# Square functions of Calderón type and applications 

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#### Abstract

We establish $L^{2}$ and $L^{p}$ bounds for a class of square functions which arises in the study of singular integrals and boundary value problems in non-smooth domains. As an application, we present a simplified treatment of a class of parabolic smoothing operators which includes the caloric single layer potential on the boundary of certain minimally smooth, non-cylindrical domains.


## 1. Introduction and notation.

In this note we prove certain square function estimates which are in the spirit of those considered by David, Journé, and Semmes [DJS, Section 11]. In particular, they (essentially) include square function estimates for solutions of the heat equation in time varying domains [HL, Theorem 3.1], but our treatment here is of a purely real variable and geometric nature, and does not depend on properties of solutions of a PDE. Our approach will be based on an idea of P. Jones [JnsP], who gave a proof of the deep result of Coifman, McIntosh, and Meyer [CMM] concerning the $L^{2}$ boundedness of the Cauchy integral operator along a Lipschitz curve, by viewing the Lipschitz curve as (locally) a pertubation of an approximating line, and then controlling the resulting error terms by a certain Carleson measure estimate. In this context see also the work of Fang [Fng], and the monograph of Christ [Ch]. We note that an important antecedent of Jones' ideas is contained in the work of Dorronsoro [Do]. We shall apply our square function estimates
to obtain an alternative proof of [ H 2 , Theorem 3], which is a regularity result for a class of parabolic smoothing operators which includes the caloric single layer on the boundary of certain non-smooth time-varying domains.

Our main application being parabolic, we shall state and prove a parabolic version of our square function estimates. The elliptic version is similar, but a bit simpler. Indeed, another application of our method has been given by D. Mitrea, M. Mitrea, and M. Taylor [MMT, Section 1], who follow our approach here to prove certain square function estimates that are useful in their work on elliptic boundary value problems in non-smooth Riemannian manifolds.

Let us now introduce some notation. Our operators are modeled on operators arising from the theory of layer potential on non-smooth, time-varying domains. The class of domains under consideration have boundaries given (at least locally) as graphs of functions $A(x, t), x \in$ $\mathbb{R}^{n-1}, t \in \mathbb{R}$, which are Lipschitz in space, uniformly in time, and which satisfy a certain half order smoothness condition in the time variable, which is related to the BMO Sobolev spaces of Strichartz [Stz]. To be more precise, we suppose that there exists a constant $\beta$ such that

$$
\begin{equation*}
|A(x, t)-A(y, t)| \leq \beta|x-y| \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\mathbb{D}_{n} A\right\|_{*} \leq \beta \tag{1.2}
\end{equation*}
$$

Here, $\|\cdot\|_{*}$ denotes the parabolic BMO norm (defined below), and, following Fabes and Riviere [FR], we have defined a half-order time derivative by

$$
\begin{equation*}
\mathbb{D}_{n} A(x, t)=\left(\frac{\tau}{\|(\xi, \tau)\|} \widehat{A}(\xi, \tau)\right)^{\vee}(x, t) \tag{1.3}
\end{equation*}
$$

where ${ }^{\wedge}$ and ${ }^{\vee}$ denote respectively the Fourier and inverse Fourier transforms on $\mathbb{R}^{n}$, and $\xi, \tau$ denote, respectively, the space and time variables on the Fourier transform side. Also, $\|z\|$ denotes the parabolic "norm" of $z$. We recall that this "norm" satisfies the non-isotropic dilation invariance property $\left\|\left(\delta x, \delta^{2} t\right)\right\| \equiv \delta\|(x, t)\|$. Indeed, $\|(x, t)\|$ is defined as the unique positive solution $\rho$ of the equation

$$
\begin{equation*}
\sum_{i=1}^{n-1} \frac{x_{i}^{2}}{\rho^{2}}+\frac{t^{2}}{\rho^{4}}=1 \tag{1.4}
\end{equation*}
$$

We note that the class of functions $A(x, t)$ satisfying (1.1) and (1.2), has been introduced (with a somewhat different, albeit equivalent formulation) in [LM], and considered further in [H1], [H2], and [HL]. In particular, it is shown in [H1] that this class of functions is the natural sharp parabolic analogue, of the class of Lipschitz functions in the elliptic theory, for the development of a Calderón type singular integral theory [Ca1], [Ca2]. Indeed, in [H1] it is shown that

$$
\left\|\left(\sqrt{\Delta-\frac{\partial}{\partial t}}, A\right)\right\|_{\mathrm{op}} \approx\left\|\nabla_{x} A\right\|_{\infty}+\left\|\mathbb{D}_{n} A\right\|_{*}
$$

where $\approx$ means the two quantities are bounded by constant multiples of each other. Moreover, $\|\cdot\|_{\text {op }}$ denotes the operator norm on $L^{2}\left(\mathbb{R}^{n-1}\right)$, and

$$
\begin{equation*}
\nabla_{x} \equiv\left(\frac{\partial}{\partial x_{1}}, \ldots, \frac{\partial}{\partial x_{n-1}}\right) . \tag{1.5}
\end{equation*}
$$

Since $(\sqrt{\Delta-\partial / \partial t}, A)$ is the parabolic version of the first Calderón commutator, we define the "commutator" norm of $A$ by

$$
\begin{equation*}
\|A\|_{\text {comm }} \equiv\left\|\nabla_{x} A\right\|_{\infty}+\left\|\mathbb{D}_{n} A\right\|_{*} . \tag{1.6}
\end{equation*}
$$

Of course, (1.1) and (1.2) say that this quantity is finite. In [H1] it is also shown that finiteness of (1.6) implies the parabolic Lipschitz condition

$$
\begin{equation*}
|A(x, t)-A(y, s)| \leq c \beta\|(x, t)-(y, s)\| \approx c \beta\left(|x-y|+|t-s|^{1 / 2}\right) \tag{1.7}
\end{equation*}
$$

We recall now that parabolic BMO is the space of all locally integrable functions modulo constants satisfying

$$
\begin{equation*}
\|b\|_{*} \equiv \sup _{B} \frac{1}{|B|} \int_{B}\left|b(z)-m_{B} b\right| d z<\infty \tag{1.8}
\end{equation*}
$$

Here, $z=(x, t)$ and $B$ denotes the parabolic ball

$$
\begin{equation*}
B \equiv B_{r}\left(z_{0}\right) \equiv\left\{z \in \mathbb{R}^{n}:\left\|z-z_{0}\right\|<r\right\} \tag{1.9}
\end{equation*}
$$

where $|B|$ denotes the Lesbegue $n$ measure of $B$ and

$$
m_{B} b \equiv \frac{1}{|B|} \int_{B} b(z) d z
$$

We note that $\left|B_{r}\left(z_{0}\right)\right| \equiv c r^{d}$ where $c$ is a constant and $d=n+1$ is the homogeneous dimension of $\mathbb{R}^{n}$ endowed with the metric induced by $\|\cdot\|$, as defined in (1.4). We observe that $\mathbb{R}^{n}$ so endowed is a space of homogeneous type in the sense of Coifman and Weiss [CW]. Indeed, there is a polar decomposition

$$
\begin{align*}
& z \equiv(x, t) \equiv\left(\rho \theta_{1}, \ldots, \rho \theta_{n-1}, \rho^{2} \theta_{n}\right) \\
& d z \equiv d x d t \equiv \rho^{d-1}\left(1+\theta_{n}^{2}\right) d \rho d \theta \tag{1.10}
\end{align*}
$$

where $\theta=\left(\theta_{1}, \ldots, \theta_{n}\right),|\theta|=1$, and $d \theta$ denotes surface area on the unit sphere.

Finally, we note that througout the sequel, we shall use the convenient notation

$$
z=(x, t) \in \mathbb{R}^{n}, \quad v=(y, s) \in \mathbb{R}^{n}
$$

and we shall denote the parabolic dilations by the convenient notation

$$
\delta^{\alpha} z \equiv\left(\delta x, \delta^{2} t\right),
$$

where $\alpha$ will always denote the $n$-dimensional multi-index $(1, \ldots, 1,2)$.
In the next section, we introduce the class of operators which we shall consider, and state our results.

## 2. Statement of results.

We begin by defining our square functions. To this end, let $H \in$ $C^{1}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ satisfy the homogeneity condition

$$
\begin{equation*}
H\left(\delta^{\alpha} z\right)=\delta^{-d-1} H(z), \quad \text { for } z=(x, t), d=n+1 \tag{2.1}
\end{equation*}
$$

and assume that $F \in C^{1}(\mathbb{R})$ with

$$
\begin{align*}
& |F(r)| \leq c_{F} \frac{1}{1+|r|^{d+1}}  \tag{2.2}\\
& \left|F^{\prime}(r)\right| \leq c_{F} \frac{1}{1+|r|^{d+2}}
\end{align*}
$$

whenever $r \in \mathbb{R}$. For $F, H$ as above define a square function $G$ of "Calderón type" by setting

$$
\begin{gather*}
R_{\lambda} f(z) \equiv \lambda \int_{\mathbb{R}^{n}} H(z-v) F\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v) d v  \tag{2.3}\\
G f(z)=\left(\int_{0}^{\infty}\left|R_{\lambda} f(z)\right|^{2} \frac{d \lambda}{\lambda}\right)^{1 / 2} \tag{2.4}
\end{gather*}
$$

Let $\omega$ be a parabolic $A_{2}$ weight (these are defined in the usual way, in this case with respect to parabolic balls, or cubes), and $f \in L_{\omega}^{2}\left(\mathbb{R}^{n}\right)$. As usual,

$$
\|f\|_{2, \omega} \equiv\left(\int|f(x)|^{2} \omega(x) d x\right)^{1 / 2}
$$

We shall work with weighted $L^{2}$, because, when dealing with square functions, this is a particularly suitable way to obtain $L^{p}$ bounds (via extrapolation - see [GR]). Furthermore our main application is to rough singular integral operators which do not satisy the standard CalderónZygmund kernel estimates, and thus cannot be shown to be bounded on $L^{p}$ via the standard program. However, as usual, it is really our unweighted $L^{2}$ bounds which are the heart of the matter - the extension to the weighted case is routine. We shall prove the following theorem.

Theorem 2.5. Suppose that for $H, F$ as above (see (2.1) and (2.2)) we have either $F$ is odd and $H(x, t)$ is odd in $x$ for each fixed $t$; or else that $F$ is even, $H(x, t)$ is even in $x$ for each fixed $t$, and also that $\int_{\mathbb{R}} F(r) d r=0$. If $\|A\|_{\text {comm }} \leq \beta<\infty$, and $\omega \in A_{2}$, then there exists a positive integer $N$ depending only on $d$ such that

$$
\|G f\|_{2, \omega} \leq c_{F, H, \omega}(1+\beta)^{N}\|f\|_{2, \omega} .
$$

Remark. Here and in the sequel, when we indicate that a constant depends on $\omega$, we mean that it actually depends only on the $A_{2}$ constant of $\omega$, so that $L^{p}$ bounds follow by extrapolation [GR].

Theorem 2.5 is easily generalized, in a way that is useful for some applications. Indeed, let $H, F$, be as in (2.1), (2.2), and let $B: \mathbb{R}^{n} \longrightarrow \mathbb{R}$ with

$$
\|B\|_{\text {comm }} \leq \beta_{0}<\infty
$$

Let $A$ be as in Theorem 2.5 and put

$$
\begin{array}{rl}
\widetilde{R}_{\lambda} f(z) \equiv \lambda \int_{\mathbb{R}^{n}} & H(z-v) \frac{B(z)-B(v)}{\|z-v\|} \\
& \cdot F\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v) d v \\
\widetilde{G} f(z)=( & \left.\int_{0}^{\infty}\left|\widetilde{R}_{\lambda} f(z)\right|^{2} \frac{d \lambda}{\lambda}\right)^{1 / 2} \tag{2.7}
\end{array}
$$

We then have
Theorem 2.8. Let $H, F$, and $A$ be as in Theorem 2.5, and let $B$ satisfy $\|B\|_{\text {comm }} \leq \beta_{0}<\infty$. Suppose that either $F$ is odd and $H(x, t)$ is even in $x$ for each fixed $t$; or else that $F$ is even, $H(x, t)$ is odd in $x$ for each fixed $t$, and also that $\int_{\mathbb{R}} F(r) d r=0$. If $\omega \in A_{2}$, then there exists a positive integer $N$ depending only on $d$ such that

$$
\|\widetilde{G} f\|_{2, \omega} \leq c_{F, H, \omega} \beta_{0}(1+\beta)^{N}\|f\|_{2, \omega}
$$

In our applications the square functions defined in (2.3)-(2.4) and (2.6)-(2.7) model the second derivatives of the single layer potential mapped to $\mathbb{R}_{+}^{n+1}$. We shall also describe here a model for higher order derivatives. We refrain from stating the most general result of this type as it would lead us too far astray from the purposes of this paper. Suppose $L \in C^{1}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ with

$$
\begin{equation*}
L\left(\delta^{\alpha} z\right)=\delta^{-d-2} L(z), \quad z \in \mathbb{R}^{n} \tag{2.9}
\end{equation*}
$$

and let $E \in C^{1}(\mathbb{R})$ with

$$
\begin{align*}
& |E(r)| \leq c_{E} \frac{1}{1+|r|^{d+2}}  \tag{2.10}\\
& \left|E^{\prime}(r)\right| \leq c_{E} \frac{1}{1+|r|^{d+3}}
\end{align*}
$$

whenever $r \in \mathbb{R}$. Suppose that either $E$ is even with $\int_{\mathbb{R}} E(r) d r=0$ and $L(x, t)$ is odd in $x$ for each fixed $t$; or else that $E$ is odd, with $\int_{\mathbb{R}} r E(r) d r=0$, and $L(x, t)$ is even in $x$ for each fixed $t$. Next assume
that $\widetilde{L} \in C^{1}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ satisfies (2.9) and $\widetilde{E} \in C^{1}(\mathbb{R})$ satisfies (2.10). Suppose that either $\widetilde{E}$ is even with $\int_{\mathbb{R}} \widetilde{E}(r) d r=0$ while $\widetilde{L}(x, t)$ is even in $x$ for each fixed $t$; or else that $\widetilde{E}$ is odd with $\int_{\mathbb{R}} r \widetilde{E}(r) d r=0$, while $\widetilde{L}(x, t)$ is odd in $x$ for each fixed $t$. We set

$$
\begin{gather*}
T_{\lambda} f(z) \equiv \lambda^{2} \int_{\mathbb{R}^{n}} L(z-v) E\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v) d v \\
\widetilde{T}_{\lambda} f(z) \equiv \lambda^{2} \int_{\mathbb{R}^{n}} \widetilde{L}(z-v) \frac{B(z)-B(v)}{\|z-v\|}  \tag{2.11}\\
\cdot \widetilde{E}\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v) d v
\end{gather*}
$$

where $\|A\|_{\text {comm }} \leq \beta<\infty,\|B\|_{\text {comm }} \leq \beta_{0}<\infty$, and

$$
\begin{align*}
& g(f)(z)=\left(\int_{0}^{\infty}\left|T_{\lambda} f(z)\right|^{2} \frac{d \lambda}{\lambda}\right)^{1 / 2}  \tag{2.12}\\
& \widetilde{g}(f)(z)=\left(\int_{0}^{\infty}\left|\widetilde{T}_{\lambda} f(z)\right|^{2} \frac{d \lambda}{\lambda}\right)^{1 / 2}
\end{align*}
$$

With this notation we have
Theorem 2.13. Let $E, L, \widetilde{E}, \widetilde{L}, g, \widetilde{g}, A, B$, be as above. Then there exists a positive integer $N=N(d)$ such that if $f, \omega$ are as in Theorem 2.5, we have

$$
\|g(f)\|_{2, \omega}+\beta_{0}^{-1}\|\widetilde{g}(f)\|_{2, \omega} \leq c(1+\beta)^{N}\|f\|_{2, \omega}
$$

where $c$ depends on $\omega, E, L, \widetilde{E}, \widetilde{L}$, and d.
We shall not bother to give the proof of Theorem 2.13 in this note, as the interested reader could easily supply it after reading the proofs of Theorems 2.5 and 2.8.

To conclude this section, we now describe the parabolic smoothing operators which are our main application. Let $J$ denote a kernel which satisfies the homogeneity property

$$
\begin{equation*}
J\left(\delta^{\alpha} z\right) \equiv \delta^{-d+1} J(z) \tag{2.14}
\end{equation*}
$$

where $d=n+1$ and $z \in \mathbb{R}^{n}$. We also assume that $J$ is sufficiently smooth away from the origin, i.e., $J \in C^{m}\left(\mathbb{R}^{n} \backslash\{0\}\right)$, for some large
$m$. With this notation, let $E$ denote either sine or cosine, and define "smoothing operators of Calderón type" by

$$
\begin{gather*}
S_{A} f(z) \equiv \int_{\mathbb{R}^{n}} J(z-v) E\left(\frac{A(z)-A(v)}{\|z-v\|}\right) f(v) d v \\
U_{A, B} f(z) \equiv \int_{\mathbb{R}^{n}} J(z-v) E\left(\frac{A(z)-A(v)}{\|z-v\|}\right)  \tag{2.15}\\
\cdot \frac{B(z)-B(v)}{\|z-v\|} f(v) d v
\end{gather*}
$$

We shall give a simpler proof of the following result of the first author [H2, Theorem 3]. Let $L_{1,1 / 2}^{p}$ denote the parabolic Sobolev space defined as the collection of all f having a spatial gradient and $1 / 2$ a time derivative in $L^{p}$, i.e., those $f$ for whom the following norm is finite

$$
\|f\|_{L_{1,1 / 2}^{p}} \equiv\left\|\nabla_{x} f\right\|_{p}+\left\|\mathbb{D}_{n} f\right\|_{p}
$$

Theorem 2.16. Let $\|A\|_{\text {comm }},\|B\|_{\text {comm }}<\infty$ and $f \in L^{p}\left(\mathbb{R}^{n}\right), 1<$ $p<\infty$. Suppose that $J$ is sufficiently smooth away from the origin. If $J(x, t)$ has the same parity in $x$ as does $E$, then for some large positive $N$, we have

$$
\left\|S_{A} f\right\|_{L_{1,1 / 2}^{p}} \leq c_{p, J}\left(1+\|A\|_{\text {comm }}\right)^{N}\|f\|_{p}
$$

Similarly if $J(x, t)$ has opposite parity in $x$ to that of $E$, then

$$
\left\|U_{A, B} f\right\|_{L_{1,1 / 2}^{p}} \leq c_{p, J}\|B\|_{\text {comm }}\left(1+\|A\|_{\text {comm }}\right)^{N}\|f\|_{p}
$$

Remarks. 1) Using the method of [CDM], one can immediately replace the trigonometric function $E$ by any sufficiently smooth function defined on $\mathbb{R}$ with the same parity as $E$. One can also treat layer potentials via this method.
2) Theorem 3 in $[\mathrm{H} 2]$ is stated for $A_{2}$ weights but implies our Theorem 2.16 by extrapolation.

In the next section (3), we treat our square functions (theorems 2.5 and 2.8). In the last section (4), we give the alternative proof of Theorem 2.16.

## 3. Proofs of theorems 2.5, 2.8.

We begin with a simple lemma. For $(\lambda, z),(\lambda, v) \in \mathbb{R}_{+}^{n+1}$, let $K_{\lambda}(z, v)$ be a family of real valued kernels satisfying

$$
\begin{gather*}
\left|K_{\lambda}(z, v)\right| \leq c_{K} \frac{\lambda}{(\lambda+\|z-v\|)^{d+1}},  \tag{3.1}\\
\left|K_{\lambda}(z, v)-K_{\lambda}(z, \widetilde{v})\right| \\
\leq c_{K}\|v-\widetilde{v}\| \min \left\{\frac{1}{\lambda^{d}\|z-v\|}, \frac{\lambda}{\|z-v\|^{d+2}}\right\},
\end{gather*}
$$

whenever $2\|v-\widetilde{v}\| \leq\|z-v\|$. Let $\omega$ be a parabolic $A_{2}$ weight. Put

$$
K_{\lambda} f(z)=\int_{\mathbb{R}^{n}} K_{\lambda}(z, v) f(v) d v, \quad z \in \mathbb{R}^{n}
$$

The following result is standard, and we omit the proof.
Lemma 3.3. Let $\left(K_{\lambda}\right)$ satisfy (3.1), (3.2) and let $\omega$, $f$ be as above. If $K_{\lambda} 1 \equiv 0$ for each $\lambda>0$, then

$$
\int_{\mathbb{R}_{+}^{n+1}}\left(K_{\lambda} f\right)^{2}(z) \omega(z) \frac{d z d \lambda}{\lambda} \leq c_{K, \omega}\|f\|_{2, \omega}^{2}
$$

In Lemma 3.3, $c_{K, \omega}$ denotes a constant depending only on $K, d$, and the $A_{2}$ constant of $\omega$, which is the same convention we used in Section 2. Lemma 3.3 is stated in [Ch, p. 69, Theorem 20] for $\omega=1$ (see also [CJ]) under slightly weaker hypotheses.

Proof of Theorem 2.5. Let $P \in C_{0}^{\infty}\left(B_{1}(0)\right)$ be an even function with $\int_{\mathbb{R}^{n}} P_{\lambda}(z) d z \equiv 1$, where as usual $P_{\lambda}(z) \equiv \lambda^{-d} P\left(\lambda^{-\alpha} z\right)$ and let $f \longrightarrow P_{\lambda} f$ be the convolution operator whose kernel is $P_{\lambda}(z)$. Put

$$
Q_{\lambda}^{*} f(z) \equiv \lambda \int_{\mathbb{R}^{n}} H(z-v) F\left(\frac{\left\langle\nabla_{z^{\prime}} P_{\lambda} A(z), z^{\prime}-v^{\prime}\right\rangle+\lambda}{\|z-v\|}\right) f(v) d v
$$

where $z^{\prime}=x, v^{\prime}=y$ if $z=(x, t)$ and $v=(y, s)$. Then

$$
\begin{align*}
G f(z) & \leq\left(\int_{0}^{\infty}\left|\left(R_{\lambda}-Q_{\lambda}^{*}\right) f(z)\right|^{2} \frac{d \lambda}{\lambda}\right)^{1 / 2}+\left(\int_{0}^{\infty}\left|Q_{\lambda}^{*} f(z)\right|^{2} \frac{d \lambda}{\lambda}\right)^{1 / 2}  \tag{3.4}\\
& =G_{1} f(z)+G_{2} f(z)
\end{align*}
$$

We set $V_{\lambda} \equiv R_{\lambda}-Q_{\lambda}^{*}$ and observe from (2.1) and (2.2) that the kernel $V_{\lambda}(z, v)$ of $V_{\lambda}$ satisfies

$$
\begin{align*}
\left|V_{\lambda}(z, v)\right| \leq & c(1+\beta)^{d+2} \frac{\lambda}{(\lambda+\|z-v\|)^{d+2}}  \tag{3.5}\\
& \cdot\left|A(z)-A(v)-\left\langle\nabla_{z^{\prime}} P_{\lambda} A(z), z^{\prime}-v^{\prime}\right\rangle\right|
\end{align*}
$$

where $c$ depends on $F, H, d$. Using (3.5) and (1.7) we deduce that $V_{\lambda}$ satisfies (3.1) with $K$ replaced by $V$ and $c_{K}$ replaced by $c(1+\beta)^{d+3}$. Also by the same argument we see that the kernel of $Q_{\lambda}^{*}$ satisfies (3.1) with $Q^{*}=K$ and the same constants as $V$. Moreover, since $H \in$ $C^{1}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ we find in addition from (2.1), (2.2) and (1.7), that the kernels of $V_{\lambda}, Q_{\lambda}^{*}$ satisfy (3.2) with the same constants as in (3.1).

First we consider $G_{1}$ in (3.4). This term will be treated using the main idea in [JnsP], but with the particular details closer to the exposition in [Ch]. From the above discussion we see that we may follow the standard approach, as in $[\mathrm{CM}]$, to handle $K_{\lambda}=V_{\lambda}-\left(V_{\lambda} 1\right) P_{\lambda}$, via Lemma 3.3 since $K_{\lambda} 1 \equiv 0$ for each $\lambda>0$. Thus to show

$$
\begin{equation*}
\left\|G_{1} f\right\|_{2, \omega} \leq c_{F, H, \omega}(1+\beta)^{N}\|f\|_{2, \omega} \tag{3.6}
\end{equation*}
$$

we need only prove

$$
\int_{0}^{\infty} \int_{\mathbb{R}^{n}}\left(V_{\lambda} 1 P_{\lambda} f\right)^{2} \omega \frac{d z d \lambda}{\lambda} \leq c_{F, H, \omega}(1+\beta)^{2 N}\|f\|_{2, \omega}
$$

i.e., that

$$
\begin{equation*}
d \nu(\lambda, z)=\left(V_{\lambda} 1(z)\right)^{2} \omega(z) \frac{d z d \lambda}{\lambda} \tag{3.7}
\end{equation*}
$$

is a weighted Carleson measure with norm comparable to the constants in Theorem 2.5. To this end let $z_{0} \in \mathbb{R}^{n}, r>0$, and let $\chi, \chi^{*}$ denote the characteristic functions of $B_{10 r}\left(z_{0}\right), \mathbb{R}^{n} \backslash B_{10 r}\left(z_{0}\right)$, respectively. Fixing this ball, and using (3.1) for $V_{\lambda}$ we deduce, as usual, that it suffices to replace 1 by $\chi$ in (3.7). Next we put $\widetilde{A}(z)=\psi\left(\left\|z-z_{0}\right\|\right)\left(A(z)-A\left(z_{0}\right)\right)$, $z \in \mathbb{R}^{n}$, where $\psi \in C_{0}^{\infty}(-20 r, 20 r)$ is an even function with $\psi \equiv 1$ on $[-15 r, 15 r]$. Then $V_{\lambda} \chi(z)$ is unchanged for $z \in B_{10 r}\left(z_{0}\right), 0<\lambda<r$, if we replace $A$ in its definition by $\widetilde{A}$. Moreover from [H2, Section 6 , Lemma 2] we have
i) $\|\widetilde{A}\|_{\text {comm }} \leq c\|A\|_{\text {comm }}$.
ii) For $1<p<\infty,\|\mathbb{D} \tilde{A}\|_{p}^{p} \leq c_{p} \beta^{p} r^{d}$,
where the parabolic fractional derivative operator $\mathbb{D}$ is defined by the Fourier multiplier

$$
\widehat{\mathbb{D} f} \equiv\|\zeta\| \widehat{f}
$$

Using (3.5), Schwarz's inequality, and the change of variable $\lambda \longrightarrow \lambda / 2^{l}$ we obtain, for $N$ large enough that

$$
\begin{align*}
& (1+\beta)^{-2 N} \int_{0}^{r} \int_{B_{r}\left(z_{0}\right)}\left(V_{\lambda} \chi\right)^{2}(z) \omega(z) \frac{d z d \lambda}{\lambda} \\
& \leq c \sum_{l=0}^{\infty} 2^{-l} \int_{0}^{\infty} \int_{\mathbb{R}^{n}} \lambda^{-d-2}\left(\int_{B_{\lambda}(z)} \mid \widetilde{A}(z)-\widetilde{A}(v)\right.  \tag{3.9}\\
& \\
& \left.\quad-\left.\left\langle\nabla_{z^{\prime}} P_{2^{-l} \lambda} \widetilde{A}(z), z^{\prime}-v^{\prime}\right\rangle\right|^{2} d v\right) \\
& \quad \cdot \omega(z) \frac{d z d \lambda}{\lambda} \\
& \leq c_{\omega} \beta^{2} \omega\left(B_{r}\left(z_{0}\right)\right),
\end{align*}
$$

where the last inequality follows from (3.8) and an argument involving Plancherel's Theorem in the case $\omega \equiv 1$ (see [H2, Section 5] for more details) or else the argument of [H2, Section 6, Lemma 3] in the weighted case. Thus (3.6) holds.

To prove the analogue of (3.6) with $G_{1}$ replaced by $G_{2}$ we note that (3.1), (3.2) for $Q_{\lambda}^{*}$, and Lemma 3.3 imply that it is enough to show that $Q_{\lambda}^{*} 1 \equiv 0$. To do this we introduce the parabolic polar coordinates defined in (1.10) to get

$$
Q_{\lambda}^{*} 1(z)=\lambda \int_{S}\left(\int_{0}^{\infty} F\left(\left\langle\vec{a}, \sigma^{\prime}\right\rangle+\frac{\lambda}{\rho}\right) \frac{d \rho}{\rho^{2}}\right) H(\sigma) \Phi(\sigma) d \sigma
$$

where $\vec{a}=\nabla_{z^{\prime}} P_{\lambda} A(z), \Phi(\sigma)=\left(1+\sigma_{n}^{2}\right)$, and $\sigma=\left(\sigma^{\prime}, \sigma_{n}\right) \in S=$ the unit sphere in $\mathbb{R}^{n}$. We change variables in the above integral by $\rho \longrightarrow \lambda \rho$, then $r=1 / \rho$, then $r \longrightarrow r-\left\langle\vec{a}, \sigma^{\prime}\right\rangle$, to obtain

$$
Q_{\lambda}^{*} 1(z)=\int_{S}\left(\int_{\left\langle\vec{a}, \sigma^{\prime}\right\rangle}^{\infty} F(r) d r\right) H(\sigma) \Phi(\sigma) d \sigma=0
$$

since our hypotheses in Theorem 2.5 guarantee that this last expression is zero. Indeed $\int_{\left\langle\vec{a}, \sigma^{\prime}\right\rangle}^{\infty} F(r) d r$ is a function of $\sigma^{\prime}$ having opposite parity to $H(\sigma) \Phi(\sigma)$, for each fixed non-zero $\vec{a}$. The case $\vec{a}=0$ is much simpler:
if $H$ is odd in $\sigma^{\prime}$, then clearly $\int_{S} H(\sigma) \Phi(\sigma) d \sigma=0$, and if $F$ is even with $\int_{-\infty}^{\infty} F(r) d r=0$, then $\int_{0}^{\infty} F(r) d r=0$. Thus (3.6) holds also for $G_{2}$, and the conclusion of Theorem 2.5 follows.

Proof of Theorem 2.8. We shall be brief, since the ideas are now familiar. Put

$$
\begin{aligned}
& U_{\lambda} f(z) \\
\equiv & \lambda \int_{\mathbb{R}^{n}} H(z-v) \frac{\left\langle\nabla_{z^{\prime}} P_{\lambda} B(z), z^{\prime}-v^{\prime}\right\rangle}{\|z-v\|} F\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v) d v .
\end{aligned}
$$

Then as in (3.4)

$$
\begin{align*}
\widetilde{G} f(z) & \leq\left(\int_{0}^{\infty}\left|\left(\widetilde{R}_{\lambda}-U_{\lambda}\right) f(z)\right|^{2} \frac{d \lambda}{\lambda}\right)^{1 / 2}+\left(\int_{0}^{\infty}\left|U_{\lambda} f(z)\right|^{2} \frac{d \lambda}{\lambda}\right)^{1 / 2}  \tag{3.10}\\
& =\widetilde{G}_{1} f(z)+\widetilde{G}_{2} f(z)
\end{align*}
$$

If $\widetilde{V}_{\lambda}=\widetilde{R}_{\lambda}-U_{\lambda}$, then as in (3.5) we deduce

$$
\begin{aligned}
\left|\widetilde{V}_{\lambda}(z, v)\right| \leq & c(1+\beta)^{d+2} \min \left\{\frac{\lambda}{\|z-v\|^{d+2}}, \frac{1}{\lambda^{d}\|z-v\|}\right\} \\
& \cdot\left|B(z)-B(v)-\left\langle\nabla_{z^{\prime}} P_{\lambda} B(z), z^{\prime}-v^{\prime}\right\rangle\right|,
\end{aligned}
$$

where $c$ depends on $F, H, d$. Using this inequality in place of (3.5) we can now repeat the argument following (3.5) through (3.9) to get that (3.6) holds with $G_{1}$ replaced by $\widetilde{G}_{1}$ and constants as in Theorem 2.8. As for $\widetilde{G}_{2}$ we note from (1.7) that the kernel of $U_{\lambda}$ can be written as a sum of $L^{\infty}$ functions (the components of $\nabla_{z^{\prime}} P_{\lambda} B(z)$ ) times operators whose kernels satisfy the hypotheses of Theorem 2.5. Thus (3.6) holds with $G_{1}$ replaced by $\widetilde{G}_{2}$ and constants as in Theorem 2.8, and we are done.

## 4. Alternative proof of Theorem 2.16.

Next we shall use Theorems 2.5, 2.8, and 2.13, to give an alternate proof of Theorem 2.16 (i.e. essentially [H2, Theorem 3]). Our reduction of the proof of Theorem 2.16 to the square function estimates which we have proved in the previous theorems, will be in the spirit of some
recent work of Li, McIntosh, and Semmes [LiMS, Section 4]. To begin, we consider the operator $S=S_{A}$ of Theorem 2.16. For specificity, we consider

$$
S f(z) \equiv \int_{\mathbb{R}^{n}} J(z-v) \cos \left(\frac{A(z)-A(v)}{\|z-v\|}\right) f(v) d v
$$

where
a) $J(x, t)$ is even in $x$, for each fixed $t$,
b) $J\left(\lambda^{\alpha} z\right) \equiv \lambda^{1-d} J(z), z \in \mathbb{R}^{n}$,
c) $J \in C_{0}^{N}\left(\mathbb{R}^{n} \backslash\{0\}\right)$, for some large $N$.

Our goal is to show that for some large $N$ and for each $j, 1 \leq j \leq n$, we have

$$
\begin{equation*}
\left\|\mathbb{D}_{j} S f\right\|_{p} \leq c_{J, p}(1+\beta)^{N}\|f\|_{p} . \tag{4.2}
\end{equation*}
$$

whenever $f \in L^{p}\left(\mathbb{R}^{n}\right)$, and $1<p<\infty$. Here, $\mathbb{D}_{j}=\partial / \partial x_{j}$ for $1 \leq$ $j \leq n-1$, and $\mathbb{D}_{n}$ is the $1 / 2$ order time derivative defined in Section 1. Since $\nabla_{x} A \in L^{\infty}\left(\mathbb{R}^{n}\right)$, we have that, modulo pointwise multiplication by a bounded function, each $\mathbb{D}_{j} S, 1 \leq j \leq n-1$, gives rise to a standard parabolic Calderón-Zygmund operator which falls under the scope of [H2, Theorem 1] (to see this, just differentiate formally under the integral sign in the definition of $S f$ - this formal computation may be justified by smoothly truncating the kernel $J$, and obtaining bounds independent of the truncation). Thus it suffices to prove the case $j=n$ of (4.2). In fact if $\omega$ is an $A_{2}$ weight and $f \in L_{\omega}^{2}\left(\mathbb{R}^{n}\right)$, we shall show that

$$
\begin{equation*}
\left\|\mathbb{D}_{n} S f\right\|_{2, \omega} \leq c_{J, \omega}(1+\beta)^{N}\|f\|_{2, \omega} \tag{4.3}
\end{equation*}
$$

Once (4.3) is proved, the Theorem then follows from extrapolation (see [GR, Chapter 4, Theorem 5.19]). We remark that the operator $\mathbb{D}_{n} S$ cannot be viewed as a standard Calderón-Zygmund operator (modulo multiplication by a bounded function), and hence does not fall under the scope of [H2, Theorem 1], nor can one use the classical CalderónZygmund theory to pass from $L^{2}$ bounds to $L^{p}$. The failure of the standard C-Z estimates in this case is related to the fact that the chain rule does not hold for fractional derivatives like $\mathbb{D}_{n}$.

To make our arguments rigorous, we observe that since

$$
|A(z)-A(v)| \leq c\|A\|_{\text {comm }}\|z-v\|
$$

(see (1.7)), we can replace the cosine in the definition of $S f$ by $\Lambda$ where $\Lambda(r)=\phi(r) \cos (r)$ and $\phi \in C_{0}^{\infty}(\mathbb{R})$ is an even function with $\phi \equiv 1$ on $[-c \beta, c \beta]$. Clearly we can also choose $\phi$ so that $\int_{-\infty}^{\infty} \Lambda(r) d r=0$. We make the a priori assumption that $f \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right), A \in C^{\infty}\left(\mathbb{R}^{n}\right)$, and that $J$ has been smoothly truncated so that it is supported on a parabolic annulus. These assumptions allow us easily to justify repeated differentiations and integrations by parts in the argument which follows. In the rest of the proof we shall systematically suppress the truncation, so as not to tire the reader with routine details. This means that we shall be ignoring certain error terms which arise as a result of the truncation, but these are not difficult to handle. Of course, our estimates will not have any quantitative dependence upon our a priori assumptions.

Under these assumptions we first use a construction of Kenig and Stein (which appeared first in a paper of Dahlberg [D]; see also [DKPV] and [HL] for applications related to the present paper), to write $S f(z)=$ $\lim _{\lambda \rightarrow 0} S_{\lambda} f(z)$, where

$$
S_{\lambda} f(z) \equiv \int_{\mathbb{R}^{n}} J(z-v) \Lambda\left(\frac{P_{\gamma \lambda} A(z)+\lambda-A(v)}{\|z-v\|}\right) f(v) d v, \quad z \in \mathbb{R}^{n}
$$

and $P_{\gamma \lambda}$ is defined as follows. Let $P \in C_{0}^{\infty}\left(B_{1}(0)\right)$ be an even function with $\int_{\mathbb{R}^{n}} P_{\lambda}(z) d z \equiv 1$, where as usual $P_{\lambda}(z) \equiv \lambda^{-d} P\left(\lambda^{-\alpha} z\right)$, and let $f \longrightarrow P_{\lambda} f$ be the convolution operator whose kernel is $P_{\lambda}(z)$. We choose $\gamma$ to be a small, fixed number, depending only on $\|A\|_{\text {comm }}$, such that

$$
\left|\frac{\partial}{\partial \lambda} P_{\gamma \lambda} A(z)\right| \leq \frac{1}{2}
$$

Next let $g \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ with $\|g\|_{2,1 / \omega}=1$ and observe that

$$
\left\|\mathbb{D}_{n} S f\right\|_{2, \omega}=\sup \left|\int_{\mathbb{R}^{n}} \mathbb{D}_{n} S f g d z\right|
$$

where the supremum is taken over all such $g$. Moreover,

$$
\begin{align*}
-\int_{\mathbb{R}^{n}} \mathbb{D}_{n} S f g d z= & \int_{0}^{\infty} \int_{\mathbb{R}^{n}} \frac{\partial}{\partial \lambda}\left(\mathbb{D}_{n} S_{\lambda} f P_{\lambda} g\right) d z d \lambda \\
= & \int_{0}^{\infty} \int_{\mathbb{R}^{n}} \mathbb{D}_{n} \frac{\partial}{\partial \lambda} S_{\lambda} f P_{\lambda} g d z d \lambda  \tag{4.4}\\
& +\int_{0}^{\infty} \int_{\mathbb{R}^{n}} \mathbb{D}_{n} S_{\lambda} f \frac{\partial}{\partial \lambda} P_{\lambda} g d z d \lambda \\
= & I+I I .
\end{align*}
$$

We recall that we have defined a parabolic fractional derivative operator $\mathbb{D}$ by the Fourier multiplier

$$
\begin{equation*}
\widehat{\mathbb{D} f} \equiv\|\zeta\| \widehat{f} \tag{4.5}
\end{equation*}
$$

We observe that $\partial P_{\lambda} / \partial \lambda=\mathbb{D} \widetilde{Q}_{\lambda}$ where $\widetilde{Q}_{\lambda}$ is an approximation to the zero operator (i.e., $\widetilde{Q}_{\lambda} 1 \equiv 0$ ) whose convolution kernel satisfies the standard kernel estimates (3.1) and (3.2). We leave the details of this routine estimate to the reader, noting only that to prove it, one uses the fact that the kernel of $\partial P_{\lambda} / \partial \lambda$ has not only mean value zero, but also vanishing first moments, since we have chosen $P(z)$ to be an even function. Thus since $\mathbb{D}_{n}=i \mathbb{D}^{-1}(\partial / \partial t)$, we have

$$
|I I|=\left|\int_{0}^{\infty} \int_{\mathbb{R}^{n}} \frac{\partial}{\partial t} S_{\lambda} f \widetilde{Q}_{\lambda} g d z d \lambda\right|
$$

Since $\|g\|_{2,1 / \omega}=1$, weighted Littlewood-Paley theory implies that

$$
\int_{0}^{\infty} \int_{\mathbb{R}^{n}}\left(\widetilde{Q}_{\lambda} g\right)^{2}\left(\frac{1}{\omega}\right) d z \frac{d \lambda}{\lambda} \leq c_{\omega}
$$

Hence, by Schwarz's inequality,

$$
\begin{equation*}
|I I|^{2} \leq c_{\omega} \int_{0}^{\infty} \int_{\mathbb{R}^{n}}\left|\frac{\partial}{\partial t} S_{\lambda} f\right|^{2} \omega \lambda d z d \lambda \tag{4.6}
\end{equation*}
$$

Now let

$$
w\left(x_{0}, z\right) \equiv \int_{\mathbb{R}^{n}} J(z-v) \Lambda\left(\frac{x_{0}-A(v)}{\|z-v\|}\right) f(v) d v
$$

and define the Kenig-Stein mapping

$$
\begin{equation*}
\rho(\lambda, z)=\left(\lambda+P_{\gamma \lambda} A(z), z\right) . \tag{4.7}
\end{equation*}
$$

Since $w \circ \rho(\lambda, z)=S_{\lambda} f(z)$, we have for $z=(x, t)$ that

$$
\begin{align*}
\frac{\partial}{\partial t} S_{\lambda} f(z) & =\frac{\partial}{\partial t}(w \circ \rho)(\lambda, x, t)  \tag{4.8}\\
& =w_{t} \circ \rho(\lambda, x, t)+w_{x_{0}} \circ \rho(\lambda, x, t) \frac{\partial}{\partial t} P_{\gamma \lambda} A(x, t)
\end{align*}
$$

To handle the contribution of $w_{t} \circ \rho$ to the integral in (4.6) we use the change of variable

$$
\begin{equation*}
\widetilde{\lambda} \equiv \lambda+P_{\gamma \lambda} A(z)-A(z), \tag{4.9}
\end{equation*}
$$

which defines a mapping $(\lambda, z) \longrightarrow(\widetilde{\lambda}, z)$ of $\mathbb{R}_{+}^{n+1}$ with Jacobian

$$
1+\frac{\partial}{\partial \lambda} P_{\gamma \lambda} A(z)=\eta(\lambda, z)
$$

Since $\left|(\partial / \partial \lambda) P_{\gamma \lambda} A(z)\right| \leq 1 / 2$ for $\gamma$ small enough depending only on $\|A\|_{\text {comm }}$, and $\lim _{\lambda \rightarrow 0} P_{\gamma \lambda} A=A$, we deduce first that $1 / 2 \leq \eta \leq 3 / 2$ and thereupon that the above mapping is 1-1 and onto $\mathbb{R}_{+}^{n+1}$. Changing variables as in (4.9) we find that by Theorem 2.5,

$$
\begin{equation*}
\int_{0}^{\infty} \int_{\mathbb{R}^{n}}\left(w_{t} \circ \rho\right)^{2} \omega \lambda d z d \lambda \leq c_{J, \omega}(1+\beta)^{2 N}\|f\|_{2, \omega}^{2} \tag{4.10}
\end{equation*}
$$

as desired.
To handle the contribution of the second term in (4.8) to the integral in (4.6), we claim that the non-tangential maximal function of

$$
w_{x_{0}} \circ \rho(\lambda, x, t)
$$

is bounded on $L_{\omega}^{2}$ with norm on the order of $\left(1+\|A\|_{\text {comm }}\right)^{N}$. Indeed, the operator

$$
f \longrightarrow w_{x_{0}} \circ \rho(0, x, t)
$$

is of the form

$$
T_{A} f(z) \equiv \text { p.v. } \int_{\mathbb{R}^{n}} K(z-v) F\left(\frac{A(z)-A(v)}{\|z-v\|}\right) f(v) d v
$$

where

$$
K\left(\delta x, \delta^{2} t\right) \equiv \delta^{-d} K(x, t),
$$

$K \in C^{m}\left(\mathbb{R}^{n} \backslash\{0\}\right)$, for some large $m, F \in C^{k}\left(\mathbb{R}^{1}\right)$, for some large $k$, and where the parity of $K(x, t)$ in the $x$ variable is opposite to that of $F$. It is essentially the conclusion of [H2, Theorem 1], that such operators are bounded on $L^{2}$, and hence on $L_{\omega}^{2}$, with norm on the order of $\left(1+\|A\|_{\text {comm }}\right)^{N}$. The claim now follows by applying a standard argument involving Cotlar's inequality for maximal singular integrals,
to pass from the singular integral on the boundary to the non-tangential maximal function. Furthermore

$$
\left\|\frac{\partial}{\partial t} P_{\gamma \lambda} A(z)\right\|^{2} \lambda d \lambda d z
$$

is a parabolic Carleson measure with a norm no larger than $\left\|\mathbb{D}_{n} A\right\|_{*}^{2}$. The desired bound for (4.6) now follows by the usual properties of Carleson measures.

We now turn to $I$ in (4.4). We integrate by parts in the integral defining $I$ to get

$$
\begin{align*}
-I= & \int_{0}^{\infty} \int_{\mathbb{R}^{n}} \frac{\partial^{2}}{\partial \lambda^{2}} S_{\lambda} f P_{\lambda} g \lambda d z d \lambda \\
& +\int_{0}^{\infty} \int_{\mathbb{R}^{n}} \frac{\partial}{\partial \lambda} S_{\lambda} f \frac{\partial}{\partial \lambda} P_{\lambda} g \lambda d z d \lambda  \tag{4.11}\\
= & I_{1}+I_{2} .
\end{align*}
$$

Arguing as in the proof of (4.6) we find

$$
\begin{align*}
\left|I_{2}\right|^{2} & =\left|\int_{0}^{\infty} \int_{\mathbb{R}^{n}} \frac{\partial}{\partial t} \frac{\partial}{\partial \lambda} S_{\lambda} f \widetilde{Q}_{\lambda} g \lambda d z d \lambda\right|^{2}  \tag{4.12}\\
& \leq c_{\omega} \int_{\mathbb{R}^{n}}\left|\frac{\partial^{2}}{\partial t} S_{\lambda} f\right|^{2} \omega \lambda^{3} d z d \lambda .
\end{align*}
$$

Again

$$
\begin{align*}
\frac{\partial^{2}}{\partial t \partial \lambda} S_{\lambda} f= & \frac{\partial^{2}}{\partial t \partial \lambda} w \circ \rho \\
= & \frac{\partial}{\partial t}\left(\left(w_{x_{0}} \circ \rho\right)\left(1+\frac{\partial}{\partial \lambda} P_{\gamma \lambda} A\right)\right) \\
= & \left(w_{x_{0} t} \circ \rho\right)\left(1+\frac{\partial}{\partial \lambda} P_{\gamma \lambda} A\right)  \tag{4.13}\\
& +\left(w_{x_{0} x_{0}} \circ \rho\right)\left(\frac{\partial}{\partial t} P_{\gamma \lambda} A\right)\left(1+\frac{\partial}{\partial \lambda} P_{\gamma \lambda} A\right) \\
& +\left(w_{x_{0}} \circ \rho\right)\left(\frac{\partial^{2}}{\partial t \partial \lambda} P_{\gamma \lambda} A\right) \\
= & \Lambda_{1}+\Lambda_{2}+\Lambda_{3} .
\end{align*}
$$

Since $\left|(\partial / \partial \lambda) P_{\gamma \lambda} A\right| \leq 1 / 2$, we have $\Lambda_{1} \leq 2\left|w_{x_{0} t} \circ \rho\right|$. We now use the change of variable (4.9), and invoke Theorem 2.13, to handle the contribution of $\Lambda_{1}$. As for $\Lambda_{2}$, since

$$
\left|\frac{\partial}{\partial t} P_{\gamma \lambda} A\right| \leq c(1+\beta)^{2} \lambda^{-1}
$$

we can use Theorem 2.5 to handle $w_{x_{0} x_{0}}$ in the same way that we treated $w_{t}$ above. Finally, we may handle the contribution of $\Lambda_{3}$, by the usual nontangential maximum-Carleson measure arguments, i.e., by exactly the same method that we used previously to treat the contribution of the second term on the right hand side of (4.8). Altogether, we obtain the desired bound for the term $I_{2}$.

It remains to estimate $I_{1}$. We note that $\lambda \mathbb{D}_{n} P_{\lambda} \equiv \widetilde{\widetilde{Q}}_{\lambda}$ where $\widetilde{\widetilde{Q}}_{\lambda}$ is an approximation to the zero operator whose kernel satisfies (3.1) and (3.2). Thus arguing as in the proof of (4.6), we obtain

$$
\begin{align*}
\left|I_{1}\right| & =\left|\int_{0}^{\infty} \int_{\mathbb{R}^{n}} \frac{\partial^{2}}{\partial \lambda^{2}} S_{\lambda} f \widetilde{\widetilde{Q}}_{\lambda} g d z d \lambda\right|  \tag{4.14}\\
& \leq c_{\omega}\left(\int_{0}^{\infty} \int_{\mathbb{R}^{n}}\left|\frac{\partial^{2}}{\partial \lambda^{2}} S_{\lambda} f\right|^{2} \omega \lambda d z d \lambda\right)^{1 / 2}
\end{align*}
$$

But

$$
\frac{\partial^{2}}{\partial \lambda^{2}} S_{\lambda} f=\left(w_{x_{0} x_{0}} \circ \rho\right)\left(1+\frac{\partial}{\partial \lambda} P_{\gamma \lambda} A\right)^{2}+\left(w_{x_{0}} \circ \rho\right)\left(\frac{\partial^{2}}{\partial \lambda^{2}} P_{\gamma \lambda} A\right),
$$

and these terms can each be handled by our earlier arguments. This concludes the proof of Theorem 2.16 for $S=S_{A}$. The proof for the second class of operators, $U_{A, B}$, is similar, and we leave the details to the interested reader.

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# Solutions des équations de Navier-Stokes incompressibles dans un domaine exterieur 

Nicolas Depauw

## 0. Introduction.

Le mouvement d'un fluide incompressible visqueux, de viscosité $\varepsilon$, remplissant un ouvert $\Omega \subset \mathbb{R}^{n}$ à bord $\partial \Omega$ régulier est modélisé par les équations de Navier-Stokes

$$
\begin{cases}\partial_{t} u+(u \cdot \nabla) u-\varepsilon \Delta u+\nabla p=f, & \left.u\right|_{\partial \Omega}=0 \\ \nabla \cdot u=0, & \left.u\right|_{t=0}=u_{0}\end{cases}
$$

La dimension est $n \geq 3$. Les inconnues sont le champ vectoriel de vitesses $u(t, x) \in \mathbb{R}^{n}$ et le champ scalaire pression $p(t, x) \in \mathbb{R}$; les variables sont $t \in \mathbb{R}$ et $x \in \Omega$. $\nabla$ est l'opérateur différentiel ( $\partial_{x_{1}}, \ldots, \partial_{x_{n}}$ ), noté comme un vecteur. Ainsi $\nabla \cdot u$ est la divergence du champ $u$, tandis que $(u \cdot \nabla)$ est l'opérateur de dérivation partielle $u_{1} \partial_{x_{1}}+\cdots+u_{n} \partial_{x_{n}}$. $\Delta$ est l'opérateur de Laplace $(\nabla \cdot \nabla)$.

L'équation de Navier-Stokes présente deux types de difficultés. D'une part elle est non linéaire. Dans la résolution de cette équation, le terme $(u \cdot \nabla) u$ est souvent traité comme un terme de perturbation. D'autre part, en ignorant le terme non linéaire, cette équation ressemble beaucoup à l'équation de la chaleur avec condition de Dirichlet

$$
\begin{cases}\partial_{t} u-\varepsilon \Delta u=0=f, & \left.u\right|_{\partial \Omega}=0 \\ & \left.u\right|_{t=0}=u_{0}\end{cases}
$$

mais en diffère par l'incompressibilité. Prenant modèle sur l'équation de la chaleur, on peut résoudre l'équation de Stokes avec second membre

$$
\begin{cases}\partial_{t} u-\varepsilon \Delta u+\nabla p=f, & \left.u\right|_{\partial \Omega}=0 \\ \nabla \cdot u=0, & \left.u\right|_{t=0}=u_{0}\end{cases}
$$

à l'aide d'un semi-groupe d'opérateur, après avoir défini correctement les espaces de champs de vitesses à divergence nulle et tangent au bord.

Dans le cas de $\mathbb{R}^{n}$, de $\Omega_{b}$ borné, du demi-espace $\mathbb{R}_{+}^{n}$ et enfin d'un domaine extérieur $\Omega_{e}$, c'est-à-dire le complémentaire d'un compact, il a été démontré que le semi-groupe en question est analytique et borné uniformément en temps sur des espaces construits à partir d'espaces de Lebesgue $L^{p}(\Omega)$.

Pour traiter de la perturbation non linéaire, on la considère comme un second membre et on résout l'équation de point fixe qui en résulte par le théorème du point fixe dans les espaces de Banach. Afin d'en vérifier les hypothèses, Fujita et Kato [6] ont introduits les puissances fractionnaires du générateur du semi-groupe considéré sur $L^{2}$ et étudié leurs domaines. D'un autre côté Weissler [32] a considéré le semi-groupe sur les espaces $L^{p}$. Les différentes adaptations aux quatre cas ( $\mathbb{R}^{n}, \Omega_{b}$, $\mathbb{R}_{+}^{n}$ et $\Omega_{e}$ ) de ces techniques de semi-groupe, puissances fractionnaires et espaces de Lebesgue ont été poursuivis, entre autre grâce aux travaux de Giga [9].

Dans le cas de l'équation de Navier-Stokes sur $\mathbb{R}^{n}$, Cannone [4] a étudié l'existence globale de solutions à données petites dans l'espace de Besov homogène $\dot{B}_{p \infty}^{-1+n / p}\left(\mathbb{R}^{n}\right)$. Ce type d'argument semble avoir été poussé au maximum par Kozono et Yamazaki dans [18]. Cette étude est justifiée parce qu'une donnée initiale peut être à la fois petite dans $\dot{B}_{p}^{-1+n / p}\left(\mathbb{R}^{n}\right)$ et grande dans $L^{n}\left(\mathbb{R}^{n}\right)$ à condition d'être suffisamment oscillante. De plus, l'existence de solutions à données dans de tels espaces de Besov est essentielle pour l'étude des solutions autosimilaires de l'équation de Navier-Stokes telle que l'ont menée Cannone et Planchon [27], [5], [28]. Les méthodes utilisées par ces auteurs reposent de manière essentielle sur l'expression explicite du noyau de la chaleur par transformation de Fourier.

Nous exposons dans cet article l'analogue de ces résultats d'existence pour l'équation de Navier-Stokes, mais sur un domaine extérieur $\Omega_{e}$, complémentaire d'un compact à bord lisse. Les deux difficultés nouvelles qui se présentent sont l'absence d'une représentation explicite en Fourier du semi-groupe associé à l'opérateur de Stokes et la nécessité de transposer la notion d'espace de Besov homogène.

La méthode de point fixe utilisée depuis Weissler, reprise par Kato [16] et plus récemment par Cannone et Planchon, présente le défaut de n'assurer l'unicité d'une solution continue à valeur $L^{n}\left(\mathbb{R}^{n}\right)$, pour une donnée initiale dans $L^{n}$ petite dans $\dot{B}_{p \infty}^{-1+n / p}$, que dans une boule d'un sous-espace de l'ensemble $C\left(L^{n}\left(\mathbb{R}^{n}\right)\right.$ ) des fonctions continue de $[0, \infty[$ dans $L^{n}\left(\mathbb{R}^{n}\right)$.

Récemment, Furioli, Lemarie-Rieusset et Terraneo [7], [8] ont obtenu un remarquable résultat d'unicité des solutions locales $C\left(L^{3}\left(\mathbb{R}^{3}\right)\right)$, à l'aide de ces espaces de Besov homogènes. Signalons au passage que Lions et Masmoudi [21] ont annoncé une autre démonstration, encore valable dans le cas d'un domaine $\Omega$ à bord régulier.

Pour le cas de $\Omega_{e}$ domaine extérieur, nous retrouvons le résultat d'unicité des solutions locales $C\left(L^{n}(\Omega)\right)$. Nous preferrons utiliser là les espaces de Besov non homogènes de Kobayashi et Muramutu [17], pour éviter la détérioration de l'estimation du gradient du semi-groupe en temps grand quand $\Omega=\Omega_{e}$ mise en évidence par Maremonti et Solonnikov [22].

Un certain nombre d'auteurs ont utilisé des espaces de Besov non homogènes sur un domaine extérieur. Grubb et Solonnikov [13], [12] ont introduit de tels espaces pour résoudre sur $I \times \Omega_{b}$ ( $I$ intervalle de $\mathbb{R}, \Omega_{b}$ ouvert borné) les équations de Navier-Stokes avec toute une variété de conditions au bord et un second membre. Il s'agit d'espaces de Besov avec des régularités différentes en temps et en espace, ce qui donne des résultats très précis sur les conditions de compatibilité que doivent vérifier les données pour obtenir existence et unicité de solutions régulières. Grubb [11] a récemment adapté cette méthode au cas de $\Omega_{e}$ domaine extérieur, mais en n'utilisant que des espaces non homogènes, son résultat n'est que local en temps. De mème pour Kobayashi et Muramutu [17] qui ont obtenu sur $\Omega_{e}$ un résultat d'existence locale en temps pour une donnée initiale dans un espace de Besov abstrait non homogène construit par interpolation réelle à partir du générateur du semi-groupe. Encore pour $\Omega_{e}$, mentionnons que Borchers et Myakawa [2] avaient utilisé des espaces d'interpolation complexe définis à partir du générateur du semi-groupe (qui sont en quelque sorte l'analogue des espaces de Bessel homogènes) pour obtenir des estimations cœrcives homogènes optimales.

Pour les espaces homogènes, signalons en plus de [4] et [27], que Kozono et Yamazaki dans [19] présentent un résultat d'existence globale à donnée petite dans l'espace de Lorentz $L^{n \infty}\left(\Omega_{e}\right)$ pour un domaine $\Omega$, par la méthode de Kato. Meyer dans [24] donne des résultats
de continuité pour le terme non-linéaire dans $L^{n \infty}\left(\mathbb{R}^{n}\right)$, qui sont faux dans $L^{3}\left(\mathbb{R}^{3}\right)$ d'après Oru [26], et qui permettent tout à la fois de prouver l'existence globale à donnée petite dans $L^{n \infty}\left(\mathbb{R}^{n}\right)$ et de retrouver l'unicité des solutions $C\left(L^{3}\left(\mathbb{R}^{3}\right)\right)$.

Plan de l'article. Dans les préliminaires, après des notations générales nous étudions l'opérateur de Stokes, son semi-groupe associé, puis décrivons les espaces fonctionnels construits avec. La section suivante rassemble les énoncés des résultats importants de l'article : existence à donnée petite, avec un exemple, et unicité. La troisième section est consacrée à la démonstration des théorèmes, après étude de la continuité du terme non linéaire. La dernière section expose en détail l'exemple.

## 1. Préliminaires.

### 1.1. Notations générales.

On note $\left.\mathbb{R}_{+} \dot{=}\right] 0, \infty\left[\right.$ et $\overline{\mathbb{R}}_{+} \doteq\left[0, \infty\left[. C(I ; X)\right.\right.$ et $C_{b}(I ; X)$ désignent respectivement les fonctions continues et continues bornées de $I$ dans $X$.

Notons $\mathcal{L}(X ; Y)$ l'espace de Banach des applications linéaires continues d'un espace de Banach $X$ dans un espace de Banach $Y$, et $\|T\|_{\mathcal{L}(X ; Y)}$ la norme d'un opérateur $T$ élément de cet espace. Si $X=Y$, on écrit seulement $\mathcal{L}(X)$. Si $X$ est inclus dans $Y$ et si l'injection est continue, on écrit $X \hookrightarrow Y$.

Nous considérons un ouvert $\Omega$ de $\mathbb{R}^{n}$ dont le bord $\partial \Omega$ est lisse et dont le complémentaire $K$ est compact. Pour $p \in] 1, \infty[$ et $k \in \mathbb{N}$, on note $W_{p}^{k}(\Omega)$ l'espace de Sobolev des distributions dont les dérivées jusqu'à l'ordre $k$ sont dans $L^{p}(\Omega)$, et $W_{p, 0}^{k}(\Omega)$ l'adhérence dans $W_{p}^{k}(\Omega)$ des fonctions test $C_{0}^{\infty}(\Omega)$. On note $L_{\mathrm{loc}}^{p}(\Omega)$ les distributions sur $\Omega$ dont la restriction à $B \cap \Omega$, pour toute boule $B$ de $\mathbb{R}^{n}$, est dans $L^{p}(B \cap \Omega)$. On note $\nabla^{k} u$ le gradient itéré $k$ fois d'une distribution, c'est-à-dire la collection des $\partial_{x}^{\alpha} u=\partial_{x_{\alpha_{1}}} \cdots \partial_{x_{\alpha_{k}}} u$ pour $\alpha$ décrivant l'ensemble des applications de $\{1, \ldots, k\}$ dans $\{1, \ldots, n\}$, de sorte que, par exemple, $\sum_{r=0}^{k}\left\|\nabla^{r} u\right\|_{p}$ est une norme équivalente sur $W_{p}^{k}(\Omega)$.

Nous aurons aussi besoin quelquefois :

- des espaces de Lorentz $L^{p q}(\Omega)$ pour $q \in[1, \infty]$, obtenus par interpolation réelle à partir $\operatorname{des} L^{p}(\Omega)$;
- des espaces de Besov $B_{p q}^{s}(\Omega)$ pour $s \in \mathbb{R}_{+} \backslash \mathbb{N}$ et $q \in[1, \infty]$, obtenus par interpolation réelle à partir des $W_{p}^{k}(\Omega)$;
- des espaces de Sobolev sur le bord $W_{p}^{k}(\partial \Omega), k \in \mathbb{N}$ et même des espaces de Slobodetskii $W_{p}^{s}(\partial \Omega)$ (dits aussi espaces de traces) pour $s \in \mathbb{R}_{+} \backslash \mathbb{N}$ obtenus par interpolation réelle à partir des précédents.

Il est possible d'étendre ces définitions aux indices $k \in \mathbb{Z}$ et $s \in \mathbb{R}$ (voir [30]).

On note $\nu$ le vecteur normal (unitaire rentrant) au bord de $\Omega$ et $\partial_{\nu}$ l'opérateur différentiel associé. On note $\gamma_{0} u$ la restriction à $\partial \Omega$ d'une fonction continue sur $\bar{\Omega}$ et $\gamma_{1} u \doteq \gamma_{0} \partial_{\nu} u$ si $u$ est continûment dérivable. On sait étendre l'action de ces opérateurs à certains espaces de distributions. Par exemple $\gamma_{0}$ est continu de $W_{p}^{1}(\Omega)$ dans $W_{p}^{1-1 / p}(\partial \Omega)$, mais n'est pas continu sur $L^{p}(\Omega)$. Pour une distribution $u$ sur $\partial \Omega$ à valeur vectorielle, on définit l'opérateur de projection orthogonale $\pi_{\nu} u=(u \cdot \nu)$ sur $\nu$. On note $\gamma_{\nu} \doteq \pi_{\nu} \gamma_{0}$ l'opérateur de trace normale au bord.

On note $X_{p}$ (respectivement $X_{p q}$ ) l'adhérence dans $L^{p}(\Omega)$ (respectivement $\left.L^{p q}(\Omega)\right)$ des champs de vecteurs $C_{0}^{\infty}(\Omega)$ à divergence nulle. Il est bien connu que $X_{p}$ coïncide avec le sous-espace fermé des $u \in L^{p}(\Omega)$ tels que $\nabla \cdot u=0$ et $\gamma_{\nu} u=0$. Ici $\nabla$. désigne l'opérateur différentiel de divergence. La nullité de la divergence permet d'étendre l'opérateur de trace normale $\gamma_{\nu}$, au moyen d'une intégration par partie, en un opérateur continu du sous-espace fermé de $L^{p}(\Omega)$ d'équation $\nabla \cdot u=0$ dans $W_{p}^{-1 / p}(\partial \Omega)$. On sait aussi qu'on peut décomposer en somme directe l'espace de Banach

$$
\begin{equation*}
L^{p}(\Omega)=X_{p} \oplus\left\{\nabla p \in L^{p}(\Omega): p \in L_{\mathrm{loc}}^{p}(\Omega)\right\} \tag{1.1}
\end{equation*}
$$

et qu'il existe un opérateur linéaire $P$, continu pour tout $1<p<\infty$ sur les champs de vecteurs $L^{p}(\Omega)$, qui est une projection (en particulier $P^{2}=P$ ) sur $P L^{p}(\Omega)=X_{p}$ parallèlement aux champs gradients, orthogonale pour la structure euclidienne de $L^{2}(\Omega)$. Pour le cas relativement général qui nous intéresse, à savoir $1<p<\infty$ et $\Omega$ non borné, mais cependant $\partial \Omega$ compact, la preuve de ces trois affirmations est présentée en détail dans [25].

On note $p^{\prime}$ l'exposant conjugué de $p$ défini par $1 / p+1 / p^{\prime}=1$. On peut identifier $X_{p^{\prime}}$ au dual de $X_{p}$, qui est donc réflexif. L'adjoint du projecteur $P: L^{p}(\Omega) \longrightarrow L^{p}(\Omega)$ est le projecteur $P: L^{p^{\prime}}(\Omega) \longrightarrow$ $L^{p^{\prime}}(\Omega)$, tandis que si on considère $P: L^{p}(\Omega) \longrightarrow X_{p}$, son adjoint est simplement l'injection canonique $I: X_{p^{\prime}} \longrightarrow L^{p^{\prime}}(\Omega)$.

On note $A_{p}$ l'opérateur dans $X_{p}$, de domaine

$$
\begin{equation*}
\mathcal{D}\left(A_{p}\right)=W_{p}^{2} \cap W_{p, 0}^{1} \cap X_{p} \tag{1.2}
\end{equation*}
$$

et agissant comme l'opérateur de Stokes $A=-P \Delta$. On remplace alors la résolution de l'équation linéaire de Stokes pour la viscosité $\varepsilon=1$

$$
\left\{\begin{array}{l}
\partial_{t} u-\Delta u+\nabla p=0, \\
\nabla \cdot u=0,
\end{array} \quad \gamma_{0} u=u_{\mid \partial \Omega}=0,\right.
$$

par l'équation différentielle abstraite écrite dans $X_{p}$ qu'on obtient en appliquant le projecteur $P$ à l'équation de Stokes

$$
\partial_{t} u+A u=0 .
$$

Pour la commodité du lecteur, nous avons rassemblé dans le tableau 1 tous les autres espaces de fonctions définis dans la suite de l'article.

| Espace | Norme | Référence | Remarque |
| :--- | :--- | :--- | :--- |
| $\mathcal{A}$ |  | Déf. 16 |  |
| $B_{p q}^{s}(A)$ | $\left\\|u \mid B_{p q}^{s}\right\\|$ | Déf.13, (1.11) |  |
| $\mathcal{B}_{r}^{q}$ | $\left\\|u \mid \mathcal{B}_{r}^{q}\right\\|$ | Déf. 32 | $=B_{r \infty}^{-n / q+n / r}(A)$ |
| $\mathcal{B}_{r}$ | $\left\\|u \mid \mathcal{B}_{r}\right\\|$ | Déf. 32 | $=\mathcal{B}_{r}^{n}$ |
| $\dot{B}_{p q}^{s}(A)$ | $\left\\|u \mid \dot{B}_{p q}^{s}\right\\|$ | Déf. 16, (1.14) |  |
| $\dot{\mathcal{B}}_{r}$ | $\left\\|u \mid \dot{\mathcal{B}}_{r}\right\\|$ | Déf. 27 | $=\dot{B}_{r \infty}^{-1+n / r}(A)$ |
| $E_{p}$ | $\\|u\\|_{E_{p}}$ | Déf. 27 |  |
| $E_{p ; 0}(0), E_{p ; 0}(\infty)$ | Déf. 27 |  |  |
| $F_{p}^{T}$ | $\\|u\\|_{F_{p}^{T}}$ | Déf. 32 |  |
| $G_{p}^{T}$ | $\\|u\\|_{G_{p}^{T}}$ | Déf. 32 | $=L_{T}^{\infty}\left(\mathcal{B}_{p}\right)$ |
| $\mathcal{H}$ | $\\|u\\|_{\mathcal{H}}$ | (4.4) |  |
| $L_{T}^{\infty}(X)$ |  | Déf. 32 |  |
| $L^{p *}$ | $\\|f\\|_{p *}$ | Not. 11 |  |
| $L^{p * 0}$ | $\\|f\\|_{p * 0}$, | Not. 11 |  |
| $L^{p * \infty}$ | $\\|f\\|_{p * \infty}$ | Not. 11 |  |
| $W_{p}^{0}(A)$ | $\\|u\\|_{p}$ | Déf. 12 | $=X_{p}$ |
| $W_{p}^{2}(A)$ | $\left\\|u \mid W_{p}^{2}\right\\|$ | Déf. 12 | $=\mathcal{D}\left(A_{p}\right)$ |
| $W_{p}^{-2}(A)$ | $\left\\|u \mid W_{p}^{-2}\right\\|$ | Déf. 12 |  |

Tableau 1. Liste des espaces fonctionnels.

### 1.2. Opérateur de Stokes.

### 1.2.1. Décomposition de la résolvante.

Nous décrivons ici comment on peut trouver $u \in X_{p}$ solution du problème de Stokes $(\lambda+A) u=f$. Précisément : étant donné $\lambda \in$ $\mathbb{C} \backslash]-\infty, 0]$ et $f \in X_{p}$, on cherche $u$ tel que

$$
(\lambda-\Delta) u+\nabla p=f, \quad \nabla \cdot u=0, \quad \gamma_{0} u=0
$$

Nous suivons les exposés de [9], [25].
Soit $f \in X_{p}$. On note $\widetilde{f} \in L^{p}\left(\mathbb{R}^{n}\right)$ son prolongement par zéro hors de $\Omega$. On voit facilement que la distribution $\nabla \cdot \tilde{f}$ est nulle parce que $\nabla \cdot f$ et $\gamma_{\nu} f$ sont nulles. On note $\widetilde{u}$ la solution du problème de Stokes dans $\mathbb{R}^{n}$

$$
(\lambda-\Delta) \widetilde{u}+\nabla \widetilde{p}=\widetilde{f}, \quad \nabla \cdot \widetilde{u}=0 .
$$

Puisque $\nabla \cdot \tilde{f}=0$, en fait $\widetilde{p}=0$ et on peut calculer $\widetilde{u}=\widetilde{E}_{\lambda} \widetilde{f}$ au moyen de l'opérateur $\widetilde{E}_{\lambda}$ de convolution par le potentiel volume $\widetilde{e}_{\lambda}$ sur $\mathbb{R}^{n}$ qui est la transformée de Fourier inverse de $\xi \longmapsto\left(\lambda+|\xi|^{2}\right)^{-1}$. Par le théorème de Mihlin, $\widetilde{E}_{\lambda}$ est continu de $L^{p}\left(\mathbb{R}^{n}\right)$ dans $W_{p}^{2}\left(\mathbb{R}^{n}\right)$ pour chaque $1<p<\infty$.

La restriction $\mathrm{r}_{\Omega} \widetilde{u}$ vérifie l'équation intérieure de Stokes sur $\Omega$

$$
(\lambda-\Delta) \mathrm{r}_{\Omega} \widetilde{u}=f, \quad \nabla \cdot \mathrm{r}_{\Omega} \widetilde{u}=0
$$

puisque $\mathrm{r}_{\Omega} \widetilde{p}=0$, mais pas la condition au bord. On élimine la composante normale $\gamma_{\nu}=\pi_{\nu} \gamma_{0}=\nu \cdot \gamma_{0}$ au moyen du projecteur de Leray $P$. On sait que pour $v$ dans $L^{p}(\Omega)$ à divergence nulle, la projection $P v$ s'exprime au moyen de l'opérateur solution du problème de Neumann. Fixons les notations. Pour $\phi \in W_{p}^{-1 / p}(\partial \Omega)$ avec $\int_{\partial \Omega} \phi=0$, on note $w=N \phi$ la solution du problème

$$
\left\{\begin{array}{l}
\Delta w=0, \quad \nabla w \in L^{p}(\Omega) \\
\gamma_{1} w=\phi, \quad \lim _{|x| \rightarrow \infty}|w(x)|=0,
\end{array}\right.
$$

obtenue au moyen du potentiel simple couche associé à l'opérateur de Laplace $\Delta$. Voir [25] et, en annexe, la définition du potentiel simple couche et son application au problème de Neumann à la Section B.3. Remarquons que $\nabla \cdot \nabla=\Delta$ et $\gamma_{\nu} \nabla=\gamma_{1}$. L'opérateur $\nabla N$ est continu
du sous-espace fermé de $W_{p}^{-1 / p}(\partial \Omega)$ d'équation $\int_{\partial \Omega} \phi=0$ dans $L^{p}(\Omega)$. La restriction $P_{\nu}$ de $P$ aux champs à divergence nulle s'écrit $P_{\nu}=$ $1-\nabla N \gamma_{\nu}$.

Ainsi $u_{\nu}=P \mathrm{r}_{\Omega} \widetilde{u}$ est solution de

$$
\left\{\begin{array}{l}
(\lambda-\Delta) u_{\nu}+\nabla p_{\nu}=f, \quad \nabla \cdot u_{\nu}=0 \\
\gamma_{\nu} u_{\nu}=0
\end{array}\right.
$$

avec $p_{\nu}=\lambda N \gamma_{\nu} \mathrm{r}_{\Omega} \widetilde{u}$. Évidemment la trace $\gamma_{0} u_{\nu}$ n'est pas nécessairement nulle, mais elle est tangentielle.

Soit $u_{\tau}=V_{\lambda} \psi$ la solution du problème tangentiel

$$
\begin{cases}(\lambda-\Delta) u_{\tau}+\nabla p_{\tau}=0, & \nabla \cdot u_{\tau}=0  \tag{1.3}\\ \gamma_{\nu} u_{\tau}=0, & \gamma_{0} u_{\tau}=\psi\end{cases}
$$

avec $\psi$ un champ de vecteurs tangentiel au dessus de $\partial \Omega$. Le fait qu'on puisse définir $\gamma_{0} u_{\tau}$ dans $W_{p}^{-1 / p}(\partial \Omega)$ pour un $u$ dans $L^{p}(\Omega)$ qui vérifie les trois autres équations de (1.3) est justifié par un argument de dualité dans [9]. D'après [9, Proposition 2.2] et le théorème du graphe fermé, $V_{\lambda}$ est continu de $W_{p}^{-1 / p}(\partial \Omega)$ dans $L^{p}(\Omega)$.

Si on choisit $\psi=-\gamma_{0} u_{\nu}$, on vérifie enfin que $u=G_{\lambda} f=u_{\nu}+u_{\tau}$ est la solution du problème de Stokes

$$
\left\{\begin{array}{l}
(\lambda-\Delta) u+\nabla p=f, \quad \nabla \cdot u=0 \\
\gamma_{0} u=0
\end{array}\right.
$$

Remarque 1. Les résultats de continuité rappelés ci-dessus sont vrais pour chaque $\lambda \in \mathbb{C} \backslash]-\infty, 0]$ fixé. Ils découlent des propriétés des systèmes elliptiques aux limites. L'amélioration principale due à [9] consiste en des estimations uniformes par rapport à $\lambda$.

### 1.2.2. Résolvante et semi-groupe.

Dans toute la suite, $\mathcal{V}_{\varepsilon}$ désigne, pour un $\varepsilon$ donné dans $[0, \pi]$, l'ensemble

$$
\mathcal{V}_{\varepsilon}=\left\{\lambda \in \mathbb{C} \backslash \overline{\mathbb{R}}_{-}:|\arg \lambda|<\varepsilon\right\}
$$

Voici un résultat important dans l'étude du comportement en temps long des solutions de l'équation de Stokes.

Théorème 3. Pour tout $p \in] 1, \infty\left[\right.$, l'opérateur $-A_{p}$ est $\pi$-sectoriel : étant donné $\left.\varepsilon^{\prime} \in\right] 0, \pi[$, il existe $C$ tel que

$$
\begin{equation*}
\left\|(\lambda+A)^{-1} v\right\|_{p} \leq C|\lambda|^{-1}\|v\|_{p}, \quad \text { pour tous } \lambda \in \mathcal{V}_{\varepsilon^{\prime}}, v \in X_{p} . \tag{1.4}
\end{equation*}
$$

Ce théorème permet de construire des solutions à l'équation de Stokes au moyen du semi-groupe engendré par $-A$.

Corollaire 4 ([15, p. 487-492]). Pour tout $p \in] 1, \infty[$, l'intégrale de Dunford

$$
U(t) \doteq \frac{1}{2 i \pi} \int_{\Gamma} e^{t \lambda}(\lambda+A)^{-1} d \lambda
$$

où $\Gamma$ est un chemin dans $\mathbb{C}$ contournant $]-\infty, 0]$ dans le sens positif, $d e \infty \cdot e^{-i \theta} \grave{a} \infty \cdot e^{i \theta}$ pour $\left.\theta \in\right] \pi / 2, \pi[$, définit une application $t \longmapsto U(t)$ de $\mathcal{V}_{\pi} \cup\{0\}$ dans $\mathcal{L}\left(X_{p}\right)$, indépendante de $\theta$, vérifiant sur cet ensemble la règle de composition $U(t) U(s)=U(t+s)$, holomorphe sur $\mathcal{V}_{\pi}$, et telle que pour tout $\varepsilon^{\prime}<\pi$, il existe une constante $C$ avec

$$
\begin{gathered}
\|U(t) x\|_{p} \leq C\|x\|_{p}, \quad \text { pour tous } t \in \mathcal{V}_{\varepsilon^{\prime}} \cup\{0\}, x \in X, \\
\lim _{\substack{t \rightarrow 0 \\
t \in \mathcal{V}_{s^{\prime}}}}\|U(t) x-x\|_{p}=0, \quad \text { pour tout } x \in X .
\end{gathered}
$$

Pour tout $k \in \mathbb{N}$, l'application $t \longmapsto t^{k} A^{k} U(t)$ est continue bornée de $\mathbb{R}_{+}$dans $\mathcal{L}(X)$ et $\partial_{t} U(t)=-A U(t)$.

Rappelons que le domaine $\mathcal{D}\left(A^{\prime}\right)$ du dual d'un opérateur $A$ non borné dans $X$ de domaine $\mathcal{D}(A)$ dense dans $X$ est l'ensemble des $x^{\prime} \in X^{\prime}$ tel qu'il existe $y^{\prime} \in X^{\prime}$ vérifiant pour tout $x \in \mathcal{D}(A)$ l'égalité $\left\langle x^{\prime}, A x\right\rangle=$ $\left\langle y^{\prime}, x\right\rangle$.

Théorème 5 ([25]). On a les identifications d'espaces et d'opérateurs duaux

$$
\begin{equation*}
\mathcal{D}\left(A_{p}{ }^{\prime}\right)=\mathcal{D}\left(A_{p^{\prime}}\right), \quad A_{p}{ }^{\prime}=A_{p^{\prime}} \tag{1.5}
\end{equation*}
$$

Proposition 6. Soit $p$ et $q$ dans $] 1, \infty[$ tels que $0 \leq 1 / p-1 / q \leq 1 / n$. Alors l'application $t \longmapsto(1+A)^{-1} U(t) P \nabla$. est continue bornée de $\mathbb{R}_{+}$ dans $\mathcal{L}\left(L^{p}(\Omega) ; X_{q}\right)$.

DÉmonstration. On sait déjà que $(1+A)^{-1}$ et $U(t)$ commutent et que $U(t) \in C_{b}\left(\mathbb{R}_{+} ; \mathcal{L}\left(X_{q}\right)\right)$. Montrons que $(1+A)^{-1} P \nabla \cdot \in \mathcal{L}\left(L^{p}(\Omega) ; X_{q}\right)$. Par définition de $\mathcal{D}\left(A_{q^{\prime}}\right)$ et injection de Sobolev de $W_{q^{\prime}}^{1}(\Omega)$ dans $L^{p^{\prime}}(\Omega)$ pour $0 \leq 1 / q^{\prime}-1 / p^{\prime} \leq 1 / n$, on obtient $\nabla(1+A)^{-1} \in \mathcal{L}\left(X_{q^{\prime}} ; L^{p^{\prime}}(\Omega)\right)$. Or $\nabla(1+A)^{-1}=\nabla I(1+A)^{-1}$ est bien l'opérateur dual de $(1+A)^{-1} P \nabla \cdot$.

Pour le semi-groupe engendré par l'opérateur de Stokes, on dispose d'estimations $L^{p}-L^{q}$.

Définition 7. On note

$$
[p, q]=\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right) .
$$

Théorème 8 (Estimations $\left.L^{p}-L^{q}[2],[14],[22]\right)$.

1) Étant donné $1<p \leq q<\infty$, il existe $C$ tel que

$$
\begin{equation*}
\|U(t) v\|_{q} \leq C\|v\|_{p} t^{-[p, q]}, \quad \text { pour tout } v \in X_{p} . \tag{1.6}
\end{equation*}
$$

2) Étant donné $1<p \leq q \leq n$, il existe $C$ tel que

$$
\begin{equation*}
\|\nabla U(t) v\|_{q} \leq C\|v\|_{p} t^{-1 / 2-[p, q]}, \quad \text { pour tout } v \in X_{p} . \tag{1.7}
\end{equation*}
$$

3) Étant donné $1<p \leq q<\infty$ avec $n \leq q$, il existe $C$ tel que

$$
\begin{equation*}
\|\nabla U(t) v\|_{q} \leq C\|v\|_{p} t^{-1 / 2-[p, q]}(1+t)^{[n, q]}, \quad \text { pour tout } v \in X_{p} . \tag{1.8}
\end{equation*}
$$

Remarque 9 . Dans le cas de $\mathbb{R}^{n}$, (1.7) est valable pour tout $p, q$ dans $] 1, \infty[$ avec $p \leq q$. [16] en déduit un théorème d'existence globale en temps d'une solution à l'équation de Navier-Stokes dans $C\left(L^{n}\left(\mathbb{R}^{n}\right)\right)$ quand la donnée initiale est suffisamment petite dans $L^{n}$.

Pour le domaine extérieur $\Omega$, le cas limite $n=p$ est atteint d'après [14]. Cela lui permet d'étendre le résultat de [16] à ce cas. [22] montre que (1.8) est optimale. Notons que pour $t$ grand, l'exposant est

$$
-\frac{1}{2}-[p, q]+[n, q]=-\frac{n}{2 p} .
$$

Corollaire 10. Soit $p$ et $q$ dans $] 1, \infty[$ tels que $p \leq q$, et $k$ dans $\mathbb{N}$.

1) Si $n^{\prime} \leq p$, alors l'application $t \longmapsto t^{k+1 / 2+[p, q]} A^{k} U(t) P \nabla \cdot$ est continue bornée de $\mathbb{R}_{+}$dans $\mathcal{L}\left(L^{p}(\Omega) ; X_{q}\right)$.
2) Pour tout $T>0$, l'application $t \longmapsto t^{k+1 / 2+[p, q]} A^{k} U(t) P \nabla \cdot$ est continue bornée de $] 0, T\left[\right.$ dans $\mathcal{L}\left(L^{p}(\Omega) ; X_{q}\right)$.

Démonstration. Considérons d'abord le premier point, cas $k=0$. On procède par dualité comme pour la Proposition 6. L'opérateur dual de $\nabla U(t)=\nabla I U(t)$ est $U(t) P \nabla \cdot$. La borne dans $\mathcal{L}\left(L^{p}(\Omega) ; X_{q}\right)$ vient de (1.7). Pour $k>0$, on écrit $A^{k} U(t) P \nabla \cdot=\left(A^{k} U(t / 2)\right)(U(t / 2) P \nabla \cdot)$, puis on applique le cas $k=0$ à $U(t / 2) P \nabla \cdot$ et le Corollaire 4 à $A^{k} U(t / 2)$ Le second point est très semblable : on utilise (1.8) et la borne sur $t$. La continuité vient du Corollaire 4.

### 1.3. Description des espaces fonctionnels.

Notation 11. On munit $\mathbb{R}_{+}$de la mesure de Haar $d t / t$ associée à la structure de groupe multiplicatif. Alors on note $L^{p *}$ l'espace des fonctions de puissance $p$-ième intégrable, avec la norme

$$
\|f\|_{p *} \doteq\left(\int_{0}^{\infty}|f(\tau)|^{p} \frac{d \tau}{\tau}\right)^{1 / p}
$$

pour $1 \leq p<\infty$. Pour $p=\infty$, l'extension usuelle de la définition coïncide avec $L^{\infty}\left(\mathbb{R}_{+}\right)$. Si on considère seulement l'intervalle $\left.] 0,1\right]$ (respectivement $\left[1, \infty\left[\right.\right.$ ), on notera respectivement $L^{p * 0}$ et $\|f\|_{p * 0}$ (respectivement $L^{p * \infty}$ et $\left.\|f\|_{p * \infty}\right)$.

Si $f$ est à valeur dans un espace de Banach $(X,\|\cdot\|)$, on se place dans le cadre de la théorie de l'intégrale de Bochner en supposant que $f$ est fortement mesurable. D'après le théorème de Pettis, il suffit que $f$ soit faiblement mesurable (i.e. pour tout $f^{\prime} \in X^{\prime}, s \longmapsto\left\langle f^{\prime}, f(s)\right\rangle$ est mesurable) et presque partout à valeur séparable (i.e. il existe un ensemble $S$ de mesure nulle telle que $f\left(\mathbb{R}_{+} \backslash S\right)$ soit séparable).

On rappelle en Annexe A la définition et les propriétés principales de l'interpollation réelle $\left(A_{0}, A_{1}\right)_{\theta, \infty}$ entre deux espaces de Banach $A_{0}$ et $A_{1}$.

### 1.3.1. Extrapolation.

Définition 12. Notons $W_{p}^{0}(A) \doteq X_{p}$, et $W_{p}^{2}(A) \doteq \mathcal{D}\left(A_{p}\right)$ le domaine de l'opérateur de Stokes défini par (1.2), muni de la norme $\left\|u \mid W_{p}^{2}\right\| \doteq$ $\|(1+A) u\|_{p}$. Comme dans [17], on définit aussi des espaces d'indice négatif: $W_{p}^{-2}(A)$ est l'espace complété à partir de $X_{p}$ pour la norme $\left\|u \mid W_{p}^{-2}\right\| \stackrel{\stackrel{\rho}{=}}{=}\left\|(1+A)^{-1} u\right\|_{p}$.

Puisque le graphe $\mathcal{G}_{A_{p}}$ est fermé et que le domaine $\mathcal{D}\left(A_{p}\right)$ est dense dans $X_{p}$, cet espace coïncide avec la construction plus abstraite par quotient : $\left(X_{p} \times X_{p}\right) / \mathcal{G}_{A_{p}}$, introduite dans [10]. Ceci est expliqué en détail dans [31]. On y apprend aussi qu'en lien avec la dualité (1.5), cet espace est le dual de $\mathcal{D}\left(A_{p^{\prime}}\right)$.

On peut étendre $A$ en un opérateur non borné dans $W_{p}^{-2}(A)$, fermé de domaine $W_{p}^{0}(A)$ dense. Il hérite des propriétés spectrales de l'opérateur $A$ dans $X_{p}$. On peut aussi étendre à $W_{p}^{-2}(A)$ tout opérateur pris dans $\mathcal{L}\left(X_{p}\right)$ qui commute avec la résolvante $(1+A)^{-1}$. En particulier $U(t)$ admet une telle extension, qui coïncide avec le semi-groupe engendré par l'extension de $A$. Le domaine du carré de l'extension de $A$ est $W_{p}^{2}(A)$. Pour tout $t>0$, l'opérateur $U(t)$ est continu de $W_{p}^{-2}(A)$ dans $W_{p}^{2}(A)(c f .[17])$.

### 1.3.2. Espaces de Besov.

Comme $W_{p}^{2}(A) \hookrightarrow W_{p}^{-2}(A)$, on peut leur appliquer l'interpolation réelle.

Définition 13. Pour $s=-2(1-\theta)+2 \theta$, avec $0<\theta<1$ et donc $|s|<2$, pour $p \in] 1, \infty[$ et $q \in[1, \infty]$

$$
B_{p q}^{s}(A) \doteq\left(W_{p}^{-2}(A), W_{p}^{2}(A)\right)_{\theta, q}
$$

Il est bien connu (voir [20]) que

$$
\begin{equation*}
W_{p}^{2}(A) \hookrightarrow B_{p q}^{s}(A) \hookrightarrow W_{p}^{-2}(A) . \tag{1.9}
\end{equation*}
$$

De plus on peut montrer (voir [10]) que $B_{p}^{0}(A) \hookrightarrow W_{p}^{0}(A) \hookrightarrow B_{p \infty}^{0}(A)$. Du théorème de réitération, il vient alors $B_{p q}^{s}(A)=\left(W_{p}^{0}(A), W_{p}^{2}(A)\right)_{\theta, q}$ pour $s=2 \theta \in] 0,2[$.
[10] donne une expression de la norme dans ces espaces à l'aide des puissances de la résolvante $(\lambda+A)^{-n}$. On trouve dans [17] la définition équivalente suivante, valable pour $|s|<2$

$$
B_{p q}^{s}(A)=\left\{u \in W_{p}^{-2}(A):\left\|\lambda^{s / 2}\right\| \lambda A(\lambda+A)^{-2} u\left\|_{p}\right\|_{q * \infty}<\infty\right\}
$$

muni de la norme

$$
\begin{equation*}
\left\|u \mid W_{p}^{-2}\right\|+\left\|\lambda^{s / 2}\right\| \lambda A(\lambda+A)^{-2} u\left\|_{p}\right\|_{q * \infty} . \tag{1.10}
\end{equation*}
$$

Le lien avec les solutions de l'équation d'évolution est plus sensible quand on exprime ces normes avec le semi-groupe engendré par l'opérateur.

Lemme 14. Pour $|s|<2, B_{p q}^{s}(A)$ est l'ensemble des éléments de $W_{p}^{-2}(A)$ tels que la quantité

$$
\begin{equation*}
\left\|u \mid B_{p q}^{s}\right\| \doteq\|U(1) u\|_{p}+\left\|t^{-s / 2}\right\| t A U(t) u\left\|_{p}\right\|_{q * 0} \tag{1.11}
\end{equation*}
$$

soit finie. Cette quantité définit alors une norme équivalente.
Démonstration. Notons provisoirement $\|u\|_{X}$ la quantité définie par (1.11), et $X$ l'ensemble des éléments de $W_{p}^{-2}(A)$ qui vérifient $\|u\|_{X}<$ $\infty$.

Montrons d'abord $B_{p q}^{s}(A) \hookrightarrow X$. Soit $u \in B_{p q}^{s}(A)$. On peut écrire (voir Proposition 48) pour $\tau \in \mathbb{R}_{+}$,

$$
\begin{equation*}
u=u_{-}(\tau)+u_{+}(\tau) \tag{1.12}
\end{equation*}
$$

avec, quand on a fixé $s_{-}$et $s_{+}$deux réels tels que

$$
\begin{equation*}
s_{-} s_{+}<0, \quad(1-\theta) s_{-}+\theta s_{+}=0 \tag{1.13}
\end{equation*}
$$

comme définition d'une norme équivalente

$$
\left\|u \mid B_{p q}^{s}\right\|=\inf \left(\left\|\tau^{s_{-}}\right\| u_{-}(\tau)\left|W_{p}^{-2}\| \|_{q^{*}}+\left\|\tau^{s_{+}}\right\| u_{+}(\tau)\right| W_{p}^{2}\| \|_{q^{*}}\right)
$$

où la borne inférieure est prise sur toutes les décompositions (1.12). Or $U(1) \in \mathcal{L}\left(W_{p}^{-2}(A) ; W_{p}^{0}(A)\right)$ donc (1.9) implique $\|U(1) u\|_{p} \leq C \| u \mid$ $B_{p q}^{s} \|$. On calcule pour $t \leq 1$

$$
\begin{aligned}
\|t A U(t) u\|_{p} \leq & \left\|t A U(t) u_{-}(t)\right\|_{p}+\left\|t A U(t) u_{+}(t)\right\|_{p} \\
\leq & \left\|t A(1+A) U(t)(1+A)^{-1} u_{-}(t)\right\|_{p} \\
& +\left\|t A(1+A)^{-1} U(t)(1+A) u_{+}(t)\right\|_{p} \\
\leq & C t^{-1}\left\|u_{-}(t)\left|W_{p}^{-2}\|+t\| u_{+}(t)\right| W_{p}^{2}\right\|
\end{aligned}
$$

parce que $t A U(t)$ et $t^{2} A^{2} U(t)$ sont bornés sur $X_{p}$ (et $t$ est borné), de même que $A(1+A)^{-1}$ et $U(t)$. En multipliant par $t^{-s / 2}$, on voit que

$$
\begin{aligned}
& \left\|t^{-s / 2}\right\| t A U(t) u\left\|_{p}\right\|_{q * 0} \\
& \quad \leq C\left(\left\|t^{s_{-}}\right\| u_{-}(t)\left|W_{p}^{-2}\| \|_{q^{*}}+\left\|t^{s_{+}}\right\| u_{+}(t)\right| W_{p}^{2}\| \|_{q^{*}}\right)
\end{aligned}
$$

en choisissant $s_{-}=-1-s / 2, s_{+}=1-s / 2$, qui vérifient les conditions (1.13) d'après la liaison entre $s$ et $\theta$ et $-2<s<2$. En passant à la borne inférieure, ceci montre bien que $\|u\|_{X} \leq C\left\|u \mid B_{p q}^{s}\right\|$.

Montrons ensuite $X \hookrightarrow B_{p q}^{s}(A)$. Puisque $X \subset W_{p}^{-2}(A)$, on peut écrire, dans cet espace, en vertu des propriétés analytiques du semigroupe,

$$
\begin{aligned}
u & =\int_{0}^{2} \tau A^{2} U(\tau) u d \tau+(1+2 A) U(2) u \\
& =\int_{0}^{2} \tau^{2} A^{2} U(\tau) u \frac{d \tau}{\tau}+c_{a} \int_{2}^{\infty} \tau^{-a}(1+2 A) U(2) u \frac{d \tau}{\tau}
\end{aligned}
$$

pour n'importe quel réel $a$ positif. Ainsi donc on a défini une fonction $u(\tau)$ telle que dans $W_{p}^{-2}(A)$ l'intégrale $\int_{0}^{\infty} u(\tau) / \tau d \tau$ converge, et vaut $u$. Pour $\tau \geq 2$, on majore, grâce à l'effet régularisant de $U(1)$,

$$
\begin{aligned}
\left\|u(\tau) \mid W_{p}^{-2}\right\| & \leq \tau^{-a}\|U(1) u\|_{p} \\
\left\|u(\tau) \mid W_{p}^{2}\right\| & \leq \tau^{-a}\|U(1) u\|_{p}
\end{aligned}
$$

Pour $\tau \leq 2$, on calcule

$$
\begin{aligned}
\left\|u(\tau) \mid W_{p}^{-2}\right\| & =\left\|\tau^{2} A^{2}(1+A)^{-1} U(\tau) u\right\|_{p} \leq C \tau\left\|\frac{\tau}{2} A U\left(\frac{\tau}{2}\right) u\right\|_{p} \\
\left\|u(\tau) \mid W_{p}^{2}\right\| & =\left\|\tau^{2} A^{2}(1+A) U(\tau) u\right\|_{p} \leq C \tau^{-1}\left\|\frac{\tau}{2} A U\left(\frac{\tau}{2}\right) u\right\|_{p}
\end{aligned}
$$

parce que $2 A(1+A)^{-1} U(\tau / 2)$ est borné sur $X_{p}$, de même que $\tau^{2} A^{2} U(\tau / 2)$ et $\tau^{3} A^{3} U(\tau / 2)$ (et $\tau$ est borné). Prenant par exemple $a=2$, on en déduit que

$$
\left\|\tau^{s_{-}}\right\| u(\tau)\left|W_{p}^{-2}\| \|_{q *}+\left\|\tau^{s_{+}}\right\| u(\tau)\right| W_{p}^{2}\| \|_{q *} \leq C\|u\|_{X}
$$

où $s_{-}$et $s_{+}$sont encore $-1-s / 2$ et $1-s / 2$, donc respectivement dans $]-2,0[$ et $] 0,2\left[\right.$. On a utilisé $s_{+}-a\left(\right.$ et $\left.s_{-}-a\right)$ négatif pour la
convergence des intégrales en $\tau \rightarrow \infty$. Comme les conditions (1.13) sont vérifiées, ceci montre bien que $\left\|u \mid B_{p q}^{s}\right\| \leq C\|u\|_{X}$ (voir Proposition 48).

Remarque 15. Rappelons la définition des espaces de Besov usuels sur $\mathbb{R}^{n}$ au moyen d'une décomposition dyadique spectrale (voir par exemple [30]). On se donne $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$, nulle hors de la boule de rayon 2 et égale à 1 sur la boule de rayon 1 . On pose $\varphi(\xi)=\chi\left(2^{-1} \xi\right)-\chi(\xi)$. Puis on note $\varphi_{-1}=\chi$ et, pour $j \in \mathbb{N}, \chi_{j}(\xi)=\chi\left(2^{-j} \xi\right)$ et $\varphi_{j}(\xi)=\varphi\left(2^{-j} \xi\right)$, de sorte que pour tout $k \in \mathbb{N}$

$$
1=\sum_{j \geq-1} \varphi_{j}(\xi)=\chi_{k}(\xi)+\sum_{j \geq k} \varphi_{j}(\xi), \quad \chi_{j}(\xi)=\sum_{k=-1}^{j-1} \varphi_{k}(\xi)
$$

On définit ensuite les opérateurs dyadiques $\Delta_{j}$ et $S_{j}$ par

$$
\widehat{\Delta_{j} u}(\xi)=\varphi_{j}(\xi) \widehat{u}(\xi), \quad \widehat{S_{j} u}(\xi)=\chi_{j}(\xi) \widehat{u}(\xi)
$$

où $\widehat{u}$ désigne la transformée de Fourier de $u$.
L'espace de Besov $B_{p q}^{s}\left(\mathbb{R}^{n}\right)$ est alors l'ensemble des distributions tempérées $u$ telles que

$$
\left\|S_{0} u\right\|_{p}+\left(\sum_{j=0}^{\infty} 2^{s j q}\left\|\Delta_{j} u\right\|_{p}^{q}\right)^{1 / q}<\infty
$$

La comparaison de (1.11) avec cette définition des espaces de Besov sur $\mathbb{R}^{n}$ révèle que $U(1)$ joue le rôle du filtre basse fréquence $S_{0}$, et que l'échelle $2^{j}$ (avec $j \geq 0$ ) et le filtre $\Delta_{j}$ associé à cette fréquence correspondent respectivement à la quantité $t^{-1 / 2}$ (avec $t \leq 1$ ) et à l' opérateur $t A U(t)$.

Vue leur définition, ces normes peuvent jouer un rôle important pour traiter de l'existence de solutions locales en temps. Le choix du temps 1 est arbitraire.

### 1.3.3. Espaces de Besov "homogènes".

Nous nous intéresserons au problème de l'existence globale. Il paraît alors judicieux d'introduire des normes qui prennent en compte
les temps $t \longrightarrow \infty$. Soit $u$ dans l'un des espaces $W_{p}^{-2}(A)$. On sait faire opérer $U(t)$ sur $u$, et donc exprimer une condition portant sur $U(t) u$.

Définition 16. On note $\mathcal{A} \doteq \bigcup_{p \in] 1, \infty[ } W_{p}^{-2}(A)$, et $I(2, p) \doteq$ ]-2, $\min \{2, n / p\}[$.

Pour $s \in I(2, p)$, on note $\dot{B}_{p q}^{s}(A)$ l'ensemble des $u \in \mathcal{A}$ tels que

$$
\left\|u \mid \dot{B}_{p q}^{s}\right\| \doteq\left\|t^{-s / 2}\right\| t A U(t) u\left\|_{p}\right\|_{q^{*}}<\infty
$$

Remarque 17. Pour les espaces de Besov homogènes définis sur $\mathbb{R}^{n}$ à partir d'une décomposition spectrale dyadique, on utilise alors $\dot{\Delta}_{j}$ pour $j \in \mathbb{Z}$.

Avec cette définition, il n'est pas évident que $\dot{B}_{p q}^{s}(A)$ soit un espace vectoriel, car $\mathcal{A}$ n'est pas stable par addition.

Proposition 18. $\dot{B}_{p q}^{s}(A)$ muni de $\left\|\cdot \mid \dot{B}_{p q}^{s}\right\|$ est un espace de Banach pour $s \in I(2, p)$.
$\dot{B}_{p_{1} q}^{s_{1}}(A) \hookrightarrow \dot{B}_{p_{2} q}^{s_{2}}(A)$ pour $s_{i} \in I\left(2, p_{i}\right), p_{1} \leq p_{2}$ et $s_{1}-n / p_{1}=$ $s_{2}-n / p_{2}$.
$\dot{B}_{p q}^{s}(A) \hookrightarrow B_{p q}^{s}(A)$ pour $\left.s \in\right]-2,0[$.
$B_{p q}^{s}(A) \hookrightarrow \dot{B}_{p q}^{s}(A)$ pour $\left.s \in\right] 0, \min \{2, n / p\}[$.
Démonstration. Nous affirmons tout d'abord que pour $s \in]-2,0[$, $\dot{B}_{p q}^{s}(A)$ est en fait inclus dans $W_{p}^{-2}(A)$. En effet, si $u \in \dot{B}_{p q}^{s}(A)$, il existe $r$ tel que $u \in W_{r}^{-2}(A)$, et on peut donc écrire en vertu des propriétés analytiques du semi-groupe

$$
\begin{aligned}
u & =\int_{0}^{2} \tau A^{2} U(\tau) u d \tau+(1+2 A) U(2) u \\
& =\int_{0}^{2} \tau^{2} A^{2} U(\tau) u \frac{d \tau}{\tau}+(1+2 A) \int_{0}^{\infty} \tau A U(\tau) u \frac{d \tau}{\tau}
\end{aligned}
$$

On estime la norme de chacun des deux termes dans $W_{p}^{-2}(A)$

$$
\begin{aligned}
\|(1+2 A) \int_{2}^{\infty} \tau A U(\tau) & \left.u \frac{d \tau}{\tau} \right\rvert\, W_{p}^{-2} \| \\
& \leq\left\|\int_{2}^{\infty} \tau A U(\tau) u \frac{d \tau}{\tau}\right\|_{p}
\end{aligned}
$$

$$
\begin{gathered}
\leq \int_{2}^{\infty} \tau^{s / 2}\left(\tau^{-s / 2}\|\tau A U(\tau) u\|_{p}\right) \frac{d \tau}{\tau} \\
\leq\left\|\mathbf{1}_{[2, \infty[ } \tau^{s / 2}\right\|_{q^{\prime} *}\left\|u \mid \dot{B}_{p q}^{s}\right\| \\
\left\|\left.\int_{0}^{2} \tau^{2} A^{2} U(\tau) u \frac{d \tau}{\tau} \right\rvert\, W_{p}^{-2}\right\| \\
\left.\leq \int_{0}^{2} \tau^{2}\left\|(1+A)^{-1} A^{2} U(\tau) u\right\|_{p}\right) \frac{d \tau}{\tau} \\
\leq \int_{0}^{2} C \tau^{1+s / 2}\left(\left(\frac{\tau}{2}\right)^{-s / 2}\left\|\frac{\tau}{2} A U\left(\frac{\tau}{2}\right) u\right\|\right) \frac{d \tau}{\tau} \\
\leq C\left\|\mathbf{1}_{] 0,2} \tau^{1+s / 2}\right\|_{q^{\prime} *}\left\|u \mid \dot{B}_{p q}^{s}\right\|
\end{gathered}
$$

où l'on a utilisé que $A(1+A)^{-1} U(\tau / 2)$ est borné sur $L^{p}$ uniformément en $\tau$. Les intégrales convergent précisément parce qu'on a supposé $s<0$ et $-2<s$.

Ainsi, quand $s \in]-2,0\left[\right.$, puisque $\dot{B}_{p q}^{s}(A) \subset W_{p}^{-2}(A)$, on en déduit que $\dot{B}_{p q}^{s}(A)$ est un espace vectoriel et que $\left\|u \mid \dot{B}_{p q}^{s}\right\|$ est une norme. L'inclusion correspond à une injection continue. De plus, on vient de voir que $\|U(2) u\|_{p} \leq C\left\|u \mid \dot{B}_{p q}^{s}\right\|$. Le même calcul montre que cette majoration vaut aussi pour $U(1) u$. On en déduit $\left\|u\left|B_{p q}^{s}\|\leq C\| u\right|\right.$ $\dot{B}_{p q}^{s} \|$ et $\dot{B}_{p q}^{s}(A) \hookrightarrow B_{p q}^{s}(A)$.

Soient $s_{i} \in I\left(2, p_{i}\right), i=1,2$. D'après les inégalité $L^{p_{1}}-L^{p_{2}}$ sur le semi-groupe, pour $p_{1} \leq p_{2}$,

$$
\begin{aligned}
t^{-s_{2} / 2}\|t A U(t) u\|_{p_{2}} & =t^{-s_{2} / 2} 2\left\|U\left(\frac{t}{2}\right) \frac{t}{2} A U\left(\frac{t}{2}\right) u\right\|_{p_{2}} \\
& \leq C t^{-s_{2} / 2-\left[p_{1}, p_{2}\right]}\left\|\frac{t}{2} A U\left(\frac{t}{2}\right) u\right\|_{p_{1}} .
\end{aligned}
$$

Et comme $-s_{2} / 2-\left[p_{1}, p_{2}\right]=-\left(s_{2}-n / p_{2}+n / p_{1}\right) / 2$, on obtient en prenant la norme dans $L^{q *}$

$$
\left\|u\left|\dot{B}_{p_{2} q}^{s_{2}}\|\leq C\| u\right| \dot{B}_{p_{1} q}^{s_{1}}\right\|, \quad \text { pour } s_{2}-\frac{n}{p_{2}}=s_{1}-\frac{n}{p_{1}} .
$$

Cela implique $\dot{B}_{p_{1} q}^{s_{1}}(A) \subset \dot{B}_{p_{2} q}^{s_{2}}(A)$. Vu que $s_{1}-n / p_{1}<0$, on peut toujours trouver $\left.s_{2} \in\right]-2,0\left[\right.$ et $p_{2} \in\left[p_{1}, \infty\left[\right.\right.$ vérifiant $s_{2}-n / p_{2}=$
$s_{1}-n / p_{1}$. Alors $\dot{B}_{p_{1} q}^{s_{1}}(A) \subset W_{p_{2}}^{-2}(A)$; comme précédemment cela entraîne que $\dot{B}_{p_{1} q}^{s_{1}}(A)$ est un espace vectoriel normé et que les inégalités correspondent à des injections continues. À ce stade, on vérifie sans peine que les espaces $\dot{B}_{p q}^{s}(A)$ pour $s \in I(2, p)$ sont complets.

Enfin, soit $0<s<\min \{2, n / p\}$ et $u \in B_{p q}^{s}(A)$. Par (1.9) et le Corollaire 4,

$$
t^{1-s / 2}\|A U(t) u\|_{p} \leq C t^{-s / 2}\|u\|_{p} \in L^{q * \infty}
$$

car $s>0$. Donc $u \in \dot{B}_{p q}^{s}(A)$.

Ajoutons deux résultats qui renforcent le lien avec le semi-groupe.

Lemme 19. Soit $s<0$. La quantité $\left\|t^{-s / 2}\right\| U(t) u\left\|_{p}\right\|_{q *}$ définit sur $\dot{B}_{p q}^{s}(A)$ une norme équivalente.

