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The resolution of the Navier-Stokes equations in anisotropic spaces

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Abstract. In this paper we prove global existence and uniqueness for solutions of the 3-dimensional Navier-Stokes equations with small initial data in spaces which are H^{δ_i} in the i-th direction, $\delta_1 + \delta_2 + \delta_3 = 1/2$, $-1/2 < \delta_i < 1/2$ and in a space which is L^2 in the first two directions and $B^{1/2}_{2,1}$ in the third direction, where H and B denote the usual homogeneous Sobolev and Besov spaces.

Résumé. Dans cet article on montre l'existence et l'unicité globale des solutions des équations de Navier-Stokes tridimensionnelles pour des données initiales petites dans des espaces qui sont H^{δ_i} dans la i$^{\text{ème}}$ direction, $\delta_1 + \delta_2 + \delta_3 = 1/2$, $-1/2 < \delta_i < 1/2$ ou dans un espace qui est L^2 dans les deux premières directions et $B^{1/2}_{2,1}$ dans la troisième direction, où H et B sont les espaces de Sobolev et de Besov homogènes habituels.

0. Introduction.

In this paper we study the problem of global existence and uniqueness for solutions of the 3-dimensional Navier-Stokes equations. These equations are the following

$$\text{(N-S)} \qquad \begin{cases} \partial_t U + U\cdot\nabla U - \nu\,\Delta U = -\nabla P\,, \\ \operatorname{div} U(t,\cdot) = 0\,, \qquad \text{for all } t \geq 0\,, \\ U|_{t=0} = U_0\,. \end{cases}$$

Here, $U(t,x)$ is a time-dependent three-dimensional vector-field.

The goal of this work is to solve these equations in the spaces

$$\mathcal{H}^{\delta_1,\delta_2,\delta_3}\,, \qquad \delta_1+\delta_2+\delta_3=\frac{1}{2}\,, \qquad -\frac{1}{2}<\delta_i<\frac{1}{2}\,,$$

and in the space

$$HB^{0,0,1/2}\,,$$

where the first space is H^{δ_i} in the i-th direction and the second space is L^2 in the first two directions and $B^{1/2}_{2,1}$ in the third direction, where H^s, respectively $B^s_{p,q}$, denote the usual homogeneous Sobolev, respectively Besov, spaces. We are using homogeneous spaces because they are more easy to handle in the case of the Navier-Stokes equations and, in addition, they are larger than the classical ones, so we obtain more general results.

By solving (N-S) in the space X we mean proving the global existence and uniqueness of solutions for small initial data in X and the local existence and uniqueness of solutions for arbitrary initial data in X.

The first paragraph is devoted to the study of the spaces $\mathcal{H}^{s_1,s_2,s_3}$, essentially the proof of a product theorem in these spaces. A somewhat similar theorem was proved by M. Sablé-Tougeron in [9] for the Hörmander spaces.

The second paragraph contains the resolution of (N-S) in

$$\mathcal{H}^{\delta_1,\delta_2,\delta_3}\,, \qquad \delta_1+\delta_2+\delta_3=\frac{1}{2}\,, \qquad -\frac{1}{2}<\delta_i<\frac{1}{2}\,.$$

The methods used here are inspired from a paper of J.-Y. Chemin and N. Lerner (see [4]). The case when one of the δ_i equals $1/2$ is important but it cannot be studied through our results because $H^{1/2}(\mathbb{R})$ is not an algebra. This difficulty is partially avoided by replacing $H^{1/2}(\mathbb{R})$ with $B^{1/2}_{2,1}(\mathbb{R})$ which has the property to cancel this critical case. And this is how we come to solve (N-S) in the space $HB^{0,0,1/2}$ during the third paragraph. The same method of replacing H^s with $B^s_{2,1}$ may be used in the resolution of general hyperbolic symmetric systems. These systems can be solved in the space $H^s(\mathbb{R}^d)$, $s > d/2+1$ but the case $s = d/2+1$ cannot be proved unless we replace $H^{d/2+1}$ with $B^{d/2+1}_{2,1}$(a short proof is given in the Appendix).

Finally, the last paragraph makes a comparison between this article and the results which are known. We shall see there that the space $HB^{0,0,1/2}$ is not imbedded in any of the spaces introduced by H. Kozono and M. Yamazaki in [7], $\mathcal{N}^{-1+3/p}_{p,q,\infty}$, provided that $1 \leq q \leq p < 3\,q/2$, $p > 3$. We are not able to prove an imbedding or a nonimbedding if $p \geq 3\,q/2$. The space $\mathcal{H}^{\delta_1,\delta_2,\delta_3}$ is also interesting if we remark, for instance, that we allow negative values for δ_i.

The results of this article can be easily extended to an arbitrary dimension, here we consider $\mathbb{R}^3$ only for sake of simplicity. In fact, if we work in $\mathbb{R}^d$, we can solve (N-S) in the spaces

$$\mathcal{H}^{\delta_1,\delta_2,\dots,\delta_d}\,, \qquad \delta_1+\delta_2+\cdots+\delta_d = \frac{d}{2}-1\,, \qquad -\frac{1}{2} < \delta_i < \frac{1}{2}\,,$$

and in the space

$$HB^{0,\dots,0,1/2}\,,$$

where the first space is H^{δ_i} in the i-th direction and the second space is L^2 in the first $n-1$ directions and $B^{1/2}_{2,1}$ in the last one. For instance, we can solve the 2D Navier-Stokes equations with small initial data in $H^{\delta,-\delta}$, $0 < \delta < 1/2$, that is in a space of functions which are not square-integrable.

1. Study of the anisotropic spaces and preliminary results.

We work in $\mathbb{R}^3$ and we denote by $\overline{x} = (x_1, x_2, x_3)$ the variable in $\mathbb{R}^3$. If $\overline{q} = (q_1, q_2, q_3) \in \mathbb{Z}^3$ and $\overline{s} = (s_1, s_2, s_3) \in \mathbb{R}^3$ then we define $\overline{q}\cdot\overline{s} = q_1s_1 + q_2s_2 + q_3s_3$. Also, if $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ then we note $\overline{\lambda}^{\overline{s}} = |\lambda_1|^{s_1}|\lambda_2|^{s_2}|\lambda_3|^{s_3}$.

Let

$$\begin{aligned}\mathcal{L}^{\overline{p}} &= \mathcal{L}^{p_1,p_2,p_3}\\ &= \{u \text{ such that } \|u\|_{\mathcal{L}^{\overline{p}}} \overset{\text{def}}{=} \|\,\|\,\|u(x_1,x_2,x_3)\|_{L^{p_3}_{x_3}}\|_{L^{p_2}_{x_2}}\|_{L^{p_1}_{x_1}} < \infty\}\end{aligned}$$

and $\ell^{\overline{p}}$ be the analogous space for sequences. Also, when $p=q=r$ we shall note $\ell^{p,p,p}=\ell^p$ and $\mathcal{L}^{p,p,p}=L^p$. If u is a function $u:(0,T)\times\mathbb{R}^n\longrightarrow\mathbb{C}$ then we note

$$\|u\|_{L^p_T(L^q)}\stackrel{\text{def}}{=}\|\,\|u(t,x)\|_{L^q(\mathbb{R}^n)}\|_{L^p(0,T)}\,.$$

The order of integrations is important, as the following remark shows it:

REMARK 1.1. Let (X_1,μ_1), (X_2,μ_2) be two measure spaces, $1\le p\le q$ and $f:X\times Y\longrightarrow\mathbb{R}$. Then

$$\|\,\|f(\cdot,x_2)\|_{L^p(X_1,\mu_1)}\|_{L^q(X_2,\mu_2)}\le\|\,\|f(x_1,\cdot)\|_{L^q(X_2,\mu_2)}\|_{L^p(X_1,\mu_1)}\,.$$

Indeed

$$\begin{aligned}\|\,\|f(\cdot,x_2)\|_{L^p(X_1,\mu_1)}\|_{L^q(X_2,\mu_2)}&=\Big(\Big\|\int_{X_1}f^p(\cdot,x_2)\,d\mu_1\Big\|_{L^{q/p}(X_2,\mu_2)}\Big)^{1/p}\\&\le\Big(\int_{X_1}\|f^p(x_1,\cdot)\|_{L^{q/p}(X_2,\mu_2)}\,d\mu_1\Big)^{1/p}\\&=\|\,\|f(x_1,\cdot)\|_{L^q(X_2,\mu_2)}\|_{L^p(X_1,\mu_1)}\,.\end{aligned}$$

The Hölder and Young inequalities for the $\mathcal{L}^{\overline{q}}$ spaces take the form

$$\|f\,g\|_{\mathcal{L}^{\overline{p}}}\le\|f\|_{\mathcal{L}^{\overline{q}}}\|g\|_{\mathcal{L}^{\overline{r}}}\,,$$

where

$$\frac{1}{p_i}=\frac{1}{q_i}+\frac{1}{r_i}\,,$$

for all $i\in\{1,2,3\}$, and

$$\|f*g\|_{\mathcal{L}^{\overline{a}}}\le\|f\|_{\mathcal{L}^{\overline{b}}}\|g\|_{\mathcal{L}^{\overline{c}}}\,,$$

where

$$1+\frac{1}{a_i}=\frac{1}{b_i}+\frac{1}{c_i}\,,$$

for all $i\in\{1,2,3\}$.

We can prove a variant of the Littlewood-Paley lemma for the $\mathcal{L}^{\overline{q}}$ spaces:

Lemma 1.1. *If*

$$\begin{aligned}\operatorname{supp} \hat{u} \subset B(0, r\,\lambda_1, r\,\lambda_2, r\,\lambda_3) \\ \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3 \ \textit{such that}\ |\xi_1| < r\,\lambda_1, |\xi_2| < r\,\lambda_2, |\xi_3| < r\,\lambda_3\}\end{aligned}$$

and $a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3$, $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ *is a multi-index, then*

$$\|\partial^\alpha u\|_{\mathcal{L}^{\overline{b}}} \leq C\,\lambda_1^{\alpha_1+1/a_1-1/b_1}\,\lambda_2^{\alpha_2+1/a_2-1/b_2}\,\lambda_3^{\alpha_3+1/a_3-1/b_3}\,\|u\|_{\mathcal{L}^{\overline{a}}}\,.$$

PROOF. Let $\phi \in C_0^\infty(\mathbb{R})$, ϕ equal to 1 near the ball of center 0 and radius r, $g = \mathcal{F}^{-1}(\phi)$. Then

$$\hat{u}(\xi) = \phi\Big(\frac{\xi_1}{\lambda_1}\Big)\phi\Big(\frac{\xi_2}{\lambda_2}\Big)\phi\Big(\frac{\xi_3}{\lambda_3}\Big)\hat{u}(\xi)\,,$$

and thus

$$u(x) = \lambda_1\,\lambda_2\,\lambda_3 \int_{\mathbb{R}^3} g(\lambda_1\,y_1)\,g(\lambda_2\,y_2)\,g(\lambda_3\,y_3)\,u(x-y)\,dy\,.$$

Differentiating and using Young's inequality ends the proof.

Before introducing our functional spaces let us recall that the homogeneous Besov spaces are defined to be the closure of compactly supported smooth functions under the norm

$$\|u\|_{B^s_{p,q}} \stackrel{\text{def}}{=} \|2^{is}\|\Delta_i u\|_{L^p}\|_{\ell^q}\,.$$

The need of taking the closure of compactly supported smooth functions comes from the fact that the quantity above is only a semi-norm since the "norm" of a polynomial vanishes. Another way of defining these homogeneous spaces is to take equivalence classes of distributions modulo polynomials and to remark that we obtain in that way a real norm. For further details on Besov spaces (homogeneous or not) see [12].

Definition 1.1. *We denote by* $\mathcal{H}^{s_1,s_2,s_3} = \mathcal{H}^{\overline{s}}$ *the closure of compactly supported smooth functions under the norm*

$$|u|_{s_1,s_2,s_3} \stackrel{\text{def}}{=} |u|_{\overline{s}} \stackrel{\text{def}}{=} \|\overline{\xi}^{\overline{s}}\hat{u}(\overline{\xi})\|_{L^2}\,.$$

The space $\mathcal{H}^{s_1,s_2,s_3}$ is a Banach space of distributions if $s_1 < 1/2$, $s_2 < 1/2$ and $s_3 < 1/2$.

We denote by ψ a dyadical partition of unity in $\mathbb{R}$, that is a smooth function supported in the ring of center 0, small radius 3/4, large radius 8/3 and such that $\sum_{q\in\mathbb{Z}} \psi(2^{-q}\,\xi) = 1$ for all $\xi \neq 0$ (see [1], [5]). Define

$$\Delta_q^i = \psi(2^{-q}\,D_i)\,,$$

$$S_q^i = \sum_{p\leq q-1} \Delta_p^i\,,$$

$$S_{\overline{q}} = S_{q_1,q_2,q_3} = S_{q_1}^1 S_{q_2}^2 S_{q_3}^3\,,$$

$$\Delta_{\overline{q}} = \Delta_{q_1,q_2,q_3} = \Delta_{q_1}^1 \Delta_{q_2}^2 \Delta_{q_3}^3\,,$$

$$S_q = S_{q,q,q}\,,$$

$$\Delta_q = S_{q+1} - S_q\,.$$

The following lemmas are easy to prove:

Lemma 1.2. *If $u \in \mathcal{H}^{\overline{s}}$ then*

$$|u|_{\overline{s}} \sim \|2^{\overline{q}\cdot\overline{s}}\,\|\Delta_{\overline{q}}\,u\|_{L^2}\|_{\ell^2}\,.$$

Lemma 1.3. *If $u_{\overline{p}}$ is a sequence of functions such that*

$$\operatorname{supp}\hat{u}_{\overline{p}} \subset \Big\{\frac{1}{\gamma}\,2^{p_1} \leq |\xi_1| \leq \gamma\,2^{p_1}, \frac{1}{\gamma}\,2^{p_2} \leq |\xi_2| \leq \gamma\,2^{p_2}, \frac{1}{\gamma}\,2^{p_3} \leq |\xi_3| \leq \gamma\,2^{p_3}\Big\}$$

and

$$\|2^{\overline{p}\cdot\overline{s}}\,\|u_{\overline{p}}\|_{L^2}\|_{\ell^2} < \infty\,,$$

then

$$u = \sum_{\overline{p}} u_{\overline{p}} \in \mathcal{H}^{s_1,s_2,s_3}$$

and

$$|u|_{s_1,s_2,s_3} \leq C\,\|2^{\overline{p}\cdot\overline{s}}\,\|u_{\overline{p}}\|_{L^2}\|_{\ell^2}\,.$$

If $s_1 > 0$ it is enough to assume that

$$\operatorname{supp} \hat{u}_{\overline{p}} \subset \left\{ |\xi_1| \leq \gamma\, 2^{p_1}, \frac{1}{\gamma}\, 2^{p_2} \leq |\xi_2| \leq \gamma\, 2^{p_2}, \frac{1}{\gamma}\, 2^{p_3} \leq |\xi_3| \leq \gamma\, 2^{p_3} \right\}.$$

If $s_1 > 0$ and $s_2 > 0$ it is enough to assume that

$$\operatorname{supp} \hat{u}_{\overline{p}} \subset \left\{ |\xi_1| \leq \gamma\, 2^{p_1}, |\xi_2| \leq \gamma\, 2^{p_2}, \frac{1}{\gamma}\, 2^{p_3} \leq |\xi_3| \leq \gamma\, 2^{p_3} \right\}.$$

If $s_1 > 0$, $s_2 > 0$ and $s_3 > 0$ it is enough to assume that

$$\operatorname{supp} \hat{u}_{\overline{p}} \subset \{ |\xi_1| \leq \gamma\, 2^{p_1}, |\xi_2| \leq \gamma\, 2^{p_2}, |\xi_3| \leq \gamma\, 2^{p_3} \}.$$

The next theorem studies the problem of products in the $\mathcal{H}^{s_1,s_2,s_3}$ spaces.

Theorem 1.1. *Let $u \in \mathcal{H}^{\overline{s}}$, $v \in \mathcal{H}^{\overline{t}}$ such that $s_i < 1/2$, $t_i < 1/2$, $s_i + t_i > 0$, $i \in \{1,2,3\}$. Then*

$$u\, v \in \mathcal{H}^{\overline{s}+\overline{t}-(1/2,1/2,1/2)}$$

and

$$|u\, v|_{\overline{s}+\overline{t}-(1/2,1/2,1/2)} \leq C\, |u|_{\overline{s}}\, |v|_{\overline{t}}\ .$$

PROOF. We shall give a proof which imitates the argument for the classical Sobolev spaces. This will be done by introducing 3-dimensional paraproduct operators. We recall the definition of Bony's decomposition

$$u\, v = T(u,v) + R(u,v) + \widetilde{T}(u,v)\,,$$

where

$$T(u,v) = \sum_q S_{q-1} u\, \Delta_q v\,,$$

$$R(u,v) = \sum_{|p-q|\leq 1} \Delta_p u\, \Delta_q v\,,$$

$$\widetilde{T}(u,v) = T(v,u)$$

(see [1], [5]). It is well-known that $T : H^s(\mathbb{R}) \times H^t(\mathbb{R}) \longrightarrow H^{s+t-1/2}(\mathbb{R})$ is well-defined and continous if $s < 1/2$. The same is true for R if $s + t > 0$. Here we use the analogous of this decomposition

$$u\,v = (T^1 + R^1 + \widetilde{T}^1)\,(T^2 + R^2 + \widetilde{T}^2)\,(T^3 + R^3 + \widetilde{T}^3)(u, v)$$

understood as the sum of 3^3 terms. The definition of each term is a straightforward extension of the classical paraproduct and remainder. The reader may give the definition of each term; we give, for instance, the one of the term $T^1R^2\widetilde{T}^3(u, v)$

$$T^1R^2\widetilde{T}^3(u, v) = \sum_{i=-1}^{1} \sum_{\overline{p}} S^1_{p_1-1}\,\Delta^2_{p_2}\,\Delta^3_{p_3} u\,\Delta^1_{p_1}\,\Delta^2_{p_2-i}\,S^3_{p_3-1} v\,.$$

We shall prove that each of the 3^3 operators we find is continuous

$$\mathcal{H}^{\overline{s}} \times \mathcal{H}^{\overline{t}} \longrightarrow \mathcal{H}^{\overline{s}+\overline{t}-(1/2,1/2,1/2)}\,,$$

under weaker hypothesis than those given in the theorem. More precisely, the conditions to assume are given by the composition of the term in the following manner: if the term contains T^i then we have to assumme $s_i < 1/2$; if the term contains R^i then we have to assume $s_i + t_i > 0$; if the term contains $\widetilde{T}^i$ then we have to assume $t_i < 1/2$. For instance if we want the term $T^1R^2\widetilde{T}^3$ to be continous then we have to assume that $s_1 < 1/2, s_2 + t_2 > 0, t_3 < 1/2$. This term is the most difficult to handle so we prove the assertion only on it. We have

$$T^1R^2\widetilde{T}^3(u, v) = \sum_{i=-1}^{1} \sum_{\overline{p}} w^i_{\overline{p}}\ ,$$

where

$$w^i_{\overline{p}} = S^1_{p_1-1}\,\Delta^2_{p_2}\,\Delta^3_{p_3} u\,\Delta^1_{p_1}\,\Delta^2_{p_2-i}\,S^3_{p_3-1} v\,.$$

Using several times the anisotropic form of Hölder's inequality, the definition of the operators S^1 and S^3 as well as the anisotropic Littlewood-

Paley Lemma 1.1 one can show that

$$
\begin{aligned}
&\|\Delta_{\overline{q}}\, w^i_{\overline{p}}\|_{L^2} \\
&\le 2^{q_2/2}\, \|w^i_{\overline{p}}\|_{\mathcal{L}^{2,1,2}} \\
&\le 2^{q_2/2}\, \|S^1_{p_1-1}\, \Delta^2_{p_2}\, \Delta^3_{p_3} u\|_{\mathcal{L}^{\infty,2,2}} \|\Delta^1_{p_1}\, \Delta^2_{p_2-i}\, S^3_{p_3-1} v\|_{\mathcal{L}^{2,2,\infty}} \\
&\le 2^{q_2/2} \sum_{\substack{r_1 \le p_1-2 \\ r_3 \le p_3-2}} \|\Delta^1_{r_1}\, \Delta^2_{p_2}\, \Delta^3_{p_3} u\|_{\mathcal{L}^{\infty,2,2}} \|\Delta^1_{p_1}\, \Delta^2_{p_2-i}\, \Delta^3_{r_3} v\|_{\mathcal{L}^{2,2,\infty}} \\
&\le 2^{q_2/2} \sum_{\substack{r_1 \le p_1-2 \\ r_3 \le p_3-2}} 2^{r_1/2+r_3/2} \|\Delta^1_{r_1}\, \Delta^2_{p_2}\, \Delta^3_{p_3} u\|_{L^2} \|\Delta^1_{p_1}\, \Delta^2_{p_2-i}\, \Delta^3_{r_3} v\|_{L^2}\ .
\end{aligned} \tag{1.1}
$$

Let us introduce

$$
a_{\overline{q}} = 2^{\overline{q}\cdot\overline{s}}\, \|\Delta_{\overline{q}}\, u\|_{L^2}
$$

and

$$
b_{\overline{q}} = 2^{\overline{q}\cdot\overline{t}}\, \|\Delta_{\overline{q}}\, v\|_{L^2}\ .
$$

Since $s_1 < 1/2$ and $t_3 < 1/2$, inequality (1.1) implies

$$
\begin{aligned}
\|\Delta_{\overline{q}}\, w^i_{\overline{p}}\|_{L^2} \le C\, 2^{q_2/2}\, 2^{p_1(1/2-s_1-t_1)}\, 2^{-p_2(s_2+t_2)}\, 2^{p_3(1/2-s_3-t_3)} \\
\cdot \|a_{\overline{p}}\|_{\ell^\infty_{p_1}} \|b_{p_1,p_2-i,p_3}\|_{\ell^\infty_{p_3}}\ ,
\end{aligned} \tag{1.2}
$$

whence

$$
\begin{aligned}
2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\, &\|\Delta_{\overline{q}}\, w^i_{\overline{p}}\|_{L^2} \\
&\le C\, 2^{(q_1-p_1)(s_1+t_1-1/2)}\, 2^{(q_3-p_3)(s_3+t_3-1/2)} \\
&\quad \cdot 2^{(q_2-p_2)(s_2+t_2)}\, \|a_{\overline{p}}\|_{\ell^2_{p_1}} \|b_{p_1,p_2-i,p_3}\|_{\ell^2_{p_3}}\ .
\end{aligned} \tag{1.3}
$$

Since $|p_1 - q_1| \le 1$, $q_2 < p_2$, $|p_3 - q_3| \le 1$ we obtain

$$
\begin{aligned}
&2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\, \|\Delta_{\overline{q}} T^1 R^2 \widetilde{T}^3(u,v)\|_{L^2} \\
&\quad \le C \sum_{i=-1}^{1} \sum_{\substack{|p_1-q_1|\le 1 \\ |p_3-q_3|\le 1}} \sum_{p_2>q_2} 2^{(q_2-p_2)(s_2+t_2)} \|a_{\overline{p}}\|_{\ell^2_{p_1}} \|b_{p_1,p_2-i,p_3}\|_{\ell^2_{p_3}}\ .
\end{aligned}
$$

Taking the $\ell^2_{q_1,q_3}$ norm gives

$$\begin{aligned}\|2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\|\Delta_{\overline{q}}T^1R^2\widetilde{T}^3(u,v)\|_{L^2}\|_{\ell^2_{q_1,q_3}}\\ \leq C\sum_{i=-1}^{1}\sum_{p_2>q_2}2^{(q_2-p_2)(s_2+t_2)}\|a_{\overline{p}}\|_{\ell^2_{p_1,p_3}}\|b_{p_1,p_2-i,p_3}\|_{\ell^2_{p_1,p_3}}\ .\end{aligned}$$

Taking the $\ell^2_{q_2}$ norm, applying Young's inequality and using that $s_2 + t_2 > 0$ yields

$$\begin{aligned}\|2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\|\Delta_{\overline{q}}T^1R^2\widetilde{T}^3(u,v)\|_{L^2}\|_{\ell^2}\\ \leq C\sum_{i=-1}^{1}\|\,\|a_{\overline{p}}\|_{\ell^2_{p_1,p_3}}\|b_{p_1,p_2-i,p_3}\|_{\ell^2_{p_1,p_3}}\|_{\ell^1_{p_2}}\ .\end{aligned}$$

Finally, Hölder's inequality implies

$$\|2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\|\Delta_{\overline{q}}T^1R^2\widetilde{T}^3(u,v)\|_{L^2}\|_{\ell^2}\leq C\,\|a_{\overline{p}}\|_{\ell^2}\,\|b_{\overline{p}}\|_{\ell^2}\ ,$$

that is

$$|T^1R^2\widetilde{T}^3(u,v)|_{\overline{s}+\overline{t}-(1/2,1/2,1/2)}\leq C\,|u|_{\overline{s}}\,|v|_{\overline{t}}\ .$$

This completes the proof.

We shall now adjust this study to the case of the spaces $HB^{s_1,s_2,s_3} = HB^{\overline{s}}$ defined as the closure of compactly supported smooth functions under the norm

$$|u|_{HB^{\overline{s}}}\stackrel{\text{def}}{=}\|2^{\overline{q}\cdot\overline{s}}\|\Delta_{\overline{q}}u\|_{L^2}\|_{\ell^{2,2,1}}\ .$$

REMARK 1.2. In this definition, when we apply the $\ell^{2,2,1}$ norm, we first take the ℓ^1 norm and afterwards the others, but all the work we do is valid for the spaces HB obtained by appling the $\ell^{2,2,1}$ norm in an arbitrary manner. We choosed this order because, according to Remark 1.1, this space is the largest.

REMARK 1.3. For all real numbers s_1, s_2, s_3 the space $HB^{\overline{s}}$ is strictly included into the space $\mathcal{H}^{\overline{s}}$. Moreover, $HB^{\overline{s}}$ is a Banach space of distributions for $s_1 < 1/2$, $s_2 < 1/2$ and $s_3 \leq 1/2$.

The lemmas 1.2 and 1.3 will modify in an obvious way, only the product theorem is relevant for the (N-S) equations.

Theorem 1.2. *Let $u \in HB^{\overline{s}}$, $v \in HB^{\overline{t}}$ such that $s_i < 1/2$, $t_i < 1/2$, $s_i + t_i > 0$, $i \in \{1,2\}$ and $s_3 \leq 1/2$, $t_3 \leq 1/2$, $s_3 + t_3 > 0$. Then*

$$u\, v \in HB^{\overline{s}+\overline{t}-(1/2,1/2,1/2)}$$

and

$$|u\, v|_{HB^{\overline{s}+\overline{t}-(1/2,1/2,1/2)}} \leq C\, |u|_{HB^{\overline{s}}}\, |v|_{HB^{\overline{t}}} \ .$$

PROOF. The proof is almost identical to the preceding one, the modification, which allows us to take into account the case $s_3 = 1/2$ or $t_3 = 1/2$ is that the classical paraproduct $T : B^s_{2,1}(\mathbb{R}) \times B^t_{2,1}(\mathbb{R}) \longrightarrow B^{s+t-1/2}_{2,1}(\mathbb{R})$ is well-defined and continous if $s \leq 1/2$. Hence, we shall prove that each of the 3^3 operators is continous under the same assumptions as above, with the modification that if a paraproduct in the third direction is involved, then we can allow s_3 or t_3, depending on the paraproduct, to be equal to $1/2$. The only problem in the proof is that at the end we have to commute some norms which give raise to the wrong inequality. We have to restart from inequality (1.1)

$$\begin{aligned} \|\Delta_{\overline{q}}\, w^i_{\overline{p}}\|_{L^2} \leq 2^{q_2/2} \sum_{\substack{r_1 \leq p_1-2 \\ r_3 \leq p_3-2}} & 2^{r_1/2+r_3/2} \|\Delta^1_{r_1}\, \Delta^2_{p_2}\, \Delta^3_{p_3} u\|_{L^2} \\ & \cdot \|\Delta^1_{p_1}\, \Delta^2_{p_2-i}\, \Delta^3_{r_3} v\|_{L^2} \ . \end{aligned} \tag{1.4}$$

Recall that

$$a_{\overline{q}} = 2^{\overline{q}\cdot\overline{s}}\, \|\Delta_{\overline{q}}\, u\|_{L^2}$$

and

$$b_{\overline{q}} = 2^{\overline{q}\cdot\overline{t}}\, \|\Delta_{\overline{q}}\, v\|_{L^2} \ .$$

We use that $|p_1 - q_1| \leq 1$, $|p_3 - q_3| \leq 1$ to rewrite the last inequality as

$$\begin{aligned} & 2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\, \|\Delta_{\overline{q}}\, w^i_{\overline{p}}\|_{L^2} \\ & \leq C\, 2^{(s_2+t_2)(q_2-p_2)} \sum_{\substack{r_1 \leq p_1-2 \\ r_3 \leq p_3-2}} 2^{(r_1-p_1)(1/2-s_1)+(r_3-p_3)(1/2-t_3)} \\ & \qquad\qquad \cdot a_{r_1,p_2,p_3}\, b_{p_1,p_2-i,r_3} \ . \end{aligned} \tag{1.5}$$

Now we sum on $i, \overline{p}$ and q_3 to obtain

$$\sum_{q_3} 2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\|\Delta_{\overline{q}}T^1R^2\widetilde{T}^3(u,v)\|_{L^2}$$

$$\leq C\sum_{i=-1}^{1}\sum_{\substack{|p_1-q_1|\leq 1\\ p_2>q_2}} 2^{(s_2+t_2)(q_2-p_2)} \cdot \sum_{r_1\leq p_1-2} 2^{(r_1-p_1)(1/2-s_1)} \cdot \sum_{p_3}\sum_{r_3\leq p_3-2} 2^{(r_3-p_3)(1/2-t_3)} \cdot a_{r_1,p_2,p_3}\, b_{p_1,p_2-i,r_3}$$

$$\leq C\sum_{i=-1}^{1}\sum_{\substack{|p_1-q_1|\leq 1\\ p_2>q_2}} 2^{(s_2+t_2)(q_2-p_2)}\sum_{r_1\leq p_1-2} 2^{(r_1-p_1)(1/2-s_1)} \cdot \|a_{r_1,p_2,p_3}\|_{\ell^1_{p_3}}\|b_{p_1,p_2-i,r_3}\|_{\ell^1_{r_3}}$$

$$\leq C\sum_{i=-1}^{1}\sum_{\substack{|p_1-q_1|\leq 1\\ p_2>q_2}} 2^{(s_2+t_2)(q_2-p_2)}\,\|\,\|a_{r_1,p_2,p_3}\|_{\ell^1_{p_3}}\|_{\ell^2_{r_1}}\|b_{p_1,p_2-i,r_3}\|_{\ell^1_{r_3}}\ .$$

Since $|p_1-q_1|\leq 1$, applying Holdër's inequality gives

$$\begin{aligned}&\|\,\|2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\|\Delta_{\overline{q}}T^1R^2\widetilde{T}^3(u,v)\|_{L^2}\|_{\ell^1_{q_3}}\|_{\ell^2_{q_1}}\\ &\leq C\sum_{i=-1}^{1}\sum_{p_2>q_2} 2^{(s_2+t_2)(q_2-p_2)}\,\|\,\|a_{p_1,p_2,p_3}\|_{\ell^1_{p_3}}\|_{\ell^2_{p_1}}\|\,\|b_{p_1,p_2-i,p_3}\|_{\ell^1_{p_3}}\|_{\ell^2_{p_1}}\ .\end{aligned} \tag{1.6}$$

Using that $q_2<p_2$ and applying Young's inequality yields

$$\begin{aligned}\|2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}&\|\Delta_{\overline{q}}T^1R^2\widetilde{T}^3(u,v)\|_{L^2}\|_{\ell^{2,2,1}}\\ &\leq C\sum_{i=-1}^{1}\|\,\|\,\|a_{p_1,p_2,p_3}\|_{\ell^1_{p_3}}\|_{\ell^2_{p_1}}\|\,\|b_{p_1,p_2-i,p_3}\|_{\ell^1_{p_3}}\|_{\ell^2_{p_1}}\|_{\ell^1_{p_2}}\ .\end{aligned}$$

Finally, we apply Hölder's inequality to obtain

$$\|2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\|\Delta_{\overline{q}}T^1R^2\widetilde{T}^3(u,v)\|_{L^2}\|_{\ell^{2,2,1}} \\ \leq C\,\|a_{p_1,p_2,p_3}\|_{\ell^{2,2,1}}\|b_{p_1,p_2,p_3}\|_{\ell^{2,2,1}}\ ,$$

which implies

$$|T^1R^2\widetilde{T}^3(u,v)|_{HB^{\overline{s}+\overline{t}-(1/2,1/2,1/2)}} \leq C\,|u|_{HB^{\overline{s}}}\,|v|_{HB^{\overline{t}}}\ .$$

This completes the proof for $T^1R^2\widetilde{T}^3$.

Since the third variable plays a special role in the definition of the HB spaces, we show how the same estimates can be modified for other terms. We consider first the term $T^1R^2R^3$. We have

$$T^1R^2R^3(u,v) = \sum_{i,j=-1}^{1}\sum_{\overline{p}} z_{\overline{p}}^{i,j}\ ,$$

where

$$z_{\overline{p}}^{i,j} = S^1_{p_1-1}\,\Delta^2_{p_2}\,\Delta^3_{p_3}u\,\Delta^1_{p_1}\,\Delta^2_{p_2-i}\,\Delta^3_{p_3-j}\,v\,.$$

As above, we deduce the following inequalities

$$\begin{aligned}
&\|\Delta_{\overline{q}}\,z_{\overline{p}}^{i,j}\|_{L^2}\\
&\leq 2^{q_2/2+q_3/2}\,\|z_{\overline{p}}^{i,j}\|_{\mathcal{L}^{2,1,1}}\\
&\leq 2^{q_2/2+q_3/2}\,\|S^1_{p_1-1}\,\Delta^2_{p_2}\,\Delta^3_{p_3}u\|_{\mathcal{L}^{\infty,2,2}}\|\Delta^1_{p_1}\,\Delta^2_{p_2-i}\,\Delta^3_{p_3-j}v\|_{\mathcal{L}^{2,2,2}}\\
&\leq 2^{q_2/2+q_3/2}\sum_{r_1\leq p_1-2}\|\Delta^1_{r_1}\,\Delta^2_{p_2}\,\Delta^3_{p_3}u\|_{\mathcal{L}^{\infty,2,2}}\|\Delta^1_{p_1}\,\Delta^2_{p_2-i}\,\Delta^3_{p_3-j}v\|_{\mathcal{L}^{2,2,2}} \qquad (1.7)\\
&\leq 2^{q_2/2+q_3/2}\sum_{r_1\leq p_1-2}2^{r_1/2}\,\|\Delta^1_{r_1}\,\Delta^2_{p_2}\,\Delta^3_{p_3}u\|_{L^2}\|\Delta^1_{p_1}\,\Delta^2_{p_2-i}\,\Delta^3_{p_3-j}v\|_{L^2}\ .
\end{aligned}$$

Since $|p_1-q_1|\leq 1$, it follows that

$$2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\,\|\Delta_{\overline{q}}\,z_{\overline{p}}^{i,j}\|_{L^2} \\ \leq C\,2^{(s_2+t_2)(q_2-p_2)+(s_3+t_3)(q_3-p_3)} \\ \cdot\sum_{r_1\leq p_1-2}2^{(r_1-p_1)(1/2-s_1)}\,a_{r_1,p_2,p_3}\,b_{p_1,p_2-i,p_3-j}\ .$$

Now we sum on $i, j, \overline{p}$ and q_3 to obtain

$$\sum_{q_3} 2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))} \|\Delta_{\overline{q}} T^1 R^2 R^3(u,v)\|_{L^2}$$

$$\leq C \sum_{i,j=-1}^{1} \sum_{\substack{|p_1-q_1|\leq 1 \\ p_2>q_2}} 2^{(s_2+t_2)(q_2-p_2)}$$

$$\cdot \sum_{r_1\leq p_1-2} 2^{(r_1-p_1)(1/2-s_1)}$$

$$\cdot \sum_{q_3} \sum_{p_3>q_3} 2^{(s_3+t_3)(q_3-p_3)}$$

$$\cdot\, a_{r_1,p_2,p_3}\, b_{p_1,p_2-i,p_3-j} \ .$$

Applying Young's inequality gives

$$\sum_{q_3} \sum_{p_3>q_3} 2^{(s_3+t_3)(q_3-p_3)} a_{r_1,p_2,p_3}\, b_{p_1,p_2-i,p_3-j}$$

$$\leq C \,\|a_{r_1,p_2,p_3}\, b_{p_1,p_2-i,p_3}\|_{\ell^1_{p_3}}$$

$$\leq C \,\|a_{r_1,p_2,p_3}\|_{\ell^1_{p_3}} \|b_{p_1,p_2-i,p_3}\|_{\ell^1_{p_3}} .$$

It follows that

$$\sum_{q_3} 2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))} \|\Delta_{\overline{q}} T^1 R^2 R^3(u,v)\|_{L^2}$$

$$\leq C \sum_{i=-1}^{1} \sum_{\substack{|p_1-q_1|\leq 1 \\ p_2>q_2}} 2^{(s_2+t_2)(q_2-p_2)}$$

$$\cdot \sum_{r_1\leq p_1-2} 2^{(r_1-p_1)(1/2-s_1)} \|a_{r_1,p_2,p_3}\|_{\ell^1_{p_3}} \|b_{p_1,p_2-i,p_3}\|_{\ell^1_{p_3}}$$

$$\leq C \sum_{i=-1}^{1} \sum_{\substack{|p_1-q_1|\leq 1 \\ p_2>q_2}} 2^{(s_2+t_2)(q_2-p_2)} \| \|a_{r_1,p_2,p_3}\|_{\ell^1_{p_3}} \|_{\ell^2_{r_1}} \|b_{p_1,p_2-i,p_3}\|_{\ell^1_{p_3}} \ .$$

Since $|p_1 - q_1| \leq 1$, applying Holdër's inequality gives

$$
\begin{aligned}
&\| \|2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))} \|\Delta_{\overline{q}} T^1 R^2 R^3(u,v)\|_{L^2} \|_{\ell^1_{q_3}} \|_{\ell^2_{q_1}} \\
&\qquad \leq C \sum_{i=-1}^{1} \sum_{p_2 > q_2} 2^{(s_2+t_2)(q_2-p_2)} \\
&\qquad\qquad \cdot \| \|a_{p_1,p_2,p_3}\|_{\ell^1_{p_3}} \|_{\ell^2_{p_1}} \| \|b_{p_1,p_2-i,p_3}\|_{\ell^1_{p_3}} \|_{\ell^2_{p_1}} \ .
\end{aligned}
$$

This inequality is similar to (1.6), so we can continue likewise to obtain the result on $T^1 R^2 R^3$.

Finally, we give the proof for the term $T^1 \widetilde{T}^2 R^3$. As above we have

$$
T^1 \widetilde{T}^2 R^3(u,v) = \sum_{i=-1}^{1} \sum_{\overline{p}} \alpha^i_{\overline{p}} \ ,
$$

where

$$
\alpha^i_{\overline{p}} = S^1_{p_1-1}\, \Delta^2_{p_2}\, \Delta^3_{p_3} u\, \Delta^1_{p_1}\, S^2_{p_2-1}\, \Delta^3_{p_3-i} v \ .
$$

As above, we deduce the following inequalities

$$
\begin{aligned}
&\|\Delta_{\overline{q}}\, \alpha^i_{\overline{p}}\|_{L^2} \\
&\leq 2^{q_3/2}\, \|\alpha^i_{\overline{p}}\|_{\mathcal{L}^{2,2,1}} \\
&\leq 2^{q_3/2}\, \|S^1_{p_1-1}\, \Delta^2_{p_2}\, \Delta^3_{p_3} u\|_{\mathcal{L}^{\infty,2,2}} \|\Delta^1_{p_1}\, S^2_{p_2-1}\, \Delta^3_{p_3-i} v\|_{\mathcal{L}^{2,\infty,2}} \\
&\leq 2^{q_3/2} \sum_{\substack{r_1 \leq p_1-2 \\ r_2 \leq p_2-1}} \|\Delta^1_{r_1}\, \Delta^2_{p_2}\, \Delta^3_{p_3} u\|_{\mathcal{L}^{\infty,2,2}} \|\Delta^1_{p_1}\, \Delta^2_{r_2}\, \Delta^3_{p_3-i} v\|_{\mathcal{L}^{2,\infty,2}} \\
&\leq 2^{q_3/2} \sum_{\substack{r_1 \leq p_1-2 \\ r_2 \leq p_2-1}} 2^{r_1/2+r_2/2}\, \|\Delta^1_{r_1}\, \Delta^2_{p_2}\, \Delta^3_{p_3} u\|_{L^2} \|\Delta^1_{p_1}\, \Delta^2_{r_2}\, \Delta^3_{p_3-i} v\|_{L^2} \ .
\end{aligned} \tag{1.8}
$$

Since $|p_1 - q_1| \leq 1$ and $|p_2 - q_2| \leq 1$ it follows that

$$
\begin{aligned}
&2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\, \|\Delta_{\overline{q}}\, \alpha^i_{\overline{p}}\|_{L^2} \\
&\qquad \leq C\, 2^{(s_3+t_3)(q_3-p_3)} \sum_{\substack{r_1 \leq p_1-2 \\ r_3 \leq p_3-2}} 2^{(r_1-p_1)(1/2-s_1)+(r_2-p_2)(1/2-t_2)} \\
&\qquad\qquad \cdot a_{r_1,p_2,p_3}\, b_{p_1,r_2,p_3-i} \ .
\end{aligned}
$$

Now we sum on i, $\overline{p}$ and q_3 to obtain

$$\sum_{q_3} 2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))} \|\Delta_{\overline{q}} T^1 \widetilde{T}^2 R^3(u,v)\|_{L^2}$$

$$\leq C \sum_{i=-1}^{1} \sum_{\substack{|p_1-q_1|\leq 1\\ |p_2-q_2|\leq 1}} \sum_{\substack{r_1\leq p_1-2\\ r_2\leq p_2-2}} 2^{(r_1-p_1)(1/2-s_1)+(r_2-p_2)(1/2-t_2)}$$
$$\cdot \sum_{q_3} \sum_{p_3>q_3} 2^{(s_3+t_3)(q_3-p_3)} \, a_{r_1,p_2,p_3} \, b_{p_1,r_2,p_3-i} \ .$$

Applying Young's inequality gives

$$\begin{aligned}\sum_{q_3} \sum_{p_3>q_3} 2^{(s_3+t_3)(q_3-p_3)} \, a_{r_1,p_2,p_3} \, b_{p_1,r_2,p_3-i} &\leq C \, \|a_{r_1,p_2,p_3} \, b_{p_1,r_2,p_3-i}\|_{\ell^1_{p_3}} \\ &\leq C \, \|a_{r_1,p_2,p_3}\|_{\ell^1_{p_3}} \, \|b_{p_1,r_2,p_3}\|_{\ell^1_{p_3}} \ .\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{q_3} 2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))} \|\Delta_{\overline{q}} T^1 \widetilde{T}^2 R^3(u,v)\|_{L^2} &\leq C \sum_{\substack{|p_1-q_1|\leq 1\\ |p_2-q_2|\leq 1}} \sum_{\substack{r_1\leq p_1-2\\ r_2\leq p_2-1}} 2^{(r_1-p_1)(1/2-s_1)+(r_2-p_2)(1/2-t_2)} \\ &\qquad \cdot \|a_{r_1,p_2,p_3}\|_{\ell^1_{p_3}} \|b_{p_1,r_2,p_3}\|_{\ell^1_{p_3}} \\ &\leq C \sum_{\substack{|p_1-q_1|\leq 1\\ |p_2-q_2|\leq 1}} \| \, \|a_{r_1,p_2,p_3}\|_{\ell^1_{p_3}} \|_{\ell^2_{r_1}} \, \| \, \|b_{p_1,r_2,p_3}\|_{\ell^1_{p_3}} \|_{\ell^2_{r_2}} \ .\end{aligned}$$

Using again that $|p_1-q_1|\leq 1$, $|p_2-q_2|\leq 1$ and taking the $\ell^2_{q_1,q_2}$ norm yields

$$\|2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))} \|\Delta_{\overline{q}} T^1 \widetilde{T}^2 R^3(u,v)\|_{L^2}\|_{\ell^{2,2,1}} \leq C \, \|a_{\overline{p}}\|_{\ell^{2,2,1}} \|b_{\overline{p}}\|_{\ell^{2,2,1}} \ ,$$

that is

$$|T^1 \widetilde{T}^2 R^3(u,v)|_{HB^{\overline{s}+\overline{t}-(1/2,1/2,1/2)}} \leq C \, |u|_{HB^{\overline{s}}} \, |v|_{HB^{\overline{t}}} \ .$$

This completes the proof.

2. Resolution of (N-S) in the $\mathcal{H}^{s_1,s_2,s_3}$ spaces.

Let $-1/2 < \delta_i < 1/2$, $i \in \{1,2,3\}$, $\delta_1 + \delta_2 + \delta_3 = 1/2$. Then there exist nonnegative numbers a_1, a_2, a_3 such that

$$(2.1)\quad 0 < \delta_i + a_i < \frac{1}{2}\,, \qquad \text{for all } i \in \{1,2,3\} \text{ and } a_1 + a_2 + a_3 = \frac{1}{2}$$

(one can choose $a_i = 1/4 - \delta_i/2$). We shall prove the following theorems:

Theorem 2.1 (global existence and uniqueness). *There exists $C > 0$ such that if $\operatorname{div} u_0 = 0$, $u_0 \in \mathcal{H}^{\overline{\delta}}$ and $|u|_{\overline{\delta}} < C\nu$ then the* (N-S) *equations have a unique solution in*

$$L^4(]0,\infty[\,;\mathcal{H}^{\overline{\delta}+\overline{a}}) \cap L^\infty(]0,\infty[\,;\mathcal{H}^{\overline{\delta}})\,.$$

Moreover, the solution satisfies $u \in \mathcal{C}([0,\infty[\,;\mathcal{H}^{\overline{\delta}})$.

Theorem 2.2 (local existence and uniqueness). *If $\operatorname{div} u_0 = 0$ and $u_0 \in \mathcal{H}^{\overline{\delta}}$ then a time $T>0$ and a unique solution of* (N-S) *on $[0,T]$ exist so that*

$$u \in L^4(]0,T[\,;\mathcal{H}^{\overline{\delta}+\overline{a}}) \cap \mathcal{C}([0,T[\,;\mathcal{H}^{\overline{\delta}})\,.$$

The uniqueness is proved at the end. The global existence is proved in the same time with the local existence. In fact, we shall prove a better result valid for the space H_T defined as the closure of compactly supported smooth functions under the norm

$$\|u\|_{H_T} \stackrel{\text{def}}{=} \|\,\|2^{\overline{q}\cdot(\overline{\delta}+\overline{a})}\Delta_{\overline{q}} u\|_{L^4_T(L^2)}\|_{\ell^2}\,.$$

Theorem 2.3. *Let $\operatorname{div} u_0 = 0$ and $u_0 \in \mathcal{H}^{\overline{\delta}}$. Then there exist $T > 0$ and a solution of* (N-S) *on $[0,T]$ which verifies $u \in H_T$.*

REMARK 2.1. We have $H_T \hookrightarrow L^4(]0,T[\,;\mathcal{H}^{\overline{\delta}+\overline{a}})$.

Indeed, from Remark 1.1 we infer

$$\begin{aligned}\|u\|_{L^4(]0,T[;\mathcal{H}^{\overline{\delta}+\overline{a}})} &= \|\,\|2^{\overline{q}\cdot(\overline{\delta}+\overline{a})}\|\Delta_{\overline{q}}u\|_{L^2}\|_{\ell^2}\|_{L^4}\\ &\leq \|\,\|2^{\overline{q}\cdot(\overline{\delta}+\overline{a})}\Delta_{\overline{q}}u\|_{L^4_T(L^2)}\|_{\ell^2}\\ &= \|u\|_{H_T}\,.\end{aligned}$$

PROOF OF THEOREM 2.3. We approach u_0 with the sequence $u_0^n = S_n u_0$, where S_n is the classical S_n in $\mathbb{R}^3$. Let u_n be the local regular solution of (N-S) with initial data u_0^n (for the existence of u_n see [6], [11]). For each n we apply $\Delta_{\overline{q}}$ at (N-S) and we multiply by $\Delta_{\overline{q}}u_n$ to obtain

$$\begin{aligned}\frac{d}{dt}\|\Delta_{\overline{q}}u_n\|_{L^2}^2 + \nu\,\|\nabla\Delta_{\overline{q}}u_n\|_{L^2}^2 &\leq C\,|\langle\Delta_{\overline{q}}(u_n\nabla u_n)|\Delta_{\overline{q}}u_n\rangle|\\ &= C\,|\langle\Delta_{\overline{q}}(\operatorname{div}(u_n\otimes u_n))|\Delta_{\overline{q}}u_n\rangle|\,.\end{aligned}\tag{2.2}$$

The localization of the Fourier transform of $\Delta_{\overline{q}}u_n$ enables us to say that

$$\begin{aligned}\|\nabla\Delta_{\overline{q}}u_n\|_{L^2}^2 &= \|\partial_1\Delta_{\overline{q}}u_n\|_{L^2}^2 + \|\partial_2\Delta_{\overline{q}}u_n\|_{L^2}^2 + \|\partial_3\Delta_{\overline{q}}u_n\|_{L^2}^2\\ &\geq C\,4^{q_1}\,\|\Delta_{\overline{q}}u_n\|_{L^2}^2 + C\,4^{q_2}\,\|\Delta_{\overline{q}}u_n\|_{L^2}^2 + C\,4^{q_3}\,\|\Delta_{\overline{q}}u_n\|_{L^2}^2\\ &= C\,(4^{q_1}+4^{q_2}+4^{q_3})\,\|\Delta_{\overline{q}}u_n\|_{L^2}^2\,.\end{aligned}$$

Moreover, we have from Theorem 1.1 that if $u_n \in \mathcal{H}^{\overline{\delta}+\overline{a}}$, then $u_n\otimes u_n \in \mathcal{H}^{2\overline{\delta}+2\overline{a}-(1/2,1/2,1/2)}$. Thus we can write

$$\operatorname{div}(u_n\otimes u_n) = \sum_{j=1}^{3} w_j\ ,$$

where

$$\begin{aligned}|w_1|_{2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2)} &\leq C\,|u_n\otimes u_n|_{2\overline{\delta}+2\overline{a}-(1/2,1/2,1/2)} \leq C\,|u_n|_{\overline{\delta}+\overline{a}}^2\ ,\\ |w_2|_{2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2)} &\leq C\,|u_n\otimes u_n|_{2\overline{\delta}+2\overline{a}-(1/2,1/2,1/2)} \leq C\,|u_n|_{\overline{\delta}+\overline{a}}^2\ ,\\ |w_3|_{2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2)} &\leq C\,|u_n\otimes u_n|_{2\overline{\delta}+2\overline{a}-(1/2,1/2,1/2)} \leq C\,|u_n|_{\overline{\delta}+\overline{a}}^2\ .\end{aligned}$$

It follows that

$$
\begin{aligned}
&|\langle \Delta_{\overline{q}}\,(u_n \nabla u_n)\,|\Delta_{\overline{q}}\,u_n\rangle| \\
&\qquad \leq C\,(2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))} + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))} \\
&\qquad\qquad + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))})\, a_{\overline{q}}\,\|\Delta_{\overline{q}}\,u_n\|_{L^2}\,|u_n|^2_{\overline{\delta}+\overline{a}}\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
a_{\overline{q}} = &\ \frac{2^{\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))}\,\|\Delta_{\overline{q}}\,w_1\|_{L^2}}{|w_1|_{2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2)}} \\
&+ \frac{2^{\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))}\,\|\Delta_{\overline{q}}\,w_2\|_{L^2}}{|w_2|_{2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2)}} \\
&+ \frac{2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))}\,\|\Delta_{\overline{q}}\,w_3\|_{L^2}}{|w_3|_{2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2)}}
\end{aligned}
$$

so $\|a_{\overline{q}}(\tau)\|_{\ell^2} \leq 3$ for all τ. Using this in (2.2) leads to

$$
\begin{aligned}
&\frac{d}{dt}\,\|\Delta_{\overline{q}}\,u_n\|^2_{L^2} + C\,\nu\,(4^{q_1} + 4^{q_2} + 4^{q_3})\,\|\Delta_{\overline{q}}\,u_n\|^2_{L^2} \\
&\qquad \leq C\,(2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))} + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))} \\
&\qquad\qquad + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))}) \\
&\qquad\quad \cdot\, a_{\overline{q}}\,|u_n|^2_{\overline{\delta}+\overline{a}}\,\|\Delta_{\overline{q}}\,u_n\|_{L^2}\ .
\end{aligned}
$$

By Gronwall's lemma we have

$$
\begin{aligned}
&\|\Delta_{\overline{q}}\,u_n(t)\|_{L^2} \\
&\quad \leq \|\Delta_{\overline{q}}\,u_0^n\|_{L^2}\,\exp\left(-C\,\nu\,(4^{q_1} + 4^{q_2} + 4^{q_3})\,t\right) \\
&\quad + C\,(2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))} + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))} \\
&\qquad + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))}) \\
&\qquad \cdot \int_0^t \exp\left(-C\,\nu\,(4^{q_1} + 4^{q_2} + 4^{q_3})\,(t-\tau)\right) a_{\overline{q}}(\tau)\,|u_n(\tau)|^2_{\overline{\delta}+\overline{a}}\,d\tau\ .
\end{aligned}
\tag{2.3}
$$

Taking the $L^4(0,T)$ norm and using Young's inequality gives

$$\begin{aligned}
&\|\Delta_{\overline{q}}\, u_n(t)\|_{L^4_T(L^2)} \\
&\le C\,\nu^{-1/4}\,(4^{q_1}+4^{q_2}+4^{q_3})^{-1/4}\,\|\Delta_{\overline{q}}\, u_0^n\|_{L^2} \\
&\quad\cdot (1-\exp(-C\,\nu\,T\,(4^{q_1}+4^{q_2}+4^{q_3})))^{1/4} \\
&\quad + C\,(2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))} + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))} \\
&\qquad + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))}) \\
&\qquad \cdot \|\exp(-C\,\nu\,(4^{q_1}+4^{q_2}+4^{q_3})(\cdot))\|_{L^{4/3}(0,T)}\,\|a_{\overline{q}}\,|u_n|^2_{\overline{\delta}+\overline{a}}\|_{L^2(0,T)} \\
&\le C\,\nu^{-1/4}\,(4^{q_1}+4^{q_2}+4^{q_3})^{-1/4}\,\|\Delta_{\overline{q}}\, u_0^n\|_{L^2} \\
&\quad\cdot (1-\exp(-C\,\nu\,T\,(4^{q_1}+4^{q_2}+4^{q_3})))^{1/4} \\
&\quad + C\,\nu^{-3/4}\,(2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))} + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))} \\
&\qquad + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))}) \\
&\qquad \cdot (4^{q_1}+4^{q_2}+4^{q_3})^{-3/4}\,\|a_{\overline{q}}\,|u_n|^2_{\overline{\delta}+\overline{a}}\|_{L^2(0,T)}\;.
\end{aligned}$$

Young's inequality along with relation (2.1) imply

$$\begin{aligned}
2^{\overline{q}\cdot\overline{a}} &= 2^{q_1 a_1}\,2^{q_2 a_2}\,2^{q_3 a_3} \\
&\le 2\,a_1\,2^{2q_1} + 2\,a_2\,2^{2q_2} + 2\,a_3\,2^{2q_3} \\
&\le (4^{q_1}+4^{q_2}+4^{q_3})\,,
\end{aligned}$$

$$\begin{aligned}
&2^{q_1(1-2(a_1+\delta_1)/3)}\,2^{q_2(1/3-2(a_2+\delta_2)/3)}\,2^{q_3(1/3-2(a_3+\delta_3)/3)} \\
&\qquad \le \Big(1-\frac{2}{3}\,(a_1+\delta_1)\Big)2^{q_1} + \Big(\frac{1}{3}-\frac{2}{3}\,(a_2+\delta_2)\Big)2^{q_2} \\
&\qquad\quad + \Big(\frac{1}{3}-\frac{2}{3}\,(a_3+\delta_3)\Big)2^{q_3} \\
&\qquad \le 2^{q_1}+2^{q_2}+2^{q_3}\,,
\end{aligned}$$

and two similar inequalities. Therefore

$$(4^{q_1}+4^{q_2}+4^{q_3})^{-1/4} \le 2^{-\overline{q}\cdot\overline{a}}\,,$$

$$
\begin{aligned}
&2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))}\,(2^{q_1}+2^{q_2}+2^{q_3})^{-3/2}\\
&\qquad\le 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))}\,2^{-3(\overline{q}\cdot((1,1/3,1/3)-2(\overline{a}+\overline{\delta})/3))/2}\\
&\qquad= 2^{-\overline{q}\cdot(\overline{a}+\overline{\delta})}\,,
\end{aligned}
$$

$$
\begin{aligned}
&2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))}\,(2^{q_1}+2^{q_2}+2^{q_3})^{-3/2}\\
&\qquad\le 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))}\,2^{-3(\overline{q}\cdot((1/3,1,1/3)-2(\overline{a}+\overline{\delta})/3))/2}\\
&\qquad= 2^{-\overline{q}\cdot(\overline{a}+\overline{\delta})}\,,
\end{aligned}
$$

$$
\begin{aligned}
&2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))}\,(2^{q_1}+2^{q_2}+2^{q_3})^{-3/2}\\
&\qquad\le 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))}\,2^{-3(\overline{q}\cdot((1/3,1/3,1)-2(\overline{a}+\overline{\delta})/3))/2}\\
&\qquad= 2^{-\overline{q}\cdot(\overline{a}+\overline{\delta})}\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
2^{\overline{q}\cdot(\overline{a}+\overline{\delta})}\,\|\Delta_{\overline{q}}\,u_n\|_{L^4_T(L^2)} &\le C\,\nu^{-1/4}\,2^{\overline{q}\cdot\overline{\delta}}\,\|\Delta_{\overline{q}}\,u_0^n\|_{L^2}\\
&\quad\cdot(1-\exp(-C\,\nu\,T\,(4^{q_1}+4^{q_2}+4^{q_3})))^{1/4}\\
&\quad+C\,\nu^{-3/4}\,\|a_{\overline{q}}\,|u_n|^2_{\overline{\delta}+\overline{a}}\|_{L^2(0,T)}\;.
\end{aligned}
\tag{2.4}
$$

Taking the ℓ^2 norm gives

$$
\begin{aligned}
&\|u_n\|_{H_T}\\
&\le C\,\nu^{-1/4}\,\|2^{\overline{q}\cdot\overline{\delta}}\,\|\Delta_{\overline{q}}\,u_0^n\|_{L^2}(1-\exp(-C\,\nu\,T\,(4^{q_1}+4^{q_2}+4^{q_3})))^{1/4}\|_{\ell^2}\\
&\quad+C\,\nu^{-3/4}\,\|u_n(\tau)\|^2_{L^4(]0,T[;\mathcal{H}^{\overline{\delta}+\overline{a}})}\\
&\le \nu^{-1/4}f_n(T)+C\,\nu^{-3/4}\,\|u_n\|^2_{H_T}\,,
\end{aligned}
\tag{2.5}
$$

where

$$
f_n(T)=C\,\|2^{\overline{q}\cdot\overline{\delta}}\,\|\Delta_{\overline{q}}\,u_0^n\|_{L^2}\,(1-\exp(-C\,\nu\,T\,(4^{q_1}+4^{q_2}+4^{q_3})))^{1/4}\|_{\ell^2}\,.
$$

We shall need to have $f_n(T)$ small. In order to obtain that, we use Lebesgue's dominated convergence theorem. The particular form of u_0^n implies

$$
\|\Delta_{\overline{q}}\,u_0^n\|_{L^2}\le\|\Delta_{\overline{q}}\,S_n\,u_0\|_{L^2}\le\|S_n\,\Delta_{\overline{q}}\,u_0\|_{L^2}\le\|\Delta_{\overline{q}}\,u_0\|_{L^2}
$$

and the estimate

$$2^{\overline{q}\cdot\overline{\delta}}\,\|\Delta_{\overline{q}}\, u_0^n\|_{L^2}\,(1-\exp\left(-C\,\nu\, T\,(4^{q_1}+4^{q_2}+4^{q_3})\right))^{1/4}\leq 2^{\overline{q}\cdot\overline{\delta}}\,\|\Delta_{\overline{q}}\, u_0\|_{L^2}$$

fulfills the domination requirement since the right side is an ℓ^2 sequence that is independent of T and n. As for the pointwise convergence, for fixed $\overline{q}$ one has

$$\begin{aligned}2^{\overline{q}\cdot\overline{\delta}}\,\|\Delta_{\overline{q}}\, u_0^n\|_{L^2}^2\,&(1-\exp\left(-C\,\nu\, T\,(4^{q_1}+4^{q_2}+4^{q_3})\right))^{1/4}\\&\leq |u_0|_{\overline{\delta}}\,(1-\exp\left(-C\,\nu\, T\,(4^{q_1}+4^{q_2}+4^{q_3})\right))^{1/4}\overset{T\to 0}{\longrightarrow} 0\,.\end{aligned}$$

So, by Lebesgue, $\lim_{T\to 0} f_n(T)=0$ uniformly with respect to n. We choose T small enough such that $f_n(T)<\nu/(4\,C)$, where C is the constant from inequality (2.5). It follows that

$$\|u_n\|_{H_T}<\frac{\nu^{3/4}}{4\,C}+C\,\nu^{-3/4}\,\|u_n\|_{H_T}^2\,.$$

We deduce that $\|u_n\|_{H_T}\leq\nu^{3/4}/(2\,C)$ if we take into account that $\|u_n\|_{H_T}$ is continuous in T, $\|u_n\|_{H_0}=0$ and

$$\frac{\nu^{3/4}}{2\,C}=\frac{\nu^{3/4}}{4\,C}+C\,\nu^{-3/4}\Big(\frac{\nu^{3/4}}{2\,C}\Big)^2\,.$$

This allows us to take the limit and to find the existence of the solution on $[0,T]$.

Proof of the global existence. We start again from inequality (2.5) and we estimate $f_n(t)\leq C|u_0|_{\overline{\delta}}$. We find in the same way the existence of a solution in $L^4(]0,\infty[\,;\mathcal{H}^{\overline{\delta}+\overline{a}})$. Next we prove that such a solution belongs to $L^\infty(]0,\infty[\,;\mathcal{H}^{\overline{\delta}})$.

We start again from inequality (2.3), we apply the L^∞ norm and making similar computations we find

$$\begin{aligned}2^{\overline{q}\cdot\overline{\delta}}\,&\|\Delta_{\overline{q}}\, u\|_{L^\infty_T(L^2)}\\&\leq C\,2^{\overline{q}\cdot\overline{\delta}}\,\|\Delta_{\overline{q}}\, u_0\|_{L^2}+C\,\nu^{-1/2}\,\|a_{\overline{q}}\,|u|_{\overline{\delta}+\overline{a}}^2\|_{L^2(0,T)}\,.\end{aligned}\tag{2.6}$$

Taking the ℓ^2 norm yields

$$\|u\|_{L^\infty(]0,\infty[;\mathcal{H}^{\overline{\delta}})}\leq|u_0|_{\overline{\delta}}+C\,\nu^{-1/2}\,(\|u\|_{L^4(]0,\infty[;\mathcal{H}^{\overline{\delta}+\overline{a}})})^2\,.\tag{2.7}$$

Finally, the continuity in time follows from Lebesgue's dominated convergence theorem since the map $t \longrightarrow \|\Delta_{\overline{q}} u_n\|_{L^2}$ is continous and the domination requirement is given in relations (2.6) and (2.7).

Let us now prove the uniqueness.

Theorem 2.4 (uniqueness). *Let u_1 and u_2 be two solutions of* (N-S) *which belong to the space $L^4(]0,T[\,;\mathcal{H}^{\overline{\delta}+\overline{\alpha}}) \cap \mathcal{C}([0,T[\,;\mathcal{H}^{\overline{\delta}})$ with the same initial data in $\mathcal{H}^{\overline{\delta}}$. Then $u_1 = u_2$.*

Proof. We subtract the equations verified by u_1 and u_2 to obtain

$$\partial_t(u_1-u_2)-\nu\,\Delta(u_1-u_2)+u_1\cdot\nabla(u_1-u_2)+(u_1-u_2)\,\nabla u_2 = \nabla(p_1-p_2)\,.$$

The same computations as in Theorem 2.3 yield

$$\begin{aligned}&\|u_1-u_2\|_{L^4(]0,t[;\mathcal{H}^{\overline{\delta}+\overline{\alpha}})}\\ &\quad\leq C\,\|u_1-u_2\|_{L^4(]0,t[;\mathcal{H}^{\overline{\delta}+\overline{\alpha}})}\big(\|u_1\|_{L^4(]0,t[;\mathcal{H}^{\overline{\delta}+\overline{\alpha}})}+\|u_2\|_{L^4(]0,t[;\mathcal{H}^{\overline{\delta}+\overline{\alpha}})}\big)\,.\end{aligned}$$

Thus, if t is small enough, we have

$$\|u_1-u_2\|_{L^4(]0,t[;\mathcal{H}^{\overline{\delta}+\overline{\alpha}})} \leq \frac{1}{2}\,\|u_1-u_2\|_{L^4(]0,t[;\mathcal{H}^{\overline{\delta}+\overline{\alpha}})}\,,$$

so we get local uniqueness that is global uniqueness, since the map $t \longrightarrow \|u_1-u_2\|_{L^4(]0,t[;\mathcal{H}^{\overline{\delta}+\overline{\alpha}})}$ is continuous.

3. Resolution of (N-S) in the HB^{s_1,s_2,s_3} spaces.

Let us introduce the spaces $HB_{T,p,s_1,s_2,s_3} = HB_{T,p,\overline{s}}$ defined as the closure of compactly supported smooth functions under the norm

$$\|u\|_{HB_{T,p,\overline{s}}} \overset{\text{def}}{=} \|\,\|2^{\overline{q}\cdot\overline{s}}\Delta_{\overline{q}}u\|_{L^p_T(L^2)}\|_{\ell^{2,2,1}}\,.$$

As for the $\mathcal{H}^{\overline{s}}$ spaces we shall prove a theorem of global existence and uniqueness and a local existence and uniqueness one. Let a and b be two positive real numbers such that $a+b=1/2$.

Theorem 3.1 (global existence and uniqueness). *There exists $C>0$ such that if $\operatorname{div} u_0 = 0$, $u_0 \in HB^{0,0,1/2}$ and $|u|_{HB^{0,0,1/2}} < C\nu$ then the* (N-S) *equations have a unique global solution which belongs to*

$$HB_{\infty,4,a,b,1/2} \cap L^\infty(]0,\infty[\,;HB^{0,0,1/2}) \cap \mathcal{C}([0,\infty[\,;HB^{0,0,1/2})\,.$$

Theorem 3.2 (local existence and uniqueness). *If* $\operatorname{div} u_0 = 0$ *and* $u_0 \in HB^{0,0,1/2}$ *then there exist* $T > 0$ *and a unique solution of* (N-S) *on* $[0,T]$ *which belongs to* $HB_{T,4,a,b,1/2} \cap \mathcal{C}([0,T[\,; HB^{0,0,1/2})$.

REMARK. We have $HB_{T,4,a,b,1/2} \hookrightarrow L^4(]0,\infty[\,; HB^{a,b,1/2})$.

Indeed, Remark 1.1 implies

$$\begin{aligned}\|u\|_{L^4(]0,\infty[;HB^{a,b,1/2})} &= \|\,\|2^{q_1 a+q_2 b+q_3/2}\,\|\Delta_{\overline{q}}\, u\|_{L^2}\|_{\ell^{2,2,1}}\|_{L^4}\\ &\leq \|\,\|2^{q_1 a+q_2 b+q_3/2}\|\Delta_{\overline{q}}\, u\|_{L^2}\|_{L^4}\|_{\ell^{2,2,1}}\\ &= \|u\|_{HB_{T,4,a,b/2}}\,.\end{aligned}$$

We first prove

Lemma 3.1. *Let* $s_i < 1/2$, $t_i < 1/2$, $s_i + t_i > 0$, *for all* $i \in \{1,2\}$, $s_3 \leq 1/2$, $t_3 \leq 1/2$, $s_3 + t_3 > 0$ *and* $p, q \geq 1$, $r = p\,q/(p+q) \geq 1$. *Then*

$$\|u\,v\|_{HB_{T,r,\overline{s}+\overline{t}-(1/2,1/2,1/2)}} \leq \|u\|_{HB_{T,p,\overline{s}}}\,\|v\|_{HB_{T,q,\overline{t}}}\ .$$

PROOF. We shall copy the proof of Theorem 1.2 and prove this lemma for each of the 27 terms of the Littlewood-Paley decomposition. Let us take, for instance, the $T^1 R^2 \widetilde{T}^3$ term. We start again from inequality (1.4)

$$\begin{aligned}\|\Delta_{\overline{q}}\, w^i_{\overline{p}}(t)\|_{L^2} \leq 2^{q_2/2} \sum_{\substack{r_1 \leq p_1-2\\ r_3\leq p_3-2}} & 2^{r_1/2+r_3/2}\,\|\Delta^1_{r_1}\,\Delta^2_{p_2}\,\Delta^3_{p_3} u(t)\|_{L^2}\\ &\cdot \|\Delta^1_{p_1}\,\Delta^2_{p_2-i}\,\Delta^3_{r_3} v(t)\|_{L^2}\ .\end{aligned}$$

Taking the $L^r(0,T)$ norm and applying Hölder's inequality gives

$$\begin{aligned}\|\Delta_{\overline{q}}\, w^i_{\overline{p}}(t)\|_{L^r_T(L^2)} \leq 2^{q_2/2} \sum_{\substack{r_1 \leq p_1-2\\ r_3\leq p_3-2}} & 2^{r_1/2+r_3/2}\,\|\Delta^1_{r_1}\,\Delta^2_{p_2}\,\Delta^3_{p_3} u(t)\|_{L^p_T(L^2)}\\ &\cdot \|\Delta^1_{p_1}\,\Delta^2_{p_2-i}\,\Delta^3_{r_3} v(t)\|_{L^q_T(L^2)}\ .\end{aligned}$$

If we define

$$A_{\overline{q}} = 2^{\overline{q}\cdot\overline{s}}\,\|\Delta_{\overline{q}}\, u\|_{L^p_T(L^2)}$$

and

$$B_{\overline{q}} = 2^{\overline{q}\cdot\overline{t}}\, \|\Delta_{\overline{q}}\, u\|_{L^q_T(L^2)} \ ,$$

it follows that

$$\begin{aligned}
&2^{\overline{q}\cdot(\overline{s}+\overline{t}-(1/2,1/2,1/2))}\, \|\Delta_{\overline{q}}\, w^i_{\overline{p}}(t)\|_{L^r_T(L^2)} \\
&\qquad \le C\, 2^{q_1(s_1-1/2)+(s_2+t_2)(q_2-p_2)+q_3(t_3-1/2)} \\
&\qquad\quad \cdot \sum_{\substack{r_1\le p_1-2\\ r_3\le p_3-2}} 2^{r_1(1/2-s_1)+r_3(1/2-t_3)}\, A_{r_1,p_2,p_3}\, B_{p_1,p_2-i,r_3} \ .
\end{aligned}$$

This inequality is entirely similar to (1.5) so the proof continues in exactly the same way we did after that inequality.

PROOF OF THE LOCAL EXISTENCE. It is obvious that if $\overline{\delta} = (0,0,1/2)$ and $a_1 = a$, $a_2 = b$, $a_3 = 0$ then hypothesis (2.1) is verified excepted for the condition $\delta_3 + a_3 < 1/2$. This is precisely where we use that $B^{1/2}_{2,1}(\mathbb{R})$ is an algebra. Hence, we can follow the same line of proof as in Theorem 2.3, replacing the ℓ^2 norms by the $\ell^{2,2,1}$ norms and the $\mathcal{H}^{\overline{s}}$ spaces with the $HB^{\overline{s}}$ spaces. There is one fact which doesn't allow us to give an identical proof: the deduction of inequality (2.5) from inequality (2.4) which is not possible because the switch of the L^2 and $\ell^{2,2,1}$ norms yields an inequality in the opposite sens of the wanted one. To avoid that we have to give up the estimate

$$\begin{aligned}
\|\Delta_{\overline{q}}\,(u\nabla u)\|_{L^2} \le C\,(&2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(3/2,1/2,1/2))} + 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,3/2,1/2))} \\
&+ 2^{-\overline{q}\cdot(2\overline{\delta}+2\overline{a}-(1/2,1/2,3/2))})\, a_{\overline{q}}\, |u|^2_{1/2+\delta,1/2-\delta} \ ,
\end{aligned}$$

and to use, for the deduction of inequality (2.5), Lemma 3.1. As in Theorem 2.3 we find the following inequality

$$\begin{aligned}
&\frac{d}{dt}\, \|\Delta_{\overline{q}}\, u_n\|^2_{L^2} + C\,\nu\,(4^{q_1}+4^{q_2}+4^{q_3})\, \|\Delta_{\overline{q}}\, u_n\|^2_{L^2} \\
&\qquad\qquad \le \|\Delta_{\overline{q}}\,(u_n\nabla u_n)\|_{L^2}\, \|\Delta_{\overline{q}}\, u_n\|_{L^2} \ .
\end{aligned}$$

Gronwall's lemma implies

$$\begin{aligned}
&\|\Delta_{\overline{q}}\, u_n(t)\|_{L^2} \\
&\le \|\Delta_{\overline{q}}\, u^n_0\|_{L^2} \exp(-C\,\nu\,(4^{q_1}+4^{q_2}+4^{q_3})\,t) \\
&\quad + C\,(\exp(-C\,\nu\,(4^{q_1}+4^{q_2}+4^{q_3})(\cdot))\,(\|\Delta_{\overline{q}}\,(u_n(\cdot)\,\nabla u_n(\cdot))\|_{L^2}))(t)\,.
\end{aligned}$$

Taking the $L^4(0,T)$ norm and using Young's inequality gives

$$
\begin{aligned}
&\|\Delta_{\overline{q}}\, u_n(t)\|_{L^4_T(L^2)}\\
&\quad\le C\,\nu^{-1/4}\,(4^{q_1}+4^{q_2}+4^{q_3})^{-1/4}\,\|\Delta_{\overline{q}}\, u_0^n\|_{L^2}\\
&\qquad\cdot\big(1-\exp\big(-C\,\nu\, T\,(4^{q_1}+4^{q_2}+4^{q_3})\big)\big)^{1/4}\\
&\qquad+C\,\|\exp\big(-C\,\nu\,(4^{q_1}+4^{q_2}+4^{q_3})(\cdot)\big)\|_{L^{4/3}(0,T)}\,\|\Delta_{\overline{q}}\,(u_n\nabla u_n)\|_{L^2_T(L^2)}\\
&\quad\le C\,\nu^{-1/4}\,(4^{q_1}+4^{q_2}+4^{q_3})^{-1/4}\,\|\Delta_{\overline{q}}\, u_0^n\|_{L^2}\\
&\qquad\cdot\big(1-\exp\big(-C\,\nu\, T\,(4^{q_1}+4^{q_2}+4^{q_3})\big)\big)^{1/4}\\
&\qquad+C\,\nu^{-3/4}\,(4^{q_1}+4^{q_2}+4^{q_3})^{-3/4}\,\|\Delta_{\overline{q}}\,(u_n\nabla u_n)\|_{L^2_T(L^2)}\,.
\end{aligned}
$$

Again by Young's inequality we have

$$
(4^{q_1}+4^{q_2}+4^{q_3})^{-1/4}\le 2^{q_3/2}\,2^{-\overline{q}\cdot(a,b,1/2)}\,.
$$

It follows that

$$
\begin{aligned}
&2^{\overline{q}\cdot(a,b,1/2)}\|\Delta_{\overline{q}}\, u_n(t)\|_{L^4_T(L^2)}\\
&\quad\le C\,\nu^{-1/4}\,2^{q_3/2}\,\|\Delta_{\overline{q}}\, u_0^n\|_{L^2}\\
&\qquad\cdot\big(1-\exp\big(-C\,\nu\, T\,(4^{q_1}+4^{q_2}+4^{q_3})\big)\big)^{1/4}\\
&\qquad+C\,\nu^{-3/4}\,(4^{q_1}+4^{q_2}+4^{q_3})^{-3/4}\,2^{\overline{q}\cdot(a,b,1/2)}\\
&\qquad\quad\cdot\|\Delta_{\overline{q}}\,(u_n\nabla u_n)\|_{L^2_T(L^2)}\,.
\end{aligned}
\tag{3.1}
$$

Now we use the Lemma 3.1 to deduce that

$$
\begin{aligned}
&\|\Delta_{\overline{q}}\,(u_n\nabla u_n)\|_{L^2_T(L^2)}\\
&\qquad=\|\Delta_{\overline{q}}\,\mathrm{div}\,(u_n\otimes u_n)\|_{L^2_T(L^2)}\\
&\qquad\le C\,c_{\overline{q}}\,\big(2^{-\overline{q}\cdot(2a-3/2,2b-1/2,1/2)}+2^{-\overline{q}\cdot(2a-1/2,2b-3/2,1/2)}\\
&\qquad\qquad+2^{-\overline{q}\cdot(2a-1/2,2b-1/2,-1/2)}\big)\,\|u_n\|^2_{HB_{T,4,a,b,1/2}}\,,
\end{aligned}
$$

where $\|c_{\overline{q}}\|_{\ell^{2,2,1}}=1$. Young's inequality implies

$$
(4^{q_1}+4^{q_2}+4^{q_3})^{-3/4}\,2^{\overline{q}\cdot(a,b,1/2)}\,2^{-\overline{q}\cdot(2a-3/2,2b-1/2,1/2)}\le 1\,,
$$

$$
(4^{q_1}+4^{q_2}+4^{q_3})^{-3/4}\,2^{\overline{q}\cdot(a,b,1/2)}\,2^{-\overline{q}\cdot(2a-1/2,2b-3/2,1/2)}\le 1\,,
$$

$$
(4^{q_1}+4^{q_2}+4^{q_3})^{-3/4}\,2^{\overline{q}\cdot(a,b,1/2)}\,2^{-\overline{q}\cdot(2a-1/2,2b-1/2,-1/2)}\le 1\,.
$$

Hence inequality (3.1) may be written as

$$
\begin{aligned}
&2^{\overline{q}\cdot(a,b,1/2)}\,\|\Delta_{\overline{q}}\,u_n\|_{L^4_T(L^2)}\\
&\leq C\,\nu^{-1/4}\,2^{q_3/2}\,\|\Delta_{\overline{q}}\,u_0^n\|_{L^2}\,(1-\exp\left(-C\,\nu\,T\,(4^{q_1}+4^{q_2}+4^{q_3})\right))^{1/4}\\
&\quad +C\,\nu^{-3/4}\,c_{\overline{q}}\,\|u_n\|^2_{HB_{T,4,a,b,1/2}}\,.
\end{aligned}
$$

Taking the $\ell^{2,2,1}$ norm gives

$$
\begin{aligned}
&\|u_n\|_{HB_{T,4,a,b,1/2}}\\
&\leq C\,\|2^{q_3/2}\,\|\Delta_{\overline{q}}\,u_0\|_{L^2}\,(1-\exp\left(-C\,\nu\,T\,(4^{q_1}+4^{q_2}+4^{q_3})\right))^{1/4}\|_{\ell^{2,2,1}}\\
&\quad +C\,\|u_n(\tau)\|^2_{HB_{T,4,a,b,1/2}}\\
&\leq g_n(T)+C\,\|u_n\|^2_{HB_{T,4,a,b,1/2}}\,,
\end{aligned}
$$

where

$$
g_n(t)=C\,\|2^{q_3/2}\,\|\Delta_{\overline{q}}\,u_0\|_{L^2}\,(1-\exp\left(-C\,\nu\,T\,(4^{q_1}+4^{q_2}+4^{q_3})\right))^{1/4}\|_{\ell^{2,2,1}}\,.
$$

We conclude as in Theorem 2.3. The fact that $u\in\mathcal{C}([0,T[\,;HB^{0,0,1/2})$ is proved as in the case of $\mathcal{H}^{\overline{s}}$ spaces.

Proof of the global existence. Same proof as above by estimating

$$
g_n(t)\leq C\,|u_0|_{HB^{0,0,1/2}}\,.
$$

The uniqueness theorem is also similar to the one of the case $\mathcal{H}^{\overline{s}}$.

Theorem 3.3 (uniqueness). *Let u_1 and u_2 be two solutions of* (N-S) *which belong to $HB_{T,4,a,b,1/2}\cap\mathcal{C}([0,T[\,;HB^{0,0,1/2})$ with the same initial data in $HB^{0,0,1/2}$. Then $u_1=u_2$.*

Proof. Making the same computations as in Theorem 2.4, replacing the ℓ^2 norms with the $\ell^{2,2,1}$ norms and using Lemma 3.1 as shown above we find

$$
\begin{aligned}
&\|u_1-u_2\|_{HB_{t,4,a,b,1/2}}\\
&\qquad\leq C\,\|u_1-u_2\|_{HB_{t,4,a,b,1/2}}(\|u_1\|_{HB_{t,4,a,b,1/2}}+\|u_2\|_{HB_{t,4,a,b,1/2}})\,.
\end{aligned}
$$

We conclude as in Theorem 2.4.

4. Some imbeddings and nonimbeddings.

In this section we prove some imbeddings and some nonimbeddings which are used to compare the results from the previous sections with the results already known. We recall that one can solve (N-S) in the spaces $H^{1/2}$, $B^{-1+3/p}_{p,\infty}$, $2 \le p < \infty$ (see [2], [3], [7], [9]) and it seems very difficult to do it in C^{-1} ($\|u\|_{B^s_{p,q}} \stackrel{\text{def}}{=} \|2^{is}\,\|\Delta_i u\|_{L^p}\|_{\ell^q}$ and $C^{-1} = B^{-1}_{\infty,\infty}$). It is also proved by H. Kozono and M. Yamazaki in [8] that one can solve (N-S) in the homogeneous spaces $\mathcal{N}^{3/p-1}_{p,q,\infty}$, $1 \le q \le p < \infty$, $p > 3$, where $\mathcal{N}^s_{p,q,r}$ is defined to be the closure of the compactly supported smooth functions under the norm

$$\|u\|_{\mathcal{N}^s_{p,q,r}} = \|2^{sj} \sup_{x_0 \in \mathbb{R}^3} \sup_{R>0} R^{3/p-3/q}\,\|\Delta_j u\|_{L^q(B(x_0,R))}\|_{\ell^r} ,$$

where $B(x_0,R)$ denotes the closed ball in $\mathbb{R}^3$ with center x_0 and radius R. Let us remark that $B^s_{p,r} = \mathcal{N}^s_{p,p,r}$. We can prove the following proposition:

Proposition 4.1. i) *If* $\delta_1 + \delta_2 + \delta_3 = 1/2$, $-1/2 < \delta_i < 1/2$ *for all* $i \in \{1,2,3\}$ *and* $p > \max_{1\le i\le 3}(2/(1-2\,\delta_i))$ *then*

$$\mathcal{H}^{\overline{\delta}} \hookrightarrow B^{-1+3/p}_{p,\infty} \hookrightarrow C^{-1}\,.$$

ii) $L^2 \cap \mathcal{H}^{0,0,1/2} \not\subset C^{-1}$.

iii) *If* $1 \le q \le p < 3\,q/2$, $p > 3$, *then* $HB^{0,0,1/2} \not\subset \mathcal{N}^{3/p-1}_{p,q,\infty}$ *hence* $HB^{0,0,1/2} \not\subset B^{-1+3/p}_{p,\infty}$ *for all* $2 \le p < \infty$.

iv) $HB^{0,0,1/2} \hookrightarrow C^{-1}$.

Property i) shows that solutions of (N-S) were already constructed by M. Cannone [2], F. Planchon [9] and H. Kozono, M. Yamazaki [8]. Property ii) suggests that the space $\mathcal{H}^{0,0,1/2}$ is very interesting as space of initial data; unfortunately we cannot include it in our results. Finally, property iii) shows that $HB^{0,0,1/2}$ is not included in the space considered by H. Kozono and M. Yamazaki at least for some p and q; it implies that it is not included in any of the spaces used by M. Cannone and F. Planchon. The author doesn't know if the non-imbedding of iii) still holds for the other values of p and q.

PROOF OF PROPOSITION 4.1. i) First we remark that if $s < 0$, then we can replace Δ_i with S_i in the definition of the $B^s_{p,\infty}$ space. By Lemma 1.1 we have

$$
\begin{aligned}
&2^{q(-1+3/p)} \|S_q u\|_{L^p} \\
&\leq C\, 2^{q(-1+3/p)} \sum_{\substack{q_1 \leq q \\ q_2 \leq q \\ q_3 \leq q}} \|\Delta_{\overline{q}} u\|_{L^p} \\
&\leq C\, 2^{q(-1+3/p)} \sum_{\substack{q_1 \leq q \\ q_2 \leq q \\ q_3 \leq q}} 2^{q_1(1/2-1/p)+q_2(1/2-1/p)+q_3(1/2-1/p)} \|\Delta_{\overline{q}} u\|_{L^2} \\
&\leq C\, 2^{q(-1+3/p)} \sum_{\substack{q_1 \leq q \\ q_2 \leq q \\ q_3 \leq q}} 2^{q_1(1/2-1/p-\delta_1)+q_2(1/2-1/p-\delta_2)+q_3(1/2-1/p-\delta_3)} \\
&\quad \cdot 2^{\overline{q}\cdot\overline{\delta}} \|\Delta_{\overline{q}} u\|_{L^2} \\
&\leq C\, 2^{q(-1+3/p)} \sum_{\substack{q_1 \leq q \\ q_2 \leq q \\ q_3 \leq q}} 2^{q_1(1/2-1/p-\delta_1)+q_2(1/2-1/p-\delta_2)+q_3(1/2-1/p-\delta_3)} |u|_{\overline{\delta}}\,.
\end{aligned}
$$

As $1/2 - 1/p - \delta_i > 0$ for any $i \in \{1,2,3\}$, one deduces

$$
\begin{aligned}
\sum_{\substack{q_1 \leq q \\ q_2 \leq q \\ q_3 \leq q}} &2^{q_1(1/2-1/p-\delta_1)+q_2(1/2-1/p-\delta_2)+q_3(1/2-1/p-\delta_3)} \\
&\leq C\, 2^{q(3/2-3/p-\delta_1-\delta_2-\delta_3)} \\
&= C\, 2^{q(1-3/p)}\,.
\end{aligned}
$$

Hence $\|u\|_{B^{-1+3/p}_{p,\infty}} \leq C\, |u|_{\overline{\delta}}$ and the first imbedding is proved. In order to obtain the second imbedding it is enough to apply the classical Littlewood-Paley inequality

$$
\|\Delta_q u\|_{L^\infty} \leq 2^{3q/p} \|\Delta_q u\|_{L^p}\ ,
$$

to multiply by 2^{-q} and to take the upper bound on q.

ii) As $L^2 \cap \mathcal{H}^{0,0,1/2}$ and C^{-1} are distribution spaces, the closed graph theorem shows that it is enough to prove $L^2 \cap \mathcal{H}^{0,0,1/2} \not\hookrightarrow C^{-1}$. Assume by absurd that $L^2 \cap \mathcal{H}^{0,0,1/2} \hookrightarrow C^{-1}$. Then

$$
2^{-q} \|S_q u\|_{L^\infty} \leq C\, \|u\|_{L^2 \cap \mathcal{H}^{0,0,1/2}}\ , \qquad \text{for all } q\,.
$$

We choose $u = f \otimes g$ where $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $g : \mathbb{R} \longrightarrow \mathbb{C}$. It is obvious that $S_q u = S_q f \otimes S_q g$ and $\|u\|_{L^2 \cap \mathcal{H}^{0,0,1/2}} = \|f\|_{L^2} \|g\|_{H^{1/2}}$, where in $S_q f$, S_q is the 2D S_q and in $S_q g$, S_q is the 1D S_q. Hence

$$2^{-q} \|S_q f\|_{L^\infty} \|S_q g\|_{L^\infty} \leq C \|f\|_{L^2} \|g\|_{H^{1/2}} , \qquad \text{for all } q . \tag{4.1}$$

For each fixed q we use the function $f_q(x) = f_0(2^q x)$, where f_0 is chosen with $\operatorname{supp} \widehat{f_0}$ sufficiently small to get $S_q f_q = f_q$, that gives $\|S_q f_q\|_{L^\infty} = \|f_q\|_{L^\infty} = \|f_0\|_{L^\infty}$ and $\|f_q\|_{L^2} = 2^{-q} \|f_0\|_{L^2}$ since we work in two dimensions. Therefore, it comes from relation (4.1)

$$\|S_q g\|_{L^\infty} \leq C \|g\|_{H^{1/2}} ,$$

that is $H^{1/2}(\mathbb{R}) \subset L^\infty$ which is false.

iii) As above we assume by absurd that $HB^{0,0,1/2} \hookrightarrow \mathcal{N}^{3/p-1}_{p,q,\infty}$ and we remark that if $s < 0$, then we can replace Δ_j with S_j in the definition of the norm of the space $\mathcal{N}^s_{p,q,\infty}$.

Again, we choose $u = f \otimes g$ where $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $g : \mathbb{R} \longrightarrow \mathbb{C}$. It is not difficult to see that the norm

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{R>0} R^{3/p-3/q} \|f \otimes g\|_{L^q(B(x_0,R))}$$

is equivalent to the norm

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{R>0} R^{3/p-3/q} \|f\|_{L^q(B^2(x_0',R))} \|g\|_{L^q(B^1(x_0^3,R))} ,$$

where B^1 and B^2 denote the one-dimensional, respectively two-dimensional balls. This is done by including a cube of size R into the ball $B(x_0, R)$, applying Fubini's theorem, then including balls of radius $R/2$ into the one-dimensional and the two-dimensional cubes of size R and finally taking the upper bound on R.

It follows that

$$\begin{aligned} 2^{j(-1+3/p)} \sup_{x_0 \in \mathbb{R}^3} \sup_{R>0} R^{3/p-3/q} \|S_j f\|_{L^q(B^2(x_0',R))} \|S_j g\|_{L^q(B^1(x_0^3,R))} \\ \leq C \|f\|_{L^2} \|g\|_{B^{1/2}_{2,1}} , \end{aligned}$$

for all j, where the constant C does not depend on j. Choosing $x_0 = 0$ yields

$$\begin{aligned} 2^{j(-1+3/p)} \sup_{R>0} R^{3/p-3/q} \|S_j f\|_{L^q(B^2(0,R))} \|S_j g\|_{L^q(B^1(0,R))} \\ \leq C \|f\|_{L^2} \|g\|_{B^{1/2}_{2,1}} , \end{aligned}$$

for all j. Now we fix j and we choose $f_j(x) = f_0(2^j\, x)$, the same example as above. We also choose g to be a function whose Fourier transform is a compactly supported smooth function. This implies that $S_j f_j = f_j$ and $S_j g = g$ for j large enough. Moreover, we have that

$$\|S_j f_j\|_{L^q(B^2(0,R))} = \|f_j\|_{L^q(B^2(0,R))} = 2^{-2j/q}\|f_0\|_{L^q(B^2(0,2^jR))}\ ,$$

and

$$\|f_j\|_{L^2} = 2^{-j}\, \|f_0\|_{L^2}\ .$$

It follows that, for j large enough, we have

$$2^{j(3/p-2/q)} \sup_{R>0} R^{3/p-3/q}\|f_0\|_{L^q(B^2(0,2^jR))}\, \|g\|_{L^q(B^1(0,R))} \\ \leq C\, \|f_0\|_{L^2}\, \|g\|_{B^{1/2}_{2,1}}\ ,$$

which implies

$$2^{j(3/p-2/q)} \sup_{R>0} R^{3/p-3/q}\|f_0\|_{L^q(B^2(0,R))}\, \|g\|_{L^q(B^1(0,R))} \\ \leq C\, \|f_0\|_{L^2}\, \|g\|_{B^{1/2}_{2,1}}\ ,$$

for all $j > j_0$. Taking the limit on $j \longrightarrow \infty$ gives a contradiction.

iv) We write

$$\begin{aligned} 2^{-q}\, \|S_q u\|_{L^\infty} &\leq 2^{-q} \sum_{\substack{q_1\leq q\\ q_2\leq q\\ q_3\leq q}} \|\Delta_{\overline{q}}\, u\|_{L^\infty} \\ &\leq 2^{-q} \sum_{\substack{q_1\leq q\\ q_2\leq q\\ q_3\leq q}} 2^{q_1/2+q_2/2+q_3/2}\, \|\Delta_{\overline{q}}\, u\|_{L^2} \\ &\leq 2^{-q} \sum_{\substack{q_1\leq q\\ q_2\leq q}} 2^{q_1/2+q_2/2}\, \|2^{q_3/2}\, \|\Delta_{\overline{q}}\, u\|_{L^2}\|_{\ell^1_{q_3}} \\ &\leq \|2^{q_3/2}\, \|\Delta_{\overline{q}}\, u\|_{L^2}\|_{\ell^{\infty,\infty,1}} \\ &\leq \|2^{q_3/2}\, \|\Delta_{\overline{q}}\, u\|_{L^2}\|_{\ell^{2,2,1}} \\ &= \|u\|_{HB^{0,0,1/2}}\ . \end{aligned}$$

This completes the proof.

One could ask whether the divergence free condition has an influence on the choice of the spaces where we can take the initial data or not. The answer is negative because, if we look to the proofs above, we see that the scalar counterexamples f we deduce have the property that $\partial_1 f$ and $\partial_2 f$ are again good counterexamples (differentiating f_0 only diminishes the support of its Fourier transform), so we can take as initial data $u_0 = (\partial_2 f, -\partial_1 f, 0)$.

Appendix.

In this paragraph we show how a general d-dimensional hyperbolic symmetric system can be solved in $B^{1+d/2}_{2,1}(\mathbb{R}^d)$. By general hyperbolic symmetric system we mean a system of the form

$$\text{(S)} \qquad \begin{cases} \partial_t U + A(U)\cdot\nabla U = 0\,, \\ U|_{t=0} = U_0\ , \end{cases}$$

where

$$A(U) = (A_j(U))_{1\le j\le d}$$

and, for all j, $A_j(U)$ is a symmetric smooth globally Lipschitz matrix and U is a time dependent vector field in $\mathbb{R}^d$.

Proposition. *Assume that $U_0 \in L^2 \cap B^{1+d/2}_{2,1}$. Then there exist a time T and a unique solution of* (S) *on $[0,T]$ in the space $L^\infty(]0,T[\,; B^{1+d/2}_{2,1})$. Moreover, there exists a constant $C>0$ such that the maximal time existence of such a solution may be bounded from below by*

$$T > \frac{C}{\|U_0\|_{B^{1+d/2}_{2,1}}}\,.$$

Proof. The proof relies on the fact that $B^{d/2}_{2,1}$ is imbedded in L^∞ and on the following estimate:

Lemma. *For all vector fields U in $B^{1+d/2}_{2,1}$ there exists a sequence $\{c_q\}_{q\in\mathbb{N}}$ such that*

$$|\langle \Delta_q(A(U)\cdot\nabla U)|\Delta_q U\rangle| \le C\, 2^{-q(d/2+1)} c_q\, \|\Delta_q U\|_{L^2}\, \|U\|_{B^{1+d/2}_{2,1}}\, \|\nabla U\|_{L^\infty}\ ,$$

where

$$\sum_q c_q = 1\,.$$

This lemma is well-known in the case of the Sobolev spaces and the extension to the Besov spaces is simple. Decomposing the product $A(U)\cdot\nabla U$ in the usual sum of two paraproducts and a remainder, using the classical product theorem for Besov spaces, we see that the only term where a critical case appears is

$$\langle \Delta_q(T_{A(U)}\nabla U)|\Delta_q U\rangle\,.$$

Some easy computations done integrating by parts show that

$$\begin{aligned}
&\langle \Delta_q(T_{A(U)}\nabla U)|\Delta_q U\rangle\\
&\qquad = \sum_{p,j}\langle[\Delta_q, S_{p-1}A_j(U)]\partial_j\Delta_p U, \Delta_q U\rangle\\
&\qquad\quad - \frac{1}{2}\sum_p S_{p-1}\,\mathrm{div}\,A(U)\,\Delta_q\,\Delta_p U\,\Delta_q U\\
&\qquad\quad - \frac{1}{2}\sum_{p,p',j}(S_{p-1}-S_{p'-1})\,A_j(U)\,\Delta_q\,\Delta_p U\,\partial_j\,\Delta_q\,\Delta_{p'}U\,.
\end{aligned}$$

The last two terms are very easy to estimate, we need only to apply the definition of the Besov spaces. The first term is estimated by remarking that Δ_q is an operator of convolution with the function

$$2^{qd}\,h(2^q\cdot)\,,$$

where $h=\mathcal{F}^{-1}\phi$. Therefore

$$\begin{aligned}
&[S_{p-1}A_j(U), \Delta_q]\,a(x)\\
&\quad = 2^{qd}\int (S_{p-1}A_j(U)(x)-S_{p-1}A_j(U)(y))\,h(2^q(x-y))\,a(y)\,dy\,.
\end{aligned}$$

Hence

$$|[S_{p-1}A_j(U),\Delta_q]\,a(x)| \le C\,2^{q(d-1)}\,\|\nabla U\|_{L^\infty}\,|y\,h|\,(2^q\cdot)\,|a|\,.$$

Young's inequality now gives

$$\|[S_{p-1}A_j(U),\Delta_q]\,a(x)\|_{L^2} \le C\,2^{-q}\,\|\nabla U\|_{L^\infty}\|a\|_{L^2}\,.$$

This proves the lemma.

We return to the proof of the proposition. We apply Δ_q to (S) and we take the scalar product with $\Delta_q U$ to obtain

$$\begin{aligned}\frac{d}{dt}\|\Delta_q U\|_{L^2}^2 &\leq |\langle \Delta_q(A(U)\cdot\nabla U)|\Delta_q U\rangle| \\ &\leq C\,2^{-q(d/2+1)}\,c_q\,\|U\|_{B^{1+d/2}_{2,1}}\,\|\nabla U\|_{L^\infty}\|\Delta_q U\|_{L^2}\,.\end{aligned}$$

It follows that

$$\begin{aligned}&2^{q(d/2+1)}\,\|\Delta_q U(t)\|_{L^2} \\ &\leq 2^{q(d/2+1)}\,\|\Delta_q U_0\|_{L^2} + C\int_0^t c_q(\tau)\,\|U(\tau)\|_{B^{1+d/2}_{2,1}}\,\|\nabla U(\tau)\|_{L^\infty}\,d\tau\,.\end{aligned}$$

Summing on q yields

$$\|U(t)\|_{B^{1+d/2}_{2,1}} \leq \|U_0\|_{B^{1+d/2}_{2,1}} + C\int_0^t \|U(\tau)\|_{B^{1+d/2}_{2,1}}\,\|\nabla U(\tau)\|_{L^\infty}\,d\tau\,.$$

Applying Gronwall's lemma we find

$$\|U(t)\|_{B^{1+d/2}_{2,1}} \leq \|U_0\|_{B^{1+d/2}_{2,1}}\exp\Big(C\int_0^t \|\nabla U(\tau)\|_{L^\infty}\Big)\,d\tau\,.$$

Next we use that $B^{d/2}\subset L^\infty$ to write

$$\|\nabla U(t)\|_{L^\infty} \leq \|U(t)\|_{B^{1+d/2}_{2,1}} \leq \|U_0\|_{B^{1+d/2}_{2,1}}\exp\Big(C\int_0^t \|\nabla U(\tau)\|_{L^\infty}\,d\tau\Big)\,.$$

If we note

$$f(t) = C\int_0^t \|\nabla U(\tau)\|_{L^\infty}\,d\tau\,,$$

we obtain

$$f'(t) \leq C\,\|U_0\|_{B^{1+d/2}_{2,1}}\exp\left(f(t)\right).$$

Again by Gronwall's lemma it follows

$$\exp\left(-f(t)\right) \geq \exp\left(-f(0)\right) - C\,t\,\|U_0\|_{B^{1+d/2}_{2,1}}\,.$$

Hence, as long as

$$C\,t\,\|U_0\|_{B^{1+d/2}_{2,1}} < 1\,,$$

we have

$$\int_0^t \|\nabla U(\tau)\|_{L^\infty}\,d\tau < \infty\,.$$

Standard L^2 estimates and the inequality above imply uniqueness of solutions. This completes the proof.

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References.

[1] Bony, J.-M., Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup.* **14** (1981), 209-246.

[2] Cannone, M., *Ondelettes, Paraproduits et Navier-Stokes.* Diderot éditeur, Arts et Sciences, 1995.

[3] Chemin, J.-Y., Remarques sur l'existence globale pour le systeme de Navier-Stokes incompressible. *SIAM J. Math. Anal.* **23** (1992), 20-28.

[4] Chemin, J.-Y., Lerner, N., Flot de champs de vecteurs non-lipschitziens et équations de Navier-Stokes. *Journal of Differential Equations* **121** (1995), 314-328.

[5] Chemin, J.-Y., Fluides parfaits incompressibles. *Astérisque* **230**, 1995.

[6] Constantin, P., Foias, C., *Navier-Stokes Equations.* The University of Chicago Press, 1988.

[7] Fujita, H., Kato, T., On the Navier-Stokes initial value problem I. *Arch. Rational Mech. Anal.* **16** (1964), 269-315.

[8] Kozono, H., Yamazaki, M., Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. *Comm. Partial Diff. Equations* **19** (1994), 959-1014.

[9] Planchon, F., Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$. *Ann. Inst. H. Poincare, Analyse non linéaire* **13** (1996), 319-336.

[10] Sablé-Tougeron, M., Régularité microlocale pour des problèmes aux limites non linéaires. *Ann. Inst. Fourier* **36** (1986), 39-82.

[11] Temam, R., *Navier-Stokes Equations.* North-Holland, 1984.

[12] Triebel, H., *Theory of function spaces.* Birkhäuser Verlag, 1983.

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Construction of non separable dyadic compactly supported orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity

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Abstract. By means of simple computations, we construct new classes of non separable QMF's. Some of these QMF's will lead to non separable dyadic compactly supported orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity.

1. Introduction.

In the most general sense, wavelet bases consist of discrete families of functions obtained by dilations and translations of well chosen fundamental functions [8], [9]. In this paper we will focus on compactly supported dyadic orthonormal wavelet bases for $L^2(\mathbb{R}^2)$, they are of the form

$$\{2^j\, \psi_i(2^j x_1 - k_i, 2^j x_2 - l_i) :\ j, k_i, l_i \in \mathbb{Z},\ i = 1, 2, 3\}\, .$$

I. Daubechies has constructed compactly supported wavelet bases for $L^2(\mathbb{R})$ of arbitrarily high regularity, generalising the classic Haar basis

[6]. The most commonly used method to construct compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity, is the tensor product method [9]. It leads to the scaling function $\varphi(x_1, x_2) = \varphi_1(x_1)\,\varphi_2(x_2)$ and to the fundamental wavelets

$$\psi_a(x_1, x_2) = \varphi_1(x_1)\,\psi_2(x_2)\,,$$

$$\psi_b(x_1, x_2) = \psi_1(x_1)\,\varphi_2(x_2)\,,$$

and

$$\psi_c(x_1, x_2) = \psi_1(x_1)\,\psi_2(x_2)\,,$$

(where φ_1 (respectively φ_2) is a scaling function for $L^2(\mathbb{R})$ and ψ_1 (respectively ψ_2) is the corresponding fundamental wavelet). The scaling functons and the wavelets that result from the tensor product method are called separable. In this paper, we will also call separable the scaling functions and the wavelets that are the images of separable scaling functions and wavelets by an isometry of $L^2(\mathbb{R}^2)$ of the type $f(x) \longmapsto f(Bx)$ $(B \in SL(2, \mathbb{Z}))$.

Let us now give an outline of the present article.

In the second section, by means of simple computations we construct new classes of bidimensional non separable QMF's (Theorems 2.2 and 2.3).

In the third section, we show that some of these QMF's generate non separable, compactly supported, orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity. These wavelets will be constructed by two methods:

The first method consists in perturbing the separable I. Daubechies QMF's (Theorem 3.4). Thus it leads to wavelets that are close to the I. Daubechies separable wavelets with the same number of vanishing moments (for the L^∞ norm).

The second method permits to construct wavelets that are not near to the I. Daubechies separable wavelets (Theorem 3.5).

All the results of the second section and some of the results of the third section may be adapted to multidimensional compactly supported orthonormal wavelets bases for $L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ of dilation matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where for $i = 1, 2$, A_i is a matrix $d_i \times d_i$ such that all the eigenvalues λ of A_i satisfy $|\lambda| > 1$ and $A_i\,\mathbb{Z}^{d_i} \subset \mathbb{Z}^{d_i}$ [2].

2. New classes of non separable QMF's.

A d-dimensional QMF is a trigonometric polynomial on $\mathbb{R}^d$, $M_0(\xi)$ such that

$$\begin{cases} M_0(0) = 1\,, \\ \displaystyle\sum_{s\in\{0,1\}^d} |M_0(\xi + \pi\, s)|^2 = 1\,. \end{cases} \tag{2.1}$$

The conjugate filters are $2^d - 1$ trigonometric polynomials on $\mathbb{R}^d$,

$$M_1(\xi), \ldots, M_{2^d-1}(\xi)$$

such that the matrix

$$U(\xi) = (M_k(\xi + \pi s))_{s\in\{0,1\}^d, k\in\{0,\ldots,2^d-1\}}\ ,$$

is unitary for all $\xi \in \mathbb{R}^d$.

When $d = 1$, we will take $M_1(\xi) = -e^{-i\xi}\,\overline{M_0(\xi + \pi)}$.

If $\varphi(x)$ is a compactly supported scaling function with d variables, there exists a unique d-dimensional QMF $M_0(\xi)$ such that the Fourier transform of $\varphi(x)$ satisfies $\hat{\varphi}(\xi) = M_0(2^{-1}\xi)\,\hat{\varphi}(2^{-1}\xi)$. Thus, we have a one-to-one correspondance between the multiresolution analyses for $L^2(\mathbb{R}^d)$ with a compactly supported scaling function and the d-dimensional QMF's that satisfy the A. Cohen's criterion [3], [4].

A. Cohen's criterion is satisfied when $|M_0(\xi)| > 0$, for all $\xi \in [-\pi/2, \pi/2]^d$.

One can notice that, in general, it is not clear that one may always associate compactly supported wavelets to a multidimensional multiresolution analysis, even if its scaling function is compactly supported [7]. In this paper this difficulty will be solved by ad hoc constructions (Theorems 2.2 and 2.3).

The bidimensional QMF that corresponds to a separable wavelet basis is also said separable. It can be written

$$M(\xi_1, \xi_2) = m_1(a_{11}\,\xi_1 + a_{12}\,\xi_2)\, m_2(a_{21}\,\xi_1 + a_{22}\,\xi_2)\,, \tag{2.2}$$

where (a_{ij}) belongs to $SL(2,\mathbb{Z})$ and $m_1(x), m_2(x)$ are two monodimensional QMF's.

Let us now study a class of bidimensional QMF's , which is rather easy to construct. This class seems to be a natural extension of the class of the separable QMF's for $(a_{ij}) = I_2$.

2.1. The class of the semi separable QMF's.

Theorem 2.1. *Let $a(x)$, $b(x)$, $m(x)$ be three monodimensional* QMF*'s and $\tilde{a}(x), \tilde{b}(x), \tilde{m}(x)$ their conjugate filters. If $c(x)$ is a trigonometric polynomial, we set*

$$c_e(x) = \frac{1}{2}\left(c(x) + c(x+\pi)\right)$$

and

$$c_o(x) = \frac{1}{2}\left(c(x) - c(x+\pi)\right).$$

Then

$$M_0(\xi_1,\xi_2) = a(\xi_1)\, m_e(\xi_2) + b(\xi_1)\, m_o(\xi_2)\,, \tag{2.3}$$

is a bidimensional QMF *called a semi separable* QMF *and its conjugate filters are*

$$\begin{cases} M_1(\xi_1,\xi_2) = a(\xi_1)\, \{\tilde{m}\}_e(\xi_2) + b(\xi_1)\, \{\tilde{m}\}_o(\xi_2)\,, \\ M_2(\xi_1,\xi_2) = \tilde{a}(\xi_1)\, m_e(\xi_2) + \tilde{b}(\xi_1)\, m_o(\xi_2)\,, \\ M_3(\xi_1,\xi_2) = \tilde{a}(\xi_1)\, \{\tilde{m}\}_e(\xi_2) + \tilde{b}(\xi_1)\, \{\tilde{m}\}_o(\xi_2)\,. \end{cases} \tag{2.4}$$

If $a \neq b$ and if $m(x)$ has at least three non vanishing coefficients, then $M_0(\xi_1,\xi_2)$ is non separable.

Proof. An obvious calculus shows that $M_0(\xi_1,\xi_2)$ is a QMF and that $M_1(\xi_1,\xi_2)$, $M_2(\xi_1,\xi_2)$, $M_3(\xi_1,\xi_2)$ are its conjugate filters.

It is a bit technical to prove that $M_0(\xi_1,\xi_2)$ is non separable. Suppose that $M_0(\xi_1,\xi_2) = m_1(a_{11}\,\xi_1 + a_{12}\,\xi_2)\, m_2(a_{21}\,\xi_1 + a_{22}\,\xi_2)$ where (a_{ij}) belongs to $SL(2,\mathbb{Z})$ and $m_1(x)$, $m_2(x)$ are two monodimensional QMF's.

Taking $\xi_1 = 0$, we get

$$m(\xi_2) = m_1(a_{12}\,\xi_2)\, m_2(a_{22}\,\xi_2)\,. \tag{$*$}$$

If $a_{12} \equiv a_{22} + 1 \pmod 2$, we will suppose that a_{12} is even and a_{22} is odd, the other case being similiar, then

$$|m_1(a_{12}\,\xi_2)|^2 = |m(\xi_2)|^2 + |m(\xi_2+\pi)|^2 = 1\,,$$

thus we have $a_{12} = 0$.

If $a_{12} \equiv a_{22} \pmod 2$, since $(a_{ij}) \in SL(2, \mathbb{Z})$ then a_{12} and a_{22} are odd and

$$\begin{aligned}|m_1(a_{12}\,\xi_2)|^2\,|m_2(a_{22}\,\xi_2)|^2 + |m_1(a_{12}\,\xi_2 + \pi)|^2\,|m_2(a_{22}\,\xi_2 + \pi)|^2 \\ = |m(\xi_2)|^2 + |m(\xi_2 + \pi)|^2 = 1\,,\end{aligned}$$

but this cannot be true since

$$(|m_1(a_{12}\,\xi_2)|^2 + |m_1(a_{12}\,\xi_2 + \pi)|^2)\,(m_2(a_{22}\,\xi_2)|^2 + |m_2(a_{22}\,\xi_2 + \pi)|^2) = 1\,.$$

Therefore $(*)$ implies that $a_{12}\,a_{22} = 0$.

We will only study the case

$$\begin{pmatrix} \varepsilon_1 & 0 \\ a_{21} & \varepsilon_2 \end{pmatrix}, \qquad \varepsilon_i = \pm 1\,,$$

the case

$$\begin{pmatrix} a_{11} & \varepsilon_1 \\ \varepsilon_2 & 0 \end{pmatrix}$$

is similar. We have $m(x) = m_2(\varepsilon_2\,x)$, thus

$$M_0(\xi_1, \xi_2) = m_1(\varepsilon_1\,\xi_1)\,m(\varepsilon_2\,a_{21}\,\xi_1 + \xi_2)\,.$$

Taking $\xi_2 = 0$ and then $\xi_2 = \pi$ we get

$$\frac{1}{2}\,(a(\xi_1) + b(\xi_1)) = m_1(\varepsilon_1\,\xi_1)\,m(\varepsilon_2\,a_{21}\,\xi_1)\,,$$

$$\frac{1}{2}\,(a(\xi_1) - b(\xi_1)) = m_1(\varepsilon_1\,\xi_1)\,m(\varepsilon_2\,a_{21}\,\xi_1 + \pi)\,,$$

hence

$$a(\xi_1) = 2\,m_1(\varepsilon_1\,\xi_1)\,m_e(\varepsilon_2\,a_{21}\,\xi_1)\,,$$

$$b(\xi_1) = 2\,m_1(\varepsilon_1\,\xi_1)\,m_o(\varepsilon_2\,a_{21}\,\xi_1)\,.$$

Since we must have

$$|a(\xi_1)|^2 + |a(\xi_1 + \pi)|^2 = 1\,,$$

$$|b(\xi_1)|^2 + |b(\xi_1 + \pi)|^2 = 1\,,$$

it follows that

$$4\,|m_e(a_{21}\,\xi_1)|^2 = 1\,,$$
$$4\,|m_o(a_{21}\,\xi_1)|^2 = 1\,.$$

If $a_{21} \neq 0$ the trigonometric polynomials $m_e(a_{21}\,\xi_1)$ and $m_o(a_{21}\,\xi_1)$ are inversible and thus of the type

$$m_e(a_{21}\,\xi_1) = \frac{1}{2}\,e^{ik\xi_1}$$

and

$$m_o(a_{21}\,\xi_1) = \frac{1}{2}\,e^{il\xi_1}\,.$$

If $a_{21} = 0$ we have $a = b$.

When the QMF's $a(x)$ and $m(x)$ satisfy A. Cohen's criterion and when the norm $\|a - b\|_\infty$ is small enough, the corresponding semi separable QMF satisfies obviously this criterion. However, the constraint on $\|a - b\|_\infty$ does not seem necessary. Indeed, if for example

$$a(x) = m(x) = \frac{1}{2}\,(1 + e^{-ix})$$

and

$$b(x) = \frac{1}{2}\,(1 + e^{-i3x})\,.$$

We have $\|a - b\|_\infty = 1$ but the corresponding semi separable QMF satisfies A. Cohen's criterion [2]. This example also shows that it is even not necessary that both $a(x)$ and $b(x)$ satisfy A. Cohen's for the associated semi separable QMF satisfies it. In [2] we have however established that $m(x)$ must satisfy this criterion.

2.2. Other classes of non separable QMF's.

It is clear that many QMF's are not semi separable, even in the weak sense. This means that they are not of the form

$$a(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_e(c_{21}\,\xi_1 + c_{22}\,\xi_2) + b(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_o(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,,$$

where $a(x)$, $b(x)$, $m(x)$ are three monodimensional QMF's and (c_{ij}) is a matrix of $SL(2,\mathbb{Z})$.

Let $\eta \in]0,1[$ and $\alpha_\eta(x)$, $\beta_\eta(x)$ the two trigonometric polynomials in one variable defined by

$$\alpha_\eta(x) = 1 - \eta\, q(x)\,, \tag{2.5}$$

where $q(x)$ is a trigonometric polynomial vanishing in zero, with values in $[0,1]$ and such that $q \neq 0$, and by

$$|\alpha_\eta(x)|^2 + |\beta_\eta(x)|^2 = 1\,. \tag{2.6}$$

The existence of $\beta_\eta(x)$ is given by the Fejer-Riesz Lemma.

Theorem 2.2. *Let $S_0(\xi_1,\xi_2) = a(\xi_1)\, b(\xi_2)$ be a separable* QMF *and let $S_1(\xi_1,\xi_2)$, $S_2(\xi_1,\xi_2)$, $S_3(\xi_1,\xi_2)$ be its conjugate filters ($\tilde{a}(x) = -e^{-ix}\,\overline{a(x+\pi)}$ and $\tilde{b}(x) = -e^{-ix}\,\overline{b(x+\pi)}$ will be the conjugate filters of $a(x)$ and $b(x)$). If $\lambda(\xi_1,\xi_2)$ and $\mu(\xi_1,\xi_2)$ are two trigonometric polynomials π-periodic in ξ_1 and in ξ_2 and such that*

$$\begin{cases} \lambda(0,0) = 1\,, \\ |\lambda(\xi_1,\xi_2)|^2 + |\mu(\xi_1,\xi_2)|^2 = 1\,. \end{cases}$$

Then

$$R_0(\xi_1,\xi_2) = \lambda(\xi_1,\xi_2)\, S_0(\xi_1,\xi_2) + \mu(\xi_1,\xi_2)\, S_1(\xi_1,\xi_2)\,, \tag{2.7}$$

is a QMF *and its conjugate filters are*

$$\begin{cases} R_1(\xi_1,\xi_2) = \overline{\mu(\xi_1,\xi_2)}\, S_0(\xi_1,\xi_2) - \overline{\lambda(\xi_1,\xi_2)}\, S_1(\xi_1,\xi_2)\,, \\ R_2(\xi_1,\xi_2) = \lambda(\xi_1,\xi_2)\, S_2(\xi_1,\xi_2) + \mu(\xi_1,\xi_2)\, S_3(\xi_1,\xi_2)\,, \\ R_3(\xi_1,\xi_2) = \overline{\mu(\xi_1,\xi_2)}\, S_2(\xi_1,\xi_2) - \overline{\lambda(\xi_1,\xi_2)}\, S_3(\xi_1,\xi_2)\,. \end{cases} \tag{2.8}$$

Moreover

i) *If $\lambda(\xi_1,\xi_2) = \alpha_\eta(2\,\xi_1)$ and $\mu(\xi_1,\xi_2) = \beta_\eta(2\,\xi_1)$ (as defined by* (2.5) *and* (2.6)*), the* QMF *$R_0(\xi_1,\xi_2)$ is non separable when $S_1(\xi_1,\xi_2)$ is not the filter $\tilde{a}(\xi_1)\, b(\xi_2)$ and $R_0(\xi_1,\xi_2)$ has zeros of order greater or equal than 2 in $(\pi,0)$, $(0,\pi)$ and (π,π).*

ii) *If $\lambda(\xi_1,\xi_2) = \alpha_\eta(2\,(\xi_1+\xi_2))$ and $\mu(\xi_1,\xi_2) = \beta_\eta(2\,(\xi_1+\xi_2))$ or if $\lambda(\xi_1,\xi_2) = \alpha_\eta(2\,(\xi_1-\xi_2))$ and $\mu(\xi_1,\xi_2) = \beta_\eta(2\,(\xi_1-\xi_2))$, then the* QMF *$R_0(\xi_1,\xi_2)$ is non separable.*

iii) *If* $S_1(\xi_1,\xi_2) = \tilde{a}(\xi_1)\, b(\xi_2)$, $\lambda(\xi_1,\xi_2) = \alpha_\eta(2\,\xi_1)$ *and* $\mu(\xi_1,\xi_2) = \beta_\eta(2\,\xi_1)\, e^{-i2\xi_2}$, *then the* QMF $R_0(\xi_1,\xi_2)$ *is non separable.*

Proof. One sees immediately that $R_0(\xi_1,\xi_2)$ is a QMF and that $R_1(\xi_1,\xi_2)$, $R_2(\xi_1,\xi_2)$, $R_3(\xi_1,\xi_2)$ are its conjugate filters.

Let us show i).

We will begin by the case where $S_1(\xi_1,\xi_2) = a(\xi_1)\, \tilde{b}(\xi_2)$. Suppose that

$$R_0(\xi_1,\xi_2) = m_1(c_{11}\,\xi_1 + c_{12}\,\xi_2)\, m_2(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,,$$

where (c_{ij}) belongs to $SL(2,\mathbb{Z})$ and $m_1(x)$, $m_2(x)$ are two monodimensional QMF's. Taking successively $(\xi_1,\xi_2) = (x,0)$, $(0,x)$ and (x,π) where x is an arbitrary real one obtains

$$\text{(a)} \qquad \alpha(2\,x)\, a(x) = m_1(c_{11}\,x)\, m_2(c_{21}\,x)\,,$$

$$\text{(b)} \qquad b(x) = m_1(c_{12}\,x)\, m_2(c_{22}\,x)\,,$$

$$\text{(c)} \qquad \beta(2\,x)\, a(x) = m_1(c_{11}\,x + c_{12}\,\pi)\, m_2(c_{21}\,x + c_{22}\,\pi)\,.$$

Since the product of two QMF's in the same variables is never a QMF (see the proof of the Theorem 2.1), it results from (b) that $c_{12}\,c_{22} = 0$. This implies that

$$(c_{ij}) = \begin{pmatrix} \varepsilon_1 & 0 \\ c_{21} & \varepsilon_2 \end{pmatrix}$$

or

$$(c_{ij}) = \begin{pmatrix} c_{11} & \varepsilon_1 \\ \varepsilon_2 & 0 \end{pmatrix}$$

with $\varepsilon_i = \pm 1$. We will suppose that we are in the first case, the second case being similar.

We notice that whatever the value of the integer c_{21} may be, one cannot have for all x, $|\alpha(2\,x)|^2 = |m_2(c_{21}\,x)|^2$. Indeed, if $c_{21} = 0$ then $\beta = 0$ and else $\alpha(2\pi/c_{21}) = 0$. In both cases the hypotheses are contradicted.

It follows from (a) and (c) that

$$\begin{aligned} |a(x)|^2 &= |\alpha(2\,x)|^2\, |a(x)|^2 + |\beta(2\,x)|^2\, |a(x)|^2 \\ &= |m_1(\varepsilon_1\,x)|^2\, |m_2(c_{21}\,x)|^2 + |m_1(\varepsilon_1\,x)|^2\, |m_2(c_{21}\,x + \pi)|^2 \\ &= |m_1(\varepsilon_1 x)|^2\,. \end{aligned}$$

Thus by using (a) we obtain the contradiction $|\alpha(2\,x)|^2 = |m_2(c_{21}\,x)|^2$ for all $x \in \mathbb{R}$.

Let us study now the case where $S_1(\xi_1, \xi_2) = \tilde{a}(\xi_1)\,\tilde{b}(\xi_2)$. As previously, we will suppose that

$$R_0(\xi_1, \xi_2) = m_1(c_{11}\,\xi_1 + c_{12}\,\xi_2)\, m_2(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,.$$

Taking successively $(\xi_1, \xi_2) = (x, 0)$, $(0, x)$ and (x, π) where x is an arbitrary real, one obtains

$$\text{(a)} \qquad \alpha(2\,x)\, a(x) = m_1(c_{11}\,x)\, m_2(c_{21}\,x)\,,$$

$$\text{(b)} \qquad b(x) = m_1(c_{12}\,x)\, m_2(c_{22}\,x)\,,$$

$$\text{(c)} \qquad \beta(2\,x)\, \tilde{a}(x) = m_1(c_{11}\,x + c_{12}\,\pi)\, m_2(c_{21}\,x + c_{22}\,\pi)\,.$$

It follows from (b), as previously, that $c_{12}\,c_{22} = 0$ and we can suppose that

$$(c_{ij}) = \begin{pmatrix} \varepsilon_1 & 0 \\ c_{21} & \varepsilon_2 \end{pmatrix}$$

with $\varepsilon_i = \pm 1$. (a) and (c) imply then that

$$\alpha(2\,x)\, a(x) + \beta(2\,x)\, \tilde{a}(x) = 2\, m_1(\varepsilon_1\,x)\, m_{2,e}(c_{21}\,x)\,,$$

$$\alpha(2\,x)\, a(x) - \beta(2\,x)\, \tilde{a}(x) = 2\, m_1(\varepsilon_1\,x)\, m_{2,o}(c_{21}\,x)\,,$$

where

$$m_{2,e}(c_{21}\,x) = \frac{1}{2}\,(m_2(c_{21}\,x) + m_2(c_{21}\,x + \pi))$$

and

$$m_{2,o}(c_{21}\,x) = \frac{1}{2}\,(m_2(c_{21}\,x) - m_2(c_{21}\,x + \pi))\,.$$

As

$$|\alpha(2\,x)\, a(x) + \beta(2\,x)\, \tilde{a}(x)|^2 + |\alpha(2\,x)\, a(x+\pi) + \beta(2\,x)\, \tilde{a}(x+\pi)|^2 = 1\,,$$

$$|\alpha(2\,x)\, a(x) - \beta(2\,x)\, \tilde{a}(x)|^2 + |\alpha(2\,x)\, a(x+\pi) - \beta(2\,x)\, \tilde{a}(x+\pi)|^2 = 1\,,$$

we have

$$4\,|m_{2,e}(c_{21}\,x)|^2 = 1\,,$$

$$4\,|m_{2,o}(c_{21}\,x)|^2 = 1\,.$$

If $c_{21} \neq 0$, it follows from the two last equalities that the QMF $m_2(x)$ has only two non vanishing coefficients. This is impossible since $R_0(\xi_1, \xi_2)$ has zeros of order greater or equal than 2 in $(\pi, 0)$, $(0, \pi)$ and (π, π). If $c_{21} = 0$, it follows from (c) that for all x, $\beta(2\,x)\,\tilde{a}(x) = 0$. This is impossible.

Let us show ii).

As the variables ξ_1 and ξ_2 play the same role we will only study the case where $S_1(\xi_1, \xi_2) = e(\xi_1)\,\tilde{b}(\xi_2)$ with $e = \tilde{a}$ or $e = a$. As previously, we will suppose that

$$R_0(\xi_1, \xi_2) = m_1(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_2(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,.$$

Taking successively $(\xi_1, \xi_2) = (x, 0)$, $(0, x)$ and (x, π) one obtains

$$\text{(a)} \qquad \alpha(2\,x)\,a(x) = m_1(c_{11}\,x)\,m_2(c_{21}\,x)\,,$$

$$\text{(b)} \qquad \alpha(2\,x)\,b(x) = m_1(c_{12}\,x)\,m_2(c_{22}\,x)\,, \qquad \text{when } e = \tilde{a}\,,$$

$$\text{(b')} \quad \alpha(2\,x)\,b(x) + \beta(2\,x)\,\tilde{b}(x) = m_1(c_{12}\,x)\,m_2(c_{22}\,x)\,, \quad \text{when } e = a\,,$$

$$\text{(c)} \qquad \beta(2\,x)\,e(x) = m_1(c_{11}\,x + c_{12}\,\pi)\,m_2(c_{21}\,x + c_{22}\,\pi)\,.$$

When $e = \tilde{a}$, it follows from (a) and (b) that c_{11}, c_{12}, c_{21} and c_{22} are all odd. Indeed, suppose for example that c_{11} is even, c_{21} would necessarily be odd and then (a) would imply that

$$\begin{aligned}|\alpha(2\,x)|^2 &= |\alpha(2\,x)|^2\,|a(x)|^2 + |\alpha(2\,x)|^2\,|a(x+\pi)|^2 \\ &= |m_1(c_{11}\,x)|^2\,|m_2(c_{21}\,x)|^2 + |m_1(c_{11}\,x)|^2\,|m_2(c_{21}\,x+\pi)|^2 \\ &= |m_1(c_{11}\,x)|^2\,.\end{aligned}$$

But we never have $|\alpha(2\,x)|^2 = |m_1(c_{11}\,x)|^2$ for all x. Thus it results from (c) that

$$\begin{aligned}|\beta(2\,x)|^2 &= |\beta(2\,x)|^2\,|e(x)|^2 + |\beta(2\,x)|^2\,|e(x+\pi)|^2 \\ &= |m_1(c_{11}\,x+\pi)|^2\,|m_2(c_{21}\,x+\pi)|^2 + |m_1(c_{11}\,x)|^2\,|m_2(c_{21}\,x)|^2\,,\end{aligned}$$

and it results from (a) that

$$|\alpha(2\,x)|^2 = |m_1(c_{11}\,x)|^2\,|m_2(c_{21}\,x)|^2 + |m_1(c_{11}\,x+\pi)|^2\,|m_2(c_{21}\,x+\pi)|^2\,.$$

This leads to the contradiction $|\alpha|^2 = |\beta|^2$.

When $e = a$, since $\alpha(2\,x)\,b(x) + \beta(2\,x)\,\tilde{b}(x)$ is a QMF, it follows from (b') that $c_{12}\,c_{22} = 0$. So, as previously we can suppose that

$$(c_{ij}) = \begin{pmatrix} \varepsilon_1 & 0 \\ c_{21} & \varepsilon_2 \end{pmatrix}$$

with $\varepsilon_i = \pm 1$. Moreover (a) implies that c_{21} is odd.

It results then from (a) and (c) that

$$\begin{aligned} |a(x)|^2 &= |\alpha(2\,x)|^2\,|a(x)|^2 + |\beta(2\,x)|^2\,|a(x)|^2 \\ &= |m_1(\varepsilon_1\,x)|^2\,|m_2(c_{21}\,x)|^2 + |m_1(\varepsilon_1\,x)|^2\,|m_2(c_{21}\,x+\pi)|^2 \\ &= |m_1(\varepsilon_1\,x)|^2\,. \end{aligned}$$

Thus (a) implies that $|\alpha(2\,x)|^2 = |m_2(c_{21}\,x)|^2$ for all x, which is impossible.

We can prove by the same method that

$$R_0(\xi_1,\xi_2) = \alpha_\eta(2\,(\xi_1-\xi_2))\,S_0(\xi_1,\xi_2) + \beta_\eta(2\,(\xi_1-\xi_2))\,S_1(\xi_1,\xi_2)\,,$$

where $S_1(\xi_1,\xi_2)$ is any conjugate filter of $S_0(\xi_1,\xi_2)$, is non separable.

Let us show iii).

As previously, we will suppose that

$$(*) \qquad R_0(\xi_1,\xi_2) = m_1(c_{11}\,\xi_1 + c_{12}\,\xi_2)\,m_2(c_{21}\,\xi_1 + c_{22}\,\xi_2)\,.$$

Taking $(\xi_1,\xi_2) = (x,\pi)$ one obtains

$$R_0(x,\pi) = m_1(c_{11}\,x + c_{12}\,\pi)\,m_2(c_{21}\,x + c_{22}\,\pi) = 0\,,$$

and it follows that $c_{11}\,c_{21} = 0$. Thus we may suppose that

$$(c_{ij}) = \begin{pmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & c_{22} \end{pmatrix}\,,$$

where $\varepsilon_i = \pm 1$, the other case being similar. Then taking successively in $(*)$, $(\xi_1,\xi_2) = (x,0)$ and $(0,x)$ one obtains

$$\alpha(2\,x)\,a(x) + \beta(2\,x)\,\tilde{a}(x) = m_2(\varepsilon_2\,x)\,,$$

$$b(x) = m_1(\varepsilon_1\,x)\,m_2(c_{22}\,x)\,.$$

The last equality implies that $c_{22} = 0$ and $b(x) = m_1(\varepsilon_1 x)$. Thus we have

$$R_0(\xi_1, \xi_2) = m_1(\varepsilon_1 \xi_2)\, m_2(\varepsilon_2 \xi_2) = b(\xi_2)\, (\alpha(2\,\xi_1)\, a(\xi_1) + \beta(2\,\xi_1)\, \tilde{a}(\xi_1))\,,$$

and it follows that

$$\begin{aligned} b(\xi_2)\, (\alpha(2\,\xi_1)\, a(\xi_1) &+ \beta(2\,\xi_1)\, \tilde{a}(\xi_1)\, e^{-i2\xi_2}) \\ &= b(\xi_2)\, (\alpha(2\,\xi_1)\, a(\xi_1) + \beta(2\,\xi_1)\, \tilde{a}(\xi_1))\,, \end{aligned}$$

which leads to the contradiction: for all ξ_2, $e^{-i2\xi_2} = 1$.

3. Some of the previous QMF's lead to wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity.

In this section we give two methods for constructing non separable orthonormal compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity.

In all this section the norm on $\mathbb{R}^2$ will be $|(\xi_1, \xi_2)| = \sup\{|\xi_1|, |\xi_2|\}$.

3.1. The method by perturbing the I. Daubechies QMF's.

Proposition 3.1. *For all $L \geq 1$, let $D_L(\xi_1, \xi_2) = d_L(\xi_1)\, d_L(\xi_2)$ be the separable I. Daubechies* QMF *such that*

$$|d_L(x)|^2 = c_L \int_x^{\pi} \sin^{2L-1} t\, dt\,.$$

For all $\varepsilon > 0$, one can construct a non separable QMF *$D_{L,\varepsilon}(\xi_1, \xi_2)$ that satisfies*

i) $\|D_{L,\varepsilon} - D_L\|_\infty \leq \varepsilon$,

ii) *$D_{L,\varepsilon}(\xi_1, \xi_2)$ has zeros of order L on $(\pi, 0)$, $(0, \pi)$ and (π, π),*

iii) *the size of $D_{L,\varepsilon}(\xi_1, \xi_2)$ is independent on ε.*

Moreover $D_{L,\varepsilon}(\xi_1, \xi_2)$ may be chosen of the type (2.3) or of the type (2.7). φ_L and $\varphi_{L,\varepsilon}$ will be the scaling functions that correspond to $D_L(\xi_1, \xi_2)$ and $D_{L,\varepsilon}(\xi_1, \xi_2)$.

PROOF. See [2, Chapter 3] and see also [1].

It is clear that for $\varepsilon > 0$ small enough $D_{L,\varepsilon}(\xi_1, \xi_2)$ will satisfy the A. Cohen's criterion.

Proposition 3.1 remains valid, if we replace the QMF $D_L(\xi_1, \xi_2)$ by any other separable QMF $a(\xi_1)\, b(\xi_2)$ that has zeros of order L on $(\pi, 0)$, $(0, \pi)$ and (π, π).

We can now state the main result of this subsection.

Theorem 3.2. *The* QMF*'s $D_{L,\varepsilon}(\xi_1, \xi_2)$ generate for $\varepsilon > 0$ small enough non separable orthonormal compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity. We will say that these wavelets are obtained by perturbing the I. Daubechies* QMF*'s.*

PROOF. The critical Sobolev exponent of $f \in L^2(\mathbb{R}^2)$ is by definition

$$\alpha(f) = \sup\Big\{\alpha : \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2\, (1 + |\xi|^{\alpha})^2\, d\xi < \infty\Big\}.$$

R. Q. Jia has shown in [8] that if $M(\xi)$ is a QMF that satisfies A. Cohen's criterion and that has zeros of order L in $(\pi, 0)$, $(0, \pi)$ and (π, π), then the critical Sobolev exponent of φ the corresponding scaling function is

$$(*) \qquad \alpha(\varphi) = -\log_4 \Big(\rho\Big(\frac{T_M}{\tau_{2L}}\Big)\Big),$$

where $\rho(T_M/\tau_{2L})$ is the spectral radius of the restriction of the transfer operator

$$T_M f(\xi) = \sum_{\nu \in \{0,1\}^2} \Big|M\Big(\frac{\xi}{2} + \pi\,\nu\Big)\Big|^2 f\Big(\frac{\xi}{2} + \pi\,\nu\Big)$$

to the vector space τ_{2L} of the trigonometric polynomials that have a zero of order greater or equal than $2L$ in $(0, 0)$.

Let $\alpha(\varphi_L)$ and $\alpha(\varphi_{L,\varepsilon})$ the critical Sobolev exponent of the scaling functions φ_L and $\varphi_{L,\varepsilon}$ (as defined in the Proposition 3.1). Since the regularity of the I. Daubechies scaling function φ_L can arbitrarily high when L is big enough, we have $\lim_{L\to\infty} \alpha(\varphi_L) = +\infty$. At last, it follows from $(*)$ and from the continuity of the spectral radius, that $\lim_{\varepsilon\to 0} \alpha(\varphi_{L,\varepsilon}) = \alpha(\varphi_L)$, which implies the Theorem 3.2.

We have solved the open theorical problem of establishing the existence of non separable orthonormal compactly supported wavelet bases for $L^2(\mathbb{R}^2)$ of arbitarily high regularity. However the wavelets we have obtained are probabely very similar to the I. Daubechies separable wavelets since $\lim_{\varepsilon\to 0}\varphi_{L,\varepsilon} = \varphi_L$ (for the L^∞ norm) (see [2, Chapter 4]).

3.2. Another method of construction.

The aim of this subsection is to construct non separable orthonormal compactly supported wavelets of arbitrarily high regularity that are not near to the I. Daubechies bidimensional wavelets with the same number of vanishing moments (for the L^∞ norm).

Let us first give a condition ensuring the decrease at infinite of the Fourier transform of a scaling function.

Theorem 3.3. *Given two reals $\delta \in]0,1[$ and $C \geq (2\pi)^{-1}$, there exists an exponent $\alpha = \alpha(\delta, C) > 0$ having the following property. If $M(\xi_1,\xi_2)$ is a* QMF *that satisfies for some integer $N \geq 1$*

a) $|M(\xi_1,\xi_2)| \leq \delta^N$ *when $\xi_1 \in [2\pi/3, 4\pi/3]$ or $\xi_2 \in [2\pi/3, 4\pi/3]$,*

b) $|M(\xi_1,\xi_2)| \leq C^N |(\xi_1 - s_1\pi, \xi_2 - s_2\pi)|^N$ *for all ξ_1, ξ_2 for all $(s_1,s_2) \in \{0,1\}^2$ and $(s_1,s_2) \neq (0,0)$,*

c) $|M(\xi_1,\xi_2)| = |M(-\xi_1,-\xi_2)|$ *for all ξ_1, ξ_2.*

Then φ the scaling function that corresponds to $M(\xi_1,\xi_2)$, satisfies $\hat{\varphi}(\xi_1,\xi_2) = O\left(|(\xi_1,\xi_2)|^{-\alpha N}\right)$.

To prove the Theorem 3.3 we need the following lemma.

Lemma 3.4. *Given two reals $\delta \in]0,1[$ and $C \geq 1$, there exists an exponent $\alpha = \alpha(\delta, C) > 0$ having the following property. If $f(s,t)$ is a continuous function from $\mathbb{R}^2$ to $[0,1]$, 1-periodic in s and t, which satisfies*

i) $0 \leq f(s,t) \leq \delta < 1$, *when $s \in [1/3, 2/3]$ or $t \in [1/3, 2/3]$,*

ii) $f(s,t) \leq C\,|(s - \nu_1/2, t - \nu_2/2)|$, *for all s,t, for all $(\nu_1,\nu_2) \in \{0,1\}^2$ and $(\nu_1,\nu_2) \neq (0,0)$,*

iii) $f(s,t) = f(-s,-t)$, *for all s,t.*

Then if $j \geq 1$ and if (s,t) satisfies $1/4 \leq |(s,t)| \leq 1/2$ we have the inequality

$$f(s,t)\, f(2\,s, 2\,t) \cdots f(2^j\,s, 2^j\,t) \leq 2^{-\alpha j}\,.$$

PROOF. First we set $h(s,t) = f(s,t)\, f(2\,s, 2\,t)$. The function $f(s,t)$ satisfies the property ii) and the inequality i) when $s \in [1/6, 5/6]$ or when $t \in [1/6, 5/6]$. Moreover this function being with values in $[0,1]$ we have

$$\prod_{k=0}^{j-1} h(2^k\,s, 2^k\,t) \geq \Big(\prod_{k=0}^{j} f(2^k\,s, 2^k\,t)\Big)^2,$$

thus it is sufficient to show that for some $\beta = \beta(\delta, C) > 0$ for all $j \geq 1$

$$\prod_{k=0}^{j-1} h(2^k\,s, 2^k\,t) \leq 2^{-\beta j}\,.$$

Consider now (s,t) satisfying $|(s,t)| = |s| \in [1/4, 1/2]$. Because of the periodicity of $f(s,t)$ and because of iii), one can suppose that $(s,t) \in [1/4, 1/2] \times [0,1]$. It follows that

$$s = \frac{1}{4} + \frac{\alpha_3}{8} + \cdots\,, \qquad t = \frac{\beta_1}{2} + \frac{\beta_2}{4} + \cdots\,,$$

where α_j, $\beta_j \in \{0,1\}$.

We then define $q_1, \ldots, q_r$ the transition indices of the finite vectorial sequence $(\alpha_0, \beta_0), \ldots, (\alpha_{j+1}, \beta_{j+1})$ where $(\alpha_0, \beta_0) = (0,0)$ as follows: r will be the number of the indices q that satisfy, $0 \leq q \leq j-1$ and $(\alpha_{q+1}, \beta_{q+1}) \neq (\alpha_{q+2}, \beta_{q+2})$, $q_1 = 0$ and for all l, $2 \leq l \leq r$,

$$q_l = \min\{n :\; q_{l-1} < n \leq j-1 \text{ and } (\alpha_{n+1}, \beta_{n+1}) \neq (\alpha_{n+2}, \beta_{n+2})\}\,.$$

For $m = 1, \ldots, r$ we set $l_m = q_{m+1} - q_m$ ($q_{r+1} = j$ by convention), thus we have $\sum_{m=1}^{r} l_m = j$.

We have introduced the transition indices in order to get the inequalities

$$(M_1) \qquad h(2^{q_m}\,s, 2^{q_m}\,t) \leq \delta\,,$$

$$(M_2) \qquad h(2^{q_m}\,s, 2^{q_m}\,t) \leq C\, 2^{-l_m}\,.$$

We have $2^{q_m} s \in [1/6, 5/6]$ or $2^{q_m} t \in [1/6, 5/6]$ therefore i) implies (M_1). To prove (M_2) we will suppose that $\alpha_{q_m+1} \neq \alpha_{q_m+2}$, we then have $|2^{q_m} s - 1/2| \leq 2^{-(l_m+1)}$. So, if $\beta_{q_m+1} = \beta_{q_m+2}$ it follows that $|2^{q_m} t - \beta_{q_m+1}| \leq 2^{-(l_m+1)}$ and else that $|2^{q_m} t - 1/2| \leq 2^{-(l_m+1)}$, in the both cases ii) implies (M_2).

At last, for all $j \geq 1$ we have

$$h(s,t)\, h(2\,s, 2\,t) \cdots h(2^{j-1}\,s, 2^{j-1}\,t) \leq h(2^{q_1}\,s, 2^{q_1}\,t) \cdots h(2^{q_r}\,s, 2^{q_r}\,t)\,.$$

So, if A and β are two reals such that $2^A \geq C$, $2^{-\beta A} \geq \delta$ and $2^{A(1-\beta)} \geq C$ (for example $A = \log_2 C + \log_2(1/\delta)$ and $\beta = \log_2(1/\delta)/A$) then we will have

$$(*) \qquad h(2^{q_m}\,s, 2^{q_m}\,t) \leq 2^{-\beta l_m}\,,$$

indeed, when $l_m < A$ (M_1) implies that $h(2^{q_m} s, 2^{q_m} t) \leq 2^{-\beta A} < 2^{-\beta l_m}$ and when $l_m \geq A$ (M_2) implies that $h(2^{q_m} s, 2^{q_m} t) \leq C\, 2^{-l_m} \leq 2^{-\beta l_m}$.

Since $\sum_1^r l_m = j$ it results from $(*)$ that

$$h(2^{q_1}\,s, 2^{q_1}\,t) \cdots h(2^{q_r}\,s, 2^{q_r}\,t) \leq 2^{-\beta j}\,.$$

Proof (of the Theorem 3.3). If $M(\xi_1, \xi_2)$ is a QMF satisfying (a), (b) and (c) the function $f(s,t) = |M(2\pi\, s, 2\pi\, t)|^{1/N}$ satisfies the conditions i), ii) and iii) of the Lemma 3.4. Let $(\xi_1, \xi_2) \in \mathbb{R}^2$ such that $|(\xi_1, \xi_2)| \geq 2\pi$ and let $j \geq 1$ the integer such that $2^j \pi \leq |(\xi_1, \xi_2)| \leq 2^{j+1}\,\pi$. Thus, if

$$(s,t) = \frac{1}{2^{j+2}\pi}\,(\xi_1, \xi_2)\,,$$

we have $1/4 \leq |(s,t)| \leq 1/2$ and it results from the Lemma 3.4 that

$$\begin{aligned} |\hat{\varphi}_M(\xi_1, \xi_2)| &\leq (f(s,t)\, f(2\,s, 2\,t) \cdots f(2^j\,s, 2^j\,t))^N \\ &\leq 2^{-\alpha N j} \\ &\leq C(\alpha, N)\, |(\xi_1, \xi_2)|^{-\alpha N}\,. \end{aligned}$$

From now on, our aim will be to construct a sequence of non separable QMF's $\{A_L(\xi_1, \xi_2)\}_{L \geq 1}$ such that for all L big enough,

i) $A_L(\xi_1, \xi_2)$ satisfies the conditions (a), (b) and (c) of the Theorem 3.3,

ii) $A_L(\xi_1, \xi_2)$ satisfies A. Cohen's criterion,

iii) $A_L(\xi_1, \xi_2)$ is not near to the I. Daubechies QMF $D_L(\xi_1, \xi_2)$ as defined in the Proposition 3.1. More precisely we will have

$$\liminf_{L\to\infty} \|A_L - D_L\|_\infty \geq \frac{1}{4}\,.$$

Let $A_{L,\eta}(\xi_1, \xi_2)$ be a QMF of the form

$$A_{L,\eta}(\xi_1,\xi_2) = d_L(\xi_2)\,(\alpha_\eta(2\,\xi_1)\,d_L(\xi_1) + \beta_\eta(2\,\xi_1)\,\tilde{d}_L(\xi_1)\,e^{-i2\xi_2})\,, \tag{3.1}$$

where,

- $\eta \in\,]0, 1[\,$,
- $d_L(x)$ is the monodimensional I. Daubechies QMF such that

$$|d_L(x)|^2 = c_L \int_x^\pi \sin^{2L-1} t\, dt$$

and $\tilde{d}_L(x) = -e^{-ix}\,\overline{d_L(x+\pi)}$ is its conjugate filter,

- $\alpha_\eta(x) = 1 - \eta\, q(x)$ and $\beta_\eta(x)$ are the trigonometric polynomials as defined by (2.5) and (2.6).

Let us first give some useful properties of the QMF $d_L(x)$.

Proposition 3.5. *The monodimensional I. Daubechies* QMF $d_L(x)$ *satisfies:*

i) *for all real $\alpha \in\,]0, \pi/4[\,$, one can find a real $\delta \in\,]0, 1[$ such that for all L big enough, for all $x \in [\pi/2 + \alpha, 3\pi/2 - \alpha]$, $|d_L(x)| \leq \delta^L$,*

ii) *there exists a real $C \geq (2\pi)^{-1}$ such that for all $x \in \mathbb{R}$, $|d_L(x)| \leq C^L\,|x - \pi|^L$,*

iii) *for all $x \in\,]-\pi/2, \pi/2[\,$,*

$$\lim_{L\to\infty} |d_L(x)| = 1 \qquad \textit{and} \qquad \lim_{L\to\infty} |\tilde{d}_L(x)| = 0\,.$$

PROOF. The function $|d_L(x)|$ being even one can suppose that $x \in [0, \pi]$. One can notice that $c_L = O(\sqrt{L})$. For all $\alpha > 0$, we have for all $x \in [\pi/2 + \alpha, \pi]$, $|d_L(x)|^2 \leq |\pi/2 - \alpha|\, c_L\, |\sin^{2L-1}(\pi/2 + \alpha)|$ which implies i).

We have obviously ii) since

$$|d_L(x)|^2 \le c_L \int_x^{\pi} (\pi - t)^{2L-1}\, dt \le C' \, |x - \pi|^{2L} .$$

iii) is a consequence of i) and of $|d_L(x)|^2 + |d_L(x+\pi)|^2 = 1$.

The following proposition will permit us to make an appropriate choice of the real η that occurs in (3.1).

Proposition 3.6. *If* $\eta_0 = 1/4$ *then for all* $L \ge 1$ *the* QMF $A_{L,\eta_0}(\xi_1, \xi_2)$, *as defined by* (3.1) *satisfies A. Cohen's criterion.*

Proof. Let us show that for all $(\xi_1, \xi_2) \in [-\pi/2, \pi/2]^2$ and for all $L \ge 1$ we have $|A_{L,\eta_0}(\xi_1, \xi_2)| > 0$. As $|d_L(\xi_2)| > 0$ it is sufficient to show that

$$|\alpha_{\eta_0}(2\,\xi_1)\, d_L(\xi_1) + \beta_{\eta_0}(2\,\xi_1)\, \tilde{d}_L(\xi_1)\, e^{-i2\xi_2}| > 0 .$$

We have

$$|\alpha_{\eta_0}(2\,\xi_1)\, d_L(\xi_1) + \beta_{\eta_0}(2\,\xi_1)\, \tilde{d}_L(\xi_1)\, e^{-i2\xi_2}| \ge \frac{\sqrt{2}}{2} \left(|\alpha_{\eta_0}(2\,\xi_1)| - |\beta_{\eta_0}(2\,\xi_1)| \right) .$$

At last, since $|\alpha_{\eta_0}(2\,\xi_1)| \ge 3/4$ and $|\alpha_{\eta_0}(2\,\xi_1)|^2 + |\beta_{\eta_0}(2\,\xi_1)|^2 = 1$, it follows that $|\alpha_{\eta_0}(2\,\xi_1)| > |\beta_{\eta_0}(2\,\xi_1)|$.

The following lemma will permit us to make an appropriate choice of the trigonometric polynomial $q(x)$ that occurs in (3.1).

Lemma 3.7. *For all reals* δ *and* α *satisfying* $\delta \in]0,1[$ *and* $\alpha \in]0, \pi/6[$ *there exists* $\{q_L(x)\}_{L\ge 1}$ *a sequence of trigonometric polynomials in one variable with values in* $[0,1]$ *and with real coefficients, having the following properties*:

i) $\|q_L\|_\infty = 1$,

ii) $q_L(2\,x) \le \delta^L$ *for all* $x \in [0, \pi] - [\pi/3, 2\pi/3]$,

iii) $q_L(2\,x)$ *converges uniformly to* 1 *on* $[\pi/3 + \alpha, 2\pi/3 - \alpha]$,

iv) *there exists a real* $C \ge 1$ *such that* $q_L(2\,x) \le C^{2L}\, |x|^{2L}$ *for all* x.

PROOF. Consider T an even, π-periodic, C^1 function with values on $[0,1]$ such that

a) $T(0)=0$ and for all $x \in [0,\pi/3]$, $0 \le T'(x) < \beta$ (where $\beta\,\pi/3 < \sqrt{\delta}\,$),

b) for all $x \in [\pi/3+\alpha, \pi/2]$, $T(x)=1$,

c) for all $x \in [\pi/2,\pi]$, $T(x)=T(\pi-x)$.

Let $K_N(x)$ be the Fejer kernel, $K_N(x)$ is the trigonometric polynomial

$$K_N(x) = \frac{1}{2\pi\,(N+1)} \Big| \sum_{k=0}^{N} e^{ikx} \Big|^2 = \frac{1}{2\pi\,(N+1)} \frac{\sin^2\left(\dfrac{(N+1)}{2}\,x\right)}{\sin^2\left(\dfrac{x}{2}\right)}\,.$$

For every function $f \in L^2[0,2\pi]$,

$$K_N * f(x) = \int_{-\pi}^{\pi} K_N(x-y)\, f(y)\, dy$$

will be the convolution product of K_N and f. Let $Q_N(x) = K_N * T(x) - K_N * T(0)$ and

$$R_N(x) = \frac{Q_N}{\|Q_N\|_\infty}\,(x)\,.$$

Since T is even and π-periodic the trigonometric polynomial R_N is with real coefficients and π-periodic.

The sequences $\{R_N\}$ and $\{R'_N\}$ converge uniformly to the functions T and T'. Thus it follows from a) that:

- There exists $C \ge 1$ such that for all x, for all N, $|R_N(x)| \le C\,|x|$.
- For all $N \ge N_0$ and for all $x \in [0,\pi] - [\pi/3, 2\pi/3]$, $|R_N(x)| \le \sqrt{\delta}$.

At last, one can extract a sequence $\{R_{N_L}\}_{L\ge 1}$ satisfying $N_1 \ge N_0$ and $\|T - R_{N_L}\|_\infty \le e^{-L}$. We will take $q_L(2\,x) = |R_{N_L}(x)|^{2L}$.

Definition 3.8. *$A_L(\xi_1,\xi_2)$ will be a* QMF *of the type* (3.1) *such that $\eta = 1/4$ and $q(x) = q_L(x)$, where $q_L(x)$ is the trigonometric polynomial we have constructed in the Lemma* 3.7.

We can now state the main result of this subsection.

Theorem 3.9. *The* QMF*'s* $\{A_L(\xi_1,\xi_2)\}_{L\geq 1}$ *generate non separable orthonormal compactly supported wavelet bases for* $L^2(\mathbb{R}^2)$ *of arbitrarily high regularity. Moreover these wavelets are not near to the separable I. Daubechies wavelets with the same number of vanishing moments since* $\liminf_{L\to\infty} \|A_L - D_L\|_\infty \geq 1/4$.

PROOF. It follows from the Proposition 3.5, the Lemma 3.7 and the inequality

$$|A_L(\xi_1,\xi_2)| \leq |d_L(\xi_1)|\,|d_L(\xi_2)| + \sqrt{\frac{q_L(2\,\xi_1)}{2}}\,|\tilde{d}_L(\xi_1)|\,|d_L(\xi_2)|\,,$$

that, for L big enough the QMF $A_L(\xi_1,\xi_2)$ satisfies the conditions a) and b) of the Theorem 3.3. This QMF also satisfies the condition c) of the same theorem since its coefficients are reals.

Let us show that $\liminf_{L\to\infty} \|A_L - D_L\|_\infty \geq 1/4$.

We have

$$\begin{aligned}\frac{1}{4}\,q_L(2\,\xi_1)\,|d_L(\xi_1)|\,|d_L(\xi_2)| &- |\beta_{1/4}(2\,\xi_1)|\,|\tilde{d}_L(\xi_1)|\,|d_L(\xi_2)| \\ &\leq |A_L(\xi_1,\xi_2) - D_L(\xi_1,\xi_2)| \\ &\leq \frac{1}{4}\,q_L(2\,\xi_1)\,|d_L(\xi_1)|\,|d_L(\xi_2)| + |\beta_{1/4}(2\,\xi_1)|\,|\tilde{d}_L(\xi_1)|\,|d_L(\xi_2)|\,.\end{aligned}$$

It follows from the Propositions 3.5.iii) and from the Lemma 3.7.iii) that for all $(\xi_1,\xi_2)\in [\pi/3+\alpha, \pi/2\,[\,\times\,]-\pi/2,\pi/2\,[$

$$\lim_{L\to\infty} |A_L(\xi_1,\xi_2) - D_L(\xi_1,\xi_2)| = \frac{1}{4}\,,$$

therefore

$$\liminf_{L\to\infty} \|A_L - D_L\|_\infty \geq \frac{1}{4}\,.$$

4. Conclusion.

Some of the techniques we have used to construct non separable, dyadic, compactly supported, orthonormal, wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity, may be adapted to other types of wavelet bases.

In [2] we have constructed non separable, dyadic, compactly supported, biorthogonal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity by perturbing separable biorthogonal filters.

We have found recently a method for constructing QMF's that generate compactly supported, orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of dilation matrix

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(R is a rotation of $\pi/4$ and a dilation of $\sqrt{2}$). This method is inspired from the Theorem 2.2 Let $\alpha(x)$ and $\beta(x)$ two trigonometric polynomials in one variable such that $\alpha(0) = 1$ and $|\alpha(x)|^2+|\beta(x)|^2+1$. Let $m(x)$ be a monodimensional QMF (*i.e.* $m(0) = 1$ and $|m(x)|^2+|m(x+\pi)|^2 = 1$) and $\tilde{m}(x)$ its conjugate filter ($\tilde{m}(x) = -e^{-ix}\,\overline{m(x+\pi)}$). If $P(\xi_1, \xi_2)$ is one of the trigonometric polynomials

$$u(\xi_1, \xi_2) = \alpha(2\,\xi_1)\, m(\xi_2) + \beta(2\,\xi_1)\, \tilde{m}(\xi_2)\,,$$

$$v(\xi_1, \xi_2) = \alpha(\xi_1 + \xi_2)\, m(\xi_2) + \beta(\xi_1 + \xi_2)\, \tilde{m}(\xi_2)\,,$$

$$w(\xi_1, \xi_2) = \alpha(\xi_1 - \xi_2)\, m(\xi_2) + \beta(\xi_1 - \xi_2)\, \tilde{m}(\xi_2)\,,$$

then we have

$$\begin{cases} P(0,0) = 1\,, \\ |P(\xi_1, \xi_2)|^2 + |P(\xi_1 + \pi, \xi_2 + \pi)|^2\,. \end{cases}$$

This means that when $P(\xi_1, \xi_2)$ satisfies A. Cohen's criterion it generates a compactly supported, orthonormal wavelet basis for $L^2(\mathbb{R}^2)$ of dilation matrix R. We do not know yet whether the regularity of such wavelets could be made arbitrarily high.

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References.

[1] Ayache, A., Construction de bases orthonormées d'ondelettes de $L^2(\mathbb{R}^2)$ non séparables, à support compact et de régularité arbitrairement grande. *C. R. Acad. Sci. Paris* **325** Série I (1997), 17-20.

[2] Ayache, A., Bases multivariées d'ondelettes, orthonormales, à support compact et de régularité arbitraire. Ph. D. Thesis. Ceremade, University of Paris Dauphine, 1997.

[3] Cohen, A., Ondelettes, analyses multirésolution et filtres miroir en quadrature. *Ann. Inst. H. Poincaré, Analyse non linéaire* **7** (1990), 439-459.

[4] Cohen, A., Ondelettes et traitement numérique du signal. Ph. D. Thesis. Ceremade. University Paris Dauphine, 1990.

[5] Cohen, A., Daubechies, I., Non seperable bidimensional wavelet bases. *Revista Mat. Iberoamericana* **9** (1993), 51-137.

[6] Daubechies, I., Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* **41** (1988) 909-996.

[7] Gröchenig, K., Analyse multiéchelle et bases d'ondelettes. *C. R. Acad. Sci. Paris* **305** Série I (1987), 13-17.

[8] Jia, R. Q., Characterization of smoothness of multivariate refinable functions in sobolev spaces. Preprint, 1995.

[9] Meyer, Y., Ondelettes, fonctions splines et analyses graduées. *Cahiers du Ceremade* **8703** (1986).

[10] Meyer, Y., *Ondelettes et opérateurs*. Vol. I. Hermann, 1990.

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Some Dirichlet spaces obtained by subordinate reflected diffusions

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In this paper we want to show how well-known results from the theory of (*regular*) *elliptic boundary value problems, function spaces and interpolation, subordination in the sense of Bochner,* and *Dirichlet forms* can be combined and how one can thus get some new aspects in each of these fields.

Let $A = L(x, D)$ be a second-order elliptic differential operator with smooth coefficients on a bounded domain G with smooth boundary ∂G and with Dirichlet or Neumann boundary conditions. Assume that the operator is symmetric. Under Neumann boundary conditions, it generates a reflected diffusion process $\{X_t\}_{t\geq 0}$ which is associated with a Dirichlet form $\mathcal{E}$ with domain $H^1(G)$. It is clear that A defined on

$$D(A) = H^2_{\{\partial/\partial\nu\}}(G) := \Big\{u \in H^2(G) : \ \frac{\partial}{\partial \nu}\, u\Big|_{\partial G} = 0\Big\}$$

is also the generator of a sub-Markovian semigroup $\{T_t\}_{t\geq 0}$ on $L^2(\overline{G})$. Denote by f_α, $0 < \alpha < 1$, the Bernstein function $f_\alpha(x) = x^\alpha$. By subordination in the sense of Bochner it is possible to construct for each $\alpha \in (0, 1)$ four new objects, $A^{(\alpha)} := -(-A)^\alpha$, $\{T_t^{(\alpha)}\}_{t\geq 0}$, the semigroup generated by $A^{(\alpha)}$, $\mathcal{E}^{(\alpha)}(\cdot, \cdot)$, the Dirichlet form associated with $A^{(\alpha)}$ (and also with $\{T_t^{(\alpha)}\}_{t\geq 0}$), and the subordinate (with respect to the Bernstein function x^α) stochastic process $\{X_t^{(\alpha)}\}_{t\geq 0}$. These constructions are of a somewhat abstract nature and some work has to be done if one wants to determine $D(A^{(\alpha)})$ and $D(\mathcal{E}^{(\alpha)})$ explicitly in terms of function spaces. In fact, this work has already been done by R. Seeley [18] for $D(A^{(\alpha)})$, and for $D(\mathcal{E}^{(\alpha)})$ the results are even longer known, *cf.* J. L. Lions and E. Magenes [13], and, as reference for both cases, the monograph [22] by H. Triebel.

In our first section we collect some fundamental results on the Dirichlet and Neumann problems for second-order elliptic differential operators (with smooth coefficients in a domain with smooth boundary) and the associated diffusion processes. Subordination in the sense of Bochner will be discussed in Section 2, both from the analytic and probabilistic point of view. In the third section we study $D(A^{(\alpha)})$ and $D(\mathcal{E}^{(\alpha)})$ under Dirichlet and Neumann conditions. In both cases the domains are certain fractional order Sobolev spaces. Under Neumann boundary conditions we have

$$D(\mathcal{E}^{(\alpha)}) = H^{\alpha}(G)\,, \qquad \text{if } \alpha \in (0,1)$$

and

$$D(A^{(\alpha)}) = H^{2\alpha}_{\{\partial/\partial\nu\}}(G)\,, \qquad \text{if } \alpha \in \Big(\frac{3}{4},1\Big)\,,$$

$$D(A^{(\alpha)}) = H^{2\alpha}(G)\,, \qquad \text{if } \alpha \in \Big(0,\frac{3}{4}\Big)\,,$$

under Dirichlet boundary conditions we have

$$D(\mathcal{E}^{(\alpha)}) = H^{\alpha}(G)\,, \qquad \text{if } \alpha \in \Big(0,\frac{1}{2}\Big)\,,$$

$$D(\mathcal{E}^{(\alpha)}) = H^{\alpha}_0(G)\,, \qquad \text{if } \alpha \in \Big(\frac{1}{2},1\Big)\,,$$

and

$$D(A^{(\alpha)}) = H^{2\alpha}(G)\,, \qquad \text{if } \alpha \in \Big(0,\frac{1}{4}\Big)\,,$$

$$D(A^{(\alpha)}) = H^{2\alpha}_D(G)\,, \qquad \text{if } \alpha \in \Big(\frac{1}{4},1\Big)\,.$$

Here,

$$H^s_D(G) := \{u \in H^s(G) :\ \gamma\, u = 0\}$$

with the trace operator γ. One should note that these are well-known results in the theory of elliptic boundary value problems, but they seem to be rather ignored in the theory of Dirichlet forms.

Section 4 deals with the decomposition of the (Neumann) Dirichlet space $(\mathcal{E}_\lambda^{(\alpha)}, H^\alpha(G))$. We show that $H^\alpha(G)$ can be written as an orthogonal sum $H_0^\alpha(G) \oplus \mathcal{H}_\lambda^\alpha(G)$ where the functions $u \in \mathcal{H}_\lambda^\alpha(G)$ are the harmonic functions with respect to the form $\mathcal{E}_\lambda^{(\alpha)}$– *i.e.* $\mathcal{E}_\lambda^{(\alpha)}(u,w) = 0$ for all $w \in H_0^\alpha(G)$. Moreover, we show that there is an isomorphism $\Pi_\lambda^{(\alpha)}$ from $H^{\alpha-1/2}(\partial G)$ to $\mathcal{H}_\lambda^\alpha(G)$. This map establishes a unitary equivalence between $(\mathcal{E}_\lambda^{(\alpha)}, \mathcal{H}^\alpha(G))$ and $(\mathcal{C}_\lambda^{(\alpha)}, H^{\alpha-1/2}(\partial G))$, where $\mathcal{C}_\lambda^{(\alpha)}$ is – at least for $\lambda = 0$ – the analogue of the classical Douglas integral. This correspondence is further investigated in Section 5. In particular, we show that $(\mathcal{C}_\lambda^{(\alpha)}, H^{\alpha-1/2}(\partial G))$ is a regular Dirichlet space and that $\mathcal{C}_\lambda^{(\alpha)}$ is equivalent to the canonical scalar product on $H^{\alpha-1/2}(\partial G)$ which itself is a Dirichlet form. The precise knowledge of $D(\mathcal{C}_\lambda^{(\alpha)})$ allows us, for example, to derive certain L^p-estimates for $\mathcal{C}_\lambda^{(\alpha)}$ and thus L^1-L^∞-estimates for the associated semigroup.

In Section 6 we construct the associated boundary processes and show that the process generated by $(\mathcal{C}_\lambda^{(\alpha)}, H^{\alpha-1/2}(\partial G))$ can indeed be obtaind by an appropriate time-change of the process generated by $(\mathcal{C}_\lambda, H^{1/2}(\partial G))$.

The final section takes up the Skorokhod representation of the reflected diffusion which was already discussed in the first section. We use now Bochner's subordination (with respect to fractional powers) in order to derive a representation for the subordinate reflected process. Note, that subordination is one possibility to construct a reflected symmetric stable process in a unique and natural way. However, in [24] S. Watanabe pointed out that there are several methods of getting processes which one could call *reflected symmetric stable processes.*

1. Dirichlet forms generated by elliptic differential operators with boundary conditions.

In this section we summarize some results on Dirichlet forms that are generated by second-order elliptic differential operators satisfying Neumann or Dirichlet boundary conditions. In particular, we recall some conditions that allow to associate stochastic processes to these Dirichlet forms, *reflected diffusions* under Neumann boundary conditions and *absorbing diffusions* under Dirichlet boundary conditions. Since we want to present our ideas as clearly as possible (and do not want to get entangled in technical details) we will restrict our considerations to rather smooth objects – thus getting at best sub-optimal conditions from the point of view of Dirichlet forms, but keeping *full* compatibility with existing (analytic) literature. Our exposition will, later on, rely heavily on results from the theory of function spaces and interpolation theory.

The main reference for this section is the monograph [8] by M. Fukushima, Y. Oshima, and M. Takeda. For the Neumann problem we refer especially to the paper [9] by M. Fukushima and M. Tomisaki. We should, however, mention that the crux of that paper was to consider a situation with rather weak regularity assumptions – which is somehow an opposite point of view. Nevertheless we think it might be convenient for the reader to have a state-of-the-art and easily accessible reference.

Let $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary ∂G, *i.e.*, ∂G is assumed to be a C^∞-manifold. We consider the second order differential operator

$$L(x,D) = \sum_{k,\ell=1}^{n} \frac{\partial}{\partial x_k}\Big(a_{k\ell}(x)\,\frac{\partial}{\partial x_\ell}\Big)\,, \tag{1.1}$$

with coefficients $a_{k\ell} = a_{\ell k} \in C^\infty(\overline{G})$. Moreover, we assume that

$$\lambda_0^{-1}|\xi|^2 \le \sum_{k,\ell=1}^{n} a_{k\ell}(x)\,\xi_k\,\xi_\ell \le \lambda_0\,|\xi|^2\,, \tag{1.2}$$

for some $\lambda_0 > 0$ and all $x \in \overline{G}$, $\xi \in \mathbb{R}^n$. It is well known that the quadratic form

$$\mathcal{E}(u,v) := \int_G \sum_{k,\ell=1}^{n} a_{k\ell}(x)\,\frac{\partial u}{\partial x_k}\,(x)\,\frac{\partial v}{\partial x_\ell}\,(x)\,dx\,, \tag{1.3}$$

with domain $H^1(G) \subset L^2(\overline{G})$ is a regular Dirichlet form, see [8, Example 1.6.1], where the regularity problem is carefully discussed if $a_{k\ell}(x) = \delta_{k\ell}$. Therefore, *cf.* [9], there exists a conservative diffusion process $\mathbf{X} = (\{X_t\}_{t\ge 0}, \mathbb{P}^x, \{\mathfrak{F}_t\}_{t\ge 0})$ on $\overline{G}$ which is associated with the Dirichlet form (1.3). For each $t \ge 0$ and $x \in \overline{G}$ the transition function $p_t(x,\cdot)$ of $\mathbf{X}$ is known to be absolutely continuous with respect to Lebesgue measure, and $\mathbf{X}$ is a strong Feller process.

Due to our regularity assumptions, the domain $D(A)$ of the generator A of the Dirichlet form $\mathcal{E}$ is given by

$$D(A) = \left\{ u \in H^2(G) : \ \frac{\partial}{\partial \nu}\, u = 0 \right\},$$

where $\partial/\partial\nu$ denotes the derivative in direction of the outer normal $\nu = (\nu_1, \dots, \nu_n)$ to the boundary ∂G. Sometimes we will also write

$$H^2_{\{\partial/\partial\nu\}}(G) = D(A)\,.$$

On $D(A)$ we have $A = L(x,D)$ which can be interpreted to hold in strong L^2-sense, but, of course, also in the sense of distributions. Let us observe for later applications that $(L(x,D), \partial/\partial\nu)$ forms a *regular elliptic boundary value problem* in the sense of S. Agmon, A. Douglis, and L. Nirenberg [1], see also [22] which will be our standard reference.

The general theory of Dirichlet forms shows that we can always associate a sub-Markovian semigroup $\{T_t\}_{t\geq 0}$ on $L^2(\overline{G})$ with $(\mathcal{E}, H^1(G))$. In our case, this semigroup enjoys the strong Feller property, it is conservative, *i.e.* $T_t 1 = 1$, and its transition kernels have densities with respect to Lebesgue measure on $\overline{G}$, *i.e.*, we have

$$T_t u(x) = \mathbb{E}^x\left(u(X_t)\right) = \int_{\overline{G}} p_t(x,y)\, u(y)\, dy\,.$$

We call $\{T_t\}_{t\geq 0}$ the *Neumann semigroup* associated with the Dirichlet form $(\mathcal{E}, H^1(\overline{G}))$.

One of the major aims in [9] was to obtain a *Skorokhod representation* of the process $\mathbf{X}$ under minimal smoothness conditions. Of course, this result remains valid in the situation considered here and reads as follows: Let X_t^k denote the k-th coordinate of X_t, $1 \leq k \leq n$. For $t \geq 0$ and $x \in \overline{G}$ one has almost surely $(\mathbb{P}^x)$

$$\begin{aligned} X_t^k - X_0^k = M_t^k &+ \sum_{\ell=1}^n \int_0^t \frac{\partial a_{k\ell}}{\partial x_\ell}\,(X_s)\, ds \\ &+ \sum_{\ell=1}^n \int_0^t a_{k\ell}(X_s)\, \nu_\ell(X_s)\, dL_s\,. \end{aligned} \tag{1.4}$$

Here, M_t^k, $1 \leq k \leq n$, are continuous additive functionals in the strict sense, see [8, p. 181, p. 326] for the definition, satisfying

$$\mathbb{E}^x(M_t^k) = 0 \qquad \text{and} \qquad \mathbb{E}^x(M_t^k M_t^\ell) = 2\,\mathbb{E}^x\Big(\int_0^t a_{k\ell}(X_s)\, ds\Big), \tag{1.5}$$

for $t \geq 0$ and $x \in \overline{G}$. The processes M_t^k are continuous martingales (under $\mathbb{P}^x$) with co-variation

$$(1.6) \qquad \langle M^k, M^\ell \rangle_t = 2 \int_0^t a_{k\ell}(X_s)\, ds\,, \qquad \text{almost surely } (\mathbb{P}^x)\,,$$

for all $x \in \overline{G}$. Moreover, L_t is a unique positive continuous additive functional in the strict sense with Revuz measure σ and supported by ∂G and one has

$$L_t = \int_0^t \mathbf{1}_{\partial G}(X_s)\, dL_s\ .$$

Let $G \subset \mathbb{R}^n$ and $L(x, D)$ be as above. We consider now the quadratic form $\mathcal{E}_D := \mathcal{E}$ on the domain $H_0^1(G)$,

$$H_0^1(G) := \overline{C_0^\infty(G)}^{\|\cdot\|_1}\,, \qquad \text{where } \|\cdot\|_1 = \|\cdot\|_{L^2} + \|\nabla\cdot\|_{L^2}\ .$$

Clearly, $(\mathcal{E}_D, H_0^1(G))$ is a regular Dirichlet form and its generator A_D has the domain

$$D(A_D) = \{u \in H^2(G):\ \gamma u = 0\}\,,$$

where $\gamma : H^1(G) \to H^{1/2}(\partial G)$ is the *trace operator*. As usual, γ is the continuous extension of the map $u \longmapsto u\big|_{\partial G}$ when $u \in C^\infty(\overline{G})$. Thus, $\gamma u = 0$ means that u attains 0 as boundary value. The space $H_0^1(G)$ can now be characterized by

$$H_0^1(G) = \{u \in H^1(G):\ \gamma u = 0\}\,.$$

(In Section 4 below, we will have a closer look at the trace operator.) The Markov process associated with the Dirichlet form $(\mathcal{E}_D, H_0^1(G))$ is known to be an *absorbing* (elliptic) diffusion process. Since G is bounded, the following Poincaré inequality holds

$$\int_G |u(x)|^2\, dx \leq c_0 \int_G |\nabla u(x)|^2\, dx\,, \qquad u \in H_0^1(G)\,.$$

By (1.2), (1.3) we get

$$\lambda_0^{-1} \int_G |\nabla u(x)|^2\, dx \leq \mathcal{E}(u, u)\,,$$

and therefore

$$(c_0\,\lambda_0)^{-1}\int_G |u(x)|^2\,dx \le \mathcal{E}(u,u)\,. \tag{1.7}$$

This, however, means that on $H_0^1(G)$ the form $\mathcal{E}(\cdot,\cdot)$ is a scalar product which is equivalent to the canonical one $(\cdot,\cdot)_1$.

Let us return to the Dirichlet form $(\mathcal{E}, H^1(G))$. We introduce the space

$$\mathcal{H}^1(G) := \left\{u \in H^1(G):\ \mathcal{E}(u,\phi) = 0 \text{ for all } \phi \in C_0^\infty(G)\right\},$$

or, equivalently,

$$\mathcal{H}^1(G) = \left\{u \in H^1(G):\ \mathcal{E}(u,v) = 0 \text{ for all } v \in H_0^1(G)\right\}. \tag{1.8}$$

Since $\mathcal{H}^1(G)$ is a closed subspace of $H^1(G)$ there is an orthogonal decomposition

$$H^1(G) = \mathcal{H}^1(G) \oplus_{\mathcal{E}} H_0^1(G)\,, \tag{1.9}$$

and it is clear that $\mathcal{H}^1(G)$ consists of all solutions of the equation $L(x,D)\,u = 0$ in G such that u and its first order partial derivatives belong to $L^2(G)$. In particular, the elements of $\mathcal{H}^1(G)$ are arbitrarily often differentiable on G.

If $a_{k\ell}(x) = \delta_{k\ell}$, (1.9) is exactly the Weyl decomposition. Let us mention a special case when $n = 2$ and $G = B_1(0)$ is the open unit disk with boundary $\partial G = S^1$. It is well known that one can construct a Dirichlet space $(\mathcal{C}, D(\mathcal{C}))$ on the boundary such that there is a one-to-one correspondence between $(\mathcal{C}, D(\mathcal{C}))$ and the classical Dirichlet space $(\mathcal{E}/2, \mathcal{H}^1(G))$. Here

$$\mathcal{E}(u,v) = \int_{B_1(0)} \nabla u(x)\cdot\nabla v(x)\,dx\,,$$

and the form $\mathcal{C}$ is explicitly given by the Douglas integral

$$\mathcal{C}(\phi,\psi) = \frac{1}{16\pi}\int_0^{2\pi}\int_0^{2\pi}\big(\phi(\theta)-\phi(\theta')\big)\big(\psi(\theta)-\psi(\theta')\big)\sin^{-2}\Big(\frac{\theta-\theta'}{2}\Big)\,d\theta\,d\theta'\,,$$

compare [8, pp. 12-13]. In Section 5 we will give a generalization of this result.

2. Subordination in the sense of Bochner.

Definition 2.1. *An arbitrarily often differentiable function* $f : (0,\infty) \longrightarrow \mathbb{R}$ *is called Bernstein function if* $f \geq 0$ *and* $(-1)^n f^{(n)} \leq 0$ *hold for all* $n \in \mathbb{N}$.

Bernstein functions can be fully characterized by a Lévy-Khinchin formula,

$$f(x) = a + b\,x + \int_0^\infty (1 - e^{-sx})\,\mu(ds)\,, \tag{2.1}$$

with $a, b \geq 0$ and a non-negative measure μ on $(0,\infty)$ such that

$$\int_0^\infty s\,(s+1)^{-1}\,\mu(ds) < \infty\,.$$

The representation (2.1) shows that f has an analytic continuation onto the complex half-plane $\operatorname{Re} z > 0$ and is continuous up to the boundary. These and many other properties can be found in the monograph [3] by C. Berg and G. Forst. We will need one more fact about Bernstein functions (*e.g.* [3, Theorem 9.8]).

Theorem 2.2. *Every convolution semigroup* $\{\eta_t\}_{t\geq 0}$ *of sub-probability measures on* $[0,\infty)$ *is uniquely characterized by some Bernstein function* f, *and vice versa. This correspondence is given by*

$$\int_0^\infty e^{-sx}\,\eta_t(ds) = e^{-tf(x)}\,.$$

Some of the most prominent Bernstein functions are the fractional powers,

$$x^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-sx})\,s^{-1-\alpha}\,ds\,, \qquad x \geq 0\,,\ 0 < \alpha < 1\,.$$

The corresponding convolution semigroup is the *one-sided stable semigroup* of order α.

Definition 2.3. *Let* $\{T_t\}_{t\geq 0}$ *be a sub-Markovian semigroup on* $L^2(X,m)$ *where* X *is a locally compact Hausdorff space and* m *is a Borel* *measure such that* $\operatorname{supp} m = X$. *Denote by* $\{\eta_t\}_{t\geq 0}$ *the convolution semigroup with Bernstein function* f. *The semigroup* $\{T_t^f\}_{t\geq 0}$ *defined on* $L^2(X,m)$ *by the Bochner integral*

$$T_t^f u = \int_0^\infty T_s u \, \eta_t(ds)$$

is called the subordinate semigroup of $\{T_t\}_{t\geq 0}$ *with respect to* $\{\eta_t\}_{t\geq 0}$ *or with respect to* f.

It is known that the subordinate semigroup is sub-Markovian and/ or Fellerian if the original semigroup is. A lot of results concerning the domain of the (subordinate) generator A^f of $\{T_t^f\}_{t\geq 0}$ and related functional calculi are known, see *e.g.* [11], [2], [16], [17]. In the next section we will use a characterization of $D(A^{\cdot^{\alpha}})$ as interpolation spaces.

Assume that $\{T_t\}_{t\geq 0}$ is a sub-Markovian semigroup with generator $(A, D(A))$ and corresponding Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. By subordination – as above f is a Bernstein function – we get the subordinate objects, $\{T_t^f\}_{t\geq 0}$, its generator $(A^f, D(A^f))$, and Dirichlet form $(\mathcal{E}^f, D(\mathcal{E}^f))$. Let us assume that f is a *complete* Bernstein function, which means that the representing measure μ in (2.1) is of the form

$$\mu(ds) = \int_0^\infty e^{-sr}\,\rho(dr)\,ds\,,$$

where ρ is a measure on $(0,\infty)$ such that

$$\int_0^\infty \frac{\rho(dr)}{r\,(1+r)} < \infty\,.$$

Note that fractional powers are complete Bernstein functions. (Sometimes complete Bernstein functions are also called *operator monotone functions*, see E. Heinz [10]). From [16, Theorem 5.3] it follows that

$$\|u\|_{L^2}^2 \leq c\,\mathcal{E}(u,u) \qquad \text{implies} \qquad \|u\|_{L^2}^2 \leq \frac{c}{f(1)}\,\mathcal{E}^f(u,u)\,, \tag{2.3}$$

for all $u \in D(\mathcal{E})$. The latter holds also on $D(\mathcal{E}^f)$, since we have the dense inclusions $D(A) \subset D(\mathcal{E})$ and $D(A) \subset D(\mathcal{E}^f)$.

Let us now discuss some probabilistic aspects of subordination. Denote again by f a Bernstein function, by $\{\eta_t\}_{t\geq 0}$ the associated convolution semigroup on $[0,\infty)$, and assume that $f(0)=0$, thus $\eta_0 = \delta_0$.

We may interpret $\{\eta_t\}_{t\geq 0}$ as transition probabilities of a stochastic process $\{Y_t\}_{t\geq 0}$ with stationary and independent increments and càdlàg trajectories. Since $\eta_0 = \delta_0$ and since the measures η_t are supported on $[0,\infty)$, we have almost surely $Y_0 = 0$ and almost surely increasing paths $t \longmapsto Y_t$. The converse assertion is also true: every such process defines (uniquely) a convolution semigroup of probability measures on $[0,\infty)$. We will call $\{Y_t\}_{t\geq 0}$ *subordinator*.

Let $\{X_t, \mathfrak{F}_t\}_{t\geq 0}$ be a Markov process with Polish state space $(E, \mathfrak{B})$ and $\{Y_t\}_{t\geq 0}$ be a subordinator which is stochastically independent of $\{X_t\}_{t\geq 0}$. Then

$$X_t^f(\omega) := X_{Y_t}(\omega) := X_{Y_t(\omega)}(\omega)\,, \qquad t \geq 0\,, \tag{2.4}$$

defines a new process $\{X_t^f\}_{t\geq 0}$ with filtration $\left\{\mathfrak{F}_{Y_t}\right\}_{t\geq 0}$. We say that $\{X_t^f\}_{t\geq 0}$ is obtained from $\{X_t\}_{t\geq 0}$ by subordination with respect to $\{Y_t\}_{t\geq 0}$ and call it *subordinate process* to $\{X_t\}_{t\geq 0}$.

Theorem 2.4. *Let $\{X_t\}_{t\geq 0}$ be a Markov process, $\{T_t\}_{t\geq 0}$ the associated operator semigroup, $\{Y_t\}_{t\geq 0}$ a subordinator (independent of $\{X_t\}_{t\geq 0}$), and f the corresponding Bernstein function. For all Borel sets $B \in \mathfrak{B}$, $x \in E$, and $t \geq 0$ we have*

$$\mathbb{P}^x(X_{Y_t} \in B) = T_t^f \mathbf{1}_B(x) = \mathbb{P}^x(X_t^f \in B)\,,$$

where $\{X_t^f\}_{t\geq 0}$ stands for the Markov process corresponding to the subordinate semigroup $\{T_t^f\}_{t\geq 0}$.

This result can be found in [5].

3. Subordinate Neumann and Dirichlet semigroups.

Let us return to the situation of Section 1 and consider the Dirichlet form $\mathcal{E}$ on $H^1(G) \subset L^2(\overline{G})$ and generator $A = L(x, D)$ with domain

$$D(A) = \left\{ u \in H^2(G) : \ \frac{\partial}{\partial \nu}\, u = 0 \right\}.$$

The semigroup associated with it, the Neumann semigroup, is denoted by $\{T_t\}_{t\geq 0}$.

For any Bernstein function f the subordinate semigroup $\{T_t^f\}_{t\geq 0}$ is again sub-Markovian. Thus, by the general theory of Dirichlet forms, there exists a corresponding Dirichlet form $\mathcal{E}^f$ with domain $D(\mathcal{E}^f)$ and generator $(A^f, D(A^f))$. As usual,

$$D(\mathcal{E}^f) = D((-A^f)^{1/2}),$$

and, if f is written in terms of its representation (2.1), the subordinate generator A^f is given by

$$A^f u = -a\,u + b\,A\,u + \int_0^\infty (T_s u - u)\,\mu(ds), \qquad u \in D(A).$$

This formula is due to R. Phillips [15] and refinements thereof are, *e.g.*, given in [11], [2], [17]. These results are, however, of an abstract nature. We want to determine $D(A^f)$ and $D(\mathcal{E}^f)$ in terms of function spaces. To do so, we will restrict ourselves to the case where $f(x) = x^\alpha$, $0 < \alpha < 1$, and write $\{T_t^{(\alpha)}\}_{t\geq 0}$, $\mathcal{E}^{(\alpha)}$, $A^{(\alpha)}$ instead of the clumsier $\{T_t^{\cdot^\alpha}\}_{t\geq 0}$ etc. In fact, we have to deal with fractional powers of the operator $-A$. Using complex interpolation, R. Seeley determined in [18] the domains of fractional powers of elliptic differential operators under regular boundary conditions.

For $G \in \mathbb{R}^n$, ∂G smooth, and $s > 0$ we define the space

$$H^s(G) = \left\{ u\big|_G : \ u \in H^s(\mathbb{R}^n) \right\}$$

normed by

$$\|u\|_{H^s(G)} = \inf\left\{ \|w\|_s : \ w\big|_G = u \text{ in } \mathcal{D}', \ w \in H^s(\mathbb{R}^n) \right\},$$

where $H^s(\mathbb{R}^n)$, $s \geq 0$, is the space

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \ \|u\|_s^2 = \int_{\mathbb{R}}^n (1 + |\xi|^2)^s\,|\widehat{u}(\xi)|^2\,d\xi < \infty \right\}.$$

For any $s \geq 0$, $H^s(G)$ is a Hilbert space and $C^\infty(\overline{G})$ is a dense subspace. Let us finally define for $s > 3/2$

$$H^s_{\{\partial/\partial\nu\}}(G) := \left\{ u \in H^s(G) : \ \frac{\partial}{\partial \nu}\,u = 0 \right\}. \tag{3.2}$$

Observe that $H^2_{\{\partial/\partial\nu\}}(G)$ coincides with $D(A)$. Denote by $[\cdot\,;\cdot]_\alpha$ complex interpolation between the spaces inside the brackets, see *e.g.* [22], [21]. It is well known that

$$D(A^{(\alpha)}) = [L^2(G); D(A)]_\alpha\,, \qquad 0 < \alpha < 1\,, \tag{3.3}$$

holds. The following precise characterization is due to R. Seeley [18, Theorem 4.1]:

Theorem 3.1. *Let $\{T_t^{(\alpha)}\}_{t\geq 0}$, $A^{(\alpha)}$, and $\mathcal{E}^{(\alpha)}$ be as above.*

A) *For $0 < \alpha < 1$ we have $D(\mathcal{E}^{(\alpha)}) = H^\alpha(G)$.*

B) *For $3/4 < \alpha < 1$ we have $D(A^{(\alpha)}) = H^{2\alpha}_{\{\partial/\partial\nu\}} = (G)$.*

C) *For $0 < \alpha < 3/4$ we have $D(A^{(\alpha)}) = H^{2\alpha}(G)$.*

There is a similar result for the Dirichlet form $(\mathcal{E}_D, H^1_0(G))$. Denote by $\{S_t\}_{t\geq 0}$ the sub-Markovian semigroup given by this Dirichlet form. As above, let $\{S_t^{(\alpha)}\}_{t\geq 0}$ be the subordinate semigroup with respect to fractional powers x^α, $0 < \alpha < 1$. We will need some facts on Sobolev spaces, see [22], [20], [21] as standard references. For any $s \geq 0$ let $H^s_0(G) := \overline{C_0^\infty(G)}^{\|\cdot\|_s}$. Then

$$H^s(G) = H^s_0(G)\,, \qquad \text{if } 0 \leq s < \frac{1}{2}\,. \tag{3.4}$$

If $s > 1/2$, we define

$$H^s_D(G) := \big\{u \in H^s(G):\ \gamma\, u = 0\big\} \tag{3.5}$$

and one has, *cf.* [22, p. 210],

$$H^s_D(G) = H^s_0(G)\,, \qquad \text{if } \frac{1}{2} < s \leq 1\,.$$

Here $\gamma : H^s(G) \longrightarrow H^{s-1/2}(\partial G)$ is again the trace operator, *cf.* Section 1. We can now state the analogue of Theorem 3.1 which is also due to R. Seeley [18, Theorem 4.1].

Theorem 3.2. *Let $\mathcal{E}_D^{(\alpha)}$ and $A_D^{(\alpha)}$ be the Dirichlet form and the generator associated with the sub-Markovian semigroup $\{S_t^{(\alpha)}\}_{t\geq 0}$.*

A) *For* $1/2 < \alpha < 1$ *we have* $D(\mathcal{E}_D^{(\alpha)}) = H_D^\alpha(G)$.

B) *For* $0 < \alpha < 1/2$, *we have* $D(\mathcal{E}_D^{(\alpha)}) = H^\alpha(G)$.

C) *For* $1/4 < \alpha < 1$ *we have* $D(A_D^{(\alpha)}) = H_D^{2\alpha}(G)$.

D) *For* $0 < \alpha < 1/4$ *we have* $D(A_D^{(\alpha)}) = H^{2\alpha}(G)$.

In view of (3.4)–(3.6) we may restate the above assertions in the form $D(\mathcal{E}_D^{(\alpha)}) = H_0^\alpha(G)$ for $0 < \alpha < 1$ but $\alpha \neq 1/2$, and $D(A_D^{(\alpha)}) = H_0^{2\alpha}(G)$ for $0 < \alpha < 1$ but $\alpha \neq 1/4$. The values $\alpha = 1/4, 1/2$ – and also the case $\alpha = 3/4$ of Theorem 3.1 – must be treated separately. We will not do this here.

Recall that on $H_0^1(G)$ the form $\mathcal{E}$ satisfies Poincaré's inequality (1.7). By (2.3) we see that

$$\|u\|_{L^2}^2 \leq c_0\, \lambda_0\, \mathcal{E}^{(\alpha)}(u,u) \tag{3.7}$$

holds for all $u \in H_0^\alpha(G)$. Thus, $\mathcal{E}^{(\alpha)}$ defines a scalar product that is equivalent to the canonical one $(\cdot,\cdot)_\alpha$ of $H_0^\alpha(G)$.

Suppose (just for the next few lines) that the coefficients of $L(x,D)$ are defined on the whole space $\mathbb{R}^n$ and that the fractional powers of *this* operator – *i.e.* acting on functions defined on $\mathbb{R}^n$ – are considered. One should note that, in this case, the Dirichlet problem for the fractional powers of $L(x,D)$ is different from the *subordinated Dirichlet problem* discussed above, see [12].

4. A Weyl decomposition of $(\mathcal{E}^{(\alpha)}, H^\alpha(G))$.

Let $\mathcal{E}$ be the Dirichlet form (1.3) with domain $H^1(G)$ and generator A under Neumann boundary conditions, *i.e.*, with domain $H^2_{\{\partial/\partial\nu\}}(G)$. For $0 < \alpha < 1$ denote by $\mathcal{E}^{(\alpha)}$ the Dirichlet form obtained by subordination with respect to the fractional powers $f_\alpha(x) = x^\alpha$; by Theorem 3.1 its form domain is the space $H^\alpha(G)$. The aim of this section is to show how one can get a Weyl-type decomposition of $H^\alpha(G)$ with respect to the Dirichlet form.

We put

$$\mathcal{E}_\lambda^{(\alpha)}(u,v) := \big((\lambda - A)^{\alpha/2}u, (\lambda - A)^{\alpha/2}v\big)_{L^2}\,, \qquad \lambda \geq 0\,,$$

and identify $\mathcal{E}_0^{(\alpha)}$ and $\mathcal{E}^{(\alpha)}$. Clearly, $(\mathcal{E}_\lambda^{(\alpha)}, H^\alpha(G))$ is again a Dirichlet form, and for $\lambda > 0$ the form $\mathcal{E}_\lambda^{(\alpha)}(\cdot,\cdot)$ is a scalar product, *cf.* (2.3), that is equivalent to the one on $H^\alpha(G)$. Moreover, the quadratic forms $\mathcal{E}_\lambda^{(\alpha)}(\cdot,\cdot)$ and $\big(A^{\alpha/2}\cdot, A^{\alpha/2}\cdot\big)_{L^2} + \lambda\big(\cdot,\cdot\big)_{L^2}$ are equivalent. On the space $H_0^\alpha(G)$ this remains true even for $\mathcal{E}_0^{(\alpha)}(\cdot,\cdot)$, *cf.* (3.7).

