(\delta_1) \in \mathbb{C} \setminus \gamma$.*

If $z(\delta_0) \in \mathbb{C} \setminus \sigma(S)$, then

$$\mu(z(\delta_1))\, Q(\delta_1) = 0\,. \tag{5.1}$$

If $\overline{w(\delta_0)} \in \mathbb{C} \setminus \sigma(S)$, then

$$\mu(z(\delta_1))\, Q(\delta_1) = Q(\delta_1)\,. \tag{5.2}$$

PROOF. We remind that for a domain G with piecewise smooth boundary, the Smirnov class $E^p(G)$ consists of functions f analytic in G such that

$$\sup_n \int_{\partial G_n} |f(z)|^p\, |dz| < \infty\,,$$

for some increasing sequence $\{G_n\}$ of domains with smooth boundaries such that $\cup G_n = G$; here $0 < p < \infty$. We refer to [2], [6] for basic properties of Smirnov classes. The Cauchy integral of any finite measure supported in $\mathbb{C} \setminus G$ belongs to $E^p(G)$ for any $p < 1$. So it follows from (1.4) that for each $p < 1$ and each connected component Ω of $\mathbb{C} \setminus \gamma$, μ belongs to $E^p(\Omega \longrightarrow \mathcal{L}(M))$. By (1.4) and the Plemelj "jump" formula [7], the interior and exterior boundary values μ_i, μ_e of μ satisfy

$$\mu_i(z) - \mu_e(z) = 2\pi i\, (z - \Lambda)\, \frac{de(z)}{|dz|}\, \frac{|dz|}{dz}\,,$$

almost everywhere on γ_c with respect to the arc length measure. Here $de(z)/|dz|$ is the Radon-Nikodim density of the absolutely continuous part of $de(z)$ with respect to $|dz|$. By (1.3), it follows that

$$\big(C(z - \Lambda)^{-1} + \Lambda^* - \overline{z}\big)\big(\mu_i(z) - \mu_e(z)\big) = 0\,, \tag{5.3}$$

almost everywhere on γ_c. We may assume that $U \setminus z^{-1}(\gamma_c)$ consists of a finite number of connected components $\Omega_0, \dots, \Omega_k$, that $\partial\Omega_j$ and $\partial\Omega_{j+1}$ have a common arc for $j = 0, \dots, k-1$ and $\delta_0 \in \Omega_0$, $\delta_1 \in \Omega_k$. Take any Ω_j and any $\delta = (z, w) \in \partial\Omega_j \cap \partial\Omega_{j+1} \cap \Delta_0$. Then we have $w \neq \overline{z}$ by the hypothesis.

Let ψ_δ be an analytic function in a neighbourhood of $\sigma(C(z - \Lambda)^{-1} + \Lambda^*)$ such that $\psi_\delta(u) = (u - \overline{z})^{-1}$ on a small neighbourhood of w and $\psi_\delta(u) = 0$ outside this neighbourhood. Putting $\Psi(\delta) = \psi_\delta(C(z - \Lambda)^{-1} + \Lambda^*)$, we obtain from the Riesz-Dunford calculus that

$$\Psi(\delta)\big(C\,(z-\Lambda)^{-1} + \Lambda^* - \bar{z}\big) = Q(\delta)\,,$$

so that (5.3) and (3.3) give

$$(\mu_i(z) - \mu_e(z))\,Q(\delta) = Q(\delta)\,(\mu_i(z) - \mu_e(z)) = 0\,, \tag{5.4}$$

almost everywhere on γ_c. Consider first the case $z(\delta_0) \in \mathbb{C} \setminus \sigma(S)$. Put

$$\varphi(\delta) = \mu(z(\delta))\,Q(\delta)\,, \qquad \delta \in U\,.$$

Since $\mu(z) \equiv 0$ in a neighbourhood of $z(\delta_0)$, it follows that $\varphi \equiv 0$ in Ω_0. Then (5.4) implies that $\varphi|\Omega_1$ has zero boundary values on $\partial\Omega_1 \cap \partial\Omega_0$. By the Privalov uniqueness theorem [7], $\varphi|\Omega_1 \equiv 0$. Continuing in the same way, we see that $\varphi \equiv 0$ in U, and this implies (5.1).

Now assume that $\overline{w(\delta_0)} \in \mathbb{C} \setminus \sigma(S)$ and let us prove (5.2). Our arguments are motivated by the proof of [10, Lemma 7.8]. Xia proves in [9] that the function

$$S(z, w) = \int \frac{de(u)}{(u - z)\,(\overline{u} - w)}\,, \tag{5.5}$$

defined for $z, \overline{w} \in \mathbb{C} \setminus \gamma$, for $(z, w) \notin \Delta$ has a representation

$$\begin{aligned} S(z, w) = &-\big(C - (w - \Lambda^*)\,(z - \Lambda)\big)^{-1}(1 - \mu(z)) \\ &+ \mu(\overline{w})^*\big(C - (w - \Lambda^*)\,(z - \Lambda)\big)^{-1}\,. \end{aligned} \tag{5.6}$$

It follows that if $(z, w) \in \mathbb{C}^2 \setminus \Delta$ is such that $\mu(\overline{w}) = 0$, then

$$(C\,(z - \Lambda)^{-1} + \Lambda^* - w)\,(z - \Lambda)\,S(z, w) = -(1 - \mu(z))\,. \tag{5.7}$$

By continuity, we can assert that this equality also holds for $(z, w) \in \Delta$ if $z \in \mathbb{C} \setminus (\gamma \cup \sigma(\Lambda))$, $\bar{w} \in \mathbb{C} \setminus \gamma$ and $\mu(\overline{w}) = 0$. In particular, (5.7) holds por points (z, w), $(z, w) \in \Delta_0$, in a neighbourhood of δ_0. By (1.5), the operator $C\,(z-\Lambda)^{-1} + \Lambda^*$ has an invariant subspace $(1-\mu(z))M$. It follows from (3.1) that

$$\text{(5.8)}\quad \Pi_0\big(C\,(z-\Lambda)^{-1} + \Lambda^* - w\big|(1-\mu(z))\,M\big) = (1-\mu(z))\,Q((z,w))\,.$$

On the other hand, by (5.7) and (1.5),

$$(C\,(z-\Lambda)^{-1} + \Lambda^* - w)\,(1-\mu(z))\,(z-\Lambda)\,S(z,w) = -(1-\mu(z))\,,$$

which shows that $C\,(z-\Lambda)^{-1} + \Lambda^* - w\big|(1-\mu(z))M$ is invertible. By (5.8),

$$(1-\mu(z))\,Q((z,w)) \equiv 0\,,$$

for points $(z, w) \in \Delta_0$ in a neighbourhood of δ_0. Putting $\varphi(\delta) = (1 - \mu(z(\delta))\,Q(\delta)$, $\delta \in U$ and proceeding as above, we obtain (5.2) in the same way as (5.1). The proof of the Lemma is complete.

PROOF OF THEOREM 1. SUFFICIENCY. This follows from the above Theorem A by Xia. Indeed, (4.1) and (2.6) imply that $\mu(z) \equiv 0$ in the unbounded component of $\mathbb{C} \setminus (\gamma \cup \sigma(\Lambda))$. Therefore, letting $z \to \infty$ in (4.2) we get $e(\mathbb{C}) = I$. Now one gets that (4.2) is equivalent to

$$\mu(z) = \int \frac{u-\Lambda}{u-z}\,de(u)\,, \qquad z \in \mathbb{C} \setminus \gamma\,.$$

So all the hypotheses of Theorem A are satisfied. In the model for S, given by this theorem,

$$\text{(5.9)}\qquad P_M f = \int de(u)\,f(u)\,, \qquad f \in \mathcal{L}^2(e)\,.$$

Formulas (5.9) and (1.8) imply that representation (1.1) of e holds.

NECESSITY. Now we start with a subnormal operator S of finite type and put $C = C(S)$, $\Lambda = \Lambda(S)$. If Δ is not separated, then there is a nondegenerate component $\widehat{\Delta}_k$ of $\widehat{\Delta}$ such that $\widehat{\Delta}_k \setminus \widehat{\Delta}_{\mathbb{R}}$ is connected. Take $U = \widehat{\Delta}_k \setminus \widehat{\Delta}_{\mathbb{R}}$; then U has points δ_0 with $z(\delta_0) \in \mathbb{C} \setminus \sigma(S)$ and points δ_0 with $\overline{w(\delta_0)} \in \mathbb{C} \setminus \sigma(S)$. We conclude from Lemma 1 that

$$0 = \mu(z(\delta))\,Q(\delta) = Q(\delta)\,,$$

for all δ in U, which is a contradiction.

So we may assume Δ to be separated. For any nondegenerate component $\widehat{\Delta}_l$ of $\widehat{\Delta}$, $\widehat{\Delta}_l \cap \widehat{\Delta}_+$ is the connected component of $\widehat{\Delta}_l \setminus \widehat{\Delta}_{\mathbb{R}}$ containing (all) points $\delta_0 \sim (z, \infty)$, where $z \in \sigma(\Lambda)$. This follows from (2.6) and the fact that $\widehat{\Delta}_l \setminus \widehat{\Delta}_{\mathbb{R}}$ has only two connected components. Moreover, $\widehat{\Delta}_l \cap \widehat{\Delta}_+$ contains neighbourhoods of the points of the above type. A similar fact about $\widehat{\Delta}_l \cap \widehat{\Delta}_-$ takes place. By applying Lemma 1 to connected sets $\widehat{\Delta}_l \cap \widehat{\Delta}_+$ and $\widehat{\Delta}_l \cap \widehat{\Delta}_-$, we see that for $\delta \in \widehat{\Delta}_l$,

$$\mu(z(\delta))\, Q(\delta) = \begin{cases} 0\,, & \delta \in \widehat{\Delta}_-\,, \\ Q(\delta)\,, & \delta \in \widehat{\Delta}_+\,. \end{cases} \tag{5.10}$$

Now let $\widehat{\Delta}_l$ be a degenerate component of $\widehat{\Delta}'$: $\widehat{\Delta}_l = \{w \equiv w_0\}$. Then there is a point $\delta_0 \in \widehat{\Delta}_l$ with $\delta_0 \sim (\infty, w_0)$. We conclude from Lemma 1 that $\mu(z(\delta))\, Q(\delta) \equiv 0$ on $\widehat{\Delta}_l$ in this case. Therefore for $z \notin \sigma(\Lambda) \cup z(\Delta_s)$

$$\mu(z) = \mu(z) \sum_{w:(z,w)\in\widehat{\Delta}'} Q((z,w)) = \sum_{w:(z,w)\in\Delta_+} Q((z,w))\,.$$

So the righthand part of (4.1) coincides with Xia's mosaic of S. Let $e(\,\cdot\,)$ be defined by (1.1), then (4.3) follows from Theorem A, and (4.2) from (1.4).

PROOF OF THEOREM 2. First we remark that conditions i')-v') of Theorem 2 imply conditions i), ii) of Theorem 1. Indeed, if i')-v') hold, then the measure de, defined by (4.5), is finite, positive-valued, and (4.2) holds. Condition v') implies (4.2) for the discrete part of $de(\,\cdot\,)$. Since the pair (C, Λ) is non-exceptional, one has

$$(C\,(z-\Lambda)^{-1} + \Lambda^*)\, Q((z,w)) = 0$$

identically for $(z,w) \in \widehat{\Delta}_{\mathrm{ndeg}}$. This and v') give (4.3).

Conversely, let us suppose that i), ii) hold, so that $C = C(S)$, $\Lambda = \Lambda(S)$ for an operator S of finite type. First we observe that (4.1) and (1.4) imply

$$(\Lambda - z)^{-1} \sum_{(z,w)\in\widehat{\Delta}'\setminus\widehat{\Delta}_+} Q((z,w)) = (\Lambda - z)^{-1}(1 - \mu(z)) = \int \frac{de(u)}{u - z}\,.$$

It follows that there exists a finite subset F of γ_c such that

$$de\big|\gamma_c \backslash F = \frac{1}{2\pi i}\,(u-\Lambda)^{-1}Q((u,\overline{u}))\,du = \frac{1}{2\pi}\,\xi(u)\,(u-\Lambda)^{-1}Q((u,\overline{u}))\,|du|\,.$$

Therefore iii') holds.

Put $R = F \cup \sigma_C(\Lambda)$. Then there exist positive matrices A_ζ, $\zeta \in R$, such that (4.5) holds. By (4.3),

$$(C\,(u-\Lambda)^{-1} + \Lambda^* - \overline{u})\,Q((u,\overline{u})) \equiv 0\,,$$

for all $u \in \gamma_c \setminus R$. By the definition of Q, the matrix $C\,(u-\Lambda)^{-1} + \Lambda^*$ for these u has no non-trivial Jordan blocks corresponding to eigenvalue $\overline{u}$.

Fix any nondegenerate component Δ_k of Δ. Then, since $\Delta_{\mathbb{R}} \cap \Delta_k$ contains an arc, there are infinitely many points $(z,w) \in \Delta_k$ such that all Jordan blocks of $C(z-\Lambda)^{-1} + \Lambda^*$ that correspond to the eigenvalue w are trivial. From a simple algebraic argument one sees that this property holds for all but a finite number of points (z,w) in Δ_k. Thus (C,Λ) is not exceptional. We conclude that all properties i')-v') take place.

6. Some additional results.

Proposition 2. *Let $C > 0$ and Λ be two operators on a finite-dimensional space M. Then there exists an operator S with $C = C(S)$, $\Lambda = \Lambda(S)$ if and ony if there exist a two-sided sequence of spaces $\{M_n\}_{n\in\mathbb{Z}}$ and operators $\Lambda_n \in \mathcal{L}(M_n)$, $R_n : M_{n-1} \to M_n$ such that*

1) $M_0 = M$,

2) $\operatorname{Range} R_n = M_n$ *for* $n > 0$ *and* $\operatorname{Range} R_n^* = M_{n-1}$ *for* $n \leqslant 0$,

3) $R_{n+1}^*\, R_{n+1} = R_n\, R_n^* + \Lambda_n\, \Lambda_n^* - \Lambda_n^*\, \Lambda_n$,

4) $R_{n+1}^*\, \Lambda_{n+1} = \Lambda_n\, R_{n+1}^*$;

5) $\Lambda_0 = \Lambda$, *and* $C = R_0\, R_0^*$,

6) $\sup_{n\in\mathbb{Z}} \|\Lambda_n\| < \infty$ *and* $\sup_{n\in\mathbb{Z}} \|R_n\| < \infty$.

For any such operators, put

$$K = \bigoplus_{n=-\infty}^{\infty} M_n\ , \qquad H = \bigoplus_{n=0}^{\infty} M_n\ ,$$

and define N, S, S' by the following two-diagonal block matrix

$$N = \left(\begin{array}{c|c} S'^* & 0 \\ \hline 0 & S \end{array}\right)$$

$$= \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & & & & & \\ & 0 & R_{-2} & \Lambda_{-2} & 0 & & & & & \\ & & 0 & R_{-1} & \Lambda_{-1} & 0 & & & & \\ & & & 0 & R_0 & \Lambda_0 & 0 & & & \\ & & & & 0 & R_1 & \Lambda_1 & 0 & & \\ & & & & & 0 & R_2 & \Lambda_2 & 0 & \\ & & & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$= \left(\begin{array}{ccccc|ccccc} \ddots & \ddots & \ddots & \ddots & & & & & & \\ & 0 & R_{-2} & \Lambda_{-2} & 0 & & & & & \\ & & 0 & R_{-1} & \Lambda_{-1} & 0 & & & & \\ \hline & & & 0 & R_0 & \Lambda_0 & 0 & & & \\ & & & & 0 & R_1 & \Lambda_1 & 0 & & \\ & & & & & 0 & R_2 & \Lambda_2 & 0 & \\ & & & & & & \ddots & \ddots & \ddots & \ddots \end{array}\right),$$

so that $S = N|H$. Then S is pure subnormal, N is its minimal normal extension, and $C(S) = C$, $\Lambda(S) = \Lambda$.

This proposition may be known to specialists. A similar fact about hyponormal operators is contained in [8]. Therefore we omit the proof, and make only the following observations.

If N, S, S' are defined in the above way, then S' is also subnormal; it is called dual to S. Conditions 3), 4) comprise to the equality $N^*N = NN^*$, and 5) follows from the definition of $C(S)$ and $\Lambda(S)$. Without loss of generality, one can assume that $M_n \subset M$ for all $n \in \mathbb{Z}$, and that $R_n = \widetilde{R}_n|M_n$, with $\widetilde{R}_n \in \mathcal{L}(M)$ and $\widetilde{R}_n^* = \widetilde{R}_n \geqslant 0$. Then 3), 4) permit one to define $\widetilde{R}_n$, Λ_n by forward and backward inductive processes in a unique way. Namely, if $n \geqslant 0$ and $(\widetilde{R}_n, \Lambda_n)$ have been determined, then $\widetilde{R}_{n+1}^2$ is defined by 3) and $\Lambda_{n+1}\colon M_{n+1} \longrightarrow M_{n+1}$ is uniquely determined from 4), because $\operatorname{Ker}\big(\widetilde{R}_{n+1}|M_{n+1}\big) = 0$. On each inductive step, either 3) or 4) may fail to produce $\widetilde{R}_{n+1}, \Lambda_{n+1}$ (for instance, if the $\widetilde{R}_{n+1}^2$ obtained fails to be non-negative). One has similar inductive definitions for $n < 0$. The subnormal S exists if and only if this two-sided inductive process fails nowhere and additionally, 6) holds.

For a point $\delta \in \widehat{\Delta}$, define the index $\tilde{\varkappa}(\delta)$ by $\tilde{\varkappa}(\delta) = \alpha_j$ if $\delta \in \widehat{\Delta}_j$; here $\widehat{\Delta}_j$ are the components of $\widehat{\Delta}$ and α_j the corresponding multiplicities, see (2.3), (2.4). Let $\zeta \in \mathbb{C} \setminus \gamma$ be such that $\widehat{\Delta}$ as a Riemann surface over the z-plane has no branching points that project into ζ. Define the index $\varkappa(\zeta)$ at ζ by

$$\varkappa(\zeta) = \sum \left\{ \tilde{\varkappa}(\delta) : \; z(\delta) = \zeta, \; \delta \in \widehat{\Delta}_+ \right\}. \tag{6.1}$$

Proposition 3. *For any $z \in \mathbb{C} \setminus \gamma$,*

$$\dim \operatorname{Ker} (S^* - \overline{z}\, I) = (\textit{the number of positive eigenvalues of } C - (\overline{z} - \Lambda^*)\,(z - \Lambda)). \tag{6.2}$$

If z is not a projection of a branching point of $\widehat{\Delta}$, then the above two quantities also coincide with $\varkappa(z)$.

PROOF. We have $(\partial/\partial w)\, \tau_j(z,w) \neq 0$ for $(z,w) \in \Delta_j$, except for a finite number of points (z,w), by virtue of the irreducibility of τ_j. Therefore $\det\bigl(C\,(z-\Lambda)^{-1} + \Lambda^* - \omega\bigr)$ has an α_j-th order zero at $\omega = w$ for $(z,w) \in \Delta_j$ for all but a finite number of points (z,w). For these points (z,w), $\operatorname{rank} Q\bigl((z,w)\bigr) = \tilde{\varkappa}\bigl((z,w)\bigr)$, and we conclude that

$$\dim \operatorname{Ker} (S^* - \overline{z}\, I) = \operatorname{rank} \mu(z) = \operatorname{rank} \sum_{(z,w) \in \widehat{\Delta}_+} Q((z,w)) = \varkappa(z)\,,$$

the first equality is from Xia's work [9, p. 277].

Let $z \in \mathbb{C} \setminus \gamma$, then by substituting $w = \overline{z}$ in (5.6), one gets

$$S(z, \overline{z}) = -A(1 - \mu(z)) + \mu(z)^* A\,,$$

where $A = \bigl(C - (\overline{z} - \Lambda^*)\,(z - \Lambda)\bigr)^{-1}$. Therefore

$$\mu(z)^* S(z, \overline{z})\, \mu(z) = \mu(z)^* A \mu(z)\,.$$

Set $n = \dim M$, and let k be the number of the positive eigenvalues of A (which equals to the right hand part of (6.2)). Since $S(z, \overline{z}) > 0$ by

(5.5), we obtain that the quadratic form (Ax, x) on M is positive on $\mu(z)M$. It follows that $\dim \mu(z)M \leqslant k$.

Similarly,

$$(1-\mu(z)^*)\, S(z,\overline{z})\, (1-\mu(z)) = -(1-\mu(z)^*)\, A(1-\mu(z))\,,$$

gives $\dim\,(1-\mu(z))\, M \leqslant n-k$. These two inequalities imply (6.2).

REMARK. If S, S' are subnormals of finite type and their essential spectra coincide, then the nondegenerate parts of their discriminant curves Δ, Δ' also coincide (we do not assert anything here about the corresponding multiplicities α_j).

Indeed, let γ_c, γ, etc. correspond to S and γ_c', γ', etc. to S'. Two subarcs of $\partial\widehat{\Delta}_+$ cannot project into the same arc in the z-plane. It follows from the last statement of Proposition 3 that $\gamma_c \subset \sigma_{\mathrm{ess}}(S) \subset \sigma(N) \subset \gamma$. Since $\gamma \setminus \gamma_c$, $\gamma' \setminus \gamma_c'$ are finite and $\sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(S')$, we conclude that $\gamma_c = \gamma_c'$. Take any nondegenerate component Δ_j of Δ. By Theorem 1, there is an arc β of γ_c such that Δ_j contains $\beta^\# = \{(z,\overline{z}) : z \in \beta\}$. Since $\beta \subset \gamma_c'$, $\beta^\# \subset \Delta_j \cap \Delta'$. By standard algebraic geometry, this implies $\Delta_j \subset \Delta'$.

If $\sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(S')$ and, moreover,

$$\dim\,\mathrm{Ker}\,(S^* - \overline{\lambda}) = \dim\,\mathrm{Ker}\,(S'^* - \bar{\lambda})\,,$$

for $\lambda \notin \sigma_{\mathrm{ess}}(S)$, then Proposition 3 and (6.1) imply that nondegenerate parts of Δ, Δ' coincide, and the multiplicities α_j of nondegenerate components also coincide.

AN EXAMPLE. The choice of the orientation of γ_c, made before Theorem 2, does not guarantee automatically that the matrix-valued function $\xi(z)\,(z-\Lambda)^{-1}Q((z,\overline{z}))$ is non-negative on γ_c. To see this, set $C = \begin{pmatrix} 26 & 5 \\ 5 & 15 \end{pmatrix} > 0$ and $\Lambda = \begin{pmatrix} 1 & -5 \\ 5 & 0 \end{pmatrix}$. The polynomial (2.1) takes the form

$$\tau(z,w) = (w + z - z\,w)\,(-10 - z\,w) + (5z - 5w)^2\,.$$

Since $\tau_z'(0,0) = \tau_w'(0,0) = -10 \neq 0$, $z = 0$ is a nonsingular point of γ_c. The implicit function $w = w(z)$, whose graph near $(0,0)$ is given by equation $\tau(z,w) = 0$, has the form $w = -z + 9z^2 + \overline{\overline{o}}(z^2)$. In particular, $|w'(z)| > 1$ for negative z with small $|z|$ and $|w'(z)| < 1$ for small positive z. Comparing with our choice of the orientation of γ_c, we conclude that $\xi((0,0)) = 1$. But from (3.2) one calculates that $Q((0,0)) = \begin{pmatrix} 1 & 0 \\ 5 & 0 \end{pmatrix}$, so that the matrix $\xi((0,0))\,(0-\Lambda)^{-1}Q((0,0)) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ fails to be non-negative.

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Maximal functions and Hilbert transforms associated to polynomials

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1. Introduction.

Let M denote the classical Hardy-Littlewood maximal function

$$Mf(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x-t)|\, dt$$

and H the classical Hilbert transform

$$Hf(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x-t)\, \frac{dt}{t}\,,$$

on $\mathbb{R}^1$. The mapping properties of these functions are very well-known (see for example [S1]), as are those of their higher dimensional analogues the Hardy-Littlewood-Wiener maximal function and the Calderón-Zygmund singular integral operators. Analogues of M and H associated to certain submanifolds of positive codimension in $\mathbb{R}^n$, $n \geq 2$, have also been extensively studied. These are the so-called maximal functions and singular integrals along surfaces, or maximal and singular Radon transforms. See for example [SW], [S2], [Ch], [PS1], [RS2], [CWW1], [CWW2]. One approach to these general problems is to model them on translation-invariant problems in certain homogeneous Lie groups so that the basic translation operation $(x,t) \longmapsto x-t$ on $\mathbb{R}^n \times \mathbb{R}^n$ is replaced by $(x,t) \longmapsto x\,t^{-1}$ on the Lie group. When written in terms of canonical coordinates, this multiplication is a polynomial mapping. Another approach, at least for the singular integral problems, is via oscillatory integrals and Fourier integral operators. In certain model cases a partial Fourier transform may be used to reduce the problem to a less singular one but with the familiar difference or inner product replaced by a more general mapping on $\mathbb{R}^n \times \mathbb{R}^n$. Once again, polynomial mappings provide substantial model cases in this setting. Thus an understanding of the classical operators of harmonic analysis with translation and inner product replaced by more general polynomial mappings is an important step in the study of higher dimensional problems associated to submanifolds.

However, very little seems to have been done systematically in this direction, with the principal exception of [RS1], [RS2], [PS2] and [HP]. In the present paper we take up this point in the context of the most classical one-dimensional operators of harmonic analysis, the Hardy-Littlewood maximal function and the Hilbert transform. While we do not believe our results will have any direct bearing on the higher dimensional problems mentioned above, it nevertheless seems a reasonable starting point to consider the one-dimensional setting first.

Thus we let $\mathfrak{p} : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a polynomial mapping $\mathfrak{p} : (x,t) \longmapsto \mathfrak{p}(x,t)$. We shall assume that $\mathfrak{p}$ has degree $n \geq 1$ in the second variable and that $\mathfrak{p}(x,0) = x$. (That this condition cannot be entirely dispensed with is discussed below, and is natural in so far as the averages occurring in $M_{\mathfrak{p}}$ below are then concerned with the local behaviour of f near x.) We define the maximal function and Hilbert transform associated to $\mathfrak{p}$ as

$$M_{\mathfrak{p}} f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(\mathfrak{p}(x,t))|\, dt$$

and

$$H_{\mathfrak{p}} f(x) = \text{p.v.} \int_{-\infty}^{\infty} f(\mathfrak{p}(x,t))\, \frac{dt}{t}\ ,$$

when these make sense. (Indeed, as a consequence of Theorem 2.4 below, $H_{\mathfrak{p}}$ can be realised as a principal-value distribution.) When $\mathfrak{p}(x,t) = x - p(t)$ – with p a polynomial of degree n of one real variable t satisfying $p(0) = 0$ – we sometimes write these as M_p and H_p. The main object of this paper is to begin to study the mapping properties of these operators.

The principal results are as follows:

Theorem 1. *If $\mathfrak{p}$ has degree n in t, then $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ are bounded on $L^p(\mathbb{R})$ when $p > n$. $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ may not be bounded on $L^n(\mathbb{R})$ for certain $\mathfrak{p}$ of degree n in t.*

Theorem 2. *If $\mathfrak{p}$ is quadratic in t, then the mapping properties of $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ can be precisely given terms of the behaviour of the coefficients of t and t^2 in $\mathfrak{p}$. (See Theorem 3.2 for full details.)*

Theorem 3. *If $\mathfrak{p}(x,t) = x - p(t)$, then the operators M_p and H_p are of weak-type 1-1 with bounds depending only on the degree n of p, and not otherwise on the coefficients.*

These theorems are proved in subsections 3.1, 3.2 and 3.3 respectively of Section 3.

As the conditions of Theorem 1.1 place no constraints on the (polynomial) coefficients of t whatsoever, it is natural to consider the situation when these coefficients of t are completely arbitrary functions of x. Thus we are lead to what we term the supermaximal function and superhilbert transform, which seem to be of independent interest. These are defined as

$$\mathcal{M}_n f(x) = \sup_{p \in \mathfrak{P}_n} M_p f(x) = \sup_{\substack{h>0 \\ p \in \mathfrak{P}_n}} \frac{1}{2h} \int_{-h}^{h} |f(x - p(t))| \, dt$$

and

$$T_n f(x) = \sup_{p \in \mathfrak{P}_n} |H_p f(x)| = \sup_{p \in \mathfrak{P}_n} \left| \int_{-\infty}^{\infty} f(x - p(t)) \, \frac{dt}{t} \right|,$$

where $\mathfrak{P}_n$ is the class of polynomials p of degree at most n in t with $p(0) = 0$. The result about these operators, proved in Section 2, is the following:

Theorem 1.4. *$\mathcal{M}_n$ and T_n are bounded on $L^q(\mathbb{R})$ if and only if $q > n$.*

An interesting lemma that we use to prove these results is that $|H_p f(x)|$ is pointwise dominated by $M_p f(x)$ plus the maximal Hilbert transform $H^* f(x)$ with constants depending only on the degree of p. ($H^* f(x)$ is defined as

$$\sup_{0<a<b<\infty} \left| \int_{a<|t|<b} f(x - t) \, \frac{dt}{t} \right|,$$

and it is well-known (see for example [S1]) that this operator is of weak-type 1-1.)

We comment upon the condition $\mathfrak{p}(x,0)=x$. This comes from the analogues in higher dimensions where one wants to think geometrically of $S(x,t)$ as, for each fixed x, some surface passing through x when $t=0$. However there is no particular reason to assume $\mathfrak{p}(x,0)=x$ in our setting other than that a necessary condition for any $L^p(p<\infty)$ boundedness of $M_\mathfrak{p}$ is that $\mathfrak{p}(x,0)$ have no critical points. (To see this, suppose $\mathfrak{p}(x,0)$ has a critical point at say zero. Then for δ sufficiently small, $|x|\le C\delta^{1/2}$ and $|t|\le C\delta$ implies $|\mathfrak{p}(x,t)|\le C'\delta$. Thus for $f=\chi_{(-\delta,\delta)}$,

$$\frac{1}{2h}\int_{-h}^{h} f(\mathfrak{p}(x,t))\,dt \ge 1\,, \qquad \text{if } |x|\le C\delta^{1/2} \text{ and } h\le C\delta\,.$$

Hence $\|M_\mathfrak{p} f\|_p \ge C\delta^{1/(2p)}$ while $\|f\|_p\sim\delta^{1/p}$. This is a contradiction unless $p=\infty$). If $\mathfrak{p}(x,0)$ does have no critical points, then one can in principle change variables to reduce to the case $\mathfrak{p}(x,0)=x$, but for modified maximal functions and Hilbert transforms whose coefficients are no longer polynomials. It is partly for this reason that we have stated Theorem 3.2 below for coefficients which are not necessarily polynomials.

Finally we make some remarks about possible higher-dimensional analogues of our results. We first note that there is no interesting supermaximal function or superhilbert transform, even of degree 1, in $\mathbb{R}^d$, $d\ge 2$. This is because the putative supermaximal function contains the universal maximal function associated to averages in arbitrary directions in $\mathbb{R}^d$, which is well known to be unbounded on all L^p, $p<\infty$, by the Perron tree example. (See [deG] for example.) On the other hand one can study operators such as

$$f \longmapsto \sup_{\substack{a,b\\ h>0}} \frac{1}{h}\Big|\int_0^h f(x-(a\,t,b\,t^2))\,dt\Big|$$

on $\mathbb{R}^2$ and indeed Marletta and Ricci [MR] have done so. Note that these operators arise in connection with Stein's and Bourgain's circular maximal function. Secondly, while it may well be true that there is an analogue of our Theorem 1.3 above in higher dimensions (indeed the L^p, $1<p<\infty$, variant is true in all dimensions) there is at present a serious obstacle to proving it, which is the fact that the weak-type 1-1 of the Hilbert transform and maximal function along a parabola in $\mathbb{R}^2$ are unknown. That is, while the operators

$$f \longmapsto \sup_{h>0} \frac{1}{h}\Big| \int_0^h f(x-(t,t^2))\, dt\Big|$$

and

$$f \longmapsto \int_{-\infty}^{\infty} f(x-(t,t^2))\, \frac{dt}{t}\,,$$

are known to be bounded on $L^p(\mathbb{R}^2)$, $1 < p < \infty$, it is not known whether they are of weak-type 1-1. See [SW]. However if $p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a polynomial which satisfies certain nondegeneracy conditions at 0 and ∞, then the higher-dimensional versions of M_p and H_p are of weak-type 1-1; moreover the same is true if we replace the additive structure of $\mathbb{R}^n$ by the group structure in any homogeneous Lie group. We plan to return to this matter in a forthcoming paper.

2. The supermaximal function and the superhilbert transform.

Let $\mathfrak{P}_n$ be the class of all real polynomials p, of a single real variable, of degree at most $n \geq 1$, such that $p(0) = 0$. Define

$$\mathcal{M}_n f(x) = \sup_{\substack{h>0\\ p\in\mathfrak{P}_n}} \frac{1}{2h}\int_{-h}^{h} |f(x-p(t))|\, dt = \sup_{p\in\mathfrak{P}_n} M_p f(x)$$

(the "supermaximal" function of degree n.)

Theorem 2.1. *Let $1 < q < \infty$. Then $\mathcal{M}_n$ is bounded on $L^q(\mathbb{R})$ if and only if $q > n$. Moreover $\mathcal{M}_n$ is of restricted weak-type n-n.*

Remark. When $n = 1$, $\mathcal{M}_1$ is the classical Hardy-Littlewood maximal operator in one variable, and so there is nothing to prove in this case. We shall appeal to the result for $\mathcal{M}_1$ in the cases of higher n.

The failure of boundedness when $q \leq n$ may be seen as follows. Let $\lambda > 0$ be large and let $p_\lambda(t) = \lambda(1-(1-t)^n)$. Let $f_\beta(t) =$

$|t|^{-1/n}\,|\log|t|\,|^{-\beta}\,\chi_{[0,1]}$. Then $f_\beta \in L^n$ if $\beta > 1/n$. Now for $x \gg 1$ we take $\lambda = x$ and $h = 1$ and observe that

$$\begin{aligned}
\int_0^1 f_\beta(x - p_\lambda(t))\,dt &= \int_0^1 f_\beta(x\,(1-t)^n)\,dt \\
&= \int_0^1 f_\beta(x\,t^n)\,dt \\
&= \frac{1}{x^{1/n}} \int_0^{x^{1/n}} f_\beta(s^n)\,ds \qquad (\text{if } x \gg 1) \\
&= \frac{1}{x^{1/n}} \int_0^1 s^{-1}(\log|s^n|)^{-\beta}\,ds \\
&= \infty\,, \qquad \text{if } \beta \le 1\,.
\end{aligned}$$

Furthermore, for each $r > 1$ we can find a $\beta \le 1$ such that $f_\beta \in L^{n,r}$. Indeed, $f_\beta \in L^{n,r}$ if and only if $\beta > 1/r$. Thus $\mathcal{M}_n$ does not map $L^{n,r}$ to any Lebesgue-Lorentz space for any $r > 1$. (See [StW] for a discussion of Lorentz spaces and related topics.)

PROOF OF THEOREM 2.1. We only need consider the restricted weak-type $n - n$ result as the case $q > n$ follows by interpolation with the trivial L^∞ result, and the negative result has been established in the discussion above.

Let $S \subseteq \mathbb{R}$ be a measurable set, and let $f = \chi_S$. It suffices to prove that $\|\mathcal{M}_n f\|_{n,\infty} \le C_n \|f\|_n$, by standard arguments from Lorentz spaces. Let $p \in \mathfrak{P}_n$ and $h > 0$ and consider

$$\frac{1}{h}\int_{-h}^{h} f(x - p(t))\,dt = \int_{I_h} f(x-u)\,g(u)\,du\,,$$

where $I_h = p([-h,h])$,

$$g(u) = \frac{1}{h}\sum_j \chi_{E_j}(u)\,\frac{1}{|p'(p_j^{-1}(u))|}\,, \tag{1}$$

where $\{E_j\}$ are the images under p of the intervals upon which p is monotonic, and where p_j^{-1} is the inverse to p on E_j. Then

$$\begin{aligned}\int_{I_h} f(x-u)\, g(u)\, du & \\ &\le \|f(x-\cdot)\|_{L^{n,1}(I_h)}\, \|g\|_{L^{n',\infty}(I_h)} \\ &= \|f(x-\cdot)\|_{L^{n}(I_h)}\, \|g\|_{L^{n',\infty}(I_h)} \qquad (\text{since } f=\chi_S) \\ &\le \sup_{h>0} \Big(\frac{1}{|I_h|}\int_{I_h} |f(x-u)|^n\, du\Big)^{1/n} |I_h|^{1/n}\, \|g\|_{L^{n',\infty}(I_h)}\, .\end{aligned}$$

Now $p(0)=0$, so $0\in I_h$, and thus

$$\Big(\frac{1}{|I_h|}\int_{|I_h|} |f(x-u)|^n\, du\Big)^{1/n}$$

is dominated by $(Mf^n)^{1/n}(x)$ where $M=M_1$ is the Hardy-Littlewood maximal function. Since

$$|\{x:\ (Mf^n)^{1/n}(x)>\alpha| = |\{x:\ Mf^n(x)>\alpha^n\}| \le \frac{2}{\alpha^n}\int f^n\, ,$$

the result now follows once we have established the following lemma.

Lemma 2.2. *There is an absolute constant C_n, depending only upon n, such that for all $h>0$, all $p\in\mathfrak{P}_n$,*

$$|I_h|^{1/n}\, \|g\|_{L^{n',\infty}(I_h)} \le C_n\ ,$$

(*Here g is defined as in* (1).)

PROOF. For $\lambda>0$ fixed,

$$\begin{aligned}|\{u\in I_h:\ |g(u)|>\lambda\}| &= \int_{I_h} \chi_{\{u\,:\,g(u)>\lambda\}}\, du \\ &= \int_{I_h} \chi_{\left\{u\,:\,\sum_j \chi_{E_j}(u)\, |p'(p_j^{-1}(u))|^{-1}>\lambda h\right\}}\, du \\ &= \int_{-h}^{h} \chi_{\{t\,:\,1/|p'(t)|>\lambda h\}} |p'(t)|\, dt \\ &\le \frac{1}{\lambda h}\Big|\Big\{t\in[-h,h]:\ |p'(t)|\le \frac{1}{\lambda h}\Big\}\Big|\, .\end{aligned}$$

On the other hand,

$$|I_h| = \int \chi_{I_h}(u)\,du = \int_{-h}^{h} |p'(t)|\,dt\,.$$

Thus, to establish the lemma, it is enough to show

$$\left|\left\{t \in [-h,h] :\ |p'(t)| \le \frac{1}{\lambda h}\right\}\right| \le \frac{C_n \lambda h}{\lambda^{n/(n-1)}\Big(\int_{-h}^{h} |p'(t)|\,dt\Big)^{1/(n-1)}}\,.$$

By scaling we may assume that $h = 1$ and that $\int_{-1}^{1} |p'| = 1$ and so we are reduced to showing

$$|\{t \in [-1,1] :\ |p'(t)| \le \alpha\}| \le C_n\, \alpha^{1/(n-1)} \tag{2}$$

under the normalization condition $\int_{-1}^{1} |p'| = 1$.

Consider the functional $\|\,|\cdot|\,\|$ on the class Q_{n-1} of polynomials of degree at most $n-1$ given by

$$\|\,|q|\,\| = \max_{0\le j\le n-1}\ \inf_{-1\le t\le 1} |q^{(j)}(t)|\,.$$

This is a continuous function of q, positively homogeneous of degree 1, which does not vanish on the unit sphere of Q_{n-1}, (measured, say, with respect to the L^1 norm on $[-1,1]$). For if $q(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}$ and $\|\,|q|\,\| = 0$, we have successively that $a_{n-1}, a_{n-2}, \ldots, a_0$ are all zero. Thus there is a constant m_n depending only upon n such that

$$\|\,|q|\,\| \ge m_n \int_{-1}^{1} |q(t)|\,dt\,.$$

Applying this to p', we see that for some j, $0 \le j \le n-1$, $|(p')^{(j)}(t)| \ge m_n$ for all $t \in [-1,1]$. The mean-value theorem now yields (2) for small α.

Remark. (2) is an endpoint version of a result of Ricci and Stein [RS1] which states that a polynomial of degree $n-1$ (in this case p') is in the Muckenhoupt A_q class, $q > n$, with constants independent of the coefficients. Inequalities such as (2) and variants in higher dimensions are also studied in [CCW].

We now turn to the superhilbert transform of degree n. Let

$$T_n f(x) = \sup_{p\in\mathfrak{P}_n} \left| \int_{-\infty}^{\infty} f(x-p(t))\,\frac{dt}{t} \right| = \sup_{p\in\mathfrak{P}_n} |H_p f(x)|\,.$$

Theorem 2.3. *Let $1 < q < \infty$. Then T_n is bounded on $L^q(\mathbb{R})$ if and only if $q > n$. Moreover T_n is of restricted weak-type n-n.*

Remark. Again, when $n = 1$, T_1 is the classical Hilbert transform and so there is nothing to prove.

The negative result can be seen in a similar manner to the corresponding result for $\mathcal{M}_n$. Indeed, with the same f_β as above, the nonintegrable singularity of f_β when $\beta \le 1$ guarantees that

$$\int_{-\infty}^{\infty} f_\beta(x + p_\lambda(t))\,\frac{dt}{t}$$

will be $+\infty$ when λ is taken to be x, at least for large x.

The positive part of Theorem 2.3 follows from the following result, which is also useful in other contexts.

Theorem 2.4. *Let $p \in \mathfrak{P}_n$. Then there is the pointwise estimate*

$$|H_p f(x)| \le A_n\, M_p f(x) + B_n\, H^* f(x)\,,$$

where H^* is the maximal Hilbert transform and A_n and B_n are constants depending only upon n.

Proof. Let $p \in \mathfrak{p}_n$, and assume without loss of generality that p has degree n and has leading coefficient 1. We also assume (although this is not strictly speaking necessary) that all the complex roots of p are distinct. Let $0 = t_1, t_2, \ldots, t_n$ be the n complex roots of p ordered so that

$$0 < |t_2| \le |t_3| \le \cdots \le |t_n|\,.$$

The second and third parts of the next lemma say that the zeros of p' are strongly attracted to the zeros of p.

Lemma 2.5 *There are constants $C(n) \ge 1$ and $\varepsilon_0(n)$ depending only on n, such that if $A > C(n)$ and j and ℓ are such that $\ell - j \ge 3$ and are such that for some $k \in \{1, \ldots, n-1\}$*

$$|t_k| < A^j < A^\ell < |t_{k+1}|\,,$$

then

a) *If* $A^{j+1} \leq |t| \leq A^{\ell-1}$,

$$\Big(1-\frac{1}{A}\Big)^{n-1}|t|^k|t_{k+1}|\cdots|t_n| \leq |p(t)| \\ \leq \Big(1+\frac{1}{A}\Big)^{n-1}|t|^k|t_{k+1}|\cdots|t_n|\,,$$

b) $|t\,p'(t)/p(t)| \geq \varepsilon_0(n)$ *whenever* $A^{j+1} \leq |t| \leq A^{\ell-1}$,

c) $|p(t)|$ *is strictly increasing on* $[A^{j+1}, A^{\ell-1}]$ *and strictly decreasing on* $[-A^{\ell-1}, -A^{j+1}]$.

PROOF. a) This part is trivial since $p(t) = \prod_{m=1}^n (t-t_m)$ and, when $A^{j+1} \leq |t| \leq A^{\ell-1}$,

$$\Big(1-\frac{1}{A}\Big)|t| \leq |t-t_m| \leq \Big(1+\frac{1}{A}\Big)|t|\,, \qquad \text{for } 2 \leq m \leq k\,,$$

while

$$\Big(1-\frac{1}{A}\Big)|t_m| \leq |t-t_m| \leq \Big(1+\frac{1}{A}\Big)|t_m|\,, \qquad \text{for } k+1 \leq m \leq n\,.$$

(Note that only $A > 1$ is required here.)

b) Observe first that

$$\frac{p'(t)}{p(t)} = \sum_{m=1}^n \frac{1}{t-t_m}\ ,$$

so

$$\Big|\frac{p'(t)}{p(t)}\Big| \geq \Big|\sum_{m=1}^k \frac{1}{t-t_m}\Big| - \sum_{m=k+1}^n \frac{1}{|t-t_m|} \geq \Big|\sum_{m=1}^k \frac{1}{t-t_m}\Big| - \frac{(n-k)}{(A-1)\,|t|}\ ,$$

since $|t_m| \geq A\,|t|$ if $m \geq k+1$ and $|t| \in [A^{j+1}, A^{\ell-1}]$.

Assume for simplicity that $t > 0$ and consider, for $m \leq k$

$$\operatorname{Re}\frac{1}{t-t_m} = \frac{t-\operatorname{Re} t_m}{|t-t_m|^2} > \frac{\Big(1-\dfrac{1}{A}\Big)t}{\Big(1+\dfrac{1}{A}\Big)^2 t^2} = \frac{\Big(1-\dfrac{1}{A}\Big)}{\Big(1+\dfrac{1}{A}\Big)^2}\,\frac{1}{t}\ ,$$

since $t > A\,|t_m|$.

Therefore

$$\Big|\frac{p'(t)}{p(t)}\Big| \geq \left(k\,\frac{\Big(1-\dfrac{1}{A}\Big)}{\Big(1+\dfrac{1}{A}\Big)^2} - \frac{n-k}{A-1}\right)\frac{1}{t}\,.$$

Now if A is sufficiently large, the coefficient of $1/t$ is positive, which implies that $|t\,p'(t)/p(t)|$ is bounded below by an absolute constant.

c) We have in fact shown that

$$\frac{p'(t)}{p(t)} = \operatorname{Re}\frac{p'(t)}{p(t)} > 0\,, \qquad \text{for } t > 0\,,$$

that is, $\log|p(t)|$ is increasing on $[A^{j+1}, A^{\ell-1}]$. Thus, $|p(t)|$ is strictly increasing on $[A^{j+1}, A^{\ell-1}]$ and similarly is strictly decreasing on $[-A^{\ell-1}, -A^{j+1}]$.

In particular, if $A^j \leq |t_2| \leq A^{j+1}$, then $|t\,p'(t)/p(t)|$ is bounded below and p is monotonic on $[-A^{j-1}, A^{j-1}]$. One simply has to observe that, since 0 is a simple root, p is monotonic through 0.

Furthermore, implicit in the proof of Lemma 2.5 is that if $|t_n| < A^{j_*}$ and $|t| \geq A^{j_*+1}$ then

$$\Big(1-\frac{1}{A}\Big)^{n-1}|t|^n \leq |p(t)| \leq \Big(1+\frac{1}{A}\Big)^{n-1}|t|^n$$

and $|t\,p'(t)/p(t)|$ is bounded below, and $|p(t)|$ is strictly increasing on $[A^{j_*+1}, \infty)$ and strictly decreasing on $(-\infty, -A^{j_*+1}]$.

A maximal set of the form $[-A^{\ell-1}, -A^{j+1}] \cup [A^{j+1}, A^{\ell-1}]$ with $\ell - j \geq 3$ and such that for some $k \in \{2, \dots, n-1\}$,

$$|t_k| < A^j < A^\ell < |t_{k+1}|$$

is called a *gap*. There are at most $n-2$ such gaps. In addition there are two *special gaps*, $[-A^{j-1}, A^{j-1}]$ where $A^j \leq |t_2| \leq A^{j+1}$, and $(-\infty, -A^{j_*+1}] \cup [A^{j_*+1}, \infty)$, where j_* is the least integer such that $|t_n| < A^{j_*}$.

Two consecutive gaps are separated by a pair of “dyadic” intervals, symmetric with respect to the origin. In fact each of these “dyadic” intervals can contain at most $3n$ intervals of the form $[A^m, A^{m+1}]$

or $[-A^{m+1}, -A^m]$. The idea of the remainder of the proof is that such dyadic intervals are harmless since the contribution to $\int_{-\infty}^{\infty} f(x - p(t))\,dt/t$ arising from such an interval is clearly controlled by a constant times $M_p f(x)$, while on the gaps – where p' and p (except at 0) have no zeros – one can try to change variables as in the proof of Theorem 2.1. However this is not entirely straightforward because of the nature of the cancellation in the problem.

We now indicate how to handle the contribution to $\int_{-\infty}^{\infty} f(x - p(t))\,dt/t$ arising from an (ordinary) gap; the minor changes of detail required for the special gaps are left to the reader. Suppose the gap is $[-A^\ell, -A^j] \cup [A^j, A^\ell]$ with $\ell - j \geq 1$ and with $|t_k| < A^{j-1} < A^{\ell+1} < |t_{k+1}|$, $(2 \leq k \leq n-1)$. (Note that there is a slight change of notation here.) Of course A is chosen so that Lemma 2.5 is valid.

By part a) of Lemma 2.5,

$$\begin{aligned} |p(A^\ell)| &\geq \Big(1 - \frac{1}{A}\Big)^{n-1} A^{\ell k} \prod_{m=k+1}^{n} |t_m| \\ &> \Big(1 + \frac{1}{A}\Big)^{n-1} A^{jk} \prod_{m=k+1}^{n} |t_m| \\ &\geq |p(-A^j)| \end{aligned}$$

(if also $A > ((A+1)/(A-1))^{n-1}$) and similarly $|p(-A^\ell)| > |p(A^j)|$. Thus the intervals $[\,|p(A^j)|, |p(A^\ell)|\,]$ and $[\,|p(-A^j)|, |p(-A^\ell)|\,]$ have a nonempty intersection $[a, b]$, say. Then, by Lemma 2.5.c), there is a unique $\alpha \in [A^j, A^\ell]$ such that $|p(\alpha)| = a$ and a unique $\beta > \alpha$, $\beta \in [A^j, A^\ell]$ such that $|p(\beta)| = b$. Similarly there are unique $-\delta < -\gamma \in [-A^\ell, -A^j]$ such that $|p(-\gamma)| = a$, $|p(-\delta)| = b$. Observe that the set

$$([A^j, A^\ell]\backslash[\alpha, \beta]) \cup ([-A^\ell, -A^j]\backslash[-\delta, -\gamma])$$

is the union of two intervals whose logarithmic measure is bounded above by an absolute constant. (This follows again by Lemma 2.5.a); we suggest the reader draw a picture.) Therefore the integral over this set is dominated by $M_p f(x)$.

We have thus reduced matters to estimating

$$\int_\alpha^\beta f(x - p(t))\,\frac{dt}{t} + \int_{-\delta}^{-\gamma} f(x - p(t))\,\frac{dt}{t}\,.$$

We distinguish between two cases:

i) p has the same sign on both intervals $[\alpha,\beta]$ and $[-\delta,-\gamma]$; say $p>0$,

ii) p has opposite signs on the two intervals; say $p>0$ for $t>0$.

Case i). We observe that

$$\int_\alpha^\beta \frac{p'(t)}{p(t)}\,dt = -\int_{-\delta}^{-\gamma}\frac{p'(t)}{p(t)}\,dt$$

since $p(\alpha)=p(-\gamma)$ and $p(\beta)=p(-\delta)$. Thus it is enough to estimate two similar integrals separately, one of which is

$$\int_\alpha^\beta f(x-p(t))\Big(\frac{1}{t}-\frac{p'(t)}{k\,p(t)}\Big)\,dt\,,$$

where $|t_k|<A^{j-1}<A^{\ell+1}<|t_{k+1}|$.

Now, for $t\in[\alpha,\beta]\subseteq[A^j,A^\ell]$,

$$\begin{aligned}
\Big|\frac{1}{t}-\frac{p'(t)}{k\,p(t)}\Big| &= \Big|\frac{1}{t}-\frac{1}{k}\sum_{m=1}^{n}\frac{1}{t-t_m}\Big| \\
&\le \frac{1}{k}\sum_{m=1}^{k}\Big|\frac{1}{t}-\frac{1}{t-t_m}\Big| + \frac{1}{k}\sum_{m=k+1}^{n}\frac{1}{|t-t_m|} \\
&= \frac{1}{k}\sum_{m=1}^{k}\frac{|t_m|}{|t|\,|t-t_m|} + \frac{1}{k}\sum_{m=k+1}^{n}\frac{1}{|t-t_m|} \\
&\le \frac{c_1A^j}{t^2}+c_2A^{-\ell}\,,
\end{aligned}$$

where c_1 and c_2 depend upon n and A.

Therefore

$$\begin{aligned}
\Big|\int_\alpha^\beta f(x-p(t))\Big(\frac{1}{t}-\frac{p'(t)}{k\,p(t)}\Big)\,dt\Big| &\le c_1A^j\int_{A^j}^\infty |f(x-p(t))|\,\frac{dt}{t^2} \\
&\quad + c_2A^{-\ell}\int_0^{A^\ell}|f(x-p(t))|\,dt \\
&\le c_3M_pf(x)\,.
\end{aligned}$$

Case ii). It is here that we finally use the cancellation in the operator. Indeed,

$$
\begin{aligned}
&\int_\alpha^\beta f(x-p(t))\,\frac{dt}{t}+\int_{-\delta}^{-\gamma} f(x-p(t))\,\frac{dt}{t}\\
&\quad=\int_\alpha^\beta f(x-p(t))\Big(\frac{1}{t}-\frac{p'(t)}{k\,p(t)}\Big)\,dt+\int_{-\delta}^{-\gamma} f(x-p(t))\Big(\frac{1}{t}-\frac{p'(t)}{k\,p(t)}\Big)\,dt\\
&\quad\quad+\frac{1}{k}\int f(x-p(t))\,\frac{p'(t)}{p(t)}\,(\chi_{[\alpha,\beta]}(t)+\chi_{[-\delta,-\gamma]}(t))\,dt\,.
\end{aligned}
$$

The first two integrals are treated exactly as in case i), while for the third we change variables separately on $[\alpha,\beta]$ and $[-\delta,-\gamma]$ to obtain

$$
\frac{1}{k}\int_{a\le|u|\le b} f(x-u)\,\frac{du}{u}\,,
$$

which is controlled by the maximal Hilbert transform as desired. This concludes the proof of Theorem 2.4.

3. $\mathfrak{p}(x,t)$ as a polynomial in t.

Let $\mathfrak{p}:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ be a polynomial such that $\mathfrak{p}(x,0)=x$. Let

$$
M_{\mathfrak{p}}f(x)=\sup_{h>0}\frac{1}{2h}\int_{-h}^{h} f(\mathfrak{p}(x,t))\,dt
$$

and

$$
H_{\mathfrak{p}}f(x)=\int_{-\infty}^{\infty} f(\mathfrak{p}(x,t))\,\frac{dt}{t}
$$

be the maximal function and Hilbert transform respectively associated to $\mathfrak{p}$. We write

$$
\mathfrak{p}(x,t)=x+A_1(x)\,t+A_2(x)\,t^2+\cdots+A_n(x)\,t^n\,,
$$

so that $\mathfrak{p}$ has degree at most n as a polynomial in t; $A_1,\dots,A_n$ are for the moment arbitrary polynomial functions of x.

3.1. Results with no conditions on the coefficients.

In view of the negative parts of Theorems 2.1 and 2.3, the only possible general positive result (with no conditions placed on the coefficients) is:

Theorem 3.1. *For $\mathfrak{p}(x,t)$ an arbitrary polynomial of degree n in t such that $\mathfrak{p}(x,0)=x$, the operators $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ are bounded on $L^q(\mathbb{R})$ for $q>n$ and are of restricted weak-type n-n.*

This result is sharp in so far as for each n there exists a $\mathfrak{p}$ of degree n in t as in the statement of the theorem with $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ unbounded on $L^n(\mathbb{R})$. Indeed, letting $\mathfrak{p}(x,t)=x\,(1-t)^n$, the proof of the sharpness of Theorems 2.1 and 2.3 applies here also. When $n=2$ we give below in Corollary 3.7 a complete analysis of the L^q boundedness problem for each $\mathfrak{p}$.

3.2. Many coefficients vanishing - the quadratic case.

When all but one of the A_j's is identically zero and the remaining one is a completely arbitrary function of x, then $H_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ are dominated by the standard Hilbert transform and maximal function respectively and so are of weak-type 1-1 and are L^q bounded, $1<q<\infty$. (If j is even and $A_j(x)$ is the nonzero coefficient, then $H_{\mathfrak{p}}\equiv 0$.)

The situation when all but two of the A_j's are identically zero is already considerably more complicated; the first special case of this is

$$\mathfrak{p}(x,t)=x+A_1(x)\,t+A_2(x)\,t^2$$

corresponding to polynomials of degree 2 in t.

In Theorem 3.2 we give an analysis of this quadratic case. We have carried out a similar but much lengthier analysis of the cubic case which we do not propose to present here; the interested reader is invited to contact one of the authors for details. (We estimate that merely a statement of the result would fill several printed pages and so we have chosen not to unecessarily burden the reader at this moment.)

We set up some notation. Let p and q be arbitrary C^1 functions of x. We write $A_1=p$ and $A_2=q$ so that

$$\mathfrak{p}(x,t)=x+t\,p(x)+t^2\,q(x)\,.$$

We let $\Delta(x) = p^2(x) - 4\,x\,q(x)$ be the discriminant of $\mathfrak{p}(x,\cdot)$ as a quadratic in t, and when $q(x) \neq 0$ we let $\psi(x) = \Delta(x)/4\,q(x)$. We shall require ψ to have some smoothness. It turns out that the critical points of ψ play a decisive role. We say that ψ has a *monotonic critical point* at $\pm\infty$ if $\lim_{x\to\pm\infty} \psi'(x) = 0$ and ψ' is single signed as $x \longrightarrow \pm\infty$. We say that ψ has a *critical point of finite order* $k \geq 2$ at $x_0 \in \mathbb{R}$ if

$$\psi(x) = \psi(x_0) + \delta\,(x - x_0)^k + O(|x - x_0|^{k+1})$$

with $\delta \neq 0$.

Theorem 3.2. *With the notation as above, let* $\mathfrak{p}(x,t) = x + t\,p(x) + t^2\,q(x)$ *with* $p, q \in C^1$ *such that* $Z_q = \{q(x) = 0\}$ *is finite.*

i) *If* $|\psi'|$ *is bounded below on* $\mathbb{R}\setminus Z_q$ *then* $M_{\mathfrak{p}}$ *and* $H_{\mathfrak{p}}$ *are of weak-type* 1-1 *and are bounded on* L^r, $1 < r < \infty$.

ii) *If* $|\psi'|$ *is bounded below at* $\pm\infty$ *and near* Z_q, *if* ψ *has finitely many critical points of finite order at each of which* $\psi(x)+x$ *is nonzero, then* $M_{\mathfrak{p}}$ *and* $H_{\mathfrak{p}}$ *are bounded on* L^r *if and only if* $r \geq 2\,(k-1)/k$, *where* k *is the maximum of the orders of the critical points. When* $k = 2$ *this must be modified to read as* $M_{\mathfrak{p}}$ *and* $H_{\mathfrak{p}}$ *are of weak-type* 1-1 *and bounded on* L^r, $1 < r < \infty$.

iii) *If either*

a) ψ *has a monotonic critical point at* $\pm\infty$, *or*

b) ψ *has a critical point of finite order at* x_0 *such that*

$$\psi(x_0) + x_0 = 0\,,$$

then $M_{\mathfrak{p}}$ *and* $H_{\mathfrak{p}}$ *are unbounded on* L^2, *and bounded on* L^r *for* $r > 2$.

Before proving this theorem we first give some lemmas.

Lemma 3.3.

$$\sup_{\substack{p,q\in\mathbb{R}\setminus\{0\}\\ h>0}} \frac{1}{h}\int_{[h,2h]\cap\{|t+p/(2q)|\geq|p|/(4|q|)\}} |f(x+p\,t+q\,t^2)|\,dt \leq C\,Mf(x)\,,$$

where Mf *is the ordinary Hardy-Littlewood maximal function of* f.

PROOF. By scaling it is enough to take $h = 1$. Assume without loss of generality that $q > 0$. We split the integral into two pieces, the first over $|p|/(4q) \leq |t+p/(2q)| \leq 10\,|p|/q$, and the second over $|t+p/(2q)| \geq 10\,|p|/q$. Let $u = u(t) = p\,t + q\,t^2$; then

$$|u'(t)| = |p + 2\,q\,t| = 2\,q\Big|t + \frac{p}{2q}\Big| \approx |p|$$

in the first case and

$$|u'(t)| = |p + 2\,q\,t| = 2\,q\Big|t + \frac{p}{2q}\Big| \approx q\,|t| \approx q^{1/2}\,u^{1/2}$$

in the second case. Thus,

$$\begin{aligned}
\int_{[1,2]\cap\{|p|/(4q)\leq|t+p/(2q)|\leq 10|p|/q\}} &|f(x + p\,t + q\,t^2)|\,dt \\
&\leq \int_{\{|u|\leq C|p^2|/q\}\cap u[1,2]} |f(x+u)|\,\frac{du}{|p|} \\
&= \Big|\frac{p}{q}\Big|\,\Big|\frac{q}{p^2}\Big| \int_{\{|u|\leq C|p^2|/q\}\cap u[1,2]} |f(x+u)|\,du \\
&\leq C M f(x)\,,
\end{aligned}$$

since if $1 \leq t \leq 2$, we get a nonzero contribution only when $|p|/|q| \approx C$. For the second piece

$$\begin{aligned}
\int_{[1,2]\cap\{|t+p/(2q)|\geq 10|p|/q\}} &|f(x + p\,t + q\,t^2)|\,dt \\
&\leq C \int_{u\sim q} |f(x+u)|\,\frac{du}{q^{1/2}\,u^{1/2}} \\
&\leq C M f(x)
\end{aligned}$$

since if $1 \leq t \leq 2$ and $|t + p/(2q)| \geq 10|p|/q$then $|u(t)| \approx q\,t^2 \approx q$.

Corollary 3.4.

$$\sup_{\substack{p,q\in\mathbb{R}\setminus\{0\}\\ h>0}} \frac{1}{h} \int_{[-h,h]\cap\{|t+p/(2q)|\geq|p|/4|q|\}} |f(x + p\,t + q\,t^2)|\,dt \leq C M f(x)\,.$$

PROOF. Break up $[-h, h]$ into dyadic intervals $\pm[2^{-k}h, 2^{-k+1}h]$ and use Lemma 3.3 on each to obtain a convergent geometric series.

Thus for the maximal function problem, all we need consider is

$$\tilde{M}_{\mathfrak{p}} f(x) = \sup_{h>0} \frac{1}{h} \int_{\mathcal{H}} |f(x + p(x)\,t + q(x)\,t^2)|\,dt\,, \tag{3}$$

where

$$\mathcal{H} = [-h, h] \cap \Big\{ \Big| t + \frac{p(x)}{2\,q(x)} \Big| \le \frac{|p(x)|}{4\,|q(x)|} \Big\}.$$

Now when $p(x)$ or $q(x)$ (or both) are zero, $M_{\mathfrak{p}} f(x) \le CMf(x)$ and $|H_{\mathfrak{p}} f(x)| \le C\,|Hf(x)|$, so that we may assume here and in what follows that we need consider only x with $p(x)$, $q(x) \ne 0$. By virtue of Theorem 2.4, we have

$$|H_{\mathfrak{p}} f(x)| \le A_2 M_{\mathfrak{p}} f(x) + B_2 H^* f(x) \le C\,(\tilde{M}_{\mathfrak{p}} f(x) + Mf(x) + H^* f(x))$$

and so to control the Hilbert transform we again only need consider $\tilde{M}_{\mathfrak{p}} f(x)$. Furthermore it is easily seen (using arguments from Lemma 3.3 and Theorem 2.4) that

$$|H_{\mathfrak{p}} f(x) - \tilde{H}_{\mathfrak{p}} f(x)| \le C\,(Mf(x) + H^* f(x))\,,$$

where

$$\tilde{H}_{\mathfrak{p}} f(x) = \int_{|t+p(x)/(2q(x))| \le |p(x)|/(4|q(x)|)} f(x + p(x)\,t + q(x)\,t^2)\,\frac{dt}{t}\,. \tag{4}$$

Since for each fixed x, the integral in (4) is over a dyadic interval, there is no further cancellation in the operator $\tilde{H}_{\mathfrak{p}}$ and indeed $\tilde{H}_{\mathfrak{p}}$ is essentially a contribution to $\tilde{M}_{\mathfrak{p}}$ where h takes the value $2\,|p(x)|/|q(x)|$. On the other hand this value of h is the only interesting one contributing to $\tilde{M}_p$, and so the operators $\tilde{M}_{\mathfrak{p}}$ and $\tilde{H}_{\mathfrak{p}}$ are both essentially equivalent to

$$\begin{aligned} &R_{\mathfrak{p}} f(x) \\ &\quad = \Big| \frac{q(x)}{p(x)} \Big| \int_{|t+p(x)/(2q(x))| \le |p(x)|/4|q(x)|} f(x + p(x)\,t + q(x)\,t^2)\,dt\,, \end{aligned} \tag{5}$$

which therefore governs the behaviour of both $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$.

At this point it is appropriate to comment upon the simple averaging operator

$$S_{\mathfrak{p}} f(x) = \int_1^2 f(x + p(x)\,t + q(x)\,t^2)\,dt\,. \tag{6}$$

Clearly $S_{\mathfrak{p}}$ is dominated by $M_{\mathfrak{p}}$, and if $H_{\mathfrak{p}}$ has certain boundedness property, so does $S_{\mathfrak{p}}$ (see for example [CG].) On the other hand, making the change of variables $t = u\,p(x)/q(x)$ in (5) gives

$$R_{\mathfrak{p}} f(x) = \int_{|u+1/2|\le 1/4} f(x + \tilde{p}(x)\,u + \tilde{q}(x)\,u^2)\,du\,, \tag{7}$$

where $\tilde{p}(x) = p^2(x)/q(x)$ and $\tilde{q}(x) = p^2(x)/q(x)$ also. Thus $R_{\mathfrak{q}}$ arises essentially as $S_{\widetilde{\mathfrak{p}}}$ where

$$\widetilde{\mathfrak{p}}(x,t) = x + \frac{p^2(x)}{q(x)}\,t + \frac{p^2(x)}{q(x)}\,t^2\,.$$

Thus positive results for $S_{\widetilde{\mathfrak{p}}}$ imply corresponding ones for $S_{\mathfrak{p}}$ although there is no formal invariance property from which this follows. Notice that if we define $\tilde{\Delta} = \tilde{p}^2 - 4\,x\,\tilde{q}(x)$ and $\tilde{\psi} = \tilde{\Delta}/4\tilde{q}$, then $\tilde{\psi} = \psi$ and $\tilde{p}^2/4\tilde{q} = p^2/4q$; that is, the quantities arising in the statement of Theorem 3.2 remain invariant, which is natural since the basic problems for $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ are invariant under

$$(p,q) \longmapsto (p(x)h(x), q(x)h(x)^2) = (\tilde{\tilde{p}}, \tilde{\tilde{q}})$$

for any $h(x) \ne 0$. Indeed, the basic problem for $M_{\mathfrak{p}}$ is equivalent to that for $S_{\tilde{\tilde{\mathfrak{p}}}}$ with arbitrary $h(x)$, as can be seen by linearising $M_{\mathfrak{p}}$ with $h(x)$.

Performing the further changes of variables $u = v - 1/2$ and then $v = \{s/\tilde{p}(x)\}^{1/2}$ (assuming that $\tilde{p}(x) > 0$ without loss of generality) yields in (7)

$$T_{\widetilde{\mathfrak{p}}} f(x) = \frac{1}{\tilde{p}(x)^{1/2}} \int_{0\le s\le \tilde{p}(x)} f(s - \tilde{\psi}(x))\,\frac{ds}{s^{1/2}} \tag{8}$$

as the operator determining the behaviour of $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$.

Lemma 3.5. *For* $0 \leq x \leq 1$, $\alpha \geq 2$ *and* $\beta \geq 0$ *define*

$$T_{\alpha,\beta} f(x) = \frac{1}{x^{\beta/2}} \int_0^{x^\beta} f(s - x^\alpha)\, \frac{ds}{s^{1/2}} \, .$$

Then for $p < \infty$, $T_{\alpha,\beta}$ *is bounded from* $L^p(\mathbb{R})$ *to* $L^p(0,1)$.

i) *If* $\beta = 0$ *for* $p \geq 2\,(\alpha - 1)/\alpha$, *except when* $\alpha = 2$, *in which case for* $p > 1$; *moreover* $T_{2,0}$ *is of weak-type* 1-1,

ii) *if* $0 < \beta < 1$ *for* $p \geq 2\,(\alpha - 1)/(\alpha - \beta)$,

iii) *if* $\beta = 1$ *for* $p > 2$.

In all other cases, or if $\beta > 1$, $T_{\alpha,\beta}$ *is unbounded.*

PROOF. Let $\psi(x) = x^\alpha$. Then

$$\begin{aligned} \int_0^1 |T_{\alpha,\beta} f(x)|^p \, dx &\leq \int_0^1 \frac{1}{x^{p\beta/2}} \, |I_{1/2} f(\psi(x))|^p \, dx \\ &= \int_0^1 \frac{1}{\psi^{-1}(u)^{p\beta/2}} \, |I_{1/2} f(u)|^p \, \frac{du}{\psi'(\psi^{-1}(u))} \, , \end{aligned}$$

where $I_{1/2}$ is the standard fractional integral of order $1/2$. Now $\psi'(x) = \alpha\, x^{\alpha-1}$ and $\psi^{-1}(u) = u^{1/\alpha}$. So

$$\psi^{-1}(u)^{-p\beta/2} \psi'(\psi^{-1}(u))^{-1} = \alpha^{-1} u^{-p\beta/(2\alpha)} u^{-1+1/\alpha}$$

which belongs to the space $L^{r,\infty}(0,1)$, $1 \leq r \leq \infty$, precisely when $1 \leq r \leq 2\alpha/(2\,\alpha + p\,\beta - 2)$. Thus

$$\int_0^1 |T_{\alpha,\beta} f(x)|^p \, dx \leq C \, \| \, |I_{1/2} f|^p \, \|_{L^{r',1}} = C \, \|I_{1/2} f\|^p_{L^{r'p,p}} \, ,$$

provided $1 < r = 2\,\alpha/(2\,\alpha + p\,\beta - 2)$. Now, by the Marcinkiewicz interpolation theorem (see [StW]), $I_{1/2} : L^{p,p} \longrightarrow L^{q,p}$ for $1/q = 1/p - 1/2$, $1/2 < 1/p < 1$, and so $T_{\alpha,\beta}$ is bounded on L^p if $1/(r'p) = 1/p - 1/2$, *i.e.* $1/(pr) = 1/2$, *i.e.* $p = 2\,(\alpha - 1)/(\alpha - \beta)$ if this number lies in (1,2), which when $\beta = 0$ is when $\alpha > 2$, when $\beta \in (0,1)$ is for all $\alpha \geq 2$ and for $\beta = 1$ does not occur. We have thus proved the positive assertions of the lemma with the exception of the case $\alpha = 2$, $\beta = 0$ and α arbitrary, $\beta = 1$. The results for $p > 1$ and $p > 2$ respectively follow from (nonsharp) $L^p \longrightarrow L^q$ mapping properties of $I_{1/2}$, while the weak-type 1-1 result for $T_{2,0}$ follows since

$$\begin{aligned}
|\{x:\ |T_{2,0}f(x)|>\lambda\}| &= \int \chi_{\{x\,:\,|T_{2,0}f(x)|>\lambda\}}\,dx \\
&= \int \chi_{\{u\,:\,|I_{1/2}f(u)|>\lambda\}}\,\frac{du}{\psi'(\psi^{-1}(u))} \\
&\le \Big\|\frac{1}{u^{1/2}}\Big\|_{L^{2,\infty}}\,\big\|\chi_{\{u\,:\,|I_{1/2}f(u)|>\lambda\}}\big\|_{L^{2,1}} \\
&= C\,|\{u:\ |I_{1/2}f(u)|>\lambda\}|^{1/2} \\
&\le \frac{C\,\|f\|_1}{\lambda}\,,
\end{aligned}$$

as $I_{1/2}: L^1 \longrightarrow L^{2,\infty}$.

$T_{2,0}$ is clearly not bounded on L^1 (test on $f=\delta_0$). For the other necessary conditions, first let $f=\chi_{(-\delta,0)}$. Then, for $x^\alpha<\delta$,

$$T_{\alpha,\beta}f(x) = \frac{1}{x^{\beta/2}}\int_0^{x^{\alpha\vee\beta}} \frac{ds}{s^{1/2}} = \begin{cases} C\,, & \beta\ge\alpha\,,\\ C\,x^{(\alpha-\beta)/2}\,, & \beta\le\alpha\,,\end{cases}$$

and so $T_{\alpha,\beta}f$ has L^p norm bounded below by

$$\begin{cases} \delta^{1/(\alpha p)}\,, & \beta\ge\alpha\,,\\ \delta^{(\alpha-\beta)/(2\alpha)+1/(\alpha p)}\,, & \beta\le\alpha\,.\end{cases}$$

Hence, when $\beta\ge\alpha$, α is forced to be at most 1, (violating our assumption $\alpha\ge 2$) and when $\beta\le\alpha$, we must have

$$\frac{\alpha-\beta}{2\,\alpha}+\frac{1}{\alpha\,p}\ge\frac{1}{p}\,,$$

i.e. $p\ge 2\,(\alpha-1)/(\alpha-\beta)$. Secondly, to see $\beta\le 1$ is necessary, assume $\beta<\alpha$ (for when $\beta\ge\alpha$ we have already seen there are no p for which $T_{\alpha,\beta}$ is bounded on L^p), and observe that, for $f\ge 0$,

$$T_{\alpha,\beta}f(x) \ge \frac{1}{x^{\beta/2}}\int_{x^\alpha}^{x^\beta} f(s-x^\alpha)\,\frac{ds}{s^{1/2}} = \frac{1}{x^{\beta/2}}\int_0^{x^\beta-x^\alpha} f(s)\,\frac{ds}{(s+x^\alpha)^{1/2}}\,.$$

Now set $f=\chi_{(0,\delta)}$ and observe that for $x^\beta<\delta$,

$$T_{\alpha,\beta}f(x) = \frac{1}{x^{\beta/2}}\int_0^{x^\beta-x^\alpha}\frac{ds}{(s+x^\alpha)^{1/2}} = \frac{1}{x^{\beta/2}}\int_{x^\alpha}^{x^\beta}\frac{ds}{s^{1/2}} \approx C\,.$$

Thus $\|T_{\alpha,\beta}f\|_p \geq c\,\delta^{1/\beta p}$, while $\|f\|_p \sim \delta^{1/p}$. Hence indeed $\beta \leq 1$. Finally, to see that $T_{\alpha,1}$ is not bounded on L^2, (nor indeed of weak-type 2-2), let

$$f(s) = \frac{\chi_{(0,1/2)}(s)}{s^{1/2}\log\left(\frac{1}{s}\right)} \in L^2\,.$$

Then

$$\begin{aligned}
T_{\alpha,1}f(x) &\geq \frac{1}{x^{1/2}}\int_0^{x-x^\alpha} \frac{1}{s^{1/2}\log\left(\frac{1}{s}\right)}\,\frac{ds}{(s+x^\alpha)^{1/2}} \\
&= \frac{1}{x^{1/2}}\int_{x^\alpha}^{x} \frac{1}{(s-x^\alpha)^{1/2}\log\left(\frac{1}{s-x^\alpha}\right)}\,\frac{ds}{s^{1/2}} \\
&\geq \frac{1}{x^{1/2}}\int_{x^\alpha}^{x} \frac{ds}{s\log\left(\frac{1}{s}\right)} \\
&\geq \frac{c}{x^{1/2}}
\end{aligned}$$

which is not in L^2.

$T_{\alpha,1}$ is also of restricted weak-type 2-2. This follows from the proof of Theorem 3.2.

Lemma 3.6. *Suppose $\psi'(x) \longrightarrow 0$ as $x \longrightarrow \infty$, and that $\psi'(x) \geq 0$ for sufficiently large x. Let*

$$T_\psi f(x) = \frac{1}{x^{1/2}}\int_0^x f(t-\psi(x))\,\frac{dt}{t^{1/2}}\,.$$

Then T_ψ is unbounded from $L^2(\mathbb{R})$ to $L^2((0,1))$.

Proof. We may assume that $\psi(x) > 0$ for sufficiently large x. Then

$$\begin{aligned}
\frac{1}{x^{1/2}}\int_0^x f(t-\psi(x))\,\frac{dt}{t^{1/2}} &= \frac{1}{x^{1/2}}\int_{-\psi(x)}^{x-\psi(x)} f(s)\,\frac{ds}{(s+\psi(x))^{1/2}} \\
&\geq \frac{1}{x^{1/2}}\int_0^{x-\psi(x)} f(s)\,\frac{ds}{(s+\psi(x))^{1/2}}\,,
\end{aligned}$$

for $f \geq 0$. Let $f = \chi_{(0,A)}$ with A large. Then

$$T_\psi f(x) \geq \frac{1}{x^{1/2}} \int_0^A \frac{ds}{(s+\psi(x))^{1/2}}\,, \qquad \text{for } x \gg A\,,$$

since $\psi' \longrightarrow 0$ implies

$$\frac{\psi(x)}{x} = \frac{\psi(x_0)}{x} + \frac{x-x_0}{x}\,\frac{1}{x-x_0}\int_{x_0}^{x} \psi'(u)\,du$$

goes to zero as $x \longrightarrow \infty$. Hence, for such x,

$$T_\psi f(x) \geq \frac{1}{x^{1/2}} \int_{\psi(x)}^{A+\psi(x)} \frac{ds}{s^{1/2}} \sim \frac{A}{x^{1/2}(A+\psi(x))^{1/2}}\,.$$

Therefore, for appropriate constants C_1 and C_2,

$$\Big(\int |T_\psi f(x)|^2\,dx\Big)^{1/2} \geq C A^{1/2} \Big(\int_{\{x\,:\,x\geq C_1 A,\,\psi(x)\leq C_2 A\}} \frac{dx}{x}\Big)^{1/2} \gg A^{1/2}\,,$$

while $\|f\|_2 \sim A^{1/2}$.

PROOF OF THEOREM 3.2. By the discussion between Corollary 3.4 and Lemma 3.5, it is sufficient to study the operators given by (8), that is

$$T_{\tilde{\mathfrak{p}}} f(x) = \frac{1}{\tilde{p}(x)^{1/2}} \int_{0\leq s\leq \tilde{p}(x)} f(s-\tilde{\psi}(x))\,\frac{ds}{s^{1/2}}\,,$$

where $\tilde{p} = \tilde{q} = p^2/q$, $\tilde{\psi} = \tilde{\Delta}/4\tilde{q}$, $\tilde{\Delta} = \tilde{p}^2(x) - 4\,x\,\tilde{q}(x)$, so that $\psi(x) = \tilde{\psi}(x) = \tilde{p}(x)/4 - x$. Thus $\psi(x)+x$ vanishes if and only if $\tilde{p}(x)$ vanishes. (Of course it is *neighbourhoods* of such points rather than the points themselves which concern us in obtaining L^r estimates.) We change notation; we replace $\tilde{\psi}$ by ψ, $\tilde{p}$ by p and $\tilde{\mathfrak{p}}$ by $\mathfrak{p}$.

i) Let us first assume $\psi' \geq C > 0$ on $\mathbb{R}$. Then, since we always have $|T_{\mathfrak{p}} f(x)| \leq C M f(\psi(x))$ where M is the ordinary Hardy-Littlewood maximal function, we can write

$$\begin{aligned}
|\{x:\ |T_{\mathfrak{p}}f(x)|>\lambda\}| &\le \Big|\Big\{x:\ Mf(\psi(x))>\frac{\lambda}{C}\Big\}\Big| \\
&= \int \chi_{\{x\,:\,Mf(\psi(x))>\lambda/C\}}\,dx \\
&= \int \chi_{\{u\,:\,Mf(u)>\lambda/C\}}\,\frac{du}{\psi'(\psi^{-1}(u))} \\
&\le c'\,\Big|\Big\{u:\ Mf(u)>\frac{\lambda}{C}\Big\}\Big| \\
&\le c''\,\frac{\|f\|_1}{\lambda}\,.
\end{aligned}$$

Notice that the same argument controls the behaviour of $T_{\mathfrak{p}}f(x)$ on any interval of x upon which $|\psi'|$ is bounded below.

ii) By the proof of i) it is enough to consider the behaviour of $T_{\mathfrak{p}}$ near a critical point, say 0, of maximal order k. Now $p(0)\neq 0$ implies that by taking a small enough neighbourhood of zero, we can assume $p(x)\sim\varepsilon>0$. After a translation of f we can assume, then, that

$$T_{\mathfrak{p}}f(x)\approx\int_0^{\varepsilon} f(s-\delta x^k+O(x^{k+1}))\,\frac{ds}{s^{1/2}}\,,$$

which is essentially the situation of Lemma 3.5, case $\beta=0$, $\alpha=k$. (The proof of Lemma 3.5 can be easily modified to give the variant required here.)

iii) Suppose first that ψ has a monotonic critical point at ∞. Then $\lim_{x\to\infty}p'(x)=4$ and thus $p(x)\sim x$ for large x. So in this case,

$$T_{\mathfrak{p}}f(x)\sim\frac{1}{x^{1/2}}\int_0^{x} f(s-\psi(x))\,\frac{ds}{s^{1/2}}\,,$$

which is unbounded on L^2 by Lemma 3.6.

If ψ has a critical point of finite order at x_0, then $0=\psi'(x_0)=p'(x_0)/4-1$ which implies that

$$p(x)=p(x_0)+4\,(x-x_0)+O(x-x_0)^2$$

near x_0. Assuming that $x_0=0$, and making a translation of f, we have

$$T_{\mathfrak{p}}f(x)\sim\frac{1}{x^{1/2}}\int_0^{x} f(s-\psi(x))\,\frac{ds}{s^{1/2}}\,,$$

which is the case $\beta = 1$ of Lemma 3.5. Thus $T_{\mathfrak{p}}$ is unbounded on L^2 in this case too.

Corollary 3.7. *Let $p(x)$ and $q(x)$ be polynomials in x, $\mathfrak{p}(x,t) = x + t\,p(x) + t^2\,q(x)$, $\Delta(x) = p^2(x) - 4\,x\,q(x)$ and $\psi = \Delta/(4q)$.*

i) *If $\deg\Delta > \deg q$ and ψ has no critical points then M_p and H_p are of weak-type 1-1 and bounded on L^r, $1 < r < \infty$.*

ii) *If $\deg\Delta > \deg q$, $p^2/(4q)$ does not vanish at any of the critical points of ψ, the largest of the orders of which is k, then $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ are bounded on L^r if and only if $r \geq 2\,(k-1)/k$, except when $k = 2$, in which case they are of weak-type 1-1 and bounded on L^r, $1 < r < \infty$.*

iii) *If $\deg\Delta > \deg q$ and $p^2/(4q)$ vanishes at some critical point of ψ, if $\deg\Delta \leq \deg q$, or if $\Delta \equiv 0$, then $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ are unbounded on L^2.*

Proof. When $\psi \neq 0$, ψ' vanishes at infinity if and only if $\deg\Delta \leq \deg q$; when $\deg\Delta > \deg q$, ψ' is bounded below at infinity. Moreover ψ' is bounded below near Z_q anyway. The result now follows from Theorem 3.2.

3.3. Constant coefficients.

When each of the A's is constant, then $H_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ are bounded on $L^q(\mathbb{R})$, $1 < q < \infty$, and are of weak-type 1-1. Moreover when $q > 1$ the bounds may be taken to be independent of the A's. This latter statement for $H_{\mathfrak{p}}$ follows trivially from Theorem 2.3; for both $H_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ it is also a special case of [S2, Chapter XI, Section 2, Propositions 1 and 2]. However since the method of [S2] involves lifting to a higher dimensional setting $\mathbb{R}^k$, $k \geq 2$, where the lifted operators are now associated to curves in $\mathbb{R}^k$, the weak-type 1-1 estimate does not follow. We now present in Theorem 3.9 the result that the weak-type 1-1 bounds of $H_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ may be taken to be independent of the coefficients, and depend only on the degree. The following lemma is closely related to Lemma 3.3. It is also useful in examining higher degree analogues of Theorem 3.2.

Lemma 3.8. *Let p be a real polynomial of degree at most n, with $p(0) = 0$ and leading coefficient 1. Let G be the union of the gaps of p*

as in Section 2 *above. Then*

$$\sup_{h>0}\Big|\frac{1}{h}\int_{[h,2h]\cap G} f(x-p(t))\,dt\Big| \le C_n Mf(x)\,,$$

where C_n *depends only upon* n *and* Mf *is the ordinary Hardy-Littlewood maximal function of* f.

Proof. By scaling we may assume $h=1$. (Note that J is a gap for p if and only if J/h is a gap for $h^{-n}p(h\,\cdot)$). By Lemma 2.5 we may change variables to obtain

$$\begin{aligned}\Big|\int_{[1,2]\cap G} f(x-p(t))\,dt\Big| &= \Big|\int_{p([1,2]\cap G)} f(x-u)\,\frac{du}{|p'(p^{-1}(u))|}\Big|\\ &\le \frac{C}{|p'(t_0)|}\int_{-|p(t_1)|}^{|p(t_1)|}|f(x-u)|\,du\,,\end{aligned}$$

where $|p|$ attains its maximum on $[1,2]\cap G$ at t_1 and $|p'|$ attains its minimum on $[1,2]\cap G$ at t_0. Now $|p(t_1)|\le C\,|p(t_0)|$ by Lemma 2.5.a) , and by Lemma 2.5.b), $|p(t_0)|\le 2\,\varepsilon_0(n)^{-1}|p'(t_0)|$; so $|p(t_1)|\le C\,|p'(t_0)|$. Thus the integral above is dominated, independently of the coefficients of p, by the Hardy-Littlewood maximal function of f.

Theorem 3.9. *Let* $\mathfrak{p}(x,t)=x+\sum_{j=1}^n A_j\,t^j$ *with* A_j *constants. Then there exists* $C(n)$ *depending only upon* n *and not on* $\{A_j\}$ *such that*

$$|\{x:\ M_{\mathfrak{p}}f(x)>\alpha\}|\le C(n)\,\frac{\|f\|_1}{\alpha}$$

and

$$|\{x:\ |H_{\mathfrak{p}}f(x)|>\alpha\}|\le C(n)\,\frac{\|f\|_1}{\alpha}\,.$$

Proof. By Theorem 2.4 it is enough to prove the estimate for $M_{\mathfrak{p}}$. Let $p(t)=\sum_{j=1}^n A_j\,t^j$. Without loss of generality, assume $A_n=1$. It is enough to obtain the weak-type estimate for

$$\sup_{k\in\mathbb{Z}}\Big|\frac{1}{2^k}\int_{[2^k,2^{k+1}]} f(x-p(t))\,dt\Big|\,.$$

For all except boundedly may k (with the bound depending only upon n) we can use Lemma 3.8 to dominate the integrals by $Mf(x)$. The remaining k's correspond to a bounded number of finite measures of mass 1 and hence play no role.

It is interesting to note that one may also prove the quadratic case of Theorem 3.9 by dominating $M_{\mathfrak{p}}f(x)$ pointwise by $Mf(x) + Mf(x \pm p(t_x))$ where t_x is the critical point of p. The proof proceeds along the lines of that of Theorem 3.2, uses Lemma 3.3 and dominates $T_{\tilde{\mathfrak{p}}}f(x)$ by $Mf(x \pm p(t_x))$. It also suggests that it is really the gaps of p which are also gaps of p' which are crucial in Lemma 3.8.

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Weighted Weyl estimates near an elliptic trajectory

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Abstract. Let $\psi_j^\hbar$ and $E_j^\hbar$ denote the eigenfunctions and eigenvalues of a Schrödinger-type operator $H_\hbar$ with discrete spectrum. Let $\psi_{(x,\xi)}$ be a coherent state centered at a point (x, ξ) belonging to an elliptic periodic orbit, γ of action S_γ and Maslov index σ_γ. We consider "weighted Weyl estimates" of the following form: we study the asymptotics, as $\hbar \longrightarrow 0$ along any sequence

$$\hbar = \frac{S_\gamma}{2\pi l - \alpha + \sigma_\gamma}\,,$$

$l \in \mathbb{N}$, $\alpha \in \mathbb{R}$ fixed, of

$$\sum_{|E_j - E| \leq c\hbar} |\ (\psi_{(x,\xi)}, \psi_j^h)\ |^2\ .$$

We prove that the asymptotics depend strongly on α-dependent arithmetical properties of c and on the angles θ of the Poincaré mapping of γ. In particular, under irrationality assumptions on the angles, the limit exists for a non-open set of full measure of c's. We also study the regularity of the limit as a function of c.

1. Introduction and results.

Consider a Schrödinger operator $H = -\hbar^2\Delta + V(x)$ with V smooth, either on $M = \mathbb{R}^m$ (in which case we assume V tends to infinity at infinity and therefore H has discrete spectrum) or on a compact Riemannian manifold, M. In [7] we considered "trace formulae" associated to projectors on coherent states in the following sense. For $(x,\xi) \in \mathbb{R}^{2m}$ and $a \in \mathcal{S}(\mathbb{R}^m)$ define the coherent state $\psi^a_{x\xi}$ as:

$$(1) \quad \psi^a_{(x,\xi)}(y) = \rho(y-x)\,(2\pi\hbar)^{-3m/4}\,2^{-m/4}\,e^{-ix\xi/2\hbar}\,e^{i\xi y/\hbar}\,\hat{a}\Big(\frac{y-x}{\sqrt{\hbar}}\Big)\,.$$

Here ρ is a cut-off function near zero and $\hat{a}$ is the Fourier transform of a, (in the manifold case $(x,\xi) \in T^*M$ and the above definition is in local coordinates near x). Let ψ_j and E_j the eigenfunctions and eigenvalues of H. Then if φ is a Schwartz function whose Fourier transform is compactly supported and $E = |\xi|^2 + V(x)$, we have

$$(2) \qquad \sum_j \varphi\Big(\frac{E_j - E}{\hbar}\Big)\,|\,(\psi_{(x,\xi)}, \psi_j)\,|^2 \sim \sum_{j=0} c_j^{\varphi}(x,\xi)\,\hbar^{-m+1/2+j}\,,$$

for $\hbar \longrightarrow 0$. (If $E \neq |\xi|^2 + V(x)$, the left-hand side tends to 0 rapidly in $\hbar$.) Although the form of the asymptotic expansion does not depend on (x,ξ), the coefficient $c_0(x,\xi)$ is highly sensitive to the point (x,ξ) being periodic or not with respect to the classical flow. In case (x,ξ) is either not periodic or is on a *hyperbolic* trajectory, we proved in [7] (using a Tauberian theorem) that, for every $c \in \mathbb{R}$,

$$(3) \qquad \sum_{|E_j - E| \le c\hbar} |\,(\psi_{(x,\xi)}, \psi_j)\,|^2 = c_0^{\chi_{[-c,c]}}(x,\xi)\,\hbar^{-m+1/2} + o(\hbar^{-m+1/2})\,,$$

as $\hbar \to 0$ possibly along certain sequence. (Here $\chi_{[-c,c]}$ is the characteristic function of the interval $[-c,c]$.) The main goal of this paper is to study the case where (x,ξ) belongs to an elliptic closed trajectory.

Our results are related to the existence of quasi-modes near an elliptic trajectory. Recall that if H is as before and γ is a closed elliptic trajectory of the Hamiltonian $|\xi|^2 + V(x)$ with energy E, period T_γ, action S_γ, Maslov index σ_γ and Poincaré mapping of angles θ_j, $j = 1, \dots, m-1$, then one can construct (see [9], [3], [8], [7]) quasi-modes of H (namely solutions of the Schrödinger equation modulo a remainder), microlocalized near γ, of quasi-energies

$$(4) \qquad E^{k,l}_{QM} = E + \frac{\hbar}{T_\gamma}\Big(\Big(2\pi l - \frac{S_\gamma}{\hbar}\Big) + \sum_{j=1}^{m-1}\Big(k_j + \frac{1}{2}\Big)\theta_j + \sigma_\gamma\Big)\,,$$

for $(k,l) \in \mathbb{Z}^m$, l large. The remainder is $O(\hbar^2)$ uniformly as

$$\Big|2\pi l - \frac{S\gamma}{\hbar}\Big| \qquad \text{and} \qquad |k| := \sum k_j$$

remain bounded. The existence of these quasi-modes implies that part of the spectral density of H concentrates near the quasi-energies defined by (4), but this doesn't say anything about $E_{QM}^{k,l}$ as $|\, k \,| \longrightarrow \infty$ and does not involve the rest of the spectrum. The results of this paper will indicate that the rescaled localized spectral density

$$\sum_j \delta\Big(\frac{E_j - \lambda}{\hbar}\Big) \mid (\psi_{(x,\xi)}, \psi_j) \mid^2 \tag{5}$$

(which is the rescaled spectral density microlocalized at the point in phase space (x,ξ)) has a certain semiclassical limit whose singularities are indeed precisely the quasi-energies (4), and this time with no restriction on $|\, k \,|$.

We will now state our results, valid for more general quantum Hamiltonians: Let $H_\hbar = \sum_{l=0}^{L} \hbar^l P_l(x, D_x)$ where P_l is a differential operator of order l on $\mathbb{R}^m$ (or M) of principal symbol P_l^0, sub-principal symbol P_l^{-1} (formally P_l is regarded as acting on half-densities) and smooth coefficients. Let $\mathcal{H}(x,\xi) = \sum_{l=0}^{L} P_l^0(x,\xi)$ and $\mathcal{H}_{\rm sub}(x,\xi) = \sum_{l=0}^{L} P_l^{-1}(x,\xi)$ be the principal and sub-principal symbols of $H_\hbar$. We assume that P_L is elliptic, $\mathcal{H}$ is positive, and in case $M = \mathbb{R}^m$, that $\mathcal{H}$ tends polynomially to infinity at infinity. We will also suppose for simplicity that $\mathcal{H}_{\rm sub}(x,\xi) = 0$.

Let $E_j^\hbar$ and $\psi_j^\hbar$ denote the eigenvalues and eigenvectors of $H_\hbar$. Let us suppose that (x,ξ) belongs to an elliptic trajectory of period T_γ, action S_γ, Maslov index σ_γ and Poincaré mapping of angles $\theta = (\theta_1, \ldots, \theta_{m-1})$. We will use throughout the notations

$$k = (k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1}\,,$$

$$k\,\theta := \sum_{j=1}^{m-1} k_j\theta_j \qquad \text{and} \qquad \Big(k + \frac{1}{2}\Big)\theta := \sum_{j=1}^{m-1} \Big(k_j + \frac{1}{2}\Big)\theta_j\,. \tag{6}$$

Theorem 1.1. *Assume that $\theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi)$ are rational. Then, for every $\alpha \in [0, 2\pi)$, as $\hbar \to 0$ along the sequence*

$$\hbar = \frac{S_\gamma}{2\pi l - \alpha + \sigma_\gamma}\,, \qquad l \in \mathbb{N}, \tag{7}$$

one has

$$\sum_{|E_j - E| \le c\hbar} |\,(\psi_{(x,\xi)}, \psi_j)\,|^2 = \hbar^{-m+1/2} \mathcal{L}_\alpha(c) + o(\hbar^{-m+1/2})\,, \tag{8}$$

for all c such that

$$c \neq \pm \frac{1}{T_\gamma}\Big(2\pi j + \Big(k + \frac{1}{2}\Big)\theta + \alpha\Big)\,, \qquad \textit{for all } j \in \mathbb{Z}\,,\ k \in \mathbb{N}^{m-1}\,. \tag{9}$$

Moreover, as a function of c the limit $\mathcal{L}_\alpha(c)$ is a step function constant on the intervals defined by (9).

Next we consider the irrational case:

Theorem 1.2. *Assume that $1, \theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi)$ are linearly independent over the rationals. Then there exists a set $\mathcal{M}^\alpha$ of values of c, of full Lebesgue measure, such that for all $c \in \mathcal{M}^\alpha$*

$$\sum_{|E_j - E| \le c\hbar} |\,(\psi_{(x,\xi)}, \psi_j)\,|^2 = \hbar^{m-1/2} \mathcal{L}_\alpha(c) + o(\hbar^{m-1/2})\,, \tag{10}$$

for $\hbar$ as in (7). Moreover, as a function of c, $\mathcal{L}_\alpha(c)$ is locally Lipschitz on $\mathcal{M}^\alpha$ in the sense that for all $c \in \mathcal{M}^\alpha$ there exists $\beta_c > 0$ such that,

$$|\,\mathcal{L}_\alpha(c') - \mathcal{L}_\alpha(c)\,| \le \beta_c \,|\, c' - c\,|\,, \qquad \textit{for all } c' \in \mathcal{M}^\alpha\,. \tag{11}$$

Finally there exists a rapidly decreasing family $\{g_k\}_{k \in \mathbb{N}^{m-1}}$ (related to the microlocalization of the symbol a of $\psi_{(x,\xi)}$) such that

$$\{c :\ \textit{for all } k \in \mathbb{N}^{m-1}\ \ |\,1 - e^{i(cT_\gamma + (k+1/2)\theta + \alpha)}\,| > \varepsilon\, g_k\} \subset \mathcal{M}^\alpha\,, \tag{12}$$

for all $\varepsilon > 0$. (For a precise definition of the set $\mathcal{M}^\alpha$ see Lemma 3.3.).

Remark. In the rational case the discontinuities of the function $\mathcal{L}_\alpha$ are located exactly at the values of the $E_{QM}^{k,l}$ defined before by (4), for the values of $\hbar$ given by (7). In the irrational case in order to prove that $\mathcal{L}_\alpha(c)$ exists we need that c be at some distance from the quasi-energies $E_{QM}^{k,l}$ (unless the symbol a of the quasi-mode is chosen very judiciously, in which case we can work with c in the complement of the set of all quasi-energies). In all cases this suggests that the weighted spectral measure, (5), in the semi-classical limit, is particularly singular exactly at the values of the $E_{QM}^{k,l}$ defined before. We hope to provide a rigorous proof of a precise statement of this elsewhere.

The paper is organized as follows: In Section 3 we prove the existence of the functions $\mathcal{L}_\alpha$ which are studied in Section 4. In Section 5 we finish the proof of the main Theorems, using a Tauberian argument that we recall in Section 2. Finally, in the appendix we review and extend slightly a result on Hölder continuity of function such as $\mathcal{L}_\alpha$ using wavelets.

2. A Tauberian lemma.

In this section we refine the Tauberian lemma of [2] and [7].

Consider an expression of the following form

$$\Upsilon^w_{E,\hbar}(\varphi) = \sum_j w_j(\hbar)\, \varphi\Big(\frac{E_j(\hbar) - E}{\hbar}\Big), \tag{13}$$

defined for all $\varphi \in \mathcal{R}$ where $\mathcal{R}$ will henceforth denote the set of all Schwartz functions on the line with compactly supported Fourier transform.

Let $\mathcal{M}^\alpha$ a subset of $\mathbb{R}^+$ of full Lebesgue measure in a bounded interval.

We introduce the following notations. Fix a positive function $f \in \mathcal{R}$ satisfying $f(0) = 1$ and $\hat{f}(0) = 1$. For every $a > 0$, define

$$f_a(r) := a^{-1} f\Big(\frac{r}{a}\Big) \tag{14}$$

and for every $a > 0$ and $c > 0$

$$\varphi_{a,c} := f_a * \chi_{[-c,c]}, \tag{15}$$

where $\chi_{[-c,c]}$ is the characteristic function of the interval $[-c,c]$.

The Tauberian lemma in question is:

Theorem 2.1 (See [2] and [7]). *Let $\mathcal{M}^\alpha$ a subset of $\mathbb{R}^+$ of full Lebesgue measure in a bounded interval. Suppose $w_j(\hbar)$, $E_j(\hbar)$, E and $\Upsilon^w_\hbar$ itself satisfy all of the following:*

1) *There exists a positive function $\omega(\hbar)$, defined on an interval $(0,\hbar_0)$, and a functional $\mathcal{F}_0$ on $\mathcal{R}$, such that for all $\varphi \in \mathcal{R}$*

$$\Upsilon^w_{E,\hbar}(\varphi) = \mathcal{F}_0(\varphi)\,\omega(\hbar) + o(\omega(\hbar)), \qquad \hbar \longrightarrow 0\,. \tag{16}$$

2) *for all $c \in \mathcal{M}^\alpha$ the limit*

$$\mathcal{L}_\alpha(c) = \lim_{a\to 0} \mathcal{F}_0(\varphi_{a,c})$$

exists.

3) *$\mathcal{L}_\alpha$ is a continuous function on $\mathcal{M}^\alpha$.*

4) *There exists a $k \in \mathbb{Z}$ such that $\hbar^k = \mathcal{O}(\omega(\hbar))$, $\hbar \to 0$.*

5) *There exists an $\varepsilon > 0$ such that for every φ there is a constant C_φ such that for all $E' \in [E-\varepsilon, E+\varepsilon]$*

$$|\Upsilon^w_{E',\hbar}(\varphi)| \le C_\varphi\,\omega(\hbar) \tag{17}$$

(rough uniformity in E).

6) *The $w_j(\hbar)$ are non-negative and bounded: there exists a constant $C \ge 0$ such that for all j and all $\hbar, 0 < \hbar < \hbar_0$*

$$0 \le w_j(\hbar) \le C\,. \tag{18}$$

7) *The eigenvalues $E_j(\hbar)$ satisfy the following rough estimate: for each C_1 there exist constants C_2, N_0 such that for all k*

$$\#\{j : E_j(\hbar) \le C_1 + k\hbar\} \le C_2(\hbar^{-1}k)^{N_0}\,. \tag{19}$$

Define the weighted counting function by

$$N^w_{E,c}(\hbar) = \sum_{j\,;\,|x_j(\hbar)|\le c} w_j(\hbar)\,, \tag{20}$$

where

$$x_j(\hbar) := \frac{E_j(\hbar) - E}{\hbar} \,. \tag{21}$$

Then the conclusion is: for all $c \in \mathcal{M}^\alpha$,

$$N^w_{E,c}(\hbar) = \mathcal{L}_\alpha(c)\,\omega(\hbar) + o(\omega(\hbar))\,, \qquad \hbar \longrightarrow 0\,. \tag{22}$$

PROOF. Except for the fact that the set $\mathcal{M}^\alpha$ of allowed c's is not $\mathbb{R}^+$, this theorem is precisely [2, Theorem 6.3]. Proceeding exactly as in the proof of the [2, inequalities (188)], one shows that for all $R > 0$, for all $N \in \mathbb{N}$ exists $C > 0$, $C_N > 0$ such that for all $a \in (0, R)$ and for all η, $0 < \eta < c$,

$$\begin{aligned} \frac{1}{\omega(\hbar)}\Big(1 - C\,\frac{a}{\eta}\Big)N_{E,c-\eta}(\hbar) &\le \frac{1}{\omega(\hbar)}\,\Upsilon_{E,\hbar}(\varphi_{a,c}) \\ &\le \frac{1}{\omega(\hbar)}\,N_{E,c+\eta}(\hbar) + C_N\Big(\frac{a}{\eta}\Big)^N . \end{aligned} \tag{23}$$

Let $c \in \mathcal{M}^\alpha$ be given. We begin by observing that by the first of the inequalities (23)

$$\frac{1}{\omega(\hbar)}\,N_{E,c}(\hbar) \le \frac{1}{\omega(\hbar)}\,\Upsilon_{E,\hbar}(\varphi_{a,c+\eta}) + C_1\,\frac{a}{\eta}\,, \tag{24}$$

where we have also used the fact that $N_{E,c}(\hbar)/\omega(\hbar)$ is bounded (a trivial consequence of (16)). For every η such that $0 < \eta < c$ one can take the limit in (24) as $\hbar \longrightarrow 0$ to obtain that

$$\limsup_{\hbar\to 0} \frac{1}{\omega(\hbar)} N_{E,c}(\hbar) \le \mathcal{F}_0(\varphi_{a,c+\eta}) + C_1\,\frac{a}{\eta}\,. \tag{25}$$

If we now assume that $\eta + c \in \mathcal{M}^\alpha$ we can take the limit as $a \longrightarrow 0$ to obtain

$$\limsup_{\hbar\to 0} \frac{1}{\omega(\hbar)}\,N_{E,c}(\hbar) \le \mathcal{L}_\alpha(c+\eta)\,. \tag{26}$$

By the assumption that $\mathcal{M}^\alpha$ has full measure, we can find a sequence $\{\eta_j\}$ such that for all j, $c + \eta_j \in \mathcal{M}^\alpha$ and $\eta_j \longrightarrow 0$. Taking the limit in

(26) of $\mathcal{L}_\alpha(c+\eta_j)$ as $j \longrightarrow \infty$ and using the fact that $\mathcal{L}_\alpha$ is continuous at c we obtain

$$\limsup_{\hbar\to 0} \frac{1}{\omega(\hbar)} N_{E,c}(\hbar) \leq \mathcal{L}_\alpha(c)\,. \tag{27}$$

A similar argument starting with the second inequality (23) shows that

$$\liminf_{\hbar\to 0} \frac{1}{\omega(\hbar)} N_{E,c}(\hbar) \geq \mathcal{L}_\alpha(c)\,, \tag{28}$$

which finishes the proof.

3. The existence of $\mathcal{L}_\alpha(c)$.

In this section we prove the existence of the coefficients $\mathcal{L}_\alpha(c)$ in the limits (8) and (10) (see (36) below).

Lemma 3.1. *There exists a rapidly decreasing family of non-negative numbers, $\{c_k\}_{k\in\mathbb{N}^{m-1}}$, such that for all $\varphi \in \mathcal{R}$ the first coefficient $c_k^\varphi(x,\xi)$ in (2) can be written as*

$$c_0^\varphi(x,\xi) = \sum_{n=-\infty}^{+\infty} \sum_{k\in\mathbb{N}^{m-1}} \hat{\varphi}(n\,T_\gamma)\, c_k\, e^{in((k+1/2)\theta+\alpha)}\,. \tag{29}$$

PROOF. In [7] we proved that the first coefficient $c_0^\varphi(x,\xi)$ in (2) can be written as

$$\begin{aligned} &2^{2n}\,\pi^{(3n+1)/2}\,c_0^\varphi(x,\xi) \\ &\qquad = \sum_{n=-\infty}^{+\infty} \hat{\varphi}(nT_\gamma)\, e^{inS_\gamma/\hbar+\sigma_\gamma} \int_{-\infty}^{+\infty} (a, Z((s\,\dot{x}, s\,\dot{\xi}))\,U^n a)\,ds\,, \end{aligned} \tag{30}$$

where $(\dot{x},\dot{\xi})$ is the tangent vector to the classical flow at (x,ξ), Z is the Weyl/Heisenberg operator defined by

$$Z(e,f)(a)(\eta) = e^{-ief/2}\, e^{ie\eta}\, a(\eta - f) \tag{31}$$

and U is the metaplectic representation of the linearized flow at time T_γ. (We should point out that in the manifold case a defines intrinsically a smooth vector in the metaplectic representation of $T_{(x,\xi)}(T^*M)$, and U and Z are operators in that representation space.) Denoting by S the linearized flow at time T_γ, we also showed that one can find a symplectic mapping R such that $R^{-1}SR$ is block-diagonal of the form

$$R^{-1}SR = \begin{pmatrix} 1 & \mu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A_\theta \end{pmatrix}, \tag{32}$$

where $\mu \in \mathbb{R}$ and A_θ is the direct sum of rotations of angles $\theta_1, \dots, \theta_{m-1}$. Furthermore, the transformation R maps the vector $(s\,\dot{x}, s\,\dot{\xi})$ to the vector $(s, 0)$.

Let us denote $a' := \mathrm{Mp}(R)^{-1}a$ and $V := \mathrm{Mp}(R^{-1}SR)$, where $\mathrm{Mp}(R)$ denotes the metaplectic representation of the mapping R. Then, letting $Z(s) := Z((s\,\dot{x}, s\,\dot{\xi}))$ and

$$W(s) := \mathrm{Mp}(R)^{-1}\, Z(s)\, \mathrm{Mp}(R) = Z(s, 0, 0, 0)\,,$$

one has

$$\Big(a, Z(s)\, U^n a\Big) = \Big(a', W(s)\, V^n a'\Big)\,.$$

Denote the variables of a' by (η_1, η_2) where $\eta_1 \in \mathbb{R}$ and $\eta_2 \in \mathbb{R}^{m-1}$, and let $e^{i\theta(D^2_{\eta_2} + {\eta_2}^2)/2}$ denote the direct sum of the propagators of one-dimensional Harmonic oscillators at times $\theta_1, \dots \theta_{m-1}$, acting on a' by acting on the η_2 variables. If $e^{in\mu\partial^2_{\eta_1}/2}$ denotes the metaplectic quantization of

$$\begin{pmatrix} 1 & n\mu \\ 0 & 1 \end{pmatrix}, \tag{33}$$

we get that (30) becomes

$$\begin{aligned} &2^{2n}\, \pi^{(3n+1)/2}\, c_0^{\varphi}(x,\xi) \\ &\quad = \sum_{n=-\infty}^{+\infty} \hat{\varphi}(n\, T_\gamma)\, e^{in\alpha} \\ &\qquad\qquad \cdot \int \overline{a'(\eta)}\, (e^{in\mu\partial^2_{\eta_1}/2}\, e^{in\theta(D^2_{\eta_2} + {\eta_2}^2)/2}\, (a'))\, (\eta_1 - s, \eta_2)\, d\eta\, ds\,. \end{aligned}$$

The integral over ds is a convolution and the integral over $d\eta_1$ is the integral of that convolution. Therefore, using the Fourier inversion formula plus the fact that on the Fourier transform side the operator $e^{in\mu\partial_{\eta_1}^2/2}$ is multiplication by $e^{-in\mu\zeta^2/2}$ (ζ being the dual variable), one gets

$$
\begin{aligned}
&2^{2n}\,\pi^{(3n+1)/2}\,c_0^{\varphi}(x,\xi) \\
(34)\quad &= \sum_{n=-\infty}^{+\infty} \hat{\varphi}(n\,T_\gamma)\,e^{in\alpha}\int \overline{a'^{\wedge}(0,\eta_2)}\,e^{in\theta(D_{\eta_2}^2+{\eta_2}^2)/2}a'^{\wedge}(0,\eta_2)\,d\eta_2\ ,
\end{aligned}
$$

where $a'^{\wedge}$ is the Fourier transform of a' with respect to η_1. Let $b(x) := a'^{\wedge}(0,x)$ and let us decompose b on the Hermite basis, h_k, of eigenfunctions of the harmonic oscillator

$$
b = \sum_{k\in\mathbb{N}^{m-1}} b_k\,h_k\ . \tag{35}
$$

Then, letting $c_k := |b_k|^2$ we get (29) and the family $\{c_k\}$ is non-negative. It is also rapidly decreasing since the function b is Schwartz.

REMARK. For a given quantum Hamiltonian H, the coefficients $\{c_k\}$ depend only on the symbol a of the coherent state. Observe that the proof shows that given any rapidly decreasing family $\{c_k\}$ one can find an a giving rise to it.

We next prove the existence of the limit

$$
\mathcal{L}_\alpha(c) := \lim_{a\to 0} c_0^{(f_a*\chi_{[-c,c]})}(x,\xi)\,, \tag{36}
$$

for f as in the Tauberian lemma and $c_0^{\varphi}(x,\xi)$ as in (29). Let $\phi_a(c) := c_0^{(f_a*\chi_{[-c,c]})}(x,\xi)$, that is

$$
\phi_a(c) :=\ c\ +\ \sum_{n\neq 0,k} \hat{f}(a\,n)\,\frac{\sin{(n\,c\,T_\gamma)}}{n\,T_\gamma}\,c_k\,e^{in((k+1/2)\theta+\alpha)}\ . \tag{37}
$$

We must then prove that the limit $\mathcal{L}_\alpha(c) = \lim_{a\to 0}\phi_a(c)$ exists.

To lighten up the notation a bit, let us define

$$
d_k := \Big(k+\frac{1}{2}\Big)\theta+\alpha\,, \qquad k\in\mathbb{N}^{m-1}\ , \tag{38}
$$

keeping in mind the notation (6). Let $0 < a < 1$, then

$$\begin{aligned}\phi_1(c) - \phi_a(c) &= \frac{1}{T_\gamma} \sum_{(n,k)\in\mathbb{Z}\times\mathbb{N}^{m-1}} \sin(c\, n\, T_\gamma)\, c_k\, e^{ind_k} \int_a^1 \hat{f}'(t\,n)\, dt \\ &= \frac{1}{T_\gamma} \sum_{(n,\,k)} \Big(\frac{e^{incT_\gamma} - e^{-incT_\gamma}}{2\,i}\Big) c_k\, e^{ind_k} \int_a^1 \hat{f}'(t\,n)\, dt\,. \end{aligned} \tag{39}$$

Applying the Poisson summation formula to the series over n, we get (after a calculation)

$$\begin{aligned}\phi_1(c) - \phi_a(c) = \frac{-\pi}{T_\gamma} \sum_{(j,\,k)\in\mathbb{Z}\times\mathbb{N}^{m-1}} c_k \int_a^1 &\Big(g\Big(\frac{1}{t}\,(2\pi j + c\,T_\gamma + d_k)\Big) \\ &- g\Big(\frac{1}{t}\,(2\pi j - c\,T_\gamma + d_k)\Big)\Big)\frac{dt}{t}\,, \end{aligned} \tag{40}$$

where $g(x) := x\, f(x)$.

Lemma 3.2. *Define*

$$\mathcal{M}_0^\alpha = \Big\{ c \in \mathbb{R} : \text{ for all } (j,k) \in \mathbb{Z}\times\mathbb{N}^{m-1}\,,\ c \neq \pm\frac{1}{T_\gamma}\,(2\pi j + d_k)\Big\}\,.$$

If $\theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi)$ are rational and $c \in \mathcal{M}_0^\alpha$, then each of the limits

$$\lim_{a\to 0} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}^{m-1}} c_k \int_a^1 g\Big(\frac{1}{t}\,(2\pi j \pm c\,T_\gamma + d_k)\Big)\frac{dt}{t}\,, \tag{41}$$

exists (and is finite). Moreover, the convergence is locally uniform in c.

PROOF. By the rationality assumption the complement of $\mathcal{M}_0^\alpha$ is discrete. Therefore, if $c \in \mathcal{M}_0^\alpha$ there exists ε such that

$$0 < \varepsilon \le |2\pi j \pm c\,T_\gamma + d_k|\,, \qquad \text{for all } (j,k) \in \mathbb{Z}\times\mathbb{N}^{m-1}\,.$$

The function g is rapidly decreasing: for all $N \in \mathbb{N}$ $\exists C_N > 0$ such that for all $x \in \mathbb{R}$, $|g(x)| \le C_N\,(1+|x|)^{-N}$. Therefore

$$\begin{aligned}\Big| g\Big(\frac{2\pi j \pm c\,T + d_k}{t}\Big)\Big| &\le C_N\, \frac{t^N}{t^N + (2\pi j \pm c\,T + d_k)^N} \\ &\le C_N\, \frac{t^N}{(2\pi j \pm c\,T + d_k)^N}\,, \end{aligned} \tag{42}$$

and so for all $(j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1}$ and for all $a \in (0,1)$

$$\int_a^1 \left| g\Big(\frac{1}{t}\,(2\pi j \pm cT_\gamma + d_k)\Big)\right| \frac{dt}{t} \le \frac{C_N}{N}\,\frac{1-a^{N+1}}{|2\pi j \pm cT_\gamma + d_k|^N}\,. \tag{43}$$

This shows that each of the integrals in the series (41) extends to a continuous function of $a \in [0,1)$. Moreover, since the family

$$M_{k,j} := \frac{c_k}{|2\pi j \pm cT_\gamma + d_k|^N}\,, \qquad (j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1}$$

is absolutely convergent (for N sufficiently large) and it dominates the absolute values of the terms of (41), we are done.

We now turn to the irrational case.

Lemma 3.3. *Assume that $1, \theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi)$ are linearly independent over the rationals. Let*

$$\mathcal{M}^\alpha_\pm := \Big\{ c \in \mathcal{M}^\alpha_0 : \sum_{k\in\mathbb{N}^{m-1}} c_k \Big(\pm \Big(d_k + \frac{cT}{2\pi}\Big)\Big)^{-2} < \infty \Big\}\,, \tag{44}$$

where $\{x\}$ denotes the fractional part of x, and let

$$\mathcal{M}^\alpha := \mathcal{M}^\alpha_+ \cap \mathcal{M}^\alpha_-\,. \tag{45}$$

Then, if $c \in \mathcal{M}^\alpha$, each of the limits

$$\lim_{a\to 0} \sum_{j,k} c_k \int_a^b g\Big(\frac{1}{t}\,(2\pi j \pm cT_\gamma + d_k)\Big)\frac{dt}{t}$$

exists and is finite. Moreover, the convergence is locally uniform in c.

PROOF. It is enough to consider one of the series above, say the one with the plus sign. Let $c \in \mathcal{M}^\alpha$ and define

$$O^+ := \{(j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1} :\ 2\pi j + cT_\gamma + d_k > 0\}\,,$$

and

$$O^- := \{(j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1} :\ 2\pi j + cT_\gamma + d_k < 0\}\,.$$

Since $c \in \mathcal{M}_0^\alpha$, $\mathbb{Z} \times \mathbb{N}^{m-1} = O^+ \cup O^-$. Recalling that $g(x) = x\,f(x)$ and that f as well as the c_k are non-negative, we see that the terms with $(j,k) \in O^\pm$ have the sign $\pm$ and therefore each of

$$\sum_{(j,k)\in O^\pm} c_k \int_a^1 g\Big(\frac{1}{t}\,(2\pi j + c\,T_\gamma + d_k)\Big)\,\frac{dt}{t}$$

is a decreasing function of a. It therefore suffices to show that

$$\lim_{a\to 0} \sum_{(j,k)\in O^+} c_k \int_a^1 g\Big(\frac{1}{t}\,(2\pi j + c\,T_\gamma + d_k)\Big)\,\frac{dt}{t} < \infty$$

and similarly for the series over O^-.

Specializing (43) to $N = 2$, we see that exists $C > 0$ such that for all $a \in (0,1)$ and for all $(j,k) \in O^+$

$$\int_a^1 g\Big(\frac{2\pi j + c\,T + d_k}{t}\Big)\,\frac{dt}{t} \le \frac{C}{(2\pi j + c\,T + d_k)^2}\;. \tag{46}$$

(The last denominator is not zero if $(j,k) \in O^+$.) Therefore, the Lemma will be proved provided we show the convergence of the double series of scalars

$$\sum_{(j,k)\in O^+} M_{k,j}\;, \tag{47}$$

where

$$M_{k,j} = c_k\,\Big(j + \frac{c\,T + d_k}{2\pi}\Big)^{-2}\,,$$

that is

$$M_{k,j} = \frac{c_k}{(j + k\,\xi + \beta)^2}\;, \tag{48}$$

where

$$\xi = \Big(\frac{\theta_1}{2\pi}, \ldots, \frac{\theta_{m-1}}{2\pi}\Big) \qquad \text{and} \qquad \beta = \frac{1}{2\pi}\Big(c\,T + \alpha + \sum_{j=1}^{m-1} \frac{\theta_j}{2}\Big)\,. \tag{49}$$

Since the terms in (47) are positive, we can prove its convergence by first summing over j with k fixed, and then summing over $k \in \mathbb{N}^{m-1}$. Observe that

$$(j,k) \in O^+ \qquad \text{if and only if} \qquad j \ge [-k\,\xi - \beta] + 1\,, \tag{50}$$

where $[x]$ denotes the greatest integer less than or equal to x. For every k consider the series

$$\sum_{j=-[k\xi+\beta]}^{+\infty} M_{k,j} \ . \tag{51}$$

(If $x \notin \mathbb{Z}$, then $[-x] = -[x] - 1$, and since $c \in \mathcal{M}_0^\alpha$, for all $k \in \mathbb{N}^{m-1}$, $k\,\xi + \beta \notin \mathbb{Z}$.) Comparing this series with the integral

$$\int_{-[k\xi+\beta]}^{+\infty} \frac{dx}{(x + k\,\xi + \beta)^2} \ , \tag{52}$$

we find that

$$\sum_{j=-[k\xi+\beta]}^{+\infty} M_{k,j} \leq M_{k,[-k\xi+\beta]} + \frac{c_k}{-[k\xi+\beta] + k\,\xi + \beta} \ , \tag{53}$$

or with the notation $\{x\} =$ fractional part of $x = x - [x]$,

$$\sum_{j=-[k\,\xi+\beta]+1}^{+\infty} M_{k,j} \leq \frac{c_k}{\{k\,\xi+\beta\}^2} + \frac{c_k}{\{k\,\xi+\beta\}} \ . \tag{54}$$

Therefore convergence of (47) follows from the convergence of

$$\sum_{k\in\mathbb{N}^{m-1}} \frac{c_k}{\{k\,\xi+\beta\}^2} \ .$$

But since by assumption $c \in \mathcal{M}_+^\alpha$, this series converges.

In conclusion we have shown that $\mathcal{L}_\alpha(c)$ exists for c as defined by the Lemmas.

REMARKS. In the irrational case:

1) To find examples of numbers c in $\mathcal{M}_\pm^\alpha$, it suffices to find a family $\{g_k\}_{k\in\mathbb{N}^{m-1}}$ of positive numbers such that $\sum g_k^{-2}\,c_k < \infty$. Then if

$$\left|2\pi j \pm c\,T_\gamma + \left(k + \frac{1}{2}\right)\theta + \alpha\right| > g_k \ , \qquad \text{for all } (j,k) \in \mathbb{Z}\times\mathbb{N}^{m-1} \ , \tag{55}$$

then $c \in \mathcal{M}^\alpha$. Defining $\hat{g}_k = \varepsilon g_k$, $\varepsilon > 0$, one see that still $\sum \hat{g}_k^{-2} c_k < \infty$ and therefore associated to $\hat{g}_k$ (by (55)) is a subset of $\mathcal{M}_\alpha$ whose intersection with any interval I has a co-measure in I arbitrary small as $\varepsilon \longrightarrow 0$; therefore $\mathcal{M}^\alpha$ has full measure.

2) The set $\mathcal{M}^\alpha$ is related to the rate of decay of the c_k (that is to the properties of the symbol, a, of the coherent states), as well as to irrationality properties of $\theta/(2\pi)$. At one extreme, we can choose a such that only finitely-many of the coefficients c_k are non-zero (see the remark following Lemma 3.1). In that case $\mathcal{M}^\alpha = \mathcal{M}_0^\alpha$ is just the complement of the set of quasi-energies of the quasi-modes associated with the trajectory.

4. Properties of the function $\mathcal{L}_\alpha$.

Having established the existence of the function $\mathcal{L}_\alpha(c)$, we now derive some of its properties.

Rational case. Let us go back to the identity $\mathcal{L}_\alpha(c) = \lim_{a\to 0} \phi_a(c)$ where ϕ_a is defined in (37). Applying in (37) the Poisson summation formula to the series over n with k fixed one obtains

$$\mathcal{L}_\alpha(c) = \lim_{a\to 0} \frac{1}{a}\Big(F_c * f\Big(\frac{\cdot}{a}\Big)\Big)(0)\,, \tag{56}$$

where

$$F_c(y) = \int_{-c}^{c} \sum_{j,k} c_k\, \delta(T_\gamma(x-y) - 2\pi j - d_k)\, dx\,. \tag{57}$$

For each $c > 0$ the function F_c is a step function; indeed

$$F_c = \sum_{j,k} c_k\, \chi_{[-c-(2\pi j + d_k)/T\,,\, c-(2\pi j+d_k)/T]}\,. \tag{58}$$

Since $f(\cdot/a)/a \longrightarrow \delta$, we obtain

$$\mathcal{L}_\alpha(c) = \sum_{\{j,k\,:\, -cT<2\pi j+d_k<cT\}} c_k\,, \qquad \text{for all } c \in \mathcal{M}_0^\alpha\,, \tag{59}$$

which is clearly a step function (*i.e.* a locally constant function) of $c \in \mathcal{M}_0^\alpha$.

Irrational case. To study the function $\mathcal{L}_\alpha(c)$ on $\mathcal{M}^\alpha$ as defined by (36), we will use a wavelet decomposition.

Let $g \in L^2$ be a function satisfying $\int g(x)\,dx = 0$ and $\int x\,g(x)\,dx = 0$. If it exists, the wavelet coefficient of $\mathcal{L}_\alpha(c) - c$ is

$$T(a,b) = \frac{1}{a}\int g\Big(\frac{x-b}{a}\Big)(\mathcal{L}_\alpha(x) - x)\,dx\,. \tag{60}$$

Plugging in (60) the expression

$$\mathcal{L}_\alpha(x) - x = \sum_{\substack{n\neq 0\\ k}} \frac{\sin(n\,x\,T_\gamma)}{n\,T_\gamma}\,e^{ind_k}\,c_k\,, \tag{61}$$

one finds, supposing $\hat{g}$ even

$$T(a,b) = \frac{1}{2\,i}\sum_{\substack{n\neq 0\\ k}} \frac{1}{n\,T_\gamma}\,\hat{g}(a\,n)\sin(n\,b\,T_\gamma)\,c_k\,e^{in(d_k)}\,. \tag{62}$$

The following result shows that such a decomposition is indeed valid.

Proposition 4.1. *Let g as before, $\hat{g}$ being compactly supported and even, and let us suppose that φ is a compactly supported function satisfying*

$$\int \overline{\hat{\varphi}}(a)\,\hat{g}(a)\,\frac{da}{a} = \int \overline{\hat{g}}(-a)\,\hat{\varphi}(-a)\,\frac{da}{a} = 1\,. \tag{63}$$

Then, for all $c \in \mathcal{M}^\alpha$,

$$\mathcal{L}_\alpha(c) - c = \lim_{\varepsilon\to 0}\int_\varepsilon^{+\infty}\frac{da}{a}\int_{-\infty}^{+\infty}\varphi\Big(\frac{c-b}{a}\Big)T(a,b)\,db\,, \tag{64}$$

where

$$T(a,b) = \frac{1}{2\,i}\sum_{\substack{n\neq 0\\ k}} \frac{1}{n\,T_\gamma}\,\hat{g}(a\,n)\sin(n\,b\,T_\gamma)\,c_k\,e^{ind_k}\,. \tag{65}$$

PROOF.

$$\int_\varepsilon^{+\infty}\frac{da}{a}\int_{-\infty}^{+\infty}db\,\varphi\Big(\frac{c-b}{a}\Big)T(a,b)$$

$$= \int_\varepsilon^{+\infty}\frac{da}{a}\sum_{\substack{n\neq 0\\ k}}\frac{1}{n\,T_\gamma}\,(\hat\varphi(a\,n)\,e^{incT_\gamma}-\hat\varphi(-a\,n)\,e^{-incT_\gamma})$$

$$\cdot\, e^{ind_k}\,\hat g(a\,n)\,c_k$$

$$= \int_\varepsilon^{+\infty}\frac{da}{a}\sum_{\substack{n\neq 0\\ k}}\hat\varphi(a\,n)\,\hat g(a\,n)\sin(n\,c\,T_\gamma)\,e^{in((k+1/2)\theta+\alpha)}\,c_k \tag{66}$$

$$= \sum_{\substack{n\neq 0\\ k}}\psi(\varepsilon\,n)\sin(n\,c\,T_\gamma)\,e^{ind_k}\,c_k\ ,$$

where

$$\psi(\varepsilon) := \int_\varepsilon^{+\infty}\frac{da}{a}\,\hat\varphi(a)\,\hat g(a)\,.$$

Noting that $\psi'(a)=\hat\varphi(a)\,\hat g(a)/a$ is compactly supported and $\psi(0)=1$ by hypothesis one get the result, thanks to Lemma 3.3.

The next result, thanks to the result of the Appendix will enable us to prove the Lipschitz continuity on $\mathcal{M}^\alpha$.

Proposition 4.2.

(67) $T(a,b)=O(a)\,,$ *near* 0 *almost everywhere and uniformly in* b.

PROOF. Since $\int x\,g(x)\,dx=\int g(x)\,dx=0$, $g'(0)=0$. So one can find a C^∞ function f such that $\hat g(\xi)=\xi\,f(\xi)$ and $f(0)=0$. Then

$$T(a,b) = a\sum_{\substack{n\neq 0\\ k}} f(a\,n)\sin(b\,n\,T_\gamma)\,e^{ind_k}\,c_k\ , \tag{68}$$

and it is easy to check, by the same argument as in Lemma 3.3, that if $b\in\mathcal{M}^\alpha$,

$$\sum_{\substack{n\neq 0\\ k}} f(a\,n)\sin(b\,n\,T_\gamma)\,e^{in((k+1/2)\theta+\alpha)}\,c_k$$

is bounded.

5. End of proofs.

The convergence statements in both theorems are immediate consequences of the Tauberian lemma of Section 2, applied to the following objects

$$\Upsilon_\hbar(a,c) = \sum_j w_j(\hbar)\,\varphi\Big(\frac{E_j(\hbar)-E}{\hbar}\Big)\,, \tag{69}$$

where

$$w_j(\hbar) = |(\psi_{(x,\xi)},\psi_j)|^2\,. \tag{70}$$

The weighted counting function is therefore

$$\sum_{\substack{j \\ |E_j(\hbar)-E|\leq c\hbar}} |(\psi_{(x,\xi)},\psi_j)|^2\,. \tag{71}$$

The functional of the Tauberian lemma is

$$\mathcal{F}_0(\varphi) := c_0^\varphi(x,\xi) \tag{72}$$

as defined by (29). We must check that the above objects satisfy the assumptions of the Tauberian lemma.

a) Theorem 1.1. It is easy to see that the functional $\mathcal{F}_0$ defined where $c_0^\varphi(x,\xi)$ is defined by (29) satisfies the hypothesis 2 of the Tauberian Lemma of Section 2 if we take for $\mathcal{M}^\alpha$ the set defined by (9). Moreover the other hypotheses are satisfied as in [7]. Then just apply the Tauberian Lemma.

b) Theorem 1.2. The Lipschitz continuity of $\mathcal{F}_0$ is an immediate consequence of Proposition 4.2 together with Theorem A.1 below. The fact that $\mathcal{M}^\alpha$ is of full Lebesgue measure, is a classical result of Diophantine analysis (recall that the sequence $\{g_k\}$ in the remark 1, Section 3 is rapidly decreasing).

Appendix. Wavelets and Hölder continuity.

Int his appendix we will prove an easy extension of results of [6], [5] and [4].

Let $\mathcal{M}^\alpha$ a bounded subset of $\mathbb{R}$ of full Lebesgue measure.

Theorem A.1. *Let g be a be a continuously differentiable compactly supported function. Let f defined and bounded on $\mathcal{M}^\alpha$. Let us suppose that f admits a "scale-space coefficient T(a,b)" decomposition with respect to g, namely*

$$f(x) = \int_0^\infty \int_{-\infty}^{+\infty} g\Big(\frac{x-b}{a}\Big) T(a,b)\, \frac{da}{a}\, db\,, \qquad \textit{for all } x \in \mathcal{M}^\alpha\,. \tag{73}$$

Let us suppose moreover that

$$T(a,b) = o(a^\alpha)\,, \tag{74}$$

near 0 *almost everywhere and uniformly in b. Then F is α-Hölder continuous on $\mathcal{M}^\alpha$; by this we mean*

$$|f(x_1) - f(x_2)| = O_{x_1}(|x_2 - x_1|^\alpha)\,, \qquad \textit{for all } x_1, x_2 \in \mathcal{M}^\alpha\,. \tag{75}$$

PROOF. The proof is absolutely equivalent to the one in [4], so we will only sketch it. Let us write first:

$$\begin{aligned} f(x) &= \Big(\int_0^1 \frac{da}{a} + \int_1^\infty \frac{da}{a}\Big) \int db\, g\Big(\frac{x-b}{a}\Big) T(a,b) \\ &= f_s(x) + f_l(x)\,, \end{aligned} \tag{76}$$

f_l is obviously C^∞. We concentrate on f_s.

Let $x_1, x_2 \in \mathcal{M}^\alpha$, $x_1 < x_2$, we cut f_s in three pieces.

$$\begin{aligned} f_s(x_1) - f_s(x_2) = &\int_0^{x_2 - x_1} \frac{da}{a} \int db\, g\Big(\frac{x_2-b}{a}\Big) T(a,b) \\ &- \int_0^{x_1 - x_1} \frac{da}{a} \int db\, g\Big(\frac{x_2-b}{a}\Big) T(a,b) \\ &+ \int_{x_2-x_1}^1 da \int db\, \Big(\frac{1}{a}\, g\Big(\frac{x_2-b}{a}\Big) - \frac{1}{a}\, g\Big(\frac{x_1-b}{a}\Big)\Big) \\ &\qquad \cdot T(a,b) \end{aligned} \tag{77}$$

$$=: T_1 - T_2 + T_3\,. \tag{78}$$

We now analyze each term:

- T_1 and T_2. Since $T(a,b) = O(a^\alpha)$ almost everywhere, we have

$$
\begin{aligned}
|T_i| &= \int_0^{x_2-x_1} \frac{da}{a} \int db \left| \frac{1}{a}\, g\Big(\frac{x_i - b}{a}\Big) \right| C\, a^\alpha \\
&= O(|x_2 - x_1|^\alpha)\, \|g\|_{L_1} \frac{C}{\alpha} \,.
\end{aligned}
\tag{79}
$$

- T_3. If g is continuously differentiable let us write

$$
g\Big(\frac{x_2 - b}{a}\Big) - g\Big(\frac{x_1 - b}{a}\Big) = \frac{x_2 - x_1}{a}\, g'\Big(\frac{x' - b}{a}\Big)
\tag{80}
$$

with $x_1 \leq x' \leq \overline{x}_2$. So

$$
\begin{aligned}
|T_3| &\leq \int_{x_2-x_1}^1 \frac{da}{a} \int db \left| \frac{1}{a^2}\, g'\Big(\frac{x' - b}{a}\Big) \right| |T(a,b)|\, |x_2 - x_1| \\
&= O(|x_2 - x_1|)\, \|g'\|_{L_1} \int_{x_2-x_1}^1 \frac{da}{a}\, a^{\alpha-1} \\
&= O(|x_2 - x_1|^\alpha)\,.
\end{aligned}
\tag{81}
$$

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On Bernoulli identities and applications

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Part I

Abstract. Bernoulli numbers appear as special values of zeta functions at integers and identities relating the Bernoulli numbers follow as a consequence of properties of the corresponding zeta functions. The most famous example is that of the special values of the Riemann zeta function and the Bernoulli identities due to Euler. In this paper we introduce a general principle for producing Bernoulli identities and apply it to zeta functions considered by Shintani, Zagier and Eie. Our results include some of the classical results of Euler and Ramanujan. Kummer's congruences play important roles in the investigation of p-adic interpolation of the classical Riemann zeta function. It asserts congruence relations among Bernoulli numbers, *i.e.*

$$(1-p^{m-1})\,\frac{B_m}{m} \equiv (1-p^{n-1})\,\frac{B_n}{n} \pmod{p^{N+1}}$$

if $m \equiv n \pmod{(p-1)\,p^N}$ and $(p-1)$ is not a divisor of m. In the second part of this paper, we use a simple Bernoulli identity to prove that

$$\begin{aligned}(1-p^{m-1})\,\frac{B_m}{m}&\\ &\equiv \frac{p^{-(N+1)}}{m} \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^m - \frac{1}{2} \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^{m-1} \pmod{p^{N+1}}\,.\end{aligned}$$

We then deduce from this Kummer's congruence by using von Staudt's theorem and Euler's generalization of Fermat's theorem

$$a^m \equiv a^n \pmod{p^{N+1}},$$

if a is relative prime to p and $m \equiv n \pmod{(p-1)\,p^N}$. Our argument can be applied to derive congruences among Bernoulli polynomials and in general the special values at negative integers of zeta functions associated with rational functions considered by Eie.

1. Introduction.

Let $m_1, \ldots, m_r$ be positive integers and $P(T)$ be a polynomial in T with complex coefficients of degree less than $m_1 + \cdots + m_r$. For $|T| < 1$, we let

$$F(T) = \frac{P(T)}{(1 - T^{m_1}) \cdots (1 - T^{m_r})} = \sum_{k=0}^{\infty} a(k)\, T^k .$$

Such functions occur as generating functions of partition numbers (*cf.* Hardy and Wright [5, Chapter XIX]) and dimensions of spaces of automorphic forms – *e.g.* if we let $a(k)$ be the dimension of the space of Siegel modular forms of genus 2 and weight k, then

$$\sum_{k=0}^{\infty} a(k)\, T^k = \frac{1 + T^{35}}{(1 - T^4)\,(1 - T^6)\,(1 - T^{10})\,(1 - T^{12})}$$

(*cf.* Igusa [6]). The value of $a(k)$ is determined by F via the residue theorem as

$$a(k) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(z)\, dz}{z^{k+1}} ,$$

where $\mathcal{C}$ is a sufficiently small circle centered at the origin going counterclockwise.

The generating function of the numbers $a(k)$ is the Dirichlet series

$$Z_F(s) = \sum_{k=1}^{\infty} a(k)\, k^{-s}$$

(*cf.* Hardy and Wright [5, Chapter XVII]). This zeta function is related to $F(T)$ via a Mellin transform

$$Z_F(s)\,\Gamma(s) = \int_0^{\infty} t^{s-1}\,(F(e^{-t}) - F(0))\, dt ,$$

for Re s sufficiently large. Our underlying principle is to evaluate $F(T)$ in two ways, yielding a Bernoulli identity, with special values of the zeta functions of Shintani [8], Zagier [9] and Eie [2], [3] on the one hand, the special values of classical zeta functions of Riemann and Hurwitz and sums of residues on the other. One gets easily this way Euler's identity: if $n \geq 2$,

$$\sum_{k=1}^{n-1} \frac{(2\,n)!}{(2\,k)!\,(2\,n-2\,k)!}\, B_{2k}\, B_{2n-2k} = -(2\,n+1)\, B_{2n}\ ,$$

(*cf.* [1, Part I, p. 122]) and Ramanujan's identities ($\alpha, \beta > 0$ with $\alpha\,\beta = \pi^2$),

1) if $n > 1$,

$$\alpha^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\alpha k}-1} - (-\beta)^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\beta k}-1} = (\alpha^n - (-\beta)^n)\, \frac{B_{2n}}{4\,n}\ ,$$

2) if $n \in \mathbb{Z}$,

$$\begin{aligned} \alpha^{-n} \Big(\frac{1}{2}\, \zeta(2\,n+1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\alpha k}-1} \Big) \\ - (-\beta)^{-n} \Big(\frac{1}{2}\, \zeta(2\,n+1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\beta k}-1} \Big) \\ = -2^{2n} \sum_{k=0}^{n+1} (-1)^k\, \frac{B_{2k}}{(2\,k)!}\, \frac{B_{2n+2-2k}}{(2\,n+2-2\,k)!}\, \alpha^{n+1-k}\, \beta^k\ , \end{aligned}$$

3) if $n \geq 1$,

$$\begin{aligned} \alpha^{-n} \sum_{k=1}^{\infty} (-1)^{k+1}\, \frac{\operatorname{csch}(\alpha\, k)}{k^{2n+1}} - (-\beta)^{-n} \sum_{k=1}^{\infty} (-1)^{k+1}\, \frac{\operatorname{csch}(\beta\, k)}{k^{2n+1}} \\ = 2^{2n+1} \sum_{k=0}^{n+1} (-1)^k\, \frac{B_{2k}\Big(\frac{1}{2}\Big)}{(2\,k)!}\, \frac{B_{2n+2-2k}\Big(\frac{1}{2}\Big)}{(2\,n+2-2\,k)!}\, \alpha^{n+1-k}\, \beta^k\ , \end{aligned}$$

(*cf.* [1, Part II, Chapter 14]).

In the first part of this paper we present some new Bernoulli identities. In view of the current motivic interest in special values of zeta functions, one cannot help from wondering if there is an abstract framework giving a unified explanation of these identities as in the case of polylogarithms (*cf.* Zagier [10]).

In the second part of the paper the Bernoulli identities are used to give new proofs of classical Kummer congruences. The Bernoulli numbers B_n $(n = 0, 1, 2, \dots)$ and Bernoulli polynomials $B_n(x)$ $(n = 0, 1, 2, \dots)$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n\, t^n}{n!}\,, \qquad |t| < 2\pi\,,$$

and

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)\, t^n}{n!}\,, \qquad |t| < 2\pi\,.$$

Suppose that m, n are positive even integers, p is an odd prime with $p - 1$ not a divisor of m and N is a non-negative integer. Kummer's congruences asserted that if

$$m \equiv n \pmod{(p-1)\, p^N}\,,$$

then

$$(1 - p^{m-1})\, \frac{B_m}{m} \equiv (1 - p^{n-1})\, \frac{B_n}{n} \pmod{p^{N+1}}\,.$$

Kummer's congruences play important roles in the p-adic interpolation of the classical Riemman zeta function. Indeed if we consider the function

$$\zeta_p(s) = (1 - p^{-s})\, \zeta(s) = \sum_{\substack{n=1 \\ (n,p)=1}} n^{-s}\,, \qquad \operatorname{Re} s > 1\,.$$

Then the congruences tell us that $\zeta_p(s)$ is a continuous function on the ring of p-adic integers $\mathbb{Z}_p$, *i.e.*,

$$\zeta_p(1 - m) \equiv \zeta_p(1 - n) \pmod{p^{N+1}}\,,$$

if $m \equiv n \pmod{(p-1)\, p^N}$.

One can construct a p-adic measure μ on $\mathbb{Z}_p$ and express $\zeta_p(1-m)$ as a constant multiple of the p-adic integration

$$\int x^{m-1}\, d\mu(x)\,,$$

where the integration is over $\mathbb{Z}_p^*$ (see for example Koblitz [9]). Note that for $x \in (\mathbb{Z}/p^{N+1}\mathbb{Z})^*$, the set of invertible elements of the quotent ring $\mathbb{Z}/p^{N+1}\mathbb{Z}$, one has

$$x^{m-1} \equiv x^{n-1} \pmod{p^{N+1}},$$

if

$$m \equiv n \pmod{(p-1)\,p^N}.$$

So that Kummer's congruences follow as easy consequences by a simple argument (*cf.* [6]).

Here we shall develop another elementary proof of Kummer's congruences by a simple identity among Riemann zeta function and Hurwitz zeta functions,

$$\text{(I)}\qquad (1-p^{-s})\,\zeta(s) = p^{-(N+1)s} \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} \zeta\Big(s; \frac{j}{p^{N+1}}\Big),$$

where the Hurwitz zeta function is defined as

$$\zeta(s;\delta) = \sum_{n=0}^{\infty} (n+\delta)^{-s}, \qquad \operatorname{Re} s > 1,\ \delta > 0.$$

Such an identity follows easily from the consideration of zeta functions associated with rational functions of the form

$$F(T) = \frac{P(T)}{(1-T^{m_1})\cdots(1-T^{m_r})}$$

(see Part I).

Note that both the Riemann zeta function $\zeta(s)$ and Hurwitz zeta function $\zeta(s;\delta)$ have analytic continuations in the whole complex plane. Moreover, their special values at non-positive integers are given by Bernoulli numbers and Bernoulli polynomials, respectively. Specifically, one has

$$\zeta(1-m) = (-1)^{m-1}\,\frac{B_m}{m} \qquad \text{and} \qquad \zeta(1-m;\delta) = -\frac{B_m(\delta)}{m}.$$

Set $s = 1-m$ in the identity (I), we get

$$\text{(II)}\qquad (1-p^{m-1})\,\frac{B_m}{m} = \frac{1}{m} \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} \sum_{l=0}^{m} \binom{m}{l} B_l\, j^{m-l}\, p^{(N+1)(l-1)}.$$

Here

$$\binom{m}{l} = \frac{m!}{l!\,(m-l)!}$$

is the binomial coefficient.

On the other hand, von Staudt's theorem ([2, Chapter 5, Theorem 4]) implies that pB_l is alway p-integral, *i.e.* it contains no divisor of p in the denominator of pB_l. So after modulo p^{N+1}, we get

$$\text{(III)} \qquad \begin{aligned}(1-p^{m-1})\,\frac{B_m}{m} \equiv \frac{1}{m} &\sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^m\, p^{-(N+1)} \\ &- \frac{1}{2} \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^{m-1} \pmod{p^{N+1}}\,.\end{aligned}$$

Next we evaluate the sum

$$\sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^m\,,$$

in the multiplicative group $(\mathbb{Z}/p^{N+1}\,\mathbb{Z})^*$ by decomposing it into a direct product of finite cyclic groups and we obtain Kummer's congruences by assuming von Staudt's Theorem; finally we give a proof of von Staudt's theorem by using the Bernoulli identity (II) with $N=0$.

At the end of the paper we extend Kummer's congruences on Bernoulli numbers to congruences on Bernoulli polynomials.

2. Special values of zeta functions.

2.1. Bernoulli numbers and Bernoulli polynomials.

We recall some results on special values of zeta functions.

For the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\,, \qquad \operatorname{Re} s > 1$$

and the Hurwitz zeta function

$$\zeta(s;\delta)=\sum_{n=0}^{\infty}(n+\delta)^{-s}\,,\qquad \delta>0\,,\ \operatorname{Re}s>1\,,$$

it is well known that for an integer $m\geq 0$,

$$\zeta(-m)=(-1)^m\,\frac{B_{m+1}}{m+1}\qquad\text{and}\qquad \zeta(-m;\delta)=-\frac{B_{m+1}(\delta)}{m+1}\,.$$

2.2. Zeta functions associated with linear forms.

Let $\beta=(\beta_1,\ldots,\beta_r)$ be an r-tuple of nonnegative integers and $L(x)=a_1\,x_1+\cdots+a_r\,x_r+\delta$ be a linear form with

$$\operatorname{Re}a_j>0\qquad\text{and}\qquad \operatorname{Re}\Big(\delta+\sum_{j=1}^{r}a_j\Big)>0\,.$$

For $\operatorname{Re}s>r+|\beta|$, define the zeta function associated with L as

$$\begin{aligned}Z(L,\beta,s)&=\sum_{n\in\mathbb{N}^r}n^\beta L(n)^{-s}\\&=\sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}n_1^{\beta_1}\cdots n_r^{\beta_r}\,(a_1\,n_1+\cdots+a_r\,n_r+\delta)^{-s}\,,\end{aligned}$$

where we use the notation $n^\beta=n_1^{\beta_1}\cdots n_r^{\beta_r}$.

These zeta functions were first considered in more general context by Eie in [2]. In particular, they have meromorphic continuations in the whole complex s-plane. Furthermore, their special values at non-positive integers are given explicitly there. Here we summarize the results we need from [3].

For any polynomial $f(x)$ of p variables and degree k

$$f(x)=\sum_{|\alpha|=0}^{k}a_\alpha\,x_1^{\alpha_1}\cdots x_p^{\alpha_p}\ ,$$

we let

$$J^p(f(x))=\sum_{|\alpha|=0}^{k}a_\alpha\,\zeta(-\alpha_1)\cdots\zeta(-\alpha_p)=\sum_{|\alpha|=0}^{k}a_\alpha\prod_{j=1}^{p}\frac{(-1)^{\alpha_j}B_{\alpha_j+1}}{\alpha_j+1}\ ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ ranges over all p-tuples of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_p$.

Also for any nonempty subset S of the index set $I = \{1, 2, \ldots, r\}$, we let

$$L_S(x) = \sum_{i \in I-S} a_i\, x_i + \delta = L(x) - \sum_{j \in S} a_j\, x_j$$

and $|S|$ be the cardinal number of S.

The following proposition is an immediate consequence of the main theorem in [3].

Proposition 1. *For any integer $m \geq 0$, the special value at $s = -m$ of $Z(L, \beta; s)$ is given by*

$$\begin{aligned} Z(L, \beta; -m) &= J^r(x^\beta L^m(x)) \\ &\quad + \sum_S \Big(\prod_{j \in S} \frac{(-1)^{\beta_j + 1}\, \beta_j!}{a_j^{\beta_j + 1}} \Big) \frac{1}{\alpha(S)!}\, J^{r-|S|} \\ &\quad \cdot \Big(\prod_{i \notin S} x_i^{\beta_i}\, L_S^{\alpha(S)}(x) \Big), \end{aligned}$$

where S ranges over all non-empty subset of $I = \{1, 2, \ldots, r\}$ in the summation and

$$\alpha(S) = m + |S| + \sum_{j \in S} \beta_j\ .$$

Here we describe the analytic continuation of $Z(L, \beta; s)$. For $\operatorname{Re} s > r + |\beta|$, we have

$$\begin{aligned} &Z(L, \beta; s)\, \Gamma(s) \\ &\quad = \sum_{n_1 = 1}^{\infty} \cdots \sum_{n_r = 1}^{\infty} n_1^{\beta_1} \cdots n_r^{\beta_r} \int_0^\infty t^{s-1}\, e^{-(a_1 n_1 + \cdots + a_r n_r + \delta)t}\, dt \\ &\quad = \int_0^\infty e^{-\delta t} \prod_{j=1}^{r} \Big(\sum_{n=1}^{\infty} n^{\beta_j}\, e^{-a_j n t} \Big)\, dt\,. \end{aligned}$$

Set

$$F_j(t) = \sum_{n=1}^{\infty} n^{\beta_j}\, e^{-a_j n t} \qquad \text{and} \qquad F(t) = e^{-\delta t} \prod_{j=1}^{r} F_j(t)\,.$$

A term by term differentiation of the identity

$$\sum_{n=1}^{\infty} e^{-a_j n t} = \frac{1}{e^{a_j t} - 1}\,, \qquad t > 0\,,$$

we get

$$F_j(t) = (-a_j)^{-\beta_j} \Big(\frac{d}{dt}\Big)^{\beta_j} \Big(\frac{1}{e^{a_j t} - 1}\Big)\,.$$

Thus around $t = 0$, $F_j(t)$ has the asymptotic expansion

$$\frac{\beta_j!}{(a_j\, t)^{\beta_j+1}} + (-1)^{\beta_j} \sum_{n_j \ge \beta_j+1} \frac{B_n (a_j\, t)^{n-\beta_j-1}}{n(n-\beta_j-1)!}\,.$$

It follows that at $t = 0$, $F(t)$ has an asymptotic expansion of the form

$$\sum_{n \ge -(|\beta|+r)} C_n\, t^n\,.$$

Consequently, the analytic continuation of $Z(L, \beta; s)$ and its special values at negative integers follow from Lemma 7 in Section 4.

When $\beta = 0$, we have the following

Corollary. *For any integer $m \ge r$, one has*

$$Z(L, 0; r-m) = \sum_{|\alpha|=m} \frac{(-1)^{m-r-\alpha_{r+1}}\,(m-r)!}{\alpha_1! \cdots \alpha_r!\, \alpha_{r+1}!} B_{\alpha_1} \cdots B_{\alpha_r}\, a_1^{\alpha_1-1} \cdots a_r^{\alpha_r-1}\, \delta^{\alpha_{r+1}}\,.$$

2.3. Shintani zeta functions.

Next we consider another kind of zeta function which were investigated first by Shintani in [8] and then Eie in [3]. Here we reformulate the main result in [3].

Let $A = (a_1, \ldots, a_r)$ and $u = (u_1, \ldots, u_r)$ be r-tuples of complex numbers such that $\mathrm{Re}\, a_j > 0$ and $u_j > 0$. Define the zeta function

$$Z(A, u; s) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} (a_1(n_1+u_1) + \cdots + a_r(n_r+u_r))^{-s}\,,$$

where $\operatorname{Re} s > r$.

Proposition 2. *For any integer $m \geq r$, one has*

$$Z(A, u; r-m) = (-1)^r \sum_{|p|=m} \frac{(m-r)!}{p_1! \cdots p_r!} B_{p_1}(u_1) \cdots B_{p_r}(u_r)\, a_1^{p_1-1} \cdots a_r^{p_r-1} .$$

Here the summation is over all p-tuples of non-negative integers such that and $|p| = p_1 + \cdots + p_r = m$.

3. Euler's Identity.

If we start from the fraction

$$F(T) = \frac{1}{(1-T)^2} = \sum_{k=0}^{\infty} (k+1)\, T^k ,$$

we obtain the identity

$$\zeta(s-1) + \zeta(s) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} (n_1 + n_2)^{-s} + 2\,\zeta(s) ,$$

from the Dirichlet series $Z_F(s)$. Setting $s = 2 - 2\,n$, we get Euler's identity

$$\sum_{k=1}^{n-1} \frac{(2\,n)!}{(2\,k)!\,(2\,n-2\,k)!} B_{2k}\, B_{2n-2k} = -(2\,n+1)\, B_{2n} , \qquad n \geq 2 .$$

In this section we shall establish a new identity analogous to that of Euler and then as an illustration of our method we give an extension of the Euler identity to Bernoulli polynomials. We state a lemma.

Lemma 3. *Given*

$$P(T) = \sum_{j=0}^{m} b_j\, T^j$$

and

$$F(T) = \frac{P(T)}{(1-T^{m_1}) \cdots (1-T^{m_r})}$$

with $m_1 + \cdots + m_r > m$*, then, for* $|T| < 1$ *we have*

$$F(T) = \sum_{j=0}^{m} b_j \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} T^{n_1 m_1 + \cdots + n_r m_r + j}$$

and hence

$$\begin{aligned} Z_F(s) = b_0 & \sum_{\substack{n_1, \ldots, n_r \geq 0 \\ |n| > 0}} (n_1 m_1 + \cdots + n_r m_r)^{-s} \\ & + \sum_{j=1}^{m} b_j \sum_{n_1, \ldots, n_r \geq 0} (n_1 m_1 + \cdots + n_r m_r + j)^{-s} . \end{aligned}$$

To illustrate our principle we consider as a first example, a fraction related to the generating function of the dimensions of Siegel modular forms of genus two,

$$F(T) = \frac{1}{(1-T^2)\,(1-T^3)\,(1-T^5)\,(1-T^6)}$$

and we derive a new Bernoulli identity.

Proposition 4. *For any integer* $m \geq 3$*,*

$$\begin{aligned} \sum_{|p|=2m} & \frac{(2\,m-4)!}{p_1!\,p_2!\,p_3!\,p_4!}\, B_{p_1}\, B_{p_2}\, B_{p_3}\, B_{p_4}\, 2^{p_1-1}\, 3^{p_2-1}\, 5^{p_3-1}\, 6^{p_4-1} \\ & = -\frac{1}{1080}\, \frac{B_{2m}}{2\,m} - \Big(\frac{17}{432} + \frac{1}{48}\, 2^{2m-2} + \frac{257}{360}\, 3^{2m-4}\Big) \frac{B_{2m-2}}{2\,m-2} \\ & \quad - \frac{197}{180}\, \frac{6^{2m-4}}{2\,m-2} \Big(B_{2m-2}\Big(\frac{1}{6}\Big) + B_{2m-2}\Big(\frac{1}{3}\Big)\Big) \\ & \quad + \frac{1}{54}\, \frac{6^{2m-4}}{2\,m-3} \Big(25\, B_{2m-3}\Big(\frac{1}{6}\Big) - 16\, B_{2m-3}\Big(\frac{1}{3}\Big)\Big) \\ & \quad - \frac{5^{2m-3}}{2\,m-3}\, B_{2m-3}\Big(\frac{1}{5}\Big) . \end{aligned}$$

Proof. Let

$$F(T) = \frac{1}{(1-T^2)\,(1-T^3)\,(1-T^6)\,(1-T^5)} \, .$$

By Lemma 3, we have for $\operatorname{Re} s > 4$

$$
\begin{aligned}
Z_F(s) &= \sum_{\substack{n_1,\dots,n_4\geq 0\\ |n|>0}} (2\,n_1+3\,n_2+6\,n_3+5\,n_4)^{-s}\\
&= \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\sum_{n_4=1}^{\infty}(2\,n_1+3\,n_2+6\,n_3+5\,n_4)^{-s}\\
&\quad+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\big((2\,n_1+3\,n_2+6\,n_3)^{-s}+(2\,n_1+6\,n_2+5\,n_3)^{-s}\\
&\qquad\qquad+(2\,n_1+3\,n_2+5\,n_3)^{-s}+(3\,n_1+6\,n_2+5\,n_3)^{-s}\big)\\
&\quad+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\big((2\,n_1+3\,n_2)^{-s}+(2\,n_1+6\,n_2)^{-s}+(2\,n_1+5\,n_2)^{-s}\\
&\qquad\qquad+(3\,n_1+6\,n_2)^{-s}+(3\,n_1+5\,n_2)^{-s}+(6\,n_1+5\,n_2)^{-s}\big)\\
&\quad+(2^{-s}+3^{-s}+6^{-s}+5^{-s})\,\zeta(s)\,.
\end{aligned}
$$

On the other hand, we decompose $F(T)$ into partial fractions

$$
\begin{aligned}
F(T) &= \frac{1}{180\,(1-T)^4}+\frac{1}{30\,(1-T)^3}+\frac{1}{48\,(1+T)^2}+\frac{1-T^4}{5\,(1-T^5)}\\
&\quad+\frac{19-332\,T-23\,T^2-54\,T^3+15\,T^4+144\,T^5}{720\,(1-T^6)}\\
&\quad+\frac{514+274\,T+514\,T^2+514\,T^3+274\,T^4+514\,T^5}{720\,(1-T^6)^2}\\
&= \frac{1}{1080}\sum_{k=0}^{\infty}(k+1)\,(k+2)\,(k+3)\,T^k+\frac{1}{60}\sum_{k=0}^{\infty}(k+1)\,(k+2)\,T^k\\
&\quad+\frac{1}{48}\sum_{k=0}^{\infty}(-1)^k(k+1)\,T^k+\frac{1}{5}\sum_{k=0}^{\infty}(1-T^4)\,T^{5k}\\
&\quad+\frac{1}{720}\sum_{k=0}^{\infty}(19-332\,T-23\,T^2-54\,T^3+15\,T^4+144\,T^5)\,T^{6k}\\
&\quad+\frac{1}{720}\sum_{k=0}^{\infty}(k+1)\,(514+274\,T+514\,T^2
\end{aligned}
$$

$$+ 514\,T^3 + 274\,T^4 + 514\,T^5)\,T^{6k}\,.$$

So the corresponding zeta function is

$$\begin{aligned}
Z_F(s) = {} & \frac{1}{1080}\,(\zeta(s-3) + 6\,\zeta(s-2) + 11\,\zeta(s-1) + 6\,\zeta(s)) \\
& + \frac{1}{60}\,(\zeta(s-2) + 3\,\zeta(s-1) + 2\,\zeta(s)) \\
& + \frac{1}{48}\,((2^{2-s}-1)\,\zeta(s-1) + (2^{1-s}-1)\,\zeta(s)) \\
& + \frac{1}{5}\Big(\sum_{k=1}^{\infty}(5\,k)^{-s} - \sum_{k=0}^{\infty}(5\,k+4)^{-s}\Big) \\
& + \frac{1}{720}\Big(19\sum_{k=1}^{\infty}(6\,k)^{-s} \\
& \qquad -\sum_{k=0}^{\infty}\big(332\,(6\,k+1)^{-s} + 23\,(6\,k+2)^{-s} \\
& \qquad\qquad +54\,(6\,k+3)^{-s} - 15\,(6\,k+4)^{-s} - 144\,(6\,k+5)^{-s}\big)\Big) \\
& + \frac{1}{720}\Big(514\sum_{k=1}^{\infty}(6\,k)^{-s} \\
& \qquad +\sum_{k=0}^{\infty}(k+1)\big(274\,(6\,k+1)^{-s} + 514\,(6\,k+2)^{-s} \\
& \qquad\qquad + 514\,(6\,k+3)^{-s} + 274\,(6\,k+4)^{-s} \\
& \qquad\qquad + 514\,(6\,k+5)^{-s}\big)\Big)\,.
\end{aligned}$$

Set $s = 4 - 2\,m$ with $m \geq 3$, we get that $Z_F(4-2\,m)$ is equal to the right hand side of our identity after an elementary calculation.

Consider $Z_F(s)$ as a sum of zeta functions associated with linear forms, we have

$$\begin{aligned}
& Z_F(4-2\,m) \\
& \quad = \sum_{|p|=2m} \frac{(2\,m-4)!}{p_1!\,p_2!\,p_3!\,p_4!}\,B_{p_1}\,B_{p_2}\,B_{p_3}\,B_{p_4}\,2^{p_1-1}\,3^{p_2-1}\,6^{p_3-1}\,5^{p_4-1}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{|p|=2m-1}\frac{(2\,m-4)!}{p_1!\,p_2!\,p_3!}\,B_{p_1}\,B_{p_2}\,B_{p_3}\\
&\qquad\cdot\big(2^{p_1-1}\,3^{p_2-1}\,5^{p_3-1}+2^{p_1-1}\,3^{p_2-1}\,6^{p_3-1}\\
&\qquad\quad+2^{p_1-1}\,6^{p_2-1}\,5^{p_3-1}+3^{p_1-1}\,6^{p_2-1}\,5^{p_3-1}\big)\\
&+\sum_{|p|=2m-2}\frac{(2\,m-4)!}{p_1!\,p_2!}\,B_{p_1}\,B_{p_2}\\
&\qquad\cdot\big(2^{p_1-1}\,3^{p_2-1}+2^{p_1-1}\,6^{p_2-1}+2^{p_1-1}\,5^{p_2-1}\\
&\qquad\quad+3^{p_1-1}\,6^{p_2-1}+3^{p_1-1}\,5^{p_2-1}+6^{p_1-1}\,5^{p_2-1}\big)\,.
\end{aligned}
$$

In the second summation, $p=(p_1,p_2,p_3)$ ranges over all non-negative integers p_1,p_2,p_3 such that $p_1+p_2+p_3=2\,m-1$. So at least one of p_j must be odd. But Bernoulli numbers of odd index are zero except $B_1=-1/2$. Hence we have

$$
\begin{aligned}
&\sum_{|p|=2m-1}\frac{(2\,m-4)!}{p_1!\,p_2!\,p_3!}\,B_{p_1}\,B_{p_2}\,B_{p_3}\,a_1^{p_1-1}\,a_2^{p_2-1}\,a_3^{p_3-1}\\
&\qquad=-\frac{1}{2}\sum_{|p|=2m-2}\frac{(2\,m-4)!}{p_1!\,p_2!}\,B_{p_1}\,B_{p_2}\\
&\qquad\qquad\cdot\big(a_1^{p_1-1}\,a_2^{p_2-1}+a_1^{p_1-1}\,a_3^{p_2-1}+a_2^{p_1-1}\,a_3^{p_2-1}\big)\,.
\end{aligned}
$$

Therefore, the second sum in the summation cancels the third sum. Hence our identity follows.

REMARK. Different decompositions of $F(T)$ into partial fractions may lead to different expressions of $Z_F(s)$ in terms of finite sums of Riemann zeta functions and Hurwitz zeta functions. However, one can prove that the resulting identities are the same by employing well known identities such as

$$
B_m(k\,\delta)=k^{m-1}\sum_{j=1}^{k}B_m\Big(\delta+\frac{j}{k}\Big)\,.
$$

The formula in the next proposition is an analogue of Euler's identity.

Proposition 5. *For each positive integer $n \geq 4$, one has*

$$\sum_{k=2}^{n-2} \frac{(2\,n-2)!}{(2\,k-2)!\,(2\,n-2\,k-2)!}\,\frac{B_{2k}}{2\,k}\,\frac{B_{2n-2k}}{2\,n-2\,k} = \Big(-\frac{B_{2n}}{2\,n}\Big)\frac{(2\,n+1)\,(2\,n-6)}{6\,(2\,n-2)\,(2\,n-3)}\,.$$

Proof. Let

$$F(T) = \frac{T^2}{(1-T)^4}\,.$$

Then for $|T| < 1$,

$$F(T) = \frac{1}{3!}\sum_{k=0}^{\infty}(k+1)\,(k+2)\,(k+3)\,T^{k+2} = \frac{1}{6}\sum_{m=0}^{\infty}(m^3-m)\,T^m\,.$$

The corresponding zeta function $Z_F(s)$ is then

$$\frac{1}{6}\,(\zeta(s-3)-\zeta(s-1))\,.$$

Also we can express $Z_F(s)$ as a sum of zeta functions associated with linear forms. By Lemma 3 we have

$$Z_F(s) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}(n_1+n_2+n_3+n_4+2)^{-s}\,.$$

After a change of variables $n_1+n_2+1 = p_1$, $n_3+n_4+1 = p_2$ in the summation we get

$$Z_F(s) = \sum_{p_1=1}^{\infty}\sum_{p_2=1}^{\infty} p_1\,p_2\,(p_1+p_2)^{-s}\,.$$

Set $s = 4-2\,n$ with $n \geq 4$. The identity

$$\sum_{k=0}^{2n-4}\frac{(2\,n-4)!}{k!\,(2\,n-k-4)!}\,\frac{B_{k+2}}{k+2}\,\frac{B_{2n-k-2}}{2\,n-k-2}+\Big(-\frac{B_{2n}}{2\,n}\Big)\frac{2}{(2\,n-2)\,(2\,n-3)} = \frac{1}{6}\Big(-\frac{B_{2n}}{2\,n}+\frac{B_{2n-2}}{2\,n-2}\Big),$$

follows from Proposition 1 and a simple calculation yields our assertion.

REMARK. The identity of Proposition 5 appears in [7] as a consequence of an identity among Eisenstein series. Similar identities follow from different consideration of generating functions. For example, if we consider $F(T) = T^3/(1-T)^6$, we get the following identity for $n \geq 6$,

$$\sum_{\substack{p+q+r=n \\ p,q,r\geq 2}} \frac{(2\,n-6)!}{(2\,p-2)!\,(2\,q-2)!\,(2\,r-2)!}\,\frac{B_{2p}\,B_{2q}\,B_{2r}}{8\,p\,q\,r}$$

$$= \Big(-\frac{B_{2n}}{2\,n}\Big)\Big(\frac{1}{120} - \frac{2\,n^2-5\,n}{(2\,n-2)\,(2\,n-3)\,(2\,n-4)\,(2\,n-5)}\Big)$$

$$+ \frac{1}{80}\Big(\frac{B_{2n-4}}{2\,n-4}\Big)\,.$$

Proposition 6. *For any integer $n \geq 2$*

$$\sum_{k=0}^{2n} \frac{(2\,n)!}{k!\,(2\,n-k)!}\,(-1)^k\,B_k(u)\,B_{2n-k}(u) = -(2\,n-1)\,B_{2n}\,.$$

PROOF. Writing the fraction $F(T) = T/(1-T)^2$ in two ways we get the identity

$$\sum_{k=0}^{\infty} k\,T^k = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} T^{n_1+n_2+1}\,.$$

Hence for $\operatorname{Re} s > 2$, we have

$$\zeta(s-1) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} (n_1+n_2+1)^{-s}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} ((n_1+\delta)+(n_2+1-\delta))^{-s}\,, \qquad 0<\delta<1\,.$$

This is just the function $Z((1,1),(\delta,1-\delta);s)$ of Proposition 2.

Set $s = 2-2\,n$, we get

$$\sum_{k=0}^{2n} \frac{(2\,n)!}{k!\,(2\,n-k)!}\,B_k(\delta)\,B_{2n-k}(1-\delta) = -(2\,n-1)\,B_{2n}\,.$$

In light of the identity (*cf.* [14, p. 31])

$$B_{2n-k}(1-\delta) = (-1)^k \, B_{2n-k}(\delta)\,,$$

we have proved that the identity holds for $0 < u < 1$.

However, as functions of variable u, both sides of the identity are analytic functions of u. If it holds for $0 < u < 1$, it must hold for all u.

Remark. In exactly the same way, we get the following identity

$$\begin{aligned}
&\sum_{p+q+r=2n} \frac{(2\,n)!}{p!\,q!\,r!}\, B_p(u)\, B_q(v)\, B_r(w) \\
&\quad = (2\,n-1)\,(2\,n-2)\, B_{2n}(u+v+w) \\
&\qquad + (3-2\,(u+v+w))\, 2\,n\,(2\,n-2)\, B_{2n-1}(u+v+w) \\
&\qquad + ((u+v+w)^2 - 3\,(u+v+w) + 2)\, 2\,n\,(2\,n-1)\, B_{2n-2}(u+v+w)\,.
\end{aligned}$$

4. Identities in Ramanujan's notebooks.

In Chapter 14 of Ramanujan's notebooks II [1], there are many interesting identities on Bernoulli numbers. We shall use here Cauchy's formula for Taylor series coefficients. First we prove a new identity analogous to those of Ramanujan and then we make some remarks on the proof of Ramanujan's identities by our method.