DÉMONSTRATION. De la majoration $\|t A U(t) u\|_{p} \leq C\|U(t / 2) u\|_{p}$ uniforme par rapport à $t$ on tire directement

$$
\left\|u \mid \dot{B}_{p q}^{s}\right\| \leq C\left\|t^{-s / 2}\right\| U(t) u\left\|_{p}\right\|_{q *}
$$

Inversement, si $u \in \dot{B}_{p q}^{s}(A)$, comme $s<0$ on sait que $u \in W_{p}^{-2}(A)$ et on peut écrire

$$
U(t) u=\int_{t}^{\infty} \tau A U(\tau) u \frac{d \tau}{\tau}
$$

Donc

$$
\begin{aligned}
t^{-s / 2}\|U(t) u\|_{p} & \leq \int_{t}^{\infty}\left(\frac{t}{\tau}\right)^{-s / 2}\left(\tau^{-s / 2}\|\tau A U(\tau) u\|_{p}\right) \frac{d \tau}{\tau} \\
& \leq\left(\mathbf{1}_{] 0,1]} \tau^{-s / 2}\right) *\left(\tau^{-s / 2}\|\tau A U(\tau) u\|_{p}\right)
\end{aligned}
$$

où ici $*$ désigne la convolution sur le groupe $\left(\mathbb{R}_{+}, \times, d t / t\right)$. L'inégalité de Hölder-Young donne alors

$$
\begin{aligned}
\left\|t^{-s / 2}\right\| U(t) u\left\|_{p}\right\|_{q *} & \leq\left\|\mathbf{1}_{] 0,1]} \tau^{-s / 2}\right\|_{1 *}\left\|\tau^{-s / 2}\right\| \tau A U(\tau) u\left\|_{p}\right\|_{q *} \\
& \leq C\left\|u \mid \dot{B}_{p q}^{s}\right\|
\end{aligned}
$$

puisque $s<0$.

Remarque 20. Dans le cas de $\mathbb{R}^{3}$, en prenant $A=-\Delta$ et donc $U(t)=$ $\exp (t \Delta)$, ce lemme correspond à [4, Lemme 3.3.3]. C'est un point crucial dans la relecture par [4] du résultat de [16].

Lemme 21. Soit $s \in I(2, p)$ et $q \in[1, \infty]$. L'application $t \longmapsto U(t)$ est continue bornée de $\mathbb{R}_{+}$dans $\mathcal{L}\left(\dot{B}_{p q}^{s}(A)\right)$. Si de plus $s-n / p+n / r \in$ $]-2,0\left[\right.$ et $p \leq r$ alors $\lim _{t \rightarrow 0} U(t) u=u$ dans $W_{r}^{-2}(A)$ pour tout $u \in$ $\dot{B}_{p q}^{s}(A)$.

Démonstration. La continuité en 0 dans $W_{r}^{-2}(A)$ vient des injections

$$
\dot{B}_{p q}^{s}(A) \hookrightarrow \dot{B}_{r q}^{\varsigma}(A) \hookrightarrow B_{r q}^{\varsigma}(A) \hookrightarrow W_{r}^{-2}(A), \quad \varsigma=s-\frac{n}{p}+\frac{n}{r}
$$

et de la continuité de $U(t)$ dans cet espace.
La borne et la continuité de $U(t)$ dans $\mathcal{L}\left(\dot{B}_{p q}^{s}(A)\right)$ découlent des même propriétés dans $\mathcal{L}\left(X_{p}\right)$, par commutation de $U(t)$ et $\theta^{1-s / 2} A U(\theta)$.

## 2. Énoncés.

On considère l'équation de Navier-Stokes dans $\Omega$ pour la viscosité $\varepsilon=1$

$$
\begin{cases}\partial_{t} u-\Delta u+\nabla \cdot(u \otimes u)+\nabla p=0,  \tag{2.1}\\ \nabla \cdot u=0, & \gamma_{0} u=u_{\mid \partial \Omega}=0,\end{cases}
$$

avec la donnée initiale $u_{\mid t=0}=u_{0}$. On a écrit $\nabla \cdot(u \otimes u)$ pour

$$
\sum_{k} \partial_{k}\left(u_{k} u_{i}\right) .
$$

En appliquant le projecteur $P$, on se ramène à

$$
\partial_{t} u+A u+P \nabla \cdot(u \otimes u)=0, \quad u_{\mid t=0}=u_{0}, \gamma_{0} u=0,
$$

dont on cherche les solutions sous la forme

$$
\begin{equation*}
u(t)=U(t) u_{0}-\int_{0}^{t} U(t-\tau) P \nabla \cdot(u \otimes u)(\tau) d \tau \tag{2.2}
\end{equation*}
$$

Considérant le second terme du membre de droite comme un opérateur (quadratique) appliqué à $u$, on lit (2.2) comme une équation de point fixe.

Définition 22. On note $U_{0} \doteq U(t) u_{0}$. On note $\Theta$ l'opérateur bilinéaire

$$
\Theta(u, v)(t, x) \doteq \int_{0}^{t} U(t-\tau) P \nabla \cdot(u \otimes v)(\tau, x) d \tau
$$

ò̀ $\nabla \cdot(u \otimes v)$ désigne le champ de vecteurs $\sum_{k} \partial_{k}\left(u^{k} v^{i}\right)$.
Ainsi on cherche à résoudre l'équation de point fixe

$$
\begin{equation*}
u=U_{0}-\Theta(u, u) \tag{2.2}
\end{equation*}
$$

Théorème 23. Il existe $\eta \in C(] n, \infty\left[; \mathbb{R}_{+}\right)$vérifiant ceci. Pour tout $u_{0} \in X_{n}$, s'il existe $p>n$ tel que $\left\|u_{0} \mid \dot{B}_{p \infty}^{-1+n / p}\right\|<\eta(p)$, alors il existe $u \in C_{b}\left(\overline{\mathbb{R}}_{+} ; X_{n}\right)$ solution de l'équation (2.2) avec $u(0)=u_{0}$.
$u$ est l'unique solution de (2.2) parmi les fonctions $v$ de $C\left(\mathbb{R}_{+} ; X_{n}\right)$ vérifiant $\sup _{t>0} t^{(1-n / p) / 2}\|v(t)\|_{p}<2 \eta(p)$.

Pour le cas de $\mathbb{R}^{3}$, ce résultat est dû à $[4,28]$ L'intérêt de n'imposer la petitesse que sur la norme dans un espace de Besov se voit bien dans le lemme suivant, qui est une adaptation au cas de notre domaine extérieur d'un résultat semblable sur $\mathbb{R}^{3}$ de [14].

Lemme 24. Soit $n<p$. Il existe une suite $u^{k} \in X_{n}$ telle que

$$
1<\inf _{k}\left\|u^{k}\right\|_{n} \quad \text { et } \quad \lim _{k \rightarrow \infty}\left\|u^{k} \left\lvert\, \dot{B}_{p \infty}^{-1+\frac{n}{p}}\right.\right\|=0 .
$$

On peut étendre l'ensemble des données initiales de la façon suivante :

Théorème 25. Il existe $\eta \in C(] n, \infty\left[; \mathbb{R}_{+}\right)$vérifiant ceci. Pour tout $r \in] n, \infty\left[\right.$ et $u_{0} \in \dot{B}_{r \infty}^{-1+n / r}(A)$, s'il existe $p \geq r$ tel que $\left\|u_{0} \mid \dot{B}_{p \infty}^{-1+n / p}\right\|$ $<\eta(p)$, alors il existe $u \in C_{b}\left(\mathbb{R}_{+} ; \dot{B}_{r \infty}^{-1+n / r}(A)\right)$ solution de l'équation $(2.2)$ avec $u(t) \longrightarrow u_{0}$ dans $W_{r}^{-2}(A)$ quand $t \longrightarrow 0$.
$u$ est l'unique solution de (2.2) parmi les fonctions $v$ de $C_{b}\left(R_{+} ; \dot{B}_{r \infty}^{-1+n / r}(A)\right)$ vérifiant $\sup _{t>0} t^{(1-n / p) / 2}\|v(t)\|_{p}<2 \eta(p)$.

Comme dans le cas de $\mathbb{R}^{3}$, ces théorèmes d'existence, obtenus par un point fixe dans un espace de Banach plus petit que celui induit naturellement par la donnée initiale, ne sont pas satisfaisants pour leur assertion sur l'unicité. On y demande la condition supplémentaire $\sup _{t>0} t^{(1-n / p) / 2}\|v(t)\|_{p}<2 \eta(p)$. Ceci est relié au fait qu'on ne sait pas si l'opérateur bilinéaire $\Theta$ est continu dans $C_{b}\left(\mathbb{R}_{+}, X_{n}\right)$. D'après [26], il ne l'est pas dans $L^{3}\left(\mathbb{R}^{3}\right)$, tandis qu'il l'est dans $L^{3, \infty}\left(\mathbb{R}^{3}\right)$, d'après [24]. Cette difficulté est contournée dans [7], [8] par l'utilisation de normes différentes pour les deux arguments de l'opérateur bilinéaire. Nous avons adapté leur résultat au cas de $\Omega$ ouvert extérieur.

Théorème 26. Soit $u_{0} \in X_{n}$ et $U_{0} \doteq U(t) u_{0}$. Soit $u^{1}$ et $u^{2}$ dans $C_{b}\left(\left[0, T\left[; X_{n}\right)\right.\right.$, solutions de l'équation (2.2). Alors $u^{1}=u^{2}$ sur $[0, T[$.

## 3. Démonstrations des théorèmes.

Nous rassemblons d'abord les résultats de continuité sur $\Theta$ utilisés ensuite. Puis nous prouvons les deux théorèmes d'existence. Enfin le théorème d'unicité.

### 3.1. Continuité de l'opérateur bilinéaire $\Theta$.

### 3.1.1. Pour l'existence.

Définition 27. On définit

$$
[p]=[n, p]=\frac{1}{2}\left(1-\frac{n}{p}\right),
$$

de sorte que $[p, q]=[q]-[p]$.
On définit à la manière de [27] l'espace de Banach $E_{p}$ comme l'ensemble des $u$ continus de $\mathbb{R}_{+}$dans $X_{p}$ tels que

$$
\|u\|_{E_{p}} \doteq \sup _{t>0}\left\|t^{[p]} u(t)\right\|_{p}<\infty
$$

Pour $T_{*}=0$ ou $\infty$, on définit aussi le sous-espace (fermé) $E_{p ; 0}\left(T_{*}\right)$ de Epar

$$
E_{p ; 0}\left(T_{*}\right)=\left\{u(t, x) \in E_{p}: \lim _{t \rightarrow T_{*}}\left\|t^{[p]} u(t)\right\|_{p}=0\right\}
$$

Enfin on abrège $\dot{B}_{p \infty}^{-1+n / p}(A)$ en $\dot{\mathcal{B}}_{p}$.
Proposition 28. Étant donnés $p, q$ et $r$ dans $] 1, \infty[$ tels que

$$
\begin{equation*}
\frac{1}{r} \leq \frac{1}{p}+\frac{1}{q} \leq \frac{1}{r}+\frac{1}{n} \tag{3.1}
\end{equation*}
$$

l'opérateur bilinéaire $\Theta$ est continu de $E_{p} \times E_{q}$ dans $C\left(\mathbb{R}_{+} ; W_{r}^{-2}(A)\right)$, et

$$
\begin{equation*}
\left\|\Theta(u, v)(t) \mid W_{r}^{-2}\right\| \leq C t^{n(1 / p+1 / q) / 2}\|u\|_{E_{p}}\|v\|_{E_{q}} . \tag{3.2}
\end{equation*}
$$

En particulier, $\lim _{t \rightarrow 0} \Theta(u, v)(t)=0$ dans $W_{r}^{-2}(A)$.
Proposition 29. Étant donnés $p, q$ et $r$ dans $] 1, \infty[$ tels que

$$
\begin{equation*}
\frac{1}{p}+\frac{1}{q}-\frac{1}{n}<\frac{1}{r} \leq \frac{1}{p}+\frac{1}{q} \leq \frac{1}{n^{\prime}} \tag{3.3}
\end{equation*}
$$

l'opérateur bilinéaire $\Theta$ est continu de $E_{p} \times E_{q}$ dans $E_{r}$.
De plus, pour $T_{*}=0$ ou $\infty$, si $u \in E_{p ; 0}\left(T_{*}\right)$ ou $v \in E_{q ; 0}\left(T_{*}\right)$ alors $\Theta(u, v) \in E_{r ; 0}\left(T_{*}\right)$.

Proposition 30. Étant donnés $p, q$ et $r$ dans $] 1, \infty[$ tels que

$$
\begin{equation*}
\frac{1}{r} \leq \frac{1}{p}+\frac{1}{q}<\frac{2}{n} \tag{3.4}
\end{equation*}
$$

l'opérateur bilinéaire $\Theta$ est continu de $E_{p} \times E_{q}$ dans $C_{b}\left(\mathbb{R}_{+} ; \dot{\mathcal{B}}_{r}\right)$.
Notation 31. On note $s$ le nombre tel que $1 / p+1 / q=1 / s$, et $a \doteq$ $-[p]-[q]=-1+n /(2 s)$. Pour $(u, v) \in E_{p} \times E_{q}$, on note $\widetilde{u}(t) \doteq$ $t^{[p]} u(t)$ et $\widetilde{v}(t) \doteq t^{[q]} v(t)$, si bien que $\widetilde{u} \otimes \widetilde{v} \in C_{b}\left(\mathbb{R}_{+} ; L^{s}(\Omega)\right)$ avec $\sup _{t}\|(\widetilde{u} \otimes \widetilde{v})(t)\|_{s} \leq\|u\|_{E_{p}}\|v\|_{E_{q}}$ et $(u \otimes v)(t)=t^{a}(\widetilde{u} \otimes \widetilde{v})(t)$.

On note $\widetilde{U}(t) \doteq t^{-b} U(t) P \nabla \cdot$, où l'exposant $b$ s'adapte à chaque proposition, et $\widetilde{w}(t, \tau) \doteq \widetilde{U}(t(1-\tau))(\widetilde{u} \otimes \widetilde{v})(t \tau)$.

Enfin on note

$$
f(t, \tau) \doteq t U(t(1-\tau)) P \nabla \cdot(u \otimes v)(t \tau)
$$

de sorte que

$$
\Theta(u, v)(t)=\int_{0}^{1} f(t, \tau) d \tau
$$

et

$$
f(t, \tau)=t^{1+a+b} \tau^{a}(1-\tau)^{b} \widetilde{w}(t, \tau)
$$

Preuve de la Proposition 28. D'après la Proposition 6, pour $s, r$ dans $] 1, \infty[$ tels que $0 \leq 1 / s-1 / r \leq 1 / n$ et $b=0, \widetilde{U}(t) \in$ $C_{b}\left(\mathbb{R}_{+} ; \mathcal{L}\left(L^{s}(\Omega) ; W_{r}^{-2}(A)\right)\right)$. Alors $\widetilde{w} \in C_{b}\left(\mathbb{R}_{+} \times\right] 0,1\left[; W_{r}^{-2}(A)\right)$, et comme $a>-1$, le résultat découle du théorème de continuité sous le signe $\int$ appliquée à $\int_{0}^{1} f(t, \tau) d \tau$.

Preuve de la Proposition 29. D'après le Corollaire 10, pour $s, r$ dans $] 1, \infty\left[\right.$ tels que $1 / r \leq 1 / s \leq 1 / n^{\prime}$ et $b=-1 / 2-[s, r], \widetilde{U}(t) \in$ $C_{b}\left(\mathbb{R}_{+} ; \mathcal{L}\left(L^{s}(\Omega) ; X_{r}\right)\right)$. Alors $\widetilde{w} \in C_{b}\left(\mathbb{R}_{+} \times\right] 0,1\left[; X_{r}\right)$. Comme $[r]+1+$ $a+b=0$ et $a>-1$, on obtient le résultat par le théorème de continuité sous le signe $\int$ pourvu que $b>-1$, ce qui s'écrit encore $1 / s-1 / n<1 / r$.

Si $\lim _{t \rightarrow T_{*}} \widetilde{u}=0$ dans $X_{p}$ (respectivement $\widetilde{v}, X_{q}$ ), alors

$$
\lim _{t \rightarrow T_{*}} \widetilde{w}(t, \tau)=0
$$

dans $X_{r}$ uniformément en $\tau$ sur tout compact de $] 0,1[$. On en déduit facilement que $\lim _{t \rightarrow T_{*}} t^{[r]} \Theta(\widetilde{u}, \widetilde{v})=0$ dans $X_{r}$.

Preuve de la Proposition 30. On va montrer que, pour $b=$ $-n /(2 s)$ et $r, s$ dans $] 1, \infty\left[\right.$ tels que $1 / r \leq 1 / s \leq 1 / n^{\prime}$ et $1 / r \leq 3 / n$, $\widetilde{U}(t) \in C_{b}\left(\mathbb{R}_{+} ; \mathcal{L}\left(L^{s}(\Omega) ; \dot{\mathcal{B}}_{r}\right)\right)$. Alors $\widetilde{w} \in C_{b}\left(\mathbb{R}_{+} \times\right] 0,1\left[; \dot{\mathcal{B}}_{r}\right)$. Comme $1+a+b=0$ et $a>-1$, on obtient la proposition par le théorème de continuité sous le signe $\int$ pourvu que $b>-1$, ce qui s'écrit encore $1 / s<2 / n$.

Commençons par la borne. Soit $c \doteq 1+[r]$ et $w \in L^{s}(\Omega)$. Par définition

$$
\sup _{t>0}\left\|\widetilde{U}(t) w \mid \dot{\mathcal{B}}_{r}\right\|=\sup _{t, \theta>0} \theta^{c} t^{-b}\|A U(t+\theta) P \nabla \cdot w\|_{r}
$$

D'après le Corollaire 10, $\|A U(t+\theta) P \nabla \cdot w\|_{r} \leq C(t+\theta)^{b-c}\|w\|_{s}$ pour $n^{\prime} \leq s \leq r$, car $1+1 / 2+[s, r]=c-b$. Or $\theta^{c} t^{-b}(t+\theta)^{b-c} \leq 1$ pour $b \leq 0 \leq c$ et $(t, \theta) \in \mathbb{R}_{+} \times \mathbb{R}_{+}$. On en déduit la borne dès que $n / r \leq 3$.

Finissons par la continuité, qui découle de

$$
\widetilde{U}(t+h)-\widetilde{U}(t)=\left(U\left(\frac{t}{2}+h\right)-U\left(\frac{t}{2}\right)\right) \widetilde{U}\left(\frac{t}{2}\right)
$$

et de $U(t) \in C_{b}\left(\mathbb{R}_{+} ; \mathcal{L}\left(\dot{\mathcal{B}}_{r}\right)\right)$ vu au Lemme 21.

### 3.1.2. Pour l'unicité.

Définition 32. On pose $\bar{\Theta}(u, v) \doteq \Theta(v, u)$. On note $L_{T}^{\infty}(X)$ les fonctions fortement mesurables bornées de $] 0, T[$ dans un espace de $B a$ nach $X$. On définit l'espace de Banach $F_{p}^{T}$ comme l'ensemble des $u$ mesurables de $] 0, T\left[\right.$ dans $X_{p}$ tels que

$$
\|u\|_{F_{p}^{T}} \doteq \sup _{0<t<T}\left\|t^{[p]} u(t)\right\|_{p}<\infty
$$

On abrège $B_{p \infty}^{-n / q+n / p}(A)$ en $\mathcal{B}_{p}^{q}$, et $\mathcal{B}_{p}^{n}$ en $\mathcal{B}_{p}$. Enfin $G_{p}^{T} \doteq L_{T}^{\infty}\left(\mathcal{B}_{p}\right)$.
Proposition 33. Étant donnés $p, q$ et $r$ dans $] 1, \infty[$ tels que

$$
\begin{equation*}
\frac{1}{r}-\frac{1}{q} \leq \frac{1}{p} \leq \frac{1}{n} \tag{3.5}
\end{equation*}
$$

il existe $C$ tel que pour tout $T \in] 0,1[$, les opérateurs bilinéaires $\Theta$ et $\bar{\Theta}$ sont continus de $F_{p}^{T} \times L_{T}^{\infty}\left(X_{q}\right)$ dans $L_{T}^{\infty}\left(\mathcal{B}_{r}^{q}\right)$ avec une norme majorée par $C$.

Démonstration. Posons $c=1+[q, r]$. Par définition de $\| \Theta(u, v)(t) \mid$ $\mathcal{B}_{r}^{q} \|$, on cherche $C$ tel que pour $u \in F_{p}^{T}, v \in L_{T}^{\infty}\left(X_{q}\right)$ et $0<t<T<1$,

$$
\begin{align*}
&\|U(1) \Theta(u, v)(t)\|_{r}+\sup _{0<\theta<1} \| \theta^{c} A U(\theta) \Theta(u, v)(t) \|_{r}  \tag{3.6}\\
& \leq C\|u\|_{F_{p}^{T}}\|v\|_{L_{T}^{\infty}\left(X_{q}\right)}
\end{align*}
$$

Or

$$
\|(u \otimes v)(\tau)\|_{s} \leq \tau^{a}\|u\|_{F_{p}^{T}}\|v\|_{L_{T}^{\infty}\left(X_{q}\right)}
$$

avec $1 / s=1 / p+1 / q$ et $a=-[p]$.
Et

$$
U(1) \Theta(u, v)(t)=\int_{0}^{t} U(1+t-\tau) P \nabla \cdot(u \otimes v)(\tau) d \tau
$$

D'après le Corollaire $10, U(\vartheta) P \nabla$. est borné dans $\mathcal{L}\left(L^{s}(\Omega) ; X_{r}\right)$ pour $\vartheta=1+t-\tau \in] 1,2\left[\right.$ et $s \leq r$. Comme $\int_{0}^{t} \tau^{a} d \tau \leq 2$, on obtient la première partie de (3.6).

Et

$$
\theta^{c} A U(\theta) \Theta(u, v)(t)=\int_{0}^{t} \theta^{c} A U(\theta+t-\tau) P \nabla \cdot(u \otimes v)(\tau) d \tau
$$

D'après le Corollaire $10, \vartheta^{-b} U(\vartheta) P \nabla \cdot$ est borné dans $\mathcal{L}\left(L^{s}(\Omega) ; X_{r}\right)$ avec $b=-3 / 2-[s, r]$, pour $\vartheta=\theta+t-\tau \in] 0,2[$ et $s \leq r$. Donc

$$
\begin{align*}
& \left\|\theta^{c} A U(\theta) \Theta(u, v)(t)\right\|_{r} \\
& \leq C\|u\|_{F_{p}^{T}}\|v\|_{L_{T}^{\infty}\left(X_{q}\right)} \int_{0}^{t} \theta^{c}(\theta+t-\tau)^{b} \tau^{a} d \tau . \tag{3.7}
\end{align*}
$$

L'intégrale devient

$$
I=\int_{0}^{1} \xi^{c}(\xi+1-\tau)^{b} \tau^{a} d \tau
$$

si on pose $\theta=t \xi$, car $a+b+c=-1$. Si $-1<a<0 \leq c$, alors

$$
I \leq \int_{0}^{1}(1-\tau)^{-1-a} \tau^{a} d \tau
$$

car alors, pour $(\xi, \tau) \in \mathbb{R}_{+} \times \mathbb{R}_{+}, \xi^{c}(\xi+1-\tau)^{b}(1-\tau)^{1+a}<1$. Si $a=0<c$, alors $I<1 / c$. Or (3.5) implique $a \in]-1 / 2,0]$ et $1 / 2 \leq c$. On obtient donc la seconde partie de (3.6).

Corollaire 34. Étant donnés $p, q$ et $r$ dans $] 0,1[$ tel que

$$
\begin{equation*}
0<\frac{1}{r}-\frac{1}{n}<\frac{1}{p} \leq \frac{1}{n} \tag{3.8}
\end{equation*}
$$

il existe $C$ tel que pour tout $T \in[0,1]$, les opérateurs bilinéaires $\Theta$ et $\bar{\Theta}$ sont continus de $F_{p}^{T} \times G_{r}^{T}$ dans $G_{r}^{T}$ avec une norme majorée par $C$.

DÉmonstration. D'une part $L^{n \infty}(\Omega)=\left(L^{q}(\Omega), L^{o}(\Omega)\right)_{\theta, \infty}$ dès que

$$
\begin{equation*}
\frac{1}{n}=\frac{1-\theta}{q}+\frac{\theta}{o} . \tag{3.9}
\end{equation*}
$$

Quand on se restreint aux champs à divergence nulle et tangents au bord, on obtient $X_{n \infty}=\left(X_{q}, X_{o}\right)_{\theta, \infty}$ (voir [19]). D'autre part, $\mathcal{B}_{r}=$ $\left(\mathcal{B}_{r}^{q}, \mathcal{B}_{r}^{o}\right)_{\theta, \infty}$ d'après le théorème de réitération. Par ailleurs, pour $r<n$,
$B_{r \infty}^{-1+n / r}(\Omega) \hookrightarrow L^{n \infty}(\Omega)$ par injection de Sobolev et interpolation réelle (voir [30]), et $\mathcal{B}_{r} \hookrightarrow X_{r} \cap B_{r \infty}^{-1+n / r}(\Omega)$ d'après [17, Lemma 4.4]. On en déduit $\mathcal{B}_{r} \hookrightarrow X_{n \infty}$. Enfin, la théorie de l'interpolation (voir par exemple [20]) nous enseigne que, pour tout $\theta \in] 0,1\left[, L_{T}^{\infty}\left(\left(A_{0}, A_{1}\right)_{\theta, \infty}\right)=\right.$ $\left(L_{T}^{\infty}\left(A_{0}\right), L_{T}^{\infty}\left(A_{1}\right)\right)_{\theta, \infty}$. Donc

$$
\left(L_{T}^{\infty}\left(\mathcal{B}_{r}^{q}\right), L_{T}^{\infty}\left(\mathcal{B}_{r}^{o}\right)\right)_{\theta, \infty}=G_{r}^{T} \hookrightarrow L_{T}^{\infty}\left(X_{n \infty}\right)=\left(L_{T}^{\infty}\left(X_{q}\right), L_{T}^{\infty}\left(X_{o}\right)\right)_{\theta, \infty}
$$

On fixe maintenant $u \in F_{p}^{T}$, et on considère les opérateurs linéaires $v \longmapsto \Theta(u, v)$ et $v \longmapsto \bar{\Theta}(u, v)$. On leur applique la Proposition 33 et la propriété fondamentale de l'interpolation (voir Théorème 49). Il reste à choisir $o \leq q$ vérifiants (3.9) et (3.5), ce qui est possible d'après (3.8) en prenant $q$ assez proche de $n$.

### 3.2. Existence globale à donnée petite.

### 3.2.1. Point fixes.

Lemme 35. Soit $E$ un espace de Banach, $U_{0} \in E$ et $\Theta: E \times E \longrightarrow E$ une application bilinéaire avec $\|\Theta(u, v)\|_{E} \leq C\|u\|_{E}\|v\|_{E}$. On note $f: E \longrightarrow$ E l'application continue $f(u)=U_{0}-\Theta(u, u)$.

1) Si $\left\|U_{0}\right\|_{E}<(4 C)^{-1}$, alors l'équation de point fixe $f(u)=u$ admet une solution dans la boule fermée de rayon

$$
(2 C)^{-1}\left(1-\sqrt{1-4 C\left\|U_{0}\right\|_{E}}\right) .
$$

2) L'équation de point fixe $f(u)=u$ admet au plus une solution dans la boule ouverte de rayon $(2 C)^{-1}$.

La preuve est élémentaire.
Voici maintenant la partie existence globale à donnée petite. Rappelons que $u_{0} \in \mathcal{A}$ s'il existe $\left.p \in\right] 1, \infty\left[\right.$ tel que $u_{0} \in W_{p}^{-2}(A)$. Dans ce cas, on sait faire opérer $U(t)$ sur $u_{0}$, et donc exprimer une hypothèse du genre $U(t) u_{0} \in E_{p}$. Comme le point fixe fait intervenir $U_{0}$ plutôt que $u_{0}$, on énonce le résultat en ces termes:

Proposition 36 (Existence). Soit $n<p$, et $u_{0} \in \mathcal{A}$ tel que $U_{0} \doteq$ $U(t) u_{0} \in E_{p}$ avec

$$
\begin{equation*}
\left\|U_{0}\right\|_{E_{p}}<(4 C)^{-1} \tag{3.10}
\end{equation*}
$$

où $C$ est la norme de l'opérateur bilinéaire $\Theta: E_{p} \times E_{p} \longrightarrow E_{p}$. Alors il existe une solution $u \in E_{p}$ à l'équation (2.2) qui est unique dans la boule de cet espace de rayon $(2 C)^{-1}$. De plus $\|u\|_{E_{p}} \leq 2\left\|U_{0}\right\|_{E_{p}}$.

Pour $T_{*}=0$ ou $\infty$, si $U_{0} \in E_{p ; 0}\left(T_{*}\right)$ alors $u \in E_{p ; 0}\left(T_{*}\right)$.
La preuve est une application immédiate du lemme précédent et de la continuité de l'opérateur bilinéaire énoncé dans la Proposition 29. Ajoutons seulement que $p>n \geq 3$ implique $2 / p<2 / n \leq 1 / n^{\prime}$. Et que pour $x \leq(4 C)^{-1},(2 C)^{-1}(1-\sqrt{1-4 C x}) \leq 2 x$.

Remarque 37. Le Lemme 39 ci-dessous permettra de transférer l'hypothèse de petitesse de $\left\|U_{0}\right\|_{E_{p}}$ à $\left\|u_{0} \mid \dot{\mathcal{B}}_{p}\right\|$. Mais il n'est pas complètement vain de garder à l'esprit que, plus que $u_{0}$, c'est $U_{0}$ qui est réellement la donnée du problème. Si on étudiait par exemple l'équation de Navier-Stokes avec un terme de force (un $f$ au second membre de (2.1)), on s'arrangerait pour le faire rentrer dans $U_{0}$. Il resterait ensuite à trouver des conditions sur $f$ suffisantes pour que le $U_{0}$ ainsi déterminé vérifie les hypothèses du théorème.

On va maintenant exploiter plus largement la Proposition 29 pour obtenir des renseignements supplémentaires sur la solution. Il s'agit de résultats de régularité, qui rappellent ceux de l'équation de la chaleur.

Proposition 38 (Régularité). Soit $n<p$, et $u_{0} \in \mathcal{A}$ tel que $U_{0} \doteq$ $U(t) u_{0} \in E_{p}$. Alors $U_{0} \in E_{s}$ pour $p \leq s$.

Si de plus $u \in E_{p}$ est une solution de l'équation (2.2) alors $u \in E_{s}$ pour $p \leq s$ et $u-U_{0} \in E_{s}$ pour $p / 2 \leq s$.

Si de plus $U_{0} \in E_{q}$ pour un $q \in\left[n, p\left[\right.\right.$, alors $u$ et $U_{0}$ sont dans $E_{s}$ pour $q \leq s$, et $u-U_{0} \in E_{s}$ pour $q / 2 \leq s$.

Pour $T_{*}=0$ ou $\infty$, tout ceci est encore valable quand on remplace les espaces $E_{\star}$ par leur variante $E_{\star<0}\left(T_{*}\right)$.

DÉmonstration. On remarque que $n \geq 3$ entraîne $2 / n \leq 1 / n^{\prime}$.
Tout d'abord, on constate que si $U_{0} \in E_{p}$, alors $U_{0} \in E_{s}$ pour $1 / s \in] 0,1 / p]$, par l'estimation $L^{p}-L^{s}$ du semi-groupe (1.6).

Ensuite, comme $n<p$, on vérifie $2 / p<2 / n \leq 1 / n^{\prime}$, et donc $u \in E_{p}$ implique d'après la Proposition 29 que $\Theta(u, u) \in E_{r}$ pour $1 / r \in] \max \{0,2 / p-1 / n\}, 2 / p]$, donc $u=U_{0}+\Theta(u, u) \in E_{s}$ pour $1 / s \in] \max \{0,2 / p-1 / n\}, 1 / p]$. On montre ensuite par récurrence sur $k$ que pour tout $k$ l'assertion : " $u \in E_{s}$ pour tout $s$ vérifiant $1 / s \in$
$\left.\max \left\{0,2^{k} / p-\left(2^{k}-1\right) / n\right\}, 1 / p\right]$ " est vraie. En effet pour $k=1$, on vient de le voir. Et pour passer de $k$ à $k+1$, on écrit encore $u=U_{0}+\Theta(u, u)$ et on applique à nouveau la Proposition 29. Voir sur la Figure 1 la flèche issue de $1 / p_{1}$. Chaque segment vertical de la flèche représente l'intervalle des valeurs de $1 / s$ qu'on ajoute à chaque itération. Le fait que $u-U_{0} \in E_{s}$ pour $1 / s \in[1 / p, 2 / p]$ découle de l'assertion déjà démontrée " $\Theta(u, u) \in E_{r}$ pour $\left.\left.1 / r \in\right] \max \{0,2 / p-1 / n\}, 2 / p\right]$ ". Les autres valeurs de $s$ s'obtiennent en considérant $u$ et $U_{0}$ séparément.


Figure 1. Propriétés supplémentaires.
On utilise le même argument pour les valeurs de $s$ plus petites que $p$, avec l'hypothèse que $U_{0} \in E_{q}$ et $n \leq q<p$. Si $U_{0} \in E_{q}$ alors, par l'estimation $L^{q}-L^{s}$ du semi-groupe (1.6), $U_{0} \in E_{s}$ pour $\left.\left.1 / s \in\right] 0,1 / q\right]$. On a déjà dit que $u \in E_{p}$ et $1 / p<1 / n$ impliquent $\Theta(u, u) \in E_{r}$ pour $1 / r \in] \max \{0,2 / p-1 / n\}, 2 / p]$. Donc $u=U_{0}+\Theta(u, u) \in E_{s}$ pour $1 / s \in[1 / p, \min \{2 / p, 1 / q\}]$. On montre ensuite par récurrence sur $k$ que, pour tout $k$ vérifiant la condition $2^{k} / p \leq 2 / q$, l'assertion : " $u \in E_{s}$ pour tout s vérifiant $1 / s \in\left[1 / p, \min \left\{2^{k} / p, 1 / q\right\}\right]$ " est vraie. Pour $k=1$, la condition est vérifiée puisque $q<p$, et l'assertion est vraie comme on vient de le voir. Pour passer de $k$ à $k+1$, on note que l'assertion au rang $k$ implique $u \in E_{s}$ pour $1 / p \leq 1 / s \leq \min \left\{2^{k} / p, 1 / q\right\}$. La condition au rang $k+1$ implique alors $1 / s \leq 2^{k} / p$ puis $2 / s \leq 2 / q \leq 1 / n^{\prime}$, car $n \leq q$. On peut donc appliquer à nouveau la Proposition 29. Quand la récurrence s'arrête, on obtient $k=K$ tel que l'assertion soit vraie au rang $K$ et $2^{K} / p>1 / q$, donc $u \in E_{s}$ pour $1 / s \in[1 / p, 1 / q]$. Comme $2 / q \leq 1 / n^{\prime}$, on applique une dernière fois la Proposition 29 qui donne $\Theta(u, u) \in E_{s}$ pour $1 / s \in[1 / p, 2 / q]$. Voir sur la Figure 1 la flèche issue de $1 / p_{2}$.

### 3.2.2. Preuve des théorèmes d'existence.

Lemme 39. Soit $u_{0} \in X_{n}$. Alors

1) $U(t) u_{0} \in E_{p}$ pour $n \leq p$, avec $\left\|U(t) u_{0}\right\|_{E_{p}} \leq C\left\|u_{0}\right\|_{n}$.
2) $u_{0} \in \dot{\mathcal{B}}_{p}$ pour $n \leq p$ avec $\left\|u_{0} \mid \dot{\mathcal{B}}_{p}\right\| \leq C\left\|u_{0}\right\|_{n}$.
3) $\lim _{t \rightarrow 0} t^{[p]}\left\|U(t) u_{0}\right\|_{p}=0$ pour $n<p$, autrement dit $U(t) u_{0} \in$ $E_{p ; 0}(0)$.

Démonstration. La première assertion découle des inégalités $L^{n}-L^{p}$.
La seconde assertion est une conséquence des inégalités $L^{n}-L^{p}$, de l'estimation uniforme de $t A U(t)$ sur $X_{n}$ et de la définition de $\dot{B}_{p \infty}^{s}(A)$ par (1.14).

La dernière assertion s'obtient par un raisonnement classique : si $u_{0} \in W_{n}^{2}(A)$, alors $\|(U(\varepsilon)-$ Id $) u_{0}\left\|_{n} \leq \varepsilon\right\| A u_{0} \|_{n}$. Donc

$$
\begin{aligned}
t^{[p]}\left\|(U(t+\varepsilon)-U(t)) u_{0}\right\|_{p} & \leq t^{[p]}\left\|U(t)(U(\varepsilon)-\mathrm{Id}) u_{0}\right\|_{p} \\
& \leq C\left\|(U(\varepsilon)-\mathrm{Id}) u_{0}\right\|_{n} \leq C \varepsilon\left\|A u_{0}\right\|_{n}
\end{aligned}
$$

donc $t^{[p]}\left\|(U(t+\varepsilon)-U(t)) u_{0}\right\|_{p} \longrightarrow 0$ quand $\varepsilon \longrightarrow 0$, uniformément par rapport à $t$. D'autre part,

$$
t^{[p]}\left\|U(t+\varepsilon) u_{0}\right\|_{p} \leq\left(\frac{t}{t+\varepsilon}\right)^{[p]}\left\|u_{0}\right\|_{n}
$$

Pour tout $\varepsilon>0$ cela tend vers 0 quand $t \longrightarrow 0$, car $[p]>0$ pour $p>n$. En additionnant, $\lim _{t \rightarrow 0} t^{[p]}\left\|U(t) u_{0}\right\|_{p}=0$ pour tout $u_{0} \in W_{n}^{2}(A)$. Or $t^{[p]} U(t)$ est borné dans $\mathcal{L}\left(X_{n}, X_{p}\right)$, et $W_{n}^{2}(A)$ est dense dans $X_{n}$. Donc la convergence a lieu pour tout $u_{0} \in X_{n}$.

Démonstration du Théorème 23. Soit $u_{0} \in X_{n}$. D'après le Lemme $39, U_{0} \in E_{q}$ pour $q \geq n$ et $U_{0} \in E_{q ; 0}(0)$ pour $q>n$. De plus on dispose de $p>n$ tel que

$$
\left\|u_{0} \mid \dot{\mathcal{B}}_{p}\right\| \simeq \sup _{t>0} t^{[p]}\left\|U_{0}\right\|_{p}=\left\|U_{0}\right\|_{E_{p}} \leq \eta(p) \doteq\left(4 C_{p}\right)^{-1}
$$

où l'équivalence des normes vient du Lemme 19 et où $C_{p}$ est la constante de continuité de $\Theta: E_{p} \times E_{p} \longrightarrow E_{p}$ fournie par la Proposition 29. D'après la Proposition 36, il existe une solution $u \in E_{p ; 0}(0)$ de (2.2),
et $\|u\|_{E_{p}} \leq 2\left\|U_{0}\right\|_{E_{p}}$. C'est l'unique solution dans la boule de $E_{p}$ de rayon $2 \eta(p)$.

La Proposition 38 assure que $u \in E_{q}$ pour $q \geq n$ et $u \in E_{q ; 0}(0)$ pour $q>n$. On en déduit que $\Theta(u, u) \in E_{n ; 0}$. Comme par ailleurs on sait que $\lim _{t \rightarrow 0} U_{0}=u_{0}$ dans $X_{n}$, on a bien $u \in C_{b}\left(\overline{\mathbb{R}}_{+}, X_{n}\right)$ et $u(0)=u_{0}$.

Démonstration du Théorème 25. Soit $u_{0} \in \dot{\mathcal{B}}_{r}$ et $r>n$. Par la Proposition 38 et le Lemme 19, $U_{0}=U(t) u_{0} \in E_{q}$ pour $q \geq r$. De plus on dispose comme ci-dessus de $p \geq r$ tel que

$$
\left\|u_{0} \mid \dot{\mathcal{B}}_{p}\right\| \simeq \sup _{t>0} t^{[p]}\left\|U_{0}\right\|_{p}=\left\|U_{0}\right\|_{E_{p}} \leq \eta(p) \doteq\left(4 C_{p}\right)^{-1}
$$

Donc il existe une solution $u \in E_{p}$ de l'équation (2.2), unique dans la boule de $E_{p}$ de rayon $2 \eta(p)$, et $\|u\|_{E_{p}} \leq 2\left\|U_{0}\right\|_{E_{p}}$.

La Proposition 38 assure que $u \in E_{q}$ pour $q \geq r$. D'après les propositions 30 et $28, \Theta(u, u) \in C_{b}\left(\mathbb{R}_{+} ; \dot{\mathcal{B}}_{r}\right)$ et $\Theta(u, u)(t) \longrightarrow 0$ dans $W_{r}^{-2}(A)$ quand $t \longrightarrow 0$. Or $U_{0} \in C_{b}\left(\mathbb{R}_{+} ; \dot{\mathcal{B}}_{r}\right)$ et $U_{0} \longrightarrow u_{0}$ dans $W_{r}^{-2}(A)$ quand $t \longrightarrow 0$, d'après le Lemme 21. Donc $u$ aussi.

### 3.2.3. Quand $t \longrightarrow \infty$.

Ayant des solutions globales, on s'intéresse à leur comportement quand $t$ tend vers l'infini. Nous avons introduit les espaces $E_{p ; 0}(\infty)$ dans ce but. Comme [28] dans le cas de $\mathbb{R}^{3}$, nous montrons que deux solutions ont le même comportement à l'infini s'il en va de même de leurs parties linéaires.

Théorème 40. Soit $n<p$, et $u^{1}$, $u^{2}$ deux solutions dans $E_{p} d e$ l'équation de point fixe

$$
u^{1}=U_{0}^{1}-\Theta\left(u^{1}, u^{1}\right), \quad u^{2}=U_{0}^{2}-\Theta\left(u^{2}, u^{2}\right)
$$

Étant donné $n \leq q$, on suppose que $\left\|u^{1}\right\|_{E_{p}}+\left\|u^{2}\right\|_{E_{p}}<C^{-1}$ où $C$ est la constante de continuité de $\Theta$ de $E_{p} \times E_{q}$ où $E_{q} \times E_{p}$ dans $E_{q}$. Si $U_{0}^{2}-U_{0}^{1} \in E_{q ; 0}(\infty)$ alors $u^{2}-u^{1} \in E_{q ; 0}(\infty)$.

Démonstration. Notons $w=u^{2}-u^{1}$ et $W_{0}=U_{0}^{2}-U_{0}^{1}$. Alors $w=W_{0}-\Theta\left(w, u^{2}\right)-\Theta\left(u^{1}, w\right)$. Cette équation de point fixe $w=g(w)$
a un sens dans $E_{q}$ dès que $W_{0} \in E_{q}$ et $\Theta$ est continu de $E_{p} \times E_{q}$ et $E_{q} \times E_{p}$ dans $E_{q}$. Elle admet une solution unique puisque $g$ est affine et contractante de rapport $k=C\left(\left\|u^{1}\right\|_{E_{p}}+\left\|u^{2}\right\|_{E_{p}}\right)<1$. w est nécessairement la limite dans $E_{q}$ de la suite définie par $w_{0}=W_{0}$, $w_{i+1}=g\left(w_{i}\right)$. On montre alors par récurrence que $w_{i} \in E_{q ; 0}(\infty)$ dès que $W_{0} \in E_{q ; 0}(\infty)$, grâce à la Proposition 29. Il s'ensuit que $w \in$ $E_{q ; 0}(\infty)$.

### 3.3. Unicité des solutions locales $C\left(X_{n}\right)$.

Lemme 41. Soit $u_{0} \in X_{n}$ et $U_{0} \doteq U(t) u_{0}$. Soit $u^{1}$ et $u^{2}$ dans $C\left(\left[0, T\left[; X_{n}\right)\right.\right.$, solutions de l'équation (2.2). Alors il existe $\varepsilon>0$ tel que $u^{1}=u^{2}$ sur $[0, \varepsilon] \cap[0, T[$.

Démonstration. La différence $w \doteq u^{2}-u^{1}$ des deux solutions de (2.2) vérifie, au moins formellement,

$$
\begin{aligned}
w & =\Theta\left(u^{1}, u^{1}\right)-\Theta\left(u^{2}, u^{2}\right) \\
& =-\Theta\left(w, u^{1}-U_{0}\right)-\Theta\left(u^{2}-U_{0}, w\right)-\Theta\left(w, U_{0}\right)-\Theta\left(U_{0}, w\right)
\end{aligned}
$$

Par le Corollaire 34, on sait que $\Theta$ est bilinéaire continu de $G_{r}^{\varepsilon} \times F_{p}^{\varepsilon}$ et $F_{p}^{\varepsilon} \times G_{r}^{\varepsilon}$ dans $G_{r}^{\varepsilon}$, pour $r$ et $p$ vérifiant (3.8), avec une norme indépendante de $\varepsilon \leq 1$. Fixons $p$ et $r$ tels que $0<1 / r-1 / n<1 / p<$ $1 / n$. On a alors

$$
\|w\|_{G_{r}^{\varepsilon}} \leq C\|w\|_{G_{r}^{\varepsilon}}\left(2\left\|U_{0}\right\|_{F_{p}^{\varepsilon}}+\left\|u^{1}-U_{0}\right\|_{F_{n}^{\varepsilon}}+\left\|u^{2}-U_{0}\right\|_{F_{n}^{\varepsilon}}\right) .
$$

Le premier terme dans la parenthèse tend vers 0 quand $\varepsilon \longrightarrow 0$, d'après le Lemme 39. Les deux autres termes tendent aussi vers 0 car $u^{i}$ et $U_{0}$ sont continues à valeurs dans $X_{n}$ et égales en 0 . Pour $\varepsilon$ assez petit, $\|w\|_{G_{r}^{\varepsilon}}<\|w\|_{G_{r}^{\varepsilon}}$ et donc $w=0$ sur $[0, \varepsilon]$.

Démonstration du Théorème 26. Soit $J \doteq\left\{t \in\left[0, T\left[: u^{1}=\right.\right.\right.$ $u^{2}$ sur $\left.[0, t]\right\}$. Par continuité des fonctions $u^{i}, J$ est fermé. Pour tout $T^{\prime} \in\left[0, T\left[, \widetilde{u}^{i}: t \longmapsto u^{i}\left(T^{\prime}+t\right)\right.\right.$ appartient à $C\left(\left[0, T-T^{\prime}\left[; X_{n}\right)\right.\right.$, et résout l'équation (2.2) pour la donnée initiale $\widetilde{u}_{0}^{i}=u^{i}\left(T^{\prime}\right)$. Le lemme appliqué à $\widetilde{u}^{i}$ assure que $J$ est ouvert. Or $0 \in J \operatorname{car} u^{1}(0)=u_{0}=u^{2}(0)$, donc $J=[0, T[$.

## 4. L'exemple.

Il s'agit de démontrer le Lemme 24 en exhibant un exemple de suite $u^{k}$. Comme dans [4], nous allons montrer deux points. D'une part que le produit de $u_{0}$ par une suite $w^{k}$ très oscillante tend vers 0 dans un espace de Besov homogène d'indice négatif. D'autre part qu'en choisissant convenablement $w^{k}$, le produit ne tend pas vers 0 en norme $L^{p}$. Plus précisément, voici les énoncés qui tiennent compte du fait que le produit $u_{0} w^{k}$ n'est pas en général dans $X_{n}$ :

Proposition 42. Soit $u_{0} \in L^{n}(\Omega)$ et $\left\{w^{k}: k \in \mathbb{N}\right\} \subset L^{\infty}(\Omega)$ une suite bornée qui tend faiblement vers 0 . Alors pour $n<p$

$$
\lim _{k \rightarrow \infty} \sup _{t>0} t^{[p]}\left\|U(t) P\left(u_{0} w^{k}\right)\right\|_{p}=0
$$

Proposition 43. Soit $u_{0} \in L^{n}(\Omega)$ et $w^{k}(x)=\phi(x) \exp (i k \xi \cdot x)$, où $\phi \in C^{\infty}(\Omega)$ est nulle au voisinage de $\partial \Omega$ et vaut 1 au voisinage de $\infty$ et $\xi \in \mathbb{R}^{n}$ avec $|\xi|=1$. Si $\left\|\phi u_{0}\right\|_{p}>0$ alors on peut choisir $\xi$ de sorte que

$$
\liminf _{k \in \mathbb{N}}\left\|P\left(u_{0} w^{k}\right)\right\|_{p}>0
$$

### 4.1. Convergence dans un espace de Besov d'indice négatif.

La preuve de la Proposition 42 consiste à se ramener au
Lemme 44. Soit $u_{1} \in C_{0}^{\infty}(\Omega)$ et $\left\{w^{k}\right\}$ comme dans l'énoncé de la proposition. Soit $K$ un compact de $\mathbb{C} \backslash]-\infty, 0]$. Pour tout $\mu \in K$,

$$
\lim _{k \rightarrow \infty}\left\|(\mu+A)^{-1} P\left(u_{1} w^{k}\right)\right\|_{p}=0
$$

Preuve que le Lemme 44 implique la Proposition 42. Fixons $\varepsilon>0$. On peut toujours écrire $u_{0}=u_{1}+u_{2}$ avec $u_{1} \in C_{0}^{\infty}(\Omega)$ et $\left\|u_{2}\right\|_{n}<\varepsilon$, et donc, d'après (1.6),

$$
\sup _{k \in \mathbb{N}} \sup _{t>0} t^{[p]}\left\|U(t) P\left(u_{2} w^{k}\right)\right\|_{p} \leq C \sup _{k \in \mathbb{N}}\left\|P\left(u_{2} w^{k}\right)\right\|_{n} \leq C \varepsilon .
$$

Pour $u_{1}$, on dit d'abord qu'il existe $T_{\varepsilon}$ tel que

$$
\sup _{k \in \mathbb{N}} \sup _{t \notin\left[T_{\varepsilon}^{-1}, T_{\varepsilon}\right]} t^{[p]}\left\|U(t) P\left(u_{1} w^{k}\right)\right\|_{p} \leq C \varepsilon .
$$

En effet, pour $t<T_{\varepsilon}^{-1}$, on invoque que $u_{1}$ appartient à $L^{p}(\Omega)$, donc la suite des $P\left(u_{1} w^{k}\right)$ est bornée dans $X_{p}$. Comme le semi-groupe $U(t)$ est borné sur cet espace, il ne reste plus qu'à majorer $t^{[p]}$. Or pour $p>n$, on sait que $[p]>0$. Pour $t>T_{\varepsilon}$, on tient le même raisonnement en partant de $u_{1} \in L^{q}(\Omega)$, en choisissant $q<n$. Notons simplement que $[p]-[q, p]=[q]<0$.

Sur l'intervalle compact $\left[T_{\varepsilon}^{-1}, T_{\varepsilon}\right]$ les fonctions

$$
t \longmapsto t^{[p]}\left\|U(t) P\left(u_{1} w^{k}\right)\right\|_{p}
$$

sont uniformément équicontinues. En effet on sait qu'il existe des constantes telles que

$$
\partial_{t} U(t) u=-A U(t) u \quad \text { et } \quad \sup _{t>0}\|t A U(t) u\|_{p} \leq C\|u\|_{p}
$$

pour tout $u \in X_{p}$,

$$
\sup _{t>0}\left\|t^{1+[p]} A U(t) u\right\|_{p} \leq C\|u\|_{n},
$$

pour tout $u \in X_{n}$. En appliquant ceci à $u=u_{1} w^{k}$, on obtient

$$
\sup _{k \in \mathbb{N}}\left\|\partial_{t}\left(t^{[p]} U(t) P\left(u_{1} w^{k}\right)\right)\right\|_{p} \leq C t^{-1}\left\|u_{1}\right\|_{n}
$$

pour tout $t \in\left[T_{\varepsilon}^{-1}, T_{\varepsilon}\right]$. La norme $\|\cdot\|_{p}$ est une application lipschitzienne. On en déduit l'équicontinuité uniforme des fonctions

$$
t^{[p]}\left\|U(t) P\left(u_{1} w^{k}\right)\right\|_{p}
$$

D'après l'une des versions du théorème d'Ascoli, une suite équicontinue de fonctions qui converge simplement converge uniformément sur tout compact. Il nous suffit donc de vérifier que pour tout $t \in\left[T_{\varepsilon}^{-1}, T_{\varepsilon}\right]$,

$$
\lim _{k \rightarrow \infty}\left\|U(t) P\left(u_{1} w^{k}\right)\right\|_{p}=0
$$

On fixe donc maintenant $t$ dans cet intervalle.
Il est plus facile de contrôler l'influence du bord $\partial \Omega$ sur la résolvante. On écrit donc le semi-groupe à l'aide de l'intégrale de Dunford sur le contour $\Gamma=\Gamma_{\theta}$, qui est le bord dans $\left.\left.\mathbb{C} \backslash\right]-\infty, 0\right]$ de l'ouvert contenant
les $\lambda$ tels que $\pi-\theta<|\arg (\lambda)|<\pi$ ou $|\lambda|<\theta$ (où on a fixé un $\theta$ tel que $\pi / 2<\pi-\theta<\pi$ )

$$
\begin{align*}
U(t) u & =\frac{1}{2 i \pi} \int_{\Gamma} e^{t \lambda}(\lambda+A)^{-1} u d \lambda \\
& =\frac{1}{2 i \pi} \int_{\Gamma} e^{\lambda} t^{-1}\left(t^{-1} \lambda+A\right)^{-1} u d \lambda \tag{4.1}
\end{align*}
$$

La seconde égalité vient du changement de variable $t \lambda \longmapsto \lambda$. On peut conserver le même contour $\Gamma$ d'intégration grâce au théorème de Cauchy. On sait grâce l'estimation (1.4) sur la résolvante que

$$
\sup _{k \in \mathbb{N}}\left\|(\lambda+A)^{-1} P\left(u_{1} w^{k}\right)\right\|_{p} \leq C|\lambda|^{-1}\left\|u_{1}\right\|_{p}
$$

Il existe donc $R_{\varepsilon}$ tel qu'en notant $B_{\varepsilon}$ la boule fermée dans $\mathbb{C}$ de rayon $R_{\varepsilon}$, on ait

$$
\sup _{k \in \mathbb{N}}\left\|\frac{1}{2 i \pi} \int_{\Gamma \backslash B_{\varepsilon}} e^{\lambda} t^{-1}\left(t^{-1} \lambda+A\right)^{-1} P\left(u_{1} w^{k}\right) d \lambda\right\|_{p} \leq C \varepsilon
$$

Appelons $K$ le compact de $\mathbb{C} \backslash]-\infty, 0$ ] que décrit $t^{-1} \lambda$ quand $t$ varie dans $\left[T_{\varepsilon}^{-1}, T_{\varepsilon}\right]$ et $\lambda$ varie dans $\Gamma \cap B_{\varepsilon}$. Grâce au théorème de convergence dominée appliquée à l'intégrale sur $\Gamma \cap B_{\varepsilon}$, il nous suffit donc de montrer que pour tout $\mu \in K$,

$$
\lim _{k \rightarrow \infty}\left\|(\mu+A)^{-1} P\left(\mu_{1} w^{k}\right)\right\|_{p}=0
$$

Nous décomposons la démonstration du Lemme 44 en trois étapes qui s'enchaînent naturellement.

Lemme 45 (Première étape). Soit $u_{1} \in C_{0}^{\infty}(\Omega)$ et $\left\{w^{k}\right\}$ une suite bornée dans $L^{\infty}(\Omega)$ qui tend vers 0 pour la topologie $\star$-faible. Alors $f^{k}=P\left(u_{1} w^{k}\right)$ est bornée dans $X_{p}$, tend faiblement vers 0 , et est uniformément p-intégrable à l'infini : pour tout $\varepsilon$ il existe $R_{\varepsilon}$ tel que pour tout $k \in \mathbb{N}$

$$
\begin{equation*}
\left\|f^{k}\right\|_{p, \Omega \backslash B_{\varepsilon}}<\varepsilon \tag{4.2}
\end{equation*}
$$

où $B_{\varepsilon}$ est la boule fermée de rayon $R_{\varepsilon}$.

Démonstration. La borne dans $X_{p}$ et la convergence faible viennent de la continuité (forte et donc aussi faible) de $P$.

Soit $K^{\prime}$ un compact de $\Omega$ contenant le support de $u_{1}$. Pour tout $u$ dans $L^{p}\left(K^{\prime}\right)$ étendu à $\Omega$ par 0 hors de $K^{\prime}$, on sait que $(1-P) u=\nabla p$ où $p$ est l'unique solution du problème de Laplace-Neumann

$$
\begin{equation*}
\Delta p=\nabla \cdot u, \quad \gamma_{1} p=0, \quad \nabla p \in L^{p}(\Omega) \tag{4.3}
\end{equation*}
$$

En effet, d'après la décomposition de Helmholtz (1.1), il existe un $p$ dans $L_{\mathrm{loc}}^{p}(\Omega)$, unique à une constante près, tel que $(1-P) u=\nabla p$. Ceci implique que $p$ vérifie $\Delta p=\nabla \cdot u$ et $\gamma_{1} p=0$, puisque le support de $u$ ne rencontre pas $\partial \Omega$. L'unicité de $p$ sous la condition $\nabla p \in L^{p}(\Omega)$ vient de [25, Lemma 1.4]. Soit $B$ une boule de rayon $R$ si grand que $\left(K^{\prime} \cup \partial \Omega\right) \subset(1 / 2) B$. Il est clair que $p$ et donc $\nabla p$ sont harmoniques sur $\Omega \backslash B$. D'après le Théorème 50 de représentation des fonctions harmoniques, on peut développer $p$ en série de Laurent

$$
p(x)=\sum_{j=0}^{\infty}|x|^{2-n-2 j} H_{j}(x)+\sum_{j=0}^{\infty} H_{j}^{\prime}(x)
$$

où $H_{j}$ et $H_{j}^{\prime}$ sont des polynômes harmoniques homogènes de degré $j$. La première série converge uniformément sur $\Omega \backslash B$, tandis que la seconde converge uniformément sur les couronnes $B_{r} \backslash B$ pour tout $r \geq R$, avec $B_{r}$ la boule de rayon $r$. La série est dérivable terme à terme

$$
\nabla p(x)=\sum_{j=0}^{\infty} \nabla\left(|x|^{2-n-2 j} H_{j}(x)\right)+\sum_{j=0}^{\infty} \nabla H_{j}^{\prime}(x)
$$

Le terme $\nabla H_{0}^{\prime}$ est nul, et comme on sait que $\nabla p \in L^{p}(\Omega)$, nécéssairement $H_{j}^{\prime}(x)=0$ pour $j>0$. Dans la première série, le terme d'indice $j$ est en $O\left(|x|^{1-n-j}\right)$, et le terme d'indice 0 est exactement $c_{n} H_{0}|x|^{-n} x$. Par ailleurs le théorème de Stokes implique, pour tout $r \geq R$,

$$
\int_{\partial B_{r}} \gamma_{1} p=\int_{\Omega \cap B_{r}} \nabla \cdot u-\int_{\partial \Omega} \gamma_{1} p=\int_{\partial \Omega \cup \partial B_{r}} \gamma_{\nu} u=0 .
$$

Si on remplace $p$ par la série dans l'intégrale, quand on fait tendre $r$ vers l'infini, on voit que nécessairement $H_{0}(x)=0$. On obtient $\nabla p(x)=$ $O\left(|x|^{-n}\right)$ quand $x \longrightarrow \infty$. Notons $\mathcal{H}$ l'espace des champs harmoniques sur $\Omega \backslash B$, continus sur $\overline{\Omega \backslash B}$ et telles que

$$
\|u\|_{\mathcal{H}} \doteq \sup _{x \in \Omega \backslash B}|x|^{n}|u(x)|<\infty .
$$

Il est immédiat que $\mathcal{H}$ muni de $\|u\|_{\mathcal{H}}$ est un espace de Banach. On considère l'opérateur linéaire $Q$ de $L^{p}\left(K^{\prime}\right)$ dans $\mathcal{H}$ qui associe à $u$ la restriction à $\Omega \backslash B$ de $u-P u=\nabla p$. Nous affirmons que $Q$ est un opérateur fermé. En effet soit $u^{k} \longrightarrow u$ dans $L^{p}\left(K^{\prime}\right)$ et $Q u^{k} \longrightarrow v$ dans $\mathcal{H}$. Ceci implique en particulier que $Q u^{k} \longrightarrow v$ dans $L^{p}(\Omega \backslash B)$. Par ailleurs, la continuité de $1-P$ sur $L^{p}(\Omega)$ implique que $Q u^{k} \longrightarrow Q u$ dans $L^{p}(\Omega \backslash B)$. Donc $v=Q u$. Par le théorème du graphe fermé, on a donc une constante $C$ telle que pour tout $u \in L^{p}\left(K^{\prime}\right)$,

$$
\|Q u\|_{\mathcal{H}} \leq C\|u\|_{p, K^{\prime}} .
$$

Revenons à $f^{k}=P\left(u_{1} w^{k}\right)=u_{1} w^{k}-(1-P)\left(u_{1} w^{k}\right)$. On voit que

$$
\sup _{x \in \Omega \backslash B}|x|^{n}\left|f^{k}(x)\right| \leq C\left\|u_{1}\right\|_{p, K^{\prime}}
$$

Cela implique bien que $f^{k}$ est uniformément $p$-intégrable à l'infini.
Avant d'énoncer le deuxième point, rappelons que $\widetilde{E}_{\lambda}$ a été défini à la Section 1.2.1 pour $\lambda \in \mathbb{C} \backslash]-\infty, 0]$, comme l'opérateur de convolution par le potentiel volume $\widetilde{e}_{\lambda}$, transformée de Fourier inverse de $\left(\lambda+|\xi|^{2}\right)^{-1}$.

Lemme 46 (Deuxième étape). Soit $\lambda \in K$ et $f^{k}$ une suite bornée dans $X_{p}$, qui tend faiblement vers 0 et est uniformément $p$-intégrable à l'infini (i.e. vérifie (4.2)). On note $\tilde{f}^{k}$ l'extension de $f^{k}$ par 0 hors de $\Omega$, et $\widetilde{u}^{k}=\widetilde{E}_{\lambda} \widetilde{f}^{k}$. Alors $\widetilde{u}^{k}$ est bornée dans $W_{p}^{1}\left(\mathbb{R}^{n}\right)$, à divergence nulle, et tend vers 0 dans $L^{p}\left(\mathbb{R}^{n}\right)$.

Démonstration. Il est immédiat que $\tilde{f}^{k}=\mathrm{e}_{\Omega} f^{k}$ est bornée dans $L^{p}\left(\mathbb{R}^{n}\right)$, et on a déjà vu que $\nabla \cdot \widetilde{f}^{k}=0$. Par le théorème Mihlin, on en déduit que $\widetilde{u}^{k}$ est bornée dans $W_{p}^{1}\left(\mathbb{R}^{n}\right)$ et à divergence nulle. Soit $\chi_{R}=\chi(x / R)$ avec $\chi$ une fonction régulière égale à 1 sur la boule de rayon 1 et nulle hors de la boule de rayon 2 . Quel que soit $R>0$, la suite $\chi_{2 R} \widetilde{u}^{k}$ est donc compacte dans $L^{p}\left(\mathbb{R}^{n}\right)$, d'après l'injection de Sobolev. Soit $\phi$ dans $C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$. $\widetilde{E}_{\lambda}$ étant continu sur $L^{p}\left(\mathbb{R}^{n}\right)$, la convergence faible supposée de $f^{k}$ implique que $\widetilde{u}^{k}$ converge faiblement vers 0 dans $L^{p}\left(\mathbb{R}^{n}\right)$, de même que $\chi_{2 R} \widetilde{u}^{k}$. Par compacité, $\chi_{2 R} \widetilde{u}^{k}$ tend fortement vers 0 .

Si $|x|>2 R$, alors

$$
\left(\chi_{R} \widetilde{e}_{\lambda} * \widetilde{f}^{k}\right)(x)=\left(\chi_{R} \widetilde{e}_{\lambda} *\left(1-\chi_{R / 2}\right) \widetilde{f}^{k}\right)(x)
$$

Donc, d'après (4.2), pour tout $\varepsilon$ il existe $R_{\varepsilon}$ tel que pour tout $R>R_{\varepsilon}$ et tout $k \in \mathbb{N}$,

$$
\left\|\left(1-\chi_{2 R}\right)\left(\chi_{R} \widetilde{e}_{\lambda} * \widetilde{f}^{k}\right)\right\|_{p} \leq C\left\|\left(1-\chi_{R / 2}\right) \widetilde{f}^{k}\right\|_{p}<\varepsilon .
$$

D'un autre côté, comme $\xi \longmapsto \partial_{\xi}^{\alpha}\left(\lambda+|\xi|^{2}\right)^{-1}$ est $L^{1}\left(\mathbb{R}^{n}\right)$ pour $|\alpha|=$ $n+2$, il existe $C$ tel que

$$
\left|\left(1-\chi_{R}\right) \widetilde{e}_{\lambda}(x)\right| \leq C\langle R\rangle^{-1}\langle x\rangle^{-n-1}, \quad\left\|\left(1-\chi_{R}\right) \widetilde{e}_{\lambda}\right\|_{1} \leq C\langle R\rangle^{-1}
$$

et donc, quitte à augmenter $R_{\varepsilon}$, pour tout $R>R_{\varepsilon}$ et tout $k \in \mathbb{N}$,

$$
\left\|\left(1-\chi_{2 R}\right)\left(\left(1-\chi_{R}\right) \widetilde{e}_{\lambda} * \widetilde{f}^{k}\right)\right\|_{p} \leq C\langle R\rangle^{-1}\left\|\widetilde{f}^{k}\right\|_{p}<\varepsilon .
$$

Finalement on a montré que

$$
\lim _{R \rightarrow \infty} \sup _{k \in \mathbb{N}}\left\|\left(1-\chi_{2 R}\right) \widetilde{u}^{k}\right\|_{p}=0
$$

ce qui implique que $\widetilde{u}^{k}$ tend vers 0 en norme $L^{p}$.
Avant d'énoncer la troisième étape, rappelons la décomposition de la résolvante vue à la Section 1.2.1. Soit $f \in X_{p}$. On note $\widetilde{f}=\mathrm{e}_{\Omega} f$ et $\widetilde{u}=\widetilde{E}_{\lambda} \widetilde{f}$. Alors on peut écrire

$$
\begin{gathered}
(\lambda+A)^{-1} f=u_{\nu}+u_{\tau}=\left(1-V_{\lambda} \gamma_{0}\right) P \mathrm{r}_{\Omega} \widetilde{u}, \\
u_{\nu}=P \mathrm{r}_{\Omega} \widetilde{u}=P_{\nu} \mathrm{r}_{\Omega} \widetilde{u}=\left(1-\nabla N \gamma_{\nu}\right) \mathrm{r}_{\Omega} \widetilde{u}, \\
u_{\tau}=-V_{\lambda} \gamma_{0} u_{\nu}, \quad \gamma_{0} u_{\nu}=\gamma_{0} \mathrm{r}_{\Omega} \widetilde{u}-\left(\gamma_{0} \nabla N\right) \gamma_{\nu} \mathrm{r}_{\Omega} \widetilde{u},
\end{gathered}
$$

où $V_{\lambda}$ est l'opérateur solution du problème de Stokes tangentiel (1.3).
Lemme 47 (Troisième étape). Soit $\widetilde{u}^{k}$ une suite bornée dans $W_{p}^{1}\left(\mathbb{R}^{n}\right)$, $\grave{a}$ divergence nulle, et qui tend vers 0 dans $L^{p}\left(\mathbb{R}^{n}\right)$. Alors $u^{k}=(1-$ $\left.V_{\lambda} \gamma_{0}\right) P r_{\Omega} \widetilde{u}^{k}$ tend vers 0 dans $L^{p}(\Omega)$.

DÉmonstration. Il est clair que l'hypothèse implique $\mathrm{r}_{\Omega} \widetilde{u}^{k}$ tend vers 0 dans $L^{p}$. Comme $P$ est continu sur $L^{p}(\Omega)$, la conclusion s'étend à $u_{\nu}^{k}=P \mathrm{r}_{\Omega} \widetilde{u}^{k}$. La borne sur $\nabla \widetilde{u}^{k}$ dans $L^{p}\left(R^{n}\right)$ et l'inégalité de trace

$$
\left\|\gamma_{0} \mathrm{r}_{\Omega} \widetilde{u}\right\|_{p, \partial \Omega} \leq C\|\widetilde{u}\|_{p, \mathbb{R}^{n}}^{1-1 / p}\|\nabla \widetilde{u}\|_{p, \mathbb{R}^{n}}^{1 / p}
$$

montrent que $\gamma_{0} \mathrm{r}_{\Omega} \widetilde{u}^{k}$ tend vers 0 dans $L^{p}(\partial \Omega)$. D'après [9, Lemma 2.3], $\gamma_{0} \nabla N$ est un opérateur continu sur cet espace, donc $\psi^{k}=-\gamma_{0} u_{\nu}^{k}=$ $\left(1-\gamma_{0} \nabla N \pi_{\nu}\right) \gamma_{0} \mathrm{r}_{\Omega} \widetilde{u}^{k}$ tend encore vers 0 dans cet espace. Finalement la continuité de $V_{\lambda}$ de $L^{p}(\partial \Omega)$ dans $L^{p}(\Omega)$ implique que $u_{\tau}^{k}=V_{\lambda} \psi^{k}$ tend vers 0 .

Résumons la preuve du Lemme 44. On écrit $(\lambda+A)^{-1} P\left(u_{1} w^{k}\right)=$ $u^{k}$ avec

$$
\begin{gathered}
u^{k}=\left(1-V_{\lambda} \gamma_{0}\right) P \mathrm{r}_{\Omega} \widetilde{u}^{k}, \\
\widetilde{u}^{k}=\widetilde{E}_{\lambda} \widetilde{f}^{k}, \quad \widetilde{f}^{k}=\mathrm{e}_{\Omega} f^{k}, \\
f^{k}=P\left(u_{1} w^{k}\right),
\end{gathered}
$$

et on applique successivement les lemmes 45,46 et 47 .

### 4.2. Non convergence vers 0 dans un espace de Lebesgue.

Nous donnons maintenant la preuve de la Proposition 43.
Rappelons tout d'abord un résultat de [4]. On se place sur $\mathbb{R}^{n}$. On note $\widetilde{P}$ l'opérateur de projection sur les champs à divergence nulle. C'est le multiplicateur de Fourier associé à la fonction

$$
\widetilde{P}(\xi)=\left(1-\frac{\xi \otimes \xi}{|\xi|^{2}}\right),
$$

régulière hors de 0 et homogène de degré 0 . Soit $\widetilde{u} \in L^{p}\left(\mathbb{R}^{n}\right)$ pour $1<p<\infty$, et $\widetilde{w}^{k}=e^{i k \xi \cdot x}$ avec $|\xi|=1$. Alors

$$
\lim _{k \rightarrow \infty}\left\|\widetilde{P}\left(\widetilde{u} \widetilde{w}^{k}\right)-\widetilde{w}^{k} \widetilde{P}(\xi) \widetilde{u}\right\|_{p}=0
$$

Revenons à la situation de la Proposition $43, u_{0} w^{k}=u_{0} \phi \exp (i k \xi \cdot x)$. On peut considérer $\widetilde{u}=u_{0} \phi$ comme un fonction dans $L^{n}\left(\mathbb{R}^{n}\right)$. $\widetilde{P}(\xi) \widetilde{u}$ vaut alors $\phi \widetilde{P}(\xi) u_{0}$. On obtient

$$
\lim _{k \rightarrow \infty}\left\|\widetilde{P}\left(u_{0} w^{k}\right)-w^{k} \widetilde{P}(\xi) u_{0}\right\|_{p}=0
$$

Comme $|\exp (i k \xi \cdot x)|=1$, on sait que $\left\|w^{k} \widetilde{P}(\xi) u_{0}\right\|_{p}=\left\|\phi \widetilde{P}(\xi) u_{0}\right\|_{p}$, qui ne dépend pas de $k$ et qu'on peut supposer être non nul en choisissant bien $\xi$, puisque par hypothèse $\left\|\phi u_{0}\right\|_{p}>0$. On en déduit, d'après (4.5), que $\left\|\mathrm{r}_{\Omega} \widetilde{P}\left(u_{0} w^{k}\right)\right\|_{p}$ ne tend pas vers 0 .

Soit $\mathcal{V}_{\partial \Omega}$ un voisinage de $\partial \Omega$, compact dans $\mathbb{R}^{n}$ et qui ne rencontre pas le support de $\phi$. Il est clair que la restriction de $\phi \widetilde{P}(\xi) u_{0}$ à $\mathcal{V}_{\partial \Omega}$ est nulle, ce qui implique $\left\|\phi \widetilde{P}(\xi) u_{0}\right\|_{p, \mathcal{V}_{\partial \Omega}}=0$. D'après (4.5), on en déduit que $\left\|\mathrm{r}_{\Omega} \widetilde{P}\left(u_{0} w^{k}\right)\right\|_{p, \nu_{\partial \Omega} \cap \Omega}$ tend vers 0 . Ceci va nous permettre de passer de $\mathrm{r}_{\Omega} \widetilde{P}\left(u_{0} w^{k}\right)$ à $P\left(u_{0} w^{k}\right)$.

En effet pour un champ $f \in L^{p}(\Omega)$ à divergence nulle, on a déjà vu que $\operatorname{Pf}=P_{\nu} f=\left(1-\nabla N \gamma_{\nu}\right) f$. Et on a déjà vu que $\gamma_{\nu}$ est continu de l'ensemble des champs à divergence nulle de $L^{p}(\Omega)$ dans $W_{p}^{-1 / p}(\partial \Omega)$. Il est clair que $\gamma_{\nu} u$ ne dépend que de la restriction de $u$ à un voisinage de $\partial \Omega$, donc en fait $\gamma_{\nu}$ est continu de l'ensemble des champs à divergence nulle de $L^{p}\left(\mathcal{V}_{\partial \Omega} \cap \Omega\right)$ dans $W_{p}^{-1 / p}(\partial \Omega)$. Donc $\gamma_{\nu} \mathrm{r}_{\Omega} \widetilde{P}\left(u_{0} w^{k}\right)$ tend vers 0 dans $W_{p}^{-1 / p}(\partial \Omega)$. Comme en fait il s'agit de la trace normale sur $\partial \Omega$ d'une fonction à divergence nulle sur $\mathbb{R}^{n}$ tout entier, le théorème de Stokes implique que son intégrale sur $\partial \Omega$ est nulle.

Ensuite $\nabla N$ est continu de l'ensemble de fonctions $W_{p}^{-1 / p}(\partial \Omega)$ d'intégrale égale à 0 dans $L^{p}(\Omega)$. Cela montre que $\nabla N \gamma_{\nu} \mathrm{r}_{\Omega} \widetilde{P}\left(u_{0} w^{k}\right)$ tend vers 0 dans $L^{p}(\Omega)$. Enfin,

$$
\liminf _{k \rightarrow \infty}\left\|\mathrm{r}_{\Omega} \widetilde{P}\left(u_{0} w^{k}\right)\right\|_{p}>0
$$

et

$$
\lim _{k \rightarrow \infty}\left\|\nabla N \gamma_{\nu} \mathrm{r}_{\Omega} \widetilde{P}\left(u_{0} w^{k}\right)\right\|_{p}=0
$$

impliquent que $P\left(u_{0} w^{k}\right)=\left(1-\nabla N \gamma_{\nu}\right) \mathrm{r}_{\Omega} \widetilde{P}\left(u_{0} w^{k}\right)$ ne tend pas vers 0 dans $X_{n}$.

## A. Interpolation réelle.

Rappelons deux caractérisations des espaces d'interpolation réelle.
Proposition 48. Soit $\left(A_{i},\|\cdot\|_{i}\right), i=0$, 1 , deux espaces de Banach qui s'injecte continûment chacun dans un espace vectoriel topologique séparé $\mathcal{A}$. Soit $q \in[1, \infty]$ et $\theta \in] 0,1[$. Les deux définitions suivantes définissent le même espace de Banach avec des normes équivalentes.

1) Soit $\xi_{i}$ deux nombres réels tels que $\xi_{0} \xi_{1}<0$ et $(1-\theta) \xi_{0}+\theta \xi_{1}=$ 0 . On considère l'ensemble des éléments $u \in A_{0}+A_{1}$ qui s'écrivent

$$
\begin{equation*}
u=\int_{0}^{\infty} u(t) \frac{d t}{t} \tag{A.1}
\end{equation*}
$$

où l'intégrale converge dans $\mathcal{A}$ et où $u(t)$ est une fonction de $\mathbb{R}_{+}$dans $A_{0} \cap A_{1}$ vérifiant

$$
\begin{equation*}
\left\|t^{\xi_{0}}\right\| u(t)\left\|_{0}\right\|_{q^{*}}+\left\|t^{\xi_{1}}\right\| u(t)\left\|_{1}\right\|_{q^{*}}<\infty \tag{A.2}
\end{equation*}
$$

La norme de u est la borne inférieure de l'ensemble des quantités (A.2) pour toutes les fonctions $u(t)$ réalisant (A.1).
2) Soient $\xi_{i}$ deux nombres réels tels que $\xi_{0} \xi_{1}<0$ et $(1-\theta) \xi_{0}+$ $\theta \xi_{1}=0$. On considère l'ensemble des éléments $u \in A_{0}+A_{1} q u i$ s'écrivent

$$
\begin{equation*}
u=u_{0}(t)+u_{1}(t), \quad \text { pour tout } t>0 \tag{A.3}
\end{equation*}
$$

ò̀ $u_{i}(t)$ est une fonction de $\mathbb{R}_{+}$dans $A_{i}$ et

$$
\left\|t^{\xi_{0}}\right\| u_{0}(t)\left\|_{0}\right\|_{q^{*}}+\left\|t^{\xi_{1}}\right\| u_{1}(t)\left\|_{1}\right\|_{q^{*}}<\infty
$$

La norme de u est la borne inférieure de l'ensemble des quantités (A.4) pour tous les couples de fonctions $\left(u_{0}(t), u_{1}(t)\right)$ réalisant (A.3).

On note $\left(A_{0}, A_{1}\right)_{\theta, q}$ cet espace.
Pour la démonstration de cette proposition, d'autres définitions de ces espaces et l'étude de leurs propriétés, nous renvoyons aux ouvrages tels que [1] ou [29], très complets avec de nombreuses références historiques.

L'intérêt principal de l'interpolation pour les équations aux dérivées partielles réside dans la propriété suivante :

Théorème 49. Soit $\left(A_{i},\|\cdot\|_{i}\right), i=0$, 1, deux espaces de Banach qui s'injectent continûment chacun dans un espace vectoriel topologique séparé $\mathcal{A}$. De même $\left(B_{i},\|\cdot\|_{i}\right)$ et $\mathcal{B}$.

Soit $T$ un opérateur linéaire de $A_{0}+A_{1}$ dans $B_{0}+B_{1}$, continu de $A_{0}$ dans $B_{0}$ et de $A_{1}$ dans $B_{1}$.

Alors, pour tout $\theta \in] 0,1\left[\right.$ et $q \in[1, \infty]$, $T$ est continu de $\left(A_{0}, A_{1}\right)_{\theta, q}$ dans $\left(B_{0}, B_{1}\right)_{\theta, q}$ et

$$
\|T\|_{\mathcal{L}\left(\left(A_{0}, A_{1}\right)_{\theta, q} ;\left(B_{0}, B_{1}\right)_{\theta, q}\right)} \leq\|T\|_{\mathcal{L}\left(A_{0} ; B_{0}\right)}^{1-\theta}\|T\|_{\mathcal{L}\left(A_{1} ; B_{1}\right)}^{\theta} .
$$

## B. Éléments de théorie du potentiel.

Soit $\Omega \subset \mathbb{R}^{n}$ un ouvert à bord $\partial \Omega$ lisse. Rappelons que $\gamma_{0}$ est l'opérateur de trace au bord relatif à $\Omega$ : si $u$ est continu sur $\bar{\Omega}, \gamma_{0} u(x)$ est la limite de $u(y)$ quand $y \in \Omega$ tend vers $x \in \partial \Omega$. On oriente le bord $\partial \Omega$ à l'aide du vecteur normal unitaire $\nu$ rentrant. Les opérateurs de trace au bord $\gamma_{\nu}$ et $\gamma_{1}$ sont définis en conséquence, avec en particulier $\gamma_{1}=\nu \cdot \gamma_{0} \nabla=\gamma_{\nu} \nabla$. Les formules de Stokes et Green s'écrivent donc

$$
\begin{align*}
\int_{\Omega}-\nabla \cdot u & =\int \gamma_{\nu} u  \tag{B.1}\\
\int_{\Omega}(u(-\Delta v)-(-\Delta u) v) & =\int_{\partial \Omega}\left(\gamma_{0} u \gamma_{1} v-\gamma_{1} u \gamma_{0} v\right) . \tag{B.2}
\end{align*}
$$

On note $\Omega^{\prime}$ l'ouvert tel que $\mathbb{R}^{n}=\Omega \cup \partial \Omega \cup \Omega^{\prime}$. On note $\gamma_{0}^{\prime}$ l'opérateur de trace de $\bar{\Omega}^{\prime}$ sur $\partial \Omega^{\prime}=\partial \Omega$. Il s'agit de la limite de $u(y)$ quand $y$, dans $\Omega^{\prime}$, tend vers $x \in \partial \Omega$. De même, on définit $\gamma_{\nu}^{\prime}=\nu \cdot \gamma_{0}^{\prime}$ et $\gamma_{1}^{\prime}=\gamma_{\nu}^{\prime} \nabla$. Notons que ces deux opérateurs sont les opposés des opérateurs $\gamma_{\nu}$ et $\gamma_{1}$ relatifs à $\Omega^{\prime}$.

On note $E(x)=c_{n}|x|^{2-n}$ la solution fondamentale de l'équation de Laplace en dimension $n \geq 3:-\Delta E=\delta$, avec $\delta$ la masse de Dirac en 0 . On note $G_{x}$ la fonction de Green associée qui à $y$ fait correspondre $G_{x}(y)=E(x-y)$.

Dans la suite de cette section, on convient que $\nabla G_{x}$ désigne le gradient de la fonction de Green par rapport à sa deuxième variable. De même pour la trace $\gamma_{0} G_{x}$ : c'est la seconde variable qu'on astreint à rester sur $\partial \Omega$. Pour noter qu'une dérivée ou une trace se rapporte à la première variable, on ajoutera un indice : $\nabla_{(x)} G_{x}, \gamma_{0(x)} G_{x}$. En particulier $-\Delta_{(x)} G_{x}=-\Delta G_{x}=\delta_{x}$ où $\delta_{x}$ est la masse de Dirac au point $x$, et
(B.3) $\gamma_{1} G_{x}(y)=-\nu(y) \cdot \nabla E(x-y)=\nu(y) \cdot \nabla E(y-x)=\gamma_{1(y)} G_{y}(x)$.

## B.1. Potentiels simple couche et double couche.

Soit $u$ une fonction $C^{\infty}$ au voisinage d'un ouvert borné $\Omega$. On suppose de plus que $u$ est harmonique, c'est-à-dire qu'elle vérifie $\Delta u=$
0. Alors en remplaçant $v$ par $G_{x}$ dans la formule de Green (B.2), on obtient

$$
\begin{equation*}
u(x)=\int_{\partial \Omega}\left(\gamma_{0} u \gamma_{1} G_{x}-\gamma_{1} u \gamma_{0} G_{x}\right) . \tag{B.4}
\end{equation*}
$$

Inversement, donnons-nous un ouvert régulier $\Omega$ et une fonction $\psi$ dans $C_{c}^{\infty}(\partial \Omega)$. On définit les potentiels simple couche $V_{\psi}^{(\mathrm{I})}$ et double couche $V_{\psi}^{(\mathrm{II})}$ par

$$
V_{\psi}^{(\mathrm{I})}=\int_{\partial \Omega} \psi \gamma_{0} G_{x}, \quad V_{\psi}^{(\mathrm{II})}=\int_{\partial \Omega} \psi \gamma_{1} G_{x}
$$

La théorie classique du potentiel dit que les restrictions à $\Omega$ (respectivement $\Omega^{\prime}$ ) de ces deux potentiels sont des fonctions $C^{\infty}$ jusqu'au bord de chacun de ces ouverts.

Le potentiel simple couche est continu à travers la surface, mais pas sa dérivée normale au bord. Elle est donnée par la formule

$$
\begin{align*}
& \gamma_{1} V_{\psi}^{(\mathrm{I})}(x)=-\frac{1}{2} \psi(x)+\int_{\partial \Omega} \psi \gamma_{1_{(x)}} G_{x} \\
& \gamma_{1}^{\prime} V_{\psi}^{(\mathrm{I})}(x)=+\frac{1}{2} \psi(x)+\int_{\partial \Omega} \psi \gamma_{1_{(x)}} G_{x} \tag{B.6}
\end{align*}
$$

L'intégrale au second membre est une intégrale impropre. On sait que

$$
\nabla_{(x)} G_{x}(y)=c_{n}(2-n)|x-y|^{1-n} \frac{x-y}{|x-y|}
$$

est homogène de degré $1-n$ en $x-y$ à $x$ fixé. Mais si $x$ et $y$ sont astreints à rester sur $\partial \Omega$, quand $y$ tend vers $x,(x-y) /|x-y|$ tend vers le plan tangent à $x$, orthogonal à $\nu(x)$. Et donc $\gamma_{1(x)} G_{x}(y)$ est en $O\left(|x-y|^{2-n}\right)$. Comme $\partial \Omega$ est de dimension $n-1$, l'intégrale impropre est convergente.

Le potentiel double couche n'est pas continu au travers de la surface $\partial \Omega$. Ses traces intérieure et extérieure sont données par

$$
\begin{align*}
& \gamma_{0} V_{\psi}^{(\mathrm{II})}(x)=+\frac{1}{2} \psi(x)+\int_{\partial \Omega} \psi \gamma_{1} G_{x}  \tag{B.7}\\
& \gamma_{0}^{\prime} V_{\psi}^{(\mathrm{II})}(x)=-\frac{1}{2} \psi(x)+\int_{\partial \Omega} \psi \gamma_{1} G_{x}
\end{align*}
$$

Même remarque sur l'intégrale singulière au second membre.
Pour le cas de la dimension d'espace $n=3$, on trouvera ces formules et d'autres encore dans [23].

## B.2. Décomposition en série de Laurent.

Voici l'équivalent pour les fonctions harmoniques dans $\mathbb{R}^{n}$ de la décomposition en série de Laurent pour les fonctions holomorphes dans $\mathbb{C}$.

Théorème 50. Soit $u \in \mathcal{D}^{\prime}(\mathcal{C})$ une distribution sur la couronne $\mathcal{C}$ définie par $\left\{x \in \mathbb{R}^{n}: r<|x|<R\right\}$. Si $\Delta u=0$ (on dit que $u$ est harmonique) alors on peut représenter $u$ par un unique développement en séries de Laurent

$$
\begin{equation*}
u(x)=\sum_{j \in \mathbb{N}} H_{j}^{\prime}(x)+\sum_{j \in \mathbb{N}}|x|^{-2 j+2-n} H_{j}(x) \tag{B.8}
\end{equation*}
$$

où $H_{j}^{\prime}$ et $H_{j}$ sont des polynômes harmoniques homogènes de degré j. La première (respectivement la seconde) série converge, ainsi que toutes ses dérivées, uniformément sur les boules de rayon inférieur à $R$ (respectivement hors des boules de rayon supérieur à r).

Indications pour la preuve. Comme $\Delta u=0$, le théorème de régularité elliptique implique qu'en réalité $u$ est $C^{\infty}(\mathcal{C})$. Donnons nous une autre couronne $\mathcal{C}_{1} \doteq\left\{x \in \mathbb{R}^{n}: r_{1}<|x|<R_{1}\right\}$ avec $r<r_{1}<R_{1}<$ R. $u$ est alors $C^{\infty}$ au voisinage de $\mathcal{C}_{1}$. D'après (B.4), pour tout $x \in \mathcal{C}_{1}$,

$$
u(x)=\int_{\partial \mathcal{C}_{1}}\left(\gamma_{0} u \gamma_{1} G_{x}-\gamma_{1} u \gamma_{0} G_{x}\right)=u_{r_{1}}(x)+u_{R_{1}}(x)
$$

où $u_{r_{1}}$ est la fonction donnée par l'intégrale quand on restreint le domaine d'intégration à la composante $|y|=r_{1}$ de $\partial \mathcal{C}_{1}$, et de même avec $R_{1}$. Puisque $\gamma_{0} u(y)$ et $\gamma_{1} u(y)$ sont $C^{\infty}, u_{R_{1}}$ et $u_{r_{1}}$ le sont sur $\overline{\mathcal{C}}$.

La représentation intégrale de $u_{R_{1}}$ permet d'étendre cette fonction en une fonction harmonique sur la boule $B_{R_{1}}, C^{\infty}$ sur la boule fermée. En décomposant la restriction de $u_{R_{1}}$ à la sphère $S_{R_{1}}=\partial B_{R_{1}}$ en harmoniques sphériques, on obtient

$$
u_{R_{1}}(x)=\sum_{j \in \mathbb{N}} \phi_{j}^{\prime}\left(\frac{x}{R_{1}}\right), \quad|x|=R_{1},
$$

où $\phi_{j}^{\prime}$ est une harmonique sphérique d'ordre $j$ et où la série converge uniformément. Alors la fonction

$$
x \longmapsto \sum_{j \in \mathbb{N}}\left(\frac{|x|}{R_{1}}\right)^{j} \phi_{j}^{\prime}\left(\frac{x}{R_{1}}\right)=\sum_{j \in \mathbb{N}} H_{j}^{\prime}(x), \quad|x| \leq R_{1},
$$

est continue sur la boule fermée, harmonique à l'intérieur (car $|x|^{j} \phi_{j}^{\prime}(x)$ est un polynôme harmonique) et prend les mêmes valeurs au bord que $u_{R_{1}}$. C'est donc elle. On a ainsi obtenu un développement de $u_{R_{1}}$, qui donne la première série de (B.8).

On tient un raisonnement similaire pour $u_{r_{1}}$ sur le complémentaire de la boule de rayon $r_{1}$ : la formulation intégrale assure que la fonction tend vers 0 à l'infini, qu'elle est harmonique sur $|x|>r_{1}$ et $C^{\infty}$ sur $|x| \geq r_{1}$. On décompose sa restriction à $|x|=r_{1}$ en harmoniques sphériques.

$$
u_{r_{1}}(x)=\sum_{j \in \mathbb{N}} \phi_{j}\left(\frac{x}{R_{1}}\right), \quad|x|=R_{1}
$$

où $\phi_{j}$ est une harmonique sphérique d'ordre $j$ et où la série converge uniformément. Alors la fonction
$x \longmapsto \sum_{j \in \mathbb{N}}\left(\frac{|x|}{R_{1}}\right)^{-j+2-n} \phi_{j}\left(\frac{x}{R_{1}}\right)=\sum_{j \in \mathbb{N}}|x|^{-2 j+2-n} H_{j}(x), \quad|x| \leq R_{1}$,
est continue sur $|x| \geq r_{1}$, harmonique sur $|x|>r_{1}\left(\right.$ car $H_{j}(x)=$ $|x|^{j} \phi_{j}(x)$ est un polynôme harmonique homogène de degré $j$ et donc $|x|^{-2 j+2-n} H_{j}(x)$ est encore harmonique) et prend les mêmes valeurs au bord que $u_{R_{1}}$. C'est donc elle. On a ainsi obtenu le développement de $u_{R_{1}}$ qui donne la seconde série de (B.8).

L'unicité du développement s'obtient en fixant une direction $\xi$. On écrit la série en $x=\rho \xi$. Elle se présente alors comme une série de Laurent en la variable réelle $\rho$. Si on fait varier $\rho$ dans $\mathbb{C}$, la série est convergente sur les compacts de la couronne $r<|\rho|<R$. La somme de la série est une fonction holomorphe, nulle sur la partie de la droite réelle incluse dans cette couronne, donc elle est nulle. L'unicité du développement montre en particulier qu'il ne dépend pas de $r_{1}$ et $R_{1}$.

L'assertion sur les séries dérivées découle de l'application du théorème de Harnack pour les fonctions harmoniques aux sommes partielles des séries.

## B.3. Problème de Neumann extérieur.

Nous expliquons maintenant ce que signifie résoudre "à l'aide d'un potentiel simple couche" le problème de Neumann pour $\Omega$ domaine extérieur. Il s'agit de résoudre, pour $\phi \in C^{\infty}(\partial \Omega)$,

$$
\begin{equation*}
\Delta w=0, \quad \gamma_{1} w=\phi, \quad \lim _{|x| \rightarrow 0} w(x)=0 . \tag{B.9}
\end{equation*}
$$

Pour éviter la difficulté liée au fait que $\Omega$ n'est pas compact, on va se ramener à un problème sur $\partial \Omega$, qui est compact par définition d'un domaine extérieur.

On cherche $w$ sous la forme d'un potentiel simple couche, c'est-àdire qu'on cherche en réalité $\psi$ sur $\partial \Omega$ tel que $w$ définit par $w=V_{\psi}^{(\mathrm{I})}$ (voir (B.5)) soit solution de (B.9). On a déjà vu que pour n'importe quel $\psi$, un tel $w$ est harmonique dans $\Omega$. On montre qu'il tend vers 0 quand $|x|$ tend vers l'infini par une majoration directe puisque $G_{x}(y) \leq$ $C|x-y|^{2-n}$ et que $y$ varie dans $\partial \Omega$.

La condition au bord donne l'équation dont $\psi$ doit être solution

$$
\Pi \psi \doteq \gamma_{1} V_{\psi}^{(\mathrm{I})}=\phi
$$

L'opérateur $\Pi$ est un opérateur pseudodifférentiel sur $\partial \Omega$. Cela vient de $G_{x}(y)=E(x-y)$ et du fait que la convolution par $E$ est, modulo un opérateur à noyau $C^{\infty}$, un opérateur $Q$ pseudodifférentiel sur $\mathbb{R}^{n}$ paramétrix de $-\Delta$. Ce dernier a un symbole polynômial donc $Q$ possède la propriété de transmission par rapport à toutes les hypersurfaces. Nous affirmons que le symbole principal de $\Pi$ est égal à la fonction constante $-1 / 2$. Cela peut se calculer, en se ramenant par localisation dans des cartes au bord de $\partial \Omega$ au cas du demi-espace. On peut aussi s'en convaincre d'après (B.6), comme nous avons déjà fait remarquer que le noyau de l'intégrale au second membre de (B.6) a une singularité sur la diagonale $x=y$ en $O\left(|x-y|^{2-n}\right)$ (et on est sur $\partial \Omega$ de dimension $n-1$ ). On en déduit que non seulement l'opérateur $\Pi$ est elliptique, mais en plus que son indice est le même que celui de l'identité, 0 .

Pour affirmer que $\Pi$ est inversible, montrons qu'il est surjectif. Il nous suffit de montrer que son transposé ${ }^{t} \Pi$ est injectif. Calculons formellement cet opérateur. Soit $\psi$ et $\psi^{\prime}$ dans $C^{\infty}(\partial \Omega)$.

$$
\begin{aligned}
\left\langle^{t} \Pi \psi, \psi^{\prime}\right\rangle & =\left\langle\psi, \Pi \psi^{\prime}\right\rangle \\
& =\int_{\partial \Omega} \psi(x)\left(-\frac{1}{2} \psi^{\prime}(x)+\int_{\partial \Omega} \psi^{\prime} \gamma_{1(x)} G_{x}\right) d x
\end{aligned}
$$

$$
-\frac{1}{2} \int_{\partial \Omega} \psi \psi^{\prime}+\iint_{\partial \Omega^{2}} \psi(x) \psi^{\prime}(y) \gamma_{1(x)} G_{x}(y) d x d y
$$

d'après (B.6). Or $\gamma_{1(x)} G_{x}(y)=\gamma_{1} G_{y}(x)$ d'après (B.3). Donc en appliquant le théorème de Fubini,

$$
\left\langle^{t} \Pi \psi, \psi^{\prime}\right\rangle=-\frac{1}{2} \int_{\partial \Omega} \psi \psi^{\prime}+\int_{\partial \Omega}\left(\int_{\partial \Omega} \psi \gamma_{1} G_{y}\right) \psi^{\prime}(y) d y=\int_{\partial \Omega} \gamma_{0}^{\prime} V_{\psi}^{(\mathrm{II})} \psi^{\prime}
$$

d'après (B.7). Finalement on a montré que ${ }^{t} \Pi \psi=\gamma_{0}^{\prime} V_{\psi}^{(\mathrm{II)})}$.
Alors l'injectivité de ${ }^{t} \Pi$ découle de l'unicité pour le problème de Dirichlet sur l'ouvert borné $\Omega^{\prime}$. En effet, $w^{\prime}=V_{\psi}^{(\mathrm{II})}$ est une fonction harmonique sur $\Omega^{\prime}$, et prend au bord $\partial \Omega^{\prime}=\partial \Omega$ la valeur $\gamma_{0}^{\prime} V_{\psi}^{(\text {II })}$. Si ceci est nul, nécessairement $w^{\prime}=0$, par exemple par le principe du maximum. On a montré que le noyau de ${ }^{t} \Pi$ est réduit à 0 . Donc $\Pi$ est surjectif. Comme son indice est nul, il est bijectif.

Puisque $\Pi$ est un opérateur pseudodifférentiel de degré 0 , elliptique et inversible, son inverse est encore un opérateur pseudodifférentiel de degré 0 . Cela permet de résoudre l'équation $\Pi \psi=\phi$ dans, par exemple, $W_{p}^{-1 / p}(\partial \Omega)$. Le potentiel simple couche $w=V_{\psi}^{(\mathrm{I})}$ a encore un sens, puisqu'on peut le voir comme la convolution de $E$ avec la distribution à support compact $\psi \otimes \delta_{\partial \Omega}$.

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# Paraproduit sur le groupe de Heisenberg et applications 

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Résumé. En adaptant au cas inhomogène la décomposition de Little-wood-Paley homogène sur le groupe de Heisenberg introduite par H. Bahouri, P. Gérard et C.-J. Xu dans [4], on construit des opérateurs de paraproduit analogues à ceux définis par J.-M. Bony dans [5]; malgré le fait que l'on ne dispose pas de formule simple pour la transformée de Fourier d'un produit, des propriétés de localisation spectrale du cas classique sont préservées sur le groupe de Heisenberg après passage au produit. A partir du découpage dyadique et du paraproduit, on démontre l'inégalité de Gagliardo-Nirenberg sur le groupe de Heisenberg, et l'on étudie la régularité des solutions de systèmes sous-elliptiques semilinéaires, ainsi que des équations d'ondes semi-linéaires.


#### Abstract

We adapt the homogeneous Littlewood-Paley decomposition on the Heisenberg group constructed by H. Bahouri, P. Gérard and C.-J. Xu in [4] to the inhomogeneous case, which enables us to build paraproduct operators, similar to those defined by J.-M. Bony in [5]; although there is no simple formula for the Fourier transform of the product of two functions, some spectral localization properties of the classical case are preserved on the Heisenberg group after the product has been taken. Using the dyadic decomposition and the paraproduct algorithm, we prove the Gagliardo-Nirenberg inequality on the Heisenberg group; the smoothness of solutions of subelliptic, semi-linear systems is also studied, as well as semi-linear wave equations.


## 1. Introduction.

Ce travail vise à donner des applications de la théorie de Little-wood-Paley sur le groupe de Heisenberg, à partir de la construction de la décomposition homogène menée par H . Bahouri, P. Gérard et C.-J. Xu dans [4]. Une adaptation de la méthode de [4], utilisant aussi un résultat de [11], conduit à la construction d'une décomposition inhomogène, ce qui nous permet de transposer au groupe de Heisenberg divers résultats connus dans le cas classique (que l'on peut trouver par exemple dans [5], [6] ou dans [7]), concernant la théorie de LittlewoodPaley. On démontre notamment des résultats concernant le coût de la dérivation pour des fonctions dont la transformée de Fourier est localisée dans une boule ou dans une couronne, ainsi que des estimations concernant l'action des applications homogènes et la composition par des fonctions $C^{\infty}$. Enfin cette décomposition de Littlewood-Paley permet de démontrer les inclusions de Sobolev $H^{s} \subset L^{p}$ - ces inclusions sont également démontrées dans [4], par une méthode différente.

On définit ensuite, dans la Section 4, les opérateurs de paraproduit sur le groupe de Heisenberg, et l'on étudie leurs propriétés. La définition de J.-M. Bony (voir [5]) dans le cas classique s'avère opérante dans ce cadre, une fois vérifié (par la Proposition 4.1) que certaines propriétés de localisation dans l'espace de Fourier sont préservées après passage au produit. Ces opérateurs, comme dans le cas classique, permettent de démontrer des lois de produit et des estimations douces dans les espaces de Besov.

Enfin la dernière section est consacrée à des applications de cette théorie. La première concerne la démonstration de l'inégalité de Ga-gliardo-Nirenberg par utilisation du découpage dyadique et des fonctions maximales sur le groupe de Heisenberg. La seconde application vise à démontrer un résultat de régularité pour les solutions d'équations sous-elliptiques semi-linéaires, par utilisation du paraproduit. La méthode de démonstration suit la démarche de J.-Y. Chemin et C.-J. Xu dans [8]. Enfin la dernière application concerne des équations d'ondes semi-linéaires sur le groupe de Heisenberg: en utilisant les estimations douces démontrées plus haut, associées aux estimations de Strichartz généralisées de [4], on démontre un théorème analogue à un résultat de G. Ponce et T. Sideris (voir [17]) dans le cas classique.

Remarquons que dans [12], P.-G. Lemarié construit une base d'ondelettes sur les groupes de Lie nilpotents stratifiés abstraits, à partir de laquelle on peut déduire une formule de paraproduit. L'intérêt de
la construction que nous présentons ici est qu'elle est adaptée au cadre Heisenberg, et ainsi directement utilisable pour des applications.

## 2. Notations et rappels.

Nous allons rappeller ici quelques résultats sur la théorie de Little-wood-Paley sur le groupe de Heisenberg. Pour des détails concernant le groupe de Heisenberg, nous renvoyons à [9], [14], [15], [16], [18], [19], et pour le découpage dyadique homogène sur le groupe de Heisenberg, on consultera le travail de H. Bahouri, P. Gérard et C.-J. Xu dans [3] et [4].

### 2.1. Rappels de définitions.

Le groupe de Heisenberg $\mathbb{H}^{n}$ est l'ensemble $\mathbb{C}^{n} \times \mathbb{R}$ muni de la loi de produit suivante

$$
(z, s) \cdot\left(z^{\prime}, s^{\prime}\right)=\left(z+z^{\prime}, s+s^{\prime}+2 \operatorname{Im} z \cdot \bar{z}^{\prime}\right),
$$

pour tous $\left((z, s),\left(z^{\prime}, s^{\prime}\right)\right) \in \mathbb{H}^{n} \times \mathbb{H}^{n}$. Le groupe $\mathbb{H}^{n}$ étant non commutatif, la transformée de Fourier sur $\mathbb{H}^{n}$ est définie à l'aide des représentations irréductibles unitaires de $\mathbb{H}^{n}$. Nous choisissons ici les représentations définies à partir des espaces de Bargmann,

$$
\mathcal{H}_{\lambda}=\left\{F \text { holomorphe sur } \mathbb{C}^{n},\|F\|_{\mathcal{H}_{\lambda}}<\infty\right\}
$$

où l'on a noté

$$
\|F\|_{\mathcal{H}_{\lambda}}^{2} \stackrel{\text { def }}{=}\left(\frac{2|\lambda|}{\pi}\right)^{n} \int_{\mathbb{C}^{n}} e^{-2|\lambda||\xi|^{2}}|F(\xi)|^{2} d \xi
$$

et les représentations irréductibles unitaires $\left(u^{\lambda}, \mathcal{H}_{\lambda}\right)_{\lambda \neq 0}$ sont alors

$$
u_{z, s}^{\lambda} F(\xi)=F(\xi-\bar{z}) e^{i \lambda s+2 \lambda\left(\xi \cdot z-|z|^{2} / 2\right)}, \quad \text { pour } \lambda>0
$$

et

$$
u_{z, s}^{\lambda} F(\xi)=F(\xi+z) e^{i \lambda s+2 \lambda\left(\xi \cdot \bar{z}-|z|^{2} / 2\right)}, \quad \text { pour } \lambda<0
$$

Notons que l'on a une base orthornormée de l'espace de Hilbert $\mathcal{H}_{\lambda}$, formée de

$$
F_{\alpha, \lambda}(\xi)=\frac{(\sqrt{2|\lambda|} \xi)^{\alpha}}{\sqrt{\alpha}}, \quad \alpha \in \mathbb{N}
$$

On définit la transformée de Fourier d'une fonction $f \in L^{1}\left(\mathbb{H}^{n}\right)$ par

$$
\mathcal{F}(f)(\lambda)=\int_{\mathbb{H}^{n}} f(z, s) u_{z, s}^{\lambda} d z d s
$$

Dans le cas particulier des fonctions radiales, telles que $f(z, s)=g(|z|, s)$, la proposition suivante, démontrée dans [15], nous sera d'une grande utilité.

Proposition 2.1. Si $f \in L^{2}\left(\mathbb{H}^{n}\right)$ est radiale, alors $\mathcal{F}(f)(\lambda) F_{\alpha, \lambda}=$ $R_{|\alpha|}(\lambda) F_{\alpha, \lambda}$, ò̀
$R_{m}(\lambda)=\binom{m+n-1}{m}^{-1} \int f(z, s) e^{i \lambda s} L_{m}^{(n-1)}\left(2|\lambda||z|^{2}\right) e^{-|\lambda||z|^{2}} d z d s$, et où les $L_{m}^{(n)}(t)$ sont les polynômes de Laguerre

$$
L_{m}^{(n)}(t)=\sum_{k=0}^{m}(-1)^{k}\binom{m+n}{m-k} \frac{t^{k}}{k!}
$$

Réciproquement, s'il existe des scalaires $R_{m}(\lambda)$ tels que $\mathcal{F}(f)(\lambda) F_{\alpha, \lambda}=$ $R_{|\alpha|}(\lambda) F_{\alpha, \lambda}$, et

$$
\sum_{m}\binom{m+n-1}{m}^{-1} \int\left|R_{m}(\lambda)\right|^{2}|\lambda|^{n} d \lambda<\infty
$$

alors $f \in L^{2}\left(\mathbb{H}^{n}\right)$ est radiale, et l'on a presque partout

$$
f(z, s)=\frac{2^{n-1}}{\pi^{n+1}} \sum_{m} \int e^{-i \lambda s} R_{m}(\lambda) L_{m}^{(n-1)}\left(2|\lambda||z|^{2}\right) e^{-|\lambda||z|^{2}}|\lambda|^{n} d \lambda
$$

Enfin rappellons qu'il existe une base de champs de vecteurs invariants à gauche sur le groupe de Heisenberg, notés
$X_{j}=\partial_{x_{j}}+2 y_{j} \partial_{s} \quad$ et $\quad Y_{j}=\partial_{y_{j}}-2 x_{j} \partial_{s}, \quad$ pour tous $j \in\{1, \ldots, n\}$, où l'on a écrit, pour tout $z_{j} \in \mathbb{C}^{n}, z_{j}=x_{j}+i y_{j}$. On notera

$$
\Delta_{\mathbb{H}^{n}} \stackrel{\text { def }}{=} \sum_{j=1}^{n}\left(X_{j}^{2}+Y_{j}^{2}\right),
$$

et pour tout $\gamma \in \mathbb{N}, \mathcal{X}^{\gamma}$ sera un produit de $\gamma$ champs de vecteurs, du type

$$
\begin{equation*}
\mathcal{X}^{\gamma}=X_{j_{1}} \cdots X_{j_{\ell}} \tag{2.1}
\end{equation*}
$$

où $j_{k} \in\{1, \ldots, 2 n\}$, et l'on convient que $X_{j+n}=Y_{j}$, pour $j \in\{1, \ldots, n\}$. Remarquons que pour toute fonction $f \in \mathcal{S}\left(\mathbb{H}^{n}\right)$

$$
\begin{equation*}
\mathcal{F}\left(\Delta_{\mathbb{H}^{n}} f\right)(\lambda) F_{\alpha, \lambda}=-4|\lambda|(2|\alpha|+n) \mathcal{F}(f)(\lambda) F_{\alpha, \lambda} . \tag{2.2}
\end{equation*}
$$

On peut enfin définir l'opérateur suivant

$$
\begin{equation*}
\mathcal{F}\left(\left(-\Delta_{\mathbb{H}_{n}}\right)^{\rho / 2} f\right)(\lambda) F_{\alpha, \lambda}=(4|\lambda|(2|\alpha|+n))^{\rho / 2} \mathcal{F}(f)(\lambda) F_{\alpha, \lambda}, \tag{2.3}
\end{equation*}
$$

pour tous $\rho \in \mathbb{R}, f \in \mathcal{S}\left(\mathbb{H}^{n}\right)$.

### 2.2. Théorie de Littlewood-Paley sur le groupe de Heisenberg.

Nous allons rappeller tout d'abord la définition de la décomposition de Littlewood-Paley homogène construite dans [3] et [4]. Nous préciserons à la fin de ce paragraphe comment cette construction peut s'adapter pour obtenir une décomposition inhomogène.