For $0 < \alpha \le 1$ and $\lambda \ge 0$ we call the functions in

$$\mathcal{H}_\lambda^\alpha(G) := \big\{u \in H^\alpha(G) :\ \mathcal{E}_\lambda^{(\alpha)}(u,v) = 0 \text{ for all } v \in H_0^\alpha(G)\big\} \tag{4.1}$$

$\mathcal{E}_\lambda^{(\alpha)}$*-harmonic functions.* Since $\overline{C_0^\infty(G)}^{\|\cdot\|_\alpha} = H_0^\alpha(G)$, one has also

$$\mathcal{H}_\lambda^\alpha(G) = \big\{u \in H^\alpha(G) :\ \mathcal{E}_\lambda^{(\alpha)}(u,\phi) = 0 \text{ for all } \phi \in C_0^\infty(G)\big\}\,.$$

We can now state the main result of this section.

Theorem 4.1. *Let* $(\mathcal{E}_\lambda^{(\alpha)}, H^\alpha(G))$ *be as above. For all* $0 < \alpha \le 1$ *and* $\lambda \ge 0$ *one has the orthogonal decomposition*

$$H^\alpha(G) = \mathcal{H}_\lambda^\alpha(G) \oplus_{\mathcal{E}_\lambda^{(\alpha)}} H_0^\alpha(G)\,. \tag{4.2}$$

If $\alpha > 1/2$*, this decomposition is non-trivial in the sense that* $\mathcal{H}_\lambda^\alpha(G) \supsetneqq \{0\}$ *and there is a canonical isomorphism*

$$\Pi_\lambda^{(\alpha)} : H^{\alpha-1/2}(\partial G) \longrightarrow \mathcal{H}_\lambda^\alpha(G)\,. \tag{4.3}$$

Proof. We will, first of all, consider the case $0 < \alpha < 1/2$. Then $H^\alpha(G) = H_0^\alpha(G)$, and the condition in (4.1)

$$\mathcal{E}_\lambda^{(\alpha)}(u,v) = 0\,, \qquad \text{for all } v \in H_0^\alpha(G)$$

implies that $u \equiv 0$. This means that we cannot expect any non-trivial decomposition of type (4.2) if $\alpha < 1/2$.

Assume now that $1/2 < \alpha \le 1$ – as already mentioned, the limiting case $\alpha = 1/2$ will not be considered here. Note, however, that $\alpha = 1$ does not play any special rôle and will always be included in the following considerations. Now $H_0^\alpha(G)$ and $\mathcal{H}_\lambda^\alpha(G)$ are closed subspaces of $H^\alpha(G)$, and since for all $u \in H^\alpha(G)$ the condition

$$\mathcal{E}_\lambda^{(\alpha)}(u,v) = 0\,, \qquad \text{for all } v \in H_0^\alpha(G)$$

implies that $u \equiv 0$, the decomposition (4.2) is orthogonal. (These considerations are still valid for $\alpha = 1/2$.)

In order to show that $\Pi_\lambda^{(\alpha)}$ is an isomorphism we have to recall some properties of the trace operator γ. Again, [22, in particular Section 4.7] will be our standard reference. For $1/2 < s < 3/2$ we define γ as above, *cf.* Section 1. Then $\gamma : H^s(G) \longrightarrow H^{s-1/2}(\partial G)$ is continuous and onto, and there exists a bounded linear operator $\tilde{\gamma} : H^{s-1/2}(\partial G) \longrightarrow H^s(G)$ such that $\gamma \circ \tilde{\gamma} = \mathrm{id}$ on $H^{s-1/2}(\partial G)$. The kernel of γ, *i.e.*, its null-space is just $H_0^s(G)$. Thus, for any $u \in H^s(G)$, $1/2 < s < 3/2$, the trace $\gamma u \in H^{s-1/2}(\partial G)$ exists and $\gamma u = 0$ implies that $u \in H_0^s(G)$. Conversely, for $\phi \in H^{s-1/2}(\partial G)$ there is a $u_\phi := \tilde{\gamma}\,\phi \in H^s(G)$ such that $\gamma\, u_\phi = \phi$. However, the mappings are not canonical, in the sense that $\gamma\, u = \gamma\, w$ does not imply $u = w$.

Our aim is to construct a continuous, bijective linear map from $\mathcal{H}_\lambda^\alpha(G)$ to $H^{\alpha-1/2}(\partial G)$, $\alpha > 1/2$. By the results of the preceding paragraph we find for every $\phi \in H^{\alpha-1/2}(\partial G)$ some $f \in H^\alpha(G)$ such that $\gamma f = \phi$. Define a linear functional $\Lambda_{\lambda,f}^\alpha$ on $H^\alpha(G)$ by

$$\Lambda_{\lambda,f}^\alpha(v) := \mathcal{E}_\lambda^{(\alpha)}(f, v)\,, \qquad v \in H^\alpha(G)\,.$$

By our assumptions, $\mathcal{E}_\lambda^{(\alpha)}(\cdot,\cdot)$ is for all $\lambda \geq 0$ a scalar product which is equivalent to $(\cdot,\cdot)_\alpha$ on $H_0^\alpha(G)$. An application of the Lax-Milgram theorem shows that there exists a unique element $\omega_{\lambda,f} \in H_0^\alpha(G)$ such that

$$\mathcal{E}_\lambda^{(\alpha)}(\omega_{\lambda,f}, v) = \Lambda_{\lambda,f}^\alpha(v)\,, \qquad v \in H_0^\alpha(G)\,,$$

holds. We define

$$u_{\lambda,f} := \omega_{\lambda,f} - f\,.$$

Claim 1. *$u_{\lambda,f}$ is contained in $\mathcal{H}_\lambda^\alpha(G)$.* Indeed, for any $v \in C_0^\infty(G)$ we get

$$\begin{aligned}
\mathcal{E}_\lambda^{(\alpha)}(u_{\lambda,f}, v) &= \mathcal{E}_\lambda^{(\alpha)}(\omega_{\lambda,f}, v) - \mathcal{E}_\lambda^{(\alpha)}(f, v) \\
&= \Lambda_{\lambda,f}^\alpha(v) - \mathcal{E}_\lambda^{(\alpha)}(f, v) \\
&= \mathcal{E}_\lambda^{(\alpha)}(f, v) - \mathcal{E}_\lambda^{(\alpha)}(f, v) \\
&= 0\,.
\end{aligned}$$

Claim 2. *$u_{\lambda,f}$ depends only on $\phi = \gamma f$ and the map $\phi \longmapsto u_{\lambda,\phi} := u_{\lambda,f}$ is linear.* Let $f_1, f_2 \in H^\alpha(G)$ such that $f_1 \neq f_2$ but $\gamma f_1 = \gamma f_2 = \phi$. Thus, $f_1 - f_2 \in H_0^\alpha(G)$ and each f_j has an orthogonal decomposition

$$f_j = u_{\lambda,f_j} + \omega_{\lambda,f_j}\ , \qquad j = 1, 2, \tag{4.4}$$

where $u_{\lambda,f_j} \in \mathcal{H}_\lambda^\alpha(G)$ and $\omega_{\lambda,f_j} \in H_0^\alpha(G)$. For every $v \in H_0^\alpha(G)$ we get

$$\begin{aligned}\mathcal{E}_\lambda^{(\alpha)}(f_1 - f_2, v) &= \mathcal{E}_\lambda^{(\alpha)}(u_{\lambda,f_1} - u_{\lambda,f_2}, v) + \mathcal{E}_\lambda^{(\alpha)}(\omega_{\lambda,f_1} - \omega_{\lambda,f_2}, v) \\ &= \mathcal{E}_\lambda^{(\alpha)}(\omega_{\lambda,f_1} - \omega_{\lambda,f_2}, v)\,.\end{aligned}$$

Since $f_1 - f_2 \in H_0^\alpha(G)$ and $\omega_{\lambda,f_1} - \omega_{\lambda,f_2} \in H_0^\alpha(G)$, we find $f_1 - f_2 = \omega_{\lambda,f_1} - \omega_{\lambda,f_2}$, hence $u_{\lambda,f_1} = u_{\lambda,f_2}$. The linearity of $\phi \longrightarrow u_{\lambda,\phi}$ is obvious.

We have seen so far, that

$$\Pi_\lambda^{(\alpha)} : H^{\alpha-1/2}(\partial G) \longrightarrow \mathcal{H}_\lambda^\alpha(G)\,, \qquad \phi \longmapsto u_{\lambda,\phi}\ ,$$

is a well-defined linear operator.

Claim 3. *The mapping $\Pi_\lambda^{(\alpha)}$ is bijective.* Suppose that $\Pi_\lambda^{(\alpha)}(\phi) = 0$ for some $\phi \in H^{\alpha-1/2}(\partial G)$. But $0 = \Pi_\lambda^{(\alpha)}(\phi) \in H_0^\alpha(G)$, thus $\phi = 0$, *i.e.* $\Pi_\lambda^{(\alpha)}$ is injective.

In order to see surjectivity, choose any $u \in \mathcal{H}_\lambda^\alpha(G) \subset H^\alpha(G)$ and observe that there is a $\phi \in H^{\alpha-1/2}(\partial G)$ such that $\gamma\, u = \phi$. We can thus define $u_{\lambda,\phi} := \Pi_\lambda^{(\alpha)}(\phi)$. Since $\gamma\, u_{\lambda,\phi} = \phi$, we find $u_{\lambda,\phi} - u \in H_0^\alpha(G) \cap \mathcal{H}_\lambda^\alpha(G)$, therefore $u_{\lambda,\phi} = u$. This is but to say that $\Pi_\lambda^{(\alpha)}$ is onto.

Claim 4. *The mapping $\Pi_\lambda^{(\alpha)} : H^{\alpha-1/2}(\partial G) \longrightarrow \mathcal{H}_\lambda^\alpha(G)$ is continuous* (the Hilbert spaces are equipped with their canonical, respectively, induced canonical scalar products). Since the Hilbert space $H^\alpha(G)$ is the orthogonal sum of two closed subspaces, the projections

$$\pi_1 : H^\alpha(G) \longrightarrow H_0^\alpha(G) \qquad \text{and} \qquad \pi_2 : H^\alpha(G) \longrightarrow \mathcal{H}_\lambda^\alpha(G)$$

are orthogonal projections, hence continuous. By definition, $\tilde\gamma$ is also continuous, and so is the composition $\Pi_\lambda^{(\alpha)} = \pi_2 \circ \tilde\gamma$.

5. An analogue of the Douglas integral.

As in the preceding sections, G denotes a bounded domain with boundary ∂G which shall be a C^∞-manifold. Let us have a closer look at the spaces $H^{\alpha-1/2}(\partial G)$, $1/2 < \alpha \leq 1$. Following J. Wloka [25, Chapter 4.2] we can define on $H^{\alpha-1/2}(\partial G)$ an equivalent norm in the following way: Choose a finite cover $\{U_j\}_{j=1}^M$, $U_j \subset \partial G$, of ∂G by coordinate patches, and denote by $\{\rho_j\}_{j=1}^M$ a partition of unity relative to this covering. For any $\phi \in H^{\alpha-1/2}(\partial G)$ we put $\phi_j := \rho_j \phi$. Then $\|\cdot\|_{H^{\alpha-1/2}(\partial G)}$ is equivalent to the norm $|||\cdot|||_{H^{\alpha-1/2}(\partial G)}$ which is given by

$$|||\phi|||^2_{H^{\alpha-1/2}(\partial G)} := \sum_{j=1}^M |||\phi_j|||^2_{H^{\alpha-1/2}(\partial G)} \,. \tag{5.1}$$

Here,

$$\begin{aligned}|||\phi_j|||^2_{H^{\alpha-1/2}(\partial G)} :=& \int_{\partial G} |\phi_j(x)|^2 \,\sigma(dx) \\ &+ \int_{\partial G}\int_{\partial G} \frac{|\phi_j(x)-\phi_j(y)|^2}{|x-y|^{n-2+2\alpha}} \,\sigma(dx)\,\sigma(dy)\,,\end{aligned} \tag{5.2}$$

where σ is the surface measure on ∂G. Let us denote by $\big(\mathcal{S}^{(\alpha)}, D(\mathcal{S}^{(\alpha)}\big) = H^{\alpha-1/2}(\partial G)$ the quadratic form

$$\begin{aligned}&\mathcal{S}^{(\alpha)}(\phi,\psi) \\ &:= \sum_{j=1}^M \int_{\partial G} \phi_j(x)\,\psi_j(x)\,\sigma(dx) \\ &\quad + \sum_{j=1}^M \int_{\partial G}\int_{\partial G} \frac{(\phi_j(x)-\phi_j(y))\,(\psi_j(x)-\psi_j(y))}{|x-y|^{n-2+2\alpha}}\,\sigma(dx)\,\sigma(dy)\,.\end{aligned} \tag{5.3}$$

It is obvious from (5.3) that $(\mathcal{S}^{(\alpha)}, D(\mathcal{S}^{(\alpha)}))$ is a regular Dirichlet form on $L^2(\partial G)$. In particular, the unit contraction operator $N_{\partial G}(\phi) := (0 \vee \phi) \wedge 1$, leaves the form domain $D(\mathcal{S}^{(\alpha)}) = H^{\alpha-1/2}(\partial G)$ invariant and operates continuously thereon, *i.e.*,

$$\mathcal{S}^{(\alpha)}(N_{\partial G}(\phi), N_{\partial G}(\phi)) \leq \mathcal{S}^{(\alpha)}(\phi,\phi)\,, \qquad \phi \in D(\mathcal{S}^{(\alpha)})\,.$$

Denote by N_G, $N_G(u) := (0 \vee u) \wedge 1$, the unit contraction defined for functions $u : \overline{G} \longrightarrow \mathbb{R}$. Since $H^\alpha(G)$ is a Dirichlet space with respect to its canonical scalar product, we find as above that $N_G : H^\alpha(G) \longrightarrow H^\alpha(G)$ is continuous and operates on any Dirichlet form with domain $H^\alpha(G)$.

Lemma 5.1. *Let γ be the trace operator and $N_{\partial G}$, N_G unit contractions on ∂G and G. For $u \in H^\alpha(G)$ we have*

$$\gamma(N_G(u)) = N_{\partial G}(\gamma\, u)\,. \tag{5.4}$$

PROOF. For $h \in C(\overline{G}) \cap H^\alpha(G)$ the assertion (5.4) is straightforward. Since γ, $N_{\partial G}$, and N_G are continuous operators, so are their compositions $\gamma \circ N_G : H^\alpha(G) \longrightarrow H^{\alpha-1/2}(\partial G)$ and $N_{\partial G} \circ \gamma : H^\alpha(G) \longrightarrow H^{\alpha-1/2}(\partial G)$, and (5.4) follows from the density of $C(\overline{G}) \cap H^\alpha(G)$ in $H^\alpha(G)$.

Let $\mathcal{E}_\lambda^{(\alpha)}(\cdot,\cdot)$, $\Pi_\lambda^{(\alpha)}$, and $\mathcal{H}_\lambda^\alpha(G)$ be as in the preceding section. Then

$$\mathcal{C}_\lambda^{(\alpha)}(\phi,\psi) := \mathcal{E}_\lambda^{(\alpha)}\big(\Pi_\lambda^{(\alpha)}(\phi), \Pi_\lambda^{(\alpha)}(\psi)\big)\,, \qquad \phi,\psi \in H^{\alpha-1/2}(\partial G)\,,$$

defines on $H^{\alpha-1/2}(\partial G)$ a bilinear form. We know already that $\Pi_\lambda^{(\alpha)} : H^{\alpha-1/2}(\partial G) \longrightarrow \mathcal{H}_\lambda^\alpha(G)$ is a linear, continuous, and bijective operator. Since $\big(\mathcal{H}_\lambda^\alpha(G), (\cdot,\cdot)_\alpha\big)$ is a closed subspace of $H^\alpha(G)$, it is itself a Hilbert space and there exist constants $c_1, c_2 \geq 0$ such that

$$c_1\, \|\phi\|_{H^{\alpha-1/2}(\partial G)} \leq \|\Pi_\lambda^{(\alpha)}(\phi)\|_{H^\alpha(G)} \leq c_2\, \|\phi\|_{H^{\alpha-1/2}(\partial G)} \tag{5.5}$$

holds. Hence, $\mathcal{C}_\lambda^{(\alpha)}$ is a closed form on $H^{\alpha-1/2}(\partial G)$.

Theorem 5.2. *The bilinear form $(\mathcal{C}_\lambda^{(\alpha)}, H^{\alpha-1/2}(\partial G))$ is a Dirichlet form.*

PROOF. $\mathcal{C}_\lambda^{(\alpha)}$ being a closed form, it remains to prove the contraction property for the unit contraction $N_{\partial G}$,

$$\mathcal{C}_\lambda^{(\alpha)}(N_{\partial G}(\phi), N_{\partial G}(\phi)) \leq \mathcal{C}_\lambda^{(\alpha)}(\phi,\phi)\,, \qquad \phi \in H^{\alpha-1/2}(\partial G)\,.$$

In order to see this, we show first that

$$N_G(\Pi_\lambda^{(\alpha)}(\phi)) = \Pi_\lambda^{(\alpha)}(N_{\partial G}(\phi)) + g_\phi\,, \qquad \phi \in H^{\alpha-1/2}(\partial G)\,, \tag{5.6}$$

where $\Pi_\lambda^{(\alpha)}(N_{\partial G}(\phi)) \in \mathcal{H}_\lambda^\alpha(G)$ and $g_\phi \in H_0^\alpha(G)$. Since the decomposition (5.6) is necessarily unique, it is sufficient to prove that the traces satisfy

$$\gamma(N_G(\Pi_\lambda^{(\alpha)}(\phi))) = \gamma(\Pi_\lambda^{(\alpha)}(N_{\partial G}(\phi)))\,,$$

that is, since $\gamma \circ \Pi_\lambda^{(\alpha)} = \mathrm{id}$ on $H^{\alpha-1/2}(\partial G)$,

$$\gamma(N_G(\Pi_\lambda^{(\alpha)}(\phi))) = N_{\partial G}(\phi)\,.$$

This, however, is just the assertion of Lemma 5.1. Using (5.6) we find

$$\begin{aligned}
\mathcal{C}_\lambda^{(\alpha)}(\phi,\phi) &= \mathcal{E}_\lambda^{(\alpha)}\big(\Pi_\lambda^{(\alpha)}(\phi), \Pi_\lambda^{(\alpha)}(\phi)\big) \\
&\geq \mathcal{E}_\lambda^{(\alpha)}\big(N_G(\Pi_\lambda^{(\alpha)}(\phi)), N_G(\Pi_\lambda^{(\alpha)}(\phi))\big) \\
&= \mathcal{E}_\lambda^{(\alpha)}\big(\Pi_\lambda^{(\alpha)}(N_{\partial G}(\phi)), \Pi_\lambda^{(\alpha)}(N_{\partial G}(\phi))\big) \\
&\quad + 2\,\mathcal{E}_\lambda^{(\alpha)}\big(\Pi_\lambda^{(\alpha)}(N_{\partial G}(\phi)), g_\phi\big) + \mathcal{E}_\lambda^{(\alpha)}\big(g_\phi, g_\phi\big) \\
&\geq \mathcal{C}_\lambda^{(\alpha)}\big(N_{\partial G}(\phi), N_{\partial G}(\phi)\big)\,,
\end{aligned}$$

and we are done.

Let us return to the Dirichlet form $(\mathcal{S}^{(\alpha)}, H^{\alpha-1/2}(\partial G))$. Since $\mathcal{S}^{(\alpha)}$ is a closed form on $H^{\alpha-1/2}(\partial G)$, $\mathcal{S}_\mu^{(\alpha)}(\cdot,\cdot) := \mathcal{S}^{(\alpha)}(\cdot,\cdot) + \mu(\cdot,\cdot)_{L^2}$ is for any $\mu > 0$ a scalar product which is equivalent to $(\cdot,\cdot)_{H^{\alpha-1/2}(\partial G)}$. Similarly, $\mathcal{C}_{\lambda,\mu}^{(\alpha)}(\cdot,\cdot) := \mathcal{C}_\lambda^{(\alpha)}(\cdot,\cdot) + \mu(\cdot,\cdot)_{L^2}$ is also a scalar product which is equivalent to $(\cdot,\cdot)_{H^{\alpha-1/2}(\partial G)}$, thus $\mathcal{S}_\mu^{(\alpha)}$ and $\mathcal{C}_{\lambda,\mu}^{(\alpha)}$ are equivalent to each other. Since both are Dirichlet forms, we can associate with each of them a Hunt process with state space ∂G. One may expect that the comparability of the forms carries over to the processes. Let us briefly explain this point for L^1-L^∞-estimates of the semigroups

$$\Big\{T_t^{\mathcal{S}_\mu^{(\alpha)}}\Big\}_{t\geq 0} \qquad \text{and} \qquad \Big\{T_t^{\mathcal{C}_{\lambda,\mu}^{(\alpha)}}\Big\}_{t\geq 0}\,.$$

It is known that on the spaces $H^{\alpha-1/2}(\partial G)$ a Sobolev inequality holds, that is

$$\|u\|_{L^p(\partial G)} \leq c\,\|u\|_{H^{\alpha-1/2}(\partial G)}\,, \qquad p = \frac{2\,(n-1)}{n-\alpha-\dfrac{1}{2}}\,. \tag{5.7}$$

Note that $p > 2$ if $1/2 < \alpha \leq 1$. By (5.3) we get for $\mu > 0$

$$\|u\|^2_{L^p(\partial G)} \leq c\,\mathcal{S}^{(\alpha)}_{\mu}(u,u) \qquad \text{and} \qquad \|u\|^2_{L^p(\partial G)} \leq c'\,\mathcal{C}^{(\alpha)}_{\lambda,\mu}(u,u)\,.$$

This implies, *cf.* Varopoulos et al. [23], that both semigroups satisfy the estimates

$$\left\| T_t^{\mathcal{S}^{(\alpha)}_{\mu}} \right\|_{L^1-L^\infty} \leq c'_\mu \frac{e^{\mu t}}{t^{(2(n-1))/(2\alpha-1)}}$$

(5.8) and

$$\left\| T_t^{\mathcal{C}^{(\alpha)}_{\lambda,\mu}} \right\|_{L^1-L^\infty} \leq c''_\mu \frac{e^{\mu t}}{t^{(2(n-1))/(2\alpha-1)}}\,.$$

In this section we have constructed the boundary Dirichlet form associated with the subordinate process $\{X^{(\alpha)}_t\}_{t\geq 0}$ and, likewise, with the Dirichlet form $(\mathcal{E}^{(\alpha)}, H^\alpha(G))$. In the case of a Brownian motion, this was first done by M. Fukushima [6], and in a rather general (but abstract) way for general regular symmetric Dirichlet forms by M. Silverstein [19]. Here, as in the whole paper, we provide explicit constructions which allow us to determine precisely the domains in terms of function spaces. This yields additional information for studying the Dirichlet forms and/or the corresponding (boundary) process.

6. The process associated with $\mathcal{C}^{(\alpha)}_\lambda$.

We will now study the stochastic process which is generated by the Dirichlet form $C^{(\alpha)}_\lambda$ on the boundary ∂G. We will closely follow the ideas of [8], in particular Chapter 6.2. Notice, however, that the process $\{X_{\lambda,t}\}_{t\geq 0}$ generated by $L(x,D) - \lambda$ under Neumann boundary conditions is a *nice* Feller process with *smooth densities.* We may, therefore, do without the exceptional sets which frequently occur within the framework of Dirichlet forms – for a discussion of this point in the general theory we refer to M. Fukushima's paper [7]. In order to avoid technical complications we will always assume $\lambda > 0$. We conclude from this, that the extended Dirichlet space and the original one $(\mathcal{E}^{(\alpha)}_\lambda, H^\alpha(G))$ coincide.

Let us begin with $\alpha = 1$, *i.e.*, the reflected diffusion process

$$\{X_{\lambda,t}\}_{t\geq 0}$$

(with filtration $\{\mathfrak{F}_{\lambda,t}\}_{t\geq 0}$) associated with the Dirichlet form

$$(\mathcal{E}_\lambda, H^1(G)),$$

where $\lambda > 0$ and $\mathcal{E}_\lambda(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + \lambda(\cdot,\cdot)_{L^2}$ with $\mathcal{E}$ as in (1.3). If ∂G is smooth, the surface measure σ is a *smooth* measure in the sense of [8, p. 80], because by (5.15) – take $\alpha = 1$ – and the finiteness of σ the surface measure σ is even a measure of *finite energy integral* [8, p. 74], hence smooth. Thus, there is a unique positive continuous additive functional $\{L_{\lambda,t}\}_{t\geq 0}$ such that σ is its Revuz measure, see [8, pp. 187-188]. One can check that $\{L_{\lambda,t}\}_{t\geq 0}$ is the boundary local time, *i.e.*,

$$L_{\lambda,t} = \int_0^t \mathbf{1}_{\partial G}(X_{\lambda,s})\, dL_{\lambda,s} \tag{6.1}$$

holds, and that the support of $\{L_{\lambda,t}\}_{t\geq 0}$ equals ∂G. Write $\{\tau_{\lambda,t}\}_{t\geq 0}$ for the generalized right-inverse of $\{L_{\lambda,t}\}_{t\geq 0}$,

$$\tau_{\lambda,t}(\omega) := \inf\big\{s > 0 :\ L_{\lambda,s}(\omega) > t\big\}. \tag{6.2}$$

Clearly, $\{\tau_{\lambda,t}\}_{t\geq 0}$ is a subordinator. We may now apply [8, Theorem 6.2.1].

Theorem 6.1. *Let $L(x, D)$ be as before and denote by $\{X_{\lambda,t}, \mathfrak{F}_{\lambda,t}\}_{t\geq 0}$ the Feller process corresponding to the Dirichlet form $(\mathcal{E}_\lambda, H^1(G))$. The time-changed process $\{X_{\lambda,\tau_{\lambda,t}}, \mathfrak{F}_{\lambda,\tau_{\lambda,t}}\}_{t\geq 0}$ is given by the Dirichlet form $(\mathcal{C}_\lambda, H^{1/2}(\partial G))$, $\lambda > 0$.*

This theorem implies, in particular, that the boundary process $\{X_{\lambda,\tau_{\lambda,t}}\}_{t\geq 0}$ is comparable (on the level of Dirichlet forms) with the process on ∂G being associated with the form

$$\big((\cdot,\cdot)_{H^{1/2}(\partial G)} + \lambda(\cdot,\cdot)_{L^2}, H^{1/2}(\partial G)\big).$$

The latter, however, should be thought of as a perturbation of a Cauchy process on the boundary.

Let us now discuss the subordinate processes, *i.e.*, the processes associated with $(\mathcal{E}_\lambda^{(\alpha)}, H^\alpha(G))$ and $(\mathcal{C}_\lambda^{(\alpha)}, H^{\alpha-1/2}(\partial G))$, $\alpha > 1/2$ and $\lambda > 0$. Denote by $\{Y_t^{(\alpha)}\}_{t\geq 0}$ a one-sided α-stable subordinator with Bernstein function $f_\alpha(x) = x^\alpha$, $0 < \alpha \leq 1$. As for (2.4) we may choose a version of $\{Y_t^{(\alpha)}\}_{t\geq 0}$ that is independent of $\{X_{\lambda,t}\}_{t\geq 0}$. Then

$$X_{\lambda,t}^{(\alpha)}(\omega) := X_{\lambda,Y_t^{(\alpha)}}(\omega) := X_{\lambda,Y_t^{(\alpha)}(\omega)}(\omega), \qquad t \geq 0$$

is the *subordinate* (*reflected diffusion*) *process* (in the sense of Section 2) given by $(\mathcal{E}_\lambda^{(\alpha)}, H^\alpha(G))$. Its filtration is $\{\mathfrak{F}_{\lambda, Y_t^{(\alpha)}}\}_{t\geq 0}$.

Due to a result of St. Orey, [14, p. 123], ∂G will be a zero-capacity set if and only if $\alpha < 1/2$. Therefore, the assumption $\alpha \geq 1/2$ is necessary in order to obtain a smooth boundary measure σ and, thus, a positive continuous additive functional $\{L_{\lambda,t}^{(\alpha)}\}_{t\geq 0}$ with Revuz measure σ. As above, the finiteness of σ and (5.5) prove that for $\alpha > 1/2$ the measure σ is indeed smooth. Again, $L_{\lambda,t}^{(\alpha)}$ can be identified with the boundary local time for $X_{\lambda,t}^{(\alpha)}$ – *i.e.*, (6.1) holds with some obvious changes – with support in ∂G, and $\{\tau_{\lambda,t}^{(\alpha)}\}_{t\geq 0}$ will be its generalized right-inverse.

Theorem 6.2. *Let* $\{X_{\lambda,t}^{(\alpha)}, \mathfrak{F}_{\lambda, Y_t^{(\alpha)}}\}_{t\geq 0}$ *be the subordinate reflected diffusion process corresponding to the Dirichlet form* $(\mathcal{E}_\lambda^{(\alpha)}, H^\alpha(G))$, $\alpha > 1/2$. *The time-changed process* $\{X_{\lambda,\tau_{\lambda,t}^{(\alpha)}}^{(\alpha)}, \mathfrak{F}_{\lambda, Y_\bullet^{(\alpha)}\circ\tau_{\lambda,t}^{(\alpha)}}\}_{t\geq 0}$ *is given by the Dirichlet form* $(\mathcal{C}_\lambda^{(\alpha)}, H^{\alpha-1/2}(\partial G))$, $\lambda > 0$.

Starting with $(\mathcal{E}_\lambda, H^1(G))$ and $\{X_{\lambda,t}\}_{t\geq 0}$ we have, so far, constructed three new Dirichlet forms and stochastic processes.

- The associated boundary Dirichlet form/process

$$(\mathcal{C}_\lambda, H^{1/2}(\partial G)) \qquad \text{and} \qquad X_{\lambda,\tau_{\lambda,t}}, \qquad t \geq 0,$$

where $\tau_{\lambda,t}$ is the generalized inverse of the boundary local time L_t of the original process.

- The subordinate Dirichlet form/process

$$(\mathcal{E}_\lambda^{(\alpha)}, H^\alpha(G)) \qquad \text{and} \qquad X_{\lambda,t}^{(\alpha)} := X_{\lambda, Y_t^{(\alpha)}}, \qquad t \geq 0,$$

where $\alpha > 1/2$ and $Y_t^{(\alpha)}$ is a one-sided α-stable subordinator.

- The boundary Dirichlet form/process associated with the subordinate form/process

$$(\mathcal{C}_\lambda^{(\alpha)}, H^{\alpha-1/2}(\partial G)) \qquad \text{and} \qquad X_{\lambda,\tau_{\lambda,t}^{(\alpha)}}^{(\alpha)} := X_{\lambda, Y_\bullet^{(\alpha)}\circ\tau_{\lambda,t}^{(\alpha)}}, \qquad t \geq 0,$$

where $\alpha > 1/2$ and $\tau_{\lambda,t}^{(\alpha)}$ is the generalized inverse of the boundary local time $L_{\lambda,t}^{(\alpha)}$ of $X_{\lambda,t}^{(\alpha)}$.

It is natural to ask, whether the boundary process $X^{(\alpha)}_{\lambda,\tau^{(\alpha)}_{\lambda,t}}$ of the subordinate process can be directly obtained as subordinate process to $X_{\lambda,\tau_{\lambda,t}}$, *i.e.*, to the boundary process of the original process. A partial answer to this question is given below.

Theorem 6.3. *Let $L_{\lambda,t}, \tau_{\lambda,t}, L^{(\alpha)}_{\lambda,t}, \tau^{(\alpha)}_{\lambda,t}$, and $Y^{(\alpha)}_t$ be as above. Denote by $\{X_{\lambda,\tau_{\lambda,t}}\}_{t\geq 0}$ and $\{X^{(\alpha)}_{\lambda,\tau^{(\alpha)}_{\lambda,t}}\}_{t\geq 0}$ the boundary processes induced by the Dirichlet forms $(\mathcal{C}_\lambda, H^{1/2}(\partial G))$ and $(\mathcal{C}^{(\alpha)}_\lambda, H^{\alpha-1/2}(\partial G))$, $\alpha > 1/2$. Then*

$$\rho_{\lambda,t} := L_{\lambda,\bullet} \circ Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}\,, \qquad t \geq 0 \tag{6.4}$$

defines a time-change for the process $\{X_{\lambda,\tau_{\lambda,t}}\}_{t\geq 0}$, and we have

$$\tau_{\lambda,\bullet} \circ \rho_{\lambda,t} = Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t} \qquad \text{and} \qquad X^{(\alpha)}_{\lambda,\tau^{(\alpha)}_{\lambda,t}} = X_{\lambda,\tau_{\lambda,\bullet}\circ\rho_{\lambda,t}}\,, \tag{6.5}$$

i.e., the boundary process of the subordinate process can be represented as time-changed boundary process of the original process.

Proof. Clearly, the process $\rho_{\lambda,t}$ is an almost surely positive, increasing càdlàg process such that $\rho_{\lambda,0} = 0$ almost surely. Note that $Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}$ is an $\mathfrak{F}_{\lambda,t}$-stopping time. Once (6.5) is established, we see from

$$\{\rho_{\lambda,t} < s\} = \{\tau_{\lambda,\bullet} \circ \rho_{\lambda,t} < \tau_{\lambda,s}\} = \{Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t} < \tau_{\lambda,s}\} \in \mathfrak{F}_{\lambda,\tau_{\lambda,s}}\,,$$

where $s > 0$, that $\{\rho_{\lambda,t}\}_{t\geq 0}$ is a family of $\mathfrak{F}_{\lambda,\tau_{\lambda,t}}$-stopping times, hence a time-change.

It is therefore enough to prove (6.5). Since $\tau_{\lambda,t}$ is a right-inverse, we have always $L_{\lambda,\bullet} \circ \tau_{\lambda,t} = t$, but $\tau_{\lambda,\bullet} \circ L_{\lambda,t} = t$ holds only at increase times t of $L_{\lambda,t}$. In order to check that $Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}$ is almost surely an increase time of $L_{\lambda,t}$ we have to prove that

$$X_{\lambda, Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}} \in \operatorname{supp}\{L_{\lambda,t}\} := \{x \in \overline{G} :\ \mathbb{P}^x(\tau_{\lambda,0} = 0) = 1\}\,.$$

For any $\tilde{\omega} \in \Omega$ and $t > 0$

$$\mathbb{P}^{X_{\lambda, Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}}(\tilde{\omega})}\big(\tau^{(\alpha)}_{\lambda,0} = 0\big)$$

$$
\begin{aligned}
&= \mathbb{P}\big(\theta^{-1}_{Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}} \{\tau^{(\alpha)}_{\lambda,0} = 0\} \,|\, \mathfrak{F}_{\lambda, Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}+}\big)\\
&= \mathbb{P}\big(\inf\{s>0:\ L^{(\alpha)}_{\lambda, s+\tau^{(\alpha)}_{\lambda,t}} > L^{(\alpha)}_{\lambda, \tau^{(\alpha)}_{\lambda,t}}\} = 0) \,|\, \mathfrak{F}_{\lambda, Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}+}\big)\\
&= \mathbb{P}\big(\inf\{s>0:\ L^{(\alpha)}_{\lambda, s+\tau^{(\alpha)}_{\lambda,t}} > t\} = 0 \,|\, \mathfrak{F}_{\lambda, Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}+}\big)\\
&= 1
\end{aligned}
$$

since $\tau^{(\alpha)}_{\lambda,t}$ is by its definition the right endpoint of every interval of constancy of $L^{(\alpha)}_{\lambda,t}$. We have thus seen that up to an exceptional (*i.e.* capacity zero) set, say N_1,

$$
X_{\lambda, Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}} \in \operatorname{supp}\{L^{(\alpha)}_{\lambda,t}\}\ .
$$

Since $\operatorname{supp}\{L^{(\alpha)}_{\lambda,t}\}$ is a quasi-support of the Revuz measure σ, *cf.* [8, Theorem 5.1.5], we have $\operatorname{supp}\{L^{(\alpha)}_{\lambda,t}\} = \partial G$ up to another exceptional set, $N^{(\alpha)}_2$, say. Thus,

$$
X_{\lambda, Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}} \in \operatorname{supp}\{L_{\lambda,t}\} \cup N_1 \cup N^{(\alpha)}_2 \cup N^{(1)}_2\ .
$$

The set $N = N_1 \cup N^{(\alpha)}_2 \cup N^{(1)}_2$ is again exceptional and, under our smoothness assumptions, even polar with respect to $\{X^{(\alpha)}_{\lambda,t}\}_{t\geq 0}$, see [8, Theorem 4.1.2]. Therefore we have for all $t>0$ and $x \in \overline{G}$

$$
\mathbb{P}^x\big(X^{(\alpha)}_{\lambda, \tau^{(\alpha)}_{\lambda,t}} \in N\big) = \mathbb{E}^x\big(\mathbf{1}_N(X^{(\alpha)}_{\lambda, \tau^{(\alpha)}_{\lambda,t}})\big) \leq \mathbb{E}^x\big(\sup_{s>0} \mathbf{1}_N(X^{(\alpha)}_{\lambda,s})\big) = 0\,.
$$

Consequently, $X_{\lambda, Y^{(\alpha)}_\bullet \circ \tau^{(\alpha)}_{\lambda,t}} \in \operatorname{supp}\{L_{\lambda,t}\}$ holds almost surely $(\mathbb{P}^x)$ for every x, and (6.5) follows.

In general, it seems to be wrong that the *boundary process of a subordinate process* is some *subordinate to the boundary process of the original process*, since, in general, $\rho_{\lambda,t}$ is neither a Lévy process nor an independent process.

7. The subordinate reflected diffusion process.

Let $\{X_t\}_{t\geq 0}$ be the reflected diffusion considered in Section 1 above. Recall that the corresponding Dirichlet space is $(\mathcal{E}, H^1(G))$ where $H^1(G) \subset L^2(\overline{G})$, and that one has the Skorokhod representation (1.4),

$$X_t^k - X_0^k = M_t^k + \sum_{\ell=1}^n \int_0^t \frac{\partial a_{k\ell}}{\partial x_\ell}\,(X_s)\,ds + \sum_{\ell=1}^n \int_0^t a_{k\ell}(X_s)\,\nu_\ell(X_s)\,dL_s\,. \tag{7.1}$$

As in the preceding section, let $\{Y_t^{(\alpha)}\}_{t\geq 0}$ denote an α-stable subordinator and $X_t^{(\alpha)} = X_{0,t}^{(\alpha)}$ the subordinate reflected diffusion.

Theorem 7.1. *Let $\{X_t^{(\alpha)}\}_{t\geq 0}$ be the process that is obtained from the reflected diffusion $\{X_t\}_{t\geq 0}$ through subordination with respect to a one-sided stable subordinator $\{Y_t^{(\alpha)}\}_{t\geq 0}$ of order $\alpha \in (0,1]$. Then the following Skorokhod representation holds*

$$\begin{aligned}\big(X_t^{(\alpha)}\big)^k - \big(X_0^{(\alpha)}\big)^k &= (N_t^{(\alpha)})^k + \sum_{\ell=1}^n \sum_{r\leq t} \int_0^1 \frac{\partial a_{k\ell}}{\partial x_\ell}\,\big(X_{Y_{r-}^{(\alpha)}+s\Delta Y_r^{(\alpha)}}\big)\,d_s(Y_{r-}^{(\alpha)} + s\,\Delta Y_r^{(\alpha)}) \\ &\quad + \sum_{\ell=1}^n \sum_{r\leq t} \int_0^1 a_{k\ell}\big(X_{Y_{r-}^{(\alpha)}+s\Delta Y_r^{(\alpha)}}\big)\nu_\ell\big(X_{Y_{r-}^{(\alpha)}+s\Delta Y_r^{(\alpha)}}\big)\,d_sL_{Y_{r-}^{(\alpha)}+s\Delta Y_r^{(\alpha)}}\,,\end{aligned} \tag{7.2}$$

where $N_t^{(\alpha)} = M_{Y_t^{(\alpha)}}$ is a pure jump martingale (with respect to the time-changed filtration), M_t is the continuous martingale part of the Skorokhod representation of $\{X_t\}_{t\geq 0}$, and L_t is the boundary local time of the diffusion $\{X_t\}_{t\geq 0}$.

Proof. In order to keep notation to a minimum, we will sometimes omit the superscripts $^{(\alpha)}$. A change of time in (7.1) with respect to the subordinator $\{Y_t\}_{t\geq 0}$ yields

$$\begin{aligned}&\big(X_t^{(\alpha)}\big)^k - \big(X_0^{(\alpha)}\big)^k \\ &\qquad = M_{Y_t}^k + \sum_{\ell=1}^n \int_0^{Y_t} \frac{\partial a_{k\ell}}{\partial x_\ell}\,(X_s)\,ds + \sum_{\ell=1}^n \int_0^{Y_t} a_{k\ell}(X_s)\,\nu_\ell(X_s)\,dL_s\,,\end{aligned}$$

where M_{Y_t}, $t \geq 0$, is just the subordinate to the continuous martingale M_t in the Skorokhod representation of $\{X_t\}_{t\geq 0}$, see (1.5), (1.6). It is obvious that M_{Y_t} is again a martingale (with respect to the time-changed natural filtration of $\{X_t\}_{t\geq 0}$) and that it is of pure jump type (since the subordinator is of this type).

In order to study the integral expressions in the above formula, we need a change-of-variable formula for Stieltjes integrals. Recall that $\{X_t\}_{t\geq 0}$ is a continuous process and that $t \longmapsto L_t(\omega)$ is a continuous, almost surely increasing function. The main difficulty is that Y_t may have almost surely *countably* many jumps in finite time. By a well-known approximation technique for Lévy processes – *cf.* L. Breiman [4, Theorem 14.27 and Proposition 8.36] – we can approximate Y_t by processes Y_t^ε whose paths are almost surely step functions with finitely many jumps in finite time,

$$\lim_{\varepsilon\to 0} Y_t^\varepsilon(\omega) = Y_t(\omega), \qquad \text{almost surely } (\mathbb{P}^0).$$

(Y_t^ε can be chosen to be the subordinator with

$$c_\alpha \int_\varepsilon^\infty (1 - e^{-x\xi})\, x^{-1-\alpha}\, dx \longrightarrow \xi^\alpha, \qquad \varepsilon \longrightarrow 0,$$

as characteristic exponent.) Therefore,

$$\lim_{\varepsilon\to 0} \int_0^{Y_t^\varepsilon} \frac{\partial a_{k\ell}}{\partial x_\ell}(X_s)\, ds = \int_0^{Y_t} \frac{\partial a_{k\ell}}{\partial x_\ell}(X_s)\, ds, \tag{7.3}$$

almost surely $(\mathbb{P}^x)$ and

$$\lim_{\varepsilon\to 0} \int_0^{Y_t^\varepsilon} a_{k\ell}(X_s)\, \nu_\ell(X_s)\, dL_s = \int_0^{Y_t} a_{k\ell}(X_s)\, \nu_\ell(X_s)\, dL_s, \tag{7.4}$$

almost surely $(\mathbb{P}^x)$.

Assume that $s \longmapsto A_s(\omega)$ is a function which is for almost all ω continuous and increasing – this includes, in particular, the functions $s \longmapsto s$ and $s \longmapsto L_s(\omega)$ of (7.3), (7.4). We consider the pathwise defined Stieltjes integral

$$\int_0^{Y_t^\varepsilon(\omega)} u(\omega, s)\, dA_s(\omega),$$

for those ω where $s \longmapsto A_s(\omega)$ is continuous and increasing; $u(\omega,\cdot)$ is any continuous function. For fixed ω and t we may assume (if necessary, we remove another negligible ω-set) that the function $s \longmapsto Y_s^\varepsilon$ has only finitely many jumps σ_ℓ^ε, $\ell = 1, 2, \ldots, k^\varepsilon(\omega)$, on $[0,t]$. Thus,

$$\int_0^{Y_t^\epsilon(\omega)} u(\omega,s)\, dA_s(\omega) = \sum_{\ell=1}^{k^\varepsilon(\omega)} \int_{Y^\epsilon_{\sigma_\ell^\epsilon -}(\omega)}^{Y^\epsilon_{\sigma_\ell^\epsilon}(\omega)} u(\omega,s)\, dA_s(\omega)\,,$$

where $Y^\epsilon_{\sigma_\ell^\epsilon -}(\omega) = \lim_{r\uparrow\sigma_\ell^\varepsilon(\omega)} Y_r^\epsilon(\omega)$ denotes the left limit. In an appendix we will prove the following technical Lemma.

Lemma 7.2. *Denote by $\Delta Y_r^\epsilon = Y_r^\epsilon - Y_{r-}^\epsilon$. Then*

$$\begin{aligned}\int_{Y^\epsilon_{\sigma_\ell^\epsilon -}(\omega)}^{Y^\epsilon_{\sigma_\ell^\epsilon}(\omega)} & u(\omega,s)\, dA_s(\omega) \\ &= \int_0^1 u(\omega, Y^\epsilon_{\sigma_\ell^\epsilon -} + s\,\Delta Y^\epsilon_{\sigma_\ell^\epsilon})\, d_s A_{Y^\epsilon_{\sigma_\ell^\epsilon -} + s\Delta Y^\epsilon_{\sigma_\ell^\epsilon}}(\omega)\,.\end{aligned} \tag{7.5}$$

An application of Lemma 7.2 shows

$$\begin{aligned}\int_0^{Y(\omega)} & u(\omega,s)\, dA_s(\omega) \\ &= \lim_{\varepsilon\to 0} \sum_{\ell=1}^{k^\varepsilon(\omega)} \int_0^1 u(\omega, Y^\epsilon_{\sigma_\ell^\epsilon -} + s\,\Delta Y^\epsilon_{\sigma_\ell^\epsilon})\, d_s A_{Y^\epsilon_{\sigma_\ell^\epsilon -} + s\Delta Y^\epsilon_{\sigma_\ell^\epsilon}}(\omega)\,.\end{aligned}$$

Since $\{Y_t^\varepsilon\}_{t\ge 0}$ is a pure jump process, we obtain

$$\begin{aligned}\sum_{\ell=1}^{k^\varepsilon(\omega)} & \int_0^1 u(\omega, Y^\epsilon_{\sigma_\ell^\epsilon -} + s\,\Delta Y^\epsilon_{\sigma_\ell^\epsilon})\, d_s A_{Y^\epsilon_{\sigma_\ell^\epsilon -} + s\Delta Y^\epsilon_{\sigma_\ell^\epsilon}}(\omega) \\ &= \sum_{r\le t} \int_0^1 u(\omega, Y^\epsilon_{r-} + s\,\Delta Y^\epsilon_r)\, d_s A_{Y^\epsilon_{r-} + s\Delta Y^\epsilon_r}(\omega)\end{aligned}$$

and the proof of Theorem 7.1 is finished by the following lemma.

Lemma 7.3.

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{r\le t}\int_0^1 u(\omega, Y^{\epsilon}_{r-} + s\,\Delta Y^{\epsilon}_r)\,d_s A_{Y^{\epsilon}_{r-}+s\Delta Y^{\epsilon}_r}(\omega)\\ = \sum_{r\le t}\int_0^1 u(\omega, Y_{r-} + s\,\Delta Y_r)\,d_s A_{Y_{r-}+s\Delta Y_r}(\omega)\,.\end{aligned}\tag{7.6}$$

PROOF OF LEMMA 7.3. We fix $r\in(0,t]$ and set

$$v(s,\varepsilon) := u(\omega, Y^{\epsilon}_{r-} + s\,\Delta Y^{\epsilon}_r)\,,$$

$$a^{\varepsilon}_s := A_{Y^{\epsilon}_{r-}+s\Delta Y^{\epsilon}_r}\,,$$

$$v(s) := u(\omega, Y_{r-} + s\,\Delta Y_r)\,,$$

and

$$a_s := A_{Y_{r-}+s\Delta Y_r}\,.$$

Then

$$\begin{aligned}&\Big|\int_0^1 v(s,\varepsilon)\,da^{\varepsilon}_s - \int_0^1 v(s)\,da_s\Big|\\ &\le \Big|\int_0^1 v(s,\varepsilon)\,da^{\varepsilon}_s - \int_0^1 v(s,\varepsilon)\,da_s\Big| + \Big|\int_0^1 v(s,\varepsilon)\,da_s - \int_0^1 v(s)\,da_s\Big|\\ &\le \sup_{\xi\le Y_t(\omega)}|u(\omega,\xi)|\int_0^1 d(a_s - a^{\varepsilon}_s) + \int_0^1 |v(s,\varepsilon)-v(s)|\,da_s\,,\end{aligned}$$

where we have used the fact that $\Delta Y^{\varepsilon}_s(\omega)\le\Delta Y_s(\omega)$, hence $a^{\varepsilon}_s\le a_s$. Since $s\longmapsto u(\omega,s)$ is continuous, the second integral tends to 0 as $\varepsilon\longrightarrow 0$. The first integral tends also to 0 as $\varepsilon\longrightarrow 0$, because a^{ε}_s increases to the continuous function a_s, hence, by Dini's theorem, this convergence is uniform. Therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^1 u(\omega, Y^{\epsilon}_{r-} + s\,\Delta Y^{\epsilon}_r)\,d_s A_{Y^{\epsilon}_{r-}+s\Delta Y^{\epsilon}_r}(\omega)\\ = \int_0^1 u(\omega, Y_{r-} + s\,\Delta Y_r)\,d_s A_{Y_{r-}+s\Delta Y_r}(\omega)\,.\end{aligned}$$

Since

$$\int_0^1 u(\omega, Y^\epsilon_{r-} + s\,\Delta Y^\epsilon_r)\, d_s A_{Y^\epsilon_{r-}+s\Delta Y^\epsilon_r}(\omega) \\ \leq \sup_{\xi \leq Y_t(\omega)} |u(\omega,\xi)| \int_0^1 d_s A_{Y_{r-}+s\Delta Y_r}(\omega)\,,$$

and since

$$\sum_{r\leq t} \int_0^1 d_s A_{Y_{r-}+s\Delta Y_r}(\omega) = \sum_{r\leq t}(A_{Y_r} - A_{Y_{r-}}) < \infty\,,$$

we may invoke Lebesgue's dominated convergence theorem which enables us to interchange the limit $\varepsilon \longrightarrow 0$ and the summation on $r \leq t$ on the left hand side of (7.6). This finally shows Lemma 7.3 and also Theorem 7.1.

8. Concluding remarks.

Many of our results do extend in an obvious way to subordination with respect to the larger class of *complete* Bernstein functions (*cf.* [16] for a definition) containing the fractional powers $f_\alpha(x) = x^\alpha$ which were considered throughout our paper. This greater generality has to be paid for by the fact that it is not possible to obtain exact characterizations of domains etc. in terms of function spaces. If, however, a (complete) Bernstein function f is comparable from above or below or from both sides with some fractional power f_α or f_β, that is, if for some $\alpha, \beta \in (0,1]$ and large x

$$f(x) \leq C f_\alpha(x)\,,$$

$$c f_\beta(x) \leq f(x)\,,$$

or

$$c f_\beta(x) \leq f(x) \leq C f_\alpha(x)\,,$$

are satisfied, one can use some comparison result from [17] in order to identify for suitable values of α and β the domains $D(A^f)$ or $D(\mathcal{E}^f)$ etc. with subspaces of $H^\beta(G)$ (or $H_0^\beta(G)$) or to prove that they contain the space $H^\alpha(G)$ or $H_0^\alpha(G)$.

Furthermore, this allows us to give some rough characterization of the Dirichlet form for the corresponding boundary process – provided it exists, *i.e.*, $\beta > 1/2$.

A. Appendix.

We will give here the proof of Lemma 7.2. To keep notation to a minimum we will write $Y(t), A(t), \sigma_\ell, \dots$ instead of $Y_t^\varepsilon, A_t, \sigma_\ell^\varepsilon, \dots$

It is clearly enough to check (7.5) for fixed ω and for (deterministic) indicator functions $u(s,\omega) = \mathbf{1}_{(a,b]}(s)$

$$\begin{aligned}\text{(A.1)}\quad \int_{Y(\sigma_\ell-)}^{Y(\sigma_\ell)} \mathbf{1}_{(a,b]}(s)\,dA(s) &= \int_0^\infty \mathbf{1}_{(a,b]}(s)\,\mathbf{1}_{(Y(\sigma_\ell-),Y(\sigma_\ell)]}\,dA(s)\\ &= \big(A(b\wedge Y(\sigma_\ell)) - A(a\vee Y(\sigma_\ell-))\big)\vee 0\,.\end{aligned}$$

On the other hand, we have (with the convention that $(a,b] = \varnothing$ if $a \ge b$)

$$\begin{aligned}&\int_0^1 \mathbf{1}_{(a,b]}\,(Y(\sigma_\ell-) + s\,\Delta Y(\sigma_\ell))\,d_sA(Y(\sigma_\ell-) + s\,\Delta Y(\sigma_\ell))\\
&= \int_0^1 \mathbf{1}_{((a-Y(\sigma_\ell-))/(Y(\sigma_\ell)-Y(\sigma_\ell-)),(b-Y(\sigma_\ell-))/(Y(\sigma_\ell)-Y(\sigma_\ell-))]}(s)\\
&\qquad\cdot d_sA(Y(\sigma_\ell-) + s\,\Delta Y(\sigma_\ell))\\
&= \int_{-\infty}^\infty \mathbf{1}_{((a-Y(\sigma_\ell-))/(Y(\sigma_\ell)-Y(\sigma_\ell-))\vee 0,(b-Y(\sigma_\ell-))/(Y(\sigma_\ell)-Y(\sigma_\ell-))\wedge 1]}(s)\\
\text{(A.2)}\qquad&\qquad\cdot d_sA(Y(\sigma_\ell-) + s\,\Delta Y(\sigma_\ell))\\
&= A\Big(Y(\sigma_\ell-) + \Delta Y(\sigma_\ell)\,\frac{b-Y(\sigma_\ell-)}{Y(\sigma_\ell)-Y(\sigma_\ell-)}\wedge 1\Big)\\
&\quad - A\Big(Y(\sigma_\ell-) + \big(a - Y(\sigma_\ell-)\big)\vee 0\Big)\,.\end{aligned}$$

It remains to check (A.1) = (A.2) for all admissible permutations of $(a, b, Y(\sigma_\ell-), Y(\sigma_\ell))$. These are

1) $a \le Y(\sigma_\ell-) \le Y(\sigma_\ell) \le b\,,$ 2) $a \le Y(\sigma_\ell-) \le b \le Y(\sigma_\ell)\,,$

3) $a \le b \le Y(\sigma_\ell-) \le Y(\sigma_\ell)\,,$ 4) $Y(\sigma_\ell-) \le a \le b \le Y(\sigma_\ell)\,,$

5) $Y(\sigma_\ell-) \le a \le Y(\sigma_\ell) \le b\,,$ 6) $Y(\sigma_\ell-) \le Y(\sigma_\ell) \le a \le b\,,$

and this is an elementary – but somewhat tedious – exercise.

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References.

[1] Agmon, S., Douglis, A., Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, II. *Comm. Pure Appl. Math.* **12** (1959), 623-727, **17** (1964), 35-92.

[2] Berg, C., Boyadzhiev, Kh., deLaubenfels, R., Generation of generators of holomorphic semigroups. *J. Austral. Math. Soc. Ser. A* **55** (1993), 246-269.

[3] Berg, C., Forst, G., *Potential Theory on Locally Compact Abelian Groups.* Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete, II. Ser. Bd. **87**, 1975.

[4] Breiman, L., *Probability.* SIAM, Classics in Applied Mathematics. Vol. **7**, Philadelphia, 1992.

[5] Feller, W., *An Introduction to Probability Theory and its Applications.* Vol. 2. Wiley, Series in Probab. and Math. Stat., 1971.

[6] Fukushima, M., On boundary conditions for multidimensional Brownian motions with symmetric resolvent densities. *J. Math. Soc. Japan* **21** (1969), 58-93.

[7] Fukushima, M., On a decompositon of additive functionals in the strict sense for a symmetric Markov process. In: Ma, Z. M., Röckner, M., Yan, J. A., (Eds.). *Dirichlet Forms and Stochastic Processes.* Proc. Intnl. Conf. Dirichlet Forms and Stochastic Processes, Beijing 1993. de Gruyter, (1995), 155-169.

[8] Fukushima, M., Oshima, Y., Takeda, M., *Dirichlet Forms and Symmetric Markov Processes.* de Gruyter, Studies in Math. Vol. **19**, 1994.

[9] Fukushima, M., Tomisaki, M., Reflecting diffusions on Lipschitz domains with cusps - Analytic construction and Skorohod representation. *Potential Analysis* **4** (1995), 377-408.

[10] Heinz, E., Beiträge zur Störungstheorie der Spektralzerlegung. *Math. Ann.* **123** (1951), 415-438.

[11] Hirsch, F., Domaines d'opérateurs représentés comme intégrales de résolvantes. *J. Funct. Anal.* **23** (1976), 239-264.

[12] Hoh, W., Jacob, N., On the Dirichlet problem for pseudodifferential operators generating Feller semigroups. *J. Funct. Anal.* **137** (1996), 19-48.

[13] Lions, J. L., Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications.* Vol. I. Springer, Grundlehren Math. Wiss. Bd. **181**, 1972.

[14] Orey, St., Polar sets for processes with stationary independent increments. In: Chover, J., (Ed.). *Markov Processes and Potential Theory.* Proceedings of a Symposion held at the Univ. of Wisconsin, Madison, 1967. Wiley, 1970, 117-126.

[15] Phillips, R. S., On the generation of semigroups of linear operators. *Pacific. J. Math.* **2** (1952), 343-369.

[16] Schilling, R. L., On the domain of the generator of a subordinate semigroup. In: Král, J., et al. (Eds.). *Potential Theory* – ICPT 94. Proceedings Intnl. Conf. Potential Theory, Kouty (CR), 1994. de Gruyter, 1996, 449-462.

[17] Schilling, R. L., Subordination in the sense of Bochner and a related functional calculus. *J. Austral. Math. Soc. Ser. A* **64** (1998), 368-396.

[18] Seeley, R., Interpolation in L^p with boundary conditions. *Studia Math.* **44** (1972), 47-60.

[19] Silverstein, M., *Boundary theory for symmetric Markov processes.* Springer, Lecture Notes Math. **516**, 1976.

[20] Triebel, H., *Theory of Function Spaces.* Birkhäuser, Monographs in Mathematics **78**, 1983.

[21] Triebel, H., *Theory of Function Spaces II.* Birkhäuser, Monographs in Mathematics **84**, 1992.

[22] Triebel, H., *Interpolation Theory, Function Spaces, Differential Operators.* Joh. Ambrosius Barth, 1995.

[23] Varopoulos, N., Saloff-Coste, L., Coulhon, T., *Analysis and Geometry on Groups.* Cambridge University Press, Cambridge Tracts in Math. **100**, 1992.

[24] Watanabe, S., On stable processes with boundary conditions. *J. Math. Soc. Japan* **14** (1962), 170-198.

[25] Wloka, J., *Partielle Differentialgleichungen.* Teubner, Math. Leitfäden, 1982.

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The angular distribution of mass by Bergman functions

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Abstract. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane and denote by $d\mathcal{A}$ two-dimensional Lebesgue measure on $\mathbb{D}$. For $\varepsilon > 0$ we define $\Sigma_\varepsilon = \{z : |\arg z| < \varepsilon\}$. We prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if f is analytic, univalent and area-integrable on $\mathbb{D}$, and $f(0) = 0$, then

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f| \, d\mathcal{A} > \delta \int_{\mathbb{D}} |f| \, d\mathcal{A} \, .$$

This problem arose in connection with a characterization by Hamilton, Reich and Strebel of extremal dilatations for quasiconformal homeomorphisms of $\mathbb{D}$.

1. Introduction.

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane and denote by $d\mathcal{A}$ two-dimensional Lebesgue measure on $\mathbb{D}$. The Bergman space L^1_a consists of functions that are analytic on $\mathbb{D}$ and integrable with respect to $d\mathcal{A}$. It is a Banach space with norm

$$\|f\|_1 = \int_{\mathbb{D}} |f| \, d\mathcal{A} \, .$$

Each $f \in L^1_a$ induces a Borel measure μ_f on the plane defined by

$$\mu_f(E) = \int_{f^{-1}E} |f| \, d\mathcal{A} \, .$$

The problem considered in this paper concerns the angular distribution of mass by such a measure. For $\varepsilon > 0$ we define

$$\Sigma_\varepsilon = \{z : \ |\arg z| < \varepsilon\}.$$

Theorem 1.1. *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $f \in L^1_a$ is univalent and $f(0) = 0$, then*

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f|\, d\mathcal{A} > \delta \, \|f\|_1 \,. \tag{1.1}$$

Since (1.1) will then hold for all rotations of Σ_ε, Theorem 1.1 says that the measure μ_f cannot be too asymmetric. This theorem can not be extended to L^p_a for any $p > 1$; Example 4.3 at the end of the paper shows that (1.1) fails when $p > 1$, $|f|$ is replaced by $|f|^p$, and $\|f\|_1$ is replaced by $\|f\|_p^p$.

As is explained below, it is known that there exist positive constants C and η such that

$$C \int_{f^{-1}(\Sigma_{\pi/2-\eta})} |f|\, d\mathcal{A} \geq \|f\|_1 \,, \qquad \text{for all } f \in L^1_a \text{ with } f(0) = 0\,, \tag{1.2}$$

and it is an open problem to prove (1.1) without the restriction that f be univalent. This is equivalent to a conjecture regarding quasiconformal mappings made by M. Ortel and the second author in [OS]. We now briefly review the relevant parts of this theory, and indicate the consequences that a solution to the open problem would have.

A bounded area-measurable function κ on $\mathbb{D}$ with $\|\kappa\|_\infty < 1$ is said to be a dilatation. It is a theorem in Ahlfors [A1] that to any dilatation κ there is associated a unique quasiconformal homeomorphism f^κ of $\mathbb{D}$ that fixes the points 1, i, and -1, and satisfies $\overline{\partial} f^\kappa = \kappa\, \partial f^\kappa$. We say that κ is an *extremal dilation* if κ is a dilatation and $\|\kappa\|_\infty \leq \|\kappa_1\|_\infty$ whenever $f^\kappa(e^{i\theta}) = f^{\kappa_1}(e^{i\theta})$, $-\pi < \theta \leq \pi$. The following characterization of extremal dilatations is due to R. Hamilton, S. L. Krushkal, E. Reich and K. Strebel.

Theorem 1.2 ([Ha], [K], [RS])**.** *Suppose κ is a dilatation. Then κ is an extremal dilatation if and only if one of the following statements holds*:

1) *There exist $f \in L^1_a$ and $k \in [0, 1)$ such that $\kappa(z) = k\, \overline{f(z)}/|f(z)|$ almost everywhere $d\mathcal{A}(z)$.*

2) *There is a sequence* $\{f_n\} \subset L^1_a$, *converging to* 0 *uniformly on compact subsets of* $\mathbb{D}$, *such that* $\|f_n\|_1 = 1$ *and*

$$\lim_{n\to\infty} \int_{\mathbb{D}} f_n \, \kappa \, d\mathcal{A} = \|\kappa\|_\infty \, .$$

Checking whether condition 2) holds for a particular dilatation can be difficult, and for this reason a more explicit characterization of extremal dilatations would be valuable. In this regard, we note it is not difficult to construct extremal dilatations that assume only countably many values and satisfy condition 2). Such a construction, based on an example in [OS], appears below. M. Ortel and the second author investigated the arguments of an extremal dilatation, and proved the following theorem.

Theorem 1.3 ([OS]). *Suppose* κ *is a bounded measurable function on* $\mathbb{D}$, $\pi/2 < \theta < \pi/2 + \arctan(1/2\, C_0)$, *and* $\kappa(z) \in \Sigma_\theta \cup \{0\}$ *for almost all* $z \in \mathbb{D}$. *Then* κ *is an extremal dilatation if and only if there exist* $k \in [0,1)$ *and* $f \in L^1_a$ *such that* $\kappa(z) = k\,\overline{f(z)}/|f(z)|$ *for almost all* $z \in \mathbb{D}$.

Here C_0 is the infimum of those constants C such that

$$\int_{\mathbb{D}} |f| \, d\mathcal{A} \le C \int_{\mathbb{D}} |\mathrm{Re}\, f| \, d\mathcal{A} \, ,$$

for all $f \in L^1_a$ satisfying $\mathrm{Im}\, f(0) = 0$. Subsequently, X. Huang [Hu] showed that this theorem remains valid when the number $\pi/2 + \arctan(1/2\, C_0)$ is replaced by the larger number $\pi/2 + \arcsin(1/(2\, C_0 - 1))$. It was conjectured in [OS] that in the theorem, the number $\pi/2 + \arctan(1/2\, C_0)$ can in fact be replaced by π. In other words, if κ is an extremal dilatation not of the form $k\,\overline{f}/|f|$, with $f \in L^1_a$, then the arguments of κ were conjectured to be dense in the unit circle.

This conjecture is equivalent to the extension of Theorem 1.1 from univalent functions to all functions in L^1_a. To see this, first suppose that (1.1) holds for all $f \in L^1_a$ and that κ is a dilatation satisfying

$$\kappa(z) \in \mathbb{C} \setminus \Sigma_{2\varepsilon} \, ,$$

for almost all $z \in \mathbb{D}$. Let f be such that $\|f\|_1 = 1$ and $f(0) = 0$. If $f(z) \in \Sigma_\varepsilon$, then $f(z)\,\kappa(z) \in \mathbb{C} \setminus \Sigma_\varepsilon$. Thus

$$\mathrm{Re} \int_{f^{-1}(\Sigma_\varepsilon)} f \, \kappa \, d\mathcal{A} \le \cos(\varepsilon) \, \|\kappa\|_\infty \int_{f^{-1}(\Sigma_\varepsilon)} |f| \, d\mathcal{A} \, ,$$

and so

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{D}} f\,\kappa\,d\mathcal{A} &\le \|\kappa\|_\infty + (\cos(\varepsilon)-1)\,\|\kappa\|_\infty \int_{f^{-1}(\Sigma_\varepsilon)} |f|\,d\mathcal{A} \\ &< \|\kappa\|_\infty + (\cos(\varepsilon)-1)\,\|\kappa\|_\infty\,\delta \\ &< \|\kappa\|_\infty\,,\end{aligned}$$

where $\delta > 0$ comes from (1.1). Thus, condition (2) of Theorem 1.2 can not hold and κ is not extremal unless it is of the form $k\,\overline{f}/|f|$, where $f \in L^1_a$. Hence if the arguments of κ are not dense in the circle, then condition 2) of Theorem 1.2 fails. For the converse, suppose there is a sequence $\{f_n\} \subset L^1_a$ with $\|f_n\|_1 = 1$, $f_n(0) = 0$, and such that

$$\lim_{n\to\infty} \int_{f_n^{-1}(\Sigma_\varepsilon)} |f_n|\,d\mathcal{A} = 0\,.$$

It is easy to check that if f is a normal limit of $\{f_n\}$, then

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f|\,d\mathcal{A} = 0\,.$$

Since also $f(0) = 0$, it follows that f is identically 0, and so $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Then, by approximating $\overline{f_n}/|f_n|$ on an appropriate sequence of annuli in $\mathbb{D}$ while omitting values in Σ_ε, it is possible to construct a dilatation κ that satisfies condition 2) of Theorem 1.2 and which assumes no values in Σ_ε.

Thus we have shown that the conjecture from [OS] of the density in the unit circle of the arguments of an extremal dilatation κ not of the form $k\,\overline{f}/|f|$, with $f \in L^1_a$, is equivalent to the conjecture that the conclusion of Theorem 1.1 is valid for all $f \in L^1_a$ with $f(0) = 0$. We also note that the argument sketched above, together with the theorem quoted from [OS], shows that there exist positive constants C and η such that (1.2) holds.

In the next section we collect facts and background material on the hyperbolic metric and harmonic measure that will be used to prove Theorem 1.1 in Section 3. Finally, some examples have been included in the last section. These examples illustrate how some of the difficulties encountered were addressed in the proof, and that Theorem 1.1 can not be extended to L^p_a for any $p > 1$.

2. Background.

The main tools we will use in the proof of Theorem 1.1 are the hyperbolic distance and harmonic measure. The *hyperbolic distance* on $\mathbb{D}$ is defined by (see [A2, p. 2])

$$\rho_{\mathbb{D}}(z_1, z_2) = \inf\left\{ \int_\gamma \frac{2\,|dz|}{1-|z|^2} : \ \gamma \text{ is an arc in } \mathbb{D} \text{ from } z_1 \text{ to } z_2 \right\}.$$

For example, the shortest distance from 0 to any other point is along a radius, and

$$\rho_{\mathbb{D}}(0, |z|) = \log\left(\frac{1+|z|}{1-|z|}\right).$$

This distance is invariant under conformal self-maps of $\mathbb{D}$ and thus the hyperbolic geodesics are diameters of the disk together with circles orthogonal to the unit circle. This distance also transfers to a natural conformally invariant distance on any simply connected proper subset $G \subset \mathbb{C}$. If $\varphi : \mathbb{D} \longrightarrow G$ is any conformal map, the *hyperbolic distance* on G is given by $\rho_G(w_1, w_2) = \rho_{\mathbb{D}}(z_1, z_2)$, where $w_i = \varphi(z_i)$ for $i = 1, 2$. The shortest arc in $\mathbb{D}$ from z_1 to z_2 is the arc of the unique circle orthogonal to $\partial\,\mathbb{D}$ passing through z_1 and z_2. The shortest arc in G from w_1 to w_2 is the image of this arc in $\mathbb{D}$ by the map φ. If $E \subset G$, then the hyperbolic distance from z_1 to E will be denoted by $\rho_G(z_1, E)$.

The *harmonic measure* of a set E contained in the closure of a region Ω evaluated at $z \in \Omega$ is denoted by $\omega(z, E, \Omega)$. It is (roughly) the function which is harmonic on $\Omega \setminus E$, equal to 1 on E and equal 0 on $\partial\Omega \setminus E$. See [GM] for a precise definition.

2.1. Area Estimates.

We can use both the hyperbolic distance and harmonic measure to estimate the Euclidean area $\mathcal{A}(E)$ of a measurable set $E \subset \mathbb{D}$:

$$\mathcal{A}(E) \le C e^{-\rho_{\mathbb{D}}(0,E)} \omega(0, E, \mathbb{D})\,, \tag{2.1}$$

for some universal constant $C < \infty$. To see this, let $E^* = \{z/|z| : z \in E\}$ denote the radial projection of E onto $\partial\,\mathbb{D}$. Then E is contained in the set

$$\left\{ z \in \mathbb{D} : \ \rho_{\mathbb{D}}(0, z) \ge \rho_{\mathbb{D}}(0, E) \text{ and } \frac{z}{|z|} \in E^* \right\},$$

which has area at most $2\,e^{-\rho_{\mathbb{D}}(0,E)}\,|E^*|$, where $|E^*|$ denotes the length of the projection E^*. Equation (2.1) now follows from Hall's Lemma [GM].

If E is a hyperbolic ball, then a similar lower estimate is available for the area: If E is a hyperbolic ball with hyperbolic radius at least ρ_0, then

$$\mathcal{A}(E) \geq C\,(1-e^{-\rho_0})^2\,e^{-\rho_{\mathbb{D}}(0,E)}\,\omega(0,E,\mathbb{D})\,, \tag{2.2}$$

for some universal constant $C > 0$. Each quantity in the right-hand side of (2.2) can be computed explicitly using conformal invariance. Another way to make the lower estimate in (2.2) is to set $d = \inf\{|z| : z \in E\}$, so that

$$e^{-\rho_{\mathbb{D}}(0,E)} = \frac{1-d}{1+d}\,.$$

Inequality (2.2) is easy to prove if $0 \in E$, so assume that $0 \notin E$, and let Γ denote the circle orthogonal to the unit circle separating E from 0 with $\rho_{\mathbb{D}}(0,\Gamma) = \rho_{\mathbb{D}}(0,E)$, and let I denote the subarc of $\partial\,\mathbb{D}$ subtended by Γ and separated from 0 by Γ. Then

$$\omega(z,I,\mathbb{D}) \equiv \frac{1}{2}\,,$$

for all $z \in \Gamma$, and hence

$$\omega(0,E,\mathbb{D}) \leq \omega(0,\Gamma,\mathbb{D}) = 2\,\omega(0,I,\mathbb{D}) = \frac{|I|}{\pi} \leq C\,(1-d)\,.$$

A short computation shows that

$$\operatorname{diam}(E) \geq C\,(1-d)\,(1-e^{-\rho_0})\,,$$

for some universal constant C. Since $\mathcal{A}(E)$ is comparable to $\operatorname{diam}(E)^2$, inequality (2.2) follows.

To use the inequalities (2.1) and (2.2) we shall need some estimates of hyperbolic distance and harmonic measure.

2.2. Distortion theorems.

A fundamental result about univalent functions is the Koebe Distortion Theorem. The following estimates, which we have stated in a form convenient for our purposes, are easy consequences of this theorem; see [P, pp. 9, 10]. We warn the reader that the hyperbolic metric defined in [P] differs from ρ by a factor of 2. For $w \in G$, let $\delta_G(w)$ denote the Euclidean distance from w to ∂G.

Theorem 2.1 (Koebe). *Let $f : \mathbb{D} \longrightarrow G$ be a Riemann map and let $a, b \in G$. Then*

$$1)\ \delta_G(f(0))\, e^{-\rho_{\mathbb{D}}(0,z)} \leq (1+|z|)^2\, |f'(z)| \leq 4\, \delta_G(f(0))\, e^{3\rho_{\mathbb{D}}(0,z)},$$

and

$$2)\ |b-a| \leq 4\, \delta_G(a)\, e^{2\rho_G(a,b)}.$$

The hyperbolic distance is not explicitly computable in terms of the geometry of G alone. A useful substitute is the *quasi-hyperbolic distance* on G, introduced by Gehring and Palka [GP]. The quasi-hyperbolic distance from w_1 to w_2 in G is defined to be

$$k_G(w_1, w_2) = \inf \left\{ \int_\gamma \frac{|dw|}{\delta_G(w)} : \quad \gamma \text{ is an arc in } G \text{ from } w_1 \text{ to } w_2 \right\}.$$

It is an easy consequence (see [P, p. 92]) of the Koebe Distortion Theorem that

$$\frac{1}{2}\, \rho_G \leq k_G \leq 2\, \rho_G\ . \tag{2.3}$$

2.3. Estimates of Harmonic Measure.

One estimate of harmonic measure which will be used in the proof of Theorem 1.1 is the following Theorem. Let $C_r = \{z : |z| = r\}$ be the circle of radius r centered at 0. If $C_r \cap \partial\Omega \neq \varnothing$, define $\theta(r)$ to be the angular measure of the longest component of $C_r \cap \Omega$. In other words, $r\,\theta(r)$ is the length of the longest arc in $C_r \cap \Omega$. If $C_r \cap \partial\Omega = \varnothing$, set $\theta(r) = \infty$.

Theorem 2.2 (Carleman-Tsuji). *For $\varepsilon > 0$ and $r > (1+\varepsilon)^2\, |z|$,*

$$\omega(z, C_r, \Omega) \leq C(\varepsilon) \exp\left(-\pi \int_{(1+\varepsilon)|z|}^{r/(1+\varepsilon)} \frac{dr}{r\,\theta(r)} \right),$$

where $C(\varepsilon)$ is a constant depending only on ε.

The above result is based on [C]. Tsuji [T, p. 116] gave the explicit polar coordinate version above, but using the total length of $C_r \cap \Omega$, and $2\,|z|$ for the lower limit in the integral, with $C(\varepsilon)$ comparable to $\varepsilon^{-1/2}$. The same proof, using $(1+\varepsilon)\,|z|$ as the lower limit, gives the result above with $C(\varepsilon)$ comparable to $((1+\varepsilon)/\varepsilon)^{3/2}$. Several authors have observed that the proof depends only on the length of the longest arc in $C_r \cap \Omega$. See, for instance [HW, p. 123] or [GM], which contains improvements of this theorem.

Another related estimate is based on extremal distance, and is due to Beurling. Let Γ be the collection of curves in a region Ω which connect sets $E \subset \overline{\Omega}$ and $F \subset \overline{\Omega}$. The extremal distance in Ω from E to F is defined to be

$$d_\Omega(E,F) = \sup_\lambda \frac{\left(\inf\limits_{\gamma\in\Gamma} \displaystyle\int_\gamma \lambda\,|dz|\right)^2}{\int_\Omega \lambda^2\, d\mathcal{A}}\ ,$$

where the supremum is taken over all non-negative Borel functions λ with $0 < \int_\Omega \lambda^2\, d\mathcal{A} < \infty$. Extremal distance is a conformally invariant method of measuring the distance between two sets.

Theorem 2.3 (Beurling). *Suppose Ω is simply connected and $E \subset \overline{\Omega}$. Let σ be an arc in Ω connecting z_0 to $\partial\Omega$. Then*

$$\omega(z_0, E, \Omega) \le \frac{8}{\pi}\, e^{-\pi d_\Omega(\sigma,E)}\ .$$

See, for example [GM]. We will apply this result with σ replaced by a disk containing z_0 which intersects $\partial\Omega$. Since the extremal distance decreases as σ is increased, the inequality remains true.

These two preceding theorems are closely related, though there are circumstances where one gives better estimates than the other. For example, the extremal distance between two circles centered at the origin is not changed if radial slits are removed from the region, though this may greatly reduce $\theta(r)$. In this case the Carleman-Tsuji Theorem gives a better estimate. On the other hand, if a curve increasing in modulus and connecting the two bounding circles of the annulus, is removed from the annulus,then the Carleman-Tsuji estimate has $\theta(r) =$

2π. However, if this curve is not a radial slit, then the Beurling extremal distance estimate gives a better estimate (see the proof below).

The proof of Theorem 1.1 also requires the following elementary estimate. Suppose Γ_1 is a circle orthogonal to $\partial\mathbb{D}$ separating 0 from a set $E \subset \mathbb{D}$, and with $e^{-\rho_{\mathbb{D}}(0,\Gamma_1)} \le 1/4$. If $R = [0, e^{i\alpha}]$ is the radius orthogonal to Γ_1, let Γ_0 be the circle orthogonal to $\partial\mathbb{D}$ and orthogonal to R with

$$e^{-\rho_{\mathbb{D}}(0,\Gamma_0)} = 2\, e^{-\rho_{\mathbb{D}}(0,\Gamma_1)} .$$

Thus Γ_0 separates 0 from Γ_1 and E. Let $\zeta_0 = \Gamma_0 \cap R$, so that $\rho_{\mathbb{D}}(0,\zeta_0) = \rho_{\mathbb{D}}(0,\Gamma_0)$.

Proposition 2.4. *There is a universal constant $C < \infty$ so that*

$$\sup_{\zeta\in\Gamma_0} \omega(\zeta, E, \mathbb{D}) \le C\, \omega(\zeta_0, E, \mathbb{D}) . \tag{2.4}$$

PROOF. By conformal invariance, we may suppose that $e^{-\rho_{\mathbb{D}}(0,\Gamma_1)} = 1/4$ and $\zeta_0 > 0$, which determines Γ_0 and Γ_1. Note that the Euclidean distance from $\Gamma_0 \cap \mathbb{D}$ to $\Gamma_1 \cap \mathbb{D}$ is positive. (This is easiest to see using orthogonality and a self-map of the disk which sends $\zeta_0 = \Gamma_0 \cap R$ to 0). Let U denote the region in $\mathbb{D}$ bounded by $\partial\mathbb{D}$ and Γ_1, containing Γ_0. Let φ be a conformal map of U onto $\mathbb{D}$ with $\varphi(\zeta_0) = 0$ and set $I = \varphi(\Gamma_1 \cap \mathbb{D}) \subset \partial\mathbb{D}$. Note that the Euclidean distance from $\varphi(\Gamma_0 \cap \mathbb{D})$ to I is positive. Since $\omega(\varphi^{-1}(z), E, \mathbb{D})$ is a positive harmonic function on $\mathbb{D}$, vanishing on $\partial\mathbb{D} \setminus I$, we have

$$\omega(\varphi^{-1}(z), E, \mathbb{D}) = \int_I \frac{1 - |z|^2}{|e^{i\theta} - z|^2}\, d\mu(\theta) ,$$

for some positive measure $d\mu$. Since the distance from $\varphi(\Gamma_0 \cap \mathbb{D})$ to I is positive, if $z \in \varphi(\Gamma_0 \cap \mathbb{D})$ and $e^{i\theta} \in I$, then

$$\frac{1 - |z|^2}{|e^{i\theta} - z|^2} \le C ,$$

for some positive constant C. Integrating over I proves (2.4), since $\zeta_0 = \varphi^{-1}(0)$.

3. Proof of Theorem 1.1.

Let $\Omega = f(\mathbb{D})$ and, for $z \in \Omega$, define $\delta_\Omega(z)$ to be the Euclidean distance from z to $\partial\Omega$, the boundary of Ω. Multiplying f by a constant, we may assume that $\delta_\Omega(0) = 1$. For $n \geq 1$, we define

$$A_n = \{z : \ (1+\varepsilon)^{n-1} < |z| < (1+\varepsilon)^n\}$$

and $A_0 = \mathbb{D}$, the unit disk. The size of the annuli was chosen so that the intersection $A_n \cap \Sigma_\varepsilon$ is roughly a rectangle with Euclidean dimensions comparable to $\varepsilon\,(1+\varepsilon)^{n-1}$. Choose, if possible, a Euclidean square $Q_n \subset A_n \cap \Sigma_\varepsilon \cap \Omega$ with

$$\operatorname{diam}(Q_n) \geq \frac{\varepsilon}{4}\,(1+\varepsilon)^{n-1} \tag{3.1}$$

and

$$\frac{1}{2} \leq \frac{\operatorname{dist}(Q_n, \partial(A_n \cap \Sigma_\varepsilon \cap \Omega))}{\operatorname{diam}(Q_n)} \leq 2\,. \tag{3.2}$$

Note that there are constants $C_j(\varepsilon)$, depending only on ε so that

$$C_1(\varepsilon) \leq \frac{\operatorname{dist}(Q_n, \partial\Omega)}{\operatorname{diam}(Q_n)} \leq C_2(\varepsilon)\,.$$

Squares satisfying the inequalities in the display above are called *Whitney squares.* We call the $\{Q_n\}$ *dominant* Whitney squares, as it turns out that the integral of $|f|$ over their inverse images dominates $\|f\|_1$; see Lemma 3.1 below. We remark that many annuli may not contain one of these dominant Whitney squares. Let z_n denote the center of Q_n.

Define a covering $\{\Omega_n\}$, $n \geq 0$, of Ω as follows. For $z \in \Omega$, let γ_z denote the curve from z to 0, lying on a hyperbolic geodesic. Let $N(Q_n)$ denote the hyperbolic neighborhood of Q_n given by

$$N(Q_n) = \left\{z : \ \rho_\Omega(z, z_n) < \frac{100}{\varepsilon}\right\}.$$

Put z in Ω_n if $N(Q_n)$ is the first such neighborhood encountered while tracing the path γ_z starting at z. More precisely, $z \in \Omega_n$ provided

1) $\rho_\Omega(z_n, \gamma_z) < 100/\varepsilon$ and

2) if γ_z^n denotes the component of $\gamma_z \setminus N(Q_n)$ containing z, then either $\rho_\Omega(z_m, \gamma_z^n) \geq 100/\varepsilon$ for all $m \neq n$ or else γ_z^n is empty.

If there is no Q_n in A_n, set $\Omega_n = \varnothing$.

A few remarks are in order at this point.

i) $N(Q_n) \subset \Omega_n$. In particular, the regions Ω_n are not necessarily pairwise disjoint.

ii) $\cup_n \Omega_n = \Omega$. Since $\delta_\Omega(0) = 1$, it is easy to check that there is a dominant Whitney square $Q_0 \subset A_0 = \mathbb{D}$, with $\operatorname{diam}(Q_0) \geq \varepsilon/5$. If $z \in Q_0$, then $\delta_\Omega(z) \geq \varepsilon/10$, from (3.2). Thus $\delta_\Omega(w) \geq \varepsilon/10$ for w on the radial line segment from 0 to z. Integrating the quasi-hyperbolic metric $|dw|/\delta_\Omega(w)$ along this segment we have that $\sup_{z\in Q_0} \rho_\Omega(0,z) \leq 20/\varepsilon$, and so $N(Q_0)$ contains a neighborhood of 0. Hence each γ_z eventually passes through $N(Q_0)$, which means that $\{\Omega_n\}$ covers Ω.

iii) The need for the large hyperbolic radius of $100/\varepsilon$ will become apparent in the proof, and in Example 4.1 in the last section. It is comparable to the quasi-hyperbolic length in A_n of a central circle separating the two bounding circles of A_n.

Perhaps it is easier to picture the corresponding sets on $\mathbb{D}$. The sets $\{f^{-1}(N(Q_n))\}$ are disks in $\mathbb{D}$. Let $U = \cup_n f^{-1}(N(Q_n))$. If $z \in \mathbb{D} \setminus U$ then $z \in \Omega_n$ if the radial line segment from z to 0 first meets ∂U at a point of $\partial f^{-1}(N(Q_n))$.

Since $\cup_n Q_n \subset \Sigma_\varepsilon$ and $\mathbb{D} = \cup_n f^{-1}(\Omega_n)$, Theorem 1.1 is as immediate consequence of the following lemma.

Lemma 3.1. *There exists a constant $C(\varepsilon)$ such that if $f \in L_a^1$ is univalent with $f(0) = 0$, then*

$$\int_{f^{-1}(\Omega_n)} |f| \, d\mathcal{A} \leq C(\varepsilon) \int_{f^{-1}(Q_n)} |f| \, d\mathcal{A}\,. \tag{3.3}$$

PROOF. Fix $\varepsilon > 0$, with $\varepsilon < 1/10$. Throughout the proof we will use C to denote various constants that may change from one use to the next, but are independent of any parameters. Similarly, $C(\varepsilon)$ will denote various constants depending only on ε. We emphasize that C and $C(\varepsilon)$ will always be *positive.*

First we will prove the lemma when $n = 0$. We saw in ii) above that $\rho_\Omega(0,z) \leq 20/\varepsilon$, for $z \in Q_0$. Thus

$$\inf_{z\in f^{-1}(Q_0)} (1 - |z|) \geq C(\varepsilon) > 0\,.$$

By Theorem 2.1.1),

$$|f'(z)| \le C(\varepsilon)$$

on $f^{-1}(Q_0)$. Also by (3.1) and (3.2) we have that $|f| > C(\varepsilon)$ on Q_0. Combining these observations,

$$\begin{aligned}\int_{f^{-1}(Q_0)} |f|\, d\mathcal{A} &\ge C(\varepsilon) \int_{f^{-1}(Q_0)} d\mathcal{A} \\ &\ge C(\varepsilon) \int_{f^{-1}(Q_0)} |f'|^2\, d\mathcal{A} \\ &= C(\varepsilon)\, \mathcal{A}(Q_0)\end{aligned}$$

and hence

$$\int_{f^{-1}(Q_0)} |f|\, d\mathcal{A} \ge C(\varepsilon)\,. \tag{3.4}$$

To estimate the left side of (3.3) when $n = 0$, note that by (2.1) and the definition of A_j,

$$\int_{f^{-1}(\Omega_0)} |f|\, d\mathcal{A} \le C \sum_{j=0}^{\infty} (1+\varepsilon)^j\, e^{-\rho_\Omega(0, A_j \cap \Omega_0)}\, \omega(0, A_j \cap \Omega_0, \Omega)\,.$$

Thus it suffices to show that

$$e^{-\rho_\Omega(0, A_j \cap \Omega_0)}\, \omega(0, A_j \cap \Omega_0, \Omega) \le C(\varepsilon)\, (1+\varepsilon)^{-j(1+C\varepsilon)}\,. \tag{3.5}$$

Let $f(z) \in A_j$. Then, using Theorem 2.1.2) with $a = 0$ and the normalization $\delta_\Omega(0) = 1$,

$$(1+\varepsilon)^{j-1} \le |f(z)| \le 4\, e^{2\rho_\Omega(0, f(z))}\,.$$

Thus

$$e^{-\rho_\Omega(0, A_j)} \le C\, (1+\varepsilon)^{-j/2}\,. \tag{3.6}$$

To prove the estimate (3.5) we consider several cases. In the first case we will use the extremal length estimate of harmonic measure in Theorem 2.3, in the second case we will estimate harmonic measure using the Carleman-Tsuji Theorem 2.2, and in the remaining case hyperbolic distance alone will increase rapidly enough to obtain (3.5).

For $1 \le k \le j-1$, put k in $\mathcal{K}$ if there exists a component E_k of $\partial\Omega \cap A_k$ and $\theta_k \in [0, 2\pi]$ such that

$$E_k \cap \{r\, e^{i\theta} :\ 0 < r < \infty\} \neq \varnothing\,, \qquad \text{when } |\theta - \theta_k| \le \frac{\varepsilon}{100}\,. \tag{3.7}$$

Case 1. Cardinality$(\mathcal{K}) \ge (j-1)/3$.

For $k \in \mathcal{K}$, let S_k denote a small polar coordinate square centered in the annulus along the ray $\arg z = \theta_k$. More precisely, set

$$S_k = \Big\{ r\, e^{i\theta} :\ |\theta - \theta_k| < \frac{\varepsilon}{10^5} \text{ and } \Big| \log \frac{r}{(1+\varepsilon)^{k-1/2}} \Big| < \frac{\varepsilon}{10^5} \Big\}\,.$$

For $k \notin \mathcal{K}$, set $S_k = \varnothing$. We claim that if σ_k is any curve in $A_k \setminus E_k$ connecting the two boundary circles of A_k, then

$$\int_{\sigma_k \setminus S_k} \frac{1}{|z|}\, |dz| \ge \log\,(1+\varepsilon)\,. \tag{3.8}$$

Inequality (3.8) is clearly true if $\sigma_k \cap S_k = \varnothing$. If $\sigma_k \cap S_k \neq \varnothing$, then by (3.7), for at least one component σ_k' of $\sigma_k \setminus S_k$, we have

$$\sup_{z, w \in \sigma_k'} |\arg z - \arg w| \ge \frac{\varepsilon}{100} - \frac{\varepsilon}{10^5}\,.$$

Thus

$$\int_{\sigma_k \setminus S_k} \frac{1}{|z|}\, |dz| \ge \sqrt{\Big(\log\,(1+\varepsilon) - \frac{2\,\varepsilon}{10^5}\Big)^2 + \Big(\frac{\varepsilon}{100} - \frac{\varepsilon}{10^5}\Big)^2} \ge \log\,(1+\varepsilon)\,,$$

which establishes (3.8). The above inequality is perhaps easiest to see by using the change of variable $w = \log z$, so that $|dz|/|z| = |dw|$.

Now define a metric λ on $\Omega \cap \cup_{k=1}^{j-1} A_k$ by

$$\lambda(z) = \frac{1}{|z|} \Big(\sum_{k=1}^{j-1} \chi_{A_k}(z) - \sum_{k \in \mathcal{K}} \chi_{S_k}(z) \Big)\,,$$

where χ_F denotes the characteristic function of a set F. If $\sigma \subset \Omega$ is a curve connecting $\partial\,\mathbb{D}$ to A_j then by (3.8)

$$\int_\sigma \lambda(z)\, |dz| \ge (j-1) \log\,(1+\varepsilon)\,.$$

Because

$$\int_{S_k} \frac{1}{|z|^2}\, d\mathcal{A}(z) \geq C\,\varepsilon^2\,,$$

and Cardinality $(\mathcal{K}) \geq (j-1)/3$, we have that

$$\int_{\Omega\cap(1+\varepsilon)^{j-1}\mathbb{D}} \lambda^2(z)\, d\mathcal{A}(z) \leq (j-1)\,(2\pi\log(1+\varepsilon) - C\,\varepsilon^2)\,.$$

Thus

$$d_\Omega(\partial\,\mathbb{D}, A_j) \geq \frac{\left(\inf\limits_\sigma \int_\sigma \lambda(z)\,|dz|\right)^2}{\int_\Omega \lambda(z)^2\, d\mathcal{A}} \geq \frac{j-1}{2\pi}\,(1+C\,\varepsilon)\log(1+\varepsilon)\,.$$

By Theorem 2.3

$$\omega(0, A_j\cap\Omega_0, \Omega) \leq C\,e^{-\pi d_\Omega(\partial\mathbb{D},A_j)} \leq C\,(1+\varepsilon)^{-(j/2)(1+C\varepsilon)}\,, \tag{3.9}$$

and from this and (3.6) we conclude that (3.5) holds in this case.

Recall that $\theta(r)$ is the angular measure of the longest component of $\{z:\,|z|=r\}\cap\Omega$. For $1\leq m\leq j-1$, set

$$F_m = \Big\{r:\ (1+\varepsilon)^{m-1} < r < (1+\varepsilon)^m \text{ and } \theta(r) \leq 2\pi - \frac{\varepsilon}{100}\Big\}.$$

and

$$\mathcal{M} = \Big\{m:\ 1\leq m\leq j-1 \text{ and } |F_m| \geq \frac{\varepsilon\,(1+\varepsilon)^{m-1}}{100}\Big\}.$$

Case 2. Cardinality $(\mathcal{M}) \geq (j-1)/3$.

By the definition of $\mathcal{M}$, if $m\in\mathcal{M}$ then

$$\begin{aligned}\int_{(1+\varepsilon)^{m-1}}^{(1+\varepsilon)^m} \frac{1}{r\,\theta(r)} dr &\geq \int_{(1+\varepsilon)^{m-1}}^{(1+\varepsilon)^m} \frac{1}{2\pi r}\, dr + \int_{F_m} \Big(\frac{1}{2\pi-\varepsilon/100} - \frac{1}{2\pi}\Big)\frac{dr}{r}\\ &\geq \frac{1}{2\pi}\log(1+\varepsilon) + C\,\varepsilon^2\,.\end{aligned}$$

Since Cardinality $(\mathcal{M}) \geq (j-1)/3$, we have that

$$\int_1^{(1+\varepsilon)^{j-2}} \frac{1}{r\,\theta(r)}\, dr \geq \frac{1}{2\pi}\,(j-2)\log(1+\varepsilon) + j\,C\,\varepsilon^2\,.$$

Thus by the Carleman-Tsuji Theorem 2.2,

$$\begin{aligned}\omega(0, A_j \cap \Omega_0, \Omega) &\le C(\varepsilon)\, \exp\Big(-\pi \int_1^{(1+\varepsilon)^{j-2}} \frac{1}{r\,\theta(r)}\, dr\Big) \\ &\le C(\varepsilon)\,(1+\varepsilon)^{-(j/2)(1+C\varepsilon)}\,.\end{aligned} \tag{3.10}$$

By (3.6) and (3.10) we conclude that (3.5) holds in this case.

Case 3. Cardinality $(\mathcal{K}) < (j-1)/3$ and Cardinality $(\mathcal{M}) < (j-1)/3$. Set

$$\mathcal{I} = \{i:\ 1 \le i \le j-1,\ i \notin \mathcal{K} \text{ and } i \notin \mathcal{M}\}\,.$$

Then Cardinality $(\mathcal{I}) \ge (j-1)/3$. Let $i \in \mathcal{I}$. Choose a continuum $L_i \subset \partial\Omega \cap A_i$ such that L_i connects the two bounding circles of A_i. Since $i \notin \mathcal{K}$, there is a θ_i such that

$$L_i \subset e^{i\theta_i}\Sigma_{\varepsilon/100}\,.$$

Then $\partial\Omega$ does not intersect most of the middle of A_i. Indeed, let V denote the annular region given by

$$V = \Big\{z:\ (1+\varepsilon)^{i-1}\Big(1+\frac{\varepsilon}{100}\Big) < |z| < (1+\varepsilon)^{i-1}\Big(1+\frac{99\,\varepsilon}{100}\Big)\Big\}$$

and suppose

$$a \in \partial\Omega \cap (V \setminus e^{i\theta_i}\Sigma_{\varepsilon/20})\,.$$

Then there is a component σ_i of $\partial\Omega \cap A_i$ connecting a to one of the bounding circles of A_i. Since $i \notin \mathcal{K}$,

$$\sigma_i \subset e^{i\theta}\,\Sigma_{\varepsilon/100}\,,$$

for some θ. Note that since $i \notin \mathcal{K}$ the angular distance from σ_i to L_i is at least

$$\frac{\varepsilon}{20} - \Big(\frac{\varepsilon}{50} + \frac{\varepsilon}{100}\Big) = \frac{\varepsilon}{50}\,.$$

Since the length of σ_i is at least $(1+\varepsilon)^{i-1}\,\varepsilon/100$, this contradicts $i \notin \mathcal{M}$. Thus

$$\partial\Omega \cap A_i \subset e^{i\theta_i}\,\Sigma_{\varepsilon/20} \cup (A_i \setminus V)\,.$$

Since $i \notin \mathcal{M}$, $\Omega \cap A_i \not\subset e^{i\theta_i}\,\Sigma_{\varepsilon/20}$, and hence there exists a $Q_i \in A_i \cap \Omega$.

By (3.6)

$$\rho_\Omega(z_0, A_i) \geq \rho_\Omega(0, A_i) - \rho_\Omega(z_0, 0) > \frac{100}{\varepsilon},$$

for i sufficiently large. This implies that $A_i \cap N(Q_0) = \varnothing$ for $i \geq i_0$, where i_0 depends only on ε.

Suppose $z \in A_j \cap \Omega_0$ and suppose γ_z is the curve from z to 0 lying on a hyperbolic geodesic. We claim that if $w \in \gamma_z \cap A_i$, with $i \geq i_0$ and $\delta_{A_i}(w) \geq (\varepsilon/10)\,(1+\varepsilon)^{i-1}$, then

$$\delta_\Omega(w) \leq \frac{\varepsilon}{10}\,(1+\varepsilon)^{i-1}. \tag{3.11}$$

By (3.1) and (3.2), $\operatorname{dist}(Q_i, \partial\Omega) \geq \varepsilon\,(1+\varepsilon)^{i-1}/8$ and hence $Q_i \subset V \setminus e^{i\theta_i}\Sigma_{\varepsilon/20}$. Suppose such a w does not satisfy (3.11). Let σ be the curve in $V \setminus e^{i\theta_i}\Sigma_{\varepsilon/20}$ connecting w to z_i (the center of Q_i), consisting of a radial line segment from w to the circle of radius $(1+\varepsilon)^{i-1}\,(1+\varepsilon/2)$, then an arc on this circle, followed by a radial line segment to z_i. Note that on σ, the distance to $\partial(A_i \cap \Omega)$ is at least $(\varepsilon/20)\,(1+\varepsilon)^{i-1}$, and along most of the circle this distance is $(\varepsilon/2)\,(1+\varepsilon)^{i-1}$. Hence by the comparison of the hyperbolic and quasi-hyperbolic distance (2.3)

$$\rho_\Omega(w, z_i) \leq \rho_{A_i\cap\Omega}(w, z_i) \leq 2\int_\sigma \frac{|d\zeta|}{\delta_{A_i\cap\Omega}(\zeta)} < \frac{100}{\varepsilon}.$$

Thus $w \in N(Q_i) \cap A_i$. Now $A_i \cap N(Q_0) = \varnothing$, since $i \geq i_0$, and thus $N(Q_0)$ cannot be the first such neighborhood encountered along γ_z. This contradicts $z \in \Omega_0$ and completes the proof of (3.11).

Now when $i \geq i_0$, by (3.11),

$$\int_{\gamma_z \cap A_i} \frac{|d\zeta|}{\delta_\Omega(\zeta)} \geq \int_{(1+\varepsilon)^{i-1}(1+\varepsilon/10)}^{(1+\varepsilon)^{i-1}(1+9\varepsilon/10)} \frac{|d\zeta|}{\dfrac{\varepsilon}{10}\,(1+\varepsilon)^{i-1}} \geq 8.$$

Using the lower estimate in (2.3), this implies that for $z \in A_j \cap \Omega_0$,

$$\rho_\Omega(0, z) \geq \frac{1}{2}\sum_{\substack{i\in\mathcal{I}\\ i\geq i_0}} \int_{\gamma_z\cap A_i} \frac{|d\zeta|}{\delta_\Omega(\zeta)} \geq 4\Big(\frac{j-1}{3} - i_0\Big). \tag{3.12}$$

Hence

$$e^{-\rho_\Omega(0, A_j\cap\Omega_0)}\,\omega(0, A_j\cap\Omega_0, \Omega) \leq e^{-4(j-3i_0-1)/3} \leq C(\varepsilon)\,(1+\varepsilon)^{-j(1+C\varepsilon)},$$

since i_0 depends only on ε, and (3.5) holds. This completes the proof of the case $n=0$.

The proof of Lemma 3.1 for $n>0$ is very similar. To begin with, we see from (3.1) that Q_n contains a disk of Euclidean radius at least $\varepsilon\,(1+\varepsilon)^{n-1}/8\sqrt{2}$. Thus the quasi-hyperbolic length of any curve γ from the center of this disk to its boundary is at least

$$\int_\gamma \frac{|dz|}{\delta_\Omega(z)} \geq \frac{C}{(1+\varepsilon)^n}\,|\gamma| \geq C\,\varepsilon\,.$$

Hence Q_n contains a hyperbolic ball of radius at least $C\,\varepsilon$, by (2.3), and so we get from (2.2) that

$$\begin{aligned}\int_{f^{-1}(Q_n)} |f|\,d\mathcal{A} &\geq (1+\varepsilon)^{n-1}\,\mathcal{A}(f^{-1}(Q_n)) \\ &\geq C\,(1+\varepsilon)^n\,\varepsilon^2\,e^{-\rho_\Omega(0,Q_n)}\,\omega(0,Q_n,\Omega)\,.\end{aligned} \tag{3.13}$$

Next we consider the integral over $f^{-1}(\Omega_n\cap A_j)$, where allowance must be made for both $j<n$ and $j\geq n$. We have

$$\begin{aligned}\int_{f^{-1}(\Omega_n\cap A_j)} |f|\,d\mathcal{A} &\leq (1+\varepsilon)^j\,\mathcal{A}(f^{-1}(\Omega_n\cap A_j)) \\ &\leq C\,(1+\varepsilon)^j\,e^{-\rho_\Omega(0,\Omega_n\cap A_j)}\,\omega(0,\Omega_n\cap A_j,\Omega)\,,\end{aligned} \tag{3.14}$$

by (2.1). Using the triangle inequality and the definition of Ω_n, we see that

$$\rho_\Omega(0,Q_n) + \rho_\Omega(z_n,\Omega_n\cap A_j) \leq \rho_\Omega(0,\Omega_n\cap A_j) + C(\varepsilon)\,,$$

and hence

$$e^{-\rho_\Omega(0,\Omega_n\cap A_j)} \leq C(\varepsilon)\,e^{-\rho_\Omega(0,Q_n)}\,e^{-\rho_\Omega(z_n,\Omega_n\cap A_j)}\,. \tag{3.15}$$

To estimate $\omega(0,\Omega_n\cap A_j,\Omega)$, first suppose that $\rho_\Omega(0,z_n)\leq C(\varepsilon)$. We observed above that Q_n contains a hyperbolic ball of radius at least $C\,\varepsilon$, and so $\omega(0,Q_n,\Omega)\geq C(\varepsilon)$. Hence

$$\omega(0,\Omega_n\cap A_j,\Omega) \leq C(\varepsilon)\,\omega(0,Q_n,\Omega)\,\omega(z_n,\Omega_n\cap A_j,\Omega)\,, \tag{3.16}$$

by Harnack's inequality. We now show that (3.16) holds for all n. We may assume that $\exp(-\rho_\Omega(0,N(Q_n)))\leq 1/4$, since (3.16) has been established when $\rho_\Omega(0, z_n) \le C(\varepsilon)$. Let Γ_n denote the hyperbolic geodesic in $\mathbb{D}$ that separates 0 from $f^{-1}(N(Q_n))$, is orthogonal to the radius of $\mathbb{D}$ through $f^{-1}(z_n)$ and satisfies $\exp(-\rho_{\mathbb{D}}(0, \Gamma_n)) = 2\exp(-\rho_{\mathbb{D}}(0, N(Q_n)))$. Then

$$\begin{aligned}\omega(0, \Omega_n \cap A_j, \Omega) &= \omega(0, f^{-1}(\Omega_n \cap A_j), \mathbb{D})\\ &\le \omega(0, \Gamma_n, \mathbb{D}) \sup_{\zeta \in \Gamma_n} \omega(\zeta, f^{-1}(\Omega_n \cap A_j), \mathbb{D}),\end{aligned}$$

by the maximum principle. For the first factor, observe that

$$\omega(0, \Gamma_n, \mathbb{D}) \le C(\varepsilon)\, \omega(0, f^{-1}(Q_n), \mathbb{D}) = C(\varepsilon)\, \omega(0, Q_n, \Omega),$$

since the harmonic measures of these sets in $\mathbb{D}$ are comparable to the diameters of the sets. Next, we use first Proposition 2.4 and then Harnack's inequality to get that

$$\begin{aligned}\sup_{\zeta \in \Gamma_n} \omega(\zeta, f^{-1}(\Omega_n \cap A_j), \mathbb{D}) &\le C\, \omega(\zeta_n, f^{-1}(\Omega_n \cap A_j), \mathbb{D})\\ &\le C(\varepsilon)\, \omega(f^{-1}(z_n), f^{-1}(\Omega_n \cap A_j), \mathbb{D}),\end{aligned}$$

where $\zeta_n \in \Gamma_n$ is determined by $\rho_{\mathbb{D}}(0, \zeta_n) = \rho_{\mathbb{D}}(0, \Gamma_n)$. The last three displayed inequalities now combine to complete the proof of (3.16).

Putting together (3.13), (3.14), (3.15) and (3.16), we get that

$$\begin{aligned}\int_{f^{-1}(\Omega_n \cap A_j)} |f|\, d\mathcal{A} &\le C(\varepsilon)\,(1+\varepsilon)^j\, e^{-\rho_\Omega(0, Q_n)}\, \omega(0, Q_n, \Omega)\\ &\qquad \cdot e^{-\rho_\Omega(z_n, \Omega_n \cap A_j)}\, \omega(z_n, \Omega_n \cap A_j, \Omega)\\ &\le C(\varepsilon)\,(1+\varepsilon)^{j-n} \int_{f^{-1}(Q_n)} |f|\, d\mathcal{A}\\ &\qquad \cdot e^{-\rho_\Omega(z_n, \Omega_n \cap A_j)}\, \omega(z_n, \Omega_n \cap A_j, \Omega).\end{aligned} \tag{3.17}$$

We claim that, for all positive integers j and n, we have the inequality

$$e^{-\rho_\Omega(z_n, \Omega_n \cap A_j)}\, \omega(z_n, \Omega_n \cap A_j, \Omega) \le C(\varepsilon)\,(1+\varepsilon)^{-|j-n|(1+C\varepsilon)}. \tag{3.18}$$

This has been proved when $n = 0$ and z_n is replaced by 0, and the proof for $n > 0$ is similar.

For $n > 0$ the estimate of hyperbolic distance, as for $n = 0$, is based on the Distortion Theorem. Let $z \in A_j$, and assume first that $j > n + 1$. Then

$$\begin{aligned}\frac{\varepsilon\,(1+\varepsilon)^j}{2} &\le (1+\varepsilon)^{j-1} - (1+\varepsilon)^n \\ &\le |z - z_n| \\ &\le 4\,\delta_\Omega(z_n)\, e^{2\rho_\Omega(z_n,z)} \\ &\le C\,\varepsilon\,(1+\varepsilon)^n\, e^{2\rho_\Omega(z_n,z)}\,,\end{aligned}$$

where the upper bound for $|z - z_n|$ came from applying Theorem 2.1.2) with $a = z_n$ and $b = z$. For $j < n$ we estimate

$$\varepsilon\,(1+\varepsilon)^n \le C\ \mathrm{diam}\,(Q_n) \le C\,|z - z_n| \le C\,(1+\varepsilon)^j\, e^{2\rho_\Omega(z_n,z)}\,,$$

where now Theorem 2.1.2) was used with $a = z$, noting that $\delta_\Omega(z) \le 1 + (1+\varepsilon)^j$, to get the last inequality. Hence

$$e^{-\rho_\Omega(z_n,\Omega_n\cap A_j)} \le C(\varepsilon)\,(1+\varepsilon)^{-|j-n|/2}\,, \qquad 0 \le j < \infty\,, \tag{3.19}$$

after an increase in the constant $C(\varepsilon)$ to handle the cases $j = n$ and $j = n + 1$.

The harmonic measure estimates we need for the general case are also very similar to those made in the case $n = 0$. As before, we consider cases 1, 2 and 3 separately. The Case 1 estimate involving extremal distance is made in exactly the same way, yielding

$$\omega(z_n, \Omega_n \cap A_j, \Omega) \le C\,(1+\varepsilon)^{-|j-n|(1+C\varepsilon)/2}\,, \tag{3.20}$$

in place of (3.9). As above, the absolute values are required in the exponent to allow for the possibility that $j < n$.

When we use the Carleman-Tsuji estimate for harmonic measure in Case 2 with $j > n$, the integral in (3.10) is replaced by

$$\int_{(1+\varepsilon)^{n+1}}^{(1+\varepsilon)^{j-2}} \frac{1}{r\,\theta(r)}\,dr\,.$$

When $j < n$, we first invert Ω using the map $z \longrightarrow 1/z$ (which preserves harmonic measure) to put Ω in the proper form to apply Theorem 2.2. This results in the estimate

$$\omega(z_n, \Omega_n \cap A_j, \Omega) \le C(\varepsilon)\,(1+\varepsilon)^{-|j-n|(1+C\varepsilon)/2}\,, \tag{3.21}$$

in place of (3.10).

The estimates (3.19), (3.20), and (3.21) now combine to prove claim (3.18) in cases 1 and 2. Finally, the hyperbolic distance estimate in Case 3 is made just as before to get

$$\rho_\Omega(z_n, A_j \cap \Omega_n) \geq \frac{4}{3}\,(|j-n| - 3\,i_0 - 1)$$

instead of (3.12). Hence

$$e^{-\rho_\Omega(z_n, A_j \cap \Omega_n)} \leq e^{-4(|j-n|-3i_0-1)/3} \leq C(\varepsilon)\,(1+\varepsilon)^{-|j-n|(1+C\varepsilon)}\,,$$

and (3.18) has been established in this last case as well.

Combining (3.17) and (3.18), we now get

$$\begin{aligned}
\int_{f^{-1}(\Omega_n)} |f|\,d\mathcal{A} &= \sum_{j=1}^{\infty} \int_{f^{-1}(\Omega_n \cap A_j)} |f|\,d\mathcal{A} \\
&\leq C(\varepsilon) \int_{f^{-1}(Q_n)} |f|\,d\mathcal{A} \sum_{j=1}^{\infty} (1+\varepsilon)^{j-n-|j-n|(1+C\varepsilon)} \\
&\leq C(\varepsilon) \int_{f^{-1}(Q_n)} |f|\,d\mathcal{A}\,,
\end{aligned}$$

and the proof is complete.

4. Examples.

Our first example shows that, in general, infinitely many dominant Whitney squares Q_n are required in the proof of Theorem 1.1, and also that $N(Q_n)$ must be defined so that its hyperbolic radius tends to infinity as $\varepsilon \longrightarrow 0$.

Example 4.1. For $R > 0$, let $\Omega^R = \{z : |z| < R\} \setminus [1, R)$ and let f_R be the Riemann map from $\mathbb{D}$ onto Ω^R such that $f_R(0) = 0$ and $f_R'(0) > 0$. Clearly $f_R \in L^1_a$, and $\lim_{R\to\infty} \|f_R\|_1 = \infty$, since as $R \longrightarrow \infty$, f_R converges uniformly on compact subsets of $\mathbb{D}$ to $f(z) = 4\,z\,(1+z)^{-2} \notin L^1_a$. It is clear that there is a dominant Whitney square Q_n in every annulus A_n with $(1+\varepsilon)^n \leq R$. Since

$$\int_{f_R^{-1}(Q_n)} |f_R|\,d\mathcal{A} \leq (1+\varepsilon)^n\,\mathcal{A}(\mathbb{D})$$

and $\lim_{R\to\infty} \|f_R\|_1 = \infty$, as $R \longrightarrow \infty$ we must use Q_j with j arbitrarily large in the proof of Theorem 1.1.

Next, we show that it was necessary to have the hyperbolic radius of $N(Q_n)$ tending to infinity as $\varepsilon \longrightarrow 0$. We show that if $M > 0$ is any fixed constant, then the neighborhoods $\mathcal{N}(Q_n) = \{z : \rho_\Omega(z, z_n) < M\}$ will not work in the proof. Let $z \in \Omega^R$ be a point with $\operatorname{Re} z < 0$. Then $|z_n - z| \geq |z_n|$, and $\delta(z_n)$ is comparable to $\varepsilon\,|z_n|$, and so

$$\frac{1}{\varepsilon} \leq C\,\frac{|z_n - z|}{\delta_\Omega(z_n)} \leq C\,e^{2\rho_\Omega(z_n,z)}\,,$$

by Theorem 2.1.2). Thus, if ε is sufficiently small, independent of R, then $\mathcal{N}(Q_n)$ is contained in the right half plane for all $n \geq 0$. This means that if $(\Omega^R)_n$ is defined using $\mathcal{N}(Q_n)$ in place of $N(Q_n)$, then $\{(\Omega^R)_n\}$ does not cover Ω^R, and so the proof of Theorem 1.1 does not work. We now show that even if Q_0 is replaced by $\mathbb{D} \cap \Sigma_\varepsilon$, so that now $\{(\Omega^R)_n\}$ covers Ω^R, there still is a problem.

Since f_R maps $(-1, 0]$ to $(-R, 0]$, it follows that a hyperbolic neighborhood in $\mathbb{D}$ of $(-1, -1/2]$ must belong to $f_R^{-1}((\Omega^R)_0)$, when $(\Omega^R)_n$ is defined using $\mathcal{N}(Q_n)$ in place of $N(Q_n)$. This hyperbolic neighborhood contains an angle

$$\Gamma = \left\{z \in \mathbb{D} : \,|1 + z| < (1 + \eta)\,(1 - |z|) \text{ and } |1 + z| < \frac{1}{2}\right\},$$

where $\eta > 0$, in $\mathbb{D}$ with vertex at -1. Since f_R converges uniformly on compact subsets of $\mathbb{D}$ to $4\,z\,(1 + z)^{-2}$, which has a pole of order 2 at -1, it follows that

$$\lim_{R\to\infty} \int_{f_R^{-1}((\Omega^R)_0)} |f_R|\,d\mathcal{A} \geq \lim_{R\to\infty} \int_\Gamma |f_R|\,d\mathcal{A} = \infty\,.$$

Thus, if $N(Q_n)$ is replaced by $\mathcal{N}(Q_n)$, then there is no constant $C(\varepsilon)$ depending only on ε such that

$$\int_{f_R^{-1}((\Omega^R)_0)} |f_R|\,d\mathcal{A} \leq C(\varepsilon) \int_{f_R^{-1}(Q_0)} |f_R|\,d\mathcal{A}\,,$$

since the integral on the right is bounded by $1 \cdot \mathcal{A}(\mathbb{D}) = \pi$.

EXAMPLE 4.2. It might seem at first thought that the proof of Theorem 1.1 could be simplified by using circular symmetrization. However, this does not seem to be the case. One problem is that the symmetrization of an L^1_a function may not be in L^1_a. For example, let $g(z) = (f(z^2))^{1/2}$ be the square root transform of the function f from Example 4.1. Then $g(z) = 2\,z/(1+z^2)$, and g maps the disk onto the plane slit along the real axis from $-\infty$ to -1 and from 1 to ∞. Clearly $g \in L^1_a$, since its poles are simple, but its circular symmetrization is $f \notin L^1_a$.

Let g_R be the Riemann map of $\mathbb{D}$ onto

$$\{z:\ |z|<R\} \setminus ((-R,-1] \cup [1,R))$$

with

$$g_R(0) = 0$$

and

$$g_R'(0) > 0\,.$$

Then $\|g_R\|_1 \le \|g\|_1$, since g_R is subordinant to g, and so the integrals of $|g_R|$ over the inverse image of Σ_ε are uniformly bounded by $\|g\|_1$. The symmetrization of g_R is f_R, and

$$\lim_{R\to\infty} \int_{f_R^{-1}(\Sigma_\varepsilon)} |f_R|\, d\mathcal{A} \ge \lim_{R\to\infty} \delta \int_{\mathbb{D}} |f_R|\, d\mathcal{A} = \infty\,,$$

where Theorem 1.1 was used to get the inequality. Thus even when the symmetrized function is in L^1_a, its integral over the inverse image of Σ_ε cannot be bounded by the integral of the original function.

Example 4.3. This example shows that in Theorem 1.1, L^1_a can not be replaced by L^p_a, for any $p>1$. Let $p>1$ be fixed and set

$$\varepsilon_n = \pi\Big(\frac{p-1}{p}\Big) + \frac{1}{n}\,,$$

for all n sufficiently large so that $\varepsilon_n \le \pi$. Let

$$f_n : \mathbb{D} \longrightarrow \mathbb{C} \setminus (1 + \Sigma_{\varepsilon_n})$$

be the Riemann map with $f_n(0) = 0$ and $f_n'(0) > 0$. It is easy to verify that $f_n \in L^p_a$, since $\varepsilon_n > \pi\,(p-1)/p$, but

$$\|f_n\|_p \longrightarrow \infty\,, \qquad \text{as } n \longrightarrow \infty\,.$$

On the other hand, some elementary trigonometry shows that

$$\Sigma_{\pi(p-1)/2p} \cap (\mathbb{C} \setminus (1 + \Sigma_{\pi(p-1)/p}))$$

is contained in the disk of radius 2 centered at the origin. Hence

$$\int_{f_n^{-1}(\Sigma_{\pi(p-1)/2p})} |f_n|^p \, d\mathcal{A} \leq 2^p \, \mathcal{A}(\mathbb{D}) = 2^p \, \pi \,,$$

and this can not be used to dominate $\|f_n\|_p$ as $n \longrightarrow \infty$.

References.

[A1] Ahlfors, L. V., *Lectures on quasiconformal mappings.* Van Nostrand, 1966.

[A2] Ahlfors, L. V., *Conformal Invariants: Topics in Geometric Function Theory.* McGraw-Hill, 1973.

[C] Carleman, T., Sur une inégalité différentielle dans la théorie des fonctions analytiques. *C. R. Acad. Sci. Paris* **196** (1933), 995-997.

[GM] Garnett, J. B., Marshall, D. E., *Harmonic Measure.* Cambridge Univ. Press. Forthcoming.

[GP] Gehring, F. W., Palka, B. P., Quasiconformally homogeneous domains. *J. Analyse Math.* **30** (1976), 172-199.

[Ha] Hamilton, R. S., Extremal quasiconformal mappings with prescribed boundary values. *Trans. Amer. Math. Soc.* **138** (1969), 399-406.

[HW] Hayman, W. K., Weitsman, A., On the coefficients and means of functions omitting values. *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 119-137.

[Hu] Huang, X., The image domain of an extremal dilatation. *Adv. in Math.* (China) **22** (1993), 435-440.

[K] Krushkal, S. L., Teichmüller's theorem on extremal quasi-conformal mappings. *Siberian Math. J.* **8** (1967), 231-244. Translated from Sibirsk. *Math. Z.* **8** (1967), 313-332.

[OS] Ortel, M., Smith, W., The argument of an extremal dilatation. *Proc. Amer. Math. Soc.* **104** (1988), 498-502.

[P] Pommerenke, Ch., *Boundary Behavior of Conformal Maps.* Springer-Verlag, 1992.

[RS] Reich, E., Strebel, K., Extremal quasiconformal mappings with given boundary values. *Contributions to Analysis, a collection of papers dedicated to Lipman Bers.* Academic Press, 1974, 375-391.

[S] Strebel, K., On quadratic differentials and extremal quasiconformal mappings. *Proc. Internat. Congress Math.* Vancouver, 1974. *Canad. Math. Congr.* **2** (1975), 223-227.

[T] Tsuji, M., *Potential Theory in Modern Function Theory.* Chelsea, 1959.

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Local limit theorems on some non unimodular groups

Emile Le Page and Marc Peigné

Abstract. Let G_d be the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$, $d \geq 1$ and let us consider the product group $G_{d,N} = G_d \times \mathbb{R}^N$, $N \geq 1$. For a large class of probability measures μ on $G_{d,N}$, one proves that there exists $\rho(\mu) \in]0,1]$ such that the sequence of finite measures

$$\Big\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \, \mu^{*n} \Big\}_{n \geq 1}$$

converges weakly to a non-degenerate measure.

Résumé. Soit G_d le produit semi-direct de $\mathbb{R}^{*+}$ et de $\mathbb{R}^d$ et $G_{d,N}$ le groupe produit $G_d \times \mathbb{R}^N$, $N \geq 0$. Pour une large classe de mesures de probabilité sur $G_{d,N}$ nous montrons qu'il existe $\rho(\mu) \in]0,1]$ tel que la suite de mesures finies

$$\Big\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \, \mu^{*n} \Big\}_{n \geq 1}$$

converge vaguement vers une mesure non nulle.

1. Introduction.

Fix two integers $d \geq 1$, $N \geq 0$ and choose a norm $\| \cdot \|$ on $\mathbb{R}^d$ and $\mathbb{R}^N$ (when $N \geq 1$). Let $G_{d,N}$ be the connected group $\mathbb{R}^{*+} \times \mathbb{R}^d \times \mathbb{R}^N$

with the composition law

$$\text{for all } g = (a, u, b), \text{ for all } g' = (a', u', b') \in G\,,$$
$$g \cdot g' = (a\,a', a\,u' + u, b + b')\,.$$

We will note $g = (a(g), u(g), b(g))$ (or $g = (a, u, b)$ when there is no ambiguity). The group $(G_{d,N}, \cdot)$ is a non unimodular solvable group with exponential growth and the right Haar measure m_D on $G_{d,N}$ is

$$m_D(da\,du\,db_1 \cdots db_N) = \frac{da\,du\,db_1 \cdots db_N}{a}\,.$$

Note that $G_{d,0}$ is the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$; in particular $G_{1,0}$ is the affine group of the real line.

We consider a probability measure μ on G; we denote by μ^{*n} its n^{th} power of convolution. Under quite general assumptions on μ we show that there exists $\rho(\mu) \in]0, 1[$ such that the sequence

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n}\, \mu^{*n} \right\}_{n \geq 0}$$

converges weakly to a non-degenerate measure. This problem has already been tackled by Ph. Bougerol in [3] where were established local limit theorems on some solvable groups with exponential growth; in particular, for a class R of probability measures μ on the affine group of the real line (that is $d = 1$ and $N = 0$) he showed that the sequence

$$\left\{ \frac{n^{3/2}}{\rho(\mu)^n} \mu^{*n} \right\}_{n \geq 0}$$

converges weakly to a non-degenerate measure. In [7] we extend this result to a quite large class of probability measures; the new ingredient in our proof was the fact that there exists closed connections between this problem and the theory of the fluctuations of a random walk on the real line. In the present paper, we extend this result to the case $N \geq 1$; we first obtain uniform upperbounds in the Local limit theorem for a random walk on $\mathbb{R}^d$ and, secondly, we use a generalisation of the Wiener-Hopf's factorisation due to Ch. Sunyach [9].

This study is also related with the work by N. T. Varopoulos [10], [11] where upperbounds and lowerbounds for the asymptotic behaviour of the convolution powers μ^{*n} of a large class of probability measures are given.

From now on, we will suppose that $N \geq 1$ and we set $G = G_{d,N}$. We introduce the following conditions on μ:

Hypothesis G1. *There exists* $\alpha > 0$ *such that*

$$\int_G (e^{\alpha|\log a|} + \|u\|^\alpha + \|b\|^2)\, \mu(da\, du\, db) < +\infty\,.$$

Hypothesis G2. $\displaystyle\int_G \operatorname{Log} a\, \mu(da\, du\, db) = 0$ *and* $\displaystyle\int_G b\, \mu(da\, du\, db) = 0.$

Hypothesis G3. *The support of* μ *is included in* $\mathbb{R}^{*+} \times (\mathbb{R}^+)^d \times \mathbb{R}^N$, *the image of* μ *by the mapping* $(a, u, b) \longmapsto (\operatorname{Log} a, b)$ *is aperiodic in* $\mathbb{R}^{N+1}$ (*see Definition* 2.1) *and there exists* $\beta > 0$ *such that*

$$\int_G \|u\|^{-\beta}\, \mu(da\, du\, db) < +\infty\,.$$

Hypothesis G'3. *The measure* μ *is absolutely continuous with respect to the Haar measure* m_D *on* G *and its density* ϕ_μ *satisfies*

$$\int_{]0,1]\times\mathbb{R}^N} \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a, u, b)\, du}\, \frac{da\, db}{a^\gamma} < +\infty\,.$$

for some γ *and* q *in* $]1, +\infty[$.

We have the

Theorem 1.1. *Let* μ *be a probability measure on* G *satisfying hypotheses* G1, G2 *and* G3 (*or* G'3). *Then, the sequence of finite measures* $\{n^{(N+3)/2}\mu^{*n}\}_{n\geq 0}$ *converges weakly to a non-degenerate Radon measure on* G.

Note that the asymptotic behavior of the sequence $\{\mu^{*n}\}_{n\geq 1}$ does not depend on d.

When μ is not centered, that is

$$\int_G \operatorname{Log} a\, \mu(da\, du\, db) \neq 0$$

or

$$\int_G b\,\mu(da\,du\,db) \neq 0\,,$$

we bring back the study to the centered case as in [7]. We introduce the following conditions on μ:

Hypothesis G*1. *There exists $\alpha > 0$ such that*

$$\int_G \left(a^t + \|u\|^\alpha + \exp\left(t\,\|b\|\right)\right) \mu(da\,du\,db) < +\infty$$

for any $t \in \mathbb{R}$.

Hypothesis G*2. *One has*

$$\int_G \operatorname{Log} a\,\mu(da\,du\,db) \neq 0$$

with $\mu\{g \in G: \ a(g) < 1\} > 0$ and $\mu\{g \in G: \ a(g) > 1\} > 0$.

When μ satisfies these two conditions, there exists a unique $(s_0, t_0) \in \mathbb{R} \times \mathbb{R}^N$ such that

$$\int_G a^{s_0} e^{\langle t_0, b\rangle}\,\mu(da\,du\,db) = \inf_{(s,t)\in\mathbb{R}\times\mathbb{R}^N} \int_G a^s\, e^{\langle t, b\rangle}\,\mu(da\,du\,db)\,.$$

Furthermore,

$$\rho(\mu) = \int_G a^{s_0} e^{\langle t_0, b\rangle}\,\mu(da\,du\,db)$$

belongs to $]0,1]$. Note that the probability measure

$$\mu_0(dg) = \frac{1}{\rho(\mu)}\, a(g)^{s_0}\, e^{\langle t_0, b(g)\rangle}\,\mu(dg)$$

satisfies hypotheses G1 and G2. The following condition is the equivalent of Hypothesis G'3 in the non centered case:

Hypothesis G*3. *The measure μ is absolutely continuous with respect to the Haar measure m_D on G and its density ϕ_μ satisfies*

$$\int_{]0,1]\times\mathbb{R}^N} \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a,u,b)\,du}\; \frac{da\,db}{a^\gamma} < +\infty$$

for some $q \in]1, +\infty[$ *and* $\gamma \in]1 - s_0, +\infty[$.

Theorem 1.2. *Let* μ *be a probability measure on* G *satisfying conditions* G*1, G*2 *and* G3 (*or* G*3) *and let*

$$\rho(\mu) = \inf_{(s,t)\in\mathbb{R}\times\mathbb{R}^N} \int_G a^s\, e^{\langle t,b\rangle}\, \mu(da\, du\, db)\, .$$

Then, the sequence of finite measures

$$\Big\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n}\, \mu^{*n} \Big\}_{n\geq 1}$$

weakly converges to a non-degenerate Radon measure on G.

The demonstration of Theorem 2.1 is closely related to the study of the fluctuations of a random walk $(X_1^n, Y_1^n)_{n\geq 0}$ on $\mathbb{R}^{N+1}$. In Section 2, we first state the classical local limit theorem on $\mathbb{R}^{N+1}$ but we add in its statement uniform upperbounds relatively to the starting point of the random walk $(X_1^n, Y_1^n)_{n\geq 0}$. This result is thus very usefull to obtain a precise equivalent in Theorem 2.5 of the joint law of the random walk $(X_1^n, Y_1^n)_{n\geq 0}$ with its first entrance time T_+ in the half space $\mathbb{R}^+ \times \mathbb{R}^N$; a local limit theorem for the process

$$(X_1^n, \max\{0, X_1^1, \ldots, X_1^n\}, Y_1^n)_{n\geq 0}$$

is thus obtained (Theorem 2.6). In Section 3 we give the main steps of the proof of Theorem 1.1.

2. Fluctuations of a random walk on $\mathbb{R}^{N+1}$.

Fix an integer $N \geq 1$ and let $(X_1, Y_1), (X_2, Y_2), \ldots$ be independent $\mathbb{R}\times\mathbb{R}^N$-valued random variables with distribution p defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X_1^n, Y_1^n)_{n\geq 0}$ be the associated random walk on $\mathbb{R}\times\mathbb{R}^N$ starting from $(0,0)$ and defined by $X_1^0 = 0, Y_1^0 = 0$ and $X_1^n = X_1 + \cdots + X_n, Y_1^n = Y_1 + \cdots + Y_n$ for $n \geq 1$; the distribution of the couple (X_1^n, Y_1^n) is the n^{th} power of convolution p^{*n} of the measure p. Denote by $\mathcal{F}_n$ the σ-algebra generated by $(X_1, Y_1), \ldots, (X_n, Y_n), n \geq 1$.

Let us first recall the

Definition 2.1. *Let p be a probability measure on $\mathbb{R}^k$, $k \geq 1$. The measure p is aperiodic on $\mathbb{R}^k$ if there is no closed and proper subgroup H of $\mathbb{R}^k$ and no $\alpha \in \mathbb{R}^k$ such that $p(\alpha + H) = 1$.*

Denote by $\hat{p}$ the characteristic function of p defined by $\hat{p}(u, v) = \mathbb{E}[e^{iuX_1 + i\langle v, Y_1\rangle}]$ for any $(u, v) \in \mathbb{R} \times \mathbb{R}^N$. Recall that the probability measure p is aperiodic if and only if $|\hat{p}(u, v)| < 1$ for $(u, v) \neq (0, 0)$.

For any $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}^N$ let $\{T_{\mathcal{A}}^{(k)}\}_{k\geq 0}$ be the the successive times of visit of the random walk $(X_1^n, Y_1^n)_{n\geq 1}$ to the set $\mathcal{A}$; one has $T_{\mathcal{A}}^{(0)} = 0, T_{\mathcal{A}}^{(1)} = \inf\{n \geq 1 : (X_1^n, Y_1^n) \in \mathcal{A}\}$ and $T_{\mathcal{A}}^{(k+1)} = \inf\{n \geq T_{\mathcal{A}}^{(k)} + 1 : (X_1^n, Y_1^n) \in \mathcal{A}\}$. Note that the $T_{\mathcal{A}}^{(k)}$ are stopping times with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$. We will associate to $(p, \mathcal{A})$ the transition kernel $P_{\mathcal{A}}$ defined by

$$P_{\mathcal{A}}((x, y), \mathcal{B}) = \int_{\mathbb{R}\times\mathbb{R}^N} \mathbf{1}_{\mathcal{A}^c\cap\mathcal{B}}(x + x', y + y')\, p(dx'dy')\,,$$

for any Borel set $\mathcal{B}$ in $\mathbb{R} \times \mathbb{R}^N$; note that for any $k \geq 1$ one has $P_{\mathcal{A}}^k((0, 0), \mathcal{B}) = \mathbb{E}[[T_{\mathcal{A}} > k]; (X_1^k, Y_1^k) \in \mathcal{B}]$. In order to simplify the notations we will set $T_- = T_{\mathbb{R}^-\times\mathbb{R}^N}, P_- = P_{\mathbb{R}^-\times\mathbb{R}^N}$ and $T_-^{(k)} = T_{\mathbb{R}^-\times\mathbb{R}^N}^{(k)}$; similar notations will hold, with obvious modifications, when $\mathcal{A} = \mathbb{R}^{*-} \times \mathbb{R}^N, \mathbb{R}^+ \times \mathbb{R}^N$ and $\mathbb{R}^{*+} \times \mathbb{R}^N$.

Troughout this paragraph, for any $k \geq 1$, we denote by λ_k the Lebesgue measure on $\mathbb{R}^k$. Furthermore, for any $\delta > 0$, $\mathcal{H}_\delta(\mathbb{R}^k)$ is the space of $\mathbb{C}$-valued functions φ on $\mathbb{R}^k$ such that

$$\sup_{x\in\mathbb{R}^k} (1 + \|x\|^\delta)^k\, |\varphi(x)| < +\infty\,.$$

2.1. Preliminaries.

The local limit theorem gives the asymptotic behaviour of the sequence $\{p^{*n}(\varphi)\}_{n\geq 1}$ for any continuous function φ with compact support on $\mathbb{R}^{N+1}$; we state it here and we precise some uniform upperbound for the sequence $\{p^{*n}(\varphi)\}_{n\geq 1}$ when φ belongs to $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ with $\delta > 4$.

Theorem 2.2. *Assume that*:

i) *the common distribution p of the variables (X_n, Y_n), $n \geq 1$, is aperiodic on $\mathbb{R}^{N+1}$,*

ii) $\mathbb{E}[|X_1|^2 + \|Y_1\|^2] < +\infty$ *and* $\mathbb{E}[X_1] = 0$, $\mathbb{E}[Y_1] = 0$.

Then:

i) *for any continuous function* φ *with compact support on* $\mathbb{R}^{N+1}$ *one has*

$$\lim_{n\to+\infty} n^{(N+1)/2}\,\mathbb{E}[\varphi(X_1^n, Y_1^n)] = \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \int_{\mathbb{R}^{N+1}} \varphi(x,y)\,\lambda_1(dx)\,\lambda_N(dy)\,,$$

where $|C|$ *denotes the determinant of the positive definite quadratic form*

$$C(u,v) = \mathbb{E}[(u\,X_1 + \langle v, Y_1\rangle)^2]\,.$$

ii) *For any function* φ *in* $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ *with* $\delta > 4$, *the sequence* $\{n^{(N+1)/2}\,\mathbb{E}[\varphi(x + X_1^n, y + Y_1^n)]\}_{n\geq 1}$ *is bounded uniformly in* $(x,y) \in \mathbb{R}\times\mathbb{R}^N$.

PROOF. The first assumption is the classical local limit theorem. To obtain the second claim, fix a non negative function ϕ whose Fourier transform has a compact support $K(\hat{\phi})$. Recall that

$$\hat{p}(u,v) = 1 - \frac{1}{2}\,C(u,v)\,(1 + \varepsilon(u,v))$$

with $\lim_{(u,v)\to(0,0)} \varepsilon(u,v) = 0$; so there exists $\delta > 0$ such that for $|u| + \|v\| < \delta$ one has

$$|\hat{p}(u,v)| \leq 1 - \frac{1}{4}\,C(u,v) \leq e^{-C(u,v)/4}\,.$$

On the other hand, by the aperiodicity of p there exists $\rho = \rho(p, K(\hat{\phi}))$ such that $|\hat{p}(u,v)| \leq \rho$ as soon as (u,v) belongs to $K(\hat{\phi})$ and $|u| + \|v\| \geq \delta$. It follows that

$$\begin{aligned}
&(2\pi\,n)^{(N+1)/2}\,\mathbb{E}[\phi(X_1^n, Y_1^n)] \\
&\quad\leq n^{(N+1)/2} \int_{|u|+\|v\|<\delta} |\hat{\phi}(u,v)|\,|\hat{p}(u,v)|^n\,\lambda_1(du)\,\lambda_N(dv) \\
&\qquad+ n^{(N+1)/2} \int_{|u|+\|v\|\geq\delta} |\hat{\phi}(u,v)|\,|\hat{p}(u,v)|^n\,\lambda_1(du)\,\lambda_N(dv)
\end{aligned}$$

$$\le n^{(N+1)/2} \int_{|u|+\|v\|<\delta n^{(N+1)/2}} \Big|\hat\phi\Big(\frac{u}{\sqrt n}, \frac{v}{\sqrt n}\Big)\Big| e^{-(n/4)C(u/\sqrt n, v/\sqrt n)} \cdot \lambda_1(du)\,\lambda_N(dv)$$

$$+ n^{(N+1)/2}\,\rho^n\,\|\hat\phi\|_1$$

$$\le \|\hat\phi\|_\infty \int_{\mathbb{R}\times\mathbb{R}^N} e^{-C(u,v)/4}\lambda_1(du)\,\lambda_N(dv) + n^{(N+1)/2}\,\rho^n\,\|\hat\phi\|_1 \,.$$

Now set $\phi_{x,y}(x',y') = \phi(x+x', y+y')$ for any $(x,y)\in\mathbb{R}\times\mathbb{R}^N$ and note that $\hat\phi_{x,y}(u,v) = e^{iux+i\langle v,y\rangle}\hat\phi(u,v)$; the functions $\hat\phi_{x,y}$ and $\hat\phi$ thus have the same compact support and satisfies the equalities $\|\hat\phi_{x,y}\|_1 = \|\hat\phi\|_1$ and $\|\hat\phi_{x,y}\|_\infty = \|\hat\phi\|_\infty$. For any $(x,y)\in\mathbb{R}\times\mathbb{R}^N$ one thus has

$$|(2\pi\, n)^{(N+1)/2}\,\mathbb{E}[\phi_{x,y}(X_1^n, Y_1^n)]|$$

$$\le \|\hat\phi\|_\infty \int_{\mathbb{R}\times\mathbb{R}^N} e^{-C(u,v)/4}\,\lambda_1(du)\,\lambda_N(dv) + n^{(N+1)/2}\,\rho^n\,\|\hat\phi\|_1 \,.$$

The assertion ii) thus holds for any function ϕ whose Fourier transform has a compact support. To achieve the proof of ii) it suffices to show that for any function φ in $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ with $\delta > 4$ there exists a function ϕ whose Fourier transform has a compact support and $|\varphi| \le \phi$. It is an immediate consequence of the following result; we thank here J. P. Conze for helpfull discussions about this fact.

Lemma 2.3. *Set*

$$h_\varepsilon(x) = \frac{1}{1+|x|^{4+\varepsilon}}\ ,$$

for any $x\in\mathbb{R}$. If $\varepsilon > 0$ there exists a function $\overline{h_\varepsilon}$ greater than h_ε and whose Fourier transform has a compact support in $\mathbb{R}$.

Proof. Set

$$\overline{h_\varepsilon}(x) = C\Big(\frac{\sin^2 x}{x^2} + \frac{\sin^2 \alpha\, x}{x^2}\Big)$$

for some α and C in $\mathbb{R}^{*+}$ which will depend on ε. Assume $\alpha\notin\mathbb{Q}$, the function $\overline{h_\varepsilon}$ is strictly positive on $\mathbb{R}$; it thus suffices to show that there exists $\alpha\notin\mathbb{Q}$ such that

$$\lim_{x\to+\infty} x^{2+\varepsilon}(\sin^2 x + \sin^2(\alpha\, x)) = +\infty\,.$$

If such a real did not exist, then for any $\alpha \notin \mathbb{Q}$ there should exist a sequence $\{x_n\}_{n\geq 1}$ which tends to $+\infty$ and a constant $C_\varepsilon > 0$ such that for all $n \geq 1$,

$$\sin^2 x_n + \sin^2(\alpha\, x_n) \leq \frac{C}{x_n^{2+\varepsilon}}\,.$$

So there should exist two strictly increasing sequences of integers $\{k_n\}_{n\geq 1}$ and $\{l_n\}_{n\geq 1}$ such that

$$|x_n - k_n\,\pi| \leq \frac{C'}{x_n^{1+\varepsilon/2}}\,, \qquad |\alpha\, x_n - l_n\,\pi| \leq \frac{C'}{x_n^{1+\varepsilon/2}}$$

which implies

$$\left|\alpha - \frac{l_n}{k_n}\right| \leq \frac{C''}{k_n^{2+\varepsilon/2}}$$

for some positive constants C' and C''. This leads to a contradiction because for almost all $\alpha \in \mathbb{R}$ (with respect with the Lebesgue measure), this last inequality has at most a finite number of solutions in $\mathbb{N}^2$ [2]. The lemma is proved.

2.2. A local limit theorem for a killed random walk on a half space.

In [7], we proved the following

Theorem 2.4. *Let the hypotheses of Theorem 2.2 hold. Then for any continuous function with compact support φ on $\mathbb{R}^-$ we have*

$$\lim_{n\to+\infty} n^{3/2}\,\mathbb{E}\,[[T_+ > n]; \varphi(X_1^n)] = \frac{1}{\sigma(X_1)\,\sqrt{2\pi}} \int_{-\infty}^{0} \varphi(x)\,\lambda_1^- * U^{*-}(dx)\,,$$

where λ_1^- denotes the restriction of the Lebesgue measure on $\mathbb{R}^-$ and U^{*-} is the σ-finite measure on $\mathbb{R}^-$ defined by

$$U^{*-}(\mathcal{B}) = \sum_{k=1}^{+\infty} \mathbb{E}\,[\mathbf{1}_{\mathcal{B}}(X_1^{T_{*-}^{(k)}})]$$

for any Borel set $\mathcal{B}$. In the same way, one has

$$\lim_{n\to+\infty} n^{3/2}\,\mathbb{E}\,[[T_{*+} > n]; \varphi(X_1^n)] = \frac{1}{\sigma(X_1)\,\sqrt{2\pi}} \int_{-\infty}^{0} \varphi(x)\,\lambda_1^- * U^{-}(dx)\,,$$

where U^- is the σ-finite measure on $\mathbb{R}^-$ defined by

$$U^-(\mathcal{B}) = \sum_{k=1}^{+\infty} \mathbb{E}\,[\mathbf{1}_{\mathcal{B}}(X_1^{T_-^{(k)}})]$$

for any Borel set $\mathcal{B}$.

Recall that the random walks $\{X_1^{T_-^{(k)}}\}_{k\geq 1}$ and $\{X_1^{T_{*-}^{(k)}}\}_{k\geq 1}$ are transient on $\mathbb{R}^-$; it follows that the series $\sum_{k=0}^{+\infty} \mathbb{E}\,[[T_+ > k]; \varphi(x+X_1^k)]$ and $\sum_{k=0}^{+\infty} \mathbb{E}\,[[T_{*+} > k]; \varphi(x + X_1^k)]$ do converge. Furthermore one has

$$\sum_{k=0}^{+\infty} \mathbb{E}\,[[T_+ > k]; \varphi(x + X_1^k)] = \int_{-\infty}^0 \varphi(x)\, U^{*-}(dx)$$

and

$$\sum_{k=0}^{+\infty} \mathbb{E}\,[[T_{*+} > k]; \varphi(x + X_1^k)] = \int_{-\infty}^0 \varphi(x)\, U^{-}(dx)\,.$$

Let us now state the following

Theorem 2.5. *Let the hypotheses of Theorem* 2.2 *hold. Then*:

i) *For any continuous function φ with compact support on $\mathbb{R}^- \times \mathbb{R}^N$ one has*

$$\lim_{n\to+\infty} n^{(N+3)/2}\, \mathbb{E}\,[[T_+ > n]; \varphi(X_1^n, Y_1^n))] = \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \int_{\mathbb{R}^-\times\mathbb{R}^N} \varphi(x,y)\, \lambda_1^- * U^{*-}(dx)\, \lambda_N(dy)\,.$$

ii) *For any continuous function f with compact support on $\mathbb{R}$ and any g in $\mathcal{H}_\delta(\mathbb{R}^N)$ with $\delta > 4$, the sequence*

$$\{n^{(N+3)/2}\, \mathbb{E}\,[[T_+ > n]; f(X_1^n)\, g(y + Y_1^n)]\}_{n\geq 1}$$

is bounded, uniformly in $y \in \mathbb{R}^N$.

In the same way, one has

$$\lim_{n\to+\infty} n^{(N+3)/2}\, \mathbb{E}\,[[T_{*+} > n]; \varphi(X_1^n, Y_1^n)] = \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \int_{\mathbb{R}^-\times\mathbb{R}^N} \varphi(x,y)\, \lambda_1^- * U^{-}(dx)\, \lambda_N(dy)$$

and the sequence

$$\{n^{(N+3)/2}\,\mathbb{E}\,[[T_+ > n]; f(X_1^n)\,g(y+Y_1^n)]\}_{n\geq 1}$$

is bounded, uniformly in $y \in \mathbb{R}^N$.

PROOF. We prove this theorem by induction over N. Theorem 2.2 deals with the case $N=0$; we will suppose that this result hold for some $N \geq 0$ and we consider a sequence $(X_n, Y_n, Z_n)_{n\geq 1}$ of independent identically distributed random variables on $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$. By a classical argument in probability theory, it suffices to show the above convergence hold for $\varphi(x,y,z) = e^{ax}\,\mathbf{1}_{\mathbb{R}^-}(x)\,\phi(y)\,\psi(z)$ where $a \in \mathbb{R}^{*+}$ and ϕ, ψ are $\mathbb{C}$-valued functions whose Fourier transform are continuous with compact supports. By the inverse Fourier transform one has

$$\begin{aligned} I_n &= \mathbb{E}\,[[T_+ > n]; e^{aX_1^n}\,\phi(Y_1^n)\,\psi(Z_1^n)] \\ &= \frac{1}{(2\pi)^{(N+1)/2}} \int_{\mathbb{R}^N\times\mathbb{R}} \hat\phi(v)\,\hat\psi(w)\,\alpha_n(a,v,w)\,\lambda_N(dv)\,\lambda_1(dw) \end{aligned}$$

with $\alpha_n(a,v,w) = \mathbb{E}\,[[T_+ > n]; e^{aX_1^n + i\langle v, Y_1^n\rangle + iwZ_1^n}]$.

The Spitzer's factorisation for random walks on $\mathbb{R}$ gives for all $a>0$, for all $s\in[0,1[$

$$\sum_{n=0}^{+\infty} s^n\,\mathbb{E}\,[[T_+ > n]; e^{aX_1^n}] = \exp\Big(\sum_{n=1}^{+\infty}\frac{s^n}{n}\,\mathbb{E}\,[[X_1^n < 0]; e^{aX_1^n}]\Big).$$

Using the fact that $\mathbb{R}^+\times\mathbb{R}^{N+1}$ and $\mathbb{R}^{*-}\times\mathbb{R}^{N+1}$ are semi-groups in $\mathbb{R}^{N+2}$, Ch. Sunyach extended this factorisation to the multidimensionnal case ([9, Corollary 3, p. 553 and Theorem 5, p. 556]); for any $a>0$, $v\in\mathbb{R}^N$, $w\in\mathbb{R}$ and $s\in[0,1[$ one thus has

$$\begin{aligned} &\sum_{n=0}^{+\infty} s^n\,\mathbb{E}\,[[T_+ > n]; e^{aX_1^n + i\langle v, Y_1^n\rangle + iwZ_1^n}] \\ &\qquad\qquad = \exp\Big(\sum_{n=1}^{+\infty}\frac{s^n}{n}\,\mathbb{E}\,[[X_1^n < 0]; e^{aX_1^n + i\langle v, Y_1^n\rangle + iwZ_1^n}]\Big) \end{aligned}$$

that is

$$(n+1)\,\alpha_{n+1}(a,v,w) = \sum_{k=0}^{n} \beta_{n+1-k}(a,v,w)\,\alpha_k(a,v,w)$$

with $\beta_n(a,v,w) = \mathbb{E}[[X_1^n < 0]; e^{aX_1^n + i\langle v, Y_1^n\rangle + iwZ_1^n}]$. Finally

$$I_n = \frac{1}{n+1}\sum_{k=0}^{n} I_{n,k}$$

with

$$I_{n,k} = \frac{1}{(2\pi)^{(N+1)/2}}\int_{\mathbb{R}^N\times\mathbb{R}} \beta_{n+1-k}(a,v,w)\,\alpha_k(a,v,w)$$
$$\cdot\,\hat{\phi}(v)\,\hat{\psi}(w)\,\lambda_N(dv)\,\lambda_1(dw)\,.$$

Set

$$I = \frac{1}{(2\pi)^{(N+2)/2}\sqrt{|C|}}\int_{\mathbb{R}^N\times\mathbb{R}}\sum_{k=0}^{+\infty}\mathbb{E}\Big[[T_+ > k];\frac{e^{aX_1^k}}{a}\Big]$$
$$\cdot\,\phi(y)\,\psi(z)\lambda_N(dy)\,\lambda_1(dz)\,,$$

since

$$I = \lambda_1^- * U^{*-}(e^{a\cdot})\,\lambda_N(\phi)\,\lambda_1(\psi)\,,$$

it suffices to show that $\{n^{(N+4)/2}I_n\}_{n\geq 1}$ converges to I, that is

1) for all $k>0$, $\lim_{n\to+\infty} n^{(N+2)/2}I_{n,k} = I_{*k}$,

2) $\sum_{k=0}^{+\infty}|I_{*k}| < +\infty$ and $\sum_{k=0}^{+\infty} I_{*k} = I$,

3) $\limsup_{l\to+\infty}\limsup_{n\to+\infty} n^{(N+2)/2}\sum_{k=l}^{n}|I_{n,k}| = 0$.

To prove the assertion 1), note that

$$I_{n,k} = \mathbb{E}[[T_+ > k]\cap[X_{k+1}^{n+1} > 0]; e^{aX_1^{n+1}}\,\phi(Y_1^{n+1})\,\psi(Z_1^{n+1})]\,,$$

by the local limit theorem on $\mathbb{R}^{N+2}$ the assertion 1) follows with

$$I_{*k} = \frac{1}{2\pi^{(N+2)/2}\sqrt{|C|}}\,\frac{\mathbb{E}[[T_+ > k]; e^{aX_1^k}]}{a}$$
$$\cdot\int_{\mathbb{R}^N}\phi(y)\,\lambda_N(dy)\int_{\mathbb{R}}\psi(z)\,\lambda_1(dz)\,.$$

The fact that the series $\sum_{k=0}^{+\infty} |I_{*k}|$ converges is a direct consequence of Theorem 2.4. To prove the assertion 3), note that

$$\begin{aligned}
|I_{n,k}| &\leq \mathbb{E}\,[[T_+ > k] \cap [X_{k+1}^{n+1} < 0]; e^{aX_1^{n+1}}\, |\phi(Y_1^{n+1})|\, |\psi(Z_1^{n+1})|] \\
&\leq \mathbb{E}\Big[[T_+ > k]; e^{aX_1^k} \int_{\mathbb{R}^- \times \mathbb{R}^N \times \mathbb{R}} e^{ax}\, |\phi(y + Y_1^k)| \\
&\qquad\qquad \cdot |\psi(z + Z_1^k)|\, p^{*(n+1-k)}(dx\, dy\, dz)\Big] \\
&\leq \frac{C(a,\phi,\psi)}{(n+1-k)^{(N+2)/2}}\, \mathbb{E}\,[[T_+ > k]; e^{aX_1^k}] \qquad \text{by Theorem 2.2.ii)} \\
&\leq \frac{C_1}{(n+1-k)^{(N+2)/2}\, k^{3/2}} \qquad \text{by Theorem 2.4.}
\end{aligned}$$

On the other hand

$$\begin{aligned}
|I_{n,k}| &\leq \|\psi\|_\infty \int_{\mathbb{R}^- \times \mathbb{R}^N \times \mathbb{R}} \mathbb{E}[[T_+ > k]; e^{aX_1^k}\, |\psi(y + Y_1^k)| \\
&\qquad\qquad \cdot e^{ax}\, p^{*(n+1-k)}(dx\, dy\, dz)] \\
&\leq \frac{\|\psi\|_\infty\, C(a,\phi)}{k^{(N+3)/2}} \\
&\qquad \cdot \mathbb{E}\,[[X_{k+1}^{n+1} < 0]; e^{aX_{k+1}^{n+1}}] \qquad \text{by hypothesis of induction} \\
&\leq \frac{C_2}{k^{(N+3)/2}\sqrt{n+1-k}}\,.
\end{aligned}$$

The assertion 3) follows since for any $\varepsilon > 0$ one has

$$\begin{aligned}
n^{(N+2)/2} \sum_{k=l}^{n} |I_{n,k}| &\leq C_1 \sum_{k=l}^{[n(1-\varepsilon)]} \frac{n^{(N+2)/2}}{k^{3/2}\,(n+1-k)^{(N+2)/2}} \\
&\quad + C_2 \sum_{[n(1-\varepsilon)+1]}^{n} \frac{n^{(N+2)/2}}{k^{(N+3)/2}\sqrt{n+1-k}} \\
&\leq \frac{C_1}{\varepsilon^{(N+2)/2}} \sum_{k=l}^{[n(1-\varepsilon)]} \frac{1}{k^{3/2}} \\
&\quad + \frac{C_2}{\sqrt{n}\,(1-\varepsilon)^{(N+3)/2}} \sum_{[n(1-\varepsilon)+1]}^{n} \frac{1}{\sqrt{n+1-k}}
\end{aligned}$$

$$\leq C\Big(\frac{1}{\varepsilon^{(N+2)/2}\sqrt{l}}+\frac{\sqrt{\varepsilon}}{(1-\varepsilon)^{(N+3)/2}}\Big)\,.$$

Since ε is arbitrarily small, the assertion 3) follows.

The proof of ii) is also made by induction over N. If $g\in\mathcal{H}_\delta(\mathbb{R}^{N+1})$ there exist $\phi\in\mathcal{H}_\delta(\mathbb{R}^N)$ and $\psi\in\mathcal{H}_\delta(\mathbb{R}^1)$ such that $|g|\leq\phi\otimes\psi$. We set

$$I_n(y,z)=\mathbb{E}\,[[T_+>n];e^{aX_1^n}\,\phi(y+Y_1^n)\,\psi(z+Z_1^n)]$$

and we have

$$I_n(y,z)=\frac{1}{n+1}\sum_{k=0}^{n}I_{n,k}(y,z)$$

with

$$I_{n,k}(y,z)=\mathbb{E}\,[[T_+>k]\cap[X_{k+1}^{n+1}<0];e^{aX_1^{n+1}}\,\phi(y+Y_1^{n+1})\,\psi(z+Z_1^{n+1})]\,.$$

As above, one has

$$|I_{n,k}(y,z)|\leq\inf\Big\{\frac{C_1}{(n+1-k)^{(N+2)/2}\,k^{3/2}},\frac{C_2}{k^{(N+3)/2}\sqrt{n+1-k}}\Big\}$$

which proves that the sequence

$$\Big\{n^{(N+2)/2}\sum_{k=0}^{n}|I_{n,k}(y,z)|\Big\}_{n\geq1}$$

is uniformly bounded in y,z. This achieves the proof of ii).

The convergence of the sequence

$$\{n^{(N+3)/2}\mathbb{E}[[T_{*+}>n];\varphi(X_1^n,Y_1^n)]\}_{n\geq1}$$

is obtained with similar arguments.

2.3. Behaviour of the process $((X_1^n,\max\{0,X_1^1,\dots,X_1^n\},Y_1^n))_{n\geq0}$.

For any $n\geq0$ set $\mathcal{X}_1^n=\max\{0,X_1^1,\dots,X_1^n\}$ and let T_n be the random variable defined on $(\Omega,\mathcal{F},\mathbb{P})$ by $T_n=\inf\{0\leq k\leq n:\ \mathcal{X}_1^n=$

$X_1^k\}$; for any continuous function φ with compact support on $\mathbb{R}^{N+1}$ we have

$$\begin{aligned}
&\mathbb{E}\,[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)] \\
&= \sum_{k=0}^{n} \mathbb{E}\,[[T_n = k]; \varphi(X_1^k, -X_{k+1}^n, Y_1^n)] \\
&= \sum_{k=0}^{n} \mathbb{E}\,[[0 < X_1^k, X_1^1 < X_1^k, \dots, X_1^{k-1} < X_1^k, \\
&\qquad\qquad X_1^{k+1} \le X_1^k, \dots, X_1^n \le X_1^k]; \varphi(X_1^k, -X_{k+1}^n, Y_1^n)] \\
&= \sum_{k=0}^{n} \mathbb{E}\,[[X_1^1 > 0, \dots, X_1^k > 0] \cap [X_{k+1}^{k+1} \le 0, \dots, X_{k+1}^n \le 0]; \\
&\qquad\quad \varphi(X_1^k, -X_{k+1}^n, Y_1^n)]\,.
\end{aligned}$$

One obtains the following factorisation

$$\mathbb{E}\,[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)] = \sum_{k=0}^{n} J_{n,k}(\varphi)$$

with

$$\begin{aligned}
&J_{n,k}(\varphi) \\
&\quad = \int_{\mathbb{R}^{N+1}} \varphi(x, -x', y + y')\, P_-^k((0,0), dx\, dy)\, P_{*+}^{n-k}((0,0), dx'\, dy')\,.
\end{aligned}$$

The behaviour of the process $(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)$ is thus closely related to the one of the iterates of the transition kernels P_- and P_{*+}. Using this factorisation one proves the

Theorem 2.6. *Suppose that the hypotheses of Theorem 2.2 hold.*

Then, for any continuous function with compact support on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N$ the sequence

$$\{n^{(N+3)/2}\, \mathbb{E}\,[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)]\}_{n \ge 1}$$

converges to

$$\frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}}\int_{\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^N}\varphi(s,-t,y)\,U^{*+}(ds)\,\lambda_1^-*U^-(dt)\,\lambda_N(dy)$$
$$+\frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}}$$
$$\cdot\int_{\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^N}\varphi(s,-t,y)\,\lambda_+*U^{*+}(ds)\,U^-(dt)\,\lambda_N(dy)\,.$$

Furthermore, for any continuous function f with compact support on $\mathbb{R}^+\times\mathbb{R}^+$ and any g in $\mathcal{H}_\delta(\mathbb{R}^N)$, the sequence

$$\{n^{(N+3)/2}\,\mathbb{E}[f(\mathcal{X}_1^n,\mathcal{X}_1^n-X_1^n)\,g(y+Y_1^n)]\}_{n\geq1}$$

is bounded, uniformly in $y\in\mathbb{R}^N$.

Proof. We only proof the first assertion; the second one may obtained with obvious modifications as in Theorem 2.5. Set $\varphi(x,t,y)=\varphi_1(x)\,\varphi_2(t)\,\varphi_3(y)$ where φ_1,φ_2 and φ_3 are continuous with compact support. Fix $k\geq0$; by Theorem 2.5, the sequence

$$\Big\{n^{(N+3)/2}\int_{\mathbb{R}^-\times\mathbb{R}^N}\varphi_2(x')\,\varphi_3(y+y')\,P_{*+}^{n-k}((0,0),dx'\,dy')\Big\}_{n\geq1}$$

is bounded uniformly in $y\in\mathbb{R}^N$ and converges to

$$\frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}}\int_{-\infty}^0\varphi_2(-t)\,\lambda_1^-*U^-(dt)\,\lambda_N(\varphi_3)\,.$$

By the dominated convergence theorem, one thus obtains, for any fixed $i\geq1$

$$\lim_{n\to+\infty}n^{(N+3)/2}\sum_{k=0}^{i}J_{n,k}(\varphi)$$
$$=\frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}}\sum_{k=0}^{i}\mathbb{E}[[T_->k];\varphi_1(X_1^k)]$$
$$\cdot\int_{-\infty}^0\varphi_2(-t)\,\lambda_1^-*U^-(dt)\,\lambda_N(\varphi_3)\,.$$

In the same way one has

$$\lim_{n\to+\infty} n^{(N+3)/2} \sum_{k=n-i+1}^{n} J_{n,k}(\varphi)$$
$$= \frac{1}{(2\pi)^{(N+1)/2}\sqrt{|C|}} \sum_{k=0}^{i} \mathbb{E}\,[[T_{*+} > k]; \varphi_2(-X_1^k)]$$
$$\cdot\, \lambda_1^+ * U^{*+}(\varphi_1)\, \lambda_N(\varphi_3)\,.$$

Note that the sums $\sum_{k=0}^{i} \mathbb{E}\,[[T_- > k]; \varphi_1(X_1^k)]$ and $\sum_{k=0}^{i} \mathbb{E}\,[[T_{*+} > k]; \varphi_2(X_1^k)]$ converges respectively to $U^{*+}(\varphi_1)$ and $\int_{-\infty}^{0} \varphi_2(-t)\, U^-(dt)$. To obtain the theorem it suffices to check that

$$\limsup_{i\to+\infty} \limsup_{n\to+\infty} \Big| n^{(N+3)/2} \sum_{k=i+1}^{n-i} J_{n,k}(\varphi) \Big| = 0\,,$$

one has

$$\Big| n^{(N+3)/2} \sum_{k=i+1}^{[n/2]} J_{n,k}(\varphi) \Big|$$
$$\le n^{(N+3)/2} \sum_{k=i+1}^{[n/2]} \mathbb{E}\,[[T_- > k]; |\varphi_1(X_1^k)|]$$
$$\cdot \int_{\mathbb{R}^-\times\mathbb{R}^N} |\varphi_2(x')|\, |\varphi_3(y+y')|\, P_{*+}^{n-k}((0,0), dx'\, dy')$$
$$\le C(\varphi_2,\varphi_3) \sum_{k=i+1}^{[n/2]} \mathbb{E}\,[[T_- > k]; |\varphi_1(X_1^k)|]$$
$$\cdot\, \frac{n^{(N+3)/2}}{(n-k)^{(N+3)/2}} \qquad \text{by Theorem 2.5.ii)}$$
$$\le C(\varphi) \sum_{k=i+1}^{+\infty} \frac{1}{k^{3/2}}\,.$$

The same upperbound holds for the term

$$n^{(N+3)/2} \sum_{k=[n/2]+1}^{n-i} J_{n,k}(\varphi)\,.$$

This achieves the proof.

3. A local limit theorem for a particular class of solvable groups.

Recall that $G = G_{d,N} = \mathbb{R}^{*+} \times \mathbb{R}^d \times \mathbb{R}^N$ with the composition law

$$g \cdot g' = (a\,a', a\,u' + u, b + b')\,,$$

for all $g = (a, u, b)$, for all $g' = (a', u', b') \in G_{d,N}$.

The proof of Theorem 1.1 is closed to the one of the local limit theorem for the affine group of the real line given in [7]; we just give here the main steps of the demonstration.

Let us first introduce some helpfull notations. Let $g_n = (a_n, u_n, b_n)$, $n = 1, 2, \ldots$ be independent and identically distributed random variables with distribution μ. Denote by $\mathcal{F}_n$ the σ-algebra generated by the variables $g_1, g_2, \ldots, g_n$, $n \geq 1$. For any $n \geq 1$, set $G_1^n = g_1 \cdots g_n = (A_1^n, U_1^n, B_1^n)$; we have $A_1^n = a_1 \cdots a_n$, $U_1^n = \sum_{k=1}^n a_1 \cdots a_{k-1}\, u_k$ and $B_1^n = b_1 + \cdots + b_n$. More generally, if $1 \leq m \leq n$, set $A_m^n = a_m \cdots a_n$, $U_m^n = \sum_{k=m}^n a_m \cdots a_{k-1}\, u_k$, $B_m^n = b_m + \cdots + b_n$ and set $A_m^n = 1$, $U_m^n = 0$, $B_m^n = 0$ otherwise.

Let $\tilde{\mu}$ be the image of μ by the map

$$g = (a, u, b) \longmapsto \tilde{g} = \Big(\frac{1}{a}, \frac{u}{a}, b\Big)\,,$$

if $\tilde{g}_n = (\tilde{a}_n, \tilde{u}_n, \tilde{b}_n)$, $n = 1, 2, \ldots$ are independent and identically distributed random variables with distribution $\tilde{\mu}$ on G, set $\tilde{G}_m^n = \tilde{g}_m \cdots \tilde{g}_n = (\tilde{A}_m^n, \tilde{U}_m^n, \tilde{B}_m^n)$.

In order to obtain the asymptotic behaviour of the power of convolution μ^{*n} we use the fact that the sequence $\{U_1^n\}_{n\geq 1}$ behaves like the maximum of the variables $A_1^1, \ldots, A_1^n$. These idea was already used in [7]. Set $\mathcal{A} = \{g = (a, u, b) \in G : \; a > 1\}$ and consider the transition kernel $P_{\mathcal{A}}$ associated with $(\mu, \mathcal{A})$ and defined by

$$P_{\mathcal{A}}(g, \mathcal{B}) = \int_G \mathbf{1}_{\mathcal{A}^c \cap \mathcal{B}}(g\,h)\, \mu(dh)$$

for any Borel set $\mathcal{B} \subset G$ and any $g \in G$. The probabilistic interpretation of $P_{\mathcal{A}}$ is the following one: if $T_{\mathcal{A}} = \inf\{n \geq 1 : \; G_1^n \in \mathcal{A}\}$ is the first entrance time in $\mathcal{A}$ of the random walk $\{G_1^n\}_{n\geq 0}$ then

$$P_{\mathcal{A}}^n(e, \mathcal{B}) = \mathbb{P}\left[[T_{\mathcal{A}} > n] \cap [G_1^n \in \mathcal{B}]\right], \qquad \text{for all } n \geq 1\,.$$

In the same way, set $\mathcal{A}' = \{g \in G : \ a(g) \geq 1\}$, let $\tilde{P}_{\mathcal{A}'}$ be the operator associated with $(\tilde{\mu}, \mathcal{A}')$ and denote by $\tilde{T}_{\mathcal{A}'}$ the first entrance time in $\mathcal{A}'$ of the random walk $\{\tilde{G}_1^n\}_{n\geq 1}$; one has

$$\tilde{P}^n_{\mathcal{A}'}(e, \mathcal{B}) = \mathbb{P}\left[[\tilde{T}_{\mathcal{A}'} > n] \cap [\tilde{G}_1^n \in \mathcal{B}]\right], \qquad \text{for all } n \geq 1\,.$$

As in Section 2.3, we introduce the first time at which the random walk $\{A_1^n\}_{n\geq 1}$ reaches its maximun on $\mathbb{R}^{*+}$; for any continuous function φ with compact support on G, we thus obtain

$$\mathbb{E}\left[\varphi(G_1^n)\right] = \sum_{k=0}^{n} I_{n,k}(\varphi)\,,$$

where

$$I_{n,k}(\varphi) = \int_{G\times G} \varphi\Big(\frac{a'}{a}, \frac{u+u'}{a}, b+b'\Big)\, \tilde{P}^k_{\mathcal{A}'}(e, da\, du\, db)\, P^{n-k}_{\mathcal{A}}(e, da'\, du'\, db')\,.$$

We now give the main steps of the proof of Theorem 1.1 under hypothesis G1, G2 and G3.

First step. *Control of the central terms of the sum* $\sum_{k=0}^n I_{n,k}(\varphi)$.

We show here that

$$\limsup_{i\to+\infty}\, \limsup_{n\to+\infty} \sum_{k=i}^{n-i} I_{n,k}(\varphi) = 0\,.$$

Without loss of generality, one may suppose that the support of φ is included in $\mathbb{R}^{*+} \times (\mathbb{R}^{*+})^d \times \mathbb{R}^N$; for any $\varepsilon > 0$ there exist a constant $C > 0$ and a positive function ϕ with compact support on $\mathbb{R}^N$ such that

$$\varphi(a, u, b) \leq C\, \frac{a^\varepsilon}{\|u\|^{2\varepsilon}}\, \phi(b)\,,$$

it follows that for any (α, β) in $\mathbb{R}^{*+} \times \mathbb{R}^N$

$$\begin{aligned}
&\mathbb{E}\Big[[T_{\mathcal{A}} > l]; \varphi\Big(\frac{A_1^l}{\alpha}, \frac{u + U_1^l}{\alpha}, \beta + B_1^l\Big)\Big] \\
&\leq C\,\alpha^\varepsilon\, \mathbb{E}\Big[[a_1 \leq 1] \cap \Big[\max\{A_2^2, \ldots, A_2^l\} \leq \frac{1}{a_1}\Big]; \frac{(A_1^l)^\varepsilon}{\|u + U_1^l\|^{2\varepsilon}}\, \phi(\beta + B_1^l)\Big] \\
&\leq C\,\alpha^\varepsilon \int_G \mathbb{E}\Big[\frac{(A_2^l)^\varepsilon}{\max\{A_2^2, \ldots, A_2^l\}^{2\varepsilon}}\, \phi(\beta + b + B_2^l)\Big]\, \frac{\mu(da\, dv\, db)}{a^\varepsilon\, \|v\|^{2\varepsilon}}\,,
\end{aligned}$$

the last inequality being a consequence of the fact that $\|u+U_1^l\| \geq \|u_1\|$ $\mathbb{P}$-almost surely and

$$\mathbf{1}_{\{\max\{A_2^2,\ldots,A_2^l\}\leq 1/a_1\}} \leq \frac{1}{a_1^{2\varepsilon}\max\{A_2^2,\ldots,A_2^l\}^{2\varepsilon}}\,.$$

By Theorem 2.6 one obtains

$$l^{(N+3)/2}\,\mathbb{E}\Big[[T_{\mathcal{A}}>l];\varphi\Big(\frac{A_1^l}{\alpha},\frac{u+U_1^l}{\alpha},\beta+B_1^l\Big)\Big] \leq C_1(\varphi)\,\alpha^{\varepsilon}\,.$$

The same upperbound holds under hypotheses G1, G2 and G'3 (see [7, Lemma 3.1]).

It readily follows that

$$\begin{aligned}
n^{(N+3)/2}\sum_{k=i}^{[n/2]} I_{n,k}(\varphi) &\leq 2^{(N+3)/2}\sum_{k=i}^{[n/2]}(n-k)^{(N+3)/2} I_{n,k}(\varphi)\\
&\leq C_1(\varphi)\sum_{k=i}^{[n/2]}\mathbb{E}\,[[\tilde{T}_{\mathcal{A}'}>k];(\tilde{A}_1^k)^{\varepsilon}]\\
&\leq C_2(\varphi)\sum_{k=i}^{[n/2]}\frac{1}{k^{3/2}}
\end{aligned}$$

and so

$$\limsup_{i\to+\infty}\limsup_{n\to+\infty} n^{(N+3)/2}\sum_{k=i}^{[n/2]} I_{n,k}(\varphi)=0\,.$$

The control of the sum $\sum_{k=[n/2]}^{n-i} I_{n,k}(\varphi)$ goes along the same lines.

Second step. *Convergence of the sequence*

$$l^{(N+3)/2}\,\mathbb{E}\Big[[T_{\mathcal{A}}>l];\varphi\Big(\frac{A_1^l}{\alpha},\frac{u+U_1^l}{\alpha},\beta+B_1^l\Big)\Big]$$

for any $(\alpha,u,\beta)\in\,]0,1]\times(\mathbb{R}^{*+})^d\times\mathbb{R}^N$.

It is the more technical part of the proof and it uses and idea due to Afanasev [1]. Without loss of generality, one may suppose $\alpha=1$, $u=0$ and $\beta=0$. For any $n\geq 1$, set

$$\mathbb{E}_n(\varphi)=n^{(N+3)/2}\,\mathbb{E}\,[[T_A>n];\varphi(A_1^n,U_1^n,B_1^n)]\,.$$

Fix $i \in \mathbb{N}$ such that $1 \leq i \leq n/2$ and consider

$$\mathbb{E}_n(\varphi, i) = n^{(N+3)/2}\, \mathbb{E}\,[[T_A > n]; \varphi(A_1^n, U_1^i + A_1^{n-i} U_{n-i+1}^n, B_1^n)]\,.$$

To obtain the claim, it suffices to prove that

a) $\limsup_{i\to+\infty} \limsup_{n\to+\infty} |\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| = 0$,

b) for any fixed $n \in \mathbb{N}$, the sequence $\{\mathbb{E}_n(\varphi, i)\}_{n\geq 1}$ converges to a finite limit.

PROOF OF CONVERGENCE a). We use the equality

$$U_1^n = U_1^i + A_1^i\, U_{i+1}^{n-i} + A_1^{n-i}\, U_{n-i+1}^n\ ,$$

without loss of generality one may suppose that φ is continuously differentiable, and so, for any $\varepsilon > 0$ there exists $C > 0$ and a positive function ϕ with compact support on $\mathbb{R}^N$ such that

$$|\varphi(a, u, b) - \varphi(a, v, b)| \leq C\, a^\varepsilon \|u - v\|^\varepsilon \phi(b)\,,$$

consequently

$$\begin{aligned}
&|\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| \\
&\qquad \leq C\, n^{(N+3)/2}\, \mathbb{E}\,[[T_A > n]; (A_1^n)^\varepsilon\, (A_1^i)^\varepsilon\, \|U_{i+1}^{n-i}\|^\varepsilon\, \phi(B_1^n)] \\
&\qquad \leq C\, n^{(N+3)/2} \sum_{k=i+1}^{n-i} \mathbb{E}\,[[T_A > n]; (A_1^n)^\varepsilon\, (A_1^{k-1})^\varepsilon\, \|u_k\|^\varepsilon\, \phi(B_1^n)]\,.
\end{aligned}$$

Note that for $i \leq k \leq [n/2]$ one has

$$\begin{aligned}
&\mathbb{E}\,[[T_A > n]; (A_1^n)^\varepsilon\, (A_1^{k-1})^\varepsilon\, \|u_k\|^\varepsilon\, \phi(B_1^n)] \\
&\leq \mathbb{E}\,\Big[[T_A > k-1] \cap \Big[\max\{A_{k+1}^{k+1}, \ldots, A_{k+1}^n\} \leq \frac{1}{A_1^k}\Big]; \\
&\qquad (A_1^n)^\varepsilon\, (A_1^{k-1})^\varepsilon\, \|u_k\|^\varepsilon\, \phi(B_1^n)\Big] \\
&\leq \mathbb{E}\,[[T_A > k-1]; (A_1^{k-1})^{\varepsilon/2}\, a_k^{-\varepsilon/2}\, \|u_k\|^\varepsilon \\
&\qquad\qquad \cdot \max\{A_{k+1}^{k+1}, \ldots, A_{k+1}^n\}^{-3\varepsilon/2}\, (A_{k+1}^n)^\varepsilon\, \phi(B_1^n)]\,.
\end{aligned}$$

By Theorem 2.6,

$$(n-k)^{(N+3)/2}\, \mathbb{E}\,[\max\{A_{k+1}^{k+1}, \ldots, A_{k+1}^n\}^{-3\varepsilon/2}\, (A_{k+1}^n)^\varepsilon\, \phi(\beta + B_{k+1}^n)]$$

is bounded, uniformly in $\beta \in \mathbb{R}^N$ and so

$$(n-k)^{(N+3)/2}\, E\,[[T_A > n]; (A_1^n)^\varepsilon\, (A_1^{k-1})^\varepsilon\, \|u_k\|^\varepsilon\, \phi(B_1^n)] \le \frac{C_1}{k^{3/2}}\,.$$

When $[n/2] \le k \le n-i$ one obtains by a similar argument

$$k^{(N+3)/2}\, \mathbb{E}\,[[T_A > n]; (A_1^n)^\varepsilon\, (A_1^{k-1})^\varepsilon\, \|u_k\|^\varepsilon\, \phi(B_1^n)] \le \frac{C_2}{(n-k)^{3/2}}\,.$$

Finally one has

$$|\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| \le C_3\, \frac{1}{\sqrt{i}}\,,$$

convergence a) follows.

Proof of convergence b). Fix an integer i; we have

$$\begin{aligned}
&\mathbb{E}_n(\varphi, i)\\
&\qquad = \int_G E_n(\varphi, g, h_1, h_2, \dots, h_i)\, P_{\mathcal{A}}^i(e, dg)\, \mu(dh_1)\, \mu(dh_2) \cdots \mu(dh_i)
\end{aligned}$$

with

$$\begin{aligned}
&E_n(\varphi, g, h_1, h_2, \dots, h_i)\\
&= \mathbb{E}\Big[\Big[\max\{A_{i+1}^{i+1}, \dots, A_{i+1}^{n-i}\} \le \frac{1}{a(g)}\Big]\\
&\qquad \cap \Big[A_{i+1}^{n-i} \le \min\Big\{\frac{1}{a(g)}, \frac{1}{a(g)a(h_1)}, \dots, \frac{1}{a(g)a(h_1)\cdots a(h_i)}\Big\}\Big];\\
&\qquad \varphi(a(g)\, A_{i+1}^{n-i}\, a(h_1)\cdots a(h_i), u(g) + a(g)\, A_{i+1}^{n-i}\, u(h_1\cdots h_i),\\
&\qquad B_1^{n-i} + b(h_1) + \cdots + b(h_i)\big]\,.
\end{aligned}$$

Using Theorem 2.6, one may see that, for any $g, h_1, \dots, h_i \in G$, the sequence

$$\{n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \cdots, h_i)\}_{n\ge 1}$$

converges to a finite limit. To obtain the convergence b), we have to use Lebesgue dominated convergence theorem and therefore, we have to obtain an appropriate upperbound for $n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \dots, h_i)$.

Using the fact that for any $\varepsilon > 0$ there exist $C > 0$ and a positive continuous function ϕ with compact support on $\mathbb{R}^N$ such that $|\varphi(a,u,b)| \leq C\, a^{\varepsilon}\phi(b)$, one thus obtains

$$n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \dots, h_i) \leq C_1\, a(g)^{-3\varepsilon/2}\, a(h_1)^{\varepsilon} \cdots a(h_i)^{\varepsilon}$$

which allows us to use the Lebesgue dominated convergence theorem for ε small enough; convergence b) follows.

Consequently, $\{n^{(N+3)/2}\, I_{n,0}(\varphi)\}_{n\geq 1}$ converges to a finite limit; furthermore, for any $i \geq 1$ and any compact set $K \subset \mathbb{R}^{*+} \times \mathbb{R}^N$, the dominated convergence theorem ensures the existence of a finite limit as n goes to $+\infty$ for

$$\Big\{ n^{(N+3)/2} \sum_{k=0}^{i} I_{n,k}(\varphi, K) \Big\}_{n\geq 1},$$

where

$$\begin{aligned} I_{n,k}(\varphi, K) = \int_G & \mathbf{1}_K(g) \\ & \cdot \Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}, \frac{u(g)+u(h)}{a(g)}, b(g)+b(h) \Big) P_{\mathcal{A}}^{n-k}(e, dh) \Big) \\ & \cdot \tilde{P}_{\mathcal{A}'}^{k}(e, dg)\,. \end{aligned}$$

The following step shows that the indicator function $\mathbf{1}_K$ does not disturb too much the behaviour of theses integrals.

Third step. *Control of the residual terms.*

In the first step of the present proof, we have shown that, for any $\varepsilon > 0$ there exists $C_1 > 0$ such that

$$\begin{aligned} (n-k)^{(N+3)/2}\, \mathbb{E}\Big[[T_{\mathcal{A}} > n-k]; \varphi\Big(\frac{A_1^{n-k}}{\alpha}, \frac{u + U_1^{n-k}}{\alpha}, \beta + B_1^{n-k} \Big) \Big] & \\ \leq C_1(\varphi)\, \alpha^{\varepsilon}\,. & \end{aligned}$$

It follows that for any $0 < \delta < 1$

$$\sum_{k=1}^{i} \int_{\{g \in G : a(g) \leq \delta\}} \Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}, \frac{u(g)+u(h)}{a(g)}, b(g)+b(h) \Big) P_{\mathcal{A}}^{n-k}(e, dh) \Big)$$

$$
\begin{aligned}
&\cdot \tilde{P}^k_{\mathcal{A}'}(e,dg) \\
&\qquad \le C_1 \sum_{k=1}^{i} \frac{1}{(n-k)^{(N+3)/2}}\, \mathbb{E}\,[[\tilde{T}_{\mathcal{A}'} > k]; (\tilde{A}_1^k)^{\varepsilon}] \\
&\qquad \le C_1 \sum_{k=1}^{i} \frac{1}{(n-k)^{(N+3)/2}\, k^{3/2}}\ .
\end{aligned}
$$

On the other hand for any fixed $U > 0$, one has

$$
\begin{aligned}
\sum_{k=1}^{i} \int_{\{g\in G:\|u(g)\|\ge U\}} & \Big(\int_G \varphi\Big(\frac{a(h)}{a(g)}, \frac{u(g)+u(h)}{a(g)}, b(h)+b(g)\Big) P^{n-k}_{\mathcal{A}}(e,dh)\Big) \\
&\cdot \tilde{P}^k_{\mathcal{A}'}(e,dg) \\
&\le \frac{C_1}{U^{\varepsilon/2}} \sum_{k=1}^{i} \frac{1}{(n-k)^{(N+3)/2}}\, E\,[[\tilde{T}_{\mathcal{A}'} > k]; (\tilde{A}_1^k)^{\varepsilon}\, \|\tilde{U}_k\|^{\varepsilon/2}] \\
&\le \frac{C_1}{U^{\varepsilon/2}} \sum_{k=1}^{i} \frac{1}{(n-k)^{(N+3)/2}\, k^{3/2}}\ .
\end{aligned}
$$

The last inequality being guaranteed by standart estimations. (see [7, Lemma 3.3] for more details).

References.

[1] Afanas'ev, V. I., On a maximum of a transient random walk in random environment. *Theory Probab. Appl.* **35** (1987), 205-215.

[2] Billingsley, P., *Ergodic theory and information.* Wiley Series in Probability and Mathematical statistics, 1964.

[3] Bougerol, Ph., Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré* **19** (1983), 369-391.

[4] Breiman, L., *Probability.* Addison-Wesley Publishing Company, 1964.

[5] Grincevicius, A. K., A central limit theorem for the group of linear transformation of the real axis. *Soviet Math. Doklady* **15** (1974), 1512-1515.

[6] Iglehart, D. L., Random walks with negative drift conditioned to stay positive. *J. Appl. Probab.* **11** (1974), 742-751.

[7] Le Page, E., Peigné, M., A local limit theorem on the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$. *Ann. Inst. H. Poincaré* **2** (1997), 223-252.

[8] Spitzer, F., *Principles of random walks.* D. Van Nostrand Company, 1964.

[9] Sunyach, Ch., Sur les fluctuations des marches aléatoires sur un groupe. *Lecture Notes in Math.* **1096** (1984), 549-558.

[10] Varopoulos, N. Th., Wiener-Hopf theory and nonunimodular groups. *J. Funct. Anal.* **120** (1994), 467-483.

[11] Varopoulos, N. Th., Analysis on Lie groups. *Revista Mat. Iberoamaricana* **12** (1996), 791-917.

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Harnack inequalities on a manifold with positive or negative Ricci curvature

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Summary. Several new Harnack estimates for positive solutions of the heat equation on a complete Riemannian manifold with Ricci curvature bounded below by a positive (or a negative) constant are established. These estimates are sharp both for small time, for large time and for large distance, and lead to new estimates for the heat kernel of a manifold with Ricci curvature bounded below.

1. Introduction and main results.

The main purpose of this paper is to present several new Harnack estimates for non-negative solutions of the heat equation on a complete manifold with Ricci curvature bounded below by a constant which may be positive or negative. To obtain Harnack inequalities, we first deduce gradient estimates, that is upper bounds of the gradient of the logarithm of a solution of the heat equation by a concave function of the time and the time derivative of the same quantity. Then, by standard methods, these bounds lead to Harnack inequalities and then to bounds on the heat kernel.

In this context, we obtain quite strong Harnack inequalities, which are improvements of the famous Li-Yau's estimate in [6], [12]. Although our methods are similar in both cases of positive and negative lower bound on the Ricci curvature, our results are completely new in the positive case, and are improvements of previous results in the negative one.

In order to state our results, we first introduce some basic notations: let M be a complete Riemannian manifold of dimension n, and let Δ be the Laplace-Beltrami operator. Let u be a positive solution of the heat equation

$$(\Delta - \partial_t)\, u = 0\,, \qquad \text{on } [0,\infty) \times M \tag{1}$$

and let $f = \log u$. Denote by ∇f the gradient of the function f and by f_t the time derivative of f.

In 1975, Yau [10] proved a Harnack inequality via Ricci curvature bounds for harmonic functions on a complete manifold. In their paper [6], Li and Yau have established a sharp Harnack inequality for parabolic harmonic functions on a complete manifold with non-negative Ricci curvature. Namely,

$$|\nabla f|^2 - f_t \le \frac{n}{2\,t}\,, \qquad \text{for all } t > 0\,. \tag{2}$$

They also proved the following gradient estimate for a manifold with Ricci curvature bounded below by $-K$; $K \ge 0$

$$|\nabla f|^2 - \alpha\, f_t \le \frac{n\,\alpha^2}{2\,t} + \frac{n\,\alpha^2 K}{2\,(\alpha - 1)}\,, \qquad \text{for all } t > 0\,,\ \alpha > 1\,. \tag{3}$$

In his book [6], Davies improved the previous inequality under the same assumption to the following one

$$|\nabla f|^2 - \alpha\, f_t \le \frac{n\,\alpha^2}{2\,t} + \frac{n\,\alpha^2 K}{4\,(\alpha - 1)}\,, \qquad \text{for all } t > 0,\ \alpha > 1\,. \tag{4}$$

Recently Yau [12] (also see Yau [11]) further established, among other things, the following gradient estimate: if $\mathrm{Ric} \ge -K$; $K \ge 0$, then

$$|\nabla f|^2 - f_t \le \sqrt{2\,n\,K}\,\sqrt{|\nabla f|^2 + \frac{n}{2\,t} + 2\,n\,K} + \frac{n}{2\,t}\,, \quad \text{for all } t > 0\,. \tag{5}$$

With the method described in Section 4, it is standard to deduce from this a Harnack inequality close to (12) (see below), but with different constants. In Appendix A, with the method described in this paper, and under the same assumption, we will improve this inequality to

$$(6)\quad |\nabla f|^2 - f_t \le \sqrt{n\,K}\,\sqrt{|\nabla f|^2 + \frac{n}{2\,t} + \frac{n\,K}{4}} + \frac{n}{2\,t}\,, \qquad \text{for all } t > 0\,,$$

which yields a Harnack inequality essentially similar to (12) for small time and large distance.

Let us also mention that Hamilton [8] has obtained a Harnack inequality for negative curvature manifolds.

The path to obtain Harnack inequalities is to first establish gradient estimates as (4) or (5). To begin with, let us state the main results of this paper. To this end, we first introduce two functions X and $\widetilde{X}$ as follows: let $K \ge 0$, $n > 0$ be two constants. Then the functions X and $\widetilde{X}$ are defined on $(0,\infty)\times\mathbb{R}$ by

$$(7)\qquad X(t,Y) = \begin{cases} -\dfrac{n\,K}{2} + Y \\ +\sqrt{n\,K}\,\sqrt{\dfrac{n\,K}{4} - Y} \\ \quad\cdot\text{cotanh}\,\dfrac{2\,t}{n}\,\sqrt{n\,K}\,\sqrt{\dfrac{n\,K}{4} - Y}\,, & Y \le \dfrac{n\,K}{4}\,, \\ -\dfrac{n\,K}{2} + Y \\ +\sqrt{n\,K}\,\sqrt{Y - \dfrac{n\,K}{4}} \\ \quad\cdot\text{cotan}\,\dfrac{2\,t}{n}\sqrt{n\,K}\,\sqrt{Y - \dfrac{n\,K}{4}}\,, & Y > \dfrac{n\,K}{4}\,, \end{cases}$$

$$(8)\qquad \widetilde{X}(t,Y) = \begin{cases} \dfrac{n\,K}{2} + Y \\ +\sqrt{n\,K}\,\sqrt{Y + \dfrac{n\,K}{4}} \\ \quad\cdot\text{cotanh}\,\dfrac{2\,t}{n}\,\sqrt{n\,K}\,\sqrt{Y + \dfrac{n\,K}{4}}\,, & Y \ge -\dfrac{n\,K}{4}\,, \\ \dfrac{n\,K}{2} + Y \\ +\sqrt{n\,K}\,\sqrt{-Y - \dfrac{n\,K}{4}} \\ \quad\cdot\text{cotan}\,\dfrac{2\,t}{n}\sqrt{n\,K}\,\sqrt{-Y - \dfrac{n\,K}{4}}\,, & Y < -\dfrac{n\,K}{4}\,, \end{cases}$$

respectively. Indeed, as we will see later, X and $\widetilde{X}$ are solutions of some simple differential equations. The key inequalities obtained in this paper are the following gradient estimates

Theorem 1. *Suppose the Ricci curvature is bounded below by a constant $-K$; $K \geq 0$, then we have:*

1) $|\nabla f|^2 \leq \widetilde{X}(t, f_t)$, *on* $f_t \geq -n\,K/4$.

2) *For any* $Y_0 \geq -n\,K/4$, *we have*

$$|\nabla f|^2 \leq \partial_Y \widetilde{X}(t, Y_0)\,(f_t - Y_0) + \widetilde{X}(t, Y_0)\,, \qquad \textit{for all } t > 0\,.$$

3) *There is a universal constant $c > 0$, such that*

$$|\nabla f|^2 \leq \widetilde{X}(t, f_t)\,, \qquad \textit{for all } 0 < t \leq \frac{c}{K}\,.$$

See Theorem 4 below for the precise value of the constant c.

As a consequence, if $\mathrm{Ric} \geq -K$; $K \geq 0$, then

$$-f_t \leq \frac{n}{2\,t} + \frac{n\,K}{4}\,, \qquad \text{for all } t > 0\,,$$

which is very close to the best possible one could expect, since $(n-1)\,K/4$ is the spectral gap of the space form with Ricci curvature $-K$. Indeed we will prove a better but slightly more complicated estimate than (9).

Then, by standard methods, we deduce from Theorem 1 the following Harnack inequality

$$\text{(10)} \quad \frac{u(t,x)}{u(t+s,y)} \leq \exp\Big(\frac{\rho^2}{4\,s} + \int_t^{t+s} \Big(C\Big(\sigma, \frac{\rho^2}{4\,s^2}\Big) - A\Big(\sigma, \frac{\rho^2}{4\,s^2}\Big)\Big)\,d\sigma\Big)\,,$$

for all $t \geq 0, s > 0$, $x, y \in M$, where $\rho = d(x, y)$ is the geodesic distance from x to y, and

$$\begin{aligned} C(t,Y) &= \widetilde{X}(t,Y) - Y \\ &= \frac{n\,K}{2} + \sqrt{n\,K}\,\sqrt{Y + \frac{n\,K}{4}}\,\operatorname{cotanh}\frac{2\,t}{n}\,\sqrt{n\,K}\,\sqrt{Y + \frac{n\,K}{4}}\,, \\ A(t,Y) &= \frac{\partial_Y C(t,Y)\,(C(t,Y) - Y\partial_Y C(t,Y))}{1 + \partial_Y C(t,Y)}\,. \end{aligned}$$

In particular, since $A(t,Y)>0$ when $Y\geq 0$, we have

$$\frac{u(t,x)}{u(t+s,y)} \leq \left(\frac{\sinh\left(\frac{2}{n}\,(t+s)\sqrt{n\,K}\;\sqrt{\frac{\rho^2}{4\,s^2}+\frac{n\,K}{4}}\;\right)}{\sinh\left(\frac{2}{n}\,t\,\sqrt{n\,K}\;\sqrt{\frac{\rho^2}{4\,s^2}+\frac{n\,K}{4}}\;\right)} \right)^{n/2} \cdot \exp\left(\frac{\rho^2}{4\,s}+\frac{n\,K}{2}\,s\right). \tag{11}$$

By an elementary computation, inequality (10) yields the following

$$\frac{u(t,x)}{u(t+s,y)} \leq \left(\frac{t+s}{t}\right)^{n/2} \cdot \exp\left(\frac{(\rho+\sqrt{K\,n}\;s)^2}{4\,s}+\frac{\sqrt{n\,K}}{4}\,\min\{\rho,\sqrt{n\,K}\;s\}\right), \tag{12}$$

for all $s>0$, $x,y\in M$. We will give an independent proof of this in Section 4. We have been informed by Professor S. T. Yau that he already obtained a Harnack inequality in this context, see [15], [14].