We quote the following classical result from [9]:

Lemma 7. *Let $\{\lambda\}$ be a sequence of positive real numbers tending $+\infty$. Suppose that the Dirichlet series*

$$\psi(s) = \sum_{\lambda>0} a_\lambda \, \lambda^{-s}\,,$$

converges for sufficiently large $\operatorname{Re} s$. *Let*

$$f(t) = \sum_{\lambda>0} a_\lambda^{-\lambda t}$$

be the corresponding exponential series. If at $t = 0$, $f(t)$ has an expansion of the form

$$\sum_{n \geq n_0} C_n \, t^n\,, \qquad n_0 \text{ being integer},$$

then

1) $\psi(s)$ *has a meromorphic continuation in the whole complex plane, and*

2) $\psi(-m) = (-1)^m \, m! \, C_m$ *for each integer* $m \geq 0$.

Proposition 8. *For* α, $\beta > 0$ *with* $\alpha\,\beta = \pi^2$ *and each positive integer* n,

$$\begin{aligned}
&\sum_{k=0}^{n+1} \frac{B_{2k}}{(2\,k)!} \, \frac{B_{2n+2-2k}}{(2\,n+2-2\,k)!} \, (2\,k-1)\,(2\,n-2\,k+1)\,(-\beta)^k \, \alpha^{n+1-k} \\
&\qquad = (2\,n-1)\,2^{-1-2n}\,\alpha^{1-n} \sum_{k=1}^{\infty} \frac{\operatorname{csch}^2 k\alpha}{k^{2n}} \\
&\qquad\quad + 2^{-2n}\,\alpha^{2-n} \sum_{k=1}^{\infty} \frac{\operatorname{csch}^2 k\alpha \operatorname{cotanh} k\,\alpha}{k^{2n-1}} \\
&\qquad\quad + (2\,n-1)\,2^{-1-2n}\,(-\beta)^{1-n} \sum_{k=1}^{\infty} \frac{\operatorname{csch}^2 k\beta}{k^{2n}} \\
&\qquad\quad - 2^{-2n}\,\beta^{2-n} \sum_{k=1}^{\infty} \frac{\operatorname{csch}^2 k\beta \operatorname{cotanh} k\beta}{k^{2n-1}} \; .
\end{aligned}$$

Proof. For any positive number ε, consider the zeta function

$$Z_\varepsilon(s) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} n_1 n_2 \big(\sqrt{\alpha}\, n_1 + \big(\varepsilon + \sqrt{\beta}\, i\big) n_2\big)^{-s} , \qquad \operatorname{Re} s > 4 \, .$$

By Proposition 1, $Z_\varepsilon(s)$ has an analytic continuation and

$$\begin{aligned}
&Z_\varepsilon(2-2\,n) \\
&\quad = \sum_{k=0}^{2n-2} \frac{(2\,n-2)!}{k!\,(2\,n-2-k)!} \, \frac{B_{k+2}}{k+2} \, \frac{B_{2n-k}}{2\,n-k} \big(\sqrt{\alpha}\big)^k \big(\varepsilon + \sqrt{\beta}\, i\big)^{2n-2-k} \\
&\qquad + \Big(\frac{\big(\varepsilon + \sqrt{\beta}\, i\big)^{2n}}{\alpha^2} + \frac{\alpha^n}{\big(\varepsilon + \sqrt{\beta}\, i\big)^2} \Big) \Big(- \frac{B_{2n+2}}{2\,n+2} \Big) \, \frac{1}{2\,n\,(2\,n-1)} \; .
\end{aligned}$$

It follows that

$$\begin{aligned} I &= \lim_{\varepsilon\to 0} Z_\varepsilon(2-2\,n) \\ &= \sum_{k=0}^{n+1} \frac{(2\,n-2)!}{(2\,k)!\,(2\,n+2-2\,k)!}\,(2\,k-1)\,(2\,n-2\,k+1) \\ &\qquad \cdot B_{2k}\,B_{2n+2-2k}\,(-\beta)^{k-1}\,\alpha^{n-k}\,. \end{aligned}$$

On the other hand, let

$$\begin{aligned} F_\varepsilon(t) &= \Big(\sum_{n_1=1}^{\infty} n_1 e^{-\sqrt{\alpha} n_1 t}\Big)\Big(\sum_{n_2=1}^{\infty} n_2\, e^{-(\varepsilon+\sqrt{\beta}\, i) n_2 t}\Big) \\ &= \frac{e^{(\sqrt{\alpha}+\varepsilon+i\sqrt{\beta})t}}{(e^{\sqrt{\alpha}t}-1)^2\,(e^{(\varepsilon+i\sqrt{\beta})t}-1)^2} \end{aligned}$$

and

$$F(t) = \lim_{\varepsilon\to 0} F_\varepsilon(t) = \frac{e^{(\sqrt{\alpha}+i\sqrt{\beta})t}}{(e^{\sqrt{\alpha}t}-1)^2\,(e^{i\sqrt{\beta}t}-1)^2}\,.$$

Note that for $\operatorname{Re} s > 4$,

$$Z_\varepsilon(s)\,\Gamma(s) = \int_0^\infty t^{s-1} F_\varepsilon(t)\,dt\,.$$

It follows from Lemma 7 that

$$Z_\varepsilon(2-2\,n) = (2\,n-2)!\,\frac{1}{2\pi i}\int_{|z|=\delta} z^{1-2n}\,F_\varepsilon(z)\,dz\,,$$

where $0<\delta<1$ and the direction on the circle $|z|=\delta$ is counterclockwise. As $\varepsilon \longrightarrow 0$, we get

$$I = (2\,n-2)!\,\frac{1}{2\pi i}\int_{|z|=\delta} z^{1-2n}\,F(z)\,dz\,.$$

Let C_N be the contour in the complex plane consisting of the rectangle with vertices $(2N+1)(\sqrt{\alpha}+\sqrt{\beta}\ i)$, $(2N+1)(\sqrt{\alpha}-\sqrt{\beta}\ i)$, $(2N+1)(-\sqrt{\alpha}-\sqrt{\beta}\ i)$, $(2N+1)(-\sqrt{\alpha}+\sqrt{\beta}\ i)$ in counterclock direction. Note that $F(z)$ is bounded on the rectangle by a constant independent of N. Thus we have

$$\lim_{N\to\infty}\int_{C_N} z^{1-2n}\,F(z)\,dz = 0\,.$$

This implies in particular that the sum of residues of $z^{1-2n}F(z)$ inside C_N approaches zero as $N \longrightarrow \infty$. Thus the residue at zero is equal to the negative of the sum of residues elsewhere. It follows that

$$I = -(2\,n-2)!\sum_{k\neq 0}\left\{\text{Residues of } z^{1-2n}F(z) \text{ at } z = 2\,k\,\sqrt{\beta}\,i,\ 2\,k\,\sqrt{\alpha}\right\}.$$

Our assertion now follows .

REMARK 1. When $\alpha = \beta = \pi$ and n is odd, we get

$$\sum_{k=0}^{n+1}\frac{B_{2k}\,B_{2n+2-2k}}{(2\,k)!\,(2\,n+2-2\,k)!}\,(2\,k-1)\,(2\,n+1-2\,k)\,(-1)^k$$
$$= (2\,n-1)\,(2\pi)^{-2n}\sum_{k=1}^{\infty}\frac{\operatorname{csch}^2 k\pi}{k^{2n}} + (2\pi)^{1-2n}\sum_{k=1}^{\infty}\frac{\operatorname{csch}^2 k\pi\,\operatorname{cotanh} k\pi}{k^{2n-1}}\ .$$

REMARK 2. If we consider instead the zeta function

$$Z_\varepsilon(s) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sqrt{\alpha}\,(n_1+u)+\left(\varepsilon+i\sqrt{\beta}\right)(n_2+v)\right)^{-s}, \qquad \operatorname{Re} s > 2\,,$$

with $0 < u,\ v \leq 1$, we find that for all positive integers n

$$2^{2n}\sum_{k=0}^{n+1}\frac{B_{2k}(u)}{(2\,k)!}\ \frac{B_{2n-2k+2}(v)}{(2\,n-2\,k+2)!}\,\alpha^{n-k+1}\,(-\beta)^k$$
$$= -\frac{1}{2}\,\alpha^{-n}\sum_{k=1}^{\infty}\frac{\cos(2\,k\,\pi\,v)\,\left(e^{2ku\alpha}+e^{2k(1-u)\alpha}\right)}{k^{2n+1}\left(e^{2k\alpha}-1\right)}$$
$$+\frac{1}{2}\,(-\beta)^{-n}\sum_{k=1}^{\infty}\frac{\cos(2\,k\,\pi u)\,\left(e^{2kv\beta}+e^{2k(1-v)\beta}\right)}{k^{2n+1}\left(e^{2k\beta}-1\right)}\ .$$

Setting $u = v = 1/2$, we obtain the identity

$$\alpha^{-n}\sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{\operatorname{csch}(\alpha\,k)}{k^{2n+1}} - (-\beta)^{-n}\sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{\operatorname{csch}(\beta\,k)}{k^{2n+1}}$$
$$= 2^{2n+1}\sum_{k=0}^{n+1}(-1)^k\,\frac{B_{2k}\left(\frac{1}{2}\right)}{(2\,k)!}\ \frac{B_{2n+2-2k}\left(\frac{1}{2}\right)}{(2\,n+2-2\,k)!}\,\alpha^{n+1-k}\,\beta^k\ .$$

As u, v approach 0, we get the identity

$$\alpha^{-n}\Big(\frac{1}{2}\,\zeta(2\,n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2k\alpha}-1}\Big)-(-\beta)^{-n}\Big(\frac{1}{2}\,\zeta(2\,n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\Big)$$
$$=-2^{2n}\sum_{k=0}^{n+1}(-1)^k\,\frac{B_{2k}}{(2\,k)!}\,\frac{B_{2n+2-2k}}{(2\,n+2-2\,k)!}\,\alpha^{n+1-k}\,\beta^k\,,$$

with n a positive integer. The right hand side of the identity we obtained is a constant multiple of

$$\frac{1}{2\pi i}\int_{|z|=\delta}\frac{e^{-(\sqrt{\alpha}u+i\sqrt{\beta}v)z}\,dz}{z^{2n+1}\,(1-e^{-\sqrt{\alpha}z})\,(1-e^{-i\sqrt{\beta}z})}\,.$$

It is zero if $n<-1$. This yields the identity ([1, Chapter 14, p. 261])

$$\alpha^n\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2\alpha k}-1}-(-\beta)^n\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2\beta k}-1}=(\alpha^n-(-\beta)^n)\,\frac{B_{2n}}{4\,n}\,,\qquad n>1\,,$$

if we let u, v approach zero.

Remark 3. For each rational function $F(T)$ of the form

$$\frac{P(T)}{(1-T^{m_1})\cdots(1-T^{m_r})}\,,$$

where $\deg P(T)<m_1+\cdots+m_r$. The possible poles of $F(e^{-z})$ lie in the imaginary axis of the complex plane. By a direct verification, we can find a sequence of contours C_N ($N=1,2,\dots$) such that the following conditions hold:

1) C_N is the rectangle with vertices $x_N+i\,y_N$, $x_N-i\,y_N$, $-x_n+i\,y_N$, $-x_N-i\,y_N$, $x_N>0$, $y_N>0$ with direction counterclockwise,

2) $\lim_{N\to\infty}x_N=\lim_{N\to\infty}y_N=+\infty$,

3) C_N does not pass through any pole of $F(e^{-z})$, and

4) $\max_{z\in C_N}|F(e^{-z})|$ is bounded by a constant independent of N.

It follows that for any positive integer n,

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{C_N}z^{-(n+1)}F(e^{-z})\,dz=0\,.$$

This also implies that the residue of $z^{-(n+1)}F(e^{-z})$ at 0 is equal to the negative of the sum of residues of $z^{-(n+1)}F(e^{-z})$ at $z = 2k\pi i/m_j$, $k \in \mathbb{Z}$, $k \neq 0$, $j = 1, \ldots, r$. Note that the former is a constant multiple of $Z_F(-n)$ while the latter is an infinite series in general. This produces an identity between Bernoulli numbers and sums of infinite series. Here we give two examples.

I) For positive integers m and N with $N \geq 3$,

$$\sum_{\substack{n=1 \\ n\not\equiv 0 \pmod N}}^{\infty} \frac{\operatorname{cotan}\left(\dfrac{n\pi}{N}\right)}{n^{2m+1}} = \frac{(-1)^{m+1}(2\pi)^{2m+1}}{N\,(2m+1)!} \sum_{j=1}^{N-2} (N-j-1)\, B_{2m+1}\left(\frac{j}{N}\right).$$

II) For positive integer m and even integer $N \geq 3$,

$$\sum_{\substack{n=1 \\ n\not\equiv N/2 \pmod N}}^{\infty} \frac{\tan\left(\dfrac{n\pi}{N}\right)}{n^{2m+1}} = \frac{(-1)^{m}(2\pi)^{2m+1}}{N\,(2m+1)!} \sum_{j=1}^{N} (-1)^j\,(N-j-1)\, B_{2m+1}\left(\frac{j}{N}\right).$$

5. Generalizations to several variables.

It is possible to extend our arguments to the cases when $F(T)$ is a particular type of rational functions of several variables.

Suppose that $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn})$, $j = 1, \ldots, r$ are n-tuples of non-negative integers with $|\alpha_j| = \alpha_{j1} + \cdots + \alpha_{jn} > 0$ and $P(T) = P(T_1, \ldots, T_n)$ is a polynomial in n variables $T_1, \ldots, T_n$ with $\deg P(T) < |\alpha_1| + \cdots + |\alpha_r|$. We use the notation

$$T^\beta = \prod_{j=1}^{n} T_j^{\beta_j}\,, \qquad \text{if } \beta = (\beta_1, \ldots, \beta_n)\,.$$

Consider the rational function $F(T)$ of the form

$$F(T) = \frac{P(T)}{(1-T^{\alpha_1})\cdots(1-T^{\alpha_r})} = \frac{P(T_1, \ldots, T_n)}{(1-T_1^{\alpha_{11}}\cdots T_n^{\alpha_{1n}})\cdots(1-T_1^{\alpha_{r1}}\cdots T_n^{\alpha_{rn}})}\,.$$

For $|T_1| < 1, \ldots, |T_n| < 1$, $F(T)$ has a power series expansion

$$\sum_{|\beta|=0}^{\infty} a(\beta)\, T^{\beta} = \sum_{|\beta|=0}^{\infty} a(\beta)\, T_1^{\beta_1} \cdots T_n^{\beta_n} \,.$$

For sufficiently large $\operatorname{Re} s$, the zeta function associated with $F(T)$ is given by

$$Z_F(s) = \sum_{\beta_1=1}^{\infty} \cdots \sum_{\beta_n=1}^{\infty} a(\beta)\, (\beta_1 \beta_2 \cdots \beta_n)^{-s} \,.$$

Another expression for $Z_F(s)$ as a sum of zeta functions associated with linear forms was given by Eie in [2]. This leads to an identity in zeta functions. Using the special values at negative integers, we obtain a family of Bernoulli identities. Here we give an example to illustrate the general procedure.

Consider the rational function

$$F(T) = \frac{1}{(1 - T_1\, T_2)\, (1 - T_1\, T_2^2)} \,.$$

For $|T_1| < 1$ and $|T_2| < 1$, we have

$$F(T) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (T_1\, T_2)^{n_1} (T_1\, T_2^2)^{n_2} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} T_1^{n_1+n_2}\, T_2^{n_1+2n_2} \,.$$

It follows for $\operatorname{Re} s > 1$,

$$\begin{aligned} Z_F(s) &= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} ((n_1 + n_2)\, (n_1 + 2\, n_2))^{-s} + \sum_{n_1=1}^{\infty} n_1^{-2s} + \sum_{n_2=1}^{\infty} (2\, n_2^2)^{-s} \\ &= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} ((n_1 + n_2)\, (n_1 + 2\, n_2))^{-s} + (1 + 2^{-s})\, \zeta(2\, s) \,. \end{aligned}$$

On the other hand, as a rational function of T_2, we have the following decomposition of $F(T)$ into partial fraction

$$\begin{aligned}
F(T) &= \frac{1}{(1-T_1)\,(1-T_1\,T_2^2)} + \frac{T_1\,T_2}{(1-T_1)\,(1-T_1\,T_2^2)} \\
&\quad - \frac{T_1}{(1-T_1)\,(1-T_1\,T_2)} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} T_1^{n_1+n_2}\,T_2^{2n_2} + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} T_1^{n_1+n_2+1}\,T_2^{2n_2+1} \\
&\quad - \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} T_1^{n_1+n_2+1}\,T_2^{n_2}\,.
\end{aligned}$$

Consequently we get another expression for $Z_F(s)$ as

$$\begin{aligned}
Z_F(s) &= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} (n_1+n_2)^{-s}\,(2\,n_2)^{-s} \\
&\quad + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} (n_1+n_2+1)^{-s}\,(2\,n_2+1)^{-s} \\
&\quad - \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} (n_1+n_2+1)^{-s}\,n_2^{-s} \\
&= (2^{-s}-1)\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} (n_1+n_2)^{-s}\,n_2^{-s} + 2^{-s}\,\zeta(2\,s) \\
&\quad + 2^{-s}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Big(\Big(n_1+\frac{1}{2}\Big)+\Big(n_2+\frac{1}{2}\Big)\Big)^{-s}\Big(n_2+\frac{1}{2}\Big)^{-s} \\
&= \frac{1}{2}\,(2^{-s}-1)\,(\zeta^2(s)-\zeta(2\,s)) + 2^{-s}\,\zeta(2\,s) \\
&\quad + 2^{-s}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Big(\Big(n_1+\frac{1}{2}\Big)+\Big(n_2+\frac{1}{2}\Big)\Big)^{-s}\Big(n_2+\frac{1}{2}\Big)^{-s}.
\end{aligned}$$

Here we use the identity

$$\zeta(s)^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m\,n)^{-s}$$

$$= \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} (m\,n)^{-s} + \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} (m\,n)^{-s} + \zeta(2\,s)$$

$$= 2 \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} (n_1 + n_2)^{-s}\, n_2^{-s} + \zeta(2\,s)\,.$$

Now it remains to evaluate the zeta functions at negative integers. We need the following proposition from Eie [3].

Proposition 9. *Let* $Q = a\,x^2 + b\,x\,y + c\,y^2$ *with* $a,\ b,\ c > 0$ *and* $D = b^2 - 4\,a\,c > 0$. *Suppose that*

$$Z_Q(s) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} (a\,n_1^2 + b\,n_1\,n_2 + c\,n_2^2)^{-s}\,, \qquad \mathrm{Re}\, s > 1\,.$$

Then $Z_Q(s)$ *has an analytic continuation and its special value at each negative integer* $s = -m$ $(m = 1, 2, \dots)$ *is given by*

$$Z_Q(-m) = \sum_{p+q+r=m} \frac{m!}{p!\,q!\,r!}\, a^p\, b^q\, c^r\, \frac{B_{2p+q+1}}{2\,p+q+1}\, \frac{B_{2r+q+1}}{2\,r+q+1}$$
$$+ \Big(- \frac{B_{2m+2}}{2\,m+2} \Big) \Big(\int_0^{-b/(2a)} (a\,x^2 + b\,x + c)^m\, dx$$
$$+ \int_0^{-b/(2c)} (a + b\,y + c\,y^2)^m\, dy \Big)\,.$$

Proposition 10. *Suppose that*

$$Z(s) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (n_1 + n_2 + 2\,\delta)^{-s}\, (n_2 + \delta)^{-s}\,, \qquad \delta > 0,\ \mathrm{Re}\, s > 2\,.$$

Then $Z(s)$ *has an analytic continuation and its special value at the negative integer* $s = -m$ $(m = 1, 2, \dots)$ *is given by*

$$\sum_{k=1}^{m+1} \frac{m!}{(m+1-k)!\,k!\,(2\,m+2-k)}\, B_k(\delta)\, B_{2m+2-k}(\delta)$$
$$+ \Big(\frac{(-1)^m\, (m!)^2}{2\,(2\,m+2)!} + \frac{1}{2\,(m+1)^2} \Big) B_{2m+2}(\delta)\,.$$

PROOF. For $\operatorname{Re} s > 2$, we have

$$
\begin{aligned}
&Z(s)\,(\Gamma(s))^2 \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int_0^{\infty}\int_0^{\infty}(t_1\,t_2)^{s-1}\,e^{-(n_1+n_2+2\delta)t_1}\,e^{-(n_2+\delta)t_2}\,dt_1\,dt_2 \\
&= \int_0^{\infty}\int_0^{\infty}(t_1\,t_2)^{s-1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}e^{-(n_1+n_2+2\delta)t_1}\,e^{-(n_2+\delta)t_2}\,dt_1\,dt_2 \\
&= \int_0^{\infty}\int_0^{\infty}(t_1\,t_2)^{s-1}\,\frac{e^{(1-\delta)t_1+t_2}\,e^{(1-\delta)t_1}}{(e^{t_1+t_2}-1)\,(e^{t_1}-1)}\,dt_1\,dt_2 \\
&= \int_0^{\infty}\frac{t^{2s-1}\,e^{(1-\delta)t}}{e^t-1}\,dt\int_0^1\frac{(u\,(1-u))^{s-1}\,e^{(1-\delta)tu}}{e^{tu}-1}\,du\,.
\end{aligned}
$$

Rewrite the above formula as

$$
Z(s)\,\Gamma(s) = \int_0^{\infty} t^{2s-3}\,\frac{t\,e^{(1-\delta)t}}{e^t-1}\,dt\,\frac{1}{\Gamma(s)}\int_0^1 (u\,(1-u))^{s-1}\,\frac{e^{(1-\delta)tu}}{e^{tu}-1}\,du\,.
$$

It follows from a standard process as given in the proof of the main theorem in [3] that

$$
Z(-m) = \frac{(-1)^m\,m!}{2}\sum_{\beta=0}^{2m+2}\frac{B_{2m+2-\beta}(1-\delta)\,B_\beta(1-\delta)}{(2\,m+2-\beta)!\,\beta!}\,F_\beta(-m)\,,
$$

where

$$
F_\beta(s) = \frac{1}{\Gamma(s)}\int_0^1 (u\,(1-u))^{s-1}\,u^{\beta-1}\,du = \frac{\Gamma(s+\beta-1)}{\Gamma(2\,s+\beta-1)}\,.
$$

An elementary calculation shows that

$$
F_\beta(-m) = \begin{cases} m!\,, & \text{if } \beta = 2\,m+2\,, \\ 2\,(-1)^m\,\dfrac{(2\,m+1-\beta)!}{(m+1-\beta)!}\,, & \text{if } 0 \le \beta \le m+1\,, \\ 0\,, & \text{if } m+2 \le \beta \le 2\,m+1\,. \end{cases}
$$

Hence our assertion follows.

Using the identity

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}((n_1+n_2)\,(n_1+2\,n_2))^{-s}+(1+2^{-s})\,\zeta(2\,s)$$
$$=\frac{1}{2}\,(2^{-s}-1)\,(\zeta^2(s)-\zeta(2\,s))+2^{-s}\,\zeta(2\,s)$$
$$+\,2^{-s}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Big(\Big(n_1+\frac{1}{2}\Big)+\Big(n_2+\frac{1}{2}\Big)\Big)^{-s}\Big(n_2+\frac{1}{2}\Big)^{-s}$$

and propositions 9 and 10, we get the following Bernoulli identity

$$\sum_{p+q+r=m}\frac{m!\,3^q\,2^r}{p!\,q!\,r!}\,\frac{B_{2p+q+1}}{2\,p+q+1}\,\frac{B_{2r+q+1}}{2\,r+q+1}$$
$$+\Big(\frac{-B_{2m+2}}{2\,m+2}\Big)\Big(\int_0^{-3/2}(x^2+3\,x+2)^m\,dx$$
$$+\int_0^{-3/4}(1+3\,y+2\,y^2)^m\,dy\Big)$$
$$=\frac{1}{2}\,(2^m-1)\Big(\frac{B_{m+1}}{m+1}\Big)^2$$
$$+\,2^{m-1}\Big(\frac{(-1)^m\,(m!)^2}{(2\,m+2)!}+\frac{1}{(m+1)^2}\Big)B_{2m+2}\Big(\frac{1}{2}\Big)$$
$$+\,2^m\sum_{k=1}^{[(m+1)/2]}\frac{m!}{(m+1-2\,k)!\,(2\,k)!\,(2\,m+2-2\,k)}$$
$$\cdot\,B_{2k}\Big(\frac{1}{2}\Big)B_{2m+2-2k}\Big(\frac{1}{2}\Big)\,.$$

Remark 1. As shown above, the consideration of cases of several variables leads to zeta functions with products of linear forms. Though we have no general formula to evaluate their special values at negative integers, it is possible to calculate these values case by case.

Remark 2. It is possible to further extend our arguments in this section to the cases that

$$F(T)=\frac{P(T)}{(1-T^{\alpha_1})\cdots(1-T^{\alpha_r})}\ ,$$

with $P(T)$ not necessarily a polynomial and $\alpha_1, \ldots, \alpha_r$ are not necessarily n-tuples of non-negative integers. Indeed, we only need the following considerations.

I) $P(T)$ is a finite complex linear conbination of $T_1^{\beta_1} \cdots T_n^{\beta_n}$ with $\operatorname{Re} \beta_j \geq 0$.

II) For all $1 \leq j \leq r$, $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn})$ with $\alpha_{ji} = 0$ or Re $\alpha_{ji} > 0$, but $\alpha_j \neq 0$.

Under the second condition, for $0 \leq T_j < 1$, we have the expansion

$$\begin{aligned} P(T) &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} P(T)\, T^{m_1\alpha_1 + \cdots + m_r\alpha_r} \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} P(T) \prod_{j=1}^{n} T_j^{m_1\alpha_{j1} + \cdots + m_r\alpha_{jr}} . \end{aligned}$$

Hence it is easy to write down $Z_F(s)$ as a sum of zeta functions associated with products of n linear forms. By employing the same arguments as in Proposition 8 we obtain more identities. As an example we consider the function

$$F(T) = \frac{T^{(u\sqrt{\alpha} + v(\varepsilon + i\sqrt{\beta}))}}{(1 - T^{\sqrt{\alpha}})\,(1 - T^{\varepsilon + i\sqrt{\beta}})} ,$$

where $\varepsilon > 0$, $0 < u$, $v < 1$ and α, $\beta > 0$ with $\alpha\beta = \pi^2$. Calculating the residues and separating the real and imaginary parts we obtain the following known (*cf.* [1, volume II, p. 276]) identity

$$\begin{aligned} 2^{2m} \sum_{k=0}^{m+1} & \frac{B_{2n-2k+2}(u)\, B_{2k}(v)}{(2\,n - 2\,k + 2)!\,(2\,k)!}\, \alpha^{m+1-k}\, (-\beta)^k \\ &= -\frac{1}{2}\, \alpha^{-m} \sum_{k=1}^{\infty} \frac{\cos(2\,k\,\pi\,v)\,(e^{2ku\alpha} + e^{2k(1-u)\alpha})}{k^{2m+1}\,(e^{2k\alpha} - 1)} \\ &\quad + \frac{1}{2}\,(-\beta)^m \sum_{k=1}^{\infty} \frac{\cos(2\,k\,\pi\,u)\,(e^{2kv\beta} + e^{2k(1-v)\beta})}{k^{2m+1}\,(e^{2k\beta} - 1)} . \end{aligned}$$

Part II

Minking Eie

Throughout the rest of the paper, we use the following notations: p is an odd prime number, m, n are integers such that $p-1$ is not a divisor of m, N is a positive integer or zero.

6. An identity for zeta functions.

We apply the method of Part I to establish an identity for zeta functions.

Proposition 11. *For any prime number p and complex number s with $\operatorname{Re} s > 1$, one has*

$$(1-p^{-s})\,\zeta(s) = p^{-(N+1)s} \sum_{\substack{(j,p)=1 \\ 1\le j<p^{N+1}}} \zeta\Big(s; \frac{j}{p^{N+1}}\Big)\,.$$

Proof. Consider the zeta function $Z_F(s)$ associated with the rational function

$$F(T) = \frac{1}{1-T} - \frac{1}{1-T^p}\ .$$

It is easy to see that for $\operatorname{Re} s > 1$

$$Z_F(s) = \sum_{k=1}^{\infty} k^{-s} - \sum_{k=1}^{\infty}(k\,p)^{-s} = (1-p^{-s})\,\zeta(s)\,.$$

On the other hand, we have for any nonegative integer N,

$$\begin{aligned} F(T) &= \frac{T - T^p}{(1-T)\,(1-T^p)} \\ &= \frac{T + T^2 + \cdots + T^{p-1}}{1 - T^p} \\ &= \frac{(T + T^2 + \cdots + T^{p-1})\,(1 + T^p + T^{2p} + \cdots + T^{p(p^N - 1)})}{\left(1 - T^{p^{N+1}}\right)} \\ &= \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} \sum_{k=0}^{\infty} T^{j + kp^{N+1}}\,. \end{aligned}$$

It follows that for $\operatorname{Re} s > 1$,

$$Z_F(s) = p^{-(N+1)s} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} \zeta\Big(s; \frac{j}{p^{N+1}}\Big)\,.$$

Note that $Z_F(s)$ is determined by $F(T)$ uniquely through the integral formula

$$Z_F(s)\,\Gamma(s) = \int_0^\infty t^{s-1} F(e^{-t})\,dt\,, \qquad \operatorname{Re} s > 1\,,$$

where $\Gamma(s)$ is the classical gamma function. Thus our identity follows.

As a consequence, we have the following.

Proposition 12. *Suppose m is a positive even integer and p is an odd prime with $p-1$ not a divisor of m. Then*

$$(1 - p^{m-1})\,\frac{B_m}{m} \equiv C_0(m) + C_1(m) \pmod{p^{N+1}}\,,$$

where

$$C_l(m) = \frac{1}{m} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} j^{m-l} \binom{m}{l} B_l\, p^{(N+1)(l-1)}\,, \qquad 0 \le l \le m\,.$$

PROOF. We begin with the identity in Proposition 11. Both the Riemann zeta function $\zeta(s)$ and Hurwitz zeta functions

$$\zeta\Big(s; \frac{j}{p^{N+1}}\Big), \qquad j = 1, \ldots, p^{N+1},$$

have analytic continuations in the whole complex plane. So that the identity I) is true for all s. In particular, we can set $s = 1 - m$ in the identity to yield

$$\begin{aligned}
(1 - p^{m-1})\,\frac{B_m}{m} &= \frac{1}{m} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} p^{(N+1)(m-1)}\, B_m\Big(\frac{j}{p^{N+1}}\Big) \\
&= \frac{1}{m} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} \sum_{l=0}^{m} \binom{m}{l} B_l\, j^{m-l}\, p^{(N+1)(l-1)} \\
&= \sum_{l=0}^{m} C_l(m)\,.
\end{aligned}$$

Note that the exponent of p occurs in $l!$ is not greater than

$$\frac{l}{p} + \frac{l}{p^2} + \cdots + \frac{l}{p^l} + \cdots = \frac{l}{p-1} \le \frac{l}{2}\,.$$

Also $p\,B_l$ is p-integral for all l and

$$C_l(m) = (m-1)\cdots(m-l+1) \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} j^{m-l}\, \frac{1}{l!}\,(p\,B_l)\, p^{(N+1)(l-1)-1}\,.$$

Thus $C_l(m) \equiv 0 \pmod{p^{N+1}}$ provides that

$$-\frac{l}{2} + (N+1)\,(l-1) - 1 \ge N + 1\,.$$

This is equivalent to

$$(N+1)\,(l-2) \ge \frac{l}{2} + 1\,.$$

But N is a nonnegative integer, the inequality holds provides that

$$l-2 \geq \frac{l}{2}+1\,.$$

This is equivalent to $l \geq 6$. Thus it follows

$$\begin{aligned}(1-p^{m-1})\,\frac{B_m}{m} &\equiv \sum_{l=0}^{5} C_l(m) \pmod{p^{N+1}} \\ &\equiv C_0(m)+C_1(m)+C_2(m)+C_4(m) \pmod{p^{N+1}}\,.\end{aligned}$$

Next we prove

$$C_2(m) \equiv 0 \pmod{p^{N+1}}$$

and

$$C_4(m) \equiv 0 \pmod{p^{N+1}}\,.$$

Note that

$$C_2(m) = \frac{m-1}{2}\,B_2 \sum_{(j,p)=1} j^{m-2}\,p^{N+1} = \frac{m-1}{12}\,p^{N+1} \sum_{(j,p)=1} j^{m-2}\,.$$

If $p \neq 3$, then

$$\frac{m-1}{12}\,p^{N+1}\,,$$

is p-integral and divisible by p^{N+1}. However the case $p=3$ is impossible under the assumption that $p-1$ is not a divisor of m. This proves that

$$C_2(m) \equiv 0 \pmod{p^{N+1}}\,.$$

Now consider the case $l=4$,

$$\begin{aligned}C_4(m) &= \frac{(m-1)\,(m-2)\,(m-3)}{24}\,B_4 \sum_{(j,p)=1} j^{m-4}\,p^{3N+3} \\ &= -\frac{(m-1)\,(m-2)\,(m-3)}{720}\,p^{3N+3} \sum_{(j,p)=1} j^{m-4} \\ &= -\frac{(m-1)\,(m-2)\,(m-3)}{2^4\,3^2\,5}\,p^{3N+3} \sum_{(j,p)=1} j^{m-4}\,.\end{aligned}$$

Obviously $C_4(m)$ is p-integral and divisible by p^{N+1} for any odd prime p. Hence we can drop the last two terms in our congruence relation and it completes our proof.

7. Congruence relations of $C_0(m)$.

Recall that for $0 \le l \le m$,

$$C_l(m) = \frac{1}{m} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} j^{m-l} \binom{m}{l} B_l \, p^{(N+1)(l-1)} .$$

As shown in Proposition 12, Kummer's congruences are equivalent to

$$C_0(m) + C_1(m) \equiv C_0(n) + C_1(n) \pmod{p^{N+1}} .$$

However

$$C_1(m) = -\frac{1}{2} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} j^{m-1} .$$

So it is easy to see that if $m \equiv n \pmod{(p-1)p^N}$, then

$$C_1(m) \equiv C_1(n) \pmod{p^{N+1}} ,$$

since

$$j^{m-1} \equiv j^{n-1} \pmod{p^{N+1}} ,$$

for all integer j relative prime to p. Consequently, Kummer's congruences are equivalent to

$$\frac{p^{-(N+1)}}{m} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} j^m \equiv \frac{p^{-(N+1)}}{n} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} j^n \pmod{p^{N+1}} .$$

To simplify the notation we write

$$\sum_{(j,p)=1}$$

for
$$\sum_{\substack{(j,p)=1 \\ 1\le j<p^{N+1}}} .$$

Our proof that
$$C_0(m) \equiv C_0(n) \pmod{p^{N+1}},$$
employs the classical theorems of Fermat ([6, Theorems 71, 88]).

Proposition 13. *Suppose that m, n are positive even integers and p is an odd prome with $p-1$ not a divisor of m. Then*
$$C_0(m) \equiv C_0(n) \pmod{p^{N+1}},$$
if $m \equiv n \pmod{(p-1)\,p^N}$.

Proof. By the fundamental theorem of finite abelian group ([11]), we can decompose the multiplicative group $G = (\mathbb{Z}/p^{N+1}\mathbb{Z})^*$ into a direct product
$$G_0 \prod_{i=1}^{\mu} G_i\,,$$
where G_0 is a cyclic group of order $p-1$ and G_i $(i = 1, \ldots, \mu)$ is a cyclic group of order p^{e_i} with
$$e_1 + \cdots + e_\mu = N\,.$$
Such a decomposition is possible since $\mathbb{Z}/p^{N+1}\mathbb{Z}$ contains $\mathbb{Z}/p\,\mathbb{Z}$ as a subfield and the multiplicative group $(\mathbb{Z}/p\,\mathbb{Z})^*$ is a cyclic group of order $p-1$ ([8]).

Suppose that $g, g_1, \ldots, g_\mu$ are generators of $G_0, G_1, \ldots, G_\mu$; respectively. It follows
$$\begin{aligned} C_0(m) &= \frac{p^{-(N+1)}}{m} \sum_{(j,p)=1} j^m \\ &= \frac{p^{-(N+1)}}{m}\,(1+g^m+\cdots+g^{m(p-2)}) \prod_{i=1}^{\mu}(1+g_i^m+\cdots+g_i^{m(p^{e_i}-1)})\,. \end{aligned}$$
Note that $g^m \neq 1$ since $p-1$ is not a divisor of m. So
$$1 + g^m + \cdots + g^{m(p-2)} = \frac{g^{(p-1)m}-1}{g^m-1}\,.$$

For each $1 \le i \le \mu$, if $g_i^m = 1$, then we automatically have

$$1 + g_i^m + \cdots + g_i^{m(p^{e_i}-1)} = p^{e_i}\,.$$

If $g_i^m \neq 1$, we have

$$1 + g_i^m + \cdots + g_i^{m(p^{e_i}-1)} = \frac{g_i^{mp^{e_i}} - 1}{g_i^m - 1}\,.$$

But

$$g_i^{mp^{e_i}} \equiv 1 \pmod{p^{e_i}}\,.$$

Consequently the sum

$$1 + g_i^m + \cdots + g_i^{m(p^{e_i}-1)}\,,$$

always has the divisor p^{e_i}.

With a possible permutation in the indices, we suppose that $g_i^m = 1$ for $1 \le i < q$ and $g_i^m \neq 1$ for $q \le i \le \mu$. Then we rewrite $C_0(m)$ as

$$C_0(m) = \frac{p^{-(N+1)}}{m}\,\frac{g^{m(p-1)} - 1}{g^m - 1}\prod_{i=1}^{q-1} p^{e_i} \prod_{i=q}^{\mu} \frac{g_i^{mp^{e_i}} - 1}{g_i^m - 1}\,.$$

Suppose that $g^{p-1} = 1 + k\,p$, then it is a direct verification that $(g^{m(p-1)} - 1)/(m\,p)$ is p integral and

$$\frac{1}{m\,p}\,(g^{m(p-1)} - 1) \equiv k \pmod{p}\,.$$

It follows

$$\frac{1}{m\,p}\,(g^{m(p-1)} - 1) \equiv \frac{1}{n\,p}\,(g^{n(p-1)} - 1) \pmod{p}\,.$$

Also $g^m - 1$ and $g_i^m - 1$ $(i = q, \ldots, \mu)$ are invertible elements of $\mathbb{Z}/p^{N+1}\mathbb{Z}$ and

$$g^m - 1 \equiv g^n - 1 \pmod{p^{N+1}}\,,$$

$$g_i^m - 1 \equiv g_i^n - 1 \pmod{p^{N+1}}\,.$$

So that

$$(g^m - 1)^{-1} \equiv (g^n - 1)^{-1} \pmod{p^{N+1}},$$
$$(g_i^m - 1)^{-1} \equiv (g_i^n - 1)^{-1} \pmod{p^{N+1}}.$$

Multiply all these congruences together, we get

$$C_0(m) \equiv C_0(n) \pmod{p^{N+1}}.$$

8. von Staudt's Theorem.

Our proof of Proposition 13 is analogous to the proof of von Staudt's Theorem in [2, p. 384]. Indeed we are able to give another proof of von Staudt's Theorem by the identity (II) with $N = 0$. In other words, we are able to kill two birds with one stone.

Proposition 14 (von Staudt's Theorem). *Suppose that m is a positive even integer and p is an odd prime. Then*

a) *B_m is p-integral if $p-1$ is not a divisor of m,*

b) *if $p-1$ is a divisor of m, then $p\,B_m$ is p-integral and*

$$p\,B_m \equiv -1 \pmod{p}.$$

PROOF. We begin with the identity (II) with $N = 0$.

$$(1 - p^{m-1})\,\frac{B_m}{m} = \frac{1}{m}\sum_{j=1}^{p-1}\sum_{l=0}^{m}\binom{m}{l} B_l\, j^{m-l}\, p^{l-1}.$$

Multiply both sides by m, we get

$$\begin{aligned}(1 - p^{m-1})\,B_m &= \sum_{j=1}^{p-1}\sum_{l=0}^{m}\binom{m}{l} B_l\, j^{m-l}\, p^{l-1}\\ &= \sum_{j=1}^{p-1}\sum_{l=0}^{m-1}\binom{m}{l} B_l\, j^{m-l}\, p^{l-1} + p^{m-1}(p-1)\,B_m.\end{aligned}$$

It follows

$$(1-p^m)\,B_m = \sum_{j=1}^{p-1}\sum_{l=0}^{m-1}\binom{m}{l} B_l\, j^{m-l}\, p^{l-1}\,.$$

Now we shall prove our assertion by induction on m.

Suppose that $p\,B_l$ is p-integral for all $1 \le l < m-1$. Then

$$\binom{m}{l} B_l\, p^{l-1} = \binom{m}{l} (p\,B_l)\, p^{l-2}$$

is p-integral provide that $l \ge 2$. Hence we have

$$(1-p^m)\,B_m \equiv \frac{1}{p}\sum_{j=1}^{p-1} j^m - \frac{m}{2}\sum_{j=1}^{p-1} j^{m-1} + \frac{m\,(m-1)}{12}\, p \sum_{j=1}^{p-1} j^{m-2} \pmod{p}\,.$$

Note that $p \ne 2$ or 3, so that the third term on the right hand side is also p-integral and divisible by p. So we can drop it in our consideration.

If $(p-1)$ is a divisor of m, then $j^m = 1$ for all $1 \le j \le p-1$. It follows

$$(1-p^m)\,B_m \equiv \frac{p-1}{p} - \frac{m}{2}\sum_{j=1}^{p-1} j^{m-1} \pmod{p}\,.$$

Thus $p\,B_m$ is p-integral and

$$p\,B_m \equiv -1 \pmod{p}\,.$$

On the other hand, if $p-1$ is not a divisor of m, we choose an element g of order $p-1$ in $(\mathbb{Z}/p\,\mathbb{Z})^*$. Then

$$\begin{aligned}(1-p^m)\,B_m &\equiv \frac{1}{p}\sum_{j=0}^{p-2} g^{mj} - \frac{m}{2}\sum_{j=1}^{p-1} g^{(m-1)j} \pmod{p}\\ &= \frac{1}{p}\,\frac{g^{(p-1)m}-1}{g^m-1} - \frac{m}{2}\sum_{j=1}^{p-1} g^{(m-1)j} \pmod{p}\,.\end{aligned}$$

Suppose that $g^{p-1} = 1 + \alpha\,p$. Then

$$g^{(p-1)m} \equiv 1 + m\,\alpha\,p \pmod{p^2}\,.$$

Thus

$$\frac{1}{p}\,\frac{g^{(p-1)m}-1}{g^m-1}$$

is p-integral. This proves that B_m is p-integral.

9. A slight generalization of Kummer's congruences.

Here we reformulate Kummer's congruences in a general form.

Theorem 15. *Suppose that m, n are positive even integers and k is a positive integer such that $p-1$ is not a divisor of m for all prime divisor p of k. Then*

$$\frac{B_m}{m}\prod_{p|k}(1-p^{m-1}) \equiv \frac{B_n}{n}\prod_{p|k}(1-p^{n-1}) \pmod{k},$$

if $m \equiv n \pmod{\varphi(k)}$, here φ is the Euler φ-function.

Proof. Suppose that

$$k = \prod_{i=1}^{\mu} p_i^{N_i+1},$$

with $p_1, \ldots, p_\mu$ are distinct prime numbers and $N_1, \ldots, N_\mu$ are non-negative integers.

Consider the zeta function

$$\zeta_k(s) = \sum_{\substack{(n,k)=1 \\ n\geq 1}} n^{-s}, \qquad \operatorname{Re} s > 1.$$

$\zeta_k(s)$ has the Euler product

$$\prod_{p|k}(1-p^{-s})\,\zeta(s).$$

As usual, $\zeta_k(s)$ has its analytic continuation and its special value at $s = 1-m$ is given by

$$\zeta_k(1-m) = -\frac{B_m}{m}\prod_{p|k}(1-p^{m-1}).$$

On the other hand, $\zeta_k(s)$ is the zeta function associated with the rational function

$$F(T) = \frac{1}{1-T} - \sum_{i=1}^{\mu}\frac{1}{1-T^{p_i}} + \sum_{\substack{1\leq i \\ l\leq\mu}}\frac{1}{1-T^{p_ip_l}} + \cdots + (-1)^{\mu}\frac{1}{1-T^{p_1\cdots p_\mu}}$$

by the well known inclusion-exclusion principle. Also for $|T|<1$, $F(T)$ has the power series expansion

$$F(T)=\sum_{\substack{(j,k)=1\\ 1\le j<k}}\sum_{l=0}^{\infty}T^{j+lk}\,.$$

Thus it follows

$$\zeta_k(s)=k^{-s}\sum_{\substack{(j,p)=1\\ 1\le j<k}}\zeta\Big(s;\frac{j}{k}\Big)$$

and hence

$$\zeta_k(1-m)=-\frac{k^{m-1}}{m}\sum_{\substack{(j,k)=1\\ 1\le j<k}}B_m\binom{j}{k}=-\frac{1}{m}\sum_{\substack{(j,k)=1\\ 1\le j<k}}\sum_{l=0}^{m}\binom{m}{l}j^{m-l}\,B_l\,k^{l-1}\,.$$

Set

$$C_l(m)=\frac{1}{m}\sum_{\substack{(j,k)=1\\ 1\le j<k}}\binom{m}{l}j^{m-l}\,B_l\,k^{l-1}\,.$$

Note that for each $1\le i\le\mu$

$$C_l(m)\equiv\frac{1}{m\,p_i^{N_i+1}}\sum_{\substack{(j,k)=1\\ 1\le j<p^{N_i+1}}}\binom{m}{l}j^{m-l}\,B_l k^{l-1}\pmod{p_i^{N_i+1}}\,.$$

By our proof Proposition 2, we have for $l\ge 2$,

$$C_l(m)\equiv 0\pmod{p_i^{N_i+1}}\,,\qquad i=1,\dots,\mu\,.$$

By Chinese remainder's theorem, we get for $l\ge 2$.

$$C_l(m)\equiv 0\pmod{k}\,.$$

This implies

$$\frac{B_m}{m}\prod_{p|k}(1-p^{m-1})\equiv\frac{1}{m\,k}\sum_{\substack{(j,k)=1\\ 1\le j<k}}j^m-\frac{1}{2}\sum_{\substack{(j,k)=1\\ 1\le j<k}}j^{m-1}\pmod{k}\,.$$

Consequently our assertion is equivalent to prove

$$\frac{1}{mk}\sum_{\substack{(j,k)=1\\1\le j<k}} j^m \equiv \frac{1}{n\,k}\sum_{\substack{(j,k)=1\\1\le j<k}} j^n \pmod{k}\,.$$

But it follows from the fact that

$$\frac{1}{m\,k}\sum_{\substack{(j,k)=1\\1\le j<k}} j^m \equiv \frac{p_i^{-(N_i+1)}}{m}\sum_{\substack{(j,k)=1\\1\le j<p^{N_i+1}}} j^m \pmod{p^{N_i+1}}$$

and our previous identity

$$\frac{p_i^{-(N_i+1)}}{m}\sum_{\substack{(j,k)=1\\1\le j<p^{N_i+1}}} j^m \equiv \frac{p_i^{-(N_i+1)}}{n}\sum_{\substack{(j,k)=1\\1\le j<p^{N_i+1}}} j^n \pmod{p^{N_i+1}}\,,$$

for all $1 \le i \le \mu$.

10. p-adic interpolation.

Let p be a prime number. $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are the ring of p-adic integers and the field of p-adic numbers, respectively. Ω_p is the algebra completion of $\mathbb{Q}_p$. For a fixed positive integer k, we let X_k be the inverse projective limit of $\mathbb{Z}/k\,p^N\,\mathbb{Z}$, *i.e.*

$$X_k = \varprojlim \mathbb{Z}/k\,p^N\,\mathbb{Z}\,,$$

where the map from $\mathbb{Z}/k\,p^M\,\mathbb{Z}$ to $\mathbb{Z}/k\,p^N\,\mathbb{Z}$ for $M \ge N$ is the reduction modulo $k\,p^N$. Denote by $a + k\,p^N\,\mathbb{Z}_p$ the set of x in X_k which map to a in $\mathbb{Z}/k\,p^N\,\mathbb{Z}$ under the natural projection map from X_k to $\mathbb{Z}/k\,p^N\,\mathbb{Z}$.

Fix a r-th root of unity ε with r relative prime to k. Also suppose that ε is not a p^N-th root of unity for any N. Define

$$\mu_\varepsilon(a + k\,p^N\,\mathbb{Z}_p) = \frac{\varepsilon^a}{1-\varepsilon^{kp^N}}$$

and

$$\mu(a + k\,p^N\,\mathbb{Z}_p) = \sum_{\substack{\varepsilon^r=1\\ \varepsilon\ne 1}} \mu_\varepsilon(a + k\,p^N\,\mathbb{Z}_p)\,.$$

The above p-adic measure was given in [6] and it is also known as Mazure measure.

Note that

$$X_k = \bigcup_{0\le a<k} (a + k\,\mathbb{Z}_p)$$

is a disjoint union of k topological spaces isomorphic to $\mathbb{Z}_p$. Also we have

$$a + k\,p^N\,\mathbb{Z}_p = \bigcup_{0\le b<p} ((a + b\,k\,p^N) + k\,p^{N+1}\,\mathbb{Z}_p)\,.$$

The above is a disjoint union of p compact open sets. It is easy to verify directly that

$$\mu(a + k\,p^N\,\mathbb{Z}) = \sum_{b=0}^{p-1} \mu((a + b\,k\,p^N) + k\,p^{N+1}\,\mathbb{Z}_p)\,.$$

For any continuous function $f : X_k \longrightarrow \Omega_p$, we define

$$\int_{X_k} f(x)\,d\mu(x) = \lim_{N\longrightarrow\infty} \sum_{0\le a<k\,p^N} f(a)\,\mu(a + k\,p^N\,\mathbb{Z}_p)\,.$$

Consider the integration of the exponential function e^{tx} and follow the general procedure of [6], we obtain the following.

Proposition 16. *For any positive integers m and k, we have*

$$\int_{X_k} x^{m-1}\,d\mu(x) = (1 - r^m)\,\frac{B_m}{m}\,.$$

Proposition 17. *Let X_k^* be elements of X_k which map onto $(\mathbb{Z}/k\,\mathbb{Z})^*$, the invertible elements of $\mathbb{Z}/k\,\mathbb{Z}$. Then for any positive integer m,*

$$\int_{X_k^*} x^{m-1}\,d\mu(x) = (1 - r^m)\,\frac{B_m}{m}\prod_{p|k}(1 - p^{m-1})\,.$$

Proof. By the inclusion-exclusion principle, we decompose the integration into the following:

$$\int_{X_k^*} = \int_{X_k} - \sum_{p_i|k} \int_{p_i X_k} + \sum_{p_i p_j|k} \int_{p_i p_j X_k} + \cdots + (-1)^{\mu} \int_{p_1\cdots p_\mu X_k}\,.$$

Here $p_1, \ldots, p_\mu$ are distinct prime divisors of k. To prove the proposition, it suffices to prove that

$$\int_{\alpha X_k} x^{m-1}\, d\mu(x) = (1 - r^m)\, \alpha^{m-1}\, \frac{B_m}{m}\ ,$$

for any integer α which is a prime divisor or a product of distinct prime divisors of k.

Again we consider the integration of e^{tx},

$$\begin{aligned}\int_{\alpha X_k} e^{tx}\, d\mu_\varepsilon(x) &= \lim_{N\to\infty} (1 - \varepsilon^{kp^N})^{-1} \sum_{0\le b< kp^N/\alpha} (\varepsilon\, e^t)^{\alpha b} \\ &= \lim_{N\to\infty} (1 - \varepsilon^{kp^N})^{-1} (1 - (\varepsilon\, e^t)^{kp^N})\, (1 - \varepsilon^\alpha\, e^{\alpha t})^{-1} \\ &= (1 - \varepsilon^\alpha\, e^{\alpha t})^{-1}\,.\end{aligned}$$

Since r is relative prime to α, the mapping ε to ε^α causes a permutation among r-th roots of unity. Hence

$$\begin{aligned}\int_{\alpha X_k} e^{tx}\, d\mu(x) &= \frac{r - (1 + e^{\alpha t} + \cdots + e^{(r-1)\alpha t})}{1 - e^{r\alpha t}} \\ &= \frac{r}{1 - e^{r\alpha t}} - \frac{1}{1 - e^{\alpha t}} \\ &= \sum_{m=1}^{\infty} \frac{(1 - r^m)\, B_m (\alpha\, t)^{m-1}}{m!}\ .\end{aligned}$$

By comparing the coefficients of t, we get our assertion.

Now we are ready to given another proof of the theorem in Section 9.

Proof of Theorem 15. For any element x in $(\mathbb{Z}/k\,\mathbb{Z})^*$, we have the congruence relation

$$x^{m-1} \equiv x^{n-1} \pmod{k}\,,$$

since $m - n$ is a multiple of $\varphi(k)$. Hence for any prime divisor p of k, with the p-adic measure $\mu(x)$ defined on X_k, we have

$$\int_{X_k^*} x^{m-1}\, d\mu(x) \equiv \int_{X_k^*} x^{n-1}\, d\mu(x) \pmod{p^\alpha}\,,$$

where $\alpha = \nu_p(k)$ is the highest power of p dividing k. On the other hand, we have $r^m - 1 \in (\mathbb{Z}/k\,\mathbb{Z})^*$ since $r^m - 1 \in (\mathbb{Z}/p^\alpha\,\mathbb{Z})^*$ for any prime divisor p of k. Also we have

$$r^m - 1 \equiv r^n - 1 \pmod{k},$$

since $(r, n) = 1$ and $m \equiv n \pmod{\varphi(k)}$. Hence

$$(1 - r^m)^{-1} \int_{X_k^*} x^{m-1}\, d\mu(x) \equiv (1 - r^n)^{-1} \int_{X_k^*} x^{n-1}\, d\mu(x) \pmod{p^\alpha}.$$

This is equivalent to

$$\frac{B_m}{m} \prod_{p|k} (1 - p^{m-1}) \equiv \frac{B_n}{n} \prod_{p|k} (1 - p^{n-1}) \pmod{p^\alpha}.$$

Thus it follows

$$\frac{B_m}{m} \prod_{p|k} (1 - p^{m-1}) \equiv \frac{B_n}{n} \prod_{p|k} (1 - p^{n-1}) \pmod{k}.$$

11. Congruences among Bernoulli polynomials.

We are able to apply our previous arguments in Section 6 to derive congruences among Bernoulli polynomials or in general, the special values at negative integers of zeta functions associated with rational functions as considered before. Here we give a simple example to illustrate the general procedure.

Proposition 18. *For a fixed prime odd number p ($p \geq 5$) and any positive integer k relative prime to p. Suppose that α, β are positive integers such that $1 \leq \alpha,\ \beta < k$ and $\alpha + j_0\, n = \beta\, p$ for some positive integer j_0 with $1 \leq j_0 \leq p - 1$. Then for all complex number s with $Re\, s > 1$,*

$$\zeta\Big(s; \frac{\alpha}{k}\Big) - p^{-s} \zeta\Big(s; \frac{\beta}{k}\Big) = (p^{N+1})^{-s} \sum_{\substack{1 \leq j \leq kp^{N+1} \\ (j,p)=1 \\ j \equiv \alpha \pmod{k}}} \zeta\Big(s; \frac{j}{k\, p^{N+1}}\Big).$$

PROOF. Consider the zeta function $Z_F(s)$ associated with the rational function
$$F(T) = \frac{T^\alpha}{1 - T^k}\,.$$
Obviously, we have
$$Z_F(s) = k^{-s}\zeta\Big(s; \frac{\alpha}{k}\Big)\,.$$
Also from the identity
$$F(T) = \frac{T^\alpha(1 + T^k + \cdots + T^{(p-1)k})}{1 - T^{kp}}\,,$$
we conclude that
$$\begin{aligned} Z_F(s) &= (k\,p)^{-s}\sum_{j=0}^{p-1}\zeta\Big(s; \frac{\alpha + j\,k}{k\,p}\Big) \\ &= (k\,p)^{-s}\zeta\Big(s; \frac{\beta}{k}\Big) + (k\,p)^{-s}\sum_{\substack{j=0 \\ j\neq j_0}}^{p-1}\zeta\Big(s; \frac{\alpha + j\,k}{k\,p}\Big)\,. \end{aligned}$$
On the other hand, we also have
$$F(T) = \frac{T^\alpha\,(1 + T^k + \cdots + T^{(p-1)k})\,(1 + T^{kp} + \cdots + T^{kp(p^N-1)})}{1 - T^{kp^{N+1}}}\,.$$
Thus it follows also that
$$\begin{aligned} k^{-s}\,\zeta\Big(s; \frac{\alpha}{k}\Big) - (k\,p)^{-s}\,\zeta\Big(s; \frac{\beta}{k}\Big) &= (k\,p)^{-s}\sum_{\substack{j=0 \\ j\neq j_0}}^{p-1}\zeta\Big(s; \frac{\alpha + j\,k}{k\,p}\Big) \\ &= (k\,p^{N+1})^{-s}\sum_{\substack{1\le j<kp^{N+1} \\ (j,p)=1 \\ j\equiv\alpha \pmod{k}}}\zeta\Big(s; \frac{j}{k\,p^{N+1}}\Big)\,. \end{aligned}$$
Multiply the factor k^{-s} on both sides, we get our assertion.

To simplify notation we write

$$\sum_{j\equiv\alpha \pmod k}$$

for

$$\sum_{\substack{1\le j<kp^{N+1}\\ (j,p)=1\\ j\equiv\alpha \pmod k}}.$$

Proposition 19. *Under the assumptions of the previous proposition and suppose that m,n are positive integers such that $p-1$ is not a divisor of m. Then*

$$\begin{aligned}&\frac{1}{m}\Big(B_m\Big(\frac{\alpha}{k}\Big)-p^{m-1}\,B_m\Big(\frac{\beta}{k}\Big)\Big)\\ &\qquad\qquad\equiv\frac{1}{n}\Big(B_n\Big(\frac{\alpha}{k}\Big)-p^{n-1}\,B_m\Big(\frac{\beta}{k}\Big)\Big) \pmod{p^{N+1}},\end{aligned}$$

if $m\equiv n \pmod{(p-1)\,p^N}$.

Proof. Set $s=1-m$ in the identity of Proposition 7, we get

$$\begin{aligned}&\frac{1}{m}\Big(B_m\Big(\frac{\alpha}{k}\Big)-p^{m-1}B_m\Big(\frac{\beta}{k}\Big)\Big)\\ &\qquad=\frac{1}{m}\sum_{j\equiv\alpha \pmod k}\sum_{l=0}^{m}\binom{m}{l}B_l\,j^{m-l}\,p^{(N+1)(l-1)}\,k^{l-m}.\end{aligned}$$

With exact the same argument as in Proposition 2, we get

$$\begin{aligned}&\frac{1}{m}\Big(B_m\Big(\frac{\alpha}{k}\Big)-p^{m-1}B_m\Big(\frac{\beta}{k}\Big)\Big)\\ &\equiv\frac{1}{m\,k^m\,p^{N+1}}\sum_{j\equiv\alpha \pmod k}j^m-\frac{1}{2}\sum_{j\equiv\alpha \pmod k}j^{m-1}\,k^{1-m} \pmod{p^{N+1}}.\end{aligned}$$

Thus our congruences are equivalent to

$$\frac{1}{m\,k^m\,p^{N+1}}\sum_{j\equiv\alpha \pmod k}j^m\equiv\frac{1}{n\,k^n\,p^{N+1}}\sum_{j\equiv\alpha \pmod k}j^n \pmod{p^{N+1}}.$$

Note that k is relative prime to p, so the mapping $x \longmapsto kx+\alpha$ is an one to one mapping from $\mathbb{Z}/p^{N+1}\mathbb{Z}$ into $\mathbb{Z}/p^{N+1}\mathbb{Z}$. Thus we have

$$\sum_{j\equiv\alpha \pmod k} j^m \equiv \sum_{\substack{1\le j<p^{N+1}\\(j,p)=1}} j^m \pmod{p^{N+1}}.$$

Hence our congruences follow by the same argument as in Proposition 13.

REMARK. It is possible to construct another p-adic measure on the space $\mathbb{Z}_p$ so that the integration of the monomial x^{m-1} over $\mathbb{Z}_p^*$ yields a sum of Bernoulli polynomials. Hence, we have the p-adic interpolation of Kummer's congruences on Bernoulli polynomials. We'll discuss this in another paper.

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Differential equations driven by rough signals

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1. Preliminaries.

1.1. Introduction.

1.1.1. Inhomogeneous differential equations.

Time inhomogeneous (or non-autonomous) systems of differential equations are often treated rather formally as extensions of the homogeneous (or autonomous) case by adding an extra parameter to the system; however this can be a travesty. Consider an equation of the kind

$$dy_t = \sum_i f^i(y_t)\, dx_t^i \,, \tag{1.1}$$

where the f^i are vector fields, x_t represents some (multi-dimensional) forcing or controlling term and the trajectory y_t represents some filtered effect thereof. In this case the effect of such a reduction produces an equation whose expression involves a derivative of the term x_t. In problems from control, or where noise is involved, or even in algebra (developing a path from a Lie algebra into a group) this path will rarely be smooth, so the resulting autonomous system will have a defining vector field which will frequently not be continuous; perhaps it will only exist as a distribution. In this case the classical theory does not suggest the correct approach to identifying solutions; and even in highly oscillatory but smooth situations suggests inefficient algorithms for numerical approximation to classical solutions.

1.1.2. Objectives.

This paper aims to provide a systematic approach to the treatment of differential equations of the type described by (1.1) where the driving signal x_t is a rough path. Such equations are very common and occur particularly frequently in probability where the driving signal might be a vector valued Brownian motion, semi-martingale or similar process.

However, our approach is deterministic, is totally independent of probability and permits much rougher paths than the Brownian paths usually discussed. The results here are strong enough to treat the main probabilistic examples and significantly widen the class of stochastic processes which can be used to drive stochastic differential equations. (For a simple example see [10], [1]).

We hope our results will have an influence on infinite dimensional analysis on path spaces, loop groups, etc. as well as in more applied situations. Variable step size algorithms for the numerical integration of stochastic differential equations [8] have been constructed as a consequence of these results.

1.1.3. The Itô map.

Suppose every vector field f^i in (1.1) is Lipschitz with respect to some complete metric on a manifold M and that the driving signal x_t is continuous and piece-wise smooth; then classical solutions to (1.1) exist for all time and are unique; by fixing y_0, we may regard (1.1) as defining a functional (which we will refer to as the Itô map) taking each smooth path x_t (in a certain vector space V) to a unique path based at y_0 in a manifold M. By varying the starting point y_0 and taking the induced flow, one may also regard (1.1) as defining a map taking the path x_t to a path in the group of homeomorphisms of M.

We would like to extend this Itô map to a far richer class of paths. Our intention is to identify a family of metric topologies on smooth paths for which the Itô map is uniformly continuous (and even differentiable although we cannot show this here [17], [18], [19]). A point in the completion of the smooth paths in one of these metrics corresponds to a path in V with proscribed low order integrals and having finite p-variation for some $p < \infty$. As a first application we have the theorem that the solution to a Stratonovich stochastic differential equation of the classical type is a continuous function of the driving Wiener process and Lévy area taken as a pair.

1.1.4. The fundamental problem: Lack of continuity.

Before we proceed to develop the technology required to prove the main results it is useful to consider a simple example which highlights the obstruction we must overcome.

There is in general no natural extension of the Itô map to all continuous paths x_t. The following very simple example shows that the Itô map is rarely a continuous function in the uniform topology.

Example 1.1.1. *Some of the simplest differential equations are those whose solutions can be expressed as exact integrals of the driving term x_t. The simplest nontrivial example is the second iterated integral*

$$\begin{aligned} X^2(0,t) &= \int_{t>u_2>0} \Big(\int_{u_2>u_1>0} dx_{u_1} \Big) \, dx_{u_2} \\ &= \iint_{t>u_2>u_1>0} dx_{u_1} \, dx_{u_2} \; . \end{aligned} \tag{1.2}$$

In the one dimensional case, where x_t is real valued, we see that $X^2(0,t) = (x_t - x_0)^2/2$ and so the functional $x. \longrightarrow X^2(0,\cdot)$ clearly is continuous in the uniform topology.

The multi-dimensional case is quite different. Let $\boldsymbol{x}_t = (x_t^1, \dots, x_t^d)$ be vector valued and interpret the second iterated integral as the $d \times d$-matrix defined by

$$(\boldsymbol{X}^2(0,t))^{ij} = \iint_{t>u_2>u_1>0} dx_{u_1}^i \, dx_{u_2}^j \; ,$$

or better, as a 2-tensor

$$\iint_{t>u_2>u_1>0} d\boldsymbol{x}_{u_1} \otimes d\boldsymbol{x}_{u_2} \; . \tag{1.3}$$

Now decompose this integral into it's symmetric and anti-symmetric components $\boldsymbol{S}^{ij}$, $\boldsymbol{A}^{ij}$. We see that the symmetric part has a form differing little from the one dimensional situation

$$\boldsymbol{S}^{ij} = \frac{1}{2}\left(x_t^i - x_0^i\right)\left(x_t^j - x_0^j\right), \tag{1.4}$$

in particular, it is continuous in the uniform topology. The anti-symmetric part, which only arises in dimension two and higher, has the form

$$\boldsymbol{A}^{ij} = \frac{1}{2}\iint_{t>u_2>u_1>0} dx_{u_1}^i\, dx_{u_2}^j - dx_{u_1}^j\, dx_{u_2}^i \tag{1.5}$$

and has a well known geometric interpretation. For any two distinct coordinates i, j, the projection (x_t^i, x_t^j) of the path into $\mathbb{R}^2$ is a directed planar curve. The integral $\boldsymbol{A}^{ij}$ is the area between that curve $(x_\cdot^i, x_\cdot^j)$ and the chord from (x_t^i, x_t^j) to (x_0^i, x_0^j) where multiplicity and orientation are taken into account in the calculation.

Using this obvious geometric remark, it is trivial to see that $\boldsymbol{A}(0,t)$ is not a continuous function of $\boldsymbol{x}_\bullet$ in the uniform topology. Take

$$\boldsymbol{x}_t^n = \left(\frac{\cos(n^2 t)}{n}, \frac{\sin(n^2 t)}{n}\right),$$

then as n converges to infinity, the area integral converges locally uniformly to $\pi\, t$ whereas the paths $\boldsymbol{x}_t^n$ converge uniformly to the zero path.

However, closer examination of the example shows that $\boldsymbol{x}_t^n$ is converging to zero in p-variation norm for $p > 2$, and a more complicated example could be given showing that $\boldsymbol{A}$ is discontinuous even for the 2-variation norm. This and other considerations suggest that we should restrict attention to the case where $p < 2$. It is shown in [14], [15] that the Itô map extends uniquely as a continuous function to all paths of finite p-variation norm with $p < 2$ providing the vector fields f^i are smooth enough. In this case one can indeed develop a theory very similar to the classical one.

Nevertheless, there are important formal examples of equations of our basic type (1.1) in which the driving signal fails to have finite 2-variation and these have motivated several attempts to treat equations driven by rougher signals. Easily the most important and successful up to now has been the approach originating with Itô; which treats equations driven by Brownian motion or more generally by semi-martingales

(Brownian paths have finite p-variation norm for every $p > 2$ but do not have finite 2-variation norm) [12]. Although Itô's approach only constructs solutions as random variables it has lead to an enormous range of applications and must be regarded as a major achievement of 20th century mathematics.

Although Itô's approach is not path-wise, it makes it clear that any deterministic approach to interpreting (1.1) that only treats paths of finite p-variation norm with $p < 2$ is missing its target and failing to explain the richest class of examples we have.

We have just seen that iterated integrals provide the obstruction to the continuous extension of the Itô map. The remainder of the paper is dedicated to showing that they are also lead to the solution of the problem. *We will show that the solution is a continuous function of the path and its low order iterated integrals in an appropriate variation norm.* The rougher the path the more iterated integrals required and the more smoothness required of the vector fields.

1.1.5. Summary of existing approaches.

The main approaches to the solution of differential equations seem to have two key features:

- A notion of integral (Riemann, Itô, Stratonovich or Skorohod)

- An understanding of change of variable (Fundamental Theorem of Calculus, Itô's formulae, etc.)

These together allow one to use integral equations to define what one means by a solution. At this point existence can sometimes be shown via fixed point arguments, but in any case one usually wishes to add a method for *constructing* solutions (power series, Picard iteration) which will work under slightly stronger regularity conditions on the vector fields f^i and which usually gives the bonus of uniqueness of solution under these improved regularity assumptions.

Finally one needs to complete the discussion with the observation that characterisations of differential equations via integrals depend on a choice of coordinates for the underlying space where y_t takes its values. So although the equation (1.1) gives the impression of being coordinate independent, the definition of a solution may not be. The issue is a real one; in probability theory the Stratonovich equation has co-ordinate invariant solutions, while Itô equations do not.

In this paper we will concentrate on developing the co-ordinate invariant theory, the full theory is more mathematically challenging, and although we hope to return to it later we do not have a complete description at the current time.

1.1.6. History.

A number of authors have tried to develop deterministic theories of *integration* appropriate for rough paths, attempting to make sense of $\int Y\,dX$. L. C. Young [32] showed that such integrals make sense providing both paths are continuous, X has finite p-variation and Y has finite q-variation and $1/p + 1/q > 1$. For some reason he did not clinch the nonlinear question and show the existence of solutions of differential equations driven by paths with p-variation less than 2 and this was closed off in [14], [15]. Föllmer [6], [7] has written a number of interesting papers giving deterministic meaning to Itô's change of variable formula. Föllmer also made a verbal conjecture at an Oberwolfach meeting several years ago that knowing the Lévy area would be sufficient to construct solutions to SDE's. In some sense we prove his conjecture below.

The case where x_t is one dimensional or 2-dimensional and of the form $(\tilde{x}_t, t)$ is special. In this case the stochastic functional is continuous in the uniform topology - this was established by [26], [27], [28], [29].

1.1.7. Advantages to a probabilist.

A probabilist, interested in stochastic differential equations, might be tempted to believe that this article has little interest for him (except as a theoretical curiosity) because he can do everything that he wanted to do using Itô calculus. So we briefly mention a few situations where we believe that the results we develop here have consequences.

The first is conceptual, until now the probabilist's notion of a solution to an SDE has been as a function defined on path space and lying in some measure class or infinite dimensional Sobolev space. As such, the solution is only defined off an unspecified set of paths of capacity or measure zero. It is never defined at a given path. Given the results below, the solutions to all differential equations can be computed simultaneously for a path with an area satisfying certain Hölder conditions.

The set of Brownian paths with their Lévy area satisfying this condition has full measure. Therefore and with probability one, one may simultaneously solve *all* differential equations[1] over a given driving noise (the content of this remark is in the fact that there are uncountably many different differential equations).

Related consequences include:

1) Stochastic flows can be constructed simply. Changing the starting point in the differential equation is a special case of changing the differential equation. With a little more work one gets continuity, and with increasing smoothness of the vector fields, increasing smoothness of the flow.

2) It can be interesting to solve differential equations subject to boundary conditions other than initial conditions and the construction of a flow often allows one to find an initial value so that the resulting solution satisfies the boundary condition. However, in the classical framework, it is tricky to be precise about the sense in which this "solution" really is a solution. It does not satisfy the predictability condition necessary for the definition of an Itô integral to make sense; the standard approach involving changing the measure is quite deep. We have no such problems of interpretation because we use no probability, (although there will always be a problem of existence of solutions to nonlinear boundary problems - and this can be easy or difficult depending on the precise problem).

3) Stroock and Varadhan established a support theorem for solutions to stochastic differential equations. In one strong and non-trivial form it says that if we fix a smooth path in V and look at the solution to the SDE (1.1) when the driving noise is Brownian motion conditioned to be uniformly close to the smooth path; then the random solution converges in distribution to the deterministic one obtained by driving the equation with the fixed path. It is clear that all such theorems will follow if one establishes the continuity of (1.1) and that Brownian motion conditioned to be uniformly close to the smooth path converges in probability in the metric topology involving the area. Therefore our results below reduce the problem to one about Brownian motion alone.

4) Not all interesting stochastic dynamical systems are semi-martingales. It seems completely natural that there are nonlinear systems forced by random processes that may be Markov or Gaussian but are

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[1] The vector fields should be Lipschitz of order greater than two.

certainly not inside the normal framework. One thinks immediately of diffusion processes associated with elliptic operators in divergence form where the coefficients are not differentiable, or of diffusions on fractals. Both of these frequently have area processes and satisfy our hypotheses although they are not semi-martingales. Since this article was written work of Bass, Hambly and Lyons has established that the class of reversible processes to which this theory applies is really much wider than the class for which semi-martingale methods can be used. The iterated Brownian motion (IBM) studied by Burdzy and Adler is another example [1].

5) Numerical algorithms for solving differential equations which adapt their step sizes can be vastly more efficient than fixed step algorithms in certain settings. However, the decisions about step size are most efficiently made on the basis of previous rough approximations to the solution, and identification of the sensitive areas where accurate solution is required (*e.g.* before the trajectory approaches a critical point to ensure it passes on the correct side). The choice of step is typically based on knowledge of the *future evolution* of the solution and is therefore not predictable and constitute illicit information. For example if non-predictable infomation is used to determine the step size in classical approaches to solving SDEs numerically then in general these schemes will converge nicely to the wrong answer. Using the ideas set out below, and ensuring approximations to the path and area of the driving noise are correct over every interval it is possible to have a genuine variable step algorithm that converges to the correct answer for any choice of the intervals of approximation as the mesh size of the dissection goes to zero [8].

6) Stochastic filtering is concerned with the estimation of the conditional law of a Markov process, given observations of some function of it. The normal formulation (due to Zakai) looks at the case where the process is of diffusion type and splits into a first part (known as the signal) and a second part, known as the observation process with values in a vector space, and whose martingale part has stationary increments independent of the signal. In this case, Zakai showed that it was possible to completely describe the conditional density of the signal given knowledge of the observation process. In fact, the density evolves according to an infinite dimensional SDE of parabolic type. It is a commutative equation, and so the relationship between the observation process and the conditional density is a relatively stable one. On the other hand, it is really rather rare that real filtering problems present

themselves with the noise in the observation process being independent of the signal. And the transformation involved in making it so involves the solution of a generic SDE which will not commute. It follows that to do robust and stable filtering it is important to measure the "area" process as well as the values of the observation process.

7) Finally we hope that by solving the one dimensional differential equation without using predictability, our ideas might produce a few pointers to the correct way to treat PDE's driven by spatial noise. Of course in that situation predictability assumptions are quite inappropriate - at least in the initial assumptions and final conclusions. But at the moment this remains pure speculation.

1.2. Background.

1.2.1. Preliminaries: Groups and differential equations.

We set out some basic material and notation.

The logarithm of a flow. Throughout this paper we will make implicit use of the standard identification of autonomous differential equations, flows, and vector fields. If f is a Lipschitz vector field for some choice of complete Riemannian metric on a manifold then the autonomous differential equation

$$dy_t = f(y_t)\,dt\,, \qquad y_0 = a\,, \tag{1.6}$$

has a unique solution defined for all time. By varying the initial condition, one may associate with it a flow π_t defined by $\pi_t(y_0) = y_t$. The assumptions ensure the flow is defined for all positive and negative times and is a homeomorphism. We may use the notation $\pi_t = \exp(tf)$ to emphasise that vector fields should be regarded, at least formally, as elements of the Lie algebra of the group of homeo(diffeo)morphisms of the underlying manifold.

If a homeomorphism π can be realized by flowing along a fixed vector field ϕ so that $\pi = \exp\phi$, we say $\phi = \log\pi$. In general, it is not possible for one to construct a logarithmic vector field even for the smoothest diffeomorphisms homotopic to the identity; equally the logarithm need not be unique when it exists. If one has a time varying differential equation such as (1.6), and one looks at the flow obtained by solving it over a short time then it is useful to be able to determine if the resulting flow has a logarithm and express that logarithmic vector field directly in terms of x_s and the f^i, [3]. (*e.g.* in numerical analysis, to solve the time varying and rough equation over a short interval it would be sufficient to solve the smooth and time independent differential equation determined by the logarithmic vector field).

Determining this logarithm as a vector is in fact an analytic extension of the Dynhin-Campbell-Baker-Hausdorff formula (which in its algebraic form considers the effect of flowing for unit time along one left invariant vector field on a group and then a second, and tries to find an expression for the logarithm of the result). In this paper, we will be able to construct the logarithm of a flow driven by a rough signal for a short period under the hypotheses that the vector fields are invariant vector fields on a finite dimensional group.

Matrix groups. Recall some very basic facts about Lie groups. Suppose that a topological group G has a connected *finite dimensional* manifold structure, then it is very well known that it is a Lie group and can always be represented as a real analytic group of matrices, or a quotient thereof by a discrete group. In this representation, the tangent space to a point in the group is a linear space of matrices.

The tangent space g at the identity can be made into a Lie algebra in two equivalent ways.

If a is an element of the tangent space at the identity of a matrix group, then $t \longrightarrow \exp t\, a$ (where exp is the power series in the the matrix) defines a smooth path in the group (and hence a direction in the tangent space over the identity) starting at the identity element. Consider any other element ρ of the group. The map $t \longrightarrow \exp t\, a\, \rho$ defines a path and hence a direction in the tangent space over ρ; clearly the induced vector field a^* on the group is right invariant, depends linearly on a, and defines an isomorphism between right invariant fields and the tangent space over the identity. We may take the Lie bracket of these fields in the sense of vector fields and as this yields another right invariant vector field we define a Lie algebra structure on the tangent space. Alternatively, we can use the matrix representation and simply define $[A, B] = AB - BA$. They give the same results. The Lie algebra of a Lie group is important in many ways and we cannot recall them all here. However, we mention a couple of basic facts that will be essential. A group is abelian if $[a, b] \equiv 0$, and has nilpotency rank at most n if $[a_1, [a_2, [a_3, \ldots [a_{n-1}, a_n]] \ldots]] \equiv 0$ for all elements in the Lie algebra.

A homomorphism ν of one Lie group to another induces (by differentiation) a Lie map $d\nu$ from the Lie algebra of the first group to the Lie algebra of the second. These two maps intertwine with the exponential map (applied to the vector field or the matrix) and so

$$\nu(\exp t\, a) = \exp t\, d\nu(a)\,. \tag{1.7}$$

Conversely, to every finite dimensional Lie algebra we may associate a unique simply connected Lie group, and to every Lie algebra map from such a finite dimensional Lie algebra to the Lie algebra of a Lie group is associated a unique homomorphism whose derivative is the Lie algebra map.

Differential equations on matrix groups. Suppose we have a smooth path X_t in the Lie algebra of our matrix group, we may develop it onto the group. That is we solve the differential equation for the path ρ_t in the group which at time t is always tangential to $(dX_t/dt)^*$. The differential equation has the form

$$d\rho_t = (dX_t)^*(\rho_t) \tag{1.8}$$

and since * is a linear map from the vector space carrying X_t to vector fields on a manifold (the group) it falls into the general category of time inhomogeneous differential equations we introduced in (1.6).

Any time inhomogeneous differential equation can be regarded, at least formally, in the same way if one is prepared to consider the group of homeomorphisms (or diffeomorphisms) of the manifold. Any vector field defines a parameterised flow on the manifold (1.6) and hence a tangent vector to the identity map on the group of homeomorphisms. Consider the flow π_t on that same group defined by the inhomogeneous equation

$$dy_t = \sum_i f^i(y_t)\, dx_t^i\,, \qquad x_t \in V\,, \qquad f(y) : V \longrightarrow TM_y\,. \tag{1.9}$$

Now $f(\,)\, x_t$ is a path in the space of vector fields, and π_t its development onto the group of homeomorphisms.

Although there are very big differences between this formal infinite dimensional setting and the finite dimensional one (the vector fields will not in general be smooth enough to form a Lie algebra, etc.) the abstract picture is very helpful in the following two ways. It suggests that there might be a universal object, and also that we could learn something about the general problem by studying the simpler case of development of a rough path on a finite dimensional Lie group.

Definition 1.2.1. *A Lie algebra $\mathfrak{g}$ containing V is said to be free over V if it has the universal property that any linear map f of V into a Lie algebra $\mathfrak{h}$ extends in a unique way to a Lie algebra map $\tilde{f}$ of $\mathfrak{g}$ to $\mathfrak{h}$. Such a Lie algebra exists but is infinite dimensional.*

Now suppose we consider again our basic differential equation. That is, we have a path x_t in a vector space V and a linear map f of V into the Lie algebra $\mathfrak{h}$ of a Lie group H and we would like to develop a path y_t in H tangential to $(f(y_t)\,dx_t)^*$.

Pretend for a minute that we could associate a simply connected group G with the free Lie algebra $\mathfrak{g}$, and that there was a group homomorphism from it to H induced by the Lie algebra map. *It would be sufficient to develop x_t in the simply connected group G with Lie algebra $\mathfrak{g}$ and use the homomorphism*

$$G \xrightarrow{\tilde{f}} H$$

to produce a path in H. It follows that it would be both necessary and sufficient to solve our problem in general if we could develop rough paths from V to this Lie group G alone. However, there is a problem with this picture - there is no simple analytic object we can call the free group - but still the picture definitely points one in the correct direction.

Linear differential equations. Suppose that Y_t takes its values in a vector space W and that for each x the vector fields $y \longrightarrow f(y)\,x : W \longrightarrow W$ is linear in y, then we say the standard equation (1.1) is linear, and observe that the sum of two solutions is a solution; the flow is therefore a linear map (which by solving the equation backwards in time is invertible), and the solution flow takes its values in a matrix group.

Thus we see that to solve a time inhomogeneous *linear* equation (which are certainly not linear in the relationship between x and y) is essentially the same problem as to develop a path in a *finite dimensional* Lie algebra onto the associated *finite dimensional* Lie group using the right invariant extensions of the vector fields.

More generally, we can re-parameterise our problem and reduce it to a finite dimensional linear problem whenever the vector fields in the range of f are smooth enough that one can take Lie brackets and the resulting Lie algebra is finite dimensional. Although this is not the generic case, the equation

$$dy_t = f(y_t)\,dx_t\ , \qquad x_t \in V\ , \qquad f(y) : V \longrightarrow TM_y\ , \tag{1.10}$$

where the dimension of V is one satisfies the finite dimensionality hypotheses in a rather trivial way. In this case let $d\theta_t = f(\theta_t)\,dt$ be the flow defined by the autonomous equation. One readily sees that for smooth x_t the solution of (1.10) can easily be expressed as $y_t = \theta_{x_t}(y_0)$ and that this is uniformly continuous in the forcing term x_t. It is generally true that (1.10) is uniformly continuous in this way if and only if the Lie algebra is trivial and the vector fields commute. In Section 1.1.4 we showed that the iterated integral for the area produced a discontinuous Itô map. The associated differential equation has a Lie algebra of the simplest non-commutative type - nilpotent of rank 2.

Einstein expansions. Consider a linear differential equation. Let $x \longrightarrow A(\,)\,x : V \longmapsto \hom(W,W)$ be a bounded linear map (of Banach spaces) and consider the linear equations

$$dy_t = A(y)\,dx_t\ , \tag{1.11}$$

$$d\pi_t = A(\cdot)\,dx_t\,\pi_t\ , \tag{1.12}$$

for the trajectory and flow. If the path x_t is smooth and y_t is the classical solution, then one may construct a Taylor series expansion for it (and the operator π_t) in terms of iterated integrals of x_t.

$$\begin{aligned} y_t &= y_s + \int_s^t dy_u \\ &= y_s + \int_s^t A(y_u)\,dx_u \end{aligned} \tag{1.13}$$

$$\begin{aligned} &= y_s + A(y_s) \int\limits_{s<u<t} dx_u \\ &\quad + \iint\limits_{s<u_1<u_2<t} A(A(y_{u_1}))\,dx_{u_1}\,dx_{u_2} \end{aligned} \tag{1.14}$$

$$= \sum_{i=0}^{n} A(A(\cdots A(y_s))) \int \cdots \int_{s<u_1<u_2<\cdots<u_i<t} dx_{u_1}\, dx_{u_2} \cdots dx_{u_i}$$

$$+ \int \cdots \int_{s<u_1<u_2<\cdots<u_{n+1}<t} A(A(\cdots A(y_{u_1})))\, dx_{u_1}\, dx_{u_2} \cdots dx_{u_{n+1}} \tag{1.15}$$

and using the boundedness of y on $[s,t]$, the factorial decay of the iterated integrals, and the geometric growth of the norm of the product of operators, one quickly shows that the remainder goes to zero with n and so we have the convergent series

$$\pi_{s,t} = I + A \int_{s<u<t} dx_u + AA \iint_{s<u_1<u_2<t} dx_{u_1}\, dx_{u_2} + \cdots \tag{1.16}$$

and observe that the solution can be expressed as a inner product of a sequence of iterated integrals and "powers" of A.

This expansion (which occurs regularly in the literature over the last 50 years or so) underlines the importance of iterated integrals. We will see later that we will be able to associate infinite and rapidly decaying sequences of iterated integrals in settings where the paths are not smooth. In this case the series above can be used as a definition of the solution. However, it does not directly extend from the finite dimensional linear setting (1.11) to the fully nonlinear one (1.1) (for in this case the operators in the range of A are unbounded and do not have a common core). Additional ideas will be required at that point.

1.2.2. Preliminaries: Rough paths and smooth functions.

In this section we remind the reader of some basic analytic concepts. For our purposes a very convenient way of measuring the smoothness of rough paths is via the p-variation norm first introduced by Wiener. If we are to solve differential equations driven by rough paths, then it transpires that we must balance this by taking progressively smoother vector fields. For unique solutions in the classical case it suffices that the fields be Lipschitz. For our uniqueness results we will require that the fields are Lipschitz of order $\gamma > p$. Using the obvious definition, one might conclude that there were no non-constant functions satisfying the hypothesis. The definition we use follows Stein and seems particularly well adapted to the problem in hand. Any bounded function with n bounded derivatives is Lip$[\gamma]$ for $\gamma \leq n$.

Paths of finite p-variation. Suppose X_t is a path taking its values in a metric space. Following Wiener, one says that the p-variation of X_t on J is

$$\|X\|_{p,J}^p = \sup\Big\{\sum_j d(X_{t_j}, X_{t_{j+1}})^p,\ t_{j_1} < \cdots < t_{j_r} \in J\Big\}. \tag{1.17}$$

Definition 1.2.2. *We say that X_t has p-variation controlled by $\omega(s,t)$ if*

$$\|X\|_{p,[s,t]}^p \leq \omega(s,t)\,, \qquad \text{for all } s < t\,. \tag{1.18}$$

A path is said to be of regular finite p-variation if ω can be chosen to be continuous near the diagonal, and zero on the diagonal.

Note that

$$\|X\|_{p,[s,t]}^p + \|X\|_{p,[t,u]}^p \leq \|X\|_{p,[s,u]}^p \tag{1.19}$$

and so in this paper we only consider controlling ω that satisfy the inequality

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u)\,. \tag{1.20}$$

It makes sense to introduce a distance between two paths. Let Y_t denote a second path.

Definition 1.2.3. *We define the distance*[2] *between two paths to be finite if*

$$\|X,Y\|_{p,J} = \max\Big\{\sup_{t_{j_1}<\cdots<t_{j_r}\in J}\Big(\sum_j |d(X_{t_j}, X_{t_{j+1}}) - d(Y_{t_j}, Y_{t_{j+1}})|^p\Big)^{1/p},\ \sup_{t\in J} d(X_t, Y_t)\Big\} < \infty\,.$$

[2] In the more restricted situation where X takes its values in a Banach space there is a smaller norm where $|d(Y,Y) - d(X,X)|$ is replaced by $\|(X_{t_{j+1}} - X_{t_j}) - (Y_{t_{j+1}} - Y_{t_j})\|$. In fact it is this distance that we will use later.

As before we may talk about the distance being controlled by ω.

It is obvious from standard facts about sequence spaces that this distance is indeed a metric (and a norm if the original space were a Banach space) and that the space of paths of finite p-variation is complete in this metric providing the original metric space was complete. The space of regular paths is a closed subspace. The p-variation of a path and distance between two paths are monotone decreasing with increasing p. If X is continuous and of finite p-variation then X is regular for all $p' \geq p$. If X is not continuous the local p-variation never goes to zero and the path is never regular.

Example 1.2.1. *A path of bounded variation on a closed interval has finite 1-variation. Almost all Brownian paths $X_t(\omega)$ are of regular p-variation for all $p > 2$ but do not have finite 2-variation although the map $t \longrightarrow X_t(\cdot)$, $\mathbb{R}^+ \longrightarrow L^2(\Omega, \mathbb{P})$ does have finite 2-variation.*

Lipschitz functions. In [24, Chapter VI] Stein looked at the general problem of extending smooth functions from subsets of Euclidean space to the whole space. In particular, he considers the Whitney theorem which extends in a norm bounded way the space $\operatorname{Lip}(\gamma, F)$ of Lipschitz functions on a closed set F to the whole Euclidean space. In doing so he introduces a definition of $\operatorname{Lip}(\gamma, F)$ which is valid for any $\gamma > 0$ and not just for $\gamma \leq 1$. We recall a modification of this definition here; although we modify it slightly to be compatible with our notations; the resulting norms are equivalent.

Definition 1.2.4. *Suppose that V, W are normed vector spaces, k is a non-negative integer, and that $k < \gamma \leq k+1$. A function $f = f_0$ defined on a closed subset $F \subset V$ and taking values in W belongs to $\operatorname{Lip}(\gamma, F)$ if there exist symmetric multi-linear functions (formal derivatives) $f^{(j)}(\boldsymbol{x})$, $0 \leq j \leq k$ taking $\overset{j}{\underset{1}{\otimes}} V$ to W and satisfying the natural Taylor expansion type condition*

$$f^{(j)}(\boldsymbol{x}_t)(v) = \sum_{j+l \leq k} f^{(j+l)}(\boldsymbol{x}_s)\Big(v \otimes \int \underset{s<u_1<\cdots<u_l<t}{\int} d\boldsymbol{x}_{u_1} \cdots d\boldsymbol{x}_{u_l}\Big) + R_j(\boldsymbol{x}_s, \boldsymbol{x}_t)(v)\,. \tag{1.21}$$

for $v \in \overset{j}{\underset{1}{\otimes}} V$ and where, as operators on the tensor product, the deriva-

tives and remainder satisfy

$$\|f^{(j)}(\boldsymbol{x})\| \leq M\,, \tag{1.22}$$

$$\|R_j(\boldsymbol{x},\boldsymbol{y})\| \leq M\,|\boldsymbol{x}-\boldsymbol{y}|^{\gamma-j}\,, \qquad \boldsymbol{x},\boldsymbol{y} \in F\,. \tag{1.23}$$

We define the smallest M to be the $\operatorname{Lip}(\gamma, F)$ *norm of the sequence* $f^{(j)}(\boldsymbol{x})$, $0 \leq j \leq k$.

Some remarks are in order.

The terms

$$f^{(j+l)}(\boldsymbol{x}_s)\Big(v \otimes \int \underset{s<u_1<\cdots<u_l<t}{\cdots} \int d\boldsymbol{x}_{u_1} \cdots d\boldsymbol{x}_{u_l}\Big)\,, \tag{1.24}$$

are, for smooth paths/conventional integrals, independent of the choice of path and only depend on the values $(\boldsymbol{x}_s, \boldsymbol{x}_t)$. To prove this, observe first, that the dimension of W is irrelevant. Now consider the polynomial $p(\boldsymbol{x})$ of degree k whose Taylor expansion at $\boldsymbol{x}_s$ agrees with $\{f^{(j)}(\boldsymbol{x}_s)\}_{j=0,\ldots,k}$. Expanding $p(\boldsymbol{x}_t)$ in terms of iterated integrals, as in the last section, we see that the expansion formulae is exact at level k and

$$p(\boldsymbol{x}_t) = \sum_{0<l\leq k} f^{(l)}(\boldsymbol{x}_s)\Big(\int \underset{s<u_1<\cdots<u_l<t}{\cdots} \int d\boldsymbol{x}_{u_1} \cdots d\boldsymbol{x}_{u_l}\Big)\,. \tag{1.25}$$

and as the left hand expression does not depend on the path nor can the right hand side. Similar arguments can be applied to the derivatives of $p(\boldsymbol{x})$ to obtain the invariance of the other expressions. An alternative, more algebraic proof of the result is to observe that the symmetric nature of the $\{f^{(j)}(\boldsymbol{x}_s)\}_{j=0,\ldots,k}$ annihilates the antisymmetric components of a tensor and only these change when one perturbs the path. Either way, the observation is clear, and will be crucial to us.

The functions $\{f^{(j)}\}_{j=1,\ldots,k}$ will not in general be unique given $f = f_0$. One only expects this if the set F is thick enough. In other words a function in $\operatorname{Lip}(\gamma, F)$ *is not a function on F but a sequence of functions representing* formal derivatives and satisfying these complex Taylor type bounds relating one term with the next. We will see that an essentially dual idea occurs when one considers paths of finite p-variation where $p > 2$. The definition we give above for p-variation, is in some sense wrong, as it fails to specify enough information.

Definition 1.2.5. *We have defined a* $\operatorname{Lip}(\gamma, F)$ *function; this definition easily extends to i-forms. A sequence* $f^{(j)}(\boldsymbol{x})$, $i \leq j \leq k$ *is a* $\operatorname{Lip}(\gamma, F)$ *i-form if all the higher Taylor expressions* (1.21) *satisfy the estimates set out in the definition above whenever* $i \leq j \leq k$.

Both definitions make sense if the functions or forms are vector valued.

Example 1.2.2. *If* θ *is a* 1*-form on* F *and* $1 < \gamma \leq 2$, *then we say it is in* $\operatorname{Lip}(\gamma, F)$ *if one has defined a* 2*-form* $d\theta$

$$\left\| \theta(X_t) - \theta(X_s) - \frac{1}{2}\,(d\theta)\,(X_s)\,(X_t - X_s) \right\| < M\, \|X_t - X_s\|^{\gamma} \tag{1.26}$$

and

$$\|d\theta(X_t) - d\theta(X_s)\| < M\, \|X_t - X_s\|^{\gamma - 1}\,. \tag{1.27}$$

However, some caution is now required as the resulting multi-linear maps are only required to have full symmetry in the $x_1, \ldots, x_l$ coordinates. One may compare this approach to defining $\operatorname{Lip}(\gamma, F)$ j-forms with the alternative approach which simply says a form valued function is a matrix valued function, and so we have already defined what we mean by $\operatorname{Lip}(\gamma, F)$. The two approaches give the same result.

2. The Finite-Dimensional Case - Linear Differential Equations.

2.1. Multiplicative Functionals - Introduction.

2.1.1. Multiplicative functionals - Introductory material and definitions.

Let V be a vector space, and suppose that X_t is a smooth path in V. The k-th iterated integral $\boldsymbol{X}^k_{s,t}$ of X_t over a fixed time interval $[s, t]$ is an element of the tensor product $V^{\otimes k}$. The sequence of iterated integrals

$$(\boldsymbol{X}^k_{s,t})_{k=0}^{\infty}$$

is far from being a generic collection of tensors; there are complicated algebraic dependencies between the terms in the sequence. To fully understand the collection of iterated integrals one must treat them as a single object.

The tensor algebra. We start by recalling some rather elementary facts about tensor algebras. Consider the space T of sequences $\boldsymbol{a} = (a_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots)$ with $\boldsymbol{a}_k \in V^{\otimes k}$. That is

$$T = \bigoplus_{k=0}^{\infty} V^{\otimes k}. \tag{2.1}$$

(We take the zero order tensor product to be the field of scalars.) Then T is an associative algebra with unit, which we shall refer to as the tensor algebra over V. If $\boldsymbol{a} = (a_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots)$ and $\boldsymbol{b} = (b_0, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots)$ are two elements of T then we may define their sum, (tensor) product, and the action of scalars in the obvious way

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= (a_0 + b_0, \boldsymbol{a}_1 + \boldsymbol{b}_1, \boldsymbol{a}_2 + \boldsymbol{b}_2, \dots), \\ (\boldsymbol{a} \otimes \boldsymbol{b})_i &= \sum_{0 \le j \le i} \boldsymbol{a}_j \otimes \boldsymbol{b}_{i-j}, \\ \alpha\, \boldsymbol{a} &= (\alpha\, a_0, \alpha\, \boldsymbol{a}_1, \alpha\, \boldsymbol{a}_2, \dots). \end{aligned} \tag{2.2}$$

The space T with these operations is an associative algebra. Suppose that $\boldsymbol{a} = (a_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \dots)$ is any element of the algebra with $a_0 > 0$ then $\boldsymbol{a}$ is invertible using the usual geometric power series approach

$$\begin{aligned} \boldsymbol{a} &= a_0\,(1, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots) = a_0\,(1 + \boldsymbol{c}), \\ \boldsymbol{a}^{-1} &= \frac{1 - \boldsymbol{c} + \boldsymbol{c}^2 - \boldsymbol{c}^3 + \cdots}{a_0}, \qquad a_0 \in \mathbb{R}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \boldsymbol{b}_i &= \frac{\boldsymbol{a}_i}{a_0}, \\ \mathbf{1} &= (1, \mathbf{0}, \mathbf{0}, \dots), \\ \boldsymbol{c} &= (0, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots). \end{aligned} \tag{2.4}$$

Now $c_0 = 0$, hence $\{\boldsymbol{c}^j\}_k = 0$ providing $k < j$; therefore the k-tensor component of any power series in $\boldsymbol{c}$ and in particular $1 + \boldsymbol{c} + \boldsymbol{c}^2 + \boldsymbol{c}^3 + \cdots$ is a sum including only finitely many non zero terms and so has meaning. Similarly, providing $a_0 > 0$, we may define the logarithm of $\boldsymbol{a}$ by

$$\log \boldsymbol{a} = \log a_0 + \mathbf{1} + \boldsymbol{c} - \frac{\boldsymbol{c}^2}{2} + \frac{\boldsymbol{c}^3}{3} + \cdots \tag{2.5}$$

Both of these definitions are pure algebra, and no analysis is required. The exponential function is defined for all elements of T, but the series defining the k-tensor component involves a genuinely infinite sum (which always converges).

$$\exp \boldsymbol{a} = 1 + \boldsymbol{a} + \frac{\boldsymbol{a}^2}{2!} + \frac{\boldsymbol{a}^3}{3!} + \cdots \tag{2.6}$$

One can check that $\exp(-\boldsymbol{a}) = (\exp \boldsymbol{a})^{-1}$ and that $\exp \log \boldsymbol{a} = \boldsymbol{a}$, $\log \exp \boldsymbol{a} = \boldsymbol{a}$, etc. Because the space

$$D_n = \bigotimes_{k=n+1}^{\infty} V^{\otimes k} \tag{2.7}$$

of tensors of degree greater than n form an ideal we may also study the truncated tensor algebra $T^{(n)}$ obtained by quotienting out by D_n. We make the identification

$$T^{(n)} = \bigoplus_{k=0}^{n} V^{\otimes k} . \tag{2.8}$$

The full tensor algebra is an adequate algebraic object, but because it ignores any notion of convergence of the infinite sequences it is a rather poor analytic object. We will mainly work with the truncated tensor algebras $T^{(n)}$ where the analytic and algebraic structures are completely compatible, the fine analytic information will come from understanding the way objects in these finite dimensional quotients piece together.

At this point we record only the basic facts. If $m > n$, then there is a natural projection π of $T^{(m)}$ onto $T^{(n)}$ given by

$$\pi : (a_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m) \longmapsto (a_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n) . \tag{2.9}$$

The map π *is an algebra homomorphism.* Moreover, the definitions of log, exp, $\boldsymbol{a}^{-1}$ extend to $T^{(n)}$ and their actions commute with that of π

so that for example $\pi(\exp \boldsymbol{a}) = \exp(\pi(\boldsymbol{a}))$. The inclusion ι of $T^{(n)}$ into $T^{(m)}$ given by

$$\iota : (a_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n) \longmapsto (a_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \mathbf{0}, \mathbf{0}, \ldots)\,. \tag{2.10}$$

is linear *but is not an algebra homomorphism.*

The free Lie algebra and free nilpotent groups. One can build certain Lie algebras inside $T^{(n)}$ and T. The product

$$[\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}\,. \tag{2.11}$$

defines a Lie Bracket on T and $T^{(n)}$. Of particular interest is the Lie algebra generated by V. This is comprised of linear combinations of finite sequences of Lie brackets of elements of V

$$\mathfrak{A} = 0 \oplus V \oplus [V, V] \oplus [V, [V, V]] \oplus \cdots$$

where for example $[V, [V, V]]$ is the linear subspace of $V^{\otimes 3}$ spanned by

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]]\,, \qquad \boldsymbol{v}_i \in V\,.$$

One may trivially prove that it has the special property that if S is a *linear* map from V into a Lie algebra $\mathfrak{B}$ then there is a unique extension of the map to a Lie algebra map from $\mathfrak{A}$ to $\mathfrak{B}$. In other words it is the free Lie algebra we identified earlier. The corresponding Lie algebra $\mathfrak{A}^{(n)} \subset T^{(n)}$ has the same extension property providing one restricts attention to maps into Lie algebras of nilpotency rank at most n (*i.e.* all Lie products involving n or more elements of the algebra are identically zero).

Theorem 2.1.1. *Let $G^{(n)} = \exp \mathfrak{A}^{(n)} \subset T^{(n)}$ then $G^{(n)}$ is a group called the free nilpotent group of step n. $G^{(0)} = \mathbb{R}$ and $G^{(1)} = V$. The exponential map from the Lie algebra $\mathfrak{A}^{(n)}$ to the Lie group $G^{(n)}$ is one to one and onto. The restriction of the map π to a map from $G^{(m)} \longrightarrow T^{(n)}$, $m > n$ defines a group homomorphism from $G^{(m)} \longrightarrow G^{(n)}$, $m > n$. On the other hand the map ι takes $G^{(n)} \longrightarrow T^{(m)}$, $m > n$ but intersects $G^{(m)}$ only at the identity.*

Remark 2.1.1. The above theorem and indeed everything in 2.1.1 is standard, proofs can be found in, for example, [22]. The properties of $\mathfrak{A}^{(n)}$ are by no means all easy to derive, and for example it is an interesting, nontrivial, and a numerically worthwhile exercise to compute the dimension of $\mathfrak{A}^{(n)}$, to find explicit bases for the space (such calculations go back to Hall and Linden), and even to decompose the space into $\mathrm{GL}(V)$-invariant subspaces [22] according to the different irreducible representations.

Every element of the free Lie algebra is of finite degree and an element of $T^{(n)}$ for some n. We may exponentiate the free Lie algebra into the full tensor algebra and the map is injective, but the range is not a group or even multiplicatively closed. On the other hand, we can introduce the (highly non-separable) Lie algebra of infinite sequences of Lie elements. In this case, we see that the exponential map has a range comprising solely of elements of the tensor algebra which are carried by each of the projections $\pi : T \longrightarrow T^{(n)}$ into the corresponding group $G^{(n)}$. This subset of the full tensor algebra is the inverse limit of our nilpotent groups and is clearly itself a group which we denote $G^{(*)}$.

Definition 2.1.1. *We say an element of the full tensor algebra is group like if it is an element of* $G^{(*)}$.

Unfortunately this group is very big and its Lie algebra is no longer the free algebra.

Any attempt to use a linear map from V into the Lie algebra of a Lie group H to define a homomorphism of this enormous Lie group $G^{(*)}$ (or some part thereof) into the group H in a unique way must involve analysis. This paper can be viewed as an attempt to provide this analytic content.

Paths and multiplicative functionals - the definition. Let $\boldsymbol{X}_t$ be a fixed smooth path in V, and consider the sequence of iterated integrals

$$\begin{aligned} \boldsymbol{X}_{s,t}^{(n)} = 1 + \int\limits_{s<u<t} dx_u + \int\cdots\int\limits_{s<u_1<u_2<t} dx_{u_1} \otimes dx_{u_2} + \cdots \\ + \int\cdots\int\limits_{s<u_1<u_2<\cdots<u_n<t} dx_{u_1} \otimes dx_{u_2} \otimes \cdots \otimes dx_{u_n} \in T^{(n)} \,. \end{aligned} \tag{2.13}$$

Let $\boldsymbol{X}_{s,t}$ denote the infinite sequence. Suppose now that one wants to describe in detail the relationship between the iterated integrals over

$[r,t]$ and those over $[r,s]$ and $[s,t]$ where $s \in [r,t]$. If one starts to calculate in coordinates one will quickly become engulfed in terms and conclude that this is a horribly complicated thing to do, however this is really because the main features are best derived without taking coordinates. Now K. T. Chen [5] observed two essential features of the process $\boldsymbol{X}_{s,t}$ which we now state as a theorem.

Theorem 2.1.2. *For smooth paths and conventional integrals, the process $\boldsymbol{X}_{s,t}$ is multiplicative. That is to say*

$$\boldsymbol{X}_{r,s} \otimes \boldsymbol{X}_{s,t} = \boldsymbol{X}_{r,t} \; . \tag{2.14}$$

Moreover, it is group like, so that for each n,

$$\boldsymbol{X}_{s,t}^{(n)} \in G^{(n)} \,, \qquad \log\big(\boldsymbol{X}_{s,t}^{(n)}\big) \in A^{(n)} \,. \tag{2.15}$$

PROOF. The proof that $\boldsymbol{X}_{s,t}$ is multiplicative is instructive. Let the i-th component of $\boldsymbol{X}_{s,t}$ be denoted by $\boldsymbol{X}_{s,t}^{i}$, etc. Then

$$\begin{aligned}
\boldsymbol{X}_{r,t}^{i} &= \int \underset{r<u_1<u_2<\cdots<u_i<t}{\cdots\int} dx_{u_1}\, dx_{u_2} \cdots dx_{u_i} \\
&= \sum_{0\le j\le i} \int \underset{s<u_{j+1}<\cdots<u_i<t}{\cdots\int} \Big(\int \underset{r<u_1<\cdots<u_j<s}{\cdots\int} dx_{u_1} \cdots dx_{u_j} \Big) \\
&\qquad\qquad \cdot dx_{u_{j+1}} \cdots dx_{u_i} \\
&= \sum_{0\le j\le i} \Big(\int \underset{r<u_1<\cdots<u_j<s}{\cdots\int} dx_{u_1} \cdots dx_{u_j} \Big) \\
&\qquad\qquad \cdot \int \underset{s<u_{j+1}<\cdots<u_i<t}{\cdots\int} dx_{u_{j+1}} \cdots dx_{u_i} \\
&= \sum_{0\le j\le i} \boldsymbol{X}_{rs}^{j} \otimes \boldsymbol{X}_{st}^{i-j} \; ,
\end{aligned} \tag{2.16}$$

which establishes the multiplicative identity.

To prove that the iterated integral sequence is group like one needs a different approach. Because iterated integrals are integrals and our paths are smooth, it is an easy consequence of the fundamental theorem of calculus that they satisfy the system of differential equations

$$\left\{\begin{array}{l} d\boldsymbol{X}_{0,t}^{(n)} = \boldsymbol{X}_{0,t}^{(n)} \otimes dX_t \in T^{(n)}\,, \\ \boldsymbol{X}_{0,0}^{(n)} = \mathbf{1} = (1, \mathbf{0}, \mathbf{0}, \dots)\,. \end{array}\right. \tag{2.17}$$

Suppose g is an element of the group $G^{(n)}$ thought of as a sub-manifold of $T^{(n)}$. Then left tensor multiplication by g is a linear map of $T^{(n)}$ which takes the group $G^{(n)}$ to itself, and 1 to g. It follows that the derivative of this map takes the tangent space to the group $G^{(n)}$ at 1 to the tangent space to g. However the derivative of a linear map is the map itself, and V is in the tangent space to $G^{(n)}$ at $\mathbf{1}$. Hence any solution to the differential equation $d\boldsymbol{g}_t = \boldsymbol{g}_t \otimes dX_t \in T^{(n)}$ will remain in the group $G^{(n)}$ if it starts there. It follows that $\boldsymbol{X}_{s,t}$ is a group like element.

REMARKS 2.1.1. The proof of the above result yields a certain amount of extra information.

1) From the differential equation (2.17) (which of course is of a very fundamental kind) we observe that the iterated integrals over a fixed time interval are insensitive to re-parameterisation of the underlying path, and by solving the differential equation backwards in time we see that the inverse group element is produced. The map from piecewise smooth path segment to iterated integral sequence is a homomorphism of the semi-group of path segments (multiplication is concatenation) to the group like elements. Identify re-parameterisations of paths, and the inverse of path segment with the path run in the reverse direction and one makes the path segments into a group. Chen proved that in this case the map into the group like elements is injective. Therefore, the infinite algebraic sequence $\boldsymbol{X}_{s,t}$ contains (in code!) all the information from x_u, $u \in [s,t]$ required to determine the solution y_t from y_s.

2) The proof of the first part of the theorem holds in wide generality. The first integral identity relies only on additivity of the integral over disjoint domains of a nice kind. The second term depends on a multiplicative linearity of the integral. In fact these properties (of linearity and additivity) are so basic that (2.14) is true for any sensible choice of integral (Itô, etc.) and in some sense captures what one means when one talks about an integral. Because the multiplicative property

seems so widely characteristic of integrals we make it our basic object of study.

Definition 2.1.2. *A multiplicative functional is a map from pairs (s,t) of real numbers to $\boldsymbol{X}_{st} = (X^0_{st}, \boldsymbol{X}^1_{st}, \boldsymbol{X}^2_{st}, \dots)$ in $T^{(n)}$ satisfying $\boldsymbol{X}_{rs} \otimes \boldsymbol{X}_{st} = \boldsymbol{X}_{rt}$ and $X^0_{st} \equiv 1$. We say a multiplicative functional is geometric if it takes its values in the group like elements.*

Suppose that $\boldsymbol{X}_t = (1, \boldsymbol{X}^1_t, \boldsymbol{X}^2_t, \dots, \boldsymbol{X}^n_t) \in T^{(n)}$ is a path in the space of n-tensors with unit scalar component. Then we say that $\boldsymbol{X}_{s,t} = (\boldsymbol{X}_s)^{-1} \otimes \boldsymbol{X}_t$ is the multiplicative functional determined by $\boldsymbol{X}_t$. Conversely, given a multiplicative functional $\boldsymbol{X}_{s,t}$ and a point $\boldsymbol{x}$ in $T^{(n)}$, we say that $\boldsymbol{X}_t = \boldsymbol{x} \otimes \boldsymbol{X}_{0,t}$ is the path in $T^{(n)}$ starting at $\boldsymbol{x}$ determined by $\boldsymbol{X}_{s,t}$. Given this almost one to one correspondence between paths and multiplicative functionals in $T^{(n)}$ it is reasonable to question the sense of introducing the concept of multiplicative functional at all. However, we will see later that it will be fundamental to the process of constructing an integral or of solving a differential equation that one can go from an almost multiplicative functional to a multiplicative functional and hence to a path. Almost multiplicative functionals will have no direct path-wise interpretation.

The logarithmic flow. As a simple application of the algebraic ideas set out so far, we go back to a question we raised earlier, suppose that one would like to know how to construct the logarithm of a flow. We can easily derive an asymptotic formulae for the logarithm of the flow (proving that it converges to a Lie element is of course a different question). Recall our basic equation

$$dy_t = f(y_t)\, dx_t \ , \tag{2.18}$$

where f is the linear map from V to a space of vector fields and suppose the fields form a Lie algebra (*e.g.* they are smooth). Can we construct a fixed vector field which, if we flow along it for unit time, gives the same homeomorphism as solving the inhomogeneneous differential equation over the interval $[s,t]$? Now f is a linear map from V into the smooth vector fields on some general target space. Because of the universal property of $\mathfrak{A}$ the map f extends to a unique Lie map f_* from $\mathfrak{A}$ into the vector fields with $f_*([v_1,[v_2,v_3]]) = [f(v_1),[f(v_2),f(v_3)]]$. The logarithm of the flow should be given by $f_*(\log(\boldsymbol{X}_{0,t}))$. However, this calculation is formal because one quietly slips from finite to infinite sequences. On the other hand one can always compute

$$f_*(\log(\boldsymbol{X}_{0,t}^{(n)}))\,, \qquad \text{where } \log(\boldsymbol{X}_{0,t}^{(n)})$$

is regarded as an element of $\mathfrak{A}^n \subset T^{(n)}$. These form a sequence of explicit and readily calculable vector fields providing an asymptotic expansion for the logarithmic vector field. A number of the optimal algorithms for solving sde's numerically are based on this idea [4].

Rough and smooth multiplicative functionals. Although our prime examples were obtained by computing the iterated integrals of a smooth path, the underlying definition of a multiplicative functional is at present a purely algebraic one. We now wish to consider rough and smooth multiplicative functionals. Equivalently we wish to consider rough or smooth monic paths in the truncated tensor algebras. For this we need a notion of distance between tensors in $T^{(n)}$. For all further discussion, suppose that V, and more generally $V^{\otimes n}$ are Banach spaces and that they have compatible norms $\|\cdot\|$ so that $\|\boldsymbol{u}\otimes\boldsymbol{v}\| \le \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$, and that the norms are invariant under permutations of the indices of the tensors. (Given a norm on V there are many norms one could take on the tensor products so that this property holds). Let $\boldsymbol{c} = (0, \boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_n)$ be an element of the radical

$$D_0^{(n)} = \bigoplus_{k=1}^{n} V^{\otimes k}$$

of $T^{(n)}$, then for any sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of strictly positive weights we may define a homogeneous distance function

$$|||\boldsymbol{c}|||_\lambda = \max\{(\lambda_i\,\|\boldsymbol{c}_i\|)^{1/i} :\ 1 \le i \le n\}\,. \tag{2.19}$$

It is clear that $|||\boldsymbol{c}+\boldsymbol{d}|||_\lambda \le |||\boldsymbol{c}|||_\lambda + |||\boldsymbol{d}|||_\lambda$ and so we define a metric on the radical by $d(\boldsymbol{c},\boldsymbol{d}) = |||\boldsymbol{c}-\boldsymbol{d}|||_\lambda$. The metrics on $D_0^{(n)}$ are uniformly equivalent for alternative choices of the constants λ, however this is only true for fixed and finite n. Although the metric is not a norm if $n > 1$ it has the very important property that it has the same homogeneity properties as our sequence of iterated integrals when we scale the underlying path.

Consider the element of the radical $\boldsymbol{X}_{s,t}^{(n)}(\eta) - \mathbf{1}$ generated by the sequence of iterated integrals of a smooth path η_s. Now scale the path, then the individual iterated integrals transform according to their degree and

$$|||\boldsymbol{X}^{(n)}_{s,t}(\varepsilon\,\eta) - \mathbf{1}||| = \varepsilon\,|||\boldsymbol{X}^{(n)}_{s,t}(\eta) - \mathbf{1}|||\,.$$

If we are only interested in fixed n we will frequently take $\lambda \equiv 1$ to avoid complicated expressions. If we wish to prove that the Einstein expansion for the solution of a linear equation converges one will need to control the behaviour as n goes to infinity. For this one requires a choice of λ very well adapted to the problem. In this more critical work we find $\lambda_i = \beta\,(i/p)!$ to be an excellent choice, where $\beta > 0$, $p > 1$ are to be chosen later. For notational convenience, we will use the notation $|||\cdot|||$ to denote either metric. It will not cause significant confusion.

Suppose that we have a monic path $\boldsymbol{X}_t$ in the truncated tensor algebra and its associated multiplicative functional $\boldsymbol{X}_{s,t}$. Then we could introduce a distance $\rho(\boldsymbol{X}_s, \boldsymbol{X}_t) = |||\boldsymbol{X}_{s,t} - \mathbf{1}|||$. In general this will not be a metric (although it is good enough) because it fails the symmetry condition and the triangle inequality. However, it is clear from the neo-classical inequality (see later) that if $\beta \geq 2^p\,p^2$ then it will satisfy the triangle inequality. For group like elements, it is obvious from Remark 2.1.1., the inverse being obtained from the path run backwards and the invariance of the norm under re-ordering of the tensors, that the inverse of a group like element has the same modulus as the original element. In this case it is clearly a metric.

We ignore the fact that this distance is a metric or not (because it follows that it is always equivalent to one). In any case we may follow section 2.1.1., and use it to define monic paths and multiplicative functionals of finite p-variation (controlled by a regular super additive function $\omega(s,t)$ etc.) and to provide a distance between two paths.

Lemma 2.1.1. *A multiplicative functional $\boldsymbol{X}_{s,t}$ in $T^{(n)}$ is of finite p-variation controlled by ω if and only if it satisfies the inequality*

$$\|\boldsymbol{X}^i_{s,t}\| < \frac{\omega(s,t)^{i/p}}{\beta\,(i/p)!}\,, \qquad i \leq n\,.$$

The proof is immediate from the definition. We include it as a convenient formulation.

2.2. Multiplicative functionals - the first main theorems.

Overview. We have introduced the idea of a multiplicative functional in $T^{(n)}$ of finite p-variation without making any direct connection between the degree n of the multiplicative functional and the roughness of the path as described by $p \geq 1$.

The theorems in this section, which are fundamental to our approach, demonstrate the central role played by the class of multiplicative functionals for which the degree n is the integer part $[p]$ of the variation p.

We have already observed that if we take a smooth path in a vector space and take its first k iterated integrals then we have constructed a multiplicative functional of degree k; computing the next iterated integral gives a method of extending the multiplicative functional to (a geometric) one of the next degree. This extension map is continuous as a function of the underlying path in p-variation metric if and only if $p < 2$.

By way of an extension of this result, the theorems in this section show by restriction that, for any $p \geq 1$, if we regard as our basic object the smooth path *and its iterated integrals of degree up to* $[p]$ then the higher iterated integrals are uniformly continuous functions in the metric of finite p-variation. The uniform continuity allows one to extend the definition of iterated integral to this class.

These results are the first step towards our main theorem that the Itô map[1] is uniformly continuous as a function of the sequence comprising a smooth path and its iterated integrals of degree up to $[p]$ where one takes the metric of finite p-variation. So providing a natural analytic extension of the Itô map to the class of geometric paths of finite p-variation and degree $[p]$.

The application to stochastic Stratonovich differential equations is realized by taking $3 > p > 2$; where these results reduce to the statement that the Itô map is continuous in the pair comprising the path and its Lévy area.

2.2.1. The First Theorem.

Theorem 2.2.1. *Let $\boldsymbol{X}_{s,t}^{(n)}$ be a multiplicative functional in $T^{(n)}$ of finite p-variation controlled by a regular $\omega(s,t)$ on an interval J where*

[1] defined by a differential equation with smooth enough coefficients

$n = [p]$. There exists a multiplicative extension $\boldsymbol{X}_{s,t}^{(m)}$ to $T^{(m)}$, $m > n$ which is of finite p-variation, the extension is unique in this class.

Moreover, this unique extension satisfies a rather precise estimate. Suppose that the p-variation norm of $\boldsymbol{X}_{s,t}^{(n)}$ is controlled by $\omega(s,t)$ so that for all pairs of times in an interval we have

$$\|\boldsymbol{X}_{s,t}^{i}\| < \frac{\omega(s,t)^{i/p}}{\beta\,(i/p)!}\,, \qquad i \leq p\,, \tag{2.20}$$

then, providing β is large enough the same inequality

$$\|\boldsymbol{X}_{s,t}^{i}\| < \frac{\omega(s,t)^{i/p}}{\beta\,(i/p)!}\,, \qquad i > p\,, \tag{2.21}$$

holds in all degrees and p-variation norm of $\boldsymbol{X}_{s,t}^{(m)}$ is controlled by $\omega(s,t)$ without any sort of factor for all m.[2]

Remarks 2.2.1. 1) It suffices for the above theorem that

$$\beta > p^2\Big(1 + 2^{([p]+1)/p}\Big(\zeta\Big(\frac{[p]+1}{p}\Big) - 1\Big)\Big)\,, \tag{2.22}$$

where

$$\zeta(z) = \sum_{1}^{\infty} \frac{1}{n^z} \tag{2.23}$$

is the traditional Riemann zeta function.

2) It is a more or less trivial remark that, in the case where $n < [p]$, if a multiplicative functional of degree n and finite p-variation has an extension to a multiplicative functional of degree m of finite p-variation, then the extension will never be unique. On the other hand in the case where $n > [p]$ the above theorem shows by restriction the existence and uniqueness of an extension of finite p-variation.

In the remainder of this section we outline the proof.

Two key results under-pin our argument. The first is completely elementary.

---

[2] where $x! = \Gamma(x+1)$

Lemma 2.2.1. *Suppose*

$$D = \{s = t_0 < t_1 < \cdots < t_r = t\}$$

is a dissection of $[s,t]$. *Then there is a* j *such that*

$$\omega(t_{j-1}, t_{j+1}) \le \begin{cases} \dfrac{2\,\omega(s,t)}{r-1}\,, & r > 2\,, \\ \omega(s,t)\,, & r = 2\,. \end{cases} \tag{2.24}$$

PROOF. ω is super-additive and so when $r > 2$

$$\sum_{1}^{r-1} \omega(t_{j-1}, t_{j+1}) \le 2\,\omega(s,t) \tag{2.25}$$

and at least one term in a sum is dominated by the mean so the result is clear. On the other hand when $r = 2$

$$\omega(t_{j-1}, t_{j+1}) = \omega(s,t)\,, \qquad \text{if } j = 1\,.$$

A neo-classical inequality.

Lemma 2.2.2. *The following extension of the binomial theorem holds*

$$\left(\frac{1}{p}\right)^2 \sum_{j=0}^{n} \frac{x^{j/p}}{(j/p)!}\,\frac{y^{(n-j)/p}}{((n-j)/p)!} \le \frac{(x+y)^{n/p}}{(n/p)!}\,, \tag{2.27}$$

where $n \in \mathbb{N}$, $x, y > 0$, $p \ge 1$.

We postpone the proof of this inequality which is quite non-trivial. Notice that since $(x/p)!$ is roughly $(x!)^{1/p}$, the lemma loosely asserts that we have a sequence of numbers satisfying $\sum a_j = b$ from the binomial theorem and $\sum a_j^{1/p} \le b^{1/p}$. In general the inequality would be reversed.

PROOF. **Existence.** Our intention is to proceed by induction. Fix $m \ge [p]$. As initial data consider a multiplicative functional

$$\boldsymbol{X}^{(m)}_{s,t} = (1, \boldsymbol{X}^1_{st}, \ldots, \boldsymbol{X}^{[p]}_{s,t}, \boldsymbol{X}^{[p]+1}_{s,t}, \ldots, \boldsymbol{X}^m_{s,t})$$

satisfying (2.32), we wish to construct a multiplicative functional $\boldsymbol{X}^{(m+1)}_{s,t}$ satisfying the same constraints.

Consider

$$\widehat{\boldsymbol{X}}_{s,t} = i(\boldsymbol{X}^{(m)}_{s,t}) = (1, \boldsymbol{X}^1_{s,t}, \dots, \boldsymbol{X}^m_{s,t}, \mathbf{0}) \,. \tag{2.28}$$

Of course $\widehat{\boldsymbol{X}}_{st}$ is not multiplicative, but at least it is in $T^{(m+1)}$. Fix a dissection $D = \{s \le t_1 \le \cdots \le t_{i-1} \le t\}$ of $[s,t]$ and define

$$\widehat{\boldsymbol{X}}^{D}_{s,t} = \widehat{\boldsymbol{X}}_{s,t_1} \otimes \widehat{\boldsymbol{X}}_{t_1,t_2} \otimes \cdots \otimes \widehat{\boldsymbol{X}}_{t_{i-1},t} \tag{2.29}$$

using the multiplication in $T^{(m+1)}$. It suffices to show the existence of $\lim_{\mathrm{mesh}(D)\to 0} \widehat{\boldsymbol{X}}^{D}_{s,t}$, for this limit, if it exists, will surely be multiplicative.

To check this last point observe that if the limit exists over $[s,u]$, then it can be attained via dissections D all of which include a fixed $t \in (s,u)$, and so we have

$$\widehat{\boldsymbol{X}}^{D}_{s,u} = \widehat{\boldsymbol{X}}^{D\cap[s,t]}_{s,t} \otimes \widehat{\boldsymbol{X}}^{D\cap[t,u]}_{t,u} \,. \tag{2.30}$$

Taking this limit as the mesh size of D converges to zero we see that we have

$$\big(\lim_{D\to 0} \widehat{\boldsymbol{X}}^{D}_{s,t}\big) = \big(\lim_{D\to 0} \widehat{\boldsymbol{X}}^{D\cap[s,t]}_{s,t}\big) \otimes \big(\lim_{D\to 0} \widehat{\boldsymbol{X}}^{D\cap[t,u]}\big) \,. \tag{2.31}$$

To prove the convergence of $\widehat{\boldsymbol{X}}^{D}$ we see that the difficulty rests in understanding the terms $\big(\widehat{\boldsymbol{X}}^{D}_{s,t}\big)^{m+1}$ for $\big(\widehat{\boldsymbol{X}}^{D}_{s,t}\big)^{j} = \boldsymbol{X}^j_{s,t}$ for all $j \le m$ since $\boldsymbol{X}^{(m)}_{s,t}$ is multiplicative.

The heart of our argument is a maximal inequality, the existence of the limit follows by a secondary argument. Our aim is to prove, under the induction hypothesis

$$\begin{aligned}
&\boldsymbol{X}^{(m)}_{u,v} = \{1, \boldsymbol{X}^1_{u,v}, \dots, \boldsymbol{X}^m_{u,v}\} \in T^{(m)} \,, \\
&\boldsymbol{X}^{(m)}_{u,w} = \boldsymbol{X}^{(m)}_{u,v} \otimes \boldsymbol{X}^{(m)}_{v,w} \,, \\
&\|\boldsymbol{X}^i_{u,v}\| \le \Big(\frac{(\omega(u,v))^{i/p}}{\beta (i/p)!}\Big) , \qquad \text{for all } u < v,\ i \le m \,,
\end{aligned} \tag{2.32}$$

that for *any* dissection D of $[s,t]$

$$\left\|\left(\overset{D}{\widehat{\boldsymbol{X}}}_{s,t}\right)^{j}\right\| \leq \frac{(\omega(s,t))^{j/p}}{\beta(j/p)!}\,, \qquad \text{for all } j \leq m+1\,. \tag{2.33}$$

The case where $j < m+1$ is a trivial consequence of our induction hypothesis. The $(m+1)$-tensor $\left(\overset{D}{\widehat{\boldsymbol{X}}}_{s,t}\right)^{m+1}$ is the focus of our attention. Now from the triangle inequality

$$\|(\boldsymbol{X}^{D}_{st})^{m+1}\| \leq \|(\boldsymbol{X}^{D}_{s,t} - \boldsymbol{X}^{D'}_{s,t})^{m+1}\| + \|(\boldsymbol{X}^{D'}_{st})^{m+1}\|\,, \tag{2.34}$$

where D' is any other dissection. Suppose that it is obtained from D by dropping a single point from the dissection (this trick seems to be due to L. C. Young). By choosing the point to omit from the dissection carefully, and repeating this deletion procedure until we have the trivial dissection we will obtain our result.

Fix

$$D = \{s = t_0 < t_1 < \cdots < t_r = t\}$$

and use Lemma 2.2.1 to choose j so that

$$\omega(t_{j-1}, t_{j+1}) \leq \begin{cases} \left(\dfrac{2}{r-1}\right)\omega(s,t)\,, & r \geq 3\,, \\ \omega(s,t)\,, & r = 2\,. \end{cases} \tag{2.35}$$

Let D' be $D \setminus \{t_j\}$ and consider $\overset{D}{\widehat{\boldsymbol{X}}}_{s,t} - \overset{D'}{\widehat{\boldsymbol{X}}}_{s,t}$. Now[3]

$$\begin{aligned} \overset{D}{\widehat{\boldsymbol{X}}}_{s,t} &= \left(\widehat{\boldsymbol{X}}_{s,t_1} \cdots \widehat{\boldsymbol{X}}_{t_{j-2},t_{j-1}}\right)\widehat{\boldsymbol{X}}_{t_{j-1},t_j}\,\widehat{\boldsymbol{X}}_{t_j,t_{j+1}} \\ &\qquad \cdot \left(\widehat{\boldsymbol{X}}_{t_{j+1},t_{j+2}} \cdots \widehat{\boldsymbol{X}}_{t_{j-1},t_r}\right) \\ &= \overset{D^-}{\widehat{\boldsymbol{X}}}_{s,t_{j-1}} \widehat{\boldsymbol{X}}_{t_{j-1},t_j} \widehat{\boldsymbol{X}}_{t_j,t_{j+1}} \overset{D^+}{\widehat{\boldsymbol{X}}}_{t_{j+1},t}\,, \end{aligned} \tag{2.36}$$

while

$$\overset{D'}{\widehat{\boldsymbol{X}}}_{st} = \overset{D^-}{\widehat{\boldsymbol{X}}}_{s,t_{j-1}}\,\widehat{\boldsymbol{X}}_{t_{j-1},t_{j+1}}\,\overset{D^+}{\widehat{\boldsymbol{X}}}_{t_{j+1},t} \tag{2.37}$$

[3] To shorten expressions we henceforth drop the use of the $\otimes$ to denote multiplication in the tensor algebra.

and so

$$\widehat{\boldsymbol{X}}^{D}_{st} - \widehat{\boldsymbol{X}}^{D'}_{st} = \widehat{\boldsymbol{X}}^{D^-}_{s,t_{j-1}} \boldsymbol{Z}_{t_{j-1}\,t_j\,t_{j+1}} \widehat{\boldsymbol{X}}^{D^+}_{t_{j+1},t}\,, \tag{2.38}$$

where

$$\boldsymbol{Z}_{t_{j-1}t_jt_{j+1}} = \widehat{\boldsymbol{X}}_{t_{j-1},t_j}\,\widehat{\boldsymbol{X}}_{t_{j-1},t_{j+1}} - \widehat{\boldsymbol{X}}_{t_{j-1},t_{j+1}} \tag{2.39}$$

and using the definition of $\widehat{\boldsymbol{X}}$ and the multiplicative nature of $\boldsymbol{X}$ one has

$$\boldsymbol{Z}_{t_{j-1}t_jt_{j+1}} = \Big(0,\ldots,\mathbf{0},\sum_1^m \boldsymbol{X}^i_{t_{j-1},t_j}\,\boldsymbol{X}^{m+1-i}_{t_j,t_{j+1}}\Big)\,. \tag{2.40}$$

The only products which yield nonzero results in this tensor multiplication are those where the sum of the degrees of the individual factors is at most m; it follows that we have the reasonably simple expression for the difference

$$\begin{aligned}\widehat{\boldsymbol{X}}^{D}_{s,t} - \widehat{\boldsymbol{X}}^{D'}_{s,t} &= (1,\ldots)\Big(0,\ldots,\mathbf{0},\sum_1^m \boldsymbol{X}^i_{t_{j-1},t_j}\,\boldsymbol{X}^{(m+1)-i}_{t_j,t_{j+1}}\Big)(1,\ldots)\\ &= \Big(0,\ldots,\mathbf{0},\sum_1^m \boldsymbol{X}^i_{t_{j-1},t_j}\,\boldsymbol{X}^{(m+1)-i}_{t_j,t_{j+1}}\Big)\,. \end{aligned}\tag{2.41}$$

We can estimate this difference

$$\Big\|\sum_1^m \boldsymbol{X}^i_{t_{j-1}t_j}\boldsymbol{X}^{m+1-i}_{t_jt_{j+1}}\Big\| \le \sum_1^m \|\boldsymbol{X}^i_{t_{j-1}t_j}\|\,\|\boldsymbol{X}^{m+1-i}_{t_jt_{j+1}}\| \tag{2.42}$$

and so using our a priori bound (2.32) for the magnitudes of these tensors

$$\begin{aligned}&\Big\|\sum_{i=1}^m \boldsymbol{X}^i_{t_{j-1},t_j}\boldsymbol{X}^{m+1-i}_{t_j,t_{j+1}}\Big\|\\ &\qquad\le \sum_{i=0}^{m+1}\Big(\frac{\omega(t_{j-1},t_j)^{i/p}}{\beta(i/p)!}\Big)\Big(\frac{\omega(t_j,t_{j+1})^{(m+1-i)/p}}{\beta((m+1-i)/p)!}\Big)\end{aligned}\tag{2.43}$$

and by the Neo-Classical inequality, Lemma 2.2.2, and superadditivity this is

$$\leq \frac{p^2}{\beta^2}\,\frac{(\omega(t_{j-1},t_j)+\omega(t_jt_{j+1}))^{(m+1)/p}}{((m+1)/p)!} \tag{2.44}$$

$$\leq \frac{p^2}{\beta^2}\,\frac{(\omega(t_{j-1},t_{j+1}))^{(m+1)/p}}{((m+1)/p)!}\;. \tag{2.45}$$

We now recall that we chose our j carefully so that (2.35) held so that if $r>2$ one has

$$\Big\|\sum_{i=1}^{m}\boldsymbol{X}^{i}_{t_{j-1}t_j}\boldsymbol{X}^{m+1-i}_{t_jt_{j+1}}\Big\| \leq \Big(\frac{2}{r-1}\Big)^{(m+1)/p}\,\frac{p^2}{\beta}\,\frac{\omega(s,t)^{(m+1)/p}}{\beta((m+1)/p)!} \tag{2.46}$$

and if $r=2$ one has the similar

$$\Big\|\sum_{i=1}^{m}\boldsymbol{X}^{i}_{t_{j-1}t_j}\boldsymbol{X}^{m+1-i}_{t_jt_{j+1}}\Big\| \leq \frac{p^2}{\beta}\,\frac{\omega(s,t)^{(m+1)/p}}{\beta((m+1)/p)!}\;. \tag{2.47}$$

Successively dropping points we see that

$$\begin{aligned}
&\big\|\big(\overset{D}{\widehat{\boldsymbol{X}}}_{st}\big)^{m+1}-\big(\widehat{\boldsymbol{X}}_{s,t}\big)^{m+1}\big\|\\
&\qquad\leq \frac{p^2}{\beta}\Big(1+\sum_{r=3}^{\infty}\Big(\frac{2}{r-1}\Big)^{(m+1)/p}\Big)\Big(\frac{\omega(s,t)^{(m+1)/p}}{\beta((m+1)/p)!}\Big)\\
&\qquad= \frac{p^2}{\beta}\Big(1+2^{(m+1)/p}\Big(\xi\Big(\frac{m+1}{p}\Big)-1\Big)\Big)\Big(\frac{\omega(s,t)^{(m+1)/p}}{\beta((m+1)/p)!}\Big)\;.
\end{aligned} \tag{2.48}$$

Observing that as $\big(\widehat{\boldsymbol{X}}_{st}\big)^{m+1}=\mathbf{0}$ and

$$\Big(1+\sum_{r=3}^{\infty}\Big(\frac{2}{r-1}\Big)^{(m+1)/p}\Big)$$

is monotone in m and finite because $m+1>p$ we have

$$\begin{aligned}
&\big\|\big(\overset{D}{\widehat{\boldsymbol{X}}}_{s,t}\big)^{m+1}\big\|\\
&\qquad\leq \frac{p^2}{\beta}\Big(1+2^{([p]+1)/p}\Big(\zeta\Big(\frac{[p]+1}{p}\Big)-1\Big)\Big)\,\frac{\omega(s,t)^{(m+1)/p}}{\beta((m+1)/p)!}
\end{aligned} \tag{2.49}$$

(where $\zeta(z) = \sum_1^\infty 1/n^z$ is the traditional Riemann zeta function). Thus if we choose

$$\beta \geq p^2\Big(1 + 2^{([p]+1)/p}\Big(\zeta\Big(\frac{[p]+1}{p}\Big) - 1\Big)\Big)\,, \tag{2.50}$$

we get the estimate

$$\big\|\big(\widehat{\boldsymbol{X}}^D_{s,t}\big)^{m+1}\big\| \leq \frac{\omega(s,t)^{(m+1)/p}}{\beta((m+1)/p)!}\,, \tag{2.51}$$

for *all choices of dissection* D. This completes the proof of the maximal inequality.

Now we must show convergence of the products. It is at this point that we require our control ω on the p-variation to be regular. We will show that our sequence $\widehat{\boldsymbol{X}}^D$ satisfies a Cauchy convergence principle. Consider two dissections D, $\tilde{D}$ both having mesh size less than δ. We can always find a common refinement $\hat{D}$ of D and $\tilde{D}$. We fix some interval in $[t_j, t_{j+1}] \in D$; then the refinement $\hat{D}$ breaks the interval up into a number of pieces $t_j \leq s_{j_1} \leq \cdots \leq s_{j_r} = t_{j+1}$; call the dissection $\hat{D}_j$. Then, we know from the maximal inequality, how to estimate

$$\big(\widehat{\boldsymbol{X}}^{\hat{D}_j}_{t_j t_{j+1}} - \widehat{\boldsymbol{X}}_{t_j t_{j+1}}\big)^{m+1}$$

and all terms of degree less than $m+1$ in the difference are zero because $\boldsymbol{X}$ is multiplicative. Therefore

$$\big\|\widehat{\boldsymbol{X}}^{\hat{D}_j}_{t_j t_{j+1}} - \widehat{\boldsymbol{X}}_{t_j t_{j+1}}\big\| \leq \frac{\omega(t_j,t_{j+1})^{m+1/p}}{\beta((m+1)/p)!}\,. \tag{2.52}$$

So the total difference[4]

$$\big(\widehat{\boldsymbol{X}}^{\hat{D}_j} - \widehat{\boldsymbol{X}}^{D}\big)^{m+1} \tag{2.53}$$

is controlled in norm by

$$\begin{aligned} &\Big(\sum_D \frac{\omega(t_j,t_{j+1})^{(m+1)/p}}{\beta((m+1)/p)!}\Big) \\ &\qquad \leq \frac{1}{\beta((m+1)/p)!}\,\max_D(\omega(t_j,t_{j+1}))^{(m+1)/p-1}\sum_D \omega(t_j,t_{j+1}) \\ &\qquad \leq \frac{1}{\beta((m+1)/p)!}\,\max_D(\omega(t_j,t_{j+1}))^{(m+1)/p-1}\,\omega(s,t)\,, \end{aligned} \tag{2.54}$$

[4] using a simple extension of the argument used in (2.38)-(2.40) which drastically limited the range of terms which contribute to the difference of the products.

which is independent of $\hat{D}$ and by the regularity of ω this converges uniformly to zero as the mesh size of D converges to zero. Applying the triangle inequality, we have a uniform bound on $\boldsymbol{X}^{\tilde{D}} - \boldsymbol{X}^{D}$ as required. It follows that we have established the existence of a multiplicative functional satisfying all the requirements of the induction.

Uniqueness. We must show that if $\boldsymbol{X}_{s,t}$ and $\boldsymbol{Y}_{s,t}$ are two multiplicative functionals which agree up to the m-th degree, so that $\boldsymbol{X}_{st}^{i} = \boldsymbol{Y}_{st}^{i}$, $i \leq m$, and which are both of regular finite p-variation where $(m+1)/p > 1$ then they agree. The following algebraic lemma makes the situation clear.

Lemma 2.2.3. *Suppose that $\boldsymbol{X}_{s,t}$ and $\boldsymbol{Y}_{s,t}$ are multiplicative functionals in $T^{(m+1)}$ which agree up to the m-th degree so that $\boldsymbol{X}_{st}^{i} = \boldsymbol{Y}_{st}^{i}$, $i \leq m$. The difference function $\boldsymbol{\Psi}_{s,t}$*

$$\boldsymbol{\Psi}_{s,t} = \boldsymbol{X}_{s,t}^{m+1} - \boldsymbol{Y}_{s,t}^{m+1} \in \bigotimes_{i=1}^{m+1} V \tag{2.55}$$

is additive

$$\boldsymbol{\Psi}_{s,t} + \boldsymbol{\Psi}_{t,u} = \boldsymbol{\Psi}_{s,u} \ . \tag{2.56}$$

Conversely, if $\boldsymbol{X}_{s,t}$ is a multiplicative functional in $T^{(m+1)}$ and $\boldsymbol{\Psi}_{s,t}$ is additive in $V^{\otimes m+1}$ then $\boldsymbol{X}_{s,t} + \boldsymbol{\Psi}_{s,t}$ is also a multiplicative functional.

REMARK 2.2.1. This easy result reflects the nilpotent nature of the algebraic structures we are interested in, the function $\boldsymbol{\Psi}_{s,t}$ lies in the centre.

PROOF. Use the multiplicative property for $\boldsymbol{X}_{s,t}$ and $\boldsymbol{Y}_{s,t}$ to observe that

$$\begin{aligned}
(\boldsymbol{Y}_{s,u})^{m+1} &= (\boldsymbol{Y}_{s,t} \otimes \boldsymbol{Y}_{t,u})^{m+1} \\
&= \boldsymbol{Y}_{s,t}^{m+1} + \boldsymbol{Y}_{t,u}^{m+1} + (\boldsymbol{X}_{s,t} \otimes \boldsymbol{X}_{t,u})^{m+1} - \boldsymbol{X}_{s,t}^{m+1} - \boldsymbol{X}_{t,u}^{m+1} \\
&= (\boldsymbol{Y}_{s,u})^{m+1} \\
&= (\boldsymbol{X}_{s,u})^{m+1} + (\boldsymbol{Y}_{s,t}^{m+1} - \boldsymbol{X}_{s,t}^{m+1}) + (\boldsymbol{Y}_{t,u}^{m+1} - \boldsymbol{X}_{t,u}^{m+1})
\end{aligned} \tag{2.57}$$

and so our claim is verified

$$\boldsymbol{\Psi}_{s,u} = \boldsymbol{\Psi}_{s,t} + \boldsymbol{\Psi}_{t,u} \ . \tag{2.58}$$

The same identity also makes it clear that if $\boldsymbol{X}_{s,t}$ is multiplicative on $T^{(n+1)}$ and $\boldsymbol{\Psi}$ satisfies (2.56) and is in $\otimes_i^{n+1} V$, then $\boldsymbol{X}_{s,t} + \boldsymbol{\Psi}_{s,t}$ is also multiplicative.

Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ have finite p-variation controlled by a regular ω where $(m+1)/p > 1$. By assumption, there is a constant so that

$$\|\boldsymbol{\Psi}_{st}\| \leq c\,\omega(s,t)^{(m+1)/p}\,. \tag{2.59}$$

and so $\boldsymbol{\Psi}_{0,t}$ is a conventional path of finite $(m+1)/p$-variation. If $(m+1)/p > 1$ and ω is regular, it follows that $\boldsymbol{\Psi}$ is identically zero and uniqueness follows.

These calculations also establish the remarks we made on the non-uniqueness of extensions of multiplicative functionals if $(m+1)/p \leq 1$ as in this case perturbing an extension by a continuous additive $\boldsymbol{\Psi}_{s,t}$ of bounded variation will produce a different extension of finite p-variation.

2.2.2. Continuity.

We have shown that the high order multiplicative functionals are uniquely determined by the low order ones if we impose a p-variation condition. We also defined a natural distance between paths of finite p-variation. The map we have defined is continuous, and there is a very explicit estimate for the modulus of continuity.

Theorem 2.2.2. *Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are multiplicative functionals in $T^{(n)}$ of finite p-variation controlled by ω where $(n+1)/p > 1$. Suppose further that for some $\varepsilon < 1$ one has*

$$\|\boldsymbol{X}^i_{s,t} - \boldsymbol{Y}^i_{s,t}\| \leq \varepsilon\,\frac{\omega(s,t)^{i/p}}{\gamma(i/p)!}\,, \tag{2.60}$$

for all $i \leq n$. Then for a suitable choice of γ,

$$\gamma \geq 3\,p^2\Big(1 + 2^{([p]+1)/p}\Big(\zeta\Big(\frac{[p]+1}{p}\Big) - 1\Big)\Big) \tag{2.61}$$

will do, one has

$$\|\boldsymbol{X}^i_{s,t} - \boldsymbol{Y}^i_{s,t}\| \leq \varepsilon\,\frac{\omega(s,t)^{i/p}}{\gamma(i/p)!}\,, \tag{2.62}$$

for all $i < \infty$ where $\boldsymbol{X}^i$ and $\boldsymbol{Y}^i$ are, for $i > n$, the components in $V^{\otimes i}$ of the multiplicative extension of finite p-variation.

Proof. Proceed by induction. Suppose $n+1 > p$. Recall how we constructed $\boldsymbol{X}^{m+1}$ and $\boldsymbol{Y}^{m+1}$ from $\boldsymbol{X}^{(m)}$ and $\boldsymbol{Y}^{(m)}$ by taking the limit of the products $\boldsymbol{X}^D$, $\boldsymbol{Y}^D$. Recall in particular, that our choice of dissection in the proof of the maximal inequality depended on ω alone and not on $\boldsymbol{X}^{(n)}$ or $\boldsymbol{Y}^{(n)}$. So we may select the same coarsening sequence of dissections in the analysis bounding $\boldsymbol{X}^D$ and $\boldsymbol{Y}^D$. We may also use this sequence of dissections to estimate $\|\boldsymbol{X}^D - \boldsymbol{Y}^D\|$. As we coarsen the dissection we have

$$\begin{aligned} \|(\boldsymbol{X}^D - \boldsymbol{Y}^D)^{m+1}\| &\leq \|(\boldsymbol{X}^D - \boldsymbol{X}^{D'})^{m+1} - (\boldsymbol{Y}^D - \boldsymbol{Y}^{D'})^{m+1}\| \\ &\quad + \|(\boldsymbol{X}^{D'} - \boldsymbol{Y}^{D'})^{m+1}\|\,. \end{aligned} \tag{2.63}$$

Estimate the first term on the right side of the expression.

$$(\boldsymbol{X}^{D'}_{s,t} - \boldsymbol{X}^{D}_{s,t})^{n+1} = \sum_{1\leq j\leq n} \boldsymbol{X}^{j}_{t_{i-1}t_i}\,\boldsymbol{X}^{n+1-j}_{t_i t_{i+1}} \tag{2.64}$$

and

$$\boldsymbol{Y}^j_{s,t} = \boldsymbol{X}^j_{s,t} + \boldsymbol{R}^j_{s,t}\,, \tag{2.65}$$

so

$$\begin{aligned} &((\boldsymbol{X}^{D'}_{s,t} - \boldsymbol{X}^{D}_{s,t}) - (\boldsymbol{Y}^{D'}_{s,t} - \boldsymbol{Y}^{D}_{s,t}))^{n+1} \\ &\quad = \sum_{1\leq j\leq n} (\boldsymbol{X}^{j}_{t_{i-1},t_i}\,\boldsymbol{X}^{n+1-j}_{t_i,t_{i+1}} \\ &\qquad\qquad - (\boldsymbol{X}^{j}_{t_{i-1},t_i} + \boldsymbol{R}^{j}_{t_{i-1},t_i})\,(\boldsymbol{X}^{n+1-j}_{t_i,t_{i+1}} + \boldsymbol{R}^{n+1-j}_{t_i,t_{i+1}})) \end{aligned} \tag{2.66}$$

and by exploiting induction and the neo-classical inequality one has

$$\begin{aligned} &\|((\boldsymbol{X}^{D'}_{s,t} - \boldsymbol{X}^{D}_{s,t})\,(\boldsymbol{Y}^{D'}_{st} - \boldsymbol{Y}^{D}_{s,t}))^{n+1}\| \\ &\qquad\qquad \leq (2\,\varepsilon + \varepsilon^2)\,p^2\,\frac{\omega(t_{i-1},t_{i+1})^{(n+1)/p}}{\gamma^2((n+1)/p)!} \end{aligned} \tag{2.67}$$

and as before, summing over our carefully chosen and successively coarsening partitions one has

$$\begin{aligned} \|\boldsymbol{X}^D_{s,t} - \boldsymbol{Y}^D_{s,t}\| &\leq \frac{1 + 2^{(n+1)/p}(\zeta((n+1)/p) - 1)}{\gamma^2} \\ &\qquad \cdot p^2\,\frac{\omega(s,t)^{(n+1)/p}}{((n+1)/p)!}\,(2\,\varepsilon + \varepsilon^2)\,. \end{aligned} \tag{2.68}$$

So for

$$\gamma \geq \Big(1 + 2^{([p]+1)/p}\Big(\xi\Big(\frac{[p]+1}{p}\Big) - 1\Big)\Big)p^2\,(2\,\varepsilon + \varepsilon^2)$$

the result follows. In particular if $\varepsilon < 1$ the required estimate holds. This completes the induction step and this rather explicit continuity result follows.

Remark 2.2.2. It might be thought that we have introduced a variety of topologies on the space of paths of finite p-variation in the above theorem; however, they can all be pasted together in the most natural way.

Definition 2.2.1. *We say a pair of paths $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $T^{(n)}$ which have regular finite p-variation are at most a distance ε apart if*

$$\begin{aligned}&\omega^{X,Y}(s,t)\\ &\quad = \sup_{s,t\in J}\Big\{\Big(\sum_j |||\boldsymbol{X}_{t_j,t_{j+1}} - \boldsymbol{Y}_{t_j,t_{j+1}}|||^p\Big),\ s \leq t_{j_1} < \cdots < t_{j_r} \leq t\Big\} \leq \varepsilon\,,\end{aligned}$$

$$\sup\{|||\boldsymbol{X}_t - \boldsymbol{Y}_t|||,\ t \in J\} \leq \varepsilon\,.$$

It is elementary that such a distance is complete, and that if a sequence converges in the sense that we introduced and exploited in the preceding lemma then it also converges in this new sense.

Consider a sequence $U_t^{(n)}$ of paths converging to a path U_t^0. Then the p-variation of $U_t^{(n)}$, denoted by $\omega^{U^{(n)}}(s,t)$, and $\omega^{U^{(n)},U^{(0)}}(s,t)$ are continuous and zero on the diagonal because of the regularity of the paths. We may choose and re-label a subsequence so that

$$\sup_{s,t}\,\omega^{U^{(n)},U^{(0)}}(s,t) < 4^{-n}$$

on J. Consider the new superadditive functional

$$\Psi(s,t) = \sup_n\,\omega^{U^{(n)}}(s,t) + \sum_n 2^n\,\omega^{U^{(n)},U^{(0)}}(s,t) \tag{2.69}$$

and observe that it is continuous (note that the supremum of a sequence of continuous and uniformly converging functions is itself continuous), it is obviously superadditive and zero on the diagonal; it therefore provides a regular control on the p-variation of all the paths we are considering, and most importantly, satisfies $\omega^{U^{(n)},U^{(0)}}(s,t) \leq 2^{-n}\,\psi(s,t)$. This essentially concludes the remark. Every convergent sequence in the weaker sense has a subsequence converging in this stronger dominated sense, and so we see that the notions of convergent sequence must correspond.

In a metric space, the topology is determined by the convergent sequences.

2.2.3. The neo-classical inequality: a proof.

Theorem 2.2.3. *The following inequality holds uniformly in $p \geq 1$, n*

$$\frac{1}{p^2}\sum_{j=0}^{n} \frac{a^{j/p}\,b^{(n-j)/p}}{(j/p)!\,((n-j)/p)!} \leq \frac{(a+b)^{n/p}}{(n/p)!}\,, \qquad a,b>0\,. \tag{2.70}$$

REMARK 2.2.3. For our application we only require this inequality with some constant in place of $1/p^2$ which is independent of a and n. However, it is interesting to ask what is the best uniform estimate in all the variables. All numerical evidence and proofs of special cases suggest the inequality is true with $1/p$ in place of $1/p^2$ and that in this form the inequality is very strongly saturated (with equality to the n-th degree as p approaches one if $a=b$. When $p=1$, we have equality of the left and right expressions by the binomial theorem in either form. When $p=n$ we can prove the result in its strengthened form with $1/p$.

PROOF. To prove the inequality in the form stated, it suffices to establish that

$$\frac{1}{p}\sum_{j=0}^{n} x^{j/p}\,(1-x)^{(n-j)/p}\frac{(n/p)!}{(j/p)!\,((n-j)/p)!} \leq p\,, \tag{2.71}$$

because the expression (2.70) is homogeneous under scaling of a and b. Moreover, we have an integral expression for the special functions

$$\begin{aligned}\Big(\frac{x!\,y!}{(x+y)!}\Big)^{-1} &= \frac{1}{(x+y+1)\,\beta(x+1,y+1)}\\ &= \frac{1}{(x+y+1)\displaystyle\int_0^1 u^x(1-u)^y\,du}\,.\end{aligned} \tag{2.72}$$

We may rewrite the left hand of the expression (2.71)

$$\begin{aligned}\frac{1}{p}\sum_0^n x^{j/p}\,(1-x)^{(n-j)/p}\,&\frac{(n/p)!}{(j/p)!\,((n-j)/p)!}\\&=\frac{1}{n+p}\sum_0^n\frac{x^{j/p}\,(1-x)^{(n-j)/p}}{\displaystyle\int_0^1\left(u^j\,(1-u)^{n-j}\right)^{1/p}du}\\&=\frac{1}{n+p}\sum_0^n\frac{1}{\displaystyle\int_0^1\left(\frac{u}{x}\right)^{j/p}\left(\frac{1-u}{1-x}\right)^{(n-j)/p}du}\,.\end{aligned}$$

We now make a substitution: $v=p/n$, $\theta_j=j/n$. Then the individual terms in the above sum are derived from

$$F_\theta(x,v)=\frac{1}{n\,(v+1)}\,\frac{x^{\theta/\nu}\,(1-x)^{(1-\theta)/\nu}}{\displaystyle\int_0^1\left(u^\theta\,(1-u)^{1-\theta}\right)^{1/v}du}\,. \tag{2.73}$$

By the binomial theorem

$$\sum_0^n F_{\theta_j}\Big(x,\frac{1}{n}\Big)\equiv 1\,, \tag{2.74}$$

for all n and all $x\in[0,1]$. If we could also prove that

$$\sum_0^n F_{\theta_j}(x,v)\le 1\,, \tag{2.75}$$

for all $v>1/n$, and for all x then we would have established the stronger result which we believe is true. To do this it would suffice to show that

$$\Big(\frac{\partial}{\partial x}\,(x\,(1-x))\,\frac{\partial}{\partial x}-\frac{\partial}{\partial v}\Big)F_\theta\ge 0\,, \tag{2.76}$$

for in this case we could use the maximum principle for sub-parabolic functions to deduce that any positive linear combination of F_θ, taken over varying θ, attains its maximum over the region $v>1/n$, $x\in(0,1)$ on its parabolic boundary. In particular we could conclude that $\sum_0^n F_{\theta_j}(x,v)\le 1$, for all x and for all $v>1/n$.

But F_θ is not a subsolution. On the positive side, we can prove that

$$\frac{1}{v}\, x^{\theta/v}\,(1-x)^{(1-\theta)/v}\Big(\int (u^\theta\,(1-u)^{1-\theta})^{1/v}\,du\Big)^{-1} \tag{2.77}$$

is a subsolution for any choice of θ. We can therefore apply a maximum principle argument to prove that if $\theta_j = j/n$ then

$$\begin{aligned}\sum_{j=0}^{n}\frac{v+1}{v}\,F_{\theta_j}(v,u) &\le \sup_{u\in[0,1]}\sum_{j=0}^{n}\frac{\frac{1}{n}+1}{\frac{1}{n}}\,F_{\theta_j}\Big(\frac{1}{n},u\Big)\\ &= \frac{\frac{1}{n}+1}{\frac{1}{n}}\\ &= n+1\,,\end{aligned} \tag{2.78}$$

for $v > 1/n$. We may cross-multiply and substitute to obtain

$$\sum_{j=0}^{n} F_{\theta_j}(v,u) \le \frac{v\,(n+1)}{v+1} = \frac{p\,(n+1)}{p+n}\,. \tag{2.79}$$

As the inequality $v > 1/n$ is equivalent to $p \ge 1$, we may deduce that for $v > 1/n$ and $u \in [0,1]$ the inequality

$$\sum_{j=0}^{n} F_{\theta_j}(v,u) \le p \tag{2.80}$$

holds, concluding our main argument.

However, it remains to prove that our expression (2.77) is indeed a subsolution to the parabolic equation (2.76). This is elementary, but relatively delicate.

Because our expression is positive, we may work with its logarithm. Observe that as a general fact a parabolic operator applied to an exponential has a simple form

$$\begin{aligned}Le^U &:= \frac{\partial}{\partial u}\,\varphi\,\frac{\partial}{\partial u}\,e^U - \frac{\partial e^U}{\partial v}\\ &= \Big(\frac{\partial}{\partial u}\,\varphi\,\frac{\partial}{\partial u}\,U + |\nabla_\varphi U|^2 - \frac{\partial U}{\partial v}\Big)e^U\,,\end{aligned} \tag{2.81}$$

where we define

$$|\nabla_\varphi u|^2 = \varphi\,|\nabla u|^2\,. \tag{2.82}$$

To show that the exponential e^U is a subsolution it suffices to show that

$$\Big(\frac{\partial}{\partial u}\,\varphi\,\frac{\partial}{\partial u}\,U + |\nabla_\varphi U|^2 - \frac{\partial U}{\partial v}\Big) \geq 0\,. \tag{2.83}$$

The log of the expression (2.77) is

$$\begin{aligned} &-\log v + \frac{\theta}{v}\log x + \frac{1-\theta}{v}\log(1-x) \\ &-\log\int_0^1 (u^\theta(1-u)^{1-\theta})^{1/v}du\,. \end{aligned} \tag{2.84}$$

Let us apply our identity for Le^U, one term at a time, with U given by the expression (2.84) above.

$$\frac{\partial}{\partial x}\,x\,(1-x)\,\frac{\partial}{\partial x}\,U = \frac{\partial}{\partial x}\Big(\frac{\theta}{v}\,(1-x)\Big) - \frac{\partial}{\partial x}\Big(\frac{1-\theta}{v}\,x\Big) = -\frac{1}{v} \tag{2.85}$$

and

$$\begin{aligned} x\,(1-x)\,\Big|\frac{\partial}{\partial x}\,U\Big|^2 &= x\,(1-x)\Big(\frac{\theta}{v}\,\frac{1}{x} - \frac{1-\theta}{v}\,\frac{1}{1-x}\Big)^2 \\ &= \Big(\frac{\theta-x}{v}\Big)^2\,\frac{1}{x\,(1-x)}\,. \end{aligned} \tag{2.86}$$

On the other hand the expression (2.84) can also be rewritten as

$$-\log v - \log\Big(\int\Big(\Big(\frac{u}{x}\Big)^\theta\Big(\frac{1-u}{1-x}\Big)^{1-\theta}\Big)^{1/v}du\Big)\,. \tag{2.87}$$

So

$$\begin{aligned} &-\frac{\partial}{\partial v}\,U \\ &= \frac{1}{v} - \frac{1}{v^2}\,\frac{\int\Big(\theta\log\Big(\frac{u}{x}\Big) + (1-\theta)\log\Big(\frac{1-u}{1-x}\Big)\Big)\Big(\Big(\frac{u}{x}\Big)^\theta\Big(\frac{1-u}{1-x}\Big)^{1-\theta}\Big)^{1/v}du}{\int\Big(\Big(\frac{u}{x}\Big)^\theta\Big(\frac{1-u}{1-x}\Big)^{1-\theta}\Big)^{1/v}du} \\ &= \frac{1}{v} - \frac{1}{v^2}\Big(\theta\log\Big(\frac{\theta}{x}\Big) + (1-\theta)\log\Big(\frac{1-\theta}{1-x}\Big)\Big) \\ &\quad + \frac{1}{v^2}\,\frac{\int\Big(\theta\log\Big(\frac{\theta}{u}\Big) + (1-\theta)\log\Big(\frac{1-\theta}{1-u}\Big)\Big)\Big(\Big(\frac{u}{x}\Big)^\theta\Big(\frac{1-u}{1-x}\Big)^{1-\theta}\Big)^{1/v}du}{\int\Big(\Big(\frac{u}{x}\Big)^\theta\Big(\frac{1-u}{1-x}\Big)^{1-\theta}\Big)^{1/v}du} \end{aligned} \tag{2.88}$$

and applying Jensen's inequality to the convex function $x \log x$,

$$\left(\theta \log\left(\frac{\theta}{u}\right) + (1-\theta)\log\left(\frac{1-\theta}{1-u}\right)\right) \tag{2.89}$$

in the last integral, and hence the integral itself is always positive. Collecting the terms together we see that (2.83) will hold providing we can show

$$f(\theta,x) = \frac{(\theta-x)^2}{x\,(1-x)} - \left(\theta \log\left(\frac{\theta}{x}\right) + (1-\theta)\log\left(\frac{1-\theta}{1-x}\right)\right) \geq 0\,, \tag{2.90}$$

for all pairs θ, $x \in [0,1]$. This will follow through a study of $\partial f(\theta,x)/\partial x$. This derivative is 0 at $x=\theta$. If we prove it to be positive for $x \geq \theta$ and negative for $x < \theta$ then the result follows since $f(\theta,\theta)=0$. But

$$\begin{aligned}\frac{\partial f}{\partial x} &= (x-\theta)\,\frac{x\,(1-x) + (1-2\,x)\,(\theta-x)}{(x\,(1-x))^2}\\ &= (x-\theta)\,\frac{(x-\theta)^2+(\theta-\theta^2)}{(x\,(1-x))^2}\end{aligned} \tag{2.91}$$

and the second factor in the last expression is positive because $\theta \in [0,1]$.

This completes the proof of the neo-classical inequality.

2.3. Multiplicative functionals - The basic spaces of paths.

We can now identify the basic classes of objects which drive differential equations.

Definition 2.3.1. *A p-multiplicative functional is a multiplicative functional of degree $[p]$ and finite p-variation, taking its values in $T(V)^{([p])}$. We denote the set of such paths by $\Omega(V)^p$. The elements of $\Omega(V)^p$ with $\boldsymbol{X}_{s,t} \in G^{([p])}$ for all pairs of times s, t are the geometric p-multiplicative functionals denoted by $\Omega G(V)^p$.*

Within these spaces, we will often refine our interest and consider only multiplicative functionals which are controlled by a given regular ω.

The constraint defining $\Omega G(V)^p$ as a subspace of $\Omega(V)^p$ is a purely algebraic one; and for this reason it is obvious that it defines a closed subset. On the other hand $\Omega G(V)^p$ has a very important analytic interpretation. The class $S(V)$ of piecewise smooth paths can be lifted to *a subset $S(V)_p$ of $\Omega(V)^p$ in a canonical way using the first $[p]$ iterated integrals and, as we have shown, Chen observed that the embedding is actually into $\Omega G(V)^p$.*

Lemma 2.3.1. *The closure of $S(V)_p$ in $\Omega(V)^p$ is $\Omega G(V)^p$.*

The proof of this lemma is quite routine and so we only sketch it. Fix a group-like multiplicative functional $\boldsymbol{X}$. Suppose that it has finite p-variation controlled by a regular ω. We must construct piecewise smooth paths whose iterated integrals approximate it. However, given an element g of the group $G^{(n)}$ there is always a smooth path whose first n iterated integrals at time one agree with g. Among these paths the one with shortest projected distance in V has been closely studied [30]. In any case, its p-variation in a compact neighborhood of the identity in $G^{(n)}$ will be uniformly comparable[5] with $|||g-1|||$. As a consequence, we see that the paths obtained by taking the original multiplicative functional, fixing a dissection, and then replacing the intermediate segments of the multiplicative functional by these "chords" re-parameterised so that they are transversed according to the times in our dissection provide an approximating family of piecewise smooth multiplicative functionals. The regularity of ω ensures convergence.

The class of geometric multiplicative functionals will be of great importance later. A number of questions that remain open relate to the possible extension of theorems from $\Omega G(V)^p$ to $\Omega(V)^p$. Such an extension corresponds to the extension from Stratonovich to Itô in the classical probabilistic setting. In this paper, we will frequently use the above lemma to obtain results for the geometric p-functionals that we do not know how to prove more generally. We hope to understand matters better, and return to this issue in a later paper.

2.3.1. Inhomogeneous degrees of smoothness.

Consider the equation

$$dy_t = \sum_i f^i(y_t)\,dx_t^i + f^0(y_t)\,dt\,, \tag{2.92}$$

by taking our driving signal to be (x_t, t) everything we said previously applies. However, this is an analytically wasteful approach as we fail

[5] The bound will depend on the values of n and p.

to take advantage of the smoother character of one of the co-ordinates in contrast with the others. So we remark now that at the price of increased notational complexity, one may introduce a notion of multiplicative functional $\boldsymbol{X}_{s,t}$ of finite $p = (p_1, \ldots, p_d)$ variation controlled by ω.

Definition 2.3.2. *A path $\boldsymbol{X}_{s,t}$ in $T(V^1 \oplus \cdots \oplus V^d)$ is of finite $p = (p_1, \ldots, p_d)$ variation controlled by ω providing the component*

$$\boldsymbol{X}_{s,t}^{(r_1,\ldots,r)} \in V^{r_1} \otimes \cdots \otimes V^{r_l}\,, \tag{2.93}$$

where $r_i \in \{1, \ldots, d\}$ satisfies

$$\|\boldsymbol{X}_{s,t}^{(r_1,\ldots,r)}\| \leq \frac{\omega(s,t)^{l_1/p_1+\cdots+l_d/p_d}}{\beta^d\,(l_1/p_1)!\cdots(l_d/p_d)!}\,, \tag{2.94}$$

where

$$l_j = \frac{|\{i :\ r_i = j\}|}{l}\,. \tag{2.95}$$

In this case it is easy to see that essentially the same arguments and definitions can be applied to get existence, uniqueness and continuity theorems. The crucial point is that to get existence, and a uniqueness theorem, one must know all the components of the multiplicative functional for which $l_1/p_1 + \cdots + l_d/p_d < 1$. The arguments vary scarcely at all.

2.4. Differential equations driven by rough signals - The linear case.

2.4.1. The flow induced by a rough multiplicative functional.

We now draw out some applications of our first theorems on multiplicative functionals.

Recall that a linear differential equation is one where the target manifold (where y_t takes its values) is a Banach space, and the linear map from the space V carrying the driving signal x_t has as its range vector fields that are *bounded linear* maps

$$x \longrightarrow A(\,)\,x : V \longmapsto \hom(W, W)\,. \tag{2.96}$$

In our general form an equation can be reparameterised in this way if the vector fields define a finite dimensional Lie algebra.

If x_t is a smooth path then, as we saw previously, the linear flow associated to the linear equation

$$\begin{cases} dy_t = A(y)\,dx_t\ , \\ d\pi_t = A(\cdot)\,dx_t\,\pi_t\ , \end{cases} \tag{2.97}$$

can be recovered as the sum of the convergent Einstein series

$$\pi_{s,t} = I + A \int\limits_{s<u<t} dx_u + AA \iint\limits_{s<u_1<u_2<t} dx_{u_1}\,dx_{u_2} + \cdots \tag{2.98}$$

The theorems in the last section associate to any element X in $\Omega(V)^p$ a unique multiplicative functional $\boldsymbol{X}_{s,t} = (1, \boldsymbol{X}^1_{s,t}, \ldots, \boldsymbol{X}^{[p]}_{s,t}, \boldsymbol{X}^{[p]+1}_{s,t}, \ldots)$ of arbitrarily high (and hence of infinite) degree and finite p-variation. Because the terms $\boldsymbol{X}^i_{s,t}$ decay like

$$\frac{1}{(i/p)!} \tag{2.99}$$

and this is faster than any geometric series grows, the series

$$\pi_{s,t} = I + A\boldsymbol{X}^1_{s,t} + AA\boldsymbol{X}^2_{s,t} + \cdots \tag{2.100}$$

converges absolutely to an operator in $\hom(W, W)$. Moreover the mapping is obviously continuous from $\Omega(V)^p$.

Lemma 2.4.1. *The map*

$$\pi_{s,t} = I + A\boldsymbol{X}^1_{s,t} + AA\boldsymbol{X}^2_{s,t} + \cdots \tag{2.101}$$

from $\Omega(V)^p$ *to* $\hom(W, W)$ *respects multiplication. That is to say* $\pi_{s,t}\,\pi_{t,u} = \pi_{s,u}$.

REMARK 2.4.1. The fact that we can find a multiplicative extension of our map from geometric paths to all p-multiplicative paths indicates that the role of $\Omega(V)^p$ relative to $\Omega G(V)^p$ is very similar to that of the enveloping algebra to the Lie group.