Dorénavant nous noterons $\mathcal{C}_{0}$ la couronne $\{\tau \in \mathbb{R}: 1 / 2 \leq|\tau| \leq 4\}$, $\mathcal{B}_{0}$ la boule $\{\tau \in \mathbb{R}:|\tau| \leq 2\}$, et nous considèrerons une fonction $R^{*} \in C_{0}^{\infty}\left(\mathcal{C}_{0}\right)$, telle que

$$
\sum_{j \in \mathbb{Z}} R^{*}\left(2^{-2 j} \tau\right)=1, \quad \text { pour tout } \tau \in \mathbb{R}^{*}
$$

D'autre part, on définit la fonction $\widetilde{R}^{*} \in C_{0}^{\infty}\left(\mathcal{B}_{0}\right)$, identiquement égale à 1 près de 0 , telle que

$$
\widetilde{R}^{*}(\tau)+\sum_{j \geq 0} R^{*}\left(2^{-2 j} \tau\right)=1, \quad \text { pour tout } \tau \in \mathbb{R}
$$

Dans ce qui suit, nous noterons $R_{m}^{*}(\tau)=R^{*}((2 m+n) \tau)$. La Proposition 2.1 nous permet de définir la fonction radiale

$$
\varphi(z, s)=\frac{2^{n-1}}{\pi^{n+1}} \sum_{m} \int e^{-i \lambda s} R_{m}^{*}(\lambda) L_{m}^{(n-1)}\left(2|\lambda||z|^{2}\right) e^{-|\lambda||z|^{2}}|\lambda|^{n} d \lambda
$$

et [4, Proposition 2.2] indique que $\varphi$ est un élément de $\mathcal{S}\left(\mathbb{H}^{n}\right)$. Ce résultat est aussi une conséquence de [11]. Ainsi l'on peut écrire en particulier

$$
\mathcal{F}(\varphi)(\lambda) F_{\alpha, \lambda}=R_{|\alpha|}^{*}(\lambda) F_{\alpha, \lambda},
$$

et si $\varphi_{j}(z, s)=2^{N j} \varphi\left(2^{j} z, 2^{2 j} s\right)$, où $N=2 n+2$ est la dimension homogène de $\mathbb{H}^{n}$, alors la série

$$
f=\sum_{j \in \mathbb{Z}} \dot{\Delta}_{j} f, \quad \text { avec } \quad \dot{\Delta}_{j} f=f * \varphi_{j},
$$

est la décomposition de Littlewood-Paley de $f \in \mathcal{S}\left(\mathbb{H}^{n}\right)$ sur le groupe de Heisenberg.

La convergence de la série $\sum_{j \in \mathbb{Z}} \dot{\Delta}_{j} f$ est démontrée dans [4]. Notons que cette série ne converge pas dans $\mathcal{S}^{\prime}\left(\mathbb{H}^{n}\right)$ (à cause des fonctions polynômiales, qui vérifient $\dot{\Delta}_{j} f=0$, pour tout $\left.j \in \mathbb{Z}\right)$. D'autre part, notons que cette décomposition dyadique est bien une décomposition de Littlewood-Paley, puisque (voir [4, Proposition 2.3]) si $f \in \mathcal{S}^{\prime}\left(\mathbb{H}^{n}\right)$ vérifie $\sum_{j \in \mathbb{Z}} \dot{\Delta}_{j} f=f$, alors $f \in L^{p}\left(\mathbb{H}^{n}\right)$ est équivalent à $\left\|\dot{\Delta}_{j} f\right\|_{\ell^{2}(\mathbb{Z})} \in$ $L^{p}\left(\mathbb{H}^{n}\right)$.

Notre but étant d'écrire une théorie du paraproduit sur le groupe de Heisenberg, il convient à présent de construire une décomposition inhomogène de Littlewood-Paley; en d'autres termes, nous allons à présent chercher à montrer que la fonction $\psi$ définie par

$$
\begin{align*}
& \psi(z, s) \\
& \quad=\frac{2^{n-1}}{\pi^{n+1}} \sum_{m} \int e^{-i \lambda s} \widetilde{R}_{m}^{*}(\lambda) L_{m}^{(n-1)}\left(2|\lambda||z|^{2}\right) e^{-|\lambda||z|^{2}}|\lambda|^{n} d \lambda \tag{2.4}
\end{align*}
$$

vérifie $\psi \in \mathcal{S}\left(\mathbb{H}^{n}\right)$. Alors on écrira, pour $f \in L^{2}\left(\mathbb{H}^{n}\right)$,

$$
\begin{align*}
& f=\sum_{j \geq-1} \Delta_{j} f, \quad \text { avec pour tout } j \in \mathbb{N}, \\
& \Delta_{j} f=\dot{\Delta}_{j} f, \quad \Delta_{-1} f=f * \psi  \tag{2.5}\\
& \text { et pour tout } j<-1, \quad \Delta_{j} f=0 .
\end{align*}
$$

On définit aussi $S_{j} f=\psi_{j} * f$, avec $\psi_{j}(z, s)=2^{N j} \psi\left(2^{j} z, 2^{2 j} s\right)$, pour tout $j \in \mathbb{N}$.

Il s'agit donc de vérifier que $\psi$ définie en (2.4) est un élément de $\mathcal{S}\left(\mathbb{H}^{n}\right)$. Une adaptation de [4, Proposition 2.2] nous permet de démontrer le résultat suivant.

Proposition 2.2. La fonction $\psi$ définie par

$$
\psi(z, s)=\frac{2^{n-1}}{\pi^{n+1}} \sum_{m} \int e^{-i \lambda s} \widetilde{R}_{m}^{*}(\lambda) L_{m}^{(n-1)}\left(2|\lambda||z|^{2}\right) e^{-|\lambda||z|^{2}}|\lambda|^{n} d \lambda
$$

où $\widetilde{R}^{*} \in C_{0}^{\infty}\left(\mathcal{B}_{0}\right)$ est identiquement égale à 1 près de 0 , vérifie

$$
\begin{array}{lc}
\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{k} \psi\right\|_{L^{2}\left(\mathbb{H}^{n}\right)} \leq C_{k}, & \text { pour tout } k \in \mathbb{N}, \\
\left\|\left(i s-|z|^{2}\right)^{\ell} \psi\right\|_{L^{2}\left(\mathbb{H}^{n}\right)} \leq C_{\ell}, & \text { pour tout } \ell \in \mathbb{N}, \tag{2.7}
\end{array}
$$

et donc $\psi$ est dans $\mathcal{S}\left(\mathbb{H}^{n}\right)$.
Démonstration de la Proposition. Nous allons commencer par démontrer (2.6). Rappellons qu'il est démontré dans [4] que la transformée de Fourier sur le groupe de Heisenberg réalise un isomorphisme du sous-espace des fonctions radiales de $L^{2}\left(\mathbb{H}^{n}\right)$ sur les opérateurs $A$ à un paramètre, définis par

$$
A(\lambda) F_{\alpha, \lambda}=Q_{|\alpha|}(\lambda) F_{\alpha, \lambda},
$$

avec

$$
\frac{2^{n-1}}{\pi^{n+1}} \sum_{m}\binom{m+n-1}{m} \int_{-\infty}^{\infty}\left|Q_{m}(\lambda)\right|^{2}|\lambda|^{n} d \lambda<\infty
$$

Alors la Proposition 2.1, associée à (2.3), donne le résultat, puisque

$$
\begin{gathered}
\sum_{m}\binom{m+n-1}{m} \int_{-\infty}^{\infty}|-4((2 m+n)|\lambda|)|^{k}\left|\widetilde{R}^{*}((2 m+n) \lambda)\right|^{2}|\lambda|^{n} d \lambda \\
=\sum_{m}\binom{m+n-1}{m}(2 m+n)^{-n-1} \int_{-\infty}^{\infty}\left|\widetilde{R}^{*, k}(\lambda)\right|^{2}|\lambda|^{n} d \lambda
\end{gathered}
$$

où $\widetilde{R}^{*, k}$ est une fonction de $C_{0}^{0}(\mathbb{R})$, donc cette série est convergente.
Pour ce qui est de (2.7), on peut reprendre les calculs de [4]: si $Q$ est une fonction de $C_{0}^{\infty}\left(\mathbb{R}^{*}\right)$, et si $Q_{m}(\lambda)=Q((2 m+n) \lambda)$, alors il est montré dans [4] que la fonction

$$
f(z, s)=\frac{2^{n-1}}{\pi^{n+1}} \sum_{m} \int e^{-i \lambda s} Q_{m}(\lambda) L_{m}^{(n-1)}\left(2|\lambda||z|^{2}\right) e^{-|\lambda||z|^{2}}|\lambda|^{n} d \lambda
$$

est dans $L^{2}\left(\mathbb{H}^{n}\right)$, mais aussi toutes les fonctions (is $\left.-|z|^{2}\right)^{\ell} f$ pour $\ell \in \mathbb{N}$. On constate facilement (nous n'entrerons pas dans les détails ici ), au vu de la démonstration de ce résultat dans [4], que la condition $Q \in C_{0}^{\infty}\left(\mathbb{R}^{*}\right)$ peut être relaxée en $Q \in C_{0}^{\infty}(\mathbb{R})$, et $Q$ constante près de $\lambda=0$. C'est en particulier le cas pour la fonction $\widetilde{R}^{*}$, et donc le point (2.7) est démontré.

Reste donc à vérifier que (2.6) et (2.7) impliquent bien que $\psi \in$ $\mathcal{S}\left(\mathbb{H}^{n}\right)$. Cela résulte du lemme suivant.

Lemme 2.1. Soit $f \in \mathcal{S}^{\prime}\left(\mathbb{H}^{n}\right)$ telle que

$$
\left(-\Delta_{\mathbb{H}^{n}}\right)^{k} f \in L^{2}\left(\mathbb{H}^{n}\right), \quad \text { pour tout } k \in \mathbb{N}
$$

et

$$
\left(\text { is }-|z|^{2}\right)^{\ell} f \in L^{2}\left(\mathbb{H}^{n}\right), \quad \text { pour tout } \ell \in \mathbb{N} .
$$

Alors $f \in \mathcal{S}\left(\mathbb{H}^{n}\right)$.
Démonstration. Nous allons démontrer ce résultat uniquement dans le cas $\ell=1$; le cas général s'en déduit sans difficulté. La sous-ellipticité de $\left(-\Delta_{\mathbb{H}^{n}}\right)^{k}$ implique (voir [4, Lemme 2.1]) qu'il suffit de démontrer que

$$
\left(-\Delta_{\mathbb{H}^{n}}\right)^{k}\left(\left(i s-|z|^{2}\right) f\right) \in L^{2}\left(\mathbb{H}^{n}\right), \quad \text { pour tout } k \in \mathbb{N}
$$

Il est facile de voir, par la formule de Leibnitz et avec la notation (2.1), que

$$
\mathcal{X}^{2 k}\left(\left(i s-|z|^{2}\right) f\right)=\left(i s-|z|^{2}\right) \mathcal{X}^{2 k} f+\sum_{k^{\prime}<2 k} P_{k^{\prime}}(z) \mathcal{X}^{k^{\prime}} f,
$$

pour tout $k \in \mathbb{N}^{*}$, où $P_{k^{\prime}}$ est un polynôme. Mais l'hypothèse (is $\left.|z|^{2}\right)^{\ell} f \in L^{2}\left(\mathbb{H}^{n}\right)$ implique que pour tout $\beta \in \mathbb{N}$, on a $z^{\beta} f \in L^{2}\left(\mathbb{H}^{n}\right)$, par conséquent il vient

$$
\begin{aligned}
\int_{\mathbb{H}^{n}} \mathcal{X}^{2 k}\left(\left(i s-|z|^{2}\right) f\right) \mathcal{X}^{2 k}\left(\left(i s-|z|^{2}\right) f\right) d z d s \\
\quad=C \int_{\mathbb{H}^{n}}\left(i s-|z|^{2}\right) f \mathcal{X}^{4 k}\left(\left(i s-|z|^{2}\right) f\right) d z d s \\
\quad=C \int_{\mathbb{H}^{n}}\left(i s-|z|^{2}\right)^{2} f \mathcal{X}^{4 k} f d z d s
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{k^{\prime}<4 k} \int_{\mathbb{H}^{n}} P_{k^{\prime}}(z)\left(i s-|z|^{2}\right) f \mathcal{X}^{k^{\prime}} f d z d s \\
\leq & C \sup _{\beta \leq 2 k}\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\beta} f\right\|_{L^{2}\left(\mathbb{H}^{n}\right)}\left\|\left(i s-|z|^{2}\right)^{\beta} f\right\|_{L^{2}\left(\mathbb{H}^{n}\right)} .
\end{aligned}
$$

Le lemme est démontré, et avec lui, la proposition.

## 3. Lemme de localisation et applications.

## 3.1. Énoncé du lemme et démonstration.

Le résultat suivant est l'analogue du [6, Lemme 2.1.1] dans le cas classique. Il décrit le coût de la dérivation pour une fonction dont la transformée de Fourier est localisée.

Lemme 3.1. Soient $p$ et $q$ deux éléments de $[1, \infty]$, avec $p \leq q$, et soit $u \in L^{p}\left(\mathbb{H}^{n}\right)$ une fonction telle que $u * f=0$ pour toute fonction radiale $f \in \mathcal{S}\left(\mathbb{H}^{n}\right)$ vérifiant, pour tout $\alpha \in \mathbb{N}^{n}$,

$$
\begin{equation*}
\mathcal{F}(f)(\lambda) F_{\alpha, \lambda}=0, \quad \text { pour } \lambda \in(2|\alpha|+n)^{-1} 2^{2 j} \mathcal{B}_{0} . \tag{3.8}
\end{equation*}
$$

Alors on a

$$
\begin{equation*}
\sup _{\beta=k}\left\|\mathcal{X}^{\beta} u\right\|_{L^{q}\left(\mathbb{H}^{n}\right)} \leq C_{k} 2^{N j(1 / p-1 / q)+k j}\|u\|_{L^{p}\left(\mathbb{H}^{n}\right)} \tag{3.9}
\end{equation*}
$$

pour tout $k \in \mathbb{N}$. D'autre part, si $u * g=0$ pour toute fonction radiale $g \in \mathcal{S}\left(\mathbb{H}^{n}\right)$ vérifiant, pour tout $\alpha \in \mathbb{N}^{n}$,

$$
\begin{equation*}
\mathcal{F}(g)(\lambda) F_{\alpha, \lambda}=0, \quad \text { pour } \lambda \in(2|\alpha|+n)^{-1} 2^{2 j} \mathcal{C}_{0} \tag{3.10}
\end{equation*}
$$

alors

$$
\begin{equation*}
C_{\rho}^{-1} 2^{-j \rho}\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} u\right\|_{L^{p}\left(\mathbb{H}^{n}\right)} \leq\|u\|_{L^{p}\left(\mathbb{H}^{n}\right)} \tag{3.11}
\end{equation*}
$$

pour tout $\rho \in \mathbb{R}$.
Remarques. Dans le cas où la fonction $u$ est un élément de $\mathcal{S}\left(\mathbb{H}^{n}\right)$, alors les hypothèses (3.8) et (3.10) se traduisent respectivement en

$$
\mathcal{F}(u)(\lambda) F_{\alpha, \lambda}=\mathbf{1}_{(2|\alpha|+n)^{-1} 2^{2 j} \mathcal{B}_{0}}(\lambda) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda},
$$

et

$$
\mathcal{F}(u)(\lambda) F_{\alpha, \lambda}=\mathbf{1}_{(2|\alpha|+n)^{-1} 2^{2 j} \mathcal{G}_{0}}(\lambda) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda} .
$$

Notons en outre que le second résultat de ce lemme ne concerne que le cas où l'opérateur de dérivation est du type $\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2}$. Cela est dû au fait que dans le cas des opérateurs $X_{j}$, on ne dispose pas de décomposition dans la base des $F_{\alpha, \lambda}$ aussi simple que celle donnée par (2.2) pour $-\Delta_{\mathbb{H}^{n}}$.

Démonstration du Lemme. Nous allons nous placer dans le cas où la fonction $u$ est un élément de $\mathcal{S}\left(\mathbb{H}^{n}\right)$; le lemme suit par densité. Soit $R \in C_{0}^{\infty}(\mathbb{R})$, identiquement égale à 1 près de $\mathcal{B}_{0}$. Alors on a

$$
\mathcal{F}(u)(\lambda) F_{\alpha, \lambda}=R_{|\alpha|}\left(2^{-2 j} \lambda\right) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda},
$$

où l'on a posé $R_{|\alpha|}(\lambda)=R((2|\alpha|+n) \lambda)$. Mais alors d'après les propositions 2.1 et 2.2 , il existe une fonction $g \in \mathcal{S}\left(\mathbb{H}^{n}\right)$ radiale, telle que

$$
\mathcal{F}(g)(\lambda) F_{\alpha, \lambda}=R_{|\alpha|}(\lambda) F_{\alpha, \lambda} .
$$

En écrivant $g_{j}(z, s)=2^{N j} g\left(2^{j} z, 2^{2 j} s\right)$, on a alors

$$
\mathcal{F}(u)(\lambda) F_{\alpha, \lambda}=\mathcal{F}\left(g_{j}\right)(\lambda) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda},
$$

et donc $u=g_{j} * u$. Mais on a alors

$$
\mathcal{X}^{\beta} u=2^{j(N+\beta)} \mathcal{X}^{\beta} g\left(\delta_{2^{j}} \cdot\right) * u,
$$

où pour tout $a, \delta_{a}$ est la dilatation homogène définie par $\delta_{a}(z, s)=$ $\left(a z, a^{2} s\right)$. Comme dans [6], il suffit alors d'appliquer l'inégalité de Young pour obtenir (3.9).

Démontrons à présent (3.11). Soit $R^{\prime} \in C_{0}^{\infty}\left(\mathbb{R}^{*}\right)$, identiquement égale à 1 près de $\mathcal{C}_{0}$. Alors

$$
\mathcal{F}(u)(\lambda) F_{\alpha, \lambda}=R_{|\alpha|}^{\prime}\left(2^{-2 j} \lambda\right) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda},
$$

où $R_{|\alpha|}^{\prime}(\lambda)=R^{\prime}((2|\alpha|+n) \lambda)$, donc

$$
\mathcal{F}(u)(\lambda) F_{\alpha, \lambda}=2^{-j \rho} \frac{R_{|\alpha|}^{\prime}\left(2^{-2 j} \lambda\right)}{\left(42^{-2 j}|\lambda|(2|\alpha|+n)\right)^{\rho / 2}} \mathcal{F}\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} u\right)(\lambda) F_{\alpha, \lambda} .
$$

Définissons alors la fonction

$$
\theta^{\rho}(\lambda)=\frac{R^{\prime}(\lambda)}{2^{\rho}|\lambda|^{\rho / 2}},
$$

à laquelle on associe $\theta_{|\alpha|}^{\rho}(\lambda)=\theta^{\rho}((2|\alpha|+n) \lambda)$. Alors $\theta^{\rho} \in C_{0}^{\infty}\left(\mathbb{R}^{*}\right)$, et d'après [4], il existe une fonction $g^{\rho} \in \mathcal{S}\left(\mathbb{H}^{n}\right)$ telle que

$$
\mathcal{F}\left(g^{\rho}\right)(\lambda) F_{\alpha, \lambda}=\theta_{|\alpha|}^{\rho}(\lambda) F_{\alpha, \lambda},
$$

et l'on conclut comme précédemment. Le lemme est démontré.

### 3.2. Applications.

### 3.2.1. Lemme de caractérisation.

Cette première application du Lemme 3.1 permet de caractériser l'appartenance d'une fonction à un espace de Besov; la définition des espaces de Besov sur le groupe de Heisenberg est identique au cas classique (voir [4]). Rappellons simplement que l'espace $B_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$, pour $\rho \in \mathbb{R}$ et $(p, r) \in[1, \infty]^{2}$ est défini comme l'espace des distributions tempérées vérifiant

$$
u=\sum_{j} \Delta_{j} u \quad \text { et } \quad\|u\|_{B_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)} \stackrel{\text { def }}{=}\left(\sum_{j \geq-1} 2^{j r \rho}\left\|\Delta_{j} u\right\|_{L^{p}\left(\mathbb{H}^{n}\right)}^{r}\right)^{1 / r}<\infty
$$

et l'espace de Besov homogène $\dot{B}_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$, pour $\rho<N / p$ est l'espace des distributions tempérées telles que $u=\sum_{j} \dot{\Delta}_{j} u$, et que la norme suivante soit finie

$$
\|u\|_{\dot{B}_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)} \stackrel{\text { def }}{=}\left(\sum_{j \in \mathbb{Z}} 2^{j r \rho}\left\|\dot{\Delta}_{j} u\right\|_{L^{p}\left(\mathbb{H}^{n}\right)}^{r}\right)^{1 / r}
$$

Remarque. On définit aussi, comme dans le cas classique, les espaces de Hölder $C^{\rho}$, que l'on identifie pour tout $\rho \in \mathbb{R}-\mathbb{N}$ à $B_{\infty, \infty}^{\rho}$, ainsi que les espaces de Sobolev $H^{s}$ et leurs versions homogènes, pour tout $s \in \mathbb{R}$.

Lemme 3.2. Soit $\rho>0$ et $(p, r) \in[1, \infty]^{2}$. Les deux assertions suivantes sont équivalentes.
i) $u \in B_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$.
ii) Il existe $\left\{u_{j}\right\}_{j \geq 0}$ telle que $u=\sum u_{j}$, et pour tout $\gamma \in \mathbb{R}$,

$$
\left.\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\gamma} u_{j}\right\|_{L^{p}\left(\mathbb{H}^{n}\right.}\right) \leq C_{\gamma} c_{j} 2^{-j(\rho-2 \gamma)}, \quad \text { avec } \rho>0
$$

où $C_{\gamma}$ ne dépend que de $\gamma$, et $\left\{c_{j}\right\}_{j \geq 0}$ est une suite de $\ell^{r}(\mathbb{N})$.
Démonstration. Le Lemme 3.1 précédent implique clairement que i) implique ii). Démontrons donc que ii) implique i). Soit $j^{\prime} \in \mathbb{N}$, et écrivons

$$
\sum_{j} \Delta_{j^{\prime}} u_{j}=\sum_{j>j^{\prime}} \Delta_{j^{\prime}} u_{j}+\sum_{j \leq j^{\prime}} \Delta_{j^{\prime}} u_{j} .
$$

Alors

$$
\begin{aligned}
& 2^{j^{\prime} \rho}\left\|\Delta_{j^{\prime}} u\right\|_{L^{p}} \\
& \quad \leq 2^{j^{\prime} \rho} \sum_{j>j^{\prime}}\left\|\Delta_{j^{\prime}} u_{j}\right\|_{L^{p}}+C_{\rho} 2^{j^{\prime} \rho} \sum_{j \leq j^{\prime}} 2^{-2 j^{\prime} \beta}\left\|\Delta_{j^{\prime}}\left(-\Delta_{\mathbb{H}^{n}}\right)^{\beta} u_{j}\right\|_{L^{p}},
\end{aligned}
$$

par le Lemme de localisation 3.1, avec $\beta$ à fixer. Mais alors on a, par hypothèse,

$$
2^{j^{\prime} \rho}\left\|\Delta_{j^{\prime}} u\right\|_{L^{p}} \leq C \sum_{j>j^{\prime}} 2^{\left(j^{\prime}-j\right) \rho} c_{j}+C_{\rho} \sum_{j \leq j^{\prime}} 2^{\left(j-j^{\prime}\right)(2 \beta-\rho)} c_{j},
$$

où $\left\{c_{j}\right\}_{j \geq 0}$ est une suite de $\ell^{r}(\mathbb{N})$. Il suffit alors de prendre la norme $\ell^{r}$ en $j^{\prime}$, en choisissant $\beta$ tel que $2 \beta>\rho$. Comme l'on a en outre supposé que $\rho>0$, on a le résultat.

Remarque. Un résultat analogue s'énonce bien sûr dans le cas des espaces homogènes.

Le Lemme 3.1 permet de démontrer de manière évidente la continuité des opérateurs $X_{j}$ dans les espaces de Besov. Les notations sont comme en (2.1).

Lemme 3.3. Soit $\rho<N / p$, et soit le couple $(p, r) \in[1, \infty]$. Si $u$ est un élément de $\dot{B}_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$, alors pour tout $j \in\{1, \ldots, 2 n\}$, on a $X_{j} u \in \dot{B}_{p, r}^{\rho-1}\left(\mathbb{H}^{n}\right)$, et

$$
\left\|X_{j} u\right\|_{\dot{B}_{p, r}^{\rho-1}\left(\mathbb{H}^{n}\right)} \leq C\|u\|_{\dot{B}_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)}
$$

Démonstration. On a par définition de $\dot{B}_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$,

$$
\left\|\dot{\Delta}_{k} u\right\|_{L^{p}\left(\mathbb{H}^{n}\right)} \leq c_{k} 2^{-k \rho}, \quad \text { pour tout } k \in \mathbb{Z}
$$

où $\left\{c_{k}\right\}_{k \in \mathbb{Z}}$ est une suite de $\ell^{r}(\mathbb{Z})$. Par conséquent, on a par le lemme d'échantillonage 3.1

$$
\left\|X_{j} \dot{\Delta}_{k} u\right\|_{L^{p}\left(\mathbb{H}^{n}\right)} \leq C c_{k} 2^{-k(\rho-1)}, \quad \text { pour tous } j \in\{1, \ldots, 2 n\}
$$

Les opérateurs $X_{j}$ et $\dot{\Delta}_{k}$ commutent, ce qui démontre le lemme.
Remarque. Le même résultat est évidemment vrai dans le cas inhomogène.

### 3.2.2. Estimations douces.

Une autre application du lemme de localisation 3.1 consiste en la démonstration d'estimations douces, du type suivant. Remarquons que les énoncés sont les mêmes dans le cas des espaces homogènes.

Lemme 3.4. Soit $s>0$. Siu et $v$ sont deux éléments de $L^{\infty} \cap H^{s}\left(\mathbb{H}^{n}\right)$, alors $u v$ est un élément de $L^{\infty} \cap H^{s}\left(\mathbb{H}^{n}\right)$, et

$$
\|u v\|_{L^{\infty} \cap H^{s}\left(\mathbb{H}^{n}\right)} \leq C\left(\|u\|_{L^{\infty}}\|v\|_{H^{s}}+\|v\|_{L^{\infty}}\|u\|_{H^{s}}\right) .
$$

Nous ne démontrons pas ce lemme ici, car il peut être obtenu aussi comme un corollaire des lois d'opérance des opérateurs de paraproduit que nous définirons plus bas. Par contre, démontrons le résultat suivant, dont la démonstration dans le cas classique peut être trouvée dans $[1$, p. 100] par exemple.

Lemme 3.5. Soit $k \in \mathbb{N}$, et soient $u$ et $v$ deux fonctions de $L^{\infty}\left(\mathbb{H}^{n}\right) \cap$ $H^{k}\left(\mathbb{H}^{n}\right)$. Alors pour tout couple $(\beta, \gamma) \in \mathbb{N}^{2}$ tel que $\beta+\gamma=k$, on a

$$
\left\|\mathcal{X}^{\beta} u \mathcal{X}^{\gamma} v\right\|_{L^{2}\left(\mathbb{H}^{n}\right)} \leq C\left(\|u\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|v\|_{H^{k}\left(\mathbb{H}^{n}\right)}+\|v\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|u\|_{H^{k}\left(\mathbb{H}^{n}\right)}\right) .
$$

Démonstration. Supposons par exemple que $\beta \geq 1$ (puisque si $\beta=$ $\gamma=0$, alors le résultat est trivial). Par la formule de Leibnitz, on peut écrire

$$
\mathcal{X}^{\beta} u \mathcal{X}^{\gamma} v=\sum_{j} c_{j} X_{j}\left(\mathcal{X}^{\beta_{j}} u \mathcal{X}^{\gamma_{j}} v\right)+c_{k} u \mathcal{X}^{\beta+\gamma} v
$$

où les $c_{j}$ sont des constantes, et où $\beta_{j}+\gamma_{j}=k-1$. Il suffit donc de démontrer que
$\left\|\mathcal{X}^{\beta_{j}} u \mathcal{X}^{\gamma_{j}} v\right\|_{H^{1}\left(\mathbb{H}^{n}\right)} \leq C\left(\|u\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|v\|_{H^{k}\left(\mathbb{H}^{n}\right)}+\|v\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|u\|_{H^{k}\left(\mathbb{H}^{n}\right)}\right)$.
Mais on a

$$
\mathcal{X}^{\beta_{j}} u \mathcal{X}^{\gamma_{j}} v=\sum_{q \geq 0}\left(S_{q} \mathcal{X}^{\beta_{j}} u\right)\left(\Delta_{q} \mathcal{X}^{\gamma_{j}} v\right)+\sum_{q \geq 0}\left(\Delta_{q} \mathcal{X}^{\beta_{j}} u\right)\left(S_{q+1} \mathcal{X}^{\gamma_{j}} v\right)
$$

et le Lemme 3.1 implique que

$$
\begin{aligned}
\left\|\left(S_{q} \mathcal{X}^{\beta_{j}} u\right)\left(\Delta_{q} \mathcal{X}^{\gamma_{j}} v\right)\right\|_{L^{2}} & \leq\left\|S_{q} \mathcal{X}^{\beta_{j}} u\right\|_{L^{\infty}}\left\|\Delta_{q} \mathcal{X}^{\gamma_{j}} v\right\|_{L^{2}} \\
& \leq C 2^{q \beta_{j}}\|u\|_{L^{\infty}} v_{q}\|v\|_{H^{k}} 2^{-q\left(k-\gamma_{j}\right)}
\end{aligned}
$$

où $\left\{v_{q}\right\}_{q \geq 0}$ est une suite de $\ell^{2}(\mathbb{N})$, de norme 1 . On peut alors conclure que

$$
\left\|\left(S_{q} \mathcal{X}^{\beta_{j}} u\right)\left(\Delta_{q} \mathcal{X}^{\gamma_{j}} v\right)\right\|_{L^{2}} \leq C v_{q} 2^{-q}\|u\|_{L^{\infty}}\|v\|_{H^{k}}
$$

ce qui démontre le résultat.

### 3.2.3. Action des applications homogènes.

La proposition suivante décrit l'action des applications homogènes dans les espaces de Besov, et est la traduction au groupe de Heisenberg [7, Théorème 1.3.2]. Avant d'énoncer le résultat, donnons la définition suivante.

Définition 3.1. Pour toute fonction $f \in C^{\infty}\left(\mathbb{R}^{*}\right)$, on appellera $f\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right)$ l'opérateur défini par

$$
\mathcal{F}\left(f\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right) u\right)(\lambda) F_{\alpha, \lambda}=f\left((4|\lambda|(2|\alpha|+n))^{1 / 2}\right) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda},
$$

pour tout $u \in \mathcal{S}\left(\mathbb{H}^{n}\right)$.

Proposition 3.1. Si $f \in C^{\infty}\left(\mathbb{R}^{*}\right)$ est homogène de degré $m$, alors pour tout $\rho<N / p$, pour tout $(p, r) \in[1, \infty]^{2}$ et pour tout $u \in \dot{B}_{p, r}^{\rho}$,

$$
\lim _{k \rightarrow \infty}\left\|\sum_{j \leq k} \Delta_{j} f\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right) u\right\|_{\dot{B}_{p, r}^{p-r}} \leq C\|u\|_{\dot{B}_{p, r}^{p}} .
$$

Démonstration. Nous allons démontrer que pour toute fonction $u \in$ $\mathcal{S}\left(\mathbb{H}^{n}\right)$, on a

$$
\left\|f\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right) u\right\|_{\dot{B}_{p, r}^{\rho-m}} \leq C\|u\|_{\dot{B}_{p, r}^{\rho}} .
$$

La proposition suit alors par densité. Avec les notations rappellées au Paragraphe 2.2, on a

$$
\begin{align*}
\mathcal{F}\left(\dot{\Delta}_{j} f(( \right. & \left.\left.\left.-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right) u\right)(\lambda) F_{\alpha, \lambda} \\
& =R_{|\alpha|}^{*}\left(2^{-2 j} \lambda\right) f\left((4|\lambda|(2|\alpha|+n))^{1 / 2}\right) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda} \tag{3.12}
\end{align*}
$$

ce qui, par l'homogénéité de $f$, conduit à

$$
\begin{aligned}
& \mathcal{F}\left(\dot{\Delta}_{j} f\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right) u\right)(\lambda) F_{\alpha, \lambda} \\
& \quad=2^{j m} R_{|\alpha|}^{*}\left(2^{-2 j} \lambda\right) f\left(\left(4\left|2^{-2 j} \lambda\right|(2|\alpha|+n)\right)^{1 / 2}\right) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda} .
\end{aligned}
$$

Définissons alors $\theta^{*}(\lambda)=R^{*}(\lambda) f\left(2|\lambda|^{1 / 2}\right)$, et soit $h$ la fonction radiale, dans $\mathcal{S}\left(\mathbb{H}^{n}\right)$ par [4], telle que

$$
\mathcal{F}(h)(\lambda) F_{\alpha, \lambda}=\theta_{|\alpha|}^{*}(\lambda) F_{\alpha, \lambda},
$$

où comme précédemment, on a noté $\theta_{|\alpha|}^{*}(\lambda)=\theta^{*}((2|\alpha|+n) \lambda)$. La condition de support de $R^{*}$ nous permet d'écrire

$$
\begin{aligned}
& \mathcal{F}\left(\dot{\Delta}_{j} f\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right) u\right)(\lambda) F_{\alpha, \lambda} \\
&=2^{j m} \theta_{|\alpha|}^{*}\left(2^{-2 j} \lambda\right) \sum_{\left|j-j^{\prime}\right| \leq 1} R_{|\alpha|}^{*}\left(2^{-2 j^{\prime}} \lambda\right) \mathcal{F}(u)(\lambda) F_{\alpha, \lambda} .
\end{aligned}
$$

On en conclut que

$$
\begin{equation*}
\dot{\Delta}_{j} f\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right) u=2^{j m+j N} h\left(\delta_{2^{j}} \cdot\right) * \sum_{\left|j-j^{\prime}\right| \leq 1} \dot{\Delta}_{j^{\prime}} u \tag{3.13}
\end{equation*}
$$

et donc le lemme est démontré, par application de l'inégalité de Young comme dans [7].

### 3.2.4. Composition par des fonctions $C^{\infty}$.

Le Lemme 3.1 a enfin pour conséquence le résultat suivant, sur la composition par des fonctions de classe $C^{\infty}$.

Proposition 3.2. Soit $u \in L^{\infty} \cap B_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$ une fonction à valeurs réelles, avec $\rho>0$. Soit $F$ une fonction dans l'espace $C^{\infty}(\mathbb{R})$ telle que $F(0)=0$. Alors $F(u)$ est dans $L^{\infty} \cap B_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$, et

$$
\|F(u)\|_{B_{p, r}^{p}\left(\mathbb{H}^{n}\right)} \leq C\|u\|_{B_{p, r}^{p}\left(\mathbb{H}^{n}\right)},
$$

où $C$ ne dépend que de $F$ et de $\|u\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}$.
Démonstration. La démonstration de cette proposition repose sur le lemme de caractérisation 3.2, et est identique au cas classique (voir [1], [13]). Rappellons brièvement la méthode: on écrit $F(u)$ comme la série

$$
\begin{equation*}
F(u)=\sum_{j \geq-1} v_{j}, \quad \text { où } \quad v_{j}=F\left(\sum_{k \leq j} \Delta_{k} u\right)-F\left(\sum_{k \leq j-1} \Delta_{k} u\right), \tag{3.14}
\end{equation*}
$$

en se souvenant que

$$
F(u)=\lim _{j \rightarrow \infty} F\left(\sum_{k \leq j} \Delta_{k} u\right) .
$$

Il suffit alors de démontrer que

$$
\begin{equation*}
\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\gamma} v_{j}\right\|_{L^{p}\left(\mathbb{H}^{n}\right)} \leq C_{\gamma} c_{j} 2^{-j(\rho-2 \gamma)}, \quad \text { pour tout } \gamma \in \mathbb{N} \tag{3.15}
\end{equation*}
$$

où $C_{\gamma}$ ne dépend que de $\gamma$, et $\left\{c_{j}\right\}_{j \geq 0}$ est une suite de $\ell^{r}(\mathbb{N})$, ce qui par le Lemme 3.2 donnera le résultat.

L'estimation (3.15) s'obtient en écrivant la formule de Taylor avec reste intégral, à l'ordre 1, qui fournit

$$
F\left(\sum_{k \leq j} \Delta_{k} u\right)-F\left(\sum_{k \leq j-1} \Delta_{k} u\right)=\Delta_{j} u \int_{0}^{1} F^{\prime}\left(\sum_{k \leq j-1} \Delta_{k} u+t \Delta_{j} u\right) d t
$$

La fin de la preuve consiste alors à estimer des dérivées successives du terme dans l'intégrale, par application de la formule de Faa-di-Bruno: nous renvoyons à [13] pour des détails.

Remarque. On a le même type de résultat pour les espaces de Sobolev homogènes $\dot{H}^{s}$, pour tout $s>0$.

### 3.2.5. Inclusions de Sobolev.

Nous allons dans cette section présenter une démonstration des inclusions de Sobolev utilisant le découpage dyadique.

Théorème 3.1. Soit $p \in[1, \infty]$ et soit $\rho \in \mathbb{R}$ tel que $0<\rho<N / r$. Alors l'inclusion

$$
B_{r, r}^{\rho}\left(\mathbb{H}^{n}\right) \subset L^{p}\left(\mathbb{H}^{n}\right), \quad \text { avec } p=\frac{r N}{N-r \rho}
$$

est continue.
DÉmonstration. Soit $f \in \mathcal{S}\left(\mathbb{H}^{n}\right)$. On a

$$
\|f\|_{L^{p}\left(\mathbb{H}^{n}\right)}^{p}=p \int_{0}^{\infty} a^{p-1} \mu(\{|f|>a\}) d a,
$$

où $\mu$ est la mesure de Haar sur $\mathbb{H}^{n}$ (égale à la mesure de Lebesgue). Soit alors $A$ un réel strictement positif à fixer, et écrivons

$$
f=f_{1, A}+f_{2, A}, \quad \text { avec } f_{1, A}=\sum_{2^{j}<A} \Delta_{j} f \text { et } f_{2, A}=\sum_{2^{j} \geq A} \Delta_{j} f .
$$

On a, en utilisant le Lemme 3.1,

$$
\begin{align*}
\left\|f_{1, A}\right\|_{L^{\infty}\left(\mathbb{H}^{n}\right)} & \leq \sum_{2^{j}<A}\left\|\Delta_{j} f\right\|_{L^{\infty}\left(\mathbb{H}^{n}\right)} \\
& \leq \sum_{2^{j}<A} 2^{j \rho}\left\|\Delta_{j} f\right\|_{L^{r}\left(\mathbb{H}^{n}\right)} 2^{j(N / r-\rho)}  \tag{3.16}\\
& \leq C A^{N / r-\rho}\|f\|_{B_{r, r}^{\rho}\left(\mathbb{H}^{n}\right)}
\end{align*}
$$

Choisissons à présent $A=A_{a}$ tel que

$$
C A_{a}^{N / r-\rho}\|f\|_{B_{r, r}^{p}\left(\mathbb{H}^{n}\right)}=\frac{a}{4} .
$$

Comme

$$
\mu(\{|f|>a\}) \leq \mu\left(\left\{\left|f_{1, A}\right|>\frac{a}{2}\right\}\right)+\mu\left(\left\{\left|f_{2, A}\right|>\frac{a}{2}\right\}\right),
$$

on en déduit, avec le choix $A=A_{a}$, que

$$
\mu(\{|f|>a\}) \leq \mu\left(\left\{\left|f_{2, A}\right|>\frac{a}{2}\right\}\right)
$$

Mais on a, par l'inégalité de Bienaymé-Tchebytchev,

$$
\mu\left(\left\{\left|f_{2, A}\right|>\frac{a}{2}\right\}\right) \leq 2^{r} a^{-r}\left\|f_{2, A}\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{r},
$$

et

$$
\begin{aligned}
\left\|f_{2, A_{a}}\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{r} & =\int_{\mathbb{H}^{n}}\left|\sum_{2^{j} \geq A_{a}} \Delta_{j} f\right|^{r} d z d s \\
& \leq \int_{\mathbb{H}^{n}}\left(\sum_{2^{j} \geq A_{a}} 2^{j \varepsilon r}\left|\Delta_{j} f\right|^{r} d z d s\right)\left(\sum_{2^{j} \geq A_{a}} 2^{-j \varepsilon r^{\prime}}\right)^{r / r^{\prime}}
\end{aligned}
$$

où $1 / r+1 / r^{\prime}=1$, et donc

$$
\left\|f_{2, A_{a}}\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{r} \leq C A_{a}^{-\varepsilon r} \sum_{2^{j} \geq A_{a}} 2^{j \varepsilon r}\left\|\Delta_{j} f\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{r} .
$$

Par conséquent, on peut écrire, en utilisant le théorème de Fubini, que

$$
\begin{aligned}
\|f\|_{L^{p}\left(\mathbb{H}^{n}\right)}^{p} \leq & C \int_{0}^{\infty} a^{p-r-1} A_{a}^{-\varepsilon r} \sum_{2^{j} \geq A_{a}} 2^{j \varepsilon r}\left\|\Delta_{j} f\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{r} d a \\
\leq & C \sum_{j \geq-1}\left(\int_{0}^{C 2^{j(N / r-\rho)}\|f\|_{B_{r, r}^{\rho}}} a^{p-r-1-\varepsilon r(r /(N-r \rho))} d a\right) \\
& \cdot\left(4 C\|f\|_{B_{r, r}^{p}}^{p}\right)^{\varepsilon r^{2}(N-r \rho)^{-1}} 2^{j \varepsilon r}\left\|\Delta_{j} f\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{r} \\
\leq & C\|f\|_{B_{r, r}^{p}}^{p-r} \sum_{j \geq-1} 2^{j(N / r-\rho)(p-r)}\left\|\Delta_{j} f\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{r} \\
\leq & C\|f\|_{B_{r, r}^{p}}^{p-r} \sum_{j \geq-1} 2^{j \rho r}\left\|\Delta_{j} f\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{r} .
\end{aligned}
$$

Le théorème est démontré.
Remarque. La même démonstration permet d'obtenir l'inégalité de Sobolev précisée suivante (voir [10])

$$
\|f\|_{L^{p}\left(\mathbb{H}^{n}\right)} \leq C\|f\|_{B_{\infty, \infty}^{(-N / r-\rho)}\left(\mathbb{H}^{n}\right)}^{1-r / p}\|f\|_{B_{r, r}\left(\mathbb{H}^{n}\right)}^{r / p}
$$

Il suffit en effet de modifier le calcul (3.16): si l'on n'utilise pas le Lemme 3.1, il vient

$$
\left\|f_{1, A}\right\|_{L^{\infty}} \leq C A^{N / r-\rho}\|f\|_{B_{\infty}^{-(N / r-\rho)}}
$$

et les calculs sont alors identiques, en choissant $A=A_{a}$ avec

$$
\frac{a}{4}=C A_{a}^{N / r-\rho}\|f\|_{B_{\infty, \infty}^{-(N / r-\rho)}} .
$$

## 4. Paraproduit sur le groupe de Heisenberg.

L'objet de cette section est d'adapter au groupe de Heisenberg l'algorithme de paraproduit introduit par J.-M. Bony dans [5].

Par rapport au cas classique, une difficulté apparaît, due au fait que l'on ne dispose pas d'écriture simple pour la transformée de Fourier du produit de deux fonctions. Notamment il n'est pas évident a priori, et contrairement au cas classique, que si deux fonctions ont une transformée de Fourier supportée dans des couronnes suffisamment éloignées l'une de l'autre, alors la transformée de Fourier de leur produit reste supportée dans une couronne. Néanmoins ce résultat est conservé pour le groupe de Heisenberg, comme le montre la proposition suivante.

Proposition 4.1. Soient $j$ et $j^{\prime}$ deux entiers, et soient $f$ et $g$ deux fonctions de $\mathcal{S}^{\prime}\left(\mathbb{H}^{n}\right)$ telles que $f * \widetilde{f}_{j}=0$ et $g * \widetilde{g}_{j^{\prime}}=0$ pour toutes les fonctions radiales $\widetilde{f}_{j}$ et $\widetilde{g}_{j^{\prime}}$ dans $\mathcal{S}\left(\mathbb{H}^{n}\right)$, telles que

$$
\begin{array}{ll}
\mathcal{F}\left(\widetilde{f}_{j}\right)(\lambda) F_{\alpha, \lambda}=0, & \text { pour } \lambda \in(2|\alpha|+n)^{-1} 2^{2 j} \mathcal{C}_{0}, \\
\mathcal{F}\left(\widetilde{g}_{j^{\prime}}\right)(\lambda) F_{\alpha, \lambda}=0, & \text { pour } \lambda \in(2|\alpha|+n)^{-1} 2^{2 j^{\prime}} \mathcal{C}_{0}
\end{array}
$$

Alors si $j^{\prime}-j>1$, il existe une couronne $\mathcal{C}_{0}^{\prime}$ telle que $\mathrm{fg} * \widetilde{h}_{j^{\prime}}=0$ pour toutes les fonctions radiales $\widetilde{h}_{j^{\prime}}$ dans $\mathcal{S}\left(\mathbb{H}^{n}\right)$, telles que

$$
\mathcal{F}\left(\widetilde{h}_{j^{\prime}}\right)(\lambda) F_{\alpha, \lambda}=0, \quad \text { pour } \lambda \in(2|\alpha|+n)^{-1} 2^{2 j^{\prime}} \mathcal{C}_{0}^{\prime}
$$

D'autre part, si $\left|j^{\prime}-j\right| \leq 1$, alors il existe une boule $\mathcal{B}_{0}^{\prime}$ telle que $f g *$ $\widetilde{h}_{j^{\prime}}^{\prime}=0$ pour toutes les fonctions radiales $\widetilde{h}_{j^{\prime}}^{\prime}$ dans $\mathcal{S}\left(\mathbb{H}^{n}\right)$, telles que

$$
\mathcal{F}\left(\widetilde{h}_{j^{\prime}}^{\prime}\right)(\lambda) F_{\alpha, \lambda}=0, \quad \text { pour } \lambda \in(2|\alpha|+n)^{-1} 2^{2 j^{\prime}} \mathcal{B}_{0}^{\prime}
$$

Remarque. De la même manière que pour le Lemme 3.1 vu plus haut, dans le cas de fonctions dans $\mathcal{S}\left(\mathbb{H}^{n}\right)$, cette proposition s'écrit plus simplement de la façon suivante.

Proposition 4.2. Soient $j$ et $j^{\prime}$ deux entiers, et soient $f$ et $g$ deux fonctions de $\mathcal{S}\left(\mathbb{H}^{n}\right)$ telles que

$$
\begin{aligned}
& \mathcal{F}(f)(\lambda) F_{\alpha, \lambda}=\mathbf{1}_{(2|\alpha|+n)^{-1} 2^{2 j} \mathcal{C}_{0}}(\lambda) \mathcal{F}(f)(\lambda) F_{\alpha, \lambda}, \\
& \mathcal{F}(g)(\lambda) F_{\alpha, \lambda}=\mathbf{1}_{(2|\alpha|+n)^{-1} 2^{2 j^{\prime}} \mathcal{C}_{0}}(\lambda) \mathcal{F}(g)(\lambda) F_{\alpha, \lambda},
\end{aligned}
$$

avec $j^{\prime}-j>1$. Alors il existe une couronne $\mathcal{C}_{0}^{\prime}$ telle que

$$
\mathcal{F}(f g)(\lambda) F_{\alpha, \lambda}=\mathbf{1}_{(2|\alpha|+n)^{-1} 2^{2 j^{\prime}} \mathcal{C}_{0}^{\prime}}(\lambda) \mathcal{F}(f g)(\lambda) F_{\alpha, \lambda} .
$$

D'autre part, si $\left|j^{\prime}-j\right| \leq 1$, alors il existe une boule $\mathcal{B}_{0}^{\prime}$ telle que

$$
\mathcal{F}(f g)(\lambda) F_{\alpha, \lambda}=\mathbf{1}_{(2|\alpha|+n)^{-1} 2^{2 j^{\prime}} \mathcal{B}_{0}^{\prime}}(\lambda) \mathcal{F}(f g)(\lambda) F_{\alpha, \lambda} .
$$

Démonstration de la Proposition 4.1. Nous supposerons dans la suite que $f$ et $g$ sont deux fonctions de $\mathcal{S}\left(\mathbb{H}^{n}\right)$, la Proposition 4.1 s'obtenant par densité. On est donc ramené à démontrer la Proposition 4.2.

Pour simplifier nous ne traiterons dans la suite que le cas $\lambda>0$. Par définition de $\mathcal{F}(f)(\lambda)$, on a

$$
\begin{aligned}
\mathcal{F}(f)(\lambda) F_{\alpha, \lambda}(\xi) & =\int_{\mathbb{H}^{n}} f(z, s) u_{z, s}^{\lambda} F_{\alpha, \lambda}(\xi) d z d s \\
& =\int_{\mathbb{H}^{n}} f(z, s) \frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\alpha}}{\sqrt{\alpha!}} e^{i \lambda s+2 \lambda\left(\xi \cdot z-|z|^{2} / 2\right)} d z d s .
\end{aligned}
$$

En écrivant $\xi=\xi_{a}+i \xi_{b}$ et $z=z_{a}+i z_{b}$, il vient

$$
\mathcal{F}(f)(\lambda) F_{\alpha, \lambda}(\xi)=\left(A_{\lambda, \xi}^{\alpha} f\right) \hat{f}\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)
$$

où l'on a écrit $\widehat{f}$ pour la transformée de Fourier usuelle de toute fonction $f$, et où

$$
A_{\lambda, \xi}^{\alpha} f(z, s)=\frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\alpha}}{\sqrt{\alpha!}} e^{-\lambda\left(|\xi-\bar{z}|^{2}-|\xi|^{2}\right)} f(z, s)
$$

Remarque. Par ce calcul, on a fait le lien entre la transformée de Fourier "Heisenberg" et la transformée de Fourier usuelle. C'est ce lien qui est la clef de la démonstration du résultat.

On peut à présent écrire

$$
\mathcal{F}(f g)(\lambda) F_{\alpha, \lambda}(\xi)=\left(A_{\lambda, \xi}^{\alpha} f g\right) \widetilde{)}\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)
$$

Soit alors $\beta$ un multi-indice tel que $\beta \leq \alpha$, et $|\beta|=E(|\alpha| / 2)$, où $E$ est la partie entière. Définissons

$$
B_{\lambda, \xi}^{\beta} f(z, s)=\frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\beta}}{\sqrt{\beta!}} f(z, s)
$$

Alors on a

$$
\begin{aligned}
& \left(A_{\lambda, \xi}^{\alpha} f g\right)^{\Upsilon}\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right) \\
& \quad=\binom{\alpha}{\beta}^{-1 / 2}\left(B_{\lambda, \xi}^{\beta} f\right)^{\wedge} *\left(A_{\lambda, \xi}^{\alpha-\beta} g\right)^{\wedge}\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)
\end{aligned}
$$

Il reste donc à étudier les supports de ces deux fonctions en convolution. On sait que

$$
\begin{aligned}
\left(A_{\lambda, \xi}^{\alpha-\beta} g\right)^{\sim} & \left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right) \\
& =\mathcal{F}(g)(\lambda) F_{\alpha-\beta, \lambda}(\xi) \\
& =\mathbf{1}_{(2|\alpha-\beta|+n)^{-1} 2^{2 j^{\prime}} c_{0}}(\lambda)\left(A_{\lambda, \xi}^{\alpha-\beta} g\right)^{\top}\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)
\end{aligned}
$$

donc le support en $\lambda$ de la fonction $\left(A_{\lambda, \xi}^{\alpha-\beta} g(z, s)\right)\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)$ est inclus dans la couronne $(2|\alpha-\beta|+n)^{-1} 2^{2 j^{\prime}} \mathcal{C}_{0}$.


$$
\begin{aligned}
\left(B_{\lambda, \xi}^{\beta} f\right) \mathcal{f}\left(-2 \lambda \xi_{b}\right. & \left.,-2 \lambda \xi_{a},-\lambda\right) \\
& =\mathbf{1}_{(2|\beta|+n)^{-1} 2^{2 j} \mathcal{B}_{0}^{\prime \prime}}(\lambda)\left(B_{\lambda, \xi}^{\beta} f\right)^{\mathcal{(}}\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)
\end{aligned}
$$

ò̀ $\mathcal{B}_{0}^{\prime \prime}=\{\tau \in \mathbb{R}:|\tau| \leq 4\}$.
Supposons un instant ce lemme démontré. Alors la Proposition 4.1 suit immédiatement, puisque le fait que $j^{\prime}-j>1$ implique que la couronne $\mathcal{C}_{0}$ et la boule $\mathcal{B}_{0}^{\prime \prime}$ sont disjointes. De même, on a le résultat cherché dans le cas où $\left|j^{\prime}-j\right| \leq 1$.

Démonstration du Lemme 4.1. Écrivons

$$
\begin{aligned}
\left(B_{\lambda, \xi}^{\beta} f\right)(- & \left.2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right) \\
= & \int f(z, s) e^{i \lambda s+2 i \lambda\left(\xi_{b} z_{a}+\xi_{a} z_{b}\right)} \frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\beta}}{\sqrt{\beta!}} d z d s \\
= & \int f(z, s) e^{-\lambda\left(|\xi-\bar{z}|^{2}-|\xi|^{2}\right)} \frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\beta}}{\sqrt{\beta!}} \\
& \cdot e^{\lambda\left(|\xi-\bar{z}|^{2}-|\xi|^{2}\right)} e^{i J_{\lambda}(s, z, \xi)} d z d s,
\end{aligned}
$$

avec $J_{\lambda}(s, z, \xi)=\lambda s+2 \lambda\left(\xi_{b} z_{a}+\xi_{a} z_{b}\right)$. Mais il existe une suite $\left\{c_{k}\right\}_{k \in \mathbb{N}^{n}}$ telle que

$$
e^{\lambda|\xi-\bar{z}|^{2}}=\sum_{k \in \mathbb{N}^{n}} c_{k}\left(\prod_{i=1}^{n}\left(\bar{\xi}_{i}-z_{i}\right)^{k_{i}}\right) \frac{\lambda^{|k|}(\xi-\bar{z})^{k}}{(2 k)!} .
$$

La fonction $\left(B_{\lambda, \xi}^{\beta} f\right)\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)$ est donc égale à

$$
\begin{aligned}
& \int e^{i J_{\lambda}(s, z, \xi)} e^{-\lambda\left(|\xi-\bar{z}|^{2}-|\xi|^{2}\right)} \frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\beta}}{\sqrt{\beta!}} \\
& \cdot f(z, s) \sum_{k \in \mathbb{N}^{n}} c_{k}\left(\prod_{i=1}^{n}\left(\bar{\xi}_{i}-z_{i}\right)^{k_{i}}\right) \frac{\lambda^{|k|}(\xi-\bar{z})^{k}}{(2 k)!} d z d s .
\end{aligned}
$$

En écrivant

$$
\frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\beta} \lambda^{|k|}(\xi-\bar{z})^{k}}{\sqrt{\beta!}(2 k)!}=\frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\beta+k}}{\sqrt{(\beta+k)!}} \lambda^{|k| / 2} d_{k, \beta}
$$

 $-\lambda)$

$$
\begin{aligned}
\sum_{k \in \mathbb{N}^{n}} \lambda^{|k| / 2} d_{k, \beta} c_{k} \int & e^{i J_{\lambda}(s, z, \xi)} f(z, s)\left(\prod_{i=1}^{n}\left(\bar{\xi}_{i}-z_{i}\right)^{k_{i}}\right) \\
& \cdot e^{-\lambda\left(|\xi-\bar{z}|^{2}-|\xi|^{2}\right)} \frac{(\sqrt{2 \lambda}(\xi-\bar{z}))^{\beta+k}}{\sqrt{(\beta+k)!}} d z d s,
\end{aligned}
$$

d'où finalement

$$
\left(B_{\lambda, \xi}^{\beta} f\right)=\sum_{k \in \mathbb{N}^{n}} \lambda^{|k| / 2} d_{k, \beta} c_{k}\left(A_{\lambda, \xi}^{\beta+k}\left(\prod_{i=1}^{n}\left(\bar{\xi}_{i}-z_{i}\right)^{k_{i}}\right) f\right)^{\hat{)}}
$$

Étudions séparément chacun des termes de cette série. Il est facile de voir que pour tout $\gamma$ tel que $\gamma_{i} \neq 0$,

$$
\begin{aligned}
\partial_{\xi_{i}}(\mathcal{F}(f)(\lambda)) & F_{\gamma, \lambda}(\xi) \\
& =c_{\gamma} \sqrt{\lambda} \mathcal{F}(f)(\lambda) F_{\gamma-1_{i}, \lambda}(\xi)+2 \lambda\left(\mathcal{F}\left(z_{i} f\right)(\lambda)\right) F_{\gamma, \lambda}(\xi)
\end{aligned}
$$

où l'on a noté $1_{i}$ pour le vecteur de $\mathbb{R}^{n}$ dont toutes les composantes sont nulles sauf la composante $i$, égale à 1 . Dans le cas où $\gamma_{i}=0$, on a simplement

$$
\partial_{\xi_{i}}(\mathcal{F}(f)(\lambda)) F_{\gamma, \lambda}(\xi)=2 \lambda\left(\mathcal{F}\left(z_{i} f\right)(\lambda)\right) F_{\gamma, \lambda}(\xi)
$$

Donc le support en $\lambda$ de $\mathcal{F}\left(\left(\bar{\xi}_{i}-z_{i}\right) f\right)(\lambda) F_{\gamma, \lambda}(\xi)$ est inclus dans la réunion suivante

$$
2^{2 j}(2|\gamma|+n)^{-1} \mathcal{C}_{0} \cup 2^{2 j}(2(|\gamma|-1)+n)^{-1} \mathcal{C}_{0} .
$$

Une récurrence immédiate implique que le support en $\lambda$ de

$$
\mathcal{F}\left(\prod_{i=1}^{n}\left(\bar{\xi}_{i}-z_{i}\right)^{k_{i}} f\right)(\lambda) F_{\gamma, \lambda}(\xi)
$$

est inclus dans

$$
2^{2 j}(2|\gamma|+n)^{-1} \mathcal{C}_{0} \cup \cdots \cup 2^{2 j}\left(2\left(|\gamma|-k_{1}-\cdots-k_{n}\right)+n\right)^{-1} \mathcal{C}_{0}
$$

Mais comme $\gamma=\beta+k$, on obtient finalement que le support en $\lambda$ de

$$
\left(A_{\lambda, \xi}^{\beta+k}\left(\prod_{i=1}^{n}\left(\bar{\xi}_{i}-z_{i}\right)^{k_{i}}\right) f\right)\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)
$$

est inclus dans

$$
2^{2 j}(2|\beta|+n)^{-1} \mathcal{C}_{0} \cup \cdots \cup 2^{2 j}\left(2\left(|\beta|+k_{1}+\cdots+k_{n}\right)+n\right)^{-1} \mathcal{C}_{0}
$$

c'est-à-dire dans la boule, indépendante de $k$,

$$
(2|\beta|+n)^{-1} 2^{2 j} \mathcal{B}_{0}^{\prime \prime}, \quad \text { où } \quad \mathcal{B}_{0}^{\prime \prime}=\{\tau \in \mathbb{R}:|\tau| \leq 4\} .
$$

Mais alors chacun des termes de la série, qui converge vers

$$
\left(B_{\lambda, \xi}^{\beta} f\right)\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)
$$

est supporté dans une boule fixe, ce qui implique que

$$
\left(B_{\lambda, \xi}^{\beta} f\right)\left(-2 \lambda \xi_{b},-2 \lambda \xi_{a},-\lambda\right)
$$

est supporté dans cette même boule.
Le lemme est donc démontré, et avec lui, la Proposition 4.1.

### 4.2. L'algorithme de paraproduit.

### 4.2.1. Définitions.

Définition 4.1. On appelle paraproduit de $f$ par $g$, et l'on note $T_{f} g$, l'opérateur bilinéaire suivant

$$
T_{f} g \stackrel{\text { def }}{=} \sum_{j^{\prime} \leq j-2} \Delta_{j^{\prime}} f \Delta_{j} g=\sum_{j} S_{j-1} f \Delta_{j} g,
$$

où l'on a défini

$$
S_{j} f=\sum_{j^{\prime} \leq j-1} \Delta_{j^{\prime}} f
$$

On appelle reste du produit $f g$, et l'on note $R(f, g)$, l'opérateur bilinéaire symétrique suivant

$$
R(f, g) \stackrel{\text { def }}{=} \sum_{\left|j-j^{\prime}\right| \leq 1} \Delta_{j^{\prime}} f \Delta_{j} g .
$$

Remarque. La Proposition 4.1 implique en particulier que pour tout $j \geq 0$ et pour tout $\mu \in\{-1,0,1\}$, on a

$$
\begin{aligned}
& \mathcal{F}\left(S_{j-1} f \Delta_{j} g\right)(\lambda) F_{\alpha, \lambda}=\mathbf{1}_{(2|\alpha|+n)^{-1} 2^{2 j} \mathcal{C}_{0}^{\prime}}(\lambda) \mathcal{F}\left(S_{j-1} f \Delta_{j} g\right)(\lambda) F_{\alpha, \lambda}, \\
& \mathcal{F}\left(\Delta_{j-\mu} f \Delta_{j} g\right)(\lambda) F_{\alpha, \lambda}=\mathbf{1}_{(2|\alpha|+n)^{-1} 2^{2 j} \mathcal{B}_{0}^{\prime}}(\lambda) \mathcal{F}\left(S_{j-\mu} f \Delta_{j} g\right)(\lambda) F_{\alpha, \lambda}
\end{aligned}
$$

### 4.2.2. Lois de produit.

Les démonstrations des résultats ci-dessous s'obtiennent exactement comme dans le cas classique, par application du Lemme 3.1. Nous renvoyons à [7] pour les détails.

Théorème 4.1. Soient $\rho$ et $\rho^{\prime}$ deux réels, et $p$ et $r$ deux éléments de $[1,+\infty]$. Alors si a est un élément de $L^{\infty}\left(\mathbb{H}^{n}\right)$, l'opérateur $T_{a}$ est continu de $B_{p, r}^{\rho}$ dans $B_{p, r}^{\rho}$, et si $a \in C^{\rho^{\prime}}\left(\mathbb{H}^{n}\right)$ avec $\rho^{\prime}<0$, alors $T_{a}$ est continu de $B_{p, r}^{\rho}$ dans $B_{p, r}^{\rho+\rho^{\prime}}$, et l'on a

$$
\left\|T_{a} b\right\|_{B_{p, r}^{p}\left(\mathbb{H}^{n}\right)} \leq C\|a\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|b\|_{B_{p, r}^{p}\left(\mathbb{H}^{n}\right)},
$$

et

$$
\left\|T_{a} b\right\|_{B_{p, r}^{\rho+\rho^{\prime}}\left(\mathbb{H}^{n}\right)} \leq C\|a\|_{C \rho^{\prime}\left(\mathbb{H}^{n}\right)}\|b\|_{B_{p, r}^{p}\left(\mathbb{H}^{n}\right)}, \quad \rho^{\prime}<0 .
$$

D'autre part, pour tous réels $\rho_{1}$ et $\rho_{2}$ tels que $\rho_{1}+\rho_{2}>0$, et pour tous les $p_{1}, p_{2}, p, r_{1}, r_{2}$ dans $[1,+\infty]$ tels que $1 / p \leq 1 / p_{1}+1 / p_{2}$ et $1 / r \stackrel{\text { def }}{=}$ $1 / r_{1}+1 / r_{2} \leq 1$, l'opérateur $R$ est bilinéaire continu de

$$
B_{p_{1}, r_{1}}^{\rho_{1}} \times B_{p_{2}, r_{2}}^{\rho_{2}} \quad \text { dans } \quad B_{p, r}^{\rho_{12}},
$$

où

$$
\rho_{12}=\rho_{1}+\rho_{2}-N\left(\frac{1}{p_{1}}+\frac{1}{p_{2}}-\frac{1}{p}\right) .
$$

Enfin si $\rho_{1}+\rho_{2} \geq 0$ et $1 / p \leq 1 / p_{1}+1 / p_{2}$ et $1 / r_{1}+1 / r_{2}=1$, alors $R$ est bilinéaire continu de

$$
B_{p_{1}, r_{1}}^{\rho_{1}} \times B_{p_{2}, r_{2}}^{\rho_{2}} \quad \text { dans } \quad B_{p, \infty}^{\rho_{12}} .
$$

Corollaire 4.1. Soient $\rho>0$ et $(p, r) \in[1,+\infty]^{2}$ trois réels. Si $u$ et $v$ sont deux éléments de $L^{\infty} \cap B_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$, alors uv $\in B_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$, et

$$
\|u v\|_{B_{p, r\left(\mathbb{H}^{n}\right)}^{\rho}} \leq C\left(\|u\|_{L^{\infty}}\|v\|_{B_{p, r}^{\rho}}+\|v\|_{L^{\infty}}\|u\|_{B_{p, r}^{\rho}}\right) .
$$

Si $\rho_{1}+\rho_{2}>0$ et si $p_{1}$ est tel que $\rho_{1}<N / p_{1}$, alors pour tout couple $\left(p_{2}, r_{2}\right) \in[1,+\infty]^{2}$, on a pour tous $u$ et $v$ dans $B_{p_{1}, \infty}^{\rho_{1}} \cap B_{p_{2}, r_{2}}^{\rho_{2}}\left(\mathbb{H}^{n}\right)$,

$$
\|u v\|_{B_{p_{2}, r_{2}}\left(\mathbb{H}^{n}\right)} \leq C\left(\|u\|_{B_{p_{1}, \infty}^{\rho_{1}}}\|v\|_{B_{p_{2}, r_{2}}^{\rho_{2}}}+\|v\|_{B_{p_{1}, \infty}^{\rho_{1}}}\|u\|_{B_{p_{2}, r_{2}}^{\rho_{2}}}\right),
$$

ò̀ $\rho=\rho_{1}+\rho_{2}-N / p_{1}$. D'autre part, si $\rho_{1}+\rho_{2} \geq 0, \rho_{1}<N / p_{1}$ et $1 / r_{1}+1 / r_{2}=1$, alors pour $u$ et $v$ dans $B_{p_{1}, r_{1}}^{\rho_{1}} \cap B_{p_{2}, r_{2}}^{\rho_{2}}\left(\mathbb{H}^{n}\right)$, on a

$$
\|u v\|_{B_{p, \infty}\left(\mathbb{H}^{n}\right)}^{p^{\prime}} \leq C\left(\|u\|_{B_{p_{1}, r_{1}}^{\rho_{1}}}\|v\|_{B_{p_{2}, r_{2}}^{\rho_{2}}}+\|v\|_{B_{p_{1}, r_{1}}^{\rho_{1}}}\|u\|_{B_{p_{2}, r_{2}}^{\rho_{2}}}\right) .
$$

Enfin si $\rho_{1}+\rho_{2}>0, \rho_{j}<N / p_{j}$ et $p \geq \max \left\{p_{1}, p_{2}\right\}$, alors pour tout $\left(r_{1}, r_{2}\right)$,

$$
\|u v\|_{B_{p, r}^{\rho_{1}}\left(\mathbb{\mathbb { H } ^ { n }}\right)} \leq C\|u\|_{B_{p_{1}, r_{1}}^{\rho_{1}}}\|v\|_{B_{p_{2}, r_{2}}^{\rho_{2}}},
$$

avec

$$
\rho_{12}=\rho_{1}+\rho_{2}-N\left(\frac{1}{p_{1}}+\frac{1}{p_{2}}-\frac{1}{p}\right)
$$

et $r=\max \left\{r_{1}, r_{2}\right\}$, et si $\rho_{1}+\rho_{2} \geq 0$, avec $\rho_{j}<N / p_{j}$ et $1 / r_{1}+1 / r_{2}=1$, alors pour tout $p \geq \max \left\{p_{1}, p_{2}\right\}$,

$$
\|u v\|_{B_{p, \infty}^{\rho_{1}}\left(\mathbb{H} \mathbb{H}^{n}\right)} \leq C\|u\|_{B_{p_{1}, r_{1}}^{\rho_{1}}}\|v\|_{B_{p_{2}, r_{2}}^{\rho_{2}}} .
$$

Remarque. Les résultats correspondant au cas des espaces de Besov homogènes s'énoncent de manière identique.

Un second corollaire à ce théorème est démontré dans [8], la démonstration ici est identique.

Proposition 4.3. Soient $\rho$ et $r$ deux réels tels que $\rho \geq 2$ et $1 / 2<r<$ 1. Il existe alors une constante $C$ telle que pour toutes fonctions $u$, $v$, et $w$ avec

$$
u \in B_{1, \infty}^{\rho} \cap C^{r}\left(\mathbb{H}^{n}\right) \quad \text { et } \quad(v, w) \in\left(B_{1, \infty}^{\rho-1} \cap C^{r-1}\right)^{2}\left(\mathbb{H}^{n}\right)
$$

on a

$$
\|u v w\|_{B_{1, \infty}^{p-2+r}\left(\mathbb{H}^{n}\right)} \leq C\|u\|_{B_{1, \infty}^{\rho} \cap C^{r}}\|v\|_{B_{1, \infty}^{p-1} \cap C^{r-1}}\|w\|_{B_{1, \infty}^{p-1} \cap C^{r-1}} .
$$

Enfin les opérateurs de paraproduit permettent de préciser la Proposition 3.2, sur la composition par une fonction $C^{\infty}$, de la manière suivante.

Proposition 4.3. Soit u une fonction à valeurs réelles telle que $u \in$ $B_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)$, avec $\rho>N / p$. Soit enfin $F \in C^{\infty}(\mathbb{R})$. Alors

$$
F(u)=T_{F^{\prime}(u)} u+R, \quad o \grave{u} \quad R \in B_{p, r}^{2 \rho-N / p}\left(\mathbb{H}^{n}\right) .
$$

Démonstration. Nous n'allons pas donner ici les détails des calculs conduisant à ce résultat, et renvoyons au livre de Y. Meyer ([13]) pour la démonstration de cette proposition. Rappellons simplement que la démonstration consiste à écrire la série téléscopique (3.14) déjà employée pour démontrer la Proposition 3.2, et d'utiliser alors la formule de Taylor avec reste intégral à l'ordre deux.

## 5. Applications.

### 5.1. L'inégalité de Gagliardo-Nirenberg.

On définit les espaces $W^{\sigma, r}\left(\mathbb{H}^{n}\right)$ comme la complétion de $\mathcal{S}\left(\mathbb{H}^{n}\right)$ pour la norme

$$
\|u\|_{W^{\sigma, r}\left(\mathbb{H}^{n} n\right.}=\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} u\right\|_{L^{r}\left(\mathbb{H}^{n}\right)} .
$$

L'objet de cette section est de démontrer, par application des résultats présentés précédemment, le théorème suivant.

Théorème 5.1. Soit $f$ une fonction de $L^{q} \cap W^{\sigma, r}\left(\mathbb{H}^{n}\right)$, avec $q$ et $r$ strictement supérieurs à 1 et $\sigma \geq 0$. Alors $f \in W^{\rho, p}\left(\mathbb{H}^{n}\right)$, et

$$
\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} f\right\|_{L^{p}\left(\mathbb{H}^{n}\right)} \leq C\|f\|_{L^{q}\left(\mathbb{H}^{n}\right)}^{\theta}\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} f\right\|_{L^{r}\left(\mathbb{H}^{n}\right)}^{1-\theta},
$$

où $1 / p=\theta / q+(1-\theta) / r, \rho=(1-\theta) \sigma$, et $\theta \in] 0,1[$.
Démonstration. Commençons par rappeller la définition et les principales propriétés de la fonction maximale (voir [18, Chapitre XIII, p. 638], pour des détails).

Rappellons que la distance homogène sur le groupe de Heisenberg est définie par

$$
\left\||(z, s) \|| \stackrel{\text { def }}{=}\left(|z|^{4}+|s|^{2}\right)^{1 / 4}\right.
$$

Les "boules" associées à cette distance sont notées

$$
B(z, s, R)=\left\{\left(z^{\prime}, s^{\prime}\right) \in \mathbb{H}^{n}:\| \|\left(z^{\prime}, s^{\prime}\right)(z, s)^{-1} \| \mid \leq R\right\}
$$

et leur mesure est notée $m(B(z, s, R))$.

Définition 5.1. Soit $f \in L_{\mathrm{loc}}^{1}\left(\mathbb{H}^{n}\right)$. La fonction maximale de $f$ est définie par

$$
M f(z, s) \stackrel{\text { def }}{=} \sup _{R>0} \frac{1}{m(B(z, s, R))} \int_{B(z, s, R)}\left|f\left(z^{\prime}, s^{\prime}\right)\right| d z^{\prime} d s^{\prime}
$$

Proposition 5.1. Si $f \in L^{p}\left(\mathbb{H}^{n}\right)$, avec $1<p \leq \infty$, alors $M f \in$ $L^{p}\left(\mathbb{H}^{n}\right)$, et

$$
\begin{equation*}
\|M f\|_{L^{p}\left(\mathbb{H}^{n}\right)} \leq A_{p}\|f\|_{L^{p}\left(\mathbb{H}^{n}\right)} \tag{5.1}
\end{equation*}
$$

où $A_{p}$ est une constante dépendant de $p$ et de $n$.
D'autre part, soit $\varphi \in L^{1}\left(\mathbb{H}^{n}\right)$, et supposons que le plus petit majorant radial de $\varphi$, noté $\psi$ et défini par

$$
\psi(z, s)=\sup _{\| \|\left(z^{\prime}, s^{\prime}\right)\|\geq\|\|(z, s)\| \|} \varphi\left(z^{\prime}, s^{\prime}\right)
$$

est dans $L^{1}\left(\mathbb{H}^{n}\right)$. Alors pour tout $f \in L^{p}\left(\mathbb{H}^{n}\right)$, avec $1 \leq p \leq \infty$, on a

$$
\begin{equation*}
|f * \varphi(z, s)| \leq\|\psi\|_{L^{1}\left(\mathbb{H}^{n}\right)} M f(z, s) . \tag{5.2}
\end{equation*}
$$

Démontrons à présent l'inégalité proposée pour une fonction $f \in$ $\mathcal{S}\left(\mathbb{H}^{n}\right)$. On peut écrire

$$
\begin{aligned}
& \left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} f \\
& \quad=\sum_{j \leq A}\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} \dot{\Delta}_{j} f+\sum_{j>A}\left(-\Delta_{\mathbb{H}^{n}}\right)^{(\rho-\sigma) / 2} \dot{\Delta}_{j}\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{(\sigma / 2)} f\right),
\end{aligned}
$$

où $A$ est une constante à fixer. Nous avons vu en (3.12) que

$$
\begin{aligned}
& \mathcal{F}\left(a\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{1 / 2}\right) \dot{\Delta}_{j} f\right)(\lambda) F_{\alpha, \lambda} \\
& \quad=2^{j m} R_{|\alpha|}^{*}\left(2^{-2 j} \lambda\right) a\left(\left(4\left|2^{-2 j \lambda}\right|(2|\alpha|+n)\right)^{1 / 2}\right) \mathcal{F}(f)(\lambda) F_{\alpha, \lambda},
\end{aligned}
$$

dès que $a \in C^{\infty}\left(\mathbb{R}^{*}\right)$ est homogène de degré $m$. Alors comme en (3.13), on a

$$
a\left(-\Delta_{\mathbb{H}^{n}}\right) \dot{\Delta}_{j} f=2^{j m+j N} h\left(\delta_{2^{j}} \cdot\right) * \sum_{\left|j-j^{\prime}\right| \leq 1} \dot{\Delta}_{j^{\prime}} f,
$$

où $h$ est la fonction de $\mathcal{S}\left(\mathbb{H}^{n}\right)$ telle que

$$
\mathcal{F}(h)(\lambda) F_{\alpha, \lambda}=R_{|\alpha|}^{*}(\lambda) a\left(\left(4\left|2^{-2 j} \lambda\right|\right)^{1 / 2}\right) F_{\alpha, \lambda} .
$$

Mais alors comme dans [2], il existe une fonction $\widetilde{h}$ radiale, intégrable et décroissante en la distance à l'origine, qui majore $h$, ce qui par application de (5.2), donne

$$
\left|a\left(-\Delta_{\mathbb{H}^{n}}\right) \dot{\Delta}_{j} f(z, s)\right| \leq C 2^{j m} M f(z, s) .
$$

En appliquant cette inégalité à $a(D)=\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2}$ puis à $a\left(-\Delta_{\mathbb{H}^{n}}\right)=$ $\left(-\Delta_{\mathbb{H}^{n}}\right)^{(\rho-\sigma) / 2}$, il vient $\left|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} f(z, s)\right|$

$$
\begin{aligned}
& \leq C\left(\sum_{j \leq A} 2^{\rho j} M f(z, s)+\sum_{j>A} 2^{(\rho-\sigma) j} M\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} f\right)(z, s)\right) \\
& \leq C 2^{\rho A} M f(z, s)+C 2^{(\rho-\sigma) A} M\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} f\right)(z, s),
\end{aligned}
$$

puisque $\sigma>\rho$. En optimisant sur $A$, il vient

$$
\left|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} f(z, s)\right| \leq C(M f(z, s))^{1-\rho / \sigma}\left(M\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} f\right)(z, s)\right)^{\rho / \sigma} .
$$

Il suffit alors d'appliquer l'inégalité de Hölder, qui donne

$$
\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} f\right\|_{L^{p}\left(\mathbb{H}^{n}\right)} \leq C\|M f\|_{L^{q}\left(\mathbb{H}^{n}\right)}^{\theta}\left\|M\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} f\right)\right\|_{L^{r}}^{1-\theta},
$$

avec $\theta=1-\rho / \sigma$.
L'inégalité maximale (5.1) termine la démonstration.
Remarque. Dans le cas où $p=q=r=1$, l'inégalité correspondante

$$
\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} f\right\|_{L^{1}\left(\mathbb{H}^{n}\right)} \leq C\|f\|_{L^{1}\left(\mathbb{H}^{n}\right)}^{\theta}\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} f\right\|_{L^{1}\left(\mathbb{H}^{n}\right)}^{1-\theta},
$$

où $\rho=(1-\theta) \sigma$, et $\theta \in] 0,1[$, se démontre simplement par le calcul suivant

$$
\begin{aligned}
\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} f\right\|_{L^{1}\left(\mathbb{H}^{n}\right)} \leq & \left\|\sum_{j \leq A}\left(-\Delta_{\mathbb{H}^{n}}\right)^{\rho / 2} \dot{\Delta}_{j} f\right\|_{L^{1}\left(\mathbb{H}^{n}\right)} \\
& +\left\|\sum_{j>A}\left(-\Delta_{\mathbb{H}^{n}}\right)^{(\rho-\sigma) / 2} \dot{\Delta}_{j}\left(\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} f\right)\right\|_{L^{1}\left(\mathbb{H}^{n}\right)} \\
\leq & C 2^{(1-\theta) \sigma A}\|f\|_{L^{1}\left(\mathbb{H}^{n}\right)} \\
& +C 2^{-\theta \sigma A}\left\|\left(-\Delta_{\mathbb{H}^{n}}\right)^{\sigma / 2} f\right\|_{L^{1}\left(\mathbb{H}^{n}\right)}
\end{aligned}
$$

ce qui, en optimisant sur $A$, conduit au résultat.

## 5.2. Équations semi-linéaires sous-elliptiques.

Dans [21], C.-J. Xu et C. Zuily démontrent un résultat de régularité des solutions faibles d'équations quasi-linéaires sous-elliptiques. Nous nous proposons ici, dans le cas où l'opérateur sous-elliptique est $-\Delta_{\mathbb{H}^{n}}$ et dans un cadre semi-linéaire, d'en présenter une démonstration plus élémentaire, reposant sur le paraproduit et les espaces de Besov. Cette démonstration dans le cas classique est due à J.-Y. Chemin et C.-J. Xu, voir [8].

Considérons donc l'équation semi-linéaire sous-elliptique suivante, où $N_{0} \in \mathbb{N}$, et $b_{i j, k \ell}^{k^{\prime}}$ est une fonction indéfiniment différentiable

$$
\begin{equation*}
-\Delta_{\mathbb{H}^{n}} u^{k^{\prime}}+\sum_{i, j=1}^{2 n} \sum_{k, \ell=1}^{N_{0}} b_{i j, k \ell}^{k^{\prime}}(u) X_{i} u^{k} X_{j} u^{\ell}=0 \tag{5.3}
\end{equation*}
$$

pour $k^{\prime} \in\left\{1, \ldots, N_{0}\right\}$. On peut alors démontrer le théorème suivant.
Théorème 5.2. Si $\rho$ est un réel tel que $\rho>1 / 2$, et si u est solution faible de (5.3) telle que $u \in H^{1}\left(\mathbb{H}^{n}\right) \cap C^{\rho}\left(\mathbb{H}^{n}\right)$, alors $u \in C^{\infty}\left(\mathbb{H}^{n}\right)$.

Démonstration. Les résultats obtenus jusqu'ici permettent de reprendre à l'identique la démonstration de [8]; nous la reproduisons ici pour la commodité du lecteur.

On peut supposer que $\rho \leq 1$. Commençons par remarquer que si $u \in H^{1}\left(\mathbb{H}^{n}\right) \cap C^{\rho}\left(\mathbb{H}^{n}\right)$, alors

$$
b_{i j, k \ell}^{k^{\prime}}(u) X_{i} u^{k} X_{j} u^{\ell} \in L^{1}\left(\mathbb{H}^{n}\right),
$$

et donc par l'injection continue de $L^{1}\left(\mathbb{H}^{n}\right)$ dans $B_{1, \infty}^{0}\left(\mathbb{H}^{n}\right)$, on en déduit que

$$
\Delta_{\mathbb{H}^{n}} u^{k^{\prime}} \in B_{1, \infty}^{0}\left(\mathbb{H}^{n}\right) .
$$

Mais l'opérateur $\left(-\Delta_{\mathbb{H}^{n}}\right)$ est un isomorphisme de $B_{p, r}^{\sigma}\left(\mathbb{H}^{n}\right)$ dans $B_{p, r}^{\sigma-2}\left(\mathbb{H}^{n}\right)$ pour tout $\sigma \in \mathbb{R}$, et pour tous $(p, r) \in[1, \infty]$ (voir [4]), par conséquent on obtient que $u \in B_{1, \infty}^{2}\left(\mathbb{H}^{n}\right)$.

On raisonne alors par récurrence, en montrant que pour tout $k \in \mathbb{N}$,

$$
\begin{equation*}
u \in B_{1, \infty}^{2+k \rho} \cap C^{\rho}\left(\mathbb{H}^{n}\right) \quad \text { implique } \quad u \in B_{1, \infty}^{2+(k+1) \rho}\left(\mathbb{H}^{n}\right) \tag{5.4}
\end{equation*}
$$

ce qui démontrera le théorème.
Pour démontrer (5.4), il suffit d'utiliser la Proposition 3.2 ci-dessus, qui implique que

$$
b_{i j, k \ell}^{k^{\prime}}(u) \in B_{1, \infty}^{2+k \rho} \cap C^{\rho}\left(\mathbb{H}^{n}\right),
$$

et donc par le Corollaire 4.2, on a

$$
b_{i j, k \ell}^{k^{\prime}}(u) X_{i} u^{k} X_{j} u^{\ell} \in B_{1, \infty}^{(k+1) \rho}\left(\mathbb{H}^{n}\right) .
$$

On en conclut alors que

$$
u \in B_{1, \infty}^{2+(k+1) \rho}\left(\mathbb{H}^{n}\right),
$$

ce qui achève la démonstration.

## 5.3. Équations d'ondes semi-linéaires.

Considérons l'équation d'ondes semi-linéaire suivante

$$
\left\{\begin{array}{l}
\partial_{t t} u-\Delta_{\mathbb{H}^{n}} u=|X u|^{2} F(u), \quad \text { dans } \mathbb{R} \times \mathbb{H}^{n}  \tag{5.5}\\
\left(u_{\mid t=0}, \partial_{t} u_{\mid t=0}\right)=\left(u_{0}, u_{1}\right),
\end{array}\right.
$$

où $F \in C^{\infty}(\mathbb{R})$. On a noté $X u=\left(X_{1} u, \ldots, X_{2 n} u\right)$, et l'on notera dorénavant $D u=\left(\partial_{t} u, X u\right)$. Démontrons le théorème suivant.

Théorème 5.3. Soit $s>N / 2+3 / 4$, et $\left(u_{0}, u_{1}\right) \in \dot{H}^{s} \times \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)$. Alors il existe un temps $T>0$ tel que (5.5) possède une unique solution u, avec

$$
u \in L^{\infty}\left([0, T], \dot{H}^{s}\left(\mathbb{H}^{n}\right)\right),
$$

et

$$
D u \in L^{\infty}\left([0, T], \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)\right) \cap L^{4}\left([0, T], L^{\infty}\left(\mathbb{H}^{n}\right)\right)
$$

Remarque. Ce théorème est l'analogue sur le groupe de Heisenberg du [17, Théorème 1] du cas classique. Notons toutefois que la restriction sur l'indice $s$ est plus forte dans notre cadre (quand $d=3$, la restriction dans [17] est $s>2$ ). Comme nous le verrons dans la démonstration, cela est dû au fait que le domaine de validité des estimations de Strichartz
généralisées, sur le groupe de Heisenberg, est moins étendu sur $\mathbb{H}^{n}$ que sur $R^{n}$ (voir [4]).

Démonstration du Théorème. Nous allons commencer par rappeler [4, Théorème 4.1], donnant les estimations de Strichartz généralisées, sur le groupe de Heisenberg, vérifiées par la solution $u$ de

$$
\begin{cases}\partial_{t t} u-\Delta_{\mathbb{H}^{n}} u=f, & \text { dans } \mathbb{R} \times \mathbb{H}^{n}, \\ \left(u_{\mid t=0}, \partial_{t} u_{\mid t=0}\right)=\left(u_{0}, u_{1}\right) . & \end{cases}
$$

Notons que dans [4], le théorème est démontré pour $\left(u_{0}, u_{1}\right) \in \dot{H}^{1} \times$ $L^{2}\left(\mathbb{H}^{n}\right)$, mais on obtient de manière identique le cas $\left(u_{0}, u_{1}\right) \in \dot{H}^{s} \times$ $\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)$.

Proposition 5.2. Soient trois réels s, $\rho_{1}$ et $\rho_{2}$, et soient $p_{i}, r_{i}$, pour $i \in$ $\{1,2\}$, tels que

$$
\begin{gathered}
\frac{2}{p_{i}} \leq \frac{1}{2}-\frac{1}{r_{i}} \quad \text { et } \quad 2 \leq r_{i} \leq \infty \\
\rho_{1}+N\left(\frac{1}{2}-\frac{1}{r_{1}}\right)-\frac{1}{p_{1}}=s \quad \text { et } \quad \rho_{2}+N\left(\frac{1}{2}-\frac{1}{r_{2}}\right)-\frac{1}{p_{2}}=1-s .
\end{gathered}
$$

Supposons que $\left(u_{0}, u_{1}\right) \in \dot{H}^{s} \times \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)$. Alors pour tout temps $T$, on a

$$
\begin{aligned}
& \|u\|_{L^{p_{1}}\left([0, T], \dot{B}_{r_{1}, 2}^{\rho_{1}}\left(\mathbb{H}^{n}\right)\right)}+\left\|\partial_{t} u\right\|_{L^{p_{1}}\left([0, T], \dot{B}_{r_{1}, 2}^{\rho_{1}-1}\left(\mathbb{H}^{n}\right)\right)} \\
& \quad \leq C\left\|\left(u_{0}, u_{1}\right)\right\|_{\dot{H}^{s} \times \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}+C\|f\|_{L^{\bar{p}_{2}}\left([0, T], \dot{B}_{\bar{r}_{2}, 2}^{-\rho_{2}}\left(\mathbb{H}^{n}\right)\right)},
\end{aligned}
$$

où pour tout $r$, on a noté $\bar{r}$ pour son conjugué, défini par $1 / r+1 / \bar{r}=1$.
Pour alléger les notations, on notera $L_{T}^{p}\left(\dot{B}_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)\right)$ l'espace $L^{p}\left([0, T], \dot{B}_{p, r}^{\rho}\left(\mathbb{H}^{n}\right)\right)$. D'autre part, on supposera dans la suite que $s \leq N / 2+1$ puisque dans le cas $s>N / 2+1$, la résolution du système est simplement due à la théorie classique des systèmes symétriques hyperboliques. Par la proposition ci-dessus, on a donc

$$
\begin{aligned}
& \|u\|_{L_{T}^{\infty}\left(\dot{H}^{s}\left(\mathbb{H}^{n}\right)\right)}+\left\|\partial_{t} u\right\|_{L_{T}^{\infty}\left(\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)\right)} \\
& \quad \leq C\left\|\left(u_{0}, u_{1}\right)\right\|_{\dot{H}^{s} \times \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}+C\left\||X u|^{2} F(u)\right\|_{L_{T}^{1}\left(\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)\right)},
\end{aligned}
$$

et, pour tout $\varepsilon>0$, puisque $s>N / 2+3 / 4$, on a

$$
\|u\|_{L_{T}^{p_{1}}\left(\dot{B}_{\infty, 2}^{\varepsilon+1}\left(\mathbb{H}^{n}\right)\right)}+\left\|\partial_{t} u\right\|_{L_{T}^{p_{1}}\left(\dot{B}_{\infty, 2}^{\varepsilon}\left(\mathbb{H}^{n}\right)\right)}
$$

$$
\begin{equation*}
\leq C\left\|\left(u_{0}, u_{1}\right)\right\|_{\dot{H}^{s} \times \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}+C\left\||X u|^{2} F(u)\right\|_{L_{T}^{1}\left(\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)\right)}, \tag{5.6}
\end{equation*}
$$

avec

$$
\frac{1}{p_{1}}=1+\varepsilon+\frac{N}{2}-s
$$

et l'on peut choisir $p_{1}=4$.
Remarque. C'est ici que la restriction sur $s$ intervient, et elle est due au fait que sur le groupe de Heisenberg, on a nécessairement $p_{1} \geq 4$, alors que dans le cas classique, la limitation sur $p_{1}$ est $p_{1} \geq 2$ (en dimension $d \geq 3$ ). Cette limitation dans le cas classique permet donc de résoudre notre problème pour $s>(n+1) / 2$ dès que la dimension d'espace est $n \geq 3$, et pour $s>1+3 / 4$ en dimension 2 . Ici en toute dimension, on demande que

$$
s>\frac{N}{2}+\frac{3}{4} .
$$

Le Lemme 3.3 indique que les $X_{j}$ opèrent sur les espaces de Besov, par conséquent l'estimation (5.6) s'écrit aussi

$$
\begin{aligned}
& \|D u\|_{L_{T}^{4}\left(\dot{B}_{\infty, 2}^{\varepsilon}\left(\mathbb{H}^{n}\right)\right)} \\
& \quad \leq C\left\|\left(u_{0}, u_{1}\right)\right\|_{\dot{H}^{s} \times \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}+C\left\||X u|^{2} F(u)\right\|_{L_{T}^{1}\left(\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)\right)} .
\end{aligned}
$$

En outre, comme $\dot{B}_{\infty, 2}^{\varepsilon} \subset L^{\infty}$, on a finalement

$$
\begin{aligned}
& \|D u\|_{L_{T}^{4}\left(L^{\infty}\left(\mathbb{H}^{n}\right)\right)} \\
& \quad \leq C\left\|\left(u_{0}, u_{1}\right)\right\|_{\dot{H}^{s} \times \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}+C\left\||X u|^{2} F(u)\right\|_{L_{T}^{1}\left(\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)\right)} .
\end{aligned}
$$

Il reste à estimer $|X u|^{2} F(u)$ dans $\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)$. Pour cela, on va faire appel aux estimations douces obtenues en Section 3.3.2 et à la Proposition 3.2 sur la composition par des fonctions $C^{\infty}$, ainsi qu'à l'algorithme de paraproduit introduit dans la Définition 4.1. Commençons par définir la fonction $G \in C^{\infty}(\mathbb{R})$ par

$$
G(u)=F(u)-F(0) .
$$

Alors, par les estimations douces de la Section 3.3.2, on a

$$
\begin{aligned}
& \left\||X u|^{2} F(u)\right\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} \\
& \quad \leq C\|X u\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|X u\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}+\left\||X u|^{2} G(u)\right\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} .
\end{aligned}
$$

Démontrons finalement le lemme suivant:
Lemme 5.1. Si $X u \in L^{\infty}$ et $u \in \dot{H}^{s}\left(\mathbb{H}^{n}\right)$, avec $s>N / 2$, alors

$$
\left\||X u|^{2} G(u)\right\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} \leq C\|X u\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|u\|_{\dot{H}^{s}\left(\mathbb{H}^{n}\right)}^{2} .
$$

Démonstration du Lemme. Commençons par remarquer que la Proposition 3.2 sur la composition par des fonctions $C^{\infty}$ implique, puisque $G(0)=0$, que

$$
\|G(u)\|_{\dot{H}^{s}} \leq C\|u\|_{\dot{H}^{s}} .
$$

En outre, les estimations douces de la Section 3.2.2 fournissent

$$
\begin{aligned}
\left\||X u|^{2} G(u)\right\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} \leq & C\|X u\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}\||X u| G(u)\|_{L^{\infty}\left(\mathbb{H}^{n}\right)} \\
& +\|X u\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\||X u| G(u)\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} .
\end{aligned}
$$

Le premier terme de cette estimation se majore en utilisant que

$$
\||X u| G(u)\|_{L^{\infty}\left(\mathbb{H}^{n}\right)} \leq C\|X u\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|G(u)\|_{\dot{H}^{s}\left(\mathbb{H}^{n}\right)},
$$

puisque $s>N / 2$, ce qui donne

$$
\|X u\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}\||X u| G(u)\|_{L^{\infty}\left(\mathbb{H}^{n}\right)} \leq C\|u\|_{\dot{H}^{s}\left(\mathbb{H}^{n}\right)}^{2}\|X u\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}
$$

Quant au second terme, on écrit l'algorithme de paraproduit

$$
|X u| G(u)=T_{G(u)}|X u|+T_{|X u|} G(u)+R(G(u),|X u|) .
$$

Le Théorème 4.1 implique que

$$
\begin{aligned}
\left\|T_{G(u)}|X u|\right\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} & \leq C\|G(u)\|_{L^{\infty}\left(\mathbb{H}^{n}\right)}\|X u\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} \\
& \leq C\|u\|_{\dot{H}^{s}\left(\mathbb{H}^{n}\right)}^{2} .
\end{aligned}
$$

De même, on peut écrire que

$$
\begin{aligned}
\left\|T_{|X u|} G(u)\right\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} & \leq C\|X u\|_{\dot{C}^{-1}\left(\mathbb{H}^{n}\right)}\|G(u)\|_{\dot{H}^{s}\left(\mathbb{H}^{n}\right)} \\
& \leq C\|u\|_{\dot{H}^{s}\left(\mathbb{H}^{n}\right)}\|G(u)\|_{\dot{H}^{s}\left(\mathbb{H}^{n}\right)},
\end{aligned}
$$

puisque $s>N / 2$, ce qui donne l'estimation voulue. Enfin pour le terme de reste, on écrit que

$$
\begin{aligned}
\|R(G(u),|X u|)\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)} & \leq C\|X u\|_{\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}\|G(u)\|_{\dot{B}_{\infty, \infty}^{0}\left(\mathbb{H}^{n}\right)} \\
& \leq C\|u\|_{\dot{H}^{s}\left(\mathbb{H}^{n}\right)}\|G(u)\|_{L^{\infty}\left(\mathbb{H}^{n}\right)} .
\end{aligned}
$$

Le lemme est donc démontré.
On peut à présent achever la démonstration de la proposition. En désignant $\|\cdot\|_{s, T}$ pour la norme

$$
\|u\|_{s, T}=\|u\|_{L_{T}^{\infty}\left(\dot{H}^{s}\left(\mathbb{H}^{n}\right)\right)}+\left\|\partial_{t} u\right\|_{L_{T}^{\infty}\left(\dot{H}^{s-1}\left(\mathbb{H}^{n}\right)\right)}+\|D u\|_{L_{T}^{4}\left(L^{\infty}\left(\mathbb{H}^{n}\right)\right)},
$$

on a finalement montré que

$$
\left.\|u\|_{s, T} \leq C\left\|\left(u_{0}, u_{1}\right)\right\|_{\dot{H}^{s} \times \dot{H}^{s-1}\left(\mathbb{H}^{n}\right)}+C\|u\|_{s, T}^{2} T^{3 / 4}+C\|u\|_{s, T}^{3} T^{3 / 4}\right) .
$$

Il est classique que ce type d'estimation conduit à l'existence de solutions en temps petit (dépendant de $\left(u_{0}, u_{1}\right)$ ). L'unicité est une conséquence du fait que

$$
D u \in L^{1}\left([0, T], L^{\infty}\left(\mathbb{H}^{n}\right)\right)
$$

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## The maximal quartile operator

## Christoph Thiele

## 1. Introduction.

Let $C^{\Delta}(\mathbb{R})$ denote the set of all functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ that are finite linear combinations of characteristic functions of dyadic intervals, i.e., intervals of the form $\left[2^{k} n, 2^{k}(n+1)\right)$ with $k, n \in \mathbb{Z}$. We define the Walsh function $W_{l} \in C^{\Delta}(\mathbb{R})$ for $l \in \mathbb{N}_{0}$ by the following recursive formulas

$$
\begin{equation*}
W_{0}=\mathbf{1}_{[0,1)}, \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
W_{2 l}=W_{l}(2 x)+W_{l}(2 x-1), \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
W_{2 l+1}=W_{l}(2 x)-W_{l}(2 x-1) . \tag{3}
\end{equation*}
$$

For $k, n \in \mathbb{Z}, l \in \mathbb{N}_{0}$ we define the Walsh wave packet $w_{k, n, l}$ by

$$
w_{k, n, l}(x)=2^{-k / 2} W_{l}\left(2^{-k} x-n\right)
$$

The quartile operator $H_{W}$ and the maximal quartile operator $H_{W}^{\max }$ are then defined by

$$
\begin{aligned}
& H_{W}(f, g):=\sum_{\substack{k, n \in \mathbb{Z} \\
l \in \mathbb{N}_{0}}} 2^{-k / 2}\left\langle f, w_{k, n, 4 l}\right\rangle\left\langle g, w_{k, n, 4 l+1}\right\rangle w_{k, n, 4 l+2}, \\
& H_{W}^{\max }(f, g)(x) \\
& \quad:=\sup _{K \in \mathbb{Z}}\left|\sum_{\substack{k, n \in \mathbb{Z} \\
l \in \mathbb{N}_{0} \\
k \leq K}} 2^{-k / 2}\left\langle f, w_{k, n, 4 l}\right\rangle\left\langle g, w_{k, n, 4 l+1}\right\rangle w_{k, n, 4 l+2}(x)\right| .
\end{aligned}
$$

In this paper we prove the following theorem:
Theorem 1. Let p, q, r satisfy

$$
\begin{equation*}
\frac{1}{p}=\frac{1}{q}+\frac{1}{r}, \quad \frac{2}{3}<p<\infty, 1<q, r \leq \infty . \tag{4}
\end{equation*}
$$

Then there is a constant $C$ such that for all functions $f, g \in C^{\Delta}(\mathbb{R})$

$$
\begin{gathered}
\left\|H_{W}(f, g)\right\|_{p} \leq C\|f\|_{q}\|g\|_{r} \\
\left\|H_{W}^{\max }(f, g)\right\|_{p} \leq C\|f\|_{q}\|g\|_{r}
\end{gathered}
$$

Only the estimates for $H_{W}^{\max }$ are new, but our approach gives the estimates for $H_{W}$ without extra work.

The quartile operator has been introduced in [11] as a discrete model for the bilinear Hilbert transform. The bilinear Hilbert transform $H$ is defined as a bilinear operation from $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ into $C(\mathbb{R})$ by

$$
H(f, g)(x):=\text { p.v. } \int f(x-t) g(x+t) \frac{d t}{t} .
$$

It has been shown in [5] and [7], see also [6] and [8] for a survey and [11] for a condensed proof, that the bilinear Hilbert transform satisfies the a priori estimates

$$
\begin{equation*}
\|H(f, g)\|_{p} \leq C_{q, r}\|f\|_{q}\|g\|_{r} \tag{5}
\end{equation*}
$$

provided $p, q, r$ satisfy (4). More recently, M. Lacey has shown (see [4]) that also the maximal truncation of the bilinear Hilbert transform,

$$
H^{\max }(f, g)(x):=\sup _{\varepsilon>0}\left|\int_{\mathbb{R} \backslash[-\varepsilon, \varepsilon]} f(x-t) g(x+t) \frac{d t}{t}\right|
$$

satisfies estimates as in (5), (4). By the same method he has observed that the maximal operator

$$
M(f, g)(x):=\sup _{\varepsilon>0}\left|\frac{1}{\varepsilon} \int_{[-\varepsilon, \varepsilon]} f(x-t) g(x+t) d t\right|
$$

satisfies estimates as in (5), (4). For the operator $M$, these estimates are nontrivial only if $p \leq 1$.

The current paper is an adaption of the ideas in [4] to the discrete model of the quartile operator. As in [4], the main ingredient that is needed to pass from estimates for $H_{W}$ to estimates for $H_{W}^{\max }$ is a version of a lemma by Bourgain (see [1]) for certain maximal averages.

We use analysis in the Walsh phase plane as in [11]. We give all the necessary definitions, but at some places we refer to results in [11].

## 2. The main lemma.

The main issue in proving Theorem 1 is to efficiently make use of orthogonality of wave packets. For this we have to identify appropriate large sets of pairwise orthogonal wave packets. We will associate to each wave packet a rectangle in the half plane, so that disjoint rectangles correspond to orthogonal wave packets. Then the combinatorial issue is to identify sets of pairwise disjoint rectangles. This is the main idea behind the following Lemma 1. In the proof of this lemma one has to identify sets of pairwise disjoint rectangles so that we can use the second hypothesis of the lemma. This lemma already appears implicitly in [7].


Table 1. Subdivision of quartiles.
A tile $p$ is a rectangle $p=I_{p} \times \omega_{p}$ of area one in the upper half plane, such that $I_{p}$ and $\omega_{p}$ are dyadic. Hence for each tile $p$ there are integers $k, n, l$ with $l \geq 0$ such that

$$
I_{p}=\left[2^{k} n, 2^{k}(n+1)\right), \quad \omega_{p}=\left[2^{-k} l, 2^{-k}(l+1)\right) .
$$

Similarly, a quartile $P$ is a rectangle $I_{P} \times \omega_{P}$ of area four in the upper half plane, such that $I_{P}$ and $\omega_{P}$ are dyadic. Each quartile $P$ is the union of four tiles $p_{1}(P), p_{2}(P), p_{3}(P)$, and $p_{4}(P)$, as in Figure 1.

If $p, q$ are two tiles, then we write $p<q$ if $I_{p} \subset I_{q}$ and $\omega_{q} \subset \omega_{p}$. This defines a partial ordering of the set of tiles. Let $\imath \in\{1,2,3\}$. A set $T$ of quartiles is called a tree of type $\imath$, if $\left\{p_{\imath}(P)\right\}_{P \in T}$ contains exactly one element which is maximal in $\left\{p_{\imath}(P)\right\}_{P \in T}$. If $p_{\imath}\left(P_{T}\right)$ is this maximal element, we write $p_{\imath}\left(P_{T}\right)=I_{T} \times \omega_{T}=p_{T}$ and call $p_{T}$ the top of the tree.

Lemma 1. Assume that we are given exponents $1 \leq s_{\jmath}<\infty$ for $\jmath \in\{1,2,3\}$ such that $1 / s_{1}+1 / s_{2}+1 / s_{3}>1$, and we are given a constant $B>0$. Then there is a constant $C>0$ such that the following holds:

Let $\mathbf{P}$ be a finite set of quartiles. For each $\jmath \in\{1,2,3\}$ let

$$
a_{\jmath}: \mathbf{P} \longrightarrow \mathbb{R}^{+}
$$

be a function such that the following two hypotheses are satisfied:

1) Let $\imath \neq \jmath$. If $T \subset \mathbf{P}$ is a tree of type $\imath$, then

$$
\left\|\left(\sum_{P \in T} \frac{a_{\jmath}(P)^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{1} \leq B\left|I_{T}\right|
$$

2) Let $\imath \neq \jmath, m \in \mathbb{Z}$. Let $\mathbf{P}^{\prime} \subset \mathbf{P}$ be a disjoint union of trees of type $\imath$

$$
\mathbf{P}^{\prime}:=\bigcup_{T \in \mathcal{F}} T
$$

such that the set $\left\{p_{\jmath}(P): P \in \mathbf{P}^{\prime}\right\}$ is a set of pairwise disjoint rectangles and, for each $T \in \mathcal{F}$, we have

$$
\left\|\left(\sum_{P \in T} \frac{a_{\jmath}(P)^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{1} \geq 2^{m / s_{\jmath}}\left|I_{T}\right|
$$

Then

$$
\sum_{T \in \mathcal{F}}\left|I_{T}\right| \leq B 2^{-m}
$$

Then we have the estimate

$$
\sum_{P \in \mathbf{P}}\left|I_{P}\right|^{-1 / 2} a_{1}(P) a_{2}(P) a_{3}(P) \leq C .
$$

We prove the lemma. Let $m \in \mathbb{Z}$ and $\imath, \jmath \in\{1,2,3\}$ with $\imath \neq \jmath$. We say that a tree $T$ satisfies the size condition $(m, \imath, \jmath)$, if $T$ is of type $\imath$ and

$$
\begin{equation*}
\left\|\left(\sum_{P \in T} \frac{a_{\jmath}(P)^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{1} \geq 2^{m / s_{\jmath}+3}\left|I_{T}\right| . \tag{6}
\end{equation*}
$$

We say that a tree $T$ satisfies the size condition $(m, \imath, \imath)$, if $T$ is of type $\imath$ and

$$
\begin{equation*}
\frac{a_{\imath}(P)}{\sqrt{\left|I_{P}\right|}} \geq 2^{m / s_{2}} \tag{7}
\end{equation*}
$$

for all $P \in T$.
The size of a tree $T$ is the maximal $m \in \mathbb{Z}$ such that $T$ satisfies a size condition ( $m, \imath, \jmath$ ) for some $\imath, \jmath \in\{1,2,3\}$.

We partition the set $\mathbf{P}$ into trees $T_{0}, \ldots, T_{N}$ as follows. Let $\nu \in \mathbb{N}_{0}$ and assume by induction that $T_{\nu^{\prime}}$ is already chosen for all $\nu^{\prime}$ with $\nu^{\prime} \leq \nu$. Define

$$
\mathbf{P}_{\nu}:=\mathbf{P} \backslash \bigcup_{\nu^{\prime} \leq \nu} T_{\nu^{\prime}}
$$

We can assume $\mathbf{P}_{\nu}$ is not empty. Let $m_{\nu}$ be the maximal integer for which there exists a tree $T \subset \mathbf{P}_{\nu}$ of size $m_{\nu}$, and let $\mathcal{F}_{\nu}$ be the set of all trees $T \subset \mathbf{P}_{\nu}$ of size $m_{\nu}$. Define $\mathcal{F}_{\nu}^{\max }$ to be the set of trees in $\mathcal{F}_{\nu}$ which are maximal in $\mathcal{F}_{\nu}$ with respect to set inclusion. Let $\mathcal{F}_{\nu,<}$ be the set of all trees in $\mathcal{F}_{\nu}^{\max }$ which satisfy a size condition $(m, \imath, \jmath)$ with $\imath<\jmath$. If $\mathcal{F}_{\nu,<}$ is nonempty, choose $T_{\nu+1} \in \mathcal{F}_{\nu,<}$ such that the center of $\omega_{T_{\nu+1}}$ is maximal. If $\mathcal{F}_{\nu,<}$ is empty, choose $T_{\nu+1} \in \mathcal{F}_{\nu}^{\max }$ such that the center of $\omega_{T_{\nu+1}}$ is minimal.

Since $\mathbf{P}$ is finite, the algorithm stops with a finite partition of $\mathbf{P}$ into $\left\{T_{1}, \ldots, T_{N}\right\}$. Define $\mathcal{F}:=\left\{T_{1}, \ldots, T_{N}\right\}$.

In the following estimates, $C$ will denote a constant depending on $s_{\jmath}$ and $B$. The precise value of $C$ may change from line to line.