As usual, Harnack inequalities lead to lower bounds of the heat kernel. Let $H(t,x,y)$ be the heat kernel: the fundamental solution of the heat equation (1). Then (12) implies that

$$H(t,x,y) \geq \frac{1}{(4\pi\,t)^{n/2}} \cdot \exp\left(-\,\frac{(\rho+\sqrt{K\,n}\;t)^2}{4\,t}-\frac{\sqrt{n\,K}}{4}\,\min\{\rho,\sqrt{n\,K}\;t\}\right), \tag{13}$$

for all $(t,x,y)\in(0,\infty)\times M\times M$, $\rho=d(x,y)$. See [3], [8] for a comparison theorem for heat kernels.

Notice that the leading term in (13) for small time is

$$\frac{1}{(4\pi\,t)^{n/2}}\exp\left(-\,\frac{\rho^2}{4\,t}\right),$$

for large time is

$$\exp\left(-\,\frac{n\,K}{4}\,t\right),$$

for large distance is

$$\exp\left(-\,\frac{\rho^2}{4\,t}-\frac{\sqrt{n\,K}}{2}\,\rho\right).$$

They are all very close to those best possibles we could expect.

For positive Ricci curvature manifolds, we will prove the following gradient estimate

Theorem 2. *Let* $\mathrm{Ric} \geq K; K \geq 0$. *Then*

1) $|\nabla f|^2 \leq X(t, f_t)$, *on* $f_t \leq n\,K/4$.

2) *If* $t \geq 2/K$, *then* $|\nabla f|^2 \leq X(t, f_t)$ *and* $-n/(2\,t) \leq f_t \leq n\,K/4$.

3) *If* $t \leq 2/K$, *then*

$$|\nabla f|^2 \leq X(t, f_t)\,, \qquad \text{on } f_t \leq Y_0\,,$$

and

$$|\nabla f|^2 \leq \partial_Y X(t, Y_0)\,(f_t - Y_0) + X(t, Y_0)\,, \qquad \text{on } f_t > Y_0\,,$$

where

$$Y_0 = \Big(1 + \frac{\pi^2}{64}\Big)\frac{n\,K}{4}\,.$$

It turns out that both $|\nabla f|^2$ and f_t are uniformally bounded for each $t \geq 2/K$ if $\mathrm{Ric} \geq K > 0$.

We could also deduce from this a Harnack inequality, but it takes a more complicated form than in the negative curvature case, and we will therefore omit it in this paper.

The main tool used in this paper is the maximum principle, which plays a fundamental rôle in Partial Differential Equations theory, see for example [5]. Although the basic idea adopted in this paper is to apply the maximum principle and Bochner identity to some nice test functions; this has been developed in a series of papers by Yau [9], [10], [12], Cheng and Yau [4], Li and Yau [6] etc. (see [7] for more references); the main difficulty with this method relies on the fact that, for any family of test functions, one gets different kind of results, and therefore the test functions in use are related to the results one is looking for. But it is not always easy (and indeed quite hard in general) to device what is the best estimate one could expect from a given differential inequality. Our main contribution in this context is to develop a *method* which produces the best possible estimates and to show how to construct good test functions in order to prove the expected estimates via the maximum principle. This method applies to a more general setting than the one described here, and it could be used in different contexts for more general equations.

Let us explain our main idea as following. Everything relies on the three following equations satisfied by $|\nabla f|^2$ and f_t

$$\Delta f = f_t - |\nabla f|^2\,, \tag{14}$$

$$(\widetilde{\Delta} - \partial_t) f_t = 0\,, \tag{15}$$

and

$$(\widetilde{\Delta} - \partial_t)\,|\nabla f|^2 = 2\,|\mathrm{Hess} f|^2 + 2\,\mathrm{Ric}\,(\nabla f, \nabla f)\,,$$

where $\widetilde{\Delta} = \Delta + 2\,\nabla f$ which is an elliptic operator. See Section 2 for detail.

(16) comes from the Bochner identity. Therefore, if K is a lower bound of the Ricci curvature, then we have the following inequality

$$(\widetilde{\Delta} - \partial_t)\,|\nabla f|^2 \geq \frac{2}{n}\,(\Delta f)^2 + 2\,K\,|\nabla f|^2\,. \tag{17}$$

Inserting (14) into (17), we end up two differential inequalities

$$\begin{aligned} &(\widetilde{\Delta} - \partial_t)\,|\nabla f|^2 \geq \frac{2}{n}\,(|\nabla f|^2 - f_t)^2 + 2\,K\,|\nabla f|^2\,, \\ &(\widetilde{\Delta} - \partial_t) f_t = 0\,. \end{aligned} \tag{18}$$

The main point of this paper is to compare $(|\nabla f|^2, f_t)$ with the solution (X, Y) of the following system of differential equations

$$\begin{aligned} -\,\partial_t X &= \frac{2}{n}\,(X - Y)^2 + 2\,KX\,, \\ -\,\partial_t Y &= 0\,, \end{aligned} \tag{19}$$

with the condition that $X(0) = \infty$. Since $Y =$ constant, we regard it as a parameter, and write the solution as $X = X(t, Y)$. It is easy to see that if $K \geq 0$, then $X(t, Y)$ is the function defined by (7), and if $K \leq 0$, then $X(t, Y) = \widetilde{X}(t, Y)$ with $-K$ in (8).

In fact, we were not able to prove (and we do not think it is true) that $|\nabla f|^2 \leq X(t, f_t)$ everywhere, and this comes from the lack of concavity of the curve $Y \longrightarrow X(t, Y)$. What we show is that this inequality holds on the most part of the curve. More precisely, our main result asserts that if $\mathrm{Ric} \geq K$, then the most part of the curve $(X(t, f_t), f_t)$ is above the curve $(|\nabla f|^2, f_t)$. In other words, $|\nabla f|^2 \leq X(t, f_t)$ for most of the values of f_t, and we have a linear upper bound on the remaining part.

The paper is organised as follows. In Section 2 we establish gradient estimates and some consequences for manifolds with Ricci curvature bounded below. Section 3 deals with the case of positive Ricci curvature manifolds. We deduce Harnack inequalities in Section 4. In Section 5, we describe several extensions to other diffusion operators, and, in the end, we give an improved form of Yau's gradient estimate.

The results obtained in this paper have been announced in [2].

2. Gradient estimates for complete manifolds.

The main purpose of this section is to prove Theorem 1. Thus throughout this section it will be assumed that $\mathrm{Ric} \geq -K$, where $K \geq 0$ is a constant.

Let u be a positive solution of the heat equation

$$(\Delta - \partial_t)\, u = 0\,, \qquad \text{on } [0, \infty) \times M \tag{20}$$

and let $f = \log u$. One can easily see that

$$(\Delta - \partial_t)\, f = -\Gamma(f, f)\,, \tag{21}$$

where $\Gamma(f, f) = |\nabla f|^2$. In general, if Δ is replaced by any sub-elliptic differential operator, we may define

$$\Gamma(g, h) = \frac{1}{2}\,(\Delta(hg) - h\Delta g - g\Delta h)\,, \qquad \text{for all } g, h \in C^\infty(M)\,,$$

and therefore $\Gamma(g, h)$ will stand for $\langle \nabla g, \nabla h\rangle$.

Differentiating (21) in t, we obtain the first fundamental equation

$$\Delta f_t + 2\,\langle \nabla f, \nabla f_t\rangle - \partial_t f_t = 0\,. \tag{22}$$

Then, define the bilinear operator Γ_2 by iterating the previous definition of Γ

$$\Gamma_2(g, h) = \frac{1}{2}\,(\Delta\Gamma(g, h) - \Gamma(g, \Delta h) - \Gamma(\Delta g, h))\,, \tag{23}$$

for all $g, h \in C^\infty(M)$. Using Γ_2, we may rewrite the classical Bochner identity as

$$\Gamma_2(g,h) = \langle \operatorname{Hess} g, \operatorname{Hess} h\rangle + \operatorname{Ric}(\nabla g, \nabla h)\,, \tag{24}$$

for all $g, h \in C^\infty(M)$. Since $\operatorname{Ric} \geq -K$ and $|\operatorname{Hess} g| \geq (\Delta g)^2/n$, Bochner identity yields the following curvature-dimension inequality

$$\Gamma_2(g,g) \geq \frac{1}{n}\,(\Delta g)^2 - K\,\Gamma(g,g)\,, \qquad \text{for all } g \in C^\infty(M)\,. \tag{25}$$

This is the only form in which the Ricci curvature will appear in what follows. Then, the fundamental remark is that, using (21) and (23) and the previous definition of Γ_2, we get another fundamental equation

$$\Delta\Gamma(f,f) + 2\,\langle \nabla f, \nabla\Gamma(f,f)\rangle - \partial_t \Gamma(f,f) = 2\,\Gamma_2(f,f)\,. \tag{26}$$

For simplicity, introduce a differential operator: $L = \Delta + 2\,\nabla f - \partial_t$. Then the basic equations (22) and (26) can be rewritten

$$L|\nabla f|^2 = 2\,\Gamma_2(f,f)\,, \qquad L f_t = 0\,. \tag{27}$$

If we notice that (21) can be rewritten as

$$-\Delta f = |\nabla f|^2 - f_t\,, \tag{28}$$

then the curvature-dimension inequality, implies that

$$L\,|\nabla f|^2 \geq \frac{2}{n}\,(|\nabla f|^2 - f_t)^2 - 2\,K\,|\nabla f|^2\,. \tag{29}$$

We next look for a smooth function B on $(0,\infty) \times \mathbb{R}$ such that

$$|\nabla f|^2 - f_t \leq B(t, f_t)\,, \qquad \text{for all } t > 0\,.$$

To this end, we set $F = |\nabla f|^2 - f_t - B(t, f_t)$, and $G = t\,F$. By the fact that $L f_t = 0$, we have

$$LB(t,f_t) = -\partial_t B(t,f_t) + \partial_Y^2 B(t,f_t)\,|\nabla f_t|^2\,.$$

Therefore

$$\begin{aligned} LF &= L\,|\nabla f|^2 - LB(t,f_t) \\ &= L\,|\nabla f|^2 - \partial_Y^2 B(t,f_t)\,|\nabla f_t|^2 + \partial_t B(t,f_t) \\ &= 2\,\Gamma_2(f,f) - \partial_Y^2 B(t,f_t)\,|\nabla f_t|^2 + \partial_t B(t,f_t)\,. \end{aligned} \tag{30}$$

Thus if $\partial_Y^2 B \le 0$ (which means B is concave in Y), then

$$\begin{aligned} LF &\ge \frac{2}{n}\,(|\nabla f|^2 - f_t)^2 - 2\,K\,|\nabla f|^2 + \partial_t B(t, f_t) \\ &= \frac{2}{n}\,(F+B)^2 - 2\,K\,|\nabla f|^2 + \partial_t B(t, f_t) \\ &= \frac{2}{n}\,F^2 + \frac{4}{n}\,BF + \frac{2}{n}\,B^2 - 2\,K\,|\nabla f|^2 + \partial_t B(t, f_t) \qquad (31) \\ &= \frac{2}{n}\,F^2 + \Big(\frac{4}{n}\,B - 2\,K\Big)F + \partial_t B + \frac{2}{n}\,B^2 - 2\,K\,(f_t + B)\,. \qquad (32) \end{aligned}$$

Hence

$$\begin{aligned} LG &= -F + t\,LF \\ &\ge \frac{2\,t}{n}\,F^2 + \Big(\frac{4\,t}{n}\,B - 2\,K\,t - 1\Big)F \\ &\quad - 2\,K\,t\,(f_t + B) + \frac{2\,t}{n}\,B^2 + t\,\partial_t B(t, f_t)\,. \end{aligned} \tag{33}$$

Next we specify the function B, so that

$$|\nabla f|^2 - f_t \le B(t, f_t)\,, \qquad \text{for all } t > 0\,.$$

More precisely, for any $Y_0 > -n\,K/4$, we shall produce a function B depending on the parameter Y_0 for which we shall prove the above upper bound.

To this end, consider the solution C of the differential equation on the half line $(0, \infty)$ with a parameter $Y \in \mathbb{R}$

$$\partial_t C + \frac{2}{n}\,C^2 - 2\,K\,(Y + C) = 0\,, \qquad C(0) = \infty\,. \tag{34}$$

Then if $Y > -n\,K/4$, we find that

$$C(t, Y) = \widetilde{X}(t, Y) - Y = \frac{n\,K}{2} + \frac{n}{2\,t}\,\frac{b(t, Y)}{2}\,\mathrm{cotanh}\,\frac{b(t, Y)}{2}\,, \tag{35}$$

where

$$b(t, Y) = \frac{4\,t}{n}\,\sqrt{n\,K}\,\sqrt{Y + \frac{n\,K}{4}}\,. \tag{36}$$

It is easily seen that

$$\partial_Y b(t,Y) = \Big(\frac{4\,t}{n}\Big)^2 \frac{n\,K}{2}\,\frac{1}{b(t,Y)}\,, \tag{37}$$

so that

$$\begin{aligned}\partial_Y C &= \frac{n}{4\,t}\Big(\partial_Y b + \frac{2\,\partial_Y b}{e^b-1} - \frac{2\,b\,\partial_Y b\,e^b}{(e^b-1)^2}\Big)\\ &= 2\,K\,t\,\Big(\frac{1}{b} + \frac{2}{b\,(e^b-1)} - \frac{2\,e^b}{(e^b-1)^2}\Big)\,.\end{aligned}$$

Therefore

$$\lim_{Y\to -nK/4} C(t,Y) = \frac{n}{2\,t} + \frac{n\,K}{2} \tag{38}$$

and

$$\lim_{Y\to -nK/4} \partial_Y C(t,Y) = \frac{2\,K}{3}\,t\,, \qquad \lim_{b\to\infty} \partial_Y C(t,Y) = 0\,. \tag{39}$$

Moreover, for each $t>0$, the function $Y \longrightarrow C(t,Y)$ is concave on the interval $(-n\,K/4,\infty)$.

Taking derivative with respect to Y in (34) we get that

$$\partial_t\,\partial_Y C + \frac{4}{n}\,C\partial_Y C - 2\,K\,(1+\partial_Y C) = 0\,. \tag{40}$$

Let $Y_0 > -n\,K/4$, and take B to be the linearization of C at Y_0, *i.e.*

$$B(t,Y) = \partial_Y C(t,Y_0)\,(Y-Y_0) + C(t,Y_0)\,. \tag{41}$$

Then

$$\partial_t B = \partial_t\,\partial_Y C(t,Y_0)\,(Y-Y_0) + \partial_t C(t,Y_0)\,,$$

and

$$\begin{aligned}\frac{2}{n}\,B^2 &= \frac{2}{n}\,(\partial_Y C(t,Y_0)\,(Y-Y_0) + C(t,Y_0))^2\\ &= \frac{2}{n}\,(\partial_Y C(t,Y_0)\,(Y-Y_0))^2 + \frac{4}{n}\,C(t,Y_0)\,\partial_Y C(t,Y_0)\,(Y-Y_0)\\ &\quad + \frac{2}{n}\,C(t,Y_0)^2\,,\end{aligned}$$

$$2\,K\,(Y+B) = 2\,K\,(Y_0 + C(t,Y_0)) + 2\,K\,(Y-Y_0)\,(1+\partial_Y C(t,Y_0))\,.$$

Therefore, by (34) and (40), we get that

$$\partial_t B + \frac{2}{n} B^2 - 2K(Y+B) = \frac{2}{n}\left(\partial_Y C(t,Y_0)\,(Y-Y_0)\right)^2 . \tag{42}$$

Now define $F = |\nabla f|^2 - f_t - B(t, f_t)$, where the constant $Y_0 > -n\,K/4$ in the definition of the function B. Let $G = t\,F$. Then by (33), we have

$$\begin{aligned}
LG &\geq \frac{2}{n\,t} G^2 + \Big(\frac{4B}{n} - 2K - \frac{1}{t}\Big)G + t\,\Big(\partial_t B + \frac{2}{n} B^2 - 2K(f_t + B)\Big)\\
&= \frac{2}{n\,t} G^2 + \Big(\frac{4}{n}\,\partial_Y C(t,Y_0)\,(f_t - Y_0)\Big)G + \Big(\frac{4}{n}\,C(t,Y_0) - 2K - \frac{1}{t}\Big)G\\
&\quad + \frac{2\,t}{n}\left(\partial_Y C(t,Y_0)\,(f_t - Y_0)\right)^2\\
&= \frac{2}{n\,t}\left(G + t\,\partial_Y C(t,Y_0)\,(f_t - Y_0)\right)^2 + \Big(\frac{4}{n}\,C(t,Y_0) - 2K - \frac{1}{t}\Big)G .
\end{aligned} \tag{43}$$

However,

$$\frac{4\,t}{n}\,C(t,Y_0) - 2K\,t = b(t,Y_0) + \frac{2\,b(t,Y_0)}{e^{b(t,Y_0)} - 1} ,$$

and therefore by the elementary inequality

$$b + \frac{2\,b}{e^b - 1} \geq 2 , \qquad \text{for all } b > 0 ,$$

we have

$$\frac{4}{n}\,C(t,Y_0) - 2K - \frac{1}{t} \geq \frac{1}{t} , \qquad \text{for all } t > 0,\ Y_0 \geq -\frac{n\,K}{4} . \tag{44}$$

Hence by (43)

$$LG \geq \frac{2}{n\,t}\left(G + t\,\partial_Y C(t,Y_0)\,(f_t - Y_0)\right)^2 + \frac{1}{t}\,G . \tag{45}$$

If the manifold is compact, consider a point (t_0, x) at which the maximum of G on $[0,t]\times M$ is attained: then, at this point, by the maximum principle, $\widetilde{\Delta}(G) \leq 0$. Moreover, $\partial G/\partial t \geq 0$ and $\nabla G = 0$. From these we conclude that $G \leq 0$. In this case we have

$$|\nabla f|^2 - f_t \leq \partial_Y C(t,Y_0)\,(f_t - Y_0) + C(t,Y_0) .$$

In the case where the manifold M is non-compact, we will get the same conclusion with a slight modification in the arguments. Since Ricci curvature is bounded below, we may use a generalised maximum principle (see [12], [5]) as following: replace K by any $\widetilde{K} > K$ in the definition of the function C. Then the same argument yields the following inequality

$$\begin{aligned} LG &\geq \frac{2}{n\,t}\,(G + t\,\partial_Y C(t, Y_0))^2 + \frac{1}{t}\,G + 2\,t\,(\widetilde{K} - K)\,|\nabla f|^2 \\ &\geq \frac{1}{t}\,G + 2\,t\,(\widetilde{K} - K)\,|\nabla f|^2 . \end{aligned} \tag{46}$$

Using then the Li-Yau's estimate (3), we may check that

$$G(t, \cdot) \leq \frac{n}{2}\,(1 + \partial_Y C(t, Y_0))^2 + \frac{n\,K}{4}\,(1 + \partial_Y C(t, Y_0))^2\,\frac{t}{\partial_Y C(t, Y_0)} .$$

However,

$$\lim_{t \to 0} \frac{t}{\partial_Y C(t, Y_0)} = \frac{3}{2\,\widetilde{K}} .$$

Therefore for any $t > 0$,

$$\sup_{[0,t] \times M} G < \infty .$$

Thus we can use the generalised maximum principle to the function G on $[0, t] \times M$ for any fixed $t > 0$: if $\sup_{[0,t]\times M} G > 0$, then we may find a point $t_0 \in [0, t]$ and a sequence of points $\{x_k\} \in M$, such that

$$\Delta G(t_0, x_k) \leq \frac{1}{k}\,, \qquad |\nabla G|(t_0, x_k) \leq \frac{1}{k} .$$

$G(0, \cdot) \leq 0$, and therefore $t_0 > 0$. Also,

$$\partial_t G(t_0, x_k) \geq 0\,, \qquad \lim_{k \to \infty} G(t_0, x_k) = \sup_{[0,t]\times M} G .$$

Hence we have

$$LG(t_0, x_k) \leq \frac{1}{k} + \frac{2}{k}\,|\nabla f|\,,$$

which together with (46) implies that

$$\begin{aligned} \frac{1}{k} &\geq LG(t_0, x_k) - \frac{2}{k}\,|\nabla f| \\ &\geq \frac{1}{t_0}\,G(x_k) + 2\,t_0\,(\widetilde{K} - K)\,|\nabla f|^2 - \frac{2}{k}\,|\nabla f| \\ &\geq \frac{1}{t_0}\,G(x_k) - \frac{1}{2\,(\widetilde{K} - K)\,t_0}\,\frac{1}{k^2} . \end{aligned}$$

Letting $k \longrightarrow \infty$ we get that

$$0 \geq \frac{1}{t_0} \sup_{[0,t]\times M} G\,,$$

which is a contradiction to the assumption that $\sup_{[0,t]\times M} G > 0$. Therefore $G \leq 0$. Since $\widetilde{K} > K$ is arbitrary, we have proved the main result of this section:

Theorem 3. *Let* $\mathrm{Ric} \geq -K$*;* $K \geq 0$*, and let* $f = \log u$*, where* u *is a positive solution of the heat equation. Then*

$$|\nabla f|^2 \leq \widetilde{X}(t, f_t)\,, \qquad \text{on } f_t > -\frac{n\,K}{4}\,, \tag{47}$$

$$\begin{aligned} |\nabla f|^2 - f_t &\leq \frac{2\,K}{3}\,t\left(f_t + \frac{n\,K}{4}\right) + \frac{n}{2\,t} + \frac{n\,K}{2} \\ &\leq \frac{n}{2\,t} + \frac{n\,K}{2}\,, \qquad \text{on } f_t \leq -\frac{n\,K}{4} \end{aligned} \tag{48}$$

and

$$|\nabla f|^2 - f_t \leq \inf_{Y_0 > -nK/4} (\partial_Y C(t, Y_0)\,(f_t - Y_0) + C(t, Y_0))\,. \tag{49}$$

PROOF. We have proved (49). By taking $Y_0 = f_t$ and noticing that $\widetilde{X}(t, Y) = Y + C(t, Y)$ we get (47). Letting $Y_0 \longrightarrow -n\,K/4$ we get (48). So we completed the proof.

REMARK. In the above proof, we in fact used a "parabolic version" of Yau's generalised maximum principle. Indeed, if we apply Yau's argument in [12] to the product manifold $[0, t] \times M$ (with boundary), and use the Hopf's maximum principle by Hopf (*i.e.* maximum principle with boundaries), since $[0, t]$ is compact, we can easily obtain the parabolic version of the generalised maximum principle.

Corollary 1. *Let* $\mathrm{Ric} \geq -K$*,* $K \geq 0$*, and let* $f = \log u$*, where* u *is a positive solution of the heat equation. Then we have*

$$-f_t \leq \frac{1}{1 + \frac{2}{3}K\,t}\left(\frac{n}{2\,t} + \frac{n\,K}{4}\right) + \frac{n\,K}{4} \leq \frac{n}{2\,t} + \frac{n\,K}{4}\,. \tag{50}$$

PROOF. For any $Y_0 > -n\,K/4$, we have

$$-f_t \le \frac{C(t,Y_0) - Y_0\,\partial_Y C(t,Y_0)}{1+\partial_Y C(t,Y_0)}\,. \tag{51}$$

By letting $Y_0 \longrightarrow -n\,K/4$, we get the conclusion.

We note that

$$\frac{n}{2\,t} \le \frac{1}{1+\dfrac{2}{3}\,K\,t}\Big(\frac{n}{2\,t}+\frac{n\,K}{4}\Big)+\frac{n\,K}{4}\,,$$

that is, the upper bound of $-f_t$ would not be better than $n/(2\,t)$ for negative curvature manifolds. However one would expect that the best upper bound of $-f_t$ should be $n/(2\,t)+(n-1)\,K/4$, as $(n-1)\,K/4$ is the spectral gap of the heat semi-group of the constant curvature space form with Ricci curvature $-K$. But we can see that

$$\frac{n}{2\,t}+\frac{(n-1)\,K}{4} \le \frac{1}{1+\dfrac{2}{3}\,K\,t}\,\Big(\frac{n}{2\,t}+\frac{n\,K}{4}\Big)+\frac{n\,K}{4}$$

if and only if $t \ge (n-3)/(2\,K)$. Therefore, if the dimension of M is bigger than 3, then our upper bound is even better than the expected one: $n/(2\,t)+(n-1)\,K/4$, within the time range $(0,(n-3)/(2\,K))$.

Corollary 2. *Let* Ric $\ge -K$; $K \ge 0$, *and let* $H(t,x,y)$ *be the heat kernel. Then*

$$H(t,x,x) \ge \frac{1}{(4\pi\,t)^{n/2}}\,\Big(1+\frac{2}{3}\,K\,t\Big)^{n/8}\,e^{-nKt/4}\,, \tag{52}$$

for all $t>0$, $x \in M$.

PROOF. By Corollary 1, we have

$$\begin{aligned}
-\partial_t \log\,(4\pi\,t)^{n/2}H &= -\partial_t \log H - \frac{n}{2\,t}\\
&\le \frac{1}{1+\dfrac{2}{3}\,K\,t}\,\Big(\frac{n}{2\,t}+\frac{n\,K}{4}\Big) - \frac{n}{2\,t}+\frac{n\,K}{4}\\
&= -\frac{n\,K}{4}\,\frac{1}{(3+2\,K\,t)}+\frac{n\,K}{4}\,.
\end{aligned}$$

Using the fact that

$$\lim_{t\to 0}(4\pi\, t)^{n/2}H(t,x,x)=1\,,$$

and integrating both sides over $[0,t]$, we get the conclusion.

Corollary 3. *Let* $\mathrm{Ric}\geq -K$*;* $K\geq 0$*. Then*

$$(53)\quad |\nabla f|^2-f_t\leq \sqrt{n\,K}\,\sqrt{f_t+\frac{n\,K}{4}}+\frac{n}{2\,t}+\frac{n\,K}{2}\,,\qquad \textit{on } f_t\geq -\frac{n\,K}{4}\,,$$

and

$$(54)\qquad |\nabla f|^2-f_t\leq \frac{2\,K}{3}\,t\left(f_t+\frac{n\,K}{4}\right)+\frac{n}{2\,t}+\frac{n\,K}{2}\,.$$

PROOF. We only need to prove the first inequality. Since

$$\begin{aligned}\widetilde{X}(t,Y)&\leq Y+\frac{n\,K}{2}+\frac{n}{2\,t}\left(1+\frac{b}{2}\right)\\ &=Y+\frac{n}{2\,t}+\frac{n\,K}{2}+\sqrt{n\,K}\,\sqrt{Y+\frac{n\,K}{4}}\,,\end{aligned}$$

(54) follows immediately from Theorem 3.

By estimate (51), we have

$$-f_t\leq\frac{n}{2\,t}+\frac{n\,K}{4}\,.$$

With this estimate, we can prove a better gradient estimate.

Indeed, let C be the solution of the differential equation (34) on the half line $(0,\infty)$ with a parameter $Y<-n\,K/4$ and $C(0)=\infty$. Then

$$(55)\qquad C(t,Y)=\frac{n\,K}{2}+\frac{n}{2\,t}\,\frac{b(t,Y)}{2}\,\mathrm{cotan}\,\frac{b(t,Y)}{2}$$

with

$$b(t,Y)=\frac{4t}{n}\,\sqrt{n\,K}\,\sqrt{-Y-\frac{n\,K}{4}}\,.$$

Note that the function C, defined by (35) for $Y > -n\,K/4$ and by (55) for $Y < -n\,K/4$ and

$$C\Big(t, -\frac{n\,K}{4}\Big) = \frac{n\,K}{2} + \frac{n}{2\,t}\,,$$

is a smooth function on $(0, \infty) \times \mathbb{R}$. However, $Y \longrightarrow C(t, Y)$ is not concave on $(-\infty, -n\,K/4)$.

It is easy to see that $\widetilde{X}(t, Y) = Y + C(t, Y)$ for all $Y \in \mathbb{R}$.

Up to now, we restricted our attention to the part $Y \geq -n\,K/4$ of the curve $Y \longrightarrow C(t, Y)$. In what follows, we are going to improve the previous estimate for any value of Y provided that the time t is not too big.

Let c_K be the positive constant

$$\frac{\pi^2}{32}\,\frac{1}{K}\,.$$

Then, we have

Theorem 4. *Let* Ric $\geq -K$*;* $K \geq 0$*, and let* $f = \log u$*, where* u *is a positive solution of the heat equation. Then for any* $0 < t \leq c_K$*,*

$$|\nabla f|^2 \leq \widetilde{X}(t, f_t)\,. \tag{56}$$

In fact, fix any $0 < t \leq c_K$, and let $s \in (0, t]$. Let

$$Y_0 \in \Big[-\frac{n}{2\,t} - \frac{nK}{4}, -\frac{n\,K}{4}\Big)\,,$$

and let

$$B(s, Y) = \partial_Y C(s, Y_0)\,(Y - Y_0) + C(s, Y_0)\,.$$

Define a test function as usual: $F = |\nabla f|^2 - f_s - B(s, f_s)$ and $G = sF$. Then the same argument as above yields that

$$LG \geq \frac{2}{n\,s}\,(G + s\,\partial_Y C(s, Y_0)\,(f_s - Y_0))^2 + \Big(\frac{4}{n}\,C(s, Y_0) - 2\,K - \frac{1}{s}\Big)G\,. \tag{57}$$

Notice that

$$\frac{4\,s}{n}\,C(s, Y_0) - 2\,K\,s = b(s, Y_0)\,\mathrm{cotan}\,\frac{b(s, Y_0)}{2}\,,$$

as $Y_0 < -n\,K/4$, where

$$\frac{b(s,Y_0)}{2} = \frac{2\,s}{n}\,\sqrt{n\,K}\,\sqrt{-Y_0 - \frac{n\,K}{4}}\;.$$

Since $s \le t \le c_K$, we have

$$\frac{b(s,Y_0)}{2} \le \frac{2\,s}{n}\,\sqrt{n\,K}\,\sqrt{\frac{n}{2\,t}} \le \frac{2\,t}{n}\,\sqrt{n\,K}\,\sqrt{\frac{n}{2\,t}} \le \frac{\pi}{4}\;.$$

Therefore, for any $s \le t \le c_K$, and

$$Y_0 \in \Big[-\frac{n}{2\,t} - \frac{n\,K}{4}, -\frac{n\,K}{4} \Big)\,,$$

we have

$$b(s,Y_0)\,\mathrm{cotan}\,\frac{b(s,Y_0)}{2} \ge 2\cos\frac{b(s,Y_0)}{2} \ge \sqrt{2}\;.$$

Hence, for those s and Y_0, we have

$$LG \ge \frac{2}{n\,s}\,(G + s\,\partial_Y C(s,Y_0)\,(f_s - Y_0))^2 + \frac{\sqrt{2}-1}{s}\,G\,,$$

and by applying the maximum principle to G on $[0,t]\times M$, we conclude that

$$|\nabla f|^2 - f_s \le \partial_Y C(s,Y_0)\,(f_s - Y_0) + C(s,Y_0)\,, \tag{58}$$

for any $0 < s \le t \le c_K$ and

$$Y_0 \in \Big[-\frac{n}{2\,t} - \frac{n\,K}{4}, -\frac{n\,K}{4} \Big)\,.$$

In particular if $f_t \le -n\,K/4$, $t \le c_K$, since $f_t > -n/(2\,t) - n\,K/4$, we can take $Y_0 = f_t$ in (58) to get that

$$|\nabla f|^2 - f_t \le C(t,f_t)\,.$$

Thus we completed the proof.

By the above proof, we also proved in fact the following

Theorem 5. *Let* Ric $\geq -K$*;* $K \geq 0$*, and let* $f = \log u$*, where* u *is a positive solution of the heat equation. Then for any* $Y_0 \leq -n\,K/4$*,*

$$|\nabla f|^2 - f_t \leq \partial_Y C(t, Y_0)\,(f_t - Y_0) + C(t, Y_0)\,, \tag{59}$$

for any

$$t \leq \frac{\pi\, n}{8\sqrt{n\,K}\sqrt{-Y_0 - \dfrac{n\,K}{4}}}\,.$$

Corollary 4. *Let* Ric $\geq -K$*;* $K \geq 0$*, and let* $f = \log u$*, where* u *is a positive solution of the heat equation. Then*

1) *If* $0 < t \leq c_K$*, then* $f_t \geq Y_1(t)$*, where* $Y_1(t)$ *is the unique solution of the equation*

$$Y + C(t, Y) = 0\,, \qquad Y < 0\,.$$

2) *For any* $t > 0$*,*

$$-f_t \leq \frac{C(t, Z_t) - Z_t\,\partial_Y C(t, Z_t)}{1 + \partial_Y C(t, Z_t)}\,,$$

where

$$-Z_t = \frac{n\,K}{4} + \frac{\pi^2}{64}\,\frac{n}{K\,t^2}\,.$$

3. Positive curvature manifold.

The goal of this section is to prove Theorem 2. The method follows exactly the same lines as in the previous section, although the conclusions are quite different.

Let M be a Riemannian manifold with dimension n, such that Ric $\geq K$, where K is a positive constant.

Let $f = \log u$, and u be a positive solution of the heat equation. In this case we have Li-Yau's estimate

$$|\nabla f|^2 - f_t \leq \frac{n}{2\,t}\,. \tag{60}$$

Let $U(t, Y)$ be the solution of the differential equation on the half line $(0, \infty)$ with a parameter Y

$$\partial_t U + \frac{2}{n}\,U^2 + 2\,K\,(Y + U) = 0\,, \qquad U(0) = \infty\,. \tag{61}$$

If $Y < n\,K/4$, then

$$U(t,Y) = X(t,Y) - Y = -\frac{n\,K}{2} + \frac{n}{2\,t}\,\frac{h(t,Y)}{2}\,\mathrm{cotanh}\,\frac{h(t,Y)}{2}\,, \tag{62}$$

with

$$h(t,Y) = \frac{4\,t}{n}\,\sqrt{n\,K}\,\sqrt{\frac{n\,K}{4} - Y}\,.$$

If $Y > n\,K/4$, then

$$U(t,Y) = X(t,Y) - Y = -\frac{n\,K}{2} + \frac{n}{2\,t}\,\frac{h(t,Y)}{2}\,\mathrm{cotan}\,\frac{h(t,Y)}{2}$$

with

$$h(t,Y) = \frac{4\,t}{n}\,\sqrt{n\,K}\,\sqrt{Y - \frac{n\,K}{4}}\,.$$

Therefore

$$\lim_{Y\to nK/4} U(t,Y) = \frac{n}{2\,t} - \frac{n\,K}{2}\,,$$

and

$$\lim_{Y\to nK/4} \partial_Y U(t,Y) = -\frac{2\,K\,t}{3}\,.$$

Moreover, U is a smooth function on $(0,\infty)\times\mathbb{R}$, and for any $t>0$, the function $Y \longrightarrow U(t,Y)$ is concave on $(-\infty, n\,K/4)$. But it is not concave on $(n\,K/4,\infty)$.

For any $Y_0 < n\,K/4$, we define a test function $G = t\,F$, $F = |\nabla f|^2 - f_t - B(t,f_t)$, where

$$B(t,Y) = \partial_Y U(t,Y_0)\,(Y - Y_0) + U(t,Y_0)\,.$$

Then by Bochner inequality, we get that

$$\begin{aligned} LG &\geq \frac{2}{n\,t}\,G^2 + \Big(\frac{4B}{n} + 2\,K - \frac{1}{t}\Big)G + t\,\Big(\partial_t B + \frac{2}{n}\,B^2 + 2\,K\,(f_t + B)\Big) \\ &= \frac{2}{n\,t}\,(G + t\,\partial_Y U(t,Y_0)\,(f_t - Y_0))^2 + \Big(\frac{4}{n}\,U(t,Y_0) + 2\,K - \frac{1}{t}\Big)G\,. \end{aligned} \tag{63}$$

By the fact that

$$\frac{4}{n}\,U(t,Y_0) + 2\,K - \frac{1}{t} \geq \frac{1}{t}\,, \qquad \text{for all } Y_0 < \frac{n\,K}{4}\,,\ t > 0\,,$$

we conclude by the maximum principle that $G \le 0$. Therefore we have

Theorem 6. *Let* $\mathrm{Ric} \ge K \ge 0$, *and let* $f = \log u$, *where* u *is a positive solution of the heat equation. Then*

$$|\nabla f|^2 - f_t \le \inf_{Y_0 < nK/4} \left(\partial_Y U(t, Y_0)\,(f_t - Y_0) + U(t, Y_0)\right), \tag{64}$$

for any $t > 0$. *In particular, we have*

$$|\nabla f|^2 \le X(t, f_t), \qquad \text{on } f_t \le \frac{n\,K}{4}, \tag{65}$$

and

$$|\nabla f|^2 - f_t \le \frac{2}{3}\,K\,t\left(\frac{n\,K}{4} - f_t\right) + \frac{n}{2\,t} - \frac{n\,K}{2}. \tag{66}$$

Define a function $V(t, Y)$ by

$$\begin{aligned} V(t, Y) &= X(t, Y) \\ &= -\frac{n\,K}{2} + Y \\ &\quad + \sqrt{n\,K}\sqrt{\frac{n\,K}{4} - Y}\,\mathrm{cotanh}\,\frac{2\,t}{n}\sqrt{n\,K}\sqrt{\frac{n\,K}{4} - Y}, \end{aligned}$$

when $Y < n\,K/4$, and

$$V(t, Y) = -\frac{n\,K}{2} + Y - \frac{2\,K\,t}{3}\left(Y - \frac{n\,K}{4}\right),$$

when $Y \ge n\,K/4$. Then we can rewrite the estimates in Theorem 6 to be

$$|\nabla f|^2 \le V(t, f_t), \qquad \text{for all } t > 0. \tag{67}$$

It is easily seen that there is a unique zero point of $V(t, Y)$ in $(-\infty, 0)$ for each $t > 0$, denoted it by $Y_1(t)$. Then by the fact that $|\nabla f|^2 \ge 0$, we have $f_t \ge Y_1(t)$.

If $n/(2\,t) - n\,K/4 < 0$, that is, if $t > 2/K$, then there is a unique zero point of $V(t, Y)$ in $(0, n\,K/4]$, denoted by $Y_2(t)$, and again by the fact that $|\nabla f|^2 \ge 0$, the estimate (67) yields that $f_t \le Y_2(t)$.

If $n/(2\,t) - n\,K/4 > 0$ and $-2\,K\,t/3 + 1 < 0$, that is, if $3/(2\,K) < t < 2/K$, then we can see that the unique zero point of $V(t,Y)$ in $(0,\infty)$ is

$$\frac{n\,K}{4} + \Big(\frac{2\,K\,t}{3} - 1\Big)^{-1}\Big(\frac{n}{2\,t} - \frac{n\,K}{4}\Big)$$

and by the same reasoning as the above, we have

$$f_t \le \frac{n\,K}{4} + \frac{1}{\dfrac{2\,K\,t}{3} - 1}\Big(\frac{n}{2\,t} - \frac{n\,K}{4}\Big).$$

In these two cases, that is, if $t > 3/(2\,K)$, there is a unique maximum value of $V(t,Y)$, attending at some point in $(Y_1(t), n\,K/4]$, which is denoted by $V_0(t)$. Then by (67), we have $|\nabla f|^2 \le V_0(t)$.

Thus we have proved the following

Theorem 7. *Let* $\mathrm{Ric} \ge K \ge 0$, *and let* $f = \log u$, *where* u *is a positive solution of the heat equation. Then*

$$Y_1(t) \le f_t \le Y_2(t) \le \frac{n\,K}{4}\,, \qquad \text{for all } t > \frac{2}{K}\,, \tag{68}$$

$$Y_1(t) \le f_t \le \frac{n\,K}{4} + \frac{1}{\dfrac{2\,K\,t}{3} - 1}\Big(\frac{n}{2\,t} - \frac{n\,K}{4}\Big), \ \text{for all } t > \frac{3}{2\,K}\,, \tag{69}$$

and

$$|\nabla f|^2 \le V_0(t)\,, \qquad \text{for all } t > \frac{3}{2\,K}\,. \tag{70}$$

Now let us estimate $Y_1(t)$. To this end, let $\widetilde U$ be the solution of the differential equation

$$\partial_t \widetilde U + \frac{2}{n}\,\widetilde U^2 + 2\,\widetilde K\,(Y + \widetilde U) = 0\,, \qquad \widetilde U(0) = \infty\,.$$

Then $\widetilde U$ is given by the formula as for U instead of K by $\widetilde K$.

Let $W = t\,(U - \widetilde U)$. Then for any $Y < n\,K/4$, we have

$$\begin{aligned}
-\partial_t W &= \frac{2}{n\,t}\,W^2 + \Big(\frac{4}{n}\,U + 2\,K - \frac{1}{t}\Big)W + 2\,(K - \widetilde K)\,(Y + \widetilde U)\,t \\
&\ge \frac{2}{n\,t}\,W^2 + \frac{1}{t}\,W + 2\,(K - \widetilde K)\,(Y + \widetilde U)\,t\,.
\end{aligned}$$

In particular, if $K \geq \widetilde{K} = 0$, then $\widetilde{U} = n/(2\,t)$, and

$$-\partial_t W \geq \frac{2}{n\,t}\,W^2 + \frac{1}{t}\,W + 2\,K\left(Y + \frac{n}{2\,t}\right)t\,.$$

Applying the maximum principle to W on $[0,t] \times \mathbb{R}$ for any $0 < t \leq (-n/(2\,Y)) \vee 0$, we conclude that $W \leq 0$ for any $t \leq (-n/(2\,Y) \vee 0$. Therefore we have proved the following

Proposition 1. *If $K \geq 0$, then for any $Y, t > 0$ such that $Y < n\,K/4$, $Y + n/(2\,t) \geq 0$, we have*

$$U(t,Y) \leq \frac{n}{2\,t}\,.$$

As a consequence, we have

$$Y_1(t) \geq -\frac{n}{2\,t}\,, \qquad \text{for all } t > 0\,.$$

Our next goal is to bound $|\nabla f|^2 - f_t$ for small time t.

Theorem 8. *Let $\mathrm{Ric} \geq K > 0$, and let $f = \log u$, where u is a positive solution of the heat equation. Then for any $Y_0 > n\,K/4$, we have*

$$|\nabla f|^2 - f_t \leq \partial_Y U(t,Y_0)\,(f_t - Y_0) + U(t,Y_0)\,, \tag{71}$$

for

$$0 < t \leq \frac{n\,\pi}{8\,\sqrt{n\,K}\,\sqrt{Y_0 - \dfrac{n\,K}{4}}}\,.$$

Proof. Let $G = t\,F$, $F = |\nabla f|^2 - f_t - B(t,f_t)$, where

$$B(t,Y) = \partial_Y U(t,Y_0)\,(Y - Y_0) + U(t,Y_0)\,.$$

Then by (63), we have

$$LG \geq \frac{2}{n\,t}\,(G + t\,\partial_Y U(t,Y_0)\,(f_t - Y_0))^2 + \left(\frac{4}{n}\,U(t,Y_0) + 2\,K - \frac{1}{t}\right)G\,.$$

However, when $Y_0 > n\,K/4$ and

$$t \leq \frac{n\,\pi}{8\,\sqrt{n\,K}\,\sqrt{Y_0 - \dfrac{n\,K}{4}}}\,,$$

we have

$$\begin{aligned}
&\frac{4\,t}{n}\,U(t,Y_0)+2\,K\,t\\
&\qquad\qquad = h\,\mathrm{cotan}\,\frac{h}{2}\\
&\qquad\qquad = \frac{4\,t}{n}\,\sqrt{n\,K}\,\sqrt{Y_0-\frac{n\,K}{4}}\,\mathrm{cotanh}\,\frac{2\,t}{n}\,\sqrt{n\,K}\,\sqrt{Y_0-\frac{n\,K}{4}}\\
&\qquad\qquad \geq 2\cos\frac{2\,t}{n}\,\sqrt{n\,K}\,\sqrt{Y_0-\frac{n\,K}{4}}\\
&\qquad\qquad \geq 2\cos\frac{\pi}{4}\\
&\qquad\qquad = \sqrt{2}\,.
\end{aligned}$$

Therefore

$$LG \geq \frac{\sqrt{2}-1}{t}\,G\,,$$

and by the maximum principle, we get the conclusion.

Corollary 5. *Let*

$$Y_0=\Big(1+\frac{\pi^2}{64}\Big)\frac{n\,K}{4}\,.$$

Then for any $0<t\leq 2/K$*, we have*

$$|\nabla f|^2-f_t\leq U(t,f_t)\,,\qquad \text{on } f_t\leq Y_0\,, \tag{72}$$

and

$$|\nabla f|^2-f_t\leq \partial_Y U(t,Y_0)\,(f_t-Y_0)+U(t,Y_0)\,. \tag{73}$$

Therefore, if

$$E(t,Y)=X(t,Y)=Y+U(t,Y)\,,\qquad \text{if } Y\leq Y_0\,,$$

and

$$E(t,Y)=\partial_Y U(t,Y_0)\,(Y-Y_0)+U(t,Y_0)+Y\,,\qquad \text{if } Y>Y_0\,,$$

then

$$(74) \qquad |\nabla f|^2 \le E(t, f_t)\,, \qquad \text{for all } t \le \frac{2}{K}\,,$$

where Y_0 is defined in Corollary 5.

Putting all discussions together, we have the following

Theorem 9. *Let* $\mathrm{Ric} \ge K > 0$*, and let* $f = \log u$*, where* u *is a positive solution of the heat equation. Then we have the following conclusions.*

1) *If* $t \ge 2/K$*, then* $|\nabla f|^2 \le X(t, f_t)$ *and* $-n/(2\,t) \le f_t \le n\,K/4$*.*

2) *If* $t \le 2/K$*, then we have*

$$|\nabla f|^2 \le X(t, f_t)\,, \qquad \text{on } f_t \le Y_0\,,$$

and

$$|\nabla f|^2 - f_t \le \partial_Y U(t, Y_0)\,(f_t - Y_0) + U(t, Y_0)\,, \qquad \text{on } f_t > Y_0\,,$$

where

$$Y_0 = \Big(1 + \frac{\pi^2}{64}\Big)\frac{n\,K}{4}\,.$$

4. Harnack inequalities.

In this section we first show how to deduce a Harnack inequality from a gradient estimate, although it is very standard, see [9]. Then we prove the main Harnack estimates.

The link between Harnack inequalities and gradient estimates is given in the following

Proposition 2. *Let* M *be a complete Riemannian manifold, and let* $f = \log u$*; where* u *is a positive solution of the heat equation. Suppose that*

$$(75) \qquad |\nabla f|^2 \le \psi(t, f_t)\,, \qquad \text{for all } t > 0\,,$$

where $\psi : (0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function, then*

$$(76) \qquad \frac{u(t,x)}{u(t+s, y)} \le \exp\Big(\int_t^{t+s} K\Big(\sigma, \frac{\rho}{s}\Big)\,d\sigma\Big)\,,$$

where ρ is the geodesic distance between x and y, and

$$K(t,\alpha)=\sup_{\{Y:\,\psi(t,Y)\geq 0\}}\left(\alpha\sqrt{\psi(t,Y)}-Y\right).$$

PROOF. The proof of this proposition is straightforward. Let γ be a minimal geodesic joining x and y, so that $\gamma(0)=y$ and $\gamma(1)=x$. If $\rho=d(x,y)$, then $|\dot\gamma|=\rho$. Define

$$p(\sigma)=(\gamma(\sigma),(1-\sigma)\,t_2+t_1\,\sigma)\,,\qquad t_2=t+s\,,\ t_1=t\,.$$

Then $p(0)=(y,t_2)$ and $p(1)=(x,t_1)$. Set $\eta(\sigma)=f(p(\sigma))$. It is clear that

$$f(t_1,x)-f(t_2,y)=\eta(1)-\eta(0)=\int_0^1\dot\eta(\sigma)\,d\sigma=\int_0^1(\langle\nabla f,\dot\gamma\rangle-s\,f_t)\,d\sigma\,,$$

with $t=(1-\sigma)\,t_2+t_1\,\sigma$. We end up with

$$\begin{aligned}f(t_1,x)-f(t_2,y)&\leq\int_0^1(\rho\,|\nabla f|-s\,f_t)\,d\sigma\\&\leq\int_0^s\Big(\frac{\rho}{s}\,|\nabla f|-f_t\Big)\,d\sigma\\&\leq\int_0^s K\Big(\sigma,\frac{\rho}{s}\Big)\,d\sigma\,.\end{aligned}$$

From this result and the previous gradient estimates, we may now prove Harnack inequalities: we shall first establish the simplest one, for which the computations are easy: it follows from the gradient estimate (53).

Theorem 10. *Let* $\mathrm{Ric}\geq -K$, $K\geq 0$, *and let u be a positive solution of the heat equation. Then*

$$\frac{u(t,x)}{u(t+s,y)}\leq\Big(\frac{t+s}{t}\Big)^{n/2}$$
$$\cdot\exp\Big(\frac{(\rho+\sqrt{n\,K}\ s)^2}{4\,s}+\frac{\sqrt{n\,K}}{2}\min\Big\{(\sqrt2-1)\,\rho,\frac{\sqrt{n\,K}}{2}\,s\Big\}\Big)\,,\tag{77}$$

for all $s>0$, $t\geq 0$, $x,y\in M$, where $\rho=d(x,y)$.

PROOF. Let $f = \log u$, and $\alpha > 0$ be fixed.

If $f_t > -n\,K/4$, then the gradient estimate (53) yields that

$$|\nabla f|^2 \le \Big(\gamma + \sqrt{\frac{n\,K}{4}}\,\Big)^2 + \frac{n}{2\,t}\,, \tag{78}$$

where $\gamma = \sqrt{f_t + n\,K/4}$, so that

$$-f_t = \frac{n\,K}{4} - \gamma^2\,.$$

Denote by

$$u = \sqrt{\Big(\gamma + \sqrt{\frac{n\,K}{4}}\,\Big)^2 + \frac{n}{2\,t}} \ge 0\,.$$

Then $|\nabla f| \le u$ and

$$\begin{aligned} -f_t &= \frac{n\,K}{4} - \Big(\sqrt{u^2 - \frac{n}{2\,t}} - \sqrt{\frac{n\,K}{4}}\,\Big)^2 \\ &= -u^2 + \frac{n}{2\,t} + \sqrt{n\,K}\,\sqrt{u^2 - \frac{n}{2\,t}} \\ &\le -u^2 + \frac{n}{2\,t} + \sqrt{n\,K}\,u\,. \end{aligned}$$

Hence in this case we have

$$\alpha\,|\nabla f| - f_t \le (\alpha + \sqrt{n\,K}\,)\,u - u^2 + \frac{n}{2\,t} \le \frac{(\alpha + \sqrt{n\,K}\,)^2}{4} + \frac{n}{2\,t}\,. \tag{79}$$

If $f_t \le -n\,K/4$, then on one hand the estimate (54) implies that

$$-f_t \le -|\nabla f|^2 + \frac{n}{2\,t} + \frac{n\,K}{2}\,,$$

so that

$$\begin{aligned} \alpha\,|\nabla f| - f_t &\le \frac{\alpha^2}{4} + \frac{n}{2\,t} + \frac{n\,K}{2} \\ &= \frac{(\alpha + \sqrt{n\,K}\,)^2}{4} + \frac{n}{2\,t} + \frac{n\,K}{4} - \frac{\sqrt{n\,K}}{2}\,\alpha\,\sigma\,. \end{aligned} \tag{80}$$

On other hand, by the estimate (54), we have

$$|\nabla f|^2 - f_t \leq \frac{2\,K\,t}{3}\left(f_t + \frac{n\,K}{4}\right) + \frac{n}{2\,t} + \frac{n\,K}{2}\,.$$

Therefore

$$\begin{aligned}
\alpha\,|\nabla f| - f_t &\leq |\alpha\,\nabla f| - \frac{1}{1+\frac{2\,K}{3}\,t}\,|\nabla f|^2 \\
&\quad + \frac{1}{1+\frac{2\,K}{3}\,t}\left(\frac{n}{2\,t} + \frac{n\,K}{4}\right) + \frac{n\,K}{4} \\
&\leq \frac{\alpha^2}{4} + \left(1 - \frac{1}{\sqrt{1+\frac{2\,K}{3}\,t}}\right)\alpha\,|\nabla f| \\
&\quad + \frac{1}{1+\frac{2\,K}{3}\,t}\left(\frac{n}{2\,t} + \frac{n\,K}{4}\right) + \frac{n\,K}{4} \\
&\leq \frac{\alpha^2}{4} + \left(1 - \frac{1}{\sqrt{1+\frac{2\,K}{3}\,t}}\right)\sqrt{\frac{n}{2\,t} + \frac{n\,K}{2}}\,\alpha \\
&\quad + \frac{n}{2\,t} + \frac{n\,K}{4} - \frac{n\,K}{12}\,\frac{1}{1+\frac{2\,K}{3}\,t} \\
&\leq \frac{\alpha^2}{4} + \frac{\frac{2\,K}{3}\,t\sqrt{\frac{n}{2\,t} + \frac{n\,K}{2}}}{\sqrt{1+\frac{2\,K}{3}\,t}\left(1+\sqrt{1+\frac{2\,K}{3}\,t}\right)}\,\alpha + \frac{n}{2\,t} + \frac{n\,K}{4} \\
&\leq \frac{\alpha^2}{4} + \sqrt{\frac{n\,K}{2}}\,\alpha + \frac{n}{2\,t} + \frac{n\,K}{4} \\
&= \frac{(\alpha+\sqrt{n\,K}\,)^2}{4} + \frac{n}{2\,t} + \frac{\sqrt{2}-1}{2}\,\sqrt{n\,K}\,\alpha\,.
\end{aligned}$$

Hence, if $f_t + n\,K/4 < 0$, then we have

$$\begin{aligned}
\alpha\,|\nabla f| - f_t &\leq \frac{(\alpha+\sqrt{n\,K}\,)^2}{4} + \frac{n}{2\,t} \\
&\quad + \frac{\sqrt{n\,K}}{2}\,\min\left\{(\sqrt{2}-1)\,\alpha, \frac{\sqrt{n\,K}}{2}\right\}.
\end{aligned} \tag{81}$$

Therefore, by (79) and (81), we always have the following estimate

$$\alpha\,|\nabla f| - f_t \le \frac{(\alpha + \sqrt{n\,K}\,)^2}{4} + \frac{n}{2\,t} + \frac{\sqrt{n\,K}}{2}\min\Big\{(\sqrt{2}-1)\,\alpha, \frac{\sqrt{n\,K}}{2}\Big\}, \tag{82}$$

for any $\alpha \ge 0$.

Using now the previous proposition, we get

$$f(t_1, x) - f(t_2, y) \le \int_0^1 \Big(\frac{(\rho + \sqrt{n\,K}\,s)^2}{4\,s} + \frac{n}{2\,t}\,s + \frac{\sqrt{n\,K}}{2}\min\Big\{(\sqrt{2}-1)\,\rho, \frac{\sqrt{n\,K}}{2}\,s\Big\}\Big)\,d\sigma\,,$$

which yields the Harnack inequality.

REMARK. As we pointed out in the introduction, S. T. Yau mentioned to us that he obtained a similar Harnack inequality.

Now we turn to prove the main Harnack estimate. Let $\mathrm{Ric} \ge -K$ for some constant $K \ge 0$. We have seen that the main point is to estimate $\alpha\,|\nabla f| - f_t$ for $\alpha > 0$.

For any $Y_0 > -n\,K/4$, we have

$$|\nabla f|^2 - f_t \le C(t, Y_0) + \partial_Y C(t, Y_0)\,(f_t - Y_0)\,,$$

and therefore

$$-f_t \le -\frac{|\nabla f|^2}{1 + \partial_Y C(t, Y_0)} + \frac{C(t, Y_0) - Y_0\,\partial_Y C(t, Y_0)}{1 + \partial_Y C(t, Y_0)}\,.$$

Hence for any $\alpha > 0$, we have

$$\begin{aligned}
\alpha\,|\nabla f| - f_t &\le -\frac{1}{1 + \partial_Y C(t, Y_0)}\,|\nabla f|^2 + \alpha\,|\nabla f| + \frac{C(t, Y_0) - Y_0\,C(t, Y_0)}{1 + \partial_Y C(t, Y_0)}\\
&\le \frac{\alpha^2}{4}\,(1 + \partial_Y C(t, Y_0)) + \frac{C(t, Y_0) - Y_0\,\partial_Y C(t, Y_0)}{1 + \partial_Y C(t, Y_0)}\\
&= \frac{\alpha^2}{4} + \Big(\partial_Y C(t, Y_0)\Big(\frac{\alpha^2}{4} - Y_0\Big) + C(t, Y_0)\Big)\\
&\quad - \frac{\partial_Y C(t, Y_0)}{1 + \partial_Y C(t, Y_0)}\,(C(t, Y_0) + \partial_Y C(t, Y_0)\,(0 - Y_0))\,.
\end{aligned} \tag{83}$$

Letting $Y_0 = \alpha/4$ in (83), we get that

$$\alpha\,|\nabla f| - f_t \le \frac{\alpha^2}{4} + C\Big(t, \frac{\alpha^2}{4}\Big) - A\Big(t, \frac{\alpha^2}{4}\Big)\,, \tag{84}$$

where

$$A(t,Y) = \frac{\partial_Y C(t,Y)\,(C(t,Y) - Y\partial_Y C(t,Y))}{1 + \partial_Y C(t,Y)}\,.$$

Notice that $A(t,Y) \ge 0$ when $Y \ge 0$. Therefore we have proved the following

Theorem 11. *Let* Ric $\ge -K$; $K \ge 0$, *and let* u *be a positive solution of the heat equation. Then*

$$\frac{u(t,x)}{u(t+s,y)} \le \exp\left(\frac{\rho^2}{4\,s} + \int_t^{t+s} \Big(C\Big(\sigma, \frac{\rho^2}{4\,s^2}\Big) - A\Big(\sigma, \frac{\rho^2}{4\,s^2}\Big)\Big)\,d\sigma\right), \tag{85}$$

for any $t \ge 0, s > 0$ *and* $x, y \in M$, *where* ρ *is the geodesic distance between* x *and* y.

REMARK. Although we have the simple fact that

$$C(t,Y) \le \sqrt{n\,K}\,\sqrt{Y + \frac{n\,K}{4}} + \frac{n}{2\,t} + \frac{n\,K}{2}\,,$$

for any $Y > -n\,K/4$, however, unlike $C(t,Y)$ whose linearization at any point $Y \ge -n\,K/4$ is an upper bound of $|\nabla f|^2 - f_t$, the linearization of

$$\sqrt{n\,K}\,\sqrt{Y + \frac{n\,K}{4}} + \frac{n\,K}{2} + \frac{n}{2\,t}$$

at some points may not be an upper bound of $|\nabla f|^2 - f_t$. In this sense, therefore, the analysis via $C(t,Y)$ is even simpler and yields much stronger conclusions. This is also the reason why we give an independent proof of Theorem 10.

Since

$$C(t,Y)=\frac{n\,K}{2}+\frac{n}{2\,t}\,\frac{b(t,Y)}{2}\operatorname{cotanh}\frac{b(t,Y)}{2}$$
$$=\frac{n\,K}{2}+\sqrt{n\,K}\,\sqrt{Y+\frac{n\,K}{4}}\operatorname{cotanh}\Big(\frac{2\,t}{n}\,\sqrt{n\,K}\,\sqrt{Y+\frac{n\,K}{4}}\Big)\,,$$

$$\int_t^{t+s}C(\sigma,Y)\,d\sigma=\frac{n\,K}{2}\,s+\frac{n}{2}\int_{(2/n)t\sqrt{nK}\sqrt{Y+nK/4}}^{(2/n)(t+s)\sqrt{nK}\sqrt{Y+nK/4}}\operatorname{cotanh}\sigma\,d\sigma$$
$$=\frac{n\,K}{2}\,s+\frac{n}{2}\log\left(\frac{\sinh\frac{2}{n}\,(t+s)\,\sqrt{n\,K}\,\sqrt{Y+\frac{n\,K}{4}}}{\sinh\frac{2}{n}\,t\,\sqrt{n\,K}\,\sqrt{Y+\frac{n\,K}{4}}}\right),$$

so that we have the following

Corollary 6. *Let* Ric $\geq -K$*;* $K\geq 0$*, and let* u *be a positive solution of the heat equation. Then*

$$\frac{u(t,x)}{u(t+s,y)}\leq\left(\frac{\sinh\Big(\frac{2}{n}\,(t+s)\,\sqrt{n\,K}\,\sqrt{\frac{\rho^2}{4\,s^2}+\frac{n\,K}{4}}\Big)}{\sinh\Big(\frac{2}{n}\,t\,\sqrt{n\,K}\,\sqrt{\frac{\rho^2}{4\,s^2}+\frac{n\,K}{4}}\Big)}\right)^{n/2}E(\rho,s,t)\,,\tag{86}$$

where

$$E(\rho,s,t)=\exp\Big(\frac{\rho^2}{4\,s}+\frac{n\,K}{2}\,s-\int_t^{t+s}A\Big(\sigma,\frac{\rho^2}{4\,s^2}\Big)\,d\sigma\Big)\,.$$

Applying Corollary 6 to the heat kernel we have

$$H(t,x,y)\geq\left(\frac{\frac{2}{n}\,\sqrt{n\,K}\,\sqrt{\frac{\rho^2}{4\,t^2}+\frac{n\,K}{4}}}{4\pi\sinh\Big(\frac{2\,t}{n}\,\sqrt{n\,K}\,\sqrt{\frac{\rho^2}{4\,t^2}+\frac{n\,K}{4}}\Big)}\right)^{n/2}E(\rho,t,0)^{-1}$$

$$=\frac{1}{(4\pi\,t)^{n/2}}\left(\frac{\sqrt{\frac{K}{n}}\,\sqrt{\rho^2+n\,K\,t^2}}{\sinh\Big(\sqrt{\frac{K}{n}}\,\sqrt{\rho^2+n\,K\,t^2}\Big)}\right)^{n/2}E(\rho,t,0)^{-1}\,,\tag{87}$$

where

$$E(\rho,t,0)^{-1}=\exp\Big(-\frac{\rho^2}{4t}-\frac{nK}{2}t+\int_0^t A\Big(\sigma,\frac{\rho^2}{4t^2}\Big)\,d\sigma\Big)\,.$$

Remark. Proposition 2 together with the gradient estimates for the positive Ricci curvature manifolds yields a Harnack inequality. However, its form is quite complicated. Since the upper bound function $\psi(t,Y)$ is in general nonlinear , we can improve the Harnack inequality in Proposition 2 by varying the time speed, that is, replacing the straight line joining t and $t+s$ by a curve. Therefore we decided to write down the explicit Harnack inequalities for positive Ricci curvature manifolds together with the compact manifold case in a separate paper.

5. Extensions.

The same arguments in previous sections can be applied to the case when the manifold M with convex boundary ∂M; the second fundamental form π of the boundary ∂M is nonnegative. This is because of the fact that if $\partial u/\partial\nu=0$ on the boundary; where ν denotes the pointed out normal vector field; then

$$\frac{\partial|\nabla u|^2}{\partial\nu}=-2\pi\,(\nabla u,\nabla u)\,,$$

so that we can use the Hopf maximum principle when $\pi\geq 0$. We only write down a theorem in this case.

Theorem 12. *Let M be a complete Riemannian manifold with a convex boundary ∂M, and let u be a positive solution of the heat equation*

$$(\Delta-\partial_t)\,u=0\,,\qquad \textit{on } [0,\infty)\times M\,,$$

$$\partial_\nu u=0\,,\qquad \textit{on } (0,\infty)\times\partial M\,.$$

Let $f=\log u$. Then

$$|\nabla f|^2-f_t\leq C(t,f_t)\,,\qquad \textit{on } f_t\geq-\frac{nK}{4}\,, \tag{88}$$

and

$$|\nabla f|^2-f_t\leq\partial_Y C(t,Y_0)\,(f_t-Y_0)+C(t,Y_0)\,,$$

for all $t > 0$, $Y_0 \geq -n\,K/4$.

There is a further generalisation of our gradient estimates and Harnack inequalities to a general elliptic operator. We only state a result.

Let M be a complete manifold (without or with a convex boundary), and let $\Delta_B = \Delta + B$ be an elliptic operator; where B is a C^2-vector field. Assume that Δ_B satisfies a curvature-dimension inequality (see [1])

$$\Gamma_2(g,g) \geq \frac{1}{m}\,(\Delta_B g)^2 - K\,\Gamma(g,g)\,, \qquad \text{for all } g \in C^\infty(M)\,,$$

for some constants $m > 0$ and $K \geq 0$, where by definition $\Gamma(f,g) = \langle \nabla f, \nabla g\rangle$, and

$$\Gamma_2(f,g) = \frac{1}{2}\,(\Delta_B(fg) - \Gamma(\Delta_B f, g) - \Gamma(f, \Delta_B\, g))\,.$$

This condition is satisfied if and only if

$$\mathrm{Ric} - \nabla_B^s - \frac{1}{m-n}\,B \otimes B \geq -K\,,$$

where $m \geq n$, $n = \dim M$, Ric denotes the Ricci curvature and

$$\nabla_B^s(\xi,\eta) = \frac{1}{2}\,(\langle \nabla_\xi B, \eta\rangle + \langle \nabla_\eta B, \xi\rangle)\,, \qquad \text{for all } \xi, \eta \in TM\,.$$

If $f = \log u$, u is a positive solution of the heat equation

$$(\Delta_B - \partial_t)\,u = 0\,, \qquad \text{on } [0,\infty) \times M\,,$$

(in the case that the boundary $\partial M \neq 0$, we further assume that u satisfies the Neumann boundary condition), then

$$|\nabla f|^2 - f_t \leq C(t, f_t)\,, \qquad \text{on } f_t \geq -\frac{m\,K}{4}\,,$$

and

$$|\nabla f|^2 - f_t \leq \partial_Y C(t, Y_0)\,(f_t - Y_0) + C(t, Y_0)\,, \qquad \text{for all } Y_0 \geq -\frac{m\,K}{4}\,,$$

where C is the solution of the differential equation

$$\partial_t C + \frac{2}{m}\,C^2 - 2\,K\,(Y + C) = 0\,, \qquad C(0) = \infty\,.$$

6. Appendix.

The goal of this appendix is to give a proof of Yau's estimate (6). We will use the same notations as in Section 2.

Let $F = |\nabla f|^2 - f_t - Q(t, |\nabla f|^2)$ for some positive function Q which will be given later, and $G = t\,F$.

It is easily seen that

$$\begin{aligned} LF &= (1-\partial_X Q)\, L\,|\nabla f|^2 + \partial_t Q - \partial_X^2 Q\, |\nabla|\nabla f|^2|^2 \\ &= 2\,(1-\partial_X Q)\,\Gamma_2(f,f) - \partial_X^2 Q\,|\nabla|\nabla f|^2|^2 + \partial_t Q\,, \end{aligned}$$

and therefore if $\partial_X^2 Q \le 0$, $1 - \partial_X Q \ge 0$, then we have

$$\begin{aligned} LG &= -F + 2\,t\,(1-\partial_X Q)\,\Gamma_2(f,f) - t\,\partial_X^2 Q\,|\nabla|\nabla f|^2|^2 + t\,\partial_t Q \\ &\ge -F + 2\,t\,(1-\partial_X Q)\Big(\frac{1}{n}\,(|\nabla f|^2 - f_t)^2 - K\,|\nabla f|^2\Big) + t\,\partial_t Q\,. \end{aligned}$$

However $|\nabla f|^2 - f_t = F + Q$, so that

$$\begin{aligned} LG &\ge -F + 2\,t\,(1-\partial_X Q)\Big(\frac{1}{n}\,(F+Q)^2 - K\,|\nabla f|^2\Big) + t\,\partial_t Q \\ &= 2\,t\,(1-\partial_X Q)\,\frac{F^2}{n} + \Big(-1 + 2\,t\,(1-\partial_X Q)\,\frac{2\,Q}{n}\Big)F \\ &\quad + 2\,t\,(1-\partial_X Q)\Big(\frac{Q^2}{n} - K\,|\nabla f|^2\Big) + t\,\partial_t Q\,. \end{aligned}$$

Thus, if

$$\partial_X^2 Q \le 0\,, \qquad \lim_{t\to 0} t\,Q(t,X) \le 0\,, \tag{89}$$

$$\frac{4\,t\,Q}{n}\,(1-\partial_X Q) \ge 1\,, \qquad 1 - \partial_X Q > 0\,, \tag{90}$$

and

$$\partial_t Q + 2\,(1-\partial_X Q)\Big(\frac{Q^2}{n} - K\,X\Big) \ge 0\,, \tag{91}$$

for all $t > 0$, $X \ge 0$, then, if the manifold is compact,

$$|\nabla f|^2 - f_t \le Q(t, |\nabla f|^2)\,, \qquad \text{for all } t > 0\,. \tag{92}$$

Theorem 13. *Let* Ric $\geq -K$ *for some non-negative constant* K, *and let* u *be a positive solution of the heat equation and* $f = \log u$. *Then*

$$|\nabla f|^2 - f_t \leq \sqrt{n\,K}\,\sqrt{|\nabla f|^2 + \frac{n}{2\,t} + \frac{n\,K}{4}} + \frac{n}{2\,t}\,. \tag{93}$$

PROOF. For simplicity, let $a = n/(2\,t) + n\,K/4$. Then

$$Q(t,X) = m\,\sqrt{X+a} + \frac{n}{2\,t}\,, \qquad m = \sqrt{n\,K}\,.$$

Therefore

$$\partial_X^2 Q = -\frac{m}{4\,(X+a)\,\sqrt{X+a}} \leq 0\,,$$

and

$$\begin{aligned}
\partial_t Q &+ 2\,(1-\partial_X Q)\Big(\frac{Q^2}{n} - K\,X\Big) \\
&= \frac{2\,m\,\sqrt{X+a}}{t} - \frac{m^2}{t} - \frac{n\,m}{4\,t^2\sqrt{X+a}} + 2\Big(1 - \frac{m}{2\,\sqrt{X+a}}\Big)\frac{a}{n} \\
&\geq \frac{m}{t}\Big((2\,\sqrt{a} - m) - \frac{n}{4\,t\,\sqrt{a}}\Big) \\
&= \frac{m}{t}\Big(\frac{4\,a - m^2}{2\,\sqrt{a} + m} - \frac{n}{4\,t\,\sqrt{a}}\Big) \\
&= \frac{m}{t}\Big(\frac{2\,n}{t\,(2\,\sqrt{a} + m)} - \frac{n}{4\,\sqrt{a}\,t}\Big) \\
&\geq \frac{m}{t}\Big(\frac{2\,n}{t\,(4\,\sqrt{a})} - \frac{n}{4\,\sqrt{a}\,t}\Big) \\
&> 0\,.
\end{aligned}$$

Thus condition (91) is satisfied.

Let us now check the condition (90). It is easily seen that

$$\begin{aligned}
\frac{4\,Q}{n}\,(1-\partial_X Q) &= \frac{4\,m}{n}\Big(\sqrt{X+a} - \frac{m}{2}\Big) + \frac{1}{t}\Big(2 - \frac{m}{\sqrt{X+a}}\Big) \\
&\geq \frac{4\,m}{n}\Big(\sqrt{\frac{n}{2\,t} + \frac{n\,K}{4}} - \frac{\sqrt{n\,K}}{2}\Big) + \frac{1}{t}\left(2 - \frac{m}{\sqrt{\frac{n}{2\,t} + \frac{n\,K}{4}}}\right).
\end{aligned}$$

Therefore, when $n/(2t) \geq 3nK/4$,

$$2 - \frac{m}{\sqrt{\frac{n}{2t} + \frac{nK}{4}}} \geq 1$$

so that

$$\frac{4Q}{n}(1 - \partial_X Q) \geq \frac{1}{t} ,$$

and when $n/(2t) \leq 3nK/4$,

$$\frac{4Q}{n}(1 - \partial_X Q) \geq \frac{2m}{t} \frac{1}{\sqrt{\frac{n}{2t} + \frac{nK}{4}} + \frac{\sqrt{nK}}{2}} \geq \frac{4}{3t} > \frac{1}{t} .$$

Hence condition (90) is satisfied. Therefore we proved Theorem 13 for compact manifolds. If the manifold is non-compact, we may use the generalised maximum principle to go through the proof.

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References.

[1] Bakry, D., Emery, M., Diffusions hypercontractives. *Sém. de Probab. XIX.* Lecture Notes in Math. **1123**, 177-206, 1985.

[2] Bakry, D., Qian, Z., Gradient estimates and Harnack inequality on a complete manifold. Preprint, 1996.

[3] Bakry, D., Qian, Z., On Harnack estimates for positive solutions of the heat equation on a complete manifold. *C. R. Acad. Sci. Paris* **324** (1997), 1037-1042.

[4] Cheeger, J., Yau,S.-T., A lower bound for the heat kernel. *Comm. Pure Appl. Math.* **34** (1981), 465-480.

[5] Cheng, S. Y., Yau, S.-T., Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.* **28** (1975), 333-354.

[6] Davies, E. B., *Heat Kernels and Spectral Theory.* Cambridge Tracts in Math. Cambridge Univ. Press **92**, 1989.

[7] Gilbarg, D., Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 1977.

[8] Hamilton, R. S., A matrix Harnack estimate for the heat equation. *Comm. in Analysis and Geometry.* **1** (1993), 113-126.

[9] Li, P., Yau S.-T., On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156** (1986), 153-201.

[10] Schoen, R., Yau, S.-T., *Lectures on Differential Geometry.* International Press, 1995.

[11] Sturm, K.-T., Heat kernel bounds on manifolds. *Math. Ann.* **292** (1992), 149-162.

[12] Yau, S.-T., Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.* **28** (1975), 201-228.

[13] Yau, S.-T., Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. *Indiana Univ. Math. J.* **25** (1976), 659-670.

[14] Yau, S.-T., On the Harnack inequalities of partial differential equations. *Comm. in Analysis and Geometry* **2** (1994), 431-450.

[15] Yau, S.-T., Harnack inequality for non-self-adjoint evolution equations. *Math. Research Lett.* **2** (1995), 387-399.

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Parabolic Harnack inequality and estimates of Markov chains on graphs

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Abstract. On a graph, we give a characterization of a parabolic Harnack inequality and Gaussian estimates for reversible Markov chains by geometric properties (volume regularity and Poincaré inequality).

1. Introduction.

Consider the standard random walk with kernel $p_n(x,y)$ on a graph Γ with polynomial volume growth. Under which conditions does one have the following Gaussian estimates?

$$\frac{c}{V(x,\sqrt{n})}\, e^{-Cd(x,y)^2/n} \le p_n(x,y) \le \frac{C}{V(x,\sqrt{n})}\, e^{-cd(x,y)^2/n}\,,$$

where $V(x,n)$ is the cardinal of the ball of center x and radius n. Note first that $p_n(x,y)$ may be null for $d(x,y) > n$ or for $d(x,y) \not\equiv n \pmod 2$. Thus we will consider only $d(x,y) \le n$ and graphs where all vertices are loops. With these precisions, the Gaussian estimates were proved when Γ is a group in [15]. They were also proved for linear volume growth in [5] and it was there conjectured that they were true for polynomial growth under an isoperimetric assumption such as Poincaré inequality. Indeed in the continuous setting of Riemannian manifolds, they were proved first for non-negative Ricci curvature in [19] and then under Poincaré inequality assumption in [12], [30]. All these proofs are based on a parabolic Harnack inequality and [30] shows that the Poincaré inequality is the good isoperimetric assumption.

The aim of this paper, which is announced in [10], is to prove the conjecture in [5] and more precisely to give a characterization of the parabolic Harnack inequality or the Gaussian estimates by geometric properties (volume regularity and Poincaré inequality) which is the discrete counterpart of the main result in [30]. A precise statement is proposed in Section 1.4 after some definitions. The main part, the proof of the Harnack inequality, is an application of J. Moser's method [21], [22], [23]. His approach is presented on Euclidian spaces $\mathbb{R}^n$ but shows clearly the contribution of Poincaré and Sobolev inequalities. That's why it has been adapted to many different settings.

As far as graphs are concerned, elliptic versions (without the time variable) of the Harnack inequality have been proved in [20] with a special isoperimetric assumption and in [9], [16], [26] by J. Moser's iterative method with Poincaré inequality. The discretization of the space raised some technical problems but the proof could go through. It is much more intricate to deal with both discretizations (space and time), especially to obtain Cacciopoli inequalities. Section 1.5 is an attempt to show these difficulties and their origin. Because of these criss-crossing discretizations difficulties, we have tried to prove a continuous-time parabolic Harnack inequality on graphs and this raised only solvable technical problems like the elliptic version.

One application of the Harnack inequality is another proof of Hölder regularity (see Section 4.1) for solutions of the elliptic/parabolic equation (theorem of J. Nash [24]). Another application is that it yields Gaussian estimates. The study of these estimates in the mixed setting (discrete geometry, continuous time) in [8], [25] has been helpful because at first we only prove the Harnack inequality in this setting.

At this Gaussian estimates step, it is possible to deduce discrete-time results from the continuous-time ones. This is the crucial point of this paper because all other steps are more or less adaptations of known technics which are fully reviewed in [30]. This strategy –to work on the continuous-time setting and to compare with the discrete-time setting– is also employed in [31, Section 1.4.1]. One side of the comparison is stated in Theorem 3.6 and may be used again. The other side will depend closely on the problem considered.

To deduce Gaussian estimates from the Harnack inequality is the classical (chronological after J. Moser's work) way to introduce this theory. The reverse order based on J. Nash's ideas [24] and completed in [11] is also useful here because our discrete-time Gaussian estimates yield the discrete-time Harnack inequality, which is finally proven after a return trip to Gaussian estimates.

Let us note that we aimed at not using any algebraic structure (and we get an equivalence, which proves that this structure plays no role in fact). Similarly, the authors of [3] tried to extend related results to a more general class than Cayley graphs (strongly convex subgraphs of homogeneous graphs) and in [33] some estimates are obtained still on the particular graphs $\mathbb{Z}^n$ (with a continuous time) but for non-uniform transitions.

1.1. The geometric setting.

Let Γ be an infinite set and $\mu_{xy} = \mu_{yx} \geq 0$ a symmetric weight on $\Gamma\times\Gamma$. It induces a graph structure if we call x and y neighbours ($x \sim y$) when $\mu_{xy} \neq 0$ (note that loops are allowed). We will assume that this graph is connected and locally uniformly finite –this means there exists N, such that for all $x \in \Gamma$, $\#\{y : \ y \sim x\} \leq N$ and it is implied by the geometric conditions (see below) $DV(C_1)$ or $\Delta(\alpha)$. Vertices are weighted by $m(x) = \sum_{y\sim x} \mu_{xy}$. The graph is endowed with its natural metric (the smallest number of edges of a path between two points). We define balls (for r real) $B(x,r) = \{y : \ d(x,y) \leq r\}$ and the volume of a subset A of Γ, $V(A) = \sum_{x\in A} m(x)$. We will write $V(x,r)$ for $V(B(x,r))$.

We shall consider the following geometric conditions:

Definition 1.1. *The weighted graph* (Γ,μ) *satisfies the volume regularity* (*or doubling volume property*) $DV(C_1)$ *if*

$$V(x,2\,r) \leq C_1\, V(x,r)\,, \qquad \textit{for all } x\in\Gamma\,,\ \textit{for all } r\in\mathbb{R}^+\,.$$

This implies for $r \geq s$ that (the square brackets denote the integer part)

$$\begin{aligned} V(x,r) &\leq V(x, 2^{[\log(r/s)/\log 2]+1}\, s) \\ &\leq C_1\, C_1^{\log(r/s)/\log 2}\, V(x,s) \\ &= C_1 \Big(\frac{r}{s}\Big)^{\log C_1/\log 2} V(x,s)\,. \end{aligned}$$

Definition 1.2. *The weighted graph* (Γ, μ) *satisfies the Poincaré inequality* $P(C_2)$ *if*

$$\sum_{x \in B(x_0,r)} m(x)\, |f(x) - f_B|^2 \leq C_2\, r^2 \sum_{x,y \in B(x_0,2r)} \mu_{xy}\, (f(y) - f(x))^2\,,$$

for all $f \in \mathbb{R}^\Gamma$, *for all* $x_0 \in \Gamma$, *for all* $r \in \mathbb{R}^+$, *where*

$$f_B = \frac{1}{V(x_0,r)} \sum_{x \in B(x_0,r)} m(x)\, f(x)\,.$$

Some methods to obtain this Poincaré inequality on a graph are proposed in [6].

Definition 1.3. *Let* $\alpha > 0$, (Γ, μ) *satisfies* $\Delta(\alpha)$ *if*

$$x \sim y \text{ implies } \mu_{xy} \geq \alpha\, m(x)\,, \qquad \text{for all } x \in \Gamma\,,\ x \sim x\,.$$

Two assertions are contained in this definition. The fact that $\mu_{xy} \neq 0$ implies $\mu_{xy} \geq \alpha\, m(x)$ is a local ellipticity property –it may be understood as a local volume regularity if we see the graph as a network. It implies that the graph is locally uniformly finite with $N = 1/\alpha$ (so does also $DV(C_1)$ with $N = C_1^2$). Only this first assertion is needed for continuous-time results (such as Theorem 2.1). The second assertion is that $\mu_{xx} \geq \alpha\, m(x)$ (or $p(x,x) \geq \alpha$ with the notations of next section). It will be used in Section 3.2 to compare the discrete-time and continuous-time Markov kernels. This condition appears in [4] where the authors prove that it implies the analyticity of the Markov operator. This fact is used for instance in [27] to obtain temporal regularity.

If one considers a weighted graph (Γ, μ) which satisfies only the first assertion, for instance the standard random walk on $\mathbb{Z}$ ($\mu_{mn} = 1$ if $|m - n| = 1$, $\mu_{mn} = 0$ otherwise), one can study the graph $(\Gamma, \mu^{(2)})$ where

$$\mu^{(2)}_{xy} = \sum_z \frac{\mu_{xz}\, \mu_{zy}}{m(z)}\,.$$

With the notations of next section, this gives the iterated kernel

$$p^{(2)}(x,y) = p_2(x,y) \qquad \text{and} \qquad m^{(2)}(x) = m(x)\,.$$

The point is that

$$\mu^{(2)}_{xx} = \sum_z \frac{(\mu_{xz})^2}{m(z)} \geq \sum_{z\sim x} \alpha\, \mu_{xz} = \alpha\, m(x)\,.$$

Thus $(\Gamma, \mu^{(2)})$ satisfies the complete assumption $\Delta(\alpha^2)$. Indeed, if $\mu^{(2)}_{xy} \neq 0$, there exists z_0 such that $\mu_{xz_0} \neq 0$ and $\mu_{z_0 y} \neq 0$ and

$$\mu^{(2)}_{xy} \geq \frac{\mu_{xz_0}\, \mu_{z_0 y}}{m(z_0)} \geq \alpha\, \mu_{xz_0} \geq \alpha^2\, m(x)\,.$$

To extract afterwards from the results for $(\Gamma, \mu^{(2)})$ some consequences for (Γ, μ), we must be careful that $(\Gamma, \mu^{(2)})$ may not be connected (see the standard random walk on $\mathbb{Z}$).

1.2. Markov chains and parabolic equations.

To the weighted graph we associate discrete-time and continuous-time reversible Markov kernels. Set $p(x,y) = \mu_{xy}/m(x)$, the discrete kernel $p_n(x,y)$ is defined by

$$\begin{cases} p_0(x,z) = \delta(x,z)\,, \\ p_{n+1}(x,z) = \sum_y p(x,y)\, p_n(y,z)\,. \end{cases} \tag{1.1}$$

This kernel is not symmetric but

$$\frac{p_n(x,y)}{m(y)} = \frac{p_n(y,x)}{m(x)}\;.$$

We keep this notation which represents the probability to go from x to y in n steps but it may also be interesting to think to the density

$$h_n(x,y) = \frac{p_n(x,y)}{m(y)}$$

which is symmetric and is the right analog of a kernel on a continuous space.

We will say that u satisfies the (discrete-time) parabolic equation on (n,x) if

$$m(x)\, u(n+1,x) = \sum_y \mu_{xy}\, u(n,y)\,. \tag{1.2}$$

It is the case of $p.(\cdot, y)$.

Note that we have a weighted geometry and we consider only the canonic parabolic equation on it. Imagine we had a non-weighted geometry (this means volume regularity and Poincaré inequality without the weights $m(x)$ and μ_{xy}) and any parabolic equation with a uniform ellipticity constant, then we would consider the geometry weighted by the coefficients of the equation. Because of the ellipticity, the geometric assumptions on the non-weighted geometry would yield those on the weighted geometry. Our background is a little more general since the ellipticity constant ($\Delta(\alpha)$) is not uniform but above all, as A. Grigor'yan pointed out to us, its presentation is cleaner because these geometric assumptions are always applied with the weights.

Note also that every reversible Markov chain can be obtained as above: starting from the Markov kernel p and its invariant measure m, one constructs $\mu_{xy} = p(x, y)\, m(x)$.

Definition 1.4. *The weighted graph (Γ, μ) satisfies the Gaussian estimates $G(c_l, C_l, C_r, c_r)$ (all constants are positive) if*

$$d(x,y) \le n \ \textit{implies}\ \frac{c_l\, m(y)}{V(x,\sqrt{n})}\, e^{-C_l d(x,y)^2/n} \le p_n(x,y) \le \frac{C_r\, m(y)}{V(x,\sqrt{n})}\, e^{-c_r d(x,y)^2/n}\,.$$

Of course, if $d(x, y) > n$ then $p_n(x, y) = 0$. On Euclidian spaces, these estimates were first proved for fundamental solutions of parabolic equations in [1].

The continuous-time Markov kernel may be defined by

$$\mathcal{P}_t(x,z) = e^{-t} \sum_{k=0}^{+\infty} \frac{t^k}{k!}\, p_k(x,z)\,.$$

Like the discrete kernel, it satisfies

$$\frac{\mathcal{P}_t(x,y)}{m(y)} = \frac{\mathcal{P}_t(y,x)}{m(x)}\,.$$

It is also the solution for $(t, x) \in \mathbb{R}^+ \times \Gamma$ of

$$\begin{cases} \mathcal{P}_0(x,z) = \delta(x,z)\,, \\ m(x)\, \dfrac{\partial}{\partial t}\, \mathcal{P}_t(x,z) = \displaystyle\sum_y \mu_{xy}\, (\mathcal{P}_t(y,z) - \mathcal{P}_t(x,z))\,. \end{cases}$$

Indeed,

$$
\begin{aligned}
\frac{\partial}{\partial t}\,\mathcal{P}_t(x,z) &= e^{-t}\Big(\sum_{k=1}^{+\infty}\frac{k\,t^{k-1}}{k!}\,p_k(x,z)-\sum_{k=0}^{+\infty}\frac{t^k}{k!}\,p_k(x,z)\Big)\\
&= e^{-t}\Big(\sum_{k=1}^{+\infty}\frac{t^{k-1}}{(k-1)!}\sum_y p(x,y)\,p_{k-1}(y,z)-\sum_{k=0}^{+\infty}\frac{t^k}{k!}\,p_k(x,z)\Big)\\
&= \sum_y p(x,y)\,(\mathcal{P}_t(y,z)-\mathcal{P}_t(x,z))\,.
\end{aligned}
$$

Therefore we will say that u satisfies the (continuous-time) parabolic equation on (t,x) if

$$
m(x)\,\frac{\partial}{\partial t}\,u(t,x)=\sum_y \mu_{xy}\,(u(t,y)-u(t,x))\,. \tag{1.3}
$$

1.3. Parabolic Harnack inequalities.

These inequalities apply to positive solutions of the parabolic equations on cylinders (products of a time interval and a ball). Let us make this precise on the boundary of the cylinders. We shall say that u is a non-negative solution on $Q=I\times B(x_0,r)$ if it is the trace of a non-negative solution on $I\times B(x_0,r+1)$ which satisfies (1.2) or (1.3) everywhere on Q. For instance in the continuous case, this implies: for all $(t,x)\in Q$,

$$
u(t,x)\geq 0\,,
$$

for all $t\in I$, for all $x\in B(x_0,r-1)$,

$$
m(x)\Big(\frac{\partial}{\partial t}\,u(t,x)+u(t,x)\Big)=\sum_y \mu_{xy}\,u(t,y)\,,
$$

for all $t\in I$, $d(x_0,x)=[r]$ imply

$$
m(x)\Big(\frac{\partial}{\partial t}\,u(t,x)+u(t,x)\Big)\geq \sum_{y\in B(x_0,r)} \mu_{xy}\,u(t,y)\,. \tag{1.4}
$$

Definition 1.6. *Set $\eta \in]0,1[$ and $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$, (Γ,μ) satisfies the continuous-time parabolic Harnack inequality*

$$\mathcal{H}(\eta,\theta_1,\theta_2,\theta_3,\theta_4,C)$$

if for all x_0, s, r and every non-negative solution on $Q = [s, s+\theta_4 r^2] \times B(x_0,r)$, we have

$$\sup_{Q_\ominus} u \leq C \inf_{Q_\oplus} u\,,$$

where $Q_\ominus = [s+\theta_1\, r^2, s+\theta_2\, r^2] \times B(x_0,\eta\, r)$ and $Q_\oplus = [s+\theta_3\, r^2, s+\theta_4\, r^2] \times B(x_0,\eta\, r)$.

Let us explain the choice of the boundary condition. For $r<1$, Q has no interior so we just have (1.4) but this is sufficient to obtain the inequality since it gives a lower bound for $(\partial/\partial t)\, u(t,x_0)$.

If we assume $\Delta(\alpha)$ and $\mathcal{H}(\eta,\theta_1,\theta_2,\theta_3,\theta_4,C)$, then for all $\eta' \in]0,1[$ and $0<\theta_1'<\theta_2'<\theta_3'<\theta_4'$, there exists C' such that

$$\mathcal{H}(\eta',\theta_1',\theta_2',\theta_3',\theta_4',C')$$

is true. Take two points $(t_\ominus,x_\ominus)$ and $(t_\oplus,x_\oplus)$ in $Q'_\ominus$ and $Q'_\oplus$. For r big enough, there is a decomposition $x_\ominus = x_0,\ldots,x_n = x_\oplus$ and $t_\ominus = t_0 < \cdots < t_n = t_\oplus$ –where n depends only on the η, η', θ_i and θ_i''s– such that we can obtain $u(t_i,x_i) \leq C\, u(t_{i+1},x_{i+1})$. So we can take $C' = C^n$. For r bounded, the condition $\Delta(\alpha)$ gives the inequality. For simplicity, we will denote

$$\mathcal{H}(C_{\mathcal{H}}) = \mathcal{H}(0.99, 0.01, 0.1, 0.11, 100, C_{\mathcal{H}})\,.$$

These coefficients have been chosen for convenience when we apply this inequality (see typically Propositions 3.1 or 3.4). We will write $u(t_\ominus,x_\ominus) \leq C_{\mathcal{H}}\, u(t_\oplus,x_\oplus)$ as soon as $x_\oplus$ and $x_\ominus$ are in the contraction of a ball on which u is a solution and "there is time" between $t_\ominus$ and $t_\oplus$ as well as before $t_\ominus$. We will not have to bother with technical coefficients if they don't exceed 10.

Definition 1.6. *Set $\eta \in]0,1[$ and $0<\theta_1<\theta_2<\theta_3<\theta_4$, (Γ,μ) satisfies the discrete-time parabolic Harnack inequality $H(\eta,\theta_1,\theta_2,\theta_3,\theta_4,C)$ if for all $x_0\in\Gamma$, $s\in\mathbb{R}$, $r\in\mathbb{R}^+$ and every non-negative solution on $Q = (\mathbb{Z}\cap[s,s+\theta_4 r^2]) \times B(x_0,r)$, we have*

$$(n_\ominus,x_\ominus)\in Q_\ominus, (n_\oplus,x_\oplus)\in Q_\oplus \text{ and } d(x_\ominus,x_\oplus) \leq n_\oplus - n_\ominus$$

implies

$$u(n_\ominus, x_\ominus) \le C\, u(n_\oplus, x_\oplus)\,,$$

where $Q_\ominus = (\mathbb{Z} \cap [s + \theta_1\, r^2, s + \theta_2\, r^2]) \times B(x_0, \eta\, r)$ *and* $Q_\oplus = (\mathbb{Z} \cap [s + \theta_3\, r^2, s + \theta_4\, r^2]) \times B(x_0, \eta\, r)$.

If the condition $d(x_\ominus, x_\oplus) \le n_\oplus - n_\ominus$ is not satisfied, $u(n_\ominus, x_\ominus)$ has no influence on $u(n_\oplus, x_\oplus)$. It is always satisfied if $r \ge 2\,\eta/(\theta_3 - \theta_2)$ and in this case we can write

$$\sup_{Q_\ominus} u \le C \inf_{Q_\oplus} u\,.$$

The same remark as above holds for this inequality and we will denote

$$H(C_H) = H(0.99, 0.01, 0.1, 0.11, 100, C_H)\,.$$

1.4. Statement of the results.

Here is our main result:

Theorem 1.7. *The three following properties are equivalent.*

i) *There exist* $C_1, C_2, \alpha > 0$ *such that* $DV(C_1)$, $P(C_2)$ *and* $\Delta(\alpha)$ *are true.*

ii) *There exists* $C_H > 0$ *such that* $H(C_H)$ *is true.*

iii) *There exist* $c_l, C_l, C_r, c_r > 0$ *such that* $G(c_l, C_l, C_r, c_r)$ *is true.*

Theorem 3.8 states that i) implies iii), Theorem 3.10 that iii) implies ii) and Theorem 3.11 that ii) implies i).

The first part (i) implies iii)) is the most difficult and an intermediate result is:

ii)' There exists $C_{\mathcal{H}} > 0$ such that $\mathcal{H}(C_{\mathcal{H}})$ is true,

which is proven in Section 2 by a Moser type iteration argument. In fact, ii)' is also equivalent to the three properties, since we can prove ii)' implies i) the same way we prove ii) implies i). In Section 3, we complete the proof. First ii)' implies estimates for $\mathcal{P}_t$, which yield estimates iii) for p_n by comparison. Then iii) implies ii) and ii) implies i). In the last section, two properties about Hölder regularity and Green function are deduced for graphs which satisfy these properties.

Let us note that it is straightforward that ii) and iii) imply the hypothesis $\Delta(\alpha)$. For iii), just apply the lower bound to $p_1(x,y)$ where $y \in B(x,1)$. And for ii), set $n = -0.5$ to obtain $p_0(x,x) \leq C_H\, p_1(x,y)$.

This result connects with [15] and [5] since in groups or in graphs with linear volume growth, Poincaré inequality is always satisfied. In Euclidian graphs $\mathbb{Z}^n$, the estimates which are well-known for uniform transitions (this is for instance a consequence of the result in groups) are here proved for non-uniform transitions (μ_{xy} doesn't depend on $y - x$).

1.5. Continuous or discrete time.

To prove the Harnack inequality, we will use analytic methods which yield Cacciopoli inequalities. For these methods, the continuous time is naturally more convenient. We may have an idea of the problems if we look what happens on the two points graph $\Gamma = \{a,b\}$. Choose $p(a,a) = p(b,b) = \alpha$ and $p(a,b) = p(b,a) = 1 - \alpha$, this may be done if we set $\mu_{aa} = \mu_{bb} = \alpha$ and $\mu_{ab} = 1 - \alpha$. This gives

$$\begin{cases} p_n(a,a) = \dfrac{1 + (2\,\alpha - 1)^n}{2}\,, \\[2ex] p_n(a,b) = \dfrac{1 - (2\,\alpha - 1)^n}{2}\,, \end{cases} \qquad \begin{cases} \mathcal{P}_t(a,a) = \dfrac{1 + e^{(2\alpha - 2)t}}{2}\,, \\[2ex] \mathcal{P}_t(a,b) = \dfrac{1 - e^{(2\alpha - 2)t}}{2}\,. \end{cases}$$

Of course if $\alpha = 1$, there is no link between the two points. Now, if $\alpha \neq 1$, $\mathcal{P}$ is always a simple relaxation ($\mathcal{P}_t(a,a) > \mathcal{P}_t(a,b)$) whereas, for $\alpha < 1/2$, p is an oscillating relaxation or worse (for $\alpha = 0$) a pure oscillation.

The first conclusion is that we have to force a minimum value on the diagonal of the Markov kernel if we want a discrete-time parabolic Harnack inequality or estimates from below. Indeed, they are not satisfied by this example for $\alpha = 0$. This has nothing to do with the fact that the graph is finite, take the standard random walk on $\mathbb{Z}$ and observe its effect on $u(0,z) = z \bmod 2$. We obtain $u(n,z) = (n + z) \bmod 2$.

What plays a role is condition $\Delta(\alpha)$ and particulary the fact that $p(x,x) \geq \alpha$. We have extended this condition to $p(x,y) \geq \alpha$ for $x \sim y$ so that our results are true for low values of n, think for instance to the lower bound of $p_1(x,y) = p(x,y)$. Besides, lower bounds for $d(x,y) = n$

are the consequence of lower bounds for $n = 1$, see the proof of Theorem 3.8.

The second conclusion is that the behaviour of the discrete-time Markov chain is more difficult to control. In addition to the usual heat relaxation, there may be another phenomenon of relaxation of the oscillating errors due to the discretization of the time.

One more attempt to show the difficulty of adapting the analytic methods to the discrete time. Consider the proof of the Cacciopoli inequality. When time and space are continuous, take u such that $\partial u/\partial t = \triangle u$ and a compactly supported cut-off function ψ to integrate by parts.

$$\begin{aligned}\frac{1}{2}\iint \psi^2 \,\frac{\partial(u^2)}{\partial t} &= \iint \psi^2\, u \,\triangle u \\ &= -\iint \nabla(\psi^2\, u)\cdot \nabla u \\ &= -\iint \psi^2\, |\nabla u|^2 - \iint 2\,\psi\, u\, \nabla\psi \cdot \nabla u\,.\end{aligned}$$

Since

$$-\iint 2\,\psi\, \nabla\psi\, u\, \nabla u \le \frac{1}{2}\iint \psi^2\, |\nabla u|^2 + 2\iint |\nabla\psi|^2\, u^2$$

one gets

$$\frac{1}{2}\iint \psi^2\,\frac{\partial(u^2)}{\partial t} + \frac{1}{2}\iint \psi^2\,|\nabla u|^2 \le 2\iint |\nabla\psi|^2\, u^2\,.$$

This inequality is essential to estimate $\|\nabla u\|_2$ with $\|u\|_2$, which with the Sobolev inequality gives estimates between mean values for exponents of the same sign. Let us try to adapt this argument to discrete time. Note that (1.2) may be written in the following way

$$m(x)\,(u(n+1,x) - u(n,x)) = \sum \mu_{xy}\,(u(n,y) - u(n,x))\,.$$

For simplicity, we will forget about the cut-off function (take $u(n,\cdot)$

compactly supported). Write

$$\begin{aligned}
2\sum_x m(x)\,u(n,x)\,(u(n+1,x)-u(n,x))\\
&= 2\sum_{x,y}\mu_{xy}\,u(n,x)\,(u(n,y)-u(n,x))\\
&= \sum_{x,y}\mu_{xy}\,u(n,x)\,(u(n,y)-u(n,x))\\
&\quad + \sum_{x,y}\mu_{yx}\,u(n,y)\,(u(n,x)-u(n,y))\\
&= -\sum_{x,y}\mu_{xy}\,(u(n,y)-u(n,x))^2\,.
\end{aligned}$$

This is nice but we have not taken the exact time differentiation of $\sum_x m(x)\,u^2(n,x)$, that is,

$$\begin{aligned}
\sum_x m(x)\,(u^2(n+1,x)-u^2(n,x))\\
&= 2\sum_x m(x)\,u(n,x)\,(u(n+1,x)-u(n,x))\\
&\quad + \sum_x m(x)\,(u(n+1,x)-u(n,x))^2\\
&= -\sum_{x,y}\mu_{xy}\,(u(n,y)-u(n,x))^2\\
&\quad + \sum_x m(x)\,(u(n+1,x)-u(n,x))^2\,.
\end{aligned}$$

Fortunately, if we suppose that $\mu_{xx}\geq\alpha\,m(x)$, then

$$\begin{aligned}
\sum_x m(x)\,(u(n+1,x)-u(n,x))^2\\
&= \sum_x\frac{1}{m(x)}\Big(\sum_y\mu_{xy}\,(u(n,y)-u(n,x))\Big)^2\\
&\leq \sum_x\frac{1}{m(x)}\Big(\sum_{y\neq x}\mu_{xy}\Big)\Big(\sum_y\mu_{xy}\,(u(n,y)-u(n,x))^2\Big)\\
&\leq (1-\alpha)\sum_{x,y}\mu_{xy}\,(u(n,y)-u(n,x))^2\,.
\end{aligned}$$

This yields

$$\sum_x m(x)\left(u^2(n+1,x)-u^2(n,x)\right) \le -\alpha \sum_{x,y} \mu_{xy}\left(u(n,y)-u(n,x)\right)^2 .$$

The constant α has been used to control the errors due to the discrete time. But these manipulations seem far more intricate when we deal with subsolutions or the logarithm of u (and cut-off functions). Therefore, we won't try to apply Moser's iterative technique directly to solutions of the discrete-time parabolic equation.

2. Harnack inequality for solutions of the continuous-time parabolic equation.

Theorem 2.1. *Assume* (Γ,μ) *satisfies* $DV(C_1)$, $P(C_2)$ *and* $\Delta(\alpha)$. *Then, there exists* $C_{\mathcal{H}}$ *such that* $\mathcal{H}(C_{\mathcal{H}})$ *is true.*

The proof is an adaptation of [30]. The strategy is Moser's iterative technique [22], that is to prove inequalities involving the mean values

$$\mathcal{M}(u,p,[s_1,s_2]\times B) = \Big(\frac{1}{(s_2-s_1)\,V(B)} \sum_{x\in B} \int_{s_1}^{s_2} m(x)\, u^{2p}(t,x)\, dt\Big)^{1/p} .$$

The idea is we get the infimum when $p \longrightarrow -\infty$ and the supremum when $p \longrightarrow +\infty$. Thus we want to prove a series of inequalities between $-\infty$ and $+\infty$. To improve the exponent of a mean value, the Sobolev inequality proved in Section 2.1 is helpful. One application of this inequality yields an elementary step of the iterative technique proved in Section 2.2. The iteration gives inequalities between the extrema and mean values as stated in Section 2.3. The most difficult step is between negative and positive values. Here we use an improvement of the initial version [22] proposed in [23] with an idea of E. Bombieri [2]. This is the object of Section 2.4 and needs a weighted Poincaré inequality stated in Section 2.1.

Throughout this section devoted to the proof of Theorem 2.1, $DV(C_1)$, $P(C_2)$ and $\Delta(\alpha)$ are assumed and $u>0$. The theorem for $u \ge 0$ is then straightforward.

2.1. Poincaré and Sobolev inequalities.

Proposition 2.2 (Weighted Poincaré inequality)**.** *There exists C depending on C_1, C_2 and α such that for all $x_0 \in \Gamma$, $R \in \mathbb{N}$ and $f \in \mathbb{R}^{B(x_0,R)}$,*

$$\sum_{x \in B(x_0,R)} m(x)\, \psi^2(x)\, (f(x) - f_{B\psi})^2 \\ \leq C\, R^2 \sum_{x,y \in B(x_0,R)} \mu_{xy} \min\{\psi^2(x), \psi^2(y)\}\, (f(y) - f(x))^2\,,$$

where $\psi(x) = 1 - d(x_0,x)/R$ and $f_{B\psi}$ is such that the term on the left is minimal, that is

$$f_{B\psi} = \frac{\displaystyle\sum_{x \in B(x_0,R)} m(x)\, \psi^2(x)\, f(x)}{\displaystyle\sum_{x \in B(x_0,R)} m(x)\, \psi^2(x)}\,.$$

Proof. We refer to the proof in [32] based on [17]. Consider $\mathcal{F}$ a collection of balls with the following tree structure: denote one ball B_1 –the root of the tree– and assume that there is a function $B \longmapsto \overline{B}$ from $\mathcal{F} \setminus \{B_1\}$ to $\mathcal{F}$ (denote $B^{(i)}$ its iteration) such that for all $B \in \mathcal{F}$, $\operatorname{rg} B = \inf\{k : B^{(k-1)} = B_1\} < \infty$. Denote $r(B)$ the radius of B, $A[B] = \{\tilde{B} \in \mathcal{F} :\ \text{exists } k \in \mathbb{N},\ \tilde{B}^{(k)} = B\}$ and $B^* = 1.001B$. For our discrete setting, we will need this version of Poincaré inequality where C_2' depends on C_1, C_2 and α.

$$\sum_{x \in B(x_0,r)} m(x)\, |f(x) - f_B|^2 \leq C_2'\, r^2 \sum_{x,y \in B(x_0,1.001r)} \mu_{xy}\, (f(y) - f(x))^2\,,$$

for all $f \in \mathbb{R}^\Gamma$, for all $x_0 \in \Gamma$, for all $r \in \mathbb{R}^+$. It is obtained by an easy covering argument. Again, there may be some problems for small r but then $\Delta(\alpha)$ gives the inequality.

The following lemma will be applied to

$$\mu'_{xy} = \mu_{xy}\, \min\{\psi^2(x), \psi^2(y)\}\,.$$

The notations m' or f'_B should be understood with respect to μ'.

Lemma 2.3. *Assume there exists C such that, for all $B \in \mathcal{F}$, there exists c_B such that*

$$\text{(2.5)} \qquad \begin{cases} \dfrac{c_B}{C}\, m(x) \le m'(x) \le C\, c_B\, m(x)\,, & \textit{for all } x \in B^*\,, \\ \dfrac{c_B}{C}\, \mu_{xy} \le \mu'_{xy} \le C\, c_B\, \mu_{xy}\,, & \textit{for all } x, y \in B^*\,, \end{cases}$$

$$\text{(2.6)} \qquad \#\{B \in \mathcal{F}:\ x \in B^*\} \le C\,, \qquad \textit{for all } x \in \Gamma\,,$$

$$\text{(2.7)} \qquad \mu(B \cap \overline{B}) \ge \frac{\max\{\mu(B), \mu(\overline{B})\}}{C}\,, \qquad \textit{for all } B \in \mathcal{F}\,.$$

Then for every function f,

$$\sum_{x \in \cup_{B\in\mathcal{F}} B} m'(x)\,(f(x) - f'_{B_1})^2 \le 4\, C'_2\, C^8 \sup_{B\in\mathcal{F}} \Big(r^2(B) \sum_{\tilde{B} \in A[B]} \frac{\mu(\tilde{B})}{\mu(B)}\, \mathrm{rg}\, \tilde{B} \Big) \cdot \sum_{x,y \in \cup_{B\in\mathcal{F}} B^*} \mu'_{xy}\,(f(x) - f(y))^2\,.$$

PROOF. In fact condition (2.7) is needed for μ' but because of (2.5), (2.7) implies that

$$\mu'(B \cap \overline{B}) \ge \frac{\max\{\mu'(B), \mu'(\overline{B})\}}{C^3}\,, \qquad \text{for all } B \in \mathcal{F}\,.$$

First note that

$$\begin{aligned} &(f'_B - f'_{\overline{B}})^2 \\ &= \frac{\sum_{x \in B\cap\overline{B}} m'(x)\,((f'_B - f(x)) + (f(x) - f'_{\overline{B}}))^2}{\mu'(B \cap \overline{B})} \\ &\le \frac{2}{\mu'(B \cap \overline{B})} \Big(\sum_{x\in B} m'(x)\,(f(x) - f'_B)^2 + \sum_{x \in \overline{B}} m'(x)\,(f(x) - f'_{\overline{B}})^2 \Big)\,. \end{aligned}$$

These terms $\sum_{x\in B} m'(x)\,(f(x) - f'_B)^2$ satisfy

$$\sum_{x\in B} m'(x)\,(f(x) - f'_B)^2 \le \sum_{x\in B} m'(x)\,(f(x) - f_B)^2$$

$$
\begin{aligned}
&\leq C\, c_B \sum_{x\in B} m(x)\,(f(x)-f_B)^2\\
&\leq C\, c_B\, C_2'\, r^2(B) \sum_{x,y\in B^*} \mu_{xy}\,(f(x)-f(y))^2\\
&\leq C^2\, C_2'\, r^2(B) \sum_{x,y\in B^*} \mu'_{xy}\,(f(x)-f(y))^2\,.
\end{aligned}
$$

We can now prove the lemma,

$$
\begin{aligned}
&\sum_{x\in\cup_{B\in\mathcal{F}}B} m'(x)\,(f(x)-f'_{B_1})^2\\
&\leq \sum_{B\in\mathcal{F}}\sum_{x\in B} m'(x)\,\mathrm{rg}\,B\Big((f(x)-f'_B)^2+\sum_{i=1}^{\mathrm{rg}\,B-1}(f'_{B^{(i-1)}}-f'_{B^{(i)}})^2\Big)\\
&\leq \sum_{B\in\mathcal{F}}\Big(\sum_{\tilde B\in A[B]} 2\,\mu'(\tilde B)\,\mathrm{rg}\,\tilde B\,\frac{2}{\mu'(B)/C^3}\Big)\sum_{x\in B} m'(x)\,(f(x)-f'_B)^2\\
&\leq 4\,C^3\sum_{B\in\mathcal{F}}\Big(\sum_{\tilde B\in A[B]}\frac{\mu'(\tilde B)}{\mu'(B)}\,\mathrm{rg}\,\tilde B\Big)C^2\,C_2'\,r^2(B)\sum_{x,y\in B^*}\mu'_{xy}\,(f(x)-f(y))^2\\
&\leq 4\,C_2'\,C^5\sum_{x,y\in\cup_{B\in\mathcal{F}}B^*} C\sup_{B\in\mathcal{F}}\Big(r^2(B)\sum_{\tilde B\in A[B]}\frac{\mu'(\tilde B)}{\mu'(B)}\,\mathrm{rg}\,\tilde B\Big)\\
&\qquad\qquad\cdot\,\mu'_{xy}\,(f(x)-f(y))^2\,.
\end{aligned}
$$

To finish the proof, we replace $\mu'(\tilde B)$ and $\mu'(B)$ by $\mu(\tilde B)$ and $\mu(B)$ so that another factor C^2 appears.

END OF PROOF OF PROPOSITION 2.2. We will construct $\mathcal{F}$ as a Whitney covering of $B(x_0,R-1)$ by selecting $W_n\subset\{x:\ d(x_0,x)=R-2^n\}$ for $0\leq n\leq N=[\log R/\log 2]$.

$$
\mathcal{F}=\Big\{B(x,r):\ \text{exists } n, x\in W_n \text{ and } r=\frac{2^n}{1.01}\Big\}\cup\{B_1\}\,,
$$

where $B_1=B(x_0,R/1.01)$. For these balls, (2.5) is satisfied. The tree structure will be constructed this way: if $B=B(x,r)$ with $x\in W_n$ we will choose $\overline{B}$ of center $\overline{x}\in W_{n+1}$ such that $d(x,\overline{x})\leq(3/2)\,2^n$ (see the construction of the W_n's below). Thus, $B(x,(2/1.01-3/2)\,2^n)\subset B\cap\overline{B}$ and condition (2.7) is satisfied.

We must check, in order to apply Lemma 2.3, that it was possible to select W_n so that

$$\{x: \ R-2^{n+1} \le d(x_0,x) \le R-2^n\} \subset \bigcup_{x\in W_{n+1}} B\Big(x, \frac{3}{2}\, 2^n\Big)\,,$$

while (2.6) is satisfied. It is a standard Besicovitch covering argument, we choose a minimal W_{n+1} for this property. The key is that the radius of any ball B^* such that $x \in B^*$ is comparable to $R-d(x_0,x)$.

Now let us consider the term

$$\sup_{B\in\mathcal{F}} \Big(r^2(B) \sum_{\tilde B\in A[B]} \frac{\mu(\tilde B)}{\mu(B)} \operatorname{rg} \tilde B\Big)\,.$$

The first point is that $d(x,\overline{x}) \le (3/2)\, 2^n$ implies that $2\,B \subset 2\,\overline{B}$. For $B \in \mathcal{F}\setminus\{B_1\}$, set $n \le N$ such that W_n contains B's center. If we denote $\mathcal{F}_k = \{\tilde B : \tilde B^{(k)} = B\}$ and $A_k = \bigcup_{\tilde B\in\mathcal{F}_k} 2\tilde B$, $A_{k+1} \subset A_k$. But there is more than this inclusion, a ball $\tilde B = B(\tilde x, \tilde r)$ in $\mathcal{F}_k$ is such that $d(x_0,\tilde x) = R-2^{n-k}$ and $\tilde r = 2^{n-k}/1.01$, so that there is a ball of radius $\tilde r/100$ which is included in $2\tilde B$ and in the area $\{y : d(x_0,y) < R-2^{n-1-k}-2\cdot 2^{n-1-k}/1.01\}$ never reached by A_{k+1}. This yields $\mu(A_k\setminus A_{k+1}) \ge \varepsilon\,\mu(A_k)$ and consequently $\mu(A_k) \le e^{-ck}\,\mu(A_0) \le C\,e^{-ck}\,\mu(B)$. Thus,

$$r^2(B) \sum_{\tilde B\in A[B]} \frac{\mu(\tilde B)}{\mu(B)} \operatorname{rg}\tilde B \le 2^{2n} \sum_{k\ge 0} C\,e^{-ck}\,(N+1-n+k) \le C\,2^{2N} \le C\,R^2\,.$$

For the case $B = B_1$, the proof is identical but we refer to A_1 instead of A_0.

To finish the proof, let us compare $m(x)\psi^2(x)$ and $m'(x)$. For $x \ne x_0$, the condition $\Delta(\alpha)$ gives $m(x)\,\psi^2(x) \le m'(x)/\alpha$, we just have to consider $y \sim x$ such that $d(x_0,x) = d(x_0,y)+1$.

Proposition 2.4 (Sobolev-Poincaré inequality)**.** *There exist $\theta > 1$ depending on C_1 and S depending on C_1, C_2 and α such that for every function f on B of radius r,*

$$\begin{aligned}&\Big(\frac{1}{V(B)} \sum_{x\in B} m(x)\, f^{2\theta}(x)\Big)^{1/\theta} \\ &\qquad\le \frac{S}{V(B)} \Big(r^2 \sum_{x,y\in B} \mu_{xy}\,(f(y)-f(x))^2 + \sum_{x\in B} m(x)\, f^2(x)\Big)\,.\end{aligned}$$

In the setting of manifolds, this result which was proven in [29] was the key of the proof of the Harnack inequality after the work [28]. It is adapted to graphs in [9]. A nice abstract version can also be found in [14]. In the notation of this paper, the chain condition will be satisfied for $\lambda < 2$ (like the preceding section where $\lambda = 1.001$). Indeed, consider x next to the boundary of B, the smallest ball B_i of the chain not centered at x must contain x and satisfy $\lambda B_i \subset B$.

2.2. Elementary step of Moser's iterative technique.

As in Section 1.3, we will say that u is a positive sub/supersolution on $Q = I \times B(x_0, r)$ if it is the trace of a positive function on $I \times B(x_0, r+1)$ which is a sub/supersolution everywhere on Q. Precisely, we say that u is a positive subsolution on Q if it is positive and

$$m(x)\,\frac{\partial}{\partial t}\,u(t,x) \le \sum_y \mu_{xy}\,(u(t,y) - u(t,x))\,,$$

for all $t \in I$, for all $x \in B(x_0, r-1)$. And u is a positive supersolution on Q if it is positive and

$$m(x)\,\frac{\partial}{\partial t}\,u(t,x) \ge \sum_y \mu_{xy}\,(u(t,y) - u(t,x))\,,$$

for all $t \in I$, for all $x \in B(x_0, r-1)$,

$$m(x)\Big(\frac{\partial}{\partial t}\,u(t,x) + u(t,x)\Big) \ge \sum_{y \in B(x_0,r)} \mu_{xy}\,u(t,y)\,,$$

for all $t \in I$, for all x such that $d(x_0, x) = [r]$. Let us show the elementary step of Moser's iterative technique. If $Q = I \times B$ where $I = [s_1, s_2]$ and $B = B(x, r)$, note

$$B_\sigma = (1-\sigma)B = B(x, (1-\sigma)\,r)\,,$$

$$I_\sigma = [(1-\sigma^2)\,s_1 + \sigma^2 s_2, s_2]\,,$$

$$I'_\sigma = [s_1, \sigma^2\,s_1 + (1-\sigma^2)\,s_2]\,,$$

$$I''_\sigma = [(1-\sigma^2)\,s_1 + \sigma^2\,s_2, \sigma^2\,s_1 + (1-\sigma^2)\,s_2]\,,$$

$$Q_\sigma = I_\sigma \times B_\sigma,\ Q'_\sigma = I'_\sigma \times B_\sigma \text{ and } Q''_\sigma = I''_\sigma \times B_\sigma\,.$$

Note that

$$(2.8)\quad Q_{\sigma_1+\sigma_2} \subset (Q_{\sigma_1})_{\sigma_2},\ Q'_{\sigma_1+\sigma_2} \subset (Q'_{\sigma_1})'_{\sigma_2} \text{ and } Q''_{\sigma_1+\sigma_2} \subset (Q''_{\sigma_1})''_{\sigma_2}\,.$$

Lemma 2.5. *There is an exponent $\kappa = 2 - 1/\theta$ and a constant $A = A(C_1, S) = A(C_1, C_2, \alpha)$ such that if $B = B(x_0, r)$, $Q = [0, r^2] \times B$, u a positive subsolution in Q and $1/r \le \sigma \le 1/2$, then*

$$\mathcal{M}(u, \kappa, Q_\sigma) \le \Big(\frac{A}{\sigma^2}\Big)^{1/\kappa} \mathcal{M}(u, 1, Q)\,.$$

If u is a supersolution with the same assumptions, then

$$\mathcal{M}(u, \kappa, Q'_\sigma) \le \Big(\frac{A}{\sigma^2}\Big)^{1/\kappa} \mathcal{M}(u, 1, Q)\,.$$

PROOF. Consider the first part, u is a subsolution. Let ψ be a non-negative function in B, with $d(x_0, x) = r$ implies $\psi(x) = 0$, then

$$\begin{aligned}
&\sum_{x\in B} m(x)\,\psi^2(x)\,u(t,x)\,\frac{\partial}{\partial t}\,u(t,x)\\
&\le \sum_{x,y\in B} \mu_{xy}\,\psi^2(x)\,u(t,x)\,(u(t,y)-u(t,x))\\
&= \frac{1}{2}\sum_{x,y\in B} \mu_{xy}\,(\psi^2(x)\,u(t,x)-\psi^2(y)\,u(t,y))\,(u(t,y)-u(t,x))\\
&= \frac{1}{2}\sum_{x,y\in B} \mu_{xy}\,\psi^2(x)\,(u(t,x)-u(t,y))\,(u(t,y)-u(t,x))\\
&\quad + \frac{1}{2}\sum_{x,y\in B} \mu_{xy}\,(\psi^2(x)-\psi^2(y))\,u(t,y)\,(u(t,y)-u(t,x))\,.
\end{aligned}$$

In the last term, we use the inequality $a\,b \le a^2/4 + b^2$.

$$\begin{aligned}
(\psi^2(x) - \psi^2(y))\,u(t,y)\,(u(t,y)-u(t,x))&\\
= \psi(x)\,(\psi(x)-\psi(y))\,u(t,y)\,(u(t,y)-u(t,x))&\\
+ \psi(y)\,(\psi(x)-\psi(y))\,u(t,y)\,(u(t,y)-u(t,x))&\\
\le \frac{1}{4}\,(\psi^2(x)+\psi^2(y))\,(u(t,y)-u(t,x))^2&\\
+ 2\,u^2(t,y)\,(\psi(x)-\psi(y))^2\,.&
\end{aligned}$$

Note that because of the symmetry of the weights μ_{xy},

$$\sum_{x,y\in B} \mu_{xy}\,\psi^2(y)\,(u(t,y)-u(t,x))^2 = \sum_{x,y\in B} \mu_{xy}\,\psi^2(x)\,(u(t,y)-u(t,x))^2 .$$

Thus, (2.9) yields

$$\sum_{x\in B} m(x)\,\psi^2(x)\,u(t,x)\,\frac{\partial}{\partial t}\,u(t,x) + \frac{1}{4}\sum_{x,y\in B} \mu_{xy}\,\psi^2(x)\,(u(t,y)-u(t,x))^2 \tag{2.10}$$
$$\leq \sum_{x,y\in B} \mu_{xy}\,u^2(t,y)\,(\psi(x)-\psi(y))^2 .$$

For u supersolution, the result would be

$$\begin{aligned}&\sum_{x\in B} m(x)\,\psi^2(x)\,u(t,x)\,\frac{-\partial}{\partial t}\,u(t,x)\\ &+\frac{1}{4}\sum_{x,y\in B} \mu_{xy}\,\psi^2(x)\,(u(t,y)-u(t,x))^2\\ &\qquad\leq \sum_{x,y\in B} \mu_{xy}\,u^2(t,y)\,(\psi(x)-\psi(y))^2 .\end{aligned}$$

And then, the same arguments work dealing with I'_σ instead of I_σ. Return now to (2.10), if χ is a smooth function of t, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(\sum_{x\in B} m(x)\,(\chi(t)\,\psi(x)\,u(t,x))^2\Big)\\ &+\frac{\chi^2(t)}{2}\sum_{x,y\in B} \mu_{xy}\,\psi^2(x)\,(u(t,y)-u(t,x))^2\\ &\qquad\leq 2\,\chi^2(t)\sum_{x,y\in B} \mu_{xy}\,u^2(t,y)\,(\psi(x)-\psi(y))^2\\ &\qquad\quad+\sum_{x\in B} m(x)\Big(\frac{\partial}{\partial t}\,\chi^2(t)\Big)u^2(t,x) .\end{aligned}$$

Now we choose $\chi(t)=t/(\sigma\,r)^2\wedge 1$ and ψ so that $d(x_0,x)=r$ implies $\psi(x)=0$ and $\psi\equiv 1$ in B_σ. For this purpose, we took $1/r\leq\sigma$.

Integrating over I yields

$$\begin{cases} \sup_{t\in I_\sigma}\Big(\sum_{x\in B_\sigma} m(x)\,u^2(t,x)\Big) \le \dfrac{10}{(\sigma\,r)^2}\displaystyle\int_I \sum_{x\in B} m(x)\,u^2(t,x)\,dt\,, \\ \dfrac{1}{2}\displaystyle\int_{I_\sigma}\sum_{x,y\in B_\sigma} \mu_{xy}\,(u(t,y)-u(t,x))^2\,dt \\ \qquad\qquad \le \dfrac{10}{(\sigma\,r)^2}\displaystyle\int_I\sum_{x\in B} m(x)\,u^2(t,x)\,dt\,. \end{cases}$$

We have used $|\chi'|\le 1/(\sigma\,r)^2$ and $|\psi(x)-\psi(y)|\le 2/(\sigma\,r)$ when $x\sim y$.

This result (of Cacciopoli type) allows us to use Proposition 2.4 (Sobolev). Note θ' such that $1/\theta+1/\theta'=1$ and $\kappa=1+1/\theta'$.

$$\begin{aligned} \mathcal{M}(u,\kappa,Q_\sigma)^\kappa &= \frac{1}{V(B_\sigma)\,r^2\,(1-\sigma^2)}\int_{I_\sigma}\sum_{x\in B_\sigma} m(x)\,u^{2\kappa}(t,x)\,dt \\ &\le \frac{1}{r^2\,(1-\sigma^2)}\int_{I_\sigma}\Big(\frac{1}{V(B_\sigma)}\sum_{x\in B_\sigma} m(x)\,u^2(t,x)\Big)^{1/\theta'} \\ &\quad \cdot\Big(\frac{1}{V(B_\sigma)}\sum_{x\in B_\sigma} m(x)\,u^{2\theta}(t,x)\Big)^{1/\theta} dt \\ &\le \frac{\sup_{t\in I_\sigma}\Big(\sum_{x\in B_\sigma} m(x)\,u^2(t,x)\Big)^{1/\theta'}}{r^2\,(1-\sigma^2)\,V(B_\sigma)^{1/\theta'}} \\ &\quad \cdot\int_{I_\sigma}\frac{S}{V(B_\sigma)}\Big(r^2\sum_{x,y\in B_\sigma}\mu_{xy}\,(u(t,y)-u(t,x))^2 \\ &\qquad\qquad + \sum_{x\in B_\sigma} m(x)\,u^2(t,x)\Big)\,dt \\ &\le \frac{1}{r^2(1-\sigma^2)}\Big(\frac{10}{(\sigma\,r)^2}\Big)^{1/\theta'}\Big(\frac{20}{\sigma^2}+1\Big)\frac{S}{V(B_\sigma)^{1/\theta'+1}} \\ &\quad \cdot\Big(\int_I\sum_{x\in B} m(x)\,u^2(t,x)\,dt\Big)^{1/\theta'+1}. \end{aligned}$$

This yields

$$\mathcal{M}(u,\kappa,Q_\sigma)\le\Big(\frac{A}{\sigma^2}\Big)^{1/\kappa}\mathcal{M}(u,1,Q)\,,$$

for a constant A, because $\sigma \leq 1/2$ so that $V(B) \leq C_1 V(B_\sigma)$ and $1-\sigma^2 \geq 3/4$.

2.3. Mean value inequalities.

Lemma 2.6. *If u is a positive solution on $I \times B$, then*

- *u^p is a subsolution on $I \times B$ for $p \leq 0$ and $p \geq 1$,*
- *u^p is a supersolution on $I \times B$ for $0 \leq p \leq 1$.*

PROOF. Let $f(x) = x^p$, if $p \leq 0$ or $p \geq 1$, f is convex and

$$f'(a)\,(b-a) \leq f(b) - f(a)\,.$$

This yields

$$\begin{aligned} m(x)\,\frac{\partial}{\partial t}\,f(u(t,x)) &= m(x)\,f'(u(t,x))\,\frac{\partial}{\partial t}\,u(t,x) \\ &= \sum_y \mu_{xy}\,f'(u(t,x))\,(u(t,y)-u(t,x)) \\ &\leq \sum_y \mu_{xy}\,(f(u(t,y)) - f(u(t,x)))\,. \end{aligned}$$

Lemma 2.7. *Let B be a ball of radius r, $Q = [0, r^2] \times B$, u a positive solution on Q and $0 < \delta \leq 1/2$. Then, for all $p > 0$,*

$$\mathcal{M}(u, -p, Q) \leq C\,(C\,\delta^{-\gamma})^{1/p} \inf_{Q_\delta} u^2\,, \tag{2.11}$$

$$\sup_{Q''_\delta} u^2 \leq C\,(C\,\delta^{-\gamma})^{1/p} \mathcal{M}(u, p, Q)\,, \tag{2.12}$$

where C and γ depend only on C_1, C_2 and α.

PROOF. We will prove (2.11). Consider first the case $\delta\, r < 2$ (the difference between B's radius and B_δ's is less than 2, this includes the cases $B = B_\delta$ when one can not apply the elementary step, Lemma 2.5). Take $u^2(t,z) = \inf_{Q_\delta} u^2$ and note that, for all $0 \leq \tau \leq \delta^2\, r^2$, $t - \tau \in I$

and $u^2(t-\tau,z) \le e^{2\tau}\, u^2(t,z) \le e^8 \inf_{Q_\delta} u^2$. This is a consequence of (1.4). Counting only these values,

$$\mathcal{M}(u,-p,Q)^{-p} \ge \frac{m(z)\,\delta^2\, r^2}{r^2\, V(B)} \left(e^8 \inf_{Q_\delta} u^2\right)^{-p}$$

implies

$$\mathcal{M}(u,-p,Q) \le e^8 \Big(\frac{V(B)}{V(z,1/2)\,\delta^2}\Big)^{1/p} \inf_{Q_\delta} u^2 \,.$$

Applying first $DV(C_1)$ between $B(z,2\,r) \supset B$ and $B(z,1/2)$ then $r < 2\,\delta^{-1}$, we obtain (2.11).

Consider now $\delta\, r \ge 2$. Set $\sigma_i = 2^{-i}\delta$, $Q(0) = Q$ and $Q(i) = Q(i-1)_{\sigma_i}$ so that for all i, $Q_\delta \subset Q(i)$. Fix n the integer such that $2^{n+1} \le \delta\, r < 2^{n+2}$. We can apply Lemma 2.5 between $Q(i-1)$ and $Q(i)$ for $i \le n$ since u^{-q} is a subsolution, the radius of the cylinder $Q(i-1)$ is bigger than $r/2$ and $\sigma_i \ge 2/r$.

$$\mathcal{M}(u,-q\,\kappa,Q(i))^{-1/q} \le \Big(\frac{A}{\sigma_i^2}\Big)^{1/\kappa} \mathcal{M}(u,-q,Q(i-1))^{-1/q}$$

implies

$$\mathcal{M}(u,-q,Q(i-1)) \le \Big(\frac{A}{\sigma_i^2}\Big)^{1/(q\kappa)} \mathcal{M}(u,-q\,\kappa,Q(i)) \,.$$

This yields

$$\mathcal{M}(u,-p,Q) \le \Big(\prod_{i=1}^{n} \Big(\frac{A}{(2^{-i}\delta)^2}\Big)^{1/\kappa^i}\Big)^{1/p} \mathcal{M}(u,-p\,\kappa^n,Q(n)) \,.$$

To obtain (2.11), we may first check that

$$\prod_{i=1}^{+\infty} \Big(\frac{A}{(2^{-i}\delta)^2}\Big)^{1/\kappa^i} \le C\,\delta^{-\gamma} \,.$$

Then, we estimate $\mathcal{M}(u,-p\,\kappa^n,Q(n))$ as in the case $\delta\, r < 2$. Take $u^2(t,z) = \inf_{Q(n+1)} u^2$ and note that for all $0 \le \tau \le (\sigma_{n+1}\, r/2)^2$, $t-\tau \in I(n)$ and $u^2(t-\tau,z) \le e^{2\tau}\, u^2(t,z) \le e^2 \inf_{Q(n+1)} u^2$. We use $1 \le \sigma_{n+1}\, r < 2$. This yields

$$\mathcal{M}(u,-p\,\kappa^n,Q(n)) \le e^2 \left(\frac{r^2\, V(B)}{m(z)\,\frac{1}{4}}\right)^{1/(p\kappa^n)} \inf_{Q_\delta} u^2 \,.$$

Since $n > \log(\delta\, r/4)/\log 2$, we may estimate

$$\left(\frac{r^2\, V(B)}{m(z)\,\dfrac{1}{4}}\right)^{1/\kappa^n} \leq e^{(4/(\delta\, r))^c \log(C\, r^C)}\,.$$

Again C is a constant which depends on C_1 and C_2. To obtain $C'\,\delta^{-\gamma}$ as in (2.11), we must check

$$\left(\frac{4}{\delta\, r}\right)^c \log(C\, r^C) \leq C' - \gamma \log \delta\,,$$

for $\delta\, r \geq 2$. This may be done this way: either $\delta \leq r^{-1/2}$ and it suffices to note that $4/(\delta\, r) \leq 4/2$ and use the term $-\gamma \log \delta$, either $\delta > r^{-1/2}$ and we use

$$C' \geq \left(\frac{4}{r^{1/2}}\right)^c \log(C\, r^C)\,.$$

The proof of (2.12) is identical, except that u^q may be a supersolution, that's why we take Q''_δ instead of Q_δ. We also use $u(t+\tau, z) \geq e^{-\tau} u(t,z)$, that's another reason to cut I by the highest values. In fact, having in mind the all-continuous result ([30, Corollary 3, p. 447]), we could keep Q_δ. First, we should use a covering argument ([30, p. 448]) to avoid the use of Lemma 2.5 on u^q for $q < 1$. Then, instead of picking up the sup on the values $u(t+\tau, z)$, we could get it from the $u(t-\tau, z')$ where $z' \sim z$. But this is only possible when $\delta\, r \geq 1$. Taking Q''_δ is somehow artificial but it has the great technical advantage that at this point of the proof, we have no more conditions like $\delta\, r \geq 1$ in Lemma 2.5 which compel us to treat separately cases when it is no longer possible to cut the space.

2.4. About $\log u$, linking negative and positive exponents.

Let us define the measure ν on $\mathbb{R} \times \Gamma$ as the product of Lebesgue measure and V. The next lemma states that the values of $\log u$ are glued to their (space) mean value at a time τ somehow like functions with BMO norm bounded, they cannot be much bigger on a large part before or much lower after. J. Moser's improvement in [23] is that this property and the (time and space) mean value inequalities are sufficient to link extrema to the (space) mean value of $\log u$ at a fixed time between $Q_\ominus$ and $Q_\oplus$, and thus to link extrema together. This last idea is the meaning of the abstract Lemma 2.9.

Lemma 2.8. *Let $\overline{\eta}, \tau \in]0,1[$, $B = B(x_0, r)$ and u any positive supersolution on $Q = [s, s + r^2] \times B$, there is a constant $m(u,\tau)$ such that for all $\lambda > 0$,*

$$\nu(\{(t,z) \in K_\oplus :\ \log u(t,z) < m - \lambda\}) \leq \frac{C\,\nu(Q)}{\lambda}$$

and

$$\nu(\{(t,z) \in K_\ominus :\ \log u(t,z) > m + \lambda\}) \leq \frac{C\,\nu(Q)}{\lambda}\,,$$

where $K_\oplus = [s + \tau\, r^2, s + r^2] \times \overline{\eta}\, B$, $K_\ominus = [s, s + \tau\, r^2] \times \overline{\eta}\, B$ and C depends only on $\overline{\eta}$, τ, C_1 and C_2.

PROOF. Let

$$\psi(z) = 1 - \frac{d(x_0, z)}{[r] + 1}$$

($[r]$ denotes the integer part of r) and $\overline{m}(x) = \sum_{y \notin B} \mu_{xy}$ so that for all $x \in B$,

$$m(x)\, \frac{\partial}{\partial t}\, u(t,x) \geq \sum_{y \in B} \mu_{xy}\, (u(t,y) - u(t,x)) - \overline{m}(x)\, u(t,x)\,.$$

$$\begin{aligned}
&\frac{\partial}{\partial t} \sum_{x \in B} m(x)\, \psi^2(x)\, (-\log u(t,x)) \\
&\qquad = \sum_{x \in B} -\psi^2(x)\, \frac{m(x)\, \frac{\partial}{\partial t}\, u(t,x)}{u(t,x)} \\
&\qquad \leq \sum_{x,y \in B} \mu_{xy}\, \frac{-\psi^2(x)}{u(t,x)}\, (u(t,y) - u(t,x)) + \sum_{x \in B} \psi^2(x)\, \overline{m}(x) \qquad (2.13)\\
&\qquad = \frac{1}{2} \sum_{x,y \in B} \mu_{xy} \Big(\frac{\psi^2(y)}{u(t,y)} - \frac{\psi^2(x)}{u(t,x)}\Big)(u(t,y) - u(t,x)) \\
&\qquad\quad + \sum_{x \in B} \psi^2(x)\, \overline{m}(x)\,.
\end{aligned}$$

Now we show that

$$\begin{aligned}
&\Big(\frac{\psi^2(y)}{u(t,y)} - \frac{\psi^2(x)}{u(t,x)}\Big)(u(t,y) - u(t,x)) \\
&\leq 36\, (\psi(y) - \psi(x))^2 - \frac{1}{2} \min\{\psi^2(x), \psi^2(y)\}\, \frac{(u(t,y) - u(t,x))^2}{u(t,x)\, u(t,y)}\,. \qquad (2.14)
\end{aligned}$$

We may assume $u(t,x) \geq u(t,y)$ for that purpose.

Either

$$\psi^2(x) \leq \frac{\psi^2(y)}{2}\Big(1+\frac{u(t,x)}{u(t,y)}\Big)\,,$$

then

$$\begin{aligned}
&\Big(\frac{\psi^2(y)}{u(t,y)}-\frac{\psi^2(x)}{u(t,x)}\Big)(u(t,y)-u(t,x)) \\
&\qquad \leq \left(\frac{\psi^2(y)}{u(t,y)}-\frac{\dfrac{\psi^2(y)}{2}\Big(1+\dfrac{u(t,x)}{u(t,y)}\Big)}{u(t,x)}\right)(u(t,y)-u(t,x)) \\
&\qquad = -\frac{1}{2}\,\psi^2(y)\,\frac{(u(t,y)-u(t,x))^2}{u(t,x)\,u(t,y)}\,,
\end{aligned}$$

and there is no need to use the other non-negative term $36\,(\psi(y)-\psi(x))^2$.

Or

$$\psi^2(x) \geq \frac{\psi^2(y)}{2}\Big(1+\frac{u(t,x)}{u(t,y)}\Big)\,.$$

First we estimate $u(t,y)-u(t,x)$ with $\psi(y)-\psi(x)$.

$$\begin{aligned}
\frac{u(t,x)-u(t,y)}{u(t,y)} &= \frac{u(t,x)}{u(t,y)}-1 \\
&\leq \frac{2\,\psi^2(x)}{\psi^2(y)}-2 \\
&= 2\,\frac{\psi(x)+\psi(y)}{\psi^2(y)}\,(\psi(x)-\psi(y))\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi^2(y)\,\frac{(u(t,x)-u(t,y))^2}{u(t,x)\,u(t,y)} &\leq \psi^2(y)\,\frac{(u(t,x)-u(t,y))^2}{u^2(t,y)} \\
&\leq \psi^2(y)\Big(2\,\frac{\psi(x)+\psi(y)}{\psi^2(y)}\,(\psi(x)-\psi(y))\Big)^2 \\
&\leq 36\,(\psi(x)-\psi(y))^2\,.
\end{aligned}$$

Because the function ψ is such that $\psi(y) \le \psi(x) \le 2\,\psi(y)$ when $x \sim y$. We also obtain

$$\begin{aligned}
\Big(\frac{\psi^2(y)}{u(t,y)} - \frac{\psi^2(x)}{u(t,x)}\Big)&(u(t,y)-u(t,x)) \\
&\le \Big(\frac{\psi^2(y)}{u(t,y)} - \frac{\psi^2(x)}{u(t,y)}\Big)(u(t,y)-u(t,x)) \\
&= (\psi(x)+\psi(y))\,\frac{u(t,x)-u(t,y)}{u(t,y)}\,(\psi(x)-\psi(y)) \\
&\le 2\,\frac{(\psi(x)+\psi(y))^2}{\psi^2(y)}\,(\psi(x)-\psi(y))^2 \\
&\le 18\,(\psi(x)-\psi(y))^2\,.
\end{aligned}$$

Inequality (2.14) is proven because $18\,(\psi(x)-\psi(y))^2$ controls the two other terms.

We can now change (2.13) using the inequality (2.14),

$$\begin{aligned}
&\frac{\partial}{\partial t}\sum_{x\in B} m(x)\,\psi^2(x)\,(-\log u(t,x)) \\
&+\frac{1}{2}\sum_{x,y\in B}\mu_{xy}\,\min\{\psi^2(x),\psi^2(y)\}\,\frac{(u(t,y)-u(t,x))^2}{u(t,x)\,u(t,y)} \\
&\qquad\le C\sum_{x,y\in B}\mu_{xy}\,(\psi(y)-\psi(x))^2+\sum_{x\in B}\psi^2(x)\,\overline{m}(x)\,.
\end{aligned}$$

Since

$$(\log u(t,y)-\log u(t,x))^2 \le \frac{(u(t,y)-u(t,x))^2}{u(t,x)\,u(t,y)}$$

(just check $(\log a)^2 \le (a-1)^2/a$ by differentiating two times), $x \sim y$ implies $|\psi(y)-\psi(x)| \le 1/r$ and $\overline{m}(x) \ne 0$ implies $|\psi(x)| \le 1/r$, this yields

$$\begin{aligned}
&\frac{\partial}{\partial t}\sum_{x\in B} m(x)\,\psi^2(x)\,(-\log u(t,x)) \\
&+\frac{1}{2}\sum_{x,y\in B}\mu_{xy}\,\min\{\psi^2(x),\psi^2(y)\}\,(\log u(t,y)-\log u(t,x))^2 \\
&\qquad\qquad\le C\,\frac{V(B)}{r^2}\,.
\end{aligned}$$

Now use the weighted Poincaré inequality of Proposition 2.2 to estimate

$$\sum_{x\in B} m(x)\,\psi^2(x)\,(-\log u(t,x) - W(t))^2$$
$$\leq C\,r^2 \sum_{x,y\in B} \mu_{xy}\ \min\{\psi^2(x),\psi^2(y)\}\,(\log u(t,y) - \log u(t,x))^2\,,$$

where

$$W(t) = \frac{\displaystyle\sum_{x\in B} m(x)\,\psi^2(x)\,(-\log u(t,x))}{\displaystyle\sum_{x\in B} m(x)\,\psi^2(x)}\,.$$

Use also

$$\sum_{x\in B} m(x)\,\psi^2(x) \geq \sum_{x\in B/2} m(x)\Big(\frac{1}{2}\Big)^2 \geq C\,V\Big(\frac{B}{2}\Big) \geq C'\,V(B)$$

and $x \in \overline{\eta}\,B$ implies $\psi(x) \geq 1 - \overline{\eta}$. This way, we obtain two constants c and C depending only on $\overline{\eta}$, C_1 and C_2 such that

$$\frac{\partial}{\partial t}\,W(t) + \frac{c}{\nu(Q)} \sum_{x\in\overline{\eta}B} m(x)\,(-\log\,(u(t,x)) - W(t)) \leq C\,r^{-2}\,.$$

Setting $m = -W\,(s+\tau\,r^2)$, this yields the result (for precisions, follow litterally the argument on [30, p. 452]).

Lemma 2.9. *Let U_σ for $0 \leq \sigma \leq \varsigma \leq 1/2$ be subsets of a space with a measure ν such that $\sigma \leq \sigma'$ implies $U_\sigma \supset U_{\sigma'}$ and $\nu(U_0) \leq C\nu(U_\varsigma)$, f a positive measurable function on U_0 which satisfies*

$$\sup_{U_{\sigma'}} f^2 \leq C\,(C\,(\sigma'-\sigma)^{-\gamma})^{1/p}\mathcal{M}(f,p,U_\sigma)\,, \tag{2.15}$$

for all $0 < \sigma < \sigma' \leq \varsigma$ and $p > 0$ and

$$\nu(\{\log f > \lambda\}) \leq \frac{C}{\lambda}\,\nu(U_0)\,,$$

for all $\lambda > 0$. Then

$$\sup_{U_\varsigma} f \leq A\,,$$

where A depends only on ς, γ and C.

PROOF. Set $\psi(\sigma) = \log(\sup_{U_\sigma} f^2)$. Dividing U_σ into two sets $\log(f^2) \le \psi(\sigma)/2$ and $\log(f^2) \ge \psi(\sigma)/2$ yields

$$\begin{aligned}\mathcal{M}(f,p,U_\sigma)^p &\le (e^{\psi(\sigma)/2})^p + \frac{1}{\nu(U_\sigma)}\,\frac{C}{\psi(\sigma)/4}\,\nu(U_0)\sup_{U_\sigma} f^{2p} \\ &\le e^{p\psi(\sigma)/2} + \frac{4\,C^2}{\psi(\sigma)}\,e^{p\psi(\sigma)} \\ &\le 2\,e^{p\psi(\sigma)/2}\,,\end{aligned}$$

if we choose

$$p = \frac{2}{\psi(\sigma)}\,\log\frac{\psi(\sigma)}{4\,C^2}$$

so that the two terms are equal. Then we apply (2.15),

$$\begin{aligned}\psi(\sigma') &\le \log C + \frac{1}{p}\log\big(2\,C\,(\sigma'-\sigma)^{-\gamma}e^{\psi(\sigma)/2}\big) \\ &\le \log C + \frac{\psi(\sigma)}{2}\Big(\frac{\log(2\,C\,(\sigma'-\sigma)^{-\gamma})}{\log(\psi(\sigma)/(4\,C^2))} + 1\Big)\,.\end{aligned}$$

If

$$\frac{\psi(\sigma)}{4\,C^2} \ge (2\,C\,(\sigma'-\sigma)^{-\gamma})^2$$

and

$$\log C \le \frac{\psi(\sigma)}{8}$$

then

$$\psi(\sigma') \le \frac{7}{8}\,\psi(\sigma)\,.$$

Thus, we always have

$$\psi(\sigma') \le \frac{7}{8}\,\psi(\sigma) + C'(\sigma'-\sigma)^{-2\gamma}\,.$$

Take a positive decreasing sequence $\varsigma = \sigma_0 > \cdots > \sigma_i > \sigma_{i+1} > \cdots$,

$$\psi(\varsigma) \le C' \sum_{i=0}^{+\infty}\Big(\frac{7}{8}\Big)^i(\sigma_{i+1}-\sigma_i)^{-2\gamma} \le \text{constant} \quad (= \log(A^2))$$

if we set $\sigma_i = \varsigma/(1+i)$.

2.5. Proof of Theorem 2.1.

Recall the notations of Definition 1.5 and Lemma 2.8, we set $\eta = 1/2$, $\overline{\eta} = 3/4$, $\theta_1 = 1/6$, $\theta_2 = 1/3$, $\tau = 1/2$, $\theta_3 = 3/4$ and $\theta_4 = 1$. Lemma 2.8 gives a reference value m so that one can apply Lemma 2.9 to $f = e^{-m}u$ on $U_0 = [s, s+\tau r^2] \times \overline{\eta} B$ with $U_\sigma = (U_0)''_\sigma$ for $\sigma \leq \varsigma = \overline{\eta} - \eta = 1/4$. This way, $Q_\ominus \subset U_\varsigma$ and (2.15) is satisfied because of Lemma 2.7 and (2.8). This yields $\sup_{Q_\ominus}(e^{-m}u) \leq A$. Applying again Lemma 2.9 to $f = e^m u^{-1}$ on $U_0 = [s+\tau r^2, s+\theta_4 r^2] \times \overline{\eta} B$ with $U_\sigma = (U_0)_\sigma$ yields $\sup_{Q_\oplus}(e^m u^{-1}) \leq A$ and the Harnack inequality.

3. Kernel estimates, discrete-time Harnack inequality and necessity of Poincaré inequality.

3.1. Continuous-time estimates.

First, we give on-diagonal estimates. The regularity coming from the Harnack inequality shows that if one starts at x, one diffuses after a time t on the ball $B(x, \sqrt{t})$. This is well known since the papers of D. G. Aronson [1] or of P. Li and S. T. Yau [19].

Proposition 3.1 (On-diagonal estimates). *Assume* (Γ, μ) *satisfies* $\mathcal{H}(C_{\mathcal{H}})$, *then*

$$\mathcal{P}_t(x,y) \leq \frac{C_{\mathcal{H}}\, m(y)}{V(x, \sqrt{t})}, \qquad \textit{for all } x, y, t,$$

$$d(x,y)^2 \leq t \ \textit{implies}\ \mathcal{P}_t(x,y) \geq \frac{C_{\mathcal{H}}^{-2}\, m(y)}{V(x,\sqrt{t})}.$$

PROOF. Applying the Harnack inequality to $\mathcal{P}_\cdot(\cdot, y)$ yields $\mathcal{P}_t(x,y) \leq C_{\mathcal{H}} \mathcal{P}_{2t}(z,y)$ for $z \in B(x, \sqrt{t})$. Thus,

$$\mathcal{P}_t(x,y) \leq \frac{C_{\mathcal{H}}}{V(x,\sqrt{t})} \sum_{z \in B(x,\sqrt{t})} m(z)\, \mathcal{P}_{2t}(z,y)$$

$$
\begin{aligned}
&= \frac{C_{\mathcal{H}}\, m(y)}{V(x,\sqrt{t})} \sum_{z\in B(x,\sqrt{t})} \mathcal{P}_{2t}(y,z) \\
&\le \frac{C_{\mathcal{H}}\, m(y)}{V(x,\sqrt{t})} \,.
\end{aligned}
$$

For the lower bound, we will use similarly $\mathcal{P}_{t/2}(z,y) \le C_{\mathcal{H}} \mathcal{P}_t(x,y)$ for $z \in B(x,\sqrt{t})$. But first, we define a function $u(\xi,\tau)$ solution of the parabolic equation in $[0,t]\times B(x,\sqrt{t})$ this way

$$
\begin{aligned}
u(\xi,\tau) &= 1\,, \qquad \text{for all } \tau \in \left[0,\frac{t}{2}\right], \\
u(\xi,\tau) &= \sum_{z\in B(x,\sqrt{t})} \mathcal{P}_{\tau-t/2}(\xi,z)\,, \qquad \text{for all } \tau \in \left[\frac{t}{2},t\right].
\end{aligned}
$$

Applied to u the Harnack inequality yields

$$
\begin{aligned}
C_{\mathcal{H}}^{-1} = C_{\mathcal{H}}^{-1} u\Big(\frac{t}{2},x\Big) &\le u(t,y) \\
&= \sum_{z\in B(x,\sqrt{t})} \mathcal{P}_{t/2}(y,z) \\
&= \sum_{z\in B(x,\sqrt{t})} \frac{m(z)}{m(y)} \mathcal{P}_{t/2}(z,y) \\
&\le \sum_{z\in B(x,\sqrt{t})} \frac{C_{\mathcal{H}}\, m(z)}{m(y)} \mathcal{P}_t(x,y) \\
&= \frac{C_{\mathcal{H}} V(x,\sqrt{t})}{m(y)} \mathcal{P}_t(x,y)\,.
\end{aligned}
$$

These on-diagonal estimates yield the volume regularity.

Proposition 3.2. *Assume* (Γ,μ) *satisfies* $\mathcal{H}(C_{\mathcal{H}})$. *Then* $DV(C_{\mathcal{H}}^4)$ *is true.*

PROOF.

$$
\frac{C_{\mathcal{H}}^{-2}\, m(x)}{V(x,r)} \le \mathcal{P}_{r^2}(x,x) \le C_{\mathcal{H}} \mathcal{P}_{4r^2}(x,x) \le C_{\mathcal{H}} \frac{C_{\mathcal{H}}\, m(x)}{V(x,2\,r)}\,.
$$

Now we prove an off-diagonal upper bound which is more precise for x and y far apart. We still use the parabolic Harnack inequality as in Lemma 3.1 to estimate one term by a mean value and a second tool is the integrated maximum principle (see [13]).

Theorem 3.3 (Integrated maximum principle)**.** *If u is a solution on $I \times \Gamma$ and $K(t,x)$ a positive and decreasing in t function such that for all $t \in I$ and $x \sim y$,*

$$\begin{aligned}(3.16)\qquad & (K(t,x)+K(t,y))^2 \\ & \qquad \leq \Big(\frac{\partial K}{\partial t}(t,x) - 2\,K(t,x)\Big)\Big(\frac{\partial K}{\partial t}(t,y) - 2\,K(t,y)\Big),\end{aligned}$$

then the quantity

$$I(t) = \sum_{x\in\Gamma} m(x)\,u^2(t,x)\,K(t,x)$$

is decreasing in $t \in I$.

Proof.

$$\begin{aligned} I'(t) &= \sum_{x\in\Gamma} m(x)\,u^2(t,x)\,\frac{\partial K}{\partial t}(t,x) \\ &\quad + \sum_{x,y\in\Gamma} 2\,\mu_{xy}\,(u(t,y)-u(t,x))\,u(t,x)\,K(t,x)\,. \end{aligned}$$

Since the weights μ_{xy} are symmetric,

$$\begin{aligned} &\sum_{x\in\Gamma} m(x)\,u^2(t,x)\,\frac{\partial K}{\partial t}(t,x) \\ &\qquad = \sum_{x,y\in\Gamma} \frac{\mu_{xy}}{2}\Big(u^2(t,x)\,\frac{\partial K}{\partial t}(t,x) + u^2(t,y)\,\frac{\partial K}{\partial t}(t,y)\Big) \end{aligned}$$

and

$$\begin{aligned} &\sum_{x,y\in\Gamma} 2\,\mu_{xy}\,(u(t,y)-u(t,x))\,u(t,x)\,K(t,x) \\ &\quad = \sum_{x,y\in\Gamma} \mu_{xy}\,(u(t,y)-u(t,x))\,(u(t,x)\,K(t,x) - u(t,y)\,K(t,y))\,. \end{aligned}$$

This yields

$$\begin{aligned} I'(t) = \sum_{x,y\in\Gamma} \mu_{xy} \Big(& \frac{u^2(t,x)}{2} \Big(\frac{\partial K}{\partial t}(t,x) - 2\,K(t,x) \Big) \\ & + u(t,x)\,u(t,y)\,(K(t,x)+K(t,y)) \\ & + \frac{u^2(t,y)}{2} \Big(\frac{\partial K}{\partial t}(t,y) - 2\,K(t,y) \Big) \Big) \\ \le 0\,, & \end{aligned}$$

because of (3.16).

Construction of a function K. Take $\xi(t,x) = \log(K(t,x))$, (3.16) becomes

$$\chi\,(\xi(t,x)-\xi(t,y)) + \frac{\partial\xi}{\partial t}(t,x) + \frac{\partial\xi}{\partial t}(t,y) \le \frac{1}{2}\,\frac{\partial\xi}{\partial t}(t,x)\,\frac{\partial\xi}{\partial t}(t,y)\,, \tag{3.17}$$

where $\chi(s) = \cosh(s) - 1$. Note that $\chi(s) \sim s^2/2$ for s small so that (3.17) may be connected to the following eikonal inequation for the heat equation on a continuous geometry

$$\frac{\partial\xi}{\partial t} + \frac{1}{2}\,|\nabla\xi|^2 \le 0\,,$$

with a solution $K = e^{\xi} = e^{d^2/t}$ where d is a distance function of x. Our parabolic equation should have been normalized to obtain the same coefficients. This difference and differential inequation (3.17) contains only first-order terms, that's why we get nice solutions considering its Legendre associate. For instance,

$$\xi(t,x) = \zeta(t,d(x)) = \max_{\lambda}\{\lambda\,d(x) - \chi(\lambda)\,t\}$$

is a solution if $x \sim y$ implies $|d(x)-d(y)| \le 1$. Indeed, note $\lambda(t,x)$ a value for which the maximum is reached,

$$\frac{\partial\xi}{\partial t}(t,x) = -\chi(\lambda(t,x))$$

and

$$|\xi(t,x) - \xi(t,y)| \le \max\{\lambda(t,x), \lambda(t,y)\}\,.$$

We obtain

$$\lambda(t,x) = \arg\sinh\left(\frac{d(x)}{t}\right),$$

$$\zeta(t,d) = d\arg\sinh\left(\frac{d}{t}\right) - t\Big(\sqrt{1+\frac{d^2}{t^2}} - 1\Big).$$

It will be useful to note that, since $\partial\zeta/\partial d = \lambda$,

$$\text{(3.18)}\qquad \begin{cases} \zeta(t,d) \le \dfrac{1}{2}\,\dfrac{d^2}{t}\,, \\[2ex] d \le C\,t \text{ implies } \zeta(d,t) \ge \dfrac{\arg\sinh C}{2\,C}\,\dfrac{d^2}{t}\,. \end{cases}$$

Denote $E[t,d] = e^{\zeta(t,d)}$ in the sequel. This function has already been introduced by E. B. Davies in [8] with his semigroup perturbation argument (see [7]). With this argument and Harnack inequality, L. Saloff-Coste proves Gaussian upper bounds in [28] using ideas of [38], [39]. We adapt this proof to the use of the integrated maximum principle in the next proposition.

Proposition 3.4 (Off-diagonal upper bound). *Assume* (Γ,μ) *satisfies* $\mathcal{H}(C_{\mathcal{H}})$*, then for all* x, y, t,

$$\mathcal{P}_t(x,y) \le \frac{C\,m(y)}{\sqrt{V(x,\sqrt{t})\,V(y,\sqrt{t})\,E\left[6\,t, d(x,y)\right]}}\,,$$

where C *depends only on* $C_{\mathcal{H}}$*.*

Proof. Consider the following solution of the parabolic equation,

$$u(\tau,\xi) = \sum_{\eta\in B(y,\sqrt{t})} p_t(\eta,x)\,p_\tau(\xi,\eta)\,.$$

This will be useful to estimate $A(t) = \sum_{\eta\in B(y,\sqrt{t})} m(\eta)\,p_t^2(\eta,x)$. Indeed we apply Theorem 3.3 between 0 and $2\,t$ to

$$I(\tau) = \sum_{\xi\in\Gamma} m(\xi)\,u^2(\tau,\xi)\,E\left[t+\tau, d(y,\xi)\right].$$

Since $u(0,\xi) = 0$ if $\xi \notin B(y,\sqrt{t})$ and $u(0,\xi) = p_t(\xi,x)$ if $\xi \in B(y,\sqrt{t})$, we have

$$I(0) = \sum_{\xi \in B(y,\sqrt{t})} m(\xi)\, p_t^2(\xi,x)\, E\,[t, d(y,\xi)] \le C\, A(t)\,.$$

Just note that $d \le \sqrt{t}$ implies $E\,[t,d] \le e^{1/2}$, see (3.18).

In order to give a lower bound for $I(2\,t)$, we use

$$p_{2t}(\xi,\eta) \ge C_{\mathcal{H}}^{-1} p_t(x,\eta)$$

so that

$$u(2\,t,\xi) \ge \frac{C_{\mathcal{H}}^{-1}}{m(x)}\, A(t)$$

for $\xi \in B(x,\sqrt{t})$. This yields

$$\begin{aligned} I(2\,t) &= \sum_{\xi \in \Gamma} m(\xi)\, u^2(2\,t,\xi)\, E\,[3\,t, d(y,\xi)] \\ &\ge C_{\mathcal{H}}^{-2} \sum_{\xi \in B(x,\sqrt{t})} \frac{m(\xi)}{m(x)^2}\, A(t)^2\, E\,[3\,t, d(y, B(x,\sqrt{t}))]\,. \end{aligned}$$

Since $I(2\,t) \le I(0)$, we get

$$A(t) \le \frac{C\, m(x)^2}{V(x,\sqrt{t})\, E\,[3\,t, d(y, B(x,\sqrt{t}))]}\,.$$

Again the Harnack inequality gives

$$\begin{aligned} p_t^2(x,y) &= \frac{m(y)^2}{m(x)^2}\, p_t^2(y,x) \\ &\le \frac{C_{\mathcal{H}}^2\, m(y)^2}{m(x)^2\, V(y,\sqrt{t})} \sum_{\eta \in B(y,\sqrt{t})} m(\eta)\, p_{2t}^2(\eta,x) \\ &= \frac{C_{\mathcal{H}}^2\, m(y)^2}{m(x)^2\, V(y,\sqrt{t})}\, A(2\,t) \\ &\le \frac{C\, m(y)^2}{V(x,\sqrt{2\,t})\, V(y,\sqrt{t})\, E\,[6\,t, d(y, B(x,\sqrt{2\,t}))]}\,. \end{aligned}$$

The proposition follows because of the volume regularity and

$$E\,[6\,t, d(y, B(x, \sqrt{2\,t}))] \geq c\,E[6\,t, d(x,y)]\,.$$

REMARK 1. Instead of $E\,[6\,t, d(x,y)]$, we could get $E\,[\lambda\,t, d(x,y)]$ for any $\lambda > 1$, the constant C depending also on λ. We would just have to apply the Harnack inequality between t and $(1+\varepsilon)t$. Furthermore, we could have been more precise for the choice of a function E (using both $\chi(\lambda(x))$ and $\chi(\lambda(y))$ instead of only the biggest). All of these manipulations tend to obtain the analog of $e^{d^2/((4+\varepsilon)t)}$ for d/t small. Don't forget the normalization of the parabolic equation to compare.

REMARK 2. One might find the function E too complicated, but [25] explained that it is not purely technical. To understand it, one can take it as $e^{cd^2/t}$ for d/t small and $(d/t)^d\,e^{-d}$ for d/t huge glued together. The second value is adapted to the fact that when $t \longrightarrow 0$,

$$\mathcal{P}_t(x,y) \sim \frac{p_{d(x,y)}(x,y)\,t^{d(x,y)}}{d(x,y)!}\,,$$

so at least in this case the function E gives an optimal upper bound.

3.2. Discrete-time estimates.

Assume $\Delta(\alpha)$ is true so that we can consider the positive sub-markovian kernel $\overline{p} = p - \alpha\,\delta$ (this means $\overline{p}(x,y) = p(x,y) - \alpha\,\delta(x,y)$, then $\overline{p}_n(x,y)$ is defined as in (1.1)). Now compute $\mathcal{P}_n$ and p_n with $\overline{p}$

$$\mathcal{P}_n(x,y) = e^{(\alpha-1)n} \sum_{k=0}^{+\infty} \frac{n^k}{k!}\,\overline{p}_k(x,y) = \sum_{k=0}^{+\infty} a_k\,\overline{p}_k(x,y)\,, \tag{3.19}$$

$$p_n(x,y) = \sum_{k=0}^{n} C_n^k\,\alpha^{n-k}\,\overline{p}_k(x,y) = \sum_{k=0}^{n} b_k\,\overline{p}_k(x,y)\,.$$

To compare the two sums we study $c_k = b_k/a_k$ for $0 \leq k \leq n$,

$$c_k = \frac{n!\,\alpha^{n-k}}{(n-k)!\,e^{(\alpha-1)n}\,n^k}\,.$$

Lemma 3.5.

$$0 \le k \le n \textit{ implies } c_k \le C(\alpha)\,,$$

$$n \ge \frac{a^2}{\alpha^2} \textit{ and } |k-(1-\alpha)\,n| \le a\sqrt{n} \textit{ imply } c_k \ge C(a,\alpha) > 0\,.$$

The condition $n \ge a^2/\alpha^2$ ensures that $a\sqrt{n} \le \alpha\, n$. We shall consider only $\alpha \le 1/4$ so that we always have $n/2 \le k \le n$ in the second assertion.

PROOF. The c_k's follow the recurrence formula $c_{k+1} = c_k(n-k)/(\alpha\, n)$, so they reach a maximum for k around the real $(1-\alpha)\,n$. Let us use the Gamma function, $\Gamma(n+1) = n!$ and $t^t\, e^{-t}\sqrt{t}/C \le \Gamma(t+1) \le C\, t^t\, e^{-t}\sqrt{t}$. Set $c_t = \Gamma(n+1)\,\alpha^{n-t}/\Gamma(n-k+1)\,e^{(\alpha-1)t}\,n^t$. Similarly, it reaches its maximum for $t = (1-\alpha)\,n$. Thus,

$$\begin{aligned}
c_k &\le \frac{\Gamma(n+1)\,\alpha^{\alpha n}}{\Gamma(\alpha\, n+1)\,e^{(\alpha-1)n}\,n^{(1-\alpha)n}} \\
&= \frac{\Gamma(n+1)}{n^n\,e^{-n}}\,\frac{(\alpha\, n)^{\alpha n}\,e^{-\alpha n}}{\Gamma(\alpha\, n+1)} \\
&\le C\sqrt{n}\,\frac{C}{\sqrt{\alpha\, n}} \\
&= \frac{C^2}{\sqrt{\alpha}}\,.
\end{aligned}$$

Next, we prove the second assertion of the lemma. Again, because of c_k's variations, we only check this for $k \approx (1-\alpha)\,n \pm a\sqrt{n}$. For instance, for $(1-\alpha)\,n - a\sqrt{n} \le k \le (1-\alpha)\,n$,

$$\begin{aligned}
c_k &= \frac{n!\,\alpha^{n-k}}{(n-k)!\,e^{(\alpha-1)n}\,n^k} \\
&\ge \frac{\Gamma(n+1)\,\alpha^{\alpha n + a\sqrt{n}}}{\Gamma(\alpha\, n + a\sqrt{n}+1)\,e^{(\alpha-1)n}\,n^{(1-\alpha)n - a\sqrt{n}}} \\
&\ge \frac{1}{C^2}\,\frac{e^{a\sqrt{n}}}{\left(1+\dfrac{a\sqrt{n}}{\alpha\, n}\right)^{\alpha n + a\sqrt{n}}}\,\frac{\sqrt{n}}{\sqrt{\alpha\, n + a\sqrt{n}}} \\
&\ge \frac{1}{C^2}\,e^{a\sqrt{n} - (\alpha n + a\sqrt{n})\log(1+a/(\alpha\sqrt{n}))}\,1
\end{aligned}$$

$$\geq \frac{1}{C^2}\, e^{a\sqrt{n}-(\alpha n+a\sqrt{n})a/(\alpha\sqrt{n})}$$
$$= \frac{1}{C^2}\, e^{-a^2/\alpha}\,.$$

This technical result is the key to compare p and $\mathcal{P}$. One side is easy now.

Theorem 3.6. *Assume (Γ,μ) satisfies $p(x,x)\geq\alpha>0$ for all x in Γ. Then, for all x,y,n,*

$$p_n(x,y)\leq C(\alpha)\,\mathcal{P}_n(x,y)\,.$$

It is the case when $\Delta(\alpha)$ is true. This theorem may be applied in many other situations (with any volume growth) when it is easier to work on $\mathcal{P}$. When no hypothesis is assumed on $p(x,x)$, see the comment after Definition 1.3 about $(\Gamma,\mu^{(2)})$. For instance, on a locally uniformly finite (by N) non-weighted graph ($\mu_{xy}\in\{0,1\}$), $p_2(x,x)\geq 1/N$.

The other side of the comparison is more intricate.

Proposition 3.7 (On-diagonal estimates)**.** *Assume (Γ,μ) satisfies $DV(C_1)$, $P(C_2)$ and $\Delta(\beta)$. Then, there exist c_d, $C_d>0$, depending only on C_1, C_2 and β, such that*

$$p_n(x,y)\leq\frac{C_d\,m(y)}{V(x,\sqrt{n})}\,,\qquad \textit{for all } x,y,n\,,$$
$$d(x,y)^2\leq n \textit{ implies } p_n(x,y)\geq\frac{c_d\,m(y)}{V(x,\sqrt{n})}\,.$$

PROOF. The first assertion follows from Theorem 3.6 and the upper bound in Proposition 3.1. To deduce the second assertion from the lower bound in Proposition 3.1, we will have to prove that in the sum (3.19), the terms for $|k-(1-\alpha)\,n|\leq a\sqrt{n}$ contain half of the whole sum. First we will set $\alpha=\beta/2$ (this will be useful later when we apply the upper bound to a Markov kernel p'). Now, to prove that the lower bound for $\mathcal{P}$ implies one for p, it will be sufficient to prove that for all $\varepsilon>0$, there exists a,

$$\sum_{|k-(1-\alpha)n|>a\sqrt{n}} a_k\,\overline{p}_k(x,y)\leq\frac{\varepsilon\,m(y)}{V(x,\sqrt{n})}\,.$$

We will take $\varepsilon = C_{\mathcal{H}}^{-2}/2$, the desired lower bound will be proved for $n \geq N = a^2/\alpha^2$. For $n \leq N$, the condition $\Delta(\beta)$ gives $p_n(x,y) \geq \beta^N$.

We can apply the upper bound to the Markov kernel $p' = \overline{p}/(1-\alpha)$. Indeed, it is generated by weights μ'_{xy}

$$\mu'_{xx} = \frac{\mu_{xx} - \alpha\, m(x)}{1-\alpha} \geq \alpha\, m(x)\,,$$

$$\mu'_{xy} = \frac{\mu_{xy}}{1-\alpha}\,, \qquad \text{if } x \neq y\,,$$

$$m'(x) = m(x)\,.$$

Thus, the volume is identical and $P(C_2)$ is still satisfied because weights μ_{xy} for $x \neq y$ have increased. This yields $p'_k(x,y) \leq C'_d\, m(y)/V(x,\sqrt{k})$, hence $\overline{p}_k(x,y) \leq C'_d\, m(y)\,(1-\alpha)^k/V(x,\sqrt{k})$. Next, we have to get the estimate

$$\sum_{|k-(1-\alpha)n| > a\sqrt{n}} e^{(\alpha-1)n}\, \frac{((1-\alpha)\,n)^k}{k!}\, \frac{1}{V(x,\sqrt{k})} \leq \frac{\varepsilon'}{V(x,\sqrt{n})}\,.$$

The sum for $k > (1-\alpha)\,n + a\sqrt{n}$ is easier because we simply use $V(x,\sqrt{k}) \geq V(x,\sqrt{n/2}) \geq V(x,\sqrt{n}/2) \geq V(x,\sqrt{n})/C_1$. Then, we obtain the $k+1^{\text{th}}$ term of the sum if we multiply the k^{th} term by $(1-\alpha)\,n/(k+1)$. So we estimate this part by a geometric sum,

$$\begin{aligned}
&\sum_{k > (1-\alpha)n + a\sqrt{n}} e^{(\alpha-1)n}\, \frac{((1-\alpha)\,n)^k}{k!}\, \frac{1}{V(x,\sqrt{k})} \\
&\leq e^{(\alpha-1)n}\, \frac{((1-\alpha)\,n)^{(1-\alpha)n + a\sqrt{n}}}{\Gamma((1-\alpha)\,n + a\sqrt{n} + 1)}\, \frac{C_1}{V(x,\sqrt{n})}\, \frac{1}{1 - \dfrac{(1-\alpha)\,n}{(1-\alpha)\,n + a\sqrt{n}}} \\
&\leq C\, C_1\, e^{a\sqrt{n} - ((1-\alpha)n + a\sqrt{n}) \log(1 + a/((1-\alpha)\sqrt{n}))} \\
&\qquad \cdot \frac{1}{V(x,\sqrt{n})}\, \underbrace{\frac{1}{\sqrt{(1-\alpha)\,n + a\sqrt{n}}}\, \frac{(1-\alpha)\,n + a\sqrt{n}}{a\sqrt{n}}}_{\leq \frac{1}{a} \text{ because } n \geq \frac{a^2}{\alpha^2}} \\
&\leq \frac{\varepsilon'/2}{V(x,\sqrt{n})}\,,
\end{aligned}$$

with a good choice of a. Indeed, since $\log(1+u) \geq u/(u+1)$, the argument of the exponential function appears to be negative.

To deal with $k < (1-\alpha)\, n - a\sqrt{n}$, we must be careful with the factors $V(x,\sqrt{k})/V(x,\sqrt{k-1})$ when we compute the $k-1^{\text{th}}$ term from the k^{th}. A rough application of the volume regularity gives $V(x,\sqrt{k}) \leq C_1 V(x,\sqrt{k-1})$. So for the terms $k \leq (1-\alpha)\, n/(2\, C_1)$, the $k-1^{\text{th}}$ term is less than one half of the k^{th} term and the estimation is straightforward. Now for the other terms we bound all $1/V(x,\sqrt{k})$ by $1/V(x,\sqrt{(1-\alpha)\, n/(2\, C_1)})$, then the same computation as for $k > (1-\alpha)\, n + a\sqrt{n}$ shows the estimate with $1/V(x,\sqrt{(1-\alpha)\, n/(2\, C_1)})$ which is less than $C/V(x,\sqrt{n})$ if we apply many times the volume regularity.

Now we prove off-diagonal upper and lower bounds.

Theorem 3.8 (Off-diagonal estimates). *Assume (Γ,μ) satisfies $DV(C_1)$, $P(C_2)$ and $\Delta(\alpha)$. Then, there exist positive c_l, C_l, C_r and c_r depending only on C_1, C_2 and α such that $G(c_l, C_l, C_r, c_r)$ is true.*

PROOF OF THE UPPER BOUND. It is a consequence of Theorem 3.6 and Proposition 3.4.

$$\begin{aligned} p_n(x,y) &\leq \frac{C\, m(y)}{\sqrt{V(x,\sqrt{n})\, V(y,\sqrt{n})\, E\,[6\, n, d(x,y)]}} \\ &\leq \frac{C\, m(y)}{\sqrt{V(x,\sqrt{n})\, V(y,\sqrt{n})}}\, e^{-c d(x,y)^2/n}\,, \end{aligned}$$

for $d(x,y) \leq n$ because of (3.18).

Now use

$$\begin{aligned} V(x,\sqrt{n}) &\leq V(y, d(x,y)+\sqrt{n}) \\ &\leq C_1 \Big(\frac{d(x,y)+\sqrt{n}}{\sqrt{n}}\Big)^{\log C_1/\log 2} V(y,\sqrt{n})\,, \end{aligned}$$

$$\begin{aligned} p_n(x,y) \leq \frac{C\sqrt{C_1}\ m(y)}{V(x,\sqrt{n})} &\Big(\frac{d(x,y)+\sqrt{n}}{\sqrt{n}}\Big)^{\log C_1/2\log 2} \\ &\cdot e^{-(c/2) d(x,y)^2/n}\, e^{-(c/2) d(x,y)^2/n}\,. \end{aligned}$$

It is clear that the factor

$$\Big(\frac{d(x,y)+\sqrt{n}}{\sqrt{n}}\Big)^{\log C_1/2\log 2} e^{-(c/2) d(x,y)^2/n}$$

is bounded.

Proof of the lower bound. It is well-known that the Gaussian lower bound follows from the on-diagonal one. So let us apply many times the second assertion of Proposition 3.7. We set $n = n_1 + \cdots + n_j$, $x = x_0, x_1, \ldots, x_j = y$ and $B_0 = \{x\}, B_i = B(x_i, r_i), B_j = \{y\}$ such that

$$\begin{cases} j - 1 \leq C\,\dfrac{d(x,y)^2}{n}\ , \\ r_i \geq c\,\sqrt{n_{i+1}}\ , \qquad\qquad \text{so that } z \in B_i \text{ imply } V(z, \sqrt{n_{i+1}}) \leq AV(B_i)\ , \\ \displaystyle\sup_{\substack{z \in B_{i-1} \\ z' \in B_i}} d(z,z')^2 \leq n_i\ , \text{ so that } p_{n_i}(z,z') \geq \frac{c_d\, m(z')}{V(z, \sqrt{n_i})}\ . \end{cases}$$

We will see below how to construct this decomposition. It is a purely technical problem (cutting in a discrete context).

It will be sufficient to prove the Gaussian lower bound since

$$\begin{aligned} & p_n(x,y) \\ & \geq \sum_{(z_1,\ldots,z_{j-1}) \in B_1 \times \cdots \times B_{j-1}} p_{n_1}(x,z_1)\, p_{n_2}(z_1,z_2) \cdots p_{n_j}(z_{j-1},y) \\ & \geq \sum_{(z_1,\ldots,z_{j-1}) \in B_1 \times \cdots \times B_{j-1}} \frac{c_d\, m(z_1)}{V(x,\sqrt{n_1})}\, \frac{c_d\, m(z_2)}{V(z_1,\sqrt{n_2})} \cdots \frac{c_d\, m(y)}{V(z_{j-1},\sqrt{n_j})} \\ & \geq c_d^j\, A^{1-j} \sum_{(z_1,\ldots,z_{j-1}) \in B_1 \times \cdots \times B_{j-1}} \frac{m(z_1)}{V(x,\sqrt{n_1})}\, \frac{m(z_2)}{V(B_1)} \cdots \frac{m(y)}{V(B_j)} \\ & = \frac{c_d\, m(y)}{V(x,\sqrt{n_1})} \Big(\frac{c_d}{A}\Big)^{j-1}\ . \end{aligned}$$

We just have to choose $C_l \geq C \log(A/c_d)$.

Decomposition. Consider three cases.

If $d(x,y) \geq n/100$, then we can set $j = n$, $n_i \equiv 1$, $B_i = \{x_i\}$ (for instance, $r_i \equiv 1/2$) and choose $d(x_i, x_{i+1}) \leq 1$.

If $d(x,y)^2 \leq n$, then we can set $j = 1$ (in fact Proposition 3.7 has not to be iterated).

Otherwise, set

$$j = \Big[10\,\frac{d(x,y)^2}{n}\Big] \geq 10\,.$$

This way, n/j and $d(x,y)/j$ are bigger than 10 and

$$\Big(\frac{d(x,y)}{j}\Big)^2 \leq \frac{n}{9\,j}\,,$$

so we can choose $n_i \approx n/j$ (*i.e.* $[n/j]$ or $[n/j]+1$) and

$$d(x_i, x_i+1) \leq r_i \approx \frac{d(x,y)}{j}\,.$$

3.3. Discrete-time Harnack inequality.

We will prove the discrete-time Harnack inequality thanks to the Gaussian estimates. The method is based on [11, Section 3]. Denote $B = B(x_0, R)$ where $R \in \mathbb{N}^*$, with boundary $\partial B = \{x : d(x_0, x) = R\}$. The idea of the proof is that for $(\nu, \xi) \in \partial Q$ and $(n, x) \in Q_\ominus$ or $Q_\oplus$, $p_{n-\nu}(x,\xi)$'s lower and upper bounds differ only by a constant. The difficulty is that the solution u on Q is not a combination of $p_{n-\nu}(x,\xi)$ but of $U_{n-\nu}(x,\xi)$ where $U_n(x,y)$ is the solution for $(n,x) \in \mathbb{N} \times B$ satisfying $U_0(x,y) = \delta(x,y)$ and $U_n(x,y) = 0$ for $n > 0$ and $x \in \partial B$. Obviously $U_n(x,y) \leq p_n(x,y)$, so only the lower bound needs some work.

Lemma 3.9. *Assume* (Γ, μ) *satisfies* $G(c_l, C_l, C_r, c_r)$. *Then, there exist* $\varepsilon, c > 0$ *depending only on* c_l, C_l, C_r *and* c_r *such that*

$$U_n(x,y) \geq \frac{c\,m(y)}{V(x_0, 2\,\varepsilon\,R)}\,,$$

whenever

$$\begin{cases} (\varepsilon\,R)^2 \leq n \leq (2\,\varepsilon\,R)^2\,, \\ x \in B(x_0, \varepsilon\,R)\,, \\ y \in B(x_0, 2\,\varepsilon\,R)\,, \\ d(x,y) \leq n\,. \end{cases}$$

PROOF. The idea (see [11, Lemma 5.1]) is that if $d(x, \partial B)$ is big enough, $p - U$ is small and the lower bound for p applies to U. First note that

$$p_n(x, y) \geq \frac{2\, c\, m(y)}{V(x_0, 2\,\varepsilon\, R)}\ ,$$

where $c = c_l\, e^{-9C_l}/2$. Now, write

$$r(n, x) = p_n(x, y) - U_n(x, y) = \sum_{\substack{\xi \in \partial B \\ \nu \leq n}} a(\nu, \xi)\, p_{n-\nu}(x, \xi)\ ,$$

where $a(\xi, \nu) \geq 0$. These coefficients may be constructed by recurrence on ν. Another point of view is that $(m(\xi)/m(y))\, a(\nu, \xi)$ is the probability to reach ∂B for the first time at ξ after ν steps. That's why

$$\sum_{\nu, \xi} \frac{m(\xi)}{m(y)}\, a(\nu, \xi) \leq 1\,.$$

We can check it this way

$$\begin{aligned}
1 &= \sum_x \frac{m(x)}{m(y)}\, p_n(x, y) \\
&\geq \sum_x \frac{m(x)}{m(y)}\, r(n, x) \\
&= \sum_{x, \nu, \xi} \frac{m(x)}{m(y)}\, a(\nu, \xi)\, p_{n-\nu}(x, \xi) \\
&= \sum_{\nu, \xi} \Big(a(\nu, \xi) \underbrace{\sum_x \frac{m(x)}{m(y)}\, p_{n-\nu}(x, \xi)}_{= m(\xi)/m(y)} \Big)\,.
\end{aligned}$$

To estimate $r(n, x)$ we use the Gaussian upper bound.

$$\begin{aligned}
\frac{m(y)}{m(\xi)}\, p_{n-\nu}(x, \xi) &\leq \frac{C_r\, m(y)}{V(x, \sqrt{n-\nu})}\, e^{-c_r\, d(x, y)^2/(n-\nu)} \\
&\leq \Big(C_r\, \frac{V(x, 2\,\varepsilon\, R)}{V(x, \sqrt{n-\nu})}\, e^{-c_r((1-\varepsilon)R)^2/(n-\nu)} \Big) \frac{m(y)}{V(x, 2\,\varepsilon\, R)} \\
&\leq \frac{c\, m(y)}{V(x_0, 2\,\varepsilon\, R)}\ ,
\end{aligned}$$

with a good choice of ε. The lemma follows.

Theorem 3.10. *Assume (Γ,μ) satisfies $G(c_l,C_l,C_r,c_r)$, then there exists $C_H>0$ such that $H(C_H)$ is true.*

PROOF. Let us first point out that the Gaussian lower bound yields a volume regularity. The following argument,

$$\begin{aligned}1 &\geq \sum_{y\in B(x,2r)} p_{r^2}(x,y)\\ &\geq \sum_{y\in B(x,2r)} \frac{c_l\, m(y)}{V(x,r)}\, e^{-C_l(2r)^2/r^2}\\ &= c_l\, e^{-4C_l}\, \frac{V(x,2\,r)}{V(x,r)}\ ,\end{aligned}$$

is correct for r integer and $r\geq 2$ (because we need $d(x,y)\leq r^2$). This extends to other values thanks to $\Delta(\alpha)$ (which is an immediate consequence of $G(c_l,C_l,C_r,c_r)$).

Now we prove the Harnack inequality for $\eta=\varepsilon$, $\theta_1=\varepsilon^2/2$, $\theta_2=\varepsilon^2$, $\theta_3=2\,\varepsilon^2$, $\theta_4=4\,\varepsilon^2$ and $r=R\in\mathbb{N}^*$ in the notations of Definition 1.6.

Let u be a solution on Q, there is a decomposition

$$v(n,x)=\sum_{\substack{\nu\leq n\\ \xi\in\partial B(x_0,2\varepsilon R)\\ \text{or}\\ \nu=0\\ \xi\in B(x_0,2\varepsilon R)}} a(\nu,\xi)\, U_{n-\nu}(x,\xi)\ ,$$

with non-negative $a(\nu,\xi)$ such that $u(n,x)=v(n,x)$ if $x\in B(x_0,2\,\varepsilon\,R)$. Again the coefficients may be constructed by recurrence on ν, the key is to keep $v\leq u$ everywhere.

Thus, it will be sufficient to prove the Harnack inequality for the terms $U_{\cdot-\nu}(\cdot,\xi)$, this means $U_{n_\ominus-\nu}(x_\ominus,\xi)\leq C\,U_{n_\oplus-\nu}(x_\oplus,\xi)$ for $(n_\ominus,x_\ominus)\in Q_\ominus$, $(n_\oplus,x_\oplus)\in Q_\oplus$, $\nu\leq\theta_2\,R^2$ and $d(x_\ominus,x_\oplus)\leq n_\oplus-n_\ominus$. The lower bound is a consequence of Lemma 3.9 if $d(x_\oplus,\xi)\leq n_\oplus-\nu$,

$$U_{n_\oplus-\nu}(x_\oplus,\xi)\geq\frac{c\,m(\xi)}{V(x_0,2\,\varepsilon\,R)}\ ,$$

for $x_\oplus \in B(x_0, \eta R) = B(x_0, \varepsilon R)$ and $\theta_3 R^2 \le n_\oplus \le \theta_4 R^2$. If $d(x_\oplus, \xi) > n_\oplus - \nu$, then

$$d(x_\ominus, \xi) \ge d(x_\oplus, \xi) - d(x_\oplus, x_\ominus) > (n_\oplus - \nu) - (n_\oplus - n_\ominus) = n_\ominus - \nu$$

and $U_{n_\ominus - \nu}(x_\ominus, \xi) = 0$.

The upper bound looks alike either because of time regularization in the case $\nu = 0$ and $\xi \in B(x_0, 2\varepsilon R)$ or because of space regularization in the case $\xi \in \partial B(x_0, 2\varepsilon R)$. In the first case, for $x_\ominus \in B(x_0, \varepsilon R)$ and $\theta_1 R^2 \le n_\ominus \le \theta_2 R^2$,

$$\begin{aligned} U_{n_\ominus}(x_\ominus, \xi) &\le p_{n_\ominus}(x_\ominus, \xi) \\ &\le \frac{C_r\, m(\xi)}{V(x_\ominus, \sqrt{n_\ominus})} \\ &\le \frac{C_r\, m(\xi)}{V(x_\ominus, \theta_1 R)} \\ &\le \frac{C\, m(\xi)}{V(x_0, 2\,\varepsilon\, R)}\ , \end{aligned}$$

where $C = C_r\, C_1^N$, we must apply the volume regularity N times, N depending on ε and θ_1. In the second case, we use the Gaussian coefficient and $d(x_\ominus, \xi) \ge [2\,\varepsilon\, R] - [\varepsilon\, R]$,

$$\begin{aligned} U_{n_\ominus - \nu}(x_\ominus, \xi) &\le p_{n_\ominus - \nu}(x_\ominus, \xi) \\ &\le \frac{C_r\, m(\xi)}{V(x_\ominus, \sqrt{n_\ominus - \nu})}\, e^{-c_r\, d(x_\ominus, \xi)^2/(n_\ominus - \nu)} \\ &\le \frac{C\, m(\xi)}{V(x_0, 2\,\varepsilon\, R)}\ . \end{aligned}$$

3.4. Poincaré inequality.

Theorem 3.11. *Assume $H(C_H)$, then there exist C_1, C_2 and $\alpha > 0$ such that $DV(C_1)$, $P(C_2)$ and $\Delta(\alpha)$ are true.*

PROOF. In the comments after Theorem 1.7, we already mentioned that $H(C_H)$ implies a property $\Delta(\alpha)$. Then $DV(C_1)$ is proven as in Section 3.1. The discrete version raises new difficulties only for small radii but then $\Delta(\alpha)$ is sufficient. Thus we also obtain, as in Proposition 3.1,

$$d(x,y)^2 \leq n \text{ implies } p_n(x,y) \geq \frac{c\, m(y)}{V(x,\sqrt{t})}\,.$$

The fact that parabolic Harnack inequality implies Poincaré inequality is proven on manifolds in [30] with ideas of [18]. Take f defined on $B(x_0,2\,r)$ and consider the Neumann problem on $B(x_0,2\,r)$. It may be defined this way: consider the graph $B(x_0,2\,r)$ with the restriction $\mu|_{B(x_0,2r)\times B(x_0,2r)}$, it gives a kernel $p'(x,y)$. The crucial point is that $p'(x,y)$ has increased (comparing to $p(x,y)$) for $x \in \partial B(x_0,2\,r)$. Set P the Markov operator

$$Pg(x) = \sum_y p'(x,y)\, g(y)$$

and denote the iteration $Q = P^{[4r^2]}$. For any positive g, $P^n g(x)$ is a positive solution on $B(x_0,2\,r)$ of the parabolic equation (of Γ). Thus, for $x \in B(x_0,r)$,

$$\begin{aligned}(Q(f-(Qf)(x))^2)(x) &\geq \sum_{y\in B(x_0,r)} \frac{c\, m(y)}{V(x,2\,r)}(f(y)-(Qf)(x))^2 \\ &\geq \frac{c}{V(x_0,3\,r)} \sum_{y\in B(x_0,r)} m(y)\,(f(y)-f_{B(x_0,r)})^2\end{aligned}$$

because $\sum_{y\in B(x_0,r)} m(y)\,(f(y)-\lambda)^2$ is minimal for $\lambda = f_{B(x_0,r)}$. This yields

$$\begin{aligned}\sum_{y\in B(x_0,r)} m(y)\,|f(y)-f_{B(x_0,r)}|^2 &\leq C \sum_{x\in B(x_0,2r)} (Q(f-(Qf)(x))^2)(x) \\ &= C\,(\|f\|_2^2 - \|Qf\|_2^2) \qquad (3.20)\\ &\leq C\,(4\,r^2\,\|\nabla f\|_2^2)\,, \qquad (3.21)\end{aligned}$$

where

$$\|f\|_2^2 = \sum_{x\in B(x_0,2r)} m'(x)\, f(x)^2$$

and

$$\|\nabla f\|_2^2 = \sum_{x,y\in B(x_0,2r)} \mu_{xy}\,|f(x)-f(y)|^2\,.$$

The line (3.20) is a variance formula and the line (3.21) is justified by the two properties

$$\|Pf\|_2^2 \le \|f\|_2^2 \qquad \text{and} \qquad \|f\|_2^2 - \|Pf\|_2^2 \le \|\nabla f\|_2^2 \, .$$

We give the proof of the second one which is not so widely known as the first. Note that $a^2 - b^2 \le 2\,a\,(a-b)$,

$$\begin{aligned}
&\sum_x \Big(m'(x) \Big(f(x)^2 - \Big(\sum_y p'(x,y)\, f(y) \Big)^2 \Big) \Big) \\
&\qquad \le \sum_x \Big(2\, m'(x)\, f(x) \Big(f(x) - \Big(\sum_y p'(x,y)\, f(y) \Big) \Big) \Big) \\
&\qquad = 2 \sum_{x,y} \Big(\underbrace{m'(x)\, p'(x,y)}_{\mu_{xy}}\, f(x)\, (f(x) - f(y)) \Big) \\
&\qquad = \sum_{x,y} (\mu_{xy}\, (f(x) - f(y))\, (f(x) - f(y))) \, .
\end{aligned}$$

This ends the proof of Theorem 1.7.

4. Some consequences of Harnack inequality and Gaussian estimates.

4.1. Hölder regularity.

Among the immediate consequences of Harnack inequality are Liouville theorem stated in [9] because only the elliptic version is needed and Hölder regularity of solutions of the discrete parabolic equation.

Proposition 4.1. *Assume (Γ, μ) satisfies the properties of Theorem 1.7. Then there exists $h > 0$ and C such that for all $x_0 \in \Gamma$, $n_0 \in \mathbb{Z}$ and $R \in \mathbb{N}$, if u is a solution on $Q = (\mathbb{Z} \cap [n_0 - 2\,R^2, n_0]) \times B(x_0, 2\,R)$, $x_1, x_2 \in B(x_0, R)$ and $n_1, n_2 \in \mathbb{Z} \cap [n_0 - R^2, n_0]$, then*

$$|u(n_2, x_2) - u(n_1, x_1)| \le C \Big(\frac{\sup \{ \sqrt{|n_2 - n_1|}, d(x_1, x_2) \}}{R} \Big)^h \sup_Q |u| \, .$$

PROOF. Fix $n_2 \geq n_1$ and set $Q(i) = (\mathbb{Z} \cap [n_2 - 2^{2i}, n_0]) \times B(x_2, 2^i)$, $M(i) = \sup_{Q(i)} u$, $m(i) = \inf_{Q(i)} u$ and $\omega(i) = M(i) - m(i)$, $2^{i_1 - 1} \leq \sup\{\sqrt{|n_2 - n_1|}, d(x_1, x_2)\} < 2^{i_1}$ and $2^{i_2} \leq R < 2^{i_2+1}$. This way, $\omega(i_1) \geq |u(n_2, x_2) - u(n_1, x_1)|$ and $\omega(i_2) \leq 2 \sup_Q |u|$.