PROOF. If s, t, u are in V then

$$(A\,v)(AA\,t \otimes u) = AAA\,v \otimes t \otimes u\,, \qquad \text{etc.} \tag{2.102}$$

From this observation, the multiplicative property of X, and the absolute convergence of all the series the result is immediate.

We could state a more abstract form of the above result.

Corollary 2.4.1. *Suppose A is a bounded map from a Banach space V into any Banach algebra Q then the map*

$$d\pi_{t_0,t} = \pi_{t_0,t}\,A\,dx_t\ , \qquad \pi_{t_0,t_0} = 1\,, \tag{2.103}$$

defined on smooth paths in V extends in a unique continuous way to the geometric multiplicative functionals of finite p-variation in $\Omega G(V)^p$ and more generally to any regular multiplicative functional of p-variation. The map is multiplicative on $\Omega(V)^p$.

Although this allows us to give a meaning to (2.97) for elements of $\Omega(V)^p$, we only feel 100% confident about calling it a *solution* in the case where $\boldsymbol{X}$ is an element of $\Omega G(V)^p$. The reason for our nervousness is that if we apply the functional that we have just identified to an element of $\Omega(V)^p$ that is not geometric, then the resulting operator is no longer a path in the underlying Lie group, but an element of the enveloping algebra. In other words, the natural solution to an Itô equation is not a randomly evolving flow on the manifold, but rather an evolving differential operator. Only the use of a connection can bring it back to a flow.

Iterated integrals for solutions to linear equations.[6] We have established that the Itô functional associated to a linear differential equation can be extended to a continuous multiplicative function from $\Omega G(V)^p$ in a unique way. But our solution was a flow, or a path π_t in the algebra of linear homomorphisms of W to itself. By evaluating it against a single vector w we get the solution $y_t = \pi_t w$ which starts at w. At least in the linear case it would seem that all is complete. But this is not really the case. The point is that we would like the

[6] The remarks in this section are far more significant than the reader might appreciate on first inspection.

solutions to our equation to be of the same class as the driving signal. It is obvious from our estimates that the solution y_t is a path in W of finite p-variation. But we have seen that such paths are not the correct objects with which to drive differential equations, we also require the iterated integrals of low degree.

For smooth driving paths x_t we can obviously construct all the iterated integrals of y_t and the joint iterated integrals of x_t with y_t. This defines a map from $S(V)$ into $\Omega G(V \oplus W)^p$. The question we aim to answer in this section is the following: can we extend that definition to one valid for any path in $\Omega G(V)^p$, or even to any path in $\Omega(V)^p$? We only have a general answer in the former case which we now explain. (Understanding how to make the extension to $\Omega(V)^p$ is the key to generalising Itô's type of differential equation to rougher paths).

Consider the equation (2.97) driven by a piecewise smooth path. The solution is again piecewise smooth, moreover the series solution converges locally uniformly at the level of derivatives. Therefore we have the expression for the iterated integrals of y

$$
\begin{aligned}
\boldsymbol{Y}_{s,t}^i &= \iint\limits_{s<u_1<\cdots<u_i<t} dy_{u_1} \cdots dy_{u_i} \\
&= \iint\limits_{s<u_1<\cdots<u_i<t} \sum_{l_1=1}^{\infty} A^{l_1}(d\boldsymbol{X}_{s,u_1}^{l_1}) \cdots \sum_{l_1=1}^{\infty} A^{l_i}(d\boldsymbol{X}_{s,u_i}^{l_i})\ y_s^{\otimes i}\ .
\end{aligned} \tag{2.104}
$$

Providing we can justify changing the order of summation of the series we have the alternative expression

$$
\sum_{S=r}^{\infty} \sum_{\substack{l_1+\cdots+l_i=S \\ l_j \geq 1}} A^{l_1} \otimes \cdots \otimes A^{l_i} \iint\limits_{s<u_1<\cdots<u_i<t} d\boldsymbol{X}_{s,u_1}^{l_1} \cdots d\boldsymbol{X}_{s,u_i}^{l_i}\, y_s^{\otimes i} \tag{2.105}
$$

where

$$
A^{l_1} \otimes \cdots \otimes A^{l_i} : V^{\otimes(l_1+\cdots+l_i)} \longrightarrow \hom\left(W^{\otimes i}, W^{\otimes i}\right) \tag{2.106}
$$

is the obvious induced map.

To obtain the absolute convergence of the series, and the continuous extension of the map to $\Omega G(V)^p$, we must look a bit more closely at the expression for

$$\iint_{s<u_1<\cdots<u_i<t} d\boldsymbol{X}^{l_1}_{s,u_1} \cdots d\boldsymbol{X}^{l_i}_{s,u_i} \; . \tag{2.107}$$

At this point we exploit in a critical way the fact that we are dealing with iterated integrals of the classical kind and are not working with abstract multiplicative functionals. Now

$$\begin{aligned} &\iint_{s<u_1<\cdots<u_i<t} d\boldsymbol{X}^{l_1}_{s,u_1} \cdots d\boldsymbol{X}^{l_i}_{s,u_i} \\ &\qquad = \iint_{\substack{s<u_1<\cdots<u_i<t \\ s<u_{1,1}<\cdots<u_{l_1}=u_{1,l_1} \\ \cdots}} dx_{u_{1,1}} \cdots dx_{u_{i,l_i}} \; , \end{aligned} \tag{2.108}$$

and the domain of integration in this second expression can be partitioned into disjoint simplexes. Given a sequence of distinct real numbers $u_{1,1} = v_1, \ldots, u_{i,l_i} = v_S$ let π be the unique rearrangement of $1, \ldots, S$ so that $v_{\pi j}$ are monotone decreasing. More generally, consider the set of all rearrangements Π_l of $1, \ldots, S$ that arise as one reorders sequences $u_{1,1}, \ldots, u_{i,l_1}$ satisfying $s < u_1 < \cdots < u_i < t$, $s < u_{1,1} < \cdots < u_1 = u_{1,l_1}$, etc. until $s < u_{i,1} < \cdots < u_i = u_{i,l_i}$. These are in one to one correspondence with the number of ways to partition $1, \ldots, S$ into exactly i components. The correspondence with $(l_1, \ldots, l_i)$ is achieved by ordering the components according to their last surviving element, (the component that becomes extinct first is the first component etc.) and putting l_j equal to the number of elements in the j-th component. Each element $\pi \in \Pi_l$ induces a linear map of $V^{\otimes(l_1+\cdots+l_i)}$ to itself, and this map π^* is an isometry. Because the domain of integration is the sum of the disjoint simplexes associated with the rearrangements, and the integral is the sum of the integrals over these disjoint domains, we have

$$\iint_{s<u_1<\cdots<u_i<t} d\boldsymbol{X}^{l_1}_{s,u_1} \cdots d\boldsymbol{X}^{l_i}_{s,u_1} = \sum_{\pi\in\Pi_l} \pi^* \boldsymbol{X}^S_{s,t} \; . \tag{2.109}$$

As we will see, this expression is easy to estimate, and we can readily conclude that the expression (2.105) converges absolutely. So for smooth paths we have the identity

$$\boldsymbol{Y}^i_{s,t} = \sum_{S=i}^{\infty} \sum_{\substack{l_1+\cdots+l_i=S \\ l_j \geq 1}} A^{l_1} \otimes \cdots \otimes A^{l_i} \sum_{\pi \in \Pi_l} \pi^* \boldsymbol{X}^S_{s,t}\, y_s^{\otimes i} \,, \tag{2.110}$$

which has the considerable attraction that the right hand side involves x_t only through it's associated multiplicative functional and is essentially a function on the infinite tensor algebra.

However, *this expression should carry a government health warning.* Certainly, the right hand side is (as we shall see) defined for any multiplicative functional in $\Omega(V)^p$ and is a continuous function on that space. For piecewise smooth paths, it defines a multiplicative functional because it coincides with the iterated integrals of the piecewise smooth path y_t, using the continuity of the map it also defines a multiplicative functional for any element of $\Omega G(V)^p$; indeed that path is geometric.

It is therefore tempting to assume the expression has a natural interpretation for any multiplicative functional in $\Omega(V)^p$, but this is a mistake. The result will not be multiplicative, and so fails the most basic property we expect of iterated integrals, and their substitutes in the rougher case. The point is that the expression on the right in (2.110) is the unique *linear* function yielding the desired value on group-like elements in the tensor algebra. However, although the functions on smooth paths obtained by taking iterated integrals are linearly independent (when regarded as elements of the space of functions on the space of smooth paths), they are certainly not algebraically independent. There are many different algebraic expressions that agree on the sequences of iterated integrals corresponding to geometric multiplicative paths.

Observe that (2.110) defines a multiplicative map from the group-like elements in the tensor algebra of infinite degree into an associative algebra. Arguing formally, we may differentiate to induce a Lie map from the Lie elements of the tensor algebra into the associative algebra. Again arguing formally, the tensor algebra is the enveloping algebra of this embedded Lie algebra, and so exploiting the universal property of enveloping algebras, there should exist a unique multiplicative extension of the Lie map to the full tensor algebra.

If there is a unique extension of (2.110) to a continuous and multiplicative map from $T(W)$ it will not be linear. It's construction would allow us to give a unified treatment of differential equations of Itô and Stratonovich type. We would then be confident that there was good sense in extending the Itô functional beyond geometric paths, and allowing any multiplicative functional in $\Omega(V)^p$ to be the driving signal.

At the time of writing, we believe we understand the correct approach to the identification of such an extension in an analytically useful form. (In the piecewise smooth case each iterated integral of y_t solves a differential equation over x_t, and we may compute the Lie algebra associated to it. In fact this Lie algebra is always finite dimensional. Therefore, after a non-linear change of co-ordinates, we may express the iterated integral as a Taylor series as we have mapped out earlier. By computing these changes of co-ordinates the new expression would be multiplicative for all $\boldsymbol{X}_{s,t}$ in $\Omega(V)^p$); confirmation and explicit determination of the formulae one obtains requires calculations we have not carried through and must wait for a later paper.

Theorem 2.4.1. *The series* (2.110) *and* (2.105) *converge absolutely for any multiplicative functional* $\boldsymbol{X}_{s,t}$ *in* $\Omega(V)^p$ *and define continuous functions. The resulting sequence* $\mathbf{Y}_{s,t} = \{\mathbf{Y}^i_{s,t}\}^n_{i=0}$ *is of finite p-variation and*

$$\|\mathbf{Y}^i_{s,t}\| \leq K^i \,\frac{i^i}{i!}\, \frac{\omega(s,t)^{i/p}}{\beta\,(i/p)!} \sum_{S=0}^{\infty} K^S\, i^S\, \frac{\omega(s,t)^{S/p}}{(S/p)!}\, \|y_s\|^i \,. \tag{2.111}$$

If $\boldsymbol{X}_{s,t}$ *in* $\Omega G(V)^p$ *is multiplicative, then* $\mathbf{Y}_{s,t} = \{\mathbf{Y}^i_{s,t}\}^n_{i=0}$ *is multiplicative, and we have the asymptotically improved bound*

$$\|\mathbf{Y}^i_{s,t}\| \leq \frac{(U_p\, \omega(s,t))^{i/p}}{\beta\,(i/p)!}\,. \tag{2.112}$$

Proof. Let $K = \|A\|$ be the operator norm of A regarded as a linear map $A : V \longrightarrow \hom(W, W)$. The number of partitionings of an ordered set of S elements into exactly i non-empty subsets is bounded above by

$$\frac{i^S}{i!} \tag{2.113}$$

and so

$$\|\mathbf{Y}^i_{s,t}\| \leq \sum_{S=i}^{\infty} K^S\, \frac{i^S}{i!}\, \|\boldsymbol{X}^S_{s,t}\|\, \|y_s\|^i \,. \tag{2.114}$$

If $\boldsymbol{X}_{s,t}$ is in $\Omega(V)^p$ and has variation controlled by ω then one has the estimate

$$
\begin{aligned}
\|\boldsymbol{Y}^i_{s,t}\| &\le \sum_{S=i}^{\infty} K^S \frac{i^S}{i!} \frac{\omega(s,t)^{S/p}}{\beta\,(S/p)!} \|y_s\|^i \\
&= K^i \frac{i^i}{i!} \frac{\omega(s,t)^{i/p}}{\beta\,(i/p)!} \sum_{S=0}^{\infty} K^S i^S \frac{\omega(s,t)^{S/p}\,(i/p)!}{((S+i)/p)!} \|y_s\|^i \\
&\le K^i \frac{i^i}{i!} \frac{\omega(s,t)^{i/p}}{\beta\,(i/p)!} \sum_{S=0}^{\infty} K^S i^S \frac{\omega(s,t)^{S/p}}{(S/p)!} \|y_s\|^i ,
\end{aligned}
\tag{2.115}
$$

showing the series converges absolutely and bounding the individual terms in a way that makes it clear that $\boldsymbol{Y}^{(n)}_{s,t}$ has finite p-variation controlled by a multiple of ω on any interval where ω is bounded. A virtually identical argument shows the uniform continuity of the sequence under variation of $\boldsymbol{X}_{s,t}$. However, the constants in these estimates explode with the degree.

On the positive side, the continuity ensures that if $\boldsymbol{X}_{s,t}$ is in $\Omega G(V)^p$ then $\boldsymbol{Y}^{(n)}_{s,t}$ is multiplicative; our results in Section 2.2 and particularly Theorem 2.2.1 then give the much stronger and more useful estimate that for $n > p$, ω bounded by L, and with

$$
U_p = \max_{j \le [p]} \|y_s\|^j K^p \frac{j^p}{(j!)^{p/j}} \Big(\sum_{S=0}^{\infty} \frac{K^S j^S L^{S/p}}{(S/p)!} \Big)^{p/j} , \tag{2.116}
$$

choosing β large enough, we have

$$
\|\boldsymbol{Y}^i_{s,t}\| \le \frac{(U_p\,\omega(s,t))^{i/p}}{\beta\,(i/p)!} \tag{2.117}
$$

completing the proof of the theorem.

Cross terms. We have therefore seen that for linear equations the Itô functional can be extended in a unique continuous way as a map from $\Omega G(V)^p$ to $\Omega G(W)^p$. However, for technical reasons that will become apparent later, we would like also to know that the iterated integrals between solution and driving noise also exist. This is readily done by extending the original differential equation, in other words we solve the equation

$$
\left\{
\begin{aligned}
dc_t &= 0\, c_t\, dx_t \ , \\
d\hat{x}_t &= c_t\, dx_t \ , \\
dy_t &= A(y)\, dx_t \ ,
\end{aligned}
\right.
\tag{2.118}
$$

with $c_0 = 0$, $\hat{x}_0 = x_0$. The equation is still linear and so we can use the approach above to construct the iterated integrals of $\hat{x}$ and y and see that they have unique continuous extension to $\Omega G(V)^p$.

In this way we see that if we wish to record the full structure associated to our differential equation *we should regard the Itô map as an extension map lifting paths in $\Omega G(V)^p$ to paths in $\Omega G(V \oplus W)^p$.*

2.4.2. The stochastic example.

What do the results we have proved so far say in the context of Brownian motion and stochastic differential equations?

Suppose that $X_t \in V$ is a continuous path in Euclidean space, chosen randomly according to Wiener measure (in which case we say it is a Brownian path) or more generally according to some measure which makes the underlying stochastic process a martingale or semimartingale (when we say X_t is a martingale or semimartingale path). Then it is standard [11] that, with probability one, the forward and symmetric Riemann sums

$$\begin{aligned} \boldsymbol{X}^{2,\mathrm{ito}}_{s,t} &= \lim_{n\to\infty} \sum_{s<k/2^n}^{k/2^n<t} X_{k/2^n} \otimes \left(X_{k/2^n} - X_{(k+1)/2^n}\right), \\ \boldsymbol{X}^{2,\mathrm{strat}}_{s,t} &= \lim_{n\to\infty} \sum_{s<k/2^n}^{k/2^n<t} \frac{X_{k/2^n} + X_{(k+1)/2^n}}{2} \otimes \left(X_{k/2^n} - X_{(k+1)/2^n}\right), \end{aligned} \tag{2.119}$$

converge uniformly in the time co-ordinates and define two distinct multiplicative functionals

$$\begin{aligned} \boldsymbol{X}^{\mathrm{ito}}_{s,t} &= \left(1, \boldsymbol{X}_s - \boldsymbol{X}_t, \boldsymbol{X}^{2,\mathrm{ito}}_{s,t}\right), \\ \boldsymbol{X}^{\mathrm{strat}}_{s,t} &= \left(1, \boldsymbol{X}_s - \boldsymbol{X}_t, \boldsymbol{X}^{2,\mathrm{strat}}_{s,t}\right), \end{aligned} \tag{2.120}$$

corresponding to the Itô and Stratonovich integrals. A simple Borel-Cantelli lemma shows that with probability one they are both in $\Omega(V)^p$ for every $p > 2$. The two multiplicative functionals agree in degree one, so their difference is an additive function with values in two tensors. It is referred to by probabilists as the *quadratic variation process*

$$\langle X, X\rangle_{s,t} = \frac{1}{2}\lim_{n\to\infty}\sum_{s<k/2^n}^{k/2^n<t}\left(X_{k/2^n} - X_{(k+1)/2^n}\right) \otimes \left(X_{k/2^n} - X_{(k+1)/2^n}\right), \tag{2.121}$$

it has finite variation with probability one. Exploiting the Itô and Stratonovich integrals further, one may construct higher order iterated integrals. These sequences $\boldsymbol{X}_{s,t}^{\text{ito}}$ and $\boldsymbol{X}_{s,t}^{\text{strat}}$ define multiplicative functionals of finite p-variation and arbitrarily high degree.

By our theorems these higher iterated integrals etc. are continuous functions of the path and its second iterated integral. *The difference between the Itô and Stratonovich equations driven by Brownian motion depends entirely on the choice of multiplicative functional of degree two that we use to extend Brownian motion.*

To understand clearly the possibilities and choices made in extending our Brownian path to a multiplicative functional of degree two and finite p-variation where $2 < p < 3$, we must look more carefully at the symmetric and anti-symmetric components of $\boldsymbol{X}_{st}^2$.

Decomposing the second integral - the area or anti-symmetric part. In our discussion of the iterated integrals of a smooth path, we saw that the symmetric part of the classical second iterated integral of a smooth path is

$$\frac{1}{2}\left(X_t - X_s\right) \otimes \left(X_t - X_s\right) \tag{2.122}$$

and as this is a continuous function in the uniform topology this relation will hold true for any geometric path. (One readily checks that for the Stratonovich integral the symmetric component of the second integral is precisely this continuous extension.)

To create a geometric multiplicative functional of degree two it is therefore sufficient to construct the anti-symmetric two tensor process, and to be multiplicative this must satisfy the algebraic relationship

$$\boldsymbol{A}_{s,u} = \boldsymbol{A}_{s,t} + \boldsymbol{A}_{t,u} + \operatorname{Area}\left(\overline{X_s\, X_t\, X_u}\right), \tag{2.123}$$

where

$$\operatorname{Area}\left(\overline{PQR}\right), \tag{2.124}$$

is the area of the triangle interpolating the three points P, Q, R. (Observe that the *area* associated to a loop formed by taking a chord and the trajectory of the path along the time interval $[s,t]$ would obviously satisfy this relationship).

Definition 2.4.1. *We call an anti-symmetric two tensor process satisfying* (2.123) *an area process relative to the path* X_s.

Suppose $2 < p < 3$, then for any path of finite p-variation in V, and associated area process $\boldsymbol{A}_{s,t}$ (having the correct modulus of continuity) the multiplicative functional

$$\left(1, \boldsymbol{X}_t - \boldsymbol{X}_t, \frac{1}{2}\left(\boldsymbol{X}_t - \boldsymbol{X}_s\right) \otimes \left(\boldsymbol{X}_t - \boldsymbol{X}_s\right) + \boldsymbol{A}_{s,t}\right) \tag{2.125}$$

defines a *geometric* multiplicative functional in $\Omega G(V)^2$. The geometric condition does not imply any sort of uniqueness or canonical choice for the the area process given the underlying path, this is in contrast to the unique continuous choice for symmetric component. Even if X_t is smooth, there are many elements of $\Omega G(V)^2$ lying over the path. Consider the multiplicative functional $\boldsymbol{Y}_{s,t} = (1, 0, \psi(t) - \psi(s))$ constructed by taking the limit of the increments and second integrals of the smooth paths $\exp\left(n^2 \pi\, i\, \psi(t)\right)/(n\pi)$. The result is geometric, non-trivial, and for smooth enough ψ will be in $\Omega G(\mathbb{R}^2)^2$, however it projects to the constant path.

The key, then, to defining stochastic differential equations is the choice of this area integral. It really is a choice even in the Brownian case, the work [25] demonstrates just how tenuous the connection between Lévy area and geometric area of smooth paths really is.

The Itô and Stratonovich second iterated integrals only differ in the symmetric bracket process, they share a common area process - the *Lévy Area.* The Stratonovich multiplicative functional is geometric.

Theorem 2.4.2. *Let* X_t *be a semi-martingale and* $\boldsymbol{A}_{s,t}$ *be its Lévy area. The linear stochastic differential equation*

$$dy_t = A(y_t, dX_t) + B(y_t)\, dt\,, \tag{2.126}$$

where $x \longrightarrow A(\ ,x)$ *in* $\hom(V, \hom(W,W))$*, and* $B(\,)$ *in* $\hom(W,W)$*, are bounded operators which can be regarded as the composition of a continuous function on* $\Omega(V,\mathbb{R})^{2+\varepsilon,1}$ *and the random multiplicative functional*

$$\left(1, \boldsymbol{X}_t - \boldsymbol{X}_s, \frac{1}{2}\left(\boldsymbol{X}_t - \boldsymbol{X}_s\right) \otimes \left(\boldsymbol{X}_t - \boldsymbol{X}_s\right) + \boldsymbol{A}_{s,t}\right). \tag{2.127}$$

In particular, all equations can be solved simultaneously with only a single null set. The equations can be chosen to depend on the path, end point of the solution etc.

Proof. There is little to say. The driving signal is (X_t, t), so that if we consider the inhomogeneous p-variation introduced in (2.3.1), (The cross-iterated integrals against t are all canonically defined) we deduce that the differential equation can be extended from the class of smooth paths in an unique way to $\Omega G(V, \mathbb{R})^{2+\varepsilon,1}$. The multiplicative functional (2.127), with probability one, takes its values in $\Omega G(V, \mathbb{R})^{2+\varepsilon,1}$ [23]. We claim that this construction obtained by taking the composition of the two maps coincides with the Stratonovich solution which probabilists construct.

Fortunately, the very continuity of the map from $\Omega G(V, \mathbb{R})^{2+\varepsilon,1}$ ensures this. It is well known that one may solve a Stratonovich differential equation in probability, by replacing the semimartingale path by its dyadic piecewise linear approximations, and then taking the solutions to the equation driven by these piecewise linear equations [11].

On the other hand, our definition of the Lévy area makes it clear that it is the limit of the areas associated to these piecewise linear paths, a Borel-Cantelli argument ([8], Sipiläinen) shows that the rate of convergence is fast enough for the piecewise linear paths, and their iterated integrals to converge in $\Omega G(V, \mathbb{R})^{2+\varepsilon,1}$. By our continuity results, we see that our solution and the conventional probabilistic one agree with probability one.

Finally observe that our solution is obtained by composing a deterministic function depending on the coefficients of the equation with a random multiplicative functional constructed almost surely, but with a null set that is independent of the coefficients of the equation. In particular we may solve all such equations simultaneously and can choose the equation so as to depend on the path without difficulty of interpretation. No predictability condition is involved.

Remarks 2.4.1. Generalizing the equation. We will in due course prove that we can develop continuity results in the fully nonlinear situation where the vector fields in the differential equation are $\mathrm{Lip}\,(2+\varepsilon, V)$ so the remarks above apply in much greater generality than the linear case proven so far.

Remarks 2.4.2. Generalizing the noise. There are a number of directions in which one could generalize the noise. One should certainly consider jumps; in general these are a little easier, because pure jump random processes tend to have finite variation for $p < 2$. The area integral does not come into the picture [31]. In another direction, one could look at other Markov processes as driving processes for dynamical systems. Here, matters still seem relatively open, except that one can say there are wide classes of Markov processes which extend, like Brownian motion, to admit Lévy area processes, and hence Stratonovich differential equations; but which are definitely not semi-martingales and cannot be attacked via the standard Itô theory.

In these situations where the usual theory simply does not apply [10] an alternative approach is required to construct the Lévy area. Now, Lévy proved, if one takes the piecewise linear approximation to the path X_s that agrees at 2^n equally spaced points and look at the sequence of areas as one refines the dyadic partitions. Then if X_s is Brownian motion, *this sequence forms a martingale* over the filtration obtained by revealing X_s at the 2^n equally spaced points. In many other situations, one can still show that it is a convergent semi-martingale. The classical martingale techniques are still important - but not the time ordered filtration.

The symmetric part of $\mathbf{X}^2_{s,t}$ and Itô Equations. By now a persistent reader might understand enough to guess that constructing different second order multiplicative extensions to Brownian motion is essentially equivalent to varying our notion of solution to our stochastic differential equation. Even so we are at least superficially surprised that the distinction between Itô and Stratonovich second integrals is not in the discontinuous Lévy area, but in the symmetric part; the part which has a natural continuous choice for all continuous paths!

The difference between the Itô and Stratonovich approaches lie in the symmetric additive functional known as the quadratic variation or bracket process.

The earlier results about iterated integrals apply to the Itô equation and it is easy to write down series solutions etc. but now those series involve the bracket process. To apply our approach we express our equations in co-ordinate invariant form. An Itô equation

$$dy_t = f^i(y_t)\,dx^i_t + f^0(y_t)\,dt\,, \tag{2.128}$$

always requires a connection before it makes good sense, but can then be rewritten in the Stratonovich form

$$dy_t = f^i(y_t)\,dx^i_t + f^0(y_t)\,dt + \nabla_{f^i} f^j\,d\langle X, X\rangle^{i,j}_{0,t} \tag{2.129}$$

and one can deduce all the theorems one had before, but now the dependence includes the bracket process separately.

Theorem 2.4.3. *Consider the linear Itô stochastic differential equation* (2.126)

$$dy_t = A(y_t, dx_t) + B(y_t)\,dt\,, \tag{2.130}$$

where $x \longrightarrow A(\ ,x)$ *in* $\hom(V, \hom(W,W))$, *and* $B(\ )$ *in* $\hom(W,W)$, *are bounded operators. This map can be regarded as the composition of a continuous function on* $\Omega(V, V\hat{\otimes}V, \mathbb{R})^{2+\varepsilon,1,1}$ *and a random multiplicative functional depending only on the path, its Lévy area, and its bracket process. In particular, all equations can be solved simultaneously with only a single null set. The equations can be chosen to depend on the path, end point of the solution etc.*

In particular, this perspective suggests that for robust numerical solution of stochastic differential equations, one should not try to implicitly simulate the bracket process locally as the quadratic variation of the path, (as one does when one solves an Itô equation directly using Euler type methods) but treat it separately as a known quantity and go via Stratonovich methods. We think this is definitely true in some cases, although it is not the whole story, and a complete understanding of differential equations driven by (non-geometric) multiplicative functionals will be required to give a better answer.

3. Integration against a rough path.

In this section we move from the linear/real analytic setting to the truly non-linear/rough setting. Our objective is to define the integral of a rough path against a one form.

3.1. Almost multiplicative functionals - The construction of an integral.

We have shown in Section 2.2 that if $\boldsymbol{X}_{st} \in T^{(n)}$ is p-multiplicative (where we will use the convention $n = [p]$) then it extends in a unique way to a multiplicative functional $\boldsymbol{X}_{st}$ of finite p -variation in $T^{(m)}$ for all $m \geq n$ and if $\boldsymbol{X}_{st}$ is controlled by ω in $T^{(n)}$ so that

$$\|\boldsymbol{X}_{st}^{j}\| \leq \frac{\omega(s,t)^{j/p}}{\beta\,(j/p)!}\,, \qquad \text{for all } j \leq n\,, \tag{3.1}$$

then one has the same estimate for all $j < \infty$. (Here β is an appropriately chosen constant depending only on p). Similar estimates reflect the continuity of this extension map.

We will now explain how, with some loss of quantitative control, this result can be seen as a special case of a more general one concerning almost multiplicative functionals.

Definition 3.1.1. *Suppose $\boldsymbol{X}_{st}$ is any functional taking values in $T^{(n)}$, we say it is of finite p-variation controlled by ω if, for all s,t,*

$$\|\boldsymbol{X}_{st}^{j}\| \leq \frac{\omega(s,t)^{j/p}}{\beta\,(j/p)!}\,, \qquad \textit{for all } j \leq n\,, \tag{3.2}$$

In addition we say that such an $\boldsymbol{X}_{st}$ is an almost multiplicative functional if for any compact interval J there is a θ and a K such that for all s,t and u in J we have

$$\|(\boldsymbol{X}_{st}\,\boldsymbol{X}_{tu} - \boldsymbol{X}_{su})^{j}\| \leq K\,\omega(s,u)^{\theta}\,, \qquad \textit{for all } j \leq n,\ \theta > 1\,. \tag{3.3}$$

OBSERVATIONS 3.1.1. We have already seen an almost multiplicative functional. The lift $\hat{\boldsymbol{X}}_{st} = (1, \boldsymbol{X}_{st}^{1}, \ldots, \boldsymbol{X}_{st}^{n}, \mathbf{0})$ defined in the proof of Theorem 2.2.1 is an almost multiplicative functional controlled by ω providing $\boldsymbol{X}_{s,t} = (1, \ldots, \boldsymbol{X}_{st}^{n})$ is a multiplicative functional of finite p-variation where $n \geq [p]$. We see therefore that (ignoring the quality of the estimates) Theorem 2.2.1 is a special case of the following.

Theorem 3.3.1. *Suppose $\boldsymbol{X}_{st}$ is a bounded almost multiplicative functional controlled by ω on the compact interval J of degree n. Then there exists a unique multiplicative functional $\hat{\boldsymbol{X}}_{st}$ on J and a constant*

$$C\,(L, K, \theta, \max_{s,t\in J} \omega(s,t), n)\,,$$

such that

$$\|(\hat{\boldsymbol{X}}_{st} - \boldsymbol{X}_{st})^{i}\| \leq C\,(L, K, \theta, \max_{s,t\in J} \omega(s,t), n)\,\omega(s,t)^{\theta}\,, \tag{3.4}$$

for all $i \leq n$. *There is at most one multiplicative functional* $\hat{\boldsymbol{X}}_{st}$ *that can satisfy* (3.4) *regardless of the choice of* C. *Here* θ, K *and* ω *are the terms in the definition of almost multiplicative and* L *is the uniform bound on the components of* $\boldsymbol{X}_{st}$.

Corollary 3.1.1. *In addition, if* $\boldsymbol{X}_{st}$ *has finite p-variation controlled by* ω *then* $\hat{X}$ *has p-variation controlled by* $C_1\,\omega$ *where* C_1 *only depends on* $K, \theta,$ $\max\{\omega(s,t),\ s,t \in J\}$ *and* n.

Proof of the theorem. We proceed by induction and suppose the projection of $\boldsymbol{X}_{st}$ into $T^{(j)}$ has the multiplicative property. Presuming for a moment existence of the limit, define $\tilde{X}$ as follows

$$(\tilde{\boldsymbol{X}}_{st})^{j+1} = \lim_{\operatorname{mesh}(D)\to 0} (\boldsymbol{X}_{st_1}\boldsymbol{X}_{t_1t_2}\cdots\boldsymbol{X}_{t_{r-1}t})^{j+1} \tag{3.5}$$

and for all $i \neq j+1$ take $(\tilde{\boldsymbol{X}}_{st})^i = (\boldsymbol{X}_{st})^i$. In this case it is clear that $\tilde{X}$ will be multiplicative on $T^{(j+1)}$. If we show the existence of $(\tilde{\boldsymbol{X}}_{st})^{j+1}$, establish that $\tilde{X}$ is almost multiplicative, and compare it with $\boldsymbol{X}_{st}$ we will have established the induction step. Iterating it completes the proof.

We proceed in a similar way to before. Let

$$\boldsymbol{X}^D_{s,t} = \boldsymbol{X}_{s,t_1}\boldsymbol{X}_{t_1,t_2}\cdots\boldsymbol{X}_{t_{r-1},t}\ ,$$

where $D = \{s, t_1, \ldots, t_{r-1}, t\}$ is a dissection of $[s,t]$. First we bound

$$(\boldsymbol{X}^D_{st} - \boldsymbol{X}_{st})^{j+1}\,, \tag{3.6}$$

independently of the choice of dissection D, and then we will show the convergence of the products as the mesh size of the dissections tends to zero, always providing ω is regular. Observe first that in the case where the dissection is trivial, $r = 2$, the difference in (3.6) is zero. Assume the dissection is nontrivial, and choose an interior point t_i of the dissection D so that

$$\omega(t_{i-1}, t_{i+1}) \leq \frac{2}{(r-2)}\,\omega(s,t)$$

or equals $\omega(s,t)$ in the case where $r = 3$. Let $D' = D - \{t_i\}$. If we estimate $((\boldsymbol{X}^D_{st} - \boldsymbol{X}^{D'}_{st}))^{j+1}$ and the similar terms as we successively remove all the interior points of the dissection, we may use the triangle inequality to estimate (3.6); we will obtain a bound, which in analogy with our previous arguments, is easily seen to be dissection independent. Now

$$\begin{aligned}(3.7)\qquad &((\boldsymbol{X}^D_{st} - \boldsymbol{X}^{D'}_{st}))^{j+1}\\ &\quad = (\boldsymbol{X}^{D\cap[s,t_{i-1}]}_{s,t_{i-1}}(\boldsymbol{X}_{t_{i-1},t_i}\boldsymbol{X}_{t_i,t_{i+1}}\\ (3.8)\qquad &\qquad - \boldsymbol{X}_{t_{i-1},t_{i+1}})\boldsymbol{X}^{D\cap[t_{i+1},t]}_{t_{i+1},t})^{j+1}\end{aligned}$$

and the multiplicative nature of $\boldsymbol{X}$ ensures that

$$(3.9)\qquad \boldsymbol{X}_{t_{i-1}t_i}\boldsymbol{X}_{t_it_{i+1}} - \boldsymbol{X}_{t_it_{i+1}} = (\underbrace{0,\dots,0}_{j+1 \text{ terms}}, \boldsymbol{R}^{j+1}_{t_{i-1},t_i,t_{i+1}},\dots)$$

and so

$$(3.10)\qquad ((\boldsymbol{X}^D_{st} - \boldsymbol{X}^{D'}_{st}))^{j+1} = \boldsymbol{R}^{j+1}_{t_{i-1},t_i,t_{i+1}}\,.$$

But the almost multiplicative property then gives the estimate

$$(3.11)\qquad \|\boldsymbol{R}^{j+1}_{t_{i-1},t_i,t_{i+1}}\| \le K\,\omega(t_{i-1},t_{i+1})^\theta \le K\Big(\frac{2}{r-2}\Big)^\theta\,\omega(s,t)^\theta\,,$$

for $r > 3$ and the similar estimate for $r = 3$. Summing these error estimates as one drops points from the dissection leads to the, by now, familiar estimate

$$(3.12)\qquad \|(\boldsymbol{X}^D_{s,t} - \boldsymbol{X}_{s,t})^{j+1}\| \le K\,(2^\theta(\zeta(\theta)-1)+1)\,\omega(s,t)^\theta\,,$$

and the consequential argument that if ω is regular, then the X^D converge as the mesh size of the dissection goes to zero. In particular we may define

$$(\tilde{\boldsymbol{X}}_{s,t})^{j+1} = \lim_{\mathrm{mesh}(D)\to 0}(\boldsymbol{X}^D_{st})^{j+1}\,.$$

It follows that if

$$(3.13)\qquad \boldsymbol{R}^{j+1}_{st} = (\tilde{\boldsymbol{X}}_{st} - \boldsymbol{X}_{st})^{j+1}\,,$$

then

$$(3.14)\qquad \|\boldsymbol{R}^{j+1}_{st}\| \le K\,(2^\theta(\zeta(\theta)-1)+1)\,\omega(s,t)^\theta\,.$$

To see that $\tilde{\boldsymbol{X}}$ is almost multiplicative, observe that

$$(3.15)\qquad (\tilde{\boldsymbol{X}}_{st}\,\tilde{\boldsymbol{X}}_{tu}-\tilde{\boldsymbol{X}}_{su})^i=\begin{cases}0\,, & i\le j+1\,,\\ (\boldsymbol{X}_{st}\boldsymbol{X}_{tu}-\boldsymbol{X}_{su})^i & \\ +\boldsymbol{R}_{st}^{j+1}\,\boldsymbol{X}_{tu}^{i-(j+1)} & \\ +\boldsymbol{X}_{st}^{i-(j+1)}\,\boldsymbol{R}_{tu}^{j+1}\,, & i>j+1\,.\end{cases}$$

and providing $\|\boldsymbol{X}_{st}^i\|\le L$ for all $i\le n$, $s,t\in J$ we have the estimate

$$(3.16)\qquad \begin{aligned}\|(\tilde{\boldsymbol{X}}_{st}\tilde{\boldsymbol{X}}_{tu}-\tilde{\boldsymbol{X}}_{su})^i\| &\le K\,\omega(s,u)^\theta\\ &\qquad +2\,L\,K(2^\theta(\varsigma(\theta)-1)+1)\,\omega(s,u)^\theta\,,\end{aligned}$$

completing the proof that $\tilde{\boldsymbol{X}}$ is almost multiplicative, but with the new constant

$$(3.17)\qquad \tilde{K}\le K\,(1+2\,L\,(2^\theta(\xi(\theta)-1)+1))\,,$$

and a new uniform bound

$$(3.18)\qquad \tilde{L}\le L+K\,(2^\theta(\zeta(\theta)-1)+1)\,\max_{s,t\in J}\,\omega(s,t)^\theta\,.$$

As $\tilde{\boldsymbol{X}}$ is also almost multiplicative controlled by a multiple of ω and bounded on J this completes the basic induction step. Observing that the theorem is trivial if $j=0$ and repeating the step n times completes the construction of the multiplicative functional.

To see uniqueness of the functional, it is enough to show that if one has two multiplicative functionals $\widehat{\boldsymbol{X}}_{st}$, $\tilde{\tilde{\boldsymbol{X}}}_{st}$ and they satisfy

$$(3.19)\qquad \|(\widehat{\boldsymbol{X}}_{st}-\tilde{\tilde{\boldsymbol{X}}}_{st})^i\|\le C\,\omega(s,t)^\theta\,,\qquad \text{for all } s,t\in J,\ i\le n\,,$$

then they agree for all $i\le n$. The proof is also an induction argument, fix the smallest i for which the two multiplicative functionals differ. Then putting

$$\psi(s,t)=(\widehat{\boldsymbol{X}}_{st}-\tilde{\tilde{\boldsymbol{X}}}_{st})^i\,,$$

one obtains from the multiplicative property that

$$\psi(s,u)=\psi(s,t)+\psi(t,u)$$

and hence for any dissection one has the estimate that

$$\psi(s,t) \le (\max_D \omega(t_i,t_{i+1}))^{\theta-1}\,\omega(s,t)\,.$$

For regular ω on an interval J this forces $\psi(s,t) \equiv 0$ contradicting the induction hypothesis.

The theorem and its proof only require a boundedness assumption on $\boldsymbol{X}$ and regularity assumption on ω.

PROOF OF THE COROLLARY. Suppose now that $\boldsymbol{X}$ is of finite p-variation controlled by ω on $T^{(n)}$, where $n/p \le \theta$. Then it is a simple application of the triang le inequality to see that $\hat{\boldsymbol{X}}$ is also of finite p-variation on $T^{(n)}$. If $n > p$ then one may repeat the uniqueness induction argument we have just given to deduce that the new multiplicative functional we have constructed in this theorem agrees with the unique multiplicative extension of finite p-variation we constructed in Theorem 2.2.1.

3.3.1. Applications and extensions.

A) The map from p-almost multiplicative functional to p-multiplicative functional is a uniformly continuous one. However, this is not a consequence of the result so much as of the proof. Suppose that $\boldsymbol{X}$, $\boldsymbol{Y}$ are two almost multiplicative functionals controlled by the same K, ω, θ. And suppose that they are close to each other in the sense that

$$\|(\boldsymbol{X}_{s,t} - \boldsymbol{Y}_{s,t})^i\| < \varepsilon\,\omega(s,t)^{i/p}\,, \qquad i \le [p]\,, \tag{3.20}$$

then of course by the triangle inequality

$$\|(\hat{\boldsymbol{X}}_{s,t} - \hat{\boldsymbol{Y}}_{s,t})^i\| < \varepsilon\,\omega(s,t)^{i/p} + C\,\omega(s,t)^{\theta} \tag{3.21}$$

and for $i \le [p]$ this looks adequate. But for $\varepsilon < C\,\omega(s,t)^{\theta-[p]/p}$ or less seriously $\omega(s,t) \gg 1$ the estimate deteriorates. The key to the proof of a continuity result is to observe that at each stage in the construction of the multiplicative functional out of the almost multiplicative functional, we can control the difference between the two approximations. We then obtain the following theorem.

Theorem 3.1.2. *Suppose that $\boldsymbol{X}$, $\boldsymbol{Y}$ are two almost multiplicative functionals controlled by the same K, ω, θ, and that $\omega(s,t) < L$ for $s, t \in J$. Suppose further that $\boldsymbol{X}, \boldsymbol{Y}$ are close in the p-variation sense so that*

$$\|(\boldsymbol{X}_{s,t} - \boldsymbol{Y}_{s,t})^i\| < \varepsilon\, \omega(s,t)^{i/p}\,, \qquad i \leq [p]\,, \tag{3.22}$$

then there is a continuous, increasing function $\delta(\varepsilon)$ depending only on K, L, θ, p and satisfying $\delta(0) = 0$ so that the associated multiplicative functionals satisfy

$$\|(\hat{\boldsymbol{X}}_{s,t} - \hat{\boldsymbol{Y}}_{s,t})^i\| < \delta(\varepsilon)\, \frac{\omega(s,t)^{i/p}}{(i/p)!}\,, \tag{3.23}$$

for all i.

Proof. Because of Theorem 3.1.1, it is sufficient that we deal with the case $i \leq [p]$.

Suppose $\boldsymbol{X}_{st}$, $\boldsymbol{Y}_{st}$ are almost multiplicative and multiplicative up to degree $j < [p]$; and that they satisfy the hypotheses of the theorem. Define $\tilde{\boldsymbol{X}}$, $\tilde{\boldsymbol{Y}}$ by

$$(\tilde{\boldsymbol{X}}_{st})^{j+1} = \lim_{\mathrm{mesh}(D)\to 0} (\boldsymbol{X}_{st_1} \boldsymbol{X}_{t_1 t_2} \cdots \boldsymbol{X}_{t_{r-1} t})^{j+1} \tag{3.24}$$

and for all $i \neq j+1$ take $(\tilde{\boldsymbol{X}}_{st})^i = (\boldsymbol{X}_{st})^i$, and similarly for $\tilde{\boldsymbol{Y}}$. We will show that these are close in the sense of the conclusion. Repeating the argument the required finite number of times, the result will follow.

Define $\boldsymbol{X}^D_{s,t} = \boldsymbol{X}_{s,t_1} \boldsymbol{X}_{t_1,t_2} \cdots \boldsymbol{X}_{t_{r-1},t}$, etc. where

$$D = \{s, t_1, \ldots, t_{r-1}, t\}\,.$$

We will estimate $\|(\tilde{\boldsymbol{X}}_{s,t} - \tilde{\boldsymbol{Y}}_{s,t})^{j+1}\|$ by controlling $(\boldsymbol{X}^D_{st} - \boldsymbol{Y}^D_{st})^{j+1}$ in a uniform way and passing to the limit. Now as before we may successively drop points from the dissection. By making a careful choice of the point to drop (but note that the choice depends on ω alone, and can be common for both functionals) we have the following two estimates; because of the almost multiplicativeness we have

$$\|(\boldsymbol{X}^D_{st} - \boldsymbol{X}^{D'}_{st})^{j+1}\| \leq K \left(\frac{2}{r-2}\right)^{\theta} \omega(s,t)^{\theta}\,. \tag{3.25}$$

for $r > 3$ and the similar estimate for $r = 3$. Combining the estimates for $\boldsymbol{Y}$ we have

$$\text{(3.26)}\quad \|((\boldsymbol{X}_{st}^{D} - \boldsymbol{Y}_{st}^{D}) - (\boldsymbol{X}_{st}^{D'} - \boldsymbol{Y}_{st}^{D'}))^{j+1}\| \le 2\,K \left(\frac{2}{r-2}\right)^{\theta} \omega(s,t)^{\theta}\,,$$

while using the closeness hypothesis, (and some crude version of the neo-classical inequality) one obtains

$$\text{(3.27)}\quad \begin{aligned} &\|((\boldsymbol{X}_{st}^{D} - \boldsymbol{Y}_{st}^{D}) - (\boldsymbol{X}_{st}^{D'} - \boldsymbol{Y}_{st}^{D'}))^{j+1}\| \\ &\qquad \le A(p)\,(\varepsilon + \varepsilon^2) \left(\frac{1}{r-2}\right)^{(j+1)/p} \omega(s,t)^{(j+1)/p}\,, \end{aligned}$$

so combining the two and using the uniform bound that $\omega(s,t) < L$ one has

$$\text{(3.28)}\quad \begin{aligned} &\|((\boldsymbol{X}_{st}^{D} - \boldsymbol{Y}_{st}^{D}) - (\boldsymbol{X}_{st}^{D'} - \boldsymbol{Y}_{st}^{D'}))^{j+1}\| \\ &\qquad \le \Bigg(A(p)\,(\varepsilon + \varepsilon^2) \left(\frac{1}{r-2}\right)^{(j+1)/p} \\ &\qquad\qquad \wedge\, 2\,K\,L^{\theta-(j+1)/p} \left(\frac{2}{r-2}\right)^{\theta}\Bigg)\, \omega(s,t)^{(j+1)/p} \end{aligned}$$

and summing this over r yields the required uniform estimate.

B) As a second simple, but rather important corollary of Theorem 3.1.1, we see that it is possible to vary one multiplicative functional in the direction of a second. In particular, suppose that $\boldsymbol{X}_{st}$ is a multiplicative functional of finite p-variation controlled by ω and that $\boldsymbol{H}_{st}$ is a second; and suppose further that $\|(\boldsymbol{H}_{st})^j\| < K(\omega(s,t))^{\phi}$ for all $j \le [p]$ and for some $\phi > 1 - 1/p$. In this case, the neo-classical inequality shows $\boldsymbol{H}_{st}\boldsymbol{X}_{st}$ to be of finite p-variation an d more elementary considerations show it to be almost multiplicative. Moreover $\|(\boldsymbol{H}_{st}\boldsymbol{X}_{st} - \boldsymbol{X}_{st}\boldsymbol{H}_{st})^j\| < K(\omega(s,t))^{\phi+1/p}$, and so Theorem 3.1.1 shows that the multiplicative functional associated to the left or right hand perturbations of $\boldsymbol{X}_{st}$ coincide. We denote this modification by $\boldsymbol{X}_{s,t}^{H}$. Although we do not have time in this paper to pursue the matter, it will be useful if we want to differentiate functionals on path space.

3.2. Integrating a one-form - A most important almost multiplicative functional.

Our intention is to solve equations of the type

$$dY = f(Y)\, dX\,, \qquad Y_0 = a\,, \tag{3.29}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are multiplicative functionals and where $Y_s = \boldsymbol{Y}_{0s} + Y_0$. We wish to adopt an approach based on Picard iteration, in other words we treat our equation as an integral equation and construct a solution by iterating the function F

$$F(\boldsymbol{Y})_t = a + \int_0^t f(Y_s)\, d\boldsymbol{X}_s\,. \tag{3.30}$$

Although such an approach is almost universal, it is apparently unnatural from a geometric perspective. Every term in our differential equation is meaningful without a choice of co-ordinates for the space where Y takes its values and one would hope that the solution had the same properties. However, the functional in (3.30) certainly involves a choice of co-ordinate chart, and different choices produce different maps F.

To succeed in our Picard iteration we now follow up these two separate but closely related points. We must make sense of the concept of an integral, and we must understand its behaviour under changes of variable.

3.2.1. Integrating a one form.

We will now prove that a one form can be integrated against a multiplicative functional in a natural way. We do this via the construction of an almost multiplicative functional. The reader should be warned that our methods are currently limited, and in general we can only treat *geometric* multiplicative functionals of finite p-variation. However, for the case where the paths have p-variation satisfying $p < 3$ (and so degree is $n \leq 2$) then the next section will extend these results to all multiplicative functionals. This improvement in the case $n = 2$ is important because it allows one to treat the Itô approach to differential equations in common with the Stratonovich approach. We believe our failure to extend the result to all multiplicative functionals in the general case reflects a lack of understanding on our part, inspection and guess work allow one to treat $n = 2$ but do not point to the general picture.

But before explaining the analysis, for the sake of precision, we need some simple notation.

Definition 3.2.1. *We say that a multiplicative functional $\boldsymbol{X}_{s,t} \in T^{(n)}$ lies above a path $X_t \in V$ if $\boldsymbol{X}^1_{s,t} = X_t - X_s$.*

It is clear that there always is such a path under any multiplicative functional and that it is unique, once we have determined its value at a single time. In what follows we will use the notation (vector font, normal font) to express this relationship without further mention.

Main Lemma. Notation. A W-valued 1-form ϑ on V is a function on V whose value at any point is a linear homomorphism from V to W, that is $\vartheta(v) \in \hom(V, W)$. Suppose that ϑ is smooth enough that one can differentiate it. Denote by

$$\begin{cases} \vartheta^1 = \vartheta\,, & \vartheta^1(v) \in \hom(V, W)\,, \\ \vartheta^2 = d\vartheta\,, & \vartheta^2(v) \in \hom(V, \hom(V, W)) \\ & \qquad \cong \hom(V \otimes V, W)\,, \\ \vartheta^k = d\vartheta^{k-1}\,, & \vartheta^k(v) \in \hom(\bigotimes_1^k V, W)\,. \end{cases} \tag{3.31}$$

Now, the multilinear map $\vartheta^k(v)$ is not symmetric in all its coefficients - and so one must have some convention on the order in which they appear. We adopt the convention that $\vartheta^k(v)\,(v_1, v_2, \dots, v_k)$ is defined so that for smooth paths and conventional integrals

$$\begin{aligned} \int_{s<u<t} & \vartheta^k(x_u)\,(dx_u, v_2, \dots, v_k) \\ & = \vartheta^{k-1}(x_t)\,(v_2, \dots, v_k) - \vartheta^{k-1}(x_s)\,(v_2, \dots, v_k)\,. \end{aligned} \tag{3.32}$$

Recall that ϑ is a Lip$(\gamma - 1)$ one form with norm at most M providing that for $1 \le j < \gamma$ one has the Taylor series style expression

$$\begin{aligned} \vartheta^j(x_t)\,(v_1, v_2, \dots, v_j) = & \sum_{0 \le i < \gamma - j} \vartheta^{j+i}(x_0)\,(x^i_{0,t}, v_1, v_2, \dots, v_j) \\ & + R^j(x_0, x_t)\,(v_1, v_2, \dots, v_j)\,, \end{aligned} \tag{3.33}$$

where $\vartheta^i(x)$ and $R^i(x,y)$ are bounded in operator norm on

$$\hom\Big(\bigotimes_1^i V, W\Big)$$

with the controls

$$\begin{aligned} &\|\vartheta^i(x)\| \le M\,, \\ &\|R^i(x,y)\| \le M\|x-y\|^{\gamma-i}\,. \end{aligned} \tag{3.34}$$

As we noted (1.2.2), the remainder only depends on x_0, and x_t, and not on the intermediate smooth path segment. Exploiting this point, and taking a limit, we see that the identity (3.33) and estimate (3.34) hold for any sequence $(1, \boldsymbol{x}_t = \boldsymbol{x}^i_{0,t}, \ldots, \boldsymbol{x}^i_{0,t})$ arising from a geometric multiplicative functional.

We are now in a position to define the crucial almost multiplicative functional which will give us the integral. We start with a definition which is understandable for smooth paths, and then transform it in a combinatorial way so that it is clear that the functional is the restriction of a uniformly continuous function defined on all paths in $\Omega(V)^p$. This extension is easily seen to define an almost multiplicative functional when evaluated on $\Omega G(V)^p$ and this completes the definition/theorem. As a warning, this functional (which is linear) definitely does not give an almost multiplicative functional for a general element of $\Omega(V)^p$.

Predefinition 3.2.1. *For ϑ a* Lip$(\gamma-1)$ *one form with values in W, and $\boldsymbol{X}_{s,t}$ a geometric multiplicative functional of finite p-variation* (*obtained by taking a sequence of iterated integrals of a smooth path*), *define*

$$\begin{aligned} \boldsymbol{Y}^i_{s,t} = \iint\limits_{s<u_1<\cdots<u_i<t} &\sum_{l_1=1}^{[p]} \vartheta^{l_1}(X_s)\,(d\boldsymbol{X}^{l_1}_{s,u_1}) \\ &\cdots \sum_{l_i=1}^{[p]} \vartheta^{l_i}(X_s)\,(d\boldsymbol{X}^{l_i}_{s,u_i})\,. \end{aligned} \tag{3.35}$$

Because the $\vartheta^{l_i}(X_s)$ are constants we may equate the expression with

$$\begin{aligned} \boldsymbol{Y}^i_{s,t} = \sum_{l_1,\ldots,l_i=1}^{[p]} &\vartheta^{l_1}(X_s) \otimes \cdots \otimes \vartheta^{l_i}(X_s) \\ &\cdot \iint\limits_{s<u_1<\cdots<u_i<t} d\boldsymbol{X}^{l_1}_{s,u_1} \cdots d\boldsymbol{X}^{l_i}_{s,u_i}\,. \end{aligned} \tag{3.36}$$

Focus attention on $\iint_{s<u_1<\cdots<u_i<t} d\boldsymbol{X}^{l_1}_{s,u_1}\cdots d\boldsymbol{X}^{l_i}_{s,u_i}$. *For our smooth path one has*

$$\begin{aligned}\iint\limits_{s<u_1<\cdots<u_i<t} & d\boldsymbol{X}^{l_1}_{s,u_1}\cdots d\boldsymbol{X}^{l_i}_{s,u_i} \\ &= \iint\limits_{V} dX_{u_{1,1}}\cdots dX_{u_{1,l_1}}\cdots dX_{u_{i,1}}\cdots dX_{u_{i,l_i}}\ ,\end{aligned} \tag{3.37}$$

where the domain of integration V *is given by*

$$\begin{aligned} V = s < u_1 < \cdots < u_i < t\,, \\ s < u_{1,1} < \cdots < u_{1,l_1} = u_1\ , \\ \vdots \\ s < u_{i,1} < \cdots < u_{i,l_i} = u_i\ . \end{aligned} \tag{3.38}$$

But this domain of integration is a product of simplexes and can be represented as a union of disjoint simplexes obtained by shuffling. Fix $\underline{l} = (l_1,\ldots,l_i)$ *and let* $\underline{u} = (u_{1,1},\ldots,u_{1,l_1},\ldots,u_{i,1},\ldots,u_{i,l_i})$ *be any distinct sequence satisfying the constraints of* (3.38). *Let* $\pi_{\underline{u}}$ *denote the permutation that would reorder the numbers* $\underline{u}$ *to be increasing, and let* $\Pi_{\underline{l}}$ *denote the range of this function as a subset of the group* $\Sigma_{\|\underline{l}\|}$ *of permutations of* $\|\underline{l}\|$ *elements where* $\|\underline{l}\| = \sum_{j=1,\ldots,i} l_j$. *We can expand our integral as a sum*

$$\begin{aligned}\iint\limits_{s<u_1<\cdots<u_i<t} & d\boldsymbol{X}^{l_1}_{s,u_1}\cdots d\boldsymbol{X}^{l_i}_{s,u_i} \\ &= \sum_{\pi\in\Pi_{\underline{l}}}\ \iint\limits_{s<v_1<\cdots<v_{\|\underline{l}\|}<t} dX_{v_{\pi(1)}}\cdots dX_{v_{\pi(\|\underline{l}\|)}}\ .\end{aligned} \tag{3.39}$$

Now the group Σ_n *acts on* $\bigotimes_n V$ *in the obvious way taking* $(v_1,\ldots,v_n)$ *to* $(v_{\pi(1)},\ldots,v_{\pi(n)})$. *It follows that*

$$\begin{aligned}\iint\limits_{s<u_1<\cdots<u_i<t} & d\boldsymbol{X}^{l_1}_{s,u_1}\cdots d\boldsymbol{X}^{l_i}_{s,u_i} \\ &= \sum_{\pi\in\Pi_{\underline{l}}} \pi\Big(\iint\limits_{s<v_1<\cdots<v_{\|l\|}<t} dX_{v_1}\cdots dX_{v_{\|\underline{l}\|}}\Big) \\ &= \sum_{\pi\in\Pi_{\underline{l}}} \pi(\boldsymbol{X}^{\|\underline{l}\|}_{s,t})\end{aligned} \tag{3.40}$$

and so finally we have reduced the integral to an expression involving only the multiplicative functional. Regarding this calculation as motivation, we give our formal definition for $\boldsymbol{Y}_{s,t}$.

Definition 3.2.2. *For any multiplicative functional $\boldsymbol{X}_{s,t}$ in $\Omega G(V)^p$ define,*

$$\boldsymbol{Y}^i_{s,t} = \sum_{l_1,\ldots,l_i=1}^{[p]} \vartheta^{l_1}(X_s) \otimes \cdots \otimes \vartheta^{l_i}(X_s) \sum_{\pi \in \Pi_{\underline{l}}} \pi(\boldsymbol{X}^{\|\underline{l}\|}_{s,t}) . \tag{3.41}$$

Theorem 3.2.1. *For any multiplicative functional $\boldsymbol{X}_{s,t}$ in $\Omega G(V)^p$ and any one-form $\theta \in \mathrm{Lip}[\gamma - 1, \{X_u, u \in [s,t]\}]$ with $\gamma > p$ the sequence $\boldsymbol{Y}_{s,t} = (1, \boldsymbol{Y}^1_{s,t}, \ldots, \boldsymbol{Y}^{[p]}_{s,t})$ defined above is almost multiplicative and of finite p-variation; if $\boldsymbol{X}_{s,t}$ is controlled by ω on J where ω is bounded by L, and the $\mathrm{Lip}[\gamma - 1]$ norm of θ is bounded by M, then the almost multiplicative and p-variation properties of $\boldsymbol{Y}$ are controlled by multiples of ω which depend only on γ, p, L, M.*

PROOF. Note that we also have the trivial estimate based on the size of the permutation group that

$$\begin{aligned} \left\| \iint\limits_{s<u_1<\cdots<u_i<t} d\boldsymbol{X}^{l_1}_{s,u_1} \cdots d\boldsymbol{X}^{l_i}_{s,u_i} \right\| &< |\Pi_{\underline{l}}| \frac{\omega(s,t)^{\|\underline{l}\|/p}}{\beta\left(\|l\|/p\right)!} \\ &< \|\underline{l}\|! \frac{\omega(s,t)^{\|\underline{l}\|/p}}{\beta\left(\|\underline{l}\|/p\right)!} . \end{aligned} \tag{3.42}$$

We must now prove that $\boldsymbol{Y}_{s,t}$ is almost multiplicative when restricted to $\Omega G(V)^p$. For motivation of our calculations we again start by formally regarding our multiplicative functional as a sequence of iterated integrals.

$$\begin{aligned} \boldsymbol{Y}^i_{s,u} &= \iint\limits_{s<u_1<\cdots<u_i<u} \sum_{l_1=1}^{[p]} \vartheta^{l_1}(X_s)\,(d\boldsymbol{X}^{l_1}_{s,u_1}) \cdots \sum_{l_i=1}^{[p]} \vartheta^{l_i}(X_s)\,(d\boldsymbol{X}^{l_i}_{s,u_i}) \\ &= \sum_{r=1,\ldots,i} \iint\limits_{t<u_{r+1}<\cdots<u_i<u} \Bigg(\iint\limits_{s<u_1<\cdots<u_r<t} \sum_{l_1=1}^{[p]} \vartheta^{l_1}(X_s)\,(d\boldsymbol{X}^{l_1}_{s,u_1}) \\ &\qquad \cdots \sum_{l_r=1}^{[p]} \vartheta^{l_r}(X_s)\,(d\boldsymbol{X}^{l_r}_{s,u_r}) \Bigg) \end{aligned}$$

$$
\begin{aligned}
&\otimes \sum_{l_{r+1}=1}^{[p]} \vartheta^{l_{r+1}}(X_s)\,(d\boldsymbol{X}^{l_{r+1}}_{s,u_{r+1}}) \\
&\qquad\cdots \sum_{l_i=1}^{[p]} \vartheta^{l_i}(X_s)\,(d\boldsymbol{X}^{l_i}_{s,u_i}) \\
= \Bigl(\boldsymbol{Y}_{s,t} \otimes \Bigl(\iint_{t<u_1<\cdots<u_j<u} &\sum_{l_1=1}^{[p]} \vartheta^{l_1}(X_s)\,(d\boldsymbol{X}^{l_1}_{s,u_1}) \\
&\qquad\cdots \sum_{l_j=1}^{[p]} \vartheta^{l_j}(X_s)\,(d\boldsymbol{X}^{l_j}_{s,u_j})\Bigr)_{j=0}^{[p]}\Bigr)^{i}
\end{aligned}
\tag{3.43}
$$

This expression looks close to our target, but we must move the reference point in the second half of the expression from the time point s to the time point t. This follows from the Taylor type expression. Consider the terms $\vartheta^l(X_s)\, d\boldsymbol{X}^l_{s,u}$ where $u > t$. Then again by linearity of tensor multiplication one gets $d\boldsymbol{X}^l_{s,u} = (\boldsymbol{X}_{s,t} \otimes d\boldsymbol{X}_{t,u})^l$ and so

$$
\begin{aligned}
\sum_{l=1}^{[p]} \vartheta^l(X_s)\,(d\boldsymbol{X}^l_{s,u_1}) &= \sum_{l=1}^{[p]} \sum_{i=0}^{l-1 \text{ or } l} \vartheta^l(X_s)\,(\boldsymbol{X}^i_{s,t} \otimes d\boldsymbol{X}^{l-i}_{t,u}) \\
&= \sum_{j=1}^{[p]} \sum_{i=0}^{[p]-j} \vartheta^{i+j}(X_s)\,(\boldsymbol{X}^i_{s,t} \otimes d\boldsymbol{X}^j_{t,u}) \qquad (3.44) \\
&= \sum_{j=1}^{[p]} \vartheta^j(X_t)\,(d\boldsymbol{X}^j_{t,u}) + \sum_{j=1}^{[p]} R^j(X_0, X_t)\,(d\boldsymbol{X}^j_{t,u})
\end{aligned}
$$

[1] and so we have

$$
\begin{aligned}
\boldsymbol{Y}_{s,u} &= \boldsymbol{Y}_{s,t} \otimes \boldsymbol{Y}_{t,u} \\
&\qquad + \boldsymbol{Y}_{s,t} \otimes \Bigl(\sum_{l_1,\ldots,l_i=1}^{[p]} \Bigl(\sum_{\beta} \beta^{l_1}(X_s, X_t) \otimes \cdots \otimes \beta^{l_i}(X_s, X_t)
\end{aligned}
$$

---

[1] It is exactly at this point that one is assuming the derivatives of the one form contract with the iterated integrals to produce a result that depends on the path chosen only through it's initial and terminal values. In other words, the iterated integrals differ from those of the chord $X_t - X_0$ by an element in the enveloping algebra.

(3.45)

$$\cdot \iint_{t<u_1<\cdots<u_i<u} d\boldsymbol{X}^{l_1}_{t,u_1} \cdots d\boldsymbol{X}^{l_i}_{t,u_i} \Big)\Big)_{i=0}^{i=[p]}$$

where the sum is over all sequences β where

$$\beta^l \in \{\vartheta^l(X_s), R^l(X_0, X_t)\} \tag{3.46}$$

and where for each l, one has, for at least one of l_j, that $\beta^l = R^l(X_0, X_t)$. It is then an easy matter to estimate the size of this term and see that the functional is almost multiplicative.

One has that

$$\begin{aligned}
&\Big\| \sum_{l_1,\ldots,l_i=1}^{[p]} \Big(\sum_{\beta} \beta^{l_1}(X_s, X_t) \otimes \cdots \otimes \beta^{l_i}(X_s, X_t) \\
&\qquad\qquad \cdot \iint_{t<u_1<\cdots<u_i<u} d\boldsymbol{X}^{l_1}_{t,u_1} \cdots d\boldsymbol{X}^{l_i}_{t,u_i} \Big) \Big\| \\
&\le M^i \sum_{l_1,\ldots,l_i=1}^{[p]} (2^i - 1)\, |\Pi_{\underline{l}}|\, (1 + |X_s - X_t|^{\gamma-1})^{i-1}\, |X_s - X_t|^{\gamma-1} \\
&\qquad\qquad \cdot \frac{\omega(t,u)^{(l_1+\cdots+l+\cdots l_i)/p}}{\beta\,(|\underline{l}|/p)!} \\
&\le M^i\, |X_s - X_t|^{\gamma-1}\, \omega(t,u)^{(l+i-1)/p} \\
&\quad \cdot \sum_{l_1,\ldots,l_i=1}^{[p]} (2^i - 1)\, |\Pi_{\underline{l}}|\, (1 + |X_s - X_t|^{\gamma-1})^{i-1}\, (1 + \omega(t,u)^{(\gamma-1)/p})^{i-1}
\end{aligned}$$

where the passage from the first to second expression is based on the estimate given above for the iterated integral of iterated integrals, counting the number of β, and by exploiting the inequality

$$\beta^{l_j}(\boldsymbol{X}_s, \boldsymbol{X}_t) < M\,(1 + \|\boldsymbol{X}_s - \boldsymbol{X}_t\|^{\gamma-1}) \tag{3.47}$$

in all but one of the terms in the product, in the latter one uses the fact that the remainder type term appears at least, once to be more precise

$$\le M^i \Big(\frac{\omega(t,u)^{1/p}}{(1/p)!} \Big) \omega(t,u)^{l+i-1/p} \Big(1 + \Big(\frac{\omega(t,u)^{1/p}}{(1/p)!} \Big)^{\gamma-1} \Big)^{i-1}$$

$$\cdot \left(1+\omega(t,u)^{(\gamma-1)/p}\right)^{i-1} \sum_{l_1,\dots,l_i=1}^{[p]} \frac{(2^i-1)\,|\Pi_{\underline{l}}|}{\beta\,(\|\underline{l}\|/p)!}$$

$$\le M^i \Big(\frac{\omega(t,u)^{1/p}}{(1/p)!}\Big)^{\gamma-l} \omega(t,u)^{l+i-1/p} \Big(1+\Big(\frac{\omega(t,u)^{1/p}}{(1/p)!}\Big)^{\gamma-1}\Big)^{i-1}$$

$$\cdot \left(1+\omega(t,u)^{(\gamma-1)/p}\right)^{i-1} \sum_{m=1}^{i[p]} \frac{(2^i-1)\,m!}{\beta\,(m/p)!}$$

$$\le \frac{(i\,[p]+1)!\,(2^i-1)}{\beta\,(1/p)!} M^i \Big(\frac{\omega(t,u)^{1/p}}{(1/p)!}\Big)^{\gamma-l} \omega(t,u)^{l+i-1/p}$$

$$\cdot \Big(1+\Big(\frac{\omega(t,u)^{1/p}}{(1/p)!}\Big)^{\gamma-1}\Big)^{i-1} \left(1+\omega(t,u)^{(\gamma-1)/p}\right)^{i-1}$$

$$\le K(p,\beta)\,M^i\,\omega(s,u)^{\gamma+i-1/p} \Big(1+\Big(\frac{\omega(t,u)}{(1/p)!}\Big)^{(\gamma-1)/p}\Big)^{2(i-1)}$$

and since $\gamma > p$ and $i \geq 1$ we have the estimate. The functional $\boldsymbol{Y}_{st}$ is almost multiplicative with power γ/p. It is interesting that the const ant grows so rapidly with the roughness of the path.

To finalize the argument, recall that we did some manipulations of $\boldsymbol{Y}_{st}$ where we used the representation of the terms in the iterated integral to motivate certain manipulations which were obvious for classical smooth integrals because of their general properties of linearity and additivity over disjoint simplexes. It is necessary to convince oneself that an integrated form of (3.44) holds when a geometric multiplicative functional is substituted for the iterated integrals of the smooth path. This is obvious for geometric multiplicative functionals because the algebraic identities clearly hold on a closed set containing the lifts of the smooth paths. By definition this includes $\Omega G(V)^p$.

As a consequence of this result we can define the integral of a 1-form.

Definition 3.2.3. *We say that $\hat{\boldsymbol{Y}}_{s,t}$ is the integral of the one form θ against X if $\hat{\boldsymbol{Y}}_{s,t}$ is the multiplicative functional associated to the almost multiplicative functional we defined above. In this case we write*

$$\hat{\boldsymbol{Y}}_{s,t} = \int_{s<u<t} \theta(X_u)\,\delta \boldsymbol{X}_u\,, \qquad (3.48)$$

$$\delta \hat{\boldsymbol{Y}} = \theta(X)\,\delta \boldsymbol{X}\,.$$

We now have an integral. We also have a change of variable formula.

Corollary 3.2.1. *Suppose f is a* Lip$[\gamma]$ *map from $V \longrightarrow U$ then it induces a natural map of $\Omega G(V)^p \longrightarrow \Omega G(U)^p$ providing $\gamma > p$.*

Proof. Apply the above theorem to the differential of f.

So just as semimartingales as a class are preserved by smooth maps, so is $\Omega G(V)^p$.

3.2.2. The two step case - p-variation less than 3.

The reader may be particularly interested in the special case which includes stochastic differential equations. For this reason we treat independently the case where one has a multiplicative functional of degree two, the more explicit approach developed here permits a stronger result. We show that it is possible to integrate any p-multiplicative functional against a Lip$[\gamma - 1]$ one form providing $\gamma > p$. Again our approach is to construct an almost multiplicative functional and although the result is almost contained in the previous section it seems worth the effort of doing the calculation explicitly in this important special case to identify the constants and (perhaps?) get a feel for how to generalise to the general case.

Even in this case there are many terms and the algebra is relatively complex. Mathematica was used by the author to keep track of some of the terms in the calculations.

Our basic idea can be summarized by saying we start with a multiplicative functional $\boldsymbol{X}$ of degree two which we think of as representing the integral and second iterated integral of a path X. We write down the obvious approximation to the integral and iterated integral of the integral of X against a 1-form. This is not multiplicative, but it is almost multiplicative. The unique multiplicative functional that is appropriately close is regarded as the integral of X against the 1-form.

Fix $2 \le p < 3$. Then $\boldsymbol{X}_{st}$ is a multiplicative functional on J with p-variation controlled by ω if

$$\|\boldsymbol{X}^1_{ts}\| \le \omega(t,s)^{1/p} \qquad \text{and} \qquad \|\boldsymbol{X}^2_{ts}\| \le \omega(t,s)^{2/p} .$$

Suppose that θ is a 1-form that is Lip$[\gamma]$ where $p - 1 < \gamma < 2$.

By Taylor's theorem

$$\left\|\theta(X_t) - \theta(X_s) - \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^1_{st})\right\| < M\,\omega(t,s)^{\gamma/p}\,. \tag{3.49}$$

So if we wish to approximate the iterated integrals of $\boldsymbol{Y}$ the "integral" of $\boldsymbol{X}$ against θ, it makes sense to consider

$$\boldsymbol{Y}_{st} = \left\{1, \theta(X_s)\,(\boldsymbol{X}^1_{st}) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{st}), \theta(X_s)\otimes\theta(X_s)(\boldsymbol{X}^2_{st})\right\}.$$

Clearly, $\boldsymbol{Y}_{s,t}$ has finite p-variation controlled by $2\,M\,\omega$. We will now establish the claim that it is also an almost multiplicative functional

$$\boldsymbol{Y}_{st} = \left\{1, \theta(X_s)\,(\boldsymbol{X}^1_{st}) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{st}), \theta(X_s)\otimes\theta(X_s)\,(\boldsymbol{X}^2_{st})\right\},$$

$$\boldsymbol{Y}_{tu} = \left\{1, \theta(X_t)\,(\boldsymbol{X}^1_{tu}) + \frac{1}{2}\,(d\theta)\,(X_t)\,(\boldsymbol{X}^2_{tu}), \theta(X_t)\otimes\theta(X_t)\,(\boldsymbol{X}^2_{tu})\right\},$$

$$\begin{aligned}\boldsymbol{Y}_{st}\otimes\boldsymbol{Y}_{tu} = \Big\{1, &\,\theta(X_s)\,(\boldsymbol{X}^1_{st}) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{st}) + \theta(X_t)\,(\boldsymbol{X}^1_{tu})\\ &+ \frac{1}{2}\,(d\theta)\,(X_t)\,(\boldsymbol{X}^2_{tu}),\\ &(\theta(X_s)\,(\boldsymbol{X}^1_{st}) + \frac{1}{2}\,(d\theta)\,(X_s)(\boldsymbol{X}^2_{st}))\\ &\otimes(\theta(X_t)\,(\boldsymbol{X}^1_{tu}) + \frac{1}{2}\,(d\theta)\,(X_t)\,(\boldsymbol{X}^2_{tu}))\\ &+\theta(X_s)\otimes\theta(X_s)\,(\boldsymbol{X}^2_{st}) + \theta(X_t)\otimes\theta(X_t)\,(\boldsymbol{X}^2_{tu})\Big\},\end{aligned} \tag{3.51}$$

$$\begin{aligned}\boldsymbol{Y}_{su} - \boldsymbol{Y}_{st}\otimes\boldsymbol{Y}_{tu} = \Big\{0, &\,\theta(X_s)\,(\boldsymbol{X}^1_{su}) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{su})\\ &- (\theta(X_s)\,(\boldsymbol{X}^1_{su}) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{st})\\ &+\theta(X_t)\,(\boldsymbol{X}^1_{tu}) + \frac{1}{2}\,(d\theta)\,(X_t)\,(\boldsymbol{X}^2_{tu}))\,,\\ &\theta(X_s)\otimes\theta(X_s)\,(\boldsymbol{X}^2_{su})\\ &-((\theta(X_s)\,(\boldsymbol{X}^1_{st}) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{st}))\\ &\otimes(\theta(X_s)\,(\boldsymbol{X}^1_{su}) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{su}))\\ &+\theta(X_s)\,(\boldsymbol{X}^1_{st}) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{st})\\ &+\theta(X_t)\,(\boldsymbol{X}^1_{tu}) + \frac{1}{2}\,(d\theta)\,(X_t)\,(\boldsymbol{X}^2_{tu}))\Big\},\end{aligned} \tag{3.52}$$

now recalling Taylor's theorem

$$\theta(X_t) = \theta(X_s) + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^1_{st}) + r_1(t,s)\,,$$
$$\|r_1(t,s)\| < M\,\omega(t,s)^{\gamma/p}$$

and

$$\begin{aligned} d\theta(X_t) &= d\theta(X_s) + r_2(t,s)\,,\\ \|r_2(t,s)\| &< M\,\omega(t,s)^{(\gamma-1)/p}\,. \end{aligned} \tag{3.53}$$

We use these approximations to estimate $\boldsymbol{Y}_{su} - \boldsymbol{Y}_{st} \otimes \boldsymbol{Y}_{tu}$. Substituting both approximations into the 1-tensor component of $\boldsymbol{Y}_{st} \otimes \boldsymbol{Y}_{tu}$, substituting only the first into the 2-tensor component, and expanding out each term in $\boldsymbol{X}^i_{s,u}$ within $\boldsymbol{Y}_{su}$ in terms of $\boldsymbol{X}^j_{st}, \boldsymbol{X}^j_{tu}$ using the multiplicative proper ty for $\boldsymbol{X}$ one has after a tedious calculation with many terms (or using Mathematica after a relatively complex set of manipulations) the three terms of different tensor degree in $\boldsymbol{Y}_{su} - \boldsymbol{Y}_{st} \otimes \boldsymbol{Y}_{tu}$ in increasing order of complexity.

The zero'th order term is clearly zero.

The first order term is $r_1(s,t)\boldsymbol{X}^1_{t,u} + r_2(s,t)\,\boldsymbol{X}^2_{t,u}$ and

$$\begin{aligned} &\|r_1(s,t)\,\boldsymbol{X}^1_{t,u} + r_2(s,t)\boldsymbol{X}^2_{t,u}\|\\ &\qquad \le M\,(\omega(s,t)^{\gamma/p}\,\omega(t,u)^{1/p} + \omega(s,t)^{(\gamma-1)/p}\,\omega(t,u)^{2/p})\\ &\qquad \le 2\,M\,\omega(s,u)^{(\gamma+1)/p}\,, \end{aligned} \tag{3.54}$$

giving the required estimate.

The second order term breaks naturally (if somewhat painfully) into a sum of 15 terms, which under our assumptions are of 5 different magnitudes.

$$\begin{aligned} &(\theta(X_s) \otimes d\theta(X_s))\,(\boldsymbol{X}^1_{s,t} \otimes \boldsymbol{X}^2_{t,u})\\ &+ (\theta(X_s) \otimes d\theta(X_t))\,(\boldsymbol{X}^1_{s,t} \otimes \boldsymbol{X}^2_{t,u})\\ &+ (d\theta(X_s) \otimes \theta(X_s))\,(\boldsymbol{X}^1_{s,t} \otimes \boldsymbol{X}^2_{t,u} + \boldsymbol{X}^2_{s,t} \otimes \boldsymbol{X}^1_{t,u})\\ &+ (\theta(X_s) \otimes d\theta(X_s))\,(\boldsymbol{X}^1_{s,t} \otimes \boldsymbol{X}^1_{s,t} \otimes \boldsymbol{X}^1_{t,u})\\ &+ (\theta(X_s) \otimes r_1(s,t))\,(\boldsymbol{X}^2_{t,u} + \boldsymbol{X}^1_{t,u} \otimes \boldsymbol{X}^1_{s,t}) \end{aligned} \tag{3.55}$$

$$+\,(r_1(s,t)\otimes\theta(X_s))\,(\boldsymbol{X}^2_{t,u}) \tag{3.56}$$

$$\begin{aligned}&+\,(d\theta(X_s)\otimes d\theta(X_t))\,(\boldsymbol{X}^2_{s,t}\otimes\boldsymbol{X}^2_{t,u})\\&+\,(d\theta(X_s)\otimes d\theta(X_s))\,(\boldsymbol{X}^1_{s,t}\otimes\boldsymbol{X}^1_{s,t}\otimes\boldsymbol{X}^2_{t,u})\\&+\,(d\theta(X_s)\otimes d\theta(X_s))\,(\boldsymbol{X}^1_{s,t}\otimes\boldsymbol{X}^2_{s,t}\otimes\boldsymbol{X}^1_{t,u})\end{aligned} \tag{3.57}$$

$$\begin{aligned}&+\,(d\theta(X_s)\otimes r_1(s,t))\,(\boldsymbol{X}^1_{t,u}\otimes\boldsymbol{X}^2_{s,t})\\&+\,(d\theta(X_s)\otimes r_1(s,t))\,(\boldsymbol{X}^1_{s,t}\otimes\boldsymbol{X}^2_{t,u})\\&+\,(r_1(s,t)\otimes d\theta(X_s))\,(\boldsymbol{X}^1_{s,t}\otimes\boldsymbol{X}^2_{t,u})\end{aligned} \tag{3.58}$$

$$+\,(r_1(s,t)\otimes r_1(s,t))\,(\boldsymbol{X}^2_{t,u}) \tag{3.59}$$

and so one has that the norm of the expression above is less than

$$\begin{gathered}M^2\,(5\,\omega(s,u)^{3/p}+3\,\omega(s,u)^{(\gamma+2)/p}+3\,\omega(s,u)^{4/p}\\+\,3\,\omega(s,u)^{(\gamma+3)/p}+\omega(s,u)^{(2\gamma+2)/p})\end{gathered} \tag{3.60}$$

and providing $\omega(s,u)<1$ we have the simpler bound

$$15\,M^2\,\omega(s,u)^{3/p}\,,\qquad \omega(s,u)<1\,. \tag{3.61}$$

Recalling our assumption that θ is a 1-form that is Lip$[\gamma]$ where $p-1<\gamma<2$ we see that both errors are controlled to a degree greater than one in ω. This leads us to conclude that Y is an almost multiplicative functional. Our approach used the multiplicative property, but never required the geometric property of $\boldsymbol{X}$. As above we define the integral

$$\int\limits_{s<u<t}\theta(X_u)\,\delta\boldsymbol{X}\,, \tag{3.62}$$

to be the associated multiplicative functional.

3.2.3. Continuity of the integral.

It is an immediate corollary of our results so far, that the integral

$$\int_{s<u<t} \theta(X_u)\,\delta\boldsymbol{X} \tag{3.63}$$

is a continuous map from (geometric) multiplicative functionals and $\mathrm{Lip}[\gamma-1]$ one forms to p-multiplicative functionals. Since it is clear that the integral of a smooth path produces a geometric functional, it follows from the continuity of the map that the integral against any element of $\Omega G(V)^p$ produces a multiplicative functional in $\Omega G(W)^p \subset \Omega(W)^p$.

In more detail, the almost multiplicative functional associated with a geometric functional

$$\boldsymbol{Y}^i_{s,t} = \sum_{l_1,\dots,l_i=1}^{[p]} \vartheta^{l_1}(X_s)\otimes\cdots\otimes\vartheta^{l_i}(X_s)\sum_{\pi\in\Pi_{\underline{l}}}\pi(\boldsymbol{X}^{\|\underline{l}\|}_{s,t}) \tag{3.64}$$

is clearly continuous in the sense that if $\boldsymbol{X},\boldsymbol{X}'$, are multiplicative functionals controlled by ω and satisfying

$$\|(\boldsymbol{X}_{s,t}-\boldsymbol{X}'_{s,t})^i\| < \varepsilon\,\frac{\omega(s,t)}{\beta\,(i/p)!}\ , \tag{3.65}$$

moreover the $\|X_u - X'_u\| \le \varepsilon$, and the finitely many functions $x \longrightarrow \vartheta^{l_1}(x)\otimes\cdots\otimes\vartheta^{l_i}(x)$ have a uniform modulus of continuity $\sigma(\varepsilon,M,p)$, so one has the estimate on the almost multiplicative functionals

$$\begin{aligned}
&\|\boldsymbol{Y}^i_{s,t} - \boldsymbol{Y}'^{i}_{s,t}\| \\
&\qquad \le \Big(\sum_{l_1,\dots,l_i=1}^{[p]} (\varepsilon\, M^{\|\underline{l}\|}\,|\Pi_{\underline{l}}| \\
&\qquad\qquad + \sigma(\varepsilon,M,p))\,\frac{\omega(s,t)^{|l|/p-i/p}}{\beta\,(|l|/p)!}\Big)\omega(s,t)^{i/p}
\end{aligned} \tag{3.66}$$

and so we can apply the continuity theorem for the construction of a multiplicative functional from an almost multiplicative functional to deduce the following

Theorem 3.2.2. *If $\boldsymbol{X}, \boldsymbol{X}'$, are geometric multiplicative functionals of finite p-variation controlled by ω with $\omega(s,t) < L$ for $s, t \in J$, and θ is a one form with a* $\mathrm{Lip}[\gamma - 1]$ *norm at most M then there is a function $\delta(\varepsilon, L, M, p)$ continuous and zero if $\varepsilon = 0$ such that if*

$$\|(\boldsymbol{X}_{s,t} - \boldsymbol{X}'_{s,t})^i\| < \varepsilon \frac{\omega(s,t)^{i/p}}{\beta\,(i/p)!}\,, \qquad i \leq [p] \tag{3.67}$$

and

$$\|X_u - X'_u\| \leq \varepsilon\,, \tag{3.68}$$

then for $i \leq [p]$

$$\begin{aligned}\Big\| \Big(\int\limits_{s<u<t} \theta(X_u)\,\delta\boldsymbol{X} - \int\limits_{s<u<t} \theta(X'_u)\,\delta\boldsymbol{X}' \Big)^i \Big\| \\ < \delta(\varepsilon, L, M, p)\,\frac{\omega(s,t)^i}{(i/p)!}\,.\end{aligned} \tag{3.69}$$

Similar estimates apply to the variation of the one form.

Corollary 3.2.2. *If $\boldsymbol{X} \in \Omega G(V)^p$ then*

$$\int\limits_{s<u<t} \theta(X_u)\,\delta\boldsymbol{X} \qquad \textit{is in } \Omega G(W)^p \subset \Omega(W)^p\,. \tag{3.70}$$

Continuity in the case $p < 3$ for non-geometric functionals. In the situation where $p < 3$ we have the alternative description of our almost multiplicative functional valid for any $\boldsymbol{X} \in \Omega(V)^p$, and we may check the continuity directly in this case as well. We explicitly compute the changes to the almost multiplicative functional

$$\begin{aligned}\boldsymbol{Y}_{st} = \Big\{ 1, \theta(X_s)\,(\boldsymbol{X}^1_{st}) + \frac{1}{2}\,(d\theta)\,(\boldsymbol{X}_s)\,(\boldsymbol{X}^2_{st}), \\ \theta(X_s) \otimes \theta(X_s)\,(\boldsymbol{X}^2_{st}) \Big\}\,.\end{aligned} \tag{3.71}$$

Suppose that

$$\boldsymbol{X}_s = \hat{\boldsymbol{X}}_s + e\,0_s\,, \qquad \boldsymbol{X}^1_{st} = \hat{\boldsymbol{X}}^1_{st} + e\,1_{st}\,, \qquad \boldsymbol{X}^2_{st} = \hat{\boldsymbol{X}}^2_{st} + e\,2_{st}\,, \tag{3.72}$$

where the approximation errors satisfy

$$e\,0_s < \varepsilon\,, \qquad e\,1_{st} < \varepsilon\,\omega(s,t)^{1/p}\,, \qquad e\,2_{st} < \varepsilon\,\omega(s,t)^{2/p}\,. \tag{3.73}$$

Then

$$\begin{aligned}\boldsymbol{Y}_{st} - \hat{\boldsymbol{Y}}_{st} = \Big\{0,\quad & \\ & \theta(X_s)\,(\boldsymbol{X}^1_{st}) - \theta(\hat{X}_s)\,(\hat{\boldsymbol{X}}^1_{st}) \\ & + \frac{1}{2}\,(d\theta)\,(X_s)\,(\boldsymbol{X}^2_{st}) - \frac{1}{2}\,(d\theta)\,(\hat{X}_s)\,(\hat{\boldsymbol{X}}^2_{st}), \\ & \theta(X_s)\otimes\theta(X_s)\,(\boldsymbol{X}^2_{st}) - \theta(\hat{X}_s)\otimes\theta(\hat{X}_s)\,(\hat{\boldsymbol{X}}^2_{st})\Big\}\end{aligned} \tag{3.74}$$

and so

$$\begin{aligned}\|(\boldsymbol{Y}_{st} - \hat{\boldsymbol{Y}}_{st})^i\| \le \Big\{0,\quad & \\ & M\,\varepsilon\,\omega(s,t)^{1/p} + M\,\varepsilon\,\omega^{1/p}\,\omega(s,t)^{1/p} \\ & + \frac{1}{2}\,M\,(\varepsilon\,\omega(s,t)^{1/p})^{\gamma-1}\,\omega(s,t)^{2/p} \\ & + \frac{1}{2}\,M\,\varepsilon\,\omega(s,t)^{2/p}, \\ & M^2\,((\varepsilon^2+\varepsilon)\,\omega(s,t)^{2/p} \\ & \qquad + 2\,(\varepsilon^2+\varepsilon)\,\omega(s,t)^{3/p} \\ & \qquad + \varepsilon^3\,\omega(s,t)^{4/p})\Big\}\,,\end{aligned} \tag{3.75}$$

with $i = 0,1,2$, and providing $\omega(s,t) < 1$, $\varepsilon < 1$, one has the more intelligible inequality

$$\|(\boldsymbol{Y}_{st} - \hat{\boldsymbol{Y}}_{st})^i\| \le \{0, 3\,M\,\varepsilon^{\gamma-1}\,\omega(s,t)^{1/p}, 7\,M^2\,\varepsilon\,\omega(s,t)^{2/p}\}\,, \tag{3.76}$$

with $i = 0,1,2$, establishing the continuity of the map into almost multiplicative functionals.

4. Differential equations, putting it all together.

In this section we achieve our main objective of showing that the Itô functional extends uniquely to a continuous map defined on the rough paths in $\Omega G(V)^p$ providing the defining vector fields are $\mathrm{Lip}[\gamma]$

and $\gamma > p$. This permits, in a reasonably complete way, the solution of differential equations driven by rough (but geometric) multiplicative functionals. It completely removes the finite dimensional Lie algebra assumption.

The key estimate will be the one we established for the integration of one forms; this together with a reasonably delicate exploitation of inhomogeneity will show Picard's iteration scheme converges. The argument will be split into a number of distinct steps. But first we must be precise about our concept or definition of a solution!

4.1. Giving the differential equation meaning.

Take a smooth path X_t in V and a linear map f from V into the Lipschitz vector fields on a vector space W, then one may use schoolboy integration to define a solution to our basic equation. Classically, one could say the path Y_t solves the equation

$$dY_t = f(Y_t)\, dX_t \,, \qquad Y_0 = a\,, \tag{4.1}$$

providing Y_t satisfies the integral equation

$$Y_t = a + \int_{0<u<t} f(Y_u)\, dX_u \,. \tag{4.2}$$

Observe that we can reformulate this integral identity in a trivially different way

$$\begin{aligned} X_t &= X_0 + \int_{0<u<t} dX_u \,, \\ Y_t &= a + \int_{0<u<t} f(Y_u)\, dX_u \,. \end{aligned} \tag{4.3}$$

Consider the one form on $V \oplus W$ with values in $V \oplus W$ defined by

$$h((x,y))\,(dX, dY) = (dX, f(y)\, dX)\,. \tag{4.4}$$

Then for smooth paths the integral equation (4.3) can be rewritten as

$$(X_t, Y_t) = (X_0, a) + \int_0^t h(X_u, Y_u)\,(dX_u, dY_u)\,. \tag{4.5}$$

Putting $Z_t = (X_t, Y_t)$ we can say that a solution to (4.1) is a lift of the path X_t to a path in $V \oplus W$ satisfying

$$
\begin{aligned}
Z_t - Z_0 &= \int_{0<u<t} h(Z_u)\, dZ_u \;, \\
Z_0 &= (X_0, a)\,.
\end{aligned}
\tag{4.6}
$$

Although this transformation may seem essentially trivial in the classical setting, for us it is not really so. We have no difficulty extending this characterisation to rough signals.

Definition 4.1.1. *Let $\boldsymbol{X} \in \Omega G(V)^p$ be a geometric multiplicative functional projecting onto the path X_t, and let f be a linear map from V into the* Lip$[\gamma - 1, W]$ *vector fields. A solution to the equation*

$$
d\boldsymbol{Y} = f(Y_t)\, d\boldsymbol{X}\,, \qquad Y_0 = a\,, \tag{4.7}
$$

is an extension of $\boldsymbol{X}$ to $\boldsymbol{Z} \in \Omega G(V \oplus W)^p$ such that $\boldsymbol{Z}$ projects onto $Z_t = (X_t, Y_t)$, $Y_0 = a$, and such that $\boldsymbol{Z}$ satisfies $\delta \boldsymbol{Z} = h(Z_t)\, \delta \boldsymbol{Z}$.

The main point to notice is that we do not treat the solution as an independent object, but rather as an extension of the original driving signal. In particular, we require the existence of cross iterated integrals between driving signal and solution to be constructed. On the one hand this seems a bonus, if we can construct integrals between solution and driving signal so much the better; on the other hand it is essential, we could not make sense of the integral at all for rough signals without some cross information between integrand and integrator. The author is remi nded of those induction arguments which only work if you prove a stronger result than you were aiming for. In any case, the definition is clearly consistent with the classical one. If $\boldsymbol{X} \in \Omega G(V)^p$ is a smooth path with its iterated integrals, the classical solution, its iterated integrals, together with the cross integrals with the driving signal, together satisfy the extended equation.

Our approach requires that the vector fields in the equation have a smoothness related to the roughness of the path. This was necessary for the integral to make sense. However, as in the classical situation, the smoothness required of the vector fields in the definition is less than that required for uniqueness.

The main purpose of this part of the paper, and indeed of the entire paper is to prove the following theorem.

Theorem 4.1.1. *Suppose that $f : V \longrightarrow \mathrm{Lip}[\gamma, W, W]$ is a linear map into Lipschitz vector fields. Then consider the Itô map $X \longrightarrow (X, Y)$ defined for smooth paths by*

$$dY_t = f(Y_t)\, dX_t\,, \qquad Y_0 = a\,. \tag{4.8}$$

Define the one form h by

$$h((x, y))\,(dX, dY) = h(y)\,(dX, dY) = (dX, f(y)\, dX)\,.$$

For any geometric multiplicative functional $\boldsymbol{X} \in \Omega G(V)^p$ with $1 \leq p < \gamma$ there is exactly one geometric multiplicative functional extension $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y}) \in \Omega G(V \oplus W)^p$ such that if $Y_t = \boldsymbol{Y}^1_{0,t} + a$ then $\boldsymbol{Z}$ satisfies the rough differential equation

$$\delta \boldsymbol{Z} = h(Y_t)\, \delta \boldsymbol{Z}\,. \tag{4.9}$$

Moreover this solution to the rough differential equation is constructed by Picard iteration, there is a small interval $[0, T]$ whose length can be controlled entirely in terms of the control on the roughness of X and of f and the rate so that the convergence of this iteration scheme is faster than the given exponential rate on the interval. The Itô map is uniformly continuous and the map $\boldsymbol{X} \longrightarrow \boldsymbol{Z}$ is the unique continuous extension of the Itô map from $\Omega G(V)^p$ to $\Omega G(V \oplus W)^p$.

Our convergence theorem for Picard iteration requires that $\gamma > p$, and constructively produces a unique solution; the extension of Peano's theorem to show existence under the weaker hypothesis $\gamma > p - 1$ is open (except in the case where $p < 2$; here a fixed point argument can be applied to show existence and A. M. Davie (Edinburgh - private communication) has given the author examples to show that the solution need not be unique $\gamma < p$ [14], [15]).

We may define Picard iteration as follows

$$\begin{aligned} \boldsymbol{Z}^{n+1}_{s,t} &= \int_{s<u<t} h(Z^n_u)\, \delta \boldsymbol{Z}^n\,, \\ Z^n_0 &= (b, a)\,, \end{aligned} \tag{4.10}$$

where Z^n is uniquely determined by $Z^n_0 = (b, a)$; the choice of b is irrelevant to the definition as h does not depend in any way on the first coordinate of $\boldsymbol{Z}$. If we can prove that the multiplicative functionals $\boldsymbol{Z}^n$ converge in $\Omega G(V)^p$, then it is routine from our result about the continuity of the integration against one forms that the limit will be a fixed point of the functional and so our desired solution.

However, in contrast to the normal contraction mapping argument, it seems essential to consider a more complicated iteration so that we might keep track of the joint interactions of more terms.

Step 1. **Norms on tensor algebras over finite sums of vector spaces.** There are many different equivalent norms one could use on the tensor algebra over the space $V \oplus W$; we will use an induction argument where a choice adapted to the possibilities for independently scaling the different coordinates will simplify the proof.[1]

The tensors of fixed degree over a vector space admit a further direct sum decomposition if the underlying vector space is already a direct sum

$$\begin{aligned} T^{(n)}(V \oplus W) &= \bigoplus_{j=0}^{n} (V \oplus W)^{\otimes j}\,, \\ (V \oplus W)^{\otimes j} &= Z^{j,0} \oplus Z^{j-1,1} \oplus Z^{j-2,2} \oplus \cdots \oplus Z^{0,j}\,, \end{aligned} \tag{4.11}$$

where $Z^{j-k,k}$ comprises those tensors that are homogeneous of degree $j-k$ in V and k in W in whatsoever order.

Remark 4.1.1.–Requirement. Let $\boldsymbol{z} = \boldsymbol{z}^{j,0} + \boldsymbol{z}^{j-1,1} + \boldsymbol{z}^{j-2,2} + \cdots + \boldsymbol{z}^{0,j}$ represent the decomposition of an element $\boldsymbol{z} \in (V \oplus W)^{\otimes j}$, then the norm on $(V \oplus W)^{\otimes j}$ should be chosen to have the property that $\|\boldsymbol{z}\| = \sup_{k \le j} \|\boldsymbol{z}^{j-k,k}\|$.

Definition 4.1.2. *A multiplicative functional $\boldsymbol{Z}$ in $\Omega(V \oplus W)^p$ is controlled by ω if*

$$\|\boldsymbol{Z}_{s,t}^{j-k,k}\| \le \frac{\omega(s,t)^{j/p}}{\beta\,(j-k/p)!\,(k/p)!}\,, \tag{4.12}$$

for all $j \le [p]$.

Of course this control is comparable with the one that ignores the inhomogeneity.

[1] See also the earlier section inhomogeneous degrees of smoothness.

Step 2. **Rescaling and Tensor Algebras.** If S is a linear automorphism on V then it induces a natural graded algebra homomorphism $\tilde{S}$ on the tensor algebra, taking $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ to $Sv_1 \otimes Sv_2 \otimes \cdots \otimes Sv_n$. Apply this to the scaling operators $S_\varepsilon(v) = \varepsilon\, v$. Their extensions act by multiplying the tensors of degree k by ε^k so that $\tilde{S}_\varepsilon(\boldsymbol{a}) = (1, \varepsilon\, \boldsymbol{a}_1, \varepsilon^2\, \boldsymbol{a}_2, \ldots, \varepsilon^n\, \boldsymbol{a}_n)$. These operators are very important to us, but the general notation is clumsy, so we shorten it.

Definition 4.1.2. *We will use the notation* $\varepsilon\, \boldsymbol{X}_{s,t}$ *for* $\tilde{S}_\varepsilon(\boldsymbol{X}_{s,t})$.

Because $\tilde{S}$ is always an algebra homomorphism $\varepsilon\, \boldsymbol{X}_{s,t}$ is also a multiplicative functional, leading to the slightly peculiar but correct notation $\varepsilon\, \boldsymbol{X}_{s,t} \otimes \varepsilon\, \boldsymbol{X}_{t,u} = \varepsilon\, \boldsymbol{X}_{s,u}$.

Consider the linear projections $P_V : V \oplus W \longrightarrow V$, and $P_W : V \oplus W \longrightarrow W$; then if $\boldsymbol{Z}$ is a multiplicative functional in the tensor algebra over $V \oplus W$, let $\boldsymbol{X} = P_V \boldsymbol{Z}$ and $\boldsymbol{Y} = P_W \boldsymbol{Z}$ be the associated multiplicative functionals. We will frequently use the notation $(\boldsymbol{X}, \boldsymbol{Y})$ for $\boldsymbol{Z}$ to remind the reader of the direct sum structure, however the multiplicative functional $(\boldsymbol{X}, \boldsymbol{Y})$ is *not determined* by $\boldsymbol{X}, \boldsymbol{Y}$ separately, as it involves cross terms.

It is possible to scale the complementary subspaces of a direct sum differently and we use the shorthand $(\varepsilon\, \boldsymbol{X}, \phi\, \boldsymbol{Y})_{st}$ for the multiplicative functional $\tilde{S}_{\varepsilon\phi}(\boldsymbol{X}, \boldsymbol{Y})_{st}$ where $S_{\varepsilon\phi}(\boldsymbol{v} + \boldsymbol{w}) = \varepsilon\, \boldsymbol{v} + \phi\, \boldsymbol{w}$.

Consider how this inhomogeneous scaling interacts with a control on the p-variation:

Lemma 4.1. *Let* $\boldsymbol{X} \in \Omega(V)^p$ *be controlled by* $\omega(s,t)$ *so that*

$$\|\boldsymbol{X}^j_{s,t}\| \leq \frac{\omega(s,t)^{j/p}}{\beta\,(j/p)!} \tag{4.13}$$

and $(\boldsymbol{X}, \boldsymbol{Y}) \in \Omega(V \oplus W)^p$ *be an extension of* $\boldsymbol{X}$. *Suppose* $(\boldsymbol{X}, \boldsymbol{Y})_{st}$ *is controlled by* $K\,\omega(s,t)$. *Then* $(\boldsymbol{X}, \phi\boldsymbol{Y})_{st}$ *is controlled by*

$$\max\{1, \phi^{kp/j}\, K :\ 1 \leq k \leq j \leq [p]\}\, \omega(s,t)\,. \tag{4.14}$$

In particular, if $\phi < K^{-[p]/p} < 1$ *then* $(\boldsymbol{X}, \phi\, \boldsymbol{Y})_{st}$ *is controlled by* $\omega(s,t)$.

PROOF. Let $\boldsymbol{Z}^j_{s,t} = (\boldsymbol{X}, \boldsymbol{Y})^j_{s,t}$ be the component of the multiplicative functional of degree j and let $\boldsymbol{Z}^{j-k,k}_{s,t}$ denote the component of this tensor of degree $j-k$ in V and k in W. Then by assumption

$$\|\boldsymbol{Z}_{s,t}^{j-k,k}\| \le \frac{(K\,\omega(s,t))^{j/p}}{\beta\,(j-k/p)!\,(k/p)!}\ , \tag{4.15}$$

therefore

$$\|\tilde{S}_{1\phi}(\boldsymbol{Z}_{s,t}^{j-k,k})\| \le \phi^k\,\frac{(K\,\omega(s,t))^{j/p}}{\beta\,(j-k/p)!\,(k/p)!}\ , \tag{4.16}$$

but $\boldsymbol{Z}_{s,t}^{j,0} = \boldsymbol{X}_{s,t}^{j}$ and so

$$\|\boldsymbol{Z}_{s,t}^{j,0}\| \le \frac{\omega(s,t)^{j/p}}{\beta\,(j/p)!}\ , \tag{4.17}$$

without any constant. It follows that $(\boldsymbol{X}, \phi\,\boldsymbol{Y})_{st}$ is controlled by

$$\max\{1, \phi^{kp/j}\,K:\ 1\le k\le j\le [p]\}\,\omega(s,t) \tag{4.18}$$

as required.

Step 3. **The boundedness of the Picard integral operator.** As a simple application of the scaling lemma we have just established, we prove the following a priori bound.

Lemma 4.1.2. *Let $\boldsymbol{Z}^{(0)}$ be the initial multiplicative functional in the Picard iteration scheme defined recursively by* (4.10). *Suppose $\boldsymbol{Z}^{(0)}$ is controlled by ω_0. Then all iterates $\boldsymbol{Z}^{(j)}$ are uniformly controlled by*

$$\omega = \max\{1, K\,(M,p,\gamma)^{[p]}\}\,\omega_0$$

on the time interval $J = \{u:\ \omega(0,u) < 1\}$.

Here M is the Lip$[\gamma-1]$ *norm of f on*

$$\Big\{\omega:\ \|\omega - a\| \le \frac{1}{\beta}\,\Big(\frac{1}{p}\Big)!\Big\}\,,$$

and K is the constant introduced in Theorem 3.2.1.

PROOF. First we condition the problem. Suppose that the initial point $\boldsymbol{Z}_{s,t}^{(0)} = (\boldsymbol{X}, \boldsymbol{Y}^{(0)})_{s,t}$ in our Picard iteration is of finite p-variation controlled by ω_0. For any $\varepsilon > 0$ we may choose a regular

$$\omega = \max\{\varepsilon^{-p}, 1\}\,\omega_0$$

so that $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}^{(0)})_{s,t}$ is controlled by ω, and a short interval depending on ω where $\omega < 1$. We choose $\varepsilon = K(M,p,\gamma)^{-[p]/p}$ where K is the function derived in Theorem 3.2.1, and M, is defined to be the $\mathrm{Lip}[\gamma - 1]$ norm of the one form $h(x,y)$ restricted to the domain

$$V \times \left\{ w : \ \|w - a\| < \frac{1}{\beta}\left(\frac{1}{p}\right)! \right\}.$$

We now proceed by induction. Suppose that $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}^{(0)})_{s,t}$ is controlled by ω where $\omega < 1$. The control on $(\boldsymbol{Y}_{0,t}^{(0)})^1$ ensures that its projection onto the path $Y_u^{(0)}$ starting at a remains in the ball of radius $(1/\beta)\,(1/p)!$ centred on a. Observe that the multiplicative functional

$$\int_{s<u<t} h((\varepsilon^{-1}X_u, Y_u^{(0)}))\,\delta(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}^{(0)})\,(\varepsilon^{-1}X_0, Y_0^{(0)})_t \qquad = (\varepsilon^{-1}\,b, a) \tag{4.19}$$

equals $(\varepsilon^{-1}\boldsymbol{X}, \varepsilon^{-1}\boldsymbol{Y}^{(1)})_{s,t}$ where $(\boldsymbol{X}, \boldsymbol{Y}^{(1)})_{s,t}$ is the Picard iterate of $(\boldsymbol{X}, \boldsymbol{Y}^{(0)})_{s,t}$ defined in (4.10). By Theorem 3.2.1, it is controlled by $K(M,p,\gamma)\,\omega$ on the chosen time interval; here K depends only on the explicit variables (we have arranged that $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}^{(0)})_{s,t}$ is controlled by ω where $\omega < 1$).

The difference in homogeneity between $(\varepsilon^{-1}\boldsymbol{X}, \varepsilon^{-1}\boldsymbol{Y}^{(1)})_{s,t}$ and our starting data $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}^{(0)})_{s,t}$ is crucial to the analysis. If the reader finds the unfamiliar notation difficult then the equivalent formulation for smooth paths is

$$\begin{aligned} \varepsilon^{-1}Y_{st}^{(1)} &= \int_s^t f\,(Y_u^{(0)})\,d\varepsilon^{-1}X_u\ , \\ \varepsilon^{-1}X_{st} &= \int_s^t d\varepsilon^{-1}X_u\ . \end{aligned} \tag{4.20}$$

By assumption $\varepsilon \leq K\,(M,p,\gamma)^{-[p]/p}$, so we may apply Lemma 4.1.1 to prove that the rescaled functional $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}^{(1)})_{s,t}$ is controlled by $\omega < 1$. This concludes the induction. We deduce that all the Picard iterates $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}^{(n)})_{s,t}$ are uniformly controlled by this same $\omega < 1$ on this fixed time interval.

An obvious extension of the same idea shows that

$$(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}^{(n)}, \boldsymbol{Y}^{(n+1)})_{s,t}$$

is also uniformly bounded for a different fixed choice of ε, ω and the time interval. This observation will be useful to us later.

This result only requires the minimal smoothness condition required to make sense of the equation. It can be interpreted as a compactness result and can probably be used to deduce a Peano theorem in the general case although we have not pursued the matter.

The main existence result is a more subtle and complicated version of the same approach.

Step 4. **A division lemma.** Suppose that f is a $\mathrm{Lip}[\gamma]$ vector field on W, then there exists a function g which is $\mathrm{Lip}[\gamma-1]$ on $W \times W$ and such that

$$f^i(x) - f^i(y) = \sum_j (x-y)^j \, g^{ij}(x,y) \,. \tag{4.21}$$

The function g is not uniquely defined, but for example the mean value of df along the ray from x to y [2] will do perfectly well. Thus we can rewrite the classical Picard iteration in the more useful form

$$\begin{aligned}(Y_t^{(n+1)} &- Y_t^{(n)}) \\ &= \int\limits_{0<u<t} (Y_u^{(n)} - Y_u^{(n-1)}) \, g(Y_u^{(n)}, Y_u^{(n-1)}) \, dX_u \,.\end{aligned} \tag{4.22}$$

The crucial difference between the earlier formulation of Picard iteration and the approach here is that we have introduced an expression which is quasi-linear in $(Y_n - Y_{n-1})$. We will really be able to take advantage of this and push the scaling arguments we introduced above.

Interpreting the integral (4.22) requires the extra smoothness we assume for our main theorem on the convergence of Picard's iterative scheme.

Step 5. **Defining the correct iteration.** In fact we consider recursively, a sequence containing a wider series of interrelated objects

$$(\boldsymbol{Z}^{(n)}, \boldsymbol{Y}^{(n)}, \boldsymbol{Y}^{(n-1)}, \boldsymbol{X})_{s,t} \,. \tag{4.23}$$

[2] Note that if the function f is only defined on a subset of $\mathbb{R}^d$ then one would need to apply the extension theorem for Lipschitz functions to use this approach.

For smooth paths the iteration is defined by

$$
\begin{aligned}
dZ_t^{(n+1)} &= Z_u^{(n)}\, g(Y_u^{(n)}, Y_u^{(n-1)})\, dX_u \,, \\
dY_t^{(n+1)} &= dY_u^{(n)} + dZ_u^{(n)} \,, \\
dY_t^{(n)} &= dY_t^{(n)} \,, \\
dX_t &= dX_t \,,
\end{aligned}
\tag{4.24}
$$

where $dZ_t^{(1)} = f(a)\, dX_t$, $Z_0^{(n)} = 0$, $Y^{(0)} \equiv a$, and $Y_0^{(n)} = a$. Now (4.24) defines a one-form; we can use this to extend the iteration, in the now obvious way, to functionals $(Z^{(n)}, Y^{(n)}, Y^{(n-1)}, X)_{s,t}$ in $\Omega G(W \oplus W \oplus W \oplus V)^p$. The iteration step makes sense because g (and hence the full one-form) is $\mathrm{Lip}[\gamma - 1]$.

It is obvious for smooth driving paths X and smooth initial estimates for the solution, that projection onto the last two co-ordinates gives the Picard iteration we studied in Step 3. The continuity of the iteration procedure makes it clear that this identity extends to geometric functionals.

We must prove that the sequence of iterations converge as a multiplicative functional to a functional $(\boldsymbol{I}, \boldsymbol{Y}, \boldsymbol{Y}, \boldsymbol{X})$, the continuity will then show that this is a fixed point for the equation. The argument will rely on a careful exploitation of the homogeneity of the various components.

Step 6. **The conditioning.** The first step is to rescale the coordinates and condition the problem.

For any choice of $\beta > 1$, and $\varepsilon < 1$ there is a choice of ω (depending on both parameters) so that if

$$
\boldsymbol{U}^{(0)} = (\tilde{\boldsymbol{Z}}^{(1)}, \boldsymbol{Y}^{(1)}, \boldsymbol{Y}^{(0)}, \varepsilon^{-1}\boldsymbol{X})
\tag{4.25}
$$

where $\tilde{\boldsymbol{Z}}^{(1)} = \beta \boldsymbol{Z}^{(1)}$, then $\boldsymbol{U}^{(0)}$ is controlled by ω.

We now use our estimates to study what happens when we replace the top line in (4.24) by

$$
\tilde{\boldsymbol{Z}}_t^{(n+1)} = \varepsilon\, \beta \int_{0<u<t} \tilde{\boldsymbol{Z}}_u^{(n)}\, g(Y_u^{(n)}, Y_u^{(n-1)})\, \varepsilon^{-1}\, d\boldsymbol{X}_u
$$

and use the new one-form to define a changed recursion involving $\tilde{\boldsymbol{Z}}^{(n)} = \beta^n \boldsymbol{Z}^{(n)}$ etc. In other words we recursively define

$$
\boldsymbol{U}^{(n-1)} = (\tilde{\boldsymbol{Z}}^{(n)}, \boldsymbol{Y}^{(n)}, \boldsymbol{Y}^{(n-1)}, \varepsilon^{-1}\boldsymbol{X})\,.
\tag{4.26}
$$

We will prove by induction that, for any choice of $\beta > 1$ there is a suitably small choice of $\varepsilon < 1$, chosen to depend on $K(M,p,\gamma)^{-[p]/p}$ and β alone, so that the sequence of elements in the sequence $\boldsymbol{U}^{n-1}$ are uniformly controlled by our ω on our predetermined time interval. By rescaling, it will be clear that the increments in the original iteration converge to zero with a geometric rate giving the overall result.

Step 7. **The induction step.** First fix the time interval so that $\omega < 1$ and assume that

$$\varepsilon < K_1(M,p,\gamma)^{-[p]/p}, \tag{4.27}$$

where M will be chosen later, but only depends on the Lip norms of various one forms and will be independent of other parameters in this problem.

We assume as our induction hypotheses that $\boldsymbol{U}^{n-1}$ is controlled by ω. Consider the form we must integrate to go from

$$\boldsymbol{U}^{n-1} \qquad \text{to} \qquad ((\varepsilon\beta)^{-1}\tilde{\boldsymbol{Z}}^{n+1}, \boldsymbol{Y}^{n+1}, \boldsymbol{Y}^{n}, \varepsilon^{-1}\boldsymbol{X}) : \tag{4.28}$$

$$\begin{aligned} d(\varepsilon\,\beta)^{-1}\,\tilde{\boldsymbol{Z}}_t^{(n+1)} &= \tilde{\boldsymbol{Z}}_u^{(n)}\,g(Y_u^{(n)}, Y_u^{(n-1)})\,d\varepsilon^{-1}\boldsymbol{X}_u\,,\\ d\boldsymbol{Y}_t^{(n+1)} &= d\boldsymbol{Y}_u^{(n)} + \beta^{-n}\,d\tilde{\boldsymbol{Z}}_u^{(n)}\,,\\ d\boldsymbol{Y}_t^{(n)} &= d\boldsymbol{Y}_t^{(n)}\,,\\ d\varepsilon^{-1}\,\boldsymbol{X}_t &= d\varepsilon^{-1}\,\boldsymbol{X}_t\,. \end{aligned} \tag{4.29}$$

Although examination of the second line in the expression shows this form varies with n the effect of increasing n is to decrease the Lipschitz norm. Hence, and because g is $\mathrm{Lip}[\gamma-1]$ there is a uniform bound M on the $\mathrm{Lip}[\gamma-1]$ norms of the forms on the range of paths under $\boldsymbol{U}^{n-1}$. (Recall that the $\boldsymbol{U}^{n-1}$ are controlled by ω and this in turn is uniformly bounded by one).

Hence there exists $K(M,p,\gamma)$, independent of our particular multiplicative functionals, time interval, etc., so that

$$((\varepsilon\beta)^{-1}\tilde{\boldsymbol{Z}}^{n+1}, \boldsymbol{Y}^{n+1}, \boldsymbol{Y}^{n}, \varepsilon^{-1}\boldsymbol{X}) \tag{4.30}$$

is of finite p-variation controlled by $K(M,p,\gamma)\,\omega$. By Step 3 we observe that providing $\varepsilon < K_1(M,p,\gamma)^{-[p]/p}$ then $(Y^{(n)}, Y^{(n-1)}, \varepsilon^{-1}X)$ is controlled by ω on any interval where $\omega < 1$ without any sort of factor.

Therefore we can apply the rescaling lemma again. Choose ε so that $\beta\,\varepsilon < K\,(M,p,\gamma)^{-[p]/p}$ and $\varepsilon < K_1\,(M,p,\gamma)^{-[p]/p}$. Then

$$(\tilde{\boldsymbol{Z}}^{n+1}, \boldsymbol{Y}^{n+1}, \boldsymbol{Y}^{n}, \varepsilon^{-1}\boldsymbol{X}) \tag{4.31}$$

is also controlled by ω, without a constant. This establishes the induction step.

Step 8. **Convergence.** At the level of paths it is now trivial that we have convergence. Let

$$(\tilde{Z}^{n+1}, Y^{n+1}, Y^{n}, \varepsilon^{-1}X)$$

be the path under

$$(\tilde{\boldsymbol{Z}}^{n+1}, \boldsymbol{Y}^{n+1}, \boldsymbol{Y}^{n}, \varepsilon^{-1}\boldsymbol{X})$$

satisfying the initial condition

$$(\tilde{Z}_0^{n+1}, Y_0^{n+1}, Y_0^{n}, \varepsilon^{-1}X_0) = (0, a, a, 0)\,.$$

Then it is clear that for smooth paths, and by continuity, for elements of ΩG^p (and geometric multiplicative functionals are all that one will ever see) the algebraic identity

$$Y_t^{(n+1)} = Y_u^{(n)} + \beta^{-n}\,\tilde{Z}_u^{(n)} \tag{4.32}$$

holds. But $\beta > 1$ and we have just proved that the difference process $Z_t^{(n)}$ is bounded independently of n on our time interval and so we have uniform convergence. The convergence is in p-variation, and as the sequence $Y_t^{(n)}$, $Y_u^{(n+1)}$ is uniformly bounded in p-variation norm that bound goes over to the limit.

However, our real objective is not just to construct a path in W and call it the solution, we want to construct a multiplicative functional. In other words we want to show that the multiplicative functionals $(\boldsymbol{Y}^{(n)}, \boldsymbol{X})$ converge in $\Omega G(W \oplus V)^p$. This is essentially trivial as well. Consider the projection $(\tilde{Z}^{(n)}, Y^{(n)}, \varepsilon^{-1}X)$ of $\boldsymbol{U}^{(n-1)}$ and $(Y^{(n+1)}, \varepsilon^{-1}X)$ of $\boldsymbol{U}^{(n-1)}$. Let Π_n be the linear map $(z,\ y,\ x) \longrightarrow (\beta^{-n}z + y, x)$ then the induced map Π_n on the tensor algebra takes $(\tilde{\boldsymbol{Z}}^{(n)}, \boldsymbol{Y}^{(n)}, \varepsilon^{-1}\boldsymbol{X})$ to $(\boldsymbol{Y}^{(n+1)}, \varepsilon^{-1}\boldsymbol{X})$ (again this is obvious for smooth sequences, and algebraic identities hold on closed sets, and hence extend to geometric functionals). But now the convergence is clear. and uniformly controlled by the β. The uniform nature of the estimates here on the convergence of Picard iteration prove the Itô map is continuous since our earlier arguments demonstrate that the finite iterations are certainly continuous.

4.2. Uniqueness.

To see uniqueness is also relatively straightforward and we do not dwell on it. We did not need to start our new Picard iteration with the function that was constant at a and its integral. We could have started it at two of our "solutions", in this case our iteration would have compared the difference and shown that it went to zero.

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Beta-gamma random variables and intertwining relations between certain Markov processes

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1. Introduction.

In this paper, we study particular examples of the *intertwining relation*

$$Q_t \, \Lambda = \Lambda \, P_t \tag{1.a}$$

between two Markov semi-groups $(P_t, \, t \geq 0)$ and $(Q_t, \, t \geq 0)$ defined respectively on $(E, \mathcal{E})$ and $(F, \mathcal{F})$, via the Markov kernel

$$\Lambda : (E, \mathcal{E}) \longrightarrow (F, \mathcal{F}) \, .$$

A number of examples of (1.a) have already attracted the attention of probabilists for quite some time; see, for instance, Dynkin [14] and Pitman-Rogers [41]. Some more recent study by Diaconis-Fill [11] has been carried out in relation with strong uniform times.

In Section 2, a general filtering type framework for intertwining is presented which includes a fair proportion of the different examples of intertwining known up to now.

In Section 3, we prove that the relation (1.a) holds when $P_t = Q_t^{\alpha+\beta}$, $Q_t = Q_t^{\alpha}$, with $\alpha > 0$, $\beta > 0$, where $(Q_t^{\alpha+\beta})$ (respectively

(Q_t^α)) is the semi-group of the square of the Bessel process of dimension $2\,(\alpha+\beta)$ (respectively $2\,\alpha$), and $\Lambda \equiv \Lambda_{\alpha,\beta}$ is defined by

$$\text{(1.b)}\quad \Lambda f(y) = \mathbb{E}\left[f(y\,Z)\right], \quad \text{where } Z \text{ is a beta}(\alpha,\beta) \text{ random variable}$$

(in the sequel, we shall say, in general, that Λ is the multiplication kernel associated with Z).

The intertwining relation

$$Q_t^{\alpha+\beta}\,\Lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}\,Q_t^\alpha \tag{1.c}$$

may then be considered as an extension to the semi-group level of the well-known fact that the product of a beta(α,β) variable by an independent gamma$(\alpha+\beta)$ variable is a gamma(α) variable.

Changing the order in which the product of these two random variables is performed, we show the existence of a semi-group $(\Pi_t^{\alpha,\beta},\ t \geq 0)$ such that

$$\Pi_t^{\alpha,\beta}\,\Lambda_{\alpha+\beta} = \Lambda_{\alpha+\beta}\,Q_t^\alpha\,, \qquad \alpha > 0,\ \beta > 0,\ \alpha+\beta \geq 1\,, \tag{1.d}$$

where $\Lambda_{\alpha+\beta}$ is the multiplication kernel associated with a gamma$(\alpha+\beta)$ variable and $(\Pi_t^{\alpha,\beta},\ t \geq 0)$ is the semi-group of a piecewise linear Markov process $X^{\alpha,\beta}$ taking values in $\mathbb{R}_+$.

In Section 4, it is shown that the $X^{\alpha,\beta}$ processes possess a number of properties which are reminiscent of those enjoyed by the squares of Bessel processes X^α.

In Section 5, we compare the intertwining relation (1.a) and the notion of duality of two Markov processes with respect to a function h defined on their product space (see Liggett [33]). The intertwining relationships discussed in Section 3 are then translated in terms of this notion of duality. With the help of some (local time) perturbations of the reflecting Brownian motion, some other intertwining relations have been obtained in [7]; these are briefly discussed at the end of Section 5.

It would be interesting to be able, in the examples of intertwining discussed in this paper (Section 3, in particular) to obtain a joint realization of the two Markov processes (X_t) and (Y_t), with respective semi-groups which satisfy (1.a). In many cases (see Siegmund [46], Diaconis-Fill [11]), there exists a pathwise construction of Y in terms of X for instance (possibly allowing some extra randomization). So far, we have been able to obtain such a construction of the $X^{\alpha,\beta}$ process in terms of X^α only in the case $\alpha+\beta = 1$.

It may well be that, if such a pathwise construction can be obtained for any (α,β), then most of the properties of the $X^{\alpha,\beta}$ processes which are being discovered in Section 4, mainly by analogy with their Bessel counterparts, will then appear in a more straightforward manner.

A summary, without proofs, of the main results contained in this paper has been presented in [58]

2. A filtering type framework for intertwining.

The following set-up provides a fairly general framework for intertwining. $(X_t,\, t\geq 0)$ and $(Y_t,\, t\geq 0)$ are two measurable processes, defined on the same probability space $(\Omega,\mathcal{F},P)$ taking values respectively in E and F, two measurable spaces; furthermore, $(X_t,\, t\geq 0)$ and $(Y_t,\, t\geq 0)$ satisfy the following properties:

1) there exist two filtrations $(\mathcal{G}_t,\, t\geq 0)$ and $(\mathcal{F}_t,\, t\geq 0)$ such that:

a) for every t, $\mathcal{G}_t\subset\mathcal{F}_t\subset\mathcal{F}$,

b) $(X_t,\, t\geq 0)$ is $(\mathcal{F}_t)$ adapted and $(Y_t,\, t\geq 0)$ is $(\mathcal{G}_t)$ adapted;

2) $(X_t,\, t\geq 0)$ is Markovian with respect to $(\mathcal{F}_t)$, with semi-group $(P_t,\, t\geq 0)$, and $(Y_t,\, t\geq 0)$ is Markovian with respect to $(\mathcal{G}_t)$, with semi-group $(Q_t,\, t\geq 0)$;

3) there exists a Markov kernel $\Lambda: E\longrightarrow F$ such that for every $f: E\longrightarrow\mathbb{R}_+$,

$$\mathbb{E}[f(X_t)|\mathcal{G}_t]=\Lambda f(Y_t)\,,\qquad \text{for every } t\geq 0\,.$$

We then have:

Proposition 2.1. *For every function $f: E\longrightarrow\mathbb{R}_+$, for every $t,s\geq 0$,*

$$Q_t\,\Lambda\, f(Y_s)=\Lambda\, P_t\, f(Y_s)\,,\qquad \textit{almost surely}\,. \tag{2.a}$$

Consequently, under some mild (continuity) assumptions, one obtains the identity

$$Q_t\,\Lambda=\Lambda P_t\,,\qquad t\geq 0\,. \tag{2.b}$$

Proof. The result (2.a) is obtained by computing

$$\mathbb{E}[f(X_{t+s})|\mathcal{G}_s]$$

in two different ways.

On one hand, we have

$$\begin{aligned}\mathbb{E}[f(X_{t+s})|\mathcal{G}_s] &= \mathbb{E}[\mathbb{E}[f(X_{t+s})|\mathcal{G}_{t+s}]|\mathcal{G}_s] \\ &= \mathbb{E}[\Lambda f(Y_{t+s})|\mathcal{G}_s] \\ &= Q_t \Lambda f(Y_s)\,.\end{aligned}$$

On the other hand,

$$\mathbb{E}[f(X_{t+s})|\mathcal{G}_s]=\mathbb{E}[\mathbb{E}[f(X_{t+s})|\mathcal{F}_s]|\mathcal{G}_s]=\mathbb{E}[P_t f(X_s)|\mathcal{G}_s]=\Lambda P_t f(Y_s)\,.$$

We now present six classes of examples of intertwining where the hypotheses made in Proposition 2.1 are in force.

2.1. Dynkin's criterion.

This is, undoubtedly, one of the best known, and oldest, examples of intertwining between two Markov processes (see [14]). Here, we start with a Markov process $(Y_t,\, t \geq 0)$ taking its values in a measurable space F; Y is Markovian with respect to $(\mathcal{G}_t)$, with semi-group $(Q_t,\, t \geq 0)$. We assume that there exists a measurable application $\phi : F \longrightarrow E$ such that for every measurable function $f : E \longrightarrow \mathbb{R}_+$, the quantity

$$Q_t(f \circ \phi)(y) \text{ only depends, through } y, \text{ on } \phi(y)\,.$$

Now, if $x = \phi(y)$, we define $P_t f(x) = Q_t(f \circ \phi)(y)$. It is now easy to see that the process $(X_t \stackrel{\text{def}}{=} \phi(Y_t),\, t \geq 0)$ is Markovian with respect to $(\mathcal{F}_t) = (\mathcal{G}_t)$, and has semi-group $(P_t,\, t \geq 0)$. Moreover, by definition of $(P_t,\, t \geq 0)$, we have

$$Q_t \Lambda = \Lambda P_t\,, \qquad \text{with } \Lambda f(y) = f(\phi(y))\,,$$

so that the hypotheses of Proposition 2.1 are satisfied.

A particularly important example of this situation is obtained by taking Brownian motion in $\mathbb{R}^n$ for $(Y_t,\, t \geq 0)$, and $(X_t = |Y_t|,\, t \geq 0)$, the radial part of $(Y_t,\, t \geq 0)$, so called Bessel process of dimension n. Here, $F = \mathbb{R}^n$, $E = \mathbb{R}_+$ and $\phi(y) = |y|$.

2.2. Filtering theory.

Consider the canonical realization of a nice Markov process $(X_t,\, t \geq 0)$, taking values in E, with semi-group $(P_t,\, t \geq 0)$, and distribution $\mathbf{P}_\mu$ associated with the initial probability measure μ on E. Define

$$\mathbb{P}_\mu = W \times \mathbf{P}_\mu \ ,$$

where W denotes the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^n)$, which makes $(B_t,\, t \geq 0)$, the process of coordinates on $C(\mathbb{R}_+, \mathbb{R}^n)$, an n-dimensional Brownian motion. Next, define (on the product probability space), the *observation process*

$$Y_t = B_t + \int_0^t h(X_s)\, ds \,,$$

where $h : E \longrightarrow \mathbb{R}^n$ is a bounded Borel function.

Define $\mathcal{G}_t = \sigma(Y_s,\, s \leq t)$, and the filtering process $(\Pi_t^\mu,\, t \geq 0)$ by

$$\Pi_t^\mu(f) = \mathbb{E}_\mu[f(X_t)|\mathcal{G}_t] \,,$$

for every bounded measurable $f : E \longrightarrow \mathbb{R}$. Then, $(\Pi_t^\mu,\, t \geq 0)$ is a $((\mathcal{G}_t,\, t \geq 0),\, \mathbb{P}_\mu)$ Markov process, with transition semi-group

$$Q_t(\nu, \Gamma) = \mathbb{P}_\nu(\Pi_t^\nu \in \Gamma)$$

which satisfies the following intertwining relationship with $(P_t,\, t \geq 0)$

$$\text{(2.c)} \qquad Q_t \Lambda = \Lambda P_t \ , \qquad \text{where } \Lambda\phi(\nu) = \langle \nu, \phi \rangle \,.$$

PROOF OF (2.C).

$$Q_t\, \Lambda\, \phi(\nu) = \mathbb{E}_\nu[\Pi_t^\nu(\phi)] = \mathbb{E}_\nu[\phi(X_t)] = \Lambda\, P_t\, \phi(\nu) \,.$$

NOTE. A deep study of the measure-valued process $(\Pi_t^\mu,\, t \geq 0)$ has been made in [16] (see also [26] and [54]).

2.3. Pitman's representation of BES(3).

Consider $(B_t,\, t \geq 0)$ a one-dimensional Brownian motion starting from 0. In this example, we take $(X_t = |B_t|,\, t \geq 0)$ and $(Y_t = |B_t| + l_t,\, t \geq 0)$, where $(l_t,\, t \geq 0)$ is the local time at 0 of $(B_t,\, t \geq 0)$. Then, it follows from [40] that $(Y_t,\, t \geq 0)$ is a 3-dimensional Bessel process starting from 0, and a key to this result is that, if $(\mathcal{G}_t = \sigma(Y_s,\, s \leq t),\, t \geq 0)$, then, for every Borel function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, one has

$$\mathbb{E}[f(X_t)|\mathcal{G}_t] = \int_0^1 f(x\, Y_t)\, dx\,,$$

so that the hypotheses made in Proposition 2.1 are satisfied with

$$\Lambda f(y) = \int_0^1 f(x\, y)\, dx\,.$$

Several variants of this result, in different contexts, have now been obtained, starting with Pitman and Rogers [41].

2.4. Age-processes.

Let $(X_t,\, t \geq 0)$ be a real-valued diffusion such that 0 is regular for itself, and let $\mathbf{n}$ be the characteristic measure of excursions of X away from 0. Define $g_t = \sup\{s \leq t : X_s = 0\}$; $(A_t = t - g_t,\, t \geq 0)$ is called the *age-process*.

$(A_t,\, t \geq 0)$ is a Markov process in the filtration $(\mathcal{G}_t = \mathcal{F}_{g_t},\, t \geq 0)$, and its semi-group $(\Pi_t,\, t \geq 0)$ satisfies

$$\Pi_t\, \Lambda = \Lambda P_t\;, \qquad \text{where } \Lambda f(a) = \mathbf{n}(f(X_a)|V > a)\,,$$

with V the life-time of the generic excursion under $\mathbf{n}$. The identity

$$\mathbb{E}[f(X_t)|\mathcal{F}_{g_t}] = \Lambda f(A_t)$$

(which corresponds to the third hypothesis in Proposition 2.1) may be proved by excursion theory. In the particular case where $(X_t,\, t \geq 0)$ is a Bessel process with dimension $d \in (0, 2)$ and index $-\nu$ (the dimension and the index are related by $d = 2\,(1 - \nu)$, so that $0 < \nu < 1$), we now identify Λ.

We simply write g for g_1, and define the Bessel meander of index ν, $(m_\nu(u),\ u \le 1)$, by the formula

$$m_\nu(u) = \frac{1}{\sqrt{1-g}}\, X_{g+u(1-g)}\ , \qquad u \le 1$$

(this process is called the Brownian meander in the case $\nu = 1/2$). Then, we have the following

Lemma 2.2. *Let* $0 < \nu < 1$.

1) m_ν *is independent of* $\mathcal{F}_g$.

2) M_ν, *the distribution of* m_ν *on* $C([0,1], \mathbb{R}_+)$, *and* $P_0^{(\nu)}$, *the distribution on* $C([0,1], \mathbb{R}_+)$ *of* BES(d'), *with* $d' = 2(1+\nu)$, *satisfy the absolute continuity relationship*

$$M_\nu = \frac{c_\nu}{X_1^{2\nu}}\, P_0^{(\nu)}\ , \qquad \textit{with } c_\nu = \frac{\Gamma(1+\nu)}{2^{1+\nu}}\ . \tag{2.d}$$

The relation (2.d) is a generalization of Imhof's relation for $\nu = 1/2$. A proof of this relation involving enlargement of filtrations and change of probabilities, may be found in [2]; another proof is given in [61, Chapter 3].

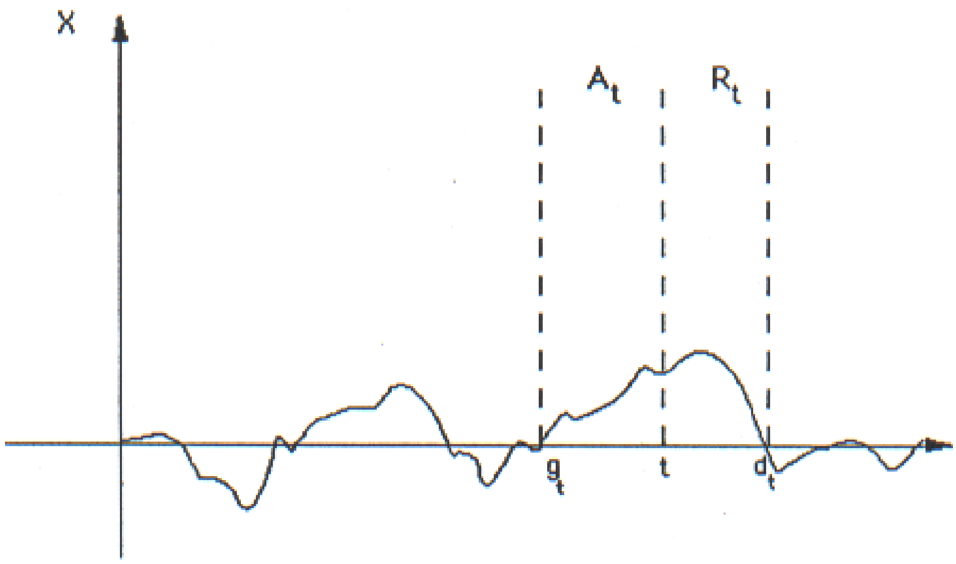

Figure 1. Age and Residual-life processes.

As a consequence of (2.d), it is easily seen that the distributions M_ν are all distinct as ν varies in $(0,1)$, but that, nonetheless, the one-dimensional marginal $X_1(M_\nu)$ does not depend on ν; we have

$$X_1(M_\nu)(d\rho) = \mathbb{P}(m_\nu(1) \in d\rho) = \rho\, e^{-\rho^2/2}\, d\rho\,,$$

so that,

$$\Lambda f(u) = \mathbb{E}[f(\sqrt{u}\, m_\nu(1))] = \int_0^\infty d\rho\, \rho\, e^{-\rho^2/2}\, f(\sqrt{u}\, \rho)\,.$$

REMARK. The age-process and the intertwining relationship corresponding to $\nu = 1/2$ have been considered in [1].

2.5. Residual-life processes.

Consider again $(X_t,\, t \geq 0)$ a real valued-diffusion such that 0 is regular for itself. Define $d_t = \inf\{s > t : X_s = 0\}$. The process $(R_t = d_t - t,\, t \geq 0)$ is called the residual-life process.

The random times $(d_t,\, t \geq 0)$ are obviously $(\mathcal{F}_t)$-stopping times, and $(R_t,\, t \geq 0)$ is a Markov process in the filtration $(\mathcal{F}_{d_t})$ with semigroup $\hat{\Pi}$ given by

$$\hat{\Pi}_u f(t) = \begin{cases} \mathbb{E}[E_{X_{u-t}}[f(T_0)]]\,, & \text{if } u \geq t\,, \\ f(t-u)\,, & \text{if } u < t\,, \end{cases}$$

where $\mathbb{E}$ denotes the expectation with respect to P_0, and $T_0 = \inf\{t > 0 : X_t = 0\}$. This is a classical result in regenerative systems theory (see [34] and [10]), the proof of which relies only on the strong Markov property of X.

Indeed, let f be a positive Borel function, and $T = d_s$. We want to establish the formula

$$\mathbb{E}[f(R_{t+s})|\mathcal{F}_T] = \hat{\Pi}_t\, f(R_s)\,.$$

On the event $\{t < R_s\} \in \mathcal{F}_T$, we have $d_{t+s} = d_s$ so that $R_{t+s} = R_s - t$.

On the event $\{t \geq R_s\}$, we can write

$$f(R_{t+s})(\omega) = g(\omega, \theta_T\, \omega)\,,$$

with $g(\omega,\omega') = f(R_{t-R_s(\omega)}(\omega'))$, a $\mathcal{F}_T \times \mathcal{F}$ measurable function. The strong Markov property taken at time $T = d_s$ yields, on $\{t \geq R_s\}$

$$\begin{aligned}\mathbb{E}[f(R_{t+s})|\mathcal{F}_T] &= \mathbb{E}[g(\omega, \theta_T\,\omega)|\mathcal{F}_T] \\ &= E_{X_T}[g(\omega,\cdot\,)] \\ &= \mathbb{E}[f(R_{t-R_s(\omega)}(\,\cdot\,))] \\ &= \mathbb{E}[E_{X_{t-R_s(\omega)}}[f(T_0)]]\,,\end{aligned}$$

(recall that for all t: $R_t = T_0 \circ \theta_t$).

The semi-group $(\hat{\Pi}_t)$ satisfies

$$P_t\,\Lambda = \Lambda\,\hat{\Pi}_t\,, \qquad \text{where } \Lambda f(x) = E_x[f(T_0)]\,,$$

and $(P_t,\, t \geq 0)$ denotes the semi-group of X. Indeed, for all positive Borel functions f, we have

$$\mathbb{E}[f(R_t)|\mathcal{F}_t] = \mathbb{E}[f(T_0 \circ \theta_t)|\mathcal{F}_t] = E_{X_t}[f(T_0)]\,.$$

In the case where $(X_t,\, t \geq 0)$ is a Bessel process with dimension $d < 2$ and index $(-\nu)$ (recall that $d = 2\,(1-\nu)$), the law of T_0 is well-known (see, *e.g.* [62])

$$T_0 \stackrel{\mathrm{d}}{=} \frac{x^2}{2\,Z_\nu}\,,$$

so that

$$\Lambda f(x) = \mathbb{E}\Big[f\Big(\frac{x^2}{2\,Z_\nu}\Big)\Big]\,.$$

Furthermore, if $u \geq t$

$$\hat{\Pi}_u f(t) = \mathbb{E}\Big[f\Big(\frac{X_{u-t}^2}{2\,Z_\nu}\Big)\Big] = \mathbb{E}\Big[f\Big(\frac{Z_{1-\nu}}{Z_\nu}\,(u-t)\Big)\Big]\,.$$

Consequently, the semi-group $\hat{\Pi}$ is given by

$$\hat{\Pi}_u f(t) = \mathbb{E}\Big[f\Big(\frac{Z_{1-\nu}}{Z_\nu}\,(u-t)^+ + (t-u)^+\Big)\Big]\,.$$

2.6. Brownian (or Bochner) subordination.

We present now an example of intertwining where $P_t \equiv K_t$ is the semi-group of the standard symmetric Cauchy process $(C_t,\, t \geq 0)$, and Λ is the kernel of multiplication by N, a centered, reduced, Gaussian variable (see Section 3.3 below for some general definition), *i.e.* for any Borel $f : \mathbb{R} \longrightarrow \mathbb{R}_+$,

$$\Lambda f(x) = \mathbb{E}[f(Nx)]\,.$$

Consider $(B_t, \beta_t,\, t \geq 0)$ a two dimensional Brownian motion starting from zero, and let

$$\mathcal{B}_t = \sigma(B_s, \beta_s,\, s \leq t)$$

be its natural filtration. Furthermore, let $(\tau_t,\, t \geq 0)$ be the inverse of the local time at zero of B.

Then, as is well-known (see, *e.g.*, Spitzer [47]), the process $(C_t \stackrel{\mathrm{def}}{=} \beta_{\tau_t},\, t \geq 0)$ is a standard symmetric Cauchy process; furthermore, if we define $\mathcal{G}_t = \sigma(\tau_s, s \leq t)$ and $\mathcal{F}_t \stackrel{\mathrm{def}}{=} \mathcal{B}_{\tau_t}$, then all the hypotheses at the beginning of this section are in force, with: $X_t = C_t$, and $Y_t = \sqrt{\tau_t}$.

Thus, if $(\theta_t^{(1/2)},\, t \geq 0)$ denotes the semi-group of $(\sqrt{\tau_t},\, t \geq 0)$, we deduce, from Proposition 2.1, the intertwining relationship

$$\theta_t^{(1/2)}\Lambda = \Lambda K_t\,. \tag{2.e}$$

More generally, if, for $0 < \alpha < 2$, $(C_t^\alpha,\, t \geq 0)$ denotes a symmetric stable process of index α starting from zero, this process may be represented as

$$C_t^\alpha = B_{T_t^{(\beta)}}\,, \qquad t \geq 0\,,$$

where $(T_t^{(\beta)},\, t \geq 0)$ denotes a one-sided subordinator of index $\beta \equiv \alpha/2$, independent from $(B_u, u \geq 0)$.

Then, just as above, if we call $(\theta_t^{(\beta)},\, t \geq 0)$ the semi-group of $(\sqrt{T_t^{(\beta)}},\, t \geq 0)$ and $(K_t^\alpha,\, t \geq 0)$ the semi-group of $(C_t^\alpha,\, t \geq 0)$, we obtain the following intertwining relationship

$$\theta_t^{(\alpha/2)}\Lambda = \Lambda K_t^\alpha\,. \tag{2.f}$$

More generally, we could also represent $(C_t^\alpha,\, t \geq 0)$ using a time change of another symmetric stable process $(C_u^\gamma, u \geq 0)$, by a suitable one-sided stable subordinator $(T_t^{(\delta)},\, t \geq 0)$, thus obtaining a more general family of intertwinings relating the symmetric stable processes to the one-sided stable subordinators.

We intend to develop such studies more thoroughly in a forthcoming paper.

Remark. After the presentation of these six classes of examples, the following instructive remark may be made: in the set-up of Proposition 2.1, it is wrong to think of $(Y_t,\, t \geq 0)$ as a (Markov) process which would carry less information than the process $(X_t,\, t \geq 0)$, so that one would have

$$\text{(2.g)} \qquad \sigma(Y_s,\, s \leq t) \subset \sigma(X_s,\, s \leq t)\,, \qquad \text{for every } t \geq 0\,.$$

Indeed, in Section 2.1, it is X which, generally, carries less information than Y; in Section 2.2, the natural filtrations of X and Y cannot, in general, be compared; in sections 2.3 and 2.4, Y carries less information than X. Instead of (2.g), the important assumption in Proposition 2.1 is that X is Markovian with respect to $(\mathcal{F}_t)$, and Y is Markovian with respect to $(\mathcal{G}_t)$, with $\mathcal{G}_t \subset \mathcal{F}_t$; this is quite different from asserting (2.g).

3. The algebra of beta-gamma variables and its relationship with intertwining.

3.1. The beta-gamma algebra.

In order to facilitate the reading of the main Section, 3.3, we need to recall a few well-known facts about beta and gamma distributed random variables.

Let a and b be two strictly positive real numbers. We shall consider three families of random variables, which we denote respectively by Z_a, $Z_{a,b}$, $Z^{(2)}_{a,b}$, and which are distributed as follows

$$P(Z_a \in dx) = \gamma_a(dx) = x^{a-1}\, e^{-x}\, \frac{dx}{\Gamma(a)}\,, \qquad x > 0\,,$$

$$P(Z_{a,b} \in dx) = \beta_{a,b}(dx) = x^{a-1}\, (1-x)^{b-1}\, \frac{dx}{B(a,b)}\,, \qquad 0 < x < 1\,,$$

$$P(Z^{(2)}_{a,b} \in dx) = \beta^{(2)}_{a,b}(dx) = \frac{x^{a-1}\, dx}{(1+x)^{a+b} B(a,b)}\,, \qquad x > 0$$

(recall that: $B(a,b) = \Gamma(a)\,\Gamma(b)/\Gamma(a+b)$).

There exist important (well-known) algebraic relations between the laws of these different variables (see *e.g.* [22]; for some applications of these relations, see [12] which also refers to [23]).

We first remark that

$$Z^{(2)}_{a,b} \stackrel{\mathrm{d}}{=} \frac{Z_{a,b}}{1 - Z_{a,b}} \,. \tag{3.a}$$

The main relation is the following

$$(Z_{a,b}, Z_{a+b}) \stackrel{\mathrm{d}}{=} \left(\frac{Z_a}{Z_a + Z_b}, Z_a + Z_b\right), \tag{3.b}$$

where, on the left hand side, the two variables are assumed to be independent, while on the right hand side, Z_a and Z_b are assumed to be independent and, as a consequence of (3.b), $Z_a/(Z_a + Z_b)$ and $Z_a + Z_b$ are independent.

Here is an interesting consequence of (3.b): if $Z_{a,b}$ and $Z_{a+b,c}$ are independent, then

$$Z_{a,b}\, Z_{a+b,c} \stackrel{\mathrm{d}}{=} Z_{a,b+c} \,. \tag{3.c}$$

Proof of (3.c). From (3.b), the pair of variables $(Z_{a,b}, Z_{a+b,c})$ may be realized as the pair

$$\left(\frac{Z_a}{Z_a + Z_b}, \frac{Z_a + Z_b}{Z_a + Z_b + Z_c}\right)$$

with Z_a, Z_b, Z_c independent; then

$$Z_{a,b}\, Z_{a+b,c} \stackrel{\mathrm{d}}{=} \frac{Z_a}{Z_a + Z_b + Z_c} \stackrel{\mathrm{d}}{=} \frac{Z_a}{Z_a + Z_{b+c}} \stackrel{\mathrm{d}}{=} Z_{a,b+c} \,.$$

We now remark that, as a consequence of (3.a) and (3.b), we obtain

$$Z^{(2)}_{a,b} \stackrel{\mathrm{d}}{=} \frac{Z_a}{Z_b} \,, \tag{3.d}$$

where Z_a and Z_b are assumed to be independent.

Finally, we remark that if $Z_{a,b}$ and $Z^{(2)}_{a+b,c}$ are independent, then

(3.e) $$Z_{a,b}\, Z^{(2)}_{a+b,c} \overset{\mathrm{d}}{=} Z^{(2)}_{a,c}\ .$$

Proof of (3.e). From (3.b) and (3.d), the pair of variables

$$(Z_{a,b}, Z^{(2)}_{a+b,c})$$

may be realized as the pair

$$\Big(\frac{Z_a}{Z_a+Z_b}, \frac{Z_a+Z_b}{Z_c}\Big)\,,$$

with Z_a, Z_b, Z_c independent. We then obtain

$$Z_{a,b}\, Z^{(2)}_{a+b,c} \overset{\mathrm{d}}{=} \frac{Z_a}{Z_a+Z_b}\,\frac{Z_a+Z_b}{Z_c} \overset{\mathrm{d}}{=} \frac{Z_a}{Z_c}\ .$$

3.2. Notation.

All the intertwining kernels Λ which will be featured in this Section 3 act from $\mathbb{R}_+$ to $\mathbb{R}_+$, and are of the form

$$\Lambda f(x) = \mathbb{E}[f(x\,Z)]\,,$$

for some positive random variable Z; it will be convenient to say that Λ is the kernel of multiplication by Z.

More precisely, we shall encounter the multiplication kernels listed in the following table

Z	$2\,Z_\alpha$	$1/(2\,Z_\alpha)$	$Z_{\alpha,\beta}$	$1/Z_{\alpha,\beta}$	$Z^{(2)}_{\alpha,\beta}$
Λ	Λ_α	$\tilde\Lambda_\alpha$	$\Lambda_{\alpha,\beta}$	$\tilde\Lambda_{\alpha,\beta}$	$\Lambda^{(2)}_{\alpha,\beta}$

Table 1. Multiplication Kernels.

3.3. Markovian extensions of the beta-gamma algebra.

In this section, (Q_t^α) denotes the semi-group of the square of the Bessel process of dimension $2\,\alpha$. Then, we have the following

Theorem 3.1. *For every $\alpha > 0$, $\beta > 0$ and every t,*

$$Q_t^{\alpha+\beta}\Lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}\, Q_t^\alpha \,. \tag{3.f}$$

REMARKS. 1) The identity (3.f) may be understood as a Markovian extension of the relation (3.b), since we deduce, in particular, from (3.f), that,

$$Q_t^{\alpha+\beta}\Lambda_{\alpha,\beta} f(0) = \Lambda_{\alpha,\beta}\, Q_t^\alpha f(0)\,,$$

which is equivalent to

$$\mathbb{E}\,[f(2\,t\,Z_{\alpha,\beta}\,Z_{\alpha+\beta})] = \mathbb{E}\,[f(2\,t\,Z_\alpha)]\,, \tag{3.g}$$

where, on the left hand side, $Z_{\alpha,\beta}$ and $Z_{\alpha+\beta}$ are assumed to be independent.

The relation (3.g) is another way to write the following main consequence of (3.b)

$$Z_\alpha \overset{\mathrm{d}}{=} Z_{\alpha,\beta}\, Z_{\alpha+\beta}\,.$$

2) We have already encountered the relation (3.f) in the particular case: $\alpha = 1/2$, $\beta = 1$, in Section 2.3.

3) As a consequence of (3.g), the infinitesimal generators are intertwined

$$L^{\alpha+\beta}\,\Lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}\, L^\alpha\,.$$

This relation corresponds, in the language of differential equations, to the transmutation of differential operators (see *e.g.* Trimèche [48]).

PROOF OF THEOREM 3.1. The identity (3.f) may be obtained as a consequence of Proposition 2.1; indeed, if (X_t^α) and (X_t^β) are independent squares of Bessel processes, with respective dimensions $2\,\alpha$ and $2\,\beta$, starting at 0, then $(X_t^{\alpha+\beta} \overset{\text{def}}{=} X_t^\alpha + X_t^\beta,\ t \geq 0)$ is the square of a Bessel process of dimension $2\,(\alpha+\beta)$, and the hypotheses which are in force in Proposition 2.1 are satisfied with

$$\mathcal{F}_t = \sigma(X_s^\alpha, X_s^\beta,\ s \leq t)\,, \qquad \mathcal{G}_t = \sigma(X_s^{\alpha+\beta},\ s \leq t)\,,$$
$$X_t = X_t^\alpha\,, \qquad\qquad Y_t = X_t^{\alpha+\beta}\,.$$

Indeed, by time-inversion, the processes $(t^2 X^\alpha_{1/t},\, t \geq 0)$ and $(t^2 X^\alpha_{1/t},\, t\geq 0)$ are independent squares of Bessel processes of respective dimensions $2\,\alpha$ and $2\,\beta$, starting from zero.

Let H be a non-negative measurable functional, and let f be a positive Borel function; we have

$$\mathbb{E}\,[H(Y_u,\, u \leq t)\, f(X^\alpha_t)] = \mathbb{E}\,[H(u^2\, Y_{1/u},\, u \leq t)\, f(t^2 X_{1/t})]\,.$$

We note $H_t = H(u^2 Y_{1/u},\, u \leq t)$. Since $(t^2 Y_{1/t},\, t \geq 0)$ is Markovian with respect to the filtration $\sigma(X^\alpha_{1/u}, X^\beta_{1/u},\, u \leq t)$,

$$\mathbb{E}\,[H(Y_u, u \leq t) f(X^\alpha_t)] = \mathbb{E}\,[H_t f(t^2 X^\alpha_{1/t})] = \mathbb{E}\,[\mathbb{E}\,[H_t|Y_{1/t}]\, f(t^2 X^\alpha_{1/t})]\,.$$

We now use (3.b) and the fact that $\Lambda_{\alpha,\beta}$ is a multiplication kernel to obtain

$$\begin{aligned}
\mathbb{E}\,[H(Y_u,\, u \leq t) f(X^\alpha_t)] &= \mathbb{E}\,[\mathbb{E}\,[H_t|Y_{1/t}]\Lambda_{\alpha,\beta} f(t^2\, Y_{1/t})] \\
&= \mathbb{E}\,[H_t\, \Lambda_{\alpha,\beta}\, f(t^2\, Y_{1/t})] \\
&= \mathbb{E}\,[H(Y_u,\, u \leq t)\, \Lambda_{\alpha,\beta} f(Y_t)]\,.
\end{aligned}$$

By comparing the two extreme terms, and letting H vary, we get

$$\mathbb{E}\,[f(X^\alpha_t)|\mathcal{G}_t] = \Lambda_{\alpha,\beta}\, f(Y_t)\,.$$

We consider again the relation (3.g) which we write in a more concise form as

$$\Lambda_\alpha = \Lambda_{\alpha,\beta}\, \Lambda_{\alpha+\beta}\,.$$

Since multiplication kernels commute, we also have

$$\Lambda_\alpha = \Lambda_{\alpha+\beta}\, \Lambda_{\alpha,\beta}$$

and this identity admits the following extension

Theorem 3.2. *Let $\alpha > 0$, $\beta > 0$, such that $\alpha + \beta \geq 1$. Then*:

1) *There exists a semi-group on $\mathbb{R}_+$, which we denote $(\Pi^{\alpha,\beta}_t)$, such that*

$$\Pi^{\alpha,\beta}_t \Lambda_{\alpha+\beta} = \Lambda_{\alpha+\beta}\, Q^\alpha_t\,. \tag{3.h}$$

2) *This semi-group is characterized by*

$$\int_0^\infty \Pi_t^{\alpha,\beta}(y,dz)\,(1+\lambda\,z)^{-(\alpha+\beta)} = \frac{(1+\lambda\,t)^\beta}{(1+\lambda\,(t+y))^{\alpha+\beta}}\,, \tag{3.i}$$

for all $t, \lambda, y \geq 0$.

3) *Suppose* $\alpha+\beta > 1$. *Then every* C^1*-function* $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, *with compact support, belongs to the domain of the infinitesimal generator* $L_{\alpha,\beta}$ *of* $(\Pi_t^{\alpha,\beta})$, *and*

$$L_{\alpha,\beta} f(x) = f'(x) + \frac{\beta(\alpha+\beta-1)}{x}\int_0^1 y^{\alpha+\beta-2}\,(f(x\,y)-f(x))\,dy\,.$$

COMMENTS. 1) The particular case $\alpha+\beta = 1$ of the relation (3.h) was already encountered in Section 2.4 (up to some elementary modification, since in that example we considered the Bessel process of dimension $2\,\alpha$, instead of the square). More precisely, the square $(A_t^2,\, t \geq 0)$ of the age process of the Bessel process of dimension $2\,\alpha$ is a realization (starting from 0) of the process $X^{\alpha,1-\alpha}$. On the contrary, in the case $\alpha+\beta > 1$, we do not know whether the relation (3.h) may be obtained as a consequence of Proposition 2.1 and our proof of (3.h) consists in showing the existence of $(\Pi_t^{\alpha,\beta})$ via (3.i). The relation (3.i) is deduced from (3.h) by applying both sides to the function

$$e_\lambda(y) \stackrel{\text{def}}{=} \exp\Big(-\frac{\lambda}{2}\,y\Big)\,, \qquad y \geq 0$$

and using the relations

$$\begin{cases} \Lambda_{\alpha+\beta}(e_\lambda)(z) = c_{\alpha+\beta}\,(1+\lambda\,z)^{-(\alpha+\beta)}\,, \\ Q_t^\alpha(e_\lambda)(z) = (1+\lambda\,t)^{-\alpha}\,\exp\Big(-\dfrac{\lambda\,z}{2\,(1+\lambda\,t)}\Big)\,. \end{cases} \tag{3.j}$$

2) The third part of the theorem follows from the second when one considers the functions

$$\phi_\lambda(z) = (1+\lambda\,z)^{-\alpha}\,.$$

3) In the case $\alpha+\beta > 1$, the following pathwise description of a Markov process $X^{\alpha,\beta}$ with semi-group $\Pi_t^{\alpha,\beta}$ is easily deduced from part

3) of the theorem: the trajectories of $X^{\alpha,\beta}$ are ascending sawteeth of constant slope 1.

More precisely, starting from $x_0 > 0$,

$$X_t^{\alpha,\beta} = x_0 + t\,, \qquad 0 \le t < S\,,$$

where $S = x_0\,(e^{\sigma} - 1)$, and σ is an exponential random variable of parameter β. Then, $X^{\alpha,\beta}$ has a negative jump of magnitude $(1 - e^{-T})\,X_{S_-}^{\alpha,\beta}$, where T is an exponential random variable of parameter $\alpha + \beta - 1$, independent of S; then, $X^{\alpha,\beta}$ starts anew from $x_1 = X_S^{\alpha,\beta}$. We draw a typical trajectory of $X^{\alpha,\beta}$.

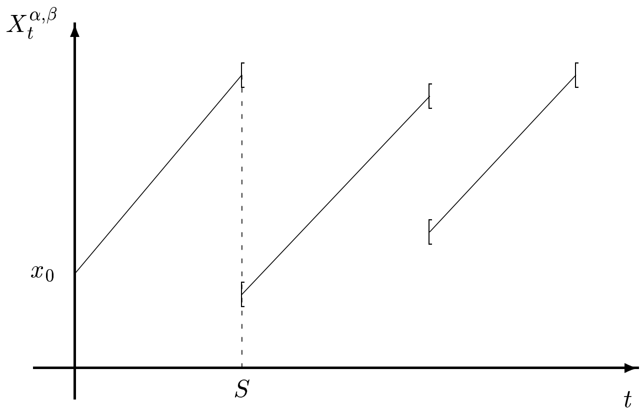

Figure 2. Trajectories of $X^{\alpha,\beta}$.

We will show in Section 3.4 the existence of a positive measure $\Pi_t^{\alpha,\beta}(y, dz)$, which is characterized by (3.i); this existence is assumed for the moment. We now discuss duality properties for the semi-groups (Q_t^{α}) and $(\Pi_t^{\alpha,\beta})$; this will be important in the sequel, both in order to discover some new intertwining relations (see theorems 3.4 and 3.5 below) and also to express some results of time reversal for $X^{\alpha,\beta}$ (see Section 4.5 below). We begin by recalling the

Definition. *Two Markov semi-groups (P_t) and $(\hat{P}_t)$ on E are said to be in duality with respect to a σ-finite positive measure μ (in short: they are in μ-duality), if for every pair of measurable functions $f, g : E \longrightarrow \mathbb{R}_+$,*

$$\langle P_t f, g\rangle_{\mu} = \langle f, \hat{P}_t\, g\rangle_{\mu}\,.$$

We now have the following

Theorem 3.3. *Let $\alpha > 0$ and $\mu(dx) = x^{\alpha-1}\,dx$. Then*:

1) *Q_t^α is self-dual with respect to μ.*

2) *Let $\beta > 0$, such that $\alpha+\beta > 1$. There is a unique Markovian semi-group $(\hat{\Pi}_t^{\alpha,\beta})$ on $\mathbb{R}_+$, which is in μ-duality with $(\Pi_t^{\alpha,\beta})$.*

3) *Every C^1-function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with compact support, belongs to the domain of the infinitesimal generator $\hat{L}_{\alpha,\beta}$ of $(\hat{\Pi}_t^{\alpha,\beta})$, and we have*

$$\hat{L}_{\alpha,\beta} f(x) = -f'(x) + \beta\,\frac{\alpha+\beta-1}{x}\int_1^\infty \frac{f(x\,y)-f(x)}{y^{1+\beta}}\,dy\,.$$

Remarks. 1) Suppose $\alpha < 1$. If we let β decrease to $1-\alpha$, we obtain in the limit a semi-group $\hat{\Pi}^{\alpha,1-\alpha}$. A realization of this semi-group is given by the square $(R_t^2,\, t \geq 0)$ of the residual-life process of a Bessel process of dimension $2\,\alpha$ (see Section 2.5).

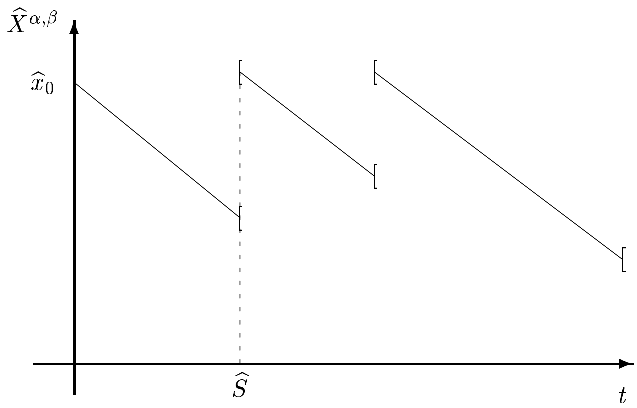

Figure 3. Trajectories of $\hat{X}^{\alpha,\beta}$.

2) Again, in the case $\alpha+\beta > 1$, the following pathwise description of a Markov process $\hat{X}^{\alpha,\beta}$ with semi-group $\hat{\Pi}_t^{\alpha,\beta}$ is easily deduced from the form of the infinitesimal generator $\hat{L}_{\alpha,\beta}$: the trajectories of $\hat{X}^{\alpha,\beta}$ are descending sawteeth of constant slope -1.

More precisely, starting from $\hat{x}_0 > 0$,

$$\hat{X}_t^{\alpha,\beta} = \hat{x}_0 - t\,, \qquad 0 \leq t < \hat{S}\,,$$

where $\hat{S} = \hat{x}_0\,(1 - e^{-\sigma})$, and σ is an exponential random variable of parameter $\alpha + \beta - 1$. Then, $\hat{X}^{\alpha,\beta}$ has a positive jump of magnitude $(1 - e^{-\hat{T}})\,\hat{X}^{\alpha,\beta}_{\hat{S}_-}$, where $\hat{T}$ is an exponential random variable of parameter β, independent of $\hat{S}$; then, $\hat{X}^{\alpha,\beta}$ starts anew from $\hat{x}_1 = \hat{X}^{\alpha,\beta}_{\hat{S}}$. We draw a typical trajectory of $\hat{X}^{\alpha,\beta}$.

From Theorem 3.3, we easily deduce two other intertwining relations, namely (3.k) and (3.l) below.

Theorem 3.4. *Let $\alpha > 0, \beta > 0$ such that $\alpha + \beta > 1$. Then, we have*

$$Q_t^\alpha \tilde{\Lambda}_\beta = \tilde{\Lambda}_\beta\, \hat{\Pi}_t^{\alpha,\beta}\;.$$

Proof. We start from the intertwining relation (3.h)

$$\Pi_t^{\alpha,\beta}\, \Lambda_{\alpha+\beta} = \Lambda_{\alpha+\beta}\, Q_t^\alpha\;,$$

and consider the adjoint operators in $L^2(\mu)$, where $\mu(dx) = x^{\alpha-1}\,dx$, as in Theorem 3.3. We obtain

$$Q_t^\alpha\, \hat{\Lambda}_{\alpha+\beta} = \hat{\Lambda}_{\alpha+\beta}\, \hat{\Pi}_t^{\alpha,\beta}\;,$$

since Q_t^α is self-adjoint with respect to μ (obviously $\hat{\Lambda}_{\alpha+\beta}$ denotes the adjoint of $\Lambda_{\alpha+\beta}$ with respect to μ). It remains to compute explicitly $\hat{\Lambda}_{\alpha+\beta}$; one finds

$$\hat{\Lambda}_{\alpha+\beta}\, g(y) = \frac{\Gamma(\beta)}{2^\alpha\,\Gamma(\alpha+\beta)}\, \mathbb{E}\left[g\Big(\frac{y}{2\,Z_\beta}\Big)\right] = c_{\alpha,\beta}\, \tilde{\Lambda}_\beta\, g(y)\,.$$

Theorem 3.5. *Let $\alpha > 0, \beta > 0$ such that $\alpha + \beta > 1$. Then, we have*

$$\Pi_t^{\alpha,\beta}\, \Lambda^{(2)}_{\alpha+\beta,\beta} = \Lambda^{(2)}_{\alpha+\beta,\beta}\, \hat{\Pi}_t^{\alpha,\beta}\,. \tag{3.l}$$

Proof. Remark that, from (3.d): $\Lambda^{(2)}_{\alpha+\beta,\beta} = \Lambda_{\alpha+\beta}\, \tilde{\Lambda}_\beta$. The result (3.l) now follows immediately from the intertwining relations (3.h) and (3.k).

As was already pointed out, Theorems 3.1 and 3.2 may be understood as Markovian extensions of the relation (3.b). Likewise, the next theorem is a Markovian extension of the relation

$$Z_{a,b+c} \stackrel{\mathrm{d}}{=} Z_{a,b}\, Z_{a+b,c}\ , \tag{3.c}$$

with the notation of Section 3.1.

Theorem 3.6. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$, such that $\alpha + \beta > 1$. Then*

$$\Pi_t^{\alpha,\beta}\, \Lambda_{\alpha+\beta,\gamma} = \Lambda_{\alpha+\beta,\gamma}\, \Pi_t^{\alpha,\beta+\gamma} \tag{3.m}$$

and

$$\hat{\Pi}_t^{\alpha,\beta+\gamma}\, \Lambda_{\beta,\gamma} = \Lambda_{\beta,\gamma}\, \hat{\Pi}_t^{\alpha,\beta}\ . \tag{3.n}$$

Proof. A kernel Λ is said to be determining if, when considered as a linear operator from $C_0(\mathbb{R}_+)$ to $C_0(\mathbb{R}_+)$, it is injective.

1) Since the kernel $\Lambda_{\alpha+\beta+\gamma}$ is determining, it suffices, in order to prove (3.m), to show the relation

$$\Pi_t^{\alpha,\beta}\, \Lambda_{\alpha+\beta,\gamma}\, \Lambda_{\alpha+\beta+\gamma} = \Lambda_{\alpha+\beta,\gamma}\, \Pi_t^{\alpha,\beta+\gamma}\, \Lambda_{\alpha+\beta+\gamma}\ . \tag{3.o}$$

Now, the left-hand side of (3.o) is equal to $\Pi_t^{\alpha,\beta}\, \Lambda_{\alpha+\beta}$, with the help of (3.g). The right-hand side of (3.o) is equal to

$$\Lambda_{\alpha+\beta,\gamma}\, \Lambda_{\alpha+\beta+\gamma}\, Q_t^{\alpha} = \Lambda_{\alpha+\beta}\, Q_t^{\alpha} = \Pi_t^{\alpha,\beta}\, \Lambda_{\alpha+\beta}\ ,$$

using first Theorem 3.2, then (3.g), and again Theorem 3.2.

2) To prove (3.n), we consider the adjoint operators in $L^2(\mu)$, where $\mu(dx) = x^{\alpha-1}\, dx$, of the kernels featured in (3.m).

By Theorem 3.3, the adjoint of $\Pi_t^{\alpha,\beta}$ (respectively $\Pi_t^{\alpha,\beta+\gamma}$) is $\hat{\Pi}_t^{\alpha,\beta}$ (respectively $\hat{\Pi}_t^{\alpha,\beta+\gamma}$), and it is easily shown that the adjoint of $\Lambda_{\alpha+\beta,\gamma}$ is a multiple of $\Lambda_{\beta,\gamma}$. The relation (3.n) is now proved.

Remarks. 1) Assuming that the different intertwining relations obtained in this chapter may be realized in such a way that they fit in the filtering framework discussed in Section 2.1, Theorem 3.6 suggests that, for α fixed, and as β increases, the process $X^{\alpha,\beta}$ is Markovian with respect to a filtration $(\mathcal{F}_t^{(\beta)},\, t \geq 0)$ which increases with β; roughly speaking, more information seems to be required as β increases in order to construct $X^{\alpha,\beta}$, and the case $\beta = \infty$ corresponds to BES $Q(2\,\alpha)$; see Section 4.5 for a more precise result formulated as a limit in law.

2) Transforming the relation (3.1) in Theorem 3.5 by duality with respect to the measure $\mu(dx) = x^{\alpha-1}\,dx$ does not yield any new relation since $\Lambda^{(2)}_{\alpha+\beta,\beta}$ is its own adjoint (up to a multiplicative constant).

3.4. Explicit computation of the semi-group $\Pi_t^{\alpha,\beta}$.

This section is devoted to the proof of the existence of a probability measure $\Pi_t^{\alpha,\beta}(y, dz)$ which satisfies formula (3.i); we have not found an elegant way to avoid the technical computations of this section.