Lemma 2. If $T \in \mathcal{F}$ and the size of $T$ is $m$, then, for $\imath \neq \jmath$,

$$
\left(\sum_{P \in T} a_{\jmath}(P)^{2}\right)^{1 / 2} \leq C 2^{m / s_{\jmath}}\left|I_{T}\right|^{1 / 2}
$$

and

$$
\sup _{P \in T} \frac{a_{\imath}(P)}{\sqrt{\left|I_{P}\right|}} \leq C 2^{m / s_{2}}
$$

Proof. Let $T \in \mathcal{F}$ be of size $m$. Define

$$
f:=\left(\sum_{P \in T} \frac{a_{\jmath}(P)^{2}}{\left|I_{P}\right|} 1_{I_{P}}\right)^{1 / 2}
$$

To prove the first estimate of the lemma, we have to bound the $L^{2}$ norm of $f$. We prove $\|f\|_{\text {BMO }} \leq C 2^{m / s_{j}}$, which gives the appropriate bound on $\|f\|_{2}$, because $f$ is supported on $I_{T}$.

Let $J$ be a dyadic interval. We have to show

$$
\begin{equation*}
\inf _{c} \frac{1}{|J|} \int_{J}(f(x)-c) d x \leq C 2^{m / s_{J}} \tag{8}
\end{equation*}
$$

We split the sum in the definition of $f$ into the sum over those $P$ with $I_{P} \subset J$ and the sum over those $P$ with $I_{P} \not \subset J$. The second sum is constant on the interval $J$. Hence, using the inequality

$$
(a+b)^{1 / 2}-b^{1 / 2} \leq a^{1 / 2}
$$

which holds for any two positive numbers $a, b$, we can estimate the left hand side of inequality (8) by

$$
\frac{1}{|J|} \int_{J}\left(\sum_{P \in T: I_{P} \subset J} \frac{a_{\jmath}^{2}(P)}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}(x)\right)^{1 / 2} d x
$$

By passing to subintervals, if necessary, one observes that it suffices to bound this expression under the assumption that there is a $P^{\prime} \in T$ such that $I_{P^{\prime}}=J$. But then the set $T_{J}:=\left\{P \in T: I_{P} \subset J\right\}$ is a tree of type $\imath$. The size of this tree is at most $m$ by construction of the tree $T$. The size estimate for $T_{J}$ then shows that (9) is bounded by $C 2^{m / s_{j}}$. This finishes the desired BMO estimate and therefore the proof of the first estimate of the lemma. The second estimate follows immediately from the observation that the set $\{P\}$ is a tree of type $\imath$ for all $P \in T$ and has size less than or equal $m$. This finishes the proof of Lemma 2.

Lemma 3. Let $\mathcal{F}_{m}$ be the set of trees in $\mathcal{F}$ with size $m$. Then

$$
\sum_{T \in \mathcal{F}^{m}}\left|I_{T}\right| \leq C 2^{-m}
$$

Proof. Fix $\imath, \jmath \in\{1,2,3\}$. It suffices to show the desired estimate for the sum over the set $\mathcal{F}_{m, \imath, j}$ of those $T \in \mathcal{F}_{n}$ which satisfy the size condition ( $m, \imath, \jmath$ ) but no size condition $\left(m, \imath, \jmath^{\prime}\right)$ with $\jmath<\jmath^{\prime}$. We first consider the case $\imath=\jmath$. Pick $\imath^{\prime} \neq \imath$ and consider the set $\mathbf{p}_{m, \imath, \jmath}$ of all tiles which are tops of trees in $\mathcal{F}_{m, 2, v}$. Then $\mathbf{p}_{m, \imath, \jmath}$ is a set of pairwise disjoint rectangles. To see this assume to the contrary that the tops of two trees $T, T^{\prime} \in \mathcal{F}_{m, \imath, \jmath}$ intersect. We can assume that $T$ has been selected before $T^{\prime}$. Then the union $T \cup T^{\prime}$ is a tree containing $T$, which contradicts the maximality of $T$ at the time it was selected. Since each set $\{P\}$ with $P \in \mathbf{P}^{\prime}$ is both a tree of type $\imath^{\prime}$ and of type $\imath$, we can apply (7) and the second hypothesis of the proposition to conclude the desired estimate.

Now assume $\imath<\jmath$. For a tree $T$ define $T^{\text {red }}$ to be the set of $P \in T$ such that $I_{P}$ is not minimal in $\left\{I_{P^{\prime}}: P^{\prime} \in T\right\}$. If $T^{\text {red }}$ is nonempty, it is again a tree. Define $T^{\text {Red }}=\left(T^{\text {red }}\right)^{\text {red }}$. If $T \in \mathcal{F}_{m, \imath, j}$, then

$$
\begin{aligned}
& \left\|\left(\sum_{P \in T^{\mathrm{Red}}} \frac{a_{\jmath}(P)^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{1} \\
& \quad \geq\left\|\left(\sum_{P \in T} \frac{a_{\jmath}(P)^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{1}-\left\|\left(\sum_{P \in T \backslash T^{\mathrm{Red}}} \frac{a_{\jmath}(P)^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{1} .
\end{aligned}
$$

Since the size of each tree $\{P\}$ with $P \in T \backslash T^{\text {Red }}$ is less than or equal $m$ and the intervals $I_{P}$ with $P \in T \backslash T^{\mathrm{red}}$ as well as those with $P \in T^{\mathrm{red}} \backslash T^{\mathrm{Red}}$ are pairwise disjoint, we can bound this expression by

$$
\geq 2^{m / s_{\jmath}+3}\left|I_{T}\right|-2^{m / s_{\jmath}+1}\left|I_{T}\right|-2^{m / s_{\jmath}+1}\left|I_{T}\right| \geq 2^{m / s_{\jmath}}\left|I_{T}\right|
$$

The desired estimate now follows from the second hypothesis of the proposition as soon as we prove that for any $T, T^{\prime} \in \mathcal{F}_{m, 2, j}$ and any $P \in T^{\mathrm{red}}, P^{\prime} \in T^{\text {red }}$ with $P \neq P^{\prime}$ we have that $p_{\jmath}(P)$ and $p_{\jmath}\left(P^{\prime}\right)$ are disjoint. To prove this assume to the contrary that $\omega_{p_{\jmath}(P)} \subset \omega_{p_{\jmath}\left(P^{\prime}\right)}$, $\omega_{p_{\jmath}(P)} \neq \omega_{p_{\jmath}\left(P^{\prime}\right)}$. Since $\imath<\jmath$, it is easy to see that the center of $\omega_{p_{\imath}(P)}$ is greater than the center of $\omega_{p_{2}\left(P^{\prime}\right)}$. Hence $T$ has been selected before $T^{\prime}$. Pick $P^{\prime \prime}, P^{\prime \prime \prime} \in T^{\prime} \backslash T^{\prime \text { Red }}$ such that $p_{\imath}\left(P^{\prime \prime}\right)<p_{\imath}\left(P^{\prime \prime \prime}\right)<p_{\imath}\left(P^{\prime}\right)$. Then we have

$$
\omega_{p_{\imath}(P)} \subset \omega_{p_{\imath}\left(P^{\prime \prime}\right)}, \quad I_{p_{\imath}\left(P^{\prime \prime}\right)} \subset I_{p_{\imath}(P)}
$$

Hence $P^{\prime \prime}$ qualifies to be in the tree $T$, a contradiction to the maximality of $T$. This finishes the proof of Lemma 3, since the case $\imath>\jmath$ is done similarly to the case $\imath<\jmath$.

The size of a tree in $\mathcal{F}$ is bounded by a constant $C$. This is immediate in the case of size conditions $(m, \imath, \jmath)$ with $\imath \neq \jmath$ from the first hypothesis of the lemma. For $\imath=\jmath$ we apply, as we have done before, the first hypothesis of the lemma to trees containing just one element.

Hence we have

$$
\begin{aligned}
\sum_{P \in \mathbf{P}} \frac{1}{\sqrt{\left|I_{P}\right|}} a_{1}(P) & a_{2}(P) a_{3}(P) \\
& =\sum_{\substack{m \in \mathbb{Z} \\
m \leq C}} \sum_{u, j=1}^{3} \sum_{T \in \mathcal{F}_{m, 2,3}} \sum_{P \in T} \frac{1}{\sqrt{\left|I_{P}\right|}} a_{1}(P) a_{2}(P) a_{3}(P)
\end{aligned}
$$

Applying Hölder's inequality gives

$$
\cdots \leq \sum_{m \leq m_{0}} \sum_{\imath, j=1}^{3} \sum_{T \in \mathcal{F}_{m, 2, j}} \sup _{P \in T} \frac{a_{\imath}(P)}{\sqrt{\left|I_{P}\right|}} \prod_{l \neq \imath}\left(\sum_{P \in T} a_{l}(P)^{2}\right)^{1 / 2} .
$$

Now Lemma 2 gives

$$
\cdots \leq \sum_{m \leq C} \sum_{\imath, j=1}^{3} \sum_{T \in \mathcal{F}_{(m, 2, j)}} C 2^{\left(1 / s_{1}+1 / s_{2}+1 / s_{3}\right) m}\left|I_{T}\right| .
$$

Finally Lemma 3 gives

$$
\leq \sum_{m \leq C} \sum_{i, j=1}^{3} C 2^{\left(1 / s_{1}+1 / s_{2}+1 / s_{3}-1\right) m}
$$

This is a convergent geometric series and hence bounded by a constant $C$. This finishes the proof of Lemma 1.

## 3. The maximal quartile operator.

If $p$ is the tile $\left[2^{k} n, 2^{k}(n+1)\right) \times\left[2^{-k} l, 2^{-k}(l+1)\right)$, then we denote by $w_{p}$ the Walsh wave packet given by

$$
w_{p}(x):=w_{k, n, l}(x)=2^{-k / 2} W_{l}\left(2^{-k} x-n\right) .
$$

The significance of this identification is that if $p$ and $p^{\prime}$ are two disjoint tiles, then $w_{p}$ and $w_{p^{\prime}}$ are orthogonal. Moreover if a $p<q$ for two tiles $p$ and $q$, then then on the interval $I_{p}$ the functions $w_{p}$ and $w_{q}$ are multiples of each other. For a proof of these easy facts see [11].

Let $\mathbf{P}$ denote the set of all quartiles. Then the maximal quartile operator $H_{W}^{\max }$ can be written as

$$
\begin{aligned}
& H_{W}^{\max }(f, g)(x) \\
& \quad:=\sup _{k \in \mathbb{Z}}\left|\sum_{P \in \mathbf{P}:\left|I_{P}\right| \geq 2^{k}} \frac{1}{\sqrt{\left|I_{P}\right|}}\left\langle w_{p_{1}(P)}, f\right\rangle\left\langle w_{p_{2}(P)}, g\right\rangle w_{p_{3}(P)}(x)\right| .
\end{aligned}
$$

Now let $\kappa \in C^{\Delta}(\mathbb{R})$. Then the linearized maximal quartile operator $H_{W}^{\kappa}$ is defined by

$$
H_{W}^{\kappa}(f, g)(x):=\sum_{P \in \mathbf{P}:\left|I_{P}\right| \geq 2^{\kappa(x)}} \frac{1}{\sqrt{\left|I_{P}\right|}}\left\langle w_{p_{1}(P)}, f\right\rangle\left\langle w_{p_{2}(P)}, g\right\rangle w_{p_{3}(P)}(x)
$$

By standard arguments, an $L^{p}$-bound on $H_{W}^{\kappa}$ that does not depend on the function $\kappa$ implies the corresponding bound for $H_{W}^{\max }$. We fix the function $\kappa$ and write

$$
H_{W}^{\kappa}(f, g)(x)=\sum_{P \in \mathbf{P}} \frac{1}{\sqrt{\left|I_{P}\right|}}\left\langle v_{1, P}, f\right\rangle\left\langle v_{2, P}, g\right\rangle v_{3, P}(x),
$$

where

$$
\begin{gathered}
v_{1, P}:=w_{p_{1}(P)}, \quad v_{2, P}:=w_{p_{2}(P)}, \\
v_{3, P}(x)
\end{gathered}:=\left\{\begin{array}{ll}
w_{p_{3}(P)}(x), & \text { if }\left|I_{P}\right| \geq 2^{\kappa(x)}, \\
0, & \text { if }\left|I_{P}\right|<2^{\kappa(x)} .
\end{array} .\right.
$$

By integrating against a third function $f_{3}$, we obtain a trilinear form

$$
\begin{gathered}
T_{W}^{\kappa}: C^{\Delta}(\mathbb{R}) \times C^{\Delta}(\mathbb{R}) \times C^{\Delta}(\mathbb{R}) \longrightarrow \mathbb{R}, \\
T_{W}^{\kappa}\left(f_{1}, f_{2}, f_{3}\right)=\sum_{P \in \mathbf{P}} \frac{1}{\sqrt{\left|I_{P}\right|}}\left\langle v_{1, P}, f_{1}\right\rangle\left\langle v_{2, P}, f_{2}\right\rangle\left\langle v_{3, P}, f_{3}\right\rangle .
\end{gathered}
$$

For each permutation $\sigma$ of the set $\{1,2,3\}$ we obtain the bilinear operator $H_{W}^{\kappa, \sigma}$ defined by

$$
\int H_{W}^{\kappa, \sigma}\left(f_{1}, f_{2}\right)(x) f_{3}(x) d x=T_{W}^{\kappa}\left(f_{\sigma^{-1}(1)}, f_{\sigma^{-1}(2)}, f_{\sigma^{-1}(3)}\right)
$$

We will prove Theorem 1 in two steps: The first step is to prove the following proposition:

Proposition 1. Let $1<r_{1}, r_{2}<2$ and assume

$$
\frac{1}{r}:=\frac{1}{r_{1}}+\frac{1}{r_{2}}>\frac{3}{2} .
$$

Then there is a constant $C$ such that for all $\kappa, \sigma$ as above and all $f_{1}, f_{2} \in C^{\Delta}(\mathbb{R})$

$$
\begin{equation*}
\left\|H_{W}^{\kappa, \sigma}\left(f_{1}, f_{2}\right)\right\|_{r} \leq C\left\|f_{1}\right\|_{r_{1}}\left\|f_{2}\right\|_{r_{2}} \tag{10}
\end{equation*}
$$

The second step consists of an interpolation argument which is given in the appendix.

## 4. Proof of Proposition 1.

By Marcinkiewicz interpolation (see [3]) it suffices to prove the corresponding weak type estimate instead of (10). By homogeneity here we use that $\kappa$ was arbitrary - and linearity it suffices to prove that for $\left\|f_{1}\right\|_{r_{1}}=\left\|f_{2}\right\|_{r_{2}}=1$ we have

$$
\left|\left\{x: H_{W}^{\kappa, \sigma}\left(f_{1}, f_{2}\right)(x)>1\right\}\right| \leq C .
$$

Fix such $f_{1}$ and $f_{2}$ and define

$$
E:=\left\{x: \max \left\{M_{r_{1}}^{\Delta} f_{1}(x), M_{r_{2}}^{\Delta} f_{2}(x)\right\} \geq 1\right\} .
$$

Here we have set

$$
M_{p}^{\Delta} f(x):=\left(\sup _{I: \text { dyadic }, x \in I} \frac{1}{|I|} \int_{I}|f(x)|^{p} d x\right)^{1 / p}
$$

By the maximal theorem the measure of $E$ is bounded by a universal constant, hence it suffices to prove a weak type estimate outside the set $E$; i.e., since each $v_{\imath, P}$ is supported on $I_{P}$, it suffices to prove a universal bound on the measure of the set

$$
F:=\left\{x: \sum_{P \in \mathbf{P}: I_{P} \not \subset E} \frac{1}{\sqrt{\left|I_{P}\right|}}\left\langle v_{\sigma(1), P}, f_{1}\right\rangle\left\langle v_{\sigma(2), P}, f_{2}\right\rangle v_{\sigma(3), P}(x)>1\right\} .
$$

For this we can assume that the measure of $F$ is larger than 1 . Let $f_{3}$ be the characteristic function of $F$, divided by $|F|^{1 / 2}$. It is easy to see that $f_{3} \in C^{\Delta}(\mathbb{R})$ and we have

$$
|F|^{1 / 2} \leq \sum_{P \in \mathbf{P}: I_{P} \not \subset E} \frac{1}{\sqrt{\left|I_{P}\right|}}\left\langle v_{\sigma(1), P}, f_{1}\right\rangle\left\langle v_{\sigma(2), P}, f_{2}\right\rangle\left\langle v_{\sigma(3), P}, f_{3}\right\rangle .
$$

Now the following lemma, applied with $r_{3}:=2$ and $f_{1}, f_{2}, f_{3}, r_{1}, r_{2}$ as above, implies that $|F|$ is bounded. Observe that for these data $E_{3}=\varnothing$, hence the set $E$ in the lemma coincides with the set $E$ above.

Lemma 4. Let $1<r_{1}, r_{2}, r_{3} \leq 2$ with

$$
1<\frac{1}{r_{1}}+\frac{1}{r_{2}}+\frac{1}{r_{3}}<2 .
$$

Then there is a constant $C$ such that the following holds: Let $f_{1}, f_{2}, f_{3} \in$ $C^{\Delta}(\mathbb{R})$ with

$$
\left\|f_{1}\right\|_{r_{1}}=\left\|f_{2}\right\|_{r_{2}}=\left\|f_{3}\right\|_{r_{3}}=1
$$

Define

$$
E_{\imath}:=\left\{x: \quad M_{r_{\imath}}^{\Delta} f_{\imath}(x) \geq 1\right\}
$$

and $E:=E_{1} \cup E_{2} \cup E_{3}$. Then

$$
\sum_{P \in \mathbf{P}: I_{P} \not \subset E} \frac{1}{\sqrt{\left|I_{P}\right|}}\left|\left\langle v_{\sigma(1)}, f_{1}\right\rangle\left\langle v_{\sigma(2)}, f_{2}\right\rangle\left\langle v_{\sigma(3)}, f_{3}\right\rangle\right| \leq C .
$$

It remains to prove this lemma. By symmetry we can assume that $\sigma$ is the identity. First observe that under the hypotheses of the lemma it suffices to prove that for any finite subset $\mathbf{Q} \subset\left\{P \in \mathbf{P}: I_{P} \not \subset E\right\}$, such that $\left\langle v_{1, P}, f_{1}\right\rangle\left\langle v_{2, P}, f_{2}\right\rangle\left\langle v_{3, P}, f_{3}\right\rangle \neq 0$ for all $P \in \mathbf{Q}$, we have

$$
\sum_{P \in \mathbf{Q}}\left|I_{P}\right|^{-1 / 2}\left|\left\langle v_{1, P}, f_{1}\right\rangle\left\langle v_{2, P}, f_{2}\right\rangle\left\langle v_{3, P}, f_{3}\right\rangle\right| \leq C
$$

This inequality is the conclusion of Lemma 1 applied to the set $\mathbf{Q}$ and the functions $a_{\jmath}$ defined by

$$
a_{\jmath}(P)=\left|\left\langle v_{\jmath, P}, f_{\jmath}\right\rangle\right| .
$$

It remains to verify the two hypotheses of Lemma 1 with $s_{\jmath}:=r_{\jmath}{ }^{\prime}+\varepsilon$ for some small $\varepsilon$, and $B$ some number which will evolve from the estimates below.

## 5. Verification of Lemma 1.1).

Let $\imath \neq \jmath$. Fix a tree $T$ as in Hypothesis 1. It suffices to prove

$$
\left\|\left(\sum_{P \in T} \frac{\left|\left\langle f, v_{\jmath, P}\right\rangle\right|^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{t} \leq C\|f\|_{t}
$$

for all $1<t \leq 2$. Namely, if this is true, we apply it to $f=f_{J} \mathbf{1}_{I_{T}}$ and obtain with Hölder's inequality

$$
\begin{equation*}
\left\|\left(\sum_{P \in T} \frac{\left|\left\langle f, v_{\jmath, P}\right\rangle\right|^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{1} \leq C\left|I_{T}\right| \inf _{z \in I_{T}} M_{t}^{\Delta} f(z) \tag{11}
\end{equation*}
$$

If we set $t:=r_{J}$, then the right hand side is bounded by $C\left|I_{T}\right|$, since $I_{T} \not \subset E$. Hence Hypothesis 1 is satisfied.

By standard square function techniques it suffices to prove the estimate

$$
\begin{equation*}
\left\|\sum_{P \in T} \varepsilon(P)\left\langle f, v_{\jmath, P}\right\rangle w_{p_{\jmath}(P)}\right\|_{t} \leq C\|f\|_{t} \tag{12}
\end{equation*}
$$

uniformly for all functions $\varepsilon: T \longrightarrow\{-1,1\}$ and all functions $f \in$ $C^{\Delta}(\mathbb{R})$.

First we assume that $\jmath \neq 3$ and prove this estimate by real interpolation. For $t=2$ it follows simply from the fact that the rectangles $\left\{p_{\jmath}(P): P \in T\right\}$ are pairwise disjoint. It remains to prove the weak type estimate

$$
\begin{equation*}
\left|\left\{x: \sum_{P \in T} \varepsilon(P)\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}(x) \geq \lambda\right\}\right| \leq C\|f\|_{1} \lambda^{-1} \tag{13}
\end{equation*}
$$

We fix $\lambda>0$ and split $f$ into a good function $g$ and a bad function $b$ as follows: Let $E$ be the set where the maximal function $M_{1}^{\Delta} f$ is larger than $\lambda$. Let $\left\{I_{n}\right\}_{n=1}^{N}$ be the set of maximal dyadic intervals contained in $E$. Define

$$
b_{n}:=\mathbf{1}_{I_{n}}\left(f-\lambda_{n} w_{p_{T}}\right),
$$

where $p_{T}$ is the top of the tree $T$ and $\lambda_{n}$ is chosen so that $b_{n}$ is orthogonal to $w_{p_{T}}$. Define $b=\sum_{n=1}^{N} b_{n}$ and $g=f-b$. It suffices to prove estimate (13) for $g$ and $b$ separately. Since $g$ is obviously bounded by $C \min \left\{\lambda, M_{1}^{\Delta} f(x)\right\}$, the estimate for $g$ follows from the previously proved $L^{2}$ estimate.

On the other hand,

$$
\sum_{P \in T} \varepsilon(P)\left\langle b_{n}, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}
$$

is supported on $4 I_{n}$. This is because if $I_{p_{\jmath}(P)}$ is larger than $4 I_{n}$, then there is a tile $q$ with $I_{q}=I_{n}, q<p_{\jmath}(P)$, and $q<p_{\jmath}\left(P_{T}\right)$. Hence $w_{p_{\jmath}(P)}$ and $w_{p_{T}}$ are multiples of each other on the interval $I_{q}$, and therefore $w_{p_{j}(P)}$ and $b_{n}$ are orthogonal.

This proves the weak type estimate for the bad function and thus finishes the proof of Hypothesis 1 in the case $\jmath \neq 3$.

Now assume $\jmath=3$. Instead of (12) we prove the dual estimate

$$
\left\|\sum_{P \in T} \varepsilon(P)\left\langle f, w_{p_{\jmath}(P)}\right\rangle v_{\jmath, P}\right\|_{t^{\prime}} \leq C\|f\|_{t^{\prime}}
$$

If we replace $f$ by

$$
\sum_{P \in T} \varepsilon(P)\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}
$$

which by the ideas used in the case $\jmath \neq 3$ satisfies

$$
\left\|\sum_{P \in T} \varepsilon(P)\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}\right\|_{t^{\prime}} \leq C\|f\|_{t^{\prime}}
$$

we see that it suffices to prove

$$
\left\|\sum_{P \in T}\left\langle f, w_{p_{\jmath}(P)}\right\rangle v_{\jmath, P}\right\|_{t^{\prime}} \leq C\|f\|_{t^{\prime}} .
$$

This in turn follows by the maximal theorem from the pointwise estimate

$$
\left|\sum_{P \in T}\left\langle f, w_{p_{\jmath}(P)}\right\rangle v_{\jmath, P}(x)\right| \leq C M_{2}^{\Delta} f(x) .
$$

To prove this pointwise estimate, it suffices to prove

$$
\begin{equation*}
\left|f_{k}\right| \leq C M_{2}^{\Delta} f \tag{14}
\end{equation*}
$$

for all $k \in \mathbb{Z}$, where

$$
f_{k}:=\sum_{P \in T:\left|I_{P}\right| \geq 2^{k}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)} .
$$

Fix $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let $I_{k}$ and $I_{k-2}$ be the dyadic intervals containing $x$ of length $2^{k}$ and $2^{k-2}$ respectively. Then the functions $w_{p_{\jmath}(P)}$ with $I \subset I_{P},\left|I_{P}\right| \geq 2^{k}$ are multiples of each other on the interval $I_{k-2}$. Hence $f_{k}$ is of constant modulus on $I_{k-2}$, and we have

$$
\begin{equation*}
\left|I_{k}\right|^{1 / 2}\left|f_{k}(x)\right| \leq C\left\|f_{k}\right\|_{L^{2}\left(I_{k}\right)} \tag{15}
\end{equation*}
$$

It is easy to see that $f_{k}$ is orthogonal to $f-f_{k}$ on the interval $I_{k}$. Hence the right hand side of (15) can be estimated by $\|f\|_{L^{2}(I)}$. This proves (14) and completes the verification of Lemma 1.1).

## 6. Verification of Lemma 1.2).

Let $\mathbf{P}^{\prime} \subset \mathbf{Q}$ be a set of quartiles as in Hypothesis 2, i.e., $\mathbf{P}^{\prime}$ is a disjoint union of trees of type $\imath$

$$
\mathbf{P}^{\prime}:=\bigcup_{T \in \mathcal{F}} T
$$

such that the set $\left\{p_{\jmath}(P): P \in \mathbf{P}^{\prime}\right\}$ is a set of pairwise disjoint rectangles and, for each $T \in \mathcal{F}$, we have

$$
\begin{equation*}
\left\|\left(\sum_{P \in T} \frac{\left|\left\langle f_{j}, v_{\jmath, P}\right\rangle\right|^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}\right)^{1 / 2}\right\|_{1} \geq 2^{m / s_{j}}\left|I_{T}\right| \tag{16}
\end{equation*}
$$

Here $\jmath \neq \imath$ and $m \in \mathbb{Z}$. Define the counting function

$$
N:=\sum_{T \in \mathcal{F}} \mathbf{1}_{I_{T}}
$$

We have to estimate the $L_{1}$-norm of the counting function $N$. Fix $\lambda \geq 0$ and consider the set

$$
E_{\lambda}:=\left\{x: M_{r_{\jmath}-\delta}^{\Delta} f_{\jmath}(x) \leq \gamma \lambda^{1 / r_{\jmath}} \text { and } N(x) \geq \lambda\right\}
$$

for some small constants $\gamma=\gamma\left(m, s_{\jmath}\right)>0$ and $\delta>0$ to be specified later. The set $E_{\lambda}$ is clearly contained in the set

$$
F_{\lambda}:=\left\{N(x) \geq \frac{\lambda}{4}\right\} .
$$

Let $I$ be a maximal dyadic interval contained in $F_{\lambda}$ and assume $I \cap E_{\lambda} \neq$ $\varnothing$. Define

$$
N_{I}=\sum_{T \in \mathcal{F}: I_{T} \subset I} \mathbf{1}_{I_{T}}
$$

Then $N-N_{I}$ is constant on $I$ and bounded by $\lambda / 4$, since otherwise the double of $I$ was also contained in $F_{\lambda}$, a contradiction to the maximality of $I$.

We assume the following inequality, which we will prove later

$$
\begin{equation*}
\left|\left\{N_{I}(x) \geq \frac{\lambda}{4}\right\}\right| \leq C|I|\left(2^{-m / s_{J}} \lambda^{-1 / s_{J}} \inf _{x \in I} M_{r_{j}-\delta}^{\Delta} f(x)\right)^{r_{J}-\delta} \tag{17}
\end{equation*}
$$

Since $I \cap E_{\lambda} \neq \varnothing$, the infimum on the right hand side is bounded by $\gamma \lambda^{1 / r_{3}}$. Moreover it is bounded by 1 , since otherwise $I$ was contained in the set where the maximal function $M_{r_{3}}^{\Delta} f$ is larger than 1 , and hence $N_{I}=0$, which is impossible because $I \cap E_{\lambda} \neq \varnothing$.

Maximizing the expression on the right hand side of the previous inequality over $\lambda$ gives

$$
\left|\left\{N_{I}(x) \geq \frac{\lambda}{4}\right\}\right| \leq C|I|\left(2^{-m / s_{3}} \gamma^{r_{3} / s_{3}}\right)^{r_{3}-\delta} .
$$

Now we pick $\gamma$ smaller than $C 2^{m / r_{J}}$ for an appropriate small constant $C$, then we have

$$
\left|\left\{N_{I}(x) \geq \frac{\lambda}{4}\right\}\right| \leq \frac{|I|}{100}
$$

Taking unions we obtain

$$
\left|E_{\lambda}\right| \leq \frac{\left|F_{\lambda}\right|}{100} .
$$

Now we have

$$
\begin{aligned}
\|N\|_{1} & =\int_{\lambda=0}^{\infty}|\{N(x) \geq \lambda\}| d \lambda \\
& \leq \int\left|\left\{M_{r_{j}-\delta}^{\Delta} f_{\jmath}(x) \geq \gamma \lambda^{1 / r_{j}}\right\}\right| d \lambda+\int\left|E_{\lambda}\right| d \lambda \\
& \leq \gamma^{-r_{j}}\left\|M_{r_{j}-\delta}^{\Delta} f_{j}\right\|_{r_{j}}^{r_{j}}+\frac{1}{100} \int\left|\left\{N(x) \geq \frac{\lambda}{4}\right\}\right| d \lambda \\
& \leq C 2^{-m}\left\|f_{j}\right\|_{r_{j}}+\frac{1}{10}\|N\|_{1} .
\end{aligned}
$$

This gives the appropriate bound on the counting function. Therefore it remains to prove (17).

Pick a $\mu \geq 1$ and define

$$
E_{\mu}:=\left\{x: N_{I}(x) \geq \mu\right\} .
$$

Define

$$
\begin{gathered}
\mathcal{F}_{I, \mu}:=\left\{T \in \mathcal{F}: \quad I_{T} \subset I, I_{T} \not \subset E_{\mu}\right\}, \\
\mathbf{P}_{I, \mu}:=\bigcup_{T \in \mathcal{F}_{I, \mu}} T, \quad N_{I, \mu}:=\sum_{T \in T_{I, \mu}} \mathbf{1}_{I_{T}} .
\end{gathered}
$$

It is easy to see from the dyadic property of all intervals $I_{P}$ that $\left\|N_{I, \mu}\right\|_{\infty} \leq \mu$.

We introduce some measure spaces: The first one is the set $\mathbf{P}_{I, \mu}$ endowed with counting measure. The second one, $\mathcal{I}$, is as a set the abstract disjoint union of the sets $I_{T}, T \in \mathcal{F}_{I, \mu}$, where each of the $I_{T}$ is endowed with Lebesgue measure normalized such that $I_{T}$ has measure 1. The third one is $\mathcal{F}_{I, \mu}$ with counting measure. The fourth space is simply $\mathbb{R}$ with Lebesgue measure.

Now we consider functions on the cartesian product of these measure spaces,

$$
f: \mathbb{R} \times \mathcal{F}_{I, \mu} \times \mathcal{I} \times \mathbf{P}_{I, \mu} \longrightarrow \mathbb{R}
$$

and define norms on these functions by

$$
\|f\|_{p, q, r, s}:=\| \|\| \| f\left\|_{L^{s}\left(\mathbf{P}_{I, \mu}\right)}\right\|_{L^{r}(\mathcal{I})}\left\|_{L^{q}\left(\mathcal{F}_{I, \mu}\right)}\right\|_{L^{p}(\mathbb{R})}
$$

Define the linear operator $S$ mapping functions on $\mathbb{R}$ to functions on $\mathbb{R} \times \mathcal{F}_{I, \mu} \times \mathcal{I} \times \mathbf{P}_{I, \mu}$ by $S f(y, T, x, P)= \begin{cases}\frac{\left\langle f, w_{p_{j}(P)}\right\rangle}{\left|I_{P}\right|^{1 / 2}}, & \text { if } y \in I_{T}, P \in T, \text { and } x \in I_{P} \subset I_{T}, \\ 0, & \text { otherwise } .\end{cases}$
Here the condition $x \in I_{P} \subset I_{T}$ means that $x$ is contained in the piece $I_{T}$ of $\mathcal{I}$ and in addition $x \in I_{P}$, where $I_{P}$ is naturally identified with a subset of this piece $I_{T}$. We have
$\|S f\|_{2,2,2,2}=\left(\int_{\mathbb{R}} \sum_{T \in \mathcal{F}_{I, \mu}} \mathbf{1}_{I_{T}}(y) \frac{1}{\left|I_{T}\right|} \int_{I_{T}} \sum_{P \in T} \frac{\left|\left\langle f, v_{\jmath, P}\right\rangle\right|^{2}}{\left|I_{P}\right|} \mathbf{1}_{I_{P}}(x) d x d y\right)^{1 / 2}$

$$
\begin{align*}
& =\left(\sum_{P \in \mathbf{P}_{I, \mu}}\left|\left\langle f, v_{\jmath, P}\right\rangle\right|^{2}\right)^{1 / 2}  \tag{18}\\
& \leq C \log (1+\mu)^{2}\|f\|_{2} .
\end{align*}
$$

If $\jmath \neq 3$, then the last inequality follows simply from the orthogonality of the $v_{\jmath, P}=w_{p_{\jmath}(P)}$. We postpone the proof of inequality (18) in the case $\jmath=3$ to the next section.

Moreover we have for small $\delta>0$

$$
\begin{aligned}
& \|S f\|_{1+2 \delta, \infty, 1,2} \\
& =\left(\int _ { \mathbb { R } T \in \mathcal { F } _ { I , \mu } : y \in I _ { T } } \left(\frac { 1 } { | I _ { T } | } \int _ { I _ { T } } \left(\sum_{P \in T} \frac{\left.\left|\left\langle f, v_{, ~}\right\rangle\right\rangle\right|^{2}}{\left|I_{P}\right|}\right.\right.\right. \\
& \left.\left.\cdot \mathbf{1}_{I_{P}(x)}()^{1 / 2} d x\right)^{1+2 \delta} d y\right)^{1 /(1+2 \delta)} .
\end{aligned}
$$

Using (11) with $t=1+\delta$ we can bound this by

$$
\begin{aligned}
& \leq C\left(\int_{\mathbb{R} T \in \mathcal{F}_{I, \mu}: y \in I_{T}}\left(\inf _{z \in I_{T}} M_{1+\delta}^{\Delta} f(z)\right)^{1+2 \delta} d y\right)^{1 /(1+2 \delta)} \\
& \leq C\left(\int_{\mathbb{R}}\left(M_{1+\delta}^{\Delta} f(y)\right)^{1+2 \delta} d y\right)^{1 /(1+2 \delta)} \\
& \leq C\|f\|_{1+2 \delta}
\end{aligned}
$$

The last line followed from the maximal theorem.
Interpolation and Hölder's inequality in the third exponent gives for a different small $\delta$

$$
\begin{equation*}
\|S f\|_{r_{\jmath}-\delta, s_{\jmath}-\delta, 1,2} \leq C \log (1+\mu)^{2}\|f\|_{r_{\jmath}-\delta} \tag{19}
\end{equation*}
$$

We replace in this inequality $f$ by $f \mathbf{1}_{I}$, which does not change the left hand side of this inequality. With the assumption (16) this gives

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\left(\sum_{T \in \mathcal{F}}\left(\mathbf{1}_{I_{T}}(y) 2^{m / s_{\jmath}}\right)^{s_{\jmath}-\delta}\right)^{\left(r_{3}-\delta\right) /\left(s_{3}-\delta\right)} d x\right)^{1 /\left(r_{3}-\delta\right)} \\
&=\left\|N_{I, \mu}^{1 /\left(s_{\jmath}-\delta\right)}\right\|_{r_{3}-\delta} 2^{m / s_{j}} \\
& \leq C \log (1+\mu)^{2}\left\|f \mathbf{1}_{I}\right\|_{r_{3}-\delta}
\end{aligned}
$$

This gives the weak type estimate

$$
\begin{aligned}
\left|\left\{x: N_{I} \geq \mu\right\}\right| & =\left|\left\{x: N_{I, \mu} \geq \mu\right\}\right| \\
& \leq C\left(\mu^{-1 /\left(s_{\jmath}-\delta\right)} \log (1+\mu)^{2} 2^{-m / s_{j}}\left\|f \mathbf{1}_{I}\right\|_{r_{j}-\delta}\right)^{r_{3}-\delta} \\
& \leq C\left(\mu^{-1 / s_{j}} 2^{-m / s_{j}}\left\|f \mathbf{1}_{I}\right\|_{r_{J}-\delta}\right)^{r_{j}-\delta} .
\end{aligned}
$$

Since $\mu \geq 1$ was arbitrary and $N_{I}$ takes only integer values, this proves (17) and finishes the verification of Lemma 1.2).

It remains to prove inequality (18) in the case $\jmath=3$.
Proof of inequality (18) in the case $\jmath=3$. It suffices to prove for all functions $f \in C^{\Delta}(\mathbb{R})$

$$
\begin{equation*}
\left\|\sum_{P \in \mathbf{P}_{I, \mu}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle v_{\jmath, P}\right\|_{2} \leq C \log (1+\mu)^{2}\|f\|_{2} . \tag{20}
\end{equation*}
$$

Namely, this implies by duality

$$
\left\|\sum_{P \in \mathbf{P}_{I, \mu}}\left\langle f, v_{\jmath, P}\right\rangle w_{p_{\jmath}(P)}\right\|_{2} \leq C \log (1+\mu)^{2}\|f\|_{2} .
$$

which implies (18) by orthogonality of the $w_{p_{\jmath}(P)}$.
We prove (20). Let $\mathcal{I}$ be the set of intervals $I_{T}$ with $T \in \mathcal{F}_{I, \mu}$. Let $\mathcal{I}_{1}$ be the set of maximal intervals in $\mathcal{I}$ with respect to set inclusion, and define $\mathcal{I}_{\nu}$ for $\nu=2,3, \ldots$ to be the set of maximal intervals in

$$
\mathcal{I} \backslash \bigcup_{\nu^{\prime}<\nu} I_{\nu^{\prime}}
$$

From the dyadic property of the intervals $I_{T}$ with $T \in \mathcal{F}_{I, \mu}$ we conclude that for every $J \in \mathcal{I}_{\nu}, \nu>1$, there is a $J^{\prime} \in \mathcal{I}_{\nu-1}$ with $J \subset J^{\prime}$. Since the counting function $N_{I, \mu}$ is bounded by $\mu$, we conclude that $I_{\nu}$ is empty for $\mu>\nu$.

Let $\mathbf{P}_{\nu}$ be the set of all tiles $p \in \mathbf{P}_{I, \mu}$ with $I_{p} \subset J$ for some $J \in \mathcal{I}_{\nu}$, but $I_{p} \not \subset J^{\prime}$ for all $J^{\prime} \in \mathcal{I}_{\nu+1}$. Define $\nu(x)$ so that the left hand side of (20) is bounded by

$$
\begin{aligned}
&\left\|\sup _{\nu}\left|\sum_{n=1}^{\nu}\left(\sum_{P \in \mathbf{P}_{n}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}\right)\right|\right\|_{2} \\
&+\left\|\sum_{P \in \mathbf{P}_{\nu(x)}:\left|I_{p}\right| \geq 2^{\kappa(x)}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}\right\|_{2} .
\end{aligned}
$$

By Rademacher-Menshov, the first term in this sum is bounded by $C \log (\mu+1)\|f\|_{2}$, which is the desired estimate for this summand. The second summand can be estimated by

$$
\left(\sum_{\nu}\left\|\sup _{k}\left|\sum_{P \in \mathbf{P}_{\nu}:\left|I_{p}\right| \geq 2^{k}}\left\langle f_{\nu}, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}\right|\right\|_{2}^{2}\right)^{1 / 2},
$$

where

$$
f_{\nu}:=\sum_{P \in \mathbf{P}_{\nu}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)} .
$$

Since the functions $f_{\nu}$ are orthogonal as $\nu$ varies, it suffices to prove for a fixed $\nu$

$$
\begin{equation*}
\left\|\sup _{k}\left|\sum_{P \in \mathbf{P}_{\nu}:\left|I_{p}\right|>2^{k}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}\right|\right\|_{2} \leq C \log (\mu+1)^{2}\|f\|_{2} . \tag{21}
\end{equation*}
$$

We split the set $\mathbf{P}_{\nu}$ further. Let $J$ be an interval in $\mathcal{I}_{\nu}$. By a trivial splitting of $\mathbf{P}_{\nu}$ we can assume that all $P \in \mathbf{P}_{\nu}$ satisfy $I_{P} \subset J$. Then, if $P \in \mathbf{P}_{\nu}$, we necessarily have $P \in T$ for some tree $T \in \mathcal{F}_{I, \mu}$ with $J \subset I_{T}$. Hence we can find a collection of at most $\mu$ trees $T \in \mathcal{F}_{I, \mu}$ such that $P_{\nu}$ is contained in the union of these trees. For each tree $T$ in this collection pick a top frequency $\xi \in \omega_{T}$, and let $\Xi$ be the set of these frequencies.

For each integer $k$ with $2^{k}<|J|$ consider the collection $\Omega_{k}$ of all dyadic intervals of length $2^{-k}$ which have nonempty intersection with $\Xi$. Call $k$ an exceptional value if if the cardinality of $\Omega_{k+4}$ is larger than the cardinality of $\Omega_{k-4}$. There are at most $8 \mu$ exceptional values. Pick a chain of integers $k_{0}<k_{1}<k_{2}<\cdots<k_{8 \mu}$ such that all exceptional values appear in this chain.

We can estimate the left hand side of (21) by

$$
\begin{aligned}
&\left\|\left.\sup _{m}\right|_{p \in \mathbf{P}_{\nu}:\left|I_{P}\right|>2^{k_{m}}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)} \mid\right\|_{2} \\
&+\left\|\sup _{m} \sup _{k_{m-1}<k \leq k_{m}}\left|\sum_{p \in \mathbf{P}_{\nu}: 2^{k_{m} \geq\left|I_{P}\right|>2^{k}}}\left\langle f, w_{\left.p_{\jmath}(P)\right\rangle}\right\rangle w_{p_{\jmath}(P)}\right|\right\|_{2} .
\end{aligned}
$$

Again by Rademacher-Menshov the first summand is bounded by $C \log (\mu+1)\|f\|_{2}$.

To estimate the second summand it suffices by a similar argument as before to prove for each $m$

$$
\begin{align*}
\left\|\sup _{k_{m-1}<k \leq k_{m}}\left|\sum_{p \in \mathbf{P}_{\nu}: 2^{k_{m}} \geq\left|I_{P}\right|>2^{k}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)}\right|\right\|_{2}  \tag{22}\\
\leq C \log (\mu+1)^{2}\|f\|_{2} .
\end{align*}
$$

If $k_{m-1}=k_{m}-1$, then this estimate is trivial. Therefore assume that $k_{m-1}<k_{m}-1$.

We claim that for $k_{m-1}<k \leq k_{m}$ we have

$$
\begin{aligned}
& \sum_{p \in \mathbf{P}_{\nu}: 2^{k_{m}} \geq\left|I_{P}\right|>2^{k}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)} \\
&=\Pi_{k+k_{0}} \sum_{p \in \mathbf{P}_{\nu}: 2^{k_{m}} \geq\left|I_{P}\right|>2^{k_{m-1}}}\left\langle f, w_{p_{\jmath}(P)}\right\rangle w_{p_{\jmath}(P)} .
\end{aligned}
$$

Here $\Pi_{k+k_{0}}$ denotes the projection onto the subspace of $L^{2}(\mathbb{R})$ corresponding to all points in the Walsh phase plane whose frequency coordinate is contained in the union of intervals in $\Omega_{k-k_{0}}$, where $k_{0} \in\{0,1\}$ depends only on $\jmath$ and $\imath$. For the definition and properties of subspaces associated to sets in the Walsh phase plane (see [11]).

We prove the claim in the case $\imath=1$ and $\jmath=2$, the other cases being similar. In this case we have $k_{0}=0$. Let $F_{k}$ be the union of all intervals in $\Omega_{k}$. To prove inequality (23) we have to show that for all $P \in \mathbf{P}_{\nu}$ with $I_{P}=2^{k}$ we have $\omega_{p_{\jmath}(P)} \subset F_{k-1}$ and $\omega_{p_{\jmath}(P)} \cap \subset F_{k}=\varnothing$. However it is clear that $\omega_{p_{\imath}(P)}$ contains a $\xi \in \Xi$, hence $\omega_{\imath(P)} \cup \omega_{\jmath(P)}$ is a dyadic interval of length $2^{-k+1}$ having nonempty intersection with $\Xi$ and therefore being contained in $F_{k-1}$. Moreover, $\omega_{p_{\jmath}(P)} \cap \subset F_{k}=$ $\varnothing$, because $k$ is not exceptional and therefore the two neighbouring intervals $\omega_{p_{2}(P)}$ and $\omega_{p_{\jmath}(P)}$ can not be both in $\Omega_{k}$.

Now the claim (23) shows that inequality (22) is a direct consequence of the following Lemma which is a version of a lemma by Bourgain (see [11]):

Lemma 5 (Bourgain). Let $\Xi \subset \mathbb{R}^{+}$. For each integer $k$ define $\Omega_{k}$ to be the set of dyadic intervals of length $2^{-k}$ which have nonempty intersection with $\Xi$. Define $\Pi_{k}$ to be the orthogonal projection onto the subspace of $L^{2}(\mathbb{R})$ associated to the set of all points in the phase plane whose frequency coordinate is contained in the union of the intervals in $\Omega_{k}$. Let $k<k^{\prime}$ be two integers such that $\Omega_{k}$ and $\Omega_{k^{\prime}}$ have the same cardinality. Define

$$
M_{\Xi} f(x):=\sup _{k<\kappa \leq k^{\prime}}\left|\Pi_{\kappa} f(x)\right| .
$$

Then

$$
\left\|M_{\Xi} f\right\|_{2} \leq C \log (\sharp \Xi)^{2}\|f\|_{2} .
$$

Proof of Lemma 5. Following [1], we present a series of lemmata that leads to a proof of Lemma 5. The first lemma is a version of Doob's oscillation lemma for martingales and is obtained by methods of stopping times and square functions

Lemma 6 (Doob). Let $1<r<\infty$ and $f \in L^{r}(\mathbb{R})$. For each dyadic interval $I$ let $m_{I} f$ denote the mean of $f$ on $I$. For $\lambda>0$ and $x \in \mathbb{R}$ let $M_{\lambda}(x)$ be the maximal number such that there is an increasing chain of dyadic intervals $x \in I_{1} \subset I_{2} \subset \cdots \subset I_{M_{\lambda}(x)}$ with

$$
\left|m_{I_{j}} f-m_{I_{j+1}} f\right| \geq \lambda .
$$

Then

$$
\left\|\lambda M_{\lambda}^{1 / 2}\right\|_{r} \leq C_{r}\|f\|_{r}
$$

For a proof of this Lemma we refer to [10]. With Lemma 6 we prove the following lemma due to Lépingle (see [9]):

Lemma 7 (Lépingle). Let $\varepsilon$ be small and $2-\varepsilon<p<s<2+\varepsilon, 2<s$. Let $f \in L^{p}(\mathbb{R})$. Then

$$
\int \sup \left\{\left(\sum_{j=1}^{J}\left|m_{I_{j}} f-m_{I_{j+1}} f\right|^{s}\right)^{1 / s}: J \in \mathbb{N}_{0}\right.
$$

$$
\begin{align*}
x \in I_{1} \subset I_{2} & \left.\subset \cdots \subset I_{J+1}\right\}^{p} d x  \tag{24}\\
& \leq C(s-2)^{-1}\|f\|_{p}^{p}
\end{align*}
$$

Proof. By interpolation it suffices to prove this for $f$ being the characteristic function of a set $A$. Using the numbers $M_{\lambda}(x)$ defined in Lemma 6 we can estimate (24) by

$$
\int\left(\sum_{n=0}^{\infty} 2^{-n s} M_{2^{-n}}(x)\right)^{p / s} d x
$$

Now an easy calculation shows that this is bounded by

$$
\sum_{n=0}^{\infty} 2^{-n p} 2^{2 n p / s} \int 2^{-2 n p / s}\left(M_{2^{-n}}^{1 / 2}(x)\right)^{2 p / s} d x
$$

By Lemma 6 this is bounded by

$$
\sum_{n=0}^{\infty} 2^{-n p(1-2 / s)}|A| \leq C\left(\frac{1}{2}-\frac{1}{s}\right)\|f\|_{p}^{p}
$$

This proves Lemma 7.
Next, we prove a vector valued version of Lemma 7.
Lemma 8 (Lépingle, vector valued). Consider the Euclidean space $\mathbb{R}^{n}$ and let $f=\left(f_{1}, \ldots, f_{n}\right) \in L^{2}\left(\mathbb{R}, \mathbb{R}^{n}\right)$. Let $2<s$. For $\lambda>0, x \in \mathbb{R}$ let $M_{\lambda}(x)$ denote the minimal number of $\lambda$-balls necessary to cover the set $\left\{m_{I} f\right\}_{I \text { dyadic }: x \in I}$. Then

$$
\left\|\sup _{\lambda>0}\left(\lambda M_{\lambda}^{1 / s}\right)\right\|_{2}^{2} \leq C(s-2)^{-1} \sum_{\alpha=1}^{n}\left\|f_{\alpha}\right\|_{2}^{2} .
$$

For a proof of this lemma we calculate with $p=2$ in the previous lemma

$$
\begin{aligned}
& \left\|\sup _{\lambda>0}\left(\lambda M_{\lambda}^{1 / s}\right)\right\|_{2}^{2} \\
& \qquad \begin{array}{l}
\leq C \int \sup \left\{\left(\sum_{j=1}^{J}\left(\sum_{\alpha=1}^{n}\left|m_{I_{j}} f_{\alpha}-m_{I_{j+1}} f_{\alpha}\right|^{2}\right)^{s / 2}\right)^{1 / s}:\right. \\
\left.J \in \mathbb{N}_{0}, x \in I_{1} \subset I_{2} \subset \cdots \subset I_{J+1}\right\}^{2} d x \\
\leq C \sum_{\alpha=1}^{n} \int \sup \left\{\left(\sum_{j=1}^{J}\left|m_{I_{j}} f_{\alpha}-m_{I_{j+1}} f_{\alpha}\right|^{s}\right)^{1 / s}:\right. \\
\left.\quad J \in \mathbb{N}_{0}, x \in I_{1} \subset I_{2} \subset \cdots \subset I_{J+1}\right\}^{2} d x \\
\leq C(s-2)^{-1} \sum_{\alpha=1}^{n}\left\|f_{\alpha}\right\|_{2}^{2} .
\end{array}
\end{aligned}
$$

This proves Lemma 8.
Now we proceed to prove Lemma 5. By passing to a subset of $\Xi$ if necessary we can assume that $\Omega_{k}$ and $\Omega_{k^{\prime}}$ have the same cardinality as $\Xi$. We enumerate $\Xi$ as $\xi_{1}, \ldots, \xi_{n}$.

Let $x \in \mathbb{R}$. We use the following equality, which is an easy result of Walsh phase plane analysis as in [11]

$$
\Pi_{k} f(x)=\sum_{\alpha=1}^{n} m_{k}\left(f w_{\xi_{\alpha}}\right) w_{\xi_{\alpha}},
$$

where $w_{\xi}$ is the Walsh function of modulus 1 on $\mathbb{R}$ associated to the frequency $\xi$ and $m_{k}(f)(x)$ is the mean of $f$ over the dyadic interval of length $2^{k}$ which contains $x$.

Let $J$ be a dyadic interval of length $2^{k}$. For $s \in \mathbb{Z}$ pick a minimal collection $\mathbf{B}_{s, J}$ of $2^{s}$-balls covering the set

$$
\begin{equation*}
\left\{\left(m_{\kappa} f_{1}(x), \ldots, m_{\kappa} f_{n}(x)\right): k<\kappa \leq k^{\prime}\right\} \tag{25}
\end{equation*}
$$

where $x \in J$ and the set clearly does not depend on the choice of $x$.
Define the function

$$
G:=\left(\sum_{\alpha=1}^{n}\left(M^{\Delta}\left(m_{k}\left(f w_{\xi_{\alpha}}\right)\right)\right)^{2}\right)^{1 / 2}
$$

where $M^{\Delta}$ denotes the dyadic Hardy Littlewood maximal function. The function $G$ is constant on dyadic inetrvals of length $J$, and we write $G_{J}$ for the value of $G$ on $J$.

If $s$ is larger that $2+\log _{2} G_{J}$, then the ball of radius $2^{s}$ centered at the origin covers the set (25), and we pick $\mathbf{B}_{s, J}$ to just consist of this ball. For each ball $B \in \mathbf{B}_{s, J}$ pick a ball $B^{\prime} \in \mathbf{B}_{s+1, J}$ which has nonempty intersection with $B$. Let $d(B)=c(B)-c\left(B^{\prime}\right)$, where $c(B)$ denotes the center of $B$. Clearly the length of the vector $d(B)$ is less than $2^{s+2}$. We write $d_{\alpha}(B)$ for the $\alpha$-th coordinate of $d(B)$.

For each $k<\kappa \leq k^{\prime}$ we can find balls $B_{\kappa, s, J} \in \mathbf{B}_{s, J}$ such that for each $x \in J, 1 \leq \alpha \leq n$

$$
m_{\kappa}\left(f w_{\xi_{\alpha}}\right)(x):=\sum_{s \in \mathbb{Z}} d_{\alpha}\left(B_{\kappa, s, J}\right) .
$$

Then we have

$$
\begin{aligned}
\sup _{k<\kappa \leq k^{\prime}} \mid \sum_{\alpha=1}^{n} & m_{k}\left(f w_{\xi_{\alpha}}\right) w_{\xi_{\alpha}}(x) \mid \\
& \leq \sup _{k<\kappa \leq k^{\prime}}\left|\sum_{\alpha=1}^{n} \sum_{s \in \mathbb{Z}} d_{\alpha}\left(B_{\kappa, s, J}\right) w_{\xi_{\alpha}}(x)\right|
\end{aligned}
$$

$$
\begin{aligned}
& \leq \sum_{s \in \mathbb{Z}} \max _{B \in \mathbf{B}_{s, J}}\left|\sum_{\alpha=1}^{n} d_{\alpha}(B) w_{\xi_{\alpha}}(x)\right| \\
& \leq C \sum_{s \in \mathbb{Z}} \min \left\{2^{s} n^{1 / 2},\left(\sum_{B \in \mathbf{B}_{s, J}}\left|\sum_{\alpha=1}^{n} d_{\alpha}(B) w_{\xi_{\alpha}}(x)\right|^{2}\right)^{1 / 2}\right\} .
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
& \left\|\sup _{k<\kappa \leq k^{\prime}}\left|\sum_{\alpha=1}^{n} m_{\kappa}\left(f_{\alpha}\right) w_{\xi_{\alpha}}\right|\right\|_{L^{2}(J)} \\
& \quad \leq C \sum_{s \in \mathbb{Z}} \min \left\{2^{k / 2} 2^{s} n^{1 / 2},\left(\sum_{B \in \mathbf{B}_{s, J}}\left\|\sum_{\alpha=1}^{n} d_{\alpha}(B) w_{\xi_{\alpha}}\right\|_{L^{2}(J)}^{2}\right)^{1 / 2}\right\} .
\end{aligned}
$$

The functions $w_{\xi_{\alpha}}$ restricted to $J$ are pairwise orthogonal for $1 \leq \alpha \leq n$, hence we can estimate the previously displayed expression by

$$
\begin{aligned}
& \leq C 2^{k / 2} \sum_{\substack{s \in \mathbb{Z} \\
s \leq 2+\log _{2} G_{J}}} \min \left\{2^{s} n^{1 / 2}, 2^{s}\left|\mathbf{B}_{s, J}\right|^{1 / 2}\right\} \\
& \leq C\left\|\int_{0}^{4 G_{J}} \min \left\{n^{1 / 2}, M_{\lambda}^{1 / 2}\right\} d \lambda\right\|_{L^{2}(J)} .
\end{aligned}
$$

Hence

$$
\left\|\sup _{k<\kappa \leq k^{\prime}}\left|\sum_{\alpha=1}^{n} m_{\kappa}\left(f_{\alpha}\right) w_{\xi_{\alpha}}\right|\right\|_{L^{2}} \leq C\left\|\int_{0}^{4 G} \min \left\{n^{1 / 2}, M_{\lambda}^{1 / 2}\right\} d \lambda\right\|_{2} .
$$

Moreover,

$$
\begin{aligned}
\int_{0}^{4 G(x)} \min & \left\{n^{1 / 2}, M_{\lambda}^{1 / 2}\right\} d \lambda \\
& \leq G(x)+\int_{n^{-1 / 2} G(x)}^{4 G(x)} n^{1 / 2-1 / s} M_{\lambda}(x)^{1 / s} d \lambda \\
& \leq G(x)+C n^{1 / 2-1 / s} \log (1+n) \sup _{\lambda>0} \lambda M_{\lambda}(x)^{1 / s} .
\end{aligned}
$$

If we pick $s$ such that $1 / 2-1 / s$ is $\log (n+1)^{-1}$, then taking the $L^{2}$ norm in $x$ of the previously displayed expression and using Lemma 8 proves Lemma 5.

This completes the proof of Proposition 1.

## 7. Appendix: Interpolation.

So far we have only proved some of the estimates which are claimed in Theorem 1. Now we prove the remaining estimates by interpolation.

Recall the definition of the trilinear form

$$
\begin{aligned}
& T_{W}^{\kappa}: C^{\Delta}(\mathbb{R}) \times C^{\Delta}(\mathbb{R}) \times C^{\Delta}(\mathbb{R}) \longrightarrow \mathbb{R}, \\
& T_{W}^{\kappa}\left(f_{1}, f_{2}, f_{3}\right)=\sum_{P \in \mathbf{P}} \frac{1}{\sqrt{\left|I_{P}\right|}}\left\langle v_{1, P}, f_{1}\right\rangle\left\langle v_{2, P}, f_{2}\right\rangle\left\langle v_{3, P}, f_{3}\right\rangle .
\end{aligned}
$$

Also recall that for each permutation $\sigma$ of the set $\{1,2,3\}$ we have the bilinear operator $H_{W}^{\kappa, \sigma}$ defined by

$$
\int H_{W}^{\kappa, \sigma}\left(f_{1}, f_{2}\right)(x) f_{3}(x) d x=T_{W}^{\kappa}\left(f_{\sigma^{-1}(1)}, f_{\sigma^{-1}(2)}, f_{\sigma^{-1}(3)}\right)
$$

Let $p_{1}, p_{2}, p_{3} \in \mathbb{R} \cup\{\infty\}$. We say that $T_{W}^{\kappa}$ is of type $\left(p_{1}, p_{2}, p_{3}\right)$ if there is a permutation $\sigma$ such that

$$
\begin{equation*}
0<p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}{ }^{\prime} \leq \infty \tag{26}
\end{equation*}
$$

and there is a constant $C$ such that

$$
\begin{equation*}
\left\|H_{W}^{\kappa, \sigma}(f, g)\right\|_{p_{\sigma(3)}^{\prime}} \leq C\|f\|_{p_{\sigma(1)}}\|g\|_{p_{\sigma(2)}} \tag{27}
\end{equation*}
$$

for all functions $f, g \in C^{\Delta}(\mathbb{R})$. Here $p^{\prime}$ denotes the conjugate exponent of $p$ defined by

$$
\frac{1}{p^{\prime}}+\frac{1}{p}=1
$$

We claim the following theorem, which implies Theorem 1.
Theorem 2. Let $\kappa \in C^{\Delta}(\mathbb{R})$. If

$$
\frac{1}{p_{1}}+\frac{1}{p_{2}}+\frac{1}{p_{3}}=1, \quad-\frac{1}{2}<\frac{1}{p_{1}}, \frac{1}{p_{2}}, \frac{1}{p_{3}}<1,
$$

then $T_{W}^{\kappa}$ is of type $\left(p_{1}, p_{2}, p_{3}\right)$.

The following diagram shows the plane of all points $\left(1 / p_{1}, 1 / p_{2}\right.$, $\left.1 / p_{3}\right)$ with $1 / p_{1}+1 / p_{2}+1 / p_{3}=1$.


Let $A$ be the open interior of the convex hull of the six large filled circles in the above diagram. Theorem 2 states that $T_{W}^{\kappa}$ is of type $\left(p_{1}, p_{2}, p_{3}\right)$ for all $\left(1 / p_{1}, 1 / p_{2}, 1 / p_{3}\right) \in A$. The closed convex hull $B$ of the three large empty circles is the region in which condition (26) is satisfied for all permutations $\sigma$ and thus the type estimates (27) are equivalent for all six bilinear operators $H_{W}^{\kappa, \sigma}$. The remainder set $A \backslash B$ splits into three connected regions $D_{\jmath}$ such that the exponent $p_{\jmath}$ is negative in the region $D_{\jmath}$ for $\jmath=1,2,3$. In each of these regions, only two permutations $\sigma$ satisfy (26) and thus the estimate (27) makes sense only for the two corresponding bilinear operators $H_{W}^{\kappa, \sigma}$.

Proposition 1 proves the type estimates in each of the three triangles which are spanned by two adjacent large filled circles and the adjacent small filled circle in the above diagram.

Hence Theorem 2 follows from Proposition 1 and the following convexity lemma:

Lemma 9. Let $\left(1 / p_{1}, 1 / p_{2}, 1 / p_{3}\right)$ and $\left(1 / q_{2}, 1 / q_{2}, 1 / q_{3}\right)$ be two points in the region $A$ such that $p_{\imath}=q_{\imath}$ for some $\imath \in\{1,2,3\}$ and assume that $T_{W}^{\kappa}$ is of type $\left(p_{1}, p_{2}, p_{3}\right)$ and of type $\left(q_{1}, q_{2}, q_{3}\right)$. Then $T_{W}^{\kappa}$ is of type $\left(u_{1}, u_{2}, u_{3}\right)$ for all $\left(1 / u_{1}, 1 / u_{2}, 1 / u_{3}\right)$ on the line segment connecting $\left(1 / p_{1}, 1 / p_{2}, 1 / p_{3}\right)$ and $\left(1 / q_{2}, 1 / q_{2}, 1 / q_{3}\right)$.

We prove the lemma. The conclusion of the lemma follows immediately by complex interpolation as in [2], if there exists a $\sigma$ such that type ( $p_{1}, p_{2}, p_{3}$ ) and type ( $q_{1}, q_{2}, q_{3}$ ) can be expressed as estimates for $H_{W}^{\kappa, \sigma}$ (i.e., all $p_{\imath}, q_{\imath}$ are in $(1, \infty]$. This is the case if there is a
$\jmath \in\{1,2,3\}$ such that both $\left(1 / p_{1}, 1 / p_{2}, 1 / p_{3}\right)$ and $\left(1 / q_{1}, 1 / q_{2}, 1 / q_{3}\right)$ are contained in the region $B \cup D_{\jmath}$.

Therefore we can assume that

$$
\left(\frac{1}{p_{1}}, \frac{1}{p_{2}}, \frac{1}{p_{3}}\right)
$$

and

$$
\left(\frac{1}{q_{1}}, \frac{1}{q_{2}}, \frac{1}{q_{3}}\right)
$$

are in different regions $D_{\jmath}$. Let $\sigma$ and $\tau$ be permutations such that

$$
\begin{align*}
\left\|H_{W}^{\kappa, \sigma}(f, g)\right\|_{p_{\sigma(3)}^{\prime}} & \leq C_{p}\|f\|_{p_{\sigma(1)}}\|g\|_{p_{\sigma(2)}}  \tag{28}\\
\left\|H_{W}^{\kappa, \tau}(f, g)\right\|_{q_{\tau(3)}^{\prime}}^{\prime} & \leq C_{q}\|f\|_{q_{\tau(1)}}\|g\|_{q_{\tau(2)}} \tag{29}
\end{align*}
$$

for all functions $f, g \in C^{\Delta}(\mathbb{R})$. By symmetry we can assume that $\sigma(1)=\tau(1)$, and then we necessarily have

$$
p_{\sigma(1)}=q_{\tau(1)}
$$

Let $C_{p}$ and $C_{q}$ be the optimal constants in the above estimates.
Pick two different points $\left(1 / u_{1}, 1 / u_{2}, 1 / u_{3}\right)$ and $\left(1 / v_{1}, 1 / v_{2}, 1 / v_{3}\right)$ on the line segment connecting the points $\left(1 / p_{1}, 1 / p_{2}, 1 / p_{3}\right)$ and $\left(1 / q_{1}\right.$, $\left.1 / q_{2}, 1 / q_{3}\right)$ such that $\left(1 / u_{1}, 1 / u_{2}, 1 / u_{3}\right)$ and $\left(1 / v_{1}, 1 / v_{2}, 1 / v_{3}\right)$ are both in the open interior of the region $B$ and the distance between $\left(1 / p_{1}\right.$, $\left.1 / p_{2}, 1 / p_{3}\right)$ and $\left(1 / u_{1}, 1 / u_{2}, 1 / u_{3}\right)$ is smaller than the distance between the points $\left(1 / p_{1}, 1 / p_{2}, 1 / p_{3}\right)$ and $\left(1 / v_{1}, 1 / v_{2}, 1 / v_{3}\right)$. It is easy to see that such points exist, because $\left(1 / p_{1}, 1 / p_{2}, 1 / p_{3}\right)$ and $\left(1 / q_{1}, 1 / q_{2}, 1 / q_{3}\right)$ are in different regions $D_{\jmath}$.

Let $f \in C^{\Delta}(\mathbb{R})$ be fixed. It is easy to see that there are constants $C_{u}$ and $C_{v}$, possibly depending on $f$, such that

$$
\begin{align*}
\left\|H_{W}^{\kappa, \sigma}(f, g)\right\|_{u_{\sigma(3)}^{\prime}} \leq C_{u}\|f\|_{u_{\sigma(1)}}\|g\|_{u_{\sigma(2)}}  \tag{30}\\
\left\|H_{W}^{\kappa, \sigma}(f, g)\right\|_{v_{\sigma(3)}^{\prime}} \leq C_{v}\|f\|_{v_{\sigma(1)}}\|g\|_{v_{\sigma(2)}} \tag{31}
\end{align*}
$$

for all functions $g \in C^{\Delta}(\mathbb{R})$. Let $C_{u}$ and $C_{v}$ be the best constants in these inequalities. Assume to get a contradiction that $C_{v}$ is larger than $C_{p}$ and $C_{q}$. Then it follows by interpolation as in [2] between the
estimates (28) and (31) that $C_{u}$ is smaller than $C_{v}$. However, we have by duality

$$
\begin{align*}
&\left\|H_{W}^{\kappa, \tau}(f, g)\right\|_{u_{\tau(3)}^{\prime}} \leq C_{u}\|f\|_{u_{\tau(1)}}\|g\|_{u_{\tau(2)}}  \tag{32}\\
&\left\|H_{W}^{\kappa, \tau}(f, g)\right\|_{v_{\tau(3)}^{\prime}} \leq C_{v}\|f\|_{v_{\tau(1)}}\|g\|_{v_{\tau(2)}} \tag{33}
\end{align*}
$$

for all $g \in C^{\Delta}(\mathbb{R})$, where the same constants $C_{u}$ and $C_{v}$ as above are optimal. Hence it follows by interpolation between the estimates (33) and (29) that $C_{v}$ is smaller than $C_{u}$ or $C_{q}$, a contradiction.

Hence $C_{v}$ is smaller than $C_{p}$ or $C_{q}$, which are independent of $f$. Hence $T_{W}^{\kappa}$ is of type ( $v_{1}, v_{2}, v_{3}$ ), and now the Lemma follows by interpolation between (28) and (31), and by interpolation between (29) and (33).

This completes the proof of Lemma 9, and therefore also the proof of theorems 2 and 1 .

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# Multi-multifractal decomposition of digraph recursive fractals 

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#### Abstract

In many situations, both deterministic and probabilistic, one is interested in measure theory in local behaviours, for example in local dimensions, local entropies or local Lyapunov exponents. It has been relevant to study dynamical systems, since the study of multifractal can be further developped for a large class of measures invariant under some map, particularly when there exist strange attractors or repelers (hyperbolic case). Multifractal refers to a notion of size, which emphasizes the local variations of the weight of a measure, of the entropy or the Lyapunov exponents. All these notions are explicited in the case of digraph recursive fractal studied by Edgar \& Mauldin where some questions are given. We study the extremal measures and introduce the notion of multi-multifractality which may be useful in problems of rigidity.


## 1. Introduction.

In many situations implicated the dimension of measures, singular measures are investigated, and more precisely how densely the singularities of a measure are distributed.

Let $(X, d)$ be a compact metric space and $\mu$ be a Borel probability measure. The decay rates of the measure $\mu$ of small balls are determined in order to define local dimensions. The singularities of the measure $\mu$
are specified by

$$
\begin{equation*}
\underline{d}_{\mu}(x)=\varliminf_{r \rightarrow 0} \frac{\ln \mu(B(x, r))}{\ln r} \quad \text { and } \quad \bar{d}_{\mu}(x)=\varlimsup_{r \rightarrow 0} \frac{\ln \mu(B(x, r))}{\ln r}, \tag{1}
\end{equation*}
$$

and when $\underline{d}_{\mu}(x)=\bar{d}_{\mu}(x)=d_{\mu}(x)$, the measure $\mu$ has pointwise dimension $d_{\mu}(x)$, and it is said that $\mu$ is exact dimensional [Si1], [Y] if for $\mu$ almost every point $x$ we have $d_{\mu}(x)=d_{\mu}=$ constant.

Even for nice measures, it is not expected that this pointwise dimension exists or the measure $\mu$ to be exact dimensional [LM], [S]. The singularity sets are then defined for any real number $\alpha \geq 0$ by

$$
\begin{align*}
& C_{\alpha}^{-}=\left\{x \in X: \underline{d}_{\mu}(x)=\alpha\right\}, \\
& C_{\alpha}^{+}=\left\{x \in X: \bar{d}_{\mu}(x)=\alpha\right\},  \tag{2}\\
& C_{\alpha}=C_{\alpha}^{+} \cap C_{\alpha}^{-},
\end{align*}
$$

which is called the multifractal decomposition.
This concept first appeared in a paper of physicists [HJKPS] where it was suggested to study the so-called dimension spectrum $f(\alpha)$, i.e.

$$
\begin{equation*}
f(\alpha)=\operatorname{HD}\left(C_{\alpha}\right) \quad \text { and } \quad f(\alpha) \equiv-\infty, \quad \text { if } C_{\alpha}=\varnothing \tag{3}
\end{equation*}
$$

There exist many definitions of dimension [F2], [P2]: Packing-dimension, Box-dimension... For theoretical purposes the Hausdorff dimension is prefered: for any Borel set $A$ and any positive real number $\tau$, put

$$
\operatorname{HD}_{\tau, \varepsilon}(A)=\inf _{\substack{A \subset \cup A_{i} \\\left|A_{i}\right|<\varepsilon}}\left\{\sum_{i \geq 0}\left|A_{i}\right|^{\tau}\right\}
$$

and

$$
\operatorname{HD}_{\tau}(A)=\lim _{\varepsilon \rightarrow 0} \operatorname{HD}_{\tau, \varepsilon}(A) \in[0,+\infty]
$$

We obtain finally the Hausdorff dimension (which derives from a measure) by the following

$$
\operatorname{HD}(A)=\sup \left\{\tau: \operatorname{HD}_{\tau}(A)=+\infty\right\}=\inf \left\{\tau: \operatorname{HD}_{\tau}(A)=0\right\}
$$

and the Hausdorff measure of $A$ is the value $\operatorname{HD}_{\mathrm{HD}(A)}(A) \in[0,+\infty]$.
We define the dimension of a Borel measure $\mu$ by

$$
\operatorname{HD}(\mu)=\inf \{\operatorname{HD}(A): A \text { a Borel set and } \mu(A)=1\} .
$$

In fact it has been found relevant information in a large class of measures, namely dynamical systems $(X, \mu, T)$ where the map $T: X \hookleftarrow$ is ergodic and the measure $\mu$ is $T$-invariant. The first rigorous result [CLP] was the multifractal analysis of $C^{2}$ one-dimensional Markov maps. Many articles appeared on this subject: $[\mathrm{R}]$ for Cookie-cutters, [Lo] for hyperbolic Julia sets, [Si1] for Axiom A surface diffeomorphisms. Other models have been developped: multiplicative chaos (tree structure) which is a model of the phase transition of a system with random interactions in physics and chemistry, in polymers, turbulence, thermodynamics, rainfall distribution - random measures with fixed supports [HW] or with random supports [F1]; iterated function systems [BPS1], [BPS2], [BMP], [CM], [CLP], [EM], [K], [Lo], [O1], [O2], [Si1]. There are now many references that may be found in particular in [P2], especially in the very well-known case of self-similarity for sets or measures [Mo], [MR].

One physical motivation is when ergodic-time averages along the process converge to a measure $\mu:=\lim _{n \rightarrow+\infty}(1 / n) \sum_{i=0}^{n-1} \delta_{T^{i}(x)}$ which describes the occupation of the attractor under iterations of $T$. This measure $\mu$ is the one that can be seen on the screen when computing the iterates of a point under the dynamical system. This is the case for SBR (Sinaï-Bowen-Ruelle) measures of diffeomorphisms of smooth Riemannian manifolds which contain a compact hyperbolic attractor $\Lambda$ of $T$. The limit measure $\mu$ has absolutely continuous conditional measures on unstable manifolds [HY], and the measure $\mu$ describes the orbit distribution of points in a basin $B \supset \Lambda$. Clearly, one sees how densely the singularities of $\mu$ are distributied - areas are darker and darker when there are more and more visits.

Most of the measures in the literature are equilibrium measures Gibbs measures - and therefore are very common and typical in physics. In some cases explicit formulae can be obtained [BPS1], [BPS2], [R], [Si1], and in all the cases the dimension spectrum $f$ is proved to be real analytic.

A new approach is suggested when looking at the distribution along orbits. We define for any $x \in X$ and any integer $q \geq 2$ the quantity [GHP], [HP], [P1], [PT],

$$
\begin{array}{r}
C(x, q, r, n)=\frac{1}{n^{q}} \#\left\{\left(i_{1}, \ldots, i_{q}\right): d\left(T^{i_{j}}(x), T^{i_{k}}(x)\right)<r\right. \\
\text { for } \left.0 \leq i_{j}<i_{k}<n\right\} .
\end{array}
$$

If the measure $\mu$ is ergodic, we have for $\mu$ allmost every $x$,

$$
\lim _{n \rightarrow+\infty} C(x, q, r, n)=\int_{X} \mu(B(y, r))^{q-1} d \mu(y)
$$

Provided the limit exists, we define the HP spectrum

$$
\begin{align*}
(1-q) C_{q}(x) & =\lim _{r \rightarrow 0} \lim _{n \rightarrow+\infty} \frac{\ln C(x, q, r, n)}{\ln r} \\
& \stackrel{\text { a.e. }}{=} \frac{\ln \left(\int_{X} \mu(B(y, r))^{q-1} d \mu(y)\right)}{\ln r} . \tag{4}
\end{align*}
$$

In [O1], [O2], [P1], [Si2] this function is generalized to real numbers and is called the correlation dimension,

$$
C(\beta)=\lim _{r \rightarrow 0} \frac{\ln \left(\int_{X} \mu(B(y, r))^{\beta} d \mu(y)\right)}{\ln r}, \quad \text { for all } \beta \in \mathbb{R},
$$

provided the limit exists, which is for $\beta=1$ the average of the singularities of $\mu[\mathrm{Si} 2]$.

This function can be seen in the following way (order two approach) suggested by D. Ruelle and described in [P1]. Consider the product metric space $Y=X \times X$ equipped with the metric

$$
d^{\prime}\left(\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right)=d\left(x_{1}, y_{1}\right)+d\left(x_{2}, y_{2}\right)
$$

and define the direct product measure $\nu=\mu \times \mu$. Define the diagonal

$$
D=\{(x, x) \in Y\} \text { and for } r>0, D_{r}=\left\{y \in Y: d^{\prime}(y, D)<r\right\}
$$

We then obtain

$$
\nu\left(D_{r}\right)=\int_{X} \mu(B(x, r)) \mu(d x)
$$

and therefore we have

$$
\frac{\ln \nu\left(D_{r}\right)}{\ln r}=\frac{\ln \int_{X} \mu(B(x, r)) \mu(d x)}{\ln r} \underset{r \rightarrow 0}{\longrightarrow} C(1) .
$$

This function $C$ plays an important role in the numerical investigation of some models and the procedure is simple and runs fast [GHP], [P1].

In multifractal analysis there are two methods: the first one comes from the theory of operators (Perron-Froenius) and gives the existence, uniqueness and regularity of the solution [EM]. The other one is based on large deviations and thermodynamics, and leads to explicit formulae [CLP], [Si1]. The latter is described in the following.

Using large deviations and under suitable assumptions, we have the multifractal formalism, i.e. the dimension spectrum $f$ is the LegendreFenchel transform of a function $F$, called free energy function, concave and at least $C^{1}$, i.e.

$$
\begin{equation*}
f(\alpha)=\inf _{t \in \mathbb{R}}\{t \alpha-F(t)\} \tag{5}
\end{equation*}
$$

where $F$ is derived from a sequence of partition functions $\left\{Z_{n}\right\}_{n \geq 1}$
(6) $F(\beta)=\lim _{n \rightarrow+\infty}-\frac{1}{n} \ln _{b(\beta, n)} Z_{n}(\beta) \quad\left(:=F_{n}(\beta)\right), \quad$ for all $\beta \in \mathbb{R}$.

These partition functions are defined by the following

$$
\begin{equation*}
Z_{n}(\beta)=\sum_{\substack{U \in Q_{n} \\ \mu(U)>0}} \mu(U)^{\beta}, \quad \text { for all } \beta \in \mathbb{R}, \tag{7}
\end{equation*}
$$

where $\left\{Q_{n}\right\}_{n \geq 1}$ is a well chosen sequence of partitions (typically the Markov partition $\left\{\mathcal{P}_{n}\right\}_{n \geq 1}$ generated by the dynamics and the iterates under $T[\mathrm{Bo}],[\mathrm{Ru}]$ ) whose diameters tend to 0 when $n$ goes to $+\infty$ (for $b(n(\beta))$ see (31) and (32)).

There is another intrinsic free energy function [CLP], [RU], [Si1] associated to the Markov partition $\left\{\mathcal{P}_{n}\right\}_{n \geq 1}$ defined on $\mathbb{R}^{2}$ by (see Theorem A and (35))

$$
\begin{equation*}
G_{\mathcal{D}}(x, y)=\lim _{n \rightarrow+\infty} \frac{1}{n} \ln G_{\mathcal{D}}^{(n)}(x, y), \quad \text { for all }(x, y) \in \mathbb{R}^{2} \tag{8}
\end{equation*}
$$

with

$$
G_{\mathcal{D}}^{(n)}(x, y)=\sum_{A \in \mathcal{P}_{n}} \mu(A)^{x}|A|^{y}, \quad \text { for all }(x, y) \in \mathbb{R}^{2}
$$

For these thermodynamic quantities it is proved that [O1], [O2], [Si2]

$$
C(\beta)=F(\beta)+1, \quad \text { for all } \beta \in \mathbb{R},
$$

and this equality holds if and only if $F$ can be associated to a sequence of uniform partitions. It is also proved that [CLP], [Si2]

$$
\operatorname{HD}(\mu)=\inf \{\operatorname{HD}(A): A \text { a Borel set and } \mu(A)=1\}=d_{\mu}=F^{\prime}(1) .
$$

The main result in multifractal analysis is the following: $f$ is smooth (real analytic or $C^{\infty}$ ) and strictly concave on an interval $] \alpha_{\min }, \alpha_{\min }[\subset$ $\mathbb{R}^{+*}$ and is the Legendre-Fenchel transform of a function $F$ of same regularity, except in the degenerate case where it is defined at one point (this case can be described).

There exist also multifractal decompositions for (Kolmogorov-Sinaï) entropy and Lyapunov exponents - decompositions into level sets.

For the entropy spectrum, let $\{\xi\}$ be a generating partition, i.e. if $\mathcal{B}(X)$ is the Borel algebra, then $\mathcal{B}(X)=\bigvee_{i>0} T^{-i}(\xi) \mu \bmod 0$ (for example the Markov partition) and $\xi_{n}(x)$ be the element of the partition $\xi_{n}$ at rank $n$,

$$
\xi_{n}=\bigvee_{i=0}^{n-1} T^{-i}(\xi)
$$

which contains the point $x$. Then define local entropy,

$$
\begin{equation*}
h_{\mu}(x)=h_{\mu}(x, \xi, T)=\lim _{n \rightarrow+\infty}-\frac{1}{n} \ln \mu\left(\xi_{n}(x)\right), \tag{9}
\end{equation*}
$$

provided the limit exists (it exists for $\mu$ almost every point $x$ in the ergodic case), and for $\mu$ almost every point $x, h_{\mu}(x)=h_{\mu}$ ( $\mu$ is exact for the entropy in the ergodic case), the entropy of the dynamical system (the exact value).

We define the level sets for entropy for any real $\eta \geq 0$ by

$$
\begin{equation*}
E(\eta)=\left\{x: h_{\mu}(x)=\eta\right\} \quad \text { and } \quad E_{n}(\eta)=\operatorname{HD}(E(\eta)), \tag{10}
\end{equation*}
$$

which is the entropy spectrum.
For the local Lyapunov exponent, let $M$ be a smooth manifold, $T: M \hookleftarrow$ a $C^{2}$ conformal expanding map leaving invariant a compact subset $\Lambda$ of $M$. Let $\mu$ be a $T$-invariant probability measure on $\Lambda$. We have for any tangent vector $\vec{v} \in T_{x}(\Lambda)$,

$$
\begin{equation*}
\chi_{\mu}(x)=\lim _{n \rightarrow+\infty} \frac{1}{n} \ln \left\|d T_{x}^{n}(\vec{v})\right\| \tag{11}
\end{equation*}
$$

provided the limit exists (it exists $\mu$ almost everywhere), and for $\mu$ alomost every point $x, \chi(x)=\chi_{\mu}$, the Lyapunov exponent of the dynamical system (the exact value).

We define the level sets for Lyapunov exponents: for any real $\vartheta \geq 0$, consider

$$
\begin{equation*}
L(\vartheta)=\left\{x: \chi_{\mu}(x)=\vartheta\right\} \quad \text { and } \quad L_{y}(\vartheta)=\operatorname{HD}(L(\vartheta)), \tag{12}
\end{equation*}
$$

which is the Lyapunov spectrum.
We then have the following multifractal decompositions

$$
\left\{\begin{aligned}
\Lambda= & \left\{x: h_{\mu}(x) \text { does not exist }\right\} \cup\left\{x: h_{\mu}(x)=h_{\mu}\right\} \\
& \bigcup_{\alpha \neq h_{\mu}}\left\{x: h_{\mu}(x)=\alpha\right\} \\
\Lambda= & \left\{x: \chi_{\mu}(x) \text { does not exist }\right\} \cup\left\{x: \chi_{\mu}(x)=\chi_{\mu}\right\} \\
& \bigcup_{\alpha \neq \chi_{\mu}}\left\{x: \chi_{\mu}(x)=\alpha\right\}
\end{aligned}\right.
$$

and the corresponding spectra. Notice that the existence of the exact values for the different spectra are given by: the Eckmann-Ruelle conjecture [BPS1] for dimension, the Shannon-McMillan-Breiman theorem for entropy and the Kingman theorem for Lyapunov exponents.

Notice that in general we have

$$
\operatorname{HD}\left(\left\{x \in X: h_{\mu}(x) \text { does not exist }\right\}\right)>0
$$

and similarly

$$
\operatorname{HD}\left(\left\{x \in X: \chi_{\mu}(x) \text { does not exist }\right\}\right)>0(=\operatorname{dim}(X)) .
$$

Our aim is to answer to questions found in [EM]: completness of the dimension spectrum (and finally the other spectra), problems at the bounds of the interval of definition of the spectra, case where the transition matrix is not irreducible...

Results found in [EM] are given in Section 2. We find again these results and generalize them in a different framework (Section 3). Then using notations and results of Section 3, let us define the following.

In the case of expanding Markov maps, a map $T \in C^{1+\delta}(\Lambda)$ is given, and for $x \in \Lambda, J(x)=-\ln T^{\prime}(x)<0\left(\in C^{\delta}(\Lambda)\right)$. The $T$ invariant measure $\mu$ is a Gibbs measure associated to the potential $\xi \in C^{\delta}(\Lambda)<0$. Since the set $\Lambda$ is compact the functions $\xi$ and $J$ take their values in compacts sets $[a, b]$ and $[c, d]$ since there are continuous.

For any real number $\beta$ we define a Gibbs measure $\mu_{\beta}$ associated to the potential $\xi_{\beta}=\beta \xi-F(\beta) J$ (and $\mu_{ \pm \infty}$ are limits when $\beta \rightarrow \pm \infty$ ).

Consider
(13) $\alpha_{\min }=\frac{\int_{\Lambda} \xi d \mu_{+\infty}}{\int_{\Lambda} J d \mu_{+\infty}}=\alpha_{+\infty} \quad$ and $\quad \alpha_{\max }=\frac{\int_{\Lambda} \xi d \mu_{-\infty}}{\int_{\Lambda} J d \mu_{-\infty}}=\alpha_{-\infty}$.

We then have the following results.
Theorem A. For any $(\beta, s, t) \in \mathbb{R}^{3}$ we have

$$
\begin{gathered}
G(s, t)=P(s \xi+t J) \\
F(\beta)=\frac{h_{\mu_{\beta}}+\beta \int_{\Lambda} \xi d_{\mu_{\beta}}}{\int_{\Lambda} J d_{\mu_{\beta}}}, \\
G(\beta,-F(\beta))=0
\end{gathered}
$$

and

$$
F(\beta)=-\psi(\beta)
$$

In the degenerate case the different spectra are reduced to points. Otherwise we can associate a family of probability measures $\left\{\mu_{\beta}\right\}_{\beta \in \overline{\mathbb{R}}}$, and we have the following.

Theorem B. We have in the general case

- $C_{\alpha} \neq \varnothing$ if and only if $\alpha \in\left[\alpha_{\min }, \alpha_{\max }\right]$ where $0<\alpha_{\min }<\alpha_{\max }<$ $+\infty$.
- For all $\alpha \in\left[\alpha_{\min }, \alpha_{\text {max }}\right]$ there exists a unique $\beta=f^{\prime}(\alpha) \in \overline{\mathbb{R}}$ such that $\mu_{\beta}$ is exact dimensional, and

$$
f(\alpha)=\operatorname{HD}\left(C_{\alpha}\right)=\operatorname{HD}\left(\mu_{\beta}\right)=d_{\mu_{\beta}}=\frac{\int_{\Lambda}-\xi d \mu_{\beta}}{\int_{\Lambda}-J d \mu_{\beta}}=\frac{h_{\mu_{\beta}}(T)}{\chi_{\mu_{\beta}}(T)} .
$$

- $\mu$ is exact dimensional: $\operatorname{HD}(\mu)=d_{\mu}=f(\alpha(1))$ where $\alpha(1)=$ $F^{\prime}(1)$.

Theorem C. We have in the general case:

1) For the entropy spectrum (9) and (10):

- $E(\eta) \neq \varnothing$ if and only if $\eta \in\left[\eta_{\min }, \eta_{\max }\right]$ where $0<\eta_{\min }<\eta_{\max }<$ $+\infty$.
- For all $\eta \in\left[\eta_{\min }, \eta_{\max }\right]$ there exists a unique $\beta \in \overline{\mathbb{R}}$ such that $\eta=\int_{\Lambda}-\xi d \mu_{\beta}=h_{\mu_{\beta}}\left(\mu_{\beta}\right.$ is exact dimensional), and

$$
E_{n}(\eta)=\operatorname{HD}(E(\eta))=\operatorname{HD}\left(\mu_{\beta}\right)=d_{\mu_{\beta}}=\frac{\int_{\Lambda}-\xi d \mu_{\beta}}{\int_{\Lambda}-J d \mu_{\beta}}=\frac{h_{\mu_{\beta}}}{\chi_{\mu_{\beta}}}=f(\alpha)
$$

where $\alpha=F^{\prime}(\beta)$.

- $\mu$ is exact dimensional: for $\eta=h_{\mu}(\beta=1)$, we have $\mu(E(\eta))=1$ and

$$
E_{n}(\eta)=\operatorname{HD}\left(E\left(h_{\mu}\right)\right)=d_{\mu}=\frac{h_{\mu}}{\chi_{\mu}}
$$

2) For the Lyapunov spectrum (11) and (12):

- $L(\vartheta) \neq \varnothing$ if and only if $\vartheta \in\left[\vartheta_{\min }, \vartheta_{\max }\right]$ where $0<\vartheta_{\min }<$ $\vartheta_{\text {max }}<+\infty$.
- For all $\vartheta \in\left[\vartheta_{\min }, \vartheta_{\max }\right]$ there exists a unique $\beta \in \overline{\mathbb{R}}$ such that $\vartheta=\int_{\Lambda}-J d \mu_{\beta}=\chi_{\mu_{\beta}}\left(\mu_{\beta}\right.$ is exact dimensional $)$, and

$$
L_{y}(\vartheta)=\operatorname{HD}(L(\vartheta))=\operatorname{HD}\left(\mu_{\beta}\right)=d_{\mu_{\beta}}=\frac{\int_{\Lambda}-\xi d \mu_{\beta}}{\int_{\Lambda}-J d \mu_{\beta}}=\frac{h_{\mu_{\beta}}}{\chi_{\mu_{\beta}}}=f(\alpha)
$$

where $\alpha=F^{\prime}(\beta)$.

- $\mu$ is exact dimensional: for $\vartheta=\chi_{\mu}(\beta=1)$, we have $\mu(L(\vartheta))=1$ and

$$
L_{y}(\vartheta)=\operatorname{HD}\left(L\left(\chi_{\mu}\right)\right)=d_{\mu}=\frac{h_{\mu}}{\chi_{\mu}} .
$$

Theorem D. The extremal measures $\mu_{ \pm \infty}$ are uniform on their Can-tor-like fractal supports.

In Section 2 we define the model and the results (theorems 1 and 2) obtained in [EM].

In Section 3 we give a short exposition concerning the thermodynamic formalism that we use for our computations in the next sections.

In Section 4 we find again and generalize the results in [EM] by proving theorems A and B.

Section 5 deals with the multifractal spectra, entropy and Lyapunov exponents, which correspond to the level sets (10) and (12), and we prove Theorem C.

In Section 6 we develop a new concept: multi-multifractality, which allows us to give answers concerning extremal points (the points $\alpha_{ \pm \infty}$ ) in a quite simple fashion and we prove Theorem D. In particular we give some graphs of the functions we have studied.

Section 7 is devoted to discussion and new questions.

## 2. The model and the operator theory.

We start from a directed multigraph $(V, E)$ [EM]. The set $E=$ $\left\{e_{1}, \ldots, e_{k}\right\}$ consists of the edges of the graph, and the elements of $V=$ $\{u, v, \ldots, w\}$ are the vertices. This graph is supposed to be strongly connected, that means there is a path from any vertex to any other along the edges of the path (if not we decompose it into connex components).

Now we define notions of length and measure (mass) in order to compute local dimensions (1).

A path of length $k$ in the graph is a finite string

$$
\gamma=e_{1} e_{2} \cdots e_{k}
$$

of edges, and to each edge $e$ correspond a ratio $r(e) \in] 0,1[$ (parameter of a homethety in $\mathbb{R}^{n}$ ), and $r(\gamma)=r\left(e_{1}\right) r\left(e_{2}\right) \cdots r\left(e_{k}\right)$. The subset $E_{u v}$, the edges from $u$ to $v$, is a partition of $E$ for $(u, v) \in V^{2}$. The set $E_{u v}^{(k)}$ is composed of all the paths of length $k$ that start at $u$ and end at $v$, $E_{u}^{(k)}$ is the set of paths of length $k$ starting at $u$, and $E_{u}$ is the set of infinite paths starting at $u$.

For any vertex $u \in V$, let $J_{u}$ be a nonempty compact subset of $\mathbb{R}^{n}$. Actually we may assume for simplicity that the diameter of the set $\left|J_{u}\right|=1$ for any $u \in E$.

A digraph recursive fractal, based on seed set $J_{u}$ and ratios $r(e)$,
is the set

$$
\begin{equation*}
K_{u}=\bigcap_{k \geq 0}\left(\bigcup_{\gamma \in E_{u}^{(k)}} J(\gamma)\right), \tag{14}
\end{equation*}
$$

where the sets $J(\gamma)$ are choosen recursively:
i) $J\left(\Lambda_{u}\right)=J_{u}$ where $\Lambda_{u}$ is the empty path from $u$ to $u$.
ii) For $\gamma$ of length $k$ with terminal vertex $v$, the set $J(\gamma)$ is geometrically similar to $J_{v}$ with reduction ratio $r(\gamma)$.
iii) For $\gamma$ of length $k$ with terminal vertex $v$, the sets $J(\gamma e), e \in$ $E_{v}$, are nonoverlapping subset of $J(\gamma)$ (they intersect at most at their boundaries: "open set condition").

There are many choices to place the sets $J(\gamma e)$ in $J(\gamma)$, and for example consider the "self-similar graph" fractals using similarities $H_{e}$ : $\mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$, one for each edge $e \in E$. Define for any $\gamma=e_{1} e_{2} \cdots e_{k} \in$ $E_{u v}^{(k)}$

$$
J(\gamma)=H_{e_{1}} H_{e_{2}} \cdots H_{e_{k}}\left(J_{v}\right),
$$

where the seed set $J_{v}$ must be choosen such that iii) is satisfied.
We now define the measure of Markov type $\mu_{u}$ on $K_{u}$ recursively: we start with $\mu_{u}\left(J_{u}\right)=1$, and the mass is distributed among the subsets $J(e), e \in E$, so that $J(e)$ has mass $p(e)$. Once the mass of a set $J(\gamma)$ has been assigned, then it is distributed among the subsets $J(\gamma e)$ according to the values of $p(e)$. With (14) we get finally a unique probability measure depending on the choice of the number $\left((p(e))_{e \in E}\right.$. As for the definition of $r(\gamma)$, we get $p(\gamma)=p\left(e_{1}\right) \cdots p\left(e_{k}\right)$ for $\gamma=e_{1} e_{2} \cdots e_{k}$.

It implies that $p$ is defined on "cylinders", and then by the Kolmogorov consistency theorem a unique measure $\mu_{u}$ on $K_{u}$ is defined.

Let for $(\sigma, k)=\left(e_{1} e_{2} \cdots e_{k}\right)$ the finite string of length $k$,

$$
\begin{align*}
h_{u}: E_{u} & \longrightarrow K_{u} \\
\sigma & \longrightarrow \bigcap_{k \geq 1} J\left(\sigma_{\mid k}\right) \tag{15}
\end{align*}
$$

(representation of the coding sequences of the trajectories, one-to-one at least on a set of $\mu$ measure 1 - the points with more than two representations have no local dimension). We have $\mu=\nu_{u} \circ h_{u}^{-1}$ where $\nu_{u}$ is defined on $E_{u}$ (it is defined on the cylinders).

Let $A$ be the transition matrix associated to the Markov partition given by the iterations of the sets $J_{v}, v \in E$, by the map $H$ which determines the distribution of the $J(v e), e \in E$, inside $J(v)$ (for example in the case of "self-similar graph fractals", $H$ is composed of similarities $H_{v}: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$, for each edge $v$ ).

Define the matrix $B$,

$$
B_{u v}(\beta, s)=\sum_{e \in E_{u v}} p(e)^{\beta} r(e)^{s}, \quad(\beta, s) \in \mathbb{R}^{2}
$$

(compare with (35) and the function $G_{\mathcal{D}}^{(0)}(\beta, s)$ ), and let $\phi(\beta, s)$ be the spectral radius of $B$. By the Perron-Frobenius theory of nonnegative matrices, $\phi$ is real analytic in both variables, and given any real number $\beta$, there exists a unique real number $s=\psi(\beta)$ such that $\phi(\beta, \psi(\beta))=1$. We get in particular $\operatorname{HD}\left(K_{u}\right)=\psi(0)=d$ which is independant of $u$.

Here are the results obtained in [EM].
Theorem 1. The function $\psi$ is real analytic, strictly decreasing from $+\infty$ to $-\infty$ and convex.

Let for any real number $\beta$,

$$
\begin{equation*}
\alpha=\psi^{\prime}(\beta)>0 \quad \text { and } \quad f=\beta \alpha+\psi(\beta) \tag{16}
\end{equation*}
$$

and for $\gamma=e_{1} e_{2} \cdots e_{k}$,

$$
\delta(\gamma)=\frac{\ln p(\gamma)}{\ln r(\gamma)}=\frac{\ln \left(p\left(e_{1}\right) p\left(e_{2}\right) \cdots p\left(e_{k}\right)\right)}{\ln \left(r\left(e_{1}\right) r\left(e_{2}\right) \cdots r\left(e_{k}\right)\right)}
$$

and $\alpha_{\text {min }}=\inf \{\delta(\gamma): \gamma$ is a simple cycle $\}\left(\alpha_{\max }=\sup \right)$.
Let $\left\{x_{v}\right\}_{v \in V}$ be the Perron numbers and consider the pairs $\left(\lambda_{v}, \rho_{v}\right)_{v \in V}$. We have, for all $v \in V, x_{v}>0$ and for all $u \in V$,

$$
\sum_{v \in V} \sum_{e \in E_{u v}} r(e)^{d} x_{v}^{d}=x_{u}^{d},
$$

for all $u \in V$,

$$
\sum_{v \in V} \sum_{e \in E_{u v}} P(e)=1,
$$

where $P(e)=\rho_{u}^{-1} p(e)^{\beta} r(e)^{\psi(\beta)} \rho_{v}$. The real numbers $\pi_{u}$ define a stationary distribution for the Markov chain: given $X_{k}=u$, the conditional probability that $X_{k+1}=v$ is $\sum_{e \in E_{u v}} P(e)$.

These are the transition probabilities for some stationary measure on $E_{u}, \nu_{u}^{(\beta)}$, a measure of Markov type defined on the cylinders of $E_{u}$. With the map $h_{u}$ it corresponds to a measure $\mu_{u}^{(\beta)}$ on $K_{u}$,

$$
\nu_{u}^{(\beta)}(\gamma)=\rho_{u}^{-1} p(\gamma)^{\beta} r(\gamma)^{\psi(\beta)} \rho_{v} \quad \text { and } \quad \mu_{u}^{(\beta)}=\nu_{u}^{(\beta)} \circ h_{u}^{-1} .
$$

We then have defined for all $u \in V$ measures $\nu_{u}^{(\beta)}, \beta \in \mathbb{R}$, on the sets $E_{u}$ by its transition probabilities, and therefore measures $\mu_{u}^{(\beta)}, \beta \in \mathbb{R}$, on the sets $K_{u}, \mu_{u}^{(\beta)}=\nu_{u}^{(\beta)} \circ h_{u}^{-1}$.

Consider for any $u \in V$,

$$
\left\{\begin{array}{l}
K_{u}^{(\alpha)}=\left\{x \in K_{u}: \lim _{r \rightarrow 0} \frac{\ln \mu_{u}(B(x, r))}{\ln r}=\alpha\right\}  \tag{17}\\
E_{u}^{(\alpha)}=\left\{\sigma \in E_{u}: \lim _{k \rightarrow+\infty} \frac{\ln p\left(\sigma_{\mid k}\right)}{\ln r\left(\sigma_{\mid k}\right)}=\alpha\right\}
\end{array}\right.
$$

then $E_{u}^{(\alpha)}=h_{u}^{-1}\left(K_{u}^{(\alpha)}\right)$. It is proved that we have for $f$ given by (16)

$$
\mu_{u}^{(\beta)}\left(K_{u}^{(\alpha)}\right)=\nu_{u}^{(\beta)}\left(E_{u}^{(\alpha)}\right)=1
$$

and

$$
\operatorname{HD}\left(K_{u}^{(\alpha)}\right)=\operatorname{HD}\left(E_{u}^{(\alpha)}\right)=f=\operatorname{HD}\left(\mu^{(\beta)}\right)=\operatorname{HD}\left(\nu^{(\beta)}\right) .
$$

Finally there are two cases for the multifractal analysis.

## Theorem 2.

i) In the degenerate case: for all $(u, v) \in V^{2}$, for all $e \in E_{u v}$, $p(e)=\left(x_{u}^{-1} r(e) x_{v}\right)^{d}$. Then $\psi$ is linear and for all $\beta \in \mathbb{R}, \psi(\beta)=$ $d(1-\beta), \operatorname{HD}\left(K_{u}\right)=d=d_{\mu_{u}}$ and $K_{u}^{(\alpha)} \neq \varnothing$ if and only if $\alpha=d$.
ii) In the nondegenerate case: there exists $e \in E_{u v}, p(e) \neq$ $\left(x_{u}^{-1} r(e) x_{v}\right)^{d}$. Then $\psi$ is real analytic and strictly convex; $\alpha$ is a strictly decreasing function of $\beta$, i.e. $\alpha: \mathbb{R} \longrightarrow] \alpha_{\min }, \alpha_{\max }[; f$ is a strictly concave function of $\alpha$ and $K_{u}^{(\alpha)} \neq \varnothing$ if and only if $\alpha \in$ $\left[\alpha_{\text {min }}, \alpha_{\text {max }}\right]$.

## 3. Thermodynamic formalism.

This is a useful theory developped in [Bo], [Ru]. It allows to transport some problems from the dynamical system $(\Lambda, \mu, T)$, where $T$ is for
example a picewise $C^{1+\delta}$ expanding Markov map [R], onto a symbolic dynamical system $\left(\Sigma_{A}^{+}, \nu, \sigma\right)$ by a coding map.

### 3.1. Symbolic dynamics.

We introduce Markov partitions to make an analogy with the symbolic dynamical systems. In a sense, we replace small balls in the definition of dimension by small elements of the iterations of this partition by the expanding Markov map.

Let $\Lambda$ be a basic set, a $T$-invariant compact metric set. A Markov partition is a finite cover of $\Lambda: \mathcal{U}_{0}=\left(U_{1}, \ldots, U_{m}\right)$, consisting of proper rectangles (compact sets $R$ such that $R=\overline{\operatorname{int}(R)}$ ) which satisfy

- $\operatorname{int}\left(U_{i}\right) \cap \operatorname{int}\left(U_{j}\right)=\varnothing$ for $i \neq j$.
- Each $T\left(U_{i}\right)$ is a union of rectangles $U_{j}$.

We can construct Markov partitions of arbitrary small diameter. We then define the partition at the rank $n$ by

$$
\mathcal{U}_{n}=\bigvee_{i=0}^{n-1} T^{-i}\left(\mathcal{U}_{0}\right)
$$

We associate to this partition the transition matrix $A$ defined by

$$
A_{i, j}=\left\{\begin{array}{ll}
1, & \text { if } T^{-1}\left(\stackrel{\circ}{U_{j}}\right) \cap \stackrel{\circ}{U}_{i} \neq \varnothing,  \tag{18}\\
0, & \text { otherwise }
\end{array} \quad 1 \leq i, j \leq p\right.
$$

which is irreducible (for all $(i, j)$, there exists $n$ such that $\left(A^{n}\right)_{i j}>0$ : you reach any $U_{i}$ from any $U_{j}$ ).

Consider the subshift of finite type associated to the matrix $A$

$$
\Sigma_{A}^{+}=\left\{\underline{x}=\left\{x_{n}\right\}_{n \geq 0} \in\{1, \ldots, m\}^{\mathbb{N}}: A_{x_{i} x_{i+1}}=1\right\}
$$

which is the set of admissible sequences.
We define the metric on $\Sigma_{A}^{+}($for $0<\lambda<1)$

$$
d^{\prime \prime}(\underline{x}, \underline{y})= \begin{cases}\lambda^{k}, & \text { if } k=\sup \left\{j: x_{i}=y_{i}, \text { for all } i, 0 \leq i<j\right\}, \\ 0, & \text { if } \underline{x}=\underline{y},\end{cases}
$$

which is a compact set, and the shift $\sigma(\underline{x})=\underline{y}$, where for all $n \in \mathbb{N}$, $y_{n}=x_{n+1}$.

We then define a continuous (Lipschitz) surjection $\pi$,

$$
\begin{aligned}
& \pi: \Sigma_{A}^{+} \longrightarrow \Lambda \\
& \quad \underline{x} \longmapsto \bigcap_{j \geq 0} T^{-j}\left(U_{x_{j}}\right)
\end{aligned}
$$

which is one-to-one on the set of points whose trajectories do not intersect the boundaries of the elements of the Markov partition (if not these points have no local dimension), a set of $\mu$ measure 1 when $\mu$ is a Gibbs measure. Nevertheless, it is bounded-to-one and satisfies $\pi \circ \sigma^{n}=T^{n} \circ \pi$.

### 3.2. Thermodynamics.

Let us define the following sets.

- Consider $M(\Lambda)$ (respectively $M\left(\Sigma_{A}^{+}\right)$) the set of Borel probability measures defined on $\Lambda$ (respectively $M\left(\Sigma_{A}^{+}\right)$).
- Let $M_{T}(\Lambda)$ (respectively $\left.M_{\sigma}\left(\Sigma_{A}^{+}\right)\right)$be the set of $T$-invariant Borel probability measures on $\Lambda$ (respectively $\sigma$-invariant on $\Sigma_{A}^{+}$).
- Let $C(\Lambda)$ (respectively $C\left(\Sigma_{A}^{+}\right)$) be the set of continuous functions defined on $\Lambda$ (respectively $\Sigma_{A}^{+}$) and $C^{\delta}(\Lambda)$ (respectively $C^{\delta}\left(\Sigma_{A}^{+}\right)$) be the set of $\delta$-Hölder continuous functions.

The pressure of a function $\varphi \in C^{\delta}(\Lambda)$ (respectively $\bar{\varphi} \in C^{\delta}\left(\Sigma_{A}^{+}\right)$) is defined by

$$
\begin{equation*}
P_{\varphi}=P_{T}(\varphi)=\sup _{\rho \in M_{T}(\Lambda)}\left(h_{\rho}+\int_{\Lambda} \varphi d \rho\right) \quad\left(=P_{\sigma}(\varphi \circ \pi)=P_{\sigma}(\bar{\varphi})\right), \tag{19}
\end{equation*}
$$

and the measures which achieve this supremum are called equilibrium measures. The entropy $h_{\rho}(T)$ - the Kolmogorov-Sinaï entropy of the map $T$ - is the following: define the set

$$
B(x, n, r)=\left\{y \in \Lambda: d\left(T^{i}(x), T^{i}(y)\right)<r, \text { for } 0 \leq i \leq n-1\right\},
$$

the set of points that cannot be distinguished from $x$ at the small distance $r$ after $(n-1)$ iterations. Then we get for an ergodic $T$-invariant probability measure $\mu$,

$$
h_{\mu}(T) \stackrel{\mu \text { a.s. }}{=}-\lim _{r \rightarrow 0} \varlimsup_{n \rightarrow+\infty} \frac{1}{n} \ln \mu(B(x, n, r)),
$$

which is a nonnegative real number in our case. Notice that the larger the entropy, the greater the rate of decrease of the indeterminacy of the dynamical system.

In our case, there exists a unique measure $\mu_{\varphi}$ (respectively $\nu_{\bar{\varphi}}, \mu_{\varphi}=$ $\pi^{*} \nu_{\bar{\varphi}}$ ) which is the Gibbs measure of the potential $\varphi$ (respectively $\bar{\varphi}$ ). The map $\pi:\left(\Sigma_{A}^{+}, \nu_{\bar{\varphi}}, s\right) \longrightarrow\left(\Lambda, \mu_{\varphi}, T\right)$ is an isomorphism of dynamical systems.

This means that the pullback of any Gibbs measure $\mu_{\varphi}$ on $\Lambda$ is a Gibbs measure $\nu_{\bar{\varphi}}$ on $\Sigma_{A}^{+}$. Conversely the pushforward of any Gibbs measure $\nu_{\bar{\varphi}}$ on $\Sigma_{A}^{+}$is a Gibbs measure $\mu_{\varphi}$ on $\Lambda$, and their thermodynamic quantities are equal: $P_{T}(\varphi)=P_{\sigma}(\varphi \circ \pi), h_{\mu_{\varphi}}(T)=h_{\nu_{\bar{\varphi}}}(\sigma)$.