Set $m_{\ominus}(i) = u(n_2 - 2^{2i+1}, x_2)$ and apply Harnack inequality in $Q(i+1)$ to $u - m\,(i+1)$ and $M\,(i+1) - u$

$$m_{\ominus}(i) - m(i+1) \leq C_H\,(m(i) - m(i+1))\,,$$

$$M(i+1) - m_{\ominus}(i) \leq C_H\,(M(i+1) - M(i))\,.$$

This yields $\omega(i) \leq (1 - C_H^{-1})\,\omega(i+1)$. Thus,

$$\omega(i_1) \leq (1 - C_H^{-1})^{i_2 - i_1}\,\omega(i_2)$$

and the proposition follows.

4.2. Green function.

With the Gaussian estimates for p_n, one easily proves estimates for the Green function.

Proposition 4.2. *Assume (Γ, μ) satisfies the properties of Theorem 1.7. Then the Green function $G(x,y) = \sum_{n=0}^{+\infty} p_n(x,y)$ is finite if and only if*

$$\sum_{n=0}^{+\infty} \frac{n}{V(x,n)} < +\infty\,, \tag{4.22}$$

and it satisfies the estimates

$$\begin{aligned} C^{-1}\,m(y) \sum_{n=d(x,y)}^{+\infty} \frac{n}{V(x,n)} &\leq G(x,y) \\ &\leq C\,m(y) \sum_{n=d(x,y)}^{+\infty} \frac{n}{V(x,n)}\,. \end{aligned} \tag{4.23}$$

Note that condition (4.22) is satisfied or not uniformly for $x \in \Gamma$. Indeed, for $n \geq d(x,x')$, $C_1^{-1}\,V(x,n) \leq V(x',n) \leq C_1\,V(x,n)$. On manifolds, the necessity of (4.22) was proved in [37]. The sufficiency and the estimates (4.23) were studied in [19], [34], [35], [36] with assumptions on the curvature. With the work [30], L. Saloff-Coste obtained them with Poincaré inequality assumption.

PROOF. We use the Gaussian estimates $G(c_l, C_l, C_r, c_r)$. They yield

$$\begin{aligned} C^{-1}\, m(y) \sum_{n=d^2(x,y)}^{+\infty} \frac{1}{V(x,\sqrt{n})} &\leq G(x,y) \\ &\leq C\, m(y) \sum_{n=d^2(x,y)}^{+\infty} \frac{1}{V(x,\sqrt{n})}\,. \end{aligned} \tag{4.24}$$

The lower bound is a consequence of

$$G(x,y) = \sum_{n=0}^{+\infty} p_n(x,y) \geq \sum_{n=d^2(x,y)}^{+\infty} p_n(x,y) \geq \sum_{n=d^2(x,y)}^{+\infty} \frac{c_l\, m(y)}{V(x,\sqrt{n})}\, e^{-C_l}\,.$$

The upper bound is obtained by dividing the sum $G(x,y)$ into two parts

$$\sum_{n=d^2(x,y)}^{+\infty} p_n(x,y) \leq C_r\, m(y) \sum_{n=d^2(x,y)}^{+\infty} \frac{1}{V(x,\sqrt{n})}\,,$$

and

$$\begin{aligned} \sum_{n=0}^{d^2(x,y)} p_n(x,y) &\leq \sum_{n=d(x,y)}^{d^2(x,y)} \frac{C_r\, m(y)}{V(x,\sqrt{n})}\, e^{-c_r d^2(x,y)/n} \\ &\leq C_r\, m(y) \sum_{n=d(x,y)}^{d^2(x,y)} \frac{1}{V(x, 2\,d(x,y))} \\ &\qquad \cdot \underbrace{C_1 \Big(\frac{2\,d(x,y)}{\sqrt{n}}\Big)^{\log C_1/\log 2} e^{-c_r d^2(x,y)/n}}_{\leq \text{constant}} \\ &\leq C\, m(y) \sum_{n=d^2(x,y)}^{2d^2(x,y)} \frac{1}{V(x,\sqrt{n})}\,. \end{aligned}$$

The proposition follows from (4.24) since

$$\sum_{n=d^2(x,y)}^{+\infty} \frac{1}{V(x,\sqrt{n})} = \sum_{k=d(x,y)}^{+\infty} \underbrace{\#\{n \in \mathbb{N} : \ k \le \sqrt{n} < k+1\}}_{=2k+1} \frac{1}{V(x,k)} .$$

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References.

[1] Aronson, D. G., Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.* **73** (1967), 890-896.

[2] Bombieri, E., Theory of minimal surfaces and a counter-example to the Bernstein conjecture in high dimensions. *Mimeographed Notes of Lectures held at Courant Institute.* New-York University, 1970.

[3] Chung, F. R. K., Yau S. T., A Harnack inequality for homogeneous graphs and subgraphs. *Comm. Anal. Geom.* **2** (1994), 628-639.

[4] Coulhon, T., Saloff-Coste, L., Puissances d'un opérateur régularisant. *Ann. Inst. H. Poincaré, Probabilités et Statistique* **26** (1990), 419-436.

[5] Coulhon, T., Saloff-Coste, L., Minorations pour les chaînes de Markov unidimensionnelles. *Probab. Theor. Relat. Fields* **97** (1993), 423-431.

[6] Coulhon, T., Saloff-Coste, L., Isopérimétrie sur les groupes et les variétés. *Revista Mat. Iberoamericana* **9** (1993), 293-314.

[7] Davies, E. B., *Heat kernels and spectral theory.* Cambridge University Press, 1989.

[8] Davies, E. B., Large deviations for heat kernels on graphs. *J. London Math. Soc.* **47** (1993), 65-72.

[9] Delmotte, T., Inégalité de Harnack elliptique sur les graphes. *Colloquium Mathematicum* **72** (1997), 19-37.

[10] Delmotte, T., Estimations pour les chaînes de Markov réversibles. *C. R. Acad. Sci. Paris* **324** (1997), 1053-1058.

[11] Fabes, E., Stroock, D., A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. *Arch. Rational Mech. Anal.* **96** (1986), 327-338.

[12] Grigor'yan, A., The heat equation on noncompact Riemannian manifolds. *Math. USSR Sb.* **72** (1992), 47-77.

[13] Grigor'yan, A., Gaussian upper bounds for the heat kernel on arbitrary manifolds. *J. Differential Geom.* **45** (1997), 33-52.

[14] Hajłasz, P., Koskela, P., Sobolev meets Poincaré. *C. R. Acad. Sci. Paris* **120** (1995), 1211-1215.

[15] Hebisch, W., Saloff-Coste, L., Gaussian estimates for Markov chains and random walks on groups. *Ann. Probab.* **21** (1993), 673-709.

[16] Holopainen, I., Soardi, P. M., A strong Liouville theorem for p-harmonic functions on graphs. *Ann. Acad. Sci. Fennicae Math.* **22** (1997), 205-226.

[17] Jerison, D., The Poincaré inequality for vector fields satisfying the Hörmander 's condition. *Duke Math. J.* **53** (1986), 503-523.

[18] Kusuoka, S., Stroock, D., Application of Malliavin calculus, part 3. *J. Fac. Sci. Univ. Tokyo, Série IA, Math.* **34** (1987), 391-442.

[19] Li, P., Yau, S. T., On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156** (1986), 153-201.

[20] Merkov, A. B., Second-order elliptic equations on graphs. *Math. USSR Sb.* **55** (1986), 493-509.

[21] Moser, J., On Harnack's Theorem for elliptic differential equations. *Comm. Pure Appl. Math.* **14** (1961), 577-591.

[22] Moser, J., A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.* **17** (1964), 101-134. Correction in **20** (1967), 231-236.

[23] Moser, J., On a pointwise estimate for parabolic differential equations. *Comm. Pure Appl. Math.* **24** (1971), 727-740.

[24] Nash, J., Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* **80** (1958), 931-954.

[25] Pang, M. M. H., Heat kernels of graphs. *J. London Math. Soc.* **47** (1993), 50-64.

[26] Rigoli, M., Salvatori, M., Vignati, M., A global Harnack inequality on graphs and some related consequences. Preprint.

[27] Russ, E., Riesz transforms on graphs for $1 \leq p \leq 2$. To appear in *Math. Scandinavia.*

[28] Saloff-Coste, L., Uniformly elliptic operators on Riemannian manifolds. *J. Differential Geom.* **36** (1992), 417-450.

[29] Saloff-Coste, L., A note on Poincaré, Sobolev and Harnack inequalities. *Internat. Math. Res. Notices* **2** (1992), 27-38.

[30] Saloff-Coste, L., Parabolic Harnack inequality for divergence form second order differential operators. *Potential analysis* **4** (1995), 429-467.

[31] Saloff-Coste, L., *Lectures on finite Markov chains.* Cours de l'Ecole d'Eté de Probabilités de Saint-Flour, 1996.

[32] Saloff-Coste, L., Stroock, D., Opérateurs uniformément sous-elliptiques sur les groupes de Lie. *J. Funct. Anal.* **98** (1991), 97-121.

[33] Stroock, D. W., Zheng W., Markov chain approximations to symmetric diffusions. *Ann. Inst. H. Poincaré* **33** (1997), 619-649.

[34] Varopoulos, N., The Poisson kernel on positively curved manifolds. *J. Funct. Anal.* **44** (1981), 359-380.

[35] Varopoulos, N., Green's function on positively curved manifolds. *J. Funct. Anal.* **45** (1982), 109-118.

[36] Varopoulos, N., Green's function on positively curved manifolds, II. *J. Funct. Anal.* **49** (1982), 170-176.

[37] Varopoulos, N., *Potential theory and diffusion on Riemaniann manifolds.* Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II, Wadsworth Math. Ser., Wadsworth, Belmont, Calif. (1983), 821-837.

[38] Varopoulos, N., Small time Gaussian estimates of the heat diffusion kernel, Part 1: the semigroup technique. *Bull. Sci. Math.* **113** (1989), 253-277.

[39] Varopoulos, N., Coulhon, T., Saloff-Coste, L., *Analysis and geometry on groups.* Cambridge University Press, 1993.

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Singular integral operators with non-smooth kernels on irregular domains

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Abstract. Let $\mathcal{X}$ be a space of homogeneous type. The aims of this paper are as follows.

i) Assuming that T is a bounded linear operator on $L_2(\mathcal{X})$, we give a sufficient condition on the kernel of T so that T is of weak type $(1,1)$, hence bounded on $L_p(\mathcal{X})$ for $1 < p \leq 2$; our condition is weaker than the usual Hörmander integral condition.

ii) Assuming that T is a bounded linear operator on $L_2(\Omega)$ where Ω is a measurable subset of $\mathcal{X}$, we give a sufficient condition on the kernel of T so that T is of weak type $(1,1)$, hence bounded on $L_p(\Omega)$ for $1 < p \leq 2$.

iii) We establish sufficient conditions for the maximal truncated operator T_*, which is defined by $T_* u(x) = \sup_{\varepsilon > 0} |T_\varepsilon u(x)|$, to be L_p bounded, $1 < p < \infty$. Applications include weak $(1,1)$ estimates of certain Riesz transforms, and L_p boundedness of holomorphic functional calculi of linear elliptic operators on irregular domains.

1. Introduction.

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, equipped with a metric d and a measure μ. Let T be a bounded linear operator on

$L_2(\mathcal{X})$ with an associated kernel $k(x,y)$ in the sense that

$$(Tf)(x) = \int_{\mathcal{X}} k(x,y)\, f(y)\, d\mu(y)\,, \tag{1}$$

where $k(x,y)$ is a measurable function, and the above formula holds for each continuous function f with compact support, and for almost all x not in the support of f.

One important result of Calderón–Zygmund operator theory is the well known Hörmander integral condition on the kernel $k(x,y)$, see [Hör], which is a sufficient condition for the operator T to be of weak type $(1,1)$. It states that T satisfies weak $(1,1)$ estimates if there exist constants C and $\delta > 1$ so that

$$\int_{d(x,y)\geq \delta d(y_1,y)} |k(x,y) - k(x,y_1)|\, d\mu(x) \leq C\,,$$

for all $y, y_1 \in \mathcal{X}$.

In practice, many operators satisfy the Hörmander integral condition, but there are numerous examples of operators which do not, and certain classes of such operators can be proved to be of weak type $(1,1)$. See, for example [F], [Ch1], [CR], [Hof], [Se]. However, in these papers, the authors investigate specific classes of operators and do not give sufficient conditions on kernels for general operators to be of weak type $(1,1)$.

A natural question is whether one can weaken the Hörmander integral condition and still conclude that T is of weak type $(1,1)$. Although Calderón–Zygmund operator theory is now well established, to our best knowledge, no such condition is known. Our first aim is to give a positive answer to this open question.

There is another limitation of the usual Calderón–Zygmund theory. It is only established for spaces of homogeneous type. The main feature of these spaces is that they satisfy the doubling property. Measurable subsets of $\mathbb{R}^n$ which do not possess any smoothness of their boundaries, do not satisfy the doubling property, hence they are not spaces of homogeneous type. Such measurable sets, however, do appear naturally in partial differential equations. Our second aim is to present a sufficient condition on the kernel of a bounded operator T on $L_2(\Omega)$, where Ω is a measurable subset of a space of homogeneous type, so that T is of weak type $(1,1)$ on Ω.

The paper is organised as follows. In Section 2, we assume that T is a bounded linear operator on $L_2(\mathcal{X})$, where $\mathcal{X}$ is a space of homogeneous type. We then prove a sufficient condition on the kernel $k(x,y)$ of T so that T is of weak type $(1,1)$ (Theorem 1). Roughly speaking, T is of weak type $(1,1)$ if there exists a class of operators A_t with kernels $a_t(x,y)$, which play the role of approximations to the identity, so that the kernels $k_t(x,y)$ of the composite operators TA_t satisfy the condition

$$\int_{d(x,y)\geq ct^{1/m}} |k(x,y)-k_t(x,y)|\, d\mu(x) \leq C\,,$$

for some positive constants m, c, C, uniformly in $y \in \mathcal{X}$ and $t>0$. The freedom in choosing A_t is important. In particular circumstances we may require them to commute with T, or we may wish to allow the kernels a_t to be discontinuous.

It is not difficult to check that our condition is a consequence of the Hörmander integral condition (Proposition 1).

In Section 3, we assume that Ω is a measurable subset of a space of homogeneous type with no smoothness on the boundary. We then present a sufficient condition on the kernel $k(x,y)$ which is somewhat stronger than that of Theorem 1, so that the operator T is of weak type $(1,1)$ on Ω (Theorem 2). Our result gives new criteria to investigate the L_p boundedness of singular integrals on measurable sets. The results on Ω are made possible by the fact that no smoothness is required on the kernels $a_t(x,y)$ in Theorem 1.

In Section 4, we extend the results in sections 2 and 3 to establish sufficient conditions on the kernel $k(x,y)$ which ensure the L_p boundedness of the maximal truncated operator T_*, where $T_*u(x) = \sup_{\varepsilon>0}|T_\varepsilon u(x)|$ and

$$T_\varepsilon u(x) = \int_{d(x,y)\geq\varepsilon} k(x,y)\, u(y)\, d\mu(y)\,.$$

Our assumptions on the kernel $k(x,y)$ are somewhat stronger than those used in Theorems 1 and 2, but are essentially weaker than the usual ones on spaces of homogeneous type (Theorem 3). The result is new for measurable subsets of spaces of homogeneous type (Theorem 4).

Applications are given in Section 5. We first establish weak $(1,1)$ estimates for certain Riesz transforms and similar types of operators (Theorem 5). This allows us, for example, to simplify the proof of the L_p boundedness of the Riesz transforms on Lie groups which was given by Saloff–Coste when $1<p\leq 2$ [SC].

Finally, we prove that every operator L with a bounded holomorphic functional calculus in $L_2(\Omega)$, which generates a semigroup with suitable upper bounds on its heat kernels, also has a bounded holomorphic functional calculus in $L_p(\Omega)$ when $1 < p < \infty$ (Theorem 6). Here Ω is a measurable subset of a space $\mathcal{X}$ of homogeneous type. It is this result which prompted our investigation, so let us outline its background.

In the case when the heat kernels also satisfy Hölder bounds, then this result follows from the usual Calderón–Zygmund theory, because the operators $f(L)$ in the functional calculus satisfy standard Calderón–Zygmund bounds. This is the approach developed by Duong in the case of those elliptic operators having such heat kernels, which are defined by boundary conditions on strongly Lipschitz domains. See his thesis [Du] and also [DM^c]. This method does not work for those elliptic operators whose heat kernels satisfy pointwise bounds but not Hölder bounds. In [DR], Duong and Robinson showed how to proceed in such cases, provided still that the operators are defined on strongly Lipschitz domains. There they proved the first part of Theorem 6 of this paper in the case when Ω is a space of homogeneous type, though the last part, namely the L_p boundedness of the maximal truncated operators, is new. In [AE], Arendt and ter Elst applied this theorem to the Dirichlet problem for certain elliptic operators defined on subsets of $\mathbb{R}^n$ whose boundary has null measure, by extending the functional calculus to that of an operator defined on all of $\mathbb{R}^n$. They asked whether the assumption concerning the null measure of the boundary could be dropped. This is what we do in Theorem 6.

As can be seen, our investigations into removing the assumption of Hölder continuity from the kernels have led to the formulation of general conditions on singular integral operators which are applicable in a variety of situations.

2. Weak (1,1) estimates of singular integral operators.

Let $\mathcal{X}$ be a topological space equipped with a measure μ and a metric d which is a measurable function on $\mathcal{X} \times \mathcal{X}$. We define $\mathcal{X}$ to be *a space of homogeneous type* if the balls $B(x;r) = \{y \in \mathcal{X} : d(x,y) < r\}$ satisfy the doubling property

$$\mu(B(x;2\,r)) \le c\,\mu(B(x;r)) < +\infty\,,$$

for some $c \ge 1$ uniformly for all $x \in \mathcal{X}$ and $r > 0$. A more general definition can be found in [CW, Chapter 3].

Note that the doubling property implies the following strong homogeneity property,

$$\mu(B(x;\lambda r)) \le c\,\lambda^n \mu(B(x;r))\,,$$

for some $c, n > 0$ uniformly for all $\lambda \ge 1$. The parameter n is a measure of the dimension of the space. There also exist c and N, $0 \le N \le n$ so that

$$\text{(U)} \qquad \mu(B(y;r)) \le c\Big(1 + \frac{d(x,y)}{r}\Big)^N \mu(B(x;r))$$

uniformly for all $x, y \in \mathcal{X}$ and $r > 0$. Indeed, the property (U) with $N = n$ is a direct consequence of triangle inequality of the metric d and the strong homogeneity property. In the cases of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, N can be chosen to be 0.

Let T be a bounded linear operator mapping $L_2(\mathcal{X})$ into $L_2(\mathcal{X})$. Assume the operator T is given by a kernel $k(x,y)$ in the sense of (1).

We shall work with a class of integral operators A_t, $t > 0$, which plays the role of approximations to the identity. We assume the operators A_t can be represented by kernels $a_t(x,y)$ in the sense that

$$A_t u(x) = \int_{\mathcal{X}} a_t(x,y)\,u(y)\,d\mu(y)\,,$$

for every function $u \in L_2(\mathcal{X}) \cap L_1(\mathcal{X})$, and the kernels $a_t(x,y)$ satisfy the following conditions

$$\text{(2)} \qquad |a_t(x,y)| \le h_t(x,y)\,,$$

for all $x, y \in \mathcal{X}$ where $h_t(x,y)$ is a function satisfying

$$\text{(3)} \qquad h_t(x,y) = (\mu(B(x;t^{1/m})))^{-1} s(d(x,y)^m\,t^{-1})$$

in which m is a positive constant and s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\kappa}\,s(r^m) = 0\,,$$

for some $\kappa > N$, where N is the power which appeared in property (U), and n the "dimension" entering the strong homogeneity property.

It then follows that

$$
\begin{aligned}
h_t(x,y) &\le c \min\Big\{\frac{1}{\mu(B(x;t^{1/m}))}, \frac{1}{\mu(B(y;t^{1/m}))}\Big\} \\
&\qquad \cdot \Big(1+\frac{d(x,y)}{t^{1/m}}\Big)^N s(d(x,y)^m\, t^{-1}) \qquad (4)\\
&\le c \min\Big\{\frac{1}{\mu(B(x;t^{1/m}))}, \frac{1}{\mu(B(y;t^{1/m}))}\Big\} s_1(d(x,y)^m\, t^{-1}),
\end{aligned}
$$

where s_1 is a function similar to s with some $\kappa > 0$.

We also note that there exist positive constants c_1 and c_2 so that

$$c_1 \le \int_{\mathcal{X}} h_t(x,y)\, d\mu(x) \le c_2$$

uniformly in t and y.

The existence of such a class of operators A_t in a space of homogeneous type, is not a problem. We can first choose a function s satisfying the decay condition in (3), define h_t as in (3), and let $a_t = h_t$, hence conditions (2) and (3) are automatically satisfied. The kernels a_t then possess the smoothness of the function s.

For any $m > 0$, we can also construct $a_t(x,y)$ with the following additional properties

$$a_t(x,y) = 0\,, \qquad \text{when } d(x,y) \ge c_0\, t^{1/m}\,, \tag{5}$$

$$\int_{\mathcal{X}} a_t(x,y)\, d\mu(x) = 1\,, \tag{6}$$

for all $y \in \mathcal{X}$, $t > 0$. This can be achieved by choosing

$$a_t(x,y) = (\mu(B(y;t^{1/m})))^{-1} \chi_{B(y;t^{1/m})}(x)\,,$$

where $\chi_{B(y;t^{1/m})}$ denotes the characteristic function on the ball $B(y;t^{1/m})$. Then let A_t be the operators which are given by the kernels $a_t(x,y)$.

These operators A_t constructed as above exist in the space $\mathcal{X}$ independently of the operator T. However, for certain operators T, it is useful to construct operators A_t which are related to T. This is of interest since the analysis of A_t, TA_t and A_tT is useful for establishing the boundedness of T in an L_p space. Examples of this are given in Section 5.

The following lemma is needed in the proof of Theorem 1. For its proof, see [DR, Proposition 2.5].

Lemma 1. *Given functions $h_t(x,z)$ which satisfy* (3), *and $\nu > 0$, there exist positive constants c and θ such that*

$$\sup_{z\in B(y,r)} h_t(x,z) \le c \inf_{z\in B(y,r)} h_{\theta t}(x,z)$$

uniformly for $x, y \in \mathcal{X}$, and $r, t > 0$ with $r^m \le \nu\, t$.

We now present the main result of this section. The proof is based on that used by Duong and Robinson in proving [DR, Theorem 3.1]. It relies upon the idea of Hebisch [He] of using L_2-estimates to obtain weak type $(1,1)$ bounds. Related ideas also appeared earlier in [F].

Theorem 1. *Let T be a bounded linear operator from $L_2(\mathcal{X})$ to $L_2(\mathcal{X})$ with an associated kernel $k(x,y)$. Assume there exists a class of operators A_t, $t > 0$, which satisfy the conditions* (2) *and* (3) *so that the composite operators TA_t have associated kernels $k_t(x,y)$ in the sense of* (1) *and there exist constants C and $c > 0$ so that*

$$\int_{d(x,y)\ge ct^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x) \le C\,, \tag{7}$$

for all $y \in \mathcal{X}$.

Then the operator T is of weak type $(1,1)$. Hence, T can be extended from $L_2(\mathcal{X}) \cap L_p(\mathcal{X})$ to a bounded operator on $L_p(\mathcal{X})$ for all $1 < p \le 2$.

Proof. We need to prove that T satisfies weak type $(1,1)$ estimates. Boundedness of T on $L_p(\mathcal{X})$ then follows from the Marcinkiewicz interpolation theorem.

Our proof makes use of the Calderón–Zygmund decomposition to decompose an integrable function into "good" and "bad" parts (see, for example, [CW]), then each part is analysed separately.

Given $f \in L_1(\mathcal{X}) \cap L_2(\mathcal{X})$ and $\alpha > \|f\|_1(\mu(\mathcal{X}))^{-1}$, then there exist a constant c independent of f and α, and a decomposition

$$f = g + b = g + \sum_i b_i\ ,$$

so that

a) $|g(x)| \le c\,\alpha$ for almost all $x \in \mathcal{X}$,

b) there exists a sequence of balls Q_i so that the support of each b_i is contained in Q_i and

$$\int |b_i(x)|\, d\mu(x) \le c\,\alpha\, \mu(Q_i)\,,$$

c) $\displaystyle\sum_i \mu(Q_i) \le \frac{c}{\alpha} \int |f(x)|\, d\mu(x)\,,$

d) each point of $\mathcal{X}$ is contained in at most a finite number N of the balls Q_i.

Note that if $\mu(\mathcal{X}) = \infty$, then $\|f\|_1(\mu(\mathcal{X}))^{-1}$ means 0. Besides that, the functions b_i are usually chosen to satisfy $\int b_i\, d\mu(x) = 0$ as well, but we do not need this property.

Conditions b) and c) also imply that $\|b\|_1 \le c\,\|f\|_1$ and hence that $\|g\|_1 \le (1+c)\,\|f\|_1$.

We have

$$\begin{aligned} &\mu(\{x:\ |Tf(x)| > \alpha\}) \\ &\qquad \le \mu\Big(\Big\{x:\ |Tg(x)| > \frac{\alpha}{2}\Big\}\Big) + \mu\Big(\Big\{x:\ |Tb(x)| > \frac{\alpha}{2}\Big\}\Big)\,. \end{aligned}$$

It is not difficult to check that $g \in L_2(\mathcal{X})$. Using the facts that T is bounded on $L_2(\mathcal{X})$ and that $|g(x)| \le c\,\alpha$, we obtain

$$\mu\Big(\Big\{x:\ |Tg(x)| > \frac{\alpha}{2}\Big\}\Big) \le 4\,\alpha^{-2}\,\|Tg\|_2^2 \le c_1\,\alpha^{-2}\|g\|_2^2 \le \frac{c_2}{\alpha}\,\|f\|_1\,. \tag{8}$$

Concerning the "bad" part $b(x)$, we temporarily fix a b_i whose support is contained in Q_i, then choose $t_i = r_i^m$ where m is the constant appearing in (3) and r_i is the radius of the ball Q_i. We then decompose

$$Tb_i(x) \;=\; TA_{t_i} b_i(x) + (T - TA_{t_i})\, b_i(x)\,.$$

To analyse $TA_{t_i} b_i(x)$, we first estimate the function $A_{t_i} b_i$. Since

$$A_{t_i} b_i(x) = \int_{\mathcal{X}} a_{t_i}(x,y)\, b_i(y)\, d\mu(y)\,,$$

it follows from Lemma 1 that

$$\begin{aligned} |A_{t_i} b_i(x)| &\le \int_{\mathcal{X}} h_{t_i}(x,y)\,|b_i(y)|\,d\mu(y) \\ &\le \|b_i\|_1 \sup_{y\in Q_i} h_{t_i}(x,y) \\ &\le c\,\alpha\,\mu(Q_i) \inf_{y\in Q_i} h_{\theta t_i}(x,y) \\ &\le c\,\alpha \int_{\mathcal{X}} h_{\theta t_i}(x,y)\,\chi_i(y)\,d\mu(y)\,, \end{aligned}$$

where χ_i denotes the characteristic function of the ball Q_i.

Denoting by M the Hardy-Littlewood maximal operator, we then have for any $u \in L_2(\mathcal{X})$

$$\begin{aligned} |\langle |u|, A_{t_i} b_i\rangle| &\le c\,\alpha \int_{\mathcal{X}}\int_{\mathcal{X}} |u(x)|\,h_{\theta t_i}(x,y)\,\chi_i(y)\,d\mu(y)\,d\mu(x) \\ &\le c\,\alpha\,\langle M\,|u|, \chi_i\rangle\,. \end{aligned}$$

Note that the second inequality follows from properties 3) and 4'). Since the Hardy-Littlewood maximal operator is bounded on $L_2(\mathcal{X})$, (see for example [Ch2]), it follows that

$$\Big\| \sum_i A_{t_i} b_i \Big\|_2 \le c\,\alpha\,\Big\| \sum_i \chi_i \Big\|_2\,. \tag{9}$$

We now use properties c) and d) of the Calderón–Zygmund decomposition to obtain the estimate

$$\Big\| \sum_i A_{t_i} b_i \Big\|_2 \le c\,\alpha \Big(\sum_i \mu(Q_i) \Big)^{1/2} \le c\,\alpha^{1/2} \|f\|_1^{1/2}\,. \tag{10}$$

Therefore

$$\begin{aligned} \mu\Big(\Big\{x :\ \Big|\sum_i T A_{t_i} b_i(x)\Big| > \frac{\alpha}{4}\Big\}\Big) &\le 16\,\alpha^{-2} \Big\| \sum_i T A_{t_i} b_i \Big\|_2^2 \\ &\le c\,\alpha^{-2} \Big\| \sum_i A_{t_i} b_i \Big\|_2^2 \\ &\le \frac{c}{\alpha}\,\|f\|_1\,. \end{aligned} \tag{11}$$

On the other hand

$$\mu\Big(\Big\{x : \Big|\sum_i (T - TA_{t_i})\, b_i(x)\Big| > \frac{\alpha}{4}\Big\}\Big)$$
$$\leq \sum_i \mu(B_i) + \sum_i \frac{4}{\alpha} \int_{^cB_i} |(T - TA_{t_i})\, b_i(x)|\, d\mu(x)\,,$$

where ^cB_i denotes the complement of B_i which is the ball with the same centre y_i as that of the ball Q_i in the Calderón–Zygmund decomposition but with radius increased by the factor $(1+c)$, where c is the constant in (7). Because of property c) of the decomposition and the doubling property of $\mathcal{X}$, we have

$$\sum_i \mu(B_i) \leq c \sum_i \mu(Q_i) \leq c\,\alpha^{-1} \|f\|_1\,. \tag{12}$$

By assumption (7), we have

$$\int_{^cB_i} |(T - TA_{t_i})\, b_i(x)|\, d\mu(x)$$
$$\leq \int_{^cB_i} \Big| \int_{\mathcal{X}} k(x,y) - k_{t_i}(x,y)\, b_i(y)\, d\mu(y) \Big|\, d\mu(x)$$
$$\leq \int_{\mathcal{X}} |b_i(y)| \Big(\int_{d(x,y) \geq c t_i^{1/m}} |k(x,y) - k_{t_i}(x,y)|\, d\mu(x) \Big)\, d\mu(y)$$
$$\leq C\, \|b_i\|_1\,,$$

because $B(y; c\,t_i^{1/m}) \subset B_i$.

Therefore

$$\sum_i \frac{1}{\alpha} \int_{^cB_i} |(T - TA_{t_i})\, b_i(x)|\, d\mu(x) \leq \sum_i \frac{C}{\alpha}\, \|b_i\|_1 \leq \frac{C\, \|f\|_1}{\alpha}\,. \tag{13}$$

Combining the estimates (8), (11), (12) and (13), the theorem is proved.

REMARK.

i) It is straightforward from the proof of Theorem 1, that the existence of both the kernels $k(x,y)$ of T and $k_t(x,y)$ of TA_t is not necessary. We only need to assume that the difference operator $T - TA_t$ has an associated kernel so that this kernel (in place of $k(x,y) - k_t(x,y)$) satisfies Condition 7. This remark also applies to Theorem 2.

ii) In Theorem 1, the assumption on boundedness of T on the space $L_2(\mathcal{X})$ can be replaced by boundedness of T on a space $L_{p_o}(\mathcal{X})$ for some $p_0 > 1$. The proof would need only minor changes to show that T is of weak type $(1,1)$, hence bounded on $L_p(\mathcal{X})$ for all $1 < p \leq p_o$.

iii) Theorem 1 and a standard duality argument give the following result.

Let T be a bounded linear operator from $L_2(\mathcal{X})$ into $L_2(\mathcal{X})$ with an associated kernel $k(x,y)$ in the sense of (1). Assume there exists a class of operators B_t whose kernels satisfy the conditions (2) and (3) so that the composite operators B_tT have associated kernels $K_t(x,y)$ in the sense of (1), and there exist constants $c > 0$ and C so that

$$\int_{d(x,y)\geq ct^{1/m}} |k(x,y) - K_t(x,y)|\, d\mu(y) \leq C\,, \tag{14}$$

for all $x \in \mathcal{X}$.

Then the adjoint operator T^* is of weak type $(1,1)$. Hence, T can be extended from $L_2(\mathcal{X}) \cap L_p(\mathcal{X})$ to a bounded operator on $L_p(\mathcal{X})$ for all $2 \leq p < \infty$.

Natural questions about Theorem 1 are how strong is the assumption (7), and what is its relation with the Hörmander integral condition. We shall show that, for suitably chosen A_t, our condition (7) is actually a consequence of the Hörmander integral condition for spaces of homogeneous type.

Proposition 1. *Assume that T has an associated kernel $k(x,y)$ which satisfies the Hörmander integral condition, i.e. there exist constants C and $\delta > 1$, so that*

$$\int_{d(x,y)\geq \delta d(y,z)} |k(x,y) - k(x,z)|\, d\mu(x) \leq C\,,$$

for all $y, z \in \mathcal{X}$. Let A_t be approximations to the identity which are represented by kernels $a_t(x,y)$ satisfying conditions (2), (3), (5) *and* (6).

Then the kernels $k_t(x,y)$ of TA_t satisfy condition (7) *of Theorem* 1. *More precisely, there exist constants c and β so that*

$$\int_{d(x,y)\geq \beta t^{1/m}} |k(x,y)-k_t(x,y)|\, d\mu(x) \leq c\,,$$

for all $y \in \mathcal{X}$.

PROOF. Choose $\delta > 1$ and let $\beta = c_0\delta$ where c_0 is the constant so that $a_t(x,y) = 0$ when $d(x,y) \geq c_0\, t^{1/m}$. Then, for $x, y \in \mathcal{X}$ so that $d(x,y) \geq \beta\, t^{1/m}$,

$$k_t(x,y) = \int_{\mathcal{X}} k(x,z)\, a_t(z,y)\, d\mu(z)\,.$$

For all $y \in \mathcal{X}$,

$$\begin{aligned}
&\int_{d(x,y)\geq \beta t^{1/m}} |k(x,y)-k_t(x,y)|\, d\mu(x)\\
&= \int_{d(x,y)\geq \beta t^{1/m}} \left|k(x,y) - \int_{\mathcal{X}} k(x,z)\, a_t(z,y)\, d\mu(z)\right| d\mu(x)\\
&= \int_{d(x,y)\geq \beta t^{1/m}}\\
&\quad \cdot \left|k(x,y)\int_{d(z,y)\leq c_0 t^{1/m}} a_t(z,y)\, d\mu(z)\right.\\
&\qquad\qquad\qquad \left. - \int_{d(z,y)\leq c_0 t^{1/m}} k(x,z)\, a_t(z,y)\, d\mu(z)\right| d\mu(x)\\
&\leq \sup_{d(z,y)\leq c_0 t^{1/m}} \Big(\int_{d(x,y)\geq \beta t^{1/m}} |k(x,y)-k(x,z)|\, d\mu(x)\Big)\\
&\qquad \cdot \Big(\int_{d(z,y)\leq c_0 t^{1/m}} |a_t(z,y)|\, d\mu(z)\Big)\\
&\leq c_1 \sup_{d(z,y)\leq c_0 t^{1/m}} \Big(\int_{d(x,y)\geq \delta c_0 t^{1/m}} |k(x,y)-k(x,z)|\, d\mu(x)\Big)\\
&\leq c\,.
\end{aligned}$$

Note that the second equality follows from property (6), the second inequality is using the estimate

$$\int_{\mathcal{X}} |a_t(z,y)|\, d\mu(z) \leq \int_{\mathcal{X}} h_t(z,y)\, d\mu(z) \leq c_1\ ,$$

and the last inequality follows from the Hörmander integral condition.

3. Singular integral operators on measurable subsets of a space of homogeneous type.

We assume in this section that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$. An example of Ω is an open domain of the Euclidean space $\mathbb{R}^n$. If Ω possesses certain smoothness on its boundary, for example Lipschitz boundary, then it is a space of homogeneous type and results of Section 2 are applicable. However, a general measurable set Ω needs not satisfy the doubling property, hence it is not a space of homogeneous type. Such a measurable set Ω appears naturally in boundary value problems, for example partial differential equations with Dirichlet boundary conditions.

Given a bounded linear operator T on $L_2(\Omega)$ with an associated kernel $k(x, y)$, the question is to find a sufficient condition on $k(x, y)$ for T to be of weak type $(1, 1)$. The main problem in this case is the fact that the Calderón–Zygmund theory is not directly applicable. For example, the Calderón–Zygmund decomposition is not valid on Ω, nor is the Hardy-Littlewood maximal operator bounded, as was needed in proving the estimate (9).

A key observation to solve this problem is surprisingly simple. Given a linear operator T which maps $L_p(\Omega)$ into itself for some p, define an associated operator $\widetilde{T}$ on $L_p(\mathcal{X})$ by

$$\widetilde{T}(u)(x) = \begin{cases} T(\chi_\Omega u)(x)\,, & x \in \Omega\,, \\ 0\,, & x \notin \Omega\,, \end{cases}$$

where χ_Ω is the characteristic function on Ω. Then T *is bounded on* $L_p(\Omega)$ *if and only if* $\widetilde{T}$ *is bounded on* $L_p(\mathcal{X})$, also T *is of weak type* $(1, 1)$ *on* Ω *if and only if* $\widetilde{T}$ *is of weak type* $(1, 1)$ *on* $\mathcal{X}$.

It is straightforward to check the above equivalences, so we leave them to reader. Note that if T has an associated kernel $k(x, y)$ in the sense of (1), then $\widetilde{T}$ also has an associated kernel $\tilde{k}(x, y)$ in the sense of (1), given by

$$\tilde{k}(x, y) = \begin{cases} k(x, y)\,, & \text{when } x \in \Omega \text{ and } y \in \Omega\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

We can see immediately that the condition that the kernel $k(x, y)$ of T satisfies the Hörmander condition is not sufficient for the kernel $\tilde{k}(x, y)$

of $\widetilde{T}$ to satisfy the Hörmander condition. By using $\widetilde{T}$, what we do is to transform the question of boundedness of T on a measurable set Ω to the boundedness of $\widetilde{T}$ on a better space (of homogeneous type) $\mathcal{X}$, but the kernel of $\widetilde{T}$ could be discontinuous. However, the proof of Theorem 1 makes use of the upper bounds on $a_t(x,y)$ and condition (7), and does not require any continuity assumptions on $k(x,y)$.

From now on, to differentiate between a ball in $\mathcal{X}$ and a ball in Ω, we use the notations $B^{\mathcal{X}}$ and B^{Ω}.

The main theorem of this section is the following.

Theorem 2. *Let T be a bounded linear operator from $L_2(\Omega)$ to $L_2(\Omega)$ with an associated kernel $k(x,y)$ in the sense of* (1). *Assume there exists a class of operators A_t, $t>0$, with kernels $a_t(x,y)$ defined on $L_2(\Omega)$ so that*:

a)

$$A_t u(x) = \int_{\mathcal{X}} a_t(x,y)\, u(y)\, d\mu(y)\,,$$

for any function $u \in L_2(\Omega)\cap L_1(\Omega)$, and the kernels $a_t(x,y)$ satisfy the following conditions

$$|a_t(x,y)| \le h_t(x,y)\,, \tag{15}$$

for all $x, y \in \Omega$, where $h_t(x,y)$ is defined on $\mathcal{X}\times\mathcal{X}$ by

$$h_t(x,y) = (\mu(B^{\mathcal{X}}(x;t^{1/m})))^{-1} s(d(x,y)^m t^{-1})\,, \tag{16}$$

and s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\kappa}\, s(r^m) = 0\,,$$

for some $\kappa > 0$ with n the "dimension" entering the strong homogeneity property of $\mathcal{X}$,

b) *the composite operators TA_t have associated kernels $k_t(x,y)$ in the sense of* (1) *and there exist constants C and $c>0$ so that*

$$\int_{d(x,y)\ge ct^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x) \le C\,, \tag{17}$$

for all $y \in \Omega$.

Then the operator T is of weak type $(1,1)$. Hence, T can be extended from $L_2(\Omega) \cap L_p(\Omega)$ to a bounded operator on $L_p(\Omega)$ for all $1 < p \leq 2$.

PROOF. First observe that $\widetilde{TA_t} = \widetilde{T}\widetilde{A}_t$ where $\widetilde{T}$, $\widetilde{A}_t$ and $\widetilde{TA_t}$ are defined on $L_2(\mathcal{X})$ as described above. Moreover $\widetilde{T}$ and $\widetilde{TA_t}$ have kernels $\tilde{k}(x,y)$ and $\tilde{k}_t(x,y)$ in the sense of (1), where $\tilde{k}$ was defined above and

$$\tilde{k}_t(x,y) = \begin{cases} k_t(x,y)\,, & \text{when } x \in \Omega \text{ and } y \in \Omega\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Further, $\widetilde{A}_t$ is represented by the kernel

$$\tilde{a}_t(x,y) = \begin{cases} a_t(x,y)\,, & \text{when } x \in \Omega \text{ and } y \in \Omega\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

which is readily seen to satisfy conditions (2) and (3) on $\mathcal{X} \times \mathcal{X}$.

Conditions (15), (16), and (17) imply that the operator $\widetilde{T}$ satisfies the hypotheses of Theorem 1, hence it is of weak type $(1,1)$ on $\mathcal{X}$. Therefore, T is of weak type $(1,1)$ on Ω and Theorem 2 is proved.

REMARK. Assume there exist B_t which satisfy (15) and (16) so that the composite operators B_tT satisfy property (14). A standard duality argument shows that the adjoint operator of T is bounded on $L_p(\Omega)$ for $1 < p \leq 2$, hence T is bounded on $L_p(\Omega)$ for $2 \leq p < \infty$.

4. Boundedness of maximal truncated operators on L_p spaces.

4.1 The case of spaces of homogeneous type.

In this subsection, we assume that $\mathcal{X}$ is a space of homogeneous type equipped with a metric d and a measure μ. Let T be a bounded operator on $L_2(\mathcal{X})$ with an associated kernel $k(x,y)$ in the sense of (1). Our aim is to investigate the maximal truncated operator T_* which is defined by

$$T_*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|\,,$$

where T_ε is the truncated singular operator defined by

$$T_\varepsilon f(x) = \int_{d(x,y)\geq\varepsilon} k(x,y)\, f(y)\, d\mu(y)\,.$$

The main result of this section is the following theorem.

Theorem 3. *We assume the following conditions.*

a) *T is a bounded operator on $L_2(\mathcal{X})$ with an associated kernel $k(x,y)$.*

b) *There exists a class of operators A_t which satisfy the conditions (2) and (3) so that the composite operators TA_t have associated kernels $k_t(x,y)$ in the sense of (1). Also assume that there exist constants c_1 and $\delta > 0$ so that*

$$\int_{d(x,y)\geq \delta t^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x) \leq c_1 \,, \tag{7}$$

for all $y \in \mathcal{X}$.

c) *There exists a class of operators B_t represented by kernels $b_t(x,y)$ which satisfy the upper bounds $h_t(x,y)$ defined by (3), and the composite operators B_tT have kernels $K_t(x,y)$. Also assume that there exist positive constants α, c_2, c_3 and c_4 so that*

$$|K_t(x,y)| \leq c_2\, (\mu(B(x;t^{1/m})))^{-1} \,, \qquad \text{when } d(x,y) \leq c_3\, t^{1/m} \tag{18}$$

and

$$|K_t(x,y) - k(x,y)| \leq c_4\, (\mu(B(x;d(x,y))))^{-1}\, \frac{t^{\alpha/m}}{d(x,y)^{\alpha}} \,, \tag{19}$$

when $d(x,y) \geq c_3\, t^{1/m}$. Then T_* is bounded on $L_p(\mathcal{X})$ for all p, $1 < p < \infty$.

PROOF. It follows from conditions (5) and (19), Theorem 1 and a duality argument that T is bounded on $L_p(\mathcal{X})$ for $1 < p < \infty$. Without any loss of generality, we prove the theorem with $c_3 = 1$. For a fixed $\varepsilon > 0$, we write

$$T_\varepsilon u(x) = B_{\varepsilon^m} Tu(x) - (B_{\varepsilon^m} T - T_\varepsilon)\, u(x) \,.$$

Since the class of operators B_t satisfies conditions (2) and (3), we have

$$|B_{\varepsilon^m} Tu(x)| \leq c\, M\, (|Tu(x)|) \,, \tag{20}$$

where M is the Hardy-Littlewood maximal operator, and c is a constant independent of ε.

The kernel of the operator $(B_{\varepsilon^m}T - T_\varepsilon)$ is given by $(K_{\varepsilon^m}(x,y) - k_\varepsilon(x,y))$ where $k_\varepsilon(x,y) = k(x,y)$ if $d(x,y) \geq \varepsilon$ and $k_\varepsilon(x,y) = 0$ otherwise. There are two cases:

Case 1. $d(x,y) < \varepsilon$, then $k_\varepsilon(x,y) = 0$ and it follows from (18) that

$$|K_{\varepsilon^m}(x,y) - k_\varepsilon(x,y)| = |K_{\varepsilon^m}(x,y)| \leq c_2 \,\frac{1}{\mu(B(x;\varepsilon))}\;.$$

Case 2. $d(x,y) \geq \varepsilon$, then $k_\varepsilon(x,y) = k(x,y)$ and it follows from (19) that

$$|K_{\varepsilon^m}(x,y) - k_\varepsilon(x,y)| \leq c_4 \,\frac{1}{\mu(B(x;d(x,y)))} \left(\frac{\varepsilon}{d(x,y)}\right)^\alpha,$$

for some $\alpha > 0$.

Therefore

$$\begin{aligned}
&|(B_{\varepsilon^m}T - T_\varepsilon)u(x)| \\
&\qquad \leq c \int_{d(x,y)\leq\varepsilon} \frac{1}{\mu(B(x;\varepsilon))}\,|u(y)|\,d\mu(y) \\
&\qquad\quad + c\int_{d(x,y)>\varepsilon} \frac{1}{\mu(B(x;d(x,y)))}\left(\frac{\varepsilon}{d(x,y)}\right)^\alpha |u(y)|\,d\mu(y) \\
&\qquad \leq c\,\frac{1}{\mu(B(x;\varepsilon))}\int_{d(x,y)\leq\varepsilon} |u(y)|\,d\mu(y) \\
&\qquad\quad + c\sum_{k=1}^\infty \frac{1}{2^{k\alpha}}\,\frac{1}{\mu(B(x;2^{k+1}\varepsilon))}\int_{d(x,y)\leq 2^{k+1}\varepsilon} |u(y)|\,d\mu(y) \\
&\qquad \leq c\,M\left(|u|(x)\right),
\end{aligned}$$

where again M is the Hardy-Littlewood maximal operator, and c is a constant independent of ε. Therefore

$$\sup_{\varepsilon>0} |(B_{\varepsilon^m}T - T_\varepsilon)u(x)| \leq c\,M\left(|u|(x)\right). \tag{21}$$

Combining estimates (20), (21) with boundedness of T and the Hardy-Littlewood maximal operator on $L_p(\mathcal{X})$, we obtain boundedness of T_* on $L_p(\mathcal{X})$, $1 < p < \infty$.

In the next proposition, we show that, for suitably chosen B_t, our condition (19) is a consequence of the Hölder continuity estimates on the kernel.

Proposition 2. *Assume that for some $\alpha > 0$ and $c_1 > 1$, the kernel $k(x,y)$ associated with T satisfies the condition*

$$|k(z,y) - k(x,y)| \le c\,\frac{1}{\mu(B(x;d(x,y))}\Big(\frac{d(x,z)}{d(x,y)}\Big)^{\alpha}, \tag{22}$$

when $d(x,y) \ge c_1\, d(x,z)$. Let B_t be approximations to the identity which are represented by kernels $b_t(x,y)$ which satisfy (3), (5) and $\int_{\mathcal{X}} b_t(x,y)\, d\mu(y) = 1$, for all $x \in \mathcal{X}$, $t > 0$.

Then the kernels $K_t(x,y)$ associated with B_tT satisfy condition (19), i.e. there exists a constant c so that

$$|K_t(x,y) - k(x,y)| \le c\,(\mu(B(x;d(x,y)))^{-1}\,\frac{t^{\alpha/m}}{d(x,y)^{\alpha}}\,,$$

for $d(x,y) \ge c_0\, c_1\, t^{1/m}$ where c_0 is the constant appearing in condition (5).

PROOF. Suppose that $d(x,y) \ge c_0\, c_1\, t^{1/m}$. Then

$$\begin{aligned}
&|k(x,y) - K_t(x,y)| \\
&\qquad = \Big| k(x,y) - \int_{\mathcal{X}} k(x,z)\, b_t(z,y)\, d\mu(z)\Big| \\
&\qquad = \Big| k(x,y) \int_{d(z,y)\le c_0 t^{1/m}} b_t(z,y)\, d\mu(z) \\
&\qquad\qquad - \int_{d(z,y)\le c_0 t^{1/m}} k(x,z)\, b_t(z,y)\, d\mu(z)\Big| \\
&\qquad \le \int_{d(z,y)\le c_0 t^{1/m}} |k(x,y) - k(x,z)|\, |b_t(z,y)|\, d\mu(z) \\
&\qquad \le c\,(\mu(B(x;d(x,y)))^{-1}\,\frac{t^{\alpha/m}}{d(x,y)^{\alpha}} \int_{d(z,y)\le c_0 t^{1/m}} |b_t(z,y)|\, d\mu(z) \\
&\qquad \le c\,(\mu(B(x;d(x,y)))^{-1}\,\frac{t^{\alpha/m}}{d(x,y)^{\alpha}}\,.
\end{aligned}$$

Note that the second equality is using condition (6) and the second inequality follows from (22).

Remark. Propositions 1 and 2 show that conditions (7) and (19) are weaker than the usual conditions which guarantee L_p boundedness of maximal truncated operators. See, for example [St2, Chapter 1]. However, we need the extra assumption (18). In the case of functional calculi of generators of semigroups with suitable heat kernel bounds, condition (18) is satisfied without extra regularity conditions on the kernel of T. See Theorem 6.

4.2. The case of measurable subsets of a space of homogeneous type.

We now assume that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$ as in Section 3. By strengthening the assumptions on the kernel of T in Theorem 3, we can obtain boundedness of maximal truncated operators on L_p spaces as follows.

Theorem 4. *Let T be a bounded operator on $L_2(\Omega)$ with an associated kernel $k(x,y)$. We assume the following conditions.*

a) *There exists a class of operators A_t, $t > 0$, represented by kernels $a_t(x,y)$ which satisfy conditions* (15) *and* (16) *so that the composite operators TA_t have associated kernels $k_t(x,y)$ in the sense of* (1)*, and there exist constants c and c_1 so that*

$$\int_{d(x,y)\geq ct^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x) \leq c_1 \,, \tag{17}$$

for all $y \in \Omega$.

b) *There exists a class of operators B_t, $t > 0$, represented by kernels $b_t(x,y)$ which satisfy conditions* (15) *and* (16) *so that the composite operators B_tT have kernels $K_t(x,y)$, and $K_t(x,y)$ satisfy the following conditions*

$$|K_t(x,y)| \leq c\,(\mu(B^{\mathcal{X}}(x;t^{1/m})))^{-1}\,, \tag{23}$$

for all $x, y \in \Omega$ such that $d(x,y) \leq c_2\, t^{1/m}$,

$$|K_t(x,y) - k(x,y)| \leq c\,(\mu(B^{\mathcal{X}}(x;d(x,y))))^{-1} \frac{t^{\alpha/m}}{d(x,y)^{\alpha}}\,, \tag{24}$$

for all $x, y \in \Omega$ *such that* $d(x,y) \geq c_2\, t^{1/m}$.

Then T_* *is bounded on* $L_p(\Omega)$ *for all* p, $1 < p < \infty$.

PROOF. There is no loss of generality in proving the theorem with $c_2 = 1$.

It follows from Theorem 2 and a duality argument that T is bounded on $L_p(\Omega)$ for $1 < p < \infty$.

Given a function $u \in L_1(\Omega) \cap L_2(\Omega)$ and $\varepsilon > 0$, write

$$T_\varepsilon u(x) = B_{\varepsilon^m} T u(x) + (B_{\varepsilon^m} T - T_\varepsilon)\, u(x)\, .$$

Consider the term $B_{\varepsilon^m} T u(x)$. Let v be the extension of Tu from Ω to $\mathcal{X}$ by putting $v(x) = 0$ for $x \notin \Omega$, then $\|Tu\|_{L_p(\Omega)} = \|v\|_{\mathcal{L}_p(\mathcal{X})}$ for $1 \leq p \leq \infty$. Similarly, let w_ε be the extension of $B_{\varepsilon^m} T u$ from Ω to $\mathcal{X}$ by putting $w(x) = 0$ for $x \notin \Omega$. Since

$$B_{\varepsilon^m} T u(x) = \int_{\mathcal{X}} b_{\varepsilon^m}(x,y)\, T u(y)\, d\mu(y)$$

and the kernels $b_t(x,y)$ of $B_t(x,y)$ satisfy (15) and (16), we have for $x \in \mathcal{X}$,

$$|w_\varepsilon(x)| \leq c\, M\, (|v|(x))\, , \tag{25}$$

where M is the Hardy-Littlewood maximal operator, and c is a constant independent of ε. This gives

$$\sup_{\varepsilon > 0} |w_\varepsilon(x)| \leq c\, M\, (|v|(x))\, . \tag{26}$$

The next step is to extend u to $\mathcal{X}$ by putting $u(x) = 0$ for $x \notin \Omega$, and denote the extension by $u^{\mathcal{X}}$. Then $\|u\|_{L_p(\Omega)} = \|u^{\mathcal{X}}\|_{L_p(\mathcal{X})}$. It then follows from assumption b) and the argument of (21) that

$$\sup_{\varepsilon > 0} |(B_{\varepsilon^m} T - T_\varepsilon)\, u(x)| \leq c\, M\, (|u^{\mathcal{X}}|(x))\, . \tag{27}$$

Estimates (26) and (27) imply the boundedness of T_* on $L_p(\Omega)$ for all p, $1 < p < \infty$.

A consequence of the boundedness of the maximal truncated operator T_* is pointwise almost everywhere convergence of the limit

$$\lim_{\varepsilon \to 0} T_\varepsilon u(x)\, .$$

More precisely, we have the following lemma.

Lemma 2. *Assume that the operator T satisfies the conditions of Theorem 3. Let $1 < p < \infty$, and assume that $\lim_{\varepsilon\to 0} T_\varepsilon u(x)$ exists almost everywhere for every u in a dense subspace of $L_p(\Omega)$, then $\lim_{\varepsilon\to 0} T_\varepsilon u(x)$ exists almost everywhere for every function $u \in L_p(\Omega)$.*

Proof. Lemma 2 follows from a standard argument of proving the existence of almost everywhere pointwise limits as a consequence of the corresponding maximal inequality. See, for example [St1, p. 45].

5. Applications: Riesz transforms and holomorphic functional calculi of elliptic operators.

5.1 Definitions.

We first give some preliminary definitions. References are [M^c], [CDM^cY], [ADM^c].

For $0 \le \omega < \nu < \pi$, define the closed sector in the complex plane $\mathbb{C}$

$$S_\omega = \{\zeta \in \mathbb{C} : \ |\arg \zeta| \le \omega\} \cup \{0\}$$

and denote its interior by S_ω^0.

We employ the following subspaces of the space $H(S_\nu^0)$ of all holomorphic functions on S_ν^0.

$$H_\infty(S_\nu^0) = \{f \in H(S_\nu^0) : \ \|f\|_\infty < \infty\}\,,$$

where $\|f\|_\infty = \sup\{|f(\zeta)| : \zeta \in S_\nu^0\}$,

$$\Psi(S_\nu^0) = \{\psi \in H(S_\nu^0) : \text{ exists } s > 0,\ |\psi(\zeta)| \le C\,|\zeta|^s\,(1 + |\zeta|^{2s})^{-1}\}$$

and

$$F(S_\nu^0) = \{f \in H(S_\nu^0) : \text{ exists } s > 0,\ |f(\zeta)| \le C\,(|\zeta|^{-s} + |\zeta|^s)\}\,,$$

so that

$$\Psi(S_\nu^0) \subset H_\infty(S_\nu^0) \subset F(S_\nu^0)\,.$$

Let $0 \le \omega < \pi$. A closed operator L in $L_p(\mathcal{X})$ is said to be of type ω if $\sigma(L) \subset S_\omega$ and, for each $\nu > \omega$, there exists C_ν such that

$$\|(L - \zeta I)^{-1}\| \le C_\nu\,|\zeta|^{-1}\,, \qquad \zeta \notin S_\nu\,.$$

By the Hille–Yoshida Theorem, an operator L of type ω with $\omega < \pi/2$ is the generator of a bounded holomorphic semigroup e^{-zL} on the sector S^0_ν with $\nu = \pi/2 - \omega$.

Suppose that L is a one–one operator of type ω with dense domain and dense range in $L_p(\mathcal{X})$. We can define a functional calculus of L as follows.

If $\psi \in \Psi(S^0_\nu)$, then

$$\psi(L) = \frac{1}{2\pi i} \int_\gamma (L - \zeta I)^{-1}\, \psi(\zeta)\, d\zeta\,, \tag{28}$$

where γ is the contour $\{\zeta = r\, e^{\pm i\theta} : r \geq 0\}$ parametrised clockwise around S_ω, and $\omega < \theta < \nu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(\mathcal{X})$, and it is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\theta \in (\omega, \nu)$.

Let $f \in F(S^0_\nu)$, so that for some c and k, $|f(\zeta)| \leq c\,(|\zeta|^k + |\zeta|^{-k})$ for every $\zeta \in S^0_\nu$. Let

$$\psi(\zeta) = \Big(\frac{\zeta}{(1+\zeta)^2}\Big)^{k+1}.$$

Then ψ, $f\,\psi \in \Psi(S^0_\nu)$ and $\psi(L)$ is one–one. So $(f\,\psi)(L)$ is a bounded operator on $\mathcal{X}$, and $\psi(L)^{-1}$ is a closed operator in $\mathcal{X}$. Define $f(L)$ by

$$f(L) = (\psi(L))^{-1}\, (f\,\psi)(L)\,. \tag{29}$$

An important feature of this functional calculus is the following Convergence Lemma.

Convergence Lemma. *Let $0 \leq \omega < \nu \leq \pi$. Let L be an operator of type ω which is one–one with dense domain and range. Let $\{f_\alpha\}$ be a uniformly bounded net in $H_\infty(S^0_\nu)$, which converges to $f \in H_\infty(S^0_\nu)$ uniformly on compact subsets of S^0_ν, such that $\{f_\alpha(L)\}$ is a uniformly bounded net in $\mathcal{L}(\mathcal{X})$. Then $f(L) \in \mathcal{L}(\mathcal{X})$, $f_\alpha(L)u \longrightarrow f(L)u$ for all $u \in \mathcal{X}$, and $\|f(L)\| \leq \sup_\alpha \|f_\alpha(L)\|$.*

For the proof of the Convergence Lemma, see [M$^{\rm c}$], or [ADM$^{\rm c}$].

5.2. L_p boundedness of Riesz Transforms.

In this subsection, we assume that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$ in Section 3. Let L be a linear operator of type ω on $L_2(\Omega)$ with $\omega < \pi/2$, so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \le |\mathrm{Arg}\,(z)| < \theta$, $\theta = \pi/2 - \omega$. Assume that for real $t > 0$, the distribution kernels $a_t(x,y)$ of e^{-tL} belong to $L_\infty(\Omega \times \Omega)$ and satisfy the estimate

$$|a_t(x,y)| \le h_t(x,y)\,,$$

for $x, y \in \Omega$ where h_t is defined on $\mathcal{X} \times \mathcal{X}$ by (3) and.

For $0 \le \alpha < 1$, $\nu > 0$, define $F_\alpha(S_\nu^0)$ as follows

$$F_\alpha(S_\nu^0) = \{f \in H(S_\nu^0) : \text{ exists } c,\ |f(\zeta)| \le C\,|\zeta|^{-\alpha}\}\,.$$

Assume that $g \in F_\alpha(S_\nu^0)$ for some $\nu > \pi/2$, and that D is a densely defined linear operator on $L_2(\Omega)$ which possesses the following two properties:

a) $Dg(L)$ is bounded on $L_2(\Omega)$,

b) the function Da_t, $t > 0$, obtained by the action of D on $a_t(x,y)$ with respect to the variable x, satisfies the estimate

$$|Da_t(x,y)| \le c\,t^{-\alpha} h_t(x,y)\,,$$

for all $x, y \in \Omega$.

The main result of this section is the following theorem.

Theorem 5. *Under the above assumptions* a) *and* b)*, the operator $Dg(L)$ is of weak type $(1,1)$, hence it can be extended to a bounded operator on $L_p(\Omega)$ for $1 < p \le 2$.*

Before giving the proof of the theorem, we give examples which are applications of our results. Some specific operators which satisfy the assumptions of Theorem 5 are as follows.

i) Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra. Assume that

$$\mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i$$

as a vector space, where $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all i, j, and $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra.

Let G be the associated connected, simply connected Lie group. Then G has homogeneous dimension d given by the formula

$$d = \sum_{j=1}^{m} j \dim(\mathfrak{g}_j)\,,$$

where $\dim(\mathfrak{g}_j)$ denotes the dimension of $\mathfrak{g}_j$.

Consider any finite basis $\{X_k\}$ of $\mathfrak{g}_1$. Each X_k can be identified with a unique left invariant vector field on G. Define

$$L = -\sum_k X_k^2\,.$$

Then the *sub-Laplacian* L is a left invariant second order differential operator, which is a non-negative self-adjoint operator in $L_2(G)$. The Banach spaces $L_p(G)$ are defined with respect to Haar measure.

Note that G is a Lie group of polynomial growth, hence it is a space of homogeneous type. Consider the Riesz transforms $X_k L^{-1/2}$ which are special cases of our operator $Dg(L)$ when $D = X_k$ and $g(L) = L^{-1/2}$. It is not difficult to check that $X_k L^{-1/2}$ is bounded on $L_2(G)$. Gaussian upper bounds on heat kernels and their space derivatives are well known (see, for example [SC]), hence our condition b) is satisfied with $\alpha = 1/2$. It follows that the Riesz transforms $X_k L^{-1/2}$ are bounded on $L_p(G)$ for $1 < p \leq 2$ and are of weak type $(1,1)$. Thus we have simplified the proof of the L_p boundedness of the Riesz transforms given by Saloff-Coste [SC], because we have not used the smoothness of the heat kernels in the variable y.

In the same setting of G, we can also consider the case when L is a $2\,m$-th order strongly elliptic operator with constant coefficients (plus a sufficiently large constant c_0), and $D = X_{i_1} X_{i_2} \cdots X_{i_n}$ for some $n < 2\,m$. Then the operator $DL^{-n/2m}$ is bounded on $L_2(G)$. The condition (b) is also satisfied with $\alpha = n/2\,m$. See, for example, [R, Chapter 4]. Our theorem then shows that $DL^{-n/2m}$ can be extended to a bounded operator on each $L_p(G)$ for $1 < p \leq 2$.

ii) Let $\mathcal{X}$ be a complete Riemannian manifold with non-negative Ricci curvature, L the Laplace-Beltrami operator, and D a vector field. Then the Riesz transform $DL^{-1/2}$ is bounded on $L_2(\mathcal{X})$. Upper bounds on the heat kernels and their derivatives can be found, for example, in

[CLY], [Da2]. Thus the assumptions of Theorem 5 are satisfied (with $\alpha = 1/2$), so $DL^{-1/2}$ can be extended to a bounded operator on each $L_p(\mathcal{X})$ for $1 < p \leq 2$.

We now proceed to prove Theorem 5. The following off-diagonal estimate is proved in [DR, Proposition 2.1].

Lemma 3. *Let $h_t(x,y)$ be given by* (3)*, then for each δ, $0 < \delta < \kappa - N*$, where $\kappa, N*$ are the constants in* (4)*, there exists $c > 0$ so that*

$$\int_{d(x,y)\geq r} h_t(x,y)\, d\mu(x) \leq c\,(1 + r^m\, t^{-1})^{-\delta}\,,$$

uniformly for all $r \geq 0$, $t > 0$ and $y \in \mathcal{X}$.

PROOF OF THEOREM 5. Observe that, for each positive integer k, the powers of the resolvent $(L - \lambda\, I)^{-k}$ are given by

$$(L - \lambda\, I)^{-k} = c_k \int_0^\infty t^{k-1}\, e^{\lambda t}\, e^{-tL}\, dt\,,$$

when $\lambda < 0$. Therefore, the operators $(L - \lambda\, I)^{-k}$ are represented by kernels $g_\lambda^k(x,y)$ where

$$g_\lambda^k(x,y) = c_k \int_0^\infty t^{k-1}\, e^{\lambda t}\, a_t(x,y)\, dt\,.$$

It follows from this representation and the estimates (3) and (4) on the heat kernels that for a sufficiently large integer k, the kernels $g_\lambda^k(x,y)$ possess upper bounds which are similar to those of the heat kernels. More specifically, there exist $h_t(x,y)$ satisfying (3) and (4), possibly with a different function s, so that

$$|\lambda^k g_\lambda^k(x,y)| \leq h_t(x,y)\,,$$

for all $x, y \in \Omega$, where $t = 1/|\lambda|$.

Similarly, we also have the bound

$$|D\lambda^k g_\lambda^k(x,y)| \leq c\, t^{-\alpha}\, h_t(x,y)\,,$$

for all $x, y \in \Omega$.

Choose the class of operators $A_t = (t\,L + I)^{-k}$. By Theorem 2, it suffices to show that condition (17) is satisfied. The kernels $(k(x,y) - k_t(x,y))$ in condition (17) are associated with operators $Dg(L)(I - (t\,L + I)^{-k})$. Let $g(L)(I - (t\,L - I)^{-k}) = f(L)$ where $f(z) = g(z)\,(1 - (t\,z + 1)^{-k})$. Using the upper bounds on $g(z)$, we see that f belongs to the class $\Psi(S^0_\nu)$.

We next represent the operator $f(L)$ by using the semigroup e^{-zL}. By (28), $f(L)$ (acting on $L_2(\mathcal{X})$) is given by

$$f(L) = \frac{1}{2\pi i}\int_\gamma (L - \lambda\,I)^{-1} f(\lambda)\,d\lambda\,,$$

where the contour $\gamma = \gamma_+ \cup \gamma_-$ is given by $\gamma_+(t) = t\,e^{i\nu}$ for $t \geq 0$ and $\gamma_-(t) = -t\,e^{-i\nu}$ for $t \leq 0$, and $\nu > \pi/2$.

For $\lambda \in \gamma$, substitute

$$(L - \lambda\,I)^{-1} = \int_0^\infty e^{\lambda s}\,e^{-sL}\,ds\,.$$

Changing the order of integration gives

$$f(L) = \int_0^\infty e^{-sL}\,n(s)\,ds\,, \tag{30}$$

where

$$n(s) = \frac{1}{2\pi i}\int_\gamma e^{\lambda s}\,f(\lambda)\,d\lambda\,. \tag{31}$$

Consequently, the kernel $k_f(x,y)$ of $f(L)$ is given by

$$k_f(x,y) = \int_0^\infty a_s(x,y)\,n(s)\,ds\,. \tag{32}$$

It follows from the upper bound on $g(z)$ and assumption b) that

$$\begin{aligned}
&\int_{d(x,y)\geq ct^{1/m}} |k(x,y) - k_t(x,y)|\,d\mu(x) \\
&\leq c\int_0^\infty s^{-\alpha}\Big(\int_0^\infty |\lambda^{-\alpha}\,e^{s\lambda}\,(1 - (t\,\lambda + 1)^{-k})| \\
&\qquad\qquad \cdot\Big(\int_{d(x,y)\geq ct^{1/m}} h_s(x,y)\,d\mu(x)\Big)\,d|\lambda|\Big)\,ds \\
&\leq c\int_0^\infty s^{-\alpha}\Big(\int_0^\infty |\lambda^{-\alpha}\,e^{s\lambda}\,(1 - (t\,\lambda + 1)^{-k})|\,(1 + t\,s^{-1})^{-\delta}\,d|\lambda|\Big)\,ds\,,
\end{aligned}$$

by Lemma 3. Observe that $|1-(t\,\lambda+1)^{-k}| \le c$ and $|1-(t\,\lambda+1)^{-k}| \le c\,t\,|\lambda|$ when $t\,|\lambda| \le 1$. We then split the integral on the right hand side into two parts, I_1 and I_2, corresponding to integration over $t\,|\lambda| > 1$ and $t\,|\lambda| \le 1$. Then

$$I_1 \le \int_0^\infty s^{-\alpha} \int_{1/t}^\infty v^{-\alpha}\, e^{-\beta s v}\,(1+t\,s^{-1})^{-\delta}\, dv\, ds$$

with $\beta > 0$. Changing variables $t\,v \longrightarrow v$ and $s/t \longrightarrow s$, and choosing a positive $\varepsilon < \delta$, we have

$$\begin{aligned}
I_1 &\le c\int_0^\infty s^{-\alpha}\Big(\int_1^\infty v^{-\alpha}\, e^{-\beta s v}\,(1+s^{-1})^{-\delta}\, dv\Big)\, ds\\
&= c\int_0^\infty \frac{s^\delta}{(1+s)^\delta}\Big(\int_1^\infty \frac{1}{s^{1+\varepsilon}}\,\frac{1}{v^{1+\varepsilon}}\,(s\,v)^{1+\varepsilon-\alpha}\, e^{-\beta s v}\, dv\Big)\, ds\\
&\le c\int_0^\infty \frac{s^\delta}{(1+s)^\delta\, s^{1+\varepsilon}}\Big(\int_1^\infty \frac{1}{v^{1+\varepsilon}}\, dv\Big)\, ds\\
&\qquad \text{(since } (s\,v)^{1+\varepsilon-\alpha}\, e^{-\beta s v} \text{ is bounded)}\\
&\le C\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &\le c\int_0^\infty s^{-\alpha}\int_0^{1/t} v^{-\alpha}\, t\,v\, e^{-\beta s v}\, dv\,(1+t\,s^{-1})^{-\delta}\, dv\, ds\\
&\le c\int_0^\infty s^{-\alpha}\,(1+s^{-1})^{-\delta}\int_0^1 v^{1-\alpha}\, e^{-\beta s v}\, dv\, ds\\
&\le c\int_0^1 dv\int_0^\infty w^{-\alpha}\, e^{-\beta w}\, dw\\
&\le C\,.
\end{aligned}$$

Therefore, condition (17) is satisfied and Theorem 5 follows from Theorem 2.

REMARKS.

α) In the assumption b), we do not assume any regularity of $a_t(x,y)$ in the variable y.

β) The theorem is still true for $g \in F_\alpha(S^0_\nu)$ with $\nu < \pi/2$ if the upper bounds on $a_t(x,y)$ in condition b) hold for all complex $t \in S^0_\theta$ with $\theta > \pi/2 - \nu$. This can be achieved by first choosing $\mu = \nu - \varepsilon$, (with ε to be specified later), using the formula

$$f(L) = \frac{1}{2\pi i} \int_\gamma (L - \lambda\, I)^{-1} f(\lambda)\, d\lambda\,,$$

where the contour $\gamma = \gamma_+ \cup \gamma^-$ is given by $\gamma_+(t) = t\, e^{i\mu}$ for $t \geq 0$ and $\gamma_-(t) = -t\, e^{-i\mu}$ for $t \leq 0$.

We then substitute

$$(L - \lambda\, I)^{-1} = \int_\Gamma e^{\lambda z}\, e^{-zL}\, dz\,,$$

for $\lambda \in \gamma_+$, where Γ is given by $\Gamma(t) = t\, e^{i(\theta-\varepsilon)}$ for $t \geq 0$, and ε is chosen sufficiently small so that $(\theta + \mu - \varepsilon) > \pi/2$. We also have similar expression for $\lambda \in \gamma_-$. Thus we obtain a similar representation of $f(L)$ to that of (30), and the rest of the proof is the same as before.

γ) The pointwise bound in condition (b) can be replaced by a weaker condition on the L_2 norm with a suitable weight of Da_t (with respect to x variable). See [CD].

5.3 Holomorphic functional calculi of elliptic operators.

We again assume that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$ as in Section 3.

Let L be a linear operator of type ω on $L_2(\Omega)$ with $\omega < \pi/2$, hence L generates a holomorphic semigroup e^{-zL}, $0 \leq |\mathrm{Arg}\,(z)| < \pi/2 - \omega$.

Theorem 6. *Assume the following two conditions.*

a) *The holomorphic semigroup e^{-zL}, $|\mathrm{Arg}\,(z)| < \pi/2 - \omega$, is represented by kernels $a_z(x,y)$ which satisfy, for all $\theta > \omega$, an estimate*

$$|a_z(x,y)| \leq c_\theta\, h_{|z|}(x,y)\,,$$

for $x, y \in \Omega$ and $|\mathrm{Arg}\,(z)| < \pi/2 - \theta$, where h_t is defined on $\mathcal{X} \times \mathcal{X}$ by (3).

b) *The operator L has a bounded holomorphic functional calculus in $L_2(\Omega)$. That is, for any $\nu > \omega$ and $f \in H_\infty(S^0_\nu)$, the operator $f(L)$ satisfies*

$$\|f(T)\|_2 \le c_\nu \, \|f\|_\infty \, .$$

Then the operator L has a bounded holomorphic functional calculus in $L_p(\Omega)$, $1 < p < \infty$, that is,

$$\|f(L)\|_p \le c_{p,\nu} \, \|f\|_\infty \, ,$$

for all $f \in H_\infty(S^0_\nu)$.

When $p = 1$, the operator $f(L)$ is of weak-type $(1,1)$.

If we denote $T = f(L)$, then the maximal truncated operator T_* is bounded on $L_p(\Omega)$ for all p, $1 < p < \infty$.

Proof. Given $\pi/2 > \nu > \omega$, choose θ and μ such that $\omega < \theta < \mu < \nu$. For $f \in \Psi(S^0_\nu)$, represent the operator $f(L)$ by using the semigroup e^{-zL} as before. This gives

$$f(L) = \int_{\Gamma_+} e^{-zL} \, n_+(z) \, dz + \int_{\Gamma_-} e^{-zL} \, n_-(z) \, dz \, ,$$

where we choose the contour $\Gamma_+(s) = s \, e^{i(\pi/2-\theta)}$ for $s \ge 0$ and $\Gamma_-(s) = -s \, e^{-i(\pi/2-\theta)}$ for $s \le 0$. The functions $n_\pm(z)$ are given by

$$n_\pm = \frac{1}{2\pi i} \int_{\gamma_\pm} e^{\lambda z} \, f(\lambda) \, d\lambda \, ,$$

where $\gamma_+(s) = s \, e^{i\mu}$ for $t \ge 0$ and $\gamma_-(t) = -t \, e^{-i\mu}$ for $t \le 0$.

This implies the bound

$$|n_\pm(z)| \le c \, \|f\|_\infty \, |z|^{-1} \, .$$

Consequently, the kernel $k_f(x,y)$ of $f(L)$ is given by

$$k_f(x,y) = \int_{\Gamma_+} a_z(x,y) \, n_+(z) \, dz + \int_{\Gamma_-} a_z(x,y) \, n_-(z) \, dz \, .$$

Choose operators $A_t = e^{-tL}$. Using the upper bounds on the heat kernels and Lemma 3, similar estimates to the terms I_1 and I_2 in the proof of Theorem 5 shows that condition (17) of Theorem 2 is satisfied. Therefore, $f(L)$ is bounded on $L_p(\Omega)$. The Convergence Lemma then allows us to extend L_p boundedness of $f(L)$ to all $f \in H_\infty(S^0_\nu)$, hence the operator L has a bounded holomorphic function calculus in $L_p(\Omega)$. Although the extension of the weak type (1,1) estimates from $f(L)$ for $f \in \Psi(S^0_\nu)$ to $f(L)$ for $f \in H_\infty(S^0_\nu)$ does not follow from the Convergence Lemma, it is not difficult. See for example [ADM$^{\mathrm{c}}$, Lecture 7, Section N].

To prove the L_p boundedness of the maximal truncated operator T_*, first choose $B_t = A_t = e^{-tL}$. We then just need to verify conditions (23) and (24) of Theorem 4.

To verify (23), we use the commutative property of functional calculus:

$$e^{-tL} f(L) = e^{-tL/2} f(L)\, e^{-tL/2}\,.$$

Since e^{-tL} maps $L_1(\Omega)$ into $L_1(\Omega)$ with the operator norm less than a constant, and e^{-tL} maps $L_1(\Omega)$ into $L_\infty(\Omega)$ with the operator norm less than $(\mu(B^{\mathcal{X}}(x;t^{-1/m})))^{-1}$, interpolation and duality gives

$$\begin{aligned}\|e^{-tL/2}\|_{L_1(\Omega)\to L_2(\Omega)} &= \|e^{-tL/2}\|_{L_2(\Omega)\to L_\infty(\Omega)}\\ &\le c\,(\mu(B^{\mathcal{X}}(x;t^{-1/m})))^{-1/2}\,.\end{aligned}$$

These estimates, combined with the fact that $f(L)$ is bounded on $L_2(\Omega)$, imply condition (23).

The proof of (24) is straightforward. Consider $d(x,y) \ge c\,t^{1/m}$, we have

$$|k(x,y) - k_t(x,y)| \le c \int_0^\infty |h_z(x,y)|\, d|z| \int_0^\infty |\lambda^{-\alpha}\, e^{z\lambda}\,(1-e^{-t\lambda})\,|d|\lambda|\,.$$

Observe that $|1-e^{-t\lambda}| \le c$ since $\mathrm{Re}\,(\lambda) \ge 0$ and $|1-e^{-t\lambda}| \le c\,t\,|\lambda| \le c\,(t\,|\lambda|)^\alpha$ for $0<\alpha<1$ when $t\,|\lambda| \ge 1$. We then split the integral on the left hand side into two parts, I_1 and I_2, corresponding to integration over $t\,|\lambda| > 1$ and $t\,|\lambda| \le 1$. Using the heat kernel bounds and elementary integration, similar estimates to those of I_1 and I_2 of Theorem 5 show that

$$|K_t(x,y) - k(x,y)| \le c\,(\mu(B^{\mathcal{X}}(x;d(x,y)))^{-1}\,\frac{t^{\beta/m}}{d(x,y)^\beta}\,,$$

for some $\beta > 0$, $x,y \in \Omega$, $d(x,y) \ge t^{1/m}$. We leave details of these estimates to reader.

Remarks.

a) Condition a) of Theorem 6 can be replaced by a more general condition as follows. Assume that L is an operator of type ω and that there exists a positive integer k so that the kernels $g_\lambda^k(x,y)$ of the power of the resolvents $(\lambda L - I)^{-k}$ satisfy the following estimate

$$|g_\lambda^k(x,y)| \leq (\mu(B^{\mathcal{X}}(x;|\lambda|^{-1/m})))^{-1}\, s(d(x;y)^m|\lambda|)\,,$$

where s is a function which satisfies the decay condition in (3). The proof under this assumption is still the same as that of Theorem 6, with the operators $(\lambda L - I)^{-k}$ replacing the semigroup e^{-zL}. The advantage of this assumption is that the operator L can be of type ω with $\omega > \pi/2$, or of type ω on a double sector.

b) When $\mathcal{X}$ is a space of homogeneous type, the result on boundedness of holomorphic functional calculi of Theorem 6 was first presented in [DR, Theorem 3.1]. Note that the Hörmander integral condition is not applicable when we have no control on smoothness of heat kernels in the space variables.

c) Heat kernel bounds are known for a large class of elliptic and subelliptic operators. Also see [AM$^{\rm c}$T], [A] for recent results on heat kernel bounds for second order elliptic operators with non-smooth coefficients.

d) Theorem 6 gives new results when Ω is a measurable set with no smoothness on its boundary. An example of an operator on such a domain, which possesses Gaussian bounds on its heat kernels, is the Laplacian on an open subset of Euclidean space $\mathbb{R}^n$ subject to Dirichlet boundary conditions. Gaussian upper bounds can be obtained in this case by a simple argument using the comparison principle. More general operators on open domains of $\mathbb{R}^n$ which possess Gaussian bounds can be found in [Da1] and [AE]. Indeed Theorem 6 can be applied to prove the general statement of Theorem 5.5 in [AE] on boundedness of holomorphic functional calculi in L_p spaces, without the assumption that the boundary has null measure.

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References.

[ADM^c] Albrecht, D., Duong, X., M^{c}Intosh, A., Operator Theory and Harmonic Analysis. *Workshop in Analysis and Geometry.* 1995, Proceedings of the Centre for Mathematics and its Applications, ANU **34** (1996), 77-136.

[AE] Arendt, W., ter Elst, A. F. M., *Gaussian estimates for second order elliptic operators with boundary conditions.* Reports on Applied and Numerical Analysis, Eindhoven University of Technology, 1995.

[A] Auscher, P., Regularity theorems and heat kernels for elliptic operators. *J. London Math. Soc.* **54** (1996), 284-296.

[AM^{c}T] Auscher, P., M^{c}Intosh, A., Tchamitchian, Ph., Heat kernels of second order complex elliptic operators and applications. *J. Funct. Anal.* **152** (1998), 22-73.

[CLY] Cheng, S. Y., Li, P., Yau, S. T., On the upper estimate of the heat kernel of a complete Riemannian manifold. *Amer. J. Math.* **103** (1981), 1021-1063.

[Ch1] Christ, M., Weak-type (1,1) bounds for rough operators. *Ann. Math.* **128** (1988), 19-42.

[Ch2] Christ, M., *Lectures on Singular Integral Operators.* CBMS, 1991.

[CR] Christ, M., Rubio de Francia, J. L., Weak-type (1,1) bounds for rough operators II. *Invent. Math.* **93** (1988), 225-237.

[CW] Coifman, R. R., Weiss, G., Analyse harmonique non-commutative sur certains espaces homogènes. Lecture Notes in Math. Springer-Verlag **242**, 1971.

[CD] Coulhon, T., Duong, X. T., Riesz transforms for $1 \le p \le 2$. *Trans. Amer. Math. Soc.* To appear.

[CDM^{c}Y] Cowling, M., Doust, I., M^{c}Intosh, A., Yagi, A., Banach space operators with a bounded H^∞ functional calculus. *J. Austral. Math. Soc. Ser. A* **60** (1996), 51-89.

[Da1] Davies, E. B., *Heat Kernels and Spectral Theory.* Cambridge University Press, 1989.

[Da2] Davies, E. B., Pointwise bounds on the space and time derivatives of heat kernels. *J. Operator Theory* (1989), 367-378.

[Du] Duong, X. T., H_∞ functional calculus of elliptic partial differential operators in L_p spaces. Ph. D. Thesis. Macquarie University, 1990.

[DM^c] Duong, X. T., M^{c}Intosh, A., Functional calculi of second order elliptic partial differential operators with bounded measurable coefficients. *J. Geom. Anal.* **6** (1996), 181-205.

[DR] Duong, X. T., Robinson, D., Semigroup kernels, Poisson bounds and holomorphic functional calculus. *J. Funct. Anal.* **142** (1996), 89-128.

[F] Fefferman, C., Inequalities for strongly singular convolution operators. *Acta Math.* **124** (1970), 9-36.

[He] Hebisch, W., A multiplier theorem for Schrödinger operators. *Coll. Math.* **60** (1990), 659-664.

[Hof] Hofmann, S., Weak (1,1) boundedness of singular integrals with nonsmooth kernel. *Proc. Amer. Math. Soc.* **103** (1988), 260-264.

[Hör] Hörmander, L., Estimates for translation invariant operators on L^p spaces. *Acta Math.* **104** (1960), 93-139.

[M^c] M^{c}Intosh, A., Operators which have an H^∞ functional calculus. *Miniconference on Operator Theory and Partial Differential Equations.* Proceedings of the Centre for Mathematical Analysis, ANU **14** (1986), 210-231.

[R] Robinson, D. W., *Elliptic Operators and Lie Groups.* Oxford University, 1991.

[SC] Saloff-Coste, L., Analyse sur les groupes de Lie à croissance polynomiale. *Arkiv. Mat.* **28** (1990), 315-331.

[Se] Seeger, A., Singular integral operators with rough convolution kernels. *J. Amer. Math. Soc.* **9** (1996), 95-105.

[St1] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970.

[St2] Stein, E. M., *Harmonic Analysis.* Princeton University, 1993.

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Catching sets with quasicircles

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Abstract. We show how certain geometric conditions on a planar set imply that the set must lie on a quasicircle, and we give a geometric characterization of all subsets of the plane that are quasiconformally equivalent to the usual Cantor middle-third set.

0. Introduction.

Theorem 1. *For a subset E of $\mathbb{C}$ the following are equivalent*:

i) *E has empty interior and uniform complement.*

ii) *E is uniformly disconnected.*

iii) *E is quasiconformally equivalent to a porous subset of $\mathbb{R}$.*

The various constants depend only on each other.

One immediate consequence of this theorem is that any set E satisfying either i) or ii) lies on a quasicircle. Indeed, the main part of the proof consists of demonstrating this fact.

An NUD set is a compact set having no interior and whose complement is a uniform domain. Väisälä considered the family of NUD sets in [V], where he showed that if such a set is removed from a uniform domain, then the domain that remains is still uniform. Hence the name: NUD stands for nullsets for uniform domains. As a corollary of Theorem 1, we obtain the following characterisation of NUD sets in the plane.

Corollary 2. *E is an* NUD *set in $\mathbb{C}$ if and only if E is quasiconformally equivalent to a compact, porous subset of $\mathbb{R}$.*

We can also characterize those sets that are quasiconformally equivalent to the usual Cantor middle-third set.

Theorem 3. *For a compact set K in $\mathbb{C}$ whose interior is empty the following are equivalent*:

i) *K is uniformly perfect and has uniform complement.*

ii) *K is both uniformly perfect and uniformly disconnected.*

iii) *K is quasiconformally equivalent to the usual Cantor middle-third set.*

The various constants depend only on each other.

A result of David and Semmes [DS] says basically that any uniformly disconnected, uniformly perfect, compact metric space can be mapped quasisymmetrically to the middle-third set. In particular, if K satisfies ii) above then there is a quasisymmetric map from K to the Cantor set. The preceding theorem extends this by showing that the map can actually be taken to be a quasiconformal map of $\mathbb{C}$.

1. Preliminaries.

By a quasiconformal map of $\mathbb{C}$ we mean a quasiconformal map from $\mathbb{C}$ onto itself. Such maps can also be viewed as maps of $\overline{\mathbb{C}}$ onto itself that fix infinity. Two subsets of $\mathbb{C}$ are said to be quasiconformally equivalent if there is a quasiconformal map of $\mathbb{C}$ that maps one onto the other. A quasiconformal arc is the image of a closed sub-interval of $\mathbb{R}$ under a quasiconformal map of $\mathbb{C}$. If this map is L-quasiconformal, then we will say that the quasiconformal arc is an L-quasiconformal arc.

We call a set A *c-uniform* ($1 \le c$) under the following condition: A contains at least two points and for each pair of distinct points a and b of A, there exists a continuum F containing a and b such that $\operatorname{diam} F \le c\,|a-b|$ and such that

$$\bigcup_{z\in F\setminus\{a,b\}} B(z, c^{-1}\, r(z)) \subset A\,,$$

where $r(z) = \min\{|z-a|, |z-b|\}$. A set is c-uniform if and only if there is a c-uniform domain D for which $D \subseteq A \subseteq \overline{D}$. See [V] for a proof of this last remark and for alternative definitions of uniformity.

Let $1 \le m$. A subset E of $\mathbb{C}$ that contains at least two points is said to be *m-uniformly perfect* if at each point z_0 of E the closed annulus

$$A = \left\{ z : \frac{r}{m} \le |z - z_0| \le r \right\}$$

has non-empty intersection with E whenever $0 < r < \operatorname{diam} E$.

A subset E of $\mathbb{R}$ is said to be *λ-porous* $(1 \le \lambda)$ if every interval I centred on E contains an interval of length $|I|/\lambda$ that lies in $\mathbb{R} \setminus E$.

We say that E is *τ-uniformly disconnected* $(\tau \ge 1)$ if for each $x \in E$ and each $r > 0$ we can find a subset A of E containing x, of diamter no more than r, and for which $d(A, E \setminus A) \ge \tau^{-1} r$. This concept was introduced recently in [DS].

$M, M_0, \ldots$ and $\varepsilon, \varepsilon_0, \ldots$ will denote constants that depend only on the relevant data (*e.g.* uniformity constants) associated to the set in question; the former are used for constants that are at least 1, and the latter are used for constants that are less than 1. The same symbol may be used to denote different constants. When we write $A \sim B$, we mean that the ratio of A to B is bounded above and below by a constant that depends, once again, only on the relevant data.

We are going to use a result from [M] on building quasiconformal arcs. In order to state this result we need to introduce the idea of a *chain*. A standard rectangle R is a closed rectangle whose major axis lies on the real line. Let $\{R_i\}_{i=1}^N$ be a family of disjoint standard rectangles of height h with each R_i at least a distance h to the left of R_{i+1}. Take I to be a closed interval in $\mathbb{R}$ whose left endpoint is at least a distance h to the left of R_1 and whose right endpoint is at least a distance h to the right of R_N. The union of I with the R_i is called a standard h-chain. Each R_i is referred to as a rectangle of the chain. The closed intervals joining R_i to R_{i+1}, along with the two closed intervals joining the endpoints of I to the nearest rectangle, are called the links of the chain, and the points where the links meet the rectangles are called the weld points of the chain. An (M, h)-chain is any M bi-Lipschitz image of a standard h-chain. The various parts of an (M, h)-chain are given the same names as their pre-images in the standard chain.

These chains can be used to build quasiconformal arcs. Assume that for each n we have a family $\mathcal{F}_n$ of (M, h_n)-chains with the following properties:

i) $\mathcal{F}_1$ contains only one chain.

ii) Each chain in $\mathcal{F}_{n+1}$ is contained in one of the rectangles of some chain in $\mathcal{F}_n$, and the endpoints of a chain in $\mathcal{F}_{n+1}$ are the same as the weld points of the associated rectangle.

iii) Each rectangle of each chain in $\mathcal{F}_n$ contains exactly one chain from $\mathcal{F}_{n+1}$.

iv) The h_n converge to zero.

Take T_n to be the union of every chain in $\mathcal{F}_n$ with the links of every chain in $\mathcal{F}_k$ for $1 \leq k \leq n-1$. Each T_n is a continuum and $T_{n+1} \subseteq T_n$.

Theorem. *Under the assumptions* i)-iv), $\Gamma_0 = \bigcap_{n=1}^{\infty} T_n$ *is a* $C(M)$ *quasiconformal arc.*

This is [M, Theorem 5.1] and it can be read independently of the rest of that paper.

2. Proofs of the Theorems.

We start with Theorem 1. It is easily confirmed that any quasiconformal map of $\mathbb{C}$ preserves both uniform domains and the property of being uniformly disconnected. It is also easy to check that a subset of the line is porous (as a subset of the line) if and only if its complement in the plane is uniform, and if and only if it is uniformly disconnected. Thus, iii) implies both i) and ii). Furthermore, in order to prove the opposite implications it suffices to show that E lies on a quasiconformal arc.

$\mathcal{G}_\delta$ is the square grid whose vertices are the points $(m\,\delta, n\,\delta)$ where m and n are any integers, and Σ_δ is the associated family of (closed) squares. A useful way of thickening up a set is the following. Let W be a bounded subset of the plane. W^δ is the union of the elements of Σ_δ that intersect W. Let $\mathcal{T}_\delta(W) = (W^{4\delta})^\delta$. The following facts are easily confirmed.

Lemma 2.1. *If W is a bounded subset of the plane, then the boundary of $\mathcal{T}_\delta(W)$ is a finite, disjoint union of Jordan curves, each of which is a subset of $\mathcal{G}_\delta$. Furthermore, the distance from any boundary point of $\mathcal{T}_\delta(W)$ to W is less than $8\,\delta$ and greater than δ.*

The next lemma will allow us to break the set up into manageable chunks at all scales.

Lemma 2.2. *Suppose that E satisfies* i) *or* ii) *in Theorem* 1. *Then for any $x \in E$ and any positive λ, there is a loop γ separating x from infinity for which* $\operatorname{diam} \gamma \sim d(E, \gamma) \sim \lambda$.

Proof. Let us first assume that E satisfies i). Fix $x \in E$ and $\lambda > 0$. Set $r = \lambda/4$. Choose points $\{y_1, y_2, \ldots, y_n\}$ that are equally spaced with distance $\varepsilon\, r$ on the circle C_r of radius r centred at x. We will see as we go along how small an ε we need to choose. We will use the convention that $y_{n+1} = y_1$.

For each y_i there is $z_i \in E^c$ whose distance to E is at least $M^{-1} \varepsilon\, r$ and whose distance to y_i is at most $\varepsilon\, r$. There is a path γ_i joining z_i to z_{i+1} whose diameter is at most $M \varepsilon\, r$ and whose distance to E is at least $M^{-1} \varepsilon\, r$. Let γ denote the loop (*i.e.*, closed curve) obtained by joining up the γ_i in the obvious way. The next few statements hold for small enough ε. The diameter of γ is at most $3\, r$, which is less than λ, and the distance from γ to E is least $M^{-1} \varepsilon\, r = M_1^{-1} \lambda$. Furthermore, each γ_i is freely homotopic to the segment $[y_i, y_{i+1}]$ in $\mathbb{C} \setminus \{x\}$, and so γ is freely homotopic in $\mathbb{C} \setminus \{x\}$ to the circle C_r. As a result, γ must separate x from infinity.

Now let us assume that E satisfies ii). Once again fix $x \in E$ and $\lambda > 0$. There is a subset A of E containing x, of diameter no more than λ, and for which $d(A, E \setminus A) \geq \tau^{-1} \lambda$. Let $\delta = (20\, \tau)^{-1} \lambda$, and set $X = \mathcal{T}_\delta(A)$. Then X contains x, has diameter comparable to λ, and all of its boundary points are at least a distance δ, and no more than $8\, \delta$, from A. This information about the boundary combined with the estimate on the distance between A and $E \setminus A$ implies that the distance to E of every point on the boundary is comparable to λ. The boundary of X is a finite, disjoint, collection of Jordan curves. One of these must enclose the point x and this is the loop we seek.

It is clear that if E satisfies ii), then any subset of E also satisfies ii) with the same constant. This also holds for i), and depends on the simple observation that the complement of E is uniform and dense in $\mathbb{C}$. These facts combined with a standard limiting argument imply that it suffices to prove Theorem 1 for sets consisting of a finite number of points. We will assume from here on that that E contains only a finite number of points. This is by no means necessary but it means that we do not have to worry about possible technical difficulties.

Set $\mathcal{D}_\delta$ to be the set of closures of all Jordan domains whose boundaries are both subsets of $\mathcal{G}_\delta$ and at least a distance δ from E.

Lemma 2.3. *Suppose that E satisfies* i) *or* ii) *in Theorem* 1. *Then for any positive δ there is a finite, disjoint collection $\{\Delta_1, \Delta_2, \dots\}$ of elements of $\mathcal{D}_\delta$ with the properties listed below.*

i) $E \cap \Delta_k \neq \varnothing$ *for all k, and* $E \subseteq \bigcup \Delta_k$.

ii) $\operatorname{diam} \Delta_k \leq M_0\, \delta$, *for all k.*

Proof. Fix $\delta > 0$. It follows from Lemma 2.2 that for each $x \in E$ there is a loop γ_x of diameter comparable to δ that separates x from infinity and whose distance to E is at least $10\,\delta$. The diameter of C_x, the component of γ_x^c that contains x, is no more than $M\,\delta$.

Let $G = \bigcup_{x\in E} \gamma_x$. The distance from G to E is at least $10\,\delta$. The boundary of $\mathcal{T}_\delta(G)$ is a finite, disjoint union of Jordan curves, each of which lies on $\mathcal{G}_\delta$ and is at least a distance δ from E. As a result, the set

$$\begin{aligned}\mathcal{C} = \{\overline{U} :\ & U \text{ is the bounded domain determined} \\ & \text{by some component of } \partial\mathcal{T}_\delta(G)\}\end{aligned}$$

is contained in $\mathcal{D}_\delta$, and any two elements of $\mathcal{C}$ are either disjoint or one is contained in the other.

Now, the boundary of any bounded component V of $(\mathcal{T}_\delta(G))^c$ is a union of components of $\partial\mathcal{T}_\delta(G)$ and so there is a unique element of $\mathcal{C}$ that contains V and whose boundary is contained in the boundary of V.

For each $x \in E$, define V_x to be the component of $(\mathcal{T}_\delta(G))^c$ that contains x, and Δ_x to be the element of $\mathcal{C}$ that corresponds to V_x. Let $\{\Delta_1, \Delta_2, \dots\}$ be the maximal elements among the Δ_x. It is clear that they have the required properties, except perhaps ii). Each V_x must be a subset of C_x, as $\gamma_x \subseteq \mathcal{T}_\delta(G)$. Thus the diameter of V_x is less than $M\delta$. Consequently, we have that $\operatorname{diam} \Delta_x = \operatorname{diam} V_x \leq M\,\delta$, which is ii).

We are now ready to build the quasiconformal arc containing the set E. For convenience, we will assume that E lies in the unit disc. Let ε be a suitably small constant. Abbreviate $\mathcal{D}_{\varepsilon^n}$ to $\mathcal{D}_n$. For $n \geq 1$, the subset of $\mathcal{D}_n$ obtained in Lemma 2.3 by setting $\delta = \varepsilon^n$ is called $\mathcal{S}_n$. We define $\mathcal{S}_0$ to be the square of sidelength 2 that is centred at the origin. Every element of $\mathcal{S}_{n+1}$ is a subset of some element of $\mathcal{S}_n$, and every element of $\mathcal{S}_n$ contains at least one element of $\mathcal{S}_{n+1}$. For any $S \in \mathcal{S}_n$, we have $\operatorname{diam} S \sim \varepsilon^n$. Furthermore, if $\widehat{S} \in \mathcal{S}_{n+1}$ lies in S, then $d(\partial S, \widehat{S}) \sim \varepsilon^n$. If we set E_n to be the union of the elements of $\mathcal{S}_n$, then $E = \bigcap_n E_n$. We now have nicely nested coverings of E, whose infinite intersection is E. All that remains in the proof of Theorem 1 is to use the $\mathcal{S}_n$ to construct families of chains satisfying conditions i)-iv) on pages 3-4. For all n, and for each $S \in \mathcal{S}_n$, choose two boundary points whose distance apart is the diameter of S. We will refer to these as the distinguished points of S.

Lemma 2.4. *Suppose that $S \in \mathcal{S}_{n-1}$. Denote the elements of $\mathcal{S}_n$ that lie in S by $\mathcal{C}(S)$. Then S contains an (M, ε^n)-chain whose endpoints are the distinguished points of S, whose rectangles are precisely $\mathcal{C}(S)$, and whose weld points are the distinguished points of the elements of $\mathcal{C}(S)$.*

This result is just a special case of [M, Corollary 4.2], but since this case is particularly simple we briefly outline the proof here.

Sketch of Proof. S and the elements of $\mathcal{C}(S)$ are all elements of $\mathcal{D}_n$. Their boundaries are disjoint and each has at most M_1 edges. Take Q to be either S^c or one of the elements of $\mathcal{C}(S)$ and define $\hat{Q}$ to be the $\varepsilon^n/8$ neighbourhood of Q. There is a bi-Lipschitz map that is the identity outside $\hat{Q}$, that sends ∂Q to the boundary of a square, and that sends the distinguished points to the midpoints of opposite sides. The bi-Lipschitz constant will depend only on M_1. The regions $\hat{Q}$ are disjoint, so the composition of all the maps just described does not increase the bi-Lipschitz constant and it yields a bi-Lipschitz map F that sends S and all the elements of $\mathcal{C}(S)$ to squares and that also sends the distinguished points to the midpoints of opposite sides. It is easy to find a suitable chain for $F(S)$ and $F(\mathcal{C}(S))$, and the pullback of this chain by F^{-1} is the chain we seek.

Now let $\mathcal{F}_n$ be the family of chains obtained by applying the previous lemma to each element of $\mathcal{S}_{n-1}$. These satisfy conditions i)-iv). Consequently, E is contained in an M-quasiconformal arc.

We turn next to the proof of Theorem 3. Quasiconformal maps of $\mathbb{C}$ preserve uniform domains, uniformly disconnected sets, and uniformly perfect sets. As the middle-third set is uniformly perfect, uniformly disconnected, and has uniform complement, part iii) implies both parts i) and ii). Suppose now that K is compact and satisfies either i) or ii). Theorem 1 implies that we can assume that K is a porous and uniformly perfect subset of the real line. We need to find a quasiconformal map of $\mathbb{C}$ that maps K onto the middle-third Cantor set. We will only sketch the proof as the details are quite routine.

Set $\mathcal{O}$ to be the collection of disjoint open intervals that make up $\mathbb{R} \setminus K$. We will say that we *split* an interval I when we remove the largest subinterval that is an element of $\mathcal{O}$. Define K_0 to be the smallest closed interval containing K. Let K_1 be the union of the two intervals obtained by splitting K_0. Next split each of these to obtain another set, K_2, that is the union of four closed intervals. Continue indefinitely in this way. We summarize the properties of the sets K_n:

i) Each K_n is a finite union of disjoint closed intervals with end-points in K, and K_0 consists of just one interval.

ii) $K_{n+1} \subseteq K_n$.

iii) Each of the intervals I that make up K_n contains exactly two of the intervals, I_l and I_r, that make up K_{n+1}, and $|I| \sim |I_l| \sim |I_r| \sim d(I_l, I_r)$.

iv) $K = \bigcap_{n=0}^{\infty} K_n$.

The key property is iii), and this is a consequence of K being porous and uniformly perfect. Suppose that I is a component of K_n and that J is a component of K_m. Let us say that $I < J$ if either $n < m$ or $n = m$ and J lies to the right of I. Label the collection of all components of all the K_n as $I_1, I_2, I_3, \dots$ where $I_k < I_{k+1}$ for all k. Then $I_1 = K_0$, $I_2 \cup I_3 = K_1$, and so on.

From iii) above we deduce that there is a small constant ε, which we fix now, that ensures the validity of the statements that follow. To any $I_k = [x - r, x + r]$ we associate the rectangle $[x - (1 + \varepsilon)\, r, x + (1 + \varepsilon)\, r] \times [-r, r]$. Rectangles from the same level are disjoint, and the families consisting of the rectangles from each level are nested and nest nicely down to K. The next lemma is an easy consequence of iii) above. Here I, I_l, and I_r are as in iii), and R, R_l, and R_r are the corresponding rectangles.

Lemma 2.5. *There is an M_1-quasiconformal map of $\mathbb{C}$, which is the identity in R^c, is a similarity on R_l and on R_r, and which maps I_l to the left third of I and I_r to the right third of I.*

Denote by g_i the map we get from the lemma when $I = I_i$. Let τ be a similarity mapping K_0 onto $[0,1]$. Set $G = \tau \circ g_1 \circ g_2 \circ \cdots$. Then G is an M_1-quasiconformal map of $\mathbb{C}$ that maps K onto the Cantor set. The composition does not increase the dilatation because the maps g_i only have non-trivial dilatation in the doubly connected region between the three corresponding rectangles and these regions are disjoint.

A natural question now is: do Theorems 1 and 3 hold in higher dimensions? The first point to note is that sets with uniform complement are no longer the same as uniformly disconnected sets. As an example, consider the compact set in $\mathbb{R}^3$ consisting of the line segment joining $(0,0,0)$ and $(0,0,1)$ and the family of line segments joining $(2^{-n},0,0)$ and $(2^{-n},0,1)$ for $n \geq 1$. This set is obviously not uniformly disconnected, yet its complement is a uniform domain. However, it is always true that any uniformly disconnected set has a uniform complement. This can be shown by the compactness method of Väisälä; see [V, Theorem 3.6]. Thus we have that ii) always implies i) in Theorem 1.

It is not reasonable to ask for quasiconformal equivalence in higher dimensions as topological issues complicate and cloud the issue. We saw in the planar case that the key to the whole problem is showing that the given set lies on a quasiconformal arc. A concept that makes sense in all dimensions and that agrees with that of a quasiconformal arc in the plane is that of an arc of bounded turning, *i.e.*, an arc with the property that the diameter of every sub-arc is comparable to the distance between its endpoints. Such arcs are precisely the quasisymmetric images of line-segments (see [TV, Section 4]). Thus a better formulation of the problem is: do uniformly disconnected sets or sets with uniform complement lie on an arc of bounded turning? The example in the previous paragraph shows that there are sets with uniform complement that do not lie on *any* Jordan arc. In contrast, it turns out that the answer is yes for uniformly disconnected sets. It follows immediately that a subset E of $\mathbb{R}^n$ is uniformly disconnected if and only if there is a quasisymmetric map from $[0,1]$ into $\mathbb{R}^n$ that maps a porous subset of $[0,1]$ onto E. Once we have this theorem, we find, following the planar case, that a subset of $\mathbb{R}^n$ is uniformly disconnected and uniformly perfect if and only if there is quasisymmetric map from

$[0,1]$ into $\mathbb{R}^n$ that maps the middle-third Cantor set onto E.

The proof that a uniformly disconnected set in any Euclidean space lies on an arc of bounded turning follows essentially the same scheme as that of the planar case. We give a brief justification of this result. Lemma 2.3 will hold with $\mathcal{D}_\delta$ being those sets which are finite unions of cubes from the grid of sidelength δ, and which have connected interiors and complements. This part is straightforward. Define the families $\mathcal{S}_n$ as in the discussion preceding Lemma 2.4. We can uniformly bound the number of cubes in each element of any $\mathcal{S}_n$. Using the notation of Lemma 2.4, we define A to be

$$S \setminus \bigcup_{\Delta \in \mathcal{C}(S)} \Delta .$$

The uniform bound on the number of cubes that make up both S and the elements of $\mathcal{C}(S)$ and the fact that these latter have connected interiors and complements allow us to show that S contains an arc γ of bounded turning with the following properties: the endpoints of γ are the distinguished points of S, the intersection of γ with any element Δ of $\mathcal{C}(S)$ consists of a sub-arc of γ whose end points are the distinguished points of Δ, and for any point on any component of $\gamma \cap A$ the distance to the nearest endpoint (of the component) and the distance to the boundary of A are comparable. We will refer to the union of γ and $\mathcal{C}(S)$ as a chain, although it does not fit our former definition. We now define $\mathcal{F}_n$ as before. The proof of [M, Theorem 5.1] shows that when such chains are nested (as described earlier in Section 1) they converge, as in the planar case, to an arc of bounded turning.

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References.

[DS] David, G., Semmes, S., *Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure.* Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, 1997.

[M] MacManus, P., Bi-Lipschitz Extensions in the Plane. *J. d'Analyse* **66** (1995), 85-115.

[TV] Tukia, P., Väisälä, J., Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **5** (1980), 97-114.

[V] Väisälä, J., Uniform domains. *Tôhoku Math. J.* **40** (1988), 101-118.

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Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality

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Abstract. Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by a Schrödinger operator $-A = \Delta - V$, where V is a nonnegative potential that belongs to a certain reverse Hölder class. We define a Hardy space H^1_A be means of a maximal function associated with the semigroup $\{T_t\}_{t>0}$. Atomic and Riesz transforms characterizations of H^1_A are shown.

1. Introduction and main results.

Let $A = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where $V \not\equiv 0$ is a nonnegative potential. We will assume that V belongs to reverse Hölder class $\mathcal{H}_q$ for some $q \geq d/2$, that is, V is locally integrable and

$$(1.0) \qquad \Big(\frac{1}{|B|}\int_B V^q\,dx\Big)^{1/q} \leq C\Big(\frac{1}{|B|}\int_B V\,dx\Big), \qquad \text{for every ball } B\,.$$

Trivially, $\mathcal{H}_q \subset \mathcal{H}_p$ provided $1 < p \leq q < \infty$. It is well known, *cf.* [Ge], that if $V \in \mathcal{H}_q$, then there is $\varepsilon > 0$ such that $V \in \mathcal{H}_{q+\varepsilon}$. Moreover, the measure $V(x)\,dx$ satisfies the doubling condition

$$\int_{B(y,2r)} V(x)\,dx \le C \int_{B(y,r)} V(x)\,dx\,.$$

We note that if V is a polynomial then $V \in \mathcal{H}_q$ for every $1 < q < \infty$.

Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by $-A$ and $T_t(x,y)$ be their kernels. Since V is nonnegative the Feynman-Kac formula implies that

$$0 \le T_t(x,y) \le \tilde{T}_t(x,y) = (4\pi\, t)^{-d/2} \exp\Big(- \frac{|x-y|^2}{4\,t}\Big)\,. \tag{1.1}$$

Obviously, by (1.1) the maximal operator

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)| \tag{1.2}$$

is of weak-type (1,1).

We say that a function f is in the Hardy space H^1_A if

$$\|f\|_{H^1_A} = \|\mathcal{M}f\|_{L^1} < \infty\,.$$

The aim of this article is to present an atomic characterization of H^1_A.

For $n \in \mathbb{Z}$ we define the sets $\mathcal{B}_n$ by

$$\mathcal{B}_n = \{x:\ 2^{n/2} \le m(x,V) < 2^{(n+1)/2}\}\,, \tag{1.3}$$

where

$$m(x,V) = \Big(\sup\Big\{r > 0 :\ \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\,dy \le 1\Big\}\Big)^{-1}\,. \tag{1.4}$$

For more details concerning the function $m(x,V)$ and its applications in studying the Schrödinger operator A we refer the reader to [Fe] and [Sh].

Since $0 < m(x,V) < \infty$, we have $\mathbb{R}^d = \bigcup \mathcal{B}_n$.

A function a is an atom for the Hardy space H^1_A associated to a ball $B(x_0,r)$ if

(i) $$\operatorname{supp} a \subset B(x_0,r),$$

(ii) $$\|a\|_{L^\infty} \le \frac{1}{|B(x_0,r)|}\,,$$

(iii) $$\text{if } \ x_0 \in \mathcal{B}_n \text{ then } r \le 2^{1-n/2}\,,$$

(iv) $$\text{if } x_0 \in \mathcal{B}_n, \text{ and } r \le 2^{-1-n/2} \text{ then } \int a(x)\,dx = 0\,.$$

The atomic norm in H^1_A is defined by

$$\|f\|_{A-\mathrm{atom}} = \inf\left\{\sum |c_j|\right\},$$

where the infimum is taken over all decompositions $f = \sum c_j\, a_j$, where a_j are H^1_A atoms.

The main result of this article is the following

Theorem 1.5. *Assume that $V \not\equiv 0$ is a nonnegative potential such that $V \in \mathcal{H}_{d/2}$, then the norms $\|f\|_{H^1_A}$ and $\|f\|_{A-\mathrm{atom}}$ are equivalent, that is, there exists a constant $C > 0$ such that*

$$C^{-1}\,\|f\|_{H^1_A} \leq \|f\|_{A-\mathrm{atom}} \leq C\,\|f\|_{H^1_A}\,.$$

For $j = 1, 2, \dots, d$, let us define the Riesz transforms R_j setting

$$R_j f = \frac{\partial}{\partial x_j} A^{-1/2}\,. \tag{1.6}$$

It was proved in [Sh] that if $V \in \mathcal{H}_{d/2}$ then the operators R_j are are bounded on L^p for $1 < p \leq d$. It turns out that these operators characterize our Hardy space H^1_A, that is the following theorem holds.

Theorem 1.7. *If $V \in \mathcal{H}_{d/2}$ is a nonnegative potential, $V \not\equiv 0$, then there is a constant $C > 0$ such that*

$$C^{-1}\,\|f\|_{H^1_A} \leq \|f\|_{L^1} + \sum_{j=1}^{d} \|R_j f\|_{L^1} \leq C\,\|f\|_{H^1_A}\,. \tag{1.8}$$

2. Auxiliary lemmas.

Lemma 2.0. *There is a constant C such that for every $R > 2$ and every n if $x \in \mathcal{B}_n$, then*

$$\{k:\ B(x, 2^{-n/2}R) \cap \mathcal{B}_k \neq \varnothing\} \subset [n - C\log_2 R, n + C\log_2 R]\,.$$

PROOF. [Sh, Lemma 1.4] asserts that there exist constants $C > 0$, $c > 0$, and $k_0 > 0$ such that for every $x, y \in \mathbb{R}^d$ we have

$$m(y,V) \le C\,(1 + |x-y|\,m(x,V))^{k_0}\,m(x,V) \tag{2.1}$$

and

$$m(y,V) \ge \frac{c\,m(x,V)}{(1+|x-y|\,m(x,V))^{k_0/(k_0+1)}}\,. \tag{2.2}$$

If $x \in \mathcal{B}_n$ and $y \in B(x, 2^{-n/2}R)$ then $|x-y|\,m(x,V) \le 2\,R$ and by (2.1)

$$m(y,V) \le C\,(1+2\,R)^{k_0}\,2^{n/2} \le C\,2^{(n+C\log_2 R)/2}\,.$$

On the other hand applying (2.2), we obtain

$$m(y,V) \ge \frac{c\,2^{n/2}}{(1+2\,R)^{k_0/(k_0+1)}} \ge c\,2^{(n-C\log_2 R)/2}\,.$$

This completes the proof of the lemma.

Lemma 2.3. *There is a constant C and a collection of balls $B_{(n,k)} = B(x_{(n,k)}, 2^{2-n/2})$, $n \in \mathbb{Z}$, $k = 1, 2, \dots,$ such that $x_{(n,k)} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \bigcup_k B(x_{(n,k)}, 2^{-n/2})$, and*

$$\#\{(n',k') :\ B(x_{(n,k)}, R\,2^{-n/2}) \cap B(x_{(n',k')}, R\,2^{-n'/2}) \neq \varnothing\} \le R^C\,,$$

for every (n,k) and $R \ge 2$.

From Lemma 2.3, we deduce

Corollary 2.4. *There exist constants $C > 0$ and $l_0 > 0$ such that for $l \ge l_0$ and every $x_{(n',k')}$ we have*

$$\sum_{(n,k)} (1+2^{n/2}|x_{(n',k')} - x_{(n,k)}|)^{-l} + \sum_{(n,k)} (1+2^{n'/2}|x_{(n',k')} - x_{(n,k)}|)^{-l} \le C\,.$$

Another consequence of Lemma 2.3 is

Lemma 2.5. *There are nonnegative functions $\psi_{(n,k)}$ such that*

$$\psi_{(n,k)} \in C_c^\infty(B(x_{(n,k)}, 2^{1-n/2}))\,, \tag{2.6}$$

$$\sum_{(n,k)} \psi_{(n,k)}(x) = 1\,, \tag{2.7}$$

$$\|\nabla \psi_{(n,k)}\|_{L^\infty} \le C\, 2^{n/2}\,. \tag{2.8}$$

3. Local maximal functions.

Lemma 3.0. *For ever $l > 0$ there is a constant C_l such that*

$$T_t(x,y) \le C_l\,(1+m(x,V)\,|x-y|)^{-l}\,|x-y|^{-d}\,, \qquad \text{for } x,y \in \mathbb{R}^d\,. \tag{3.1}$$

Moreover, there is an $\varepsilon > 0$ such that for every $C' > 0$ there exists C such that

$$|T_t(x,y) - \tilde{T}_t(x,y)| \le C\,\frac{(|x-y|\,m(x,V))^\varepsilon}{|x-y|^d}\,, \tag{3.2}$$

for $|x-y| \le C'\,m(x,V)^{-1}$.

PROOF. Let $\Gamma(x,y,\tau)$, $\tilde{\Gamma}(x,y,\tau)$ be the kernels of the operators $(A + i\,\tau)^{-1}$ and $(-\Delta + i\,\tau)^{-1}$, $\tau \in \mathbb{R}$. It is proved in [Sh] (see [Sh, Theorem 2.7]) that for every $l > 0$ there is a constant C_l such that

$$\begin{aligned}|\Gamma(x,y,\tau)| \le\; & \frac{C_l}{(1+|\tau|^{1/2}\,|x-y|)^l\,(1+m(x,V)\,|x-y|)^l} \\ & \cdot \frac{1}{|x-y|^{d-2}}\,.\end{aligned} \tag{3.3}$$

By the functional calculus, $T_t(x,y) = c\int_{\mathbb{R}} e^{it\tau}\,\Gamma(x,y,\tau)\,d\tau$. Thus (3.1) is easily deduced from (3.3).

It follows from [Sh], see [Sh, Lemma 4.5 and its proof], that for every $l, C' > 0$ there exists a constant $C > 0$ such that

$$|\Gamma(x,y,\tau) - \tilde{\Gamma}(x,y,\tau)| \le \frac{C_l}{(1+|\tau|^{1/2}\,|x-y|)^l}\,\frac{(|x-y|\,m(x,V))^\varepsilon}{|x-y|^{d-2}}\,, \tag{3.4}$$

for $|x-y| \le C' \, m(x,V)^{-1}$. Now the estimate (3.2) is a consequence of (3.4) and the formula $T_t - \tilde{T}_t = c \int_{\mathbb{R}} e^{it\tau} \, (\Gamma - \tilde{\Gamma}) \, d\tau$.

Since $T_t(x,y)$ is a symmetric function, we also have

$$T_t(x,y) \le C_l \, (1+m(y,V)\, |x-y|)^{-l} \, |x-y|^{-d}, \qquad \text{for } x,y \in \mathbb{R}^d . \tag{3.5}$$

We define the local maximal operators M_n $\widetilde{\mathcal{M}}_n$, and $\mathcal{M}_n$ putting

$$M_n f(x) = \sup_{0<t\le 2^{-n}} |\tilde{T}_t f(x) - T_t f(x)| , \tag{3.6}$$

$$\widetilde{\mathcal{M}}_n f(x) = \sup_{0<t\le 2^{-n}} |\tilde{T}_t f(x)| , \tag{3.7}$$

$$\mathcal{M}_n f(x) = \sup_{0<t\le 2^{-n}} |T_t f(x)| . \tag{3.8}$$

Lemma 3.9. *There exists a constant $C>0$ such that for every (n,k)*

$$\|M_n(\psi_{(n,k)} f)\|_{L^1} \le C \, \|f \, \psi_{(n,k)}\|_{L^1} \, .$$

PROOF. Set $B^*_{(n,k)} = B(x_{(n,k)}, 2^{(8-n)/2})$. Then by (3.2)

$$\|M_n(\psi_{(n,k)} \, f)\|_{L^1(B^*_{(n,k)})} \le C_{(n,k)} \, \|\psi_{(n,k)} \, f\|_{L^1} \, ,$$

where

$$C_{(n,k)} \le \sup_{y\in B_{(n,k)}} \int_{B^*_{(n,k)}} \frac{(|x-y| \, m(x,V))^{\varepsilon}}{|x-y|^d} \, dx \, .$$

It is easy to check that $C_{(n,k)} \le C$.

The task is now to estimate $\|M_n(\psi_{(n,k)} f)\|_{L^1((B^*_{(n,k)})^c)}$. According to (1.1), we obtain

$$\|M_n(\psi_{(n,k)} \, f)\|_{L^1((B^*_{(n,k)})^c)} \le C'_{(n,k)} \, \|f \, \psi_{(n,k)}\|_{L^1} \, ,$$

where

$$C'_{(n,k)} = 2 \sup_{y\in B_{(n,k)}} \int_{(B^*_{(n,k)})^c} \big(\sup_{0<t\le 2^{-n}} \tilde{T}_t(x,y) \big) \, dx \le C' \, .$$

This finishes the proof of the lemma.

Let

$$\mathcal{M}_{(n,k)}f(x) = \sup_{0<t\leq 2^{-n}} |T_t(\psi_{(n,k)}\, f)(x) - \psi_{(n,k)}(x)\, T_t f(x)|\,. \tag{3.10}$$

Lemma 3.11. *There is a constant C such that*

$$\sum_{(n,k)} \|\mathcal{M}_{(n,k)}f\|_{L^1} \leq C\, \|f\|_{L^1}\ . \tag{3.12}$$

PROOF.

$$[\psi_{(n,k)}, T_t]f(x) = \sum_{(n',k')} T_{t,(n,k),(n',k')}f(x)\,,$$

where

$$T_{t,(n,k),(n',k')}f(x) = \int f(y)\, T_t(x,y)(\psi_{(n,k)}(x) - \psi_{(n,k)}(y))\, \psi_{(n',k')}(y)\, dy\,.$$

Let

$$\mathcal{M}_{(n,k),(n',k')}f(x) = \sup_{0<t\leq 2^{-n}} |T_{t,(n,k),(n',k')}f(x)|\,.$$

Set $J_{(n,k)} = \{(n',k') : |x_{(n',k')} - x_{(n,k)}| \leq C'\, 2^{-n/2}\}$, and $I_{(n,k)} = \{(n',k') : |x_{(n',k')} - x_{(n,k)}| > C'\, 2^{-n/2}\}$. Note that the number of elements in $J_{(n,k)}$ is bounded by a constant independent of (n,k). Moreover, taking C' is sufficiently large we see that if $(n',k') \in I_{(n,k)}$ then $B^{**}_{(n,k)} \cap B^{**}_{(n',k')} = \varnothing$, where $B^{**}_{(n,k)} = B(x_{(n,k)}, 2^{8-n/2})$. Furthermore, $|x-y| \sim |x_{(n,k)} - x_{(n',k')}|$ for $x \in B_{(n,k)}$, $y \in B_{(n',k')}$, provided $(n',k') \in I_{(n,k)}$.

Obviously,

$$\|\mathcal{M}_{(n,k),(n',k')}f\|_{L^1} \leq C_{(n,k),(n',k')}\, \|f\|_{L^1(B_{(n',k')})}\ ,$$

where

$$\begin{aligned} &C_{(n,k),(n',k')} \\ &\qquad \leq \sup_{y\in B_{(n',k')}} \int \Big(\sup_{0<t\leq 2^{-n}} |T_t(x,y)\, (\psi_{(n,k)}(x) \\ &\qquad\qquad\qquad - \psi_{(n,k)}(y))\, \psi_{(n',k')}(y)|\Big)\, dx\,. \end{aligned}$$

If $(n', k') \in J_{(n,k)}$ then, by (1.1) and (2.8), we have

$$C_{(n,k),(n',k')} \leq C \sup_{y \in B_{(n',k')}} \int \Big(\sup_{0<t\leq 2^{-n}} 2^{n/2} \, |x-y| \, \tilde{T}_t(x,y) \Big) \, dx \leq C \, .$$

If $(n', k') \in I_{(n,k)}$ then using (3.1), we get

$$\begin{aligned} &C_{(n,k),(n',k')} \\ &\quad \leq \sup_{y \in B_{(n',k')}} \int \frac{C_{2l} \, \psi_{(n,k)}(x) \, \psi_{(n',k')}(y) \, dx}{|x-y|^d \, (1 + m(x,V) \, |x-y|)^{2l}} \\ &\quad \leq \sup_{y \in B_{(n',k')}} \int \frac{C_{2l} \, \psi_{(n,k)}(x) \, \psi_{(n',k')}(y) \, dx}{|x-y|^d \, (1 + 2^{n/2} \, |x-y|)^l \, (1 + 2^{n/2} \, |x_{(n,k)} - x_{(n',k')}|)^l} \\ &\quad \leq \frac{C}{(1 + 2^{n/2} \, |x_{(n,k)} - x_{(n'k')}|)^l} \, . \end{aligned}$$

Applying the above estimates, we obtain

$$\begin{aligned} &\sum_{(n,k)} \|\mathcal{M}_{(n,k)} f\|_{L^1} \\ &\quad \leq \sum_{(n,k)} \sum_{(n',k')} \|\mathcal{M}_{(n,k),(n',k')} f\|_{L^1} \\ &\quad \leq C \sum_{(n,k)} \sum_{(n',k') \in J_{(n,k)}} \|f\|_{L^1(B(x_{(n,k)}, C 2^{-n/2}))} \\ &\qquad + C \sum_{(n,k)} \sum_{(n',k') \in I_{(n,k)}} (1 + 2^{n/2} \, |x_{(n,k)} - x_{(n',k')}|)^{-l} \, \|f\|_{L^1(B_{(n',k')})} \, . \end{aligned}$$

Finally, by Corollary 2.4, we get (3.12).

4. Proof of Theorem 1.5.

In this section we prove our main theorem. First we recall some results from the theory of local Hardy spaces, *cf.* [Go].

We say that a function f is in the local Hardy space $\mathbf{h}_n^1$ if

$$\|f\|_{\mathbf{h}_n^1} = \|\widetilde{\mathcal{M}}_n f\|_{L^1} < \infty \, . \tag{4.0}$$

A function $\tilde{a}$ is an atom for the local Hardy space $\mathbf{h}_n^1$ if there is a ball $B(x_0, r)$, $r \le 2^{1-n/2}$ such that

$$\operatorname{supp} \tilde{a} \subset B(x_0, r), \tag{4.1}$$

$$\|\tilde{a}\|_{L^\infty} \le |B(x_0, r)|^{-1}, \tag{4.2}$$

$$\text{if } r \le 2^{-1-n/2}, \text{ then } \int \tilde{a}(x)\, dx = 0\,. \tag{4.3}$$

The atomic norm in $\mathbf{h}_n^1$ is defined by

$$\|f\|_{\mathbf{h}^1_{\tilde{a},n}} = \inf \Big(\sum_j |c_j| \Big), \tag{4.4}$$

where the infimum is taken over all decompositions $f = \sum c_j\, \tilde{a}_j$, where $\tilde{a}_j$ are $\mathbf{h}_n^1$ atoms.

Theorem 4.5 ([Go])**.** *The norms $\|\cdot\|_{\mathbf{h}_n^1}$ and $\|\cdot\|_{\mathbf{h}^1_{\tilde{a},n}}$ are equivalent with constants independent of $n \in \mathbb{Z}$.*

Moreover, if $f \in \mathbf{h}_n^1$, $\operatorname{supp} f \subset B(x, 2^{1-n/2})$, then there are $\mathbf{h}_n^1$ atoms $\tilde{a}_j$ such that $\operatorname{supp} \tilde{a}_j \in B(x, 2^{2-n/2})$ and

$$f = \sum_j c_j\, \tilde{a}_j\ , \qquad \sum_j |c_j| \le C\, \|f\|_{\mathbf{h}_n^1} \tag{4.6}$$

with a constant C independent of n.

PROOF OF THEOREM 1.5. We first assume that $f \in H^1_A$. Lemma 3.9 implies

$$\begin{aligned}
&\|\widetilde{\mathcal{M}}_n(\psi_{(n,k)} f)\|_{L^1} \\
&\qquad \le C\, (\|\mathcal{M}_n(\psi_{(n,k)} f)\|_{L^1} + \|\psi_{(n,k)} f\|_{L^1}) \\
&\qquad \le C\, (\|\psi_{(n,k)}(\mathcal{M}_n f)\|_{L^1} + \|\mathcal{M}_{(n,k)} f\|_{L^1} + \|\psi_{(n,k)} f\|_{L^1})\,.
\end{aligned}$$

From Lemma 3.11 we conclude that

$$\sum_{(n,k)} \|\widetilde{\mathcal{M}}_n(\psi_{(n,k)} f)\|_{L^1} \le C\, (\|\mathcal{M} f\|_{L^1} + \|f\|_{L^1})\,. \tag{4.7}$$

Application of Theorem 4.5 gives

$$\psi_{(n,k)}f = \sum_j c_j^{(n,k)}\, a_j^{(n,k)}\,, \qquad \text{where } a_j^{(n,k)} \text{ are } H^1_A \text{ atoms}, \tag{4.8}$$

and

$$\sum_j |c_j^{(n,k)}| \le C\, \|\widetilde{\mathcal{M}}_n(\psi_{(n,k)}f)\|_{L^1}\,. \tag{4.9}$$

Finally, using (4.7) and (4.8), we obtain the required H^1_A atomic decomposition

$$f = \sum_{(n,k)} \sum_j c_j^{(n,k)}\, a_j^{(n,k)} \text{ and } \sum_{(n,k)} \sum_j |c_j^{(n,k)}| \le C\, \|\mathcal{M}f\|_{L^1}\,, \tag{4.10}$$

and the inequality $\|f\|_{A-\mathrm{atom}} \le C\,\|f\|_{H^1_A}$ is proved.

In order to prove the converse inequality we need only to show that that there exists a constant $C > 0$ such that for every H^1_A atom a

$$\|\mathcal{M}a\|_{L^1} \le C\,. \tag{4.11}$$

Let a be an H^1_A atom associated to a ball $B(x_0, r)$. Assume that $x_0 \in \mathcal{B}_n$. Then, by definition, $r \le 2^{1-n/2}$. Theorem 4.5 combined with Lemma 3.9 implies that $\|\mathcal{M}_n a\|_{L^1} \le C$. Therefore what is left is to show that

$$\Big\| \sup_{t>2^{-n}} |T_t a(x)| \Big\|_{L^1(dx)} \le C\,.$$

If $x \in B(x_0, 2^{(8-n)/2})$ then

$$\sup_{t>2^{-n}} |T_t a(x)| \le \sup_{t>2^{-n}} \int \tilde{T}_t(x,y)\, |a(y)|\, dy \le C\, 2^{nd/2}\,,$$

and, consequently,

$$\Big\| \sup_{t>2^{-n}} |T_t a(x)| \Big\|_{L^1(B(x_0, 2^{(8-n)/2}))} \le C\,.$$

If $x \notin B(x_0, 2^{(8-n)/2})$ and $y \in B(x_0, 2^{1-n/2})$ then $|x-y| \ge 2^{(2-n)/2}$. Moreover, $m(y,V) \sim 2^{n/2}$. Applying (3.5) we get

$$\begin{aligned}
&\int_{B(x_0, 2^{(8-n)/2})^c} \sup_{t>2^{-n}} |T_t a(x)|\, dx \\
&\le \int_{B(x_0, 2^{(8-n)/2})^c} \int |a(y)|\, C_l\, (1 + m(y,V)\,|x-y|)^{-l}\, \frac{1}{|x-y|^d}\, dy\, dx \\
&\le C_l \int 2^{dn/2} (1 + 2^{n/2}\,|x|)^{-l}\, dx \le C\,.
\end{aligned}$$

5. Characterization of H^1_A by the Riesz transforms.

In this section we prove Theorem 1.7. Our proof of it is very much in the spirit of the proof of Theorem 1.5.

First we recall the characterization of the local Hardy spaces $\mathbf{h}^1_n$ by means of local Riesz transforms. Let ζ be a C^∞ function on $\mathbb{R}^d$ such that $\zeta(x) = 0$ for $|x| \geq 1$ and $\zeta(x) = 1$ for $|x| < 1/2$. We define the local Riesz transforms $\mathcal{R}^{[n]}_j$ by

$$\mathcal{R}^{[n]}_j f = f * \mathcal{R}^{[n]}_j \,, \tag{5.0}$$

where

$$\mathcal{R}^{[n]}_j(x) = c_d\,\zeta(2^{n/2}\,x)\,\frac{x_j}{|x|^{d+1}}\,.$$

We have

Theorem 5.1 *There is a constant $C > 0$ such that for every integer n*

$$C^{-1}\,\|f\|_{\mathbf{h}^1_n} \leq \|f\|_{L^1} + \sum_{j=1}^{d} \|f * \mathcal{R}^{[n]}_j\|_{L^1} \leq C\,\|f\|_{\mathbf{h}^1_n}\,. \tag{5.2}$$

Throughout this section we shall assume that $V \in \mathcal{H}_{d/2}$ is a non-negative potential, $V \not\equiv 0$.

Let us denote by $R_j(x,y)$ the integral kernel of the operator

$$\frac{\partial}{\partial x_j}\,A^{-1/2}\,.$$

Lemma 5.3 *There exists a constant $C > 0$ such that for every (n,k)*

$$C_{(n,k)} = \sup_{y\in B_{(n,k)}} \int_{B(x_{(n,k)},2^{8-n/2})^c} |R_j(x,y)|\,dx \leq C\,.$$

PROOF. By [Sh, p. 538] we have that for every $l > 0$ there is a constant C_l such that

$$|R_j(x,y)| \leq \frac{C_l}{(1+m(y,V)\,|x-y|)^l}$$
$$\cdot\Big(\frac{1}{|x-y|^{d-1}}\int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz + \frac{1}{|x-y|^d}\Big)\,. \tag{5.4}$$

Let us note that if $y \in B_{(n,k)}$ and $x \notin B^{**}_{(n,k)} = B(x_{(n,k)}, 2^{8-n/2})$, then $|x-y| \sim |x - x_{(n,k)}|$. Thus

$$
\begin{aligned}
C_{(n,k)} \le C_l \sup_{y \in B_{(n,k)}} \int_{(B^{**}_{(n,k)})^c} & \frac{1}{(1 + 2^{n/2}\,|x - x_{(n,k)}|)^l\,|x - x_{(n,k)}|^{d-1}} \\
& \cdot \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz\,dx \\
+ C_l \int_{(B^{**}_{(n,k)})^c} & \frac{1}{(1 + 2^{n/2}\,|x - x_{(n,k)}|)^l\,|x - x_{(n,k)}|^{d}}\,dx \\
= C'_{(n,k)} + C''_{(n,k)}\,. &
\end{aligned}
$$

Obviously $C''_{(n,k)} \le C$. We now turn to estimate $C'_{(n,k)}$.

$$
\begin{aligned}
C'_{(n,k)} &\le C_l \int_{(B^{*}_{(n,k)})^c} \Big(\frac{V(z)}{(1 + 2^{n/2}\,|z - x_{(n,k)}|)^l\,|z - x_{(n,k)}|^{d-1}} \\
&\qquad \cdot \int_{B(z,|x_{(n,k)}-z|/2)} \frac{1}{|z-x|^{d-1}}\,dx \Big)\,dz \\
&\le C_l \int_{(B^{*}_{(n,k)})^c} \frac{V(z)}{(1 + 2^{n/2}\,|z - x_{(n,k)}|)^l\,|z - x_{(n,k)}|^{d-2}}\,dz\,.
\end{aligned}
$$

[Sh, Lemma 1.8] asserts that if $\rho\, m(x,V) \ge 1$ then

$$
\frac{1}{\rho^{d-2}} \int_{B(x,\rho)} V(z)\,dz \le C\,(\rho\, m(x,V))^{k_0}\,, \tag{5.5}
$$

for some $k_0 > 0$. Therefore

$$
\begin{aligned}
C'_{(n,k)} &\le C_l \sum_{i=0}^{\infty} \int_{B(x_{(n,k)},2^{i+1-n/2})} \frac{V(z)}{(1+2^i)^l\,(2^{i-n/2})^{d-2}}\,dz \\
&\le C_l \sum_{i=0}^{\infty} \frac{1}{(1+2^i)^l}\,(2^{i+1-n/2}\,2^{n/2})^{k_0} \\
&\le C\,.
\end{aligned}
$$

Corollary 5.6. *There is a constant $C > 0$ such that for every (n,k) we have*

$$\|R_j(\psi_{(n,k)} f)\|_{L^1((B^{**}_{(n,k)})^c)} \le C\,\|\psi_{(n,k)} f\|_{L^1}\,. \tag{5.7}$$

Lemma 5.8. *There exists a constant C such that*

$$\sum_{(n,k)} \|R_j(\psi_{(n,k)} f) - \psi_{(n,k)} R_j f\|_{L^1} \le C\,\|f(x)\|_{L^1(dx)}\,. \tag{5.9}$$

Proof. For fixed (n,k) we have

$$\|[\psi_{(n,k)}, R_j] f\|_{L^1} \le \sum_{(n',k')} C_{(n',k')}\,\|f\|_{L^1(B_{(n',k')})}\,, \tag{5.10}$$

where

$$\begin{aligned} C_{(n',k')} \le \sup_{y\in B_{(n',k')}} \int |R_j(x,y)(\psi_{(n,k)}(x) \\ - \psi_{(n,k)}(y))\,\psi_{(n',k')}(y)|\,dx\,. \end{aligned} \tag{5.11}$$

Let $J_{(n,k)}$ and $I_{(n,k)}$ be as in the proof of Lemma 3.11.

If $(n',k') \in J_{(n,k)}$ then, by Lemma 2.5,

$$\begin{aligned} C_{(n',k')} &\le \sup_{y\in B_{(n,k)}} \int_{(B^{**}_{(n,k)})^c} |R_j(x,y)|\,dx \\ &\qquad + \sup_{y\in B_{(n',k')}} \int_{B^{**}_{(n,k)}} C\,|R_j(x,y)|\,2^{n/2}\,|x-y|\,dx \\ &= S_1 + S_2\,. \end{aligned}$$

Lemma 5.3 clearly forces $S_1 \le C$.

Applying (5.4) and the theorem on fractional integrals we obtain

$$S_2 \le \sup_{y\in B_{(n',k')}} \int_{B^{**}_{(n,k)}} \frac{C_l}{(1+m(y,V)\,|x-y|)^l} \frac{2^{n/2}\,|x-y|}{|x-y|^{d-1}} \cdot \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz\,dx$$
$$+ \sup_{y\in B_{(n',k')}} \int_{B^{**}_{(n,k)}} \frac{C_l}{(1+m(y,V)\,|x-y|)^l} \frac{2^{n/2}\,|x-y|}{|x-y|^{d}}\,dx$$
$$\le \sup_{y\in B_{(n',k')}} C\int_{B^{**}_{(n,k)}} \Big(\frac{1}{|x-y|^{d-1}}\int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz + \frac{2^{n/2}}{|x-y|^{d-1}}\Big)\,dx$$
$$\le \sup_{y\in B_{(n',k')}} C\int_{B(x_{(n,k)},C2^{-n/2})} \frac{V(z)}{|z-y|^{d-2}}\,dz + \sup_{y\in B_{(n',k')}} C\int_{B^{**}_{(n,k)}} \frac{2^{n/2}}{|x-y|^{d-1}}\,dx\,.$$

Let us note that the Hölder inequality and the fact that $V\in\mathcal{H}_{d/2+\varepsilon}$ for some $\varepsilon>0$ imply that

$$\int_{B(x,\rho)} \frac{V(z)}{|z-x|^{d-2}}\,dz \le \frac{C}{\rho^{d-2}}\int_{B(x,\rho)} V(z)\,dz\,. \tag{5.12}$$

Therefore $S_2\le C$.

If $(n',k')\in I_{(n,k)}$, then

$$C_{(n',k')} \le \sup_{y\in B_{(n',k')}} \int_{B_{(n,k)}} |R_j(x,y)|\,dx\,.$$

Using (5.4) we get

$$C_{(n',k')} \le \sup_{y\in B_{(n',k')}} \int_{B_{(n,k)}} \frac{C_l}{(1+m(y,V)\,|x-y|)^l\,|x-y|^{d-1}} \cdot \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz\,dx$$
$$+ \sup_{y\in B_{(n',k')}} \int_{B_{(n,k)}} \frac{C_l}{(1+m(y,V)\,|x-y|)^l\,|x-y|^{d}}\,dx\,.$$

Since $|x-y| \sim |x_{(n,k)} - x_{(n',k')}|$ for $x \in B_{(n,k)}$, $y \in B_{(n',k')}$, we have

$$C_{(n',k')} \leq \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l} \cdot \sup_{y \in B_{(n',k')}} \int_{B(x_{(n,k)},|x_{(n,k)}-y|/2)} \frac{V(z)}{|y-z|^{d-2}}\,dz + \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l}\,.$$

It is not difficult to check that $B(x_{(n,k)}, |x_{(n,k)}-y|/2) \subset B(y, C|x_{(n,k)} - x_{(n',k')}|)$ for $y \in B_{(n',k')}$, with C independent of (n,k) and (n',k'). Thus

$$C_{(n',k')} \leq \frac{C_l}{(1+2^{n'/2}|x_{(n,k)} - x_{(n',k')}|)^l} \cdot \Big(1 + \sup_{y \in B_{(n',k')}} \int_{B(y,C|x_{(n,k)}-x_{(n',k')}|)} \frac{V(z)}{|y-z|^{d-2}}\,dz\Big)\,.$$

Now using (5.12) we obtain

$$\begin{aligned} &C_{(n',k')} \\ &\leq \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l} \\ &\quad + \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l} \\ &\quad\cdot \sup_{y \in B_{(n',k')}} \frac{1}{|x_{(n,k)} - x_{(n',k')}|^{d-2}} \int_{B(y,C|x_{(n,k)}-x_{(n',k')}|)} V(z)\,dz\,. \end{aligned}$$

By virtue of (5.5) we get

$$\begin{aligned} C_{(n',k')} &\leq \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^l} \cdot (1 + C\,(2^{n'/2}\,|x_{(n',k')} - x_{(n,k)}|)^{k_0}) \\ &\leq \frac{C_l}{(1+2^{n'/2}\,|x_{(n,k)} - x_{(n',k')}|)^{l-k_0}}\,. \end{aligned}$$

Now (5.9) follows easily from (5.10), Corollary 2.4, and Lemma 2.3.

Let $\tilde{R}_j f = (\partial/\partial x_j)\,\Delta^{-1/2} f$ denote the classical Riesz transforms and let $\tilde{R}_j(x,y)$ be their kernels.

Lemma 5.13. *There exists a constant $C>0$ such that*

$$\|R_j(\chi_{B_{(n,k)}} f) - \tilde{R}_j(\chi_{B_{(n,k)}} f)\|_{L^1(B^{**}_{(n,k)})} \le C\,\|\chi_{B_{(n,k)}} f\|_{L^1}\,. \tag{5.14}$$

PROOF. The left-hand side of (5.14) is estimated by

$$C_{(n,k)}\,\|\chi_{B_{(n,k)}} f\|_{L^1}\,,$$

where

$$C_{(n,k)} \le \sup_{y\in B_{(n,k)}} \int_{B^{**}_{(n,k)}} |R_j(x,y) - \tilde{R}_j(x,y)|\,dx\,.$$

[Sh, Estimate (5.9)] says that for every $C'>0$ there is a constant $C>0$ such that

$$|R_j(x,y) - \tilde{R}_j(x,y)| \le \frac{C}{|x-y|^{d-1}}\Big(\int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{d-1}}\,dz + \frac{1}{|x-y|}\,(|x-y|\,m(y,V))^{\varepsilon}\Big)\,,$$

for $|x-y|\le C'/m(y,V)$ and some $\varepsilon>0$. (In [Sh] this estimate is shown with $C'=1$. Actually the proof works for any C'). The theorem on fractional integrals leads to

$$C_{(n,k)} \le C \sup_{y\in B_{(n,k)}} \int_{B(y,C2^{-n/2})} \frac{V(z)}{|y-z|^{d-2}}\,dz + C \sup_{y\in B_{(n,k)}} \int_{B^{**}_{(n,k)}} \frac{(2^{n/2}\,|x-y|)^{\varepsilon}}{|x-y|^{d}}\,dx\,.$$

By virtue of (5.12) we have $C_{(n,k)}\le C$, and the proof is complete.

PROOF OF THEOREM 1.7. Assume first that $\|f\|_{L^1} + \sum_{j=1}^{d} \|R_j f\|_{L^1} < \infty$. Lemmas 5.8 and 5.13 imply that

$$\sum_{(n,k)} \|\tilde{R}_j(\psi_{(n,k)} f)\|_{L^1(B^{**}_{(n,k)})} \le C\,(\|f\|_{L^1} + \|R_j f\|_{L^1})\,.$$

Now using Theorem 5.1, we obtain the required atomic decomposition

$$f = \sum_{(n,k)} \psi_{(n,k)} f = \sum_{(n,k)} \sum_i c_i^{(n,k)} \, a_i^{(n,k)} \,,$$

$$\sum_{(n,k)} \sum_i |c_i^{(n,k)}| \leq C\Big(\sum_{j=1}^{d} \|R_j f\|_{L^1} + \|f\|_{L^1} \Big) \,,$$

where $a_i^{(n,k)}$ are H_A^1 atoms.

To prove the converse inequality we only, by Theorem 1.5, need to show that

$$\|R_j a\|_{L^1} \leq C \,,$$

for every H_A^1 atom a with a constant C independent of a. Assume that a is an H_A^1 atom associated to a ball $B(x_0, r)$. If $x_0 \in \mathcal{B}_n$ then by definition $r \leq 2^{1-n/2}$ and there exists k such that $B(x_0, r) \subset B(x_{(n,k)}, 2^{2-n/2})$. By Lemma 5.3 we have

$$\|R_j a\|_{L^1((B^{**}_{(n,k)})^c)} \leq C \,.$$

On the other hand, since a is an atom for $\mathbf{h}_n^1$, Theorem 5.1 implies that $\|\tilde{R}_j a\|_{L^1(B^{**}_{(n,k)})} \leq C$. Applying Lemma 5.13, we get

$$\|R_j a\|_{L^1(B^{**}_{(n,k)})} \leq C \,,$$

which finishes the proof of the theorem.

References.

[DZ] Dziubański, J., Zienkiewicz, J., Hardy spaces associated with some Schrödinger operators. *Studia Math.* **126** (1997), 149-160.

[Fe] Fefferman, C., The uncertainty principle. *Bull. Amer. Math. Soc.* **9** (1983), 129-206.

[FeS] Fefferman, C., Stein, E., H^p spaces of several variables. *Acta Math.* **129** (1972), 137-193.

[Ge] Gehring, F., The L^p-integrability of the partial derivatives of a quasi-conformal mapping. *Acta Math.* **130** (1973), 265-277.

[Go] Goldberg, D., A local version of real Hardy spaces. *Duke Math. J.* **46** (1979), 137-193.

[Sh] Shen, Z., L^p estimates for Schrödinger operators with certain potentials. *Ann. Inst. Fourier* **45** (1995), 513-546.

[S] Stein, E., *Harmonic Analysis.* Princeton University Press, 1993.

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L^p-estimates for the wave equation on the Heisenberg group

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Abstract. Let $\mathcal{L}$ denote the sub-Laplacian on the Heisenberg group $\mathbb{H}_m$. We prove that $e^{i\sqrt{-\mathcal{L}}}/(1-\mathcal{L})^{\alpha/2}$ extends to a bounded operator on $L^p(\mathbb{H}_m)$, for $1 \le p \le \infty$, when $\alpha > (d-1)\,|1/p - 1/2|$.

0. Introduction.

On the Heisenberg group $\mathbb{H}_m$, which is $\mathbb{C}^m \times \mathbb{R}$ endowed with the group law

$$(z,t)\cdot(z',t') := \Big(z+z', t+t' - \frac{1}{2}\,\mathrm{Im}\, z\cdot\overline{z'}\Big),$$

the vector fields

$$X_j := \frac{\partial}{\partial x_j} - \frac{1}{2}\, y_j\, \frac{\partial}{\partial t}\,, \qquad Y_j := \frac{\partial}{\partial y_j} + \frac{1}{2}\, x_j\, \frac{\partial}{\partial t}\,,$$

$j = 1, \dots, m$, and $T := \partial/\partial t$ form a natural basis for the Lie algebra of left-invariant vector fields. The only non-trivial commutation relations among those are $[X_j, Y_j] = T$, $j = 1, \dots, m$. Due to these relations, the non-elliptic *sub-Laplacian*

$$\mathcal{L} := \sum_{j=1}^{m} (X_j^2 + Y_j^2)$$

on $\mathbb{H}_m$ is still hypoelliptic, and provides one of the simplest examples of a non-elliptic "sum of squares operator" in the sense of Hörmander (see *e.g.* [K], [Hö]). Moreover, $\mathcal{L}$ takes over in many respects of analysis on $\mathbb{H}_m$ the role which the Laplacian plays on Euclidian space.

Consider the following Cauchy problem for the wave equation on $\mathbb{H}_m \times \mathbb{R}$ associated to $\mathcal{L}$

$$\text{(CP)} \qquad \frac{\partial^2 u}{\partial \tau^2} - \mathcal{L}u = 0\,, \qquad u|_{\tau=0} = f\,, \qquad \frac{\partial u}{\partial \tau}\Big|_{\tau=0} = g\,,$$

where $\tau \in \mathbb{R}$ denotes time.

If we put $L := -\mathcal{L}$, then the solution to this problem is formally given by

$$u(x,\tau) = \Big(\frac{\sin(\tau\sqrt{L})}{\sqrt{L}}\, g\Big)(x) + (\cos(\tau\sqrt{L})f)(x)\,, \qquad (x,\tau) \in \mathbb{H}_m \times \mathbb{R}.$$

In fact, if $L^p(\mathbb{H}_m), 1 \leq p \leq \infty$, denotes the L^p-Lebesgue space on $\mathbb{H}_m$ with respect to the bi-invariant Haar measure (which incidentally agrees with the Lebesgue measure on $\mathbb{C}^m \times \mathbb{R}$), then the above expression for u makes perfect sense at least for $f, g \in L^2(\mathbb{H}_m)$, if one defines the functions of L involved by the spectral theorem (notice that L is essentially selfadjoint on $C_0^\infty(\mathbb{H}_m)$).

If one decides to measure smoothness properties of the solution $u(x,\tau)$ to (CP) for fixed time τ in terms of Sobolev norms of the form $\|f\|_{L^p_\alpha} := \|(1+L)^{\alpha/2} f\|_{L^p}$, one is naturally led to study the mapping properties of operators such as

$$\frac{e^{i\tau\sqrt{L}}}{(1+L)^{\alpha/2}}$$

or

$$\frac{\sin(\tau\sqrt{L})}{\sqrt{L}(1+L)^{\alpha/2}}$$

as operators on $L^p(\mathbb{H}_m)$ into itself.

For the classical wave equation on Euclidian space, sharp estimates for the corresponding operators have been established by Peral [P] and Miyachi [Mi].

In particular, if Δ denotes the Laplacian on $\mathbb{R}^d$, then

$$(1-\Delta)^{-\alpha/2}\, e^{i\tau\sqrt{-\Delta}}$$

is bounded on $L^p(\mathbb{R}^d)$, if $\alpha \geq \alpha(d,p) := (d-1)\,|1/p - 1/2|$, for $1 < p < \infty$. Moreover, $(1-\Delta)^{-((d-1)/2)/2}\, e^{i\tau\sqrt{-\Delta}}$ is bounded from the real Hardy space $H^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

Local analogues of these results hold true for solutions to strictly hyperbolic differential equations (see *e.g.* [CF], [P], [B], [Mi], [SSS]).

Indeed, as has been shown in [B] and [SSS], the estimates in [P] and [Mi] locally hold true more generally for large classes of Fourier integral operators, and solutions to strictly hyperbolic equations can be expressed in terms of such operators.

The problem in studying the wave equation associated to the sub-Laplacian on the Heisenberg group is the lack of strict hyperbolicity, since $\mathcal{L}$ is degenerate-elliptic, and Fourier integral operator technics do not seem to be available any more.

Interesting information about solutions to (CP) have been obtained by Nachman [N]. Among other things, Nachman showed that the wave operator on $\mathbb{H}_m$ admits a fundamental solution supported in a "forward light cone", whose singularities lie along the cone Γ formed by the bicharacteristics through the origin. Moreover, he computed the asymptotic behaviour of the fundamental solution as one approaches a generic singular point on Γ. His method does, however, not provide uniform estimates on these singularities, so that it cannot be used to prove L^p-estimates for solutions to (CP). What his results do reveal, however, is that Γ is by far more complex for $\mathbb{H}_m$ than the corresponding cone in the Euclidian case. This is related to the underlying, more complex sub-Riemannian geometry.

Nevertheless, in this article we shall prove the following theorem: Let $d = m+1$ denote the Euclidian dimension of $\mathbb{H}_m$.

Theorem. $e^{i\sqrt{L}}/(1+L)^{\alpha/2}$ *extends to a bounded operator on* $L^p(\mathbb{H}_m)$, *for* $1 \leq p \leq \infty$, *when* $\alpha > (d-1)\,|1/p - 1/2|$.

REMARK. One can see below that the same result holds for

$$\frac{\sin\sqrt{L}}{\sqrt{L}\,(1+L)^{(\alpha-1)/2}}\ ,$$

or with the factors $(1+L)^{-\alpha/2}$ (respectively $(1+L)^{-(\alpha-1)/2}$) replaced by $(1+\sqrt{L})^{-\alpha}$, (respectively $(1+\sqrt{L})^{-(\alpha-1)}$).

Notice that the restriction to time $\tau = 1$ in our theorem is inessential, since L is homogeneous of degree 2 with respect to the automorphic dilations $(z,t) \longmapsto (r\,z, r^2\,t)$, $r > 0$.

Our theorem is slightly weaker than what one would expect in direct analogy with the afore mentioned result of Peral and Miyachi. It would be interesting to know whether the condition $\alpha = \alpha(d, p)$ does already suffice, if $1 < p < \infty$, and if there is an endpoint result for $p = 1$.

Finally we would like to mention that the spectral multiplier theorem for L in [MS] (see also [H]) can easily be deduced from our theorem by means of the method of subordination.

Our approach to the theorem is based on harmonic analysis on $\mathbb{H}_m$, in the sense expressed by Strichartz in [St], as the joint spectral theory of the two operators L and iT. We shall closely follow the notation in [St], and freely make use of the results of that paper, as well as of those in [MRS1,2].

1. Basic reductions and dyadic decomposition of $e^{i\sqrt{L}}$.

In order to prove the theorem and the subsequent remark, we first observe that it suffices to prove the case $p = 1$. This follows from a standard interpolation argument. Namely, if we assume that the case $p = 1$ was true, and define the analytic family of operators $T_\alpha := e^{i\sqrt{L}}/(1 + L)^{\alpha/2}$, then we had

$$\|T_\alpha f\|_2 \le C_\alpha \, \|f\|_2 \, , \qquad \text{if } \operatorname{Re} \alpha = 0 \, ,$$

$$\|T_\alpha f\|_1 \le C_\alpha \, \|f\|_1 \, , \qquad \text{if } \operatorname{Re} \alpha > \frac{d-1}{2} \, .$$

The latter inequality remains true even if $\operatorname{Im} \alpha \neq 0$, since the operators $(1+L)^{-\varepsilon+i\gamma}$, γ real, are known to be bounded on L^1, for any $\varepsilon > 0$, with norm growing at most polynomially in γ. This can in fact also be seen by a slight modification of the proof of Corollary 1.2 to follow. Hence one can use the analytic interpolation theorem in [S1] and a standard duality argument to deduce the theorem for arbitrary p (the results in the remark can be obtained similarly).

For any bounded function ψ on $\mathbb{R}^+$ we define the operator $\psi(L)$ by the spectral theorem, and denote by $M_\psi \in \mathcal{S}'(\mathbb{H}_m)$ the corresponding Schwartz convolution kernel, so that $\psi(L) = f * M_\psi$ whenever $f \in \mathcal{S}$. We also write $M_\psi = \psi(L)\, \delta_0$, where δ_0 is the Dirac measure at the origin.

The results for the case $p = 1$ are proved by showing that the corresponding convolution kernels belong to $L^1(\mathbb{H}_m)$.

1.1. Reduction to an estimate for the local part of the convolution kernel.

Let η be an even $C_0^\infty(\mathbb{R})$ function, so that $\eta(\xi) = 1$ for small $|\xi|$, and $\eta(\xi) = 0$, if $|\xi| \geq 1$. For some large constant $N > 1$, to be chosen later, put $\eta_N(\xi) := \eta(\xi/N)$. Consider the function

$$h(\xi) := (1 - \eta_N)(\xi)\, \xi^{-\alpha/2}\, e^{i\sqrt{\xi}}\,, \qquad \xi > 0\,. \tag{1.1}$$

We let M denote the corresponding convolution kernel, so that $h(L)f = f * M$.

Proposition 1.1. *To prove the theorem, it suffices to show that $\chi_{B_2} M$ belongs to $L^1(\mathbb{H}_m)$.*

Here B_r denotes the ball of radius r centered at 0 with respect to the optimal control distance on $\mathbb{H}_m$. (For the definition of this distance see *e.g.* [VCS]).

The proof of the proposition is based on the following two facts. The first deals with the speed of propagation of the wave equation and can be found in [Me].

(1.2). *The support of the distribution* $\cos(t\sqrt{L})\,\delta_0$ *is contained in* $B_{|t|}$.

The second fact guarantees that for certain multipliers ψ the corresponding kernel M_ψ is in $L^1(\mathbb{H}_m)$.

(1.3). *Suppose $\psi \in C^{(k)}(\mathbb{R}^+)$, with k assumed to be sufficiently large. If ψ satisfies the inequalities*

$$\begin{cases} |\xi^\ell\, \psi^{(\ell)}(\xi)| \leq A\, \xi^{1/2}\,, & \textit{when } 0 < \xi \leq 1\,, \\ |\xi^\ell\, \psi^{(\ell)}(\xi)| \leq A\, \xi^{-1/2}\,, & \textit{when } 1 \leq \xi < \infty\,, \end{cases}$$

for $0 \leq \ell \leq k$, then $M_\psi = \psi(L)\,\delta_0$ is in $L^1(\mathbb{H}_m)$.

REMARKS. 1) It actually suffices to take $k > (d-1)/2$; also the exponent $1/2$ can be reduced to $\varepsilon > 0$. However the above special case suffices for our purposes.

2) The proof gives the bound $\|M_\psi\|_{L^1(\mathbb{H}_m)} \leq \text{constant}\, A$, with A as in (1.3).

To prove (1.3), we let

$$1=\sum_{j=-\infty}^{\infty}\chi_j(x)$$

be a standard dyadic partition of unity for $\mathbb{R}^+$, with $\chi_j(x):=\chi(2^{-j}x)$, where $\chi\in C_0^\infty(\mathbb{R})$ is non-negative and supported in $[1/2,2]$.

We write $\psi_j := 2^{|j|/2}\chi_j\psi$. Then $\psi(\xi)=\sum_j 2^{-|j|/2}\psi_j(\xi)$, and $\|M_{\psi_j}\|_{L^1(\mathbb{H}_m)}\le A$, uniformly in j.

In fact, with $\tilde\psi_j(\xi):=\psi_j(2^j\xi)$, each $\tilde\psi_j$ is supported in $[1/2,2]$, and the $\tilde\psi_j$ satisfy the inequalities

$$\sup_{j,\xi}|\tilde\psi_j^{(\ell)}(\xi)|\le A\,,\qquad \text{for } 0\le\ell\le k\,.$$

Thus the key step in the proof of the Marcinkiewicz-Mikhlin-Hörmander multiplier theorem for $\mathbb{H}_m$ (for which see *e.g.* [FoS], [C], [MM]; also [MS], [H], [MRS1,2]) shows that

$$\sup_j\|M_{\psi_j}\|_{L^1(\mathbb{H}_m)}=\sup_j\|M_{\tilde\psi_j}\|_{L^1(\mathbb{H}_m)}<\infty\,,$$

and the assertion (1.3) is proved, since $M_\psi=\sum 2^{-|j|/2}M_{\psi_j}$.

Now for $\alpha>0$ and $|t|\le 1$ set

$$f_{\alpha,t}(\xi):=(1-\eta_N)(\xi)\,\xi^{-\alpha/2}\cos(t\sqrt{\xi})\,,$$

$$g_{\alpha,t}(\xi):=(1-\eta_N)(\xi)\,\xi^{-\alpha/2}\sin(t\sqrt{\xi})\,,$$

so that by (1.1) $h(\xi)=f_{\alpha,1}(\xi)+i\,g_{\alpha,1}(\xi)$.

It is easily seen that for $\xi>0$

$$(1-\eta_N)(\xi)\,|\xi|^{-\alpha/2}=\int\varphi(\tau)\cos(\tau\sqrt{\xi})\,d\tau\,,$$

where φ is such that $\eta\,\varphi\in L^1$ and $(1-\eta)\,\varphi\in\mathcal{S}$.

Hence

$$f_{\alpha,t}(\xi)=\int(\eta\,\varphi)(\tau)\cos(\tau\sqrt{\xi})\cos(t\sqrt{\xi})\,d\tau+\Phi(\sqrt{\xi})\cos(t\sqrt{\xi})\,,$$

with $\Phi \in \mathcal{S}$.

Now the support of the distribution corresponding to

$$\int (\eta\,\varphi)(\tau)\cos(\tau\sqrt{L})\cos(t\sqrt{L})\,d\tau$$

lies in B_2. This is because $2\cos(\tau\sqrt{L})\cos(t\sqrt{L}) = \cos((\tau+t)\sqrt{L}) + \cos((\tau-t)\sqrt{L})$ and the result of fact (1.2). However the kernel corresponding to $\Phi(\sqrt{L})\cos(t\sqrt{L})$ is in $L^1(\mathbb{H}_m)$, uniformly for $|t| \le 1$, as long as $\Phi \in \mathcal{S}$.

This can be seen by applying the result (1.3) to the function $\psi(\xi) := \Phi(\sqrt{\xi})\cos(t\sqrt{\xi}) - \Phi(0)\,e^{-\xi}$, and recalling that the kernel corresponding to $e^{-\xi}$ is the heat-kernel, which is in $L^1(\mathbb{H}_m)$.

Thus we have that the $f_{\alpha,t}(L)\,\delta_0$ are uniformly in $L^1(\mathbb{H}_m)$ in the complement of the ball B_2.

As for $g_{\alpha,1}$, we observe that

$$g_{\alpha+1,1}(\xi) = \int_0^1 f_{\alpha,t}(\xi)\,dt\,, \tag{1.4}$$

and thus $g_{\alpha+1,1}(L)(\delta_0)$ is in $L^1(\mathbb{H}_m)$ outside the ball B_2, if $\alpha > 0$. As a result

$$h(L)\,\delta_0 = f_{\alpha,1}(L)\,\delta_0 + i\,g_{\alpha,1}(L)\,\delta_0 = M$$

is in $L^1(\mathbb{H}_m)$ outside the ball B_2, if $\alpha > 1$. Thus, if we knew that $\chi_{B_2} M$ was in $L^1(\mathbb{H}_m)$, we could conclude that $M \in L^1(\mathbb{H}_m)$.

The conditional assertion for $(1+L)^{-\alpha/2}\,e^{i\sqrt{L}}$ can now be obtained as follows. We write $(1+\xi)^{-\alpha/2}\,e^{i\sqrt{L}}$ as

$$\eta_N(\xi)\,(1+\xi)^{-\alpha/2}\,e^{i\sqrt{\xi}} + \frac{\xi^{\alpha/2}}{(1+\xi)^{\alpha/2}}(1-\eta_N)(\xi)\,\xi^{-\alpha/2}\,e^{i\sqrt{\xi}}\,.$$

The function $\eta_N(\xi)\,(1+\xi)^{-\alpha/2}\,e^{i\sqrt{\xi}} - e^{-\xi}$ satisfies the hypothesis of (1.3), and $e^{-\xi}$ corresponds to the heat kernel, thus

$$\eta_N(L)\,(1+L)^{-\alpha/2}\,e^{i\sqrt{L}}$$

has an $L^1(\mathbb{H}_m)$ kernel. Next the function $\xi^{\alpha/2}/(1+\xi)^{\alpha/2} - 1 + e^{-\xi}$ satisfies the hypothesis of (1.3), thus $L^{\alpha/2}/(1+L)^{\alpha/2}$ is the identity operator plus a convolution operator whose kernel is in $L^1(\mathbb{H}_m)$. Combining this with the previous assertion about $(1-\eta_N)(L)\,L^{-\alpha/2}\,e^{i\sqrt{L}}$ proves the proposition. The further conclusions in the remark are proved similarly.

We have reduced the proof of our theorem to showing that

$$\|\chi_{B_2} M\|_{L^1(\mathbb{H}_m)} < \infty ,$$

with $M = M_h$ and h given by (1.1), provided $\alpha > m = (d-1)/2$.

A further reduction is given as follows: We let $\tilde{\chi}_{B_2}$ be a smooth variant of χ_{B_2}; that is, $\tilde{\chi}_{B_2}$ is in $C_0^\infty(\mathbb{H}_m)$, with $\tilde{\chi}_{B_2}(x) = 1$, if $x \in B_2$.

Corollary 1.2. *To prove the theorem, it suffices to prove that the operator $f \longmapsto f * (\tilde{\chi}_{B_2} M)$ is bounded on $L^p(\mathbb{H}_m)$ to itself, for all p, $1 < p < \infty$.*

Proof. Write $M = M_\alpha$ to indicate the dependence on α. Now, if $\alpha > (d-1)/2$, we can write $\alpha = \alpha' + \varepsilon$, $\varepsilon > 0$, $\alpha' > (d-1)/2$. We know from the above that $(1 - \tilde{\chi}_{B_2})M_{\alpha'}$ is in $L^1(\mathbb{H}_m)$, so if $f \longmapsto f * (\tilde{\chi}_{B_2} M_{\alpha'})$ is bounded on L^p, $1 < p < \infty$, so is $f \longmapsto f * M_{\alpha'}$. But $M_\alpha = M_{\alpha'} * ((1+L)^{-\varepsilon}\,\delta_0)$.

However, $(1+L)^{-\varepsilon}\,\delta_0$ is in $L^p(\mathbb{H}_m)$ for some $p > 1$, if $\varepsilon > 0$ (we shall prove this momentarily). We would then have $M_\alpha \in L^p(\mathbb{H}_m)$, and hence $\chi_{B_2} M_\alpha \in L^1(\mathbb{H}_m)$. However, $(1 - \chi_{B_2})M_\alpha$ was already shown to be in $L^1(\mathbb{H}_m)$, and so this would imply that $M_\alpha \in L^1(\mathbb{H}_m)$.

To see that $(1+L)^{-\varepsilon}\,\delta_0 \in L^p(\mathbb{H}_m)$ for some $p > 1$, we write

$$(1+L)^{-\varepsilon}\,\delta_0 = \frac{1}{\Gamma(\varepsilon)} \int_0^\infty e^{-s}\, s^{-m-1}\, p(s^{-1/2}z, s^{-1}\,t)\, s^{-1+\varepsilon}\, ds ,$$

where $p(z,t)$ denotes the heat kernel associated to L at unit time. As is well-known, $p(z,t) = O\left((1 + |z|^2 + |t|)^{-N}\right)$, for every $N \geq 0$, and as a result

$$\begin{aligned} &(1+L)^{-\varepsilon}\,\delta_0(z,t) \\ &\qquad = \begin{cases} O\left((|z|^2 + |t|)^{-m-1+\varepsilon}\right), & \text{if } |z|^2 + |t| \leq 1 , \\ O\left((|z|^2 + |t|)^{-N}\right), & \text{if } |z|^2 + |t| > 1 , \text{ for all } N \geq 0 . \end{cases} \end{aligned}$$

From this it follows that $(1+L)^{-\varepsilon}\delta_0 \in L^p(\mathbb{H}_m)$, if $(-m-1+\varepsilon)\,p > -m-1$.

In order to verify the assumption in the corollary, we shall invoke the Gelfand transform $\mathcal{G}$ for the algebra of radial functions on $\mathbb{H}_m$ (compare [MRS1]). For $f \in L^1(\mathbb{H}_m)$ radial, we have

$$\mathcal{G}(h(L)f)(\lambda, n) = h((m + 2\,n)\,|\lambda|)\,\mathcal{G}f(\lambda, n)\,,$$

$\lambda \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$.

If χ_j, $j \in \mathbb{Z}$, denotes again our dyadic decomposition of unity on $\mathbb{R}^+$, we put

$$\varphi^\varepsilon_{k,j}(\lambda, n) := h((m + 2\,n)|\,\lambda|)\,\chi_{2k-j}(\varepsilon\,\lambda)\,\chi_j(m + 2\,n)\,,$$

for $j \geq 0$, $k \in \mathbb{Z}$, $\varepsilon \in \{-1, 1\}$. We also set

$$K^\varepsilon_{k,j} = \mathcal{G}^{-1}(\varphi^\varepsilon_{k,j})\,.$$

By [MRS1], since $\varphi^\varepsilon_{k,j}$ is smooth and supported away from the axes, one has $K^\varepsilon_{k,j} \in \mathcal{S}$. Observe also that

$$2^{k-1} \leq \sqrt{(m + 2\,n)\,|\lambda|} \leq 2^{k+1} \text{ on } \operatorname{supp} \varphi^\varepsilon_{k,j}\,, \tag{1.5}$$

so that $\varphi^\varepsilon_{k,j} = 0$, unless $2^k \geq N/4$. So, if we fix any $k_0 \gg 1$, we may choose N sufficiently large so that

$$M = \sum_{\substack{\varepsilon=\pm1\\ k\geq k_0\\ j\geq 0}} K^\varepsilon_{k,j}\,, \tag{1.6}$$

for instance in the sense of distributions.

The proof of the theorem is then reduced to showing the following

Proposition 1.3. *If $\alpha > m$, then*

$$\sum_{\substack{\varepsilon=\pm1\\ k\geq k_0\\ j\geq 0}} \|\tilde{\chi}_{B_2} K_{k,j}\|_{(p,p)} < \infty\,,$$

for every p, $1 < p < \infty$, where $\|K\|_{(p,p)}$ denotes the norm of the convolution operator $f \longmapsto f * K$ on $L^p(\mathbb{H}_m)$.

1.2. Formulas for $K_{k,j}$.

In order to compute $K^{\varepsilon}_{k,j}$, we first observe that

$$\varphi^{-1}_{k,j}(\lambda, n) = \varphi^{1}_{k,\varepsilon}(-\lambda, n)\,,$$

hence $K^{-1}_{k,j}(z,u) = K^{1}_{k,j}(z,-u)$ (compare [MRS2]). This allows us to reduce to the case $\varepsilon = 1$, and we shall from now on suppress the suffix ε, assuming that it is 1.

Next, observe that $\mathcal{G}(i\,\mathcal{L}\,T^{-1} f)(\lambda, n) = (m+2\,n)\,\mathcal{G}f(\lambda, n)$, if $\lambda > 0$. Therefore, by [St, Corollary 2.5],

$$\varphi(i\,\mathcal{L}\,T^{-1})\, f = \sum_n \varphi(m + 2\,n)\, f * \tilde{P}_n \ ,$$

for any bounded multiplier φ, if $\mathcal{G}f$ is supported in $\lambda > 0$, where $\tilde{P}_n = c_n\,\delta_0 + \text{p.v.}\,P_n$, and where P_n is the Calderón-Zygmund kernel

$$P_n(z,t) = 2^{m-1}\,\pi^{-m-1}\,(-1)^n\,\frac{(m+n)!}{n!} \cdot \frac{(|z|^2 - 4\,i\,t)^n}{(|z|^2 + 4\,i\,t)^{m+1+n}} \Big(1 + \frac{n}{m+n}\,\frac{|z|^2 + 4\,i\,t}{|z|^2 - 4\,i\,t}\Big)$$

(see [St, Lemma 2.1 and (2.25)]).

If we define "polar coordinates" by putting

$$r := (|z|^4 + 16\,t^2)^{1/4}\,, \qquad 4\,t + i\,|z|^2 =: r^2\,e^{i\theta/2}\,, \qquad 0 \le \theta < 2\pi\,,$$

then we have

$$P_n = c_m\,(Q_n - Q_{n-1})\,, \tag{1.7.a}$$

with

$$\begin{aligned} Q_n &:= \frac{(m+n)!}{n!}\,\frac{e^{i(m+2\,n+1)\theta/2}}{r^{2m+2}} \\ &= \frac{(m+n)!}{n!}\,\frac{(4\,t + i\,|z|^2)^n}{(4\,t - i\,|z|^2)^{m+n+1}} \\ &= i^{m+1}\,(-1)^n\,\frac{(m+n)!}{n!}\,\frac{(|z|^2 - 4\,i\,t)^n}{(|z|^2 + 4\,i\,t)^{m+1+n}}\,. \end{aligned} \tag{1.7.b}$$

Next, observe that

$$\begin{aligned}\varphi_{k,j}(\lambda,n) &:= \varphi^1_{k,j}(\lambda,n)\\ &= ((m+2\,n)\,|\lambda|)^{-\alpha/2}\,\chi_{2k-j}(\lambda)\,\chi_j(m+2\,n)\,e^{i\sqrt{(m+2n)|\lambda|}}\,,\end{aligned}$$

if k is sufficiently large (in that case we may in fact delete the factor $(1-\eta)\,(\cdot/N)$ in h), which we may assume. Putting

$$\tilde\chi(x) := x^{-\alpha/2}\,\chi(x)\,,$$

this may be written as

$$\varphi_{k,j}(\lambda,n) = 2^{-\alpha k}\,\tilde\chi_{2k-j}(\lambda)\,\tilde\chi_j(m+2\,n)\,e^{i\sqrt{(m+2n)|\lambda|}}\,.$$

Since $\tilde\chi$ is of similar type as χ, we shall again write χ in place of $\tilde\chi$, and then get, for k,j fixed,

$$\varphi_{k,j}(\lambda,n') = 2^{-\alpha k}\sum_n \chi_j(m+2\,n)\,\gamma_n(\lambda)\,\delta_n(n')\,,$$

where $\delta_n(n')=1$ if $n=n'$ and $\delta_n(n')=0$ otherwise, with

$$\gamma_n(\lambda) := \chi_{2k-j}(\lambda)\,e^{i\sqrt{(m+2n)|\lambda|}}\,.$$

This implies

$$\begin{aligned}f*K_{k,j} &= 2^{-\alpha k}\sum_n \chi_j(m+2\,n)\,\gamma_n(-i\,T)\,\delta_n\Big(\frac{i\,\mathcal{L}\,T^{-1}-m}{2}\Big)f\\ &= 2^{-\alpha k}\sum_n \chi_j(m+2\,n)\,\gamma_n(-i\,T)\,(f*\tilde P_n)\,.\end{aligned}$$

In the sequel, we shall often use the following abbreviation

$$\ell := 2\,k-j\,.$$

We put

$$\Phi_{\ell,n}(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\sqrt{(m+2n)\lambda}}\,\chi_\ell(\lambda)\,e^{i\lambda t}\,d\lambda\,.$$

Then, away from $z=0$, $K_{k,j}$ is given by

$$K_{k,j}(z,t) = 2^{-\alpha k} \sum_n \chi_j(m+2\,n) \int P_n(z,t-s)\, \Phi_{2k-j,n}(s)\, ds \tag{1.8}$$

(we do know that $K_{k,j} \in \mathcal{S}$, although this is not evident from this formula).

Putting $\lambda = 2^\ell\, x^2$ in the integral defining $\Phi_{\ell,n}$, we write

$$\Phi_{\ell,n}(t) = \frac{2^\ell}{\pi} \int_{-\infty}^{\infty} e^{i(\sqrt{(m+2n)\,2^\ell}x - 2^\ell t x^2)}\, \chi(x)\, x\, dx\,,$$

which shows that the asymptotics of $\Phi_{\ell,n}$ can be computed by the stationary phase method of

Lemma 1.4. *Let $f \in C_0^\infty(\mathbb{R})$ be supported in $[1/2,2]$. For every $N \in \mathbb{N}$ there exist functions $f_0,\dots,f_N \in C_0^\infty(\mathbb{R})$ supported in $[1/4,4]$ and $E_N \in C^\infty(\mathbb{R}^2)$, such that for $(a,b) \in \mathbb{R}^2$ with $|(a,b)|>1$*

$$\int_{-\infty}^{\infty} e^{i(ax-bx^2/2)} f(x)\, dx = e^{ia^2/(2b)} \sum_{\nu=0}^{N} b^{-1/2-\nu}\, f_\nu\Big(\frac{a}{b}\Big) + E_N(a,b)\,,$$

where E_N satisfies

$$E_N^{(\alpha)}(a,b) = O\left(|(a,b)|^{-N/2-1}\right),$$

for every $\alpha \in \mathbb{N}^2$.

PROOF. We have

$$\int_{-\infty}^{\infty} e^{i(ax-bx^2/2)}\, f(x)\, dx = e^{ia^2/(2b)} \int_{-\infty}^{\infty} e^{-ibx^2/2}\, f\Big(x+\frac{a}{b}\Big)\, dx\,.$$

Now, in the region where $1/4 \le a/b \le 4$, the result follows easily from the proof of [S2, Proposition 3, Chapter VIII] and the remarks in [S2, Chapter VIII, 1.3.4], since the critical point in the integral on the right is $x=0$. The functions f_ν do in fact arise as linear combinations of derivatives of f.

In the remaining region, the result is obtained by integrating by parts in the integral on the left.

We may apply the Lemma to $\Phi_{\ell,n}(t)$, since $\sqrt{(m+2\,n)\,2^\ell} \sim 2^k \gg 1$, and obtain

$$\begin{aligned}\Phi_{\ell,n}(t) = e^{i(m+2n)/(4t)} \sum_{\nu=0}^{N} (2^{\ell+1}\,t)^{-1/2-\nu}\,2^\ell\,f_\nu\Big(\sqrt{\frac{m+2\,n}{2^\ell}}\,\frac{1}{2\,t}\Big) \\ + 2^\ell\,E_N\Big(\sqrt{(m+2\,n)\,2^\ell}, 2^{\ell+1}\,t\Big)\,, \end{aligned} \tag{1.9}$$

with f_ν and E_N as in the lemma.

Put

$$a_{n,\ell} := \sqrt{(m+2\,n)\,2^\ell}\,.$$

Since $a_{n,\ell} \sim 2^k$, and since

$$(2^{\ell+1}\,t)^{-1/2-\nu} = a_{n,\ell}^{-1/2-\nu}\Big(\sqrt{\frac{m+2\,n}{2^\ell}}\,\frac{1}{2\,t}\Big)^{1/2+\nu}\,,$$

the ν-th term in (1.9) is of the form

$$c_\nu\,\tilde{f}_\nu\Big(\sqrt{\frac{m+2\,n}{2^\ell}}\,\frac{1}{2\,t}\Big)\,,$$

with $c_\nu = O\,(2^{-(1/2+\nu)k})$, $\tilde{f}_\nu(x) = x^{1/2+\nu}\,f_\nu(x)$.

Consequently, we may reduce in (1.8) to the case where $\Phi_{\ell,n}$ is either of the form

$$\Phi_{\ell,n}(t) = 2^\ell\,a_{n,\ell}^{-1/2-\nu}\,f\Big(\sqrt{\frac{m+2\,n}{2^\ell}}\,\frac{1}{4\,t}\Big)\,e^{i(m+2n)/(4t)} \tag{a}$$

with $f \in C_0^\infty(\mathbb{R})$ supported in [1/8,2] and $\nu \geq 0$, or of the form

$$\Phi_{\ell,n}(t) = 2^\ell\,E_N(a_{n,\ell}, 2^{\ell+1}t) \tag{b}$$

with E_N as in the lemma and N sufficiently large.

Case (b) is easily dealt with by the Marcinkiewicz multiplier theorem in [MRS1, Theorem 2.2].

If $\Phi_{\ell,n}$ is of the form (b), then its inverse Fourier transform $\eta_{\ell,n}$ is of the form

$$\eta_{\ell,n}(\lambda) = \Psi(a_{n,\ell}, 2^{-\ell}\,\lambda)\,,$$

where

$$\Psi(a,\lambda) := \int_{-\infty}^{\infty} E_N(a,t)\,e^{-i\lambda t}\,dt$$

is of class C^M for $M = [N/2] - 1$ and satisfies

$$\partial_a^\alpha \, \partial_\lambda^\beta \, \Psi(a, \lambda) = O\left(a^{-M+\beta} \, (1 + |\lambda|)^{-K}\right), \tag{1.10}$$

for every α, $K \in \mathbb{N}$ and every $\beta \leq M$, as can easily be seen by integration by parts.

Now, since our "original" function $\varphi_{\ell,n}$ had its Fourier transform supported where $\lambda \sim 2^\ell$, we may also localize the support of $\eta_{\ell,n}$ in this region. And, in the region where $n \sim 2^j$ and $\lambda \sim 2^\ell = 2^{2k-j}$, we see from (1.10) that $\eta_{\ell,n}(\lambda) = \psi(\sqrt{(m + 2\,n)\, 2^{2k-j}}, 2^{j-2k}\,\lambda)$ satisfies estimates of the form

$$|\partial_n^\alpha \, \partial_\lambda^\beta \, \eta_{\ell,n}(\lambda)| \leq C_{\alpha,\beta} \, 2^{-Mk+(\beta-\alpha)(j-k)} \leq C_{\alpha,\beta} \, 2^{-\alpha j} \, 2^{-\beta(2k-j)},$$

if $\alpha + \beta \leq M$.

Thus, if we define $K_{k,j}$ by (1.8), with $\Phi_{\ell,n}$ as in (b), but Fourier transform localized in $\lambda \sim 2^\ell$, and choose N sufficiently large, we see that for any $\alpha \geq 0$ in (1.8)

$$K := \sum_{\substack{k \geq k_0 \\ j \geq 0}} K_{k,j}$$

is a kernel whose Gelfand transform satisfies the multiplier condition in [MRS1, Theorem 2.2], and thus satisfies the kernel estimates of [MRS1, Theorem 3.1], for sufficiently many derivatives. But then one checks easily that the same is true of the truncated kernel $\tilde{\chi}_{B_2} K$, and consequently the operator $f \longmapsto f * (\tilde{\chi}_{B_2} K)$ is L^p-bounded for $1 < p < \infty$ by [MRS1, Theorem 4.4].

Moreover, since

$$a_{n,\ell}^{-1/2} \, \chi_j(m + 2\,n) = 2^{-k/2} \, \tilde{\chi}_j(m + 2\,n)$$

with $\tilde{\chi}(x) = x^{-1/2} \, \chi(x)$, by modifying χ we may assume that the factor $a_{n,\ell}$ in (a) equals 2^k. We thus find that, in order to prove Proposition 1.3, it suffices to prove the following

Proposition 1.5. *Suppose that $K_{k,j}$ is given by*

$$K_{k,j}(z, t) = 2^{-mk} \sum_n \chi_j(m + 2\,n) \int P_n\Big(z, \frac{t-s}{4}\Big) \, \Phi_{k,j,n}(s) \, ds$$

with

$$\Phi_{k,j,n}(t) := 2^{k/2+k-j} f\Big(\sqrt{\frac{m+2\,n}{2^{2k-j}}}\;\frac{1}{t}\Big)\, e^{i(m+2n)/t}\,,$$

$f \in C_0^\infty(\mathbb{R})$ *supported in* $[1/8,2]$, $k \geq 0$ *sufficiently large. Then*

$$\sum_{j\geq 0} \|K_{k,j}\|_{L^1(B_2)} = O\,(k^2)\,.$$

2. Integral formulas for $K_{k,j}$.

In order to sum the series for $K_{k,j}$ in Proposition 1.5, we first observe that $\chi_j(m+2\,n)\,\Phi_{k,j,n}(s) = 0$, unless $1/4 \leq 2^{k-j}\,s \leq 16$. Thus, if we choose $\tilde\chi \in C_0^\infty(\mathbb{R})$ such that $\tilde\chi(x) = 1$ for $1/4 \leq x \leq 16$ and $\operatorname{supp}\tilde\chi \subset [1/8, 32]$, then we may replace $\Phi_{k,j,n}(t)$ by $\tilde\chi(2^{k-j}\,t)\,\Phi_{k,j,n}(t)$.

Moreover, writing

$$f\Big(\sqrt{\frac{m+2\,n}{2^{2k-j}}}\;\frac{1}{t}\Big) = g\Big(\log\Big(\frac{m+2\,n}{2^{2k-j}}\,t^{-2}\Big)\Big)\,,$$

with g smooth on $[\log 1/16, \log 2] \subset [-\pi,\pi]$ and, say, supported in $]-\pi,\pi[$, and developping g into a Fourier series on $[-\pi,\pi]$, we see that

$$\tag{2.1} \begin{aligned} \Phi_{k,j,n}(t) &= \sum_{\nu\in\mathbb{Z}} a_\nu \Big(\frac{m+2\,n}{2^{2k-j}\,t^2}\Big)^{i\nu}\, 2^{3k/2-j}\,\tilde\chi(2^{k-j}t)\, e^{i(m+2n)/t} \\ &=: \sum_{\nu\in\mathbb{Z}} a_\nu\, \Phi_{k,j,n,\nu}(t)\,, \end{aligned}$$

where

$$\tag{2.2} a_\nu = O\,(|\nu|^{-N})\,, \qquad \text{for every } N \in \mathbb{N}\,.$$

We also put

$$K_{k,j,\nu} := 2^{-mk} \sum_n \chi_j(m+2\,n) \int P_n\Big(z, \frac{t-s}{4}\Big)\Phi_{k,j,n,\nu}(s)\,ds\,,$$

so that

$$K_{k,j} = \sum_\nu a_\nu\, K_{k,j,\nu}\;.$$

Writing
$$\tilde\chi_{(\nu)}(t) := t^{-2i\nu}\,\tilde\chi(t)\,,$$
we still have $\tilde\chi_{(\nu)} \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\tilde\chi_{(\nu)} \subset [1/8, 32]$.

Moreover,
$$\|\tilde\chi_{(\nu)}^{(\alpha)}\|_\infty = O\left((|\nu|+1)^\alpha\right), \qquad \alpha \in \mathbb{N}, \tag{2.3}$$
and
$$\Phi_{k,j,n,\nu}(t) = 2^{3k/2-j}\,2^{ij}\,(m+2\,n)^{i\nu}\,\tilde\chi_{(\nu)}(2^{k-j}\,t)\,e^{i(m+2n)/t}\,.$$
Consequently, by (1.7.b),
$$\begin{aligned}\sum_n \chi_j(m+2\,n)\,Q_n\Big(z,\frac{t-s}{4}\Big)\Phi_{k,j,n,\nu}(s) &= 2^{ij}\,2^{3k/2-j}\sum_n \chi_{(\nu),j}(m+2\,n)\,\frac{(m+n)!}{n!}\,e^{i(m+1+2n)(\theta'/2+1/s)}\\ &\qquad\cdot\frac{e^{-i/s}}{r'^{2m+2}}\,\tilde\chi_{(\nu)}(2^{k-j}\,s)\,,\end{aligned}$$
where r' and θ' are defined by
$$t - s + i\,|z|^2 = r'^2\,e^{i\theta'/2}\,,$$
and where $\chi_{(\nu),j}(x) = \chi_{(\nu)}(2^{-j}\,x)$, with
$$\chi_{(\nu)}(x) := x^{i\nu}\,\chi(x)\,.$$
Let us put
$$\zeta_{\nu,j}(\omega) := \sum_n \chi_{(\nu),j}(m+2\,n)\,\frac{(m+n)!}{n!}\,2^{-mj}\,e^{i(2n+m+1)\omega}\,.$$
For fixed ν, $\varrho_j := |\zeta_{\nu,j}|$ has the following properties, as can easily be seen by applying Poisson's summation formula:
$$\begin{aligned}&\text{i) } \varrho_j \text{ is } \pi\text{-periodic}\,,\\ &\text{ii) } \varrho_j(\omega) \le C_N\,\frac{2^j}{(1+2^j\,|\omega|)^N}\,, \qquad \text{for } |\omega| \le \frac{\pi}{2}\,,\end{aligned} \tag{2.4}$$

for every $N \in \mathbb{N}$, $j \in \mathbb{N}$. ii) means that $\zeta_{\nu,j}$ is essentially supported in $\{|\omega| \le 2^{-j}\}$, and implies that $\|\zeta_{\nu,j}\|_1$ is uniformly bounded in j. Notice also that the constants C_N in (2.4.ii) will grow with ν, however, only polynomially, namely

$$C_N = O\left((|\nu|+1)^{N+1}\right). \tag{2.5}$$

With $\zeta_{\nu,j}$ as above, we have

$$\begin{aligned}\sum_n \chi_j\,(m+2\,n)\, Q_n\Big(z, \frac{t-s}{4}\Big)\,\Phi_{k,j,n,\nu}(s)\\ = 2^{ij}\, 2^{3k/2+(m-1)j}\, \zeta_{\nu,j}\Big(\frac{\theta'}{2}+\frac{1}{s}\Big)\frac{e^{-i/s}}{r'^{2m+2}}\,\tilde{\chi}_{(\nu)}\,(2^{k-j}\, s)\,.\end{aligned}$$

And, since

$$\frac{\theta'}{2} = \arctan\Big(\frac{|z|^2}{t-s}\Big)\,,$$

where arctan denotes the branch of $\tan^{-1}$ taking values in $[0,\pi]$, we obtain

$$\begin{aligned}2^{-2mk}\sum_n \chi_j\,(m+2\,n)\int Q_n\Big(z, \frac{t-s}{4}\Big)\,\Phi_{k,j,n,\nu}(s)\,ds\\ = 2^{ij}\,2^{k/2}\,2^{(m-1)(j-k)}\\ \cdot\int \frac{\zeta_{\nu,j}\Big(\arctan\Big(\frac{|z|^2}{t-s}\Big)+\frac{1}{s}\Big)}{(|z|^4+(t-s)^2)^{m+1/2}}\, e^{-i/s}\,\tilde{\chi}_{(\nu)}\,(2^{k-j}s)\,ds\,.\end{aligned} \tag{2.6$'$}$$

Since $P_n = c_m(Q_n - Q_{n-1})$, this allows to establish an integral formula for $K_{k,j,\nu}$.

In order to simplify the notation, we shall do this only for the case $\nu = 0$. In fact, we shall see that the estimates of $K_{k,j,\nu}$ will only depend on the constants C_N in (2.4) for a finite number of N's and on the norms of a finite number of derivatives of $\tilde{\chi}_{(\nu)}$. Therefore, in view of (2.3) and (2.5), we shall get the same type of estimate for $\|K_{k,j,\nu}\|_1$ as for $\|K_{k,j,0}\|_1$, except possibly for a factor which grows like a power of $|\nu|+1$. But, because of (2.2), it will then be clear that

$$\|K_{k,j}\|_1 \le \sum_\nu |a_\nu|\,\|K_{k,j,\nu}\|_1\,,$$

which leads to an estimate of the same type as for $\|K_{k,j,0}\|_1$.

So, from now on we shall assume that $\Phi_{k,j,n} = \Phi_{k,j,n,0}$, *i.e.* that

$$\Phi_{k,j,n}(t) = 2^{3k/2-j}\,\chi(2^{k-j}\,t)\,e^{i(m+2n)/t}\,, \tag{2.7}$$

with $\chi \in C_0^\infty(\mathbb{R})$ supported in $[1/8, 32]$. Then, by (2.6′),

$$\begin{aligned} &2^{-mk}\sum_n \chi_j(m+2\,n)\int Q_n\Big(z,\frac{t-s}{4}\Big)\,\Phi_{k,j,n}(s)\,ds \\ &\qquad = 2^{ij}\,2^{k/2}\,2^{(m-1)(j-k)} \\ &\qquad\quad \cdot\int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big)+\dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\,e^{-i/s}\,\tilde{\chi}(2^{k-j}s)\,ds\,, \end{aligned} \tag{2.6}$$

where we have used the abbreviations $\zeta_j = \zeta_{0,j}$ and

$$R := |z|^2\,.$$

Now, observe that if we replace Q_n by Q_{n-1} in the left hand side of (2.6), we have to sum

$$\begin{aligned} &\sum_n \chi_j(m+2\,n)\,Q_{n-1}\Big(z,\frac{t-s}{4}\Big)\,\Phi_{k,j,n}(s) \\ &\quad = 2^{ij}\,2^{3k/2-j}\sum_n \chi_j(m+2\,n)\,\frac{(m+n-1)!}{(n-1)!}\,e^{i(m+1+2(n-1))(\theta'/2+1/s)} \\ &\qquad \cdot\frac{e^{i/s}}{r'^{2m+2}}\,\chi(2^{k-j}s)\,. \end{aligned}$$

Replacing $\chi_j(m+2\,n)$ in this sum by $\chi_j(m+2\,(n-1)) + 2^{-j}\,\tilde{\chi}_j(m+2\,(n-1))$, with

$$\begin{aligned} \tilde{\chi}_j(m+2\,(n-1)) &= 2^j\Big(\chi\Big(\frac{m+2\,n}{2^j}\Big)-\chi\Big(\frac{m+2\,n-2}{2^j}\Big)\Big) \\ &= -2\int_0^1 \chi'\Big(\frac{m+2\,n-2\,t}{2^j}\Big)\,dt \end{aligned}$$

having similar properties as χ_j, we find that

$$\sum_n \chi_j(m+2n)\, Q_{n-1}\Big(z, \frac{t-s}{4}\Big)\Phi_{k,j,n}(s)$$
$$= 2^{ij}\, 2^{k/2}\, 2^{(m-1)(j-k)}$$
$$\cdot \left(\int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big)+\dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\, e^{i/s}\, \chi(2^{k-j}s)\, ds \right.$$
$$\left. + 2^{-j} \int \frac{\tilde{\zeta}_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big)+\dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\, e^{i/s}\, \chi(2^{k-j}s)\, ds \right),$$

with the same ζ_j as in (2.6), and $\tilde{\zeta}_j$ of the same type as ζ_j.