We first reduce the problem to the inversion of a certain Laplace transform. Let t, y be given and define $\kappa = t/(t+y)$. Then, from formula (3.i), there exists a measure $\mu^\kappa(du)$ on $\mathbb{R}_+$ which depends only on κ (and α, β) such that

$$\int \Pi_t^{\alpha,\beta}(y; dz)\, f(z) = \int \mu^\kappa(du)\, f((t+y)\,u)$$

and, from formula (3.i) again, μ^κ is the only probability measure on $\mathbb{R}_+$ such that, for every $\lambda \geq 0$

$$\int_0^\infty \mu^\kappa(du)\,(1+\lambda\,u)^{-(\alpha+\beta)} = \frac{(1+\lambda\,\kappa)^\beta}{(1+\lambda)^{\alpha+\beta}}\,. \tag{3.p}$$

In fact, from the comments following Theorem 3.2, we see that μ^κ must be carried by $[0,1]$.

We shall then deduce from formula (3.p) the following Laplace transform identity

$$\begin{aligned} &\int_0^1 \mu^\kappa(du)\, u^{-(\alpha+\beta)} \exp\Big(-s\Big(\frac{1}{u}-1\Big)\Big) \\ &\qquad\qquad = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\Big(\frac{\kappa}{s}\Big)^\beta\, \Phi\Big(-\beta, \alpha; -\frac{\overline{\kappa}\,s}{\kappa}\Big)\,, \end{aligned} \tag{3.q}$$

where $\overline{\kappa} = 1-\kappa = y/(t+y)$, and $\Phi(a,b;z)$ is the confluent hypergeometric function defined by

$$\Phi(a,b;z) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k}\,\frac{z^k}{k!}\,,$$

where $(a)_k = a\,(a+1)\cdots(a+k-1)$. The hypergeometric function

$$F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

shall also play a prominent role in the sequel (see *e.g.* Lebedev [30]). Now, the key to the explicit computation of $\Pi_t^{\alpha,\beta}$ is the

Proposition 3.7. *Let $\alpha > 0$, $\beta > 0$ and $\alpha + \beta > 1$. Then:*

1) *there exists a unique function $g_{\alpha,\beta} : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+$ such that for all $s \geq 0$*

$$1 + \int_0^\infty du\, g_{\alpha,\beta}(u)\, e^{-su} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\, s^{-\beta}\, \Phi(-\beta,\alpha;-s)\,,$$

2) *the function $g_{\alpha,\beta}$ may be expressed as follows in terms of F*

$$g_{\alpha,\beta}(u) = \begin{cases} c_+\, u^{\beta-1}\, F\Big(\beta, 1-\beta, \alpha; \dfrac{1}{u}\Big)\,, & \text{if } u > 1\,, \\ c_-\, F(2-\alpha-\beta, 1-\beta, 2; u)\,, & \text{otherwise}, \end{cases}$$

where

$$c_+ = \frac{1}{B(\alpha,\beta)} \qquad \text{and} \qquad c_- = (\alpha+\beta-1)\,\beta\,.$$

It is now easy to express μ^κ and $\Pi_t^{\alpha,\beta}(y;dz)$ in terms of $g_{\alpha,\beta}$. We obtain the

Theorem 3.8. *Let $\alpha > 0$, $\beta > 0$ and $\alpha+\beta > 1$. Then*

$$\mu^\kappa(du) = \overline{\kappa}^\beta\, \varepsilon_1(du) + \Big(\frac{\overline{\kappa}}{\kappa}\Big)\, \overline{\kappa}^\beta\, g_{\alpha,\beta}\Big(\frac{\kappa\,\overline{u}}{\overline{\kappa}\,u}\Big) u^{\alpha+\beta-2}\, \mathbf{1}_{\{0<u<1\}}\, du$$

and the semi-group $\Pi_t^{\alpha,\beta}$ is given by the formula

$$\begin{aligned} \int \Pi_t^{\alpha,\beta}(y;dz)\, f(z) &= \Big(\frac{y}{t+y}\Big)^\beta f(t+y) \\ &\quad + \int_0^1 du\, u^{\alpha+\beta-2} \Big(\frac{y}{t+y}\Big)^\beta g_{\alpha,\beta}\Big(\frac{t}{y}\Big(\frac{1}{u}-1\Big)\Big) f((t+y)\,u)\,. \end{aligned}$$

For the sake of clarity, we have postponed the proofs of formula (3.q) and Proposition 3.7 until now.

Proof of formula (3.q). If we apply the formula

$$\frac{1}{a^{\alpha+\beta}} = \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty dx\, x^{\alpha+\beta-1}\, e^{-ax}$$

to $a = 1 + \lambda\, u$, the left-hand side of (3.p) becomes

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+\beta)} \int_0^1 \mu^\kappa(du) \int_0^\infty dx\, x^{\alpha+\beta-1}\, e^{-x-\lambda u x} \\ &\qquad = \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty d\xi\, e^{-\lambda\xi}\, \xi^{\alpha+\beta-1} \int_0^1 \mu^\kappa(du)\, u^{-(\alpha+\beta)}\, e^{-\xi/u} \,. \end{aligned}$$

We shall now identify the right-hand side of (3.p) as a Laplace transform in λ. Since formula (3.i) follows from (3.h), we know that

$$\text{(3.r)} \qquad \frac{(1+\lambda\, t)^\beta}{(1+\lambda\,(t+y))^{\alpha+\beta}} = \mathbb{E}[Q_t^\alpha(2\, y\, Z_{\alpha+\beta}; e_\lambda)] \,,$$

where, keeping with our notation, $Z_{\alpha+\beta}$ is a gamma variable with parameter $\alpha+\beta$. We introduce the density $p_t^\alpha(a,b)$ of Q_t^α which is known to be (see [37])

$$\text{(3.s)} \quad p_t^\alpha(a,b) = \frac{1}{2\,t}\Big(\frac{b}{a}\Big)^{(\alpha-1)/2} \exp\Big(-\frac{a+b}{2\,t}\Big) I_{\alpha-1}\Big(\frac{\sqrt{a\,b}}{t}\Big) \,, \quad a \neq 0 \,.$$

Making an elementary change of variable in (3.r), we obtain the identity

$$\frac{(1+\lambda\,\kappa)^\beta}{(1+\lambda)^{\alpha+\beta}} = 2\,(t+y) \int_0^\infty d\xi\, e^{-\lambda\xi}\, \mathbb{E}[p_t^\alpha(2\,y\,Z_{\alpha+\beta}; 2\,(t+y)\,\xi)] \,.$$

Comparing the new forms we have just obtained for the two sides of (3.p), we get the identity

$$\begin{aligned} \text{(3.t)} \qquad & \frac{\xi^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^1 e^{-\xi/u}\, u^{-(\alpha+\beta)}\, \mu^\kappa(du) \\ &\qquad\qquad = 2\,(t+y)\, \mathbb{E}[p_t^\alpha(2\,y\,Z_{\alpha+\beta}; 2\,(t+y)\,\xi)] \,. \end{aligned}$$

Using formula (3.s), we obtain

$$\begin{aligned}
&2\,(t+y)\,\mathbb{E}\,[p_t^\alpha\,(2\,y\,Z_{\alpha+\beta};2\,(t+y)\,\xi)]\\
&\quad=\Big(\kappa\,\overline{\kappa}\,\frac{\alpha-1}{2}\Big)^{-1}\mathbb{E}\Big[\Big(\frac{\xi}{Z_{\alpha+\beta}}\Big)^{(\alpha-1)/2}\\
&\qquad\qquad\cdot I_{\alpha-1}\Big(\frac{2\,\sqrt{\overline{\kappa}\,Z_{\alpha+\beta}\,\xi}}{\kappa}\Big)\exp\Big(-\frac{\overline{\kappa}\,Z_{\alpha+\beta}+\xi}{\kappa}\Big)\Big]
\end{aligned}$$

and, developing this expectation, we find that the formula (3.t) may be written as

$$\begin{aligned}
&\frac{\xi^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^1 e^{-\xi/u}\,u^{-(\alpha+\beta)}\,\mu^\kappa(du)\\
&\quad=\frac{\xi^{\alpha-1}\,e^{-\xi/\kappa}}{2\,\kappa\overline{\kappa}\,\dfrac{\alpha-1}{2}}\int_0^\infty d\eta\,\eta^{\alpha+\beta-(\alpha-1)/2}\,e^{-\eta/\kappa}\,I_{\alpha-1}\Big(\frac{2\sqrt{\overline{\kappa}\,\xi\,\eta}}{\kappa}\Big)\,. \qquad \text{(3.u)}
\end{aligned}$$

Now, with the help of the integral representation

$$\Phi(a,b;z)=\frac{\Gamma(b)}{\Gamma(b-a)}\,e^z\,z^{(1-b)/2}\int_0^\infty dt\,e^{-t}\,t^{((b-1)/2)-a}\,J_{b-1}(2\,\sqrt{z\,t})\,,$$

which is valid for $\mathrm{Re}\,(b-a)>0$, $|\arg(z)|<\pi$, $b\neq 0,1,2,\dots$ (see [30, p. 278]) together with the relation

$$I_\nu(\xi)=e^{-i\pi\nu/2}\,J_\nu(\xi\,e^{i\pi/2})\,,$$

we find that (3.r) may be written as

$$\int_0^1 e^{-\xi/u}\,u^{-(\alpha+\beta)}\,\mu^\kappa(du)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\,\Big(\frac{\kappa}{\xi}\Big)^\beta\,e^{-\xi}\,\Phi\Big(-\beta,\alpha;-\frac{\overline{\kappa}}{\kappa}\,\xi\Big)\,,$$

which is obviously equivalent to (3.q).

Proof of Proposition 3.7. i) The case when β is an integer n is easy, since $\Phi(-n,\alpha;-s)$ is a polynomial of degree n in s and the inversion of the Laplace transform

$$s^{-n}\,\Phi(-n,\alpha;-s)$$

is elementary.

ii) It then remains to prove the proposition when $0 < \beta < 1$, and then, when $1 < \beta < 2$, etc ...

In fact, from the definition of $g_{\alpha,\beta}$ as presented in Proposition 3.7.1), we deduce the recurrence relation

$$g_{\alpha,\beta}(x) = \frac{x^{\beta-1}}{B(\alpha,\beta)} + \beta \int_0^\infty dt\, t^{-\beta}\, g_{\alpha+1,\beta-1}(t\,x) \tag{3.v}$$

(more precisely, assuming that $g_{\alpha+1,\beta-1}$ exists, then if we define $g_{\alpha,\beta}$ by (3.v), it satisfies part 1) of the proposition).

On the other hand, we also show that the expression of $g_{\alpha,\beta}$ as presented in Proposition 3.7.2) satisfies the same recurrence relation; consequently, using a recurrence argument, it will be sufficient to prove the proposition in the case $0 < \beta < 1$.

iii) We start with the proof of the recurrence relation (3.v). We denote by $g^*_{\alpha,\beta}(x)$ the right-hand side of (3.v). We easily obtain the formula

$$\begin{aligned} 1 + \int_0^\infty du\, g^*_{\alpha,\beta}(u)\, e^{-su} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\, s^\beta} + \frac{\beta\, \Gamma(\alpha+\beta)}{s^\beta\, \Gamma(\alpha+1)} \int_0^s dv\, \Phi(1-\beta, \alpha+1; -v) \end{aligned}$$

and, in order to prove (3.v), it suffices to show that the right-hand side in the last equality is, in fact

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\, s^\beta}\, \Phi(-\beta, \alpha; -s)\,,$$

or, equivalently

$$\Phi(-\beta, \alpha; -s) = 1 + \frac{\beta}{\alpha} \int_0^s dv\, \Phi(1-\beta, \alpha+1; -v)\,.$$

But this follows from the identity

$$\frac{d}{dx}\, \Phi(-\beta, \alpha; -x) = \frac{\beta}{\alpha}\, \Phi(1-\beta, \alpha+1; -x)$$

(see [30, formula 9.9.4, p. 261]).

iv) We now prove the same recurrence relation (3.v) between $\tilde{g}_{\alpha,\beta}$ (the function defined in part 2) of the proposition in terms of F) and

$\tilde{g}_{\alpha+1,\beta-1}$. It is elementary to transform the desired relation (3.v) between $\tilde{g}_{\alpha,\beta}$ and $\tilde{g}_{\alpha+1,\beta-1}$ into the following relation

$$\tilde{g}_{\alpha,\beta}\Big(\frac{1}{y}\Big) = y^{1-\beta}\Big(\frac{1}{B(\alpha,\beta)} + \beta\int_0^y d\eta\,\eta^\beta\,\tilde{g}_{\alpha+1,\beta-1}\Big(\frac{1}{\eta}\Big)\Big)\,. \tag{3.w}$$

Consequently, in order to prove (3.w) for $y < 1$, we need to verify the identity

$$F(-\beta,1-\beta,\alpha;y) = 1 + \frac{B(\alpha,\beta)\,\beta}{B(\alpha+1,\beta-1)}\int_0^y d\eta\,F(1-\beta,2-\beta,\alpha+1;\eta)\,,$$

which follows from the classical identity

$$\frac{d}{dz}\,F(a,b,c;z) = \frac{a\,b}{c}\,F(a+1,b+1,c+1;z)$$

(see [30, formula 9.2.2, p. 241]).

At this point, it remains to verify the relation (3.v) between $\tilde{g}_{\alpha,\beta}$ and $\tilde{g}_{\alpha+1,\beta-1}$ only for $x < 1$. We write (3.v) in the equivalent form

$$\tilde{g}_{\alpha,\beta}(x) = x^{-1+\beta}\Big(\frac{1}{B(\alpha,\beta)} + \beta\int_x^\infty d\xi\,\xi^{-\beta}\,\tilde{g}_{\alpha+1,\beta-1}(\xi)\Big)$$

which implies

$$\tilde{g}_{\alpha,\beta}(x) = \frac{\beta-1}{x}\,\tilde{g}_{\alpha,\beta}(x) - \frac{\beta}{x}\,\tilde{g}_{\alpha+1,\beta-1}(x)\,.$$

Since the value of $\tilde{g}_{\alpha,\beta}(1)$ is known, the above difference equation determines $\tilde{g}_{\alpha,\beta}$ uniquely. Hence, all we have to verify is the following relationship

$$\begin{aligned} &c_-\,\frac{a\,b}{c}\,F(a+1,b+1,c+1;x) \\ &\quad = c_-\,\frac{\beta-1}{x}\,F(a,b,c;x) - \frac{\beta}{x}\,(\alpha+\beta-1)\,(\beta-1)\,F(a,b+1,c;x)\,, \end{aligned}$$

where $c_- = (\alpha+\beta-1)\,(\beta-1)$, $a = 2-\alpha-\beta$, $b = 1-\beta$, $c = 2$. This relationship is equivalent to

$$\frac{a\,x}{c}\,F(a+1,b+1,c+1;x) = -F(a,b,c;x) + F(a,b+1,c;x)\,,$$

which is precisely [30, formula 9.2.13, p. 243].

v) We finally prove the proposition when $0 < \beta < 1$. The first part of the Proposition will now follow from the relationship

$$\frac{d}{ds}(s^{-\beta}\,\Phi(-\beta,\alpha,-s)) = \beta\, s^{-\beta-1}\,\Phi(-\beta+1,\alpha,-s)$$

and the integral representations

$$\Gamma(1+\beta)\,y^{-\beta-1} = \int_0^\infty dt\, e^{-yt}\, t^\beta$$

and

$$\Phi(-\beta+1,\alpha,-y) = \frac{1}{B(-\beta+1,\alpha+\beta-1)}\int_0^1 dt\, e^{-yt}\, t^{-\beta}\,(1-t)^{\alpha+\beta-2}\,.$$

We now obtain that part 1) of the proposition is satisfied with the function $g = g_{\alpha,\beta}$ defined by

$$u\,g(u) = \frac{c\,\beta}{B(-\beta+1,\alpha+\beta-1)\,\Gamma(1+\beta)}\,h(u)\,,$$

where

$$h(u) = \int_0^{u\wedge 1} dt\, t^{-\beta}\,(1-t)^{\alpha+\beta-2}\,(u-t)^\beta\,, \qquad \text{and } c = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\,.$$

The expression of h, hence of g, in terms of F, is then deduced from the integral representation

$$F(a,b,c;u) = \frac{1}{B(b,c-b)}\int_0^1 dt\, t^{b-1}\,(1-t)^{c-b-1}\,(1-u\,t)^{-a}\,,$$

which is valid for $\mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0$ and $u < 1$ (see [30, formula 9.1.4, p. 239]).

4. Some properties of the $X^{\alpha,\beta}$ processes.

The family of processes $X^{\alpha,\beta}$ enjoys a number of properties which are the counterparts of properties of the squares of Bessel processes. In the eight following sections, we shall compare such properties for both classes of processes.

4.1. Time-changing.

a) Here are two transformations of Bessel processes which are most useful in some computations:

i) if $(R_t,\, t \geq 0)$ is a BES$\,(d)$, with $d \geq 2$, starting at $r_0 > 0$, there exists a real-valued Brownian motion $(\beta_t,\, t \geq 0)$ such that

$$\log(R_t) = \beta_u + \nu\, u\,, \qquad \text{where } u = \int_0^t \frac{ds}{R_s^2}\,,\ \nu = \frac{d}{2} - 1\,.$$

In the literature, this relation is also found in the form of a representation of the geometric Brownian motion with drift ν, *i.e.* $(\exp(\beta_u + \nu\, u),\, u \geq 0)$, in terms of a Bessel process R_ν with dimension $d = 2\,(1+\nu)$, as follows

$$\exp(\beta_u + \nu\, u) = R_\nu\Big(\int_0^u ds\, \exp(2\,(\beta_s + \nu\, s))\Big)\,, \qquad u \geq 0\,,$$

(see, for example, [52], and for some applications, [62] and [60]),

ii) for convenience, $(R_\mu(t),\, t \geq 0)$ now denotes the Bessel process with index μ, *i.e.* with dimension $d = 2\,(1+\mu)$. Let p and q such that $1/p + 1/q = 1$. Then, under suitable conditions on μ and p, we have

$$q\, R_\mu^{1/q}(t) = R_{\mu q}\Big(\int_0^t ds\, R_\mu^{-2/p}(s)\Big) \tag{4.a}$$

(see [5, Lemma 3.1] and [43, Chapter XI]).

b) Here are some similar results for the processes $X^{\alpha,\beta}$.

Theorem 4.1. i) *If $X \equiv X^{\alpha,\beta}$ starts from $x > 0$, and $\alpha \geq 1$, there exists a process with stationary independent increments $\xi \equiv \xi^{\alpha,\beta}$ such that*

$$\log(X_t) = \xi\Big(\int_0^t \frac{ds}{X_s}\Big)\,, \qquad t \geq 0\,.$$

The generator of ξ is given by

$$\mathcal{L}^{\alpha,\beta}\phi(z) = \phi'(z) + \beta\,(\alpha+\beta-1)\int_0^\infty dy\, e^{-y(\alpha+\beta-1)}\,(\phi(z-y) - \phi(z))\,.$$

ii) *Let* $m > 0$, *then*

$$X_t = X_t^{\alpha,\beta} = X^{\alpha_{(m)},\beta_{(m)}}\Big(\int_0^t du\, m\, X_u^{m-1}\Big), \tag{4.b}$$

where $\alpha_{(m)} = ((\alpha - 1)/m) + 1$, *and* $\beta_{(m)} = (\beta/m) + 1$.

REMARKS. 1) There are some similar results for the processes $\hat{X}^{\alpha,\beta}$ introduced via Theorem 3.3, the discussion of which is postponed until Section 4.4.

2) In fact, both Bessel processes and the processes $X^{\alpha,\beta}$ are examples of a particular class of $\mathbb{R}_+$-valued Markov processes X which enjoy the following scaling property: there exists $c > 0$ such that, for $a \geq 0$, $\lambda > 0$, the law of $(X_{\lambda t},\, t \geq 0)$ under P_a is that of

$$(\lambda^c X_t,\, t \geq 0)\,, \qquad \text{under } P_{a/\lambda^c}\ .$$

Lamperti ([29]) has studied these processes, which he calls semi-stable Markov processes, and has shown that, if P_a almost surely, $(X_t,\, t \geq 0)$ does not visit 0, then one has

$$\log(X_t) = \xi\Big(\int_0^t \frac{du}{X_u}\Big)\,, \qquad t \geq 0 \tag{4.c}$$

(here, we have assumed, for simplicity $c = 1$) for some process ξ with stationary independent increments. Several studies of such processes have been made in recent years (see [19], [20], [50]).

3) Let $(X_t^{(m)},\, t \geq 0)$ be the semi-stable Markov process associated with the Lévy process $(m\,\xi_t,\, t \geq 0)$. It is easy, using relation (4.c), to show that

$$X_t^m = X^{(m)}\Big(\int_0^t X_u^{m-1}\, du\Big)\,.$$

Thus, the relations (4.a) and (4.b) are easy consequences of the representation (4.c).

4.2. Absolute continuity relations.

Fix $x > 0$. As α varies in $[1, \infty[$, the laws Q_x^α of BES $Q_x(2\,\alpha)$ are locally mutually equivalent. The following explicit formula holds

$$\text{(4.d)} \quad Q_x^\alpha|_{\mathcal{F}_t} = \Big(\frac{X_t}{x}\Big)^{\nu/2} \exp\Big(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\Big) Q_x^1|_{\mathcal{F}_t}\,, \qquad \nu = \alpha - 1\,.$$

From this relation, one deduces the important formula

$$\Big(\frac{y}{x}\Big)^{\nu/2} Q_x^1\Big(\exp\Big(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\Big)\Big| X_t = y\Big) = \frac{p_t^\alpha(x,y)}{p_t^1(x,y)}\,.$$

It implies (see (3.s))

$$Q_x^1\Big(\exp\Big(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\Big)\Big| X_t = y\Big) = \frac{I_{|\nu|}}{I_0}\Big(\frac{\sqrt{x\,y}}{t}\Big)\,.$$

This formula plays a key role in the study of the winding number of complex Brownian motion around 0 (see [47], [55], for instance); it has also found applications in mathematical finance ([17]).

The counterpart of (4.d) for the laws $\Pi_x^{\alpha,\beta}$ of the processes $X^{\alpha,\beta}$ starting from x is the following

Theorem 4.2. *Let* $\lambda \geq 0$, $\alpha_\lambda + \beta_\lambda = \alpha + \beta + \lambda$, $\alpha_\lambda\,(\alpha + \beta + \lambda - 1) = \lambda\,(\alpha + \beta + \lambda) + (\alpha + \beta - 1)\,(\alpha + \lambda)$. *Then, one has*

$$\text{(4.e)} \qquad \Pi_x^{\alpha_\lambda,\beta_\lambda}|_{\mathcal{G}_t} = \Big(\frac{X_t}{x}\Big)^{\lambda} \exp\Big(-\mu\int_0^t \frac{ds}{X_s}\Big)\Pi_x^{\alpha,\beta}|_{\mathcal{G}_t}\,,$$

where

$$\mu = \lambda\,\frac{\alpha - 1 + \lambda}{\alpha + \beta - 1 + \lambda}\,.$$

REMARKS. 1) *Beware*: the notation $(\alpha_\lambda, \beta_\lambda)$ has nothing to do with the notation $(\alpha_{(m)}, \beta_{(m)})$ introduced in Theorem 4.1.

2) The absolute continuity relations (4.d) and (4.e) are obvious consequences of the representation (4.c) of a Markov semi-stable process as the time-change of the exponential of a Lévy process.

Since the Lévy process ξ^α, associated with the BES $Q(2\,\alpha)$ process X^α, is a Brownian motion with drift, precisely

$$\xi_t^\alpha = 2\,((\alpha - 1)\,t + B_t)\,,$$

we see that the relation (4.d) may be obtained, by time-changing, from the Cameron-Martin Formula, which relates the laws of Brownian motion and Brownian motion with drift.

The Lévy process associated with $X^{\alpha,\beta}$ is

$$\xi_t^{\alpha,\beta} = t - \mathrm{Pois}\,(\beta, \alpha+\beta-1)_t \ ,$$

where $\mathrm{Pois}\,(\beta,\alpha+\beta-1)$ is the compound Poisson process of parameter β whose jumps are distributed as exponentials of parameter $\alpha+\beta-1$ (see the preceding section to identify $\xi^{\alpha,\beta}$ with the help of its infinitesimal generator). Thus, formula (4.e) may be obtained, by time-changing, from the Girsanov Formula, when we make the change of probabilities associated with the martingale

$$\exp\,(\lambda\,\xi_t^{\alpha,\beta} - t\,\psi^{\alpha,\beta}(\lambda))\,, \qquad t \geq 0\,,$$

where $\psi^{\alpha,\beta}$ is the Lévy exponent of $\xi^{\alpha,\beta}$

$$\mathbb{E}\,[\exp -\lambda\,\xi_t^{\alpha,\beta}] = \exp\,(t\,\psi^{\alpha,\beta}(\lambda)) = \exp\left(t\,\lambda\,\frac{\alpha+\lambda-1}{\alpha+\beta+\lambda-1}\right).$$

4.3. First passage times.

4.3.1. First passage times for $\mathrm{BES}\,Q(d)$.

If $(X_t,\, t \geq 0)$ denotes $\mathrm{BES}\,Q(d)$, *i.e.* the square of a d-dimensional Bessel process, we recall ([24], [18], [42]) that

$$\phi(\lambda\,X_t)\,e^{-\lambda t} \text{ is a local martingale, for } \phi = \phi_+ \text{ or } \phi_- \ ,$$

with

$$\phi_+(x) = x^{-\nu/2}\,I_\nu(\sqrt{2\,x}\,) \qquad \text{and} \qquad \phi_-(x) = x^{-\nu/2}\,K_\nu(\sqrt{2\,x}\,)\,.$$

This implies

$$\mathbb{E}_a[e^{-\lambda T_b}] = \frac{\phi(\lambda\,a)}{\phi(\lambda\,b)}\ , \qquad \text{with } \phi = \begin{cases} \phi_+\ , & \text{if } a \leq b\,, \\ \phi_-\ , & \text{if } a \geq b\,, \end{cases}$$

where $T_b = \inf\,\{t > 0 : X_t = b\}$.

4.3.2. Intertwining and martingales.

The following lemma will be useful in the sequel:

Lemma 4.3. *Assume that* $Q_t\Lambda = \Lambda P_t$. *Then*:

1) *if* $\phi(X_t)\, e^{-\lambda t}$ *is a* P_x *martingale, for every* x, *then*

$$\Lambda\, \phi(Y_t)\, e^{-\lambda t} \text{ is a } Q_y \text{ martingale, for every } y\,.$$

2) *More generally, if* L *(respectively* $\tilde{L}$*) denotes the infinitesimal generator of* X *(respectively* Y*), then* $\tilde{L}\Lambda = \Lambda L$, *and if* $f \in D(L)$, *then* $\Lambda f \in D(\tilde{L})$ *and* $f(X_t) - \int_0^t Lf(X_s)\, ds$ *is a* P_x*-martingale, while*

$$\Lambda f(Y_t) - \int_0^t ds\, \Lambda L f(Y_s) \text{ is a } Q_y\text{-martingale}.$$

REMARK. The first result may be understood as a particular case of the second one, since the function ϕ satisfies $L\phi = \lambda\phi$, and hence, $\tilde{L}\Lambda\phi = \lambda\, \Lambda\phi$.

4.3.3. First passage times for $X^{\alpha,\beta}$.

For convenience, we write $\gamma = \alpha + \beta$, and $X \equiv X^{\alpha,\beta}$. From the above paragraphs, we deduce that

$$\Lambda_\gamma\, \phi_\pm(\lambda\, X_t)\, e^{-\lambda t} \text{ is a } \Pi_y^{\alpha,\beta} \text{ martingale}\,,$$

which yields

$$\Phi(\gamma, \alpha; \lambda\, X_t) \text{ and } \Psi(\gamma, \alpha; \lambda\, X_t)\, e^{-\lambda t} \text{ are } \Pi_y^{\alpha,\beta} \text{ martingales}\,.$$

Hence

$$\text{(4.f)} \quad \Pi_a^{\alpha,\beta}(e^{-\lambda T_b}) = \frac{H(\gamma, \alpha; \lambda\, a)}{H(\gamma, \alpha; \lambda\, b)}\,, \qquad \text{where } H = \begin{cases} \Phi\,, & \text{if } a < b\,, \\ \Psi\,, & \text{if } a > b\,. \end{cases}$$

In the particular case $a = 0, b = 1$, we obtain

$$\Pi_0^{\alpha,\beta}(e^{-\lambda T_1}) = \frac{1}{\Phi(\gamma, \alpha; \lambda)}\,.$$

Hence, the function $\log(\Phi(\gamma, \alpha; \lambda))$ admits the Lévy-Khintchine representation

$$\log \Phi(\gamma, \alpha; \lambda) = c\,\lambda + \int_0^\infty (1 - e^{-\lambda x})\, d\nu(x)\,,$$

for some measure ν to be determined. Taking derivatives with respect to λ, and using the relations

$$\frac{d}{d\lambda}\,\Phi(\gamma, \alpha; \lambda) = \frac{\gamma}{\alpha}\,\Phi(\gamma+1, \alpha+1; \lambda) = \Phi(\gamma, \alpha; \lambda) + \frac{\gamma-\alpha}{\gamma}\,\Phi(\gamma, \alpha+1; \lambda)$$

(see [30, formula 9.9.13, p. 262]), we obtain

$$1 + \frac{(\gamma-\alpha)\,\Phi(\gamma, \alpha+1; \lambda)}{\alpha\,\Phi(\gamma, \alpha; \lambda)} = c + \int_0^\infty x\, e^{-\lambda x}\, d\nu(x)\,.$$

From the asymptotic result ([30, formula 9.12.8, p. 271])

$$\Phi(\gamma, \alpha; \lambda) \sim C_{\gamma,\alpha}\, e^{\lambda}\, \lambda^{-(\gamma-\alpha)}\,, \qquad \lambda \longrightarrow \infty\,,$$

we deduce that $c = 1$ and there exists a probability $\mu(dx)$ on $\mathbb{R}_+$ such that

$$\frac{\Phi(\gamma, \alpha+1; \lambda)}{\Phi(\gamma, \alpha; \lambda)} = \int_0^\infty e^{-\lambda x}\, \mu(dx) \qquad \text{and} \qquad \mu(dx) = \frac{\alpha}{\gamma-\alpha}\, x\, \nu(dx)\,.$$

Another interpretation of the probability μ will be given in Section 4.7.

4.3.4. First passage times for $\xi^{\alpha,\beta}$.

The results in this paragraph follow essentially from the absolute continuity relation obtained in Theorem 4.2 for the processes $\xi^{\alpha,\beta}$.

First, we have (recall that $\gamma = \alpha + \beta$)

$$E_0[e^{\lambda \xi_t}] = e^{t\psi(\lambda)}\,, \qquad \text{where } \psi(\lambda) = \lambda\, \frac{\alpha - 1 + \lambda}{\gamma - 1 + \lambda} = \lambda\, \frac{a - b + \lambda}{a + \lambda}$$

and we have defined $a = \gamma - 1 = \alpha + \beta - 1$ and $b = \gamma - \alpha = \beta$.

We then deduce (or we could appeal again to Theorem 4.2) that, with the notation $\tau_v = \inf\{u : \xi_u = v\}$,

$$E_0[e^{-\mu \tau_v}] = e^{-v\psi^{-1}(\mu)}\,,$$

where

$$\psi^{-1}(\mu) = \frac{1}{2}\left(\mu - (a-b) + \sqrt{(\mu-(a-b))^2 + 4\,a\,\mu}\right).$$

It is interesting to study the Lévy-Khintchine representation of ψ^{-1}; we find

$$\psi^{-1}(\mu) = \mu + \int_0^\infty (1 - e^{-\mu u})\,\nu(du)\,,$$

where

$$\nu(du) = \frac{\sqrt{a\,b}}{u}\, I_1(2\sqrt{a\,b}\,u)\, e^{-(a+b)u}\, du\,. \tag{4.g}$$

PROOF OF FORMULA (4.g). We first remark that

$$(\mu-(a-b))^2 + 4\,a\,\mu = (\mu+a+b)^2 - 4\,a\,b\,.$$

We now seek a constant c and a positive measure ν on $\mathbb{R}_+$ such that

$$\mu - a + b + \sqrt{(\mu+a+b)^2 - 4\,a\,b} = 2\Big(c\,\mu + \int_0^\infty (1-e^{-\mu u})\,\nu(du)\Big)\,.$$

Taking derivatives with respect to μ, we obtain

$$1 + \frac{\mu+a+b}{\sqrt{(\mu+a+b)^2 - 4\,a\,b}} = 2\Big(c + \int_0^\infty e^{-\mu u}\, u\,\nu(du)\Big)\,,$$

from which we deduce, by letting $\mu \longrightarrow \infty$, that $c = 1$. It remains to find the measure ν which is specified by the equality

$$-1 + \frac{\mu+a+b}{\sqrt{(\mu+a+b)^2 - 4\,a\,b}} = 2\int_0^\infty e^{-\mu u}\, u\,\nu(du)\,.$$

Making the change of variables: $\mu + a + b = 2\sqrt{a\,b}\,\eta$, and using the following relation, valid for $\eta \geq 1$ ([15, p. 414])

$$\frac{\eta}{\sqrt{\eta^2-1}} - 1 = \int_0^\infty dx\, I_1(x)\, e^{-\eta x}\,,$$

we obtain

$$-1 + \frac{\mu+a+b}{\sqrt{(\mu+a+b)^2 - 4\,a\,b}} = 2\sqrt{a\,b}\int_0^\infty dy\, I_1(2\sqrt{a\,b}\,y)\, e^{-\mu y}\, e^{-(a+b)y}$$

and formula (4.g) follows.

Note. These computations appear to be closely related to recent work by J. Pellaumail et al in Queuing Theory ([32]).

4.3.5. Laguerre polynomials and hypergeometric polynomials.

e.i) Let (X_t^α) denote the square of BES(d'), with $d' = 2\,\alpha = 2\,(1+\nu')$. (X_t^α) may be characterized (in law) as the unique solution of the martingale problem

(4.h) for every $\lambda > 0$, $\phi(\lambda\, X_t^\alpha)\, e^{-\lambda t}$ is a martingale,

where $\phi(x) = x^{-\nu'/2}\, I_{\nu'}(\sqrt{2\,x})$.

We recall the hypergeometric functions notation (see [30, p. 275])

$$ {}_0F_1(-, 1+\nu'; z) = \Gamma(\nu'+1)\, z^{-\nu'/2}\, I_{\nu'}(2\sqrt{z})\,, $$

which implies

$$ \text{(4.i)} \quad {}_0F_1\Big(-, 1+\nu'; \frac{z}{2}\Big) = c_{\nu'}\, \phi(z)\,, \qquad \text{where } c_{\nu'} = \Gamma(\nu'+1)\, 2^{\nu'/2}\,. $$

The Laguerre polynomials with parameter ν': $L_n^{(\nu')}(x)$ may be defined as the coefficients of the generating function (in y)

$$ {}_0F_1(-, 1+\nu'; -x\,y)\, e^y = \sum_{n=0}^{\infty} \frac{L_n^{(\nu')}(x)\, y^n}{(1+\nu')_n} $$

([35, p. 39]).

It then follows from formula (4.i) that

$$ \text{(4.j)} \quad \begin{aligned} c_{\nu'}\, \phi(\lambda\, x)\, e^{-\lambda t} &= \sum_{n=0}^{\infty} \frac{1}{(1+\nu')_n}\, L_n^{(\nu')}\Big(\frac{x}{2\,t}\Big)(-\lambda\, t)^n \\ &= \sum_{n=0}^{\infty} \lambda^n\, P_n(x,t)\,, \end{aligned} $$

where we have defined

$$ P_n(x,t) = \frac{(-t)^n}{(1+\nu')_n}\, L_n^{(\nu')}\Big(\frac{x}{2\,t}\Big) = \frac{(-t)^n}{n!}\, \Phi\Big(-n, \nu'+1; \frac{x}{2\,t}\Big)\,, $$

since the expression of $L_n^{(\nu')}$ in terms of the confluent hypergeometric function Φ is

$$L_n^{(\nu')}(z) = \frac{(1+\nu')_n}{n!}\,\Phi(-n, 1+\nu'; z) \qquad ([30, \text{p. } 273])$$

(we recall that, with our notation, $\alpha = 1+\nu'$). We deduce from (4.h) and (4.j) that

$$\text{(4.k)} \qquad \text{for every } n \in \mathbb{N}, \qquad \left(t^n\, L_n^{(\nu')}\left(\frac{X_t^\alpha}{2\,t}\right),\, t \geq 0\right) \text{ is a martingale.}$$

e.ii) We now discuss similar results for the process $X^{\alpha,\beta}$. This process may be characterized (in law) as the unique solution of the martingale problem (recall that $\gamma = \alpha + \beta$)

$$\text{(4.l)} \qquad \text{for every } \lambda > 0 \ \Lambda_\gamma \phi(\lambda\,\cdot\,)(X_t^\alpha)\, e^{-\lambda t} \text{ is a martingale.}$$

Define

$$\psi(y) = \Lambda_\gamma \phi(y) = \frac{1}{\Gamma(\gamma)} \int_0^\infty da\, a^{\gamma-1}\, e^{-a}\, \phi(2\,y\,a)$$

and

$$Q_n(y,t) = \frac{1}{c_\nu}\, \Lambda_\gamma(P_n(\,\cdot\,,t))(y)\,.$$

It follows from (4.j) that

$$\text{(4.m)} \qquad c_{\nu'}\, \psi(\lambda y)\, e^{-\lambda t} = \sum_{n=0}^{\infty} \lambda^n\, Q_n(y,t)\,.$$

We now identify ψ and Q_n.

We remark that, in general, if $F(z) = \sum_{n=0}^\infty f_n z^n$ (with $f_n \geq 0$, for every n), then

$$F^\gamma(z) \stackrel{\text{def}}{=} \Lambda_\gamma F(z) = \sum_{n=0}^{\infty} (\gamma)_n\, f_n z^n\,.$$

In particular, the application $F \longrightarrow F^\gamma$ transforms ${}_pF_q(a_r, b_s; z)$ into

$${}_{p+1}F_q(\gamma, a_r, b_s; z)\,.$$

Consequently, we obtain

$$\psi(y) = \Lambda_\gamma \phi(y) \underset{\text{from (4.j)}}{=} \frac{1}{c_{\nu'}}\, {}_0F_1(-,\alpha;\,\cdot\,)^\gamma(z) = \frac{1}{c_{\nu'}}\,\Phi(\gamma,\alpha;z)\,.$$

Likewise,

$$\begin{aligned} Q_n(y,t) &= \frac{1}{c_{\nu'}}\,\Lambda_\gamma(P_n(\,\cdot\,,t))(y) \\ &= \frac{(-t)^n}{c_{\nu'}n!}\,\Phi\Big(-n,\alpha;\frac{\cdot}{t}\Big)^\gamma(y) \\ &= \frac{(-t)^n}{c_{\nu'}\,n!}\,F\Big(-n,\gamma,\alpha;\frac{y}{t}\Big)\,. \end{aligned}$$

Hence, the series (4.m) may be written in the form

$$\Phi(\gamma,\alpha;\lambda\,y)\,e^{-\lambda t} = \sum_{n=0}^\infty \frac{\lambda^n}{n!}\,(-t)^n\,F\Big(-n,\gamma,\alpha;\frac{y}{t}\Big)\,, \tag{4.n}$$

the polynomials $F(-n,\gamma,\alpha;y/t)$ are the so-called hypergeometric polynomials.

The assertions similar to (4.h) and (4.k) are (recall that $\gamma=\alpha+\beta$)

$$\text{for every } \lambda>0\,,\qquad \Phi(\gamma,\alpha;\lambda\,X_t^{\alpha,\beta})\,e^{-\lambda t} \text{ is a martingale} \tag{4.o}$$

and

$$\text{for every } \lambda>0\,,\qquad t^n\,F\Big(-n,\gamma,\alpha;\frac{X_t^{\alpha,\beta}}{t}\Big) \text{ is a martingale}\,. \tag{4.p}$$

e.iii) We have just seen that, in analytic terms, the intertwining of the processes X^α and $X^{\alpha,\beta}$ with respect to the kernel Λ_γ translates as the transformation of Laguerre polynomials $\Phi(-n,\alpha;\,\cdot\,)$ into hypergeometric polynomials $F(-n,\gamma,\alpha;\,\cdot\,)$ via the formula

$$F(-n,\gamma,\alpha;y) = \frac{1}{\Gamma(\gamma)}\int_0^\infty da\,a^{\gamma-1}\,e^{-a}\,\Phi(-n,\alpha;a\,y)\,.$$

Likewise, the intertwining of the processes X^α and $X^{\alpha+\beta}$ with respect to the kernel $\Lambda_{\alpha,\beta}$ translates, in analytic terms, as the transformation of Laguerre polynomials with parameter $\nu' = \alpha - 1$: $L_n^{(\nu')}(x)$ into Laguerre polynomials with parameter $\nu = \gamma - 1 = \alpha + \beta - 1$; $L_n^{(\nu)}(x)$ via *Koshlyakov's formula* ([30, p. 94])

$$L_n^{(\nu)}(x) = \frac{\Gamma(n+\alpha+\beta)}{\Gamma(\beta)\,\Gamma(n+\alpha)} \int_0^1 dt\, t^{\alpha-1}\,(1-t)^{\beta-1}\, L_n^{(\nu')}(x\,t)\,.$$

In the same spirit, the integral relation (see [30, p. 277])

$$F(a,b,c;z) = \frac{1}{B(d,c-d)} \int_0^1 dt\, t^{d-1}\,(1-t)^{c-d-1}\, F(a,b,d;z\,t)$$

may be considered as a translation, in analytic terms, of the intertwining relations which hold between the different processes $X^{\alpha,\beta}$ (see Theorem 3.6).

e.iv) We now consider two other fundamental generating functions for $(L_n^{(\nu')}(x),\, n \geq 0)$ and $(F(-n,\gamma;\alpha,z),\, n \geq 0)$ respectively, which have a clear meaning in terms of martingale properties of X^α and $X^{\alpha,\beta}$ respectively. These generating functions are

$$\text{(4.q)}\qquad \begin{aligned} (1-t)^{-(\nu+1)}\, e^{-xt/(1-t)} &= \sum_{n=0}^{\infty} L_n^{(\nu)}(x)\, t^n\,, \\ (1-t)^{\beta}\,(1-t+x\,t)^{-\gamma} &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}\, F(-n,\gamma,\alpha;x)\, t^n \end{aligned}$$

([30, p. 77 and 277 respectively]).

Let $t = \lambda\, s/(1+\lambda)$, with $s<1$, $x = z/(2\,s)$, and $u(\lambda) = (1+\lambda)^{-\gamma}$. The two left-hand sides of (4.q) become

$$u(\lambda)\,(1+\lambda-\lambda\,s)^{-\gamma} \exp\Big(-\frac{\lambda\,z}{2\,(1+\lambda\,(1-s))}\Big)$$

and

$$u(\lambda)\,(1+\lambda)^{-\beta}\,(1+\lambda-\lambda\,s)^{\beta}\,\Big(1+\lambda\,\Big(1-s+\frac{z}{2}\Big)\Big)^{-\gamma}\,.$$

Both expressions played a key role in the explicit computation of $\Pi_t^{\alpha,\beta}$ (see formula (3.i)). Indeed, these expressions are in fact respectively equal to

$$u(\lambda)\, Q_{1-s}^{\gamma}(e_\lambda)(z) \equiv u(\lambda) \sum_{n=0}^{\infty} L_n^{(\nu)}\Big(\frac{z}{2\,s}\Big)\, s^n \Big(\frac{\lambda}{1+\lambda}\Big)^n$$

and

$$u(\lambda)\,(1+\lambda)^{-\beta}\,\Pi^{\alpha,\beta}_{1-s}(\phi_\lambda)\Big(\frac{z}{2}\Big)$$
$$\equiv (1+\lambda)^{-\beta}\,u(\lambda)\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\,F\Big(-n,\gamma,\alpha;\frac{z}{2\,s}\Big)s^n\Big(\frac{\lambda}{1+\lambda}\Big)^n .$$

Now, replacing z respectively by X^α_s and $Y^{\alpha,\beta}_s$, we obtain two martingales which are in correspondence via the intertwining kernel Λ_γ, since, by formula (3.j)

$$\Lambda_\gamma(e_\lambda)(z) = c_\gamma\,\phi_\lambda(z) = c_\gamma\,(1+\lambda\,z)^{-\gamma}\,.$$

4.4. Time reversal.

In this section, we apply the following general result on time-reversal successively to X^α, a BES $Q(2\,\alpha)$ process, and $X^{\alpha,\beta}$, at their last exit time from $b>0$, when $\alpha>1$. This result was originally proved by Nagasawa [38]; for another proof see [44], or [39].

Theorem 4.4. *Let X and $\hat{X}$ be standard Markov processes in E, which are in duality with respect to μ (see Section 3.3 for the definition). Let $u(x,y)$ denote the potential kernel density of X relative to μ, so that*

$$\mathbb{E}_x\Big[\int_0^\infty f(X_t)\,dt\Big] = \int u(x,y)\,f(y)\,\mu(dy)\,.$$

Let L be a cooptional time for X, that is a positive random variable satisfying: $L\le\zeta$ (ζ is the lifetime of X), and $L\circ\theta_t=(L-t)^+$. Define $\tilde{X}_t$ by

$$\tilde{X}_t = \begin{cases} X_{(L-t)-}\,, & \text{on } 0<L<\infty \text{ for } 0<t<L\,,\\ \Delta\,, & \text{otherwise}.\end{cases}$$

Then, for any initial law λ, the process $(\tilde{X}_t,\ t>0)$ under P_λ, is an homogeneous Markov process with transition semi-group $(\tilde{P}_t)$ given by

$$\tilde{P}_t f(y) = \begin{cases} \dfrac{\hat{P}_t(f\,v)(y)}{v(y)}\,, & \text{if } 0<v(y)<\infty\,,\\ 0\,, & \text{if } v(y)=0 \text{ or } \infty\,.\end{cases}$$

In case $\lambda = \varepsilon_x$, $v(y) = u(x, y)$.

For our application, we take: $x = 0$, $\alpha > 1$, $L = L_b$ the last exit time from $b > 0$, for either X^α or $X^{\alpha,\beta}$. We can take $\mu(dx) = x^{\alpha-1}\,dx$ and use the results of Theorem 3.3. However, a more natural choice is $\mu(dx) = dx$, the Lebesgue measure on $\mathbb{R}_+$, since it will yield: $v(y) = u(0, y) = c$, a constant.

Indeed, it is obvious that a Lévy process ξ is in duality with $\hat{\xi} = -\xi$, with respect to the Lebesgue measure on $\mathbb{R}$. The representation

$$\log(X_t) = \xi\Big(\int_0^t \frac{du}{X_u}\Big), \qquad t \ge 0,$$

implies that the semi-stable Markov process X associated with ξ, is in dx-duality with the semi-stable Markov process $\hat{X}$ associated with $\hat{\xi} = -\xi$. Furthermore, thanks to the scaling property enjoyed by X, we have $v(y) = u(0, y) = c$, a constant, as shown by the following computation

$$\mathbb{E}_0\Big[\int_0^\infty dt\, f(X_t)\Big] = \mathbb{E}_0\Big[\int_0^\infty dt\, f(tX_1)\Big] = \int_0^\infty du\, f(u)\, \mathbb{E}_0\Big[\frac{1}{X_1}\Big].$$

Since $\xi_t^\alpha = 2\,((\alpha-1)\,t + B_t)$ and $\xi_t^{\alpha,\beta} = t - \mathrm{Pois}\,(\beta, \alpha+\beta-1)_t$, we have the following

Theorem 4.5. *Let* $\alpha > 2$, $\beta > 0$, *and* (X_t^α) *and* $(X_t^{\alpha,\beta})$ *start at* 0; *then for* $b > 0$

a) $(X_t^\alpha,\, t \le L_b) \stackrel{\mathrm{d}}{=} (X_t^{2-\alpha},\, t \le T_0)$,

b) $(X_t^{\alpha,\beta},\, t \le L_b) \stackrel{\mathrm{d}}{=} (\hat{X}_t^{\alpha+\beta-1,\beta},\, t \le T_0)$,

where, on both right hand sides, it is assumed that the processes start at b.

4.5. Some limit theorems.

In this section, we obtain several limit theorems concerning the processes $X^{\alpha,\beta}$ and $\xi^{\alpha,\beta}$, some of which are then applied to the study of the asymptotics of the functional

$$\int_0^t \frac{ds}{X_s^{\alpha,\beta}},$$

as $t \longrightarrow \infty$, when $X_0^{\alpha,\beta} \neq 0$. In the sequel, we use the notation (fd) to denote the convergence in law of finite-dimensional distributions of processes indexed by $\mathbb{R}_+$.

The main result of this section is the following

Theorem 4.6. *Let $\alpha > 0$ and $\alpha+\beta > 1$. Define $\nu' = \alpha - 1$, $\nu = \alpha+\beta-1$, and let $(X_t^\alpha,\, t \geq 0)$ denote a* BES $Q(2\,\alpha)$, *and $(B_t,\, t\geq 0)$ a 1-dimensional Brownian motion. Then:*

i) *for fixed α,*

$$(X^{\alpha,\beta}_{(\alpha+\beta)t},\, t \geq 0) \underset{\beta\to\infty}{\xrightarrow{\text{(fd)}}} (X_t^\alpha,\, t\geq 0)\,,$$

ii) *for fixed α,*

$$(\xi^{\alpha,\beta}_{(\alpha+\beta)t},\, t \geq 0) \underset{\beta\to\infty}{\xrightarrow{\text{(fd)}}} (2\,(B_t+\nu'\,t),\, t\geq 0)\,,$$

iii) *for fixed α and β with $\alpha > 1$,*

$$\frac{1}{\lambda}\,\xi^{\alpha,\beta}_{\lambda t} \underset{\lambda\to\infty}{\xrightarrow{\text{(P)}}} \frac{\nu'}{\nu}\,t\,,$$

iv) *for fixed $\alpha = 1$, $\beta > 0$,*

$$\Big(\frac{1}{\sqrt{\lambda}}\,\xi^{1,\beta}_{\lambda t},\, t\geq 0\Big) \underset{\lambda\to\infty}{\xrightarrow{\text{(fd)}}} \Big(\sqrt{\frac{2}{\nu}}\;B_t,\, t\geq 0\Big)\,.$$

Remarks. 1) The result in ii) is in agreement with i) and the time-change formula (see Section 4.1)

$$\log X_t^{\alpha,\beta} = \xi^{\alpha,\beta}\Big(\int_0^t \frac{ds}{X_s^{\alpha,\beta}}\Big)\,.$$

Hence, we have

$$\log X^{\alpha,\beta}_{(\alpha+\beta)t} = \xi^{\alpha,\beta}\Big((\alpha+\beta)\int_0^t \frac{ds}{X^{\alpha,\beta}_{(\alpha+\beta)s}}\Big)\,,$$

and we remark that the result i) fits in well with the time-change representation of $(\log X_t^\alpha,\, t\geq 0)$ as

$$\log X_t^\alpha = 2\,(B_u + \nu'\,u)\,, \qquad \text{with } u = \int_0^t \frac{ds}{X_s^\alpha}\,.$$

2) In the case where $X^{\alpha,\beta}(0) = 0$, the following scaling property holds

$$(X^{\alpha,\beta}(\lambda t),\, t \geq 0) \stackrel{\mathrm{d}}{=} (\lambda X_t^{\alpha,\beta},\, t \geq 0) \tag{4.r}$$

and we may write i) in the equivalent form

$$((\alpha+\beta)\, X_t^{\alpha,\beta},\, t \geq 0) \underset{\beta\to\infty}{\overset{\text{(fd)}}{\longrightarrow}} (X_t^{\alpha},\, t \geq 0)\,.$$

The result for one-dimensional marginals is easily understood, since we know that

$$X_1^{\alpha,\beta} \stackrel{\mathrm{d}}{=} Z_{\alpha,\beta} \stackrel{\mathrm{d}}{=} \frac{X_1^{\alpha}}{X_1^{\alpha} + X_1^{\beta}}\,,$$

where X^α and X^β are independent squares of Bessel processes with respective dimensions $2\,\alpha$ and $2\,\beta$. We then deduce from the law of large numbers that $(\alpha+\beta)/(X_1^\alpha + X_1^\beta)$ converges in probability to 1, as $\beta \longrightarrow \infty$, which implies the desired result.

3) iv) is obviously a refinement of iii) in the case $\alpha = 1$ (which implies $\nu' = 0$).

4) Inspection of infinitesimal generators easily yields the following identity in law

$$\Big(\frac{1}{\lambda}\,\xi_{\lambda t}^{\alpha,\beta},\, t \geq 0\Big) \stackrel{\mathrm{d}}{=} (\xi_t^{\alpha_\lambda,\beta_\lambda},\, t \geq 0)\,, \tag{4.s}$$

where the couple $(\alpha_\lambda, \beta_\lambda)$ is defined by

$$\beta_\lambda = \beta\,, \qquad \alpha_\lambda + \beta_\lambda - 1 = \lambda\,(\alpha+\beta-1)\,,$$

or, in terms of indices instead of dimensions

$$\nu_\lambda = \lambda\,\nu \qquad \text{and} \qquad \nu'_\lambda = \nu' + \nu\,(\lambda - 1)\,.$$

The identity in law (4.s) allows to recast the limit results in ii), iii), and iv) in terms of ξ-processes, both indices of which increase to ∞ as $\lambda \longrightarrow \infty$, in the manner we have just indicated.

PROOF OF THEOREM 4.6. 1) The infinitesimal generator of $(X^{\alpha,\beta}_{(\alpha+\beta)t}, t \geq 0)$, applied to $\phi \in C^2(\mathbb{R}_+)$, is, in terms of α and β

$$\begin{aligned} &2\,(\alpha+\beta)\Big(\phi'(y) + \beta\,\frac{\alpha+\beta-1}{y}\int_0^1 dz\, z^{\alpha+\beta-2}\,(\phi(z\,y)-\phi(y))\Big) \\ &\quad = 2\,(\alpha+\beta)\Big(\phi'(y) + \frac{\beta}{y}\int_0^\infty dv\, e^{-v}\,(\phi(e^{-v/(\alpha+\beta-1)}\,y)-\phi(y))\Big)\,, \end{aligned}$$

after an elementary change of variables.

It is now easy to justify that, as α is fixed, and β goes to ∞, we may replace

$$\phi(e^{-v/(\alpha+\beta-1)}y) - \phi(y)\,,$$

by

$$y\,\phi'(y)\,(e^{-v/(\alpha+\beta-1)}-1) + \frac{y^2}{2}\,\phi''(y)\,(e^{-v/(\alpha+\beta-1)}-1)^2\,.$$

Then, the coefficient of $\phi'(y)$, respectively $\phi''(y)$, converges, as β increases to ∞, towards $2\,\alpha$, respectively $2\,y$, which implies i).

2) The same kind of argument may be applied to prove the results ii), iii) and iv). We give only the details for ii):

the infinitesimal generator of $(\xi^{\alpha,\beta}_{(\alpha+\beta)t}, t \geq 0)$, applied to $\phi \in C^1(\mathbb{R})$ is, in terms of α and β

$$2\,(\alpha+\beta)\Big(\phi'(y) + \beta\,\frac{\alpha+\beta-1}{y}\int_0^\infty du\, e^{-u(\alpha+\beta-1)}\,(\phi(y-u)-\phi(y))\Big)\,.$$

We then replace: $\phi(y-u)-\phi(y)$ by: $-u\,\phi'(y) + u^2\,\phi''(y)/2$; then, the coefficient of $\phi'(y)$, respectively $\phi''(y)$, is

$$\frac{2\,(\alpha+\beta)}{\alpha+\beta-1}\,(\alpha-1)\,, \qquad \text{respectively } \frac{2\,(\alpha+\beta)\,\beta}{(\alpha+\beta-1)^2}$$

and they converge, as β increases to ∞, to $2\,\nu'$, respectively 2, which implies ii).

We begin by recalling the following asymptotic results for the BES $Q(2\,\alpha)$ process X^α, when $X^\alpha_0 \neq 0$

$$\text{(4.t)} \qquad \frac{4}{(\log t)^2}\int_0^t \frac{ds}{X^1_s} \xrightarrow[t\to\infty]{\mathrm{d}} \sigma\,, \qquad \text{if } \alpha = 1\,,$$

where $\sigma = \inf\{t : B_t = 1\}$, and B is a 1-dimensional Brownian motion starting from 0, and

$$\text{(4.u)} \qquad \frac{2}{\log t}\int_0^t \frac{ds}{X_s^\alpha} \xrightarrow[t\to\infty]{\text{a.s.}} \frac{1}{\nu'}\,, \qquad \text{if } \alpha > 1\,.$$

We now prove similar results for the processes $X^{\alpha,\beta}$:

Theorem 4.7. *We consider the process $X^{\alpha,\beta}$ with $\alpha \geq 1$ and $X_0^{\alpha,\beta} \neq 0$. Then*

i) *if $\alpha = 1$,*

$$\frac{1}{(\log t)^2}\int_0^t \frac{du}{X_u^{1,\beta}} \xrightarrow[t\to\infty]{\text{d}} \frac{\nu}{2}\,\sigma\,,$$

where $\nu = \alpha+\beta-1$, and $\sigma = \inf\{u : B_u = 1\}$, with the same notation as in (4.t) *above;*

ii) *if $\alpha > 1$,*

$$\frac{1}{\log t}\int_0^t \frac{ds}{X_s^{\alpha,\beta}} \xrightarrow[t\to\infty]{\text{a.s.}} \frac{\nu}{\nu'}\,,$$

where $\nu = \alpha+\beta-1$ and $\nu' = \alpha - 1$.

At least, three different proofs of (4.t) are known ; they hinge respectively on:

1) Laplace's asymptotic method (see [13], [57], [31]),

2) a pinching argument ([53], [36]), and finally:

3) the explicit computation of the law of $\int_0^t ds/X_s^1$ (see [47], [21], [55]).

We now see that the three methods admit variants from which part i) of Theorem 4.7 follows.

4.5.1. Laplace's method.

From the formula

$$\log X_t^{\alpha,\beta} = \xi^{\alpha,\beta}\Big(\int_0^t \frac{du}{X_u^{\alpha,\beta}}\Big)\,,$$

we deduce

$$\int_0^t \frac{du}{X_u^{\alpha,\beta}} = \inf\Big\{v : \int_0^v ds \, \exp(\xi_s^{\alpha,\beta}) > t\Big\}.$$

Let $\lambda = \log t$. We have, after some elementary transformations

$$\text{(4.v)} \quad \frac{1}{\lambda^2}\int_0^t \frac{du}{X_u^{\alpha,\beta}} = \inf\Big\{u : \frac{1}{\lambda}\log\Big(\lambda^2\int_0^u ds\, \exp\Big(\lambda\,\frac{1}{\lambda}\,\xi_{\lambda^2 s}^{\alpha,\beta}\Big)\Big) > 1\Big\}.$$

Using Theorem 4.6.iv), we now deduce from (4.v) that

$$\frac{1}{\lambda^2}\int_0^t \frac{du}{X_u^{\alpha,\beta}} \xrightarrow[t\to\infty]{\mathrm{d}} \inf\Big\{u : \sqrt{\frac{2}{\nu}}\, B_u > 1\Big\} \stackrel{\mathrm{d}}{=} \frac{\nu}{2}\,\sigma\,,$$

which proves Theorem 4.7.i).

4.5.2. Pinching method.

Let $T_a = \inf\{t : X_t^{\alpha,\beta} = a\}$ and $\tau_b = \inf\{t : \xi_t^{\alpha,\beta} = b\}$. The main ingredients of the proof (see [53]) are

$$\text{(4.w)} \qquad \frac{1}{(\log t)^2}\int_t^{T_t} \frac{du}{X_u^{\alpha,\beta}} \xrightarrow[t\to\infty]{\mathrm{d}} 0\,,$$

and

$$\int_0^{T_t} \frac{du}{X_u^{\alpha,\beta}} = \tau_{\log t}\,.$$

The latter equality is immediate from the time change formula

$$\log X_t^{\alpha,\beta} = \xi^{\alpha,\beta}\Big(\int_0^t \frac{du}{X_u^{\alpha,\beta}}\Big).$$

Moreover, from Theorem 4.6.iv), we obtain

$$\frac{1}{(\log t)^2}\,\tau_{\log t} \xrightarrow[t\to\infty]{\mathrm{d}} \frac{\nu}{2}\,\sigma\,.$$

This could also be deduced from the explicit formula

$$\mathbb{E}[\exp(-\mu\,\tau_b)] = \exp\Big(-\frac{b}{2}\big(\mu + \sqrt{\mu^2 + 4\,\mu\,\nu}\big)\Big),$$

see Section 4.3.4.

It now remains to prove the convergence result (4.w). We have

$$\int_t^{T_t} \frac{du}{X_u^{\alpha,\beta}} = \int_1^{\tilde{T}_1} \frac{du}{\tilde{X}_u^{\alpha,\beta}} ,$$

where $\tilde{X}_u^{\alpha,\beta} = X_{tu}^{\alpha,\beta}/t$, which, thanks to the scaling property of $X^{\alpha,\beta}$, converges in law, as $t \longrightarrow \infty$, towards $(\overline{X}_v^{\alpha,\beta}, v \geq 0)$, a $X^{\alpha,\beta}$ process starting from zero. Consequently, we have

$$\int_t^{T_t} \frac{du}{X_u^{\alpha,\beta}} \xrightarrow[t\to\infty]{\mathrm{d}} \int_1^{\bar{T}_1} \frac{du}{\overline{X}_u^{\alpha,\beta}} ,$$

and the result (4.w) follows a fortiori.

4.5.3. Explicit computation.

In the case of Bessel processes, this computation follows from the conditional expectation formula given in Section 4.2, as a consequence of the Girsanov relationship (4.g). Likewise, for the processes $X^{\alpha,\beta}$, we deduce from Theorem 4.2 the following

$$\Pi_t^{\alpha_\lambda,\beta_\lambda}(y,dz) = \Pi_y^{\alpha,\beta}\Big(\exp\Big(-\mu\int_0^t \frac{ds}{X_s^{\alpha,\beta}}\Big)\Big| X_t^{\alpha,\beta} = z\Big)\Big(\frac{z}{y}\Big)^{\lambda} \Pi_t^{\alpha,\beta}(y,dz) ,$$

where

$$\mu = \lambda\, \frac{\alpha - 1 + \lambda}{\alpha + \beta - 1 + \lambda} .$$

Then, using the explicit forms of the semi-groups $\Pi_t^{\alpha,\beta}(y,dz)$ presented in Section 3.4, we obtain a closed form expression for the above conditional expectation, from which one should be able to deduce the limit results announced in Theorem 4.7.

4.6. A Ciesielski-Taylor type theorem.

a) Let X^α and $X^{\alpha+1}$ be two squares of Bessel processes with respective dimensions $2\,\alpha$ and $2\,(1+\alpha)$, with $\alpha > 0$, starting from 0. Let

$$T_{(\alpha)} = \inf\{u :\ X_u^\alpha \geq 1\} \qquad \text{and} \qquad S_{(\alpha+1)} = \int_0^\infty du\, \mathbf{1}_{\{X_u^{\alpha+1} \leq 1\}} .$$

Ciesielski and Taylor (see [8], and also [18]) have proved that

$$T_{(\alpha)} \overset{\mathrm{d}}{=} S_{(\alpha+1)} \,. \tag{4.x}$$

For an extension of this result to a large class of diffusions and functionals, see Biane [4].

b) We now prove a result similar to (4.x) when the Bessel processes are replaced by the processes $X^{\alpha,\beta}$ with $\alpha > 0$ and $\alpha + \beta > 1$.

Theorem 4.8. *Define* $T_x^{\alpha,\beta} = \inf\{u : X_u^{\alpha,\beta} \geq x\}$. *Then*

a)

$$\mathbb{E}[\exp(-\lambda\, T_x^{\alpha,\beta})] = \frac{1}{\Phi(\alpha+\beta, \alpha; \lambda\, x)} \,,$$

b) *if* $\alpha > 1$,

$$\mathbb{E}\Big[\exp\Big(-\lambda \int_0^\infty ds\, \mathbf{1}_{\{X_s^{\alpha,\beta} \leq x\}}\Big)\Big] = \frac{1}{\Phi(\alpha+\beta, \alpha-1; \lambda\, x)} \,.$$

Consequently, for every $x \geq 0$, *we have*

$$T_x^{\alpha,\beta} \overset{\mathrm{d}}{=} \int_0^\infty ds\, \mathbf{1}_{\{X_s^{\alpha+1,\beta-1} \leq x\}} \,. \tag{4.y}$$

Proof. Part a) was already proved in Section 4.3.3.

To prove part b), we may take $x = 1$, using the scaling property. We simply note X for $X^{\alpha,\beta}$, starting from zero, and T_x for $T_x^{\alpha,\beta}$. We now remark that, if there exists a C^1-function $(u(x),\, x \geq 0)$ such that $L^{\alpha,\beta}\, u(x) = \lambda\, \mathbf{1}_{\{x \leq 1\}} u(x)$, then

$$\mathbb{E}\Big[\exp\Big(-\lambda \int_0^{T_a} ds\, \mathbf{1}_{\{X_s \leq 1\}}\Big)\Big] = \frac{u(0)}{u(a)} \,,$$

so that, letting a increase to ∞, we obtain

$$\mathbb{E}\Big[\exp\Big(-\lambda \int_0^{\infty} ds\, \mathbf{1}_{\{X_s \leq 1\}}\Big)\Big] = \frac{u(0)}{u(\infty)} \,. \tag{4.z}$$

The function

$$u(x) = \begin{cases} \Phi(\alpha+\beta, \alpha; \lambda\, x)\,, & x < 1\,, \\ a + b\, x^{1-\alpha}\,, & x > 1\,, \end{cases}$$

satisfies $L^{\alpha,\beta} u(x) = \lambda \, \mathbf{1}_{\{x \le 1\}} \, u(x)$, on $(0,1)$ and $(1,\infty)$. It remains to find a and b such that u is C^1. This will be so if and only if

$$\text{(4.za)} \qquad \begin{cases} a + b = \Phi(\alpha+\beta, \alpha; \lambda) \,, \\ (1-\alpha)\, b = \lambda \, \dfrac{\alpha+\beta}{\alpha} \, \Phi(\alpha+\beta+1, \alpha+1; \lambda) \,, \end{cases}$$

where, in order to find the second equality, we have used

$$\frac{d}{dx} \, \Phi(\alpha+\beta, \alpha; x) = \frac{\alpha+\beta}{\alpha} \, \Phi(\alpha+\beta+1, \alpha+1; x)$$

([30, formula 9.9.4, p. 261]). The solution of the system (4.za) is

$$b_\lambda = \frac{\lambda\,(\alpha+\beta)}{\alpha\,(1-b\,\alpha)} \, \Phi(\alpha+\beta+1, \alpha+1; \lambda) \,,$$

$$a_\lambda = \Phi(\alpha+\beta, \alpha; \lambda) - \frac{\lambda\,(\alpha+\beta)}{\alpha\,(1-\alpha)} \, \Phi(\alpha+\beta+1, \alpha+1; \lambda) \,.$$

Hence, we have: $u(0) = 1$, $u(\infty) = a_\lambda$, so that, from (4.z)

$$\mathbb{E}\Big[\exp\Big(-\lambda \int_0^\infty du \, \mathbf{1}_{\{X_u \le 1\}} \Big) \Big] = \frac{1}{a_\lambda} \,.$$

We now show, with the help of the recurrence relations satisfied by Φ, that $a_\lambda = \Phi(\alpha+\beta, \alpha-1; \lambda)$, which implies b). Indeed, we find in [30, (9.9.12), p. 262], that

$$\frac{\lambda}{\alpha} \, \Phi(\alpha+\beta+1, \alpha+1; \lambda) = \Phi(\alpha+\beta+1, \alpha; \lambda) - \Phi(\alpha+\beta, \alpha; \lambda) \,,$$

whence

$$\begin{aligned} a_\lambda &= \Phi(\alpha+\beta, \alpha; \lambda) - \frac{\alpha+\beta}{(1-\alpha)} \left(\Phi(\alpha+\beta+1, \alpha; \lambda) - \Phi(\alpha+\beta, \alpha; \lambda) \right) \\ &= \frac{1}{\alpha-1} \left((\alpha+\beta) \, \Phi(\alpha+\beta+1, \alpha; \lambda) - (\beta+1) \, \Phi(\alpha+\beta, \alpha; \lambda) \right) \\ &= \Phi(\alpha+\beta, \alpha-1; \lambda) \,, \end{aligned}$$

(from [30, formula 9.9.6, p. 262]).

We now notice, using jointly parts a) and b) of Theorem 4.8 that

$$\Pi_x^{\alpha+1,\beta}\Big(\exp\Big(-\lambda\int_0^\infty ds\,\mathbf{1}_{\{X_s\le x\}}\Big)\Big)=\frac{\Phi(\alpha+\beta,\alpha+1;\lambda\,x)}{\Phi(\alpha+\beta,\alpha;\lambda\,x)}\ ,$$

so that the probability measure μ defined in Section 4.3.3, now appears as the distribution of $\int_0^\infty ds\,\mathbf{1}_{\{X_s\le x\}}$ under $\Pi_x^{\alpha+1,\beta}$.

Again, there exists similar results for Bessel processes (see [42], [18]) and Bessel functions (see [24]).

NOTE. An explanation of the Ciesielski-Taylor identity (4.x) is given in [59], using jointly Ray-Knight theorems for local times of Bessel processes and a stochastic integration by parts formula.

It would be interesting to derive such an approach to explain the identity in law (4.y).

4.7. Affine boundaries.

This problem has been considered by Breiman [6].

a) Let $\alpha>1$, and consider $\tilde T_c=\inf\{u: X_u^\alpha=c\,(1+u)\}$, where X^α is a BES $Q_a(2\,\alpha)$, with $a<c$.

Following a method due to Shepp [45] in the case $\alpha=1$, it has been shown in [56] that

$$E_a^\alpha[(1+\tilde T_c)^{-\kappa}]=\frac{\Phi\Big(\kappa,\alpha;\dfrac{a}{2}\Big)}{\Phi\Big(\kappa,\alpha;\dfrac{c}{2}\Big)}\ . \tag{4.zb}$$

REMARK. It may be interesting to compare this formula with

$$\Pi_a^{\alpha,\beta}(e^{-\lambda T_c})=\frac{\Phi(\alpha+\beta,\alpha;\lambda\,a)}{\Phi(\alpha+\beta,\alpha;\lambda\,c)}\ ,$$

a formula obtained in the above Section 4.3.3.

b) We shall now obtain a formula similar to (4.zb) for

$$\tilde T_c=\inf\{u:\ X_u^{\alpha,\beta}=c\,(1+u)\}\,,$$

when $X_0^{\alpha,\beta}=a$, and $a<c<1$.

Under these conditions, we prove the formula

$$\Pi_a^{\alpha,\beta}((1+\tilde{T}_c)^{-\kappa}) = \frac{F(\kappa,\alpha+\beta,\alpha;a)}{F(\kappa,\alpha+\beta,\alpha;c)} \,. \tag{4.zc}$$

PROOF. Following Shepp [45] again, we use the two next arguments jointly (we drop the superscripts α, β since there is no risk of confusion)

i) $\Phi(\alpha+\beta,\alpha;\lambda\, X_t)\, e^{-\lambda t}$ is a martingale,

ii) $F(\kappa,\alpha+\beta,\alpha;y) = \dfrac{1}{\Gamma(\kappa)} \displaystyle\int_0^\infty d\lambda\, \lambda^{\kappa-1}\, e^{-\lambda}\, \Phi(\alpha+\beta,\alpha;\lambda\, y)\,.$

From i), we deduce

$$\Pi_a(\Phi(\alpha+\beta,\alpha;\lambda\, c\,(1+\tilde{T}_c))\, e^{-\lambda \tilde{T}_c}) = \Phi(\alpha+\beta,\alpha;\lambda\, a)$$

and then, integrating both sides with respect to

$$\frac{d\lambda}{\Gamma(\kappa)}\, \lambda^{\kappa-1}\, e^{-\lambda}\,,$$

we obtain

$$\Pi_a\Big(\frac{1}{\Gamma(\kappa)}\int_0^\infty d\lambda\, \lambda^{\kappa-1}\, e^{-\lambda(1+\tilde{T}_c)}\, \Phi(\alpha+\beta,\alpha;\lambda\, c\,(1+\tilde{T}_c))\Big) = F(\kappa,\alpha+\beta,\alpha;a)\,.$$

Making the change of variables $\xi = \lambda\,(1+\tilde{T}_c)$ in the above integral in $(d\lambda)$, we obtain formula (4.zc).

5. Some final remarks.

5.1. Duality and intertwinings.

5.1.1. μ-duality and h-duality.

There are presently, in the Markovian literature, two notions of duality which have little in common; they are:

• the notion of duality of two Markov semi-groups (P_t) and $(\hat{P}_t)$ on E, with respect to a σ-finite measure μ on E: this notion, which has already been presented in Section 3.3 plays, as we have seen in Section 4.5, a crucial role in time reversal;

• the notion of duality of two Markov semi-groups (R_t) and (S_t) on E and F respectively, with respect to a function $h : E \times F \longrightarrow \mathbb{R}_+$; we borrow this notion from [33]: (R_t) and (S_t) are said to be in h-duality if for every $(\xi, \eta) \in E \times F$,

$$R_t(h_\eta)(\xi) = S_t(h^\xi)(\eta)\,,$$

where $h_\eta(\xi) = h^\xi(\eta) = h(\xi, \eta)$.

5.1.2. Comparison of intertwining and h-duality.

The following proposition shows, under adequate assumptions, the equivalence between a property of intertwining and a property of h-duality.

Proposition 5.1. *Suppose that the semi-groups (S_t) and $(\hat{S}_t)$ are in μ-duality. Then:*

1) *if the semi-groups (R_t) and (S_t) are in h-duality, then*

$$R_t H_\mu = H_\mu \, \hat{S}_t\,,$$

with $H_\mu f(\xi) = \int d\mu(\eta)\, h(\xi, \eta)\, f(\eta)$;

2) *conversely, if $R_t H_\mu = H_\mu \, \hat{S}_t$, then (R_t) and (S_t) are in almost h-duality, that is: for all ξ,*

$$R_t(h_\eta)(\xi) = S_t(h^\xi)(\eta)\,, \qquad d\mu(\eta) \text{ almost surely}.$$

Proof. For every positive Borel function f, we have

$$\begin{aligned} R_t H_\mu f(\xi) &= \int R_t(\xi, dz)\, H_\mu f(z) \\ &= \int R_t(\xi, dz) \int h(z, \eta)\, f(\eta)\, d\mu(\eta) \\ &= \int d\mu(\eta)\, f(\eta) \int R_t(\xi, dz)\, h(z, \eta) \\ &= \int d\mu(\eta)\, f(\eta)\, R_t(h_\eta)(\xi)\,. \end{aligned}$$

On the other hand, by definition of the dual semi-group, one has

$$H_\mu \hat{S}_t f(\xi) = \int d\mu(\eta)\, \hat{S}_t f(\eta)\, h^\xi(\eta) = \int d\mu(\eta)\, f(\eta)\, S_t(h^\xi)(\eta)\,.$$

Consequently, the first part of the theorem is proven.

The second part of the theorem is also immediate. Suppose $R_t H_\mu = H_\mu \hat{S}_t$. Then, for all positive Borel functions f, we have

$$\int d\mu(\eta)\, f(\eta)\, R_t(h_\eta)(\xi) = \int d\mu(\eta)\, f(\eta)\, S_t(h^\xi)(\eta)\,.$$

Thus, for all ξ

$$R_t(h_\eta)(\xi) = S_t(h^\xi)(\eta)\,, \qquad d\mu(\eta) \text{ almost surely}\,.$$

In the particular case where $S_t = \Pi_t^{\alpha,\beta}$, $\hat{S}_t = \hat{\Pi}_t^{\alpha,\beta}$, $R = Q_t^\alpha$, and $\mu(dx) = x^{\alpha-1}\, dx$, the intertwining relation is given by

$$Q_t^\alpha\, \tilde{\Lambda}_\beta = \tilde{\Lambda}_\beta\, \hat{\Pi}_t^{\alpha,\beta}\,.$$

Consequently, the semi-groups are in $\mu(dx)$ almost h-duality, where

$$h(\xi,\eta) = \frac{\xi^\beta}{\eta^{\alpha+\beta}} \exp\Big(-\frac{\xi}{2\,\eta}\Big)\,.$$

This function is much more complex than the function that appears in classical duality

$$h(\xi,\eta) = \mathbf{1}_{\{\xi\le\eta\}}\,.$$

5.2. More intertwinings.

A more complete list of intertwinings of Markov processes is presented in [7], making important use of the reflecting Brownian motion $(|B_t|,\, t\ge 0)$ perturbed by a multiple of its local time at zero $(l_t,\, t\ge 0)$, *i.e.* $(|B_t| - \mu\, l_t,\, t\ge 0)$, for some $\mu > 0$.

The new Markov processes are constructed explicitly in terms of this perturbed reflecting Brownian motion, which gives hope that the intertwining relations described in the present paper and in [7] may have a pathwise interpretation, that is, we hope these processes have joint realizations that fit into the filtering framework of Section 2.1.

Although we have not yet been able to achieve this program, we introduced another framework (see [7, Theorem 3.2]) which enabled us to prove these intertwining relations. We saw in Section 4.6 how these relations can be used to prove Ciesielski-Taylor identities between semi-stable Markov processes of the same family.

Furthermore, the technique developed in [7] to compute the distributions of the exponential functionals

$$A_t = \int_0^t e^{\xi_s}\, ds\,, \qquad t \geq 0\,,$$

where ξ is the Lévy process associated with a semi-stable Markov process X, consists in determining a family of random variables (H_p) such that

$$P(H_p > t) = E_1\left[\frac{1}{X_t^p}\right].$$

The intertwining relation $Q_t\,\Lambda = \Lambda P_t$ implies that the families (H_p) and (K_p) associated respectively to the processes X (with semi-group (P_t)) and Y (with semi-group (Q_t)), are connected by

$$P(K_p > t) = \frac{1}{E\,[Z^{-p}]}\,\mathbb{E}[Z^{-p}\,\mathbf{1}_{\{ZH_p > t\}}]\,,$$

if Λ is the kernel of multiplication by Z. Thus, the intertwining relations enabled us to infer the distributions of random variables related to a family of processes (*e.g.* $Y = X^{\alpha,\beta}$) from the distributions of random variables related to another family of processes (*e.g.* $X = X^{\alpha}$), therefore avoiding tedious computations.

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Unrectifiable 1-sets have vanishing analytic capacity

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Résumé. On complète la démonstration d'une conjecture de Vitushkin: si E est une partie compacte du plan complexe de mesure de Hausdorff unidimensionelle nulle, alors E est de capacité analytique nulle (toute fonction analytique bornée sur le complémentaire de E est constante) si et seulement si E est totalement non rectifiable (l'intersection de E avec toute courbe de longueur finie est de mesure de Hausdorff nulle). Comme dans un papier précédent avec P. Mattila, la démonstration repose sur un critère de rectifiabilité utilisant la courbure de Menger, et une extension d'une construction de M. Christ. L'élément nouveau principal est une généralisation du théorème $T(b)$ sur certains espaces qui ne sont pas nécessairement de type homogène.

Abstract. We complete the proof of a conjecture of Vitushkin that says that if E is a compact set in the complex plane with finite 1-dimensional Hausdorff measure, then E has vanishing analytic capacity (*i.e.*, all bounded analytic functions on the complement of E are constant) if and only if E is purely unrectifiable (*i.e.*, the intersection of E with any curve of finite length has zero 1-dimensional Hausdorff measure). As in a previous paper with P. Mattila, the proof relies on a rectifiability criterion using Menger curvature, and an extension of a construction of M. Christ. The main new part is a generalization of the $T(b)$-Theorem to some spaces that are not necessarily of homogeneous type.

1. Introduction.

The main goal of this paper is to complete the proof of Vitushkin's conjecture on 1-sets of vanishing analytic capacity.

Let E be a compact set in the complex plane. We say the E has vanishing analytic capacity if all bounded analytic functions on $\mathbb{C}\backslash E$ are constant. Ahlfors ([Ah]) proved that E has vanishing analytic capacity if and only if it is removable for bounded analytic functions, *i.e.*, if for all choices of an open set $\Omega \subset E$ and a bounded analytic function f on $\Omega\backslash E$, f has an analytic extension to Ω.

It was conjectured by Vitushkin ([Vi]) that if E is a compact set such that $0 < H^1(E) < +\infty$, then E has vanishing analytic capacity if and only if E is totally unrectifiable (or irregular in the terminology of Besicovitch), which means that $H^1(E \cap G) = 0$ for all rectifiable curves G. Here H^1 denotes one-dimensional Hausdorff measure. Actually, Vitushkin's conjecture also said something about the case when $H^1(K) = +\infty$, but this part turned out to be false ([Ma1]).

The first half of this conjecture was obtained as a consequence of A. P. Calderón's result on the boundedness of the Cauchy integral operator on $L^2(\Gamma)$ when Γ is a C^1-curve (or even a Lipschitz graph with small constant) in the plane ([Ca]). Indeed, if E is a compact subset of a rectifiable curve and $H^1(E) > 0$, there is a C^1-curve Γ such that $H^1(E \cap F) > 0$, and one can use Calderón's theorem and a nice duality argument of Uy ([Uy]) or Havin and Havinson ([HH]) to find non constant bounded analytic functions on $\mathbb{C}\backslash(E \cap F)$. Thus E cannot be removable for bounded analytic functions if $H^1(E \cap G) > 0$ for some rectifiable curve G. See for instance [Ch1] for a recent treatment of this result.

Our main result is as follows.

Theorem 1.1. *Let $E \subset \mathbb{C}$ be a compact set such that $H^1(E) < +\infty$ and E is totally unrectifiable. Then E has vanishing analytic capacity.*

Progress in the direction of Theorem 1.1 has been quite slow for some time, because one was not able to relate nicely information on the Cauchy kernel (typically, the existence of a bounded function on E whose Cauchy integral is bounded on $\mathbb{C}\backslash E$) to the geometry of E. Then M. Melnikov introduced "Menger curvature" in connection to analytic capacity ([Me]). This was rapidly followed by a result on the Cauchy operator ([MV]) and the proof of Theorem 1.1 in the special case when

E is Ahlfors-regular ([MMV]). This last means that there is a constant $C > 0$ such that

$$C^{-1}\, r \le H^1(E \cap B(x,r)) \le C\, r\,, \tag{1.2}$$

for all $x \in E$ and $0 < r \le \operatorname{diam} E$.

H. Pajot ([Pa]) observed that Ahlfors-regularity can be replaced with the weaker condition that

$$\begin{cases} \liminf\limits_{r\to 0}\, (r^{-1} H^1(E \cap B(x,r))) > 0\,, \\ \limsup\limits_{r\to 0}\, (r^{-1} H^1(E \cap B(x,r))) < +\infty\,, \qquad \text{for all } x \in E\,. \end{cases} \tag{1.3}$$

(This last is a sufficient condition for E to be contained in a countable union of Ahlfors-regular sets.) The method for these papers uses the miraculous positivity properties of Menger curvature, but also relies on standard Calderón-Zygmund techniques such as the $T(1)$-theorem. For these it is very useful to know that E is Ahlfors-regular, or at least that the restriction of H^1 to E is doubling, *i.e.*, that $H^1(E \cap B(x, 2\,r)) \le C H^1(E \cap B(x,r))$ for all $x \in E$ and $0 < r \le \operatorname{diam} E$ ([Li]).

It turns out that the general Calderón-Zygmund techniques used by [Ch2] and [MMV] do not fail in the general case when $0 < H^1(E) < +\infty$, but merely become much more painful to apply. This was (partially) observed in [DM], where the analogue of Theorem 1.1 for Lipschitz harmonic functions (instead of bounded analytic) is proved. The present paper will rely on the construction of [DM].

Before we start a short description of the argument, let us observe that it is very easy to show that E is removable for bounded analytic functions if $H^1(E) = 0$ (apply Cauchy's formula on curves of arbitrarily small lengths that surround E). Also, compact sets of dimension $d > 1$ are not removable: one can construct bounded analytic functions by taking Cauchy integrals of positive measures μ such that $\mu(B(x,r)) \le C\, r^{d'}$ for some $d' \in (1,d)$; such measures can be obtained from Frostman's lemma. Thus the only unclear situation left is when E has dimension 1 and $H^1(E) = +\infty$. See for instance [Ga], [Ch1], [Ma2], or [Vi] for general information about analytic capacity.

Let us now describe our strategy for proving Theorem 1.1. More details will be given in the course of the paper, but the reader may want to use this description to avoid getting lost in unimportant complications.

Let $E \subset \mathbb{C}$ be compact, and assume that $H^1(E) < +\infty$ and E does not have vanishing analytic capacity; we want to prove that E has a non trivial rectifiable piece. By easy manipulations, we can find a bounded analytic function h on $\mathbb{C}\setminus E$ such that $h(\infty) = 0$ and $h'(\infty) = \lim_{z\to\infty} z\, h(z) =: a > 0$. It is not hard to show that

$$h(z) = \int_E \frac{f(y)\, d\mu(y)}{z-y}\,, \qquad \text{for } z \in \mathbb{C}\setminus E\,, \tag{1.4}$$

where μ denotes the restriction of H^1 to E (*i.e.*, $\mu(A) = H^1(A \cap E)$ for all Borel sets A) and f is some bounded measurable function on E. This is Theorem 19.9 in [Ma2]. To prove it one surrounds E by a sequence of (finitely connected) curves Γ_n and one applies Cauchy's formula to them; eventually $f\, d\mu$ comes out as a weak limit of measures $h(y)\, dy$ on curves Γ_n.

The first stage of our argument consists in replacing $f\, d\mu$ with a new finite measure $g\, d\nu$ with the following properties:

$$0 \le \nu(B(x,r)) \le C\, r\,, \qquad \text{for all } x \in \mathbb{C} \text{ and } r > 0\,, \tag{1.5}$$

g is bounded acccretive, *i.e.*,

$$|g(x)| \le C\,, \qquad \mathrm{Re}\, g(x) \ge C^{-1} \text{ for all } x \in \mathbb{C}\,, \tag{1.6}$$

$$\int g\, d\nu = \int f\, d\mu = a > 0\,, \tag{1.7}$$

there is a Borel set $F \subset E$ such that

$$C^{-1}\mu \le \nu \le \mu \text{ on } F \text{ and } \nu(F) \ge \frac{a}{2}\,, \tag{1.8}$$

(the first half means that $C^{-1}\mu(A) \le \nu(A) \le \mu(A)$ for all Borel subsets A of F), and

$$\begin{array}{c}\text{the Cauchy integral of } g\, d\nu \text{ lies}\\ \text{in an appropriate space } \mathrm{BMO}(d\nu)\,.\end{array} \tag{1.9}$$

The measure $g\, d\nu$ will be imported directly from [DM], where it was constructed for very similar reasons (see in particular Theorem 2.4 in [DM]); the properties (1.5)-(1.8) are the same as (2.5)-(2.8) in [DM], and (1.9) will have to be made more precise and proved, starting from the corresponding L^2-estimate (2.9) in [DM]. The construction of $g\, d\nu$ is very similar in spirit to a construction of M. Christ ([Ch2]), who used it to show that if E is a regular set with positive analytic capacity, then there is another Ahlfors-regular set G such that $H^1(E \cap G) > 0$ and for which the Cauchy integral defines a bounded operator on $L^2(G)$. At that time, [MMV] did not exist, and so M. Christ could not conclude that G is uniformly rectifiable. The proof of boundedness of the Cauchy operator on $L^2(G)$ was directly deduced from the analogues of (1.6) and (1.9) by the $T(b)$-theorem (on G).

The construction of $g\,d\nu$ in [Ch2] and [DM] relies on the existence on E of an analogue of the decomposition of $\mathbb{R}^n$ into dyadic cubes. The general scheme is to replace $f\,d\mu$ by measures that live on small circles on (maximal) "cubes" $Q \subset E$ where $\mathrm{Re} \int f\,d\mu$ is a little too small. The construction is less pleasant in [DM] than in [Ch2], because one has to find slightly different ways to deal with the "small boundary property" of the constructed "dyadic cubes" when μ is not doubling. Nonetheless the spirit is the same.

In [DM] we could not continue as in [Ch2], because we did not have an appropriate $T(b)$-theorem. This is the reason why we restricted to Lipschitz harmonic capacity. If $H^1(E) < +\infty$ and E has positive Lipschitz harmonic capacity, then we can get $f\,d\mu$ (and then $g\,d\nu$) as above, but with f real-valued (and hence $g(x) \geq C^{-1}$) in addition. Then we do not need Stage 2 below, and we can use the argument of Stage 3 below to find that F is rectifiable (and hence that E is not totally unrectifiable).

In the present situation, g is not necessarily positive and we cannot apply directly the positivity argument with Menger curvature from [MMV] (see below), as in [DM]. So we'll prove a $T(b)$-theorem on $\tilde{E} = \mathrm{supp}\,(\nu)$ and apply it to the truncated operators T_ε given by

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{f(y)\,d\nu(y)}{x-y}\,, \tag{1.10}$$

to get uniform bounds on the norm of T_ε on $L^2(\tilde{E}, d\nu)$. Once again, the proof of the $T(b)$-theorem of Section 3 will follow rather classical outlines: we shall use the dyadic cubes from [DM], construct a version of the Haar system adapted to those cubes and the accretive function $b = g$, remove a "paraproduct" that takes care of Tb and $T^t b$, and prove that the matrix of the remaining operator in the modified Haar system has sufficient decay away from the diagonal to allow a use of Schur's lemma. This is the same program as in the proof of the (standard) $T(b)$-theorem by Coifman-Semmes ([CJS]) or Auscher-Tchamitchian ([AT]). See also [Da] or [My] for a presentation of this scheme and [DJS] for the original

$T(b)$ paper. Here again, the fact that μ is not necessarily doubling will create trouble, but altogether nothing dramatic. See sections 2-9 for the details.

We shall also need to spend some time checking that our $T(b)$-theorem applies to the space $(\tilde{E}, d\nu)$ and the function $b = g$ (see sections 10-13). In particular we'll have to build cubes adapted to $d\nu$, and then check the appropriate version of (1.9).

At the end of this (call it Stage 2), we know that the truncated Cauchy operators T_ε are bounded on $L^2(d\nu)$, with bounds that do not depend on ε. In particular,

$$\|T_\varepsilon 1\|^2_{L^2(d\nu)} \leq C\,, \tag{1.11}$$

where C does not depend on $\varepsilon > 0$. A brutal expansion of (1.11) gives that

$$\int_{x\in\tilde{E}} \Big(\int_{|x-y|>\varepsilon} \frac{d\nu(y)}{x-y}\Big)\Big(\int_{|x-z|>\varepsilon} \overline{\frac{d\nu(z)}{x-z}}\Big)\, d\nu(x) \leq C\,. \tag{1.12}$$

(There is no problem of convergence here and in the lines that follows, because ν is a finite measure.) The domain of integration in (1.12) is $U(\varepsilon) \cup V(\varepsilon)$, where

$$U(\varepsilon) = \big\{(x,y,z) \in \tilde{E}^3 :\ \varepsilon < |x-y|,\ |x-z|,\ |y-z|\big\} \tag{1.13}$$

and

$$V(\varepsilon) = \big\{(x,y,z) \in \tilde{E}^3 :\ |x-y| > \varepsilon,\ |x-z| > \varepsilon \text{ and } |y-z| \leq \varepsilon\big\}\,. \tag{1.14}$$

A fairly brutal computation gives that

$$\iiint_{V(\varepsilon)} \frac{d\nu(x)\, d\nu(y)\, d\nu(z)}{|x-y|\,|x-z|} \leq C\,, \tag{1.15}$$

see [MV, (5)], and note that the (very short) proof only uses (1.5). Thus

$$\Big|\iiint_{U(\varepsilon)} \frac{d\nu(x)\, d\nu(y)\, d\nu(z)}{(x-y)\,(\overline{x-z})}\Big| \leq C\,. \tag{1.16}$$

Now we want to use the following nice formula [Me]: for each triple (z_1, z_2, z_3) of distinct points of $\mathbb{C}$,

$$\sum_{\sigma\in G_3} \frac{1}{(z_{\sigma(1)} - z_{\sigma(2)})\,(\overline{z_{\sigma(1)} - z_{\sigma(3)}})} = c^2(z_1, z_2, z_3)\,, \tag{1.17}$$

where we sum over the group G_3 of permutations of $\{1, 2, 3\}$ and $c(z_1, z_2, z_3)$ denotes the Menger curvature of the triple (z_1, z_2, z_3), *i.e.*, the inverse of the radius of the circle that goes through z_1, z_2, z_3. (When the three points are on a line, set $c(z_1, z_2, z_3) = 0$.) This is [Me, (19), p. 842]. Because the integral in (1.16) is invariant under permutations of x, y, z, we can use (1.17) to get that

$$\iiint_{U(\varepsilon)} c^2(x, y, z)\, d\nu(x)\, d\nu(y)\, d\nu(z) \le C\,, \tag{1.18}$$

still with a constant C that does not depend on ε. Hence (by positivity),

$$c^2(\nu) =: \iiint_{\tilde{E}^3} c^2(x, y, z)\, d\nu(x)\, d\nu(y)\, d\nu(z) \le C\,. \tag{1.19}$$

We shall call $c(\nu)$ the Melnikov curvature of the measure ν.

At this point we can use a theorem of David and Léger ([Lé]), which says that if ν is a finite measure on $\mathbb{C}$ such that (1.5) holds, $c^2(\nu) < +\infty$, and if $\tilde{E}$, the support of ν, has finite H^1-measure, then ν is rectifiable. This means that $\tilde{E}$ is contained in a countable union of rectifiable curves, plus possibly a set of ν-measure zero. The set $\tilde{E} \cap F$, where F is as in (1.8), is also rectifiable, and hence meets some rectifiable curve on a set of H^1-measure greater than 0. This third stage completes the (sketch of) proof of Theorem 1.1.

Theorem 1.1 leaves open the characterization of vanishing analytic capacity for compact subsets of the plane such that $H^1(E) = +\infty$ but dimension $(E) = 1$. The obvious generalization of Vitushkin's conjecture where one would demand that

$$H^1(\pi_\theta(E)) = 0\,, \qquad \text{for almost every } \theta \in \mathbb{R}\,, \tag{1.20}$$

where π_θ denotes the orthogonal projection onto the line of direction $e^{i\theta}$, does not work. P. Mattila ([Ma1]) showed that (1.20) is not preserved when we replace E with its image under conformal mappings, while vanishing analytic capacity is. P. Jones and T. Murai ([JM]) later found examples of compact sets $E \subset \mathbb{C}$ with positive analytic capacity and such that (1.20) holds. It is not known yet whether there are compact sets of vanishing analytic capacity for which (1.20) does not hold. M. Melnikov likes to conjecture that compact sets E have positive analytic capacity if and only if there is a (nonzero) positive measure ν supported on E and such that $\nu(B(x, r)) \le C\, r$ for all $x \in E$ and $r > 0$, and $c^2(\nu) < +\infty$. Note that the "if" part of this conjecture is proved in [Me].

2. Construction of a Haar system.

In this section we are given a Borel subset E of some $\mathbb{R}^N$ and a finite Borel measure μ on E. We are also given a sequence of partitions of E into Borel subsets Q, $Q \in \Delta_k$, $k \geq 0$, with the following properties:

2.1) for each integer $k \geq 0$, E is the disjoint union of the sets Q, $Q \in \Delta_k$,

2.2) if $0 \leq k < \ell$, $Q \in \Delta_k$, and $R \in \Delta_\ell$, then either $Q \cap R = \varnothing$ or else $R \subset Q$,

2.3) $\mu(Q) > 0$ for all $Q \in \Delta_k$ and all $k \geq 0$,

2.4) $\operatorname{diam} Q \leq C_0 \, A^{-k}$ for all $k \geq 0$ and $Q \in \Delta_k$,

2.5) for each $k \geq 0$ and each $Q \in \Delta_k$, the number of $R \in \Delta_{k+1}$ such that $R \subset Q$ is $\leq C_0$.

Here C_0 and $A > 1$ are two constants that do not depend on k or Q, and $\operatorname{diam} Q$ is the diameter of Q. The sets Q, $Q \in \bigcup_k \Delta_k$, will be called cubes, or dyadic cubes (even though they should probably be called A-adic.) In the later sections, more will be required from these cubes, but the properties 2.1)-2.5) will be enough for the moment.

For each cube Q, we shall denote by $k(Q)$ the integer k such that $Q \in \Delta_k$, and by $d(Q) = A^{-k(Q)}$ its official approximate size. We should mention now that $\operatorname{diam} Q$ may be much smaller than $d(Q)$, and also that a given subset of E could be equal to Q for a few different cubes Q coming from different generations $k(Q)$. When we talk about a cube Q, we shall always mean both the set Q itself and the knowledge of the generation $k(Q)$.

If Q is a cube of generation $k(Q) \geq 1$, then there is a unique cube $\hat{Q} \in \Delta_{k(Q)-1}$ which contains Q, and which we'll call the parent of Q. The children of Q are the cubes $R \in \Delta_{k(Q)+1}$ that are contained in Q. We shall denote by $F(Q)$ the set of children of Q. Note that in some instances $F(Q)$ will be reduced to only one child, the set Q itself. At any rate, 2.5) says that $F(Q)$ never has more than C_0 elements.

In this section we want to construct a Riesz basis of $L^2(E, d\mu)$ which is adapted to the above decomposition of E into cubes, and a given accretive function b. This Riesz basis will be analogous to the

Haar basis, which corresponds to the case of $E = [0,1] \subset \mathbb{R}$, equipped with the Lebesgue measure, the usual dyadic intervals, and $b \equiv 1$. The construction given below is very similar to one initially used for [CJS] or [AT], but we shall need to repeat the argument to convince the reader that nothing more than 2.1)-2.5) is needed. For personal convenience reasons, we shall stay pretty close to the argument given in [Da].

Our function b is Borel-measurable, complex-valued, and bounded and accretive. This means that

$$|b(x)| \le C \qquad \text{and} \qquad \operatorname{Re} b(x) \ge C^{-1}\,, \qquad \text{for all } x \in E\,. \tag{2.6}$$

In fact, we shall only use the paraaccretivity condition that b is bounded and

$$\Big| \int_Q b\, d\mu \Big| \ge C^{-1} \mu(Q)\,, \qquad \text{for all cubes } Q\,, \tag{2.7}$$

but this will not matter for our only application.

We start our construction with the definition of a few projection operators. For $x \in E$ and $k \ge 0$, denote by $Q_k(x)$ the cube of Δ_k that contains x. Then set, for each $f \in L^2(E, d\mu)$,

$$E_k f(x) = \mu(Q_k(x))^{-1} \int_{Q_k(x)} f\, d\mu\,. \tag{2.8}$$

This is the standard orthogonal projection on the set of functions that are constant on each cube $Q \in \Delta_k$. Also set

$$D_k = E_{k+1} - E_k\,, \qquad k \ge 0\,, \tag{2.9}$$

and then define the corresponding twisted operators F_k and Z_k by

$$F_k f(x) = \Big(\int_{Q_k(x)} b\, d\mu \Big)^{-1} \int_{Q_k(x)} f\, b\, d\mu \tag{2.10}$$

and

$$Z_k = F_{k+1} - F_k\,. \tag{2.11}$$

We need a few easy facts concerning these operators. First,

$$\int_Q (F_k f)\, b\, d\mu = \int_Q f\, b\, d\mu\,, \qquad \text{for all } Q \in \Delta_k\,, \tag{2.12}$$

which is clear from (2.10). Next,

$$F_j F_k = F_{j\wedge k}\ , \tag{2.13}$$

with $j\wedge k = \min\{j,k\}$. When $j \geq k$, we observe that $F_k f$ is constant on all cubes $Q \in \Delta_k$, and hence also on cubes of Δ_j. Then $F_j F_k f = F_k f$. When $j < k$, (2.12) says that

$$\int_Q (F_k f)\, b\, d\mu = \int_Q f\, b\, d\mu\,,$$

for all $Q \in \Delta_k$, and hence all cubes $Q \in \Delta_j$. Then $F_j F_k f = F_j f$, by definition of F_j. This proves (2.13). Next

$$Z_j Z_k = \delta_{j,k}\, Z_j\ , \tag{2.14}$$

because

$$Z_j Z_k = (F_{j+1} - F_j)\,(F_{k+1} - F_k) = F_{j+1}F_{k+1} - F_j F_{k+1} - F_{j+1}F_k + F_j F_k\ .$$

A brutal computation using (2.13) gives the result.

Let us also check that

$$\int (Z_k\, u)\,(Z_\ell\, v)\, b\, d\mu = 0\,, \qquad \text{for } u, v \in L^2(d\mu) \text{ and } k \neq \ell\,. \tag{2.15}$$

We can assume that $k > \ell$. Since $Z_\ell\, v$ is constant on each cube of Δ_k, it is enough to show that

$$\int_Q (Z_k\, u)\, b\, d\mu = 0\,, \qquad \text{for all } Q \in \Delta_k\ . \tag{2.16}$$

This last holds because

$$\int_Q (F_k\, u)\, b\, d\mu = \int_Q (F_{k+1}\, u)\, b\, d\mu = \int_Q f\, b\, d\mu$$

by (2.12).

Next we check that E_0 and the D_k, $k \geq 0$, provide an orthonormal decomposition of $L^2(d\mu)$. First observe that if $\mathcal{E}$ denotes the set of (finite) linear combinations of characteristic functions of cubes, then

$$\mathcal{E} \text{ is dense in } L^2(d\mu)\,. \tag{2.17}$$

This is an easy consequence of (2.4), or more precisely of the fact that we can decompose E into disjoint unions of cubes of arbitrarily small diameters, because continuous functions are dense in $L^2(d\mu)$. Then

$$f = \lim_{k\to\infty} E_k f \qquad (\text{with convergence in } L^2(d\mu))\,, \tag{2.18}$$

for all $f \in L^2(d\mu)$, because this is obviously true when $f \in \mathcal{E}$, and the operators E_k are uniformly bounded. Also, the decomposition

$$E_k f = E_0 f + \sum_{\ell=0}^{k-1} D_\ell\, f \tag{2.19}$$

is orthogonal. The orthogonality of the D_ℓ's among themselves comes for instance from (2.15) with $b \equiv 1$, and they are orthogonal to E_0 by (2.16) with $b \equiv 1$. Because of this and (2.18),

$$\|f\|_2^2 = \|E_0 f\|_2^2 + \sum_{\ell\geq 0} \|D_\ell f\|_2^2\,, \tag{2.20}$$

for all $f \in L^2(d\mu)$.

We want to prove similar estimates for F_0 and the Z_ℓ's, but first we need a few facts about Carleson measures.

Definition 2.21. *A Carleson measure on $E \times \mathbb{N}$ is a measure $\nu = \{\nu_k\}_{k\geq 0}$ on $E \times \mathbb{N}$ such that*

$$\nu(Q \times \{k \in \mathbb{N} :\ k \geq k(Q)\}) =: \sum_{k\geq k(Q)} \nu_k(Q) \leq C\,\mu(Q)\,, \tag{2.22}$$

for all cubes Q, and with a constant C that does not depend on Q.

Recall that $k(Q)$ denotes the generation of Q. The definition is very analogous to the definition of discrete Carleson measures on the upper half space; one should not be disturbed by the fact that the role of $t > 0$ is played by A^{-k}, $k \in \mathbb{N}$, in our situation. Here is Carleson's theorem in our context.

Lemma 2.23. *Let $\nu = \{\nu_k\}_{k\geq 0}$ be a Carleson measure on $E \times \mathbb{N}$. Also let $f \in L^2(d\mu)$ and a sequence $\{f_k\}_{k\in\mathbb{N}}$ of functions be given. If*

$$|f_k(x)| \leq \mu(Q_k(x))^{-1} \int_{Q_k(x)} |f|\, d\mu\,, \tag{2.24}$$

for all $k \geq 0$ and $x \in E$, then

$$\int (|f_k|^2)_k \, d\nu =: \sum_k \int |f_k|^2 \, d\nu_k \leq C \, \|f\|_2^2 \, . \tag{2.25}$$

To prove this, we first need estimates on the maximal function

$$f^*(x) = \sup_k \mu(Q_k(x))^{-1} \int_{Q_k(x)} |f| \, d\mu \, . \tag{2.26}$$

We start with the usual weak-L^1 estimate. Let $f \in L^1(d\mu)$ and $\lambda > 0$ be given, and set $\mathcal{O}(\lambda) = \{x \in E : \ f^*(x) > \lambda\}$. Also denote by $\mathcal{M}_\lambda$ the collection of maximal cubes Q with the property that

$$\int_Q |f| \, d\mu > \lambda \, \mu(Q) \, . \tag{2.27}$$

(These are the cubes such that (2.27) holds and either $Q \in \Delta_0$ or else none of the ancestors of Q satisfies (2.27).) By definitions, the cubes Q are disjoint (because they are maximal) and cover exactly $\mathcal{O}(\lambda)$. Then

$$\mu(\mathcal{O}(\lambda)) = \sum_{Q \in \mathcal{M}_\lambda} \mu(Q) \leq \lambda^{-1} \sum_{Q \in \mathcal{M}_\lambda} \int_Q |f| \, d\mu \leq \lambda^{-1} \, \|f\|_1 \, . \tag{2.28}$$

Thus the maximal operator $f \longrightarrow f^*$ maps $L^1(d\mu)$ boundedly into weak-$L^1(d\mu)$. Since it is also clearly bounded on $L^\infty(d\mu)$, real interpolation gives that

$$\|f^*\|_2 \leq C \, \|f\|_2 \, , \qquad \text{for } f \in L^2(d\mu) \, . \tag{2.29}$$

Now let f and $\{f_k\}$ be as in the lemma, and set

$$\mathcal{U}(\lambda) = \{(x,k) \in E \times \mathbb{N} : \ |f_k(x)| > \lambda\} \, , \tag{2.30}$$

for each $\lambda > 0$. If $(x,k) \in \mathcal{U}(\lambda)$, then

$$\mu(Q_k(x))^{-1} \int_{Q_k(x)} |f| \, d\mu \geq |f_k(x)| > \lambda$$

by (2.24), and hence $Q_k(x)$ is contained in one of the cubes of $\mathcal{M}_\lambda$. Thus

$$U(\lambda) \subset \bigcup_{Q \in \mathcal{M}_\lambda} Q \times \{k \geq k(Q)\} \tag{2.31}$$

and then

$$
\begin{aligned}
\nu(\mathcal{U}(\lambda)) &\le \sum_{Q\in\mathcal{M}_\lambda} \nu(Q\times\{k\ge k(Q)\}) \\
&\le C \sum_{Q\in\mathcal{M}_\lambda} \mu(Q) = C\,\mu(\mathcal{O}(\lambda))\,,
\end{aligned} \tag{2.32}
$$

where $\mathcal{O}(\lambda)$ is as above, and by (2.22) and the first part of (2.28).

Thus the function of repartition of $\{f_k\}_{k\ge 0}$ for the measure ν is dominated by the function of repartition of f^* for μ; the desired estimate (2.25) follows from this and the maximal theorem (2.29). This proves Lemma 2.23.

Lemma 2.33. *For every $f\in L^2(d\mu)$,*

$$
f = F_0 f + \sum_{k\ge 0} Z_k f\,, \tag{2.34}
$$

where the series converges in $L^2(d\mu)$, and

$$
C^{-1}\|f\|_2^2 \le \|F_0 f\|_2^2 + \sum_{k\ge 0}\int |Z_k f|^2\,d\mu \le C\,\|f\|_2^2\,. \tag{2.35}
$$

Of course the constant C is not allowed to depend on f; it depends only on the accretivity constant in (2.6).

The formula (2.34) obviously holds when $f\in\mathcal{E}$ (and then the sum is finite), because $F_k f = f$ as soon as f is constant on all the cubes of Δ_k. The general case follows by density of $\mathcal{E}$, plus the fact that the operators F_k are uniformly bounded on L^2, by their definition (2.10) and the accretivity condition (2.6). (Look at the effect of F_k on each cube $Q\in\Delta_k$ separately.)

Now we want to prove the second inequality in (2.35). Write

$$
\begin{aligned}
Z_k f &= F_{k+1} f - F_k f \\
&= (E_{k+1}\,b)^{-1} E_{k+1}(bf) - (E_k\,b)^{-1} E_k(bf) \\
&= ((E_{k+1}\,b)^{-1} - (E_k\,b)^{-1})\,E_{k+1}(bf) \\
&\quad + (E_k\,b)^{-1}\,(E_{k+1}(bf) - E_k(bf))\,,
\end{aligned} \tag{2.36}
$$

and then use the fact that $(E_{k+1}\, b)^{-1} (E_k\, b)^{-1}$ is bounded because of (2.7) to get that

$$|Z_k f|^2 \le C\, |D_k\, b|^2\, |E_{k+1}(bf)|^2 + C\, |D_k(bf)|^2 . \tag{2.37}$$

We can easily take care of the second piece, because

$$\sum_{k \ge 0} \int |D_k(bf)|^2\, d\mu = \sum_k \|D_k\, bf\|_2^2 \le \|bf\|_2^2 \le C\, \|f\|_2^2 , \tag{2.38}$$

by (2.20). For the first piece, we want to use Lemma 2.23 with the sequence $\{f_k\}$ given by $f_k = E_k(bf)$, $k \ge 1$. Obviously

$$|E_k(bf)(x)| \le \mu(Q_k(x))^{-1} \int_{Q_k(x)} |bf|\, d\mu ,$$

for all $x \in E$, and so (2.24) holds (modulo an inessential constant).

We also want to take $\nu_k = |D_{k-1}\, b|^2\, d\mu$ for $k \ge 1$, and we have to check that this is a Carleson measure. Thus we take a cube Q and try to estimate

$$\sum_{k \ge k(Q)} \int_Q |D_{k-1}\, b|^2\, d\mu .$$

When $k > k(Q)$, $D_{k-1}\, b = D_{k-1}(b\, \mathbf{1}_Q)$ on Q by definitions, and so

$$\begin{aligned} \sum_{k > k(Q)} \int_Q |D_{k-1}\, b|^2\, d\mu &\le \sum_k \int |D_{k-1}(b\, \mathbf{1}_Q)|^2\, d\mu \\ &\le \|b\, \mathbf{1}_Q\|_2^2 \\ &\le C\, \mu(Q) \end{aligned} \tag{2.39}$$

by (2.20) and the fact that b is bounded. The last term $\int_Q |D_{k(Q)}\, b|^2\, d\mu$ is at most $C\, \mu(Q)$ because $\|D_{k(Q)}\, b\|_\infty \le 2\, \|b\|_\infty$, and so $\{\nu_k\}_{k \ge 1}$ defines a Carleson measure. By Lemma 2.23,

$$\sum_{k \ge 1} \int |D_{k-1}\, b|^2\, |E_k(bf)|^2\, d\mu \le C\, \|f\|_2^2 . \tag{2.40}$$

We are left with a last term, $k = 0$. For this one,

$$\int |D_0\, b|^2\, |E_1(bf)|^2\, d\mu \le C\, \|E_1(bf)\|_2^2 \le C\, \|f\|_2^2 , \tag{2.41}$$

by a brutal estimate. From (2.37), (2.38), (2.40) and (2.41) we deduce that

$$\sum_{k\geq 0} \int |Z_k f|^2 \, d\mu \leq C \, \|f\|_2^2 \, . \tag{2.42}$$

Since we also have that

$$\begin{aligned} \|F_0 f\|_2^2 &= \sum_{Q\in\Delta_0} \Big| \Big(\int_Q b \, d\mu \Big)^{-1} \Big(\int_Q f \, b \, d\mu \Big) \Big|^2 \mu(Q) \\ &\leq C \sum_{Q\in\Delta_0} \int_Q |f \, b|^2 \, d\mu \\ &\leq C \, \|f\|_2^2 \, , \end{aligned}$$

by Cauchy-Schwarz, we get the second half of (2.35).

The first half of (2.35) will now follow by duality. We write

$$f = F_0 f + \sum_k Z_k \, f$$

and

$$b^{-1} \, \overline{f} = F_0(b^{-1} \, \overline{f}) + \sum_k Z_k(b^{-1} \, \overline{f})$$

as in (2.34), and then

$$\|f\|_2^2 = \int f \, (b^{-1} \, \overline{f}) \, b \, d\mu \, , \tag{2.43}$$

which we expand as suggested above. Note that for $k \neq \ell$,

$$\int (Z_k f) \, (Z_\ell(b^{-1} \, \overline{f})) \, b \, d\mu = 0$$

by (2.15), and also that

$$\int (F_0 f) \, Z_k(b^{-1} \, \overline{f}) \, b \, d\mu = \int F_0(b^{-1} \, \overline{f}) \, Z_k(f) \, b \, d\mu = 0 \, ,$$

for all k because $F_0(f)$ and $F_0(b^{-1}\overline{f})$ are constant on cubes of Δ_0 and by (2.16). Thus

$$
\begin{aligned}
\|f\|_2^2 &\leq \Big|\int (F_0 f)\,(F_0(b^{-1}\overline{f}))\,b\,d\mu\Big| + \sum_k \Big|\int (Z_k f)\,(Z_k(b^{-1}\overline{f}))\,b\,d\mu\Big| \\
&\leq C\,\|F_0 f\|_2\,\|F_0(b^{-1}\overline{f})\|_2 + C\sum_k \|Z_k f\|_2\,\|Z_k(b^{-1}\overline{f})\|_2 \\
&\leq C\Big(\|F_0 f\|_2^2 + \sum_k \|Z_k f\|_2^2\Big)^{1/2} \\
&\qquad \cdot \Big(\|F_0(b^{-1}\overline{f})\|_2^2 + \sum_k \|Z_k(b^{-1}\overline{f})\|_2^2\Big)^{1/2} \\
&\leq C\Big(\|F_0 f\|_2^2 + \sum_k \|Z_k f\|_2^2\Big)^{1/2} \|b^{-1}\overline{f}\|_2
\end{aligned}
\tag{2.44}
$$

by Cauchy-Schwarz (twice) and the second half of (2.35) (applied to $b^{-1}\overline{f}$). Of course $\|b^{-1}\overline{f}\|_2 \leq C\,\|f\|_2$, so we may divide both sides of (2.44) by $\|f\|_2$ (if $f \neq 0$) and get the first half of (2.35).

This completes the proof of Lemma 2.33.

For each cube Q, denote by $W^+(Q)$ the vector space of all functions f that are supported on Q and constant on each of the children of Q. Also let $W(Q)$ be the set of functions $f \in W^+(Q)$ such that

$$\int_Q f\,b\,d\mu = 0\,. \tag{2.45}$$

Let r denote the number of children of Q; thus $1 \leq r \leq C_0$ by (2.5). The dimension of $W^+(Q)$ is obviously r. Since the condition (2.45) is not degenerate on $W^+(Q)$ (because $\mathbf{1}_Q$ does not satisfy (2.45)), $W(Q)$ is an $(r-1)$-dimensional space.

We want to find an appropriate basis of $W(Q)$. If $r = 1$, *i.e.*, if Q has only one child, then $W(Q) = \{0\}$ and there is nothing to do. Otherwise we set $D = D(Q) = \{1, 2, \ldots, r-1\}$ and look for a basis $\{h_Q^\varepsilon\}_{\varepsilon \in D}$ of $W(Q)$ such that

$$\int_Q h_Q^\varepsilon\, h_Q^{\varepsilon'}\, b\,d\mu = \delta_{\varepsilon,\varepsilon'}\,, \tag{2.46}$$

for $\varepsilon, \varepsilon' \in D$, and where $\delta_{\varepsilon,\varepsilon'} = 1$ if $\varepsilon = \varepsilon'$ and 0 otherwise. It will be convenient for us to add the function

$$h_Q^0 = \Big(\int_Q b\,d\mu\Big)^{-1/2} \mathbf{1}_Q\ , \tag{2.47}$$

where the choice of square root is irrelevant, to get a basis of $W^+(Q)$. With this choice of h_Q^0, we'll even have (2.46) for all $\varepsilon, \varepsilon' \in D^+ = \{0, 1, \ldots, r-1\}$, because $\int_Q h_Q^\varepsilon\, b\,d\mu = 0$ if $h_Q^\varepsilon \in W(Q)$, by (2.45). Denote by $\alpha_{\varepsilon,R}\,\mu(R)^{-1/2}$ the constant value of h_Q^ε on the child $R \in F(Q)$ of Q. Thus we want to look for h_Q^ε under the form

$$h_Q^\varepsilon = \sum_{R\in F(Q)} \alpha_{\varepsilon,R}\ \mu(R)^{-1/2}\,\mathbf{1}_R\ . \tag{2.48}$$

We have already decided that

$$\alpha_{0,R} = \Big(\int_Q b\,d\mu\Big)^{-1/2} \mu(R)^{1/2}\,.$$

Set $b_R = \mu(R)^{-1}\int_R b\,d\mu$ for all $R \in F(Q)$. Note that these numbers are bounded and bounded away from 0 by (2.7). With all these notations, our constraints (2.46) are equivalent to

$$\sum_{R\in F(Q)} \alpha_{\varepsilon,R}\ \alpha_{\varepsilon',R}\ b_R = \delta_{\varepsilon,\varepsilon'}\ , \qquad \text{for } \varepsilon, \varepsilon' \in D^+\,. \tag{2.49}$$

Lemma 2.50. *We can find complex numbers $\alpha_{\varepsilon,R}$, $1 \le \varepsilon \le r-1$ and $R \in F(Q)$, such that* (2.49) *holds and $|\alpha_{\varepsilon,R}| \le C$ for some constant C that depends only on the accretivity constant in* (2.6) *and C_0 in* (2.5).

To prove the lemma, some additional notation will be useful. Define a bilinear form $\langle \cdot\,,\cdot\rangle_b$ on $\mathbb{C}^r$ (indexed by the set $F(Q)$ of children of Q) by

$$\langle v, w\rangle_b = \sum_R v_R\, w_R\, b_R\ ,$$

where $v = (v_R)$ and $w = (w_R)$.

Now suppose we already chose coefficients $\alpha_{\varepsilon,R}$, $0 \le \varepsilon \le k-1$, for some $k \in \{1, \ldots r-1\}$, in such a way that the equations in (2.49) hold for $0 \le \varepsilon$, $\varepsilon' \le k-1$. (We already did this with $k = 1$.) Call v_ε,

$0 \leq \varepsilon \leq k-1$, the vector of $\mathbb{C}^r$ with coordinates $\alpha_{\varepsilon,R}$, $R \in F(Q)$. With our new notations,

$$\langle v_\varepsilon, v_{\varepsilon'} \rangle_b = \delta_{\varepsilon,\varepsilon'}\ , \qquad \text{for } 0 \leq \varepsilon,\ \varepsilon' \leq k-1\,. \tag{2.51}$$

We want to define a new vector v_k. Set

$$V = \{v \in \mathbb{C}^r : \langle v, v_\varepsilon \rangle = 0 \ \text{ for } 0 \leq \varepsilon \leq k-1\}\,. \tag{2.52}$$

Because $k \leq r-1$, V is at least one-dimensional and in particular is not empty. Select a first vector $z \neq 0$ in V. Because the numbers b_R are all $\neq 0$, we can find $w \in \mathbb{C}^r$ such that $\langle z, w \rangle_b \neq 0$. Since the $|b_R|$ are bounded from below, we can even choose z and w with bounded coefficients, and with $\langle z, w \rangle_b = 1$.

We want to modify w to get a vector in V. Set

$$v = w - \sum_{\varepsilon \leq k-1} \langle w, v_\varepsilon \rangle_b \, v_\varepsilon\ . \tag{2.53}$$

Then

$$\langle v, v_{\varepsilon'} \rangle_b = \langle w, v_{\varepsilon'} \rangle_b - \sum_\varepsilon \langle w, v_\varepsilon \rangle_b \, \langle v_\varepsilon, v_{\varepsilon'} \rangle_b = 0\,, \tag{2.54}$$

for all $\varepsilon' \leq k-1$, because of (2.51). Hence $v \in V$, as desired. Also,

$$\langle z, v \rangle_b = \langle z, w \rangle_b - \sum_{\varepsilon \leq k-1} \langle w, v_\varepsilon \rangle_b \, \langle z, v_\varepsilon \rangle_b = \langle z, w \rangle_b = 1\,, \tag{2.55}$$

because $z \in V$.

Choose among z, v, and $z+v$ the vector x for which $|\langle x, x \rangle_b|$ is largest. Note that if $|\langle z, z \rangle_b|$ and $|\langle v, v \rangle_b|$ are less than $1/2$, then

$$|\langle z+v, z+v \rangle_b| = |\langle z, z \rangle_b + \langle v, v \rangle_b + 2\,\langle z, w \rangle_b| \geq 1\,,$$

by (2.55), so that $|\langle x, x \rangle_b| \geq 1/2$ in all cases. We take

$$v_k = (\langle x, x \rangle_b)^{-1/2}\, x\,.$$

It is easy to see that v_k has coefficients $\alpha_{k,R}$, $R \in F(Q)$, that can be bounded in terms of the $|\alpha_{\varepsilon,R'}|$, $\varepsilon \leq k-1$ and $R' \in F(Q)$, and the accretivity constant for b. With this choice of v_k, we now have the identities in (2.49) for $\varepsilon, \varepsilon' \le k$. The lemma follows by induction.

Let us choose the coefficients $\alpha_{\varepsilon,R}$ as in Lemma 2.50. This defines functions h_Q^ε, $\varepsilon \in D = D(Q)$, that lie in $W(Q)$ and satisfy (2.46). Set

$$\langle f, g\rangle_b = \int f\, g\, b\, d\mu\,, \qquad \text{for } f, g \in L^2(d\mu)\,. \tag{2.56}$$

With this notation, (2.46) is the same as

$$\langle h_Q^\varepsilon, h_Q^{\varepsilon'}\rangle_b = \delta_{\varepsilon,\varepsilon'}\,, \qquad \text{for } \varepsilon, \varepsilon' \in D(Q)\,. \tag{2.57}$$

Lemma 2.58. *The functions h_Q^ε, $\varepsilon \in D(Q)$, form a basis of $W(Q)$, and*

$$f = \sum_{\varepsilon\in D(Q)} \langle f, h_Q^\varepsilon\rangle_b\, h_Q^\varepsilon\,, \qquad \textit{for all } f \in W(Q)\,. \tag{2.59}$$

In addition,

$$C^{-1}\|f\|_2^2 \le \sum_{\varepsilon\in D(Q)} |\langle f, h_Q^\varepsilon\rangle_b|^2 \le C\,\|f\|_2^2\,, \tag{2.60}$$

for all $f \in W(Q)$, with a constant C that depends only on the constants in (2.5) *and* (2.6).

Indeed, if $f \in W(Q)$ can be written as $f = \sum_{\varepsilon\in D} c_\varepsilon\, h_Q^\varepsilon$, then

$$\langle f, h_Q^\varepsilon\rangle_b = \sum_{\varepsilon'} c_{\varepsilon'}\, \langle h_Q^{\varepsilon'}, h_Q^\varepsilon\rangle_b = c_\varepsilon\,,$$

by (2.57). Applying this with $f = 0$ gives the independence of the functions h_Q^ε; we then deduce that they form a basis of $W(Q)$ because we know that dimension $(W(Q)) = r - 1$. Thus all $f \in W(Q)$ can be written as $f = \sum_{\varepsilon\in D} c_\varepsilon\, h_Q^\varepsilon$, and the computation above shows that the c_ε are as in (2.59).

From the formula (2.48) and the fact that the coefficients $\alpha_{\varepsilon,R}$ are bounded, we deduce at one that

$$|h_Q^\varepsilon| \le C \sum_{R\in F(Q)} \mu(R)^{-1/2}\, \mathbf{1}_R\,. \tag{2.61}$$

In particular,

$$\|h_Q^\varepsilon\|_2 \le C\,. \tag{2.62}$$

If $f \in W(Q)$, then (2.59) implies that

$$\|f\|_2 \le \sum_{\varepsilon\in D} |\langle f, h_Q^\varepsilon\rangle_b|\, \|h_Q^\varepsilon\|_2 \le C\Big(\sum_{\varepsilon\in D} |\langle f, h_Q^\varepsilon\rangle_b|^2\Big)^{1/2},$$

by the equivalence of the ℓ^1 and ℓ^2-norms in $\mathbb{C}^{r-1}$, and the fact that $r \le C_0$. Similarly,

$$\sum_{\varepsilon\in D} |\langle f, h_Q^\varepsilon\rangle_b|^2 \le C_0\, \|f\|_2^2\,,$$

by Schwarz and (2.62). This completes our proof of Lemma 2.58.

Proposition 2.63. *Every function $f \in L^2(d\mu)$ can be written as*

$$f = F_0 f + \sum_{k\ge 0}\sum_{Q\in\Delta_k}\sum_{\varepsilon\in D(Q)} \langle f, h_Q^\varepsilon\rangle_b\, h_Q^\varepsilon\,, \tag{2.64}$$

where

$$\langle f, h_Q^\varepsilon\rangle_b = \int_Q f\, h_Q^\varepsilon\, b\, d\mu \tag{2.65}$$

is as in (2.56), *and the convergence of the series in k occurs in $L^2(d\mu)$. Moreover,*

$$C^{-1}\|f\|_2^2 \le \|F_0 f\|_2^2 + \sum_{k\ge 0}\sum_{Q\in\Delta_k}\sum_{\varepsilon\in D(Q)} |\langle f, h_Q^\varepsilon\rangle_b|^2 \le C\,\|f\|_2^2\,. \tag{2.66}$$

Finally, the decomposition in (2.64) *is unique: if there is a decomposition*

$$f = f_0 + \sum_{k}\sum_{Q\in\Delta_k}\sum_{\varepsilon\in D(Q)} c_Q^\varepsilon\, h_Q^\varepsilon\,, \tag{2.67}$$

where f_0 is constant on each cube of Δ_0 and the series (in k) converges in $L^2(d\mu)$, then $f_0 = F_0 f$ and $c_Q^\varepsilon = \langle f, h_Q^\varepsilon\rangle_b$ for all $Q \in \bigcup_k \Delta_k$ and $\varepsilon \in D(Q)$.

Recall from (2.10) that F_0 is a harmless projection onto the subspace of functions that are constant on each cube of Δ_0.

We start with the proof of the existence of the decomposition and the estimate (2.66). We already have a decomposition of f as $f = F_0 f + \sum_k Z_k f$, with a control on the norms, that comes from Lemma 2.33. Because of this, it will be enough to show that for all $k \geq 0$,

$$Z_k f = \sum_{Q \in \Delta_k} \sum_{\varepsilon \in D(Q)} \langle f, h_Q^\varepsilon \rangle_b \, h_Q^\varepsilon \tag{2.68}$$

and

$$\|Z_k f\|_2^2 \sim \sum_{Q \in \Delta_k} \sum_{\varepsilon \in D(Q)} |\langle f, h_Q^\varepsilon \rangle_b|^2 \,. \tag{2.69}$$

Obviously, $Z_k f = \sum_{Q \in \Delta_k} Z_k^Q f$, where $Z_k^Q f = \mathbf{1}_Q \, Z_k f$, and

$$\|Z_k f\|_2^2 = \sum_{Q \in \Delta_k} \|Z_k^Q f\|_2^2 \,.$$

Thus it is enough to show that

$$Z_k^Q f = \sum_{\varepsilon \in D(Q)} \langle f, h_Q^\varepsilon \rangle_b \, h_Q^\varepsilon \tag{2.70}$$

and

$$\|Z_k^Q f\|_2^2 \sim \sum_{\varepsilon \in D(Q)} |\langle f, h_Q^\varepsilon \rangle_b|^2 \,, \tag{2.71}$$

for each cube $Q \in \Delta_k$, and with constants in (2.71) that do not depend on f, k, or Q. In view of Lemma 2.58, it is enough to show that $Z_k^Q f \in W(Q)$ and that

$$\langle Z_k^Q f, h_Q^\varepsilon \rangle_b = \langle f, h_Q^\varepsilon \rangle_b \,, \tag{2.72}$$

for all $\varepsilon \in D(Q)$.

It is clear that $Z_k^Q f = \mathbf{1}_Q \, (F_{k+1} f - F_k f)$ is supported on Q and constant on each child of Q. (See the definitions (2.10) and (2.11).)

Also, $\int_Q (Z_k^Q f)\, b\, d\mu = 0$ by (2.16), and hence $Z_k^Q f \in W(Q)$. (See near (2.45) for the definition of $W(Q)$.) Finally, let $\varepsilon \in D(Q)$ be given. Then

$$\begin{aligned} \langle Z_k^Q f, h_Q^\varepsilon \rangle_b &= \int_Q (Z_k^Q f)\, h_Q^\varepsilon\, b\, d\mu \\ &= \int_Q (F_{k+1} f - F_k f)\, h_Q^\varepsilon\, b\, d\mu \\ &= \int_Q (F_{k+1} f)\, h_Q^\varepsilon\, b\, d\mu\,, \end{aligned} \tag{2.73}$$

by definitions (and in particular (2.11)), the fact that $F_k f$ is constant on Q, and because

$$\int_Q h_Q^\varepsilon\, b\, d\mu = 0\,, \qquad \text{for all } Q \text{ and } \varepsilon \in D(Q) \tag{2.74}$$

(because $h_Q^\varepsilon \in W(Q)$). Next h_Q^ε is constant on each cube of Δ_{k+1}, and so (2.12) (applied with $k+1$) tells us that

$$\int_Q (F_{k+1} f) h_Q^\varepsilon\, b\, d\mu = \int_Q f\, h_Q^\varepsilon\, b\, d\mu = \langle f, h_Q^\varepsilon \rangle_b\,.$$

This completes the proof of (2.64)-(2.66), and we are left with the uniqueness result to prove. To this effect, let us first check that

$$\langle h_Q^\varepsilon, h_{Q'}^{\varepsilon'} \rangle_b = \delta_{(Q,\varepsilon),(Q',\varepsilon')} \tag{2.75}$$

(that is, 1 if $Q = Q'$ and $\varepsilon = \varepsilon'$ and 0 otherwise) for all choices of Q, $Q' \in \bigcup_k \Delta_k$, $\varepsilon \in D(Q)$, and $\varepsilon' \in D(Q')$.

We already know this when $Q = Q'$. When Q and Q' both lie in a same Δ_k but $Q \neq Q'$, then (2.75) holds because h_Q^ε and $h_Q^{\varepsilon'}$ have disjoint supports. Finally assume that $Q \in \Delta_k$ and $Q' \in \Delta_\ell$, and that $\ell < k$. Then $h_{Q'}^{\varepsilon'}$ is constant on Q and $\langle h_Q^\varepsilon, h_Q^{\varepsilon'} \rangle_b = 0$ by (2.74). Thus (2.75) holds in all cases.

Now let $f \in L^2(d\mu)$, and suppose that f has a decomposition (2.67) as in the proposition. For each choice of $Q' \in \bigcup_k \Delta_k$ and $\varepsilon' \in D(Q')$, $\langle f_0, h_{Q'}^{\varepsilon'} \rangle_b = 0$ by (2.74) and because f_0 is constant on Q'. Then (2.75) tells us that

$$\Big\langle f_0 + \sum_{k=0}^{\ell} \sum_{Q \in \Delta_k} \sum_{\varepsilon \in D(Q)} c_Q^\varepsilon\, h_Q^\varepsilon, h_{Q'}^{\varepsilon'} \Big\rangle_b = c_{Q'}^{\varepsilon'}\,, \tag{2.76}$$

for ℓ large enough. Thus $c_{Q'}^{\varepsilon'} = \langle f, h_{Q'}^{\varepsilon'} \rangle_b$ by taking limits. A comparison of (2.67) with (2.64) now gives that $f_0 = F_0 f$ because we know that the series are the same.

This completes our proof of Proposition 2.63.

3. A $T(b)$-theorem.

Let E be a compact subset of the plane, and let μ be a finite positive Borel measure, with support $(\mu) = E$. We shall assume that

$$\mu(B(x,r)) \le C_0\, r\,, \qquad \text{for all } x \in E \text{ and } r > 0 \tag{3.1}$$

and some constant $C_0 > 0$. We want to state (and later prove) a T(b)-theorem on the space $(E, d\mu)$ for one-dimensional singular integral operators; unfortunately, our statement will already require the existence of a collection of "dyadic cubes on E" with properties somewhat stronger than those of Section 2. We shall assume that E is equipped with collections Δ_k, $k \ge 0$, of Borel subsets (which we'll call cubes) with the following properties.

First we ask for the same combinatorial properties as in (2.1) and (2.2):

$$\begin{aligned} &\text{for each } k \ge 0,\ E \text{ is the disjoint union}\\ &\text{of the cubes } Q,\ Q \in \Delta_k, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\text{if } k < \ell,\ Q \in \Delta_k \text{ and } R \in \Delta_\ell,\\ &\text{then either } Q \cap R = \varnothing \text{ or else } R \subset Q. \end{aligned} \tag{3.3}$$

We also require that for each integer $k \ge 0$ and each $Q \in \Delta_k$, there be a ball $B(Q) = B(x(Q), r(Q))$ centered on E and such that

$$A^{-k} \le r(Q) \le C_1 A^{-k} \tag{3.4}$$

and

$$E \cap B(Q) \subset Q \subset E \cap (30\, B(Q))\,, \tag{3.5}$$

where $30\, B(Q) = B(x(Q), 30\, r(Q))$. Here A and C_1 are positive constant, and we shall assume (mostly for security reasons) that $A > 10^4\, C_1$. It will be convenient for us to demand also that

$$\Delta_0 \text{ has only one element}\,, \tag{3.6}$$

because it will make some of the algebra easier. This is also easy to arrange, because E is bounded and we could always add a first generation of cubes with only one element, or group all the cubes of Δ_0 into a single one. (This would make the constants C_1 and A slightly worse, though.)

We shall also need "small boundary" properties for our cubes. Set

$$\begin{aligned} N_t(Q) = \{x \in Q: \ \operatorname{dist}(x, E \backslash Q) \le t A^{-k(Q)}\} \\ \cup \{x \in E \backslash Q: \ \operatorname{dist}(x, Q) \le t A^{-k(Q)}\}, \end{aligned} \tag{3.7}$$

for all $Q \in \Delta = \bigcup_k \Delta_k$ and $0 < t \le 1$, and where $k(Q)$ denotes, as in Section 2, the integer such that $Q \in \Delta_{k(Q)}$. We require the existence of an exponent $\tau \in [9/10, 1]$ and positive numbers $\xi(Q)$, $Q \in \Delta$, with the following properties. First,

$$\mu(N_t(Q)) \le C_0 \, t^\tau \, \xi(Q), \qquad \text{for all } Q \in \Delta \text{ and } 0 < t \le 1. \tag{3.8}$$

Also,

$$\mu(91\, B(Q)) \le C_0 \, \xi(Q) \le C_0^2 A^{-k(Q)}, \tag{3.9}$$

and

$$\sum_{\substack{R \in \Delta_k \\ R \subset 91 B(Q)}} \xi(R) \le C_0 \, \xi(Q), \tag{3.10}$$

for all $k > k(Q)$. These are coherence relations that will be useful when we try to apply Shur's lemma (much later). A reasonable choice would be $\xi(Q) = \mu(92\, B(Q))$, say, but this will not suffice for our application to Theorem 1.1 because we shall be working at the same time with some other measure.

Our condition (3.8) will be even more useful for cubes Q such that

$$\xi(Q) \le C_0 \, \mu(Q). \tag{3.11}$$

Let us call these cubes good cubes. Denote by $\mathcal{G}$ the set of good cubes. We also assume that the only cube of Δ_0 is a good cube (which would be fairly easy to arrage anyway), and add a last requirement on the numbers $\xi(Q)$ that will allow a better control on the bad cubes. We demand that

$$\xi(Q) \le A^{-10} \xi(\hat{Q}), \tag{3.12}$$

whenever Q is a bad cube and $\hat{Q}$ is its parent (*i.e.*, the cube of $\Delta_{k(Q)-1}$ that contains it).

The reader may be worried by this long list of requirements. Indeed this will make it rather unpleasant to check all the hypotheses of Theorem 3.20 below, but nonetheless it is always possible to construct cubes with the properties above when $E = \operatorname{supp} \mu$ and μ satisfies (3.1). Such a construction is done in [DM], and we shall encounter it when we try to apply Theorem 3.20 to analytic capacity.

We shall also assume that we are given a Borel function b on E, and that b is bounded accretive, *i.e.*, satisfies (2.6).

Now we want to describe the singular integral operators that we want to study. Denote by $\mathcal{E}$ the vector space of (finite) complex linear combinations of characteristic functions of cubes $Q \in \Delta$. Also let $b\,\mathcal{E}$ be the set of products bf, $f \in \mathcal{E}$. It will be easier to define our operators as operators from $b\,\mathcal{E}$ to its dual, or equivalently as bilinear operators from $b\,\mathcal{E} \times b\,\mathcal{E} \longrightarrow \mathbb{C}$. We shall denote by $\langle Tbf, b\,g\rangle$, $f, g \in \mathcal{E}$, the effect of $T(bf)$ on $b\,g$ (or equivalently the image of $(bf, b\,g)$ under the bilinear operator). In particular, we drop the parentheses around bf intentionnally, to simplify notations.

We shall assume that T is associated to a "standard kernel", as follows. By standard kernel, we mean a continuous function $K(x, y)$ on $\{(x, y) \in \mathbb{C}^2 : x \neq y\}$ such that

$$(3.13) \qquad |K(x,y)| \le C_2 |x-y|^{-1}\,, \qquad \text{for } x \neq y$$

and

$$(3.14) \qquad |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le C_2 \frac{|z-y|}{|x-y|^2}\,,$$

whenever $|z - y| < |x - y|/2$.

The Cauchy kernel $K(x, y) = (x - y)^{-1}$ is obviously a very good example of standard kernel.

The relation between T and K is that

$$(3.15) \qquad \langle Tf, g\rangle = \iint K(x,y)\, f(x)\, g(y)\, d\mu(x)\, d\mu(y)\,,$$

whenever $f, g \in b\,\mathcal{E}$ have disjoint supports.

By disjoint supports we mean that we can write f and g as $f = \sum_Q \lambda_Q\, b\,\mathbf{1}_Q$ and $g = \sum_R \eta_R\, b\,\mathbf{1}_R$, with all the cubes Q disjoint from the cubes R. The reader should not worry about the convergence of the integral in (3.15). We shall see later that

$$\int_Q \int_R \frac{d\mu(x)\, d\mu(y)}{|x-y|} < +\infty\,, \tag{3.16}$$

for all cubes Q, R such that $Q \cap R = \varnothing$. This will come as a fairly easy consequence of (3.1) and (3.8), but we prefer not to check it now and try to state our main theorem soon. See (8.7) and the relevant definition (7.9) for a proof.

We shall also demand that T satisfy the following analogue of the "weak boundedness property": there is a constant $C_3 \geq 0$ such that

$$|\langle Tb\,\mathbf{1}_Q, b\,\mathbf{1}_Q\rangle| \leq C_3\, \mu(Q)\,, \text{for all } Q \in \Delta\,. \tag{3.17}$$

Our last conditions will be that $Tb \in$ BMO and $T^t b \in$ BMO. Since E is in general far from being a space of homogeneous type, there is some ambiguity as to which definition of BMO we should take. The following version of "dyadic-BMO" based on L^2-oscillation will be best suited to our needs.

Definition 3.18. *We denote by* BMO *the set of functions* $\beta \in L^2(d\mu)$ *such that*

$$\int_Q |\beta(x) - m_Q\beta|^2\, d\mu(x) \leq C^2 \mu(Q)\,, \tag{3.19}$$

for all cubes $Q \in \Delta$ *and some* $C \geq 0$.

Here
$$m_Q\beta = \frac{1}{\mu(Q)} \int_Q \beta\, d\mu\,.$$

We shall denote by $\|\beta\|_{\mathrm{BMO}}$ the smallest constant $C \geq 0$ such that (3.19) holds for all $Q \in \Delta$. As usual, BMO is a Banach space of functions defined modulo an additive constant, the mean value of β on the unique cube of Δ_0, or equivalently the value of the constant function $E_0\beta$, where E_0 is as in Section 2. We are now ready to state our $T(b)$-theorem.

Theorem 3.20. *Let* $E \subset \mathbb{C}$ *be a compact set and* μ *a finite positive Borel measure such that* $E = \operatorname{supp}\mu$ *and* (3.1) *holds. Let* b *be a bounded* *accretive function on E, as in* (2.6). *Let $(\Delta_k)_{k\geq 0}$ be collections of "dyadic cubes", with the properties* (3.2)-(3.12). *Finally let $T : b\,\mathcal{E} \times b\,\mathcal{E} \longrightarrow \mathbb{C}$ be an operator that satisfies* (3.13)-(3.15) *and* (3.17), *and suppose that there are functions β and $\tilde{\beta}$ in* BMO *such that*

$$\langle Tb, b\,g\rangle = \int \beta\, b\, g\, d\mu \tag{3.21}$$

and

$$\langle Tb\,g, b\rangle = \int \tilde{\beta}\, b\, g\, d\mu\,, \tag{3.22}$$

for all $g \in \mathcal{E}$. Then T extends to a bounded operator on $L^2(d\mu)$.

A few comments on this statement will be useful.

The conditions (3.21) and (3.22) are just a dual way to say that $Tb = \beta$ and $T^t b = \tilde{\beta}$, where T^t denotes the transposed operator. Recall that $\mathcal{E}$ is dense in $L^2(d\mu)$, as in (2.17). Since $C^{-1} \leq |b| \leq C$ by (2.6), $b\,\mathcal{E}$ also is dense in $L^2(d\mu)$ and the $\langle Tb, b\,g\rangle$, $g \in \mathcal{E}$, determine Tb.

REMARK 3.23. Because $b\,\mathcal{E}$ is dense in $L^2(d\mu)$, it is easy to see that T extends to a bounded operator on $L^2(d\mu)$ (or, if we see T as a bilinear operator, that T extends to a bounded bilinear operator from $L^2(d\mu) \times L^2(d\mu)$ to $\mathbb{C}$) if and only if there is a constant $C \geq 0$ such that

$$|\langle Tbf, b\,g\rangle| \leq C\,\|f\|_2\,\|g\|_2\,, \qquad \text{for all } f, g \in \mathcal{E}\,. \tag{3.24}$$

REMARK 3.25. Although this was not said explicitely in the statement, our proof will give a bound on the norm of T (or equivalently on the best constant C in (3.24)) that depends only C_0, C_1, C_2, C_3, A, $\|\beta\|_{\mathrm{BMO}}$ and $\|\tilde{\beta}\|_{\mathrm{BMO}}$.

Here we work with a compact set E, and this has the small advantage that we did not need to define Tb and $T^t b$ as "distributions modulo additive constants". Our hypothesis (3.17), applied to the only cube of Δ_0, gives a control on the integrals of Tb and $T^t b$ against b (*i.e.*, the constant piece $F_0(Tb) = F_0(T^t b)$, with the notations of Section 2). Thus it is not surprising that we only need to control $\|\beta\|_{\mathrm{BMO}}$ and $\|\tilde{\beta}\|_{\mathrm{BMO}}$ once we have (3.17).

Remark 3.26. As far as the main goal of this paper is concerned, the reader should not pay too much attention to the (slightly complicated) general definition of singular integral operators given here: Theorem 3.20 will be applied to operators T_ε that can be defined brutally by integration against the very integrable kernels

$$\frac{1}{x-y}\,\varphi\Big(\frac{|x-y|}{\varepsilon}\Big)\,,$$

where φ is a smooth cut-off function that vanishes in a neighborhood of 0. Also see the beginning of the discussion about principal value operators associated to antisymmetric standard kernels in the next section.

Remark 3.27. In our statement we have assumed that $E = \operatorname{supp}\mu$ because this was natural and simple. However, Theorem 3.20 is still true if we only assume instead that E is a bounded Borel set contained in the support of μ and such that $\mu(\mathbb{C}-E) = 0$. This will not make any difference in the proof below, and it may make the hypotheses a little bit easier to check, because we could be given partitions of E (rather than $\operatorname{supp}\mu$) into dyadic cubes. This is not a very serious issue anyway, because it is fairly easy to see that such a partition can be extended to a partition of $\operatorname{supp}\mu$ with the same properties. See the argument a little below (3.57) in [DM].

Remark 3.28. Our condition (3.17) is clearly necessary for T to have a bounded extension to $L^2(d\mu)$, and we wish to claim without proof (essentially, because we shall not need this fact) that our main conditions $Tb \in \mathrm{BMO}$ and $T^t b \in \mathrm{BMO}$ are necessary as well. The verification should amount to checking that

$$\int_Q |T((1-\mathbf{1}_Q)b)(x) - T((1-\mathbf{1}_Q)\,b)(x(Q))|^2\,d\mu(x) \le C\,\mu(Q)\,, \tag{3.29}$$

for all $Q \in \Delta$, and this would follow from

$$\int_Q \Big(\int_{E\setminus Q} \Big|\frac{1}{x-y} - \frac{1}{x(Q)-y}\Big|\,d\mu(y)\Big)^2\,d\mu(x) \le C\,\mu(Q)\,. \tag{3.30}$$

We shall prove similar (only a little more complicated) estimates later; see in particular the proof of (13.65) to reduce to

$$\int_Q \Big(\int_{2Q\setminus Q} \frac{d\mu(y)}{|x-y|}\Big)^2\,d\mu(x)\,,$$

and then the proof of (13.75), where we can define $h(x)$ as in (13.70) and (13.55) but with $r(x) = 0$, because μ satisfies (3.1).

REMARK 3.31. Our statement of Theorem 3.20 is clearly not optimal. We can replace our accretivity condition (2.6) with the slightly weaker requirement that b be bounded and satisfy (2.7). Our choice of $\tau = 9/10$ in (3.8) is not optimal; probably a weaker definition of standard kernels would work as well and E should not need to be bounded. Our hypothesis that E and K live in the plane (as opposed to some $\mathbb{R}^n$) is not needed (see Remark 9.112); quite possibly E and K do not need to be one-dimensional either. However the modifications needed to take care of all these details could be quite painful (if they exist), and our proof is already complicated enough without them. Since we only have one clear application in mind so far, it is probably wiser not to think too much about extensions now.

A more unpleasant aspect of Theorem 3.20 is that we have to use cubes with the properties (3.2)-(3.12). This will even create some trouble in the present paper, because the cubes that are given to us will come from a different measure and will not be directly adapted to the measure on which we want to apply Theorem 3.20.

It seems that F. Nazarov, S. Treil, and A. Volberg were able to prove a $T(b)$-theorem for measures that satisfy (3.1) without using our machinery with dyadic cubes [NTV]. It would be interesting to see whether their proof can be adapted to give Theorem 1.1.

In the next section we want to say a few words about the "principal value operator" associated to a given antisymmetric standard kernel. After this we'll discuss shortly how to verify that Tb and $T^t b$ lie in BMO with the help of the Haar system of Section 2.

4. Antisymmetric standard kernels.

Let K be a standard kernel, and suppose that

$$K(x,y) = -K(y,x)\,, \qquad \text{when } x \neq y\,. \tag{4.1}$$

We want to define a singular integral operator $T : b\,\mathcal{E} \times b\,\mathcal{E} \longrightarrow \mathbb{C}$ such that (3.15) and (3.17) hold.

We start with the easy case when

$$\int_{E\setminus\{x\}} |K(x,y)|\, d\mu(y) \leq C\,, \tag{4.2}$$

for all $x \in E$ and some $C \geq 0$. Then we can set

$$
Tf(x) = \int K(x,y)\, f(y)\, d\mu(y)\,, \tag{4.3}
$$

for all $f \in b\,\mathcal{E}$ and $x \in E$; Tf is a bounded function and

$$
\begin{aligned}
\langle Tf, g\rangle &= \int Tf(x)\, g(x)\, d\mu(x) \\
&= \iint K(x,y)\, f(y)\, g(x)\, d\mu(y)\, d\mu(x)\,,
\end{aligned} \tag{4.4}
$$

with a nicely convergent integral, for all $g \in b\,\mathcal{E}$. By Fubini and antisymmetry,

$$
\langle Tb\,\mathbf{1}_Q, b\,\mathbf{1}_Q\rangle = 0\,, \qquad \text{for all } Q \in \Delta \tag{4.5}
$$

in this case. If $f, g \in \mathcal{E}$, then for k large enough we can write

$$
f = \sum_{Q\in\Delta_k} \lambda_Q\, \mathbf{1}_Q \qquad \text{and} \qquad g = \sum_{R\in\Delta_k} \eta_R\, \mathbf{1}_R\,. \tag{4.6}
$$

Then (4.4) and (4.5) imply that

$$
\begin{aligned}
&\langle Tbf, b\,g\rangle \\
&\quad = \sum_{Q,R\in\Delta_k}\ \sum_{Q\neq R} \lambda_Q\, \eta_R \int_R \int_Q K(x,y)\, b(y)\, b(x)\, d\mu(y)\, d\mu(x)\,,
\end{aligned} \tag{4.7}
$$

when (4.6) holds.

When we no longer assume (4.2), the simplest is probably to get T as a limit of operators T_ε, as follows. Select a nice C^1 cut-off function φ such that $\varphi(t) = 0$ for $0 \leq t \leq 1$ and $\varphi(t) = 1$ for $t \geq 2$, and then set

$$
K_\varepsilon(x,y) = \varphi\Big(\frac{|x-y|}{\varepsilon}\Big) K(x,y)\,,
$$

for all (small) $\varepsilon > 0$. The kernels K_ε are still uniformly standard and antisymmetric, and they satisfy (4.2), so we can define singular integral operators T_ε as in the discussion above.

Lemma 4.8. *For every antisymmetric standard kernel K we can define a singular integral operator $T : b\,\mathcal{E} \times b\,\mathcal{E} \longrightarrow \mathbb{C}$ by*

$$\langle Tbf, b\,g \rangle = \lim_{\varepsilon \to 0} \langle T_\varepsilon bf, b\,g \rangle \,, \qquad \text{for all } f, g \in \mathcal{E}\,. \tag{4.9}$$

Moreover T satisfies (3.15) *and* (4.5)*, and* (4.7) *holds whenever f, g are as in* (4.6)*.*

We shall refer to T as the principal value operator associated to (the antisymmetric standard kernel) K. Note that we shall only use (3.13), and not (3.14).

Our proof of Lemma 4.8 will rely on (3.16), which will only be proved later (see (8.7) and the definition (7.9)) but is fairly simple.

Because of (3.16), the integrals in (4.7) converge, and we could have taken (4.7) as our definition of T. It is slightly easier to proceed as we do because we won't have to check that different expressions for f and g in (4.6) give the same result in (4.7). Let us return to the lemma. The existence of a limit in (4.9) follows from the dominated convergence theorem, applied to the kernels K_ε (that converge pointwise to K) in the formula (4.7) (which is satisfied by all the T_ε's as soon as (4.6) holds). We also get the formula (4.7) for T at the same time. From (4.9) and the linearity of each T_ε we get that T is linear. The formula (4.5) for T follows directly from (4.9) and the fact that each T_ε satisfies it. Finally (3.15) is an easy consequence of (4.7) (and the existence of decompositions as in (4.6)), or can be obtained directly from its analogue for the T_ε's and the dominated convergence theorem.

This completes our discussion of the principal value operator associated to antisymmetric standard kernels. Note that they satisfy the weak boundedness property (3.17) automatically, because they satisfy the stronger (4.5).

5. $Tb \in \mathrm{BMO}$ and the Haar system.

In this section we want to see how to use the modified Haar system of Section 2 to check our conditions that $Tb \in \mathrm{BMO}$ and $T^t b \in \mathrm{BMO}$.

First observe that our cubes Q, $Q \in \Delta$, satisfy the conditions (2.1)-(2.5) required for the construction of Section 2: (2.1) and (2.2) are the same as (3.2) and (3.3), (2.3) follows from (3.5) and the fact that $B(Q)$ is centered on $\operatorname{supp}\mu$, (2.4) is a consequence of (3.4) and (3.5) (although with a slightly larger constant), and finally (2.5) (again with a larger constant) follows from the fact that for each r,

$$\text{the number of cubes of } \Delta_k \text{ that meet a ball of radius } r \text{ is always } \leq 1 + CA^{2k}\, r^2\,. \tag{5.1}$$

This last is an easy consequence of (3.4), (3.5), and the fact that the balls $B(Q)$ are centered on E, because this implies that $|x(Q)-x(Q')| \geq A^{-k}$ when $Q, Q' \in \Delta_k$, with $Q \neq Q'$.

So we can apply the construction of Section 2 to our cubes $Q \in \Delta$ and our function b. We do this and get a modified Haar system $\{h_Q^\varepsilon\}_{Q,\varepsilon}$. It will be simpler to call

$$H = \{(Q,\varepsilon) : \; Q \in \Delta \text{ and } \varepsilon \in D(Q)\} \tag{5.2}$$

the set of indices that show up.

For each function $\beta \in L^2(d\mu)$, set

$$\beta_Q^\varepsilon = \langle \beta, h_Q^\varepsilon \rangle_b = \int \beta\, h_Q^\varepsilon\, b\, d\mu\,, \tag{5.3}$$

for all $(Q,\varepsilon) \in H$. These coefficients do not determine β entirely, but only modulo the piece $F_0\beta$ (see (2.64) and (2.65)). Here, because Δ_0 has only one cube, $F_0\beta$ is simply the constant

$$F_0\beta = \Big(\int_E b\, d\mu\Big)^{-1} \int_E \beta\, b\, d\mu\,. \tag{5.4}$$

(See the definition (2.10).) Nonetheless, the coefficients β_Q^ε are enough to determine whether $\beta \in \mathrm{BMO}$.

Lemma 5.5. *Let $\beta \in L^2(d\mu)$ be given, and define the β_Q^ε, $(Q,\varepsilon) \in H$, by* (5.3)*. Then $\beta \in \mathrm{BMO}$ if and only if the β_Q^ε satisfy the following quadratic Carleson measure condition: there is a constant $C \geq 0$ such that*

$$\sum_{Q \subset R} \sum_{\varepsilon \in D(Q)} |\beta_Q^\varepsilon|^2 \leq C^2 \mu(R)\,, \qquad \text{for all } R \in \Delta\,. \tag{5.6}$$

Moreover the best constant in (5.6) *is equivalent to* $\|\beta\|_{\mathrm{BMO}}$.

To prove the lemma, let $\beta \in L^2(d\mu)$ and $R \in \Delta$ be given. Set

$$m_R\beta = \frac{1}{\mu(R)} \int_R \beta \, d\mu \,,$$

as in Definition 3.18, and then apply Proposition 2.63 to $f = (\beta - m_R\beta)\, \mathbf{1}_R$. For all cubes $Q \subset R$ and all $\varepsilon \in D(Q)$,

$$\langle f, h_Q^\varepsilon \rangle_b = \int_Q f \, h_Q^\varepsilon \, b \, d\mu = \beta_Q^\varepsilon \tag{5.7}$$

(the extra term $\int_Q m_R \, \beta \, h_Q^\varepsilon \, b \, d\mu$ disappears because of (2.74)). Then

$$\sum_{Q \subset R} \sum_{\varepsilon \in D(Q)} |\beta_Q^\varepsilon|^2 \leq C \, \|f\|_2^2 \leq C \int_R |\beta - m_R\beta|^2 \, d\mu \,, \tag{5.8}$$

by the second half of (2.66).

Denote by λ the constant value on R of $F_0\beta + \sum_{Q,\varepsilon} \langle \beta, h_Q^\varepsilon \rangle_b \, h_Q^\varepsilon$, where the sum is restricted to the pairs (Q, ε) such that Q contains R and is of a generation $k(Q) < k(R)$. It would be easy to check that λ is the value of $F_{k(R)}\beta$ on R, but we don't need this fact. Because of (2.64),

$$(\beta - \lambda) \, \mathbf{1}_R = \sum_{Q \subset R} \sum_{\varepsilon \in D(Q)} \langle \beta, h_Q^\varepsilon \rangle_b \, h_Q^\varepsilon \,. \tag{5.9}$$

Apply the uniqueness result in Proposition 2.63, and then (2.66), to the function $(\beta - \lambda)\mathbf{1}_R$. This gives

$$\int_R |\beta - \lambda|^2 \, d\mu \leq C \sum_{Q \subset R} \sum_{\varepsilon \in D(Q)} |\beta_Q^\varepsilon|^2 \tag{5.10}$$

(recall (5.3)). Finally observe that

$$\int_R |\beta - m_R\beta|^2 \, d\mu \leq \int_R |\beta - \lambda|^2 \, d\mu \,. \tag{5.11}$$

This would be true for any constant λ: it follows from the pythagorean theorem, or the fact that $m_R\beta$ is the orthogonal projection of β on the vector space of constant functions in $L^2(R, d\mu)$.

When we compare (5.8), (5.11), and (5.10), we find that the quantities in (5.8) are equivalent. Lemma 5.5 follows by taking the supremum over all cubes R.

Lemma 5.12. *Let $T : b\,\mathcal{E} \times b\,\mathcal{E} \longrightarrow \mathbb{C}$ be a bilinear operator. Set*

$$\beta_Q^\varepsilon = \langle Tb, b\,h_Q^\varepsilon \rangle \tag{5.13}$$

and

$$\tilde{\beta}_Q^\varepsilon = \langle Tb\,h_Q^\varepsilon, b \rangle\,, \tag{5.14}$$

for all $(Q,\varepsilon) \in H$. Then there are functions β and $\tilde{\beta} \in \mathrm{BMO}$ such that (3.21) and (3.22) hold if and only if the sequences $\{\beta_Q^\varepsilon\}$ and $\{\tilde{\beta}_Q^\varepsilon\}$ both satisfy the Carleson condition (5.6).

Indeed if $\beta \in \mathrm{BMO}$ is such that (3.21) holds, then (3.21) with $g = h_Q^\varepsilon$ says that the numbers β_Q^ε in (5.13) are the same as the ones in (5.3). Lemma 5.5 then gives the desired control on the β_Q^ε. Conversely, suppose that the β_Q^ε in (5.13) satisfy (5.6). For each integer $k \geq 0$, set

$$\beta_k = \sum_{Q \in \Delta_k} \sum_{\varepsilon \in D(Q)} \beta_Q^\varepsilon\, h_Q^\varepsilon\,. \tag{5.15}$$

Note that

$$\Big\| \sum_{k=m}^{n} \beta_k \Big\|_2^2 \leq C \sum_{k=m}^{n} \sum_{Q \in \Delta_k} \sum_{\varepsilon \in D(Q)} |\beta_Q^\varepsilon|^2\,, \tag{5.16}$$

by Proposition 2.63. Since the right-hand side of (5.16) tends to 0 when m and n tend to ∞ (because $\sum_H |\beta_Q^\varepsilon|^2 < +\infty$, by (5.6) applied to the only cube of Δ_0), the series $\sum_{k=0}^{\infty} \beta_k$ converges in $L^2(d\mu)$. Denote its limit by β^*. By the uniqueness part of Proposition 2.63,

$$\langle \beta^*, h_Q^\varepsilon \rangle_b = \beta_Q^\varepsilon\,, \qquad \text{for all } (Q,\varepsilon) \in H\,, \tag{5.17}$$

and $\beta^* \in \mathrm{BMO}$ by (5.6) and Lemma 5.5.

Denote by W the subspace of $\mathcal{E}$ spanned by the h_Q^ε, $(Q,\varepsilon) \in H$. By (5.13) and (5.17),

$$\langle Tb, b\,g \rangle = \langle \beta^*, g \rangle_b\,, \qquad \text{for all } g \in W\,. \tag{5.18}$$

From Proposition 2.63 and the description of F_0 in (5.4) we see that W is a subspace of codimension 1 in $\mathcal{E}$ and the one-dimensional space of constant functions is a complementary space for W in $\mathcal{E}$. Thus, even though (5.18) does not imply that β^* satisfies (3.21), this will be easy to fix. Set

$$\beta = \beta^* + \Big(\int_E b \, d\mu \Big)^{-1} (\langle Tb, b\rangle - \langle \beta^*, b\rangle) \tag{5.19}$$

(note that $\int_E b \, d\mu \neq 0$ by accretivity.) Obviously, adding a constant to β^* does not modify $\langle \beta^*, g\rangle_b$ for $g \in W$, because of (2.74). Therefore (5.18) yields

$$\langle Tb, b\, g\rangle = \langle \beta, g\rangle_b = \int \beta \, b \, g \, d\mu \,, \tag{5.20}$$

for all $g \in W$. Since we also have that

$$\int \beta \, b \, d\mu = \langle \beta, b\rangle = \langle \beta^*, b\rangle + (\langle Tb, b\rangle - \langle \beta^*, b\rangle) = \langle Tb, b\rangle \,, \tag{5.21}$$

by (5.19), we see that (5.20) holds for all $g \in \mathcal{E}$, *i.e.*, (3.21) holds. Note that β lies in BMO because β^* does. This proves the converse.

The story for the transposed operator, *i.e.*, with (3.22) and the numbers $\tilde{\beta}_Q^{\varepsilon}$ is the same. This completes our proof of Lemma 5.12.

The proof of Theorem 3.20 will (continue to) keep us busy for the next few sections. The argument will follow roughly the same lines as in the Coifman-Semmes or Auscher-Tchamitchian proofs of $T(b)$. See [CJS], [AT], [Da] or [My].

6. Paraproducts.

In this section we want to construct bounded operators P such that Pb and $P^t b$ are prescribed functions in BMO. We shall call them paraproducts because they look like other operators that actually looked like Bony paraproducts.

In the standard situation for the regular $T(1)$-theorem, say, these operators are bounded singular integral operators, and we can use them to substract them from the operator T of Theorem 3.20; this allows one to reduce to the situation where $T1$ and $T^t 1$ are equal to 0 (instead of just lying in BMO.) Here this approach will not work brutally, because our paraproducts will have a fairly bad kernel. We shall have to use them in the following slightly more subtle way. The boundedness of these operators, which will not be so trivial because it will use Carleson's theorem, will be used to show that their matrices in the modified Haar system of Section 2 define bounded operators on $\ell^2(H)$. These bounded matrices will then be substracted from the matrices of operators T from Theorem 3.20, and we shall be able to prove that the resulting differences of matrices are small enough to be handled by just looking at the size of their coefficients.

In this section we construct the paraproducts, prove their boundedness, and compute their matrices. For the results of this section, none of the small boundary conditions on our cubes will be used: the weaker structure of Section 2 is still enough.

For each sequence $\{\beta_Q^\varepsilon\}_{(Q,\varepsilon)\in H}$ of complex numbers that satisfies the Carleson condition (5.6) we define an operator P on $\mathcal{E}$ by

$$Pf = \sum_{(Q,\varepsilon)\in H} \beta_Q^\varepsilon \, \langle f, h_Q^\varepsilon\rangle_b \, \theta_Q \, , \tag{6.1}$$

where

$$\theta_Q = \Big(\int_Q b \, d\mu \Big)^{-1} \mathbf{1}_Q \, . \tag{6.2}$$

The sum in (6.1) has only finitely many terms, because only finitely many coefficients $\langle f, h_Q^\varepsilon\rangle_b$ can be different from 0 when $f \in \mathcal{E}$. Thus (6.1) makes sense, and even $Pf \in \mathcal{E}$.

We shall also be interested in the operator $\tilde{P}$ that we get from P by "b-transposition", as follows: $\tilde{P}$ is the linear operator from $\mathcal{E}$ to the dual of $b\,\mathcal{E}$ defined by

$$\langle \tilde{P} g, bf\rangle = \langle Pf, b\, g\rangle \, , \tag{6.3}$$

or equivalently

$$\langle \tilde{P} g, f\rangle_b = \langle Pf, g\rangle_b \, , \qquad \text{for all } f, g \in \mathcal{E} \, . \tag{6.4}$$

Lemma 6.5. *The operator $\tilde{P}$ is also given by*

$$\tilde{P} g = \sum_{(Q,\varepsilon)\in H} \beta_Q^\varepsilon \Big(\int_Q b \, d\mu \Big)^{-1} \Big(\int_Q g \, b \, d\mu \Big) h_Q^\varepsilon \, , \tag{6.6}$$

for all $g \in \mathcal{E}$ and where the series in (6.6) *converges in $L^2(d\mu)$.*

Let $g \in \mathcal{E}$ be given, and set

$$c_Q^\varepsilon = \beta_Q^\varepsilon \Big(\int_Q b \, d\mu \Big)^{-1} \Big(\int_Q g \, b \, d\mu \Big) . \tag{6.7}$$

By the paraaccretivity conditions (2.6),

$$\Big| \int_Q b \, d\mu \Big|^{-1} \le C \, \mu(Q)^{-1}$$

and, since g is obviously bounded, $|c_Q^\varepsilon| \le C \, |\beta_Q^\varepsilon|$ for all Q and ε. The constant C may depend wildly on g, but we don't care. In particular, $\sum_{Q,\varepsilon} |c_Q^\varepsilon|^2 < +\infty$ by (5.6), and the same argument as in Lemma 5.12 (see around (5.15)) shows that the series in (6.6) converges in $L^2(d\mu)$. Call $h \in L^2(d\mu)$ the limit; we want to check that h can be taken as $\tilde{P} g$, *i.e.*, that

$$\langle h, f \rangle_b = \langle P f, g \rangle_b \, , \qquad \text{for all } f \in \mathcal{E} \, . \tag{6.8}$$

When f is a constant, $\langle h, f \rangle_b = 0$ because h is a limit in L^2 of finite linear combinations of functions h_Q^ε and $\langle h_Q^\varepsilon, f \rangle_b = 0$ by (2.74). Since $Pf = 0$ because all the $\langle f, h_Q^\varepsilon \rangle_b$ are equal to 0, we get (6.8) for constant functions. Since all functions in $\mathcal{E}$ are linear combinations of some constant and functions h_Q^ε (by Proposition 2.63 and (5.4)), it is enough to prove (6.8) when $f = h_Q^\varepsilon$. But

$$\langle P h_Q^\varepsilon, g \rangle_b = \beta_Q^\varepsilon \, \langle \theta_Q, g \rangle_b = \beta_Q^\varepsilon \Big(\int_Q b \, d\mu \Big)^{-1} \Big(\int_Q g \, b \, d\mu \Big) = \langle h, h_Q^\varepsilon \rangle_b \, ,$$

by (6.1), (2.75), (6.2), the definition of h as the right-hand side of (6.6), and (2.75) again. This proves Lemma 6.5.

Proposition 6.9. *The operators P and $\tilde{P}$ both extend to bounded operators on $L^2(d\mu)$, with norms less than C' times the constant C in the Carleson condition* (5.6)*.*

First observe that P extends to a bounded operator on $L^2(d\mu)$ if and only if there is a constant $C \ge 0$ such that

$$|\langle P f, b \, g \rangle| \le C \, \|f\|_2 \, \|g\|_2 \, , \qquad \text{for all } f, g \in \mathcal{E} \, . \tag{6.10}$$

This follows easily from the density of $\mathcal{E}$ in $L^2(d\mu)$ and the fact that $C^{-1} \leq |b| \leq C$ by (2.6). This condition is also equivalent to the existence of an extension of $\tilde{P}$ to a bounded operator on $L^2(d\mu)$, because of (6.3). Thus it will be enough to prove the boundedness of (an extension of) $\tilde{P}$ to $L^2(d\mu)$.