The measure $\nu_{\bar{\varphi}}$ is well defined on the cylinders which generate the topology of $\Sigma_{A}^{+}$. There exist nonnegative constants $c$ and $C$ such that

$$
\begin{equation*}
c \leq \frac{\nu_{\bar{\varphi}}\left\{\underline{y} \in \Sigma_{A}^{+}: y_{0}=x_{0}, \ldots, y_{n-1}=x_{n-1}\right\}}{\exp \left(-n P_{\bar{\varphi}}+\sum_{k=0}^{n-1} \bar{\varphi}\left(\sigma^{k}(\underline{x})\right)\right)} \leq C, \tag{20}
\end{equation*}
$$

uniformly in $n$.
The pressure function $P: C^{\delta}\left(\Sigma_{A}^{+}\right) \longrightarrow \mathbb{R}$ is real analytic (not true for arbitrary symbolics). Consider for $(\bar{\xi}, \bar{\zeta}) \in C^{\delta}\left(\Sigma_{A}^{+}\right)^{2}$, the map

$$
\begin{align*}
& Q: \mathbb{R}^{2} \longrightarrow \mathbb{R} \\
& \quad(x, y) \longmapsto P(x \bar{\xi}+y \bar{\zeta}) . \tag{21}
\end{align*}
$$

It is real analytic in both variables, convex and strictly convex if and only if the functions $\bar{\xi}$ et $\bar{\zeta}$ are not conjugate to constants $c$ and $c^{\prime}$, i.e. $\bar{\xi} \neq c+\bar{\varphi}-\bar{\varphi} \circ \sigma, \bar{\varphi} \in C^{\delta}\left(\Sigma_{A}^{+}\right)$(respectively $\bar{\zeta}$ and $\left.c^{\prime}\right)$.

Let $\nu_{x_{0} \bar{\xi}+y_{0} \bar{\zeta}}$ be the Gibbs measure of the function $x_{0} \bar{\xi}+y_{0} \bar{\zeta} \in$ $C^{\delta}\left(\Sigma_{A}^{+}\right)$, then we have [M], [Ma], [MC], [R], [Ru], [Si1]

$$
\left\{\begin{array}{l}
\frac{\partial Q}{\partial x}\left(x_{0}, y_{0}\right)=\int_{\Sigma_{A}^{+}} \bar{\xi} d \nu_{x_{0}} \bar{\xi}+y_{0} \bar{\zeta}  \tag{22}\\
\frac{\partial Q}{\partial y}\left(x_{0}, y_{0}\right)=\int_{\Sigma_{A}^{+}} \bar{\zeta} d \nu_{x_{0}} \bar{\xi}+y_{0} \bar{\zeta}
\end{array}\right.
$$

## 4. Dimension spectrum and the thermodynamic theory.

### 4.1. Idea of the computation.

Consider the Markov partition

$$
\mathcal{P}_{n}=\bigvee_{i=0}^{n-1} T^{-j}(\mathcal{P}),
$$

where $\mathcal{P}=\left(K_{1}, K_{2}, \ldots, K_{q}\right)$ (see just below). The idea for computation of local dimensions (1) is to replace small balls $B(x, r)$ by elements $V=T^{-n}(U) \in \mathcal{P}_{n}(U \in \mathcal{P})$ which are in the set $B_{j(\beta, n)}^{n}$ (see (29)) which cover at the limit the singularity set $C_{\alpha}$ for $\alpha=F^{\prime}(\beta)$. Those elements generate a measure $\mu_{\beta}$ (of course singular to each other) which is ergodic. We use the assumptions on $T$ and $\mu$ :

- For any $V=T^{-n}(U):=V(U) \in \mathcal{P}_{n}$ there exists an element $y(U) \in U$ such that

$$
\begin{align*}
|V(U)| & =\left|T^{-n}(U)\right| \\
& =\left|\left(T^{-n}\right)^{\prime}(y(U))\right||U| \\
& =\exp \left(\sum_{j=0}^{n-1} J\left(T^{j}(y(U))\right)\right) \underbrace{|U|}_{\approx 1} \tag{23}
\end{align*}
$$

(where the sign $\approx$ expresses that the ratios of both sides are uniformly bounded by constants), expression which controls the length of $V(U)$.

- Since the measure $\mu$ is a Gibbs measure we have following (20)

$$
\begin{equation*}
\mu(V(U)) \approx \exp \left(\sum_{j=0}^{n-1} \xi\left(T^{j}(y(U))\right)\right. \tag{24}
\end{equation*}
$$

expression which controls the mass of $V(U)$.
It follows from the Birkhoff's sums and the ergodicity of the dy-
namical system that

$$
\begin{aligned}
\frac{\ln \mu(B(x, r))}{\ln r} & \sim \frac{\ln \mu(V(U))}{\ln |V(U)|} \\
& \sim \frac{\frac{1}{n} \sum_{j=0}^{n-1} \xi\left(T^{j}(y(U))\right)}{\frac{1}{n} \sum_{j=0}^{n-1} J\left(T^{j}(y(U))\right)} \\
& \underset{\substack{\mu_{\beta} \text { a.s. } \\
n \rightarrow+\infty}}{\int_{\Lambda} \xi d \mu_{\beta}} \\
& =\alpha d \mu_{\beta} \\
& =F^{\prime}(\beta),
\end{aligned}
$$

which gives the existence and the value of the local dimension for points covered by the sets of the type $B_{j(\beta, n)}^{n}$ (29). Otherwise it suffices to prove for the points which do not have this property that they do not have local dimension.

Note that it is not always possible to replace balls by elements of the partition [O2].

### 4.2. Dimension spectrum.

The Markov measures that are used are in fact a special case of Gibbs measures. These measures are associated to potentials $\bar{\varphi}$ depending only on the first coordinate, i.e. $\bar{\varphi}(\underline{x})=g\left(x_{0}\right)$ for $\underline{x}=\left(x_{i}\right)_{i \geq 0}$. For this purpose, consider the transfer operator

$$
\begin{aligned}
L_{\bar{\varphi}}: & C^{\delta}\left(\Sigma_{A}^{+}\right) \longrightarrow C^{\delta}\left(\Sigma_{A}^{+}\right) \\
& f \longmapsto \sum_{\underline{y} \in \sigma^{-1}(\underline{x})} \exp (\bar{\varphi}(\underline{y}) f(\underline{y})),
\end{aligned}
$$

and the corresponding operator defined on measures $L_{\bar{\varphi}}^{*}: M\left(\Sigma_{A}^{+}\right) \longrightarrow$ $M\left(\Sigma_{A}^{+}\right)$.

Then there exist (see [Ru]):
i) $\lambda>0(=\exp (P(\bar{\varphi})))$,
ii) $h \in C^{0}\left(\Sigma_{A}^{+}\right)$such that $h>0$,
iii) $\rho \in M\left(\Sigma_{A}^{+}\right)$,
such that $L_{\bar{\varphi}}(h)=\lambda h, L_{\bar{\varphi}}^{*}(\rho)=\lambda \rho$ and $\nu_{\bar{\varphi}}=h \rho \in M_{\sigma}\left(\Sigma_{A}^{+}\right)$ ( $d \nu_{\bar{\varphi}}=h\left(x_{0}\right) d \rho$ ) which is the Gibbs measure for $\bar{\varphi}$ and can be represented on the cylinder sets by

$$
\begin{align*}
& \nu_{\bar{\varphi}}\left\{\underline{y} \in \Sigma_{A}^{+}: y_{0}=x_{0}, \ldots, y_{n}=x_{n}\right\}  \tag{25}\\
& \quad=R\left(x_{0}, x_{1}\right) R\left(x_{1}, x_{2}\right) \cdots R\left(x_{n-1}, x_{n}\right) p\left(x_{n}\right),
\end{align*}
$$

where we have

$$
R\left(x_{i}, x_{j}\right)=\frac{A_{i j} h\left(x_{i}\right) \exp \left(\bar{\varphi}\left(x_{i}\right)\right)}{\lambda h\left(x_{j}\right)}
$$

and $p$ is an invariant probability vector: $\sum_{i} p_{i}=1$ and $R(p)=p$.
These equations define all the Markov measures $\nu_{u}$ and a fortiori all the measures $\mu_{u}$.

We compute the partition functions (7) for any pair $(k, s) \in \mathbb{N}^{*} \times \mathbb{R}$,

$$
Z_{k}(s)=\sum_{V(U) \in \mathcal{P}_{k}} \mu(V(U))^{s}=\sum_{\substack{u \in E \\ p(u)>0}} \sum_{\gamma \in E_{u}^{(k)}} p(\gamma)^{s}
$$

Let $C(s)=\max _{u \in E} \mathbb{P}\left(X_{0}=u\right)^{s}$ and for any pair $(k, m)$ of integers, we have

$$
Z_{k}(s)=\sum_{u \in E} \sum_{\gamma \in E_{u}^{(k)}} p(\gamma)^{s} \mathbb{P}\left(X_{0}=u\right)^{s}
$$

and

$$
Z_{m}(s)=\sum_{v \in E} \sum_{\gamma^{\prime} \in E_{v}^{(m)}} p\left(\gamma^{\prime}\right)^{s} \mathbb{P}\left(X_{0}=v\right)^{s} .
$$

We then obtain

$$
\begin{aligned}
Z_{k}(s) & Z_{m}(s) \\
& =\sum_{u \in E} \sum_{v \in E} \sum_{\left(\gamma, \gamma^{\prime}\right) \in E_{u}^{(k)} \times E_{v}^{(m)}}\left[p(\gamma) p\left(\gamma^{\prime}\right)\right]^{s}\left[\mathbb{P}\left(X_{0}=u\right) \mathbb{P}\left(X_{0}=v\right)\right]^{s} \\
& \leq C(s) Z_{k+m}(s) \\
& =C(s) \sum_{u \in E} \sum_{v \in E} \sum_{\gamma^{\prime \prime} \in E_{u}^{(k+m)}} p\left(\gamma^{\prime \prime}\right)^{s} \mathbb{P}\left(X_{0}=u\right)^{s},
\end{aligned}
$$

where $\gamma^{\prime \prime}=\gamma \gamma^{\prime}: \gamma=u e_{2} \cdots e_{k}$ and $\gamma^{\prime}=v e_{2}^{\prime} \cdots e_{m}^{\prime}$. Finally we obtain

$$
\frac{1}{C(s)} Z_{k}(s) \frac{1}{C(s)} Z_{m}(s) \leq \frac{1}{C(s)} Z_{k+m}(s)
$$

which implies that the sequence $\left\{\ln \left(Z_{k}(s) / C(s)\right\}_{k \geq 1}\right.$ is subadditive, and that the sequence $\left\{\ln Z_{k}(s) / k\right\}_{k>1}$ converges to a concave function.

Following the same method we prove (8) that for any pair $(s, t)$ of real numbers the sequence

$$
-\frac{1}{k} \ln G_{\mathcal{D}}^{(k)}(s, t) \underset{k \rightarrow+\infty}{\longrightarrow} G(s, t),
$$

where

$$
G_{\mathcal{D}}^{(k)}(s, t)=\sum_{u \in E} \sum_{\gamma \in E_{u}^{(k)}} p(\gamma)^{s} \mathbb{P}\left(X_{0}=u\right)^{s}|J(\gamma)|^{t}
$$

(we haved assumed that $\left|J_{u}\right|=1$ for any $u \in E$ ).
Framework. The dynamical systems $\left(K_{u}, \mu_{u}, H\right)_{u \in E}$ (respectively $\left.\left(E_{u}, \nu_{u}, \sigma\right)_{u \in E}\right)$ may be studied in the same way. Define $(\mathbf{K}, \mu, T)$ (respectively $(\mathbf{E}, \nu, \sigma)$ ) be one of these sets, where the map $T$ is a picewise $C^{\infty}$ expanding Markov map ( $T=H^{-1}$, for all $e \in E, T_{e}^{-1}=H_{e}$ ).

The measure $\mu$ is the Gibbs measure of the potential $\xi \in C^{\delta}(\mathbf{K})<0$ (respectively $\bar{\xi} \in C^{\delta}(\mathbf{E})$ ), and $J=-\ln T \in C^{\delta}(\mathbf{K})<0$ (respectively $\left.\bar{J} \in C^{\delta}(\mathbf{E})\right)$. We have seen that for Markov measures the associated potentials $\bar{J}$ and $\bar{\xi}$ depends only on the first coordinate.

We now prove theorem A.
Assume that $P(\xi)=0$, if not take $\widetilde{\xi}=\xi-P(\xi)$ which is cohomologous to the potential $\xi$, which implies the equality $\mu_{\tilde{\xi}}=\mu_{\xi}=\mu$.

From the expressions (23) and (24) there exists for any set $V(U)=$ $T^{-n}(U) \in \mathcal{P}_{n}$ an element $y(U) \in U \subset \mathcal{P}$ such that

$$
\begin{align*}
& \frac{1}{n} \ln \mu(V(U)) \sim \frac{1}{n} \sum_{j=0}^{n-1} \xi\left(T^{j}(y(U))\right),  \tag{26}\\
& \frac{1}{n} \ln |V(U)| \sim \frac{1}{n} \sum_{j=0}^{n-1} J\left(T^{j}(y(U))\right) .
\end{align*}
$$

Since the functions $J$ and $\xi$ are $C^{\delta}$-Hölder, they are continuous on the compact set $K$ and therefore take their values in compact sets $[a, b]$ and $[c, d]$.

Consider for any integer $i \in \mathbb{Z} \cap[a n, b n-1]$ (linear scale) the set

$$
\begin{equation*}
A_{i}^{n}=\left\{V(U) \in \mathcal{P}_{n}: \ln \mu(V(U)) \in[i, i+1[ \},\right. \tag{27}
\end{equation*}
$$

and for any real number $\beta$, the integer $i(\beta, n)$ such that

$$
\sum_{V(U) \in A_{i}^{n}} \mu(V(U))^{\beta} \leq \sum_{V(U) \in A_{i(\beta, n)}^{n}} \mu(V(U))^{\beta} .
$$

Since there is a linear scale we have for any real number $\beta$,

$$
\begin{aligned}
\sum_{V(U) \in A_{i(\beta, n)}^{n}} \mu(V(U))^{\beta} & \leq \sum_{i} \sum_{V(U) \in A_{i}^{n}} \mu(V(U))^{\beta} \\
& =\sum_{V(U) \in \mathcal{P}_{n}} \mu(V(U))^{\beta} \\
& =Z_{n}(\beta) \\
& \leq(b-a) n \sum_{V(U) \in A_{i(\beta, n)}^{n}} \mu(V(U))^{\beta} .
\end{aligned}
$$

We get therefore for any real number $\beta$ (7),

$$
\begin{align*}
\frac{1}{n} \ln Z_{n}(\beta) & \sim \frac{1}{n} \ln \left(\sum_{V(U) \in A_{i(\beta, n)}^{n}} \mu(V(U))^{\beta}\right)  \tag{28}\\
& \sim \beta \frac{i(\beta, n)}{n}+\frac{\ln \# A_{i(\beta, n)}^{n}}{n}
\end{align*}
$$

since the elements of $A_{i(\beta, n)}^{n}$ have same mass $\approx \exp (i(\beta, n))$.
Among the elements of $A_{i(\beta, n)}^{n}$ we make a new selection for the length, in order to obtain elements of $A_{i(\beta, n)}^{n}$ with same mass and same length.

Therefore consider in the same way for all integer $j \in \mathbb{Z} \cap[c n, d n-$ 1] (linear scale) the set

$$
\begin{equation*}
B_{j}^{n}=\left\{V(U) \in A_{i(\beta, n)}^{n}: \ln |V(U)| \in[j, j+1[ \}\right. \tag{29}
\end{equation*}
$$

For any real number $\beta$, define the integer $j(\beta, n)$ such that

$$
\sum_{V(U) \in B_{j}^{n}} \mu(V(U))^{\beta} \leq \sum_{V(U) \in B_{j(\beta, n)}^{n}} \mu(V(U))^{\beta} .
$$

We then have for any real number $\beta$,

$$
\begin{aligned}
\sum_{V(U) \in B_{j(\beta, n)}^{n}} \mu(V(U))^{\beta} & \leq \sum_{j} \sum_{V(U) \in B_{j}^{n}} \mu(V(U))^{\beta} \\
& =\sum_{V(U) \in A_{i(\beta, n)}^{n}} \mu(V(U))^{\beta} \\
& \leq(d-c) n \sum_{V(U) \in B_{j(\beta, n)}^{n}} \mu(V(U))^{\beta},
\end{aligned}
$$

which implies for any real number $\beta$,

$$
\frac{1}{n} \ln \left(\sum_{V(U) \in A_{i(\beta, n)}^{n}} \mu(V(U))^{\beta}\right) \sim \frac{1}{n} \ln \left(\sum_{V(U) \in B_{j(\beta, n)}^{n}} \mu(V(U))^{\beta}\right)
$$

Finally we have

$$
\begin{align*}
-\frac{1}{n} \ln _{b(n(\beta))} Z_{n}(\beta) & \sim-\frac{1}{n} \ln _{b(n(\beta))}\left(\sum_{V(U) \in B_{j(\beta, n)}^{n}} \mu(V(U))^{\beta}\right)  \tag{30}\\
& \sim \beta \frac{i(\beta, n)}{j(\beta, n)}+\frac{\ln \# B_{j(\beta, n)}^{n}}{j(\beta, n)} .
\end{align*}
$$

Notice that the set $B_{j(\beta, n)}^{n} \subset A_{i(\beta, n)}^{n}$ consists of elements of the partition $\mathcal{P}_{n}$ with "same" measure $\exp (i(\beta, n))$ and "same" length $\exp (j(\beta, n))$ $=b(n(\beta))^{-n}$ (in the order $\left.(1 / n) \ln \right)$, where $b(n(\beta))$ is the logarithmic basis in the expression of the free energy function (6),

$$
\left\{\begin{array}{l}
\mu(V(U)) \approx \exp (i(\beta, n)),  \tag{31}\\
|V(U)| \approx \exp (j(\beta, n)),
\end{array} \quad \text { for all } V(U) \in B_{j(\beta, n)}^{n}\right.
$$

In fact it is the set where the distribution of the mass $\mu(V(U))^{\beta}$ of the function is the largest, and this is where large deviations occur.

The aim is to determine the measures $\mu_{\beta}$ whose supports are the singularity sets $C_{\alpha}$. We consider for any real number $\beta$ the following probability measures

$$
\theta_{n}(\beta)=\frac{1}{\# B_{j(\beta, n)}^{n}} \sum_{V(U) \in B_{j(\beta, n)}^{n}} \delta_{y(U)} \quad \text { and } \quad \zeta_{n}(\beta)=\frac{1}{n} \sum_{j=0}^{n-1} T^{j} \theta_{n}(\beta)
$$

(We remark that a cluster point of the sequence $\left\{\zeta_{n}(\beta)\right\}_{n \geq 1}$ is $T$-invariant.)

By our assumptions, the following sequences take their values in compact sets

$$
\begin{gathered}
\frac{1}{n} \ln \# B_{j(\beta, n)}^{n} \in[-d,-c], \quad \frac{i(\beta, n)}{n} \in[a, b], \\
\frac{j(\beta, n)}{n} \in[c, d], \quad \zeta_{n}(\beta) \in M(\mathbf{K})
\end{gathered}
$$

Then there exists a sub-sequence $\left\{n_{k}\right\}_{k \geq 1}$, that we note for simplicity $\{m\}_{m \geq 1}(m=m(\beta))$, such that

$$
\left\{\begin{array}{l}
\frac{1}{m} \ln \# B_{j(\beta, m)_{m \rightarrow+\infty}^{m}}^{\longrightarrow} \gamma(\beta) \in[-d,-c]>0  \tag{32}\\
\frac{i(\beta, m)}{m} \underset{m \rightarrow+\infty}{\longrightarrow} \eta(\beta) \in[a, b]<0 \\
\frac{j(\beta, m)}{m} \underset{m \rightarrow+\infty}{\longrightarrow}-b(\beta) \in[c, d]<0 \\
\zeta_{m}(\beta) \in M(\mathbf{K})_{m \rightarrow+\infty}^{\longrightarrow} \zeta_{\beta} \in M_{T}(\mathbf{K})
\end{array}\right.
$$

We get finally with (30) for any real number $\beta$,

$$
\begin{equation*}
-\frac{1}{m} \ln _{b(m(\beta))} Z_{m}(\beta)=F_{m}(\beta) \underset{m \rightarrow+\infty}{\longrightarrow} \frac{-1}{b(\beta)}(\gamma(\beta)+\beta \eta(\beta)), \tag{33}
\end{equation*}
$$

where $\gamma(\beta)$ and $-\eta(\beta)$ represent entropies and $b(\beta)$ a Lyapunov exponent.

Consider the functional

$$
\begin{aligned}
I: M_{T}(\mathbf{K}) & \times \mathbb{R} \longrightarrow \mathbb{R} \\
\qquad(\rho, \beta) \longmapsto & \frac{h_{\rho}(T)+\beta \int_{\mathbf{K}} \xi d \rho}{\int_{\mathbf{K}} J d \rho} .
\end{aligned}
$$

We have the following fundamental result.

Lemma 1 ([Si1]). We have for any real number $\beta$,

$$
F(\beta)=\inf _{\rho \in M_{T}(\mathbf{K})}(I(\rho, \beta))=\inf _{\substack{\rho \in M_{T}(\mathbf{K}) \\ \rho \text { ergodic }}}(I(\rho, \beta)) .
$$

The proof is given in three steps (the three following expressions):

1) For all $\beta \in \mathbb{R}, \sup _{\substack{\rho \in M_{T}(\mathbf{K}) \\ \rho \text { ergodic }}}(-I(\rho, \beta))=\sup _{\rho \in M_{T}(\mathbf{K})}(-I(\rho, \beta))$.
2) For all $\beta \in \mathbb{R}, \underset{n \rightarrow+\infty}{\lim }-F_{n}(\beta) \geq \sup _{\substack{\rho \in M_{T}(\mathbf{K}) \\ \rho \text { ergodic }}}(-I(\rho, \beta))$.
3) For all $\beta \in \mathbb{R}, \varlimsup_{n \rightarrow+\infty}-F_{n}(\beta) \leq \sup _{\rho \in M_{T}(\mathbf{K})}(-I(\rho, \beta))$.

The functional $I$ is semicontinuous since the (entropy) map $\rho \longmapsto$ $h_{\rho}(T)$ is expanding, i.e. two orbits never stay $\varepsilon$-close. Its infimum is attained since $M_{T}(\mathbf{K})$ is a compact set. Since the ergodic measures are extremal and form a $G_{\delta}$ set in the convex set $M_{T}(\mathbf{K})$, we have the first equality. The two others are much harder to prove.

For the second step we consider an ergodic Borel probability measure $\rho \in M_{T}(\mathbf{K})$. The ergodic theorem implies that for $\rho$ allmost every $x$,

$$
\frac{1}{n} \sum_{j=0}^{n} \delta_{T^{j}(x)} \underset{n \rightarrow+\infty}{\longrightarrow} \rho
$$

We know that for $\bar{\rho}$ (where $\bar{\rho} \leftrightarrow \rho$ ) allmost cylinders the ergodic measure $\bar{\rho}$ satisfies: $\bar{\rho}\left(C_{n}(\underline{x})\right) \approx e^{-n h_{\bar{\rho}}(\sigma)}$ and $\left|C_{n}(\underline{x})\right| \approx e^{-n \chi_{\bar{\rho}}(\sigma)}$. For the elements of the Markov partition (which correspond on the dynamical system to the cylinders) $V(U) \in \mathcal{P}_{n}$, we have

$$
\rho(V(U)) \approx e^{-n h_{\rho}(T)} \quad \text { and } \quad|V(U)| \approx e^{-n \chi_{\rho}(T)}
$$

Using the sets $B_{j(\beta, n)}$ (29) we see that (31)

$$
\left\{\begin{array}{l}
\frac{i(\beta, n)}{n} \underset{n \rightarrow+\infty}{\longrightarrow} \int_{\mathbf{K}} \xi d \rho=-h_{\rho}(T) \\
\frac{j(\beta, n)}{n} \underset{n \rightarrow+\infty}{\longrightarrow} \int_{\mathbf{K}} J d \rho=-\chi_{\rho}(T)
\end{array}\right.
$$

According the Shannon-McMillan-Breiman theorem [DGS, p. 81] we define for $\varepsilon>0$ the set

$$
H_{(\beta, \rho, n, \varepsilon)}=\left\{V(U) \in \mathcal{P}_{n}:-n \chi_{\rho}(T)-\varepsilon<j(\beta, n)<-n \chi_{\rho}(T)+\varepsilon\right\}
$$

for which there exists an integer $N$ such that for any integer $n \geq N$, we get
$\rho\left(H_{(\beta, \rho, n, \varepsilon)}\right) \geq 1-\varepsilon \quad$ and $\quad \# H_{(\beta, \rho, n, \varepsilon)} \geq(1-\varepsilon) \exp \left(n\left(h_{\rho}(T)-\varepsilon\right)\right)$.
We get therefore for any real number $\beta$ and any element $V(U) \in$ $H_{(\beta, \rho, n, \varepsilon)}$,

$$
\rho(V(U))^{\beta} \geq \exp \left(\beta n\left(\int_{\mathbf{K}} \xi d \rho-\varepsilon\right)\right)
$$

( $\pm \varepsilon$ according to the sign of the real number $\beta$ ), which gives for any integer $n \geq 1$,

$$
\begin{aligned}
-F_{n}(\beta) & =\frac{1}{n} \ln _{b(n(\beta))} Z_{n}(\beta) \\
& \geq \frac{1}{n} \ln _{b(n(\beta))}\left(\sum_{V(U) \in H_{(\beta, \rho, n, \varepsilon)}} \rho(V(U))^{\beta}\right) \\
& \geq \frac{\ln \# H_{(\beta, \rho, n, \varepsilon)}}{\int_{\mathbf{K}}-J d \rho+\varepsilon}+\beta \frac{\int_{\mathbf{K}} \xi d \rho+\varepsilon}{\int_{\mathbf{K}}-J d \rho+\varepsilon} \\
& \geq \frac{h_{\rho}(T)+\beta \int_{\mathbf{K}} \xi d \rho-2 \varepsilon}{\int_{\mathbf{K}}-J d \rho+\varepsilon}
\end{aligned}
$$

which implies that

$$
\lim _{n \rightarrow+\infty}-F_{n}(\beta) \geq \frac{h_{\rho}(T)+\beta \int_{\mathbf{K}} \xi d \rho}{\int_{\mathbf{K}}-J d \rho}=-I(\rho, \beta),
$$

which ends the second step since the ergodic measure $\rho$ is arbitrary.

For the third step, using (23), (24) and (26), we compute for any real number $\beta$ the following integrals

$$
\left\{\begin{array}{l}
\int_{\mathbf{K}} J d \zeta_{m}(\beta)=\frac{1}{\# B_{j(\beta, m)}^{m}} \sum_{V(U) \in B_{j(\beta, m)}^{m}}(\frac{1}{m} \underbrace{\sum_{j=0}^{m-1} J\left(T^{j}(y(U))\right)}_{=\ln |V(U)|(23)}) \\
\int_{\mathbf{K}} \xi d \zeta_{m}(\beta)=\frac{1}{\# B_{j(\beta, m)}^{m}} \sum_{V(U) \in B_{j(\beta, m)}^{m}}(\frac{1}{m} \underbrace{\sum_{j=0}^{m-1} \xi\left(T^{j}(y(U))\right)}_{=\ln \mu(V(U))(24)})
\end{array}\right.
$$

Using (32) and (33) we have

$$
\left\{\begin{array}{l}
\frac{i(\beta, m)}{m} \underset{m \rightarrow+\infty}{\longrightarrow} \eta(\beta)=\int_{\mathbf{K}} \xi d \zeta_{\beta} \\
\frac{j(\beta, m)}{m}=-b(\beta, m) \underset{m \rightarrow+\infty}{\longrightarrow}-b(\beta)=\int_{\mathbf{K}} J d \zeta_{\beta}
\end{array}\right.
$$

We get finally for any real number $\beta$,

$$
\begin{equation*}
\frac{1}{m} \ln _{b(m(\beta))} Z_{m}(\beta)=-F_{m}(\beta) \underset{m \rightarrow+\infty}{\longrightarrow} \frac{\gamma(\beta)+\beta \int_{\mathbf{K}} \xi d \zeta_{\beta}}{\int_{\mathbf{K}}-J d \zeta_{\beta}} . \tag{34}
\end{equation*}
$$

In this expression we do not know the value $\gamma(\beta)$ which satisfies the following.

Lemma 2. For all $\beta \in \mathbb{R}, \gamma(\beta) \leq h_{\zeta_{\beta}}$.
This estimate uses a standard argument of Misiurewicz [DGS, p. 145].

It implies that (34) becomes for any real number $\beta$,

$$
-F_{m}(\beta) \leq\left(-I\left(\zeta_{\beta}, \beta\right)\right)
$$

which implies that

$$
-F_{m}(\beta) \leq \sup _{\rho \in M_{T}(\mathbf{K})}(-I(\rho, \beta))
$$

Remember that the sequence $\left\{-F_{m}(\beta)\right\}_{m \geq 1}$ is a subsequence (32), which implies that

$$
\varlimsup_{n \rightarrow+\infty}-F_{n}(\beta) \leq \sup _{\rho \in M_{T}(\mathbf{K})}(-I(\rho, \beta)),
$$

which ends the third step and the proof of Lemma 1.
By the same way we prove that for any pair $(x, y) \in \mathbb{R}^{2}$ we have (35)

$$
G_{\mathcal{D}}(x, y)=P(x \xi+y J)=\sup _{\rho \in M_{T}(\mathbf{K})}\left(h_{\rho}(T)+\int_{\mathbf{K}}(x \xi+y J) d \rho\right)
$$

This function is real analytic in both variables, and by the way it is computed we have

$$
\begin{equation*}
G_{D}(s, t)=\ln \phi(s, t) . \tag{35}
\end{equation*}
$$

Finally define the Gibbs measure $\mu_{\beta}$ associated to the potential $\xi_{\beta}=$ $\beta \xi-F(\beta) J$. We verify that we have for any real number $\beta$,

$$
\begin{equation*}
P\left(\xi_{\beta}\right)=P(\beta \xi-F(\beta) J)=\sup _{\rho \in M_{T}(\mathbf{K})}\left(h_{\rho}(T)+\int_{\mathbf{K}} \xi_{\beta} d \rho\right)=0 \tag{36}
\end{equation*}
$$

It implies that the unique measure which achieves the value 0 is the Gibbs measure $\mu_{\beta}$. Replacing this result in the expression of the free energy function, we obtain
(37) $F(\beta)=\inf _{\rho \in M_{T}(\mathbf{K})}\left(\frac{h_{\rho}(T)+\beta \int_{\mathbf{K}} \xi d \rho}{\int_{\mathbf{K}} J d \rho}\right)=\frac{h_{\mu_{\beta}}(T)+\beta \int_{\mathbf{K}} \xi d \mu_{\beta}}{\int_{\mathbf{K}} J d \mu_{\beta}}$,
for all $\beta \in \mathbb{R}$. Since we have for any real number $\beta$,

$$
\begin{equation*}
G_{\mathcal{D}}(\beta, \psi(\beta))=\ln \phi(\beta, \psi(\beta))=0=G_{\mathcal{D}}(\beta,-F(\beta)), \tag{38}
\end{equation*}
$$

we have $F=-\psi$, which ends the proof of Theorem A.
Since the pressure is differentiable (36), by differentiating the following expression

$$
P(\beta \xi-F(\beta) J)=0
$$

we get for any real number $\beta$ (22),

$$
\frac{\partial P}{\partial x}(\beta,-F(\beta))=\int_{\mathbf{K}} \xi d \mu_{\beta}<0
$$

and

$$
\frac{\partial P}{\partial y}(\beta,-F(\beta))=\int_{\mathbf{K}} J d \mu_{\beta}<0 .
$$

We then obtain for any real number $\beta$,

$$
\begin{equation*}
F^{\prime}(\beta)=\frac{\int_{\mathbf{K}} \xi d \mu_{\beta}}{\int_{\mathbf{K}} J d \mu_{\beta}}>0 \tag{39}
\end{equation*}
$$

Differentiating once more, we obtain for any real number $\beta[\mathrm{M}]$, $[\mathrm{Ma}]$, [R], [Si1],

$$
F^{\prime \prime}(\beta)=\frac{F^{\prime}(\beta)^{2}\left(\frac{\partial^{2} P}{\partial y^{2}}\right)-2 F^{\prime}(\beta)\left(\frac{\partial^{2} P}{\partial x \partial y}\right)+\left(\frac{\partial^{2} P}{\partial x^{2}}\right)}{\left(\frac{\partial P}{\partial x}\right)}(\beta,-F(\beta)) \leq 0
$$

We prove that $F^{\prime \prime}<0$ if and only if the functions $\xi$ et $J$ are not cohomologous to constants [ Ru ] (if not $F$ is linear).

Consider the Legendre-Fenchel transform of $F$ (5). Since $F$ is at least $C^{1}$ (it is real analytic) and according to the theory of conjugate functions [E], we have for the function $f$ and any real number $\beta$,

$$
f(\alpha)+F(\beta)=\alpha \beta \quad \text { if and only if } \quad\left\{\begin{array}{l}
\alpha=F^{\prime}(\beta)  \tag{40}\\
\beta=f^{\prime}(\alpha)
\end{array}\right.
$$

We then obtain (37) for any real number $\beta$,

$$
\begin{equation*}
f\left(F^{\prime}(\beta)\right)=\beta F^{\prime}(\beta)-F(\beta)=\frac{h_{\mu_{\beta}}(T)}{\chi_{\mu_{\beta}}(T)}=\frac{\int_{\mathbf{K}} \xi_{\beta} d \mu_{\beta}}{\int_{\mathbf{K}} J d \mu_{\beta}}=d_{\mu_{\beta}} \tag{41}
\end{equation*}
$$

In the degenerate case, the free energy function $F$ is linear $\beta \longmapsto d_{\mu}(\beta-$ 1 ), and the dimension spectrum $f \equiv d=d_{\mu}=\operatorname{HD}(\mu)$.

If not the free energy function is strictly increasing and strictly concave. This implies in particular that the dimension spectrum $f$ is real analytic on the interval $] \alpha_{\text {min }}, \alpha_{\text {max }}[$ where

$$
\left\{\begin{array}{l}
\alpha_{\min }=\inf _{\beta \in \mathbb{R}} F^{\prime}(\beta)=\lim _{\beta \rightarrow+\infty} F^{\prime}(\beta),  \tag{42}\\
\alpha_{\max }=\sup _{\beta \in \mathbb{R}} F^{\prime}(\beta)=\lim _{\beta \rightarrow-\infty} F^{\prime}(\beta),
\end{array}\right.
$$

and strictly concave since for any $\left.\alpha=F^{\prime}(\beta) \in\right] \alpha_{\min }, \alpha_{\max }[$,

$$
f^{\prime \prime}(\alpha)=\frac{1}{F^{\prime \prime}(\beta)}<0
$$

In the expression (32) and the existence of the limit $F(\beta)$, we have for any real number $\beta, \zeta_{\beta}=\mu_{\beta}$. The sets $B_{j(\beta, n)}^{n}$ from (29) cover at the limit the singularity set $C_{\alpha}$ where $\alpha=\alpha(\beta)=F^{\prime}(\beta)$ (see Section 4.1).

We can prove directly [CLP], [Si1] that $f(\alpha)=\operatorname{HD}\left(C_{\alpha}\right)$. Here we have parametrized all the fractal sets $\left\{C_{\alpha(\beta)}\right\}_{\beta \in \mathbb{R}}$, and we have associated to the Gibbs measure $\mu$ a family of Gibbs measures $\left\{\mu_{\beta}\right\}_{\beta \in \mathbb{R}}$ (respectively $\nu$ and the family $\left\{\nu_{\beta}\right\}_{\beta \in \mathbb{R}}$ ) where $\mu_{\beta}$ has the potential $\beta \xi-F(\beta) J \in C^{\delta}(\mathbf{K})$ (respectively $\beta \bar{\xi}-F(\beta) \bar{J} \in C^{\delta}(\mathbf{E})$ ).

Let $\mu_{-\infty}$ (respectively $\mu_{+\infty}$ ) be a cluster point of the $\mu_{\beta}$ when $\beta \longrightarrow-\infty$ (respectively $\beta \longrightarrow+\infty$ ) - respectively $\nu_{-\infty}$ and $\nu_{+\infty}$ in $M_{\sigma}(\mathbf{E})$. It is clear with (13) that we obtain the extremal points $\alpha_{ \pm \infty}$ given in (39) and the corresponding singularity sets $C_{\alpha_{ \pm \infty}}$. Remark that the way there are given they may be not well defined. But in Section 6 we see that they are uniquely determined.

We have thus proved Theorem B which contains Theorem 2 (Section 2).

Remarks. 1) We have:

- $F(0)=-\operatorname{HD}(\mathbf{K})=d ; f\left(F^{\prime}(0)\right)=\sup f(\alpha)=d$.
- $F(1)=0 ; f\left(F^{\prime}(1)\right)=F^{\prime}(1)$ and the tangeant of the graph $\alpha \longmapsto$ $f(\alpha)$ at the point $\alpha=F^{\prime}(1)=d_{\mu}$ is the line $y=x$. Moreover we have $\mu_{1}=\mu$.

2) For any $\beta \in \overline{\mathbb{R}}$ and $\alpha=F^{\prime}(\beta)$ we have $\mu_{\beta}\left(C_{\alpha}\right)=1$ (therefore the $\mu_{\beta}$ are singular to each other), the measure $\mu_{\beta}$ is exact dimensional since $d_{\mu_{\beta}}=\operatorname{HD}\left(\mu_{\beta}\right)=f(\alpha)$. The tangent of the graph $\alpha \longmapsto f(\alpha)$ at the point $\alpha=\alpha(\beta)=F^{\prime}(\beta)$ is the line $y=\beta x-F(\beta)$ (41). The measure $\mu$ is also exact dimensional since $\mu_{1}=\mu$.

## 5. Multifractal spectra of entropy and Lyapunov.

The multifractal spectra of entropy, (9) and (10), and Lyapunov exponents, (11) and (12), are given by the following.

Let us define (there are same values when using the subshift $\mathbf{E}=$ $\left.\Sigma_{A}^{+}\right)$

$$
\begin{align*}
& \eta_{\min }=\inf _{\beta \in \overline{\mathbb{R}}} \int_{\mathbf{K}}-\xi d \mu_{\beta}=\eta_{+\infty}, \\
& \eta_{\max }=\sup _{\beta \in \mathbb{\mathbb { R }}} \int_{\mathbf{K}}-\xi d \mu_{\beta}=\eta_{-\infty}, \\
& \vartheta_{\min }=\inf _{\beta \in \overline{\mathbb{R}}} \int_{\mathbf{K}}-J d \mu_{\beta}=\vartheta_{+\infty},  \tag{44}\\
& \vartheta_{\max }=\sup _{\beta \in \mathbb{R}} \int_{\mathbf{K}}-J d \mu_{\beta}=\vartheta_{-\infty} .
\end{align*}
$$

In the degenerate case for the dimension spectrum, the two spectra are simultaneously degenerate: hence the functions $\xi$ and $J$ are cohomologous to constants. In this situation the two intervals $\left[\eta_{\min }, \eta_{\max }\right]$ and [ $\left.\vartheta_{\min }, \vartheta_{\max }\right]$ are reduced to points $h_{\mu}$ and $\chi_{\mu}$.

Otherwise at least one of the two spectra is not degenerate. This means that at least two of the three spectra (plus dimension spectrum) are not degenerate, and therefore one of the functions $E_{n}$ (10) and $L_{y}$ (12) is real analytic on an open interval.

Proof of Theorem C. Suppose that for some $\eta \notin\left[\eta_{\text {min }}, \eta_{\text {max }}\right]$ we have $\mathbf{E}(\eta) \neq \varnothing(10)$. The concentration of the measures $\nu_{\bar{\xi}}$ and $\nu_{\bar{J}}$ are given on $\mathbf{E}$ by expansions of the type (26)

$$
\begin{equation*}
\sum_{j=0}^{n-1} \bar{\xi}\left(\sigma^{j}(\underline{x})\right) \quad \text { and } \quad \sum_{j=0}^{n-1} \bar{J}\left(\sigma^{j}(\underline{x})\right) . \tag{45}
\end{equation*}
$$

For any $\underline{x} \in \mathbf{E}(\eta)$ we have

$$
-\frac{1}{n} \sum_{j=0}^{n-1} \bar{\xi}\left(\sigma^{j}(\underline{x})\right) \underset{n \rightarrow+\infty}{\longrightarrow} \eta,
$$

and for any $\beta \in \overline{\mathbb{R}}, \nu_{\beta}(\mathbf{E}(\eta))=0$ since $\eta \notin\left[\eta_{\min }, \eta_{\max }\right]$ because the last expression converges to

$$
\int_{\mathbf{E}}-\bar{\xi} d \nu_{\beta} \in\left[\eta_{\min }, \eta_{\max }\right]
$$

We obtain in the same way the following convergence

$$
-\frac{1}{n} \sum_{j=0}^{n-1} \bar{J}\left(\sigma^{j}(\underline{x})\right) \underset{n \rightarrow+\infty}{\longrightarrow} \vartheta \notin\left[\vartheta_{\min }, \vartheta_{\max }\right]
$$

We have on a set $\Omega$ the existence of local dimension : for all $x \in \Omega$, $d_{\nu}(\underline{x})=\eta / \vartheta$. On the other hand we have for any $\beta \in \overline{\mathbb{R}}, \nu_{\beta}(\Omega)=0$ implies $\mathbf{E}(\eta) \subset\left\{x: d_{\nu}(\underline{x})\right.$ does not exist $\}$, which gives a contradiction.

In fact the sequences in (45) are in the domain of attraction of the measure $\nu_{\beta}$, and therefore we have

$$
(\eta, \vartheta)=\left(\int_{\mathbf{E}}-\bar{\xi} d \nu_{\beta}, \int_{\mathbf{E}}-\bar{J} d \nu_{\beta}\right) \quad\left(=\left(\int_{\mathbf{K}}-\xi d \mu_{\beta}, \int_{\mathbf{K}}-J d \mu_{\beta}\right)\right)
$$

Then we obtain for $\alpha=F^{\prime}(\beta)$ the spectra (10) and (12)

$$
E(\eta)=L(\vartheta)=C_{\alpha}
$$

and

$$
E_{n}(\eta)=L_{y}(\vartheta)=\operatorname{HD}\left(C_{\alpha}\right)=f(\alpha)=d_{\mu_{\beta}}
$$

which gives Theorem C.

## 6. Multi-multifractal, extremal measures and graphs.

In the multifractal analysis of a measure $\mu$ the support $\mathbf{K}$ is decomposed into fractal sets which represent the singularity sets (level sets for local dimension or other spectra) and of course the sets of points which do not have local dimension.

The idea for multi-multifractal analysis is to iterate infinitely this process and refine the decompositions. The interesting case is when the dimension spectrum is nondegenerate (if not all the spectra are degenerate and constants). We introduce multi-multifractal analysis for dimension, but notice that the constructions for the other spectra are similar.

In the nondegenerate case we define a set of Gibbs measures (we omit the measures $\mu_{ \pm \infty}$ since we show that they are uniform on their supports, and in particular $\left.\mu_{1}=\mu\right) M_{0}(\mu)=\left\{\mu_{\beta}\right\}_{\beta \in \mathbb{R}}$ where the singularity sets $C_{\alpha}$ satisfy for $\alpha=F^{\prime}(\beta)$,

$$
\mu_{\beta}\left(C_{\alpha}\right)=1 \quad \text { and } \quad \operatorname{HD}\left(C_{\alpha}\right)=f(\alpha)=\operatorname{HD}\left(\mu_{\beta}\right)=d_{\mu_{\beta}}
$$

Then multifractal analysis can be represented by the triple $(\mu, F$, $\left.M_{0}(\mu)\right)$ 。

In fact it is possible to define many infinite sequences of multifractal spectra. Let us describe the second step.

First fix $\beta \in \mathbb{R} \backslash\{1\}$ and realize the multifractal analysis for the measure $\mu_{\beta}$. Define for $(\rho, \tau) \in M_{T}(K) \times \mathbb{R}$,
$I_{1}(\rho, \tau)=\frac{h_{\rho}(T)+\tau \int_{\mathbf{K}} \xi_{\beta} d \rho}{\int_{\mathbf{K}} J d \rho} \quad$ and $\quad F_{1}(\rho, \tau)=\inf _{\rho \in M_{T}(\mathbf{K})}\left(I_{1}(\rho, \tau)\right)$.
We have the following:

- at the first step: $\mu=\mu_{\xi} \longleftrightarrow \beta \in \mathbb{R}, F(\beta)=I\left(\mu_{\beta}, \beta\right) \longleftrightarrow \xi_{\beta}=$ $\beta \xi-F(\beta) J, \mu_{\beta}=\mu_{\xi_{\beta}} \longleftrightarrow f(\alpha)=d_{\mu_{\beta}}$ for $\alpha=F^{\prime}(\beta)$,
- at the second step: $\mu_{\beta}=\mu_{\xi_{\beta}} \longleftrightarrow \tau \in \mathbb{R}, F_{1}(\tau)=I_{1}\left(\mu_{\beta, \tau}, \tau\right) \longleftrightarrow$ $\zeta_{\tau}=\tau \xi_{\beta}-F_{1}(\tau) J, \mu_{\beta, \tau}=\mu_{\zeta_{\tau}} \longleftrightarrow f_{1}(\alpha)=d_{\mu_{\beta, \tau}}=\operatorname{HD}\left(C_{\beta, \alpha}\right)$ for $\alpha=F_{1}^{\prime}(\beta)$ and $C_{\beta, \alpha}=\left\{x \in K: d_{\mu_{\beta}}(x)=\alpha\right\}$. If $M_{1}\left(\mu_{\beta}\right)=\left\{\mu_{\beta, \tau}\right\}_{\tau \in \mathbb{R}}$, we have then defined a new triple $\left(\mu_{\beta}, F_{1}, M_{1}\left(\mu_{\beta}\right)\right)$.

We can iterate this construction step by step at any level.
Suppose that multifractal analysis has been defined at level $n$. We have then for $\left(\beta_{1}, \ldots, \beta_{n-1}\right) \in \mathbb{R}^{n-1}$ a triple

$$
\left(\mu_{\beta_{1}, \ldots, \beta_{n-1}}, F_{\beta_{1}, \ldots, \beta_{n-1}},\left\{\mu_{\beta_{1}, \ldots, \beta_{n-1}, \beta}\right\}_{\beta \in \mathbb{R}}\right)
$$

and

$$
\begin{aligned}
\mu_{\beta_{1}, \ldots, \beta_{n-1}}= & \mu_{\xi_{\beta_{1}, \ldots, \beta_{n-1}}} \\
& \longleftrightarrow \beta_{n} \in \mathbb{R}, F_{\beta_{1}, \ldots, \beta_{n-1}}\left(\beta_{n}\right) \\
& \longleftrightarrow \beta_{n} \in \mathbb{R}, \mu_{\beta_{1}, \ldots, \beta_{n-1}, \beta_{n}}, \mu_{\xi_{\beta_{1}, \ldots, \beta_{n-1}, \beta_{n}}} \\
& \longleftrightarrow f(\alpha)=\operatorname{HD}\left(C_{\beta_{1}, \ldots, \beta_{n-1}, \alpha}\right)=\operatorname{HD}\left(\mu_{\beta_{1}, \ldots, \beta_{n-1}, \beta_{n}}\right)
\end{aligned}
$$

for $\alpha=F_{\beta_{1}, \ldots, \beta_{n-1}}^{\prime}\left(\beta_{n}\right)$ where we have

$$
C_{\beta_{1}, \ldots, \beta_{n-1}, \alpha}=\left\{x: \frac{\ln \mu_{\beta_{1}, \ldots, \beta_{n-1}}(B(x, r))}{\ln r} \underset{r \rightarrow 0}{\longrightarrow} \alpha\right\} .
$$

We have then defined a new triple

$$
\left(\mu_{\beta_{1}, \ldots, \beta_{n-1}, \beta_{n}}, F_{\beta_{1}, \ldots, \beta_{n-1}, \beta_{n}},\left\{\mu_{\beta_{1}, \ldots, \beta_{n-1}, \beta_{n}, \beta}\right\}_{\beta \in \mathbb{R}}\right),
$$

where we omit the two extremal measures $\mu_{\beta_{1}, \ldots, \beta_{n-1}, \beta_{n}, \pm \infty}$.
If at the first level the spectrum is nondegenerate, then it is nondegenerate at any level. We have seen that it is degenerate at the first level if and only if the two potentials $\xi$ and $J$ are cohomologous to constants. Since at any level it is a linear combination of the functions $\xi$ and $J$ it is never degenerate.

Concerning local Lyapunov exponents this is the same behaviour than for dimension. If the multi-multifractal spectrum is nondegenerate at the first step ( $J$ is not cohomologous to a constant), then it is not degenerate at any step.

The behaviour for local entropies is different. For example at the first level it may be degenerate ( $\xi$ is cohomologous to a constant), but at the second level it may be not since for any real number $\beta \neq 1$, $\xi_{\beta}=\beta \xi-F(\beta) J$ is not cohomologous to a constant, and in fact it is not at any further level.

We omit at each step the extremal measures $\mu_{\beta_{1}, \ldots, \beta_{n-1}, \beta_{n}, \pm \infty}$ obtained at the limits when $|\beta|$ goes to $+\infty$. In fact at any level these measures are uniform on their supports and then imply degenerate spectra.

We will see it on a very simple example on the unit interval, namely a linear Markov map modeled by the full shift on 3 symbols.

Let us describe this dynamical system by the following simple model.


Figure 1. The measure $\mu$ given by $p_{1}+2 p_{0}=1$.
(for example $p_{0}=0.3$ and $p_{1}=0.4$ ), and


Figure 2. The measure $\mu$ given at the second step (and so on...).
In the computation of the partition functions (7), the different sets $B_{j(\beta, n)}^{n}$ that are selected (29) $\left(=A_{j(\beta, n)}^{n}(27)\right.$ since $J$ is constant: the partitions are uniform, $|V(U)|=3^{-n}$ ) when $\beta \longrightarrow+\infty$, are in fact the intervals where the distribution of the mass $\mu(V(U))$ is the largest. They are actually the central intervals $\left[1 / 2-1 /\left(2 \cdot 3^{n}\right), 1 / 2+1 /\left(2 \cdot 3^{n}\right)\right]$ of measure $p_{1}^{n}$ which covers at the limit the set $\{1 / 2\}$. We have then $\mu_{+\infty}=\delta_{1 / 2}$ and $d_{\mu_{+\infty}}=0$.

When $\beta \longrightarrow-\infty$, it is the set of intervals where the distribution of the mass $\mu(V(U))$ is the smallest. In fact we select the sets

$$
\bigcup_{k=0}^{3^{n-1}-1}\left(\left[\frac{3 k}{3^{n}}, \frac{3 k+1}{3^{n}}\right] \cup\left[\frac{3 k+2}{3^{n}}, \frac{3 k+3}{3^{n}}\right]\right)
$$

composed of $2^{n}$ intervals of measure $p_{0}^{n}$, which cover at the limit the tryadic Cantor set. We obtain therefore that $\mu_{-\infty}$ is the uniform measure on the Cantor set for which the dimension spectrum is degenerate at the point $d_{\mu_{-\infty}}=\ln 2 / \ln 3$.

The multifractal analysis implies the following results.

1) $\operatorname{HD}\left(\left\{x: d_{\mu}(x)\right.\right.$ does not exists $\left.\}\right)=1$. This set contains for example the set of points obtained by iterations of the boundaries: for these special points we have

$$
\underline{d}_{\mu}(x)=\alpha_{+\infty}=-\frac{\ln p_{1}}{\ln 3} \quad \text { and } \quad \bar{d}_{\mu}(x)=\alpha_{-\infty}=-\frac{\ln p_{0}}{\ln 3} .
$$

In higher dimension $n \geq 2$, this set contains iterates of the boundaries of the Markov partition (countable in dimension 1) and then has Hausdorff dimension greater or equal to 1 (equal to $n$ in general).
2) The dimension spectrum is real analytic on the interval $] \alpha_{+\infty}$, $\alpha_{-\infty}$ [.
3) For all $\beta \in \mathbb{R}, \mu_{\beta}$ is exact dimensional and $d_{\mu_{\beta}}=\beta F^{\prime}(\beta)-F(\beta)$ where we have

$$
F(\beta)=-\frac{\ln \left(2 p_{0}^{\beta}+p_{1}^{\beta}\right)}{\ln 3} \quad \text { and } \quad F^{\prime}(\beta)=-\frac{2 p_{0}^{\beta} \ln p_{0}+p_{1}^{\beta} \ln p_{1}}{\left(2 p_{0}^{\beta}+p_{1}^{\beta}\right) \ln 3}
$$

For $\beta=1, \mu_{1}=\mu$ is exact dimensional.
We see that the extremal measures $\mu_{ \pm \infty}$ are uniform measures on their supports. This phenomenon seems to be general, and it is quite clear for linear Markov maps equipped with Gibbs measures. The next step is for subshifts of finite type where things are more complicated (case of the digraph recursive fractals) in the nondegenerate case.

We have seen in (29) that for any real number $\beta$ the set $B_{j(\beta, n)}^{n}$ consists of elements of the Markov partition $\mathcal{P}_{n}$ (of "same" measure $\exp (i(\beta, n))$ and "same" length $\exp (j(\beta, n)) \sim b(\beta, n)^{-n}$ in the order $(1 / n) \ln$ ) indicates at the step $n$ the distribution of the mass $\mu(V(U))^{\beta}$ of the partition function (7) and where the large deviations occur (6).

In the order $(1 / n) \ln$ some small variations for the mass of the elements of $B_{j(\beta, n)}^{n}$ occur which imply the multifractality of the measure $\mu_{\beta}$ (multi-multifractality at the second level).

The situation is different for the extremal measures $\mu_{ \pm \infty}$ given by the limits of the measures $\mu_{\beta}$ when $|\beta| \longrightarrow+\infty$.

For the measure $\mu_{-\infty}$ the elements of $\mathcal{P}_{n}$ which cover at the limit the set $\mathbf{K}^{(-\infty)}$ are those which satisfy the following:

$$
0<\mu(V(U))=\min _{V(U) \in \mathcal{P}_{n}} \mu(V(U)) .
$$

In the same way, for the measure $\mu_{+\infty}$ the elements of $\mathcal{P}_{n}$ which cover at the limit the set $\mathbf{K}^{(+\infty)}$ are those which satisfy the following

$$
0<\mu(V(U))=\max _{V(U) \in \mathcal{P}_{n}} \mu(V(U))
$$

In our example these sets are respectively the $2^{n}$ intervals of measures $p_{0}^{n}$ and the central intervals of measures $p_{1}^{n}$.

Therefore if we want to realize the multi-multifractality analysis of the measures $\mu_{ \pm \infty}$ at the second level, we get for example $\mu_{-\infty,-\infty}=$ $\mu_{-\infty}$ and $\mu_{+\infty,+\infty}=\mu_{+\infty}$ and finally for any $\tau \in \overline{\mathbb{R}}, \mu_{-\infty, \tau}=\mu_{-\infty}$ and $\mu_{+\infty, \tau}=\mu_{+\infty}$. This gives Theorem D.

Here we present the different graphs of the functions we have studied for the particular values: $p_{0}=0.3$ and $p_{1}=0.4$ : Figure 3: the function $F$; Figure 4: the derivative $F^{\prime}$; Figure 5: the function which represents the distribution of $\beta \longmapsto d_{\mu_{\beta}}$; (see (41); Figure 6: the dimension spectrum : $\alpha \longmapsto f(\alpha)$.


Figure 3. The free energy function $F: \mathbb{R} \longrightarrow \mathbb{R}, \beta \longmapsto F(\beta)$.


Figure 4. The derivative of the free energy function

$$
\left.F^{\prime}: \mathbb{R} \longrightarrow\right] \alpha_{+\infty}, \alpha_{-\infty}\left[, \beta \longmapsto F^{\prime}(\beta) .\right.
$$



Figure 5. The parametrized dimension spectrum

$$
\left.\left.f_{\beta}: \mathbb{R} \longrightarrow\right] 0,1\right], \beta \longmapsto \beta F^{\prime}(\beta)-F(\beta) .
$$

$$
\operatorname{HD}(\operatorname{supp} \mu)=1
$$

$$
\mathrm{HD}(\mu)=d_{\mu}=\alpha_{1}
$$



Figure 6. The dimension spectrum $f:\left[\alpha_{+\infty}, \alpha_{-\infty}\right] \longrightarrow[0,1], \alpha \longmapsto f(\alpha)$.

## 7. Discussion and questions.

We may summarize the differents results concerning the measure $\mu$. The measure $\mu$ is exact dimensional, i.e. $d_{\mu}(x)=h_{\mu} / \chi_{\mu} \mu$ almost everywhere, although we have the following

$$
\operatorname{HD}\left(\left\{x: d_{\mu}(x) \text { does not exist }\right\}\right)=n .
$$

For $\alpha=F^{\prime}(1)=d_{\mu}$, we have $\mu\left(\mathbf{K}^{(\alpha)}\right)=1$ which gives the completness of the measure.

There are limiting constructions for the $\mathbf{K}^{(\alpha)}$ when $\alpha \longrightarrow \alpha_{ \pm \infty}$. The sets $\mathbf{K}^{\left(\alpha_{ \pm \infty}\right)}$ are the supports of the measures $\mu_{ \pm \infty}$ which are uniform on their supports. Therefore their multifractal and multimultifractal are reduced to points

$$
d_{\mu_{-\infty}}=\frac{h_{\mu_{-\infty}}}{\chi_{\mu_{-\infty}}} \quad \text { and } \quad d_{\mu_{+\infty}}=\frac{h_{\mu_{+\infty}}}{\chi_{\mu_{+\infty}}} .
$$

The disjointness conditions on the sets $J(\gamma)$ are those for Markov partitions, i.e. the interiors are disjoints and they intersect at most at their boundaries which are of measure 0 for any Gibbs measure. Like for the example, all the points on the boundaries belong to the set

$$
\left\{x: d_{\mu}(x) \text { does not exist }\right\}
$$

which is not countable in dimension geater or equal to 2 .
If the graph is not strongly connected, we analyse all the strongly connected components of the graph, i.e. if the matrix $A$ (see Section 3.1 ) is not reducible, we decompose it into irreducible components.

To each irreducible component $A_{1 \leq j \leq p}^{(j)}$ we associate in the same fashion as in the digraph recursive fractal sets the singularity sets and the different dimension spectra which may or not intersect with the others. For any value $\alpha \in\left[\alpha_{\min }, \alpha_{\max }\right]$, there are at most $p$ different singularity sets where $C_{\alpha}^{(j)}=\left\{x: d_{\mu}(x)=\alpha\right\}$ (which may be $=\varnothing$ ), and therefore we define

$$
\begin{aligned}
f(\alpha) & =\max _{1 \leq j \leq p} \operatorname{HD}\left(C_{\alpha}^{(j)}\right) \quad(\equiv-\infty \text { if all the singularity sets are } \varnothing) \\
& =\max _{1 \leq j \leq p} f^{(j)}(\alpha)
\end{aligned}
$$

where $f^{(j)}$ is the dimension spectrum of the measure $\mu$ restricted to the set generated by the j -th strongly connected component.

The result means that we get for any positive real number $\alpha, f(\alpha)$ to be the greatest Hausdorff dimension of the singularity sets $C_{\alpha}^{(j)}$ (since we have the following: $\operatorname{HD}(E \cup F)=\max \{\operatorname{HD}(E), \operatorname{HD}(F)\})$.

We have seen in (43) in the nondegenerate case that $F^{\prime \prime}<0$ (if and only if the Hölder continuous functions $\xi$ and $J$ are not cohomologous to constants), and we get finally that $f^{\prime \prime}<0$ on $] \alpha_{\text {min }}, \alpha_{\max }$ [ since we have $f^{\prime \prime}(\alpha)=1 / F^{\prime \prime}(\beta)$. Then we have for any real number $\beta \in \mathbb{R}$ and $\alpha \in] \alpha_{\min }, \alpha_{\max }\left[, F^{\prime \prime}(\beta)<0\right.$ and $f^{\prime \prime}(\alpha)<0$, and the value 0 is never achieved.

The challenging question at this moment comes from the concept of rigidity and the conjecture that the dimension spectrum is an invariant for dynamical systems modeled by subshifts of finite type.

Rigidity deals with an important problem which is to know if we can restore the dynamics of a dynamical system by recovering information from the different spectra. The aim is to obtain a physical classification of dynamical systems given by maps and Gibbs measures.

Let $(X, \mu, T)$ and $(Y, \rho, S)$ be two topologically equivalent dynamical systems, i.e. there exists a homeomorphism $h: X \longrightarrow Y$. The problem is to know if some of their multifractal spectra coincide then they are smoothly equivalent and $h$ is a diffeomorphism. If there exists a topological conjugacy $\zeta$ between $T$ and $S$, we want to find in all the class of conjugacies a homeomorphism $\phi$ preserving the differentiable structure, $T=S \circ \phi$, and also measure preserving, $\mu=\rho \circ \phi$.

This has been proved in [BPS2] in a very particular case, namely one-dimensional (and two-dimensional) linear Markov maps of $[0,1]$ (or $[0,1]^{2}$ ) modeled by the full shift on two symbols (where all the things work). We believe that this assertion is true for linear Markov maps of the unit interval (or $[0,1]^{2}$ ) modeled by the full shift on $p \geq 2$ symbols. The generalization of this statement will be for arbitrary subshifts of finite type $\Sigma_{A}^{+}$.

We believe that multifractal dimension spectrum is only needed to recover information, but if necessary one can use multi-multifractal analysis.

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# On certain Markov processes attached to exponential functionals of Brownian motion; application to Asian options 

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#### Abstract

We obtain a closed formula for the Laplace transform of the first moment of certain exponential functionals of Brownian motion with drift, which gives the price of Asian options. The proof relies on an identity in law between the average on $[0, t]$ of a geometric Brownian motion and the value at time $t$ of a Markov process, for which we can compute explicitly the resolvent.


## 1. Introduction.

The aim of this paper is two-fold:
i) We take up the computation of the value of a continuously averaged Asian option in a Black-Scholes setting, with initial price normalized to 1 , at maturity date $t$, and strike $k$, i.e. $E\left[\left(A_{t}^{a, b}-k\right)^{+}\right]$where $A_{t}^{a, b}=\int_{0}^{t} d s \exp \left(a B_{s}+b s\right)$ and $\left(B_{s} ; s \geq 0\right)$ denotes a one dimensional Brownian motion. The computation of the general price, which involves $r$ the instantaneous risk-free rate easily reduces to the previous computation (for details, see [4, p. 354-355]).

However, our approach in the present paper is very different from
that in Geman-Yor [3], [4], or Yor [19], which relies very heavily upon Lamperti's representation

$$
\begin{equation*}
\exp \left(B_{t}+\nu t\right)=R^{(\nu)}\left(A_{t}^{(\nu)}\right), \quad t \geq 0 \tag{1.1}
\end{equation*}
$$

where $A_{t}^{(\nu)}=\int_{0}^{t} d s \exp \left(2\left(B_{s}+\nu s\right)\right)$, and $\left(R^{(\nu)}(u) ; u \geq 0\right)$ denotes a Bessel process with index $\nu$.

In the present paper, rather than relying on (1.1), we shall use the following remark

$$
\begin{equation*}
A_{t}^{a, b} \stackrel{(\mathrm{law})}{=} \exp \left(a B_{t}+b t\right) \int_{0}^{t} d s \exp \left(-\left(a B_{s}+b s\right)\right), \tag{1.2}
\end{equation*}
$$

for any fixed $t \geq 0$, which is a very particular case of a general identity in law involving the generalized Ornstein-Uhlenbeck processes discussed in Section 2 below.

An important fact is that the right-hand side of (1.2) defines the value at time $t$ of a Markov process $\left(Y_{t}^{a, b} ; t \geq 0\right)$. This remark being made, we write

$$
\begin{equation*}
E\left[\left(A_{t}^{a, b}-k\right)^{+}\right]=E\left[\left(Y_{t}^{a, b}-k\right)^{+}\right] \tag{1.3}
\end{equation*}
$$

and we develop the right-hand side of (1.3) using Itô-Tanaka formula.
It turns out than we can compute explicitly the density of the resolvent of $Y^{a, b}$, so that, finally, we obtain another derivation of the main results of Geman-Yor (see [3], [19]).
ii) The second aim of this paper is to present, throughout the text, a more complete view of the bibliography about exponential functionals of Lévy processes than in the Monograph [21]; the incompleteness of the bibliography in [21] is the sole responsability of the third author of the present paper. In particular, we refer to Urbanik [16], [17] for the study of the law of $\int_{0}^{\infty} \exp \left(-u X_{t}\right) d t$ for a positive Lévy process $X$ and to Paulsen [14] and co-authors ([13], [5]) for computations of the laws of randomly discounted integrals $\int_{0}^{\infty} \exp \left(-X_{t}\right) d P_{t}$, where $X$ and $P$ are two independent Lévy processes.

Concerning the price of Asian options, we also mention the work of Rogers-Shi [15] which gives interesting lower and upper bounds for the price.

In a different direction, Leblanc's work [12] deals with the joint law of

$$
\left(\exp \left(B_{t}+\nu t\right), \int_{0}^{t} \exp \left(B_{S}+\nu s\right) d s, \int_{0}^{t} \exp 2\left(B_{s}+\nu s\right) d s\right) .
$$

We now hope that, together with the references found in [21] concerning exponential functionals of Lévy processes, this paper offers a more reasonably complete bibliography (than in [21]). Needless to say, any new omission of relevant work is not intended.

## 2. On generalized Ornstein-Uhlenbeck processes.

It is a remarkable fact that, if $\left\{\left(\xi_{t}, \eta_{t}\right) ; t \geq 0\right\}$ is a two-dimensional Lévy process with respect to a filtration $\left(\mathcal{F}_{t}\right)$, then the process defined by

$$
\begin{equation*}
X_{t}=\exp \left(\xi_{t}\right)\left(x+\int_{0}^{t} \exp \left(-\xi_{s^{-}}\right) d \eta_{s}\right) \tag{2.1}
\end{equation*}
$$

is also a Markov process. Cases of particular interest involve independent $\xi$ and $\eta$, but this independence hypothesis is not necessary.

Some of these processes have been studied in the literature. The case where $\xi_{s}=\lambda s$ and $\eta$ is a Brownian motion gives the usual Orn-stein-Uhlenbeck process of parameter $\lambda$. Hadjiev [7] considers the case where $\xi_{s}=\lambda s$ and $\eta$ is a Lévy process without positive jumps and determines the distribution of the hitting times for $X$. Gravereaux [6] studies the case where $\eta$ is a $d$-dimensional Lévy process and $\xi_{s}=s h$ where $h$ is a linear map on $\mathbb{R}^{d}$, and looks for the existence of an invariant measure. We also refer to Jurek [9] for the condition on $\eta$ insuring the existence of $\int_{0}^{\infty} \exp (\lambda s) d \eta_{s}$ for $\lambda<0$ and to Jacod [8] for the study of (2.1) when the initial condition $x$ is replaced by an anticipating random variable.

Yor [22] considers the process $X$ for $\xi$ and $\eta$ two independent Brownian motions with respective drifts $\mu$ and $\nu$ and deduces from Proposition 2.1 below the law of a subordinated perpetuity, a result already obtained by Paulsen [14] by a different method.

We also mention the work of de Haan-Karandikar [10] where the Markov process $X$ appears as the solution of a "SDE" of the form

$$
X_{t}=A_{t}^{s} X_{s}+B_{t}^{s}, \quad s \leq t
$$

for random variables $\left\{A_{t}^{s}, B_{t}^{s} ; s \leq t\right\}$ satisfying compatibility conditions and certain independence and stationarity properties.

These Markov processes are related to exponential functionals of Lévy processes via the following:

Proposition 2.1. Let $\xi$ and $\eta$ be two independent Lévy processes, then for fixed $t$,

$$
\left.\left.\begin{array}{rl}
\left(\exp \left(\xi_{t}\right), \exp \left(\xi_{t}\right)\right. & \int_{0}^{t}
\end{array}\right) \exp \left(-\xi_{s^{-}}\right) d \eta_{s}\right) .
$$

This identity follows from the invariance by time reversal of the distribution of a Lévy process. We refer to Carmona-Petit-Yor in [21] for applications of this result.

Corollary 2.1. Consider the Lévy process

$$
\xi_{t}=-\left(c t+\sigma B_{t}+\tau_{t}^{+}-\tau_{t}^{-}\right),
$$

where ( $\tau_{t}^{ \pm} ; t \geq 0$ ) are subordinators without drift and Lévy measures $\nu_{ \pm} ; B$ is a Brownian motion and the processes $B, \tau^{+}, \tau^{-}$are independent. We denote by $\Phi(\lambda)$ the Lévy exponent of $\xi$ determined by $E\left(\exp \left(\lambda \xi_{t}\right)\right)=\exp (-t \Phi(\lambda))$.

Let

$$
A_{t}=\int_{0}^{t} \exp \left(\xi_{s}\right) d s
$$

and

$$
X_{t}=\exp \left(\xi_{t}\right) \int_{0}^{t} \exp \left(-\xi_{s}\right) d s
$$

$T_{\alpha}$ denotes an exponential variable of parameter $\alpha$, independent of $\xi$.
i) The law $\mu_{\alpha}$ of $A_{T_{\alpha}}$ satisfies

$$
\begin{equation*}
\mu_{\alpha}=\frac{1}{\alpha} L^{*} \mu_{\alpha}, \quad \text { on }(0, \infty) \tag{2.3}
\end{equation*}
$$

where $L$ denotes the infinitesimal generator of the Markov process $X$.
ii) In particular, the moments of $A_{T_{\alpha}}$ satisfy

$$
\begin{equation*}
E\left[A_{T_{\alpha}}^{m}\right]=\frac{m}{\alpha+\Phi(m)} E\left[A_{T_{\alpha}}^{m-1}\right], \tag{2.4}
\end{equation*}
$$

for $m>0$ and $\alpha+\Phi(m)>0$.
iii) Finally, if $A_{\infty}<\infty$ almost surely, then $\mu_{0}$, the law of $A_{\infty}$, solves $L^{*} \mu_{0}=0$.

Proof. i) From Proposition 2.1, for $f \in \operatorname{Dom}(L)$,

$$
\begin{aligned}
E\left[f\left(A_{t}\right)\right] & =E\left[f\left(X_{t}\right)\right] \\
& =f(0)+E\left[\int_{0}^{t} L f\left(X_{s}\right) d s\right] \\
& =f(0)+\int_{0}^{t} E\left[L f\left(A_{s}\right)\right] d s
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E\left[f\left(A_{T_{\alpha}}\right)\right] & =f(0)+\alpha \int_{0}^{\infty} d t \exp (-\alpha t) \int_{0}^{t} E\left[L f\left(A_{s}\right)\right] d s \\
& =f(0)+\int_{0}^{\infty} d t \exp (-\alpha t) E\left[L f\left(A_{t}\right)\right]=\frac{1}{\alpha} E\left[L f\left(A_{T_{\alpha}}\right)\right]
\end{aligned}
$$

proving (2.3), as we restrict $f$ to $C_{K}^{\infty}((0, \infty))$, and use integration by parts.
ii) (2.4) has already been obtained by Carmona-Petit-Yor in [21]. We give another proof relying on (2.3).

The generator $L$ of $X$ is given by (see [21, p. 81])

$$
\begin{aligned}
L f(x)= & \frac{\sigma^{2}}{2} x^{2} f^{\prime \prime}(x)+\left(\left(\frac{\sigma^{2}}{2}-c\right) x+1\right) f^{\prime}(x) \\
& -\int_{0}^{x} f^{\prime}(u) \bar{\nu}_{+}\left(\ln \left(\frac{x}{u}\right)\right) d u+\int_{x}^{\infty} f^{\prime}(u) \bar{\nu}_{-}\left(\ln \left(\frac{u}{x}\right)\right) d u
\end{aligned}
$$

where $\bar{\nu}$ is the tail of $\nu$. (We point out a misprint in the formula given in [21, p. 81] where the sign minus before the coefficient of $f^{\prime \prime}$ must be deleted).

An easy computation shows that if $f_{m}(x):=x^{m}, m>0$, then

$$
\begin{aligned}
L f_{m}(x)= & \left(\frac{\sigma^{2}}{2} m^{2}-c m+\int_{0}^{\infty}(\exp (-m z)-1) \bar{\nu}_{+}(d z)\right. \\
& \left.+\int_{0}^{\infty}(\exp (m z)-1) \bar{\nu}_{-}(d z)\right) f_{m}(x)+m f_{m-1}(x) \\
= & -\Phi(m) f_{m}(x)+m f_{m-1}(x)
\end{aligned}
$$

Thus, from (2.3),

$$
E\left[A_{T_{\alpha}}^{m}\right]=\frac{1}{\alpha} E\left[m A_{T_{\alpha}}^{m-1}-\Phi(m) A_{T_{\alpha}}^{m}\right]
$$

hence

$$
E\left[A_{T_{\alpha}}^{m}\right]=\frac{m}{\alpha+\Phi(m)} E\left[A_{T_{\alpha}}^{m-1}\right]
$$

iii) It suffices to multiply both sides of (2.3) by $\alpha$ and to let $\alpha$ converge to 0 .
3. Application to the computation of the price of Asian options.

We take up (1.3) again, i.e.

$$
E\left[\left(A_{t}^{a, b}-k\right)^{+}\right]=E\left[\left(Y_{t}^{a, b}-k\right)^{+}\right]
$$

where

$$
Y_{t}^{a, b}(x)=\exp \left(a B_{t}+b t\right)\left(x+\int_{0}^{t} d s \exp \left(-\left(a B_{s}+b s\right)\right)\right)
$$

is a Markov process and we write simply $Y_{t}^{a, b}$ for $Y_{t}^{a, b}(0)$. It may be worth mentioning that these processes come up as an important example throughout Arnold's book [1].

Proposition 3.1. The process $Y_{t}^{a, b}(x)$ is the solution of the equation

$$
Y_{t}=x+a \int_{0}^{t} d B_{u} Y_{u}+\int_{0}^{t} d u\left(\left(\frac{a^{2}}{2}+b\right) Y_{u}+1\right) .
$$

Proof. This is immediate, using Itô's formula, and Fubini's theorem.
By scaling, we may and we shall assume that $a^{2} / 2=1$ and we set $Y_{\nu}=Y^{\sqrt{2}, \nu}$.

Theorem 3.1. We denote by $U_{\alpha}$ the resolvent of the Markov process $Y_{\nu}$, that is

$$
U_{\alpha} f(x)=\int_{0}^{\infty} \exp (-\alpha t) E_{x}\left[f\left(Y_{\nu}(t)\right)\right] d t
$$

Then, the resolvent $U_{\alpha}$ admits a density which is given by

$$
\begin{align*}
& u_{\alpha}(x, y)=\frac{\Gamma\left(\frac{\nu+\mu}{2}\right)}{\Gamma(1+\mu)} \\
& \left(\mathbf{1}_{[0, x[ }(y)\left(\frac{1}{y}\right)^{1-\nu} \exp \left(-\frac{1}{y}\right) \varphi_{1}(x) \varphi_{2}(y)\right.  \tag{3.1}\\
& \\
& \left.\quad+\mathbf{1}_{[x, \infty[ }(y)\left(\frac{1}{y}\right)^{1-\nu} \exp \left(-\frac{1}{y}\right) \varphi_{2}(x) \varphi_{1}(y)\right),
\end{align*}
$$

for $x, y>0$ where

$$
\begin{align*}
& \varphi_{1}(x)=\left(\frac{1}{x}\right)^{(\nu+\mu) / 2} \Phi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{x}\right),  \tag{3.2}\\
& \varphi_{2}(x)=\left(\frac{1}{x}\right)^{(\nu+\mu) / 2} \Psi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{x}\right), \tag{3.3}
\end{align*}
$$

with $\mu=\sqrt{\nu^{2}+4 \alpha}$ and $\Phi$ and $\Psi$ denote the confluent hypergeometric functions of first and second kind (see Lebedev [11, Section 9.9]).

Proof. Let $f$ be a bounded function on $\mathbb{R}_{+}$. The function $u(x):=$ $U_{\alpha} f(x)$ solves the differential equation $(\alpha I-L) u(x)=f(x)$ where $L$ denotes the infinitesimal generator of $Y_{\nu}$. Thus, $u$ is the bounded solution of

$$
\begin{equation*}
x^{2} \frac{d^{2}}{d x^{2}} u(x)+((1+\nu) x+1) \frac{d}{d x} u(x)-\alpha u(x)+f(x)=0 \tag{3.4}
\end{equation*}
$$

Let us consider the homogeneous equation associated to (3.4)

$$
\begin{equation*}
x^{2} y^{\prime \prime}+((1+\nu) x+1) y^{\prime}-\alpha y=0 . \tag{3.5}
\end{equation*}
$$

Then, $\varphi_{1}$ and $\varphi_{2}$, given by (3.2) and (3.3) are two independent solutions of (3.5). Moreover, $\varphi_{1}$ is bounded near $+\infty$ and $\varphi_{2}$ is bounded near $0\left(\varphi_{2}(0)=1\right)$. Now, we are looking for a solution of (3.4) of the form $u(x)=\alpha_{1}(x) \varphi_{1}(x)+\alpha_{2}(x) \varphi_{2}(x)$ with

$$
\left\{\begin{array}{l}
\alpha_{1}^{\prime}(x) \varphi_{1}(x)+\alpha_{2}^{\prime}(x) \varphi_{2}(x)=0 \\
\alpha_{1}^{\prime}(x) \varphi_{1}^{\prime}(x)+\alpha_{2}^{\prime}(x) \varphi_{2}^{\prime}(x)=-\frac{f(x)}{x^{2}}
\end{array}\right.
$$

Then,

$$
\alpha_{1}^{\prime}(x)=\frac{f(x) \varphi_{2}(x)}{W\left(\varphi_{1}, \varphi_{2}\right)(x) x^{2}}, \quad \alpha_{2}^{\prime}(x)=\frac{-f(x) \varphi_{1}(x)}{W\left(\varphi_{1}, \varphi_{2}\right)(x) x^{2}},
$$

where the Wronskian $W\left(\varphi_{1}, \varphi_{2}\right)(x)$ is given by

$$
W\left(\varphi_{1}, \varphi_{2}\right)(x)=\frac{\Gamma(1+\mu)}{\Gamma\left(\frac{\nu+\mu}{2}\right)}\left(\frac{1}{x}\right)^{1+\nu} \exp \left(\frac{1}{x}\right)
$$

(see [11, (9.10.10)]). Using the boundary conditions on $\varphi_{1}, \varphi_{2}$, the bounded solution of (3.4) is given by

$$
\begin{align*}
u(x)= & \left(\int_{0}^{x} \frac{f(t) \varphi_{2}(t)}{W\left(\varphi_{1}, \varphi_{2}\right)(t) t^{2}} d t\right) \varphi_{1}(x) \\
& +\left(\int_{x}^{\infty} \frac{f(t) \varphi_{1}(t)}{W\left(\varphi_{1}, \varphi_{2}\right)(t) t^{2}} d t\right) \varphi_{2}(x) \tag{3.6}
\end{align*}
$$

This gives formula (3.1).
Corollary 3.1. Let $T_{\alpha}$ be an exponential time with parameter $\alpha$ independent from $B$; then the density of $A_{T_{\alpha}}^{\sqrt{2}, \nu}$ is given by

$$
\begin{align*}
k_{\nu}(x)= & \alpha \frac{\Gamma\left(\frac{\nu+\mu}{2}\right)}{\Gamma(1+\mu)}\left(\frac{1}{x}\right)^{1+(\mu-\nu) / 2}  \tag{3.7}\\
& \cdot \exp \left(-\frac{1}{x}\right) \Phi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{x}\right) .
\end{align*}
$$

For $\alpha>\nu+1$, we have

$$
\begin{align*}
E\left[\left(A_{T_{\alpha}}^{\sqrt{2}, \nu}-k\right)^{+}\right]= & \frac{\Gamma\left(\frac{\nu+\mu}{2}+1\right) \Gamma\left(\frac{\mu-\nu}{2}-1\right)}{\Gamma(1+\mu) \Gamma\left(\frac{\mu-\nu}{2}\right)} \\
3.8) & \cdot k^{1-(\mu-\nu) / 2} \exp \left(-\frac{1}{k}\right) \Phi\left(\frac{\nu+\mu}{2}+2 ; 1+\mu ; \frac{1}{k}\right) . \tag{3.8}
\end{align*}
$$

Proof. 1) From (1.3), the distribution of $A_{T_{\alpha}}^{\sqrt{2}, \nu}$ is the same as the distribution of $Y_{\nu}\left(T_{\alpha}\right)$, that is $\alpha u_{\alpha}(0, y) d y$. Thus, (3.7) follows from (3.1) and $\varphi_{2}(0)=1$.

$$
\begin{aligned}
& \text { 2) } E\left[\left(A_{T_{\alpha}}^{\sqrt{2}, \nu}-k\right)^{+}\right]=\alpha U_{\alpha} f_{k}(0) \text { for } f_{k}(x)=(x-k)^{+} . \text {From (3.1), } \\
& \begin{aligned}
& U_{\alpha} f_{k}(0)=\frac{\Gamma\left(\frac{\nu+\mu}{2}\right)}{\Gamma(1+\mu)} \int_{k}^{\infty}(t-k)\left(\frac{1}{t}\right)^{1+(\mu-\nu) / 2} \\
& \cdot \exp \left(-\frac{1}{t}\right) \Phi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{t}\right) d t
\end{aligned}
\end{aligned}
$$

and the integral converges for $\alpha>\nu+1$. Now, we have the following relation
$\int_{k}^{\infty}(t-k)\left(\frac{1}{t}\right)^{1+(\mu-\nu) / 2} \exp \left(-\frac{1}{t}\right) \Phi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{t}\right) d t$

$$
\begin{equation*}
=\frac{\Gamma\left(\frac{\mu-\nu}{2}-1\right)}{\Gamma\left(\frac{\mu-\nu}{2}+1\right)} k^{1-(\mu-\nu) / 2} \exp \left(-\frac{1}{k}\right) \Phi\left(\frac{\nu+\mu}{2}+2 ; 1+\mu ; \frac{1}{k}\right) \tag{3.9}
\end{equation*}
$$

(see Lebedev [11, 21. p. 279] and Yor [19, Chapter 6] for a probabilistic proof of this relation) and (3.8) follows.

Remark. 1) Formula (3.8) yields the result of Geman and Yor [3, (2)] since, by scaling,

$$
E\left[\left(A_{T_{\lambda}}^{(\nu)}-\frac{1}{2 x}\right)^{+}\right]=\frac{1}{2} E\left[\left(A_{T_{\lambda / 2}}^{\sqrt{2}, \nu}-\frac{1}{x}\right)^{+}\right]
$$

2) We can decompose the right-hand side of (1.3) using Itô-Tanaka formula

$$
E\left[\left(Y_{t}-k\right)^{+}\right]=\int_{0}^{t} d s E\left[\left(\left(\frac{a^{2}}{2}+b\right) Y_{s}+1\right) \mathbf{1}_{\left\{Y_{s}>k\right\}}\right]+\frac{1}{2} E\left[L_{t}^{k}\right]
$$

where $\left(L_{t}^{k} ; t \geq 0\right)$ denotes the local time of $Y^{a, b}$ at level $k$. Thus, the quantity $C^{a, b}(t, k):=E\left[\left(Y_{t}-k\right)^{+}\right]$can be decomposed as

$$
C^{a, b}(t, k)=\left(\frac{a^{2}}{2}+b\right) C_{1}^{a, b}(t, k)+C_{2}^{a, b}(t, k)+\frac{1}{2} C_{3}^{a, b}(t, k),
$$

where

$$
\left\{\begin{aligned}
C_{1}^{a, b}(t, k) & =\int_{0}^{t} d s E\left[Y_{s} \mathbf{1}_{\left\{Y_{s}>k\right\}}\right] \\
C_{2}^{a, b}(t, k) & =\int_{0}^{t} d s E\left[\mathbf{1}_{\left\{Y_{s}>k\right\}}\right] \\
C_{3}^{a, b}(t, k) & =E\left[L_{t}^{k}\right]
\end{aligned}\right.
$$

We restrict ourselves to the case $a=\sqrt{2}, b=\nu$ and we delete the superscripts in $C$.

From Proposition 3.1, we can compute the Laplace transform

$$
\int_{0}^{\infty} d t \exp (-\alpha t) C_{i}(t, k)
$$

for $i=1,2,3$. For $i=3$, from the occupation density formula

$$
\int_{0}^{t} f\left(Y_{s}\right) d\langle Y\rangle_{s}=\int_{0}^{\infty} f(y) L_{t}^{y} d y
$$

it follows that

$$
\begin{aligned}
E\left[L_{T_{\alpha}}^{y}\right] & =2 y^{2} u_{\alpha}(0, y) \\
& =2 \frac{\Gamma\left(\frac{\nu+\mu}{2}\right)}{\Gamma(1+\mu)} y^{1-(\mu-\nu) / 2} \exp \left(-\frac{1}{y}\right) \Phi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{y}\right) .
\end{aligned}
$$

Now,

$$
\begin{aligned}
E & {\left[C_{2}\left(T_{\alpha}, k\right)\right] } \\
& =\int_{0}^{\infty} \exp (-\alpha t) E\left[\mathbf{1}_{\left\{Y_{t}>k\right\}}\right] d t \\
& =U_{\alpha} \mathbf{1}_{\{x>k\}}(0) \\
& =\frac{\Gamma\left(\frac{\nu+\mu}{2}\right)}{\Gamma(1+\mu)} \int_{k}^{\infty}\left(\frac{1}{t}\right)^{1+(\mu-\nu) / 2} \exp \left(-\frac{1}{t}\right) \Phi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{t}\right) d t
\end{aligned}
$$

Now, the following relation holds (see Lebedev [11, 21. p. 279])

$$
\begin{align*}
\int_{k}^{\infty} & \left(\frac{1}{t}\right)^{1+(\mu-\nu) / 2} \exp \left(-\frac{1}{t}\right) \Phi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{t}\right) d t  \tag{3.10}\\
& =\frac{2}{\mu-\nu} k^{(\nu-\mu) / 2} \exp \left(-\frac{1}{k}\right) \Phi\left(\frac{\nu+\mu}{2}+1 ; 1+\mu ; \frac{1}{k}\right) .
\end{align*}
$$

In the same way,

$$
\begin{aligned}
& E\left[C_{1}\left(T_{\alpha}, k\right)\right] \\
& =\frac{\Gamma\left(\frac{\nu+\mu}{2}\right)}{\Gamma(1+\mu)} \int_{k}^{\infty} t\left(\frac{1}{t}\right)^{1+(\mu-\nu) / 2} \exp \left(-\frac{1}{t}\right) \Phi\left(\frac{\nu+\mu}{2} ; 1+\mu ; \frac{1}{t}\right) d t
\end{aligned}
$$

which can be expressed in term of $\Phi((\nu+\mu) / 2+1 ; 1+\mu ; 1 / k)$ and $\Phi((\nu+\mu) / 2+2 ; 1+\mu ; 1 / k)$ using the two relations (3.9) and (3.10).

In terms of confluent hypergeometric functions, the equality

$$
E\left[C\left(T_{\alpha}, k\right)\right]=(1+\nu) E\left[C_{1}\left(T_{\alpha}, k\right)\right]+E\left[C_{2}\left(T_{\alpha}, k\right)\right]+\frac{1}{2} E\left[C_{3}\left(T_{\alpha}, k\right)\right]
$$

corresponds to the recurrence relation

$$
(b-a) \Phi(a-1 ; b ; z)+(2 a-b+z) \Phi(a ; b ; z)-a \Phi(a+1 ; b ; z)=0
$$

(see $[11,(9.9 .10)])$.

## 4. Some finite dimensional Markov processes.

It was shown in [2], [20] that, for $a \neq 0$, and $b>0$,
(4.1) the variable $\int_{0}^{\infty} d s \exp \left(a B_{s}-b s\right)$ is distributed as $\frac{2}{a^{2} Z_{2 b / a^{2}}}$,
where $Z_{\nu}$ is a gamma variable with parameter $\nu$, i.e.

$$
P\left(Z_{\nu} \in d t\right)=\frac{d t t^{\nu-1} e^{-t}}{\Gamma(\nu)}
$$

However, in [21], the joint law of

$$
\left\{\int_{0}^{\infty} d s \exp \left(a_{i} B_{s}-b_{i} s\right), i=1,2, \cdots, n\right\}
$$

for different constants $a_{i}, b_{i}$ could not be obtained.
In this section, using (1.2), we can express this law as the invariant measure of a Markov process. Indeed, we consider jointly the onedimensional Markov processes $\left(Y_{t}^{(i)} ; t \geq 0\right)$ defined as

$$
Y_{t}^{(i)}=\int_{0}^{t} d s \exp \left(\xi_{t}^{(i)}-\xi_{s}^{(i)}\right), \quad t \geq 0
$$

where $\xi_{t}^{(i)}=a_{i} B_{t}+b_{i} t$, and $\left(B_{t}\right)$ is a one-dimensional Brownian motion.
Of course, these processes are not independent, and jointly, they constitute a $\left(\mathbb{R}_{+}\right)^{n}$-valued Markov process, specified in the following:

Thorem 4.1. The process $Y_{t} \equiv\left(Y_{t}^{(1)}, \ldots, Y_{t}^{(n)}\right), t \geq 0$ is a Markov process, whose infinitesimal generator coincides on $C^{2}\left((0, \infty)^{n}\right)$ with

$$
\begin{aligned}
L= & \frac{1}{2}\left(\sum_{i=1}^{n} a_{i}^{2} y_{i}^{2} \frac{\partial^{2}}{\partial y_{i}^{2}}+2 \sum_{i<j} a_{i} a_{j} y_{i} y_{j} \frac{\partial^{2}}{\partial y_{i} \partial y_{j}}\right) \\
& +\sum_{i=1}^{n}\left(\left(\frac{a_{i}^{2}}{2}+b_{i}\right) y_{i}+1\right) \frac{\partial}{\partial y_{i}} .
\end{aligned}
$$

We are now interested in the case where $b_{i}<0$, for every $i$. In this case, since

$$
Y_{t}^{(i)} \stackrel{(\text { law })}{=} \int_{0}^{t} d s \exp \left(\xi_{s}^{(i)}\right)
$$

the vector $Y_{t} \equiv\left(Y_{t}^{(1)}, \cdots, Y_{t}^{(n)}\right)$ converges in law, towards

$$
U=\left(U^{(1)}, U^{(2)}, \ldots, U^{(n)}\right)
$$

where $U^{(i)}=\int_{0}^{\infty} d s \exp \left(\xi_{s}^{(i)}\right)$.
Our aim now is to describe $\mu$, the joint law of the random vector $U$.

Just as in Carmona-Petit-Yor in [21], we note that $\mu$ is the unique invariant measure of the Markov process $Y$; hence, it satisfies: $\mu L=0$, i.e.

$$
\begin{equation*}
\text { for all } f \in C_{b}^{2}\left(\left(\mathbb{R}_{+}\right)^{n}\right), \quad\langle\mu, L f\rangle=0 \tag{4.2}
\end{equation*}
$$

Let us assume that $\mu(d y)=k(y) d y$, where $y$ denotes the generic element in $\left(\mathbb{R}_{+}\right)^{n}$, and $d y$ is Lebesgue's measure on $\left(\mathbb{R}_{+}\right)^{n}$.

Now, from (4.2), it follows that

$$
\begin{equation*}
\frac{1}{2}\left(\sum_{i=1}^{n} a_{i}^{2} \frac{\partial^{2}}{\partial y_{i}^{2}}\left(y_{i}^{2} k(y)\right)+2 \sum_{i<j} a_{i} a_{j} \frac{\partial^{2}}{\partial y_{i} \partial y_{j}}\left(y_{i} y_{j} k(y)\right)\right) \tag{4.3}
\end{equation*}
$$

$$
-\sum_{i=1}^{n}\left(\left(\frac{a_{i}^{2}}{2}+b_{i}\right) \frac{\partial}{\partial y_{i}}\left(y_{i} k(y)\right)+\frac{\partial}{\partial y_{i}}(k(y))\right)=0 .
$$

We check that in the case $n=1$, we recover the result (4.1). Indeed, the density $(k(u), u>0)$ of $X \equiv c / Z_{\nu}$ is

$$
k(u)=\frac{c^{\nu}}{\Gamma(\nu) u^{\nu+1}} \exp \left(-\frac{c}{u}\right)
$$

where we have denoted $c=2 / a^{2}$, and $\nu=2 b / a^{2}$.
On the other hand, for $n=1$, and $a_{1}=a, b_{1}=-b$, (4.3) becomes

$$
\begin{equation*}
\left(y^{2} k(y)\right)^{\prime \prime}-(1+\nu)(y k(y))^{\prime}-c k^{\prime}(y)=0 \tag{4.4}
\end{equation*}
$$

and we easily verify that the density $k$ defined above solves (4.4).
Remark. 1) Unfortunately, except in the case $n=1$, it does not seem easy to solve (4.2), i.e. to find explicitly the density of $U$. It may be easier to find the Laplace transform $\psi$ of $U$. From (4.3), we can easily deduce the equation satisfied by $\psi$, that is

$$
\begin{equation*}
\widehat{L} \psi(x)=\left(\sum_{1}^{n} x_{i}\right) \psi(x), \quad x \in \mathbb{R}_{+}^{n} \tag{4.5}
\end{equation*}
$$

where
$\widehat{L}:=\frac{1}{2}\left(\sum_{i=1}^{n} a_{i}^{2} y_{i}^{2} \frac{\partial^{2}}{\partial y_{i}^{2}}+2 \sum_{i<j} a_{i} a_{j} y_{i} y_{j} \frac{\partial^{2}}{\partial y_{i} \partial y_{j}}\right)+\sum_{i=1}^{n}\left(\frac{a_{i}^{2}}{2}+b_{i}\right) y_{i} \frac{\partial}{\partial y_{i}}$
with the boundary conditions $\psi(0)=1$ and $\lim _{x \rightarrow \infty} \psi(x)=0$.
For $n=1,(4.5)$ is a particular case of Theorem 3.3 of Paulsen [14] where it is shown that the Laplace transform of a randomly discounted integral solves an integro-differential equation.
2) For $n=2$, the Laplace transform of $U=\left(U^{(1)}, U^{(2)}\right)$ is obtained in [20], [22] in the two particular cases where

1) $a_{1}=-a_{2}=2$ and $b_{1}=b_{2}=-1$.
2) $a_{1}=2 a_{2}=2$ and $b_{1}=2 b_{2}=\mu(\mu<0)$.

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# Harmonic analysis in value at risk calculations 

Claudio Albanese and Luis Seco


#### Abstract

Value at Risk is a measure of risk exposure of a portfolio and is defined as the maximum possible loss in a certain time frame, typically 1-20 days, and within a certain confidence, typically $95 \%$. Full valuation of a portfolio under a large number of scenarios is a lengthy process. To speed it up, one can make use of the total delta vector and the total gamma matrix of a portfolio and compute a Gaussian integral over a region bounded by a quadric. We use methods from harmonic analysis to find approximate analytic formulas for the Value at Risk as a function of time and of the confidence level. In this framework, the calculation is reduced to the problem of evaluating linear algebra invariants such as traces of products of matrices, which arise from a Feynmann expansion. The use of Fourier transforms is crucial to resum the expansions and to obtain formulas that smoothly interpolate between low and large confidence levels, as well as between short and long time horizons.


## 1. Introduction.

The notion of Value at Risk (VaR), introduced in the J. P. Morgan RiskMetrics document [JPM], captures the risk exposure of a portfolio in terms of the largest possible loss within a certain confidence interval.

In the RiskMetrics framework, one deals with portfolios subject to a number of risk factors whose evolution is a geometric Brownian motion with a given covariance matrix. The full valuation method
consists of repricing the portfolio under a number of scenarios by calling all the relevant pricing functions. This procedure is computationally very intensive. In typical applications with portfolios that consist of several hundred thousand instruments, not more than a few thousand scenarios can be priced overnight with current technologies. The small number of scenarios results in large inaccuracies in the Value at Risk measurement. The use of rather unsophisticated pricing models can speed up the calculation but is also at the origin of uncontrollable errors.

An alternative that has been advocated in the RiskMetrics technical document is to use the quadratic approximation for the portfolio variation as a function of the underlying risk factors. To obtain this representation, the knowledge of the total delta vector and of the total gamma matrix is required. This leads to the problem of evaluating an integral of the form

$$
\begin{equation*}
I_{0}(K)=\int_{\Delta x+(1 / 2)(x, \Gamma x) \leq K} \exp (-\pi(x, \mathbb{A} x)), \tag{1.1}
\end{equation*}
$$

for certain vectors $\Delta$ and matrix $\Gamma$. To our knowledge, the problem of estimating (1.1) was first considered by Ruben [Rub] (in the case of positive definite $\Gamma$ and zero $\Delta$ ), and then extended by a number of other authors, see [KJB] and references therein. An asymptotic expansion in the large confidence limit has been obtained by Quintanilla (see [Q]).

The notion of Value at Risk owes its popularity to the fact that it captures, with just one parameter of intuitive meaning, the risk exposure of a portfolio. However, the Value at Risk evolves with time and is subject to stochastic fluctuations which reflect the evolution of the risk factors and the evolution of the composition of the portfolio itself. The sensitivity of the Value at Risk with respect to the dynamics of the underlying risk factors depends on the relative importance of delta and gamma risks. To capture this effect, it is useful to use dual variables which give the sensitivity to the total delta and the total gamma risk of a given portfolio.

In our setting, duality transformations involve Fourier transforms. After an initial simultaneous diagonalization of the covariance matrix $\mathbb{A}$ and of the total gamma of the portfolio, we reduce the calculation to an integral of a Gaussian over a high dimensional quadric. This integral is computed using techniques from harmonic analysis, which reduce all calculations to the Fourier transform of quadrics. In the case of positive definite Gammas, this reduces to explicit Bessel functions; the general case is not much more difficult. The Paley-Wiener theorem guarantees
that our formulas have adequate computational properties. This result can be strengthened by deriving analytic formulas which give the asymptotic behaviour and determine the relevant Fourier transforms up to smooth multipliers. The moments of these transforms can be computed by means of a technique related to Feynmann diagrams. This gives rise to matrix invariants such as traces and determinants, which yield analytic formulas for the Value at Risk as a function of the time horizon and of the confidence level. In this context, the use of Fourier transforms is crucial to resum the expansions and to obtain formulas that smoothly interpolate between low and large confidence levels, as well as between short and long time horizons.

By changing coordinates, the integral in (1.1) can be reduced to a convolution of integrals of the same form but with a positive $\Gamma$. In this case, Fourier transforms give rise to the two following representations: first,

$$
\begin{aligned}
& I_{0}(K(R))=R^{n / 2}\left(\int_{-\infty}^{0} \frac{J_{n / 2}(2 R \pi \sqrt{|\widehat{\alpha}|}) G(\widehat{\alpha})}{\left(2|\widehat{\alpha}|^{n / 4}\right)} d \widehat{\alpha}\right. \\
&+\pi i \int_{-\infty}^{0} \int_{\widehat{\alpha} \tan ^{-1}\|v\|^{2}}^{0} \frac{J_{n / 2}(2 R \pi \sqrt{|\widehat{\alpha}|}) F(\widehat{\alpha}, \widehat{\beta})}{(2|\widehat{\alpha}|)^{n / 4}} \\
&\left.\cdot \frac{\cos (2 \pi \sqrt{2 \widehat{\beta}})-1}{2 \pi^{2} \widehat{\beta}} d \widehat{\alpha} d \widehat{\beta}\right),
\end{aligned}
$$

for a certain function $K(R)$ and for suitable functions $G$ and $F$, which arise as Fourier transforms of certain determinant functions. The integer $n$ represents the number of risk factors (or underlyings) in the portfolio under consideration.

In this expression, the first term corresponds to the VaR of a perfectly $\delta$-hedged portfolio, while the second captures the VaR of hedging imperfections. In fact, we will also obtain a second expression, as an asymptotic expansion of the form

$$
I_{0}(K(R))=R^{n / 2} \sum_{j=0}^{\infty} \frac{1}{j!} \int_{-\infty}^{0} \frac{J_{n / 2}(2 R \pi \sqrt{|\widehat{\alpha}|})}{(2|\widehat{\alpha}|)^{n / 4}} H^{(j)}(\widehat{\alpha}) d \widehat{\alpha}
$$

for suitable functions $H^{(j)}$. Each term in this expansion corresponds to increasing degrees of delta-hedge slippage. It is obtained by expanding the first in powers of the $\Delta$ vector and is convenient in the limit of
small $\Delta$ or of large time horizon. The functions $G(\widehat{\alpha}), F(\widehat{\alpha}, \widehat{\beta})$ and $H^{(j)}(\widehat{\alpha})$ admit an integral representation that allows one to find their asymptotic behaviour at the boundaries of their support. Moreover, the Fourier transform of these functions can be computed explicitly and the moments admit an expansion in Feynmann diagrams. This expansion can be used jointly with the asymptotic analysis to find an approximation scheme to efficiently interpolate between large and small values of the arguments based on the knowledge of linear invariants of the matrix $D$ and the vector $v$, such as $\operatorname{Tr} D^{k},\|v\|$ and $\left(v, D^{k} v\right)$.

The interest of analytical formulas for VaR of the type presented in this paper is manifold, and not unrelated to the interest of analytical expressions for traditional pricing theories. First, they allow for further analysis and calibration for different portfolio parameters. Second, they allow for VaR calibration techniques based on historical P\&L data.

This article is organized into seven sections. In the next one, we present the general framework and the main formulas in our analysis. In the third, fourth and fifth sections, which are rather technical, we provide all the details that justify our approach and the formulas it gives rise to. Based on these results, in the sixth section we derive an efficient approximation scheme for value at risk calculations. The last section contains concluding remarks.

## 2. Value at Risk.

Consider a portfolio of price $\Pi$ consisting of a combination of underlying securities $S_{j}$, for $j=1, \ldots, n$, which we assume to be lognormally distributed with covariance matrix $\mathbb{V}$. The Value at Risk of the portfolio is defined to be the number $K$ such that

$$
\begin{equation*}
\operatorname{Prob}(\Pi(0)-\Pi(t)) \geq K=\varepsilon, \tag{2.1}
\end{equation*}
$$

where $\varepsilon$ is a small number (typically 0.05 ), and $t$ is a small time window (i.e., 1 day).

In this paper we consider only portfolios that are smooth over time horizons of interest. This includes most traded securities with some exceptions as, for instance, barrier options when the price of the underlying is near the barrier. To apply our methods, a split of the portfolio into a regular and a singular sub-portfolio is necessary if such singular securities are present.

To leading order in time, we approximate the value of the portfolio by today's deltas and gammas,

$$
\Delta_{0}=\nabla_{\mathbf{S}} \Pi=\left(\frac{\partial \Pi}{\partial S_{1}}, \ldots, \frac{\partial \Pi}{\partial S_{n}}\right),
$$

and

$$
\Gamma_{0}=\operatorname{Hess}_{s} \Pi=\left\{\frac{\partial^{2} \Pi}{\partial S_{i} \partial S_{j}}\right\}
$$

in the sense that, in the near future $t$,
(2.2) $\Pi(t) \approx \Pi(0)+\Delta_{0} \cdot(\mathbf{S}(t)-\mathbf{S}(0))+\frac{1}{2}(\mathbf{S}(t)-\mathbf{S}(0)) \cdot \Gamma_{0} \cdot(\mathbf{S}(t)-\mathbf{S}(0))^{\dagger}$.

Our assumption on log-normality means that

$$
\mathbf{S}(t)=\left(S_{1}(0) e^{\eta_{1}}, \ldots, S_{n}(0) e^{\eta_{n}}\right)
$$

with $E=\left(\eta_{1}, \ldots, \eta_{n}\right)$ a normally distributed random vector,

$$
\begin{equation*}
\operatorname{Prob}\{E \in \Omega\}=\pi^{-n / 2} \int_{\Omega} e^{-(x-m) \mathbb{V}^{-1}(x-m)^{t}} \frac{d x}{\sqrt{\operatorname{det} \mathbb{V}}} \tag{2.3}
\end{equation*}
$$

Taylor-expanding the exponential, (2.2) becomes

$$
\begin{align*}
\Pi(t)= & \Pi(0)+\sum_{i=1}^{n} S_{i}(0) \Delta_{0}^{i}\left(\eta_{i}+\frac{1}{2} \eta_{i}^{2}\right) \\
& +\frac{1}{2} \sum_{i, j} S_{i}(0) S_{j}(0) \eta_{i} \eta_{j} \Gamma_{0}^{i, j}+\mathcal{O}\left(|E|^{3}\right)  \tag{2.4}\\
= & \Pi(0)+\Delta \cdot E+\frac{1}{2} E \cdot \Gamma \cdot E^{t}
\end{align*}
$$

where

$$
\Delta_{i}=S_{i}(0) \cdot \Delta_{0}^{i}, \quad \Gamma_{i, j}= \begin{cases}S_{i}(0) S_{j}(0) \Gamma_{0}^{i, j}, & \text { if } i \neq j  \tag{2.5}\\ S_{i}^{2}(0) \Gamma_{0}^{i, i}+\Delta_{i}, & \text { if } i=j\end{cases}
$$

According to this approximation, the Value at Risk $K$ of the portfolio (2.1) can be approximated by the number $K^{\prime}$ such that

$$
\operatorname{Prob}\left\{\Delta \cdot E+\frac{1}{2} E \cdot \Gamma \cdot E^{t} \leq-K^{\prime}\right\}=\varepsilon
$$

which, using (2.3), becomes

$$
\pi^{-n / 2} \int_{x \cdot \Delta+(1 / 2) x \Gamma x^{\dagger} \leq-K^{\prime}} e^{-(x-m) \mathbb{V}^{-1}(x-m)^{t}} \frac{d x}{\sqrt{\operatorname{det} V}}=\varepsilon,
$$

or

$$
\pi^{-n / 2} \int_{x \cdot \Delta^{\prime}+(1 / 2) x \Gamma x^{\dagger} \leq K^{\prime \prime}} e^{-x \mathbb{V}^{-1} x^{t}} \frac{d x}{\sqrt{\operatorname{det} \mathbb{V}}}=\varepsilon
$$

for

$$
\begin{gathered}
\Delta^{\prime}=\Delta+m \Gamma \\
K^{\prime \prime}=-K^{\prime}-m \cdot \Delta-\frac{1}{2} m \Gamma m^{t}
\end{gathered}
$$

Hence, the goal of this paper is to produce an efficient scheme to compute multidimensional integrals of the type

$$
\begin{equation*}
I_{0}(K)=\int_{x \cdot \Delta+(1 / 2)(x, \Gamma x) \leq K} e^{-\pi(x, \mathbb{A} x)} d x . \tag{2.6}
\end{equation*}
$$

Here, $\mathbb{A}$ is a symmetric positive definite matrix, while $\Gamma$ is just symmetric. We shall assume that $\Gamma$ is non-singular as this is the generic case; the singular case can be reduced to this, plus explicit erf terms.