Writing $K_{k,j}(R,t)$ instead of $K_{k,j}(z,t)$, we then get

$$K_{k,j}(R,t) = C_m\, 2^{ij}\, 2^{k/2}\, 2^{(m-1)(j-k)}$$
$$\cdot \left(\int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big)+\dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\, (e^{-i/s}-e^{i/s})\, \chi(2^{k-j}s)\, ds \right. \tag{2.8}$$
$$\left. + 2^{-j} \int \frac{\tilde{\zeta}_j\Big(\arctan\Big(\dfrac{R}{t-s}+\dfrac{1}{s}\Big)\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\, e^{i/s}\, \chi(2^{k-j}s)\, ds \right).$$

Formula (2.8) will be useful in the region where $R^2+(t-s)^2$ is large. To deal with the region where $R^2+(t-s)^2$ is small, we establish a second formula for $K_{k,j}$.

To this end, we put

$$R_n(z,t) := \frac{(m+n-1)!}{n!}\,\frac{(4\,t+i\,R)^n}{(4\,t-i\,R)^{m+n}} = \frac{(m+n-1)!}{n!}\,\frac{e^{i(m+2n)\theta/2}}{r^{2m}}\ ,$$

and observe that

$$\partial_t R_n = 4\,Q_{n-1} - 4\,Q_n\ ,$$

so that by (1.7)

$$P_n = -\frac{c_m}{4}\,\partial_t R_n\ . \tag{2.9}$$

Integrating by parts in the formula for $K_{k,j}$ in Proposition 1.5, we thus obtain

$$K_{k,j}(z,t) = -c_m\, 2^{-mk} \sum_n \chi_j\,(m+2\,n) \int R_n\Big(z, \frac{t-s}{4}\Big)\Phi'_{k,j,n}(s)\,ds\,.$$

And, one realizes easily that

$$\Phi'_{k,j,n} = 2^{\ell}\,\tilde{\Phi}_{k,j,n}\ ,$$

with $\tilde{\Phi}_{k,j,n}$ similar to $\Phi_{k,j,n}$ (only $\tilde{\chi}$ has to be modified in the definition of $\tilde{\Phi}_{k,j,n}$).

Arguing now similarly as before, we find that

$$\begin{aligned} K_{k,j}(R,t) &= C_m\, 2^{ij}\, 2^{k/2}\, 2^{(m-3)(j-k)} \\ &\quad \cdot \int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big) + \dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{m/2}}\,\chi(2^{k-j}s)\,ds\,, \end{aligned} \tag{2.10}$$

with functions ζ_j and χ similar as in (2.8), but not necessarily identical.

Notice that in passing from (2.8) to (2.10) we "gain" a factor $2^{2(k-j)}\,(R^2+(t-s)^2)^{1/2}$. In addition we should point out that the right-side of (2.8) contains factors of $e^{\pm i/s}$, which do not appear in (2.10); this is due to the extra factor $e^{i\theta/2}$ occuring in the formula (1.7.b) for Q_n, which is not present in the formula for R_n.

We shall now specialize these formulas in the cases $j \le k$ and $j > k$.

A) *The case* $j \le k+M$. Fix $M \in \mathbb{N}$ to be chosen later. If $j \le k+M$, then the variable s in (2.8) is of the order $2^{j-k} \le 2^M$, so that no cancellation in the factor $e^{-i/s} - e^{i/s}$ can be expected. We therefore estimate each of the terms appearing in (2.8) , which are all of similar type, seperately.

In order to exploit formulas (2.8) as well as (2.10), we choose a cut-off function $\varrho \in C_0^\infty(\mathbb{R})$ such that $\varrho(x) = 1$ for $|x| \le 2^{11}$, $\varrho(x) = 0$ for $|x| \ge 2^{12}$, and split $K_{k,j}$ into

$$\begin{aligned} &K_{k,j}(z,t) \\ &\qquad = 2^{-mk} \sum_n \chi_j\,(m+2\,n) \int (1-\varrho)\,(2^{4(k-j)}\,(R^2+(t-s)^2)) \end{aligned}$$

$$\cdot P_n\Big(z, \frac{t-s}{4}\Big)\, \Phi_{k,j,n}(s)\, ds$$
$$+\, 2^{-mk} \sum_n \chi_j(m + 2\,n) \int \varrho(2^{4(k-j)}\,(R^2 + (t-s)^2))$$
$$\cdot P_n\Big(z, \frac{t-s}{4}\Big)\Phi_{k,j,n}(s)\, ds\,.$$

Using (2.9), and performing an integration by parts in the second term, we then find that $K_{k,j}$ will be made up of a finite number of terms of the following types

$$\tilde F_{k,j}(z,t) := 2^{k/2} 2^{(m-1)(j-k)}$$
$$\cdot \int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big) + \dfrac{1}{s}\Big)}{(R^2 + (t-s)^2)^{(m+1)/2}} \tag{$\tilde{\text{A}}$.1}$$
$$\cdot\,(1-\varrho)\,(2^{4(k-j)}\,(R^2 + (t-s)^2))\, e^{i/s}\, \chi(2^{k-j}s)\, ds\,,$$

the complex conjugate of $\tilde F_{k,j}$,

$$\tilde G_{k,j}(z,t) := 2^{k/2}\, 2^{(m-3)(j-k)}$$
$$\cdot \int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big) + \dfrac{1}{s}\Big)}{(R^2 + (t-s)^2)^{m/2}} \tag{$\tilde{\text{A}}$.2}$$
$$\cdot\, \varrho(2^{4(k-j)}(R^2 + (t-s)^2))\, \chi(2^{k-j}s)\, ds\,,$$

and

$$\tilde H_{k,j}(z,t) := 2^{k/2} 2^{(m-5)(j-k)-j}$$
$$\cdot \int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big) + \dfrac{1}{s}\Big)}{(R^2 + (t-s)^2)^{m/2}}(t-s) \tag{$\tilde{\text{A}}$.3}$$
$$\cdot\, \varrho'(2^{4(k-j)}\,(R^2 + (t-s)^2))\, \chi(2^{k-j}s)\, ds\,.$$

Notice also that $\zeta_j(\omega)\, e^{-i\omega} =: \tilde\zeta_j(\omega)$ is a function of the same type as ζ_j, and that

$$\zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{1}{s}\Big) e^{i/s}$$
$$= \tilde\zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{1}{s}\Big) \frac{t-s-i\,R}{(R^2 + (t-s)^2)^{1/2}}\,.$$

This shows that we may put $\tilde{F}_{k,j}$ also into the form

$$\tilde{F}_{k,j}(z,t) = 2^{k/2}\,2^{(m-1)(j-k)}$$

$$(\tilde{\mathrm{A}}.1') \qquad \cdot \int \tilde{\zeta}_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{1}{s}\Big)\frac{t-s-i\,R}{(R^2+(t-s)^2)^{(m+2)/2}}$$

$$\cdot\,(1-\varrho)\,(2^{4(k-j)}\,(R^2+(t-s)^2))\,\chi(2^{k-j}s)\,ds\,.$$

Now observe that there is some $A \in \mathbb{N}$ such that

$$B_2 \subset \{(z,t) \in \mathbb{H}_m :\ |z|^2 \le 2^A,\ |t| \le 2^A\} =: Q\,.$$

This is clear since $|(z,t)| := (\max\{|z|^2, |t|\})^{1/2}$ is a homogeneous norm on $\mathbb{H}_m$, hence equivalent to the optimal control norm. Thus $\|f\|_{L^1(B_2)} \le \|f\|_{L^1(Q)}$.

Moreover, replacing R by $2^{j-k}R$, t by $2^{j-k}t$ and s by $2^{j-k}s$, we see that $\|\tilde{F}_{k,j}\|_{L^1(B_2)} \le \|F_{k,j}\|_{L^1(2^{k-j}Q,dRdt)}$, with

$$F_{k,j}(R,t) := 2^{k/2+m(j-k)}$$

$$(\mathrm{A}.1) \qquad \cdot \int \zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\Big)\frac{R^{m-1}(t-s-i\,R)}{(R^2+(t-s)^2)^{(m+2)/2}}$$

$$\cdot\,(1-\varrho)\,(2^{2(k-j)}\,(R^2+(t-s)^2))\,\chi(s)\,ds\,.$$

Similarly, instead of estimating the $L^1(B_2)$-norms of $\tilde{G}_{k,j}$ and $\tilde{H}_{k,j}$, we may estimate the $L^1(2^{k-j}Q)$-norms of $G_{k,j}$ and $H_{k,j}$, defined by

$$G_{k,j}(R,t) := 2^{k/2}\,2^{(m-1)(j-k)}$$

$$(\mathrm{A}.2) \qquad \cdot \int \zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\Big)\frac{R^{m-1}}{(R^2+(t-s)^2)^{m/2}}$$

$$\cdot\,\varrho(2^{2(k-j)}(R^2+(t-s)^2))\,\chi(s)\,ds\,,$$

$$H_{k,j}(R,t) := 2^{k/2}\,2^{(m-2)(j-k)-j}$$

$$(\mathrm{A}.3) \qquad \cdot \int \zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\Big)\frac{(t-s)R^{m-1}}{(R^2+(t-s)^2)^{m/2}}$$

$$\cdot\,\varrho'(2^{2(k-j)}\,(R^2+(t-s)^2))\,\chi(s)\,ds\,.$$

Notice that we are interested in these functions in the region

$$(2.11) \qquad 0 \le R \le 2^{k-j+A}\,, \qquad |t| \le 2^{k-j+A}\,.$$

By $\|f\|$ we shall denote the L^1-norm of f restricted to this domain.

B) *The case* $j > k+M$. In this case, we have $s \leq 2^{j-k+5}$ in the integral for $\tilde{F}_{k,j}$, and so, if $(R,t) \in Q$, then

$$2^{4(k-j)}\,(R^2+(t-s)^2) \leq C\,2^{2(k-j)} \leq 1\,,$$

if M is choosen sufficiently large. Thus, in this case $\tilde{F}_{k,j} = 0$, and consequently we shall here entirely make use of formula (2.10).

After scaling, we are thus lead to estimating the L^1-norm of

$$G_{k,j}(R,t) := 2^{k/2}\,2^{(m-1)(j-k)}$$
$$\text{(B)} \qquad \cdot \int \zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big)+\frac{2^{k-j}}{s}\Big)\frac{R^{m-1}}{(R^2+(t-s)^2)^{m/2}}\,\chi(s)\,ds$$

on the region given by (2.11).

3. The change of coordinates.

In the estimates to come, the following change of coordinates turns out to be useful

$$\text{(3.1)} \qquad x := \frac{R}{t-s}\,, \qquad y := \frac{R^2+(t-s)^2}{t-s}\,, \qquad s := s\,,$$

with inverse transformation

$$\text{(3.2)} \qquad t = \frac{y}{\langle x\rangle^2}+s\,, \qquad R = \frac{x\,y}{\langle x\rangle^2}\,, \qquad s = s\,,$$

where we have put

$$\langle x\rangle := (1+x^2)^{1/2}\,.$$

Then one verifies easily the following formulas

$$\text{(3.3)} \qquad dR\,dt\,ds = \frac{|y|}{\langle x\rangle^4}\,dy\,ds\,dx\,,$$

$$\text{(3.4)} \qquad R^2+(t-s)^2 = \frac{y^2}{\langle x\rangle^2}\,, \qquad t-s = \frac{y}{\langle x\rangle^2}\,.$$

Put

$$\psi_{k,j}(x,s) := \left|\zeta_j\Big(\arctan x + \frac{2^{k-j}}{s}\Big)\right|.$$

Let $A \in \mathbb{N}$ be as in (2.11), and fix $M \in \mathbb{N}$ such that $M \geq A + 20$.

We shall frequently make use of the following

Lemma 3.1. a) *If* $j \leq k + M$, *then*

$$\int \psi_{k,j}(x,s)\,|\chi(s)|\,ds \leq C\,,$$

with C *independent of* j, k *and* x.

b) *If* $j \geq k + M$, *then*

$$\int_{|x| \leq 2^{10+A+k-j}} \psi_{k,j}(x,s)\,dx \leq C\,,$$

with C *independent of* k, j *and* s.

Proof. Put $u := \arctan x \in [0, \pi]$. Since $\operatorname{supp}\chi \subset [1/8, 32]$, we have

$$\int \left|\zeta_j\Big(u + \frac{2^{k-j}}{s}\Big)\chi(s)\right| ds \leq C_1\, 2^{j-k} \int_{2^{k-j-4}}^{2^{k-j+3}} |\zeta_j(u+s)|\,ds\,.$$

And, if $k - j \geq -M$, then it follows easily from (2.4) that

$$\int_{2^{k-j-4}}^{2^{k-j+3}} |\zeta_j(u+s)|\,ds \leq C_2\, 2^{k-j}\,.$$

This covers parts a) of the lemma.

As for b), assume now that $16 + A + k - j \leq 10 + A - M \leq -10$, and $|x| \leq 2^{10+A+k-j} =: L \ll 1$.

Since $\arctan(-x) = \pi - \arctan x$, and since $|\zeta_j|$ is π-periodic, we have

$$\int_{-L}^{0} \psi_{k,j}(x,s)\,dx = \int_0^L \left|\zeta_j\Big(-\arctan x + \frac{2^{k-j}}{s}\Big)\right| dx\,.$$

Thus, choosing $u = \tan^{-1} x$ with $|u| \leq \tan^{-1}(L) \leq L$ as variable of integration in place of x, where $\tan^{-1}$ denotes the branch with values in $[-\pi/2, \pi/2]$, then we get

$$\int_{|x| \leq L} \psi_{k,j}(x,s)\,dx \leq C' \int_{|u| \leq L} \left|\zeta_j\Big(u + \frac{2^{k-j}}{s}\Big)\right| du \leq C\,,$$

again by (2.4).

4. Estimates for $j \leq k + M$.

4.1. Estimation of $H_{k,j}$.

Since $\tau \sim 1$ for $\tau \in \operatorname{supp} \varrho'$, we get by (A.3) and Lemma 3.1 that

$$
\begin{aligned}
\|H_{k,j}\| &\leq C\, 2^{k/2+(m-2)(j-k)-j} \\
&\quad \cdot \int_{y \sim 2^{j-k}\langle x\rangle} |\psi_{k,j}(x,s)| \left| \frac{\frac{y}{\langle x\rangle^2}\left(\frac{x\,y}{\langle x\rangle^2}\right)^{m-1}}{\left(\frac{y}{\langle x\rangle}\right)^m} \right| \frac{|y|}{\langle x\rangle^4}\, dy\, ds\, dx \\
&\leq C\, 2^{k/2+m(j-k)-j} \int \frac{|x|^{m-1}}{\langle x\rangle^{m+2}}\, dx \\
&\leq C\, 2^{k/2-mk+(m-1)j}\,.
\end{aligned}
$$

Consequently,

$$\sum_{j \leq k+M} \|H_{k,j}\|_1 \leq C\, 2^{-k/2}\,. \tag{4.1}$$

REMARK. In the above estimate, we did not make use of the condition (2.11). Notice that in the (x,y,s)-coordinates, this condition is equivalent to

$$\frac{|y|}{\langle x\rangle} \leq C\, 2^{k-j}\,. \tag{4.2}$$

This condition will be of importance in the estimations of $\|F_{k,j}\|$ and $\|G_{k,j}\|$.

In fact, arguing similarly as for $H_{k,j}$ and using (4.2), one finds that

$$\|F_{k,j}\| \leq C\,((k-j)+M)\, 2^{k/2+m(j-k)}\,, \qquad \|G_{k,j}\| \leq C\, 2^{k/2+m(j-k)}\,.$$

Thus, if one choose any $\varepsilon > 0$, one finds that

$$\sum_{j<(1-(\varepsilon+1/2)/m)k} (\|F_{k,j}\| + \|G_{k,j}\|) \leq C_\varepsilon\,. \tag{4.3}$$

In order to deal with the remaining values of $j \leq k+M$, we shall have to perform another integration by parts.

Notice, however, that by (4.3) we may from now on assume that j is sufficiently large, so that $2^{-j} \ll 1$.

4.2. Estimation of $F_{k,j}$.

For R and t fixed, let us put

$$a(s) := (R^2 + (t-s)^2)^{1/2} = |t - s - i\,R|\,.$$

Notice that (4.2) just means that

$$a(s) \leq C\,2^{k-j}\,, \tag{4.2'}$$

and that for s in the support of the integrand of (A.1) we have $a(s) \geq 2^{j-k}$, so that

$$2^{j-k} \leq a(s) \leq C\,2^{k-j}\,. \tag{4.4}$$

In the discussion to follow, we shall always assume that the estimates we shall establish are valid for s in the support of the integrals under examination without further mentioning.

We shall say that a function h of R, t and s is a *symbol of order* β, if it satisfies estimates of the form

$$|\partial_s^{(j)} h| \leq C_j\, a^{\beta-j}\,, \qquad j \in \mathbb{N},$$

at least for $j = 0,1,2$, where C_j is independent of R, t and s. Evidently a is a symbol of order 1. Similarly, $\arctan(R/(t-s))$ is a symbol of order 0. For R and t fixed, let us write

$$\varphi(s) := \arctan\Big(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\,,$$

$$\kappa(s) := \frac{t-s-i\,R}{(R^2+(t-s)^2)^{(m+2)/2}}\,(1-\varrho)\,((2^{k-j}\,a(s))^2)\,.$$

One checks easily that κ is a symbol of order $-m-1$. We may then write

$$F_{k,j}(R,t) = 2^{k/2+m(j-k)}\,R^{m-1}\int \zeta_j \circ \varphi(s)\,\kappa(s)\,\chi(s)\,ds\,. \tag{4.5}$$

In the latter integral, we can perform an integration by parts, if we observe that

$$\zeta_j = -2^{-j}\, \tilde{\zeta}_j' \,, \tag{4.6}$$

where $\tilde{\zeta}_j$ is a function of the same type as ζ_j, so that $\varrho_j = |\tilde{\zeta}_j|$ satisfies in particular (2.4). (4.6) can in fact easily be obtained by going back to the definition of ζ_j. Since

$$\zeta_j(\varphi(s)) = -2^{-j}\, \frac{d}{ds}\, (\tilde{\zeta}_j \circ \varphi)(s)\, \frac{1}{\varphi'(s)} \,,$$

we may write $F_{k,j}$ also in the form

$$F_{k,j}(R,t) = 2^{k/2+m(j-k)}\, R^{m-1} \int \tilde{\zeta}_j \circ \varphi(s)\, 2^{-j} \Big(\frac{\kappa\,\chi}{\varphi'}\Big)'(s)\, ds \,. \tag{4.7}$$

Since

$$\frac{2^{-j}\Big(\dfrac{\kappa\,\chi}{\varphi'}\Big)'}{\kappa\,\chi} = \frac{2^{-j}\,\kappa'}{\varphi'\,\kappa} + \frac{2^{-j}\,\chi'}{\varphi'\,\chi} - 2^{-j}\,\frac{\varphi''}{\varphi'^2} \,,$$

we shall gain by the integration by parts if these terms are bounded, say by $1/3$. Now, if $\tilde{\kappa}$ is formed as κ, only with ρ replaced by a function $\tilde{\rho}$ of slightly smaller support, then

$$\Big|\frac{\kappa'}{\varphi'\,\tilde{\kappa}}\Big| \le C\, \frac{a^{-1}}{|\varphi'|} \,, \tag{4.8.a}$$

and similarly, since $\arctan\,(R/(t-s))$ is a symbol of order 0,

$$|\varphi''| \le C'\Big(a^{-2} + \frac{2^{k-j}}{s^3}\Big) \le C\,(2^{(k-j)/2} + a^{-1})^2 \,,$$

hence

$$\Big|\frac{\varphi''}{\varphi'^2}\Big| \le C\,\Big(\frac{2^{(k-j)/2} + a^{-1}}{\varphi'}\Big)^2 . \tag{4.8.b}$$

Finally, if $\tilde{\chi}$ is similar to χ, only with a slightly larger support, then

$$\Big|\frac{\chi'}{\varphi'\,\tilde{\chi}}\Big| \le C\, \frac{1}{|\varphi'|} \,. \tag{4.8.c}$$

The natural condition in order to gain by the integration by parts is thus

$$\sigma := 2^{-j}\Big(\frac{2^{(k-j)/2}+a^{-1}}{\varphi'}\Big)^2 \leq 1\,. \tag{4.9}$$

Under this condition, we get

$$\Big|2^{-j}\Big(\frac{\kappa\,\chi}{\varphi'}\Big)'\Big| \leq C\,|(\sigma + (2^{-j}\,\sigma)^{1/2})\,\tilde\kappa\,\tilde\chi|\,. \tag{4.10}$$

In fact, we have

$$2^{-j}\,\sigma \geq \Big(\frac{2^{-j}\,a^{-1}}{\varphi'}\Big)^2\,, \tag{4.11}$$

so that

$$\Big|2^{-j}\,\frac{\kappa'}{\varphi'\,\tilde\kappa}\Big| \leq C\,(2^{-j}\,\sigma)^{1/2}\,.$$

Moreover,

$$\varphi'(s) = \frac{R}{R^2+(t-s)^2} - \frac{2^{k-j}}{s^2}\,,$$

so that by (4.4)

$$|\varphi'(s)| \leq C'\,(a^{-1}(s) + 2^{k-j}) \leq C\,2^{k-j}\,.$$

But then

$$\sigma \geq \frac{2^{k-2j}}{\varphi'^2} \geq C\,\frac{2^{-j}}{|\varphi'|}\,,$$

so that

$$\Big|2^{-j}\,\frac{\chi'}{\varphi'\,\tilde\chi}\Big| \leq C\,\sigma\,.$$

In order to simplify the notation, we shall often replace again $\tilde\kappa$ by κ and $\tilde\chi$ by χ in the estimates to follow.

Let us now express the relevant functions in the coordinates (x, y, s) of Section 3. Some easy computations based on (3.4) yield

$$a(s) = \frac{|y|}{\langle x\rangle}\,,$$

$$\varphi(s) = \arctan x + \frac{2^{k-j}}{s}\,,$$

$$\varphi'(s) = \frac{x}{y} - \frac{2^{k-j}}{s^2}\,,$$

$$R^{m-1}\,\kappa(s) \le C\,\frac{|x|^{m-1}}{\langle x\rangle^{m-3}}\,\frac{1}{y^2}\,(1-\varrho)\Big(\Big(2^{k-j}\,\frac{y}{\langle x\rangle}\Big)^2\Big)\,, \tag{4.12}$$

$$\begin{aligned}\sigma(s) &= 2^{-j}\left(\frac{2^{(k-j)/2} + \dfrac{\langle x\rangle}{|y|}}{\dfrac{x}{y} - \dfrac{2^{k-j}}{s^2}}\right)^2\\ &= s^4\Big(2^{-j/2}\,\frac{2^{(j-k)/2}\,|y| + 2^{j-k}\,\langle x\rangle}{y - 2^{j-k}s^2\,x}\Big)^2.\end{aligned}$$

Observe now that, due to the choice of ϱ, $|y| \ge 2\cdot 2^{j-k}s^2\langle x\rangle$ for any s with $(\chi\,\kappa)(s) \neq 0$, so that

$$\sigma(s) \sim 2^{-j}\Big(2^{(j-k)/2} + \frac{2^{k-j}\,\langle x\rangle}{|y|}\Big)^2 \le C\,2^{-j} \ll 1\,, \tag{4.13}$$

hence

$$\sigma(s) + (2^{-j}\,\sigma(s)^{1/2}) \le C\,2^{-j}\,. \tag{4.14}$$

We shall therefore estimate $\|F_{k,j}\|$ by means of formula (4.7), which, in combination with (4.10), (4.12), (4.14) and (4.4) yields

$$\begin{aligned}\|F_{k,j}\| &\le C\,2^{k/2+m(j-k)}\\ &\qquad\cdot\int_{2^{j-k}\le a\le C2^{k-j}} \psi_{k,j}(x,s)\,R^{m-1}\,|(\sigma + (2^{-j}\,\sigma)^{1/2})\,\kappa\,\chi|(s)\,ds\\ &\le C\,2^{k/2+m(j-k)-j}\\ &\qquad\cdot\int_{2^{j-k}\langle x\rangle\le|y|\le C2^{k-j}} \psi_{k,j}(x,s)\,\frac{|x|^{m-1}}{\langle x\rangle^{m+1}}\,\frac{1}{|y|}\,dy\,ds\,dx\\ &\le C\,2^{k/2+m(j-k)-j}\,k\,,\end{aligned}$$

again by Lemma 3.1.a). This implies

$$\sum_{j \le k+M} \|F_{k,j}\| \le C\, k^2\, 2^{-k/2}\,. \tag{4.15}$$

4.3. Estimation of $G_{k,j}$.

We shall proceed similarly as in the preceding section. We define $\varphi(s)$ and $a(s)$ as before, only κ has to be replaced here by

$$\kappa(s) := (R^2 + (t-s)^2)^{-m/2}\, \varrho(2^{2(k-j)}\,(R^2 + (t-s)^2))\,,$$

so that now κ is a symbol of order $-m$. Notice also that for $\kappa(s) \neq 0$ we have

$$a(s) \le C\, 2^{j-k} \tag{4.16}$$

in place of (4.4).

Then

$$G_{k,j}(R,t) = 2^{k/2+(m-1)(j-k)}\, R^{m-1} \int \zeta_j \circ \varphi(s)\, \kappa(s)\, \chi(s)\, ds\,.$$

We may perform an integration by parts as in the preceding section, and the gain by this can be estimated by the same function σ defined in (4.9). However, here we may have $\sigma \gg 1$. Therefore we fix a cut-off function ϱ_1 supported in $|x| \le 2$ and with $\varrho_1(x) = 1$ for $|x| \le 1$, and write

$$\begin{aligned} G_{k,j}(R,t) = 2^{k/2+(m-1)(j-k)}\, R^{m-1} \\ \cdot \Big(\int \zeta_j \circ \varphi(s)\, \kappa(s)\, \chi(s)\, \varrho_1(\sigma(s))\, ds \\ + \int \zeta_j \circ \varphi(s)\, \kappa(s)\, \chi(s)\, (1-\varrho_1)\,(\sigma(s))\, ds \Big)\,. \end{aligned}$$

Performing the integration by parts in the first integral, we find that

$$|G_{k,j}| \le C\,(G^1_{k,j} + G^2_{k,j} + G^3_{k,j})\,, \tag{4.17}$$

with

$$G^1_{k,j}(R,t) := 2^{k/2+(m-1)(j-k)-j}\, R^{m-1} \int_{1\le\sigma\le 2} \Big|\tilde\zeta_j \circ \varphi(s)\, \frac{\kappa\,\chi}{\varphi'}(s)\, \sigma'(s)\Big|\, ds\,,$$

$$G^2_{k,j}(R,t) := 2^{k/2+(m-1)(j-k)}\, R^{m-1} \int_{\sigma\le 2} |\tilde\zeta_j \circ \varphi(s)\, ((\sigma + (2^{-j}\sigma)^{1/2})\, \kappa\,\chi)(s)|\, ds\,,$$

$$G^3_{k,j}(R,t) := 2^{k/2+(m-1)(j-k)}\, R^{m-1} \int_{\sigma\ge 1} |\zeta_j \circ \varphi(s)\, (\kappa\,\chi)(s)|\, ds\,.$$

Notice that in the second term we have already estimated $|2^{-j}\,(\kappa\,\chi/\varphi)'|$ by $C\,|(\sigma + (2^{-j}\,\sigma)^{1/2})\,\tilde\kappa\,\tilde\chi|$. This is justified, since (4.16) remains valid here – the only property of κ made use of here is that $|\kappa'/\tilde\kappa| \le C\,a^{-1}$.

If one expresses the functions arising in these integrals in the (x,y,s)-coordinates, formulas (4.12) remain the same except for the estimate for $R^{m-1}\kappa(s)$, which here is to be replaced by

$$R^{m-1}\,\kappa(s) \le C\,\frac{|x|^{m-1}}{\langle x\rangle^{m-2}}\,\frac{1}{|y|}\,\varrho\Big(\Big(2^{k-j}\,\frac{y}{\langle x\rangle}\Big)^2\Big)\,. \tag{4.12$'$}$$

Observe also that

$$|\sigma'| \le C\,2^j\,|\varphi'|\,, \qquad \text{if } \sigma(s)\sim 1\,. \tag{4.18}$$

In fact, if $\sigma(s)\sim 1$, then by (4.8.b)

$$\begin{aligned}
|\sigma'| &\le C\,2^{-j/2}\,|\sigma|^{1/2}\Big(\frac{a^{-2}}{|\varphi'|} + (2^{(k-j)/2} + a^{-1})\,\Big|\frac{\varphi''}{\varphi'^2}\Big|\Big)\\
&\le C\Big(\Big(2^{-j}\,\frac{a^{-2}}{\varphi'^2}\Big)2^{j/2} + 2^{-j/2}\Big(\frac{2^{(k-j)/2} + a^{-1}}{|\varphi'|}\Big)^3\Big)|\varphi'|\\
&\le C\,(2^{j/2}\,\sigma + 2^j\,\sigma^{3/2})\,|\varphi'|\,.
\end{aligned}$$

And, since $|y| \le C\,2^{j-k}\,\langle x\rangle$, by (4.16), we see that

$$\sigma(s) \sim \Big(\frac{2^{-j/2+j-k}\,\langle x\rangle}{y - 2^{j-k}\,s^2\,x}\Big)^2\,. \tag{4.19}$$

This implies

$$|y - 2^{j-k}\, s^2\, x| \le C\, 2^{-j/2+j-k}\, \langle x\rangle\,, \qquad \text{if } \sigma(s) \ge 1\,. \tag{4.20}$$

Now, observe that, due to (4.18), we have $|G^1_{k,j}| \le C\, |G^3_{k,j}|$. Therefore

$$\|G^1_{k,j}\| + \|G^3_{k,j}\| \le C\, 2^{k/2+(m-1)(j-k)} \cdot \int_{\sigma\ge 1, s\sim 1} \psi_{k,j}(x,s)\, \frac{|x|^{m-1}}{\langle x\rangle^{m-1}}\, \frac{1}{|y|}\, \frac{|y|}{\langle x\rangle^4}\, dy\, ds\, dx\,, \tag{4.21}$$

which, by (4.20) and Lemma 3.1.a), can be estimated by

$$C\, 2^{k/2+(m-1)(j-k)-j/2+j-k} \int \frac{|x|^{m-1}}{\langle x\rangle^{m+1}}\, dx\,.$$

This yields

$$\|G^1_{k,j}\| + \|G^3_{k,j}\| \le C\, 2^{(m-1/2)(j-k)}\,. \tag{4.22}$$

Finally, putting $B := 2^{-j/2+j-k}\langle x\rangle$, by (4.19) we have

$$\sigma(s) \sim \Big(\frac{B}{y - 2^{j-k}\, s^2\, x}\Big)^2\,.$$

Thus, if $\sigma(s) \le C^2$, then this and (4.16) imply

$$\frac{B}{C} \le |y - 2^{j-k}\, s^2\, x| \le C'\, 2^{j-k}\, \langle x\rangle\,,$$

and then

$$\begin{aligned}
\int_{\sigma\le 2, a\le C2^{j-k}} |\sigma|\, dy
&\le \sqrt{2} \int_{\sigma\le 2, a\le C2^{j-k}} |\sigma|^{1/2}\, dy \\
&\le \sqrt{2} \int_{B/C\le |y-2^{j-k}s^2x|\le C' 2^{j-k}\langle x\rangle} \frac{B}{|y - 2^{j-k}\, s^2\, x|}\, dy \\
&\le CB \log\Big(\frac{c\, 2^{j-k}\, \langle x\rangle}{B}\Big) \\
&\le C\, 2^{-j/2+j-k}\, j\, \langle x\rangle\,.
\end{aligned}$$

Similarly,

$$\int_{\sigma \leq 2, a \leq C 2^{j-k}} |2^{-j}\,\sigma|^{1/2}\,dy \leq C\,2^{-j/2}\,2^{-j/2+j-k}\,j\,\langle x\rangle\,.$$

This implies

$$\begin{aligned} \|G^2_{k,j}\| &\leq C\,2^{k/2+(m-1)(j-k)} \\ &\quad\cdot \int_{s\sim 1}\int_{\sigma\leq 2, a\leq 2^{j-k}} \psi_{k,j}(x,s)\,(|\sigma(x,y,s)| + |2^{-j}\,\sigma(x,y,s)|^{1/2}) \\ &\qquad\qquad \cdot dy\,\frac{|x|^{m-1}}{\langle x\rangle^{m+2}}\,dx\,ds \\ &\leq C\,k\,2^{(m-1/2)(j-k)} \int \frac{|x|^{m-1}}{\langle x\rangle^{m+1}}\,dx \\ &\leq C\,k\,2^{(m-1/2)(j-k)}\,. \end{aligned}$$

In combination with (4.22) and (4.17) we thus find

$$\sum_{j\leq k+M} \|G_{k,j}\| \leq C\,k\,.$$

Put together, the estimates of this section yield

$$\sum_{j\leq k+m} \|K_{k,j}\|_{L^1(B_2)} \leq C\,k\,. \tag{4.23}$$

5. Estimates for $j > k + M$.

In order to estimate the norm of $G_{k,j}$, now given by (B), we follow the same scheme as in the preceding Section 4.3 and split G_{kj} as in (4.17) by performing an integration by parts on the region where $\sigma \leq C$. Notice, however, the following differences compared to the case $j \leq k + M$:

Firstly, since $1/8 \leq s \leq 32$ in (B), by (2.11) we have

$$2^{-5} \leq a(s) \leq 2^6 \tag{5.1}$$

in place of (4.16). Moreover, since

$$\varphi'(s) = \frac{R}{R^2 + (t-s)^2} - \frac{2^{k-j}}{s^2}\,, \qquad \varphi''(s) = \frac{R\,(t-s)}{(R^2+(t-s)^2)^2} + \frac{2^{k-j+1}}{s^3}\,,$$

by (2.11) and (5.1) we now have

$$\Big|\frac{\varphi''}{\varphi'^2}\Big| \le C\,\frac{2^{k-j}}{\varphi'^2}\,, \qquad |\varphi'| \le C\,2^{k-j}\,,$$

so that

$$2^{-j}\Big(\Big|\frac{\kappa'}{\varphi'\,\tilde\kappa}\Big| + \Big|\frac{\chi'}{\varphi'\,\tilde\chi}\Big| + \Big|\frac{\varphi''}{\varphi'^2}\Big|\Big) \le C\,\frac{2^{k-2j}}{\varphi'^2}\,. \tag{5.2}$$

We shall therefore put

$$\sigma := \frac{2^{k-2j}}{\varphi'^2}$$

here. Then (4.10) remains valid.

With this function σ, and with $\kappa = (R^2+(t-s)^2)^{-m/2} \sim 1$ here, we may define $G^\ell_{k,j}$, $\ell = 1,2,3$, as before, where, because of (5.2), we may even assume that $G^2_{k,j}$ is given by

$$2^{k/2+(m-1)(j-k)}\,R^{m-1}\int_{\sigma\le 2} |\zeta_j\circ\varphi(s)\,(\sigma\,\kappa\,\chi)(s)|\,ds\,.$$

Then (4.17) remains true.

Since

$$|\sigma'| = \Big|\frac{2^{k-2j+1}}{\varphi'^3}\,\varphi''\Big| \le \Big|\frac{2^{2k-3j}}{\varphi'^3}\Big| \le C\,2^{k/2}\,\sigma^{3/2} \le C\,2^j\,|\varphi'|\,,$$

if $\sigma \sim 1$, also (4.18) remains true, so that again $|G^1_{k,j}| \le C\,|G^3_{k,j}|$. We thus only have to estimate $\|G^3_{k,j}\|$ and $\|G^2_{k,j}\|$.

Now, by (5.1),

$$2^{-5}\langle x\rangle \le |y| \le 2^6\,\langle x\rangle\,.$$

Given this, (2.11) implies $|x/\langle x\rangle| \le 2^{5+A+k-j}$, hence

$$|x| \le 2^{10+A+k-j} \ll 1\,, \tag{5.3}$$

as well as

$$\Big|\frac{y}{\langle x\rangle^2}+s\Big| \le 2^{A+k-j}\,,$$

hence

$$|y+s\,\langle x\rangle^2| \le C\,2^{k-j}\,. \tag{5.4}$$

In particular, we find that

$$|y|\sim\langle x\rangle\sim 1\,. \tag{5.5}$$

In view of the definition of σ, this implies that, in place of (4.19),

$$\sigma\sim\frac{2^{-k}}{(y-2^{j-k}\,s^2\,x)^2}\,. \tag{5.6}$$

Now, if $\sigma\ge 1$, then

$$|y-2^{j-k}\,s^2\,x| \le C\,2^{-k/2}\,. \tag{5.7}$$

Let $\mathcal{D}$ denote the domain given by (5.3), (5.4), (5.7) and $s\sim 1$. Then, similarly as in (4.21), we get

$$\begin{aligned}\|G^3_{k,j}\| &\le C\,2^{k/2+(m-1)(j-k)}\int_{\mathcal{D}}\psi_{k,j}(x,s)\,|x|^{m-1}\,dy\,dx\,ds\\ &\le C\,2^{k/2}\int_{\mathcal{D}}\psi_{k,j}(x,s)\,dy\,dx\,ds\,.\end{aligned}$$

And, by (5.4), (5.7), we have

$$\int_{(x,y,s)\in\mathcal{D}}dy \le C\min\{2^{k-j},2^{-k/2}\}\,.$$

Moreover, by Lemma 3.1.b),

$$\int_{|x|\le 2^{10+A+k-j},\,s\sim 1}\psi_{k,j}(x,s)\,dx\,ds\le C\,,$$

hence

$$\|G^1_{k,j}\|+\|G^3_{k,j}\|\le C\min\{2^{3k/2-j},1\}\,. \tag{5.8}$$

There remains to estimate $\|G^2_{k,j}\|$, which can be done similarly as in the preceding section:

If $\sigma \leq 2$, then we have

$$\frac{1}{c}\, 2^{-k/2} \leq |y - 2^{j-k}\, s^2\, x| \leq c\,, \tag{5.9}$$

for some $c \geq 1$, hence

$$\begin{aligned}\int_{\sigma \leq 2, |y| \sim 1} |\sigma|\, dy &\leq \sqrt{2} \int_{2^{-k/2}/c \leq |y - 2^{j-k} s^2 x| \leq c} \frac{2^{-k/2}}{|y - 2^{j-k}\, s^2\, x|}\, dy \\ &\leq C\, k\, 2^{-k/2}\,.\end{aligned}$$

Moreover, by (5.4),

$$\int_{\sigma \leq 2, |y + s\langle x\rangle^2| \leq C 2^{k-j}} |\sigma|\, dy \leq \int_{|y + s\langle x\rangle^2| \leq C 2^{k-j}} 2\, dy \leq C\, 2^{k-j}\,.$$

Thus, if $\mathcal{E}$ denotes the domain given by (5.3), (5.4), (5.5) and (5.9), then

$$\begin{aligned}\|G^2_{k,j}\| &\leq C\, 2^{k/2 + (m-1)(j-k)} \int_{\mathcal{E}} \psi_{k,j}(x,s)\, |\sigma|\, |x|^{m-1}\, dy\, dx\, ds \\ &\leq C\, k 2^{k/2} \min\{2^{-k/2}, 2^{k-j}\} \int_{|x| \leq 2^{10+A+k-j}, s \sim 1} \psi_{k,j}(x,s)\, dx\, ds \\ &\leq C\, k \min\{1, 2^{3k/2 - j}\}\,.\end{aligned}$$

In combination with (5.8) and (4.17) we thus obtain

$$\sum_{j > k+M} \|K_{k,j}\| \leq C\, k^2\,. \tag{5.10}$$

Together with (4.23), this proves Proposition 1.5, which completes the proof of the theorem.

References.

[B] Beals, M., L^p boundedness of Fourier integral operators. *Mem. Amer. Math. Soc.* **264** (1982).

[C] Christ, M., L^p bounds for spectral multipliers on nilpotent Lie groups. *Trans. Amer. Math. Soc.* **328** (1991), 73-81.

[CF] Colin de Verdière, Y., Frisch, M., *Régularité Lipschitzienne et solutions de l' équation des ondes sur une variété Riemannienne compacte. Ann. Sci. Ecole Norm. Sup.* **9** (1976), 539-565.

[FoS] Folland, G., Stein, E. M., *Hardy Spaces on Homogeneous Groups.* Princeton University Press, 1982.

[H] Hebisch, W., Multiplier theorem on generalized Heisenberg groups. *Colloquium Math.* **LXV** (1993), 231-239.

[Hö] Hörmander, L., Hypoelliptic second-order differential equations. *Acta Math.* **119** (1967), 147-171.

[K] Kohn, J., Harmonic integrals on strongly pseudo convex manifolds I. *Ann. of Math.* **78** (1963), 112-147.

[MM] Mauceri, G., Meda, S., Vector-valued multipliers on stratified groups. *Revista Mat. Iberoamericana* **6** (1990), 141-154.

[Me] Melrose, R., Propagation for the wave group of a positive subelliptic second order differential operator. *Hyperbolic equations and related topics.* Academic Press, 1986.

[Mi] Miyachi, A., On some estimates for the wave equation in L^p and H^p. *J. Fac. Sci. Tokyo Sci. IA* **27** (1980), 331-354.

[MRS1] Müller, D., Ricci, F., Stein, E. M., Marcinkiewicz multipliers and multiparameter structure on Heisenberg (-type) groups I. *Invent. Math.* **119** (1995), 199-233.

[MRS2] Müller, D., Ricci, F., Stein, E. M., Marcinkiewicz multipliers and multiparameter structure on Heisenberg (-type) groups II. *Math. Z.* **221** (1996), 267-291.

[MS] Müller, D., Stein, E. M., On spectral multipliers for Heisenberg and related groups. *J. Math. Pures et Appl.* **73** (1994), 413-440.

[N] Nachman, A. I., The wave equation on the Heisenberg group. *Comm. Partial Diff. Equations* **7** (1982), 675-714.

[P] Peral, J., L^p estimates for the wave equation. *J. Funct. Anal.* **36** (1980), 114-145.

[SSS] Seeger, A., Sogge, C. D., Stein, E. M., Regularity properties of Fourier integral operators. *Ann. of Math.* **134** (1991), 231-251.

[S1] Stein, E. M., Interpolation of linear operators. *Trans. Amer. Math. Soc.* **87** (1958), 159-172.

[S2] Stein, E. M., *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, 1993.

[St] Strichartz, R., L^p-harmonic analysis and Radon transforms on the Heisenberg group. *J. Funct. Anal.* **96** (1991), 350-406.

[VCS] Varopoulos, N. Th., Saloff-Coste, L., Coulhon, Th., *Analysis and geometry on groups.* Cambridge University Press, 1992.

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Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equation without cut-off

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Abstract. We show that in the setting of the spatially homogeneous Boltzmann equation without cut-off, the entropy dissipation associated to a function $f \in L^1(\mathbb{R}^N)$ yields a control of $\sqrt{f}$ in Sobolev norms as soon as f is locally bounded below. Under this additional assumption of lower bound, our result is an improvement of a recent estimate given by P.-L. Lions, and is optimal in a certain sense.

1. Introduction.

The Boltzmann equation in the kinetic theory of gases is one of the fundamental models for nonequilibrium statistical mechanics. The gas is modelled by a density function $f(t,x,v) \geq 0$ on the extended phase space of particles, such that

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f)\,, \qquad t \geq 0\,,\ x \in \mathbb{R}^N\,,\ v \in \mathbb{R}^N\,, \tag{1}$$

where $Q(f,f)$ is the Boltzmann collision operator, which acts only on the velocity variable v. If f is a function of $v \in \mathbb{R}^N$, it is defined by

$$(2) \qquad Q(f,f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\omega \, B(v - v_*, \omega) \, (f' f'_* - f f_*) \,,$$

where S^{N-1} is the unit sphere in $\mathbb{R}^N$, $f' = f(v')$, and so on, and

$$(3) \qquad \begin{cases} v' = v - (v - v_*, \omega) \, \omega, \\ v'_* = v_* + (v - v_*, \omega) \, \omega \,. \end{cases}$$

The kernel, or cross-section, $B : \mathbb{R}^N \times S^{N-1} \longrightarrow \mathbb{R}_+$ is a weight function modelling the interaction, such that $B(z, \omega)$ depends only on $|z|$ and $(z/|z|, \omega)$.

The great majority of mathematical works upon the Boltzmann equation is based on the assumption that B is locally integrable on $\mathbb{R}^N \times S^{N-1}$. However, this assumption is often unsatisfactory from the physical point of view, since it is always false if the particles interact through forces of infinite range [20], [3], [24]. In particular, for inverse power laws, $B(z, \omega) = |z|^\gamma \, b(\cos\alpha)$ with $\cos\alpha = (z/|z|, \omega)$, $\gamma = (s - (2\,N - 1))/(s-1)$, and if $N = 3$, b has a singularity of order $(s+1)/(s-1)$ as $\cos\alpha \longrightarrow 0$. In this work, we shall precisely focus on the case where B is singular.

We shall only be concerned with the spatially homogeneous case, *i.e.* when the unknown in (1) is assumed not to depend on x, so that (1) simply reads

$$(4) \qquad \frac{\partial f}{\partial t} = Q(f,f) \,, \qquad t \geq 0 \,, \; v \in \mathbb{R}^N \,.$$

(we refer to [19] for partial results in the inhomogeneous case). For this equation, there is by now a fairly complete theory of existence in an L^1 setting for non cut-off potentials, which covers all the physically interesting potentials [3], [15], [24]. In the last reference, we also showed how one could rigorously derive the (spatially homogeneous) Landau equation for plasmas, which is the equation corresponding to (4) in the case of Coulomb interactions.

Apart from existence results, very little is known from the analytical point of view. However, it is conjectured that, due to the nonintegrable singularity in B, solutions to (4) become smooth for positive times (which is false for cut-off kernels). The likelihood of this conjecture is reinforced by the study of the Landau equation [18], [4], [13], which is obtained from the Boltzmann equation by "concentrating on grazing collisions", and has definite smoothing (and compactifying) effects. In particular, (in the homogeneous case) its solutions become C^∞ for positive times, at least for the so-called "hard potentials" (see [13] for precise statements and complete proofs).

The smoothing conjecture for the Boltzmann equation was tackled by Desvillettes [10], [11], [12] and Proutière [21], in rather particular cases, with the help of Fourier representations. The proofs are however very technical, and depend highly upon the dimension N. In the aforementioned works, the case of radially symmetric data in 2 dimensions is treated (or non radially symmetric if $\gamma = 0$; for more complicated cases, the proofs have still not been written down. Moreover, some unnatural smoothing of the kinetic cross-section is needed (while $|z|^\gamma$, $0 < \gamma < 1$, is not smooth near $z = 0$).

In [13], a different strategy was followed for proving smoothness in the Landau equation. The proof is at the same time simpler and independent on the dimension. Our aim here is to give a (loosely related) possible startpoint for a complete study of regularization effects in the Boltzmann equation, by showing that the usual estimate on the entropy dissipation automatically entails such an effect.

More precisely, let us define

$$(5)\quad \overline{D}(f) \equiv \int_{\mathbb{R}^{2N}} dv\, dv_* \int_{S^{N-1}} d\omega\, B(v-v_*,\omega)\big(\sqrt{f'f'_*} - \sqrt{ff_*}\,\big)^2 \geq 0\,.$$

The functional $\overline{D}$ is well-defined (possibly infinite) on $L^1(\mathbb{R}^N)$, for instance by the use of the joint convexity of $(x,y) \longrightarrow (\sqrt{x}-\sqrt{y})^2$ on $\mathbb{R}^2$. It is clear that $\overline{D}$ is related to the usual entropy dissipation functional,

$$(6)\quad D(f) \equiv \frac{1}{4}\int_{\mathbb{R}^{2N}} dv\, dv_* \int_{S^{N-1}} d\omega\, B(v-v_*,\omega)\,(f'f'_* - ff_*)\log\frac{f'f'_*}{ff_*}\ ,$$

since, by the classical inequality $(x-y)\log(x/y) \geq 4\,(\sqrt{x}-\sqrt{y})^2$, one has

$$D(f) \geq \overline{D}(f)\,.$$

Our main result is essentially the following. Let $f \in L^1_+(\mathbb{R}^N)$, such that $\overline{D}(f)$ is finite, and assume that

$$(7)\qquad B(z,\omega) \geq \Phi(|z|)\, b(\cos\alpha)\,,$$

where Φ is smooth and bounded below away from 0 and infinity, and b has a singularity of order $1+\nu$, $\nu > 0$. Then, if f is locally bounded from below,

$$\sqrt{f} \in H^{\nu/2}_{\mathrm{loc}} \,. \tag{8}$$

As an immediate consequence of this estimate, solutions of (4) for an initial datum which has finite entropy, and is locally bounded below, will satisfy for all $R, T > 0$

$$\int_0^T dt \, \big\| \sqrt{f}\,(t,\cdot) \big\|^2_{H^{\nu/2}(|v|<R)} < \infty \,,$$

since a lower bound is known to exist for these solutions [8] (see also [22] in the cut-off case).

Closely related results have been obtained recently by Lions [19]. Before we comment on them, it may be of interest to briefly track the idea that smoothness estimates for the Boltzmann equation should be obtained naturally for $\sqrt{f}$ instead of f. First of all, such estimates have been sought for a long time in the context of Maxwellian potentials (when B depends only on $(z/|z|, \omega)$). Indeed, it is now known that in this case, the Fisher information

$$I(f) = 4 \big\| \sqrt{f} \big\|^2_{\dot{H}^1} = 4 \int_{\mathbb{R}^N} \big| \nabla \sqrt{f} \big|^2$$

is a Lyapunov functional [16], [23], [7], [25]. Complete proofs are given in the last reference.

In a more general setting, regularity estimates for $\sqrt{f}$ and entropy dissipation estimates are associated together in works by Lions [17], [19] and the author [24] (see also Cercignani [9]). In [24], it is shown that $\overline{D}(f)$ yields sufficient regularity on the tensor product $\sqrt{f f_*}$ to give a meaning to (4) even for very singular and very soft potentials (*i.e.* $\gamma < -2$). More generally, $D(f)$ gives some control on the regularity of $\sqrt{f f_*}$. It is then a natural question whether this estimate for $\sqrt{f}$ implies a control of f itself. A first strategy to answer this problem is to introduce a well-chosen artificial weight-function $\hat{B}(v - v_*, \omega)$, and integrate the estimate for $\sqrt{f' f'_*} - \sqrt{f f_*}$ in v_* and ω after multiplication by $\hat{B}$, thus obtaining an estimate on

$$\begin{aligned} \hat{Q}(\sqrt{f}, \sqrt{f}) &= \int dv_* \, d\omega \, \hat{B}(v - v_*, \omega) \big(\sqrt{f' f'_*} - \sqrt{f f_*} \big) \\ &\equiv \hat{Q}^+(\sqrt{f}, \sqrt{f}) - \hat{Q}^-(\sqrt{f}, \sqrt{f}) \,. \end{aligned}$$

Then one uses the regularity properties of Q^+ in L^2 (*cf.* [17], [26], [5]), which is of course the natural space for $\sqrt{f}$, and the simple form of Q^-. This is what Lions does in [17] (to characterize equilibria distributions for (1) under very little assumptions) and in [19], to prove an estimate of the same kind as (8), namely

$$\sqrt{f} \in H^s_{\mathrm{loc}}\,, \qquad \text{for all } s < s_1 = \frac{\nu}{2}\left(\frac{1}{1+\dfrac{\nu}{N-1}}\right). \tag{9}$$

The exponent is hence not so good as the one in (8), but a lower bound is not needed for it. The proof by Lions is very simple, but relies on the deep result of smoothing of the positive part of Boltzmann's collision operator. It is possible that a better knowledge of the explicit constants in this result (see [5] for some of them) could lead to (8). In any case, our proof implies that $\nu/2$ is the optimal exponent, in the sense that for all $\varepsilon > 0$ one can find a function f such that

$$\overline{D}(f) < \infty \qquad \text{and} \qquad \|\sqrt{f}\|^2_{\dot{H}^{\nu/2+\varepsilon}} = \infty\,.$$

Moreover, our proof is elementary and relies only upon careful changes of variables. As far as the physical meaning is concerned, it is another illustration of the general principle that the entropy dissipation yields regularity "along the collisions", either in the tensor phase space $\mathbb{R}^{2N}$ (via estimates on $\sqrt{f'f'_*}-\sqrt{ff_*}$), or in $\mathbb{R}^N$ (via estimates on $\sqrt{f'}-\sqrt{f}$). See [14] for still another manifestation of this principle.

The only drawback of this method is the need for a lower bound. It is possible that our computation can be refined in such a way to dispend with this assumption, maybe at the loss of the optimal exponent $\nu/2$. In the end of the paper, we give possible hints for this. However, we shall not go further, since on one hand Lions's result is general enough to cover all the cases when one is not interested in the exact exponent (in particular for compactness properties associated to the complete equation (1)), and on the other hand a pointwise lower bound is available for f in realistic problems (in the homogeneous case only).

The plan of the paper is as follows. In Section 2, we give a decomposition of $\overline{D}(f)$ in two terms, one of which includes cancellations, and the other is nonnegative. The former is shown to be controlled by L^1-type estimates in Section 3, and the latter is shown to give the desired estimate via the so-called Carleman representation in Section 4. Finally, in Section 5, we give some remarks about the role of intermediate collisions.

2. Splitting of $\overline{D}(f)$ and main result.

Let us write

$$
\begin{aligned}
\sqrt{f'f'_*} - \sqrt{ff_*} = {} & \frac{1}{2}\,(\sqrt{f'} - \sqrt{f})\,(\sqrt{f'_*} + \sqrt{f_*}) \\
& + \frac{1}{2}\,(\sqrt{f'} + \sqrt{f})\,(\sqrt{f'_*} - \sqrt{f_*})\,.
\end{aligned} \tag{10}
$$

Accordingly,

$$
\begin{aligned}
& (\sqrt{f'f'_*} - \sqrt{ff_*})^2 \\
& = \frac{1}{4}\,((\sqrt{f'} - \sqrt{f})^2\,(\sqrt{f'_*} + \sqrt{f_*})^2 \\
& \quad + (\sqrt{f'} + \sqrt{f})^2\,(\sqrt{f'_*} - \sqrt{f_*})^2) \\
& \quad + \frac{1}{2}\,(\sqrt{f'} - \sqrt{f})\,(\sqrt{f'_*} + \sqrt{f_*})(\sqrt{f'} + \sqrt{f})\,(\sqrt{f'_*} - \sqrt{f_*})\,.
\end{aligned} \tag{11}
$$

Reporting in (5) and using the classical change of variables $(v, v_*) \longrightarrow (v', v'_*)$, involutive and with unit Jacobian, we obtain

$$
\overline{D}(f) = S(f) + T(f)\,,
$$

where

$$
\left\{
\begin{aligned}
S(f) &= \frac{1}{2}\int dv\,dv_*\,d\omega\,B(v - v_*, \omega) \\
& \qquad \cdot (\sqrt{f'} - \sqrt{f})^2\,(\sqrt{f'_*} + \sqrt{f_*})^2\,, \\
T(f) &= \frac{1}{2}\int dv\,dv_*\,d\omega\,B(v - v_*, \omega)\,(f' - f)\,(f'_* - f_*)\,.
\end{aligned}
\right. \tag{12}
$$

It is clear that in $T(f)$ one can expect strong cancellation effects, while $S(f)$ is nonnegative. We shall prove that $T(f)$ is well-defined without any regularity assumptions on f, while $S(f)$ is (locally) bounded below by a multiple of the square of some Sobolev norm of $\sqrt{f}$.

Before we state our results, let us discuss the assumptions for B. First of all, since $\overline{D}(f)$ is monotonic in B, it is sufficient, to obtain a general result, to treat the case when B is "small". We shall therefore assume, without real loss of generality,

Assumption A. $B(v - v_*, \omega) = \Phi(|v - v_*|)\, b(\cos\alpha)$, *where*

$$\cos\alpha = k \cdot \omega\,, \qquad k = \frac{v - v_*}{|v - v_*|}\,. \tag{13}$$

Assumption B. $\Phi \in W^{1,\infty}(\mathbb{R}_+)$ *is a positive function with a bounded derivative, and is bounded from below uniformly, except maybe near* 0 *and* $+\infty$.

The last assumption means of course that for all numbers ε, $R > 0$ there exists $K(\varepsilon, R) > 0$ such that $\Phi(|z|) \geq K\,(\varepsilon, R) > 0$ if $\varepsilon \leq |z| \leq R$. This assumption is perfectly realistic from a physical point of view.

Now, we assume b to be singular only for grazing collisions, in the sense

Assumption C. $b \in C([-1, 1] \setminus \{0\})$, *and*

$$b(\cos\alpha) \geq \frac{C}{|\cos\alpha|^{1+\nu}} \sim \frac{C}{\left(\frac{\pi}{2} - \alpha\right)^{1+\nu}}\,, \qquad 0 < \nu < 2\,. \tag{14}$$

Here $\alpha \in [-\pi/2, \pi/2)$, C stands for arbitrary positive constants, and the sign $\sim$ only denotes similar asymptotic behaviour near the singularity ($\cos\alpha = 0$). Note in particular that b is bounded below.

It is clear that the parameter ν measures the strength of the singularity of B (note that if $\alpha \sim \pi/2$, $d\omega$ and $d\alpha$ are roughly proportional). Let us comment on the assumption $\nu < 2$. In order to do so, we introduce another (classical) representation for the collision operator (2), based on the unit vector σ such that

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\,\sigma\,, \\[2mm] v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\,\sigma\,. \end{cases} \tag{15}$$

In this representation, Boltzmann's collision operator keeps the same form as in (2), except that $d\omega$ is changed into $d\sigma$, and $B(v - v_*, \omega)$ into

$$\widetilde{B}(v - v_*, \sigma) = (2\,|k \cdot \omega|)^{-(N-2)}\, B(v - v_*, \omega)\,.$$

Now, grazing collisions correspond to $k = \sigma$, *i.e* $k \cdot \sigma = 1$. Assumptions A and C have to be replaced by their counterparts

Assumption A'. $\widetilde{B}(v - v_*, \sigma) = \Phi(|v - v_*|)\, \widetilde{b}(k \cdot \sigma)$.

Assumption C'. $\widetilde{b} \in C([-1,1))$, *and*

$$\widetilde{b}(\cos\theta) \geq \frac{C}{\sin^{N-2}\theta}\,\frac{1}{\theta^{1+\nu}} \sim \frac{C}{(\cos\theta - 1)^{(N-1+\nu)/2}}\,, \quad 0 < \nu < 2\,. \tag{16}$$

Here $\theta \in [-\pi, \pi]$. Note that $\alpha = \pi/2 - \theta/2$. Since $d\sigma$ is proportional to $\sin^{N-2}\theta\, d\theta$, we see that

$$\Lambda \equiv \int_{S^{N-1}} d\sigma\, \widetilde{b}(k \cdot \sigma)\,(1 - k \cdot \sigma) = C_N \int_0^\pi d\theta\, \widetilde{b}(\cos\theta) \sin^2 \frac{\theta}{2}$$

is finite if and only if $\nu < 2$. Since Λ has the physical meaning of a total cross-section for momentum transfer, we see that our assumption on ν is physically justified. This is consistent with the state of the art concerning the existence theory for the Boltzmann equation [24].

We can now state our main result. We use the classical notation

$$\|f\|_{L^1_1} = \int_{\mathbb{R}^N} |f(v)|\,(1 + |v|)\, dv\,, \qquad L^1_1 = \{f \in L^1(\mathbb{R}^N) :\ \|f\|_{L^1_1} < \infty\}\,.$$

Theorem 1. *Let* $f \in L^1_1(\mathbb{R}^N)$, *and let* B *be a cross-section satisfying assumptions* A, B, C. *Then*

i) *There exists a constant* C, *independent of* f, *such that*

$$|T(f)| \leq C\, \|f\|_{L^1_1}\, \|f\|_{L^1}\,. \tag{17}$$

ii) *Assume in addition that there exists a strictly positive function* $\varepsilon(R)$ *such that*

$$|v| \leq R \textit{ implies } f(v) \geq \varepsilon(R)\,, \qquad \textit{for all } v \in \mathbb{R}^N\,,\ \textit{for all } R > 0\,.$$

Then there exists a strictly positive function $K(R)$, *depending only on* $\|f\|_{L^1_1}$, $\varepsilon(R)$ *and the cross-section, such that*

$$S(f) \geq K(R)\, \big\|\sqrt{f}\,\big\|^2_{\dot{H}^{\nu/2}(|v|<R)}\,. \tag{18}$$

As an immediate consequence of the monotonicity of the entropy dissipation, we then deduce the

Corollary 1.1. *Let $f \in L^1_1(\mathbb{R}^N)$ such that $D(f) < \infty$, and f satisfies the additional assumption of lower bound. Assume that $B \geq B_0$, where B_0 is a cross-section satisfying assumptions* A, B, C. *Then* $\sqrt{f} \in H^{\nu/2}_{\mathrm{loc}}$.

REMARK. After completion of this work, we became aware of two Notes by Alexandre on the same subject [1], [2], where the Carleman representation is also used, but no splitting of the entropy dissipation as ours. It seems very difficult to understand whether the results therein are comparable to ours, but Alexandre kindly informed us that he had used this splitting independently in recent work, and obtained a bound very similar to ours, as well as related results in the (very difficult) inhomogeneous case. The proofs by Alexandre rely on the theory of pseudo-differential operators. Desvillettes has also shown us some of his partial results in collaboration with Wennberg, which are consistent with both our conclusions and our method of proof, but do not start from the entropy estimate.

3. Cancellation effects for grazing collisions.

In this section, we prove the estimate (17). First, by the usual change of variables $(v, v_*) \longrightarrow (v', v'_*)$,

$$T(f) = \int dv\, dv_*\, d\omega\, B(v - v_*, \omega)\, f\,(f_* - f'_*) = \int_{\mathbb{R}^N} dv\, f(v)\, G(v)$$

with

$$G(v) = \int_{\mathbb{R}^N \times S^{N-1}} dv_*\, d\omega\, B(v - v_*, \omega)\,(f_* - f'_*)\,.$$

We now proceed to estimate G. We turn to the σ-representation (15) and use Assumption A:

$$G(v) = \int_{\mathbb{R}^N \times S^{N-1}} dv_*\, d\sigma\, \Phi(|v - v_*|)\, \widetilde{b}(k \cdot \sigma)\,(f_* - f'_*)\,.$$

Now, *for fixed* σ, the change of variables $v_* \longrightarrow v'_*$ is valid, and an easy computation yields

$$dv'_* = \frac{1}{2^N}(1 + (k \cdot \sigma))\, dv_* = \frac{1}{2^N}\,(1 + \cos\theta)\, dv_*\,. \tag{19}$$

Let $k' = (v - v'_*)/|v - v'_*|$. An elementary geometric argument shows that

$$1 + (k \cdot \sigma) = 1 + \cos\theta = 2\cos^2\frac{\theta}{2} = 2\,(k' \cdot \sigma)^2\,. \tag{20}$$

By symmetrization, one can assume that $b(\cos\theta)$ is supported in $(-\pi/2 \le \theta \le \pi/2)$, *i.e.* $(k \cdot \sigma) \ge 0$ (this can be seen as a consequence of the indiscernability of the particles), so that the Jacobian in (19) is bounded below.

For given σ, let us introduce

$$\psi_\sigma : v'_* \longmapsto v_*\,.$$

It is easy to check that for given (v, v'_*, σ) such that $(v - v'_*, \sigma) > 0$, the equation with unknown $v_* \in \mathbb{R}^N$

$$\psi_\sigma(v'_*) = v_*$$

is uniquely solvable (note that if v'_* is given by (15), then $2\,(v - v'_*, \sigma) = (v - v_*, \sigma) + |v - v_*| > 0$). Moreover, the condition $(v - v_*, \sigma) > 0$ is equivalent to

$$(k', \sigma) = \Big(\frac{v - v'_*}{|v - v'_*|}, \sigma\Big) > \frac{1}{\sqrt{2}}\,.$$

See the geometric interpretation of ψ_σ in Figure 1.

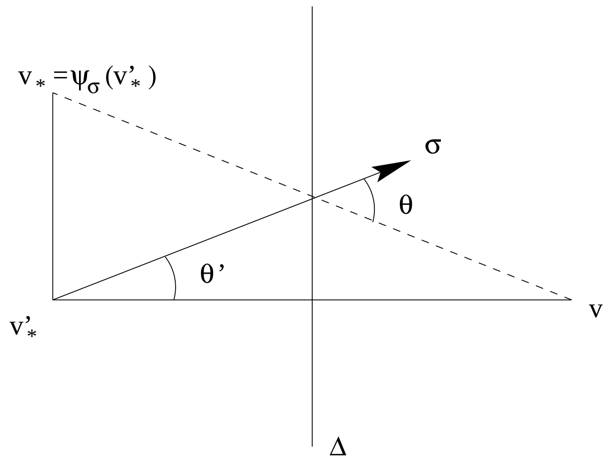

Figure 1. Δ is the mediatrix of (v, v'_*); $\cos\theta = k \cdot \sigma$; $\cos\theta' = \cos(\theta/2)$.

Accordingly, we write (using (19) and (20))

$$G(v)=\int_{k\cdot\sigma>0}dv_*\,d\sigma\,\Phi(|v-v_*|)\,\widetilde{b}(k\cdot\sigma)\,f(v_*)$$
$$-\int_{k'\cdot\sigma>1/\sqrt{2}}dv'_*\,d\sigma\,\frac{2^{N-1}}{(k'\cdot\sigma)^2}\,\Phi(|v-\psi_\sigma(v'_*)|)$$
$$\cdot\,\widetilde{b}(2\,(k'\cdot\sigma)^2-1)\,f(v'_*)\,.$$

Since the integration variable in the second integral is a dummy one, we conclude that

$$G(v)=\int_{\mathbb{R}^N}dv_*\,f(v_*)\,C(v,v_*) \tag{21}$$

with

$$C(v,v_*)=\int_{S^{N-1}}d\sigma\,\Big(\Phi(|v-v_*|)\,\widetilde{b}(k\cdot\sigma)\,\mathbf{1}_{\{k\cdot\sigma>0\}}$$
$$-\,\Phi(|v-\psi_\sigma(v_*)|)\,\frac{2^{N-1}}{(k\cdot\sigma)^2}\,\widetilde{b}(2\,(k\cdot\sigma)^2-1)\,1_{\{k\cdot\sigma>1/\sqrt{2}\}}\Big) \tag{22}$$

$$=\Phi(|v-v_*|)\int_{S^{N-1}}d\sigma\,\widetilde{b}(k\cdot\sigma)\,\mathbf{1}_{\{0<k\cdot\sigma\le 1/\sqrt{2}\}} \tag{23}$$

$$+\,\Phi(|v-v_*|)\int_{S^{N-1}}d\sigma\,\Big(\widetilde{b}(k\cdot\sigma)-\frac{2^{N-1}}{(k\cdot\sigma)^2}\,\widetilde{b}(2\,(k\cdot\sigma)^2-1)\Big)$$
$$\cdot\,\mathbf{1}_{\{k\cdot\sigma>1/\sqrt{2}\}} \tag{24}$$

$$+\int_{S^{N-1}}d\sigma\,\big(\Phi(|v-v_*|)-\Phi(|v-\psi_\sigma(v_*)|)\big)$$
$$\cdot\,\frac{2^{N-1}}{(k\cdot\sigma)^2}\,\widetilde{b}(2\,(k\cdot\sigma)^2-1)\,\mathbf{1}_{\{k\cdot\sigma>1/\sqrt{2}\}}\,. \tag{25}$$

As an immediate consequence of Assumptions B and C, the expression (23) is bounded by a constant, independently of v, v_*. Let us now consider the last term (25). Still using the notation $\cos\theta=k\cdot\sigma$, we see that

$$|v-\psi_\sigma(v_*)|=\frac{|v-v_*|}{\cos\theta}\,,$$

so that

$$|\Phi(|v-v_*|)-\Phi(|v-\psi_\sigma(v_*)|)|\le\|\Phi'\|_\infty\,|v-v_*|\Big(\frac{1}{\cos\theta}-1\Big)\,.$$

Therefore, as a consequence of Assumption C, the integral (25) is bounded by

$$\begin{aligned} C_N \, \|\Phi'\|_\infty \, |v - v_*| \int_0^{\pi/4} d\theta \, \widetilde{b}(\cos 2\,\theta) \, (\cos\theta - 1) \sin^{N-2}\theta \\ \leq C_N \, \|\Phi'\|_\infty \, (|v| + |v_*|) \int_0^{\pi/4} d\theta \, \theta^{1-\nu} \\ \leq C \, (|v| + |v_*|) \, , \end{aligned}$$

for some $\nu < 2$, where the constant C depends only on N and the constants in Assumptions B, C.

Finally, we estimate the integral in (24). By a spherical change of variables, this is

$$\begin{aligned} C_N \int_0^{\pi/4} d\theta \, \sin^{N-2}\theta \left(\widetilde{b}(\cos\theta) - \frac{2^{N-1}}{\cos^2\theta} \, \widetilde{b}(\cos 2\,\theta) \right) \\ = C_N \int_0^{\pi/4} d\theta \, \sin^{N-2}\theta \, \widetilde{b}(\cos\theta) \\ - C_N \int_0^{\pi/2} d\theta \, \sin^{N-2}\left(\frac{\theta}{2}\right) \frac{2^{N-2}}{\cos^2\left(\frac{\theta}{2}\right)} \, \widetilde{b}(\cos\theta) \, . \end{aligned}$$

By the formula $\sin^{N-2}(\theta/2)\, 2^{N-2} \cos^{N-2}(\theta/2) = \sin^{N-2}\theta$, we get

$$\begin{aligned} C_N \int_0^{\pi/4} d\theta \, \sin^{N-2}\theta \, \widetilde{b}(\cos\theta) \left(1 - \frac{1}{\cos^N\left(\frac{\theta}{2}\right)} \right) \\ - C_N \int_{\pi/4}^{\pi/2} d\theta \, \sin^{N-2}\left(\frac{\theta}{2}\right) \frac{2^{N-2}}{\cos^2\left(\frac{\theta}{2}\right)} \, \widetilde{b}(\cos\theta) \, . \end{aligned}$$

The second integral is convergent by Assumption C, and so is also the first, since for $N \geq 2$,

$$1 - \frac{1}{\cos^N\left(\frac{\theta}{2}\right)} = O\left(\theta^2\right) , \qquad \text{as } \theta \longrightarrow 0 \, .$$

In the end, we find

$$|C(v, v_*)| \leq C \, (|v| + |v_*|) \, , \tag{26}$$

whence the conclusion.

4. The Carleman representation.

We now transform $S(f)$ into an expression looking like the square of a (fractional) Sobolev norm. To this purpose, we use the so-called Carleman representation, which was actually introduced by Carleman in [6], and later reformulated by Wennberg [26]. It should be noted that the purpose of Wennberg is also to obtain regularity estimates, though in a very different context.

The idea of the Carleman representation is to replace the set of variables (v, v_*, ω) by the set (v, v', v'_*), where $v \in \mathbb{R}^N$, $v' \in \mathbb{R}^N$ and $v'_* \in E_{vv'}$, the hyperplane going through v and orthogonal to $v - v'$.

Let us recall briefly the argument. Following Wennberg, we introduce the variable $q = |v - v'|$ and note that

$$\begin{cases} v' = v + q\,\omega\,, \\ v_* = v'_* + q\,\omega\,. \end{cases} \tag{27}$$

Since ω and $E_{vv'}$ are orthogonal, the second relation entails $dv_* = dv'_*\,dq$, where dv'_* denote the Lebesgue measure on $E_{vv'}$. On the other hand, $dv' = q^{N-1} dq\,d\omega$. Hence,

$$dv_*\,d\omega = dv'_*\,dq\,d\omega = \frac{dv'_*\,dv'}{q^{N-1}} = \frac{dv'_*\,dv'}{|v - v'|^{N-1}}\ .$$

Note that dv'_* is a $(N-1)$-dimensional measure, while dv' is N-dimensional.

By Assumption A, and since $\cos\alpha = |v - v'|/|v - v_*|$, $|v - v_*| = |v' - v'_*|$, we obtain

$$\begin{aligned} S(f) = \frac{1}{2} \int_{\mathbb{R}^{2N}} & \frac{dv\,dv'}{|v - v'|^{N-1}} \\ & \cdot \Big(\int_{E_{vv'}} dv'_*\,\Phi(|v' - v'_*|)\,(\sqrt{f'_*} + \sqrt{f_*})^2\,b\Big(\frac{|v' - v|}{|v' - v'_*|}\Big)\Big) \\ & \cdot (\sqrt{f'} - \sqrt{f})^2\,. \end{aligned} \tag{28}$$

Note that (for given v, v') v'_* and v_* describe parallel planes (in fact, $v_* = (v' - v) + v'_*$).

Using Assumption C, for some positive constant K,

$$b\Big(\frac{|v'-v|}{|v'-v'_*|}\Big) \geq K\,\frac{|v'-v'_*|^{1+\nu}}{|v'-v|^{1+\nu}}\,.$$

Thus, we can write

$$S(f) \geq K \int_{\mathbb{R}^{2N}} dv\,dv'\,A(v,v')\,\frac{(\sqrt{f(v')}-\sqrt{f(v)})^2}{|v'-v|^{N+\nu}} \tag{29}$$

with

$$A(v,v') = \int_{E_{vv'}} dv'_*\,\Phi(|v'-v'_*|)\,(f'_*+f_*)\,|v'-v'_*|^{1+\nu}\,.$$

Let us set $\Psi(|z|) = |z|^{1+\nu}\,\Phi(|z|)$. By Assumption B, Ψ can vanish only near 0 and ∞. We note that

$$A(v,v') = \int_{E_{vv'}} dv'_*\,\Psi(|v'-v'_*|)\,f(v'_*) + \int_{E_{v'v}} dv_*\,\Psi(|v-v_*|)\,f(v_*)\,.$$

The estimate (29) is a Sobolev estimate as soon as A is bounded from below. This is clearly true locally if f is locally bounded from below (it suffices in fact that all integrals of f upon bounded portions of hyperplanes going through v be bounded from below, locally in v). This completes our proof.

REMARK. The coefficients $A(v,v')$ are given by Radon transforms, and therefore are likely to be smooth, in some sense; this remark, combined with the method of the next section, could help relax the assumption of local lower bound.

5. The role of intermediate collisions.

In this section, we only want to emphasize how the method applied above can be refined by the use of intermediate collisions. Indeed, the coefficients $A(v,v')$ of the previous section measure, in some sense, the number of collisions in which the particles change their velocity from v to v'. The "gain of regularity" therefore depends upon these coefficients. But particles can also gain an arbitrary velocity $v'' \in \mathbb{R}^N$ before they gain the velocity v'. We shall see how to make this vague physical idea more precise. Even though we did not find any application for it, we think it likely that this method could be useful for related problems.

Let us rewrite the estimate (29) with v' replaced by w: up to a constant,

$$\int_{\mathbb{R}^{2N}} dv\, dw\, A(v,w)\, \frac{(\sqrt{f(v)} - \sqrt{f(w)})^2}{|v-w|^{N+\nu}} \le S(f)\,.$$

As a consequence, for any C, $\varepsilon > 0$,

$$\begin{aligned} \int_{\mathbb{R}^{3N}} dv\, dw\, dv'\, A(v,w)\, \mathbf{1}_{\{|v-w|\le C|v-v'|\le C\}} & \frac{(\sqrt{f(v)} - \sqrt{f(w)})^2}{|v-v'|^{2N+\nu-\varepsilon}} \\ & \le C(S(f), C, \varepsilon) \\ & < \infty\,. \end{aligned} \tag{30}$$

Indeed, in the domain of integration, $|v'-w| \le (1+C)$, so that the integral (30) is bounded by

$$C^{N+\nu}\Big(\int_{\mathbb{R}^{2N}} dv\, dw\, A(v,w)\, \frac{(\sqrt{f(v)} - \sqrt{f(w)})^2}{|v-w|^{N+\nu}}\Big) \cdot \Big(\int_{|v'-w|\le 1+C} dv'\, \frac{1}{|v'-w|^{N-\varepsilon}}\Big)\,.$$

Similarly,

$$\begin{aligned} \int_{\mathbb{R}^{3N}} & dv\, dv'\, dw\, A(w,v')\, \mathbf{1}_{\{|w-v'|\le C|v-v'|\le C\}} \\ & \cdot \frac{(\sqrt{f(w)} - \sqrt{f(v')})^2}{|v-v'|^{2N+\nu-\varepsilon}} < \infty\,. \end{aligned} \tag{31}$$

Since

$$(\sqrt{f(v)} - \sqrt{f(v')})^2 \le 2\,(\sqrt{f(v)} - \sqrt{f(w)})^2 + 2\,(\sqrt{f(w)} - \sqrt{f(v')})^2\,,$$

we get by adding up (30) and (31)

$$\begin{aligned} \int_{\mathbb{R}^{2N}} & dv\, dv'\, \mathbf{1}_{\{|v-v'|\le 1\}} \\ & \cdot \Big(\int_{|v-w|,|v'-w|\le C|v-v'|} dw\, \min\{A(v,w), A(w,v')\}\Big) \\ & \cdot \frac{(\sqrt{f(v)} - \sqrt{f(v')})^2}{|v-v'|^{2N+\nu-\varepsilon}} < \infty\,. \end{aligned}$$

Since, of course,

$$\int_{\mathbb{R}^{2N}} dv\,dv'\,\mathbf{1}_{\{|v-v'|\geq 1\}}\,\frac{(\sqrt{f(v)}-\sqrt{f(v')})^2}{|v-v'|^{N+\nu}} \leq C\,\|f\|_{L^1} < \infty\,,$$

we see that, up to an arbitrarily small degradation in the Sobolev exponent, the coefficients $A(v,v')$ can be replaced by

$$\overline{A}(v,v') \equiv \frac{1}{|v-v'|^N}\int_{\max\{|v-w|,|v'-w|\}\leq C|v-v'|} dw\,\min\{A(v,w),A(w,v')\}\,. \tag{32}$$

Since the volume of

$$\{w \in \mathbb{R}^N\,:\,\max\{|v-w|,|v'-w|\} \leq C\,|v-v'|\}$$

behaves like $|v-v'|^N$, we see that $\overline{A}$ is a kind of *average* of A, and hence more likely to be bounded from below than A. Of course the procedure can be iterated as many times as desired. We did not go further in this investigation.

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References.

[1] Alexandre, R., Sur l'opérateur de Boltzmann non linéaire 3D sans troncature angulaire. *C. R. Acad. Sci. Paris* **326** (1998), 165-168.

[2] Alexandre, R., Sur le taux de dissipation d'entropie sans troncature angulaire. *C. R. Acad. Sci. Paris* **326** (1998), 311-315.

[3] Arkeryd, L., Intermolecular forces of infinite range and the Boltzmann equation. *Arch. Rational Mech. Anal.* **77** (1981), 11-21.

[4] Arsen'ev, A. A., Buryak, O. E., On the connection between a solution of the Boltzmann equation and a solution of the Landau-Fokker-Planck equation. *Math. USSR Sbornik* **69** (1991), 465-478.

[5] Bouchut, F., Desvillettes, L., A proof of the smoothing properties of the positive parts of Boltzmann's kernel. *Revista Mat. Iberoamericana* **14** (1998), 47-61.

[6] Carleman, T., *Problèmes Mathématiques dans la Théorie Cinétique des Gaz.* Almqvist & Wiksell, 1957.

[7] Carlen, E., Carvalho, M., Strict entropy production bounds and stability of the rate of convergence to equilibrium for the Boltzmann equation. *J. Stat. Phys.* **67** (1992), 575-608.

[8] Carlen, E., Desvillettes, L., Work in progress (personal communication).

[9] Cercignani, C., A remarkable estimate for the solutions of the Boltzmann equation. *Appl. Math. Lett.* **5** (1992), 59-62.

[10] Desvillettes, L., About the regularizing properties of the non-cut-off Kac equation. *Comm. Math. Phys.* **168** (1995), 417-440.

[11] Desvillettes, L., Regularization for the non-cutoff 2D radially symmetric Boltzmann equation with a velocity dependent cross section. *Transp. Theory Stat. Phys.* **25** (1996), 383-394.

[12] Desvillettes, L., Regularization properties of the 2-dimensional non radially symmetric non cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules. *Transp. Theory Stat. Phys.* **26** (1997), 341-357.

[13] Desvillettes, L., Villani, C., On the spatially homogeneous Landau equation for hard potentials. Part I: Existence, uniqueness and smoothness. To appear in *Comm. Partial Diff. Equations.*

[14] Desvillettes, L., Villani, C., On the spatially homogeneous Landau equation for hard potentials. Part II: H-theorem and applications. To appear in *Comm. Partial Diff. Equations.*

[15] Goudon, T., On the Boltzmann equation and its relations to the Landau-Fokker-Planck equation: influence of grazing collisions. *C. R. Acad. Sci. Paris* **324** (1997), 265-270.

[16] McKean. H. P. Jr., Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas. *Arch. Rational Mech. Anal.* **21** (1966), 343-367.

[17] Lions, P. L., Compactness in Boltzmann's equation via Fourier integral operators and applications. *J. Math. Kyoto Univ.* **34** (1994), 391-427.

[18] Lions, P. L., On Boltzmann and Landau equations. *Phil. Trans. R. Soc. Lond.* **346** (1994), 191-204.

[19] Lions, P. L., Regularity and compactness for Boltzmann collision operators without angular cut-off. *C. R. Acad. Sci. Paris* **326** (1998), 37-41.

[20] Maxwell, J. C., On the dynamical theory of gases. *Phil. Trans. R. Soc. Lond.* **157** (1867), 49-88.

[21] Proutière, A., New results of regularization for weak solutions of Boltzmann equation. Preprint, 1996.

[22] Pulvirenti, A., Wennberg, B., A Maxwellian lower bound for solutions to the Boltzmann equation. *Comm. Math. Phys.* **183** (1997), 145-160.

[23] Toscani, G., New a priori estimates for the spatially homogeneous Boltzmann equation. *Cont. Mech. Thermodyn.* **4** (1992), 81-93.

[24] Villani, C., On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Rational Mech. Anal.* **143** (1998), 273-307.

[25] Villani, C., Fisher information bounds for the Boltzmann equation. *J. Math. Pures et Appl.* **77** (1998), 821-837.

[26] Wennberg, B., Regularity in the Boltzmann equation and the Radon transform. *Comm. Partial Diff. Equations* **19** (1994), 2057-2074.

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Isoperimetric inequalities in Riemann surfaces of infinite type

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1. Introduction.

By $\mathcal{S}$ we denote a hyperbolic Riemann surface, *i.e.* a (open and connected) Riemann surface whose universal covering space is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$, endowed with its Poincaré metric (also called the hyperbolic metric), *i.e.* the metric obtained by projecting the Poincaré metric of the unit disk

$$ds = \frac{2\,|dz|}{1-|z|^2}\ .$$

With this metric, $\mathcal{S}$ is a complete Riemannian manifold with constant curvature -1. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori.

It is convenient to remark that this definition of hyperbolic Riemann surface is not universally accepted, since sometimes the word *hyperbolic* refers to the existence of Green's function.

We say that $\mathcal{S}$ satisfies the *hyperbolic isoperimetric inequality* (HII) if $\mathcal{S}$ is a hyperbolic Riemann surface and there exists a constant $h > 0$ such that for every relatively compact domain (an open and connected set) G with smooth boundary one has that

$$A_{\mathcal{S}}(G) \le h\, L_{\mathcal{S}}(\partial G)\,, \tag{1.1}$$

where $A_{\mathcal{S}}(G)$ denotes the (hyperbolic) area of G and $L_{\mathcal{S}}(\partial G)$ the (hyperbolic) length of its boundary. An approximation argument gives that if $\mathcal{S}$ satisfies HII, then (1.1) is also true for domains with finite area. We denote by $h(\mathcal{S})$ the best constant in (1.1)

It is clear that a finite area hyperbolic Riemann surface does not satisfy HII.

A Riemann surface $\mathcal{S}$ is said to be of *finite type* if its fundamental group $\Pi_1(p,\mathcal{S})$, $p \in \mathcal{S}$, is finitely generated. In other case we say that $\mathcal{S}$ is of *infinite type*. It is well known that every Riemann surface of finite type can be obtained from a compact Riemann surface by deleting p points (the punctures of $\mathcal{S}$) and n closed disks (whose boundaries represent the ideal boundaries of $\mathcal{S}$). It is also a known fact that a Riemann surface of finite type has HII if and only if $n > 0$ or, equivalently, if $\mathcal{S}$ has infinite area. Therefore, in spite of most of our results are true independently of the type of the considered Riemann surface, we will be interested in Riemann surfaces of infinite type.

There are a number of natural questions concerning the HII-property of Riemann surfaces. Particularly interesting are the stability under quasiconformal maps, its relation with other conformal invariants and its characterization for plane domains. Here the word *conformal* refers to holomorphic homeomorphisms.

Concerning the study of the stability of HII, in [FR, Theorem 1] it was proved that if two Riemann surfaces are quasiconformally equivalent and one has HII, the other has too.

One of the conformal invariants related with the HII-property is the bottom of the spectrum of the Laplace-Beltrami operator, $b(\mathcal{S})$, which can be defined in terms of Rayleigh's quotients as

$$b(\mathcal{S}) = \inf_{\varphi \in C_c^\infty(\mathcal{S})} \frac{\displaystyle\iint \|\nabla\varphi\|^2 \, dw}{\displaystyle\iint \varphi^2 \, dw} \,,$$

where $\|\cdot\|$, ∇ and dw refer to the Poincaré metric of $\mathcal{S}$.

The number $b(\mathcal{S})$ belongs to $[0,1/4]$ and a celebrated theorem of Elstrodt, Patterson and Sullivan [Su, p. 333] relates it with other important conformal invariant of $\mathcal{S}$, its exponent of convergence $\delta(\mathcal{S})$ (see *e.g.* [N, p. 21] for basic background), which can be defined as

$$\delta(\mathcal{S}) := \inf \{ t : \; U_t(p) < \infty, \text{ for some } p \in \mathcal{S} \} \,,$$

where

$$U_t(p) := \sum_{[\gamma] \in \Pi_1(p,\mathcal{S})} \exp\big(-t \, L_{\mathcal{S}}([\gamma]) \big)$$

and

$$L_{\mathcal{S}}([\gamma]) := \inf\{L_{\mathcal{S}}(g) : \; [\gamma] = [g]\} .$$

It is easy to check that if $U_t(p) < \infty$ for some $p \in \mathcal{S}$, then $U_t(q) < \infty$ for all $q \in \mathcal{S}$.

It is a well known fact that $0 \le \delta(\mathcal{S}) \le 1$ (see *e.g.* [N, p. 21]).

The theorem of Elstrodt, Patterson and Sullivan asserts that

$$b(\mathcal{S}) = \begin{cases} \dfrac{1}{4}\,, & \text{if } 0 \le \delta(\mathcal{S}) \le \dfrac{1}{2}\,, \\ \delta(\mathcal{S})\,(1-\delta(\mathcal{S}))\,, & \text{if } \dfrac{1}{2} \le \delta(\mathcal{S}) \le 1\,. \end{cases}$$

It is also well known (see *e.g.*, [Ch1, p. 95], [Che], [FR, Theorem 2]) that

$$\frac{1}{4} \le b(\mathcal{S})\, h(\mathcal{S})^2 \qquad \text{and} \qquad b(\mathcal{S})\, h(\mathcal{S}) \le C < \frac{3}{2}\,,$$

where C is an absolute constant.

Therefore $\mathcal{S}$ has the HII-property if and only if $b(\mathcal{S}) > 0$ or, equivalently, $\delta(\mathcal{S}) < 1$.

A theorem of Myrberg [T, p. 522] states that if $\delta(\mathcal{S}) < 1$ then $\mathcal{S}$ has Green's function, or equivalently, that it possesses non-constant positive superharmonic functions (see [AS, p. 204] or [T, p. 434]). Therefore if $\mathcal{S}$ has finite genus, $\mathcal{S}$ has non-constant harmonic functions with finite Dirichlet integral [AS, p. 208], [SN, p. 332]. In the general case, the conclusion is also true with additional hypothesis [Ro1]. However, there exists a Riemann surface $\mathcal{S}_0$ having infinite genus and HII such that the constants are the unique positive harmonic functions in $\mathcal{S}_0$ [Ro2]. Recall that if there exists a non-constant harmonic function with finite Dirichlet integral, then there exists a non-constant positive (in fact, bounded) harmonic function.

It is also known that $\delta(\mathcal{S})$ coincides with the Hausdorff dimension of the conical limit set of the covering group of $\mathcal{S}$ (see *e.g.* [N, p. 154]). This says us that the HII-property must be also related with the size of the "boundary" of $\mathcal{S}$.

At the moment no characterization of the HII-property is known for hyperbolic plane domains (*i.e.* subsets of the Riemann sphere whose boundary has at least three points) in *euclidean* terms of the size of its boundary. In [FR, Theorems 3 and 4] a sufficient condition and a necessary condition were obtained so that a hyperbolic plane domain satisfies HII, but none of them constitutes a characterization of the HII-property, although these conditions are quite close.

As an example of the difficulties involving the problem, recall that a plane domain Ω has Green function if and only if its boundary has positive logarithmic capacity (see [AS, p. 249], [T, p. 440] or [SN, p. 332-333]). But, for example, $\mathbb{D} \setminus \{0\} \setminus \{1/2^n\}_{n=1}^{\infty}$ has not HII, while $\mathbb{D}\setminus\{1-1/2^n\}_{n=0}^{\infty}$ has it (these facts are consequence of [FR, Theorems 3 and 4] or Theorem 1 below). Hence, this shows that the problem of deciding whether a hyperbolic plane domain has the HII-property or not is delicate. Observe that if Ω is a hyperbolic plane domain and $\partial\Omega$ has zero logarithmic capacity, then Ω has not HII.

The main results of this paper are Theorems 1, 3, 5, 9 and 10. Theorem 1 shows that for any hyperbolic Riemann surface the HII-property is preserved by removing a sufficiently separated set. Theorem 3 relates simple euclidean conditions with the HII-property in Denjoy domains. Theorem 5 gives an euclidean characterization of Denjoy domains satisfying the HII-property. Finally, Theorems 9 and 10 give localization results for the HII-property in general planar domains.

In the next section we give some definitions needed to state our results.

2. The main results.

We say that a domain $G \subset \hat{\mathbb{C}}$ is *modulated* if there is an upper bound for the modulus of every doubly connected domain contained in G which separates the boundary of G. In particular, every simply connected domain is modulated (since in this case there are not such doubly connected domains). Also, if the boundary of G consists of a finite number of continua, G is modulated. On the other hand, if the boundary of G has an isolated point, G is not modulated.

These are the domains in the plane that as far as Function Theory is concerned behave almost like simply connected domains (see for example [BP] and the references therein).

In [FR, Theorem 3] it was proved that if $G \subset \hat{\mathbb{C}}$ is modulated (and therefore G has HII) then $H = G \setminus \{a_n\}$ has also HII if the sequence $\{a_n\}$ is *uniformly separated* in the hyperbolic metric of G, *i.e.* if there exists a positive constant c such that

$$d_G(a_n, a_m) > c\,, \qquad \text{for all } n \neq m\,,$$

where d_G denotes the hyperbolic distance in G. This result is not true if G is not modulated (see Theorem 1 below). Obviously, every finite sequence is uniformly separated, and a sequence converging to a point of G is not uniformly separated.

Conversely, also in [FR, Theorem 4], it was proved that if $H \subset \hat{\mathbb{C}}$ has HII, and $G = H \cup I$, where I is the set of isolated points of ∂H, then I is uniformly separated in the hyperbolic metric of G.

In this work we reduce the study of the HII-property of H to that of G, not only for hyperbolic plane domains, but for general hyperbolic Riemann surfaces.

To state our result, we need a previous definition.

Definition. *A subset I of a hyperbolic Riemann surface $\mathcal{S}$ is strongly uniformly separated in $\mathcal{S}$, if there exists a positive constant r_0 such that the hyperbolic balls $B_{\mathcal{S}}(p, r_0)$, where $p \in I$, are simply connected and pairwise disjoint.*

Theorem 1. *Let $\mathcal{S}$ be a hyperbolic Riemann surface, let I be a closed and countable subset of $\mathcal{S}$ and $\mathcal{R} = \mathcal{S} \setminus I$. Then, $\mathcal{R}$ has* HII *if and only if $\mathcal{S}$ has* HII *and I is strongly uniformly separated in $\mathcal{S}$.*

We also have obtained a relationship between the isoperimetric constants on $\mathcal{R}$ and $\mathcal{S}$ (see Section 3 below).

We want to remark that Theorem 1 is a new result even in the case of plane domains.

Corollary 1. *Let $\mathcal{S}$ be a hyperbolic Riemann surface, let I be a closed and countable subset of $\mathcal{S}$ and let $\mathcal{R} = \mathcal{S} \setminus I$. If I has an accumulation point in $\mathcal{S}$, then $\mathcal{R}$ has not* HII*.*

Observe that Theorem 1 and [FR, Theorem 3] give that every discrete set which is uniformly separated in a modulated domain G is also strongly uniformly separated in G.

As we mentioned above, at the moment no characterization of the HII-property is known for hyperbolic plane domains in euclidean terms of the size of its boundary. In [FR] it was obtained a necessary condition and a sufficient condition so that a hyperbolic plane domain has HII, but we know that none of them is, in fact, a characterization of the HII-property for this type of Riemann surfaces. In this paper we obtain a characterization of the HII-property for the case of *Denjoy domains*, *i.e.* hyperbolic plane domains whose boundary is contained in $\hat{\mathbb{R}}$, in euclidean terms of the size of their boundaries.

Since the HII-property is a quasiconformal invariant between general Riemann surfaces [FR, Theorem 1] our results characterize the HII-property for subsets of $\hat{\mathbb{C}}$ whose boundary is contained in a quasicircle. In fact we can prove a more general result (see Section 7).

Definition. *Let Ω be a plane domain, let I be the set of isolated points of $\partial\Omega$ and $\Omega_0 = \Omega \cup I$. We say that Ω is admissible if Ω_0 is a hyperbolic plane domain and I is strongly uniformly separated in Ω_0.*

Observe that if Ω is admissible, then there are no isolated points in $\partial\Omega_0$; therefore $\partial\Omega_0$ has infinitely many points and Ω has infinite area.

Now we can restate Corollary 1 for hyperbolic plane domains.

Corollary 2. *If a hyperbolic plane domain is not admissible, then it has not* HII.

In what follows $\Omega \subset \hat{\mathbb{C}}$ will usually be a Denjoy domain. In order to establish our characterization of the HII-property for Denjoy domains (Theorem 5) we need some preliminary background.

For $\alpha < \beta$, (β, α) denotes the set $\{x \in \mathbb{R} : \, x < \alpha \text{ or } x > \beta\} \cup \{\infty\}$. Also we mean that $(\infty, \alpha) = \{x \in \mathbb{R} : x < \alpha\}$ and as usual $(\alpha, \infty) = \{x \in \mathbb{R} : x > \alpha\}$. Along the paper we mean that the point at infinity is the greatest of the numbers in $\hat{\mathbb{R}}$.

Definition. *We say that a finite subset $A = \{a_1, \dots, a_{2n}\}$ $(n \geq 2)$ of points of $\partial\Omega \subset \hat{\mathbb{R}}$ is a border set of $\partial\Omega$ if A verifies the following two conditions:*

i) *A is "ordered" in $\hat{\mathbb{R}}$, i.e. there exists $j \in \mathbb{Z}_{2n} = \mathbb{Z}/(2\,n\,\mathbb{Z})$ such that $a_{j+1} < \cdots < a_{j+2n}$, where the subscripts belong to $\mathbb{Z}_{2n}$.*

ii) *The open set $\cup_{k=1}^{n} (a_{2k-1}, a_{2k})$ is contained in Ω.*

Obviously every finite subset $A = \{a_1, \dots, a_{2n}\}$ of $\hat{\mathbb{R}}$ can be "ordered" in such a way that the condition i) is satisfied. So ii) is the significant condition in the definition above.

Example. Let us consider the Denjoy domain Ω whose boundary is $\partial\Omega = \{\infty\} \cup (\cup_{n=1}^{\infty} [2\,n - 1, 2\,n])$. It is clear that the ordered sets $\{2, 3, 6, 7, 10, 11\}$ and $\{4, 5, \infty, 1\}$ are border sets of $\partial\Omega$, but $\{1, 4, 5, \infty\}$ is not. In fact, the ordered set of real numbers $\{a_1, \dots, a_{2n}\}$ is a border set if and only if $a_{2k} = 1 + a_{2k-1}$ and $a_{2k-1} \in 2\,\mathbb{Z}$ for $k = 1, \dots, n$. On the other hand, the ordered set $\{a_1, \dots, a_{2n-1}, \infty\}$ never is a border set. The ordered set $\{a_1, \dots, a_{2n-2}, \infty, a_{2n}\}$, with $n \geq 3$ and $a_1 < \cdots < a_{2n-2}$ is a border set if and only if $\{a_1, \dots, a_{2n-2}\}$ is a border set and $a_{2n} = 1$.

Observe that the set of four "consecutive points", $\{a_{2k-1}, a_{2k}, a_{2k+1}, a_{2k+2}\}$ with $k \in \mathbb{Z}_{2n}$, of a border set of Ω is also a border set of Ω. Besides, observe that if $\partial\Omega$ has not any border set, then Ω is some of the three following trivial domains (up conformal equivalence): $\mathbb{C}\setminus\{0,1\}$ (which has not HII), $\mathbb{C}\setminus[0,1]$ (which has HII), $\hat{\mathbb{C}}\setminus[0,1]$ (which has HII).

If $B = \{b_1, b_2, b_3, b_4\}$, we denote by $r(B)$ the cross ratio

$$r(B) = \frac{(b_2 - b_1)(b_4 - b_3)}{(b_3 - b_2)(b_4 - b_1)} \,. \tag{2.1}$$

In the following by $\Phi_1, \Phi_2 : (0,\infty) \longrightarrow (0,\infty)$ we denote any fixed continuous functions satisfying the following properties:

a) $\Phi_1(r) \asymp \Phi_2(r) \asymp \dfrac{1}{\log r}$ as $r \longrightarrow \infty$,

b) $\Phi_1(r) \asymp \log \dfrac{1}{r}$ and $\Phi_2(r) \asymp \log\log \dfrac{1}{r}$ as $r \longrightarrow 0$.

After these preliminaries we can state the following partial result which gives a necessary condition and a sufficient condition for the HII-property of a Denjoy domain Ω. In many cases these conditions give an answer to the question of whether or not Ω has HII, since they are very close.

Theorem 3. *Let Ω be an admissible Denjoy domain, let I be the set of isolated points of $\partial\Omega$ and $\Omega_0 = \Omega \cup I$.*

a) *If Ω has* HII, *then there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that*

$$\frac{1}{n} \sum_{j=1}^{n} \Phi_1(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c \,.$$

b) *If there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that*

$$\frac{1}{n} \sum_{j=1}^{n} \Phi_2(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c \,,$$

then Ω *has* HII.

Besides, we have a characterization of the Denjoy domains with HII in euclidean terms of the size of their boundaries. This characterization has the disadvantage that the function which appears instead of Φ_1 and Φ_2 (in Theorem 3) is more complicated and depends on the domain.

Theorem 5. *Let* Ω *be a Denjoy domain, let* I *be the set of isolated points of* $\partial\Omega$ *and let* $\Omega_0 = \Omega \cup I$. *Then,* Ω *has* HII *if and only if* Ω *is admissible and there exists a positive constant* c *such that for any border set of* $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ *with* $n \geq 3$, *we have that*

$$\frac{1}{n} \sum_{j=1}^{n} \Psi_{\Omega_0}(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}) > c\,,$$

where Ψ_{Ω_0} *is the function appearing in Theorem* 4 (*see Section* 5).

Roughly speaking, this function Ψ_{Ω_0} "counts" in some sense the number of annuli which intersect $\partial\Omega_0$.

If $\mathcal{S}_2$ is a hyperbolic Riemann surface, we will consider (open and connected) subsurfaces $\mathcal{S}_1 \subset \mathcal{S}_2$, endowed with its own hyperbolic metric (recall that any subsurface of a hyperbolic Riemann surface is also hyperbolic). Of course, with this metric $\mathcal{S}_1$ is a complete Riemannian manifold.

As a direct consequence of Corollary 7 (see Section 7 below) we obtain two localization theorems.

Theorem 9. *Given a closed subset* E *of* $\hat{\mathbb{C}}$ *with infinitely many points, the following conditions are equivalent*:

1) $\hat{\mathbb{C}} \setminus E$ *satisfies* HII.

2) $\Omega \setminus E$ *satisfies* HII, *for any subdomain* Ω *of* $\hat{\mathbb{C}}$ *of finite type such that* E *is contained in* Ω.

3) $\Omega \setminus E$ *satisfies* HII, *for some subdomain* Ω *of* $\hat{\mathbb{C}}$ *of finite type such that* E *is contained in* Ω.

Theorem 10. *Let* $E_1, \dots, E_n$ *be pairwise disjoint closed subsets in* $\hat{\mathbb{C}}$ *with infinitely many points such that* $\Omega = \hat{\mathbb{C}} \setminus \cup_k E_k$ *is connected. Then, we have that* Ω *satisfies* HII *if and only if* $\hat{\mathbb{C}} \setminus E_k$ *satisfies* HII *for* $k = 1, \dots, n$.

In fact, we prove in Section 7 a general version of theorems 9 and 10 about Riemann surfaces (see Theorem 7). We should remark that we have also obtained other results on localization (see for example Lemmas 3.1 and 7.1 or Corollary 5).

2.1. Notations and background.

As usual, $\mathbb{R}$ and $\hat{\mathbb{R}}$ will denote the real line and the extended real line. Similarly, $\mathbb{C}$ and $\hat{\mathbb{C}}$ will denote, respectively, the complex plane and the Riemann sphere. The simbol $A \setminus B$ denotes the difference of the sets A and B. The expression $A(r) \asymp B(r)$ will mean that there exists a positive constant C such that

$$C^{-1} \le \frac{A(r)}{B(r)} \le C\,,$$

for the values of r indicated in each case. We denote by $[x]$ the greatest natural number which is less or equal than x.

By $d_{\mathcal{S}}$ and $B_{\mathcal{S}}$ we shall denote, respectively, the distance and the balls in the Poincaré metric of $\mathcal{S}$. By d and B we shall denote, respectively, the distance and the balls in the euclidean metric of $\mathbb{C}$. $B^*_{\mathcal{S}}$ and B^* will denote the corresponding balls without its centers. If Ω is a hyperbolic plane domain, $\delta_\Omega(z)$ will be the euclidean distance of z to the boundary of Ω. By λ_Ω we shall denote the conformal density of the Poincaré metric in Ω, *i.e.* the function such that $ds = \lambda_\Omega(z)\,|dz|$ is the Poincaré metric in Ω. For $\alpha < \beta$, (β, α) denotes the set $\{x \in \mathbb{R} : \ x < \alpha \text{ or } x > \beta\} \cup \{\infty\}$. Also we mean that $(\infty, \alpha) = \{x \in \mathbb{R} : \ x < \alpha\}$ and as usual $(\alpha, \infty) = \{x \in \mathbb{R} : \ x > \alpha\}$. We define the corresponding closed intervals in a similar way. Along the paper we mean that the point at infinity is the greatest of the numbers in $\hat{\mathbb{R}}$.

Finally, we denote by c positive constants which can assume different values from line to line and even in the same line. On the other hand, the constants t_0 and r_0 will have always the same value.

In order to prove our results we shall need some well known facts concerning the Poincaré metric:

1) A conformal map between two hyperbolic Riemann surfaces is an isometry.

2) If $\mathcal{S}_1$ is a subsurface of the hyperbolic Riemann surface $\mathcal{S}_2$, then $d_{\mathcal{S}_1}(p,q) \ge d_{\mathcal{S}_2}(p,q)$, for all $p, q \in \mathcal{S}_1$.

3) Let $\mathcal{S}_1$ be a subsurface of the hyperbolic Riemann surface $\mathcal{S}_2$ and let σ be a simple closed curve in $\mathcal{S}_1$. Denote by γ_j the simple closed geodesic (if exists) freely homotopic to σ in $\mathcal{S}_j$ $(j = 1, 2)$. Then $L_{\mathcal{S}_1}(\gamma_1) \geq L_{\mathcal{S}_2}(\gamma_2)$.

4) If Ω is a hyperbolic plane domain, $\Omega \subset \mathbb{C}$, then $\lambda_\Omega(z) \leq 2/\delta_\Omega(z)$, for all $z \in \Omega$ (recall that $\delta_\Omega(z)$ denotes the euclidean distance of z to the boundary of Ω).

5) A hyperbolic plane domain Ω, $\Omega \subset \mathbb{C}$, is modulated if and only $\lambda_\Omega(z)\, \delta_\Omega(z) \asymp 1$, for $z \in \Omega$ (see [BP, Corollary 1]). The constant in $\asymp$ depends on Ω.

6) For $\Omega \subset \mathbb{C}$, define $\beta_\Omega(z)$ as the function

$$\beta_\Omega(z) = \inf \left\{ \left| \log \left| \frac{z-a}{b-a} \right| \right| : \ a, b \in \partial\Omega, \ |z-a| = \delta_\Omega(z) \right\} . \tag{2.2}$$

In [BP, Theorem 1] it was proved that

$$\lambda_\Omega(z)\, \delta_\Omega(z)\, (1 + \beta_\Omega(z)) \asymp 1 , \qquad \text{for } z \in \Omega , \tag{2.3}$$

up to universal constants. See (6.1) below for a precise estimate.

7) If $F : \mathbb{D} \longrightarrow \Omega$ is a universal covering map, then we have

$$\lambda_\Omega(F(z))\, |F'(z)| = \lambda_{\mathbb{D}}(z) , \qquad \text{for all } z \in \mathbb{D} .$$

The organization of the paper is as follows. In sections 3 and 4 we prove, respectively, theorems 1 and 3. Theorems 4 and 5 will be proved in Section 5. Section 6 contains a proposition relating balls and collars of punctures. In Section 7 we develop some useful technology to prove theorems 9 and 10 and other further results. In Section 8 we discuss the relationship between the HII-property, polarization and circular symmetrization. Finally we discuss about the possibility to improve Theorem 5 in sections 9 and 10.

3. Proof of Theorem 1.

Theorem 1. *Let $\mathcal{S}$ be a hyperbolic Riemann surface, let I be a closed and countable subset of $\mathcal{S}$ and $\mathcal{R} = \mathcal{S} \setminus I$. Then, $\mathcal{R}$ has* HII *if and only if $\mathcal{S}$ has* HII *and I is strongly uniformly separated in $\mathcal{S}$.*

More precisely, if r_0 *is a positive number such that* $\{B_{\mathcal{S}}(p,r_0)\}_{p\in I}$ *is a family of pairwise disjoint and simply connected balls in* $\mathcal{S}$*, then we have that*

$$h(\mathcal{R}) \le \frac{h(\mathcal{S})}{\tanh^2\left(\frac{r_0}{4}\right)} + \frac{2\pi}{r_0 \log \dfrac{\tanh r_0}{\tanh\left(\frac{r_0}{4}\right)}} \,.$$

The difficult implication in this theorem is to prove that $\mathcal{R}$ has HII. Our proof of this consists of finding a relationship between the Poincaré metrics of $\mathcal{R}$ and $\mathcal{S}$. Far from the points in I both metrics are comparable (see Lemma 3.1 below). Close to these isolated points they are not comparable but, in fact, there exists a very precise relation between the $\mathcal{S}$-balls centered at points in I and its corresponding collars in $\mathcal{R}$ (see Proposition 1 in Section 6).

We start by studying the relationship between the Poincaré metrics of $\mathcal{R}$ and $\mathcal{S}$.

Lemma 3.1. *Let* $\mathcal{S}$ *be a hyperbolic Riemann surface, let* C *be a closed non-empty subset of* $\mathcal{S}$ *and* $\mathcal{S}^* = \mathcal{S}\setminus C$*. Let us consider a positive number* ε*. Then we have that*

$$\tanh\frac{\varepsilon}{2} < \frac{L_{\mathcal{S}}(\gamma)}{L_{\mathcal{S}^*}(\gamma)} < 1\,, \tag{3.1}$$

for every curve $\gamma\subset\mathcal{S}$ *with finite length in* $\mathcal{S}$ *such that* $d_{\mathcal{S}}(\gamma,C)\ge\varepsilon$*, and*

$$\left(\tanh\frac{\varepsilon}{2}\right)^2 < \frac{A_{\mathcal{S}}(D)}{A_{\mathcal{S}^*}(D)} < 1\,, \tag{3.2}$$

for every domain $D\subset\mathcal{S}$ *with finite area in* $\mathcal{S}$ *such that* $d_{\mathcal{S}}(D,C)\ge\varepsilon$*.*

Proof. We prove Lemma 3.1 in local coordinates.

Let us fix $p\in\mathcal{S}$ with $d_{\mathcal{S}}(p,C)\ge\varepsilon$ and let us consider a local chart $\phi: V\longrightarrow\mathbb{C}$ with $\phi(p)=0$.

Let $F:\mathbb{D}\longrightarrow\mathcal{S}$ be a universal covering map with $F(0)=p$. The set $C'=F^{-1}(C)$ is a closed subset of the unit disk. Obviously the euclidean ball $B(0,\tanh(\varepsilon/2)) = B_{\mathbb{D}}(0,\varepsilon)$ is a connected component of $F^{-1}(B_{\mathcal{S}}(p,\varepsilon))$; it is contained in $\mathbb{D}\setminus C'$ and the mapping $F:\mathbb{D}\setminus C'\longrightarrow\mathcal{S}^*$ is a covering map with $F(0)=p$. Let $G:\mathbb{D}\longrightarrow\mathbb{D}\setminus C'$ be a universal covering map with $G(0) = 0$. We have that $F \circ G : \mathbb{D} \longrightarrow \mathcal{S}^*$ is a universal covering map with $(F \circ G)(0) = p$.

Let us consider the Poincaré metrics $\lambda_{\mathcal{S}}(z)\,|dz|$ and $\lambda_{\mathcal{S}^*}(z)\,|dz|$ in local coordinates $(z \in \phi(V))$. Then

$$\lambda_{\mathcal{S}}((\phi \circ F)(0))\,|(\phi \circ F)'(0)| = \lambda_{\mathbb{D}}(0)\,,$$
$$\lambda_{\mathcal{S}^*}((\phi \circ F \circ G)(0))\,|(\phi \circ F \circ G)'(0)| = \lambda_{\mathbb{D}}(0)\,,$$

and this gives

$$\lambda_{\mathcal{S}}(0)\,|(\phi \circ F)'(0)| = 2\,, \qquad \lambda_{\mathcal{S}^*}(0)\,|(\phi \circ F)'(0)|\,|G'(0)| = 2\,.$$

These last equalities give Lemma 3.1 if we prove that $\tanh(\varepsilon/2) < |G'(0)| < 1$ since this is the infinitesimal version of (3.1) and (3.2).

Observe that $G : \mathbb{D} \longrightarrow \mathbb{D}$ satisfies $G(0) = 0$. Schwarz's Lemma gives the inequality $|G'(0)| < 1$.

Recall that the simply connected set $B(0, \tanh(\varepsilon/2))$ is contained in $\mathbb{D} \setminus C'$. Therefore, there exists a well defined local inverse $G^{-1} : B(0, \tanh(\varepsilon/2)) \longrightarrow \mathbb{D}$ verifying $G^{-1}(0) = 0$. Using again Schwarz's Lemma we obtain that

$$|(G^{-1})'(0)| = \frac{1}{|G'(0)|} < \operatorname{cotanh}\left(\frac{\varepsilon}{2}\right).$$

This finishes the proof of Lemma 3.1.

Proof of Theorem 1. We begin with the proof that if $\mathcal{R}$ has HII then $\mathcal{S}$ has it and I is strongly uniformly separated in $\mathcal{S}$.

We shall prove first that I is a discrete set. In fact, if this is not the case, then I is not strongly uniformly separated and, as we shall see, this implies that $\mathcal{R}$ has not HII, a contradiction.

Let us assume that I is not a discrete set. Let $F : \mathbb{D} \longrightarrow \mathcal{S}$ be a universal covering map and let J be the preimage of I by F. Then $F : \mathbb{D} \setminus J \longrightarrow \mathcal{R}$ is a covering map. Therefore, $\delta(\mathbb{D} \setminus J) \le \delta(\mathcal{R})$ (see, for example [FR, p. 181]). Obviously, J is a closed, countable and non discrete subset of $\mathbb{D}$. Let z_0 be an accumulation point of J in $\mathbb{D}$. Then, we have that $B(z_0, r) \cap \partial(\mathbb{D} \setminus J) = B(z_0, r) \cap J$ is countable, for $0 < r < 1 - |z_0|$, and therefore it has zero logarithmic capacity. [FR, Theorem 4] implies that $1 = \delta(\mathbb{D} \setminus J) \le \delta(\mathcal{R}) \le 1$. But, if $\delta(\mathcal{R}) = 1$, a fortiori, $\mathcal{R}$ has not HII.

A theorem of Patterson [P2, Theorem 4.1] gives that $\delta(\mathcal{S}) \le \delta(\mathcal{R})$, since I is discrete. Therefore $\delta(\mathcal{S}) < 1$ and $\mathcal{S}$ has HII.

Suppose that the discrete set I is not strongly uniformly separated. Let us see that, then, $\mathcal{R}$ has not HII, a contradiction. Denote again by $F : \mathbb{D} \longrightarrow \mathcal{S}$ a universal covering map and by J the preimage of I by F. As before $F : \mathbb{D} \setminus J \longrightarrow \mathcal{R}$ is a covering map and therefore, $\delta(\mathbb{D}\setminus J) \le \delta(\mathcal{R})$ (see, for example again [FR, p. 181]). We have that for each $\varepsilon > 0$, there exist points $p, q \in I$ such that either $d_{\mathcal{S}}(p,q) < \varepsilon$ or $B_{\mathcal{S}}(p,\varepsilon)$ is not simply connected. This implies that there exist $z, w \in J$ such that $d_{\mathbb{D}}(z,w) < \varepsilon$, *i.e.* that J is not uniformly separated in $\mathbb{D}$. [FR, Theorem 4] implies again that $\delta(\mathcal{R}) \ge \delta(\mathbb{D} \setminus J) = 1$.

Let us assume now that $\mathcal{S}$ has HII and I is strongly uniformly separated in $\mathcal{S}$. We want to prove that then $\mathcal{R}$ has also HII.

Let $\mathcal{D}$ be an open subset of $\mathcal{R}$ with finite area. In order to check (1.1) for $\mathcal{D}$, we can assume without loss of generality that $\mathcal{D}$ is not simply or doubly connected since this particular type of subsets always satisfy HII with constant 1 [FR, Lemma 1.1]. We can also suppose that $\partial\mathcal{D} = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ where the simple closed curves γ_j are not homotopic to the trivial loop and does not "surround" only a puncture. In fact, if this would be the case for γ_j, say, we could join to $\mathcal{D}$ the simply or doubly connected open set whose boundary is γ_j, obtaining by this way a new domain with greater area and whose boundary had less length.

Let us consider a positive number r_0 such that the balls $B_{\mathcal{S}}(p, r_0)$ with $p \in I$ are simply connected and pairwise disjoint. Let $\tilde{\mathcal{S}}$ be the subset of $\mathcal{R}$ given by $\tilde{\mathcal{S}} = \mathcal{S} \setminus \cup_{p\in I} B_{\mathcal{S}}(p, r_0/2)$. Let J, J_1, J_2, be the subsets of I defined by

$$J = \{p \in I : \mathcal{D} \cap B_{\mathcal{S}}(p, r_0/2)^* \neq \varnothing\}\,,$$

$$J_1 = \{p \in J : B_{\mathcal{S}}(p, r_0/2)^* \subset \mathcal{D}\}\,,$$

$$J_2 = \{p \in J : \partial\mathcal{D} \cap B_{\mathcal{S}}(p, r_0/2)^* \neq \varnothing\}\,.$$

It is obvious that $\{J_1, J_2\}$ is a partition of J.

First of all we remark that

$$L_{\mathcal{S}}(\partial\mathcal{D} \cap B_{\mathcal{S}}(p, r_0)) \ge r_0\,, \qquad \text{for all } p \in J_2\,. \tag{3.3}$$

To see this, suppose that $L_{\mathcal{S}}(\partial\mathcal{D} \cap B_{\mathcal{S}}(p, r_0)) < r_0$ for some $p \in J_2$. Then, we have that there exists a boundary curve γ_j with $L_{\mathcal{S}}(\gamma_j) < r_0$; such a curve must verify that $\gamma_j \subset B_{\mathcal{S}}(p, r_0)$ since

$$d_{\mathcal{S}}\Big(\partial B_{\mathcal{S}}(p, r_0), \partial B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)\Big) = \frac{r_0}{2}\,.$$

But, if $\gamma_j \subset B_{\mathcal{S}}(p, r_0)$, then γ_j is homotopic in $\mathcal{R}$ to zero or to p, and this is not possible.

Claim. There exists a constant c, which only depends on r_0 and neither on $\mathcal{S}$ nor I, such that

$$A_{\mathcal{R}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)^*\Big) \le c\,, \qquad \text{for every } p \in I\,. \tag{3.4}$$

Then we have that

$$A_{\mathcal{R}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)^*\Big) \le \frac{c}{4\pi \sinh^2\big(\frac{r_0}{4}\big)}\, A_{\mathcal{S}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)\Big)\,, \tag{3.5}$$

for every $p \in J_1$; since $B_{\mathcal{S}}(p, r_0/2)$ is simply connected and then

$$A_{\mathcal{S}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)\Big) = A_{\mathbb{D}}\Big(B_{\mathbb{D}}\Big(0, \frac{r_0}{2}\Big)\Big) = 4\pi \sinh^2\Big(\frac{r_0}{4}\Big)\,,$$

and by (3.3)

$$A_{\mathcal{R}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)^*\Big) \le \frac{c}{r_0}\, L_{\mathcal{S}}(\partial\mathcal{D} \cap B_{\mathcal{S}}(p, r_0))\,, \tag{3.6}$$

for every $p \in J_2$. Using (3.2), (3.5) and (3.6), we have that

$$\begin{aligned}
A_{\mathcal{R}}(\mathcal{D}) &\le A_{\mathcal{R}}(\mathcal{D} \cap \tilde{\mathcal{S}}) + \sum_{p \in J_1} A_{\mathcal{R}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)^*\Big) + \sum_{p \in J_2} A_{\mathcal{R}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)^*\Big) \\
&\le \frac{1}{\tanh^2\big(\frac{r_0}{4}\big)}\, A_{\mathcal{S}}(\mathcal{D} \cap \tilde{\mathcal{S}}) \\
&\quad + \frac{c}{4\pi \sinh^2\big(\frac{r_0}{4}\big)} \sum_{p \in J_1} A_{\mathcal{S}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)\Big) + \frac{c}{r_0} L_{\mathcal{S}}(\partial\mathcal{D})\,.
\end{aligned}$$

Let H be

$$H = \max\left\{ \frac{1}{\tanh^2\big(\frac{r_0}{4}\big)}, \frac{c}{4\pi \sinh^2\big(\frac{r_0}{4}\big)} \right\}.$$

Therefore

$$\begin{aligned}
A_{\mathcal{R}}(\mathcal{D}) &\le H\Big(A_{\mathcal{S}}(\mathcal{D} \cap \tilde{\mathcal{S}}) + \sum_{p \in J_1} A_{\mathcal{S}}\Big(B_{\mathcal{S}}\Big(p, \frac{r_0}{2}\Big)\Big)\Big) + \frac{c}{r_0} L_{\mathcal{S}}(\partial\mathcal{D}) \\
&\le H\, A_{\mathcal{S}}(\mathcal{D}) + \frac{c}{r_0}\, L_{\mathcal{S}}(\partial\mathcal{D}) \\
&\le H\, h(\mathcal{S})\, L_{\mathcal{S}}(\partial\mathcal{D}) + \frac{c}{r_0}\, L_{\mathcal{S}}(\partial\mathcal{D}) \\
&\le \Big(H\, h(\mathcal{S}) + \frac{c}{r_0}\Big) L_{\mathcal{R}}(\partial\mathcal{D})\,.
\end{aligned}$$

Then we have that $\mathcal{R}$ has HII with constant

$$h(\mathcal{R}) \le H\, h(\mathcal{S}) + \frac{c}{r_0}\,. \tag{3.7}$$

To finish the proof of Theorem 1 we only need to prove (3.4) with

$$c = \frac{2\pi}{\log \dfrac{\tanh r_0}{\tanh\left(\dfrac{r_0}{4}\right)}}\,,$$

since we will see below that, with this value of c, we have that

$$H = \frac{1}{\tanh^2\left(\dfrac{r_0}{4}\right)}\,.$$

Lemma 3.2. *Let $\mathcal{S}$ be a hyperbolic Riemann surface and $\{B_{\mathcal{S}}(p, r_0)\}_{p\in I}$ be a disjoint family of simply connected balls in $\mathcal{S}$. If $\mathcal{R} = \mathcal{S}\setminus I$, then we have that*

$$A_{\mathcal{R}}(B_{\mathcal{S}}(p,r)^*) \le \frac{2\pi}{\log \dfrac{\tanh r_0}{\tanh\left(\dfrac{r}{2}\right)}}\,, \qquad \textit{for } 0 < r \le r_0\,.$$

Proof. Let us fix a point $p \in I$. Let us consider a universal covering map $F : \mathbb{D} \longrightarrow \mathcal{S}$ such that $F(0) = p$. Let J be the preimage of I by F. The intersection of the ball $F^{-1}(B_{\mathcal{S}}(p, 2\,r_0)) = B_{\mathbb{D}}(0, 2\,r_0) = B(0, \tanh r_0)$ with the set J is exactly $\{0\}$. Since $F : \mathbb{D}\setminus J \longrightarrow \mathcal{R}$ is a covering map, it follows that for $0 < r \le r_0$

$$\begin{aligned}
A_{\mathcal{R}}(B_{\mathcal{S}}(p,r)^*) &= A_{\mathbb{D}\setminus J}(B_{\mathbb{D}}(0,r)^*)\\
&\le A_{B(0,\tanh r_0)^*}(B_{\mathbb{D}}(0,r)^*)\\
&= A_{B(0,\tanh r_0)^*}\Big(B\Big(0, \tanh\Big(\frac{r}{2}\Big)\Big)^*\Big)\\
&= A_{\mathbb{D}^*}\left(B\left(0, \frac{\tanh\left(\dfrac{r}{2}\right)}{\tanh r_0}\right)^*\right)\\
&= \int_{B(0,\tanh(r/2)/\tanh r_0)^*} \frac{dx\,dy}{(|z|\log|z|)^2}\\
&= \frac{2\pi}{\log \dfrac{\tanh r_0}{\tanh\left(\dfrac{r}{2}\right)}}\,.
\end{aligned}$$

This finishes the proof of Lemma 3.2.

The estimate (3.4) follows now from Lemma 3.2 with

$$c=\frac{2\pi}{\log\dfrac{\tanh r_0}{\tanh\left(\dfrac{r_0}{4}\right)}}\,, \tag{3.8}$$

and therefore (3.7) and (3.8) give the inequality in Theorem 1, if

$$H=\frac{1}{\tanh^2\left(\dfrac{r_0}{4}\right)}\,.$$

In order to prove this, we only need to check that

$$\frac{1}{\tanh^2\left(\dfrac{r_0}{4}\right)}\geq\frac{1}{2\sinh^2\left(\dfrac{r_0}{4}\right)\log\dfrac{\tanh r_0}{\tanh\left(\dfrac{r_0}{4}\right)}}\,,$$

and this follows from the fact that

$$G(x)=2\log\frac{\tanh x}{\tanh\left(\dfrac{x}{4}\right)}-\frac{1}{\cosh^2\left(\dfrac{x}{4}\right)}\geq 0\,,\qquad \text{for } x>0\,.$$

This inequality is a consequence of the fact that

$$\begin{aligned}G'(x)&=\frac{4}{2\sinh x\cosh x}-\frac{1}{2\sinh\left(\dfrac{x}{4}\right)\cosh^3\left(\dfrac{x}{4}\right)}\\&=\frac{4}{\sinh(2\,x)}-\frac{4}{2\sinh\left(\dfrac{x}{2}\right)+\sinh x}\\&\leq 0\,,\end{aligned}$$

and

$$\lim_{x\to\infty}G(x)=0\,.$$

REMARK. The inequality (3.4) can be obtained alternatively from Proposition 1. This proposition will be stated and proved in Section

6. We have used here Lemma 3.2 since it gives best estimates in this context.

4. Proof of Theorem 3.

Theorem 3. *Let Ω be an admissible Denjoy domain, let I be the set of isolated points of $\partial\Omega$ and $\Omega_0 = \Omega \cup I$.*

a) *If Ω has* HII, *then there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that*

$$\frac{1}{n}\sum_{j=1}^{n} \Phi_1(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,.$$

b) *If there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that*

$$\frac{1}{n}\sum_{j=1}^{n} \Phi_2(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,,$$

then Ω has HII.

Theorem 3 is a direct consequence of Theorem 1 and the following result.

Theorem 2. *Let Ω be a Denjoy domain such that $\partial\Omega$ has no isolated points. Then*

1) *If Ω has* HII, *then there exists a positive constant c such that for any border set of $\partial\Omega$, $B = \{b_1, \dots, b_{2n}\}$ $(n \geq 3)$, we have that*

$$\frac{1}{n}\sum_{j=1}^{n} \Phi_1(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,.$$

2) *If there exists a positive constant c such that for any border set of $\partial\Omega$, $B = \{b_1, \dots, b_{2n}\}$ $(n \geq 3)$, we have that*

$$\frac{1}{n}\sum_{j=1}^{n} \Phi_2(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,,$$

then Ω *has* HII.

The proof of Theorem 2 has three main ideas. The first one (see Lemma 4.1) is to reduce dramatically the set of domains in which we must check (1.1). Secondly, we will establish a bijective correspondence between these domains and border sets (see Lemma 4.2). Finally, we relate the length of each boundary curve of these domains with the length of some curves in some extremal domains which is given by the functions Φ_1 and Φ_2 (see Lemmas 4.3 and 4.5).

A *geodesic domain* in a Riemann surface $\mathcal{S}$ is a domain $G \subset \mathcal{S}$ (which is not simply or doubly connected) such that ∂G consists of finitely many simple closed geodesics, and $A_{\mathcal{S}}(G)$ is finite. G does not have to be relatively compact since it may "surround" finitely many punctures (isolated points in $\partial\mathcal{S}$ in the case that $\mathcal{S} \subset \hat{\mathbb{C}}$). We can think of a puncture as a boundary geodesic of zero length. Recall that if γ is a closed curve in $\mathcal{S}$ and $[\gamma]$ denotes its free homotopy class in $\mathcal{S}$, then there is a unique simple closed geodesic of minimal length in the class, unless γ is homotopic to zero or surrounds only a puncture; in these cases it is not possible to find such geodesic because there are curves in the class with arbitrary small length.

In [FR, Lemma 1.2] it was proved that if $\mathcal{S}$ verifies (1.1) for geodesic domains, then it verifies HII. In fact, if $h_g(\mathcal{S})$ is the infimum of the constants h such that the inequality

$$A_{\mathcal{S}}(G) \le h\, L_{\mathcal{S}}(\partial G)\,,$$

is true for any geodesic domain G, we have that

$$h(\mathcal{S}) \le h_g(\mathcal{S}) + 2\,.$$

We shall prove now that if a Denjoy domain Ω verifies (1.1) for geodesic domains which are symmetric with respect to the real axis (SG-*domains*), then Ω verifies (1.1) for any geodesic domain and therefore it verifies HII.

In fact, we have the following result, which is true even if $\partial\Omega$ has isolated points.

Lemma 4.1. *Let* Ω *be a Denjoy domain satisfying the inequality*

$$A_\Omega(G) \le h\, L_\Omega(\partial G)\,,$$

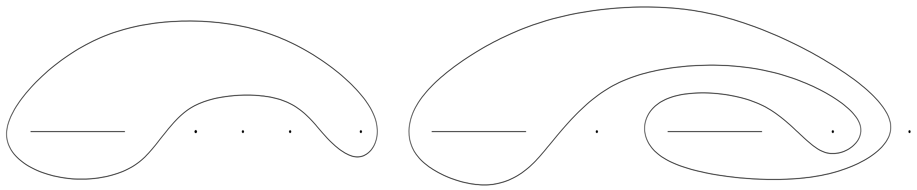
for every SG-*domain* G *in* Ω *and for a positive constant* h.

Then Ω *satisfies* HII *with*

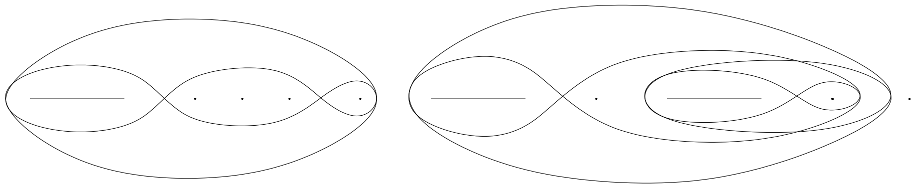
$$h_g(\Omega) \le 2\,h \qquad \textit{and} \qquad h(\Omega) \le 2\,h + 2\,.$$

Proof. Let G be a geodesic domain in Ω. Without loss of generality we can suppose that G contains the point at infinity. Let us consider the family $\mathcal{F}_1$ of subarcs of ∂G which joins two points of the real axis and are contained either in $\{z :\ \operatorname{Im} z \ge 0\}$ or in $\{z :\ \operatorname{Im} z \le 0\}$ and reflect each of them with respect to the real axis. We obtain in this way a family of closed curves $\mathcal{F}_2$. Let $\mathcal{F}_3$ be the family constituted by all simple closed geodesics in Ω which are freely homotopic to some curve of $\mathcal{F}_2$. We construct now a new family $\mathcal{F}_4$ from $\mathcal{F}_3$ in the following way: a curve γ of $\mathcal{F}_3$ belongs to $\mathcal{F}_4$ if and only if the bounded (in the euclidean sense) Jordan domain $\mathcal{J}$ such that $\partial \mathcal{J} = \gamma$ does not contain any other curve in $\mathcal{F}_3$ and $\mathcal{J} \cap \partial\Omega$ is not a finite set. Observe that the negative curvature implies that any two geodesics γ_1, γ_2 in $\mathcal{F}_3$ are disjoint; therefore either $\mathcal{J}_1$ and $\mathcal{J}_2$ are disjoint, either $\mathcal{J}_1 \subset \mathcal{J}_2$ or $\mathcal{J}_2 \subset \mathcal{J}_1$. Let G_0 be the SG-domain whose boundary is constituted by the curves in $\mathcal{F}_4$.

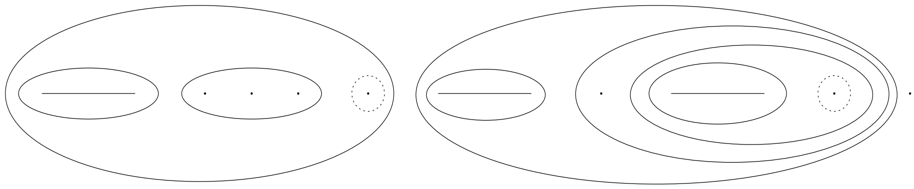
To illustrate this construction, let us consider for example the geodesic domain G shown as the exterior of the curves in this picture.



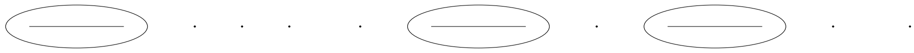
Then, the family of curves $\mathcal{F}_2$ looks like the following



The family of simple closed geodesics $\mathcal{F}_3$ is shown by



Note that the dotted curves in the last picture represent the free homotopy classes without geodesics; they are not in $\mathcal{F}_3$. Finally, the geodesic domain G_0 is the exterior of the geodesics in



It is clear that

$$L_\Omega(\partial G_0) \leq 2\, L_\Omega(\partial G)\,.$$

Let now n, p be, respectively, the number of simple closed geodesics limiting G and the number of punctures in G. Let also n_0, p_0 be the corresponding numbers for G_0. Observe that $n_0 + p_0 \geq n + p$. To see this, let us consider the set $\Gamma(G)$ of *generalized geodesics* limiting G, *i.e.* the union of the set of n geodesics in ∂G and the set of p punctures "surrounded" by G. We want to show that

$$\operatorname{card}\Gamma(G) \leq \operatorname{card}\Gamma(G_0)\,.$$

If a puncture is surrounded by G it is also surrounded by G_0. On the other hand, given a geodesic γ of ∂G let us consider the bounded (in the euclidean sense) Jordan domain $\mathcal{J}$ with $\partial\mathcal{J} = \gamma$; if the intersection of $\mathcal{J}$ with the real axis has m connected components, the geodesic γ "generates" at least m generalized geodesics of ∂G_0. This gives that $n_0 + p_0 \geq n + p$.

Gauss-Bonnet theorem gives that

$$A_\Omega(G_0) = 2\pi\,(n_0 + p_0 - 2) \geq 2\pi\,(n + p - 2) = A_\Omega(G)\,,$$

since the hyperbolic metric of Ω has curvature -1.

Therefore

$$A_\Omega(G) \leq A_\Omega(G_0) \leq h\, L_\Omega(\partial G_0) \leq 2\, h\, L_\Omega(\partial G)\,,$$

and so we have proved the first inequality in Lemma 4.1. The second inequality is a consequence of the first one and [FR, Lemma 1.2].

Given a border set of $\partial\Omega$ with four points, $B = \{b_1, b_2, b_3, b_4\}$, we denote by $\gamma(B)$ the unique simple closed geodesic in Ω which separates $[b_2, b_3]$ from $[b_4, b_1]$.

Lemma 4.2. *A Denjoy domain Ω such that $\partial\Omega$ has no isolated points has* HII *if and only if there exists a positive constant c such that for any border set of $\partial\Omega$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that*

$$\frac{1}{n} \sum_{j=1}^{n} L_\Omega(\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,.$$

Proof. Observe that we can establish a one to one correspondence between border sets of $\partial\Omega$ with $n \geq 3$ and SG-domains in Ω. Given a border set B of $\partial\Omega$, $B = \{b_1, \dots, b_{2n}\}$, let us consider the set of n geodesics ($n \geq 3$)

$$\mathcal{G} = \{\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}) : \; j = 1, \dots, n\}\,.$$

The curves in $\mathcal{G}$ limit a geodesic domain G associated to B. Observe that if $n = 2$, both geodesics are the same and then obviously they do not limit a geodesic domain.

It is clear that this process has a well defined inverse. Gauss-Bonnet theorem gives that $A(G) = 2\pi\,(n-2)$. Therefore, we have that $A(G) \asymp n$. This fact and Lemma 4.1 give Lemma 4.2.

It is clear that if we define

$$\begin{aligned} \Omega_1 &= \hat{\mathbb{C}} \setminus ([b_{2j}, b_{2j+1}] \cup [b_{2j+2}, b_{2j-1}])\,, \\ \Omega_2 &= \hat{\mathbb{C}} \setminus \{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}\,, \end{aligned}$$

then we have

$$\begin{aligned} L_{\Omega_2}(\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) &\leq L_\Omega(\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) \\ &\leq L_{\Omega_1}(\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}))\,, \end{aligned}$$

since $\Omega_1 \subset \Omega \subset \Omega_2$.

In order to prove Theorem 2 we only need to relate the length in Ω_1 and Ω_2 of the geodesic

$$\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})$$

with the functions Φ_1 and Φ_2 (see their definitions after (2.1)).

The following result gives an estimate of the hyperbolic length of the imaginary axis in some normalized Denjoy domains. This curve is important because it is the geodesic (in many symmetric cases (see, *e.g.* Lemma 4.5)) whose length we want to estimate.

We recall that $[x]$ denotes the greatest natural number which is less or equal than x.

Lemma 4.3. *Let us fix a number $0 < a < 1$ and let $0 < t < 1$. For each natural number m such that*

$$m \le N = \left[\frac{\log \dfrac{a}{t}}{\log \dfrac{1}{a}} \right] , \tag{4.1}$$

let us consider the closed set

$$D_m = D_m(t) = \{ z \in \mathbb{C} : \ a^{m+1} \le |z+t| \le a^m \ \text{ or } a^{m+1} \le |z-t| \le a^m \} .$$

Let Ω be a Denjoy domain such that $\{-1,-t,t,1\} \subset \partial\Omega \subset [-1,-t] \cup [t,1]$.

Let $n_1 < n_2 < \cdots < n_{\ell-1}$ be all the natural numbers in $(0,N)$ satisfying $D_{n_j} \cap \partial\Omega \ne \varnothing$, $n_0 = 0$ and $n_\ell = N$.

Then we have that there exists a universal constant $0 < t_0 < 1$ such that if we denote by σ the imaginary axis with the point at infinity included, then

$$L_\Omega(\sigma) \asymp \sum_{j=1}^{\ell} (1 + \log (n_j - n_{j-1})) , \qquad \text{for } \ 0 < t \le t_0 .$$

Here the constant in $\asymp$ depends only on a but neither on Ω nor t.

Proof. The idea of the proof is to estimate the length of "dyadic" segments of the curve. Over each one of these segments we shall have a precise estimate of the distance to the boundary of the domain and the function β_Ω (see (2.2)). These facts and [BP, Theorem 1] will give the lemma up to a technical detail involving the point at infinity which we solve in Lemma 4.4.

Let $I_m = \sigma \cap D_m$, $0 \le m \le N$. We are going to estimate $L_\Omega(I_m)$, the length in Ω of I_m, under the assumptions that

$$(D_{m-k} \cup D_{m+k}) \cap \partial\Omega \neq \varnothing \,,$$

$$\begin{aligned}(D_{m-k+1} &\cup D_{m-k+2} \cup \cdots \cup D_{m-1} \\ &\cup D_m \cup D_{m+1} \cup \cdots \cup D_{m+k-1}) \cap \partial\Omega = \varnothing \,,\end{aligned}$$

for $0 \le k \le \min\{m, N-m\}$ (obviously the second condition does not appear if $k = 0$).

Let $\Omega^* := \Omega \setminus \{\infty\}$; the computations in Ω^* are easier than in Ω because we can apply [BP, Theorem 1] since $\Omega^* \subset \mathbb{C}$.

Let us consider a point $b \in (D_{m-k} \cup D_{m+k}) \cap \partial\Omega$. We have four possibilities:

i) $b \in D_{m+k}$ and $a^{m+k+1} \le |b+t| \le a^{m+k}$,

ii) $b \in D_{m+k}$ and $a^{m+k+1} \le |b-t| \le a^{m+k}$,

iii) $b \in D_{m-k}$ and $a^{m-k+1} \le |b+t| \le a^{m-k}$,

iv) $b \in D_{m-k}$ and $a^{m-k+1} \le |b-t| \le a^{m-k}$.

We consider now the case i). If $z \in I_m$, it satisfies inequalities $a^{m+1} \le |z+t| \le a^m$ (in fact, z satisfies both inequalities in the definition of D_m), and then

$$\frac{1}{a^{k-1}} = \frac{a^{m+1}}{a^{m+k}} \le \frac{|z+t|}{|b+t|} \le \frac{a^m}{a^{m+k+1}} = \frac{1}{a^{k+1}} \,.$$

This implies that

$$1 + \beta_{\Omega^*}(z) \asymp (k+1) \log \frac{1}{a} \,. \tag{4.2}$$

The same result can be deduced, with similar arguments, in the cases ii), iii) and iv).

Using (4.2) and [BP, Theorem 1] we obtain that

$$\lambda_{\Omega^*}(z) \asymp \frac{1}{a^m (k+1)} \,, \qquad \text{for } k \ge 0\,. \tag{4.3}$$

Next we are going to estimate the euclidean length of I_m

$$\begin{aligned}|I_m| &= \sqrt{a^{2m} - t^2} - \sqrt{a^{2m+2} - t^2} \\ &= \frac{a^{2m}\,(1 - a^2)}{\sqrt{a^{2m} - t^2} + \sqrt{a^{2m+2} - t^2}} \,.\end{aligned} \tag{4.4}$$

Observe that (4.1) gives $t^2 \le a^{2m+2}$. This fact and (4.4) imply that $|I_m| \asymp a^m$, and therefore

$$(4.5) \qquad L_{\Omega^*}(I_m) = \int_{I_m} \lambda_{\Omega^*}(z)\,|dz| \asymp \int_{I_m} \frac{|dz|}{a^m\,(k+1)} \asymp \frac{1}{k+1}\,.$$

In order to estimate $L_\Omega(I_m)$ we only need to prove that $\lambda_{\Omega^*}(z) \asymp \lambda_\Omega(z)$ for $|z| \le 1$.

This last relation would be easy to prove (see Lemma 3.1 with $C = \{\infty\}$) if we were not interested in obtaining constants independent of Ω and t. But, to obtain universal constants, we need a more sophisticated argument.

Lemma 4.4. *Let E be a closed subset of the closed unit disk such that $\{-1,-t,t,1\} \subset E$. Then, for each $\rho > 1$ there exist constants $t_0 \in (0,1)$ and $c > 0$ which only depend on ρ such that*

$$\lambda_{\hat{\mathbb{C}}\setminus E}(z) \ge c\,\lambda_{\mathbb{C}\setminus E}(z)\,,$$

for every $0 < t \le t_0$ and $|z| \le \rho$.

Proof. By [He, Theorem 1] we have that

$$\lambda_{\hat{\mathbb{C}}\setminus\{-1,-t,t,1\}}(z) \longrightarrow \lambda_{\hat{\mathbb{C}}\setminus\{-1,0,1\}}(z)\,, \qquad \text{as } t \longrightarrow 0\,,$$

uniformly over compact subsets of $\hat{\mathbb{C}} \setminus \{-1,0,1\}$. Therefore, for each $\rho > 1$, there exist constants t_0, c_1 which only depend on ρ, such that if $0 < t \le t_0$ and γ is a curve contained in $\{w \in \hat{\mathbb{C}} :\ |w| \ge \rho\}$, then

$$L_{\hat{\mathbb{C}}\setminus\{-1,-t,t,1\}}(\gamma) \ge c_1\, L_{\hat{\mathbb{C}}\setminus\{-1,0,1\}}(\gamma)\,.$$

On the other hand, by [Br, Theorem 1], the set $\{w \in \hat{\mathbb{C}} :\ |w| \ge \rho\}$ is hyperbolically convex in every hyperbolic plane domain containing it. Hence,

$$(4.6) \qquad d_{\hat{\mathbb{C}}\setminus E}(w,\infty) \ge d_{\hat{\mathbb{C}}\setminus\{-1,-t,t,1\}}(w,\infty) \ge c_1\, d_{\hat{\mathbb{C}}\setminus\{-1,0,1\}}(w,\infty)\,,$$

if $|w| \ge \rho$. Now, it is clear that there exists a positive constant r which only depends on ρ such that

$$B_{\hat{\mathbb{C}}\setminus\{-1,0,1\}}\Big(\infty, \frac{r}{c_1}\Big) \subset \{w \in \hat{\mathbb{C}} :\ |w| > \rho\}\,.$$

This fact and (4.6) says that

$$B_{\hat{\mathbb{C}}\setminus E}(\infty, r) \subset \{w \in \hat{\mathbb{C}} : \ |w| > \rho\} ,$$

and so, if $|z| \le \rho$, we have that $d_{\hat{\mathbb{C}}\setminus E}(\infty, z) \ge r$. Therefore Lemma 3.1 (with $C = \{\infty\}$) gives that

$$c = \tanh\Big(\frac{r}{2}\Big) < \frac{\lambda_{\hat{\mathbb{C}}\setminus E}(z)}{\lambda_{\mathbb{C}\setminus E}(z)} < 1 .$$

This finishes the proof of Lemma 4.4.

In what follows we will take the fixed value $\rho = 2$ and we will consider the corresponding c and t_0. This t_0 is the constant that works in Lemma 4.3.

Now Lemma 4.4 and (4.5) give

$$L_\Omega(I_m) \asymp \frac{1}{k+1} , \qquad \text{if } 0 < t \le t_0 .$$

Therefore

$$L_\Omega(I_{n_{j-1}} \cup \cdots \cup I_{n_j}) \asymp 2\Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{2}{n_j - n_{j-1}}\Big) \asymp 1 + \log(n_j - n_{j-1})$$

and

$$L_\Omega(I_0 \cup \cdots \cup I_N) \asymp \sum_{j=1}^{\ell} (1 + \log(n_j - n_{j-1})) .$$

In order to finish the proof of Lemma 4.3 it is enough to check that

$$L_\Omega(\sigma) \asymp L_\Omega(I_0 \cup \cdots \cup I_N) ,$$

where the constant in $\asymp$ depends only on a.

This is a consequence of the following facts.

$$\begin{aligned} L_\Omega(\sigma \cap \{w \in \hat{\mathbb{C}} :\ |w| \le \sqrt{a^{2N+2} - t^2}\,\}) &\\ \le L_{\hat{\mathbb{C}}\setminus[t,-t]}(\sigma \cap \{w \in \hat{\mathbb{C}} :\ |w| \le a^{N+1}\}) &\\ \le L_{\hat{\mathbb{C}}\setminus[t,-t]}\Big(\sigma \cap \Big\{w \in \hat{\mathbb{C}} :\ |w| \le \frac{t}{a}\Big\}\Big) &\\ = L_{\hat{\mathbb{C}}\setminus[1,-1]}\Big(\sigma \cap \Big\{w \in \hat{\mathbb{C}} :\ |w| \le \frac{1}{a}\Big\}\Big)\,, & \end{aligned}$$

$$\begin{aligned} L_\Omega(\sigma \cap \{w \in \hat{\mathbb{C}} : |w| \ge \sqrt{1-t^2}\,\}) &\\ \le L_{\hat{\mathbb{C}}\setminus[-1,1]}(\sigma \cap \{w \in \hat{\mathbb{C}} :\ |w| \ge \sqrt{1-t_0^2}\,\})\,, & \end{aligned}$$

$$\sum_{j=1}^{\ell}(1+\log(n_j - n_{j-1})) \ge 1 + \log N \asymp \log\log\Big(\frac{1}{t}\Big)\,,$$

$$\begin{aligned} \sigma \subset I_0 \cup \cdots \cup I_N \cup \{w \in \hat{\mathbb{C}} :\ |w| \le \sqrt{a^{2N+2} - t^2}\,\} &\\ \cup \{w \in \hat{\mathbb{C}} :\ |w| \ge \sqrt{1-t^2}\,\}\,. & \end{aligned}$$

This finishes the proof of Lemma 4.3.

For a border set $B = \{b_1, b_2, b_3, b_4\}$, in order to obtain a more symmetric situation, as in Lemma 4.3, we consider the Möbius transformations

$$\begin{aligned} T(z) &= T_B(z) = \frac{(b_2-b_1)(z-b_3)}{(b_3-b_2)(z-b_1)}\,, \\ T_B^{-1}(z) &= \frac{b_1(b_3-b_2)z - b_3(b_2-b_1)}{(b_3-b_2)z - (b_2-b_1)}\,, \\ S(z) &= S_B(z) = \frac{z+1-\sqrt{1+r(B)}}{z+1+\sqrt{1+r(B)}}\,, \\ S_B^{-1}(z) &= \frac{-2}{1-t(B)}\ \frac{z+t(B)}{z-1} \\ &= \frac{(-1-\sqrt{1+r(B)}\,)\,z + 1 - \sqrt{1+r(B)}}{z-1}\,, \\ U(z) &= U_B(z) = (T^{-1}\circ S^{-1})(z)\,, \end{aligned} \tag{4.7}$$

where $r(B)$ is defined by (2.1) (observe that $r(B) = T_B(b_4)$) and $t(B)$ is defined by

$$t(B) = \frac{\sqrt{1+r(B)}-1}{\sqrt{1+r(B)}+1}\,, \qquad r(B) = \frac{4\,t(B)}{(1-t(B))^2}\,. \tag{4.8}$$

Observe that the images by T of b_1, b_2, b_3, b_4 are $\infty, -1, 0, r$ in this order, the images by S of $\infty, -1, 0, r$ are $1, -1, -t, t$ also in this order and therefore the images by U of $1, -1, -t, t$ are b_1, b_2, b_3, b_4.

Lemma 4.5. *For $r > 0$ let T_r be the Teichmüller annulus, i.e. $T_r = \mathbb{C} \setminus ([-1,0] \cup [r,\infty))$ and $S_r = \mathbb{C} \setminus \{-1,0,r\}$. Then we have that the simple closed geodesic σ_r which surrounds $\{-1,0\}$ and does not surround $\{r\}$ is equal to $\{z \in \mathbb{C} : \ |z+1| = \sqrt{1+r}\}$ in both domains. Moreover,*

$$L_{T_r}(\sigma_r) \asymp \Phi_1(r)\,, \qquad L_{S_r}(\sigma_r) \asymp \Phi_2(r)\,, \qquad r > 0\,.$$

Proof. Let us consider the images of the domains T_r and S_r under the Möbius transformation $S(z)$ (see (4.7)) which maps the points $-1, 0, r, \infty$ to $-1, -t, t, 1$ in this order (if r and t are related by (4.8)). It is clear, by symmetry, that the simple closed geodesic in $S(T_r)$ and $S(S_r)$ corresponding to σ_r is in both cases the imaginary axis (with the point at infinity included). Therefore, $\sigma_r = S^{-1}(\{w \in \mathbb{C} : \operatorname{Re} w = 0\} \cup \{\infty\}) = \{z \in \mathbb{C} : \ |z+1| = \sqrt{1+r}\}$.

To finish the proof we have to prove the following four facts:

1) $L_{T_r}(\sigma_r) \asymp \log(1/r)$ as $r \longrightarrow 0$,

2) $L_{S_r}(\sigma_r) \asymp \log\log(1/r)$ as $r \longrightarrow 0$,

3) $L_{T_r}(\sigma_r) \asymp 1/\log r$ as $r \longrightarrow \infty$,

4) $L_{S_r}(\sigma_r) \asymp 1/\log r$ as $r \longrightarrow \infty$.

1) follows as a direct consequence of Lemma 4.3 by observing that $\{n_0, n_1, \dots, n_\ell\} = \{0, 1, 2, \dots, N\}$ and $\ell = N \asymp \log(1/t) \asymp \log(1/r)$. Similarly, 2) follows also as a direct consequence of Lemma 4.3 since in this case $\ell \le 3$.

3) is a well-known fact (see sections 1 and 2 of [LV, Chapter II], where 1) is also proved; recall that the product of the modulus of an annulus by the length of its simple closed geodesic is constant). 4)

follows from 3), [BP, Theorem 1] and the fact that, as $r \geq 3$, the β-functions defined by (2.2) verify

$$\beta_{T_r}(z) = \beta_{S_r}(z)\,, \qquad \text{for all } z \in \sigma_r\,.$$

Lemma 4.5 has been proved.

Proof of Theorem 2. Let us consider a border set of $\partial\Omega$, $B = \{b_1, \dots, b_{2n}\}$ $(n \geq 3)$. We have that $\hat{\mathbb{C}} \setminus ([b_{2j}, b_{2j+1}] \cup [b_{2j+2}, b_{2j-1}]) \subset \Omega \subset \hat{\mathbb{C}} \setminus \{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}$. Therefore, if we denote by r the positive number

$$r = r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})\,,$$

Lemma 4.5 gives that

$$\begin{aligned} c_2\,\Phi_2(r) &\leq L_{\hat{\mathbb{C}}\setminus\{0,r,\infty,-1\}}(\gamma(\{0,r,\infty,-1\})) \\ &= L_{\hat{\mathbb{C}}\setminus\{b_{2j-1},b_{2j},b_{2j+1},b_{2j+2}\}}(\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) \\ &\leq L_\Omega(\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}))\,, \end{aligned}$$

where we should remark that in the second line of the last display, $\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})$ refers to the geodesic in the domain $\hat{\mathbb{C}} \setminus \{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}$, but the same symbol in the third line refers to the geodesic in the domain Ω.

Lemma 4.5 also gives that

$$\begin{aligned} &L_\Omega(\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) \\ &\qquad\leq L_{\hat{\mathbb{C}}\setminus\{[b_{2j},b_{2j+1}]\cup[b_{2j+2},b_{2j-1}]\}}(\gamma(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) \\ &\qquad= L_{\hat{\mathbb{C}}\setminus([-1,0]\cup[r,\infty])}(\gamma(\{0,r,\infty,-1\})) \\ &\qquad\leq c_1\,\Phi_1(r)\,, \end{aligned}$$

where we should make a remark similar to the one in the last paragraph.

These inequalities and Lemma 4.2 prove Theorem 2.

5. Length of geodesics and characterization of the HII-property in Denjoy domains.

In order to state the characterization of the HII-property for Denjoy domains we need a good estimate of the length of the simple closed geodesic $\gamma(B)$ associated to any border set B of $\partial\Omega$ with four points. This estimate, which is interesting by itself, is the statement of Theorem 4.

Let us fix a number $0 < a < 1$ and denote by D_m the closed set

$$\begin{aligned} D_m = D_m(B) = U(\{z \in \mathbb{C} :\ & a^{m+1} \le |z + t(B)| \le a^m \\ & \text{or } a^{m+1} \le |z - t(B)| \le a^m\}) \,, \end{aligned}$$

$m \in \mathbb{N}$. The intersection of D_m with the real axis is, in fact, a union of at most four closed intervals. Observe that the definition of D_m above is consistent with the one in Lemma 4.3.

We need also to define the following natural number

$$N = N_B := \left[\frac{\log \dfrac{a}{t(B)}}{\log \dfrac{1}{a}} \right] ,$$

where $[x]$ is the greatest natural number which is less or equal than x.

Theorem 4. *Let Ω be a Denjoy domain, $0 < a < 1$ and $B = \{b_1, b_2, b_3, b_4\}$ be a border set of $\partial\Omega$.*

Let $n_1 < n_2 < \cdots < n_{\ell-1}$ be the list of the natural numbers in $(0, N)$ satisfying $D_{n_j} \cap \partial\Omega \neq \varnothing$, $n_0 = 0$ and $n_\ell = N$.

Then there exists a universal constant $0 < r_0 < 1$ such that

$$L_\Omega(\gamma(B)) \asymp \Psi_\Omega(B) := \begin{cases} \dfrac{1}{\log r(B)} \,, & \text{if } r(B) > e \,, \\ 1 \,, & \text{if } r_0 < r(B) \le e \,, \\ \displaystyle\sum_{j=1}^{\ell} (1 + \log (n_j - n_{j-1})) \,, & \text{if } r(B) \le r_0 \,. \end{cases}$$

Here the constant in $\asymp$ depends only on a and neither on Ω nor B.

Observe that Theorem 4 gives a general procedure to obtain the length of a symmetric simple closed geodesic in a Denjoy domain. This theorem is a useful tool in order to study the asymptotic behaviour of the length of geodesics in domains which depend on a parameter. Also, note that the condition of admissibility of Ω does not appear in the hypotheses.

In the proof of Theorem 5 we use Theorem 4 and some of the ingredients of the proof of Theorem 2. Theorem 4 allows to relate the previous ideas with euclidean conditions on the size of $\partial\Omega$; this is the most delicate part of our argument.

We start proving an analogue of Lemma 4.3 but now for the "authentic" geodesics. This result will be the basic tool in the proof of Theorem 4.

Lemma 5.1. *Let us fix a number $0 < a < 1$ and let $0 < t < 1$. For each natural number m such that*

$$m \le N = \left[\frac{\log \dfrac{a}{t}}{\log \dfrac{1}{a}} \right] ,$$

let us consider the closed set

$$D_m = D_m(t) = \{ z \in \mathbb{C} : \ a^{m+1} \le |z+t| \le a^m \ \text{or} \ a^{m+1} \le |z-t| \le a^m \} .$$

Let Ω be a Denjoy domain such that $B = \{-t, t, 1, -1\} \subset \partial\Omega \subset [-1, -t] \cup [t, 1]$.

Let $n_1 < n_2 < \cdots < n_{\ell-1}$ be all the natural numbers in $(0, N)$ satisfying $D_{n_j} \cap \partial\Omega \neq \varnothing$, $n_0 = 0$ and $n_\ell = N$.

Then we have that there exists a universal constant $0 < t_0 < 1$ (the same constant that in Lemma 4.3) such that

$$L_\Omega(\gamma(B)) \asymp \sum_{j=1}^{\ell} (1 + \log (n_j - n_{j-1})) , \qquad \text{for} \ \ 0 < t \le t_0 .$$

Here the constant in $\asymp$ depends only on a and neither on Ω nor t.

The main ideas of the proof of this lemma are the following. First, we shall use a polarization argument (see below) in order to reduce our problem to some extremal cases (Lemma 5.2). Secondly, observe that we do not know where is the geodesic $\gamma(B)$. So, in order to obtain lower bounds for its length, we shall study the length of any curve in the same homotopy class in Ω by using again a "dyadic" argument (Lemma 5.3). We should remark that we have already upper bounds of the length of $\gamma(B)$ (Lemma 4.3).

In order to prove Lemma 5.1 it is convenient to introduce some concepts.

If z is a complex number, we consider its symmetric point with respect to the imaginary axis $z^{\#} = -\bar{z}$, with $\infty^{\#} = \infty$. The symmetric $A^{\#}$ of a set $A \subset \hat{\mathbb{C}}$ is defined as $A^{\#} = \{z^{\#} : z \in A\}$. The positive and negative parts of A are

$$A^{+} = A \cap \{z : \operatorname{Re} z \geq 0\}, \qquad A^{-} = A \cap \{z : \operatorname{Re} z \leq 0\}.$$

Let us consider a domain Ω as in Lemma 5.1. The *polarization* Ω_p of the Denjoy domain Ω is defined as

$$\Omega_p = (\Omega \cup \Omega^{\#})^{+} \cup (\Omega \cap \Omega^{\#})^{-}$$

and the *antisymmetric* Ω_{as} of the domain Ω as

$$\Omega_{as} = \big(\{z : \operatorname{Re} z \geq 0\} \setminus \{t, 1\}\big) \cup (\Omega \cap \Omega^{\#})^{-}.$$

Observe that $(\Omega_p)_p = \Omega_p$, $(\Omega_{as})_{as} = \Omega_{as}$, $(\Omega_p)_{as} = (\Omega_{as})_p = \Omega_{as}$ and $\Omega_p \subset \Omega_{as}$.

The concept of polarization appeared in a paper by Wolontis [W], who proved results on the behavior of certain extremal lengths under polarization and also symmetrization results by repeated application of polarization.

We shall need the following result about polarization [So, Theorem 9]

$$\lambda_{\Omega_p}(z) \leq \min\{\lambda_\Omega(z), \lambda_\Omega(z^{\#})\}, \qquad \text{if } \operatorname{Re} z \geq 0. \tag{5.1}$$

In particular, we have that

$$\lambda_{\Omega_p}(z) \leq \lambda_{\Omega_p}(z^{\#}), \qquad \lambda_{\Omega_{as}}(z) \leq \lambda_{\Omega_{as}}(z^{\#}), \qquad \text{if } \operatorname{Re} z \geq 0. \tag{5.2}$$

This last result is well-known [M, Theorem 3].

The results concerning the Poincaré metric that appear in [M] and [So] use as symmetry axis the real axis instead of the imaginary one, but it is obvious (as Solynin comments in [So]) that the result is true for polarization with respect to any fixed straight line.

We can prove now

Lemma 5.2. *In order to prove Lemma* 5.1 *it is enough to consider the sets* Ω_{as} *instead of* Ω.

Proof. If σ denotes the imaginary axis with the point at infinity, we have that

$$L_{\Omega_{as}}(\sigma) \asymp L_{\Omega_p}(\sigma) \asymp L_\Omega(\sigma)\,, \qquad \text{for } 0 < t \le t_0\,,$$

where t_0 is the constant in Lemma 4.3. This fact is a direct consequence of Lemma 4.3, since the expression $\sum_{j=1}^{\ell}(1+\log{(n_j-n_{j-1})})$ is exactly the same for Ω_{as}, Ω_p and Ω.

Let us consider now the simple closed geodesic γ, (respectively γ_p, γ_{as}) in Ω (respectively Ω_p, Ω_{as}) which is freely homotopic to σ. By the definition of geodesic it follows that

$$L_\Omega(\gamma) \le L_\Omega(\sigma)\,, \qquad L_{\Omega_p}(\gamma_p) \le L_{\Omega_p}(\sigma)\,, \qquad L_{\Omega_{as}}(\gamma_{as}) \le L_{\Omega_{as}}(\sigma)\,.$$

We also have that $\lambda_{\Omega_{as}}(z) \le \lambda_{\Omega_p}(z)$ for all $z \in \Omega_p$, since $\Omega_p \subset \Omega_{as}$. Therefore $L_{\Omega_{as}}(\gamma_{as}) \le L_{\Omega_p}(\gamma_p)$.

In order to finish the proof of Lemma 5.2, it is enough to see that $L_{\Omega_p}(\gamma_p) \le L_\Omega(\gamma)$.

Let us consider the curve $\tilde\gamma = \gamma^+ \cup (\gamma^-)^\#$. Obviously, $\tilde\gamma$ is freely homotopic to γ in Ω. Therefore

$$L_{\Omega_p}(\gamma_p) \le L_{\Omega_p}(\tilde\gamma) \le L_\Omega(\gamma)\,,$$

where the first inequality follows from the fact that $\tilde\gamma$ is also freely homotopic to γ_p in Ω_p, and we have the second one by (5.1).

This finishes the proof of Lemma 5.2.

Lemma 5.3. *Let us fix a number* $0 < a < 1$. *Let* Ω *be a Denjoy domain such that* $\{-1, -t, t, 1\} \subset \partial\Omega \subset [-1, -t] \cup [t, 1]$, *with* $0 < t \le t_0$, *where* t_0 *is the constant in Lemma* 4.3. *Let us consider the antisymmetric set* Ω_{as} *of* Ω. *Let* μ_m *be a curve contained in* $B_m = \{z \in \mathbb{C} :\ 0 \le \operatorname{Re} z \le (1+t)/2,\ a^{m+1} \le |z-t| \le a^m\}$ *which joins* $S_m = \{z \in \mathbb{C} :\ |z-t| = a^m\}$

with S_{m+1}. Then, there exists a positive constant c, which only depends on a, such that if

$$(D_{m-k} \cup D_{m+k}) \cap \partial\Omega_{as} \neq \varnothing \,,$$

$$(D_{m-k+1} \cup D_{m-k+2} \cup \cdots \cup D_{m-1} \\ \cup D_m \cup D_{m+1} \cup \cdots \cup D_{m+k-1}) \cap \partial\Omega_{as} = \varnothing \,,$$

then we have

$$L_{\Omega_{as}}(\mu_m) \geq \frac{c}{k+1} \,,$$

for $0 \leq k \leq \min\{m, N-m\}$ (obviously the second condition over $\partial\Omega_{as}$ does not appear if $k=0$).

PROOF. Let $\Omega^*_{as} := \Omega_{as} \setminus \{\infty\}$; the computations in Ω^*_{as} are easier than in Ω_{as} because we can apply [BP, Theorem 1] since $\Omega^*_{as} \subset \mathbb{C}$. We are going to find bounds for $\beta_{\Omega^*_{as}}(z)$, in order to estimate $\lambda_{\Omega^*_{as}}(z)$ for $z \in B_m$.

We have that $\delta_{\Omega^*_{as}}(z) = |z-t|$ and (4.1) gives that

$$(5.3) \qquad t \leq a^{m+1} \leq |z-t| \,, \qquad \text{for all } z \in B_m \text{ and } m \leq N \,.$$

Let us consider a point $b \in (D_{m-k} \cup D_{m+k}) \cap \partial\Omega^*_{as}$. We have four possibilities:

i) $b \in D_{m+k}$ and $a^{m+k+1} \leq |b+t| \leq a^{m+k}$,

ii) $b \in D_{m+k}$ and $a^{m+k+1} \leq |b-t| \leq a^{m+k}$,

iii) $b \in D_{m-k}$ and $a^{m-k+1} \leq |b+t| \leq a^{m-k}$,

iv) $b \in D_{m-k}$ and $a^{m-k+1} \leq |b-t| \leq a^{m-k}$.

We consider first the cases ii) and iv). The conditions which define these possibilities and (5.3) give that

$$\frac{1}{a^{k-1}} = \frac{a^{m+1}}{a^{m+k}} \leq \frac{|z-t|}{|b-t|} \leq \frac{a^m}{a^{m+k+1}} = \frac{1}{a^{k+1}} \,,$$

in the case ii), and

$$a^{k+1} = \frac{a^{m+1}}{a^{m-k}} \leq \frac{|z-t|}{|b-t|} \leq \frac{a^m}{a^{m-k+1}} = a^{k-1} \,,$$

in the case iv). In both cases, this implies that

$$\beta_{\Omega^*_{as}}(z) \le (k+1) \log \frac{1}{a} \, .$$

We consider now the cases i) and iii). If $b = 1$ or $b = t$ we can take $-b$ instead of b (since $-b$ also belongs to $(D_{m-k} \cup D_{m+k}) \cap \partial\Omega^*_{as})$ and we are in the cases ii) or iv); obviously, $b \ne -t$. Therefore, without loss of generality we can assume that $b < -t$. In both possibilities i) and iii) we have that

$$|b-t| \ge |b+t| \, . \tag{5.4}$$

In order to obtain upper bounds for $|b-t|$, we study separately the cases i) and iii).

In the case iii) we have that

$$|b-t| = 2\,t + |b+t| \le 2\,a^{m+1} + a^{m-k} \le 3\,a^{m-k} \, ,$$

and

$$\frac{a^{k+1}}{3} = \frac{a^{m+1}}{3\,a^{m-k}} \le \frac{|z-t|}{|b-t|} \le \frac{a^m}{a^{m-k+1}} = a^{k-1} \, .$$

This fact implies that

$$\beta_{\Omega^*_{as}}(z) \le \log 3 + (k+1) \log \frac{1}{a} \, .$$

In the case i) the condition $m + k \le N$ gives that

$$|b-t| = 2\,t + |b+t| \le 2\,a^{m+k+1} + a^{m+k} \le 3\,a^{m+k} \, ,$$

and

$$\frac{1}{3\,a^{k-1}} = \frac{a^{m+1}}{3\,a^{m+k}} \le \frac{|z-t|}{|b-t|} \le \frac{a^m}{a^{m+k+1}} = \frac{1}{a^{k+1}} \, .$$

This fact implies that

$$\beta_{\Omega^*_{as}}(z) \le \max\left\{ \left| \log \frac{1}{3\,a^{k-1}} \right|, \log \frac{1}{a^{k+1}} \right\} .$$

Therefore, there is a constant c_1, only depending on a, such that

$$\beta_{\Omega^*_{as}}(z) \le c_1\,(k+1) \, .$$

Consequently, we have in any case

$$\beta_{\Omega^*_{as}}(z) \le c_2\,(k+1)\,.$$

Therefore [BP, Theorem 1] gives that

$$\lambda_{\Omega^*_{as}}(z) \ge \frac{c_3}{|z-t|\,(k+1)\log\left(\dfrac{1}{a}\right)}\,,$$

and we deduce that

$$\begin{aligned}
L_{\Omega^*_{as}}(\mu_m) &= \int_{\mu_m} \lambda_{\Omega^*_{as}}(z)\,|dz| \\
&\ge \int_{\mu_m} \frac{c_3\,|dz|}{|z-t|\,(k+1)\log\left(\dfrac{1}{a}\right)} \\
&\ge \int_{a^{m+1}}^{a^m} \frac{c_3\,dr}{r\,(k+1)\log\left(\dfrac{1}{a}\right)} \\
&= \frac{c_3}{k+1}\,.
\end{aligned}$$

Observe that $|z-t| \le a^m \le 1$ and $t<1$. These facts imply that $|z|<2$. Lemma 4.4 (recall that we have chosen $\rho = 2$) gives that

$$L_{\Omega_{as}}(\mu_m) \ge \frac{c}{k+1}\,, \qquad \text{if } 0<t\le t_0\,.$$

This finishes the proof of Lemma 5.3.

Proof of Lemma 5.1. As Lemma 5.2 states, we only need to prove Lemma 5.1 for the domains Ω_{as}.

Let us consider any curve μ freely homotopic to $\gamma(B)$ in Ω_{as}.

We want to prove that there exists a positive constant c_1, which only depends on a, such that

$$L_{\Omega_{as}}(\mu) \ge c_1 \sum_{j=1}^{\ell} \left(1+\log\left(n_j - n_{j-1}\right)\right).$$

If we prove this inequality, then Lemma 5.1 is true since $\gamma(B)$ is one of the curves μ above. The upper bound of $L_{\Omega_{as}}(\mu)$ is a consequence of Lemma 4.3, since

$$L_{\Omega_{as}}(\gamma(B)) \le L_{\Omega_{as}}(\sigma) \le c_2 \sum_{j=1}^{\ell} (1+\log(n_j - n_{j-1})) ,$$

and the n_j's are the same for Ω_{as} and Ω.

Let us consider now the curve $\tilde\mu = \mu^+ \cup (\mu^-)^{\#}$. Obviously $\tilde\mu$ is freely homotopic to μ in Ω_{as}, and (5.2) gives that $L_{\Omega_{as}}(\tilde\mu) \le L_{\Omega_{as}}(\mu)$.

Let μ^0 be a connected component of $\tilde\mu$ contained in $\{z : 0 \le \operatorname{Re} z \le (1+t)/2\}$ which joins the interval $[0,t)$ with $\{z : \operatorname{Re} z = (1+t)/2,\ \operatorname{Im} z \ge 0\}$. The curve μ^0 meets the vertical line $\{z : \operatorname{Re} z = (1+t)/2\}$ at a point with the form $i\,b_2 + (1+t)/2$. We have that

$$\Big|\frac{1+t}{2} + i\,b_2 - t\Big| \ge \frac{1-t}{2} \ge \frac{1-t_0}{2} \,.$$

If m satisfies

$$\frac{\log \dfrac{2}{1-t_0}}{\log \dfrac{1}{a}} \le m \le \frac{\log \dfrac{a}{t}}{\log \dfrac{1}{a}}$$

then we have that $a^m \le (1-t_0)/2$ and $a^{m+1} \ge t$ and so $\mu_m = \mu^0 \cap \{z : a^{m+1} \le |z-t| \le a^m\}$ joins $S_m = \{z : |z-t| = a^m\}$ with S_{m+1}. Therefore Lemma 5.3 and the same argument used at the end of the proof of Lemma 4.3 give that

$$L_{\Omega_{as}}(\mu) \ge L_{\Omega_{as}}(\mu^0) \ge c_1 \sum_{j=1}^{\ell} (1+\log(n_j - n_{j-1})) ,$$

since the terms in the last sum corresponding to

$$0 \le m \le \frac{\log \dfrac{2}{1-t_0}}{\log \dfrac{1}{a}}$$

have bounded length.

Proof of Theorem 4. If we apply the Möbius transformation U^{-1} (which preserves the hyperbolic metric) to Ω we obtain a new domain Ω' with

$$\{-1,-t,t,1\} \subset \partial\Omega' \subset [-1,-t] \cup [t,1]\,. \tag{5.5}$$

Therefore, without loss of generality we can assume that Ω satisfies (5.5) and so

$$D_m = \{z \in \mathbb{C} : \; a^{m+1} \le |z+t| \le a^m \text{ or } a^{m+1} \le |z-t| \le a^m\}\,.$$

Let γ be the simple closed geodesic in Ω given by $\gamma = \gamma(\{-t,t,1,-1\})$.

Let us consider first the case $0 < t \le t_0$. Lemma 5.1 gives that

$$L_\Omega(\gamma) \asymp \sum_{j=1}^{\ell} (1 + \log(n_j - n_{j-1}))\,.$$

For $t_0 \le t < 1$, observe that $\Omega_1 = \hat{\mathbb{C}} \setminus ([-1,-t] \cup [t,1]) \subset \Omega \subset \Omega_2 = \hat{\mathbb{C}} \setminus \{-1,-t,t,1\}$. Then we have that

$$\lambda_{\Omega_1}(z) \ge \lambda_\Omega(z)\,, \qquad \text{for all } z \in \Omega_1\,,$$

$$\lambda_\Omega(z) \ge \lambda_{\Omega_2}(z)\,, \qquad \text{for all } z \in \Omega\,,$$

and consequently Lemma 4.5 gives that

$$\Phi_1(r) \asymp L_{\Omega_1}(\sigma) \ge L_\Omega(\gamma) \ge L_{\Omega_2}(\sigma) \asymp \Phi_2(r)\,, \qquad \text{with } r = \frac{4\,t}{(1-t)^2}\,,$$

and we have that

$$\Phi_1(r) \asymp \Phi_2(r) \asymp \begin{cases} \dfrac{1}{\log r}\,, & \text{if } r > e\,, \\ 1\,, & \text{if } r_0 \le r \le e\,, \end{cases}$$

with $r_0 = 4\,t_0/(1-t_0)^2$. Here the constant in $\asymp$ depends only on a but neither on Ω nor r. This finishes the proof of Theorem 4.

Theorem 5. *Let Ω be a Denjoy domain, let I be the set of isolated points of $\partial\Omega$ and let $\Omega_0 = \Omega \cup I$. Then, Ω has* HII *if and only if Ω*

is admissible and there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that

$$\frac{1}{n}\sum_{j=1}^{n} \Psi_{\Omega_0}(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}) > c\,.$$

PROOF OF THEOREM 5. If $\partial\Omega_0$ has isolated points, then Ω is not admissible and Theorem 1 gives that Ω has not HII. Let us assume now that $\partial\Omega_0$ has not isolated points. Theorem 1 reduces the proof of Theorem 5 to the following:

Ω_0 *has* HII *if and only if there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that*

$$\frac{1}{n}\sum_{j=1}^{n} \Psi_{\Omega_0}(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}) > c\,.$$

This fact is a consequence of Lemma 4.2 and Theorem 4.

6. Collars and balls.

Let $\mathcal{R}$ be a hyperbolic Riemann surface with a puncture p. A *collar* in $\mathcal{R}$ about p is a doubly connected domain in $\mathcal{R}$ "bounded" by p and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from p. It is well known that the length of the boundary curve is equal to the area of the collar.

A collar in $\mathcal{R}$ about p of area α will be called an α-*collar* and it will be denoted by $C_{\mathcal{R}}(p, \alpha)$. A theorem of Shimizu [S] gives that for every puncture in any hyperbolic Riemann surface, there exists an α-collar for every $0 < \alpha \leq 1$ (see also [K, p. 60-61]).

Next, we will prove a relationship (involving universal constants) between collars in $\mathcal{R}$ and balls in $\mathcal{R} \cup \{p\}$.

Proposition 1. *Let $\mathcal{S}$ be a hyperbolic Riemann surface and let $\{B_{\mathcal{S}}(p, r_0)\}_{p\in I}$ be a family of simply connected and pairwise disjoint balls. Let us denote by $\mathcal{R}$ the Riemann surface $\mathcal{R} = \mathcal{S} \setminus I$. Let $k = 4.76$ and $K = e^k$.*

a) *We have that*

$$C_{\mathcal{R}}\Big(p, \frac{2\pi}{k - \log(1 - e^{-r})}\Big) \subset B_{\mathcal{S}}(p, r)^* ,$$

for $p \in I$ *and*

$$0 < r < \min\Big\{ \log \frac{1}{1 - K\, e^{-2\pi}}, \log \frac{2}{1 + e^{-2r_0}} \Big\} .$$

b) *We have that*

$$B_{\mathcal{S}}(p, r)^* \subset C_{\mathcal{R}}\Big(p, \frac{2\pi}{\log(1 - e^{-2r_0}) - \log(e^r - 1)}\Big) ,$$

for $p \in I$ *and*

$$0 < r < \log(1 + (1 - e^{-2r_0})\, e^{-2\pi}) .$$

Observe that, in both cases, the conditions on r imply that $0 < r < r_0$.

Proof. Let $F : \mathbb{U} \longrightarrow \mathcal{S}$ be a universal covering map and $J = F^{-1}(I)$. The balls in $\{B_{\mathbb{U}}(z, r_0)\}_{z\in J} = \{F^{-1}(B_{\mathcal{S}}(p, r_0))\}_{p\in I}$ are obviously simply connected (every ball in $\mathbb{U}$ is simply connected). We also remark that these balls are pairwise disjoint. If we have that for some $z, w \in J$

$$B_{\mathbb{U}}(z, r_0) \cap B_{\mathbb{U}}(w, r_0) \neq \varnothing ,$$

this implies that $B_{\mathcal{S}}(F(z), r_0)$ is not simply connected (if $F(z) = F(w)$) or $B_{\mathcal{S}}(F(z), r_0) \cap B_{\mathcal{S}}(F(w), r_0) \neq \varnothing$ (if $F(z) \neq F(w)$), and both conclusions contradict the hypothesis on $\{B_{\mathcal{S}}(p, r_0)\}_{p\in I}$.

Since

$$F(B_{\mathbb{U}}(z, r)) = B_{\mathcal{S}}(F(z), r) , \qquad \text{for } z \in J,\ 0 < r \le r_0 ,$$

and

$$F(C_{\mathbb{U}\setminus J}(z, \alpha)) = C_{\mathcal{S}\setminus I}(F(z), \alpha) , \qquad \text{for } z \in J,\ 0 < \alpha < 1 ,$$

we have that Proposition 1 is true for all hyperbolic Riemann surface $\mathcal{S}$ if and only if it is true for the case $\mathcal{S} = \mathbb{U}$ (with the same constants).

Therefore, without loss of generality we can assume that $\mathcal{S} = \mathbb{U}$. Let $\mathbb{V}$ be the Riemann surface $\mathbb{V} = \mathbb{U} \setminus I$.

In the following we need a precise version of (2.3). It is well known that if $\Omega \subset \mathbb{C}$ is a hyperbolic plane domain then

$$\lambda_\Omega(z) \geq \frac{1}{\delta_\Omega(z)\,(4.76 + \beta_\Omega(z))}\,, \qquad \text{for } z \in \Omega\,. \tag{6.1}$$

Lemma 6.1. *Let $r > 0$, $z_1, z_2 \in \mathbb{U}$. If $d_{\mathbb{U}}(z_1, z_2) \geq 2\,r$ and $z \in B_{\mathbb{U}}(z_1, a(r))$, we have that*

$$|z - z_1| < |z - z_2|\,, \tag{6.2}$$

where

$$a(r) = \log \frac{2}{1 + e^{-2r}}\,.$$

PROOF. Since this statement is invariant under conformal automorphisms of $\mathbb{U}$, we can assume without loss of generality that $z_1 = i$ and $d_{\mathbb{U}}(i, z_2) = 2\,r$.

A computation gives that Lemma 6.1 is true if (6.2) holds for $z_2 = i\,e^{-2r}$ and z belongs to the segment joining $i\,e^{-a(r)}$ with i (this is the worse case) and this follows from our election of $a(r)$.

Using (6.1) and Lemma 6.1 we can prove the following result.

Lemma 6.2. *Let $\{B_{\mathbb{U}}(p, r_0)\}_{p \in I}$ be a family of pairwise disjoint balls. Then we have, for $p \in I$, that*

$$\lambda_{\mathbb{V}}(z) \leq \lambda_{B(p,(1-e^{-2r_0})\operatorname{Im} p)^*}(z)\,, \tag{6.3}$$

for $z \in B(p, (1 - e^{-2r_0}) \operatorname{Im} p)^*$, and

$$\lambda_{B(p,K \operatorname{Im} p)^*}(z) \leq \lambda_{\mathbb{V}}(z)\,, \qquad \text{for } \; z \in B_{\mathbb{U}}(p, a(r_0))\,, \tag{6.4}$$

where $a(r)$ is the function defined in Lemma 6.1.

PROOF. The following relationship between hyperbolic and euclidean balls is well-known.

$$B_{\mathbb{U}}(x + i\,y, r) = B(x + i\,y \cosh r, y \sinh r)\,, \qquad \text{for } x \in \mathbb{R},\ y, r > 0\,.$$

This implies that

$$(6.5) \qquad B(z,(1-e^{-r})\operatorname{Im} z) \subset B_{\mathbb{U}}(z,r) \subset B(z,(e^{r}-1)\operatorname{Im} z)\,,$$

for $z \in \mathbb{U}$, $r>0$. We deduce that

$$B(p,(1-e^{-2r_0})\operatorname{Im} p) \subset B_{\mathbb{U}}(p,2\,r_0)\,, \qquad \text{for } p \in I\,.$$

Since $d_{\mathbb{U}}(p,q) \ge 2\,r_0$ for all $p,q \in I$, $p \ne q$, we have that $B(p,(1-e^{-2r_0})\operatorname{Im} p)^* \subset \mathbb{V}$. This implies (6.3).

Using again that $d_{\mathbb{U}}(p,q) \ge 2\,r_0$ for all $p,q \in I$, $p \ne q$, and Lemma 6.1 we deduce that

$$(6.6) \qquad |z-p| < |z-q|\,, \qquad \text{for } z \in B_{\mathbb{U}}(p,a(r_0))\,.$$

A computation gives that

$$(6.7) \qquad |z-p| \le \operatorname{Im} z\,, \qquad \text{for } z \in B_{\mathbb{U}}(p,a(r_0))\,,$$

since $e^{a(r_0)} < 2$. Hence, (6.6) and (6.7) imply that

$$\delta_{\mathbb{V}}(z) = |z-p|\,, \qquad \text{for } z \in B_{\mathbb{U}}(p,a(r_0))\,.$$

Consequently,

$$(6.8) \qquad \beta_{\mathbb{V}}(z) \le \min\Big\{ \Big| \log \Big| \frac{z-p}{w-p} \Big| \Big| : \ w \in \partial\mathbb{V} \Big\} \le \log \frac{\operatorname{Im} p}{|z-p|}\,,$$

since $|z-p| \le \operatorname{Im} p$, for $z \in B_{\mathbb{U}}(p,a(r_0))$ (to see this it is enough to change the roles of z and p in (6.7)).

Now, (6.1) and (6.8) imply that

$$\lambda_{\mathbb{V}}(z) \ge \frac{1}{|z-p| \log \dfrac{K \operatorname{Im} p}{|z-p|}}\,, \qquad \text{for } z \in B_{\mathbb{U}}(p,a(r_0))\,.$$

This inequality and the well known fact

$$\lambda_{B(w,c)^*}(z) = \frac{1}{|z-w| \log \dfrac{c}{|z-w|}}\,, \qquad \text{for } z \in B(w,c)^*\,,$$

give (6.4). This finishes the proof of Lemma 6.2.

Next we will prove Proposition 1, part a). First of all we observe that $K\,e^{-2\pi} < 1$, since $k < 2\pi$. Secondly, the condition $r < -\log(1 - K\,e^{-2\pi})$ implies that

$$0 < \frac{2\pi}{k - \log(1 - e^{-r})} < 1$$

and then we can assure that there exists the collar in $\mathcal{R}$ [K, p. 60-61].

On the other hand, (6.4) and (6.5) give, for $p \in I$, that

$$\lambda_{B(p,K\,\mathrm{Im}\,p)^*}(z) \le \lambda_{\mathbb{V}}(z)\,, \tag{6.9}$$

for $z \in B(p,(1-e^{-a(r_0)})\,\mathrm{Im}\,p)^* \subset B_{\mathbb{U}}(p,a(r_0))^*$. A straightforward computation shows that, for $w \in \mathbb{C}$ and $\rho > 0$,

$$\begin{aligned} &C_{B(w,\rho)^*}(w,\alpha) = B(w,\rho\,e^{-2\pi/\alpha})^*\,, &&\text{for } \alpha > 0\,,\\ &B(w,r)^* = C_{B(w,\rho)^*}\Big(w,\frac{2\pi}{\log\rho - \log r}\Big)\,, &&\text{for } 0 < r < \rho\,. \end{aligned} \tag{6.10}$$

Therefore (6.9) and (6.10) imply that

$$C_{\mathbb{V}}(p,\alpha) \subset C_{B(p,K\,\mathrm{Im}\,p)^*}(p,\alpha) = B(p,Ke^{-2\pi/\alpha}\,\mathrm{Im}\,p)^*\,,$$

if

$$B(p,K\,e^{-2\pi/\alpha}\,\mathrm{Im}\,p)^* \subset B(p,(1-e^{-a(r_0)})\,\mathrm{Im}\,p)^* \subset B_{\mathbb{U}}(p,a(r_0))^*.$$

If we choose

$$\alpha = \frac{2\pi}{k - \log(1-e^{-r})}\,,$$

we obtain that

$$\begin{aligned} C_{\mathbb{V}}\Big(p,\frac{2\pi}{k-\log(1-e^{-r})}\Big) &\subset B(p,K\,e^{-2\pi/\alpha}\,\mathrm{Im}\,p)^*\\ &= B(p,(1-e^{-r})\,\mathrm{Im}\,p)^*\\ &\subset B_{\mathbb{U}}(p,r)^*\\ &\subset B_{\mathbb{U}}(p,a(r_0))^*\,, \end{aligned}$$

if

$$r \le a(r_0) = \log \frac{2}{1+e^{-2r_0}} \,.$$

This finishes the proof of Proposition 1, part a).

Finally, to prove part b), observe that the condition $r < \log (1 + (1-e^{-2r_0})\, e^{-2\pi})$ implies that

$$0 < \frac{2\pi}{\log (1-e^{-2r_0}) - \log (e^r - 1)} < 1 \,,$$

and then, as above, we can assure that there exists the collar in $\mathcal{R}$.

Now, for any $p \in I$, (6.10) and (6.3) give that

$$B(p, (1-e^{-2r_0})\, e^{-2\pi/\alpha} \operatorname{Im} p)^* = C_{B(p,(1-e^{-2r_0}) \operatorname{Im} p)^*}(p,\alpha) \subset C_{\mathbb{V}}(p,\alpha) \,,$$

for $0<\alpha<1$. In particular, if we choose

$$\alpha = \frac{2\pi}{\log (1-e^{-2r_0}) - \log (e^r - 1)}$$

we obtain that

$$B(p, (e^r-1) \operatorname{Im} p)^* \subset C_{\mathbb{V}}\Big(p, \frac{2\pi}{\log (1-e^{-2r_0}) - \log (e^r - 1)}\Big) \,.$$

Therefore (6.5) gives that

$$B_{\mathbb{U}}(p,r)^* \subset C_{\mathbb{V}}\Big(p, \frac{2\pi}{\log (1-e^{-2r_0}) - \log (e^r - 1)}\Big) \,.$$

This finishes the proof of Proposition 1.

We define a *generalized collar* in a hyperbolic Riemann surface $\mathcal{R}$ about a puncture p as a domain (not necessarily doubly connected) in $\mathcal{R}$ "bounded" by p and a finite number of curves (if the collar is not equal to $\mathcal{R}$) orthogonal to the pencil of geodesics emanating from p. Observe that if $\mathcal{R}$ is a punctured compact surface (with only a puncture p), when the collar "grows" it is eventually equal to $\mathcal{R}$ and then there are not such boundary curves.

In the punctured disk, $\mathcal{R} = B(z,r)^*$ we have that

$$d_{\mathcal{R}}(\partial C_{\mathcal{R}}(z,\alpha_1), \partial C_{\mathcal{R}}(z,\alpha_2)) = \Big| \log \frac{\alpha_1}{\alpha_2} \Big| \,.$$

Then, we can define for $\alpha \ge 1$ the *generalized α-collar* in $\mathcal{R}$ about p as the set

$$C_{\mathcal{R}}(p,\alpha) = C_{\mathcal{R}}\Big(p,\frac{1}{2}\Big) \cup \Big\{q \in \mathcal{R}:\ d_{\mathcal{R}}\Big(q, \partial C_{\mathcal{R}}\Big(p,\frac{1}{2}\Big)\Big) < \log{(2\,\alpha)}\Big\}\,.$$

Obviously this definition coincides with the original one if there exists the α-collar. The number $1/2$ can be changed for any number $0<\eta<1$, if $\log{(2\,\alpha)}$ is substituted by $\log{(\alpha/\eta)}$.

If $\mathcal{R}$ is not a punctured disk, it is obvious that there exists an α_0 such that there is an α-collar only for $0<\alpha\le\alpha_0$. However there always are generalized α-collars.

With this definition we can extend part b) of Proposition 1.

Corollary 3. *Let $\mathcal{S}$ be a hyperbolic Riemann surface and let $\{B_{\mathcal{S}}(p,r_0)\}_{p\in I}$ be a family of simply connected and pairwise disjoint balls. Let us denote by $\mathcal{R}$ the Riemann surface $\mathcal{R}=\mathcal{S}\setminus I$. If we denote the generalized α-collar by $C_{\mathcal{R}}(p,\alpha)$, then we have that*

$$B_{\mathcal{S}}(p,r)^* \subset C_{\mathcal{R}}\Big(p, \frac{2\pi}{\log(1-e^{-2r_0}) - \log(e^r-1)}\Big)\,,$$

for $p\in I$ and

$$0<r<\min\{\log(2-e^{-2r_0}), r_0\}\,.$$

The proof of Corollary 3 is the same as the proof of Proposition 1, part b). We do not need now the condition $\alpha<1$ but we also need $\alpha>0$; the condition on r guarantees this fact.

A computation and (6.10) give that

$$B_{B(w,\rho)}(w,r)^* = B\Big(w,\rho\tanh\Big(\frac{r}{2}\Big)\Big)^* = C_{B(w,\rho)^*}\Bigg(w, \frac{2\pi}{\log\operatorname{cotanh}\Big(\dfrac{r}{2}\Big)}\Bigg)\,,$$

for $w\in\mathbb{C}$ and $\rho,\,r>0$.