From Lemma 6.5, the uniqueness result in Proposition 2.63, and (2.66) we deduce that for every $g \in \mathcal{E}$,

$$\|\tilde{P}g\|_2^2 \leq C \sum_{(Q,\varepsilon)\in H} |c_Q^\varepsilon|^2 , \tag{6.11}$$

where c_Q^ε is as in (6.7). We want to use Lemma 2.23 (Carleson's theorem) to estimate the right-hand side of (6.11). Set

$$f_k = \sum_{Q\in\Delta_k} \mu(Q)^{-1}\Big(\int_Q |g|\,d\mu\Big)\mathbf{1}_Q\ , \tag{6.12}$$

for all $k \geq 0$. Obviously the sequence $\{f_k\}$ satisfies (2.24) with f replaced with g. Also define measures ν_k on E by

$$d\nu_k = \sum_{Q\in\Delta_k} \Big(\sum_{\varepsilon\in D(Q)} |\beta_Q^\varepsilon|^2\Big)\mu(Q)^{-1}\,\mathbf{1}_Q\,d\mu\,. \tag{6.13}$$

Let us check that $\{\nu_k\}_{k\geq 0}$ defines a Carleson measure on $E\times\mathbb{N}$, as in Definition 2.21. For each cube $R \in \bigcup_k \Delta_k$,

$$\sum_{k\geq k(R)} \nu_k(R) = \sum_{Q\subset R}\Big(\sum_{\varepsilon\in D(Q)} |\beta_Q^\varepsilon|^2\Big) \leq C\,\mu(R)\,, \tag{6.14}$$

by (5.6). In other words, (2.22) holds and $\nu = \{\nu_k\}$ is a Carleson measure. Lemma 2.23 now tells us that

$$\sum_k \int |f_k|^2\,d\nu_k \leq C\,\|g\|_2^2\,.$$

But

$$\begin{aligned}\sum_k \int |f_k|^2\,d\nu_k &= \sum_k \sum_{Q\in\Delta_k}\Big(\sum_{\varepsilon\in D(Q)} |\beta_Q^\varepsilon|^2\Big)\mu(Q)^{-2}\Big(\int_Q |g|\,d\mu\Big)^2 \\ &\geq C^{-1} \sum_{(Q,\varepsilon)\in H} |c_Q^\varepsilon|^2\,, \qquad (6.15)\end{aligned}$$

by definitions (6.12) and (6.13), the accretivity condition (2.6), and (6.7). Because of (6.11), this gives that $\|\tilde{P}g\|_2^2 \le C\,\|g\|_2^2$, proves the boundedness of $\tilde{P}$, and completes our proof of Proposition 6.9.

Next we want to talk about matrices.

Definition 6.16. *Let $T : \mathcal{E} \times b\,\mathcal{E} \longrightarrow \mathbb{C}$ be a bilinear operator. The matrix of T (relative to the system $\{h_Q^\varepsilon\}$) is the matrix $\mathcal{M}$ with coefficients*

$$(6.17)\quad M(Q,\varepsilon,R,\varepsilon') = \langle T h_Q^\varepsilon, b\, h_R^{\varepsilon'}\rangle\,, \qquad (Q,\varepsilon) \in H \text{ and } (R,\varepsilon') \in H\,.$$

The slight asymmetry of this definition cannot be a serious problem because $C^{-1} \le |b| \le C$ by (2.6); our definition is just more convenient for our paraproducts P and $\tilde{P}$. Note in particular that if $\tilde{T}$ denotes the b-transpose of T as in (6.3), *i.e.*, if $\tilde{T} : \mathcal{E} \times b\,\mathcal{E} \longrightarrow \mathbb{C}$ is defined by

$$(6.18)\qquad \langle \tilde{T} g, b\, f\rangle = \langle T f, b\, g\rangle\,, \qquad \text{for } f, g \in \mathcal{E}\,,$$

then the matrix of $\tilde{T}$ is just the transpose of $\mathcal{M}$.

We do not claim that $\mathcal{M}$ determines T, and indeed it does not say anything about $\langle T1, bf\rangle$ or $\langle Tf, b\rangle$ when $f \in \mathcal{E}$, but it will still be useful to determine when T has a bounded extension to $L^2(d\mu)$.

Lemma 6.19. *Let $T : \mathcal{E} \times b\,\mathcal{E} \longrightarrow \mathbb{C}$ be a bilinear operator and $\mathcal{M}$ denote its matrix relative to the system $\{h_Q^\varepsilon\}$. Then T admits an extension to a bounded operator on $L^2(d\mu)$ if and only if*

$$(6.20)\qquad T\,1 \in L^2(d\mu)\,,$$

$$(6.21)\qquad \tilde{T}\,1 \in L^2(d\mu)\,,$$

and

$$(6.22)\qquad \mathcal{M} \text{ defines a bounded operator on } \ell^2(H)\,.$$

Let us explain these conditions; (6.20) means that there is a function $h \in L^2(d\mu)$ such that

$$(6.23)\qquad \langle T\,1, bf\rangle = \langle h, bf\rangle = \int h\, bf\, d\mu\,, \qquad \text{for all } f \in \mathcal{E}\,.$$

Similarly, (6.21) means that there is an $\tilde{h} \in L^2(d\mu)$ such that

$$(6.24) \qquad \langle \tilde{T}\,1, bf\rangle = \langle Tf, b\rangle = \int \tilde{h}\, bf\, d\mu\,, \qquad \text{for all } f \in \mathcal{E}\,.$$

As for (6.22), let W^* denote the set of finitely supported sequences $x = \{x_Q^\varepsilon\}_{(Q,\varepsilon)\in H}$ and define a bilinear operator S from $W^* \times W^*$ to $\mathbb{C}$ by

$$(6.25) \qquad \langle Sx, y\rangle = \sum_{(Q,\varepsilon)\in H}\ \sum_{(R,\varepsilon')\in H} M(Q,\varepsilon,R,\varepsilon')\, x_Q^\varepsilon\, y_R^{\varepsilon'}\,,$$

for all $x, y \in W^*$. Then (6.22) means that there is a constant $C \geq 0$ such that

$$(6.26) \qquad |\langle Sx, y\rangle| \leq C\, \|x\|\, \|y\|\,, \qquad \text{for } x, y \in W^*\,,$$

where

$$\|x\| = \Big(\sum_{(Q,\varepsilon)\in H} |x_Q^\varepsilon|^2\Big)^{1/2}$$

and similarly for y.

The obvious mapping from W^* to $W = \operatorname{span}\{h_Q^\varepsilon : (Q,\varepsilon) \in H\}$ defined by $\varphi(x) = \sum x_Q^\varepsilon\, h_Q^\varepsilon$ is a bijection and

$$C^{-1}\, \|x\| \leq \|\varphi(x)\|_2 \leq C\, \|x\|\,,$$

by Proposition 2.63. From (6.17) and (6.25) we deduce that

$$(6.27) \qquad \langle Sx, y\rangle = \langle T\varphi(x), b\,\varphi(y)\rangle\,, \qquad \text{for all } x, y \in W^*\,.$$

Hence (6.22) holds if and only if there is a constant $C \geq 0$ such that

$$(6.28) \qquad |\langle Tf, b\,g\rangle| \leq C\, \|f\|_2\, \|g\|_2\,, \qquad \text{for all } f, g \in W\,.$$

Because of this, (6.22) is clearly necessary if we want T to have a bounded extension; (6.20) and (6.21) are necessary too, because $1 \in L^2(d\mu)$ and T has a bounded extension if and only if $\tilde{T}$ does. The converse is not much harder. Suppose that (6.20), (6.21), and (6.22) hold. By Proposition 2.63, every $f \in \mathcal{E}$ has a decomposition $f = F_0 f + \pi f$, where $F_0 f$ is a constant because Δ_0 has only one cube, $\pi f \in W$, and

$$\|F_0 f\|_2 + \|\pi f\|_2 \leq C\, \|f\|_2\,.$$

Then, for $f, g \in \mathcal{E}$,

$$\begin{aligned}
|\langle Tf, b\,g\rangle| &\le |\langle T(F_0 f), b\,g\rangle| + |\langle T(\pi f), b\,g\rangle| \\
&\le C\,|F_0 f|\,\|T1\|_2\,\|g\|_2 + |\langle T(\pi f), b\,F_0\,g\rangle| + |\langle T(\pi f), b\,\pi\,g\rangle| \\
&\le C\,\|F_0 f\|_2\,\|g\|_2 + C\,|F_0\,g|\,\|\tilde{T}1\|_2\,\|\pi f\|_2 + C\,\|\pi f\|_2\,\|\pi\,g\|_2 \\
&\le C\,\|F_0 f\|_2\,\|g\|_2 + C\,\|F_0\,g\|_2\,\|f\|_2 + C\,\|f\|_2\,\|g\|_2 \\
&\le C\,\|f\|_2\,\|g\|_2\,,
\end{aligned} \tag{6.29}$$

by (6.20), (6.21), and (6.28). Thus T has a bounded extension to L^2, as desired.

This completes the proof of Lemma 6.19.

Finally we want to compute the matrix of P.

Lemma 6.30. *Denote by $\mathcal{P} = ((P(Q,\varepsilon,R,\varepsilon')))$ the matrix of the paraproduct P defined by* (6.1) (*using the sequence $\{\beta_Q^\varepsilon\}$.*) *Then*

$$P(Q,\varepsilon,R,\varepsilon') = 0\,, \qquad \text{when } Q \cap R = \varnothing \text{ or } R \subset Q\,, \tag{6.31}$$

and

$$\begin{aligned}
&P(Q,\varepsilon,R,\varepsilon') \text{ is } \beta_Q^\varepsilon \text{ times the constant value} \\
&\text{of } h_R^{\varepsilon'} \text{ on } Q \text{ when } Q \subset R,\ Q \neq R\,.
\end{aligned} \tag{6.32}$$

Recall from (6.17) and (6.1) that

$$P(Q,\varepsilon,R,\varepsilon') = \langle P h_Q^\varepsilon, b\,h_R^{\varepsilon'}\rangle = \beta_Q^\varepsilon\,\langle \theta_Q, b\,h_R^{\varepsilon'}\rangle\,, \tag{6.33}$$

by (2.75). This is obviously 0 when $Q \cap R = \varnothing$, and also when $R \subset Q$ because θ_Q is constant on Q, and by (2.74). Thus we are left with the case when $Q \subset R$, $Q \neq R$. In this case $h_R^{\varepsilon'}$ is constant on Q and

$$\langle \theta_Q, b\rangle = \int \theta_Q\,b\,d\mu = 1\,,$$

by (6.2). The lemma follows.

7. Reduction to the study of a matrix $\mathcal{N}$.

In this section we take an operator T that satisfies the hypotheses of Theorem 3.20, compute its matrix, substract from it the matrices of appropriate paraproducts, and show that the remaining matrix defines a bounded operator if some other matrix $\mathcal{N}$ defines a bounded operator on ℓ^2. The matrix $\mathcal{N}$ will be a matrix with nonnegative coefficients, that no longer depends on the operator T but only on the size of certain integrals on E. The boundedness of (the operator defined by) $\mathcal{N}$ will be proved in later sections, with the help of Schur's lemma.

We shall not use the small boundary properties of our cubes in this section either, except for the fact that

$$\mu(\{x \in Q : \ \operatorname{dist}(x, E \backslash Q)\}) = 0\,, \qquad \text{for all } Q \in \Delta\,, \tag{7.1}$$

which follows from (3.8).

Let T be an operator that satisfies the hypotheses of Theorem 3.20. Denote by $\mathcal{T} = ((T(Q,\varepsilon,R,\varepsilon')))$ the matrix of TM_b in the modified Haar system $\{h_Q^\varepsilon\}$, and where M_b denotes the operator of pointwise multiplication by b. Since T is defined on $b\,\mathcal{E} \times b\,\mathcal{E}$, TM_b is defined on $\mathcal{E} \times b\,\mathcal{E}$, as required in Definition 6.16, and

$$T(Q,\varepsilon,R,\varepsilon') = \langle Tb\, h_Q^\varepsilon, b\, h_R^{\varepsilon'}\rangle\,, \qquad \text{for } (Q,\varepsilon),\ (R,\varepsilon') \in H\,. \tag{7.2}$$

We already know from (3.21) and (3.22) that $(TM_b)1 = \beta$ and $(\tilde{T}M_b)1 = \tilde{\beta}$ lie in BMO, hence in $L^2(d\mu)$. (Compare (3.21) and (3.22) with (6.23) and (6.24) for TM_b.) Hence Lemma 6.19 says that it will be enough to prove that $\mathcal{T}$ defines a bounded operator on $\ell^2(H)$.

Next define sequences $\{\beta_Q^\varepsilon\}$ and $\{\tilde{\beta}_Q^\varepsilon\}$ by (5.13) and (5.14). Then Lemma 5.12 says that $\{\beta_Q^\varepsilon\}$ and $\{\tilde{\beta}_Q^\varepsilon\}$ satisfy the Carleson condition (5.6).

Denote by P the paraproduct constructed in Section 6 with the sequence $\{\beta_Q^\varepsilon\}$ and by P^* the analogous operator defined with the sequence $\{\tilde{\beta}_Q^\varepsilon\}$. These two operators have bounded extensions to $L^2(d\mu)$, by Proposition 6.9. Denote by $\mathcal{P}$ the matrix of P. By Lemma 6.19, $\mathcal{P}$ defines a bounded operator on $\ell^2(H)$, and so does its transpose $\tilde{\mathcal{P}}$. Similarly, the matrix $\mathcal{P}^*$ of P^* defines a bounded operator on $\ell^2(H)$.

Set $\mathcal{M} = \mathcal{T} - \tilde{\mathcal{P}} - \mathcal{P}^*$ and denote by $M(Q,\varepsilon,R,\varepsilon')$ its generic element. The discussion above shows that

(7.3) Theorem 3.20 will follow if we can prove that $\mathcal{M}$ defines a bounded operator on $\ell^2(H)$.

Let us compute the coefficients of $\mathcal{M}$. We use (7.2), Lemma 6.30, and then (5.14) and (5.13) to get that

$$M(Q,\varepsilon,R,\varepsilon') = \langle Tb\, h_Q^{\varepsilon}, b\, h_R^{\varepsilon'}\rangle\,, \tag{7.4}$$

when $Q\cap R=\varnothing$ or $Q=R$,

$$\begin{aligned} M(Q,\varepsilon,R,\varepsilon') &= \langle Tb\, h_Q^{\varepsilon}, b\, h_R^{\varepsilon'}\rangle - P^*(Q,\varepsilon,R,\varepsilon') \\ &= \langle Tb\, h_Q^{\varepsilon}, b\, h_R^{\varepsilon'}\rangle - \tilde\beta_Q^{\varepsilon}\,(\text{value of } h_R^{\varepsilon'} \text{ on } Q) \\ &= \langle Tb\, h_Q^{\varepsilon}, b\, h_R^{\varepsilon'}\rangle - \langle Tb\, h_Q^{\varepsilon}, b\rangle\,(\text{value of } h_R^{\varepsilon'} \text{ on } Q)\,, \end{aligned} \tag{7.5}$$

when $Q\subset R$, $Q\neq R$, and

$$\begin{aligned} M(Q,\varepsilon,R,\varepsilon') &= \langle Tb\, h_Q^{\varepsilon}, b\, h_R^{\varepsilon'}\rangle - P(R,\varepsilon',Q,\varepsilon) \\ &= \langle Tb\, h_Q^{\varepsilon}, b\, h_R^{\varepsilon'}\rangle - \beta_R^{\varepsilon'}\,(\text{value of } h_Q^{\varepsilon} \text{ on } R) \\ &= \langle Tb\, h_Q^{\varepsilon}, b\, h_R^{\varepsilon'}\rangle - \langle Tb, b\, h_R^{\varepsilon'}\rangle\,(\text{value of } h_Q^{\varepsilon} \text{ on } R)\,, \end{aligned} \tag{7.6}$$

when $R\subset Q$, $R\neq Q$.

The next stage of our computation is to express the coefficients of $\mathcal{M}$ in terms of the kernel $K(x,y)$ and then estimate them in terms of some integrals on E. The following notation will be useful. Set

$$d(Q) = A^{-k(Q)}\,, \tag{7.7}$$

for all $Q\in\Delta$, where $k(Q)$ denotes the generation of Q, and also

$$2\,Q = \{x\in E:\ \operatorname{dist}(x,Q)\le d(Q)\}\,. \tag{7.8}$$

For each Borel subset V of E such that $Q\cap V=\varnothing$, set

$$I(Q,V) = \int_V\int_Q \frac{d\mu(x)\,d\mu(y)}{|x-y|}\,, \tag{7.9}$$

and

$$J(Q,V) = \int_V \frac{d(Q)\,d\mu(x)}{|x-x(Q)|^2}\,, \tag{7.10}$$

where $x(Q)$ denotes the center of the ball $B(Q)$, as in (3.5). These are the quantities that will be used to control the coefficients of $\mathcal{M}$. We still denote by $F(Q)$, $Q \in \Delta$, the set of children of Q, *i.e.*, the set of cubes $Q^* \in \Delta_{k(Q)+1}$ such that $Q^* \subset Q$. We shall try to be systematic about calling Q^* or R^* generic children of Q or R.

Lemma 7.11. *If $Q \cap R = \varnothing$, then for all choices of $\varepsilon \in D(Q)$ and $\varepsilon' \in D(R)$,*

$$|M(Q,\varepsilon,R,\varepsilon')| \leq C\,A_1(Q,R) + C\,A_2(Q,R)\,, \tag{7.12}$$

where

$$A_1(Q,R) = \sum_{Q^* \in F(Q)} \sum_{R^* \in F(R)} \mu(Q^*)^{-1/2}\,\mu(R^*)^{-1/2}\,I(Q^*, R^* \cap 2\,Q) \tag{7.13}$$

and

$$A_2(Q,R) = \mu(Q)^{1/2} \sum_{R^* \in F(R)} \mu(R^*)^{-1/2}\,J(Q, R^* \backslash 2\,Q)\,. \tag{7.14}$$

To prove the lemma, let us first observe that $Tb\,h_Q^\varepsilon(x)$ is well-defined when $\operatorname{dist}(x,Q) > 0$ and that it is given by

$$Tb\,h_Q^\varepsilon(x) = \int_Q K(x,y)\,b(y)\,h_Q^\varepsilon(y)\,d\mu(y)\,. \tag{7.15}$$

Recall from (2.48) and Lemma 2.50 that

$$h_Q^\varepsilon = \sum_{Q^* \in F(Q)} \alpha_{\varepsilon,Q^*}\,\mu(Q^*)^{-1/2}\,\mathbf{1}_{Q^*}\,, \tag{7.16}$$

where the coefficients α_{ε,Q^*} are uniformly bounded. From this description and the first standard estimate (3.13) we get that

$$|Tb\,h_Q^\varepsilon(x)| \leq C \sum_{Q^* \in F(Q)} \mu(Q^*)^{-1/2} \int_{Q^*} \frac{d\mu(y)}{|x-y|}\,, \tag{7.17}$$

when $\operatorname{dist}(x,Q) > 0$. Notice incidentally that $\operatorname{dist}(x,Q) > 0$ for μ-almost all $x \in R$, by (7.1) (or (3.8)).

This estimate is best when $x \in 2\,Q\backslash Q$, but when $x \notin 2\,Q$ we can use the second standard estimate (3.14) and the fact that $\int_Q b\,h_Q^\varepsilon\,d\mu = 0$

(by (2.74)) to get a better one. Let $x(Q)$ denote the center of $B(Q)$, as usual. (Actually, for the computation that follows, any point of Q would work equally well.) If $x \in E \backslash 2\,Q$,

$$
\begin{aligned}
|Tb\, h_Q^\varepsilon(x)| &= \Big| \int_Q K(x,y)\, b(y)\, h_Q^\varepsilon(y)\, d\mu(y) \Big| \\
&= \Big| \int_Q (K(x,y) - K(x,x(Q)))\, b(y)\, h_Q^\varepsilon(y)\, d\mu(y) \Big| \\
&\le C \int_Q \frac{|y-x(Q)|}{|x-x(Q)|^2} |\, b(y)\, h_Q^\varepsilon(y)|\, d\mu(y) \qquad (7.18)\\
&\le C\, \frac{d(Q)}{|x-x(Q)|^2} \sum_{Q^* \in F(Q)} \mu(Q^*)^{1/2} \\
&\le C\, \mu(Q)^{1/2}\, \frac{d(Q)}{|x-x(Q)|^2}\,,
\end{aligned}
$$

by (7.15), (3.5), (3.4) and (7.16). We may now use (3.15), (7.4), (7.16) and the discussion above to get that

$$
\begin{aligned}
|M(Q,\varepsilon,R,\varepsilon')| &= |\langle Tb\, h_Q^\varepsilon, b\, h_R^{\varepsilon'}\rangle| \\
&\le C \sum_{R^* \in F(R)} \mu(R^*)^{-1/2} \int_{R^*} |Tb\, h_Q^\varepsilon(x)|\, d\mu(x)\,. \qquad (7.19)
\end{aligned}
$$

On each $R^* \cap 2\,Q$ we use (7.17) to estimate $|Tb\, h_Q^\varepsilon(x)|$; when we integrate the estimate and sum over R^*, we get less than $CA_1(Q,R)$. Similarly, we use (7.18) for $x \in R^* \backslash 2\,Q$, integrate over $R^* \backslash 2\,Q$ and sum over R^*, and we get a contribution $\le CA_2(Q,R)$. This proves Lemma 7.11.

Note that our estimate is more performant when $d(Q) \le d(R)$; in the other situations, we would use a symmetric argument. We won't need to do this, because as we shall see soon we won't have to bound coefficients of $\mathcal{M}$ for which $d(Q) > d(R)$.

Lemma 7.20. *We have that*

$$
|M(Q,\varepsilon,Q,\varepsilon')| \le C + CA_3(Q)\,, \qquad (7.21)
$$

for all $Q \in \Delta$ and $\varepsilon, \varepsilon' \in D(Q)$, where

$$
A_3(Q) = \sum_{Q_1^* \in F(Q)} \sum_{\substack{Q_2^* \in F(Q) \\ Q_2^* \ne Q_1^*}} \mu(Q_1^*)^{-1/2}\, \mu(Q_2^*)^{-1/2}\, I(Q_1^*, Q_2^*)\,. \qquad (7.22)
$$

To prove the lemma we start again from (7.4) and use (7.16) to get that

$$
\begin{aligned}
M(Q,\varepsilon,Q,\varepsilon') &= \langle Tb\, h_Q^{\varepsilon}, b\, h_Q^{\varepsilon'}\rangle \\
&= \sum_{Q_1^*} \sum_{Q_2^* \in F(Q)} \alpha_{\varepsilon,Q_1^*}\, \alpha_{\varepsilon',Q_2^*}\, \mu(Q_1^*)^{-1/2}\, \mu(Q_2^*)^{-1/2} \\
&\qquad \cdot \langle Tb\, \mathbf{1}_{Q_1^*}, \mathbf{1}_{Q_2^*}\rangle\,.
\end{aligned} \tag{7.23}
$$

The terms for which $Q_1^* = Q_2^*$ are less or equal than CC_3, by our weak boundedness assumption (3.17), and so we are left with terms for which $Q_1^* \neq Q_2^*$. For each such term we use (3.15) and (3.13) to get that

$$
\begin{aligned}
|\langle Tb\, \mathbf{1}_{Q_1^*}, b\, \mathbf{1}_{Q_2^*}\rangle| &= \Big| \int_{Q_1^*} \int_{Q_2^*} K(x,y)\, b(y)\, b(x)\, d\mu(y)\, d\mu(x) \Big| \\
&\le C I(Q_1^*, Q_2^*)\,.
\end{aligned} \tag{7.24}
$$

Lemma 7.20 follows because the coefficients $\alpha_{Q,\varepsilon}$ are uniformly bounded.

Now we want to estimate the coefficients of $\mathcal{M}$ for which $Q \subset R$, $Q \neq R$. In such situations, we shall systematically denote by $R(Q)$ the child of R that contains Q.

Lemma 7.25. *For each choice of cubes $Q \subset R$, $Q \neq R$ and $\varepsilon \in D(Q)$, $\varepsilon' \in D(R)$,*

$$
|M(Q,\varepsilon,R,\varepsilon')| \le C\,(B_{11} + B_{12} + B_{21} + B_{22})\,, \tag{7.26}
$$

where

$$
\begin{aligned}
B_{11} = \sum_{Q^* \in F(Q)} \sum_{\substack{R^* \in F(R) \\ R^* \neq R(Q)}} & \mu(Q^*)^{-1/2}\, \mu(R^*)^{-1/2} \\
& \cdot I(Q^*, R^* \cap 2\,Q)\,,
\end{aligned} \tag{7.27}
$$

$$
B_{12} = \sum_{\substack{R^* \in F(R) \\ R^* \neq R(Q)}} \mu(Q)^{1/2}\, \mu(R^*)^{-1/2}\, J(Q, R^* \backslash 2\,Q)\,, \tag{7.28}
$$

$$
B_{21} = \sum_{Q^* \in F(Q)} \mu(Q^*)^{-1/2}\, \mu(R(Q))^{-1/2}\, I(Q^*, 2\,Q \backslash R(Q))\,, \tag{7.29}
$$

and

$$B_{22} = \mu(Q)^{1/2}\, \mu(R(Q))^{-1/2}\, J(Q, E\backslash(2\,Q \cup R(Q)))\,. \tag{7.30}$$

To prove the lemma, let $Q, \varepsilon, R, \varepsilon'$ be given, and denote by α the constant value of $h_R^{\varepsilon'}$ on Q. Thus $|\alpha| \le C\,\mu(R(Q))^{-1/2}$ by (7.16). This time we apply (7.5)

$$M(Q,\varepsilon,R,\varepsilon') = \langle Tb\, h_Q^\varepsilon, b\, h_R^{\varepsilon'}\rangle - \alpha\,\langle Tb\, h_Q^\varepsilon, b\rangle = B_1 - B_2\,, \tag{7.31}$$

where

$$B_1 = \langle Tb\, h_Q^\varepsilon, b\, h_R^{\varepsilon'}\, \mathbf{1}_{R\backslash R(Q)}\rangle \tag{7.32}$$

and

$$B_2 = \alpha\langle Tb\, h_Q^\varepsilon, \mathbf{1}_{E-R(Q)}\, b\rangle\,. \tag{7.33}$$

Note that the part $\langle Tb\, h_Q^\varepsilon, \alpha\,\mathbf{1}_{R(Q)}\, b\rangle$ cancelled out; this will allow us to use the kernel $K(x,y)$ again to estimate B_1 and B_2. Thus

$$|B_1| \le C \sum_{\substack{R^*\in F(R)\\ R^*\neq R(Q)}} \mu(R^*)^{-1/2} \int_{R^*} |Tb\, h_Q^\varepsilon(x)|\, d\mu(x)\,, \tag{7.34}$$

by (3.15) and (7.16) for R, and now we can estimate $|Tb\, h_Q^\varepsilon(x)|$ with (7.17) and (7.18). As before, we use (7.17) on each $R^*\cap 2\,Q$. After we integrate on $R^*\cap 2\,Q$ and sum over R^*, we get a contribution less or equal than CB_{11}. On the rest of R^* we use (7.18), and we get a total contribution less or equal than CB_{12} after integrating on $R^*\backslash 2\,Q$ and summing over R^*.

The estimates for B_2 are similar. Recall that $|\alpha| \le C\,\mu(R(Q))^{-1/2}$ and hence

$$|B_2| \le C\,\mu(R(Q))^{-1/2} \int_{E\backslash R(Q)} |Tb\, h_Q^\varepsilon(x)|\, d\mu(x)\,. \tag{7.35}$$

On $2\,Q\backslash R(Q)$ we use (7.17) and get a contribution less or equal than CB_{21}. On $E\backslash(2\,Q\cup R(Q))$ we use (7.18) and get less or equal than CB_{22}. This proves Lemma 7.25.

We are now ready to reduce the proof of Theorem 3.20 to the "verification" that a certain matrix $\mathcal{N}$ defines a bounded operator on $\ell^2(\Delta)$. Define a matrix $\mathcal{N} = ((N(Q,R)))_{Q,R\in\Delta}$ as follows. Set

$$N(Q,R) = A_1(Q,R) + A_2(Q,R)\,, \tag{7.36}$$

where $A_1(Q,R)$ and $A_2(Q,R)$ are as in (7.13) and (7.14), when

$$\begin{aligned} &Q\cap R = \varnothing \text{ and either } d(Q) < d(R) \text{ or else}\\ &d(Q) = d(R) \text{ and } \operatorname{diam} Q \le \operatorname{diam} R\,, \end{aligned} \tag{7.37}$$

$$N(Q,Q) = A_3(Q)\,, \qquad \text{for } Q\in\Delta\,, \tag{7.38}$$

and

$$N(Q,R) = B_{11} + B_{12} + B_{21} + B_{22}\,, \tag{7.39}$$

when $Q\subset R$, $Q\neq R$, where $A_3(Q)$ is as in (7.22) and the B_{ij} are as in Lemma 7.25. Finally set $N(Q,R) = 0$ in the other cases, *i.e.*, when $Q\cap R=\varnothing$ but (7.37) does not hold and when $R\subset Q$, $R\neq Q$.

Lemma 7.40. *To prove Theorem* 3.20 *it is enough to show that* $\mathcal{N}$ *defines a bounded operator on* $\ell^2(\Delta)$.

Set $\mathcal{N}^+ = \mathcal{N} + \mathcal{N}^t + \mathrm{Id}$, where $\mathcal{N}^t$ is the transpose of $\mathcal{N}$ and Id the identity matrix. Obviously $\mathcal{N}^+$ defines a bounded operator on $\ell^2(\Delta)$ if $\mathcal{N}$ does. Let us suppose that this is the case; since $\mathcal{N}^+$ is a matrix with nonnegative entries and all the sets $D(Q)$, $Q\in\Delta$, have at most C elements, we shall get that $\mathcal{M}$ defines a bounded operator on $\ell^2(H)$ if we can prove that

$$|M(Q,\varepsilon,R,\varepsilon')| \le N^+(Q,R)\,, \tag{7.41}$$

for all $Q,\varepsilon,R,\varepsilon'$, and where $N^+(Q,R)$ denotes the generic element of $\mathcal{N}^+$.

Denote by D_0 the set of (ordered) pairs (Q,R) such that $Q\subset R$ or (7.37) holds. When $(Q,R)\in D_0$, (7.41) follows from Lemma 7.11, 7.20, or 7.25. Otherwise, we shall use the transpose $\tilde T$ of T, which is defined by $\langle \tilde T b\,f, b\,g\rangle = \langle T b\,g, b\,f\rangle$ for all $f,g\in\mathcal{E}$. Notice that $\tilde T$ also satisfies the hypotheses of Theorem 3.20, only with $K(x,y)$ replaced with $K(y,x)$ and the functions β, $\tilde\beta$ exchanged. We can define a matrix

$\tilde{\mathcal{M}}$ with $\tilde{T}$ as we did for T itself, and it is clear from (7.4)-(7.6) that $\tilde{\mathcal{M}}$ is the transpose of $\mathcal{M}$. If $(Q,R) \notin D_0$, then $(R,Q) \in D_0$ and

$$|M(Q,\varepsilon,R,\varepsilon')| = |\tilde{M}(R,\varepsilon',Q,\varepsilon)| \le N^+(R,Q) = N^+(Q,R)\,,$$

by Lemma 7.11, 7.20 or 7.25 (applied to $\tilde{T}$.) Thus (7.41) holds in all cases, and $\mathcal{M}$ defines a bounded operator if $\mathcal{N}$ does. Lemma 7.40 follows, by (7.3).

We completed the task assigned to this section: we can forget singular integral operators and concentrate on the matrix $\mathcal{N}$.

8. Estimates on $I(Q,V)$.

We shall need to estimate the various coefficients of our new matrix $\mathcal{N}$. In this section we prove a few estimates on integrals like $I(Q,V)$ that will be useful later. The small boundary properties (3.8)-(3.12) will be needed here.

We start with a simple estimate that uses the density property (3.1) only. First observe that

$$\begin{aligned}\int_{|x-y|\ge d} \frac{d\mu(y)}{|x-y|^2} &\le \sum_{\ell\ge 0} \int_{2^\ell d\le |x-y|<2^{\ell+1}d} \frac{d\mu(y)}{|x-y|^2}\\ &\le C\sum_{\ell\ge 0}(2^\ell\, d)(2^\ell\, d)^{-2}\\ &\le C\, d^{-1}\,,\end{aligned}\tag{8.1}$$

for all $x\in E$ and $d>0$.

Next let $Q\in\Delta$ and $V\subset E\backslash Q$ be given. For each $x\in Q$ we use Cauchy-Schwarz to show that

$$\int_V \frac{d\mu(y)}{|x-y|} \le \mu(V)^{1/2}\Big(\int_V \frac{d\mu(y)}{|x-y|^2}\Big)^{1/2} \le C\,\mu(V)^{1/2}\, d(x)^{-1/2}\,,\tag{8.2}$$

where we set $d(x) = \operatorname{dist}(x, E\backslash Q)$. Note that $d(x)>0$ almost everywhere on Q, by (3.7)-(3.8). We may now integrate (8.2) on Q to get that

$$I(Q,V) = \int_Q\int_V \frac{d\mu(y)\,d\mu(x)}{|x-y|} \le C\,\mu(V)^{1/2}\int_Q d(x)^{-1/2}\,d\mu(x)\tag{8.3}$$

(see (7.9) for the definition of $I(Q,V)$).

Lemma 8.4. *We have that*

$$\int_Q d(x)^{-1/2}\,d\mu(x) \le C\,d(Q)^{-1/2}\,\xi(Q)\,. \tag{8.5}$$

Here $d(Q) = A^{-k(Q)}$, as in (7.7). To prove the lemma we decompose Q into a first region B_0 where $d(x) \ge d(Q)$ and annuli B_ℓ, $\ell \ge 1$, where $2^{-\ell}d(Q) \le d(x) < 2^{-\ell+1}d(Q)$. Then

$$\int_{B_0} d(x)^{-1/2}\,d\mu(x) \le d(Q)^{-1/2}\,\mu(Q) \le C\,d(Q)^{-1/2}\,\xi(Q)\,,$$

by (3.9), and

$$\begin{aligned}\int_{B_\ell} d(x)^{-1/2}\,d\mu(x) &\le 2^{\ell/2}\,d(Q)^{-1/2}\,\mu(B_\ell)\\ &\le C\,2^{\ell/2}\,d(Q)^{-1/2}\,2^{-\tau\ell}\,\xi(Q)\,,\end{aligned} \tag{8.6}$$

for $\ell \ge 1$, by (3.8). Lemma 8.4 follows by summing a convergent power series.

From (8.3) and Lemma 8.4 we deduce that

$$I(Q,V) \le C\mu(V)^{1/2}\,\xi(Q)\,d(Q)^{-1/2}\,, \tag{8.7}$$

for all cubes Q and all sets $V \subset E\backslash Q$.

We want to refine this estimate when Q is not a good cube (as in (3.11)), because getting estimates in terms of $\mu(Q)$ rather than $\xi(Q)$ will be very useful to get rid of some of the negative powers in formulae like (7.13), (7.22), (7.27) or (7.29). Recall that μ is not doubling or anything like that, and we don't have much in terms of lower bounds for μ.

Lemma 8.8. *We have that*

$$I(Q,V) \le C\mu(V)^{1/2}\,\mu(Q)^{1/2}\,\xi(Q)^{1/2}\,d(Q)^{-1/2}\,, \tag{8.9}$$

for all $Q \in \Delta$ and $V \subset E\backslash Q$.

To prove this we shall use a decomposition of Q_0 into maximal good subcubes. For each $Q \in \Delta$, denote by $S(Q)$ the set of maximal good cubes contained in Q. Obviously the cubes S, $S \in S(Q)$, are disjoint and contained in Q, but it is also true that they almost cover Q, *i.e.*, that

$$\mu\Big(Q \backslash \bigcup_{S \in S(Q)} S\Big) = 0 \,. \tag{8.10}$$

This is essentially [DM, Lemma 5.28], but the proof is quite simple and so we give it here. For each integer $\ell > 0$, let Z_ℓ denote the set of cubes $R \in \Delta_{k(Q)+\ell}$ such that $R \subset Q$ but R is not contained in any $S \in S(Q)$. Such cubes are obviously bad, as well as all their ancestors until Q and hence they satisfy

$$\mu(R) \le C_0\, \xi(R) \le C_0\, A^{-10\ell}\, \xi(Q)\,, \tag{8.11}$$

by (3.9) and repeated uses of (3.12). Because of (5.1) and (3.4), (3.5), Z_ℓ has at most $CA^{2\ell}$ elements, and so

$$\mu\Big(\bigcup_{R \in Z_\ell} R\Big) \le CA^{-8\ell}\, \xi(Q)\,, \tag{8.12}$$

where the value of C does not matter because we only need to know that $\mu(\bigcup_{R \in Z_\ell} R)$ tends to 0 when $\ell \longrightarrow +\infty$. The desired estimate (8.10) follows because

$$\Big(Q \backslash \bigcup_{S(Q)} S\Big) \subset \Big(\bigcup_{R \in Z_\ell} R\Big)\,,$$

for all $\ell > 0$.

To prove Lemma 8.8 we use (8.10) to almost-decompose Q into its maximal good subcubes S, $S \in S(Q)$ and write

$$\begin{aligned} I(Q, V) &= \int_Q \int_V \frac{d\mu(x)\, d\mu(y)}{|x-y|} \\ &= \sum_{S \in S(Q)} I(S, V) \\ &\le C\mu(V)^{1/2} \sum_{S \in S(Q)} \xi(S)\, d(S)^{-1/2} \\ &\le C\mu(V)^{1/2} \sum_{S \in S(Q)} \mu(S)\, d(S)^{-1/2}\,, \end{aligned} \tag{8.13}$$

by (8.7) and (3.11) for the good cubes S.

Lemma 8.14. *For all $Q \in \Delta$,*

$$\sum_{S \in S(Q)} \mu(S) \Big(\frac{d(Q)}{d(S)}\Big)^6 \le C\,\xi(Q)\,. \tag{8.15}$$

Of course we don't need the power 6 here, but the proof will be just as easy. Denote by $S_\ell(Q)$, $\ell \ge 0$, the set of cubes $S \in S(Q)$ such that $k(S) = k(Q) + \ell$. Because of (5.1), $S_\ell(Q)$ has at most $CA^{2\ell}$ elements. Let us check that

$$\mu(S) \le C_0\, A^{-10(\ell-1)}\, \xi(Q)\,, \tag{8.16}$$

for all $S \in S_\ell(Q)$. When $\ell = 0$ or 1, $\mu(S) \le \mu(Q) \le C_0 \xi(Q)$ by (3.9). When $\ell > 1$, $\mu(S) \le \mu(\hat{S}) \le C_0\, \xi(\hat{S}) \le C_0\, A^{-10(\ell-1)}\, \xi(Q)$ by (3.9) and repeated uses of (3.12), and where $\hat{S}$ denotes the parent of S. Here we use the fact that all the ancestors of S between $\hat{S}$ and Q are bad, by definition of $S(Q)$.

From (8.16) and the fact that $S_\ell(Q)$ has at most $CA^{2\ell}$ elements we deduce that the contribution of $S_\ell(Q)$ to the left-hand side of (8.15) is at most $CA^{2\ell}A^{-10\ell}A^{6\ell}\xi(Q) \le CA^{-2\ell}\xi(Q)$; Lemma 8.14 follows by summing over $\ell \ge 0$.

Most of the time, Lemma 8.14 will be used in combination with Cauchy-Schwarz, as follows

$$\begin{aligned}\sum_{S \in S(Q)} \mu(S) \Big(\frac{d(Q)}{d(S)}\Big)^3 &\le \Big(\sum_{S \in S(Q)} \mu(S)\Big)^{1/2} \Big(\sum_{S \in S(Q)} \mu(S) \Big(\frac{d(Q)}{d(S)}\Big)^6\Big)^{1/2} \\ &\le C\mu(Q)^{1/2}\, \xi(Q)^{1/2}\,,\end{aligned} \tag{8.17}$$

because Q is (essentially) the disjoint union of the cubes $S \in S(Q)$. A trivial consequence of (8.17) is

$$\begin{aligned}\sum_{S \in S(Q)} \mu(S)\, d(S)^{-1/2} &= d(Q)^{-1/2} \sum_S \mu(S) \Big(\frac{d(Q)}{d(S)}\Big)^{1/2} \\ &\le C\, d(Q)^{-1/2}\, \mu(Q)^{1/2}\, \xi(Q)^{1/2}\,.\end{aligned} \tag{8.18}$$

Lemma 8.8 follows from this and (8.13).

We shall need a last estimate on $I(Q,V)$, to be used when we have a larger power of $\mu(Q)$ to recuperate

$$I(Q, 2\,Q\backslash Q) \leq C\mu(Q)\Big(\frac{\xi(Q)}{d(Q)}\Big)^{1/2}. \tag{8.19}$$

To prove this we write

$$\begin{aligned} I(Q, 2\,Q\backslash Q) &= \sum_{S\in S(Q)} I(S, 2\,Q\backslash Q) \\ &\leq \sum_{S\in S(Q)} I(S, 2\,S\backslash S) + \sum_{S\in S(Q)} I(S, 2\,Q\backslash 2\,S) \\ &= I_1 + I_2\,. \end{aligned} \tag{8.20}$$

For each $S \in S(Q)$,

$$I(S, 2\,S\backslash S) \leq C\mu(2\,S)^{1/2}\,\xi(S)\,d(S)^{-1/2} \leq C\mu(S)^{3/2}\,d(S)^{-1/2}, \tag{8.21}$$

by (8.7), (3.9) and (3.11) for the good cube S. Hence

$$\begin{aligned} I_1 &\leq C \sum_{S\in S(Q)} \mu(S)^{3/2}\,d(S)^{-1/2} \\ &\leq C\mu(Q)^{1/2}\sum_{S} \mu(S)\,d(S)^{-1/2} \\ &\leq C\mu(Q)\,\xi(Q)^{1/2}\,d(Q)^{-1/2}, \end{aligned} \tag{8.22}$$

by (8.18). This takes care of I_1.

As for I_2, let us check that

$$\int_{2Q\backslash 2S} \frac{d\mu(y)}{|x-y|} \leq C\,\frac{\xi(Q)}{d(Q)}\,, \tag{8.23}$$

for all $S \in S(Q)$ and $x \in S$.

Denote by T_ℓ, $0 \leq \ell \leq k(S) - k(Q)$, the cube of $\Delta_{k(Q)+\ell}$ that contains S. This is a decreasing sequence of cubes, with $T_0 = Q$ and, $T_{k(S)-k(Q)} = S$, and $2\,Q\backslash 2\,S$ is the union of the sets $2\,T_\ell\backslash 2\,T_{\ell+1}$, $0 \leq \ell \leq k(S) - k(Q) - 1$. For these values of ℓ,

$$\mu(2\,T_\ell) \leq C_0\,\xi(T_\ell) \leq C_0\,A^{-10\ell}\,\xi(Q)\,, \tag{8.24}$$

by (3.9) and repeated uses of (3.12). Then

$$
\begin{aligned}
\int_{2Q\setminus 2S} \frac{d\mu(y)}{|x-y|} &= \sum_{\ell=0}^{k(S)-k(Q)-1} \int_{2T_\ell \setminus 2T_{\ell+1}} \frac{d\mu(y)}{|x-y|} \\
&\leq \sum_{\ell} \mu(2\,T_\ell)\, d(T_{\ell+1})^{-1} \\
&\leq C\,\xi(Q)\, d(Q)^{-1},
\end{aligned} \tag{8.25}
$$

by definition (7.8) of $2\,T_{\ell+1}$, the fact that $x \in S \subset T_{\ell+1}$, and then (8.24). This proves (8.23). Now

$$
\begin{aligned}
I_2 &= \sum_{S\in S(Q)} \int_S \int_{2Q\setminus 2S} \frac{d\mu(y)\, d\mu(x)}{|x-y|} \\
&\leq C \sum_S \mu(S)\, \xi(Q)\, d(Q)^{-1} \\
&\leq C\mu(Q)\, \xi(Q)\, d(Q)^{-1} \\
&\leq C\mu(Q) \Big(\frac{\xi(Q)}{d(Q)}\Big)^{1/2},
\end{aligned} \tag{8.26}
$$

by the definitions (8.20) and (7.9), (8.23), and (3.9) (to get that $\xi(Q) \leq C\, d(Q)$). The desired estimate (8.19) follows from (8.20), (8.22) and (8.26).

9. Bounds on $\mathcal{N}$.

In the original version of this paper, the matrix $\mathcal{N}$ was bounded with the help of Schur's lemma. This was quite tempting, but it turns out that it actually complicated the estimates. The current section was revisited in October 1997, after the author noticed that in the similar extension of $T(b)$ by Nazarov, Treil, and Volberg, the corresponding estimates were much simpler. Here is the simple trick that makes the difference; I am sure the reader will be glad that the authors of [NTV] kindly communicated it to me.

Lemma 9.1. *Let $\mathcal{N} = ((N(Q,R)))_{Q,R\in\Delta}$ be a matrix with complex coefficients. Assume that for each $Q \in \Delta$ there are at most C_1 indices*

$R \in \Delta$ such that $N(Q,R) \neq 0$, and also that

$$(9.2) \qquad \sum_{Q \in \Delta} |N(Q,R)|^2 \le C_2^2 \,, \qquad \text{for each } R \in \Delta \,.$$

Then $\mathcal{N}$ defines a bounded operator on $\ell^2(\Delta)$, with norm $|||\mathcal{N}||| \le C_1 \, C_2$.

This is easy to prove. First observe that if $\mathcal{N}$ is as in the lemma, then it is the sum of at most C_1 matrices that satisfy the hypotheses of the lemma with $C_1 = 1$ and the same constant C_2. Thus we may assume that $C_1 = 1$. For each $R \in \Delta$, denote by $v_R \in \ell^2(\Delta)$ the vector with coordinates $N(Q,R)$, $Q \in \Delta$. By (9.2), $\|v_R\|^2 \le C_2^2$, while our first hypothesis with $C_1 = 1$ says that the vectors v_R, $R \in \Delta$, are orthogonal to each other. Hence if $x = (x_R)_{R \in \Delta}$ is any vector in $\ell^2(\Delta)$,

$$(9.3) \qquad \|\mathcal{N} x\|^2 = \Big\| \sum_R x_R \, v_R \Big\|^2 = \sum_R |x_R|^2 \, \|v_R\|^2 \le C_2^2 \, \|x\|^2 \,,$$

as needed. The lemma follows.

To estimate the matrix $\mathcal{N}$ from Section 7, we want to decompose it into a sum of matrices $\mathcal{N}^k$, with $k = k(Q) - k(R)$ and prove geometrically decreasing bounds on the norms $|||\mathcal{N}^k|||$. For each integer $k \ge 0$, denote by $\mathcal{N}^k$ the matrix with coefficients $N^k(Q,R) = N(Q,R)$ when $k(Q) = k(R) + k$ and $N^k(Q,R) = 0$ otherwise. Note that $\mathcal{N} = \sum_{k \ge 0} \mathcal{N}^k$, because $N(Q,R) = 0$ when $k(Q) < k(R)$. See around (7.36)-(7.39) for the definition of $\mathcal{N}$.

At this point, and for almost all the rest of this section, we fix an integer $k \ge 0$ and we study $\mathcal{N}^k$ by cutting it into smaller pieces. As we shall see, Lemma 9.1 will be quite handy for most of them.

Case A. *Terms with $Q = R$.* Of course this only shows up when $k = 0$. Denote by $\mathcal{N}_1$ the part of $\mathcal{N}$ that lives on the diagonal, *i.e.*, set $N_1(Q,R) = 0$ when $Q \neq R$ and $N_1(Q,R) = N(Q,R) = A_3(Q)$ for $Q \in \Delta$. (See (7.38).)

Recall from (7.22) that

$$(9.4) \qquad A_3(Q) = \sum_{Q_1^*} \sum_{Q_2^*} \mu(Q_1^*)^{-1/2} \, \mu(Q_2^*)^{-1/2} \, I(Q_1^*, Q_2^*) \,,$$

where we sum over pairs of distinct children of Q. By (8.9) and (3.9),

$$
\begin{aligned}
I(Q_1^*, Q_2^*) &\le C\mu(Q_2^*)^{1/2}\,\mu(Q_1^*)^{1/2}\,\xi(Q_1^*)^{1/2}\,d(Q_1^*)^{-1/2} \\
&\le C\mu(Q_2^*)^{1/2}\,\mu(Q_1^*)^{1/2}\,,
\end{aligned} \tag{9.5}
$$

and so $A_3(Q)$ is a sum of boundedly many bounded terms. Thus $\mathcal{N}_1$ defines a bounded operator on $\ell^2(\Delta)$, with norm $|||\mathcal{N}_1||| \le C$.

Case B. *Terms coming from* $A_1(Q,R)$. Set $N_2(Q,R) = A_1(Q,R)$ when $k(Q) = k(R) + k$ and (7.37) holds, and $N_2(Q,R) = 0$ otherwise. We should perhaps have written $N_2^k(Q,R)$ instead of $N_2(Q,R)$, but k is fixed and we'll try to keep the notation simple. Note that $N_2(Q,R) = 0$ unless $2Q$ meets R; this is clear from the definitions (7.13) and (7.9). Thus for each Q there are at most C cubes $R \in \Delta_{k(Q)-k}$ such that $N_2(Q,R) \ne 0$. We can apply Lemma 9.1 to the matrix $\mathcal{N}_2$ with coefficients $\mathcal{N}_2(Q,R)$ and get that

$$
|||\mathcal{N}_2|||^2 \le C \sup_{R\in\Delta} \Sigma(R)\,, \tag{9.6}
$$

where

$$
\Sigma(R) = \sum_{Q\in\Delta(R)} N_2(Q,R)^2 \tag{9.7}
$$

and $\Delta(R)$ is the set of cubes $Q \in \Delta_{k(R)+k}$ such that $Q \cap R = \varnothing$ but $2\,Q \cap R \ne \varnothing$.

Fix $R \in \Delta$, plug (7.13) into (9.7) and then apply (8.9) to get that

$$
\begin{aligned}
\Sigma(R) &\le C \sum_{R^*\in F(R)} \sum_{Q\in\Delta(R)} \sum_{Q^*\in F(Q)} \mu(Q^*)^{-1}\,\mu(R^*)^{-1}\,I(Q^*, R^*\cap 2\,Q)^2 \\
&\le C \sum_{R^*} \sum_{Q} \sum_{Q^*} \mu(R^*)^{-1}\,\mu(R^*\cap 2\,Q)\,\xi(Q^*)\,d(Q^*)^{-1}\,.
\end{aligned} \tag{9.8}
$$

Let us fix R^* and try to bound the corresponding sum. Let us warm up with the easy case when $k \le 10$, say. Then we simply say that $\xi(Q^*)\,d(Q^*)^{-1} \le C$ by (3.9), that the $R^* \cap 2\,Q$, $Q \in \Delta_{k(R)+k}$, have bounded overlap (by (3.4), (3.5)) and are contained in R^* and then that $\Sigma(R) \le C$ after summing over boundedly many children R^* of R.

For larger k we wish to argue that since $Q \cap R = \varnothing$ by (7.37), the sets $R^* \cap 2\,Q$ only cover a small proportion of R^*. This can be implemented directly if R^* is a good cube, but in general we need to bring in the decomposition of R^* into maximal good subcubes S, $S \in S(R^*)$, as in (8.10). For each $R^* \in F(R)$ set $S_+ = \{S \in S(R^*) : k(S) \leq k(R) + k/2\}$, $R^*_+ = \bigcup_{S \in S_+} S$ and $R^*_- = R^* \setminus R^*_+$. We write

$$\Sigma(R) \leq C \sum_{R^* \in F(R)} (\sigma_+(R^*) + \sigma_-(R^*))\, \mu(R^*)^{-1}\,, \tag{9.9}$$

where

$$\sigma_\pm(R^*) = \sum_{Q \in \Delta(R)} \sum_{Q^* \in F(Q)} \mu(R^*_\pm \cap 2\,Q)\, \xi(Q^*)\, d(Q^*)^{-1}\,. \tag{9.10}$$

For $\sigma_+(R^*)$ we say that $\xi(Q^*) \leq C\, d(Q^*)$ by (3.9), so that

$$\begin{aligned} \sigma_+(R^*) &= \sum_{S \in S^+} \sum_{Q \in \Delta(R)} \sum_{Q^* \in F(Q)} \mu(S \cap 2\,Q)\, \xi(Q^*)\, d(Q^*)^{-1} \\ &\leq C \sum_{S \in S^+} \sum_{Q \in \Delta(R)} \mu(S \cap 2\,Q) \leq C \sum_{S \in S^+} \mu(A_S)\,, \end{aligned} \tag{9.11}$$

where A_S is the union of the sets $S \cap 2\,Q$, $Q \in \Delta(R)$. We used the fact that the $2\,Q$, $Q \in \Delta_{k(R)+k}$, have bounded overlap. Next all the points of A_S lie within $A^{-k(R)-k} = A^{-k}\, d(R)$ of some point of $E \setminus S$, because the cubes Q do not meet R (and even less S). (See (7.8) for the definition of $2\,Q$.) Hence A_S is contained in the set $N_t(S)$ of (3.7), with $t = A^{-k}\, d(R)\, d(S)^{-1} \leq C\, A^{-k/2}$ (because $d(R) \leq A^{k/2} d(S)$ by definition of S_+). So

$$\begin{aligned} \sigma_+(R^*) &\leq C \sum_{S \in S_+} A^{-k\tau/2} \xi(S) \\ &\leq C A^{-k\tau/2} \sum_{S \in S_+} \mu(S) \\ &\leq C A^{-k\tau/2}\, \mu(R^*)\,, \end{aligned} \tag{9.12}$$

because the cubes S are good (as in (3.11)), disjoint, and contained in R^*. This will be enough to take care of S_+.

For $\sigma_-(R^*)$ we only say that $\mu(R^*_- \cap 2\,Q) \leq \mu(R^*)$, but we use a better estimate for $\xi(Q^*)$. Let $Q \in \Delta(R)$ be such that $2\,Q$ meets R^*_-, and let z be any point of $2\,Q \cap R^*_-$. Then let H be the smallest cube that contains z such that $k(H) \le k(R) + k/2$. Since H is not contained in any cube of S_+, it is a bad cube and so are all its ancestors contained in R^*. Then

$$\begin{aligned} \xi(H) &\le A^{-10(k(H)-k(R^*))}\,\xi(R^*) \\ &\le CA^{-5k}\,\xi(R^*) \\ &\le CA^{-5k}\,d(R)\,, \end{aligned} \tag{9.13}$$

by repeated applications of (3.12), because $k(H) \ge k(R) + k/2 - 1$, and by (3.9).

Since $2\,Q$ meets H and Q is of a strictly later generation than H, Q is contained in $91B(H)$ and (3.10) says that $\xi(Q^*) \le C\xi(H) \le CA^{-5k}\,d(R)$ for all $Q^* \in F(Q)$.

Thus all the terms in the sum that defines $\sigma_-(R^*)$ (in (9.10)) are at most

$$C\mu(R^*)A^{-5k}\,d(R)\,d(Q)^{-1} \le C\mu(R^*)A^{-4k}\,.$$

Since by easy geometric considerations (like (5.1)) there are at most CA^{2k} cubes Q in $\Delta(R)$, we get that

$$\sigma_-(R^*) \le C\mu(R^*)A^{-2k}\,. \tag{9.14}$$

From this and the similar estimate (9.12) we deduce that $\sum(R) \le CA^{-\tau k/2}$ (see (9.9)), and then that $|||\mathcal{N}_2||| \le CA^{-\tau k/4}$ (by (9.6)).

Case C. *Terms from* B_{11}. Set $N_3(Q,R) = B_{11}$, where B_{11} is as in (7.27), when $Q \subset R$, $Q \ne R$, and $k(Q) = k(R) + k$. Otherwise set $\mathcal{N}_3(Q,R) = 0$. These coefficients are like the $\mathcal{N}_2(Q,R) = A_1(Q,R)$ that we just treated (compare (7.27) with (7.13)), except that now we sum over pairs Q^*, R^* such that $Q^* \in F(Q)$ and $R^* \in F(R)$ is not the cube of $F(R)$ that contains Q. The same estimates as before can be carried out, because whenever we used the fact that Q does not meet R in subsection B, we only needed to know that Q does not meet R^*. So the matrix $\mathcal{N}_3$ with coefficients $\mathcal{N}_3(Q,R)$ has a norm $|||\mathcal{N}_3||| \le CA^{-\tau k/4}$, and the proof is the same as for $\mathcal{N}_2$.

Case D. *Terms from* B_{21}. Now set $N_4(Q,R) = B_{21}$, where B_{21} is as in (7.29), when $Q \subset R$, $Q \ne R$, and $k(Q) = k(R) + k$. Otherwise set $N_4(Q,R) = 0$. These coefficients are a little like the previous ones, but with a $\mu(R^*)^{-1/2}$ replaced with $\mu(R(Q))^{-1/2}$, where $R(Q)$ is the child of R that contains Q. To accommodate this change, it will be better to use (8.19) rather than (8.9). Recall from (7.29) that

$$B_{21} = \sum_{Q^* \in F(Q)} \mu(Q^*)^{-1/2} \mu(R(Q))^{-1/2} I(Q^*, 2\,Q \backslash R(Q)), \tag{9.15}$$

and note that

$$\begin{aligned} I(Q^*, 2\,Q\backslash R(Q)) &\leq I(Q^*, 2\,Q\backslash Q^*) \\ &\leq I(Q^*, 2\,Q\backslash 2\,Q^*) + I(Q^*, 2\,Q^*\backslash Q^*), \end{aligned} \tag{9.16}$$

by definition of $I(\,\cdot\,)$ (see (7.9)).

The last term is at most

$$C\mu(Q^*)\,\xi(Q^*)^{1/2}\,d(Q^*)^{-1/2} \leq C\mu(Q^*)\,\xi(Q)^{1/2}\,d(Q)^{-1/2}$$

by (8.19) and (3.10).

The first term is

$$\begin{aligned} I(Q^*, 2\,Q\backslash 2\,Q^*) &\leq \mu(Q^*)\,\mu(2\,Q)\,\mathrm{dist}\,(Q^*, 2\,Q\backslash Q^*)^{-1} \\ &\leq C\mu(Q^*)\,\mu(2\,Q)\,d(Q)^{-1} \\ &\leq C\mu(Q^*)\,\xi(Q)\,d(Q)^{-1} \\ &\leq C\mu(Q^*)\,\xi(Q)^{1/2}\,d(Q)^{-1/2}, \end{aligned}$$

by (7.9) and (3.9). Thus

$$N_4(Q,R)^2 \leq C \sum_{Q^* \in F(Q)} \mu(Q^*)\,\mu(R(Q))^{-1}\,\xi(Q)\,d(Q)^{-1}. \tag{9.17}$$

Note that for each $Q \in \Delta$ there is at most one cube $R \in \Delta$ such that $N_4(Q,R) \neq 0$ (namely, the ancestor of order k of Q). Thus we can apply Lemma 9.1 to the matrix $\mathcal{N}_4$ with coefficients $N_4(Q,R)$, and

$$|||\mathcal{N}_4||| \leq \sup_{R \in \Delta} \Sigma(R), \tag{9.18}$$

with

$$\begin{aligned} \Sigma(R) &= \sum_{Q} N_4(Q,R)^2 \\ &\leq C \sum_{R^* \in F(R)} \sum_{Q \in \Delta(R^*)} \sum_{Q^* \in F(Q)} \mu(Q^*)\,\mu(R^*)^{-1}\,\xi(Q)\,d(Q)^{-1}, \end{aligned} \tag{9.19}$$

and where

$$\Delta(R^*) = \{Q \in \Delta_{k(R)+k} : \; Q \subset R^* \text{ and } 2\,Q \text{ meets } E\backslash R^*\} .$$

(The last condition is needed if we want $I(Q^*, 2\,Q\backslash R(Q)) \neq 0$ in (9.15).)

We shall now proceed as in Case B. As before, the case when $k \leq 10$ is easy, because we can just use (3.9) to get that

$$\Sigma(R) \leq \sum_{R^*} \sum_{Q} \sum_{Q^*} \mu(Q^*)\, \mu(R^*)^{-1} \leq C$$

(because the cubes Q^* are disjoint and contained in R^*). So we may assume $k \geq 10$.

Set

$$S_+ = \Big\{S \in S(R^*) : \; k(S) \leq k(R) + \frac{k}{2}\Big\}$$

and subdivide $\Delta(R^*)$ into Δ^+ and Δ^-, where

$$\Delta^+ = \{Q \in \Delta(R^*) : \; Q \subset S \text{ for some } S \in S_+\}$$

and $\Delta^- = \Delta(R^*)\backslash\Delta^+$. For cubes of Δ^+ we use (3.9) to get that

$$\begin{aligned} \sigma_+(R^*) &=: \sum_{Q \in \Delta^+} \sum_{Q^* \in F(Q)} \mu(Q^*)\, \xi(Q)\, d(Q)^{-1} \\ &\leq C \sum_{Q \in \Delta^+} \mu(Q) \\ &\leq C \sum_{S \in S_+} \sum_{\substack{Q \in \Delta^+ \\ Q \subset S}} \mu(Q) . \end{aligned} \tag{9.20}$$

Now for $S \in S_+$ and $Q \in \Delta^+$, $Q \subset S$, we have that $2\,Q$ meets $E\backslash R^*$ by definition of $\Delta(R^*)$ and so $Q \subset N_t(S)$, with $t = A^{-k(Q)+k(S)+1}$, say. By definition of S_+, $t \leq CA^{-k/2}$ and so (3.8) yields

$$\sum_{\substack{Q \subset S \\ Q \in \Delta^+}} \mu(Q) \leq \mu(N_t(S)) \leq CA^{-k\tau/2}\, \xi(S) \leq CA^{-k\tau/2}\, \mu(S) , \tag{9.21}$$

because S is a good cube. Altogether, (9.20) becomes

$$\sigma_+(R^*) \leq C \sum_{S \in S_+} A^{-k\tau/2}\, \mu(S) \leq CA^{-k\tau/2}\, \mu(R^*) , \tag{9.22}$$

because the maximal cubes S, $S \in S_+$, are disjoint and contained in R^*.

Next we want to estimate

$$\sigma_-(R^*) = \sum_{Q \in \Delta^-} \sum_{Q^* \in F(Q)} \mu(Q^*)\, \xi(Q)\, d(Q)^{-1}. \tag{9.23}$$

This time we shall just say that $\mu(Q^*) \leq \mu(R^*)$, but we'll use a better estimate on $\xi(Q)$. By definition of $\Delta^{\pm}$, the smallest ancestor H of Q such that $k(H) \leq k(R) + k/2$ is a bad cube, and so are all its ancestors in R^*. By repeated uses of (3.12),

$$\begin{aligned} \xi(H) &\leq A^{-10(k(H)-k(R^*))}\, \xi(R^*) \\ &\leq CA^{-5k}\, \xi(R^*) \\ &\leq CA^{-5k}\, d(R) \end{aligned} \tag{9.24}$$

(by (3.9)). Also, (3.10) says that $\xi(Q) \leq C_0 \xi(H)$. Altogether,

$$\xi(Q)\, d(Q)^{-1} \leq C\, \xi(H)\, d(Q)^{-1} \leq CA^{-4k}. \tag{9.25}$$

By (5.1), there are at most CA^{2k} cubes Q in $\Delta(R^*)$ and so $\sigma_-(R^*) \leq CA^{-2k}\mu(R^*)$. Finally

$$\Sigma(R) \leq C \sum_{R^* \in F(R)} \mu(R^*)^{-1}\, (\sigma_+(R^*) + \sigma_-(R^*)) \leq CA^{-k\tau/2}, \tag{9.26}$$

by (9.19), (9.20), (9.23), (9.22) and this, and so $|||\mathcal{N}_4||| \leq CA^{-k\tau/4}$ by (9.18).

Case E. *The far part from* $A_2(Q,R)$. Now we study the piece of $\mathcal{N}^k$ that comes from terms $A_2(Q,R)$ for which $\operatorname{dist}(Q,R) \geq d(R)$. For each $R \in \Delta$ denote by $\mathcal{A}(R)$ the set of cubes $Q \in \Delta_{k(R)+k}$ for which (7.37) holds and $\operatorname{dist}(Q,R) \geq d(R) = A^{-k(R)}$. Define $\mathcal{N}_5$ by $N_5(Q,R) = A_2(Q,R)$ when $Q \in \mathcal{A}(R)$ and $N_5(Q,R) = 0$ otherwise. When $Q \in \mathcal{A}(R)$,

$$\begin{aligned} A_2(Q,R) &= \mu(Q)^{1/2} \sum_{R^* \in F(R)} \mu(R^*)^{-1/2}\, J(Q, R^* \backslash 2\,Q) \\ &\leq \mu(Q)^{1/2} \sum_{R^* \in F(R)} \mu(R^*)^{-1/2} \int_{R^*} \frac{d(Q)\, d\mu(x)}{|x - x(Q)|^2} \\ &\leq C\mu(Q)^{1/2} \sum_{R^*} \mu(R^*)^{1/2}\, d(Q) \operatorname{dist}(Q,R^*)^{-2} \\ &\leq C\mu(Q)^{1/2}\, \mu(R)^{1/2}\, d(Q) \operatorname{dist}(Q,R)^{-2}, \end{aligned} \tag{9.27}$$

by (7.14) and (7.10).

Subdivide each $\mathcal{A}(R)$ further into the

$$(9.28) \qquad \mathcal{A}_\ell(R) = \{Q \in \mathcal{A}(R) : \ 2^\ell\, d(R) \le \operatorname{dist}(Q,R) < 2^{\ell+1}\, d(R)\},$$

$\ell \ge 0$. We want to control the norms of the corresponding pieces $\mathcal{N}_{5,\ell}$ of $\mathcal{N}_5$, and this is the only place in this revised Section 9 where it will be more pleasant to use Schur's lemma.

Lemma 9.29 (Schur)**.** *Let $\mathcal{N} = ((N(Q,R)))_{Q\in\Delta, R\in\Delta}$ be a matrix with complex coefficients, and assume that there are positive numbers $\omega(Q)$, $Q \in \Delta$, such that*

$$(9.30) \qquad \sum_{Q\in\Delta} \frac{\omega(Q)}{\omega(R)}\, |N(Q,R)| \le C\,, \qquad \text{for all } R \in \Delta$$

and

$$(9.31) \qquad \sum_{R\in\Delta} \frac{\omega(R)}{\omega(Q)}\, |N(Q,R)| \le C\,, \qquad \text{for all } Q \in \Delta\,.$$

Then $\mathcal{N}$ defines a bounded operator on $L^2(\Delta)$, with norm $|||\mathcal{N}||| \le C$.

For the very easy proof, see for instance [Da, p. 43] or [My, p. 269]. We want to apply this to $\mathcal{N}_{5,\ell}$, with $\omega(Q) = \mu(Q)^{1/2}$. Let us first check sums over Q. For $R \in \Delta$,

$$(9.32) \qquad \begin{aligned} \sum_Q \frac{\omega(Q)}{\omega(R)}\, |N_{5,\ell}(Q,R)| &\le C \sum_{Q\in\mathcal{A}_\ell(R)} \mu(Q)\, d(Q) \operatorname{dist}(Q,R)^{-2} \\ &\le C A^{-k}\, d(R)\, (2^\ell\, d(R))^{-2} \sum_{Q\in\mathcal{A}_\ell(R)} \mu(Q)\,, \end{aligned}$$

by (9.27) and definitions. Since all the cubes $Q \in \mathcal{A}_\ell(R)$ lie within $C\,2^\ell\, d(R)$ of R, their total mass is at most $C\,2^\ell\, d(R)$ by (3.1), and so

$$(9.33) \qquad \sum_Q \frac{\omega(Q)}{\omega(R)}\, |N_{5,\ell}(Q,R)| \le C A^{-k}\, 2^{-\ell}\,.$$

Next we fix Q and sum over R. Of course we need only consider those R for which $Q \in \mathcal{A}_\ell(R)$, and all these cubes R lie at distance less or equal than $C\,2^\ell\, d(R) = C\,2^\ell\, A^k\, d(Q)$ from Q. Thus

$$\begin{aligned}\sum_R \frac{\omega(R)}{\omega(Q)}\, N_{5,\ell}(Q,R) &\le C \sum_R \mu(R)\, d(Q)\, \mathrm{dist}\,(Q,R)^{-2} \\ &\le C\, d(Q)\, (A^k\, 2^\ell\, d(Q))^{-2} \sum_R \mu(R) \\ &\le C\, d(Q)\, (A^k\, 2^\ell\, d(Q))^{-1} \\ &= C\, A^{-k}\, 2^{-\ell}\,.\end{aligned} \tag{9.34}$$

Altogether, Schur's lemma yields

$$|||\mathcal{N}_5||| \le \sum_\ell |||\mathcal{N}_{5,\ell}||| \le C A^{-k}\,. \tag{9.35}$$

Case F. *The local part of* $A_2(Q,R)$ *and* B_{12}. Set $N_6(Q,R) = A_2(Q,R)$ when $k(Q) = k(R) + k$, (7.37) holds, and $\mathrm{dist}\,(Q,R) < d(R)$; set $N_6(Q,R) = B_{12}$ when $k(Q) = k(R)+k$, $Q \subset R$ and $Q \ne R$; finally set $N_6(Q,R) = 0$ otherwise. Note that

$$N_6(Q,R) = \mu(Q)^{1/2} \sum_{\substack{R^* \in F(R) \\ Q \cap R^* = \varnothing}} \mu(R^*)^{-1/2}\, J(Q, R^* \setminus 2\,Q)\,, \tag{9.36}$$

when $N_6(Q,R) \ne 0$, by (7.14) or (7.28). Also, $\mathrm{dist}\,(Q,R) \le d(R)$ when $N_6(Q,R) \ne 0$, so for each $Q \in \Delta$ there are at most C cubes $R \in \Delta$ such that $N_6(Q,R) \ne 0$. Lemma 9.1 tells us that

$$|||\mathcal{N}_6|||^2 \le C \sup_{R \in \Delta} \Sigma(R)\,, \tag{9.37}$$

where $\mathcal{N}_6$ is the matrix with coefficients $N_6(Q,R)$ and

$$\Sigma(R) = \sum_Q N_6(Q,R)^2\,.$$

For each $R \in \Delta$ and $R^* \in F(R)$, set

$$\mathcal{A}(R^*) = \{Q \in \Delta_{k(R)+k} :\ \mathrm{dist}\,(Q,R) \le d(R) \text{ but } Q \cap R^* = \varnothing\}\,. \tag{9.38}$$

Then

$$\Sigma(R) \le C \sum_{R^* \in F(R)} \mu(R^*)^{-1}\, \sigma(R^*)\,, \tag{9.39}$$

with

$$\sigma(R^*) = \sum_{Q \in \mathcal{A}(R^*)} \mu(Q)\, J(Q, R^* \backslash 2\,Q)^2\,. \tag{9.40}$$

Fix R and $R^* \in F(R)$. For each $Q \in \mathcal{A}(R^*)$ set

$$\delta(Q) = d(Q) + \operatorname{dist}\,(Q, R^*) \tag{9.41}$$

and, for notational convenience,

$$J_Q = J(Q, R^* \backslash 2\,Q)\,. \tag{9.42}$$

Our basic estimate for J_Q is

$$\begin{aligned} J_Q &= \int_{R^*\backslash 2Q} \frac{d(Q)\, d\mu(x)}{|x - x(Q)|^2} \\ &\le d(Q) \int_{|x-x(Q)|\ge \delta(Q)/2} \frac{d\mu(x)}{|x - x(Q)|^2} \\ &\le C\, \frac{d(Q)}{\delta(Q)}\,, \end{aligned} \tag{9.43}$$

which follows from (7.10), the fact that

$$\operatorname{dist}\,(x(Q), R^*) \ge \operatorname{dist}\,(x(Q), E\backslash Q) \ge d(Q)$$

(by (3.4) and (3.5)), and (8.1).

Let us first say rapidly how we would estimate $\sigma(R^*)$ if R^* were a good cube. We would first sum over the cubes Q such that $\delta(Q) \sim \delta$ for a given δ, the interesting case being when $d(Q) \le \delta < d(R^*)$. By (9.43), the contribution of Q to the sum would be at most $C\mu(Q)\,(d(Q)/\delta(Q))^2$. Also, the total mass of the cubes Q would be about

$$\Big(\frac{\delta}{d(R^*)}\Big)^{\tau} \xi(R^*) \le C \Big(\frac{\delta}{d(R)}\Big)^{\tau} \mu(R^*)$$

(if R^* is good) because each cube Q lies at distance less than $C\delta$ from R^* but does not meet R^*. Summing over Q would give less than

$$C\Big(\frac{\delta}{d(R)}\Big)^{\tau}\Big(\frac{A^{-k}\,d(R)}{\delta}\Big)^{2}\mu(R^*)\,.$$

We would then sum over δ and get that

$$\sigma(R^*) \le C A^{-k\tau}\,\mu(R^*)$$

(the largest terms are when $\delta \sim A^{-k}\,d(R)$).

In general, R^* is not a good cube and we'll have to localize to maximal good subcubes of R^* and distinguish two cases as usual. For each $Q \in \mathcal{A}(R^*)$, choose a point $z(Q)$ such that

$$z(Q) \in R^* \qquad \text{and} \qquad \operatorname{dist}(z(Q), Q) \le \delta(Q)\,. \tag{9.44}$$

Denote by $\mathcal{A}^+(R^*)$ the set of cubes $Q \in \mathcal{A}(R^*)$ such that

$$\begin{aligned}&z(Q) \text{ is contained in a maximal good}\\ &\text{cube } S_Q \in S(R^*) \text{ and } Q \subset 2\,S_Q\,.\end{aligned} \tag{9.45}$$

Also set $\mathcal{A}^-(R^*) = \mathcal{A}(R^*)\backslash\mathcal{A}^+(R^*)$ and

$$\sigma_\pm(R^*) = \sum_{Q\in\mathcal{A}^\pm(R^*)} \mu(Q)\,J_Q^2\,. \tag{9.46}$$

Let us first estimate $\sigma_+(R^*)$. A trivial estimate for J_Q is

$$\begin{aligned}J_Q &= \int_{R^*\backslash 2Q} \frac{d(Q)\,d\mu(x)}{|x-x(Q)|^2}\\ &\le \mu(R^*)\,d(Q)\operatorname{dist}(R^*, x(Q))^{-2}\\ &\le \mu(R^*)\,d(Q)^{-1}\,.\end{aligned} \tag{9.47}$$

We want to use the following weighted average of (9.47) and (9.43)

$$J_Q^2 \le C\Big(\frac{d(Q)}{\delta(Q)}\Big)^{2-\tau/2}\Big(\frac{\mu(R^*)}{d(Q)}\Big)^{\tau/2} \le C\,\frac{d(Q)}{\delta(Q)}\,\Big(\frac{\mu(R^*)}{d(Q)}\Big)^{\tau/2}. \tag{9.48}$$

For each $S \in S(R^*)$ and $\ell \geq 0$, denote by $B_\ell(S)$ the set of cubes $Q \in \mathcal{A}^+(R^*)$ such that $2^\ell\, d(Q) \leq \delta(Q) < 2^{\ell+1}\, d(Q)$ and $S = S_Q$. Obviously every $Q \in \mathcal{A}^+(R^*)$ lies in some $B_\ell(S)$ and so

$$\sigma_+(R^*) \leq \sum_{S \in S(R^*)} \sum_{\ell \geq 0} \sum_{Q \in B_\ell(S)} \mu(Q)\, J_Q^2\,. \tag{9.49}$$

If $Q \in B_\ell(S)$, then Q does not meet S (by definition (9.38) of $\mathcal{A}(R^*)$) but

$$\operatorname{dist}(Q,S) \leq \operatorname{dist}(Q, z(Q)) \leq \delta(Q) \leq 2^{\ell+1}\, d(Q)\,,$$

by definitions (see in particular (9.44) and (9.45)). Thus $Q \subset N_t(S)$, with

$$t = C\, 2^{\ell+1}\, d(Q)\, d(S)^{-1} = C\, 2^{\ell+1}\, A^{-k}\, d(R)\, d(S)^{-1}\,.$$

Note that ℓ cannot be too large: if $B_\ell(S)$ contains some Q, then $2^\ell\, d(Q) \leq \delta(Q) \leq C\, d(S)$ because $Q \subset 2\,S$ (by (9.45)). In particular, the value of t above is never more than some constant C. Set $t' = \min\{t, 1\}$. Then all cubes $Q \in B_\ell(S)$ still lie in $N_{t'}(S)$ (because $Q \subset 2\,S$ for $Q \in B_\ell(S)$). We may now apply (3.8) and get that

$$\begin{aligned}\sum_{Q \in B_\ell(S)} \mu(Q) &\leq \mu(N_{t'}(S)) \\ &\leq C\,(2^\ell\, A^{-k}\, d(R)\, d(S)^{-1})^\tau\, \xi(S) \\ &\leq C\,(2^\ell\, A^{-k}\, d(R)\, d(S)^{-1})^\tau\, \mu(S)\,,\end{aligned} \tag{9.50}$$

because S is a good cube. Next

$$\begin{aligned}&\sum_{Q \in B_\ell(S)} \mu(Q)\, J_Q^2 \\ &\qquad \leq C\,(2^\ell\, A^{-k}\, d(R)\, d(S)^{-1})^\tau \mu(S)\, 2^{-\ell} \Big(\frac{\mu(R^*)}{A^{-k}\, d(R)}\Big)^{\tau/2}\,,\end{aligned} \tag{9.51}$$

by (9.50), (9.48), and the definition of $B_\ell(S)$. We may now sum over $\ell \geq 0$, noticing that the largest term is for $\ell = 0$, and get less than

$$C A^{-k\tau/2} \Big(\frac{d(R)}{d(S)}\Big)^\tau \Big(\frac{\mu(R^*)}{d(R)}\Big)^{\tau/2} \mu(S)\,.$$

Thus (9.49) becomes

$$\sigma_+(R^*) \leq C A^{-k\tau/2} \Big(\frac{\mu(R^*)}{d(R)}\Big)^{\tau/2} \sum_{S \in S(R^*)} \Big(\frac{d(R)}{d(S)}\Big)^\tau \mu(S)\,. \tag{9.52}$$

By Hölder,

$$
\begin{aligned}
\sum_{S\in S(R^*)} \Big(\frac{d(R)}{d(S)}\Big)^{\tau} \mu(S) &\le \Big(\sum_S \mu(S)\Big)^{1-\tau/2} \Big(\sum_S \Big(\frac{d(R)}{d(S)}\Big)^2 \mu(S)\Big)^{\tau/2} \\
&\le C\mu(R^*)^{1-\tau/2}\, \xi(R^*)^{\tau/2} \qquad (9.53)\\
&\le C\mu(R^*)^{1-\tau/2}\, d(R)^{\tau/2}\,,
\end{aligned}
$$

because the cubes S, $S \in S(R^*)$, are disjoint and contained in R^*, and by Lemma 8.14 and (3.9). Hence

$$\sigma_+(R^*) \le CA^{-k\tau/2}\,\mu(R^*)\,, \tag{9.54}$$

which will be enough for our purposes. Let us now turn to $\sigma_-(R^*)$. First we want to check that

$$J_Q \le CA^{-k}\,, \qquad \text{for all } Q \in \mathcal{A}^-(R^+)\,. \tag{9.55}$$

We start with the easy case when Q is not contained in $2R^*$. If $d(Q) \ge d(R^*)$, then

$$\operatorname{dist}(x(Q), R^*) \ge \operatorname{dist}(x(Q), E\backslash Q) \ge d(Q) \ge d(R^*)\,,$$

by definition (9.38) of $\mathcal{A}(R^*)$, (3.4) and (3.5). Otherwise, $\operatorname{diam} Q < d(R^*)/2$ and, since some point of Q lies at distance $> d(R^*)$ from R^*, $\operatorname{dist}(x(Q), R^*) \ge d(R^*)/2$. In both cases

$$
\begin{aligned}
J_Q &\le \mu(R^*)\, d(Q) \operatorname{dist}(x(Q), R^*)^{-2} \\
&\le 4\,\mu\,(R^*)\, d(Q)\, d(R^*)^{-2} \\
&\le CA^{-k}\,\mu\,(R^*)\, d(R)^{-1} \\
&\le CA^{-k}\,,
\end{aligned}
$$

by (9.42), (7.10) and (3.9).

We still need to check (9.55) when $Q \subset 2R^*$. Let $H_0 = R^* \supset H_1 \supset \cdots \supset H_\ell$ be the decreasing sequence of all cubes $H \subset R^*$ that contain $z(Q)$ (the point of R^* that was chosen in (9.44)) and such that $Q \subset 2H$. Since $Q \subset 2R^*$, there is at least one such cube, and then $d(H_\ell) \le C\delta(Q)$ by minimality of H_ℓ (and (9.41)). Note also that all the cubes H_j, $0 \le j \le \ell$, are bad because $Q \in \mathcal{A}^-(R^*)$ and (9.45) does not hold. Thus (3.9) and (3.12) yield

$$\mu(2H_j) \le C\xi(H_j) \le CA^{-10j}\,\xi(R^*) \le CA^{-10j}\,d(R^*)\,. \tag{9.56}$$

Decompose $R^*\backslash 2\,Q$ into the sets $Z_j = (R^*\backslash 2\,Q) \cap (2H_j\backslash 2H_{j+1})$, $0 \le j \le \ell - 1$, and $Z_\ell = (R^*\backslash 2\,Q) \cap 2H_\ell$. When $0 \le j \le \ell - 2$ and $x \in Z_j$,

$$\begin{aligned} |x - x(Q)| &\ge \operatorname{dist}(x, Q) \\ &\ge \operatorname{dist}(x, 2H_\ell) \\ &\ge \operatorname{dist}(E\backslash 2H_{j+1}, 2H_\ell) \\ &\ge \frac{1}{2}\,d(H_{j+1})\,. \end{aligned} \tag{9.57}$$

Thus, for $0 \le j \le \ell - 2$,

$$J(Q, Z_j) \le 4\,\mu(Z_j)\,d(Q)\,d(H_{j+1})^{-2} \le CA^{-8j}\,A^{-k}\,, \tag{9.58}$$

by (7.10) and (9.56).

When $j = \ell - 1$ or $j = \ell$, we want to use the simple estimate

$$|x - x(Q)| \ge \frac{\delta(Q)}{2}\,, \qquad \text{for } x \in R^*\,, \tag{9.59}$$

which comes from the fact that $|x - x(Q)| \ge \operatorname{dist}(Q, R^*)$ trivially and $|x - x(Q)| \ge \operatorname{dist}(x(Q), E\backslash Q) \ge d(Q)$ by (9.38), (3.4) and (3.5). (See also the definition (9.41).) Thus, for $j = \ell - 1$ and $j = \ell$,

$$J(Q, Z_j) \le 4\,\mu(Z_j)\,d(Q)\,\delta(Q)^{-2} \le CA^{-10\ell}\,A^{-k}\,d(R)^2\,\delta(Q)^{-2}\,. \tag{9.60}$$

Recall that $d(H_\ell) \le C\delta(Q)$, so that

$$A^{-10\ell} = \Big(\frac{d(H_\ell)}{d(R^*)}\Big)^{10} \le C\Big(\frac{\delta(Q)}{d(R)}\Big)^{10}\,.$$

Since we also have that $\delta(Q) \le C\,d(R^*)$ because $Q \subset 2R^*$, (9.60) implies that

$$J(Q, Z_j) \le CA^{-k}\Big(\frac{\delta(Q)}{d(R)}\Big)^{8} \le CA^{-k}\,,$$

when $j = \ell - 1$ or $j = \ell$. Altogether

$$J_Q = J(Q, R^* \backslash 2\,Q) \leq \sum_{j=0}^{\ell} J(Q, Z_j) \leq C A^{-k}\,, \tag{9.61}$$

which completes our proof of (9.55).

A second estimate for J_Q is

$$J_Q \leq \mu(R^*)\, d(Q)\, \mathrm{dist}\,(x(Q), R^*)^{-2} \leq 4\,\mu(R^*)\, d(Q)\, \delta(Q)^{-2}\,, \tag{9.62}$$

which follows directly from the definitions (9.42) and (7.10), and (9.59).

Plug these two estimates into (9.46) to get

$$\sigma_-(R^*) \leq C A^{-k} \sum_{Q \in \mathcal{A}^-(R^*)} \mu(Q)\, \mu(R^*)\, d(Q)\, \delta(Q)^{-2}\,. \tag{9.63}$$

When we sum over the set of cubes Q such that $\delta(Q) \geq A^{-k/2}\, d(R)$, we get less than

$$\begin{aligned} C A^{-k} \Big(\sum_Q \mu(Q) \Big) \mu(R^*)\, A^{-k}\, d(R)\, A^k\, d(R)^{-2} & \\ & \leq C A^{-k}\, \mu(R^*) \Big(\sum_Q \mu(Q) \Big) d(R)^{-1} \\ & \leq C A^{-k}\, \mu(R^*)\,, \end{aligned}$$

by (3.1) or (3.9).

We are left with the cubes Q such that $\delta(Q) \leq A^{-k/2}\, d(R)$. These cubes are contained in $N_t(R^*)$, with $t = \min\{1, C A^{-k/2}\}$ because they are $\delta(Q)$-close to R^* but do not meet it (by (9.38)). By (3.8) and (3.9), their total mass is at most

$$C A^{-k\tau/2}\, \xi(R^*) \leq C A^{-k\tau/2}\, d(R)\,,$$

and so the corresponding piece of $\sigma_-(R^*)$ is at most

$$C A^{-k}\, A^{-k\tau/2}\, d(R)\, \mu(R^*)\, (A^{-k}\, d(R))^{-1} \leq C A^{-k\tau/2}\, \mu(R^*)\,.$$

Altogether, $\sigma_-(R^*) \leq C A^{-k\tau/2}\, \mu(R^*)$. Now

$$\sigma(R^*) = \sigma_+(R^*) + \sigma_-(R^*) \leq C A^{-k\tau/2}\, \mu(R^*)\,, \tag{9.64}$$

by (9.40), (9.46), (9.54) and this last estimate. We may now compare with (9.39) and (9.37) to get that $|||\mathcal{N}_6||| \leq CA^{-k\tau/4}$, as desired.

Case G. *The terms* B_{22}. Finally define $\mathcal{N}_7$ by taking $N_7(Q,R) = B_{22}$ when $Q \subset R$, $Q \neq R$, and $k(Q) = k(R) + k$, and $N_7(Q,R) = 0$ otherwise. This is the last piece of the matrix $\mathcal{N}^k$ that we have to study: recall that $\mathcal{N}$ was defined around (7.36)-(7.39), and that coefficients $A_1(Q,R)$ and $A_2(Q,R)$ were dealt with in subsections B, E and F respectively, while $A_3(Q,R)$ was treated in Subsection A, B_{11} in Subsection C , B_{12} in F and B_{21} in D.

Recall from (7.30) that

$$B_{22} = \mu(Q)^{1/2}\, \mu(R(Q))^{-1/2}\, J(Q, E\backslash(2\,Q \cup R(Q)))\,, \tag{9.65}$$

where $R(Q)$ is the child of R that contains Q. As usual we can apply Lemma 9.1, and

$$|||\mathcal{N}_7|||^2 \leq \sup_{R\in\Delta} \Sigma(R)\,, \tag{9.66}$$

with

$$\Sigma(R) = \sum_{R^*\in F(R)} \sum_{Q\in\mathcal{A}(R^*)} \mu(Q)\, \mu(R^*)^{-1}\, J_Q^2\,, \tag{9.67}$$

where this time we set

$$\mathcal{A}(R^*) = \{Q \in \Delta_{k(R)+k} :\ Q \subset R^*\} \tag{9.68}$$

and

$$J_Q = J(Q, E\backslash(2\,Q \cup R^*))\,. \tag{9.69}$$

Set $\delta(Q) = d(Q) + \mathrm{dist}\,(Q, E\backslash R^*)$ for $Q \in \mathcal{A}(R^*)$. Note that

$$|x - x(Q)| \geq \frac{\delta(Q)}{2}\,, \qquad \text{for } x \in E\backslash(2\,Q \cup R^*)\,, \tag{9.70}$$

because $|x - x(Q)| \geq d(Q)$ on $E\backslash 2\,Q$ and $|x - x(Q)| \geq \mathrm{dist}\,(Q, E\backslash R^*)$ on $E\backslash R^*$. Then

$$J_Q = d(Q) \int_{E\backslash(2Q\cup R^*)} \frac{d\mu(x)}{|x - x(Q)|^2} \leq C\, d(Q)\, \delta(Q)^{-1}\,, \tag{9.71}$$

by (8.1).

When $k \le 10$, say, we can simply say that $J_Q \le C$ by (9.71) and $\Sigma(R) \le C$ by summing brutally. When $k \ge 10$, we expect to win something from (9.71) when $\delta(Q) \gg d(Q)$, and otherwise to use the fact that Q stays close to the "boundary of R^*" to say that $\sum \mu(Q)$ is small. As usual we need to distinguish cases because R^* is not necessarily good.

Fix $R^* \in F(R)$ and first consider

$$\begin{aligned} \mathcal{A}^+ = \Big\{ Q \in \mathcal{A}(R^*) :\ &\text{there is a maximal good cube} \\ &S \in S(R^*) \text{ such that } k(S) \le k(R) + \frac{k}{2} \text{ and } Q \subset S \Big\} . \end{aligned} \tag{9.72}$$

For each $S \in S(R^*)$ and $\ell \ge 0$, set

$$\mathcal{A}_\ell^+(S) = \{ Q \in \mathcal{A}^+ :\ Q \subset S \text{ and } 2^\ell\, d(Q) \le \delta(Q) < 2^{\ell+1}\, d(Q) \} . \tag{9.73}$$

All these cubes lie at distance less than $2^{\ell+1}\, d(Q)$ from $E \backslash R^*$, and so they lie in $N_t(S)$, with $t = C\, 2^\ell\, d(Q)\, d(S)^{-1}$. If we get a $t \ge 1$, simply remember that $Q \in \mathcal{A}_\ell^+(S)$ is always contained in S; otherwise apply (3.8) and the fact that S is a good cube to get that

$$\sum_{Q \in \mathcal{A}_\ell^+(S)} \mu(Q)\, J_Q^2 \le C\, (2^\ell\, A^{-k}\, d(R)\, d(S)^{-1})^\tau\, \mu(S)\, 2^{-2\ell} , \tag{9.74}$$

where the $2^{-2\ell}$ comes from (9.71). When we sum this over $\ell \ge 0$, the largest term is when $\ell = 0$ and we get at most

$$C A^{-k\tau}\, d(R)^\tau\, d(S)^{-\tau}\, \mu(S) \le C A^{-k\tau/2}\, \mu(S) ,$$

because only the maximal good cubes S with $k(S) \le k(R) + k/2$ can give non empty sets $\mathcal{A}_\ell^+(S)$, by (9.72). Since every cube $Q \in \mathcal{A}^+$ lies in some $\mathcal{A}_\ell^+(S)$,

$$\begin{aligned} \sum_{Q \in \mathcal{A}^+} \mu(Q)\, \mu(R^*)^{-1}\, J_Q^2 &\le C A^{-k\tau/2}\, \mu(R^*)^{-1} \sum_{S \in S(R^*)} \mu(S) \\ &\le C A^{-k\tau/2} . \end{aligned} \tag{9.75}$$

Next we want to estimate the contribution of $\mathcal{A}^- = \mathcal{A}(R^*) \backslash \mathcal{A}^+$ to $\sum(R)$ (in (9.67)). Let $Q \in \mathcal{A}^-$ be given, and let $H_0 = R^* \supset H_1 \supset \cdots \supset H_\ell$ be the collection of all subcubes of R^* that contain Q and are of generation less or equal than $k(R)+k/2$. By the definition (9.72) of $\mathcal{A}^+$, all these cubes are bad, and so

$$\mu(2H_j) \le C\xi(H_j) \le CA^{-10j}\,\xi(R^*) \le CA^{-10j}\,d(R)\,, \tag{9.76}$$

by (3.9), (3.12), and (3.9) again. Now

$$J_Q = J(Q, E\backslash(2\,Q \cup R^*)) \le \sum_{j=0}^{\ell+1} J(Q, Z_j)\,, \tag{9.77}$$

where $Z_0 = E\backslash 2R^*$, $Z_j = 2H_{j-1}\backslash 2H_j$ for $1 \le j \le \ell$, and $Z_{\ell+1} = 2H_\ell\backslash 2\,Q$. This comes directly from the definitions (9.69) and (7.10). On Z_j, $0 \le j \le \ell$,

$$|x - x(Q)| \ge \operatorname{dist}(E\backslash 2H_j, Q) \ge d(H_j) = A^{-j}\,d(R^*)\,,$$

because $Q \subset H_j$. Thus, for $1 \le j \le \ell$,

$$J(Q, Z_j) \le d(Q)\,A^{2j}\,d(R^*)^{-2}\,\mu(Z_j) \le CA^{-8j}A^{-k}\,, \tag{9.78}$$

by (9.76). For $j = 0$, we simply have that

$$\begin{aligned} J(Q, Z_0) &= d(Q)\int_{E\backslash 2R^*} \frac{d\mu(x)}{|x - x(Q)|^2} \\ &\le C\,d(Q)\,d(R^*)^{-1} \\ &\le CA^{-k}\,, \end{aligned} \tag{9.79}$$

because $\operatorname{dist}(x(Q), E\backslash 2R^*) \ge \operatorname{dist}(Q, E\backslash 2R^*) \ge d(R^*)$ (since $Q \subset R^*$), and by (8.1). Finally, $|x - x(Q)| \ge d(Q)$ on $Z_{\ell+1}$ and so

$$J(Q, Z_{\ell+1}) \le d(Q)^{-1}\,\mu(2H_\ell) \le CA^k\,A^{-10\ell} \le CA^{-k}\,, \tag{9.80}$$

because H_ℓ is the smallest cube H containing Q and for which $k(H) \le k(R)+k/2$. Summing over ℓ now gives that $J_Q \le CA^{-k}$ for all $Q \in \mathcal{A}^-$, and then

$$\sum_{Q\in\mathcal{A}^-} \mu(Q)\,\mu(R^*)^{-1}\,J_Q^2 \le CA^{-2k}\,, \tag{9.81}$$

because all these cubes are disjoint and contained in R^*. Finally, when we add up the estimates in (9.75) and (9.81) and then sum over $R^* \in$

$F(R)$, we get that $\Sigma(R) \leq CA^{-k\tau/2}$ and $|||\mathcal{N}_7||| \leq CA^{-k\tau/4}$ (see (9.67) and (9.66)).

At this point we may collect all the estimates from the various subsections. We get that

$$|||\mathcal{N}^k||| \leq \sum_j |||\mathcal{N}_j||| \leq CA^{-k\tau/4}$$

and finally

$$|||\mathcal{N}||| \leq \sum_k |||\mathcal{N}^k||| \leq C\,.$$

This completes the proof of Theorem 3.20.

Remark 9.82. We have only used the fact that the ambient dimension is 2 a few times, when we used (8.1) to estimate the number of cubes $Q \in \Delta_{k(R)+k}$ in a ball of radius $Cd(R)$. This estimate was always beaten by a A^{-10k} that came from (3.12). If we had been working in a larger ambient dimension, we would only have needed to replace 10 with a larger constant in (3.12), which is possible. Thus Theorem 3.20 works also for one-dimensional sets $E \subset R^n$, with almost the same proof. The proof most probably also works for different dimensions of E (and corresponding homogeneities of kernel estimates) but we did not check this carefully. The authors of [NTV] did for their version.

10. A short description of [DM].

We want to use Theorem 3.20 to prove our theorem about analytic capacity. So we give ourselves a compact set $E \subset \mathbb{C}$ such that $H^1(E) < +\infty$ and E has positive analytic capacity, and we want to show that E is not totally unrectifiable. As we discussed in the introduction, we can find a bounded measurable function f on E such that $\int f\,d\mu = a > 0$ and the Cauchy integral of $f\,d\mu$ is bounded on $\mathbb{C}\backslash E$. Here μ denotes the restriction of H^1 to E.

Next we want to replace $f\,d\mu$ with a new measure $g\,d\nu$, where g has the advantage of being accretive (*i.e.*, satisfies (2.6)). We shall use the measure ν and the function g constructed in [DM] for purposes similar to those of this paper. These satisfy (1.5)-(1.8), and also a weaker analogue of (1.9), namely, the fact that the maximal Cauchy integral of $g\,d\nu$ lies in $L^2(d\nu)$. To complete the argument outlined in the introduction, we shall have to put ourselves in position to apply Theorem 3.20

to the measure ν, and in particular construct an acceptable collection of dyadic cubes on the support of ν. These cubes will be constructed as modifications of the dyadic cubes on E given by [DM]; see the next section. Once this is done and we are in positition to apply Theorem 3.20 we shall have also to check that truncated Cauchy integrals of $g\,d\nu$ lie in the relevant BMO-space (instead of just L^2) uniformly. This will only be possible after we give a reasonable description of the construction of g and ν, which is the aim of this section. It will be convenient to use references like $(*1.2)$ rather than the longer "[DM, (1.2)]".

We start with our compact set $E \subset \mathbb{C}$, $d\mu = dH^1_{|E}$, and a bounded function f such that $\|f\|_\infty \leq 1$ and $\int f\,d\mu = a > 0$. The construction of ν and g will only use these informations; it will happen that in addition the Cauchy integral of $f\,d\mu$ is bounded on $\mathbb{C}\backslash E$, and then $g\,d\mu$ will also have nice properties with respect to the Cauchy kernel, but we don't need to think about this now.

The first thing we do is construct a collection $\Delta = \bigcup_{k\geq 0} \Delta_k$ of dyadic cubes with the properties listed below. Note that μ is a finite measure, but does not necessarily satisfy (3.1); this will not be a problem. The constants C_1, C_2, A, below are absolute constants; see the discussion below. Let us describe the properties of Δ. First

$$\text{For each } k \geq 0,\ E \text{ is the disjoint union of the Borel sets } Q,\ Q \in \Delta_k\,, \tag{10.1}$$

$$\text{if } k < \ell,\ Q \in \Delta_k \text{ and } R \in \Delta_\ell\,, \text{ then } Q \cap R = \varnothing \text{ or else } R \subset Q\,, \tag{10.2}$$

and for each $k \geq 0$ and each cube $Q \in \Delta_k$, there is a ball $B(Q) = B(x(Q), r(Q))$, centered on E, and such that

$$A^{-k} \leq r(Q) \leq C_1 A^{-k}\,, \tag{10.3}$$

$$E \cap B(Q) \subset Q \subset E \cap 28B(Q)\,, \tag{10.4}$$

and

$$\text{the balls } 5B(Q),\ Q \in \Delta_k, \text{ are disjoint}\,. \tag{10.5}$$

These are the properties $(*3.3)$-$(*3.9)$ in Theorem $*3.2$. It is also easy to arrange that

$$\Delta_0 \text{ has only one element}\,. \tag{10.6}$$

This was assumed in [DM] also (see just after ($*$4.1); the construction gives this automatically if we normalize things by taking $\operatorname{diam} E = 1$. Next there is the story about small boundaries. Set

$$
\begin{aligned}
N_t(Q) = \{x \in E \backslash Q : \ \operatorname{dist}(x, Q) \le t A^{-k(Q)}\} \\
\cup \{x \in Q : \ \operatorname{dist}(x, E \backslash Q) \le t A^{-k(Q)}\},
\end{aligned}
\tag{10.7}
$$

for $Q \in \Delta$ and $0 < t \le 1$, and where $k(Q)$ denotes, as always, the generation of Q. Then

$$\mu(N_t(Q)) \le C_2 \, t^\tau \, \mu(90 B(Q)), \tag{10.8}$$

for all $Q \in \Delta$ and $0 < t \le 1$, and where we can take the constant $\tau > 1$ as close to 1 as we want. Here we shall take $\tau = 9/10$. Furthermore we can decompose Δ into the set of good cubes Q such that

$$\mu(10^4 B(Q)) \le C_1 \, \mu(Q), \tag{10.9}$$

and the set of bad cubes that do not satisfy (10.9) but for which

$$r(Q) = A^{-k(Q)} \tag{10.10}$$

and, more importantly,

$$\mu(10^4 B(Q)) \le A^{-10} \, \mu(10^4 B(\hat{Q})), \tag{10.11}$$

where $\hat{Q}$ denotes the parent of Q. Note that the only cube of Δ_0 is good by definitions, and so $\hat{Q}$ is defined for all bad cubes.

These are not exactly the condition ($*$3.13)-($*$3.16) stated in Theorem $*$3.2. First, there is the difference that we replaced $100 B(Q)$ in ($*$3.16) with $10^4 B(Q)$. This does not cause any harm; it just makes some of the constants larger. The second difference is in the phrasing of the conditions: (10.8)-(10.11) are are slightly different from ($*$3.13)-($*$3.16), even with 10^4 instead of 100, but they are fairly easy to deduce from ($*$3.13)-($*$3.16) by choosing C_1 and A large enough. In fact, this is what was done in [DM], in sections 4 and 5. Theorem $*$3.2 was stated for all choices of C_1 (which is called C_0 there) and A, provided that $C_1 > 1$ and $A > 5000\, C_1$, but then it was decided to take $A = C\, C_1^{100}$ for some absolute constant C (the one that shows up in ($*$3.13)) and then C_1 so large that ($*$3.13) and ($*$3.16) actually imply (10.8) and

(10.11). See (∗4.1) for the choice of A, (∗5.25) and (∗5.26) for a discretized version of (10.8) where $t = A^{-\ell} = (C\,C_1^{100})^{-\ell}$ and we get $\mu\,(N_t(Q)) \leq C_1^{-93\ell}\mu(90B(Q)) \leq C' t^{9/10}\mu(90B(Q))$, and (∗5.30) for (10.11). The two other relations (10.9) and (10.10) are the same as (∗3.14) and (∗3.15).

This completes our discussion of the construction of cubes in [DM]. Note that we get our implicit property that $A \gg C_1$ from earlier sections automatically here (*i.e.*, without having to skip generations artificially).

Once our collection of cubes is chosen, we run a stopping time construction, somewhat like in [Ch2]. We select collections I_1 and LI of cubes $Q \in \Delta$, with the following main properties:

(10.12) the cubes of $I_1 \cup L\,I$ are disjoint (this is (∗4.11)) and,

$$\begin{array}{l}\text{all the cubes } Q \in \Delta \text{ such that } Q \subset \mathcal{O}(M) \text{ or} \\ \operatorname{Re} \displaystyle\int_Q f\,d\mu \leq a_1\,\mu(Q) \text{ are contained in some} \\ \text{cube of } I_1 \cup L\,I\,,\end{array} \tag{10.13}$$

where $\mathcal{O}(M) = \{x \in E : \text{there is an } r > 0 \text{ such that } \mu(B(x,r)) > Mr\}$, and M and a_1 are two positive constants (that may depend wildly on E). This is Remark ∗4.12; see also (∗4.4) and (∗4.5) for the definition of $\mathcal{O}(M)$. Set

$$\begin{aligned}\Delta^0 = \{Q \in \Delta :\ Q \in I_1 \cup L\,I \ \text{ or } Q \text{ is not} \\ \text{contained in any cube of } I_1 \cup L\,I\}\,.\end{aligned} \tag{10.14}$$

These are the cubes which we shall really be working with. A fairly easy consequence of (10.13) (see (∗4.13)) is that

$$\mu(100B(Q)) \leq C A^{-k(Q)}\,, \qquad \text{for all } Q \in \Delta^0\,. \tag{10.15}$$

Denote by $PL\,I$ the set of parents of cubes of $L\,I$. This makes sense because the only cube of Δ_0 happens not to be in $L\,I$ (or I_1 either), by construction. Set $I = I_1 \cup PL\,I$. One puts a suitable order on I; this order is chosen so that cubes of earlier generations come first and, in case of equality, cubes of $I_1 \cap \Delta_k$ come before cubes of $PL\,I \cap \Delta_k$. Call Q_n, $n \geq 1$, the n^{th} cube of I for this order. We construct a sequence of measures F_n, $n \geq 0$, as follows.

All measures F_n are of the type

$$F_n = \rho_n f\,d\mu + \sum_{1 \leq m \leq n} \alpha_m\,d\nu_m \tag{10.16}$$

(see ($*4.15$)), where $\{\rho_n\}$ is a decreasing sequence of nonnegative functions on E, with $0 \le \rho_n \le 1$, the α_m's are bounded complex numbers, and $d\nu_m$ is a finite sum of multiples of restrictions of Hausdorff measure on circles.

We start with $F_0 = f\,d\mu$, $\rho_0 \equiv 1$, and no measure ν_0, and construct the F_n by induction. To go from F_{n-1} to F_n, we distinguish between two cases. When $Q_n \in I_1$, we simply replace Q_n with a circle, as follows. Take $\rho_n = \rho_{n-1}\,\mathbf{1}_{E\setminus Q_n}$ (*i.e.*, kill the part of $\rho_{n-1}\,f\,d\mu$ that lives on Q_n) and choose $\mathcal{C}_n = \mathcal{C}(Q_n)$, where

(10.17) $\mathcal{C}(Q)$ denotes the circle with center $x(Q)$ and radius $\dfrac{r(Q)}{100}$,

and $x(Q)$, $r(Q)$ are as in (10.3)-(10.5). In [DM] we chose a slightly larger radius for $\mathcal{C}(Q)$ (see ($*4.2$)), but this new choice does not make any difference in [DM], and will help us a little bit here. Finally choose

$$d\nu_n = \rho_n^*\,\frac{\mu(Q_n)}{H^1(\mathcal{C}_n)}\,dH^1_{|\mathcal{C}_n}\ ,$$

where ρ_n^* denotes the value of ρ_{n-1} on Q_n, which happens to be constant by construction. Take $\alpha_n = \mu(Q_n)^{-1}\int_{Q_n} f\,d\mu$, so as to get $\int F_n = \int F_{n-1}$.

When $Q_n \in PLI$, the construction is slightly more complicated. We want to remove the children of Q_n that lie in LI and replace them with circles, but we shall also modify the values of $\rho_{n-1}f$ on the rest of Q. Denote by $\mathcal{A}_n$ the set of children of Q that lie in LI and by $\mathcal{A}_n^*$ the set of other children of Q (*i.e.*, those that do not lie in LI). Set $H_n = \bigcup_{Q\in\mathcal{A}_n} Q$, $G_n = \bigcup_{Q\in\mathcal{A}_n^*} Q$, and then

$$\rho_n(x) = \begin{cases} \rho_{n-1}(x)\,, & \text{when } x \in E\setminus Q_n\ ,\\ 0\,, & \text{when } x \in H_n\ ,\\ (1-\theta_n)\,\rho_{n-1}(x)\,, & \text{when } x \in G_n\ ,\end{cases} \tag{10.18}$$

where the number $0 \le \theta_n < 1$ is correctly chosen (see ($*4.28$) and ($*4.32$)). Also set

$$\mathcal{C}_n = \sum_{Q\in\mathcal{A}_n} \mathcal{C}(Q) \tag{10.19}$$

and

$$d\nu_n = \sum_{Q\in\mathcal{A}_n} \rho_n^*\,\frac{\mu(Q)}{H^1(\mathcal{C}(Q))}\,dH^1_{|\mathcal{C}(Q)}\ , \tag{10.20}$$

where ρ_n^* still denotes the constant value of ρ_{n-1} on Q_n. This is slightly different from the choice given in [DM], where $\mathcal{C}_n$ was taken to be only one of the $\mathcal{C}(Q)$, $Q \in \mathcal{A}_n$, chosen at random, and on which we put the total mass of H_n. This modification will make our life a little more pleasant later (when we compare the mass repartitions of μ and ν), but it does not alter the argument in [DM]. The main point, of course, is that we still have the same mass

$$\|\nu_n\| = \rho_n^* \, \mu(H_n) \,. \tag{10.21}$$

To complete the definition of F_n when $Q_n \in PLI$, one also chooses a complex number α_n and sets

$$F_n = F_{n-1} - \mathbf{1}_{H_n} \, \rho_{n-1} \, f \, d\mu - \theta_n \, \mathbf{1}_{G_n} \, \rho_{n-1} \, f \, d\mu + \alpha_n \, d\nu_n \ , \tag{10.22}$$

We don't need to be too precise here about the way the constants α_n and θ_n were chosen. The main constraint was that

$$\int F_n = \int F_{n-1} \ , \tag{10.23}$$

our choices were such that

$$0 \leq \theta_n \leq C \, \frac{\mu(H_n)}{\mu(Q_n)} \tag{10.24}$$

and

$$|\alpha_n| \leq C \tag{10.25}$$

(see ($*$4.33) and ($*$4.38)).

It is a good idea to set $\mathcal{A}_n = \{Q_n\}$, $\mathcal{A}_n^* = \varnothing$ (say, but it does not matter) when $Q_n \in I_1$. With these conventions, we still have the properties (10.18)-(10.22) when $Q_n \in I_1$ (see ($*$4.21)-($*$4.23)).

We may also have to use later the fact that

$$\text{the sets } H_n,\ n \geq 1, \text{ are disjoint}\,, \tag{10.26}$$

which comes from (10.12) and the fact that each H_n is the (disjoint) union of the cubes of $\mathcal{A}_n$. Alternatively, see ($*$4.69) for this statement.

Since $\{\rho_n\}$ is a decreasing sequence of nonnegative functions, it has a limit ρ_∞. Set

$$E_\infty = \{x \in E : \ \rho_\infty(x) > 0\} \,. \tag{10.27}$$

By construction, E_∞ does not meet any cube of $I_1 \cup L\,I$. Then

$$(10.28)\quad \operatorname{dist}(\mathcal{C}(Q), E_\infty) \geq \operatorname{dist}(\mathcal{C}(Q), E\backslash Q) \geq \frac{99}{100}\, r(Q) \geq \frac{99}{100}\, d(Q)\,,$$

for $Q \in I_1 \cup L\,I$, by (10.17), (10.4), and (10.3).

Similarly, if Q and $Q' \in \Delta$ are such that $Q \cap Q' = \varnothing$, (10.4) says that $|x(Q) - x(Q')| \geq \max\{r(Q), r(Q')\}$, and hence

$$(10.29)\qquad \operatorname{dist}(\mathcal{C}(Q), \mathcal{C}(Q')) \geq \frac{98}{100}\, \max\{r(Q), r(Q')\}\,.$$

This is the case in particular when $Q, Q' \in I_1 \cup L\,I$ and $Q \neq Q'$.

The measure that we want to study is

$$(10.30)\qquad d\nu = \rho_\infty\, d\mu + \sum_n d\nu_n\ ,$$

which is obviously finite because μ is, and by (10.21) and (10.26). The function g is given by

$$(10.31)\qquad \begin{cases} g(x) = f(x)\,, & \text{on } E_\infty\ , \\ g(x) = \alpha_n\ , & \text{on } \mathcal{C}_n\ , \end{cases}$$

which does not cause any confusion because all these sets are disjoint by (10.28), (10.29), and (10.30).

The function g turns out to be bounded (by (10.25)) and accretive (which means that it satisfies (2.6)) by construction. This comes from the whole design of the stopping time argument (and in particular (10.13)) and the choice of the coefficients α_n, but we don't need to know precisely how it is proved to understand the rest of the present paper. See ($*$2.6) and its proof before Lemma $*$4.56 for details.

Our next task is to define a collection of cubes $\tilde{\Delta}$ on the support of ν, and then prove a $T(b)$-theorem for ν and these cubes. This is the aim of the two next sections.

11. Dyadic cubes for ν and ν^+.

The following measure ν^+ will be slightly easier to handle than ν. Set

$$(11.1)\qquad d\nu^+ = \mathbf{1}_{E_\infty}\, d\mu + \sum_n d\nu_n^+\ ,$$

where

$$dv_n^+ = (\rho_n^*)^{-1}\, d\nu_n = \sum_{Q\in\mathcal{A}_n} \frac{\mu(Q)}{H^1(\mathcal{C}(Q))}\, dH^1_{|\mathcal{C}(Q)}\,. \tag{11.2}$$

Obviously $\nu \leq \nu^+$, and ν^+ is still a finite measure because μ is finite, and by (10.21) and (10.26). Set

$$\tilde{E} = E_\infty \cup \Big(\bigcup_{n\geq 1} \mathcal{C}_n\Big) = E_\infty \cup \Big(\bigcup_{Q\in I_1\cup LI} \mathcal{C}(Q)\Big)\,. \tag{11.3}$$

This is not quite the support of ν^+, because $\operatorname{supp}\nu^+$ is closed, but on the other hand

$$\nu^+(\mathbb{C}\backslash\tilde{E}) = 0\,, \tag{11.4}$$

which will be enough for our purposes.

In this section we want to construct families $\tilde{\Delta}_k$ of partitions of $\tilde{E}$ and check that they satisfy the conditions (3.1)-(3.12) required for Theorem 3.20, with respect to the measure ν^+. Let us start with the construction of cubes.

For each cube $Q\in\Delta^0$ (see (10.14) for the definition), set

$$R(Q) = (Q\cap E_\infty) \cup \Big(\bigcup_{\substack{S\in I_1\cup LI\\ S\subset Q}} \mathcal{C}(S)\Big)\,. \tag{11.5}$$

Our first collection of cubes for ν^+ is $\tilde{\Delta}^0 = \{R(Q) : Q\in\Delta^0\}$, which we naturally split into the $\tilde{\Delta}^0_k = \{R(Q) : Q\in\Delta^0\cap\Delta_k\}$, $k\geq 0$. We need to complete $\tilde{\Delta}^0$ with cubes that come from decomposing the circles $\mathcal{C}(Q)$, $Q\in I_1\cup L\,I$.

For each cube $Q\in I_1\cup LI$ we construct a collection $\tilde{\Delta}(Q)$ of subsets of $\mathcal{C}(Q)$ as follows. We start at generation $k(Q)+1$; we cut $\mathcal{C}(Q)$ into (disjoint) arcs of circle of equal length ℓ_1, with $10A^{-k(Q)-1}\leq \ell_1\leq 20A^{-k(Q)-1}$, say, and call $\tilde{\Delta}_{k(Q)+1}(Q)$ the collection of these arcs of circle. Then we subdivide further each arc $R\in\tilde{\Delta}_{k(Q)+1}(Q)$ into smaller arcs of circle of equal length $\ell_2\in[10A^{-k(Q)-2}, 20A^{-k(Q)-2}]$, and call $\tilde{\Delta}_{k(Q)+2}(Q)$ the resulting collection of arcs of $\mathcal{C}(Q)$. We continue like this, and eventually construct a collection $\tilde{\Delta}_k(Q)$ of (disjoint) subarcs of $\mathcal{C}(Q)$ for all $k>k(Q)$, and with the usual properties of dyadic cubes. Finally set $\tilde{\Delta}(Q) = \bigcup_{k>k(Q)}\tilde{\Delta}_k(Q)$.

Our collection of cubes for ν^+ (and ν) is

$$\tilde{\Delta} = \tilde{\Delta}^0 \cup \Big(\bigcup_{Q \in I_1 \cup LI} \tilde{\Delta}(Q) \Big), \tag{11.6}$$

which we can decompose into the

$$\tilde{\Delta}_k = \tilde{\Delta}_k^0 \cup \Big(\bigcup_{\substack{Q \in I_1 \cup LI \\ k(Q) < k}} \tilde{\Delta}_k(Q) \Big). \tag{11.7}$$

First we want to check that $\tilde{\Delta}$ has the combinatorial properties (3.2) and (3.3). We start with the first one:

$$\begin{aligned} &\text{for each } k \geq 0,\ \tilde{E} \text{ is the disjoint} \\ &\text{union of the cubes } R,\ R \in \tilde{\Delta}_k\,. \end{aligned} \tag{11.8}$$

Fix $k \geq 0$. Because E_∞ does not meet the cubes of $I_1 \cup L\,I$ (see after (10.27)), it does not meet the cubes of $\Delta \backslash \Delta^0$ either (by definition (10.14)), and then (11.5) says that E_∞ is the disjoint union of the $E_\infty \cap R(Q)$, $Q \in \Delta_k^0$. So we are left with the circles $\mathcal{C}(S)$, $S \in I_1 \cup LI$. If $S \in I_1 \cup LI$ and $k(S) \geq k$, then there is exactly one cube $Q \in \Delta_k^0$ that contains S, and $\mathcal{C}(S)$ is contained in $R(Q)$ by (11.5). Moreover $\mathcal{C}(S)$ does not meet any other $R(Q')$, $Q' \in \Delta_k^0$, and it does not meet any of the circles $\mathcal{C}(Q'')$, $Q'' \in I_1 \cup LI$ and $k(Q'') < k$ (and even less the corresponding cubes of $\tilde{\Delta}_k(Q'')$). Thus the cubes of $\tilde{\Delta}_k$ partition $\mathcal{C}(S)$. If $k(S) < k$, then $\mathcal{C}(S)$ does not meet any of the $R(Q)$, $Q \in \Delta_k^0$, because all the circles contained in those circles come from cubes Q' with $k(Q') \geq k > k(S)$. It does not meet the $\tilde{\Delta}_k(S')$, $S' \neq S$, either, and it is nicely covered by the cubes of $\tilde{\Delta}_k(S)$. This completes our proof of (11.8).

Next we check (3.3). Let $R_1 \in \tilde{\Delta}_k$ and $R_2 \in \tilde{\Delta}_{k+1}$ be given, and suppose that $R_1 \cap R_2 \neq \varnothing$. If $R_1 \cap R_2 \cap E_\infty \neq \varnothing$, then $R_1 = R(Q_1)$ and $R_2 \in R(Q_2)$ for cubes $Q_1 \in \Delta_k$ and $Q_2 \in \Delta_{k+1}$, and (11.5) says that $Q_1 \cap Q_2 \supset R_1 \cap R_2 \cap E_\infty \neq \varnothing$. Then $Q_2 \subset Q_1$ and $R_2 \subset R_1$. If $R_1 \cap R_2 \cap E_\infty = \varnothing$, then $R_1 \cap R_2 \cap \mathcal{C}(S) \neq 1$ for some $S \in I_1 \cup LI$. If $k > k(S)$, then $R_1, R_2 \subset \tilde{\Delta}(S)$ and $R_2 \subset R_1$ by construction of $\tilde{\Delta}(S)$. If $k = k(S)$, then $R_1 = R(S)$ and $R_2 \in \tilde{\Delta}(S)$, whence $R_2 \subset R_1$. Finally, if $k < k(S)$, then $R_1 = R(Q_1)$ and $R_2 = R(Q_2)$ for cubes $Q_1, Q_2 \in \Delta^0$ that both contain S. In this case also $Q_2 \subset Q_1$ and $R_2 \subset R_1$. This proves (3.3) when $\ell = k + 1$; the general case follows because of (11.8).

Next we want to consider properties of our cubes that involve the measures ν and ν^+. We start with the upper bound for density (3.1)

$$\nu^+(B(x,r)) \leq C\,r\,, \qquad \text{for all } x \in \mathbb{C} \text{ and } r > 0\,. \tag{11.9}$$

This is proved in [DM], beginning of Section 4.2; unfortunately the statement $(*2.5)$ only mentions ν and not ν^+, but the proof applies to ν^+. (The only difference between ν and ν^+ comes from the size of the functions ρ_n, and the only information used in the proof of $(*2.5)$ in this respect is that $0 \leq \rho_n \leq 1$.)

We also want to relate the measures of our cubes for μ, ν^+, and ν, and to this effect we define numbers ρ_Q, $Q \in \Delta^0$, by

$$\rho_Q = \prod_{\substack{n \geq 1 : Q_n \in PLI \\ \text{and } Q \subset G_n}} (1 - \theta_n)\,. \tag{11.10}$$

Recall from (10.12) and (10.14) that if $Q \in \Delta^0$, Q is never strictly contained in a cube of $I_1 \cup LI$. Let n_0 denote the largest integer for which $k(Q_{n_0}) < k(Q)$. By construction, the function ρ_{n_0} is constant on Q, and in fact the only times ρ_n has possibly been modified on Q for $n \leq n_0$ where when $Q_n \in PLI$ and $Q \subset Q_n$ (and hence $Q \subset G_n$). Because of this, the constant value of ρ_{n_0} on Q is precisely ρ_Q (see (10.18)).

If furthermore $Q \in I_1$ and m is the integer such that $Q = Q_m$, then $\rho_{m-1} = \rho_{n_0}$ on Q because the cubes Q_ℓ, $n_0 < \ell < m$, do not meet Q. (All these cubes lie in I_1, by definition of our order.) Thus

$$\rho_{Q_m} = \rho_m^*\,, \qquad \text{when } Q_m \in I_1\,, \tag{11.11}$$

where ρ_m^* still denotes the constant value of ρ_{m-1} on Q_m.

If $Q \in LI$ and $m \geq 1$ is such that $Q \in \mathcal{A}_m$ (*i.e.*, the parent of Q is Q_m), then ρ_m is equal to ρ_{n_0} on Q, because none of the cubes Q_ℓ, $m < \ell < n_0$ meet Q_m. Thus

$$\rho_Q = \rho_m^*\,, \qquad \text{when } Q \in \mathcal{A}_m\,. \tag{11.12}$$

(We just proved this when $Q_m \in PLI$, but (11.11) says that this is also true when $Q_m \in I_1$.)

Lemma 11.13. *For all $Q \in \Delta^0$,*

$$\nu(R(Q)) \le \rho_Q \, \nu^+(R(Q)) \le \rho_Q \, \mu(Q) \le C \, \nu(R(Q)) \, . \tag{11.14}$$

We start with the first inequality. Let us even prove that for all $Q \in \Delta^0$,

$$d\nu \le \rho_Q \, d\nu^+ \, , \qquad \text{on } R(Q) \, . \tag{11.15}$$

Recall that ρ_Q is the constant value on Q of ρ_{n_0}, where n_0 denotes the largest integer such that $k(Q_{n_0}) < k(Q)$. Obviously $\rho_\infty \le \rho_{n_0} = \rho_Q$ on Q, and hence $\rho_\infty d\mu \le \rho_Q \, \mathbf{1}_{E_\infty} \, d\mu$ on $E_\infty \cap Q = E_\infty \cap R(Q)$. Thus $d\nu \le \rho_Q \, d\nu^+$ on $E_\infty \cap R(Q)$ (see the definitions (10.30) and (11.1) of ν and ν^+). Now let $\mathcal{C}(S)$ be one of the circles that compose $R(Q)$, as in (11.5). Let n denote the integer such that $S \in \mathcal{A}_n$. Then $d\nu = d\nu_n = \rho_n^* \, d\nu_n^+ = \rho_n^* \, d\nu^+$ on $\mathcal{C}(S)$, by (11.2). Since $\rho_n^* = \rho_S$ by (11.12), $S \subset Q$ by (11.5), and ρ_Q is obviously a nondecreasing function of Q, we get that $\rho_n^* \le \rho_Q$ and $d\nu \le \rho_Q \, d\nu^+$ on $\mathcal{C}(S)$. This proves (11.15).

The second inequality in (11.14) is fairly straightforward

$$\begin{aligned} \nu^+(R(Q)) &= \mu(Q \cap E_\infty) + \sum_{\substack{S \in I_1 \cup LI \\ S \subset Q}} \nu^+(\mathcal{C}(S)) \\ &= \mu(Q \cap E_\infty) + \sum_{\substack{S \in I_1 \cup LI \\ S \subset Q}} \mu(S) \\ &\le \mu(Q) \, , \end{aligned} \tag{11.16}$$

by (11.5), (11.2), (10.12), and the fact that E_∞ does not meet the cubes of $I_1 \cup L\, I$.

To prove the last inequality, we want to use the fact that the integral of $g \, d\nu$ on Q is not too small. Let us first check that

$$\operatorname{Re} \int_Q f \, d\mu > a_0 \, \mu(Q) \, , \qquad \text{for all } Q \in \Delta^0 \backslash L\, I \, , \tag{11.17}$$

where $a_0 < a_1$ is some positive constant (the same one as in [DM].) When $Q \in \Delta^0 \backslash (I_1 \cup L\, I)$, this follows directly from (10.13), the definition (10.14) of Δ^0, and the fact that $a_0 < a_1$. When $Q \in I_1$, Q is not contained strictly in any cube of $HD \cup MI$ (see the definition of I_1 in

[DM], just above (∗4.11)), because it is a maximal cube of $HD \cup MI$. Also, Q is not contained in any cube of LI (by (10.12), or the definition of I_1). Since LI is (by definition) the set of maximal cubes with the properties that

$$Q \text{ is not strictly contained in any cube of } HD \cup MI \tag{11.18}$$

and that (11.17) does not hold (see (∗4.8) and (∗4.9)), and since we know already that Q satisfies (11.18), we get that it satisfies (11.17), as promised.

Let $Q \in \Delta^0 \backslash LI$ be given, and again denote by n_0 the largest integer such that $k(Q_{n_0}) < k(Q)$. Observe that Q does not meet any of the $\mathcal{C}_n$, $n \leq n_0$; otherwise Q would meet a cube of $\mathcal{A}_n$, thus would be contained in this cube (because $k(Q_n) < k(Q)$), and even would be strictly contained in it (because $k(Q_n) < k(Q)$ if $Q_n \in I_1$ and because $Q \notin LI$ if $Q_n \in PLI$), a contradiction with the definition of Δ^0. Then

$$\int_Q F_{n_0} = \int_Q \rho_{n_0} f \, d\mu = \rho_Q \int_Q f \, d\mu \,, \tag{11.19}$$

by (10.16) and the discussion after (11.10).

Next we claim that

$$\int_{R(Q)} g \, d\nu = \int_Q F_{n_0} \,, \tag{11.20}$$

i.e., the further modifications of F_n, $n \geq n_0$, do not change the integral of F_n on (what becomes of) Q. This will follow from the fact that

$$\int_Q \rho_n f \, d\mu + \sum_{\substack{1 \leq m \leq n \\ Q_m \subset Q}} \alpha_m \, \|\nu_m\| = \int_Q F_{n_0} \,, \tag{11.21}$$

for all $n \geq n_0$ by taking limits and comparing with (11.5). (The union of the $\mathcal{C}_m$, $Q_m \subset Q$, is the same as the union of the $\mathcal{C}(S)$, $S \in I_1 \cup LI$ and $S \subset Q$, because $Q \notin LI$.) The relation (11.21) is easily proved by induction. It holds for n_0 because no Q_m, $m \leq n_0$, can be contained in Q (they are all of strictly earlier generations). If (11.21) holds for $n-1$, $n > n_0$, and if Q_n does not meet Q, then (11.21) also holds for n because the left-hand side is not modified. Otherwise, $Q_n \subset Q$ (because $k(Q_n) \geq k(Q)$), and all the modifications of the integral of F_{n-1} affect the left-hand side of (11.21). Since the sum of these modifications is zero by (10.23) (or by construction), (11.21) for n follows from (11.21) for $n-1$.

From (11.17), (11.19) and (11.20) we deduce that

$$
\begin{aligned}
a_0\,\rho_Q\,\mu(Q) &\le \rho_Q \operatorname{Re}\int_Q f\,d\mu \\
&\le \rho_Q \Big|\int_Q f\,d\mu\Big| \\
&\le \Big|\int_Q F_{n_0}\Big| \\
&= \Big|\int_{R(Q)} g\,d\nu\Big| \\
&\le C\,\nu(R(Q))\,,
\end{aligned}
\tag{11.22}
$$

because g is bounded (by (10.31) and (10.25)). This proves the last inequality in (11.14) when $Q \in \Delta^0 \backslash L\,I$.

When $Q \in L\,I$, $R(Q) = \mathcal{C}(Q)$ and $\nu(R(Q)) = \nu(\mathcal{C}(Q)) = \rho_n^*\,\mu(Q)$, where n is such that $Q \in \mathcal{A}_n$ and ρ_n^* is as in (10.20). Thus (11.12) says that $\nu(R(Q)) = \rho_Q\,\mu(Q)$, and (11.14) holds in this case as well. Lemma 11.13 follows.

Note that (11.14) implies that $\nu(R) > 0$ for all $R \in \tilde{\Delta}^0$, because $\mu(Q) > 0$ for all $Q \in \Delta$. (Recall that Q is centered on $E = \operatorname{supp}\mu$.) Thus $\nu(R) > 0$ for all $R \in \tilde{\Delta}$, and so $\tilde{E} \subset \operatorname{supp}\nu \subset \operatorname{supp}\nu^+$. (We shall see soon that $\operatorname{diam} R \le C A^{-k}$ for $R \in \tilde{\Delta}_k$.) As was observed in Remark 3.27, this and (11.4) are just as good, in view of Theorem 3.20, as knowning that $\tilde{E} = \operatorname{supp}\nu$ or $\tilde{E} = \operatorname{supp}\nu^+$.

We want to continue checking that ν^+, $\tilde{E}$, and $\tilde{\Delta}$ satisfy the hypotheses for Theorem 3.20. We already know that (3.1)-(3.3) hold, and the next verification in our list is the story about the balls $B(Q)$.

Thus we want to define a center $x(R)$ and a radius $r(R)$ for every $R \in \tilde{\Delta}$. We start with the case when $R \in \tilde{\Delta}^0$ and $R = R(Q)$ for some $Q \in \Delta^0$. First,

$$
\operatorname{dist}(x(Q), R) \le \frac{r(Q)}{100}\,. \tag{11.23}
$$

Indeed, if $x(Q)$ does not lie in E_∞, there are only two possibilities. The first one is that $x(Q) \in Q'$ for some $Q' \in I_1 \cup L\,I$ which is contained in Q. If $Q' = Q$, then (11.23) holds because $R = \mathcal{C}(Q)$. If Q' is strictly contained in Q (*i.e.*, of a strictly later generation), then

$$\operatorname{dist}(x(Q), R) \le \operatorname{dist}(x(Q), \mathcal{C}(Q')) \le 60\, r(Q') \le \frac{r(Q)}{100}\,.$$

The second possibility is that $\rho_n(x(Q))$ tends to 0 without ever being equal to 0. Indeed, $0 \le \theta_n < 1$ for all n, and hence (10.18) says that the only places where ρ_n becomes 0 are the H_n's, *i.e.*, the cubes of $I_1 \cup L\, I$. In this second case $x(Q)$ lies in infinitely many cubes $Q_n \in PL\, I$, and $\operatorname{dist}(x(Q), R) = 0$. Thus (11.23) holds in all cases.

Let us also check that

$$\text{every point of } R = R(Q) \text{ lies at distance less or equal than } \frac{r(Q)}{100} \text{ from } Q\,. \tag{11.24}$$

Of course there is nothing to check for points of $Q \cap E_\infty$; thus we are left with points of the circles $\mathcal{C}(S)$, $S \in I_1 \cup L\, I$ and $S \subset Q$ (see (11.5)). These points are within $r(S)/100$ of some center $x(S) \in Q$, by definition of $\mathcal{C}(S)$; (11.24) follows because $r(S) \le r(Q)$ when $S \subset Q$.

Let us choose a center $x(R) \in R$ at distance at most $r(Q)/100$ from $x(Q)$ and take $r(R) = r(Q)$. Then (3.4) is the same as (10.3), and

$$R \subset \tilde{E} \cap B(x(R), 29\, r(R))\,, \tag{11.25}$$

by (10.4) and (11.24). Let us also verify that

$$\tilde{E} \cap B\Big(x(R), \frac{98\, r(R)}{100}\Big) \subset R\,. \tag{11.26}$$

Let $x \in \tilde{E} \cap B(x(R), 98\, r(R)/100)$ be given. If $x \in E_\infty$, then $x \in Q$ by (10.4), and hence $x \in R$. Otherwise $x \in \mathcal{C}(S)$ for some $S \in I_1 \cup L\, I$. If $S \subset Q$ we are happy because then $\mathcal{C}(S) \subset R$ by (11.5). So let us assume this is not the case. Then $S \cap Q = \varnothing$, because Q cannot be strictly contained in S (since $Q \in \Delta^0$). We know that

$$\operatorname{dist}(x, Q) \le |x - x(Q)| < \frac{99}{100}\, r(Q)\,,$$

but on the other hand (10.28) says that

$$\operatorname{dist}(x, Q) \ge \operatorname{dist}(\mathcal{C}(S), Q) \ge \operatorname{dist}(\mathcal{C}(S), E \backslash S) \ge \frac{99}{100}\, r(S)\,,$$

and so $r(S) < r(Q)$. Then

$$\operatorname{dist}(x, S) \leq \frac{r(S)}{100} < \frac{r(Q)}{100} ,$$

and there are points of S at distance less than $|x - x(Q)| + r(Q)/100 \leq r(Q)$ from $x(Q)$. This is impossible because of (10.4), and (11.26) follows. (Note that the argument did not need to be as tight as it looks, because in the dangerous case where $r(S) \sim r(Q)$, we could use (10.5) to get a somewhat more brutal contradiction.)

Our estimates (11.25) and (11.26) are not quite the same as (3.5), because of the factor 98/100, but they are just as good for the proof of Theorem 3.20. We could also have decided to take $r(R) = 98\, r(Q)/100$; then we would have obtained (3.5), but only

$$\frac{98}{100}\, A^{-k} \leq r(Q) \leq C_1\, A^{-k}$$

instead of (3.4). This difference is even more obviously harmless (just dilate E.)

We still need to define $x(R)$ and $r(R)$ when $R \in \tilde{\Delta}\backslash\tilde{\Delta}^0$, *i.e.*, when $R \in \tilde{\Delta}_k(Q)$ for some $Q \in I_1 \cup L\, I$ and some $k > k(Q)$. In this case R is a small arc of the circle $\mathcal{C}(Q)$, with length $\ell \in [10\, A^{-k}, 20\, A^{-k}]$. We choose for $x(R)$ the center of this arc and take $r(Q) = A^{-k}$. Then (11.25) and (11.26) (and even the analogue of (3.5)) hold for R because $k > k(Q)$ and

$$(11.27) \qquad \operatorname{dist}(\mathcal{C}(Q), \tilde{E}\backslash\mathcal{C}(Q)) \geq \frac{98}{100}\, r(Q)\,, \qquad \text{for all } Q \in I_1 \cup L\, I\,,$$

by (10.28) and (10.29).

This completes our discussion of (3.4) and (3.5). Since (3.6) is the same as (10.6), we are left with the story about small boundaries. We first need to define numbers $\xi(R)$, $R \in \tilde{\Delta}$.

When $R \in \tilde{\Delta}(Q)$ for some $Q \in I_1 \cup LI$, simply take $\xi(R) = \nu^+(R)$. When $R \in \tilde{\Delta}^0$, set $\xi(R) = \mu(10^4\, B(Q))$, where $Q \in \Delta^0$ is such that $R = R(Q)$. Let us first check the auxiliary conditions (3.9)-(3.12), and then we shall return to (3.8).

When $R \in \tilde{\Delta}(Q)$, (11.27) and the fact that $k(R) > k(Q)$ imply that $\tilde{E} \cap 91B(R) = \mathcal{C}(Q) \cap 91B(R)$. The property (3.9) for R and the measure ν^+ follows from the fact that ν^+ is a bounded constant times Hausdorff measure on $\mathcal{C}(Q)$ (by (11.2) and (10.15)); (3.10) for R follows because in addition $\xi(S) = \nu^+(S)$ for all the cubes $S \in \tilde{\Delta}(Q)$. All cubes of $\tilde{\Delta}(Q)$ are good for ν^+ (*i.e.*, satisfy (3.11) for ν^+), and hence we don't need to check (3.12) for them.

Now consider $R \in \tilde{\Delta}^0$, and let $Q \in \Delta^0$ be such that $R = R(Q)$. Recall that we chose $r(R) = r(Q)$ and $x(R)$ at distance less or equal than $r(Q)/100$ from $x(Q)$. (See above (11.25)). Thus $91B(R) \subset 92B(Q)$.

Let $\mathcal{A}$ denote the set of cubes $S \in I_1 \cup LI$ such that $\mathcal{C}(S)$ meets $91B(R)$. Then

$$\begin{aligned}\nu^+(91B(R)) &\le \nu^+(E_\infty \cap 91B(R)) + \sum_{S\in\mathcal{A}} \nu^+(\mathcal{C}(S))\\ &\le \mu(E_\infty \cap 91B(R)) + \sum_{S\in\mathcal{A}} \mu(S)\\ &\le \xi(R) + \sum_{S\in\mathcal{A}} \mu(S)\,,\end{aligned}\tag{11.28}$$

by (11.3), (11.2), the facts that $\nu^+ \le \mu$ on E_∞ and $91B(R) \subset 92B(Q)$, and the definition of $\xi(R)$. If $S \in \mathcal{A}$ and S is not contained in Q, then $S \cap Q = \varnothing$ because Q cannot be strictly contained in S, since $Q \in \Delta^0$. Then (10.28) says that

$$r(S) \le \frac{100}{99}\,\mathrm{dist}\,(\mathcal{C}(S), E\backslash S) \le \frac{100}{99}\,\mathrm{dist}\,(\mathcal{C}(S), x(Q)) \le 100\, r(Q)\,.$$

Then (10.4) says that $S \subset 10^4B(Q)$. Hence

$$\sum_{S\in\mathcal{A}} \mu(S) \le \mu(10^4B(Q)) = \xi(R)$$

and (3.9) follows from (11.28) and (10.15).

Now fix $k > k(R) = k(Q)$, and denote by $\mathcal{B}_k$ the set of cubes $T \in \Delta^0_k$ such that $R(T) \subset 91B(R)$. If $T \in \mathcal{B}_k$, $T \subset 93B(Q)$, by crude estimates on diam $(T \cup R(T))$ and the fact that $k > k(Q)$. Then

$$\begin{aligned}\sum_{T\in\mathcal{B}_k} \xi(R(T)) &= \sum_T \mu(10^4B(T))\\ &\le C\mu\Big(\bigcup_T (10^4B(T))\Big)\\ &\le C\,\xi(R)\,,\end{aligned}\tag{11.29}$$

because the $10^4B(T)$, $T \in \Delta_k$, have bounded overlap and are contained in $10^4B(Q)$. This takes care of the cubes of $\tilde{\Delta}^0$ in the sum in (3.10). Now let $\mathcal{D}_k$ be the set of cubes $T \in \tilde{\Delta}_k \backslash \tilde{\Delta}^0$ that are contained in $91B(R)$. All these cubes lie in sets $\tilde{\Delta}(S)$ for cubes $S \in I_1 \cup LI$ such that $\mathcal{C}(S)$ meets $91B(R)$. Hence

$$(11.30) \qquad \sum_{T \in \mathcal{D}_k} \xi(T) = \sum_T \nu^+(T) \leq \nu^+\Big(\bigcup_{S \in \mathcal{A}} \mathcal{C}(S)\Big) = \sum_{S \in \mathcal{A}} \mu(S) \leq \xi(R)\,,$$

because the cubes $T \in \mathcal{D}_k$ are disjoint, and by the discussion above. This completes the verification of (3.10) for $R \in \tilde{\Delta}^0$.

Finally (3.11)-(3.12) follows easily from its counterpart (10.9)-(10.11) if $C_0 \geq C_1$, and also the only cube of $\tilde{\Delta}_0$ is good for ν^+ and (3.11) because the only cube of Δ_0 is good for (10.9) or (∗3.14).

We still need to check (3.8) for cubes of $\tilde{\Delta}$. For cubes $R \in \tilde{\Delta}(Q)$, $Q \in I_1 \cup L\,I$, this follows from the fact that $N_t(R) \subset \mathcal{C}(Q)$, by (11.27), and the simple structure of the cubes of $\tilde{\Delta}(Q)$.

Now let $R \in \tilde{\Delta}^0$ be given, and let $Q \in \Delta^0$ be such that $R = R(Q)$. Also set $k = k(R) = k(Q)$ and

$$(11.31) \qquad \begin{aligned} N_t = \{x \in R :\ \operatorname{dist}(x, \tilde{E}\backslash R) \leq t\,A^{-k}\} \\ \cup\, \{x \in \tilde{E}\backslash R :\ \operatorname{dist}(x, R) \leq t\,A^{-k}\}\,, \end{aligned}$$

for $0 \leq t \leq 1$. This is the set that we need to control for (3.8). Still denote by $N_t(Q)$ the set in (10.7); we want to use (10.8) to control the sets N_t. Note that because of (3.9), it is enough to prove that

$$(11.32) \qquad \nu^+(N_t) \leq C\,t^\tau\,\xi(Q) = C\,t^\tau\,\mu(10^4 B(Q))\,,$$

for $0 < t \leq 10^{-2}$, say.

So let $0 < t \leq 10^{-2}$, $y \in R \cap N_t$, and $z \in N_t\backslash R$ be given, with $|y-z| < 2\,t\,A^{-k}$. Note that for each $y \in R \cap N_t$ there is a z like this, and for each $z \in N_t\backslash R$ there is an y like this. Let us distinguish a few cases.

If y and z both lie in E_∞, then $y \in Q$ and $z \in E\backslash Q$, and so y and z both lie in $N_{2t}(Q)$.

Next consider the case when $z \in E_\infty$ (and hence $z \in E\backslash Q$) and $y \in R\backslash E_\infty$. Then (11.5) says that $y \in \mathcal{C}(S)$ for some $S \in I_1 \cup L\,I$ such that $S \subset Q$, and

$$(11.33) \qquad 2\,t\,A^{-k} \geq |y-z| \geq \operatorname{dist}(\mathcal{C}(S), E\backslash Q) \geq \frac{99}{100}\,r(S)\,,$$

by (10.28). The center $x(S)$ of S lies in $S \subset Q$, while $z \in E \backslash Q$; since

$$|x(S) - y| + |y - z| \leq \frac{r(S)}{100} + 2\,t\,A^{-k} < 3\,t\,A^{-k}\,,$$

we get that z and $x(S)$ lie in $N_{3t}(Q)$. Using (11.33) again and (10.4), we deduce from this that the whole cube S lies in $N_{100t}(Q)$.

Our next case is when $y \in R \cap E_\infty = Q \cap E_\infty$ and $z \in (\tilde{E} \backslash R) \backslash E_\infty$. Then (11.3) says that $z \in \mathcal{C}(S)$ for some $S \in I_1 \cup LI$, and (11.5) even adds that S is not contained in Q. Moreover $S \cap Q = \varnothing$, because Q cannot be strictly contained in S (since $Q \in \Delta^0$). This time

$$2\,t\,A^{-k} > |y - z| > \operatorname{dist}(\mathcal{C}(S), Q) \geq \operatorname{dist}(\mathcal{C}(S), E \backslash S) \geq \frac{99}{100}\, r(S)\,, \tag{11.34}$$

by (10.28), and

$$|x(S) - y| \leq |x(S) - z| + |z - y| \leq \frac{r(S)}{100} + 2\,t\,A^{-k} < 3\,t\,A^{-k}\,.$$

Since $y \in Q$ and $x(S) \in S \subset E \backslash Q$, we get that $y \in N_{3t}(Q)$, $x(S) \in N_{3t}(Q)$, and (by (10.4) and (11.34)) the whole S lies in $N_{100t}(Q)$.

Our last case is when y and z lie in $\tilde{E} \backslash E_\infty$. Then $y \in \mathcal{C}(S)$ for some $S \in I_1 \cup LI$ such that $S \subset Q$, and z lies in $\mathcal{C}(T)$ for some $T \in I_1 \cup LI$ such that $T \cap Q = \varnothing$. Then

$$2\,t\,A^{-k} > |y - z| \geq \operatorname{dist}(\mathcal{C}(S), \mathcal{C}(T)) \geq \frac{98}{100} \max\{r(S), r(T)\}\,, \tag{11.35}$$

by (10.29). Since $x(S) \in S \subset Q$ and $x(T) \in T \subset E \backslash Q$, and

$$|x(S) - x(T)| \leq |y - z| + \frac{r(S)}{100} + \frac{r(T)}{100} < 3\,t\,A^{-k}\,,$$

we get that $x(S), x(T) \in N_{3t}(Q)$, and then that S and T are contained in $N_{100t}(Q)$ (by (11.35) again.)

We may now summarize our discussion:

$$N_t \subset (E_\infty \cap N_{3t}(Q)) \cup \Big(\bigcup_{S \in Z} \mathcal{C}(S) \Big)\,, \tag{11.36}$$

where Z denotes the set of cubes $S \in I_1 \cup L\,I$ that are contained $N_{100t}(Q)$. Now

$$\begin{aligned} \sum_{S \in Z} \nu^+(\mathcal{C}(S)) &= \sum_{S \in Z} \mu(S) \\ &\leq 100\, C_2\, t^\tau\, \mu(90B(Q)) \\ &\leq 100\, C_2 t^\tau\, \xi(R)\,, \end{aligned} \tag{11.37}$$

by (11.2), (10.12), (10.8), and the definition of $\xi(R)$. Since $\nu^+(E_\infty \cap N_{3t}(Q)) \leq 3\,C_2\,t^\tau\,\xi(R)$ by (10.8) again, (11.32) follows from (11.36) and (11.37).

This completes our verification of the hypotheses of Theorem 3.20 for the set $\tilde{E}$, the measure ν^+, and the cubes of $\tilde{\Delta}$. In the next section we use this information to show that Theorem 3.20 also holds on $\tilde{E}$, ν, and with the cubes of $\tilde{\Delta}$, even though the hypotheses (3.8)-(3.12) are not necessarily satisfied in this case.

12. Theorem 3.20 holds for ν.

In general we do not expect that ν (equipped with the cubes of $\tilde{\Delta}$) will satisfy the conditions (3.8)-(3.12) about small boundaries. A typical bad thing that may happen is the following. For some good cubes $R = R(Q)$, $Q \in \Delta^0$, the factor ρ_Q in (11.14) could be very small, much smaller than the corresponding factors for other cubes that touch R. When this happens, we shall not have a good control on the measure for ν of the sets $N_t(R)$ in terms of $\nu(R)$, and so we may have to declare that R is bad for ν without having any compensation available in terms of (3.12). Nonetheless we want to prove that Theorem 3.20 holds for $\tilde{E}$, ν, and the cubes of $\tilde{\Delta}$.

By this we mean that if $T : b\mathcal{E} \times b\mathcal{E} \longrightarrow \mathbb{C}$ is an operator that satisfies (3.13)-(3.15) and (3.17) (with μ and Δ replaced with ν and $\tilde{\Delta}$), and if there are functions β, $\tilde{\beta} \in \mathrm{BMO}(d\nu)$ that satisfy (3.21) and (3.22) (for ν), then T extends to a bounded operator on $L^2(d\nu)$. The definition of $\mathrm{BMO}(d\nu)$ is the same as for $d\mu$: we do not use small boundaries there.

To prove our claim, we shall simply follow the proof of Theorem 3.20 and show that it applies.

All the arguments in sections 2-7 can be applied without modification; the small boundary properties are never used there, except to get qualitative information like (3.16) or (7.1). These properties are also true for ν because they hold for ν^+. Thus we can get as far as Lemma 7.40, which says that it is enough to prove that the matrix $\mathcal{N}$ (associated to the measure ν) defines a bounded operator on $\ell^2(\tilde{\Delta})$.

We already know from Section 11 that the corresponding matrix $\mathcal{N}^+$ for ν^+ defines a bounded operator, and so it will be enough to show that

$$N(Q,R) \leq C N^+(Q,R) \tag{12.1}$$

(with obvious notations). To make the comparison easier, it will be useful to define positive numbers ρ_R for all $R \in \tilde{\Delta}$. When $R \in \tilde{\Delta}^0$ and $R = R(Q)$ for some $Q \in \Delta^0$, we take $\rho_R = \rho_Q$. When $R \in \tilde{\Delta}(Q)$ for some $Q \in I_1 \cup L\,I$, we set $\rho_R = \rho_Q$. We claim that

$$d\nu \leq \rho_R \, d\nu^+ \,, \qquad \text{on } R \tag{12.2}$$

and

$$\nu(R) \geq C^{-1} \, \rho_R \, \nu^+(R) \,, \qquad \text{for all } R \in \tilde{\Delta} \,. \tag{12.3}$$

When $R \in \tilde{\Delta}^0$ and $R = R(Q)$, this follows from (11.15) and (11.14). When $R \in \tilde{\Delta}(Q)$ for some $Q \in I_1 \cup L\,I$, this is obvious because $\nu = \rho_Q \, \nu^+$ on $\mathcal{C}(Q)$, by (11.2) and (11.12).

We are now ready to prove (12.1). We shall just take the different types of coefficients $N(Q,R)$ from (7.36)-(7.39) one after the other and compare them with the corresponding ones for ν^+. We start with $A_1(Q,R)$ in (7.13). Recall that $A_1(Q,R)$ is a sum of terms

$$(\nu(Q^*) \, \nu(R^*))^{-1/2} \, I(Q^*, R^* \cap 2\,Q) \,,$$

where $Q^* \in F(Q)$ (the set of children of Q) and $R^* \in F(R)$. Note that for each choice of Q^* and R^*,

$$(\nu(Q^*) \, \nu(R^*))^{-1/2} \leq C \, (\rho_{Q^*} \, \rho_{R^*} \, \nu^+(Q^*) \, \nu^+(R^*))^{-1/2} \,, \tag{12.4}$$

by (12.3), and

$$\begin{aligned} I(Q^*, R^* \cap 2\,Q) &= \int_{Q^*} \int_{R^* \cap 2Q} \frac{d\nu(x) \, d\nu(y)}{|x-y|} \\ &\leq \rho_{Q^*} \, \rho_{R^*} \, I^+(Q^*, R^* \cap 2\,Q) \,, \end{aligned} \tag{12.5}$$

by (12.2). Here we set

$$I^+(Q,V) = \int_Q \int_V \frac{d\nu^+(x) \, d\nu^+(y)}{|x-y|} \,, \tag{12.6}$$

for $Q \in \tilde{\Delta}$ and $V \subset \tilde{E} \backslash Q$, the obvious analogue of $I(Q,V)$ for ν^+.

From (12.4) and (12.5) we deduce that $A_1(Q,R) \leq C A_1^+(Q,R)$ (with obvious notations).

Next let $A_2(Q,R)$ be as in (7.14);

$$
\begin{aligned}
A_2(Q,R) &= \nu(Q)^{1/2} \sum_{R^* \in F(R)} \nu(R^*)^{-1/2}\, J(Q, R^* \backslash 2\,Q) \\
&\le C\,\nu^+(Q)^{1/2} \sum_{R^*} \rho_{R^*}^{-1/2}\, \nu^+(R^*)^{-1/2}\, J(Q, R^* \backslash 2\,Q) \\
&\le C\,\nu^+(Q)^{1/2} \sum_{R^*} \rho_{R^*}^{1/2}\, \nu^+(R^*)^{-1/2}\, J^+(Q, R^* \backslash 2\,Q) \\
&\le C A_2^+(Q,R)\,,
\end{aligned}
\tag{12.7}
$$

by (12.2), (12.3), (12.2) again, and where J^+ and $A_2^+(Q,R)$ are the obvious analogous of J and $A_2(Q,R)$ for ν^+. (See (7.10) for the definition of J.)

The story for $A_3(Q)$ in (7.22) is similar: $A_3(Q)$ is a sum of terms

$$
\begin{aligned}
&\nu(Q_1^*)^{-1/2}\, \nu(Q_2^*)^{-1/2}\, I(Q_1^*, Q_2^*) \\
&\qquad \le C\,(\rho_{Q_1^*}\, \rho_{Q_2^*}\, \nu^+(Q_1^*)\, \nu^+(Q_2^*))^{-1/2}\, \rho_{Q_1^*}\, \rho_{Q_2^*}\, I^+(Q_1^*, Q_2^*)
\end{aligned}
\tag{12.8}
$$

and hence $A_3(Q) \le C A_3^+(Q)$. Next (7.27) says that B_{11} is a sum of terms

$$
\begin{aligned}
&(\nu(Q^*)\, \nu(R^*))^{-1/2}\, I(Q^*, R^* \cap 2\,Q)) \\
&\qquad \le C\,(\rho_{Q^*}\, \rho_{R^*}\, \nu^+(Q^*)\, \nu^+(R^*))^{-1/2}\, \rho_{Q^*}\, \rho_{R^*}\, I^+(Q^*, R^* \cap 2\,Q)
\end{aligned}
\tag{12.9}
$$

(still by (12.2) and (12.3)), and hence $B_{11} \le C\, B_{11}^+$. Similarly B_{12} in (7.28) is composed of terms

$$
\begin{aligned}
&\nu(Q)^{1/2}\, \nu(R^*)^{-1/2}\, J(Q, R^* \backslash 2\,Q) \\
&\qquad \le C\,\nu^+(Q)^{1/2}\, (\rho_{R^*}\, \nu^+(R^*))^{-1/2}\, \rho_{R^*}\, J^+(Q, R^* \backslash 2\,Q)
\end{aligned}
\tag{12.10}
$$

and is thus $\le C\, B_{12}^+$. Our next term is B_{21} in (7.29), and it is a sum of terms

$$
\begin{aligned}
&(\nu(Q^*)\, \nu(R(Q)))^{-1/2}\, I(Q^*, 2\,Q \backslash R(Q)) \\
&\qquad \le C\,(\rho_{Q^*}\, \rho_{R(Q)}\, \nu^+(Q^*)\, \nu^+(R(Q)))^{-1/2}\, \rho_{Q^*}\, I^+(Q^*, 2\,Q \backslash R(Q))\,,
\end{aligned}
\tag{12.11}
$$

which are also dominated by the corresponding terms for ν^+ because $\rho_{Q^*} \le \rho_{R(Q)}$ (since $Q^* \subset Q \subset R(Q)$ by definitions). Finally,

$$\begin{aligned} B_{22} &= \nu(Q)^{1/2}\, \nu(R(Q))^{-1/2}\, J(Q, E\backslash(2\,Q \cup R(Q))) \\ &\le \rho_Q^{1/2}\, \rho_{R(Q)}^{-1/2}\, B_{22}^+ \\ &\le B_{22}^+ \,, \end{aligned} \tag{12.12}$$

for the same reason.

This completes our verification of (12.1); Theorem 3.20 for ν and the cubes of $\tilde{\Delta}$ follows.

13. The Cauchy operator is bounded on $L^2(d\nu)$.

It will be easier for us to deal with the truncated operators T_ε, $\varepsilon > 0$, defined by

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{f(y)\, d\nu(y)}{x-y}\,, \qquad \text{for } f \in L^2(d\nu)\,. \tag{13.1}$$

Because ν is a finite measure, there is no problem in defining T_ε, or even in proving that it is a bounded operator on $L^2(d\nu)$. Of course we want to prove that T_ε is bounded on $L^2(d\nu)$ with bounds that do no depend on ε, and this will require more work.

We cannot apply Theorem 3.20 (for ν) directly to T_ε, because it does not have a standard kernel, but this will be very easy to fix. Denote by $\mathcal{X}$ the nice cut-off function such that $\mathcal{X}(t) = 0$ for $0 \le t \le 1/2$, $\mathcal{X}(t) = 2\,t - 1$ for $1/2 \le t \le 1$, and $\mathcal{X}(t) = 1$ for $t \ge 1$. Then set

$$\tilde{T}_\varepsilon f(x) = \int \mathcal{X}\Big(\frac{|x-y|}{\varepsilon}\Big)\, \frac{f(y)\, d\nu(y)}{x-y}\,, \tag{13.2}$$

for $f \in L^2(d\nu)$. We can replace T_ε with $\tilde{T}_\varepsilon$ because

$$\|\,|T_\varepsilon - \tilde{T}_\varepsilon|\,\|_{L^2(d\nu)} \le\, C\,, \tag{13.3}$$

where $\|\,|\cdot|\,\|$ denotes the operator norm, and with a constant C that does not depend on ε. This follows easily from (the continuous version of) Shur's lemma, since

$$|(T_\varepsilon - \tilde{T}_\varepsilon) f(x)| \le \int_{\varepsilon/2<|x-y|<\varepsilon} \frac{|f(y)|\, d\nu(y)}{|x-y|}\,, \tag{13.4}$$

and

$$\text{(13.5)} \qquad \begin{aligned} &\sup_x \Big(\int_{\varepsilon/2<|x-y|<\varepsilon} \frac{d\nu(y)}{|x-y|} \Big) \\ &\qquad = \sup_y \Big(\int_{\varepsilon/2<|x-y|<\varepsilon} \frac{d\nu(x)}{|x-y|} \Big) \leq C \,, \end{aligned}$$

by (11.9).

We want to prove that

$$\text{(13.6)} \qquad |||T_\varepsilon|||_{L^2(d\nu)} \leq C \,,$$

with a constant C that does not depend on ε; (13.3) tells us that it is enough to deal with $\tilde{T}_\varepsilon$ instead. We want to apply Theorem 3.20, with $\tilde{E}$, ν, and the cubes of $\tilde{\Delta}$; Section 12 says that we can do this. We choose $b = g$, where g is as in (10.31). Note that g satisfies (2.6), as was observed shortly after (10.31) (or directly by ($*$2.6)); this was the whole point of the construction in [DM].

The kernel

$$K(x,y) = \mathcal{X}\Big(\frac{|x-y|}{\varepsilon}\Big) \, \frac{1}{x-y}$$

is antisymmetric and standard with uniform estimates, and $\tilde{T}_\varepsilon$ is the singular integral operator associated with $K(x,y)$ as in Lemma 4.8. (Most of the construction is not needed, though, because $K(x,y)$ satisfies the integrability condition (4.2).) In particular, it satisfies the weak boundedness property (3.17) automatically, by antisymmetry. Hence (13.6) will follow as soon as we verify the last condition in Theorem 3.20, namely that Tg and $T^t g$ lie in BMO with uniform estimates.

Note that we don't need to be as careful as in the statement of Theorem 3.20 and define $\tilde{T}g$ and $\tilde{T}^t g$ by duality. Here, due to the fact that our kernel K is bounded, $\tilde{T}g$ and $\tilde{T}^t g$ are well defined, and even bounded, and the only thing we have to check is that they lie in BMO with uniform bounds. Also, $\tilde{T}g = -\tilde{T}^t g$ by definitions (and in particular antisymmetry), so we only need to show that $\|\tilde{T}g\|_{\mathrm{BMO}(d\nu)} \leq C$ for some C that does not depend on ε.

Note that

$$|\tilde{T}_\varepsilon g(x) - T_\varepsilon g(x)| \leq \int_{\varepsilon/2<|x-y|<\varepsilon} \frac{|g(y)| \, d\nu(y)}{|x-y|} \leq C \,,$$

by (13.4) and (11.9). Since bounded functions obviously lie in BMO, the desired estimate (13.6) will follow if we prove that

$$\|T_\varepsilon\, g\|_{\mathrm{BMO}(d\nu)} \le C\,. \tag{13.7}$$

In view of Definition 3.18, this means that

$$\int_{R_0} |T_\varepsilon\, g(x) - m_{R_0}(T_\varepsilon\, g)|^2\, d\nu(x) \le C\,\nu(R_0)\,, \tag{13.8}$$

for all $R_0 \in \tilde{\Delta}$, where $m_{R_0}(T_\varepsilon\, g)$ denotes the mean value of $T_\varepsilon\, g$ on R_0 (for ν). It is even enough to show that for each $R_0 \in \tilde{\Delta}$ there is a constant m_{R_0} such that

$$\int_{R_0} |T_\varepsilon\, g(x) - m_{R_0}|^2\, d\nu(x) \le C\,\nu(R_0)\,, \tag{13.9}$$

because we know that the left-hand side of (13.8) is always less than or equal to the left-hand side of (13.9).

Let us first take care of the cubes R_0 that are contained in circles $\mathcal{C}(Q)$, $Q \in I_1 \cup L\,I$.

Lemma 13.10. *For each $Q \in I_1 \cup L\,I$ there is a constant C_Q^ε such that*

$$|T_\varepsilon g(x) - C_Q^\varepsilon| \le C\,, \qquad \text{on } \mathcal{C}(Q)\,. \tag{13.11}$$

Recall that on $\mathcal{C}(Q)$, $g(y)$ is a bounded constant α_n (by (10.31) and (10.25)), and $d\nu = \lambda_Q\, dH^1$, where λ_Q is of the form

$$\rho_n^*\,\frac{\mu(Q)}{H^1(\mathcal{C}(Q))}\,,$$

by (10.20). Hence $\lambda_Q \le C$ as well, and

$$|T_\varepsilon(\mathbf{1}_{\mathcal{C}(Q)}\, g)(x)| \le C\,, \tag{13.12}$$

by elementary properties of truncated Cauchy integrals on circles, and it is enough to study

$$a(x) = T_\varepsilon((1-\mathbf{1}_{\mathcal{C}(Q)})\, g)(x) = \int_{\{|x-y|>\varepsilon,\, y \in \tilde{E}\setminus\mathcal{C}(Q)\}} \frac{g(y)\, d\nu(y)}{x-y}\,. \tag{13.13}$$

Recall from (11.27) that

$$\operatorname{dist}(\mathcal{C}(Q), \tilde{E}\backslash\mathcal{C}(Q)) \geq \frac{98}{100}\, r(Q)\,,$$

so that we can assume that $\varepsilon \geq r(Q)/2$, say, because otherwise we can replace ε with $r(Q)/2$ without modifying $a(x)$. Denote by x_0 the center of $\mathcal{C}(Q)$, and also set

$$D = \{y \in \tilde{E}\backslash\mathcal{C}(Q) :\ |y - x_0| > \varepsilon\}$$

(the domain of integration for $a(x_0)$) and

$$\mathcal{A} = \Big\{y \in \tilde{E} :\ \varepsilon - \frac{r(Q)}{100} \leq |y - x_0| \leq \varepsilon + \frac{r(Q)}{100}\Big\}$$

(which contains the difference between D and the domain of integration for $a(x)$ when $x \in \mathcal{C}(Q)$). Then

$$\begin{aligned}
|a(x) - a(x_0)| &\leq \Big|a(x) - \int_D \frac{g(y)\, d\nu(y)}{x-y}\Big| \\
&\quad + \Big|\int_D \Big(\frac{1}{x-y} - \frac{1}{x_0 - y}\Big)\, g(y)\, d\nu(y)\Big| \\
&\leq C \int_{\mathcal{A}} \frac{d\nu(y)}{|x-y|} \\
&\quad + C \int_{\{|y-x_0|>r(Q)/2\}} \Big|\frac{x - x_0}{(x-y)\,(x_0-y)}\Big|\, d\nu(y) \\
&\leq C\,,
\end{aligned} \tag{13.14}$$

because $\varepsilon \geq r(Q)/2$, and by the upper density estimate (11.9). (The computation for the last line is the same one as for (8.1).) Thus we can choose $C_Q^\varepsilon = a(x_0)$, and Lemma 13.10 follows.

Lemma 13.10 immediately gives (13.9) for all the cubes R_0 that are contained in a $\mathcal{C}(Q)$. Thus we are left with the cubes $R_0 \in \tilde{\Delta}^0$, and we can even suppose that $R_0 = R(Q_0)$ for some $Q_0 \in \Delta^0\backslash(I_1 \cup L\,I)$. Because of (11.15),

$$\int_{R_0} |T_\varepsilon g(x) - m_{R_0}|^2\, d\nu(x) \leq \rho_{Q_0} \int_{R_0} |T_\varepsilon g(x) - m_{R_0}|^2\, d\nu^+(x) \tag{13.15}$$

and, since $\rho_{Q_0}\mu(Q_0) \leq C\,\nu(R_0)$ by Lemma 11.13, (13.9) will follow if we can show that

$$\int_{R_0} |T_\varepsilon g(x) - m_{R_0}|^2\, d\nu^+(x) \leq C\,\mu(Q_0)\,. \tag{13.16}$$

Let us summarize what we have done so far.

Lemma 13.17. *To prove* (13.6) *with a constant that does not depend on* ε, *it is enough to show that for each* $\varepsilon > 0$ *and each cube* $Q_0 \in \Delta^0 \backslash (I_1 \cup L\,I)$, *we can find a complex number* m_0 *such that*

$$\int_{R(Q_0)} |T_\varepsilon g(x) - m_0|^2\, d\nu^+(x) \leq C\,\mu(Q_0)\,, \tag{13.18}$$

where C *does not depend on* ε *or* Q_0.

At this point we fix a cube Q_0 as in the lemma, and we want to find m_0 and eventually check (13.18). Our notations so far have been slightly different from those of [DM, Section 8], where what we call $T_\varepsilon g$ was called $T^\varepsilon(g\,d\nu)$. It will be more convenient for us now to revert to the notation of [DM], *i.e.*, to let the measure show up in the notations. Recall from (10.31), (10.30), and (10.16) that

$$g\,d\nu = \lim_{n\to\infty} F_n = f\,d\mu + \sum_{n\geq 1}(F_n - F_{n-1}) = f\,d\mu + \sum_{n\geq 1}\varphi_n\;, \tag{13.19}$$

where

$$\varphi_n = -\mathbf{1}_{H_n}\,\rho_{n-1}\,f\,d\mu - \theta_n\,\mathbf{1}_{G_n}\,\rho_{n-1}\,f\,d\mu + \alpha_n\,d\nu_n\;, \tag{13.20}$$

by (10.22). Hence

$$T^\varepsilon(g\,d\nu) = T^\varepsilon(f\,d\mu) + \sum_{n\geq 1} T^\varepsilon(\varphi_n)\,, \tag{13.21}$$

the proof of $(*2.9)$ in [DM] also gives that the series converges absolutely ν^+-almost everywhere, so we should not worry about convergence.

Fortunately we shall not need to estimate most of the terms in (13.21) in the present paper, because this was mostly done in [DM].

Denote by J the set of integers $n \geq 1$ such that $Q_n \subset Q_0$ and define a function A on $\tilde{E}$ by

$$A(x) = \sup_{\varepsilon > 0} \Big(|T^\varepsilon(f\, d\mu)(x)| + \sum_{n \in J} |T^\varepsilon(\varphi_n)(x)| \Big) , \tag{13.22}$$

for $x \in E_\infty$, and

$$A(x) = \sup_{\varepsilon \geq A^{-k(Q)}/5} \Big(|T^\varepsilon(f\, d\mu)(x)| + \sum_{n \in J} |T^\varepsilon(\varphi_n)(x)| \Big) , \tag{13.23}$$

for $x \in \mathcal{C}(Q)$, $Q \in I_1 \cup L\, I$.

Lemma 13.24. *We have that*

$$\int_{R(Q_0)} A(x)^2 \, d\nu^+(x) \leq C\, \mu(Q_0) , \tag{13.25}$$

with a constant C that does not depend on $\varepsilon > 0$ or Q_0.

When $x \in E_\infty$, [DM, ($*$4.130) and ($*$4.131)] give that

$$\begin{aligned} A(x) \leq C + C \sum_{n \in J} \sum_{Q \in \mathcal{A}_n \cup \mathcal{A}_n^*} \theta(Q)\, \mathbf{1}_{E \setminus Q}(x)\, e_Q^*(x) \\ + C \sum_{\substack{n \in J \\ Q_n \in PLI}} \sum_{Q \in \mathcal{A}_n^*} \theta(Q)\, \mathbf{1}_Q(x)\, h_Q^*(x) , \end{aligned} \tag{13.26}$$

with the notations of [DM], that we won't have to make explicit here. Thus

$$A(x) \leq C + W_1^J(x) + W_2^J(x) , \tag{13.27}$$

where W_1^J and W_2^J are as in ($*$5.1) and ($*$5.3), but where one sums only on the cubes $Q \in \mathcal{R} = I_1 \cup L\, I \cup BL\, I$ that come from indices $n \in J$, *i.e.*, cubes that lie in $\mathcal{A}_n \cup \mathcal{A}_n^*$ for some $n \in J$. By Remarks $*$5.177 and $*$5.179, and especially ($*$5.182),

$$\begin{aligned} \int_{R(Q_0) \cap E_\infty} A(x)^2 \, d\nu^+(x) &\leq C\, \nu^+(R(Q_0) \cap E_\infty) + C\, \mu\Big(\bigcup_{n \in J} Q_n \Big) \\ &\leq C\, \mu(Q_0) . \end{aligned} \tag{13.28}$$

(See (∗5.180) if you want to check that ν^+ is the same here as in [DM], and recall that $\nu^+ = \mu$ on E_∞).

Now suppose that $x \in \mathcal{C}(Q)$ for some $Q \in I_1 \cup LI$. We may use (∗4.132) and (∗4.133) to get that

$$
\begin{aligned}
A(x) &\leq C + C \sum_{n\in J} \sum_{Q\in\mathcal{A}_n\cup\mathcal{A}_n^*} \theta(Q)\, \tilde{e}_Q(x) \\
&\quad + C \sum_{\substack{n\in J \\ Q_n\in PLI}} \sum_{Q\in\mathcal{A}_n^*} \theta(Q)\, \tilde{h}_Q(x)\,.
\end{aligned}
\tag{13.29}
$$

(See (∗4.14) and a little below for the definition of k_m; indeed $k_m = k(Q)$ for the cubes $Q \in \mathcal{A}_m$.) Then

$$
A(x) \leq C + \tilde{W}_1^J(x) + \tilde{W}_2^J(x)\,, \tag{13.30}
$$

where $\tilde{W}_1^J$ and $\tilde{W}_2^J$ are defined like $\tilde{W}_1$ and $\tilde{W}_2$ in (∗5.2) and (∗5.4), but where we only sum over those cubes $Q \in \mathcal{R}$ that lie in $\mathcal{A}_n \cup \mathcal{A}_n^*$ for some $n \in J$. Now

$$
\begin{aligned}
\int_{R(Q_0)\setminus E_\infty} A(x)^2\, d\nu^+(x) &= \int_{R(Q_0)\cap(\bigcup_{Q\in I_1\cup LI}\mathcal{C}(Q))} A(x)^2\, d\nu^+(x) \\
&\leq C\,\nu^+(R(Q_0)) + C\,\mu\Big(\bigcup_{n\in J} Q_n\Big) \\
&\leq C\,\mu(Q_0)\,,
\end{aligned}
\tag{13.31}
$$

by (∗5.182) and Lemma 11.13. Lemma 13.24 follows from this and (13.28).