## 3. Diagonalizations.

Lemma 1. We have that

$$
\sqrt{\operatorname{det} \mathbb{A}} I_{0}(K)=1-\sqrt{\operatorname{det}\left(\frac{\mathbb{A}}{|\Gamma|}\right)} I(\bar{K}),
$$

where

$$
I(\bar{K})=\int_{\left|x_{+}\right|^{2}-\left|x_{-}\right|^{2} \leq \bar{K}} e^{-\pi((x-v), \mathbb{D}(x-v))} d x
$$

with $\bar{K}, v$ and $\mathbb{D}$ defined below.
Proof. The matrices $\mathbb{A}$ and $\Gamma$ are as follows. $\mathbb{A}$ is symmetric positive definite, with diagonal form given by

$$
\mathbb{A}=\mathbb{Q}^{-1} \mathbb{M} \mathbb{Q}
$$

where $\mathbb{Q}$ is orthogonal and

$$
\mathbb{M}=\left(\begin{array}{lll}
a_{1} & & \\
& \ddots & \\
& & a_{n}
\end{array}\right)
$$

We have that

$$
(x, \mathbb{A} x)=(\mathbb{Q} x, \mathbb{M} \mathbb{Q} x)
$$

$\Gamma$ is symmetric non-singular. The matrix and the vector

$$
\Gamma^{\prime}=\mathbb{M}^{-1 / 2} \mathbb{Q} \Gamma \mathbb{Q}^{-1} \mathbb{M}^{-1 / 2}, \quad \Delta^{\prime}=\mathbb{M}^{-1 / 2} \mathbb{Q}^{-1} \Delta
$$

are such that

$$
(x, \Gamma x)=\left(\mathbb{M}^{1 / 2} \mathbb{Q} x, \Gamma^{\prime} \mathbb{M}^{1 / 2} \mathbb{Q} x\right), \quad \Delta \cdot x=\Delta^{\prime} \mathbb{M}^{1 / 2} \mathbb{Q} x
$$

In terms of these matrices, the integral in (2.6) is given by

$$
I_{0}(K)=\frac{1}{\sqrt{\operatorname{det} \mathbb{A}}} \int_{\Delta^{\prime} x+(1 / 2)\left(x, \Gamma^{\prime} x\right) \leq K} \exp \left(-\pi\|x\|^{2}\right) d x
$$

Let $\mathbb{S}$ be the orthogonal transformation which diagonalizes $\Gamma^{\prime}$, i.e.

$$
\Gamma^{\prime}=\mathbb{S}^{-1} \mathbb{L} \mathbb{S}
$$

and let $\Delta^{\prime \prime}, \mathbb{D}$ be such that

$$
(x, x)=(\sqrt{|\mathbb{L}|} \mathbb{S} x, \mathbb{D} \sqrt{|\mathbb{L}|} \mathbb{S} x), \quad \Delta^{\prime} \cdot x=\Delta^{\prime \prime} \sqrt{|\mathbb{L}|} \mathbb{S} x
$$

i.e.

$$
\mathbb{D}=|\mathbb{L}|^{-1}, \quad \Delta^{\prime \prime}=|\mathbb{L}|^{-1 / 2} \mathbb{S}^{-1} \Delta^{\prime}=|\mathbb{L}|^{-1 / 2} \mathbb{S}^{-1} \mathbb{M}^{-1 / 2} \mathbb{Q}^{-1} \Delta
$$

We have that

$$
I_{0}(K)=\frac{1}{\sqrt{\operatorname{det} \Gamma}} \int_{\Delta^{\prime \prime} \cdot x+(x,(\mathbb{L} / \mathbb{\mathbb { L }} \mid) x) \leq K} \exp (-\pi(x, \mathbb{D} x)) d x .
$$

Setting

$$
v=\frac{1}{2} \frac{\mathbb{L}}{|\mathbb{L}|} \Delta^{\prime \prime}, \quad \bar{K}=K-(v, v)+\Delta^{\prime \prime} v
$$

and shifting coordinates, we arrive at an integral of the form

$$
I_{0}(K)=\frac{1}{\sqrt{\operatorname{det} \Gamma}} \int_{\left|x_{+}\right|^{2}-\left|x_{-}\right|^{2} \leq \bar{K}} \exp (-\pi(x-v, \mathbb{D}(x-v))) d x
$$

where the vector $x=\left(x_{-}, x_{+}\right)$is split into the components along the eigenspaces of $\mathbb{L} /|\mathbb{L}|$.

## 4. The positive definite case.

The goal is to compute the integral

$$
\begin{equation*}
I\left(R^{2}\right)=\int_{|x|^{2} \leq R^{2}} e^{-\pi(x-v) \mathbb{D}(x-v)^{t}} d x \tag{4.1}
\end{equation*}
$$

when $\mathbb{D}$ is an $d \times d$ positive definite matrix. We denote its (positive) diagonal elements by $\gamma_{j}, j=1, \ldots, d$.

The expansion of the Bessel function

$$
\frac{J_{d / 2}(2 z)}{|z|^{d / 2}} \cos (2 w)=\sum_{k, j} a_{k, j} z^{2 k} w^{2 j}
$$

has coefficients

$$
a_{k, j}=\frac{(-1)^{k+j} 2^{2 j}}{k!(2 j)!\Gamma\left(k+1+\frac{d}{2}\right)} .
$$

Note that we have

$$
\sum_{k, j}\left|a_{k, j}\right| z^{2 k} w^{2 j}=\frac{I_{d / 2}(2 z)}{|z|^{d / 2}} \cosh (2 w)
$$

where $I_{n / 2}$ denotes the modified Bessel function. We will also need the related hypergometric functions

$$
\mathcal{B}(z, w)=\sum_{k, j}\left|a_{k, j}\right| z^{k} w^{j}=\frac{\mathcal{Z}_{d / 2}(2 z)}{|z|^{d / 4}} \mathcal{C}(2 w)
$$

where

$$
\begin{gathered}
\mathcal{Z}_{d / 2}(2 z)= \begin{cases}I_{d / 2}(\sqrt{2 z}), & \text { if } z>0, \\
J_{d / 2}(\sqrt{2|z|}), & \text { if } z<0,\end{cases} \\
\mathcal{C}(2 z)= \begin{cases}\cosh (2 \sqrt{z}) & \text { if } z>0, \\
\cos (2 \sqrt{|z|}) & \text { if } z<0,\end{cases}
\end{gathered}
$$

and

$$
\widetilde{\mathcal{B}}(z, w)=\sum_{k, j}\left|a_{k, j+1}\right| z^{k} w^{j}=\frac{\mathcal{Z}_{d / 2}(2 z)}{|z|^{\frac{d}{4}}} \frac{\mathcal{C}(2 w)-1}{w} .
$$

The only property we will use for these functions is that they are bounded for negative arguments and grow at most exponentially for positive arguments.

Lemma 2. Define

$$
\begin{gather*}
\mathbb{N}(\alpha, \beta)=\mathbf{1}+i \alpha \mathbb{D}+i \beta \mathbb{D}^{1 / 2} v^{t} v \mathbb{D}^{1 / 2} \\
f(\alpha, \beta)=\int_{\mathbb{R}^{d}} e^{-\pi \xi \mathbb{N}(\alpha, \beta) \xi^{t}} d \xi=(\operatorname{det} \mathbb{N}(\alpha, \beta))^{-1 / 2} \tag{4.2}
\end{gather*}
$$

We have

$$
I\left(R^{2}\right)=\left.\left.\pi^{d / 2} \sum_{k, j=0} R^{n+2 k}(\pi i)^{k+j} a_{k j} \frac{\partial^{k}}{\partial \alpha^{k}}\right|_{\alpha=0} \frac{\partial^{j}}{\partial \beta^{j}}\right|_{\beta=0} f(\alpha, \beta) .
$$

The Fourier transform of the characteristic function of a ball is a Bessel function. Therefore, using Parseval's identity, in dimension $n$ we have

$$
\begin{aligned}
I\left(R^{2}\right) & =\int_{|x| \leq R} e^{-\pi(x-v) \mathbb{D}(x-v)^{t}} d x \\
& =R^{d} \int_{\mathbb{R}^{d}} \frac{J_{d / 2}(2 \pi R|\xi|)}{|R \xi|^{d / 2}} e^{-\pi\left(\xi, \mathbb{D}^{-1} \xi\right)} \cos (2 \pi \xi \cdot v) \frac{d \xi}{\sqrt{\operatorname{det} \pi \mathbb{D}}} \\
& =\sum_{k, j}^{\infty} R^{d+2 k} \pi^{k+j} a_{k j} \int_{\mathbb{R}^{d}}|\xi|^{2 k}(\xi \cdot v)^{2 j} e^{-\pi \xi \mathbb{D}^{-1} \xi} \frac{d \xi}{\sqrt{\operatorname{det} \pi \mathbb{D}}}
\end{aligned}
$$

Since

$$
(v \cdot \xi)^{2}=\xi^{t} v^{t} v \xi
$$

we have that

$$
\begin{aligned}
\int_{\mathbb{R}^{d}} & |\xi|^{2 k}(\xi \cdot v)^{2 j} e^{-\pi \xi \mathbb{D}^{-1} \xi} d \xi \\
& =\left.\left.(-i)^{-k-j} \frac{\partial^{k}}{\partial \alpha^{k}}\right|_{\alpha=0} \frac{\partial^{j}}{\partial \beta^{j}}\right|_{\beta=0} \int e^{-\pi \xi\left(\mathbb{D}^{-1}+i \alpha+i \beta v^{t} v\right) \xi} d \xi \\
& =\left.\left.\pi^{d / 2} i^{k+j} \frac{\partial^{k}}{\partial \alpha^{k}}\right|_{\alpha=0} \frac{\partial^{j}}{\partial \beta^{j}}\right|_{\beta=0}\left(\operatorname{det}\left(\mathbb{D}^{-1}+i \alpha+i \beta v^{t} v\right)\right)^{-1 / 2} .
\end{aligned}
$$

The function $f$ is smooth since the real part of $\mathbb{N}$ is positive. $f$ can easily be computed using the following elementary result in linear algebra.

Lemma 3. Let $\gamma_{j}>0$ be the eigenvalues of $\mathbb{D}$, for $j=1, \ldots, d$, and $v \in \mathbb{R}^{d}$. Then, for $\alpha, \beta \in \mathbb{C}$ we have
$\operatorname{det}\left(1+\alpha \mathbb{D}+\beta \mathbb{D}^{1 / 2} v^{t} v \mathbb{D}^{1 / 2}\right)=\left(\prod_{j=1}^{d}\left(1+\alpha \gamma_{j}\right)\right)\left(1+\sum_{j=1}^{d} \frac{\beta \gamma_{j}\left|v_{j}\right|^{2}}{1+\alpha \gamma_{j}}\right)$.
Proof. For $w \in \mathbb{R}^{d}$, we have

$$
\operatorname{det}\left(\mathbf{1}+w^{t} w\right)=1+|w|^{2}
$$

This follows simply by rotating $v$ with a unitary $\mathbb{U}$ so it is of the form $(|w|, 0, \ldots, 0)$, for which the claim is obvious.

We find

$$
\begin{aligned}
& \operatorname{det}\left(\mathbf{1}+\alpha \mathbb{D}+\beta \mathbb{D}^{1 / 2} v^{t} v \mathbb{D}^{1 / 2}\right) \\
& =\operatorname{det}(\mathbf{1}+\alpha \mathbb{D}) \operatorname{det}\left(\mathbf{1}+\beta(\mathbf{1}+\alpha \mathbb{D})^{-1 / 2} \mathbb{D}^{1 / 2} v^{t} v \mathbb{D}^{1 / 2}(\mathbf{1}+\alpha \mathbb{D})^{-1 / 2}\right) \\
& =\left(\prod_{j=1}^{d}\left(1+\alpha \gamma_{j}\right)\right)\left(1+\sum_{j=1}^{d} \frac{\beta \gamma_{j}\left|v_{j}\right|^{2}}{\left(1+\alpha \gamma_{j}\right)}\right) .
\end{aligned}
$$

Lemma 4. The function $f(\alpha, \beta)$ can be extended as an analytic function to the domain

$$
\mathcal{Q}=\left\{(\alpha, \beta) \in \mathbb{C}^{2}: \operatorname{Im} \alpha \max \gamma_{j}+\operatorname{Im} \beta v D v^{t}<1\right\}
$$

Proof. Modify (4.2) to
(4.3) $f(\alpha, \beta)=\int_{\mathbb{R}^{d}} \exp \left(-\pi \xi \cdot\left(\mathbf{1}+i \alpha \mathbb{D}+i \beta \mathbb{D}^{1 / 2} v^{t} v \mathbb{D}^{1 / 2}\right) \cdot \xi^{t}\right) d \xi$,
for complex $\alpha$ and $\beta$. Since

$$
\xi \mathbb{D} \xi^{t} \leq|\xi|^{2} \max _{j} \gamma_{j}, \quad \xi \mathbb{D}^{1 / 2} v^{t} v \mathbb{D}^{1 / 2} \xi^{t} \leq|\xi|^{2}\left(v \mathbb{D} v^{t}\right)
$$

we see that the integral defining $f$ in (4.3) is absolutely convergent for all $\alpha$ and $\beta$ in $\mathcal{Q}$.

## Lemma 5.

$$
|f(\alpha, \beta)| \leq|\operatorname{Re} \alpha|^{-(d-1) / 2}(\operatorname{det} \mathbb{D})^{-1 / 2}\left(|\operatorname{Re} \alpha|+|\operatorname{Re} \beta||v|^{2}\right)^{-1 / 2}
$$

where $(\alpha, \beta) \in \mathcal{D}$.
Proof. Let $\mathbb{A}$ and $\mathbb{B}$ be positive definite matrices.

$$
\int_{\mathbb{R}^{d}} e^{-\pi \xi \mathbb{A} \xi^{t}} e^{\pi i \xi \mathbb{B} \xi^{t}} d \xi=\frac{e^{\pi i d / 4}}{\sqrt{\operatorname{det}(\mathbb{A} \mathbb{B})}} \int_{\mathbb{R}^{d}} e^{-\pi \xi \mathbb{A}^{-1} \xi^{t}} e^{\pi i \xi \mathbb{B}^{-1} \xi^{t}} d \xi
$$

Hence,

$$
\begin{align*}
\left|\int_{\mathbb{R}^{d}} e^{-\pi \xi \mathbb{A} \xi^{t}} e^{\pi i \xi \mathbb{\mathbb { R } ^ { t }}} d \xi\right| & \leq(\operatorname{det}(\mathbb{A} \mathbb{B}))^{-1 / 2} \int_{\mathbb{R}^{d}} e^{-\pi \xi \mathbb{A}^{-1} \xi^{t}} d x  \tag{4.4}\\
& =(\operatorname{det} \mathbb{B})^{-1 / 2}
\end{align*}
$$

We apply this to (4.3) with

$$
\begin{gathered}
\mathbb{A}=1-(\operatorname{Im} \alpha) \mathbb{D}-(\operatorname{Im} \beta) v^{t} v, \\
\mathbb{B}=\sigma \mathbb{D}+t v^{t} v, \\
\sigma=\operatorname{Re} \alpha, t=\operatorname{Re} \beta,
\end{gathered}
$$

to obtain

$$
\begin{aligned}
|f(\alpha, \beta)| & \leq\left(\operatorname{det}\left(|\sigma| \mathbb{D}+|t| v^{t} v \mathbb{D}\right)\right)^{-1 / 2} \\
& =|\sigma|^{-d / 2}(\operatorname{det} \mathbb{D})^{-1 / 2} \operatorname{det}\left(1+\frac{|t|}{|\sigma|} v^{t} v\right)^{-1 / 2}
\end{aligned}
$$

We use Lemma 3 to compute the last determinant above and conclude that

$$
|f(\alpha, \beta)| \leq|\sigma|^{-d / 2}(\operatorname{det} \mathbb{D})^{-1 / 2}\left(1+\frac{|t|}{|\sigma|}|v|^{2}\right)^{-1 / 2}
$$

as claimed.
It can easily be seen using stationary phase estimates that the bound in the lemma above is sharp. An immediate consequence of this result is that the function $f$ is integrable in $\alpha$ but not in $\beta$.

Lemma 6. Fix constants $T<\|\mathbb{D}\|^{-1}$, and $U, V$ such that $U\|\mathbb{D}\|+$ $V v \mathbb{D} v^{t}<1$. then,
$\int_{-\infty}^{\infty}|f(\sigma+i T, 0)| d \sigma<\infty, \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}\left|\frac{\partial f}{\partial \beta}(\sigma+i U, t+i V)\right| d \sigma d t<\infty$.

Proof. It is enough to establish the integrability of $f$ and $f_{\beta}$ at infinity, since they are bounded inside $\mathcal{D}$. Thus the first bound follows from Lemma 5. For the second, just note that by Lemma 3 the determinant function is linear in $\beta$. Hence,

$$
\begin{align*}
\frac{\partial f}{\partial \beta} & =-\frac{i}{2} f^{3}(\alpha, \beta) \prod_{j=1}^{d}\left(1+i \alpha \gamma_{j}\right) \sum_{j=1}^{d} \frac{\gamma_{j}\left|v_{j}\right|^{2}}{1+i \alpha \gamma_{j}} \\
& =\frac{i}{2} \prod_{j=1}^{d}\left(1+i \alpha \gamma_{j}\right)^{-1 / 2} \frac{\sum_{j=1}^{d} \frac{\gamma_{j}\left|v_{j}\right|^{2}}{1+i \alpha \gamma_{j}}}{\left(1+i \beta \sum_{j=1}^{d} \frac{\gamma_{j}\left|v_{j}\right|^{2}}{1+i \alpha \gamma_{j}}\right)^{3 / 2}} . \tag{4.5}
\end{align*}
$$

Therefore,

$$
\begin{aligned}
\left|\frac{\partial f}{\partial \beta}\right| & \leq \frac{|f|^{3} v D v^{t}}{2 \min \left|1+i \alpha \gamma_{j}\right|} \prod_{j=1}^{d}\left|1+i \alpha \gamma_{j}\right| \\
& \leq \frac{1}{2}|f|^{3} v D v^{t} \max \left|1+i \alpha \gamma_{j}\right|^{d-1} \\
& \leq \frac{v D v^{t}}{2(\operatorname{det} D)^{3 / 2}} \frac{\max \left|1+i \alpha \gamma_{j}\right|^{d-1}}{|\sigma|^{3(d-1) / 2}\left(|\sigma|+|t||v|^{2}\right)^{3 / 2}},
\end{aligned}
$$

which is clearly integrable at infinity.
Our next target is to rewrite Lemma 2 using the Fourier transform of $f$. Define

$$
\begin{gathered}
F(\widehat{\alpha}, \widehat{\beta})=\int_{\mathbb{R}^{2}} e^{-2 \pi i(\widehat{\alpha} \alpha+\widehat{\beta} \beta)} \frac{\partial f}{\partial \beta}(\alpha, \beta) d \alpha d \beta \\
G(\widehat{\alpha})=\int_{\mathbb{R}} e^{-2 \pi i \widehat{\alpha} \alpha} f(\alpha, 0) d \alpha .
\end{gathered}
$$

Both are well defined due to Lemma 6.
Lemma 7 (Paley-Wiener). Let $\varepsilon>0$.

$$
|F(\widehat{\alpha}, \widehat{\beta})| \leq C_{\varepsilon} e^{-2 \pi\left(c_{1}|\widehat{\alpha}|+c_{2}|\widehat{\beta}|\right)}, \quad|G(\widehat{\alpha})| \leq C_{\varepsilon} e^{-2 \pi\left(c_{1}|\widehat{\alpha}|\right)}
$$

where

$$
\begin{aligned}
& c_{1}= \begin{cases}\|D\|^{-1}-\varepsilon, & \text { if } \widehat{\alpha}<0, \\
\text { arbitrarily large, }, & \text { if } \widehat{\alpha}>0,\end{cases} \\
& c_{2}= \begin{cases}v D v^{t}-\varepsilon, & \text { if } \widehat{\beta}<0, \\
\text { arbitrarily large, }, & \text { if } \widehat{\beta}>0 .\end{cases}
\end{aligned}
$$

Proof. Let $\alpha, \beta \in \mathcal{Q}$, with $\alpha=\alpha_{1}+i \alpha_{2}$ and $\beta=\beta_{1}+i \beta_{2}$. By Cauchy's formula,

$$
\begin{aligned}
F(\widehat{\alpha}, \widehat{\beta})= & \int_{-\infty+i \alpha_{1}}^{+\infty+i \alpha_{2}} \int_{-\infty+i \beta_{1}}^{+\infty+i \beta_{2}} e^{-2 \pi i(\alpha \widehat{\alpha}+\beta \widehat{\beta})} \frac{\partial f}{\partial \beta}(\alpha, \beta) d \alpha d \beta \\
= & e^{2 \pi\left(\alpha_{2} \widehat{\alpha}+\beta_{2} \widehat{\beta}\right)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-2 \pi i\left(\alpha_{1} \widehat{\alpha}+\beta_{1} \widehat{\beta}\right)} \\
& \cdot \frac{\partial f}{\partial \beta}\left(\alpha_{1}+i \alpha_{2}, \beta_{1}+i \beta_{2}\right) d \alpha_{1} d \beta_{1} .
\end{aligned}
$$

Therefore,

$$
|F(\widehat{\alpha}, \widehat{\beta})| \leq e^{2 \pi\left(\alpha_{2} \widehat{\alpha}+\beta_{2} \widehat{\beta}\right)} \int_{\mathbb{R}^{2}}\left|\frac{\partial f}{\partial \beta}\left(\alpha_{1}+i \alpha_{2}, \beta_{1}+i \beta_{2}\right)\right| d \alpha_{1} d \beta_{1}
$$

If $\widehat{\alpha}<0$, we use Lemma 6 , with any $0 \leq \alpha_{2}<\|D\|^{-1}$. If $\widehat{\alpha}>0$, we can use the previous argument with any negative $\alpha_{2}$. Same thing for $\beta$. Same thing for $G$.

Lemma 8. $G(\widehat{\alpha})=0$ when $\widehat{\alpha} \geq 0$. Also,

$$
\begin{equation*}
G(\widehat{\alpha})=\frac{(-2 \widehat{\alpha})^{d / 2-1}}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{\mathbf{n} \in S_{d-1}} \exp \left(2 \pi \widehat{\alpha}\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)\right) d \sigma, \quad \widehat{\alpha}<0, \tag{4.6}
\end{equation*}
$$

where $d \sigma$ denotes the usual surface measure on $S_{d-1}$.

Proof. First $G$. Let $\phi \in C_{0}^{\infty}$

$$
\begin{aligned}
I & \equiv \int_{\mathbb{R}} G(\widehat{\alpha}) \phi(\widehat{\alpha}) d \widehat{\alpha} \\
& =\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2 \pi i \alpha \widehat{\alpha}} f(\alpha, 0) d \alpha d \widehat{\alpha} \\
& =\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^{d}} e^{-\pi|\xi|^{2}} e^{-2 \pi i \alpha \widehat{\alpha}} e^{-\pi i \alpha \xi \mathbb{\infty} \xi^{t}} \phi(\widehat{\alpha}) d \xi d \alpha d \widehat{\alpha} \\
& =\int_{\mathbb{R}} \int_{\mathbb{R}^{d}} e^{-\pi|\xi|^{2}} e^{-\pi i \alpha \xi \mathbb{D} \xi^{t}} \widehat{\phi}(\alpha) d \xi d \alpha \\
& =\int_{\mathbb{R}^{d}} e^{-\pi|\xi|^{2}} \phi\left(-\frac{1}{2} \xi \mathbb{D} \xi^{t}\right) d \xi .
\end{aligned}
$$

The substitution $\eta=\xi \mathbb{D}^{1 / 2}$ leads to

$$
\begin{aligned}
I & =\frac{1}{\sqrt{\operatorname{det} \mathbb{D}}} \int e^{-\pi\left(\eta, \mathbb{D}^{-1} \eta\right)} \phi\left(-\frac{1}{2}|\eta|^{2}\right) d \eta \\
& =\frac{1}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} \int_{\mathbf{n} \in S_{d-1}} d o(\mathbf{n}) e^{-\pi r^{2}\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)} \phi\left(-\frac{1}{2} r^{2}\right) r^{d-1} d r .
\end{aligned}
$$

By substituting $r$ with $\alpha=r^{2} / 2$, we find that

$$
I=\frac{1}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} \int_{S_{d-1}} e^{-2 \pi \alpha\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)} \phi(-\widehat{\alpha})(2 \widehat{\alpha})^{(d-1) / 2} d \sigma(\mathbf{n}) \frac{d \widehat{\alpha}}{\sqrt{2 \widehat{\alpha}}}
$$

Let us introduce the polar coordinate in the ( $\widehat{\alpha}, \widehat{\beta}$ ) plane

$$
\widehat{\rho}=\sqrt{\widehat{\alpha}^{2}+\widehat{\beta}^{2}}, \quad \widehat{\theta}=\pi+\tan ^{-1}\left(\frac{\widehat{\beta}}{\widehat{\alpha}}\right)
$$

where the arc-tangent is taken with values in $(-\pi / 2, \pi / 2)$ and the angle $\widehat{\theta}$ is defined with a shift of $\pi$ to keep notation simple in what follows.

Lemma 9. The function $F(\widehat{\alpha}, \widehat{\beta})$ has support in the sector

$$
0 \leq \widehat{\theta} \leq \tan ^{-1}\|v\|^{2}
$$

Inside this sector, we have that

$$
\begin{equation*}
F(\widehat{\rho}, \widehat{\theta})=-\frac{4 \pi i}{\sqrt{\operatorname{det} \mathbb{D}}} \widehat{\rho}^{d}(2 \cos \widehat{\theta})^{d-1} \sqrt{\frac{\tan \hat{\theta}}{\|v\|^{2}-\tan \widehat{\theta}}} m(\widehat{\rho}, \widehat{\theta}), \tag{4.6}
\end{equation*}
$$

where

$$
\begin{equation*}
m(\widehat{\theta}, \widehat{\rho})=\int_{\Gamma_{\widehat{\theta}}} e^{-2 \pi \widehat{\rho} \cos \widehat{\theta}\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)} d \sigma_{\Gamma_{\widehat{\theta}}} \tag{4.7}
\end{equation*}
$$

$\Gamma_{\widehat{\theta}}$ is the sphere

$$
\begin{equation*}
\Gamma_{\widehat{\theta}}=\left\{\mathbf{n}:\|\mathbf{n}\|^{2}=1 \text { and } v \cdot \mathbf{n}=\sqrt{\tan \widehat{\theta}}\right\} \tag{4.8}
\end{equation*}
$$

and $d \sigma_{\Gamma_{\widehat{\theta}}}$ is the surface measure on $\Gamma_{\theta}$.
Proof. Denote $w=\sqrt{\mathbb{D}} v$.

$$
\begin{aligned}
I \equiv & \iint F(\widehat{\alpha}, \widehat{\beta}) \phi(\widehat{\alpha}, \widehat{\beta}) d \widehat{\alpha} d \widehat{\beta} \\
= & -\pi i \int(\xi \cdot w)^{2} e^{-\pi\left|\xi^{2}\right|} e^{-2 \pi i(\alpha \widehat{\alpha}+\beta \widehat{\beta})} e^{-\pi i \xi\left(\alpha \mathbb{D}+\beta w^{t} w\right) \xi^{t}} \\
& \cdot \phi(\widehat{\alpha}, \widehat{\beta}) d \widehat{\alpha} d \widehat{\beta} d \alpha d \beta d \xi \\
= & -\pi i \int(\xi \cdot w)^{2} e^{-\pi\left|\xi^{2}\right|} e^{-\pi i \xi\left(\alpha \mathbb{D}+\beta w^{t} w\right) \xi^{t}} \widehat{\phi}(\alpha, \beta) d \alpha d \beta d \xi \\
= & -\pi i \int(\xi \cdot w)^{2} e^{-\pi\left|\xi^{2}\right|} \phi\left(-\frac{1}{2} \xi \mathbb{D} \xi^{t},-\frac{1}{2}(w \cdot \xi)^{2}\right) d \xi
\end{aligned}
$$

In terms of the new coordinate $\eta=\xi \mathbb{D}^{1 / 2}$, we find that

$$
\begin{array}{r}
I=-\frac{\pi i}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{\mathbb{R}^{d}}(v \cdot \eta)^{2} e^{-\pi\left(\eta, \mathbb{D}^{-1} \eta\right)} \phi\left(-\frac{1}{2} \eta^{2},-\frac{1}{2}(v \cdot \eta)^{2}\right) \\
=-\frac{\pi i}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} d r \int_{\mathbf{n} \in S_{d-1}} d \sigma(\mathbf{n}) r^{d+1}(v \cdot \mathbf{n})^{2} e^{-\pi r^{2}\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)} \\
\cdot \phi\left(-\frac{1}{2} r^{2},-\frac{1}{2} r^{2}(v \cdot \mathbf{n})^{2}\right) .
\end{array}
$$

In polar coordinates,

$$
\widehat{\rho}=\frac{1}{2} r^{2} \sqrt{1+(v \cdot \mathbf{n})^{2}}, \quad \widehat{\theta}=\tan ^{-1}(v \cdot \mathbf{n})^{2}
$$

and the function

$$
\psi(\widehat{\rho}, \widehat{\theta})=\phi(\widehat{\alpha}(\widehat{\rho}, \widehat{\theta}), \widehat{\beta}(\widehat{\rho}, \widehat{\theta}))
$$

We have that

$$
\begin{gathered}
I=-\frac{\pi i}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} d r \int_{\mathbf{n} \in S_{d-1}} d \sigma(\mathbf{n}) r^{d+1}(v \cdot \mathbf{n})^{2} e^{-\pi r^{2}\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)} \\
\cdot \psi\left(\widehat{\rho}, \tan ^{-1}(v \cdot \mathbf{n})^{2}\right) .
\end{gathered}
$$

The unit sphere $S_{d-1}=\left\{\mathbf{n}:|\mathbf{n}|^{2}=1\right\}$ intersects the plane of equation $v \cdot \mathbf{n}=\sqrt{\tan \hat{\theta}}$ on a codimension 2 sphere $\Gamma_{\widehat{\theta}}$, (which might degenerate to a point or be empty). The points of $\Gamma_{\widehat{\theta}}$ have the form

$$
\mathbf{n}=\sqrt{\tan \widehat{\theta}} \frac{v}{|v|^{2}}+\zeta
$$

with

$$
\zeta \cdot v=0 .
$$

Hence, the radius is

$$
\sqrt{\left(1-\frac{\tan \widehat{\theta}}{\|v\|^{2}}\right)_{+}}
$$

and the $(d-2)$-dimensional volume of such set is

$$
\frac{2 \pi^{d / 2-1}}{(d-2) \Gamma\left(\frac{d}{2}-1\right)}\left(1-\frac{\tan \hat{\theta}}{\|v\|^{2}}\right)_{+}^{d / 2-1}
$$

The co-area formula (see Chavel [Cha]) for spheres reads as follows

$$
\int_{\mathbf{n} \in S_{d-1}} f(\mathbf{n}) d \sigma_{d-1}=\int_{-1}^{1} \int_{\substack{\mathbf{n} \cdot v_{0}=t \\ \mathbf{n} \in S_{d-1}}} f(\mathbf{n}) d A_{t} \frac{d t}{\sqrt{1-t^{2}}}, \quad v_{0} \in S_{d-1}
$$

where $d A_{t}$ is the $(d-2)$-dimensional surface measure on that sphere. In our context, $v_{0}=v /\|v\|$ and this formula implies that

$$
\int_{\mathbf{n} \in S_{d-1}} f(\mathbf{n}) d \sigma_{d-1}=\int_{0}^{\|v\|} \frac{d \sqrt{\tan \hat{\theta}}}{\sqrt{\|v\|^{2}-\tan \widehat{\theta}}} \int_{\Gamma_{\widehat{\theta}}} f(\mathbf{n}) d \sigma_{\Gamma_{\widehat{\theta}}} .
$$

Thus,

$$
\begin{aligned}
& I=-\frac{\pi i}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} d r r^{d+1} \int_{0}^{\|v\|} \tan \widehat{\theta} \frac{d \sqrt{\tan \hat{\theta}}}{\sqrt{\|v\|^{2}-\tan \hat{\theta}}} \\
& \cdot \psi(\widehat{\rho}, \widehat{\theta}) \int_{\mathbf{n} \in \Gamma_{\widehat{\theta}}} e^{-\pi r^{2}\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)} d \Gamma_{\widehat{\theta}} \\
&=-\frac{\pi i}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} d \widehat{\rho} \int_{0}^{\|v\|} \frac{d \sqrt{\tan \widehat{\theta}}}{\sqrt{\|v\|^{2}-\tan \widehat{\theta}}} \\
&=-\frac{\pi i}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} d \widehat{\rho} \int_{0}^{\tan ^{-1}\|v\|^{2}} d \widehat{\theta} \frac{(2 \widehat{\rho} \cos \widehat{\theta})^{d+1} \sqrt{\tan \hat{\theta}}}{\cos ^{2} \widehat{\theta} \sqrt{\|v\|^{2}-\tan \widehat{\theta}}} \\
& \cdot m(\widehat{\rho}, \widehat{\theta}) \psi(\widehat{\rho}, \widehat{\theta}),
\end{aligned}
$$

where $m(\widehat{\theta}, \widehat{\rho})$ is the function defined in the statement of the lemma. Since

$$
I=\int_{0}^{\infty} \int_{-\pi}^{\pi} F(\widehat{\rho}, \widehat{\theta}) \psi(\widehat{\rho}, \widehat{\theta}) \widehat{\rho} d \widehat{\rho} d \widehat{\theta}
$$

we conclude that $F$ is supported on the sector

$$
\operatorname{supp} F=\left\{(\widehat{\rho}, \widehat{\theta}):|\widehat{\theta}| \leq \tan ^{-1}\|v\|^{2}\right\}
$$

and

$$
F(\widehat{\rho}, \widehat{\theta})=-\frac{4 \pi i}{\sqrt{\operatorname{det} \mathbb{D}}} \widehat{\rho}^{d}(2 \cos \widehat{\theta})^{d-1} \sqrt{\frac{\tan \hat{\theta}}{\|v\|^{2}-\tan \widehat{\theta}}} m(\widehat{\rho}, \widehat{\theta}) .
$$

## Lemma 11.

$$
\begin{aligned}
& I\left(R^{2}\right)=R^{d / 2}\left(\int_{-\infty}^{0} d \widehat{\alpha} \frac{J_{d / 2}(2 R \pi \sqrt{|\widehat{\alpha}|}) G(\widehat{\alpha})}{(2|\widehat{\alpha}|)^{d / 4}}\right. \\
&+\pi i \int_{-\infty}^{0} d \widehat{\alpha} \int_{\widehat{\alpha} \tan ^{-1} \|\left. v\right|^{2}}^{0} d \widehat{\beta} \frac{J_{d / 2}(2 R \pi \sqrt{|\widehat{\alpha}|}) F(\widehat{\alpha}, \widehat{\beta})}{(2|\widehat{\alpha}|)^{d / 4}} \\
&\left.\cdot \frac{\cos (2 \pi \sqrt{2 \widehat{\beta}})-1}{2 \pi^{2} \widehat{\beta}}\right) .
\end{aligned}
$$

Proof. Lemma 7 allows us to continue Lemma 2 as

$$
\begin{aligned}
I\left(R^{2}\right)= & \pi^{d / 2} \sum_{k} R^{d+2 k}(\pi i)^{k} a_{k, 0} \int_{\mathbb{R}}(2 \pi i \widehat{\alpha})^{k} G(\widehat{\alpha}) d \widehat{\alpha} \\
& +\pi^{d / 2} \sum_{k, j} R^{d+2 k}(\pi i)^{k+j+1} a_{k, j+1} \\
& \cdot \int_{\mathbb{R}^{2}}(2 \pi i \widehat{\alpha})^{k}(2 \pi i \widehat{\beta})^{j} F(\widehat{\alpha}, \widehat{\beta}) d \widehat{\alpha} d \widehat{\beta} \\
= & \pi^{d / 2} \int_{\mathbb{R}} \sum_{k} R^{d+2 k}\left(-2 \pi^{2}\right)^{k} a_{k, 0} \widehat{\alpha}^{k} G(\widehat{\alpha}) d \widehat{\alpha} \\
& +\pi^{d / 2} \int_{\mathbb{R}^{2}} \sum_{k, j} R^{d+2 k}\left(-2 \pi^{2}\right)^{k+j+1} a_{k, j+1} \widehat{\alpha}^{k} \widehat{\beta}^{j} F(\widehat{\alpha}, \widehat{\beta}) d \widehat{\alpha} d \widehat{\beta} \\
= & R^{d} \pi^{d / 2} \int_{\mathbb{R}} \sum_{k}\left(-2 R^{2} \pi^{2} \widehat{\alpha}\right)^{k} a_{k, 0} F(\widehat{\alpha}, \widehat{\beta}) d \widehat{\alpha} \\
& +R^{d} \pi^{d / 2} \int_{\mathbb{R}^{2}} \sum_{k, j}\left(-2 R^{2} \pi^{2} \widehat{\alpha}\right)^{k}\left(-2 \pi^{2} \widehat{\beta}\right)^{j} a_{k, j+1} F(\widehat{\alpha}, \widehat{\beta}) d \widehat{\alpha} d \widehat{\beta} \\
= & R^{d} \pi^{d / 2} \int_{\mathbb{R}} \sum_{k, 0}\left(2 R^{2} \pi^{2} \widehat{\alpha}\right)^{k}\left|a_{k, 0}\right| G(\widehat{\alpha}) d \widehat{\alpha} \\
& +R^{d} \pi^{d / 2} \int_{\mathbb{R}^{2}} \sum_{k, j}\left(2 R^{2} \pi^{2} \widehat{\alpha}\right)^{k}\left(2 \pi^{2} \widehat{\beta}\right)^{j}\left|a_{k, j+1}\right| F(\widehat{\alpha}, \widehat{\beta}) d \widehat{\alpha} d \widehat{\beta} \\
= & R^{d} \pi^{d / 2}\left(\int_{\mathbb{R}} \mathcal{Z}\left(4 R^{2} \pi^{2} \widehat{\alpha}\right) G(\widehat{\alpha}) d \widehat{\alpha}\right. \\
& \left.+\int_{\mathbb{R}^{2}} \widetilde{\mathcal{B}}\left(4 R^{2} \pi^{2} \widehat{\alpha}, 4 \pi^{2} \widehat{\beta}\right) F(\widehat{\alpha}, \widehat{\beta}) d \widehat{\alpha} d \widehat{\beta}\right)
\end{aligned}
$$

and the last integral converges unconditionally due to the growth properties of $\mathcal{B}$, in conjunction with Lemma 7.

Lemma 12. We have that

$$
I\left(R^{2}\right)=R^{d / 2} \sum_{j=0}^{\infty} \frac{1}{j!} \int_{-\infty}^{0} \frac{J_{d / 2}(2 R \pi \sqrt{|\widehat{\alpha}|})}{(2|\widehat{\alpha}|)^{d / 4}} H^{(j)}(\widehat{\alpha}) d \widehat{\alpha},
$$

where

$$
\begin{aligned}
H^{(j)}(\widehat{\alpha})= & \frac{(2 \pi)^{2 j}(-2 \widehat{\alpha})^{(d+j) / 2-1}}{\sqrt{\operatorname{det} \mathbb{D}}} \\
& \cdot \int_{\mathbf{n} \in S_{d-1}}(\mathbf{n} \cdot v)^{j} \exp \left(2 \pi \widehat{\alpha}\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)\right) d \sigma(\mathbf{n}),
\end{aligned}
$$

with $\widehat{\alpha}<0$.

Proof. We have that

$$
H^{(j)}(\widehat{\alpha})=i 2^{3 j-1} \pi^{2 j-1} \int_{\widehat{\alpha} \tan ^{-1}\|v\|^{2}}^{0} d \widehat{\beta} F(\widehat{\alpha}, \widehat{\beta}) \widehat{\beta}^{j-1} .
$$

Hence
$I_{j} \equiv \int_{-\infty}^{\infty} d \widehat{\alpha} H^{(j)}(\widehat{\alpha}) \phi(\widehat{\alpha})=(2 \pi)^{2 j} \int d \xi(\xi \cdot w)^{2 j} e^{-\pi\left|\xi^{2}\right|} \phi\left(-\frac{1}{2} \xi \mathbb{D} \xi^{t}\right)$.
where $w=\sqrt{\mathbb{D}} v$. Proceeding as in the proof of Lemma 10, we make the substitution $\eta=\sqrt{\mathbb{D}} \xi$ and obtain

$$
\begin{aligned}
I_{j}= & \frac{(2 \pi)^{2 j}}{\sqrt{\operatorname{det} \mathbb{D}}} \int(\eta \cdot v)^{j} e^{-\pi\left(\eta, \mathbb{D}^{-1} \eta\right)} \phi\left(-\frac{1}{2}|\eta|^{2}\right) d \eta \\
= & \frac{(2 \pi)^{2 j}}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} r^{j+d-1} d r \int_{\mathbf{n} \in S_{d-1}} d \sigma(\mathbf{n})(\mathbf{n} \cdot v)^{j} \\
& \cdot e^{-\pi r^{2}\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)} \phi\left(-\frac{1}{2} r^{2}\right) .
\end{aligned}
$$

By substituting $r$ with $\alpha=r^{2} / 2$, we find that

$$
\begin{aligned}
I_{j}=\frac{(2 \pi)^{2 j}}{\sqrt{\operatorname{det} \mathbb{D}}} \int_{0}^{\infty} \int_{S_{d-1}} & (\mathbf{n} \cdot v)^{j} e^{-2 \pi \alpha\left(\mathbf{n}, \mathbb{D}^{-1} \mathbf{n}\right)} \\
& \cdot \phi(-\widehat{\alpha})(2 \widehat{\alpha})^{(j+d-1) / 2} \frac{d \widehat{\alpha}}{\sqrt{2 \widehat{\alpha}}} d \sigma(\mathbf{n}) .
\end{aligned}
$$

## 5. The hyperbolic case.

The hyperbolic case can be reduced to the positive definite case, via spherical convolutions.

We will use the following elementary identities

$$
\frac{d}{d r} \int_{|x| \leq r} f(x) d x=\int_{|x|=r} f(\sigma) d \sigma
$$

and

$$
\iint_{|x|^{2}-|y|^{2} \leq R} \varphi(x, y) d x d y=\int_{0}^{\infty} \int_{|y|=r} \int_{|x| \leq R+r^{2}} \varphi(x, y) d x d \sigma(y) d r .
$$

We apply these identities to $I$ with the notation of Lemma 1 (namely $n=n_{1}+n_{2}$ ), to obtain

$$
\begin{align*}
I(K) & =\int_{\left|x-v_{1}\right|^{2}-\left|y-v_{2}\right|^{2} \leq K} e^{-\left(x, \mathbb{D}_{1} x\right)-\left(y, \mathbb{D}_{2} y\right)} \\
& =\int_{0}^{\infty} \int_{\left|y-v_{2}\right|^{2}=r^{2}} e^{-\left(y, \mathbb{D}_{2} y\right)} \int_{\left|x-v_{1}\right|^{2} \leq r^{2}+K} e^{-\left(x, \mathbb{D}_{1} x\right)} d x d \sigma(y) d r  \tag{5.1}\\
& =\int_{0}^{\infty} I_{1}\left(r^{2}+K\right) \frac{\partial}{\partial r} I_{2}\left(r^{2}\right) d r .
\end{align*}
$$

Here, the integrals $I_{1}$ and $I_{2}$ are both of the positive definite type, with matrices $\mathbb{D}_{1}$ and $\mathbb{D}_{2}$ respectively, and offset vectors $v_{1}$ and $v_{2}$, in dimension $n_{1}$ and $n_{2}$. We can therefore deal with the methods in the preceeding section. In particular, let $H_{1}^{(j)}(\widehat{\alpha})$ and $H_{2}^{(j)}(\widehat{\alpha})$ be the functions associated to the integrals $I_{1}$ and $I_{2}$, respectively, and let

$$
H_{i}(\widehat{\alpha})=\sum_{j=0}^{\infty} \frac{1}{j!} H_{i}^{(j)}(\widehat{\alpha}) .
$$

where $i=1,2$, so that

$$
I_{i}\left(R^{2}\right)=R^{n_{i} / 2} \int_{-\infty}^{0} \frac{J_{n_{1} / 2}(2 \pi R \sqrt{|\widehat{\alpha}|})}{(2|\widehat{\alpha}|)^{n_{i} / 4}} H(\widehat{\alpha}) d \widehat{\alpha}
$$

Equation (5.1) then implies that

$$
\begin{aligned}
I(K)= & \int_{0}^{\infty} I_{1}\left(r^{2}+K\right) \frac{\partial}{\partial r} I_{2}(r) d r \\
= & \int_{0}^{\infty} \iint_{\mathbb{R}_{-}^{2}}\left(r^{2}+K\right)^{n_{1} / 4} \frac{J_{n_{1} / 2}\left(2 \pi \sqrt{\left(r^{2}+K\right)\left|\widehat{\alpha}_{1}\right|}\right)}{\left(2\left|\widehat{\alpha}_{1}\right|\right)^{n_{1} / 4}\left(2\left|\widehat{\alpha}_{2}\right|\right)^{n_{2} / 4}} \\
& \cdot \partial_{r}\left(r^{n_{2} / 2} J_{n_{2} / 2}\left(2 \pi r \sqrt{\left|\widehat{\alpha}_{2}\right|}\right)\right) H\left(\widehat{\alpha}_{1}\right) H_{2}\left(\widehat{\alpha}_{2}\right) d \widehat{\alpha}_{1} d \widehat{\alpha}_{2} \\
= & 2^{-n / 4} \int \Theta\left(\widehat{\alpha}_{1}, \widehat{\alpha}_{2}\right) H_{1}\left(\widehat{\alpha}_{1}\right) H_{2}\left(\widehat{\alpha}_{2}\right) d \widehat{\alpha}_{1} d \widehat{\alpha}_{2},
\end{aligned}
$$

where

$$
\begin{gathered}
\Theta\left(\widehat{\alpha}_{1}, \widehat{\alpha}_{2}\right)=\int_{0}^{\infty}\left(r^{2}+K\right)^{n_{1} / 4} \frac{J_{n_{1} / 2}\left(2 \pi \sqrt{\left(r^{2}+K\right)\left|\widehat{\alpha}_{1}\right|}\right)}{\left|\widehat{\alpha}_{1}\right|^{n_{1} / 4}\left|\widehat{\alpha}_{2}\right|^{n_{2} / 4}} \\
\cdot \partial_{r}\left(r^{n_{2} / 2} J_{n_{2} / 2}\left(2 \pi r \sqrt{\left|\widehat{\alpha}_{2}\right|}\right)\right) .
\end{gathered}
$$

## 6. Feynmann expansion for moments.

It will suffice to consider the positive definite case. To compute the functions $H^{(j)}(\widehat{\alpha})$ one can make use of the following ansätze which satisfy the asymptotic properties of the exact functions

$$
\begin{equation*}
H^{(j)}(\widehat{\alpha}) \approx(-2 \widehat{\alpha})^{d / 2-1} P^{(j)}(\widehat{\alpha}) e^{-m \widehat{\alpha}} \tag{6.1}
\end{equation*}
$$

where $m=2 \pi \inf \sigma\left(\mathbb{D}^{-1}\right)$ and the $P^{(j)}(\widehat{\alpha})$ are polynomials of the form

$$
\begin{equation*}
P^{(j)}(\widehat{\alpha})=\sum_{k=0}^{n} c_{k}^{(j)} \widehat{\alpha}^{j} . \tag{6.2}
\end{equation*}
$$

To estimate the coefficients in the polynomials, one can match moments.
Let

$$
\mathbb{B}=i \alpha \mathbb{D}+i \beta\|v\|^{2} \mathbb{D}^{1 / 2} \mathbb{P}_{v} \mathbb{D}^{1 / 2}
$$

To compute the determinant $\operatorname{det}(1+\mathbb{B})$, we can use a Feynmann expansion.

$$
\begin{align*}
\operatorname{det}(1+\mathbb{B})= & \exp (\operatorname{Tr} \log (1+\mathbb{B})) \\
= & \exp \left(\operatorname{Tr} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \mathbb{B}^{k}\right) \\
= & \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \operatorname{Tr} \mathbb{B}^{k}\right) \\
= & \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \operatorname{Tr} \mathbb{B}^{k}\right)^{n}  \tag{6.3}\\
= & 1+\operatorname{Tr} \mathbb{B}+\left(-\frac{1}{2} \operatorname{Tr} \mathbb{B}^{2}+\frac{1}{2}(\operatorname{Tr} \mathbb{B})^{2}\right) . \\
& +\left(\frac{2}{3} \operatorname{Tr} \mathbb{B}^{3}-(\operatorname{Tr} \mathbb{B})\left(\operatorname{Tr} \mathbb{B}^{2}\right)+\frac{1}{3}(\operatorname{Tr} \mathbb{B})^{3}\right) \\
& +\left(-\frac{3}{2} \operatorname{Tr} \mathbb{B}^{4}+\frac{3}{4}\left(\operatorname{Tr} \mathbb{B}^{2}\right)^{2}+2 \operatorname{Tr} \mathbb{B} \operatorname{Tr} \mathbb{B}^{3}\right. \\
& \left.-\frac{3}{2}(\operatorname{Tr} \mathbb{B})^{2} \operatorname{Tr} \mathbb{B}^{2}+\frac{1}{4}(\operatorname{Tr} \mathbb{B})\right) \\
& +\cdots
\end{align*}
$$

Traces of powers of $\mathbb{B}$ are given by
(6.4) $\quad \operatorname{Tr}\left(\mathbb{B}^{k}\right)=i^{k}\left(\frac{\left(v, \mathbb{D}^{k} v\right)}{\|v\|^{2}}\left(\left(\alpha+\beta\|v\|^{2}\right)^{k}-\alpha^{k}\right)+\alpha^{k} \operatorname{Tr} \mathbb{D}^{k}\right)$,
where we use the fact that the projection operator $\mathbb{P}_{v}$ is nilpotent.
Here are some moments computed by means of this formula

$$
\begin{gathered}
\int d \widehat{\alpha} H^{(0)}(\widehat{\alpha})=1, \\
\int d \widehat{\alpha} \widehat{\alpha} H^{(0)}(\widehat{\alpha})=-\frac{1}{2} \operatorname{Tr} \mathbb{D}, \\
\int d \widehat{\alpha} \widehat{\alpha}^{2} H^{(0)}(\widehat{\alpha})=\frac{1}{2} \operatorname{Tr} \mathbb{D}^{2}+\frac{1}{4}(\operatorname{Tr} \mathbb{D})^{2}, \\
\int d \widehat{\alpha} H^{(1)}(\widehat{\alpha})=-\frac{1}{2} \operatorname{Tr} \mathbb{D},
\end{gathered}
$$

$$
\begin{aligned}
& \int d \widehat{\alpha} \widehat{\alpha} H^{(1)}(\widehat{\alpha})=\frac{1}{4} \operatorname{Tr} \mathbb{D}(v, \mathbb{D} v)+\frac{1}{2}\left(v, \mathbb{D}^{2} v\right), \\
& \int d \widehat{\alpha} \widehat{\alpha}^{2} H^{(1)}(\widehat{\alpha})=-\frac{19}{8} \operatorname{Tr} \mathbb{D}(v, \mathbb{D} v)^{2}-\frac{3}{2} \frac{\left.\left(v, \mathbb{D}^{2} v\right)-\operatorname{Tr} \mathbb{D}(v, \mathbb{D} v)\right)}{(v, \mathbb{D} v)} \\
&-\frac{3}{4} \operatorname{Tr} \mathbb{D}\left(\left(v, \mathbb{D}^{2} v\right)\|v\|-(v, \mathbb{D} v)^{2}\right) \\
&+\frac{1}{2}\left(v, \mathbb{D}^{2} v\right)\|v\| \operatorname{Tr} \mathbb{D}+(v, \mathbb{D} v)\left(v, \mathbb{D}^{2} v\right) \\
&-\left(v, \mathbb{D}^{3} v\right)\|v\|
\end{aligned}
$$

The coefficients $c_{k}^{j}$ in (6.2) can be computed by matching the momenta above. This involves solving a linear system. In fact, based on the ansatz in (6.1) we have that

$$
\begin{aligned}
\int d \widehat{\alpha} \widehat{\alpha}^{n} H^{(j)}(\widehat{\alpha}) & \approx \int d \widehat{\alpha}(-2 \widehat{\alpha})^{d / 2-1} P^{(j)}(\widehat{\alpha}) e^{m \widehat{\alpha}} \\
& =\sum_{k=0}^{n} c_{k}^{(j)} \int d \widehat{\alpha}(-2 \widehat{\alpha})^{d / 2-1} \widehat{\alpha}^{j} e^{m \widehat{\alpha}} \\
& =-\sum_{k=0}^{n} c_{k}^{(j)} 2^{d / 2-1} m^{-d / 2-j} \int_{0}^{\infty} d x x^{j+d / 2-1} e^{-x} \\
& =-\sum_{k=0}^{n} c_{k}^{(j)} 2^{d / 2-1} \Gamma\left(j+\frac{d}{2}\right) m^{-d / 2-j}
\end{aligned}
$$

## 7. Conclusions.

In this article, we develop a resummed perturbation expansion for the calculation of high dimensional Gaussian integrals on sets bounded by quadrics. Such integrals arise in the calculation of Value at Risk for large portfolios in the quadratic approximation.

After an initial simultaneous diagonalization of the covariance matrix and of the total gamma of the portfolio, we reduce the calculation of the Value at Risk to an integral of a Gaussian over a high dimensional quadric. This integral is computed using techniques from harmonic analysis, which reduce all calculations to the Fourier transform of quadrics. In the case of positive definite Gammas, this reduces to
explicit Bessel functions; the general case is not much more difficult. The asymptotic behaviour of the relevant functions can be computed analytically up to factors which are smooth and bounded. The moments of these transforms can be computed by means of a Feynmann expansion and can be expressed in terms of linear invariants such as traces and determinants. This yields analytic formulas for the Value at Risk as a function of the time horizon.

Possible applications of this Fourier transform method that we can envisage include:
i) Performing real time monitoring of Value at Risk.
ii) Finding the impact of the sale of one single contract to the global risk exposure in real time, thus permitting to price against the current holdings.
iii) Identifying the risk factors which are mostly responsible for large Value at Risk.
iv) Visualizing and monitoring the risk exposure in terms of few parameters. (The Fourier transform we compute captures all the risk exposure effects in just two variables and contains information about the interplay between delta risk and gamma risk.)
v) Estimating "Bayesian" Value at Risk by integrating the covariance matrix over the Wishart distribution.
vi) VaR calibration from historical P\&L data.

These applications and extensions will be covered in forthcoming papers.

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# On pseudospheres that are quasispheres 

John L. Lewis and Andrew Vogel

Abstract. We construct bounded domains $D$ not equal to a ball in $n \geq 3$ dimensional Euclidean space, $\mathbb{R}^{n}$, for which $\partial D$ is homeomorphic to a sphere under a quasiconformal mapping of $\mathbb{R}^{n}$ and such that $n-1$ dimensional Hausdorff measure equals harmonic measure on $\partial D$.

## 1. Introduction.

Denote points in Euclidean space, $\mathbb{R}^{n}$, by $x=\left(x_{1}, \ldots, x_{n}\right)$ and let $\bar{E}, \partial E$, denote the closure and boundary of $E \subseteq \mathbb{R}^{n}$, respectively. Put $B(x, r)=\{y:|y-x|<r\}$ and $S(x, r)=\{y:|y-x|=r\}$ when $r>0$. Define $k$ dimensional Hausdorff measure, $1 \leq k \leq n$, in $\mathbb{R}^{n}$ as follows: For fixed $\delta>0$ and $E \subseteq \mathbb{R}^{n}$, let $L(\delta)=\left\{B\left(x_{i}, r_{i}\right)\right\}$ be such that $E \subseteq \cup B\left(x_{i}, r_{i}\right)$ and $0<r_{i}<\delta, i=1,2, \ldots$ Set

$$
\phi_{\delta}^{k}(E)=\inf _{L(\delta)}\left(\sum \alpha(k) r_{i}^{k}\right)
$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^{k}$. Then

$$
H^{k}(E)=\lim _{\delta \rightarrow 0} \phi_{\delta}^{k}(E), \quad 1 \leq k \leq n .
$$

Let $D$ be a bounded domain in $\mathbb{R}^{n}$ with $0 \in D$ and $H^{n-1}(\partial D)<+\infty$.

Then $\partial D$ is said to be a pseudo sphere (see [S]) if
a) $D \neq$ ball and there is a homeomorphism $f: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$ with $f(S(0,1))=\partial D$,
b) $h(0)=a \int_{\partial D} h d H^{n-1}$, whenever $h$ is harmonic in $D$ and continuous on $\bar{D}$.

In b), $a$ denotes a constant. The construction of pseudo spheres in $\mathbb{R}^{2}$, which are not circles, was first done by Keldysh and Lavrentiev to show the existence of domains not of Smirnov type (see [KL], [P, Chapter 3]). Also a completely different proof of existence in $\mathbb{R}^{2}$ has been given by Duren, Shapiro, and Shields in [DSS] (see also [Du, Chapter 10]). In higher dimensions we proved in [LV].

Theorem A. There exists a pseudo sphere in $\mathbb{R}^{n}, n \geq 3$.
Recall that a function $g: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$ is said to be $K \geq 1$ quasiconformal on $\mathbb{R}^{n}$ (see $[R],[R e]$ ) if:
i) $g$ is a homeomorphism of $\mathbb{R}^{n}$ onto $\mathbb{R}^{n}$,
ii) $g$ has distributional partial derivatives that are locally $n$-th power integrable,
iii) $\|D g(x)\|^{n} \leq K J_{g}(x)$, almost everywhere.

In iii), $D g(x)=\left(\partial g_{i}(x) / \partial x_{j}\right)$, is the Jacobian matrix of $g$ and $\|D g(x)\|$ is the norm of $D g(x)$ as a linear operator on $\mathbb{R}^{n}$. Also $J_{g}(x)$ (the Jacobian of $g$ at $x$ ) is the determinant of $D g(x)$. In [LV] we asked whether $f$ in the definition of a pseudosphere can also be chosen $K>1$ quasiconformal from $\mathbb{R}^{n}$ to $\mathbb{R}^{n}$ when $n \geq 3$. If so, then $\partial D$ is said to be a $K$ quasisphere. In $\mathbb{R}^{2}$ it follows easily from the geometric construction of Keldysh and Lavrentiev and the Ahlfors three point condition [A] that there exists pseudospheres which are quasispheres. The construction in [DSS] (see also [D, Chapter 10]) is also easily seen to produce pseudospheres that are quasispheres. In this note we answer our own conjecture by proving

Theorem 1. Given $K>1$ there exists a pseudo sphere in $\mathbb{R}^{n}, n \geq 3$, which is a $K$ quasisphere.

We note that the only 1 quasiconformal maps of $\mathbb{R}^{n}$ are Möbius transformations. Also, it was shown by [FL] that convex domains satisfying some mild smoothness conditions cannot be pseudospheres. More generally, let $G$ be Green's function for a bounded domain $D$ with pole at 0 and suppose $B(0,2 s) \subset D$. Assume that

$$
\begin{aligned}
& \text { +) }|\nabla G| \leq M<\infty \text { in } D \backslash B(0, s), \\
& ++) H^{n-1}\left(\partial D \backslash \partial D^{*}\right)=0, \text { where } \\
& \partial D^{*}=\left\{x \in \partial D: \limsup _{r \rightarrow 0} r^{-n} \min \left\{H^{n}(B(x, r) \cap D),\right.\right. \\
& \\
& \left.\left.\quad H^{n}(B(x, r) \backslash D)\right\}>0\right\} .
\end{aligned}
$$

In [LV1, Theorem 5] we showed that if b), +), ++ ) are valid, then $D$ must be a ball. Recall that $\partial D$ is said to be Ahlfors regular if for some $r_{0}>0$ and every $x \in \partial D$ we have $H^{n-1}(B(x, r) \cap \partial D) \approx r^{n-1}$ where $\approx$ means the two quantities are related by constants independent of $x$ and $r, 0<r \leq r_{0}$. This inequality and b) are easily seen to imply + ). Also if $D$ is an NTA domain in the sense of Jerison and Kenig [JK], then ++ ) is valid. We conclude that an NTA domain whose boundary is Ahlfors regular and satisfies b) must be a ball. So in particular if $f$ is a bilipschitz mapping of $\mathbb{R}^{n}$ with $f(S(0,1))=\partial D$ and b) holds, then $D=$ ball. Thus pseudospheres can be nice (quasispheres) but not too nice (Lipschitz).

To point out some of the differences between Theorem 1 and Theorem A we need to recall some details from [LV]. Suppose $a=1$ in the definition of a pseudosphere. To construct $D$, let $D_{0}=B\left(0, \rho_{0}\right)$ and let

$$
G_{0}(x)=(n(n-2) \alpha(n))^{-1}\left(|x|^{2-n}-\rho_{0}^{2-n}\right), \quad x \in B\left(0, \rho_{0}\right),
$$

be Green's function for $B\left(0, \rho_{0}\right)$, where $\rho_{0}$ is chosen so that if $x \in$ $\partial B\left(0, \rho_{0}\right)$, then

$$
\begin{equation*}
\left|\nabla G_{0}(x)\right|=(n \alpha(n))^{-1} \rho_{0}^{1-n}=2 . \tag{1.2}
\end{equation*}
$$

By induction, if $D_{m}$ has been defined for $m=$ a nonnegative integer, we added certain smooth bumps to $\partial D_{m}$ to get $D_{m+1}$ with $D_{m} \subset$ $D_{m+1}$. Then $D=\bigcup D_{m}$. To obtain $f$ we modified the identity mapping slightly in a neighborhood of each bump, to get $h_{m+1}$ a homeomorphism from $\mathbb{R}^{n}$ into $\mathbb{R}^{n}$, with $h_{m+1}\left(\partial D_{m}\right)=\partial D_{m+1}, h_{m+1}\left(D_{m}\right)=D_{m+1}$, for $m=0,1, \ldots$ Put $h_{0}(x)=\rho_{0} x$ and set $f_{k}(x)=h_{k} \circ h_{k-1} \circ \cdots \circ$
$h_{0}(x)$. Then $f=\lim _{k \rightarrow \infty} f_{k}$ uniformly in $\mathbb{R}^{n}$. The problem with our construction in [LV] was that the distortion (i.e. K) could build up under successive iterations. In the present paper we overcome this difficulty by using the so called "mickey mouse" construction which is apparently due to Thurston (oral communication to the first author by Seppo Rickman). Under this construction $h_{m+1}$ is defined in such a way that it is 1 quasi-conformal (i.e. the restriction of a Möbius transformation) in a neighborhood of $H^{n-1}$ almost every point of $\partial D_{m}$. To get $D_{m+2}$ we then only allow bumps to be added that lie in the image of the above neighborhoods. It turns out for $H^{n-1}$ almost every point $x \in \mathbb{R}^{n}$ that we can arrange it so that all functions in the composition defining $f_{k}(x)$, with one exception, are 1 quasiconformal, while the remaining function can be chosen $K$ quasiconformal for fixed $K>1$. We note that the construction of a pseudosphere in $\mathbb{R}^{2}$ given in $[\mathrm{P}]$ also uses circles, but for a different reason. To carry out the above program we have had to overcome certain problems not encountered in [LV]. For example in this paper we added $C^{\infty}$ bumps to $\partial D_{m}$ and consequently were able to use Schauder type theorems to make the desired estimates on the Green's function of $D_{m+1}$. However, to get $h_{m+1}$, as above, we are forced to add non smooth spherical bumps to $\partial D_{m}$. Hence we have to argue that our earlier program can still be used. Also in [LV] we used an important lemma of Wolff [W, Lemma 2.7] for the Green's function of a domain obtained by adding a $C^{\infty}$ bump to a half space. Again we have to verify that Wolff's lemma remains valid for spherical bumps (whose radius is large). As for the proof of Theorem 1 we follow closely the proof of Theorem A in [LV] so the reader is advised to have this paper at hand. In Section 2 we discuss adding spherical bumps to a domain and show inequality (1.1) in [LV] (see (2.3)) is still valid. In Section 3 we use the "mickey mouse construction" to get $D$ and $f$. In Section 4 we add a spherical bump to a half space and show that the conclusion of Wolff's lemma remains true. We then use this lemma in Section 5 to show that (1.3) in [LV] (see (3.14)) still holds. (1.2), (1.3), and (1.9) of [LV] imply that (1.1) b) is valid (see the discussion in Section 3 following (3.11)).

## 2. Spherical bumps.

We assume throughout this section that $\Omega$ is a bounded domain with $0 \in \Omega$. Moreover we assume $\Omega$ is locally Lipschitz. That is given
$y \in \partial \Omega$ there exists $s>0$ such that $B(y, s) \cap \partial \Omega$ is a part of the graph of a Lipschitz function defined on a hyperplane in $\mathbb{R}^{n}$ and $B(y, s) \cap \Omega$ lies above the graph. We also assume that $\partial \Omega$ is connected and the union of a finite number of closed spherical caps with centers in $\Omega$ and the property that each point of $\partial \Omega$ lies in at most two spherical caps. Thus either two caps are disjoint or their intersection is an $n-2$ dimensional "circle" (intuitively cut out by the smaller sphere from the larger sphere). Let $T$ denote the set of points in the union of these "circles". Finally we assume that $F \subset \mathbb{R}^{n}$ is a compact set with $F \cap \partial \Omega \subset T$. We remark that in our construction $F$ will be the set of points where a certain iterate is not 1 quasiconformal. Intuitively we want to avoid this set in modifying $\Omega$ to get $\Omega^{\prime}$ so that successive iterations will not increase $K$. Let $G$ be Green's function for $\Omega$ with pole at 0 . By definition,

$$
G(x)-(n(n-2) \alpha(n))^{-1}|x|^{2-n}, \quad x \in \mathbb{R}^{n},
$$

is harmonic in $\Omega$ and $G$ has boundary value 0 in the sense of Perron-Wiener-Brelot. Using the Kelvin transformation (see $[\mathrm{H}]$ ) we see that each component of

$$
\nabla G(x)=\left(\frac{\partial G}{\partial x_{1}}, \ldots, \frac{\partial G}{\partial x_{n}}\right)
$$

extends to a $C^{\infty}$ function on $\bar{\Omega} \backslash(T \cup\{0\})$. Under this assumption suppose that

$$
\begin{equation*}
|\nabla G|>1, \quad \text { on } \partial \Omega \backslash T . \tag{2.1}
\end{equation*}
$$

Given $\delta, 0<\delta \leq 10^{-20}$, we shall add smooth spherical bumps to $\partial \Omega$ by "pushing out" $\partial \Omega$ along certain small surface elements in $\{x \in \partial \Omega \backslash T$ : $|\nabla G|(x)>1+\delta\}$ of approximate side length $r^{\prime}$. Let $\Omega^{\prime}, G^{\prime}$ be the domain and Green's function with pole at 0 , obtained from this process. Then $\partial \Omega^{\prime}$ will have the same properties as $\partial \Omega$, i.e. it is locally Lipschitz, connected, and the union of a finite number of closed spherical caps with centers in $\Omega^{\prime}$ and the property that each point of $\partial \Omega^{\prime}$ lies in at most two spherical caps. Define $T^{\prime}$ relative to $\Omega^{\prime}$ in the same way that $T$ was defined relative to $\Omega$. Then $\Omega \subset \Omega^{\prime}, T \subset T^{\prime}$ and we shall choose the spherical bumps so that

$$
\begin{equation*}
\left|\nabla G^{\prime}\right|>1, \quad \text { on } \partial \Omega^{\prime} \backslash T^{\prime} \tag{2.2}
\end{equation*}
$$

Also for $t \geq \delta$, we shall have

$$
\begin{equation*}
H^{n-1}\left(\partial \Omega^{\prime}\right) \geq H^{n-1}(\partial \Omega)+\eta(t) H^{n-1}(\{x:|\nabla G(x)|>1+t\}), \tag{2.3}
\end{equation*}
$$

where $\eta$ is a nondecreasing positive function on $(0, \infty)$ which is independent of $\Omega, \Omega^{\prime}$.

Let $0<\sigma_{0}<10^{-3}$ be a small positive number to be chosen in Section 5 and let $l$ be the largest nonnegative integer such that $2^{-l} \sigma_{0}>$ $\delta>0$. Put $\sigma_{k}=2^{-k} \sigma_{0}$, for $k=0,1, \ldots$ and set

$$
\begin{gathered}
E_{k}=\left\{x \in \partial \Omega: 1+\sigma_{k}<|\nabla G(x)| \leq 1+\sigma_{k-1}\right\}, \quad 1 \leq k \leq l+1 \\
E_{0}=\left\{x \in \partial \Omega:|\nabla G(x)|>1+\sigma_{0}\right\}
\end{gathered}
$$

Let $d\left(E_{1}, E_{2}\right)$ denote the Euclidean distance between the sets $E_{1}, E_{2}$ and put

$$
U=\left\{y \in \partial \Omega: d(\{y\}, T)<10^{8} \widehat{r}_{0}\right\},
$$

where $\widehat{r}_{0}>0$ is so small that

$$
\begin{equation*}
H^{n-1}\left(E_{k} \cap U\right) \leq \frac{1}{4} H^{n-1}\left(E_{k}\right), \quad \text { for } 0 \leq k \leq l+1 \tag{2.4}
\end{equation*}
$$

which is possible since $H^{n-1}(T)=0$. Next if $0<\widehat{r}_{1}<1$ is the smallest radius of the spheres whose caps form $\partial \Omega$ we also choose

$$
\begin{equation*}
\widehat{r}_{0} \leq\left(\frac{\delta \widehat{r}_{1}}{10}\right)^{20} \tag{2.5}
\end{equation*}
$$

Let

$$
V=\left\{y \in \bar{\Omega}: d(\{y\}, T) \geq 10^{4} \widehat{r}_{0}\right\} \cap\left\{y \in \bar{\Omega}: d(\{y\},\{0\}) \geq \frac{\rho_{0}}{2}\right\}
$$

where $\rho_{0}$ is as in (1.2) and set

$$
M_{1}=\max _{x \in V} \sum\left|\partial_{\beta} G(x)\right|,
$$

where $\beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{n}\right), 0 \leq|\beta| \leq 2$, is a multiindex and $\partial_{\beta}$ denotes the corresponding partial derivative with respect to $x^{\beta}, x \in V$. We first choose $r_{0}^{\prime}, 0<r_{0}^{\prime} \leq \widehat{r}_{0}$, so that

$$
\begin{equation*}
r_{0}^{\prime} \leq \frac{1}{\left(10 n M_{1}\right)^{10}} . \tag{2.6}
\end{equation*}
$$

Given $y \in \partial \Omega \backslash U$ let $B\left(\bar{y}, \bar{r}_{0}\right)$ denote the reflection of $B\left(y, 10^{4} r_{0}^{\prime}\right)$ with respect to the sphere whose spherical cap $\subset \partial \Omega$ contains $y$. From our assumptions on $\partial \Omega$ we can choose $r_{0}^{\prime}>0$ so small that for any $y \in \partial \Omega \backslash U$,

$$
\begin{align*}
& B\left(y, 10^{4} r_{0}^{\prime}\right) \text { intersects exactly one spherical cap } \subset \partial \Omega, \\
& \text { and }\left(B\left(y, 10^{4} r_{0}^{\prime}\right) \cup B\left(\bar{y}, \bar{r}_{0}\right)\right) \cap F=\varnothing . \tag{2.7}
\end{align*}
$$

From compactness and a standard covering argument it follows for each $r^{\prime}, 0<r^{\prime} \leq r_{0}^{\prime}$, that there exists, $y^{1}, y^{2}, \ldots, y^{N} \in \partial \Omega \backslash U$, such that

$$
\begin{align*}
& \partial \Omega \backslash U \subseteq \bigcup_{i=1}^{N} B\left(y^{i}, 100 r^{\prime}\right) \cap \bar{\Omega} \subseteq V  \tag{2.8}\\
& \text { and } B\left(y^{i}, 10 r^{\prime}\right) \cap B\left(y^{j}, 10 r^{\prime}\right)=\varnothing, \quad i \neq j
\end{align*}
$$

We now construct $\Omega^{\prime}$. Let $L$ be the set of all $y \in\left\{y^{i}\right\}_{1}^{N}$ for which

$$
B\left(y, 100 r^{\prime}\right) \cap\left(\bigcup_{k=0}^{l+1} E_{k}\right) \neq \varnothing
$$

Let $\lambda_{k} \geq 2, k=0, \ldots$ be an increasing sequence of positive numbers to be specified later and set $r_{k}=r^{\prime} / \lambda_{k}$ for $k=0, \ldots, l+1$. For fixed $y \in L$, let $j=j(y)$ be the smallest nonnegative integer with

$$
\begin{equation*}
B\left(y, 100 r^{\prime}\right) \cap E_{j} \neq \varnothing \tag{2.9}
\end{equation*}
$$

We draw a sphere $S(\widetilde{y}, \widetilde{r})$ of radius $\widetilde{r}$, center $\widetilde{y} \in \Omega$ with the following properties
a) $S(\widetilde{y}, \widetilde{r}) \cap \partial \Omega=S\left(y, \sigma_{j}^{2} r_{j}\right) \cap \partial \Omega$,
b) The angle between the normals to $S(\widetilde{y}, \widetilde{r})$ and $\partial \Omega$ at points of intersection is $\sigma_{j}^{2}$,
c) $B(\widetilde{y}, \widetilde{r}) \subset \Omega \cup B\left(y, \sigma_{j}^{2} r_{j}\right)$.

Existence of $S(\widetilde{y}, \widetilde{r})$ as in (2.10) follows from (2.5) and elementary geometry. Define $\Omega^{\prime}$ by
i) $\Omega \backslash\left(\bigcup_{z \in L} B\left(z, \sigma_{j}^{2} r_{j}\right)\right)=\Omega^{\prime} \backslash\left(\bigcup_{z \in L} B\left(z, \sigma_{j}^{2} r_{j}\right)\right)$,
ii) $\partial \Omega^{\prime} \cap \bar{B}\left(y, \sigma_{j}^{2} r_{j}\right)=S(\widetilde{y}, \widetilde{r}) \backslash \Omega$ whenever $y \in\left\{y^{i}\right\}_{1}^{N}$,
iii) $\Omega^{\prime} \cap B\left(y, \sigma_{j}^{2} r_{j}\right)=B(\widetilde{y}, \widetilde{r}) \cap B\left(y, \sigma_{j}^{2} r_{j}\right)$.

From (2.10), (2.11), and (2.7) it is clear that $\partial \Omega^{\prime}$ is locally Lipschitz, connected, and the union of a finite number of closed spherical caps with centers in $\Omega^{\prime}$ and the property that each point of $\partial \Omega^{\prime}$ lies in at most two spherical caps. We now prove (2.2). If $x \in \partial \Omega^{\prime} \cap \partial \Omega$, then it follows from (2.1) and the Hopf boundary maximum principle that (2.2) is true. Otherwise, $x \in S(\widetilde{y}, \widetilde{r}) \cap\left(\partial \Omega^{\prime} \backslash T^{\prime}\right)$ for some $y \in\left\{y^{i}\right\}_{1}^{N}, S(\widetilde{y}, \widetilde{r})$, satisfying (2.7)-(2.11). Using (2.5), (2.10) a), b) and high school geometry it is easily seen for $\sigma_{0}$ small enough that

$$
\begin{equation*}
\frac{\widetilde{r}}{2} \leq r_{j} \leq 2 \widetilde{r} . \tag{2.12}
\end{equation*}
$$

From (2.12), (2.5) we deduce that $S(\widetilde{y}, \widetilde{r}) \cap \bar{\Omega} \subset V$ and thereupon from (2.6), (2.1) as well as Taylor's theorem with remainder that

$$
\begin{equation*}
\left(1-\left(r_{0}^{\prime}\right)^{1 / 2}\right)|\nabla G(y)|\langle\nu, z-y\rangle \leq G(z), \tag{2.13}
\end{equation*}
$$

whenever $z \in S(\widetilde{y}, \widetilde{r})$ and $\langle\nu, z-y\rangle \geq 8 \sigma_{j}^{4} r_{j}$. Here $\nu$ denotes the inner unit normal to $\partial \Omega$ at $y$ and $\langle\cdot, \cdot\rangle$ denotes the inner product on $\mathbb{R}^{n}$. Let $\phi$ be a $C^{\infty}$ function on $\mathbb{R}$ with $\phi=0$ in $\left(-\infty, 8 \sigma_{j}^{4} r_{j}\right), \phi(x) \leq x$, for $x \geq 0$, with equality when $x \geq 16 \sigma_{j}^{4} r_{j}$ and

$$
\left(\sigma_{j}^{4} r_{j}\right)^{i-1}\left|\frac{d^{i}}{d x^{i}}(\phi(x)-x)\right| \leq 10^{4}
$$

whenever $x \geq 0$ and $0 \leq i \leq 2$. Let $h$ be the harmonic function in $B(\widetilde{y}, \widetilde{r})$ which is continuous in $\bar{B}(\widetilde{y}, \widetilde{r})$, with boundary values $h(x)=$ $\phi(\langle\nu, x-y\rangle)$ whenever $x \in S(\widetilde{y}, \widetilde{r})$. Let

$$
H(z)=h(\widetilde{y}+\widetilde{r} z)-\langle\nu, \widetilde{y}-y+\widetilde{r} z\rangle, \quad \text { for } z \in B(0,1) .
$$

Using (2.12) and Schauder type estimates (see [GT]), or direct estimates by way of the Poisson integral for $B(0,1)$ we find that

$$
|\nabla H|(z) \leq c(n) \widetilde{r} \sigma_{j}^{3 / 2}
$$

whenever $\widetilde{y}+\widetilde{r} z \in \partial \Omega^{\prime}$. Transferring back we get

$$
\begin{equation*}
|\nabla h-\nu| \leq c(n) \sigma_{j}^{3 / 2}, \tag{2.14}
\end{equation*}
$$

in $S(\widetilde{y}, \widetilde{r}) \cap \partial \Omega^{\prime}$, where $c(n) \geq 1$ as in the sequel is a constant which only depends on $n$, not necessarily the same at each occurence. Since $G \leq G^{\prime}$ in $\Omega$, we see from (2.13) and the boundary values of $h$ that

$$
\begin{equation*}
\left(1-\left(r_{0}^{\prime}\right)^{1 / 2}\right)|\nabla G(y)| h \leq G^{\prime} \tag{2.15}
\end{equation*}
$$

on $S(\widetilde{y}, \widetilde{r})$. Using the Hopf boundary maximum principle and (2.14), (2.15) it follows that

$$
\begin{equation*}
\left|\nabla G^{\prime}\right| \geq\left(1-\left(r_{0}^{\prime}\right)^{1 / 2}\right)\left(1-c(n) \sigma_{j}^{3 / 2}\right)|\nabla G(y)| \tag{2.16}
\end{equation*}
$$

on $S(\widetilde{y}, \widetilde{r}) \cap \partial \Omega^{\prime}$. Now from (2.5), (2.6), (2.9) we deduce that

$$
|\nabla G(y)| \geq 1+\frac{\sigma_{j}}{2}
$$

Putting this inequality in (2.16) we see for $\sigma_{0}=\sigma_{0}(n)>0$ small enough that (2.2) is true for $x \in S(\widetilde{y}, \widetilde{r}) \cap \partial \Omega^{\prime}$. Hence (2.2) is true on $\partial \Omega^{\prime} \backslash T^{\prime}$.

Next we prove (2.3). To do this observe from (2.5) that since $r^{\prime} \ll \widehat{r}_{1}$ we have

$$
\begin{align*}
H^{n-1}(\partial \Omega & \left.\cap B\left(y, \sigma_{j}^{2} r_{j}\right)\right) \\
& \leq \alpha(n-1)\left(\sigma_{j}^{2} r_{j}\right)^{n-1}-c(n) \widehat{r}_{1}^{-2}\left(\sigma_{j}^{2} r_{j}\right)^{n+1} \tag{2.17}
\end{align*}
$$

Note from (2.5), (2.12), and elementary trigonometry, that the solid angle $\theta$ subtended by $B(\widetilde{y}, \widetilde{r}) \cap \partial \Omega$ with respect to $\widetilde{y}$ satisfies

$$
\begin{equation*}
\left|\theta-\sigma_{j}^{2}\right| \leq \frac{4 \widetilde{r} \sigma_{j}^{2}}{\widehat{r}_{1}} \tag{2.18.a}
\end{equation*}
$$

and

$$
\begin{equation*}
\widetilde{r} \sin \theta \geq \sigma_{j}^{2} r_{j}-\frac{\left(\sigma_{j}^{2} r_{j}\right)^{3}}{100 \widehat{r}_{1}^{2}} \tag{2.18.b}
\end{equation*}
$$

for $\sigma_{0}=\sigma_{0}(n)>0$ small enough. Now using spherical coordinates and (2.18.a) it is easily seen that

$$
H^{n-1}(S(\widetilde{y}, \widetilde{r}) \backslash \Omega) \geq \alpha(n-1)\left(1+\frac{\sigma_{j}^{4}}{c(n)}\right)(\widetilde{r} \sin \theta)^{n-1}
$$

From this inequality, ( $2.18 . \mathrm{b}$ ), and once again (2.5) we conclude that

$$
\begin{equation*}
H^{n-1}\left(\partial \Omega^{\prime} \cap B\left(y, \sigma_{j}^{2} r_{j}\right)\right) \geq\left(1+\frac{\sigma_{j}^{4}}{c(n)}\right) \alpha(n-1)\left(\sigma_{j}^{2} r_{j}\right)^{n-1} \tag{2.19}
\end{equation*}
$$

Combining (2.17), (2.19), and using $\lambda_{j} r_{j}=r^{\prime}$, we find for some $c(n) \geq 1$ that

$$
\begin{align*}
& H^{n-1}\left(\partial \Omega^{\prime} \cap B\left(y, \sigma_{j}^{2} r^{\prime} / \lambda_{j}\right)\right) \\
& \quad \geq\left(1+\frac{\sigma_{j}^{4}}{c(n)}\right) H^{n-1}\left(\partial \Omega \cap B\left(y, \sigma_{j}^{2} \frac{r^{\prime}}{\lambda_{j}}\right)\right) . \tag{2.20}
\end{align*}
$$

Let $\eta(t)=\sigma_{i}^{2 n+2} \lambda_{i}^{1-n} / c_{1}(n)$ for $\sigma_{i+1} \leq t<\sigma_{i}, i=0,1, \ldots$ and set $\eta(t)=\sigma_{0}^{2 n+2} \lambda_{0}^{1-n} / c_{1}(n)$ for $t \geq \sigma_{0}$. Then from (2.20), (2.4), and (2.8) we conclude for $c_{1}(n)$ large enough that (2.3) is true for $t \geq \delta$.

## 3. The Mickey mouse construction.

We continue with the same notation introduced in sections 1-2. Let $\Omega, \Omega^{\prime}, y \in\left\{y^{i}\right\}_{1}^{N}, r^{\prime}, \widetilde{r}=\widetilde{r}(j), S(\widetilde{y}, \widetilde{r}), \lambda_{j}$, and $\sigma_{j}^{2}$ be as in (2.7)-(2.12). Suppose that $B\left(y, 100 r^{\prime}\right) \cap \partial \Omega \subset S\left(w, \rho^{*}\right)$ with $B\left(w, \rho^{*}\right) \subset \Omega$. Choose a Möbius transformation $L$ so that
a) $L\left(B\left(w, \rho^{*}\right)\right)=H=\left\{x \in \mathbb{R}^{n}: x_{2}>0\right\}$,

乃) $L\left(S\left(w, \rho^{*}\right) \cap S(\widetilde{y}, \widetilde{r})\right)=\left\{x \in \mathbb{R}^{n}: x_{1}=x_{2}=0\right\}$,
र) $L(B(\widetilde{y}, \widetilde{r}))=\widetilde{H}$ and $L(\widetilde{H} \backslash H) \subset\left\{x: x_{1}, x_{2}<0\right\}$,
$\delta)$ The angle between the normals to $H, \widetilde{H}$ at points of $\widetilde{H} \cap H$ is $\sigma_{j}^{2}$.
(3.1) is easily proven using (2.10) b), as well as the fact that Möbius transformations preserve angles and map balls into hyperplanes or balls (see [Re, Chapter 3]). We introduce polar coordinates $x_{1}=r \cos \theta$, $x_{2}=r \sin \theta, r \geq 0,0 \leq \theta<2 \pi$. If $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ we put $\widehat{x}=$ $\left(x_{3}, \ldots, x_{n}\right)$ and write $x=\left(x_{1}, x_{2}, \widehat{x}\right)$. Next we define a quasiconformal mapping $q$ of $\mathbb{R}^{n}$ as follows

$$
\begin{gathered}
q(x)=x, \text { when } 0 \leq \theta \leq \sigma_{j}^{2}, \\
q(x)=\left(r \cos \left(\lambda\left(\theta-\sigma_{j}^{2}\right)+\sigma_{j}^{2}\right), r \sin \left(\lambda\left(\theta-\sigma_{j}^{2}\right)+\sigma_{j}^{2}\right), \widehat{x}\right) \\
\text { for } \sigma_{j}^{2}<\theta \leq \pi-\sigma_{j}^{2} \text { with } \lambda=\left(\pi-\sigma_{j}^{2}\right) /\left(\pi-2 \sigma_{j}^{2}\right), \\
q(x)=\left(r \cos \left(\theta+\sigma_{j}^{2}\right), r \sin \left(\theta+\sigma_{j}^{2}\right), \widehat{x}\right), \text { for } \pi-\sigma_{j}^{2}<\theta \leq \pi+\sigma_{j}^{2},
\end{gathered}
$$

$$
\begin{aligned}
& q(x)=\left(r \cos \left(\lambda^{*}\left(\theta-\pi-\sigma_{j}^{2}\right)+\pi+2 \sigma_{j}^{2}\right)\right. \\
& \qquad \\
& \text { for } \left.\left.\pi+\sigma_{j}^{2}<\theta \leq 2 \pi-\lambda_{j}^{2}, \text { with } \lambda^{*}=\left(\pi-\pi-\sigma_{j}^{2}\right)+\pi+2 \sigma_{j}^{2}\right), \widehat{x}\right) \\
& \qquad q(x)=x, \text { for } 2 \pi-\sigma_{j}^{2}<\theta<2 \pi
\end{aligned}
$$

From the above definition of $q$ we note that
i) $q$ maps $H$ onto $H \cup \widetilde{H}$,
ii) $q$ is the identity mapping on

$$
\begin{equation*}
\left\{x: x_{1} / \sqrt{x_{1}^{2}+x_{2}^{2}} \geq \cos \left(\sigma_{j}^{2}\right)\right\} \tag{3.2}
\end{equation*}
$$

iii) $q$ is a rotation on $\left\{x: x_{1} / \sqrt{x_{1}^{2}+x_{2}^{2}} \leq-\cos \left(\sigma_{j}^{2}\right)\right\}$,
iv) $q$ is $1+10 \sigma_{j}^{2}$ quasiconformal on $\mathbb{R}^{n}$.

Put $g(x)=L^{-1} \circ q \circ L(x)$ when $x \in \mathbb{R}^{n}$. From (2.7), (2.12), (3.1) we note that if

$$
\widetilde{F}=L^{-1} \circ q\left(\left\{x:-\cos \left(\sigma_{j}^{2}\right) \leq \frac{x_{1}}{\sqrt{x_{1}^{2}+x_{2}^{2}}} \leq \cos \left(\sigma_{j}^{2}\right)\right\}\right),
$$

then

$$
\begin{align*}
& \left(\widetilde{F} \cup g^{-1}(\widetilde{F})\right) \subset B\left(y, 10^{4} r_{0}^{\prime}\right) \cup B\left(\bar{y}, \bar{r}_{0}\right)  \tag{3.3}\\
& \text { and } \quad\left(\widetilde{F} \cup g^{-1}(\widetilde{F})\right) \cap F=\varnothing
\end{align*}
$$

where the last line follows from (2.7). From (3.2) we also conclude that
a) $g$ is the identity transformation on the unbounded component $I$ of $\mathbb{R}^{n} \backslash g^{-1}(\widetilde{F})$,
b) $g$ is a Möbius transformation on the bounded component $J$ of $\mathbb{R}^{n} \backslash g^{-1}(\widetilde{F})$,
c) $g$ is $1+10 \sigma_{j}^{2}$ quasiconformal on $\mathbb{R}^{n}$.

We do this construction for each $y \in\left\{y^{i}\right\}_{1}^{N}$ obtaining functions $g_{1}, \ldots$, $g_{N}$ and sets $\widetilde{F}_{1}, I_{1}, J_{1}, \ldots, \widetilde{F}_{N}, I_{N}, J_{N}$, corresponding to $y^{1}, y^{2}, \ldots, y^{N}$,
in such a way that (3.3), (3.4) hold with $g=g_{i}, 1 \leq i \leq N$. Define $\Phi: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$ by

$$
\Phi(x)= \begin{cases}x, & \text { when } x \in I_{1} \cap I_{2} \cdots \cap I_{N}  \tag{3.5}\\ g_{i}(x), & \text { when } x \in g_{i}^{-1}\left(\widetilde{F}_{i}\right) \cup J_{i}, 1 \leq i \leq N\end{cases}
$$

We note that $\Phi$ is well defined since from (3.3), (2.8), and (2.12) it follows that the sets $g_{i}^{-1}\left(\widetilde{F}_{i}\right) \cup J_{i}, 1 \leq i \leq N$, are pairwise disjoint. Using this note and (3.5) we conclude that
*) $\Phi(\Omega)=\Omega^{\prime}$ and $\Phi$ is a $1+10 \sigma_{j}^{2}$ quasiconformal mapping of $\mathbb{R}^{n}$ onto $\mathbb{R}^{n}$,
**) $\Phi \equiv$ a Möbius transformation in each component

$$
\begin{equation*}
\text { of } \mathbb{R}^{n} \backslash \Phi^{-1}(\widehat{F}) \text { where } \widehat{F}=\bigcup_{i=1}^{N} \widetilde{F}_{i} \tag{3.6}
\end{equation*}
$$

$$
* * *)\left(\widehat{F} \cup \Phi^{-1}(\widehat{F})\right) \cap F=\varnothing \text { and } F^{\prime}=\widehat{F} \cup F
$$ is compact with $F^{\prime} \cap \partial \Omega^{\prime} \subset T^{\prime}$.

We now construct $D, f$. Let $D_{0}=B\left(0, \rho_{0}\right)$ be as in Section 1 where $\rho_{0}$ is as in (1.2) and set $F=F_{0}=T=T_{0}=\varnothing$. Let $\delta=\delta_{0}=10^{-20} \sigma_{0}$ and put $\Omega=D_{0}$. We use the results in Section 2 to get $\Omega^{\prime}=D_{1}$ satisfying (2.2), (2.3) and $\Phi_{1}=\Phi$ satisfying (3.6) with $\Phi_{1}\left(D_{0}\right)=D_{1}$. Let $F_{1}=F^{\prime}, T_{1}=T^{\prime}$ be the sets obtained from this construction. We now proceed by induction. Suppose $D_{k}, \Phi_{k}, T_{k}, F_{k}$ have been constructed using the results in Section 2 for $m \geq 1$ with

$$
\begin{align*}
& D_{k} \subset D_{k+1}, T_{k} \subset T_{k+1}, F_{k} \subset F_{k+1}  \tag{3.7}\\
& \text { and } F_{k+1} \cap \partial D_{k+1} \subset T_{k+1}, \text { for } 0 \leq k \leq m-1,
\end{align*}
$$

in such a way that

$$
\begin{align*}
&\left|\nabla G_{k}\right|>1, \quad \text { on } \partial D_{k} \backslash T_{k}  \tag{3.8}\\
& H^{n-1}\left(\partial D_{k+1}\right) \geq H^{n-1}\left(\partial D_{k}\right)  \tag{3.9}\\
&+\eta(t) H^{n-1}\left(\left\{x:\left|\nabla G_{k}(x)\right|>1+t\right\}\right),
\end{align*}
$$

whenever $t \geq \delta_{k}=10^{-20} \sigma_{k}, 0 \leq k \leq m-1$. Here $G_{k}$ denotes the Green's function for $D_{k}$ with pole at 0 . We also assume that

$$
\begin{equation*}
\text { (3.6) holds with } \Phi, F, F^{\prime} \text { replaced by } \Phi_{k+1}, F_{k}, F_{k+1} \tag{3.10}
\end{equation*}
$$ respectively, for $0 \leq k \leq m-1$.

We put $\Omega=D_{m}, F=F_{m}, T=T_{m}$, and note from the induction hypothesis, (3.7), that $F \cap \partial \Omega \subset T$. If $\delta=\delta_{m}=10^{-20} \sigma_{m}$, then we can apply the results in Section 2 to get $\Omega^{\prime}=D_{m+1}, T^{\prime}=T_{m+1}$, for which (3.8), (3.9) hold when $k=m+1$. Also using (3.6) we get $F^{\prime}=F_{m+1}$, $\Phi=\Phi_{m+1}$, satisfying (3.10) with $k=m$. By induction we conclude that (3.7)-(3.10) holds, for each nonnegative integer $k$.

Put $D=\bigcup_{0}^{\infty} D_{i}$. We note that $f_{m}=\Phi_{m} \circ \cdots \circ \Phi_{1}$ maps $D_{0}$ onto $D_{m}$. From (3.6) $* * *$ ),$(3.7)$, and (3.10) it is clear for given $x \in \mathbb{R}^{n} \backslash T_{m-1}$ that each function in the composition defining $f_{m}$, with at most one exception, is a Möbius transformation in a neighborhood of $x$. Moreover such an exception is $1+10 \sigma_{0}^{2}$ quasiconformal in a neighborhood of $x$. Thus $f_{m}$ is $1+10 \sigma_{0}^{2}$ quasiconformal on $\mathbb{R}^{n}$ and $f_{m}\left(D_{0}\right)=D_{m}$ for $m \geq 1$. Now $\left\{f_{m}\right\}_{1}^{\infty}$ is a locally bounded sequence of $1+10 \sigma_{0}^{2}$ quasiconformal mappings on $\mathbb{R}^{n}$, so a subsequence (see [Re, Chapter 9] or $[R$, Chapter 6]) of this sequence either converges uniformly to a $1+10 \sigma_{0}^{2}$ quasiconformal $\widehat{f}$ from $\mathbb{R}^{n}$ to $\mathbb{R}^{n}$ or to a constant. Clearly a constant is ruled out. Put $f(x)=\widehat{f}\left(\rho_{0} x\right)$. Then from our construction we conclude that $f(B(0,1))=D, f(S(0,1))=\partial D$. Thus if $10 \sigma_{0}^{2}<K$, then (1.1) a) in Theorem 1 is true.

For the reader's convenience we outline the proof of b) given in [LV]. Using (2.5)-(2.12) it is intuitively clear for $\sigma_{0}$ small enough that $D$ is NTA in the sense of Jerison and Kenig [JK] with constant 1000 (see [LV, Section 4] for details). Also from Green's theorem and (3.8) we see that

$$
\begin{equation*}
H^{n-1}\left(\partial D_{k}\right) \leq \int_{\partial D_{k}}\left|\nabla G_{k}\right| d H^{n-1}=1 \tag{3.11}
\end{equation*}
$$

From (3.11) we see that $D$ is of finite perimeter in the sense of Federer (see [GE]). Thus as $k \longrightarrow \infty$,

$$
\begin{equation*}
\left.\left.H^{n-1}\right|_{\partial D_{k}} \longrightarrow H^{n-1}\right|_{\partial^{*} D}=\left.H^{n-1}\right|_{\partial D} \tag{3.12}
\end{equation*}
$$

Here the convergence is weak convergence as measures. Also $\partial^{*} D$ is the reduced boundary of $D$. To get the last inequality, we note that $\partial^{*} D$
agrees $H^{n-1}$ almost everywhere with the so called measure theoretic boundary of $D$, defined as the set of points where the Lebesgue lower $n$ densities of $D, \mathbb{R}^{n} \backslash D$ are positive. Using the fact that $D$ is NTA, it is easily seen that $\partial D$ equals the measure theoretic boundary of $D$. Hence (3.12) is true (for a more direct proof see [LV, Section 4]). Also observe from (3.9) that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} H^{n-1}\left(\left\{x \in \partial D_{k}:\left|\nabla G_{k}(x)\right|>1+\delta\right\}\right)=0 \tag{3.13}
\end{equation*}
$$

for each $\delta>0$, since otherwise we could use (3.9) and iteration to get a contradiction. Finally we shall show in Section 5 that

$$
\begin{equation*}
\int_{D_{k}}\left|\nabla G_{k}\right| \log \left|\nabla G_{k}\right| d H^{n-1} \leq c<\infty, \quad \text { for } k=0,1, \ldots \tag{3.14}
\end{equation*}
$$

From (3.14) we deduce for $\alpha>1, k=0,1, \ldots$

$$
\begin{equation*}
\log \alpha \int_{\left\{\left|\nabla G_{k}\right|>\alpha\right\}}\left|\nabla G_{k}\right| d H^{n-1} \leq c<+\infty \tag{3.15}
\end{equation*}
$$

Let $g \geq 0$ be a harmonic function in $D$ which is continuous on $\bar{D}$. Then from (3.8), (3.12), and Green's theorem we get

$$
\begin{align*}
g(0) & =\int_{\partial D_{k}} g\left|\nabla G_{k}\right| d H^{n-1}  \tag{3.16}\\
& \geq \int_{\partial D_{k}} g d H^{n-1} \longrightarrow \int_{\partial D} g d H^{n-1}
\end{align*}
$$

as $k \longrightarrow \infty$. To obtain the reverse inequality for fixed $\delta<10^{-3}$ and $\alpha>10^{3}$, put

$$
\begin{gathered}
P_{k}=\left\{x \in \partial D_{k}: 1 \leq\left|\nabla G_{k}(x)\right| \leq 1+\delta\right\} \\
Q_{k}=\left\{x \in \partial D_{k}: 1+\delta<\left|\nabla G_{k}(x)\right| \leq \alpha\right\} \\
L_{k}=\left\{x \in \partial D_{k}:\left|\nabla G_{k}(x)\right|>\alpha\right\}
\end{gathered}
$$

for $k=0,1,2, \ldots$ Then
$g(0)=\int_{\partial D_{k}} g\left|\nabla G_{k}\right| d H^{n-1}=\int_{P_{k}} \cdots+\int_{Q_{k}} \cdots+\int_{L_{k}} \cdots=I_{1}+I_{2}+I_{3}$.

Clearly,

$$
\left|I_{1}\right| \leq(1+\delta) \int_{\partial D_{k}} g d H^{n-1} .
$$

Also from (3.13) we find that

$$
\left|I_{2}\right| \leq \alpha\|g\|_{\infty} H^{n-1}\left(\left\{x \in \partial D_{k}: 1+\delta \leq\left|\nabla G_{k}\right|\right\}\right) \longrightarrow 0,
$$

as $k \longrightarrow \infty$. Here, $\|g\|_{\infty}$ denotes the maximum of $g$ in $\bar{D}$. Using (3.15) we get

$$
\left|I_{3}\right| \leq\|g\|_{\infty} \int_{\left\{\left|\nabla G_{k}\right|>\alpha\right\}}\left|\nabla G_{k}\right| d H^{n-1} \leq c(\log \alpha)^{-1}\|g\|_{\infty} .
$$

Letting $k \longrightarrow \infty$ we obtain from the above estimates and (3.12) that

$$
g(0) \leq(1+\delta) \int_{\partial D} g d H^{n-1}+c(\log \alpha)^{-1}\|g\|_{\infty}
$$

Finally letting $\delta \longrightarrow 0, \alpha \longrightarrow \infty$, we have

$$
g(0) \leq \int_{\partial D} g d H^{n-1} .
$$

In view of (3.16) we conclude that

$$
\begin{equation*}
g(0)=\int_{\partial D} g d H^{n-1}, \tag{3.17}
\end{equation*}
$$

when $g \geq 0$ is continuous on $\bar{D}$ and harmonic in $D$. From (3.17) with $g \equiv 1$ we note that, $H^{n-1}(\bar{\partial})=1$. If $g_{1}$ is continuous on $\bar{D}$, harmonic in $D$, and $g_{1}-m \geq 0$ in $\bar{D}$, then from (3.17) and the above note we deduce
$g_{1}(0)=\left(g_{1}-m\right)(0)+m=\int_{\partial D}\left(g_{1}-m\right) d H^{n-1}+m=\int_{\partial D} g_{1} d H^{n-1}$.
Finally from a simple barrier estimate it is easily seen that for each $y \in T_{1}$,

$$
\infty=\limsup _{x \rightarrow y}\left|\nabla G_{1}\right|(x) \leq \limsup _{x \rightarrow y}|\nabla G|(x) .
$$

From this inequality we conclude that $D \neq$ ball.

Thus $D$ is a pseudosphere and Theorem 1 is true once we have proved (3.14).

## 4. Lemma of Wolff.

If $x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$ we write $x=\left(x^{\prime}, x_{n}\right)$, where $x^{\prime}=$ $\left(x_{1}, \ldots, x_{n-1}\right)$. For given $\varepsilon>0,0<\varepsilon<1 / 10$, define $\phi(\cdot, \varepsilon)$ on $\mathbb{R}^{n-1}$ by

$$
\phi\left(x^{\prime}, \varepsilon\right)= \begin{cases}\varepsilon^{-1}\left(1-\sqrt{1+\varepsilon^{2}\left(1-\left|x^{\prime}\right|^{2}\right)}\right) \\ \quad \text { when } x^{\prime} \in \mathbb{R}^{n-1} \text { and }\left|x^{\prime}\right| \leq 1 \\ 0, & \text { when } x^{\prime} \in \mathbb{R}^{n-1} \text { and }\left|x^{\prime}\right|>1\end{cases}
$$

Put $K=\left\{x \in \mathbb{R}^{n}: x_{n}>0\right\}$ and set $K(\varepsilon)=\left\{x=\left(x^{\prime}, x_{n}\right) \in \mathbb{R}^{n}: x_{n}>\right.$ $\left.\phi\left(x^{\prime}, \varepsilon\right)\right\}$. We note that $\partial K(\varepsilon) \backslash \partial K$ consists of the part of the sphere with center $\left(0, \ldots, 0, \varepsilon^{-1}\right)$ and radius, $\sqrt{\varepsilon^{-2}+1}$ which lies outside $K$. Thus $K(\varepsilon)$ is obtained by adding a spherical bump to $K$. Let $g(\cdot, \varepsilon)$ be the Green's function for $K(\varepsilon)$ with pole at $\infty$. That is, $g(x, \varepsilon)-x_{n}$ is a bounded harmonic function in $K(\varepsilon)$ and $g(\cdot, \varepsilon)$ is continuous on $\bar{K}(\varepsilon)$ with $g(\cdot, \varepsilon) \equiv 0$ on $\partial K(\varepsilon)$. Set

$$
I(\varepsilon)=\int_{\partial K(\varepsilon)}|\nabla g(\cdot, \varepsilon)| \ln |\nabla g|(\cdot, \varepsilon) d H^{n-1}
$$

Next let $\widehat{\theta}\left(x^{\prime}\right)=\left(1-\left|x^{\prime}\right|^{2}\right)^{+}, x^{\prime} \in \mathbb{R}^{n-1}$, where $a^{+}=\max \{a, 0\}$. Let $\theta$ denote the bounded harmonic function on $K$ which is continuous on $\bar{K}$ with $\theta=\widehat{\theta}$ on $\partial K=\mathbb{R}^{n-1}$. Put

$$
\Lambda(\theta)=\int_{\mathbb{R}^{n-1}}\left(\left(\theta_{x_{n}}\right)^{3}-3\left|\nabla^{\prime} \theta\right|^{2} \theta_{x_{n}}\right) d H^{n-1}
$$

where $\nabla^{\prime}$ denotes the gradient in $x^{\prime}$ only. We prove
Lemma 4.1. If $\Lambda(\theta)>0$, then there exists $c^{*}=c^{*}(n) \geq 1$, such that $I(\varepsilon) \leq-\varepsilon^{3} \Lambda(\theta) / 100$, for $0<\varepsilon \leq c^{*}(n)^{-1} \min \{\Lambda(\theta), 1\}$.

Proof. The proof is essentially the same as [W, Lemma 2.12]. However this lemma was proved under the assumption that $\partial K(\varepsilon)$ is smooth $\left(C^{\infty}\right)$ where in our case $\partial K(\varepsilon)$ is just Lipschitz. Therefore we include
some details. We shall show that $I$ has continuous fourth derivatives and $\left|I^{\prime \prime \prime \prime}\right| \leq c(n)$ on $\left(0, \varepsilon_{0}\right)$ for $\varepsilon_{0}=\varepsilon_{0}(n)>0$, sufficiently small. Also it will turn out that the derivatives of $I$ can be found by differentiating under the integral sign as in [W] and $I(0)=I^{\prime}(0)=I^{\prime \prime}(0)=0$, while $I^{\prime \prime \prime}(0)=-(1 / 8) \Lambda(\theta)$. Using Taylor's theorem with remainder we then get Lemma 4.1.

To begin, let $y \in \partial K(\varepsilon)$ and suppose for some $r>0$ that $w$ is harmonic in $K(\varepsilon) \cap B(y, 2 r)$ with continuous boundary values zero on $\partial K(\varepsilon) \cap B(y, 2 r)$ and $|w| \leq M<\infty$ in $B(y, 2 r) \cap K(\varepsilon)$. From a barrier type argument we find for $0<\varepsilon \leq \varepsilon_{0} \leq 1 / 100$, sufficiently small, that

$$
\begin{equation*}
|w|(x) \leq c(n) M\left(\frac{|x-y|}{r}\right)^{9 / 10} \tag{4.2}
\end{equation*}
$$

for $x \in K(\varepsilon) \cap B(y, r)$. With $\varepsilon_{0}$ now fixed let $g(\cdot, z, \varepsilon)$ denote Green's function for $K(\varepsilon)$ with pole at $z \in K(\varepsilon)$ for $0<\varepsilon \leq \varepsilon_{0}$. We note that $g(x, z, \varepsilon) \leq c(n)|x-z|^{2-n}$ since the righthand side is a constant multiple of the Green's function for $\mathbb{R}^{n}$. Let $S=\left\{\left(x^{\prime}, 0\right) \in \mathbb{R}^{n}:\left|x^{\prime}\right|=\right.$ $1\}$ and let $\widehat{x}, \widehat{z} \in \partial K(\varepsilon) \cap(B(0,2) \backslash S)$. Let $x \in B(\widehat{x},|1-|\widehat{x}|| / 2)$, $z \in B(\widehat{z},|1-|\widehat{z}|| / 2)$, with $|1-|\widehat{x}|| \leq|1-|\widehat{z}|| / 16$. Then from (4.2) with $r=|x-z| / 2, y=\widehat{x}, w=g(\cdot, z, \varepsilon)$, and the above note it follows that

$$
\begin{equation*}
g(x, z, \varepsilon) \leq c|1-|x||^{9 / 10}|x-z|^{1-n+1 / 10} . \tag{4.3}
\end{equation*}
$$

Next suppose that $v$ is harmonic in $K(\varepsilon)$ with

$$
v(x)=\int_{\partial K(\varepsilon) \cap B(0,1)}(1-|z|)^{4 / 5}|\nabla g(x, z, \varepsilon)| d H^{n-1} z, \quad x \in K(\varepsilon)
$$

where derivatives of $g(\cdot, \cdot, \varepsilon)$ are with respect to $z$. Under these assumptions we prove for $x \in B(\widehat{x},|1-|\widehat{x}|| / 2) \cap K(\varepsilon)$, and $\widehat{x} \in \partial K(\varepsilon) \cap B(0,2)$ that there exists $\widehat{c}(n) \geq 1$ with

$$
\begin{equation*}
|v(x)| \leq \widehat{c}(n)|1-|x||^{4 / 5} . \tag{4.4}
\end{equation*}
$$

Now from Green's formula,

$$
\begin{equation*}
|v(x)| \leq c(n)|1-|x||^{4 / 5}+\int_{J}(1-|z|)^{4 / 5}|\nabla g(x, z, \varepsilon)| d H^{n-1} z \tag{4.5}
\end{equation*}
$$

where $J=\{z \in \partial K(\varepsilon) \cap B(0,1):|1-|x||<(1 / 100)(1-|z|)\}$. From the Kelvin transformation (see $[\mathrm{H}]$ ), it is easily seen that $g(x, \cdot, \varepsilon)$ extends
to a harmonic function in $B(z,|1-|z||)$ whenever $x$ is not in this ball and $z \in \partial K(\varepsilon) \backslash S$. We shall also denote this extension by $g(x, \cdot, \varepsilon)$. Using this fact, (4.3), and interior estimates for harmonic functions we see that

$$
|\nabla g(x, z, \varepsilon)| \leq c(n)|1-|x||^{9 / 10}|x-z|^{-n+1 / 10}, \quad \text { whenever } z \in J
$$

Putting this estimate in (4.5), using $2|x-z| \geq|1-|z||$ when $z \in J$ and integrating we get (4.4).