We want to remark that Proposition 1 is sharp for $r\to 0$ in the following sense

$$\lim_{r\to 0^+} \frac{\dfrac{2\pi}{k-\log{(1-e^{-r})}}}{\dfrac{2\pi}{-\log\tanh\Big(\dfrac{r}{2}\Big)}} = \lim_{r\to 0^+} \frac{\dfrac{2\pi}{\log{(1-e^{-2r_0})}-\log{(e^r-1)}}}{\dfrac{2\pi}{-\log\tanh\Big(\dfrac{r}{2}\Big)}} = 1\,.$$

Proposition 1 also gives the following result.

Corollary 4. *Let $\mathcal{S}$ be a hyperbolic Riemann surface and let $\{B_{\mathcal{S}}(p, r_0)\}_{p\in I}$ be a family of simply connected and pairwise disjoint balls. Let us denote by $\mathcal{R}$ the Riemann surface $\mathcal{R} = \mathcal{S} \setminus I$. Let $k = 4.76$ and $K = e^k$.*

a) *We have that*

$$C_{\mathcal{R}}(p, \alpha) \subset B_{\mathcal{S}}\Big(p, \log \frac{1}{1 - K\, e^{-2\pi/\alpha}}\Big)^*,$$

for $p \in I$, $0 < \alpha < 1$ and

$$\alpha \leq \frac{2\pi}{k + \log \dfrac{2}{1 - e^{-2r_0}}}\,.$$

b) *If we denote the generalized α-collar by $C_{\mathcal{R}}(p, \alpha)$, then we have that*

$$B_{\mathcal{S}}(p, \log(1 + (1 - e^{-2r_0})\, e^{-2\pi/\alpha}))^* \subset C_{\mathcal{R}}(p, \alpha)\,,$$

for $p \in I$ and

$$0 < \alpha \leq \frac{2\pi}{\log(1 + e^{-r_0}) - r_0}\,.$$

7. Further results.

We will generalize theorems 3 and 5 in this section. To do this, we shall comment some remarks:

1) If the set I in theorems 3 and 5 is not contained in $\hat{\mathbb{R}}$, these theorems are also true since Theorem 1 is a general result about hyperbolic Riemann surfaces.

2) If $\partial\Omega_0$ is contained in a quasicircle Q (the image of a straight line by a quasiconformal mapping of the Riemann sphere onto itself) our characterization of the HII-property for Denjoy domains can be yet applied (if we know the quasiconformal mapping which applies $\hat{\mathbb{R}}$ in Q) since the HII-property is preserved by quasiconformal mappings [FR, Theorem 1].

We can define in an obvious way a border set of a closed subset of a quasicircle. In this context we can generalize Theorem 3.

Theorem 6. *Let Ω_0 be a hyperbolic plane domain whose boundary is contained in a quasicircle and has not isolated points, let I be a strongly uniformly separated set in Ω_0, and let $\Omega = \Omega_0 \setminus I$. Then*

a) *If Ω has* HII, *then there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that*

$$\frac{1}{n} \sum_{j=1}^{n} \Phi_1(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,.$$

b) *If there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \geq 3$, we have that*

$$\frac{1}{n} \sum_{j=1}^{n} \Phi_2(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,,$$

then Ω has HII.

Observe that Theorem 6 follows directly from Theorem 3, [FR, Theorem 1] and the following facts: a) a quasiconformal map quasi-preserves cross ratios; b) $\Phi_i(s) \asymp \Phi_i(r)$ for $s \asymp r$, with $0 < r < \infty$ and $i = 1, 2$.

Theorem 6 gives a necessary and a sufficient condition for Ω to have HII. We shall improve this result in the remainder of the section.

If $\partial\Omega_0$ is contained in a finite union of quasicircles, we can also characterize the HII-property of Ω in many cases. We give now the details:

Let $\{E_j\}_{j=1}^n$ be a collection of pairwise disjoint closed subsets of $\hat{\mathbb{C}}$ such that each E_j is contained in a quasicircle and $\Omega_0 = \hat{\mathbb{C}} \setminus \cup_j E_j$ is connected. Let I be a strongly uniformly separated set in Ω_0 and let $\Omega = \Omega_0 \setminus I$. A necessary and sufficient condition for Ω to have HII is that each $\hat{\mathbb{C}} \setminus E_j$ has HII (see Theorem 8 below). By using remark 2) or Theorem 6 as a test, we can verify if each one of these last domains has HII or not.

Although we are interested in plane domains and closed subsets of quasicircles, many results in this section are true for general Riemann surfaces instead of $\hat{\mathbb{C}}$ and general closed sets E_j. We start with some definitions.

Definition. *Let $\mathcal{S}$ be a Riemann surface and $\varepsilon > 0$. Let E_1, E_2 be two closed disjoint subsets of $\mathcal{S}$. We say that E_1 and E_2 are weakly ε-separated in $\mathcal{S}$ if $\mathcal{S}_1 = \mathcal{S} \setminus E_1$, $\mathcal{S}_2 = \mathcal{S} \setminus E_2$ are (connected) hyperbolic Riemann surfaces and the two following subsets are disjoint:*

$$E_{1,\varepsilon} = \{q \in \mathcal{S}_2 : \; d_{\mathcal{S}_2}(q, E_1) < 2\,\varepsilon\}\,,$$

$$E_{2,\varepsilon} = \{q \in \mathcal{S}_1 : \; d_{\mathcal{S}_1}(q, E_2) < 2\,\varepsilon\}\,.$$

We say that E_1 and E_2 are weakly separated in $\mathcal{S}$ if they are weakly ε-separated in $\mathcal{S}$ for some $\varepsilon > 0$.

We say that the closed sets $E_1, E_2, \dots, E_n$ are weakly separated in $\mathcal{S}$ if the $n-1$ pairs of sets (E_1, E_2), $(E_1 \cup E_2, E_3), \dots, (E_1 \cup E_2 \cup \cdots \cup E_{n-1}, E_n)$ are weakly separated in $\mathcal{S}$.

Remark 1. It is clear that if E_1, E_2 are disjoint closed subsets of $\mathcal{S}$, E_2 is compact and $\mathcal{S} \setminus (E_1 \cup E_2)$ is connected, then E_1, E_2 are weakly separated in $\mathcal{S}$. It is also clear that if $E_1, E_2, \dots, E_n$ are pairwise disjoint compact subsets of $\mathcal{S}$ and $\mathcal{S} \setminus \cup_{j=1}^n E_j$ is connected, then $E_1, E_2, \dots, E_n$ are weakly separated in $\mathcal{S}$.

Remark 2. If E_1, E_2 are disjoint closed subsets of a plane domain Ω, it is possible that they are not weakly separated in Ω. Let Ω be the plane domain $\Omega = \mathbb{C} \setminus \{0\}$. Let us consider as E_1 a sequence $\{x_n\}$ of real numbers decreasing to 0. Let E_2 be a sequence $\{y_n\}$ such that:

a) $0 < x_{n+1} < y_n < x_n$,

b) $\lim_{n\to\infty}(x_n - y_n)/(y_n - x_{n+1}) = \lim_{n\to\infty}(x_n - y_n)/(y_{n-1} - x_n) = 0$.

Then E_1, E_2 are not weakly separated in $\mathbb{C} \setminus \{0\}$.

Remark 3. Let E_1, E_2 be closed sets in a domain $\Omega \subset \mathbb{C} \setminus \{z_0\}$. Let us assume that there is a positive constant δ_0 such that

$$|z_1 - z_2| \ge \delta_0\,|z_1 - z_0|\,, \qquad \text{for all } z_1 \in E_1\,,\; z_2 \in E_2\,.$$

Then E_1, E_2 are weakly separated in Ω.

Proof of Remark 3. Without loss of generality we can suppose $z_0 = 0$. For $w \in \mathbb{C} \setminus \{0, 1\}$, we define the function

$$e(w) := \max\big\{\varepsilon > 0 : \; B_{\mathbb{C}\setminus\{0,1\}}(w, \varepsilon) \cap B_{\mathbb{C}\setminus\{0,w\}}(1, \varepsilon) = \varnothing\big\}\,.$$

Observe that $\partial B_{\mathbb{C}\setminus\{0,1\}}(w,\varepsilon)$ and $\partial B_{\mathbb{C}\setminus\{0,w\}}(1,\varepsilon)$ vary continuously with w, since

$$\lambda_{\mathbb{C}\setminus\{0,w\}}(z) = \lambda_{\mathbb{C}\setminus\{0,1\}}\Big(\frac{z}{w}\Big)\frac{1}{|w|}$$

is a real analytic function on w.

Therefore, e is a continuous function $e:\mathbb{C}\setminus\{0,1\}\longrightarrow(0,\infty)$. On the other hand, we can deduce of (6.1) that

$$\lambda_{\mathbb{C}\setminus\{0,1\}}(z) \ge \frac{1}{|z|\,(k+\big|\log|z|\big|)}\,, \qquad \text{for } z\in\mathbb{C}\setminus\{0,1\}\,,$$

where $k=4.76$ is the constant in Section 6. This is a bad estimate if z is near 1, but it is good for z in a neighborhood of 0 or ∞. This inequality gives

$$B_{\mathbb{C}\setminus\{0,1\}}(w,\varepsilon)\subset\big\{|z|>\exp\big((k+\log|w|)\,e^{-\varepsilon}-k\big)\big\}\,,$$

and consequently,

$$B_{\mathbb{C}\setminus\{0,w\}}(1,\varepsilon)\subset\big\{|z|<|w|\exp\big(k-(k+\log|w|)e^{-\varepsilon}\big)\big\}\,,$$

for

$$|w|>1 \quad\text{and}\quad 0<\varepsilon\le\log\frac{k+\log|w|}{k}\,.$$

Therefore, $B_{\mathbb{C}\setminus\{0,1\}}(w,\varepsilon)\cap B_{\mathbb{C}\setminus\{0,w\}}(1,\varepsilon)=\varnothing$ for

$$|w|>1 \quad\text{and}\quad 0<\varepsilon\le\log\frac{k+\log|w|}{k+\frac{1}{2}\log|w|}\,.$$

Then, for any $M>1$, there is a positive constant c_0 such that $e(w)\ge c_0$ if $|w|\ge M$.

Observe that $e(1/w)=e(w)$ since the conformal map $T(z)=1/z$ is an isometry of $\mathbb{C}\setminus\{0,1\}$ onto itself. Consequently, $e(w)\ge c_0$ if $|w|\le 1/M$. These facts imply that, for any $\delta>0$, there exists $\varepsilon>0$ such that $e(w)\ge\varepsilon$ if $|w-1|\ge\delta$.

For $z_1,z_2\in\mathbb{C}\setminus\{0\}$ with $z_1\ne z_2$, we define now the function

$$E(z_1,z_2):=\max\{\varepsilon>0:\ B_{\mathbb{C}\setminus\{0,z_1\}}(z_2,\varepsilon)\cap B_{\mathbb{C}\setminus\{0,z_2\}}(z_1,\varepsilon)=\varnothing\}\,.$$

It is clear that

$$E(z_1,z_2)=E\Big(1,\frac{z_2}{z_1}\Big)=e\Big(\frac{z_2}{z_1}\Big)\,.$$

The hypothesis on E_1 and E_2 give that there is $\delta_0 > 0$ such that $|z_2 - z_1| \geq \delta_0 |z_1|$ for all $z_1 \in E_1$ and $z_2 \in E_2$, *i.e.*, $|z_2/z_1 - 1| \geq \delta_0$ for all $z_1 \in E_1$ and $z_2 \in E_2$. Consequently, there is $\varepsilon_0 > 0$ such that

$$E(z_1, z_2) \geq \varepsilon_0 \,, \qquad \text{for all } z_1 \in E_1 \text{ and } z_2 \in E_2 \,.$$

Then we have that

$$B_{\mathbb{C}\setminus\{0,z_1\}}(z_2, \varepsilon_0) \cap B_{\mathbb{C}\setminus\{0,z_2\}}(z_1, \varepsilon_0) = \varnothing \,, \quad \text{for all } z_1 \in E_1 \text{ and } z_2 \in E_2 \,.$$

In the following we will use the notation $B_\Omega(A, r) := \cup_{p \in A} B_\Omega(p, r)$ for a set A and a positive number r.

Let us fix $z_1 \in E_1$. We have that

$$B_{\mathbb{C}\setminus\{0,z_1\}}(E_2, \varepsilon_0) := \cup_{z_2 \in E_2} B_{\mathbb{C}\setminus\{0,z_1\}}(z_2, \varepsilon_0)$$

and

$$B_{\mathbb{C}\setminus\{0,E_2\}}(z_1, \varepsilon_0) \subset \cap_{z_2 \in E_2} B_{\mathbb{C}\setminus\{0,z_2\}}(z_1, \varepsilon_0) \,.$$

Therefore

$$B_{\mathbb{C}\setminus\{0,z_1\}}(E_2, \varepsilon_0) \cap B_{\mathbb{C}\setminus\{0,E_2\}}(z_1, \varepsilon_0) = \varnothing \,, \qquad \text{for all } z_1 \in E_1 \,.$$

Now, we have that

$$B_{\mathbb{C}\setminus\{0,E_1\}}(E_2, \varepsilon_0) \subset \cap_{z_1 \in E_1} B_{\mathbb{C}\setminus\{0,z_1\}}(E_2, \varepsilon_0)$$

and

$$B_{\mathbb{C}\setminus\{0,E_2\}}(E_1, \varepsilon_0) = \cup_{z_1 \in E_1} B_{\mathbb{C}\setminus\{0,E_2\}}(z_1, \varepsilon_0) \,.$$

Then

$$B_{\mathbb{C}\setminus\{0,E_1\}}(E_2, \varepsilon_0) \cap B_{\mathbb{C}\setminus\{0,E_2\}}(E_1, \varepsilon_0) = \varnothing \,.$$

Remark 4. Let E_1, E_2 be closed sets in a domain $\Omega \subset \mathbb{C}$ with $z_0 \in \partial\Omega$. Let C_1, C_2 be closed sets in $\mathbb{C}$, such that each C_j is a finite union of cones with vertex in z_0, $E_j \subset C_j$ and $C_1 \cap C_2 = \{z_0\}$. Then

$$|z_1 - z_2| \geq \delta_0 \, |z_1 - z_0| \,, \qquad \text{for all } z_1 \in E_1 \,,\ z_2 \in E_2 \,,$$

and therefore, E_1, E_2 are weakly separated in Ω.

In order to prove Theorem 8 we shall state some previous results.

Lemma 7.1. *Let $\mathcal{S}$ be a Riemann surface and $\varepsilon > 0$. Let E_1, E_2 be two closed weakly ε-separated subsets in $\mathcal{S}$. Let $\mathcal{S}_k = S \setminus E_k$ be (connected) hyperbolic Riemann surfaces for $k = 1, 2$ and let $\mathcal{R}$ be a connected component of $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S} \setminus (E_1 \cup E_2)$. Then,*

$$b(\mathcal{R}) \geq \frac{1}{2} \tanh^2 \varepsilon \min\{b(\mathcal{S}_1), b(\mathcal{S}_2)\}.$$

Proof. Let $\varphi \in C_c^\infty(\mathcal{R})$. Obviously $\varphi \in C_c^\infty(\mathcal{S}_1) \cap C_c^\infty(\mathcal{S}_2)$ and

$$\frac{\displaystyle\iint_{\mathcal{S}_k} \|\nabla\varphi\|^2 \, dw_k}{\displaystyle\iint_{\mathcal{S}_k} \varphi^2 \, dw_k} \geq b(\mathcal{S}_k), \qquad k = 1, 2,$$

where dw_1 and dw_2 denote, respectively, the area element in $\mathcal{S}_1$ and $\mathcal{S}_2$. Recall that $\|\cdot\|$ and ∇ refer also to the corresponding Poincaré metrics.

Let us consider now the open sets

$$E_{1,\varepsilon} = \{q \in \mathcal{S}_2 : \ d_{\mathcal{S}_2}(q, E_1) < 2\varepsilon\},$$

$$E_{2,\varepsilon} = \{q \in \mathcal{S}_1 : \ d_{\mathcal{S}_1}(q, E_2) < 2\varepsilon\}.$$

By hypothesis we have that $E_{1,\varepsilon} \cap E_{2,\varepsilon} = \varnothing$ and therefore $(\mathcal{S} \setminus E_{1,\varepsilon}) \cup (\mathcal{S} \setminus E_{2,\varepsilon}) = \mathcal{S}$. On the other hand, we also have as a consequence of (3.2) that

$$\iint_{\mathcal{S}\setminus E_{1,\varepsilon}} \varphi^2 \, dw \leq \operatorname{cotanh}^2\varepsilon \iint_{\mathcal{S}\setminus E_{1,\varepsilon}} \varphi^2 \, dw_2$$

and

$$\iint_{\mathcal{S}\setminus E_{2,\varepsilon}} \varphi^2 \, dw \leq \operatorname{cotanh}^2\varepsilon \iint_{\mathcal{S}\setminus E_{2,\varepsilon}} \varphi^2 \, dw_1 ,$$

where dw is the area element in $\mathcal{R}$.

Therefore, we deduce that

$$\begin{aligned}
\iint_{\mathcal{R}} \varphi^2 \, dw &\leq \iint_{\mathcal{S}\setminus E_{1,\varepsilon}} \varphi^2 \, dw + \iint_{\mathcal{S}\setminus E_{2,\varepsilon}} \varphi^2 \, dw \\
&\leq \operatorname{cotanh}^2\varepsilon \Big(\iint_{\mathcal{S}\setminus E_{1,\varepsilon}} \varphi^2 \, dw_2 + \iint_{\mathcal{S}\setminus E_{2,\varepsilon}} \varphi^2 \, dw_1 \Big) \qquad (7.1) \\
&\leq \operatorname{cotanh}^2\varepsilon \Big(\iint_{\mathcal{S}_2} \varphi^2 \, dw_2 + \iint_{\mathcal{S}_1} \varphi^2 \, dw_1 \Big).
\end{aligned}$$

Recall that $\iint \|\nabla\varphi\|^2\,dw$ is a conformal invariant, *i.e.*

$$\iint_{\mathcal{R}} \|\nabla\varphi\|^2\,dw = \iint_{\mathcal{S}_1} \|\nabla\varphi\|^2\,dw_1 = \iint_{\mathcal{S}_2} \|\nabla\varphi\|^2\,dw_2\,. \tag{7.2}$$

We obtain from (7.1) and (7.2) that

$$\begin{aligned}\frac{\displaystyle\iint_{\mathcal{R}} \|\nabla\varphi\|^2\,dw}{\displaystyle\iint_{\mathcal{R}} \varphi^2\,dw} &\geq \frac{1}{2}\,\tanh^2\varepsilon\,\frac{\displaystyle\iint_{\mathcal{S}_1} \|\nabla\varphi\|^2\,dw_1 + \iint_{\mathcal{S}_2} \|\nabla\varphi\|^2\,dw_2}{\displaystyle\iint_{\mathcal{S}_1} \varphi^2\,dw_1 + \iint_{\mathcal{S}_2} \varphi^2\,dw_2} \\ &\geq \frac{1}{2}\,\tanh^2\varepsilon\min\{b(\mathcal{S}_1), b(\mathcal{S}_2)\}\,,\end{aligned}$$

for every $\varphi \in C_c^\infty(\mathcal{R})$. This finishes the proof of Lemma 7.1.

As a consequence of this lemma one obtains the following results.

Proposition 2. *Let $\mathcal{S}$ be a Riemann surface. Let $E_1, E_2, \dots, E_n$ be weakly separated closed sets in $\mathcal{S}$ such that $\mathcal{S}_k = S \setminus E_k$ $(k = 1, \dots, n)$ are (connected) hyperbolic Riemann surfaces and let $\mathcal{R}$ be a connected component of $\cap_k \mathcal{S}_k = \mathcal{S} \setminus \cup_k E_k$. Then there exists a positive constant c such that*

$$b(\mathcal{R}) \geq c\,\min_k b(\mathcal{S}_k)\,.$$

Lemma 7.2. *Let $\mathcal{S}$ be a hyperbolic Riemann surface. Let E_1, E_2 be two disjoint closed subsets of $\mathcal{S}$ such that $\mathcal{S}_k = S \setminus E_k$ are connected surfaces for $k = 1, 2$, let $\mathcal{R}$ be a connected component of $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S} \setminus (E_1 \cup E_2)$ and let $4\,\varepsilon = d_{\mathcal{S}}(E_1, E_2)$. Then,*

$$b(\mathcal{R}) \geq \frac{1}{2}\,\tanh^2\varepsilon\,\min\{b(\mathcal{S}_1), b(\mathcal{S}_2)\}\,.$$

Lemma 7.2 is a direct consequence of Lemma 7.1, since $d_{\mathcal{S}}(E_1, E_2) = 4\,\varepsilon$ implies that E_1, E_2 are weakly ε-separated in $\mathcal{S}$.

Proposition 3. *Let $\mathcal{S}$ be a hyperbolic Riemann surface. Let $\{E_k\}_{k=1}^n$ be a collection of pairwise disjoint closed subsets of $\mathcal{S}$ such that $\mathcal{S}_k = S \setminus E_k$*

$(k = 1, \ldots, n)$ are connected surfaces, let $\mathcal{R}$ be a connected component of $\cap_k \mathcal{S}_k = \mathcal{S} \setminus \cup_k E_k$ and let $\varepsilon = \min_{j \neq k} d_{\mathcal{S}}(E_j, E_k)$. Suppose that $\varepsilon > 0$. Then, there exists a positive constant c, which only depends on ε and n (but not on $\mathcal{S}$), such that

$$b(\mathcal{R}) \geq c \min_k b(\mathcal{S}_k) .$$

Remark. Let $\{E_k\}_{k=1}^n$ be a collection of pairwise disjoint closed subsets of $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} \setminus E_k$ is a (connected) hyperbolic plane domain for $k = 1, \ldots, n$. Let $\Omega_0 = \hat{\mathbb{C}} \setminus \cup_k E_k$. Let also I be a strongly uniformly separated set in Ω_0 and let $\Omega = \Omega_0 \setminus I$. A sufficient condition for Ω to have HII is that each $\hat{\mathbb{C}} \setminus E_k$ has HII.

Definition. *Let $\mathcal{S}$ be a hyperbolic Riemann surface and let $\gamma_1, \ldots, \gamma_k$ be simple closed geodesics in $\mathcal{S}$. We say that G is a quasigeodesic domain in $\mathcal{S}$, relatively to $\gamma_1, \ldots, \gamma_k$, if G is a domain of finite area in $\mathcal{S}$ and ∂G consists of finitely many simple closed curves $\alpha_1, \ldots, \alpha_r$, where each α_i is either a simple closed geodesic or a finite union of subarcs of simple closed geodesics such that if two arcs meet at a point, one of these arcs is a subarc of some γ_j. We define $\partial_0 G$ as $\partial_0 G = \partial G \setminus \{\gamma_1 \cup \cdots \cup \gamma_k\}$.*

Obviously, we can have $\partial_0 G = \varnothing$.

Quasigeodesic domains appear in a natural way as intersection of geodesics domains: If G_1, G_2 are geodesic domains in $\mathcal{S}$, then $G_1 \cap G_2$ is a quasigeodesic domain relatively to ∂G_1.

We need to talk about collars of geodesics in any hyperbolic Riemann surface $\mathcal{S}$.

Given a simple closed geodesic γ in $\mathcal{S}$, a *collar* about γ is a doubly connected domain on $\mathcal{S}$ bounded by two simple closed curves (the boundary curves of the collar) each point of which has the same distance d from γ. The distance d is called the *width* of the collar. A collar about γ of area $2\,\beta$ is called a *β-collar*.

Randol [R] proved that there exists a collar C_γ of γ with width d_0, such that

$$\cosh d_0 \geq \coth \frac{L_{\mathcal{S}}(\gamma)}{2} , \qquad A_{\mathcal{S}}(C_\gamma) \geq 2\, L_{\mathcal{S}}(\gamma) \operatorname{cosech} \frac{L_{\mathcal{S}}(\gamma)}{2} .$$

Moreover, if γ' is a geodesic such that $\gamma \cap \gamma' = \varnothing$, we also have that $C_\gamma \cap \gamma' = \varnothing$.

Randol [R] states the Collar Lemma under the hypothesis that the surface is compact, but the same proof, without any change, works for any hyperbolic Riemann surface.

Lemma 7.3. *Let $\mathcal{S}$ be a hyperbolic Riemann surface satisfying* HII *and let $\{\gamma_1, \dots, \gamma_k\}$ be a collection of pairwise disjoint simple closed geodesics in $\mathcal{S}$. Then, there exists a positive constant c such that*

$$A_{\mathcal{S}}(G) \le c\, L_{\mathcal{S}}(\partial_0 G)\,, \tag{7.3}$$

for any quasigeodesic domain G in $\mathcal{S}$, relatively to $\gamma_1, \dots, \gamma_k$, with $L_{\mathcal{S}}(\partial_0 G) > 0$.

Proof. By the isoperimetric inequality of $\mathcal{S}$, we only need to check (7.3) for quasigeodesic domains G in $\mathcal{S}$, such that $0 < L_{\mathcal{S}}(\partial_0 G) < L_{\mathcal{S}}(\partial G)$.

First of all, let us consider the compact sets $C_{t,i} = \{p \in \mathcal{S} : d_{\mathcal{S}}(p, \gamma_i) \le t\}$ for positive t and $i \in \{1, \dots, k\}$. Given a geodesic γ_i we choose a positive and a negative side of γ_i, denoted respectively by γ_i^+ and γ_i^-. We denote by $C_{t,i}^+$ (respectively $C_{t,i}^-$) the set of points in $\mathcal{S}$ which are in some geodesic of length t which starts orthogonally to γ_i^+ (respectively γ_i^-). Obviously, we have that $C_{t,i} = C_{t,i}^+ \cup C_{t,i}^-$. It can happen that $C_{t,i}^+ \cap C_{t,i}^- \ne \gamma_i$ if the Riemann surface $\mathcal{S}$ has positive genus (of course, if $\mathcal{S} \setminus \gamma$ is connected).

Let $G_{t,i}^+$ (respectively $G_{t,i}^-$) be the geodesic domain "corresponding" to $C_{t,i}^+$ (respectively $C_{t,i}^-$): each puncture or boundary curve of $G_{t,i}^+$ is freely homotopic to a boundary curve of $C_{t,i}^+$. Denote by $G_{t,i}$ the union $G_{t,i} = \gamma_i \cup G_{t,i}^+ \cup G_{t,i}^-$. If for some $i \in \{1, \dots, k\}$ we have that $G_{t,i} = \gamma_i$ for all positive t (the two boundary curves of $C_{t,i}$ are freely homotopic to γ_i), then $k = 1$ and $\mathcal{S}$ is a doubly connected domain (an annulus), and (7.3) is true since there are not quasigeodesic domains in $\mathcal{S}$. Therefore we can assume without loss of generality that $G_{t,i}$ is not empty for $t \ge t_0$ and $i \in \{1, \dots, k\}$. Observe that $G_{t,i}^+$ is non decreasing in t. In fact, if $t_1 < t_2$ are such that $A_{\mathcal{S}}(G_{t_1,i}^+) < A_{\mathcal{S}}(G_{t_2,i}^+)$, the constant curvature -1 and Gauss-Bonnet theorem give $A_{\mathcal{S}}(G_{t_1,i}^+) + 2\pi \le A_{\mathcal{S}}(G_{t_2,i}^+)$. The same is true for $G_{t,i}^-$.

This implies that for each $i \in \{1, \dots, k\}$ either there exists a positive number T_i^+ such that $G_{t,i}^+ = G_{T_i^+,i}^+$ for all $t \ge T_i^+$, or $A_{\mathcal{S}}(G_{t,i}^+) \to \infty$ as $t \to \infty$. The same is true for $G_{t,i}^-$ with T_i^-.

Now, let G be a quasigeodesic domain in $\mathcal{S}$, such that $0 < L_{\mathcal{S}}(\partial_0 G) < L_{\mathcal{S}}(\partial G)$. Therefore, there exists $j \in \{1, \dots, k\}$ with $\partial G \cap \gamma_j \neq \varnothing$. We consider three possibilities:

Case 1. $A_{\mathcal{S}}(G) \geq 2\, h(\mathcal{S})\, \ell$, with $\ell := \sum_{i=1}^k L_{\mathcal{S}}(\gamma_i)$. In this case,

$$2\, h(\mathcal{S})\, \ell \leq A_{\mathcal{S}}(G) \leq h(\mathcal{S})\, L_{\mathcal{S}}(\partial G) \leq h(\mathcal{S})\, \big(L_{\mathcal{S}}(\partial_0 G) + \ell \big)$$

and we obtain that

$$\ell \leq L_{\mathcal{S}}(\partial_0 G)\,.$$

Therefore,

$$A_{\mathcal{S}}(G) \leq 2\, h(\mathcal{S})\, L_{\mathcal{S}}(\partial_0 G)\,.$$

Case 2. $A_{\mathcal{S}}(G) < 2\, h(\mathcal{S})\, \ell$.

For each $i \in \{1, \dots, k\}$, let t_i be a positive number verifying the two following conditions:

a) $t_i \geq T_i^+$ (if there exists T_i^+) or $A_{\mathcal{S}}(G^+_{t_i,i}) \geq 2\, h(\mathcal{S})\, \ell$,

b) $t_i \geq T_i^-$ (if there exists T_i^-) or $A_{\mathcal{S}}(G^-_{t_i,i}) \geq 2\, h(\mathcal{S})\, \ell$.

Let Ω_i be the geodesic domain $\Omega_i := G_{t_i,i}$. We define the following positive numbers

$$a := \min \big\{ L_{\mathcal{S}}(\gamma) : \ \gamma \text{ simple closed geodesic, } \ \gamma \subset \cup_i \overline{\Omega_i} \big\}\,,$$

$$b := \max \big\{ L_{\mathcal{S}}(\gamma) : \ \gamma \text{ simple closed geodesic, } \ \gamma \subset \cup_{i=1}^k \{ \gamma_i \cup \partial \Omega_i \} \big\}\,.$$

Recall that $\partial G \cap \gamma_j \neq \varnothing$. This fact, the inequalities, $A_{\mathcal{S}}(G) < 2\, h(\mathcal{S})\, \ell$, $L_{\mathcal{S}}(\partial_0 G) > 0$, and the definition of t_j give that one of the two next possibilities holds:

Case 2.1. There exists a simple closed geodesic $\gamma \subset \overline{\Omega_j} \cap \partial_0 G$. Then

$$L_{\mathcal{S}}(\partial_0 G) \geq L_{\mathcal{S}}(\gamma) \geq a\,.$$

Case 2.2. here exists a geodesic arc η in $\partial_0 G$ which meets some simple closed geodesic $\gamma \subset \partial \Omega_j \cup (\cup_{i=1}^k \gamma_i)$.

Observe that if G is not a geodesic domain we are in this situation; in fact, there is a geodesic arc η in ∂G_0 which meets some γ_i.

Collar Lemma [R] says that $L_{\mathcal{S}}(\eta) \geq d_0$, where d_0 (the width of the collar C_γ) satisfies

$$\cosh d_0 \geq \coth \frac{L_{\mathcal{S}}(\gamma)}{2} \geq \coth \frac{b}{2}\,,$$

and

$$d_0 \geq D := \operatorname{Arg} \cosh \Big(\coth \frac{b}{2} \Big)$$

(recall that if a geodesic γ' does not intersect γ then γ' does not intersect C_γ).

Therefore,

$$L_{\mathcal{S}}(\partial_0 G) \geq L_{\mathcal{S}}(\eta) \geq D\,.$$

In both cases (2.1 and 2.2) $L_{\mathcal{S}}(\partial_0 G) \geq \min\{a, D\} =: c_0$. Then

$$A_{\mathcal{S}}(G) \leq h(\mathcal{S})\big(L_{\mathcal{S}}(\partial_0 G) + \ell\big) \leq h(\mathcal{S}) \left(L_{\mathcal{S}}(\partial_0 G) + \ell \, \frac{L_{\mathcal{S}}(\partial_0 G)}{c_0} \right)$$

and

$$A_{\mathcal{S}}(G) \leq h(\mathcal{S}) \Big(1 + \frac{\ell}{c_0} \Big) L_{\mathcal{S}}(\partial_0 G)\,.$$

Obviously, $\ell \geq a \geq c_0$ and $1 + \ell/c_0 \geq 2$. Therefore, in any case,

$$A_{\mathcal{S}}(G) \leq h(\mathcal{S}) \Big(1 + \frac{\ell}{c_0} \Big) L_{\mathcal{S}}(\partial_0 G)\,.$$

Consequently, Lemma 7.3 is true with

$$c = h(\mathcal{S}) \Big(1 + \frac{\ell}{c_0} \Big)\,.$$

If $\mathcal{S}$ is a hyperbolic Riemann surface, we have considered (open and connected) subsurfaces $\mathcal{S}_1 \subset \mathcal{S}$, endowed with its own hyperbolic metric. Of course, $\mathcal{S}_1$ is a geodesically complete Riemannian manifold with this metric. In the following we will consider also bordered (connected) Riemann subsurfaces $\mathcal{S}_2 \subset \mathcal{S}$, endowed with the restriction to $\mathcal{S}_2$ of the hyperbolic metric of $\mathcal{S}$. Therefore $\mathcal{S}_2$ is not a geodesically complete Riemannian manifold with this metric.

Lemma 7.3 and [FR, Lemma 1.2] have the following consequences.

Corollary 5. *Let $\mathcal{S}_1, \dots, \mathcal{S}_m$ be hyperbolic Riemann surfaces satisfying* HII. *For $j = 1, \dots, m$, let $\mathcal{S}_j^0$ be a bordered subsurface of $\mathcal{S}_j$ whose border is a set of n_j $(1 \leq n_j < \infty)$ pairwise disjoint simple closed geodesics. Let us assume that we can paste $\mathcal{S}_1^0, \dots, \mathcal{S}_m^0$ along their boundaries, obtaining a complete* (*without boundary*) *hyperbolic Riemann surface $\mathcal{R}$*

(*recall that we can join two surfaces identifying two boundary geodesics if and only if they have the same length*). *Then,* $\mathcal{R}$ *satisfies* HII *if and only if there exists* $1 \le j \le m$ *such that* $A_{\mathcal{S}_j}(\mathcal{S}_j^0) = \infty$.

Proof. If $A_{\mathcal{S}_j}(\mathcal{S}_j^0)$ is finite for $j = 1, \dots, m$, then $\mathcal{R}$ also has finite area (since $A_{\mathcal{R}}(\mathcal{S}_j^0) = A_{\mathcal{S}_j}(\mathcal{S}_j^0)$) and therefore, it does not satisfy HII.

Let us assume now that $A_{\mathcal{S}_1}(\mathcal{S}_1^0) = \infty$. Let Λ be the union (for $j = 1, \dots, m$) of the n_j geodesics in the boundary of $\mathcal{S}_j^0$.

Let G be a geodesic domain in $\mathcal{R}$. If G was already a geodesic domain in some $\mathcal{S}_j^0$, it satisfies (1.1) with constant

$$h_1 = \max\{h_g(\mathcal{S}_1), \dots, h_g(\mathcal{S}_m)\}\,.$$

In other case, we consider the sets $G_j = G \cap \mathcal{S}_j^0$, for $j = 1, \dots, m$. Let $\partial_0 G = \partial G \setminus \Lambda$ and $\partial_j G = \partial_0 G \cap \mathcal{S}_j^0$, for $j = 1, \dots, m$. Let us consider now the set J of the indices $j \in \{1, \dots, m\}$ such that $L_{\mathcal{R}}(\partial_j G) > 0$.

If $J = \varnothing$ then ∂G is contained in Λ, and there are only a finite number of such G. These domains satisfy (1.1) with a fixed constant h_2, which only depends on $\mathcal{R}$.

If $j \in J$, then Lemma 7.3 gives that

$$A_{\mathcal{R}}(G_j) \le c_j\, L_{\mathcal{R}}(\partial_j G) \le h_3\, L_{\mathcal{R}}(\partial_j G)\,, \tag{7.4}$$

where

$$h_3 := \max\{c_1, \dots, c_m\}\,,$$

since $A_{\mathcal{R}}(G_j) = A_{\mathcal{S}_j}(G_j)$ and $L_{\mathcal{R}}(\partial_j G) = L_{\mathcal{S}_j}(\partial_j G)$.

Otherwise, Gauss-Bonnet theorem gives that

$$\sum_{j\in J} A_{\mathcal{R}}(G_j) \ge 2\pi\,. \tag{7.5}$$

Consequently, (7.4) and (7.5) give that

$$L_{\mathcal{R}}(\partial G) \ge \sum_{j\in J} L_{\mathcal{R}}(\partial_j G) \ge \frac{1}{h_3} \sum_{j\in J} A_{\mathcal{R}}(G_j) \ge \frac{2\pi}{h_3}\,. \tag{7.6}$$

Let

$$A := \sum_{A_{\mathcal{R}}(S_j^0)<\infty} A_{\mathcal{R}}(S_j^0)\,.$$

As a consequence of (7.4) and (7.6), one deduces that

$$A_{\mathcal{R}}(G) \le A + \sum_{j\in J} A_{\mathcal{R}}(G_j) \le \frac{A\,h_3}{2\pi} L_{\mathcal{R}}(\partial G) + h_3\, L_{\mathcal{R}}(\partial_0 G)\,.$$

Therefore,

$$h_g(\mathcal{R}) \le \min\Big\{ h_1, h_2, h_3\Big(1+\frac{A}{2\pi}\Big)\Big\}\,.$$

Now [FR, Lemma 1.2] finishes the proof of Corollary 5.

Corollary 6. *Let $\mathcal{S}$ be a hyperbolic Riemann surface satisfying* HII. *Let $\gamma_1,\dots,\gamma_k$ be pairwise disjoint simple closed geodesics in $\mathcal{S}$. Let $\mathcal{S}_1$ be any connected component of $\mathcal{S}\setminus\{\gamma_1\cup\dots\cup\gamma_k\}$ with $A_{\mathcal{S}}(\mathcal{S}_1)=\infty$, and let $\mathcal{S}_0$ be the Schottky double of $\mathcal{S}_1$. Then, $\mathcal{S}_0$ satisfies* HII.

The Schottky double of $\mathcal{S}_1$ is the union of $\overline{\mathcal{S}_1}$ and its "reflection" with respect to $\partial\mathcal{S}_1$. See [AS, p. 26] for a precise definition.

This corollary was proved in [Ro1, p. 245-248] with similar arguments that those in Lemma 7.3. However, we need the precise statements of Lemma 7.3 and Corollary 5, which are more general than Corollary 6.

We need some additional results. The first one is well-known (see *e.g.* [Be]).

Lemma 7.4. *Let $\mathcal{S}$ be a hyperbolic Riemann surface with a puncture p. Then, we have that*

$$C_{\mathcal{S}}(p,1)\cap\gamma=\varnothing\,,$$

for any simple closed geodesic γ in $\mathcal{S}$.

We say that a function f is in the class $C^k(F)$, where $1\le k\le\infty$ and F is a closed set, if the derivatives of f up to the order k are continuous in F where we define the derivative of f in a point $z\in F$ as the usual limit when we approach to z by taking points in F. We just consider with this purpose closed sets F which are closures of open sets with smooth boundaries.

Lemma 7.5. *Let $\mathcal{S}$ be a Riemann surface and let $\mathcal{J}$ be a simply connected domain in $\mathcal{S}$ whose boundary is an analytic simple closed curve*

η. Given a compact subset K of $\mathcal{J}$, an open subset V of $\mathcal{J}$ and a point $q \in V$, there exists a quasiconformal automorphism f of $\mathcal{S}$ such that $f|_{\mathcal{S}\setminus\mathcal{J}}$ is the identity map, $f(K) \subset V$ and $f(q) = q$.

Proof. Let us consider a universal covering map $\pi : \mathbb{D} \longrightarrow \mathcal{S}$. Let $\mathcal{J}_0$ be a connected component of $\pi^{-1}(\mathcal{J})$. In what follows by π^{-1} we mean the inverse function of $\pi|_{\mathcal{J}_0}$. Let F_1 (respectively F_2) be a conformal map of $\hat{\mathbb{C}} \setminus \overline{\mathcal{J}_0}$ (respectively $\mathcal{J}_0$) on $\{z \in \hat{\mathbb{C}} : |z| > 1\}$ (respectively $\mathbb{D}$). Observe that F_1 and F_2 have an analytic extension in a neighbourhood of $\eta_0 = \partial\mathcal{J}_0$ since η_0 and $\partial\mathbb{D}$ are analytic curves. Therefore $h = F_2 \circ F_1^{-1}$ is a homeomorphism of $\partial\mathbb{D}$ on itself which has an analytic extension.

It is well known that in this case there is a quasiconformal automorphism H of $\hat{\mathbb{C}}$ such that $H(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$, $H|_{\partial\mathbb{D}} = h$ and $H \in C^\infty(\overline{\mathbb{D}})$. This fact is a consequence of the Beurling-Ahlfors theorem (see [BA] or [A, p. 69], where they construct a quasiconformal extension $H_0 : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ of a quasisymmetric map $h_0 : \hat{\mathbb{R}} \longrightarrow \hat{\mathbb{R}}$, which preserves the differentiability properties of h_0).

We define a bijection u of $\hat{\mathbb{C}}$ on itself by

$$u(z) := \begin{cases} F_2(z)\,, & z \in \mathcal{J}_0\,, \\ (H \circ F_1)(z)\,, & z \notin \mathcal{J}_0\,. \end{cases}$$

This function is continuous in $\hat{\mathbb{C}}$ since

$$H \circ F_1|_{\eta_0} = H|_{\partial\mathbb{D}} \circ F_1|_{\eta_0} = h \circ F_1|_{\eta_0} = F_2 \circ F_1^{-1} \circ F_1|_{\eta_0} = F_2|_{\eta_0}\,,$$

and we have that $u \in C^\infty(\hat{\mathbb{C}} \setminus \eta_0) \cap C(\hat{\mathbb{C}})$. The regularity properties of F_2 and $H \circ F_1$ in η_0 gives that the distributional derivatives of u in a neighborhood of η_0 are equal to the classical derivatives (we use the differentiability properties only for this argument). Therefore u is a quasiconformal map on $\hat{\mathbb{C}}$ with the same quasiconformality constant than H.

Let M be a Möbius map which fixes $\mathbb{D}$, and such that $M(0) = u(\pi^{-1}(q)) \in \mathbb{D}$. For any $\alpha > 0$, let us consider the following quasiconformal automorphism of $\hat{\mathbb{C}}$

$$v_\alpha(z) = \begin{cases} z\,, & z \notin \mathbb{D}, \\ z\,|z|^{\alpha-1}\,, & z \in \mathbb{D}. \end{cases}$$

Let f_α be the following homeomorphism from $\mathcal{S}$ on itself.

$$f_\alpha(p) = \begin{cases} p\,, & p \notin \mathcal{J}\,, \\ (\pi \circ u^{-1} \circ M \circ v_\alpha \circ M^{-1} \circ u \circ \pi^{-1})(p)\,, & p \in \mathcal{J}\,. \end{cases}$$

Obviously, $f_\alpha|_{\mathcal{S}\setminus\mathcal{J}} = \mathrm{id}|_{\mathcal{S}\setminus\mathcal{J}}$. Observe that f_α is continuous in $\mathcal{S}$, since $v_\alpha|_{\partial\mathbb{D}} = \mathrm{id}|_{\partial\mathbb{D}}$ implies that

$$(\pi \circ u^{-1} \circ M)|_{\partial\mathbb{D}} \circ v_\alpha|_{\partial\mathbb{D}} \circ (M^{-1} \circ u \circ \pi^{-1})|_\eta = \mathrm{id}|_\eta \, .$$

The same argument used to see that u is a quasiconformal map gives that f_α is a quasiconformal automorphism of $\mathcal{S}$ for any $\alpha > 0$. Observe that $f_\alpha(q) = q$ since $(M^{-1} \circ u \circ \pi^{-1})(q) = 0$.

For a small $\varepsilon > 0$ we have that $(\pi \circ u^{-1} \circ M)(\{z \in \mathbb{C} : |z| \le \varepsilon\}) \subset V$ since $(\pi \circ u^{-1} \circ M)(0) = q \in V$. Given the compact set $K \subset \mathcal{J}$ we can choose α such that $(v_\alpha \circ M^{-1} \circ u \circ \pi^{-1})(K) \subset \{z \in \mathbb{C} : |z| \le \varepsilon\}$, since $(M^{-1} \circ u \circ \pi^{-1})(K)$ is a compact subset of $\mathbb{D}$.

Therefore we obtain that $f_\alpha(K) \subset V$ for this α.

Lemma 7.6. *Let w be a C^1 homeomorphism of $\partial\mathbb{D}$ on itself. For each $0 < r < 1$ there exists a quasiconformal automorphism f of $A = \{r \le |z| \le 1\}$ such that $f|_{\{|z|=r\}}$ is the identity map, $f|_{\partial\mathbb{D}} = w$ and $f \in C^1(A)$.*

Proof. For each $0 < r < 1$, let us consider the positive number

$$a = \frac{1}{2\pi} \log \frac{1}{r}$$

and the universal covering map

$$\pi : B = \{0 \le \operatorname{Im} z \le a\} \longrightarrow A \, , \qquad \pi(z) = r\, e^{-2\pi i z} \, .$$

The map π is a periodic function with period 1 and satisfies

$$\pi(\{z : \operatorname{Im} z = 0\}) = \{z : |z| = r\} \, , \qquad \pi(\{z : \operatorname{Im} z = a\}) = \{z : |z| = 1\} \, .$$

Therefore, we only need to prove that if v is a C^1 homeomorphism of $\{z : \operatorname{Im} z = a\}$ on itself with

$$v(x + 1 + i\,a) = v(x + i\,a) + 1 \, , \qquad x \in \mathbb{R} \, ,$$

then there exists a C^1 quasiconformal automorphism g of B on itself such that

$$g|_{\{\operatorname{Im} z = 0\}} = \mathrm{id}|_{\{\operatorname{Im} z = 0\}} \, , \qquad g|_{\{\operatorname{Im} z = a\}} = v \, ,$$

and
$$g(z+1) = g(z) + 1\,, \qquad z \in B\,.$$
Such a function g can be constructed explicitly. For example, let us consider
$$g(x+i\,y) = x\Big(1 - \frac{y}{a}\Big) + \frac{y}{a}\,v(x+i\,a)\,.$$
It is clear that $g(z+1) = g(z)+1$ for $z \in B$, that g satisfies the boundary conditions, and that g is a C^1 homeomorphism from B on itself. It is easy to check that g is a quasiconformal map since it is a C^1 sense-preserving map and $g(z+1) = g(z)+1$ for $z \in B$.

In order to state the following lemma we need a definition. Recall that any bordered Riemann surface $\mathcal{S}$ with a finitely generated fundamental group may be obtained from a compact Riemann surface of genus g by removing p distinct points (the punctures of $\mathcal{S}$), n closed disks (whose boundaries represent the ideal boundaries of $\mathcal{S}$) and m open disks (whose boundaries are the border of $\mathcal{S}$). The vector (g,p,n,m) is called the *quasiconformal type* of $\mathcal{S}$. It is well known that there exists a quasiconformal mapping between two bordered Riemann surfaces with the same quasiconformal type.

Lemma 7.7. *Let $\mathcal{S}$ be a hyperbolic Riemann surface. Let $\{g_1,\dots,g_N\}$ be a family of pairwise disjoint simple closed curves such that each g_i is not homotopic to zero or to a puncture in $\mathcal{S}$ and they are pairwise not homotopic.*

Let $S_1,\dots,S_r,S_{r+1},\dots,S_k$ $(1 \le r \le k-1)$ be the connected components of $\mathcal{S}\setminus(g_1\cup\cdots\cup g_N)$, where $S_{r+1},\dots,S_k$ are (open) surfaces of finite type. We also require that each g_j is contained in the boundary of $\overline{S_n}$ and $\overline{S_\ell}$ with $n \le r$ and $\ell > r$.

If $g_j \subset \overline{S_m}$, let γ_j be the unique simple closed geodesic in S_m freely homotopic to the ideal boundary g_j.

Let R_m $(m = 1,\dots,r)$ be the bordered surface obtained by deleting from S_m the open funnel F_j bounded by γ_j and the ideal curve g_j, for every $\gamma_j \subset S_m$.

Let R_m $(m = r+1,\dots,k)$ be a bordered surface with the same quasiconformal type than $\overline{S_m}$ such that the border of R_m is constituted by simple closed geodesics with the following condition: if g_j is an ideal boundary curve of S_n and S_m $(n \le r)$ and η_i is a boundary curve of R_m corresponding to g_j, we have that
$$L_{R_m}(\eta_i) = L_{S_n}(\gamma_j)\,.$$

Let $\mathcal{R}$ be a surface obtained by pasting $R_1, \ldots, R_k$ following the design of $S_1, \ldots, S_k$ (identifying geodesics of equal length).

Then $\mathcal{S}$ and $\mathcal{R}$ are quasiconformally equivalent.

PROOF. Let us fix $m > r$ and let $g_{j_1}, \ldots, g_{j_i}$ be the boundary curves of $\overline{S_m}$. Let us consider $M_m = \overline{S_m} \cup \overline{F_{j_1}} \cup \cdots \cup \overline{F_{j_i}} \subset S$.

It is well-known that there is a C^1 quasiconformal map f_m of R_m on M_m, since R_m and M_m have the same quasiconformal type.

If γ_j is contained in S_n $(n \leq r)$, let us consider a fixed closed collar C_j about γ_j in S_n and let K_j be the set $K_j = C_j \cap R_n$. The curve γ_j is contained in the border of R_n and M_m for some $m > r$.

Lemma 7.6 gives that there exists a C^1 quasiconformal automorphism h_j of K_j such that $h_j|_{\gamma_j} = f_m|_{\gamma_j}$ and $h_j|_{\partial K_j \setminus \gamma_j} = \mathrm{id}|_{\partial K_j \setminus \gamma_j}$.

Let us consider the homeomorphism f of $\mathcal{R}$ on $\mathcal{S}$ given by $f|_{R_m} = f_m$ for $m > r$, $f|_{K_j} = h_j$ for $1 \leq j \leq N$, and $f = \mathrm{id}$ otherwise.

It is easy to check that f is a quasiconformal map.

We will need the two following well known facts (see for example [C, Theorem 5.1] or [FR, Lemma 4.2]).

Proposition A. *Let $\mathcal{S}$ be a Riemann surface and let I and J be closed subsets of $\mathcal{S}$ such that $\mathcal{S} \setminus I$ is a hyperbolic Riemann surface and every connected component of J has a non-empty intersection with I. If $\mathcal{R}$ is a connected component of $\mathcal{S} \setminus (I \cup J)$ then we have that $\delta(\mathcal{R}) \leq \delta(\mathcal{S} \setminus I)$.*

Proposition B. *Let $\mathcal{S}_1, \mathcal{S}_2$ be two hyperbolic Riemann surfaces such that $\mathcal{S}_1 \subset \mathcal{S}_2$ and $\Pi_1(q, \mathcal{S}_1) \leq \Pi_1(q, \mathcal{S}_2)$ for some $q \in \mathcal{S}_1$. Then we have that $\delta(\mathcal{S}_1) \leq \delta(\mathcal{S}_2)$.*

Observe that Proposition A is a particular case of Proposition B. The proof of this last one is elementary; it is enough to remark that in $\mathcal{S}_1$ there are fewer curves and they are longer.

Proposition 4. *Let $\mathcal{S}$ be a hyperbolic Riemann surface with infinite area. Let $C_1, \ldots, C_n$ be pairwise disjoint compact simply connected subsets of $\mathcal{S}$. Then $\mathcal{S}$ satisfies* HII *if and only if $\mathcal{S} \setminus (C_1 \cup \cdots \cup C_n)$ satisfies* HII.

REMARK. It is easy to find examples showing that the conclusion of Proposition 4 is not true if some C_j is not compact.

Proof. We can assume without loss of generality that $n = 1$. Let p be a point in C_1. Theorem 1 gives that the statement of Proposition 4 is equivalent to the following one: $\mathcal{S} \setminus \{p\}$ *satisfies* HII *if and only if* $\mathcal{S} \setminus C_1$ *satisfies* HII.

This is trivially true if $C_1 = p$. Therefore, we can assume that C_1 has infinitely many points.

Let us assume that $\mathcal{S} \setminus \{p\}$ satisfies HII. Observe that $\mathcal{S} \setminus C_1 \subset \mathcal{S} \setminus \{p\}$ and that the fundamental groups of the two surfaces are isomorphic. Therefore, Proposition B implies that $\mathcal{S} \setminus C_1$ satisfies HII.

Let us assume now that $\mathcal{S} \setminus C_1$ satisfies HII. Let η_1 be the simple closed geodesic freely homotopic in $\mathcal{S} \setminus C_1$ to the ideal boundary ∂C_1.

Let F_1 be the open funnel in $\mathcal{S} \setminus C_1$ bounded by η_1 and the ideal boundary ∂C_1, and let J_1 be the open set $J_1 = C_1 \cup F_1 \subset \mathcal{S}$. Observe that $\partial J_1 = \eta_1$ is an analytic curve.

Let us consider (in $\mathcal{S}$) the open set $V = (\{p\} \cup C_{\mathcal{S}\setminus\{p\}}(p, 1/4)) \cap J_1$ and the compact set C_1. Lemma 7.5 gives that there exists a quasiconformal automorphism f of $\mathcal{S}$ such that $C = f(C_1) \subset V$, $f|_{\mathcal{S}\setminus J_1} = \mathrm{id}|_{\mathcal{S}\setminus J_1}$ and $f(p) = p$. Therefore, f is a quasiconformal map of $\mathcal{S} \setminus C_1$ on $\mathcal{S} \setminus C$. [FR, Theorem 1] implies that $\mathcal{S} \setminus C$ satisfies HII. We will prove that $\mathcal{S} \setminus \{p\}$ also satisfies HII.

Let η be the simple closed geodesic freely homotopic in $\mathcal{S} \setminus C$ to the ideal boundary ∂C. Let F be the open funnel in $\mathcal{S} \setminus C$ bounded by η and the ideal boundary ∂C, and let J be the open set $J = C \cup F \subset \mathcal{S}$.

Let us consider a geodesic domain G in $\mathcal{S} \setminus \{p\}$ and let G' be the corresponding geodesic domain in $\mathcal{S} \setminus C$: each boundary curve of G is freely homotopic in $\mathcal{S} \setminus \{p\}$ to a boundary curve of G'; if G contains a collar about the puncture p, the curve η is a boundary curve of G' (observe that η is freely homotopic to p in $\mathcal{S} \setminus \{p\}$).

Gauss-Bonnet theorem gives that

$$A_{\mathcal{S}\setminus\{p\}}(G) = A_{\mathcal{S}\setminus C}(G')\,. \tag{7.7}$$

Lemma 7.3 gives that there exists a positive constant c, independent of G, such that

$$A_{\mathcal{S}\setminus C}(G') \le c\, L_{\mathcal{S}\setminus C}(\partial G' \setminus \eta)\,, \tag{7.8}$$

since $\mathcal{S} \setminus C$ satisfies HII and $\partial G' \neq \eta$. We have that $\partial G' \neq \eta$ since there are only two domains in $\mathcal{S} \setminus C$ whose boundary is exactly η: F and $\mathcal{S} \setminus \overline{F}$, and both have infinite area in $\mathcal{S} \setminus C$. This last fact is a consequence of the hypothesis $A_{\mathcal{S}}(\mathcal{S}) = \infty$.

We have that $\partial G' \subset \mathcal{S} \setminus J \subset \mathcal{S} \setminus C$. Lemma 7.4 implies that $\partial G \subset \mathcal{S} \setminus C_{\mathcal{S}\setminus\{p\}}(p, 1/2) \subset \mathcal{S} \setminus C$. These facts give that $\partial G'$ and ∂G are far from C.

Then, (3.1) implies that the hyperbolic metrics of $\mathcal{S}\setminus\{p\}$ and $\mathcal{S}\setminus C$ are comparable in $(\mathcal{S} \setminus J) \cup (\mathcal{S} \setminus C_{\mathcal{S}\setminus\{p\}}(p, 1/2))$, since $p \in C$.

Therefore, $L_{\mathcal{S}\setminus C}(\partial G' \setminus \eta)$ and $L_{\mathcal{S}\setminus\{p\}}(\partial G)$ are comparable. This fact, (7.7) and (7.8) give that there is a constant $c_0 > 0$, independent of G, such that

$$A_{\mathcal{S}\setminus\{p\}}(G) \le c_0\, L_{\mathcal{S}\setminus\{p\}}(\partial G)\,,$$

and then, [FR, Lemma 1.2] gives that $\mathcal{S} \setminus \{p\}$ satisfies HII.

Definition. *We will say that a closed and connected subset C of a Riemann surface $\mathcal{S}$ is of finite type if C is a compact simply connected set or, if it has finitely generated fundamental group and ∂C is a union of simple closed curves.*

Proposition 5. *Let $\mathcal{S}$ be a hyperbolic Riemann surface with infinite area. Let $C_1, \dots, C_n$ be pairwise disjoint closed connected subsets of finite type of $\mathcal{S}$. Then, we have the following facts*:

a) *If S_0 is a connected component of $\mathcal{S} \setminus (C_1 \cup \dots \cup C_n)$ and $\mathcal{S}$ satisfies* HII*, then S_0 satisfies* HII*.*

b) *If $\mathcal{S} \setminus (C_1 \cup \dots \cup C_n)$ is connected and satisfies* HII*, then $\mathcal{S}$ satisfies* HII*.*

Remark. It is easy to construct examples showing that b) is not true if some C_j is not of finite type.

Proof. We can assume without loss of generality that $n = 1$ and C_1 is not a simply connected set (by Proposition 4).

Observe that Proposition 5 is trivial if $\mathcal{S}$ is either a simply or a doubly connected surface. Therefore, without loss of generality we can assume that $\mathcal{S}$ is neither a simply nor doubly connected surface.

Let us assume that $\mathcal{S}$ satisfies HII. Let p be a point in C_1. Theorem 1 gives $\mathcal{S} \setminus \{p\}$ also satisfies HII.

We have that $S_0 \subset \mathcal{S} \setminus \{p\}$ and the fundamental group of S_0 is a subgroup of the fundamental group of $\mathcal{S} \setminus \{p\}$. Therefore, Proposition B implies that S_0 satisfies HII since $\mathcal{S} \setminus \{p\}$ satisfies HII.

Let us assume now that $\mathcal{S} \setminus C_1$ satisfies HII. Let $g_1, \dots, g_N$ be the simple closed curves in ∂C_1.

Without loss of generality we can assume that each g_j is not homotopic to zero. In other case, we have that $\mathcal{S}\setminus C_1$ is simply connected, since $\mathcal{S}\setminus C_1$ is connected and C_1 is not simply connected. Therefore, $\mathcal{S}$ is of finite type, since C_1 is of finite type; then $\mathcal{S}$ satisfies HII, since it has infinite area.

Without loss of generality we can assume that each g_j is not homotopic to a puncture p_j in $\mathcal{S}$. In other case, Theorem 1 allows us to consider the surface $\mathcal{S}_1 = \mathcal{S} \cup \{p_j\}$ instead of $\mathcal{S}$. Therefore, we would have that g_j is homotopic to zero in $\mathcal{S}_1$. Using again the last argument we obtain that $\mathcal{S}_1$, and consequently $\mathcal{S}$, satisfies HII.

Let us assume now that there exist two different curves g_i, g_j, freely homotopic in $\mathcal{S}$. In this case, there is a doubly connected domain D in $\mathcal{S}$ such that $\partial D = g_i \cup g_j$. Then we have that $N = 2$, since $\mathcal{S}\setminus C_1$ and C_1 are connected. Therefore, we have that either the set C_1 is equal to $\overline{D}$ or $\mathcal{S}\setminus C_1$ is equal to D.

The second possibility implies that $\mathcal{S}\setminus C_1$ is a doubly connected domain and therefore, $\mathcal{S}$ is of finite type, since C_1 is of finite type; then $\mathcal{S}$ satisfies HII, since it has infinite area.

If $C_1 = \overline{D}$, we can take a closed subset C of finite type of $\mathcal{S}$ such that $C_1 \subset C$ and C is not a doubly connected set (remember that $\mathcal{S}$ is neither a simply nor a doubly connected surface). Proposition B gives that $\mathcal{S}\setminus C$ satisfies HII, since $\mathcal{S}\setminus C_1$ satisfies HII.

Therefore, we can assume without loss of generality that there are not two different curves in ∂C_1 freely homotopic.

Let $\gamma_1, \dots, \gamma_N$ be the simple closed geodesics in $\mathcal{S}\setminus C_1$ such that γ_j is freely homotopic to the ideal boundary g_j.

Then, we can apply to $\mathcal{S}$ the construction of the surface $\mathcal{R}$ of Lemma 7.7, relative to $\{g_1, \dots, g_N\}$ (with $r = 1$ and $k = 2$).

Corollary 5 implies that $\mathcal{R}$ satisfies HII since $\mathcal{S}$ has infinite area and $\mathcal{S}\setminus C_1$ satisfies HII. Finally, $\mathcal{S}$ satisfies HII since Lemma 7.7 implies that $\mathcal{R}$ and $\mathcal{S}$ are quasiconformally equivalent.

We can state now the following general version of theorems 9 and 10.

Theorem 7. *Let $\mathcal{S}$ be a Riemann surface and let E be a closed subset of $\mathcal{S}$ such that $\mathcal{S}\setminus E$ is a hyperbolic Riemann surface with $A_{\mathcal{S}\setminus E}(\mathcal{S}\setminus E) = \infty$. Then, the following conditions are equivalent*:

1) *$\mathcal{S}\setminus E$ satisfies* HII.

2) $\mathcal{S}_0 \setminus E$ *satisfies* HII, *for any subsurface* $\mathcal{S}_0$ *of* $\mathcal{S}$ *such that* E *is contained in* $\mathcal{S}_0$, $\mathcal{S}_0 \setminus E$ *is connected, and* $\mathcal{S} \setminus \mathcal{S}_0$ *is a finite union of closed sets of finite type.*

3) $\mathcal{S}_0 \setminus E$ *satisfies* HII, *for some subsurface* $\mathcal{S}_0$ *of* $\mathcal{S}$ *such that* E *is contained in* $\mathcal{S}_0$, $\mathcal{S}_0 \setminus E$ *is connected, and* $\mathcal{S} \setminus \mathcal{S}_0$ *is a finite union of closed sets of finite type.*

4) $\mathcal{S} \setminus (E \cup F)$ *satisfies* HII *for any closed subset* F *of* $\mathcal{S}$ *verifying*: a) $\mathcal{S} \setminus F$ *satisfies* HII; b) *there exists a set* M, *which is a finite union of pairwise disjoint closed sets of finite type, such that* $F \subset M$ *and* $E \cap M = \varnothing$.

5) $\mathcal{S} \setminus (E \cup F)$ *satisfies* HII *for some closed subset* F *of* $\mathcal{S}$ *verifying*: a) $\mathcal{S} \setminus F$ *satisfies* HII; b) *there exists a set* M, *which is a finite union of pairwise disjoint closed sets of finite type, such that* $F \subset M$ *and* $E \cap M = \varnothing$.

Remark. If E and F are closed subsets of a Riemann surface $\mathcal{S}$ and there exists a set M which is a finite union of pairwise disjoint closed sets of finite type such that $F \subset M$ and $E \cap M = \varnothing$, then E and F are weakly separated in $\mathcal{S}$.

Proof. Proposition 5 gives that 1), 2) and 3) are equivalent. Lemma 7.1 and the latest remark give that 1) implies 4). Therefore, since 5) follows directly from 4), we only need to prove that 5) implies 3). But this is a consequence of propositions B and 5: Proposition B gives that $(\mathcal{S} \setminus E) \setminus M$ satisfies HII and then Proposition 5 gives that $\mathcal{S} \setminus E$ satisfies HII.

Patterson proved in [P1, Theorem 4] a related result for Riemann surfaces $\mathcal{S}$ of finite area and discrete closed subsets E.

As a consequence of Theorem 7 we obtain the following result.

Corollary 7. *Given a closed subset* E *of* $\hat{\mathbb{C}}$ *with infinitely many points, the following conditions are equivalent*:

1) $\hat{\mathbb{C}} \setminus E$ *satisfies* HII.

2) $\Omega \setminus E$ *satisfies* HII, *for any subdomain* Ω *of* $\hat{\mathbb{C}}$ *of finite type such that* E *is contained in* Ω.

3) $\Omega \setminus E$ *satisfies* HII*, for some subdomain* Ω *of* $\hat{\mathbb{C}}$ *of finite type such that* E *is contained in* Ω*.*

4) $\hat{\mathbb{C}} \setminus (E \cup F)$ *satisfies* HII *for any closed subset* F *of* $\hat{\mathbb{C}}$ *such that* $\hat{\mathbb{C}} \setminus F$ *satisfies* HII *and* $E \cap F = \varnothing$*.*

5) $\hat{\mathbb{C}} \setminus (E \cup F)$ *satisfies* HII *for some closed subset* F *of* $\hat{\mathbb{C}}$ *such that* $\hat{\mathbb{C}} \setminus F$ *satisfies* HII *and* $E \cap F = \varnothing$*.*

Finally, if we apply $n-1$ times Corollary 7 (and Theorem 1), we obtain the following result which was announced at the beginning of this section.

Theorem 8. *Let* $E_1, \dots, E_n$ *be pairwise disjoint closed subsets in* $\hat{\mathbb{C}}$ *with infinitely many points such that* $\Omega_0 = \hat{\mathbb{C}} \setminus \cup_k E_k$ *is connected. Let* I *be a strongly uniformly separated set in* Ω_0 *and let* $\Omega = \Omega_0 \setminus I$*. Then, we have that* Ω *satisfies* HII *if and only if* $\hat{\mathbb{C}} \setminus E_k$ *satisfies* HII *for* $k = 1, \dots, n$*.*

8. Isoperimetric inequality, polarization and symmetrization.

In general, symmetrization arguments are at the heart of isoperimetric inequalities in Riemannian manifolds of constant sectional curvature, which is the case of hyperbolic Riemann surfaces (see *e.g.* [Ch2, Chapter 6] and the references therein).

On the other hand, the ideas used in the proof of Theorem 4 (see Section 5) can suggest that there is a relation between the HII-property of a hyperbolic plane domain Ω and this property for its polarization Ω_p. A similar question can be proposed for its circular symmetrization Ω_{cs} (see [B] or [H] for the definition and basic background), since polarization and circular symmetrization are very regular processes. Therefore one could expect that some of the following relations would be true:

a) If Ω satisfies HII, then Ω_p also satisfies HII.

b) If Ω_p satisfies HII, then Ω also satisfies HII.

c) If Ω satisfies HII, then Ω_{cs} also satisfies HII.

d) If Ω_{cs} satisfies HII, then Ω also satisfies HII.

In this section we will show that all these conjectures are false even for Denjoy domains.

1) Let us consider $E = \{a_n\}$ and $F = \{b_n\}$ two increasing sequences of positive numbers converging to 1 such that $E \cap F = \varnothing$. Let $\Omega = \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty) \cup E \cup (-F))$, where $-F = \{-b_n\}$. We have that $\Omega_p = \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty) \cup (-E) \cup (-F))$ and $\Omega_{cs} = \mathbb{C} \setminus ((-\infty, -1] \cup (-E) \cup (-F))$. Let us assume also that E and F are strongly uniformly separated in $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ and that $E \cup F$ is not. Theorem 1 gives that Ω satisfies HII but Ω_p and Ω_{cs} do not satisfy HII. This example shows that a) and c) are not true.

2) Let us consider $E = \cup_{k=0}^{\infty} [1 - 2^{-2k}, 1 - 2^{-2k-1}] \cup \{1\}$ and $F = \cup_{k=0}^{\infty} I_k \cup \{-1\}$, where each I_k is a closed interval centered in $-1 + 3 \cdot 2^{-2k-3}$ and contained in $(-1 + 2^{-2k-2}, -1 + 2^{-2k-1})$. Let $\Omega = \hat{\mathbb{C}} \setminus (E \cup F)$. If $\lim_{k\to\infty} 2^{2k} |I_k| = 0$, one can check that Ω does not satisfies HII: It is enough to apply Theorem 2 to geodesic domains "surrounding" I_n and I_{n+1}.

If $-E = \cup_{k=0}^{\infty} [-1 + 2^{-2k-1}, -1 + 2^{-2k}] \cup \{-1\}$, we have that $\Omega_{cs} = \hat{\mathbb{C}} \setminus ((-E) \cup F)$ and $\Omega_p = \Omega_{cs} \setminus \{1\}$. The following argument as in the proof of Proposition 6 (see Section 9) gives that Ω_{cs} satisfies HII: $\hat{\mathbb{C}} \setminus (-E)$ satisfies HII since it is a modulated domain. Let a_k be a point in I_k for $k \geq 0$. Theorem 1 gives that $\hat{\mathbb{C}} \setminus \big((-E) \cup (\cup_{k=0}^{\infty} \{a_k\})\big)$ satisfies HII. Therefore Proposition A implies that $\Omega_{cs} = \hat{\mathbb{C}} \setminus \big((-E) \cup F\big)$ satisfies HII. Theorem 1 gives that $\Omega_p = \Omega_{cs} \setminus \{1\}$ also satisfies HII. This example shows that b) and d) are not true.

9. Geodesic domains.

One can think that Theorem 5 could be improved by studying only border sets with six points, in the following way.

Let Ω be a Denjoy domain, let I be the set of isolated points of $\partial\Omega$ and let $\Omega_0 = \Omega \cup I$. Then, Ω has HII *if and only if Ω is admissible and there exists a positive constant c such that for any border set of $\partial\Omega_0$ with six points, $B = \{b_1, \dots, b_6\}$, we have that*

$$(9.1) \qquad \sum_{j=1}^{3} \Psi_{\Omega_0}(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\}) > c\,.$$

This statement seems to be reasonable since if we want to study a border set $B = \{b_1, \dots, b_{2n}\}$, we can "divide" it in border sets with six points.

We prove now by an example that this statement is not true.

Example. Let Ω be the Denjoy domain defined as the complement of a dyadic Cantor set, $\Omega = \hat{\mathbb{C}} \setminus K$, where K is constructed as follows.

Let $E_0 := [0,1]$ and suppose that E_n has been defined and consists of 2^n closed disjoint subintervals of E_0, say J_j, each of them with length $d_n = r_1 \cdots r_n$, with

$$r_n := \begin{cases} \dfrac{1}{3}\,, & \text{for odd } n\,, \\[2ex] \dfrac{1}{n+1}\,, & \text{for even } n\,. \end{cases}$$

We divide each subinterval J_j in three intervals, obtaining two closed subintervals J_j^1 and J_j^2 (the *children* of J_j), each of them with length $d_{n+1} = d_n\, r_{n+1}$ and removing the central interval with length $d_n - 2\, d_{n+1}$. If we denote by E_{n+1} the union of the intervals with length d_{n+1}, the Cantor set K is defined as $K := \cap_n E_n$.

Let us consider an interval J of E_n and the unique simple closed geodesic γ_n which "surrounds" J in Ω.

For odd n we have

$$L_\Omega(\gamma_n) \geq L_{\hat{\mathbb{C}}\setminus\{0,1/3,2/3,1\}}(\gamma)\,, \tag{9.2}$$

where γ is the geodesic in $\hat{\mathbb{C}}\setminus\{0,1/3,2/3,1\}$ given by $\gamma := \{\operatorname{Re} z = 1/2\}$.

We also have

$$L_\Omega(\gamma_n) \leq L_{\mathbb{C}\setminus\{(-\infty,-1/3]\cup[0,1/3]\cup[2/3,\infty)\}}(\eta)\,,$$

where η is the simple closed geodesic in $\mathbb{C} \setminus \{(-\infty,-1/3] \cup [0,1/3] \cup [2/3,\infty)\}$.

If $B := \{-1/3, 0, 1/3, 2/3\}$, we have that $r(B) = 1/3$. Therefore Lemma 4.5 gives

$$\begin{aligned} L_\Omega(\gamma_n) &\leq L_{\mathbb{C}\setminus\{(-\infty,-1/3]\cup[0,1/3]\cup[2/3,\infty)\}}(\eta) \\ &= L_{\mathbb{C}\setminus\{[-1,0]\cup[1/3,\infty)\}}(\sigma)\,, \end{aligned} \tag{9.3}$$

where σ is the simple closed geodesic in $\mathbb{C} \setminus \{[-1,0] \cup [1/3,\infty)\}$.

For even n we have

$$\begin{aligned} L_\Omega(\gamma_n) &\le L_{\mathbb{C}\setminus\{(-\infty,2-r_n^{-1}]\cup[0,1]\cup[r_n^{-1}-1,\infty)\}}(\eta_n) \\ &= L_{\mathbb{C}\setminus\{(-\infty,1-n]\cup[0,1]\cup[n,\infty)\}}(\eta_n)\,, \end{aligned}$$

where η_n is the simple closed geodesic in $\mathbb{C}\setminus\{(-\infty,1-n]\cup[0,1]\cup[n,\infty)\}$.

If $B_n := \{1-n,0,1,n\}$, we have that

$$r(B_n) = \frac{(n-1)^2}{2\,n-1} \asymp n\,.$$

Therefore Lemma 4.5 gives

$$\begin{aligned} L_\Omega(\gamma_n) &\le L_{\mathbb{C}\setminus\{(-\infty,1-n]\cup[0,1]\cup[n,\infty)\}}(\eta_n) \\ &= L_{\mathbb{C}\setminus\{[-1,0]\cup[r(B_n),\infty)\}}(\sigma_n) \\ &\asymp \Phi_1(r(B_n)) \\ &\asymp \frac{1}{\log n}\,, \end{aligned} \tag{9.4}$$

where σ_n is the simple closed geodesic in $\mathbb{C}\setminus\{[-1,0]\cup[r(B_n),\infty)\}$.

We say that a border set B of $\partial\Omega$ is *n-basic* if it has six points and the three simple closed geodesics associated with it surround an interval $J\subset E_n$ and their two children $J^1,J^2\subset E_{n+1}$. We say that a border set B of $\partial\Omega$ is *basic* if it is n-basic for some n.

For a n-basic border set B, we always have (9.1) since one (respectively two) of the three geodesics associated with B verifies (9.2) if n is odd (respectively even).

Inequalities (9.3) and (9.4) give that there is a finite upper bound l for the length of the geodesics associated with any basic border set. Then, Collar Lemma [R] gives that every geodesic which intersects a geodesic γ associated with any basic border set has length at least twice the width w of the collar C_γ and

$$w \ge \operatorname{Arg\,cosh}\left(\operatorname{cotanh}\left(\frac{l}{2}\right)\right).$$

Therefore, (9.1) is satisfied by every border set B of $\partial\Omega$ with six points, since at least one of the three geodesics associated with B intersects a geodesic associated with a basic border set.

However, Ω does not satisfy HII. To see this, let us consider the geodesic domain G_k in Ω bounded by the 2^{2k} geodesics which surround each interval of E_{2k}.

Gauss-Bonnet theorem gives $A_\Omega(G_k) = 2\pi\,(2^{2k} - 2)$. Inequality (9.4) gives, for some positive constant c_0,

$$L_\Omega(\partial G_k) \le c_0\, \frac{2^{2k}}{\log(2\,k)}\,.$$

Therefore

$$\frac{L_\Omega(\partial G_k)}{A_\Omega(G_k)} \le \frac{c_1}{\log(2\,k)} \longrightarrow 0\,, \qquad \text{as } k \longrightarrow \infty\,,$$

and this fact gives that Ω does not satisfy HII.

10. An open problem.

In this section we want to discuss about the possibility to find a simpler characterization of the HII-property. In fact, we would like to have a result of the following type:

Conjecture. *Let Ω be a Denjoy domain, let I be the set of isolated points of $\partial\Omega \subset \hat{\mathbb{C}}$ and let $\Omega_0 = \Omega \cup I$. There exists a function Φ, independent of Ω, such that Ω has* HII *if and only if Ω is admissible and there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \ge 3$, we have that*

$$\frac{1}{n} \sum_{j=1}^{n} \Phi(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,.$$

We can say something about this function Φ, if it exists.

Proposition 6. *Let Φ be a function verifying the following condition:*

If a Denjoy domain Ω has HII *then there exists a positive constant c such that for any border set of $\partial\Omega_0$, $B = \{b_1, \dots, b_{2n}\}$ with $n \ge 3$, we have that*

$$\frac{1}{n} \sum_{j=1}^{n} \Phi(r(\{b_{2j-1}, b_{2j}, b_{2j+1}, b_{2j+2}\})) > c\,.$$

Then Φ *must verify*

$$\limsup_{r\to 0} \frac{\Phi(r)}{\log\left(\dfrac{1}{r}\right)} > 0\,.$$

Proof. Let us consider the following closed subset E of $[0,1]$

$$E = \cup_{n=0}^{\infty}\{[2^{-2n-1},2^{-2n}]\cup I_n\}\cup\{0\}\,,$$

where I_n is the set of $2\,n+1$ points $\{x_{n,k}\}_{k=-n}^{n}$ in $(2^{-2n-2},2^{-2n-1})$, with $x_{n,\pm k}=(3\pm(1-2^{-k}))2^{-2n-3}$, for $k=0,1,\dots,n$.

Let I be the discrete set $I=\cup_{n=0}^{\infty} I_n$. Let Ω_1, Ω_2 be the Denjoy domains $\Omega_1=\hat{\mathbb{C}}\setminus E$ and $\Omega_2=\Omega_1\cup I$.

First we will see that Ω_1 and Ω_2 have HII:

The set Ω_2 is modulated and so [FR, Theorem 3] implies that Ω_2 has HII.

Therefore, [FR, Theorem 3] gives also that in order to prove that Ω_1 has a HII, we only need to check that I is uniformly separated in Ω_2:

The hyperbolic metrics in Ω_2 and $\Omega_2^*=\Omega_2\cup\{\infty\}$ are comparable in each euclidean ball of the complex plane. We also have [BP, Corollary 1] that there is a positive constant c such that

$$\frac{2}{d(x,E\setminus I)} \ge \lambda_{\Omega_2^*}(x) \ge \frac{c}{d(x,E\setminus I)}\,, \qquad \text{for } x\in[0,1]\cap\Omega_2\,.$$

These two facts give that

$$\lambda_{\Omega_2}(x) \asymp \frac{1}{d(x,E\setminus I)}\,, \qquad \text{for } x\in[0,1]\cap\Omega_2\,.$$

Then we have that

$$\begin{aligned} d_{\Omega_2}(x_{n,k},x_{n,k+1}) &\asymp \int_{x_{n,k}}^{x_{n,k+1}} \frac{dx}{2^{-2n-1}-x} \\ &= \log\frac{2^{-2n-1}-x_{n,k}}{2^{-2n-1}-x_{n,k+1}} \\ &= \log 2\,. \end{aligned}$$

A similar argument gives the same estimate for $d_{\Omega_2}(x_{n,-k}, x_{n,-k-1})$. This implies that I is uniformly separated in Ω_2, and consequently, that Ω_1 has HII.

For each point $x_{n,k} \in I$, let us consider the interval $J_{n,k} = [a_{n,k}, b_{n,k}]$ such that $x_{n,k} \in J_{n,k}$ and $J_{n,k}$ does not meet any interval of the form $[2^{-2m-1}, 2^{-2m}]$ or another $J_{m,l}$. We also choose $a_{n,-n} = x_{n,-n}$ and $b_{n,n} = x_{n,n}$. Let $J = \cup_{n,k} J_{n,k}$ and $\Omega = \Omega_1 \setminus J$. The length of these intervals $J_{n,k}$ have been chosen so small in such a way that the length of the geodesics $\gamma_{n,k}$ in Ω which surrounds only $J_{n,k}$ tends to zero as $n \longrightarrow \infty$ (uniformly in k).

The domain Ω has HII (in fact $\delta(\Omega) \le \delta(\Omega_1) < 1$) as a consequence of Proposition A (see Section 7).

Let us consider now the border set B_n in Ω given by

$$B_n = \{2^{-2n-2}, a_{n,-n}, b_{n,n}, 2^{-2n-1}\}.$$

We have that

$$r(B_n) = r(\{2^{-2n-2}, x_{n,-n}, x_{n,n}, 2^{-2n-1}\}) = \frac{2^{-2n}}{1-2^{-n}}.$$

Since Ω has HII, the property of Φ, with the border set

$$\{2^{-2n-2}, a_{n,-n}, b_{n,-n}, \dots, a_{n,0}, b_{n,0}, \dots, a_{n,n}, b_{n,n}, \dots, 2^{-2n-1}\},$$

implies

$$\frac{1}{2n+2}\,\Phi\Big(\frac{2^{-2n}}{1-2^{-n}}\Big) + o(1) > c, \qquad \text{for all } n \in \mathbb{N}.$$

Then we have

$$\limsup_{n\to\infty} \frac{\Phi\Big(\dfrac{2^{-2n}}{1-2^{-n}}\Big)}{\log \dfrac{1-2^{-n}}{2^{-2n}}} = \limsup_{n\to\infty} \frac{\Phi\Big(\dfrac{2^{-2n}}{1-2^{-n}}\Big)}{2\,n\log 2} \ge \frac{c}{\log 2}.$$

This finishes the proof of Proposition 6.

Proposition 6 implies that the conjecture is not true for any function Φ satisfying

$$\limsup_{r\to 0} \frac{\Phi(r)}{\log\Big(\dfrac{1}{r}\Big)} = 0.$$

In particular it is not true for the function Φ_2 in Theorem 2, but it could be true for Φ_1.

In any case, if the conjecture would be true for $\Phi = \Phi_1$, the proof should be more sophisticated that our arguments, because it is not true that

$$\Phi_1(r(B)) \asymp L_\Omega(\gamma(B))$$

for any border set B of any Denjoy domain Ω as $r \to 0$ (if $\Omega = \mathbb{C} \setminus ([-1,-1+r] \cup [-r,0] \cup [r,2\,r] \cup [2,\infty))$ and $B = \{0, r, \infty, -1\}$, then

$$r = r(B)\,,$$

$$\Phi_1(r) \asymp \log\left(\frac{1}{r}\right)$$

and Theorem 4 gives that $L_\Omega(\gamma(B)) \asymp \log\log(1/r))$.

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References.

[AS] Ahlfors, L. V., Sario, L., *Riemann surfaces.* Princeton Univ. Press, 1960.

[B] Baernstein II, A., Integral means, univalent functions and circular symmetrization. *Acta Math.* **133** (1974), 139-169.

[BP] Beardon, A. F., Pommerenke, Ch., The Poincaré metric of a plane domain. *J. London Math. Soc.* **18** (1978), 475-483.

[Be] Bers, L., *An inequality for Riemann Surfaces. Differential Geometry and Complex Analysis.* H. E. Rauch Memorial Volume. Springer-Verlag, 1985.

[BA] Beurling, A., Ahlfors, L. V., The boundary correspondence under quasiconformal mappings. *Acta Math.* **96** (1956), 125-142.

[Br] Brown Flinn, B., Hyperbolic convexity and level sets of analytic functions. *Indiana Univ. J.* **32** (1983), 831-841.

[C] Carleson, L., On a class of meromorphic functions and its associated exceptional sets. Thesis. Uppsala (1950).

[Ch1] Chavel, I., *Eigenvalues in Riemannian Geometry.* Academic Press, 1984.

[Ch2] Chavel, I., *Riemannian Geometry: A Modern Introduction.* Cambridge Univ. Press, 1993.

[Che] Cheeger, J., A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in Analysis*, Princeton Univ. Press (1970), 195-199.

[FR] Fernández, J. L., Rodríguez, J. M., The exponent of convergence of Riemann surfaces. Bass Riemann surfaces. *Ann. Acad. Sci. Fenn. Series AI* **15** (1990), 165-183.

[H] Hayman, W. K., *Subharmonic functions* (*Volume* 2). Academic Press, 1989.

[He] Hejhal, D. A., Universal covering maps for variable regions. *Math. Z.* **137** (1974), 7-20.

[K] Kra, I., *Automorphic forms and Kleinian groups.* W. A. Benjamin, 1972.

[LV] Lehto, O., Virtanen, K. I., *Quasiconformal mappings in the plane.* Springer, 1973.

[M] Minda, A reflection principle for the Poincaré metric and applications to Geometric Function Theory. *Complex Variables* **8** (1987), 129-144.

[N] Nicholls, P. J., *The Ergodic Theory of Discrete Groups.* Lecture Notes Series **143**, Cambridge Univ. Press, 1989.

[P1] Patterson, S. J., The exponent of convergence of Poincaré series. *Monatsh. Math.* **82** (1976), 297-315.

[P2] Patterson, S. J., Some examples of Fuchsian groups. *Proc. London Math. Soc.* **39** (1979), 276-298.

[R] Randol, B., Cylinders in Riemann surfaces. *Comment. Math. Helv.* **54** (1979), 1-5.

[Ro1] Rodríguez, J. M., Isoperimetric inequalities and Dirichlet functions of Riemann surfaces. *Pub. Mat.* **38** (1994), 243-253.

[Ro2] Rodríguez, J. M., Two remarks on Riemann surfaces. *Pub. Mat.* **38** (1994), 463-477.

[SN] Sario, L., Nakai, M., *Classification theory of Riemann surfaces.* Springer, 1970.

[S] Shimizu, H., On discontinuous groups operating on the product of upper half-planes. *Ann. of Math.* **77** (1963), 33-71.

[So] Solynin, A. Y., Functional inequalities via polarization. *Algebra i Analiz.* English translation in *St. Petersburg Math. J.* **8** (1996), 148-185.

[Su] Sullivan, D., Related aspects of positivity in Riemannian Geometry. *J. Diff. Geom.* **25** (1987), 327-351.

[T] Tsuji, M., *Potential Theory in Modern Function Theory.* Chelsea, 1959.

[W] Wolontis, V., Properties of conformal invariants. *Amer. J. Math.* **74** (1952), 587-606.

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On radial behaviour and balanced Bloch functions

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Abstract. A Bloch function g is a function analytic in the unit disk such that $(1-|z|^2)\,|g'(z)|$ is bounded. First we generalize the theorem of Rohde that, for every "bad" Bloch function, $g(r\,\zeta)$ $(r \longrightarrow 1)$ follows any prescribed curve at a bounded distance for ζ in a set of Hausdorff dimension almost one. Then we introduce balanced Bloch functions. They are characterized by the fact that $|g'(z)|$ does not vary much on each circle $\{|z| = r\}$ except for small exceptional arcs. We show *e.g.* that

$$\int_0^1 |g'(r\,\zeta)| dr < \infty$$

holds either for all $\zeta \in \mathbb{T}$ or for none.

1. Radial behaviour of Bloch functions.

Let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ and $\mathbb{T} = \partial\,\mathbb{D}$. The function g analytic in $\mathbb{D}$ is called a Bloch function if

$$\|g\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} \,(1-|z|^2)\,|g'(z)| < \infty\,. \tag{1.1}$$

This holds if and only if the Riemann image surface of g as a cover of $\mathbb{C}$ does not contain arbitrarily large unramified disks. We denote the family of Bloch functions by $\mathcal{B}$.

First we generalize a surprising result of Steffen Rohde [Ro93]. Let $c_1, c_2, \dots$ be positive absolute constants and let $\dim E$ denote the Hausdorff dimension [Fa85, p. 7] of $E \subset \mathbb{T}$. Note that $\dim \mathbb{T} = 1$.

Theorem 1.1. *Let $G \subset \mathbb{C}$ be a domain with $0 \in G$ and let g be a Bloch function with $\|g\|_{\mathcal{B}} \le 1$ and $g(0) = 0$. We assume that, for almost all $\zeta \in \mathbb{T}$,*

$$\lim_{r \to 1} g(r\,\zeta) \ \textit{lies in } \mathbb{C} \backslash G \ \textit{or does not exist}. \tag{1.2}$$

Let Γ be any halfopen curve in G starting at 0. If

$$c_1 < R < \operatorname{dist}(0, \partial G), \qquad \operatorname{dist}(\Gamma, \partial G) \ge 2\,R, \tag{1.3}$$

then there exists $E_\Gamma \subset \mathbb{T}$ with

$$\dim E_\Gamma \ge 1 - \frac{c_2}{R} \tag{1.4}$$

such that, for $\zeta \in E_\Gamma$, we can find a parametrization $\gamma_\zeta(r)$, $0 \le r < 1$ of Γ with $\gamma_\zeta(0) = 0$ such that

$$|g(r\,\zeta) - \gamma_\zeta(r)| \le 2\,R, \qquad \textit{for } 0 \le r < 1. \tag{1.5}$$

This theorem is due to Rohde [Ro93] for the case that $G = \mathbb{C}$. Thus the radial image follows any prescribed curve with a bounded deviation on a set of dimension almost 1. Now we apply this theorem to (injective) conformal maps f of $\mathbb{D}$ into $\mathbb{C}$. It is well-known [DuShSh66], [Be72] that

$$\begin{gathered} f \text{ conformal implies } \|\log f'\|_{\mathcal{B}} \le 6\,, \\ \|\log f'\|_{\mathcal{B}} \le 1 \text{ implies } f \text{ conformal}. \end{gathered} \tag{1.6}$$

If the radial limit $f(\zeta)$ exists and is finite (which holds for almost all $\zeta \in \mathbb{T}$), we write

$$\begin{aligned} \alpha(\zeta) &= \liminf_{r \to 1} \arg\left((r\,\zeta) - f(\zeta)\right), \\ \beta(\zeta) &= \limsup_{r \to 1} \arg\left(f(r\,\zeta) - f(\zeta)\right). \end{aligned} \tag{1.7}$$

We give a partial generalization of [CaPo97, Theorem 1].

Corollary 1.2. *Let f map $\mathbb{D}$ conformally into $\mathbb{C}$ and suppose that*

$$\limsup_{r\to 1} |f'(r\,\zeta)| \geq 1\,, \qquad \text{for almost all } \zeta \in \mathbb{T}\,, \tag{1.8}$$

$$\liminf_{r\to 1} |f'(r\,\zeta_0)| = 0\,, \qquad \text{for some } \zeta_0 \in \mathbb{T}\,. \tag{1.9}$$

Then, for $j = 1,2,3,4$, there exist sets $E_j \subset \mathbb{T}$ with $\dim E_j = 1$, such that

i) $\alpha(\zeta) = -\infty$, $\beta(\zeta) = +\infty$, *for $\zeta \in E_1$ (twist point),*

ii) $\alpha(\zeta) = \beta(\zeta) = +\infty$, *for $\zeta \in E_2$ (spiral point),*

iii) $-\infty < \alpha(\zeta) < \beta(\zeta) = +\infty$, *for $\zeta \in E_3$ (gyration point),*

iv) $-\infty < \alpha(\zeta) + 2\pi < \beta(\zeta) < +\infty$, *for $\zeta \in E_4$ (oscillation point).*

Moreover $f(\zeta)$ is well-accessible for $\zeta \in E_j$ $(j = 1,2,3,4)$.

The McMillan Twist Theorem [Mc69], [Po92, p. 142] states that, for almost all points $\zeta \in \mathbb{T}$, either ζ is a twist point or the angular derivative $f'(\zeta) \neq 0, \infty$ exists. The three sets of points satisfying ii), iii) and iv) were introduced in [Do92] and [CaPo97]. The Twist Theorem shows that these sets have measure 0. If $\lim_{r\to 1} f'(r\,\zeta)$ fails to exist on a set of positive measure then Plessner's Theorem for Bloch functions [Po92, p. 140] shows that assumption (1.9) is automatically satisfied. The special case of Corollary 1.2 that $\lim f'(r\,\zeta)$ exists almost nowhere is contained in [CaPo97, Theorem 1]. The boundary point $f(\zeta)$ is called *well-accessible* [Po92, p. 251] if there is a curve $z(t)$, $0 \leq t \leq 1$ with $z(0) = \zeta$ such that

$$\operatorname{diam}\{f(z(\tau)) :\ t \leq \tau \leq 1\} = O\left(\operatorname{dist}\left(f(z(t)), \partial f(\mathbb{D})\right)\right), \qquad \text{as } t \longrightarrow 1\,.$$

It is known [CaPo97, (3.17)] that the condition

$$-b \leq \log|f'(r\,\zeta)| \leq b\,, \qquad b > 1\,, \tag{1.10}$$

implies that $f(\zeta)$ is well-accessible and [CaPo97, (3.18)] that

$$|\arg f'(r\,\zeta) - \arg\left(f(r\,\zeta) - f(\zeta)\right)| \leq c_3\, b\,. \tag{1.11}$$

Proof of Corollary 1.2. Let $n > c_1$; see (1.3). By (1.9) there exist $r_n < 1$ such that $a_n = \log f'(r_n\,\zeta_0)$ satisfies $\operatorname{Re} a_n < -16\,n$. We define

$$(1.12)\quad \varphi_n(z) = \frac{z + r_n\,\zeta_0}{1 + r_n\,\overline{\zeta}_0\,z}\,,\ f_n = f \circ \varphi_n\,,\ g_n = \frac{1}{8}\,(\log f' \circ \varphi_n - a_n)\,.$$

Then $g_n \in \mathcal{B}$ with $g_n(0) = 0$ and $\|g_n\|_{\mathcal{B}} \le 1$ by (1.6). We apply Theorem 1.1 with $G = \{\operatorname{Re} w < |\operatorname{Re} a_n|\}$, $R = n$ and curves

$$\Gamma_j(t)\,, \qquad 0 \le t < 1 \ (j = 1, 2, 3, 4)$$

such that $\Gamma_j(0) = 0$, $\operatorname{Re}\Gamma_j(t) = 0$ and, as $t \longrightarrow 1$,

i) $\liminf \operatorname{Im}\Gamma_1(t) = -\infty$, $\limsup \operatorname{Im}\Gamma_1(t) = +\infty$,

ii) $\lim \operatorname{Im}\Gamma_2(t)) = +\infty$,

iii) $-\infty < \liminf \operatorname{Im}\Gamma_3(t) < +\infty$, $\limsup \operatorname{Im}\Gamma_3(t) = +\infty$,

iv) $\liminf \operatorname{Im}\Gamma_4(t) = 0$, $\limsup \operatorname{Im}\Gamma_4(t) = 3\pi + 2\,n + (c_3\,b_n + |a_n|)/8$,

see (1.15) below. Then (1.3) is satisfied, and (1.2) holds by (1.8) because $|\operatorname{Re} a_n| > 16\,n$. We conclude that there are sets $E_{jn} \subset \mathbb{T}$ with

$$(1.13)\qquad \dim E_{jn} \ge 1 - \frac{c_2}{n}\,, \qquad \text{for } j = 1, \dots, 4 \text{ and } n > c_1\,,$$

such that (1.5) holds for $\zeta \in E_{jn}$. We obtain from (1.12) that

$$(1.14)\qquad \log f_n'(z) = a_n + \log\left((1 - r_n^2)\,(1 + \overline{\zeta}_0\,r_n\,z)^{-2}\right) + 8\,g_n(z)\,.$$

Since $\operatorname{Re}\gamma_\zeta(r) = 0$ it follows from (1.5) that

$$(1.15)\qquad |\log|f_n'(r\,\zeta)|\,| \le b_n := |\operatorname{Re} a_n| + \log\frac{1 + r_n}{1 - r_n} + 16\,n$$

so that $f_n(\zeta)$ is well-accessible; see (1.10). We obtain from (1.5), (1.11) and (1.15) that

$$(1.16)\qquad \begin{aligned} &\limsup_{r\to 1}\,|\arg\left(f_n(r\,\zeta) - f_n(\zeta)\right) - 8\,\gamma_\zeta(r)| \\ &\qquad < 16\,n + c_3\,b_n + |\operatorname{Im} a_n| + 2 \\ &\qquad < \infty\,, \end{aligned}$$

for $\zeta \in E_{jn}$. Finally we set

$$E_j = \bigcup_n \varphi_n(E_{jn}), \qquad j = 1,2,3,4.$$

Then $\dim E_j = 1$ by (1.13), and if $\zeta \in E_j$ then $\zeta = \varphi_n(\zeta_n)$ for some $\zeta_n \in E_{jn}$.

Hence $f(\zeta) = f_n(\zeta_n)$ is well-accessible, and by the Koebe distortion theorem it is easy to deduce from (1.16) and the choice of $\operatorname{Im}\Gamma_j(t)$ that $\alpha(\zeta)$ and $\beta(\zeta)$ have the required properties.

REMARK 1. We assume now that $f(\mathbb{D})$ is bounded by a rectifiable curve. Then $f' \in H^1$ and thus [Du70, p. 24]

$$f'(z) = e^{i\alpha} \exp\Big(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log|f'(\zeta)|\,|d\zeta|\Big) \exp\Big(-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta)\Big),$$

where $\mu \geq 0$ is a singular measure. By definition $f(\mathbb{D})$ is a Smirnov domain if $\mu = 0$. Hence (1.8) holds if $|f'(\zeta)| \geq 1$ for almost all $\zeta \in \mathbb{T}$, and (1.9) holds if $f(\mathbb{D})$ is not a Smirnov domain. In particular Corollary 1.2 can be applied if $f(\mathbb{D})$ is a Keldish-Lavrentiev domain, that is a non-Smirnov domain for which $|f'(\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$; see [DuShSh66].

REMARK 2. There are local versions of Theorem 1.1 and Corollary 1.2. We can replace $\mathbb{T}$ by an open subarc A and restrict ζ and our sets E to lie in A.

2. The proof of Theorem 1.1.

We use the martingale technique introduced by Makarov [Ma90] into the theory of Bloch functions. For $n = 0,1,\ldots$ let $\mathcal{D}_n$ be the family of dyadic arcs of length $2\pi/2^n$ on $\mathbb{T}$, that is,

$$\mathcal{D}_n = \Big\{\Big\{e^{it}:\ \frac{2\pi k}{2^n} \leq t < \frac{2\pi\,(k+1)}{2^n}\Big\}:\ 0 \leq k < 2^n\Big\}. \tag{2.1}$$

If I and J are any dyadic arcs then $I \cap J = \varnothing$ or $I \subset J$ or $J \subset I$. Let $g \in \mathcal{B}$ and $n = 0,1,\ldots$ We define the *martingale associated to* g by

$$W_n(\zeta) \equiv W_n(I) = \lim_{r\to 1} \frac{1}{|I|}\int_I g(rs)\,|ds|, \qquad \text{for } \zeta \in I \in \mathcal{D}_n\ , \tag{2.2}$$

where $|\cdot|$ denotes the linear measure on $\mathbb{T}$. Let $c_1, c_2, \ldots$ denote suitable positive absolute constants. We need two known results. The first is due to Makarov [Ma90]; compare [Po92, p. 156].

Proposition 2.1 (Makarov)**.** *Let $g \in \mathcal{B}$, $\|g\|_{\mathcal{B}} \le 1$ and let W_n be the associated martingale. Then*

$$(2.3)\quad |g(r\,\zeta) - W_n(\zeta)| < c\,, \quad \textit{for } \zeta \in \mathbb{T},\ 1 - \frac{1}{2^n} \le r \le 1 - \frac{1}{2^{n+1}}\,,$$

$$(2.4)\qquad |W_{n+1}(\zeta) - W_n(\zeta)| < c\,, \qquad \textit{for } \zeta \in \mathbb{T}\,.$$

We also need the following technical result [ON95], [Do97]; compare [Ro93, p. 493].

Proposition 2.2 (O'Neill, Donaire)**.** *Let W_n be the martingale associated to $g \in \mathcal{B}$ and let $\|g\|_{\mathcal{B}} \le 1$, $0 < \alpha < \pi/2$. Let $I \in \mathcal{D}_m$ and $R > c_1(\alpha)$. If the stopping time*

$$(2.5)\qquad \tau_I(\zeta) = \inf\{n > m:\ |W_n(\zeta) - W_m(\zeta)| \ge R\}$$

is finite for almost all $\zeta \in I$, then

$$(2.6)\qquad |\{\zeta \in I:\ |\arg(W_{\tau_I(\zeta)}(\zeta) - W_m(\zeta)) - \vartheta| < \alpha\}| \ge c_2(\alpha)\,|I|\,,$$

for every ϑ. Here $c_1(\alpha)$ and $c_2(\alpha)$ only depend on α.

PROOF OF THEOREM 1.1. a) Let $\Gamma(t)$, $0 \le t < 1$ be some parametrization of our given curve Γ. Let $\mathcal{F}_0 = \{\mathbb{T}\}$ and $t_0 = 0$. We shall recursively construct families $\mathcal{F}_j$ of dyadic arcs such that each arc in $\mathcal{F}_j$ is contained in some arc of $\mathcal{F}_{j-1}$, furthermore stopping times

$$(2.7)\qquad t_j(\zeta) \equiv t_j(I) \in [0,1]\,, \qquad \text{for } \zeta \in I \in \mathcal{F}_{j-1}$$

constant on I such that $t_{j-1}(\zeta) \le t_j(\zeta)$ and

$$(2.8)\qquad \operatorname{dist}(W_m(I), \mathbb{C}\backslash G) > R + c\,, \qquad \text{for } I \in \mathcal{F}_j \cap \mathcal{D}_m\,,$$

where c is the constant of Proposition 2.1.

b) Suppose that $\mathcal{F}_j$ and t_j have already been defined. Let $\zeta \in I \in \mathcal{F}_j$. Then $I \in \mathcal{D}_m$ for some m. If $t_j(\zeta) = 1$ then we define $t_{j+1}(\zeta) = 1$, otherwise

$$(2.9) \qquad t_{j+1}(\zeta) \equiv t_{j+1}(I) = \inf\{t > t_j(\zeta) : \ |\Gamma(t) - W_m(I)| \geq R\},$$

if this set is empty we define $t_{j+1}(I) = 1$ and $A_j(I) = I$.

Now let $t_{j+1}(I) < 1$. Plessner's theorem for Bloch functions [Po92, p. 140] says that, for almost all $\zeta \in \mathbb{T}$, either the radial limit $g(\zeta)$ exists or the limit set of $g(r\,\zeta)$ as $r \longrightarrow 1$ is equal to $\hat{\mathbb{C}}$. Hence it follows from assumption (1.2) that

$$\liminf_{r\to 1} \operatorname{dist}(g(r\,\zeta), \mathbb{C}\backslash G) = 0, \qquad \text{for almost all } \zeta \in \mathbb{T},$$

so that, by (2.3),

$$\liminf_{n\to\infty} \operatorname{dist}(W_n(\zeta), \mathbb{C}\backslash G) \leq c, \qquad \text{for almost all } \zeta \in \mathbb{T}.$$

Therefore we obtain from (2.4) and (2.8) that, for almost all $\zeta \in I$, the stopping time $\tau_I(\zeta)$ defined in (2.5) is finite. By (2.4) we then have

$$(2.10) \qquad R \leq |W_{\tau_I(\zeta)}(\zeta) - W_m(\zeta)| < R + c.$$

Thus we can apply Proposition 2.2 with $\alpha = 1/4$. We see from (2.6) that, for $R > c_3 = \max\{4\,c, c_1\}$, the set

$$(2.11) \qquad \begin{aligned} A_j(I) = \Big\{\zeta \in I : \ &|\arg(W_{\tau_I(\zeta)}(\zeta) - W_m(\zeta)) \\ &- \arg(\Gamma_{t_{j+1}(I)} - W_m(I))| < \frac{1}{4}\Big\} \end{aligned}$$

satisfies $|A_j(I)| \geq c_2\,|I|$. Note that $A_j(I)$ is the union of dyadic arcs $J \in \mathcal{D}_n$ with $n > m$.

We define $\mathcal{F}_{j+1}$ as the family of the dyadic arcs J of $A_j(I)$ for all $I \in \mathcal{F}_j$. Then

$$(2.12) \qquad \sum_{\substack{J \subset I \\ J \in \mathcal{F}_{j+1}}} |J| = |A_j(I)| \geq c_2\,|I|.$$

Furthermore it follows from (2.4) and (2.10) that $\tau_I(\zeta) \geq m + R/c$. Hence

$$(2.13) \qquad J \in \mathcal{F}_{j+1}\,,\ J \subset I \in \mathcal{F}_j \text{ implies } |J| \leq 2^{-R/c}\,|I|.$$

Now we verify (2.8) for $j+1$, that is, we shall show that

$$\operatorname{dist}(W_n(J), \mathbb{C}\backslash G) > R + c\,, \tag{2.14}$$

for $J \in \mathcal{F}_{j+1}$, $\zeta \in I \in \mathcal{F}_j$, $n = \tau_I(\zeta)$; see (2.11). This is trivial by (2.8) if $t_{j+1}(I) = 1$ and thus $A_j(I) = I$. Therefore let $t_{j+1}(I) < 1$. Since $\Gamma(t)$ is continuous we see from (2.9) that $|\Gamma(t_{j+1}(I)) - W_m(I)| = R$. Hence it follows from (2.10) and (2.11) that the quantity

$$q = \frac{W_n(\zeta) - W_m(\zeta)}{\Gamma_{t_{j+1}}(I) - W_m(I)}$$

satisfies $1 \le |q| \le 1 + c/R$ and $|\arg q| < 1/4$. Since $R > c_3 \ge 4\,c$ we deduce that $|q-1| < 1/2$. Hence

$$|W_n(\zeta) - \Gamma(t_{j+1})| = |\Gamma(t_{j+1}) - W_m(\zeta)|\,|q-1| < \frac{R}{2}$$

and it follows by assumption (1.3) that

$$\operatorname{dist}(W_n(\zeta), \mathbb{C}\backslash G) \ge \operatorname{dist}(\Gamma, \partial G) - \frac{R}{2} \ge \frac{3R}{2} > R + c\,.$$

This completes our construction.

c) We define

$$E_\Gamma = \bigcap_{j \ge 1} \bigcup_{I \in \mathcal{F}_j} I\,. \tag{2.15}$$

It follows from (2.12) and (2.13) by a theorem [Po92, p. 226] of Hungerford [Hu88] and Makarov [Ma90] that

$$\dim E_\Gamma \ge \frac{\log(c_2\, 2^{R/c})}{\log 2^{R/c}} = 1 - \frac{c \log\left(\dfrac{1}{c_2}\right)}{R \log 2}\,,$$

which proves (1.4).

Now let $\zeta \in E_\Gamma$. There are two cases.

i) First we assume that $t_j(\zeta) < 1$ for all j. Let $I_j \in \mathcal{F}_j$ be the arc containing ζ. Then $I_j \in \mathcal{D}_{n_j}$ for some n_j. We define $\varphi_\zeta : [0,1) \longrightarrow [0,1)$ by $\varphi_\zeta(2^{-n_j}) = t_j(\zeta)$ and linear in between. We parametrize Γ by

$\gamma_\zeta(r) = \Gamma(\varphi_\zeta(r))$, $0 \le r < 1$. If $1 - 2^{-n_j} \le r \le 1 - 2^{-n_{j+1}}$ then $t_j(\zeta) \le \varphi_\zeta(r) \le t_{j+1}(\zeta)$ and thus

$$|g(r\,\zeta) - \gamma_\zeta(r)| \le |g(r\,\zeta) - W_{n_j}(\zeta)| + |\Gamma(\varphi_\zeta(r)) - W_{n_j}(I_j)| \le c + R \le 2R$$

by (2.3) and (2.9).

ii) Now we suppose that $t_j(\zeta) < 1$ for $j \le k$ and $t_j(\zeta) = 1$ for $j > k$. Then we define φ_ζ as in (i) for $j < k$ but linear in $[1 - 2^{-n_k}, 1]$. If $1 - 2^{-n_k} \le r < 1$ then (see (2.9))

$$|\Gamma(\varphi_\zeta(r)) - W_n(\zeta)| < R\,, \qquad \text{for } n \ge n_k$$

and (1.5) follows as above.

3. Balanced Bloch functions.

Let $\triangle(\zeta, \rho)$ denote the non-euclidean disk of center $\zeta \in \mathbb{D}$ and radius ρ. For $g \in \mathcal{B}$ we define

$$\mu_g(r) = \sup_{r \le |z| < 1} (1 - |z|^2)\, |g'(z)|\,, \qquad 0 \le r < 1\,. \tag{3.1}$$

Using the maximum principle for $|z| \le r$, we see that

$$|g'(z)| \le \max\left\{\frac{\mu_g(r)}{1 - r^2}, \frac{\mu_g(r)}{1 - |z|^2}\right\}, \qquad \text{for } z \in \mathbb{D},\ 0 \le r < 1\,. \tag{3.2}$$

By definition we have $g \in \mathcal{B}_0$ if $\mu_g(r) \longrightarrow 0$ as $r \longrightarrow 1$.

We call g a *balanced Bloch function* if there exist $a > 0$ and $\rho < \infty$ such that

$$\sup_{z \in \triangle(\zeta, \rho)} (1 - |z|^2)\, |g'(z)| \ge a\, \mu_g(|\zeta|)\,, \qquad \text{for } \zeta \in \mathbb{D}\,. \tag{3.3}$$

This condition is trivially satisfied if $0 < \alpha \le |g'(z)| \le \beta < \infty$ for $z \in \mathbb{D}$. Balanced Bloch functions for the case $g \notin \mathcal{B}_0$ were first considered by P. Jones [Jo89]; see *e.g.* also [Ro91], [BiJo97]. Jones showed that if $J = \partial f(\mathbb{D})$ is a quasicircle, then $\log f'$ is balanced and not in $\mathcal{B}_0$ if and only if

$$\inf_{w_1, w_2 \in J} \sup\left\{\frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|} :\ w \in J \text{ between } w_1 \text{ and } w_2\right\} > 1\,.$$

Curves with this property are called uniformly wiggly. The prototype of balanced Bloch functions are sufficiently regular series with Hadamard gaps.

Theorem 3.1. *Suppose that*

$$1 < \lambda \le \frac{n_{k+1}}{n_k} \le \lambda' < \infty\,, \qquad \textit{for } k = 0, 1, \dots \tag{3.4}$$

$$\frac{1}{M}\Big(\frac{n_j}{n_k}\Big)^{\alpha} |b_j| \le |b_k| \le M\,|b_j|\,, \qquad \textit{for } 0 \le j \le k \tag{3.5}$$

with constants M and $\alpha < 1$. Then

$$g(z) = \sum_{k=0}^{\infty} b_k\, z^{n_k}\,, \qquad z \in \mathbb{D}, \tag{3.6}$$

is a balanced Bloch function.

A typical example of a balanced Bloch function is

$$g(z) = \sum_{k=1}^{\infty} k^{-\gamma}\, z^{2^k}\,, \qquad 0 \le \gamma < \infty\,.$$

Proof. Let $M_1, M_2, \dots$ denote constants that depend only on $\lambda, \lambda', \alpha$ and M. If $1 - 1/n_j \le r \le 1 - 1/n_{j+1}$ and $|z| = r$ then, by (3.6),

$$\begin{aligned}|z\,g'(z)| &\le \sum_{k=0}^{j} n_k\,|b_k| + \sum_{k=j+1}^{\infty} n_k\,|b_k| \exp\Big(-\frac{n_k}{n_{j+1}}\Big)\\ &\le M\,n_j^{\alpha}\,|b_j| \sum_{k=0}^{j} n_k^{1-\alpha} + \lambda' M\,n_j\,|b_j| \sum_{k=j+1}^{\infty} \frac{n_k}{n_{j+1}} \exp\Big(-\frac{n_k}{n_{j+1}}\Big)\end{aligned}$$

by (3.5) and (3.4). Since $t\,e^{-t}$ is decreasing for $t \ge 1$ we therefore obtain from (3.4) that

$$|z\,g'(z)| \le M_1\,n_j\,|b_j| + \lambda' M\,n_j\,|b_j| \sum_{\nu=0}^{\infty} \lambda^{\nu} \exp(-\lambda^{\nu}) \le M_2\,\frac{|b_j|}{(1-r^2)}\,.$$

Using the maximum principle near $z = 0$, we thus see from (3.1) that

$$\mu_g(r) \le \sup_{k \ge j} M_3\,|b_k| \le M_4\,|b_j|\,, \qquad \text{for } 1 - \frac{1}{n_j} \le r \le 1 - \frac{1}{n_{j+1}}\,. \tag{3.7}$$

Now we apply a standard method [Bi69] to estimate the coefficients of gap series. It follows from (3.4), (3.5) and [GHPo87, Theorem 2] that

$$n_j\,|b_j| \le M_5 \sup\{|g'(z)| :\ z\in\triangle(\zeta,\rho)\}\,,$$

for $1-M_6/n_j \le |\zeta| \le 1-M_7/n_j$. Hence

$$\sup_{z\in\triangle(\zeta,\rho)} (1-|z|^2)\,|g'(z)| \ge M_8^{-1}\,(1-|\zeta|^2)\,n_j\,|b_j| \ge M_9^{-1}\,\mu_g(r)$$

by (3.7).

Further examples of balanced Bloch functions come from automorphic forms. Let Γ be a Fuchsian group with compact fundamental domain F in $\mathbb{D}$. Let h be an analytic automorphic form of weight 1, corresponding to a differential on the Riemann surface $\mathbb{D}/\Gamma$. Then $\gamma' h\circ\gamma = h$ for $\gamma\in\Gamma$ and

$$g(z)=\int_0^z h(\zeta)\,d\zeta\,,\qquad z\in\mathbb{D}$$

is a balanced Bloch function because $\overline{F}\subset\mathbb{D}$. Note that $\inf\mu_g(r)>0$.

Now we prove two results on real convex functions needed for the next section.

Lemma 3.2. *Let the real-valued functions φ and ψ be continuous and convex in the interval $I\subset\mathbb{R}$. If the function*

$$\chi(s)=\sup_{t\ge s}\,(\varphi(t)-\psi(t))+\psi(s)\,,\qquad s\in I \tag{3.8}$$

is finite, then it is also continuous and convex in I.

Proof. The function $\sup\{\varphi(t)-\psi(t) :\ t\in I,\ t\ge s\}$ is decreasing in $s\in I$. Let $I_k=[s_k,t_k]$ be its intervals of constancy with values c_k. We define

$$\chi_k(s)=\begin{cases}\varphi(s)\,, & \text{for } s\in I\backslash I_k\,,\\ c_k+\psi(s)\,, & \text{for } s\in I_k\,.\end{cases} \tag{3.9}$$

Since $\varphi(s)-\psi(s)\le c_k$ for $s\in I_k$, we have

$$\varphi(s)\le c_k+\psi(s)=\chi_k(s)\,,\qquad \text{for } s_k\le s\le t_k\,, \tag{3.10}$$

with equality for $s = s_k$ and $s = t_k$. The convex function φ has left and right derivatives $D^{\pm}\varphi$ in I and $D^{\pm}\varphi$ is increasing [HLP67, p. 91-94]. If $s < s_k$ then

$$D^+\chi_k(s) = D^+\varphi(s) \le D^+\varphi(s_k) \le D^+\psi(s_k) = D^+\chi_k(s_k)$$

by (3.10). If $s_k \le s < t_k$ then

$$D^+\chi_k(s_k) = D^+\psi(s_k) \le D^+\psi(s) = D^+\chi_k(s)$$

by (3.9). Since $D^-\psi(t_k) \le D^-\varphi(t_k)$ by (3.10), we furthermore have

$$D^+\chi_k(s) \le D^-\psi(t_k) \le D^-\varphi(t_k) \le D^+\varphi(t_k) = D^+\chi_k(t_k)\,.$$

Using again that $D^+\varphi$ and $D^+\psi$ are increasing, we deduce that $D^+\chi_k$ is increasing in I. Since χ_k is locally absolutely continuous it follows by integration that χ_k is convex. Finally $\chi = \sup_k \chi_k$ by (3.9) and (3.10), so χ is also convex.

Lemma 3.3. *The function*

$$\chi(s) = \log \mu_g(e^s) - \log(1 - e^{2s})\,, \qquad -\infty < s < 0$$

is convex and the function $u(z) = \chi(\log|z|)$ with $u(0) = \log\mu_g(0)$ is continuous and subharmonic in $\mathbb{D}$.

Proof. Let $M(r) = \max\{|g'(z)| : \ |z| = r\}$. It follows from (3.1) that (3.8) holds with

$$\varphi(s) = \log M(e^s)\,, \qquad \psi(s) = -\log(1 - e^{2s})\,.$$

The function φ is convex by the Hadamard three circles theorem [Co78, p. 137], and χ is convex because $\psi''(s) = 4\,e^{2s}\,(1-e^{2s})^{-2} > 0$. Therefore χ is convex by Lemma 3.2. It follows that u is subharmonic [HaKe76, Theorem 2.2].

4. Properties of balanced Bloch functions.

Let μ_g be defined by (3.1). We consider the open level sets

$$A_g(\varepsilon) = \{z \in \mathbb{D} : \ (1 - |z|^2)\,|g'(z)| < \varepsilon\,\mu_g(|z|)\}\,, \tag{4.1}$$

for $0 < \varepsilon \le 1$. We see from (4.1) and (3.1) that

$$|g'(z)| \ge \frac{\varepsilon\, \mu_g(r)}{1-r^2} \ge \varepsilon \max_{|\zeta|=r} |g'(\zeta)|\,, \qquad \text{for } z \notin A_g(\varepsilon)\,,\ |z| = r\,.$$

If g' is unbounded it follows that $\mathbb{T} \subset \overline{A_g(\varepsilon)}$ for all $\varepsilon > 0$. Otherwise we would have $|g'(z)| \longrightarrow \infty$ as $z \longrightarrow I$ for some arc I of $\mathbb{T}$, which is impossible by the Privalov uniqueness theorem [Po92, p. 140].

Let $M_1, \dots$ denote positive constants that depend only on a and ρ in the definition (3.3) of balanced Bloch functions. In particular, if g' is unbounded then $A_g(\varepsilon)$ is nonempty for $0 < \varepsilon \le 1$. By contrast, the example $g(z) \equiv z$ shows that $A_g(\varepsilon)$ can be empty if g' is bounded and $\varepsilon < 1$.

Proposition 4.1. *Let g be a balanced Bloch function and let $z_0 \in \mathbb{D}$. Then the harmonic measure satisfies*

$$\omega(z_1, \overline{\triangle}(z_0, 2\rho) \cap \overline{A}_g(\varepsilon), \triangle(z_0, 2\rho) \backslash \overline{A}_g(\varepsilon)) \le \frac{M_1}{\log\left(\dfrac{1}{\varepsilon}\right)}\,, \tag{4.2}$$

for some $z_1 \in \triangle(z_0, \rho)$.

Proof. We write $r = |z_0|$, $\triangle_0 = \triangle(z_0, 2\rho)$ and $A = \overline{A}_g(\varepsilon)$. It follows from (3.2) that

$$|g'(z)| \le \frac{M_2}{1-r^2}\, \mu_g(r)\,, \qquad \text{for } z \in \overline{\triangle}_0\,. \tag{4.3}$$

It follows from (4.1) that

$$|g'(z)| \le \frac{M_2}{1-r^2}\, \mu_g(r)\, \varepsilon\,, \qquad \text{for } z \in \overline{\triangle}_0 \cap A\,.$$

Hence the two-constants theorem [Ah73, p. 39] implies that

$$|g'(z)| \le \frac{M_2}{1-r^2}\, \mu_g(r)\, \varepsilon^{\omega(z, \overline{\triangle}_0 \cap A, \triangle_0 \backslash A)}\,, \tag{4.4}$$

for $z \in \triangle_0 \backslash A$. By (3.3) there exists $z_1 \in \triangle(z_0, \rho)$ such that

$$|g'(z_1)| \ge \frac{a}{1-|z_1|^2}\, \mu_g(r) \ge \frac{M_3^{-1}}{1-r^2}\, \mu_g(r)\,.$$

Hence (4.2) follows from (4.4).

Theorem 4.2. *Let g be a balanced Bloch function. Then there are $\alpha > 0$ and $\varepsilon_0 > 0$ such that every component of $A_g(\varepsilon)$ $(0 < \varepsilon < \varepsilon_0)$ lies in some disk $\triangle(z_0, \varepsilon^\alpha)$ $(z_0 \in \mathbb{D})$ and contains a zero of g'.*

PROOF. a) Let B be a component of $A_g(\varepsilon)$, let $z_0 \in B$ and let B_0 be the component of $B \cap \triangle(z_0, \rho/2)$ with $z_0 \in B_0$. Let φ map $\triangle(z_0, 2\rho)\backslash\overline{B}_0$ conformally onto $\{r < |z| < 1\}$ such that $\partial\triangle(z_0, 2\rho)$ corresponds to $\mathbb{T}$. Then

$$\omega(z,\, \overline{\triangle}(z_0, 2\rho) \cap \overline{B}_0, \triangle(z_0, 2\rho)\backslash\overline{B}_0) = \frac{\log\left(\dfrac{1}{|\varphi(z)|}\right)}{\log\left(\dfrac{1}{r}\right)} .$$

Since $B_0 \subset A_g(\varepsilon)$ it follows from Proposition 4.1 and the principle of majorization for harmonic measure [Ah73, p.39] that

$$\frac{\log\left(\dfrac{1}{|\varphi(z_1)|}\right)}{\log\left(\dfrac{1}{r}\right)} \le \frac{M_1}{\log\left(\dfrac{1}{\varepsilon}\right)} ,$$

for some $z_1 \in \triangle(z_0, \rho)$. Since $B_0 \subset \triangle(z_0, \rho/2)$ a normal family argument gives $|\varphi(z_1)| < 1 - \alpha_1$ where $\alpha_1 > 0$ depends only on a and ρ. Hence $r \le \varepsilon^{\alpha_2}$ and therefore

$$B_0 \subset \triangle(z_0, \varepsilon^\alpha) , \qquad \text{for } 0 < \varepsilon < \alpha_3 .$$

Since B is connected and contains z_0, it follows that $B = B_0$ if $\varepsilon^\alpha < \rho/2$.

b) Now we prove that every component B of $A_g(\varepsilon)$ with $\overline{B} \subset \mathbb{D}$ contains a zero of g'. Suppose that $g'(z) \ne 0$ for $z \in B$ and thus for $z \in \overline{B}$. Then $\log|g'|$ is harmonic in B and continuous in $\overline{B}$. Hence it follows from Lemma 3.3 that

$$v(z) = \log \mu_g(|z|) - \log(1 - |z|^2) - \log|g'(z)|$$

is subharmonic in B and continuous in $\overline{B}$. Since B is a component of $A_g(\varepsilon)$ and since $\overline{B} \subset \mathbb{D}$, we see from (4.1) that $v(z) = \log(1/\varepsilon)$ for $z \in \partial B$ and thus $v(z) \le \log(1/\varepsilon)$ for $z \in B$ by the maximum principle for subharmonic functions. But this contradicts (4.1).

Theorem 4.3. *Let g be a balanced Bloch function and suppose that*

$$\frac{\mu_g(r')}{\mu_g(r)} \geq \frac{1-r'}{1-r}\,\lambda\Big(\frac{1-r}{1-r'}\Big), \qquad \text{for } 0 < r < r' < 1\,, \tag{4.5}$$

where $\lambda(x) \nearrow \infty$ as $x \longrightarrow \infty$. Then there exist $\varepsilon > 0$ and $\rho^* < \infty$ such that every disk $\triangle(\zeta, \rho^*)$ ($\zeta \in \mathbb{D}$) contains a component of $A_g(\varepsilon)$.

Some (rather weak) condition like (4.5) is necessary as the balanced Bloch function $g(z) \equiv z$ shows. Note that (4.5) implies that g' is unbounded.

Proof. We claim: Given $\varepsilon > 0$ there exists $\rho' < \infty$ such that

$$\triangle(\zeta, \rho') \cap A_g(\varepsilon) \neq \varnothing\,, \qquad \text{for every } \zeta \in \mathbb{D}. \tag{4.6}$$

This claim implies the assertion of Theorem 4.3 with $\rho^* = \rho' + 2\,\varepsilon^\alpha$ and $0 < \varepsilon < \varepsilon_0$ by Theorem 4.2.

Suppose our claim is false. Then, for $0 < \varepsilon < 1$, there exist $z_n \in \mathbb{D}$ such that

$$(1-|z|^2)\,|g'(z)| > \varepsilon\,\mu_g(|z|)\,, \qquad \text{for } z \in \triangle(z_n, n)\,,\ n = 1, 2, \ldots \tag{4.7}$$

We write $r_n = |z_n|$ and consider the functions

$$h_n(s) = \frac{1-r_n^2}{\mu_g(r_n)}\,g'\Big(\frac{s+z_n}{1+\overline{z}_n s}\Big), \qquad s \in \mathbb{D}. \tag{4.8}$$

It follows from (4.8) and (3.2) that $|h_n(s)| \leq 4/(1-|s|^2)$ for $s \in \mathbb{D}$. Therefore we may assume that $h_n \longrightarrow h$ as $n \longrightarrow \infty$ locally uniformly in $\mathbb{D}$. Furthermore we may assume that $z_n \longrightarrow \zeta \in \mathbb{T}$.

Let $|s| = \sigma < 1$. By (3.1) and (4.5) we have

$$\mu_g\Big(\Big|\frac{s+z_n}{1+\overline{z}_n s}\Big|\Big) \geq \mu_g\Big(\frac{\sigma + r_n}{1 + r_n\,\sigma}\Big) \geq \frac{1-\sigma}{1+r_n\,\sigma}\,\lambda\Big(\frac{1+r_n\,\sigma}{1-\sigma}\Big)\mu_g(r_n)\,.$$

Hence it follows from (4.7) and (4.8) that

$$|h_n(s)| \geq \frac{\varepsilon\,|1+\overline{z}_n\,s|^2}{(1+\sigma)\,(1+r_n\,\sigma)}\,\lambda\Big(\frac{1+r_n\,\sigma}{1-\sigma}\Big).$$

Since $h_n \longrightarrow h$ and $\zeta_n \longrightarrow \zeta$ as $n \longrightarrow \infty$, we conclude that

$$|h(s)| \geq \frac{\varepsilon\,|1+\overline{\zeta}\,s|^2}{(1+\sigma)^2}\,\lambda\Big(\frac{1+\sigma}{1-\sigma}\Big) \geq \frac{\varepsilon}{4}\,\lambda\Big(\frac{1+\sigma}{1-\sigma}\Big)\,,$$

for $\operatorname{Re}(\overline{\zeta}\,s) > 0$. Hence

$$|h(s)| \longrightarrow \infty\,, \qquad \text{as } |s| \longrightarrow 1\,,$$

$\operatorname{Re}(\overline{\zeta}\,s) > 0$ which contradicts the Privalov uniqueness theorem [Pr56, p. 208], [Po92, p. 140].

Geometric interpretation. Let g be a balanced Bloch function that satisfies condition (4.5). Let $\varepsilon > 0$ be small but fixed. Then

$$(4.9) \qquad |g'(z)| \geq \varepsilon\,\frac{\mu_g(|z|)}{1-|z|^2} \longrightarrow \infty\,, \qquad \text{as } |z| \longrightarrow 1\,,\ z \in \mathbb{D}\backslash A_g(\varepsilon)$$

by (4.5). Theorem 4.2 says that the components of $A_g(\varepsilon)$ have small hyperbolic diameter, each containing a zero of g', whereas Theorem 4.3 says that there are many components. Hence the surface

$$\{(x,y,u):\ x+i\,y \in \mathbb{D},\ u = |g'(x+i\,y)|\}$$

rises to infinity at $\partial\,\mathbb{D}$ except for very many very small but deep holes near the zeros of g'.

Ruscheweyh and Wirths [RuWi82] have studied, for any Bloch function g, the set where $(1-|z|^2)\,|g'(z)|$ attains its maximum and its relation to the zeros of g'.

J. Becker [Be87], [PoWa82, Theorem 4.2] has shown that, for any $g \in \mathcal{B}$, the condition

$$(4.10) \qquad \int_0^1 \mu_g(r)^2\,\frac{dr}{1-r} < \infty$$

implies that $g \in$ VMOA (vanishing mean oscillation) and thus has finite radial limits $g(\zeta)$ for almost all $\zeta \in \mathbb{T}$. It follows [Pr56, p. 208] that $\operatorname{cap}\{g(\zeta):\ \zeta \in \mathbb{T},\ g(\zeta) \neq \infty \text{ exists}\} > 0$.

Now we turn to a condition stronger than (4.10), namely

$$(4.11) \qquad \int_0^1 \mu_g(r)\,\frac{dr}{1-r} < \infty\,.$$

It follows from (3.1) by integration that $\int_0^1 |g'(r\,\zeta)|\,dr < \infty$ for all $\zeta \in \mathbb{T}$ and that g is continuous in $\overline{\mathbb{D}}$. We show now that exactly the opposite happens if $g \in \mathcal{B}$ is balanced and condition (4.11) is false.

Theorem 4.4. *Let g be a balanced Bloch function with*

$$\int_0^1 \mu_g(r)\,\frac{dr}{1-r} = \infty\,. \tag{4.12}$$

If C is any curve in $\mathbb{D}$ ending on $\mathbb{T}$, then

$$\int_C |g'(z)|\,|dz| = \infty\,. \tag{4.13}$$

Furthermore g assumes every value in $\mathbb{C}$ infinitely often in $\mathbb{D}$.

Geometric interpretation. Let g be a balanced Bloch function that satisfies (4.10) and (4.12). The Riemann image surface of g over $\mathbb{C}$ then has many accessible boundary points; their projection to $\mathbb{C}$ has positive capacity. But (4.13) shows that none of these boundary points is accessible through a curve of finite length.

Proof. Let $c_1, c_2, \ldots$ denote suitable positive constants. Since C goes to $\mathbb{T}$, we can find $z_n \in C$, $r_n \nearrow 1$ and disks $\triangle_n$ such that

$$\triangle_n = \triangle(z_n, 2\rho) \subset \{r_n < |z| < r_{n+1}\}\,, \qquad \frac{1-r_{n+1}}{1-r_n} > c_1\,. \tag{4.14}$$

Let φ_n map $\triangle_n$ conformally onto $\mathbb{D}$ such that $\varphi_n(z_n) = 0$. By Proposition 4.1 there exist $\varepsilon > 0$ and $z_n^* \in \triangle(z_n, \rho)$ such that

$$\frac{M_1}{\log\left(\dfrac{1}{\varepsilon}\right)} > \omega(z_n^*, \overline{\triangle}_n \cap \overline{A}_g(\varepsilon), \triangle_n \backslash \overline{A}_g(\varepsilon)) = \omega(s_n^*, A_n, \mathbb{D}\backslash A_n)\,,$$

where $s_n^* = \varphi_n(z_n^*)$ and $A_n = \varphi_n(\overline{\triangle}_n \cap \overline{A}_g(\varepsilon))$. If p_n denotes the circular projection onto the radius from 0 to $\mathbb{T}$ opposite to s_n^*, then [Ah73, p. 43], [Ne53, p. 108]

$$\omega(s_n^*, p_n(A_n), \mathbb{D}\backslash p_n(A_n)) < \frac{M_1}{\log\left(\dfrac{1}{\varepsilon}\right)}\,.$$

Since $s_n^* \in \varphi_n(\triangle(z_n,\rho)) = \{|z|<\rho^*\}$ with $\rho^*<1$ depending only on ρ, we see that the linear measure satisfies $|p_n(A_n)| < M_4/\log(1/\varepsilon)$. Since $\varphi_n(C\cap\triangle_n)$ connects 0 and $\mathbb{T}$, we conclude that

$$|\varphi_n(C\cap\triangle_n)\backslash A_n| \geq 1-|p_n(A_n)| > 1-\frac{M_4}{\log\left(\dfrac{1}{\varepsilon}\right)} > \frac{1}{2}$$

if ε is chosen sufficiently small. It is easy to deduce that

$$|(C\cap\triangle_n)\backslash A_g(\varepsilon)| > c_1\,(1-|z_n|) > c_1\,c_2\,(1-r_n)$$

by (4.14). Hence it follows from (4.1) that

$$\int_{C\cap\triangle_n}|g'(z)|\,|dz| \geq \frac{\varepsilon\,\mu_g(r_{n+1})}{1-r_n^2}\,|(C\cap\triangle_n)\backslash A_g(\varepsilon)| > \frac{\varepsilon\,c_2}{2}\,\mu_g(r_{n+1})\,.$$

Since $\mu_g(r)$ is decreasing we have

$$\sum_n \mu_g(r_n) \geq c_1\sum_n\int_{r_n}^{r_{n+1}}\frac{\mu_g(r)}{1-r}\,dr = \infty$$

by (4.14) and (4.12). This implies (4.13).

The last assertion is an immediate consequence of (4.13) and the following proposition, where g need not be a Bloch function.

Proposition 4.5. *Let g be analytic in $\mathbb{D}$ and suppose that* (4.13) *holds for any curve C in $\mathbb{D}$ ending on $\mathbb{T}$. Then g assumes every finite value infinitely often in $\mathbb{D}$.*

Proof. a) For $w\in\mathbb{C}$ let $N(w)\leq\infty$ denote the number of zeros (with multiplicity) of $g-w$ in $\mathbb{D}$. Let $w,w'\in\mathbb{C}$ and let L be a rectifiable Jordan arc from w to w' that does not meet $\{g(z):\ z\in\mathbb{D},\ g'(z)=0\}$ except possibly in w and w'. At each point z_k of $g^{-1}(\{w\})$, we consider the maximal Jordan arcs C_k in $g^{-1}(L)$ with initial point z_k; the number of these arcs is equal to the multiplicity of the zero z_k of $g-w$. Therefore there are $N(w)$ arcs C_k altogether.

The maximal arc C_k ends either at some point $z_k'\in\mathbb{D}$ with $g(z_k')=w'$ or approaches $\mathbb{T}$. The second case cannot arise by our assumption because $|g(C_k)|\leq|L|<\infty$. The number of points z_k' that coincide is equal to the multiplicity of $g - w'$ in z'_k. Hence $N(w') \geq N(w)$ and thus $N(w') = N(w)$ by symmetry. Thus we have shown

$$N(w) \equiv m \leq \infty\,, \qquad \text{for } w \in \mathbb{C}\,. \tag{4.15}$$

b) Now we give a proof of the known fact that, for any function g analytic in $\mathbb{D}$, it is not possible that (4.15) holds with $m < \infty$. Let

$$r(\rho) = \sup\{|z| : \ |g(z)| = \rho\}\,, \qquad 0 < \rho < \infty\,. \tag{4.16}$$

We claim that $r(\rho) < 1$. Otherwise there would exist w with $|w| = \rho$ and points $z_n \in \mathbb{D}$ with $|z_n| \longrightarrow 1$ such that $g(z_n) \longrightarrow w$. But w is assumed m times in $\mathbb{D}$ so that there exist distinct z_{n_k} $(k = 1, \dots, m)$ with $g(z_{n_k}) = g(z_n)$ and $z_{n_k} \neq z_n$ for large n, which would imply $N(w) > m$.

It follows from (4.16) that $|g(z)| \neq \rho$ in $R(\rho) = \{r(\rho) < |z| < 1\}$. Since $g(R(\rho))$ is an unbounded domain we conclude that $|g(z)| > \rho$ for $z \in R(\rho)$ for any $\rho > 0$. Hence $|g(z)| \longrightarrow \infty$ as $|z| \longrightarrow 1$, which contradicts the Privalov uniqueness theorem.

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References.

[Ah73] Ahlfors, L. V., *Conformal invariants.* McGraw Hill, 1973.

[Be72] Becker, J., Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen. *J. Reine Angew. Math.* **255** (1972), 23-43.

[Be87] Becker, J., On asymptotically conform extension of univalent functions. *Complex Variables* **9** (1987), 109-120.

[Bi69] Binmore, K. G., Analytic functions with Hadamard gaps. *Bull. London Math. Soc.* **1** (1969), 211-217.

[BiJo97] Bishop, C. J., Jones, P. W., Wiggly sets and limit sets. *Arkiv. Math.* **35** (1997), 201-224.

[CaPo97] Carmona, J. J., Pommerenke, Ch., Twisting behaviour of conformal maps. *J. London Math. Soc.* **56** (1997), 16-36.

[Co78] Conway, J. B., *Functions of a complex variable.* Springer, 1978.

[Do92] Donaire, J. J., Comportamiento en la frontera de la derivada de una representación conforme. Master Thesis. Univ. Aut. Barcelona, 1992.

[Do97] Donaire, J. J., On the radial behaviour of inner functions in $\mathcal{B}_0$. Preprint. Univ. Aut. Barcelona, 1997.

[Du70] Duren, P. L., *Theory of H^p spaces.* Academic Press, 1970.

[DuShSh66] Duren, P. L., Shapiro, H. S., Shields, A. L., Singular measures and domains not of Smirnov type. *Duke Math. J.* **33** (1966), 247-254.

[Fa85] Falconer, K. J., *The geometry of fractal sets.* Cambridge Univ. Press, 1985.

[GHPo87] Gnuschke-Hauschild, D., Pommerenke, Ch., On dominance in Hadamard gap series. *Complex Variables* **9** (1987), 189-197.

[HaKe76] Hayman, W. K., Kennedy, P. B., *Subharmonic functions.* Vol. I. Academic Press, 1976.

[HLP67] Hardy, G. H., Littlewood, J. E., Pólya, G., *Inequalities.* Cambridge Univ. Press, 1967.

[Hu88] Hungerford, G. J., Boundaries of smooth sets and singular sets of Blaschke products in the little Bloch space. Thesis. CalTech, 1988.

[Jo89] Jones, P. W., Square functions, Cauchy integrals, analytic capacity and Hausdorff measure. Lecture Notes in Math. Springer **1384** (1989), 24-68.

[Ma90] Makarov, N. G., Probability methods in the theory of conformal mapping. *Leningrad Math. J.* **1** (1990), 1-56.

[Mc69] McMillan, J. E., Boundary behavior of a conformal mapping. *Acta Math.* **123** (1969), 43-67.

[Ne53] Nevanlinna, R., *Eindeutige analytische Funktionen.* Second Ed. Springer, 1953.

[ON95] O'Neill, M. D., Some results on H^∞ and the Bloch space. Thesis. Univ. of California, 1995.

[Po92] Pommerenke, Ch., *Boundary behaviour of conformal maps.* Springer, 1992.

[PoWa82] Pommerenke, Ch., Warschawski, S. E., On the quantitative boundary behavior of conformal maps. *Comm. Math. Helvetici* **57** (1982), 107-129.

[Pr56] Privalov, I. I., *Randeigenschaften analytischer Funktionen.* Deutscher Verlag Wiss., 1956.

[Ro91] Rohde, S., On conformal welding and quasicircles. *Michigan Math. J.* **38** (1991), 111-116.

[Ro93] Rohde, S., The boundary behaviour of Bloch functions. *J. London Math. Soc.* **48** (1993), 488-499.

[RuWi82] Ruscheweyh, S., Wirths, K.-J., On extreme Bloch functions with prescribed critical prints. *Math. Z.* **180** (1982), 91-105.

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Absolute values of BMOA functions

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Abstract. The paper contains a complete characterization of the moduli of BMOA functions. These are described explicitly by a certain Muckenhoupt-type condition involving Poisson integrals. As a consequence, it is shown that an outer function with BMO modulus need not belong to BMOA. Some related results are obtained for the Bloch space.

1. Introduction.

Let $\mathbb{D}$ denote the disk $\{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T}$ its boundary, and m the normalized arclength measure on $\mathbb{T}$. Further, let μ_z be the harmonic measure associated with a point $z \in \mathbb{D}$, so that

$$d\mu_z(\zeta) \stackrel{\text{def}}{=} \frac{1-|z|^2}{|\zeta - z|^2}\, dm(\zeta)\,, \qquad \zeta \in \mathbb{T}\,.$$

The space BMO consists, by definition, of all functions $f \in L^1(\mathbb{T}, m)$ satisfying

$$\|f\|_* \stackrel{\text{def}}{=} \sup_{z \in \mathbb{D}} \int |f(\zeta) - f(z)|\, d\mu_z(\zeta) < \infty\,,$$

where $f(z)$ stands for $\int f\, d\mu_z$. Alternative characterizations of BMO, as well as a systematic treatment of the subject, can be found in [G, Chapter VI] or [K, Chapter X]. Meanwhile, let us only recall that the

Garsia norm

$$\|f\|_G \overset{\text{def}}{=} \sup_{z\in\mathbb{D}} \Big(\int |f|^2 \, d\mu_z - |f(z)|^2 \Big)^{1/2},$$

defined originally for $f \in L^2(\mathbb{T}, m)$, is in fact an equivalent norm on BMO.

We shall also be concerned with the analytic subspace

$$\mathrm{BMOA} \overset{\text{def}}{=} \mathrm{BMO} \cap H^1$$

(as usual, we denote by H^p, $0 < p \le \infty$, the classical Hardy spaces of the disk). It is well known that

$$H^\infty \subset \mathrm{BMOA} \subset \bigcap_{0<p<\infty} H^p .$$

Now one of the basic facts about H^p spaces (see *e.g.* [G, Chapter II]) is this: In order that a function $\varphi \ge 0$, living almost everywhere on $\mathbb{T}$, coincide with the modulus of some nonzero H^p function, it is necessary and sufficient that $\varphi \in L^p(\mathbb{T}, m)$ and

$$\int \log \varphi \, dm > -\infty . \tag{1.1}$$

On the other hand, the very natural (and perhaps no less important) problem of characterizing the moduli of functions in BMOA seems to have been unsolved (or unposed?) until now, and the present paper is intended to fill that gap.

Thus, we look at a measurable function $\varphi \ge 0$ on $\mathbb{T}$ and ask whether

$$\varphi = |f| , \qquad \text{for some } f \in \mathrm{BMOA} ,\ f \not\equiv 0 . \tag{1.2}$$

The two immediate necessary conditions are (1.1) and

$$\varphi \in \mathrm{BMO} . \tag{1.3}$$

(To see that (1.2) implies (1.3), use the following simple fact: If for any $z \in \mathbb{D}$ there is a number $c(z)$ such that

$$\sup_{z\in\mathbb{D}} \int |\varphi(\zeta) - c(z)| \, d\mu_z(\zeta) < \infty , \tag{1.4}$$

then $\varphi \in \mathrm{BMO}$. Now, given that (1.2) holds, (1.4) is obviously fulfilled with $c(z) = |f(z)|$.) However, we shall see that (1.1) and (1.3) together are not yet sufficient for (1.2) to hold.

Assuming that (1.1) holds true, we consider the *outer function* $\mathcal{O}_\varphi$ given by

$$\mathcal{O}_\varphi(z) \stackrel{\text{def}}{=} \exp\Big(\int \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta)\, dm(\zeta)\Big), \qquad z \in \mathbb{D},$$

and note that (1.2) is equivalent to saying that

$$\mathcal{O}_\varphi \in \mathrm{BMOA}. \tag{1.5}$$

Indeed, since $|\mathcal{O}_\varphi| = \varphi$ almost everywhere on $\mathbb{T}$, the implication (1.5) implies (1.2) is obvious. The converse is also true, because the outer factor of a BMOA function must itself belong to BMOA (in fact, if $f = FI$ with $F \in H^2$ and I an inner function, then it is easy to see that $\|f\|_G \geq \|F\|_G$). The problem has thus been reduced to ascertaining when (1.5) holds.

In this paper we point out a new crucial condition (reminiscent, to some extent, of the Muckenhoupt (A_p) condition, cf. [G, Chapter VI]) which characterizes, together with (1.1) and (1.3), the non-negative functions φ with $\mathcal{O}_\varphi \in \mathrm{BMOA}$; this is contained in Section 2 below. Further, in Section 3, we exhibit an example of a BMO function $\varphi \geq 0$ with $\log \varphi \in L^1(\mathbb{T}, m)$ for which our Muckenhoupt-type condition fails. In other words, we show that the obvious necessary conditions (1.1) and (1.3) alone do not ensure the inclusion $\mathcal{O}_\varphi \in \mathrm{BMOA}$. Finally, in Section 4 we find out when an outer function with BMO modulus lies in the Bloch space $\mathcal{B}$.

2. Outer functions in BMOA.

Given a function $\varphi \in L^1(\mathbb{T}, m)$, $\varphi \geq 0$, we recall the notation

$$\varphi(z) \stackrel{\text{def}}{=} \int \varphi\, d\mu_z, \qquad z \in \mathbb{D},$$

and introduce, for a fixed $M > 0$, the level set

$$\Omega(\varphi, M) \stackrel{\text{def}}{=} \{z \in \mathbb{D} : \ \varphi(z) \geq M\}.$$

In order to avoid confusion, let us point out the notational distinction between

$$\varphi(z)^p \stackrel{\text{def}}{=} (\varphi(z))^p = \Big(\int \varphi\, d\mu_z\Big)^p$$

and

$$\varphi^p(z) \stackrel{\text{def}}{=} (\varphi^p)(z) = \int \varphi^p\, d\mu_z$$

(here $p > 0$ and $z \in \mathbb{D}$). Finally, we need the function

$$\log^- t \stackrel{\text{def}}{=} \begin{cases} \log \dfrac{1}{t}\,, & 0 < t < 1\,, \\ 0\,, & t \ge 1\,. \end{cases}$$

Our main result is

Theorem 1. *Suppose that* $\varphi \in \mathrm{BMO}$, $\varphi \ge 0$, *and*

$$\int \log \varphi\, dm > -\infty\,.$$

The following are equivalent.

i) $\mathcal{O}_\varphi \in \mathrm{BMOA}$.

ii) *For some* $M > 0$, *one has*

$$\sup\Big\{\varphi(z)^2 \int \log^- \varphi\, d\mu_z :\ z \in \Omega(\varphi, M)\Big\} < \infty\,.$$

Remark. The latter is vaguely reminiscent of the well-known Muckenhoupt (A_p) condition [G, Chapter VI] which can be written in the form

$$\sup\Big\{\varphi(z)^\tau \int \varphi^{-\tau}\, d\mu_z :\ z \in \mathbb{D}\Big\} < \infty\,,$$

where $\tau = 1/(p-1)$ and $1 < p < \infty$.

The proof of Theorem 1 makes use of the following elementary fact.

Lemma 1. *The function*

$$R(u) \stackrel{\text{def}}{=} \log \frac{1}{u} + u - 1\,, \qquad u > 0\,,$$

is nonnegative and satisfies

$$R(u) \le 2\,(u-1)^2\,, \qquad \textit{for } u \ge \frac{1}{2}\,.$$

Indeed, since $R(u)$ is the remainder term in the first order Taylor formula for $\log 1/u$, when expanded about the point $u = 1$, one has

$$R(u) = \frac{1}{2\,\xi^2}\,(u-1)^2\,,$$

where $\xi = \xi(u)$ is a suitable point between u and 1.

We also cite, as Lemma 2, the "harmonic measure version" of the classical John-Nirenberg theorem (see Section 2 and Exercise 18 in [G, Chapter VI]).

Lemma 2. *There are absolute constants $C > 0$ and $c > 0$ such that*

$$\mu_z\{\zeta \in \mathbb{T} :\ |f(\zeta) - f(z)| > \lambda\} \le C \exp\Big(-\frac{c\,\lambda}{\|f\|_*}\Big)\,,$$

whenever $z \in \mathbb{D}$, $f \in \mathrm{BMO}$ and $\lambda > 0$ (here again $f(z) \stackrel{\mathrm{def}}{=} \int f\, d\mu_z$).

PROOF OF THEOREM 1. Since $\varphi \in \mathrm{BMO}$, we know that

$$\|\varphi\|_G^2 = \sup_{z\in\mathbb{D}} \big(\varphi^2(z) - \varphi(z)^2\big) < \infty\,. \tag{2.1}$$

Similarly, condition i) of Theorem 1 is equivalent to

$$\|\mathcal{O}_\varphi\|_G^2 = \sup_{z\in\mathbb{D}} \big(\varphi^2(z) - |\mathcal{O}_\varphi(z)|^2\big) < \infty$$

and hence, in view of (2.1), to

$$\sup_{z\in\mathbb{D}} \big(\varphi(z)^2 - |\mathcal{O}_\varphi(z)|^2\big) < \infty\,. \tag{2.2}$$

In order to ascertain when (2.2) holds, we note that

$$|\mathcal{O}_\varphi(z)| = \exp\Big(\int \log\varphi\, d\mu_z\Big) = \varphi(z)\, e^{-J(z)}\,,$$

where
$$J(z) \stackrel{\text{def}}{=} \log \varphi(z) - \int \log \varphi \, d\mu_z \,,$$
and rewrite (2.2) in the form
$$\sup_{z\in\mathbb{D}} \varphi(z)^2 \left(1 - e^{-2J(z)}\right) < \infty \,. \tag{2.3}$$
We remark that $J(z) \geq 0$ by Jensen's inequality. Further, we claim that (2.3) is equivalent to the following condition
$$\sup\{\varphi(z)^2 J(z) : \ z \in \Omega(\varphi, M)\} < \infty \,, \qquad \text{for some } M > 0 \,. \tag{2.4}$$
Indeed, to deduce (2.3) from (2.4), one uses the inequality $1 - e^{-x} \leq x$ and the obvious fact that
$$\sup\{\varphi(z)^2 \left(1 - e^{-2J(z)}\right) : \ z \in \mathbb{D} \setminus \Omega(\varphi, M)\} \leq M^2 \,.$$
Conversely, to show that (2.3) implies (2.4), let K be the value of the supremum in (2.3) and put $M \stackrel{\text{def}}{=} \sqrt{2\,K}$. It then follows from (2.3) that
$$\sup\{J(z) : \ z \in \Omega(\varphi, M)\} < \infty \,,$$
and so $1 - e^{-2J(z)}$ is comparable to $J(z)$ as long as $z \in \Omega(\varphi, M)$.

We have thus reduced condition i) to (2.4), and we now proceed by looking at (2.4) more closely. To this end, we fix a point $z \in \Omega(\varphi, 2)$ and introduce the sets
$$E_1 = E_1(z) \stackrel{\text{def}}{=} \left\{\zeta \in \mathbb{T} : \ \varphi(\zeta) \geq \frac{1}{2}\, \varphi(z)\right\}$$
and
$$E_2 = E_2(z) \stackrel{\text{def}}{=} \mathbb{T} \setminus E_1 \,.$$
Using the function $R(u)$ from Lemma 1, we write
$$\begin{aligned} J(z) &= \int \log \frac{\varphi(z)}{\varphi(\zeta)} \, d\mu_z(\zeta) \\ &= \int \left(\log \frac{\varphi(z)}{\varphi(\zeta)} + \frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)}\right) d\mu_z(\zeta) \\ &= \int R\Big(\frac{\varphi(\zeta)}{\varphi(z)}\Big) \, d\mu_z(\zeta) \\ &= I_1(z) + I_2(z) \,, \end{aligned} \tag{2.5}$$

where

$$I_j(z) \stackrel{\text{def}}{=} \int_{E_j} R\Big(\frac{\varphi(\zeta)}{\varphi(z)}\Big)\, d\mu_z(\zeta)\,, \qquad j = 1, 2\,.$$

Now if $\zeta \in E_1$ then $\varphi(\zeta)/\varphi(z) \geq 1/2$, and Lemma 1 tells us that

$$R\Big(\frac{\varphi(\zeta)}{\varphi(z)}\Big) \leq 2\,\Big(\frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)}\Big)^2\,.$$

Integrating, we get

$$I_1(z) \leq \frac{2}{\varphi(z)^2} \int (\varphi(\zeta) - \varphi(z))^2\, d\mu_z(\zeta) \leq \frac{2}{\varphi(z)^2}\, \|\varphi\|_G^2\,,$$

so that

$$I_1(z) = O\Big(\frac{1}{\varphi(z)^2}\Big)\,. \tag{2.6}$$

In order to estimate $I_2(z)$, we observe that

$$\begin{aligned} \mu_z(E_2) &= \mu_z\Big\{\zeta :\ \varphi(\zeta) < \frac{1}{2}\,\varphi(z)\Big\} \\ &= \mu_z\Big\{\zeta :\ \varphi(z) - \varphi(\zeta) > \frac{1}{2}\,\varphi(z)\Big\} \\ &\leq \mu_z\Big\{\zeta :\ |\varphi(z) - \varphi(\zeta)| > \frac{1}{2}\,\varphi(z)\Big\} \\ &\leq C \exp\Big(-\frac{c\,\varphi(z)}{2\,\|\varphi\|_*}\Big)\,, \end{aligned} \tag{2.7}$$

as follows from Lemma 2. Besides, for $\zeta \in E_2$ one obviously has

$$\Big|\frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)}\Big| = 1 - \frac{\varphi(\zeta)}{\varphi(z)} \leq 1 \tag{2.8}$$

and

$$\log \frac{\varphi(z)}{\varphi(\zeta)} \geq \log 2 > 0\,. \tag{2.9}$$

Further, we set

$$E_2^+ \stackrel{\text{def}}{=} \{\zeta \in E_2 :\ \varphi(\zeta) \geq 1\}\,,$$

$$E_2^- \stackrel{\text{def}}{=} \{\zeta \in E_2 :\ \varphi(\zeta) < 1\}\,,$$

and

$$S(z) \stackrel{\text{def}}{=} \int_{E_2} \frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)} \, d\mu_z(\zeta) + \int_{E_2^+} \log \frac{\varphi(z)}{\varphi(\zeta)} \, d\mu_z(\zeta) + \mu_z(E_2^-) \log \varphi(z) \, .$$

We have then

$$I_2(z) = S(z) + \int_{E_2^-} \log \frac{1}{\varphi(\zeta)} \, d\mu_z(\zeta) \, . \tag{2.10}$$

Using (2.8) and (2.9), we see that

$$\Big| \int_{E_2} \frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)} \, d\mu_z(\zeta) \Big| \le \mu_z(E_2)$$

and

$$0 \le \int_{E_2^+} \log \frac{\varphi(z)}{\varphi(\zeta)} \, d\mu_z(\zeta) \le \mu_z(E_2^+) \log \varphi(z) \, .$$

Consequently,

$$\begin{aligned} |S(z)| &\le \mu_z(E_2) + (\mu_z(E_2^+) + \mu_z(E_2^-)) \log \varphi(z) \\ &= \mu_z(E_2) \, (1 + \log \varphi(z)) \\ &\le C \exp\Big(- \frac{c \, \varphi(z)}{2 \, \|\varphi\|_*} \Big) \, (1 + \log \varphi(z)) \, , \end{aligned} \tag{2.11}$$

where the last inequality relies on (2.7). The function

$$t \longmapsto t^2 \exp(-a \, t) \, (1 + \log t) \, , \qquad t \ge 2 \, ,$$

being bounded for any fixed $a > 0$, we conclude from (2.11) that

$$S(z) = O\Big(\frac{1}{\varphi(z)^2} \Big) \, .$$

Together with (2.10), this means that

$$I_2(z) = O\Big(\frac{1}{\varphi(z)^2} \Big) + \int_{E_2^-} \log \frac{1}{\varphi(\zeta)} \, d\mu_z(\zeta) \, . \tag{2.12}$$

A juxtaposition of (2.5), (2.6) and (2.12) now yields

$$J(z) = O\Big(\frac{1}{\varphi(z)^2}\Big) + \int_{E_2^-} \log \frac{1}{\varphi(\zeta)}\, d\mu_z(\zeta)\,. \tag{2.13}$$

Finally, recalling the assumption $z \in \Omega(\varphi, 2)$, we note that

$$E_2^- = \{\zeta \in \mathbb{T} :\ \varphi(\zeta) < 1\}$$

(indeed, if $\zeta \in \mathbb{T}$ and $\varphi(\zeta) < 1$, then $\varphi(\zeta) < \varphi(z)/2$, so that $\zeta \in E_2$). Thus, (2.13) can be rewritten as

$$J(z) = O\Big(\frac{1}{\varphi(z)^2}\Big) + \int \log^- \varphi\, d\mu_z\ ,$$

and this relation has been actually verified for $z \in \Omega(\varphi, 2)$.

It now follows that condition (2.4) (in which one can safely replace the words "for some $M > 0$" by "for some $M > 2$") holds if and only if

$$\sup \Big\{ \varphi(z)^2 \int \log^- \varphi\, d\mu_z :\ z \in \Omega(\varphi, M) \Big\} < \infty\,,$$

for some $M > 0$; we have thus arrived at ii). On the other hand, we have seen that (2.4) is a restatement of i). The desired equivalence relation is therefore established.

We proceed by pointing out a few corollaries of Theorem 1.

Corollary 1. *Let $\varphi \in \mathrm{BMO}$, $\varphi \geq 0$, and $\int \log \varphi\, dm > -\infty$. If $\mathcal{O}_\varphi \in$ BMOA and $0 < p < 1$, then $\mathcal{O}_{\varphi^p} (= \mathcal{O}_\varphi^p) \in$ BMOA.*

PROOF. Since $\varphi \in \mathrm{BMO}$, we have also $\varphi^p \in \mathrm{BMO}$ (this is easily deduced from the inequality $|a^p - b^p| \leq |a - b|^p$, valid for a, $b \geq 0$ and $0 < p < 1$). By Theorem 1, the inclusion $\mathcal{O}_\varphi \in$ BMOA yields

$$\sup \Big\{ \varphi(z)^2 \int \log^- \varphi\, d\mu_z :\ z \in \Omega(\varphi, M) \Big\} < \infty\,, \tag{2.14}$$

for some $M > 0$, and hence also for some $M \geq 1$. Hölder's inequality gives

$$\varphi^p(z) \leq \varphi(z)^p\,, \qquad z \in \mathbb{D}\,,$$

whence

$$\varphi^p(z) \le \varphi(z)\,, \qquad z \in \Omega(\varphi, 1)\,,$$

and

$$\Omega(\varphi^p, M^p) \subset \Omega(\varphi, M)\,.$$

Therefore, (2.14) with $M \ge 1$ implies the condition

$$\sup\left\{(\varphi^p(z))^2 \int \log^- \varphi^p \, d\mu_z :\; z \in \Omega(\varphi^p, M^p)\right\} < \infty\,,$$

which in turn means, by Theorem 1, that $\mathcal{O}_{\varphi^p} \in \mathrm{BMOA}$.

Corollary 2. *Let $\varphi \in \mathrm{BMO}$, $\varphi \ge 0$, and $\int \log \varphi \, dm > -\infty$. Assume, in addition, that φ possesses (after a possible correction on a set of zero measure) the following property: For some $\varepsilon > 0$, the set $\{\zeta \in \mathbb{T} :\; \varphi(\zeta) \le \varepsilon\}$ is closed and consists of continuity points for φ. Then $\mathcal{O}_\varphi \in \mathrm{BMOA}$.*

Proof. We may put $\varepsilon = 1$ (otherwise, consider the function $\varphi_1 \stackrel{\text{def}}{=} \varphi/\varepsilon$). Thus, we are assuming that the set

$$K \stackrel{\text{def}}{=} \{\zeta \in \mathbb{T} :\; \varphi(\zeta) \le 1\}$$

is closed, while φ is continuous at every point of K. We now claim that

$$K \cap \operatorname{clos} \Omega(\varphi, 2) = \varnothing\,. \tag{2.15}$$

Indeed, if $\zeta_0 \in K \cap \operatorname{clos} \Omega(\varphi, 2)$, then one could find a sequence $\{z_n\} \subset \mathbb{D}$ such that $\varphi(z_n) \ge 2$ and $z_n \longrightarrow \zeta_0$. On the other hand, since φ is continuous at ζ_0, we would have $\lim_{n\to\infty} \varphi(z_n) = \varphi(\zeta_0) \le 1$, a contradiction.

From (2.15) it follows that

$$\delta \stackrel{\text{def}}{=} \operatorname{dist}(K, \Omega(\varphi, 2)) > 0\,.$$

Hence, for $z \in \Omega(\varphi, 2)$, one has

$$\begin{aligned} \int \log^- \varphi \, d\mu_z &= \int_K \frac{1 - |z|^2}{|\zeta - z|^2} \log \frac{1}{\varphi(\zeta)} \, dm(\zeta) \\ &\le \frac{1 - |z|^2}{\delta^2} \, \|\log \varphi\|_{L^1(\mathbb{T}, m)}\,. \end{aligned} \tag{2.16}$$

An easy estimate for the Poisson integral of a BMO function gives

$$\varphi(z) = O\Big(\log \frac{2}{1-|z|}\Big), \qquad z \in \mathbb{D}. \tag{2.17}$$

Combining (2.16) and (2.17) yields

$$\begin{aligned}\varphi(z)^2 \int \log^- \varphi \, d\mu_z \\ \leq \text{const} \Big(\log \frac{2}{1-|z|}\Big)^2 \frac{1-|z|^2}{\delta^2} \, \| \log \varphi \|_{L^1(\mathbb{T},m)} ,\end{aligned} \tag{2.18}$$

for all $z \in \Omega(\varphi, 2)$. Since

$$(1-|z|^2)\Big(\log \frac{2}{1-|z|}\Big)^2 = O(1), \qquad z \in \mathbb{D},$$

the right-hand side of (2.18) is bounded by a constant independent of z. Thus,

$$\sup \Big\{ \varphi(z)^2 \int \log^- \varphi \, d\mu_z : \ z \in \Omega(\varphi, 2) \Big\} < \infty ,$$

and the desired conclusion follows by Theorem 1.

Corollary 3. *If* $\varphi \in \mathrm{BMO}$ *and* $\operatorname*{ess\,inf}_{\zeta \in \mathbb{T}} \varphi(\zeta) > 0$*, then* $\mathcal{O}_\varphi \in \mathrm{BMOA}$.

PROOF. For a suitable $\varepsilon > 0$ one has $\{\zeta \in \mathbb{T} : \varphi(\zeta) \leq \varepsilon\} = \varnothing$, so it only remains to apply Corollary 2.

3. An outer function with BMO modulus that does not belong to BMOA.

Although Theorem 1 provides a complete characterization of the moduli of BMOA functions, one may still ask whether the obvious necessary conditions (1.1) and (1.3) are also sufficient for $\mathcal{O}_\varphi$ to be in BMOA (equivalently, whether condition ii) of Theorem 1 follows automatically from (1.1) and (1.3)). An affirmative answer might parhaps seem plausible in light of corollaries 2 and 3 above. However, we are now going to construct an example that settles the question in the negative. In other words, we prove

Theorem 2. *There is a nonnegative function* $\varphi \in \mathrm{BMO}$ *with*

$$\int \log \varphi \, dm > -\infty$$

such that $\mathcal{O}_\varphi \notin \mathrm{BMOA}$.

Actually, we find it more convenient to deal with the space $\mathrm{BMO}(\mathbb{R})$ of the real line, defined as the set of functions $f \in L^1(\mathbb{R}, dt/(1+t^2))$ with

$$\|f\|_* \stackrel{\text{def}}{=} \sup_{z \in \mathbb{C}_+} \int_{\mathbb{R}} |f(t) - f(z)| \, d\mu_z(t) < \infty \,.$$

Here $\mathbb{C}_+$ denotes the upper half-plane $\{\operatorname{Im} z > 0\}$, the harmonic measure μ_z is now given by

$$d\mu_z(t) = \frac{1}{\pi} \frac{\operatorname{Im} z}{|t-z|^2} \, dt \,, \qquad z \in \mathbb{C}_+ \;, \; t \in \mathbb{R},$$

and $f(z)$ stands for $\int_{\mathbb{R}} f \, d\mu_z$. The subspace $\mathrm{BMOA}(\mathbb{C}_+)$ consists, by definition, of those $f \in \mathrm{BMO}(\mathbb{R})$ for which $f(z)$ is holomorphic on $\mathbb{C}_+$. Using the conformal invariance of BMO (see [G, Chapter VI]), one can restate Theorem 2 as follows.

Theorem 2′. *There is a nonnegative function* $\varphi \in \mathrm{BMO}(\mathbb{R})$ *with*

$$\int_{\mathbb{R}} \frac{\log \varphi(t)}{1+t^2} \, dt > -\infty$$

such that the outer function

$$\mathcal{O}_\varphi(z) \stackrel{\text{def}}{=} \exp\Big(\frac{i}{\pi} \int_{\mathbb{R}} \Big(\frac{1}{z-t} + \frac{t}{t^2+1} \Big) \log \varphi(t) \, dt \Big) , \qquad z \in \mathbb{C}_+ \;,$$

fails to belong to $\mathrm{BMOA}(\mathbb{C}_+)$.

The proof will rely on the following auxiliary result.

Lemma 3. *Let E and I be two (finite and nondegenerate) subintervals of* $\mathbb{R}$ *having the same center and satisfying*

$$\frac{|E|}{|I|} \stackrel{\text{def}}{=} \sigma < 1$$

(here $|\cdot|$ denotes length). Then there exists a function $\psi \in \mathrm{BMO}(\mathbb{R})$ such that

$$0 \le \psi \le 1\,, \qquad \textit{almost everywhere on } \mathbb{R}\,, \tag{3.1}$$

$$\psi|_E = 1\,, \qquad \psi|_{\mathbb{R}\setminus I} = 0\,, \tag{3.2}$$

and

$$\|\psi\|_* \le C\Big(\log\frac{1}{\sigma}\Big)^{-1}, \tag{3.3}$$

where $C > 0$ is some absolute constant.

PROOF OF LEMMA 3. By means of a linear mapping, the general case is reduced to the special one where $E = [-\sigma, \sigma]$ and $I = [-1, 1]$. This done, we define the function ψ by (3.2) and by

$$\psi(t) = \frac{\log|t|}{\log\sigma}\,, \qquad \sigma < |t| \le 1\,.$$

Now (3.1) is obvious, while (3.3) follows from the well-known facts that $\log|t| \in \mathrm{BMO}(\mathbb{R})$ and that $\mathrm{BMO}(\mathbb{R})$ is preserved by truncations (see Section 1 and Exercise 1 in [G, Chapter VI]).

REMARK. A more general (and much more difficult) version of Lemma 3, where E is an arbitrary measurable set contained in the middle third of I, is due to Garnett and Jones [GJ]; see also Exercise 19 in [G, Chapter VI]. We have, nonetheless, found it worthwhile to include a short proof of the version required.

PROOF OF THEOREM 2′. For $k = 1, 2, \ldots$, set $\sigma_k \stackrel{\text{def}}{=} \exp(-k^2)$ and let the numbers

$$0 = a_1 < b_1 < a_2 < b_2 < \cdots$$

be such that

$$b_k - a_k = \sigma_k \qquad \text{and} \qquad a_{k+1} - b_k = k^{-5/4}\,\sigma_k\,.$$

Consider the intervals $I_k \stackrel{\text{def}}{=} [a_k, b_k]$ and $J_k \stackrel{\text{def}}{=} [b_k, a_{k+1}]$. Further, let

$$x_k \stackrel{\text{def}}{=} \frac{a_k + b_k}{2}\,, \qquad y_k \stackrel{\text{def}}{=} \sigma_k^2 \tag{3.4}$$

and

$$E_k \stackrel{\text{def}}{=} \Big[x_k - \frac{1}{2}\,y_k,\, x_k + \frac{1}{2}\,y_k\Big]\,.$$

Since $|E_k| = \sigma_k^2 = \sigma_k\,|I_k|$, Lemma 3 provides, for every $k \in \mathbb{N}$, a function $\psi_k \in \mathrm{BMO}(\mathbb{R})$ such that

$$0 \le \psi_k \le 1\,, \qquad \text{on } \mathbb{R}\,,$$

$$\psi_k|_{E_k} = 1\,, \qquad \psi_k|_{\mathbb{R}\setminus I_k} = 0$$

and

$$\|\psi_k\|_* \le C\Big(\log\frac{1}{\sigma_k}\Big)^{-1}\,.$$

Finally, we set

$$\alpha_k \stackrel{\text{def}}{=} k^{3/4}\,, \qquad \beta_k \stackrel{\text{def}}{=} \exp\Big(-\frac{1}{\sigma_k}\Big)$$

and define the sought-after function φ by

$$\varphi \stackrel{\text{def}}{=} \chi_{\mathbb{R}\setminus\cup_k J_k} + \sum_k(\alpha_k\,\psi_k + \beta_k\,\chi_{J_k})$$

(here, as usual, χ_A stands for the characteristic function of the set A). In order to show that φ enjoys the required properties, we have to verify several claims.

Claim 1. $\varphi \in \mathrm{BMO}(\mathbb{R})$.

This follows at once from the inclusions

$$\varphi - \sum_k \alpha_k\,\psi_k \in L^\infty(\mathbb{R})$$

and

$$\sum_k \alpha_k\,\psi_k \in \mathrm{BMO}(\mathbb{R})\,,$$

where the latter holds true because

$$\sum_k \alpha_k\,\|\psi_k\|_* \le C\sum_k \alpha_k\Big(\log\frac{1}{\sigma_k}\Big)^{-1} = C\sum_k k^{-5/4} < \infty\,.$$

Claim 2. $\log\varphi \in L^1(\mathbb{R}, dt/(1+t^2))$.

Indeed, since $\varphi(t) < 1$ if and only if $t \in \bigcup_k J_k$, we have

$$\int \log^- \varphi \, dt = \sum_k \int_{J_k} \log \frac{1}{\varphi} \, dt = \sum_k |J_k| \log \frac{1}{\beta_k} = \sum_k k^{-5/4} < \infty \, .$$

Thus $\log^-\varphi \in L^1(\mathbb{R}, dt)$. Observing, in addition, that $\log\varphi = 0$ outside the finite interval

$$S \stackrel{\text{def}}{=} \bigcup_k I_k \cup \bigcup_k J_k$$

and noting that Claim 1 implies $\varphi \in L^1(S, dt)$, whence also

$$\log^+\varphi \ (= |\log\varphi| - \log^-\varphi) \in L^1(S, dt) \, ,$$

we eventually conclude that

$$\log\varphi \in L^1(\mathbb{R}, dt) \, .$$

A stronger version of Claim 2 is thus established.

Claim 3. *For every $M > 0$, one has*

$$\sup \left\{ \varphi(z)^2 \int \log^- \varphi \, d\mu_z : \ z \in \mathbb{C}_+ , \ \varphi(z) \geq M \right\} = \infty \, . \tag{3.5}$$

To verify (3.5), we set $z_k \stackrel{\text{def}}{=} x_k + i\, y_k$ (here x_k and y_k are defined by (3.4)) and show that both

$$\lim_{k\to\infty} \varphi(z_k) = \infty \tag{3.6}$$

and

$$\lim_{k\to\infty} \varphi(z_k)^2 \int \log^- \varphi \, d\mu_{z_k} = \infty \, . \tag{3.7}$$

To this end, we first note that $\mu_{z_k}(E_k) = \text{const}$, and so

$$\varphi(z_k) = \int \varphi \, d\mu_{z_k} \geq \int_{E_k} \varphi \, d\mu_{z_k} = (\alpha_k + 1)\, \mu_{z_k}(E_k) \geq \text{const}\, \alpha_k \ , \tag{3.8}$$

which proves (3.6). Further, we write

$$\int \log^- \varphi \, d\mu_{z_k} \geq \int_{J_k} \log^- \varphi \, d\mu_{z_k} = \mu_{z_k}(J_k) \log \frac{1}{\beta_k} \,. \tag{3.9}$$

Together with the simple fact that

$$\mu_{z_k}(J_k) \geq \text{const}\, |J_k| \,,$$

the inequality (3.9) gives

$$\int \log^- \varphi \, d\mu_{z_k} \geq \text{const}\, |J_k| \log \frac{1}{\beta_k} \,. \tag{3.10}$$

Finally, combining (3.8) and (3.10), we obtain

$$\varphi(z_k)^2 \int \log^- \varphi \, d\mu_{z_k} \geq \text{const}\, \alpha_k^2 \, |J_k| \log \frac{1}{\beta_k} = \text{const}\, k^{1/4} \,.$$

This proves (3.7), and hence also Claim 3. In view of Theorem 1 (which admits an obvious restatement for $\text{BMO}(\mathbb{R})$), Claim 3 is equivalent to saying that

$$\mathcal{O}_\varphi \notin \text{BMOA}(\mathbb{C}_+) \,,$$

so the proof is complete.

4. Outer functions with BMO moduli lying in the Bloch space.

Recall that the Bloch space $\mathcal{B}$ is defined to be the set of analytic functions f on $\mathbb{D}$ with

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} \sup_{z \in \mathbb{D}} (1 - |z|) \, |f'(z)| < \infty$$

(see [ACP] for a detailed discussion of this class). We now supplement Theorem 1 from Section 2 with the following result.

Theorem 3. *Let*

$$\varphi \in \text{BMO} \,, \ \ \varphi \geq 0 \,, \ \ \textit{and} \int \log \varphi \, dm > -\infty \,. \tag{4.1}$$

Suppose that, for some $M > 0$,

$$\sup \Big\{ \varphi(z) \int \log^- \varphi \, d\mu_z : \ z \in \Omega(\varphi, M) \Big\} < \infty \,. \tag{4.2}$$

Then $\mathcal{O}_\varphi \in \mathcal{B}$.

The proof hinges on

Lemma 4. *If φ satisfies* (4.1), *then*

$$(1 - |z|) \, |\mathcal{O}'_\varphi(z)| \le \mathrm{const} + 2 \, \varphi(z) \int \log^- \varphi \, d\mu_z \,, \tag{4.3}$$

whenever $z \in \Omega(\varphi, 2)$; the constant on the right depends only on φ.

PROOF OF LEMMA 4. Differentiating the equality

$$\mathcal{O}_\varphi(z) = \exp \Big(\int \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) \, dm(\zeta) \Big) \,, \qquad z \in \mathbb{D} \,,$$

gives

$$\begin{aligned} \mathcal{O}'_\varphi(z) &= \mathcal{O}_\varphi(z) \int \frac{2\,\zeta}{(\zeta - z)^2} \log \varphi(\zeta) \, dm(\zeta) \\ &= \mathcal{O}_\varphi(z) \int \frac{2\,\zeta}{(\zeta - z)^2} \log \frac{\varphi(\zeta)}{\varphi(z)} \, dm(\zeta) \,, \end{aligned} \tag{4.4}$$

where we have also used the fact that

$$\int \frac{2\,\zeta}{(\zeta - z)^2} \, dm(\zeta) = 0 \,.$$

From (4.4) one gets

$$(1 - |z|) \, |\mathcal{O}'_\varphi(z)| \le 2 \, |\mathcal{O}_\varphi(z)| \int \Big| \log \frac{\varphi(\zeta)}{\varphi(z)} \Big| \, d\mu_z(\zeta) \,, \tag{4.5}$$

and we proceed by looking at the integral on the right. Following the strategy employed in the proof of Theorem 1, we set

$$E_1 = E_1(z) \stackrel{\mathrm{def}}{=} \Big\{ \zeta \in \mathbb{T} : \ \varphi(\zeta) \ge \frac{1}{2} \, \varphi(z) \Big\}$$

and

$$E_2 = E_2(z) \stackrel{\text{def}}{=} \mathbb{T} \setminus E_1 \, .$$

Using the elementary inequality

$$|\log u| \le 2\,|u-1|\,, \qquad u \ge \frac{1}{2}\,,$$

we obtain

$$\tag{4.6} \begin{aligned} \int_{E_1} \left|\log \frac{\varphi(\zeta)}{\varphi(z)}\right| d\mu_z(\zeta) &\le 2 \int_{E_1} \left|\frac{\varphi(\zeta)}{\varphi(z)} - 1\right| d\mu_z(\zeta) \\ &\le \frac{2}{\varphi(z)} \int |\varphi(\zeta) - \varphi(z)|\, d\mu_z(\zeta) \\ &\le \frac{2}{\varphi(z)}\, \|\varphi\|_* \, . \end{aligned}$$

Repeating again some steps from the proof of Theorem 1, we introduce the sets

$$E_2^+ \stackrel{\text{def}}{=} \{\zeta \in E_2 : \ \varphi(\zeta) \ge 1\}\,,$$

$$E_2^- \stackrel{\text{def}}{=} \{\zeta \in E_2 : \ \varphi(\zeta) < 1\}\,,$$

and note that, since $z \in \Omega(\varphi, 2)$ (which is assumed from now on), we actually have

$$\tag{4.7} E_2^- = \{\zeta \in \mathbb{T} : \ \varphi(\zeta) < 1\}\,.$$

This done, we write

$$\tag{4.8} \begin{aligned} \int_{E_2} \left|\log \frac{\varphi(\zeta)}{\varphi(z)}\right| d\mu_z(\zeta) &= \int_{E_2} \log \frac{\varphi(z)}{\varphi(\zeta)}\, d\mu_z(\zeta) \\ &= \mu_z(E_2) \log \varphi(z) + \int_{E_2^+} \log \frac{1}{\varphi(\zeta)}\, d\mu_z(\zeta) \\ &\quad + \int_{E_2^-} \log \frac{1}{\varphi(\zeta)}\, d\mu_z(\zeta)\,. \end{aligned}$$

The estimate (2.7) from Section 2 says

$$\tag{4.9} \mu_z(E_2) \le C \exp\Big(-\frac{c\,\varphi(z)}{2\,\|\varphi\|_*}\Big)\,,$$

where $C > 0$ and $c > 0$ are certain absolute constants. Besides, we obviously have

$$\int_{E_2^+} \log \frac{1}{\varphi(\zeta)}\, d\mu_z(\zeta) \le 0 \tag{4.10}$$

and

$$\int_{E_2^-} \log \frac{1}{\varphi(\zeta)}\, d\mu_z(\zeta) = \int \log^- \varphi(\zeta)\, d\mu_z(\zeta) \tag{4.11}$$

(the latter relies on (4.7)). Using (4.9), (4.10) and (4.11) to estimate the right-hand side of (4.8), we get

$$\begin{aligned}\int_{E_2} \Big|\log \frac{\varphi(\zeta)}{\varphi(z)}\Big|\, d\mu_z(\zeta)& \\ \le C \exp\Big(-\frac{c\,\varphi(z)}{2\,\|\varphi\|_*}\Big) \log \varphi(z) &+ \int \log^- \varphi\, d\mu_z\,.\end{aligned} \tag{4.12}$$

Since

$$\sup_{t\ge 2} t\, e^{-at} \log t < \infty\,,$$

for any $a > 0$, (4.12) implies

$$\int_{E_2} \Big|\log \frac{\varphi(\zeta)}{\varphi(z)}\Big|\, d\mu_z(\zeta) \le \frac{\mathrm{const}}{\varphi(z)} + \int \log^- \varphi\, d\mu_z\,. \tag{4.13}$$

Combining (4.6) and (4.13) yields

$$\int_{\mathbb{T}} \Big|\log \frac{\varphi(\zeta)}{\varphi(z)}\Big|\, d\mu_z(\zeta) \le \frac{\mathrm{const}}{\varphi(z)} + \int \log^- \varphi\, d\mu_z\,. \tag{4.14}$$

Finally, substituting (4.14) into the right-hand side of (4.5) and noting that $|\mathcal{O}_\varphi(z)| \le \varphi(z)$ (say, by Jensen's inequality), one eventually arrives at (4.3).

PROOF OF THEOREM 3. Let $M \ge 2$ be a number for which (4.2) holds. Further, set $\psi \overset{\text{def}}{=} \sqrt{\varphi}$. Then $\psi \in \mathrm{BMO}$, $\log \psi \in L^1(\mathbb{T}, m)$, and

$$\psi(z)^2 \le \varphi(z)\,, \qquad z \in \mathbb{D}.$$

In particular,

$$\Omega(\psi, \sqrt{M}) \subset \Omega(\varphi, M)$$

(similar observations were made in the proof of Corollary 1 in Section 2). Condition (4.2) therefore yields

$$\sup\left\{\psi(z)^2 \int \log^- \psi \, d\mu_z : \; z \in \Omega(\psi, \sqrt{M})\right\} < \infty .$$

By Theorem 1, it follows that $\mathcal{O}_\psi \in$ BMOA. Since BMOA $\subset \mathcal{B}$, we also know that $\mathcal{O}_\psi \in \mathcal{B}$. In order to derive the required estimate

$$|\mathcal{O}'_\varphi(z)| \le \text{const}\,(1-|z|)^{-1}, \tag{4.15}$$

we distinguish two cases.

Case 1. $z \in \mathbb{D} \setminus \Omega(\varphi, M)$.

We have then

$$|\mathcal{O}_\psi(z)| \le \psi(z) \le \varphi(z)^{1/2} \le \sqrt{M}\,,$$

and so

$$|\mathcal{O}'_\varphi(z)| = |(\mathcal{O}^2_\psi)'(z)| = 2\,|\mathcal{O}_\psi(z)|\,|\mathcal{O}'_\psi(z)| \le 2\sqrt{M}\,\|\mathcal{O}_\psi\|_{\mathcal{B}}\,(1-|z|)^{-1}.$$

Case 2. $z \in \Omega(\varphi, M)$.

Since $\Omega(\varphi, M) \subset \Omega(\varphi, 2)$, a juxtaposition of (4.3) and (4.2) immediately yields

$$(1-|z|)\,|\mathcal{O}'_\varphi(z)| \le \text{const} < \infty .$$

Thus, (4.15) is established for all $z \in \mathbb{D}$, and the proof is complete.

Before proceeding with our final result, we point out two elementary facts.

Lemma 5. *Let φ satisfy* (4.1). *For any $M > 0$, the following are equivalent.*

(a) $$\sup\left\{\varphi(z)^2 \int \log^- \varphi \, d\mu_z : \; z \in \Omega(\varphi, M)\right\} < \infty .$$

(b) $$\sup\left\{\varphi^2(z) \int \log^- \varphi \, d\mu_z : \; z \in \Omega(\varphi, M)\right\} < \infty .$$

Proof. Since $\varphi(z)^2 \le \varphi^2(z)$, the implication (b) implies (a) is obvious. Conversely, let C be the value of the supremum in (a). For $z \in \Omega(\varphi, M)$, condition (a) implies

$$\int \log^- \varphi \, d\mu_z \le \frac{C}{M^2}$$

and hence

$$(\varphi^2(z) - \varphi(z)^2) \int \log^- \varphi \, d\mu_z \le C\, M^{-2} \, \|\varphi\|_G^2 \, ,$$

which leads to (b).

Lemma 6. *Let $\psi \in \mathrm{BMO}$, $\psi \ge 0$. Suppose the numbers $M > 0$ and $M_1 > 0$ are related by*

$$M_1 = M^2 + \|\psi\|_G^2 \, . \tag{4.16}$$

Then $\Omega(\psi^2, M_1) \subset \Omega(\psi, M)$.

Proof. If $\psi^2(z) \ge M_1$, then

$$\psi(z)^2 = \psi^2(z) - (\psi^2(z) - \psi(z)^2) \ge M_1 - \|\psi\|_G^2 = M^2 \, ,$$

so that $\psi(z) \ge M$.

Now we are in a position to prove

Theorem 4. *If $f \in \mathrm{BMOA}$ is an outer function with $|f|^2 \in \mathrm{BMO}$, then $f^2 \in \mathcal{B}$.*

Proof. Set $\psi \overset{\mathrm{def}}{=} |f|$ and $\varphi \overset{\mathrm{def}}{=} \psi^2$, so that $f = \mathcal{O}_\psi$ and $f^2 = \mathcal{O}_\varphi$. Since $\mathcal{O}_\psi \in \mathrm{BMOA}$, Theorem 1 yields

$$\sup \Big\{ \psi(z)^2 \int \log^- \psi \, d\mu_z : \ z \in \Omega(\psi, M) \Big\} < \infty \tag{4.17}$$

with some $M > 0$. By Lemma 5, we can replace $\psi(z)^2$ by $\psi^2(z)$ ($= \varphi(z)$); by Lemma 6, the arising condition will remain valid if we replace $\Omega(\psi, M)$ by the smaller set $\Omega(\varphi, M_1)$, where M_1 is defined by (4.16). Consequently, (4.17) implies

$$\sup \Big\{ \varphi(z) \int \log^- \varphi \, d\mu_z : \ z \in \Omega(\varphi, M_1) \Big\} < \infty \, .$$

Since $\varphi \in \mathrm{BMO}$, the desired conclusion that $\mathcal{O}_\varphi \in \mathcal{B}$ now follows by Theorem 3.

Remarks. 1) Of course, there are outer functions $f \in \mathrm{BMOA}$ with $f^2 \notin \mathcal{B}$. For example, this happens for $f(z) = \log(1-z)$, where log is the branch determined by $\log 1 = 2\pi i$.

2) Let $\varphi \geq 0$ on $\mathbb{T}$. Recalling Muckenhoupt's (A_p) condition (see Section 2 above), we have the implications

$$\varphi \in \mathrm{BMO} \cap (A_{3/2}) \text{ implies } \mathcal{O}_\varphi \in \mathrm{BMOA}$$

and

$$\varphi \in \mathrm{BMO} \cap (A_2) \text{ implies } \mathcal{O}_\varphi \in \mathcal{B}\,.$$

To see why, use Theorems 1 and 3 together with the inequality $\tau \log^- \varphi \leq \varphi^{-\tau}$ $(\tau > 0)$. It would be interesting to determine the full range of p's for which $\varphi \in \mathrm{BMO} \cap (A_p)$ implies $\mathcal{O}_\varphi \in \mathrm{BMOA}$ or $\mathcal{O}_\varphi \in \mathcal{B}$.

3) There used to be a question whether there existed a function lying in all H^p classes with $0 < p < \infty$ and in $\mathcal{B}$, but not in BMOA. Various constructions (based on different ideas) of such functions were given in [CCS], [HT] and [D2]. Our current results show how to construct an *outer* function with these properties. Namely, it suffices to find a function φ satisfying (4.1) and (4.2), with some $M > 0$, but such that

$$\sup\left\{\varphi(z)^2 \int \log^- \varphi \, d\mu_z : \ z \in \Omega(\varphi, M)\right\} = \infty\,,$$

for all $M > 0$. (An explicit example can be furnished in the spirit of Section 3 above.) This done, one has $\mathcal{O}_\varphi \in \bigcap_{0<p<\infty} H^p$ (because $\varphi \in \bigcap_{0<p<\infty} L^p$) and $\mathcal{O}_\varphi \in \mathcal{B} \setminus \mathrm{BMOA}$, as readily seen from Theorems 1 and 3.

4) While this paper deals with outer functions only, in [D1] and [D2] we have studied the interaction between the outer and inner factors of BMOA functions. Besides, we have characterized in [D3], [D4], [D5] the moduli of analytic functions in some other popular classes, such as Lipschitz and Besov spaces. In this connection, see also [Sh, Chapter II]. Finally, we mention the recent paper [D6], which is close in spirit to the current one.

References.

[ACP] Anderson, J. M., Clunie, J., Pommerenke, Ch., On Bloch functions and normal functions. *J. Reine Angew. Math.* **270** (1974), 12-37.

[CCS] Campbell, D., Cima, J., Stephenson, K., A Bloch function in all H^p classes, but not in BMOA. *Proc. Amer. Math. Soc.* **78** (1980), 229-230.

[D1] Dyakonov, K. M., Division and multiplication by inner functions and embedding theorems for star-invariant subspaces. *Amer. J. Math.* **115** (1993), 881-902.

[D2] Dyakonov, K. M., Smooth functions and coinvariant subspaces of the shift operator. *St. Petersburg Math. J.* **4** (1993), 933-959.

[D3] Dyakonov, K. M., The moduli of holomorphic functions in Lipschitz spaces. *Michigan Math. J.* **44** (1997), 139-147.

[D4] Dyakonov, K. M., Equivalent norms on Lipschitz-type spaces of holomorphic functions. *Acta Math.* **178** (1997), 143-167.

[D5] Dyakonov, K. M., Besov spaces and outer functions. *Michigan Math. J.* **45** (1998), 143-157.

[D6] Dyakonov, K. M., Multiplicative structure in weighted BMOA spaces. *J. Analyse Math.* **75** (1998), 85-104.

[G] Garnett, J. B., *Bounded Analytic Functions.* Academic Press, 1981.

[GJ] Garnett, J. B., Jones, P. W., The distance in BMO to L^∞. *Ann. of Math.* **108** (1978), 373-393.

[HT] Holland, F., Twomey, J. B., Explicit examples of Bloch functions in every H^p space, but not in BMOA. *Proc. Amer. Math. Soc.* **95** (1985), 227-229.

[K] Koosis, P., *Introduction to H^p Spaces.* Cambridge Univ. Press, 1980.

[Sh] Shirokov, N. A., *Analytic Functions Smooth up to the Boundary.* Lecture Notes in Math. **1312** Springer-Verlag, 1988.

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Maximal averages over flat radial hypersurfaces

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Let $A_t f(x) = \int_S f(x - t\,y)\,d\sigma(y)$, where S is a smooth compact hypersurface in $\mathbb{R}^n$ and $d\sigma$ denotes the Lebesgue measure on S. Let $\mathcal{A}f(x) = \sup_{t>0} |A_t f(x)|$. If the hypersurface S has non-vanishing Gaussian curvature, then

$$(*) \qquad \|\mathcal{A}f\|_{L^p(\mathbb{R}^n)} \le C_p\, \|f\|_{L^p(\mathbb{R}^n)}\,, \qquad f \in \mathcal{S}(\mathbb{R}^n)\,,$$

for $p > n/(n-1)$. Moreover, the result is sharp. See [St76], [Gr82].

If the hypersurface S is convex and the order of contact with every tangent line is finite, the optimal exponents for the inequality $(*)$ are known in $\mathbb{R}^3$, (see [IoSaSe98]), and in any dimension in the range $p > 2$, (see [IoSa96]). More precisely, the result in the range $p > 2$ is the following.

Theorem 1 ([IoSa96])**.** *Let S be a smooth convex compact finite type hypersurface, in the sense that the order of contact with every tangent line is finite. Then for $p > 2$, the following condition is necessary and sufficient for the maximal inequality $(*)$*

$$(1) \qquad (d(x, \mathcal{H}))^{-1} \in L^{1/p}(S)\,,$$

for every tangent hyperplane $\mathcal{H}$ not passing through the origin, where $d(x, \mathcal{H})$ denotes the distance from a point $x \in S$ to the tangent hyperplane $\mathcal{H}$.

In fact, the condition (1) is a necessary condition for any smooth compact hypersurface in $\mathbb{R}^n$. See [IoSa96, Theorem 2].

In this paper we shall consider convex radial hypersurfaces of the form

$$S = \{x \in B : \ x_n = \gamma(|x'|) + 1\}\,, \tag{2}$$

where B is a ball centered at the origin, $x = (x', x_n)$, γ is convex, γ, γ'' increasing, $\gamma(0) = \gamma'(0) = 0$, and γ is allowed to vanish of infinite order.

If γ'' does vanish of infinite order, the condition (1) cannot hold for any $p < \infty$. Since the condition (1) is necessary by Theorem 1 above, our only hope is to look for an inequality of the form

$$\|\mathcal{A}f\|_{L^\Phi(\mathbb{R}^n)} \le C_\Phi \, \|f\|_{L^\Phi(\mathbb{R}^n)}\,, \tag{3}$$

where $L^\Phi(\mathbb{R}^n)$ is an Orlicz space, near $L^\infty(\mathbb{R}^n)$, associated to a Young function Φ, with the norm given by

$$\|f\|_\Phi = \inf\Big\{s > 0 : \ \int \Phi\Big(\frac{|f(x)|}{s}\Big)\, dx \le 1\Big\}\,. \tag{4}$$

The following result was proved in [Bak95].

Theorem 2. *Let S be as in (2) with $n = 3$. Assume that for each $\lambda > 1$*

$$\frac{\gamma'(\lambda\, t)}{\gamma'(t)} \qquad \textit{is non-decreasing for } t > 0\,. \tag{5}$$

Put $G(t) = t^2\, \gamma'(t)$. For $\beta > 1$ and $d > 0$ let $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a non-decreasing function such that $\phi(t) = t^{-1}\,(G(t^{-d}))^{-\beta}$ if t is sufficiently large, $\phi(t) > 0$ if $t > 1$, and $\phi(t) = 0$ if $0 \le t \le 1$. Let $\Phi(u) = \int_0^u \phi(t)\, dt$. Then for every $d > 1/2$ there exists a constant C such that the estimate (3) holds.

The examples show (see [Bak95, Example 3.3]) that Theorem 2 is sharp for some surfaces, for example if $\gamma(s) = e^{-1/s^b}$, $b > 0$, but not for others, for example if $\gamma(s) = s^m$.