Now we want to take care of the $T^\varepsilon(\varphi_n)$ for which $n \notin J$. We start with the set J_1 of integers such that Q_n does not meet Q_0.

Denote by x_0 the "center of Q_0", *i.e.*, the point $x(Q_0)$ of (10.3)-(10.5). For each $n \in J_1$, set

$$
B_n(x) = |T^\varepsilon\varphi_n(x) - T^\varepsilon\varphi_n(x_0)|\,. \tag{13.32}
$$

Lemma 13.33. *We have that*

$$
\sum_{n\in J_1} B_n(x) \leq C + C\,Z(x)\,, \qquad \textit{for } x \in R(Q_0)\,, \tag{13.34}
$$

where

$$Z(x) = \int_{E\setminus Q_0} \frac{A^{-k(Q_0)}}{|x-y|\,|x_0-y|}\,d\mu(y)\,. \tag{13.35}$$

To prove the lemma, set

$$V(x) = \{y \in \mathbb{C}:\ |x-y| > \varepsilon \text{ and } |x_0-y| > \varepsilon\}\,, \tag{13.36}$$

$$W(x) = \{y \in \mathbb{C}\setminus V(x):\ |x-y| > \varepsilon \text{ or } |x_0-y| > \varepsilon\}\,, \tag{13.37}$$

and then define a function h by

$$h(y) = \begin{cases} \dfrac{|x-x_0|}{|x-y|\,|x_0-y|}\,, & \text{when } y \in V(x)\,, \\ |x-y|^{-1} + |x_0-y|^{-1}\,, & \text{when } y \in W(x)\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Obviously

$$\begin{aligned} B_n(x) &\le \int h(y)\,|\varphi_n(y)| \\ &\le \int_{H_n} \rho_{n-1}\,h\,d\mu + \theta_n \int_{G_n} \rho_{n-1} h\,d\mu + |\alpha_n| \int_{\mathcal{C}_n} h\,d\nu_n\ , \end{aligned} \tag{13.38}$$

by (13.20) and because $\|f\|_\infty \le 1$. We want to sum this over $n \in J_1$. Notice that the sets H_n are disjoint by (10.26) and contained in $E\setminus Q_0$ by definition of J_1. The $\mathcal{C}_n$'s are disjoint too, by (10.29). The sets G_n are not necessarily disjoint, but (10.18) says that

$$\theta_n\,\rho_{n-1}(x) = \rho_{n-1}(x) - \rho_n(x)\,, \qquad \text{when } x \in G_n\ , \tag{13.39}$$

so that for a given $x \in E$,

$$\sum_{n:x\in G_n} \theta_n\,\rho_{n-1}(x) \le 1\,. \tag{13.40}$$

Thus

$$\begin{aligned} \sum_{n\in J_1} B_n(x) &\le \int_{\bigcup_{n\in J_1} H_n} h\,d\mu + \int_{\bigcup_{n\in J_1} G_n} h\,d\mu + C \sum_{n\in J_1} \int_{\mathcal{C}_n} h\,d\nu \\ &\le 2\int_{E\setminus Q_0} h\,d\mu + C \sum_{n\in J_1} \int_{\mathcal{C}_n} h\,d\nu\,, \end{aligned} \tag{13.41}$$

for all $x \in R(Q_0)$.

Let us first take care of the integrals on $W(x)$. Let $x \in R(Q_0)$ be given. When $\varepsilon > 2\,\mathrm{diam}\,(R(Q_0) \cup \{x_0\})$, $W(x) \subset B(x_0, 2\varepsilon)$ and $h(x) = |x-y|^{-1} + |x_0 - y|^{-1} \leq 4\,\varepsilon^{-1}$ on $W(x)$, and hence

$$\int_{W(x)} h(y)\, d\mu(y) + \int_{W(x)} h(y)\, d\nu(y) \leq C\,, \tag{13.42}$$

be (10.15) (applied to Q_0 or to a suitable ancestor of Q_0) and (11.9).

When $\varepsilon \leq 2\,\mathrm{diam}\,(R(Q_0) \cup \{x_0\})$, $W(x) \subset B(x_0, CA^{-k(Q_0)})$, and then

$$h(y) \leq \frac{CA^{-k(Q_0)}}{|x-y|\,|x_0-y|}\,, \qquad \text{on } W(x)\,. \tag{13.43}$$

From this and (13.41) we deduce that

$$\sum_{n\in J_1} B_n(x) \leq C + C\,Z(x) + C \sum_{n\in J_1} \int_{\mathcal{C}_n} \rho(y)\, d\nu(y)\,, \tag{13.44}$$

where

$$\rho(y) = \frac{A^{-k(Q_0)}}{|x-y|\,|x_0-y|}\,. \tag{13.45}$$

We still need to control the contribution of the sets $\mathcal{C}_n$. Let $n \in J_1$ be given, and let $Q \in \mathcal{A}_n$. Since $n \in J_1$, Q_n does not meet Q_0, and neither does $Q \subset Q_n$. Then

$$\mathrm{dist}\,(x_0, \mathcal{C}(Q)) \geq \mathrm{dist}\,(Q_0, \mathcal{C}(Q)) \geq \mathrm{dist}\,(\mathcal{C}(Q), E\backslash Q) \geq \frac{99}{100}\, r(Q)\,,$$

by (10.28). Hence

$$|x_0 - z| \leq C\,|x_0 - y|\,, \qquad \text{for all } z \in Q \text{ and } y \in \mathcal{C}(Q)\,. \tag{13.46}$$

Similarly, $\mathcal{C}(Q)$ does not meet $R(Q_0)$, by (11.5) and the fact that the circles $\mathcal{C}(Q)$, $Q \in I_1 \cup LI$, are disjoint (by (10.29)). Then for all $x \in R(Q)$ we have that

$$\mathrm{dist}\,(x, \mathcal{C}(Q)) \geq \mathrm{dist}\,(R(Q), \mathcal{C}(Q)) \geq \mathrm{dist}\,(\mathcal{C}(Q), \tilde{E}\backslash \mathcal{C}(Q)) \geq \frac{98}{100}\, r(Q)\,,$$

by (11.27), and

$$|x-z| \le C\,|x-y|\,, \qquad \text{for } z \in Q \text{ and } y \in \mathcal{C}(Q)\,. \tag{13.47}$$

From (13.46) and (13.47) we deduce that $\rho(y) \le C\,\rho(z)$ whenever $y \in \mathcal{C}(Q)$ and $z \in Q$, and then

$$\begin{aligned}\sum_{n\in J_1}\int_{\mathcal{C}_n}\rho(y)\,d\nu(y) &= \sum_{n\in J_1}\sum_{Q\in\mathcal{A}_n}\int_{\mathcal{C}(Q)}\rho(y)\,d\nu(y)\\ &\le C\sum_{n\in J_1}\sum_{Q\in\mathcal{A}_n}\int_{Q}\rho(z)\,d\mu(z)\\ &\le C\int_{E\setminus Q_0}\rho(z)\,d\mu(y)\\ &= C\,Z(x)\,,\end{aligned} \tag{13.48}$$

because $\nu(\mathcal{C}(Q)) \le \mu(Q)$, the cubes Q are disjoint and do not meet Q_0, and by definition (13.35) of Z.

Lemma 13.33 follows from (13.44) and (13.48).

Lemma 13.49. *We have*

$$\int_{R(Q_0)} Z(x)^2\,d\nu^+(x) \le C\,\mu(Q_0)\,. \tag{13.50}$$

We leave the proof of Lemma 13.49 for later, and continue with the proof of (13.18). Lemmas 13.33 and 13.49 will give us enough control on the $T^\varepsilon(\varphi_n)$, $n \in J_1$ (see later). So we want to switch to the set $J_2 = \mathbb{N}^* - (J \cup J_1)$ of integers $n \ge 1$ such that Q_0 is strictly contained in Q_n. Thus $Q_0 \subset G_n$ when $n \in J_2$. For each $n \in J_2$, set

$$\begin{aligned}\psi_n &= \varphi_n + \theta_n\,\mathbf{1}_{Q_0}\,\rho_{n-1}\,f\,d\mu\\ &= -\mathbf{1}_{H_n}\,\rho_{n-1}\,f\,d\mu - \theta_n\,\mathbf{1}_{G_n\setminus Q_0}\,\rho_{n-1}\,f\,d\mu + \alpha_n\,d\nu_n\ ,\end{aligned} \tag{13.51}$$

(by (13.20)), and then set

$$B_n(x) = |T^\varepsilon\psi_n(x) - T^\varepsilon\psi_n(x_0)|\,, \qquad \text{for } x \in R(Q_0)\,. \tag{13.52}$$

We claim that

$$\sum_{n\in J_2} B_n(x) \le C + C\,Z(x)\,, \qquad \text{for } x \in R(Q_0)\,, \tag{13.53}$$

by the same proof as for Lemma 13.33. The main point is still that the sets H_n are disjoint and disjoint from $R(Q_0)$, that the integrals against $\alpha_n d\nu_n$ are controlled by the integrals on H_n, and that the integrals on the sets $G_n\backslash Q_0$ sum up by (13.39) and still concern $E\backslash Q_0$.

The last piece that we need to study is

$$\psi = \sum_{n\in J_2} \theta_n\, \rho_{n-1}\, \mathbf{1}_{Q_0} f\, d\mu = (1-\rho)\, \mathbf{1}_{Q_0}\, f\, d\mu\,, \tag{13.54}$$

where ρ denotes the constant value of ρ_{n_0} on Q_0, where n_0 is the largest integer in J_2. (If J_2 is empty, we don't need to worry but we can also take $\rho = 1$ and $\psi = 0$.) The last equality in (13.54) comes from (13.39). For each $x \in R(Q_0)$, set

$$D(x) = E \cap B(x, \mathrm{diam}\,(Q_0 \cup R(Q_0)) + A^{-k(Q_0)})\,. \tag{13.55}$$

By ($*$4.97) or ($*$4.98),

$$|T^\varepsilon(\mathbf{1}_{E\backslash D(x)}\, f\, d\mu)(x)| \le C\,, \tag{13.56}$$

because it is a $T^{\tilde\varepsilon}(f\, d\mu)(x)$ for some $\tilde\varepsilon \ge A^{-k(Q_0)}$; next

$$|T^\varepsilon(\mathbf{1}_{E\backslash D(x_0)}\, f\, d\mu)(x)| \le C\,, \tag{13.57}$$

by (13.56), and because the difference between the left-hand sides of (13.56) and (13.57) is controlled by

$$\int_\Delta \frac{d\mu(y)}{|x-y|} \le C\,,$$

where $\Delta = (D(x_0)\backslash D(x)) \cup (D(x)\backslash D(x_0))$. This last estimate uses (10.15). Now assume that $x \in R(Q_0) \cap E_\infty$ or $x \in R(Q_0)\backslash E_\infty$ and $x \in \mathcal{C}(Q)$ for some $Q \in I_1 \cup L\,I$ such that $\varepsilon \ge A^{-k(Q)}/5$. Then $|T^\varepsilon(f\, d\mu)(x)| \le C$ by ($*$4.97) or ($*$4.98), and hence

$$\begin{aligned}
|T^\varepsilon\psi(x)| &\le |T^\varepsilon(f\, d\mu)(x)| + |T^\varepsilon(\mathbf{1}_{E\backslash Q_0}\, f\, d\mu)(x)| \\
&\le C + |T^\varepsilon(\mathbf{1}_{E\backslash D(x_0)}\, f\, d\mu)(x)| + \int_{D(x_0)\backslash Q_0} \frac{d\mu(y)}{|x-y|} \\
&\le C + \int_{D(x_0)\backslash Q_0} \frac{d\mu(y)}{|x-y|}\,.
\end{aligned} \tag{13.58}$$

The following lemma will be useful; we shall prove it later, at the same time as Lemma 13.49.

Lemma 13.59. *Set*

$$Z_1(x) = \int_{D(x_0)\setminus Q_0} \frac{d\mu(y)}{|x-y|}\,, \qquad \textit{for all } x \in R(Q_0)\,.$$

Then

$$\int_{R(Q_0)} Z_1(x)^2\, d\nu^+(x) \le C\,\mu(Q_0)\,. \tag{13.60}$$

We are now ready to prove (13.18) (modulo the two lemmas). Take

$$m_0 = \sum_{n\in J_1} T^\varepsilon \varphi_n(x_0) \,+\, \sum_{n\in J_2} T^\varepsilon \psi_n(x_0)\,. \tag{13.61}$$

For each $x \in R(Q_0) \cap E_\infty = Q_0 \cap E_\infty$ and $\varepsilon > 0$,

$$\begin{aligned}
|T_\varepsilon\, g(x) - m_0| &= |T^\varepsilon(g\, d\nu)(x) - m_0| \\
&\le A(x) + \sum_{n\in J_1} |T^\varepsilon\varphi_n(x) - T^\varepsilon\varphi_n(x_0)| \\
&\quad + \sum_{n\in J_2} |T^\varepsilon\psi_n(x) - T^\varepsilon\psi_n(x_0)| + |T^\varepsilon\psi(x)| \\
&\le A(x) + \sum_{n\in J_1\cup J_2} B_n(x) + C + Z_1(x) \\
&\le A(x) + C + C\,Z(x) + Z_1(x)\,,
\end{aligned} \tag{13.62}$$

by (13.19), (13.22), (13.51) and (13.54) (to get that $\sum_{n\in J_2}\varphi_n = \sum_{n\in J_2}\psi_n + \psi$), (13.32) and (13.52), (13.58), Lemma 13.33, and (13.53).

When $x \in R(Q_0)\setminus E_\infty$ and $x \in \mathcal{C}(Q)$ for some $Q \in I_1 \cup L\,I$, and we suppose in addition that $\varepsilon \ge A^{-k(Q)}/5$, we can use (13.23) instead of (13.22), and the same computations as for (13.62) yield

$$|T_\varepsilon g(x) - m_0| \le A(x) + C + C\,Z(x) + Z_1(x)\,. \tag{13.63}$$

When $x \in \mathcal{C}(Q)$ and $\varepsilon < A^{-k(Q)}/5$, set $\varepsilon' = A^{-k(Q)}/5$ and observe that

$$\begin{aligned}
|T_\varepsilon\, g(x) - T_{\varepsilon'}\, g(x)| &= \Big| \int_{\{\varepsilon<|x-y|<\varepsilon'\}} \frac{g(y)\, d\nu(y)}{x-y} \Big| \\
&= \Big| \int_{\{y\in\mathcal{C}(Q),\varepsilon<|x-y|<\varepsilon'\}} \frac{\alpha_m\, d\nu_m(y)}{x-y} \Big| \le C\,,
\end{aligned} \tag{13.64}$$

by (11.27), (10.31), (10.25), (10.20), and elementary properties of truncated Cauchy integrals on circles, and where m denotes the integer such that $Q \in \mathcal{A}_m$. Thus (13.63) holds also when $\varepsilon < A^{-k(Q)}/5$, even though with a slightly larger constant C. Altogether, (16.63) holds for all $x \in R(Q_0)$ (and all $\varepsilon > 0$).

Now (13.18) follows from Lemmas 13.24, 13.49, 13.59, plus the fact that $\nu^+(R(Q_0)) \leq C\,\mu(Q_0)$, by Lemma 11.13. Because of Lemma 13.17, our proof of (13.6) will be complete as soon as we establish the two lemmas.

First consider the function $Z(x)$ of Lemma 13.33. We claim that

$$Z(x) \leq C + Z_1(x)\,, \qquad \text{for all } x \in R(Q_0)\,, \tag{13.65}$$

where Z_1 is as in Lemma 13.59. Let $D(x_0)$ be as in (13.55) and the definition of Z_1. Then

$$\int_{E\setminus D(x_0)} \frac{A^{-k(Q_0)}}{|x-y|\,|x_0-y|}\,d\mu(y) \leq C\,, \tag{13.66}$$

by the same computation as for (8.1), because $(|x-y|\,|x_0-y|)^{-1} \leq C\,|x_0-y|^{-2}$ on the domain of integration and by (10.15), applied to Q_0 and its ancestors.

So we may concentrate on

$$Z_2(x) = \int_{D(x_0)\setminus Q_0} \frac{A^{-k(Q_0)}}{|x-y|\,|x_0-y|}\,d\mu(y)\,.$$

But $|x_0 - y| \geq A^{-k(Q_0)}/2$ on $D(x_0)\setminus Q_0$, by (10.3) and (10.4), and so $Z_2(x) \leq 2\,Z_1(x)$. This proves our claim (13.65).

Obviously Lemma 13.49 will follow from Lemma 13.59 and (13.65), because $\nu^+(R(Q_0)) \leq \mu(Q_0)$ by Lemma 11.13.

We now prove Lemma 13.59. The argument is quite similar to estimates for functions h_Q^* that were done at the beginning of [DM, Section 5.1], but we give the argument here because some of the computations in [DM] are much more general than what we need here.

First we want to reduce to an integral on Q_0 (rather than $R(Q_0)$). For each $x \in Q_0$, set

$$r(x) = \inf\{A^{-k} : \text{ there is a cube } Q \in \Delta_k^0 \text{ that contains } x\}\,. \tag{13.67}$$

The main point of this definition is that

$$\mu(B(x,r)) \leq C\,r\,, \qquad \text{for all } r \geq r(x)\,, \tag{13.68}$$

by (10.15). Also note that

$$r(x) = 0\,, \qquad \text{on } E_\infty \cap Q_0\,, \tag{13.69}$$

because E_∞ does not meet any cube of $I_1 \cup L\,I$. Next set

$$h(x) = \mathbf{1}_{Q_0}(x) \int_{D(x_0)\backslash Q_0} \frac{d\mu(y)}{r(x) + |x-y|}\,. \tag{13.70}$$

We want to check that

$$\int_{R(Q_0)} Z_1(x)^2\,d\nu^+(x) \leq C \int_{Q_0} h(x)^2\,d\mu(x)\,. \tag{13.71}$$

For $x \in E_\infty \cap R(Q_0)$, $r(x) = 0$ and $Z_1(x) = h(x)$; for the corresponding part of the integral, there is nothing to check because $\nu^+ \leq \mu$ on E_∞.

Now let $Q \in I_1 \cup L\,I$ be given, with $Q \subset Q_0$, and let us look at the contribution of $\mathcal{C}(Q)$. For each $x \in \mathcal{C}(Q)$,

$$\operatorname{dist}(x, D(x_0)\backslash Q_0) \geq \operatorname{dist}(\mathcal{C}(Q), E\backslash Q) \geq \frac{99}{100}\,r(Q)\,,$$

by (10.28), and hence

$$r(z) + |z-y| \leq A^{-k(Q)} + |z-y| \leq 100\,r(Q) + |x-y| \leq C\,|x-y|\,, \tag{13.72}$$

for all $y \in D(x_0)\backslash Q_0$ and all $z \in Q$. Then $Z_1(x) \leq C\,h(z)$ for all $z \in Q$, and

$$\int_{\mathcal{C}(Q)} Z_1(x)^2\,d\nu^+(x) \leq C \int_Q h^2(z)\,d\mu(z)\,, \tag{13.73}$$

because $\nu^+(\mathcal{C}(Q)) = \mu(Q)$. When we sum this over the (disjoint) cubes $Q \in I_1 \cup L\,I$ that are contained in Q_0, we obtain that

$$\int_{R(Q_0)\backslash E_\infty} Z_1(x)^2\,d\nu^+(x) \leq C \int_{Q_0} h(z)^2\,d\mu(z) \tag{13.74}$$

(by (11.5)); our claim (13.71) follows from this and the trivial estimate for E_∞ mentionned above.

Because of (13.71), Lemma 13.59 will follow as soon as we show that

$$\int_{Q_0} h(x)^2\,d\mu(x) \leq C\,\mu(Q_0)\,. \tag{13.75}$$

To prove this we decompose Q_0 into its maximal good subcubes R, $R \in S(Q_0)$. The decomposition is the same as in Section 8, even though μ is a slightly different measure now (that does not satisfy (3.1)). In particular, the analogue of (8.10) in this context holds, with the same proof. (See Lemma $*5.28$.) For each $R \in S(Q_0)$, set

$$h_R(x) = \mathbf{1}_R(x) \int_{2R\setminus R} \frac{d\mu(y)}{r(x) + |x-y|}\,, \tag{13.76}$$

where $2\,R$ is as in (7.7)-(7.8) or in ($*4.79$). This is almost the same function as in [DM] (see ($*5.8$)), with the only minor difference that we may have chosen $r(x)$ a little larger than the one in [DM]. (See in particular ($*5.5$) and ($*5.7$).) This difference does not disturb us, because our function h_R is slightly smaller than the one in [DM], and the estimates from [DM] will work even better for it. Now we claim that

$$h(x) \le C + h_{Q_0}(x) \le C' + h_R(x)\,, \tag{13.77}$$

when $x \in R$, $R \in S(Q_0)$. The first inequality is an easy consequence of the fact that $|x-y| \ge A^{-k(Q_0)}$ on $D(x_0)\setminus 2Q_0$, so that

$$h(x) - h_{Q_0}(x) = \int_{D(x_0)\setminus 2Q_0} |x-y|^{-1}\, d\mu(y) \le A^{k(Q_0)}\, \mu(D(x_0)) \le C\,,$$

by (10.15). The second inequality comes directly from Lemma $*5.36$. The fairly easy proof is quite similar to arguments used earlier in this paper: because all the cubes Q such that $R \subset Q \subset Q_0$ and $Q \ne R$ are bad, the contribution to $h_{Q_0}(x)$ of the annular shells at distance $\sim A^{-k(Q_0)-\ell}$, $\ell \le k(R) - k(Q_0)$, from x decrease rapidly; the main contribution comes from $\ell = 0$ and is less than C by (10.15). (See [DM] for details.)

Next, for each $R \in S(Q_0)$ and each $x \in R$,

$$h_R(x) \le C\,(1 + \log\,(1 + A^{-k(R)}\,\mathrm{dist}\,(x, 2\,R\setminus R)^{-1}))\,. \tag{13.78}$$

This is ($*5.24$), and it follows from a rather brutal computation using dyadic annular shells and the density estimate (13.68). The logarithm is an estimate of the number of shells that we need to cover the domain of integration. Finally,

$$\int_R h_R(x)^2\, d\mu(x) \le C\,\mu(90B(R)) \le C\,\mu(R)\,. \tag{13.79}$$

This follows fairly easily from (13.78) and (10.8), plus the fact that R is a good cube. This is also a consequence of Lemma $*5.22$. Now

$$
\begin{aligned}
\int_{Q_0} h(x)^2\, d\mu(x) &= \sum_{R\in S(Q_0)} \int_R h(x)^2\, d\mu(x) \\
&\le 2 \sum_R \int_R (h_R(x)^2 + C)\, d\mu(x) \qquad (13.80)\\
&\le C \sum_R \mu(R) = C\, \mu(Q_0)\,,
\end{aligned}
$$

by (8.10) (or Lemma $*5.28$), (13.77), and (13.79).

This completes our proof of (13.75); Lemma 13.59, Lemma 13.49, and our main estimate (13.6) follow.

At this point we may return to the description given in Section 1: the estimate (1.11) follows readily from (13.6), and we may conclude as in the introduction.

This complete our proof of Theorem 1.1.

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Inverse problems in the theory of analytic planar vector fields

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Abstract. In this communication we state and analyze the new inverse problems in the theory of differential equations related to the construction of an analytic planar vector field from a given, finite number of solutions, trajectories or partial integrals.

Likewise we study the problem of determining a stationary complex analytic vector field Γ from a given, finite subset of terms in the formal power series

$$V(z,w) = \lambda\,(z^2+w^2) + \sum_{k=3}^{\infty} H_k(z,w)\,, \qquad H_k(a\,z, a\,w) = a^k H_k(z,w)\,,$$

and from the subsidiary condition

$$\Gamma(V) = \sum_{k=1}^{\infty} G_{2k}\,(z^2+w^2)^{k+1}\,,$$

where G_{2k} is the Liapunov constant. The particular case when

$$V(z,w) = f_0(z,w) - f_0(0,0)$$

and $(f_0, D \subset \mathbb{C}^2)$ is a canonic element in the neigbourhood of the origin of the complex analytic first integral F is analyzed. The results are applied to the quadratic planar vector fields. In particular we constructed the all quadratic vector field tangent to the curve

$$(y-q(x))^2 - p(x) = 0\,,$$

where q and p are polynomials of degree k and $m \leq 2\,k$ respectively. We showed that the quadratic differential systems admits a limit cycle of this tipe only when the algebraic curve is of the fourth degree. For the case when $k > 5$ it proved that there exist an unique quadratic vector field tangent to the given curve and it is Darboux's integrable.

1. Introduction.

We consider analytic planar vector fields or equivalent systems of differential equations

$$\begin{cases} \dfrac{dx}{dt} = P(x, y, t)\,, \\ \dfrac{dy}{dt} = Q(x, y, t)\,. \end{cases} \tag{1.1}$$

We shall mainly be concerned here with real systems (1.1). In order to understand such systems it is however advisable to sometimes consider the natural extension of (1.1) to the complex system

$$\begin{cases} \dfrac{dz}{dz^0} = Z(z, w, z^0)\,, \\ \dfrac{dw}{dz^0} = W(z, w, z^0)\,. \end{cases} \tag{1.2}$$

The following representation is often used instead of (1.1) and (1.2)

$$\omega \equiv P\,dy - Q\,dx = 0\,,$$

$$\Omega \equiv W\,dz - Z\,dw = \Omega_1 + i\,\Omega_2 = 0\,.$$

In the theory of differential equations (1.1) (or (1.2)) two main problems can be studied:

I) Direct problem or problem of integration (1.1) (or (1.2)).

II) Inverse problem or problem of construction (1.1) (or (1.2)) from given properties.

Before solving the direct problem, the question as to what the integration of (1.1) means must be answered. If the given equations describe the behaviour of physical phenomena, then these can be seen to change over time.

By using the theorem of existence and unicity we can determine the evolution of the phenomena in the past and future by integration. Integrating the equations without complementary information about the real situation may lead to useless results. So if integration enables us to understand the process of finding the analytical expressions for the solutions, the following question immediately arises: What character and properties must the required expression have?. It is well known that the solutions to (1.1) can be expressed though elementary functions or integrals of such functions only in some exceptional cases.

The analytical expression of linear systems is well known. However, there are few physical systems which can be described by such models. If solutions can be found for non-linear systems, the formulae for expressing them are so complicated that they are practically impossible to study. The problem of integrating (1.1) can be stated with infinite formal series. The difficulties which arise have to do with the convergence of the series which is so slow as to be useless in most cases. Finally, the problems related to the approximate calculation of the solutions to the given equations are well known. These difficulties lead the specialist to state and solve another type of problem which is that of constructing differential equations from given properties. This sort of problems are called inverse problems in the theory of differential equations. Generally speaking, by an inverse problem one usually means the problem of constructing a mathematical object from given properties. In recent years this branch of mathematics has been developing in different directions, in particular in the field of differential equations.

One of the difficulties encountered when studying such questions is that of the high degree of arbitrariness but this can be remedied by introducing subsidiary conditions inspired by the physical nature of the phenomenon.

The first inverse problem of the differential equations was stated by Newton.

Book One of Newton's Philosophiae Naturalis Mathematica is totally dominated by the idea of determining the forces capable of generating planetary orbits of the solar system.

The problem of finding the forces which generate a given motion has played a dominant role in the history of dynamics from Newton's time to the present. In fact, this problem has been studied by Bertran, Suslov, Joukovski, Darboux, Danielli, Whittaker and recently by Galiullin [1], Szebehely [2], and their followers.

Of course, this problem is essentially a problem of construction differential equations of the second order with given properties.

Another fundamental inverse problem in this theory is that of to Eruguin, who stated the problem of constructing a system of differential equations from given integral curves [3]. This idea were futher developed in [1].

The aim of this communication is to developed the Eruguin's ideas and construct the planar analytical vector field from given solutions, trajectories, partial integrals, etc. The problem posed are illustrated in a specific case. In particular, we determine all the quadratic autonomous vector fields from the given algebraic curves of the genus 2.

2. Constructing an analytic planar vector field from a given finite number of solutions.

PROBLEM 2.1. Let us specify smooth functions

$$z_j = x_j + i\,y_j : I \subset \mathbb{R} \longrightarrow \mathbb{C}$$

$$t \longmapsto z_j(t) = x_j(t) + i\,y_j(t)\,, \qquad j = 1, M\,,$$

We want to construct a differential equation

$$F\Big(z, \overline{z}, t, \frac{dz}{dt}\Big) \equiv a\,(z, \overline{z}, t)\,\frac{dz}{dt} + f\,(z, \overline{z}, t) = 0\,, \tag{2.1}$$

where $z = x + i\,y$, $\overline{z} = x - i\,y$, in such a way that

$$z = z_j(t)\,, \qquad j = 1, 2, \ldots, M \tag{2.2}$$

be its solutions.

Evidently, the sought after equation can be represented as follows: Let us denote by $\mathcal{D}$ the matrix

$$\mathcal{D} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z & z_1(t) & \ldots & z_M(t) \\ \overline{z} & \overline{z}_1(t) & \ldots & \overline{z}_M(t) \\ z^2 & z_1^2(t) & \ldots & z_M^2(t) \\ |z|^2 & |z_1(t)|^2 & \ldots & |z_M(t)|^2 \\ \overline{z}^2 & \overline{z}_1^2(t) & \ldots & \overline{z}_M^2(t) \\ \vdots & \vdots & & \vdots \\ \overline{z}^n & \overline{z}_1^n(t) & \ldots & \overline{z}_M^n(t) \\ \dfrac{dz}{dt} & \dfrac{dz_1(t)}{dt} & \ldots & \dfrac{dz_M(t)}{dt} \end{pmatrix}, \tag{2.3}$$

where $(n+1)\,(n+2)/2 = M$.

Proposition 2.1. *The differential equation admitting* (2.2) *as its solutions can be represented as follows*

$$F\Big(z, \overline{z}, t, \frac{dz}{dt}\Big) = \det \mathcal{D} - \Phi(z, \overline{z}, t) = 0\,, \tag{2.4}$$

where Φ (*which we will call Eruguin's function*) *is an arbitrary function such that*

$$\Phi(z, \overline{z}, t)|_{z=z_j(t),\, \overline{z}=\overline{z}_j(t)} \equiv 0\,, \qquad j = 1, 2, \ldots, M\,.$$

As can be seen the arbitrariness of the equations obtained is high in relation to the function Φ, but this drawback can be removed with the help of some complementary conditions. In the paper [4] we studied the problem of constructing a stationary polynomial planar vector field

$$\frac{dz}{dt} \equiv \dot{z} = \sum_{j+k=n} a_{kj}\, z^j\, \overline{z}^k\,, \qquad a_{kj} \in \mathbb{C}\,,$$

from given solutions (2.2) and with evidently subsidiary conditions which enable us to solve (2.4) with respect to $\dot{z}$.

We have proposed a method for determining the Eruguin function in [4]. In order to illustrate Proposition 2.1 and this method we shall analyze the case when the sought after vector field is quadratic. We solve the simplest problem when the given solutions are the following $z = 0$ and $z = z_0 = \text{const} \neq 0$.

We determine the Eruguin function as linear combinations of elements of the matrix H_j which we define as follows

$$\begin{aligned} &H_0(z, \overline{z}, t) = \sum_{j+k=n} B_{jk}\, z^j\, \overline{z}^k\,, \\ &H_j(z, \overline{z}, t) = [H_{j-1}(z, \overline{z}, t), H_{j-1}(z_j(t), \overline{z}_j(t), t)]\,, \end{aligned} \tag{2.5}$$

where $j = 1, 2, \ldots, M$, B_{kj} is an arbitrary matrix of order s and $[A, B] = AB - BA$ is the Lie bracket of the matrices A and B. By introducing the vector

$$L(z, \overline{z}) = (1, z, \overline{z}, z^2, z\,\overline{z}, \overline{z}^2)\,,$$

we easily obtain, for our particular case, that the sought after quadratic vector field is such that

$$\frac{dz}{dt} = (L(z_0, \overline{z}_0), K L^T(z, \overline{z})) \,, \tag{2.6}$$

where by K we denote the antisymmetrical matrix

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & -\beta_1 & 0 & \beta_5 & \beta_6 & \beta_7 \\ 0 & -\beta_2 & -\beta_5 & 0 & \beta_8 & \beta_9 \\ 0 & -\beta_3 & -\beta_6 & -\beta_8 & 0 & \beta_{10} \\ 0 & -\beta_4 & -\beta_7 & -\beta_9 & -\beta_{10} & 0 \end{pmatrix},$$

where $\beta_j \in \mathbb{C}$. The equation (2.6) determines the required quadratic vector field with two critical points. The following particular case is of interest

$$\beta_j = \begin{cases} 0\,, & \text{if } j \neq 4\,, \\ a + i\,b\,, & \text{if } j = 4\,. \end{cases}$$

and $z_0 = \varepsilon \in \mathbb{R}$. The above equation in this case take the form

$$\frac{dz}{dt} = -\beta_4\,(\varepsilon^2 z - \varepsilon\,\overline{z}^2)\,,$$

or, what amounts to the same,

$$\begin{cases} \dot{x} = -\varepsilon\,(b\,\partial_y H + a\,(\varepsilon\,x + y^2 - x^2))\,, \\ \dot{y} = \varepsilon\,(-b\,\partial_x H - a\,(\varepsilon\,y + 2\,x\,y))\,, \end{cases}$$

where

$$H = \frac{\varepsilon}{2}\,(x^2 + y^2) + x\,y^2 - \frac{1}{3}\,x^3\,.$$

3. Constructing a planar vector field from a given complex analytic first integral.

In this section we shall study two problems related with the constructing of a vector field Γ such that

$$\begin{cases} \dfrac{dz}{dz^0} = -\Lambda\,w + Z(z, w)\,, \\ \dfrac{dw}{dt} = \Lambda\,z + W(z, w)\,, \end{cases}$$

where z and z^0 are complex variables, $\Lambda \in \mathbb{C}$, Z, W are polynomial functions in the variables z and w. The first problem is the following:

PROBLEM 3.1. Let

$$V(z,w) = \frac{\Lambda}{2}\,(z^2+w^2) + \sum_{j=3}^{\infty} H_j(z,w)\,, \tag{3.2}$$

be a formal power series, where Λ is a nonzero complex parameter and H_j is a homogenous function of degree j.

The analytic vector field Γ need to be constructed in such a way that

$$\Gamma(V(z,w)) = \sum_{j=1}^{\infty} G_{2k}\,(z^2+w^2)^{k+1}\,,$$

where $G_{2k} \in \mathbb{C}$ are the Liapunov (complex) constants.

The second problem is a consequence of the Problem 3.1. Firstly we introduce the following concepts and notations [5].

Definition 3.1 *By a canonical element centered at the point $a \in \overline{\mathbb{C}}^2$ will be called a pair (U_a, f_a), where f_a is the sum of a power series with its centre at a and U_a is the domain of convergence of the power series.*

Definition 3.2. *Two canonic elements (U_a, f_a) and (V_a, g_a) are said to be equivalent if $f_a \equiv g_a$ in the neighbourhood of a.*

Definition 3.3. *The complex analytic function F with domain $\mathcal{D} \subset \mathbb{C}^2$ will be called the set of canonic elements which can be generated from the canonic element (U_a, f_a) after analytic continuation along the whole path starting from the given point $a \in U_a$.*

Definition 3.4. [5] *The system* (3.1) *will be called integrable in the Liapunov sense* (*or Liapunov integrable*) *if and only if there is an analytic first integral F which contains the canonic element (U_0, f_0) with f_0*

$$f_0(z,w) = f_0(0,0) + \frac{\Lambda}{2}\,(w^2+z^2) + \sum_{j=3}^{\infty} H_j(z,w)\,,$$

where the H_j, $j = 3, 4, \ldots,$ are homogenous functions of degree j, and Λ is a nonzero complex parameter [5].

Problem 3.2. To construct a Liapunov integrable polynomial vector field Γ of degree n such that

$$\left\{\begin{aligned} \frac{dz}{dt} &= -\Lambda\, w + Z_n(z,w)\,, \\ \frac{dw}{dt} &= \Lambda\, z + W_n(z,w)\,, \end{aligned}\right. \tag{3.4}$$

where by Z_n and W_n we denote a polynomial function of degree $n > 1$ in of the variables z and w.

We find the solutions to these problem for $n = 2$ and $n = 3$, while for $n > 3$ solutions are found by in an analogous manner.

Proposition 3.1. *Let us suppose that the function H_3 is such that*

$$\{H_2,\, H_3\} \equiv \partial_z H_2\, \partial_w H_3 - \partial_w H_2\, \partial_z H_3 \not\equiv 0\,.$$

Then the sought after quadratic stationary vector field Γ can be represented as follows

$$\Gamma_2 = \{H,\ \} + g_1\{\ ,H_2\}\,, \tag{3.5}$$

if this condition holds

$$\Gamma_2(H_{2k} + H_{2k+1}) = G_{2k}\,(z^2+w^2)^k\,,$$

or, what amounts to the same,

$$\left\{\begin{aligned} &\{H_2, H_{2k+1}\} + \{H_3, H_{2k}\} + g_1\{H_{2k}, H_2\} = 0\,, \\ &\{H_2, H_{2k+2}\} + \{H_3, H_{2k+1}\} + g_1\{H_{2k+1}, H_2\} \\ &\qquad\qquad\qquad\qquad = G_{2k+2}\,(z^2+w^2)^{k+1}\,, \end{aligned}\right. \tag{3.6}$$

where H_2, H are functions such that

$$H_2(z,w) = \frac{\Lambda}{2}\,(z^2+w^2)\,,$$

$$H(z,w) = H_2(z,w) + H_3(z,w)\,.$$

Consequence 3.1. The Liapunov constants G_{2k} for the quadratic vector field thus constructed can be calculated by the formulas:

$$G_{2k+2} = \frac{1}{2\pi}\int_0^{2\pi} \left(- g_1(z,w)\,\{H_2, H_{2k+1}\} + \{H_3, H_{2k+1}\}\right)\Big|_{\substack{z=\cos t\\ w=\sin t}} dt\,,$$

where $k \in \mathbb{N}$. From the above results we can deduce the following consequence

Consequence 3.2. Let us give the functions H_2, H_3, H_4 and the Liapunov constant G_4.

Then we can construct:

i) the quadratic vector field Γ_2,

ii) all members of the formal power series $\sum_{k=5}^{\infty} H_k(z,w)$, and

iii) the Liapunov constants G_{2k+2}, $k = 2, 3, 4, \ldots$

In order to illustrate these assertions, we shall study the following particular case. Let H_2, H_3, H_4 and G_4 be such that

$$\begin{cases} H_2 = \dfrac{\Lambda}{2}\,(z^2 + w^2)\,, \\ H_3 = \dfrac{1}{3}\,((a_6 + a_4)\,w^3 - (a_2 + a_5)\,z^3) + a_2\,z\,w^2 - a_3\,z^2\,w\,, \\ H_4 = \dfrac{1}{4}\,(a_4\,(a_3 + a_4 + a_6) - a_5(2\,a_2 + a_5)\,z^4) - a_2\,a_4\,z\,w^3\,, \\ G_4 = \dfrac{1}{8}\,a_5\,(a_3 - a_6)\,, \end{cases}$$

where a_2, a_3, a_4, a_5, a_6 are some complex parameters.

The sought after quadratic vector field can be represented as follows

$$\begin{cases} \dfrac{dz}{dt} = -\partial_w H^* - a_5\,z\,w\,, \\ \dfrac{dw}{dt} = \partial_z H^* - a_4\,z\,w\,, \end{cases}$$

where

$$H^* = \frac{\Lambda}{2}\,(z^2 + w^2) + \frac{a_2}{3}\,z^3 + a_3\,z^2\,w - a_2\,z\,w^2 - \frac{a_6}{3}\,w^3\,.$$

By using computer techniques it is easy to obtain the expression for all the terms of the power series and the Liapunov constants from the above formulas.

CONSEQUENCE 3.3. Let us suppose that the functions H_{2k} and H_{2k+1} are such that

$$\{H_2, H_{2k}\} \not\equiv 0\,, \qquad \{H_2, H_{2k+1}\} \not\equiv 0\,,$$

so we have the following relations

$$\begin{aligned} g_1(z,w) &= -\frac{\{H_2, H_{2k+1}\} + \{H_3, H_{2k}\}}{\{H_{2k}, H_2\}} \\ &= \frac{\{H_2, H_{2k+2}\} + \{H_3, H_{2k+1}\} - G_{2k+2}\,(z^2+w^2)^{k+1}}{\{H_2, H_{2k+1}\}}\,, \end{aligned} \tag{3.7}$$

where $k \in \mathbb{N}$. Likewise we can deduce the following result for cubic vector fields.

Proposition 3.2. *Let H_4 be a function such that*

$$\{H_4, H_2\} \not\equiv 0\,. \tag{3.8}$$

Then the cubic vector field Γ admits the representation below

$$\Gamma_3 = \nu(z,w)\,\{H,\ \} + g_2\,\{\ , H_2\}\,, \tag{3.9}$$

if the following relation holds

$$\begin{cases} \{H_2, H_3\} = 0\,, \\ \nu(z,w)\,\{H_{2k+1}, H_4\} + g(z,w)\,\{H_2, H_{2k+3}\} = 0\,, \\ \nu(z,w)\,\{H_{2k}, H_4\} + g(z,w)\,\{H_2, H_{2k}\} + \{H_2, H_{2k+2}\} \\ \qquad\qquad = -G_{2k+2}\,(z^2+w^2)^{k+1}\,, \end{cases}$$

where $H = H_2 + H_4$.

As an immediate consequence we find that all functions H_{2k+1} are equal to zero. Formulas analogous to (3.7) can be deduced.

From (3.10) we easily deduce that the function ν is such that

$$\nu(z,w)\,\{H_4, H_2\} + \{H_2, H_4\} = -G_4\,(z^2+w^2)^2\,.$$

Proposition 3.3. Let us suppose that the formal power series is such that

$$V(z,w)=\sum \frac{a_k}{k}\,(z^2+w^2)^k \equiv \rho(r^2)\,.$$

So the sougth after analytic vector field can be rewritten as follows

$$\tag{3.11} \begin{cases} \dfrac{dz}{dt}=\Lambda(z,w)\,w+\mathcal{R}(r^2)\,z\,, \\[2mm] \dfrac{dw}{dt}=-\Lambda(z,w)\,z+\mathcal{R}(r^2)\,w\,, \end{cases}$$

where Λ is an arbitrary analytic function and $\mathcal{R}$ is a function

$$\mathcal{R}(r^2)=\frac{r^2\displaystyle\sum_{k=0}^{\infty} G_{2k}\,r^{2k}}{\partial_{r^2}\rho(r^2)}\,.$$

Likewise we can study the problem of constructing a polynomial vector field of degree n. In order to illustrate these ideas we shall analyze the following specific case.

Let us give the functions H_k, $k=2,3,\ldots,n+1$, such that

$$\begin{cases} H_2(z,w)=\dfrac{1}{2}\,(z^2+w^2)\,, & \\[2mm] H_j(z,w)=0\,, & j=3,\ldots,n\,, \\[2mm] H_{n+1}(z,w)=\dfrac{1}{2}\,(c\,(b\,w^{n+1}+a\,z^{n+1}))\,, & c,b,a\in\mathbb{C}, \end{cases}$$

and let us suppose that $G_{2n}=G_{2n+2}=0$.

We wish to construct the polynomial vector field of degree n.

We obtain the solutions to this problem in the same way as in the above problem. Firstly it is easy to find that

$$\nu(z,w)=1\,,$$

$$g_{n-1}(z,w)=\frac{2\,n\,(c-1)}{n+1}\,(a\,z^{n-1}+b\,w^{n-1})\,,$$

$$H(z,w)=H_2(z,w)+H_{n+1}\,.$$

So the sought after vector field is

$$\begin{cases} \dfrac{dz}{dt}=-w-A\,w^n+B\,w\,z^{n-1}\,, \\[2mm] \dfrac{dw}{dt}=z+A\,z^n-B\,z\,w^{n-1}\,, \end{cases}$$

where
$$A = \frac{b\,(c\,(n+1)^2 + 4\,n)}{2\,n+2}\,, \qquad B = \frac{2\,a\,n\,(c-1)}{n+1}\,.$$
For $n = 2\,m+1$ we observe that the system obtained has the symmetry $(z,w,t) \longrightarrow (-z,w,-t)$ and $(z,w,t) \longrightarrow (z,-w,-t)$, *i.e.*, it is reversible. As a consequence there is an analytic first integral.

It is interesting to observe that the complex analytic function

$$V(z,w) = z^2\,(1 + a\,z^{n-1})^c + w^2\,(1 + b\,w^{n-1})^c$$

has the canonic element (f_0, U_0) such that

$$f_0(z,w) = z^2 + w^2 + c\,(a\,z^{n+1} + b\,w^{n+1}) + c\,(c-1)\,(a^2\,z^{2n} + b^2\,w^{2n}) + \cdots$$

The solution to this Problem 3.2 can easily be obtained from the solution to Problem 3.1, by considering the complementary condition that the Liapunov constants are zero in this case.

Lunkevich and Sibirski determine the first integral for a quadratic planar vector field with its center at the origin (see [7]). It is easy to show that these quadratic systems are Liapunov integrable (see [5]).

In order to illustrate the solution to the Problem 3.2 we shall analyze the problem of constructing a quadratic vector field from a given Lunkevich-Sibirski first integral.

We shall only study the case below. The others case can be done analogously.

Firstly, we shall suppose that we have a complex analytic integral

$$V(z,w) = \exp(-2\,w)\,(2\,z^2 + 2\,(b-1)\,w + 2\,b\,w^2 + b - 1)\,, \qquad b \in \mathbb{C}.$$

The canonic element in the neighbourhood of the origin is the following

$$\begin{cases} U_0 = \mathbb{C}^2\,, \\ f_0(z,w) = b - 1 + 2\,(z^2 + w^2) - \dfrac{4}{3}\,(2+b)\,w^3 - 4\,w\,z^2 \\ \qquad\qquad + 4\,z^2\,w^2 + 2\,(1+b)\,w^4 + \cdots \end{cases}$$

For this case it is easy to deduce that

$$\begin{cases} g_1(z,w) = \dfrac{\{H_4, H_2\}}{\{H_2, H_3\}} = 2\,w\,, \\ H(z,w) = \dfrac{1}{2}\,(z^2 + w^2) - \dfrac{1}{3}\,(2+b)\,w^3 - w\,z^2\,. \end{cases}$$

As a consequence, we obtain the following representation for the require quadratic vector field

$$\begin{cases} \dfrac{dw}{dt} = \partial_z H(z,w) + 2\,z\,w = z\,, \\ \dfrac{dz}{dt} = -\partial_w H(z,w) - 2\,w^2 = -w + z^2 + b\,w^2\,. \end{cases}$$

We shall now analyze the specific case when the complex analytic first integral V is given by the formula

$$\begin{aligned} V(z,w) = (1+2\,a\,w)^{a-1}\big(b + 3\,a - 1 + 2\,(a-1)(2\,a-1)\,(b\,w^2 - (3\,a-1)\,z^2) \\ - 2\,(a-1)\,(b+3\,a-1)\,w\big)^{a}\,, \end{aligned}$$

where $a, b \in \mathbb{C}$.

The canonic element of the given analytic function is such that

$$\begin{aligned} &f_0(z,w) \\ &= T\big((b+3\,a-1)^2 - 2\,(b+3\,a-1)\,(z^2+w^2) + \frac{4}{3}\,(b+2)\,w^3 \\ &\quad - 4\,(a-1)\,w\,z^2 + 2\,(6\,a^4 - 17\,a^3 + 14\,a^2 - 9\,a - 4\,a\,b + 2 - b^2)\,w^4 \\ &\quad + 2\,(a-1)^2\,(2\,a-1)^2\,(3\,a-1)\,z^4 \\ &\quad + 4\,(a-1)\,(6\,a^3 - 11\,a^2 + 9\,a + b - 2)\,z^2\,w^2\big) + \cdots\,, \end{aligned}$$

where

$$T \equiv (b+3\,a-1)^{a-1}\,a\,(a-1)\,(2\,a-1)\,(3\,a-1) \neq 0\,.$$

By using the proposed method we can deduce the well known quadratic vector field

$$\begin{cases} \dfrac{dw}{dt} = z + 2\,a\,w\,z\,, \\ \dfrac{dz}{dt} = -w + b\,w^2 + (1-a)\,z^2\,. \end{cases} \tag{3.12}$$

The integrability of the case when $a\,(a-1)\,(2\,a-1)\,(3\,a-1) = 0$ was deduced in [7]. The integrability of the case when $b + 3\,a - 1 = 0$ is easy to obtain (see [5]). The analytic first integral V and its canonic element are such that

$$V(z,w) = (1 + 2\,a\,w)^{(a-1)/a}\,(z^2 + w^2)\,,$$
$$f_0(z,w) = z^2 + w^2 + 2\,(a-1)\,(w^3 + z^2 w - w^4 - z^2\,w^2) + \cdots$$

4. Constructing a vector field with given trajectories.

In [4] we stated and solved the following problem

Problem 4.1 *Let*

$$\begin{aligned} w_j : \mathcal{D} \subset \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto w_j(z)\,, \qquad j = 1, \ldots, M\,, \end{aligned}$$

be a holomorphic function on $\mathcal{D}$ such that

$$K = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ w_1(z) & w_2(z) & \ldots & w_M(z) \\ {w_1}^2(z) & w_2^2(z) & \ldots & w_M^2(z) \\ \vdots & \vdots & & \vdots \\ w_1^{M-1}(z) & w_2^{M-1}(z) & \ldots & w_M^{M-1}(z) \end{pmatrix} \tag{4.1}$$

is identically nonvanishing on $\mathcal{D}$.

We need to construct an analytic vector field on $\mathcal{D}^* \subset \mathbb{C}^2$

$$\left\{ \begin{aligned} \frac{dz}{dt} &= P(z,w)\,, \\ \frac{dw}{dt} &= Q(z,w)\,, \end{aligned} \right. \tag{4.2}$$

in such a way that

$$w = w_j(z)\,, \qquad j = 1, \ldots, M\,, \tag{4.3}$$

are its trajectories. We deduced the solution to this problem from the equality

$$
\det\begin{pmatrix}
1 & 1 & 1 & \dots & 1 \\
w & w_1(z) & w_2(z) & \dots & w_M(z) \\
w^2 & w_1^2(z) & w_2^2(z) & \dots & w_M^2(z) \\
\vdots & \vdots & \vdots & & \vdots \\
w^{M-1} & w_1^{M-1}(z) & w_2^{M-1}(z) & \dots & w_M^{M-1}(z) \\
\dfrac{dw}{dz} & \dfrac{dw_1(z)}{dz} & \dfrac{dw_2(z)}{dz} & \dots & \dfrac{dw_M(z)}{dz}
\end{pmatrix}
= g(z,w)\det S\,, \tag{4.4}
$$

where by S we denote the following matrix

$$
S = \begin{pmatrix}
1 & 1 & 1 & \dots & 1 \\
w & w_1(z) & w_2(z) & \dots & w_M(z) \\
w^2 & w_1^2(z) & w_2^2(z) & \dots & w_M^2(z) \\
\vdots & \vdots & \vdots & & \vdots \\
w^{M-1} & w_1^{M-1}(z) & w_2^{M-1}(z) & \dots & w_M^{M-1}(z) \\
w^M & w_1^M(z) & w_2^M(z) & \dots & w_M^M(z)
\end{pmatrix}.
$$

g is an arbitrary analytic function on $\mathcal{D}^*$. From (4.4) we obtain the following expression for the most general vector field admitting the given curves as trajectories.

$$
\begin{cases}
\dfrac{dz}{dt} = \det A \equiv P\,, \\
\dfrac{dw}{dt} = \det B \equiv Q\,,
\end{cases} \tag{4.5}
$$

where

$$
A = \begin{pmatrix}
1 & 1 & 1 & \dots & 1 \\
w & w_1(z) & w_2(z) & \dots & w_M(z) \\
w^2 & w_1^2 & w_2^2 & \dots & w_M^2 \\
\vdots & \vdots & \vdots & & \vdots \\
w^{M-1} & w_1^{M-1} & w_2^{M-1} & \dots & w_M^{M-1} \\
K_1(z,w) & K_2(z,w) & K_3(z,w) & \dots & K_M(z,w)
\end{pmatrix},
$$

$$B = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ w & w_1(z) & w_2(z) & \dots & w_M(z) \\ w^2 & w_1^2 & w_2^2 & \dots & w_M^2 \\ \vdots & \vdots & \vdots & & \vdots \\ w^{M-1} & w_1^{M-1} & w_2^{M-1} & \dots & w_M^{M-1} \\ g_2(z,w)\, w^M & h_1(z,w) & h_2(z,w) & \dots & h_M(z,w) \end{pmatrix}$$

and

$$h_j = -\nu(z,w)\,\frac{dw_j}{dz} + g_2(z,w)\, w_j^M\,, \qquad j = 1,2,\dots,M\,,$$

$$K_1(z,w) = \nu(z,w) + g_1(z,w)\, w^M\,,$$

$$K_j(z,w) = g_1(z,w)\, w_j^M\,, \qquad j = 2,3,\dots,M\,.$$

In particular for $M = 2$ and

$$\begin{cases} w_1(z) = q(z) + \sqrt{p(z)}\,, \\ w_2(z) = q(z) - \sqrt{p(z)}\,, \end{cases}$$

we easily deduce the differential equations

$$(4.6) \qquad \begin{cases} \dfrac{dz}{dt} = 2\,p(z)\,\nu^*(z,w) + \alpha(z,w)\,((w-q(z))^2 - p(z))\,, \\ \dfrac{dw}{dt} = \nu^*(z,w)\Big(w\,\dfrac{dp(z)}{dz} + 2\,\dfrac{dq(z)}{dz}\,p(z) - q(z)\,\dfrac{dp(z)}{dz}\Big) \\ \qquad\qquad +\beta(z,w)\,((w-q(z))^2 - p(z))\,, \end{cases}$$

where $\nu = \sqrt{p(z)}\,\nu^*$, α and β are arbitrary analytic functions on $\mathcal{D}^*$.

By changing $w - q(z) \longrightarrow w$ in (4.6) we deduce the following formulas

$$(4.7) \qquad \begin{cases} \dfrac{dz}{dt} = \nu(z,w)\,\partial_w f(z,w) + \lambda_1(z,w)\,f(z,w)\,, \\ \dfrac{dw}{dt} = -\nu(z,w)\,\partial_z f(z,w) + \lambda_2(z,w)\,f(z,w)\,, \end{cases}$$

where $f(z,w) = w^2 - p(z)$ and λ_j, $j = 1,2$, are arbitrary holomorphic functions.

The specific case when the given trajectories are conic

$$q(z) = \frac{a_1\,z + a_2}{2}\,,$$

$$p(z) = b_1\,z^2 + 2\,b_2\,z + b_3\,, \qquad a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}\,,$$

was analyzed in [4] and [8].

For the subcase when $a_j = 0$, $j = 1, 2$ and $b_1 = 1$, $b_2 = 0$, $b_3 = 1$, we obtain the quadratic vector field

$$\begin{cases} \dfrac{dz}{dt} = -2\,(z^2 + 1) + \beta\,(w^2 - z^2 - 1)\,, \\ \dfrac{dw}{dt} = -2\,z\,w + \alpha\,(w^2 - z^2 - 1)\,, \qquad \alpha, \beta \in \mathbb{R}\,. \end{cases}$$

The bifurcations of the vector field on the plane (α, β) are given in [8].

Likewise, for the particulary case when $a_j = 0$, $j = 1, 2$ and $b_1 = -1$, $b_2 = 0$, $b_3 = 1$ we deduce the quadratic vector field

$$\begin{cases} \dfrac{dz}{dt} = 2\,(z^2 - 1) + \beta\,(w^2 + z^2 - 1)\,, \\ \dfrac{dw}{dt} = 2\,w\,z + \alpha\,(w^2 + z^2 - 1)\,, \qquad \alpha, \beta \in \mathbb{R}\,. \end{cases}$$

The bifurcations of the vector field on the plane (α, β) can be found in [8].

Finally, for the subcase when $a_1 = 2$, $a_2 = 0$ and $b_1 = 0$, $b_2 = 1$, $b_3 = 0$ we construct the quadratic vector field

$$\begin{cases} \dfrac{dz}{dt} = -4\,z\,(z + w) + \beta\,((w - z)^2 - 2\,z))\,, \\ \dfrac{dw}{dt} = -2\,(z + w)^2 + \alpha\,((w - z)^2 - 2z)\,, \qquad \alpha,\, \beta \in \mathbb{R}\,. \end{cases}$$

The critical points of this system are

$$O(0,0),\ N\Big(\frac{1}{2}, -\frac{1}{2}\Big),\ M\Big(\frac{\beta^2}{K_3}, \frac{\beta\,(2\,\alpha - \beta)}{K_3}\Big)\,,$$

where $K_3 = 2\,((\alpha - \beta)^2 - 2\,\alpha)$.

The bifurcation curves are given by the formulas

$$l_1 : (\alpha - \beta)^2 - 2\,\alpha = 0\,,$$

$$l_2 : \beta^2\,(4\,\alpha + 1) + 4\,\alpha\,\beta\,(1 - 2\,\alpha) + 4\,\alpha^2\,(\alpha - 1) = 0\,,$$

$$l_3 : \beta + 2\,\alpha = 0\,,$$

$$l_4 : \beta + \alpha = 0\,,$$

$$l_5 : \beta = 0\,,$$

$$l_6 : \alpha = 0\,.$$

These curves divide the plane α, β into 17 regions in which we find a change in the behaviour of the vector field. Of special interest is the region between the curves l_3 and l_4 for $\beta < 0$, where there is a stable limit cycle. The bifurcations of the vector field are given in [8].

The problem related to studying the quadratic vector field with parabola as trajectories was analyzed in particular in [9], [10] and [11].

To conclude this section it is interesting to observe that the function $\det S$ satisfies the relations

$$P(z, w)\,\partial_z \det S + Q(z, w)\,\partial_w \det S = \mathcal{R} \det S$$

along the solutions of the equations (4.5), for some function $\mathcal{R}$.

5. Constructing the planar vector field from given algebraic partial integrals.

Darboux in [12] gives a method of integration (1.2) with $P, Q \in \mathbb{C}[z, w]$ using algebraic curves. His first idea is to search for a general integral of the form

$$F(z, w) = \prod_{j=1}^{q} f_j^{\alpha_j}(z, w)\,, \tag{5.1}$$

where $\alpha_j \in \mathbb{C}$ and $f_j \in \mathbb{C}[z, w]$. This integral is called Darboux 's first integral and the system (1.2) is called Darboux integrable.

Definition 5.1 ([13]). *Let $f \in \mathbb{C}[z,w]$ and let $\gamma \subset \mathbb{C}^2 : f(z,w) = 0$ be an algebraic particular integral of* (1.2) *if and only if there exists $\lambda \in \mathbb{C}[z,w]$ such that*

$$P(z,w)\,\partial_z f(z,w) + Q(z,w)\,\partial_w f(z,w) = \lambda(z,w)\,f(z,w)\,. \tag{5.2}$$

The result below is Darboux's.

Theorem ([12]). *Consider the equation of the form* (1.2). *Let $m = \max\{\deg P, \deg Q\}$. If $q > m\,(m+1)/2$ and*

$$f_j(z,w) = 0\,, \qquad j = 1,2,\dots,q$$

are different algebraic solutions for which (5.2) *takes place, then there are complex numbers α_j, $j = 1,2,\dots,q$ such that* (5.1) *is a first integral of* (1.2).

In all of these papers the authors started with a system and asked what kind of invariant algebraic curves this system could have, but it seems interesting (by considering the argument given in the introduction) to analyze the inverse problem related to constructing the planar vector field tangent to the set of algebraic curves $f_j(z,w) = 0$, $j = 1,2,\dots,q$.

This problem was first stated by Eruguin [3] and developed by Galiullin and his followers [1]. The new different approach can be found in the papers [14] and [4]. The purpose of this section is to analyze the problem from another point of view.

We will first study the case when $q \geq 2$.

Proposition 5.1. *Let us give algebraic curves*

$$f_j(z,w) = 0\,, \qquad j = 1,2 \tag{5.3}$$

such that $\{f_1, f_2\} \not\equiv 0$ in the neighbouhood of the set (5.3).

So the the vector field tangent to the given curves can be represented as follows

$$\Gamma = \frac{\lambda_1 f_1\{\ ,f_2\} + \lambda_2 f_2\{f_1,\ \}}{\{f_1,\ f_2\}}\,, \tag{5.4}$$

where λ_j, $j = 1,2$ are arbitrary holomorphic functions on $\mathcal{D}^* \subset \mathbb{C}^2$.

The proof follows from the equalities

$$\begin{cases} \partial_z f_1(z,w)\, P(z\, w) + \partial_w f_1(z,w)\, Q(z,w) = \lambda_1(z,w)\, f_1(z,w)\,, \\ \partial_z f_2(z,w)\, P(z,w) + \partial_w f_2(z,w)\, Q(z,w) = \lambda_2(z,w)\, f_2(z,w)\,. \end{cases}$$

Of course if $P, Q \in \mathbb{C}[z,w]$, then λ_1 and λ_2 belong to $\mathbb{C}[z,w]$. The differential equations which generate (2.5) can be represented as follows

$$\begin{cases} \dfrac{dz}{dt} = \dfrac{\lambda_1 f_1\{z, f_2\} + \lambda_2 f_2\{f_1, z\}}{\{f_1, f_2\}}\,, \\ \dfrac{dw}{dt} = \dfrac{\lambda_1 f_1\{w, f_2\} + \lambda_2 f_2\{f_1, w\}}{\{f_1, f_2\}}\,. \end{cases} \tag{5.5}$$

As an immediate consequence we get the following results:

Consequence 5.1. *The vector field* (5.4) *has the following algebraic curves as complementary integrals*

$$f_j(z,w) = 0\,, \qquad j = 3, 4, \ldots, q\,,$$

if and only if

$$\lambda_1 f_1(z,w)\,\{f_j, f_2\} + \lambda_2 f_2\{f_1, f_j\} + \lambda_j f_j\{f_2, f_1\} = 0\,. \tag{5.6}$$

In fact, from the equalities

$$\Gamma(f_j) = \lambda_j f_j\;, \qquad j = 3, 4, \ldots, q\,,$$

we deduce that

$$\frac{\lambda_2 f_2\{f_1, f_j\} + \lambda_1 f_1\{f_1, f_2\}}{\{f_1, f_2\}} = \lambda_j f_j$$

and so (5.6) follows trivially.

Consequence 5.2. *Let F be function* (5.1)*. Then*

$$\Gamma(F) = \Big(\sum_{j=1}^{q} \alpha_j\, \lambda_j\Big) F\,. \tag{5.7}$$

Consequence 5.3. *Let us suppose that*

$$\prod_{\substack{j,k=1\\k\neq j}}^{q}\{f_j,f_k\}\not\equiv 0 \tag{5.8}$$

in the neighbourhood of the set $\{f_j=0,\ j=1,2,\dots,q\}$*. so the vector field tangent to the given curves admits the representations*

$$\Gamma=\frac{\lambda_j f_j\{f_{j-1},\ \}+\lambda_{j-1}f_{j-1}\{\ ,f_j\}}{\{f_j,f_{j-1}\}}\,,\qquad j=1,2,\dots,q\,, \tag{5.9}$$

if and only if the following relations hold

$$\lambda_j f_j\{f_n,f_m\}+\lambda_m f_m\{f_j,f_n\}+\lambda_n f_n\{f_m,f_j\}=0\,, \tag{5.10}$$

where $j,k,n,m=1,2,\dots,q>3$ *and* $n\neq k\neq j\neq m$*.*

Let us denote by A the matrix such that

$$A=\begin{pmatrix}0 & \{f_n,f_m\} & \{f_j,f_n\} & \{f_m,f_j\}\\ \{f_m,f_n\} & 0 & \{f_k,f_n\} & \{f_m,f_k\}\\ \{f_n,f_j\} & \{f_n,f_k\} & 0 & \{f_j,f_k\}\\ \{f_j,f_m\} & \{f_k,f_m\} & \{f_k,f_j\} & 0\end{pmatrix}.$$

Of course,

$$\begin{aligned}\det A &= \big(\{f_n,\,f_m\}\,\{f_j,\,f_k\}+\{f_k,\,f_m\}\,\{f_n,\,f_j\}\\ &\qquad+\{f_j,\,f_m\}\,\{f_k,\,f_n\}\big)^2\\ &=0\,.\end{aligned} \tag{5.11}$$

It is easy to prove that theses relations are an identity for all f_j,f_n,f_m and f_k. By using these identities we can easily deduce the following consequences

Consequence 5.4. *Let us suppose that the arbitrary functions* λ_j*,* $j=1,2,\dots,q$ *are such that*

$$\mathcal{R}\{H,f_j\}=\lambda_j f_j\ , \tag{5.12}$$

where H and $\mathcal{R}$ are arbitrary functions. Hence the vector field Γ admits the following representation

$$\Gamma = \mathcal{R}\{H,\ \}\,. \tag{5.13}$$

From here we can observe that the function $\mathcal{R}$ is an integrant factor of the 1-form $\Omega = \Gamma(z)\,dw - \Gamma(w)\,dz$. It is clear that (5.10), in view of (5.11) holds identically.

Consequence 5.5. *Let us suppose that the following development holds*

$$\mathcal{R}\{f_j, f_n\} = \sum_{m=1}^{q} C_{jn}^{m}(z,w)\, f_m\ . \tag{5.14}$$

Then the functions C_{jn}^m must satisfy the relations

$$\begin{cases} C_{jn}^m + C_{nj}^m = 0\,, \\ C_{nm}^l\, C_{lk}^s + C_{mk}^l\, C_{ln}^s + C_{kn}^l\, C_{lm}^s = 0\,. \end{cases}$$

These equalities are identities in the specific case when

$$C_{jn}^l f_l = \frac{1}{\mathcal{R}}\,(\lambda_j f_j - \lambda_n f_n)\,. \tag{5.15}$$

Consequence 5.6 *Let us suppose that*

$$f_j(z,w) = w - w_j(z)\,, \qquad f'_{jz} - f'_{(j-1)z} \neq 0\,, \qquad j = 1,2,\dots,q\,,$$

then (5.14) *holds with*

$$\mathcal{R} = \frac{\lambda_j f_j - \lambda_{j-1} f_{j-1}}{w'_{j-1}(z) - w'_j(z)}\ ,$$

$$w'_j \equiv \frac{dw}{dz}\ .$$

The differential equations which generate the vector field Γ are the following

$$\begin{cases} \dfrac{dz}{dt} = \mathcal{R}\,, \\[2mm] \dfrac{dw}{dt} = \mathcal{R}\,\dfrac{\lambda_j f_j\, w'_j(z) - \lambda_{j-1} f_{j-1}\, w'_{j-1}(z)}{w'_{j-1}(z) - w'_j(z)}\ . \end{cases}$$

To conclude this section we give the solution for the stated problem when $q = 1$.

Proposition 5.2. *The planar vector field tangent to the algebraic curve $f(z, w) = 0$ can be represented as follows*

$$\Gamma = \mu(z, w)\,\{f,\ \} + f(z, w)\,(\lambda_1(z, w)\,\partial_z + \lambda_2(z, w)\,\partial_w)\,,$$

where μ, λ_1 and λ_2 are arbitrary analytic functions, such that

$$\Gamma(f) = \lambda(z, w)\,f(z, w)\,, \qquad \textit{for all } \lambda \in \mathbb{C}[z, w]\,. \tag{5.16}$$

In order to illustrate the above assertions in the section below we shall give the solution to the stated problem for the subcase when Γ is a quadratic vector field in the variables z and w and the given algebraic curve is the following

$$f(z, w) = w^2 - 2\,w\,q(z) + v(z)\,, \qquad v(z) = q^2(z) - p(z)\,, \tag{5.17}$$

where q and p are polynomials of degree k and $m \leq 2\,k$ respectively.

6. Quadratic stationary planar vector fields with given algebraic curves (5.17).

It is well known that the domain G of a real analytic planar stationary vector field is divided into elementary regions by singular trajectories. The non singular trajectories (which are topologically equivalent) are located in these regions.

For structurally stable dynamical systems the singular trajectories can be stable simple critical points, stable limit cycles, $\alpha{-}\omega$ separatrices which may spread towards a node, a focus, a limit cycle. They may even leave the domain G.

From these facts we state and analyze the problem of constructing a planar vector field from a finite number of singular trajectories.

In this section we are going to construct a real quadratic vector field with a given real invariant algebraic curve (5.17). All the obtained results can be generalized (with the respective considerations) to the complex case.

The problem of constructing a quadratic planar vector field with a given algebraic curve of the type (5.17) has been studied by many specialists.

In 1966 A. I. Jablonski, published an article (see [15]) in which the author constructed a differential equation

$$w' = \frac{P(z,w)}{Q(z,w)}\,,$$

where P and Q are quadratics, which has an algebraic curve of fourth degree as a limit cycle. He also investigated the phase portraits of this equation.

In 1972 V. F. Filipsov, in the paper [16], showed that for the specific quadratic system studied by Jablonski there is an orbit of the form

$$w = b_0\,z^2 + b_1\,z + b_2 + (a_0\,z + a_1)\,\sqrt{-z^2 + l_1\,z + l_0}\,.$$

The author shows that for various values of the parameters there is no limit cycle and no separatrix going from one saddle point to another. In 1973 this author, in the article [17] is considering the quadratic system under the condition that

$$a_1 + a_0\,z + b_0\,w^2 + b_1\,z\,w + c_0\,z\,w^2 + c_1\,z^2\,w + c_2\,z^3 + z^4 = 0\,,$$

is a solution. The author shows that in this case a global analysis of the topology of integral curves is possible.

Later in the paper "Algebraic limit cycles" the author finds conditions under which the quadratic differential systems

$$\begin{cases} \dot z = P(z,w)\,, \\ \dot w = Q(z,w)\,, \end{cases}$$

have a limit cycle that is an algebraic curve of the fourth degree.

In 1991 Shen Boian, in the paper [18], proves that a quadratic system possesses a quartic curve solution

$$\text{(A)} \qquad (w + c\,z^2)^2 + z^2\,(z-a)\,(z-b) = 0\,, \qquad (a-b)\,a\,b\,c \neq 0\,,$$

if and only if the quadratic system can be written in the form

$$\begin{cases} \dot z = -4\,a\,b\,c\,z - (a+b)\,z + 3\,(a+b)\,c\,z^2 + 4\,z\,w\,, \\ \dot w = -(a+b)\,a\,b\,z - 4\,a\,b\,c\,w + (4\,a\,b\,c^2 - \dfrac{3}{2}\,(a+b)^2 + 4\,a\,b\,)\,z^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \cdot 8\,(a+b)\,c\,z\,w + 8\,w^2\,. \end{cases}$$

For this system a necessary and sufficient condition for the existence of a type of quartic curve limit cycle (A) and a separatrix cycle are given.

The aim of the present section is to state and solve the following

PROBLEM 6.1. Let us give the algebraic curve (5.17). We require to construct a real quadratic planar vector field which admits it as a particular integral.

Firstly we give the following aspects related to the plane curve (5.17).

Let us suppose that the algebraic curve (5.17) is found on the plane. The critical points (z_0, w_0) of this curve are the points such that

$$\begin{cases} p(z_0) = 0\,, \\ w_0 - q(z_0) = 0\,, \\ \left.\dfrac{dp(z)}{dz}\right|_{z=z_0} = 0\,. \end{cases} \tag{6.1}$$

Proposition 6.1 *The following type of critical points can be obtained for the curve* (5.17) :

i) *Isolated point. The point with coordinates* $(z_0, q(z_0))$ *where* z_0 *is the maximum of the function* p.

ii) *Knot* (*saddle*) *point. The point* $(z_0, q(z_0))$ *where* z_0 *is a minimum of* p.

iii) *If* $p''(z)|_{z=z_0} = 0$ *then the well known* 4 *configurations are possible.*

Proposition 6.2. *The relation* (5.16) *holds for the quadratic planar vector field*

$$\begin{cases} \Gamma = (\alpha(z) + \beta(z)\, w + \gamma\, w^2)\, \partial_z + (a(z) + b(z)\, w + c\, w^2)\, \partial_w\ , \\ \alpha(z) = \alpha_2\, z^2 + \alpha_1\, z + \alpha_0\ , \\ \beta(z) = \beta_1\, z + \beta_0\ , \\ a(z) = a_2\, z^2 + a_1\, z + a_0\ , \\ b(z) = b_1\, z + b_0\ , \end{cases} \tag{6.2}$$

if and only if the following equality holds

$$
\begin{aligned}
&-P(z,w)\,((2\,(w-q(z))\,q'(z)+p'(z))+Q(z,w)\,(2\,w-2\,q(z)) \\
&\qquad\qquad = (A\,z+B\,w+C)\,((w-q(z))^2-p(z))\,,
\end{aligned}
\tag{6.3}
$$

or, what amounts to the same,

$$
\begin{cases}
\gamma\,q'(z)=c-\dfrac{B}{2}\,, \\
2\,(B-c)\,q(z)-2\,\beta(z)\,q'(z)+\gamma\,v'(z)=A\,z+C-2\,b(z)\,, \\
2\,(A\,z+C-b(z))\,q(z)-2\,\alpha(z)\,q'(z)-B\,v(z)+\beta(z)\,v'(z) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -2\,a(z)\,, \\
-2\,a(z)\,q(z)-(A\,z+c)\,v(z)+\alpha(z)\,v'(z)=0\,,
\end{cases}
\tag{6.4}
$$

where $v=q^2(z)-p(z)$.

In order to solve this system we first introduce the following notations

$$
\begin{aligned}
S(z) &= ((A\,z+C-b(z))\,(A\,z+C)+B\,a(z))\,q(z) \\
&\quad -\alpha(z)\,(A\,z+C)\,\frac{dq}{dz}+a(z)\,(A\,z+C)\,, \\
D(z) &= (A\,z-B\,\alpha_2)\,z^2+(A\,\beta_0+C\,\beta_1-\alpha_1\,B)\,z+C\,\beta_0-\alpha_0\,B\,, \\
R(z) &= ((A\,z+C-b(z))\,\alpha(z)+a(z)\,b(z))\,q(z)-\alpha^2(z)\,\frac{dq}{dz}+a(z)\,\alpha(z)\,.
\end{aligned}
$$

Then for v and dv/dz from (3.1) we obtain the following relations

$$
\begin{cases}
D(z)\,v(z)=R(z)\,, \\
D(z)\,\dfrac{dv}{dz}=S(z)\,.
\end{cases}
$$

As a consequence the compatibility conditions gives us the relations

$$
\frac{dD(z)}{dz}\,R(z)=\Big(\frac{dR(z)}{dz}-S(z)\Big)D(z)\,,
\tag{6.5}
$$

where q is a polynomial such that

$$
(6.6)\quad q(z) = \begin{cases} k\,z + k_0\,, & \text{if } \gamma \neq 0\,,\\ k\,(\beta_1\,z + \beta_0)^n + k_1\,z + k_0\,, & \text{if } \gamma \neq 0,\ \beta_1 \neq 0\,,\\ k\,z^2 + k_1\,z + k_0\,, & \text{if } \gamma = B = c = \beta_1 = 0\,. \end{cases}
$$

By using computer techniques the solutions to (6.4) can be obtained.

The first case in (6.6) enables us to obtain all quadratic vector fields admitting the conics as trajectories. For the second case, we deduce that it is important when $n = 2, 3, 4, 5$. For $n > 5$ we deduce that there is only one quadratic vector field tangent to the given curve.

As Poincaré observed (see [19]) in order to recognize when the stationary planar vector field is algebraically integrable it is sufficient to find a bound for the degrees of the invariant algebraic curves which the system could have. In [14] the following problem is stated: find a bound for the degrees of the invariant algebraic curves which a system (1.1) could have.

In the development of some aspects of this problem, the results below about the construction of a quadratic vector field from given algebraic curves for $n > 5$ seems to be interesting.

6.1. Quadratic vector field with given conics.

For the case when the given algebraic curve is the following

$$
(6.7)\qquad f(z,w) = (w - k\,z - k_0)^2 - p_2\,z^2 - p_1\,z - p_0 = 0\,,
$$

we obtain all the quadratic vector fields tangent to it.

In particular, for the case when $p_1 = p_0 = 0$ and $p_2 \neq 0$ we get the following result:

Proposition 6.3 *The quadratic vector field tangent to the curve* (6.7) *with* $p_1 = p_0 = 0$ *and* $p_2 \neq 0$ *is the following*

$$
(6.8)\qquad \begin{cases}
\dfrac{dz}{dt} = -k_0\,(k_0\,\gamma + \beta_0) \\
\qquad + (\Omega\,(\beta_0 + 2\,k_0\,\gamma) + k_0\,\gamma\,(2\,\beta_1 - B) - C\,\gamma)\,\dfrac{z}{-2\,\gamma} \\
\qquad + \beta_0\,w\,\beta_1\,z\,w + \gamma\,w^2 \\
\qquad + (\Omega^2 + \Omega\,(2\,\beta_1 - B) - 2\,A\,\gamma)\,\dfrac{z^2}{-4\,\gamma}\,, \\
\\
\dfrac{dw}{dt} = \dfrac{k_0}{-2\,\gamma}\,(\Omega\,(\beta_0 + k_0\,\gamma) + C\,\gamma) \\
\qquad + (\Omega^2\,(\beta_0 + 2\,k_0\,\gamma) \\
\qquad\quad + 2\,\Omega\,(\beta_0\,\beta_1 - 2\,B\,k_0\,\gamma - B\,\beta_0 + 4\,k_0\,\gamma\,\beta_1) \\
\qquad\quad + 2\,\beta_0\,(A - 2\,b_1) - 4\,k_0\,\gamma^2\,(b_1 - A)) \\
\qquad\quad \cdot\dfrac{z}{-4\,\gamma^2}\,\dfrac{(\Omega\,(\beta_0 - k_0\,\gamma) + C\,\gamma)}{2\,\gamma}\,w \\
\qquad + (\Omega^2 + \Omega\,(2\,\beta_1 - B) - 2\,A\,\gamma)\,\dfrac{z^2}{-4\,\gamma}\,b_1\,z\,w + c\,w^2\,,
\end{cases}
$$

where $\Omega, B, \beta_1, A, \gamma, b_1$ are parameters such that

$$
(6.9)\qquad \begin{cases}
\Omega = 2\,\gamma\,k\,, \\
B = 2\,c + 2\,\gamma\,k\,, \\
\Omega\,(B - 2\,\beta_1) + 2\,\gamma\,(2\,b_1 - A) = 4\,\gamma^2\,p_2\,, \qquad p_2 \neq 0\,.
\end{cases}
$$

Of course if $p_2 > 0$ then the quadratic vector field has two invariant straight lines

$$
w = (k + \sqrt{p_2}\,)\,z + k_0\ ,
$$
$$
w = (k - \sqrt{p_2}\,)\,z + k_0\ .
$$

We can deduce the important subcase when

$$
(6.10)\qquad \begin{cases}
\beta_0 = -k_0\,\gamma\,, \\
c = 0\,, \\
A = -2\,\gamma\,, \\
B = \beta_1\ .
\end{cases}
$$

Under these restrictions we obtain the well known Darboux integrable quadratic vector field

$$\begin{cases} \dfrac{dz}{dt} = \beta_0\, w + \beta_1\, z\, w + \gamma\, w^2 - \gamma\, z^2\,, \\ \dfrac{dw}{dt} = -\beta_0\, z + \beta_1\, z\, w\,. \end{cases}$$

In this case the relations (6.10) and (6.9) take the form

$$\begin{cases} \beta_1^2 + 4\,\gamma\,(b_1 + \gamma) = 4\,\gamma^2\, p_2\,, \qquad p_2 > 0\,, \\ \Omega = 2\,\gamma\, k\,, \\ \beta_1 = B = -\Omega\,. \end{cases}$$

Likewise we deduce all the quadratic planar vector fields with given trajectories (6.7).

6.2. Quadratic planar vector fields, with a given curve of fourth degree.

We now shall analyze the above stated problem when the given curve is an algebraic curve of fourth degree

$$f(z, w) = (w - k_0\, z^2 - k_1\, z - k_2)^2 - p(z) = 0\,,$$

where p is a polynomial of degree four. This case was analyzed, in particular, in the papers refered to in the section above.

Proposition 6.4. *Let*

$$(w - k_0\, z^2 - k_1\, z - k_2)^2 + z^4 - 4\, h_3\, z^3 - 4\, h_2\, z^2 - 4\, h_0 = 0\,, \tag{6.11}$$

be a curve such that

$$\begin{cases} h_2 < 0\,, \\ 9\, h_3^2 > -8\, h_2\,. \end{cases}$$

Then the curve (6.11) *has an oval.*

Proposition 6.5. *The curve* (6.11) *with* $h_0 = 0$ *is a trajectory of the following quadratic system*

$$\begin{cases} \dot{z} = -k_0\,\beta_1\Big(\dfrac{3}{4}\,p_1\,z^2 + p_0\,z\Big) + \beta_1\Big(z - \dfrac{1}{4}\,p_1\Big)w\,, \\ \dot{w} = -\beta_1\Big(\Big(p_0 + \dfrac{3}{8}\,p_1^2 + k_0^2\,p_0\Big)z^2 + \dfrac{1}{4}\,p_0\,p_1\,z\Big) - \beta_1\,k_0\,(2\,p_1\,z + p_0)\,w\,. \end{cases}$$

The parameters A, B and C are determined as follows

$$A = -3\,k_0\,\beta_1\,b_1\,, \qquad B = 4\,\beta_1\,, \qquad C = -2\,q_0\,p_0\,\beta_1\,.$$

The existence of limit cycles can be deduced by analyzing the Liapunov function V

$$V(z,w) = w^2 + p_0\,z^2 - k_0\,w\,z^2 - p_1\,z^3 + (1 + k_0^2)\,z^4\,.$$

Of course, this function is definitively strictly positive for $p_0 > 0$. By considering that its derivative is such that

$$\dot{V} = -2\,q_0\,p_0\,\beta_1\,V + (4\,\beta_1\,w - 2\,q_0\,\beta_1\,b_1\,z)\,V\,,$$

we deduce that the origin is asymptotically stable if $q_0\,\beta_1\,p_0 > 0$ and unstable if $q_0\,\beta_1\,p_0 < 0$. On the other hand, the curve $V(z,w) = 0$ has an oval around the origin, which is evidently a limit cycle of the system.

Likewise we can analyze the problem of the construction of a quadratic vector field with algebraic with $n = 3, 4, 5$.

It should be pointed out that from the solution of the stated problem it follows that if the quadratic differential system has an algebraic limit cycle, this must be an algebraic curve of the fourth degree.

7. Quadratic vector fields with algebraic curves with $n > 5$.

With no loss of generality we shall suppose that $\beta_1 = 1$ and $\beta_0 = 0$.

By using computer techniques the following results can be easily deduced:

Proposition 7.1. *Let us suppose that* $\deg(q(z)) = n > 5$. *Then the only solutions to* (6.4) *are the following*:

i)

$$(w - K_0\, z^n - K_1\, z - K_2)^2 - (p_0\, z^n + p_1\, z + p_2)^2 = 0\,,$$
$$P(z,w) = z\,(\alpha_2\, z + w + \alpha_1)\,,$$
$$Q(z,w) = -(\alpha_2\, z + w + \alpha_1)\,((n\,\alpha_2 - b_1)z - n\,w + \alpha_1\, n)\,,$$

where $K_0, K_1, K_2, p_0, p_1, p_2$, are parameters such that

$$\begin{cases} K_1 = \dfrac{b_1 - \alpha_2}{2\,n\,(n-1)}\ , \\ K_2 = \dfrac{b_0}{2\,n}\ , \\ p_0 = K_0\ , \\ p_1 = \dfrac{n\,(2\,\alpha_2\, n - b_1 - \alpha_2)}{4\,n\,(n-1)}\ , \\ p_2 = \dfrac{(n-1)(2\,\alpha_1\, n - b_0)}{2\,n\,(n-1)}\ , \\ A = \alpha_2 + b_1\ , \\ B = 2\,n\,, \\ C = b_0\ , \end{cases}$$

and

ii)

$$(w - K_0\, z^n - K_1\, z - K_2)^2 - z^n\,(p_0\, z^n + p_1\, z + p_2) = 0\,,$$
$$P(z,w) = z\Big(\alpha_2\, z + w - \frac{4}{3}\,\alpha_1 + \frac{2\,b_0}{3\,n}\Big)\,,$$
$$\begin{aligned} Q(z,w) = \frac{z}{9\,n\,(n-1)}\,&(n\,(b_1 + (n-2)\,\alpha_2)\,(2\,b_1 - (n+2)\,\alpha_2)\,z \\ &+ (n-2)\,((n+2)\,\alpha_2 - 2\,b_1)\,(b_0 - 2\,n\,\alpha_1)) \\ + b_1\, z\, w + n\, w^2 - \frac{w}{3}\,&(b_0 - 2\,n\,\alpha_1)\,, \end{aligned}$$

where $K_0, K_1, K_2, p_0, p_1, p_2$ *are parameters such that*

$$\begin{cases} K_1 = \dfrac{(n+1)\,\alpha_2 - 2\,b_1}{3\,(n-1)}\,, \\ K_2 = \dfrac{n\,\alpha_1 - 2\,b_0}{3\,n}\,, \\ p_0 = K_0^2\,, \\ p_1 = \dfrac{2\,K_0\,((2\,n-1)\,\alpha_2 - b_1)}{3\,(n-1)}\,, \\ p_2 = -\dfrac{2\,K_0\,(b_0 - 2\,n\,\alpha_1)}{3\,n}\,, \\ A = \dfrac{2}{3}\,((n+1)\,\alpha_2 + b_1)\,, \\ B = 2\,n\,, \\ C = \dfrac{2}{3}\,(n\,\alpha_1 + b_0)\,. \end{cases}$$

The first case is trivial. A qualitative analysis of the second case gives us the following: denoting

$$\nabla = b_0 - 2\,n\,\alpha_1 \equiv \frac{3\,n\,p_2}{K_0}$$

and

$$\tau = (2\,n-1)\,\alpha_2 - b_1 \equiv \frac{3\,(n-1)\,p_1}{2\,K_0}\,,$$

the critical points are

$$(z_1, w_1) = (0, 0)\,,$$

$$(z_2, w_2) = \Big(0, \frac{\nabla}{3\,n}\Big)\,,$$

$$(z_3, w_3) = \Big(\frac{\nabla}{\tau}, \frac{((n-2)\,\alpha_2 - 2\,b_1)\,\nabla}{3\,n\,\tau}\Big)\,,$$

$$(z_4, w_4) = \Big(\frac{(n-1)\,\nabla}{n\,\tau}, \frac{((n+2)\,\alpha_2 - 2\,b_1)\,\nabla}{3\,n\,\tau}\Big)\,.$$

The quantity

$$\delta(z,w) = \partial_z P(z,w)\,\partial_w Q(z,w) - \partial_w P(z,w)\,\partial_z Q(z,w)\,,$$
$$\sigma(z,w) = \partial_z P(z,w) + \partial_w Q(z,w)\,,$$

calculated at the above points give us the following results

$$\delta(z_1,w_1) = \frac{2\,\nabla^2}{9\,n}\,, \qquad \sigma(z_1,w_1) = -\frac{(n+2)\,\nabla}{3\,n}\,,$$
$$\delta(z_2,w_2) = -\frac{\nabla^2}{9\,n}\,, \qquad \sigma(z_2,w_2) = \frac{(n-1)\,\nabla}{3\,n}\,,$$
$$\delta(z_3,w_3) = \frac{2\,\nabla^2}{9\,n^2}\,, \qquad \sigma(z_3,w_3) = 0\,,$$
$$\delta(z_4,w_4) = \frac{2\,\nabla^2}{9\,n^2}\,, \qquad \sigma(z_4,w_4) = \frac{\nabla}{n}\,.$$

Of course, we obtain the bifurcation curves from the equalities: i) $\nabla = 0$, ii) $\tau = 0$. The behaviour of the constructed planar vector field is easily obtained.

In fact, with no loss of generality we shall suppose that $K_0 = 1$ and under the change

$$\begin{cases} \alpha_1 = p_2 - K_2\,, \\ \alpha_2 = p_1 - K_1\,, \\ b_0 = \dfrac{n}{2}\,p_2 - 2\,n\,K_2\,, \\ b_1 = (1-2\,n)\,K_1 + \dfrac{(n+1)\,p_1}{2}\,, \\ z = X\,, \\ w = Y + K_1 X + K_2\,, \end{cases}$$

we deduce that the constructed differential equations coincide with the two dimensional logistic system

$$\begin{cases} \dot X = X\,(p_2 + p_1\,X + Y)\,, \\ \dot Y = Y\left(\dfrac{n}{2}\,p_2 + \dfrac{(n+1)\,p_1}{2}\,X + n\,Y\right). \end{cases} \tag{7.1}$$

The function (5.17) and the equation (6.3) in the coordinates X, Y take the form respectively

$$\text{(7.2)}\qquad \begin{cases} f(X,Y) = Y^2 - 2\,Y\,X^n - p_1\,X^{n+1} - p_2\,X^n\,, \\ \dfrac{df(X,Y)}{dt} = 2\left(\dfrac{n}{2}\,p_2 + \dfrac{(n+1)\,p_1}{2}\,X + n\,Y\right)f(X,Y)\,. \end{cases}$$

The critical points of (7.1) are the following

$$(0,0),\quad \Big(0, -\frac{p_2}{2}\Big),\quad \Big(\frac{-p_2}{p_1}, 0\Big),\quad \Big(\frac{-n\,p_2}{(n-1)\,p_1}, \frac{p_2}{n-1}\Big)\,.$$

Proposition 7.2. *If $p_2 \neq 0$ then the equations* (7.1) *do not admit the first integral which can be developed in a formal power series with respect to X and Y.*

By making a linear approximation of (7.1) we find for arbitrary set of $m_1, m_2 \in \mathbb{N}$, $m_1 + m_2 \geq 0$ and for $p_2 \neq 0$ that

$$\Big(m_1 + \frac{m_2\,n}{2}\Big)p_2 \neq 1\,.$$

Hence, using Liapunov's results, we can prove 7.2.

To study the case when $p_2 = 0$ we can apply the results obtained in [20] and [21], which are related with the arithmetic properties of the Kovalevski exponents.

For the equations (7.1) it is easy to calculate the Kovalevski exponent $\rho_1 = -1$, $\rho_2 = 1 - n$ when $p_1 \neq 0$.

Proposition 7.3. *The equations*

$$\begin{cases} \dot{X} = X(p_1\,X + Y)\,, \\ \dot{Y} = Y\Big(\dfrac{(n+1)\,p_1}{2}\,X + n\,Y\Big)\,, \end{cases}$$

do not admits polynomials first integral.

The proof follows from the fact that for this case, and for an arbitrary set of natural numbers m_1, m_2 such that $m_1 + m_2 \geq 1$ we deduce that

$$m_1\,\rho_1 + m_2\,\rho_2 = m_1 + (n-1)\,m_2 \neq 0\,.$$

By applying the results given in [20] we deduce the veracity of the above assertion.

It is important to observe that autonomous analytic vector field on the plane cannot have chaotic behaviour and so in some sense they are integrable. But under some conditions the first integral is a "bad integral". One of these integrals are Darboux's integrals.

Proposition 7.4. *The system* (7.1) *is Darboux integrable.*

In fact, in view of (7.1), (7.2) we easily get that the function

$$F(X,Y) = f(X,Y)\,Y^{-2}$$

is the Darboux's first integral. It is easy to deduce the following representation for the system (7.1)

$$\begin{cases} \dot{X} = \mu(X,Y)\,\dfrac{\partial F}{\partial Y}\,, \\ \dot{Y} = -\mu(X,Y)\,\dfrac{\partial F}{\partial X}\,, \end{cases}$$

where $\mu(X,Y) = 2\,Y^{-3}\,X^n$.

When $n = 5$, as well as the vector field constructed above, there are two complementary vector fields tangent respectively to the following curves (we suppose that $\beta_1 = 1, \beta_0 = 0$)

$$\begin{cases} (w - K_0\,x^5 + K_1\,x + K_2)^2 \\ \qquad\qquad -\dfrac{1}{6718464\,p_1^4}\,(p_0^2\,x^2 + 24\,p_0\,p_1)\,(p_0\,x^2 - 6\,p_1)^4 = 0\,, \\ p_0 = b_1 - 9\,\alpha_2\,, \\ p_1 = \alpha_0\,, \\ K_j \in \mathbb{R}, \qquad j = 0,1,2 \end{cases}$$

and

$$\begin{cases} (w - K_0\,z^5 - K_1\,z - K_2)^2 \\ \qquad\qquad -\dfrac{3}{2379293284\,p_1^4}\,(p_0^2 + 29\,p_0\,p_1)^2\,(3\,p_0\,z^2 - 58\,p_1)^3 = 0\,, \\ p_0 = b_1 - 9\,\alpha_2\,, \\ p_1 = \alpha_0\,, \\ K_j \in \mathbb{R}, \qquad j = 0,1,2\,. \end{cases}$$

The critical points are easy to find. The quantity δ and σ for these vector fields are, respectively, the following

$$\delta = -\frac{25}{162}\, p_0\, p_1 < 0\,,$$

$$\sigma = -\frac{5}{18}\,\sqrt{p_0\, p_1}\,,$$

and

$$\delta = -25\, p_0\, p_1 < 0\,,$$

$$\sigma - \frac{25}{29}\sqrt{p_0\, p_1}\,.$$

For the polynomial vector field of degree $n > 2$ we can study the problem stated above analogously.

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Average decay of Fourier transforms and geometry of convex sets

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Abstract. Let B be a convex body in $\mathbb{R}^2$, with piecewise smooth boundary and let $\widehat{\chi}_B$ denote the Fourier transform of its characteristic function. In this paper we determine the admissible decays of the spherical L^p-averages of $\widehat{\chi}_B$ and we relate our analysis to a problem in the geometry of convex sets. As an application we obtain sharp results on the average number of integer lattice points in large bodies randomly positioned in the plane.

1. Introduction.

Given a convex body B, that is, a compact convex set with non empty interior in $\mathbb{R}^n$, we denote by χ_B its characteristic function. The study of the decay of the Fourier transform

$$\widehat{\chi}_B(\xi) = \int_B e^{-2\pi i \xi \cdot x}\, dx\,,$$

as $|\xi| \longrightarrow \infty$, in terms of the geometric properties of B, is a fascinating and by now classical subject (see [18, Chapter VIII] for basic results, related problems and references). For instance, it is well known that, when the boundary is smooth with everywhere strictly positive Gauss-

Kronecker curvature, the order of decay of $\widehat{\chi}_B$ in a given direction is independent of this latter.

This situation is far from being typical, as one can easily check by considering either a cube or any convex body with a smooth boundary containing flat points. Furthermore, a number of problems requires some sort of “global information” on the decay of $\widehat{\chi}_B(\xi)$ which is not a direct consequence of the presently known directional estimates.

In this setting, the study of the spherical L^p-averages

$$\Big(\int_{\Sigma_{n-1}} |\widehat{\chi}_B(\rho\,\sigma)|^p\,d\sigma\Big)^{1/p}$$

turns out to be quite useful.

We point out that the L^2 case has been investigated by various authors, notably [14], [15], [19], [13], [12]; while for general p's and B a polyhedron, a detailed analysis with applications to problems on lattice points and on irregularities of distributions can be found in [3]. We note that the L^1 case is also naturally related with the summability of multiple Fourier integrals (see *e.g.* [2] or [4]), moreover, F. Ricci and one of us (G. Travaglini) have recently shown that the general L^p case is connected to boundedness of Radon transforms (see [16]).

Throughout this paper, unless otherwise explicitly stated, we consider convex bodies B in $\mathbb{R}^2$ with piecewise smooth boundary. More precisely, we assume that ∂B is a union of a finite number of regular arcs, each one of them being C^∞ in its interior.

According to a more general result of Podkorytov [13], (see also [19]) the L^2-average decay of $\widehat{\chi}_B$ satisfies

$$\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^2\,d\theta\Big)^{1/2} \le c\,\rho^{-3/2}\,, \tag{1.1}$$

where, from now on,

$$\Theta = (\cos\theta, \sin\theta)\,, \qquad \theta \in [0, 2\pi)\,,$$

$\rho \ge 1$ and $c, c_1, c_2, \dots,$ denote positive constants independent of ρ which may change from line to line.

It is an easy consequence of a result of Montgomery [12, p. 116] that (1.1) is sharp. Namely, for any B,

$$\limsup_{\rho\to\infty} \rho^{3/2}\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^2\,d\theta\Big)^{1/2} > 0\,. \tag{1.2}$$

We stress that in the L^2 case the order of decay is independent of B. The aim of this paper is to study the general L^p case where the results turn out to depend on the shape of B.

It is worth to begin with the case of a polygon P. It has been proved in [3] that

$$
\begin{aligned}
&\Big(\int_0^{2\pi} |\widehat{\chi}_P(\rho\,\Theta)|^p \, d\theta\Big)^{1/p} \\
&\qquad\qquad \leq \begin{cases} c\,\rho^{-2}\log(1+\rho)\,, & \text{when } p=1\,,\\ c\,\rho^{-1-1/p}\,, & \text{when } 1<p\leq\infty\,. \end{cases}
\end{aligned}
\tag{1.3}
$$

Here we prove

$$
\int_0^{2\pi} |\widehat{\chi}_P(\rho\,\Theta)| \, d\theta \geq c\,\rho^{-2}\log(1+\rho)
$$

and, for each $1<p\leq\infty$

$$
\limsup_{\rho\to\infty} \rho^{1+1/p}\Big(\int_0^{2\pi} |\widehat{\chi}_P(\rho\,\Theta)|^p \, d\theta\Big)^{1/p} > 0\,. \tag{1.4}
$$

Next, we consider the case when B is not a polygon. We show that

$$
\limsup_{\rho\to\infty} \rho^{3/2}\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p \, d\theta\Big)^{1/p} > 0\,, \tag{1.5}
$$

whenever $1\leq p\leq\infty$ (note that, when $p=2$, (1.4) and (1.5) agree with (1.2)). These results, when compared with (1.1) and (1.3), completely describe the case $1\leq p\leq 2$. As for $p\geq 2$, an easy interpolation argument between $p=2$ and $p=\infty$ gives

$$
\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p \, d\theta\Big)^{1/p} \leq c\,\rho^{-1-1/p}\,,
$$

for every $2\leq p\leq\infty$. Contrary to the case $1\leq p<2$, in the range $2<p\leq\infty$ every order of decay between $\rho^{-3/2}$ and $\rho^{-1-1/p}$ is possible. More precisely we exhibit, for any $2\leq p\leq\infty$ and $1+1/p\leq a\leq 3/2$, a corresponding convex body B such that

$$
c_1\,\rho^{-a} \leq \Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p \, d\theta\Big)^{1/p} \leq c_2\,\rho^{-a}\,.
$$

When $1 + 1/p < a < 3/2$ such examples are constructed so to have, for a suitable $\gamma > 2$, a piece of the curve of equation $y = |x|^\gamma$ in its boundary. As a side-product, we obtain a result on the average decay of the Fourier transforms of singular measures supported on the above curves (see Proposition 3.17 below).

The different results for $p < 2$ and $p > 2$ are due to the following fact. When B is not a polygon, its boundary ∂B must contain points with positive curvature and for $1 \le p \le 2$ they give the relevant contribution to

$$\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p \, d\theta \Big)^{1/p} .$$

On the other hand, when $2 \le p \le \infty$ the main contribution is given by the flat points (if any), as one may guess considering the L^∞ case.

We summarize the main results discussed so far in Figure 1. For $p > 1$ and $a > 0$ the point $(1/p, a)$ is marked black if and only if there exists B satisfying

$$\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p \, d\theta \Big)^{1/p} < c\,\rho^{-a}$$

and

$$\limsup_{\rho\to\infty} \rho^a \Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p \, d\theta \Big)^{1/p} > 0\,.$$

Figure 1.

It is natural to ask whether (1.4) and (1.5) can be turned into estimates from below. As a matter of fact, a negative answer is given by the two simplest examples of convex bodies in $\mathbb{R}^2$: the square (see Lemma 3.12) and the disc (because of the zeroes of the Bessel function J_1). On the other hand, we show that, for any $1 \le p \le \infty$ and for some polygons P, we have

$$\Big(\int_0^{2\pi} |\widehat{\chi}_P(\rho\,\Theta)|^p \, d\theta \Big)^{1/p} \ge c_2 \, \rho^{-1-1/p} \,,$$

while, if B is neither a polygon nor a body too "similar" (see Definition 3.3) to a disc, then

$$\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)| \, d\theta \ge c \, \rho^{-3/2} \,.$$

The above results are organized in our main theorem of Section 2. We stress that such general L^p estimates hold provided ∂B is piecewise smooth. In Section 4 we shall see that in the framework of arbitrary convex bodies one can find very "chaotic" situations.

A basic tool in some of our proofs is the following known fact.

Let $S_\theta = \sup_{x \in B} x \cdot \Theta$. For $\delta > 0$ sufficiently small we define, see Figure 2, the set

$$A_B(\delta, \theta) = \{ x \in B : S_\theta - \delta \le x \cdot \Theta \le S_\theta \} \,. \tag{1.6}$$

Figure 2.

Then (see Lemma 3.8, [5], or [13])

$$|\widehat{\chi}_B(\rho\,\Theta)| \le c\,(|A_B(\rho^{-1},\theta)| + |A_B(\rho^{-1},\theta+\pi)|)\,,$$

where $|K|$ denotes the Lebesgue measure of a measurable set K.

As a consequence, for each $p \ge 1$,

$$\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p\,d\theta\Big)^{1/p} \le c\Big(\int_0^{2\pi} |A_B(\rho^{-1},\theta)|^p\,d\theta\Big)^{1/p} \tag{1.7}$$

providing a way to estimate the average decay of $\widehat{\chi}_B$ from above. Moreover we shall see (*cf.* also [5]) that (1.7) can be reversed under additional assumptions on B.

Observe that the right hand side of (1.7) does not involve any Fourier transform and the problem of estimating

$$\Big(\int_0^{2\pi} |A_B(\delta,\theta)|^p\,d\theta\Big)^{1/p},$$

as $\delta \longrightarrow 0$, is indeed a genuine problem in the geometry of convex sets. To the best of our knowledge, such a problem has never been considered before and the closest area in the field is perhaps the study of floating bodies (see *e.g.* [17]). In Section 5 we shall investigate the admissible decays of

$$\Big(\int_0^{2\pi} |A_B(\delta,\theta)|^p\,d\theta\Big)^{1/p},$$

as $\delta \longrightarrow 0$, mostly as a consequence of the similar problem for $\widehat{\chi}_B$.

We end the paper by applying some of the previous results to a problem on the number of lattice points in a large convex planar body $\rho\,B$.

Elementary geometric considerations show that

$$\mathrm{card}\,(\rho B \cap \mathbb{Z}^2) \sim \rho^2|B|$$

and

$$\mathrm{card}\,(\rho B \cap \mathbb{Z}^2) - \rho^2|B| = O(\rho) \tag{1.8}$$

as $\rho \longrightarrow \infty$. The improvement of (1.8) and the related problems constitute a whole area of research (see *e.g.* [11] or [8]), where the pointwise estimate (1.8) is often substituted by mean estimates.

Here we consider a large convex body ρB randomly positioned in the plane. More precisely, for $\sigma \in SO(2)$ and $t \in \mathbb{R}^2$ we study the *discrepancy*

$$D_B(\rho, \sigma, t) = \operatorname{card}\left((\rho\, \sigma^{-1}(B) - t) \cap \mathbb{Z}^2\right) - \rho^2 |B| \,,$$

where $\rho\, \sigma^{-1}(B) - t$ is a rotated, dilated and translated copy of B. Since this function is periodic with respect to the variable t we restrict this latter to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Kendall ([10]) has proved L^2 estimates related to the above discrepancy (see also [3]). Here we prove that if B is a convex planar body with piecewise smooth boundary, different from a polygon, then, for any $1 \le p \le 2$,

$$c_1\, \rho^{1/2} \le \|D_B(\rho, \cdot, \cdot)\|_{L^p(SO(2)\times\mathbb{T}^2)} \le c_2\, \rho^{1/2} \,. \tag{1.9}$$

We do not know whether (1.9) holds for some $p > 2$. We point out that, in general, it is false when $p = \infty$. Indeed, as a consequence of Hardy's Ω-result for the circle problem (see [7] or [11]) we have, for a disc D,

$$\limsup_{\rho\to\infty} \rho^{-1/2} (\log \rho)^{-1/4} \, \|D_D(\rho, \cdot, \cdot)\|_{L^\infty(SO(2)\times\mathbb{T}^2)} > 0 \,.$$

2. Statement of the main result.

Let Σ_1 be the unit circle in $\mathbb{R}^2$. For any complex measurable function g on Σ_1 and for any $p \ge 1$, let

$$\|g\|_{L^p(\Sigma_1)} = \Big(\int_0^{2\pi} |g(\Theta)|^p \, d\theta \Big)^{1/p} ,$$

where $d\theta$ is the normalized Lebesgue measure. As usual we set

$$\|g\|_{L^\infty(\Sigma_1)} = \operatorname*{ess\,sup}_{\Theta\in\Sigma_1} |g(\Theta)| \,.$$

Let B be a convex body in $\mathbb{R}^2$; $\varphi : [1, +\infty) \longrightarrow \mathbb{R}^+$ a non-increasing function and let $1 \le p \le \infty$. We say that φ is an optimal estimate of the p-average decay of $\widehat{\chi}_B$ whenever

i) $\|\widehat{\chi}_B(\rho\, \cdot)\|_{L^p(\Sigma_1)} \le c\, \varphi(\rho)$,

ii) $\displaystyle\limsup_{\rho\to\infty} \frac{\|\widehat{\chi}_B(\rho\,\cdot)\|_{L^p(\Sigma_1)}}{\varphi(\rho)} > 0\,.$

Similarly, φ is a sharp estimate of the p-average decay of $\widehat{\chi}_B$ provided

$$c_1\,\varphi(\rho) \le \|\widehat{\chi}_B(\rho\,\cdot)\|_{L^p(\Sigma_1)} \le c_2\,\varphi(\rho)\,.$$

Our main result essentially concerns the case $\varphi(\rho) = \rho^{-a}$ and the following definition will be useful.

Definition 2.1. *When $\varphi(\rho) = \rho^{-a}$ is an optimal or sharp estimate of the p-average decay of $\widehat{\chi}_B$ we say that the p-average decay of $\widehat{\chi}_B$ has optimal order a or sharp order a respectively.*

With this preparation we state our main result.

Theorem 2.2. I) *Let $1 < p \le \infty$ and define*

$$S = \Big\{ \Big(\frac{1}{p}, a\Big) :\ 1 < p < 2,\ a = \frac{3}{2} \text{ or } a = 1 + \frac{1}{p} \Big\}\,,$$

$$T = \Big\{ \Big(\frac{1}{p}, a\Big) :\ 2 \le p \le \infty,\ 1 + \frac{1}{p} \le a \le \frac{3}{2} \Big\}\,.$$

The following are equivalent:

i) *There exists a convex body B with piecewise C^∞ boundary such that the p-average decay of $\widehat{\chi}_B$ has optimal order a.*

ii) $(1/p, a) \in S \cup T$.

II) *Let $p = 1$. If P is a polygon then $\varphi(\rho) = \rho^{-2}\log(1+\rho)$ is an optimal estimate for the 1-average decay of $\widehat{\chi}_P$. If B is any other convex body with piecewise C^∞ boundary, then the 1-average decay of $\widehat{\chi}_B$ has optimal order $3/2$.*

Moreover it will be clear from the proof that this result still holds after substituting the word "optimal" with the word "sharp".

The above theorem will be obtained as a consequence of the following somewhat more informative results.

In the first Proposition we cover the case $1 \le p \le 2$ when B is not a polygon.

Proposition 2.3. *Let $1 \leq p \leq 2$ and let B be a convex body with piecewise C^∞ boundary. Suppose B is not a polygon, then $3/2$ is the optimal order of the p-average decay of $\widehat{\chi}_B$. Moreover, $3/2$ is the sharp order of the p-average decay of $\widehat{\chi}_B$ for some, but not for all, bodies B.*

The above Proposition follows from Lemma 3.1, Lemma 3.2, Lemma 3.6 and the example of the disc.

We now consider the case of a polygon.

Proposition 2.4. *Let P be a compact convex polygon with non empty interior. Then $\varphi(\rho) = \rho^{-2}\log(1+\rho)$ is a sharp estimate of the 1-average decay of $\widehat{\chi}_P$. If $1 < p \leq \infty$, then $1 + 1/p$ is the optimal order of the p-average decay of $\widehat{\chi}_P$. Moreover, $1+1/p$ is the sharp order of the p-average decay of $\widehat{\chi}_P$ for some, but not for all, polygons P.*

This is a consequence of Lemma 3.9, Lemma 3.10, Lemma 3.11 and Lemma 3.12.

Finally, for $2 \leq p \leq \infty$, we have

Proposition 2.5. *Let $2 \leq p \leq \infty$, then the following are equivalent:*

j) *There exists a convex body B with piecewise C^∞ boundary such that the p-average decay of $\widehat{\chi}_B$ has optimal order a.*

jj) $1 + 1/p \leq a \leq 3/2$.

The above Proposition follows from Lemma 3.2, Lemma 3.13 and Lemma 3.16.

3. Lemmas.

The following lemma is contained in [13, p. 63].

Lemma 3.1. *Let B be a convex body in $\mathbb{R}^2$. Then*

$$\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^2\,d\theta\Big)^{1/2} \leq c\,\rho^{-3/2}.$$

We now prove the following result.

Lemma 3.2. *Let B be a convex body in $\mathbb{R}^2$ with piecewise C^∞ boundary ∂B. Assume B is not a polygon then, for any $p \geq 1$,*

$$\limsup_{\rho\to\infty} \rho^{3/2}\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p\,d\theta\Big)^{1/p} > 0\,.$$

Proof. It is enough to prove the lemma when $p = 1$. Let Γ be an arc in ∂B where the curvature is strictly positive. We examine two cases.

i) There exists an open interval U of angles θ such that for every $\theta \in U$ there is exactly one point $\sigma(\theta) \in \Gamma$ whose tangent is orthogonal to $\Theta = (\cos\theta, \sin\theta)$ (this may happen since ∂B is only *piecewise* smooth).

ii) There exists an open interval U of angles θ such that for every $\theta \in U$ there are exactly two points $\sigma_1(\theta), \sigma_2(\theta) \in \partial B$ whose tangent is orthogonal to Θ.

We proceed with the proof in case i).
We apply [1, Theorem 1] (see also [13]) to obtain

$$\widehat{\chi}_B(\rho\,\Theta) = -\frac{1}{2\pi i}\,\rho^{-3/2}\,e^{-2\pi i\rho\Theta\cdot\sigma(\theta)+\pi i/4}K^{-1/2}(\sigma(\theta)) + E_\rho\,, \tag{3.1}$$

where $K(P)$ denotes the curvature at $P \in \partial B$ and $|E_\rho| \leq c\,\rho^{-2}$. We remark that although [1, Theorem 1] is stated for sets with smooth boundary, in the bidimensional case it still holds true for sets having a piecewise smooth boundary. From (3.1) we have

$$\rho^{3/2}\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|\,d\theta \geq \frac{1}{2\pi}\int_U K^{-1/2}(\sigma(\theta))\,d\theta - c_1\,\rho^{-1/2} \geq c_2 > 0\,.$$

We now turn to ii).
As in the previous case we obtain

$$\widehat{\chi}_B(\rho\,\Theta) = -\frac{1}{2\pi i}\,\rho^{-3/2}\sum_{j=1}^{2} e^{-2\pi i\rho\Theta\cdot\sigma_j(\theta)+\pi i/4}K^{-1/2}(\sigma_j(\theta)) + E_\rho\,.$$

We consider three subcases.

a) Suppose first there exists a neighborhood $\widetilde{U} \subseteq U$ where

$$K(\sigma_1(\theta)) \neq K(\sigma_2(\theta))\,.$$

Then

$$\begin{aligned}\rho^{3/2}&\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|\,d\theta\\ &\geq \frac{1}{2\pi}\int_{\widetilde{U}} |K^{-1/2}(\sigma_1(\theta)) - K^{-1/2}(\sigma_2(\theta))|\,d\theta - c_1\,\rho^{-1/2} \geq c_2 > 0\,.\end{aligned}$$

b) Suppose there exists a neighborhood $\widetilde{U} \subseteq U$ where the vectors Θ and $\sigma_2(\theta) - \sigma_1(\theta)$ are not parallel. Let $A_j(\theta) = K^{-1/2}(\sigma_j(\theta))$. We have

$$\begin{aligned}\rho^{3/2}&\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|\,d\theta\\ &\geq \frac{1}{2\pi}\int_{\widetilde{U}} \Big|\sum_{j=1}^{2} e^{-2\pi i\rho\Theta\cdot\sigma_j(\theta)} A_j(\theta)\Big|\,d\theta - c_1\rho^{-1/2}\\ &= \frac{1}{2\pi}\int_{\widetilde{U}} |A_1(\theta) + A_2(\theta)\,e^{-2\pi i\rho\Theta\cdot(\sigma_2(\theta)-\sigma_1(\theta))}|\,d\theta - c_1\,\rho^{-1/2}\\ &\geq \frac{1}{2\pi}\Big|\int_{\widetilde{U}} (A_1(\theta) + A_2(\theta)\,e^{-2\pi i\rho\Theta\cdot(\sigma_2(\theta)-\sigma_1(\theta))})\,d\theta\Big| - c_1\,\rho^{-1/2}\\ &\geq M - \frac{1}{2\pi}\Big|\int_{\widetilde{U}} A_2(\theta)\,e^{-2\pi i\rho\Theta\cdot(\sigma_2(\theta)-\sigma_1(\theta))}\,d\theta\Big| - c_1\,\rho^{-1/2}\,.\end{aligned}$$

We claim that the last integral tends to zero as ρ tends to infinity. Observe that $\Theta\cdot\sigma_j'(\theta) = 0$ since Θ is normal to ∂B at the point $\sigma_j(\theta)$. Hence

$$\frac{d}{d\theta}\,(\Theta\cdot(\sigma_2(\theta) - \sigma_1(\theta))) = (-\sin\theta, \cos\theta)\cdot(\sigma_2(\theta) - \sigma_1(\theta)) \tag{3.2}$$

is different from zero since $(-\sin\theta, \cos\theta)$ is not orthogonal to $\sigma_2(\theta) - \sigma_1(\theta)$. Integration by parts shows that the integral vanishes as $\rho \longrightarrow +\infty$.

c) We suppose now that for every $\theta \in U$ the points $\sigma_1(\theta)$, $\sigma_2(\theta)$ have the same curvature and that Θ and $\sigma_2(\theta) - \sigma_1(\theta)$ are parallel. In this case the quantity (3.2) vanishes so that

$$\lambda = \Theta\cdot(\sigma_2(\theta) - \sigma_1(\theta))$$

is constant. Let $K(\theta) = K(\sigma_1(\theta)) = K(\sigma_2(\theta))$, then

$$\begin{aligned}\rho^{3/2}\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|\,d\theta &\geq \frac{1}{2\pi}\int_{\widetilde{U}} K^{-1/2}(\theta)\,|1+e^{-2\pi i\rho\lambda}|\,d\theta - c_1\,\rho^{-1/2}\\ &\geq \frac{1}{2\pi}\,|1+e^{-2\pi i\rho\lambda}|\int_{\widetilde{U}} K^{-1/2}(\theta)\,d\theta - c_1\,\rho^{-1/2}\,,\end{aligned}$$

and since

$$\limsup_{\rho\to+\infty} |1+e^{-2\pi i\rho\lambda}| > 0\,,$$

the proof is complete.

The result of the previous lemma can be strengthened under simple geometric hypothesis on the boundary. The following definition may be useful.

Definition 3.3. *We say that a convex body B is a cut disc if it is not a polygon and if its boundary ∂B is the union of a finite number of segments and of a finite number of couples of antipodal arcs of a given circle.*

We now need a technical lemma.

Lemma 3.4. *Let I and J be two neighborhoods of the origin in $\mathbb{R}$ and let $f \in C^2(I)$, $g \in C^2(J)$. Assume $f(x) < 0$, $f''(x) > 0$, for $x \in I$, $g(x) > 0$, $g''(x) < 0$ for $x \in J$; also suppose $f(0) = -1$, $g(0) = 1$, $f'(0) = g'(0) = 0$. Finally we assume the existence of a bijection $H : I \longrightarrow J$ such that*

i) $f'(x) = g'(H(x))$,

ii) *the curvature of the graph of f at $(x, f(x))$ equals the curvature of the graph of g at $(H(x), g(H(x)))$,*

iii) *the segment joining the points $(x, f(x))$ and $(H(x), g(H(x)))$ is orthogonal to the tangent lines at these points.*

Then the graphs of f and g are two (antipodal) arcs of equal length in the same circle.

Proof. By our assumptions,

i) $f'(x) = g'(H(x))$,

ii) $\dfrac{f''(x)}{(1+(f'(x))^2)^{3/2}} = \dfrac{-g''(H(x))}{(1+(g'(H(x)))^2)^{3/2}}$,

iii) $(x - H(x)) + (f(x) - g(H(x)))f'(x) = 0$.

Then i) and ii) imply $f''(x) = -g''(H(x))$, while differentiating i) one gets $f''(x) = g''(H(x))\, H'(x)$. Because of the other assumptions, this implies $H(x) = -x$ and

$$f(x) = -g(-x)\,.$$

Then iii) becomes

$$2\,x + 2\,f(x)\,f'(x) = 0\,,$$

which gives the equation of a circle.

Lemma 3.4 can be restated in the following, more geometrical, way.

Lemma 3.5. *Suppose B is a convex body with piecewise C^∞ boundary which is not a cut disc, then ∂B contains a regular point P with unit exterior normal Θ such that either there is no other regular point in ∂B with unit exterior normal $-\Theta$, or, if such a point Q exists, at least one of the following facts happens:* i) *$P - Q$ is not parallel to Θ,* ii) *the curvatures of ∂B at P and at Q differ.*

The following is a strengthened version of Lemma 3.2.

Lemma 3.6. *Suppose B is a convex body with piecewise C^∞ boundary which is neither a polygon nor a cut disc, then, for $1 \le p \le 2$,*

$$c_1\,\rho^{-3/2} \le \Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p\,d\theta\Big)^{1/p} \le c_2\,\rho^{-3/2}\,.$$

PROOF. The estimate from above is contained in Lemma 3.1. On the other hand the estimate from below holds in cases i), ii)-a, ii)-b of the proof of Lemma 3.2. Our assumptions and Lemma 3.5 exclude the case ii)-c. This ends the proof .

The forthcoming lemma is probably known. However, since we have not found a suitable reference, we provide an elementary argument.

Lemma 3.7. *Let $f : \mathbb{R} \longrightarrow [0,+\infty)$ be supported and concave in $[-1,1]$. Then, for every $|\xi| \geq 1$,*

$$|\widehat{f}(\xi)| \leq \frac{1}{|\xi|}\Big(f\Big(1 - \frac{1}{2\,|\xi|}\Big) + f\Big(-1 + \frac{1}{2\,|\xi|}\Big)\Big). \tag{3.3}$$

Proof. It is enough to prove (3.3) when $\xi > 1$. The assumption on the concavity of f allows us to integrate by parts obtaining

$$|\widehat{f}(\xi)| \leq \frac{1}{2\pi\xi}\, f(1^-) + \frac{1}{2\pi\xi}\, f(-1^+) + \frac{1}{2\pi\xi}\Big|\int_{-1}^{1} f'(t)\, e^{-2\pi i\xi t}\, dt\Big|.$$

Let α be a point where f attains its maximum. Then f will be non-decreasing in $[-1,\alpha]$ and non-increasing in $[\alpha,1]$. We can assume $0 \leq \alpha \leq 1$, so that $f(-1^+) \leq f(-1+1/(2\,\xi))$. To estimate $f(1^-)$ we observe that when $\alpha \leq 1 - 1/(2\,\xi)$, one has $f(1^-) \leq f(1 - 1/(2\,\xi))$. On the other hand, since f is concave, in case $\alpha > 1 - 1/(2\,\xi)$ we have

$$f(1^-) \leq f(\alpha) \leq 2f(0) \leq 2f\Big(1 - \frac{1}{2\,\xi}\Big).$$

To estimate the integral we observe that, by a change of variable,

$$I = \int_{-1}^{1} f'(t)\, e^{-2\pi i\xi t}\, dt = -\int_{-1+1/(2\xi)}^{1+1/(2\xi)} f'\Big(t - \frac{1}{2\,\xi}\Big)\, e^{-2\pi i\xi t}\, dt.$$

So that

$$\begin{aligned}
2I &= \int_{-1}^{1} f'(t)\, e^{-2\pi i\xi t}\, dt - \int_{-1+1/(2\xi)}^{1+1/(2\xi)} f'\Big(t - \frac{1}{2\,\xi}\Big)\, e^{-2\pi i\xi t}\, dt \\
&= \int_{-1}^{-1+1/(2\xi)} f'(t)\, e^{-2\pi i\xi t}\, dt \\
&\quad + \int_{-1+1/(2\xi)}^{1} \Big(f'(t) - f'\Big(t - \frac{1}{2\,\xi}\Big)\Big)\, e^{-2\pi i\xi t}\, dt \\
&\quad + \int_{1}^{1+1/(2\xi)} f'\Big(t - \frac{1}{2\,\xi}\Big)\, e^{-2\pi i\xi t}\, dt \\
&= I_1 + I_2 + I_3\,.
\end{aligned}$$

To estimate I_1 from above we note that

$$|I_1| \le \int_{-1}^{-1+1/(2\xi)} f'(t)\,dt = f\Big(-1+\frac{1}{2\,\xi}\Big) - f(-1^+) \le f\Big(-1+\frac{1}{2\,\xi}\Big),$$

since $0 \le \alpha \le 1$.

The estimate for I_3 is similar in case $\alpha \le 1 - 1/(2\,\xi)$. If $\alpha > 1 - 1/(2\,\xi)$, then

$$\begin{aligned}
|I_3| &\le \int_1^{\alpha+1/(2\xi)} f'\Big(t-\frac{1}{2\,\xi}\Big)\,dt - \int_{\alpha+1/(2\xi)}^{1+1/(2\xi)} f'\Big(t-\frac{1}{2\,\xi}\Big)\,dt \\
&= 2f(\alpha) - f\Big(1-\frac{1}{2\,\xi}\Big) - f(1^-) \\
&\le 2f(\alpha) \\
&\le 4f(0) \\
&\le 4f\Big(1-\frac{1}{2\,\xi}\Big).
\end{aligned}$$

As for I_2, since f' is non increasing, we have

$$\begin{aligned}
|I_2| &\le \int_{-1+1/(2\xi)}^{1} \Big(f'\Big(t-\frac{1}{2\,\xi}\Big) - f'(t)\Big)\,dt \\
&= f\Big(1-\frac{1}{2\,\xi}\Big) - f(-1^+) - f(1^-) + f\Big(-1+\frac{1}{2\,\xi}\Big) \\
&\le f\Big(1-\frac{1}{2\,\xi}\Big) + f\Big(-1+\frac{1}{2\,\xi}\Big)
\end{aligned}$$

ending the proof. Note that no constant c is missing in (3.3).

Remark. A different proof of the above lemma can be modeled on an argument similar to that of Lemma 3.15 below.

The following result is similar to [5, Theorem 6.1] (see also [13, Lemma 3]). Our proof is based on the previous lemma.

Lemma 3.8. *Let B be a convex body in $\mathbb{R}^2$, $\Theta = (\cos\theta, \sin\theta)$ and $S_\theta = \sup_{x\in B} x\cdot\Theta$. For $\rho \ge 1$ we set (see Figure 2 with ρ^{-1} in place of δ)*

$$A_B(\rho^{-1},\theta) = \{x \in B:\ S_\theta - \rho^{-1} \le x\cdot\Theta \le S_\theta\}.$$

Then

$$|\widehat{\chi}_B(\rho\,\Theta)| \le c\,(|A_B(\rho^{-1},\theta)| + |A_B(\rho^{-1},\theta+\pi)|)\,,$$

where $|E|$ *denotes the Lebesgue measure of a measurable set* E.

Proof. Without loss of generality we choose $\Theta = (1,0)$. Then

$$\begin{aligned}\widehat{\chi}_B(\xi_1,0) &= \int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}\chi_B(x_1,x_2)\,dx_2\Big)\,e^{-2\pi i x_1\xi_1}\,dx_1\\ &= \widehat{h}(\xi_1)\,,\end{aligned} \tag{3.4}$$

where $h(s)$ is the lenght of the segment obtained intersecting B with the line $x_1 = s$. Observe that h is concave on its support, say $[a,b]$. We can therefore apply Lemma 3.7 to obtain, after a change of variable,

$$\begin{aligned}|\widehat{h}(\xi_1)| &\le \frac{1}{|\xi_1|}\Big(h\Big(b-\frac{1}{2\,|\xi_1|}\Big)+h\Big(a+\frac{1}{2\,|\xi_1|}\Big)\Big)\\ &\le c\,(|A_B(|\xi_1|^{-1},0)| + |A_B(|\xi_1|^{-1},\pi)|)\,.\end{aligned}$$

We now consider polygons.

The following lemma appears in [3]; here we give a different, more geometric, argument based on the previous lemma.

Lemma 3.9. *Let* P *be a compact polygon in* $\mathbb{R}^2$. *Then*

$$\Big(\int_0^{2\pi}|\widehat{\chi}_P(\rho\,\Theta)|^p\,d\theta\Big)^{1/p} \le \begin{cases} c\,\rho^{-2}\log(1+\rho)\,, & \textit{when } p=1\,,\\ c\,\rho^{-1-1/p}\,, & \textit{when } p>1\,.\end{cases} \tag{3.5}$$

Proof. Without loss of generality we can assume that the polygon is convex, lies in the left halfplane and that the points $(0,-1)$ and $(0,1)$ are vertices. By Lemma 3.8 we reduce the problem to estimating $|A_P(1/\rho,\theta)|$ in a suitable right neighborhood of zero. A simple geometric consideration shows that

$$|A_P(\rho^{-1},\theta)| \le \begin{cases} c\,\rho^{-1}\,, & \text{for } 0\le\theta\le c_1\,\rho^{-1}\,,\\ c_2\,\rho^{-1}\theta^{-1}\,, & \text{for } c_1\,\rho^{-1}\le\theta\le c_3\,,\end{cases}$$

which implies (3.5) by integration.

We still have to check sharpness of the estimates in (3.5). This is not entirely trivial since parallel edges of P (if any) give the same contribution to the decay of $\widehat{\chi}_P$ so that cancellations may occur. Actually this does not happen for $p = 1$, but it may happen for $p > 1$, as shown in the next three lemmas.

Lemma 3.10. *Let χ_P be the characteristic function of a compact convex polygon P in $\mathbb{R}^2$ with non empty interior. Then*

$$\int_0^{2\pi} |\widehat{\chi}_P(\rho\,\Theta)|\,d\theta \geq c\,\rho^{-2}\log(1+\rho)\,.$$

Proof. Let $L_j = [P_j, P_{j+1}]$, $j = 1, \ldots, S$, be the edges of the polygon P and let l_j be their lengths. Then, with the aid of the divergence formula, we obtain

$$\begin{aligned}
\widehat{\chi}_P(\rho\,\Theta) &= \int_P e^{-2\pi i\rho\Theta\cdot t}\,dt \\
&= -\frac{1}{2\pi i\rho}\int_{\partial P} e^{-2\pi i\rho\Theta\cdot t}\,\Theta\cdot\nu(t)\,dt' \\
&= \frac{1}{4\pi^2\rho^2}\sum_{j=1}^{S}\Theta\cdot\nu_j\,\frac{e^{-2\pi i\rho\Theta\cdot P_{j+1}} - e^{-2\pi i\rho\Theta\cdot P_j}}{\Theta\cdot(P_{j+1}-P_j)}\,l_j\,,
\end{aligned}$$

where dt' is the 1-dimensional measure and ν_j is the outward unit normal to L_j. The argument is divided in three cases.

Case 1. Suppose there exists an edge, say L_1, which is not parallel to any other edge. We can suppose $P_1 = (0,-1)$ and $P_2 = (0,1)$.

Because of these assumptions there exists a right neighborhood $U(0) \subset [0, 2\pi)$ such that

$$\inf_{\theta\in U(0)} |\Theta\cdot(P_{j+1}-P_j)| \geq c > 0\,,$$

for each $j \geq 2$. Hence

$$\begin{aligned}
\int_0^{2\pi} |\widehat{\chi}_P(\rho\,\Theta)|\,d\theta &\geq \frac{c_1}{\rho^2}\int_{U(0)}\Big|\Theta\cdot\nu_1\,\frac{e^{-2\pi i\rho\Theta\cdot P_2} - e^{-2\pi i\rho\Theta\cdot P_1}}{\Theta\cdot(P_2-P_1)}\,l_1\Big|\,d\theta - \frac{c_2}{\rho^2} \\
&\geq \frac{c_3}{\rho^2}\int_{U(0)}\Big|\cos\theta\,\frac{\sin(2\pi\rho\sin\theta)}{\sin\theta}\Big|\,d\theta - \frac{c_2}{\rho^2}
\end{aligned}$$

$$\geq \frac{c_4}{\rho^2}\int_0^{c_5}\Big|\frac{\sin(2\pi\rho\, u)}{u}\Big|\,du - \frac{c_2}{\rho^2}$$
$$\geq c\,\rho^{-2}\log(1+\rho)\,.$$

Case 2. Suppose there exists a couple of parallel edges of different length. Let $M_1 = [Q_1, R_1]$ and $M_2 = [Q_2, R_2]$ be such a pair.

We can assume $Q_1 = H_1 + (0, -a_1)$, $R_1 = H_1 + (0, a_1)$, $Q_2 = H_2 + (0, -a_2)$, $R_2 = H_2 + (0, a_2)$ with $a_2 > a_1 > 0$.

Then, arguing as above,

$$\begin{aligned}
&\int_0^{2\pi}|\widehat{\chi}_P(\rho\,\Theta)|\,d\theta\\
&\quad\geq \frac{c_1}{\rho^2}\int_{U(0)}\Big|\sum_{j=1}^{2}\Theta\cdot\nu_1\,\frac{e^{-2\pi i\rho\Theta\cdot Q_j}-e^{-2\pi i\rho\Theta\cdot R_j}}{\Theta\cdot(Q_j-R_j)}\,l_j\Big|\,d\theta-\frac{c_2}{\rho^2}\\
&\quad\geq \frac{c_1}{\rho^2}\int_{U(0)}\Big|\cos\theta\sum_{j=1}^{2}e^{-2\pi i\rho\Theta\cdot H_j}\,\frac{\sin(2\pi\rho\,a_j\sin\theta)}{a_j\sin\theta}\Big|\,d\theta-\frac{c_2}{\rho^2}\\
&\quad\geq \frac{c_3}{\rho^2}\Big(\int_0^{c_4}\Big|\frac{\sin(2\pi\rho\,a_1\,u)}{a_1\,u}\Big|\,du-\int_0^{c_4}\Big|\frac{\sin(2\pi\rho\,a_2\,u)}{a_2\,u}\Big|\,du\Big)-\frac{c_2}{\rho^2}\\
&\quad\geq \frac{c_3}{\rho^2}\Big(\frac{1}{a_1}\log(a_1\,\rho)-\frac{1}{a_2}\log(a_2\,\rho)\Big)-\frac{c_5}{\rho^2}\\
&\quad\geq c\,\rho^{-2}\log(1+\rho)\,.
\end{aligned}$$

Case 3. Suppose the edges of P are pairwise parallel and with the same length. Let $M_1 = [Q_1, R_1]$ and $M_2 = [Q_2, R_2]$ be one of these couples. We can assume $Q_1 = H + (0, -1)$, $R_1 = H + (0, 1)$, $Q_2 = -H + (0, -1)$, $R_2 = -H + (0, 1)$. Then

$$\begin{aligned}
&\int_0^{2\pi}|\widehat{\chi}_P(\rho\,\Theta)|\,d\theta\\
&\quad\geq \frac{c_1}{\rho^2}\int_{U(0)}\Big|\sum_{j=1}^{2}\Theta\cdot\nu_1\,\frac{e^{-2\pi i\rho\Theta\cdot Q_j}-e^{-2\pi i\rho\Theta\cdot R_j}}{\Theta\cdot(Q_j-R_j)}\Big|\,d\theta-\frac{c_2}{\rho^2}\\
&\quad\geq \frac{c_1}{\rho^2}\int_{U(0)}\Big|\cos(2\pi\rho\,\Theta\cdot H)\,\frac{\sin(2\pi\rho\sin\theta)}{\sin\theta}\Big|\,d\theta-\frac{c_2}{\rho^2}\,.
\end{aligned}$$

Let $H = (h_1, h_2)$ and $\Theta \cdot H = h_1 \cos\theta + h_2 \sin\theta$. We choose φ so that $\Theta \cdot H = |H| \sin(\theta + \varphi)$. Since $h_1 \neq 0$ we have $\varphi \neq 0$ and for symmetry reasons we can restrict ourselves to the case $0 < \varphi \leq \pi/2$.

We obtain

$$\begin{aligned}\int_0^{2\pi} &|\widehat{\chi}_P(\rho\,\Theta)|\,d\theta \\ &\geq \frac{c_1}{\rho^2} \int_{U(0)} \Big| \cos(2\pi\rho\,|H| \sin(\theta+\varphi))\, \frac{\sin(2\pi\rho\sin\theta)}{\sin\theta} \Big|\, d\theta - \frac{c_2}{\rho^2}\,.\end{aligned}$$

Observe that choosing a sequence ρ_n so that $\rho_n |H| \sin\varphi$ is close to an integer we immediately get

$$\int_0^{2\pi} |\widehat{\chi}_P(\rho_n\,\Theta)|\,d\theta \geq \frac{c}{\rho_n^2} \int_{U(0)} \Big| \frac{\sin(2\pi\rho_n\sin\theta)}{\sin\theta} \Big|\, d\theta \geq c\,\rho_n^{-2} \log(1+\rho_n)\,,$$

that is, we have proved that $\rho^{-2}\log(1+\rho)$ is an optimal estimate of the 1-average decay of $\widehat{\chi}_P$. To get the full statement of the lemma we must deal with the values of ρ close to those annihilating $\cos(2\pi\rho\,|H|\sin\varphi)$.

We begin with the case $0 < \varphi < \pi/2$. Let $0 < \varepsilon < \pi/2 - \varphi$ such that $[0, \varepsilon] \subseteq U(0)$ and let $\{[a_j, b_j]\}$ be the collection of intervals determined by the choice

$$a_j = \arcsin\Big(\frac{j + \frac{1}{2} + \delta}{2\rho\,|H|}\Big) - \varphi\,, \qquad b_j = \arcsin\Big(\frac{j + \frac{3}{2} - \delta}{2\rho\,|H|}\Big) - \varphi$$

and $j = [2\rho\,|H|\sin\varphi] + 1, \dots, [2\rho\,|H|\sin(\varphi+\varepsilon)]$ for some sufficiently small $\delta > 0$. We observe that on each $[a_j, b_j]$ we have

$$|\cos(2\pi\rho\,|H|\sin(\theta+\varphi))| \geq \delta' > 0\,.$$

As a consequence

$$\begin{aligned}\int_{U(0)} &\Big| \cos(2\pi\rho\,|H|\sin(\theta+\varphi))\,\frac{\sin(2\pi\rho\sin\theta)}{\sin\theta}\Big|\,d\theta \\ &\geq \delta' \sum_j \frac{1}{\sin b_j} \int_{a_j}^{b_j} |\sin(2\pi\rho\sin\theta)|\,d\theta \qquad (3.6)\\ &\geq c\,\delta' \sum_j \frac{1}{\rho\sin b_j} \int_{\rho\sin a_j}^{\rho\sin b_j} |\sin(2\pi\,u)|\,du\,.\end{aligned}$$

Using the elementary inequality

$$\sin(b_j+\varphi)-\sin(a_j+\varphi)\le \sin b_j-\sin a_j$$

and the above definition of a_j and b_j we see that the quantity

$$\rho\sin b_j-\rho\sin a_j$$

is bounded away from zero and therefore

$$\int_{\rho\sin a_j}^{\rho\sin b_j}|\sin(2\pi\, u)|du>c>0\,.$$

Now the choice of b_j implies

$$\sum_j \frac{1}{\rho\sin b_j}\ge c\,\log(1+\rho)\,.$$

Indeed, let $k=j-[2\rho\,|H|\sin\varphi]$, so that we have to estimate

$$\sum_{k=1}^{c\rho}\frac{1}{\rho\sin b_{k+[2\rho|H|\,\sin\varphi]}}$$

from below. The choice of b_j shows that

$$\sin b_{k+[2\rho|H|\,\sin\varphi]}\le\frac{k+2}{2\rho\,|H|}$$

and therefore the last term in (3.6) is greater than

$$c_1\sum_{k=1}^{c\rho}\frac{1}{\rho\sin b_{k+[2\rho|H|\,\sin\varphi]}}\ge c_2\sum_{k=1}^{c\rho}\frac{1}{k+2}\ge c_3\log(1+\rho)\,.$$

The case $\varphi=\pi/2$ is similar. We fix $\varepsilon>0$ so that $[0,\varepsilon]\subset U(0)$. Next, we consider the collection of intervals $\{[a_j,b_j]\}$ with

$$a_j=\frac{\pi}{2}-\arcsin\left(\frac{j+\frac{1}{2}+\delta}{2\rho\,|H|}\right),\qquad b_j=\frac{\pi}{2}-\arcsin\left(\frac{j+\frac{3}{2}-\delta}{2\rho\,|H|}\right)$$

and $j = [2\rho\,|H|\sin(\pi/2+\varepsilon)+1], \dots, [2\rho\,|H|]$ for some sufficiently small $\delta > 0$. As before on each $[a_j, b_j]$ we have

$$|\cos(2\pi\rho\,|H|\sin(\theta+\varphi))| \geq \delta' > 0\,.$$

Using the fact that

$$\frac{\pi}{2} - \arcsin x = 2\arcsin\sqrt{\frac{1-x}{2}}$$

one deduces the estimates

$$a_j \approx \sqrt{\frac{2\rho\,|H| - j - \dfrac{1}{2} - \delta}{\rho\,|H|}}\,, \qquad b_j \approx \sqrt{\frac{2\rho\,|H| - j - \dfrac{3}{2} + \delta}{\rho\,|H|}}$$

and consequently the required result.

Lemma 3.11. *Let χ_P be the characteristic function of a compact polygon P in $\mathbb{R}^2$. For any $p > 1$*

$$\limsup_{\rho\to\infty} \rho^{1+1/p}\Big(\int_0^{2\pi} |\widehat{\chi}_P(\rho\,\Theta)|^p\,d\theta\Big)^{1/p} > 0\,.$$

Proof. We can suppose that one of the sides of P is vertical. We assume the following facts, which will be proved in the sequel:

$$\text{there exists } \rho_k \longrightarrow +\infty \text{ so that } |\widehat{\chi}_P(\rho_k, 0)| \geq \frac{c}{\rho_k}\,, \tag{3.7}$$

$$|\nabla\widehat{\chi}_P(\xi)| \leq \frac{c}{|\xi|+1}\,. \tag{3.8}$$

Next we consider

$$\int_0^{2\pi} |\widehat{\chi}_P(\rho_k\,\Theta)|^p\,d\theta \geq \int_0^{\varepsilon/\rho_k} |\widehat{\chi}_P(\rho_k\,\Theta)|^p\,d\theta\,.$$

By choosing ε sufficiently small we can make $(\rho_k\cos\theta, \rho_k\sin\theta)$ close to $(\rho_k, 0)$ so that (3.8), (3.7) and the mean value theorem imply

$$|\widehat{\chi}_P(\rho_k\,\Theta)| \geq \frac{c_1}{\rho_k}\,.$$

Hence,

$$\int_0^{2\pi} |\widehat{\chi}_P(\rho_k\,\Theta)|^p\,d\theta \ge c\int_0^{\varepsilon/\rho_k} \rho_k^{-p}\,d\theta \ge c\,\rho_k^{-p-1}\,.$$

We now prove (3.7).

First we recall, see (3.4), that

$$\widehat{\chi}_P(\xi_1,0) = \widehat{h}(\xi_1)\,,$$

where $h(t)$ is the length of the chord given by the intersection of P with the line $x_1 = t$. Observe that $h(t)$ is a piecewise linear function, continuous at any point except at least one of the extremes of the support. Split

$$h(t) = b(t) + g(t)\,,$$

where $b(t)$, $g(t)$ and $h(t)$ share the same support, $b(t)$ is linear inside the support and $g(t)$ is continuous on $\mathbb{R}$. Our choice forces $b(t)$ to be discontinuous in at least one of the extremes (recall that at least one side of P is ortogonal to $(1,0)$), while $g(t)$ must be piecewise linear.

An explicit computation gives a sequence $\rho_k \longrightarrow +\infty$ such that $|\widehat{b}(\rho_k)| \ge c\,\rho_k^{-1}$, while

$$|\widehat{g}(\xi_1)| \le c\,\frac{1}{1+\xi_1^2}\,.$$

This proves (3.7).

In order to prove (3.8) we observe that, for any unit vector u,

$$\begin{aligned}\frac{\partial}{\partial u}\,\widehat{\chi}_P(\xi) &= \frac{\partial}{\partial u}\int_P e^{-2\pi i\xi\cdot x}\,dx \\ &= -2\pi i\int_P (u\cdot x)\,e^{-2\pi i\xi\cdot x}\,dx \\ &= -2\pi i\int_{\mathbb{R}^2} ((u\cdot x)\,\chi_P(x))\,e^{-2\pi i\xi\cdot x}\,dx\,,\end{aligned}$$

and (3.8) follows since the function $x \longrightarrow (u\cdot x)\,\chi_P(x)$ has bounded variation.

The following lemma is taken from [3]. We reproduce the short proof.

Lemma 3.12. i) *Let P be a polygon having an edge not parallel to any other. Then, if* $1 \le p \le \infty$,

$$\Big(\int_0^{2\pi} |\widehat{\chi}_Q(\rho\,\Theta)|^p\, d\theta\Big)^{1/p} \ge c\,\rho^{-1-1/p}\,.$$

ii) *Let Q be the unit square* $[-1/2,1/2]^2$. *If* $1 < p \le +\infty$ *and if k is a positive integer, then*

$$\Big(\int_0^{2\pi} |\widehat{\chi}_Q(k\,\Theta)|^p\, d\theta\Big)^{1/p} \le c\,k^{-3/2-1/(2p)}\,.$$

PROOF. i) Arguing as in the first case in the proof of Lemma 3.10 we are reduced to bounding

$$\frac{1}{\rho^{2p}}\int_0^{c_1} \Big|\frac{\sin(2\pi\rho\,u)}{u}\Big|^p\, du$$

from below. A computation ends the proof of this case.

ii) We have

$$\begin{aligned}
\int_0^{2\pi} |\widehat{\chi}_Q(k\,\Theta)|^p\, d\theta &= 8\int_0^{\pi/4} \Big|\frac{\sin(\pi k\cos\theta)}{\pi k\cos\theta}\ \frac{\sin(\pi k\sin\theta)}{\pi k\sin\theta}\Big|^p\, d\theta\\
&\le c\,k^{-2p}\int_0^{\pi/4}\Big|\frac{\sin(\pi k\cos\theta)}{\sin\theta}\Big|^p\, d\theta\\
&\le c\,k^{-2p}\int_0^{\pi/4}\Big|\sin\Big(2\pi k\sin^2\Big(\frac{\theta}{2}\Big)\Big)\Big|^p\theta^{-p}\, d\theta \qquad (3.9)\\
&\le c\,k^{-2p}\int_0^{k^{-1/2}} k^p\,\theta^p\, d\theta + c\,k^{-2p}\int_{k^{-1/2}}^{\pi/4}\theta^{-p}\, d\theta\\
&\le c\,k^{-3p/2-1/2}\,.
\end{aligned}$$

The forthcoming results will be used in the proof of Proposition 2.5.

Lemma 3.13. *Let* $2 \le p \le +\infty$ *and let* $s < 1 + 1/p$. *Then the p-average decay of $\widehat{\chi}_B$ has optimal order s for no convex body B with piecewise C^∞ boundary.*

PROOF. Lemma 3.1 and the theorem on the decay of the Fourier transform of a function of bounded variation imply this lemma when $p = 2$ and $p = \infty$ respectively. When $2 < p < \infty$ we have

$$\begin{aligned}\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p \, d\theta\Big)^{1/p} &= \Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^2 \, |\widehat{\chi}_B(\rho\,\Theta)|^{p-2} \, d\theta\Big)^{1/p} \\ &\le c \Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^2 \, d\theta\Big)^{1/p} \rho^{-1+2/p} \\ &\le c\,\rho^{-1-1/p}\,.\end{aligned}$$

Lemma 3.14. *Let $P = (s_0, s_0^\alpha)$ be a given point in the graph of the function $t = s^\alpha$, with $0 < \alpha < 1$. Let $\varphi = \arctan(\alpha\, s_0^{\alpha-1})$ be the slope of the corresponding tangent line and let, for a small positive δ,*

$$t = \alpha\, s_0^{\alpha-1}(s - s_0) + s_0^\alpha - \frac{\delta}{\cos\varphi}$$

be parallel to the above tangent line, at distance δ. Here we assume that this last line and the curve $t = s^\alpha$ intersect in two points $A = (s_1, s_1^\alpha)$ and $B = (s_2, s_2^\alpha)$ (see Figure 3). We denote by $d(\delta)$ the distance between A and B. Then $d'(\delta)$ is a convex function of δ.

Figure 3.

Proof. Since $d(\delta) = (s_2 - s_1)/\cos\varphi$ it is enough to check that the functions $h(\delta) = s_2 - s_0$ and

$$k(\delta) = \frac{s_0 - s_1}{\cos\varphi} + \delta \tan\varphi$$

have convex derivatives.

We start with $h(\delta)$. By the definition of the point B we have

$$s_2^\alpha = \alpha\, s_0^{\alpha-1}\,(s_2 - s_0) + s_0^\alpha - \frac{\delta}{\cos\varphi}\,,$$

that is

$$(h(\delta) + s_0)^\alpha = \alpha\, s_0^{\alpha-1}\, h(\delta) + s_0^\alpha - \frac{\delta}{\cos\varphi}\,.$$

Differentiating the above with respect to δ we get

$$\alpha\,(h(\delta) + s_0)^{\alpha-1} h'(\delta) = \alpha\, s_0^{\alpha-1} h'(\delta) - \frac{1}{\cos\varphi}\,,$$

which implies $h'(\delta) > 0$ since $0 < \alpha < 1$. Further differentiations show that $h''(\delta) < 0$ and $h'''(\delta) > 0$.

We now turn to $k(\delta)$, which is the distance between the points A and C in Figure 3. In order to prove that the negative function $k''(\delta)$ increases with δ we observe that

$$k''(\delta) = -K(A)\,(1 + (k'(\delta))^2)^{3/2}\,, \tag{3.10}$$

where $K(A)$ denotes the curvature at the point A. Now it is easy to check that $K(A)$ decreases as A moves towards O (that is as δ grows). On the other hand, by convexity, $k'(\delta)$ decreases too. Therefore, by (3.10), $k''(\delta)$ increases and this ends the proof of the lemma.

The following result is related to [5, Lemmas 6.2 and 6.3].

Lemma 3.15. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ be supported in $[-1, 1]$, such that $f \in C^\infty(\mathbb{R}\backslash\{1\})$, $f \in C(\mathbb{R})$, f and f' are concave in $[b, 1)$ and $f'(b) = 0$, $f'(1^-) = -\infty$. Then, for $|\xi| \ge 1$,*

$$|\widehat{f}(\xi)| \ge c\,\frac{1}{|\xi|}\, f\Big(1 - \frac{1}{6\,|\xi|}\Big)\,. \tag{3.11}$$

The constant c depends only on the supremum of $|f(t)|$ on $\mathbb{R}$ and on the variation of $f'(t)$ outside a neighborhood of $t = 1$.

Proof. We write

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-1}^{1} f(t)\, e^{-2\pi i t\xi}\, dt \\
&= \frac{1}{2\pi i\,\xi} \int_{-1}^{1} f'(t)\, e^{-2\pi i t\xi}\, dt \\
&= \frac{1}{2\pi i\,\xi} \int_{-1}^{b} f'(t)\, e^{-2\pi i t\xi}\, dt + \frac{1}{2\pi i\,\xi} \int_{b}^{1} f'(t)\, e^{-2\pi i t\xi}\, dt \\
&= I_1(\xi) + I_2(\xi)\,.
\end{aligned}$$

Since f' is of bounded variation on $[-1, b]$ we have $|I_1(\xi)| \leq c\,|\xi|^{-2}$ where c depends only on the variation of f'. Morover, f concave on $[b, 1]$ and $f'(1^-) = -\infty$ imply

$$|\xi|^{-2} = o\Big(|\xi|^{-1} f\Big(1 - \frac{1}{6\,|\xi|}\Big)\Big)$$

so that

$$|I_1(\xi)| = o\Big(|\xi|^{-1} f\Big(1 - \frac{1}{6\,|\xi|}\Big)\Big)\,.$$

To analyze $I_2(\xi)$ we proceed as follows: we assume $\xi > 0$ (the case $\xi < 0$ is similar) we write $\xi = [\xi] + \eta$ and let $\sigma = (1 - 6\,\eta)/(6\,\xi)$ (this choice will be appreciated later on, while estimating $I_5(\xi)$). Then

$$\begin{aligned}
|I_2(\xi)| &= \frac{1}{2\pi\,\xi} \Big| \int_{b}^{1} (-f'(t))\, e^{2\pi i t\xi}\, dt \Big| \\
&= \frac{1}{2\pi\,\xi} \Big| \int_{b}^{1} (-f'(t))\, e^{2\pi i (t+\sigma)\xi}\, dt \Big| \\
&\geq \frac{1}{2\pi\,\xi} \Big| \int_{b}^{1} (-f'(t)) \cos\left(2\pi\,(t+\sigma)\,\xi\right) dt \Big| \\
&= \frac{1}{2\pi\,\xi}\, |I_3(\xi) + I_4(\xi) + I_5(\xi)|\,,
\end{aligned}$$

where

$$I_3(\xi) = \int_b^{j_0/(4\xi)-\sigma} (-f'(t)) \cos(2\pi\,(t+\sigma)\,\xi)\,dt\,,$$

$$I_4(\xi) = \sum_{j=j_0}^{4[\xi]-1} \int_{j/(4\xi)-\sigma}^{(j+1)/(4\xi)-\sigma} (-f'(t)) \cos(2\pi\,(t+\sigma)\,\xi)\,dt = \sum_{j=j_0}^{4[\xi]-1} A_j\ ,$$

$$I_5(\xi) = \int_{1-1/(6\xi)}^{1} (-f'(t)) \cos(2\pi\,(t+\sigma)\,\xi)\,dt\,,$$

with j_0 the smallest even integer such that $j_0/(4\,\xi) - \sigma \geq b$. First we observe that $|I_3(\xi)| \leq c/\xi$ and therefore its contribution is negligible.

We consider $I_4(\xi)$ and we show that

$$I_4(\xi) = \sum_{j=j_0}^{4[\xi]-1} A_j \geq 0\,. \tag{3.12}$$

Indeed,

i) $A_{4[\xi]-1} > 0$, $A_{4[\xi]-2} < 0$, $A_{4[\xi]-3} < 0$, $A_{4[\xi]-4} > 0$, $A_{4[\xi]-5} > 0$, $A_{4[\xi]-6} < 0, \ldots$

ii) $|A_j| \leq |A_{j+1}|$ so that $A_{4[\xi]-1} + A_{4[\xi]-2} > 0$, $A_{4[\xi]-3} + A_{4[\xi]-4} < 0$, $A_{4[\xi]-5} + A_{4[\xi]-6} > 0, \ldots$

iii) $|A_{4[\xi]-1} + A_{4[\xi]-2}| \geq |A_{4[\xi]-3} + A_{4[\xi]-4}| \geq |A_{4[\xi]-5} + A_{4[\xi]-6}| \geq \cdots$

The validity of i) is obvious, while ii) depends on the monotonicity of f'. As for iii) we note that the concavity of f' implies

$$|A_{4[\xi]-1}| - |A_{4[\xi]-3}| \geq |A_{4[\xi]-2}| - |A_{4[\xi]-4}| \geq \cdots$$

By i), ii), iii) it follows that the sum

$$(A_{j_0} + A_{j_0+1}) + (A_{j_0+2} + A_{j_0+3}) + \cdots + (A_{4[\xi]-2} + A_{4[\xi]-1})$$

shares the sign of its last term $(A_{4[\xi]-2} + A_{4[\xi]-1})$, thereby proving (3.12).

Hence,

$$I_4(\xi) + I_5(\xi) \geq I_5(\xi) = \int_{1-1/(6\xi)}^{1} (-f'(t)) \cos(2\pi\,(t+\sigma)\,\xi)\,dt \geq \frac{1}{2}\, f\Big(1 - \frac{1}{6\,\xi}\Big)\,,$$

since $\cos(2\pi\,(t+\sigma)\,\xi) \geq 1/2$ on the domain of integration.

Lemma 3.16. *For each*

$$\left(\frac{1}{p}, a\right) \in T = \left\{ \left(\frac{1}{p}, a\right) :\ 2 \leq p \leq \infty,\ 1 + \frac{1}{p} < a < \frac{3}{2} \right\}$$

there exists a convex body B with piecewise C^∞ boundary such that the p-average decay of $\widehat{\chi}_B$ has sharp order a.

Proof. Let B be a convex body symmetric with respect to the vertical axis and assume that its boundary ∂B satisfies the following conditions.

i) ∂B passes through the origin and it is of class C^∞ in any other point.

ii) ∂B coincides with the graph of the function $y = |x|^\gamma$ in a neighborhood of the origin (the exponent $\gamma = \gamma(p, a) > 2$ will be chosen later).

iii) ∂B has strictly positive curvature out of the above neighborhood.

We first prove that $|\widehat{\chi}_B(\rho\,\Theta)| \leq c\,\rho^{-1-1/\gamma}$ for any $\Theta \in \Sigma_1$. This bound seems to be quite obvious since $|\xi_2|^{-1-1/\gamma}$ is the order of decay of $\widehat{\chi}_B(0, \xi_2)$, that is, the decay associated to the flattest point in ∂B. However, a proof seems to be necessary (in order to check that the constant does not depend on Θ), and the argument will be needed in the sequel.

Let $\psi = \theta + \pi/2$. We choose $\varepsilon > 0$ sufficiently small and we assume $\varepsilon \leq |\psi| \leq \pi - \varepsilon$. Since ∂B has strictly positive curvature away from the origin, by Lemma 3.8 we have,

$$|\widehat{\chi}_B(\rho\,\Theta)| \leq c\left(\left|A_B(\rho^{-1}, \psi - \frac{\pi}{2})\right| + \left|A_B\left(\rho^{-1}, \psi + \frac{\pi}{2}\right)\right|\right) \leq c\,\rho^{-3/2}\,,$$

for $\varepsilon \leq |\psi| \leq \pi - \varepsilon$.

Symmetry enables us to consider only the case $0 \leq \psi \leq \varepsilon$. The assumptions on the curvature of ∂B show that the contribution of $|A_B(\rho^{-1}, \psi + \pi/2)|$ is not larger than $c\,\rho^{-3/2}$ so that it suffices to consider $A_B(\rho^{-1}, \psi - \pi/2)$ (which is a cap close to the origin).

We set more notation. For any $0 \leq \psi \leq \varepsilon$, we consider the straight line with slope ψ and tangent to the curve $y = x^\gamma$ at a point (x_0, x_0^γ). Then $A_B(\rho^{-1}, \psi - \pi/2)$ is the set enclosed between the line $y = r(x) =$

$\gamma\, x_0^{\gamma-1}(x - x_0) + x_0^\gamma + (\rho\cos\psi)^{-1}$ and the curve $y = x^\gamma$. Let us call x_1 and x_2 the abscissae of the two points where they intersect (see Figure 4).

Figure 4.

Since $\tan\psi = \gamma\, x_0^{\gamma-1}$ we have

$$c_1\,\psi \le x_0^{\gamma-1} \le c_2\,\psi\,. \tag{3.13}$$

We further split the interval $0 \le \psi \le \varepsilon$ into $0 \le \psi \le c\,\rho^{-1+1/\gamma}$ and $c\,\rho^{-1+1/\gamma} \le \psi \le \varepsilon$ for some suitable constant c.

Assume

$$0 \le \psi \le c\,\rho^{-1+1/\gamma}\,. \tag{3.14}$$

Since ψ is positive, $|A_B(\rho^{-1}, \psi - \pi/2)| \le c\,\rho^{-1}\,x_2$. We recall that x_2 is the largest solution of the equation

$$x^\gamma = \gamma\, x_0^{\gamma-1}(x - x_0) + x_0^\gamma + (\rho\cos\psi)^{-1}\,.$$

We now estimate x_2. This gives a bound for $|A_B(\rho^{-1}, \psi - \pi/2)|$ since the assumption $\psi \ge 0$ yields $x_2 \ge |x_1|$. To do this we observe that (3.14) implies that the above equation has no solutions for $x > k\,\rho^{-1/\gamma}$

for k sufficiently large. Indeed, (3.13) and (3.14) imply $x_0 \leq c_3\, \rho^{-1/\gamma}$ and therefore

$$\begin{aligned} x^\gamma - \gamma\, x_0^{\gamma-1}(x-x_0) - x_0^\gamma - (\rho\cos\psi)^{-1} & \\ &> x^\gamma - c_4\, \rho^{-1+1/\gamma}\, x - c_4\, \rho^{-1} - (\rho\cos\psi)^{-1} \\ &> \rho^{-1}\left((\rho^{1/\gamma}\, x)^\gamma - c_4\, \rho^{1/\gamma}\, x - c_4 - (\cos\psi)^{-1}\right) \\ &> 0\,, \end{aligned}$$

when $\rho^{1/\gamma}\, x$ is larger than a suitable k. Then $x_2 \leq k\, \rho^{-1/\gamma}$ and

$$\left| A_B\Big(\rho^{-1}, \psi - \frac{\pi}{2}\Big) \right| \leq c\, \rho^{-1-1/\gamma}\,, \qquad \text{for } 0 \leq \psi \leq c\, \rho^{-1+1/\gamma}\,. \tag{3.15}$$

Next, let $c\,\rho^{-1+1/\gamma} \leq \psi \leq \varepsilon$. Then (3.13) and a suitable choice of the constant c imply $x_1 > 0$. We want to show that

$$x^\gamma - \gamma\, x_0^{\gamma-1}(x-x_0) - x_0^\gamma - (\rho\cos\psi)^{-1}$$

becomes positive whenever $|x - x_0| > c_5\, \rho^{-1/2}\, x_0^{1-\gamma/2}$. Towards this aim one checks the inequality

$$(1+u)^\gamma - 1 - \gamma\, u \geq \frac{\gamma}{2}\, u^2\,, \tag{3.16}$$

which holds true for $\gamma > 2$ and $u \geq -1$. Then

$$\begin{aligned} x^\gamma - \gamma\, x_0^{\gamma-1}\,(x-x_0) - x_0^\gamma - (\rho\cos\psi)^{-1} & \\ &= (x_0 + (x-x_0))^\gamma - \gamma\, x_0^{\gamma-1}\,(x-x_0) - x_0^\gamma - (\rho\cos\psi)^{-1} \\ &= x_0^\gamma\Big(\Big(1 + \frac{x-x_0}{x_0}\Big)^\gamma - \gamma\,\frac{x-x_0}{x_0} - 1\Big) - (\rho\cos\psi)^{-1} \\ &\geq x_0^\gamma\, \frac{\gamma}{2}\Big(\frac{x-x_0}{x_0}\Big)^2 - (\rho\cos\psi)^{-1} \\ &\geq \frac{\gamma}{2}\, c_5^2\, \rho^{-1} - (\rho\cos\psi)^{-1} \\ &> 0\,, \end{aligned}$$

for a suitably large c_5. Consequently

$$|x - x_0| \leq c_5\, \rho^{-1/2}\, x_0^{1-\gamma/2}\,, \tag{3.17}$$

for any $x_1 \le x \le x_2$. This and (3.13) show that

$$\Big| A_B\Big(\rho^{-1}, \psi - \frac{\pi}{2}\Big)\Big| \le c_6\, \rho^{-3/2}\, \psi^{(2-\gamma)/(2(\gamma-1))}\,, \tag{3.18}$$

for $c\,\rho^{-1+1/\gamma} \le \psi \le \varepsilon$. Then (3.15), (3.18), the assumptions on the curvature of ∂B and Lemma 3.8 yield

$$|\widehat{\chi}_B(\rho\,\Theta)| \le c_7\, \rho^{-1-1/\gamma}\,. \tag{3.19}$$

for any Θ.

We now study the estimates of the L^p-norm, $2 \le p < +\infty$. Because of the symmetry of B it is enough to bound

$$\begin{aligned}
\Big(\int_{-\pi/2}^{\pi/2} |\widehat{\chi}_B(\rho\,\Theta)|^p\, d\theta\Big)^{1/p} &\le \Big(\int_{-\pi/2}^{-\pi/2+c\rho^{(1-\gamma)/\gamma}} |\widehat{\chi}_B(\rho\,\Theta)|^p\, d\theta\Big)^{1/p} \\
&\quad + \Big(\int_{-\pi/2+c\rho^{(1-\gamma)/\gamma}}^{-\pi/2+\varepsilon} |\widehat{\chi}_B(\rho\,\Theta)|^p\, d\theta\Big)^{1/p} \\
&\quad + \Big(\int_{-\pi/2+\varepsilon}^{\pi/2} |\widehat{\chi}_B(\rho\,\Theta)|^p\, d\theta\Big)^{1/p} \\
&= I_1 + I_2 + I_3\,.
\end{aligned}$$

By the assumptions on the curvature of ∂B we have $I_3 \le c_8\, \rho^{-3/2}$. Furthermore, by (3.19),

$$I_1 \le c_7\, \rho^{-1-1/\gamma}\Big(\int_0^{c\rho^{(1-\gamma)/\gamma}} d\psi\Big)^{1/p} \le c_9\, \rho^{-1-1/p-1/\gamma+1/(\gamma p)}\,.$$

In order to estimate I_2 we observe that Lemma 3.8, (3.18), the assumptions on the curvature of ∂B and the choice $\gamma > 2$ give

$$\begin{aligned}
I_2 &\le c_{10}\, \rho^{-3/2}\Big(\int_{c\rho^{(1-\gamma)/\gamma}}^{\varepsilon} \psi^{p(2-\gamma)/(2\gamma-2)}\, d\psi\Big)^{1/p} \\
&\le \begin{cases}
c_{11}\, \rho^{-3/2}\,, & \text{for } p < \dfrac{2\,\gamma-2}{\gamma-2}\,, \\[2ex]
c_{11}\, \rho^{-3/2}\, (\log\rho)^{(\gamma-2)/(2\gamma-2)}\,, & \text{for } p = \dfrac{2\,\gamma-2}{\gamma-2}\,, \\[2ex]
c_{11}\, \rho^{-1-1/p-1/\gamma+1/(\gamma p)}\,, & \text{for } p > \dfrac{2\,\gamma-2}{\gamma-2}\,.
\end{cases}
\end{aligned}$$

In particular, for $p > (2\,\gamma - 2)/(\gamma - 2)$,

$$\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p\, d\theta\Big)^{1/p} \le c_{12}\,\rho^{-1-1/p-1/\gamma+1/(\gamma p)}\,. \tag{3.20}$$

Observe that (3.20) cannot be obtained interpolating between L^2 and L^∞. Moreover, we shall see in a moment that the above estimates are sharp and therefore

$$\begin{aligned} &\|\widehat{\chi}_B(\rho\,\cdot)\|_{L^p(\Sigma_1)} \\ &\approx \begin{cases} \rho^{-3/2}\,, & \text{for } p < \dfrac{2\,\gamma - 2}{\gamma - 2}\,, \\ \rho^{-3/2}(\log\rho)^{(\gamma-2)/(2\gamma-2)}\,, & \text{for } p = \dfrac{2\,\gamma - 2}{\gamma - 2}\,, \\ \rho^{-1-1/p-1/\gamma+1/(\gamma p)}\,, & \text{for } p > \dfrac{2\,\gamma - 2}{\gamma - 2}\,. \end{cases} \end{aligned} \tag{3.21}$$

When $p < (2\,\gamma - 2)/(\gamma - 2)$ the estimate from below follows from Lemma 3.6. We shall now prove the estimates from below in (3.21) when $p \ge (2\,\gamma - 2)/(\gamma - 2)$. Indeed, (3.17) can be reversed so that, by (3.13), $x \notin (x_1, x_2)$ implies

$$|x - x_0| \ge c_{13}\,\rho^{-1/2}\,x_0^{1-\gamma/2}\,,$$

whence

$$\Big|A_B\Big(\frac{1}{\rho}, \psi - \frac{\pi}{2}\Big)\Big| \ge c_{13}\,\rho^{-3/2}\,\psi^{(2-\gamma)/(2(\gamma-1))}\,,$$

for $c\,\rho^{-1+1/\gamma} \le \psi \le \varepsilon$. To prove this, we argue as for the estimate from above, after substituting (3.16) with the inequality

$$(1+u)^\gamma - 1 - \gamma\,u \le \gamma^2\,2^\gamma\,u^2\,,$$

valid for $\gamma > 2$ and $-1 \le u \le 1$. The restriction $u \le 1$ causes no troubles since the monotonicity of the curvature of $y = x^\gamma$ implies $x_2 - x_0 \le x_0 - x_1$ for $c\,\rho^{-1+1/\gamma} \le \psi \le \varepsilon$, so that

$$-1 \le \frac{x - x_0}{x_0} \le 1$$

if $x_1 \le x \le x_2$.

To estimate $\widehat{\chi}_B$, let ψ be fixed in $c\,\rho^{-1+1/\gamma} \le \psi \le \varepsilon$ and recall that $\theta = \psi - \pi/2$. By Lemma 3.8 the decay of $\widehat{\chi}_B(\rho\,\Theta)$ depends on the shape of ∂B at the point (x_0, x_0^α), see Figure 4, and at the "opposite" point. The latter will turn out to give a negligible contribution because of our assumption on the curvature of ∂B outside the origin. Then, if $\zeta \in C_0^\infty(\mathbb{R}^2)$, $\zeta(x) = 1$ in a neighborhood of the point (x_0, x_0^α), Lemma 3.14 allows us to apply Lemma 3.15 so to obtain

$$\begin{aligned}
&\Big(\int_0^{2\pi} |\widehat{\chi}_B(\rho\,\Theta)|^p\,d\theta\Big)^{1/p} \\
&\qquad \ge \Big(\int_{c\rho^{-1+1/\gamma}}^{\varepsilon} |\widehat{\chi}_B(\rho\,\Theta)|^p\,d\psi\Big)^{1/p} \\
&\qquad \ge \Big(\int_{c\rho^{-1+1/\gamma}}^{\varepsilon} |[\zeta\,\chi_B]^\wedge\,(\rho\,\Theta)|^p\,d\psi\Big)^{1/p} \\
&\qquad\quad - \Big(\int_{c\rho^{-1+1/\gamma}}^{\varepsilon} |[(1-\zeta)\,\chi_B]^\wedge\,(\rho\,\Theta)|^p\,d\psi\Big)^{1/p} \\
&\qquad \ge c_{14}\Big(\int_{c\rho^{-1+1/\gamma}}^{\varepsilon} \Big|A_B\Big(\frac{1}{\rho}, \psi - \frac{\pi}{2}\Big)\Big|^p\,d\psi\Big)^{1/p} - c_{15}\,\rho^{-3/2} \\
&\qquad \ge c_{16}\,\rho^{-1-1/p-1/\gamma+1/(\gamma p)}\,.
\end{aligned}$$

We recall that the above holds whenever $p > (2\,\gamma - 2)/(\gamma - 2)$. This ends the proof once we observe that when $p > (2\,\gamma - 2)/(\gamma - 2)$ we have $(2\,p - 2)/(p - 2) < \gamma < \infty$ and therefore the range of the exponent

$$1 + \frac{1}{p} + \frac{1}{\gamma} - \frac{1}{\gamma\,p}$$

is the open interval $(1 + 1/p, 3/2)$.