Again from the Kelvin transformation, (4.2) with $w=g$, and interior estimates for harmonic functions we observe that

$$
\begin{equation*}
\left|\frac{\partial^{k} g}{\partial x^{\alpha}}\right|(x, \varepsilon) \leq c(k, n)|1-|x||^{-k+9 / 10}, \quad x \in B\left(\widehat{x}, \frac{1}{2}|1-|\widehat{x}||\right) \tag{4.6}
\end{equation*}
$$

whenever $\widehat{x} \in \partial K(\varepsilon) \cap(B(0,2) \backslash S), 0<\varepsilon \leq \varepsilon_{0}$, and $k=0,1, \ldots$ Here $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ is a multi index with $|\alpha|=k$ and $x^{\alpha}=x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}$. Also we have

$$
\begin{equation*}
g\left(x^{\prime}, \phi\left(x^{\prime}, \varepsilon\right), \varepsilon\right) \equiv 0, \quad x^{\prime} \in \mathbb{R}^{n-1}, \quad 0<\varepsilon \leq \varepsilon_{0} \tag{4.7}
\end{equation*}
$$

Next observe for $k=0,1, \ldots$, that $\partial^{k} \phi(\cdot, \varepsilon) / \partial \varepsilon^{k}$ is uniformly Lipschitz for $x^{\prime} \in \mathbb{R}^{n-1}, 0<\varepsilon \leq \varepsilon_{0}$, with

$$
\text { i) }\left|\nabla \frac{\partial^{k} \phi\left(x^{\prime}, \varepsilon\right)}{\partial \varepsilon^{k}}\right| \leq c(n, k),
$$

$$
\begin{equation*}
\text { ii) }\left|\frac{\partial^{k} \phi\left(x^{\prime}, \varepsilon\right)}{\partial \varepsilon^{k}}\right| \leq c(n, k)\left(1-\left|x^{\prime}\right|\right)^{+} \tag{4.8}
\end{equation*}
$$

$$
\text { iii) } \frac{\partial \phi\left(x^{\prime}, 0\right)}{\partial \varepsilon}=-\frac{\left(1-\left|x^{\prime}\right|^{2}\right)^{+}}{2}, \frac{\partial^{2} \phi\left(x^{\prime}, 0\right)}{\partial \varepsilon^{2}} \equiv 0
$$

We claim that (4.6)-(4.8) imply $g(\cdot, \varepsilon)$ has continuous mixed partials in $x, \varepsilon$ of all orders whenever, $x \in B(\widehat{x},|1-|\widehat{x}|| / 2), 0<\varepsilon \leq \varepsilon_{0}$ and $\widehat{x} \in \partial K(\varepsilon) \backslash S$. Moreover if also $|\widehat{x}|<2$,

$$
\begin{equation*}
\left|\frac{\partial^{k+l} g(x, \varepsilon)}{\partial x^{\alpha} \partial \varepsilon^{l}}\right| \leq c(k, l, n)|1-|x||^{-k+4 / 5} \tag{4.9}
\end{equation*}
$$

for $k, l=0,1, \ldots$, while

$$
\begin{equation*}
\left|\frac{\partial^{k+l}\left(g(x, \varepsilon)-x_{n}\right)}{\partial x^{\alpha} \partial \varepsilon^{l}}\right| \leq c(k, l, n)|1+|x||^{-k-n+1} \tag{4.10}
\end{equation*}
$$

with $x \in \partial K(\varepsilon),|x| \geq 2$. (4.9) for $l=0, k=0,1, \ldots$ is implied by (4.6). (4.10) follows from the fact that $g(\cdot, \varepsilon)-x_{n}$ extends to a bounded harmonic function in $\mathbb{R}^{n} \backslash B(0,3 / 2)$ which is zero on $\left\{\left(x^{\prime}, 0\right): x^{\prime} \in\right.$ $\left.\mathbb{R}^{n-1},\left|x^{\prime}\right|>3 / 2\right\}$ and the Poisson integral formula for such functions. Thus (4.9), (4.10) are true for $l=0, k=0,1, \ldots$ We can now proceed by induction to get (4.9), (4.10). We do only the case $l=1, k=0, \ldots$, in detail. From (4.7), (4.8) ii), the mean value theorem from elementary calculus, and (4.6) for $k=1$ we see that

$$
\begin{align*}
\mid g\left(x^{\prime}, \phi\left(x^{\prime}\right.\right. & \left.\left., \varepsilon_{2}\right), \varepsilon_{2}\right)-g\left(x^{\prime}, \phi\left(x^{\prime}, \varepsilon_{2}\right), \varepsilon_{1}\right) \mid \\
= & \left|g\left(x^{\prime}, \phi\left(x^{\prime}, \varepsilon_{2}\right), \varepsilon_{1}\right)-g\left(x^{\prime}, \phi\left(x^{\prime}, \varepsilon_{1}\right), \varepsilon_{1}\right)\right| \\
\leq & c(n) \max \left\{\left|\frac{\partial g}{\partial x_{n}}\right|\left(x^{\prime}, \phi\left(x^{\prime}, \varepsilon\right), \varepsilon\right): 0<\varepsilon \leq \varepsilon_{0}\right\}  \tag{4.11}\\
& \cdot\left|\phi\left(x^{\prime}, \varepsilon_{2}\right)-\phi\left(x^{\prime}, \varepsilon_{1}\right)\right| \\
\leq & c(n)\left|\varepsilon_{2}-\varepsilon_{1}\right|\left(\left(1-\left|x^{\prime}\right|\right)^{+}\right)^{4 / 5}
\end{align*}
$$

for $x^{\prime} \in \partial K\left(\varepsilon_{2}\right) \backslash S, 0<\varepsilon_{1}, \varepsilon_{2} \leq \varepsilon_{0}$. From (4.11) we deduce that $\left\{\left(\varepsilon_{2}-\varepsilon_{1}\right)^{-1}\left(g\left(\cdot, \varepsilon_{2}\right)-g\left(\cdot, \varepsilon_{1}\right)\right)\right\}$ is uniformly bounded and has a continuous extension to $\partial K\left(\varepsilon_{2}\right)$ whenever $0<\varepsilon_{1}, \varepsilon_{2} \leq \varepsilon_{0}$ and $\varepsilon_{1} \neq \varepsilon_{2}$. From the maximum principle for harmonic functions and the Kelvin transformation, it follows that this sequence is harmonic and uniformly bounded in $L\left(\varepsilon_{2}\right)=K\left(\varepsilon_{2}\right) \bigcup\left\{B(\widehat{x},|1-|\widehat{x}||): \widehat{x} \in \partial K\left(\varepsilon_{2}\right) \backslash S\right\}$. Letting $\varepsilon_{1} \longrightarrow \varepsilon_{2}$ it follows that $\partial g / \partial \varepsilon$ is uniformly continous and bounded in $L\left(\varepsilon_{2}\right)$ whenever $0 \leq \varepsilon_{2} \leq \varepsilon_{0}$. Moreover,

$$
\begin{equation*}
\frac{\partial g}{\partial \varepsilon}(x, \varepsilon)=-\frac{\partial g}{\partial x_{n}}(x, \varepsilon) \frac{\partial \phi}{\partial \varepsilon}\left(x^{\prime}, \varepsilon\right), \tag{4.12}
\end{equation*}
$$

with $x=\left(x^{\prime}, \phi\left(x^{\prime}, \varepsilon\right)\right) \in \partial K(\varepsilon) \backslash S$. Using (4.8) ii) and (4.6) with $k=1$ we get $|\partial g / \partial \varepsilon(\cdot, \varepsilon)| \leq c(n)\left((1-|x|)^{+}\right)^{4 / 5}$ on $\partial K(\varepsilon)$. Using this inequality and the maximum principle for bounded harmonic functions in $K(\varepsilon)$ we conclude first that $|\partial g / \partial \varepsilon| \leq c(n) v$, and thereupon from (4.4), the Kelvin transformation, and interior estimates for harmonic functions that (4.9) is true when $l=1, k=0,1, \ldots$ (4.10) follows for $l=1$ by the same reasoning as when $l=0$. Finally since a uniformly convergent sequence of harmonic functions has derivatives which also converge uniformly, it follows that the mixed partial derivatives consisting of one partial derivative in $\varepsilon$ and $k$ partial derivatives in the space variable $x$ are independent of the order of differentiation.

Next we use (4.12) and argue as in (4.11) to obtain that

$$
\begin{align*}
\frac{\partial^{2} g}{\partial \varepsilon^{2}}(x, \varepsilon)= & -2 \frac{\partial^{2} g}{\partial \varepsilon \partial x_{n}}(x, \varepsilon) \frac{\partial \phi}{\partial \varepsilon}\left(x^{\prime}, \varepsilon\right) \\
& -\frac{\partial g}{\partial x_{n}}(x, \varepsilon) \frac{\partial^{2} \phi}{\partial \varepsilon^{2}}\left(x^{\prime}, \varepsilon\right)-\frac{\partial^{2} g}{\partial x_{n}^{2}}(x, \varepsilon)\left(\frac{\partial \phi}{\partial \varepsilon}\right)^{2}\left(x^{\prime}, \varepsilon\right), \tag{4.13}
\end{align*}
$$

whenever $x \in \partial K(\varepsilon)$. Using (4.8) ii) and (4.9) with $l=0,1, k=1,2$, we conclude first that $\left|\partial^{2} g / \partial \varepsilon^{2}(x, \varepsilon)\right| \leq c(n)\left((1-|x|)^{+}\right)^{4 / 5}$ when $x \in \partial K(\varepsilon)$ and thereupon from (4.4), the Kelvin transformation, and interior estimates for harmonic functions, that (4.9) is true when $l=0,1,2$. (4.10) follows by the same reasoning as when $l=0,1$. As above we see that the mixed partial derivatives consisting of two partial derivatives in $\varepsilon$ and $k$ partial derivatives in the space variable $x$ are independent of the order of differentiation. Continuing by induction we get (4.9), (4.10).

Finally observe from a barrier argument that

$$
\begin{equation*}
c(n)|\nabla g(\cdot, \varepsilon)| \geq 1, \quad \text { on } \partial K(\varepsilon) \backslash S, \tag{4.14}
\end{equation*}
$$

for $0<\varepsilon \leq \varepsilon_{0}$. Using (4.9), (4.10), (4.14) we deduce that derivatives of $I$ with respect to $\varepsilon$ of all orders can be found by differentiating under the integral sign defining $I$. Doing this and letting $\varepsilon \longrightarrow 0$ we find that the argument of Wolff [W, pp. 360-362] can be used essentially verbatim. One only needs to check that the second and third partial derivatives of $\phi$ with respect to $\varepsilon$ do not add additional terms in the calculations when $\varepsilon=0$. In fact from (4.8) iii) we see that the second partial of $\phi$ with respect to $\varepsilon$ vanishes identically. Moreover all terms involving the third partial of $\phi$ with respect to $\varepsilon$ vanish at $\varepsilon=0$ (since all second partials of $g(x, 0)=x_{n}$ are identically zero and $\left.|\nabla g(x, 0)| \equiv 1\right)$. Lemma 4.1 now follows from Wolff's argument in the way mentioned at the beginning of the proof.

In order to apply Wolff's lemma we need to show that $\Lambda(\theta)>0$. In fact we shall show in [LVV] that if $\widehat{\psi} \geq 0$, is a radial, nonincreasing, Lipschitz function on $\mathbb{R}^{n-1}$ with compact support and $\widehat{\psi} \not \equiv 0$, then $\Lambda(\psi)>0$. As usual, $\psi$ denotes the bounded harmonic extension of $\widehat{\psi}$ to $K$ which is continuous on $\bar{K}$ with $\psi=\widehat{\psi}$ on $\partial K$. Clearly this result implies

$$
\begin{equation*}
\Lambda(\theta)>0 \tag{4.15}
\end{equation*}
$$

Here we outline a direct method for establishing (4.15) which gives a numerical lower bound for the integral when $n \geq 5$. Using separation
of variables or the Poisson integral formula for harmonic functions in a half space one can show for $r=\left|x^{\prime}\right|$ that

$$
\theta_{x_{n}}\left(x^{\prime}, 0\right)=-c_{n} F\left(\frac{n}{2},-\frac{1}{2}, \frac{n-1}{2}, r^{2}\right), \quad 0<r<1,
$$

where $F(a, b, c, z)$ is the usual hypergeometric function,

$$
c_{n}=\frac{2 \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{3}{2}\right)},
$$

and $\Gamma$ is the Euler gamma function. Writing the hypergeometric function in a series it is easily seen that

$$
-F\left(\frac{n}{2},-\frac{1}{2}, \frac{n-1}{2}, r^{2}\right) \geq-1+\frac{n}{n-1}\left(1-\left(1-r^{2}\right)^{1 / 2}\right) .
$$

Using this estimate and doing some arithmetic we find that

$$
\begin{aligned}
\int_{0}^{1} \theta_{x_{n}}^{3} r^{n-2} d r \geq & c_{n}^{3}(n-1)^{-3} \int_{0}^{1}\left(-1+n\left(1-r^{2}\right)^{1 / 2}\right)^{3} r^{n-2} d r \\
= & c_{n}^{3}(n-1)^{-3}\left(\frac{1}{n-1}+\frac{6 n^{2}}{n^{2}-1}\right) \\
& -6 c_{n}^{2}(n-1)^{-3}\left(1+\frac{n^{2}}{n+2}\right) \\
= & b_{n} .
\end{aligned}
$$

Also we note that

$$
\begin{aligned}
3 \int_{0}^{1} \theta_{x_{n}}\left|\nabla^{\prime} \theta\right|^{2} r^{n-2} d r & =-12 c_{n} \int_{0}^{1} F\left(\frac{n}{2},-\frac{1}{2}, \frac{n-1}{2}\right) r^{n} d r \\
& =\frac{48(n-2)}{(n+2) \pi n} .
\end{aligned}
$$

The last equality is obtained by writing out the series for the integrand and integrating term by term. The series one gets after evaluating at 1 can be written as the sum of several hypergeometric functions evaluated at 1 . Using tables one then gets the last equality. Finally using Stirling's formula and making some more estimates one can show
that $b_{n}+48(n-2) /((n+2) \pi n) \geq 0$ for $n \geq 5$, which in view of (4.16), the above equality, and the fact that $\theta_{x_{n}}(r, 0) \geq 0$ for $r \in(1, \infty)$ (by positivity of $\theta$ ) implies (4.15) for $n \geq 5$. The cases $n=3,4$ can be done separately. A more involved argument using estimates also for $\int_{1}^{\infty} \theta_{x_{n}}^{3} r^{n-2} d r$ can be used to show that for some absolute constant $c$ one has $\Lambda(\theta) \geq c / n, n=3,4, \ldots$ (more details will be supplied upon request).

Next we introduce some notation in order to state some consequences of Lemma 4.1 and (4.15). Let $\Omega_{1}$ be a bounded domain with diameter $\approx 1$ and NTA constant 1000 . Then by definition,
i) (corkscrew condition) For each $x \in \partial \Omega_{1}$,

$$
0<r<1 \text {, there are points } P_{r}(x) \in \Omega_{1}, Q_{r}(x) \in \mathbb{R}^{n} \backslash \Omega_{1},
$$

$$
\text { with }\left|P_{r}(x)-x\right| \leq 1000 r,\left|Q_{r}(x)-x\right| \leq 1000 r \text {, and }
$$

$$
\operatorname{dist}\left(P_{r}(x), \partial \Omega_{1}\right) \geq 1000^{-1} r, \quad \operatorname{dist}\left(Q_{r}(x), \partial \Omega_{1}\right) \geq 1000^{-1} r
$$

ii) (Harnack chain condition) For each $x, y \in \Omega_{1}$ there is a path $\gamma:[0,1] \longrightarrow \Omega_{1}$ with $\gamma(0)=x, \gamma(1)=y$, and with length $\leq 1000|x-y|$. Also $\operatorname{dist}\left(\gamma(t), \partial \Omega_{1}\right) \geq 1000^{-1} \min \{|\gamma(t)-x|,|\gamma(t)-y|\}$ for $t \in[0,1]$.

Next suppose that $\Omega_{1}$ is Lipschitz on scale $t$ with constant 1000. That is assume for each $z \in \partial \Omega_{1}$, there exists a coordinate system such that $\partial \Omega_{1} \cap B(z, t)$ is the graph of a Lipschitz function defined on $\mathbb{R}^{n-1}$ with Lipschitz norm $\leq 1000$. Moreover, $\Omega_{1} \cap B(z, t)$ lies above the graph of this function. Finally assume for some $w \in \partial \Omega_{1}$ and $t>0$ that after a possible rotation of coordinates,

$$
\begin{align*}
& \partial \Omega_{1} \cap B(w, t)=\left\{x: x_{n}=w_{n}\right\} \cap B(w, t) \\
& \Omega_{1} \cap B(w, t)=\left\{x: x_{n}>w_{n}\right\} \cap B(w, t) . \tag{4.17}
\end{align*}
$$

Let $\phi(\cdot, \varepsilon)$ be as defined at the beginning of Section $4, \lambda \geq 2$, and define $\Omega_{2}(\varepsilon) \supset \Omega_{1}$ for $0<\varepsilon \leq \varepsilon_{0}$, as follows:
a) $\Omega_{1} \backslash B(w, t)=\Omega_{2}(\varepsilon) \backslash B(w, t)$,
b) $\partial \Omega_{2}(\varepsilon) \cap B(w, t)=\left\{\left(x^{\prime}+w^{\prime}, w_{n}+t \lambda^{-1} \phi\left(t^{-1} \lambda x^{\prime}, \varepsilon\right)\right): \quad x^{\prime} \in\right.$ $\left.\mathbb{R}^{n-1}\right\} \cap B(w, t)$,
c) $\Omega_{2}(\varepsilon) \cap B(w, t)=\left\{\left(x^{\prime}+w^{\prime}, x_{n}\right): x_{n}>w_{n}+t \lambda^{-1} \phi\left(t^{-1} \lambda x^{\prime}, \varepsilon\right)\right\} \cap$ $B(w, t)$.

We assume

$$
\begin{equation*}
B\left(0, \rho_{0}\right) \subseteq \Omega_{1} \subseteq B(0,1), \tag{4.18}
\end{equation*}
$$

where $\rho_{0}$ is as in (1.2). Denote Green's functions for $\Omega_{1}, \Omega_{2}(\varepsilon)$, with pole at 0 , by $G_{1}, G_{2}(\cdot, \varepsilon)$ respectively, and let $\omega_{1}$ be harmonic measure on $\Omega_{1}$ with respect to 0 . With this notation we state

Lemma 4.19. Let $\Omega_{1}$ be NTA and Lipschitz on scale $t$ with constant 1000. Suppose $\Omega_{1}$ satisfies (4.17), (4.18), and $\Omega_{2}$ is obtained by adding a spherical bump to $\Omega_{1}$ as in a)-c). Let $\varepsilon_{0}=\left(2 c^{*}(n)\right)^{-1} \min \{\Lambda(\theta), 1\}$, where $c^{*}$ is as in Lemma 4.1. If $0<\widehat{\varepsilon} \leq \varepsilon_{0}$, then there exists $\lambda^{*}=$ $\lambda^{*}(\widehat{\varepsilon}, n), \bar{c}=\bar{c}(\widehat{\varepsilon}, n) \geq 2$, such that for $\lambda \geq \lambda^{*}$,

$$
\begin{aligned}
& \int_{\partial \Omega_{2}(\varepsilon)}\left|\nabla G_{2}(\cdot, \varepsilon)\right| \log \left|\nabla G_{2}\right|(\cdot, \varepsilon) d H^{n-1} \\
& \quad \leq \int_{\partial \Omega_{1}}\left|\nabla G_{1}\right| \log \left|\nabla G_{1}\right| d H^{n-1}-\frac{1}{\bar{c} \lambda^{n-1}} \omega_{1}(B(w, t))
\end{aligned}
$$

whenever $\widehat{\varepsilon} \leq \varepsilon \leq \varepsilon_{0}$.
Proof. In view of Lemma 4.1 and (4.15) we can essentially apply [W, Lemma 2.7] to get Lemma 4.19 in $\mathbb{R}^{3}$. The proof in $\mathbb{R}^{n}, n>3$, is unchanged.

## 5. Proof of Theorem 1.

Armed with Lemma 4.19 we can use the argument in [LV, Section 3] to prove (3.14) and hence complete the proof of Theorem 1. Unfortunately, in [LV, Section 3] Schauder estimates for smooth domains were again used, whereas our boundaries are only locally Lipschitz. Thus for the reader's convenience we sketch the argument in [LV, Section 3] indicating the necessary changes. We wish to apply Lemma 4.19 to $D_{m}, D_{m+1}$ constructed in Section 3, but in order to do so we need to introduce intermediary domains with flat bumps as in Lemma 4.19 and make some estimates. We shall use the same notation as in Section 2. Note that in Section 3 we constructed $D_{m+1}$ from $D_{m}$ by adding
spherical bumps as in (2.9)-(2.11). Thus we work with $\Omega, \Omega^{\prime}$ as in Section 2. We assume, as we may, that $\sigma_{0}^{2} \leq \varepsilon_{0} / 100$ where $\sigma_{0}$ is yet to be fixed and $\varepsilon_{0}$ is as in Lemma 4.19. We now define ( $\lambda_{k}$ ) introduced above (2.9). Let $\widehat{r}_{0}, \widehat{r}_{1}, \delta, M_{1}, r_{0}^{\prime}, r^{\prime},\left(E_{k}\right), L, l, r_{j}$ be as in Section 2. For fixed $y \in L$ recall from (2.9) that $j$ was the least postive integer such that $B\left(y, 100 r^{\prime}\right) \cap E_{j} \neq \varnothing, 1 \leq j \leq l+1$. Let $T$ be the tangent plane to $\partial \Omega$ at $y \in \partial \Omega$. From the above restriction on $\sigma_{0}$, (2.5), we see as in (2.18) a) that the central angle, say $\eta_{j}=\eta_{j}(y)$, subtended by $B(\widetilde{y}, \widetilde{r}) \cap T$ (relative to $\widetilde{y}$ ) satisfies $2^{-1} \sigma_{j}^{2} \leq \eta_{j} \leq 2 \sigma_{j}^{2} \leq \varepsilon_{0} / 4$, regardless of the choice of $\lambda_{j} \geq 2, y \in L$ or $r^{\prime}$. Put $\varepsilon_{j}=\tan \left(\sigma_{j}^{2} / 2\right), \lambda_{j}^{*}=\lambda^{*}\left(\varepsilon_{j}, n\right)$ and set $\lambda_{j}^{\prime}=\max \left\{\sigma_{j}^{-2}, b_{j}, \lambda_{j}^{*}\right\}, j=0,1, \ldots$, where $b_{j}=\bar{c}\left(\varepsilon_{j}, n\right)$. Let $\lambda_{k}=\max _{0 \leq j \leq k} \lambda_{j}^{\prime}, k=0,1, \ldots$, and observe that $\left(\lambda_{k}\right)_{0}^{\infty}$ depends only on $n$ once $\sigma_{0}$ is fixed.

We add flat bumps to $\Omega, \Omega^{\prime}$ as follows. Let $y, j$ be as above and as in (3.1) let $S\left(w, \rho^{*}\right)$ be such that $B\left(y, 100 r^{\prime}\right) \cap \partial \Omega \subset S\left(w, \rho^{*}\right)$ and $B\left(w, \rho^{*}\right) \subset \Omega$. After a rotation if necessary we may assume that $y=$ $\left(w^{\prime}, w_{n}-\rho^{*}\right)$, where $w=\left(w^{\prime}, w_{n}\right)$. Let $A=\rho^{*}-\sqrt{\left(\rho^{*}\right)^{2}-\left(r^{\prime}+\left(r^{\prime}\right)^{3 / 2}\right)^{2}}$ and define $\psi: \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ by

$$
\psi\left(x^{\prime}\right)=\left\{\begin{array}{l}
w_{n}-\rho^{*}, \\
\left(r^{\prime}\right)^{-3 / 2} A\left(\left|x^{\prime}-w^{\prime}\right|-r^{\prime}\right)+w_{n}-\rho^{*}, \\
\quad \text { for } r^{\prime} \leq\left|x^{\prime}-w^{\prime}\right| \leq r^{\prime}+\left(r^{\prime}\right)^{3 / 2}, \\
w_{n}-\sqrt{\left(\rho^{*}\right)^{2}-\left|x^{\prime}-w^{\prime}\right|^{2}} \\
\text { for } r^{\prime}+\left(r^{\prime}\right)^{3 / 2} \leq\left|x^{\prime}-w^{\prime}\right| \leq \rho^{*} .
\end{array}\right.
$$

Note that the graph of $\psi$ coincides with the tangent plane $T$ to $S\left(w, \rho^{*}\right)$ at $y$ when $\left|x^{\prime}-w^{\prime}\right| \leq r^{\prime}$ and "linearly" connects this tangent plane with $S\left(w, \rho^{*}\right)$ when $r^{\prime} \leq\left|x^{\prime}-w^{\prime}\right| \leq r^{\prime}+\left(r^{\prime}\right)^{3 / 2}$. Suppose that $L=$ $\left\{z_{1}, z_{2}, \ldots, z_{p}\right\}$ and put $L_{k}=\left\{z_{1}, \ldots, z_{k}\right\}, 1 \leq k \leq p$.

Define $\widehat{\Omega}_{k}, 1 \leq k \leq p$, by

$$
\begin{aligned}
& \widehat{\mathrm{I}}) \widehat{\Omega}_{k} \backslash\left(\bigcup_{z \in L_{k}} B\left(z, 10 r^{\prime}\right)\right)=\Omega \backslash\left(\bigcup_{z \in L_{k}} B\left(z, 10 r^{\prime}\right)\right), \\
& \widehat{\mathrm{II}}) \partial \widehat{\Omega}_{k} \cap B\left(y, 10 r^{\prime}\right)=\left\{\left(x^{\prime}, \psi\left(x^{\prime}\right)\right)\right\} \cap B\left(y, 10 r^{\prime}\right), \\
& \widehat{\mathrm{III}}) \widehat{\Omega}_{k} \cap B\left(y, 10 r^{\prime}\right)=\left\{\left(x^{\prime}, x_{n}\right): x_{n}>\psi\left(x^{\prime}\right)\right\} \cap B\left(y, 10 r^{\prime}\right),
\end{aligned}
$$

whenever $y \in L_{k}$. Next we define $\widetilde{\Omega}_{k} \supseteq \widehat{\Omega}_{p}, 1 \leq k \leq p$, relative to $\widehat{\Omega}_{k}$ in the same way that $\Omega^{\prime}$ was defined by adding spherical bumps to $\Omega$.

That is,
$\widetilde{I}) \widehat{\Omega}_{k} \backslash\left(\bigcup_{z \in L_{k}} B\left(z, \sigma_{j}^{2} r_{j}\right)\right)=\widetilde{\Omega}_{k} \backslash\left(\bigcup_{z \in L_{k}} B\left(z, \sigma_{j}^{2} r_{j}\right)\right)$,
$\widetilde{I I}) \partial \widetilde{\Omega}_{k} \cap \bar{B}\left(y, \sigma_{j}^{2} r_{j}\right)=\bar{B}\left(y, \sigma_{j}^{2} r_{j}\right) \cap\left(\partial \widehat{\Omega}_{k} \backslash B(\widetilde{y}, \widetilde{r}) \cup S(\widetilde{y}, \widetilde{r}) \backslash \widehat{\Omega}_{k}\right)$, whenever $y \in L_{k}$,
$\widetilde{I I I}) \widetilde{\Omega}_{k} \cap B\left(y, \sigma_{j}^{2} r_{j}\right)=\left(B(\widetilde{y}, \widetilde{r}) \cup \widehat{\Omega}_{k}\right) \cap B\left(y, \sigma_{j}^{2} r_{j}\right)$ whenever $y \in L_{k}$.
Here $\widetilde{y}, \widetilde{r}$ are defined as in (2.10) relative to $y$. From the definition of $\Omega, \Omega^{\prime}$ we see that $\widehat{\Omega}_{k} \supseteq \Omega, \widetilde{\Omega}_{p} \supseteq \Omega^{\prime}$ for $1 \leq k \leq p$. Also from the definition of $\psi$ and (2.5) it can be shown as in [LV, Section 4] that $\widehat{\Omega}_{k}, \widetilde{\Omega}_{k}, 1 \leq k \leq p$, are NTA and Lipschitz on scale $r^{\prime}$ with constant 1000. Let $\widehat{\Omega}_{0}=\Omega, \widetilde{\Omega}_{0}=\widehat{\Omega}_{p}$. From the defininion of $\left\{\lambda_{k}\right\}$ and our restriction on $\sigma_{0}$ we deduce after a possible rotation and translation that Lemma 4.19 can be applied with $\Omega_{1}=\widetilde{\Omega}_{0}, \Omega_{2}=\widetilde{\Omega}_{1}$. Next by the same reasoning we can apply Lemma 4.19 with $\Omega_{1}=\widetilde{\Omega}_{1}$ and $\Omega_{2}=$ $\widetilde{\Omega}_{2}, \ldots$, etc. Let $\widehat{G}_{k}, \widetilde{G}_{k}, \widehat{\omega}_{k}, \widetilde{\omega}_{k}$, be the Green's functions and harmonic measures relative to 0 for $\widehat{\Omega}_{k}, \widehat{\Omega}_{k}$. Applying the above argument $p$ times we obtain an inequality for $\widehat{G}_{p}=\widetilde{G}_{0}$ and $\widetilde{G}_{p}$. Using the definition of $\left\{\lambda_{k}\right\}_{0}^{\infty}$, we conclude

$$
\begin{align*}
& \int_{\partial \widetilde{\Omega}_{p}}\left|\nabla \widetilde{G}_{p}\right| \log \left|\nabla \widetilde{G}_{p}\right| d H^{n-1} \\
& \leq \int_{\partial \widehat{\Omega}_{p}}\left|\nabla \widehat{G}_{p}\right| \log \left|\nabla \widehat{G}_{p}\right| d H^{n-1}  \tag{5.1}\\
&-c(n)\left(\lambda_{l+1}\right)^{-(n-1)} \sum_{k=0}^{p-1} \widetilde{\omega}_{k}\left(B\left(z_{k+1}, 2 r^{\prime}\right)\right) .
\end{align*}
$$

To prove (3.14) we must show that $\widehat{G}_{p}, \widetilde{G}_{p}$, in (5.1) can be replaced by $G, G^{\prime}$, with a manageable error term. To do so we introduce $\Omega_{k}^{\prime}, 0 \leq$ $k \leq p$, defined by $\Omega_{0}^{\prime}=\Omega^{\prime}$, and for $1 \leq k \leq p$,

$$
\begin{aligned}
& \left.\mathrm{I}^{\prime}\right) \Omega_{k}^{\prime} \backslash\left(\bigcup_{z \in L_{k}} B\left(z, 10 r^{\prime}\right)\right)=\Omega^{\prime} \backslash\left(\bigcup_{z \in L_{k}} B\left(z, 10 r^{\prime}\right)\right), \\
& \left.\mathrm{II}^{\prime}\right) \partial \Omega_{k}^{\prime} \cap B\left(y, 10 r^{\prime}\right)=\partial \widetilde{\Omega}_{k} \cap B\left(y, 10 r^{\prime}\right), \\
& \left.\mathrm{III}^{\prime}\right) \Omega_{k}^{\prime} \cap B\left(y, 10 r^{\prime}\right)=\widetilde{\Omega}_{k} \cap B\left(y, 10 r^{\prime}\right),
\end{aligned}
$$

for each $y \in L_{k}$. Denote the corresponding Green's functions and harmonic measures relative to 0 , by $G_{k}^{\prime}, \omega_{k}^{\prime}, 1 \leq k \leq p$. We shall also need the following facts about the NTA domain $\Omega_{1}$ with constant 1000 satisfying (4.18). If $z \in \partial \Omega_{1}$ and $0<\rho \leq 10$, then

$$
\begin{align*}
\omega_{1}(B(z, \rho)) & \leq c(n) \rho^{n-2} \max _{B(z, \rho) \cap \Omega_{1}} G_{1}  \tag{5.2}\\
& \leq c(n) \rho^{n-2} G_{1}\left(P_{\rho}\right) \leq c(n) \omega_{1}(B(z, \rho))
\end{align*}
$$

where $P_{\rho}=P_{\rho}(z)$. Moreover,

$$
\begin{equation*}
\omega_{1}(B(z, 2 \rho)) \leq c(n) \omega_{1}(B(z, \rho)) \tag{5.3}
\end{equation*}
$$

(5.3) is called the doubling inequality for harmonic measure. Also, there exists $\mu=\mu(A)>0$ so that for $z, P_{\rho}$, as above, and $x \in B(z, \rho) \cap \Omega_{1}$,

$$
\begin{equation*}
G_{1}(x) \leq c(n)\left(\frac{|x-z|}{\rho}\right)^{\mu} G_{1}\left(P_{\rho}\right) \tag{5.4}
\end{equation*}
$$

From Harnack's inequality, it follows that there exists $\nu=\nu(n), 1<$ $\nu<\infty$, with

$$
\begin{equation*}
c(n)^{-1} \rho^{\nu} \leq \omega_{1}(B(z, \rho)) \leq 1, \quad 0<\rho \leq 1 . \tag{5.5}
\end{equation*}
$$

Next we note that if $z \in \partial \Omega_{1}$ and $u, v$, are two positive harmonic functions in $\Omega_{1}$ which vanish continuously on $\partial \Omega_{1} \backslash B(z, \rho)$, and $P_{\rho}=$ $P_{\rho}(z)$, then for $x \in \Omega_{1} \backslash B(z, 2 \rho)$

$$
\begin{equation*}
c(n)^{-1} \frac{u\left(P_{\rho}\right)}{v\left(P_{\rho}\right)} \leq \frac{u(x)}{v(x)} \leq c(n) \frac{u\left(P_{\rho}\right)}{v\left(P_{\rho}\right)} . \tag{5.6}
\end{equation*}
$$

Moreover, (5.6) is valid when $u, v$, vanish on $\partial \Omega_{1} \cap B(z, 2 \rho)$, and $x \in B(z, \rho) \cap \Omega_{1}$. (5.6) is called the rate inequality. Next since $\Omega_{1}$ is Lipschitz on scale $t$, we have for $0<t_{1} \leq t$,

$$
\begin{equation*}
t_{1}^{1-n} \int_{B\left(z, t_{1}\right)}\left|\nabla G_{1}\right|^{2} d H^{n-1} \leq c(n)\left(t_{1}^{1-n} \omega_{1}\left(B\left(z, t_{1}\right)\right)\right)^{2} \tag{5.7}
\end{equation*}
$$

which is called an $L^{2}$ reverse Hölder inequality. Using (5.7) and Hölder's inequality one easily deduces the following $A_{\infty}$ type condition. If $E \subset$ $B\left(z, t_{1}\right)$ is a Borel set, then

$$
\begin{equation*}
\frac{\omega_{1}(E)}{\omega_{1}\left(B\left(z, t_{1}\right)\right)} \leq c(n)\left(\frac{H^{n-1}(E)}{H^{n-1}\left(B\left(z, t_{1}\right)\right)}\right)^{1 / 2} \tag{5.8}
\end{equation*}
$$

Also using (5.5), (5.7) and Jensen's inequality one deduces,

$$
\begin{equation*}
\int_{B\left(z, t_{1}\right)}\left|\nabla G_{1}\right||\log | \nabla G_{1}| | d H^{n-1} \leq-c(n) \log t_{1} \omega_{1}\left(B\left(z, t_{1}\right)\right) \tag{5.9}
\end{equation*}
$$

For the proof of (5.2)-(5.6) see [JK, sections 4 and 5]. (5.7) follows from (5.6) and a result of Dahlberg (see [D]). Using (5.2)-(5.5) it follows as in $[\mathrm{LV},(3.10)]$ that

$$
\begin{equation*}
\sum_{k=0}^{p-1} \omega_{k}^{*}\left(B\left(z_{k+1}, 6 r^{\prime}\right)\right) \leq c(n), \tag{5.10}
\end{equation*}
$$

whenever $*$ is an element of $\left\{\wedge, \sim,{ }^{\prime}\right\}$. We show for $0 \leq k \leq p-1$ that

$$
\int_{\partial \Omega_{k}^{\prime}}\left|\nabla G_{k}^{\prime}\right| \log \left|\nabla G_{k}^{\prime}\right| d H^{n-1} \leq \int_{\partial \Omega_{k+1}^{\prime}}\left|\nabla G_{k+1}^{\prime}\right| \log \left|\nabla G_{k+1}^{\prime}\right| d H^{n-1}
$$

$$
\begin{equation*}
+c(n)\left(r^{\prime}\right)^{1 / 8} \omega_{k}^{\prime}\left(B\left(z_{k+1}, 3 r^{\prime}\right)\right), \tag{5.11}
\end{equation*}
$$

$$
\int_{\partial \widehat{\Omega}_{k+1}}\left|\nabla \widehat{G}_{k+1}\right| \log \left|\nabla \widehat{G}_{k+1}\right| d H^{n-1} \leq \int_{\partial \widehat{\Omega}_{k}}\left|\nabla \widehat{G}_{k}\right| \log \left|\nabla \widehat{G}_{k}\right| d H^{n-1}
$$

$$
\begin{equation*}
+c(n)\left(r^{\prime}\right)^{1 / 8} \widehat{\omega}_{k}\left(B\left(z_{k+1}, 3 r^{\prime}\right)\right) . \tag{5.12}
\end{equation*}
$$

Summing (5.11) and using (5.10), it then follows that

$$
\begin{align*}
\int_{\partial \Omega^{\prime}}\left|\nabla G^{\prime}\right| \log \mid & \nabla G^{\prime} \mid d H^{n-1} \\
& \leq \int_{\partial \widetilde{\Omega}_{p}}\left|\nabla \widetilde{G}_{p}\right| \log \left|\nabla \widetilde{G}_{p}\right| d H^{n-1}+c(n)\left(r^{\prime}\right)^{1 / 8}, \tag{5.13}
\end{align*}
$$

where we have used the fact that $\Omega_{0}^{\prime}=\Omega^{\prime}, \Omega_{p}^{\prime}=\widetilde{\Omega}_{p}$. Summing (5.12) and using (5.10), we find

$$
\begin{align*}
& \int_{\partial \widehat{\Omega}_{p}}\left|\nabla \widehat{G}_{p}\right| \log \left|\nabla \widehat{G}_{p}\right| d H^{n-1}  \tag{5.14}\\
& \leq \int_{\partial \Omega}|\nabla G| \log |\nabla G| d H^{n-1}+c(n)\left(r^{\prime}\right)^{1 / 8}
\end{align*}
$$

since $\widehat{\Omega}_{0}=\Omega$. Putting (5.13), (5.14) into (5.1) we get

$$
\begin{align*}
\int_{\partial \Omega^{\prime}}\left|\nabla G^{\prime}\right| \log \mid & \nabla G^{\prime} \mid d H^{n-1}  \tag{5.15}\\
& \leq \int_{\partial \Omega}|\nabla G| \log |\nabla G| d H^{n-1}+c(n)\left(r^{\prime}\right)^{1 / 8}
\end{align*}
$$

Using this inequality in the definition of $D_{m+1}$ we obtain

$$
\begin{align*}
& \int_{\partial D_{m+1}}\left|\nabla G_{m+1}\right| \log \left|\nabla G_{m+1}\right| d H^{n-1} \\
& (5.16) \quad \leq \int_{\partial D_{m}}\left|\nabla G_{m}\right| \log \left|\nabla G_{m}\right| d H^{n-1}+c(n)\left(r^{\prime}\right)^{1 / 8} \tag{5.16}
\end{align*}
$$

where $r^{\prime}=r^{\prime}(m)$. From (2.5) and the definition of $\delta_{k}$ following (3.9) we see that $\sum_{m=0}^{\infty}\left(r^{\prime}(m)\right)^{1 / 8}<\infty$. Hence (3.14) is true and the proof of Theorem 1 is complete after we prove (5.11), (5.12).

We prove only (5.11) for $k=0$, since the proof of all the other inequalities is the same. To prove (5.11) for $k=0$ let $y=z_{1}$ in the definition of $\Omega_{1}^{\prime}$ and let $\psi$ be as defined earlier relative to $y$. If $\widetilde{y}, \widetilde{r}$ are as in (2.10), put

$$
\begin{aligned}
& \tau\left(x^{\prime}\right) \\
& \quad= \begin{cases}\min \left\{\psi\left(x^{\prime}\right), \widetilde{y}_{n}-\sqrt{(\widetilde{r})^{2}-\left|w^{\prime}-x^{\prime}\right|^{2}}\right\}, & \text { for }\left|x^{\prime}-w^{\prime}\right| \leq \widetilde{r} \\
\psi\left(x^{\prime}\right), & \text { for } \widetilde{r} \leq\left|x^{\prime}-w^{\prime}\right| \leq \rho^{*}\end{cases}
\end{aligned}
$$

Then

$$
\begin{gathered}
\partial \Omega_{1}^{\prime} \cap B\left(y, 10 r^{\prime}\right)=\left\{\left(x^{\prime}, \tau\left(x^{\prime}\right)\right)\right\} \cap B\left(y, 10 r^{\prime}\right), \\
\Omega_{1}^{\prime} \cap B\left(y, 10 r^{\prime}\right)=\left\{\left(x^{\prime}, x_{n}\right): x_{n}>\tau\left(x^{\prime}\right)\right\} \cap B\left(y, 10 r^{\prime}\right) .
\end{gathered}
$$

Also if

$$
\sigma\left(x^{\prime}\right)=\left\{\begin{array}{l}
\min \left\{w_{n}-\sqrt{\left(\rho^{*}\right)^{2}-\left|x^{\prime}-w^{\prime}\right|^{2}}, \widetilde{y}_{n}-\sqrt{(\widetilde{r})^{2}-\left|x^{\prime}-w^{\prime}\right|^{2}}\right\} \\
\text { for }\left|x^{\prime}-w^{\prime}\right| \leq \widetilde{r} \\
w_{n}-\sqrt{\left(\rho^{*}\right)^{2}-\left|x^{\prime}-w^{\prime}\right|^{2}} \\
\text { for } \widetilde{r} \leq\left|x^{\prime}-w^{\prime}\right| \leq \rho^{*}
\end{array}\right.
$$

then

$$
\begin{gathered}
\partial \Omega^{\prime} \cap B\left(y, 10 r^{\prime}\right)=\left\{\left(x^{\prime}, \sigma\left(x^{\prime}\right)\right)\right\} \cap B\left(y, 10 r^{\prime}\right), \\
\Omega^{\prime} \cap B\left(y, 10 r^{\prime}\right)=\left\{\left(x^{\prime}, x_{n}\right): x_{n}>\sigma\left(x^{\prime}\right)\right\} \cap B\left(y, 10 r^{\prime}\right) .
\end{gathered}
$$

Next let

$$
\begin{gathered}
K_{1}=\left\{x^{\prime}:\left(x^{\prime}, x_{n}\right) \in S(\widetilde{y}, \widetilde{r}) \cap S\left(w, \rho^{*}\right)\right\}, \\
K_{2}=\left\{x^{\prime}:\left(x^{\prime}, w_{n}-\rho^{*}\right) \in S(\widetilde{y}, \widetilde{r})\right\}, \\
K_{3}=\left\{x^{\prime}: r^{\prime} \leq\left|x^{\prime}-w^{\prime}\right| \leq r^{\prime}+\left(r^{\prime}\right)^{3 / 2}\right\} .
\end{gathered}
$$

Let $K$ be the set of all $x^{\prime} \in \mathbb{R}^{n-1}$ whose distance from $\cup_{i=1}^{3} K_{i}$ is at most $1000\left(r^{\prime}\right)^{3 / 2}$ and set

$$
\begin{gathered}
H=\left\{x^{\prime}:\left|x^{\prime}-w^{\prime}\right|<3 r^{\prime}\right\} \backslash K, \\
K^{\prime}=\left\{\left(x^{\prime}, x_{n}\right) \in \partial \Omega^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right): x^{\prime} \in K\right\}, \\
K_{1}^{\prime}=\left\{\left(x^{\prime}, x_{n}\right) \in \partial \Omega_{1}^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right): x^{\prime} \in K\right\} .
\end{gathered}
$$

We have

$$
\begin{align*}
& \left|\int_{\partial \Omega^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right)}\right| \nabla G^{\prime}|\log | \nabla G^{\prime} \mid d H^{n-1} \\
& \quad-\int_{\partial \Omega_{1}^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right)}\left|\nabla G_{1}^{\prime}\right| \log \left|\nabla G_{1}^{\prime}\right| d H^{n-1} \mid \\
& \leq\left|\int_{K^{\prime}}\right| \nabla G^{\prime}|\log | \nabla G^{\prime}\left|d H^{n-1}-\int_{K_{1}^{\prime}}\right| \nabla G_{1}^{\prime}|\log | \nabla G_{1}^{\prime}\left|d H^{n-1}\right|  \tag{5.17}\\
& \quad+\left|\int_{\left(\partial \Omega^{\prime} \backslash K^{\prime}\right) \cap B\left(z_{1}, 3 r^{\prime}\right)}\right| \nabla G^{\prime}|\log | \nabla G^{\prime} \mid d H^{n-1} \\
& \quad-\int_{\left(\partial \Omega_{1}^{\prime} \backslash K_{1}^{\prime}\right) \cap B\left(z_{1}, 3 r^{\prime}\right)}\left|\nabla G_{1}^{\prime}\right| \log \left|\nabla G_{1}^{\prime}\right| d H^{n-1} \mid \\
& =T_{1}+T_{2}
\end{align*}
$$

To estimate $T_{1}$ we cover $K^{\prime}$ by balls of radius $10\left(r^{\prime}\right)^{3 / 2}$ with centers in $K^{\prime}$ and the property that the balls with the same centers and radius
$\left(r^{\prime}\right)^{3 / 2}$ are disjoint. Using (5.9) in each ball and (5.3), (5.8) with $E=$ $K^{\prime}, t_{1}=3 r^{\prime}$ we deduce after summing that

$$
\begin{aligned}
\left|\int_{K^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right)}\right| \nabla G^{\prime}|\log | \nabla G^{\prime}\left|d H^{n-1}\right| & \leq-c(n) \log r^{\prime} \omega^{\prime}\left(K^{\prime}\right) \\
& \leq c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right)
\end{aligned}
$$

This inequality also holds with $K^{\prime}, G^{\prime}, \omega^{\prime}$ replaced by $K_{1}^{\prime}, G_{1}^{\prime}, \omega_{1}^{\prime}$. Next we observe that it follows in the same way as (5.10) that $\omega_{1}^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right)$ $\approx \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right)$ where $\approx$ means the two quantities are constant multiples of each other (depending only on $n$ ). From the above inequalities we conclude

$$
\begin{equation*}
T_{1} \leq c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right) \tag{5.18}
\end{equation*}
$$

To begin the estimate of $T_{2}$ we write $x$ for $\left(x^{\prime}, \sigma\left(x^{\prime}\right)\right)$ and $\widehat{x}$ for $\left(x^{\prime}, \tau\left(x^{\prime}\right)\right)$ in the following integrals.

$$
\begin{align*}
T_{3}= & \left|\int_{\left(\partial \Omega^{\prime} \backslash K^{\prime}\right) \cap B\left(z_{1}, 3 r^{\prime}\right)}\right| \nabla G_{1}^{\prime}|\log | \nabla G_{1}^{\prime} \mid d H^{n-1} \\
& -\int_{\left(\partial \Omega_{1}^{\prime} \backslash K_{1}^{\prime}\right) \cap B\left(z_{1}, 3 r^{\prime}\right)}\left|\nabla G_{1}^{\prime}\right| \log \left|\nabla G_{1}^{\prime}\right| d H^{n-1} \mid \\
\leq & \int_{H}| | \nabla G_{1}^{\prime}|(x) \log | \nabla G_{1}^{\prime}| |(x) \mid \\
& \cdot \sqrt{1+\left|\nabla \sigma\left(x^{\prime}\right)\right|^{2}}-\sqrt{1+\left|\nabla \tau\left(x^{\prime}\right)\right|^{2}} \mid d x^{\prime} \\
(5.19) & +\int_{H}| | \nabla G_{1}^{\prime}\left|(\widehat{x})-\left|\nabla G_{1}^{\prime}\right|(x)\right||\log | \nabla G_{1}^{\prime}| |(x) \sqrt{1+\left|\nabla \tau\left(x^{\prime}\right)\right|^{2}} d x^{\prime}  \tag{5.19}\\
& +\int_{H}\left|\nabla G_{1}^{\prime}\right|(\widehat{x})|\log | \nabla G_{1}^{\prime}(x)|-\log | \nabla G_{1}^{\prime}(\widehat{x})| | \sqrt{1+\left|\nabla \tau\left(x^{\prime}\right)\right|^{2}} d x^{\prime} \\
= & U_{1}+U_{2}+U_{3} .
\end{align*}
$$

From the definition of $\tau, \sigma$, and (5.9) we find that

$$
\begin{equation*}
U_{1} \leq c(n)\left(r^{\prime}\right)^{1 / 8} \omega_{1}^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right) \leq c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right) \tag{5.20}
\end{equation*}
$$

To estimate $U_{2}, U_{3}$ let $\widehat{x}=\left(x^{\prime}, \tau\left(x^{\prime}\right)\right), x=\left(x^{\prime}, \sigma\left(x^{\prime}\right)\right), x^{\prime} \in H$, be as in (5.19). Then $\widehat{x} \in \partial \Omega_{1}^{\prime} \backslash K_{1}^{\prime}$ and using the Kelvin transformation it is
easily seen that $G_{1}^{\prime}$ extends to a harmonic function in $B\left(\widehat{x}, 2\left(r^{\prime}\right)^{3 / 2}\right)$. If $\rho=\left(r^{\prime}\right)^{3 / 2}$, then from standard estimates for harmonic functions in balls, (5.2), and the fact that $|x-\widehat{x}| \leq c(n)\left(r^{\prime}\right)^{2} / \rho^{*}$, we obtain

$$
\begin{align*}
\left|\nabla G_{1}^{\prime}(x)-\nabla G_{1}^{\prime}(\widehat{x})\right| & \leq c(n)|x-\widehat{x}| \rho^{-2} \max _{B(\widehat{x}, \rho)} G_{1}^{\prime} \\
& \leq c(n)\left(r^{\prime} \rho^{*}\right)^{-1} G_{1}^{\prime}\left(P_{\rho}(\widehat{x})\right)  \tag{5.21}\\
& \leq c(n)\left(r^{\prime} \rho^{*}\right)^{-1} \rho^{2-n} \omega_{1}^{\prime}(B(\widehat{x}, \rho)) .
\end{align*}
$$

Using positivity of $G_{1}^{\prime}$ and (5.3) we also find that

$$
\begin{equation*}
c(n)^{-1} \rho^{1-n} \omega_{1}^{\prime}(B(\widehat{x}, \rho)) \leq\left|\nabla G_{1}^{\prime}(\widehat{x})\right| \leq c(n) \rho^{1-n} \omega_{1}^{\prime}(B(\widehat{x}, \rho)) . \tag{5.22}
\end{equation*}
$$

Putting (5.22) in (5.21) we find in view of (2.5) that

$$
\begin{equation*}
\left|\nabla G_{1}^{\prime}(x)-\nabla G_{1}^{\prime}(\widehat{x})\right| \leq c(n)\left(r^{\prime}\right)^{1 / 4}\left|\nabla G_{1}^{\prime}\right|(\bar{x}), \tag{5.23}
\end{equation*}
$$

where $\bar{x}=x$ or $\widehat{x}$. From (5.23) and (5.9) we see that

$$
\begin{equation*}
U_{2}+U_{3} \leq c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right) \tag{5.24}
\end{equation*}
$$

Using (5.24), (5.20) in (5.19) we deduce

$$
\begin{equation*}
T_{3} \leq c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right) \tag{5.25}
\end{equation*}
$$

If $x \in \partial \Omega^{\prime} \backslash K^{\prime}, \widehat{x}$, and $\rho=\left(r^{\prime}\right)^{3 / 2}$, are as above, then again using the Kelvin transformation we deduce first that $G^{\prime}-G_{1}^{\prime}$ has a harmonic extension to $B(x, 2 \rho)$ and second that

$$
\begin{equation*}
\left|\nabla G^{\prime}-\nabla G_{1}^{\prime}\right|(x) \leq c(n) \rho^{-1} \max _{B(x, \rho)}\left|G^{\prime}-G_{1}^{\prime}\right| . \tag{5.26}
\end{equation*}
$$

We claim that

$$
\begin{align*}
& \max _{B(x, \rho)}\left|G^{\prime}-G_{1}^{\prime}\right| \leq c(n) \max _{B(x, \rho) \cap \Omega^{\prime}}\left|G^{\prime}-G_{1}^{\prime}\right|+c(n) \frac{\left(r^{\prime}\right)^{1 / 2}}{\rho^{*}} G^{\prime}\left(P_{\rho}(x)\right) \\
& (5.27) \tag{5.27}
\end{align*} \quad \leq c(n) \max _{B(x, \rho) \cap \Omega^{\prime}}\left|G^{\prime}-G_{1}^{\prime}\right|+c(n)\left(\left(r^{\prime}\right)^{1 / 2} \frac{\rho}{\rho^{*}}\right)\left|\nabla G^{\prime}\right|(x) .
$$

The second line of (5.27) follows from the first line, (5.2), (5.3), and the same argument as in (5.22). To prove the first line of (5.27) observe that if $x \in S(\widetilde{y}, \widetilde{r}) \cap K^{\prime}$, then this inequality is obvious since both functions
are extended by essentially reflecting across $S(\widetilde{y}, \widetilde{r})$. Otherwise suppose $\bar{z} \in B(x, \rho)$ and $\widetilde{z}, \widehat{z}$ denote the reflection of $\bar{z}$ with respect to the plane $\left\{u \in \mathbb{R}^{n}: u_{n}=w_{n}-\rho^{*}\right\}$ and the sphere $S\left(w, \rho^{*}\right)$, respectively. Then

$$
|\widetilde{z}-\widehat{z}| \leq c(n)\left(r^{\prime}\right)^{1 / 2} \frac{\rho}{\rho^{*}} .
$$

Using this fact, the definition of the Kelvin transformation, and standard estimates for functions vanishing on $B(x, \rho) \cap S\left(w, \rho^{*}\right)$ we obtain (5.27). As noted earlier we have

$$
\omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right) \approx \omega_{1}^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right)
$$

so from (5.2), we have

$$
G^{\prime}\left(P_{r^{\prime}}(x)\right) \approx G_{1}^{\prime}\left(P_{r^{\prime}}(x)\right)
$$

Now if $\sigma_{0}=\sigma_{0}(n)>0$ is small enough, then from this note and a barrier type estimate using interior and exterior cones, we deduce for $r^{\prime} \geq t \geq 2|x-\widehat{x}|$ that

$$
\begin{align*}
& c(n)^{-1}\left(\frac{t}{r^{\prime}}\right)^{11 / 10} \max \left\{G^{\prime}\left(P_{r^{\prime}}(x)\right), G_{1}^{\prime}\left(P_{r^{\prime}}(x)\right)\right\}  \tag{5.28.a}\\
& \leq \min \left\{G^{\prime}\left(P_{t}(x)\right), G_{1}^{\prime}\left(P_{t}(x)\right)\right\}
\end{align*}
$$

and

$$
\begin{align*}
& \max \left\{G^{\prime}\left(P_{t}(x)\right), G_{1}^{\prime}\left(P_{t}(x)\right)\right\} \\
& \quad \leq c(n)\left(\frac{t}{r^{\prime}}\right)^{9 / 10} \min \left\{G^{\prime}\left(P_{r^{\prime}}(x)\right), G_{1}^{\prime}\left(P_{r^{\prime}}(x)\right)\right\} \tag{5.28.b}
\end{align*}
$$

We observe that every point of $\partial \Omega^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right)$ lies within $c(n)\left(r^{\prime}\right)^{2} / \rho^{*}$ of a point of $\partial \Omega_{1}^{\prime}$. From this observation, the maximum principle for harmonic functions, (5.28) and (2.5) we see that

$$
\begin{align*}
\max _{B(x, \rho) \cap \Omega^{\prime}}\left|G^{\prime}-G_{1}^{\prime}\right| & \leq c(n) \max _{\partial \Omega^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right)}\left|G_{1}^{\prime}\right| \\
& \leq c(n)\left(\frac{r^{\prime}}{\rho^{*}}\right)^{9 / 10} G^{\prime}\left(P_{r^{\prime}}(x)\right)  \tag{5.29}\\
& \leq c(n)\left(r^{\prime}\right)^{1 / 4} G^{\prime}\left(P_{\rho}(x)\right) \\
& \leq c(n)\left(r^{\prime}\right)^{1 / 4} \rho\left|\nabla G^{\prime}(x)\right|
\end{align*}
$$

Using (5.27), (5.29) in (5.26) we get

$$
\begin{equation*}
\left|\nabla G^{\prime}-\nabla G_{1}^{\prime}\right|(x) \leq c(n)\left(r^{\prime}\right)^{1 / 4}\left|\nabla G^{\prime}(x)\right| . \tag{5.30}
\end{equation*}
$$

Finally from (5.30), (5.25) we conclude that

$$
\begin{align*}
T_{2} \leq & \left|\int_{\left(\partial \Omega^{\prime} \backslash K^{\prime}\right) \cap B\left(z_{1}, 3 r^{\prime}\right)}\right| \nabla G_{1}^{\prime}|\log | \nabla G_{1}^{\prime} \mid d H^{n-1} \\
& -\int_{\left(\partial \Omega^{\prime} \backslash K^{\prime}\right) \cap B\left(z_{1}, 3 r^{\prime}\right)}\left|\nabla G^{\prime}\right| \log \left|\nabla G^{\prime}\right| d H^{n-1} \mid+T_{3} \\
\leq & \int_{H}| | \nabla G_{1}^{\prime}\left|(x)-\left|\nabla G^{\prime}\right|(x)\right||\log | \nabla G_{1}^{\prime}| |(x) \sqrt{1+\left|\nabla \sigma\left(x^{\prime}\right)\right|^{2}} d x^{\prime} \\
(5.31) & +c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right)  \tag{5.31}\\
& +\int_{H}\left|\nabla G^{\prime}\right|(x)|\log | \nabla G_{1}^{\prime}(x)|-\log | \nabla G^{\prime}(x)| | \sqrt{1+\left|\nabla \sigma\left(x^{\prime}\right)\right|^{2}} d x^{\prime} \\
\leq & c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right) .
\end{align*}
$$

From (5.18), (5.31) and (5.17) we conclude that

$$
\begin{align*}
& \left|\int_{\partial \Omega_{1}^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right)}\right| \nabla G_{1}^{\prime}|\log | \nabla G_{1}^{\prime} \mid d H^{n-1} \\
& \quad-\int_{\partial \Omega^{\prime} \cap B\left(z_{1}, 3 r^{\prime}\right)}\left|\nabla G^{\prime}\right| \log \left|\nabla G^{\prime}\right| d H^{n-1} \mid  \tag{5.32}\\
& \quad \leq c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right)
\end{align*}
$$

Next we note that the argument in [LV] from (3.26) to (3.28) uses only NTA estimates (primarily (5.4) and (5.6)) so is also valid for our current domains. Thus

$$
\begin{align*}
& \left|\int_{\partial \Omega_{1}^{\prime} \backslash B\left(z_{1}, 3 r^{\prime}\right)}\right| \nabla G_{1}^{\prime}|\log | \nabla G_{1}^{\prime} \mid d H^{n-1} \\
& -\int_{\partial \Omega^{\prime} \backslash B\left(z_{1}, 3 r^{\prime}\right)}\left|\nabla G^{\prime}\right| \log \left|\nabla G^{\prime}\right| d H^{n-1} \mid  \tag{5.33}\\
& \quad \leq c(n)\left(r^{\prime}\right)^{1 / 8} \omega^{\prime}\left(B\left(z_{1}, 3 r^{\prime}\right)\right)
\end{align*}
$$

From (5.32), (5.33) we find that (5.11) is valid for $k=0$. Fix $\sigma_{0}=\sigma_{0}(n)>0$ subject to the stipulations in sections 2-5. From our earlier remarks we conclude first (3.14) and thereupon that Theorem 1 is valid.

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[^1]
# Martin boundary for homogeneous riemannian manifolds of negative curvature at the bottom of the spectrum 

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## 0. Introduction.

Let $M$ be a manifold and let $\mathcal{L}$ be a subelliptic second order differential operator on $M$. Positive $\mathcal{L}$-harmonic functions have been intensively studied for many decades. In particular, if $M$ has negative curvature and $\mathcal{L}$ is coercive (i.e. there is a positive $\varepsilon$ such that $\mathcal{L}+\varepsilon I$ admits the Green function), the Martin boundary has been described by A. Ancona [A], and earlier by M. Anderson and Schoen [AS] in the case when $\mathcal{L}$ is the Laplace-Beltrami operator. If $\mathcal{L}$ is noncoercive, the situation is much more complicated, there are no results like in $[\mathrm{A}]$, so various particular cases are of interest.

In this paper we treat noncoercive operators on simply connected homogeneous manifolds of negative curvature. J. Wolf [W] and E. Heintze [Hei] proved that such a manifold is isometric with a solvable Lie group $S=N A$, being a semi-direct product of a nilpotent Lie group $N$ and $A=\mathbb{R}^{+}$and, moreover, for a $H \in \mathcal{A}$ the Lie algebra of $A$ the eigenvalues of $\left.\operatorname{Ad}_{H}\right|_{N}$ are all greater than 0 . Conversely, every such group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature.

On $S$ we consider a second order left-invariant operator

$$
\mathcal{L}=\sum_{j=0}^{m} Y_{j}^{2}+Y,
$$

such that $Y_{0}, \ldots, Y_{m}$ generate $\mathcal{S}$. Let $\pi: S \longrightarrow A=S / N$ be the canonical homomorphism. $d \pi(\mathcal{L})$ is a second order invariant operator on $\mathbb{R}^{+}$, hence

$$
d \pi(\mathcal{L})=\left(a \partial_{a}\right)^{2}-\gamma a \partial_{a}
$$

for a $\gamma \in \mathbb{R}$. $-\gamma a \partial_{a}$ is the $\mathcal{A}$-component of $Y$ and $\mathcal{L}=\mathcal{L}_{\gamma}$ is coercive, if and only if $\gamma \neq 0$.

Let $\mu_{t}$ be the semigroup of measures generated by $\mathcal{L}_{\gamma}$. If $\gamma \geq 0$, then there is a unique (up to a constant) positive Radon measure $\nu_{\gamma}$ on $N$ such that

$$
\check{\mu}_{t}^{\gamma} * \nu_{\gamma}=\nu_{\gamma}, \quad t>0
$$

[E]. For $\gamma>0$ the measure $\nu_{\gamma}$ is bounded, while $\nu_{0}$ is unbounded. The measures $\nu_{\gamma}, \gamma>0$ have been studied in various contexts [B], [E], [G], [Ra], see also [D1], [D2], [DH2], [DHZ]. In particular, the bounded $\mathcal{L}_{\gamma^{-}}$ harmonic functions, $\gamma>0$ are described as $\nu_{\gamma}$-Poisson integrals [Ra], [D1], [DH2] of $L^{\infty}$-functions on $N$. If $\gamma=0$, the only bounded $\mathcal{L}$ harmonic functions are constants but the unbounded measure $\nu_{0}$ gives rise to non-trivial positive $\mathcal{L}_{0}$ harmonic functions.

Also $\nu_{\gamma}$ plays an essential role in description of the Martin boundary for $\mathcal{L}_{\gamma}$ (and $\mathcal{L}_{-\gamma}$ ) both in the coercive and the noncoercive case. However, while the first case can be deduced from Ancona's theory [D2], the latter requires new methods. This is the main topic of our study here.

We make use of a probabilistic method introduced in [DH1] and continued in [DHZ]. The essence of it is a decomposition of the diffusion on $S$ generated by $a^{-2} \mathcal{L}$ into the "vertical component" generated by $\left(\partial_{a}\right)^{2}-(\gamma / a) \partial_{a}$ (Bessel process) and the "horizontal component" for which the transition probabilities conditioned on a trajectory $a_{t}$ of the "vertical component" satisfy some evolution equation (Chapter 3). The idea of this decomposition is very intuitive and goes back to $[\mathrm{M}],[\mathrm{MM}]$, $c f$. also $[\mathrm{K}]$, $[\mathrm{S}]$, [Tay]. The available proofs of the properties of this decomposition are either very sketchy or quite involved. We give here a direct proof of it adapted to the situation of our interest.

The main aim of the present paper is to describe the Martin boundary for $\mathcal{L}_{\gamma}$, for all $\gamma \in \mathbb{R}$. In addition, we find lower and upper pointwise
bounds for $\nu_{\gamma} \cdot \nu_{\gamma}$ turns out to be the main building block for all minimal positive $\mathcal{L}_{\gamma}$.

In the simplest two dimensional case, i.e. when $S=" a x+b$ " the description of the Martin boundary is due to Molchanov, [Mo]. Indeed, his technique is based on properties of the Bessel process, as is ours, only in the two-dimensional case the operator in the horizontal direction can be made independent of the vertical direction which makes the decomposition mentioned above superfluous, and all the arguments are much simpler.

## 1. Preliminaries.

Let

$$
\begin{equation*}
\mathcal{S}=\mathcal{N} \oplus \mathcal{A} \tag{1.0}
\end{equation*}
$$

be a solvable Lie algebra which is the sum of its nilpotent ideal $\mathcal{N}$ and a one-dimensional algebra $\mathcal{A}=\mathbb{R}^{+}$. We assume that
there exists $H \in \mathcal{A}$ such that the real parts
of the eigenvalues of $\operatorname{ad}_{H}: \mathcal{N} \longmapsto \mathcal{N}$ are positive.
Let $N, A, S$ be the connected and simply connected Lie groups whose Lie algebras are $\mathcal{N}, \mathcal{A}, \mathcal{S}$ respectively. Then $S=N A$ is a semi-direct product of $N$ and $A=\mathbb{R}^{+}$.

On $S$ we consider a second order left-invariant operator

$$
\mathcal{L}=\sum_{j=0}^{m} Y_{j}^{2}+Y,
$$

such that $Y_{0}, \ldots, Y_{m}$ generate $\mathcal{S}$. It follows from elementary linear algebra that $Y_{0}, \ldots, Y_{m}$ can be chosen in the way that $Y_{1}(e), \ldots, Y_{m}(e) \in \mathcal{N}$.

The decomposition (1.0) is not unique, i.e. there is no canonical choice of $A$. We put $A=\exp \left\{t Y_{0}: t>0\right\}$ and assume with no loss of generality that the real parts of the eigenvalues of $\operatorname{ad}_{Y_{0}}$ are strictly positive. Moreover, multiplying $\mathcal{L}$ by a constant we may assume that the real parts of $\operatorname{ad}_{Y_{0}}$ are large. Decomposing $s \in S$ as $s=x a, x \in N$,
$a=\exp (\log a)\left(Y_{0}\right)$, we write

$$
\begin{align*}
\mathcal{L} f(x a)= & \mathcal{L}_{\gamma} f(x a) \\
= & \left(\left(a \partial_{a}\right)^{2}-\gamma a \partial_{a}\right) f(x a) \\
& +\left(\sum_{j=1}^{m} \Phi_{a}\left(X_{j}\right)^{2}+\Phi_{a}(X)\right) f(x a), \tag{1.2}
\end{align*}
$$

where $\Phi_{a}=\operatorname{Ad}_{\exp (\log a) Y_{0}}$ and $X, X_{1}, \ldots, X_{m}$ are left-invariant vector fields on $N$ and $X_{1}, \ldots, X_{m}$ generate $\mathcal{N}$. We shall keep the subscript $\gamma$ in $\mathcal{L}$ in order to stress the role of the $\mathcal{A}$-component of $Y$.
(1.1) together with the assumption on the length of $Y_{0}$ imply (see e.g. [DHZ]) that there are $m_{1}, m_{2}>2$ and $C>0$ such that

$$
\begin{equation*}
\left\|\Phi_{a}\right\|_{\mathcal{N} \rightarrow \mathcal{N}} \leq C\left(a^{m_{1}}+a^{m_{2}}\right), \quad a>0 . \tag{1.3}
\end{equation*}
$$

In $N$ we define a "homogeneous" norm $|\cdot|$. Let $(\cdot, \cdot)$ be an arbitrary fixed inner product in $\mathcal{N}$ and let

$$
\langle X, Y\rangle=\int_{0}^{1}\left(\Phi_{a}(X), \Phi_{a}(Y)\right) \frac{d a}{a}, \quad\|X\|=\sqrt{\langle X, X\rangle} .
$$

We put

$$
|\exp X|=|X|=\left(\inf \left\{a>0:\left\|\Phi_{a}(X)\right\| \geq 1\right\}\right)^{-1} .
$$

Since for $X \neq 0$

$$
\begin{aligned}
& \lim _{a \rightarrow 0}\left\|\Phi_{a}(X)\right\|=0 \\
& \lim _{a \rightarrow \infty}\left\|\Phi_{a}(X)\right\|=\infty \\
& \text { and } a \longrightarrow\left\|\Phi_{a}(X)\right\| \text { is increasing, }
\end{aligned}
$$

it follows that for every $Y \neq 0$ there is precisely one $a$ such that

$$
Y=\Phi_{a}(X), \quad|X|=1, \quad|Y|=a
$$

If the action of $A$ on $N$ is diagonal, $|\cdot|$ is the usual homogeneous norm on $N$. Finally, let

$$
\sigma_{a}(\exp X)=\exp (\log a) Y_{0} \exp X \exp (-\log a) Y_{0}
$$

i.e. $\Phi_{a}$ is the differential of $\sigma_{a}$.

The space $\mathcal{H}_{b}$ of bounded harmonic functions for $\mathcal{L}$ is well known. If $\gamma \leq 0$, then bounded harmonic functions are constant. This is a consequence of [BR] (cf. also [DH2]). If $\gamma>0, \mathcal{H}_{b}$ is in one-one correspondence with $L^{\infty}(N)$ via the Poisson integral

$$
\begin{equation*}
F(s)=\int_{N} f(s \cdot x) m_{\gamma}(x) d x \tag{1.4}
\end{equation*}
$$

where $x \longrightarrow s \cdot x$ denotes the action of $S$ on $N=S / A$ ([Ra], [DH2]). $m_{\gamma}$ is a smooth, bounded positive function with $d \nu_{\gamma}(x)=m_{\gamma}(x) d x$ whence $\int_{N} m_{\gamma}(x) d x=1$ ([D]). Moreover [D],

$$
\begin{equation*}
C^{-1}(1+|x|)^{-Q-\gamma} \leq m_{\gamma}(x) \leq C(1+|x|)^{-Q-\gamma}, \quad x \in N . \tag{1.5}
\end{equation*}
$$

For $\gamma>0$ the function $m_{\gamma}$ is uniquely defined by two conditions

$$
\int_{N} m_{\gamma}(x) d x=1
$$

and

$$
P(x a)=a^{-Q} \breve{m}_{\gamma}\left(\sigma_{a^{-1}}(x)\right) \text { is } \mathcal{L} \text {-harmonic . }
$$

It turns out that the probability measure $m_{\gamma}$ is also the basic ingredient in the description of positive harmonic functions for all $\gamma \in$ $\mathbb{R}$.

Let

$$
\begin{equation*}
Q=\operatorname{Re} \operatorname{Tr} \operatorname{ad}_{Y_{0}} \tag{1.6}
\end{equation*}
$$

and

$$
\begin{equation*}
P_{y}(x a)=a^{-Q} \breve{m}_{\gamma}\left(\sigma_{a^{-1}}\left(y^{-1} x\right)\right) \tag{1.7}
\end{equation*}
$$

If $\gamma>0$, the family $\left\{P_{y}\right\}_{y \in N}$ and the function $a^{\gamma}$ are all the minimal positive $\mathcal{L}_{\gamma}$-harmonic functions ([A], cf also [D2]). The proofs (as well as the proof of (1.5)) are based on the Ancona's potential theory on manifolds with negative curvature. Since $\mathcal{L}_{-\gamma} f=a^{-\gamma} \mathcal{L}\left(a^{\gamma} f\right)$, the minimal positive $\mathcal{L}_{-\gamma}$-harmonic functions are 1 and $a^{-\gamma} P_{y}(x a)$.

The case $\gamma=0$ is essentially different, because Ancona's theory does not apply. To examine the Martin kernel we have to estimate the Green function $\mathcal{G}_{0}$ for $\mathcal{L}_{0}$ in another way. The final description of
positive minimal $\mathcal{L}_{0}$-harmonic functions, however, is very similar to the case $\gamma \neq 0$.

Let $\mu_{t}$ be the semigroup of probability measures with the infinitesimal generator $\mathcal{L}_{0}$ and let $\mu=\mu_{1}$. The Markov chain on $N$ with the transition probability

$$
P(x, B)=\breve{\mu} * \delta_{x}(B), \quad x \in N, B \subset N
$$

is a Harris chain with the unique (up to a multiplicative constant) positive Radon measure $\nu_{0}$ such that $\breve{\mu} * \nu_{0}=\nu_{0}$, [E]. $\nu_{0}$ has a smooth density $m_{0}$ which is not integrable in contrast to $m_{\gamma}, \gamma>0$.

The aim of this paper is to show
Theorem. The minimal positive $\mathcal{L}_{0}$-harmonic functions normalized at $e$ are

$$
\begin{align*}
& \text { the constant function } 1 \\
& \text { and } P_{y}(x a)=\frac{1}{m_{0}(y)} a^{-Q} \breve{m}_{0}\left(\sigma_{a^{-1}}\left(y^{-1} x\right)\right) . \tag{1.8}
\end{align*}
$$

Moreover, we have

$$
\begin{equation*}
C^{-1}(1+|x|)^{-Q} \leq m_{0}(x) \leq C(1+|x|)^{-Q}, \quad x \in N . \tag{1.9}
\end{equation*}
$$

To prove the theorem we proceed in the following way. For $\gamma=$ $-2 \alpha \leq 0$ we define a new operator

$$
L_{\gamma}=a^{-2} \mathcal{L}_{\gamma}
$$

which is not left-invariant on $S$. We study it on the space $N \times \mathbb{R}^{+}$. However, it has some homogeneity with respect to the family of "dilations" $D_{r}, r>0$ on $N \times \mathbb{R}^{+}$

$$
D_{r}(x, a)=\left(\sigma_{r}(x), r a\right) .
$$

We have

$$
\begin{equation*}
L_{\gamma}\left(f \cdot D_{r}\right)=r^{2} L_{\gamma} f \cdot D_{r} \tag{1.10}
\end{equation*}
$$

Also $L_{\gamma}$ commutes with the natural action of $N$ on $N \times \mathbb{R}^{+}$on the left.

The Green function $G_{\gamma}$ for $L_{\gamma}$ is given by

$$
\begin{equation*}
G_{\gamma}(x, a ; y, b)=\int_{0}^{\infty} p_{t}(x, a ; y, b) d t \tag{1.11}
\end{equation*}
$$

where

$$
T_{t} f(x a)=\int_{N \times \mathbb{R}^{+}} f(y, b) p_{t}(x, a ; y, b) b^{1+2 \alpha} d y d b
$$

is the heat semigroup on $L^{2}\left(a^{2 \alpha+1}\right)$ generated by $L_{\gamma}$ (see Theorem 5.6). By (1.10)

$$
\begin{equation*}
p_{r^{2} t}(x, a ; y, b)=r^{-Q-2 \alpha-2} p_{t}\left(D_{r^{-1}}(x, a) ; D_{r^{-1}}(y, b)\right) \tag{1.12}
\end{equation*}
$$

and so

$$
\begin{equation*}
G_{\gamma}(x, a ; y, b)=r^{-Q-2 \alpha} G_{\gamma}\left(D_{r^{-1}}(x, a) ; D_{r^{-1}}(y, b)\right) \tag{1.13}
\end{equation*}
$$

The operator $L_{\gamma}^{*}$ conjugate to

$$
L_{\gamma}=\partial_{a}^{2}+(1-\gamma) a^{-1} \partial_{a}+a^{-2} \sum_{j=1}^{m} \Phi_{a}\left(X_{j}\right)^{2}+a^{-2} \Phi_{a}(X)
$$

with respect to the measure $a^{1+2 \alpha} d x d a$ is

$$
L_{\gamma}^{*}=\partial_{a}^{2}+(1-\gamma) a^{-1} \partial_{a}+a^{-2} \sum_{j=1}^{m} \Phi_{a}\left(X_{j}\right)^{2}-a^{-2} \Phi_{a}(X)
$$

Clearly,

$$
p_{t}^{*}(x, a ; y, b)=p_{t}(y, b ; x, a)
$$

and

$$
\begin{equation*}
G_{\gamma}^{*}(x, a ; y, b)=G_{\gamma}(y, b ; x, a) . \tag{1.14}
\end{equation*}
$$

Although the case $\gamma=0$ is the most interesting for us, we keep the assumption $\gamma \leq 0$ to stress that our method works for all those cases. In particular, we obtain new proofs of (1.5) and (1.7). (Again conjugating the operator by $a^{\gamma}$.)

Let $\mathcal{G}_{\gamma}$ be the Green function for $\mathcal{L}_{\gamma}, \gamma \leq 0 . \mathcal{G}_{\gamma}$ is uniquely defined by the following two conditions

$$
\begin{equation*}
\mathcal{L}_{\gamma} \mathcal{G}_{\gamma}(\cdot ; y b)=-\delta_{y b}, \quad \text { as distributions } \tag{1.15}
\end{equation*}
$$

(Functions are identified with distributions via the right Haar measure $a^{-1} d a d x$.)

$$
\begin{equation*}
\text { For every } y b \in S, \mathcal{G}_{\gamma}(\cdot, y b) \text { is a potential for } \mathcal{L}_{\gamma} . \tag{1.16}
\end{equation*}
$$

It turns out that

$$
\begin{equation*}
G_{\gamma}(x, a ; y, b) b^{-\gamma}=\mathcal{G}_{\gamma}(x a ; y b) \tag{1.17}
\end{equation*}
$$

Since the notions of potentials for $L_{\gamma}$ and $\mathcal{L}_{\gamma}$ coincide, the only condition to check is (1.15). By Theorem (5.6) we have

$$
\int G_{\gamma}(x, a ; y, b) L_{\gamma}^{*} \phi(x, a) a^{2 \alpha+1} d a d x=-\phi(y, b) .
$$

But

$$
\begin{aligned}
& \int G_{\gamma}(x, a ; y, b) L_{\gamma}^{*} \phi(x, a) a^{2 \alpha+1} d a d x \\
&=\int G_{\gamma}(x, a ; y, b) a^{2-\gamma} L_{\gamma}^{*} \phi(x, a) a^{-1} d a d x \\
&=\int G_{\gamma}(x, a ; y, b) a^{-\gamma} \mathcal{L}_{\gamma}^{*} \phi(x, a) a^{-1} d a d x
\end{aligned}
$$

which shows (1.17).
Using (1.17) we describe the Martin boundary for $\mathcal{L}_{0}$ (Theorem 6.3). The case $\gamma \neq 0$ was described in [D2]. For that we heavily use (1.13) to find appropriate estimates for Martin kernels.
(1.11) can be extended to $b=0$ (see Lemma (5.2) and (5.5)) as the limit of $G_{\gamma}\left(x, a ; y, b_{n}\right), b_{n} \longrightarrow 0$. More precisely,

$$
G_{\gamma}(x, a ; y, 0)=\lim _{b_{n} \rightarrow 0} G_{\gamma}\left(x, a ; y, b_{n}\right)
$$

as Radon measures. Then

$$
\begin{equation*}
\breve{m}_{\gamma}(x)=G_{-\gamma}(x, 1 ; e, 0), \quad \gamma \geq 0 . \tag{1.18}
\end{equation*}
$$

(1.18) follows from the fact that

$$
G_{-\gamma}(x, a ; e, 0)=a^{-Q-2 \alpha} G_{-\gamma}\left(\sigma_{a^{-1}}(x), 1 ; e, 0\right)
$$

is $L_{-\gamma^{-h}}$-harmonic. Hence $a^{-Q-2 \alpha} \breve{m}_{\gamma}\left(\sigma_{a^{-1}}(x)\right)$ is $\mathcal{L}_{-\gamma^{-}}$-harmonic, and so $a^{-Q} \breve{m}_{\gamma}\left(\sigma_{a^{-1}}(x)\right)$ is $\mathcal{L}_{\gamma^{-}}$-harmonic. But the last condition implies that for every $t$

$$
\breve{\mu}_{t} * m_{\gamma}=m_{\gamma}, \quad \gamma \geq 0,
$$

which uniquely determines $m_{\gamma}$.
Hence, from estimates on $G$ we conclude estimates for $m_{\gamma}$.

## 2. Bessel Process.

Let $b_{\alpha}(t)$ denotes the Bessel process with a parameter $\alpha \geq 0$, [RY], i.e. a continuous Markov process with state space $[0,+\infty)$ generated by $\Delta=\partial_{a}^{2}+(2 \alpha+1 / a) \partial_{a}, \alpha \geq 0$.

The transition function with respect to the measure $y^{2 \alpha+1} d y$ is given by ([RY])

$$
= \begin{cases}c(\alpha) \frac{1}{2 t} \exp \left(\frac{-x^{2}-y^{2}}{4 t}\right) I_{\alpha}\left(\frac{x y}{2 t}\right) \frac{1}{(x y)^{\alpha}}, & \text { for } x, y>0  \tag{t}\\ c(\alpha)(2 t)^{-(\alpha+1)} \exp \left(\frac{-y^{2}}{4 t}\right), & \text { for } x=0, y>0\end{cases}
$$

where

$$
I_{\alpha}(x)=\sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2 k+\alpha}}{k!\Gamma(k+\alpha+1)}
$$

is the Bessel function [L]. Therefore, for $x \geq 0$ and $B \subset(0,+\infty)$

$$
\mathbf{P}_{x}\left(b_{\alpha}(t) \in B\right)=\int_{B} p_{t}(x, y) y^{2 \alpha+1} d y
$$

The Bessel process appears as the vertical component of the diffusion generated by $L_{\gamma}, \gamma=-2 \alpha$. The aim of this chapter is to recall the basic properties of the process $b_{\alpha}(t)$. The proofs are rather standard, we sketch them briefly for reader's convenience.

Lemma 2.2. Let $\Omega$ be the space of trajectories of the Bessel process $b_{\alpha}(t)$. For $b_{\alpha} \in \Omega$ and $\lambda>0$ define $\theta_{\lambda}\left(b_{\alpha}\right)(t)=\sqrt{\lambda} b_{\alpha}(t / \lambda)$. Assume that $b_{\alpha}(t)$ starts from $x$. Then:
i) for every $\lambda>0, \widetilde{b}_{t}=\theta_{\lambda}\left(b_{\alpha}\right)(t)$ is the Bessel process (with a parameter $\alpha$ ) starting from $\sqrt{\lambda} x$,
ii) for every $\lambda>0, x \geq 0$,

$$
\mathbf{E}_{x} f \circ \theta_{\lambda}=\mathbf{E}_{\sqrt{\lambda} x} f .
$$

The Bessel process $b_{\alpha}$ on $\mathbb{R}^{+}$started at $x>0$ satisfies the following stochastic differential equation [RY, p. 416],

$$
b_{\alpha}(t)=x+\beta(t)+(2 \alpha+1) \int_{0}^{t} \frac{1}{b_{\alpha}(s)} d s
$$

where $\beta(t)$ is the one-dimensional Brownian motion started at 0 . Consequently, we have

$$
\mathrm{P}_{x}\left[b_{\alpha}(s) \leq \lambda\right] \leq \mathrm{P}_{0}\left[b_{\alpha}(s) \leq \lambda\right] \quad \text { and } \quad \mathrm{P}_{x}[b(s) \leq \lambda] \leq \mathrm{P}_{x}[\beta(s) \leq \lambda]
$$

Also, by the comparison theorem [RY, p. 364],

$$
\alpha \leq \alpha^{\prime} \text { then for all } s \geq 0, b_{\alpha}(s) \leq b_{\alpha^{\prime}}(s), \quad \text { almost everywhere },
$$

whence

$$
b_{\alpha}(s) \leq\left|\beta_{n}(s)\right|, \quad \text { where } n=[2 \alpha]+3,
$$

and $\beta_{n}$ is the $n$-dimensional Brownian motion.

## Lemma 2.3

$$
\mathrm{P}_{a}\left[\max _{0 \leq s \leq t} \beta_{\alpha}(s) \leq \lambda\right] \leq e^{-\varepsilon\left(t / \lambda^{2}\right)}
$$

Indeed, Let $q=\mathrm{P}_{0}\left[\beta_{\alpha}(1) \leq 1\right]$. Then $q<1$ and

$$
\begin{aligned}
\mathrm{P}_{a}\left[\max _{0 \leq s \leq t} b_{\alpha}(s) \leq \lambda\right] & \leq \mathrm{P}_{a / \lambda}\left[\max _{0 \leq s \leq t / \lambda^{2}} b_{\alpha}(s) \leq 1\right] \\
& \leq \mathrm{E}_{0} \prod_{k=0}^{\left[t / \lambda^{2}\right]} \mathrm{P}_{b_{\alpha}(k)}\left[b_{\alpha}(1) \leq 1\right] \\
& \leq q^{\left[t / \lambda^{2}\right]} \\
& \leq e^{-\varepsilon\left(t / \lambda^{2}\right)}
\end{aligned}
$$

Lemma 2.4. There exist constants $c_{1}, c_{2}$ such that for every $R>0$ and for every $t>0$,

$$
\mathbf{P}_{R}\left(\inf _{s \in[0, t]} b_{\alpha}(s)<\frac{R}{2}\right) \leq c_{1} e^{-c_{2} R^{2} / t}
$$

Indeed,

$$
\mathrm{P}_{R}\left[\inf _{s \in[0, t]} b_{\alpha}(s)<\frac{R}{2}\right] \leq \mathrm{P}_{R}\left[\inf _{s \in[0, t]} \beta(s)<\frac{R}{2}\right] \leq c_{1} e^{-c_{2} R^{2} / t}
$$

Lemma 2.5. There exist constants $c_{1}, c_{2}$ such that for every $x \geq 0$, for every $\lambda>0$ and for every $t>0$,

$$
\mathbf{P}_{x}\left(\sup _{s \in[0, t]} b_{\alpha}(s)>x+\lambda\right) \leq c_{1} e^{-c_{2} \lambda^{2} / t}
$$

Indeed, for $n=[2 \alpha]+3$

$$
\mathbf{P}_{x}\left(\sup _{s \in[0, t]} b_{\alpha}(s)>x+\lambda\right) \leq \mathbf{P}_{x}\left(\sup _{s \in[0, t]} \beta_{n}(s)>x+\lambda\right) \leq c_{1} e^{-c_{2} \lambda^{2} / t}
$$

Lemma 2.6. Let $\xi>0$. There are constants $\delta, c_{1}, c_{2}>0$ such that for every $a \geq 0$ and $A>0$,

$$
\mathbf{P}_{a}\left(\int_{0}^{1} b_{\alpha}^{\xi}(s) d s<A\right) \leq c_{1} e^{-c_{2} A^{-\delta}}
$$

Proof. Given positive $\delta$, we have

$$
\begin{aligned}
& \mathbf{P}_{a}\left(\int_{0}^{1} b^{\xi}(s) d s<A\right) \\
& \leq \mathbf{P}_{a}\left(\sup _{s \in[0,1]} b_{\alpha}(s) \leq 2 A^{\delta}\right) \\
& +\mathbf{P}_{a}\left(\sup _{s \in[0,1]} b_{\alpha}(s)>2 A^{\delta},\left|\left\{s: b_{\alpha}(s)>A^{\delta}\right\}\right|<A^{1-\delta \xi}\right) .
\end{aligned}
$$

By Lemma 2.3,

$$
\mathbf{P}_{a}\left(\sup _{s \in[0,1]} b_{\alpha}(s) \leq 2 A^{\delta}\right) \leq c_{1} e^{-c_{2} A^{-\delta}}
$$

To estimate the probability of

$$
\Omega=\left\{\sup _{s \in[0,1]} b_{\alpha}(s)>2 A^{\delta},\left|\left\{s: b_{\alpha}(s)>A^{\delta}\right\}\right|<A^{1-\delta \xi}\right\},
$$

we define the stopping time $\tau=\inf \left\{s: b_{\alpha}(s)=2 A^{\delta}\right\}$. Then by Lemma 2.4,

$$
\mathbf{P}_{a}(\Omega) \leq \mathbf{E}_{a} \mathbf{P}_{b_{\alpha}(\tau)}\left(\inf _{s \in\left[0, A^{1-\delta \xi]}\right.} b_{\alpha}(s)<\frac{b_{\alpha}(0)}{2}\right) \leq c_{1} e^{-c_{2} A^{2 \delta-1+\delta \xi}}
$$

We choose $\delta$ such that $2 \delta-1+\delta \xi<0$.
Corollary 2.7. Let $\xi \geq 0$. Then

$$
\sup _{a \geq 0} \mathbf{E}_{a}\left(\int_{0}^{1} b_{\alpha}^{\xi}(s) d s\right)^{-D / 2}<+\infty
$$

Proof. Since by the previous Lemma

$$
\mathbf{P}_{a}\left(\frac{1}{n+1} \leq \int_{0}^{1} b_{\alpha}^{\xi}(s) d s \leq \frac{1}{n}\right) \leq c_{1}^{-c_{2} n^{\delta}},
$$

we have

$$
\mathbf{E}_{a}\left(\int_{0}^{1} b_{\alpha}^{\xi}(s) d s\right)^{-D / 2} \leq \sum_{n}(n+1)^{D / 2} e^{-c_{2} n^{\delta}}<+\infty .
$$

## 3. Solution of a heat equation on the product $N \times \mathbb{R}^{+}$.

In this chapter we give an analytic proof of the decomposition of the diffusion on $N \times \mathbb{R}^{+}$into its components. Using it we find a convenient formula for the solution of the heat equation

$$
\left(L_{\gamma}-\partial_{t}\right) u(t, x, a)=0 .
$$

For a multi-index $\beta=\left(\beta_{1}, \ldots, \beta_{k}\right), \beta_{j} \in \mathbb{Z}^{+}$and a basis $X_{1}, \ldots, X_{n}$ of the Lie algebra $\mathcal{N}$ of the Lie group $N$ we write

$$
X^{\beta}=X_{1}^{\beta_{1}} \cdots X_{n}^{\beta_{n}}
$$

For $k=0,1, \ldots, \infty$ we define

$$
C^{k}=\left\{f: X^{\beta} f \in C(N), \text { for }|\beta|<k+1\right\}
$$

and

$$
C_{\infty}^{k}=\left\{f \in C^{k}: \lim _{x \rightarrow \infty} X^{\beta} f(x) \text { exists for }|\beta|<k+1\right\}
$$

For $k<\infty$ the space $C_{\infty}^{k}$ is a Banach space with the norm

$$
\|f\|_{C_{\infty}^{k}}=\sum_{|\beta| \leq k}\left\|X^{\beta} f\right\|_{C(N)} .
$$

Let

$$
L_{\sigma(t)}=\sigma(t)^{-2}\left(\sum\left(\Phi_{\sigma(t)}\left(X_{j}\right)\right)^{2}+\Phi_{\sigma(t)}(X)\right)
$$

For a continuous function $\sigma:[0,+\infty) \longrightarrow[0,+\infty)=A$ let $\left\{U^{\sigma}(s, t)\right.$, $0<s<t\}$ be the (unique) family of bounded operators on $C_{\infty}=C_{\infty}^{0}$ which satisfies
i) $U^{\sigma}(s, s)=I$,
ii) $U^{\sigma}(s, r) U^{\sigma}(r, t)=U^{\sigma}(s, t), s<r<t$,
iii) $\partial_{s} U^{\sigma}(s, t) f=-L_{\sigma(s)} U^{\sigma}(s, t) f$, for every $f \in C_{\infty}$,
iv) $\partial_{t} U^{\sigma}(s, t) f=U^{\sigma}(s, t) L_{\sigma(t)} f$ for every $f \in C_{\infty}$,
v) $U^{\sigma}(s, t): C_{\infty}^{2} \longrightarrow C_{\infty}^{2}$.
$U^{\sigma}(s, t)$ is a convolution operator $U^{\sigma}(s, t) f=f * p^{\sigma}(t, s)$, where $p^{\sigma}(t, s)$ is a probability measure with a smooth density. By ii) we have $p^{\sigma}(t, r) * p^{\sigma}(r, s)=p^{\sigma}(t, s)$ for $t>r>s$. Existence of $U^{\sigma}(s, t)$ follows from [T].

Let $d \mathbf{W}_{a}$ be the probability measure on the space $C\left([0,+\infty), \mathbb{R}^{+}\right)$, for the Bessel process $b_{\alpha}(t)=b_{t}$.

For $f \in C_{c}^{\infty}(N)$ we define

$$
\begin{align*}
u(t, x, a) & =\int U^{\sigma}(0, t) f(x, \sigma(t)) d \mathbf{W}_{a}(\sigma)  \tag{3.1}\\
& =\mathbf{E}_{a} U^{\sigma}(0, t) f(x, \sigma(t))
\end{align*}
$$

Theorem 3.1. Let $\gamma=-2 \alpha$ and let $u=u(t, x, a)$ be the function on $N$ defined by (3.1). Then

$$
L_{\gamma} u(t, x, a)=\partial_{t} u(t, x, a), \quad \text { on } \mathbb{R}^{+} \times N \times \mathbb{R}^{+}
$$

$u$ is continuous and

$$
\begin{equation*}
u(0, x, a)=f(x, a), \quad \text { when } t \longrightarrow 0 . \tag{3.2}
\end{equation*}
$$

Proof. First, we prove that $u=u(t, x, a)$ defined in (3.1) is a solution of the integral equation

$$
\begin{equation*}
u(t, x, a)=\mathbf{E}_{a} f\left(x, b_{t}\right)+\int_{0}^{t} \mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right) d s \tag{3.3}
\end{equation*}
$$

To do this we observe that $\mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right)$ is finite. Let $Y_{1}$, $\ldots, Y_{n}$ be a fixed basis of $\mathcal{N}$. Then

$$
\Phi_{a} X_{j}=\alpha_{1}^{j}(a) Y_{1}+\cdots+\alpha_{n}^{j}(a) Y_{n}
$$

where $\alpha_{i}^{j}$, s are continuous functions and $\left|\alpha_{i}^{j}(a)\right| \leq C\left(a^{m_{1}}+a^{m_{2}}\right)$. Moreover,

$$
Y_{k} \int f *_{N} p^{\sigma}(s, 0)\left(x, \sigma_{s}\right) d \mathbf{W}_{a}(\sigma)
$$

and

$$
Y_{k} Y_{l} \int f *_{N} p^{\sigma}(s, 0)\left(x, \sigma_{s}\right) d \mathbf{W}_{a}(\sigma)
$$

are bounded for $x$ in a compact set. We have

$$
\begin{aligned}
& L(a) u(s, x, a) \\
& \qquad \begin{array}{l}
=L(a) \int U^{\sigma}(0, s) f\left(x, \sigma_{s}\right) d \mathbf{W}_{a}(\sigma) \\
= \\
\quad L(a) \int f *_{N} p^{\sigma}(s, 0)\left(x, \sigma_{s}\right) d \mathbf{W}_{a}(\sigma) \\
\quad=a^{-2} \sum_{j, k, l} \alpha_{k}^{j}(a) \alpha_{l}^{j}(a) Y_{k} Y_{l} \int f *_{N} p^{\sigma}(s, 0)\left(x, \sigma_{s}\right) d \mathbf{W}_{a}(\sigma) \\
\quad+a^{-2} \sum_{j, k} \alpha_{k}^{j}(a) Y_{k} \int f *_{N} p^{\sigma}(s, 0)\left(x, \sigma_{s}\right) d \mathbf{W}_{a}(\sigma)
\end{array}
\end{aligned}
$$

and, by the above remarks

$$
\begin{equation*}
|L(a) u(s, x, a)| \leq C\left(a^{m_{3}}+a^{m_{4}}\right), \tag{3.5}
\end{equation*}
$$

where

$$
m_{3}=\min \left\{m_{1}, m_{2}, 2 m_{1}, 2 m_{2}, m_{1}+m_{2}\right\}-2>0
$$

and

$$
m_{4}=\max \left\{m_{1}, m_{2}, 2 m_{1}, 2 m_{2}, m_{1}+m_{2}\right\}-2 .
$$

It follows that $\mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right)$ is finite. Indeed, by (3.4) and (3.5), proceeding as before (i.e. replacing $a$ by $b_{t-s}$ ) we obtain

$$
\left|\mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right)\right| \leq C \mathbf{E}_{a}\left(b_{t-s}^{m_{3}}+b_{t-s}^{m_{4}}\right) .
$$

Now we calculate

$$
\begin{aligned}
& \mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right) \\
&=\int L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right) d \mathbf{W}_{a}(b) \\
&=\int L\left(b_{t-s}\right) \int U^{\sigma}(0, s) f\left(x, \sigma_{s}\right) d \mathbf{W}_{b_{t-s}}(\sigma) d \mathbf{W}_{a}(b) \\
&=\iint L\left(b_{t-s}\right) U^{\sigma}(0, s) f\left(x, \sigma_{s}\right) d \mathbf{W}_{b_{t-s}}(\sigma) d \mathbf{W}_{a}(b) \\
&=\int L\left(b_{t-s}\right) U^{b}(t-s, t) f\left(x, b_{t}\right) d \mathbf{W}_{a}(b) .
\end{aligned}
$$

By (3.6), and the Fubini's theorem we obtain

$$
\begin{aligned}
\int_{0}^{t} \mathbf{E}_{a} L\left(b_{t-s}\right) u(s, x, & \left.b_{t-s}\right) d s \\
& =\iint_{0}^{t} L\left(b_{t-s}\right) U^{b}(t-s, t) f\left(x, b_{t}\right) d s d \mathbf{W}_{a}(b)
\end{aligned}
$$

but

$$
\int_{0}^{t} L\left(b_{t-s}\right) U^{b}(t-s, t) f\left(x, b_{t}\right) d s=U^{b}(0, t) f\left(x, b_{t}\right)-f\left(x, b_{t}\right)
$$

Indeed by iii) we get

$$
\begin{aligned}
\frac{d}{d s} U^{b}(t-s, t) f\left(x, b_{t}\right) & =-\left.\frac{d}{d s} U^{b}(\cdot, t) f\left(x, b_{t}\right)\right|_{t-s} \\
& =-\left(-L\left(b_{t-s}\right) U^{b}(t-s, t) f\left(x, b_{t}\right)\right) \\
& =L\left(b_{t-s}\right) U^{b}(t-s, t) f\left(x, b_{t}\right) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \int_{0}^{t} \mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right) d s \\
&=\int U^{b}(0, t) f\left(x, b_{t}\right) d \mathbf{W}_{a}(b)-\int f\left(x, b_{t}\right) d \mathbf{W}_{a}(b) \\
&=u(t, x, a)-\mathbf{E}_{a} f\left(x, b_{t}\right)
\end{aligned}
$$

Now we are going to prove that $u$ is a solution of the differential equation (3.2). Since $u$ is a solution of (3.3) we have

$$
\begin{aligned}
& \frac{u(t+h, x, a)-u(t, x, a)}{h} \\
& =\frac{\mathbf{E}_{a} f\left(x, b_{t+h}\right)-\mathbf{E}_{a} f\left(x, b_{t}\right)}{h}+\frac{1}{h} \int_{0}^{t}\left(\mathbf{E}_{a} L\left(b_{t+h-s}\right) u\left(s, x, b_{t+h-s}\right)\right. \\
& \\
& \left.\quad-\mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right)\right) d s \\
& \quad+\frac{1}{h} \int_{t}^{t+h} \mathbf{E}_{a} L\left(b_{t+h-s}\right) u\left(s, x, b_{t+h-s}\right) d s .
\end{aligned}
$$

Let $\Delta$ be the infinitesimal generator of the Bessel process i.e.

$$
\Delta=\partial_{a}^{2}+\frac{2 \alpha+1}{a} \partial_{a} .
$$

Letting $h$ to 0 we get

$$
\begin{aligned}
& \partial_{t} u(t, x, a) \\
& \quad=\Delta \mathbf{E}_{a} f\left(x, b_{t}\right)+\Delta \int_{0}^{t} \mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right) d s+L(a) u(t, x, a)
\end{aligned}
$$

in a sense of distributions.

On the other hand, since $u$ is a solution of (3.3) thus

$$
\begin{aligned}
& L u(t, x, a) \\
&=(L(a)+\Delta) u(t, x, a) \\
&=L(a) u(t, x, a)+\Delta\left(\mathbf{E}_{a} f\left(x, b_{t}\right)+\int_{0}^{t} \mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right) d s\right) \\
&=L(a) u(t, x, a)+\Delta \mathbf{E}_{a} f\left(x, b_{t}\right)+\Delta \int_{0}^{t} \mathbf{E}_{a} L\left(b_{t-s}\right) u\left(s, x, b_{t-s}\right) d s .
\end{aligned}
$$

So $u$ is a solution of (3.2).
Theorem 3.2. Let

$$
T_{t} f(x, a)=\int U^{\sigma}(0, t) f\left(x, \sigma_{t}\right) d \mathbf{W}_{a}(\sigma)
$$

Then $\left\{T_{t}\right\}$ is a semigroup.
Proof.

$$
\begin{aligned}
T_{s}\left(T_{t} f\right)(x, a) & =\int U^{b}(0, s) T_{t} f\left(x, b_{s}\right) d \mathbf{W}_{a}(b) \\
& =\int U^{b}(0, s) \int U^{\sigma}(0, t) f\left(x, \sigma_{t}\right) d \mathbf{W}_{b_{s}}(\sigma) d \mathbf{W}_{a}(b) \\
& =\int U^{b}(0, s) U^{b}(s, s+t) f\left(x, b_{s+t}\right) d \mathbf{W}_{a}(b) \\
& =\int U^{b}(0, s+t) f\left(x, b_{s+t}\right) d \mathbf{W}_{a}(b) \\
& =T_{s+t} f(x, a)
\end{aligned}
$$

where in the third equality we have used the Markov property.
4. Estimate of the evolution kernels by the Nash inequality.

Let $X, X_{1}, \ldots, X_{m}$ be as in (1.2),

$$
\begin{gathered}
L_{a}=a^{-2}\left(\sum_{j=1}^{m}\left(\Phi_{a} X_{j}\right)^{2}+\Phi_{a}(X)\right) \\
\Delta_{0}=\sum_{j=1}^{m} X_{j}^{2}
\end{gathered}
$$

and

$$
\Delta=\Delta_{0}+X
$$

Let $\sigma:[0,+\infty) \longrightarrow[0,+\infty)$ be a continuous function such that $\sigma(t)>0$ for $t>0$, and $p^{\sigma}(t, s, x)=p^{\sigma}(t, s)(x), s<t$ be the evolution generated by the operator $L_{\sigma(t)}+\partial_{t}$.

The aim of this Chapter is to prove the following estimate for $p^{\sigma}(t, 0, x)$ :

Theorem 4.1. For every compact set $K \subset N$, which does not contain the identity element e of $N$, there exist positive constants $C_{1}, C_{2}, m_{3}$, $m_{4}$ and $n \leq Q$ such that for every $x \in K$ and for every $t$,

$$
p^{\sigma}(t, 0, x) \leq C_{1}\left(\int_{0}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 2} \exp \left(-\frac{C_{2}}{A(0, t)}\right)
$$

where

$$
A(s, t)=\int_{s}^{t}\left(\sigma^{m_{3}}(u)+\sigma^{m_{4}}(u)\right) d u
$$

The main tool in the proof of the above theorem is the Nash inequality (see e.g. [VSC])

$$
\begin{equation*}
\|f\|_{L^{2}}^{2+4 / n} \leq-C(\Delta f, f)\|f\|_{L^{1}}^{4 / n}=\left(\Delta_{0} f, f\right)\|f\|_{L^{1}}^{4 / n} \tag{4.2}
\end{equation*}
$$

for all $f \in C_{0}^{\infty}(N)$, where $d$ is the local dimension of $\left(N, X_{1}, \ldots, X_{m}\right)$ and $D$ is the dimension at infinity of $\left(N, X_{1}, \ldots, X_{m}\right) n$ is any number satisfying $d \leq n \leq D$ (see [VSC]). Let $Q_{t}$ be the heat semi-group generated by $\Delta_{0}$. Then

$$
\left\|Q_{t}\right\|_{L^{1} \rightarrow L^{\infty}} \leq C \begin{cases}t^{-d / 2}, & \text { if } t \leq 1 \\ t^{-D / 2}, & \text { if } t \geq 1\end{cases}
$$

(Theorem IV.4.1 in [VSC]) and so (4.1) follows by the Nash theorem (Theorem II.5.2 in [VSC]). Since we can make $Q$ arbitrarily big (see 1.6), $\xi=-2(1-Q / n)$ is positive.

Proof of Theorem 4.1. We start with some integral estimates on $f * p^{\sigma}(t, s)$.

Let $0 \leq \varphi \in C_{c}^{\infty}(N), \operatorname{supp} \varphi \subset B_{r}(e)$ and $\int \varphi=1(r$ will be fixed later). Let $\eta(x)=\tau * \varphi(x)$ where $\tau$ is a left invariant Riemannian metric
on $N$. There exists a positive constant $C$ such that if $Y_{1}, \ldots, Y_{n}$ is a fixed basis of $\mathcal{N}$ then

$$
\begin{equation*}
\left|Y_{j} \eta(x)\right| \leq C, \quad\left|Y_{i} Y_{j} \eta(x)\right| \leq C, \quad \text { for } i, j=1, \ldots, n \tag{4.3}
\end{equation*}
$$

$[\mathrm{H}]$. Moreover,

$$
\begin{equation*}
\tau(x) \leq \int\left(\tau\left(x y^{-1}\right)+\tau(y)\right) \varphi(y) d y \leq \eta(x)+r \tag{4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\eta(e)=\int \tau\left(y^{-1}\right) \varphi(y) d y \leq r \tag{4.5}
\end{equation*}
$$

For a natural number $m$ let $\eta_{m}(x)=\tau_{m} * \varphi(x)$, where

$$
\tau_{m}(x)=\min \{m, \tau(x)\}
$$

Then there exists a positive constant $C$ such that for every $m$, (4.3), (4.4) and (4.5) hold with $\eta_{m}$ and $\tau_{m}$ instead of $\eta$ and $\tau$ respectively.