In this paper we shall give a set of simple sufficient conditions for the inequality (3) for some classes of Orlicz functions Φ. We will show that our result is sharp for a wide class of both finite type and infinite type γ's.

1. Assumptions on Φ.

Assume that Φ is a Young function such that $\Phi(s) = \int_0^s \phi(t)\, dt$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a non-decreasing function such that $\phi(t) = 0$ for $0 \le t \le 1$, and $\phi(t) > 0$ for $t > 1$. Assume that there exist constants $c > 1$, C_0, and C_1 such that

$$\int_1^u \frac{\phi(t)}{t^r}\, dt \le C_0 \frac{\phi(u)}{u^{r-1}}\,, \qquad \text{for } u > 1\,, \tag{6}$$

and for every $\lambda > 1$,

$$C_1 \frac{\phi(\lambda\, t)}{\phi(t)} \ge \phi(\lambda)\,, \qquad \text{for } t \ge c\,. \tag{7}$$

Our main reason for making these assumptions about Φ is the following generalization of the Marcienkiewicz interpolation theorem due to Bak. See [Bak95, Lemma 1.1].

Lemma 3. *Let $r \in [1,\infty)$. Suppose that the operator T is simultaneously weak type $(1,1)$ and (∞,∞), namely there exist constants $A, B > 0$ such that*

$$\mu(\{x : \ |Tf(x)| > t\}) \le \Big(\frac{A\, \|f\|_r}{t}\Big)^r\,, \qquad \textit{for all } t > 0\,, \tag{8}$$

$$\|Tf\|_\infty \le B\, \|f\|_\infty\,. \tag{9}$$

Suppose that Φ satisfies the assumptions above. Then there exists a constant $C = C(\Phi, r)$ depending only on Φ and r such that

$$\|Tf\|_\Phi \le C\, B\, \Phi^{-1}\Big(\Big(\frac{A}{B}\Big)^r\Big)\, \|f\|_\Phi\,. \tag{10}$$

REMARK. Lemma 3 has the following interesting consequence. Let

$$\mathcal{A}f(x) = \sup_{t>0} \int f(x - t\,(s, s^m + 1))\, \psi(s)\, ds\,, \qquad m > 2\,,$$

where ψ is a smooth cutoff function, and let $\mathcal{A}^k f(x)$ denote the same operator with s localized to the interval $[2^{-k}, 2^{-k+1}]$. It was proved in [I94] that $\mathcal{A}^k : L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2)$, $p > 2$, with norm $C\,2^{-k}\,2^{mk/p}$. Let $\Phi_{p,\alpha}(t) = t^p \log^\alpha(t)$. It follows by Lemma 3 that $\mathcal{A} : L^{\Phi_{p,\alpha}}(\mathbb{R}^2) \longrightarrow L^{\Phi_{p,\alpha}}(\mathbb{R}^2)$ if $p = m$ and $\alpha > m$.

2. Statement of results.

Our main results are the following.

Theorem 4. *Let S be as in* (2). *Let $n \geq 3$. Suppose that Φ satisfies the conditions* (6) *and* (7) *above. Suppose that $\lim_{t\to 0} \Phi(t)/t^2 = 0$. Then the estimate* (3) *holds if*

$$\sum_{j=0}^{\infty} 2^{-j(n-1)}\, \Phi^{-1}\Big(\frac{1}{\gamma(2^{-j})}\Big) < \infty\,. \tag{11}$$

The main technical result involved in the proof of Theorem 4 is the following version of the standard stationary phase estimates.

Lemma 5. *Let $n \geq 3$. Let*

$$F_j(\xi) = \int_{\{y:1\leq|y|\leq 2\}} e^{i(\langle y,\xi'\rangle + \xi_n \gamma_j(|y|))}\, e^{i\xi_n/\gamma(2^{-j})}\, dy\,, \tag{12}$$

with $\gamma_j(s) = \gamma(2^{-j}\,s)/\gamma(2^{-j})$, where γ is as in (2). *Then*

$$|F_j(\xi)| \leq C\,(1 + |\xi|)^{-1}\,, \tag{13}$$

where C is independent of j and γ.

Moreover, if $|F_j(\xi)|$ is replaced by $|\nabla F_j(\xi)|$ then the estimate (13) *still holds with C on the right-hand side replaced by $C/\gamma(2^{-j})$.*

The main technical result used in the proof of Theorem 2 is the following. See [Bak95, Theorem 2.1].

Lemma 6. *Let $\chi \in C_0^1([0,\infty))$ be a non-negative function that is compactly supported in the interval (a,∞), where $a > 0$. Let $n = 3$ and let S be as in* (2) *where γ satisfies the condition of Theorem* 2. *Let*

$F_S(\chi)(\xi)$ denote $F_0(\xi)$ in Lemma 5 with $\chi(|y|)$ in place of the characteristic function of the annulus $\{y:\ 1 \leq |y| \leq 2\}$.

Then for every multi-index α with $|\alpha| \leq 1$ there exists a constant C independent of a, ξ, and χ such that

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^{\alpha} F_S(\chi)(\xi)\Big| \leq C\,C_\chi\,\frac{a}{\sqrt{\gamma'(a)\,\gamma'\Big(\dfrac{a}{2}\Big)}}\,(1+|\xi|)^{-1}\,, \tag{14}$$

where $C_\chi \leq \|\chi\|_\infty + \|\chi'\|_1$ if $\alpha = 0$, and $C_\chi \leq \|\chi\|_\infty + \|\chi\|_1 + \|\chi'\|_1$ if $\alpha = 1$.

3. Main idea.

The point is that even though a higher dimensional analog of Lemma 6 may be difficult to obtain, we get around the problem by using Lemma 5. We have to settle for the uniform decay of order $\max\{-(n-2)/2, -1\}$ instead of $-(n-1)/2$, but this is enough in dimension $n \geq 4$ as we shall see below. The idea is, roughly speaking, the following. We are trying to prove $L^\Phi \longrightarrow L^\Phi$ estimates for maximal operators associated to radial convex surfaces. If the surface is infinitely flat, then [IoSa96, Theorem 2] implies that $L^p \longrightarrow L^p$ estimates are not possible for $p < \infty$. So we are looking for $L^\Phi \longrightarrow L^\Phi$ estimates where L^Φ is very close to L^∞, so interpolating between L^2 and L^∞ in the right way should do the trick. However, in order to obtain L^2 boundedness of the maximal operator, we only need decay $-1/2-\varepsilon$, $\varepsilon > 0$. If $n \geq 4$, then $(n-2)/2 > 1/2$, so we should be alright. If $n = 3$ a bit more integration by parts will be required.

4. Plan.

The rest of the paper is organized as follows. In the next section we shall prove Theorem 4 assuming Lemma 5. In the following section we shall prove Lemma 5. In the final section of the paper we shall discuss the sharpness of Theorem 4 and give some examples.

5. Proof of Theorem 4.

Let

$$A_t^j f(x) = \int f(x' - t\,y, x_n - t\,(\gamma(|y|) + 1))\,\psi_0(y)\,dy\,,$$

where ψ_0 is a smooth cutoff function supported in $[1, 2]$, such that $\sum_j \psi(2^j\,s) \equiv 1$. Let $\tau_j f(x) = f(2^{-j}\,x', \gamma(2^{-j})\,x_n)$. Making a change of variables we see that

$$A_t^j f(x) = 2^{-j(n-1)}\,\tau_j^{-1}\,B_t^j\,\tau_j\,f(x)\,, \tag{15}$$

where

$$B_t^j f(x) = \int f\Big(x' - t\,y, x_n - t\Big(\frac{\gamma_j(|y|) + 1}{\gamma(2^{-j})}\Big)\Big)\,\psi_0(y)\,dy\,. \tag{16}$$

We shall prove that

$$\sup_{t>0} B_t^j : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \qquad \text{with norm } \Big(\frac{1}{\gamma(2^{-j})}\Big)^{1/2}\,. \tag{17}$$

By interpolating with the trivial estimate $\|\sup_{t>0} B_t^j f\|_\infty \le C\,\|f\|_\infty$ using Lemma 3, we shall conclude that

$$\sup_{t>0} B_t^j : L^\Phi(\mathbb{R}^n) \longrightarrow L^\Phi(\mathbb{R}^n) \qquad \text{with norm } \Phi^{-1}\Big(\frac{1}{\gamma(2^{-j})}\Big)\,. \tag{18}$$

Since the L^p norms of τ_j and τ_j^{-1} are reciprocals of each other, it follows that $\mathcal{A} : L^\Phi(\mathbb{R}^n) \longrightarrow L^\Phi(\mathbb{R}^n)$ if

$$\sum_{j=0}^{\infty} 2^{-j(n-1)}\,\Phi^{-1}\Big(\frac{1}{\gamma(2^{-j})}\Big) < \infty\,. \tag{19}$$

So it remains to prove (18). The proof follows from the standard Sobolev imbedding theorem type argument. See for example [St76]. We shall use the following version which follows from the proof of [IoSa96, Theorem 15]. See also, for example, [CoMa86], [MaRi95].

Lemma 7. *Suppose that τ is the Lebesgue measure on the hypersurface S supported in an ellipsoid with eccentricities $(1,\dots,1,R)$. Suppose that $|\hat{\tau}(\xi)|\le C$ and $\max\{|x|:\ x\in \operatorname{supp}(\tau)\}\le 10R$. Suppose that*

$$\Big(\int_1^2 |\hat{\tau}(t\,\xi)|^2\,dt\Big)^{1/2} \le C\,(1+|\xi|)^{-1/2-\varepsilon}\,, \tag{20}$$

and

$$\Big(\int_1^2 |\nabla\hat{\tau}(t\,\xi)|^2\,dt\Big)^{1/2} \le C\,R\,(1+|\xi|)^{-1/2-\varepsilon}\,, \tag{21}$$

for some $\varepsilon>0$. Let $\hat{\tau}_t(\xi)=\hat{\tau}(t\,\xi)$. Let $\mathcal{M}f(x)=\sup_{t>0}|f*\tau_t(x)|$. Then

$$\|\mathcal{M}f\|_2 \le 100\,C\,\sqrt{R}\,\|f\|_2\,. \tag{22}$$

Application of Lemma 7 immediately yields (17) since by Lemma 5 C is a universal constant and $R\le C/\gamma(2^{-j})$. This completes the proof of Theorem 4.

6. Proof of Lemma 5.

We must show that

$$\begin{aligned} |F_j(\xi)| &= \Big|\int_{\{y:1\le|y|\le 2\}} e^{i(\langle y,\xi'\rangle+\xi_n\gamma_j(|y|))}\,e^{i\xi_n/\gamma(2^{-j})}\,dy\Big| \\ &\le C\,|\xi|^{-1}\,, \end{aligned} \tag{23}$$

with C independent of γ and j.

Our plan is as follows. We will first show that if either $|\xi'|\approx|\xi_n|$, or $|\xi'|\gg|\xi_n|$, then $|F_j(\xi)|\le C\,(1+|\xi|)^{-(n-2)/2}$. If $|\xi_n|\gg|\xi'|$, we will show that $|F_j(\xi)|\le C\,(1+|\xi_n|)^{-1}$. This will complete the proof since $(n-2)/2\ge 1$ if $n\ge 4$.

Going into polar coordinates and applying stationary phase, we get

$$e^{i\xi_n/\gamma(2^{-j})}\int_1^2 e^{i\xi_n\gamma_j(r)}\,r^{n-2}\,dr\int_{S^{n-2}} e^{ir\langle\xi',\omega\rangle}\,d\omega\,. \tag{24}$$

Since the Gaussian curvature on S^{n-2} does not vanish, it is a classical result that

$$(25) \qquad \Big| \int_{S^{n-2}} e^{i\langle \xi', \omega\rangle}\, d\omega \Big| \le C\,(1+|\xi'|)^{-(n-2)/2}\,.$$

It follows that $|F_j(\xi)| \le C\,(1+|\xi|)^{-n-2/2}$ if either $|\xi'| \gg |\xi_n|$ or $|\xi'| \approx |\xi_n|$. If $|\xi_n| \gg |\xi'|$, let $h(r) = \xi_n\,\gamma_j(r) - r\,\langle \xi', \omega\rangle$. Since γ is convex, it follows that $|h'(r)| \ge |\xi_n| - |\xi'|$. Since $|\xi_n| \gg |\xi'|$, it follows by the van der Corput Lemma that the expression in (24) is bounded by $C/|\xi|$.

The estimate for ∇F_j follows in the same way if we observe that the derivative with respect to ξ_n brings down a factor of $\gamma_j(r) + 1/\gamma(2^{-j})$, and $\gamma_j(r) + 1/\gamma(2^{-j}) \le 2/\gamma(2^{-j})$. This completes the proof of Lemma 5 if $n \ge 4$.

To prove the three dimensional case we go into polar coordinates, integrate in the angular variables and use the well known asymptotics for the Fourier transform of the Lebesgue measure on the circle to obtain

$$(26) \qquad \int e^{i\phi(r)}\, r\, b(rA)\, \psi_0(r)\, dr\,,$$

where $A = |\xi'|$, $\lambda = \xi_n$, b is a symbol of order $-1/2$, ψ_0 is as above, and $\phi(r) = rA - \gamma_j(r)\,\lambda$.

Let

$$(27) \qquad G(r) = \int_r^2 e^{i\phi(s)}\, ds\,,$$

so the integral in (26) becomes

$$(28) \qquad \int G'(r)\, r\, b(rA)\, \psi_0(r)\, dr\,.$$

Integrating by parts we get

$$(29) \qquad \int G(r)\, (r\, b(rA)\, \psi_0(r))'\, dr\,.$$

Let r_0 be defined by the relation $\gamma_j'(r_0) = A/(2\,\lambda)$. We have $|\phi''(s)| \ge |\gamma_j''(s)\,\lambda| \ge |\gamma_j'(s)\,\lambda| \ge |\gamma_j'(r)\,\lambda|$. If $r_0 < r$ this quantity is bounded below by $C\,|A|$ and the van der Corput lemma gives the decay $C\,|A|^{-1/2}$ for $G(r)$. Using the fact b is a symbol of order $-1/2$ we see that (29) is bounded by $C\,|A|^{-1}$, $|A|$ large. This handles the case $|\lambda| \le C\,|A|$ and $r \le r_0$.

On the other hand, $|\phi'(s)| = |A - \gamma_j'(s)\,\lambda|$. Split up the integral that defines $G(r)$ into two pieces: $s \in [r, r_0]$ and $s \in [r_0, 2]$. The second integral was just handled above. In the first integral $|\phi'(s)| \ge |\phi'(r_0)| \ge C\,|A|$. The van der Corput lemma yields decay $C/|A|$. Taking the properties of the symbol b into account, as before, we get the decay $C\,|A|^{-1/2}/|A|$. This takes care of the case $|\lambda| \le C\,|A|$ and $r \ge r_0$.

If $|\lambda| \gg |A|$, $|\phi'(s)| \ge C\,|\lambda|$ and the van der Corput lemma yields the decay $C/|\lambda|$ for (29). This completes the proof of the three dimensional case.

7. Examples.

Example 1. Let $\gamma(s) = s^m$, $m \ge 2\,(n-1)$, and $\Phi(t) = t^p$. Theorem 4 yields boundedness for $p > m/(n-1)$. This is sharp by Theorem 1.

Example 2. Let $\gamma(s) = s^m$, $m \ge 2\,(n-1)$, and $\Phi_{p,\alpha}(s) = s^p \log^\alpha(s)$. Then Theorem 4 yields boundedness for $p = m/(n-1)$ and $\alpha > m/(n-1)$.

Example 3. Let $\gamma(s) = e^{-1/s^\alpha}$, $\alpha > 0$, and $\Phi(t) = e^{t^\beta}$, $\beta > 0$. Then Theorem 4 tells us that the maximal operator is bounded if $\alpha < \beta\,(n-1)$. Testing $A_t f(x)$ against

$$h_p(x) = \Phi^{-1}\Big(\frac{1}{|x_n|}\Big)\,\frac{1}{\log\Big(\dfrac{1}{|x_n|}\Big)}\,\chi_B(x)\,,$$

where χ_B is the characteristic function of the ball of radius $1/2$ centered at the origin, shows that this result is sharp. The same procedure establishes sharpness of the estimate given in Example 2.

In fact, testing $A_t f(x)$ against $h_p(x)$ shows that the summation condition of Theorem 4 is pretty close to being sharp. It is not hard to see that, at least up to a log factor, $\mathcal{A}$ bounded on $L^\Phi(\mathbb{R}^n)$ implies that

$$\int_{\{y:|y|\le 2\}} \Phi^{-1}\Big(\frac{1}{\gamma(|y|)}\Big)\,dy < \infty\,. \tag{30}$$

This would literally follow, without the log factor, from the proof of [IoSa96, Theorem 2] if we assumed, in addition, that $\Phi(a\,b) \ge \Phi(a)\,\Phi(b)$, for every $a, b > 0$.

The condition (30) is equivalent (after making a change of variables and going into polar coordinates) to

$$\sum_{j=0}^{\infty} 2^{-j(n-1)} \int_1^2 \Phi^{-1}\Big(\frac{1}{\gamma(2^{-j}\, r)}\Big)\, r^{n-2}\, dr < \infty\,. \tag{31}$$

The expression (31) is equivalent to the summation condition of Theorem 4 if γ does not vanish to infinite order. If γ vanishes to infinite order, the two conditions are still often equivalent, as in the Example 3 above.

Remark. It would be interesting to extend the results of this paper to a more general class of hypersurfaces. For example, one could consider hypersurfaces of the form $S = \{x \in \mathbb{R}^n : \ x_n = \gamma(\phi(x')) + 1\}$ where γ is as above and ϕ is a smooth convex finite type function. Some recent results (see *e.g.* [IoSa97], [IoSaSe98], and [WWZ97]) suggest that such an analysis should be possible. We shall address this issue in a subsequent paper ([I98]). More generally, a bigger challange would be to consider a hypersurface of the form $S = \{x \in \mathbb{R}^n : \ x_n = G(x') + 1\}$, where G is a smooth function of $n-1$ variables that vanishes of infinite order at the origin. At the moment, obtaining sharp Orlicz estimates, even in the case where the determinant of the Hessian matrix of G only vanishes at the origin, does not seem accessible.

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References.

[Bak95] Bak, J.-G., Averages over surfaces with infinitely flat points. *J. Funct. Anal.* **129** (1995), 455-470.

[B86] Bourgain, J., Averages in the plane over convex curves and maximal operators. *J. Analyse Math.* **47** (1986), 69-85.

[CoMa86] Cowling, M., Mauceri, G., Inequalities for some maximal functions II. *Trans. Amer. Math. Soc.* **296** (1986), 341-365.

[Gr82] Greenleaf, A., Principal curvature and harmonic analysis. *Indiana Math. J.* **30** (1982), 519-537.

[I94] Iosevich, A., Maximal operators associated to families of flat curves in the plane. *Duke Math. J.* **76** (1994), 633-644.

[I98] Iosevich, A., Averages over convex infinite type hypersurfaces. Preprint.

[IoSa96] Iosevich, A., Sawyer, E., Oscillatory integrals and maximal averaging operators associated to homogeneous hypersurfaces. *Duke Math. J.* **82** (1996), 103-141.

[IoSa97] Iosevich, A., Sawyer, E., Maximal averages over surfaces. *Advances in Math.* **132** (1997), 46-119.

[IoSaSe98] Iosevich, A., Sawyer, E., Seeger, A., Averages over convex surfaces of finite type in $\mathbb{R}^3$. Preprint, 1998.

[MaRi95] Marletta, G., Ricci, F., Two parameter maximal functions associated with homogeneous surfaces in $\mathbb{R}^n$. *Studia Math.* **130** (1998), 53-65.

[St76] Stein, E. M., Maximal functions: spherical means. *Proc. Nat. Acad. Sci. USA* **73** (1976), 2174-2175.

[St93] Stein, E. M., *Harmonic Analysis.* Princeton Univ. Press, 1993.

[WWZ97] Wright, J., Wainger, S., Ziesler, S., Preprint.

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Topological sectors for Ginzburg-Landau energies

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1. Introduction.

1.1. Ginzburg-Landau functionals.

Let Ω be the annulus $\{x \in \mathbb{R}^2 : 1/4 < |x| < 1\} \subset \mathbb{R}^2$. For maps $u \in H^1(\Omega, \mathbb{R}^2) = W^{1,2}(\Omega, \mathbb{R}^2)$ we consider the Ginzburg-Landau functional

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\,\varepsilon^2} \int_\Omega (1 - |u|^2)^2 \,, \tag{1.1}$$

where ε is a small parameter. For $\Lambda \in \mathbb{R}^+$ we define the energy level set E_ε^Λ as

$$E_\varepsilon^\Lambda := \{u \in H^1(\Omega, \mathbb{R}^2) : \; E_\varepsilon(u) < \Lambda\} \,. \tag{1.2}$$

One of the main purposes of this paper is to show that given $\Lambda > 0$, for ε small enough, E_ε^Λ may be multiply connected. Moreover, the connected components of E_ε^Λ may be classified by the degree of u (since u is not S^1-valued, we have to be careful in order to define its degree – this is the main technical problem of our work).

Functionals like E_ε play an important role in many low temperature physics phenomena like superfluidity. We can also find closely related functionals in the theory of superconductivity and in two-dimensional Higgs models. In our work we will consider one of these superconductivity models: the gauge-covariant Ginzburg-Landau model, where the energy functional may be written as

$$F_\varepsilon(u,A) = \frac{1}{2}\int_{\mathbb{R}^2} |dA|^2 + \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2\,,$$

where $u \in H^1(\Omega,\mathbb{R}^2)$, as before, and $A \in H^1(\mathbb{R}^2,\mathbb{R}^2)$ is the gauge potential one-form,

$$A = A_1\,dx^1 + A_2\,dx^2 \cong \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = (A_1, A_2)\,.$$

Here, as we will often do in this paper, we used the natural identification (given by the $\mathbb{R}^2$ scalar product) between the one-form A and the vector with the same components which we also denote by A. In equation (1.3) the expression $\nabla_A u$ denotes the covariant derivative of u, *i.e.* $\nabla_A u = \nabla u - \imath A u$.

This model was introduced by Ginzburg and Landau in the 50's for the study of phase transitions in superconducting materials (see the remarks on physics below).

The main feature of the functional F_ε is its invariance under gauge transformations. For a function $\phi \in W^{2,2}(\mathbb{R}^2,\mathbb{R})$, the gauge transformation associated to ϕ is the map $(u,A) \longmapsto (u_\phi, A_\phi)$ given by

$$\begin{cases} u_\phi = \exp(\imath\,\phi)\,u\,, & \text{in } \Omega\,, \\ A_\phi = A + d\phi\,, & \text{in } \mathbb{R}^2\,. \end{cases} \tag{1.4}$$

In this case we say that (u,A) is gauge-equivalent to (u_ϕ, A_ϕ) and we denote this by $(u,A) \sim (u_\phi, A_\phi)$. Saying that F_ε is gauge-invariant means that

$$F_\varepsilon(u_\phi, A_\phi) = F_\varepsilon(u,A)\,, \qquad \text{for all } \phi \in W^{2,2}(\mathbb{R}^2,\mathbb{R})\,. \tag{1.5}$$

This gauge-invariance follows easily from the facts that

$$(u_\phi, A_\phi) \in H^1(\Omega,\mathbb{R}^2) \times H^1(\mathbb{R}^2,\mathbb{R}^2)\,, \qquad |u_\phi| = |u|\,,$$

$$dA_\phi = dA + d\,d\phi = dA\,, \tag{1.6}$$

$$\nabla_{A_\phi} u_\phi = \exp(\imath\,\phi)\,\nabla_A u\,, \text{ and thus } |\nabla_{A_\phi} u_\phi| = |\nabla_A u|\,. \tag{1.7}$$

The only quantities which are significant from the physics point of view are those, like $|u|$, $\nabla_A u$ and the magnetic field $h = \star dA$, which are invariant under gauge transformations. Other important gauge-invariant quantities are the current $J = (\imath\, u, \nabla_A u)$ and, the one which we are more concerned about in this paper, the degree of u along a smooth closed curve γ, diffeomorphic to S^1, such that $|u| \neq 0$ on γ. In integral form, this degree is given by

$$\deg(u,\gamma) = \frac{1}{2\pi} \int_\gamma \frac{u}{|u|} \times \partial_\tau \Big(\frac{u}{|u|} \Big)\, d\tau\,, \tag{1.8}$$

where τ denotes the unit tangent to γ.

It is easy to see that gauge-equivalence defines an equivalence relation in $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$. A physical state of our system is associated not with an individual configuration (u, A), but with a whole equivalence class $[u, A] := \{(v, B) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2) : (v, B) \sim (u, A)\}$. We denote the physical space by $H_{gi} = [H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)]/\sim$, and also consider F_ε as a functional defined on H_{gi}.

As in the case of E_ε, we define the energy level sets of F_ε by

$$F_\varepsilon^\Lambda := \{[v, B] \in H_{gi} :\ F_\varepsilon([v, B]) < \Lambda\}\,.$$

Since the functional E_ε does not involve the connection, it is a little easier to deal with than the functional F_ε. Nevertheless, as we will see in our work, most of the mathematical difficulties are already encountered in the study of E_ε. In fact, after some additional technical arguments, we deduce the classification result for the components of the level sets of F_ε, from the corresponding result for E_ε. Therefore, we start by considering the functional E_ε given by (1.1).

1.2. Degree of a map and definition of topological sectors.

We consider a fixed number $\Lambda > 0$, and focus our attention on the level set E_ε^Λ defined by (1.2). First, we remark that since the notion of degree we define is continuous in $W^{1,2}(\Omega) \cap E_\varepsilon^\Lambda$ and that smooth maps are dense in $W^{1,2}(\Omega) = H^1(\Omega)$, it suffices to consider the case where $u \in W^{1,2}(\Omega) \cap C^\infty$. Hence, without loss of generality, we will always assume that u is smooth in this paper.

Based on the work of B. White [28] (see also the work of F. Bethuel [6]), for maps $u \in W^{1,2}(\Omega, S^1)$, *i.e.* for the case when $|u| \equiv 1$, we can define the degree of u in Ω, $\deg(u, \Omega)$, as the degree of the restriction of u to a one-dimensional skeleton of Ω – for instance, in case u is continuous, this can be any circle $S_r = \{x : |x| = r\}$, for $1/4 < r < 1$

(if u is not continuous we might need to move the circle slightly in order to have a "nice" restriction). The degree can then be written, in integral form, as

$$\deg(u,\Omega) = \deg(u,S_r) = \frac{1}{2\pi}\int_{S_r} \frac{u}{|u|} \times \partial_\tau\Big(\frac{u}{|u|}\Big)\,d\tau\,. \tag{1.9}$$

This definition of the degree will always give us an integer, and it classifies the homotopy classes of $W^{1,2}(\Omega,S^1)$. Our purpose is to extend this notion to all $u \in E_\varepsilon^\Lambda$ for ε sufficiently small. In this context, our first result is given by the following Theorem.

Theorem 1. *Given $\Lambda \in \mathbb{R}^+$, there exists $\varepsilon_0 > 0$, depending only on Λ, such that for $\varepsilon < \varepsilon_0$, we can define a continuous map*

$$\begin{aligned} \chi : E_\varepsilon^\Lambda &\longrightarrow \mathbb{Z}\,,\\ u &\longmapsto \deg(u,\Omega)\,, \end{aligned} \tag{1.10}$$

such that this map coincides with the classical notion of degree mentioned above when u has values in S^1 (i.e. when $u \in W^{1,2}(\Omega,S^1)\cap E_\varepsilon^\Lambda$).

Usually we call the map χ the global degree in Ω and, as above, we denote $\chi(u) = \deg(u,\Omega)$. For each $n \in \mathbb{Z}$, $\chi^{-1}(n) = \{u \in E_\varepsilon^\Lambda : \deg(u,\Omega) = n\}$, is an open and closed subset of E_ε^Λ which we call the n^{th} topological sector of E_ε^Λ, and we also denote it by $\mathrm{top}_n(E_\varepsilon^\Lambda)$.

REMARK. In fact, what we prove in Theorem 1 is that the degree of u is constant inside each connected component of E_ε^Λ – we do not show that different connected components correspond to different values of the degree, which would give us a complete classification of the components by the degree of its members. We will come back to this question later on.

The asymptotic behavior, when $\varepsilon \longrightarrow 0$ of critical points of the functionals E_ε and F_ε was extensively studied by many authors. Among them we would like to single out the work of F. Bethuel, H. Brezis and F. Hélein [8] regarding the functional E_ε, and those of F. Bethuel and T. Rivière [9] and [10] which concern the functional F_ε.

We will give a rough description of the proof of Theorem 1 at the end of the Introduction. This proof is rather technical and will be done in sections 2 to 8. The Euler-Lagrange equations for the functional E_ε are called the Ginzburg-Landau equations. They can be written as

$$-\Delta u = \frac{1}{\varepsilon^2}\,u\,(1-|u|^2)\,, \qquad \text{in } \Omega\,. \tag{1.11}$$

In the context of the gauge invariant model, we can also extend the definition of degree to any configuration $[v, B] \in F_\varepsilon^\Lambda$ provided ε is small enough. In fact, we prove

Theorem 2. *Given $\Lambda \in \mathbb{R}^+$, there exists $\varepsilon_0 > 0$, depending only on Λ, such that for $\varepsilon < \varepsilon_0$, we can define a continuous map*

$$\begin{aligned} \hat{\chi} : F_\varepsilon^\Lambda &\longrightarrow \mathbb{Z}\,, \\ [u, A] &\longmapsto \deg\left([u, A], \Omega\right), \end{aligned} \tag{1.12}$$

such that this map coincides with the classical notion of degree mentioned above when u has values in S^1 (i.e. when $u \in W^{1,2}(\Omega, S^1) \cap F_\varepsilon^\Lambda$). Usually we call the map $\hat{\chi}$ the global degree in Ω and, as above, we denote $\hat{\chi}(u, A) = \deg\left([u, A], \Omega\right)$.

Minimizing E_ε inside each component of E_ε^Λ (or F_ε inside each component of F_ε^Λ), we will obtain solutions of (1.11) which are locally minimizing, *i.e.* critical points of E_ε (respectively, F_ε) which are local minima. These are the solutions that should be associated with permanent currents.

Moreover, we will show in the next subsection, that as a corollary of Theorems 1 and 2, we can also prove the existence of mountain-pass points for E_ε (which correspond to mountain-pass type solutions of (1.11)). An analogous reasoning gives the existence of mountain-pass points for F_ε. This result is stated in Theorem 4. Unlike the solutions obtained minimizing the energy inside each topological sector, the solutions of (1.11) we obtain in Theorem 4 will not necessarily be local minimizers of E_ε, and are probably unstable.

1.3. Mountain-pass solutions and threshold energies.

We start by the crucial, although elementary, remark that when $\Lambda = \infty$, we have that $E_\varepsilon^\infty = H^1(\Omega)$, *i.e.* the whole affine space $H^1(\Omega, \mathbb{R}^2)$. This space has obviously an unique component and furthermore, given any two elements $u_0, u_1 \in H^1(\Omega, \mathbb{R}^2)$ there is a natural path between them: the straight line segment $\gamma : [0,1] \longrightarrow H^1(\Omega, \mathbb{R}^2)$, defined by

$$\gamma(s) := (1 - s)\, u_0 + s\, u_1\,, \qquad \text{for } s \in [0,1]\,. \tag{1.13}$$

Likewise, $F_\varepsilon^\infty = H_{gi}$, which is the projection (continuous image) of $H^1 \times H^1$, and thus is connected. Given two states $[u_0], [u_1] \in H_{gi}$ we may consider the straight line between two of their representatives, $u_0, u_1 \in H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$ and consider the projection in H_{gi} of the straight line in $H^1 \times H^1$ between u_0 and u_1.

An important example of a map of degree $n \in \mathbb{Z}$, in $H^1(\Omega, S^1) \subset H^1(\Omega, \mathbb{R}^2)$ (and for which we can thus use the classical definition of the degree), is the map

$$w_n(r,\theta) := \exp(\imath\, n\, \theta) = \frac{z^n}{|z|^n}\,. \tag{1.14}$$

Using (1.9) it is easy to check that $\deg(w_n, \Omega) = n$ and moreover, we can see that the energy, $E_\varepsilon(w_n)$, of the maps w_n, $n \in \mathbb{Z}$, is independent of ε and is given by

$$E_\varepsilon(w_n) = \frac{1}{2}\int_\Omega |\nabla w_n|^2 = \frac{1}{2}\int_{1/4}^1 r \int_0^{2\pi} \frac{n^2}{r^2}\, d\theta\, dr = \pi\, n^2\, \log 4\,. \tag{1.15}$$

Hence, given $\Lambda \in \mathbb{R}^+$, let

$$n_0 := \left[\sqrt{\frac{\Lambda}{\pi \log 4}}\,\right],$$

be the largest integer less than or equal to $\sqrt{\Lambda/(6\pi \log 4)}$. From equation (1.15) it follows that, at least for $n \in [-n_0, \dots, n_0]$, the topological sector $\mathrm{top}_n(E_\varepsilon^\Lambda)$ will be non-empty, and this independently of the value of $\varepsilon > 0$.

Likewise, for F_ε we could take $w_n(r,\theta) := [(\exp(\imath\, n\, \theta), 0)]$. All the rest of the discussion also easily extends to the case of F_ε.

Let $\Lambda \in \mathbb{R}^+$ be given, and let $\varepsilon < \varepsilon_0$ (where ε_0 is as in Theorem 1). Suppose that for some $n \in \mathbb{Z}$ both $\mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\mathrm{top}_{n+1}(E_\varepsilon^\Lambda)$ are non-empty, and consider two maps

$$u_0 \in \mathrm{top}_n(E_\varepsilon^\Lambda)\,, \qquad u_1 \in \mathrm{top}_{n+1}(E_\varepsilon^\Lambda)\,.$$

Let $\gamma : [0,1] \longrightarrow H^1(\Omega)$ be a path between u_0 and u_1 (*i.e.* $\gamma(0) = u_0$ and $\gamma(1) = u_1$). Recall that, as we mentioned above, such a path always exists because $H^1(\Omega, \mathbb{R}^2)$ is an affine space. Then, γ cannot be entirely contained in E_ε^Λ – if this were so, u_0 and u_1 would be in the same path component of E_ε^Λ, and hence also in the same component of E_ε^Λ

which contradicts our assumption (since, by Theorem 1, the topological sectors $\mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\mathrm{top}_{n+1}(E_\varepsilon^\Lambda)$ are disjoint open and closed subsets of the energy level set E_ε^Λ). Hence, there exists some $s \in (0,1)$ such that $\gamma(s) \notin E_\varepsilon^\Lambda$, which is equivalent to saying that $E_\varepsilon(\gamma(s)) \geq \Lambda$. A standard Min-Max argument will then yield the existence of generalized critical values of E_ε of the form

$$c_n := \inf_{\gamma \in \mathcal{V}} \max_{s \in [0,1]} E_\varepsilon(\gamma(s))\,. \tag{1.16}$$

where $\mathcal{V} := \{\gamma \in C^0([0,1], H^1(\Omega, \mathbb{R}^2)) : \gamma(0) = u_0,\ \text{and}\ \gamma(1) = u_1\}$, is the space of continuous paths in $H^1(\Omega)$ between u_0 and u_1. The value c_n will be a generalized critical value of E_ε. To make sure it is actually a critical value we use the following

Theorem 3. *The functionals E_ε and F_ε satisfy the Palais-Smale condition (in $H^1(\Omega, \mathbb{R}^2)$ and H_{gi}, respectively).*

This implies that c_n is a critical value of E_ε and hence, there exists a map $u \in H^1(\Omega)$ such that u is a critical point of E_ε and $E_\varepsilon(u) = c_n$. This u is probably not a local minimum of E_ε. All this discussion extends to the case of F_ε. Thus, we have proved

Theorem 4. *Suppose that for some $\Lambda \in \mathbb{R}^+$, we have that for some $\varepsilon < \varepsilon_0$ (where ε_0 is given Theorem 1) there exists $n \in \mathbb{Z}$ such that the topological sectors $\mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\mathrm{top}_{n+1}(E_\varepsilon^\Lambda)$ are both non-empty. Then, there are mountain-pass type critical points of E_ε or, equivalently, there exist mountain-pass type solutions of the Ginzburg-Landau equations (1.11).*

More precisely, consider two maps

$$u_0 \in \mathrm{top}_n(E_\varepsilon^\Lambda) \qquad \textit{and} \qquad u_1 \in \mathrm{top}_{n+1}(E_\varepsilon^\Lambda)\,,$$

and let c_n be defined as in (1.16). Then, there exists a map $u \in H^1(\Omega, \mathbb{R}^2)$ such that u is a critical point of E_ε and $E_\varepsilon(u) = c_n$.

Likewise, if we consider two states $\Phi_0 \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ and $\Phi_1 \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$, and let c_n be defined by

$$c_n := \inf_{\gamma \in \mathcal{V}} \max_{s \in [0,1]} F_\varepsilon(\gamma(s))\,, \tag{1.17}$$

where now $\mathcal{V} := \{\gamma \in C^0([0,1], H_{gi}) : \gamma(0) = \Phi_0,\ \text{and}\ \gamma(1) = \Phi_1\}$, is the space of continuous paths in H_{gi} between Φ_0 and Φ_1. Then, there *exists a state* $\Phi = [(u, A)] \in H_{gi}$ *such that* Φ *is a critical point of* F_ε *and* $F_\varepsilon(\Phi) = c_n$.

Remark 1. The number c_n defined by (1.16) is called the *threshold energy* for the transition from the state u_0 to the state u_1. It will be the infimum of the energies for which such a transition is possible. This concept will play a crucial role in the physical behavior of our system. We will come back to this point in the remarks on physics (see below).

Remark 2. In Theorem 4, for simplicity, we just considered transitions from a state $u_0 \in \mathrm{top}_n(E_\varepsilon^\Lambda)$ to a state u_1 belonging to the adjacent state $\mathrm{top}_{n+1}(E_\varepsilon^\Lambda)$. However, both the concept of threshold energy and the result stated in Theorem 4 are immediately generalizable to the case where $u_0 \in \mathrm{top}_n(E_\varepsilon^\Lambda)$ and $u_1 \in \mathrm{top}_k(E_\varepsilon^\Lambda)$, for any two distinct integers $n, k \in \mathbb{Z}$. As usual, this remark and the previous one extend to the setting of the gauge-covariant functional F_ε.

Remark 3. All these results extend to the setting of more general domains considered in Theorem 6, stated below.

1.4. Remarks on physics.

1.4.1. Ginzburg-Landau theory.

In the Ginzburg-Landau theory of superconductivity, the conducting electrons are described as a fluid existing in two phases, the superconducting one and the normal one. In the superconducting state the material has an infinite electrical conductivity and magnetic fields are repelled from the interior of the sample (this is the so called Meissner effect).

On a microscopic scale, the superconducting state is described by the theory of Bardeen, Cooper and Schrieffer (BCS). In this theory, the existence of superconductivity is due to a pairing of the conducting electrons forming the so called Cooper pairs. For small applied forces, these pairs behave as a single particle (a boson) of twice the charge of the electron. At a macroscopic scale the behavior of the Cooper pairs is described by a complex-valued function u, called the condensate wave function (or order parameter). The density $|u(x)|^2$ is proportional to the density of pairs of superconducting electrons.

The Ginzburg-Landau model is a phenomenological model which extends Landau's theory of second order phase transitions. It was proposed well before the microscopic theory (BCS) existed, but it can be obtained as an approximation to the macroscopic consequences of this theory. This model gives us a system of equations which describe the interaction between the condensate wave function, u, and the electromagnetic vector potential, A. In this model the parameter $\kappa = \varepsilon^{-1}$ (which depends on the material we consider and on the temperature) plays a crucial role in determining the behavior of our system.

If $\kappa < 1/\sqrt{2}$, the material is called a type I superconductor. If one applies an exterior magnetic field to the sample, then there is a critical value, H_c, such that when the applied magnetic field H increases beyond H_c, the sample passes suddenly from the superconducting phase to the normal phase. On the other hand, if $\kappa \geq 1/\sqrt{2}$, the behavior is quite different and the transition between the superconducting and the normal phase is done gradually. These materials are called type II superconductors and they are characterized by two critical values of the applied magnetic field: the first, H_{c1}, corresponds to the critical field above which the two phases coexist, and the second, H_{c2}, corresponds to the critical field above which all the sample will be in the normal phase. Between these two critical values the normal and superconducting phase will coexist: the normal state will be confined in vortices or filaments whose number will increase as the applied field increases. The flux lines of the magnetic field inside the material will be concentrated inside these vortices (since they are repelled by the part of the sample that is in the superconducting phase). For a detailed description of the physics involved in the phenomena of superconductivity and superfluidity see, for instance, the works of D. Saint-James, G. Sarma and E. J. Thomas [26], and of D. Tilley and T. Tilley [27]. For a more mathematical approach see the work of A. Jaffe and C. Taubes [20].

1.4.2. Permanent currents.

A very interesting phenomenon in superconductivity, that motivates our work, is the existence of permanent currents in a superconducting ring. The experiment is the following: a ring of superconducting material in the normal state is submitted to a fixed external magnetic field (subcritical), and then the temperature of the system is decreased until temperatures below the critical temperature corresponding to the applied field are attained. The applied field is then turned off and there is a current that persists inside the superconducting ring. Furthermore it was observed that such a current does not dissipate with time – there were experiments where the current persisted for several years without any dissipation, thus the name permanent current.

This behavior of the system indicates that we should be in presence of an energy functional having multiple wells (local minima) separated by very high barriers. The main purpose of our work is to show that even in the simple models considered in this paper, the energy functionals E_ε and F_ε have this type of structure.

The big height of the barriers would be associated to the "permanent" character of these currents. In fact, considering the possibility of the system tunneling through the barrier, thus moving from one energy well into another (and eventually to the ground state), the associated probability should be proportional to $\exp(-h)$, where h is the height of the barrier relative to the initial state of the system. Thus, having very high barriers will yield transition probabilities close to zero and therefore justify the "permanent" character of our currents.

1.4.3. Transitions between states and threshold energies.

The natural question is then to describe the transitions between two different sectors – thus, the notion of threshold energy for such transitions (defined in equation (1.16)) is a crucial one for the physical behavior of our system. We remark that in the setting of the gauge-invariant model, as we mentioned before, physical states of the system are represented by gauge-equivalence classes (defined by (1.4)) of configurations of our system – thus the configuration (u, A) is just a particular representative of the state $[u, A]$. Therefore, we shouldn't consider paths between configurations in the space $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$, but paths between states in the quotient space of $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$ by the gauge-equivalence relation, which we denote by H_{gi} (this is the physical space).

The threshold energy c_n for a transition between a state $[u_0, A_0] \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ and a state $[u_1, A_1] \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$ will be of the order of $|\log \varepsilon|$. It is easy to see that it is at most of this order. Indeed, we can prove the following upper bound for the transition energy.

Theorem 5. *Let c_n be the threshold energy for the transition between the state $[u_0, A_0] \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ and the state $[u_1, A_1] \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$, defined as in* (1.16). *Then,*

$$c_n \leq M_n |\log \varepsilon| + L_n \,, \tag{1.18}$$

where M_n and L_n are constants that depend only on n and our domain Ω.

We will give an intuitive proof of Theorem 5. Let $\Lambda > \pi \log(4)\,(n+1)^2$ and suppose that we want to describe a path from the configuration $(u_n, A_n) = (\exp(\imath\, n\, \theta), 0) \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ to the configuration $(u_{n+1}, A_{n+1}) = (\exp(\imath\,(n+1)\,\theta), 0) \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$. We remark that once we construct a path in the space $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$ between (u_n, A_n) and (u_{n+1}, A_{n+1}), we can obtain a path between the corresponding physical states $[u_n, A_n]$ and $[u_{n+1}, A_{n+1}]$ in the quotient space H_{gi} by projecting the original path. The general case of a transition between $(v_0, B_0) \in \mathrm{top}_n(F_\varepsilon^\Lambda)$ and $(v_1, B_1) \in \mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$ can be proved in a similar way.

Physically, the path we construct corresponds to bringing a positive unit charge of size ε from a point P arbitrarily close to infinity, to the origin. By a positive unit charge of size ε at a point $z_s \in \mathbb{C}$, we mean the map

$$f_{z_s}(z) = \frac{z - z_s}{|z - z_s|}\, \varphi_\varepsilon(z - z_s)\,, \tag{1.19}$$

where $\varphi_\varepsilon(\cdot) = \varphi(\cdot/\varepsilon)$, and $\varphi \in C_0^\infty(\mathbb{R}^2)$ is such that

$$\begin{cases} \varphi(x) = 0\,, & \text{if } |x| < 1\,, \\ \varphi(x) = 1\,, & \text{if } |x| > 2\,, \\ 0 \leq \varphi(x) \leq 1\,, & \text{for all } x\,, \\ |\nabla \varphi(x)| \leq 2\,, & \text{for all } x\,. \end{cases} \tag{1.20}$$

Hence f_{z_s} is a unit vortex at z_s which is "smoothened out" in a ball of radius $2\,\varepsilon$ around z_s. Then,

$$F_\varepsilon(f_{z_s}, 0) \leq C_1\, |\log \varepsilon| + C_2\,, \tag{1.21}$$

where C_1 and C_2 are constants.

Let $M \in \mathbb{R}^+$ be an arbitrarily big number, and let $z_s = (1 - s)\,(-M) \in \mathbb{C}$, for $s \in [0,1]$. This will be a path from the point $(-M)$ in the negative real axis, to the origin. Using z_s we construct the path in $H^1(\Omega, \mathbb{R}^2) \times H^1(\mathbb{R}^2, \mathbb{R}^2)$ defined by

$$(v_s, B_s) := (f_{z_s} u_n, 0)\,, \qquad \text{for } s \in [0,1]\,.$$

We can check that (v_0, B_0) is arbitrarily close in $H^1(\Omega)$ norm to (u_n, A_n) – in fact, we would obtain the configuration (u_n, A_n) if we chose $M = +\infty$. Hence, in particular, for big values of M, we certainly have $(v_0, B_0) \in \mathrm{top}_n(F_\varepsilon^\Lambda)$. Furthermore, $(v_1, B_1) = (u_{n+1}, A_{n+1})$ and we can obtain estimate (1.18) as a consequence of the bound (1.21).

Hence we see that the path corresponding to passing a positive unit charge "of size ε" from the outside of our annulus, to the hole inside the annulus, corresponds to increasing by one the degree of our map and requires that we go to an energy level of order $|\log \varepsilon|$. To prove that any transition between $\mathrm{top}_n(F_\varepsilon^\Lambda)$ and $\mathrm{top}_{n+1}(F_\varepsilon^\Lambda)$ also requires passing through energy levels of order $|\log \varepsilon|$, thus proving that c_n is of order $|\log \varepsilon|$, is a very delicate problem. We will show a way to solve this problem and obtain very precise estimates for the threshold energies in a forthcoming work ([1]).

1.5. The case of more general domains.

In Theorem 1 we considered a very particular domain – the annulus $\Omega = \{x \in \mathbb{R}^2 : 1/4 < |x| < 1\}$. However, once we have the result for the annulus, it is not hard to extend it to the case of a general open subset $D \subset \mathbb{R}^2$, or even the case of a domain in a Riemannian manifold $\mathcal{M}$. We define the energy functional just as in (1.1) but replacing Ω by our new domain D,

$$E_\varepsilon(u, D) = \frac{1}{2} \int_D |\nabla u|^2 + \frac{1}{4\,\varepsilon^2} \int_D (1 - |u|^2)^2\,, \tag{1.22}$$

and we define the corresponding level sets

$$E_\varepsilon^\Lambda(D) := \{u \in H^1(D, \mathbb{R}^2) : \ E_\varepsilon(u, D) < \Lambda\}\,.$$

We start by fixing a set of representatives of generators of $\pi_1(D)$ (the first homotopy group of D), $\{\gamma_j, j \in J\}$, such that each $\gamma_j : S^1 \longrightarrow D$, is an injective closed smooth curve inside our open set D. Hence, γ_j will have a tubular neighborhood $\Gamma_j \subset \Omega$. We may suppose that for each j there is a positive number, $\delta_j > 0$, such that for each j

i) $\Gamma_j = \{x \in D : \ \mathrm{dist}\,(x, \gamma_j) < \delta_j\}$.

ii) There is a diffeomorphism

$$\Phi_j : \Gamma_j \longrightarrow S^1 \times (0,1)\,,$$

such that $\gamma_j(\theta) = \Phi_j^{-1}(\theta, 1/2)$, and the Jacobian of Φ_j is uniformly bounded from above and away from zero, *i.e.* there is a constant $C_j > 0$ such that

$$\frac{1}{C_j} < |\nabla \Phi_j(x)| < C_j\,, \qquad \text{for all } x \in \Gamma_j\,. \tag{1.24}$$

Let $\hat{\Omega} := S^1 \times (1/4, 3/4)$. This set is topologically an annulus just like our standard set Ω considered before. Let $Y_j := \Phi_j^{-1}(\hat{\Omega})$. Given a map $u \in E_\varepsilon^\Lambda(D)$ we consider the map $w_j = u \circ \Phi_j^{-1} : \hat{\Omega} \longrightarrow \mathbb{R}^2$. The map w_j belongs to $E_\varepsilon^\varrho(\hat{\Omega})$, where ϱ is a constant that depends only on Λ and the constant C_j in (1.24). Thus, we can apply Theorem 1 replacing Ω and Λ by $\hat{\Omega}$ and ϱ, respectively. Hence for ε sufficiently small $\deg\,(w_j, \hat{\Omega})$ is well defined. We set, for each $j \in J$,

$$\deg\,(u, Y_j) := \deg\,(w_j, \hat{\Omega})\,. \tag{1.25}$$

Suppose that the index set J is finite ($J = \{1, \ldots, m\}$), *i.e.* suppose that we fix a finite number of (representatives of) generators of $\pi_1(D)$. We define the topological type of $u \in E_\varepsilon^\Lambda(D)$ as the m-tuple of integers

$$\chi(u) := (\deg\,(u, Y_1), \ldots, \deg\,(u, Y_m))\,. \tag{1.26}$$

By the previous argument, this $\chi(u) \in \mathbb{Z}^m$ is well defined for sufficiently small ε. The continuity of χ in $W^{1,2}(D, \mathbb{R}^2)$ topology inside $E_\varepsilon^\Lambda(D)$ (which is an immediate consequence of the continuity of $\deg\,(u, \Omega)$ proved in section 7) will then allow us to assert that, since $\mathbb{Z}^m$ is discrete, for each $P \in \mathbb{Z}^m$, its inverse image by χ, *i.e.* $\chi^{-1}(P) = \{u \in E_\varepsilon^\Lambda(D) : \chi(u) = P\}$, will be an open and closed subset of $E_\varepsilon^\Lambda(D)$. For each $P \in \mathbb{Z}^m$, we call $\chi^{-1}(P)$ the P-topological sector of $E_\varepsilon^\Lambda(D)$. We have thus proved the following Theorem which extends the classification given by Theorem 1 to this more general setting.

Theorem 6. *Let D be an open subset of $\mathbb{R}^2$ or a domain in a Riemann manifold $\mathcal{M}$. Let $\gamma_1, \ldots, \gamma_m$ be simple, closed and smooth curves which are a set of representatives of generators of $\pi_1(D)$. Given $\Lambda > 0$ there exists $\varepsilon_0 > 0$, depending on Λ, such that for $\varepsilon < \varepsilon_0$ we can define a continuous map*

$$\begin{aligned} \chi : E_\varepsilon^\Lambda(D) &\longrightarrow \mathbb{Z}^m, \\ u &\longmapsto (\deg(u, Y_1), \ldots, \deg(u, Y_m)), \end{aligned} \tag{1.27}$$

such that for the special case where $u \in E_\varepsilon^\Lambda(D) \cap W^{1,2}(D, S^1)$, we recover the classical notion of degree of a S^1 valued map. Therefore, given $P = (P_1, \ldots, P_m) \in \mathbb{Z}^m$, the subset $\chi^{-1}(P) \subset E_\varepsilon^\Lambda(D)$ will be an open and closed subset of $E_\varepsilon^\Lambda(D)$.

The same argument in the context of the superconductivity model will give a similar extension of Theorem 2.

1.6. Idea of the proof of Theorem 1.

The maps $u \in E_\varepsilon^\Lambda$ may take values close to zero, which creates big technical problems for defining their degree. However, this can only happen in a set of small measure. We will start by studying, in sections 2, 3 and 4 the set $G(\zeta)$ where $|u|$ is smaller than an appropriately chosen $\zeta \in (1/2, 3/4)$. For technical reasons (to avoid problems that may appear near the boundary $\partial\Omega$) we will concentrate on the components of $G(\zeta)$ that intersect an interior annulus

$$Y := \left\{ x \in \mathbb{R}^2 : \frac{1}{2} < |x| < \frac{3}{4} \right\}.$$

Using Sard's Lemma we will see that for sufficiently small ε, these components of G may be included in a finite number of simply-connected sets, which we denote by $W_k, k = 1, \ldots, \tilde{N}$. Their boundaries will be closed smooth curves, $V_k = \partial W_k$, and $|u| = \zeta$ on each of the V_k's.

In Section 2 we see, using the coarea formula, that the sum of the lengths of the V_k's will tend to zero when $\varepsilon \longrightarrow 0$. Furthermore, the coarea formula also gives us a bound on the L^1 norm of ∇u on $V = \bigcup V_k$. Since $|u| = \zeta > 1/2$ on V_k, it makes sense to talk about $\deg(u, V_k)$.

In Section 3 we the obtain an uniform bound on $\sum |\deg(u, V_k)|$ using the estimate for $\|\nabla u\|_{L^1(V)}$ (and consequently we will also have uniform bounds on $|\deg(u, V_k)|$ for each k). Thus, we see that for all $u \in E_\varepsilon^\Lambda$ the number of V_k's such that $\deg(u, V_k) \neq 0$ (which we call the "charged" V_k's) is uniformly bounded by a constant depending only on Λ. Suppose that the charged V_k's are $V_1, \dots, V_{N_2}$.

In Section 4 we will focus our attention on the "uncharged" V_k's (*i.e.* those for which $\deg(u, V_k) = 0$). We will see, again using the estimate for $\|\nabla u\|_{L^1(V)}$ obtained in Section 2, that the number of "uncharged" V_k's such that the oscillation of u is bigger than or equal to $\pi/3$, is also uniformly bounded. Suppose they are $V_{N_2+1}, \dots, V_N$. Moreover, for the remaining V_k's, *i.e.* the "uncharged" ones such that the oscillation of u is smaller than $\pi/3$ (which will be $V_{N+1}, \dots, V_{\tilde{N}}$), we are able to prove that the energy minimizing extension to W_k of $u_{|V_k}$ will have absolute value which is uniformly bounded away from zero – hence we will show that these sets are rather "harmless".

In Section 5, thanks to the uniform bound on N (the number of "charged" V_k's plus that of "uncharged" V_k's such that the oscillation of u is bigger than or equal to $\pi/3$), we can cover $V_1, \dots, V_N$ by a finite (uniformly bounded) number of balls, $B_1, \dots, B_m$, of radius of order at most ε^α for some $\alpha > 1/2$, and which are far away from each other (in the sense that suitable dilations of the B_i's are pairwise disjoint). Furthermore, we will see that $\deg(u, \partial B_i) = 0$, for all i. This means that though we may have individual singularities that are charged, at a scale of order $\varepsilon^{1/2}$ they cluster to form neutral structures.

In Section 6 we will finally give the good definition of the global degree of u in Ω, $\deg(u, \Omega)$. Let

$$T := \left\{ r \in \left(\frac{1}{2}, \frac{3}{4}\right) \text{ such that } S_r \cap G(\zeta) \neq \varnothing \right\},$$

and let

$$A := \left(\frac{1}{2}, \frac{3}{4}\right) \setminus T\,.$$

We show that $|T| \longrightarrow 0$, when $\varepsilon \longrightarrow 0$, and hence $|A| \longrightarrow 1/4$, when $\varepsilon \longrightarrow 0$. For $r \in A$ we define

$$f(r) := \deg(u, S_r) = \deg\Big(\frac{u}{|u|}, S_r\Big) \in \mathbb{Z}\,. \tag{1.28}$$

This function is well defined since for $r \in A, |u(r, \theta)| \geq \zeta$. As we mentioned before, for $u \in W^{1,2}(\Omega, S^1)$ this function is constant. In our case this might not be true, but by the results of Section 5, it cannot change too much: as a matter of fact, for ε sufficiently small, the value of f can only change when S_r intersects one of the balls B_i, and even when this occurs, the absolute value of f remains bounded by a constant that depends only on Λ. Outside these balls (*i.e.* when $S_r \cap B = \varnothing$, where $B := \bigcup B_i$) $f(r)$ will always have the same value (since $\deg(u, \partial B_i) = 0$). This is the value we use to define $\deg(u, \Omega)$, which will thus automatically be an integer. To recover this integer we can also integrate $f(r)$ over A and divide by the measure of A, thus defining

$$\widetilde{\text{adeg}}\,(u,\Omega) := \frac{1}{|A|}\int_A f(r)\,dr\,. \tag{1.29}$$

This quantity, $\widetilde{\text{adeg}}\,(u,\Omega)$, is called the approximate degree of u in Ω. In general, it is not an integer, but it will tend to the integer $\deg(u,\Omega)$ as $\varepsilon \longrightarrow 0$. In fact, let $Q = A \cap B = \bigcup (A \cap B_i)$. The measure of Q tends to zero when $\varepsilon \longrightarrow 0$ (it is bounded by $|B|$ which, in turn, is at most, of order $\varepsilon^\alpha < \varepsilon^{1/2}$). Furthermore, f remains uniformly bounded even inside Q, and hence, we can see that

$$|\widetilde{\text{adeg}}\,(u,\Omega) - \deg(u,\Omega)| < \frac{1}{4}\,, \tag{1.30}$$

for sufficiently small ε. Thus we can recover the integer $\deg(u,\Omega)$ as the closest integer to $\widetilde{\text{adeg}}\,(u,\Omega)$ for ε small.

In Section 7 we will prove, for sufficiently small ε, the continuity of $\widetilde{\text{adeg}}\,(u,\Omega)$ (and thus also of $\deg(u,\Omega)$) in $W^{1,2}(\Omega)$ norm, inside the level set E_ε^Λ we fixed. Using this continuity we will then conclude the proof of Theorem 1 in Section 5.

Finally, in the Appendix (Section 12) we prove a general covering Lemma of which we used a special case to obtain the balls B_i in Section 5.

1.7. Open questions and related results.

As we saw, many questions about this subject remain open, in particular in the borderline between the mathematics and the physical behavior of these systems, a considerable amount of work remains to be done. In this subsection we will discuss some of these problems shortly and mention some results of related interest. We start by mentioning a few problems we are working on at the moment.

In [1] we are able to carry out a more detailed study of the properties of the threshold energies we introduced above. In particular, using some techniques introduced by F. Bethuel and the author in [4], we can prove a more accurate version of the upper bound for the threshold energy c_n stated in Theorem 5. More precisely, we show that there exists a constant α_n, not depending on ε, such that $c_n \leq \pi\,|\log\varepsilon| + \alpha_n$.

This estimate is crucial to succeeding in obtaining (see [1]) a lower bound for c_n which is of the same order of the above, i.e. to showing that $c_n \geq \pi\,|\log\varepsilon| - \alpha_n$. Such a bound, as we mentioned, implies that the energy barriers have a height of at least $\pi\,|\log\varepsilon| - \alpha_n$, and therefore, since ε is supposed to be small, we will have very high barriers separating the wells. This agrees with what we expected considering the physical behavior of our system, as we described above.

Regarding the extension of our results to the 3-dimensional case, there is a substantial part we are able to do, but there are still some technical difficulties (which stem from the higher degree of liberty of the equivalent of the V_k's, which, in this setting, will be two-dimensional surfaces). Once we succeed in defining the degree, we can obtain mountain-pass solutions just as for the dimension 2, but proving that the threshold energy, c_n, is of order $|\log\varepsilon|$ should be considerably harder (for results on the structure of the singularities of the Abelian Higgs model in $\mathbb{R}^3$, see the works of T. Rivière [23] and [24]).

Our work was also motivated by the paper of S. Jimbo and Y. Morita [16]. In [16] the authors establish the existence of stable nontrivial solutions for the Ginzburg-Landau equations in the case the domain $\Omega \subset \mathbb{R}^3$ is a solid of revolution obtained by rotating a convex cross-section around the z-axis in $\mathbb{R}^3$. Thanks to this special geometry, they can find solutions using a separation of variables method. They show that the solutions constructed are stable for variations in a linear space that is transversal to the gauge-invariance of the problem.

Very recently, while this work was being finished, the author received a series of preprints of S. Jimbo, Y. Morita and J. Zhai [17], [18], [19] where they improve the techniques developed in [16] and introduce some new ideas to obtain very interesting results about stationary solutions of the Ginzburg-Landau equations in topologically non-trivial domains. The author also received recently a preprint J. Rubinstein and P. Sternberg [25], where the ideas of B. White and F. Bethuel concerning the homotopy classes for Sobolev functions are used, together with variational techniques, in a very ingenious way, to obtain a homotopy classification for the minimizers of the Ginzburg-Landau energy in the case the domain is topologically a torus in $\mathbb{R}^3$. One fundamental difference between these works and ours is that, since their authors are looking at critical points, they rely strongly on the Ginzburg-Landau equation to prove nice properties for these critical points, and then succeed in defining the degree of the stationary solutions using these properties. In our case, since we look at the whole level set of the energy, we cannot rely on the equation to help us define the degree. This, as we saw, poses many technical problems, but gives us a considerable amount of new information. Such information should enable us to have a better understanding about the formation of permanent currents and the transition processes between physical states.

Another important question is that of the evolution equation for Ginzburg-Landau. Recently there was some work of F. H. Lin [21], [22], and of S. Demoulini and D. Stuart [12] on the heat flow for Ginzburg-Landau. The author, F. Bethuel and Y. Guo have also obtained some results regarding the dynamical stability of symmetric vortices in the Maxwell-Higgs model (see [15] and [5]).

Remarks on notation.

• Ω is the annulus $\{x \in \mathbb{R}^2 : 1/4 < |x| < 1\} \subset \mathbb{R}^2$. Its boundary, $\partial\Omega$, has two connected components: $\partial\Omega_1 = S_{1/4}$, the inner circle, and $\partial\Omega_2 = S_1$, the exterior circle. On $\partial\Omega$, $\nu(x)$ stands for the exterior unit normal to $\partial\Omega$ at x. Hence $\nu(x) = -x/|x|$ on $\partial\Omega_1$, and $\nu(x) = x/|x|$ on $\partial\Omega_2$. For $x \in \partial\Omega$, $\tau(x)$ stands for the unit tangent vector to $\partial\Omega$ at x, pointing in the sense of increasing θ.

• $\wedge$ denotes the wedge product of differential forms, and $\times$ represents the exterior product of two vectors in $\mathbb{R}^2$ (it is considered as a real number).

• We often use the natural identification between an one-form and the associated vector (given by the scalar product in $\mathbb{R}^2$).

• Although we would normally prefer to write vectors as columns, we will often write them as rows because it makes it easier to insert them in the text.

• We identify the vector $(v^1, v^2) \in \mathbb{R}^2$ with the complex number $v^1 + \imath v^2$. The scalar product in $\mathbb{C}$ is denoted by $(\ ,\ )$. So $(u, v) = (u\,\overline{v} + v\,\overline{u})/2$. With this notation we have that $u \times u_\tau = (\imath\, u, u_\tau)$.

Although this permanent switch between the vector and the complex number notation may be slightly confusing at the beginning, later on the reader will appreciate the convenience that stems from having both notations available.

• d denotes the exterior derivative and $\star$ denotes the Hodge star operator, which in $\mathbb{R}^2$ is the linear operator on $\mathbb{R}$-valued forms defined by

$$\star 1 = dx^1 \wedge dx^2\,,\ \star dx^1 = dx^2\,,\ \star dx^2 = dx^1\,,\ \text{and}\ \star dx^1 \wedge dx^2 = 1\,.$$

We have that for k-forms on $\mathbb{R}^2$, $\star\star = I^{(k(2-k))}$, where I denotes the identity. Hence $\star \star \alpha = \alpha$, if α is a zero-form or a two-form, and $\star \star \alpha = -\alpha$, if α is a one-form.

• $d^\star$ denotes the operator $\star^{-1} d \star$, where $\star^{-1}$ stands for the inverse operator of $\star$.

• In many of the estimates we obtain during the proof of Theorem 1, there are constants which depend on the domain considered. However, since we will have fixed as domain the annulus Ω, we will usually not mention such dependence explicitly in the text.

2. Coarea formula and control of the bad set.

As we mentioned before, the bad set consists of the places where $|u|$ is close to zero. Nevertheless, the presence of the potential term in E_ε (in particular, the presence of the ε^{-2} factor), assures us that for $u \in E_\varepsilon^\Lambda$, the measure of the set $\{x : \, |u| < 1/2\}$ will be very small when $\varepsilon \longrightarrow 0$. In fact, as we will see in this section, a more careful analysis using the coarea formula will allow us to prove much more about this set.

Suppose Λ and ε given and fix an element $u \in E_\varepsilon^\Lambda \cap C^\infty(\Omega)$. For each $\zeta \in [1/2, 3/4]$, let

$$V(\zeta) = \{x \in \Omega : \ |u(x)| = \zeta\}\,.$$

By Sard's Lemma we know that for almost every ζ, $V(\zeta)$ is a one-dimensional submanifold of Ω, hence we will suppose that the ζ we choose is in these conditions. We will now define as our bad set, the set G where $|u|$ is smaller than ζ. Let

$$G(\zeta) := \{x \in \Omega : \ |u(x)| < \zeta\}\,, \qquad \zeta \in \left[\frac{1}{2}, \frac{3}{4}\right].$$

It is easy to see that for small ε, the measure of $G(\zeta)$ will be very small. In fact,

$$
\begin{aligned}
\int_{G(\zeta)} (1-|u|^2)^2 &\geq \int_{G(\zeta)} (1-\zeta^2)^2 \\
&\geq |G(\zeta)|\,(1-\zeta^2)^2 \\
&\geq \Big(\frac{7}{16}\Big)^2 |G(\zeta)|\,,
\end{aligned}
\tag{2.1}
$$

and,

$$
\int_{G(\zeta)} (1-|u|^2)^2 \leq 4\,\varepsilon^2 \,\frac{1}{4\,\varepsilon^2} \int_{\Omega} (1-|u|^2)^2 \leq 4\,\varepsilon^2\,\Lambda\,. \tag{2.2}
$$

Combining (2.1) and (2.2) we obtain the desired bound on $|G(\zeta)|$,

$$
|G(\zeta)| \leq \Big(\frac{16}{7}\Big)^2 4\,\varepsilon^2\,\Lambda = C\,\varepsilon^2 \underset{\varepsilon\to 0}{\longrightarrow} 0\,, \tag{2.3}
$$

where C is a constant depending only on the energy bound Λ.

2.1. The coarea formula.

Using the coarea formula of Federer and Flemming, we can obtain a considerable amount of information about the V_k's and the behavior of $u_{|V_k}$, for ζ conveniently chosen.

Here we will apply a special case of this formula which can be stated as follows (for a proof and more general forms of this result see, for instance, L. Evans and R. Gariepy [13]).

Theorem 7 (coarea formula (change of variables)). *Let $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ be Lipschitz. Then, for every Lebesgue summable function $g:\mathbb{R}^2 \longrightarrow \mathbb{R}$, we have that*

i) *The restriction $g_{|f^{-1}\{y\}}$ is Hausdorff $\mathcal{H}^1$-measurable for almost every y.*

ii) *For every measurable set $X\subset\mathbb{R}^2$,*

$$
\int_X g|\nabla f|\,dx = \int_{\mathbb{R}} \Big(\int_{f^{-1}\{y\}\cap X} g\,d\mathcal{H}^1\Big)\,dy\,.
$$

REMARK. By Rademacher's Theorem, since f is Lipschitz, it is differentiable almost everywhere, and hence ∇f is defined almost everywhere $x \in X$.

2.2. Upper-bound for the length of the V_k's.

We start by proving that the length (Hausdorff one-dimensional measure) of the V_k's is small for small ε. As a matter of fact, if we denote $\Xi := \{x : 1/2 \le |u| \le 3/4\}$, it follows from the co-area formula that

$$\begin{aligned} \int_{1/2}^{3/4} \mathcal{H}^1(V(\zeta))\, d\zeta &= \int_\Xi |\nabla |u|\,| \\ &\le \int_\Xi |\nabla u| \\ &\le \Big(\int_\Xi |\nabla u|^2 \Big)^{1/2} |\Xi|^{1/2}\,, \end{aligned} \tag{2.4}$$

where we used Cauchy-Schwarz for the last inequality. Moreover,

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \le E_\varepsilon(u) \le \Lambda\,,$$

hence,

$$\Big(\int_\Omega |\nabla u|^2 \Big)^{1/2} \le \sqrt{2\,\Lambda}\,. \tag{2.5}$$

On the other hand, the measure of Ξ can also be estimated using the energy bound (just like we did for $G(\zeta)$, in fact $\Xi = G(1/2)$). We obtain

$$|\Xi| \le \Big(\frac{16}{7}\Big)^2 \int_\Xi (1 - |u|^2)^2 \le \Big(\frac{32}{7}\Big)^2 \varepsilon^2 \Lambda\,. \tag{2.6}$$

From (2.4), (2.5) and (2.6), it follows that

$$\int_{1/2}^{3/4} \mathcal{H}^1(V(\zeta))\, d\zeta \le \frac{32\sqrt{2}}{7}\,\varepsilon\,\Lambda\,.$$

Hence, except for ζ in a set $Z_1 \subset [1/2, 3/4]$ of measure at most $\sqrt{2}/70 \leq 1/40$,

$$\mathcal{H}^1(V(\zeta)) \leq \frac{70}{\sqrt{2}}\,\frac{32\sqrt{2}}{7}\,\Lambda\,\varepsilon = 320\,\Lambda\,\varepsilon\,. \tag{2.7}$$

2.3. Upper-bound for the $L^1(V(\zeta))$ norm of ∇u.

A different application of the coarea formula yields

$$\int_{0<\zeta<1}\int_{V(\zeta)} |\nabla u| = \int_{\Omega} |\nabla|u||\,|\nabla u| \leq \int_{\Omega} |\nabla u|^2\,. \tag{2.8}$$

Since we assume that $u \in E_\varepsilon^\Lambda$, from (2.8) it follows that

$$\int_{0<\zeta<1}\int_{V(\zeta)} |\nabla u| \leq 2\,E_\varepsilon^\Lambda(u) \leq 2\,\Lambda\,. \tag{2.9}$$

Using Fubini's Theorem, we will then have that except for ζ in a set $Z_2 \subset [1/2, 3/4]$ of measure at most $1/40$,

$$\int_{V(\zeta)} |\nabla u| \leq 80\,\Lambda\,. \tag{2.10}$$

Thus, except when ζ belongs to the set $Z_1 \cup Z_2$, whose measure is at most $1/20$, estimates (2.7) and (2.10) will be valid. For the rest of this paper we will choose a $\zeta \in (1/2,\, 3/4)$ such that estimates (2.7) and (2.10) are valid, and that $V(\zeta)$ is a one-dimensional submanifold of Ω. Hence, $V(\zeta)$ consists of a finite number of simple curves in Ω. Let $V_1, \ldots, V_{\breve{N}}$, denote the connected components of $V(\zeta)$. Equation (2.7) gives us an upper-bound on the length of each V_k,

$$\sum_{k=1}^{\breve{N}} \mathcal{H}^1(V_k) \leq \mathcal{H}^1(V(\zeta)) \leq 320\,\Lambda\,\varepsilon\,. \tag{2.11}$$

In particular,

$$\mathcal{H}^1(V_k) \leq 320\,\Lambda\,\varepsilon\,, \qquad \text{for all } k = 1, \ldots, \breve{N}\,. \tag{2.12}$$

Hence, for small ε, the length of each V_k will be small (the same being true for the sum of their lengths).

3. Properties of the V_k's which are far from $\partial\Omega$.

We consider the interior subdomain $Y := \{(r,\theta) : \ 1/2 < r < 3/4\} \subset \Omega$, *i.e.*, the interior annulus consisting of the points whose distance to the origin lies between 1/2 and 3/4. For technical reasons, we will also have to consider a slightly enlarged subdomain, $\hat{Y} := \{(r,\theta) : 3/8 < r < 7/8\}$. Hence, $Y \Subset \hat{Y} \Subset \Omega$.

We start by proving that for ε sufficiently small, the V_k's that intersect $\hat{Y}$ are closed curves that stay away from the boundary of Ω.

Lemma 1. *If ε is sufficiently small, then $V_k \cap \hat{Y} \neq \varnothing$, implies that V_k is a closed curve and* $\operatorname{dist}(V_k, \partial\Omega) > 1/16$.

PROOF. Suppose that $V_k \cap \hat{Y} \neq \varnothing$. Then, since $\operatorname{dist}(\hat{Y}, \partial\Omega) = 1/8$, for $\operatorname{dist}(V_k, \partial\Omega)$ to be smaller than 1/16, it is necessary that $\operatorname{diam}(V_k) \geq 1/16$. However, from (2.12) it follows that

$$\operatorname{diam}(V_k) \leq \mathcal{H}^1(V_k) \leq 320\,\Lambda\,\varepsilon\,.$$

Hence, for $\varepsilon < \Lambda/5120$ we must have that $\operatorname{diam}(V_k) < 1/16$, and thus, $\operatorname{dist}(V_k, \partial\Omega) > 1/16$.

The fact that V_k is then a closed curve, follows from it being a one-dimensional submanifold of Ω which does not touch $\partial\Omega$.

Henceforth, we will always suppose that ε is chosen sufficiently small for the result in Lemma 1 to be true. Suppose that the V_k's that intersect $\hat{Y}$ are $V_1, \ldots, V_{\overline{N}}$. They will be closed curves and thus, by Jordan's Curve Theorem, we can define the domain W_k enclosed by V_k (W_k is the bounded component of $\mathbb{R}^2 \setminus V_k$, and in particular, $V_k = \partial W_k$).

Among $V_1, \ldots, V_{\overline{N}}$ we will only consider those which are maximal in the following sense: for $i, j \leq \overline{N}$, if $V_i \subset W_j$ then we disregard V_i and just keep V_j in our list (so we always keep only the exterior curves). Suppose that $V_1, \ldots, V_{\tilde{N}}$, for some $\tilde{N} \leq \overline{N}$, are the maximal curves we obtain. These are the V_k's that will interest us for the rest of this paper (unless stated otherwise, henceforth we will always assume $k \leq \tilde{N}$).

3.1. Estimates for $\deg(u, V_k)$.

By the definition of $V(\zeta)$, the restriction of u to V_k will have values in the circle of radius ζ, *i.e.* $u_{|V_k} : V_k \longrightarrow S_\zeta$, where we denote $S_\zeta = \{z \in \mathbb{R}^2 : |z| = \zeta\}$. Therefore, we can define the degree of u: as usual we consider the map

$$v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} := \frac{u}{|u|} : V_k \longrightarrow S^1 ,$$

and we define

$$\deg(u, V_k) := \deg(v, V_k) := \frac{1}{2\pi} \int_{V_k} v \times \frac{\partial v}{\partial \tau} \, d\tau , \tag{3.1}$$

where τ denotes, as usual, the arc-length parameter on V_k.

Since $u = |u|\, v$, we have that

$$\nabla u = \nabla(|u|\, v) = \begin{pmatrix} \dfrac{\partial |u|}{\partial x^1}\, v^1 + |u|\, \dfrac{\partial v^1}{\partial x^1} & \dfrac{\partial |u|}{\partial x^2}\, v^1 + |u|\, \dfrac{\partial v^1}{\partial x^2} \\[2ex] \dfrac{\partial |u|}{\partial x^1}\, v^2 + |u|\, \dfrac{\partial v^2}{\partial x^1} & \dfrac{\partial |u|}{\partial x^2}\, v^2 + |u|\, \dfrac{\partial v^2}{\partial x^2} \end{pmatrix} . \tag{3.2}$$

Thus,

$$\begin{aligned} |\nabla u|^2 = {} & |u|^2 \Big(\Big(\frac{\partial v^1}{\partial x^1}\Big)^2 + \Big(\frac{\partial v^1}{\partial x^2}\Big)^2 + \Big(\frac{\partial v^2}{\partial x^1}\Big)^2 + \Big(\frac{\partial v^2}{\partial x^2}\Big)^2 \Big) \\ & + ((v^1)^2 + (v^2)^2) \\ & \cdot \Big(\Big(\frac{\partial |u|}{\partial x^1}\Big)^2 + \Big(\frac{\partial |u|}{\partial x^2}\Big)^2 \Big) + |u|\, \frac{\partial |u|}{\partial x^1} \Big(v\, \frac{\partial v}{\partial x^1} \Big) + |u|\, \frac{\partial |u|}{\partial x^2} \Big(v\, \frac{\partial v}{\partial x^2} \Big) . \end{aligned} \tag{3.3}$$

But since $|v| = C^{te} = 1$, it follows that

$$(v^1)^2 + (v^2)^2 = |v|^2 = 1 ,$$

and,

$$v\, \frac{\partial v}{\partial x^i} = \frac{1}{2}\, \frac{\partial}{\partial x^i}\, (v\, v) = 0 .$$

Thus, (3.3) yields

$$|\nabla u|^2 = |u|^2\, |\nabla v|^2 + |\nabla |u|\,|^2 . \tag{3.4}$$

Hence, in particular,

$$|\nabla u|^2 \geq |u|^2 \, |\nabla v|^2 \, . \tag{3.5}$$

For $x \in V_k$, since $|u(x)| = \zeta \geq 1/2$, this yields

$$|\nabla u|^2 \geq \zeta^2 \, |\nabla v|^2 \geq \frac{1}{4} \, |\nabla v|^2 \, , \tag{3.6}$$

which, in turn, implies that on V_k,

$$|\nabla u| \geq \frac{1}{2} \, |\nabla v| \, . \tag{3.7}$$

From equations (3.1) and (3.7) it follows that

$$|\deg (u, V_k)| = |\deg (v, V_k)| \leq \int_{V_k} \left| v \times \frac{\partial v}{\partial \tau} \right| d\tau \leq \int_{V_k} |\nabla v| \leq 2 \int_{V_k} |\nabla u| \, .$$

Therefore, using equation (2.10), we obtain a bound on the absolute value of the degree of u in each of the V_k, for all $k = 1, \ldots, \tilde{N}$ (we remark that this bound is also valid for $\tilde{N} < k \leq \hat{N}$ as long as V_k is a closed curve – so that we have no problem defining $\deg (u, V_k)$),

$$|\deg (u, V_k)| \leq 2 \int_{V_k} |\nabla u| \leq 2 \int_{V(\zeta)} |\nabla u| \leq 160 \, \Lambda \, . \tag{3.8}$$

Moreover, we even have a bound on the sum of the absolute values of these degrees,

$$\sum_{k=1}^{\tilde{N}} |\deg (u, V_k)| \leq 2 \sum_{k=1}^{\tilde{N}} \int_{V_k} |\nabla u| \leq 2 \int_{V(\zeta)} |\nabla u| \leq 160 \, \Lambda \, , \tag{3.9}$$

which gives a bound on the number $N_2 := \#\{k \, : \, V_k \cap \hat{Y} \neq \varnothing$, and $\deg (u, V_k) \neq 0\}$, *i.e.*, the number of "charged" V_k's that intersect $\hat{Y}$. In fact, we obtain

$$N_2 \leq \sum_{k=1}^{\tilde{N}} |\deg (u, V_k)| \leq 160 \, \Lambda \, . \tag{3.10}$$

Remark. We will often refer to a V_k such that $\deg(u, V_k) \neq 0$ as a "charged" (or topologically charged) singularity of u, and to one such that $\deg(u, V_k) = 0$ as a "uncharged" (or neutral or topologically uncharged) singularity of u. This terminology is unprecise but helps convey the essential difference between the behavior of u on these two types of sets.

Using this terminology, the charged V_k's that intersect $\hat{Y}$ are $V_1, \ldots, V_{N_2}$, and the neutral ones are $V_{N_2+1}, \ldots, V_{\tilde{N}}$.

4. The "uncharged" V_k's.

Although the charged V_k's are the only ones that may change the value of $f(r) = \deg(u, S_r)$, defined in (6.1), in order to prove that these cannot be isolated, we will need some control of u on the uncharged V_k's (*i.e.*, $V_{N_2+1}, \ldots, V_{\tilde{N}}$), and on the energy minimizing extensions of u to the W_k's that lie inside them. Thus, in this section we will always suppose $k \in \{N_2 + 1, \ldots, \tilde{N}\}$.

The restriction of u to $V_k = \partial W_k$, $g_k : V_k \longrightarrow S_\zeta$, has degree zero (since we are considering only the "uncharged" V_k's) and W_k is simply connected, hence g_k can be written as

$$g_k = \zeta \exp(\imath\, \theta_k)\,, \tag{4.1}$$

where $\theta_k : V_k \longrightarrow \mathbb{R}$, is a smooth lifting of $u_{|V_k}$. For $x \in V_k$ we have that

$$|\nabla\theta_k|^2 = |\nabla(\exp(\imath\,\theta_k))|^2 = \left|\nabla\Big(\frac{u}{|u|}\Big)\right|^2 .$$

Therefore, by (3.4),

$$|\nabla u|^2 = \zeta^2\, |\nabla\theta_k|^2 + |\nabla|u|\,|^2\,, \tag{4.2}$$

and, in particular,

$$|\nabla\theta_k| \leq \frac{|\nabla u|}{\zeta}\,. \tag{4.3}$$

As usual, we define the oscillation of θ_k as

$$\operatorname{osc}(\theta_k) := \sup_{x\in V_k}(\theta_k(x)) - \inf_{x\in V_k}(\theta_k(x))\,. \tag{4.4}$$

We will prove that the number of V_k's for which θ_k can oscillate considerably, is uniformly bounded (by a constant depending only on the energy bound Λ).

Lemma 2. *Given $\Lambda \in \mathbb{R}^+$, there is a constant $M \in \mathbb{R}^+$ such that, for all $\varepsilon > 0$, for all $u \in E_\varepsilon^\Lambda$, if*

$$I := \left\{ k \in \{N_2 + 1, \dots, \tilde{N}\},\ \textit{such that}\ \operatorname{osc}(\theta_k) > \frac{\pi}{3} \right\},$$

then,

$$\#I \leq M = \frac{480\,\Lambda}{\pi}\,. \tag{4.5}$$

PROOF. By the fundamental Theorem of Calculus,

$$\operatorname{osc}(\theta_k) = \sup_{x,y \in V_k} (\theta_k(x) - \theta_k(y)) \leq \int_{V_k} \left| \frac{\partial \theta_k}{\partial \tau} \right| \leq \int_{V_k} |\nabla \theta_k|\,.$$

Then, using equations (2.10) and (4.3) we obtain

$$\begin{aligned}
\frac{\pi}{3}\,\#I &\leq \sum_{k \in I} \operatorname{osc}(\theta_k) \\
&\leq \sum_{k \in I} \int_{V_k} |\nabla \theta_k| \\
&\leq \sum_{k \in I} \frac{1}{\zeta} \int_{V_k} |\nabla u| \\
&\leq 2 \int_{V(\zeta)} |\nabla u| \\
&\leq 160\,\Lambda\,.
\end{aligned}$$

Hence,

$$\#I \leq \frac{3}{\pi}\,160\,\Lambda = \frac{480}{\pi}\,\Lambda\,. \tag{4.6}$$

Thus, we have proven Lemma 2 with $M = 480\,\Lambda/\pi$.

If $\deg(u, V_k) = 0$, we know that there exist smooth extensions of $g = u_{|V_k} : V_k \longrightarrow S_\zeta$ to $\overline{W}_k$, where $S_\zeta = \{x \in \mathbb{R}^2 : |x| = \zeta\} \simeq S^1$, and W_k is the domain inside V_k (in the sense of Jordan's curve Theorem). Let $H_g^1 := \{u \in H^1(W_k, \mathbb{C}) : u = g \text{ on } V_k\}$. Then, as in the work of F. Bethuel, H. Brezis and F. Hélein [7], we know that

$$\mu_g := \min_{u \in H_g^1} E_\varepsilon(u)\,, \tag{4.7}$$

is achieved by some map u_ε, and furthermore, u_ε satisfies the Euler equation

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\, u_\varepsilon\,(1 - |u_\varepsilon|^2)\,, & \text{in } W_k\,, \\ u_\varepsilon = g = u\,, & \text{on } V_k\,. \end{cases} \tag{4.8}$$

This elliptic system will allow us to prove some sort of maximum principle for u_ε which will give us upper and lower bounds for $|u_\varepsilon|$ in terms of the oscillation of $g = u_{|V_k}$ or, more precisely, in terms of $\operatorname{osc}(\theta_k)$. In particular, we will be able to prove that if the oscillation of θ_k is small enough, then $|u_\varepsilon|$ stays bounded away from zero in W_k. Together with Lemma 2 this will imply that the number of W_k's for which $|u_\varepsilon|$ can be close to zero, is uniformly bounded.

We start by proving an upper bound for $|u_\varepsilon|$. The following Lemma is just an adaptation of [7, Proposition 2] to our situation.

Lemma 3. *Let u_ε be a solution of* (4.8). *Then, $|u_\varepsilon| \le 1$, in W_k.*

PROOF. We start by observing that

$$\Delta(|u_\varepsilon|^2) = 2\, u_\varepsilon\, \Delta u_\varepsilon + 2\, |\nabla u_\varepsilon|^2\,.$$

Hence, by (4.8),

$$\Delta(|u_\varepsilon|^2) = \frac{2}{\varepsilon^2}\, |u_\varepsilon|^2\,(|u_\varepsilon|^2 - 1) + 2\,|\nabla u_\varepsilon|^2 \ge \frac{2}{\varepsilon^2}\, |u_\varepsilon|^2\,(|u_\varepsilon|^2 - 1)\,. \tag{4.9}$$

Therefore, $v_\varepsilon := |u_\varepsilon|^2 - 1$, will satisfy

$$\begin{cases} \Delta v_\varepsilon - \dfrac{2}{\varepsilon^2}\, |u_\varepsilon|^2\, v_\varepsilon \ge 0\,, & \text{in } W_k\,, \\ v_\varepsilon = \zeta^2 - 1\,, & \text{on } V_k = \partial W_k\,. \end{cases}$$

Since $-(2/\varepsilon^2)\,|u_\varepsilon|^2 \leq 0$, the maximum principle implies that (see, for instance, [14, Corollary 3.2])

$$\sup_{W_k} v_\varepsilon \leq \sup_{V_k} v^+ \,, \tag{4.10}$$

where $v^+(x) := \max\{v_\varepsilon(x), 0\}$. Hence, since $v^+(x) := \max\{\zeta^2 - 1, 0\} = 0$, on V_k, it follows that

$$\sup_{W_k} |u_\varepsilon|^2 - 1 = \sup_{W_k} v_\varepsilon \leq 0\,.$$

Thus,

$$\sup_{W_k} |u_\varepsilon| \leq 1\,. \tag{4.11}$$

This concludes the proof of Lemma 3.

Using this Lemma and equation (4.8), we are now able to obtain

Proposition 1. *Suppose that* $\operatorname{osc}(\theta_k) \leq \pi/3$. *Let* u_ε *be the minimizer of* (4.7). *Then,*

$$|u_\varepsilon(x)| \geq \frac{1}{2}\,\zeta \geq \frac{1}{4}\,, \qquad \text{for all } x \in W_k\,. \tag{4.12}$$

PROOF. If $\operatorname{osc}(\theta_k) \leq \pi/3$, then $u(V_k)$ is contained in an arch $\hat\beta$ of S_ζ, of amplitude at most $\pi/3$. Let a and b be the endpoints of $\hat\beta$, and let B be the domain bounded by the straight line $\hat r$ passing through a and b, and the unit circle S^1. We claim that the maximum principle implies that

$$u_\varepsilon(W_k) \subset B\,. \tag{4.13}$$

By Lemma 3 we already know that $|u_\varepsilon| \leq 1$, so it suffices to prove that $u_\varepsilon(W_k)$ and the origin lie on opposite sides of the straight line $\hat r$ defined above.

Choose coordinates y^1, y^2 in the image space such that the y^2 axis is parallel to $\hat r$ (*i.e.*, it is the straight line through the origin parallel to the segment $\overline{a\,b}$), and the y^1 axis cuts the segment $\overline{a\,b}$ perpendicularly at its midpoint. In these coordinates we may write

$$u_\varepsilon(x) = \begin{pmatrix} u_\varepsilon^1(x) \\ u_\varepsilon^2(x) \end{pmatrix} = \zeta \exp(\imath\,\theta_k)\,,$$

where, we are taking the positive y^1 axis as the origin for the angle θ_k.

Since the amplitude $\beta := \operatorname{osc}(\theta_k) \leq \pi/3$, the y^1 coordinate of the endpoints a and b satisfies

$$\begin{aligned} \ell &:= y^1(a) \\ &= y^1(b) \\ &= \min_{x \in V_k} y^1(u(x)) \\ &= \zeta \cos\left(\frac{\beta}{2}\right) \\ &\geq \zeta \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\zeta}{2} \\ &\geq \frac{1}{4}\,. \end{aligned} \tag{4.14}$$

On the other hand, since u_ε is a minimizer of E_ε, hence a critical point, it is a solution of equation (4.8). In particular u_ε^1 will satisfy

$$\begin{cases} -\Delta u_\varepsilon^1 = \dfrac{1}{\varepsilon^2}\, u_\varepsilon^1\,(1 - |u_\varepsilon|^2)\,, & \text{in } W_k\,, \\ u_\varepsilon^1 \geq \ell\,, & \text{on } V_k = \partial W_k\,. \end{cases} \tag{4.15}$$

Doing a reflection of u across the y^2 axis in order to make the image lie in the right half-plane, we obtain the map

$$\tilde{u}_\varepsilon(x) = \begin{pmatrix} \tilde{u}_\varepsilon^1(x) \\ \tilde{u}_\varepsilon^2(x) \end{pmatrix} := \begin{pmatrix} |u_\varepsilon^1(x)| \\ u_\varepsilon^2(x) \end{pmatrix}\,,$$

which satisfies

$$E_\varepsilon(\tilde{u}_\varepsilon) = E_\varepsilon(u_\varepsilon) = \min_{v \in H_g^1(W_k, \mathbb{C})} E_\varepsilon(u)\,.$$

Hence, $\tilde{u}_\varepsilon$ is also a minimizer, and thus critical point, of E_ε, and therefore, $\tilde{u}_\varepsilon^1 = |u_\varepsilon^1|$, satisfies

$$\begin{cases} -\Delta \tilde{u}_\varepsilon^1 = \dfrac{1}{\varepsilon^2}\, \tilde{u}_\varepsilon^1\,(1 - |u_\varepsilon|^2)\,, & \text{in } W_k\,, \\ \tilde{u}_\varepsilon^1 \geq \ell\,, & \text{on } V_k = \partial W_k\,. \end{cases} \tag{4.16}$$

Using Lemma 3 we see that the right-hand side of (4.16) is always non-negative. Hence, $-\Delta\tilde{u}_\varepsilon \geq 0$, and thus the maximum principle assures us that

$$\min_{\overline{W}_k} \tilde{u}^1_\varepsilon = \min_{V_k} \tilde{u}^1_\varepsilon \geq \ell\,.$$

Consequently, using (4.14) we obtain

$$\min_{\overline{W}_k} |u^1_\varepsilon| \geq \ell \geq \frac{\zeta}{2} \geq \frac{1}{4}\,. \tag{4.17}$$

Since u^1_ε is continuous and W_k is connected, $u^1_\varepsilon(W_k)$ has to be connected. Thus, using (4.17) and the fact that $u^1_\varepsilon(x) \geq \ell$ on V_k, we know that we must have

$$u^1_\varepsilon(x) \geq \ell\,, \qquad \text{for all } x \in W_k\,. \tag{4.18}$$

This, together with equation (4.11), proves claim (4.13). In particular, from (4.18) it follows that

$$|u_\varepsilon| = \sqrt{(u^1_\varepsilon)^2 + (u^2_\varepsilon)^2} \geq |u^1_\varepsilon| \geq \ell \geq \frac{\zeta}{2} \geq \frac{1}{4}\,, \quad \text{for all } x \in W_k\,, \tag{4.19}$$

which is equation (4.12).

REMARK. The same method we used to prove claim (4.13) will give us the slightly more precise result

$$u_\varepsilon(W_k) \subset A \subset B\,, \tag{4.20}$$

where A is the closed set bounded by the half-lines $\dot{0}\,a$ and $\dot{0}\,b$, the segment $\overline{a\,b}$ and the circle S^1. In fact, all we have to do to prove this result is to, instead of using a reflection relative to an axis parallel to the segment $\overline{a\,b}$, as before, we have to consider reflections with respect to axii which approach $\overline{0\,a}$ (and others which approach $\overline{0\,b}$) on the outside of the set A defined above.

5. Blow-up of the energy around an isolated "charged" singularity.

5.1. The covering argument.

For simplicity, we will do one more renumbering of the V_k's, $k = 1, \ldots, \tilde{N}$ such that

a) $\deg(u, V_k) \neq 0$ and $V_k \cap Y \neq \varnothing$ if and only if $k \in \{1, \ldots, N_1\}$.

b) $\deg(u, V_k) \neq 0$, $V_k \cap \hat{Y} \neq \varnothing$ and $V_k \cap Y = \varnothing$ if and only if $k \in \{N_1 + 1, \ldots, N_2\}$.

c) $\deg(u, V_k) = 0$, $V_k \cap \hat{Y} \neq \varnothing$ and $\operatorname{osc}(\theta_k) > \pi/3$ if and only if $k \in \{N_2 + 1, \ldots, N\}$.

d) $\deg(u, V_k) = 0$, $V_k \cap \hat{Y} \neq \varnothing$ and $\operatorname{osc}(\theta_k) \leq \pi/3$ if and only if $k \in \{N + 1, \ldots, \tilde{N}\}$.

From (3.10) it follows that

$$N_1 \leq N_2 \leq 160\,\Lambda\,. \tag{5.1}$$

On the other hand, Lemma 2 implies that

$$N = N_2 + \#I \leq 160\,\Lambda + \frac{480}{\pi}\,\Lambda \leq 320\,\Lambda\,. \tag{5.2}$$

We remark that (5.2) gives a bound for N which is valid for all $u \in E_\varepsilon^\Lambda$ and which, moreover, depends only on Λ and not on ε. We have no similar bound for $\tilde{N}$, the total number of V_k's that intersect $\hat{Y}$. However, as we will see in this section, a bound on N like (5.2) is enough since Proposition 1 will allow us to prove that the V_k's in condition d) (*i.e.*, those for which $\deg(u, V_k) = 0$ and $\operatorname{osc}(\theta_k) \leq \pi/3$ are "harmless" – in fact, Proposition 1 gives us a good enough control over the behavior of u inside these V_k's for our estimates of lower bounds on the energy of an isolated charged singularity to go through, regardless of the the presence of V_k's of type d) in its neighborhood. We will need the following two rather technical Lemmas to obtain these lower bounds.

The first one is a covering argument that will allow us to see that $W_1, \ldots, W_N$ can be subdivided into groups, each of which is contained in some ball of radius of order bigger than $\sqrt{\varepsilon}$, and that the different balls are, in some sense, far apart (this type of technique has recently been used by several authors like M. Strüwe or F. Bethuel, H. Brezis and F. Hélein or still F.H. Lin in [22] – our approach is closer to that of the latter).

The second Lemma will then serve to prove that if any of the balls B_j which intersect Y were charged, then we would have to pay a very high price (of order $|\log \varepsilon|$) in energy.

Lemma 4. *Fix $\Lambda \in \mathbb{R}^+$. Let $u \in E_\varepsilon^\Lambda$, and $W_1, \dots, W_N$ be defined as above. Then, for ε sufficiently small, there is an integer $m \leq N$, a family of numbers $\alpha_1, \dots, \alpha_m \in (1/2, 1]$, and a family of balls B_j, $j = 1, \dots, m$, of centers x_j and radii r_j such that*

i) $r_j \leq C\varepsilon^{\alpha_j}$.

ii) $\displaystyle\bigcup_{i=1}^{N} W_i \subset \bigcup_{j=1}^{m} B_j$.

iii) *The enlarged balls $\tilde{B}_j := B(x_j, \varepsilon^{-\alpha_j/(2^{N+1}+1)}\, r_j)$ are pairwise disjoint.*

PROOF. We have fixed $\Lambda \in \mathbb{R}^+$, and we are looking at maps $u \in E_\varepsilon^\Lambda$, for ε sufficiently small (to be chosen later). We define $W_1, \dots, W_N$ as above (thus they will be open, simply-connected subsets of $\Omega \subset \mathbb{R}^2$, such that $\partial W_k = V_k$). By equation (5.2) we know that there exists a uniform bound on N depending only on the energy level Λ we are considering, and not on ε – to be able to change ε while having an uniform bound on the number m of balls used in the covering is crucial for our argument to work.

On the other hand, by (2.12) we have that

$$\operatorname{diam}(W_k) \leq \frac{1}{2}\,\mathcal{H}^1(V_k) \leq 160\,\Lambda\,\varepsilon\,. \tag{5.13}$$

Hence our Lemma follows from the more general covering argument stated in Lemma 7 of the Appendix. In fact, it corresponds to the special case where $C = 160\,\Lambda$ and $\alpha = 1$.

5.2. Lower-bound for the energy around an isolated charged singularity.

Lemma 5. *Let R_1, $R_2 \in \mathbb{R}^+$ be such that $R_1 < R_2$. Let Ω be the annulus $\Omega = B(0, R_2) \setminus B(0, R_1)$, and $u \in H^1(\Omega, \mathbb{C})$ be such that exists*

$\sigma \in \mathbb{R}^+$ *such that* $|u(x)| \geq \sigma > 0$, *for all* $x \in \Omega$, *and* $\deg(u, S_{R_1}) = \deg(u, S_{R_2}) = d \neq 0$. *Then,*

$$E_\varepsilon(u) \geq \pi\, d^2\, \sigma^2 \log\left(\frac{R_2}{R_1}\right). \tag{5.4}$$

Proof. We have that

$$E_\varepsilon(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,, \qquad \text{for all } u \in H^1(\Omega, \mathbb{C})\,. \tag{5.5}$$

Hence, we will concentrate on obtaining a lower bound for the Dirichlet energy of u (the right hand side of (5.5)). Since, by hypothesis, $|u| \geq \sigma > 0$, we may define

$$v := \frac{u}{|u|} \in H^1(\Omega, S^1)\,, \text{ and } \deg(v, S_{R_1}) = \deg(v, S_{R_2}) = d \neq 0\,.$$

By (3.5) we know that

$$|\nabla u|^2 \geq |u|^2\,|\nabla v|^2 \geq \sigma^2\,|\nabla v|^2\,. \tag{5.6}$$

We define

$$\mathcal{V}_d = \{v \in H^1(\Omega, S^1) :\ \deg(v, S_{R_1}) = \deg(v, S_{R_2}) = d\}\,.$$

From (5.5) and (5.6) it follows that

$$E_\varepsilon(u) \geq \frac{1}{2}\int |\nabla u|^2 \geq \sigma^2 \inf_{v \in \mathcal{V}_d}\left(\frac{1}{2}\int |\nabla v|^2\right). \tag{5.7}$$

The problem of determining

$$\inf_{v \in \mathcal{V}_d}\left(\frac{1}{2}\int |\nabla u|^2\right)$$

has already been extensively studied. In fact we can reduce it, using an associated linear problem (see, for instance, [8, Theorems I.1 and II.1, and their Corollaries]), to determining the Dirichlet energy of a harmonic map Φ such that

$$\begin{cases} \Delta\Phi = 0\,, & \text{in } \Omega\,,\\ \Phi = 0\,, & \text{on } S_{R_2}\,,\\ \Phi = C\,, & \text{on } S_{R_1}\,,\\ \displaystyle\int_{S_{R_i}} \frac{\partial \Phi}{\partial \nu} = 2\pi\, d\,, & \end{cases} \tag{5.8}$$

where C is some constant, and ν is the outward normal to B_{R_1} and also the outward normal to B_{R_2} (so ν will point inside Ω on S_{R_1} and outside on S_{R_2}).

We can easily check that $\Phi = d\log(r/R_2)$ is a solution of (5.8). Therefore, by the proof of [8, Theorem I.1] (see step 1 of that proof – it is essentially a consequence of Poincaré's Lemma) we know that for all $v \in H^1(\Omega, S^1)$: $\deg(v, S_{R_i}) = d$, $i = 1, 2$,

$$
\begin{aligned}
\int_\Omega |\nabla v|^2 &\geq \int_\Omega |\nabla \Phi|^2 \\
&= \int_\Omega \left|\frac{d}{r}\right|^2 \\
&= \int_0^{2\pi} d\theta \int_{R_1}^{R_2} r\,\frac{d^2}{r^2}\,dr \\
&= 2\pi\, d^2 \log\left(\frac{R_2}{R_1}\right).
\end{aligned}
\tag{5.9}
$$

Combining equations (5.7) and (5.9) we obtain

$$E_\varepsilon(u) \geq \pi\,\sigma^2\,d^2 \log\left(\frac{R_2}{R_1}\right),$$

which is the desired result.

We are now ready to prove the main result of this section.

Theorem 8. *Let $\Lambda \in \mathbb{R}$ be fixed and $u \in E_\varepsilon^\Lambda$. Then, there exists $\varepsilon_0 > 0$ (depending only on Λ) such that if $\varepsilon < \varepsilon_0$, then $B_j \cap Y \neq \varnothing$ implies that* $\deg(u, \partial B_j) = 0$*, where the balls B_j are given by Lemma* 4.

PROOF. Suppose that for some ε, sufficiently small to apply Lemma 4, there exists $u \in E_\varepsilon^\Lambda$ such that in Lemma 4 we obtained a ball B_j such that $B_j \cap Y \neq \varnothing$ and $\deg(u, \partial B_j) \neq 0$. Since $B_j \cap Y \neq \varnothing$, if ε is sufficiently small (depending only on Λ) $\tilde{B}_j \subset \hat{Y}$ (because the radius of $\tilde{B}_j$ tends to zero when $\varepsilon \longrightarrow 0$). Thus, since in the covering argument we took care of all the V_k's such that $V_k \cap \hat{Y} \neq \varnothing$ and $\deg(u, V_j) \neq 0$ or $\mathrm{osc}(\theta_k) > \pi/3$, we know that the annulus $D_j := \tilde{B}_j \setminus B_j$ may only intersect uncharged V_k's such that $\mathrm{osc}(\theta_k) \leq \pi/3$ (what we called V_k's of type d) in the beginning of this section).

We may suppose, without loss of generality, that the V_k's that intersect D_j are $V_{N+1}, \dots, V_{\hat{N}}$, for some $\hat{N} \le \tilde{N}$. We know that $\operatorname{osc}(\theta_k) \le \pi/3$, $k = N+1, \dots, \hat{N}$. However, we cannot apply Lemma 5 directly to u on D_j since a priori we have no lower bound on $|u|$ inside $W_{N+1}, \dots, W_{\hat{N}}$. Nevertheless, if we replace u inside each of the W_k, $k = N+1, \dots, \hat{N}$, by the corresponding minimizer of (4.7), we will decrease the energy and, at the same time, by Proposition 1, we will have a lower bound on the absolute value of the map obtained. Let

$$\tilde{u} := \begin{cases} u\,, & \text{in } D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k\,, \\ u_\varepsilon\,, & \text{in } W_k\,,\ k = N+1, \dots, \hat{N}\,, \end{cases} \tag{5.10}$$

where u_ε is the minimizer of E_ε in W_k with boundary value u. In particular, u_ε satisfies equation (4.8). By construction, $|\tilde{u}| \ge \zeta \ge 1/2$ in $D_j \setminus \bigcup_{k=N+1}^{\hat{N}} W_k$, and by Proposition 1, $|\tilde{u}| = |u_\varepsilon| \ge 1/4$ in W_k, $k = N+1, \dots, \hat{N}$. Therefore,

$$|\tilde{u}| \ge \frac{1}{4}\,, \qquad \text{in } D_j\,. \tag{5.11}$$

Hence, $\deg(\tilde{u}, \partial\tilde{B}_j) = \deg(\tilde{u}, \partial B_j) = d \ne 0$.Thus, we may apply Lemma 5 to $\tilde{u}$ in D_j. Denoting the energy of a map w in a domain G by

$$E_\varepsilon(w, G) := \frac{1}{2}\int_G |\nabla w|^2 + \frac{1}{4\,\varepsilon^2}\int_G (1 - |w|^2)^2\,,$$

this Lemma yields

$$E_\varepsilon(\tilde{u}, D_j) \ge \pi d^2 \Big(\frac{1}{4}\Big)^2 \log\big(\varepsilon^{-\alpha_j/(2^{N+1}+1)}\big)\,. \tag{5.12}$$

Since $\alpha_j \ge 1/2$ (by Lemma 4), we have that

$$E_\varepsilon(\tilde{u}, D_j) \ge \frac{\pi\, d^2}{16} \log\big(\varepsilon^{-1/(2(2^{N+1}+1))}\big) = -\frac{\pi\, d^2}{32\,(2^{N+1}+1)} \log\varepsilon\,. \tag{5.13}$$

We claim that, for ε sufficiently small

$$E_\varepsilon(u, \Omega) \ge E_\varepsilon(\tilde{u}, D_j)\,. \tag{5.14}$$

PROOF OF CLAIM (5.14). We have that

$$
\begin{aligned}
E_\varepsilon(\tilde u, D_j) &= E_\varepsilon\Big(\tilde u, D_j \setminus \bigcup_{k=N+1}^{\hat N} W_k\Big) + \sum_{k=N+1}^{\hat N} E_\varepsilon(\tilde u, W_k \cap D_j) \\
&\le E_\varepsilon\Big(\tilde u, D_j \setminus \bigcup_{k=N+1}^{\hat N} W_k\Big) + \sum_{k=N+1}^{\hat N} E_\varepsilon(\tilde u, W_k)\,.
\end{aligned}
\tag{5.15}
$$

By construction, $\tilde u = u$ on $D_j \setminus \bigcup_{k=N+1}^{\hat N} W_k$, we have that

$$
E_\varepsilon\Big(\tilde u, D_j \setminus \bigcup_{k=N+1}^{\hat N} W_k\Big) = E_\varepsilon\Big(u, D_j \setminus \bigcup_{k=N+1}^{\hat N} W_k\Big)\,,
$$

and, on the other hand, by the definition of u_ε as the minimizer of (4.7), we also have that

$$
E_\varepsilon(\tilde u, W_k) \le E_\varepsilon(u, W_k)\,, \qquad \text{for } k = N+1, \dots, \hat N\,.
$$

Therefore, it follows from (5.15) that

$$
\begin{aligned}
E_\varepsilon(\tilde u, D_j) &\le E_\varepsilon\Big(u, D_j \setminus \bigcup_{k=N+1}^{\hat N} W_k\Big) + \sum_{k=N+1}^{\hat N} E_\varepsilon(u, W_k) \\
&= E_\varepsilon(u, D_j \cup W_{N+1} \cup \cdots \cup W_{\hat N}) \\
&\le E_\varepsilon(u, \Omega)\,,
\end{aligned}
$$

since $W_k \subset \hat Y \subset \Omega$, $k = N+1, \dots, \hat N$, if ε is sufficiently small. This concludes the proof of claim (5.14).

Combining (5.13) and (5.14) we have that for ε sufficiently small,

$$
E_\varepsilon(u, \Omega) \ge -\frac{\pi\, d^2}{32\,(2^{N+1}+1)} \log\varepsilon \ge C\, d^2\, |\log\varepsilon|\,,
\tag{5.16}
$$

where C is a positive constant only depending on Λ (in fact, using equation (5.2) we may choose $C = \pi/(32\,(2^{320\Lambda+1}+1)) > 0$).

If, as we supposed, $d \ne 0$, then, since $u \in E_\varepsilon^\Lambda$, we would have that $C\, d^2 |\log\varepsilon| \le \Lambda$, for all ε sufficiently small. However, this is clearly not true for $\varepsilon \leq \exp(-\Lambda/(C\,d^2))$. Hence, d must be zero, which concludes the proof of Theorem 8.

REMARK. Theorem 8 proves rigorously our idea that as ε gets small the charged V_k's have to cluster, giving rise to "neutral" (deg $= 0$) B_j's, or to "drift" towards the boundary $\partial\Omega$ (thus exiting the interior domain Y). Hence, in the interior of Ω, and for a distance scale of order $\varepsilon^{1/2}$, the charged singularities shouldn't be "perceptible".

6. Definition of the degree of u in Ω.

In this section we define the degree of u in Ω, which is an integer, and show that this integer is well defined.

Let

$$v := \frac{u}{|u|} : \hat{Y} \setminus \bigcup_{k=1}^{\tilde{N}} W_k \longrightarrow S^1 ,$$

and

$$A := \Big\{ r \in \Big(\frac{1}{2}, \frac{3}{4}\Big) : \ S_r \cap V_k = \varnothing\,, \ \text{for all } k = 1, \dots, \tilde{N} \Big\} .$$

As before, for $r \in A$, we define

$$f(r) := \frac{1}{2\pi} \int_{S_r} v \times \frac{\partial v}{\partial \tau} = \deg(u, S_r)\,, \tag{6.1}$$

and we define the approximate degree as

$$\operatorname{adeg}(u) := \frac{1}{2\pi|A|} \int_A \int_{S_r} v \times \frac{\partial v}{\partial \tau}\, d\tau\, dr = \frac{1}{|A|} \int_A f(r)\, dr\,. \tag{6.2}$$

The function f may only change value when we cross a charged V_k since if $r_1, r_2 \in A$, $r_2 > r_1$, then

$$f(r_2) - f(r_1) = \sum_{k \in I_{r_1, r_2}} \deg(u, V_k)\,, \tag{6.3}$$

$$I_{r_1, r_2} = \{ k : \ V_k \subset B(0, r_2) \setminus B(0, r_1) \} .$$

By (5.3), (5.2), Lemma 4 and Theorem 8, inside Y we can cover all the charged V_k's by an uniformly bounded number of balls $B_1, \dots, B_m$, with $m \leq 320\,\Lambda$, and such that $r_j = \operatorname{radius}(B_j) \leq 160\,\Lambda\,\varepsilon^{1/2}$, and $\deg(u, \partial B_j) = 0$. Hence the function f will always have the same value in $\breve{A} := A \setminus B$, where $B := \bigcup_{j=1}^{m} \{r :\ S_R \cap B_j \neq \varnothing\}$. This is the value we use to define $\deg(u, \Omega) \in \mathbb{Z}$.

When $\varepsilon \longrightarrow 0$ the approximate degree $(\operatorname{adeg}(u))$ approaches this value. In fact, from (5.2) and Lemma 4, it follows that

$$|B| \leq 2 \sum_{j=1}^{m} r_j \leq 2\, m\, 160\,\Lambda\,\varepsilon^{1/2} \leq (320\,\Lambda)^2\,\varepsilon^{1/2}\,. \tag{6.4}$$

Furthermore, even inside $A \cap B$ the value of $f(r) = \deg(u, S_r)$ is uniformly bounded – equations (3.9) and (6.3) imply that

$$|f - \deg(u, \Omega)| \leq \sum_{k=1}^{N_1} |\deg(u, V_k)| \leq 160\,\Lambda\,. \tag{6.5}$$

Thus, using (2.7), (6.4) and (6.5), we obtain

$$\begin{aligned}
|\operatorname{adeg}(u) - \deg(u)| &= \Big| \frac{1}{|A|} \int_A f(r)\,dr - \frac{1}{|A|} \int_A \deg(u, \Omega)\,dr \Big| \\
&\leq \frac{1}{|A|} \int_A |f(r) - \deg(u, \Omega)| \\
&\leq \frac{1}{|A|}\, |B|\, 160\,\Lambda \\
&\leq \frac{(320\,\Lambda)^3}{2\Big(\dfrac{1}{4} - \mathcal{H}^1(V(\zeta))\Big)}\,\varepsilon^{1/2} \\
&\leq \frac{(320\,\Lambda)^3}{\dfrac{1}{2} - 320\,\Lambda\,\varepsilon}\,\varepsilon^{1/2}\,.
\end{aligned} \tag{6.6}$$

Since this bound depends only on Λ and ε (and not on u), we will have that $\operatorname{adeg}(u)$ will converge to $\deg(u, \Omega) \in \mathbb{Z}$, uniformly in $u \in E_\varepsilon^\Lambda$. Hence, given Λ, we know that for ε sufficiently small

$$|\operatorname{adeg}(u) - \deg(u)| \leq \frac{1}{4}\,,$$

and therefore, the knowledge of $\operatorname{adeg}(u)$ will determine the integer $\deg(u)$ as desired.

REMARK. Of course we can also obtain $\deg(u,\Omega)$ by evaluating $f(r) = \deg(u,S_r)$ for any $r \in \breve{A} = A \setminus B$. The problem is that the process of obtaining the balls B_j that define B is very elaborate – hence our choice of also showing how to obtain $\deg(u,\Omega)$ using the approximate degree. We remark also that the B_j's obtained using Lemma 4, and thus also B, are not uniquely determined. However, using estimate (6.4), it is easy to check that (for sufficiently small ε, as usual) the value of $\deg(u,\Omega)$ obtained by evaluating $f(r)$ in $\breve{A}$, is independent of the particular B_j's used in the process.

7. Continuity of $\deg(u,\Omega)$.

This section is devoted to showing that the notion of $\deg(u,\Omega)$ we defined in the previous section (Section 6) is continuous in $H^1(\Omega)$ topology inside each level set of the Ginzburg-Landau energy (1.1). This result will be stated in Theorem 9 at the end of the section.

Let $\Lambda \in \mathbb{R}^+$ be given and $\varepsilon < \varepsilon_0$ (with ε_0 defined as in Theorem 8) and consider $u_1, u_2 \in E_\varepsilon^\Lambda$. Suppose $B_1^i, \ldots, B_{m_i}^i$, are the balls obtained when applying Lemma 4 to u_i, $i = 1,2$, and V_k^i, $k = 1, \ldots, \tilde{N}_i$, $i = 1,2$, denote the corresponding V_k's. We define, as before, $v_i := u_i/|u_i|$,

$$A_i := \Big\{ r \in \Big(\frac{1}{2}, \frac{3}{4}\Big) : \ S_r \cap V_k^i = \varnothing, \text{ for all } k = 1, \ldots, \tilde{N},$$
$$\text{and } S_r \cap B_j^i = \varnothing, \text{ for all } j = 1, \ldots, m_i \Big\},$$

$$f_i(r) := \frac{1}{2\pi} \int_{S_r} v_i \times \frac{\partial v_i}{\partial \tau}\, d\tau\,, \qquad \text{for } r \in A_i\,.$$

Then, denoting $A := A_1 \cap A_2$,

$$\deg(u_i,\Omega) = \frac{1}{|A_i|} \int_{A_i} f_i(r)\, dr = \frac{1}{|A|} \int_A f_i(r)\, dr\,, \tag{7.1}$$

since $f_i(r) = C^{te} = \deg(u_i,\Omega)$ in A_i (hence also in $A \subset A_i$). Therefore, denoting $G := \{(r,\theta) : \ r \in A, \ \theta \in [0,2\pi)\}$,

$$|\deg(u_1,\Omega) - \deg(u_2,\Omega)|$$
$$= \frac{1}{2\pi|A|} \Big| \int_A \int_{S_r} \Big(\frac{u_1}{|u_1|} \times \partial_\tau\Big(\frac{u_1}{|u_1|}\Big) - \frac{u_2}{|u_2|} \times \partial_\tau\Big(\frac{u_2}{|u_2|}\Big)\Big)\, d\tau\, dr \Big| \tag{7.2}$$

$$= \frac{1}{2\pi|A|} \Big| \int_A \int_{S_r} \Big(\frac{u_1}{|u_1|^2} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|^2} \times \frac{\partial u_2}{\partial \tau} \Big) \, d\tau \, dr \Big| ,$$

since

$$\begin{aligned} \frac{u_i}{|u_i|} \times \partial_\tau \Big(\frac{u_i}{|u_i|} \Big) &= \frac{u_i}{|u_i|} \times \Big(\frac{1}{|u_i|} \frac{\partial u_i}{\partial \tau} \Big) + \frac{u_i}{|u_i|} \times \Big(u_i \, \partial_\tau \Big(\frac{1}{|u_i|} \Big) \Big) \\ &= \frac{u_i}{|u_i|} \times \Big(\frac{1}{|u_i|} \frac{\partial u_i}{\partial \tau} \Big) , \end{aligned}$$

because $u_i \times u_i = 0$.

Furthermore, from equation (2.11) and Lemma 4, it follows that $|A_1|$, $|A_2|$ and $|A| \longrightarrow 1/4$ uniformly when $\varepsilon \longrightarrow 0$, and thus, in particular, we have that for ε sufficiently small (independent of the particular choice of $u_1, u_2 \in E_\varepsilon^\Lambda$), $|A| > 1/(2\pi)$. Hence, equation (7.2) yields that for all ε as above,

$$\begin{aligned} &|\deg (u_1, \Omega) - \deg (u_2, \Omega)| \\ &\quad = \frac{1}{2\pi|A|} \Big| \int_A \int_{S_r} \Big(\frac{u_1}{|u_1|^2} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|^2} \times \frac{\partial u_2}{\partial \tau} \Big) \, d\tau \, dr \Big| \\ &\quad \le \frac{1}{2\pi|A|} \int_G \Big| \frac{u_1}{|u_1|^2} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|^2} \times \frac{\partial u_2}{\partial \tau} \Big| \\ &\quad \le \Big\| \frac{u_1}{|u_1|^2} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|^2} \times \frac{\partial u_2}{\partial \tau} \Big\|_{L^1(G)} . \end{aligned} \tag{7.3}$$

We can write the integrand in (7.3) as

$$\begin{aligned} &\frac{u_1}{|u_1|^2} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|^2} \times \frac{\partial u_2}{\partial \tau} \\ &\quad = \frac{1}{|u_1|} \frac{u_1}{|u_1|} \times \frac{\partial u_1}{\partial \tau} - \frac{1}{|u_2|} \frac{u_2}{|u_2|} \times \frac{\partial u_2}{\partial \tau} \\ &\quad = \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|} \Big) \frac{u_1}{|u_1|} \times \frac{\partial u_1}{\partial \tau} - \frac{1}{|u_2|} \Big(\frac{u_1}{|u_1|} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|} \times \frac{\partial u_2}{\partial \tau} \Big) . \end{aligned} \tag{7.4}$$

Moreover, one can write the last factor in (7.4) as

$$\frac{u_1}{|u_1|} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|} \times \frac{\partial u_2}{\partial \tau}$$

$$
\begin{aligned}
&= \frac{1}{|u_1|}\Big(u_1 \times \frac{\partial u_1}{\partial \tau} - u_2 \times \frac{\partial u_2}{\partial \tau}\Big) + \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big) u_2 \times \frac{\partial u_2}{\partial \tau} \\
&= \frac{1}{|u_1|}\Big((u_1 - u_2) \times \frac{\partial u_1}{\partial \tau} + u_2 \times \frac{\partial (u_1 - u_2)}{\partial \tau}\Big) \\
&\quad + \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big) u_2 \times \frac{\partial u_2}{\partial \tau} \,.
\end{aligned} \tag{7.5}
$$

From (7.4) and (7.5) it follows that

$$
\begin{aligned}
&\frac{u_1}{|u_1|^2} \times \frac{\partial u_1}{\partial \tau} - \frac{u_2}{|u_2|^2} \times \frac{\partial u_2}{\partial \tau} \\
&\quad = \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big) \frac{u_1}{|u_1|} \times \frac{\partial u_1}{\partial \tau} + \frac{1}{|u_1|\,|u_2|}\Big((u_1 - u_2) \times \frac{\partial u_1}{\partial \tau}\Big) \\
&\qquad + \frac{1}{|u_1|}\Big(\frac{u_2}{|u_2|} \times \frac{\partial (u_1 - u_2)}{\partial \tau}\Big) + \Big(\frac{1}{|u_1|} - \frac{1}{|u_2|}\Big)\Big(u_2 \times \frac{\partial u_2}{\partial \tau}\Big) .
\end{aligned} \tag{7.6}
$$

On the other hand, since $|u_i| \geq 1/2$ in G, we have that

$$
\frac{1}{|u_i|} \leq 2\,,\ i = 1, 2\,,\ \text{ and } \frac{1}{|u_1|\,|u_2|} \leq 4\,,\ \text{ in } G\,. \tag{7.7}
$$

Furthermore, we have the following estimates for $v_i = u_i/|u_i|$,

$$
\Big\| \frac{u_i}{|u_i|} \Big\|_{L^\infty(\Omega)} = 1\,, \tag{7.8}
$$

$$
\Big\| \frac{u_i}{|u_i|} \Big\|_{L^2(G)} \leq \Big\| \frac{u_i}{|u_i|} \Big\|_{L^\infty(G)} |G|^{1/2} \leq |G|^{1/2} \leq |Y|^{1/2} = \frac{\sqrt{5\pi}}{4}\,. \tag{7.9}
$$

Regarding the tangential derivatives, we have that $|\partial u_i/\partial \tau| \leq |\nabla u_i|$, and thus,

$$
\Big\| \frac{\partial u_i}{\partial \tau} \Big\|_{L^2(G)} \leq \|\nabla u_i\|_{L^2(G)} \leq \|\nabla u_i\|_{L^2(\Omega)}\,, \tag{7.10}
$$

and also that

$$
\Big| \frac{\partial (u_1 - u_2)}{\partial \tau} \Big| \leq |\nabla (u_1 - u_2)|\,,
$$

which implies that

$$
\Big\| \frac{\partial (u_1 - u_2)}{\partial \tau} \Big\|_{L^2(G)} \leq \|\nabla (u_1 - u_2)\|_{L^2(G)} \leq \|\nabla (u_1 - u_2)\|_{L^2(\Omega)}\,. \tag{7.11}
$$

Finally, we can easily check that

$$\Big|\frac{1}{|u_1|}-\frac{1}{|u_2|}\Big| = \frac{|\,|u_1|-|u_2|\,|}{|u_1|\,|u_2|} \le \frac{|u_1-u_2|}{|u_1|\,|u_2|} \le 4\,|u_1-u_2|\,,$$

which, in turn, yields

$$\Big\|\frac{1}{|u_1|}-\frac{1}{|u_2|}\Big\|_{L^2(G)} \le 4\,\|u_1-u_2\|_{L^2(G)} \le 4\,\|u_1-u_2\|_{L^2(\Omega)}\ . \tag{7.12}$$

Moreover, since we supposed that $u_i \in E_\varepsilon^\Lambda$, we have, as in (2.5),

$$\|\nabla u_i\|_{L^2(G)} \le \|\nabla u_i\|_{L^2(\Omega)} \le \sqrt{2\,E_\varepsilon(u_i)} \le \sqrt{2\,\Lambda}\,. \tag{7.13}$$

Using the Cauchy-Schwarz inequality and equations (7.6), (7.7), (7.8), (7.9), (7.10), (7.11), (7.12) and (7.13), it follows from equation (7.3) that

$$\begin{aligned}
&|\deg(u_1,\Omega)-\deg(u_2,\Omega)|\\
&\qquad\le \Big\|\frac{u_1}{|u_1|^2}\times\frac{\partial u_1}{\partial\tau}-\frac{u_2}{|u_2|^2}\times\frac{\partial u_2}{\partial\tau}\Big\|_{L^1(G)}\\
&\qquad\le \Big\|\frac{u_1}{|u_1|}\Big\|_{L^\infty(G)}\Big\|\frac{1}{|u_1|}-\frac{1}{|u_2|}\Big\|_{L^2(G)}\|\nabla u_1\|_{L^2(G)}\\
&\qquad\quad + 4\,\|u_1-u_2\|_{L^2(G)}\,\|\nabla u_1\|_{L^2(G)}\\
&\qquad\quad + 2\,\Big\|\frac{u_2}{|u_2|}\Big\|_{L^2(G)}\|\nabla(u_1-u_2)\|_{L^2(G)}\\
&\qquad\quad + \Big\|\frac{u_2}{|u_2|}\Big\|_{L^\infty(G)}\Big\|\frac{1}{|u_1|}-\frac{1}{|u_2|}\Big\|_{L^2(G)}\|\nabla u_2\|_{L^2(G)}\\
&\qquad\le 4\,\|\nabla u_1\|_{L^2(G)}\,\|u_1-u_2\|_{L^2(G)} + 4\,\|\nabla u_1\|_{L^2(G)}\,\|u_1-u_2\|_{L^2(G)}\\
&\qquad\quad + 2\,|Y|^{1/2}\,\|\nabla(u_1-u_2)\|_{L^2(G)} + 4\,\|\nabla u_2\|_{L^2(G)}\,\|u_1-u_2\|_{L^2(G)}\\
&\qquad\le \big(8\,\|\nabla u_1\|_{L^2(\Omega)} + 4\,\|\nabla u_2\|_{L^2(\Omega)}\big)\,\|u_1-u_2\|_{L^2(\Omega)}\\
&\qquad\quad + 2\,\frac{\sqrt{5\pi}}{4}\,\|\nabla(u_1-u_2)\|_{L^2(\Omega)}\\
&\qquad\le 12\,\sqrt{2\,\Lambda}\,\|u_1-u_2\|_{L^2(\Omega)} + \frac{\sqrt{5\pi}}{2}\,\|\nabla(u_1-u_2)\|_{L^2(\Omega)}\\
&\qquad\le C\,\|u_1-u_2\|_{H^1(\Omega)}\ ,
\end{aligned}$$

where C is a constant that depends only on the energy bound Λ (we may take $C = 12\sqrt{2\,\Lambda} + \sqrt{5\pi}/2)$. Therefore, we have proven the following Theorem which is the main result of this section.

Theorem 9. *Let $\Lambda > 0$ be given and ε be sufficiently small. Then, inside the level set E_ε^Λ the degree defined as above is continuous in $H^1(\Omega)$ topology, and there is a constant C, depending only on Λ, such that for all $u_1, u_2 \in E_\varepsilon^\Lambda$*

$$|\deg(u_1, \Omega) - \deg(u_2, \Omega)| \leq C\, \|u_1 - u_2\|_{H^1(\Omega)} \, . \tag{7.14}$$

8. Proof of Theorem 1 and Theorem 6.

We start by proving Theorem 1, *i.e.* the case where Ω is of the special form we studied (the annulus $\Omega = \{x \in \mathbb{R}^2 : 1/4 < |x| < 1\}$). In this case we defined in Section 6 the map $\deg(u, \Omega)$ which has all the required properties of $\chi(u)$. Thus, we define $\chi(\cdot) := \deg(\cdot, \Omega) : E_\varepsilon^\Lambda \longrightarrow \mathbb{Z}$. Theorem 9 states that this map is continuous inside each level set of the Ginzburg-Landau energy. Since χ is a continuous map with values in the discrete set $\mathbb{Z}$, for each $k \in \mathbb{Z}$, $\chi^{-1}(k) = \{u \in E_\varepsilon^\Lambda : \chi(u) = k\}$, will be an open and closed subset of E_ε^Λ (in H^1 topology). We have thus succeeded in defining topological sectors inside E_ε^Λ. This concludes the proof of Theorem 1. Theorem 6 follows from Theorem 1 as described in the Introduction.

9. The Palais-Smale condition: proof of Theorem 3.

Suppose that u_n is a Palais-Smale sequence for E_ε, *i.e.* that there exists a constant M such that

$$E_\varepsilon(u_n) \leq M\,, \qquad \text{for all } n\,, \tag{9.1}$$

$$d\,E_\varepsilon(u_n) \longrightarrow 0 \text{ in } (H^1)^* \text{ as } n \longrightarrow +\infty\,, \tag{9.2}$$

where $(H^1)^*$ is the dual of $H^1(\Omega, \mathbb{R}^2)$, and $d\,E_\varepsilon(u_n)$ denotes the differential of E_ε at u_n. We want to show that then u_n has a strongly convergent subsequence in H^1. This shall be achieved in two steps: first we prove that u_n is bounded in $H^1(\Omega, \mathbb{R}^2)$ and then we find a convergent subsequence.

9.1. Step 1: u_n is bounded in H^1.

Equation (9.1) can be written as

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 + \frac{1}{4\,\varepsilon^2}\int_\Omega (1-|u_n|^2)^2 \le M\,, \qquad \text{for all } n\,, \tag{9.3}$$

and equation (9.2) means that there is a sequence $C_n \ge 0$, such that for all $v \in H^1(\Omega,\mathbb{R}^2)$,

$$\Big|\int_\Omega \nabla u_n \cdot \nabla v - \frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\,u_n\cdot v\Big| \le C_n\,\|v\|_{H^1(\Omega,\mathbb{R}^2)}\,, \tag{9.4}$$

which implies that there exists a sequence $b_n(v)$ such that $0 \le b_n(v) \le C_n$, for all n, v (and hence $b_n \longrightarrow 0$) and

$$\Big|\int_\Omega \nabla u_n \cdot \nabla v\Big| = b_n\,\|v\|_{H^1(\Omega,\mathbb{R}^2)} + \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\,u_n\cdot v\Big|\,. \tag{9.5}$$

Taking $v = u_n$ in (9.4) we obtain

$$\Big|\int_\Omega |\nabla u_n|^2 - \frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\,|u_n|^2\Big| \le C_n\,\|u_n\|_{H^1(\Omega,\mathbb{R}^2)}\,, \tag{9.6}$$

and thus

$$\Big|\int_\Omega |\nabla u_n|^2\Big| \le C_n\,\|u_n\|_{H^1(\Omega,\mathbb{R}^2)} + \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\,|u_n|^2\Big|\,. \tag{9.7}$$

First, using the Cauchy-Schwarz inequality and (9.3), we notice that,

$$\begin{aligned}\Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\,|u_n|^2\Big| &= \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)^2 - \frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\Big| \\ &\le 4\,M + \frac{1}{\varepsilon^2}\Big(\int_\Omega (1-|u_n|^2)^2\Big)^{1/2}|\Omega|^{1/2} \\ &\le 4\,M + \frac{2}{\varepsilon}\,M^{1/2}\,|\Omega|^{1/2}\,.\end{aligned} \tag{9.8}$$

Second, the same type of estimate yields

$$\begin{aligned}\Big|\int_\Omega |u_n|^2\Big| &= \Big|\int_\Omega 1-|u_n|^2+1\Big| \\ &\le \Big|\int_\Omega 1-|u_n|^2\Big| + |\Omega| \\ &\le 2\,M^{1/2}\,|\Omega|^{1/2}\,\varepsilon + |\Omega| \\ &= |\Omega| + o\,(\varepsilon)\,.\end{aligned} \tag{9.9}$$

From (9.7) and (9.8) it follows that

$$\int_\Omega |\nabla u_n|^2 \le C_n\,(\|u_n\|_{L^2} + \|\nabla u_n\|_{L^2}) + 4\,M + \frac{1}{\varepsilon}\,2\,M^{1/2}\,|\Omega|^{1/2}\,, \tag{9.10}$$

and, using (9.9), this yields

$$\begin{aligned} \|\nabla u_n\|_{L^2}^2 - C_n\,\|\nabla u_n\|_{L^2} &\le C_n\,(2\,M^{1/2}\,|\Omega|^{1/2}\,\varepsilon + |\Omega|)^{1/2} \\ &\qquad + 4\,M + \frac{1}{\varepsilon}\,2\,M^{1/2}\,|\Omega|^{1/2} \\ &= \hat{C}(M,\varepsilon)\,. \end{aligned} \tag{9.11}$$

Since $C_n \longrightarrow 0$ this implies that $\|\nabla u_n\|_{L^2(\Omega)}$ is bounded. Together with (9.9), which gives us a bound on $\|u_n\|_{L^2(\Omega)}$, this yields

$$\|u_n\|_{H^1(\Omega)} \le C(M,\varepsilon)\,. \tag{9.12}$$

which concludes the proof of the first step.

Step 2: u_n has a strongly convergent subsequence in H^1.

Since by (9.12) u_n is bounded in $H^1(\Omega,\mathbb{R}^2)$, it has a subsequence, which we will still denote by u_n which is weakly convergent in $H^1(\Omega,\mathbb{R}^2)$. Hence, using the fact that we have a compact embedding $H^1(\Omega,\mathbb{R}^2) \hookrightarrow L^2(\Omega)$, we know that, up to passing to a subsequence, there exists $u \in H^1(\Omega,\mathbb{R}^2)$ such that

$$u_n \longrightarrow u \text{ in } L^2(\Omega) \text{ and } \nabla u_n \rightharpoonup \nabla u \text{ in } L^2(\Omega)\,. \tag{9.13}$$

Therefore, we just need to prove strong convergence in $L^2(\Omega)$ of the gradients, $\nabla u_n \longrightarrow \nabla u$ in $L^2(\Omega)$. By (9.13) we already have weak convergence $\nabla u_n \rightharpoonup \nabla u$, thus we just need to prove the convergence of the $L^2(\Omega)$ norms in order to obtain strong convergence.

Since $H^1(\Omega) \hookrightarrow L^p(\Omega)$, for all $1 \le p < +\infty$, we have that

$$u_n \rightharpoonup u \text{ in } H^1 \text{ implies } u_n \longrightarrow u \text{ in } L^p\,, \quad \text{for all } 1 \le p < +\infty\,. \tag{9.14}$$

In particular

$$u_n \longrightarrow u \text{ in } L^4(\Omega) \text{ and } |u_n|^2 \longrightarrow |u|^2 \text{ in } L^4(\Omega)\,.$$

Thus, using Hölder's inequality,

$$(1-|u_n|^2)\,u_n \longrightarrow (1-|u|^2)\,u \text{ in } L^2(\Omega)\,,$$

$$(1-|u_n|^2)\,u_n\cdot u \longrightarrow (1-|u|^2)\,|u|^2 \text{ in } L^1(\Omega)\,, \tag{9.15}$$

and, since $u_n \longrightarrow u$ in $L^2(\Omega)$,

$$(1-|u_n|^2)\,u_n\cdot u_n \longrightarrow (1-|u|^2)\,|u|^2 \text{ in } L^1(\Omega)\,. \tag{9.16}$$

Taking $v = u \in H^1$ in equation (9.5) we obtain

$$\Big|\int_\Omega \nabla u_n\cdot\nabla u\Big| = b_n\,\|u\|_{H^1(\Omega,\mathbb{R}^2)} + \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u_n|^2)\,u_n\cdot u\Big|\,. \tag{9.17}$$

Passing to the limit $n \longrightarrow +\infty$, using the fact that $\nabla u_n \rightharpoonup u$ weakly in $L^2(\Omega)$, $b_n \longrightarrow 0$ and (9.15), inequality (9.17) yields

$$\int_\Omega |\nabla u|^2 = \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2)\,|u|^2\Big|\,. \tag{9.18}$$

On the other hand, passing to the limit in (9.7), using the fact that $C_n \longrightarrow 0$, (9.12), (9.16) and (9.18), we obtain

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^2 \le \Big|\frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2)\,|u|^2\Big| = \int_\Omega |\nabla u|^2\,. \tag{9.19}$$

Since by the lower semi-continuity of the L^2 norm in weak topology we have that

$$\int_\Omega |\nabla u|^2 \le \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^2\,,$$

equation (9.19) implies that

$$\int_\Omega |\nabla u|^2 = \lim_{n\to+\infty}\int_\Omega |\nabla u_n|^2\,, \tag{9.20}$$

which concludes the proof of Theorem 3 for E_ε. For the case of the functional F_ε the same proof will work once we fix the Coulomb gauge. The reader interested in seeing how the gauge invariance affects Palais-Smale sequences in this problem may take a look at the appendix of [4].

10. Threshold energies and components of E_ε^Λ.

We can reformulate the statement of Theorem 4 and state the following Proposition.

Proposition 2. *Suppose that for some $\Lambda \in \mathbb{R}^+$, we have that for some $\varepsilon < \varepsilon_0$ (where ε_0 is given Theorem 1) there exist $n, k \in \mathbb{Z}$, $n \neq k$, such that the topological sectors $\mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\mathrm{top}_k(E_\varepsilon^\Lambda)$ are both non-empty. Then, there are mountain-pass type critical points of E_ε or, equivalently, there exist mountain-pass type solutions of the Ginzburg-Landau equations (1.11).*

More precisely, consider two non-empty components of E_ε^Λ, $\Sigma_0 \subset \mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\Sigma_1 \subset \mathrm{top}_k(E_\varepsilon^\Lambda)$, and let $c_{n,k}(\Sigma_0, \Sigma_1)$ be defined as in (10.4). Then, there exists a map $u \in H^1(\Omega, \mathbb{R}^2)$ which is a critical point of E_ε and such that $E_\varepsilon(u) = c_{n,k}(\Sigma_0, \Sigma_1)$.

Since $H^1(\Omega)$ is locally pathwise connected and the level sets E_ε^Λ are open, their path components coincide with their components, so we can use the two concepts indistinguishably. Let $n, k \in \mathbb{Z}$ be two distinct integers, and let Σ_0 and Σ_1 be components of E_ε^Λ such that $\Sigma_0 \subset \mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\Sigma_1 \subset \mathrm{top}_k(E_\varepsilon^\Lambda)$. Then, given $u_0, u_0' \in \Sigma_0$ and $u_1, u_1' \in \Sigma_1$, we know that there exist two paths γ_i, $i = 0, 1$, such that

$$\gamma_i : [0,1] \longrightarrow \Sigma_i \ , \ \gamma_i(0) = u_i \ , \ \gamma_i(1) = u_i' \ , \ i = 0, 1 \, .$$

In particular,

$$\gamma_i(s) < \Lambda \, , \qquad \text{for all } s \in [0,1] \, . \tag{10.1}$$

As usual, we define the composition operation for paths: let γ be a path from p to q, and σ be a path from q to r, then $\varrho = \gamma\,\sigma$ is the path from p to r defined by

$$\varrho(s) := \begin{cases} \gamma(2\,s) \, , & \text{for } 0 \leq s \leq \dfrac{1}{2} \, , \\[2ex] \sigma(2\,s - 1) \, , & \text{for } \dfrac{1}{2} \leq s \leq 1 \, . \end{cases}$$

And we define the inverse path of γ, which we denote by γ^{-1}, as $\gamma^{-1}(s) := \gamma(1-s)$, for $s \in [0,1]$. Then, to any path $\gamma : [0,1] \longrightarrow H^1(\Omega)$ between u_0 and u_1, one can associate a path $\gamma' = \gamma_0^{-1}\,\gamma\,\gamma_1 : [0,1] \longrightarrow$

$H^1(\Omega)$ from u_0' to u_1'. And vice-versa, to any path $\gamma' : [0,1] \longrightarrow H^1(\Omega)$ between u_0' and u_1', one can associate a path $\gamma = \gamma_0\, \gamma'\, \gamma_1^{-1} : [0,1] \longrightarrow H^1(\Omega)$ from u_0 to u_1. With these definitions, from equation (10.1) it follows that

$$\max_{s\in[0,1]} E_\varepsilon(\gamma(s)) = \max_{s\in[0,1]} E_\varepsilon(\gamma'(s)) \geq \Lambda\,. \tag{10.2}$$

And hence,

$$\inf_{\gamma\in\mathcal{V}} \big(\max_{s\in[0,1]}(E_\varepsilon(\gamma(s)))\big) = \inf_{\gamma\in\mathcal{V}'} \big(\max_{s\in[0,1]}(E_\varepsilon(\gamma'(s)))\big) \geq \Lambda\,, \tag{10.3}$$

where,

$$\mathcal{V} := \{\gamma \in C^0([0,1], H^1(\Omega,\mathbb{R}^2)) :\ \gamma(0) = u_0,\ \text{and } \gamma(1) = u_1\}\,,$$

and

$$\mathcal{V}' := \{\gamma' \in C^0([0,1], H^1(\Omega,\mathbb{R}^2)) :\ \gamma'(0) = u_0',\ \text{and } \gamma'(1) = u_1'\}\,.$$

Thus, c_n, the threshold energy for a transition from u_0 to u_1 defined in (1.16), is well defined as a transition energy from a component Σ_0 of $\mathrm{top}_n(E_\varepsilon^\Lambda)$ to a component Σ_1 of $\mathrm{top}_k(E_\varepsilon^\Lambda)$. We can define,

$$c_{n,k}(\Sigma_0,\Sigma_1) := \inf_{\gamma\in\mathcal{V}_{n,k}(\Sigma_0,\Sigma_1)} \big(\max_{s\in[0,1]}(E_\varepsilon(\gamma(s)))\big)\,, \tag{10.4}$$

where,

$$\begin{aligned}\mathcal{V}_{n,k}&(\Sigma_0,\Sigma_1)\\ &:= \{\gamma \in C^0([0,1], H^1(\Omega,\mathbb{R}^2)) :\ \gamma(0) \in \Sigma_0 \subset \mathrm{top}_n(E_\varepsilon^\Lambda),\\ &\qquad\qquad \text{and } \gamma(1) \in \Sigma_1 \subset \mathrm{top}_k(E_\varepsilon^\Lambda)\}\,.\end{aligned}$$

By the Mountain Pass Theorem we know that $c_{n,k}(\Sigma_0,\Sigma_1)$ is a generalized critical value of E_ε and, since by Theorem 3 the functional E_ε satisfies the Palais-Smale condition, this implies that $c_{n,k}(\Sigma_0,\Sigma_1)$ is also a critical value of E_ε, thus concluding the proof of Proposition 2 and Theorem 4.

REMARK. For small ε and $n \neq k$, $c_{n,k}(\Sigma_0,\Sigma_1)$ shouldn't depend on the specific components $\Sigma_0 \subset \mathrm{top}_n(E_\varepsilon^\Lambda)$ and $\Sigma_1 \subset \mathrm{top}_k(E_\varepsilon^\Lambda)$, but only on n and k (*i.e.* only on the topological sectors themselves). This leads us back to the question of how many distinct components can there be inside a topological sector and how do they change when Λ changes. We expect that for certain values of Λ, $\mathrm{top}_n(E_\varepsilon^\Lambda)$ may not be connected, but that as we increase Λ the different components which existed at lower energies, should increase in size and eventually intersect thus becoming the same component. As a matter of fact, in [1] we will be able to prove that all the components in $\mathrm{top}_n(E_\varepsilon^\Lambda)$ can be connected by paths wich involve energies of, at most, something like $6\,\Lambda$, while to connect different topological sectors we will need energies like $\pi\,|\log\varepsilon|$, which for small enough ε is much bigger than $6\,\Lambda$. In this case $c_{n,k}(\Sigma_0,\Sigma_1)$ will depend only on n and k as we said.

Remark. As usual, similar results are valid for F_ε.

11. A model for superconductivity.

In this section we will consider the gauge-invariant Ginzburg-Landau model (1.3), and prove that inside the level sets F_ε^Λ we can define topological sectors in a similar way to the one used for defining such sectors inside the level sets E_ε^Λ in theorems 1 and 6 which we proved in Section 8.

11.1. Gauge fixing.

Given a configuration $(v,B)\in F_\varepsilon^\Lambda$, we will show in this section how to choose a gauge equivalent configuration, $(u,A)\approx(v,B)$, such that we have the necessary control on A to allow us to bound the L^2 norm of ∇u by a constant depending only on the energy level Λ. In fact, to achieve this, all we need to do is to fix a Coulomb gauge over the unit disk $D=B(0,1)=\Omega\cup B(0,1/4)$.

Proposition 3. *Given a configuration $(v,B)\in H^1$, there exists $(u,A)\approx(v,B)$ such that*

$$\text{(1.11)}\qquad \begin{cases} d^\star A=0\,, & \text{in } D\,,\\ A\cdot\nu=0\,, & \text{on } \partial D=S^1\,. \end{cases}$$

The proof is just the same as that of [9, Propositions I.1 and I.2]. Now we remark that, since D is simply-connected, (11.1) implies that there exists $\zeta \in H^2(D,\mathbb{R})$ such that writing $\hat{\zeta} = \zeta\, dx^1 \wedge dx^2 = \star\zeta$,

$$\begin{cases} A = d^\star \hat{\zeta} = \star\, d\zeta\,, & \text{in } D\,, \\ \zeta = 0\,, & \text{on } \partial D\,. \end{cases} \tag{11.2}$$

It follows from (11.1) and (11.2) that ζ satisfies

$$\begin{cases} \Delta\zeta = d^\star\, d\zeta = \star\, dA\,, & \text{in } D\,, \\ \zeta = 0\,, & \text{on } \partial D\,. \end{cases} \tag{11.3}$$

This implies, using standard elliptic estimates, that

$$\|\zeta\|_{W^{2,2}(D)} \leq \hat{C}\, \|dA\|_{L^2(D)}\ ,$$

which, together with (11.2) yields

$$\begin{aligned} \|A\|^2_{W^{1,2}(D)} &= \int_D |A|^2 + \int_D |\nabla A|^2 \\ &= \int_D |\nabla\zeta|^2 + \int_D |\nabla^2\zeta|^2 \\ &\leq \|\zeta\|^2_{W^{2,2}(D)} \\ &\leq \hat{C} \|dA\|^2_{L^2(D)} \\ &\leq \hat{C}\, F_\varepsilon(u,A) \\ &< \hat{C}\, \Lambda\,, \end{aligned} \tag{11.4}$$

where $\hat{C}$ is a constant.

11.2. Global control of $|\nabla u|^2$.

The purpose of this subsection is to show how to obtain a bound on $\|\nabla u\|_{L^2(\Omega)}$ by a constant depending only on the energy level Λ.

Lemma 6. *Given $(v,B) \in F_\varepsilon^\Lambda$, let (u,A) be as in Proposition* 3. *Then,*

$$\int_\Omega |\nabla u|^2 \leq C\,, \tag{11.5}$$

where C is a constant which only depends on Λ.

Proof. Since, by construction, $F_\varepsilon(u,A) = F_\varepsilon(v,B) \leq \Lambda$, we have that, in particular,

$$\int_\Omega |\nabla_A u|^2 = \int_\Omega |\nabla u - \imath\, A\, u|^2 \leq 2\, F_\varepsilon(u,A) \leq 2\,\Lambda\,. \tag{11.6}$$

Hence,

$$\begin{aligned}
\int_\Omega |\nabla u|^2 &= \int_\Omega |\nabla u - \imath\, A\, u + \imath\, A\, u|^2 \\
&\leq 2 \int_\Omega |\nabla u - \imath\, A\, u|^2 + 2 \int_\Omega |A\, u|^2 \\
&\leq 4\, F_\varepsilon(u,A) + 2 \int_\Omega |A|^2\, |u|^2 \\
&\leq 4\,\Lambda + 2 \int_\Omega |A|^2\, (|u|^2 - 1) + 2 \int_\Omega |A|^2 \\
&\leq 4\,\Lambda + 2 \int_\Omega |A|^2\, |1 - |u|^2| + 2 \int_\Omega |A|^2\,.
\end{aligned} \tag{11.7}$$

Using Hölder's inequality, and the fact that from the energy bound it follows that

$$\|1 - |u|^2\|^2_{L^2(\Omega)} \leq 4\,\varepsilon^2\, F_\varepsilon(u,A) \leq 4\,\varepsilon^2\,\Lambda\,,$$

we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^2 &\leq 4\,\Lambda + 2\, \|A^2\|_{L^2(\Omega)}\, \|1 - |u|^2\|_{L^2(\Omega)} + 2\, \|A\|^2_{L^2(\Omega)} \\
&\leq 4\,\Lambda + 4\,\varepsilon\,\Lambda^{1/2}\, \|A\|^2_{L^4(\Omega)} + 2\, \|A\|^2_{L^2(\Omega)}\,.
\end{aligned} \tag{11.8}$$

Since we are in a two-dimensional domain it follows from the Sobolev Embedding Theorem that $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$, for all $q < +\infty$. hence, in particular, there exists a constant $\tilde{C}$ such that

$$\|A\|_{L^4(\Omega)} \leq \tilde{C}\, \|A\|_{W^{1,2}(\Omega)}\,. \tag{11.9}$$

Furthermore, from (11.4) we know that

$$\|A\|_{W^{1,2}(\Omega)} \leq \|A\|_{W^{1,2}(D)} \leq \sqrt{\hat{C}\,\Lambda}\,. \tag{11.10}$$

From equations (11.8), (11.9) and (11.10) it follows that for $\varepsilon < 1$ (as mentioned before, it is the case where ε is small that interests us),

$$\begin{aligned}\int_\Omega |\nabla u|^2 &\le 4\,\Lambda + 4\,\varepsilon\,\Lambda^{1/2}\,\tilde{C}^2\,\|A\|^2_{W^{1,2}(D)} + 2\,\|A\|^2_{W^{1,2}(D)} \\ &\le 4\,\Lambda + 4\,\Lambda^{1/2}\,\tilde{C}^2\,\hat{C}\,\Lambda + 2\,\Lambda\,\hat{C} = C\,,\end{aligned} \tag{11.11}$$

where C is a constant depending only on Λ.

11.3. Definition of $\deg([v, B], \Omega)$ and proof of Theorem 2.

Once we have the estimate (11.5), we can define $\deg(u, \Omega)$ as in the case of the initial model (1.1), since we will have all the estimates we used in the work that culminated with the definition of the degree in Section 6. Thus, for ε sufficiently small, $\deg(u, \Omega)$ is well defined, and hence we may define

$$\deg([v, B], \Omega) := \deg(u, \Omega)\,.$$

Once we have achieved this, Theorem 2 follows from the corresponding result for $\deg(u, \Omega)$ which, thanks to estimate (11.5), can be proven in a similar way to that we used for proving Theorem 1 (therefore, we omit this proof).

The generalization of Theorem 2 to the setting of Riemannian manifolds will then follow from Theorem 2 in an analogous way as Theorem 6 followed from Theorem 1.

12. Appendix: Covering Lemma.

This section is devoted to a general covering Lemma we used to prove Lemma 4.

Lemma 7. *Let $\varepsilon > 0$ and $W_1, \dots, W_n$ be connected open subsets of $\mathbb{R}^2$ such that there exist $C, \alpha > 0$ such that* $\operatorname{diam}(W_l) \le C\,\varepsilon^\alpha$. *Then, for ε sufficiently small, there is a family of numbers $\alpha_1, \dots, \alpha_m \ge \alpha/2$, and a family of balls $B_1, \dots, B_m$, with $m \le n$, such that, denoting by x_j the center of B_j, and by r_j its radius,*

i) $r_j \le C\varepsilon^{\alpha_j}$.

ii) $\displaystyle\bigcup_{l=1}^{n} W_l \subset \bigcup_{j=1}^{m} B_j$.

iii) *The enlarged balls* $\tilde{B}_j := B(x_j, \varepsilon^{-\alpha_j/(2^{n+1}+1)}\, r_j)$ *are pairwise disjoint.*

Proof. We start by defining

$$q_n := \frac{2^{n+1}}{2^{n+1}+1}\,,$$

$$p_k := \frac{1}{\displaystyle\sum_{j=0}^{k} 2^{-j}} = \frac{2^k}{2^{k+1}-1}\,,$$

for $k = 1, \dots, n$.

The proof of this Lemma is done by induction on the number k of components of $A = \bigcup_{l=1}^{n} W_l$. For $k = 1$, it suffices to consider a unique ball of radius $r_1 = C\,\varepsilon^{\alpha_1}$, with $\alpha_1 = 2\,\alpha/3 = \alpha\, p_1$, since, for ε sufficiently small,

$$\operatorname{diam}(A) \le \sum_{l=1}^{n} \operatorname{diam}(W_l) \le n\, C\, \varepsilon^{\alpha} \le C\, \varepsilon^{2\alpha/3}\,. \tag{12.1}$$

Hence, we can find a ball B_1, of radius $r_1 \le C\varepsilon^{2\alpha/3}$ containing $\bigcup_{l=1}^{n} W_l$.

Suppose that the result is always true if A has $\overline{n}$ components, for all $\overline{n} \le k-1 \le n-1$, and, furthermore, the number m of balls obtained in the covering process is at most $\overline{n}$ and each of the α_j's obtained satisfies

$$\alpha_j \ge \frac{\alpha}{\displaystyle\sum_{j=0}^{\overline{n}} 2^{-j}} = \alpha\, p_{\overline{n}} \ge \alpha\, p_{k-1}\,. \tag{12.2}$$

To complete the induction argument, we just have to show that then the result will still be true when A has k components, and that in this case $m \le k \le n$ and we can find α_j's such that

$$\alpha_j \ge \frac{\alpha}{\displaystyle\sum_{j=0}^{k} 2^{-j}} = \alpha\, p_k\,.$$

Let $A_1, \ldots, A_k$ be the connected components of A. Suppose that

$$\operatorname{diam}(A) \leq 5\, n\, C\, \varepsilon^{\alpha q_n p_{k-1}} . \tag{12.3}$$

Then, for ε sufficiently small, we can include A in a ball B_1 of radius $r_1 \leq \varepsilon^{\alpha p_k}$. In fact, it suffices that

$$3\, n\, C\, \varepsilon^{\alpha q_n p_{k-1}} \leq C\, \varepsilon^{\alpha q_n p_k} .$$

This is always true, provided that ε is sufficiently small, since

$$\alpha\, q_n\, p_{k-1} > \alpha\, p_k \qquad \text{if and only if} \qquad \frac{p_{k-1}}{p_k} > \frac{1}{q_n} ,$$

and

$$\frac{p_{k-1}}{p_k} = 1 + \frac{2^{-k}}{p_{k-1}} > 1 + \frac{1}{2^{k+1}} = \frac{2^{k+1}}{2^{k+1}+1} \geq \frac{2^{n+1}}{2^{n+1}+1} = \frac{1}{q_n} .$$

Thus, if (12.3) is true, our proof will be completed. Hence, we may suppose that this is not so, *i.e.*, that

$$\operatorname{diam}(A) \geq 5\, n\, C\, \varepsilon^{\alpha q_n p_{k-1}} . \tag{12.4}$$

Let $y_1, y_2 \in \overline{A}$ be such that $|y_1 - y_2| = \operatorname{diam}(A)$, and consider the family of balls $B(y_1, r)$ for $r \in (0, \operatorname{diam}(A))$. Define $G_j := \{r : B(y_1, r) \cap A_j \neq \varnothing\}$, $j = 1, \ldots, k$. Each G_j will be an interval, and the sum of the lengths of the G_j's will be smaller than the sum of the diameters of the W_l's, which is at most $nC\varepsilon^\alpha$. Since $n\, C\, \varepsilon^\alpha \leq n\, C\, \varepsilon^{\alpha q_n p_{k-1}}$, for all $\varepsilon \leq 1$, it follows that the set

$$\hat{G} := (0, \operatorname{diam}(A)) \setminus \bigcup_{j=1}^{k} G_j ,$$

will have a measure of at least

$$5\, n\, C \varepsilon^{\alpha q_n p_{k-1}} - n\, C\, \varepsilon^{\alpha q_n p_{k-1}} = 4\, n\, C\, \varepsilon^{\alpha q_n p_{k-1}} .$$

Moreover, the set $\hat{G}$ is the union of, at most, $k-1$ subintervals of $(0, \operatorname{diam}(A))$ since it was obtained from the latter by removing the k open intervals G_j (among which one had endpoint 0 and another had endpoint diam(A)). Consequently, at least one of its components, which we will denote by $[a_0, b_0]$, will be such that

$$b_0 - a_0 \geq \frac{|\hat{G}|}{k-1} \geq \frac{4\,n}{k-1}\,\varepsilon^{\alpha q_n p_{k-1}} > 4\,\varepsilon^{\alpha q_n p_{k-1}}\,. \tag{12.5}$$

Let $\hat{A} = A \cap B(y_1, a_0)$, and $\tilde{A} = A \setminus B(y_1, b_0)$. Then, $A = \hat{A} \cup \tilde{A}$, and both $\hat{A}$ and $\tilde{A}$ include at least one of the A_j's. Hence, both $\hat{A}$ and $\tilde{A}$ have at most $k-1$ components and thus we can apply the induction step to each of them. It yields, since the sum of the number of components of $\hat{A}$ and $\tilde{A}$ is k, that there will be a total of $m \leq k$ balls $B_1, \ldots, B_m$, such that

a) $\hat{A} \subset B_1 \cup \cdots \cup B_{\overline{m}}$, $\tilde{A} \subset B_{\overline{m}+1} \cup \cdots \cup B_m$, for some $\overline{m} < m$.

b) Each B_j has center x_j and radius $r_j \leq C\,\varepsilon^{\alpha_j}$, where $\alpha_j \geq \alpha\,p_{k-1} \geq \alpha\,p_k$.

c) The enlarged balls $\tilde{B}_j := B(x_j, \varepsilon^{-\alpha_j/(2^{n+1}+1)}\,r_j)$ are pairwise disjoint for $j \in \{1, \ldots, \overline{m}\}$ and also for $j \in \{\overline{m}+1, \ldots, m\}$.

However, to obtain the disjointness of two $\tilde{B}_j$, one corresponding to $\hat{A}$ (*i.e.* $j \leq \overline{m}$) and the other to $\tilde{A}$ (*i.e.* $j > \overline{m}$), we need to use equation (12.5). In fact, if $j_1 \leq \overline{m}$ and $j_2 > \overline{m}$, then

$$|x_{j_1} - y_1| < a_0 + C\,\varepsilon^{\alpha_{j_1}} < a_0 + C\,\varepsilon^{\alpha q_n p_{k-1}}\,, \tag{12.6}$$

since $B_{j_1} \cap \hat{A} \neq \varnothing$, $\hat{A} \subset B(y_1, a_0)$ and by b), $\alpha_{j_1} \geq \alpha\,p_{k-1} > q_n\,\alpha\,p_{k-1}$. Similarly, we have that

$$|x_{j_2} - y_1| > b_0 - C\,\varepsilon^{\alpha_{j_2}} > b_0 - C\,\varepsilon^{\alpha q_n p_{k-1}}\,, \tag{12.7}$$

since $B_{j_2} \cap \tilde{A} \neq \varnothing$, $\tilde{A} \subset A \setminus B(y_1, b_0)$ and, by b), $\alpha_{j_2} \geq \alpha\,p_{k-1} > q_n\,\alpha\,p_{k-1}$.

Therefore, combining (12.6) and (12.7) we have

$$|x_{j_1} - x_{j_2}| > 2\,C\,\varepsilon^{\alpha q_n p_{k-1}}\,. \tag{12.8}$$

Since $\tilde{B}_{j_i}$ has radius

$$C\,\varepsilon^{q_n \alpha_{j_i}} < C\,\varepsilon^{\alpha q_n p_{k-1}}\,,$$

equation (12.8) implies that

$$\tilde{B}_{j_1} \cap \tilde{B}_{j_2} = \varnothing\,,$$

as desired. Consequently, the balls B_j obtained satisfy all the conditions required for the induction argument, and thus the proof of Lemma 7 in completed.

REMARK. Relative to the similar covering argument of Lin [22], our result has the advantage that we are able to keep the α_j always bigger than $\alpha/2$, which corresponds to keeping the balls B_j rather small – in Lin's result α_j may tend to zero when $n \longrightarrow \infty$. However, we also lose something, both because our proof is technically more complicated, but also because we obtain smaller (and more complex) expansion factors for the $\tilde{B}_j$'s. In fact, even Lin's expansion factors ($\varepsilon^{-\alpha_j/3}$) go to 1 when $n \longrightarrow \infty$, but ours ($\varepsilon^{-\alpha_j/(2^{n+1}+1)}$) will decrease to 1 considerably faster.

We prefered to privilege the scale of the balls because it enables us to assert that in our problem, at least at a scale $\varepsilon^{1/2}$, things appear neutral to an outside observer (and it also makes the energy explosion estimate (5.13) slightly neater). Using Lin's result, the scale would depend on n, and hence on Λ, which would be less satisfactory.

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References.

[1] Almeida, L., Transition energies for Ginzburg-Landau functionals. To appear in *Nonlinearity.*

[2] Almeida, L., Bethuel, F., Méthodes topologiques pour l'équation de Ginzburg-Landau. *C. R. Acad. Sci. Paris* **320** Série I (1995), 935-938.

[3] Almeida, L., Bethuel, F., Topological methods for the Ginzburg-Landau equation. *J. Math. Pures et Appl.* **77** (1998), 1-49.

[4] Almeida, L., Bethuel, F., Multiplicity results for the Ginzburg-Landau equation in the presence of symmetries. *Houston J. Math.* **23** (1997), 733-764.

[5] Almeida, L., Bethuel, F., Guo, Y., A remark on instability of the symmetric vortices with large charge and coupling constant. *Comm. Pure Appl. Math.* **50** (1997), 1295-1300.

[6] Bethuel, F., The approximation problem for Sobolev maps between manifolds. *Acta Math.* **167** (1991), 153-206.

[7] Bethuel, F., Brezis, H., Hélein, F., Asymptotics for the minimization of a Ginzburg-Landau functional. *Calc. Var. and Partial Diff. Equations* **1** (1993), 123-148.

[8] Bethuel, F., Brezis, H., Hélein, F., *Ginzburg-Landau vortices.* Birkhäuser, 1994.

[9] Bethuel, F., Rivière, T., A minimization problem related to superconductivity. *Ann. Inst. H. Poincaré, Analyse non linéaire* **12** (1995), 243-303.

[10] Bethuel, F., Rivière, T., *Vorticité dans les modèles de Ginzburg-Landau pour la supraconductivité.* Actes du Séminaire EDP 1993-1994. Ecole Polytechnique. Exposé XVI.

[11] Boutet de Monvel-Berthier, A., Georgescu, V., Purice, R., A boundary value problem related to the Ginzburg-Landau model. *Comm. Math. Phys.* **142** (1991), 1-23.

[12] Demoulini, S., Stuart, D., Gradient flow of the superconducting Ginzburg-Landau functional on the plane. *Comm. Math. Phys.* **5** (1997), 121-198.

[13] Evans, L., Gariepy, R., *Measure theory and fine properties of functions.* CRC Press, 1992.

[14] Gilbarg, D., Trudinger, N., *Elliptic partial differential equations of second order.* 2nd edition, Grundlehren Math. Wiss. **224**, Springer, 1983.

[15] Guo, Y., Instability of the symmetric vortices with large charge and coupling constant. *Comm. Pure Appl. Math.* **69** (1996), 1051-1080.

[16] Jimbo, S., Morita, Y., Ginzburg-Landau equation and stable solutions in a rotational domain. *SIAM J. Math. Anal.* **27** (1996), 1360-1385.

[17] Jimbo, S., Morita, Y., Stable solutions with zeros to the Ginzburg-Landau equation with Neumann boundary condition. *J. Diff. Equations* (1996), 596-613.

[18] Jimbo, S., Morita, Y., Zhai, J., Ginzburg-Landau equation and stable steady state solutions in a non-trivial domain. *Comm. Partial Diff. Equations* **20** (1995), 2093-2112.

[19] Jimbo, S., Zhai, J., Ginzburg-Landau equation with magnetic effect: non-simply-connected domains. Preprint Univ. Hokkaido **303**.

[20] Jaffe, A., Taubes, C., *Vortices and monopoles.* Birkhäuser, 1980.

[21] Lin, F. H., Some dynamical properties of Ginzburg-Landau vortices. *Ann. Inst. H. Poincaré, Analyse non linéaire* **12** (1995), 599-622.

[22] Lin, F. H., Solutions of Ginzburg-Landau equations and critical points of the renormalized energy. *Ann. Inst. H. Poincaré, Analyse non linéaire* **12** (1995), 599-622.

[23] Rivière, T., Lignes de tourbillon dans le modèle abélien de Higgs. *C. R. Acad. Sci. Paris* **321** Série I (1995), 73-76.

[24] Rivière, T., Line vortices in the $U(1)$ Higgs model. *ESAIM*: *COCV* **1** (1996), 77-167.

[25] Rubinstein, J., Sternberg, P., Homotopy classification of minimizers of the Ginzburg-Landau energy and the existence of permanent currents. *Comm. Math. Phys.* **179** (1996), 257-263.

[26] Saint-James, D., Sarma, G., Thomas, E. J., *Type II superconductivity.* Pergamon Press, 1969.

[27] Tilley, D., Tilley, T., Superfluidity and superconductivity. Adam Hilger Ltd., 1990.

[28] White, B., Homotopy classes in Sobolev spaces and the existence of energy minimizing maps. *Acta Math.* **160** (1988), 1-17.

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Controllability of analytic functions for a wave equation coupled with a beam

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Abstract. We consider the controllability and observation problem for a simple model describing the interaction between a fluid and a beam. For this model, microlocal propagation of singularities proves that the space of controlled functions is smaller that the energy space. We use spectral properties and an explicit construction of biorthogonal sequences to show that analytic functions can be controlled within finite time. We also give an estimate for this time, related to the amount of analyticity of the latter function.

1. Introduction.

Let Ω be the two-dimensional square $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$.

We assume that Ω is filled with an elastic, inviscid, compressible fluid whose velocity field $\vec{v}$ is given by the potential $\Phi = \Phi(x,y,t)$, $\vec{v} = \nabla \Phi$. By linearization we assume that the potential Φ satisfies the linear wave equation in $\Omega \times (0,\infty)$.

The boundary $\Gamma = \partial\Omega$ of Ω is divided in two parts $\Gamma_0 = \{(0,y) : y \in (0,1)\}$ and $\Gamma_1 = \Gamma \backslash \Gamma_0$. The subset Γ_1 is assumed to be rigid and we impose zero normal velocity of the fluid on it. The subset Γ_0 is supposed to be flexible and occupied by a Bernoulli-Euler beam that vibrates under the pressure of the fluid on the plane where Ω lies. The displacement of Γ_0 is described by the scalar function $W = W(y,t)$. On the other hand, on Γ_0 we impose the continuity of the normal velocities of the fluid and the beam. The beam is assumed to satisfy Neumann-type boundary conditions on its extremes. All deformations are supposed to be small enough so that linear theory applies. Under natural initial conditions for Φ and W the linear motion of this system is described by means of the following coupled equations

$$(1) \qquad \begin{cases} \Phi_{tt} - \Delta\Phi = 0\,, & \text{in } \Omega \times (0,\infty)\,, \\ \dfrac{\partial \Phi}{\partial \nu} = 0\,, & \text{on } \Gamma_1 \times (0,\infty)\,, \\ \dfrac{\partial \Phi}{\partial x} = -W_t\,, & \text{on } \Gamma_0 \times (0,\infty)\,, \\ W_{tt} + W_{yyyy} + \Phi_t = 0\,, & \text{on } \Gamma_0 \times (0,\infty)\,, \\ W_y(0,t) = W_y(1,t) = 0\,, & \text{for } t > 0\,, \\ W_{yyy}(0,t) = W_{yyy}(1,t) = 0\,, & \text{for } t > 0\,, \\ \Phi(0) = \Phi^0\,,\ \Phi_t(0) = \Phi^1\,, & \text{in } \Omega\,, \\ W(0) = W^0\,,\ W_t(0) = W^1\,, & \text{on } \Gamma_0\,. \end{cases}$$

By ν we denote the unit outward normal to Ω.

In (1) we have chosen to take the various parameters of the system to be equal to one.

System (1) is well-posed in the energy space $\mathcal{Y} = H^1(\Omega) \times L^2(\Omega) \times H^2_N(\Gamma_0) \times L^2(\Gamma_0)$ for the variables (Φ, Φ_t, W, W_t) where $H^2_N(\Gamma_0) = \{v \in H^2(0,1) :\ v_y(0) = v_y(1) = 0\}$. The energy

$$(2) \qquad E(t) = \frac{1}{2}\int_\Omega (|\nabla\Phi|^2 + |\Phi_t|^2)\,dx\,dy + \frac{1}{2}\int_{\Gamma_0} (|W_{yy}|^2 + |W_t|^2)\,dy$$

remains constant along trajectories.

It is easy to see that the equilibria of these systems are of the form

$$(3) \qquad (\Phi, \Phi_t, W, W_t) = (c_1, 0, c_2, 0)\,,$$

c_1 and c_2 being constant functions.

We study the controllability of system (1) under the action of an exterior force on the flexible part of the boundary Γ_0. The control is given by a scalar function $\beta = \beta(y,t)$ in the space $H^{-2}(0,T;L^2(\Gamma_0))$. Of course this is an arbitrary choice and many others make sense. However this is the most natural one when solving the control problem by means of J. L. Lions's HUM (see [6]), as we will do. The controlled system reads as follows

$$
\begin{cases}
\Phi_{tt} - \Delta\Phi = 0\,, & \text{in } \Omega \times (0,\infty)\,,\\
\dfrac{\partial \Phi}{\partial \nu} = 0\,, & \text{on } \Gamma_1 \times (0,\infty)\,,\\
\dfrac{\partial \Phi}{\partial x} = -W_t\,, & \text{on } \Gamma_0 \times (0,\infty)\,,\\
W_{tt} + W_{yyyy} + \Phi_t = \beta\,, & \text{on } \Gamma_0 \times (0,\infty)\,,\\
W_y(0,t) = W_y(1,t) = 0\,, & \text{for } t > 0\,,\\
W_{yyy}(0,t) = W_{yyy}(1,t) = 0\,, & \text{for } t > 0\,,\\
\Phi(0) = \Phi^0\,,\ \Phi_t(0) = \Phi^1\,, & \text{in } \Omega\,,\\
W(0) = W^0\,,\ W_t(0) = W^1\,, & \text{on } \Gamma_0\,.
\end{cases}
\tag{4}
$$

The problem of controllability can be formulated as follows: Given $T > 2$, find the space of initial data $(\Phi^0, \Phi^1, W^0, W^1)$ that can be driven to an equilibrium of the form (3) in time T by means of a suitable control $\beta \in H^{-2}(0,T;L^2(\Gamma_0))$.

The model under consideration is inspired in and related to that of H. T. Banks et al. in [4]. However, there are some important differences between these two models. First of all, we choose Neumann-type boundary conditions for the beam. These are compatible with those of Φ in order to develop solutions in Fourier series. Another difference is related to the nature of the controls. In [4] the control acts on the system through a finite number of piezoceramic patches located on Γ_0. This restricts very much the set of admissible controls, that are essentially second derivatives of Heaviside functions, and much weaker controllability results have to be expected. In [4] the controllability problem is not addressed. Instead, they consider a quadratic optimal control problem. More recently in [2] a Riccati equation for the optimal control is derived. The problem of the controllability of one-dimensional beams with piezoelectric actuators has been successfully addressed by M. Tucsnak [9]. However, to our knowledge, there are no rigorous results on the controllability of fluid-structure systems under such controls. In [7] the controllability problem for a similar system with a string instead of a beam was studied. It was shown that a space of analytical initial data can be controlled in any time $T > 2$. The techniques we develop in the present article can be applied to that case and allow to show that larger and larger classes of analytic functions can be controlled in finite time.

The propagation of singularities for the wave equation on any segment parallel to Γ_0 proves that the space of controlled functions will be small. It will not contain all functions of finite energy.

Let us denote by $\mathcal{X} = H^1(0,1) \times L^2(0,1) \times \mathbb{C} \times \mathbb{C}$ and by $\mathcal{X}'$ its dual space. Let also $\mathcal{Y}^n = (H^1(0,1) \times L^2(0,1) \times \mathbb{C} \times \mathbb{C}) \cos(n\,\pi\,y)$.

By the HUM method, we will first prove that if $C(n,T)$ is a sequence of constants such that any solution of the observation problem

$$(5) \qquad \begin{cases} \Phi_{tt} - \Delta\Phi = 0\,, & \text{in } \Omega \times (0,\infty)\,, \\ \dfrac{\partial \Phi}{\partial \nu} = 0\,, & \text{on } \Gamma_1 \times (0,\infty)\,, \\ \dfrac{\partial \Phi}{\partial x} = W_t\,, & \text{on } \Gamma_0 \times (0,\infty)\,, \\ W_{tt} + W_{yyyy} - \Phi_t = 0\,, & \text{on } \Gamma_0 \times (0,\infty)\,, \\ W_y(0,t) = W_y(1,t) = 0\,, & \text{for } t > 0\,, \\ W_{yyy}(0,t) = W_{yyy}(1,t) = 0\,, & \text{for } t > 0\,, \\ (\Phi, \Phi_t)|_{t=0} = (\Phi^0, \Phi^1)\,, & \text{in } \Omega\,, \\ (W, W_t)|_{t=0} = (W^0, W^1)\,, & \text{on } \Gamma_0\,, \end{cases}$$

with initial conditions in $\mathcal{Y}^n$, satisfies

$$\|(\Phi^0, \Phi^1, W^0, W^1)\|_{\mathcal{Y}}^2 \le C(n,T) \int_{-T}^{T} |W_{tt}(0,t)|^2\,dt\,,$$

then the space of initial data

$$H = \Big\{ \sum_n (\Phi^0, \Phi^1, W^0, W^1)_n \cos(n\,\pi\,y) \;|\, (\Phi^0, \Phi^1, W^0, W^1)_n \in \mathcal{X}$$
$$\text{such that } \sum_n C(n,T)\,(\|(\Phi^0, \Phi^1, W^0, W^1)_n\|_{\mathcal{X}'}^2 + |\Phi^0(0)|^2) < \infty \Big\}$$

is a subset of the space of controlled functions. Remark that the space H depends on the constants $C(n,T)$: when $C(n,T)$ "increase", H becomes smaller.

This paper aims at proving that, for T and n large enough,

$$C(n,T) \le C\, e^{\alpha(T)|n|} \tag{6}$$

with the following property

Theorem 1. *For any positive real number q, there is a constant C_q such that*

$$\alpha(T) \le \frac{C_q}{T^{1-q}}\,. \tag{7}$$

It means that any initial condition whose Fourier coefficients in y decrease like $e^{-|n|\alpha}$ can be controlled if T is larger than $T(\alpha) = \sqrt[1-q]{C_q/\alpha}$. This condition on the Fourier coefficients means that the initial condition is analytic with respect to y and that it can be continued as an holomorphic function over the complex strip $|\mathrm{Im}\, y| < \alpha$.

Now any initial condition that is analytic with respect to y can be continued as an holomorphic function over a such a strip $|\mathrm{Im}\, y| < \varepsilon$ for a positive ε that depends on this initial condition. Therefore, its Fourier coefficients with respect to y decrease like $e^{-|n|\varepsilon}$. So according to Theorem 1 and (6), it can be controlled if $T > T(\varepsilon)$.

This means that any initial condition of finite energy that is analytic with respect to y can be controlled in a finite time (which is not uniform).

It is important to notice that analyticity is required only with respect to the variable y. Therefore the space of controlled functions is not symmetric in x and y. This means that we do not use the fact that the metrics in our problem is analytic with respect to x. In [1], the boundary control problem is studied on a surface of revolution. The same kind of result is proved in that case, even if the surfece is only $\mathcal{C}^\infty$. This is posible because such surface is still “analytic” with respect to the angular variable, even if it is only $\mathcal{C}^\infty$ with respect to its axial variable.

The rest of the article is organized as follows. In Section 2 we give a direct estimate for the observation problem and, by using (6), we apply Hilbert Uniqueness Method to solve our controllability problem. We obtain that the initial data from H can be controlled in time T. In Section 3 we prove some spectral properties of the operator that will be used in the proof of the main theorem in Section 4. In the last section an explicit dependence of the space H on the time T is obtained.

2. The direct estimate and the controllability problem.

2.1. Direct estimate.

Let us consider the system

$$(8)\qquad \begin{cases} \eta_{tt} - \eta_{xx} + n^2\,\pi^2\,\eta = f\,, & \text{in } (0,1)\times(0,T)\,,\\ \eta_x(1) = 0\,, & \text{for } t\in(0,T)\,,\\ \eta_x(0) = u_t\,, & \text{for } t\in(0,T)\,,\\ u_{tt} + n^4\,\pi^4\,u - \eta_t(0) = g\,, & \text{for } t\in(0,T)\,,\\ \eta(0) = \eta^0\,,\ \eta_t(0) = \eta^1\,, & \text{in } (0,1)\,,\\ u(0) = u^0\,,\ u_t(0) = u^1\,. \end{cases}$$

The unknowns are $\eta = \eta(x,t)$ and $u = u(t)$. Of course, since the coefficients of the system depend on $n = 0, 1, \ldots$, solutions (η, u) depend on n too. However, in order to simplify the notations we will not use the index n to distinguish the solutions of (8) for the different values of n.

The energy space for system (8) is the Hilbert space $\mathcal{X} = H^1(0,1)\times L^2(0,1)\times\mathbb{C}\times\mathbb{C}$.

It is easy to see that for any $(\eta^0,\eta^1,u^0,u^1)\in\mathcal{X}$ and $(f,g)\in L^1(0,T;L^2(0,1)\times\mathbb{C})$ system (8) has a unique solution in the class

$$(9)\qquad \eta\in C([0,T];H^1(0,1))\cap C^1([0,T];L^2(0,1));\ \ u\in C^1([0,T];\mathbb{C})\,.$$

In other words $(\eta,\eta_t,u,u_t)\in C([0,T];\mathcal{X})$.

The energy of the system

$$(10)\qquad F(t) = \frac{1}{2}\int_0^1 (|\eta_t|^2 + |\eta_x|^2 + n^2\,\pi^2\,\eta^2)\,dx + \frac{1}{2}\,(|u_t|^2 + n^4\,\pi^4\,|u|^2)$$

satisfies

$$(11)\qquad \frac{dF(t)}{dt} = \int_0^1 f(x,t)\,\eta_t(x,t)\,dx + g(t)\,u_t(t)\,.$$

Therefore, when $f\equiv 0$ and $g\equiv 0$, the energy F remains constant along trajectories.

We observe that when $n\geq 1$ the square root of F defines a norm in $\mathcal{X}$ equivalent to the canonical norm $\|\cdot\|_{\mathcal{X}}$ of $\mathcal{X}$

$$(12)\qquad \|(u,v,w,z,)\|_{\mathcal{X}} = \Big(\int_0^1 (|u_x|^2 + |u|^2 + |v|^2)\,dx + w^2 + z^2\Big)^{1/2}.$$

However, when $n = 0$ this is not the case. Actually, for $n = 0, (\eta, u) = (c_1, c_2)$ with c_1, c_2 real constants are stationary solutions of (8) with $f \equiv 0$, $g \equiv 0$ for which the energy F vanishes.

We have the following "hidden regularity" result

Proposition 1. *For any $T > 0$ there exists a constant $C(T) > 0$ independent of $n = 0, 1, \dots$ such that*

$$
\begin{aligned}
\Big(\int_0^T |u_{tt}|\, dt\Big)^2 &+ \int_0^T (|u_t|^2 + (1 + n^8\,\pi^8)\,u^2 + (1 + n^2\,\pi^2)\,\eta^2(0,t))dt \\
&\le C\,(n^4+1)\,(\|(\eta^0, \eta^1, u^0, u^1)\|^2_{\mathcal{X}} \\
&\qquad + \|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)})\,,
\end{aligned}
\tag{13}
$$

for any $(\eta^0, \eta^1, u^0, u^1) \in \mathcal{X}$, $f \in L^1(0,T; L^2(0,1))$ and $g \in L^1(0,T)$.

If $g \in L^2(0,T)$, then $u \in H^2(0,T)$ and we also have

$$
\begin{aligned}
\int_0^T |u_{tt}|^2\, dt &\le C\,(n^4+1)\,(\|(\eta^0, \eta^1, u^0, u^1)\|^2_{\mathcal{X}} \\
&\qquad + \|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^2(0,T)})\,.
\end{aligned}
\tag{14}
$$

REMARK. This proposition shows that u is more smooth than what (9) guarantees. This is due to the structure of the second order (in time) equation that u satisfies. The fact that the constant c in (13) and (14) does not depend on the index n is worth mentioning.

PROOF OF PROPOSITION 1. It is enough to consider smooth solutions since a classical density argument allows to extend inequalities (13) and (14) to any solution with finite right hand side. We use a classical multiplier technique (see, for instance, [6]). We multiply the first equation in (8) by $(1-x)\,\eta_x$ and integrate over $(0,1) \times (0,T)$. Integrating by parts we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^T (|\eta_t|^2 + |\eta_x|^2 - n^2\,\pi^2\,\eta^2)(0,t)\,dt \\
&= -\int_0^1 \eta_t\,(1-x)\,\eta_x\,dx\Big|_0^T \\
&\quad + \frac{1}{2}\int_0^T\int_0^1 (\eta_t^2 + \eta_x^2 - n^2\,\pi^2\,\eta^2)\,dx\,dt + \int_0^T\int_0^1 f(1-x)\,\eta_x\,dx\,dt \\
&= X\,.
\end{aligned}
$$

In this identity we use the notation $L|_0^T = L(T) - L(0)$. The right hand side of this identity can be easily bounded as follows

$$\begin{aligned}
|X| &\le \frac{1}{2}\int_0^1 (\eta_t^2 + \eta_x^2)(x,0)\,dx + \frac{1}{2}\int_0^1 (\eta_t^1 + \eta_x^2)(x,T)\,dx \\
&\quad + \int_0^T F(t)\,dt + \frac{1}{2}\left(\|f\|^2_{L^1(0,T;L^2(0,1))} + \|\eta_x\|^2_{L^\infty(0,T;L^2(0,1))}\right) \\
&\le F(0)+F(T)+\int_0^T F(t)\,dt+\|F(t)\|_{L^\infty(0,T)}+\frac{1}{2}\,\|f\|^2_{L^1(0,T;L^2(0,1))} \\
&\le C\left(\|F\|_{L^\infty(0,T)} + \|f\|^2_{L^1(0,T;L^2(0,1))}\right),
\end{aligned}$$

with $C > 0$ independent of n.

In the sequel, if some constant in the inequalities depends on n, we will make it explicit by means of an index n on that constant.

On the other hand, from identity (11) and using Gronwall's inequality it is easy to deduce that

$$\|F\|^2_{L^\infty(0,T)} \le C\left(\|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)} + F(0)\right).$$

Since $H^1(0,1)$ is continuously embedded in $C([0,1];\mathbb{C})$ we also have

$$\begin{aligned}
\int_0^T \eta^2(0,t)\,dt &\le C\int_0^T F(t)\,dt \\
&\le C\left(\|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)} + F(0)\right).
\end{aligned}$$

Combining these inequalities we deduce that

$$\begin{aligned}
&\int_0^T (|\eta_t|^2 + |\eta_x|^2 + n^2\pi^2\eta^2)(0,t)\,dt \\
&\qquad \le C\,(n^2+1)\left(\|(\eta^0,\eta^1,u^0,u^1)\|^2_{\mathcal{X}}\right. \\
&\qquad\qquad \left. + \|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)}\right).
\end{aligned} \tag{15}$$

On the other hand

$$\begin{aligned}
n^8\pi^8\int_0^T u^2(t)\,dt &\le 2\,n^4\pi^4\int_0^T F(t)\,dt \\
&\le C\,n^4\left(\|(\eta^0,\eta^1,u^0,u^1)\|^2_{\mathcal{X}}\right. \\
&\qquad \left. + \|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)}\right).
\end{aligned} \tag{16}$$

Inequalities (13) and (14) are a direct consequence of (15) and (16) and the coupling conditions between η and u given in system (8), *i.e.*

$$\eta_y(0,t) = u_t(t), \qquad u_{tt}(t) = g(t) + \eta_t(0,t) - n^4\,\pi^4\,u(t)\,, \tag{17}$$

for $t \in (0,T)$.

2.2. A controllability result.

In this section, we solve the controllability problem (4) stated in the Introduction by using J.- L. Lions's HUM. This is done by Fourier descomposition which is possible because of the boundary conditions we have chosen for W. Indeed, W is assumed to satisfy Neumann type boundary conditions which are compatible with those of Φ to develop solutions in Fourier series.

Indeed, let us decompose the control β, the solutions Φ, W and the initial data in the following way

$$\begin{cases} \beta = \displaystyle\sum_{n=0}^{\infty} \beta_n(t)\cos(n\,\pi\,y)\,, \\ \Phi = \displaystyle\sum_{n=0}^{\infty} \Psi_n(x,t)\cos(n\,\pi\,y)\,, \\ (\Phi^0,\Phi^1) = \displaystyle\sum_{n=0}^{\infty} (\Psi_n^0(x),\Psi_n^1(x))\cos(n\,\pi\,y)\,, \\ W = \displaystyle\sum_{n=0}^{\infty} V_n(t)\cos(n\,\pi\,y)\,, \\ (W^0,W^1) = \displaystyle\sum_{n=0}^{\infty} (V_n^0,V_n^1)\cos(n\,\pi\,y)\,. \end{cases} \tag{18}$$

With this decomposition, system (4) can be split into the following sequence of one-dimensional controlled systems for $n = 0, 1, \ldots$

$$\begin{array}{ll} \Psi_{n,tt} - \Psi_{n,xx} + n^2\,\pi^2\,\Psi_n = 0\,, & \text{in } (0,1)\times(0,\infty)\,, \\ \Psi_{n,x}(1,t) = 0\,, & \text{for } t > 0\,, \\ \Psi_{n,x}(0,t) = -V_t(t)\,, & \text{for } t > 0\,, \end{array}$$

$$
(19) \qquad \begin{array}{ll}
V_{n,tt}(t) + n^4\pi^4 V_n(t) & \\
+\, \Psi_{n,t}(0,t) = \beta_n(t)\,, & \text{for } t>0\,,\\
\Psi_n(0) = \Psi_n^0,\ \Psi_{n,t}(0) = \Psi_n^1\,, & \text{in } (0,1)\,,\\
V_n(0) = V_n^0,\ V_{n,t}(0) = V_n^1\,. &
\end{array}
$$

The control β we obtain is of the form

$$\beta = \frac{\partial^2}{\partial t^2}\gamma\,,$$

with $\gamma \in L^2(\Gamma_0 \times (0,T))$ having compact support in time. Therefore $\int_0^T \beta = 0$. Taking this fact into account it is easy to see that the constants c_1, c_2 of the equilibrium we reach at time $t = T$ are determined a priori by the initial data. Indeed, integrating the first equation of (4) in Ω we obtain that

$$\int_\Omega \Phi_t\, dx\, dx - \int_{\Gamma_0} W\, dy$$

remains constant in time. Therefore, necessarily,

$$c_2 = \int_{\Gamma_0} W^0\, dy - \int_\Omega \Phi^1\, dx\, dy\,. \tag{20}$$

On the other hand, integrating the equation satisfied by W on $\Gamma_0\times(0,T)$ and taking into account that $\int_0^T \beta = 0$ we deduce that

$$\int_{\Gamma_0} W_t(T)\, dy + \int_{\Gamma_0} \Phi(0,y,T)\, dy = \int_{\Gamma_0} W^1\, dy + \int_{\Gamma_0} \Phi^0(y,0)\, dy \tag{21}$$

and therefore

$$c_1 = \int_{\Gamma_0} (W^1 + \Phi^0(0,y))\, dy\,.$$

In terms of the Fourier coefficients (18) these constants can be written in the following way

$$c_1 = V_0^1 + \Psi_0^0(0)\,, \qquad c_2 = V_0^0 - \int_0^1 \Psi_0^1(x)\, dx\,. \tag{22}$$

Therefore, the constants c_1 and c_2 of the equilibrium we may reach are uniquely determined by the Fourier coefficients of the initial data corresponding to the frequency $n = 0$ in the y-variable.

This fact is related to the different nature of systems (19) for $n = 0$ and $n \geq 1$. While for any $n \geq 1$ system (19) is exactly controllable to zero at any time $T > 2$, when $n = 0$ we can only control the system to the equilibrium given by (22) in terms of the initial data.

In this section we suppose that for any $n \in \mathbb{N}^*$ and time $T > 2$ we can find a constant $C(n,T)$ such that for any $(\Psi^0, \Psi^1, V^0, V^1) \in \mathcal{X}$, the solution of problem

$$(23)\qquad \begin{cases} \Psi_{tt} - \Psi_{xx} + n^2\,\pi^2\,\Psi = 0\,, & \text{in } (0,1)\times(0,\infty)\,,\\ \Psi_x(1,t) = 0\,, & \text{for } t > 0\,,\\ \Psi_x(0,t) = V_t(t)\,, & \text{for } t > 0\,,\\ V_{tt}(t) + n^4\,\pi^4\,V(t) - \Psi_t(0,t) = 0\,, & \text{for } t > 0\,,\\ \Psi(0) = \Psi^0\,,\ \ \Psi_t(0) = \Psi^1\,, & \text{in } (0,1)\,,\\ V(0) = V^0\,,\ \ V_t(0) = V^1\,. \end{cases}$$

satisfies

$$(24)\qquad \|(\Psi^0, \Psi^1, V^0, V^1)\|_{\mathcal{X}}^2 \leq C(n,T) \int_0^T |V_{tt}|^2\,dt\,.$$

We shall prove (24) and we shall give estimates over $C(n,T)$ in Section 4, while proving Theorem 1.

When $n \geq 1$ we have the following controllability result for (19).

Proposition 2. *Let $\mathcal{X}$ be the space $H^1(0,1)\times L^2(0,1)\times\mathbb{C}\times\mathbb{C}$. Assume that $T > 2$ and $n \geq 1$. Then, for any $(\Psi^1, \Psi^0, V^1, V^0) \in \mathcal{X}$, there exists a control $\beta \in H^{-2}(0,T)$ with compact support such that the solution (Ψ, V) of (19) satisfies*

$$(25)\qquad \Psi(T) = \Psi_t(T) \equiv 0 \ \textit{in } (0,1)\,, \qquad V(T) = V_t(T) = 0\,.$$

Remark. In the statement of Proposition 2 and in the sequel we drop the index n from the unknowns (Ψ, V) to simplify the notation.

The solution (Ψ, V) is defined by transposition. Therefore (25) has to be understood in a suitable weak sense. We will return to this question in the proof of the proposition.

The proof of Proposition 2 provides the continuous dependence of the control β on the initial data. More precisely

$$(26)\qquad \|\beta\|_{H^{-2}(0,T)}^2 \leq C(n,T)\,(\|(\Psi^1, \Psi^0, V^1, V^0)\|_{\mathcal{X}'}^2 + |\Psi^0(0)|^2)\,,$$

for any initial data $(\Psi^0, \Psi^1, V^0, V^1)$ as in the statement of Proposition 2. By $\mathcal{X}'$ we denote the dual of the space $\mathcal{X}$. The constant $C(n,T)$ in (26) is the one appearing in (24) and will be evaluated in Section 4.

Proof. We use HUM to prove this result.

Given any $(\eta^0, \eta^1, u^0, u^1) \in \mathcal{X}$ we solve the adjoint system

$$
\tag{27}
\begin{cases}
\eta_{tt} - \eta_{xx} + n^2\,\pi^2\,\eta = 0\,, & \text{in } (0,1)\times(0,T)\,,\\
\eta_x(1,t) = 0\,, & \text{for } t \in (0,T)\,,\\
\eta_x(0,t) = u_t(t)\,, & \text{for } t \in (0,T)\,,\\
u_{tt}(t) + n^4\,\pi^4\,u(t) - \eta_t(0,t) = 0\,, & \text{for } t \in (0,T)\,,\\
\eta(0) = \eta^0\,,\ \ \eta_t(0) = \eta^1\,, & \text{in } (0,1)\,,\\
u(0) = u^0\,,\ \ u_t(0) = u^1\,.
\end{cases}
$$

We fix, some non-negative smooth function $\rho(0,T) \longrightarrow \mathbb{R}$ with compact support such that $\rho \equiv 1$ in $(\varepsilon, T-\varepsilon)$ with $T - 2\,\varepsilon > 2$.