The proof of the previous lemma can be used to get a result for singular measures supported on the curve $y = |x|^\gamma$, $\gamma > 2$.

Proposition 3.17. *Let $d\sigma$ be the measure on the curve $y = |x|^\gamma$, $\gamma > 2$, induced by the Lebesgue measure on $\mathbb{R}^2$. Let $\kappa \in C_0^\infty(\mathbb{R}^2)$, $\kappa(t) = 1$ in a neighborhood of the origin and let $d\mu = \kappa(t)\,d\sigma$. Let $\ell(\rho^{-1}, \theta)$ be the* *length of the chord as in Figure 4. Let $1 \leq p \leq \infty$, then*

$$\|\ell(\rho^{-1},\cdot)\|_{L^p([0,2\pi))} \approx \|\widehat{d\mu}(\rho\,\cdot)\|_{L^p(\Sigma_1)}$$
$$\approx \begin{cases} \rho^{-1/2}\,, & \textit{for } p < \dfrac{2\gamma-2}{\gamma-2}\,, \\ \rho^{-1/2}\,(\log\rho)^{(\gamma-2)/(2\gamma-2)}\,, & \textit{for } p = \dfrac{2\gamma-2}{\gamma-2}\,, \\ \rho^{-1/p-1/\gamma+1/(\gamma p)}\,, & \textit{for } p > \dfrac{2\gamma-2}{\gamma-2}\,. \end{cases}$$

4. A remark on the average decays associated to arbitrary convex sets.

Let $\mathcal{C}$ be the space of convex bodies in $\mathbb{R}^2$ endowed with the Hausdorff metric δ^H defined by

$$\delta^H(C,D) = \max\big\{\sup_{x\in C}\inf_{y\in D}|x-y|, \sup_{y\in D}\inf_{x\in C}|x-y|\big\}\,,$$

for $C, D \in \mathcal{C}$. A weak version of Blaschke selection theorem (see [9]) shows that $(\mathcal{C},\delta^H)$ is locally compact and therefore of second category (not meager) by Baire theorem. We fix $n \in \mathbb{N}$. On $\mathcal{C}$ we consider the functional

$$\Phi_n(B) = \Big(\int_0^{2\pi} |\widehat{\chi}_B(n\,\Theta)|^p\,d\theta\Big)^{1/p}$$

and we observe that

$$\big|\,\|\widehat{\chi}_C(n\,\cdot)\|_{L^p(\Sigma_1)} - \|\widehat{\chi}_D(n\,\cdot)\|_{L^p(\Sigma_1)}\big| \leq \|\widehat{\chi}_C(n\,\cdot) - \widehat{\chi}_D(n\,\cdot)\|_{L^p(\Sigma_1)}$$
$$\leq |C\,\Delta D|\,,$$

implies continuity of Φ_n.

Next, let $1 < p < 2$ and $3/2 < \gamma < 1 + 1/p$. Let B be a convex set with piecewise smooth boundary. The results in the previous section show that the family $\{\Phi_n\}$ satisfies

$$\Phi_n(B) = o\,(n^{-\gamma})\,,$$

when B is a polygon and

$$n^{-\gamma} = o\,(\Phi_n(B))\,,$$

if B is not a polygon nor a cut disc. Therefore, the sets

$$\mathcal{A}_1 = \{B \in \mathcal{C} : \ \Phi_n(B) = o\,(n^{-\gamma})\}$$

and

$$\mathcal{A}_2 = \{B \in \mathcal{C} : \ n^{-\gamma} = o\,(\Phi_n(B))\}$$

are dense in $\mathcal{C}$. A similar argument also applies when $p > 2$.

We now use the following result due to Gruber, [6].

Lemma 4.1. *Let T be a second category topological space.*

i) *Let $\alpha_1, \alpha_2, \dots \in \mathbb{R}^+$ and let $\phi_1, \phi_2, \dots : T \longrightarrow \mathbb{R}^+$ be continuous functions such that*

$$\mathcal{A} = \{x \in T : \ \phi_n(x) = o\,(\alpha_n) \ \text{ as } n \longrightarrow +\infty\}$$

is dense in T. Then for all, but a meager subset of x's belonging to T, the inequality $\phi_n(x) < \alpha_n$ holds for infinitely many n.

ii) *Let $\beta_1, \beta_2, \dots \in \mathbb{R}^+$ and let $\psi_1, \psi_2, \dots : T \longrightarrow \mathbb{R}^+$ be continuous functions such that*

$$\mathcal{B} = \{x \in T : \ \beta_n = o\,(\psi_n(x)) \ \text{ as } n \longrightarrow +\infty\}$$

is dense in T. Then for all, but a meager subset of x's belonging to T, the inequality $\beta_n < \psi_n(x)$ holds for infinitely many n.

By way of summary we have.

Proposition 4.2. *Let $1 < p < 2$ and $3/2 < \gamma_1 \leq \gamma_2 < 1 + 1/p$ or let $2 < p < \infty$ and $1 + 1/p < \gamma_1 \leq \gamma_2 < 3/2$. Then there exists a meager set $\mathcal{E} \subset \mathcal{C}$ such that for all $B \in \mathcal{C} \setminus \mathcal{E}$ there exist two sequences n_k, m_k satisfying*

$$\Big(\int_0^{2\pi} |\widehat{\chi}_B(n_k\,\Theta)|^p \, d\theta \Big)^{1/p} \geq n_k^{-\gamma_1}$$

and

$$\Big(\int_0^{2\pi} |\widehat{\chi}_B(m_k\,\Theta)|^p \, d\theta \Big)^{1/p} \leq m_k^{-\gamma_2} \,.$$

5. A result on the geometry of convex sets.

At this point, little effort is needed to prove the following result, which may be of independent interest.

Theorem 5.1. *Let $A_B(\delta,\theta)$ be as in* (1.6) *and let $\delta \le 1/2$.*
If B is a polygon, then

$$c_1\,\delta^2 \log\Big(\frac{1}{\delta}\Big) \le \int_0^{2\pi} A_B(\delta,\theta)\,d\theta \le c_2\,\delta^2 \log\Big(\frac{1}{\delta}\Big)\,,$$

while if B is not a polygon

$$c_1\,\delta^{3/2} \le \int_0^{2\pi} A_B(\delta,\theta)\,d\theta \le c_2\,\delta^{3/2}\,.$$

Let $1 < p \le \infty$. Then the following are equivalent.

i) *There exist $a > 0$ and a convex body B with C^2 boundary such that*

$$c_1\,\delta^a \le \Big(\int_0^{2\pi} A_B(\delta,\theta)^p\,d\theta\Big)^{1/p} \le c_2\,\delta^a\,.$$

ii) *The pair $(1/p, a)$ belongs to the set $S \cup T$, where*

$$S = \Big\{\Big(\frac{1}{p}, a\Big) : \ 1 < p < 2,\ a = \frac{3}{2} \ \text{or}\ a = 1 + \frac{1}{p}\Big\}\,,$$

$$T = \Big\{\Big(\frac{1}{p}, a\Big) : \ 2 \le p \le \infty,\ 1 + \frac{1}{p} \le a \le \frac{3}{2}\Big\}\,.$$

The proof of this theorem is largely a consequence of results in the previous section. Actually, the present problem is simpler since $A_B(\delta,\theta)$ is positive and no cancellation can arise. We sketch the argument for a reader specifically interested in this result.

Proof. We split the proof into several steps. We assume $\delta > 0$ sufficiently small.

Step 1. Upper bound when $1 \le p \le 2$

$$\Big(\int_0^{2\pi} A_B(\delta,\theta)^2\,d\theta\Big)^{1/2} \le c\,\delta^{3/2}\,,$$

for any B.

This has been proved by Podkorytov in [13, p. 60].

Step 2. If P is a polygon, then

$$c_1\,\delta^2\log\Big(\frac{1}{\delta}\Big)\le\int_0^{2\pi}A_P(\delta,\theta)\,d\theta\le c_2\,\delta^2\log\Big(\frac{1}{\delta}\Big)\,,$$

$$c_1\,\delta^{1+1/p}\le\Big(\int_0^{2\pi}A_P(\delta,\theta)^p\,d\theta\Big)^{1/p}\le c_2\,\delta^{1+1/p}\,,\qquad \text{for } 1<p\le\infty\,.$$

These estimates are easy consequences of the argument in Lemma 3.9.

Step 3. Upper bound when $2\le p\le\infty$

$$\Big(\int_0^{2\pi}A_B(\delta,\theta)^p\,d\theta\Big)^{1/p}\le c_2\,\delta^{1+1/p}\,,$$

for any B.

The case $p=\infty$ is obvious; the case $2<p<\infty$ follows as in Lemma 3.13.

Step 4. Admissible decays when $2\le p\le\infty$.

For any $2\le p\le\infty$ and any $1+1/p\le a\le 3/2$ there exists B such that

$$c_1\,\delta^a\le\Big(\int_0^{2\pi}A_P(\delta,\theta)^p\,d\theta\Big)^{1/p}\le c_2\,\delta^a\,.$$

This is precisely the content of Lemma 3.16.

Step 5. Lower bound for $1\le p\le\infty$ when B is not a polygon

$$\Big(\int_0^{2\pi}A_B(\delta,\theta)^p\,d\theta\Big)^{1/p}\ge c\,\delta^{3/2}\,.$$

Indeed, if B is not a polygon, there exists a regular arc in ∂B which does not coincide with its chord. Then, at any point in this arc one can apply the following elementary observation. Let $f\in C^2[-1,1]$ be a real function satisfying $f(0)=f'(0)=0$ and $0\le f''(x)\le 2\,c$ for any $x\in[-1,1]$. Writing $f(x)=\delta$ for $x=x_1$ and $x=x_2$, we have $|x_1-x_2|\ge 2\sqrt{\delta/c}$.

6. Lattice points in large convex planar sets.

From the Introduction we recall the following:

Definition 6.1. *Let $\sigma \in SO(2)$ and $t \in \mathbb{T}^2$. The discrepancy function $D_B(\rho, \theta, t)$ is defined by*

$$\begin{aligned} D_B(\rho, \sigma, t) &= \operatorname{card}\left((\rho\sigma^{-1}(B) - t) \cap \mathbb{Z}^2\right) - \rho^2 |B| \\ &= \sum_{m \in \mathbb{Z}^2} \chi_{\rho\sigma^{-1}(B)-t}(m) - \rho^2 |B| \, . \end{aligned}$$

We prove the following result.

Theorem 6.2. *Assume B is a convex body in $\mathbb{R}^2$ with piecewise C^∞ boundary, which is not a polygon. Let $1 \le p \le 2$, then*

$$c_1 \, \rho^{1/2} \le \|D_B(\rho, \cdot, \cdot)\|_{L^p(SO(2)\times\mathbb{T}^2)} \le c_2 \, \rho^{1/2} \, .$$

PROOF. The estimate from above is easy (and essentially known). Indeed a computation gives

$$D_B(\rho, \sigma, \cdot)^\wedge(m) = \rho^2 \, \widehat{\chi}_B(\rho \, \sigma(m)) \, ,$$

for any $m \in \mathbb{Z}^2$, $m \ne 0$ (please note that the hat symbol in the left hand side and in the right hand side refer to the Fourier transform on $\mathbb{T}^2$ and on $\mathbb{R}^2$ respectively). Hence, by Lemma 3.1,

$$\begin{aligned} \int_{SO(2)} \int_{T^2} |D_B(\rho, \sigma, t)|^2 \, dt \, d\sigma &= \rho^4 \int_{SO(2)} \sum_{m \ne 0} |\widehat{\chi}_B(\rho \, \sigma(m))|^2 \, d\sigma \\ &= \rho^4 \sum_{m \ne 0} \int_{SO(2)} |\widehat{\chi}_B(\rho \, \sigma(m))|^2 \, d\sigma \\ &\le \rho^4 \sum_{m \ne 0} |\rho \, m|^{-3} \\ &= c \, \rho \, . \end{aligned}$$

Therefore, whenever $1 \le p \le 2$,

$$\|D_B(\rho, \cdot, \cdot)\|_{L^p(SO(2)\times\mathbb{T}^2)} \le \|D_B(\rho, \cdot, \cdot)\|_{L^2(SO(2)\times\mathbb{T}^2)} \le c_2 \, \rho^{1/2} \, .$$

On the other hand, for any $m \in \mathbb{Z}^2$, $m \neq 0$,

$$
\begin{aligned}
\|D_B(\rho,\cdot,\cdot)\|_{L^p(SO(2)\times\mathbb{T}^2)} &\geq \|D_B(\rho,\cdot,\cdot)\|_{L^1(SO(2)\times\mathbb{T}^2)} \\
&= \int_{SO(2)}\int_{\mathbb{T}^2} |D_B(\rho,\sigma,t)|\,dt\,d\sigma \\
&\geq \int_{SO(2)} |D_B(\rho,\sigma,\cdot)^\wedge(m)|\,d\sigma \\
&= \rho^2 \int_{SO(2)} |\widehat{\chi}_B(\rho\,\sigma(m))|\,d\sigma\,.
\end{aligned}
\tag{6.1}
$$

We split the argument for the estimate from below into three cases.

First case. Suppose B is not a cut disc (see Definition 3.3). Then, making use of Lemma 3.6, (6.1) implies

$$\|D_B(\rho,\cdot,\cdot)\|_{L^p(SO(2)\times\mathbb{T}^2)} \geq c_1\,\rho^{1/2}\,.$$

Second case. Suppose we have a disc D. First assume

$$\min_{n\in\mathbb{Z}}\left|2\,\rho - \frac{1}{4} - n\right| \geq \frac{1}{10}\,.$$

Let $m = (1,0)$, then, by (6.1), and the asymptotic of Bessel functions,

$$
\begin{aligned}
\|D_D(\rho,\cdot,\cdot)\|_{L^p(SO(2)\times\mathbb{T}^2)} &\geq \rho\,J_1(2\pi\rho) \\
&= \pi^{-1}\,\rho^{1/2}\cos\left(2\pi\rho - \frac{3}{4}\,\pi\right) + \mathcal{O}(1) \\
&\geq c\,\rho^{1/2}\,.
\end{aligned}
$$

On the other hand, when

$$\min_{n\in\mathbb{Z}}\left|2\,\rho - \frac{1}{4} - n\right| \leq \frac{1}{10}$$

we choose $m = (2,0)$, then

$$\|D_D(\rho,\cdot,\cdot)\|_{L^p(SO(2)\times\mathbb{T}^2)} \geq \pi^{-1}\,\rho^{1/2}\cos\left(4\pi\rho - \frac{3}{4}\,\pi\right) + \mathcal{O}(1) \geq c\,\rho^{1/2}\,.$$

Third case. Suppose B is a cut disc, coming from a given disc D. Without loss of generality we can assume

$$\{(\cos\theta, \sin\theta) : \ |\theta| \le \alpha \text{ or } |\pi - \theta| \le \alpha\} \subset \partial B\,,$$

for a small $\alpha > 0$. Let

$$U = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \ |\theta| < \frac{\alpha}{2} \right\}.$$

Then, for $m = (1,0)$ or $m = (2,0)$,

$$\begin{aligned} &\|D_B(\rho,\cdot,\cdot)\|_{L^p(SO(2)\times\mathbb{T}^2)} \\ &\qquad \ge \rho^2 \int_U |\widehat{\chi}_B(\rho\,\sigma(m))|\, d\sigma \\ &\qquad \ge \left| \rho^2 \int_U |\widehat{\chi}_D(\rho\,\sigma(m))|\, d\sigma - \rho^2 \int_U |\widehat{\chi}_{D\setminus B}(\rho\,\sigma(m))|\, d\sigma \right|. \end{aligned}$$

Now the third case is a consequence of the second one if we prove that

$$\int_U |\widehat{\chi}_{D\setminus B}(\rho\,\sigma(m))|\, d\sigma \le c\,\rho^{-2}\,.$$

Indeed $D\setminus B$ looks like in the following picture and therefore, by applying Lemma 3.8 to each one of the connected components of $D\setminus B$, we get

$$|\widehat{\chi}_{D\setminus B}(\rho\,\sigma(m))| \le c\,\rho^{-2}\,,$$

uniformly in $\sigma \in U$.

Figure 5.

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A mixed norm estimate for the X-ray transform

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Let G be the space of lines in $\mathbb{R}^3$, *i.e.* the 4-dimensional manifold whose elements are all lines in $\mathbb{R}^3$. We can coordinatize G in the following way

$$\ell = \ell(e, x)\,,$$

where $e \in S^2 \backslash \{\pm 1\}$ is the direction of ℓ and $x = x_\ell$ is the unique point on ℓ which is perpendicular to e. We will denote the direction e of ℓ by ℓ^*.

The distance on G can be defined using the standard distances on the sphere and in $\mathbb{R}^3$ and this identification, thus

$$d(\ell, m) = |x_\ell - x_m| + \theta(\ell, m)\,, \tag{1}$$

where $\theta(\ell, m) = \theta(\ell^*, m^*)$ is the unoriented angle ($\in [0, \pi/2]$) between ℓ and m. This distance has the following property. Let $T_\ell(a)$ be the cylinder of radius δ, axis ℓ and length 1, centered at the point $a \in \ell$, and let $T_\ell = T_\ell(x_\ell)$ where x_ℓ is as defined above. Then for $\sigma \geq \delta$,

$$\theta(\ell, m) \leq \sigma\,, \text{ and } T_\ell \cap T_m \neq \varnothing \text{ imply } d(\ell, m) \leq C_0\, \sigma\,, \tag{2}$$

where C_0 is a suitable numerical constant.

All metric quantities defined on G refer to the distance d.

We will be using mixed norms on G defined in the following way: if $F : G \longrightarrow \mathbb{R}$ then

$$\|F\|_{L^q_e(L^r_x)} \stackrel{\text{def}}{=} \Big(\int_{e \in S^2} \Big(\int_{\{x \in \mathbb{R}^3 : x \perp e\}} |F(e, x)|^r \, dx \Big)^{q/r} de \Big)^{1/q}\,,$$

where the x-integral is with respect to two dimensional Lebesgue measure. We remark that the functions we will be considering will generally be supported in the set $\{(e,x)\in G:\ \|x\|\le 1\}$.

The X-*ray transform* is the map from functions on $\mathbb{R}^3$ to functions on G defined by

$$Xf(\ell)=\int_\ell f\,.$$

Our purpose is to prove the following estimate

Theorem 1. *If $f:\mathbb{R}^3\longrightarrow\mathbb{R}$ and the support of f is contained in the unit disc then*

$$\|Xf\|_{L^q_e(L^r_x)}\le C_\varepsilon\,\|f\|_{p,\varepsilon}\,,$$

for any $\varepsilon>0$. Here $\|\ \|_{p,\varepsilon}$ is the inhomogeneous Sobolev norm with ε derivatives in L^p, and the exponents are as follows

$$p=\frac{5}{2}\,,\qquad q=\frac{10}{3}\,,\qquad r=10\,.$$

The following is an equivalent formulation of Theorem 1 which is easier to work with.

Theorem 2. *Let Ω be a subset of $S^2\setminus\pm 1$, let E be a subset of the unit disc in $\mathbb{R}^3$, and $\lambda>0$. Assume that for each $e\in\Omega$ there are m δ-separated lines ℓ with direction $\ell^*=e$ such that*

$$|T_\ell\cap E|\ge\lambda\,|T_\ell|\,. \tag{3}$$

Then

$$|E|\ge C_\varepsilon^{-1}\,\delta^{C\varepsilon}\,\lambda^{5/2}\,m^{1/4}\,|\Omega|^{3/4}\,\delta^{1/2}\,. \tag{4}$$

Of course, a subset $\{m_j\}$ of a metric space M is called *δ-separated* if $j\ne k$ implies that the distance from m_j to m_k is at least δ.

Theorems 1 and 2 are refinements of the result in [7] – the result in [7] corresponds to the case $m=1$. The argument in the present context is more subtle than the argument in [7], but the basic strategy is similar. Let $D(a,r)$ be the ball centered at a with radius r. The main work is to prove

Lemma 0. *Theorem 2 is true provided we make the following additional hypothesis on the tubes T_ℓ: for any $a \in \mathbb{R}^3$,*

$$|T_\ell \cap E \cap D(a, \delta^\varepsilon)| \leq \lambda \left(\log \frac{1}{\delta}\right)^{-10} |T_\ell| \, .$$

A version of property (5) was also used in [7]. We could call it the "two ends" condition, since it expresses the fact that $E \cap T_\ell$ is not concentrated near one end of T_ℓ.

We now explain briefly how Theorem 1 fits into the literature. There is a "space time" estimate for the X-ray transform, *i.e.* an estimate from L^p to $L^q(G)$, which in the three dimensional case says that

$$\|Xf\|_{L^4_e(L^4_x)} \lesssim \|f\|_2 \, .$$

After a result of Oberlin and Stein [6] for the Radon transform, this was proved by Drury [3] with a loss of ε derivatives and then by Christ [2] as stated. The main conjecture on the Kakeya maximal function can be stated as

$$\|Xf\|_{L^3_e(L^\infty_x)} \lesssim \|f\|_{3,\varepsilon}$$

and if one interpolates between this conjectural result and Drury's, one obtains the conjectural bound

$$\|Xf\|_{L^q_e(L^r_x)} \lesssim \|f\|_{p,\varepsilon} \, , \qquad \varepsilon > 0 \, , \tag{6}$$

for any $p \in (2, 3)$, where $q = 2\,p'$ and $1/r = 1 - 3/q$. Theorem 1 confirms (6) when $p \leq 5/2$.

In [2] it is conjectured that (6) should hold as an endpoint result, *i.e.* without the loss of ε derivatives. When $p < 5/2$ it is conceivable that this can be proved by refining the argument below, but we do not attempt that here. Nor do we attempt a generalization of Theorem 1 to higher dimensions; the natural generalization would be (6) in $\mathbb{R}^n$ with

$$p = \frac{n+2}{2} \, , \qquad q = (n-1)p' \qquad \text{and} \qquad \frac{1}{r} = 1 - \frac{n}{q} \, .$$

The plan of the paper is as follows: sections 1 and 2 are preliminaries to the proof of Lemma 0, Section 3 is the proof of Lemma 0 and Section 4 is the proof of Theorems 2 and 1.

1. Preliminaries.

Some notation and terminology is as follows: the number ε is kept fixed throughout the proof of Lemma 0. We also fix δ, although needless to say the values of all constants must be independent of δ. If ℓ is a line then the tubes $T_\ell(a)$ and T_ℓ are as defined in the introduction and in particular have cross section radius δ. We will say that tubes T_ℓ and T_m *intersect at angle* τ if $T_\ell \cap T_m \neq \varnothing$ and $\theta(\ell, m) = \tau$. If E is a set then the notation $|E|$ will be used to denote the Lebesgue measure or cardinality of E depending on the context. The characteristic function of E will be denoted by χ_E. The disc of radius r centered at x in a metric space is denoted $D(x, r)$; we remark that we use this notation regardless of whether the metric space is $\mathbb{R}^3$, G, S^2 or something else. Finally we will use a certain normalization of the entropy of a set, which in practice will be a set in G or on the 2-sphere.

Definition. *If M is a metric space and $\sigma > 0$ then $\mathcal{E}_\sigma(M) = \sigma^2 \mathcal{N}_\sigma(M)$, where $\mathcal{N}_\sigma(M)$ is the maximum possible cardinality for a σ-separated subset of M.*

In proving Lemma 0 we can assume that our lines intersect the unit ball in $\mathbb{R}^3$ and make an angle of less or equal than $1/100$ with the vertical direction, say, and will always make these assumptions in order to avoid some notational complications. We also always assume that δ is sufficiently small.

In several places we will need to use some elementary but not completely obvious facts from solid geometry. We will generally not give the proofs of these facts. However, we want to clarify our terminology. If $\ell, \ell' \in G$ are intersecting lines then the *plane spanned by ℓ and ℓ'* means of course the unique plane containing ℓ and ℓ'. In addition, if $\ell \in G$ and $e \in S^2$ then the *plane spanned by ℓ and e* is the set $\{x \in \mathbb{R}^3 : \ x = y + t\, e \text{ for some } y \in \ell \text{ and } t \in \mathbb{R}\}$. If Π and $\tilde{\Pi}$ are 2-planes, then the *angle between* Π *and* $\tilde{\Pi}$ is of course the inverse cosine of the dot product between the unit normal vectors to Π and $\tilde{\Pi}$, just as the angle between two lines is is the inverse cosine of the dot product of their direction vectors. As an example of the kind of statement we have in mind, we note the following.

Lemma 1.0. Suppose that Π is a plane, ℓ is a line contained in Π, ℓ' is a line intersecting ℓ at a point a, and that the angle between ℓ and ℓ' is less or equal than σ and the angle between Π and the plane spanned by ℓ and ℓ' is $\leq \phi$. Then $T_{\ell'}(a)$ is contained in the $C(\phi\,\sigma+\delta)$-neighborhood of Π, *i.e.*, if $x \in T_{\ell'}(a)$ then $\operatorname{dist}(x,\Pi) \leq C(\phi\,\sigma + \delta)$.

PROOF. Choose coordinates so that a is the origin, Π is the x_1x_2 plane, and ℓ is the x_2 axis. Then the assumptions mean that if $y \in \ell'$, then

$$|y_1| + |y_3| \lesssim \sigma(|y_1| + |y_2| + |y_3|)\,,$$

$$|y_3| \lesssim \phi(|y_1| + |y_3|)\,.$$

If $x \in T_{\ell'}(a)$, then there is a point $y \in \ell'$ with $|y_1| + |y_2| + |y_3| \leq C$ and with $|x-y| \leq C\,\delta$. The above equations then imply $|y_3| \lesssim \sigma\,\phi$, so $|x_3| \lesssim \sigma\,\phi + \delta$ as claimed.

One problem in adapting the argument in [7] is as follows: use was made there of the fact (perhaps due to Córdoba) that a family of tubes contained in a $C\delta$-neighborhood of a 2-plane and with δ-separated directions must satisfy an estimate $\sum_j |T_j| \approx |\cup_j T_j|$ up to δ^ε factors. Here we will be considering families of lines which are δ-separated in the Grassmannian G, but their directions may not be δ-separated. Lemma 1.2 below is an adaptation of the Córdoba argument to this situation; the form of the statement may look peculiar, but it is the one which is most useful for our purposes.

We will be considering various rectangles R relative to an orthonormal basis e_1, e_2, e_3 with respective dimensions $100 \times w \times 100\,\delta$, where we always assume that $100 > w > 100\,\delta$. Given such a rectangle R, we will call w the *width* of R and will refer to the plane through the center point of R spanned by the e_1 and e_2 directions as the *2-plane of* R and to the line through the center in the e_1 direction as the *axis of* R.

We fix a set E and number λ. If $\mathcal{A}$ is a δ-separated family of lines and if R is a $100 \times w \times 100\,\delta$-rectangle then we define the *tube density* of R, $d_{\mathcal{A}}(R)$, via

$$d_{\mathcal{A}}(R) = \frac{|\{\ell \in \mathcal{A}:\ T_\ell \subset R\}|}{\frac{w}{\delta}} \tag{7}$$

A *plate of width* w *relative to* $\mathcal{A}$ is a $100 \times w \times 100\,\delta$-rectangle R with the following property:

Plate property. *Suppose that for each $\ell \in \mathcal{A}$ with $T_\ell \subset R$, a subset $Y_\ell \subset T_\ell \cap E$ is given, satisfying*

$$|Y_\ell| \geq \Big(\log \frac{1}{\delta}\Big)^{-3} \lambda \, |T_\ell| \, .$$

Then

$$\Big| \bigcup_{T_\ell \subset R} Y_\ell \Big| \geq \Big(\log \frac{1}{\delta}\Big)^{-10} \lambda^2 \, |R| \, . \tag{8}$$

Assuming that $\mathcal{A}$ is δ-separated and the tubes $\{T_\ell\}_{\ell \in \mathcal{A}}$ satisfy (3), we define a quantity $p_\sigma(\mathcal{A})$ in the following way

$$p_\sigma(\mathcal{A}) = \sup_R d_{\mathcal{A}}(R) \, , \tag{9}$$

where R runs over all plates relative to $\mathcal{A}$ of width $\leq \sigma$. We will frequently use the fact (easy to prove) that p_σ is monotone under set inclusion,

$$\mathcal{B} \subset \mathcal{A} \text{ implies } p_\sigma(\mathcal{B}) \leq p_\sigma(\mathcal{A}) \, . \tag{10}$$

Lemma 1.1. *Assume that $\mathcal{A}$ is δ-separated and the tubes $\{T_\ell\}_{\ell \in \mathcal{A}}$ satisfy* (3). *Then*

$$p_\sigma(\mathcal{A}) = \sup_R d_{\mathcal{A}}(R) \, ,$$

where R runs over all $100 \times w \times 100\,\delta$ rectangles with $w \leq \sigma$ (not just plates).

Corollary.

i) *$p_\sigma(\mathcal{A})$ actually depends only on $\mathcal{A}$ and not on E or λ.*

ii) *Let $\overline{\sigma} = \max(100\,\delta, \delta^{2\varepsilon}\sigma)$. Then $p_{\overline{\sigma}}(\mathcal{A}) \geq \delta^{5\varepsilon}\, p_\sigma(\mathcal{A})$.*

PROOF OF THE COROLLARY. Part i) is obvious from Lemma 1.1. Part ii) follows since it is easy to see that if $w' = \max\{\delta^{2\varepsilon} w, 100\,\delta\}$ and if R is a $100 \times w \times 100\,\delta$ rectangle which contains M tubes $T_\ell, \ell \in \mathcal{A}$, then there must be a $100 \times w' \times 100\,\delta$-subrectangle containing at least $C^{-1}\,\delta^{4\varepsilon}\, M$ of these tubes.

PROOF OF LEMMA 1.1. Fix a rectangle P with essentially the maximum tube density, *i.e.*, P is a $100 \times w \times 100\,\delta$ rectangle with $w \leq \sigma$, and if R is any other such rectangle, then $d_{\mathcal{A}}(R) \leq 2\, d_{\mathcal{A}}(P)$. Let $\mathcal{C}(P)$ be the lines $\ell \in \mathcal{A}$ with $T_\ell \subset P$.

It suffices to show that P is a plate relative to $\mathcal{A}$. So fix appropriate subsets $Y_\ell \subset T_\ell$, which from the form of the statement may be assumed to have measure *exactly*

$$\frac{\lambda}{\left(\log \frac{1}{\delta}\right)^3} \left|T_\ell\right|.$$

Let $\tilde{E} = \cup_{\ell \in \mathcal{C}(P)} Y_\ell$. Then, by Córdoba's well-known calculation,

$$\begin{aligned}
\frac{\lambda}{\left(\log \frac{1}{\delta}\right)^3} \left|\mathcal{C}(P)\right| \delta^2 &\approx \sum_{\ell \in \mathcal{C}(P)} |Y_\ell| \\
&= \int_{\tilde{E}} \sum_{\ell \in \mathcal{C}(P)} \chi_{Y_\ell} \\
&\leq |\tilde{E} \cap P|^{1/2} \Big\| \sum_{\ell \in \mathcal{C}(P)} \chi_{Y_\ell} \Big\|_2 \\
&= |\tilde{E} \cap P|^{1/2} \Big(\sum_{\ell, m \in \mathcal{C}(P)} |Y_\ell \cap Y_m| \Big)^{1/2}.
\end{aligned}$$

For each ℓ and $\tau \geq \delta$, the maximality property of P implies there are $\lesssim (\tau/w)\, |\mathcal{C}(P)|$ tubes T_m with $m \in \mathcal{C}$ which intersect T_ℓ at angle between $(\tau - \delta)/2$ and τ. For each such m, $|Y_\ell \cap Y_m| \lesssim \tau^{-1}\, \delta^3$. Accordingly (the sum over τ below runs over dyadic values between δ and σ)

$$\begin{aligned}
&\lesssim |\tilde{E} \cap P|^{1/2} \Big(\sum_{\ell \in \mathcal{C}(P)} \sum_{\tau} \frac{|\mathcal{C}(P)|\, \tau}{w} \frac{\delta^3}{\tau} \Big)^{1/2} \qquad (11) \\
&\lesssim |\tilde{E} \cap P|^{1/2} \left(\frac{|\mathcal{C}(P)|^2\, \delta^3 \, \log \frac{1}{\delta}}{w} \right)^{1/2}
\end{aligned}$$

and now (8) follows by algebra.

Lemma 1.2. *Let $\mathcal{A}$ be a δ-separated subset of G and assume that the tubes T_ℓ, $\ell \in \mathcal{A}$ are contained in the intersection of a σ-neighborhood of a line and a $100\,\delta$-neighborhood of a 2-plane, where $\sigma \geq \delta$. Assume that for each $\ell \in \mathcal{A}$ a subset $Y_\ell \subset T_\ell \cap E$ is given, satisfying*

$$|Y_\ell| \geq \Big(\log\frac{1}{\delta}\Big)^{-3} \lambda\, |T_\ell|\,.$$

Let $\tilde{E} = \cup_{\ell\in\mathcal{A}} Y_\ell$. Then, with $p = p_\sigma(\mathcal{A})$,

$$|\tilde{E}| \geq \Big(\log\frac{1}{\delta}\Big)^{-10} p^{-1}\, \lambda^2\, \mathcal{E}_\delta(\mathcal{A})\,. \tag{12}$$

Proof. This is similar to the proof of Lemma 1.1. By Lemma 1.1 we know that $|\mathcal{C}(R)| \lesssim p\,w/\delta$ for *all* $100 \times w \times 100\,\delta$ rectangles R. So for any fixed $\ell \in \mathcal{A}$ and τ, there are $\lesssim p\,\tau/\delta$ tubes which intersect T_ℓ at angle less or equal than τ. Hence

$$\begin{aligned}
\frac{\lambda}{\Big(\log\frac{1}{\delta}\Big)^3}\,|\mathcal{A}|\,\delta^2 &\lesssim \sum_{\ell\in\mathcal{A}} |Y_\ell| \\
&\leq |\tilde{E}|^{1/2}\Big(\sum_{\ell m} |Y_\ell \cap Y_m|\Big)^{1/2} \\
&\lesssim |\tilde{E}|^{1/2}\Big(\sum_i \sum_\tau \frac{p\,\tau}{\delta}\,\frac{\delta^3}{\tau}\Big)^{1/2} \\
&\lesssim |\tilde{E}|^{1/2}\Big(|\mathcal{A}|\,p\,\delta^2\,\log\frac{1}{\delta}\Big)^{1/2},
\end{aligned}$$

using the same type of reasoning as before. The result follows.

The rest of this section is of a technical nature – Lemma 1.4 below will allow us to avoid some unpleasant technicalites later on. Similar issues come up elsewhere in the literature and Lemma 1.3 was suggested by some (rather more sophisticated) lemmas of the same type due to Szemeredi and Balog-Szemeredi, see [5, Section 9.3].

Assume that $\mathcal{A}$ is a set, N a number with $|\mathcal{A}| \leq N$. An *allowable relation* on $\mathcal{A}$ means a pair $\{\Pi_\mathcal{B}\}_{\mathcal{B}\subset\mathcal{A}}$, $\sim$, where

1) For each $\mathcal{B} \subset \mathcal{A}$, $\Pi_\mathcal{B}$ is a collection of subsets of $\mathcal{B}$. Also $\sim$ is a relation between points of $\mathcal{A}$ and subsets of $\mathcal{A}$ which belong to $\cup_{\mathcal{B}\subset\mathcal{A}}\Pi_\mathcal{B}$.

2) If $\mathcal{B}_1 \subset \mathcal{B}_2$ and if $S_1 \in \Pi_{\mathcal{B}_1}$, then there is $S_2 \in \Pi_{\mathcal{B}_2}$ with $S_1 \subset S_2$ such that $x \sim S_1$ implies $x \sim S_2$.

3) If $x \in \mathcal{B}$ then there is $S \in \Pi_{\mathcal{B}}$ with $x \sim S$.

If $\mathcal{B} \subset \mathcal{A}, S \in \Pi_{\mathcal{B}}$ then we define $n_{\mathcal{B}}(S) = |\{x \in \mathcal{B} : \ x \sim S\}|$; and $q(\mathcal{B}) = \max\{n_{\mathcal{B}}(S) : \ S \in \Pi_{\mathcal{B}}\}$. We note that property 2) guarantees that q is monotone under set inclusion, $\mathcal{B}_1 \subset \mathcal{B}_2$ implies $q(\mathcal{B}_1) \leq q(\mathcal{B}_2)$. Likewise property 3) guarantees that $q(\mathcal{B}) \geq 1$ for all $\mathcal{B} \subset \mathcal{A}$.

Definition. *A subset $\mathcal{A}' \subset \mathcal{A}$ is good relative to $\sim$ if the following holds: if $\mathcal{B} \subset \mathcal{A}'$ with $|\mathcal{B}| \geq (\log N)^{-10}\,|\mathcal{A}'|$ then there is a subset $\mathcal{C} \subset \mathcal{B}$ with $|\mathcal{C}| \geq |\mathcal{B}|/2$ such that $x \in \mathcal{C}$ implies there is $S \in \Pi_{\mathcal{B}}$ such that $x \in S$ and $n_{\mathcal{B}}(S) \geq N^{-\varepsilon}\, q(\mathcal{A}')$.*

In practice, we will work with several allowable relations simultaneously. Suppose then that $\{\{\Pi^j_{\mathcal{B}}\}_{\mathcal{B}\subset\mathcal{A}}, \sim_j\}_{j=1}^k$ is a family of allowable relations on a set $\mathcal{A}$ and denote the quantities $n_{\mathcal{B}}(S)$ and $q(\mathcal{B})$ defined using the relation $\sim_j$ by $n^j_{\mathcal{B}}(S)$ and $q^j(\mathcal{B})$. We say that $\mathcal{A}' \subset \mathcal{A}$ is *good with respect to all of the relations* $\sim_j$ if the preceding definition is valid for each j, with the set $\mathcal{C}$ being independent of j. More precisely,

Definition. *A subset $\mathcal{A}' \subset \mathcal{A}$ is good relative to all of the relations $\sim_j$ if the following holds: if $\mathcal{B} \subset \mathcal{A}'$ with $|\mathcal{B}| \geq (\log N)^{-10}\,|\mathcal{A}'|$ then there is a subset $\mathcal{C} \subset \mathcal{B}$ with $|\mathcal{C}| \geq |\mathcal{B}|/2$ such that $x \in \mathcal{C}$ implies that for each j there is $S \in \Pi^j_{\mathcal{B}}$ such that $x \in S$ and $n^j_{\mathcal{B}}(S) \geq N^{-\varepsilon}\, q^j(\mathcal{A}')$.*

The point is that a fairly large "good" subset will always exist:

Lemma 1.3. *If $\{\sim_j\}_{j=1}^k$ is a family of allowable relations on a set $\mathcal{A}$ with $|\mathcal{A}| \leq N$, and if N is large enough depending on ε and k, then there is a subset $\mathcal{A}' \subset \mathcal{A}$ with $|\mathcal{A}'| \geq N^{-\varepsilon}\,|\mathcal{A}|$ which is good relative to all of the relations $\sim_j$.*

Proof. Consider a subset of $\mathcal{A}$, which we denote by $\mathcal{A}_i$, which is *not* good with respect to all of the relations $\sim_j$. Then, from the definition, there is a subset $\mathcal{B} \subset \mathcal{A}_i$ with $|\mathcal{B}| \geq (\log N)^{-10}|\mathcal{A}_i|$, such that half of the elements $x \in \mathcal{B}$ satisfy $\max\{n^j_{\mathcal{B}}(S) : \ x \in S, \ S \in \Pi^j_{\mathcal{B}}\} \leq N^{-\varepsilon}\, q^j(\mathcal{A}_i)$ for some j (depending on x). Hence we can find a common value of j which works for at least $|\mathcal{B}|/(2\,k)$ elements. Defining $\mathcal{A}_{i+1}$ to be these elements, we see that $n^j_{\mathcal{B}}(S) \leq N^{-\varepsilon}\, q^j(\mathcal{A}_i)$ for all $S \in \Pi^j_{\mathcal{B}}$ such that

$S \cap \mathcal{A}_{i+1} \neq \varnothing$. Consequently, if $S \in \Pi^j_{\mathcal{A}_{i+1}}$ then $n^j_{\mathcal{A}_{i+1}}(S) \leq N^{-\varepsilon} q^j(\mathcal{A}_i)$ by property 2), and therefore $q^j(\mathcal{A}_{i+1}) \leq N^{-\varepsilon} q^j(\mathcal{A}_i)$. We conclude.

If $\mathcal{A}_i$ is not good, then there are $\mathcal{A}_{i+1} \subset \mathcal{A}_i$ with $|\mathcal{A}_{i+1}| \geq (\log N)^{-10} |\mathcal{A}_i|/(2\,k)$ and $j \in \{1, \ldots, k\}$ such that

$$q^j(\mathcal{A}_{i+1}) \leq N^{-\varepsilon}\, q^j(\mathcal{A}_i)\,.$$

Now suppose we have a string

$$\mathcal{A} = \mathcal{A}_0 \supset \cdots \supset \mathcal{A}_n \tag{13}$$

so that the above property holds for each $i = 0, \ldots, n-1$. We can pigeonhole to obtain a common value of j for at least n/k values of i. Using the monotonicity property of q^j it then follows that

$$1 \leq q^j(\mathcal{A}_n) \leq N^{-\varepsilon n/k}\, q^j(\mathcal{A}) \leq N^{-\varepsilon n/k+1}\,,$$

i.e. $n \leq k/\varepsilon$. On the other hand the last element of a maximal string (13) must be good. So we have found a good subset with at least $((\log N)^{-10}/(2\,k))^{k/\varepsilon}\, |\mathcal{A}|$ elements, which gives the result.

We now specialize to the situation we care about, namely the following situation:

$(*)$ $\mathcal{A}$ is a δ-separated subset of G and the tubes $\{T_\ell\}_{\ell \in \mathcal{A}}$ satisfy (3) with respect to some set E contained in the unit ball (and some λ).

If $\mathcal{B} \subset \mathcal{A}$ then we let $P_j(\mathcal{B})$ be the set of all plates relative to $\mathcal{B}$ of width less or equal than $\delta^{j\varepsilon}$. If ℓ is a line, then we let $P_j(\mathcal{B}, \ell)$ be the set of all plates relative to $\mathcal{B}$ of width less or equal than $\delta^{j\varepsilon}$ which contain T_ℓ. Finally, if R is a plate relative to $\mathcal{B}$ then we let $\mathcal{B}_{ir}(R)$ be the set of lines in $\mathcal{B}$ such that the following conditions hold: i) T_ℓ intersects R; and if we denote the axis direction of R by e, then ii) the angle between the direction of ℓ and the direction of e is less or equal than $\delta^{i\varepsilon}$, and iii) the angle between the 2-plane of R and the 2-plane spanned by ℓ and the e direction is less or equal than $\delta^{r\varepsilon}$.

Definition. *Suppose that $\mathcal{A}' \subset \mathcal{A}$. Then $\mathcal{A}'$ is good if for any $\mathcal{B} \subset \mathcal{A}'$ with*

$$|\mathcal{B}| \geq \Big(\log \frac{1}{\delta}\Big)^{-10} |\mathcal{A}'|\,,$$

there is $\mathcal{C} \subset \mathcal{B}$ *with* $|\mathcal{C}| \geq |\mathcal{B}|/2$ *such that if* $\ell_0 \in \mathcal{C}$ *then*

1) *For any integer* $j \leq 1/\varepsilon$*, we have*

$$\begin{aligned} &|\{m \in \mathcal{B} : \ T_m \cap T_{\ell_0} \neq \varnothing \ \textit{and} \ \theta(\ell_0, m) \leq \delta^{j\varepsilon}\}| \\ &\qquad \geq \delta^{\varepsilon} \, |\{m \in \mathcal{A}' : \ T_m \cap T_{\ell_0} \neq \varnothing \ \textit{and} \ \theta(\ell_0, m) \leq \delta^{j\varepsilon}\}| \, . \end{aligned}$$

2) *For any integer* j *with* $\delta^{j\varepsilon} \geq 100\,\delta$*, we have*

$$\max_{R \in P_j(\mathcal{B}, \ell_0)} d_{\mathcal{B}}(R) \geq \delta^{2\varepsilon} p_\sigma(\mathcal{A}') \, . \tag{14}$$

Here we have set $\sigma = \delta^{j\varepsilon}$*, and the notation* $d_{\mathcal{B}}(R)$ *and* $p_\sigma(\mathcal{A}')$ *is defined by* (7) *and* (9).

3) *For any* j *with* $\delta^{j\varepsilon} \geq 100\delta$ *and any* $i \leq 1/\varepsilon$, $r \leq 1/\varepsilon$*, we have*

$$\max_{R \in P_j(\mathcal{B}, \ell_0)} |\mathcal{B}_{ir}(R)| \geq \delta^{\varepsilon} \max_{R \in P_j(\mathcal{A}', \ell_0)} |\mathcal{A}'_{ir}(R)| \, . \tag{15}$$

Lemma 1.4. *If* $\mathcal{A}$ *is as described by* $(*)$ *then* $\mathcal{A}$ *has a good subset* $\mathcal{A}'$ *with* $|\mathcal{A}'| \geq \delta^{\varepsilon}|\mathcal{A}|$.

Proof. We will define a set of allowable relations and apply Lemma 1.3. Let $\mathcal{A}$ be our set of lines, $N = \delta^{-4}$ which is clearly an upper bound for $|\mathcal{A}|$. If $\mathcal{B} \subset \mathcal{A}$, and if R is a plate relative to $\mathcal{B}$, then we define a subset $S_{\mathcal{B}}(R) = \{m \in \mathcal{B} : \ T_m \subset R\}$. We could call this the combinatorial plate corresponding to the geometric plate R. We let $\Pi_j(\mathcal{B})$ be the set of all "combinatorial plates" relative to $\mathcal{B}$ with width less or equal than $\delta^{j\varepsilon}$, *i.e.*

$$\Pi_j(\mathcal{B}) = \{S_{\mathcal{B}}(R) : \ R \in P_j(\mathcal{B})\} \, . \tag{16}$$

The following then constitute a set of less or equal than ε^{-4} allowable relations:

$1)_j$ For each $\mathcal{B} \subset \mathcal{A}$, $\Pi_{\mathcal{B}}$ is all singleton subsets $\{m\}$, $m \in \mathcal{B}$, with the relations $\ell \sim \{m\}$ if $\theta(\ell, m) \leq \delta^{j\varepsilon}$ and $T_\ell \cap T_m \neq \varnothing$.

$2)_j$ $\Pi_{\mathcal{B}} = \Pi_j(\mathcal{B})$ is defined by (16), and $\ell \sim S_{\mathcal{B}}(R)$ if $T_\ell \subset R$.

$3)_{ijr}$ $\Pi_{\mathcal{B}} = \Pi_j(\mathcal{B})$ is defined by (16); and $\ell \sim S_{\mathcal{B}}(R)$ if T_ℓ intersects R, ℓ makes an angle less or equal than $\delta^{i\varepsilon}$ with the axis direction of

R, and the 2-plane spanned by ℓ and the axis direction of R makes an angle less or equal than $\delta^{r\varepsilon}$ with the 2-plane of R.

It is almost immediate that all these relations are allowable. We indicate the proof.

Property 2) holds for the relations 1): if $S_1 = \{m\}$, then take $S_2 = \{m\}$ also.

Property 2) holds for the relations 2) and 3): if $S_1 \in \Pi_j(\mathcal{B}_1)$ then S_1 is the combinatorial plate $S_{\mathcal{B}_1}(R)$ corresponding to some plate $R \in P_j(\mathcal{B}_1)$. Then clearly $R \in P_j(\mathcal{B}_2)$ also, and it follows that we can take $S_2 = S_{\mathcal{B}_2}(R)$.

Property 3) holds for the relations 1): if $\ell \in \mathcal{B}$ then we can take $S = \{\ell\}$.

Property 3) holds for the relations 2) and 3): for this, fix a line $\ell \in \mathcal{B}$ and set $S = S_{\mathcal{B}}(R)$ where R is a $100 \times w \times 100\,\delta$ rectangle containing T_ℓ, with axis very close to and coplanar with ℓ, and width slightly greater than $100\,\delta$. R will be a plate with respect to $\mathcal{B}$ according to our definition and clearly $\ell \sim S$ for any of the relations 2) or 3).

By Lemma 1.3, there is a subset $\mathcal{A}' \subset \mathcal{A}$ which is good with respect to all of these relations and has cardinality $\geq \delta^{\varepsilon}|\mathcal{A}|$. Let us now see that this means $\mathcal{A}'$ is good in the sense of the preceding definition. Fix an appropriate subset $\mathcal{B}$ and choose a further subset $\mathcal{C}$ using the fact that $\mathcal{A}'$ is good with respect to the relations 1), 2), 3). If $\ell_0 \in \mathcal{C}$ then properties 1) and 3) in the definition of good follow immediately using the relations 1) and 3). For example, the relation $3)_{ijr}$ leads to the conclusion

$$\max_{R \in P_j(\mathcal{B},\ell_0)} |\mathcal{B}_{ir}(R)| \geq \delta^{\varepsilon} \max_{R \in P_j(\mathcal{A}')} |\mathcal{A}'_{ir}(R)|\,,$$

which is slightly stronger than (15), and similarly $1)_j$ leads to a slightly stronger form of property 1). It remains to prove (14). The relations $2)_j$ imply in the notation (7), (9) that

$$\max_{R \in P_k(\mathcal{B},\ell_0)} w(R)\, d_{\mathcal{B}}(R) \geq \delta^{\varepsilon} \max_{R' \in P_k(\mathcal{A}')} w(R')\, d_{\mathcal{A}'}(R')\,, \tag{17}$$

for any k, where $w(R)$ is the width of R. Now let j and σ be as in (14) and choose a plate achieving $p_\sigma(\mathcal{A}')$, *i.e.* let R' be a plate relative to $\mathcal{A}'$ with width $w' \leq \sigma$ and with $p_\sigma(\mathcal{A}') = d_{\mathcal{A}'}(R')$. Choose k as large as possible subject to $\delta^{k\varepsilon} \geq w'$, and apply (17). Thus $p_\sigma(\mathcal{A}') \leq (\delta^{\varepsilon} w')^{-1} \max_{R \in P_k(\mathcal{B},\ell_0)} w(R) d_{\mathcal{B}}(R)$. Now note that $\delta^{k\varepsilon} \leq \delta^{-\varepsilon} w'$; we conclude therefore that $p_\sigma(\mathcal{A}') \leq \delta^{-2\varepsilon} \max_{R\in P_k(\mathcal{B},\ell_0)} d_{\mathcal{B}}(R)$. Clearly $k \geq j$, so $P_k(\mathcal{B},\ell_0) \subset P_j(\mathcal{B},\ell_0)$, and (14) follows.

2. First part of proof.

In this section we prove the following lemma, which is a refinement of the main lemma in [7].

Lemma 2.1. *Assume $\mathcal{A}$ is a δ-separated subset of G and for each $\ell \in \mathcal{A}$ the tube T_ℓ satisfies* (3), (5).

Then for some $\sigma \in (100\,\delta, 100)$ and for some subset $\mathcal{A}' \subset \mathcal{A}$ with $|\mathcal{A}'| \geq \delta^{C_1\varepsilon}|\mathcal{A}|$,

$$|E| \geq \delta^{C_1\varepsilon}\, p_\sigma(\mathcal{A}')^{-1/2}\,\lambda^2 \sqrt{\mathcal{E}_\delta(\mathcal{A}')\,\mathcal{E}_\sigma(\mathcal{A}')\,\frac{\delta}{\sigma}}\,. \tag{18}$$

Proof. We may assume that $\mathcal{A}$ is good in the sense of Lemma 1.4, else we pass to a suitable subset which is. (Actually, for the current argument only property 1) in the definition is needed) Let χ_E be the characteristic function of E and define

$$\mu_{\mathcal{A}}(x) = \sum_{m\in\mathcal{A}} \chi_{T_m}(x)\,. \tag{19}$$

It is easy to see that $\mu_{\mathcal{A}}(x) \lesssim \delta^{-2}$ for all x – this follows since a δ-separated family of lines passing through a fixed point has cardinality $\lesssim \delta^{-2}$. We also define

$$\mu^j_{\mathcal{A},\ell}(x) = \sum_{m\in\mathcal{A}:\delta^{j\varepsilon}\leq\theta(\ell,m)\leq\delta^{(j-1)\varepsilon}} \chi_{T_m}(x)\,. \tag{20}$$

We claim there are positive integers $j \leq 1/\varepsilon$ and $N \lesssim \delta^{-2}$ and a subset $\mathcal{A}'' \subset \mathcal{A}$ such that

$$|\mathcal{A}''| \geq \Big(\log\frac{1}{\delta}\Big)^{-2}|\mathcal{A}| \tag{21}$$

and if $\ell \in \mathcal{A}''$, then

$$|Y_\ell| \geq \frac{\lambda\,|T_\ell|}{\Big(\log\dfrac{1}{\delta}\Big)^2}\,, \tag{22}$$

where

$$Y_\ell \stackrel{\text{def}}{=} T_\ell \cap E \cap \{x:\ \mu_{\mathcal{A}}(x) \le 2N\} \cap \{x:\ \mu^j_{\mathcal{A},\ell}(x) \ge \varepsilon N\}\,. \tag{23}$$

This follows from the pigeonhole principle. Namely, if C is a suitable constant then for each ℓ there is N, a dyadic integer so that

$$|T_\ell \cap E \cap \{x:\ N \le \mu_{\mathcal{A}}(x) \le 2N\}| \ge \frac{\lambda\,|T_\ell|}{C \log \dfrac{1}{\delta}}\,. \tag{24}$$

Accordingly we can pick a value of N so that (24) holds with that value of N for at least $(C\log(1/\delta))^{-1}|\mathcal{A}|$ tubes from $\mathcal{A}$. Next, for each of these tubes there must be a value of $j \le 1/\varepsilon$ such that

$$\begin{aligned} &|T_\ell \cap E \cap \{x:\ N \le \mu_{\mathcal{A}}(x) \le 2N\} \cap \{x:\ \mu^j_{\mathcal{A},\ell}(x) \ge \varepsilon N\}| \\ &\qquad\qquad \ge \frac{\lambda\,|T_\ell|}{C \log \dfrac{1}{\delta}}\,\varepsilon \end{aligned} \tag{25}$$

and therefore (25) holds with a common value of j for at least $\varepsilon\,(C\log(1/\delta))^{-1}|\mathcal{A}|$ lines ℓ. This proves the claim. We will use similar "pigeonhole" arguments several times below without giving the details.

We clearly have

$$|E| \ge (2N)^{-1} \sum_{\ell \in \mathcal{A}''} |Y_\ell| \ge \delta^\varepsilon\, \frac{\lambda\, \mathcal{E}_\delta(\mathcal{A}'')}{N}\,. \tag{26}$$

Note that this immediately implies (18) (with $\sigma \approx \delta$) if $N\lambda \le \delta^{-12\varepsilon}$, say, so in proving (18) we may assume that $N\lambda \ge \delta^{-12\varepsilon}$.

Assuming $N\lambda \ge \delta^{-12\varepsilon}$ we now set $\sigma = \delta^{(j-1)\varepsilon}$ and let $\tilde{T}_\sigma(\ell)$ be the $3 \times 3\,\sigma$ tube concentric with T_ℓ. For each $\ell \in \mathcal{A}''$, we define $\mathcal{A}''(\ell) = \{m \in \mathcal{A}'':\ m \sim_j \ell\}$ where $\sim_j$ is the relation

$$\ell \sim_j m$$

if $T_\ell \cap T_m \ne \varnothing$ and $\theta(\ell, m) \le \delta^{(j-1)\varepsilon}$. We further define E_ℓ for $\ell \in \mathcal{A}''$ by

$$E_\ell = \bigcup_{m \in \mathcal{A}''(\ell)} Y_m\,.$$

Note E_ℓ is contained in $\tilde{T}_\sigma(\ell)$.

Lemma 2.2. *If $N \geq \delta^{-12\varepsilon}$ then there is a subset $\mathcal{A}' \subset \mathcal{A}''$ with $|\mathcal{A}'| \geq |\mathcal{A}''|/2$, such that if $\ell \in \mathcal{A}'$, then (with $p = p_\sigma(\mathcal{A})$)*

$$|E_\ell| \geq \delta^{5\varepsilon}\, p^{-1} N\, \sigma\, \delta\, \lambda^3\,.$$

Proof. Fix $\ell \in \mathcal{A}''$. If a tube T_m intersects T_ℓ at angle greater or equal than $\delta^\varepsilon \sigma$ then the intersection has measure $\lesssim \delta^{-\varepsilon}\delta^3/\sigma$. It follows using (22) that there are at least

$$C^{-1}\delta^\varepsilon N\, \frac{\sigma}{\delta}\left(\log\frac{1}{\delta}\right)^{-2}\lambda$$

lines m *in* $\mathcal{A}$ such that T_m intersects T_ℓ at angle between $\delta^\varepsilon\sigma$ and σ.

Detailed justification for the latter assertion is as follows. Let $\mathcal{B} = \{m \in \mathcal{A} :\ T_m \text{ intersects } T_\ell \text{ at angle between } \delta^\varepsilon\sigma \text{ and } \sigma\}$. Then

$$\begin{aligned}
|\mathcal{B}|\,\delta^{-\varepsilon}\,\frac{\delta^3}{\sigma} &\gtrsim \sum_{m\in\mathcal{B}} |T_m \cap T_\ell| \\
&= \int_{T_\ell} \sum_{m\in\mathcal{B}} \chi_{T_m} \\
&= \int_{T_\ell} \mu^j_{\mathcal{A},\ell} \\
&\geq \varepsilon N\,|Y_\ell| \\
&\gtrsim \frac{N\,\lambda\,\delta^2}{\left(\log\frac{1}{\delta}\right)^2}
\end{aligned}$$

as claimed. We will use this argument again in Section 3 without giving the details.

By the "goodness" property, we can choose $\mathcal{A}' \subset \mathcal{A}''$ with $|\mathcal{A}'| \geq |\mathcal{A}''|/2$ so that if $\ell \in \mathcal{A}'$ then there are at least

$$C^{-1}\delta^{2\varepsilon} N\, \frac{\sigma}{\delta}\left(\log\frac{1}{\delta}\right)^{-2}\lambda$$

lines *in* $\mathcal{A}''$ such that T_m intersects T_ℓ at angle less than σ, *i.e.*

$$|\mathcal{A}''(\ell)| \geq C^{-1} N \delta^{2\varepsilon}\, \frac{\sigma}{\delta}\left(\log\frac{1}{\delta}\right)^{-2}\lambda\,.$$

Fix $\ell \in \mathcal{A}'$. We can pick $\tau \in [\delta, \sigma]$ such that at least

$$\delta^{3\varepsilon} N \, \frac{\sigma}{\delta} \, \lambda$$

of these tubes T_m intersect T_ℓ at angle between $\tau/2$ and τ. We denote this set of lines m by $\mathcal{C}$. Thus

$$|\mathcal{C}| \geq \delta^{3\varepsilon} N \, \frac{\sigma}{\delta} \, \lambda \, . \tag{27}$$

We now repeat the argument from [7]. First we dispense with a minor technicality. Namely, we have

$$\tau \geq \delta^{-2\varepsilon} \, \delta \, . \tag{28}$$

To see this note that

$$\delta^{3\varepsilon} N \, \frac{\sigma}{\delta} \, \lambda \leq C \, \frac{\tau^4}{\delta^4} \ ,$$

since $C\,\tau^4/\delta^4$ is a bound for the number of δ-separated tubes which can intersect T_ℓ at angle less or equal than τ. Hence, since $N\lambda \geq \delta^{-12\varepsilon}$,

$$\tau \geq (C^{-1} \delta^{3\varepsilon} N \lambda)^{1/4} (\delta^3 \sigma)^{1/4} \geq \delta^{-2\varepsilon} \, \delta \, ,$$

proving (28). Now, as in [7] we choose a family of 2-planes Π_k through ℓ corresponding to a maximal (δ/τ)-separated set of directions perpendicular to ℓ and consider their $100\,\delta$-neighborhoods $\Pi_k^{100\delta}$. Then every tube T_m, $m \in \mathcal{C}$ is contained in some $\Pi_k^{100\delta}$ and a point at distance ρ from ℓ belongs to at most $C \max\{\tau/\rho, 1\}\, \Pi_k^{100\delta}$'s. This is clear geometrically, see also Lemma 3.1 below. For each k, let $\mathcal{C}_k$ be the tubes in $\mathcal{C}$ which are contained in $\Pi_k^{100\delta}$. Let Z_m be the points in Y_m which are at distance at least $\delta^\varepsilon \tau$ from the axis ℓ. Using (28) and standard geometrical facts, the complement of Z_m in Y_m is contained in a disc of radius $\approx \delta^\varepsilon$, so (22) and the "two ends property" (5) imply that

$$|Z_m| \geq \Big(\log \frac{1}{\delta} \Big)^{-3} \lambda \, |T_m| \, .$$

Lemma 1.2 implies (using (10)) that

$$\Big| \bigcup_{m \in \mathcal{C}_k} Z_m \Big| \geq \delta^\varepsilon p^{-1} |\mathcal{C}_k| \, \delta^2 \lambda^2 \, .$$

Therefore, since no point of any Z_m can belong to more than $C\,\delta^{-\varepsilon}\,\Pi_k^{100\delta}$'s, we have

$$\begin{aligned}
|E_\ell| &\ge \Big|\bigcup_m Z_m\Big| \\
&\ge \delta^{\varepsilon}\sum_k \Big|\bigcup_{m\in\mathcal{C}_k} Z_m\Big| \\
&\ge \sum_k \delta^{2\varepsilon} p^{-1} |\mathcal{C}_k|\,\delta^2\lambda^2 \\
&\ge \delta^{2\varepsilon} p^{-1} |\mathcal{C}|\,\delta^2\lambda^2 \\
&\ge \delta^{5\varepsilon} p^{-1} N\sigma\,\delta\,\lambda^3\,,
\end{aligned}$$

by (27).

We now note the following (this is the punchline!). Let C_0 be the constant in (2).

Claim. *If $x\in\mathbb{R}^3$, then there are at most $(2/\varepsilon)\,4\,C_0\,\sigma$-separated lines $\ell\in\mathcal{A}'$ such that $x\in E_\ell$.*

Namely, suppose we have M such lines ℓ. For each of them there is a line $m=m_\ell$ at distance less or equal than σ from ℓ such that $x\in Y_m$. Thus

1) $\mu_{\mathcal{A}}(x)\le 2N$.

2) $\mu^m_{\mathcal{A},j}(x)\ge\varepsilon N$ for each m.

Note the m's are $2\,C_0\,\sigma$-separated by (2), since T_ℓ intersects T_{m_ℓ} at angle less or equal than σ. It follows by (2) that no tube can intersect two different T_m's at angle less or equal than σ. Accordingly property 2) implies that $\mu_{\mathcal{A}}(x)\ge M\varepsilon N$, hence $M\le 2/\varepsilon$ by property 1) This proves the claim.

Now take a maximal $4\,C_0\,\sigma$-separated subset $\mathcal{B}\subset\mathcal{A}'$. By the claim and then Lemma 2.2, we have

$$|E|\ge\frac{2}{\varepsilon}\sum_{\ell\in\mathcal{B}}|E_\ell|\ge\delta^{6\varepsilon}p^{-1}|\mathcal{B}|\,N\,\sigma\,\delta\,\lambda^3\,,$$

or in other words

$$|E| \geq \delta^{6\varepsilon} p^{-1} N \lambda^3 \mathcal{E}_\sigma(\mathcal{A}') \, \frac{\delta}{\sigma} \,, \tag{29}$$

since of course $\mathcal{E}_\sigma(\mathcal{A})$ and $\mathcal{E}_{4C_0\sigma}(\mathcal{A})$ are comparable. If we take the geometric mean of (29) and (26) we get (18).

A *slab of thickness* ϕ is a ϕ-neighborhood of a 2-plane. What we actually use below is the following corollary of Lemma 2.1.

Lemma 2.3. *Assume $\mathcal{A}$ is a δ-separated subset of G and for each $\ell \in \mathcal{A}$ the tube T_ℓ satisfies* (3) *and* (5).

Assume in addition that all tubes T_ℓ, $\ell \in \mathcal{A}$ are contained in a slab of thickness ϕ and in a ρ-neighborhood of a line. Let $p = p_\rho(\mathcal{A})$ and define $m = m(\mathcal{A})$ via

$$\begin{aligned} &m(\mathcal{A}) = \max_{e \in S^2} m(\mathcal{A}, e) \,, \\ &\textit{where } m(\mathcal{A}, e) \stackrel{\text{def}}{=} |\{\ell \in \mathcal{A} : \ \theta(e, \ell^*) < \delta\}| \,. \end{aligned} \tag{30}$$

Then

$$|E| \geq \delta^{C_2\varepsilon} (m\, p)^{-1/2} \lambda^2 \, \mathcal{E}_\delta(\mathcal{A}) \sqrt{\frac{\delta}{\phi}} \,. \tag{31}$$

Proof. Fix a number $\sigma \geq \delta$. Note that all the lines in $\mathcal{A}$ make an angle less or equal than ϕ with a fixed 2-plane. We will use this fact to get a lower bound on $\mathcal{E}_\sigma(\mathcal{A})$. Namely, let $\mathcal{A}^*$ be the set of angles $\ell^*, \ell \in \mathcal{A}$. Clearly

$$\mathcal{E}_\delta(\mathcal{A}^*) \gtrsim \frac{\mathcal{E}_\delta(\mathcal{A})}{m} \,.$$

On the other hand, $\mathcal{A}^*$ is contained in a ϕ-neighborhood of a great circle on the 2-sphere, which implies that

$$\mathcal{E}_\sigma(\mathcal{A}^*) \gtrsim \frac{\sigma}{\phi} \, \mathcal{E}_\phi(\mathcal{A}^*) \,,$$

when $\sigma \geq \phi$. Also $\mathcal{E}_\sigma(\mathcal{A}^*) \gtrsim \mathcal{E}_\tau(\mathcal{A}^*)$ if $\sigma \geq \tau$ (this is true for any set on the 2-sphere), so we may conclude that

$$\frac{\mathcal{E}_\sigma(\mathcal{A}^*)}{\sigma} \gtrsim \frac{\mathcal{E}_\delta(\mathcal{A}^*)}{\phi} \,,$$

for all σ, and therefore

$$\frac{\mathcal{E}_\sigma(\mathcal{A}^*)}{\sigma} \gtrsim \frac{\mathcal{E}_\delta(\mathcal{A})}{m\,\phi}\,.$$

The result now follows from Lemma 2.1.

Corollary. *Under the assumptions of Lemma 2.3, suppose that for each $\ell \in \mathcal{A}$ a subset $Y_\ell \subset T_\ell \cap E$ is given, with*

$$|Y_\ell| \geq \Big(\log\frac{1}{\delta}\Big)^{-4} \lambda\,|T_\ell|\,.$$

Let $\tilde{E} = \cup_{\ell\in\mathcal{A}} Y_\ell$. Then estimate (31) holds also for $\tilde{E}$, i.e.

$$|\tilde{E}| \geq \delta^{C_2\varepsilon}(m\,p)^{-1/2}\lambda^2\,\mathcal{E}_\delta(\mathcal{A})\,\sqrt{\frac{\delta}{\phi}}\,.$$

PROOF. The idea is to apply Lemma 2.3 with E replaced by $\tilde{E}$ and λ replaced by

$$\lambda\Big(\log\frac{1}{\delta}\Big)^{-4}\,.$$

In order to do this we must make the following remarks:

- (5) does not quite hold anymore. However, it holds if we replace the exponent 10 on the right hand side by 6. The reader can easily check that this does not make any difference.

- The definition of the number $p_\sigma(\mathcal{A})$ depended in principle on λ and E as well as $\mathcal{A}$. However, in fact it depends only on $\mathcal{A}$ by the corollary to Lemma 1.1.

Accordingly we can apply Lemma 2.3 as indicated, obtaining

$$|\tilde{E}| \geq \delta^{C_2\varepsilon}(m\,p)^{-1/2}\lambda^2\Big(\log\frac{1}{\delta}\Big)^{-8}\mathcal{E}_\delta(\mathcal{A})\,\sqrt{\frac{\delta}{\phi}}\,.$$

The factor $(\log(1/\delta))^{-8}$ may of course be incorporated into the $\delta^{C_2\varepsilon}$ factor, so we are done.

REMARK. The considerations in Section 2 generalize immediately to higher dimensions. In particular, Lemma 2.1 is true in $\mathbb{R}^n$ with the same proof provided we define $\mathcal{E}_\sigma(\mathcal{A}) = \sigma^{n-1}$ times the maximum possible cardinality for a σ-separated subset of $\mathcal{A}$, define p using $100 \times w \times 100\,\delta \times \cdots \times 100\,\delta$ rectangles and replace the factor δ/σ by $(\delta/\sigma)^{n-2}$.

3. Main argument.

The argument in this section will be based on considering families of tubes which intersect a plate, rather than a tube as in the previous section. Lemmas 3.1 and 3.2 below record some geometrical facts relevant in this situation.

Lemma 3.1. *Suppose that $\phi \in (\delta, \pi/2)$, $\sigma \in (\delta, \pi/2)$, $w \le \sigma$ and R is a $100 \times w \times 100\,\delta$ rectangle. Let Π be the 2-plane of R, let $\{\theta_k\}$ be a maximal $(\phi\, w + \delta)/\sigma$-separated subset of $(\phi/2, \phi)$, and for each k let $\Pi_k^{\pm}$ be the two 2-planes through the axis of R which make an angle θ_k with Π, and $\Pi_k^{\pm, C(\phi w+\delta)}$ their $C\,(\phi\, w + \delta)$-neighborhoods. Then:*

i) *Let T_ℓ be a tube which intersects R and such that ℓ makes an angle less or equal than σ with the axis of R and the 2-plane spanned by ℓ and the axis direction of R makes an angle between $(\phi - \delta)/2$ and ϕ with the 2-plane Π. Then T_ℓ is contained in some slab $\Pi_k^{\pm, C(\phi w+\delta)}$.*

ii) *A point at distance greater or equal than ρ from Π is contained in $\lesssim \max\{\phi\,\sigma/\rho, 1\}$ slabs $\Pi_k^{\pm, C(\phi w+\delta)}$.*

Proof. i) First let Π_1 and Π_2 be 2-planes passing through the axis of R and making angle less or equal than β with each other. Let τ_σ be the σ-neighborhood of the axis of R. Then every point of $\tau_\sigma \cap \Pi_2$ will be within $\beta\,\sigma$ of Π_1. Accordingly (take $\beta = (\phi\, w + \delta)/\sigma$) it suffices to show that T_ℓ is contained in a $C\,(\phi\, w + \delta)$-neighborhood of *some* plane passing through the axis of R and making an angle between $\phi/2$ and ϕ with Π. On the other hand, let Π' be the plane spanned by ℓ and the axis direction of R. Let Π'' be the plane parallel to Π' which passes through the axis of R. The distance between Π' and Π'' is then $\lesssim \phi\, w + \delta$ and therefore T_ℓ is contained in the $C\,(\phi\, w + \delta)$-neighborhood of Π''.

ii) Choose coordinates so that Π is the xy plane and the axis of R is the y axis. Assume $a = (x, y, z)$ is in n $\Pi_k^{+, C(w\phi+\delta)}$'s, say. Then (assuming $\phi \le \pi/4$; otherwise some minor changes in the argument are required) we have

$$|z| = (\sin\theta_k)\,|x| + \mathcal{O}\,(\phi\, w + \delta)\,,$$

for each k and therefore, arranging the θ_k's in increasing order,

$$(n-1)\,\frac{\phi\, w + \delta}{\sigma}\,|x| \le (\theta_n - \theta_1)\,|x| \lesssim (\phi\, w + \delta)\,,$$

so $|x| \lesssim \sigma/(n-1)$. Then

$$|z| \lesssim \frac{\phi\,\sigma}{n-1} + \phi\, w + \delta\,,$$

which implies

$$|z| \lesssim \frac{\phi\,\sigma}{n-1}$$

since obviously

$$n-1 \lesssim \frac{\phi\,\sigma}{\phi\, w + \delta}\,.$$

This is equivalent to the statement.

Lemma 3.2. *Suppose that $\sigma \in (\delta, \pi/2)$, $\phi \in (\delta, \pi/2)$ and R is a $100 \times w \times 100\,\delta$ rectangle. Let ℓ be a line and assume that ℓ makes an angle greater or equal than $\sigma - \delta$ with the axis direction of R and that the 2-plane spanned by ℓ and the axis direction of R makes an angle greater or equal than $\phi - \delta$ with the 2-plane of R. Then*

$$|T_\ell \cap R| \lesssim \min\Big\{\delta^2\,\frac{w}{\sigma}, \frac{\delta^3}{\phi\,\sigma + \delta}\Big\}\,.$$

PROOF. Choose coordinates so the axis direction of R is the y direction, the 2-plane of R is parallel to the xy plane and the origin belongs to $T_\ell \cap R$. If $p = (x, y, z)$ is a point of T_ℓ then the assumptions mean that

$$\begin{aligned} &|x| + |y| + |z| \lesssim \sigma^{-1}\,(|x| + |z|) + \delta\,,\\ &|x| + |z| \lesssim \phi^{-1}|z| + \delta \end{aligned} \tag{32}$$

and therefore

$$|x| + |y| + |z| \lesssim (\phi\,\sigma)^{-1}|z| + \sigma^{-1}\delta\,. \tag{33}$$

If $p \in T_\ell \cap R$ then (32) and (33) imply

$$|x| + |y| + |z| \lesssim \min\{\sigma^{-1}(w+\delta), (\phi\,\sigma)^{-1}\delta\},$$

or in other words $T_\ell \cap R$ is a subset of T_ℓ with diameter

$$\lesssim \min\Big\{\frac{w}{\sigma}, \frac{\delta}{\phi\,\sigma}\Big\}.$$

The lemma follows.

The next lemma estimates the measure of the union of a "large" family of tubes intersecting a rectangle.

Lemma 3.3. *Suppose $\sigma \geq 100\,\delta$, $\phi \in (\delta, \pi/2)$, $w \leq \sigma$, R is a $100 \times w \times 100\,\delta$ rectangle and $\mathcal{C}$ is a family of lines. Assume that if $\ell \in \mathcal{C}$ then T_ℓ intersects R, ℓ makes an angle less or equal than σ with the axis direction of R, and the 2-plane spanned by ℓ and the axis direction of R makes an angle in $((\phi-\delta)/2, \phi)$ with the 2-plane of R. Assume furthermore that if $\ell \in \mathcal{C}$ then T_ℓ satisfies (3), (5). Let $p = p_\sigma(\mathcal{C})$ and define $m = m(\mathcal{C})$ via (30). Assume that for each $\ell \in \mathcal{C}$ a subset $Y_\ell \subset T_\ell \cap E$ is given, with*

$$|Y_\ell| \geq \Big(\log\frac{1}{\delta}\Big)^{-3} \lambda\, |T_\ell|.$$

Let $\tilde{E} = \cup_\ell Y_\ell$. Then $\tilde{E}$ is contained in a slab of width $C\,(\phi\,\sigma + \delta)$ and

$$|\tilde{E}| \geq \delta^{C_3\varepsilon}\,(m\,p)^{-1/2}\,\lambda^2\,\mathcal{E}_\delta(\mathcal{C})\,\sqrt{\frac{\delta}{w\,\phi+\delta}}\,. \tag{34}$$

PROOF. It follows by Lemma 1.0 that $\tilde{E}$ is contained in a slab of width $C\,(\phi\,\sigma+\delta)$ – namely, the $C\,(\phi\,\sigma+\delta)$-neighborhood of the 2-plane of R. We now prove (34). We first dispense with a couple of minor technicalities. First of all, we can assume that all the lines in $\mathcal{C}$ actually make an angle between $\delta^{-\varepsilon}\sigma$ and σ with the axis direction of R, since we can always achieve this by replacing σ by $\delta^{j\varepsilon}\sigma$ for a suitable $j \geq 0$ and replacing $\mathcal{C}$ by a subset $\mathcal{C}'$ with $\mathcal{E}_\delta(\mathcal{C}') \gtrsim \mathcal{E}_\delta(\mathcal{C})$. Second, we can assume $\phi\,\sigma \geq \delta^{-3\varepsilon}\,\delta$. To see this, suppose that $\phi\,\sigma \leq \delta^{-3\varepsilon}\,\delta$. Then all the tubes in $\mathcal{C}$ are contained in a $C\,\delta^{1-3\varepsilon}$-neighborhood of a 2-plane. Accordingly (34) follows immediately from the corollary to Lemma 2.3.

Now we consider the main case where $\phi\,\sigma \geq \delta^{-3\varepsilon}\,\delta$. Let Z_ℓ be the points in Y_ℓ which are at distance greater or equal than $\delta^{2\varepsilon}\,\phi\,\sigma$ from the 2-plane of R and $\tilde{\tilde{E}} = \cup_\ell Z_\ell$. Since $\phi\,\sigma \geq \delta^{-3\varepsilon}\,\delta$, it follows from (33) that the set of points of T_ℓ which are within $\delta^{2\varepsilon}\,\phi\,\sigma$ of the 2-plane of R is contained in a $C\,\delta^\varepsilon$-disc. Thus the complement of Z_ℓ in Y_ℓ is contained in a $C\,\delta^\varepsilon$-disc. So property (5) implies

$$|Z_\ell| \geq \lambda\Big(\log\frac{1}{\delta}\Big)^{-4}\,|T_\ell|\,.$$

Now consider a subdivision into 2-plane neighborhoods $\Pi_k^{C(w\phi+\delta)}$ as in Lemma 3.1, relative to the rectangle R, and with the given value of σ. Let $\mathcal{C}_k$ be the tubes which are contained in a given $\Pi_k^{C(\phi w+\delta)}$. By the corollary to Lemma 2.3,

$$\big|\tilde{\tilde{E}} \cap \Pi_k^{C(w\phi+\delta)}\big| \geq \delta^{C_2\varepsilon}\,(m\,p)^{-1/2}\,\lambda^2\,\mathcal{E}_\delta(\mathcal{C}_k)\,\sqrt{\frac{\delta}{w\,\phi+\delta}}\,.$$

Notice that no point of $\tilde{\tilde{E}}$ is in more than $C\,\delta^{-\varepsilon}$ sets of the form $\tilde{\tilde{E}} \cap \Pi_k^{C(w\phi+\delta)}$, by Lemma 3.1.ii). So if we sum over k we get

$$\begin{aligned}
|\tilde{E}| &\geq |\tilde{\tilde{E}}| \\
&\geq \delta^\varepsilon \sum_k \big|\tilde{\tilde{E}} \cap \Pi_k^{C(w\phi+\delta)}\big| \\
&\geq \delta^{C\varepsilon} \sum_k (m\,p)^{-1/2}\,\lambda^2\,\mathcal{E}_\delta(\mathcal{C}_k)\,\sqrt{\frac{\delta}{w\,\phi+\delta}} \\
&\geq \delta^{C\varepsilon}\,(m\,p)^{-1/2}\,\lambda^2\,\mathcal{E}_\delta(\mathcal{C})\,\sqrt{\frac{\delta}{w\,\phi+\delta}}\,.
\end{aligned}$$

In order to apply Lemma 3.3 we need to find sufficiently large families of tubes which intersect a suitable rectangle. This is done in the next lemma, which is analogous to Lemma 2.2 in Section 2. The quantities $\mu_{\mathcal{A}}$, $\mu^j_{\mathcal{A},\ell}$ were defined in (19), (20).

Lemma 3.4. *Assume that $\mathcal{A} \subset G$ is δ-separated and that the tubes T_ℓ satisfy* (3) *and* (5)*, and furthermore that $\mathcal{A}$ is good in the sense of Lemma* 1.4*. Fix j and suppose that $\mathcal{B}$ is a subset of $\mathcal{A}$ with*

$$|\mathcal{B}| \geq \Big(\log\frac{1}{\delta}\Big)^{-10}|\mathcal{A}|$$

and that for each $\ell \in \mathcal{B}$, a subset

$$Y_\ell \subset T_\ell \cap E \cap \{x: \ \mu_{\mathcal{A}}(x) \le 2N\} \cap \{x: \ \mu^j_{\mathcal{A},\ell}(x) \ge \varepsilon N\}$$

is given, with

$$|Y_\ell| \ge \left(\log \frac{1}{\delta}\right)^{-3} \lambda \, |T_\ell| \, .$$

Let $\sigma = \delta^{(j-1)\varepsilon}$ and let $m = m(\mathcal{A})$. Then for some line $\ell \in \mathcal{B}$ there are a number $\phi \in (\delta, \pi/2)$, a 2-plane Π and a set of lines $\mathcal{D} \subset \mathcal{B} \cap D(\ell, C_4 \delta^{-\varepsilon}\sigma)$ such that

$$\bigcup_{m \in \mathcal{D}} Y_m \subset \Pi^{C(\delta^{-2\varepsilon} \, \phi \, \sigma + \delta)}$$

and

$$\Big| \bigcup_{m \in \mathcal{D}} Y_m \Big| \ge \delta^{C_5 \varepsilon} \, N \, m^{-1/2} \, \lambda^{7/2} \, \delta \, \sqrt{\sigma} \, \sqrt{\phi \, \sigma + \delta} \, . \tag{35}$$

PROOF. Let $\mathcal{C}$ be associated to $\mathcal{B}$ as in the definition of "good" preceding Lemma 1.4. We will show that the conclusion holds for any $\ell \in \mathcal{C}$. So fix a line $\ell_0 \in \mathcal{C}$.

By (14) and part ii) of the corollary to Lemma 1.1, T_{ℓ_0} must be contained in a plate P relative to $\mathcal{B}$ of width $w \le \max\{100\,\delta, \delta^{2\varepsilon}\sigma\}$ and $\mathcal{B}$-tube density $d_{\mathcal{B}}(P) \ge \delta^{7\varepsilon} p$, where $p = p_\sigma(\mathcal{A})$.

Claim. *For some $\phi \ge \delta$, there is a set $\mathcal{D}_0 \subset \mathcal{A}$ with*

$$|\mathcal{D}_0| \gtrsim \delta^{12\varepsilon} \, N \Big(p \, \frac{w}{\delta}\Big)^{1/2} \lambda^{3/2} \, \max\Big\{\delta^{-1}(\phi \, \sigma + \delta), \frac{\sigma}{w}\Big\} \, ,$$

such that if $\ell \in \mathcal{D}_0$ then T_ℓ intersects P, ℓ makes an angle less or equal than σ with the axis of P, and the 2-plane spanned by ℓ and the axis direction of P makes an angle less or equal than ϕ with the 2-plane of P.

To prove the claim, let Σ be the set of lines $\ell \in \mathcal{B}$ such that $T_\ell \subset P$ and $Z = \cup\{Y_\ell : \ \ell \in \Sigma\}$. Then, since $|\Sigma| \ge \delta^{7\varepsilon} \, p \, w/\delta$, we have

$$|Z| \gtrsim \delta^{8\varepsilon} \, \lambda^2 \, \delta \, w \, , \tag{36}$$

by Lemma 1.2. We will now show that also

$$|Z| \gtrsim \delta^{8\varepsilon}\,\lambda\,\delta^2\,p\,. \tag{37}$$

Namely, let Σ^* be the set of directions of lines in Σ. It is clear that the maximum possible cardinality for a δ-separated subset of Σ^* is $\lesssim w/\delta$. Accordingly, by definition of $d_{\mathcal{B}}(P)$ there must be a direction e such that $\theta(\ell^*, e) < \delta$ for $\gtrsim \delta^{7\varepsilon}p$ lines $\ell \in \Sigma$. Denote this set of $\gtrsim \delta^{7\varepsilon}p$ lines by Σ'. It is clear that no point can belong to more than a bounded number of the (essentially parallel) tubes $T_\ell, \ell \in \Sigma'$. Accordingly

$$|Z| \gtrsim \sum_{\ell\in\Sigma'} |Y_\ell| \gtrsim \delta^{6\varepsilon} p\Big(\log\frac{1}{\delta}\Big)^{-3}\lambda\,\delta^2$$

and (37) follows.

Taking the geometric mean of (36) and (37) we conclude that

$$|Z| \gtrsim \delta^{8\varepsilon}\sqrt{\frac{p\,w}{\delta}}\,\lambda^{3/2}\,\delta^2\,.$$

Next, each point $x \in Z$ belongs to Y_m for some line m such that $T_m \subset P$. By definition of Y_m there are $\gtrsim N$ lines $\ell \in \mathcal{A}$ such that T_ℓ contains x and ℓ makes an angle between $\delta^\varepsilon\sigma$ and σ with the line m. We denote this set of lines ℓ by $\mathcal{A}(x)$. Since the width of P is less or equal than $\max\{\delta^{2\varepsilon}\sigma, 100\,\delta\}$ it follows that all lines $\ell \in \cup_{x\in Z}\mathcal{A}(x)$ make an angle between $\delta^\varepsilon\sigma/2 - C\,\delta$ and $2\,\sigma + C\,\delta$ with the axis of P. For each $\ell \in \cup_{x\in Z}\mathcal{A}(x)$, the 2-plane spanned by ℓ and the axis direction of P makes a certain angle ϕ_ℓ depending on ℓ with the 2-plane of P. We can now use the pigeonhole principle to obtain a common value of ϕ_ℓ. Namely, by the pigeonhole principle there are a number $\phi \in (\delta, \pi/2)$ and a subset $F \subset Z$ with $|F| \geq \delta^\varepsilon\,|Z|$, so that if $x \in F$ then there is a subset $\tilde{\mathcal{A}}(x) \subset \mathcal{A}(x)$ with cardinality at least $N\delta^\varepsilon$, which consists of lines ℓ such that $\phi_\ell \in ((\phi-\delta)/2, \phi)$.

To summarize: there is a subset $F \subset P$ with measure greater or equal than

$$\delta^{9\varepsilon}\sqrt{\frac{p\,w}{\delta}}\,\lambda^{3/2}\,\delta^2\,,$$

so that if $x \in F$ then there is a set (which we denoted $\tilde{\mathcal{A}}(x)$) of $N\delta^\varepsilon$ lines $\ell \in \mathcal{A}$ such that T_ℓ contains x and ℓ makes an angle in $(\delta^\varepsilon\sigma/2 - C\,\delta, 2\,\sigma + C\,\delta)$ with the axis of P and the 2-plane spanned by ℓ and the axis direction of P makes an angle in $((\phi-\delta)/2,\phi)$ with the 2-plane of P. We define $\mathcal{D}_0=\cup_{x\in F}\tilde{\mathcal{A}}(x)$. By Lemma 3.2, we have

$$|T_\ell\cap P|\lesssim\delta^{-2\varepsilon}\min\left\{\frac{\delta^3}{\phi\,\sigma+\delta},\delta^2\,\frac{w}{\sigma}\right\},$$

for any $\ell\in\mathcal{D}_0$. From this we conclude in a standard way (see the argument at the beginning of the proof of Lemma 2.2 that the cardinality of $\mathcal{D}_0$ must be

$$\gtrsim\delta^{12\varepsilon}N\Big(p\,\frac{w}{\delta}\Big)^{1/2}\lambda^{3/2}\max\Big\{\delta^{-1}(\phi\,\sigma+\delta),\frac{\sigma}{w}\Big\},$$

proving the claim.

It follows by (15) in the definition of good (applied with $\delta^{i\varepsilon}=\delta^{-\varepsilon}\sigma$, r as large as possible subject to $\delta^{r\varepsilon}\geq\phi$, and j as large as possible subject to $\delta^{j\varepsilon}\geq w$) that there is a plate P' containing T_{ℓ_0} with width $w'\leq\delta^{-\varepsilon}w$ which intersects at least

$$\delta^{13\varepsilon}\,N\Big(p\,\frac{w}{\delta}\Big)^{1/2}\lambda^{3/2}\max\Big\{\delta^{-1}\,(\phi\,\sigma+\delta),\frac{\sigma}{w}\Big\}$$

tubes T_ℓ *with* $\ell\in\mathcal{B}$ such that ℓ makes an angle less or equal than $\delta^{-\varepsilon}\sigma$ with the axis of P' and the 2-plane spanned by ℓ and the axis direction of P' makes an angle less or equal than $\delta^{-\varepsilon}\phi$ with the 2-plane of P'. We can pigeonhole to obtain a number $\tau\leq\delta^{-\varepsilon}\phi$ and a choice of

$$\delta^{14\varepsilon}\,N\Big(p\,\frac{w}{\delta}\Big)^{1/2}\lambda^{3/2}\max\Big\{\delta^{-1}\,(\phi\,\sigma+\delta),\frac{\sigma}{w}\Big\},$$

of these tubes T_ℓ for which the 2-plane spanned by ℓ and the axis direction of P' makes an angle in $((\tau-\delta)/2,\tau)$ with the 2-plane of P', and we let $\mathcal{D}$ be the lines ℓ corresponding to the latter set of tubes T_ℓ. It is easy to see that $\mathcal{D}\subset D(\ell_0,C_4\delta^{-\varepsilon}\sigma)$: this follows since i) each tube in $\mathcal{D}$ intersects the plate P' at angle less or equal than $\delta^{-\varepsilon}\sigma$ to its axis and therefore (since $w'\leq\delta^{-\varepsilon}\sigma$) also at angle $\lesssim\delta^{-\varepsilon}\sigma$ to the direction of ℓ_0 and ii) since $w'\leq\delta^{-\varepsilon}\sigma$, every point of P' is within $C\,\delta^{-\varepsilon}\,\sigma$ of ℓ_0. It remains to observe that $\cup_{m\in\mathcal{C}}Y_m\subset\Pi^{C(\delta^{-2\varepsilon}\phi\,\sigma+\delta)}$ where Π is the 2-plane of P' and to prove (35). For this we apply Lemma 3.3, with R there equal to P' and σ there replaced by $\delta^{-\varepsilon}\sigma$ and ϕ there equal to τ. We conclude in the first place that

$$\bigcup_{m\in\mathcal{D}}Y_m\subset\Pi^{C(\tau\,\delta^{-\varepsilon}\,\sigma+\delta)}\subset\Pi^{C(\delta^{-\varepsilon}\,\phi\,\delta^{-\varepsilon}\,\sigma+\delta)}\,.$$

Note also that $p_{\delta^{-\varepsilon}\sigma}(\mathcal{D}) \le \delta^{-C\varepsilon} p_\sigma(\mathcal{D}) \le \delta^{-C\varepsilon} p$ by the corollary to Lemma 1.1 and then (10). Hence by (34),

$$\begin{aligned}
\Big|\bigcup_{m\in\mathcal{C}} Y_m\Big| &\ge \delta^{C\varepsilon}(p\,m)^{-1/2}\,\lambda^2\Big(N\Big(p\,\frac{w}{\delta}\Big)^{1/2}\lambda^{3/2}\max\Big\{\delta^{-1}(\phi\,\sigma+\delta),\,\frac{\sigma}{w}\Big\}\delta^2\Big) \\
&\quad\cdot\sqrt{\frac{\delta}{w'\,\tau+\delta}} \\
&\ge \delta^{C\varepsilon}(p\,m)^{-1/2}\lambda^2\Big(N\Big(p\,\frac{w}{\delta}\Big)^{1/2}\lambda^{3/2}\max\Big\{\delta^{-1}(\phi\,\sigma+\delta),\,\frac{\sigma}{w}\Big\}\delta^2\Big) \\
&\quad\cdot\sqrt{\frac{\delta}{w\,\phi+\delta}} \\
&= \delta^{C\varepsilon}Nm^{-1/2}\lambda^{7/2}\Big(\frac{w}{\delta}\Big)^{1/2}\max\Big\{\delta^{-1}(\phi\,\sigma+\delta),\,\frac{\sigma}{w}\Big\}\delta^2 \\
&\quad\cdot\sqrt{\frac{\delta}{w\,\phi+\delta}} \\
&\ge \delta^{C\varepsilon}Nm^{-1/2}\,\lambda^{7/2}\,\delta\,\sqrt{\sigma}\,\sqrt{\phi\,\sigma+\delta}\,. \qquad (38)
\end{aligned}$$

Inequality (38) may be seen as follows: if $w\,\phi \ge \delta$ then

$$\begin{aligned}
\Big(\frac{w}{\delta}\Big)^{1/2}\max\Big\{\delta^{-1}(\phi\,\sigma+\delta),\,\frac{\sigma}{w}\Big\}\delta^2\sqrt{\frac{\delta}{w\,\phi+\delta}}& \\
&\gtrsim \Big(\frac{w}{\delta}\Big)^{1/2}\delta^{-1}\,(\phi\,\sigma+\delta)\,\delta^2\sqrt{\frac{\delta}{w\,\phi}} \\
&\ge \delta\,\sqrt{\sigma}\,\sqrt{\phi\,\sigma+\delta}\,.
\end{aligned}$$

On the other hand if $w\,\phi \le \delta$ then

$$\begin{aligned}
\Big(\frac{w}{\delta}\Big)^{1/2}\max\Big\{\delta^{-1}(\phi\,\sigma+\delta),\,\frac{\sigma}{w}\Big\}\delta^2\sqrt{\frac{\delta}{w\,\phi+\delta}}& \\
&\gtrsim \Big(\frac{w}{\delta}\Big)^{1/2}\delta^2\sqrt{\delta^{-1}\,(\phi\,\sigma+\delta)\,\frac{\sigma}{w}} \\
&\ge \delta\,\sqrt{\sigma}\,\sqrt{\phi\,\sigma+\delta}\,.
\end{aligned}$$

This proves (38), hence (35).

Now we need a simple lemma. Here we let

$$f_\delta^*(e) = \sup_{\ell:\ell^*=e} \sup_{a\in\ell} \frac{1}{|T_\ell(a)|} \int_{T_\ell(a)} f$$

be the Kakeya maximal function as defined in [1].

Lemma 3.5. *Suppose that $\delta \le \sigma \le 100$ and that $\{S_j\}_{j=1}^M$ are slabs with respective thicknesses less or equal than $C\,(\phi_j\,\sigma+\delta)$. Let $f = \sum_j \chi_{S_j}$. Fix $e_0 \in S^2$. Then*

$$|\{e \in D(e_0,\sigma):\ f_\delta^*(e) \ge \lambda\}| \lesssim \frac{\sigma \sum_j (\phi_j\,\sigma+\delta)}{\lambda} \log\frac{1}{\delta}\,.$$

PROOF. If $\sum_{j=1}^M (\phi_j\,\sigma+\delta) \ge \sigma$ there is nothing to prove. It follows that we can assume $M \le 1/\delta$.

First consider the case where there is just one slab S, with thickness $\lesssim \phi\,\sigma+\delta$. Then the set $\{e \in D(e_0,\sigma):\ f_\delta^* \ge \lambda\}$ is contained in the intersection of $D(e_0,\sigma)$ with a $(C\,(\phi\,\sigma+\delta)/\lambda)$-neighborhood of a great circle, so its measure is $\lesssim \sigma\,(\phi\,\sigma+\delta)/\lambda$. Since $M \le 1/\delta$, the general case now follows from the Stein – N. J. Weiss result on summing weak type 1 estimates.

In the next corollary we use the notation ℓ^* = direction of the line ℓ, and if $\mathcal{C}$ is a set of lines then $\mathcal{C}^* = \{\ell^*\}_{\ell\in\mathcal{C}}$.

Corollary. *Let $\{E_k\}_{k=1}^M$ be a family of subsets of the unit ball in $\mathbb{R}^3$, such that E_k is contained in a $C\,(\phi_k\,\sigma+\delta)$-slab. Let $\mathcal{C}$ be a family of lines and assume that for each $\ell \in \mathcal{C}$ a subset $\mathcal{K}(\ell) \subset \{1,\dots,M\}$ is given, and that the following holds*

$$\text{(39)} \qquad \textit{If } \operatorname{dist}(\ell_1^*, \ell_2{}^*) \ge C\,\sigma\,, \qquad \textit{then } \mathcal{K}(\ell_1)\cap\mathcal{K}(\ell_2) = \varnothing\,.$$

Let $E(\ell) = \cup_{k\in\mathcal{K}(\ell)} E_k$, and assume that for every $\ell \in \mathcal{C}$, we have $|T_\ell \cap E(\ell)| \ge \delta^\varepsilon\,\lambda\,|T_\ell|$. Then

$$\text{(40)} \qquad \sigma \sum_k (\phi_k\,\sigma+\delta) \ge \delta^{2\varepsilon}\,\lambda\,\mathcal{E}_\delta(\mathcal{C}^*)\,.$$

Proof. Because of property (39) it suffices to show (40) assuming that $\mathcal{C}^*$ is contained in a single σ-disc, and in that case it is immediate from Lemma 3.5 since $(\sum_k \chi_{E_k})^* \gtrsim \delta^\varepsilon \lambda$ on a δ-neighborhood of $\mathcal{C}^*$.

Proof of Lemma 0. We start by fixing a maximal δ-separated subset Σ^* of Ω, and for each $e \in \Sigma^*$ we choose (exactly) m δ-separated lines ℓ with $\ell^* = e$ and so that the tubes T_ℓ satisfy (3) and (5). We then choose a "good" subset by Lemma 1.4. We denote this last set of lines by $\mathcal{A}$. Note that

$$\mathcal{E}_\delta(\mathcal{A}) \geq \delta^\varepsilon \, m \, |\Omega| \tag{41}$$

and also

$$\mathcal{E}_\delta(\mathcal{C}^*) \geq \delta^{2\varepsilon} \, |\Omega| \, , \tag{42}$$

if $\mathcal{C}$ is any subset of $\mathcal{A}$ with $|\mathcal{C}| \geq \delta^\varepsilon |\mathcal{A}|$. Furthermore, the quantity $m(\mathcal{A})$ defined by (30) is $\lesssim m$.

We choose N and $\sigma = \delta^{(j-1)\varepsilon}$ as in the proof of Lemma 2.1 so that the set

$$Y_\ell^0 \stackrel{\text{def}}{=} T_\ell \cap E \cap \{x : \ \mu_{\mathcal{A}}(x) \leq 2N\} \cap \{x : \ \mu^j_{\mathcal{A},\ell}(x) \geq \varepsilon N\}$$

will have measure greater or equal than

$$\Big(\log \frac{1}{\delta} \Big)^{-2} \lambda \, |T_\ell| \, ,$$

for a set of $\ell \in \mathcal{A}$ with cardinality greater or equal

$$\Big(\log \frac{1}{\delta} \Big)^{-2} \, |\mathcal{A}|$$

and we let $\mathcal{B}$ be this set of ℓ's. We also let $\{\ell_j\}$ be a maximal $\delta^{-\varepsilon}\,\sigma$-separated subset of $\mathcal{B}$ and let τ_j be the tube of length C_6 and radius $C_6 \, \delta^{-\varepsilon} \, \sigma$ concentric with T_{ℓ_j}. Here C_6 is a large constant which is chosen as follows: let C_4 be as in Lemma 3.4 and make C_6 large enough that if $d(\ell, \ell_j) \leq (C_4 + 2) \, \delta^{-\varepsilon} \, \sigma$ and T_m intersects T_ℓ at angle less or equal than $\delta^{-\varepsilon} \, \sigma$ then T_m is contained in τ_j. It is easy to see that this is legitimate.

We will define subsets $F_k \subset E$ by a recursive construction. The logic here is similar to [1, p. 154]. The F_k will have the following properties:

1) Each F_k is *assigned* to a unique τ_j, $j = j(k)$.

2) The F_k assigned to a given τ_j are disjoint and are contained in

$$\bigcup_{\ell \in \mathcal{B} \cap D(\ell_j, (C_4+2)\delta^{-\varepsilon}\sigma)} Y_\ell^0 .$$

(In particular, this implies they are contained in τ_j, by choice of C_6).

3) Each F_k is contained in a $C\,(\delta^{-2\varepsilon}\,\phi_k\,\sigma + \delta)$-slab for a certain $\phi_k \leq \pi/2$, and satisfies

$$|F_k| \geq \delta^{C\varepsilon} N\, m^{-1/2}\, \lambda^{7/2}\, \delta\, \sqrt{\sigma}\, \sqrt{\phi_k\, \sigma + \delta}\, .$$

4) $\sigma \sum_k (\phi_k\, \sigma + \delta) \geq \delta^{C\varepsilon}\, \lambda\, |\Omega|$.

To start the recursion, let $F_0 = \varnothing$ and assign it to some arbitrary τ_j. If F_i has been defined for $i \leq k-1$, then for each tube in $\mathcal{B}$, we let

$$\begin{aligned} Y_\ell = Y_\ell^0 \backslash \cup \{F_i :\ & i < k,\ F_i \text{ assigned to } \tau_j \\ & \text{for some } j \text{ with } \ell \in D(\ell_j, (C_4+2)\,\delta^{-\varepsilon}\,\sigma)\}\, . \end{aligned} \tag{43}$$

We *throw out* all $\ell \in \mathcal{B}$ such that $|Y_\ell| \leq |Y_\ell^0|/2$. If half the lines in $\mathcal{B}$ are thrown out, we stop the induction. Otherwise, we let $\mathcal{B}_k$ be the remaining lines and note that the family $\mathcal{B}_k$ and the sets Y_ℓ satisfy the hypotheses of Lemma 3.4, since

$$|Y_\ell| \geq \frac{1}{2}\, |Y_\ell^0| \geq \Big(\log\frac{1}{\delta}\Big)^{-3}\, \lambda\, |T_\ell|\, .$$

It follows that for some $\ell \in \mathcal{B}_k$ there is a set $F_k \subset \cup\{Y_m : m \in \mathcal{B}_k \cap D(\ell, C_4\delta^{-\varepsilon}\sigma)\}$ and with property 3). We choose j so that $\ell \in D(\ell_j, 2\,\delta^{-\varepsilon}\sigma)$ and assign F_k to this τ_j. Then clearly F_k is contained in $\cup\{Y_m : m \in D(\ell_j, (C_4+2)\,\delta^{-\varepsilon}\sigma)\}$. It follows using (43) that F_k is disjoint from F_i if $i < k$ and F_i is also assigned to τ_j. This gives property 2).

It remains only to observe that when the induction stops property 4) will hold. This follows from the corollary to Lemma 3.5. Namely, if the induction stops at stage k then at stage k we have a subset $\mathcal{C} \subset \mathcal{B}$ of "thrown out" lines, with

$$|\mathcal{C}| \geq \frac{1}{2}|\mathcal{B}| \geq \frac{1}{2}\Big(\log\frac{1}{\delta}\Big)^{-2}|\mathcal{A}|\, ,$$

and therefore also $\mathcal{E}_\delta(\mathcal{C}^*) \gtrsim \delta^{2\varepsilon}|\Omega|$ by (42). If $\ell \in \mathcal{C}$, then we let

$$E(\ell) = \cup\{F_i : i < k,\ F_i \text{ assigned to } \tau_j \\ \text{for some } j \text{ with } \ell \in D(\ell_j, (C_4+2)\,\delta^{-\varepsilon}\,\sigma)\}\,.$$

Since ℓ is thrown out we have

$$|T_\ell \cap E(\ell)| \geq \frac{1}{2}\Big(\log\frac{1}{\delta}\Big)^{-2}\lambda\,|T_\ell|\,. \tag{44}$$

We say that the F_i in (44) are *used in forming* $E(\ell)$. If C_7 is a suitable constant then each set F_i is contained in a $C_7(\delta^{-2\varepsilon}\phi_i\sigma+\delta)$-slab, and if ℓ and m are lines with $\operatorname{dist}(\ell^*, m^*) \geq C_7\,\delta^{-\varepsilon}\,\sigma$, then no ℓ_j can be within $(C_4+2)\,\delta^{-\varepsilon}\,\sigma$ of both ℓ and m, so no F_i is used in forming both $E(\ell)$ and $E(m)$. This gives the property (39) (with σ replaced by $C\,\delta^{-\varepsilon}\sigma$). Accordingly (40) with σ replaced by $C\,\delta^{-\varepsilon}\,\sigma$ implies

$$\delta^{-\varepsilon}\sigma\sum_{i<k}(\delta^{-2\varepsilon}\,\phi_i\,\sigma+\delta) \gtrsim \delta^{2\varepsilon}\,\lambda\,\delta^{2\varepsilon}\,|\Omega|\,,$$

which gives 4).

Next, using properties 3) and 4) we have

$$\begin{aligned}\sum_k |F_k| &\geq \delta^{C\varepsilon} N\, m^{-1/2}\,\lambda^{7/2}\,\delta\sum_k(\sigma(\phi_k\,\sigma+\delta))^{1/2}\\ &\geq \delta^{C\varepsilon} N\, m^{-1/2}\,\lambda^{7/2}\,\delta\Big(\sum_k\sigma(\phi_k\,\sigma+\delta)\Big)^{1/2}\\ &\geq \delta^{C\varepsilon} N\, m^{-1/2}\,\lambda^{7/2}\,\delta\,(\lambda\,|\Omega|)^{1/2}\\ &= \delta^{C\varepsilon} N\, m^{-1/2}\,\lambda^{4}\,\delta\,|\Omega|^{1/2}\,.\end{aligned}$$

Let $E_j = \cup\{F_k :\ F_k \text{ assigned to } \tau_j\}$. Then

$$\sum_j |E_j| = \sum_k |F_k|$$

by the disjointness property 2). On the other hand, we have

$$E_j \subset \bigcup_{\ell\in D(\ell_j,(C_4+2)\delta^{-\varepsilon}\sigma)} Y_\ell^0$$

and since the $\{\ell_j\}$ are $\delta^{-\varepsilon}\sigma$-separated this implies (see the proof of the claim at the end of the proof of Lemma 2.1) that no point is in more than C_ε E_j's. We conclude that

$$|E| \gtrsim \sum_j |E_j| \gtrsim \delta^{C\varepsilon} N\, m^{-1/2}\, \lambda^4\, \delta\, |\Omega|^{1/2}. \tag{45}$$

As in the proof of Lemma 2.1 (see (26)), we also have

$$|E| \geq (2N)^{-1} \sum_{\ell\in\mathcal{B}} |Y_\ell^0| \gtrsim \delta^\varepsilon\, \frac{\lambda\, \mathcal{E}_\delta(\mathcal{A})}{N}\ ,$$

hence

$$|E| \geq \delta^{2\varepsilon}\, \frac{m\, \lambda\, |\Omega|}{N}$$

by (41). If we combine this with (45) we get

$$\begin{aligned} |E| &\gtrsim \delta^{C\varepsilon}\, (N m^{-1/2}\, \lambda^4\, \delta\, |\Omega|^{1/2})^{1/2} \Big(\frac{\lambda\, m\, |\Omega|}{N}\Big)^{1/2} \\ &= \delta^{C\varepsilon} \lambda^{5/2}\, m^{1/4}\, |\Omega|^{3/4}\, \delta^{1/2} \end{aligned}$$

and the proof of Lemma 0 is complete.

4. Proofs of the theorems.

PROOF OF THEOREM 2. This is essentially the same as [7, Section 3]. The argument may appear simpler here however due to our attempt at abstraction in [7].

The idea is to induct downward on δ. There is a technical point which must be dealt with first. Namely, in the preceding sections it was convenient to assume that E was contained in the unit ball but this is now inconvenient, since we will want to use a rescaling argument. We take care of this issue in the next lemma.

Lemma 4.1. *Assume that Theorem 2 is true for a certain value of δ. Then the following variant is also true for the same value of δ. Here the constants C and C_ε are the same as in (4) and β is a numerical constant.*

Let Ω be a subset of $S^2\setminus \pm 1$, let E be a subset of $\mathbb{R}^3$, and $\lambda > 0$. Assume that for each $e \in \Omega$ there are m δ-separated lines $\{\ell_j\}_{j=1}^m$, and points $\{a_j\}_{j=1}^m$ with $a_j \in \ell_j$, such that $|T_{\ell_j}(a_j) \cap E| \geq \lambda\, |T_{\ell_j}(a_j)|$. Then

$$|E| \geq \beta\, C_\varepsilon^{-1} \left(\log \frac{1}{\delta}\right)^{-1} \delta^{C\varepsilon}\, \lambda^{5/2}\, m^{1/4}\, |\Omega|^{3/4}\, \delta^{1/2}\,.$$

PROOF. Let κ and α be small constants chosen in that order. Subdivide $\mathbb{R}^3$ in cubes, $\mathbb{R}^3 = \cup_{j\in\mathbb{Z}^3} Q_j$ where Q_j is the cube centered at κj with sidelength κ. Denote $E_j = (Q_j \cap E) - \kappa\, j$, *i.e.* E_j is the part of E contained in Q_j, translated to the origin. Then E_j is contained in the unit ball, and since any tube $T_\ell(a)$ intersects only a bounded number of cubes Q_j, one has the following: let $m_j(e)$ be the maximum possible cardinality for a δ-separated set of lines ℓ in the e direction such that $|T_\ell \cap E_j| \geq \alpha\, \lambda\, |T_\ell|$. Then $\sum_j m_j(e) \geq m$ for all $e \in \Omega$.

Hence also
$$\sum_j \int_\Omega m_j(e)\, de \geq m\, |\Omega|\,.$$

Note that $m_j(e) \lesssim \delta^{-2}$ for any j and e. Accordingly there are numbers $\{\mu_j\}$ such that

$$\int_{\{e\in\Omega:\mu_j\leq m_j(e)\leq 2\mu_j\}} m_j(e)\, de \gtrsim \left(\log\frac{1}{\delta}\right)^{-1} \int_\Omega m_j(e)\, de$$

and therefore
$$\sum_j \mu_j\, |\Omega_j| \gtrsim \frac{m\, |\Omega|}{\log \dfrac{1}{\delta}}\,,$$

where $\Omega_j = \{e \in \Omega : \ \mu_j \leq m_j(e) \leq 2\, \mu_j\}$. Because of the hypothesis that Theorem 2 is true with the given δ, we then get

$$|E| = \sum_j |E_j| \gtrsim C_\varepsilon\, \delta^{C\varepsilon}\, \lambda^{5/2}\, \delta^{1/2} \sum_j \mu_j^{1/4}\, |\Omega_j|^{3/4}\,, \tag{46}$$

where the implicit constant is purely numerical. On the other hand, clearly $\mu_j \leq m$ and $|\Omega_j| \leq |\Omega|$ for any j. Accordingly

$$\sum_j \mu_j^{1/4}\, |\Omega_j|^{3/4} \geq \sum_j \frac{\mu_j\, |\Omega_j|}{m^{3/4}\, |\Omega|^{1/4}} \gtrsim \frac{m\, |\Omega|}{m^{3/4}\, |\Omega|^{1/4} \log \dfrac{1}{\delta}} = \frac{m^{1/4}\, |\Omega|^{3/4}}{\log \dfrac{1}{\delta}}\,.$$

If we substitute this into (46) we get the result.

Proof of Theorem 2. As has already been mentioned the proof is by induction on δ. By Lemma 2 we can choose C and A_ε so that if (3) and (5) hold then

$$|E| \geq A_\varepsilon^{-1}\, \delta^{C\varepsilon}\, \lambda^{5/2}\, m^{1/4}\, |\Omega|^{3/4}\, \delta^{1/2}\,.$$

Next we choose δ_0 small enough that if $\delta < \delta_0$ then

$$2^{-7/2}\, \beta \Big(\log \frac{1}{\delta}\Big)^{-26} \delta^{C\varepsilon(1-\varepsilon)} > \delta^{C\varepsilon}\,. \tag{47}$$

Theorem 2 is trivial when $\delta \geq \delta_0$ provided C_ε is large enough, so we can define a constant C_ε by the following requirements:

- Theorem 2 is true with the given constant C_ε provided $\delta \geq \delta_0$.
- $C_\varepsilon \geq 2A_\varepsilon$.

Fix $\delta < \delta_0$ and assume that Theorem 2 has been proved with this value of C_ε for parameters $\overline{\delta}$, $\overline{\delta} \geq \delta^{1-\varepsilon}$. Then under the assumptions of Theorem 2_δ, one of the following must happen:

1) There is a subset $\tilde{\Omega} \subset \Omega$ with measure greater or equal than $|\Omega|/2$, such that if $e \in \tilde{\Omega}$ then there are $m/2$ δ-separated lines ℓ with direction e such that (3) and (5) hold.

2) There is a subset $\tilde{\Omega} \subset \Omega$ with measure greater or equal than $|\Omega|/2$, such that if $e \in \tilde{\Omega}$ then there are $m/2$ δ-separated lines ℓ with direction e such that (3) holds and (5) fails.

In case 1), we simply apply Lemma 0 with Ω and m replaced by $\tilde{\Omega}$ and $m/2$ (more precisely, we use the second requirement on C_ε), obtaining the estimate

$$|E| \geq A_\varepsilon^{-1} \delta^{C\varepsilon}\, \lambda^{5/2} \Big(\frac{m}{2}\Big)^{1/4} \Big(\frac{|\Omega|}{2}\Big)^{3/4} \delta^{1/2} \geq C_\varepsilon^{-1} \delta^{C\varepsilon}\, \lambda^{5/2}\, m^{1/4}\, |\Omega|^{3/4}\, \delta^{1/2}\,,$$

which is the necessary inequality (4).

In case 2), we let $\overline{E}$ be E dilated by a factor $\delta^{-\varepsilon}$. Fix $e \in \tilde{\Omega}$ and one of the $m/2$ tubes in (2). Because of the hypothesis that (5) fails, there is a subtube of length δ^ε which intersects E in measure greater or equal than

$$\frac{1}{2}\, \lambda \Big(\log \frac{1}{\delta}\Big)^{-10} |T_\ell|\,.$$

The dilation of this tube is a tube $\overline{T}_\ell$ of length 1 and radius $\overline{\delta} \stackrel{\text{def}}{=} \delta^{1-\varepsilon}$ which intersects $\overline{E}$ in measure greater or equal than

$$\frac{1}{2}\,\lambda\left(\log\frac{1}{\delta}\right)^{-10}\delta^{\varepsilon}\,|\overline{T}_\ell|\,.$$

Thus (after dilation) for each $e \in \tilde{\Omega}$ we have $m/2$ $\delta^{1-\varepsilon}$-separated ℓ's generating such $\overline{T}_\ell$'s. By the inductive hypothesis and Lemma 4.1 we have

$$\begin{aligned}\delta^{-3\varepsilon}\,|E| = |\overline{E}| \geq \beta\, C_\varepsilon^{-1}\Big(\log\frac{1}{\delta}\Big)^{-1}\,\overline{\delta}^{\,C\varepsilon}&\\ \cdot\Big(\frac{1}{2}\,\delta^{-\varepsilon}\,\Big(\log\frac{1}{\delta}\Big)^{-10}\lambda\Big)^{5/2}\Big(\frac{m}{2}\Big)^{1/4}\Big(\frac{|\Omega|}{2}\Big)^{3/4}\,\overline{\delta}^{\,1/2}&\end{aligned}$$

or equivalently

$$\begin{aligned}|E| &\geq 2^{-7/2}\,\beta\, C_\varepsilon^{-1}\Big(\log\frac{1}{\delta}\Big)^{-26}\delta^{C\varepsilon(1-\varepsilon)}\,\lambda^{5/2}\,m^{1/4}\,|\Omega|^{3/4}\,\delta^{1/2}\\ &\geq C_\varepsilon^{-1}\,\delta^{C\varepsilon}\,\lambda^{5/2}\,m^{1/4}\,|\Omega|^{3/4}\,\delta^{1/2}\,,\end{aligned}$$

where the last line follows from (47). This finishes the proof of Theorem 2.

PROOF OF THEOREM 1. Fix δ and define

$$X_\delta f(\ell) = |T_\ell|^{-1}\int_{T_\ell} f\,.$$

The first step is to prove

$$\|X_\delta f\|_{L^q_e L^r_x} \lesssim \delta^{-C\varepsilon}\,\|f\|_p\,, \tag{48}$$

when f is supported in the unit disc, with p, q, r as in Theorem 1.

A well-known argument (in this case it can be carried out by interpolation with L^∞ and then with the result of [3]) shows that a bound like (48) which is insensitive to $\delta^{-\varepsilon}$ factors need only be proved for characteristic functions. So fix a set E, let $f = \chi_E$, and define $N = \|X_\delta f\|_{L^q_e L^r_x}$. We claim that for some M there is a set $\Omega \subset S^2$ with

$$|\Omega| \gtrsim \Big(\log\frac{1}{\delta}\Big)^{-1}\Big(\frac{N}{M}\Big)^{q},$$

such that $e \in \Omega$ implies

$$\|X_\delta f(e,\cdot)\|_{L^r_x} \geq M\,.$$

To see this, note that $X_\delta f$ is roughly constant on discs of radius δ, in the sense (say) that if $X_\delta f(\ell_0) = \rho$, then $X_\delta f(\ell) \gtrsim \rho$ on a subset of $D(\ell_0, \delta)$ with measure $\gtrsim \delta^4$. Hence also

$$\int_{D(e_0,\delta)} \|X_\delta f(e,\cdot)\|^q_{L^r_x}\, de \geq \delta^2\, \|X_\delta f(e_0,\cdot)\|^q_{L^r_x}\ ,$$

for any $e_0 \in S^2$, so that

$$\sup_e \|X_\delta f(e,\cdot)\|^q_{L^r_x} \leq \delta^{-2} N^q\,.$$

So if we let

$$J = \{e \in S^2 :\ C^{-1} N^q \leq \|X_\delta f(e,\cdot)\|^q_{L^r_x} \leq C\delta^{-2} N^q\}\,,$$

then

$$\int_J \|X_\delta f(e,\cdot)\|^q_{L^r_x}\, de \gtrsim N^q\,.$$

Split the integral over J into the regions Ω_j where $\|X_\delta f(e,\cdot)\|^q_{L^r_x} \in (2^j, 2^{j+1})$ and note that there are $\lesssim \log(1/\delta)$ relevant values of j. Hence the claim holds for some $M = 2^{j/q}$ and $\Omega = \Omega_j$.

Next, by a similar argument there are m and λ with $m\,\delta^2\,\lambda^r \gtrsim \delta^\varepsilon M^r$ and $\tilde\Omega \subset \Omega$ with $|\tilde\Omega| \gtrsim \delta^\varepsilon\,|\Omega|$ such that if $e \in \tilde\Omega$ then $X_\delta f(e,x) \geq \lambda$ for a set of x of measure at least $m\,\delta^2$. Equivalently, if $e \in \tilde\Omega$ then there are m δ-separated lines ℓ with direction e and with $|E \cap T_\ell| \geq \lambda\,|T_\ell|$. We conclude by Theorem 2 that

$$\begin{aligned} |E| &\gtrsim \delta^{C\varepsilon} \lambda^{5/2}\,(m\,\delta^2)^{1/4}\,|\Omega|^{3/4} \\ &= \delta^{C\varepsilon}\,(m\,\delta^2\,\lambda^r)^{1/4}\,|\Omega|^{3/4} \\ &\gtrsim \delta^{C\varepsilon} M^{r/4} \Big(\frac{N}{M}\Big)^{3q/4} \\ &= \delta^{C\varepsilon} N^{5/2}\,, \end{aligned}$$

so we have proved (48). To finish the proof of Theorem 1 we have to trade the δ^ε factor for ε derivatives. This is a standard argument. We choose a C_0^∞ function ψ with $\operatorname{supp}\psi \subset D(0,1/1000)$, and a Schwarz function ρ such that $\hat\rho$ has compact support not containing the origin, such that

$$\hat\eta \overset{\text{def}}{=} 1 - \sum_{j\geq 0} \widehat{\psi_j}\,\widehat{\rho_j} \in C_0^\infty\,,$$

where $\psi_j(x) = 2^{3j}\psi(2^j x)$, $\rho_j(x) = 2^{3j}\rho(2^j x)$. It is easy to see that this is possible. Here are details since we don't have a reference at hand: start with a C_0^∞ function ϕ supported in $D(0,1/2000)$ with $\hat\phi(0) \neq 0$. By multiplying ϕ by a character we can insure that $\hat\phi$ does not vanish identically on any sphere centered at 0. Let $\phi_2 = \tilde\phi * \phi$ where $\tilde\phi(x) = \overline{\phi(-x)}$. Then $\widehat{\phi_2} = |\hat\phi|^2$. Let $\psi = \sum_i \phi_2 \circ T_i$ where $\{T_i\}$ is an appropriate finite set of rotations. By a compactness argument we can arrange that $\hat\psi$ be nonzero on $D(0,2)$ say. Next choose a partition of unity of the form $\{\chi_j\}_{j=-\infty}^{\infty}$ where $\chi_j(\xi) = \chi_0(2^{-j}\xi)$. Define ρ_j via $\hat\rho_j = \chi_j/\hat\psi_j$, $j \geq 0$.

Furthermore, let γ be a C_0^∞ cutoff function equal to 1 on $D(0,1/100)$ and supported in $D(0,1/10)$.

In proving Theorem 1 we can suppose that f is supported in $D(0,1/1000)$ and $\|f\|_{p,\varepsilon} = 1$. Then in the first place,

$$\sum_j 2^{\eta j}\,\|\rho_j * f\|_p \leq C_\varepsilon\,, \tag{49}$$

if (say) $\eta < \varepsilon/2$; this follows easily using the definition of the Sobolev space and the support property of $\hat\rho$. In the second place, using the support properties we have

$$f = \eta * f + \sum_j \psi_j * (\gamma \cdot (\rho_j * f))\,,$$

on $\operatorname{supp} f$, and therefore

$$|Xf| \leq X(|\eta * f|) + \sum_j X(|\psi_j * (\gamma\cdot(\rho_j * f))|)\,.$$

The first term is less or equal than C pointwise. For the remaining terms, we use that $X(|\psi_j * g|)| \leq C X_{\delta_j}|g|$ pointwise if $\operatorname{supp} g \subset D(0,1/10)$, where $\delta_j = 2^{-j}$. This is clear from the definitions and the compact support of ψ. Applying (48) with ε in (48) taken to be small compared with the current ε, we obtain

$$\|X(|\psi_j * (\gamma\cdot(\rho_j * f))|)\|_{L^q_e L^r_x} \lesssim \|X_{\delta_j}(|\gamma\cdot\rho_j * f|)\|_{L^q_e L^r_x} \leq 2^{\eta j}\,\|\gamma\cdot\rho_j * f\|_p\,,$$

with $\eta < \varepsilon/2$. Theorem 1 now follows by (49).

CONCLUDING REMARKS. 1) The following is an easy corollary of Theorem 1 or 2:

Let E be a Borel subset of $\mathbb{R}^3$ and assume that for each $e \in S^2$, there is a Borel set $L_e \subset e^\perp$ with Hausdorff dimension at least β such that for each $x \in L_e$, some segment of the line through x in the e direction is contained in E. Then the Hausdorff dimension of E is at least $5/2 + \beta/4$.

Here $e^\perp$ is the orthogonal complement of e in $\mathbb{R}^3$. We omit the proof. It follows a standard pattern originating (to the author's knowledge) in [1].

2) We make a few remarks about the open question of whether or not the exponent $5/2$ in the Kakeya problem can be improved. For example, let E be a compact set containing a unit line segment ℓ_e in every direction e. Is its Minkowski dimension (*i.e.*

$$3 - \limsup_{\delta \to 0} \frac{\log |E^\delta|}{\log \delta} ,$$

$E^\delta = \delta$-neighborhood of E) strictly greater than $5/2$? Theorem 2 shows that the enemy is the case where the lines "stick together" in the sense that $d(\ell_e, \ell_{e'}) \approx |e - e'|$ up to $\delta^{-\varepsilon}$ factors. The reason is that if this condition fails in too dramatic a way, then the sets E^δ will contain not just one but many δ-tubes per direction and Theorem 2 will be applicable with a large value of m. For example, one can reduce in this way to the case where the following condition $(*)$ is satisfied:

$(*)$ For any $\varepsilon > 0$ there is a sequence of δ going to 0 such that the set $(e, e') \in S^2 \times S^2 : d(\ell_e, \ell_{e'}) \leq \delta$ has measure greater or equal than $\delta^\varepsilon \, \delta^2$.

At the opposite extreme, if the inequality $d(\ell_e, \ell_{e'}) \approx |e - e'|$ holds in the strict sense that

$$d(\ell_e, \ell_{e'}) \leq C \, |e - e'| \, , \tag{50}$$

for all e and e' then it is easy to show using Rademacher's theorem on almost everywhere differentiability of Lipschitz functions (*e.g.* [4]) that E will have positive measure.

We indicate the proof (in $\mathbb{R}^n$) assuming that for each e, E contains a segment of ℓ_e with length 1 which intersects the plane $x_n = 0$; only minor modifications are required to treat the general case. First let $U \subset \mathbb{R}^n$ be open and let $F : U \longrightarrow \mathbb{R}^m$ be any Lipschitz function. We claim that if $\varepsilon > 0$ then there are a subset $Y \subset U$ with positive measure, and a linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$|F(x) - F(y) - T(x - y)| \leq \varepsilon \, |x - y| \, ,$$

for all $x \in Y$, $y \in Y$. Namely, let $DF(a)$ be the derivative of F at a given by Rademacher's theorem. By the Lusin and Egoroff theorems there is a positive measure subset Y_0 such that $DF(a)$ is continuous on Y_0 as a function of a, and furthermore the difference quotients

$$\frac{|F(x) - F(a) - DF(a)(x - a)|}{|x - a|}$$

converge to 0 as $x \longrightarrow a$ uniformly over $a \in Y_0$. Let δ be small enough and let a be a point of density of Y_0. Let $Y = Y_0 \cap D(a, \delta)$. Let $T = DF(a)$. Then for $x, y \in Y$, the properties of Y_0 imply

$$\begin{aligned} &|F(x) - F(y) - T(x - y)| \\ &\qquad \leq |F(x) - F(y) - DF(y)(x - y)| + |DF(y)(x - y) - T(x - y)| \\ &\qquad \leq \varepsilon \, |x - y| \, , \end{aligned}$$

as claimed.

Now parametrize (an appropriate subset of) projective space via $e = (\xi, 1)$, $\xi \in \mathbb{R}^{n-1}$ and define a family of maps F_t from a suitable subset of $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-1}$ by letting $(F_t(\xi), t)$ be the intersection point between ℓ_e and the plane $x_n = t$. Note that $F_t(\xi) = F_0(\xi) + t \, \xi$. F_0 is Lipschitz, so we can choose Y and a linear map T so that Y has positive measure and

$$|F_0(\xi) - F_0(\eta) - T(\xi - \eta)| < \varepsilon \, |\xi - \eta| \, , \qquad \xi, \eta \in Y \, ,$$

where ε is to be determined. We then have

$$|F_t(\xi) - F_t(\eta) - (t \, I + T) \, (\xi - \eta)| \leq \varepsilon \, |\xi - \eta| \, ,$$

when $\xi, \eta \in Y$, where I is the identity map. Hence F_t is bilipschitz on Y provided that $\varepsilon \, \|(t \, I + T)^{-1}\| < 1$, which will be the case for all t except a set of measure less or equal than $C\,\varepsilon^{1/(n-1)}$. We are free to choose ε small, so the result follows using the fact that a bilipshitz image of a set of positive measure has positive measure and then Fubini's theorem.

However, it appears difficult to replace the strict sense condition (50) with a similar condition (*e.g.* $(*)$) which is weak enough to be useful, even if one asks only for the weaker conclusion $\dim(E) > 5/2$.

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Hölder quasicontinuity of Sobolev functions on metric spaces

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Abstract. We prove that every Sobolev function defined on a metric space coincides with a Hölder continuous function outside a set of small Hausdorff content or capacity. Moreover, the Hölder continuous function can be chosen so that it approximates the given function in the Sobolev norm. This is a generalization of a result of Malý [Ma1] to the Sobolev spaces on metric spaces [H1].

1. Introduction.

The classical Luzin theorem asserts that every measurable function is continuous if it is restricted to the complement of a set of arbitrary small measure. If the function is more regular, then it is natural to expect that Luzin's theorem can be refined. One important class of functions are the Sobolev functions. It is known that every Sobolev function, after a redefinition on a set of measure zero, is continuous when restricted to the complement of a set of arbitrary small capacity. This is a capacitary version of Luzin's theorem. On the other hand, if we restrict the function to the complement of a slightly larger set, we obtain more regularity, see [BH], [CZ], [Li], [MZ] and [Z]. We are interested in the Hölder continuity of the restriction. Indeed, a Sobolev function coincides with a Hölder continuous function on the complement of a set of arbitrary small capacity. Moreover, the Hölder continuous function can be chosen so that it belongs to the Sobolev space and it approximates the given function in the Sobolev norm. This phenomenon was first observed by Malý [Ma1] in the Euclidean case. Malý's result plays a crucial role in the refined versions of the change of variable formula for the Sobolev functions, see [Ma2] and [MM].

It is possible to define the first order Sobolev space and to develop a capacity theory on an arbitrary metric space which is equipped with a doubling measure, see [H1] and [KM]. Hence all notions in refined Luzin's and Malý's theorems make sense also in the metric context. Indeed, the capacitary version of Luzin's theorem holds, see [KM]. The purpose of this paper is to generalize Malý's result to metric spaces. As a by–product we obtain a new proof for the Euclidean case. Malý's argument is based on the representation of the Sobolev functions by the Bessel potentials and it does not generalize to the metric setting. Our approach is based on pointwise estimates for the Sobolev functions. In fact, we obtain slightly better estimates for the exponent of the Hölder continuity and the size of the exceptional set than Malý.

The fundamental fact in our proof is that the oscillation of a Sobolev function is controlled pointwise by the fractional maximal function of the derivative, see [H2], [HM, Lemma 4]. If the fractional maximal function is bounded, then the function is Hölder continuous. This is Morrey's lemma. A generalization of Morrey's lemma to metric spaces has been studied in [MS1]. Their main result follows from our pointwise estimates.

If the fractional maximal function is not bounded, the function is Hölder continuous when restricted to the set where the maximal function is small. The classical weak type inequalities give estimates for the Hausdorff content and for the capacity of the exceptional set. As a consequence, we obtain that, after a redefinition on a set of measure zero, a Sobolev function coincides with a Hölder continuous function outside a very small set. The obtained function can be easily extended to the Hölder continuous function on the whole space. The main problem is to construct the extension so that it belongs to the Sobolev space and approximates the given function in the Sobolev norm. Instead of using the McShane extension [Mc] we use a Whitney type extension. However, in our case it should be rather called Whitney smoothing, since instead of extending the function we smooth it in the bad set. The method of our paper can be generalized to higher order derivatives in the Euclidean case, see [BHS].

One of the most important applications of Sobolev spaces on metric spaces are the Sobolev spaces associated to the vector fields. If $V =$

$\{X_1, X_2, \dots, X_k\}$ is a family of vector fields satisfying Hörmander's condition on $\mathbb{R}^n$, and $W_V^{1,p}$ is the closure of $C^\infty(\mathbb{R}^n)$ in the norm

$$\|f\|_{W_V^{1,p}} = \left(\|f\|_{L^p}^p + \|Xf\|_{L^p}^p\right)^{1/p},$$

where

$$|Xf| = \Big(\sum_{i=1}^{k} |X_i f|^2\Big)^{1/2},$$

then $W_V^{1,p}$ is equivalent to the Sobolev space on metric space $\mathbb{R}^n$ with the Carnot–Carathéodory metric and the Lebesgue measure. This can be deduced from the Poincaré inequality of Jerison, see [CDG], [J], [FLW2], the fact that the Lipschitz functions with respect to the Carnot–Carathéodory metric belong to $W_V^{1,p}$, see [FSS], [GN2], and an approximation argument similar to that we use in the proof of our main result. For related results, see [FHK], [FLW1], [GN1], [HK2] and [Vo]. In this paper we work in general metric spaces and no knowledge in the theory of vector fields satisfying Hörmander's condition is required. For other papers related to the Sobolev spaces on metric spaces, see [HeK], [HM], [K], [KM], [KMc] and [Se].

The outline of our paper is the following. Section 2 contains some results on the maximal functions and measure theory. In Section 3 we recall the definitions of the Sobolev spaces and the capacity on metric spaces. We also give two characterizations of Sobolev spaces. The first characterization is in terms of Poincaré inequalities and the second is a generalization of a result of Calderón [C]. Section 4 is devoted to study the set of Lebesgue's points of a Sobolev function. The main result (Theorem 5.3) is presented in Section 5.

Our notation is fairly standard. By $B(x,r)$ we denote an open ball with the center x and the radius r. The symbol χ_E stands for the characteristic function of the set E. The average value of f over the ball $B(x,r)$ is denoted by

$$f_{B(x,r)} = \fint_{B(x,r)} f\,d\mu = \frac{1}{\mu(B(x,r))}\int_{B(x,r)} f\,d\mu\,.$$

Various positive constants are denoted by c; they may change even on the same line. The dependence on the parameters is expressed, for example, by $c = c(n,p)$. We say that two quantities are comparable, and denote $A \approx B$, if there is a constant $c \geq 1$ such that $A/c \leq B \leq c\,A$.

2. Measure theory.

In this section we collect some basic results concerning measure theory and the maximal functions. With minor changes all the results of this section are standard, see [Ch], [CW], [St].

Throughout the paper (X, d) is a metric space and μ a non-negative Borel regular outer measure on X which is finite on bounded sets. We also assume that μ is *doubling* in the sense that

$$\mu(B(x, 2\,r)) \le C_d\, \mu(B(x, r))\,,$$

whenever $x \in X$ and $r > 0$. The constant C_d is called the *doubling constant.*

The first result states that the doubling condition gives a lower bound for the growth of the measure of a ball.

Proposition 2.1. *Suppose that μ is a doubling measure on (X, d). If $Y \subset X$ is a bounded set, then*

$$\mu(B(x, r)) \ge (2\,\operatorname{diam} Y)^{-n}\, \mu(Y)\, r^n\,, \tag{2.2}$$

for $n = \log_2 C_d$, $x \in Y$ and $0 < r \le \operatorname{diam} Y$. Here C_d is the doubling constant of μ.

In this paper we keep the triple (X, d, μ) fixed and n always refers to the exponent in (2.2).

Let $0 \le \alpha < \infty$ and $R > 0$. The *fractional maximal function* of a locally integrable function f is defined by

$$\mathcal{M}_{\alpha,R} f(x) = \sup_{0<r<R} r^\alpha \fint_{B(x,r)} |f|\, d\mu\,,$$

If $Y \subset X$, then we denote $\mathcal{M}_{\alpha,Y} = \mathcal{M}_{\alpha,\operatorname{diam} Y}$. For $R = \infty$, we write $\mathcal{M}_{\alpha,\infty} = \mathcal{M}_\alpha$. If $\alpha = 0$, we obtain the Hardy-Littlewood maximal function and we write $\mathcal{M}_0 = \mathcal{M}$. By the Hardy-Littlewood maximal theorem $\mathcal{M}$ is bounded in L^p provided $1 < p \le \infty$. For $p = 1$ we have a weak type inequality.

Proposition 2.3. *Under the above assumptions*

$$\mu\big(\{x \in X\colon\ \mathcal{M} f(x) > \lambda\}\big) \le \frac{c}{\lambda}\, \|f\|_{L^1(X)}\,, \qquad \lambda > 0\,, \tag{2.4}$$

and

$$\|\mathcal{M}f\|_{L^p(X)} \le c\,\|f\|_{L^p(X)}\,, \qquad 1 < p \le \infty\,. \tag{2.5}$$

It is easy to verify that the set

$$E_\lambda = \{x \in X:\ \mathcal{M}_\alpha f(x) > \lambda\}\,, \qquad \lambda > 0\,,$$

is open. Next we would like to get some estimates for the size of the set E_λ. To this end, recall that the Hausdorff s-content of $E \subset X$ is defined by

$$\mathcal{H}^s_\infty(E) = \inf\Big\{\sum_{i=1}^\infty r_i^s:\ E \subset \bigcup_{i=1}^\infty B(x_i,r_i)\Big\}\,.$$

It is easy to see that $\mathcal{H}^s_\infty(E) = 0$ if and only if $\mathcal{H}^s(E) = 0$, where $\mathcal{H}^s$ denotes the Hausdorff s-measure.

By the standard Vitali covering argument [CW, p. 69] we obtain the following weak type inequality for the fractional maximal function, see [BZ, Lemma 3.2], [St, Theorem 3.3].

Lemma 2.6. *Suppose that $f \in L^1(X)$ and let $Y \subset X$ be a bounded set with $\mu(Y) > 0$. Let n be as in* (2.2) *and $0 \le \alpha < n$. Then*

$$\mathcal{H}^{n-\alpha}_\infty\big(\{x \in Y:\ \mathcal{M}_{\alpha,Y} f(x) > \lambda\}\big) \le \frac{c}{\lambda}\int_X |f|\,d\mu\,, \qquad \lambda > 0\,, \tag{2.7}$$

with $c = 5^{n-\alpha}(2\operatorname{diam} Y)^n\,\mu(Y)^{-1}$.

3. Sobolev space and capacity.

Let $u : X \longrightarrow [-\infty,\infty]$ be μ-measurable. We denote by $D(u)$ the set of all μ-measurable functions $g : X \longrightarrow [0,\infty]$ such that

$$|u(x) - u(y)| \le d(x,y)\,(g(x) + g(y))\,, \tag{3.1}$$

almost everywhere. By saying that inequality (3.1) holds almost everywhere we mean that there exists $N \subset X$ with $\mu(N) = 0$ such that (3.1) holds for every $x, y \in X \setminus N$.

A function $u \in L^p(X)$ belongs to the Sobolev space $W^{1,p}(X)$, $1 < p \leq \infty$, if $D(u) \cap L^p(X) \neq \varnothing$. The space $W^{1,p}(X)$ is endowed with the norm

$$\|u\|_{W^{1,p}(X)} = \left(\|u\|^p_{L^p(X)} + \|u\|^p_{L^{1,p}(X)}\right)^{1/p}, \tag{3.2}$$

where

$$\|u\|_{L^{1,p}(X)} = \inf_{g \in D(u)} \|g\|_{L^p(X)}. \tag{3.3}$$

With this norm $W^{1,p}(X)$ is a Banach space.

If $X = \Omega \subset \mathbb{R}^n$ is an open bounded domain with a Lipschitz boundary, d is the Euclidean metric and μ is the Lebesgue measure, then the above definition is equivalent to the standard definition of the Sobolev space $W^{1,p}(\Omega)$. Moreover, $\|u\|_{L^{1,p}} \approx \|\nabla u\|_{L^p}$, see [H1]. This explains our notation: $D(u)$ corresponds to the set of the "generalized" gradients of u.

The above definition of the Sobolev space on a metric space is due to the first author [H1]. If $p = 1$, then the above metric definition in no longer equivalent to the standard definition, see [H2]. For that reason we exclude $p = 1$.

We present two characterizations of the Sobolev space on a metric space, see [FHK], [FLW1], [HK2], [KMc] for related results. To this end, we need yet another maximal function.

Let $0 < \beta < \infty$ and $R > 0$. The *fractional sharp maximal function* of a locally integrable function f is defined by

$$f^{\#}_{\beta,R}(x) = \sup_{0<r<R} r^{-\beta} \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu,$$

If $R = \infty$ we simply write $f^{\#}_{\beta}(x)$.

Theorem 3.4. *Let $1 < p \leq \infty$. Then the following three conditions are equivalent.*

1) $u \in W^{1,p}(X)$.

2) $u \in L^p(X)$ and there is $g \in L^p(X)$, $g \geq 0$, such that the Poincaré inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq c\, r \fint_{B(x,r)} g \, d\mu, \tag{3.5}$$

holds for every $x \in X$ *and* $r > 0$.

3) $u \in L^p(X)$ *and* $u_1^{\#} \in L^p(X)$.

Moreover, we obtain

$$\|u\|_{L^{1,p}(X)} \approx \inf \{\|g\|_{L^p(X)} : \ g \text{ satisfies } (3.5)\} \approx \|u_1^{\#}\|_{L^p(X)} \,.$$

Remark. The equivalence of 1) and 2) has been proved in [FLW1] and in the classical Euclidean case the equivalence of 1) and 3) can be found in [C].

Proof. The implication 1) implies 2) follows by integrating (3.1) twice over the ball.

We prove 2) implies 3). The Poincaré inequality (3.5) implies that

$$r^{-1} \fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le c \fint_{B(x,r)} g \, d\mu \,.$$

Hence

$$u_1^{\#}(x) \le c \, \mathcal{M} g(x) \,,$$

for every $x \in X$ and the claim follows from the Hardy-Littlewood maximal theorem (Proposition 2.3).

3) implies 1). We need the following lemma which, in the Euclidean case with the Lebesgue measure, has been proved in [DS, Theorem 2.7].

Lemma 3.6. *Suppose that* $f : X \longrightarrow [-\infty, \infty]$ *is locally integrable and let* $0 < \beta < \infty$. *Then there is a constant* $c = c(\beta, C_d)$ *such that*

$$|f(x) - f(y)| \le c \, d(x,y)^{\beta} \big(f^{\#}_{\beta, 4d(x,y)}(x) + f^{\#}_{\beta, 4d(x,y)}(y) \big) \,, \tag{3.7}$$

for almost every $x, y \in X$.

Proof. Let N be the the complement of the set of Lebesgue points of f in X. Fix $x \in X \setminus N$, $0 < r < \infty$ and denote $B_i = B(x, 2^{-i} r)$,

$i = 0, 1, \dots$ By Lebesgue's theorem $\mu(N) = 0$, see [CW]. Then

$$
\begin{aligned}
|f(x) - f_{B(x,r)}| &\le \sum_{i=0}^{\infty} |f_{B_{i+1}} - f_{B_i}| \\
&\le \sum_{i=0}^{\infty} \frac{\mu(B_i)}{\mu(B_{i+1})} \fint_{B_i} |f - f_{B_i}|\, d\mu \\
&\le c \sum_{i=0}^{\infty} (2^{-i} r)^{\beta} (2^{-i} r)^{-\beta} \fint_{B_i} |f - f_{B_i}|\, d\mu \\
&\le c\, r^{\beta} f^{\#}_{\beta,r}(x)\,.
\end{aligned}
\tag{3.8}
$$

Let $y \in B(x,r) \setminus N$. Then $B(x,r) \subset B(y, 2\,r)$ and we obtain

$$
\begin{aligned}
|f(y) - f_{B(x,r)}| &\le |f(y) - f_{B(y,2\,r)}| + |f_{B(y,2\,r)} - f_{B(x,r)}| \\
&\le c\, r^{\beta} f^{\#}_{\beta,2r}(y) + \fint_{B(x,r)} |f - f_{B(y,2\,r)}|\, d\mu \\
&\le c\, r^{\beta} f^{\#}_{\beta,2r}(y) + c \fint_{B(y,2r)} |f - f_{B(y,2\,r)}|\, d\mu \\
&\le c\, r^{\beta} f^{\#}_{\beta,2r}(y)\,.
\end{aligned}
\tag{3.9}
$$

Let $x, y \in X \setminus N$, $x \ne y$ and $r = 2\, d(x,y)$. Then $x, y \in B(x,r)$ and hence (3.8) and (3.9) imply that

$$
\begin{aligned}
|f(x) - f(y)| &\le |f(x) - f_{B(x,r)}| + |f(y) - f_{B(x,r)}| \\
&\le c\, d(x,y)^{\beta} \left(f^{\#}_{\beta,4d(x,y)}(x) + f^{\#}_{\beta,4d(x,y)}(y) \right).
\end{aligned}
$$

This completes the proof.

Now the last implication in Theorem 3.4 follows immediately from Lemma 3.6 and the definition of the Sobolev space. Moreover, the equivalence of the norms follows from the proof. This completes the proof of Theorem 3.4.

Now we state some useful inequalities for the future reference.

Corollary 3.10. *Let $u \in W^{1,p}(X)$, $g \in D(u) \cap L^p(X)$ and $0 \le \alpha < 1$. Then*

$$
u^{\#}_{1-\alpha,R}(x) \le c\, \mathcal{M}_{\alpha,R}\, g(x)\,,
\tag{3.11}
$$

for every $R > 0$ and $x \in X$. Moreover, we have

$$(3.12)\quad |u(x) - u(y)| \le c\, d(x,y)^{1-\alpha}\big(\mathcal{M}_{\alpha,4d(x,y)}g(x) + \mathcal{M}_{\alpha,4d(x,y)}g(y)\big)\,,$$

for almost every $x, y \in X$.

Proof. The first assertion follows from the Poincaré inequality, since

$$r^{\alpha-1}\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le c\, r^{\alpha}\fint_{B(x,r)} g\, d\mu\,,$$

for every $x \in X$ and $0 < r < R$. Inequality (3.11) and Lemma 3.6 imply (3.12).

There is a natural capacity in the Sobolev space $W^{1,p}(X)$. The norm (3.2) enables us to define the Sobolev p-capacity of an arbitrary set $E \subset X$ by

$$(3.13)\qquad \mathrm{C}_p(E) = \inf_{u \in \mathcal{A}(E)} \|u\|^p_{W^{1,p}(X)}\,,$$

where the infimum is taken over all admissible functions

$$\mathcal{A}(E) = \big\{u \in W^{1,p}(X):\ u \ge 1 \text{ on an open neighbourhood of } E\big\}\,.$$

This capacity is a monotone and a countably subadditive set function. The rudiments of the capacity theory on metric spaces were developed by the second author with O. Martio in [KM]. By [KM, Theorem 4.6] there is a constant $c = c(p, C_d)$ such that

$$(3.14)\qquad \mathrm{C}_p(B(x,r)) \le c\, r^{-p}\mu(B(x,r))\,,\qquad 0 < r \le 1\,.$$

Using the same standard covering argument as in the proof of Lemma 2.6 together with (3.14) and the assumption that the measure is doubling we obtain the following capacitary version of Lemma 2.6.

Lemma 3.15. *Suppose that $f \in L^1(X)$ and let $1 < \alpha < \infty$. Then there is $c = c(C_d, \alpha)$ such that*

$$(3.16)\qquad \mathrm{C}_\alpha\left(\big\{x \in X: \mathcal{M}_{\alpha,1}f(x) > \lambda\big\}\right) \le \frac{c}{\lambda}\, \|f\|_{L^1(X)}\,,\qquad \lambda > 0\,.$$

4. Lebesgue points.

A Sobolev function $u \in W^{1,p}(X)$ is defined only up to a set of measure zero, but we define u *everywhere* in X by

$$\widetilde{u}(x) = \limsup_{r\to 0} \fint_{B(x,r)} u\,d\mu\,. \tag{4.1}$$

By Lebesgue's theorem not only the limit superior but the limit exists and equals to u almost everywhere. Hence $\widetilde{u}$ coincides with u almost everywhere and gives the same element in $W^{1,p}(X)$. We identify u with $\widetilde{u}$ and omit the tilde in notation.

We recall that $x \in X$ is Lebesgue's point for u if

$$\fint_{B(x,r)} |u(y) - u(x)|\,d\mu(y) \longrightarrow 0$$

as $r \longrightarrow 0$. Lebesgue's theorem states that almost all points of a $L^1_{\mathrm{loc}}(X)$ function are Lebesgue points, see [CW]. If a function belongs to the classical Sobolev space, then we can improve the result and prove that the complement of the set of the Lebesgue points has has small Hausdorff dimension, see [FZ], [Z], [EG]. We generalize this result to the Sobolev spaces on metric spaces.

By (3.8) we have for $\beta > 0$

$$\begin{aligned} &\fint_{B(x,r)} |u(y) - u(x)|\,d\mu(y) \\ &\qquad \le \fint_{B(x,r)} |u(y) - u_{B(x,r)}|\,d\mu(y) + |u_{B(x,r)} - u(x)| \\ &\qquad \le c\,r^{\beta} u^{\#}_{\beta,r}(x)\,. \end{aligned} \tag{4.2}$$

Letting $r \to 0$ we see that x is Lebesgue's point for u provided $u^{\#}_{\beta,1}(x) < \infty$.

We want to estimate the size of the set of the Lebesgue points of $u \in W^{1,p}(X)$. We begin with studying the case $p > n$, where n is the exponent in (2.2).

Proposition 4.3. *Suppose that $u \in W^{1,p}(X)$ with $n < p < \infty$. Then u is Hölder continuous on every bounded set in X. In particular, every point is a Lebesgue point of u.*

PROOF. Let $g \in D(u) \cap L^p(X)$. It follows from the Hölder inequality and (2.2) that $\mathcal{M}_{n/p}\, g$ is bounded in every ball. Hence (3.12) implies that u is Hölder continuous with the exponent $1 - n/p$ in every ball.

Then we consider the more interesting case $1 < p \le n$.

Theorem 4.4. *Suppose that $u \in W^{1,p}(X)$, $g \in D(u) \cap L^p(X)$ and $1 < p \le n$. Let $0 \le \alpha < 1$. Then every point for which $\mathcal{M}_{\alpha,1} g(x) < \infty$ is Lebesgue's point for u. Moreover, the Hausdorff dimension of the complement of the set of the Lebesgue points of u is less than or equal to $n - p$.*

PROOF. Inequalities (4.2) and (3.11) imply that

$$\fint_{B(x,r)} |u(y) - u(x)|\, d\mu(y) \le c\, r^{1-\alpha}\, u^{\#}_{1-\alpha,1}(x) \le c\, r^{1-\alpha}\, \mathcal{M}_{\alpha,1} g(x)\,,$$

when $0 < r < 1$. The term on the right side goes to zero as $r \longrightarrow 0$ if $\mathcal{M}_{\alpha,1} g(x) < \infty$. This shows that x is the Lebesgue point for u.

The set of the non-Lebesgue points is contained in

$$E_\infty = \{x \in X : \ \mathcal{M}_{\alpha,1} g(x) = \infty\}\,.$$

Note that

$$\mathcal{M}_{\alpha,1} g(x) \le (\mathcal{M}_{\alpha p,1} g^p(x))^{1/p}\,. \tag{4.5}$$

Let $1 < q < p$. Choose $\alpha = q/p$. Inequality (4.5) and the weak type estimate (2.7) yield

$$\begin{aligned} \mathcal{H}^{n-q}_\infty\big(E_\infty \cap B(y,1)\big) &\le \mathcal{H}^{n-q}_\infty\big(\{x \in B(y,1) : \ \mathcal{M}_{q,1} g^p(x) > \lambda^p\}\big) \\ &\le c\, \lambda^{-p} \int_X g^p\, d\mu\,, \end{aligned} \tag{4.6}$$

for every $\lambda > 0$. Letting $\lambda \longrightarrow \infty$ we see that

$$\mathcal{H}^{n-q}_\infty(E_\infty \cap B(y,1)) = 0\,,$$

for every ball $B(y,1)$ and hence $\mathcal{H}^{n-q}(E_\infty) = 0$ for any $q < p$. This gives the desired estimate for the Haudorff dimension. The proof is complete.

5. Hölder quasicontinuity and approximation of Sobolev functions.

In this section we assume that u coincides with the representative $\widetilde{u}$ defined pointwise by (4.1). It follows from the proof of Lemma 3.6 that for every $0 < \beta \leq 1$ the inequality

$$|u(x) - u(y)| \leq c\, d(x,y)^\beta \big(u^\#_{\beta,4d(x,y)}(x) + u^\#_{\beta,4d(x,y)}(y)\big)\,. \tag{5.1}$$

holds for *every* $x \neq y$. It may happen that the left hand side of (5.1) is of the indefinite form like $|\infty - \infty|$, then we adopt the convention $|\infty - \infty| = \infty$. In any case inequality (5.1) remains valid since $u(x) = \pm\infty$ implies that $u^\#_{\beta,R}(x) = \infty$ for every $R > 0$.

In particular, if $\|u^\#_\beta\|_\infty < \infty$, then (5.1) shows that u is Hölder continuous. This is the content of Morrey's lemma, see [MS1, Theorem 4].

Denote

$$E_\lambda = \{x \in X :\ u^\#_\beta(x) > \lambda\}\,, \qquad \lambda > 0\,. \tag{5.2}$$

Using (5.1) we see that $u|_{X\setminus E_\lambda}$ is Hölder continuous with the exponent β. We can extend this function to a Hölder continuous function on X using the McShane extension

$$u(x) = \inf\big\{u(y) + 2\,\lambda\, d(x,y)^\beta :\ y \in X \setminus E_\lambda\big\}\,,$$

for every $x \in X$, see [Mc]. However, this does not guarantee that the extended function belongs to the Sobolev space $W^{1,p}(X)$ nor that it is close to the original function in the Sobolev norm. For that we need a Whitney type extension. In fact, we do not extend the function from the set $X \setminus E_\lambda$, but we smooth the function u outside that set leaving the values of u on $X \setminus E_\lambda$ unchanged. Thus our construction should be called the Whitney smoothing.

Now we are ready for the main result of the paper.

Theorem 5.3. *Suppose that $u \in W^{1,p}(X)$ is defined pointwise by* (4.1), $1 < p \leq n$ *and let* $0 < \beta \leq 1$. *Then for every* $\varepsilon > 0$ *there is a function w and an open set O such that*

1) $u = w$ *everywhere in* $X \setminus O$,

2) $w \in W^{1,p}(X)$, *and w is Hölder continuous with the exponent β on every bounded set in* X,

3) $\|u - w\|_{W^{1,p}(X)} < \varepsilon$,

4) $\mathcal{H}_\infty^{n-(1-\beta)p}(O) < \varepsilon$.

Proof. First suppose that the support of $u \in W^{1,p}(X)$ is contained in a ball

$$\operatorname{supp} u \subset B(x_0, 1)\,,$$

for some $x_0 \in X$. The general case follows from a localization argument. Various constants c that appear in the proof do not depend on λ.

Note that there is $\lambda_0 > 0$ such that for every $r > 1$ and $x \in X$ we have

$$r^{-\beta} \fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu < \lambda_0\,, \tag{5.4}$$

and hence $E_\lambda \subset B(x_0, 2)$ when $\lambda > \lambda_0$. Indeed, if the term on the left hand side of (5.4) is positive then $B(x,r) \cap B(x_0,1) \neq \varnothing$. Thus by the doubling property we have

$$\mu(B(x,r)) \geq c\, \mu(B(x_0,1)) > 0\,,$$

for $r > 1$ and estimate (5.4) follows easily.

It is easy to verify that the set E_λ defined by (5.2) is open.

Let $g \in D(u) \cap L^p(X)$. If $x \in E_\lambda$ and $\lambda > \lambda_0$, by (5.4), (3.11) and the Hölder inequality we obtain

$$u_\beta^\#(x) = u_{\beta,1}^\#(x) \leq c\, \big(\mathcal{M}_{(1-\beta)p,1}\, g^p(x)\big)^{1/p} \leq c\, \big(\mathcal{M} g^p(x)\big)^{1/p}\,. \tag{5.5}$$

The weak type estimate (2.4) shows that

$$\mu(E_\lambda) \leq \mu\big(\{x \in X :\ \mathcal{M} g^p(x) > c\, \lambda^p\}\big) \leq c\, \lambda^{-p} \int_X g^p\, d\mu < \infty\,, \tag{5.6}$$

for every $\lambda > \lambda_0$.

We recall the following Whitney type covering theorem [MS2, Lemma 2.9] and [CW].

Lemma 5.7. *Let $O \subset X$ be an open set such that $O \neq X$ and $\mu(O) < \infty$. For given $C \geq 1$, let $r(x) = \operatorname{dist}(x, X \setminus O)/(2\,C)$. Then there is $N \geq 1$ and a sequence $\{x_i\}$ such that, denoting $r(x_i) = r_i$, the following properties are true:*

1) *The balls* $B(x_i, r_i/4)$ *are pairwise disjoint.*

2) $\bigcup_{i\in I} B(x_i, r_i) = O$.

3) $B(x_i, C\,r_i) \subset O$ *for every* $i = 1, 2, \dots$

4) *For every* i, $x \in B(x_i, C\,r_i)$ *implies that*

$$C\,r_i \le \operatorname{dist}(x, X \setminus O) \le 3\,C\,r_i\,.$$

5) *For every* i, *there is* $y_i \in X \setminus O$ *such that* $d(x_i, y_i) < 3\,C\,r_i$.

6) $\sum_{i=1}^{\infty} \chi_{B(x_i, Cr_i)} \le N$.

The previous covering lemma enables us to construct a partition of unity, see [MS2, Lemma 2.16] or [Se, Lemma C.31]. Let $B(x_i, r_i)$, $i = 1, 2, \dots$, be the Whitney covering of E_λ constructed in Lemma 5.7 with $C = 5$. Then there are non-negative functions $\{\varphi_i\}_{i=1}^{\infty}$ such that $\operatorname{supp}\varphi_i \subset B(x_i, 2\,r_i)$, $0 \le \varphi_i(x) \le 1$ for every $x \in X$, every φ_i is Lipschitz with the constant c/r_i and

$$\sum_{i=1}^{\infty} \varphi_i(x) = \chi_{E_\lambda}(x)\,, \qquad x \in X\,. \tag{5.8}$$

We define the Whitney smoothing of u by

$$w(x) = \begin{cases} u(x)\,, & x \in X \setminus E_\lambda\,, \\ \displaystyle\sum_{i=1}^{\infty} \varphi_i(x)\, u_{B(x_i, 2r_i)}\,, & x \in E_\lambda\,. \end{cases} \tag{5.9}$$

Note that since $E_\lambda \subset B(x_0, 2)$ for $\lambda > \lambda_0$, we have $\operatorname{supp} w \subset B(x_0, 2)$. We prove the theorem with $O = E_\lambda$ for sufficiently large λ.

Claim 1) is a trivial consequence of the definition of w.

Claim 2). First we show that w is Hölder continuous with the exponent β.

Suppose that $x \in E_\lambda$ and choose $\overline{x} \in X \setminus E_\lambda$ so that $d(x, \overline{x}) \le 2\operatorname{dist}(x, X \setminus E_\lambda)$. Then by (5.8) and (5.9) we have

$$\begin{aligned} |w(\overline{x}) - w(x)| &= \Big| \sum_{i=1}^{\infty} \varphi_i(x) \big(u(\overline{x}) - u_{B(x_i, 2r_i)}\big) \Big| \\ &\le \sum_{i \in I_x} \big| u(\overline{x}) - u_{B(x_i, 2r_i)} \big|\,, \end{aligned} \tag{5.10}$$

where $i \in I_x$ if and only if $x \in \operatorname{supp} \varphi_i$. A straightforward calculation, using the properties of the partition of unity, shows that for every $i \in I_x$ we have $B(x_i, 2\,r_i) \subset B(\overline{x}, 50\,r_i)$. Hence the argument similar to that in the proof of (3.9) gives

$$\big|u(\overline{x}) - u_{B(x_i,2\,r_i)}\big| \le c\,r_i^{\beta} u_{\beta}^{\#}(\overline{x})\,. \tag{5.11}$$

Since the overlap of the balls $B(x_i, 2\,r_i)$ is uniformly bounded by Lemma 5.7.6), we see that the cardinality of I_x is uniformly bounded. By Lemma 5.7.4) we see that r_i, $i \in I_x$, is comparable to $\operatorname{dist}(x, \overline{x})$. Using (5.10), (5.11) and recalling that $u_{\beta}^{\#}(\overline{x}) \le \lambda$ for $\overline{x} \in X \setminus E_\lambda$, we arrive at

$$|w(\overline{x}) - w(x)| \le c\,d(\overline{x}, x)^{\beta} u_{\beta}^{\#}(\overline{x}) \le c\,\lambda\,d(\overline{x}, x)^{\beta}\,. \tag{5.12}$$

We show that

$$|w(x) - w(y)| \le c\,\lambda\,d(x, y)^{\beta}\,.$$

for all $x, y \in X$. We divide the proof into several cases.

First suppose that $x, y \in E_\lambda$ and let

$$\gamma = \min\big\{\operatorname{dist}(x, X \setminus E_\lambda), \operatorname{dist}(y, X \setminus E_\lambda)\big\}\,. \tag{5.13}$$

If $d(x, y) \ge \gamma$, then (5.12), the fact that $\overline{x}, \overline{y} \in X \setminus E_\lambda$ and (5.1) imply

$$\begin{aligned} |w(x) - w(y)| &\le |w(x) - w(\overline{x})| + |u(\overline{x}) - u(\overline{y})| + |w(\overline{y}) - w(y)| \\ &\le c\,\lambda\,\big(d(x, \overline{x})^{\beta} + d(\overline{x}, \overline{y})^{\beta} + d(\overline{y}, y)^{\beta}\big) \\ &\le c\,\lambda\,d(x, y)^{\beta}\,. \end{aligned}$$

Suppose then that $x, y \in E_\lambda$ with $d(x, y) \le \gamma$. By (5.8) we have

$$\sum_{i=1}^{\infty} (\varphi_i(x) - \varphi_i(y)) = 0\,.$$

Hence we obtain

$$\begin{aligned} |w(x) - w(y)| &= \Big|\sum_{i=1}^{\infty} \big(\varphi_i(x)\,u_{B(x_i,2r_i)} - \varphi_i(y)\,u_{B(x_i,2r_i)}\big)\Big| \\ &= \Big|\sum_{i=1}^{\infty} \big(\varphi_i(x) - \varphi_i(y)\big)\big(u(\overline{x}) - u_{B(x_i,2r_i)}\big)\Big| \qquad (5.14) \\ &\le c\,d(x, y) \sum_{i \in I_x \cup I_y} r_i^{-1}\big|u(\overline{x}) - u_{B(x_i,2r_i)}\big|. \end{aligned}$$

Using the same argument as in (5.11) we see that

$$|w(x)-w(y)| \le c\, d(x,y)^{\beta} \sum_{i\in I_x\cup I_y} \frac{d(x,y)^{1-\beta}}{r_i^{1-\beta}}\, u^{\#}_{\beta}(\overline{x}) \le c\,\lambda\, d(x,y)^{\beta}\,.$$

The last inequality follows from the fact that r_i, $i\in I_x\cup I_y$, is comparable to γ.

If $x,y\in X\setminus E_\lambda$, then the claim follows from (5.1). If $x\in E_\lambda$ and $y\in X\setminus E_\lambda$, then

$$|w(x)-w(y)| \le |w(x)-w(\overline{x})| + |u(\overline{x})-u(y)|$$

and the claim follows from (5.12) and (5.1). This proves the Hölder continuity of w.

Then we prove that $w\in W^{1,p}(X)$. To this end, it suffices to show that $w\in L^p(X)$ and that for $g\in D(u)\cap L^p(X)$ we have $\mathcal{M}g\in D(w)\cap L^p(X)$.

First we observe that

$$\int_{E_\lambda} |w|^p\, d\mu \le c\sum_{i=1}^{\infty}\int_{B(x_i,2r_i)} |u_{B(x_i,2r_i)}|^p\, d\mu \le c\int_{E_\lambda}|u|^p\, d\mu\,. \tag{5.15}$$

In both inequalities we applied the uniform bound for the overlapping number of the balls $B(x_i,2\,r_i)\subset E_\lambda$ (Lemma 5.7.6) and 5.7.3)). Since $w(x)=u(x)$ for every $x\in X\setminus E_\lambda$, we see that $w\in L^p(X)$.

Let $g\in D(u)\cap L^p(X)$. Then for almost every $x,y\in X\setminus E_\lambda$ we have

$$|w(x)-w(y)| = |u(x)-u(y)| \le d(x,y)\,(g(x)+g(y))\,.$$

For almost every $x,y\in E_\lambda$ with $d(x,y)\le\gamma$ (*cf.* (5.13)) the calculation as in (5.14) gives

$$|w(x)-w(y)| \le c\, d(x,y)\sum_{i\in I_x\cup I_y} r_i^{-1}\big|u(x)-u_{B(x_i,2r_i)}\big|\,. \tag{5.16}$$

Since $d(x,y)$ is small enough, we have $B(x_i,2\,r_i)\subset B(x,100\,r_i)$ whenever $i\in I_x\cup I_y$. Then by the Poincaré inequality and the doubling condition we obtain

$$\big|u(x)-u_{B(x_i,2r_i)}\big| \le c\,r_i\,\mathcal{M}g(x)\,. \tag{5.17}$$

Since the cardinality of $I_x \cup I_y$ is bounded, by (5.16) we obtain

$$|w(x) - w(y)| \leq c\, d(x,y)\, \mathcal{M}g(x)\,.$$

For almost every $x, y \in E_\lambda$ with $d(x,y) \geq \gamma$, using the same argument as in (5.17), we have

$$\begin{aligned}
|w(x) - w(y)| = \Big| \sum_{i=1}^{\infty} \big(\varphi_i(x)\big(u_{B(x_i,2r_i)} - u(x)\big) \\
- \varphi_i(y)\big(u_{B(x_i,2r_i)} - u(y)\big)\big) + (u(x) - u(y)) \Big| \\
&\leq \sum_{i\in I_x} \big|u(x) - u_{B(x_i,2r_i)}\big| + \sum_{i \in I_y} \big|u(y) - u_{B(x_i,2r_i)}\big| \\
&\quad + |u(x) - u(y)| \\
&\leq c \operatorname{dist}(x, X\setminus E_\lambda)\, \mathcal{M}g(x) + c \operatorname{dist}(y, X \setminus E_\lambda)\, \mathcal{M}g(y) \\
&\quad + d(x,y)\, (g(x) + g(y)) \\
&\leq c\, d(x,y)\, (\mathcal{M}g(x) + \mathcal{M}g(y))\,.
\end{aligned}$$

If either $x \in X \setminus E_\lambda$ or $y \in X \setminus E_\lambda$, then the proof is similar. We conclude that

$$|w(x) - w(y)| \leq c\, d(x,y)\, (\mathcal{M}g(x) + \mathcal{M}g(y))\,, \tag{5.18}$$

for almost every $x, y \in X$. By the Hardy-Littlewood theorem (Proposition 2.3) we obtain $\mathcal{M}g \in L^p(X)$. This shows that $u \in W^{1,p}(X)$.

Claim 3). Then we prove that $w \longrightarrow u$ in $W^{1,p}(X)$ as $\lambda \longrightarrow \infty$. Since $\mu(E_\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$, we conclude using inequalities (5.15) and (5.6) that $\|u - w\|_{L^p(X)} \longrightarrow 0$ as $\lambda \longrightarrow \infty$. Now we take care about the "gradient" estimates.

Let $g \in D(u) \cap L^p(X)$. Inequalities (5.18) and (5.17) imply that for a suitable constant c the function $g_\lambda = c\,(\mathcal{M}g)\,\chi_{E_\lambda}$ satisfies $g_\lambda \in D(u-w) \cap L^p(X)$. Since $\|g_\lambda\|_{L^p(X)} \longrightarrow 0$, as $\lambda \longrightarrow 0$ we conclude that $\|u - w\|_{L^{1,p}(X)} \longrightarrow 0$ as $\lambda \longrightarrow \infty$.

Claim 4). The claim follows from Lemma 2.6 and the fact that for $\lambda > \lambda_0$ we have

$$E_\lambda \subset \big\{x \in B(x_0, 2) : \ \mathcal{M}_{(1-\beta)p,1} g^p(x) > c\,\lambda^p\big\}\,, \tag{5.19}$$

for some $c > 0$, see (5.2) and (5.5). This completes the proof in the case when the support of u lies in a ball.

The case of general $u \in W^{1,p}(X)$ will be deduced from the case when u has the support in a ball via a partition of unity. First we need a lemma which shows that the multiplication by a bounded Lipschitz function is a bounded operator in the Sobolev norm. The following lemma is in some sense a generalization of the Leibniz differentiation rule.

Lemma 5.20. *Let $u \in W^{1,p}(X)$ and φ be a bounded Lipschitz function. Then $u\,\varphi \in W^{1,p}(X)$. Moreover, if L is a Lipschitz constant of φ and* $\operatorname{supp}\varphi = K$, *then*

$$\big(g\|\varphi\|_\infty + L|u|\big)\,\chi_K \in D(u\varphi) \cap L^p(X)\,,$$

for every $g \in D(u) \cap L^p(X)$.

Proof. The triangle inequality implies that

$$|u(x)\,\varphi(x) - u(y)\,\varphi(y)| \le d(x,y)\big(\big(g(x)+g(y)\big)|\varphi(x)| + L|u(y)|\big)$$

and

$$|u(x)\,\varphi(x) - u(y)\,\varphi(y)| \le d(x,y)\big(\big(g(x)+g(y)\big)|\varphi(y)| + L|u(x)|\big)\,.$$

Now it suffices to consider four easy cases depending on whether each of the points x, y belongs to K or not. This completes the proof.