We then solve the backward system

$$
\tag{28}
\begin{cases}
\Psi_{tt} - \Psi_{xx} + n^2\,\pi^2\,\Psi = 0\,, & \text{in } (0,1)\times(0,T)\,,\\
\Psi_x(1,t) = 0\,, & \text{for } t \in (0,T)\,,\\
\Psi_x(0,t) = -V_t(t)\,, & \text{for } t \in (0,T)\,,\\
V_{tt} + n^4\,\pi^4\,V + \Psi_t(0,t) & \\
\qquad\qquad = -\dfrac{d^2}{dt^2}\left(\rho(t)\,u_{tt}(t)\right), & \text{for } t \in (0,T)\,,\\
\Psi(T) = \Psi_t(T) = 0\,, & \text{in } (0,1)\,,\\
V(T) = V_t(T) = 0\,.
\end{cases}
$$

The solution of (28) is defined by transposition (see [6]). If we multiply in (28) by any solution $(\widetilde{\eta}, \widetilde{u})$ of (8) and integrate (formally) by parts we obtain the following identity

$$
\tag{29}
\begin{aligned}
&\int_0^T \rho(t)\,u_{tt}(t)\,\widetilde{u}_{tt}(t)\,dt + \int_0^T\!\!\int_0^1 \widetilde{f}\,\Psi\,dx\,dt - \int_0^T \widetilde{g}\,V\,dt\\
&\qquad = \int_0^1 \left(-\Psi_t(0)\,\widetilde{\eta}(0) + \Psi(0)\,\widetilde{\eta}_t(0)\right)dx + V(0)\,\widetilde{\eta}(0,0)\\
&\qquad\quad + \Psi(0,0)\,\widetilde{u}(0) - V(0)\,\widetilde{u}_t(0) + V_t(0)\,\widetilde{u}(0)\,.
\end{aligned}
$$

Notice that when we derived (29) we have used the fact that ρ and its first derivative vanish for $t = 0$ and T.

We adopt (29) as definition of weak solution in the sense of transposition of (28). More precisely we say that (Ψ, V) solve (28) if (29) holds for any $(\widetilde{\eta}^0, \widetilde{\eta}^1, \widetilde{u}^0, \widetilde{u}^1) \in \mathcal{X}$ and $(\widetilde{f}, \widetilde{g}) \in L^1(0, T; L^2(0,1) \times \mathbb{C})$.

We observe that (29) can be rewritten in the following way

$$\begin{aligned} \int_0^T \rho(t)\, u_{tt}(t)\, \widetilde{u}_{tt}\, dt - \int_0^T \int_0^1 \widetilde{f}\, \Psi\, dx\, dt + \int_0^T \widetilde{g}\, V\, dt \\ = -\langle \Psi_t(0) + V(0)\, \delta_0, \widetilde{\eta}(0)\rangle + \langle \Psi(0), \widetilde{\eta}_t(0)\rangle \\ + (V_t(0) + \Psi(0,0))\, \widetilde{u}(0) - V(0)\, \widetilde{u}_t(0)\,, \end{aligned} \tag{30}$$

where $\langle\cdot,\cdot\rangle$ denotes both the duality pairing between $(H^1(0,1))'$ and $H^1(0,1)$ and the scalar product in $L^2(0,1)$ and $\delta_0 \in (H^1(0,1))'$ denotes the Dirac delta at $y = 0$.

We have the following existence and uniqueness result of solutions in the sense of transposition.

Proposition 3. *System* (28) *has a unique solution in the sense of transposition. More precisely, for any solution* (η, u) *of* (27) *with initial data in* $\mathcal{X}$*, there exists a unique* $(\Psi, V) \in C([0,T]; L^2(0,1)) \times L^2(0,T)$, $\rho^0 \in L^2(0,1)$, $\rho^1 \in (H^1(0,1))'$, $\mu^0 \in \mathbb{C}$, $\mu^1 \in \mathbb{C}$ *satisfying*

$$\begin{aligned} \int_0^T \rho(t)\, u_{tt}(t)\, \widetilde{u}_{tt}\, dt = \int_0^T \int_0^1 \widetilde{f}\, \Psi\, dx\, dt - \int_0^T \widetilde{g}\, V\, dt \\ + \langle \rho^1, \widetilde{\eta}(0)\rangle + \langle \rho^0, \widetilde{\eta}_t(0)\rangle + \mu^1\, \widetilde{u}(0) + \mu^0\, \widetilde{u}_t(0) \end{aligned} \tag{31}$$

for any solution $(\widetilde{\eta}, \widetilde{u})$ *of* (8) *with*

$$(\widetilde{\eta}^0, \widetilde{\eta}^1, \widetilde{u}^0, \widetilde{u}^1) \in \mathcal{X},\ \ \widetilde{f} \in L^1(0,T; L^2(0,1)),\ \ \widetilde{g} \in L^2(0,1)\,.$$

REMARK. In the identity (31) ρ^0, ρ^1, μ^0 and μ^1 play respectively the role of $\Psi(0)$, $-\Psi_t(0) + V(0)\,\delta_0$, $-V(0)$ and $V_t(0) + \Psi(0,0)$. It is easy to see that, in the frame of smooth functions, there is a one to one correspondence between $(\rho^0, \rho^1, \mu^0, \mu^1)$ and $(\Psi(0), \Psi_t(0), V(0), V_t(0))$.

PROOF OF PROPOSITION 3. In view of Proposition 1 we know that the map

$$(\widetilde{\eta}^0, \widetilde{\eta}^1, \widetilde{u}^0, \widetilde{u}^1, \widetilde{f}, \widetilde{g}) \longrightarrow \int_0^T \rho(t)\, u_{tt}(t)\, \widetilde{u}_{tt}(t)\, dt \tag{32}$$

is linear and continuous from $\mathcal{X} \times L^1(0,T;L^2(0,1)) \times L^2(0,T)$ into $\mathbb{C}$. This implies the existence and uniqueness of $(\rho^1,\rho^0,\mu^1,\mu^0) \times (\Psi,V) \in \mathcal{X}' \times L^\infty(0,T;L^2(0,1)) \times L^2(0,T)$ such that (31) holds. Moreover, there exists a constant $C > 0$ such that

$$\begin{aligned} \|(\Psi,V)\|_{L^\infty(0,T;L^2(0,1))\times L^2(0,T)} + \|(\rho^1,\rho^0,\mu^1,\mu^0)\|_{\mathcal{X}'} & \\ \leq C\,\|u_{tt}\|_{L^2(0,T)} & \\ \leq C\,\|(\eta^0,\eta^1,u^0,u^1)\|_{\mathcal{X}'} \,. & \end{aligned} \tag{33}$$

The fact that $\Psi \in C([0,T];L^2(0,1))$ can be deduced from (33) by a classical density argument.

REMARK 4. When the data of (27) are smooth, the solution (η,u) is smooth too. It is easy to see that (28) has a finite energy solution. In this case one can check that the element $(\rho^0,\rho^1,\mu^0,\mu^1) \in \mathcal{X}'$ obtained in Proposition 3 is such that

$$\rho^0 = \Psi(0)\,,\ \rho^1 = -\Psi_t(0) + V(0)\,\delta_0\,,\ \mu^0 = -V(0),\ \mu^1 = V_t(0) + \Psi(0,0)\,.$$

By a density argument one can then deduce that the solution (Ψ,V) obtained in Proposition 3 is such that the traces

$$\Psi|_{t=0}\,,\ -\Psi_t + V\,\delta_0|_{t=0}\,,\ -V|_{t=0}\,,\ V_t + \Psi(0,t)|_{t=0}$$

are well defined and coincide with $(\rho^0,\rho^1,\mu^0,\mu^1)$.

The same arguments allows us to show that the traces are also well defined at $t = T$. This suffices to assert that the weak solution of (28) we have constructed by transposition is at rest at $t = T$.

We can now complete the proof of Proposition 2.

END OF THE PROOF OF PROPOSITION 2. In view of Proposition 3 and Remark 4 we can define a linear and continuous map Λ from $\mathcal{X}$ into $\mathcal{X}'$ such that

$$\Lambda(\eta^0,\eta^1,u^0,u^1) = (-\Psi_t + V\,\delta_0|_{t=0}, \Psi(0), V_t + \Psi(0,t)|_{t=0}, -V|_{t=0})\,.$$

Taking in (31), $\widetilde{f} \equiv 0, \widetilde{g} \equiv 0$ and $(\widetilde{\eta},\widetilde{u}) = (\eta,u)$, we deduce that

$$\langle \Lambda(\eta^0,\eta^1,u^0,u^1),(\eta^0,\eta^1,u^0,u^1)\rangle = \int_0^T \rho(t)\,|u_{tt}(t)|^2\,dt$$

and in view of (24) we deduce that there exists $C > 0$ such that

$$\langle \Lambda(\eta^0, \eta^1, u^0, u^1), (\eta^0, \eta^1, u^0, u^1) \rangle \geq C \, \|(\eta^0, \eta^1, u^0 u^1)\|^2_{\mathcal{X}} \,.$$

Actually, $C = (C(T,n))^{-1}$, where $C(T,n)$ is as in (24).

This implies that Λ is an isomorphism.

This shows that given any $(\rho^1, \rho^0, \mu^1, \mu^0) \in \mathcal{X}'$ there exists

$$(\eta^0, \eta^1, u^0, u^1) = \Lambda^{-1}(\rho^1, \rho^0, \mu^1, \mu^0)$$

such that the corresponding solution of (28) in the sense of transposition satisfies

$$\begin{aligned} &\Psi(0) = \rho^0 \,, \ -\Psi_t + V\,\delta_0|_{t=0} = \rho^1 \,, \\ &-V|_{t=0} = \mu^0 \,, \ V_t + \Psi(0,t)|_{t=0} = \mu^1 \,. \end{aligned} \tag{34}$$

If we want this to be equivalent to the initial data of (19) we have to take

$$\rho^0 = \Psi^0 \,, \ \rho^1 = -\Psi^1 + V^0\,\delta_0 \,, \ \mu^0 = -V^0 \,, \ \mu^1 = V^1 + \Psi^0(0) \,. \tag{35}$$

This makes sense when the data $(\Psi^0, \Psi^1, V^0, V^1)$ is in $\mathcal{X}$.

The control we have obtained is of the form

$$\beta = -\frac{d^2}{dt^2}\,(\rho\, u_{tt}) \,,$$

where u corresponds to the solution (η, u) of (27) with data

$$(\eta^0, \eta^1, u^0, u^1) = \Lambda^{-1}(\rho^1, \rho^0, \mu^1, \mu^0) \,,$$

where $(\rho^0, \rho^1, \mu^0, \mu^1)$ is given by (34).

From the identities above we see that

$$\begin{aligned} \|\beta\|^2_{H^{-2}(0,T)} &\leq \|\rho\, u_{tt}\|^2_{L^2(0,T)} \\ &\leq C\, \|(\rho^1, \rho^0, \mu^1, \mu^0)\|^2_{\mathcal{X}'} \\ &\leq C\, (\|(\Psi^1, \Psi^0, V^1, V^0)\|^2_{\mathcal{X}'} + |\Psi^0(0)|^2) \,, \end{aligned}$$

where $C = C(T,n)$ is the constant obtained in (24).

REMARK 5. In fact, in some sense, we obtain a stronger result since we prove that we can control the problem (31) for any initial data $(\rho^0, \rho^1, \mu^0, \mu^1) \in \mathcal{X}'$. In order to give an interpretation of the control problem in terms of the initial data $(\Psi^1, \Psi^0, V^1, V^0)$ we have to assure that $\Psi^0(0)$ makes sense. For this reason we consider that

$$(\Psi^1, \Psi^0, V^1, V^0) \in \mathcal{X}\,.$$

When $n = 0$ one can not expect the same controllability result due to the conservation of the quantities (22) along the trajectories. In this case the controllability result reads as follows

Proposition 4. *Assume that $T > 2$ and $n = 0$. Then, for any $(\Psi^1, \Psi^0, V^1, V^0) \in \mathcal{X}$ there exists a control $\beta \in H^{-2}(0,T)$ with compact support such that the solution (Ψ, V) of (19) satisfies*

$$\begin{aligned} &\Psi(T) = V^1 + \Psi^0(0)\,, \qquad \Psi_t(T) = 0\,, \\ &V(T) = V^0 - \int_0^1 \Psi^1\,dx\,, \qquad V_t(T) = 0\,. \end{aligned} \tag{36}$$

REMARK 6. This result asserts that, when $n = 0$, any solution of (19) can be driven to an equilibrium configuration which is a priori determined by the initial data.

PROOF. First of all we observe that proving Proposition 4 is equivalent to showing that for any initial data as in the statement of Proposition 4 and satisfying the further assumptions

$$V^1 + \Psi^0(0) = 0\,, \qquad V^0 - \int_0^1 \Psi^1(x)\,dx = 0\,, \tag{37}$$

then, there exists a control β such that

$$\Psi(T) = \Psi_t(T) \equiv 0 \text{ in } (0,1)\,, \qquad V(T) = V_t(T) = 0\,. \tag{38}$$

Indeed, this is an immediate consequence of the remark made in the introduction that shows that when β is of zero average the following identities hold

$$\begin{aligned} &V_t(T) + \Psi(0,T) = V^1 + \Psi^0(0)\,, \\ &V(T) - \int_0^1 \Psi_t(x,T) = V^0 - \int_0^1 \Psi^1(x)\,. \end{aligned} \tag{39}$$

Thus, in the sequel we focus on initial data $(\Psi^0, \Psi^1, V^0, V^1)$ satisfying (39). For the adjoint system

$$
(40) \qquad \begin{cases} \eta_{tt} - \eta_{xx} = 0\,, & \text{in } (0,1) \times (0,T)\,, \\ \eta_x(1) = 0\,, & \text{for } t \in (0,T)\,, \\ \eta_x(0) = u_t\,, & \text{for } t \in (0,T)\,, \\ u_{tt} - \eta_t(0) = 0\,, & \text{for } t \in (0,T)\,, \\ \eta(0) = \eta^0\,,\ \eta_t(0) = \eta^1\,, & \text{in } (0,1)\,, \\ u(0) = u^0\,,\ u_t(0) = u^1\,, \end{cases}
$$

we consider initial data in the following subspace $\mathcal{X}_0$ of $\mathcal{X}$

$$
(41) \quad \mathcal{X}_0 = \Big\{ (\eta^0, \eta^1, u^0, u^1) \in \mathcal{X} u^1 - \eta^0(0) = 0\,,\ \int_0^1 \eta^1\, dy + u^0 = 0 \Big\}\,.
$$

It is easy to see that the subspace $\mathcal{X}_0$ is invariant under the flow generated by (40).

Given $(\eta^0, \eta^1, u^0, u^1) \in \mathcal{X}_0$ we solve first (40) and then the backward system

$$
(42) \qquad \begin{cases} \Psi_{tt} - \Psi_{xx} = 0\,, & \text{in } (0,1) \times (0,T)\,, \\ \Psi_x(1,t) = 0\,, & \text{for } t \in (0,T)\,, \\ \Psi_x(0,t) = -V_t(t)\,, & \text{for } t \in (0,T)\,, \\ V_{tt}(t) + \Psi_t(0,t) = -\dfrac{d^2}{dt^2}\left(\rho(t)\, u_{tt}(t)\right), & \text{for } t \in (0,T)\,, \\ \Psi(T) = \Psi_t(T) = 0\,, & \text{in } (0,1)\,, \\ V(T) = V_t(T) = 0\,, \end{cases}
$$

where ρ is as in the proof of Proposition 2.

Proceeding as in the proof of Proposition 3 one can show that (42) has a unique solution defined by transposition such that the traces (38) are well defined. On the other hand, integrating the equations in (42) we deduce that

$$
(43) \qquad \int_0^1 \rho^1(x)\, dx = 0\,, \qquad \mu^1 = 0\,.
$$

Let us denote by Z the subspace of $\mathcal{X}'$ satisfying (43). More precisely,

$$Z = \{(\rho^1, \rho^0, \mu^1, \mu^0) \in \mathcal{X}' \text{ such that (43) holds}\} . \tag{44}$$

It is easy to check that Z is actually the dual of $\mathcal{X}_0$. Indeed, the dual of $\mathcal{X}_0$ is a quotient space of $\mathcal{X}'$ and there is a one-to-one correspondence between Z and this quotient space in the sense that, in Z, we have chosen the unique element of each class of the quotient space satisfying (43).

As in the proof of Proposition 2 we can define a linear and continuous operator $\Lambda : \mathcal{X}_0 \longrightarrow Z$ that associates the trace $(\rho^1, \rho^0, \mu^1, \mu^0) \in Z$ in (31) to each $(\eta^0, \eta^1, u^0, u^1) \in \mathcal{X}_0$.

We also have

$$\langle \Lambda(\eta^0, \eta^1, u^0, u^1), (\eta^0, \eta^1, u^0, u^1) \rangle = \int_0^T \rho(t)\, |u_{tt}(t)|^2 \, dt .$$

Remark that inequality (24) also holds for the case $n = 0$ if we consider initial data in $\mathcal{X}_0$. Hence there exists a constant $C > 0$ such that

$$\langle \Lambda(\eta^0, \eta^1, u^0, u^1), (\eta^0, \eta^1, u^0, u^1) \rangle \geq C \, \|(\eta^0, \eta^1, u^0, u^1)\|^2_{\mathcal{X}'} ,$$

for all $(\eta^0, \eta^1, u^0, u^1) \in \mathcal{X}_0$, since the quantity $(\|\eta^0_x\|^2_{L^2(0,1)} + \|\eta^1\|^2_{L^2(0,1)} + |u^1|^2)^{1/2}$ defines a norm in $\mathcal{X}_0$ which is equivalent to the norm induced by $\mathcal{X}$.

We deduce that $\Lambda : \mathcal{X}_0 \longrightarrow Z$ is an isomorphism.

Then, given initial data as in the statement of Proposition 4 and such that (37) holds we define $(\rho^1, \rho^0, \mu^1, \mu^0) \in Z$ by (35). The control we are looking for is

$$\beta = -\frac{d^2}{dt^2} \left(\rho(t)\, u_{tt}(t)\right) ,$$

where u is the second component of the solution (η, u) of (40) with initial data $(\eta^0, \eta^1, u^0, u^1) = \Lambda^{-1} (\rho^1, \rho^0, \mu^1, \mu^0)$.

This concludes the proof of the Proposition.

Let us now state the controllability results for the two-dimensional system (4).

We use the Fourier decomposition method described at the beginning of this section. Thus we develop the initial data $(\Phi^0, \Phi^1, W^0, W^1)$

to be controlled in Fourier series

$$(45)\qquad \begin{cases} \Phi^0 = \displaystyle\sum_{n=0}^{\infty} \Psi_n^0(x)\cos(n\,\pi\,y)\,, & \Phi^1 = \displaystyle\sum_{n=0}^{\infty} \Psi_n^1(x)\cos(n\,\pi\,y)\,, \\ W^0 = \displaystyle\sum_{n=0}^{\infty} V_n^0\cos(n\,\pi\,y)\,, & W^1 = \displaystyle\sum_{n=0}^{\infty} V_n^1\cos(n\,\pi\,y)\,. \end{cases}$$

We assume that for every $n = 0, 1, \ldots$ the initial data satisfy the assumptions of Proposition 2 and Proposition 4. We set

$$(46)\quad \rho_n^0 = \Psi_n^0\ ,\ \rho_n^1 = -\Psi_n^1 + V_n^0\,\delta_0\ ,\ \mu^0 = -V_n^0\ ,\ \mu_n^1 = V_n^1 + \Psi_n^0(0)\,.$$

We introduce the following space of initial data

$$(47)\qquad \begin{aligned} H = \Big\{(\Phi^0, \Phi^1, W^0, W^1) \in \mathcal{Y} :\ & \sum_{n=0}^{\infty} C(n,T)\,\|(\rho_n^1, \rho_n^0, \mu_n^1, \mu_n^0)\|_{\mathcal{X}'}^2 \\ & = \|(\Phi^0, \Phi^1, W^0, W^1)\|_H^2 < \infty\Big\}\,. \end{aligned}$$

where the constants $C(n,T)$ are those appearing in (24).

Proposition 5. *Assume that $T > 2$. Then, for every initial data $(\Phi^0, \Phi^1, W^0, W^1)$ in H there exists a control $\beta \in H^{-2}(0,T; L^2(0,1))$ such that the solution (Φ, W) of (4) satisfies*

$$(48)\qquad \begin{cases} \Phi(T) \equiv \mu^1 = \displaystyle\int_0^1 W^1(y)\,dy + \int_0^1 \Psi^0(0,y)\,dy\,,\ \ \Phi_t(T) \equiv 0\,, \\ W(T) \equiv \langle \rho^1, 1\rangle \\ \qquad\quad = \displaystyle\int_0^1 W^0(y)\,dy - \int_0^1\int_0^1 \Psi^1(x,y)\,dx\,dy\,,\ \ W_t(T) \equiv 0\,. \end{cases}$$

Moreover there exists a constant $C > 0$ such that

$$(49)\qquad \|\beta\|_{H^{-2}(0,T;L^2(0,1))} \le C\,\|(\Phi^0, \Phi^1, W^0, W^1)\|_H\,.$$

Remark 7. The control time $T > 2$ is optimal. Indeed, when $T < 2$ it is easy to see that the set of controllable data is not dense in the space of finite energy data. Actually, when $T < 2$ none of the one-dimensional problems (19) is approximately controllable, *i.e.* the space of controllable data is no even dense in $\mathcal{Y}'$.

Remark 8. The constants $C(n,T)$ play an important role in the controllability problem since the space H of controllable functions depends on them. The next two sections are devoted to the evaluation of these constants.

Proof. In view of propositions 2 and 4 for any $n = 0, 1, \dots$ there exists a control $\beta_n \in H^{-2}(0,T)$ such that the solution (Ψ_n, V_n) of (19) satisfies

$$\Psi_n(T) \equiv \Psi_{n,t}(T) = 0 \text{ in } (0,1)\,, \qquad V_n(T) = V_{n,t}(T) = 0\,, \tag{50}$$

for $n \geq 1$ and

$$\begin{aligned} &\Psi_0(T) = \mu^1\,,\ \Psi_{0,t}(T) = 0 \text{ in } (0,1)\,,\\ &V_0(T) = \langle \rho^1, 1\rangle\,,\ V_{0,t}(T) = 0\,. \end{aligned} \tag{51}$$

when $n = 0$.

On the other hand

$$\|\beta_n\|^2_{H^{-2}(0,T)} \leq C(n,T)\, \|(\rho_n^1, \rho_n^0, \mu_n^1, \mu_n^0)\|^2_{\mathcal{X}'}\,. \tag{52}$$

We construct the following control for the two-dimensional system

$$\beta(y,t) = \sum_{n=0}^{\infty} \beta_n \cos(n\,\pi\,y)\,. \tag{53}$$

We have, in view of (52),

$$\begin{aligned} \|\beta\|^2_{H^{-2}(0,T;L^2(0,1))} &= \sum_{n=0}^{\infty} \|\beta_n(t)\|^2_{H^{-2}(0,T)}\\ &\leq \sum_{n=0}^{\infty} C(n,T)\|(\rho_n^1, \rho_n^0, \mu_n^1, \mu_n^0)\|^2_{\mathcal{X}'}\\ &= \|(\Psi^0, \Psi^1, W^0, W^1)\|^2_H\\ &< \infty\,. \end{aligned}$$

Therefore $\beta \in H^{-2}(0,T;L^2(0,1))$. On the other hand,

$$\Psi(x,y,t) = \sum_{n=0}^{\infty} \Psi_n(y,t)\cos(n\,\pi\,y)\,, \qquad W(y,t) = \sum_{n=0}^{\infty} V_n(t)\cos(n\,\pi\,y)$$

solves (4) with the control β given in (53) and satisfies (48) at time $t = T$.

This concludes the proof of this Proposition.

3. Spectral analysis.

In this section we give some estimates on the spectrum of the differential operator corresponding to (23) that will be used in the next section to prove (24). In order to analyze the spectrum of (23) let $(\Psi(x,t), V(t))$ be solution of

$$
\begin{cases}
\Psi_{tt} - \Psi_{xx} + n^2\,\pi^2\,\psi = 0\,, & \text{in } (0,1)\times(0,\infty)\,,\\
\Psi_x(1) = 0\,, & \text{for } t\in(0,\infty)\,,\\
\Psi_x(0) = V_t\,, & \text{for } t\in(0,\infty)\,,\\
V_{tt} + n^4\,\pi^4\,V - \psi_t(0) = 0\,, & \text{for } t\in(0,\infty)\,.
\end{cases}
\tag{54}
$$

Now if we look for solutions of (54) of the form $(\Psi(x,t), V(t)) = e^{\lambda t}\,(\Psi(x), V)$, with $V \in \mathbb{R}$, it follows that the eigenvalues λ of system (1) are the roots of the equation

$$
e^{2\sqrt{\lambda^2+n^2\pi^2}} = -\frac{\lambda^2 - \sqrt{\lambda^2+n^2\,\pi^2}\,(\lambda^2+n^4\,\pi^4)}{\lambda^2 + \sqrt{\lambda^2+n^2\,\pi^2}\,(\lambda^2+n^4\,\pi^4)}\,.
\tag{55}
$$

We have the following first result

Lemma 1. *System* (1) *has a two-parameter sequence of purely imaginary eigenvalues* $\{\lambda_{n,k}\}_{n\in\mathbb{N},k\in\mathbb{Z}^*}$ *given by*

$$
\lambda_{n,k} = \sqrt{z_{n,k}^2 + n^2\,\pi^2}\;i
\tag{56}
$$

if $k > 0$ *and* $\lambda_{n,k} = -\lambda_{n,-k}$ *if* $k < 0$, *where* $\{z_{n,k}\}_{k\in\mathbb{N}^*}$ *are the roots of the equation*

$$
\tan z = \frac{z^2 + n^2\,\pi^2}{z^3 + z\,(n^2\,\pi^2 - n^4\,\pi^4)}\,.
\tag{57}
$$

Moreover, there are another two eigenvalues of (1), λ_n^* *and* λ_n^{**}, *with the modulus less than* $n\,\pi$, *given by*

$$
\lambda_n^* = \sqrt{n^2\,\pi^2 - (z_n^*)^2}\;i\,, \qquad \lambda_n^{**} = \overline{\lambda}_n^{\,*}\,,
\tag{58}
$$

where z_n^* is the unique positive root of the equation

$$e^{2z} = \frac{z^3 - z^2 + n^2\,\pi^2 + z\,(n^4\,\pi^4 - n^2\,\pi^2)}{z^3 + z^2 - n^2\,\pi^2 + z\,(n^4\,\pi^4 - n^2\,\pi^2)}\,. \tag{59}$$

In the last case, $\lambda_n^* = \lambda_n^{**} = 0$ when $n = 0$.

PROOF. We know that the eigenvalues λ are roots of (55). Considering the change of variable $\lambda = \sqrt{\zeta^2 - n^2\,\pi^2}$ equation (55) becomes

$$e^{2\zeta} = \frac{\zeta^3 - \zeta^2 + n^2\,\pi^2 + \zeta\,(n^4\,\pi^4 - n^2\,\pi^2)}{\zeta^3 + \zeta^2 - n^2\,\pi^2 + \zeta\,(n^4\,\pi^4 - n^2\,\pi^2)}\,. \tag{60}$$

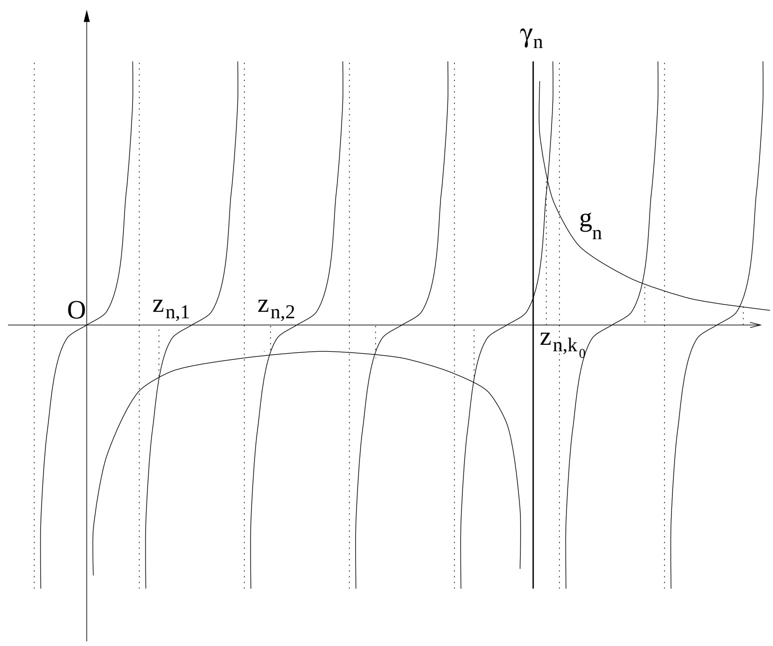

Figure 1.

Since the differential operator corresponding to (1) is conservative its eigenvalues will be all purely imaginary. Hence, we have to look only for those roots of (60) which are purely imaginary or real. It follows that the imaginary roots of (60) are the roots of the equation (57) and the real ones are roots of (59).

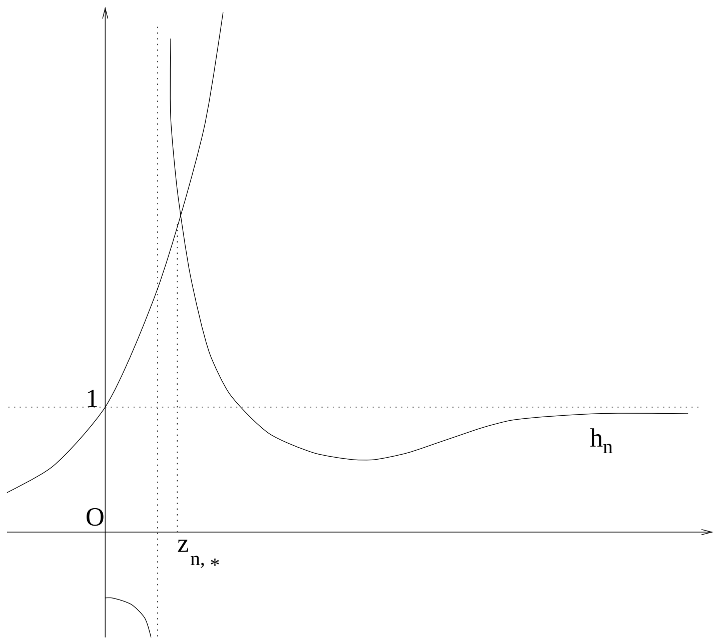

Figure 2.

Observe that the right hand side of (57) has a pole at

$$z = \sqrt{n^4\,\pi^4 - n^2\,\pi^2}\ .$$

Let us denote by $\alpha_n = n^4\,\pi^4 - n^2\,\pi^2$, $\gamma_n = \sqrt{\alpha_n}$ and let $k_0 \in \mathbb{N}$ be such that $k_0\,\pi - \pi/2 \leq \sqrt{\alpha_n} < k_0\,\pi + \pi/2$.

Equation (57) has an unique root in each interval $(k\,\pi - \pi/2, k\,\pi + \pi/2)$ for $k \in \mathbb{N} \setminus \{k_0\}$.

In $(k_0\,\pi - \pi/2, k_0\,\pi + \pi/2)$ there are two roots z_{n,k_0-1} and z_{n,k_0} of (57).

The localization of the roots $\{z_{k,m}\}_{k\in\mathbb{N}^*}$ and $z_{n,*}$ is illustrated in figures 1 and 2, where

$$g_n(z) = \frac{z^2 + n^2\,\pi^2}{z^3 + z\,(n^2\,\pi^2 - n^4\,\pi^4)}$$

and

$$h_n(z) = \frac{z^3 - z^2 + n^2\,\pi^2 + z\,(n^4\,\pi^4 - n^2\,\pi^2)}{z^3 + z^2 - n^2\,\pi^2 + z\,(n^4\,\pi^4 - n^2\,\pi^2)}\ .$$

The roots correspond to the points of intersection of the curves in the figures.

The skew adjoint operator corresponding to (1) can be diagonalised over the orthogonal basis of eigenvectors

$$\xi_{n,k} = \begin{pmatrix} \xi^1_{n,k} \\ \xi^2_{n,k} \\ \xi^3_{n,k} \\ \xi^4_{n,k} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{\lambda_{n,k}} \cosh\left(\sqrt{n^2\,\pi^2 + \lambda^2_{n,k}}\,(x-1)\right) \cos(n\,\pi\,y) \\ -\cosh\left(\sqrt{n^2\,\pi^2 + \lambda^2_{n,k}}\,(x-1)\right) \cos(n\,\pi\,y) \\ \dfrac{-\sqrt{n^2\,\pi^2 + \lambda^2_{n,k}}}{\lambda^2_{n,k}} \sinh\left(\sqrt{n^2\pi^2 + \lambda^2_{n,k}}\right) \cos(n\,\pi\,y) \\ \dfrac{\sqrt{n^2\,\pi^2 + \lambda^2_{n,k}}}{\lambda_{n,k}} \sinh\left(\sqrt{n^2\,\pi^2 + \lambda^2_{n,k}}\right) \cos(n\,\pi\,y) \end{pmatrix}$$

and the solution of (5) with initial condition $\xi_{n,k}$ is such that

$$\begin{pmatrix} \Psi(x,y,t) \\ \Psi_t(x,y,t) \\ W(y,t) \\ W_t(y,t) \end{pmatrix} = \xi_{n,k}(x,y)\, e^{\lambda_{n,k} t}\,.$$

As this basis is not normalized, we will denote $\Xi_{n,k} = \|\xi_{n,k}\|_{\mathcal{Y}}$. Notice that if n and k are integers,

$$c \le \Xi_{n,k} \le C\,. \tag{61}$$

On the other hand $z_{n,*}$ is the only positive real solution to

$$e^{2t} = h_n(t) = \frac{N(t)}{D(t)} = \frac{-t^2 + n^2\,\pi^2 + t\,(t^2 + n^4\,\pi^4 - n^2\,\pi^2)}{t^2 - n^2\,\pi^2 + t\,(t^2 + n^4\,\pi^4 - n^2\,\pi^2)}\,.$$

Let $t_0(n)$ be the real root of D. It follows that $z_{n,*} > t_0(n) > n^{-3}$. Furthermore, as $D(n^{1/2}) > 0$ and $R(n^{1/2}) \sim 1 < e^{2n^{1/2}}$, $z_{n,*} < n^{1/2}$ for large n.

Therefore, as $\lambda_{n,*} = i\sqrt{n^2\pi^2 - z_{n,*}^2}$,

$$c\,n^{-8} \le \left|\frac{\lambda_{n,*}}{i\,n} - \pi\right| \le C\,n^{-1} \tag{62}$$

and

$$c\,n^{-3} \le \Xi_{n,*} \le C\,e^{n^{2/3}}\,. \tag{63}$$

For any $(\Phi^0, \Phi^1, W^0, W^1)$ in $\mathcal{Y}$,

$$(\Phi^0, \Phi^1, W^0, W^1) = \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}^*\cup\{*,**\}}} \frac{a_{n,k}}{\Xi_{n,k}}\,\xi_{n,k}(x,y)\,,$$

with $\{a_{n,k}\}_{n,k} \in l^2$.

Let us now make some notations. We will write for any $(\Phi^0, \Phi^1, W^0, W^1)$ in $\mathcal{Y}$ that

- $(\Phi^0, \Phi^1, W^0, W^1) \in \mathcal{Y}^{n_0}$ if $n \ne n_0$ implies $a_{n,k} = 0$,
- $(\Phi^0, \Phi^1, W^0, W^1) \in \mathcal{Y}^{(1)}$ if $|k| > |n|$ implies $a_{n,k} = 0$,
- $(\Phi^0, \Phi^1, W^0, W^1) \in \mathcal{Y}^{(2)}$ if $|k| \le |n|$ or $k \in \{*, **\}$ implies $a_{n,k} = 0$,
- $(\Phi^0, \Phi^1, W^0, W^1) \in \mathcal{Y}^{i,n}$ implies $(\Phi^0, \Phi^1, W^0, W^1) \in \mathcal{Y}^{(i)} \cap \mathcal{Y}^n$.

We can denote

$$(\Phi^0, \Phi^1, W^0, W^1) = (\Phi^0, \Phi^1, W^0, W^1)^{(1)} + (\Phi^0, \Phi^1, W^0, W^1)^{(2)}\,,$$

with $(\Phi^0, \Phi^1, W^0, W^1)^{(i)} \in \mathcal{Y}^{(i)}$.

Moreover I will be the set of (k, n) such that $k \in \{*, **\}$ or $|k| \le |n|$, and we will agree that $** = -*$. To end with, we shall also denote $\nu_{n,k} = \operatorname{Im}\lambda_{n,k}$ to deal with real numbers.

4. Proof of Theorem 1.

In order to prove the theorem, we will use a proposition for low frequencies and a lemma for high ones.

Proposition 6 (Low frequencies). *For any positive ε and δ, there exists a constant $C_{\varepsilon,\delta}$, an integer $n_1(\varepsilon)$ and a positive time $T_2(\varepsilon,\delta) \le C_\delta/\varepsilon^{1+\delta}$ such that for any integer n greater than $n_1(\varepsilon)$ and any (Φ^0,Φ^1,W^0,W^1) in $\mathcal{Y}^n$, the solution of* (5) *with initial condition (Φ^0,Φ^1,W^0,W^1) satisfies*

$$\|(\Phi^0,\Phi^1,W^0,W^1)^{(1)}\|_{\mathcal{Y}}^2 \le C_{\varepsilon,\delta}\, e^{2\varepsilon|n|} \int_{-T_2(\varepsilon,\delta)}^{T_2(\varepsilon,\delta)} |W_{tt}(0,t)|^2\, dt\,.$$

This proposition will be proved in Section 4.2.

Lemma 2 (High frequencies). *There exists a constant C and a positive time T_1 such that for integer n and any (Φ^0,Φ^1,W^0,W^1) in $\mathcal{Y}^{n,2}$, the solution of* (5) *with initial condition (Φ^0,Φ^1,W^0,W^1) satisfies*

$$\|(\Phi^0,\Phi^1,W^0,W^1)\|_{\mathcal{Y}} \le \frac{C}{n^4}\, \|W_{tt}\|_{L^2((0,T_1)\times\Gamma_0)}\,. \tag{64}$$

The proof of Lemma 2 will be given in subsection 4.1.

Let us now prove how do Proposition 6 and Lemma 2 imply that Theorem 1 is true.

Proof of Theorem 1. Let ε and δ be two positive real numbers. Out of Propositions 6 and Lemma 6, we get two positive times, denoted T_1 and $T_2(\varepsilon,\delta)$. Let us define $T(\varepsilon,\delta) = \sup\{T_1, T_2(\varepsilon,\delta)\}$.

Let n be a positive integer and (Φ^0,Φ^1,W^0,W^1) any initial condition in $\mathcal{Y}^n$. Then we have

$$\begin{aligned}\|(\Phi^0,\Phi^1,W^0,W^1)\|_{\mathcal{Y}}^2 \\ &= \|(\Phi^0,\Phi^1,W^0,W^1)^{(1)}\|_{\mathcal{Y}}^2 + \|(\Phi^0,\Phi^1,W^0,W^1)^{(2)}\|_{\mathcal{Y}}^2\,.\end{aligned}$$

So by Proposition 6 and Lemma 2, for $n \ge n_1(\varepsilon)$,

$$\begin{aligned}\|(\Phi^0,\Phi^1,W^0,W^1)\|_{\mathcal{Y}}^2 &\le C_{\varepsilon,\delta}\, e^{2\varepsilon|n|} \int_{-T_2(\varepsilon,\delta)}^{T_2(\varepsilon,\delta)} |W_{tt}(0,t)|^2\, dt \\ &\quad + \frac{C}{n^4} \int_0^{T_1} |W^{(2)}{}_{tt}(0,t)|^2\, dt \\ &\le C_{\varepsilon,\delta}\, e^{2\varepsilon|n|} \int_{-T(\varepsilon,\delta)}^{T(\varepsilon,\delta)} |W_{tt}(0,t)|^2\, dt\end{aligned}$$

$$+ \frac{C}{n^4} \int_0^{T_1} |W_{tt}(0,t)|^2 \, dt$$
$$+ \frac{C}{n^4} \int_0^{T_1} |W^{(1)}{}_{tt}(0,t)|^2 \, dt \, .$$

Therefore, by the direct estimate (14),

$$\|(\Phi^0, \Phi^1, W^0, W^1)\|_{\mathcal{Y}}^2$$
$$\leq C'_{\varepsilon,\delta} \, e^{2\varepsilon|n|} \int_{-T(\varepsilon,\delta)}^{T(\varepsilon,\delta)} |W_{tt}(0,t)|^2 \, dt + C' \, \|(\Phi^0, \Phi^1, W^0, W^1)^{(1)}\|_{\mathcal{Y}}^2 \, .$$

So by Proposition 6,

$$\|(\Phi^0, \Phi^1, W^0, W^1)\|_{\mathcal{Y}}^2 \leq C'_{\varepsilon,\delta} \, e^{2\varepsilon|n|} \int_{-T(\varepsilon,\delta)}^{T(\varepsilon,\delta)} |W_{tt}(0,t)|^2 \, dt \, .$$

We can increase the constant to take care of the first $n_1(\varepsilon)$ values of n. As $T(\varepsilon, \delta) \leq T_2(\varepsilon, \delta) \leq C/\varepsilon^{1+\delta}$, if we put

$$T(\alpha(T), \delta) = T \, ,$$

we get

$$\alpha(T) \leq \frac{C_q}{T^{1-q}} \, ,$$

for any positive real number q and (7) is proved.

We pass now to prove Lemma 2.

4.1. Proof of Lemma 2.

Since $(\Phi^0, \Phi^1, W^0, W^1) \in \mathcal{Y}^{2,n}$,

$$(65) \qquad \begin{aligned} (\Phi^0, \Phi^1, W^0, W^1) &= \sum_{|k|>n} a_{n,k} \, \frac{\xi_{n,k}}{\Xi_{n,k}} \, , \\ \|(\Phi^0, \Phi^1, W^0, W^1)^{(2)}\|_{\mathcal{Y}}^2 &= \sum_{|k|>n} |a_{n,k}|^2 \, . \end{aligned}$$

On the other hand, for $T_1 > 0$,

$$\begin{aligned}
&\int_0^{T_1} |W_{tt}(t,0)|^2 \\
&= \int_0^{T_1} \Big| \nu_{n,k}^2 \sum_{|k|>n} a_{n,k} \, \frac{\xi_{n,k}^3}{\Xi_{n,k}} \, e^{\nu_{n,k} t\, i} \Big|^2 \\
&= \int_0^{T_1} \Big| \sum_{|k|>n} \frac{a_{n,k}}{\Xi_{n,k}} \sqrt{n^2\,\pi^2 - \nu_{n,k}^2} \, \sin\big(\sqrt{n^2\pi^2 - \nu_{n,k}^2}\,\big)\, e^{\nu_{n,k} ti} \Big|^2 .
\end{aligned}$$

Let us prove that there exists $c > 0$ such that, for $k > n$,

$$\nu_{n,k+1} - \nu_{n,k} \geq c\,. \tag{66}$$

Firstly, remark that $z_{n,k+1} - z_{n,k} > \pi/2$, for all $k \neq k_0 - 1$, k_0 where $k_0 \in \mathbb{N}$ is such that $(k_0 - 1)\,\pi + \pi/2 \leq \sqrt{\alpha_n} < k_0\,\pi + \pi/2$. We recall that $\alpha_n = n^4\,\pi^4 - n^2\,\pi^2$. In order to prove that there is a gap between z_{n,k_0-1} and z_{n,k_0} let us show that, if $z \in ((k_0 - 1)\,\pi + \pi/2, k_0\,\pi + \pi/2)$ then

$$\Big| \frac{z^2 + n^2\,\pi^2}{z^3 - z\,\alpha_n} \Big| \geq \frac{1}{\pi}\,. \tag{67}$$

Indeed we have

$$\begin{aligned}
&\Big| \frac{z^2 + n^2\,\pi^2}{z^3 - z\,\alpha_n} \Big| \\
&\geq \min\left\{ \left| \frac{\Big((k_0-1)\,\pi + \dfrac{\pi}{2}\Big)^2 + n^2\,\pi^2}{\Big((k_0-1)\,\pi + \dfrac{\pi}{2}\Big)^3 - \Big((k_0-1)\,\pi + \dfrac{\pi}{2}\Big)\alpha_n} \right|, \right. \\
&\qquad\quad \left. \left| \frac{\Big((k_0+1)\,\pi + \dfrac{\pi}{2}\Big)^2 + n^2\,\pi^2}{\Big((k_0+1)\,\pi + \dfrac{\pi}{2}\Big)^3 - \Big((k_0+1)\,\pi + \dfrac{\pi}{2}\Big)\alpha_n} \right| \right\} \\
&\geq \frac{\alpha_n + n^2\,\pi^2}{\alpha_n + \sqrt{\alpha_n}\,\sqrt{\alpha_n}} \\
&\quad \cdot \min\left\{ \frac{1}{\Big|(k_0+1)\,\pi + \dfrac{\pi}{2} - \sqrt{\alpha_n}\Big|}, \frac{1}{\Big|(k_0-1)\,\pi + \dfrac{\pi}{2} - \sqrt{\alpha_n}\Big|} \right\} \\
&\geq \frac{1}{\pi}\,.
\end{aligned}$$

From (67) it follows that $\max\{|\tan z_{n,k_0-1}|, |\tan z_{n,k_0}|\} > 1/\pi$.

Hence $|z_{n,k_0} - z_{n,k_0-1}| > \arctan(1/\pi)$.

We can evaluate now

$$\begin{aligned}
\nu_{n,k+1} - \nu_{n,k} &= \sqrt{n^2\,\pi^2 + z_{n,k+1}^2} - \sqrt{n^2\,\pi^2 + z_{n,k}^2} \\
&> \frac{(z_{n,k+1} - z_{n,k})\,(z_{n,k+1} + z_{n,k})}{2\,\sqrt{n^2\,\pi^2 + z_{n,k}^2}} \\
&> \arctan\frac{1}{\pi}\,\frac{n\,\pi}{4\,n\,\pi} \\
&= \frac{1}{4}\arctan\frac{1}{\pi}
\end{aligned}$$

and (66) holds with $c = (1/4)\arctan(1/\pi)$.

By using Ingham's inequality (see Ingham [5]) we obtain that, for $T_1 > 2\pi/c$,

$$\begin{aligned}
&\int_0^{T_1} |W_{tt}(t,0)|^2 \\
&\qquad \geq C \sum_{|k|>n} \left|\frac{a_{n,k}}{\Xi_{n,k}}\right|^2 \left|\sqrt{n^2\,\pi^2 - \nu_{n,k}^2}\,\sin\left(\sqrt{n^2\,\pi^2 - \nu_{n,k}^2}\right)\right|^2 .
\end{aligned} \tag{68}$$

Let us prove that

$$\left|\sqrt{n^2\,\pi^2 - \nu_{n,k}^2}\,\sin\left(\sqrt{n^2\,\pi^2 - \nu_{n,k}^2}\right)\right| = |z_{n,k}\sin z_{n,k}| \geq \frac{C}{n^4}\,, \tag{69}$$

where C is a positive constant not depending on n and k.

Firstly, from (57), we have

$$z_{n,k}\sin z_{n,k} = \frac{z_{n,k}^2 + n^2\,\pi^2}{z_{n,k}^2 + \alpha_n}\cos z_{n,k}\ .$$

Consider the following cases

i) $z_{n,k} > \sqrt{\alpha_n}$. In this case

$$\frac{z_{n,k}^2 + n^2\,\pi^2}{z_{n,k}^2 - \alpha_n} > \frac{z_{n,k}\,(z_{n,k} - \sqrt{\alpha_n}\,)}{z_{n,k}^2 - \alpha_n} = \frac{1}{1 + \dfrac{\sqrt{\alpha_n}}{z_{n,k}}} > \frac{C}{n^2}\ .$$

If $|\cos z_{n,k}| \geq 1/\sqrt{2}$ then

$$z_{n,k} \sin z_{n,k} = \frac{z_{n,k}^2 + n^2\,\pi^2}{z_{n,k}^2 + \alpha_n} \cos z_{n,k} > \frac{C}{n^2}\,.$$

If $|\cos z_{n,k}| < 1/\sqrt{2}$ then $|\sin z_{n,k}| \geq 1/\sqrt{2}$ and

$$|z_{n,k} \sin z_{n,k}| \geq \frac{z_{n,k}}{\sqrt{2}} > \frac{\sqrt{\alpha_n}}{\sqrt{2}}\,.$$

ii) $z_{n,k} < \sqrt{\alpha_n}$. Now we have

$$|\tan z_{n,k}| > \inf_{z<\sqrt{\alpha_n}} \frac{z_{n,k}^2 + n^2\,\pi^2}{z_{n,k}^3 + z_{n,k}\,\alpha_n} > \frac{1}{n^6}\,.$$

It follows that

$$|z_{n,k} \sin z_{n,k}| \geq \frac{C}{n^4}\,.$$

Finally, we obtain that (69) holds.

From (65), (68) and (61) it follows that

$$\|(\Phi^0, \Phi^1, W^0, W^1)\|_{\mathcal{Y}}^2 \leq \frac{C}{n^4} \int_0^{T_1} |W_{tt}(t,0)|^2\,.$$

We still have to prove Proposition 6. This will be dealt with in the following subsection.

4.2. Proof of Proposition 6.

This proposition deals with the lowest eigenmodes of the problem. In this part of the spectrum, the Ingham techniques do not work, because the gap between frequencies goes to zero. The technique of biorthogonal sequence, that we will use, is more general. Examples of its application can be found in [3] for instance.

The idea is to find a sequence of functions $h_{n,k}$ with compact support such that $\widehat{h}_{n,k}(\nu_{n,k_0}) = \delta_{k_0}^k$, and whose L^2 norm is not too large. Indeed, will prove the following lemma:

Lemma 3. *For any odd integer q and any positive real number ε, there exists a time $T_2(q,\varepsilon)$ smaller than $C_q\,\varepsilon^{(q+1)/(1-q)}$ such that for any (n,k_0) in $\mathbb{N}^*\times(\mathbb{Z}^*\cup\{*,**\})$ there exists a function $h_{\varepsilon,q}^{k_0,n}$ that satisfies*

i) *$h_{\varepsilon,q}^{k_0,n}$ is supported by $[-T_2(q,\varepsilon),T_2(q,\varepsilon)]$.*

ii) *For $(k_0,n)\in I$, $\|h_{\varepsilon,q}^{k_0,n}\|_{L^2}^2\le C\,e^{2\varepsilon|n|}$.*

iii) *If $k\neq\pm k_0$,*

$$\int h_{\varepsilon,q}^{k_0,n}(t)\,e^{it\nu_{n,k}}\,dt=0\,.$$

iv) *If $n\ge n_1(\varepsilon,q)$ and $(k_0,n)\in I$,*

$$\Big|\int h_{\varepsilon,q}^{k_0,n}(t)\,e^{t\nu_{n,\pm k_0}}\,dt\Big|\ge\frac{c}{n^{N_q}}\,.$$

The constants depend only on q and ε. Moreover the functions h can be chosen as even or odd. We will denote them $h_{e\,\varepsilon,q}^{\;k_0,n}$ or $h_{o\,\varepsilon,q}^{\;k_0,n}$.

Let us show at first how to prove Proposition 6 out of this lemma.

Let n be an integer greater than $n_1(\varepsilon)$, and (Φ^0,Φ^1,W^0,W^1) an initial condition in $\mathcal{Y}^n$. Let us denote (Ψ,V) the solution of (5) with these data.

We will denote K the operator that maps (Φ^0,Φ^1,W^0,W^1) in $\mathcal{Y}^n$ to $W_{tt}(y=0,\cdot)$. If we denote $a_{n,k}=\langle(\Phi^0,\Phi^1,W^0,W^1),\xi_{n,k}/\Xi_{n,k}\rangle$, we notice that

$$W(y,t)=\sum_{k\in\mathbb{Z}^*\cup\{*,**\}}a_{n,k}\,\frac{\xi_{n,k}^3}{\Xi_{n,k}}\cos(\pi\,n\,y)\,e^{i\nu_{n,k}t}\,.$$

Thus

$$K(\Phi^0,\Phi^1,W^0,W^1)(t)=-\sum_{k\in\mathbb{Z}^*\cup\{*,**\}}a_{n,k}\,\frac{\xi_{n,k}^3}{\Xi_{n,k}}\,\nu_{n,k}^2\,e^{i\nu_{n,k}t}\,.$$

Now for (k_0,n) in I and L in $\mathbb{N}^*$, as $\widehat{h_e}$ is even,

$$\begin{aligned}\int h_{e\,\varepsilon,q}^{\;k_0,n}(t)\,K\Big(\sum_{\substack{|k|=*\\|k|\le L}}a_{n,k}\,\frac{\xi_{n,k}}{\Xi_{n,k}}\Big)(t)\,dt\\ =-\sum_{\substack{k=*\\1\le k\le L}}(a_{n,k}+a_{n,-k})\,\frac{\xi_{n,k}^3}{\Xi_{n,k}}\,\nu_{n,k}^2\int h_{e\,\varepsilon,q}^{\;k_0,n}(t)\,e^{i\nu_{n,k}t}\,dt\,.\end{aligned}$$

So, out of iii), if $L \geq k_0$,

$$\int h_{e_{\varepsilon,q}}^{k_0,n}(t)\, K\Big(\sum_{\substack{|k|=*\\|k|\leq L}} a_{n,k}\, \frac{\xi_{n,k}}{\Xi_{n,k}}\Big)(t)\, dt = -(a_{n,k_0} + a_{n,-k_0})\, \frac{\xi^3_{n,k_0}}{\Xi_{n,k_0}}\, \nu^2_{n,k_0} \int h_{e_{\varepsilon,q}}^{k_0,n}(t)\, e^{i\nu_{n,k_0} t}\, dt\,.$$

So out of iv), we get that

$$\begin{aligned}\Big|\int h_{e_{\varepsilon,q}}^{k_0,n}(t)\, K\Big(\sum_{\substack{|k|=*\\|k|\leq L}} a_{n,k}\, \frac{\xi_{n,k}}{\Xi_{n,k}}\Big)(t)\, dt\Big| &\geq |a_{n,k_0} + a_{n,-k_0}|\, \Big|\frac{\xi^3_{n,k_0}}{\Xi_{n,k_0}}\Big|\, |\nu_{n,k_0}|^2\, \frac{c}{n^{N_q}}\\ &\geq |a_{n,k_0} + a_{n,-k_0}|\, c\, e^{-n^{2/3}}\,.\end{aligned}$$

out of (63) and because, as we have already seen,

$$|\xi^3_{n,k_0}| = \frac{|z_{n,k_0}|}{|\nu_{n,k_0}|}\, |\sin z_{n,k_0}| \geq \frac{C_\gamma}{n^N}\,.$$

If we take the limit with $L \longrightarrow +\infty$,

$$\Big|\int h_{e_{\varepsilon,q}}^{k_0,n}(t)\, K(\Phi^0, \Phi^1, W^0, W^1)(t)\, dt\Big| \geq |a_{n,k_0} + a_{n,-k_0}|\, c\, e^{-n^{2/3}}\,.$$

We can show the same way that

$$\Big|\int h_{o_{\varepsilon,q}}^{k_0,n}(t)\, K(\Phi^0, \Phi^1, W^0, W^1)(t)\, dt\Big| \geq |a_{n,k_0} - a_{n,-k_0}|\, c\, e^{-n^{2/3}}\,.$$

So, by summing conveniently,

$$\begin{aligned}|a_{n,k_0}| \leq C\, e^{n^{2/3}} \Big(\Big|\int h_{e_{\varepsilon,q}}^{k_0,n}(t)\, K(\Phi^0, \Phi^1, W^0, W^1)(t)\, dt\Big| \\ + \Big|\int h_{o_{\varepsilon,q}}^{k_0,n}(t)\, K(\Phi^0, \Phi^1, W^0, W^1)(t)\, dt\Big|\Big)\,.\end{aligned} \tag{70}$$

So for any n greater than $n_1(\varepsilon)$,

$$\|(\Phi^0, \Phi^1, W^0, W^1)^{(1)}\|_{\mathcal{Y}}^2 = \sum_{\substack{|k|=* \\ |k|\le |n|}} |a_{n,k}|^2 \,.$$

So out of (70),

$$\begin{aligned}&\|(\Phi^0, \Phi^1, W^0, W^1)^{(1)}\|_{\mathcal{Y}}^2 \\ &\qquad \le C \sum_{\substack{|k|=* \\ |k|\le |n|}} e^{n^{2/3}} \Big| \int h_{e\varepsilon,q}^{k,n}(t)\, W_{tt}(0,t)\, dt \Big|^2 + \text{same with } h_o \,.\end{aligned}$$

Thus, out of i),

$$\begin{aligned}&\|(\Phi^0, \Phi^1, W^0, W^1)^{(1)}\|_{\mathcal{Y}}^2 \\ &\qquad \le C\, e^{n^{2/3}} \sum_{\substack{|k|=* \\ |k|\le |n|}} \int |h_{\varepsilon,q}^{k,n}(t)|^2\, dt \int_{-T_2(q,\varepsilon)}^{T_2(q,\varepsilon)} |W_{tt}(0,t)|^2\, dt \,.\end{aligned}$$

Thus out of ii),

$$\|(\Phi^0, \Phi^1, W^0, W^1)^{(1)}\|_{\mathcal{Y}}^2 \le C\, e^{Cn^{2/3}}\, e^{2\varepsilon |n|} \int_{-T_2(q,\varepsilon)}^{T_2(q,\varepsilon)} |W_{tt}(0,t)|^2\, dt \,.$$

When q goes to the infinity, if $(q+1)/(1-q) = -1-\delta$, δ goes to 0. So we have proved Proposition 6.

We still have to prove Lemma 3.

First, we will introduce a sequence of functions $f^{k_0,n}$, that will satisfy conditions i), iii) and iv), but which L^2 norms will behave like $e^{n\pi}$, that is too large for ii). We will notice though that these norms will be mostly concentrated within $[-\pi\, n, \pi\, n]$, on the Fourier side.

Then we will build a sequence of functions g of which we will know, by stationary phases computations, that their norms, on the Fourier side, are exponentially small over $[-\pi\, n, \pi\, n]$, and reasonably bounded outside.

We will then put $h = f * g$, and show that h satisfy i) to iv), for suitable parameters.

4.3. Proof of Lemma 3.

In order to prove this lemma, we will build two sequences of functions, denoted f and g, and put $h = f * g$. The functions f will have the right zeroes (on the Fourier side), but too large an L^2 norm. The functions g will be small where f is large, in order to get controlled L^2 norms. We will have to ensure also that they behave properly at the zeroes of f.

Namely, we will prove he following lemmas:

Lemma 4. *For any* (n, k_0) *in* $\mathbb{N}^* \times (\mathbb{Z}^* \cup \{*, **\})$, *there is an even* L^2 *function* $f^{k_0,n}$ *that satisfies*:

i) $f^{k_0,n}$ *is supported by* $[-3\pi, 3\pi]$.

ii) *For* $z \in [-\pi\, n, \pi\, n]$, $|\widehat{f}^{n,k_0}(z)| \le C\, e^{3n\sqrt{\pi^2-(z/n)^2}}$ *and for* $z \notin [-\pi\, n, \pi\, n]$, $|\widehat{f}^{n,k_0}(z)| \le P(n,k)$, *where* P *is a polynomial.*

iii) *If* $k \neq \pm k_0$, $\widehat{f}^{n,k_0}(\nu_{n,k}) = 0$.

iv) *If* $n \ge n_1(\varepsilon, q)$ *and* $(k_0, n) \in I = \{(k,n) : |k| = * \text{ or } |k| \le n\}$, $|\widehat{f}^{n,k_0}(\nu_{n,k_0})| \ge c/n^{N_q}$.

Lemma 5. *For large enough* T, *for any real number* $\delta > 1$, *close to 1, and any odd integer* q, *we can find three positive constants* C_q^1, $C_{q,T}^2$, $c_{q,T,\delta}^3$ *and two integers* $r_q, n(q,\delta)$ *such that for any integer* n, *there is a function* $g_{T,q,\delta}^n$ *in* $L^2(\mathbb{R})$ *such that*:

i) $g_{T,q,\delta}^n$ *is supported by* $[-T, T]$.

ii) $|\widehat{g}_{T,q,\delta}^n|_{L^\infty} \le 2\,T$, *and for any real number* τ *such that* $|\tau| \le n/\delta$,

$$|\widehat{g}_{T,q,\delta}^n(\tau)| \le C_{q,T}^2\, e^{-TnC_q^1 \min\{(1/\delta-\tau/n)^{q/(q-1)},1\}}\,.$$

iii) *For any integer* n *greater than* $n(q,\delta)$, *if* $k_0 = *$ *or* $1 \le k_0 \le n$, *there is a time* T_{n,k_0} *in* $[T, T+1]$ *such that*

$$\Big|\widehat{g_+}^{\,n}_{T_{n,k_0},q}\Big(\frac{|\nu_{n,k_0}|}{\pi}\Big)\Big| \ge \frac{c_{q,T,\delta}^3}{\sqrt{n}}\,.$$

The constants depend only on q *and* ε. *Moreover the functions* g *can be chosen as even or odd. We will denote them* ${g_e}_{T,q,\delta}^n$ *or* ${g_o}_{T,q,\delta}^n$.

Let us prove Lemma 3 out of those two results.

Let ε be a positive real number. Let us choose δ_ε such that

$$3\pi\sqrt{1-\Big(\frac{1}{\delta_\varepsilon}\Big)^2}=\frac{\varepsilon}{2}$$

and T^ε such that

$$\sup_{\beta\in[0,1/\delta_\varepsilon]}\Big(3\pi\sqrt{1-\beta^2}-C_q^1\,T^\varepsilon\Big(\frac{1}{\delta_\varepsilon}-\beta\Big)^{q/(q-1)}\Big)\le\varepsilon\,. \tag{71}$$

The derivative is

$$\frac{-3\pi\beta}{\sqrt{1-\beta^2}}+\frac{q}{q-1}\,T^\varepsilon\,C_q^1\,\Big(\frac{1}{\delta_\varepsilon}-\beta\Big)^{1/(q-1)}\,,$$

we choose T^ε such that it is 0 for β_ε such that $3\pi\sqrt{1-\beta_\varepsilon}$.

We have

$$\delta_\varepsilon=1+\frac{\varepsilon^2}{72\pi^2}+o\,(\varepsilon^2)\,,$$

$$\beta_\varepsilon=1-\frac{\varepsilon^2}{18\pi^2}+o\,(\varepsilon^2)\,,$$

so

$$\frac{1}{\delta_\varepsilon}-\beta_\varepsilon\sim\frac{\varepsilon^2}{24\pi^2}\,,$$

hence $T^\varepsilon\sim c_q\,\varepsilon^{(q+1)/(1-q)}$.

Let us define positive times T^ε_{n,k_0} as follows. For integers k_0 such that $|k_0|\le|n|$ or $|k_0|=*$, we take the time T^ε_{n,k_0} given by Lemma 5 with $T=T^\varepsilon$; and for $|k_0|>|n|$, we put $T^\varepsilon_{n,k_0}=T^\varepsilon$.

$$T^\varepsilon_{n,k_0}\in[T^\varepsilon,T^\varepsilon+1]\,,\qquad\text{so } c_q^1\,\varepsilon^{(q+1)/(1-q)}\le T^\varepsilon_{n,k_0}\le c_q^2\,\varepsilon^{(q+1)/(1-q)}\,.$$

Let us denote

$$\widehat{h}^{\,k_0,n}_{e\varepsilon,q}(\tau)=\widehat{f}^{\,k_0,n}(\tau)\,\widehat{g}^{\,n}_{eT^\varepsilon_{n,k_0},q,\delta_\varepsilon}\Big(\frac{\tau}{\pi}\Big)\,,$$

$$\widehat{h}^{\,k_0,n}_{o\varepsilon,q}(\tau)=\widehat{f}^{\,k_0,n}(\tau)\,\widehat{g}^{\,n}_{oT^\varepsilon_{n,k_0},q,\delta_\varepsilon}\Big(\frac{\tau}{\pi}\Big)\,.$$

The subscript meaning that h is even or odd. We will not write this subscript when not necessary.

We shall now prove step by step that $h^{k_0,n}_{\varepsilon,q}$ satisfies all the properties of Lemma 3.

PROOF OF i). By Lemma 4.i), the support of $f^{k_0,n}$ is located within $[-3\pi, 3\pi]$.

By Lemma 5.i), $g^n_{T^\varepsilon_{n,k_0},q,\delta}$ is supported by $[-T^\varepsilon_{n,k_0}, T^\varepsilon_{n,k_0}]$.

As $h^{k_0,n}_{\varepsilon,q}$ is the convolution product of those two functions, it is supported by $[-T_2(q,\varepsilon), T_2(q,\varepsilon)]$ with $T_2(q,\varepsilon) = 3\pi + c_q^2\, \varepsilon^{(q+1)/(1-q)}$. The estimates on T^ε_{n,k_0} insures that $T_2(q,\varepsilon) \le C_q\, \varepsilon^{(q+1)/(1-q)}$.

PROOF OF ii). We will use results about the small size of $\|g\|$ that will compensate $\|f\|$.

By Lemma 5.ii),

$$|\widehat{g}|_{L^\infty} \le 2\, T_2(q,\varepsilon)\,.$$

Furthermore, outside of $[-\pi\, n, \pi\, n]$, the L^2 norm of f is bounded by a polynomial in n, so the problems are located within this interval.

We must estimate $\int_{-n}^{n} |\widehat{h}^{k_0,n}_{\varepsilon,q}(\tau)|^2\, d\tau$.

Now, out of Lemma 4.ii), we know that if τ/n belongs to $[-\pi, \pi]$, we have

$$|f^{k_0,n}(\tau)|^2 \le C\, e^{6n\sqrt{\pi^2-|\tau/n|^2}} = C\, e^{6\pi n\sqrt{1-|\tau/(\pi n)|^2}}\,.$$

Thus if $|\tau/(\pi\, n)| \ge 1/\delta_\varepsilon$, $|\widehat{h}^{k_0,n}_{\varepsilon,q}(\tau)|^2 \le C\, e^{\varepsilon n}$.

Moreover, out of Lemma 5.ii), if $|\tau/(\pi\, n)|$ is smaller than $1/\delta_\varepsilon$, we have

$$\left|\widehat{g}^n_{T^\varepsilon_{n,k_0},q,\delta_\varepsilon}\left(\frac{\tau}{\pi}\right)\right|^2 \le C\, e^{-2T^\varepsilon_{n,k_0}\, n\, C^1_q (1/\delta_\varepsilon - |\tau/(\pi n)|)^{q/(q-1)}}\,.$$

So out of (71), we get $|\widehat{h}^{k_0,n}_{\varepsilon,q}(\tau)|^2 \le C\, e^{2\varepsilon n}$. Thus

$$\|\widehat{h}^{k_0,n}_{\varepsilon,q}\|^2_{L^2} \le C\, e^{2\varepsilon n}\,.$$

PROOF OF iii). This is a simple consequence of Lemma 4.iii). Indeed for any integer k different from k_0, $\widehat{f}^{k_0,n}(|\nu_{n,k}|) = 0$. So by definition of h, we also have $\widehat{h}^{k_0,n}_{\varepsilon,q}(|\nu_{n,k}|) = 0$, which is exactly the Fourier transcription of iii).

PROOF OF iv). For any couple (n,k_0) in I, out Lemma 4.iv) and Lemma 5.iii), we get

$$|\widehat{h}^{k_0,n}_{\varepsilon,q}(\pm|\nu_{n,k_0}|)| \geq \frac{C}{n^N}\,\frac{c_{q,T_\varepsilon,\delta_\varepsilon}}{\sqrt{n}} \geq \frac{C_{q,\varepsilon}}{n^{N'}}\ ,$$

which is once again the Fourier transcription of the needed result.

Now we have to prove Lemmas 4 and 5.

4.3.1. Proof of Lemma 4: construction of f.

Put

$$F^n(z) = ((z^3 + z\,(n^2\,\pi^2 - n^4\,\pi^4))\tan z - z^2 - n^2\,\pi^2)\cos z\,,$$

$$G^n(z) = \sqrt{z^2 - n^2\,\pi^2}\ ,$$

and

$$f^n(z) = F^n(G^n(z))\,.$$

The following properties hold for these functions:

f-i) $f_0^n \in \mathcal{O}(\mathbb{C})$.

f-iii) For any k in $\mathbb{Z}^* \cup \{*,**\}$, $f_0^n(\nu_{n,k}) = (z_{n,k}^3\tan z_{n,k} - z_{n,k}^2 - n^2\,\pi^2)\cos z_{n,k} = 0$ out of (57).

Let us evaluate $f^{n\prime}(\nu_{n,k})$.

$$f^{n\prime}(\nu_{n,k}) = G^{n\prime}(\nu_{n,k})F^{n\prime}(\underbrace{G^n(\nu_{n,k})}_{z_{n,k}})\,.$$

Now $|G^{n\prime}(\nu_{n,k})| = |\nu_{n,k}/z_{n,k}| \geq 1$. So to bound $|f^{n\prime}(\nu_{n,k})|$ from bellow, we only have to bound $|F^{n\prime}(z_{n,k})|$ from bellow. To simplify the notation, put $\alpha_n = n^4\,\pi^4 - n^2\,\pi^2$.

$$\begin{aligned}
&F^{n\prime}(z_{n,k})\\
&= \cos z_{n,k}\\
&\quad\cdot \underbrace{(-2\,z_{n,k} + (z_{n,k}^3 - \alpha_n z_{n,k})\,(1+\tan^2 z_{n,k}) + (3\,z_{n,k}^2 - \alpha_n)\tan z_{n,k})}_{h(z_{n,k})}
\end{aligned}$$

(see pictures 1 and 2.)

We recall that the first value of k for which $z_{n,k}$ is larger than $\sqrt{\alpha_n}$ is denoted by k_0. If $k \neq k_0$ and $k \neq k_0 - 1$, $|z_{n,k} - \sqrt{\alpha_n}| \geq \pi/2$.As we also have $z_{n,k} \geq \pi/2$ and $z_{n,k}$ is a root of $\tan z_{n,k} = (z^2 + n^2\,\pi^2)/(z^3 - \alpha_n\, z)$, we get $|\cos z_{n,k}| \geq 1/P(n,k)$ where P is a polynomial.

Let us consider h. For any positive ε and $z = z_{n,k} \geq \sqrt{\alpha_n} + \varepsilon$,

$$\begin{aligned} h(z) &= -2\,z + (z^3 - \alpha_n\, z)\,(1 + \tan^2 z) + (3\,z^2 - \alpha_n) \tan z \\ &\geq -2\,z + (z - \sqrt{\alpha_n})\,(z + \sqrt{\alpha_n})\,z \\ &\geq (2\,\varepsilon\,\alpha_n - 2)\,z \\ &> 1 \qquad \text{for large } n\,. \end{aligned}$$

For $\sqrt{\alpha_n/3} \leq z = z_{n,k} < \sqrt{\alpha_n}$, $h(z) \leq -2\,z$ so $|h(z)| \geq 1$.

And for $z = z_{n,k} \leq \sqrt{\alpha_n/3}$,

$$h(z) \leq -2\,z - \left(|z^3 - \alpha_n\, z|\,|1 + \tan^2 z| - |3\,z^2 - \alpha_n|\,|\tan z|\right),$$

now $|1 + \tan^2 z| > |\tan z|$ and as $\pi/2 \leq z \leq \sqrt{\alpha_n/3}$, $|z^3 - \alpha_n\, z| \geq |3\,z^2 - \alpha_n|$.

So $|h(z_{n,k})| \geq |2\,z_{n,k}| \geq 1$.

Hence we know that if $k \neq k_0$ and $k \neq k_0 - 1$,

$$|F^{n\prime}(z_{n,k})| \geq \frac{1}{P(n,k)}\,.$$

Now if $k = k_0$ or $k = k_0 - 1$, $z_{n,k} \in [\sqrt{\alpha_n} - \pi/2, \sqrt{\alpha_n} + \pi/2]$ so for large n, $z_{n,k} \sim \sqrt{\alpha_n}$. Now

$$z_{n,k} - \sqrt{\alpha_n} = \underbrace{\frac{z_{n,k}^2 + n^2\,\pi^2}{z\,(z + \sqrt{\alpha_n})}}_{\sim 1/2}\,\frac{\cos z_{n,k}}{\sin z_{n,k}}\,.$$

So for a small fixed η, either $|\cos z_{n,k}| \geq \eta$, then

$$|z_{n,k} - \sqrt{\alpha_n}| \geq \varepsilon(\eta)$$

and in that case we know that $|h(z_{n,k})| \geq 1$, hence $|F^{n\prime}(z_{n,k})| \geq \eta$.

Either $|\cos z_{n,k}| < \eta$, now

$$F^{n\prime}(z_{n,k}) = \underbrace{-2\, z_{n,k} \cos z_{n,k}}_{|\cdot| \le 2(\eta+\varepsilon)\sqrt{\alpha_n}} + \underbrace{(3\, z_{n,k}^2 - \alpha_n) \sin z_{n,k}}_{|\cdot| \ge 2\alpha_n \sqrt{1-\eta^2}}$$
$$+ \underbrace{\frac{z_{n,k} - \sqrt{\alpha_n}}{\cos z_{n,k}}\, z_{n,k}\, (z_{n,k} + \sqrt{\alpha_n})}_{|\cdot| \le (1/2+\varepsilon)(2/\sqrt{1-\eta^2})\alpha_n} .$$

Now for small η,

$$2\sqrt{1-\eta^2} > \frac{2}{\sqrt{1-\eta^2}} \Big(\frac{1}{2} + \varepsilon\Big) ,$$

so

$$|F^{n\prime}(z_{n,k})| \ge c\, \alpha_n \ge 1 .$$

So we have proved that for any n, k,

$$|f^{n\prime}(\nu_{n,k})| \ge \frac{1}{P(n,k)} .$$

Let us put for any k in $\mathbb{Z}^* \cup \{*, **\}$,

$$\widehat{f}^{\,k,n}(z) = f_0^n(z)\, \frac{1}{z^2 - |\nu_{n,k}|^2} \Big(\frac{\sin\sqrt{z^2 - \pi^2 n^2}}{\sqrt{z^2 - \pi^2 n^2}}\Big)^2$$

(the last term ensures that f remains in L^2).

Let us show that these functions satisfy the properties of Lemma 4: by construction, they are even.

As f_0 has got zeroes at $\pm|\nu_{n,k}|$, $\widehat{f}^{\,k,n} \in \mathcal{O}(\mathbb{C})$. Moreover $\widehat{f}^{\,k,n} \in L^2(\mathbb{R})$ and for any complex number z, $|\widehat{f}_0^{\,k,n}(z)| \le C\, e^{3|\mathrm{Im}\, z|}$.

So by the Paley Wienner theorem, we have property i).

Property ii) is straightforward, due to the explicit value of $\widehat{f}^{n,k}$.

As by f-iii), $\nu_{n,k}$ is a zero of f_0^n for any k, it is by definition a zero of $\widehat{f}^{n,k_0}$ if $k \ne k_0$, so iii) holds

Furthermore,

$$f_0^{k,n}(\pm|\nu_{n,k}|) = f_0^{n\prime}(\pm|\nu_{n,k}|) \Big(\frac{\sin z_{n,k}}{z_{n,k}}\Big)^2 \frac{1}{\mp 2\, |\nu_{n,k}|} ,$$

thus

$$|f_0^{k,n}(\pm|\nu_{n,k}|)| \geq \frac{C}{(1+n^2)\,|k|^3}\,|\sin z_{n,k}|^2 \geq \frac{C}{(1+n^2+k^2)^{N_2}}\,.$$

So iv) holds.

4.3.2. Proof of Lemma 5: construction of functions g.

Let q be an odd integer and let us denote $h_q(x)$ the solution of $y' = 1 + y^{q-1}$ that satisfies $y(0) = 0$. This function is defined over $(-x_q, x_q)$ for a positive x_q. It is odd, strictly increasing and analytic. Moreover, we have $h_q(x) = x + \alpha_q\, x^q + o\,(x^q)$ when x is near 0, with a positive α_q and when x goes to x_q, h_q goes to the infinity.

We shall denote H_q the reciprocal function to h_q. It is defined over $\mathbb{R}$, odd, strictly increasing, bounded by x_q. We have $H_q(x) = x - \alpha_q\, x^q + o\,(x^q)$ if x is close to 0.

Let δ be a real number, greater than 1, and close to 1, that will be fixed later.

Let us define functions g as follows

$$g_{+T,q}^{\,n}(t) = \mathbf{1}_{(-T,T)}\, e^{in(T/\delta x_q) h_q((x_q/T)t)}\,, \tag{72}$$

$$\widehat{g_{+T,q}^{\,n}}(\tau) = \int_{-T}^{T} e^{in(T/\delta x_q) h_q((x_q/T)t) - i\tau t}\, dt\,. \tag{73}$$

Let us write $\Psi_q(s) = (T/x_q)\, H_q((\delta\, x_q/T)\, s)$,

$$\widehat{g_{+T,q}^{\,n}}(\tau) = \int_{-\infty}^{+\infty} e^{ins - i\tau\Psi_q(s)}\, \Psi_q'(s)\, ds\,.$$

If we denote

$$\theta_q(s) = \frac{1}{x_q}\, H_q(\delta\, x_q\, s)\,,$$

we have

$$\begin{aligned}\widehat{g_{+T,q}^{\,n}}(\tau) &= \int_{-\infty}^{+\infty} \theta_q'\Big(\frac{s}{T}\Big)\, e^{inT(s/T - (\tau/n)\theta_q(s/T))}\, ds \\ &= T \int_{-\infty}^{+\infty} \theta_q'(v)\, e^{inT(v - (\tau/n)\theta_q(v))}\, dv\,.\end{aligned}$$

Let us put $\alpha = n\,T$ and $\beta = \tau/n$. We will estimate

$$\psi(\alpha,\beta) = \int \theta'_\delta(v)\, e^{i\alpha(v-\beta\theta_\delta(v))}\, dv\,,$$

for α going to the infinity.

There will be two kinds of estimates depending upon the value of β as compared to $1/\delta$.

• If $\beta < 1/\delta$. In this zone, the phases is non-stationary. So we will get and exponential decrease.

Let us shift slightly in the imaginary direction. For any real number v, any β smaller than $1/\delta$ and any little ε, we get

$$\begin{aligned}
&\operatorname{Im}(v + i\,\varepsilon - \beta\,\theta_q(v+i\,\varepsilon)) \\
&\qquad = \varepsilon - \beta \operatorname{Im}\theta_q(v+i\,\varepsilon) \\
&\qquad = \varepsilon - \beta \operatorname{Im}(\theta_q(v+i\,\varepsilon) - \theta_q(v)) \\
&\qquad = \varepsilon - \beta \operatorname{Im}\int_v^{v+i\varepsilon} \theta'_q(z)\, dz \\
&\qquad = \varepsilon - \beta \operatorname{Im}\int_v^{v+i\varepsilon} \frac{\delta\, dz}{1+\delta^{q-1}\, x_q^{q-1}\, z^{q-1}} \\
&\qquad = \varepsilon - \beta\,\varepsilon\,\delta \operatorname{Re}\int_0^1 \frac{du}{1+\delta^{q-1}\, x_q^{q-1}\,(v+i\,\varepsilon\, u)^{q-1}} \\
&\qquad \geq \varepsilon \qquad \text{if } \beta \leq 0\,.
\end{aligned}$$

If β is positive,

$$\operatorname{Im}(v + i\,\varepsilon - \beta\,\theta_q(v+i\,\varepsilon)) \geq \varepsilon - \beta\,\varepsilon\,\delta\Big|\int_0^1 \frac{du}{1+\delta^{q-1}\, x_q^{q-1}\,(v+i\,\varepsilon\, u)^{q-1}}\Big|\,.$$

Now for any real v,

$$\underbrace{\Big|\int_0^1 \frac{du}{1+\delta^{q-1}\, x_q^{q-1}\,(v+i\,\varepsilon\, u)^{q-1}}\Big|}_{I} \leq \frac{1}{1-c_q\,\varepsilon^{q-1}}\;,$$

because either $v \gg \varepsilon$ and then

$$I \leq \frac{c}{1+v^{q-1}} \leq 1\,,$$

or $v \le M_q\,\varepsilon$ and then

$$|v + i\,\varepsilon\,u|^{q-1} \le C_q\,\varepsilon^{q-1}$$

implies

$$|1 + \delta^{q-1}\,x_q^{q-1}\,(v + i\,\varepsilon\,u)^{q-1}| \ge 1 - c_q\,\varepsilon^{q-1}$$

implies

$$I \le \frac{1}{1 - c_q\,\varepsilon^{q-1}}\,.$$

Thus

$$\begin{aligned}\operatorname{Im}(v + i\,\varepsilon - \beta\,\theta_q(v + i\,\varepsilon)) &\ge \varepsilon - \frac{\beta\,\delta\,\varepsilon}{1 - c_q\,\varepsilon^{q-1}}\\ &\ge \varepsilon\,(1 - \delta\,\beta) - c'_q\,\beta\,\varepsilon^q\\ &\ge \varepsilon\Big(\frac{1}{\delta} - \beta\Big) - c'_q\,\beta\,\varepsilon^q\,.\end{aligned}$$

Now

$$\max_{\varepsilon}\varepsilon\,\Big(\frac{1}{\delta}-\beta\Big)-c_q\,\beta\,\varepsilon^q = c'_q\Big(\frac{1}{\delta}-\beta\Big)^{q/(q-1)}\beta^{1/(1-q)} \ge c''_q\Big(\frac{1}{\delta}-\beta\Big)^{q/(q-1)}.$$

We can choose a real number ε and a very small c_q such that for any real number v,

$$\begin{cases}\operatorname{Im}(v + i\,\varepsilon - \beta\,\theta_q(v + i\,\varepsilon)) \ge c_q\Big(\dfrac{1}{\delta} - \beta\Big)^{q/(q-1)}, & \text{if } \beta \in \Big(0, \dfrac{1}{\delta}\Big]\,,\\ \operatorname{Im}(v + i\,\varepsilon - \beta\,\theta_q(v + i\,\varepsilon)) \ge c_q\,, & \text{if } \beta \le 0\,.\end{cases}$$

Now we can shift the integration line over v from $\mathbb{R}$ to $\mathbb{R} + i\,\varepsilon$

$$\psi(\alpha,\beta) = \int \theta'_q(v + i\,\varepsilon)\,e^{i\alpha(v+i\varepsilon-\beta\theta_q(v+i\varepsilon))}\,dv\,.$$

To end with, as

$$\theta'_q(v + i\,\varepsilon) = \frac{\delta}{1 + (\delta\,x_q\,(v + i\,\varepsilon))^q - 1}\,,$$

we get

$$|\theta'_q(v + i\,\varepsilon)| \le \frac{C_q}{1 + v^{q-1}}\,,$$

hence for any real number α and any $\beta \le 1/\delta$,

$$|\psi(\alpha,\beta)| \le \int \frac{C_q}{1+v^{q-1}}\, e^{-\alpha c_q \min\{(1/\delta-\beta)^{q/(q-1)},1\}}\, dv$$
$$\le C_q\, e^{-\alpha c_q \min\{(1/\delta-\beta)^{q/(q-1)},1\}}\,.$$

So if $\tau/n \le 1/\delta$,

$$|\widehat{g_{+T,q}^{n}}(\tau)| \le C_q\, T\, e^{-nTc_q \min\{(1/\delta-\tau/n)^{q/(q-1)},1\}}\,. \tag{74}$$

• If $\beta \ge 1$ $(> 1/\delta)$. Through the stationary phase formula (see [7, p. 431]), we get

$$\psi(\alpha,\beta) = C\,(|H_{\beta,\delta}|\cos\alpha\, p_0(\beta,\delta))\Big(\frac{\theta_q'(v_0(\beta,\delta))}{\sqrt{\alpha}} + \sum_{j=1}^{N}\frac{a_j(\beta,\delta)}{\alpha^j\sqrt{\alpha}}\Big) + r_{\beta,\delta}(\alpha)\,,$$

where $r_{\beta,\delta}(\alpha) \le C_\beta/\alpha^{N+1}$ and $\alpha \ge A_{\beta,\delta}$; $H_{\beta,\delta}$ denoting the square root of the Hessian at the critical points.

Moreover, in this formula, C and A are continuous with respect to β and δ, and $a_j(\beta,\delta)$ depends on the first $2\,j+1$ derivatives of $v \longmapsto \theta_q(v)$ at $v = v_0(\beta,\delta)$.

Let us compute $p_0(\beta,\delta)$.

$$\begin{aligned}
\frac{\partial}{\partial v}\,(v-\beta\,\theta_q(v)) = 0 \qquad & \text{if an only if} \qquad 1 - \frac{\beta\,\delta}{1+\delta^{q-1}\,x_q^{q-1}\,v^{q-1}} = 0\\
& \text{implies} \qquad 1+\delta^{q-1}\,x_q^{q-1}\,v_0^{q-1}(\beta,\delta) = \beta\,\delta\\
& \text{implies} \qquad v_0(\beta,\delta) = \frac{1}{\delta\,x_q}\,(\delta\,\beta-1)^{1/(q-1)}\,.
\end{aligned}$$

If β takes the values $|\nu_{n,k}|/(n\,\pi)$ for any couple (n,k) such that $|k| \le n$, we have $1 \le \beta \le \pi\sqrt{2}$.

Moreover, if $\beta = |\nu_{n,*}|/(n\,\pi)$, by (62),

$$\beta \ge 1 - \frac{C}{\sqrt{n}} \ge \frac{1}{2}\Big(1+\frac{1}{\delta}\Big)$$

as soon as $n \ge n_0(\delta)$.

So for any n greater than $n_0(\delta)$, if (n,k) belongs to I and $\beta = |\nu_{n,k}|/(n\,\pi)$, $C \ge v_0(\beta,\delta)$, $|p_0(\beta,\delta)|$, $|H_{\beta,\delta}| \ge c_\delta$, thus $1 \ge \theta_q'(v_0(\beta,\delta)) \ge c_q$. Moreover $a_j(\beta,\delta) \le C_{j,\delta}$.

Let T be a positive real time. As $|p_0(\beta,\delta)| \geq c_\delta$, for any n greater than $n_0(\delta)$, and k_0 such that (n,k_0) belongs to I, one can pick a time T_{n,k_0} in $[T,T+1]$ such that

$$\cos\Big(n\,T_{n,k_0}\,p_0\Big(\frac{|\nu_{n,k_0}|}{n\,\pi},\delta\Big)\Big) \geq c'_\delta\ .$$

Thus for $T > T_u$, $n \geq n(q,\delta)$, $\alpha = T\,n$, $(k_0,n) \in I$ and $\beta = |\nu_{n,k_0}|/(n\,\pi)$,

$$\Big|\frac{\theta'_q(v_0(\beta,\delta))}{\sqrt{\alpha}} + \sum_{j=1}^{N}\frac{a_j(\beta,\delta)}{\alpha^j\sqrt{\alpha}}\Big| \geq \frac{|\theta'_q(v_0(\beta,\delta))|}{2\sqrt{\alpha}}\ ,$$

$$|r_{\beta,\delta}(\alpha)| \leq c'_\delta\,\frac{|H_{\beta,\delta}|\,\theta'_q(v_0(\beta,\delta))}{4\sqrt{\alpha}}\ .$$

And in the same conditions, there is a time T_{n,k_0} in $[T,T+1]$ such that

$$\Big|\psi\Big(n\,T_{n,k_0},\frac{|\nu_{n,k_0}|}{n\,\pi}\Big)\Big| \geq \frac{c'_\delta\,|H|\,\theta'_q\Big(v_0\Big(\frac{\nu_{n,k_0}}{n\,\pi},\delta\Big)\Big)}{4\sqrt{n}\,\sqrt{T_{n,k_0}}} \geq \frac{c}{\sqrt{n}}\ .$$

We have proved that for any time T greater than T_u, for any n larger than $n(q,\delta)$ and k_0 such that $|k_0| = *$ or $|k_0| \leq n$, there is a time T_{n,k_0} in $[T,T+1]$ such that

$$\Big|\widehat{g_{+}}^{\,n}_{T_{n,k_0},q}\Big(\frac{|\nu_{n,k_0}|}{\pi}\Big)\Big| \geq \frac{C_{T,q,\delta}}{\sqrt{n}}\ . \tag{75}$$

By changing t into $-t$, we can prove two estimates similar to (74) and (75) for the functions

$$g_{-}{}^{n}_{T,q,\delta}(t) = \mathbf{1}_{(-T,T)}\,e^{in(T/\delta x_q)h_q((x_q/T)t)}\ .$$

As $g_{-}{}^{n}_{T,q,\delta} = \overline{g_{+}{}^{n}_{T,q,\delta}}$, we have: $T_{n,k_0,+} = T_{n,k_0,-}$.

So if we put

$$g_{e}{}^{n}_{T,q,\delta}(t) = \mathbf{1}_{(-T,T)}\cos\Big(n\,\frac{T}{\delta\,x_q}\,h_q\Big(\frac{x_q}{T}\,t\Big)\Big)\,,$$

we have

$$g_{e}{}^{n}_{T,q,\delta}(t) = \operatorname{Re} g_{+}{}^{n}_{T,q,\delta}(t) = \frac{1}{2}\,(g_{+}{}^{n}_{T,q,\delta}(t) + g_{-}{}^{n}_{T,q,\delta}(t))\,.$$

Let us show that this even function satisfies the properties of Lemma 5.

i) By definition, it is supported by $[-T,T]$.

ii) is an easy consequence of the definition and (73) for the L^∞ estimate, and (74) for the other one.

iii) If $n \geq n(q,\delta)$ and ($|k_0| \leq n$ or $|k_0| = *$), $C^2_{q,T}\, e^{-nTC^1_q} \leq c^3_{q,T,\delta}/(2\sqrt{n})$, so , if n is large enough, by (74) and (75),

$$|\widehat{g_{\pm}}^n_{T_{n,k_0},q,\delta}(\tau)| \leq |\widehat{g_{\mp}}^n_{T_{n,k_0},q,\delta}(\tau)|\,, \qquad \text{for } \tau = \mp\frac{|\nu_{n,k_0}|}{\pi}\,.$$

As we can increase the constants to cope with the finite number of (n,k) in I for which n is not large enough, we get for (n,k_0) in I and $\tau = \pm|\nu_{n,k_0}|/\pi$,

$$|\widehat{g_e}^n_{T_{n,k_0},q,\delta}(\tau)| \geq \frac{c_q,T,\delta}{\sqrt{n}}\,.$$

Of course, similar results hold for the odd function

$$g_o{}^n_{T,q,\delta}(t) = \mathbf{1}_{(-T,T)} \sin\Big(n\,\frac{T}{\delta\, x_q}\, h_q\Big(\frac{x_q}{T}\,t\Big)\Big)\,.$$

This ends the proof of Lemma 5.

References.

[1] Allibert, B., Contrôle analytique de l'équation des ondes et de l'équation de Schrodinger sur des surfaces de revolution. *Comm. Partial Diff. Equations* **23** (1998), 1493-1556.

[2] Avalos, G., Lasiecka, I., A differential Riccati equation for the active control of a problem in structural acoustics. *J. Optim. Theory Appl.* **91** (1996), 695-728.

[3] Avdonin, S. A., Ivanov, S. A., *Families of exponentials. The method of moments in controllability problems for distributed parameter systems.* Cambridge Univ. Press, 1995.

[4] Banks, H. T., Fang, W., Silcox, R. J., Smith, R. C., Approximation Methods for Control of Acustic/Structure Models with Piezoceramic Actuators. *Journal of Intelligent Material Systems and Structures.* **4** (1993), 98-116.

[5] Ingham, A. E., Some trigonometrical inequalities with applications to the theory of series. *Math. Z.* **41** (1936), 367-369.

[6] Lions, J. L., *Contrôlabilité exacte, perturbations et stabilization de systèmes distribués.* Tome 1. *Contrôlabilité exacte.* Masson, 1988.

[7] Micu, S., Zuazua, E., Boundary controllability of a linear hybrid system arising in the control of noise. *SIAM J. Control Optim.* **35** (1997), 1614-1637.

[8] Trèves, F., *Introduction to pseudodifferential and Fourier integral operators.* Plenum Press, The university series in mathematics. 2 vol., 1980.

[9] Tucsnak, M., Regularity and Exact Controllability for a Beam with Piezoelectric Actuators. *SIAM J. Control Optim.* **34** (1996), 922-930.

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A Lieb-Thirring bound for a magnetic Pauli Hamiltonian, II

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Abstract. We establish a Lieb-Thirring type estimate for Pauli Hamiltonians with non-homogeneous magnetic fields. Besides of depending on the size of the field, the bound also takes into account the size of the field gradient. We then apply the inequality to prove stability of non-relativistic quantum mechanical matter coupled to the quantized ultraviolet-cutoff electromagnetic field for arbitrary values of the fine structure constant.

1. Introduction.

We continue here our analysis of Lieb-Thirring type estimates for Pauli Hamiltonians, which we begun in [1] (henceforth called I) and present its applications to the stability of matter coupled to the (ultraviolet-cutoff) quantized electromagnetic field. The one-particle Hamiltonian we consider describes a spin 1/2 electron and is once more

$$H = \not{D}^2 - V\,, \tag{1.1}$$

acting on $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, where $D = p - A$ and $\not{D} = D \cdot \sigma$. Here, $A(x)$ is the magnetic vector potential, σ is the vector of Pauli matrices, and $V(x) \geq 0$ is a scalar potential. In I, the paradigm was given by the well-known Lieb-Thirring estimate [11] for the case $B = \nabla \wedge A = 0$ and our estimate (I.1.2) aimed at estimating the effect of $B \neq 0$ (see

[4], [9], [17], [18], [5], [15] for other results in this direction). Here, by contrast, the starting point is the following bound, due to Lieb, Solovej and Yngvason [10], on the sum of the negative eigenvalues $-e_i$ of (1.1),

$$\sum e_i \le C \int V(x)^{3/2} \, (V(x) + B) \, d^3x \,, \tag{1.2}$$

which holds for the case in which the field B is constant. Our goal is to generalize it to the case where B is not constant, or, more precisely, that of estimating the effect of $\nabla \otimes B = (\partial_i B_j)_{i,j=1,2,3} \neq 0$ on (1.2). We remark that an estimate having the same purpose, but quite different assumptions on B, has been derived in [5], [6].

In I, the role of $B(x)$ was expressed by means of a length scale $r(x)$ defined through $B(x)$ non-locally (incorporating insight of [4], [17], [18]). Similarly here, the role of $\nabla \otimes B$ will be reflected in a second length scale $l(x)$. These two length scales satisfy

$$\int r(x)^{-4} \, d^3x \le C \int B(x)^2 \, d^3x \,, \tag{1.3}$$

$$\int l(x)^{-6} \, d^3x \le C \int (\nabla \otimes B(x))^2 \, d^3x \,, \tag{1.4}$$

as well as some local variants thereof. We can now state our generalization of (1.2).

Theorem 1. *For sufficiently small $\varepsilon > 0$ there are constants C', $C'' > 0$ such that for any vector potential $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$*

$$\begin{aligned} \sum e_i \le C' &\int V(x)^{3/2} \, (V(x) + \widehat{B}(x)) \, d^3x \\ &+ C'' \int V(x) \, P(x)^{1/2} \, (P(x) + \widehat{B}(x)) \, d^3x \,, \end{aligned} \tag{1.5}$$

where $\widehat{B}(x)$ is the average of $|B(y)|$ over a ball of radius $\varepsilon \, l(x)$ centered at x, and

$$P(x) = l(x)^{-1} \, (r(x)^{-1} + l(x)^{-1}) \,.$$

As noticed in [5], (1.5) yields, by the variational principle, a bound on the density $n(x) = E(x,x)$ of zero modes of $\not{D}$, where $E(x,y)$ is the integral kernel of the spectral projection E corresponding to the possible [13] eigenvalue 0 of $\not{D}$. The bound is

$$n(x) \le C'' \, P(x)^{1/2} \, (P(x) + \widehat{B}(x)) \,,$$

and, as it should, it vanishes in the case of a homogeneous magnetic field, where $l = \infty$.

In Section 2 we discuss the properties of the two length scales mentioned above. The main part of the proof of Theorem 1 is given in Section 3, while some more technical aspects are deferred to Section 4. In order to keep these sections reasonably short we shall be brief on details which have already been discussed at length in I.

We now turn to the implications of estimate (1.5) regarding stability of non-relativistic matter coupled to quantum electromagnetic field. We recover a result of [8] establishing stability for any value of the fine structure constant α, with a bound depending however on the ultraviolet cutoff $\Lambda < \infty$. The details of the model are as follows. The electromagnetic vector potential is (in appropriate units [2])

$$\begin{aligned} A_\Lambda(x) \equiv A(x) &= A_-(x) + A_+(x)\,, \qquad A_+(x) = A_-(x)^*\,, \\ A_-(x) &= \frac{\alpha^{1/2}}{2\pi} \int \kappa(k)\, |k|^{-1/2} \sum_{\lambda=\pm} a_\lambda(k)\, e_\lambda(k)\, e^{ikx}\, d^3k\,. \end{aligned} \tag{1.6}$$

The cutoff function $\kappa(k)$ satisfies $|\kappa(k)| \le 1$ and $\operatorname{supp}\kappa \subset \{k \in \mathbb{R}^3 : |k| \le \Lambda\}$; the operators $a_\lambda(k)^*$ and $a_\lambda(k)$ are creation and annihilation operators on the bosonic Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ (with $\mathbb{C}^2$ accounting for the helicity states of the photon) and satisfy canonical commutation relations

$$[a_\lambda(k)^\#, a_{\lambda'}(k')^\#] = 0\,, \qquad [a_\lambda(k), a_{\lambda'}(k')^*] = \delta_{\lambda\lambda'}\, \delta(k - k')\,.$$

Moreover, for each k, the direction of propagation $\hat{k} = k/|k|$ and the polarizations $e_\pm(k) \in \mathbb{C}^3$ are orthonormal. The free photon Hamiltonian is

$$H_f = \alpha^{-1} \int |k| \sum_{\lambda=\pm} a_\lambda(k)^*\, a_\lambda(k)\, d^3k\,.$$

Matter consists of K nuclei of charge $Z > 0$ with arbitrary positions R_k, $(k = 1, \ldots, K)$ and N electrons obeying the Pauli principle. The Hamiltonian for both matter and field, acting on $(\wedge^N \mathcal{H}) \otimes \mathcal{F}$, is

$$H = H_m + H_f\,,$$

where

$$H_m = \sum_{i=1}^{N} \not{p}_i^2 + V_C \ ,$$

$$V_C = \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{1}{|x_i - x_j|} - \sum_{i,k=1}^{N,K} \frac{Z}{|x_i - R_k|} + \sum_{\substack{k,l=1\\ k<l}}^{K} \frac{Z^2}{|R_k - R_l|} \ .$$

The energy per particle is bounded below as shown by the following result, previously established in [8].

Theorem 2. *The Hamiltonian H satisfies*

$$H \geq -C\,(Z, \alpha, \Lambda)\,(N + K)\,,$$

where

$$C(Z, \alpha, \Lambda) = \text{const}\, {z^*}^5 \log\left(1 + z^*\right) Z^* \left(\Lambda + {z^*}^{-2} Z^*\right),$$

with $z^* = 1 + Z^* \alpha^2$ and $Z^* = Z + 1$.

The proof, given in Section 6, rests on a stability result [7] for matter coupled to a classical magnetic field, which is here established in Section 5. This is actually where estimate (1.5) enters.

2. The basic length scales.

We define the length scales we mentioned in the introduction as the solutions $r = r(x) > 0$ respectively $l = l(x) > 0$ of the equations

$$r \int \varphi\Big(\frac{y - x}{r}\Big)\, B(y)^2\, d^3y = 1\,, \tag{2.1}$$

$$l^3 \int \varphi\Big(\frac{y - x}{l}\Big) (\nabla \otimes B(y))^2\, d^3y = 1\,. \tag{2.2}$$

The function $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $\varphi(z) = (1 + z^2/2)^{-2}$ is the same as in I and satisfies

$$z \cdot \nabla \varphi(z) \leq 0\,, \tag{2.3}$$

$$|D_1 \cdots D_n \varphi| \lesssim \varphi\,, \qquad n \in \mathbb{N}\,, \tag{2.4}$$

where $D_j = \partial_i$, $(i = 1, 2, 3)$ or $D_j = z \cdot \nabla$. Here and in the following $X \lesssim Y$ means $X \leq C\,Y$ for some constant C independent of the data, *i.e.*, of A, V.

The solutions of (2.1) and (2.2) exist and are unique, except for the case $B \equiv 0$ (almost everywhere), respectively $\nabla \otimes B \equiv 0$ (almost everywhere), where we set $r \equiv \infty$, respectively $l \equiv \infty$. They are smooth as a function of $x \in \mathbb{R}^3$ (see Section I.2).

We first discuss how these length scales are semi-locally controlled by the original quantities B and $\nabla \otimes B$. To this end let $\Omega_R = \{x : \operatorname{dist}(x, \Omega) < R\}$ for $R > 0$ and $\Omega \subset \mathbb{R}^3$.

Lemma 3. *The length scales $r(x)$ and $l(x)$ satisfy* (1.3), (1.4). *Moreover, for any $R > 0$ and $\Omega \subset \mathbb{R}^3$ there is a function $\Phi_{\Omega,R}(x) \geq 0$ satisfying $\|\Phi_{\Omega,R}\|_\infty \lesssim 1$ and $\|\Phi_{\Omega,R}\|_1 \lesssim |\Omega_R|$, uniformly in Ω, R, such that*

$$\int_{\Omega_R} r(x)^{-4}\, d^3x \lesssim \int \Phi_{\Omega,R}(x)\, B(x)^2\, d^3x + |\Omega_R|\, R^{-4}\,, \tag{2.5}$$

$$\int_{\Omega_R} l(x)^{-6} d^3x \lesssim \int \Phi_{\Omega,R}(x)\, (\nabla \otimes B(x))^2\, d^3x + |\Omega_R|\, R^{-6}\,. \tag{2.6}$$

PROOF. Estimates (1.3) and (2.5) were proven in Lemmas I.2 and I.12. The same proofs are valid for the remaining two estimates once the following remark about the proof of Lemma I.2 has been made: We replace there $r(x)$ by $l(x)$. Because of $g_+(|x|) \geq 1$, (I.2.6) implies

$$g_+(|x|)^3\, \varphi\Big(\frac{z - x}{g_+(|x|)}\Big) \geq \varphi(z)\,,$$

which after integration against $(\nabla \otimes B(z))^2\, d^3z$ implies $l(x) \leq g_+(|x|)$. Then the proof continues as before.

The length scales $r(x)$ and $l(x)$ are tempered in the following sense:

Lemma 4.

$$|\partial^\alpha l(x)| \lesssim l(x)^{-(|\alpha|-1)}\,, \qquad |\alpha| \geq 0\,, \tag{2.7}$$

$$|\partial^\alpha r(x)| \lesssim r(x)^{-(|\alpha|-1)} \min\Big\{1, \Big(\frac{r(x)}{l(x)}\Big)^{3/2}\Big\}\,, \qquad |\alpha| \geq 1\,, \tag{2.8}$$

where $\alpha \in \mathbb{N}^3$ is a multiindex.

Proof. We omit the proof of (2.7) since it consists of a minor adaptation of that of (I.2.9). For $r(x) > l(x)$ (2.8) reduces to (I.2.9), so that we may assume $r(x) < l(x)$. We discuss this case using a variant of the argument given in I. We recall that it was based on the equation

$$(1 - m(x))\, \partial_i r(x) = m_i(x)\,, \tag{2.9}$$

where

$$m(x) = r(x) \int z \cdot \nabla\varphi(z)\, U(y)\, d^3y\,, \quad m_i(x) = r(x) \int (\partial_i \varphi)(z)\, U(y)\, d^3y\,,$$

with $z = (y - x)/r(x)$. Moreover, we denoted by V_n, $n \in \mathbb{N}$, the space of finite sums of functions of the form

$$f(x) = r(x)^{-(n-1)}\, P(\{\partial^\alpha r\}) \int \psi(z)\, B(y)^2\, d^3y\,,$$

where ψ is of the form $D_1 \cdots D_k \varphi$ and P is a monomial in the derivatives $\{\partial^\alpha r\}_{|\alpha| \le n}$ of order 0 in the sense that it contains as many powers of ∂ as of r. In addition we consider here the subspace $\widetilde{V}_n \subset V_n$ obtained by restricting f to satisfy: i) some $\partial^\alpha r$ with $1 \le |\alpha| \le n$ occurs among the factors of P; or else ii) $D_1 = \partial_i$, *i.e.*, $\psi = \partial_i \widetilde{\psi}$ with $\widetilde{\psi}$ of the form previously stated for ψ. One verifies that $\partial_i V_n \subset \widetilde{V}_{n+1}$ and $r^{-1}\widetilde{V}_n \subset \widetilde{V}_{n+1}$.

The induction assumption states that (2.8) holds for $1 \le |\alpha| \le n$. (It is empty for $n = 0$). We now prove it for $n + 1$ instead of n. First, we claim that $f \in \widetilde{V}_n$ satisfies

$$|f(x)| \lesssim r(x)^{-n} \Big(\frac{r(x)}{l(x)}\Big)^{3/2}.$$

In case i) this follows directly from the induction assumption; in case ii) by integration by parts

$$\int \partial_i \widetilde{\psi}(z)\, B(y)^2\, d^3y = 2\, r(x) \int \widetilde{\psi}(z)\, B(y) \cdot \partial_i B(y)\, d^3y\,,$$

which by (2.4) and the Cauchy-Schwarz inequality is bounded in absolute value by

$$2\, r(x) \Big(\int \varphi(z) B(y)^2\, d^3y\Big)^{1/2} \Big(\int \varphi(z)\, (\nabla \otimes B(y))^2\, d^3y\Big)^{1/2} \lesssim r(x)^{-1} \Big(\frac{r(x)}{l(x)}\Big)^{3/2}.$$

In the last estimate we used that the first integral equals $r(x)^{-1}$, whereas the second may be estimated by replacing z by $(y-x)/l(x)$, since $r(x)^{-1} > l(x)^{-1}$ and $\varphi(z)$ is radially decreasing. Hence that integral is bounded by $l(x)^{-3}$. We can turn to (2.8): Applying ∂^α, ($|\alpha| = n$) to (2.9) and using $m \in V_0$ we obtain $(1 - m(x))\, \partial^\alpha \partial_i r(x) \in \partial^\alpha m_i + \widetilde{V}_n$. The last set is $\widetilde{V}_n$ (even for $|\alpha| = n = 0$), since $m_i \in \widetilde{V}_0$. The result follows with $m \le 0$.

We remark that (2.7) implies (see (I.2.13))

$$|x - y| \le \varepsilon\, l(x) \qquad \text{implies} \qquad \frac{1}{2} \le \frac{l(y)}{l(x)} \le 2 \tag{2.10}$$

for $\varepsilon > 0$ small enough. A partition of unity based on the length scale $l(x)$ is

$$j_y(x) = (\varepsilon\, l(x))^{-3/2}\, \chi\left(\frac{x-y}{\varepsilon\, l(x)}\right), \qquad y \in \mathbb{R}^3\,,$$

where $0 < \varepsilon \le 1$ and $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp} \chi \subset \{z : |z| \le 1\}$ and $\int \chi(z)^2\, d^3 z = 1$. Analogously to Lemma I.4 we have

Lemma 5.

$$\int j_y(x)^2\, d^3 y = 1\,, \tag{2.11}$$

$$\int |\partial^\alpha j_y(x)\, \partial^\beta j_y(x)|\, d^3 y \lesssim (\varepsilon\, l(x))^{-(|\alpha| + |\beta|)}\,, \tag{2.12}$$

for any α, $\beta \in \mathbb{N}^3$, where $\partial = \partial / \partial x$.

The length scale $l(x)$ will be the one most frequently used in the following sections. At one point however (in the proof of Lemma 8), we will use the length scale $\lambda(x)$ defined by $\lambda(x)^{-1} = r(x)^{-1} + l(x)^{-1}$. It also satisfies (2.7) and (2.10) (with l replaced by λ), and Lemma 5 applies accordingly to the partition based on $\lambda(x)$.

Finally we point out that Lemma 4 (in particular, the improvement of (2.8) over (I.2.9)) implies

$$|\nabla P(x)| \lesssim P(x)\, l(x)^{-1}\,, \qquad |\Delta P(x)| \lesssim P(x)^2\,. \tag{2.13}$$

Combining (2.13) with (2.10) we also find that for $|x-y| \le \varepsilon\, l(x)$ we have $|\log P(y) - \log P(x)| \lesssim \varepsilon$, and hence

$$\frac{1}{2} \le \frac{P(y)}{P(x)} \le 2\,, \tag{2.14}$$

for $\varepsilon > 0$ small enough.

3. The eigenvalue sum.

In this section we present the framework of the proof of (1.5), with large parts of it deferred to the next section. We begin by applying, as in I, the Birman-Schwinger principle [14]

$$\sum e_i \le 2\int_0^\infty n((\not{p}^2 + E)^{-1/2}\,(V-E)_+^{1/2}, 1)\, dE\,, \tag{3.1}$$

where $n(X,\mu)$ is the number of singular values $\lambda \ge \mu > 0$ of a compact operator X, *i.e.*, the number of eigenvalues $\lambda^2 \ge \mu^2$ of X^*X. We then decompose the operator in (3.1) as $K_>(E) + K_<(E)$ with

$$\begin{aligned} K_>(E) &= (\not{p}^2 + \varepsilon^{-3}P + E)^{-1/2}\,(V-E)_+^{1/2}\,,\\ K_<(E) &= ((\not{p}^2 + E)^{-1/2} - (\not{p}^2 + \varepsilon^{-3}P + E)^{-1/2})\,(V-E)_+^{1/2}\,, \end{aligned}$$

for some sufficiently small $\varepsilon > 0$, and note that (see *e.g.* [3], [19])

$$n(K_> + K_<, s_1 + s_2) \le n(K_>, s_1) + n(K_<, s_2)\,, \tag{3.2}$$

(we take $s_1 = s_2 = 1/2$). For the last term we shall prove the bound

$$n\Big(K_<(E), \frac{1}{2}\Big) \lesssim n((\not{p}^2 + \varepsilon^{-3}P)^{-1}\,\varepsilon^{-3}PV^{1/2}, \mathrm{const}\, E^{1/2})\,. \tag{3.3}$$

For the purpose of estimating $n(K_<, 1/2)$ and $n(K_>, 1/2)$ we introduce some auxiliary objects, starting with the Hilbert space $\widehat{\mathcal{H}} = \int_{\mathbb{R}^3}^\oplus \mathcal{H}\, d^3y$ and the linear map

$$J : \mathcal{H} \longrightarrow \widehat{\mathcal{H}}\,, \qquad J = \int_{\mathbb{R}^3}^\oplus j_y\, d^3y\,,$$

(see also Section I.3). Next we define

$$\widehat{H} : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}\,, \qquad \widehat{H} = \int_{\mathbb{R}^3}^{\oplus} e^{if_y}\, H_y\, e^{-if_y}\, d^3y\,,$$

where $H_y = H(B_y) + \varepsilon^{-3}P(y)$, $H(B) = ((p - (1/2)\, B \wedge x) \cdot \sigma)^2$, $f_y(x)$ is a function to be specified later and $B_y = |K_y|^{-1} \int_{K_y} B(x)\, d^3x$ is the average magnetic field in the ball $K_y = \{x : \ |x - y| < 2\,\varepsilon\, l(y)\}$. In summary, $\widehat{H}$ acts on fibers of $\widehat{\mathcal{H}}$ as a Pauli Hamiltonian with constant magnetic field. The Pauli operator $\not{p}^2$ compares to the above construction as

$$(\not{p}^2 + \varepsilon^{-3}P)^2 \gtrsim J^* \widehat{H}^2 J\,. \tag{3.4}$$

This inequality, which is at the center of our analysis, is obtained by first localizing $(\not{p}^2 + \varepsilon^{-3}P)^2$ and then by locally replacing the fields $B = \nabla \wedge A$ by a constant magnetic field and P by a constant. Indeed, (3.4) results from the combination of the following two inequalities.

Lemma 6.

$$(\not{p}^2 + \varepsilon^{-3}P)^2 \geq \int j_y \Big(\not{p}^4 + \frac{1}{2}\,\varepsilon^{-6}P^2\Big) j_y\, d^3y\,, \tag{3.5}$$

$$j_y \Big(\not{p}^4 + \frac{1}{2}\,\varepsilon^{-6}P^2\Big) j_y \gtrsim j_y\, H_y^2\, j_y\,. \tag{3.6}$$

Let us point out that (3.4) implies the weaker inequality (see (I.3.4))

$$\not{p}^2 + \varepsilon^{-3}P \gtrsim J^* \widehat{H} J\,. \tag{3.7}$$

PROOF OF (1.5). Let

$$\widehat{H}^0 : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}\,, \qquad \widehat{H}^0 = \int_{\mathbb{R}^3}^{\oplus} e^{if_y}\, H(B_y)\, e^{-if_y}\, d^3y\,.$$

Then $\widehat{H} \geq \widehat{H}^0$ and, as in I, we obtain from (3.7)

$$n\Big(K_>(E), \frac{1}{2}\Big) \leq n((\widehat{H}^0 + E)^{-1/2} J (V - E)_+^{1/2}, \mathrm{const}) \tag{3.8}$$

by means of (3.7). From now on the computation closely follows the line given in [10], where the contribution of the lowest Landau band is split from that of the higher bands. We set

$$\widehat{\Pi} : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}\,, \qquad \widehat{\Pi} = \int_{\mathbb{R}^3}^{\oplus} e^{if_y}\, \Pi(B_y)\, e^{-if_y}\, d^3y\,,$$

where $\Pi(B)$ is the projection in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ onto the lowest band of $H(B)$. Its integral kernel is

$$\Pi(B)(x,x') = \frac{|B|}{2\pi} \exp\Big(i\,(x_\perp \wedge x'_\perp)\,\frac{B}{2} - (x_\perp - x'_\perp)^2\,\frac{|B|}{4}\Big)\delta(x_3 - x'_3)\,\mathcal{P}^{\downarrow}\,, \tag{3.9}$$

in coordinates $x = (x_\perp, x_3)$ where $B = (0, |B|)$, and $\mathcal{P}^{\downarrow} = (1 + \sigma_3)/2$ is the projection in $\mathbb{C}^2$ onto the subspace where $B \cdot \sigma = |B|$. We remark that $\widehat{\Pi}$ commutes with $\widehat{H}^0$. The operator appearing on the right hand side of (3.8) is then split as $(\widehat{H}^0 + E)^{-1/2} J\,(V - E)_+^{1/2} = K_0(E) + K_1(E)$, with

$$\begin{aligned} K_0(E) &= (\widehat{H}^0 + E)^{-1/2}\, \widehat{\Pi} J (V - E)_+^{1/2}\,, \\ K_1(E) &= (\widehat{H}^0 + E)^{-1/2}\, (1 - \widehat{\Pi}) J (V - E)_+^{1/2}\,, \end{aligned}$$

so that by (3.2) it suffices to estimate $n(K_i(E), \mathrm{const})$, $i = 0, 1$, separately. The first term is bounded by

$$\begin{aligned} n(K_0(E), \mathrm{const}) &\lesssim \mathrm{tr}\, K_0(E)^* K_0(E) \\ &= \int d^3y\, \mathrm{tr}\,\big(j_y\, (V - E)_+^{1/2}\, \Pi(B_y)\, (H(B_y) + E)^{-1} \\ &\qquad\qquad \cdot \Pi(B_y)\, (V - E)_+^{1/2}\, j_y\big) \\ &= (4\pi E^{1/2})^{-1} \int d^3y\, d^3x\, (V(x) - E)_+\, j_y(x)^2\, |B_y| \end{aligned} \tag{3.10}$$

where the last estimate is [10, (2.15)]. Note that the gauge transformation e^{if_y} disappeared from the trace by cyclicity. For the second term we use the inequality before [10, (2.18)], which states that

$3\,H(B_y)/2 \geq D_y^2 \equiv (p - (1/2)\,B_y \wedge x)^2$ on the orthogonal complement $\mathrm{Ran}\,(1 - \Pi(B_y))$ of the lowest Landau band. We hence get

$$\widehat{H}^0 \geq \frac{2}{3} \int_{\mathbb{R}^3}^{\oplus} e^{if_y}\, D_y^2\, e^{-if_y}\, d^3y \equiv \widehat{H}_S \tag{3.11}$$

on $\mathrm{Ran}\,(1 - \widehat{\Pi})$, as well as $(1 - \widehat{\Pi})\,(\widehat{H}^0 + E)^{-1}\,(1 - \widehat{\Pi}) \leq (\widehat{H}_S + E)^{-1}$, because $\widehat{\Pi}$ and $\widehat{H}_S$ commute. Together with $n(X, 1) \leq \mathrm{tr}\,((X^*X)^2)$ this yields

$$\begin{aligned}
&n(K_1(E), \mathrm{const})\\
&\quad\lesssim \mathrm{tr}\,((V - E)_+^{1/2} J^*(\widehat{H}_S + E)^{-1} J(V - E)_+\\
&\qquad\quad \cdot J^*(\widehat{H}_S + E)^{-1} J(V - E)_+^{1/2})\\
&\quad= \int \mathrm{tr}\,\Big(j_y\, j_{y'}\, e^{i(f_y - f_{y'})}\,(V - E)_+\Big(\frac{2}{3}\,D_y^2 + E\Big)^{-1}\\
&\qquad\qquad \cdot j_y\, j_{y'}\, e^{-i(f_y - f_{y'})}\,(V - E)_+\Big(\frac{2}{3}\,D_{y'}^2 + E\Big)^{-1}\Big)\, d^3y\, d^3y'\,.
\end{aligned}$$

Using the pointwise diamagnetic inequality [16] for the resolvent kernel

$$\Big|\Big(\frac{2}{3}\,D_y^2 + E\Big)^{-1}(x, x')\Big| \leq \Big(\frac{2}{3}\,p^2 + E\Big)^{-1}(x - x')\,, \tag{3.12}$$

the trace under the integral is bounded as in (I.3.9) by

$$\frac{3}{8\pi}\Big(\frac{3}{2E}\Big)^{1/2} \int (V(x) - E)_+^2\, j_y(x)^2\, j_{y'}(x)^2\, d^3x\,.$$

This leads to $n(K_1(E), \mathrm{const}) \lesssim E^{-1/2} \int (V(x) - E)_+^2\, d^3x$ by (2.11) and, together with (3.10), to

$$\begin{aligned}
&\int_0^\infty n\Big(K_>(E), \frac{1}{2}\Big)\, dE\\
&\qquad\qquad \lesssim \int d^3x\, V(x)^{3/2}\Big(V(x) + \int d^3y\, |B_y|\, j_y(x)^2\Big)\,.
\end{aligned} \tag{3.13}$$

We now turn to $K_<$. The inequality

$$\int_0^\infty n\Big(K_<(E), \frac{1}{2}\Big)\, dE \lesssim \varepsilon^{-6}\, \mathrm{tr}\,(V^{1/2} P J^* \widehat{H}^{-2} J P V^{1/2})$$

follows from (3.3), from $\int_0^\infty n(X,\mu^{1/2})\,d\mu = \mathrm{tr}\; X^*X$, and from (3.4). We then split $\widehat{H}^{-2} = \widehat{\Pi}\widehat{H}^{-2}\widehat{\Pi} + (1-\widehat{\Pi})\widehat{H}^{-2}(1-\widehat{\Pi})$. The contribution of the first term is

$$\begin{aligned}\int d^3y\ \mathrm{tr}\,(j_y\,V^{1/2}P\,\Pi(B_y)\,(H(B_y)+\varepsilon^{-3}P(y))^{-2}\,\Pi(B_y)\,P\,V^{1/2}\,j_y)&\\ = \frac{1}{8\pi}\int (\varepsilon^{-3}P(y))^{-3/2}\,|B_y|\,P(x)^2\,V(x)\,j_y(x)^2\,d^3y\,d^3x\,,&\end{aligned}$$

because of (3.9) and of $\Pi(B)\,(H(B)+E)^{-2} = \Pi(B)\,(p_3^2+E)^{-2}$ in the coordinates used there. For the second term we use (see (3.11)) $\widehat{H}^2 \geq (\widehat{H}_S + \widehat{P})^2$ on $\mathrm{Ran}\,(1-\widehat{\Pi})$, since $\widehat{H}$ and $\widehat{H}_S+\widehat{P}$ commute, where $\widehat{P} = \varepsilon^{-3}\int_{\mathbb{R}^3}^{\oplus} P(y)\,d^3y$. This yields a contribution bounded by

$$\begin{aligned}\int \mathrm{tr}\,\Big(j_y\,V^{1/2}\,P\Big(\frac{2}{3}\,D_y^2+\varepsilon^{-3}P(y)\Big)^{-2}P\,V^{1/2}\,j_y\Big)\,d^3y&\\ \leq \frac{3}{8\pi}\int\Big(\frac{3}{2\,\varepsilon^{-3}P(y)}\Big)^{1/2}P(x)^2\,V(x)\,j_y(x)^2\,d^3y\,d^3x\,,&\end{aligned}$$

where we used again (3.12). Taking into account (2.14) and (2.11) we thus obtain

$$\begin{aligned}&\int_0^\infty n\Big(K_<(E),\frac{1}{2}\Big)\,dE\\ &\quad\lesssim \int d^3x\,V(x)\Big(\varepsilon^{-9/2}P(x)^{3/2}+\varepsilon^{-3/2}P(x)^{1/2}\int d^3y\,|B_y|\,j_y(x)^2\Big)\,.\end{aligned}\tag{3.14}$$

In order to put the result, *i.e.*, the sum of (3.13) and (3.14), into the form given in Theorem 1 we estimate

$$|B_y| \leq |K_y|^{-1}\int_{K_y}|B(z)|\,d^3z = |K_y|^{-1}\int |B(z)|\,\theta(|z-y|<2\,\varepsilon\,l(y))\,d^3z\,,$$

where $\theta(A)$ is the characteristic function of the set A, so that

$$\begin{aligned}&\int d^3y\,|B_y|\,j_y(x)^2\\ &\quad\leq \int d^3z\,|B(z)|\int d^3y\,|K_y|^{-1}\,\theta(|z-y|<2\,\varepsilon\,l(y))\,j_y(x)^2\,.\end{aligned}\tag{3.15}$$

We recall that $\operatorname{supp} j_y \subset \{x : |x-y| \le \varepsilon\, l(x)\}$. Using again (2.10) and the triangle inequality $|x-z| \le |x-y| + |z-y|$ we bound (3.15) by a constant times

$$\begin{aligned} |K_x|^{-1} \int d^3z\, |B(z)|\, \theta(|x-z| < 5\,\varepsilon\, l(x)) \int d^3y\, j_y(x)^2 \\ = |K_x|^{-1} \int_{|x-z|<5\varepsilon l(x)} d^3z\, |B(z)|\,, \end{aligned}$$

i.e., by $\widehat{B}(x)$ after a redefinition of ε.

At this point Theorem 1 is proven, except for Lemma 6 and (3.3).

4. Proofs.

In this section we give all the proofs we omitted in the previous one in order to complete the derivation of (1.5).

Lemma 7. *Let $U \in L^{3/2}(\mathbb{R}^3)$. Then*

$$U \le \frac{1}{3}\Big(\frac{\pi}{2}\Big)^{-4/3} \|U\|_{3/2}\, D^2\,. \tag{4.1}$$

For a proof, see Lemma I.7 and subsequent remark.

Lemma 8.

$$D l^{-2} D \lesssim \not{p}^2 P + P \not{p}^2 + \varepsilon^{-2} P^2\,. \tag{4.2}$$

PROOF. The first step towards (4.2) consists in showing

$$D l^{-2} D \lesssim \not{p}^2 l^{-2} + l^{-2} \not{p}^2 + \varepsilon^{-2} P^2\,. \tag{4.3}$$

This statement is closely related to Lemma I.8 and, similarly, its proof reduces to that of

$$l^{-2}\, |B| \lesssim \varepsilon^{1/2}\, (D l^{-2} D + \varepsilon^{-2} P^2)\,. \tag{4.4}$$

This is again proven as in I, except for the fact that we use here (and only here) a partition of unity based on the length scale $\varepsilon\lambda(x)$ as discussed at the end of Section 2, with $\lambda(x)^{-1} = r(x)^{-1} + l(x)^{-1}$. In particular, we now set $\widetilde{K}_y = \{x : |x-y| < \varepsilon\,\lambda(x)\}$ with characteristic function $\widetilde{\chi}_y$. It then still holds that

$$\begin{aligned}\|l^{-2}\,|B|\,\widetilde{\chi}_y\|_{3/2} &\le \|l^{-2}\,\widetilde{\chi}_y\|_\infty\,\|B\widetilde{\chi}_y\|_2\,\|\widetilde{\chi}_y\|_6\\ &\lesssim l(y)^{-2}\,r(y)^{-1/2}\,(\varepsilon\,r(y))^{1/2}\\ &= \varepsilon^{1/2}\,l(y)^{-2}\,,\end{aligned}$$

where: we used $\lambda(x) \le l(x)$ in estimating the first factor; $\lambda(x) \le r(x)$ and (2.1) in the second; and again $\lambda(x) \le r(x)$ in the last one. We hence obtain, just as in I,

$$l^{-2}\,|B| \lesssim \varepsilon^{1/2}\Big(Dl^{-2}D + l^{-2}\int(\nabla j_y)^2\,d^3y\Big)$$

with the integral bounded by $(\varepsilon\,\lambda(x))^{-2}$ due to (2.12). The proof of (4.4), and hence of (4.3), is completed by noticing that $l^{-2}\lambda^{-2} = P^2$. We now come back to (4.2). We have

$$\pm(\not{D}^2 f + f\not{D}^2 - 2\,\not{D} f\,\not{D}) \lesssim \varepsilon\,\not{D}P\not{D} + \varepsilon^{-1}P^2\,,$$

for $f = l^{-2}$ or $f = P$. Indeed, the left hand side is

$$\pm[\not{D},[\not{D},f]] = \mp i\,[\not{D},\nabla f\cdot\sigma] = -X^*X + \varepsilon\,\not{D}P\not{D} + \varepsilon^{-1}P^{-1}(\nabla f)^2$$

with $X = (\varepsilon\,P)^{1/2}\not{D} \pm i\,(\varepsilon\,P)^{-1/2}\,\nabla f\cdot\sigma$ and $(\nabla f)^2 \lesssim P^3$ due to (2.7) respectively (2.13). Taking $f = l^{-2}$ we first obtain from (4.3)

$$Dl^{-2}D \lesssim \not{D}l^{-2}\not{D} + \varepsilon\,\not{D}P\not{D} + \varepsilon^{-1}P^2 + \varepsilon^{-2}P^2 \le 2\,(\not{D}P\not{D} + \varepsilon^{-2}P^2)\,,$$

and then, with $f = P$, we obtain (4.2).

Proof of (3.5). The localization argument begins as that given for (I.3.2), with b replaced by P, *i.e.*, we have

$$\not{D}^4 = \int\Big(j_y\,\not{D}^4 j_y + \frac{1}{2}\,([j_y,[j_y,\not{D}^2]],\not{D}^2) + [j_y,\not{D}^2]^2\Big)\,d^3y\,,$$

with the estimate

$$-\int\frac{1}{2}\,([j_y,[j_y,\not{D}^2]],\not{D}^2)\,d^3y \le \frac{1}{2}\,\varepsilon^{-3}\,(\not{D}^2P + P\not{D}^2) + \varepsilon^{-5}P^2$$

for the first localization error. The other one is estimated similarly

$$
\begin{aligned}
-\int [j_y, \not{D}^2]^2 \, d^3y &\le \text{const}\, (\varepsilon^{-2} D l^{-2} D + \varepsilon^{-4} l^{-4}) \\
&\le \frac{1}{2} \varepsilon^{-3} (\not{D}^2 P + P \not{D}^2) + \varepsilon^{-5} P^2 \,,
\end{aligned}
$$

by using (4.2). The conclusion then is as in I.

Lemma 9 ([7])**.** *Let $K = \{x : |x| < 1\}$ be the unit ball, and $K^* = 2\,K$. Let $B \in L^2(K^*, \mathbb{R}^3)$ be a vector field with $\nabla \cdot B = 0$ (as a distribution) and*

$$\int_K B(x)\, d^3x = 0\,. \tag{4.5}$$

Then there is a vector field A such that

$$\nabla \wedge A = B\,, \qquad \nabla \cdot A = 0\,, \tag{4.6}$$

and

$$\|A\|_{\infty,K} \lesssim \|\nabla \otimes B\|_{2,K^*} \,. \tag{4.7}$$

PROOF. A solution A to (4.6) is constructed as in I, *i.e.*, as $A = \nabla \wedge F$, where F is the solution of $-\Delta F = B$ with boundary conditions (I.4.11). By $\|F\|_{2,K^*} \lesssim \|B\|_{2,K^*}$ and the elliptic estimate

$$\|\nabla^{\otimes 3} F\|_{2,K} \lesssim \|F\|_{2,K^*} + \|\Delta F\|_{2,K^*} + \|\nabla \otimes \Delta F\|_{2,K^*}$$

we have

$$\|\nabla^{\otimes 2} A\|_{2,K} \lesssim \|B\|_{2,K^*} + \|\nabla \otimes B\|_{2,K^*} \lesssim \|\nabla \otimes B\|_{2,K^*} \,.$$

In establishing the last inequality we used that a Poincaré inequality (see *e.g.* [20, Theorem 4.4.2]) applies to $\|B\|_{2,K^*}$, due to (4.5). Another Poincaré type inequality ([20, Corollary 4.2.3]) yields

$$\|A - \alpha - \beta\, x\|_{\infty,K} \lesssim \|\nabla^{\otimes 2} A\|_{2,K} \,,$$

for $\alpha_i = |K|^{-1} \int_K A_i(x)\, d^3x$ and $\beta_{ij} = |K|^{-1} \int_K \partial_j A_i(x)\, d^3x$. This proves (4.7) for $A - \alpha - \beta\, x$ instead of A. Equation (4.6) is preserved under this replacement, since it implies $\beta_{ij} - \beta_{ji} = 0$ and $\operatorname{tr} \beta = 0$.

PROOF OF (3.6). Let $B_y = |K_y|^{-1} \int_{K_y} B(x)\, d^3x$ be the average magnetic field over $K_y = \{x : |x-y| < 2\varepsilon\, l(y)\}$. It is generated by the vector potential $A_y(x) = (1/2)\, B_y \wedge (x-y)$. On the other hand, let $\widetilde{A}_y(x)$ be the vector potential of $\widetilde{B}_y(x) = B(x) - B_y$, which by scaling corresponds to the one constructed in the previous lemma. It satisfies

$$|\widetilde{A}_y(x)| \lesssim \varepsilon^{1/2}\, l(y)^{-1}\,, \tag{4.8}$$

for $x \in K_y$ because of (2.2), (4.7). Since $B = \nabla \wedge (A_y + \widetilde{A}_y)$, we may assume, upon making a gauge transformation, $A = A_y + \widetilde{A}_y$. The Pauli operators corresponding to $\not{D}_y = (p - A_y)\cdot\sigma$ and $\not{D}$ are related as

$$\begin{aligned} \not{D}_y^2 = (\not{D} + \widetilde{A}_y \cdot \sigma)^2 &= \not{D}^2 + (\widetilde{A}_y)^2 + \{\widetilde{A}_y \cdot \sigma, \not{D}\} \\ &= \not{D}^2 + (\widetilde{A}_y)^2 + \{\widetilde{A}_y, D\} + \widetilde{B}_y \cdot \sigma\,. \end{aligned}$$

This and $\nabla \cdot \widetilde{A}_y = 0$ yield

$$\not{D}_y^4 \le 4\,(\not{D}^4 + (\widetilde{A}_y)^4 + 4\, D\, (\widetilde{A}_y)^2 D + (\widetilde{B}_y)^2)\,.$$

After multiplying from both sides with j_y we may replace $\widetilde{A}_y$ by $\chi_y \widetilde{A}_y$ and similarly for $\widetilde{B}_y$, where $\chi_y(x)$ is the characteristic function of K_y. Note that, besides of (4.8), we have by (2.2) and $\|\chi_y\|_3 \lesssim \varepsilon\, l(y)$

$$\|\widetilde{B}_y^2\, \chi_y\|_{3/2} \le \|\widetilde{B}_y^2\, \chi_y\|_3\, \|\chi_y\|_3 \lesssim \|(\nabla \otimes B)^2\, \chi_y\|_1\, \|\chi_y\|_3 \lesssim \varepsilon\, l(y)^{-2}\,.$$

We can thus estimate, using (4.1),

$$j_y\, \not{D}_y^4\, j_y \lesssim j_y\, (\not{D}^4 + \varepsilon^2\, l(y)^{-4} + \varepsilon\, D l(y)^{-2} D)\, j_y$$

and finally, using (2.10), (2.14), (4.2),

$$\begin{aligned} j_y\, (\not{D}_y^2 + \varepsilon^{-3} P(y))^2\, j_y &\le 2\, j_y\, (\not{D}_y^4 + \varepsilon^{-6} P(y)^2)\, j_y \\ &\lesssim j_y \Big(\not{D}^4 + \frac{1}{2}\, \varepsilon^{-6} P(x)^2 + \varepsilon\, D l(x)^{-2} D\Big) j_y \\ &\le j_y\, (\not{D}^2 + \varepsilon^{-3} P)^2\, j_y\,. \end{aligned}$$

PROOF OF (3.3). The proof can be taken over literally from that of (I.3.6), after replacing b by P. To be checked however is that $f = \log P$ satisfies $(\nabla f)^2 \lesssim l^{-2} \leq P$ and $|\Delta f| \lesssim P$, as well as $D\,(\nabla f)^2 D \lesssim \not{p}^2 P + P \not{p}^2 + \varepsilon^{-2} P^2$. This follows from (2.13), (4.2).

5. Stability of matter.

As an application of (1.5), we state and prove a stability estimate for matter coupled to a classical magnetic field. It is essentially identical to a result of [7], except for exhibiting a somewhat more explicit dependence of the stability bound on the parameters involved. The system we consider consists of N spin 1/2 electrons (with Hilbert space $\wedge^N \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$) interacting with K static nuclei, having positions R_k and charges $Z > 0$, and with a classical magnetic field B. The theorem then reads:

Theorem 10. *Let $\mathcal{R} = \{R_k\}_{k=1}^K$ and R, Z, Γ, $\gamma > 0$. There is $C(Z, \Gamma, \gamma)$ and a function $\Phi_{\mathcal{R}}(x) \geq 0$ with*

$$\|\Phi_{\mathcal{R}}\|_\infty \lesssim 1\,, \qquad \|\Phi_{\mathcal{R}}\|_1 \lesssim R^3 K\,, \tag{5.1}$$

uniformly in $\mathcal{R}$, Z, such that the N-body Hamiltonian

$$\begin{aligned} H_N = \sum_{i=1}^N \not{p}_i^2 + V_C \\ + \Gamma \int \Phi_{\mathcal{R}}(x)\,(B(x)^2 + \gamma\, R^2\,(\nabla \otimes B)(x)^2)\, d^3x\,, \end{aligned} \tag{5.2}$$

$$V_C = \sum_{\substack{i,j=1 \\ i<j}}^N \frac{1}{|x_i - x_j|} - \sum_{i,k=1}^{N,K} \frac{Z}{|x_i - R_k|} + \sum_{\substack{k,l=1 \\ k<l}}^K \frac{Z^2}{|R_k - R_l|}\,,$$

acting on $\wedge^N \mathcal{H}$, satisfies

$$H_N \geq -C(Z, \Gamma, \gamma)\,(Z+1)\,R^{-1}\,(N+K) \tag{5.3}$$

for arbitrary $R \leq (Z+1)^{-1}$. For $\Gamma \leq Z+1$ and $1 \leq \gamma \leq z^4$ one can take

$$C(Z, \Gamma, \gamma) = \text{const}\,(z^3 + z^5 \gamma^{-1/2} \log(z^5 \gamma^{-1/2})) \tag{5.4}$$

with $z = 1 + (Z+1)\,\Gamma^{-1}$.

Remark. One may modify the definition (2.2) of $l(x)$ by replacing $(\nabla \otimes B)^2$ by $(\nabla \otimes B)^2 + R^{-6}$ for some $R > 0$. Theorem 1 continues to hold. On the right hand side of (2.6) a term R^{-6} should also be added to $(\nabla \otimes B)^2$, but it can be absorbed into the last term. The purpose of this variant is to ensure

$$l(x) \lesssim R\,. \tag{5.5}$$

Proof. By monotonicity, it will be enough to prove the theorem for $Z \geq 1$, $\Gamma \leq Q$ and $\gamma \leq z^4$. We partition [9] $\mathbb{R}^3$ into Voronoi cells $\Gamma_j = \{x : |x - R_j| \leq |x - R_k| \text{ for } k = 1, \dots, K\}$, $j = 1, \dots, K$. Let $D_j = \min\{|R_j - R_k| : j \neq k\}/2$. For any $\nu > 0$ the reduction to a one-body problem reads [9], [12]

$$\begin{aligned} H_N \geq \sum_{i=1}^{N} h_i - \nu\,N + \frac{Z^2}{8}\sum_{j=1}^{K} D_j^{-1} \\ + \Gamma \int \Phi_{\mathcal{R}}(x)\,(B(x)^2 + \gamma\,R^2\,(\nabla \otimes B)(x)^2)\,d^3x\,, \end{aligned} \tag{5.6}$$

where $h = \not{p}^2 - (W - \nu)_+$ and W is a potential satisfying $W(x) \leq Q\,|x - R_j|^{-1}$ for $x \in \Gamma_j$, with $Q = Z + \sqrt{2\,Z} + 2.2$.

We choose $\nu = Q\,R^{-1}$ and apply Theorem 1 (in the variant discussed above) to obtain

$$\begin{aligned} \sum_{i=1}^{N} h_i \gtrsim -\int V^{5/2}\,d^3x - \int P^{3/2} V\,d^3x \\ - \int \widehat{B}\,V^{3/2}\,d^3x - \int \widehat{B}\,P^{1/2}\,V\,d^3x\,, \end{aligned} \tag{5.7}$$

where $V = (W - Q\,R^{-1})_+$. Comparing with (5.6) it appears to be enough to show that each of the integrals (5.7), which we shall denote by i)-iv) below, is bounded by the bound (5.3) or by a small (universal) constant times

$$\frac{Z^2}{8}\sum_{j=1}^{K} D_j^{-1} + \Gamma \int \Phi_{\mathcal{R}}(x)\,(B(x)^2 + \gamma\,R^2\,(\nabla \otimes B)(x)^2)\,d^3x\,. \tag{5.8}$$

i) Note that $\operatorname{supp} V \subset \Omega_R$ for $\Omega = \{R_j : j = 1, \ldots, K\}$. This integral is thus bounded by $\text{const}\, Q^{5/2} R^{1/2} K \lesssim Q\, R^{-1} K$.

ii) We note that for any $\beta_1 > 0$

$$P^{3/2} \le \sqrt{2}\, l^{-3/2}\,(r^{-3/2} + l^{-3/2}) \le \sqrt{2}\,\frac{\beta_1}{2}\, r^{-3} + \sqrt{2}\Big(1 + \frac{\beta_1^{-1}}{2}\Big) l^{-3} \tag{5.9}$$

and we estimate the contributions to ii) of the two terms separately. For the first one we use that

$$\int_{\Omega_R} r(x)^{-3}\, V(x)\, d^3x \lesssim Q \int \Phi_{\mathcal{R}}(x)\, B(x)^2\, d^3x + Q \sum_{j=1}^{K} D_j^{-1} + Q R^{-1} K\,,$$

as was shown in Section I.5. This is consistent with the bound (5.3) if $\beta_1 \ll \min\{Q^{-1}\,\Gamma, 1\}$. (By $a \ll b$ we mean $a = \text{const}\, b$ for some sufficiently small universal constant). For the last term in (5.9) we use instead

$$\begin{aligned}
&\int_{\Omega_R} l(x)^{-3}\, V(x)\, d^3x \\
&\qquad \le \frac{\beta_2}{2} \int_{\Omega_R} l(x)^{-6}\, d^3x + \frac{\beta_2^{-1}}{2} \int_{\Omega_R} V(x)^2\, d^3x \\
&\qquad \lesssim \beta_2 \int \Phi_{\mathcal{R}}(x) (\nabla \otimes B)(x)^2\, d^3x + (\beta_2\, R^{-3} + \beta_2^{-1}\, Q^2\, R)\, K\,,
\end{aligned}$$

due to (2.6). The desired bound holds provided we pick $z \cdot \beta_2 \ll \Gamma\, \gamma\, R^2$.

iii) We split the integral into K inner integrals over $U_j = \{x : |x - R_j| \le \widehat{D}_j\}$, $\widehat{D}_j = \min\{D_j, \varepsilon\, l(R_j), R\}$ for some small $\varepsilon > 0$; and one outer integral over $\mathbb{R}^3 \setminus \bigcup_{j=1}^{K} U_j$. The inner integrals can be estimated as

$$\begin{aligned}
\int_{U_j} \widehat{B}(x)\, V(x)^{3/2}\, d^3x &\lesssim (\sup_{x \in U_j} \widehat{B}(x))\, \widehat{D}_j^{3/2} Q^{3/2} \\
&\le \frac{\beta}{2}\, \widehat{D}_j^3\, (\sup_{x \in U_j} \widehat{B}(x)^2) + \frac{\beta^{-1}}{2}\, Q^3\,.
\end{aligned}$$

Because of (2.10) we have $l(R_j)/2 \le l(x) \le 2\,l(R_j)$ for $x \in U_j$ and thus

$$
\begin{aligned}
\widehat{B}(x)^2 &= |K_x|^{-2}\Big(\int_{K_x} |B(y)|\, d^3y\Big)^2 \\
&\le |K_x|^{-1} \int_{K_x} B(y)^2\, d^3y \qquad (5.10)\\
&\lesssim (\varepsilon\, l(R_j))^{-3} \int \theta(|y-R_j| \le 3\,\varepsilon\, l(R_j))\, B(y)^2\, d^3y\,.
\end{aligned}
$$

Altogether we find for any $\beta > 0$

$$
\int_{\cup_{j=1}^K U_j} \widehat{B}(x)\, V(x)^{3/2}\, d^3x \lesssim \beta \int \Phi(y)\, B(y)^2\, d^3y + \beta^{-1} Q^3 K\,,
$$

$$
\Phi(y) = \sum_{j=1}^K \widehat{D}_j^3 (\varepsilon\, l(R_j))^{-3}\, \theta(|y-R_j| \le 3\,\varepsilon\, l(R_j))\,.
$$

For $\beta \ll \Gamma$ this will be bounded as claimed once we show that

$$
\Phi \lesssim \theta_{\Omega_R}\ .
$$

First, $\operatorname{supp}\Phi \subset \Omega_R$ for small $\varepsilon > 0$ because of (5.5). It thus suffices to show $\|\Phi\|_\infty \lesssim 1$: from $\widehat{D}_j \le \varepsilon\, l(R_j)$, the triangle inequality and (2.10) we find

$$
\begin{aligned}
\|\Phi\|_\infty &\le \sup_y \sum_{j=1}^K (\varepsilon\, l(R_j))^{-3}\, \theta(|y-R_j| \le 3\,\varepsilon\, l(R_j)) \\
&\qquad\qquad \cdot \int_{U_j} \theta(|x-R_j| \le \varepsilon\, l(R_j))\, d^3x \\
&\lesssim \sup_y \sum_{j=1}^K (\varepsilon\, l(y))^{-3} \int_{U_j} \theta(|x-y| \le 8\,\varepsilon\, l(y))\, d^3x \\
&\lesssim 1\,,
\end{aligned}
$$

since the U_j are disjoint.

The outer integral can be written and estimated as

$$
\int_{\Omega_R \setminus (\cup_{j=1}^K U_j)} d^3x\, V(x)^{3/2}\, |K_x|^{-1} \int d^3y\, |B(y)|\, \theta(|x-y| < \varepsilon\, l(x))
$$

$$\begin{aligned}
&\leq \frac{\beta_1}{2} \int_{\Omega_R \times \mathbb{R}^3} d^3x\, d^3y\, |B(y)|^2\, |K_x|^{-1}\, \theta(|x-y| < \varepsilon\, l(x)) \\
&+ \frac{\beta_1^{-1}}{2} \int_{\Omega_R \setminus (\cup_{j=1}^K U_j) \times \mathbb{R}^3} d^3x\, d^3y\, V(x)^3\, |K_x|^{-1}\, \theta(|x-y| < \varepsilon\, l(x))\,.
\end{aligned} \tag{5.11}$$

By the usual argument (2.10), the first integral is bounded by a constant times $\int \Phi(y)\, |B(y)|^2\, d^3y$ for

$$\Phi(y) = |K_y|^{-1} \int_{\Omega_R} \theta(|x-y| < 2\,\varepsilon\, l(y))\, d^3x \lesssim 1\,.$$

Moreover, $\operatorname{supp} \Phi \subset \Omega_{2R}$ as before. It thus suffices to take $\beta_1 \ll \Gamma$. In the second term on the right hand side of (5.11) the integration over y is explicit, and the integral is

$$\begin{aligned}
\int_{\Omega_R \setminus (\cup_{j=1}^K U_j)} V(x)^3\, d^3x &\lesssim \sum_{j=1}^{K} Q^3 \log R\, \widehat{D}_j^{-1} \\
&\leq \beta_2\, Q^3 \sum_{j=1}^{K} R\, \widehat{D}_j^{-1} + (\log \beta_2^{-1})\, Q^3\, K\,,
\end{aligned} \tag{5.12}$$

where we used that $\log t \leq \beta_2\, t + \log \beta_2^{-1}$ for $t,\ \beta_2 > 0$. We shall take $\Gamma^{-1} \cdot \beta_2\, Q^2 R \ll 1$, so that the last term is of the desired form. The first one reduces to an arbitrarily small constant times $Q \sum_{j=1}^K \widehat{D}_j^{-1}$. Note that

$$\widehat{D}_j^{-1} \lesssim \varepsilon^{-2} \Big(\int_{U_j} l(x)^{-6}\, d^3x \Big)^{1/3} + D_j^{-1} + R^{-1}\,. \tag{5.13}$$

In fact, by (2.10), the integral is bounded below by a constant times $(\varepsilon\, l(R_j))^{-2}\, \widehat{D}_j$, and thus the whole right hand side by

$$\widehat{D}_j^{-1} \Big(\Big(\frac{\widehat{D}_j}{\varepsilon\, l(R_j)} \Big)^2 + \frac{\widehat{D}_j}{D_j} + \frac{\widehat{D}_j}{R} \Big) \geq \widehat{D}_j^{-1}\,,$$

by definition of $\widehat{D}_j$. The contribution of the last two terms of (5.13) are then controlled by the first term (5.8), respectively by (5.3). For the integral, I, we use $I^{1/3} \leq 2\, \beta_3^{-1/2}/3 + \beta_3\, I/3$ and choose $Q \cdot \beta_3\, \varepsilon^{-2} \ll$

$\Gamma\, z^{-4} R^2$. Note that the U_j are disjoint, allowing for the application of (2.6).

iv) Using

$$P^{1/2} \le l^{-1/2}\,(r^{-1/2} + l^{-1/2}) \le \frac{\beta_1}{2}\, r^{-1} + \Big(1 + \frac{\beta_1^{-1}}{2}\Big) l^{-1}\,, \tag{5.14}$$

we estimate the contributions to iv) of the two terms separately. The first integral is

$$\begin{aligned} &\int_{\Omega_R} d^3x\, r(x)^{-1}\, V(x)\, |K_x|^{-1} \int d^3y\, |B(y)|\, \theta(|x-y| < \varepsilon\, l(x)) \\ &\le \frac{Q}{2} \int_{\Omega_R \times \mathbb{R}^3} d^3x\, d^3y\, |B(y)|^2\, |K_x|^{-1}\, \theta(|x-y| < \varepsilon\, l(x)) \\ &\quad + \frac{Q^{-1}}{2} \int d^3x\, d^3y\, r(x)^{-2}\, V(x)^2\, |K_x|^{-1}\, \theta(|x-y| < \varepsilon\, l(x))\,. \end{aligned} \tag{5.15}$$

The first term on the right hand side is like the corresponding one in (5.11) and hence acceptable provided $\beta_1 \cdot Q \ll \Gamma$. The second integral, $Q^{-1} \int r(x)^{-2} V(x)^2 d^3x$, is dealt with by splitting it with respect to $\widetilde{U}_j = \{x :\, |x - R_j| < \widetilde{D}_j\}$, $\widetilde{D}_j = \min\{D_j, \varepsilon\, r(R_j), R\}$ (see Section I.5). Then

$$\int_{\widetilde{U}_j} r(x)^{-2}\, V(x)^2\, d^3x \lesssim r(R_j)^{-2} \int_{\widetilde{U}_j} V(x)^2\, d^3x \lesssim \varepsilon^2\, Q^2\, \widetilde{D}_j^{-1}\,,$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3 \setminus (\cup_{j=1}^K \widetilde{U}_j)} r(x)^{-2}\, V(x)^2\, d^3x \\ &\qquad \le \frac{\varepsilon^2\, Q^{-2}}{2} \int_{\mathbb{R}^3 \setminus (\cup_{j=1}^K \widetilde{U}_j)} V(x)^4\, d^3x + \frac{\varepsilon^{-2}\, Q^2}{2} \int_{\Omega_R} r(x)^{-4}\, d^3x\,. \end{aligned}$$

Since the first integral is bounded above by const $Q^4 \sum_{j=1}^K \widetilde{D}_j^{-1}$ we have that

$$\begin{aligned} &Q^{-1} \int r(x)^{-2}\, V(x)^2\, d^3x \\ &\qquad \lesssim Q \sum_{j=1}^K \widetilde{D}_j^{-1} + Q \int_{\Omega_R} r(x)^{-4}\, d^3x \\ &\qquad \lesssim Q \sum_{j=1}^K D_j^{-1} + Q \int \Phi_{\mathcal{R}}(x)\, B(x)^2\, d^3x + Q\, R^{-1}\, K \end{aligned}$$

due (I.5.4) (augmented by R^{-1}) and (2.5). These terms fit (5.3) for our choice of β_1.

The integral corresponding to the last term in (5.14) is estimated similarly to iii) and is split accordingly. The inner integrals can be estimated as

$$
\begin{aligned}
&\int_{U_j} \widehat{B}(x)\, l(x)^{-1}\, V(x)\, d^3x \\
&\qquad \lesssim \big(\sup_{x\in U_j} \widehat{B}(x)\, l(x)^{-1}\big)\, \widehat{D}_j^2 Q \\
&\qquad \le \frac{2\,\beta_2^{1/2}}{3}\, \widehat{D}_j^3 \big(\sup_{x\in U_j} \widehat{B}(x)\, l(x)^{-1}\big)^{3/2} + \frac{\beta_2^{-1}}{3}\, Q^3\,,
\end{aligned}
\tag{5.16}
$$

where

$$
(\widehat{B}\, l^{-1})^{3/2} \le \frac{1}{4}\, \gamma^{-1/4}\, R^{-1/2}\, (3\, \widehat{B}^2 + \gamma R^2\, l^{-6})\,. \tag{5.17}
$$

The term coming from $\widehat{B}^2$ will be dealt with by (5.10), the other one by using

$$
\widehat{D}_j^3 \sup_{x\in U_j} l(x)^{-6} \lesssim \int_{U_j} l(x)^{-6}\, d^3x\,.
$$

Choosing $z\cdot\beta_2^{1/2}\,\gamma^{-1/4}\,R^{-1/2} \ll \Gamma$ ensures that both terms (5.17) are controlled by (5.8) and (5.3). The contribution of the last term (5.16) is then of order $z\cdot\beta_2^{-1}\,Q^3\,K \lesssim z^5\,\gamma^{-1/2}\,Q\,R^{-1}K$. The estimate of the outer integral follows the line of (5.15)

$$
\begin{aligned}
&\int_{\Omega_R\setminus(\cup_{j=1}^K U_j)} d^3x\, l(x)^{-1}\, V(x)\, |K_x|^{-1} \int d^3y\, |B(y)|\, \theta(|x-y| < \varepsilon\, l(x)) \\
&\le \frac{\beta_3}{2} \int_{\Omega_R\times\mathbb{R}^3} d^3x\, d^3y\, |B(y)|^2\, |K_x|^{-1}\, \theta(|x-y| < \varepsilon\, l(x)) \\
&\quad + \frac{\beta_3^{-1}}{2} \int_{\Omega_R\setminus(\cup_{j=1}^K U_j)\times\mathbb{R}^3} d^3x\, d^3y\, l(x)^{-2}\, V(x)^2\, |K_x|^{-1} \\
&\qquad\qquad \cdot\, \theta(|x-y| < \varepsilon\, l(x))\,.
\end{aligned}
$$

The first term just requires $z\beta_3 \ll \Gamma$. The second one is

$$
\begin{aligned}
&\int_{\mathbb{R}^3\setminus(\cup_{j=1}^K U_j)} l(x)^{-2}\, V(x)^2\, d^3x \\
&\qquad \le \frac{2}{3}\, \beta_4^{-1/2} \int_{\mathbb{R}^3\setminus(\cup_{j=1}^K U_j)} V(x)^3\, d^3x + \frac{1}{3}\, \beta_4 \int_{\Omega_R} l(x)^{-6}\, d^3x\,.
\end{aligned}
$$

To accomodate the last term, after application of (2.6), we require $z^2\,\Gamma^{-1}\cdot\beta_4 \ll \Gamma\, z^{-4}\, R^2$. The first term is dealt as in (5.12), with $\beta_2 \ll z^{-7}$ there.

6. Proof of Theorem 2.

We split the total Hamiltonian into two parts [8], [2]

$$H = H_{\mathrm{I}} + H_{\mathrm{II}}\ ,$$

with

$$H_{\mathrm{I}} = \sum_{i=1}^{N} \not{p}_i^2 + V_C + \Gamma \int \Phi_{\mathcal{R}}(x)\,(B(x)^2 + \gamma\, R^2\,(\nabla\otimes B)(x)^2)\, d^3x\,,$$

$$H_{\mathrm{II}} = H_f - \Gamma \int \Phi_{\mathcal{R}}(x)\,(B(x)^2 + \gamma\, R^2\,(\nabla\otimes B)(x)^2)\, d^3x\,,$$

where $B = \nabla\wedge A$, and $\Phi_{\mathcal{R}}$ is the positive function appearing in Theorem 10. Γ and γ will be chosen later.

All the fields appearing in H_{I} are multiplication operators in the same Schrödinger representation of $\mathcal{F}$ [8]. Thus Theorem 10 applies and yields

$$H_{\mathrm{I}} \geq -C(Z,\Gamma,\gamma)\,(Z+1)\,R^{-1}\,(N+K)\,. \tag{6.1}$$

We now turn to H_{II}. Let $F(x)$ be either $B(x)$ or $\nabla\otimes B(x)$. As in (1.6), we may write $F(x) = F_-(x) + F_+(x)$ and obtain

$$\begin{aligned} F(x)^2 &\leq F(x)^2 + (F_-(x) - F_+(x))^*(F_-(x) - F_+(x)) \\ &\leq 2\,(2\,F_+(x)\,F_-(x) + [F_-(x), F_+(x)])\,, \end{aligned}$$

where the commutator is a multiple of the identity, independent of x. We then integrate against $f(x)\,d^3x$ with $f \geq 0$ and bound the first term using $f(x) \leq \|f\|_\infty$ and Parseval's identity. This yields

$$\begin{aligned} &\int f(x)\,B(x)^2\,d^3x \\ &\qquad \leq 8\pi\,\alpha\,\|f\|_\infty \int d^3k\,|k|\,|\kappa(k)|^2 \sum_{\lambda=\pm} a_\lambda(k)^*\,a_\lambda(k) + \frac{\alpha\,\Lambda^4}{\pi}\,\|f\|_1\ , \end{aligned}$$

respectively

$$\int f(x)\,(\nabla \otimes B)(x)^2\,d^3x$$
$$\leq 8\pi\,\alpha\,\|f\|_\infty \int d^3k\,|k|^3\,|\kappa(k)|^2 \sum_{\lambda=\pm} a_\lambda(k)^*\,a_\lambda(k) + \frac{2\,\alpha\Lambda^6}{3\pi}\,\|f\|_1\,.$$

Note that the integrals on the right hand side are bounded by αH_f and $\alpha\,\Lambda^2 H_f$, respectively. In particular, for $f = \Phi_{\mathcal{R}}$ we find

$$\Gamma \int \Phi_{\mathcal{R}}(x)\,(B(x)^2 + \gamma\,R^2\,(\nabla\otimes B)(x)^2)\,d^3x$$
$$\leq \mathrm{const}\,\Gamma\,\alpha^2\,(1+\gamma\,(\Lambda\,R)^2)\,(H_f + \alpha^{-1}\,\Lambda^4\,R^3\,K)\,.$$

We may now optimize over Γ, γ, R, within the ranges allowed by Theorem 10, in such a way that the factor in front of H_f is less than 1. The resulting choice is as follows: We pick $\Gamma \ll Z^*\,(1+Z^*\,\alpha^2)^{-1}$ and $R = \gamma^{-1/2}\,(\Lambda + Z^*\,(Z^*\,\alpha^2)^{-2})^{-1}$. As a result, the factor in front of H_{f} is indeed less than 1 and

$$H_{\mathrm{II}} \gtrsim -Z^*\,\alpha\,\gamma^{-3/2}\,\Lambda\,K\,. \tag{6.2}$$

We finally choose $\gamma = z^4$ with z as in Theorem 10. Since $z \approx 1 + Z^*\,\alpha^2$ we have $R \leq {Z^*}^{-1}$, so that (6.1) applies

$$\begin{aligned} H_{\mathrm{I}} &\gtrsim -z^3\,(1+\log z)\,Z^*\,R^{-1}\,(N+K)\\ &\gtrsim -z^5\,(1+\log z)\,Z^*\,(\Lambda + Z^*\,(Z^*\,\alpha^2)^{-2})\,(N+K)\,. \end{aligned}$$

This is also a lower bound to (6.2), because of $\alpha \leq 1 + Z^*\,\alpha^2$.

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References.

[1] Bugliaro, L., Fefferman, C., Fröhlich, J., Graf, G. M., Stubbe, J., A Lieb-Thirring Bound for a Magnetic Pauli Hamiltonian. *Comm. Math. Phys.* **187** (1997), 567-582.

[2] Bugliaro, L., Fröhlich, J., Graf, G. M., Stability of quantum electrodynamics with non-relativistic matter. *Phys. Rev. Lett.* **77** (1996), 3494-3497.

[3] Dunford, N., Schwartz, J. T., *Linear Operators. Part II. Spectral Theory.* Interscience, 1967.

[4] Erdös, L., Magnetic Lieb-Thirring inequalities. *Comm. Math. Phys.* **170** (1995), 629-668.

[5] Erdös, L., Solovej, J. P., Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields. I. Non-asymptotic Lieb-Thirring type estimates. To appear in *Duke Math. J.*

[6] Erdös, L., Solovej, J. P., Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields. II. Leading order asymptotic estimates. *Comm. Math. Phys.* **188** (1997), 599-656.

[7] Fefferman, C., On electrons and nuclei in a magnetic field. *Advances in Math.* **124** (1996), 100-153.

[8] Fefferman, C., Fröhlich, J., Graf, G. M., Stability of ultraviolet-cutoff quantum electrodynamics with non-relativistic matter. *Comm. Math. Phys.* **190** (1997), 309-330.

[9] Lieb, E. H., Loss, M., Solovej, J. P., Stability of matter in magnetic fields. *Phys. Rev. Lett.* **75** (1995), 985-989.

[10] Lieb, E. H., Solovej, J. P., Yngvason, J., Asymptotics of heavy atoms in high magnetic fields: II. Semiclassical regions. *Comm. Math. Phys.* **161** (1994), 77-124.

[11] Lieb, E. H., Thirring, W., Bound for the kinetic energy of fermions which proves the stability of matter. *Phys. Rev. Lett.* **35** (1975), 687-689.

[12] Lieb, E. H., Yau, H. T., The stability and instability of relativistic matter. *Comm. Math. Phys.* **118** (1998), 177-213.

[13] Loss, M., Yau, H. T., Stability of Coulomb systems with magnetic fields: III. Zero energy bound states of the Pauli operator. *Comm. Math. Phys.* **104** (1986), 283-290.

[14] Reed, M., Simon, B., *Methods of Modern Mathematical Physics IV. Analysis of Operators.* Academic Press, 1978.

[15] Shen, Z., On the moments of negative eigenvalues for the Pauli operator. To appear in *J. Diff. Equations.*

[16] Simon, B., *Functional integration and quantum physics.* Academic Press, 1979.

[17] Sobolev, A. V., On the Lieb-Thirring estimates for the Pauli operator. *Duke Math. J.* **82** (1996), 607-635.

[18] Sobolev, A. V., *Lieb-Thirring inequalities for the Pauli operator in three dimensions.* IMA Vol. Math. Appl. **95** Springer, 1997.

[19] Yafaev, D. R., *Mathematical scattering theory: general theory.* AMS, 1992.

[20] Ziemer, W. P., *Weakly differentiable functions.* Graduate Texts in Mathematics. **120** Springer, 1989.

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On proximity relations for valuations dominating a two-dimensional regular local ring

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Abstract. The purpose of this paper is to define a new numerical invariant of valuations centered in a regular two-dimensional regular local ring. For this, we define a sequence of non-negative rational numbers $\delta_\nu = \{\delta_\nu(j)\}_{j\geq 0}$ which is determined by the proximity relations of the successive quadratic transformations at the points determined by a valuation ν. This sequence is characterized by seven combinatorial properties, so that any sequence of non-negative rational numbers having the above properties is the sequence associated to a valuation.

0. Introduction.

Valuations centered in a two-dimensional regular local ring have been studied and classified by Zariski, Abhyankar and Lipman (see for example [1]). More recently, there has been a revival of interest in this subject (see [15], [13], [7], [9], $\dots$).

The main purpose of this paper is to define a new numerical invariant of valuations centered in a regular two-dimensional local ring. One advantage of our invariant over those of [15] is that it works for a general regular local ring of dimension two; in particular, we do not assume that the residue field is algebraically closed.

The idea of proximity to classify singularities of analytically irreducible plane curves was developed by Enriques (see [6]) and can be adapted to the situation above (see [7], [9], [13], ...).

Several invariants can be associated to proximity relations (the refined proximity matrix, the multiplicity sequence, the semigroup-length sequence, ..., see [13]). Here we will introduce a new one which is a sequence of non-negative rational numbers $\delta_\nu = \{\delta_\nu(j)\}_{j\geq 0}$ (later called proximity sequence), where the proximity relations are codified.

In what follows, all rings considered will be commutative and with a unit element. For a local ring R, we will denote by $M(R)$ its maximal ideal.

Throughout this paper, R will be a two-dimensional regular noetherian local ring and we will consider a fixed sequence

$$(*) \qquad R = R_0 \subset R_1 \subset \cdots \subset R_n \subset \cdots ,$$

where R_{i+1} is a quadratic transform of R_i (*i.e.* R_{i+1} is a localization at a maximal ideal of a ring $R[x^{-1}M(R_i)]$ with $x \in M(R_i)$ and $x \notin (M(R_i))^2$.

For $i > 0$, we will denote by

$$e_{i-1} = \Big[\frac{R_i}{M(R_i)} : \frac{R_{i-1}}{M(R_{i-1})}\Big] .$$

It should be remembered that $S = \cup_{i\geq 0} R_i$ is a valuation ring. (See [1]). If ν is the valuation of S then ν is the only valuation of the quotient field of R centered at the maximal ideal of R_i for all $i \geq 0$.

The main goal of the paper is the characterization of the properties of the proximity sequence in the following sense: the properties that a sequence of non-negative rational numbers $\{\delta(j)\}_{j\geq 0}$ must satisfy in order to be the sequence associated to a valuation ν (or equivalently to a sequence $(*)$). Therefore these properties characterize the class of all valuations with the same associated sequence δ_ν. This gives rise to a notion of equisingularity of valuations.

For this, we see that all such sequence can be realized taking $R = \mathbb{Q}(t_1, \ldots, t_n, \ldots)[[X, Y]]$, $\mathbb{Q}$ being the field of rational numbers. In general, this is not possible for any R. If, in addition, the sequence satisfies that $\delta(j)$ is an integer for all $j \geq 0$ (or equivalently all rings of $(*)$ have the same residue field) then there is a valuation ν such that its associated proximity sequence is the given one.

We are also interested in other properties of the proximity sequence. In particular, if R is a complete ring then there is a non-zero principal prime ideal J of R such that J "goes through" R_n for all $n \geq 0$ (*i.e.* $J_n \neq R_n$, where J_n is the strict quadratic transform of J in R_n) if and only if there is N_0, such that $\delta_\nu(n) = 0$ for all $n \geq N_0$. In this situation, δ_ν characterizes the equisingularity classes of analytically irreducible plane curves. So we also have an explicit description of the different equisingularity classes.

The paper is organized as follows:

In Section 1 we outline some definitions and properties of proximity relations.

Section 2 is devoted to an introduction of the invariant and to study its properties. In particular we see that it is equivalent to the refined proximity matrix.

In the last section we characterize δ_ν by its properties and when δ_ν is an invariant for the equisingularity of plane curves.

1. Preliminaries.

First we will outline some concepts about the proximity relations of $(*)$.

For $j > i$ we say that R_j is *proximate* to R_i if the valuation ring $V(R_i)$ of Ord_{R_i} contains R_j, where Ord_{R_i} is the usual valuation order of R_i (*i.e.* $\mathrm{Ord}_{R_i}(x)$ is the greatest non-negative integer d such that $x \in (M(R_i))^d$, x being a non-zero element of R_i). In this case, $V(R_i) = (R_j)_{\mathfrak{p}}$, where $\mathfrak{p}$ is a height one prime ideal of R_j containing $M(R_i)R_j$ and R_k is proximate to R_i for $i < k \leq j$.

Moreover, for $j > i$ it is easy to verify that $M(R_i)R_j = t_{ij}^{a_{ij}} u_{ij}^{b_{ij}}$, where $t_{ij}R_j = M(R_{j-1})R_j$, $(t_{ij}, u_{ij})R_j = M(R_j)$, $a_{ij} > 0$ and $b_{ij} \geq 0$. (a_{ij} and b_{ij} being integers). So R_j is proximate to R_{j-1} and at most to one other ring in $(*)$. In fact if $j > i+1$ and R_j is proximate to R_i we can write

$$R_k = \left(R_{k-1}\left[\frac{u_{i,k-1}}{t_{i,k-1}}\right]\right)_{(t_{i,k}, u_{i,k})},$$

with $t_{i,k} = t_{i,k-1}$ and $u_{i,k} = u_{i,k-1}/t_{i,k-1}$, $i+2 \geq k \geq j$. So $b_{i,j} = 1$ and this is also a sufficient condition for R_j to be proximate to R_i.

One also has $a_{i,j} = j + i - 1$ and $e_{k-1} = 1$, $i+2 \leq k \leq j$.

In general, for $j > i+1$ we say that R_j is a *satellite* of R_i if $b_{ij} \neq 0$, where $M(R_i)R_j = t_{ij}^{a_{ij}} u_{ij}^{b_{ij}}$ as above. If $b_{ij} = 0$ we say that R_j is *free* with respect to R_i.

This is simply Zariski's definition of satellite and free points. (See [17]). It should be noted that R_j is a satellite of R_i if and only if $\mathrm{Ord}_{R_j}\left(\sqrt{M(R_i)R_j}\,\right) = 2$. It is also easy to verify that R_j is a satellite of R_i if and only if there is a non-negative integer q with $j-1 > q \geq i$ such that R_j is proximate to R_q.

2. The invariant.

In this section we will use the above notations.

We define the function $\gamma : \mathbb{Z}_0 \longrightarrow \mathbb{Z}_0$ as follows: $\gamma(0) = 0$ and for $j \geq 1$, $\gamma(j) = 1 + \min\{k : R_j \text{ is proximate to } R_k\}$, where $\mathbb{Z}_0$ denotes the set of non-negative integers.

Thinking geometrically, this map computes the oldest exceptional divisor that "goes through" R_j.

On the other hand, note that $\gamma(j) < j$ if and only if R_j is a satellite of R_i for some $i < j-1$. So $\gamma(j) = j$ if and only if R_j is free with respect to R_i for all $i < j-1$.

The most interesting properties of γ are given in the following results.

Proposition 2.1. *We have the following statements*:

a) $\gamma(j) \leq j$.

b) *If* $\gamma(j) < i < j$ *then* $\gamma(i) = \gamma(j)$.

c) *For all* $j \geq 0$ *there is a non-negative integer* n *such that* $\gamma^n(j) = \gamma^{n+1}(j)$, *where* $\gamma^0 = \mathbf{1}_{\mathbb{Z}_0}$ *and* $\gamma^{k+1} = \gamma \circ \gamma^k$.

d) *If* $\gamma(j) < j$ *then* $\gamma(j) = j-1$ *or* $\gamma(j) = \gamma(j-1)$.

Proof. a) Follows from the definition of γ.

b) If $m+1 = \gamma(j) < j$ then R_j is proximate to R_m and also R_i is proximate to R_m for $m+1 \leq i \leq j$. So $\gamma(i) = m+1 = \gamma(j)$.

c) By a) we have $0 \leq \cdots \leq \gamma^k(j) \leq \cdots \leq \gamma(j) \leq j$. So there is an n such that $\gamma^n(j) = \gamma^{n+1}(j)$.

d) As $\gamma(j) < j$, if $\gamma(j) \neq j-1$ then $\gamma(j) < j-1 < j$ by a). And by b) $\gamma(j) = \gamma(j-1)$.

In what follows we will denote by

$$n(j) = \min\{n \in \mathbb{Z}_0 : \ \gamma^n(j) = \gamma^{n+1}(j)\}\,.$$

Proposition 2.2. *With the above notations, let us assume that $k < j$, then we have*:

1) *If $\gamma(j) = k$ then $n(j) = n(k) + 1$.*

2) *If $\gamma(j) = \gamma(k)$ then $n(j) = n(k)$.*

Proof. Note that $\gamma^{n(k)+1}(j) = \gamma^{n(k)}(k) = \gamma^{n(k)+1}(k) = \gamma^{n(k)+2}(j)$, so $n(j) \leq n(k) + 1$.

On the other hand, $\gamma^{n(j)-1}(k) = \gamma^{n(j)}(j) = \gamma^{n(j)+1}(j) = \gamma^{n(j)}(k)$, so $n(j) \geq n(k) + 1$ and we have 1).

The proof of 2) is similar.

Now we have the conditions to define the invariant, which we will call *proximity sequence.*

We define $\delta_\nu = \{\delta_\nu(j)\}_{j\geq 0}$ as follows: $\delta_\nu(0) = 0$ and for $j \geq 1$

$$\delta_\nu(j) = n(j) + 1 - \frac{1}{e_{j-1}}\,.$$

First of all, we will see that the sequence δ_ν characterizes the proximity relations of ν (or equivalently of $(*)$).

Proposition 2.3. *With the above notations, the following statements are equivalent*:

a) *R_j is free with respect to R_i for all $i < j - 1$.*

b) *$\delta_\nu(j) = 1 - (1/e_{j-1})$.*

c) *$\delta_\nu(j) < 1$.*

Proof. R_j is free with respect to R_i for all $i < j - 1$ if and only if $\gamma(j) = j$, so if and only if $\delta_\nu(j) = 1 - (1/e_{j-1})$ or equivalently $\delta_\nu(j) < 1$.

Proposition 2.4. *With the above notations, if $i < j - 1$ the following statements are equivalent*:

a) *R_j is proximate to R_i.*

b) $n(i+1)+1 = \delta_\nu(i+1) + 1/e_i = \delta_\nu(k) = \delta(j)$, $i+2 \leq k \leq j$.

Proof. In order to see that a) implies b), we note that R_k is proximate to R_i, $i+1 \leq k \leq j$. So $\gamma(k) = i+1$ for $i+2 \leq k \leq j$. Then, by definition of δ_ν we have $\delta_\nu(i+1) + 1/e_i = 1 + n(i+1) = n(k) = \delta_\nu(k)$, $i+2 \leq k \leq j$.

On the other hand, by Proposition 2.3 R_j is proximate to R_h, $h < j-1$.

If $h < i$ then by a) implies b) we have that $\delta_\nu(k) = \delta_\nu(j)$ for $h+2 \leq k \leq j$. In particular, $\delta_\nu(i+1) = \delta_\nu(j)$. Yet $\delta_\nu(i+1) < n(i+1)+1 = \delta_\nu(j)$, which is a contradiction.

If $i < h$ then also by a) implies b) $\delta_\nu(h+1) < \delta_\nu(j)$, which is also a contradiction.

So $h = i$ and we have that b) implies a).

Proposition 2.5. *With the above notations, the proximity sequence δ_ν has the following properties*:

1) $\delta_\nu(j) \geq 0$.

2) $\delta_\nu(0) = 0$.

3) $\delta_\nu(1) < 1$.

4) *If $\delta_\nu(j) \geq 1$ then $\delta_\nu(i)$ is an integer.*

5) *If $\delta_\nu(j) < 1$ then $1/(1-\delta_\nu(j))$ is an integer.*

6) *If $\delta_\nu(j+1) < \delta_\nu(j)$ then $\delta_\nu(j+1) < 1$.*

7) $\delta_\nu(j+1) \leq 1 + \delta_\nu(j)$.

Proof. 1) and 2) follow from the definition of δ_ν.

3) As $\gamma(1) = 1$ we have $n(1) = 0$ and $\delta_\nu(1) = 1 - 1/e_1 < 1$.

4) If $\delta_\nu(j) \geq 1$ then $\gamma(j) \neq j$, so R_j is proximate to R_q, with $q < j-1$. So $e_{j-1} = 1$ and $\delta_\nu(j) = n(j)$ is an integer.

5) If $\delta_\nu(j) < 1$ then $\gamma(j) = j$ and $\delta_\nu(j) = 1 - 1/e_{j-1}$, so $e_{j-1} = 1/(1-\delta_\nu(j))$ is an integer.

6) If $\delta_\nu(j+1) < \delta_\nu(j)$ and $\delta_\nu(j+1) \geq 1$, then $\delta_\nu(j) \geq 1$. So R_{j+1} is proximate to R_q, $q < j$ and R_j is proximate to R_h, $h < j-1$. Therefore $\gamma(j+1) = q+1$, $\gamma(j) = h+1$, $e_{j-1} = e_j = 1$, $\delta_\nu(j+1) = n(j+1) = n(q+1)+1$ and $\delta_\nu(j) = n(j) = n(h+1)+1$.

If $q < j-1$ then $q = h$ and $\delta_\nu(j+1) = \delta_\nu(j)$, which is a contradiction.

So $q = j-1$ and $\gamma(j+1) = j$. Then by 2.2 we have $n(j+1) = n(j)+1$ and

$$\delta_\nu(j+1) = n(j+1)+1-\frac{1}{e_j} = n(j)+1+1-\frac{1}{e_{j-1}} = \delta_\nu(j)+1\,,$$

which is also a contradiction. So $\delta_\nu(j+1) < 1$.

7) We have three possibilities:

- $\gamma(j+1) = j+1$, in this case $\delta_\nu(j+1) < 1$ and always $\delta_\nu(j+1) \leq \delta_\nu(j)+1$.
- $\gamma(j+1) = j$, in this case we have $n(j+1) = n(j)+1$, see 2.2. So

$$\delta_\nu(j+1) = n(j+1)+1-\frac{1}{e_j} \leq \delta_\nu(j)+1\,.$$

- $\gamma(j+1) = \gamma(j)$, in this case we have $n(j+1) = n(j)$, see 2.2. So

$$\delta_\nu(j+1) = n(j+1)+1-\frac{1}{e_j}$$

and

$$\delta_\nu(j) = n(j)+1-\frac{1}{e_{j-1}}\,,$$

then

$$\delta_\nu(j+1) = \delta_\nu(j)+\frac{1}{e_{j-1}}-\frac{1}{e_j} < \delta_\nu(j)+1\,.$$

To finish this section we will compare the proximity sequence with other invariants. Namely, we will see that it defines equivalent data to the refined proximity matrix.

It should be remembered (see [13]) that the *refined proximity matrix* $P_\nu = (p_{ij})_{i,j\geq 0}$ is given by $p_{ii} = 1$,

$$p_{ij} = -\Big[\frac{R_j}{M(R_j)} : \frac{R_i}{M(R_i)}\Big]$$

if R_j is proximate to R_i and $p_{ij} = 0$ for the rest. Note that P_ν is an upper triangular matrix.

Proposition 2.6. *The proximity sequence δ_ν determines the refined proximity matrix P_ν and vice-versa.*

Proof. First we note that $p_{00} = 1$, $p_{10} = 0$ and

$$p_{01} = -\Big[\frac{R_1}{M(R_1)} : \frac{R_0}{M(R_0)}\Big] = -e_0 = \frac{1}{\delta_\nu(1) - 1} \ .$$

So p_{01} and $\delta_\nu(1)$ are the same data.

Now let us assume that δ_ν determines p_{ij} for $0 \leq i,\, j \leq n$, $n \geq 1$. We have $p_{n+1,n+1} = 1$ and $p_{n+1,k} = 0$ for $0 \leq k \leq n$.

If R_{n+1} is free with respect to R_k for all $k < n$, then $p_{k,n+1} = 0$ for $k < n$ and

$$p_{n,n+1} = -\Big[\frac{R_{n+1}}{M(R_{n+1})} : \frac{R_n}{M(R_n)}\Big] = -e_n = \frac{1}{\delta_\nu(n+1) - 1} \ .$$

If R_{n+1} is proximate to R_k with $k < n$ then

$$n(k+1) + 1 = \delta_\nu(k+2) = \delta_\nu(k+3) = \cdots = \delta_\nu(n+1) = \delta_\nu(k) + \frac{1}{e_k} \ .$$

So

$$\frac{1}{\delta_\nu(n+1) - \delta_\nu(k+1)} = p_{k,n+1} \ .$$

Now

$$p_{n,n+1} = -\Big[\frac{R_{n+1}}{M(R_{n+1})} : \frac{R_n}{M(R_n)}\Big] = -e_n = -1$$

and $p_{j,n+1} = 0$ for $j < n$, and $j \neq k$.

So δ_ν determines P_ν.

Similar reasoning proves that P_ν determines δ_ν.

3. Valuations with a given δ.

Now we will prove the main result of this paper.

Theorem 3.1. *Let $\delta = \{\delta(j)\}_{j\geq 0}$ be a sequence of non-negative rational numbers having the seven properties of Proposition 2.5. Then there is a two dimensional regular noetherian local ring R and a valuation ν centered at $M(R)$ such that its proximity sequence is δ.*

Proof. We consider $R = \mathbb{Q}(t_1, \dots, t_n, \dots)\,[[X, Y]]$, where $\mathbb{Q}$ is the field of rational numbers, $\{t_1, \dots, t_n, \dots\}$ is a set of indeterminates over $\mathbb{Q}$ and X and Y are two indeterminates over $\mathbb{Q}(t_1, \dots, t_n, \dots)$.

We define $e_{j-1} = 1$ if $\delta(j) \geq 1$ and

$$e_{j-1} = \frac{1}{1 - \delta(j)}\,, \qquad \text{if } \delta(j) < 1\,.$$

We put $R = R_0$ and

$$R_1 = \Big(R\Big[\frac{Y}{X}\Big]\Big)_{(X,(Y/X)^{e_0} - t_1)}\,.$$

Now let us assume that for $n \geq 1$ we have $R = R_0 \subset R_1 \subset \cdots \subset R_n$ such that for any valuation ν' centered at $M(R_n)$ we have that $\delta_{\nu'}(j) = \delta(j)$, for each $0 \leq j \leq n$, and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}\,[t_j^{1/e_{j-1}}]\,, \qquad \text{if } e_{j-1} > 1$$

and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}\,, \qquad \text{if } e_{j-1} = 1,\ 1 \leq j \leq n\,.$$

We have two possibilities:

1) $\delta(n+1) < 1$ (*i.e.* R_{n+1} must be free with respect to R_i for all $i < n$). In this case, let (x_n, y_n) be a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$.

We define

$$R_{n+1} = \Big(R_n\Big[\frac{y_n}{x_n}\Big]\Big)_{(x_n,(y_n/x_n)^{e_n} - t_{n+1})}\,.$$

2) $\delta(n+1) \geq 1$ (*i.e.* R_{n+1} must be a satellite). In this case, we have $1 + \delta(n) \geq \delta(n+1) \geq \delta(n)$.

• If $\delta(n+1) > \delta(n)$, then R_{n+1} must be proximate to R_{n-1}. (See 2.4). Let (x_n, y_n) be a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$.

We define

$$R_{n+1} = \Big(R_n\Big[\frac{x_n}{y_n}\Big]\Big)_{(y_n, x_n/y_n)}\,.$$

• If $\delta(n+1) = \delta(n)$, then R_{n+1} must be proximate to R_k, with $k < n - 1$. (See 2.4). In this case, we can take (x_n, y_n) a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$ and $M(R_k)R_n = x_n^a\, y_n R_n$.

We define

$$R_{n+1} = \left(R_n\left[\frac{y_n}{x_n}\right]\right)_{(x_n, y_n/x_n)} .$$

Now it is easy to see that $R = R_0 \subset R_1 \subset \cdots \subset R_n \subset R_{n+1}$ proves that for any valuation ν' centered at $M(R_{n+1})$ we have $\delta_{\nu'}(j) = \delta(j)$, for each $0 \le j \le n+1$, and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}[t_j^{1/e_{j-1}}], \qquad \text{if } e_{j-1} > 1$$

and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}, \qquad \text{if } e_{j-1} = 1,\ 1 \le j \le n+1 .$$

Now we will study the case in which $\delta_\nu(j)$ is an integer for all $j \ge 0$.

Theorem 3.2. *Let $\delta = \{\delta(j)\}_{j\ge 0}$ be a sequence of non-negative integers having the seven properties of Proposition* 2.5. *Let R be any regular noetherian local ring of dimension two. Then there is a valuation ν centered at $M(R)$ such that its proximity sequence is δ.*

PROOF. First we put $e_{j-1} = 1$ for all $j \ge 0$, $R = R_0$ and

$$R_1 = \left(R\left[\frac{y}{x}\right]\right)_{(x, y/x)} ,$$

(x, y) being any basis of $M(R)$.

Now let us assume that we have $R = R_0 \subset R_1 \subset \cdots \subset R_n$ such that for any valuation ν' centered at $M(R_n)$ we have $\delta_{\nu'}(j) = \delta(j)$, for each $0 \le j \le n$,

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})}, \qquad 1 \le j \le n .$$

We have two possibilities:

1) $\delta(n+1) = 0$ (*i.e.* R_{n+1} must be free with respect to R_i, $i < n$). In this case, let (x_n, y_n) be a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$.

We define

$$R_{n+1} = \left(R_n\left[\frac{y_n}{x_n}\right]\right)_{(x_n, (y_n/x_n))} .$$

2) $\delta(n+1) \geq 1$ (*i.e.* R_{n+1} must be a satellite). In this case, we have $1+\delta(n) \geq \delta(n+1) \geq \delta(n)$.

• If $\delta(n+1) > \delta(n)$, then R_{n+1} must be proximate to R_{n-1}. (See 2.4). Let (x_n, y_n) be a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$.

We define
$$R_{n+1} = \left(R_n\left[\frac{x_n}{y_n}\right]\right)_{(y_n, x_n/y_n)} .$$

• If $\delta(n+1) = \delta(n)$, then R_{n+1} must be proximate to R_k, with $k < n-1$. (See 2.4). In this case, we can take (x_n, y_n) a basis of $M(R_n)$, such that $M(R_{n-1})R_n = x_n R_n$ and $M(R_k)R_n = x_n^a\, y_n R_n$.

We define
$$R_{n+1} = \left(R_n\left[\frac{y_n}{x_n}\right]\right)_{(x_n, y_n/x_n)} .$$

Now it is easy to see that $R = R_0 \subset R_1 \subset \cdots \subset R_n \subset R_{n+1}$ proves that for any valuation ν' centered at $M(R_{n+1})$ we have $\delta_{\nu'}(j) = \delta(j)$, for each $0 \leq j \leq n+1$, and

$$\frac{R_j}{M(R_j)} = \frac{R_{j-1}}{M(R_{j-1})} , \qquad 1 \leq j \leq n+1 .$$

It should be noted that the above theorem is not true if δ is not a sequence of non-negative integers.

For example, let us consider $R = \mathbb{R}[[X,Y]]$, where $\mathbb{R}$ is the field of real numbers. Let $\delta = \{\delta(j)\}_{j\geq 0}$ be the sequence given by $\delta(0) = 0$, $\delta(1) = 2/3$ and $\delta(k) = 0$ for $k \geq 2$. If there is a valuation ν (or equivalently a sequence $(*)$) with δ as the proximity sequence, then $R/M(R)$ is isomorphic to $\mathbb{R}$ and

$$e_0 = \left[\frac{R_1}{M(R_1)} : \frac{R}{M(R)}\right] = 3 ,$$

which is a contradiction.

To finish the paper, we will clarify the relation between the proximity sequence and the classification of plane curve singularities.

For this, we need to assume that R is a complete ring.

Proposition 3.3. *Let us assume that there is a non-zero principal prime ideal J of $R = R_0$ such that $J_n \neq R_n$, J_n being the strict quadratic transform of J in R_n, $n \geq 0$. Then there is a non-negative integer N_0 such that $\delta_\nu(n) = 0$ for $n \geq N_0$.*

Proof. By [2, Proposition 9.4 and Theorem 10.7], there is an N_0 such that JR_n has a normal crossing for $n \geq N_0$, that is $JR_n = x_n^{a_n} y_n^{b_n} R_n$, where (x_n, y_n) is a basis of R_n and a_n and b_n are non-negative integers.

On the other hand, by definition of strict quadratic transform of J we have

$$JR_n = \Big(\prod_{i=0}^{n-1} (M(R_i))^{d_i} \Big) J_n R_n \ ,$$

where $d_i = \mathrm{Ord}_{R_i}(J_i)$, $0 \leq i \leq n-1$.

We can thus assume that $J_n = y_n R_n$, with $b_n = 1$ and

$$\prod_{i=0}^{n-1} (M(R_i))^{d_i} R_n = x_n^{a_n} R_n \ .$$

Therefore R_n is free for $n \geq N_0$.

As $J_{n+1} \neq R_{n+1}$ we have

$$R_{n+1} = \Big(R_n \Big[\frac{y_n}{x_n} \Big] \Big)_{(x_n, y_n/x_n)} \ ,$$

so $e_n = 1$, for $n \geq N_0$.

Now, we have $\gamma(n) = n$ and $\delta(n) = 0$ for $n \geq N_0$.

Proposition 3.4. *With the above notations, let us assume that there is a non-negative integer N_0 such that $\delta_\nu(n) = 0$ for $n \geq N_0$. Then, there is a non-zero principal prime ideal J of $R = R_0$ such that $J_n \neq R_n$, J_n being the strict quadratic transform of J in R_n for all $n \geq 0$.*

Proof. As $\delta_\nu(n) = 0$ for $n \geq N_0$ we have that R_n is free and $e_n = 1$, for $n \geq N_0$.

So we can write

$$R_{n+1} = \Big(R_n \Big[\frac{y_n}{x_n} \Big] \Big)_{(x_n, (y_n/x_n)+a)} \ ,$$

where (x_n, y_n) is a basis of $M(R_n)$ and $a_n \in R_{N_0}$, $n \geq N_0$.

Let us consider the ideal

$$J_{N_0} = (y_{N_0} + a_{N_0} \, x_{N_0} + a_{N_0+1} \, x_{N_0}^2 + \cdots) \, (R_{N_0})^* \ ,$$

where $(R_{N_0})^*$ is the complection of R_{N_0}.

It is now easy to see that $J = J_{N_0} \cap R$ is the required non-zero principal prime ideal of R.

It should be noted that Propositions 3.3 and 3.4 characterize the proximity sequences such that there is an analytically irreducible plane curve that "goes through" all the rings of $(*)$.

In addition, it is easy to verify that δ_ν is an invariant of the equisingularity class of such a curve. For a more specific treatment of proximity relations and plane curve singularities refer to [12].

References.

[1] Abhyankar, S. S., On the valuations centered in a local domain. *Amer. J. Math.* **78** (1956), 321-348.

[2] Abhyankar, S. S., Desingularization of plane curves. In *Proceedings Symposia Pure Mathematics, Part* 1. **40** (1983), 1-45.

[3] Aparicio, J., Clasificación de Singularidades de Curvas Planas y Valoraciones Divisoriales. Ph. D. Thesis, University of Valladolid, 1992.

[4] Bourbaki, N., *Algebre Commutative.* Chapitres 8 et 9. Masson, 1983.

[5] Bourbaki, N., *Algebre Commutative.* Chapitres 5 a 7. Masson, 1985.

[6] Enriques, F., Chisini, O., *Lezioni sulla teoria geometrica delle equazioni e delle funcioni algebriche.* Vol. II, N. Zanichelli, 1918.

[7] Campillo, A., Gonzalez-Sprinberg, G., Lejeune-Jalabert, M., Amas, ideaux a support fini et chaines toriques. *C.R. Acad. Sci. Paris Serie I.* **315** (1992), 987-990.

[8] Campillo, A., Gonzalez-Sprinberg, G., Lejeune-Jalabert, M., Clusters of infinitelly near points. *Math. Ann.* **306** (1996), 169-194.

[9] Campillo, A., Reguera, A. J., Combinatorial aspects of sequences of point blowing ups. *Manuscripta Math.* **84** (1994), 29-46.

[10] Granja, A., Coefficient fields for plane curves and equisingularity. *Comm. Algebra* **18** (1990), 193-208.

[11] Granja, A., Equisingularity invariants of plane curves. In: *Algebraic Geometry and its Applications.* Springer-Verlag, 1994, 89-12.

[12] Granja, A., Sánchez-Giralda, T., Enriques graphs of plane curves. *Comm. Algebra* **20** (1992), 527-562.

[13] Lipman, J., Proximity inequalities for complete ideals in two-dimensional regular local rings. *Contemporary Math.* **159** (1994), 293-306.

[14] Northcott, D. G., Abstract dilatations and infinitely near points. *Proc. Cambridge Phil. Soc.* **52** (1956), 176-197.

[15] Spivakovsky, M., Valuations in function fields of surfaces. *Amer. J. Math.* **112** (1990), 107-156.

[16] Zariski, O., Studies in equisingularity I. *Amer. J. Math.* **87** (1965), 507-536.

[17] Zariski, O., General Theory of Saturation and of Saturated Local Rings II. *Amer. J. Math.* **93** (1971), 872-964.

[18] Zariski, O., Samuel, P., *Commutative Algebra.* Vol. I and II. Springer-Verlag, 1986.

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