Now we are ready to complete the proof of Theorem 5.3. Let $B(x_i, 1/4)$, $i = 1, 2, \ldots$, be a maximal family of pairwise disjoint balls in X. Then by maximality $X \subset \bigcup_{i=1}^\infty B(x_i, 1/2)$. Let $\{\varphi_i\}_{i=1}^\infty$ be a partition of unity such that $\operatorname{supp}\varphi_i \subset B(x_i, 1)$, $0 \le \varphi_i(x) \le 1$ for every $x \in X$,

$$\sum_{i=1}^\infty \varphi_i(x) = 1\,,$$

for every $x \in X$, and φ_i, $i = 1, 2, \ldots$, are Lipschitz continuous with the same Lipschitz constant.

Suppose that $u \in W^{1,p}(X)$. Then

$$u(x) = \sum_{i=1}^\infty u\,\varphi_i(x)\,,$$

for every $x \in X$. Lemma 5.20 implies that the series converges also in the Sobolev norm. Let $\varepsilon > 0$. Clearly $\operatorname{supp} u\,\varphi_i \subset B(x_i, 1)$, $i = 1, 2, \dots$ Let $w_i \in W^{1,p}(X)$, $i = 1, 2, \dots$, be a Hölder continuous function with the exponent β such that

$$\mathcal{H}_\infty^{n-(1-\beta)p}\big(\{x \in X : \ w_i(x) \neq u\,\varphi_i(x)\}\big) \le 2^{-i}\,\varepsilon\,,$$

$$\big\|w_i - u\,\varphi_i\big\|_{W^{1,p}(X)} \le 2^{-i}\varepsilon\,,$$

and

$$\operatorname{supp} w_i \subset B(x_i, 2)\,.$$

Then it is easy to see that

$$w = \sum_{i=1}^{\infty} w_i$$

has the desired properties. The proof of Theorem 5.3 is complete.

Remarks 5.21. 1) The case $\beta = 1$ of Theorem 5.3 has been previously proved in [H1]. This case is much easier, since it suffices to use the McShane extension [Mc]. Indeed, a locally Lipschitz function belongs to the Sobolev space $W^{1,p}(X)$ by the definition.

2) Using Lemma 3.15 and (5.19) we see that estimate (4) in Theorem 5.3 may be replaced by

$$\mathrm{C}_{(1-\beta)p}(O) < \varepsilon\,,$$

provided $\beta < 1 - 1/p$.

3) If the measure is *Ahlfors-David regular*, which means that there are $n > 0$ and $c \ge 1$ so that

$$c^{-1}\,r^n \le \mu(B(x,r)) \le c\,r^n\,, \qquad x \in X\,,\ 0 < r \le \operatorname{diam}(X)\,, \tag{5.22}$$

then the function w in Theorem 5.3 can be chosen to be globally Hölder continuous on X. Indeed, then the boundedness assumption in Lemma 2.6 is not needed. In addition, observe that we do not require that the space is bounded in (5.22).

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Subnormal operators of finite type II. Structure theorems

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Abstract. This paper concerns pure subnormal operators with finite rank self-commutator, which we call subnormal operators of finite type. We analyze Xia's theory of these operators [21]-[23] and give its alternative exposition. Our exposition is based on the explicit use of a certain algebraic curve in $\mathbb{C}^2$, which we call the discriminant curve of a subnormal operator, and the approach of dual analytic similarity models of [26]. We give a complete structure result for subnormal operators of finite type, which corrects and strenghtens the formulation that Xia gave in [23]. Xia claimed that each subnormal operator of finite type is unitarily equivalent to the operator of multiplication by z on a weighted vector H^2-space over a "quadrature Riemann surface" (with a finite rank perturbation of the norm). We explain how this formulation can be corrected and show that, conversely, every "quadrature Riemann surface" gives rise to a family of subnormal operators. We prove that this family is parametrized by the so-called characters. As a departing point of our study, we formulate a kind of scattering scheme for normal operators, which includes Xia's model as a particular case.

0. Introduction.

This paper is devoted to an alternative exposition of some aspects of Xia's theory of subnormal operators from a different viewpoint. We make use of the results of [25] and the approach of [26] and give new results and new connections. The ideas of Xia are exploited much throughout the paper, but our exposition is independent.

We develop a scattering scheme for normal operators, whose particular case is Xia's model, explain the role of the discriminant curve and the involution on it, and prove a complete structure result, which gives a two-sided connection between subnormal operators of finite type and real algebraic curves of a certain class.

The structure theorem we obtain in this paper allows one to prove an interesting relationship between subnormal operators of finite type and a certain class of vector analytic Toeplitz operators. This relationship gives rise to a new characterization of quadrature domains. These results will be presented elsewhere.

Let H be a Hilbert space and $\mathcal{L}(H)$ the space of bounded linear operators on H. An operator $S \in \mathcal{L}(H)$ is called subnormal if there is a Hilbert space K, $K \supset H$ and a normal operator $N \in \mathcal{L}(K)$ such that $NH \subset H$ and $S = N|H$. S is called *pure* if it has no nonzero reducing subspace on which it is normal. We will say that S is *of finite type* if it is pure and $\operatorname{rank}[S^*, S] < \infty$ (here $[S^*, S] = S^*S - SS^*$).

Let S be pure subnormal, and put

$$\begin{aligned} M &= \operatorname{clos}\operatorname{Range}[S^*, S], \\ C &= [S^*, S]|M, \\ \Lambda &= (S^*|M)^*. \end{aligned} \tag{0.1}$$

Xia has shown in [21] that M is invariant for S^* and that the pair of operators C, Λ on M completely determines S up to the unitary equivalence. Operators C, Λ play an essential role in Xia's model. For the case of a subnormal operator of finite type, the set of matrix parameters (C, Λ) has been described completely in [25]. The answer was formulated in terms of the algebraic curve

$$\Delta = \big\{(z, w) : \ \det\big(C - (w - \Lambda^*)\,(z - \Lambda)\big) = 0\big\}. \tag{0.2}$$

If C, Λ correspond to a subnormal operator S, then Δ is called *the discriminant curve of* S.

Here we define a certain class of algebraic curves in $\mathbb{C}^2$, which we call admissible separated curves. An algebraic curve Δ is in this class if it has a prescribed behavior at infinity and the real linear manifold $w = \bar{z}$ divides each of its irreducible components into two connected parts. For such curve Δ, there is a canonical way to define its "halves"

Δ_+, Δ_-. Let $\widehat{\Delta}$ be the blow-up of Δ. Each connected component of $\widehat{\Delta}$ is a compact Riemann surface, obtained from an irreducible component of Δ by deleting its singular points and then adding a finite number of "ideal" points [11].

For any admissible separated algebraic curve $\widehat{\Delta}$ and a matrix-valued function Ω on $\partial\widehat{\Delta}_+$ with integrable $\log\|\Omega\|$, $\log\|\Omega^{-1}\|$, we introduce the weighted Hardy class $H^2(\widehat{\Delta}_+, \Omega)$. We show that the operator of multiplication by the variable z on $H^2(\widehat{\Delta}_+, \Omega)$ is subnormal of finite type and that its discriminant curve is Δ. (We call such subnormal operators simple.) We deduce from this fact that an algebraic curve is the nondegenerate part of the discriminant curve of a subnormal operator of finite type if and only if this curve is admissible and separated.

The main structure result we get shows that any subnormal operator of finite type is obtained from a simple subnormal operator by "glueing" finitely many points of $\widehat{\Delta}_+$ and then performing a finite rank perturbation of the Hilbert space structure. Conversely, any such procedure gives a subnormal operator of finite type.

A criterion for unitary equivalence of subnormal operators of finite type is given. Roughly speaking, it consists in equality of certain characters (homeomorphisms of fundamental groups of the components of $\widehat{\Delta}_+$ into suitable groups of unitary matrices). This criterion generalizes a result by McCarthy and Yang [16], who considered the rationally cyclic case.

In order to understand better Xia's model, in sections 1–3 we introduce its generalization. It has an operator theory face and a complex analysis face, and we study them separately.

The operator theory part of the construction has the form of a scattering type scheme for normal operators. We say that a tuple (N, K, H', H, M) is *a scattering tuple* if K is a Hilbert space, H', H, M are its subspaces, the operator $N : K \longrightarrow K$ is similar to a normal operator, a direct sum decomposition $K = H' \dotplus H$ holds, and $NH \subset H$, $NH' \subset H' + M$, $M \subset H$, $\dim M < \infty$. With each such tuple we associate the operator $S = N|H$.

To formalize the complex analysis context, we introduce what we call mosaic tuples. A mosaic tuple consists of three matrix-valued functions and a scalar measure, interrelated in a certain way. Each mosaic tuple gives rise to a projection-valued mosaic μ and serves as a prerequisite for defining functional model spaces, which consist of analytic and antianalytic M-valued functions on $\mathbb{C} \setminus \sigma(N)$. The conclusion of sections 1–3 is that the two settings are equivalent: to each mosaic tuple corresponds a unique scattering tuple, and vice versa. In a sense, the mosaic tuple plays the role of the characteristic function in these constructions. The consideration of a generalized Xia's model has the advantage that one can understand well the freedom in choosing parameters of the mosaic tuple (see Section 5). The class of operators S which one gets in this way is much more general than the class of subnormal operators of finite type. For instance, the essential spectrum of a subnormal operator of finite type always lies on an algebraic curve, whereas the essential spectrum of an operator of the type considered in sections 1–3 can be any reasonable finite union of piecewise C^1-smooth curves.

In sections 1–3, the ideas and approach of dual bundle shift models [26] are used. The connection with dual bundle shift models is explained in Section 4. These models have been used in [26] for studying Toeplitz operators and in [27] to study hyponormal operators. The results of Section 4 are not used in the sequel.

Then we use the scattering scheme of sections 1–3 to study Xia's original model. We show how the properties $H' = H^\perp$, $M = [S^*, S]H$, which distinguish it, are connected with the existence of the antianalytic involution on $\widehat{\Delta}$. One of the outcomes of our exposition is a concrete explicit construction of a subnormal S of finite type from matrices C and Λ, if it exists.

The proof of the structure results we give consists in two reductions (whose idea is due to Xia). First we replace the mosaic model space $E^2(\mu)$ of functions on $\mathbb{C} \setminus \sigma(N)$ by a space of cross-sections of a certain analytic bundle over $\widehat{\Delta}_+$. Then, after trivializing this bundle and characterizing the space of its cross-sections (Section 10), we obtain our main structure results in Section 12. Necessary facts about weighted vector Hardy spaces over Riemann surfaces, characters and related topics are given in Section 9.

The relationship between subnormal operators and separated algebraic curves is most clear from Lemma 11.4 and its proof. The reader who just wishes to get an idea of the subject can first look at this lemma.

It is worth noticing that subnormal operators of finite type turn out to be unexpectedly close to Toeplitz operators with rational and similar symbols, which were studied in [18], [24]. In particular, the results about spectral multiplicity [18] and invariant and hyperinvariant subspaces [24] extend to subnormal operators of finite type.

Algebraic curves and Hardy classes over these curves also have been used intensively in works of Alpay, Fedorov, Livšic, Vinnikov and others (see [3], [8], [9], [14]). It would be interesting to know a connection between subnormal operators and the subject of those works.

At the end of the paper, an index of mathematical notation is given.

1. Mosaic tuples and mosaic model spaces.

Suppose we are given a compactly supported positive Borel measure ν on the complex plane, a finite-dimensional Hilbert space M and $\mathcal{L}(M)$-valued measurable functions F, $\mathcal{E}$, G on $\mathbb{C}$ such that $\mathcal{E} = \mathcal{E}^* \geqslant 0$ ν-almost everywhere. Put $\gamma = \operatorname{supp} \nu$,

$$de(\cdot) = \mathcal{E}(\cdot)\, d\nu(\cdot)\,,$$

and

$$p(u) = F(u)\,\mathcal{E}(u)\,G(u)\,. \tag{1.1}$$

Consider the space

$$L^2(e) = \Big\{ f:\ \|f\|^2 = \int \langle \mathcal{E}(u)\, f(u), f(u)\rangle\, d\nu(u) < \infty \Big\}\,.$$

After factoring by the set of functions f with $\|f\|^2 = 0$, $L^2(e)$ becomes a Hilbert space. Each element of $L^2(e)$ has a unique representative f such that $f(\cdot) \in \operatorname{Range} \mathcal{E}(\cdot)$ ν-almost everywhere. Two functions f, g are equal in $L^2(e)$ if and only if $\mathcal{E}f = \mathcal{E}g$.

In the setting of sections 1–3, there will be no loss of generality if we assume that $\mathcal{E}$ is a projection-valued function and $F = F\mathcal{E}$, $G = \mathcal{E}G$. Then $L^2(e)$ is the direct integral of the spaces $\mathcal{E}(\cdot)M$. We choose the formally more general setting in order to include the original Xia's mosaic.

We make the following assumptions.

M1) The function

$$\mu(z) = \int_\gamma \frac{p(u)}{u - z}\, d\nu(u)\,, \qquad z \in \mathbb{C} \setminus \gamma\,, \tag{1.2}$$

is projection-valued.

M2) $F^*(\cdot)\, m \in L^2(e)$ and $G(\cdot)\, m \in L^2(e)$ for any $m \in M$. The operators $F(u)|\mathcal{E}(u)\, M$, $G^*(u)|\mathcal{E}(u)\, M$ are one-to-one for ν-almost every u.

M3) The family of functions $(\cdot - \lambda)^{-1} G(\cdot)\, m$, $m \in M$, $\lambda \in \mathbb{C} \setminus \gamma$ and the family of functions $(\bar{\cdot} - \bar{\lambda})^{-1} F^*(\cdot)\, m$, $m \in M$, $\lambda \in \mathbb{C} \setminus \gamma$ are complete in $L^2(e)$.

The function μ is "piecewise analytic", that is, it is analytic on $\mathbb{C} \setminus \gamma$. We call it a *generalized Xia's mosaic*. Since $\mu(\infty) = 0$, it follows that $\mu \equiv 0$ in the unbounded component of $\mathbb{C} \setminus \gamma$.

Define the Cauchy integral

$$\mathcal{K} f(z) = \int_\gamma \frac{f(t)}{t - z}\, d\nu(t)\,, \qquad z \in \mathbb{C} \setminus \gamma\,.$$

By (M2) and (M3), the map $f \longmapsto \mathcal{K} F \mathcal{E} f$, $f \in L^2(e)$ is one-to-one. Let $\mathcal{K} F \mathcal{E} L^2(e) = \{\mathcal{K} F \mathcal{E} f : f \in L^2(e)\}$ be the image of this map, with the norm inherited from $L^2(e)$. We need also the following assumption.

M4) The operator

$$\big(P_\mu u\big)(z) = \mu(z)\, u(z)\,, \qquad z \in \mathbb{C} \setminus \gamma \tag{1.3}$$

acts on $\mathcal{K} F \mathcal{E} L^2(e)$ and is bounded.

Definition. *We say that* $(M, F, \mathcal{E}, G, \nu, \mu)$ *is a mosaic tuple if* M1)–M4) *hold.*

EXAMPLE. Let $M = \mathbb{C}^1$, $\nu = |dz|$ on $\partial\, \mathbb{D}$, where $\mathbb{D} = \{|z| \leqslant 1\}$, $p(z) = z/2\pi$, so that $\mu(z) = 1$ for $|z| < 1$ and $\mu(z) = 0$ for $|z| > 1$. Put $\mathcal{E} \equiv 1/2\pi$. Then F, G have to satisfy $F(z)\, G(z) \equiv z$ on $\partial\, \mathbb{D}$. Let P_+ be the Riesz projection, that is, the orthogonal projection of $L^2(\partial\, \mathbb{D})$ onto H^2. Then, obviously, $P_\mu \mathcal{K} = \mathcal{K} P_+$. Therefore M4) holds in this case if and only if P_+ extends to a bounded operator on $|F|\, L^2(d\nu)$, that is, if and only if $|F|$ satisfies the Muckenhoupt condition (A_2).

So M4) has the sense of a vector Muckenhoupt condition. It implies that P_μ is a bounded projection on $\mathcal{K} F \mathcal{E} L^2(e)$. In a recent work [20] by Treil and Volberg, a very explicit form of this condition on $\partial\, \mathbb{D}$ has been found.

To construct more general mosaic tuples, one has to start from a piecewise analytic projection-valued function μ and then find F, $\mathcal{E}$, G, ν (see Section 5 below).

We introduce the following Smirnov type (closed) subspaces of $\mathcal{K}F\mathcal{E}L^2(e)$, which will be called *mosaic model spaces*

$$E^2(\mu) = P_\mu \mathcal{K}F\mathcal{E}L^2(e) = \left\{u \in \mathcal{K}F\mathcal{E}L^2(e) :\ u = \mu\, u \text{ on } \mathbb{C} \setminus \gamma\right\},$$

$$\begin{aligned} E_0^2(1-\mu) &= (I - P_\mu)\, \mathcal{K}F\mathcal{E}L^2(e) \\ &= \left\{u \in \mathcal{K}F\mathcal{E}L^2(e) :\ u = (1-\mu)\, u \text{ on } \mathbb{C} \setminus \gamma\right\} \end{aligned}$$

(we use the notation $|\cdot|$, 1 for the norm and the identity operator on a finite-dimensional space).

Functions in $E^2(\mu)$ and $E_0^2(1-\mu)$ are analytic on $\mathbb{C} \setminus \gamma$ and take value 0 at infinity; moreover, the functions in $E^2(\mu)$ are identically zero in the unbounded component of $\hat{\mathbb{C}} \setminus \gamma$. In general, spaces $E^2(\mu)$, $E_0^2(1-\mu)$ depend on the whole mosaic tuple rather than only on μ.

Associated to (1.1) is the factorization

$$p^*(u) = G^*(u)\, \mathcal{E}^*(u)\, F^*(u)\,.$$

Put

$$\overline{\mathcal{K}}g(z) = \int \frac{g(t)}{\overline{t} - \overline{z}}\, d\nu(t)\,, \qquad g \in G^*\mathcal{E}L^2(e)$$

and $\overline{\mathcal{K}}G^*\mathcal{E}L^2(e) = \{\overline{\mathcal{K}}G^*\mathcal{E}g : g \in L^2(e)\}$. Then

$$\overline{\mathcal{K}}G^*\mathcal{E}L^2(e) = \big(\mathcal{K}F\mathcal{E}L^2(e)\big)^*$$

if we use the pairing

$$\langle \mathcal{K}F\mathcal{E}f, \overline{\mathcal{K}}G^*\mathcal{E}g\rangle_d \overset{\text{def}}{=} \langle f, g\rangle\,, \qquad f, g \in L^2(e)\,. \tag{1.4}$$

The following fact will be proved later.

Proposition 1.1. *The projection* $P_\mu^* : \overline{\mathcal{K}}G^*\mathcal{E}L^2(e) \longrightarrow \overline{\mathcal{K}}G^*\mathcal{E}L^2(e)$ *is given by*

$$\big(P_\mu^* v\big)(z) = (1 - \mu^*(z))\, v(z)\,, \qquad z \in \mathbb{C} \setminus \gamma\,.$$

The subspaces

$$\begin{aligned} \overline{E}_0^2(1-\mu^*) &= P_\mu^* \overline{\mathcal{K}}G^*\mathcal{E}L^2(e) \\ &= \left\{u \in \overline{\mathcal{K}}G^*\mathcal{E}L^2(e) :\ u = (1-\mu^*)\, u \text{ on } \mathbb{C} \setminus \gamma\right\}, \end{aligned}$$

$$\begin{aligned} \overline{E}^2(\mu^*) &= (I - P_\mu^*)\, \overline{\mathcal{K}}G^*\mathcal{E}L^2(e) \\ &= \left\{u \in \overline{\mathcal{K}}G^*\mathcal{E}L^2(e) :\ u = \mu^* u \text{ on } \mathbb{C} \setminus \gamma\right\}, \end{aligned}$$

are completely analogous to the model spaces $E^2(\mu)$, $E_0^2(1-\mu)$. Functions in $\overline{E}^2(\mu^*)$, $\overline{E}_0^2(1-\mu^*)$ are antianalytic on $\mathbb{C}\setminus\gamma$. Since the maps $f \longmapsto \mathcal{K}F\mathcal{E}f$, $f \longmapsto \overline{\mathcal{K}}G^*\mathcal{E}f$, $f \in L^2(e)$ are one-to-one, we will regard the four mosaic model spaces as embedded into $L^2(e)$.

The most simple case is the above example, where one puts $F(z) \equiv z$, $G(z) \equiv 1$. Then $E^2(\mu) = H^2$, $E_0^2(1-\mu) = H_0^2(\hat{\mathbb{C}}\setminus \operatorname{clos}\mathbb{D}) \stackrel{\text{def}}{=} \{z^{-1}f(z^{-1})\colon f\in H^2\}$, $\overline{E}^2(\mu^*) = \bar{H}^2$, $\overline{E}_0^2(1-\mu^*) = \bar{H}_0^2(\hat{\mathbb{C}}\setminus \operatorname{clos}\mathbb{D})$.

Definition. *Subspaces $\mathcal{K}^{-1}E^2(\mu)$, $\mathcal{K}^{-1}E_0^2(1-\mu)$ of $F\mathcal{E}L^2(e)$ and $\overline{\mathcal{K}}^{-1}\overline{E}^2(\mu^*)$, $\overline{\mathcal{K}}^{-1}\overline{E}_0^2(1-\mu^*)$ of $G^*\mathcal{E}L^2(e)$ will be called the spaces of boundary values of functions in corresponding model classes $E^2(\mu)$, $E_0^2(1-\mu)$, $\overline{E}^2(\mu^*)$, $\overline{E}_0^2(1-\mu^*)$.*

Put

$$L(z) = (1-\mu(z))\,M\,, \qquad L'(z) = \mu(z)\,M\,, \tag{1.5}$$

then

$$L(z)^\perp = \mu^*(z)\,M\,, \qquad L'(z)^\perp = (1-\mu^*(z))\,M\,,$$

and for each $z\in\mathbb{C}\setminus\gamma$, we have direct sum decompositions

$$M = L(z)\dotplus L'(z) = L(z)^\perp \dotplus L'(z)^\perp\,. \tag{1.6}$$

Xia uses the notation $M(z)$, $M'(z)$ instead of $L(z)$, $L'(z)$.

The functions

$$\varphi_{t,m}(z) = \frac{\mu(z)-\mu(t)}{z-t}\,m\,, \qquad \varphi_{*,t,n}(z) = \frac{\mu^*(z)-\mu^*(t)}{\bar{z}-\bar{t}}\,n\,, \tag{1.7}$$

$t\in\mathbb{C}\setminus\gamma$, $m,n\in M$, will be called *the Cauchy reproducing kernels.*

Basic facts we need about the model spaces are collected in the following theorem.

Theorem 1.2. *Let $(M,F,\mathcal{E},G,\nu,\mu)$ be a mosaic tuple. Then*

a) *Direct sum decompositions*

$$\mathcal{K}F\mathcal{E}L^2(e) = E_0^2(1-\mu)\dotplus E^2(\mu)\,, \tag{1.8}$$

$$\overline{\mathcal{K}}G^*\mathcal{E}L^2(e) = \overline{E}^2(\mu^*)\dotplus \overline{E}_0^2(1-\mu^*)\,, \tag{1.9}$$

hold, P_μ is the parallel projection onto $E^2(\mu)$ with respect to the decomposition (1.8) *and P_μ^* is the parallel projection onto $\overline{E}_0^2(1-\mu^*)$ with respect to the decomposition* (1.9).

b) *The following equalities with respect to duality* (1.4) *hold for annihilator spaces*

$$E^2(\mu)^\perp = \overline{E}^2(\mu^*)\,, \qquad E_0^2(1-\mu)^\perp = \overline{E}_0^2(1-\mu^*)\,.$$

c) *The duality* (1.4) *gives rise to the following representations of duals*

$$\big(E^2(\mu)\big)^* = \overline{E}_0^2(1-\mu^*)\,, \qquad \big(E_0^2(1-\mu)\big)^* = \overline{E}^2(\mu^*)\,.$$

d) *The Cauchy kernels $\varphi_{t,m}$, $t \in \mathbb{C}\setminus\gamma$, $m \in M$ are in $\mathcal{K}F\mathcal{E}L^2(e)$ and generate it. The Cauchy kernels $\varphi_{*,t,n}$ generate $\overline{\mathcal{K}}G^*\mathcal{E}L^2(e)$. The reproducing formulas*

$$\langle u, \varphi_{*,t,n}\rangle_d = \langle u(t), n\rangle\,, \qquad \langle \varphi_{t,m}, v\rangle_d = \langle m, v(t)\rangle\,, \tag{1.10}$$

hold for all $t \in \mathbb{C}\setminus\gamma$, $m,n \in M$, $u \in \mathcal{K}F\mathcal{E}L^2(e)$, $v \in \overline{\mathcal{K}}G^*\mathcal{E}L^2(e)$.

e) *Moreover,*

$$\begin{aligned}
&\overline{\operatorname{span}}\big\{\varphi_{t,m}:\ t\in\mathbb{C}\setminus\gamma,\ m\in L'(t)\big\} = E_0^2(1-\mu)\,,\\
&\overline{\operatorname{span}}\big\{\varphi_{t,m}:\ t\in\mathbb{C}\setminus\gamma,\ m\in L(t)\big\} = E^2(\mu)\,,\\
&\overline{\operatorname{span}}\big\{\varphi_{*,t,m}:\ t\in\mathbb{C}\setminus\gamma,\ m\in L(t)^\perp\big\} = \overline{E}_0^2(1-\mu^*)\,,\\
&\overline{\operatorname{span}}\big\{\varphi_{*,t,m}:\ t\in\mathbb{C}\setminus\gamma,\ m\in L'(t)^\perp\big\} = \overline{E}^2(\mu^*)\,.
\end{aligned} \tag{1.11}$$

f) *The operators $M_z u(z) = z\,u(z)$ on $E^2(\mu)$ and $M_{\overline{z}} u(z) = \overline{z}\,u(z)$ on $\overline{E}^2(\mu^*)$ are subnormal. Their adjoints are given by*

$$M_z^* v(z) = \overline{z}\,v(z) - (1-\mu^*(z))\big(\overline{z}\,v(z)\big|_{z=\infty}\big)\,, \tag{1.12}$$

where $v \in \overline{E}_0^2(1-\mu^*)$,

$$M_{\overline{z}}^* v(z) = z\,v(z) - (1-\mu(z))\big(z\,v(z)\big|_{z=\infty}\big)\,, \tag{1.13}$$

where $v \in E_0^2(1-\mu)$.

PROOFS OF PROPOSITION 1.1 AND THEOREM 1.2. It follows from (1.2) and (1.7) that

$$\begin{aligned}\varphi_{t,m}(z) &= \int \frac{p(s)\,m}{(s-z)\,(s-t)}\,d\nu(s) \\ &= \mathcal{K}\big(F(\cdot)\,\mathcal{E}(\cdot)\,(\cdot-t)^{-1}\,G(\cdot)\,m\big)(z)\,.\end{aligned} \tag{1.14}$$

Since $G(\cdot)\,m \in L^2(e)$ by M2), it follows that $\varphi_{t,m} \in \mathcal{K}F\mathcal{E}L^2(e)$. Similarly, $\varphi_{*,t,m} \in \overline{\mathcal{K}}G^*\mathcal{E}L^2(e)$. Take any $v \in \mathcal{K} = G^*\mathcal{E}L^2(e)$, then $v = \overline{\mathcal{K}}G^*\mathcal{E}g$ for some $g \in L^2(e)$, and by (1.14) we have

$$\begin{aligned}\langle \varphi_{t,m}, v\rangle_d &= \big\langle (\cdot - t)^{-1} G\,m, g\big\rangle_{L^2(e)} \\ &= \int \big\langle \mathcal{E}(s)\,(s-t)^{-1} G(s)\,m, g\big\rangle\,d\nu(s) \\ &= \langle m, v(t)\rangle\,,\end{aligned}$$

which gives the second identity in (1.10). Now suppose that v is orthogonal to all reproducing kernels $\varphi_{t,m}$. Then $v(z) \equiv 0$ on $\mathbb{C}\setminus\gamma$, so that $G^*\mathcal{E}g \equiv 0$, which implies $g \equiv 0$ by M2). This shows the completeness of the $\varphi_{t,m}$. The first equality in (1.10) and the completeness of the $\varphi_{*,t,m}$ are proved in the same way. Thus, d) holds true.

Next, one gets from (1.7) that

$$P_\mu \varphi_{t,m} = \varphi_{t,(1-\mu(t))m}\,.$$

For every $v \in \overline{\mathcal{K}}G^*\mathcal{E}L^2(e)$ and $m \in M$, $t \in \mathbb{C}\setminus\gamma$,

$$\begin{aligned}\langle m, (P_\mu^* v)(t)\rangle &= \langle \varphi_{t,m}, P_\mu^* v\rangle \\ &= \langle \varphi_{t,(1-\mu(t))m}, v\rangle \\ &= \langle m, (1-\mu^*(t))\,v\rangle\,,\end{aligned}$$

which proves Proposition 1.1.

Now assertion a) of Theorem 1.2 is a direct consequence of the definitions of the model spaces. Next, it is obvious that

$$\varphi_{t,m}(z) = \frac{\mu(z)-\mu(t)}{z-t}\,m = -\frac{(1-\mu(z))-(1-\mu(t))}{z-t}\,m$$

is in $E^2(\mu)$ if $m \in L(t)$ and in $E^2_0(1-\mu)$ if $m \in L'(t)$ (see (1.5)). Since $\{\varphi_{t,m}\}$ generate $\mathcal{K}F\mathcal{E}L^2(e)$, $\{P_\mu\varphi_{t,m}\}$ generate $E^2(\mu)$ and so on; in this way we obtain (1.11).

Next, $v \in E^2(\mu)^\perp$ if and only if $\langle v(t), m\rangle = \langle v, \varphi_{t,m}\rangle = 0$, $t \in \mathbb{C}\setminus\gamma$,$m \in L(t)$, if and only if $v(t) \in L(t)^\perp = \mu(t)^* M$, $t \in \mathbb{C}\setminus\gamma$, if and only if $v \in \overline{E}^2(\mu^*)$. Similarly, $E_0^2(1-\mu) = \overline{E}_0^2(1-\mu^*)^\perp$, so that b) has been checked. The first equality in c) is a direct consequence of b) and the identity

$$E^2(\mu)^* = \big(\mathcal{K}F\mathcal{E}L^2(e)\big)^*/E^2(\mu)^\perp\,.$$

We obtain that $E_0^2(1-\mu)^* = \overline{E}^2(\mu^*)$ in the same way.

To prove f), we observe that

$$M_z = \mathcal{M}_z\big|_{E^2(\mu)} = P_\mu \mathcal{M}_z P_\mu\ ,$$

where

$$\mathcal{M}_z\mathcal{K}f \overset{\text{def}}{=} \mathcal{K}(zf)\,, \qquad f \in F\mathcal{E}L^2(e)\,.$$

One sees that $\mathcal{M}_z^* = \mathcal{M}_{\overline{z}}$, where

$$\mathcal{M}_{\overline{z}}\overline{\mathcal{K}}g \overset{\text{def}}{=} \overline{\mathcal{K}}(\overline{z}\,g)\,, \qquad g \in G^*\mathcal{E}L^2(e)\,.$$

Since $\mathcal{M}_z$ and $\mathcal{M}_{\overline{z}}$ are normal, M_z and $M_{\overline{z}}$ are subnormal. Let $v = \overline{\mathcal{K}}g \in \overline{E}_0^2(1-\mu^*)$. Then

$$\begin{aligned} M_z^* v(z) &= P_\mu^* \mathcal{M}_{\bar z} P_\mu^* v(z) \\ &= \big(P_\mu^* \overline{\mathcal{K}}(\overline{z}\,g)\big)(z) \\ &= P_\mu^*\big(\overline{z}\,v - (1-\mu^*(z))\,(\overline{z}\,v)\big|_{z=3D\infty}\big)\,, \end{aligned}$$

which proves (1.12). One proves (1.13) in the same way.

2. A scattering type scheme.

Suppose that K is a Hilbert space and $N : K \longrightarrow K$ is a linear operator. We will apply our scheme only to situations when $\sigma(N) \overset{\text{def}}{=} \gamma$ has empty interior. Suppose that

Sc1) $K = H' \dotplus H$, where $NH \subset H$.

Sc2) There exists a finite-dimensional subspace M of H such that

$$NH' \subset H' \dotplus M\,. \tag{2.1}$$

We put $S = N|H$.

In this section, we discuss tuples (N, K, H', H, M) subject to Sc1), Sc2). Two more requirements will be added in Section 3. Our aim is to establish a relationship between tuples (N, K, H', H, M) and mosaic tuples from Section 1.

The original Xia's model corresponds to the case when $S \in \mathcal{L}(H)$ is pure subnormal, $N \in \mathcal{L}(K)$ is its minimal normal extension, $H' = H^{\perp}$ and $M = [S^*, S]H$ (see Section 6.2).

If R is a linear space and $\mathcal{W}$ an open subset in $\widehat{\mathbb{C}}$, then we denote by $\mathrm{Hol}(\mathcal{W}, R)$ the space of all holomorphic functions $f : \mathcal{W} \longrightarrow R$ and by $\overline{\mathrm{Hol}}(\mathcal{W}, R)$ the corresponding space of antiholomorphic functions. If $\infty \in \mathcal{W}$, then we put $\mathrm{Hol}_0(\mathcal{W}, R) = \{f \in \mathrm{Hol}_0(\mathcal{W}, R) \colon f(\infty) = 0\}$ and define $\overline{\mathrm{Hol}}_0(\mathcal{W}, R)$ similarly.

Any linear operator $B : K \longrightarrow R$ gives rise to an operator $W_B : K \longrightarrow \mathrm{Hol}_0(\mathbb{C} \setminus \gamma, R)$, defined by

$$(W_B x)(z) = B(N - z)^{-1} x \,, \qquad x \in K,\ z \in \mathbb{C} \setminus \gamma .$$

The operator W_B "almost diagonalizes" N in the sense that

$$(W_B N x)(z) = z\,(W_B x)(z) - \big(z\,(W_B x)(z)\big|_{z=\infty}\big) . \tag{2.2}$$

In what follows, we will see how to obtain "almost diagonalization" operators W_B with good additional properties with respect to the decomposition $K = H' \dotplus H$.

We put

$$L(z) = \{m \in M :\ (N - z)^{-1} m \in H\} ,$$
$$L'(z) = \{m \in M :\ (N - z)^{-1} m \in H'\} \qquad (z \in \mathbb{C} \setminus \gamma) .$$

Lemma 2.1. *For each $z \in \mathbb{C} \setminus \gamma$,*

$$M = L(z) \dotplus L'(z) .$$

PROOF. By the definition, $L(z) \cap L'(z) = 0$. Take any $m \in M$, and let $(N - z)^{-1} m = g_1 + g_2$, where $g_1 \in H$, $g_2 \in H'$. Put $l_j = (N - z)\, g_j$, then $l_1 + l_2 = m$, and $l_1 \in H$, $l_2 \in H' + M$. It follows that $l_2 = m - l_1 \in (H' + M) \cap H = M$, $l_1 \in M$, and we are done.

2.1. Transform $\widetilde{U}$.

Let $P_H : K \longrightarrow H$, $P_{H'} : K \longrightarrow H'$ be the coordinate projections with respect to the decomposition given in Sc1), then $P_H + P_{H'} = I$. Put

$$A = P_H N P_{H'} : K \longrightarrow M \tag{2.3}$$

and define $\rho(z) : K \longrightarrow M$ by

$$\rho(z) = A(N-z)^{-1}\,, \qquad z \in \mathbb{C} \setminus \gamma\,. \tag{2.4}$$

We define a transform $\widetilde{U} : K \longrightarrow \mathrm{Hol}(\mathbb{C} \setminus \gamma, M)$ by

$$(\widetilde{U}x)(z) = \rho(z)\,x\,, \qquad x \in K,\ z \in \mathbb{C} \setminus \gamma \tag{2.5}$$

(note that $\widetilde{U} = W_{N,A}$).

Lemma 2.2. *For all $z \in \mathbb{C} \setminus \gamma$, $\rho(z)^2 = \rho(z) P_H$.*

Proof. This is a straightforward calculation. Using that

$$P_{H'}(N-z)P_{H'} = P_{H'}(N-z)\,,$$

we get

$$\begin{aligned}
\rho(z)^2 &= P_H N P_{H'}(N-z)^{-1}(I - P_{H'})\,(N-z)P_{H'}(N-z)^{-1} \\
&= P_H N P_{H'}(N-z)^{-1} \\
&\quad - P_H N P_{H'}(N-z)^{-1} P_{H'}(N-z)P_{H'}(N-z)^{-1} \\
&= P_H N P_{H'}(N-z)^{-1} - P_H N P_{H'}(N-z)^{-1}P_{H'} \\
&= \rho(z)P_H\,.
\end{aligned}$$

Now we define the mosaic μ, associated to the tuple (N, K, H', H, M), by

$$\mu(z) = \rho(z)|M : M \longrightarrow M\,, \qquad z \in \mathbb{C} \setminus \gamma\,. \tag{2.6}$$

It follows from the above lemma that $\mu(z)^2 = \mu(z)$, that is, $\mu(z)$ is a projection. It also follows that

$$(2.7)\qquad \begin{aligned} \big(\widetilde{U}P_H x\big)(z) &= \mu(z)\big(\widetilde{U}x\big)(z)\,, \\ \big(\widetilde{U}P_{H'} x\big)(z) &= (1-\mu(z))\big(\widetilde{U}x\big)(z)\,, \qquad x \in K\,. \end{aligned}$$

Set

$$\operatorname{Hol}\big(\{L\}\big) = \big\{u \in \operatorname{Hol}(\mathbb{C}\setminus\gamma, M):\ u(z)\in L(z),\ z\in\mathbb{C}\setminus\gamma\big\}\,,$$

and define similarly spaces $\operatorname{Hol}\big(\{L'\}\big)$, $\overline{\operatorname{Hol}}\big(\{L^\perp\}\big)$, etc.

Lemma 2.3.

1) *If* $x \in H$, *then* $\widetilde{U}x \in \operatorname{Hol}\big(\{L'\}\big)$.

2) *If* $x \in H'$, *then* $\widetilde{U}x \in \operatorname{Hol}\big(\{L\}\big)$.

3) *The operator* $\widetilde{U}$ *"almost diagonalizes" N, that is, it satisfies* (2.2).

4) *It diagonalizes* $S = N|H$

$$(2.8)\qquad \big(\widetilde{U}Sx\big)(z) = z\big(\widetilde{U}x\big)(z)\,, \qquad x\in H\,.$$

PROOF. Assertions 1), 2) follow from (2.7). Equality $\widetilde{U} = W_{N,A}$ implies 3). Since $\mu\equiv 0$ in the unbounded connected component of $\mathbb{C}\setminus\gamma$, it follows that

$$\big(\widetilde{U}Nx\big)(z) = z\widetilde{U}x(z) - \big(z\widetilde{U}x(z)\big)\big|_{z=\infty} = z\widetilde{U}x(z)\,, \qquad x\in H\,.$$

Lemma 2.4. *One has*

$$\operatorname{Ker}\mu(z) = L(z)\,, \qquad \operatorname{Range}\mu(z) = L'(z)\,.$$

PROOF. Obviously, $\mu(z)\,m = 0$ for $m\in L(z)$. If $m \in L'(z)$, then

$$\mu(z)\,m = P_H N(N-z)^{-1}m = P_H(N-z)(N-z)^{-1}m = m\,.$$

It follows from Lemma 2.1 that $\mu(z)$ is the parallel projection onto $L'(z)$ that corresponds to the decomposition (2.5).

So the transform $\widetilde{U}$ "almost diagonalizes" N and has good properties with respect to the decomposition $K = H' \dot{+} H$ (see (2.8)). Now we shall construct a good "almost diagonalization" operator for N^*.

2.2. Transform $\widetilde{V}$.

The decomposition $K = H' \dot{+} H$ gives rise to a dual decomposition

$$K = H'_* \dot{+} H_* \,, \tag{2.9}$$

where we have put $H'_* \stackrel{\text{def}}{=} H^\perp$, $H_* \stackrel{\text{def}}{=} H'^\perp$. We will consider H'_*, H_* as realizations of the duals to H', H, respectively, and assume S^* to be defined on H_*. Then $P^*_{H'}$, P^*_H are parallel projections onto H'_*, H_* with respect to the decomposition (2.9). Since $A^* = P^*_{H'} N^* P^*_H$, we have

$$M_* \stackrel{\text{def}}{=} \operatorname{Range} A^* \subset H'_* \,. \tag{2.10}$$

It is easy to see that

$$N^* H'_* \subset H'_* \,, \qquad N^* H_* \subset H_* + M_* \,.$$

Therefore there is a certain symmetry: the tuple (N, K, H', H, M) can be replaced by (N^*, K, H'_*, H_*, M_*), which has the same properties Sc1), Sc2). We break (a little) this symmetry and associate with (N^*, K, H'_*, H_*, M_*) an operator

$$\big(\widetilde{V} y\big)(z) = P_M (N^* - \overline{z})^{-1} y \,, \qquad y \in K \,,\ z \in \mathbb{C} \setminus \gamma \,, \tag{2.11}$$

here P_M is the orthogonal projection onto M.

Lemma 2.5.

i) *$y \in H'_*$ implies $\mu^* V y = V y$.*

ii) *$y \in H_*$ implies $(1 - \mu^*) V y = V y$.*

iii) *The following intertwining formula holds*

$$\big(\widetilde{V} N^* y(z)\big)(z) = \overline{z} \, \big(\widetilde{V} y(z)\big)(z) \,, \qquad y \in H'_* \,,\ z \in \mathbb{C} \setminus \gamma \,.$$

PROOF. If $y \in H_*$ and $m \in M$, then $(N-z)^{-1}\mu(z)\, m \in H'$ by Lemma 2.4, which gives $\langle m, \mu(z)^*\big(\widetilde{V}y\big)(z)\rangle = \langle (N-z)^{-1}\mu(z)\, m, y\rangle = 0$, so that ii) holds. Similarly, one checks that $\big(1-\mu(z)^*\big)\widetilde{V}y(z) \equiv 0$ if $y \in H'_*$, which gives i). Assertion iii) follows from i) in the same way as in Lemma 2.3.

The following formulas are immediate

$$\begin{aligned} &\widetilde{U}(N-\lambda)^{-1}m = \varphi_{\lambda,m}\ , \\ &\widetilde{V}(N^*-\overline{\lambda})^{-1}P^*_{H'}A^*m = \varphi_{*,\lambda,m}\ , \qquad \lambda \in \mathbb{C}\setminus\gamma,\ m \in M\,. \end{aligned} \tag{2.12}$$

3. A model theorem.

In addition to Sc1), Sc2), let us assume the following conditions.

Sc3) Spaces $(N-z)^{-1}M$, $z \in \mathbb{C}\setminus\gamma$, as well as spaces $(N^*-\overline{z})^{-1}M_*$, $z \in \mathbb{C}\setminus\gamma$, are complete in K.

Sc4) N is similar to a normal operator: there exists a scalar measure $d\nu$, a $\mathcal{L}(M)$-valued Borel function $\mathcal{E}(\cdot) = \mathcal{E}(\cdot)^*$ and a linear isomorphism $W : K \longrightarrow L^2(e)$, where $de = \mathcal{E}\, d\nu$, that transforms N into M_z

$$WN = M_z W\,. \tag{3.1}$$

A condition similar to Sc3) appears in [26] as a hypothesis, which is necessary for constructing dual analytic models.

Observe that $W^{*-1} : K \longrightarrow L^2(e)$ is an isomorphism that satisfies

$$W^{*-1}N^* = M_{\overline{z}}W^{*-1}\,. \tag{3.2}$$

Definition. *We say that* (N, K, H', H, M) *is a scattering tuple if* Sc1)–Sc4) *hold.*

Choose (any) $\mathcal{L}(M)$-valued matrices $F(\cdot)$, $G(\cdot)$ such that

$$\big(Wm\big)(\cdot) = G(\cdot)\, m\,,\ \big(W^{*-1}A^*m\big)(\cdot) = F^*(\cdot)\, m\,, \qquad m \in M\,. \tag{3.3}$$

Next theorem gives a relationship between scattering tuples, mosaic tuples and corresponding mosaic model spaces.

Theorem 3.1. *Let (N, K, H', H, M) be a scattering tuple. Let $W : K \longrightarrow L^2(e)$ be the operator from* Sc4), *and let $F(\cdot)$, $G(\cdot)$ satisfy* (3.3). *Then the mosaic μ, given by* (2.6), (2.4) *coincides with the function* (1.2), *and $(M, F, \mathcal{E}, G, \nu, \mu)$ is a mosaic tuple. The operators* (2.5), (2.11) *admit representations*

$$\widetilde{U} = \mathcal{K} F \mathcal{E} W\,, \qquad \widetilde{V} = \overline{\mathcal{K}} G^* \mathcal{E} W^{*-1}\,. \tag{3.4}$$

Moreover,

$$P_\mu = \widetilde{U} P_H \widetilde{U}^{-1}\,, \tag{3.5}$$

$$\langle \widetilde{U}x, \widetilde{V}y \rangle_d = \langle x, y \rangle\,, \qquad x, y \in K\,, \tag{3.6}$$

$$\widetilde{U}H' = E_0^2(1-\mu)\,, \qquad \widetilde{U}H = E^2(\mu)\,, \tag{3.7}$$

$$\widetilde{V}H' = \overline{E}^2(\mu^*)\,, \qquad \widetilde{V}H = \overline{E}_0^2(1-\mu^*)\,. \tag{3.8}$$

In particular, $S = N|H$ is similar to the model operator

$$\widetilde{U}S\widetilde{U}^{-1} = \big(M_z \text{ on } E^2(\mu)\big)\,.$$

The meaning of this theorem is reflected in the following commutative diagram.

(3.9)

$$\begin{tikzcd}
 & K \arrow[dl, "W"'] \arrow[d, "\widetilde{U}"] \arrow[r, equal] & H' \arrow[d, "\widetilde{U}"] & \dotplus & H \arrow[d, "\widetilde{U}"] \\
L^2(e) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "*"] & \mathcal{K}F\mathcal{E}L^2(e) \arrow[r, equal] \arrow[d, leftrightarrow, "*"] & E_0^2(1-\mu) \arrow[d, leftrightarrow, "*"] & \dotplus & E^2(\mu) \arrow[d, leftrightarrow, "*"] \\
L^2(e) \arrow[r, leftrightarrow] & \overline{\mathcal{K}}G^*\mathcal{E}L^2(e) \arrow[r, equal] & \overline{E}^2(\mu^*) & \dotplus & \overline{E}_0^2(1-\mu^*) \\
 & K \arrow[ul, "W^{*-1}"] \arrow[u, "\widetilde{V}"'] \arrow[r, equal] & H'_* \arrow[u, "\widetilde{V}"'] & \dotplus & H_* \arrow[u, "\widetilde{V}"']
\end{tikzcd}$$

Here $\overset{*}{\longleftrightarrow}$ links spaces that are dual to each other.

Using the construction of Section 5, one can give many examples of subnormal operators S with $\dim [S^*, S]H = \infty$, for which this theorem produces a model with finite-dimensional "base" space M.

PROOF OF THEOREM 3.1. By (3.1)–(3.3),

$$\begin{aligned} &W(N-\lambda)^{-1}m = (\cdot-\lambda)^{-1}G(\cdot)\,m\,, \\ &W^{*-1}(N^*-\overline{\lambda})^{-1}A^*m = (\bar{\cdot}-\overline{\lambda})^{-1}F^*(\cdot)\,m\,, \qquad m\in M\,. \end{aligned} \tag{3.10}$$

Hence for any $m, n \in M$,

$$\begin{aligned} \langle \mu(\lambda)\,m, n\rangle &= \langle (N-\lambda)^{-1}m, A^*n\rangle \\ &= \langle W(N-\lambda)^{-1}m, W^{*-1}A^*n\rangle \\ &= \int \langle \mathcal{E}(z)\,G(z)\,(z-\lambda)^{-1}m, F^*(z)\,n\rangle\,d\nu(z) \\ &= \Big\langle \Big(\int p(z)\,(z-\lambda)^{-1}\,d\nu(z)\Big)m, n\Big\rangle\,. \end{aligned}$$

This implies that (1.2) holds and gives M1).

Let us show that $F(\cdot)\big|\operatorname{Range}\mathcal{E}(\cdot)$ is one-to-one ν-almost everywhere. Suppose it is not so. Then there exists $f \in L^2(e)$, $f \neq 0$ such that $F\mathcal{E}f = 0$. By (3.10), this implies

$$\langle \mathcal{E}f, W^{*-1}(N^*-\overline{\lambda})^{-1}A^*m\rangle_{L^2(e)} = 0\,,$$

for all $\lambda \in \mathbb{C}\setminus\gamma, m \in M$. Then Sc3) gives that $f \perp \operatorname{Range} W^{*-1}$, a contradiction.

The second part of M2) is proved in the same way.

Condition M3) follows from Sc4) and (3.10). Also, (3.10) and (2.12) show that

$$(\widetilde{U}x)(z) = \big(\mathcal{K}F\mathcal{E}Wx\big)(z)\,, \qquad z \in \mathbb{C}\setminus\gamma\,, \tag{3.11}$$

whenever $x = (N-\lambda)^{-1}m$, $m \in M$, $\lambda \in \mathbb{C}\setminus\gamma$. Therefore (3.11) holds for all $x \in K$. We obtain the second equality in (3.4) in the same way. In particular,

$$\widetilde{U} : K \longrightarrow \mathcal{K}F\mathcal{E}L^2(e)\,, \qquad \widetilde{V} : K \longrightarrow \overline{\mathcal{K}}G^*\mathcal{E}L^2(e)$$

are isomorphisms.

Formula (3.5) follows from (2.7). Hence M4) holds. Formula (3.6) is immediate from (3.4) and (1.4). It follows that $\widetilde{V} = \widetilde{U}^{*-1}$, and therefore $P_\mu^* = \widetilde{V} P_H^* \widetilde{V}^{-1}$. Now (3.7), (3.8) follow from the results of Section 1.

Remark. Any mosaic tuple $(M, F, \mathcal{E}, G, \nu, \mu)$ can appear as a result of an application of the above theorem. If suffices to put $K = L^2(e)$, $H = E^2(\mu)$, $H' = E_0^2(1-\mu)$ and to embed M into $E^2(\mu)$ according to the rule $m \longmapsto \mu(\cdot)\, m$, $m \in M$. Then $\widetilde{U}$ and $\widetilde{V}$ are identity maps.

4. Connection with dual bundle shift models.

Let us recall briefly the construction of [26]. Let $S : H \longrightarrow H$ be a linear operator, and choose an auxiliary operator $J : H \longrightarrow R$, where $\dim R < \infty$. Assume that

A1) $\operatorname{Ker}(S - z) = 0$ for all z in $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(S)$.

A2) $J | \operatorname{Ker}(S^* - \overline{z})$ is one-to-one for all z in $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(S)$.

Then the ultraspectrum $\mathcal{F}$ of S is defined as the antianalytic family of spaces $\mathcal{F} = \big\{ H(z) : z \notin \sigma_{\mathrm{ess}}(S) \big\}$, where $H(z) = J \operatorname{Ker}(S^* - \overline{z})$.

In the setting of [26], the ultraspectrum is an analytic family because we used there bilinear products. See [27] for a reformulation for sesquilinear products.

Next, let H_* be a realization of the dual space to H. Diagonalizing transforms

$$U : H \longrightarrow \operatorname{Hol}\big(\{H(z)^*\}\big)\,, \qquad V : H_* \longrightarrow \overline{\operatorname{Hol}}_0\big(\{(H(z)^\perp)^*\}\big)$$

were defined via the formulas

$$\langle (Ux)(z), m\rangle = \langle x, h_{z,m}\rangle\,, \qquad \langle (Vx)(z), l\rangle = \langle x, g_{z,l}\rangle\,,$$
$$x \in H,\ z \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(S),\ m \in H(z),\ l \in H(z)^\perp\,,$$

where $\{h_{z,m}\}$, $\{g_{z,l}\}$ are families of vectors in H_*, H, respectively, that are uniquely determined by the conditions

$$\begin{aligned} & h_{z,m} \in \operatorname{Ker}(S^* - \overline{z})\,, \qquad J h_{z,m} = m \in H(z)\,, \\ & (z - S)\, g_{z,l} = J^* l\,, \qquad l \in H(z)^\perp\,. \end{aligned} \tag{4.1}$$

The vectors $g_{z,l}$ were called *almost-eigenvectors of* S. If the families $\{h_{z,m}\}$ and $\{g_{z,l}\}$ are complete, then U and V are one-to-one. It has been shown in [26] that U transforms S into the operator of multiplication by the independent variable on the model space UH and V transforms S^* into the operator $v \longmapsto \overline{z}\,v - (\overline{z}\,v)(\infty)$ on the model space VH_*. There is a natural duality between UH and VH_*, which is defined in an intrinsic way.

Now let us set up the relationship between the dual analytic models and the construction of sections 1–3. Assume that (N, K, H', H, M) is a scattering tuple, and let $S = N|H$. Put

$$H_* = H'^{\perp}, \qquad R = M, \qquad J = P_M .$$

Proposition 4.1. *For this choice of* R, J, A1), A2) *are satisfied. The ultraspectrum of* S *is given by*

$$H(z) = L(z)^{\perp} = \mu^*(z)\, M . \tag{4.2}$$

It follows that the direct sum decompositions (1.6) give rise to isomorphisms

$$\begin{aligned} H(z)^* &= \big(L(z)^{\perp}\big)^* = M/L(z) \cong L'(z)\,, \\ \big(H(z)^{\perp}\big)^* &= L(z)^* = M/L(z)^{\perp} \cong L'(z)^{\perp}\,. \end{aligned}$$

We denote them as $i_z : \big(L(z)^{\perp}\big)^* \longrightarrow L'(z)$, $i_{*,z} : L(z)^* \longrightarrow L'(z)^{\perp}$. Then instead of U, V we can consider the transforms

$$\widetilde{U} : H \longrightarrow \mathrm{Hol}\big(\{L'(z)\}\big)\,, \qquad \widetilde{V} : H_* \longrightarrow \overline{\mathrm{Hol}}_0\big(\{L'(z)^{\perp}\}\big)\,,$$

acting by

$$\widetilde{U}x(z) = i_z(Ux)(z)\,, \qquad \widetilde{V}y(z) = i_{*,z}(Vy)(z)\,. \tag{4.3}$$

Proposition 4.2. *Operators* (4.3) *coincide with the transforms* $\widetilde{U}$, $\widetilde{V}$ *from Section* 2.

To prove Propositions 4.1 and 4.2, we need the following fact.

Lemma 4.3. *Let $z \in \mathbb{C} \setminus \sigma_{\rm ess}(S)$, and put*

$$h_{z,m} = \rho(z)^* m\,, \qquad m \in L(z)^\perp\,.$$

Then $\operatorname{Ker}(S^* - \overline{z}) = \big\{h_{z,m} : m \in L(z)^\perp\big\}$ *and* $J h_{z,m} = m$, *so that* $\{h_{z,m}\}$ *is exactly the family defined in* (4.1).

Proof. For any $h \in \operatorname{Ker}(S^* - \overline{z})$, $(N^* - \overline{z})h \in H'_*$, which gives

$$(N^* - \overline{z})\, h = P_{H'_*} N^* h = A^* h = A^* P_M h\ .$$

Therefore $h = \rho(z)^* P_M h = \rho(z)^* \mu(z)^* P_M h = h_{z,m}$, where $m \overset{\rm def}{=} \mu(z)^* P_M h \in L(z)^\perp$. Next, by (2.10),

$$(S^* - \overline{z})\, h_{z,m} = P_{H_*} (N^* - \overline{z})\, h_{z,m} = P_{H_*} A^* m = 0\,,$$

for all $m \in L(z)^\perp$. At last, $J h_{z,m} = P_M \rho(z)^* m = \mu(z)^* m = m$ for $m \in L(z)^\perp$.

Proof of Propositions 4.1 and 4.2. Since $S = N|H$, A1) holds. Lemma 4.3 implies A2) and (4.2). Therefore

$$\langle U x(z), m\rangle = \langle \rho(z)\, x, m\rangle = \langle \widetilde{U} x(z), m\rangle\,,$$

$$\langle V y(z), l\rangle = \langle y, (z - N)^{-1} l\rangle = \langle \widetilde{V} y(z), l\rangle\,,$$

for $m \in H(z) = L(z)^\perp$, $l \in H(z)^\perp = L(z)$. This proves Proposition 4.2.

We also obtain the following fact.

Proposition 4.4. *Eigenvectors $h_{z,m}$ of S^* $(z \in \mathbb{C} \setminus \sigma_{\rm ess}(S))$ are complete. Almost-eigenvectors $g_{z,l}$ of S $(z \in \mathbb{C} \setminus \sigma_{\rm ess}(S))$ are also complete.*

Proof. $\widetilde{U} : H \longrightarrow E^2(\mu)$ and $\widetilde{V} : H_* \longrightarrow \overline{E}^2_0(1 - \mu^*)$ are isomorphisms. Therefore U and V are one-to-one.

5. How to construct mosaic tuples?

The method we give here is not the most general, but it shows how much freedom we have.

Let γ be a finite union of arbitrary C^1-smooth arcs, intersecting only in their endpoints, such that $\mathbb{C} \setminus \gamma$ has at least two connected components. Denote the components of $\mathbb{C}\setminus\gamma$ by $\Omega_1, \dots, \Omega_t$. We assume for simplicity that for any of the arcs α that compose γ, one has $\alpha \subset \partial\Omega_j \cap \partial\Omega_k$ for some $j \neq k$. Let $\mu : \mathbb{C} \setminus \gamma \longrightarrow \mathcal{L}(M)$ be a projection-valued analytic function such that $\mu \equiv 0$ in the unbounded component and for each Ω_k, μ extends to a continuous function on $\operatorname{clos} \Omega_k$. Fix an orientation of each of the arcs of γ. For a function f in $\operatorname{Hol}(\mathbb{C} \setminus \gamma)$, we denote by f_i, f_e its interior and exterior limit values on γ.

Put $d\nu = |dz|\big|_\gamma$ and $p_1 = \mu_i - \mu_e$. Then (1.2) holds for

$$p = \frac{1}{2\pi i}\, p_1\, \frac{dz}{|dz|}\bigg|_\gamma .$$

Assume additionally that $\operatorname{rank} p$ is constant on each of the arcs $\partial\Omega_j \cap \partial\Omega_k$ and that

$$\|p(z)\big|(\operatorname{Ker} p(z))^\perp\| > \varepsilon > 0\,, \tag{5.1}$$

for z in the interior of these arcs, where ε does not depend on z. To assure that p satisfies this property one can take for μ, for instance, a small perturbation of a locally constant mosaic μ_0 "in general position".

Next let us fix a factorization (1.1) of p. By (5.1), we may assume that F, F^{-1}, G, G^{-1} are in $L^\infty(\gamma; M)$ and that $\mathcal{E}$ is projection-valued. We will use the Smirnov classes $E^2(\Omega_j)$; we refer to [6], [17] for their definition. The corresponding classes $E^2(\Omega_j; M)$ of M-valued functions are defined componentwise.

It is easy to see that

$$\begin{aligned} \mathcal{K}F\mathcal{E}L^2(e) = \big\{u \in \oplus_j E^2(\Omega_j; M) :&\\ u(\infty) = 0 \text{ and } u_i(\cdot) - u_e(\cdot) \in p(\cdot)M&\\ \text{almost everywhere on } \gamma\big\}&\,. \end{aligned}$$

Let $u \in \mathcal{K}F\mathcal{E}L^2(e)$, then $u_i - u_e = p_1 f$ for some $f \in L^2(d\nu; M)$. Therefore $\mu\, u \in \oplus_j E^2(\Omega_j; M)$, and we have

$$\mu_i\, u_i - \mu_e\, u_e = p_1\big(u_i + (1 - \mu_i) f\big)\,.$$

Hence M4) holds. It follows that all conditions M1)–M4) hold.

In this example, $E^2(\mu)$ does not depend on the choice of factorization (1.1) of p, and

$$E^2(\mu) = \big\{\{i_z u(z)\} : \; u \in \mathrm{Mod}^2(\mathcal{F})\big\},$$
$$\overline{E}^2_0(1-\mu^*) = \big\{\{i_{*,z} v(z)\} : \; v \in \overline{\mathrm{Mod}}^2_0(\mathcal{F}_\perp)\big\}\,,$$

where $\mathrm{Mod}^2(\mathcal{F})$, $\overline{\mathrm{Mod}}^2_0(\mathcal{F}_\perp)$ are, essentially, the Smirnov type model spaces from [26, Section 2] (one has to define $\overline{\mathrm{Mod}}^2_0(\mathcal{F}_\perp)$ as a space of antianalytic functions, see [27]).

We can make the following résumé. If only the family $\{H(z)\} = \{\mu(z)^* M\}$ is given, then only the dual model spaces UH, VH_* appear that correspond to the operators S on H and S^* on H_*. If the whole mosaic μ is given, then all four model spaces $E^2_0(1-\mu)$, $E^2(\mu)$, $\overline{E}^2(\mu^*)$, $\overline{E}^2_0(1-\mu^*)$ appear. They serve to model spaces H', H, H'_*, H_*.

6. Xia's model.

Here we show how to specialize the constructions of sections 2 and 3 in order to obtain Xia's results.

6.1. The discriminant surface.

For the reader's convenience, we remind the definitions and results from [25] we will need. Some of the notions discussed are reflected on the Figure.

Let $S : H \longrightarrow H$ be a subnormal operator of finite type and $N : K \longrightarrow K$ its minimal normal extension. Define M, C, Λ by (0.1) [11]. Let Δ be the discriminant curve (0.2), $\widehat{\Delta}$ its blow-up [11] and $\widehat{\Delta} = \widehat{\Delta}_1 \cup \cdots \cup \widehat{\Delta}_T$ the decomposition of $\widehat{\Delta}$ into irreducible components. In order to reflect the multiplicities of $\widehat{\Delta}_j$, sometimes we will write $\widehat{\Delta} = \widehat{\Delta}_1^{k_1} \cup \cdots \cup \widehat{\Delta}_T^{k_T}$ [25]. Each of $\widehat{\Delta}_j$ is a compact Riemann surface, and we consider them as branched coverings of the z-plane.

Let Δ_0 be the set of regular points of Δ. Then $\Delta \setminus \Delta_0$ is finite, and $\widehat{\Delta}$ is obtained from Δ_0 by adding a finite number of points. There is a natural projection of $\widehat{\Delta}$ onto Δ, which is identical on Δ_0. The point in $\widehat{\mathbb{C}}^2$ that corresponds to a point $\delta \in \widehat{\Delta}$ will be denoted as $\big(z(\delta), w(\delta)\big)$, and we write $\delta \sim \big(z(\delta), w(\delta)\big)$.

Let

$$\sigma_C(\Lambda) = \bigl\{z \in \mathbb{C} : \ \det\bigl(C - (w - \Lambda^*)\,(z - \Lambda)\bigr) = 0\,, \ \text{for all } w \in \mathbb{C}\bigr\}\,.$$

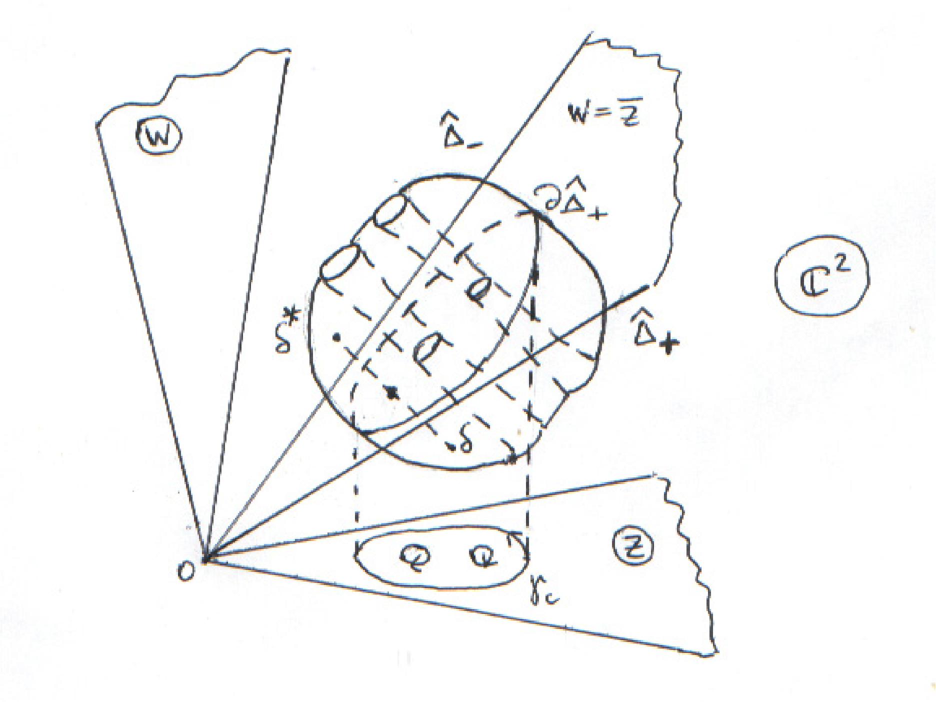

Figure.

It is immediate that $\sigma_C(\Lambda) \subset \sigma(\Lambda)$. For any $z_0 \in \sigma_C(\Lambda)$, $\widehat{\Delta}$ has irreducible components $z \equiv z_0$ and $w \equiv \overline{z}_0$. These irreducible components of $\widehat{\Delta}$ were called *degenerate* and all other irreducible components were called *nondegenerate*. Let $\widehat{\Delta}_{\mathrm{deg}}$ be the union of degenerate components of $\widehat{\Delta}$ and $\widehat{\Delta}_{\mathrm{ndeg}}$ the union of nondegenerate components.

Let

$$\widehat{\Delta}_+ = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} : \ |\eta| < 1\}\,, \qquad \widehat{\Delta}_- = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} : \ |\eta| > 1\}\,,$$

where η is a meromorphic function on $\widehat{\Delta}_{\mathrm{ndeg}}$, defined by $\eta = -dz/dw$. The map $\delta = (z, w) \longmapsto \delta^* = (\overline{w}, \overline{z})$ is an antianalytic involution on Δ_0, and it naturally extends to $\widehat{\Delta}$. It interchanges $\widehat{\Delta}_+$ and $\widehat{\Delta}_-$, because $\eta(\delta^*) = \overline{\eta(\delta)}^{\,-1}$, $\delta \in \widehat{\Delta}_{\mathrm{ndeg}}$.

The curve $\widehat{\Delta}$ was called *separated* if the set $\widehat{\Delta} \cap \{(z, \overline{z}) : z \in \mathbb{C}\}$ divides each nondegenerate component $\widehat{\Delta}_k$ into two connected components; then these two connected components are $\widehat{\Delta}_k \cap \widehat{\Delta}_+$ and $\widehat{\Delta}_k \cap$

$\widehat{\Delta}_-$. In this case, $\partial\widehat{\Delta}_+ = \{\delta \in \widehat{\Delta}\colon \delta = \delta^*\}$ (we refer to [25] for more details).

Let

$$\widehat{\Delta}' = \widehat{\Delta}_{\mathrm{ndeg}} \cup \bigcup_{\lambda \in \sigma_C(\Lambda)} \{(z,w) : \; w \equiv \bar{\lambda}\}$$

be the algebraic curve obtained from $\widehat{\Delta}$ by excluding from it the "vertical" components $z \equiv \lambda$. A projection-valued meromorphic function $\delta \longmapsto Q(\delta)$ on $\widehat{\Delta}'$ is defined by

$$Q\big((z,w)\big) = -\frac{1}{2\pi i}\int_{\partial\mathcal{D}_w} \big(C\,(z-\Lambda)^{-1} + \Lambda^* - u\big)^{-1}\,du\,, \qquad z \notin \sigma(\Lambda)\,,$$

where $\mathcal{D}_w$ is a small disc centered in w such that $\mathcal{D}_w \cap \sigma\big(C(z-\Lambda)^{-1} + \Lambda^*\big) = \{w\}$. We have

$$\begin{aligned} &Q(\delta_1)\,Q(\delta_2) = 0\,, \qquad && \delta_1 \neq \delta_2,\ z(\delta_1) = z(\delta_2)\,, \\ &\sum_{z(\delta)=\zeta} Q(\delta) = I\,, \qquad && \text{for all } \zeta \in \mathbb{C}\setminus\sigma(\Lambda)\,. \end{aligned} \tag{6.1}$$

Put

$$\gamma_c = z(\partial\widehat{\Delta}_+) \subset \mathbb{C}\,, \qquad \gamma = \{z \in \mathbb{C} : \; (z,\overline{z}) \in \Delta\}\,.$$

Then γ_c is a union of analytic arcs and $\gamma \setminus \gamma_c$ is finite. It follows from [21]–[23] that

$$\gamma_c \subset \sigma(N) \subset \gamma\,.$$

Orient the curve γ_c according to the positive orientation of the boundary $\partial\widehat{\Delta}_+$ of $\widehat{\Delta}_+$. One can define a function ξ on γ_c (except for a finite number of points) so that $dz = i\,\xi(z)\,|dz|$ and $|\xi| \equiv 1$ almost everywhere on γ_c. Then $\eta\big((z,\overline{z})\big) \equiv \xi(z)^2$ almost everywhere on γ_c.

Denote by δ_ζ the delta-measure at a point ζ of $\mathbb{C}$. The following theorem collects some of the results of [22], [25].

Theorem A. *Let (C,Λ) correspond to a subnormal operator S of finite type. Then $\widehat{\Delta}$ is separated and all Jordan blocks of $C\,(z-\Lambda)^{-1}+\Lambda^*$ that correspond to eigenvalues w such that $(z,w) \in \widehat{\Delta}_{\mathrm{ndeg}}$ are trivial for all but a finite number of values of z. The involution $\delta \longmapsto \delta^*$ maps each of the nondegenerate components of $\widehat{\Delta}$ onto itself. The mosaic* (2.6) *has a representation*

$$\mu(z) = \sum_{(z,w)\in\widehat{\Delta}_+} Q\big((z,w)\big)\,. \tag{6.2}$$

Moreover, $\operatorname{rank} Q(\delta) = k_j$, $\delta \in \widehat{\Delta}_j \cap \Delta_0$.

There is a finite subset R *of* $\mathbb{C}$ *and matrices* A_s, $s \in R$ *such that* $de(\cdot) = \mathcal{E}(\cdot)\, d\nu(\cdot)$, *where* $d\nu = |dz|\big|_{\gamma_c} + \sum_{\zeta \in R} \delta_\zeta$ *and*

$$\mathcal{E}(s) = \begin{cases} \dfrac{1}{2\pi}\, \xi(s)\,(s-\Lambda)^{-1} Q\big((s,\bar{s})\big)\,, & s \in \gamma_c \setminus R\,, \\ A_s\,, & s \in R\,. \end{cases} \tag{6.3}$$

6.2. Xia's model as a particular case of mosaic model spaces.

Let S, N be as in Section 6.1, and put $H' = H^\perp$, $M = [S^*, S]H$. It is known that $S' = N^*|H'$ also is a subnormal operator of finite type. Let $E(\cdot)$ be the spectral measure of N, and define e by $e(\cdot) = P_M E(\cdot)|M$. In [21], Xia considers a unitary operator

$$W : K \longrightarrow L^2(e)\,,$$

given by

$$W f(N)\, m = f(\cdot)\, m\,, \qquad m \in M\,, \tag{6.4}$$

where f is any bounded Borel function on $\sigma(N)$. In our terminology, we get the situation of sections 2 and 3, where now $H' \perp H$ and $W^{*-1} = W$ satisfies both (3.1), (3.2).

Lemma 6.1. (K, N, H', H, M) *is a scattering tuple.*

PROOF. Properties Sc1) and Sc4) are obvious. It is easy to see that $[S^*, S] = A^*A$, where A is defined by (2.3). Therefore $P_H N H' = [S^*, S]H$, which gives (Sc2). It follows, for instance, from [15, Chapter 2, Theorem 1.3] that $\overline{\operatorname{span}}\{S^n M \colon n \geqslant 0\} = H$. This implies the first part of Sc3). The second part of Sc3) is proved in [23, Lemma 1].

Since

$$W A^* m = W(N^* - S^*)\, m = (\bar{\cdot} - \Lambda^*)\, m\,, \qquad m \in M\,,$$

we see that (3.3) holds if we put

$$F(z) = z - \Lambda\,, \qquad G(z) \equiv I\,. \tag{6.5}$$

So now the model spaces are $\mathcal{K}(\cdot - \Lambda)\mathcal{E}L^2(e)$ and $\overline{\mathcal{K}}\mathcal{E}L^2(e)$. We arrive at the following result, which, besides the representation (6.2) of μ, is essentially contained in Xia's works [21] and [23].

Theorem 6.2. *Let S be any subnormal operator of finite type. Define F, G by* (6.5)*, μ, $\mathcal{E}$, ν from Theorem* A *and W by* (6.4)*. Then W is unitary, and all conclusions of Theorem* 3.1 *hold. In particular, μ is alternatively given by*

$$\mu(z) = \int \frac{(t-\Lambda)\, de(t)}{t-z}\,. \tag{6.6}$$

Now we have $H'_* = H'$, $H_* = H$, so that diagram (3.9) acquires the form

(6.7)

$$\begin{tikzcd}
& \mathcal{K}(\cdot-\Lambda)\mathcal{E}L^2(e) \arrow[dl, leftrightarrow] \arrow[r, equal] & E_0^2(1-\mu) \arrow[r, phantom, "\dotplus"] & E^2(\mu) \\
L^2(e) & K \arrow[l, "W"'] \arrow[u, "\widetilde{U}"'] \arrow[d, "\widetilde{V}"] \arrow[r, equal] & H' \arrow[u, "\widetilde{U}"'] \arrow[d, "\widetilde{V}"] \arrow[r, phantom, "\dotplus"] & H \arrow[u, "\widetilde{U}"'] \arrow[d, "\widetilde{V}"] \\
& \overline{\mathcal{K}}\mathcal{E}L^2(e) \arrow[ul, leftrightarrow] \arrow[r, equal] & \overline{E}^2(\mu^*) \arrow[r, phantom, "\dotplus"] & \overline{E}_0^2(1-\mu^*)
\end{tikzcd}$$

Xia calls the function $\widetilde{U}x(z)$ the analytic representation of a vector $x \in K$ and the function $\widetilde{V}x(\overline{z})$ the dual analytic representation of x.

Theorem 6.2 gives an explicit construction of a finite type subnormal operator from matrices C, Λ. The set of possible pairs (C, Λ) has been completely described in [25]. If a pair (C, Λ) satisfies the criterium that was given there, define $\mathcal{E}$, de and μ from Theorem A and F, G from (6.5). Then S will be unitarily equivalent to the operator M_z on $E^2(\mu)$.

Looking at diagram (6.7), one notices an interesting phenomenon. The operator $j : \mathcal{K}(\cdot - \Lambda)\mathcal{E}L^2(e) \longrightarrow \overline{\mathcal{K}}\mathcal{E}L^2(e)$, given by

$$j\mathcal{K}(\cdot - \Lambda)h \overset{\text{def}}{=} \overline{\mathcal{K}}h\,, \qquad h \in \mathcal{E}L^2(e)\,, \tag{6.8}$$

is obviously an isometric isomorphism. One sees from diagram (6.7) that

$$j = \widetilde{V}\,\widetilde{U}^{-1}\,.$$

Therefore the multiplication operator

$$h \longmapsto (\cdot - \Lambda)\, h$$

maps isomorphically $\bar{\mathcal{K}}^{-1}\overline{E}^2(\mu^*)$ onto $\mathcal{K}^{-1}E_0^2(1-\mu)$ and $\bar{\mathcal{K}}^{-1}\overline{E}_0^2(1-\mu^*)$ onto $\mathcal{K}^{-1}E^2(\mu)$. So the multiplication by a linear analytic matrix binomial maps certain spaces of boundary values of antianalytic functions onto spaces of boundary values of analytic functions.

The subsequent exposition is organized as follows. In Section 7, we replace the model operator M_z on $E^2(\mu)$ (see Theorem 3.1) with the operator of multiplication by $z(\cdot)$ on a function space $H^2_\mu(X_+)$, which consists of analytic cross-sections of a bundle X_+ over $\widehat{\Delta}_+$. In Section 8, we relate the above-described phenomenon with the involution on Δ. In Section 9, we will give a necessary background on weighted Hardy classes over Riemann surfaces. These facts will permit us to give a complete characterization of $H^2_\mu(X_+)$ and to prove main structure results.

7. External Riemann surface models.

Put

$$X(\delta) = \operatorname{Range} Q(\delta)\,, \qquad Y(\delta) = \operatorname{Range} Q^*(\delta)\,, \qquad \delta \in \widehat{\Delta}'\,.$$

For any component $\widehat{\Delta}_j$ of $\widehat{\Delta}'$, $X|\widehat{\Delta}_j$ ($Y|\widehat{\Delta}_j$) can be considered as an analytic (antianalytic) vector subbundle of dimension k_j of the trivial bundle $\widehat{\Delta}_j \times M$, where k_j is the multiplicity of $\widehat{\Delta}_j$. This can be deduced from the following simple fact.

Proposition 7.1. *Let Ω be a domain in $\mathbb{C}$, $\lambda_0 \in \Omega$, $k \in \mathbb{N}$, and $r_1, \dots, r_k : \Omega \to M$ be analytic functions such that $r_1(\lambda), \dots, r_k(\lambda)$ are linearly independent for some $\lambda \in \Omega$. Then there exist analytic functions $q_1, \dots, q_k$, defined in some disk $\mathcal{D}$, with $\lambda_0 \in \mathcal{D} \subset \Omega$, such that $\overline{\operatorname{span}}\{r_1(\lambda), \dots, r_k(\lambda)\} = \overline{\operatorname{span}}\{q_1(\lambda), \dots, q_k(\lambda)\}$ for $\lambda \in \mathcal{D} \setminus \{\lambda_0\}$ and $q_1(\lambda), \dots, q_k(\lambda)$ are linearly independent for all $\lambda \in \mathcal{D}$.*

PROOF. The family $\{r_j\}$ can be transformed into the family $\{q_j\}$ by taking linear combinations with constant coefficients and dividing several times by $\lambda - \lambda_0$. We omit the details.

We put $\widehat{\Delta}'_- = \widehat{\Delta}_- \cup \bigcup_{\lambda\in\sigma_C(\Lambda)}\{\delta\colon w(\delta)=\bar\lambda\}$, then $\widehat{\Delta}'$ decomposes into a disjoint union

$$\widehat{\Delta}' = \widehat{\Delta}_+ \cup \widehat{\Delta}'_- \cup \partial\widehat{\Delta}_+ \,.$$

Let

$$X_+ = X|\widehat{\Delta}_+ \,, \qquad Y_+ = Y|\widehat{\Delta}_+ \,, \qquad X_- = X|\widehat{\Delta}'_- \,, \qquad Y_- = Y|\widehat{\Delta}'_- \,.$$

Set

$$\begin{aligned} a\,u(\delta) &= Q(\delta)\,u(z(\delta))\,, \qquad u\in\mathcal{K}(\cdot-\Lambda)\mathcal{E}L^2(e)\,,\\ b\,v(\delta) &= Q^*(\delta)\,v(z(\delta))\,, \qquad v\in\overline{\mathcal{K}}\mathcal{E}L^2(e)\,, \end{aligned}$$

$(\delta\in\widehat{\Delta}')$, and

$$\begin{aligned} H^2_\mu(X_+) &= a\,E^2(\mu)\,, \qquad & H^2_\mu(X_-) &= a\,E^2_0(1-\mu)\,,\\ \overline{H}^2_{\mu^*}(Y_+) &= b\,\overline{E}^2(\mu^*)\,, \qquad & \overline{H}^2_{\mu^*}(Y_-) &= b\,\overline{E}^2_0(1-\mu^*)\,. \end{aligned}$$

It follows from (6.2) that functions in $H^2_\mu(X_+)$, $\overline{H}^2_{\mu^*}(Y_+)$ vanish on $\widehat{\Delta}'_-$ and functions in $H^2_\mu(X_-)$, $\overline{H}^2_{\mu^*}(Y_-)$ vanish on $\widehat{\Delta}_+$. We have

$$\begin{aligned} a\,\mathcal{K}(\cdot-\Lambda)\mathcal{E}L^2(e) &= H^2_\mu(X_-)+H^2_\mu(X_+)\,,\\ b\,\overline{\mathcal{K}}\mathcal{E}L^2(e) &= \overline{H}^2_{\mu^*}(Y_-)+\overline{H}^2_{\mu^*}(Y_+)\,. \end{aligned}$$

We will need notation for several exceptional sets. Let Pol_Q be the set of all poles of Q on $\widehat{\Delta}'$ and $\tilde\tau$ the maximum of orders of these poles. Let

$$B = z\,(\mathrm{Pol}_Q)\cup z\,(\widehat{\Delta}'\setminus\Delta_0)\cup R\,,$$

where R is the set from (6.3), and put $B^\# = z^{-1}(B)\cap\widehat{\Delta}'$. The sets Pol_Q, B and $B^\#$ are finite.

Lemma 7.2. *Each function in* $a\,\mathcal{K}(\cdot-\Lambda)\mathcal{E}L^2(e)$ $(b\,\overline{\mathcal{K}}\mathcal{E}L^2(e))$ *is analytic* (*antianalytic*) *on* $\widehat{\Delta}\setminus(\partial\widehat{\Delta}_+\cup\mathrm{Pol}_Q)$ *and has poles in points of* $\mathrm{Pol}_Q\setminus\partial\widehat{\Delta}_+$ *at most of order* $\tilde\tau$*.*

A priori, functions in $a\,\mathcal{K}(\cdot - \Lambda)\mathcal{E}L^2(e)$ and $b\,\overline{\mathcal{K}}\mathcal{E}L^2(e)$ may have jumps on the whole preimage curve $z^{-1}(\gamma_c)$. This lemma shows that they have jumps only on $\partial\widehat{\Delta}_+$. Hence the model spaces $H^2_\mu(X_+)$, $H^2_\mu(X_-)$ consist of meromorphic cross-sections of X_+, X_-, respectively, and $\overline{H}^2_{\mu^*}(Y_+)$, $\overline{H}^2_{\mu^*}(Y_-)$ consist of conjugate meromorphic cross-sections of Y_+, Y_-. Lemma 7.2 will be proved at the end of this section.

Operations a and b are invertible. By (6.1), the inverses are given by

$$\big(a^{-1}f\big)(\zeta) = \sum_{z(\delta)=\zeta} f(\delta)\,, \qquad \big(b^{-1}g\big)(\zeta) = \sum_{z(\delta)=\zeta} g(\delta)\,, \qquad \zeta \in \mathbb{C}\setminus\gamma\,.$$

Define Hilbert norms on $a\,\mathcal{K}(\cdot - \Lambda)\mathcal{E}L^2(e)$ and $b\,\overline{\mathcal{K}}\mathcal{E}L^2(e)$ so that a, b become unitary operators. We arrive at the following fact.

Proposition 7.3. *S is unitarily equivalent to the operator of multiplication by $z(\cdot)$ on $H^2_\mu(X_+)$, and S' is unitarily equivalent to the operator of multiplication by $\overline{z}(\cdot)$ on $\overline{H}^2_{\mu^*}(Y_+)$.*

We call these representations *the external Riemann surface representations of S, S'.* By (6.1), projections P_μ and $I - P^*_\mu$ in these representations are expressed as

$$a\,P_\mu a^{-1} = \big(M_{\chi_+} \text{ on } a\,\mathcal{K}(\cdot - \Lambda)\mathcal{E}L^2(e)\big),$$
$$b\,(I - P_\mu)\,b^{-1} = \big(M_{\chi_+} \text{ on } b\,\overline{\mathcal{K}}\mathcal{E}L^2(e)\big)\,,$$

where $\chi_+ = 1$ on $\widehat{\Delta}_+$ and $\chi_+ = 0$ on $\widehat{\Delta}'_-$.

PROOF OF LEMMA 7.2. For any domain $\mathcal{W}$ in $\mathbb{C}$, bounded by a piecewise smooth Jordan curve, the Smirnov class $E^2(\mathcal{W})$ has the following properties [6], [17]:

1)
$$f(\cdot) = \frac{1}{2\pi i}\int g(z)\,((\cdot) - z)^{-1}\,dz$$
is in $E^2(\mathcal{W})$ for every $g \in L^2(\partial\mathcal{W}, |dz|)$.

2) Each $f \in E^2(\mathcal{W})$ has boundary values almost everywhere on $\partial\mathcal{W}$, and
$$\frac{1}{2\pi i}\int f(z)\,(z - w)^{-1}\,dz$$

gives $f(w)$ for $w \in \mathcal{W}$ and 0 for $w \in \mathbb{C} \setminus \operatorname{clos} \mathcal{W}$.

3) For any smooth arc $\gamma \subset \mathcal{W}$, the map $f \longmapsto f|\gamma$ from $E^2(\mathcal{W})$ into $L^2(\gamma, |dz|)$ is bounded.

Let $f \in L^2(e)$ and $g = \mathcal{K}(\cdot - \Lambda)\, \mathcal{E} f$, $u = a\, g$. By (6.3) and the Privalov–Plemelj "jump" formula [17],

$$u_i(\delta) - u_e(\delta) = Q(\delta)\, Q\big((z, \bar{z})\big) f(z)\,, \qquad \delta = (z, w) \in z^{-1}(\gamma_c)\,.$$

Bearing in mind (6.1), we see that $u_i = u_e$ almost everywhere on $z^{-1}(\gamma_c) \setminus \partial\widehat{\Delta}_+$. Note that $\mathcal{E}^{1/2} f|\gamma_c \setminus B$ is in $L^2(|dz|\big|\gamma_c, M)$ and $(\cdot - \Lambda)\, \mathcal{E}^{1/2}$ is bounded on $\gamma_c \setminus \mathcal{U}$ for any neighbourhood $\mathcal{U}$ of B. It follows that any $\delta_0 \in z^{-1}(\gamma_c) \setminus \big(\partial\widehat{\Delta}_+ \cup B^{\#}\big)$, $\delta_0 = (z_0, w_0)$, has a small neighbourhood $\mathcal{W}$ in $\widehat{\Delta}$ that projects homeomorphically onto a disc centered in z_0 such that $u \circ (z\big|\mathcal{W}')^{-1} \in E^2\big(z(\mathcal{W}')\big)$ for any connected component $\mathcal{W}'$ of $\mathcal{W} \setminus z^{-1}(\gamma_c)$. The above fact **2)** easily implies that u is a restriction of a function, analytic in a neighbourhood of δ_0. So u is analytic on $\Delta' \setminus \big(\partial\widehat{\Delta}_+ \cup B^{\#}\big)$.

Now let $\delta_0 \in B^{\#} \setminus \partial\widehat{\Delta}_+$, $\delta_0 \sim (z_0, w_0)$. Take a small neighbourhood $\mathcal{W}$ of δ_0 with analytic boundary such that $\operatorname{clos} \mathcal{W} \cap B = \{\delta_0\}$. The above proof and **3)** show that the map $u \longmapsto u|\partial\mathcal{W}$ is bounded from $a\, \mathcal{K}(\cdot - \Lambda)\mathcal{E} L^2(e)$ into $L^2(\partial\mathcal{W}, |dz|, M)$. Pick any function s, holomorphic on $\operatorname{clos} \mathcal{W}$, such that sQ is also holomorphic on $\operatorname{clos} \mathcal{W}$. We assert that functions su are analytic at δ_0 for all $u \in a\, \mathcal{K}(\cdot - \Lambda)\mathcal{E} L^2(e)$. Indeed, it suffices to check that this is true for u in a complete set. By (1.11), the functions

$$\varphi_{t,m}(z) = \frac{\mu(z)}{z - t}\, m\,,$$

where $t \in \mathbb{C} \setminus \gamma$ and $m \in M$ are complete in $\mathcal{K}(\cdot - \Lambda)\mathcal{E} L^2(e)$. Therefore the functions $u(\delta) = Q(\delta)\big(z(\delta) - t\big)^{-1} m$ are complete in $a\, \mathcal{K}(\cdot - \Lambda)\mathcal{E} L^2(e)$. For every such u, $s\, u$ is analytic at δ_0.

It follows that the statement of Lemma is true for $a\, \mathcal{K}(\cdot - \Lambda)\mathcal{E} L^2(e)$. The proof for $b\, \overline{\mathcal{K}} \mathcal{E} L^2(e)$ is similar.

8. The role of the symmetry on $\widehat{\Delta}$.

Lemma 8.1 *The identity*

$$(8.1) \qquad (z - \Lambda)^{-1} Q(\delta) = \eta(\delta)^{-1} Q^*(\delta^*)\, (w - \Lambda^*)^{-1}\,, \qquad \delta = (z, w)$$

holds on $\widehat{\Delta}_{\rm ndeg}$.

PROOF. We use arguments similar to those of Xia [22, p. 895]. Let $\delta_0 = (z_0, w_0) \in \widehat{\Delta}_{\rm ndeg} \cap \Delta_0$ and $\mathcal{V}$ be a small neighbourhood of δ_0 in $\mathbb{C}^2$. We assume that $\mathcal{V} \cap \Delta$ is given by equations $w = \alpha(z)$, $z = \beta(w)$, where α, β are analytic functions. Then the equation of $\{\delta^* : \delta \in \mathcal{V} \cap \Delta\}$ is $w = \overline{\beta}(\overline{z})$. Since all Jordan blocks of $C(z-\Lambda)^{-1} + \Lambda^*$ that correspond to eigenvalue $\alpha(z)$ are trivial by Theorem A, analytic perturbation theory [4] gives

$$\bigl(w-\alpha(z)\bigr)\bigl(C-(w-\Lambda^*)\,(z-\Lambda)\bigr)^{-1}\big|_{(z,w)\in\mathcal{V}\cap\Delta} = -(z-\Lambda)^{-1} Q\bigl((z,\alpha(z))\bigr)\,,$$

the matrix on the left hand side being in fact analytic in $\mathcal{V}$. Let us apply this equality to a neighbourhood of δ_0^*, pass to the adjoints and then substitute $w \longmapsto \overline{z}$, $z \longmapsto \overline{w}$. We obtain that

$$\begin{aligned}\bigl(z-\beta(w)\bigr)\bigl(C-(w-\Lambda^*)\,(z-\Lambda)\bigr)^{-1}\big|_{(z,w)\in\mathcal{V}\cap\Delta}& \\ = -Q^*\bigl((\overline{w},\beta(\overline{w}))\bigr)(w-\Lambda^*)^{-1}\,.&\end{aligned}$$

It remains only to remark that $\alpha(z_0) = w_0$, $\beta(w_0) = z_0$, and

$$\lim_{\substack{(z,w)\in\mathbb{C}^2\setminus\Delta \\ (z,w)\to\delta_0}} \frac{w-\alpha(z)}{z-\beta(w)} = -\left.\frac{dw}{dz}\right|_{\delta_0} = \eta(\delta_0)^{-1}\,.$$

Let $v \in \overline{H}^2_{\mu^*}(Y_-)$ and $u \in H^2_\mu(X_-)$. We define symmetries α_1, α_2 by

$$(\alpha_1 v)(\delta) = \eta(\delta)^{-1}(z(\delta)-\Lambda)\,v(\delta^*)\,, \tag{8.2}$$

$$(\alpha_2 u)(\delta) = \overline{\eta}(\delta)^{-1}(\overline{w}(\delta)-\Lambda)\,u(\delta^*)\,, \tag{8.3}$$

for $\delta \in \widehat{\Delta}_+$ (notice that then $\delta^* \in \widehat{\Delta}_- \subset \widehat{\Delta}'$). By (8.1) and Lemma 7.2, $\alpha_1 v$ is a meromorphic cross-section of X_+ and $\alpha_2 u$ is a conjugate meromorphic cross-section of Y_+.

Theorem 8.2. *Let S be a subnormal operator of finite type. Define F, G by* (6.5). *Let μ, $\mathcal{E}$, ν, e be as in Theorem* 6.2 *and j be defined by*

(6.8). *Then the diagram*

$$
\begin{tikzcd}
H^2_\mu(X_+) \arrow[dd, leftarrow, "\alpha_1"'] & E^2(\mu) \arrow[l, "a"'] \arrow[dd, "j"] \arrow[dr] & & E_0^2(1-\mu) \arrow[r, "a"] \arrow[dd, "j"] \arrow[dl] & H^2_\mu(X_-) \arrow[dd, "\alpha_2"] \\
& & L^2(e) & & \\
\overline{H}^2_{\mu^*}(Y_-) & \overline{E}^2_0(1-\mu^*) \arrow[l, "b"] \arrow[ur] & & \overline{E}^2(\mu^*) \arrow[r, "b"'] \arrow[ul] & \overline{H}^2_{\mu^*}(Y_+)
\end{tikzcd}
$$

is commutative (*the embeddings of the mosaic model spaces into* $L^2(e)$ *are defined by diagram* (6.7)). *In partucular,* j *maps isometrically* $E^2(\mu)$ *onto* $\overline{E}^2_0(1-\mu^*)$ *and* $E^2_0(1-\mu)$ *onto* $\overline{E}^2(\mu^*)$. *Spaces* $E^2(\mu)$ *and* $\overline{E}^2_0(1-\mu^*)$ *coincide as subspaces of* $L^2(e)$. *Spaces* $E^2_0(1-\mu)$ *and* $\overline{E}^2(\mu^*)$ *also coincide as subspaces of* $L^2(e)$.

The symmetries α_1, α_2 also explain the existence of the formulas for action of S and S^* in both model spaces $E^2(\mu)$, $\overline{E}^2_0(1-\mu^*)$ (see [22, Theorems 2 and 3]).

PROOF. Let $\delta \in \widehat{\Delta}_+$, $\delta \sim (z,w)$, $m \in M$, and $\lambda \in \mathbb{C}$, $(z,\overline{\lambda}) \notin \Delta$. Since $\widehat{\Delta}_+ \subset \widehat{\Delta}_{\mathrm{ndeg}}$, the assertion about Jordan blocks in Theorem A implies that
$$\big(C(z-\Lambda)^{-1} + \Lambda^* - \overline{\lambda}\big)Q(\delta) = Q(\delta)\,(w-\overline{\lambda})\,,$$
and together with (8.1) and the formula $\eta(\delta^*) = \overline{\eta(\delta)}^{\,-1}$ this gives

$$
\begin{aligned}
&Q^*(\delta)\big(C-(\overline{z}-\Lambda^*)\,(\lambda-\Lambda)\big)^{-1}m \\
&\qquad\qquad = (\overline{w}-\lambda)^{-1}Q^*(\delta)\,(\overline{z}-\Lambda^*)^{-1}m \\
&\qquad\qquad = (\overline{w}-\lambda)^{-1}\eta(\delta^*)\,(\overline{w}-\Lambda)^{-1}Q(\delta^*)\,m\,.
\end{aligned}
\tag{8.4}
$$

By the results of Section 1, the elements $\varphi_{\lambda,m}$ span $\mathcal{K}(\cdot-\Lambda)\mathcal{E}L^2(e)$. Since $\varphi_{\lambda,m} = \mathcal{K}(\cdot-\Lambda)\,(\cdot-\lambda)^{-1}m$,

$$j\,\varphi_{\lambda,m}(z) = \int \frac{de(t)}{(t-\lambda)\,(\overline{t}-\overline{z})}\,m\,. \tag{8.5}$$

We will make use of Xia's formula

$$
\begin{aligned}
\int \frac{de(t)}{(t-\lambda)\,(\overline{t}-\overline{z})} &= \mu(z)^*\big(C-(\overline{z}-\Lambda^*)\,(\lambda-\Lambda)\big)^{-1} \\
&\qquad - \big(C-(\overline{z}-\Lambda^*)\,(\lambda-\Lambda)\big)^{-1}(1-\mu(\lambda))\,,
\end{aligned}
\tag{8.6}
$$

which is valid whenever λ, $z \notin \sigma(N)$ and $(\lambda, \overline{z}) \notin \Delta$ (see [22]). Consider two subfamilies of the family of functions $\{\varphi_{\lambda,m}\}$.

1) Let $\lambda \notin \sigma(N)$ and $m \in (1-\mu(\lambda))M$. Then $\varphi_{\lambda,m} = (z-\lambda)^{-1}\mu(z)\, m$ by (1.7), and by (8.5), (8.6),

$$j\,\varphi_{\lambda,m}(z) = -(1-\mu(z)^*)\big(C - (\overline{z}-\Lambda^*)\,(\lambda-\Lambda)\big)^{-1} m\,.$$

Since $j\,\varphi_{\lambda,m} \in \overline{\mathcal{K}}\mathcal{E}L^2(e)$, it follows that $j\,\varphi_{\lambda,m} \in \overline{E}_0^2(1-\mu^*)$. Moreover, by (6.2), $Q^*(\delta)\big(1-\mu(z(\delta))^*\big) = Q^*(\delta)$ for $\delta = (z,w) \in \widehat{\Delta}_+$. By (8.4),

$$\begin{aligned}\big(b\,j\,\varphi_{\lambda,m}\big)(\delta) &= Q^*(\delta)\big(C - (\overline{z}(\delta)-\Lambda^*)\,(\lambda-\Lambda)\big)^{-1} m \\ &= (\overline{w}-\lambda)^{-1}\eta(\delta^*)\,(\overline{w}-\Lambda)^{-1}Q(\delta^*)\, m\,,\end{aligned}$$

so that

$$\big(\alpha_1 b\,j\,\varphi_{\lambda,m}\big)(\delta) = (z-\lambda)^{-1}Q(\delta)\, m = \big(a\,\varphi_{\lambda,m}\big)(\delta)\,, \qquad \delta \in \widehat{\Delta}_+\,.$$

Since the family of functions $\varphi_{\lambda,m}$ we are considering is complete in $E^2(\mu)$, we conclude that $jE^2(\mu) \subset \overline{E}_0^2(1-\mu^*)$ and that the left rectangle in the diagram is commutative.

2) Let $\lambda \notin \sigma(N)$ and $m \in \mu(\lambda)M$, $\varphi_{\lambda,m} = -(z-\lambda)^{-1}\big(1-\mu(z)\big)m$. Then

$$j\,\varphi_{\lambda,m}(z) = \mu(z)^*\big(C - (\overline{z}-\Lambda^*)\,(\lambda-\Lambda)\big)^{-1} m\,,$$

so that $j\,\varphi_{\lambda,m} \in \overline{E}^2(\mu^*)$. Now we obtain from (8.4) that for $\delta \in \widehat{\Delta}_+$, $\delta \sim (z,w)$,

$$\begin{aligned}\big(b\,j\,\varphi_{\lambda,m}(\delta)\big) &= Q^*(\delta)\big(C - (\overline{z}-\Lambda^*)\,(\lambda-\Lambda)\big)^{-1} m \\ &= (\overline{w}-\lambda)^{-1}\eta(\delta^*)\,(\overline{w}-\Lambda)^{-1}Q(\delta^*)\, m \\ &= \alpha_2\, a\,\varphi_{\lambda,m}(\delta)\,.\end{aligned}$$

This proves that $jE_0^2(1-\mu) \subset \overline{E}^2(\mu^*)$ and that the right rectangle in the diagram is commutative.

At last, note that $\mathcal{K}(\cdot-\Lambda)\mathcal{E}L^2(e) = E^2(\mu) \oplus E_0^2(1-\mu)$ implies $\overline{\mathcal{K}}\mathcal{E}L^2(e) = jE^2(\mu) \oplus jE_0^2(1-\mu)$. Since $jE^2(\mu) \subset \overline{E}_0^2(1-\mu^*)$ and $jE_0^2(1-\mu) \subset \overline{E}^2(\mu^*)$, we have $jE^2(\mu) = \overline{E}_0^2(1-\mu^*)$ and $jE_0^2(1-\mu) = \overline{E}^2(\mu^*)$.

REMARKS. Suppose we start from a pair (C, Λ) that satisfies the criterium of [25, Theorem 2] and our aim is to construct the corresponding S. Then (6.2), (6.6) define the same function μ, and model spaces $E^2(\mu)$, $\overline{E}^2(\mu^*)$, etc. arise. The proof of (8.6) in [21] uses only (6.6) and the formula

$$\big(C - (\overline{z} - \Lambda^*)\,(z - \Lambda)\big)\, de(z) \equiv 0\,,$$

which follows from the hypotheses on C, Λ. So the fact that $E^2(\mu)$ and $\overline{E}_0^2(1-\mu^*)$ define the same subspace of $L^2(e)$ can be deduced directly from the hypotheses of [25, Theorem 2]. Therefore we can put S to be equal to M_z on $E^2(\mu)$, and we will get all its necessary properties.

It is easier to understand the sense of Theorem 8.2 when $\widehat{\Delta}$ has no degenerate components. Then formulas (8.2), (8.3) permit us to define α_1^{-1}, α_2^{-1} as well. But α_1, α_2 are isomorphisms even if there are degenerate components. In this case, we conclude from (8.2), (8.3) that the value of every function in $\overline{H}^2_{\mu^*}(Y_-)$ or $H^2_\mu(X_-)$ on degenerate components $w \equiv \text{const}$ is determined by its values on other components.

9. Analytic functional classes over Riemann surfaces.

This section has an auxiliary character. Our exposition uses the approaches of [1], [10].

Let R be a (connected) branched Riemann surface over $\mathbb{C}$, whose boundary ∂R is a finite union of analytic arcs. We assume that $\overline{R} = R \cup \partial R$ is compactly imbedded into a larger Riemann surface $\widetilde{R}$ and that $\partial R = \partial \overline{R}$. Let $\delta \longmapsto z(\delta)$ be the projection of R onto $\mathbb{C}$. We assume that $z(\overline{R})$ is a compact set in $\mathbb{C}$.

From now on, let us fix a base point $\delta_0 \in R$, and let ω be the harmonic measure for ∂R at δ_0. It is easy to see that ω and the arc length measure $|dz(\cdot)|$ are mutually absolute continuous.

The Nevanlinna class of R is defined as

$$\mathcal{N}(R) = \big\{ f \in \mathrm{Hol}(R):\ \log|f| \text{ has a harmonic majorant} \big\}\,.$$

The hypotheses on R imply that the unit disc $\mathbb{D}$ is its universal covering space. Let

$$\mathbf{T}\colon \mathbb{D} \longrightarrow R$$

be a covering map, normalized so that $\mathbf{T}(0) = \delta_0$. The boundary of R has a finite number of connected components, and the fundamental group $\pi_1(R)$ of R is finitely generated. It follows that there exists a relatively open and dense subset $\mathcal{A}$ of $\partial\,\mathbb{D}$ such that $\mathbf{T}$ extends continuously to a map from $\mathcal{A}$ onto ∂R. (see [7]). This map will be also denoted by $\mathbf{T}$. Moreover, $\omega(E) = (2\pi)^{-1} m_1(\mathbf{T}^{-1}(E))$ for each Borel subset E of ∂R; here m_1 is the arc length measure on $\partial\,\mathbb{D}$.

It is easy to see (using the techniques of [10, Section 3.4]) that $f \in \mathcal{N}(R)$ if and only if $f \circ \mathbf{T} \in \mathcal{N}(\mathbb{D})$. Since $f \in \mathcal{N}(R)$ implies $f|\mathcal{D} \in \mathcal{N}(\mathcal{D})$ for every connected subdomain $\mathcal{D}$ of R, it follows that each function f in $\mathcal{N}(R)$ has non-tangential limit values ω-almost everywhere on ∂R, and that $\mathbf{T}$ lifts these boundary values to boundary values of $f \circ \mathbf{T}$ on $\partial\,\mathbb{D}$. Let

$$\mathcal{N}^+(\mathbb{D}) = \left\{ gh^{-1} :\ g, h \in H^\infty,\ g \text{ is outer in } \mathbb{D} \right\}$$

be the Smirnov subclass of the Nevanlinna class on $\mathbb{D}$, and define the Smirnov class of R by

$$\mathcal{N}^+(R) = \left\{ f \in \mathcal{N}(R) :\ f \circ \mathbf{T} \in \mathcal{N}^+(\mathbb{D}) \right\}.$$

For each natural number k, we denote by $\mathcal{N}^+(R, \mathbb{C}^k)$ the set of $\mathbb{C}^k$-valued functions on R, whose components are in $\mathcal{N}^+(R)$.

Let Ω be a Borel measurable selfadjoint nonnegative $k \times k$ matrix function on ∂R, which is log-integrable, that is,

$$\int \log^+ \max\left\{ \|\Omega\|, \|\Omega^{-1}\| \right\} d\omega < \infty \tag{9.1}$$

(here $\log^+(\cdot) = \max\{\log(\cdot), 0\}$). Consider the weighted space

$$L^2(\partial R, \omega, \Omega)$$

with the norm given by $\|f\|^2 = \int \langle \Omega f, f \rangle \, d\omega$ and the corresponding weighted H^2-space

$$H^2(R; \Omega) = \left\{ f \in \mathcal{N}^+(R, \mathbb{C}^k) :\ f|\partial R \in L^2(\partial R, \omega, \Omega) \right\}.$$

Let $\widetilde{\Omega} = \Omega \circ \mathbf{T}$ be the matrix weight on $\partial\,\mathbb{D}$ that corresponds to Ω; then (9.1) gives

$$\int \log^+ \max\left\{ \|\widetilde{\Omega}\|, \|\widetilde{\Omega}^{-1}\| \right\} |dz| < \infty.$$

Hence there exists a matrix function $\mathcal{O}$ on $\mathbb{D}$ such that $\mathcal{O}, \mathcal{O}^{-1} \in \mathcal{N}^{+}(\mathbb{D}, \mathbb{C}^{k\times k})$ and

$$\widetilde{\Omega} = \mathcal{O}^{*}\mathcal{O} \text{ on } \partial\,\mathbb{D} \tag{9.2}$$

(see [19, Chapter 5, Section 7]). The matrix function $\mathcal{O}$ is determined uniquely, up to a constant left unitary factor.

Let Möb$\,(\mathbb{D})$ be the group of all linear fractional transformations of $\mathbb{D}$, and let G be the group of deck transformations, that is,

$$G = \left\{\varphi \in \text{Möb}\,(\mathbb{D}) :\ \mathbf{T}\circ\varphi = \mathbf{T}\right\}.$$

It is known that G is a discrete group and is isomorphic to $\pi_1(R)$. Denote by $\mathbf{U}(\mathbb{C}^k)$ the group of unitary linear transformations of $\mathbb{C}^k$. Since $\widetilde{\Omega}\circ\varphi = \widetilde{\Omega}$ for all $\varphi \in G$, one deduces from (9.2) that for every $\varphi \in G$ there exists a constant matrix $\alpha(\varphi) \in \mathbf{U}(\mathbb{C}^k)$ such that

$$\mathcal{O}\circ\varphi = \alpha(\varphi)\,\mathcal{O}\,.$$

It follows that $\alpha : G \longrightarrow \mathbf{U}(\mathbb{C}^k)$ is a group homomorphism.

Definition. *α is called the character that corresponds to the matrix weight Ω.*

Denote by $\text{Char}_k(R)$ the set of all group homomorphisms $\alpha : G \longrightarrow \mathbf{U}(\mathbb{C}^k)$.

Let α be an arbitrary element of $\text{Char}_k(R)$. If an analytic $\mathbb{C}^k$ – or $\mathbb{C}^{k\times k}$ – valued function f satisfies $f\circ\varphi = \alpha(\varphi)f$, $\varphi \in G$, then f is called *α-automorphic*. Put

$$H^2_\alpha(R) = \left\{f \in H^2(\mathbb{D}, \mathbb{C}^k) :\ f \text{ is } \alpha\text{-automorphic}\right\},$$

it is a closed subspace of $H^2(\mathbb{D}, \mathbb{C}^k)$. Informally, we interpret an element of $H^2_\alpha(R)$ as a multivalued analytic function f on R such that $|f|$ is single-valued.

Lemma 9.1. *Let Ω, $\widetilde{\Omega}$, $\mathcal{O}$ be as above, and let α be the character that corresponds to Ω. Then the map*

$$f \longmapsto \mathcal{O}\cdot(f\circ\mathbf{T})$$

defines an isometric isomorphism of $H^2(R,\Omega)$ onto $H^2_\alpha(R)$.

PROOF. Let $f \in \mathrm{Hol}(R, \mathbb{C}^k)$. Then $f \in \mathcal{N}^+(R, \mathbb{C}^k)$ if and only if $\mathcal{O} \cdot (f \circ \mathbf{T}) \in \mathcal{N}^+(\mathbb{D}, \mathbb{C}^k)$. Since for $f \in \mathcal{N}^+(R, \mathbb{C}^k)$,

$$\int_{\partial R} \langle \Omega f, f \rangle \, d\omega = \int_{\partial \mathbb{D}} \langle \widetilde{\Omega}(f \circ \mathbf{T}), (f \circ \mathbf{T}) \rangle \frac{|dz|}{2\pi} = \int_{\partial \mathbb{D}} \| \mathcal{O} \cdot (f \circ \mathbf{T}) \|^2 \frac{|dz|}{2\pi},$$

the above map is an isomorphic isomorphism of $H^2(R, \Omega)$ onto its image in $H^2(\mathbb{D}, \mathbb{C}^k)$. A function $g \in \mathcal{N}^+(\mathbb{D}, \mathbb{C}^k)$ is α-automorphic if and only if $g = \mathcal{O} \cdot (f \circ \mathbf{T})$ for some $f \in \mathcal{N}^+(R, \mathbb{C}^k)$ (here we use that the least harmonic majorant of a G-invariant subharmonic function is also G-invariant). Therefore the image of our map is exactly $H^2_\alpha(R)$.

Lemma 9.2. *The operator $M_{z(\cdot)}$ of multiplication by $z(\cdot)$ on*

$$L^2(\partial R, \omega, \Omega)$$

is the minimal normal extension of the operator of multiplication by $z(\cdot)$ on $H^2(R, \Omega)$.

It suffices to prove that

$$\overline{\operatorname{span}}_{\lambda \notin z(\partial R)} \left\{ \left(z(\cdot) - \lambda \right)^{-1} f \big| \partial R : \ f \in H^2(R, \Omega) \right\} = L^2(\partial R, \omega, \Omega) . \tag{9.3}$$

Let $\psi \in L^2(\partial R, \omega, \Omega)$ be orthogonal to all functions in the left hand part. Then, by the Hartogs–Rosenthal theorem [12],

$$\sum_{\substack{\delta \in \partial R \\ z(\delta) = \zeta}} \left\langle \Omega(\delta)\, \psi(\delta), f(\delta) \right\rangle = 0 ,$$

for almost every $\zeta \in z(\partial R)$ and all $f \in H^2(R, \Omega)$. Take any $g \in H^2(R, \Omega)$ and put here $f = \varphi\, g$, $\varphi \in H^\infty(R)$. Since for each $\zeta \in z(\partial R)$ we can choose φ which is analytic on a neighbourhood of $\overline{R}$ and has arbitrarily prescribed values in points of $z^{-1}(\zeta) \cap \partial R$, we conclude that $\psi = 0$ almost everywhere on ∂R.

Define an operator $S_{R,\alpha}$ on $H^2_\alpha(R)$ by

$$S_{R,\alpha} f = (z \circ \mathbf{T}) \cdot f .$$

It is obviously subnormal. In our interpretation of $H^2_\alpha(R)$ as a space of multivalued functions on R, $S_{R,\alpha}$ acts as multiplication by $z(\cdot)$. Sometimes we will write S_{R,α,δ_0} to show the dependence on the base point δ_0.

Functions in $H^2_\alpha(R)$, considered as multivalued functions on R, have boundary limit values on ∂R, which are also multivalued. Let us fix a Borel subset E of $\partial\mathbb{D}$ such that $\mathbf{T}|E$ is an isomorphism of E onto ∂R, and put $\mathbf{T}^{[-1]} = (\mathbf{T}|E)^{-1}$. We associate with each $f \in H^2_\alpha(R)$ the single-valued function $f \circ \mathbf{T}^{[-1]}$ on ∂R.

One has

$$\|f\|^2_{H^2_\alpha(R)} = \int_{\partial R} |f \circ \mathbf{T}^{[-1]}|^2 \, d\omega \,,$$

(we use here the norm in H^2, given by $\|x\|^2 = (2\pi)^{-1} \int |x|^2 \, |dz|$.) So the map $f \longmapsto f \circ \mathbf{T}^{[-1]}$ allows us to consider $H^2_\alpha(R)$ as embedded isometrically into $L^2(\partial R, \omega, \mathbb{C}^k)$.

With this agreement, the map $f \longmapsto \mathcal{O} \circ \mathbf{T}^{[-1]} \cdot f$ extends the map of Lemma 9.1 to an isometric isomorphism of $L^2(\partial R, \omega, \Omega)$ onto $L^2(\partial R, \omega, \mathbb{C}^k)$. Therefore, by Lemma 9.2, the operator $M_{z(\cdot)}$ of multiplication by $z(\cdot)$ on $L^2(\partial R, \omega, \mathbb{C}^k)$ is the minimal normal extension of the operator $S_{R,\alpha}$.

From now on, we assume that the Riemann surface $\widetilde{R}$, which contains $\bar{R}$, is such that the imbedding $R \subset \widetilde{R}$ induces an isomorphism between the fundamental groups $\pi_1(R)$ and $\pi_1(\widetilde{R})$.

Lemma 9.3. *For any $\alpha \in \mathrm{Char}_k(R)$, there exists an α-automorphic function $A : \mathbb{D} \longrightarrow \mathbb{C}^k$ such that $\|A\|$, $\|A^{-1}\|$ are bounded in $\mathbb{D}$.*

The proof of Lemma 9.3 will be given a little bit later.

Let $\mathbb{I}$ be the unit element of $\mathrm{Char}_k(R)$, that is, $\mathbb{I}(\varphi) = I$ for all $\varphi \in G$. Then $H^2_{\mathbb{I}}(R)$ is the set of G-invariant functions in $H^2(\mathbb{D})$; this space is naturally isometrically isomorphic to the unweighted Hardy space $H^2(R) = H^2(R, I)$. In the situation of Lemma 9.3, the map $f \longmapsto A \cdot f$, $f \in H^2_{\mathbb{I}}(R)$, defines a (not necessarily isometric) isomorphism of $H^2_{\mathbb{I}}(R)$ onto $H^2_\alpha(R)$. Since this isomorphism commutes with $M_{z(\cdot)}$, we obtain the following fact.

Corollary. *All the operators $S_{R,\alpha}$, $\alpha \in \mathrm{Char}_k(R)$, are mutually similar.*

Putting $\mathcal{O} = A$ in the construction preceding Lemma 9.1, we also see that each operator $S_{R,\alpha}$ is unitarily equivalent to an operator $M_{z(\cdot)}$ on $H^2(R,\Omega)$ for a matrix weight Ω such that $\|\Omega\|$, $\|\Omega^{-1}\|$ are uniformly bounded on R.

Definition. *Characters α, β in $\operatorname{Char}_k(R)$ are called equivalent ($\alpha \sim \beta$) if there is a constant u in $\mathbf{U}(\mathbb{C}^k)$ such that $\beta(\varphi) = u\,\alpha(\varphi)\,u^*$ for all $\varphi \in G$.*

The following statement is in a sense analogous to [1, Theorem 6].

Lemma 9.4. *Suppose that there is a point in $\mathbb{C}$ whose preimage on R consists of exactly one point. Let $\alpha, \beta \in \operatorname{Char}_k(R)$. The operators $S_{R,\alpha}$ and $S_{R,\beta}$ are unitarily equivalent if and only if $\alpha \sim \beta$.*

PROOF OF LEMMA 9.3. Let $P : \mathcal{A} \longrightarrow \widetilde{R}$ be a universal covering map for $\widetilde{R}$; we assume $\mathcal{A}$ to be a simply connected domain in $\mathbb{C}$. Put $\widetilde{\mathbb{D}} = P^{-1}(R)$, then $\widetilde{\mathbb{D}}$ is connected and simply connected and $P|\,\widetilde{\mathbb{D}}$ is the universal covering map for R. If $\varphi : \mathbb{D} \longrightarrow \widetilde{\mathbb{D}}$ is a conformal map such that $P\big(\varphi(0)\big) = \delta_0$, then we can set $\mathbf{T} = P \circ \varphi$. Let $\widetilde{G}$ be the group of deck transformations of $\widetilde{R}$. It is easy to see that the map $g \longmapsto \varphi^{-1} \circ (g|\,\widetilde{\mathbb{D}}) \circ \varphi$ is an isomorphism of $\widetilde{G}$ onto G; it gives rise to a canonical isomorphism between $\operatorname{Char}_k(R)$ and $\operatorname{Char}_k(\widetilde{R})$.

Let γ be the element of $\operatorname{Char}_k(\widetilde{R})$ that corresponds to α. There exists a $\mathbb{C}^{k\times k}$-valued γ-automorphic function Γ on $\mathcal{A}$ such that $\det \Gamma \neq 0$ in $\mathcal{A}$; this assertion in fact is a restatement of the fact that every two analytic bundles over $\widetilde{R}$ are isomorphic. This fact follows from the Grauert theorem [13]. The function $A \stackrel{\text{def}}{=} \Gamma \circ \varphi$ is α-automorphic on $\mathbb{D}$. Since the functions $\|\Gamma \circ P^{-1}\|$, $\|\Gamma^{-1} \circ P^{-1}\|$ are single-valued and continuous on $\overline{R}$, it follows that A, A^{-1} are in $H^\infty(\mathbb{D}, \mathbb{C}^{k\times k})$.

PROOF OF LEMMA 9.4. Let A, B be the matrix functions that correspond to characters α, β as in the above proof of Lemma 9.3, so that $H^2_\alpha(R) = AH^2_{\mathbb{I}}(R)$, $H^2_\beta(R) = BH^2_{\mathbb{I}}(R)$. Let $\Phi : H^2_\alpha(R) \longrightarrow H^2_\beta(R)$ be the isometric isomorphism such that $\Phi S_{R,\alpha} = S_{R,\beta}\Phi$. Define a function ρ in $H^2(\mathbb{D}, \mathbb{C}^{k\times k})$ by $\Phi(Ac) = \rho \cdot Ac$, $c \in \mathbb{C}^n$. The hypothesis implies that there exists a subdomain $\mathcal{W}$ in $\mathbb{D}$ such that each point in $(z \circ \mathbf{T})(\mathcal{W})$ has only one preimage on R. For any $\zeta \in \mathcal{W}$, if $g \in H^2_\alpha(R)$ and $g(\zeta) = 0$, then $g = \big(z \circ \mathbf{T} - (z \circ \mathbf{T})(\zeta)\big)h$ for a certain $h \in H^2_\alpha(R)$, which implies $(\Phi g)(\zeta) = 0$. It follows that $\Phi(f)|\mathcal{W} = \rho \cdot f|\mathcal{W}$ for all

$f \in H^2_\alpha(R)$. One easily deduces that

$$\Phi(f) = \rho \cdot f\,, \qquad f \in H^2_\alpha(R)\,.$$

Since $\rho \cdot A$ is β-automorphic, it follows that

$$\rho \circ \varphi = \beta(\varphi)\,\rho\,\alpha(\varphi)^{-1}\,, \qquad \varphi \in G\,. \tag{9.4}$$

The map Φ extends to a unitary equivalence of minimal normal extensions [5]. We see from Lemma 9.1 that there exists a unitary operator Ψ on $L^2(\partial R, \omega, \mathbb{C}^k)$ such that $\Psi | H^2_\alpha(R) = \Phi$ and $\Psi M_{z(\cdot)} = M_{z(\cdot)}\Psi$. Looking at the action of Ψ on functions $\big(z(\cdot) - \lambda\big)^{-1}(f \circ \mathbf{T}^{[-1]})$, $f \in H^2_\alpha(R)$ and bearing in mind (9.3), we conclude that $\Psi f = (\rho \circ \mathbf{T}^{[-1]})f$ for all $f \in L^2(\partial R, \omega, \mathbb{C}^k)$. Therefore $\rho \circ \mathbf{T}^{[-1]}$ is unitary almost everywhere on ∂R, that is, ρ is unitary almost everywhere on E. Since $\bigcup_{\varphi \in G} \varphi(E)$ has a full measure in $\partial\,\mathbb{D}$, it follows that ρ is unitary almost everywhere on $\partial\,\mathbb{D}$.

We can repeat the whole argument for the operator $\Phi^{-1} f = \rho^{-1} \cdot f$, $f \in H^2_\beta(R)$. Hence $\rho, \rho^{-1} \in H^2(\mathbb{D}, \mathbb{C}^{k \times k})$, and $\rho^* \rho \equiv I$ on $\partial\,\mathbb{D}$. Consider $\tilde\rho(\overline{z}^{\,-1}) = \rho^{*-1}(z)$, $z \in \mathbb{D}$. We conclude that $\rho(z) \equiv u$, $z \in \mathbb{D}$ for a unitary constant $u \in \mathbf{U}(\mathbb{C}^k)$. Then (9.4) gives $\beta(\varphi) = u\,\alpha(\varphi)\,u^*$, $\varphi \in G$.

The converse "if" part of the statement is obvious.

Lemma 9.5. *Suppose Ω satisfies* (9.1) *and $\int \|\Omega\|\,d\omega < \infty$. Then the set of functions on $\widetilde{R}$ that are holomorphic on $\overline{R}$ is dense in $H^2(R;\Omega)$.*

PROOF. Let $f \in H^2(R;\Omega)$ and $\varepsilon > 0$; then $f \circ \mathbf{T} \in H^2(R;\widetilde{\Omega})$, where $\widetilde{\Omega} = \Omega \circ \mathbf{T}$. We can assume that $\widetilde{R}$ is contained in the double $\widehat{R}$ of R [7]. We make use of the conditional expectation operator E [7]. It is easy to see that E maps $H^2(\mathbb{D}, \widetilde{\Omega})$ onto $H^2(R, \Omega) \dotplus N$, where N is a finite-dimensional defect space [7]. If $g \in H^2(R;\Omega)$, then $E(g \circ \mathbf{T}) = g$.

Let $\tilde r$ be a rational function on $\widehat{\mathbb{C}}$, analytic on $\widehat{\mathbb{C}} \setminus \mathbb{D}$, with $\|\tilde r - f \circ \mathbf{T}\|_{H^2(R;\widetilde{\Omega})} < \varepsilon$. Then $E\tilde r \in H^2(R, \Omega) \dotplus N$. Put r to be the component of $\tilde r$ in $H^2(R, \Omega)$. Then $E\tilde r$ is meromorphic on $\overline{R}$ and r is analytic on $\overline{R}$. We have $\|E\tilde r - f\|_{H^2(R;\Omega)} < \varepsilon$ and $\|r - E\tilde r\| \approx \operatorname{dist}\big(E\tilde r, H^2(R;\Omega)\big) \leqslant \|E\tilde r - f\| < \varepsilon$. Hence $\|r - f\|_{H^2(R;\Omega)} < C\varepsilon$, where C is an absolute constant.

10. Characterization of $H^2_\mu(X_+)$.

We assume here that the spectral measure of N has no point masses. Then $d\nu = |dz|\big|\gamma_c$. We will use the following notation. If f is a function on γ_c and g its lifting to $\partial\widehat{\Delta}_+$, that is, $g\big((z,\overline{z})\big) \equiv f(z)$, then we will write $g = f^\#$, $f = g^\flat$ (note that z-projections of different subarcs of $\partial\widehat{\Delta}_+$ cannot coincide). If g is a function on $\widehat{\Delta}$ or $\widehat{\Delta}_+$, we put $g^\flat = (g|\partial\widehat{\Delta}_+)^\flat$; in the latter case $g|\partial\widehat{\Delta}_+$ denotes the boundary limit values of g. By (6.3), $de(\cdot) = \mathcal{E}(\cdot)\,|dz|$, and

$$\begin{aligned}\mathcal{E}^\#(\delta) &= \frac{1}{2\pi}\,\xi\big(z(\delta)\big)\big(z(\delta)-\Lambda\big)^{-1}Q(\delta)\\ &= \frac{1}{2\pi}\,\eta(\delta)^{1/2}\big(z(\delta)-\Lambda\big)^{-1}Q(\delta)\,,\end{aligned}\tag{10.1}$$

for $\delta \in \partial\widehat{\Delta}_+$. Since $|\eta| \equiv 1$ on $\partial\widehat{\Delta}_+$, the last expression allows us to consider $\mathcal{E}^\#$ as a function, defined and meromorphic on a neighbourhood of $\partial\widehat{\Delta}_+$ in $\widehat{\Delta}$. We use the sets B, $B^\#$, Pol_Q and the natural number $\tilde{\tau}$ that were introduced in Section 7.

Proposition 10.1.

1) *For each $f \in L^2(e)$, the equality $f = Q^\flat f$ holds in $L^2(e)$.*

2) *If $\delta \in \partial\widehat{\Delta}_+ \setminus \mathrm{Pol}_Q$ and $m \in M$, $m = Q(\delta)\,m$, then $\mathcal{E}(\delta)\,m = 0$ implies $m = 0$.*

PROOF. By (10.1), $\mathcal{E}f = \mathcal{E}Q^\flat f$, which gives 1). Statement 2) also is obvious from (10.1).

Proposition 10.2. *The embedding $H^2_\mu(X_+) \longrightarrow L^2(e)$, which is the composition of the isomorphism $a^{-1} : H^2_\mu(X_+) \longrightarrow E^2(\mu)$ with the canonical embedding of $E^2(\mu)$ into $L^2(e)$, defined by (6.7), is given by $f \in H^2_\mu(X_+) \longmapsto f^\flat$.*

PROOF. Let $f = a\,g$, $g \in E^2(\mu)$, and h be the image of g in $L^2(e)$, that is, $g = \mathcal{K}(\cdot - \Lambda)\,\mathcal{E}\,h$. By (10.1) and the Plemelj "jump" formula, $g_i - g_e = Q^\flat h$. Therefore the boundary values of f on $\partial\widehat{\Delta}_+$ are given by

$$f^\flat = Q^\flat g_i = Q^\flat(g_i - g_e) = Q^\flat h\,.\tag{10.2}$$

By Proposition 10.1, $h = Q^\flat h = f^\flat$ in $L^2(e)$.

From now on, let us fix points p_j on nondegenerate components $\widehat{\Delta}_j^+$, and let ω_j be the harmonic measure for $\widehat{\Delta}_j^+$ at p_j. We define the harmonic measure ω for $\partial\widehat{\Delta}_+$ by $\omega|\partial\widehat{\Delta}_j^+ = \omega_j$.

Lemma 10.3. *For each $\delta_0 \in \partial\widehat{\Delta}_+$, there exists $c < 1$ such that $\|Q(\delta)\| \leqslant |z(\delta) - z(\delta_0)|^{-c}$ in a neighbourhood of δ_0.*

Proof. By (10.1), $\|\mathcal{E}^\#(\delta)\| \sim |z(\delta) - z(\delta_0)|^{-c}$ for some rational c. But $\int \mathcal{E}(z)\,|dz| = I$, which implies that $c < 1$. Now we remark that

$$Q(\delta) = \eta(\delta)^{-1/2}\big(z(\delta) - \Lambda\big)\mathcal{E}^\#(\delta)\,,$$

which gives the statement of the Lemma.

Definition. *Any branching point δ of $\widehat{\Delta}_+$ has a neighbourhood $\mathcal{W}$ such that $\mathcal{W}\setminus\{\delta\}$ projects j-to-one onto the z-plane for some $j \geqslant 2$. We put the order of the branching point δ to be equal to $j-1$.*

Let Br be the set of branching points of $\widehat{\Delta}_+$. Choose $\tau \in \mathbb{N}$ such that the orders of these points do not exceed τ, and $\tau \geq \tilde{\tau}$. Put

$$\mathcal{P} = \mathrm{Br} \cup (\mathrm{Pol}_Q \cap \widehat{\Delta}_+)\,.$$

Fix an analytic function q on a neighbourhood of $\operatorname{clos}\widehat{\Delta}_+$, which has simple zeros in points of $\mathcal{P}$ and no other zeros on $\operatorname{clos}\widehat{\Delta}_+$.

Let $\widehat{\Delta}_j$ be a nondegenerate component of $\widehat{\Delta}$ of multiplicity k_j. Fix a function $\widetilde{X}_j \in \mathrm{Hol}\big(\widehat{\Delta}_j^+, \mathcal{L}(\mathbb{C}^{k_j}, M)\big)$ such that $\widetilde{X}_j(\delta)\,\mathbb{C}^{k_j} = X(\delta)$ for all $\delta \in \widehat{\Delta}_j^+$. Since analytic vector bundles $X|\widehat{\Delta}_j^+$ are trivial by the Grauert theorem [13], such an $\widetilde{X}_j$ exists. Moreover, we will assume $\widetilde{X}_j$ to be analytic on a neighbourhood of $\operatorname{clos}\widehat{\Delta}_j^+$ in $\widehat{\Delta}_j$. The map $f \longmapsto \widetilde{X}_j f$ is an isomorphism between analytic cross-sections of the trivial vector bundle $\widehat{\Delta}_j^+ \times \mathbb{C}^{k_j}$ and analytic cross-sections of the bundle $X_+|\widehat{\Delta}_j^+$.

Define the weight Ω_j by

$$\Omega_j = \widetilde{X}_j^*\,\mathcal{E}^\#\widetilde{X}_j|\partial\widehat{\Delta}_j^+\,.$$

We put

$$H^2_\tau(\widehat{\Delta}_j^+, \Omega_j) = \big\{f \in q^{-\tau}\mathcal{N}^+(\widehat{\Delta}_j^+; \mathbb{C}^{k_j}) :\ (f|\partial\widehat{\Delta}_j^+)^\flat \in L^2(z(\partial\widehat{\Delta}_j^+), \nu, \Omega_j)\big\}\,.$$

It follows from (10.1) that $\Omega_j|\partial\widehat{\Delta}_j^+$ is real analytic, except for a finite number of power-type singularities. The well-known formula

$$d\omega = -(2\pi)^{-1}\frac{\partial g}{\partial n}\, ds\,,$$

which expresses the harmonic measure in terms of the Green function [2], [10] implies that the same is true for the function $d\nu^{\#}/d\omega$. By Proposition 10.1.2), Ω_j is invertible in points of $\partial\widehat{\Delta}_j^+ \setminus B^{\#}$. Put

$$\widetilde{\Omega}_j = |q|^{\tau}\frac{d\nu^{\#}}{d\omega}\,\Omega_j\ .$$

Then there is a scalar function s, analytic in a neighbourhood of $\operatorname{clos}\widehat{\Delta}_j^+$, such that $s\,\widetilde{\Omega}_j$ and $s\,\widetilde{\Omega}_j^{-1}$ are bounded on $\partial\widehat{\Delta}_+$. This implies that $\widetilde{\Omega}_j$ meets the log-integrability condition (9.1). Therefore we can rewrite the above definition as

$$H^2_\tau(\widehat{\Delta}_j^+,\Omega_j) = q^{-\tau}H^2\Big(\widehat{\Delta}_j^+, |q|^{\tau}\frac{d\nu^{\#}}{d\omega}\Omega_j\Big)\,. \tag{10.3}$$

Functions in this class may have at most poles of order τ in points of $\mathcal{P}$.

Let $\{\Delta_j\}$ be all nondegenerate components of Δ. We define $\widetilde{X}$, Ω by

$$\widetilde{X}|\widehat{\Delta}_j^+ = \widetilde{X}_j\ , \qquad \Omega|\partial\widehat{\Delta}_j^+ = \Omega_j\ ,$$

and put

$$\operatorname{Hol}(\widehat{\Delta}_+) = \bigoplus_j \operatorname{Hol}(\widehat{\Delta}_j^+, \mathbb{C}^{k_j})\,, \qquad \widetilde{X}H^2_\tau(\widehat{\Delta}_+,\Omega) = \bigoplus_j \widetilde{X}_j H^2_\tau(\widehat{\Delta}_j^+,\Omega_j)\,.$$

We give $\widetilde{X}H^2_\tau(\widehat{\Delta}_+,\Omega)$ the Hilbert direct sum norm, defined by norms of H^2-spaces that figure in (10.3).

Theorem 10.4. *Suppose that the spectral measure of N has no point masses. Let $\mathcal{K}$ be a compact subset of $\widehat{\Delta}_+$ such that $\mathcal{P}\subset \operatorname{int}\mathcal{K}$, $z(\partial\mathcal{K})\cap z(\mathcal{P}) = \varnothing$. Then*

$$\begin{aligned} H^2_\mu(X_+) = \Big\{ f \in \widetilde{X}H^2_\tau(\widehat{\Delta}_+,\Omega) :& \\ (f)_{\mathcal{K}}(z) \overset{\text{def}}{=} \sum_{\substack{z(\delta)=z\\ \delta\in\mathcal{K}}} f(\delta) \ \textit{is bounded near}\ z(\mathcal{P})\Big\}\,,& \end{aligned} \tag{10.4}$$

and the norms in these two spaces coincide.

Some of the points of $z^{-1}\big(z(\mathcal{P})\big)$ may lie on $\partial\widehat{\Delta}_+$, but the values of f in neighbourhoods of such points are not taken into account in the above expression for $(f)_{\mathcal{K}}$. Therefore for each $f \in \widetilde{X} H^2_\tau(\widehat{\Delta}_+, \Omega)$, $(f)_{\mathcal{K}}$ is meromorphic on $z(\mathcal{P})$ and has on it at most poles of order τ. So the above condition on f reduces to a finite number of linear conditions.

Spaces $H^2_\mu(X_-)$ and $\overline{H}^2_{\mu^*}(Y_\pm)$ can be characterized in the same way.

Denote the right hand part of (10.4) by $\widetilde{H}^2_\mu(X_+)$.

Proposition 10.5. *The map $f \longmapsto f^\flat$ defines isometries of $H^2_\mu(X_+)$, $\widetilde{H}^2_\mu(X_+)$ into $L^2(e)$ (the norm in $\widetilde{H}^2_\mu(X_+)$ is inherited from $H^2_\tau(\widehat{\Delta}_+, \Omega)$).*

Proof. The first map is an isometry by Proposition 10.2. The second map is isometric because for $f = \widetilde{X} v \in \widetilde{X} H^2_\tau(\widehat{\Delta}_+, \Omega)$,

$$\|f\|^2_{\widetilde{H}^2_\mu(X_+)} = \int \langle \Omega^\flat v, v\rangle \, d\nu = \int \langle \mathcal{E} f, f\rangle \, d\nu = \|f\|^2_{L^2(e)} \, .$$

Proof of Theorem 10.4. By the above proposition, to prove that $H^2_\mu(X_+) = \widetilde{H}^2_\mu(X_+)$, it suffice to check that a complete subset of $H^2_\mu(X_+)$ is contained in $\widetilde{H}^2_\mu(X_+)$, and vice versa.

1) By (1.11), the functions

$$\varphi_{t,m}(z) = \frac{\mu(z)}{z-t}\, m \, ,$$

with $t \in \mathbb{C} \setminus \gamma$ and $\mu(t)\, m = 0$ are complete in $E^2(\mu)$. Therefore the corresponding functions

$$f(\delta) = (a\, \varphi_{t,m})(\delta) = Q(\delta)\big(z(\delta) - t\big)^{-1} m$$

are complete in $H^2_\mu(X_+)$. Take any such f. Fix a point $z_0 \in z(\mathcal{P})$, and let us prove that $(f)_{\mathcal{K}}$ does not have pole in z_0 (the most difficult case is when $z_0 \in \gamma_c$). We have

$$\varphi_{t,m}(z) = a^{-1} f(z) = (f)_{\mathcal{K}}(z) + (f)_{\widehat{\Delta}_+ \setminus \mathcal{K}}(z) \, , \qquad z \in \mathbb{C} \setminus \gamma \, . \tag{10.5}$$

By Lemma 10.3, $|(f)_{\widehat{\Delta}_+ \setminus \mathcal{K}}(z)| \leqslant |z - z_0|^{-c}$ in a neighbourhood of z_0 for some $c < 1$. On the other hand, from (6.6) and the fact that

$(z-\Lambda)\mathcal{E} \in L^1(|dz|)$ one can conclude that $|\varphi_{t,m}(z)| = o\big(\operatorname{dist}(z,\gamma_c)\big)$, $z \longrightarrow z_0$. By (10.5), $(f)_{\mathcal{K}}(z)$ cannot have a pole in z_0.

Let $f = \widetilde{X}g$, then g is meromorphic on a domain containing $\operatorname{clos}\widehat{\Delta}_+$. Therefore $f \in q^{-N} \oplus \mathcal{N}^+(\widehat{\Delta}_j^+, \mathbb{C}^{k_j})$. Since

$$\|g\|_{L^2(\partial\widehat{\Delta}_+, d\nu, \Omega)} = \big\|\big((\cdot - t)\big)^{-1} m\big\|_{L^2(e)} < \infty\,,$$

we get that $f \in \widetilde{H}^2_\mu(X_+)$.

2) To prove the converse inclusion, take any $f \in \widetilde{H}^2_\mu(X_+)$, which has the form $f = \widetilde{X}g$ for a function $g \in q^{-N} \oplus \operatorname{Hol}(\widetilde{\Delta}_j^+, \mathbb{C}^{k_j})$, where $\widetilde{\Delta}_j^+$ are neighbourhoods of $\operatorname{clos}\widehat{\Delta}_j^+$. Then, obviously,

$$\int \langle \Omega^\flat g^\flat, g^\flat\rangle\, d\nu = \int \langle \mathcal{E}\widetilde{X}^\flat g^\flat, \widetilde{X}^\flat g^\flat\rangle\, d\nu < \infty\,.$$

By Lemma 9.5, such functions $f = \widetilde{X}g$ are dense in $\widetilde{H}^2_\mu(X_+)$.

The function $a^{-1}f$ is bounded and analytic on $\mathbb{C}\setminus\gamma_c$ and takes value 0 at ∞. Writing down its Cauchy representations in the components of $\mathbb{C}\setminus\gamma_c$ and summing them up, we arrive at the formula

$$a^{-1}f = \frac{1}{2\pi i}\,\mathcal{K}(i\,\xi\, f^\flat)\,,$$

because $f^\flat$ is the jump of $a^{-1}f$ on γ_c. Since $f^\flat \in L^2(e)$ and $(a^{-1}f)(\cdot) = \mu(\cdot)\,(a^{-1}f)(\cdot)$ on $\mathbb{C}\setminus\gamma_c$ (see (6.1)), we conclude that $a^{-1}f \in E^2(\mu)$. Thus $f \in a\,E^2(\mu) = H^2_\mu(X_+)$.

11. Simple subnormal operators.

Definitions. *A polynomial $P(z,w)$ will be called admissible if it has a form*

$$P(z,w) = \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}\, z^i w^j\,,$$

where $n > 0$ is an integer, $a_{nn} = 1$, and $a_{ji} = \overline{a}_{ij}$, $0 \leqslant i,j \leqslant n$, and P has no irreducible factors of the type $z - z_0$, $w - w_0$. An algebraic curve Δ in $\mathbb{C}P^2$ is called admissible if it is given by an equation $P(z,w) = 0$ for some admissible P. We put $\operatorname{rank}\Delta = n$.

If $\Delta = \Delta_1^{k_1} \cup \cdots \cup \Delta_T^{k_T}$ is the decomposition of Δ into the union of irreducible curves, then $\operatorname{rank} \Delta = \sum k_j \operatorname{rank} \Delta_j$.

One sees from (0.1) that each discriminant curve of the type studied above without degenerate components is admissible. Also, the product of two admissible polynomials is admissible.

Lemma 11.1. *For each admissible algebraic curve Δ, the sets $z(\widehat{\Delta}_+)$ and $w(\widehat{\Delta}_-)$ are bounded.*

Proof. Let $w = w_k(z)$ be all (possibly multivalued) analytic functions that are defined in a neighbourhood of $z = \infty$ by the implicit equation $P(z, w) = 0$. Put $K = \sum_{i,j} |a_{ij}|$. Then $(z, w) \in \Delta$ and $|z| > K$ imply $|w| \leqslant K$ (by the triangle inequality). It follows that

$$w_k(z) = \lambda_k + \rho_k\, \varphi_k(z^{-1})^{c_k}\,,$$

where $\lambda_k, \rho_k \in \mathbb{C}$, $c_k > 0$ are rational and φ_k are analytic in 0, with $\varphi_k(0) = 0$ and $\varphi_k'(0) = 1$. An easy calculus now shows that

$$-\eta\big((z, w)\big)^{-1} = \frac{dw}{dz} \longrightarrow 0$$

if $w = w_k(z)$ and $z \longrightarrow \infty$. This shows that $z(\widehat{\Delta}_+)$ is bounded. The set $w(\widehat{\Delta}_-)$ is also bounded, because $w(\widehat{\Delta}_-) = \overline{z(\Delta_+)}$.

Now let $\widehat{\Delta}$ be the discriminant surface of a subnormal S of finite type such that the spectral measure of N has no point masses (but we allow $\widehat{\Delta}$ to have degenerate components). As before, let $\widehat{\Delta}_j$ be the nondegenerate components of $\widehat{\Delta}$, pick points $p_j \in \widehat{\Delta}_j^+$ and denote by ω_j the harmonic measures of $\widehat{\Delta}_j^+$ with respect to p_j. Put $\omega = \sum_j \omega_j$. The corresponding measure $\omega^\flat$ on γ_c is absolutely continuous with respect to the arc length measure.

We can define an $H^\infty(\widehat{\Delta}_+)$-calculus for N: we put

$$f(N) = (f|\widehat{\Delta}_+)^\flat(N)\,, \qquad f \in H^\infty(\widehat{\Delta}_+)\,. \tag{11.1}$$

Definition. *Let N have no point masses. Then S is called simple if it admits the $H^\infty(\widehat{\Delta}_+)$-calculus, that is, $f(N)H \subset H$ for all $f \in H^\infty(\widehat{\Delta}_+)$.*

Choose a function r, which is holomorphic on a neighbourhood of $\operatorname{clos}\widehat{\Delta}_+$ and has zeros exactly in branching points of $\widehat{\Delta}_+$ of orders that are equal to orders of these points. Let $\mathcal{K}$ be the subset of $\widehat{\Delta}_+$ from Theorem 10.4.

Proposition 11.2.

i) *For any* $f \in \operatorname{Hol}(\widehat{\Delta}_+)$, $(r^{-1}f)_{\mathcal{K}}$ *is bounded near* $z(\mathcal{P})$.

ii) *If* f *is meromorphic on* $\widehat{\Delta}_+$ *and* $(\varphi f)_{\mathcal{K}}$ *is bounded near* $z(\mathcal{P})$ *for all* $\varphi \in H^\infty(\widehat{\Delta}_+)$, *then* $rf \in \operatorname{Hol}(\widehat{\Delta}_+)$.

The proof of i) is elementary. When proving ii), one can use the fact that if $\delta_0, \ldots, \delta_l$ are points of $\widehat{\Delta}_+$ that project into the same point z_0 and $\mathcal{W}_0, \ldots, \mathcal{W}_l$ are their small neighbourhoods and ψ is analytic in $\mathcal{W}_0$, then for any $s \in \mathbb{N}$ there is φ in $H^\infty(\widehat{\Delta}_+)$ such that $\varphi - \psi|\mathcal{W}_0$ and $\varphi|\mathcal{W}_j$, $j \geqslant 1$ have zeros of orders $\geqslant s$ in δ_j. We omit the details.

Note that Proposition 11.2 is true for vector-valued functions $f \in \operatorname{Hol}(\widehat{\Delta}_+, M)$ as well.

Proposition 11.3. *The following are equivalent.*

1) S *is simple.*

2) rQ *is analytic on* $\widehat{\Delta}_+$.

3) $H^2_\mu(X_+) = r^{-1}\widetilde{X} H^2(\widehat{\Delta}_+, \Omega')$, *where*

$$\Omega' = |r| \, \frac{d\nu^\#}{d\omega} \, \Omega \, .$$

PROOF. 2) implies 1). If 2) holds, then for every $f \in H^2_\mu(X_+)$, rf is analytic on $\widehat{\Delta}_+$ (because functions $Q(\cdot)\big(z(\cdot) - t\big)^{-1} m$ with $m \in M$, $t \in \mathbb{C} \setminus \gamma$, $\mu(t)\, m = 0$ are complete in $H^2_\mu(X_+)$). By Proposition 11.2.i) and (10.4), $H^\infty(\widehat{\Delta}_+)$ acts by multiplication on $H^2_\mu(X_+)$. Therefore S is simple.

1) implies 3). By Proposition 11.2.i) and (10.4),

$$r^{-1}\widetilde{X} H^2(\widehat{\Delta}_+, \Omega') \subset H^2_\mu(X_+) \, .$$

If $f \in H^2_\mu(X_+)$, then $(\varphi f)_{\mathcal{K}}$ has to be bounded near $z(\mathcal{P})$ for all $\varphi \in H^\infty(\widehat{\Delta}_+)$. By Proposition 11.2.ii), f is in $r^{-1}\,\mathrm{Hol}(\widehat{\Delta}_+)$, and so by Theorem 10.4, $f \in r^{-1}\widetilde{X} H^2(\widehat{\Delta}_+, \Omega')$.

3) implies 2). Trivially, because $Q(\cdot)\, m \in H^2_\mu(X_+)$ for all $m \in M$.

These arguments imply that for a simple subnormal S, the order of pole of Q at each branching point of $\widehat{\Delta}_+$ equals to the order of this branching point.

We remark that if 1)–3) hold, then the range norm on

$$r^{-1}\widetilde{X} H^2\Big(\widehat{\Delta}_+, |r|\, \frac{d\nu^\#}{d\omega}\, \Omega\Big)$$

coincides with the norm on $H^2_\mu(X_+)$.

The following lemma extends a result by McCarthy and Yang [16, Theorem 1.12].

Lemma 11.4. *Let Δ be any irreducible admissible separated algebraic curve, and Ω a matrix $\mathbb{C}^{k\times k}$-valued* log*-integrable weight. Define an operator S on $H^2(\widehat{\Delta}_+, \Omega)$ by $Sf(\delta) = z(\delta)\, f(\delta)$ (to define the harmonic measure, we choose any base point). Then S is a simple subnormal operator of finite type and* $\mathrm{rank}\,[S^*, S] = k\,\mathrm{rank}\,\Delta$.

PROOF. Lemma 11.1 implies that S is bounded. By Lemmas 9.1 and 9.2, it is subnormal, and its minimal normal extension is given by $Nf(\delta) = z(\delta)\, f(\delta)$, $f \in L^2(\partial\widehat{\Delta}_+, \omega, \Omega)$. The operator S^* acts by

$$S^* f = P_{H^2(\widehat{\Delta}_+, \Omega)}(\overline{z} f)\,, \qquad f \in H^2(\widehat{\Delta}_+, \Omega)\,,$$

which gives

$$\begin{aligned} \mathrm{Ker}[S^*, S] &= \big\{ f :\ \|Sf\| = \|S^* f\| \big\} \\ &= \big\{ f \in H^2(\widehat{\Delta}_+, \Omega) :\ \overline{z} f \in H^2(\widehat{\Delta}_+, \Omega) \big\} \qquad (11.2) \\ &= \big\{ f \in H^2(\widehat{\Delta}_+, \Omega) :\ w(\cdot)\, f(\cdot) \in \mathrm{Hol}(\widehat{\Delta}_+, \mathbb{C}^k) \big\}\,. \end{aligned}$$

The sum of orders of all poles of $w(\cdot)$ on $\widehat{\Delta}$ equals $\mathrm{rank}\,\Delta$. By Lemma 11.1, $w(\cdot)$ has no poles on $\widehat{\Delta}_-$. Hence the last expression in (11.2) gives $\mathrm{Codim}\,\mathrm{Ker}[S^*, S] = k\,\mathrm{rank}\,\Delta$. By the remark in [25, Section 6], the nondegenerate part of the discriminant surface of S coincides with Δ. As S trivially admits $H^\infty(\widehat{\Delta}_+)$-calculus, it is simple.

Xia's exposition in [23] uses the notion of a "quadrature domain on a Riemann surface". He defines it as a domain $\mathcal{D}$ with a boundary, consisting of rectifiable Jordan curves on a branched Riemann surface equipped with a projection $z(\cdot)$ onto $\mathbb{C}$ such that $w(\cdot) = \overline{z(\cdot)}$ for some function $w(\cdot)$, meromorphic on $\operatorname{clos}\mathcal{D}$. It is easy to see that $\big(\mathcal{D}, z(\cdot)\big)$ is a quadrature domain on a Riemann surface if and only if $\mathcal{D} = \widehat{\Delta}_+$ for some irreducible admissible separated algebraic curve $\widehat{\Delta}$ in $\mathbb{C}^2$.

Theorem 11.5. *A Hilbert space operator is simple subnormal of finite type if and only if there exist an admissible separated algebraic curve* $\Delta = \cup\Delta_j^{k_j}$, *points* $p_j \in \widehat{\Delta}_j^+$ *and homomorphisms* $\alpha_j \in \operatorname{Char}_{k_j}(\widehat{\Delta}_j^+)$ *such that* S *is unitarily equivalent to*

$$\bigoplus_j S_{\Delta_j^+,\alpha_j,p_j} \tag{11.3}$$

(*see the notation in Section* 9). *In this case,* $\operatorname{rank}[S^*, S] = \operatorname{rank}\Delta$.

PROOF. If S is simple subnormal, then by Proposition 11.3, S is unitarily equivalent to the operator of multiplication by $z(\cdot)$ on the space

$$H^2(\widehat{\Delta}_+, \Omega') = \bigoplus_j H^2(\widehat{\Delta}_j^+, \Omega'_j)\,,$$

where Δ is the discriminant curve of S and Δ_j are its nondegenerate components. Then $\cup\Delta_j^{k_j}$ is an admissible separated algebraic curve. By Lemma 9.1, S is unitarily equivalent to an operator of the form (11.3).

The converse follows from Lemma 11.4.

REMARK. The discriminant curve of a simple subnormal operator has no degenerate components.

Indeed, let S be simple, and let $\Delta = \bigcup \Delta_j^{k_j} \cup \bigcup \Delta_l'^{c_l}$ be the the discriminant curve of S, where the Δ_l are nondegenerate components of Δ and Δ'_l are degenerate. Put $n = \operatorname{rank}\Delta$. By Lemma 11.4,

$$\sum_j k_j \operatorname{rank}\Delta_j + \sum_l c_l = n = \operatorname{rank}[S^*, S] = \sum_j k_j \operatorname{rank}\Delta_j\ .$$

The following statement answers the questions of simiarity and unitary equivalence of simple subnormal operators.

Proposition 11.6.

1) *Two simple subnormal operators are similar if and only if their discriminant surfaces coincide and the multiplicities of their irreducible components also agree.*

2) *Let* $S = \oplus_j S_{\widehat{\Delta}_j^+, \alpha_j, p_j}$ *and* $S' = \oplus_j S_{\widehat{\Delta}_j^+, \alpha'_j, p_j}$ *be two simple subnormal operators with the same discriminant curve* $\Delta = \cup \Delta_j^{k_j}$ *and the same choice of base points* $p_j \in \widehat{\Delta}_j^+$*. Then* S *and* S' *are unitarily equivalent if and only if* $\alpha_j \sim \alpha'_j$ *for all* j*.*

PROOF. 1). The "if" part follows from Theorem 11.5 and Corollary to Lemma 9.3. The converse follows from the Remark in [25, Section 6].

Let S, S' be as in 2), and let $S' = L^{-1}SL$, where L is a unitary isomorphism. Then $f(S') = L^{-1}f(S)L$ for all $f \in H^\infty(\widehat{\Delta}_+)$, which implies that L splits: $L = \oplus L_j$, where for each j, L_j is a unitary isomorphism between $H^2_{\alpha_j}(\widehat{\Delta}_j^+)$ and $H^2_{\alpha'_j}(\widehat{\Delta}_j^+)$. Now the statement follows from Lemma 9.4.

Let S be rationally cyclic and irreducible. It is easy to see that then $\widehat{\Delta}$ has only one nondegenerate component $\widehat{\Delta}_1$, $\widehat{\Delta}_1^+$ projects homeomorphically into $\mathbb{C}$ and $k_1 = 1$. For this case, the above fact has been proved by McCarthy and Yang ([16, Theorem 2.2]).

12. Internal Riemann surface models and general structure theorems.

12.1. Elimination of point masses.

Since N is unitarily equivalent to the operator of multiplication by z on $L^2(e)$, S has no point masses if and only if $e(\cdot)$ is absolutely continuous with respect to the arc length measure. In general, as follows from Theorem A, $e(\cdot) = e_a(\cdot) + e_s(\cdot)$, where e_a is absolutely continuous and e_s is a finite sum of matrix point masses. Since N is unitarily equivalent to M_z on $L^2(e)$, we have the corresponding orthogonal sum decompositions

$$K = K_a \oplus K_s \,, \qquad N = N_a \oplus N_s \,.$$

Put $H_0 = P_{K_a} H$ and consider an operator $L\colon H \to H_0$, given by

$$Lx = P_{K_a} x \,, \qquad x \in H \,.$$

The following simple fact will be used in the sequel: if T is a subnormal operator and T_1 its restriction to an invariant subspace of finite codimension, then $\operatorname{rank}[T^*, T] < \infty$ if and only if $\operatorname{rank}[T_1^*, T_1] < \infty$.

Lemma 12.1. *The space H_0 is closed and L is invertible. The operator $S_0 \stackrel{\text{def}}{=} LSL^{-1}$ on H_0 is pure subnormal without point masses, and* $\operatorname{rank}[S_0^*, S_0] < \infty$.

PROOF. We have

$$\langle Lx, Lx \rangle = \big\langle (I - P_H P_{K_s} P_H)\, x, x \big\rangle \,, \qquad x \in H \,.$$

Let t be the maximal eigenvalue of the finite rank self-adjoint operator $P_H P_{K_s} P_H$. Since $H \cap K_s = 0$, it follows that L is one-to-one, and therefore $t < 1$. Hence L is an isomorphism onto its range. Obviously, $LSx = LNx = NLx$ for $x \in H$, which implies that $N|K_a$ is a normal extension of S_0 with absolutely continuous spectrum. So S_0 has no point masses.

At last, put $H_1 = H \cap K_a$ and $S_1 = S|H_1$. Since $\operatorname{rank} H \ominus H_1 < \infty$, we have $\operatorname{rank} H_0 \ominus H_1 < \infty$, and thus $\operatorname{rank}[S_1^*, S_1] < \infty$, $\operatorname{rank}[S_0^*, S_0] < \infty$.

So for any operator S we have defined in a canonical way a subnormal operator S_0 without point masses, which is similar to S.

12.2. Passage to a simple subnormal operator: a finite-dimensional extension.

Here we assume S to be a subnormal operator of finite type without point masses. Define a linear manifold

$$\widetilde{H} = \operatorname{span}\big\{ f(N)H : \; f \in H^\infty(\widehat{\Delta}_+) \big\} \,, \tag{12.1}$$

clearly, $\widetilde{H} \supset H$.

Proposition 12.2.

1)$\widetilde{H}$ *is a closed invariant subspace of* N, *and* $\dim \widetilde{H} \ominus H$ *is finite.*

2) $\widetilde{S} \stackrel{\mathrm{def}}{=} N|\widetilde{H}$ *is a simple subnormal operator.*

PROOF. Let $f \in H^\infty(\widehat{\Delta}_+)$, and let $\widetilde{U}$, W be the operators from Section 6.2. By Theorem 10.4, the multiplication by f sends $H^2_\mu(X_+)$ into $\widetilde{X} H^2_\tau(\widehat{\Delta}_+, \Omega)$. This action agrees with the action of $f(N)$, that is, if $x \in H$ and $u = a\,Ux \in H^2_\mu(X_+)$, then

$$f^\flat u^\flat = W f(N)\, x \tag{12.2}$$

(see Proposition 10.2). Let

$$\mathcal{H}(X_+) = \operatorname{span}\bigl\{ f\,u : \ f \in H^\infty(\widehat{\Delta}_+), \ u \in H^2_\mu(X_+)\bigr\}\,,$$

it follows from (12.1), (12.2) that

$$\widetilde{H} = \bigl\{ W^{-1} u^\flat : \ u \in \mathcal{H}(X_+)\bigr\}\,.$$

Since $H^2_\mu(X_+) \subset \mathcal{H}(X_+) \subset \widetilde{X} H^2_\tau(\widehat{\Delta}_+, \Omega)$, $\mathcal{H}(X_+)$ is closed and

$$\dim H^2_\mu(X_+) \ominus \mathcal{H}(X_+) < \infty\,.$$

By Proposition 10.2, the map $u \longmapsto W^{-1} u^\flat$ is an isometry from

$$\widetilde{X} H^2_\tau(\widehat{\Delta}_+, \Omega)$$

into $L^2(e)$. This implies both assertions of 1). Since N is a normal extension of $\widetilde{S}$ and $f \cdot \mathcal{H}(X_+) \subset \mathcal{H}(X_+)$ for all $f \in H^\infty(\widehat{\Delta}_+)$, we obtain 2).

It is clear that $\widetilde{H}$ is the minimal closed subspace of K such that $\widetilde{H} \supset H$ and $N|\widetilde{H}$ is simple subnormal.

12.3. Main structure result.

Let S be a subnormal operator of finite type. Following the above procedures, we can construct from it an operator S_0 without point masses and then a simple operator $\widetilde{S}_0$. The operator $\widetilde{S}_0$ may be called *the canonical simple operator that corresponds to* S.

Theorem 12.3. *Let S be a subnormal operator of finite type. Let S_0 be the operator obtained from S by eliminating point masses, and $\widetilde{S}_0$ be the canonical simple operator corresponding to S_0; suppose that $\widetilde{S}_0$ acts on a space $\widetilde{H}_0$. Then*

1) *There exist eigenvalues $\overline{\lambda}_k$, $1 \le k \le r$ of $\widetilde{S}_0^*$ and corresponding Jordan chains $\{\psi^j_{\lambda_k}\}_{j=0}^{m_k}$ of generalized eigenvectors: $\big(\widetilde{S}_0^* - \overline{\lambda}_k\big)\psi^0_{\lambda_k} = 0$, $\big(\widetilde{S}_0^* - \overline{\lambda}_k\big)\psi^j_{\lambda_k} = \psi^{j-1}_{\lambda_k}$, $j = 1, \dots, m_k$, such that $S_0 = \widetilde{S}_0 | H_0$, where*

$$H_0 = \big\{x \in \widetilde{H}_0 : \ \langle x, \psi^j_{\lambda_k}\rangle = 0\,, \qquad 1 \leqslant k \leqslant r,\ 0 \leqslant j \leqslant m_k\big\}$$

(*the λ_k's are not necessarily distinct*).

2) *There is a finite set $\{\mu_j\}$ and operators $l_j : H_0 \longrightarrow \mathbb{C}^{t_j}$, $t_j \in \mathbb{N}$, $1 \leqslant j \leqslant m$, with $\big(S_0^* - \bar{\mu}_j\big)l_j^* = 0$ such that the operator S coincides with S_0, acting on the renormed space $(H_0, \|\cdot\|_1)$, where*

$$\|x\|_1^2 \overset{\text{def}}{=} \|x\|^2 + \sum_{j=1}^{m} \|l_j x\|^2\,. \tag{12.3}$$

Conversely, let $\widetilde{S}_0$ be any simple subnormal operator of finite type and let S be obtained from $\widetilde{S}_0$ by applying the above procedure, where $\{\psi^j_{\lambda_j}\}$ and $\{l_j\}$ are arbitrary finite families with the above properties. Then S is a pure subnormal of finite type.

The first part of this theorem is a corrected version of Xia's Theorem 3 in [23]. Xia's formulation is not accurate, because it would follow from it that every subnormal operator of finite type is simple (see example 1) below).

If one combines this theorem with Theorem 11.5, he will obtain a functional model representation of an arbitrary subnormal operator of finite type. One can call it *the internal Riemann surface representation.*

REMARK. If S is obtained from $\widetilde{S}_0$ in the way described in the theorem and S' is the canonical simple subnormal that corresponds to S, then S' may be different from $\widetilde{S}_0$. For instance, if $\widetilde{S}_0$ is the standard shift operator $\widetilde{S}_0 f(z) = zf(z)$ on H^2, then each its restriction $S = S_0$ to an invariant subspace of finite codimension is simple itself, so that $S' = S$ and $S' \neq \widetilde{S}_0$.

EXAMPLES. 1) Let $S = M_z \oplus M_z$ on $H = H^2(\mathbb{D}) \oplus H^2(\mathbb{D} + 1/2)$, then S is a simple subnormal operator with rank $[S^*, S] = 2$. Put

$$\langle (f, g), \psi \rangle = f\Big(\frac{3}{4}\Big) - g\Big(\frac{3}{4}\Big), \qquad (f, g) \in H.$$

It is immediate that $(S^* - 3/4)\,\psi = 0$, so that

$$H_0 = (\psi)^\perp = \Big\{(f, g) : \; f\Big(\frac{3}{4}\Big) = g\Big(\frac{3}{4}\Big)\Big\}$$

is an invariant subspace of S, and $S_0 = S|H_0$ is a subnormal of finite type. The discriminant curve of S is

$$\Delta = \{z\,w = 1\} \bigcup \Big\{\Big(z - \frac{1}{2}\Big)\, w = 1\Big\}$$

which implies that Δ_+, as a Riemann surface over the z-plane, consists of two sheaves that cover in a bijective way, respectively, $\mathbb{D}$ and $\mathbb{D}+1/2$. Since H_0 is not $H^\infty(\widehat{\Delta}_+)$-invariant, S_0 is not simple. The canonical simple subnormal operator that corresponds to S_0 coincides with S. One can say that S_0 is obtained from S by "glueing" the points of Δ_+ over $3/4$.

2) Suppose S is a simple subnormal operator, $\widehat{\Delta}$ has only one non-degenerate component of multiplicity one, and $\widehat{\Delta}_+$ is simply connected. By Theorem 11.5, S is unitarily equivalent to the multiplication operator M_z on $H^2(\widehat{\Delta}_+)$ (for any base point in $\widehat{\Delta}_+$), and let us identify S with this model. Suppose $\widehat{\Delta}_+$ has a branching point δ_0 of order 1. For instance, one can take

$$Sf(\zeta) = \Big(\zeta + \frac{1}{2}\Big)^2 f(\zeta)$$

on H^2. Then $\widehat{\Delta}_+$ can be identified with the unit disc, with the z-projection given by $z(\zeta) = (\zeta + 1/2)^2$, $\zeta \in \mathbb{D}$. Fix a branch of the function $\psi(\delta) = \big(z(\delta) - z(\delta_0)\big)^{1/2}$, then ψ is an analytic homeomorphism of a neighbourhood of δ_0 onto a neighbourhood of 0. Put

$$l\,u = \big(u \circ \psi^{-1}\big)'(0)\,, \qquad u \in H^2(\widehat{\Delta}_+)\,.$$

Then $l\big(S - z(\delta_0)\big) = l M_{\psi^2} = 0$, so that we can put $\|u\|_1^2 = \|u\|^2 + |lu|^2$, $u \in H^2(\widehat{\Delta}_+)$ (see (12.3)). This example shows that the expression for $\|l_j x\|^2$ in (12.3) may fail to have the form

$$\sum_{z(\delta_s)=z(\delta_t)=\mu} \langle e_{st}\, u(\delta_s), u(\delta_t)\rangle\,,$$

which was given in [23,Theorem 3].

PROOF OF THEOREM 12.3. 2) We use the notation of Section 12.1. Put $\|x\|_1 = \|L^{-1}x\|$, $x \in H_0$. By the spectral theorem, there exist $t_j \in \mathbb{N}$, $G_j : K \longrightarrow \mathbb{C}^{t_j}$ and $\mu_j \in \mathbb{C}$ such that $G_j N y = \mu_j\, y$ and $\|P_{K_s} y\|^2 = \sum_1^r \|G_j y\|^2$ for all y in K. Put $l_j = G_j L^{-1}$. Then

$$l_j(S_0 - \mu_j) = G_j(S - \mu_j)L^{-1} = G_j(N - \mu_j)L^{-1} = 0\,,$$

$$\|x\|_1^2 = \|x\|^2 + \|P_{K_s} L^{-1} x\|^2 = \|x\|^2 + \sum_{j=1}^{r} \|l_j x\|^2\,, \qquad x \in H_0\,.$$

This proves the statement.

1) By Section 12.2, S_0 is a restriction of $\widetilde{S}_0$ to its invariant subspace H_0 such that $\dim \widetilde{H}_0 \ominus H_0$ is finite. Put $R = \widetilde{H}_0 \ominus H_0$, then $\widetilde{S}_0^* R \subset R$. So the statement follows from the linear algebra theorem on the Jordan structure, applied to the operator $\widetilde{S}_0^* | R$.

Let S^1, S^2 be two subnormals of finite type. Let S_0^1, S_0^2 be the corresponding subnormals without point masses and $\widetilde{S}_0^1$, $\widetilde{S}_0^2$ the corresponding simple subnormals. Then S^1, S^2 are unitarily equivalent if and only if $\widetilde{S}_0^1$, $\widetilde{S}_0^2$ are unitarily equivalent, S_0^1, S_0^2 are obtained from $\widetilde{S}_0^1$, $\widetilde{S}_0^2$ by passing to the same invariant subspace (in the sense of the latter unitary equivalence), and S^1, S^2 are obtained from S_0^1, S_0^2 by the same finite rank perturbation of the norm. We remind the reader that the question of unitary equivalence of simple subnormals has been completely answered in Proposition 11.6.

Notation (some of the entries appear several times).

M, C, Δ, Λ	Section 0
F, e, $\mathcal{E}$, G, $E^2(\mu)$, $E^2_0(1-\mu)$, $\overline{E}^2(\mu^*)$, $\overline{E}^2_0(1-\mu^*)$, $L^2(e)$, $L(z)$, $L'(z)$, $\mathcal{K}$, $\overline{\mathcal{K}}$, $\mathcal{K}F\mathcal{E}L^2(e)$, $\overline{\mathcal{K}}G^*\mathcal{E}L^2(e)$, P_μ, $\varphi_{t,m}$, $\varphi_{*,t,m}$, $\mu(z)$, $\langle\cdot,\cdot\rangle_d$	Section 1
A, H, H', H_*, H'_*, K, M, M_*, $L(z)$, $L'(z)$, N, P_H, $P_{H'}$, $\widetilde{U}$, $\widetilde{V}$, γ, $\mu(z)$, $\rho(z)$	Section 2
W	Section 3
J, $H(z)$, U, V	Section 4
Q, T, $z(\cdot)$, $w(\cdot)$, γ, γ_c, $\widehat{\Delta}_j$, $\widehat{\Delta}_+$, $\widehat{\Delta}_-$, $\widehat{\Delta}'$, δ^*, $\widehat{\Delta}_{\mathrm{deg}}$, $\widehat{\Delta}_{\mathrm{ndeg}}$, $\sigma_C(\Lambda)$, $\eta(z)$, $\mu(z)$, $\xi(z)$	Section 6.1
e, $\mathcal{K}(\cdot-\Lambda)\mathcal{E}L^2(e)$, $\overline{\mathcal{K}}\mathcal{E}L^2(e)$, j, S', W	Section 6.2
a, b, B, $H^2_\mu(X_+)$, $H^2_\mu(X_-)$, $\overline{H}^2_{\mu^*}(Y_+)$, $\overline{H}^2_{\mu^*}(Y_-)$, Pol_Q, X, $X_\pm$, Y, $Y_\pm$, $\tilde{\tau}$	Section 7
α_1, α_2	Section 8
$\mathrm{Char}_k(R)$, $H^2_\alpha(R)$, $H^2(R,\Omega)$, $L^2(\partial R,\omega,\Omega)$, $\mathrm{Möb}\,(\mathbb{D})$, $\mathcal{N}(R)$, $\mathcal{N}^+(R)$, $S_{R,\alpha}$, $S_{R,\alpha,p}$, $\mathbf{T}$, $\mathbf{U}(\mathbb{C}^k)$	Section 9
Br, $\mathcal{E}^\#$, q, $Q^\flat$, $\mathcal{P}$, $H^2_\tau(\widehat{\Delta}^+_j,\Omega_j)$, $H^2_\tau(\widehat{\Delta}_+,\Omega)$, $\widetilde{X}_j$, $\widetilde{X}$, τ, ω_j, ω, Ω_j, Ω	Section 10
$\mathrm{rank}\,\Delta$, r, Ω'	Section 11
H_0, H_1, K_a, K_s, L	Section 12

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