We have

$$
\begin{equation*}
\left(\partial_{s}\left(f * p^{\sigma}(t, s), e^{\alpha \eta_{m}}\right)=-\left(f * p^{\sigma}(t, s), L_{\sigma(s)}^{*} e^{\alpha \eta_{m}}\right)\right) \tag{4.6}
\end{equation*}
$$

(4.6) is obvious, if instead of $e^{\alpha \eta_{m}}$ we put $e^{\alpha \eta_{m}} \psi$, where $\psi \in C_{0}^{\infty}(N)$. So to conclude (4.6) we take the sequence $\psi_{j}=\psi \circ \sigma_{a_{j}}$ for $\psi \in C_{0}^{\infty}(N)$ such that $\psi(0)=1$ and $a_{j} \longrightarrow 0$. Since $\sigma_{a_{j}}(x) \longrightarrow e$ for every $x \in N$ and, by (1.3), $\left|\Phi_{a_{j}}\left(X_{j}\right) \psi\right| \longrightarrow 0$, we obtain (4.6) as the limit of

$$
\partial_{s}\left(f * p^{\sigma}(t, s), e^{\alpha \eta_{m}} \psi_{j}\right)=-\left(f * p^{\sigma}(t, s), L_{\sigma(s)}^{*}\left(e^{\alpha \eta_{m}} \psi_{j}\right)\right)
$$

Therefore, by (1.2) and (4.3),

$$
\begin{aligned}
& \partial_{s}\left(f * p^{\sigma}(t, s), e^{\alpha \eta_{m}}\right) \\
& \quad \leq C\left(\alpha+\alpha^{2}\right) \sigma^{-2}(s)\left(\sigma^{m_{1}}(s)+\sigma^{m_{2}}(s)\right)^{2}\left(f * p^{\sigma}(t, s), e^{\alpha \eta_{m}}\right) \\
& \quad+C \alpha \sigma^{-2}(s)\left(\sigma^{m_{1}}(s)+\sigma^{m_{2}}(s)\right)\left(f * p^{\sigma}(t, s), e^{\alpha \eta_{m}}\right)
\end{aligned}
$$

Thus

$$
\frac{\partial_{s}\left(f * p^{\sigma}(t, s), e^{\alpha \eta_{m}}\right)}{\left(f * p^{\sigma}(t, s), e^{\alpha \eta_{m}}\right)} \leq C\left(\alpha+\alpha^{2}\right)\left(\sigma^{m_{3}}(s)+\sigma^{m_{4}}(s)\right)
$$

and so

$$
\left(f * p^{\sigma}(t, s), e^{\alpha \eta_{m}}\right) \leq\left(f, e^{\alpha \eta_{m}}\right) \exp \left(C\left(\alpha+\alpha^{2}\right) A(s, t)\right),
$$

where

$$
A(s, t)=\int_{s}^{t}\left(\sigma^{m_{3}}(u)+\sigma^{m_{4}}(u)\right)
$$

Therefore,

$$
\begin{aligned}
\left(p^{\sigma}(t, s), e^{\alpha \eta_{m}}\right) & \leq e^{\alpha \eta_{m}(e)} \exp \left(C\left(\alpha+\alpha^{2}\right) A(s, t)\right) \\
& \leq e^{\alpha r} \exp \left(C\left(\alpha+\alpha^{2}\right) A(s, t)\right)
\end{aligned}
$$

Now for $m \longrightarrow \infty$ (4.4) and (4.5) yield

$$
\begin{align*}
\left(p^{\sigma}(t, s), e^{\alpha \tau}\right) & \leq\left(p^{\sigma}(t, s), e^{\alpha(\eta+r)}\right)  \tag{4.7}\\
& \leq e^{2 \alpha r} \exp \left(C\left(\alpha+\alpha^{2}\right) A(s, t)\right)
\end{align*}
$$

The next step is the Nash inequality for $L_{a}$. Applying (4.2) to $f \circ \sigma_{a}$ we obtain

$$
\begin{aligned}
a^{-Q(1+2 / n)}\|f\|_{L^{2}}^{2(1+2 / n)} & \leq-C a^{-Q}\left(a^{2} L_{a} f, f\right) a^{-4 Q / n}\|f\|_{L^{1}}^{4 / n} \\
& =-C a^{-Q+2-4 Q / n}\left(L_{a} f, f\right)\|f\|_{L^{1}}^{4 / n} .
\end{aligned}
$$

Thus

$$
\begin{equation*}
\|f\|_{L^{2}}^{2(1+2 / n)} \leq-C a^{2(1-Q / n)}\left(L_{a} f, f\right)\|f\|_{L^{1}}^{4 / n} \tag{4.8}
\end{equation*}
$$

Now we proceed similarly as in the case of semigroups (e.g. [VSC]).
For a function $0 \leq f \in C_{c}^{\infty}(N)$ such that $\int f=1$ we define

$$
f_{s}(x)=f * p^{\sigma}(t, s)(x), \quad h_{s}(x)=\left\|f_{s}\right\|_{L^{2}}^{2} .
$$

Then

$$
\begin{aligned}
-\partial_{s} h_{s} & =-\partial_{s}\left(f_{s}, f_{s}\right) \\
& =2\left(L_{\sigma(s)} f_{s}, f_{s}\right) \\
& \leq-2 C^{-1} \sigma^{-2(1-Q / n)}(s)\left\|f_{s}\right\|_{L^{2}}^{2(1+2 / n)} \\
& =-C \sigma^{-2(1-Q / n)}(s) h_{s}^{1+2 / n} .
\end{aligned}
$$

(By (4.7) we may exchange $\partial_{s}$ with the integral.) So

$$
-\partial_{s} h_{s} h_{s}^{-1-2 / n} \leq-C \sigma^{-2(1-Q / n)}(s) .
$$

Hence

$$
-\int_{s}^{t} \partial_{u} h_{u} h_{u}^{-1-2 / n} d u=\left.\frac{n}{2} h_{u}^{-2 / n}\right|_{u=s} ^{u=t} \leq-C \int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u
$$

Thus

$$
\frac{n}{2}\left(h_{t}^{-2 / n}-h_{s}^{-2 / n}\right) \leq-C \int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u .
$$

Since $h_{t}^{-2 / n}>0$,

$$
-\frac{n}{2} h_{s}^{-2 / n} \leq-C \int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u
$$

and so

$$
\left\|f * p^{\sigma}(t, s)\right\|_{L^{2}}=h_{s}^{1 / 2} \leq C\left(\int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 2}\|f\|_{L^{1}}
$$

Therefore,

$$
\begin{align*}
\left\|p^{\sigma}(t, s)\right\|_{L^{2}} \leq & C\left(\int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 4} \\
\left\|p^{\sigma}(t, s)\right\|_{L^{\infty}} \leq & \left\|p^{\sigma}(t, u)\right\|_{L^{2}}\left\|p^{\sigma}(u, s)\right\|_{L^{2}} \\
\leq & C\left(\int_{\xi}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 4}  \tag{4.9}\\
& \cdot\left(\int_{s}^{\xi} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 4} .
\end{align*}
$$

Taking $\xi$ such that

$$
\begin{align*}
\int_{s}^{\xi} \sigma^{-2(1-Q / n)}(u) d u & =\int_{\xi}^{t} \sigma^{-2(1-Q / n)}(u) d u  \tag{4.10}\\
& =\frac{1}{2} \int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u
\end{align*}
$$

we obtain

$$
\left\|p^{\sigma}(t, s)\right\|_{L^{\infty}} \leq C\left(\int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 2}
$$

By the subadditivity of the metric $\tau$, estimates (4.7) and (4.9) we have

$$
\begin{aligned}
& p^{\sigma}(t, 0, x) e^{\alpha \tau(x)} \\
& \leq \int p^{\sigma}(t, s, x) p^{\sigma}\left(s, 0, x y^{-1}\right) e^{\alpha \tau(y)} e^{\alpha \tau\left(x y^{-1}\right)} d y \\
& \leq\left\|p^{\sigma}(t, s)\right\|_{L^{\infty}}^{1 / 2}\left\|p^{\sigma}(s, 0)\right\|_{L^{\infty}}^{1 / 2}\left(p^{\sigma}(t, s), e^{2 \alpha \tau}\right)^{1 / 2}\left(p^{\sigma}(s, 0), e^{2 \alpha \tau}\right)^{1 / 2} \\
& \leq C\left(\int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 4}\left(\int_{0}^{s} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 4} \\
& \cdot e^{4 \alpha r} \exp \left(C\left(\alpha+\alpha^{2}\right) A(s, t)\right) \exp \left(C\left(\alpha+\alpha^{2}\right) A(0, s)\right) \\
&= C\left(\int_{s}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 4}\left(\int_{0}^{s} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 4} \\
& \cdot e^{4 \alpha r} \exp \left(C\left(\alpha+\alpha^{2}\right) A(0, t)\right) .
\end{aligned}
$$

Now for the $s$ such that in the last product the first two factors are equal we obtain

$$
\begin{aligned}
& p^{\sigma}(t, 0, x) e^{\alpha \tau(x)} \\
& \quad \leq C\left(\int_{0}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 2} e^{4 \alpha r} \exp \left(C\left(\alpha+\alpha^{2}\right) A(0, t)\right)
\end{aligned}
$$

If $\alpha=\varepsilon \tau(x) / A(0, t)$, then

$$
\begin{aligned}
p^{\sigma}(t, 0, x) \leq & C\left(\int_{0}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 2} \\
& \cdot \exp \left(\frac{4 \varepsilon r \tau(x)}{A(0, t)}+C \varepsilon \tau(x)+\frac{C \varepsilon^{2} \tau^{2}(x)}{A(0, t)}-\frac{\varepsilon \tau^{2}(x)}{A(0, t)}\right)
\end{aligned}
$$

Now our assumptions on $K$ imply that we may neglect $C \varepsilon \tau(x)$ and we can find $r$ such that $r<\tau(x) / 16, x \in K$. Moreover, we assume that $C \varepsilon<1 / 4$. Then

$$
p^{\sigma}(t, 0, x) \leq C\left(\int_{0}^{t} \sigma^{-2(1-Q / n)}(u) d u\right)^{-n / 2} \exp \left(\frac{-\varepsilon \tau^{2}(x)}{2 A(0, t)}\right)
$$

and the proof is completed.
Theorem 4.11. Assume that

$$
\begin{equation*}
\lambda \leq \sigma(s) \leq \Lambda, \quad \text { for } s \in[r, r+T] \tag{4.12}
\end{equation*}
$$

Given $0<T_{1}<T_{2}<T$ and a neighborhood $B$ of $e$, we can find $C>0$ independent on $r$ such that

$$
\begin{equation*}
p^{\sigma}(r, r+t) \geq C, \quad \text { for } z \in B, 0<T_{1} \leq t \leq T_{2}<T \tag{4.13}
\end{equation*}
$$

and any $\sigma$ satisfying (4.12).
Proof. Although we have an evolution here, not a semigroup, the proof of (4.12) is the same ([SS, p. 106-108]). It is based on the Poincaré inequality and upper bound estimates we have just proved. Let $\rho_{a}$ be the optimal control metric defined by the vector fields $a^{-2} \Phi_{a}\left(X_{1}\right), \ldots$, $a^{-2} \Phi_{a}\left(X_{m}\right)$ and let $B_{r, a}=\left\{x \in N: \rho_{a}(x)<r\right\}$. Then

$$
\begin{align*}
\min _{z \in \mathbb{R}} \int_{B_{r, a}}|f(x)-z|^{2} d x & \leq \int_{B_{r, a}}\left|f(x)-f_{r, a}\right|^{2} d x \\
& \leq C r^{2} \int_{B_{(3 / 2) r, a}}|\nabla f(x)|^{2} d x \tag{4.14}
\end{align*}
$$

where,

$$
f_{r, a}=\frac{1}{\left|B_{r, a}\right|} \int_{B_{r, a}} f(y) d y \quad \text { and } \quad|\nabla f|^{2}=\sum_{j=1}^{m}\left(X_{j}\right)^{2} .
$$

The constant $C$ does not depend on $a, r$. (4.14) implies

$$
\begin{align*}
\min _{z \in \mathbb{R}} \int|f(x)-z|^{2} \Psi_{a, r}(x) d x & =\int\left|f(x)-f_{\Psi_{r, a}}\right|^{2} \Psi_{a, r}(x) d x \\
& \leq C r^{2} \int|\nabla f(x)|^{2} \Psi_{a, 2 r}(x) d x \tag{4.15}
\end{align*}
$$

where

$$
f_{\Psi_{a, r}}=\frac{\int f(y) \Psi_{a, r}(y) d y}{\int \Psi_{a, r}(y) d y}
$$

and

$$
\Psi_{a, r}(x)= \begin{cases}\left(\frac{1-\rho_{a}(x)}{r}\right)^{2}, & \text { if } \rho_{a}(x)<r \\ 0, & \text { if } \rho_{a}(x) \geq r\end{cases}
$$

and $c$ does not depend on $a$. Having (4.15) we follow the argument on [SS, p. 106-108].

## 5. Green function for $L_{\gamma}$.

Let

$$
T_{t} f(x, a)=\mathbf{E}_{a} U^{\sigma}(0, t) f\left(x, \sigma_{t}\right)
$$

be the semigroup of operators generated by $L_{\gamma}$. Since

$$
\left|\mathbf{E}_{a} U^{\sigma}(0, t) f\left(x, \sigma_{t}\right)\right| \leq\|f\|_{L^{\infty}} \text { and } \mathbf{E}_{a} U^{\sigma}(0, t) f\left(x, \sigma_{t}\right) \geq 0 \text { for } f \geq 0,
$$

for every $x \in N, a \geq 0, t>0$, there exists a probability measure $p_{t}(x, a ; \cdot, \cdot)$ such that

$$
T_{t} f(x, a)=\int_{N \times \mathbb{R}^{+}} f(y, b) p_{t}(x, a ; d y, d b) .
$$

Moreover, $p_{t}(x, a ; \cdot, \cdot) \in L^{2}\left(N \times \mathbb{R}^{+}, d x \otimes a^{2 \alpha+1} d a\right)$. Indeed,

$$
\left|U^{\sigma}(0, t) f(x, \sigma(t))\right| \leq\left\|p^{\sigma}(t, 0)\right\|_{L^{2}(d x)}\left(\int|f(x, \sigma(t))|^{2} d x\right)^{1 / 2}
$$

Therefore,

$$
\begin{aligned}
\left|T_{t} f(x, a)\right| & \leq\left(\mathbf{E}_{a}\left\|p^{\sigma}(t, 0)\right\|_{L^{2}(d x)}^{2}\right)^{1 / 2}\left(\mathbf{E}_{a} \int|f(x, \sigma(t))|^{2} d x\right)^{1 / 2} \\
& \leq c(a, t)\left(\mathbf{E}_{a}\left\|p^{\sigma}(t, 0)\right\|_{L^{2}(d x)}^{2}\right)^{1 / 2}\|f\|_{L^{2}\left(d x \otimes a^{2 \alpha+1} d a\right)}
\end{aligned}
$$

because for a fixed $t$ the kernel (2.1) is bounded as a function of space variable. By (4.9), Lemma 2.2 and Corollary 2.15, $\mathbf{E}_{a}\left\|p^{\sigma}(t, 0)\right\|_{L^{2}(d x)}^{2}<$ $\infty$ and so, for every $t, x, a$,

$$
p_{t}(x, a ; \cdot, \cdot) \in L^{2}\left(N \times \mathbf{R}^{+}, d x \otimes d a^{2 \alpha+1} d a\right) .
$$

Now a standard argument shows that for fixed $x \in N, a>0$,

$$
\begin{equation*}
\left(L^{*}-\partial_{t}\right) p .(x, a ; \cdot, \cdot)=0 . \tag{5.1}
\end{equation*}
$$

We want to have (5.1) also for $a=0$.
Lemma 5.2. Given $f \in C_{c}^{\infty}\left(N \times \mathbb{R}^{+} \times \mathbb{R}^{+}\right)$, we have

$$
\begin{align*}
\lim _{a \rightarrow 0} \int p_{t}(x, a ; y, b) & f(y, b, t) d y b^{2 \alpha+1} d b d t  \tag{5.3}\\
= & \int p_{t}(x, 0 ; y, b) f(y, b, t) d y b^{2 \alpha+1} d b d t
\end{align*}
$$

Proof. We rewrite (5.3) as

$$
\lim _{a \rightarrow 0} \mathbf{E}_{a} U^{\sigma}(0, t) f(x, \sigma(t), t)=\mathbf{E}_{0} U^{\sigma}(0, t) f(x, \sigma(t), t)
$$

Since the trajectories are continuous, it is enough to show that $U^{\sigma}(0, t) f(x, \sigma(t), t)$ is a continuous function of the trajectory $\sigma$. For an arbitrary fixed $T>0$ let

$$
d\left(\sigma, \sigma^{\prime}\right)=\sup _{t \in[0, T]}\left|\sigma(t)-\sigma^{\prime}(t)\right|
$$

We have

$$
\begin{align*}
& U^{\sigma}(s, t) f(x, \sigma(t), t)-U^{\sigma^{\prime}}(s, t) f(x, \sigma(t), t) \\
&= U^{\sigma}(s, t) f(x, \sigma(t), t)-U^{\sigma}(s, t) f\left(x, \sigma^{\prime}(t), t\right)  \tag{5.4}\\
&+U^{\sigma}(s, t) f\left(x, \sigma^{\prime}(t), t\right)-U^{\sigma^{\prime}}(s, t) f\left(x, \sigma^{\prime}(t), t\right)
\end{align*}
$$

and

$$
\begin{aligned}
\mid U^{\sigma}(s, t) f(x, \sigma(t), t)-U^{\sigma}(s, t) & f\left(x, \sigma^{\prime}(t), t\right) \mid \\
& \leq \sup _{x, t}\left|f(x, \sigma(t), t)-f\left(x, \sigma^{\prime}(t), t\right)\right|
\end{aligned}
$$

which clearly tends to 0 if $d\left(\sigma, \sigma^{\prime}\right) \longrightarrow 0$. The second term in (5.4) can be written as

$$
\begin{aligned}
& U^{\sigma}(s, t) f\left(x, \sigma^{\prime}(t), t\right)-U^{\sigma^{\prime}}(s, t) f\left(x, \sigma^{\prime}(t), t\right) \\
& \quad=\int_{s}^{t} U^{\sigma}(s, r)\left(L\left(\sigma_{r}\right)-L\left(\sigma_{r}^{\prime}\right)\right) U^{\sigma^{\prime}}(r, t) f\left(x, \sigma^{\prime}(t), t\right) d r .
\end{aligned}
$$

It also tends to 0 , because for $\xi \geq 0$

$$
\lim _{\sigma^{\prime} \rightarrow \sigma} \int_{0}^{t}\left|\sigma_{r}^{\xi}-\sigma_{r}^{\prime \xi}\right|=0
$$

which completes the proof of Lemma 5.2.
Now we are ready to study the Green function for $L_{\gamma}$ in greater detail. Let

$$
\begin{equation*}
G_{\gamma}(x, a ; y, b)=\int_{0}^{\infty} p_{t}(x, a ; y, b) d t \tag{5.5}
\end{equation*}
$$

The previous lemma, applied both to $L_{\gamma}$ and $L_{\gamma}^{*}$, says that $p_{t}(x, a ; y, b)$ is well defined also for $a \geq 0, b>0$ or for $a>0, b \geq 0$. Therefore $G_{\gamma}(x, a ; y, b)$ is defined for arbitrary $x, y$ in $N$ and $a^{2}+b^{2}>0$.

Theorem 5.6. $G_{\gamma}$ is the Green function for $L_{\gamma}$. More precisely,

$$
\begin{gather*}
G_{\gamma}(\cdot, \cdot ; y, b) \in L_{\mathrm{loc}}^{1}\left(N \times \mathbb{R}^{+}\right),  \tag{5.7}\\
L_{\gamma} G_{\gamma}(\cdot, \cdot ; y, b)=-\delta_{(y, b)}  \tag{5.8}\\
G_{\gamma}(\cdot, \cdot ; y, b) \text { is a } L_{\gamma}-\text { potential }, \tag{5.9}
\end{gather*}
$$

and

$$
\begin{gather*}
G_{\gamma}(x, a ; \cdot, \cdot) \in L_{\mathrm{loc}}^{1}\left(N \times \mathbb{R}^{+}\right),  \tag{5.10}\\
L_{\gamma}^{*} G_{\gamma}(x, a ; \cdot, \cdot)=-\delta_{(x, a)}  \tag{5.11}\\
G_{\gamma}(x, a ; \cdot, \cdot) \text { is a } L_{\gamma}^{*} \text {-potential } . \tag{5.12}
\end{gather*}
$$

In particular,

$$
\begin{align*}
& L_{\gamma}^{*} G_{\gamma}(x, 0 ; \cdot, \cdot)=0 \text { on } N \times \mathbb{R}^{+},  \tag{5.13}\\
& L_{\gamma} G_{\gamma}(\cdot, \cdot ; y, 0)=0 \text { on } N \times \mathbb{R}^{+} . \tag{5.14}
\end{align*}
$$

Finally, given $\varepsilon>0$, there exists $C>0$ such that

$$
\begin{equation*}
C^{-1} \leq G_{\gamma}(x, a ; y, b) \leq C, \tag{5.15}
\end{equation*}
$$

whenever $|x|<\varepsilon, 0 \leq a<\varepsilon,|y|=1, b \leq 1$ or $|y|<\varepsilon, 0 \leq b<\varepsilon$, $|x|=1, a \leq 1$, respectively.

Proof. Since the heat semigroup $p_{t}^{*}(x, a ; y, b)$ corresponding to $L_{\gamma}^{*}$ is given by $p_{t}^{*}(x, a ; y, b)=p_{t}(y, b ; x, a)$ it is enough to prove (5.10)-(5.12). First we notice that

$$
\int_{0}^{\infty} T_{t} \phi(x, a) d t<\infty, \quad \text { for } \phi \in C_{0}^{\infty}\left(N \times \mathbb{R}^{+}\right)
$$

Indeed, if $t<1$ then $\left|T_{t} \phi(x, a)\right| \leq\|\phi\|_{L^{\infty}}$ and the beginning of the proof of Lemma 5.1 shows that

$$
\int_{1}^{\infty} T_{t} \phi(x, a) d t<\infty
$$

To prove (5.11) we write
$\int_{\mathbb{R}^{+}} \int_{N} L_{\gamma}^{*} G_{\gamma}(x, a ; y, b) \phi(y, b) d y b^{2 \alpha+1} d b$

$$
\begin{align*}
& =\int_{\mathbb{R}^{+}} \int_{\mathbb{R}^{+}} \int_{N} p_{t}(x, a ; y, b) L_{\gamma} \phi(y, b) d y b^{2 \alpha+1} d b d t  \tag{5.16}\\
& =\lim _{\substack{t_{1} \rightarrow 0 \\
t_{2} \rightarrow \infty}} \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{+}} \int_{N} p_{t}(x, a ; y, b) L_{\gamma} \phi(y, b) d y b^{2 \alpha+1} d b d t,
\end{align*}
$$

because (5.16) is absolutely convergent. But

$$
\begin{equation*}
\int_{\mathbb{R}^{+}} \int_{N} p_{t}(x, a ; y, b) L_{\gamma} \phi(y, b) d y b^{2 \alpha+1} d b=\partial_{t} T_{t} \phi(x, a) . \tag{5.17}
\end{equation*}
$$

Moreover,

$$
\lim _{t_{1} \rightarrow 0} T_{t_{1}} \phi(x, a)=-\phi(x, a)
$$

and by (4.9), Corollary 2.7, Lemma 2.2

$$
\left|T_{t_{2}} \phi(x, a)\right| \leq C \mathbf{E}_{a}\left(\int_{0}^{t_{2}} b^{\xi}(s) d s\right)^{-D / 2}
$$

which tends to 0 , when $t_{2} \rightarrow \infty$. This proves (5.11) and (5.13). To show that $G_{\gamma}(x, a ; \cdot, \cdot)$ is $L_{\gamma}^{*}$-potential we consider an $L_{\gamma}^{*}$-harmonic function $h$ satisfying

$$
0 \leq h(y, b) \leq G_{\gamma}(x, a ; y, b)
$$

and apply $T_{r}^{*}$ to it. Then, on one hand side

$$
T_{r}^{*} h(z, c)=h(z, c),
$$

and on the other,

$$
T_{r}^{*} h(z, c) \leq \int_{0}^{\infty} p_{t+r}(x, a ; z, c) d t \longrightarrow 0, \quad \text { for }(z, c) \neq(x, a)
$$

Hence $h=0$. (5.15) is a direct consequence of the next Lemma.
Lemma 5.18. Given $\xi>0, \alpha \geq 0, D>0, a_{1}>0$, there is $C$ such that if $a \leq a_{1}, 0<b<1,0<\eta<1$, then

$$
\begin{aligned}
\int_{0}^{\infty} & \mathbf{E}_{a}\left(\int_{0}^{t} b_{\alpha}^{\xi}(s) d s\right)^{-D / 2} e^{-c / A(0, t)} \\
& \cdot \mu([b-\eta, b+\eta])^{-1} \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}} d t<C
\end{aligned}
$$

where $A(0, t)$ is defined in Theorem 4.1 and $\mu(A)=\int_{A} r^{2 \alpha+1} d r$.
Proof. Assume first that $t \geq 1$. Then, by the Markov property, it is enough to estimate

$$
\begin{align*}
\int_{1}^{\infty} & \mathbf{E}_{a}\left(\int_{0}^{t / 2} b_{\alpha}^{\xi}(s) d s\right)^{-D / 2}  \tag{5.19}\\
& \cdot \mu([b-\eta, b+\eta])^{-1} \mathbf{E}_{b_{\alpha}(t / 2)} \mathbf{1}_{\left\{\sigma_{\alpha}: \sigma_{\alpha}(t / 2) \in[b-\eta, b+\eta]\right\}}\left(\sigma_{\alpha}\right) .
\end{align*}
$$

But by (2.1) and Lemma 2.3

$$
\mathbf{E}_{b_{\alpha}(t / 2)} \mathbf{1}_{\left\{\sigma_{\alpha}: \sigma_{\alpha}(t / 2) \in[b-\eta, b+\eta]\right\}}\left(\sigma_{\alpha}\right) \leq C t^{-1-\alpha} \mu([b-\eta, b+\eta]) .
$$

On the other hand by Lemma 2.2

$$
\begin{aligned}
& \mathbf{E}_{a}\left(\int_{0}^{t / 2} b_{\alpha}^{\xi}(s) d s\right)^{-D / 2} \\
& \quad=2^{(1+\xi / 2) D / 2} t^{-(1+\xi / 2) D / 2} \mathbf{E}_{a / \sqrt{t}}\left(\int_{0}^{1} b_{\alpha}^{\xi}(s) d s\right)^{-D / 2}
\end{aligned}
$$

Now, Corollary 2.7 implies that (5.19) is dominated by a constant for every $a, b, \eta$.

Let $t<1$. First we notice that for every $M, c>0$ there is $C$ such that $e^{-c / x} \leq C x^{M}$ for every $x>0$. Therefore, it suffices to estimate $\int_{0}^{1} \mathbf{E}_{a}\left(\int_{0}^{t} b_{\alpha}^{\xi}(s) d s\right)^{-D / 2} A(0, t) \mu([b-\eta, b+\eta])^{-1} \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}}$,
where

$$
A(0, t)=\int_{0}^{t}\left(b_{\alpha}^{m_{3}}(s)+b_{\alpha}^{m_{4}}(s)\right) d s
$$

Since

$$
A(0, t)^{M} \leq C\left(\left(\int_{0}^{t} b_{\alpha}^{m_{3}}(s) d s\right)^{M}+\left(\int_{0}^{t} b_{\alpha}^{m_{4}}(s) d s\right)^{M}\right)
$$

we are left with

$$
\begin{aligned}
I=\int_{0}^{1} & \mathbf{E}_{a}\left(\int_{0}^{t} b_{\alpha}^{\xi}(s) d s\right)^{-D / 2}\left(\int_{0}^{t} b_{\alpha}^{m_{j}}(s) d s\right)^{M} \\
& \cdot \mu([b-\eta, b+\eta])^{-1} \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}}\left(b_{\alpha}\right), \quad \xi, m_{j}>0
\end{aligned}
$$

and so, in view of the Schwartz inequality, we are to estimate

$$
I_{1}=\int_{0}^{1} \mathbf{E}_{a}\left(\int_{0}^{t} b_{\alpha}^{\xi}(s) d s\right)^{-D} \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}}\left(b_{\alpha}\right),
$$

and

$$
I_{2}=\int_{0}^{1} \mathbf{E}_{a}\left(\int_{0}^{t} b_{\alpha}^{m_{j}}(s) d s\right)^{2 M} \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}}\left(b_{\alpha}\right) .
$$

By Lemma 2.2 and Corollary (2.15),

$$
\begin{aligned}
I_{1}= & t^{-(1+\xi / 2) D} \mathbf{E}_{a / \sqrt{t}}\left(\int_{0}^{1} b_{\alpha}^{\xi}(s) d s\right)^{-D} \\
& \cdot \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(1) \in[(b-\eta) / \sqrt{t},(b+\eta) / \sqrt{t}]\right\}}\left(b_{\alpha}\right) \\
\leq & t^{-(1+\xi / 2) D} \mathbf{E}_{a / \sqrt{t}}\left(\int_{0}^{1 / 2} b_{\alpha}^{\xi}(s) d s\right)^{-D} \\
& \cdot \mathbf{E}_{b_{\alpha}(1 / 2)} \mathbf{1}_{\left\{\sigma_{\alpha}: \sigma_{\alpha}(1 / 2) \in[(b-\eta) / \sqrt{t},(b+\eta) / \sqrt{t}]\right\}}\left(\sigma_{\alpha}\right) \\
\leq & C t^{(1+\xi / 2) D-1-\alpha} \mu([b-\eta, b+\eta]) .
\end{aligned}
$$

Let $\Omega_{-1}=\left\{b_{\alpha}: \sup _{s \in[0,1]} b_{\alpha}(s) \leq a_{1}\right\}$ and

$$
\Omega_{m}=\left\{b_{\alpha}: a_{1}+m<\sup _{s \in[0,1]} b_{\alpha}(s) \leq a_{1}+m+1\right\}, \quad m=0,1,2, \ldots
$$

Then

$$
I_{2}=\sum_{m=-1}^{\infty} \mathbf{E}_{a}\left(\int_{0}^{t} b_{\alpha}^{m_{j}}(s) d s\right)^{2 M} \mathbf{1}_{\Omega_{m}}\left(b_{\alpha}\right) \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}}\left(b_{\alpha}\right)
$$

We treat the cases $m=-1,0,1$ and $m \geq 2$ separately. For $m=-1,0,1$ we have

$$
\begin{aligned}
\mathbf{E}_{a}\left(\int_{0}^{t} b_{\alpha}^{m_{j}}(s) d s\right)^{2 M} \mathbf{1}_{\Omega_{-1} \cup \Omega_{0} \cup \Omega_{1}}\left(b_{\alpha}\right) & \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}}\left(b_{\alpha}\right) \\
\leq & C t^{2 M-1-\alpha} \mu([b-\eta, b+\eta]) .
\end{aligned}
$$

Let $0<\sigma_{1}<1 / 2, A=\left(\sum_{n=1}^{\infty} 2^{-n \sigma_{1}}\right)^{-1}$ Then

$$
\Omega_{m} \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n}-1} \Omega_{m, n, k}
$$

where

$$
\Omega_{m, n, k}=\left\{b_{\alpha}: b_{\alpha}\left(\frac{k t}{2^{n}}\right)-b_{\alpha}\left(\frac{(k-1) t}{2^{n}}\right)>\frac{m A}{2^{n \sigma_{1}}}\right\} .
$$

Indeed, since $b_{\alpha}(t) \leq 2$ and $\sup _{s \in[0, t]} b_{\alpha}(s)>2$, we can always find $n$ and $k<2^{n}$ such that $b_{\alpha} \in \Omega_{m, n, k}$. Therefore, by Lemma (2.6),

$$
\begin{aligned}
& \mathbf{E}_{a}\left(\int_{0}^{t} b_{\alpha}^{m_{j}}(s) d s\right)^{2 M} \mathbf{1}_{\Omega_{m, n, k}}\left(b_{\alpha}\right) \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}}\left(b_{\alpha}\right) \\
& \cdot t^{2 M}\left(a_{1}+m+1\right)^{2 M m_{j}} \mathbf{E}_{a} \mathbf{1}_{\Omega_{m, n, k}}\left(b_{\alpha}\right) \mathbf{E}_{b_{\alpha}\left(k t / 2^{n}\right)} \\
& \cdot \mathbf{1}_{\left\{\sigma_{\alpha}: s_{\alpha}\left(t-k t / 2^{n}\right) \in[b-\eta, b+\eta]\right\}}\left(\sigma_{\alpha}\right) \\
& \quad \leq C t^{2 M-1-\alpha}\left(a_{1}+m+1\right)^{2 M m_{j}} 2^{n(1+\alpha)} \mu([b-\eta, b+\eta] \\
& \quad \cdot \mathbf{E}_{a} \mathbf{E}_{b_{\alpha}\left(((k-1) t) / 2^{n}\right)} \mathbf{1}_{\left\{\sigma_{\alpha}: \sigma_{\alpha}\left(t / 2^{n}\right)>m A / 2^{n \sigma_{1}}+\sigma_{\alpha}(0)\right\}}\left(\sigma_{\alpha}\right) \\
& \quad \leq C t^{2 M-1-\alpha}\left(a_{1}+m+1\right)^{2 M m_{j}} 2^{n(1+\alpha)} \mu([b-\eta, b+\eta]) \\
& \quad \cdot \exp \left(-\frac{c_{2} m^{2} A^{2} 2^{n\left(1-2 \sigma_{1}\right)}}{t}\right) .
\end{aligned}
$$

Hence,

$$
I_{2} \leq C t^{M-\alpha-1} \mu([b-\eta, b+\eta])
$$

and finally,

$$
I \leq C \int_{0}^{1} t^{-(1+\xi / 2)(D / 2)+M-\alpha-1} d t<+\infty .
$$

Now we pass to the lower estimate for the Green function. Let $|y|=1, \eta>0$ and let $\phi_{\eta}$ be a family of smooth functions with the properties: $\operatorname{supp} \phi_{\eta} \subset\left\{z \in N:\left|y^{-1} z\right|<\eta\right\}, \phi_{\eta} \geq 0, \int \phi_{\eta}(z) d z=1$. Finally, let $\psi_{\eta}(\cdot)=\mu([b-\eta, b+\eta])^{-1} \mathbf{1}_{[b-\eta, b+\eta]}(\cdot)$.

Lemma 5.21. Given $a_{1}>0$ and a compact set $K \subset N$, there is $c>0$ such that for every $a \leq a_{1}, 0<b<1,0<\eta<1$,

$$
\int_{1}^{2} \mathbf{E}_{a} U^{b}(0, t) \varphi_{\eta}(x) \psi_{\eta}\left(b_{\alpha}(t)\right) d t \geq c, \quad x \in K
$$

Proof. Let $d, D$ be positive numbers which will be chosen later. We consider the set

$$
\Omega=\left\{b_{\alpha}: \sup _{s \in[0, t]} b_{\alpha}(s) \leq D, \inf _{s \in[t / 4,3 t / 4]} b_{\alpha}(s) \geq d\right\}
$$

and we estimate

$$
\int_{1}^{2} \mathbf{E}_{a} \varphi_{\eta} * p^{b}(t, 0)(x) \mathbf{1}_{\Omega}\left(b_{\alpha}\right) \mu([b-\eta, b+\eta])^{-1} \mathbf{1}_{\left\{b_{\alpha}: b_{\alpha}(t) \in[b-\eta, b+\eta]\right\}}\left(b_{\alpha}\right)
$$

from below. We have

$$
\begin{aligned}
\varphi_{\eta} & * p^{b}(t, 0)(x) \\
& =\iint \varphi_{\eta} * p^{b}\left(t, \frac{2 t}{3}\right)(z) p^{b}\left(\frac{2 t}{3}, \frac{t}{3}\right)\left(z^{-1} x y^{-1}\right) p^{b}\left(\frac{t}{3}, 0\right)(y) d z d y
\end{aligned}
$$

In view of (4.7), we choose a compact set $K_{1}$ such that for $b \in \Omega$ and $1 \leq t \leq 2$,

$$
\int_{K_{1}} \varphi_{\eta} * p^{b}\left(t, \frac{2 t}{3}\right)(z) d z \geq \varepsilon>0, \quad \int_{K_{1}} p^{b}\left(\frac{t}{3}, 0\right)(y) d y \geq \varepsilon>0
$$

where $\varepsilon=\varepsilon(A)$. Then, by Theorem (4.11) there is $C=C\left(D, d, K, K_{1}\right)$ such that

$$
p^{b}\left(\frac{2 t}{3}, \frac{t}{3}\right)\left(z^{-1} x y^{-1}\right) \geq C,
$$

for $z, y \in K_{1}, x \in K, b_{\alpha} \in \Omega, 1 \leq t \leq 2$. Therefore we are left with

$$
\begin{aligned}
I= & \mu([b-\eta, b+\eta])^{-1} \mathbf{P}_{a}\left(b_{\alpha}: b_{\alpha} \in \Omega, b_{\alpha}(t) \in[b-\eta, b+\eta]\right) \\
\geq & \mathbf{E}_{a} \mathbf{1}_{\left\{\sup _{s \in[0,2 t / 3]} b_{\alpha}(s) \leq D_{2}, \inf _{s \in[t / 3,2 t / 3]} b_{\alpha}(s) \geq d\right\}}\left(b_{\alpha}\right) \mu([b-\eta, b+\eta])^{-1} \\
& \cdot \mathbf{P}_{b_{\alpha}(2 t / 3)}\left(\sup _{s \in[0, t / 3]} \sigma_{\alpha}(s) \leq D, \sigma_{\alpha}\left(\frac{t}{3}\right) \in[b-\eta, b+\eta]\right)
\end{aligned}
$$

provided $D_{2}<D$. Notice that if $d \leq b_{\alpha}(2 t / 3) \leq D_{2}$,

$$
\mu([b-\eta, b+\eta])^{-1} \mathbf{P}_{b_{\alpha}(2 t / 3)}\left(\sigma_{\alpha}\left(\frac{t}{3}\right) \in[b-\eta, b+\eta]\right) \geq C=C\left(d, D_{2}\right) .
$$

But, proceeding as in the proof of the previous theorem we see that

$$
\begin{array}{r}
\mu([b-\eta, b+\eta])^{-1} \mathbf{P}_{b_{\alpha}(2 t / 3)}\left(\sup _{s \in[0, t / 3]} \sigma_{\alpha}(s) \geq D, \sigma\left(\frac{t}{3}\right) \in[b-\eta, b+\eta]\right) \\
\leq c_{1} e^{-c_{2}\left(D-D_{2}\right)^{2}}
\end{array}
$$

Therefore choosing $D$ and $D_{2}$ appropriately we have

$$
\begin{array}{r}
\mu([b-\eta, b+\eta])^{-1} \mathbf{P}_{b_{\alpha}(2 t / 3)}\left(\sup _{s \in[0, t / 3]} \sigma_{\alpha}(s) \leq D, \sigma_{\alpha}\left(\frac{t}{3}\right) \in[b-\eta, b+\eta]\right) \\
\geq C\left(d, D, D_{2}\right)
\end{array}
$$

for $1 \leq t \leq 2$. Hence for $D_{1}<D_{2}$,

$$
\begin{aligned}
I \geq & C\left(d, D, D_{2}\right) \mathbf{E}_{a} \mathbf{1}_{\left\{b_{\alpha}: \sup _{s \in[0, t / 3]} b_{\alpha}(s) \leq D_{1}, b_{\alpha}(t / 3)>2 d\right\}} \\
& \cdot \mathbf{P}_{b_{\alpha}(t / 3)}\left(\inf _{s \in[0, t / 3]} \sigma_{\alpha}(s) \geq d, \sup _{s \in[0, t / 3]} \sigma_{\alpha}(s) \leq D_{2}\right) .
\end{aligned}
$$

By Lemmas 2.12 and 2.13

$$
\begin{aligned}
& \mathbf{P}_{b_{\alpha}(t / 3)}\left(\inf _{s \in[0, t / 3]} \sigma_{\alpha}(s) \geq d, \sup _{s \in[0, t / 3]} \leq D_{2}\right) \\
& \geq 1-\mathbf{P}_{b_{\alpha}(t / 3)}\left(\inf _{s \in[0, t / 3]} \sigma_{\alpha}(s)<d\right)-\mathbf{P}_{b_{\alpha}(t / 3)}\left(\sup _{s \in[0, t / 3]} \sigma_{\alpha}(s)>D_{2}\right) \\
& \geq 1-c_{1} e^{-c_{2} d^{2}}-c_{1} e^{-c_{2}\left(D_{2}-D_{1}\right)^{2}} \\
& \geq C>0
\end{aligned}
$$

provided $d$ and $D_{2}-D_{1}$ are large enough. Finally,

$$
\begin{aligned}
\mathbf{P}_{a}\left(\sup _{s \in[0, t / 3]} b_{\alpha}(s)\right. & \left.\leq D_{1}, b_{\alpha}\left(\frac{t}{3}\right)>2 d\right) \\
& \geq 1-\mathbf{P}_{a}\left(\sup _{s \in[0, t / 2]} b_{\alpha}(s)>D_{1}\right)-\mathbf{P}_{a}\left(b_{\alpha}\left(\frac{t}{3}\right)<2 d\right) \\
& \geq c_{1} e^{-c_{2} d^{2}}-c_{1} e^{-c_{2} D_{1}^{2}} \geq C>0
\end{aligned}
$$

for sufficiently large $D_{1}$.

## 6. Estimates of the Poisson kernels and the Martin boundary.

(5.15) and (1.13) imply immediately the following estimates for $m_{\gamma}$.

Theorem 6.1. Let $m_{\gamma}$ be the Poisson kernel of $\mathcal{L}_{\gamma}, \gamma>0$. Then there exists a constant $C_{\gamma}$ such that

$$
C_{\gamma}^{-1}(|x|+1)^{-Q-\gamma} \leq m_{\gamma}(x) \leq C_{\gamma}(|x|+1)^{-Q-\gamma},
$$

for $x \in N$. In particular,

$$
C^{-1}(|x|+1)^{-Q} \leq m_{0}(x) \leq C(|x|+1)^{-Q},
$$

for $x \in N$.
Proof. Theorem 5.6 says that there is a positive constant $C_{\gamma}$ such that

$$
\begin{equation*}
C_{\gamma}^{-1} \leq G_{-\gamma}(x, a ; e, 0) \leq C_{\gamma} \tag{6.2}
\end{equation*}
$$

if $|x|=1, a \leq 1$. Let $x=\sigma_{a}(y),|x|=a \geq 1,|y|=1$. By (1.18), we have

$$
\begin{aligned}
m_{\gamma}(x) & =G_{-\gamma}\left(x^{-1}, 1 ; e, 0\right) \\
& =G_{-\gamma}\left(\sigma_{a}(y), 1 ; e, 0\right) \\
& =a^{-Q-\gamma} G_{-\gamma}\left(y, a^{-1} ; e, 0\right) \\
& =|x|^{-Q-\gamma} G_{-\gamma}\left(y, a^{1} ; e, 0\right),
\end{aligned}
$$

and the proof is completed.
Now we consider the case $\gamma=0$, i.e. we look at the operator $\mathcal{L}_{0}$. The next theorem gives description of the Martin boundary for $\mathcal{L}_{0}$.

Theorem 6.3. The Martin boundary for $\mathcal{L}=\mathcal{L}_{0}$ consists of the following functions:
a) the constant function 1 ,
b) $P_{y}(x a)=\frac{1}{m_{0}(e)} a^{-Q} \breve{m}_{0}\left(\sigma_{a^{-1}}\left(y^{-1} x\right)\right)$.

All of them are minimal.
Proof. By (1.17) we may use $G$ to write the Martin kernels. Assume that

$$
\lim _{n \rightarrow \infty} \frac{G\left(x, a ; y_{n}, b_{n}\right)}{G\left(e, 1 ; y_{n}, b_{n}\right)}=K(x, a)
$$

and $\left|y_{n}\right| \longrightarrow \infty$ or $b_{n} \longrightarrow \infty$.
Let $r_{n}=\max \left\{\left|y_{n}\right|, b_{n}\right\}$. Then

$$
G\left(x, a ; y_{n}, b_{n}\right)=r_{n}^{-Q} G\left(\sigma_{r_{n}^{-1}}(x), r_{n}^{-1} a ; \sigma_{r_{n}^{-1}}\left(y_{n}\right), r_{n}^{-1} b_{n}\right) .
$$

We take $n$ such that

$$
\left|\sigma_{r_{n}^{-1}}(x)\right|<\frac{1}{4}, \quad r_{n}^{-1} a<\frac{1}{4} .
$$

Since $\left|\sigma_{r_{n}^{-1}}\left(y_{n}\right)\right|=1$ and $r_{n}^{-1} b_{n} \leq 1$ or $\sigma_{r_{n}^{-1}}\left(y_{n}\right) \leq 1$ and $r_{n}^{-1} b_{n}=1$, by Theorem 5.4 and the Harnack inequality for $L^{*}$, there is a constant $c$ independent of $x, a$ such that

$$
\begin{gathered}
c^{-1} \leq G\left(\sigma_{r_{n}^{-1}}(x), r_{n}^{-1} a ; \sigma_{r_{n}^{-1}}\left(y_{n}\right), r_{n}^{-1} b_{n}\right) \leq c, \\
c^{-1} \leq G\left(e, r_{n}^{-1} ; \sigma_{r_{n}^{-1}}\left(y_{n}\right), r_{n}^{-1} b_{n}\right) \leq c .
\end{gathered}
$$

Therefore $K(x, a)$ is bounded and so must be constant (see [BR]).
Now we assume that $y_{n} \longrightarrow y_{0}$ and $b_{n} \longrightarrow 0$. First we prove that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{G\left(x, a ; y_{n}, b_{n}\right)}{G\left(e, 1 ; y_{n}, b_{n}\right)}=\lim _{n \rightarrow \infty} \frac{G\left(y_{0}^{-1} x, a ; e, b_{n}\right)}{G\left(e, 1 ; e, b_{n}\right)} \tag{6.4}
\end{equation*}
$$

i.e. that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{G\left(y_{n}^{-1} x, a ; e, b_{n}\right)}{G\left(y_{0}^{-1} x, a ; e, b_{n}\right)}=1 \tag{6.5}
\end{equation*}
$$

Notice that for $n$ sufficiently large (depending on $x, a), \tau\left(y_{n}^{-1} x, a\right.$; $\left.y_{0}^{-1} x, a\right)<1$. Hence by the Harnack inequality

$$
\begin{aligned}
\mid G\left(y_{n}^{-1} x, a ; e, b_{n}\right)-G\left(y_{0}^{-1} x\right. & \left., a ; e, b_{n}\right) \mid \\
& \leq G\left(y_{0}^{-1} x, a ; e, b_{n}\right) \tau\left(y_{n}^{-1} x, a ; y_{0}^{-1} x, a\right)
\end{aligned}
$$

and (6.5) follows. We have

$$
G\left(x, a ; e, b_{n}\right)=a^{-Q} G\left(\sigma_{a^{-1}}(x), 1 ; e, a^{-1} b_{n}\right) .
$$

Therefore when $b_{n} \longrightarrow 0$,

$$
\lim _{b_{n} \rightarrow 0} G\left(x, a ; e, b_{n}\right)=a^{-Q} G\left(\sigma_{a^{-1}}(x), 1 ; e, 0\right)=a^{-Q} \breve{m}\left(\sigma_{a^{-1}}(x)\right)
$$

and so

$$
\lim _{b_{n} \rightarrow 0} \frac{G\left(x, a ; e, b_{n}\right)}{G\left(e, 1 ; e, b_{n}\right)}=\frac{1}{m_{0}(e)} a^{-Q} \breve{m}_{0}\left(\sigma_{a^{-1}}(x)\right)=P_{e}(x a)
$$

1 is minimal because the only bounded $\mathcal{L}$-harmonic functions are constants, $P_{e}$ is minimal if and only if $P_{y}$ is minimal. Hence all of them are minimal.

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[^2]
## Path-wise solutions of stochastic differential equations driven by Lévy processes

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#### Abstract

In this paper we show that a path-wise solution to the following integral equation $$
Y_{t}=\int_{0}^{t} f\left(Y_{t}\right) d X_{t}, \quad Y_{0}=a \in \mathbb{R}^{d}
$$ exists under the assumption that $X_{t}$ is a Lévy process of finite $p$ variation for some $p \geq 1$ and that $f$ is an $\alpha$-Lipschitz function for some $\alpha>p$. We examine two types of solution, determined by the solution's behaviour at jump times of the process $X$, one we call geometric, the other forward. The geometric solution is obtained by adding fictitious time and solving an associated integral equation. The forward solution is derived from the geometric solution by correcting the solution's jump behaviour.

Lévy processes, generally, have unbounded variation. So we must use a pathwise integral different from the Lebesgue-Stieltjes integral. When $X$ has finite $p$-variation almost surely for $p<2$ we use Young's integral. This is defined whenever $f$ and $g$ have finite $p$ and $q$-variation for $1 / p+1 / q>1$. When $p>2$ we use the integral of Lyons. In order to use this integral we construct the Lévy area of the Lévy process and show that it has finite ( $p / 2$ )-variation almost surely.


## 0. Introduction.

In this paper we give a path-wise method for solving the following integral equation

$$
\begin{equation*}
Y_{t}=Y_{0}+\int_{0}^{t} f\left(Y_{t}\right) d X_{t}, \quad Y_{0}=a \in \mathbb{R}^{d} \tag{1}
\end{equation*}
$$

when the driving process is a Lévy process.
Typically, a Lévy process, almost surely, has unbounded variation. The integral does not exist in a Lebesgue-Stieltjes sense. However, the integral still makes sense as a random variable due to the stochastic calculus of semi-martingales developed by the Strasbourg school [14].

The semi-martingale integration theory is not complete though. There are processes of interest which do not fit into the semi-martingale framework, for example the fractional Brownian motion. An alternative integral is provided by the path-wise approach studied by Lyons [11], [12] and Dudley [3]. The basis of their papers is that of Young [21], who showed that the integral

$$
\begin{equation*}
\int_{0}^{t} f d g \tag{2}
\end{equation*}
$$

is defined in a Riemann sense whenever $f$ and $g$ have finite $p$ and $q$ variation for $1 / p+1 / q>1$ (and they have no common discontinuities). For a comprehensive overview of the theory we recommend the lecture notes of Dudley and Norvaiša in the case $p<2$, [4].

Recently in [15], a system of linear Riemann-Stieltjes integral equations is solved when the integrator has finite $p$-variation for some $0<$ $p<2$. These results are contained in Theorem 1.1 where we allow non-linearity of the vector field $f$. This is because our approach is an extension of the method of [11], [12].

The approach that we follow distinguishes two cases. The first is when the process has finite $p$-variation, almost surely, for some $p<2$. We use the Young integral [21]. In [11], (1) is solved when $X_{t}$ is a continuous path of finite $p$-variation for some $p<2$.

The second case is when the process has finite $p$-variation, almost surely, for some $p>2$. The Young integral is only defined when $f$ and $g$ have finite $p$ and $q$-variation for $1 / p+1 / q>1$. So an iteration scheme on the space of paths with finite $p$-variation does not work. However, Lyons defined an integral against a continuous function of
$p$-variation for some $p>2$, [12]. The integral is developed in the space of geometric multiplicative functionals (described in Appendix A). The key idea is that we enhance the path by adding an area function to it. If there is sufficient control of the pair, path and area, then the integral is defined. The canonical example in [12] is Brownian motion. The area process enhancing the Brownian motion is the Lévy area [10, Chapter 7, Section 55]. We show that there is an area process of a Lévy process which has finite ( $p / 2$ )-variation, almost surely.

In order to solve (1) for a discontinuous function we add fictitious time during which linear segments remove the discontinuities, creating a continuous path. By solving for the continuous path and then removing the fictitious time we recover a solution for the discontinuous path. This is called a geometric solution. A second type of solution is derived from the geometric solution which we call the forward solution. Several papers, [8], [7], and [5], have used the geometric solution to answer questions about continuity of solution for a stochastic differential equation driven by a discontinuous path.

The first section treats the case where the discontinuous driving path has finite $p$-variation for some $p<2$. The second section treats the case where the path has finite $p$-variation for some $p>2$ only. The main proofs of the second section are deferred to the third section. In the appendix we prove the homeomorphic flow property for the solutions when the driving path is continuous. This is used in proving that forward solutions can be recovered from geometric solutions.

## 1. Discontinuous processes $-p<2$.

In this section we extend the results of [11] to allow the driving path of (1) to have discontinuities. The results are applied to sample paths of some Lévy processes, those that have finite $p$-variation, almost surely, for some $p<2$. Throughout this section $p \in[1,2)$ unless otherwise stated.

First, we determine the solution's behaviour when the integrator jumps. There are two possibilities to consider: the first is an extension of the Lebesgue-Stieltjes integral; the second is based on a geometric approach.

Suppose that the discontinuous integrator has bounded variation. The solution $y$ would jump

$$
y_{t}-y_{t-}=f\left(y_{t-}\right)\left(x_{t}-x_{t-}\right),
$$

at a jump time $t$ of $x$. If $x$ has finite $p$-variation for some $1<p<2$ we insert these jumps at the discontinuities of $x$. We call a path $y$ with the above jump behaviour a forward solution.

The other jump behaviour we consider is the following: When a jump of the integrator occurs we insert some fictitious time during which the jump is traversed by a linear segment, creating a continuous path on an extended time frame. Then we solve the differential equation driven by the continuous path. Finally we remove the fictitious time component of the solution path. We call this a geometric solution because the solution has an "instantaneous flow" along an integral curve at the jump times. This jump behaviour has been considered before by [13], [8] and [5].

The disadvantage of the first approach is that the solution does not, generally, generate a flow of diffeomorphisms, [9].

In this section we prove the following theorem:
Theorem 1.1. Let $x_{t}$ be a discontinuous function of finite p-variation for some $p<2$. Let $f$ be an $\alpha$-Lipschitz vector field for some $\alpha>p$. Then there exists a unique geometric solution to the integral equation

$$
\begin{equation*}
y_{t}=y_{0}+\int_{0}^{t} f\left(y_{t}\right) d x_{t}, \quad y_{0}=a \in \mathbb{R}^{d} . \tag{3}
\end{equation*}
$$

With the above assumptions, there exists a unique forward solution as well.

Before proving the theorem we recall the definitions of $p$-variation and $\alpha$-Lipschitz:

Definition 1.1. The p-variation of a function $x(s)$ over the interval $[0, t]$ is defined as follows

$$
\|x\|_{p,[0, t]}=\left(\sup _{\pi \in \pi[0, t]} \sum_{\pi}\left|x\left(t_{k}\right)-x\left(t_{k-1}\right)\right|^{p}\right)^{1 / p},
$$

where $\pi[0, t]$ is the collection of all finite partitions of the interval $[0, t]$.
Remark. This is the strong $p$-variation. Usually probabilists use the weaker form where the supremum is over partitions restricted by a mesh size which tends to zero.

Definition 1.2. A function $f$ is in $\operatorname{Lip}(\alpha)$ for some $\alpha>1$ if

$$
\|f\|_{\infty}<\infty \quad \text { and } \quad \frac{\partial f}{\partial x_{j}} \in \operatorname{Lip}(\alpha-1), \quad j=1, \ldots, d
$$

Its norm is given by

$$
\|f\|_{\operatorname{Lip}(\alpha)} \triangleq\|f\|_{\infty}+\sum_{j=1}^{d}\left\|\frac{\partial f}{\partial x_{j}}\right\|_{\operatorname{Lip}(\alpha-1)}, \quad \text { for } \alpha>1
$$

This is Stein's [20] definition of $\alpha$-Lipschitz continuity for $\alpha>1$. It extends the classical definition: $f$ is in $\operatorname{Lip}(\alpha)$ for some $\alpha \in(0,1]$ if

$$
|f(x)-f(y)| \leq K|x-y|^{\alpha}
$$

with norm

$$
\|f\|_{\infty}+\sup _{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}
$$

### 1.1. Geometric solutions.

In this subsection we define a parametrisation for a càdlàg path $x$ of finite $p$-variation. The parametrisation adds fictitious time allowing the traversal of the discontinuities of the path $x$. We prove that the resulting continuous path $x^{\delta}$ has the same $p$-variation that $x$ has. We solve (3) driven by $x^{\delta}$ using the method of Lyons [11]. Then we get a geometric solution of (3) by removing the fictitious time (i.e. by undoing the parametrisation).

Definition 1.3. Let $x$ be a càdlàg path of finite p-variation. Let $\delta>0$, for each $n \geq 1$, let $t_{n}$ be the time of the $n$ 'th largest jump of $x$. We define a map $\tau^{\delta}:[0, T] \longrightarrow\left[0, T+\delta \sum_{i=1}^{\infty}\left|j\left(t_{i}\right)\right|^{p}\right]$ (where $j(u)$ denotes the jump of the path $x$ at time $u$ ) in the following way

$$
\begin{equation*}
\tau^{\delta}(t)=t+\delta \sum_{n=1}^{\infty}\left|j\left(t_{n}\right)\right|^{p} \chi_{\left\{t_{n} \leq t\right\}}(t) \tag{4}
\end{equation*}
$$

The map $\tau^{\delta}:[0, T] \longrightarrow\left[0, \tau^{\delta}(T)\right]$ extends the time interval into one where we define the continuous process $x^{\delta}(s)$
(5) $x^{\delta}(s)= \begin{cases}x(t), & \text { if } s=\tau^{\delta}(t), \\ x\left(t_{n}^{-}\right)+\left(s-\tau^{\delta}\left(t_{n}^{-}\right)\right) & \\ \cdot j\left(t_{n}\right) \delta^{-1}\left|j\left(t_{n}\right)\right|^{-p}, & \text { if } s \in\left[\tau^{\delta}\left(t_{n}^{-}\right) \tau^{\delta}\left(t_{n}\right)\right) .\end{cases}$

Remarks 1.1. 1) $\left(s, x_{s}^{\delta}\right), s \in\left[0, \tau^{\delta}(T)\right]$ is a parametrisation of the driving path $x$.
2) The terms $\left|j\left(t_{n}\right)\right|^{p}$ in (4) ensure that the addition of the fictitious time does not make $\tau^{\delta}(t)$ explode.
3) In Figure 1 we see an example of a parametrisation of a discontinuous path $x_{s}$ in terms of the pair $(t(s), y(s))$.

The next proposition shows that the above parametrisation has the same $p$-variation as the original path, on the extended time frame $\left[0, \tau^{\delta}(T)\right]$.



Figure 1. Parametrisation of a discontinuous path.
Let $x_{s}$ be a discontinuous path of bounded variation $(p=1)$. Define a map $\tau(s)$ inserting fictitious time for the discontinuities of $x$. Define a parametrisation $(t(s), y(s))$ in the manner of (5). $(t(s), y(s))$ traverses the jumps of $x$ during the fictitious time.

Proposition 1.1. Let $x$ be a càdlàg path of finite p-variation. Let $x^{\delta}$ be a parametrisation of $x$ as above.

$$
\left\|x^{\delta}\right\|_{p,\left[0, \tau^{\delta}(T)\right]}=\|x\|_{p,[0, T]}, \quad \text { for all } \delta>0
$$

Proof. Let $\pi_{0}$ be a partition of $\left[0, \tau^{\delta}(T)\right]$. Let

$$
V_{x^{\delta}}\left(\pi_{0}\right)=\sum_{\pi_{0}}\left|x^{\delta}\left(t_{i}\right)-x^{\delta}\left(t_{i-1}\right)\right|^{p} .
$$

We show that we increase the value of $V_{p}\left(\pi_{0}\right)$ by moving points lying on the jump segments to the endpoints of those segments.

Let $t_{i-1}, t_{i}, t_{i+1}$ be three neighbouring points in the partition $\pi_{0}$ such that $t_{i}$ lies in a jump segment. Consider the following term

$$
\begin{equation*}
\left|x_{t_{i}}^{\delta}-x_{t_{i-1}}^{\delta}\right|^{p}+\left|x_{t_{i+1}}^{\delta}-x_{t_{i}}^{\delta}\right|^{p} . \tag{6}
\end{equation*}
$$

We show that (6) is dominated by replacing $x_{t_{i}}^{\delta}$ by one of $x_{l}^{\delta}$ and $x_{r}^{\delta}$, where $l$ and $r$ denote the left and right endpoint of the jump segment containing $t_{i}$.

For simplicity we set $a=x_{t_{i-1}}^{\delta}, b=x_{t_{i+1}}^{\delta}$ and $c=x_{l}^{\delta}$. Let

$$
L \triangleq\left\{c+k x: k \in(0,1), c, x \in \mathbb{R}^{d}, x \neq 0\right\}, \quad a, b \in \mathbb{R}^{d} \backslash L
$$

Let the function $f:[0,1] \longrightarrow(0, \infty)$ be defined by

$$
f(k)=|a-d|^{p}+|d-b|^{p}, \quad d=c+k x .
$$

Then $f \in C^{2}[0,1]$ and one can show that $f^{\prime \prime} \geq 0$ on $(0,1)$ when $p \geq 1$. To conclude the proof we move along the partition replacing $t_{i}$ which lie in the jump segments by new points $t_{i}^{\prime}$ that increase $V_{x^{\delta}}\left(\pi_{0}\right)$. The partition $\pi_{0}$ is replaced by a partition $\pi_{0}^{\prime}$ whose points lie on the preimage of $\left[0, \tau^{\delta}(T)\right]$. Therefore we have

$$
V_{x^{\delta}}\left(\pi_{0}\right) \leq V_{x^{\delta}}\left(\pi_{0}^{\prime}\right)=V_{x}\left(\pi_{0}^{\prime}\right) .
$$

Hence $\left\|x^{\delta}\right\|_{p,\left[0, \tau^{\delta}(T)\right]}=\|x\|_{p,[0, T]}$.
Theorem 1.2. Let $x$ be a càdlàg path with finite p-variation for some $p<2$. Let $f$ be $a \operatorname{Lip}(\gamma)$ vector field on $\mathbb{R}^{n}$ for some $\gamma>p$. Then there
exists a unique geometric solution $y$, having finite p-variation which solves the differential equation

$$
\begin{equation*}
d y_{t}=f\left(y_{t}\right) d x_{t}, \quad y_{0}=a \in \mathbb{R}^{n} \tag{7}
\end{equation*}
$$

Proof. Let $x^{\delta}$ be the parametrisation given in (5). The theorem of [11, Section 3] proves that there is a continuous solution $y^{\delta}$ which solves (3) on $\left[0, \tau^{\delta}(T)\right]$. Then $\left(s, y_{s}^{\delta}\right)$ is a parametrisation of a càdlàg path $y$ on $[0, T]$.

The solution is well-defined. To see this, consider two parametrisations of $x$ and note that there exists a monotonically increasing function $\lambda_{s}$ such that

$$
\left(s, x_{s}^{\delta}\right)=\left(\lambda_{s}, x_{\lambda_{s}}^{\nu}\right)
$$

### 1.2. Forward solutions.

In this subsection we show how to recover forward solutions from geometric solutions. The idea behind our approach is to correct the jump behaviour of the geometric solution using a Taylor series expansion, $c f$. Lemma 1.1. The correction terms are controlled by

$$
\sum_{i=1}^{\infty}\left|x_{t_{i}}-x_{t_{i}^{-}}\right|^{2},
$$

which is finite due to the finite $p$-variation of the path $x$.
In the case where the driving path has only a finite number of jumps we note that the forward solution can be recovered trivially. It is enough to mark the jump times of $x$ and solve the differential equation on the components where $x$ is continuous, inserting the forward jump behaviour when the jumps occur. It remains to show that the forward solution exists when the driving path has a countably infinite number of jumps. The method we use requires the following property of the geometric solution:

Theorem 1.3. Let $x$ be a continuous path of finite p-variation for some $p>1$. Let $f$ be in $\operatorname{Lip}(\alpha)$ for some $\alpha>p$. The maps $\left(\pi_{t}\right)_{t \geq 0}: \mathbb{R}^{n} \longrightarrow$ $\mathbb{R}^{n}$ obtained by varying the initial condition of the following differential equation generate a flow of homeomorphisms

$$
\begin{equation*}
d \pi_{t}=f\left(\pi_{t}\right) d x_{t}, \quad \pi_{0}=\mathrm{Id}, \quad(\text { the identity map }) . \tag{8}
\end{equation*}
$$

We leave the proof of Theorem 1.3 until Appendix A. We note the uniform estimate

$$
\begin{equation*}
\sup _{0 \leq t \leq T}\left|\pi_{t}^{a}-\pi_{t}^{b}\right| \leq C(T)|a-b| \tag{9}
\end{equation*}
$$

The following lemma will enable estimates to be made when the geometric jumps are replaced by the forward jumps:

Lemma 1.1. Let $x$ be a càdlàg path with finite $p$-variation. Let $f$ be in $\operatorname{Lip}(\alpha)$ for some $\alpha>p$. Let $\Delta y_{i}$ (respectively $\left.\Delta z_{i}\right)$ denote the geometric (respectively forward) solution's jump which correspond to $\Delta x_{i}$, the $i$-th largest jump of $x$. Then we have the following estimate on the difference of the two jumps

$$
\left\|\Delta y_{i}-\Delta z_{i}\right\|_{\infty} \leq K\left|\Delta x_{i}\right|^{2}
$$

where the constant $K$ depends on $\|f\|_{\operatorname{Lip}(\alpha)}$.
Proof. Parametrise the path $x$ so that it traverses its discontinuity in unit time. Solve geometrically over this interval with the solution having initial point $a$. Note that the forward jump is the first order Taylor approximation to the geometric jump. Then

$$
\begin{align*}
y_{1}(a) & =y_{0}(a)+\left.\frac{d y_{s}(a)}{d s}\right|_{s=0}+\left.\frac{1}{2} \frac{d^{2} y_{s}(a)}{d s^{2}}\right|_{s=\theta}  \tag{10}\\
& =z_{1}(a)+\left.\frac{1}{2} \frac{d^{2} y_{s}(a)}{d s^{2}}\right|_{s=\theta},
\end{align*}
$$

for some $0<\theta<1$. We estimate the second order term by

$$
\begin{align*}
\left\|\frac{1}{2} \frac{d^{2} y_{s}(a)}{d s^{2}}\right\|_{\infty} & =\left\|\frac{1}{2} \frac{d}{d s} f\left(y_{s}(a)\right)\left(\Delta x_{i}\right)\right\|_{\infty} \\
& \leq \frac{1}{2}\|\nabla f\|_{\infty}\|f\|_{\infty}\left|\Delta x_{i}\right|^{2}  \tag{11}\\
& \leq \frac{1}{2}\|f\|_{\operatorname{Lip}(\alpha)}^{2}\left|\Delta x_{i}\right|^{2} .
\end{align*}
$$

Both $\|\nabla f\|_{\infty}$ and $\|f\|_{\infty}$ are finite because $f$ is $\operatorname{Lip}(\alpha)$ for some $\alpha>p \geq$ 1.

Theorem 1.4. Let $x$ be a càdlàg path with finite p-variation. Let $f$ be in $\operatorname{Lip}(\alpha)$ for some $\alpha>p$. Then there exists a unique forward solution to the following differential equation

$$
\begin{equation*}
d z_{t}=f\left(z_{t}\right) d x_{t}, \quad z_{0}=a \tag{12}
\end{equation*}
$$

Proof. By Theorem 1.3 there exists a unique homeomorphism $y$ which solves

$$
d y_{t}=f\left(y_{t}\right) d x_{t}, \quad y_{0}=a
$$

in a geometric sense.
Label the jumps of $x$ by $j_{x}=\left\{j_{i}\right\}_{i=1}^{\infty}$ according to their decreasing size. Let $z^{n}$ denote the path made by replacing the geometric jumps of $y$ corresponding to $\left\{j_{i}\right\}_{i=1}^{n}$ by the forward jumps $\left\{f(\cdot)\left(\Delta x_{i}\right)\right\}_{i=1}^{n}$. We show that the $\left\{z^{n}\right\}_{n \geq 1}$ have a uniform limit.

We order the corrected jumps chronologically, say $\left\{t_{i}\right\}_{i=1}^{n}$. Then we estimate the following term using Lemma 1.1 and the uniform bound on the growth of $y$ given in (9)

$$
\begin{align*}
\left|z_{s}^{n}(a)-y_{s}(a)\right| & \leq \sum_{i=1}^{n}\left|y_{t_{i}, s}\left(z_{t_{i}}^{n}(a)\right)-y_{t_{i}, s}\left(y_{t_{i-1}, t_{i}}\left(z_{t_{i-1}}^{n}(a)\right)\right)\right| \\
& \leq C(T) \sum_{i=1}^{n}\left|z_{t_{i}}^{n}(a)-y_{t_{i-1}, t_{i}}\left(z_{t_{i-1}}^{n}(a)\right)\right|  \tag{13}\\
& \leq C^{2}(T) K \sum_{i=1}^{\infty}\left|\Delta x_{i}\right|^{2} .
\end{align*}
$$

So we have the uniform estimate

$$
\begin{equation*}
\left\|z^{n}-y\right\|_{\infty} \leq K\left(C_{3}(T),\|f\|_{\operatorname{Lip}(\alpha)}\right) \sum_{i=1}^{\infty}\left|\Delta x_{i}\right|^{2}<\infty, \quad \text { for all } n \geq 1 \tag{14}
\end{equation*}
$$

We use an analogous bound to get Cauchy convergence of $\left\{z^{n}\right\}_{n \geq 1}$. Let $m, r \geq 1$.

$$
\left\|z^{m}-z^{m+r}\right\|_{\infty} \leq K\left(C\left(T, z^{m}\right),\|f\|_{\operatorname{Lip}(\alpha)}\right) \sum_{i=m+1}^{\infty}\left|\Delta x_{i}\right|^{2}
$$

One notes that $\left\{C\left(T, z^{m}\right)\right\}$ are uniformly bounded, because of the boundedness of $C(T)=C(T, y)$ and the Lipschitz condition on $f$. Therefore we have the following estimate

$$
\left\|z^{m}-z^{m+r}\right\|_{\infty} \leq L \sum_{i=m+1}^{\infty}\left|\Delta x_{i}\right|^{2}
$$

This implies that $\left\{z^{n}\right\}$ are Cauchy in the supremum norm because $x$ has finite $p$-variation $(p<2)$ which implies that $\sum_{m+1}^{\infty}\left|\Delta x_{i}\right|^{2}$ tends to zero as $m$ increases.

Remark. Theorems 1.4 and 1.2 combine to prove Theorem 1.1.
Corollary 1.1. With the above notation, z has finite p-variation.
Proof. Let $s<t \in[0, T]$.

$$
\left|z_{t}-z_{s}\right| \leq\left|\left(z_{t}-z_{s}\right)-\left(y_{t}-y_{s}\right)\right|+\left|y_{t}-y_{s}\right|,
$$

where $\left(y_{t}-y_{s}\right)$ is the increment of the geometric solution starting from $z_{s}$ driven by the path $x_{t}$ on the interval $[s, T]$. Then
$\left|\left(z_{t}-z_{s}\right)-\left(y_{t}-y_{s}\right)\right| \leq C \sum_{\left.j_{x}\right|_{[s, t]}}\left|\Delta x_{i}\right|^{2} \quad$ and $\quad\left|y_{t}-y_{s}\right| \leq\|x\|_{p,[s, t]}$,
which implies that

$$
\left|z_{t}-z_{s}\right|^{p} \leq 2^{p-1}\left(C^{p}\left(\sum_{\left.j_{x}\right|_{[s, t]}}\left|\Delta x_{i}\right|^{2}\right)^{p}+\|x\|_{p,[s, t]}^{p}\right)
$$

hence

$$
\|z\|_{p,[0, T]} \leq 2^{(p-1) / p}\left(C^{p}\left(\sum_{j_{x} \mid[0, T]}\left|\Delta x_{i}\right|^{2}\right)^{p}+\|x\|_{p,[0, T]}^{p}\right)^{1 / p}<\infty
$$

## 1.3. $p$-variation of Lévy processes.

In this subsection we apply Theorem 1.1 to Lévy processes which have finite $p$-variation, almost surely.

Lévy processes are the class of processes with stationary, independent increments which are continuous in probability. The class includes Brownian motion, although this process is atypical due to its continuous sample paths. Typically a Lévy process will be a combination of a deterministic drift, a Gaussian process and a jump process. For further information on Lévy processes we direct the reader to [1].

The regularity of the sample paths of a Lévy process has been studied intensively. In the 1960's several people worked on the question of characterising the sample path $p$-variation. The following theorem, due to Monroe, gives the characterisation:

Theorem 1.5 ([17, Theorem 2]). Let $\left(X_{t}\right)_{t>0}$ be a Lévy process in $\mathbb{R}^{n}$ without a Gaussian part. Let $\nu$ be the Lévy measure. Let $\beta$ denote the index of $X_{t}$, that is

$$
\begin{equation*}
\beta \triangleq \inf \left\{\alpha>0: \int_{|y| \leq 1}|y|^{\alpha} \nu(d y)<\infty\right\} \tag{15}
\end{equation*}
$$

and suppose that $1 \leq \beta \leq 2$. If $\gamma>\beta$ then

$$
\begin{equation*}
\mathbb{P}\left(\|X\|_{\gamma}<\infty\right)=1 \tag{16}
\end{equation*}
$$

where the $\gamma$-variation is considered over any compact interval.
Remark. Note that all Lévy processes with a Gaussian part only have finite $p$-variation for $p>2$.

Corollary 1.2. Let $\left(X_{t}\right)_{t \geq 0}$ be a Lévy process with index $\beta<2$ and no Gaussian part. Let $f$ be a vector field in $\operatorname{Lip}(\alpha)$ for some $\alpha>p$. Then, almost surely, the following stochastic differential equation has a unique forward and a unique geometric solution

$$
d Y_{t}=f\left(Y_{t}\right) d X_{t}, \quad Y_{0}=a
$$

Proof. The corollary follows immediately from Theorems 1.5 and 1.1.

## 2. Discontinuous processes $-p>2$.

The goal of this section is to extend (Corollary 1.2) to let any Lévy process be the integrator of (1).

One problem we have is that the Young integral is no longer useful because we use a Picard iteration scheme which fails condition (2) when $p>2$. However, we can use the method from [12]. To define the integral we need to provide more information about the sample path. We do this by defining an area process of the Lévy process. Then we prove that the enhanced process (path and area) has finite $p$-variation, $c f$. Definition A. 3 .

We parametrise the enhanced process in an analogous manner to (5) (adding fictitious time). Then we solve (1) in a geometric sense using the method for continuous paths $(p>2)$ given in [12]. Finally, forward solutions are obtained by jump correction as before.

Before enhancing $\left(X_{t}\right)_{t \geq 0}$ we give an example which shows that there exist Lévy measures with index two. So a Lévy process does not need a Gaussian part to have, almost surely, finite $p$-variation only for $p>2$.

Example 2.1. One can define the following measures on $\mathbb{R}$

$$
\begin{gathered}
\nu_{k}(d x) \triangleq|x|^{-3+1 / k} d x, \quad|x| \in\left((k+1)^{-3(k+1)}, k^{-3 k}\right] \triangleq J_{k} \\
\eta_{m}(d x) \triangleq \sum_{k=1}^{m} \nu_{k}\left(d x \cap J_{k} \cap\left(-J_{k}\right)\right) .
\end{gathered}
$$

We show that $\eta \triangleq \lim _{m \rightarrow \infty} \eta_{m}$ is a Lévy measure. The integrability condition

$$
\begin{equation*}
\int_{|x| \leq 1}|x|^{2} \eta(d x)<\infty \tag{17}
\end{equation*}
$$

must be satisfied.

$$
\begin{aligned}
\int_{|x| \leq 1}|x|^{2} \eta_{m}(d x) & =2 \int_{0}^{1} \sum_{k=1}^{m} x^{-1+1 / k} \chi_{J_{k}}(x) d x \\
& =2 \sum_{k=1}^{m}\left(k x^{1 / k}\right)_{(k+1)^{-3(k+1)}}^{k^{-3 k}} \\
& =2 \sum_{k=1}^{m} k\left(k^{-3}-(k+1)^{-3(1+1 / k)}\right) \\
& \leq 2 \sum_{k=1}^{m} k\left(k^{-3}-2^{-3(1+1 / k)} k^{-3(1+1 / k)}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =2 \sum_{k=1}^{m} k^{-2}\left(1-2^{-3(1+1 / k)} k^{-3 / k}\right) \\
& <C \sum_{k=1}^{\infty} k^{-2} \\
& <\infty
\end{aligned}
$$

where $C$ is some suitable constant. We take the limit as $m$ tends to infinity on the left hand side to prove (17).

Now we show that

$$
\begin{equation*}
\int_{|x| \leq 1}|x|^{\alpha} \eta(d x)=\infty \tag{18}
\end{equation*}
$$

for all $\alpha<2$. Fix $\alpha<2$. Define the following number

$$
m(\alpha) \triangleq \inf \left\{k: \alpha+\frac{1}{k}<2\right\}<\infty, \quad \text { as } \alpha<2
$$

Let $m>m(\alpha)$. Then

$$
\begin{aligned}
& \int_{|x| \leq 1}|x|^{\alpha} \eta_{m}(d x) \\
& \geq 2 \sum_{k=m(\alpha)}^{m} \frac{1}{(\alpha+1 / k-2)}\left(k^{-3 k(\alpha+1 / k-2)}-(k+1)^{-3(k+1)(\alpha+1 / k-2)}\right) \\
& =2 \sum_{k=m(\alpha)}^{m} \frac{1}{(2-(\alpha+1 / k))}\left((k+1)^{-3(k+1)(\alpha+1 / k-2)}-k^{-3 k(\alpha+1 / k-2)}\right) \\
& \geq \frac{2}{2-\alpha} \sum_{m(\alpha)}^{m}\left((k+1)^{3(k+1)(2-(\alpha+1 / k))}-k^{3 k(2-(\alpha+1 / k))}\right) \\
& \longrightarrow \infty, \quad \text { as } m \longrightarrow \infty .
\end{aligned}
$$

This proves that the index $\beta$ of $\eta$ equals two. Theorem 1.5 implies that the pure jump process associated to the Lévy measure $\eta$ almost surely has finite $p$-variation for $p>2$ only.

The following theorem gives a construction of the Lévy area of the Lévy process $\left(X_{t}\right)_{t \geq 0}$. The Lévy area process and the Lévy process
form the enhanced process which we need in order to use the method of Lyons [12].

Theorem 2.1. The d-dimensional Lévy process $\left(X_{t}\right)_{t \geq 0}$ has an antisymmetric area process
$\left(A_{s, t}\right)^{i j} \triangleq \frac{1}{2} \int_{s}^{t} X_{u-}^{i} \circ d X_{u}^{j}-X_{u-}^{j} \circ d X_{u}^{i}, \quad i, j=1,2, \quad$ almost surely.

The proof is deferred to Section 3.
Theorem 2.2. The Lévy area of the Lévy process $\left(X_{t}\right)_{t \geq 0}$, almost surely, has finite ( $p / 2$ )-variation for $p>2$. That is

$$
\sup _{\pi}\left(\sum_{\pi}\left|A_{t_{k-1}, t_{k}}\right|^{p / 2}\right)^{2 / p}<\infty, \quad \text { almost surely }
$$

where the supremum is taken over all finite partitions $\pi$ of $[0, T]$.
The proof is deferred to Section 3.
Now we parametrise the sample paths of $\left(X_{t}\right)_{t \geq 0}$ as before (5).
Proposition 2.1. Parametrising the process $\left(X_{t}\right)_{t \geq 0}$ does not affect the area process' $(p / 2)$-variation.

Proof. The proof is similar to the proof of Proposition 1.1. One can show that if $\lambda$ lies in a jump segment then

$$
\left|A_{s, \lambda}\right|^{(p / 2)}+\left|A_{\lambda, t}\right|^{(p / 2)}, \quad s<\lambda<t,
$$

is maximised when $\lambda$ is moved to one of the endpoints of the jump segment.

With the parametrisation of the path and the area we can define the integral in the sense of Lyons [12]. Consequently we have the following theorem:

Theorem 2.3. Let $\left(X_{t}\right)_{t \geq 0}$ be a Lévy process with finite $p$-variation for some $p>2$. Let $f$ be in $\operatorname{Lip}(\alpha)$ for some $\alpha>p$. Then there exists,
with probability one, a unique geometric and a unique forward solution to the following integral equation

$$
\begin{equation*}
Y_{t}=Y_{0}+\int_{0}^{t} f\left(Y_{t}\right) d X_{t}, \quad Y_{0}=a \in \mathbb{R}^{d} \tag{19}
\end{equation*}
$$

Remark. When constructing the forward solution it is necessary that the sum

$$
\sum_{n=1}^{\infty}\left|\Delta X_{n}\right|^{2}
$$

remains finite. This is guaranteed by the requirement on Lévy measures to satisfy

$$
\int_{|x| \leq 1}\left(|x|^{2} \wedge 1\right) \nu(d x)<\infty
$$

## 3. Proofs of Theorem 2.1 and Theorem 2.2.

For clarity, throughout this section we assume that the Lévy process $\left(X_{t}\right)_{t \geq 0}$ is two dimensional and takes the following form

$$
\begin{equation*}
X_{t}=B_{t}+\int_{|x| \leq 1} x\left(N_{t}(d x)-t \nu(d x)\right) . \tag{20}
\end{equation*}
$$

That is, $\left(X_{t}\right)_{t \geq 0}$ is a Gaussian process with a compensated pure jump process, whose Lévy measure is supported on $\left(x \in \mathbb{R}^{2}:|x| \leq 1\right)$.

Proposition 3.1. The d-dimensional Lévy process $\left(X_{t}\right)_{t \geq 0}$ has an anti-symmetric area process
$\left(A_{s, t}\right)^{i j} \triangleq \frac{1}{2} \int_{s}^{t} X_{u-}^{i} \circ d X_{u}^{j}-X_{u-}^{j} \circ d X_{u}^{i}, \quad i, j=1,2, \quad$ almost surely.
For fixed $s<t$ we obtain the area process by the following limiting procedure

$$
\left(A_{s, t}\right)^{i j}=\lim _{n \rightarrow \infty} \sum_{m=0}^{n} \sum_{\substack{k=1 \\ \text { odd }}}^{2^{m}-1} A_{k, m}^{i, j}, \quad \text { almost surely }
$$

where $A_{k, m}^{i j}$ is the area of the (ij)-projected triangle with vertices

$$
X\left(u_{(k+1) / 2, m-1}\right), X\left(u_{(k-1) / 2, m-1}\right), X\left(u_{k, m}\right),
$$

where $u_{k, m} \triangleq s+k 2^{-m}(t-s)$. Also we have the second order moment estimate

$$
\begin{equation*}
\mathbb{E}\left[\left(A_{s, t}^{i j}\right)^{2}\right] \leq C(\nu)(t-s)^{2} . \tag{21}
\end{equation*}
$$

Proof. We define $A_{s, t}(n)$

$$
\begin{aligned}
A_{s, t}(n) \triangleq & \triangleq \frac{1}{2} \sum_{k=0}^{2^{n}-1}\left(X^{(1)}\left(u_{k, n}\right)-X^{(1)}(s)\right)\left(X^{(2)}\left(u_{k+1, n}\right)-X^{(2)}\left(u_{k, n}\right)\right) \\
& \quad-\left(X^{(2)}\left(u_{k, n}\right)-X^{(2)}(s)\right)\left(X^{(1)}\left(u_{k+1, n}\right)-X^{(1)}\left(u_{k, n}\right)\right) \\
= & \sum_{k=0}^{2^{n}-1} B_{k, n}
\end{aligned}
$$

where $B_{k, n}$ is the (signed) area of the triangle with vertices

$$
X(s), X\left(u_{k, n}\right), X\left(u_{k+1, n}\right) .
$$

By considering the difference between $A_{s, t}(n)$ and $A_{s, t}(n+1)$ we see that

$$
B_{2 k, n+1}+B_{2 k+1, n+1}-B_{k, n}
$$

is the area of the triangle with vertices

$$
X\left(u_{k, n}\right), X\left(u_{k+1, n}\right), X\left(u_{2 k+1, n+1}\right),
$$

which we denote by $A_{k, n}$. We re-order $A_{s, t}(n)$

$$
\begin{aligned}
A_{s, t}(n)= & \frac{1}{2} \sum_{m=0}^{n} \sum_{\substack{k=1 \\
\text { odd }}}^{2^{m}-1}\left(X\left(u_{k, m}\right)-d_{k, m}\right) \\
& \cdot\left(X\left(u_{(k+1) / 2, m-1}\right)-X\left(u_{(k-1) / 2, m-1}\right)\right) \\
= & \frac{1}{2} \sum_{m=0}^{n} \sum_{\substack{k=1 \\
\text { odd }}}^{2^{m}-1} A_{k, m}
\end{aligned}
$$

where $d_{k, m} \triangleq(1 / 2)\left(X\left(u_{(k+1) / 2, m-1}\right)+X\left(u_{(k-1) / 2, m-1}\right)\right)$. The convergence to the area process is completed using martingale methods.

Let $\mathfrak{F}_{n} \triangleq \sigma\left(X\left(u_{k, n}\right): k=0, \ldots, 2^{n}\right)$. Then

## Lemma 3.1.

$$
\begin{equation*}
\mathbb{E}\left[X\left(u_{k, m}\right) \mid \mathfrak{F}_{m-1}\right]=d_{k, m}, \quad \text { almost surely } \tag{22}
\end{equation*}
$$

Proof. For ease of presentation we let

$$
\begin{aligned}
& U_{1} \triangleq X\left(u_{k, m}\right)-X\left(u_{(k-1) / 2, m-1}\right) \\
& U_{2} \triangleq X\left(\left(u_{(k+1) / 2, m-1}\right)-X\left(u_{k, m}\right)\right.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}[X & \left.\left(u_{k, m}\right)-d_{k, m} \mid \mathfrak{F}_{m-1}\right] \\
& =\mathbb{E}\left[X\left(u_{k, m}\right)-d_{k, m} \mid X\left(u_{(k-1) / 2, m-1}\right), X\left(u_{(k+1) / 2, m-1}\right)\right] \\
& =\frac{1}{2} \mathbb{E}\left[U_{1}-U_{2} \mid X\left(u_{(k-1) / 2, m-1}\right), X\left(u_{(k+1) / 2, m-1}\right)\right] .
\end{aligned}
$$

Using the stationarity and the independence of the increments of $X$ we see that $U_{1}$ and $U_{2}$ are exchangeable, that is

$$
\mathbb{P}\left(U_{1} \in A, U_{2} \in B\right)=\mathbb{P}\left(U_{2} \in A, U_{1} \in B\right), \quad \text { for all } A, B \in \mathfrak{B}\left(\mathbb{R}^{2}\right)
$$

The exchangeability extends to the random variables

$$
\left(U_{i} \mid X\left(u_{(k-1) / 2, m-1}\right), X\left(u_{(k+1) / 2, m-1}\right)\right), \quad i=1,2 .
$$

We deduce that

$$
\mathbb{E}\left[U_{1}-U_{2} \mid X\left(u_{(k-1) / 2, m-1}\right), X\left(u_{(k+1) / 2, m-1}\right)\right]=0 .
$$

Returning to the proof of Proposition 3.1, we compute the variance of $A_{k, m}$. This will be used to show that

$$
\sup _{n \geq 1} \mathbb{E}\left[A_{s, t}(n)^{2}\right]<\infty,
$$

$$
\begin{aligned}
& \mathbb{E}\left(A_{k, m}^{2}\right) \\
& =\mathbb{E}\left(\left(\left(X^{(1)}\left(u_{k, m}\right)-d_{k, m}^{(1)}\right)\left(U_{1}^{(2)}+U_{2}^{(2)}\right)\right.\right. \\
& \left.\left.\quad-\left(X^{(2)}\left(u_{k, m}\right)-d_{k, m}^{(2)}\right)\left(U_{1}^{(1)}+U_{2}^{(1)}\right)\right)^{2}\right) \\
& = \\
& =\frac{1}{4} \mathbb{E}\left[\left(\left(U_{1}^{(1)}-U_{2}^{(1)}\right)\left(U_{1}^{(2)}+U_{2}^{(2)}\right)-\left(U_{1}^{(2)}-U_{2}^{(2)}\right)\left(U_{1}^{(1)}+U_{2}^{(1)}\right)\right)^{2}\right] \\
& = \\
& \frac{1}{4} \mathbb{E}\left[\left(U_{1}^{(1)} U_{2}^{(2)}\right)^{2}-2 U_{1}^{(1)} U_{2}^{(2)} U_{2}^{(1)} U_{1}^{(2)}+\left(U_{2}^{(1)} U_{1}^{(2)}\right)^{2}\right] \\
& \triangleq \\
& (1)+(2)+(3) .
\end{aligned}
$$

We use the independence of the increments and Itô's formula for discontinuous semi-martingales to compute (1), (2) and (3).

$$
(1)=\mathbb{E}\left[\left(U_{1}^{(1)} U_{2}^{(2)}\right)^{2}\right]=\mathbb{E}\left[\left(U_{1}^{(1)}\right)^{2}\right] \mathbb{E}\left[\left(U_{2}^{(2)}\right)^{2}\right] .
$$

By applying Itô's formula and using the stationarity of the Lévy process we find that

$$
(3)=(1)=2^{-2 m}(t-s)^{2} \int_{|x| \leq 1}\left|x_{1}\right|^{2} \nu(d x) \int_{|x| \leq 1}\left|x_{2}\right|^{2} \nu(d x) .
$$

Another application of Itô's formula gives

$$
\begin{aligned}
(2) & =-2 \mathbb{E}\left[U_{1}^{(1)} U_{2}^{(2)} U_{2}^{(1)} U_{1}^{(2)}\right] \\
& =-2 \mathbb{E}\left[U_{1}^{(1)} U_{1}^{(2)}\right] \mathbb{E}\left[U_{2}^{(2)} U_{2}^{(1)}\right] \\
& =-2^{-2 m+1}(t-s)^{2}\left(\int_{|x| \leq 1} x_{1} x_{2} \nu(d x)\right)^{2} .
\end{aligned}
$$

Collecting the terms together we have the following expression

$$
\mathbb{E}\left[A_{k, m}^{2}\right]=C_{0}(\nu) 2^{-2 m+1}(t-s)^{2},
$$

where
$C_{0}(\nu) \triangleq\left(\int_{|x| \leq 1}\left|x_{1}\right|^{2} \nu(d x) \int_{|x| \leq 1}\left|x_{2}\right|^{2} \nu(d x)-\left(\int_{|x| \leq 1} x_{1} x_{2} \nu(d x)\right)^{2}\right)$.

Now we estimate the following term

$$
\mathbb{E}\left[A_{s, t}^{2}(n)\right]=\mathbb{E}\left[\left(\sum_{m=1}^{n} \sum_{\substack{k=1 \\ \text { odd }}}^{2^{m}-1} A_{k, m}\right)^{2}\right]
$$

which through conditioning and independence arguments equals

$$
\begin{aligned}
& =\mathbb{E}\left[\sum_{m=1}^{n} \sum_{\substack{k=1 \\
\text { odd }}}^{2^{m}-1} A_{k, m}^{2}\right] \\
& =C_{0}(\nu) \sum_{m=1}^{n} \sum_{\substack{k=1 \\
\text { odd }}}^{2^{m}-1} 2^{-2 m+1}(t-s)^{2} \\
& \leq C_{0}(\nu) \sum_{m=1}^{\infty} \sum_{\substack{k=1 \\
\text { odd }}}^{2^{m}-1} 2^{-2 m+1}(t-s)^{2} \\
& \triangleq C(\nu)(t-s)^{2} .
\end{aligned}
$$

We use the martingale convergence theorem to deduce that, almost surely, there is a unique limit of $A_{s, t}(n)$. Furthermore the last calculation implies that there is a moment estimate of the area process given by

$$
\mathbb{E}\left[A_{s, t}^{2}\right] \leq C(\nu)(t-s)^{2} .
$$

We note that there is another way that one could define an area process of a Lévy process. One could define the area process for the truncated Lévy processes and look for a limit as the small (compensated) jumps are put in. Using the above construction one can define $A_{s, t}^{\varepsilon}$ for a fixed pair of times, corresponding to the Lévy process $X^{\varepsilon}$. With the $\sigma$-fields $\left(\mathfrak{G}^{\varepsilon}\right)_{\varepsilon>0}$ defined by

$$
\mathfrak{G}^{\varepsilon} \triangleq \sigma\left(X^{\delta}: \delta>\varepsilon\right), \quad \text { for } \varepsilon>0
$$

we have the following proposition:
Proposition 3.2. $\left(A_{s, t}^{\varepsilon}\right)_{\varepsilon>0}$ form a $\left(\mathfrak{G}^{\varepsilon}\right)$-martingale.

Proof. Let $\eta>\varepsilon>0$. By considering the construction of the area given above for the truncated processes $X^{\eta}$ and $X^{\varepsilon}$ we look at the difference at the level of the triangles $A_{k, n}^{\eta}$ and $A_{k, n}^{\varepsilon}$.

$$
\begin{aligned}
& \mathbb{E}\left[A_{k, n}^{\varepsilon}-A_{k, n}^{\eta} \mid \mathfrak{G}^{\eta}\right] \\
& =\mathbb{E}\left(A_{k, n}^{\eta, \varepsilon}+\left(X_{k, n}^{\eta, \varepsilon}-d_{k, n}^{\eta, \varepsilon}\right) \otimes\left(X_{(k+1) / 2, n-1}^{\eta}-X_{(k-1) / 2, n-1}^{\eta}\right)\right. \\
& \left.\quad+\left(X_{k, n}^{\eta}-d_{k, n}^{\eta}\right) \otimes\left(X_{(k+1) / 2, n-1}^{\eta, \varepsilon}-X_{(k-1) / 2, n-1}^{\eta, \varepsilon}\right) \mid \mathfrak{G}^{\eta}\right),
\end{aligned}
$$

where the superscript $\eta, \varepsilon$ signifies that the process is generated by the part of the Lévy measure whose support is $(\varepsilon, \eta]$. Using the spatial independence of the underlying Lévy process we have

$$
\begin{aligned}
= & \mathbb{E}\left[A_{k, n}^{\eta, \varepsilon}\right]+\mathbb{E}\left[\left(X_{k, n}^{\eta, \varepsilon}-d_{k, n}^{\eta, \varepsilon}\right)\right] \otimes\left(X_{(k+1) / 2, n-1}^{\eta}-X_{(k-1) / 2, n-1}^{\eta}\right) \\
& +\left(X_{k, n}^{\eta}-d_{k, n}^{\eta}\right) \otimes \mathbb{E}\left[\left(X_{(k+1) / 2, n-1}^{\eta, \varepsilon}-X_{(k-1) / 2, n-1}^{\eta, \varepsilon}\right)\right] \\
= & 0
\end{aligned}
$$

With the uniform control on the second moment of the martingale

$$
\mathbb{E}\left[\left(A_{s, t}^{\varepsilon}\right)^{2}\right] \leq C(\nu)(t-s)^{2}, \quad \text { for all } \varepsilon>0,
$$

we conclude that $A_{s, t}^{\varepsilon}$ converges almost surely as $\varepsilon \longrightarrow 0$.
The algebraic identity

$$
\begin{equation*}
A_{s, u}=A_{s, t}+A_{t, u}+\frac{1}{2}\left[X_{s, t}, X_{t, u}\right], \quad s<t<u \tag{23}
\end{equation*}
$$

for the anti-symmetric area process $A$ generated by a piecewise smooth path $X$ extends to the area process of the Lévy process. This is due to (23) holding for the area processes $A^{\varepsilon}$ of the truncated Lévy processes $X^{\varepsilon}$.

Proposition 3.3. The Lévy area of the Lévy process $\left(X_{t}\right)_{t \geq 0}$ has finite ( $p / 2$ )-variation for $p>2$, almost surely. That is

$$
\sup _{\pi}\left(\sum_{\pi}\left|A_{t_{k-1}, t_{k}}\right|^{p / 2}\right)^{2 / p}<\infty, \quad \text { almost surely }
$$

where the supremum is taken over all finite partitions $\pi$ of $[0, T]$.

Proof. In Proposition 3.1 we constructed the area process for a pair of times, almost surely. This can be extended to a countable collection of pairs of times, almost surely. In the proof below we assume that the area process has been defined for the times

$$
k 2^{-n} T,(k+1) 2^{-n} T, \quad k=0,1, \ldots, 2^{n}-1, \quad n \geq 1
$$

The proof follows the method of estimation used in [6]. To estimate the area process for two arbitrary times $u<v$ we split up the interval $[u, v]$ in the following manner:

We select the largest dyadic interval $\left[(k-1) 2^{-n} T, k 2^{-n}\right]$ which is contained within $[u, v]$. Then we add dyadic intervals to either side of the initial interval, which are chosen maximally with respect to inclusion in the interval $[u, v]$. Continuing in this fashion we label the partition according to the lengths of the dyadics. We note that there are at most two dyadics of the same length in the partition which we label $\left[l_{1, k}, r_{1, k}\right]$ and $\left[l_{2, k}, r_{2, k}\right]$ where $r_{1, k} \leq l_{2, k}$. Then

$$
[u, v]=\bigcup_{k=1}^{\infty} \bigcup_{i=1,2}\left[l_{i, k}, r_{i, k}\right]
$$

We estimate $A_{u, v}$ using the algebraic formula (23).

$$
\begin{aligned}
& A_{l_{1, m}, r_{2, m}} \\
= & \sum_{k=1}^{m} \sum_{i=1,2} A_{l_{i, k}, r_{i, k}}+\frac{1}{2} \sum_{1 \leq a \leq b \leq 2} \sum_{1 \leq j<k \leq m}\left(X_{r_{a, k}}-X_{l_{a, k}}, X_{r_{b, j}}-X_{l_{b, j}}\right) .
\end{aligned}
$$

Noting that

$$
\begin{aligned}
& \sum_{1 \leq a \leq b \leq 2} \sum_{1 \leq j<k \leq m}\left|\left(X_{r_{a, k}}-X_{l_{a, k}}, X_{r_{b, j}}-X_{l_{b, j}}\right)\right| \\
& =\sum_{1 \leq a \leq b \leq 2} \sum_{1 \leq j<k \leq m} \mid\left(X_{r_{a, k}}-X_{l_{a, k}}\right) \otimes\left(X_{r_{b, j}}-X_{l_{b, j}}\right) \\
& \quad-\left(X_{r_{b, j}}-X_{l_{b, j}}\right) \otimes\left(X_{r_{a, k}}-X_{l_{a, k}}\right) \mid \\
& \leq \sum_{1 \leq a \leq b \leq 2} \sum_{1 \leq j<k \leq m}\left|X_{r_{a, k}}-X_{l_{a, k} \mid}\right|\left|X_{r_{b, j}}-X_{l_{b, j}}\right| \\
& \leq\left(\sum_{k=1}^{m} \sum_{i=1,2}\left|X_{r_{i, k}}-X_{l_{i, k}}\right|\right)^{2},
\end{aligned}
$$

we have the estimate

$$
\begin{align*}
\left|A_{u, v}\right|^{p / 2} \leq 2^{(p / 2)-1} & \left(\left(\sum_{k=1}^{\infty} \sum_{i=1,2}\left|A_{l_{i, k}, r_{i, k}}\right|\right)^{p / 2}\right. \\
& \left.+\frac{1}{2}\left(\sum_{k=1}^{\infty} \sum_{i=1,2}\left|X_{r_{i, k}}-X_{l_{i, k}}\right|\right)^{p}\right) . \tag{24}
\end{align*}
$$

Using Hölder's inequality, with $p>2$ and $\gamma>p-1$, we have

$$
\begin{aligned}
& \left|A_{u, v}\right|^{p / 2} \\
& \leq 2^{(p / 2)-1}\left(\left(\sum_{n=1}^{\infty} n^{-\gamma /((p / 2)-1)}\right)^{(p / 2)-1} \sum_{n=1}^{\infty} n^{\gamma}\left(\sum_{i=1,2}\left|A_{l_{i, k}, r_{i, k}}\right|\right)^{p / 2}\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.\quad+\frac{1}{2}\left(\sum_{n=1}^{\infty} n^{-\gamma /(p-1)}\right)^{p-1} \sum_{n=1}^{\infty} n^{\gamma}\left(\sum_{i=1,2}\left|X_{r_{i, k}}-X_{l_{i, k}}\right|\right)^{p}\right)  \tag{25}\\
& \leq C_{1}(p, \gamma) \sum_{n=1}^{\infty} n^{\gamma} \sum_{i=1,2}\left|A_{l_{i, k}, r_{i, k}}\right|^{p / 2} \\
& \quad+C_{2}(p, \gamma) \sum_{n=1}^{\infty} n^{\gamma} \sum_{i=1,2}\left|X_{r_{i, k}}-X_{l_{i, k}}\right|^{p} .
\end{align*}
$$

One can uniformly bound $\left|A_{u, v}\right|^{p / 2}$ for any pair of times $u<v \in[0, T]$ by extending the estimate in (25) over all the dyadic intervals at each level $n$, that is,

$$
\begin{aligned}
\left|A_{u, v}\right|^{p / 2} \leq & C_{1}(p, \gamma) \sum_{n=1}^{\infty} n^{\gamma} \sum_{i=1}^{2^{n}}\left|A_{l_{i, k}, r_{i, k}}\right|^{p / 2} \\
& +C_{2}(p, \gamma) \sum_{n=1}^{\infty} n^{\gamma} \sum_{i=1}^{2^{n}}\left|X_{r_{i, k}}-X_{l_{i, k}}\right|^{p} .
\end{aligned}
$$

If the right hand side is finite, almost surely, then the area can be defined for any pair of times.

The ( $p / 2$ )-variation of the Lévy area can be estimated by the same
bound.

$$
\begin{align*}
\sup _{\pi} \sum_{\pi}\left|A_{u, v}\right|^{p / 2} \leq & C_{1}(p, \gamma) \sum_{n=1}^{\infty} n^{\gamma} \sum_{i=1}^{2^{n}}\left|A_{l_{i, k}, r_{i, k}}\right|^{p / 2} \\
& +C_{2}(p, \gamma) \sum_{n=1}^{\infty} n^{\gamma} \sum_{i=1}^{2^{n}}\left|X_{r_{i, k}}-X_{l_{i, k}}\right|^{p} . \tag{26}
\end{align*}
$$

We use (21) to control the first sum

$$
\mathbb{E}\left[\left|A_{s, t}\right|^{p / 2}\right] \leq C(t-s)^{p / 2}, \quad \text { for } p \leq 4
$$

So we have

$$
\begin{aligned}
\mathbb{E}\left[\sum_{n=1}^{\infty} n^{\gamma} \sum_{i=1}^{2^{n}}\left|A_{l_{i, k}, r_{i, k}}\right|^{p / 2}\right] & \leq C \sum_{n=1}^{\infty} n^{\gamma} \sum_{i=1}^{2^{n}}\left(2^{-n} T\right)^{p / 2} \\
& =C \sum_{n=1}^{\infty} n^{\gamma} 2^{-n((p / 2)-1)} \\
& <\infty, \quad \text { for } p>2
\end{aligned}
$$

This implies that the first term in the right hand side of (26) is almost surely finite. Now we consider the second term of (26).

## Lemma 3.2.

$$
\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=0}^{2^{n}-1}\left|X_{(k+1) 2^{-n} T}-X_{k 2^{-n} T}\right|^{p} \quad<\infty, \quad \text { almost surely }
$$

Before proving the lemma we recall a result of Monroe, [16].
Definition 3.1. Let $B_{t}$ be a Brownian motion defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. A stopping time $T$ is said to be minimal if for any stopping time $S \leq T, B(T) \stackrel{(\mathrm{d})}{=} B(S)$ implies that, almost surely, $S=T$.

Theorem 3.1 [16, Theorem 11]. Let $\left(M_{t}\right)_{t \geq 0}$ be a right continuous martingale. Then there is a Brownian motion $\left(\Omega, \mathfrak{G}_{t}, B_{t}\right)$ and a family $\left(T_{t}\right)$ of $\mathfrak{G}_{t}$-stopping times such that the process $B_{T_{t}}$ has the same finite distributions as $M_{t}$. The family $T_{t}$ is right continuous, increasing, and
for each $t, T_{t}$ is minimal. Moreover, if $M_{t}$ has stationary independent increments then so does $T_{t}$.

Remark. It should be noted that the stopping times $T_{t}$ are not generally independent of $B_{t}$. However, in the case of $\alpha$-stable processes $0<\alpha<2$ one can use subordination to gain independence of the stopping times, [2].

Proof of Lemma 3.2. Let $\left(\tau_{t}\right)_{t \geq 0}$ denote the collection of minimal stopping times for which

$$
X_{t} \stackrel{(\mathrm{~d})}{=} B_{\tau_{t}} .
$$

The proof will be completed once it has been shown that
(27) $\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=0}^{2^{n}-1}\left|B_{\tau\left((k+1) 2^{-n} T\right)}-B_{\tau\left(k 2^{-n} T\right)}\right|^{p}<\infty, \quad$ almost surely.

The following inequality holds because Brownian motion is $\left(1 / p^{\prime}\right)$-Hölder continuous, almost surely, for $p^{\prime}>2$

$$
\begin{equation*}
\left|B_{\tau\left(t_{k+1, n}\right)}-B_{\tau\left(t_{k, n}\right)}\right|^{p} \leq C\left|\tau\left(t_{k+1, n}\right)-\tau\left(t_{k, n}\right)\right|^{p / p^{\prime}} \tag{28}
\end{equation*}
$$

for all $k=0, \ldots, 2^{n}-1$, and for all $n \geq 1$, almost surely, where $t_{k, n} \triangleq$ $k 2^{-n} T$ and $2<p^{\prime}<p$.
[17, Theorem 1] shows that the index of the process $\tau(s)$ is half that of the Lévy process. Therefore, with probability one, $\tau(s)$ has finite $(1+\delta)$-variation for all $\delta>0$.

Theorem 3.2. If $\tau$ is a minimal stopping time and $\mathbb{E}\left(B_{\tau}\right)=0$, then $\mathbb{E}(\tau)=\mathbb{E}\left(B_{\tau}^{2}\right)$.

Consequently the process $\left(\tau_{t}\right)_{t \geq 0}$ can be controlled in the following way

$$
\begin{equation*}
\mathbb{E}\left[\tau_{t}\right]=\mathbb{E}\left[B_{\tau_{t}}^{2}\right]=\mathbb{E}\left[X_{t}^{2}\right]=t \int_{|x|<1}|x|^{2} \nu(d x) \tag{29}
\end{equation*}
$$

where $\nu$ is the Lévy measure corresponding to the process $X_{t}$. From (29) and Theorem 3.1 we note that the process $\tau_{t}$ is a Lévy process whose Lévy measure, say $\mu$, satisfies the following

$$
\int_{0}^{1} x \mu(d x)<\infty
$$

From this result we deduce that the process $\tau_{t}$, almost surely, has bounded variation. From [18, Theorem 5] we note that there is a positive constant $A$ such that

$$
\mathbb{P}\left(\tau_{t} \leq A t, \text { for all } t \geq 0\right)=1
$$

From the above bound and using the fact that $\tau$ has stationary independent increments one can show

$$
\begin{gathered}
\mathbb{P}\left(\tau\left(t_{k+1, n}\right)-\tau\left(t_{k, n}\right) \leq A\left(t_{k+1, n}-t_{k, n}\right)=A 2^{-n} \mid \tau\left(t_{k, n}\right)\right)=1 \\
\mathbb{P}\left(\bigcap_{n \geq 1}^{2^{n}-1} \bigcap_{k \geq 0}\left(\left|\tau\left(t_{k+1, n}\right)-\tau\left(t_{k, n}\right)\right| \leq A 2^{-n}\right)\right)=1
\end{gathered}
$$

Returning to (28) we see that

$$
\left|B_{\tau\left(t_{k+1, n}\right)}-B_{\tau\left(t_{k, n}\right)}\right|^{p} \leq C\left|\tau\left(t_{k+1, n}\right)-\tau\left(t_{k, n}\right)\right|^{p / p^{\prime}} \leq C A 2^{-n\left(p / p^{\prime}\right)}
$$

which implies that

$$
\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=0}^{2^{n}-1}\left|B_{\tau\left((k+1) 2^{-n} T\right)}-B_{\tau\left(k 2^{-n} T\right)}\right|^{p} \leq C A \sum_{n=1}^{\infty} n^{\gamma} 2^{-n\left(p / p^{\prime}-1\right)}<\infty
$$

due to $p^{\prime}$ being chosen in the interval $(2, p)$.
This lemma concludes the proof that the bound in (26) is finite, which shows that the area process, almost surely, has finite ( $p / 2$ )variation.

In this section we have proved that the area process exists and has finite ( $p / 2$ )-variation when $\left(X_{t}\right)_{t \geq 0}$ has the form (20) To prove theorems 2.1, 2.2 we note that a general Lévy process has the form

$$
X_{t}=a t+B_{t}+L_{t}+\sum_{\substack{0 \leq s<t \\\left|\Delta X_{s}\right| \geq 1}} \Delta X_{s}, \quad \text { almost surely }
$$

So, we need to add area corresponding to the drift vector and the jumps of size greater than one. However, this part of the Lévy process has bounded variation and is piecewise smooth so there is no problem defining its area. Similarly, it has, almost surely, finite ( $p / 2$ )-variation.

## A. Homeomorphic flows.

In this section we give a proof that the solutions, generated by (1) as the initial condition is varied, form a flow of homeomorphisms when the integrator is a continuous function. The proof modifies the one given in [12] for the existence and uniqueness of solution to (1). The main idea is that one uniformly bounds a sequence of iterated maps which have projections giving the convergence of the solutions with two different initial points and bounding the difference of the solutions.

First, we need some notation.
Definition A.1. Let $T^{(n)}\left(\mathbb{R}^{d}\right)$ denote the truncated tensor algebra of length $n$ over $\mathbb{R}^{d}$. That is

$$
T^{(n)}\left(\mathbb{R}^{d}\right) \triangleq \bigoplus_{i=0}^{n}\left(\mathbb{R}^{d}\right)^{\otimes i}
$$

where $\left(\mathbb{R}^{d}\right)^{\otimes 0}=\mathbb{R}$ and $T^{(\infty)}\left(\mathbb{R}^{d}\right)$ denotes the tensor algebra over $\mathbb{R}^{d}$.
Let $\Delta=[0, T] \times[0, T]$. A map $X: \Delta \longrightarrow T^{(n)}\left(\mathbb{R}^{d}\right)$ will be called a multiplicative functional of size $n$ if for all times $s<t<u$ in $[0, T]$ the following relation holds in $T^{(n)}\left(\mathbb{R}^{d}\right)$

$$
X_{s t} \otimes X_{t u}=X_{s u}
$$

and $X_{s t}^{(0)} \equiv 1$.
A map $X: \Delta \longrightarrow T^{(n)}\left(\mathbb{R}^{d}\right)$ is called a classical multiplicative functional if $t \longrightarrow X_{t} \triangleq X_{0 t}^{(1)}$ is continuous and piecewise smooth and

$$
\begin{equation*}
X_{s t}^{(i)}=\iint_{s<u_{1}<\cdots<u_{i}<t} d X_{u_{1}} \cdots d X_{u_{i}} \tag{30}
\end{equation*}
$$

where the right hand side is a Lebesgue-Stieltjes integral. We denote the set of all classical multiplicative functionals in $T^{(n)}\left(\mathbb{R}^{d}\right)$ by $S^{(n)}\left(\mathbb{R}^{d}\right)$.

Definition A.2. We call a continuous function $\omega: \Delta \longrightarrow \mathbb{R}^{+}$a control function if it is super-additive and regular, that is,

$$
\begin{gathered}
\omega(s, t)+\omega(t, u) \leq \omega(s, u), \quad \text { for all } s<t<u \in[0, T], \\
\omega(s, s)=0, \quad \text { for all } s \in[0, T] .
\end{gathered}
$$

Let $X$ be a path of strong finite p-variation. Then we can define the following control function

$$
\omega(s, t) \triangleq\|X\|_{p,[s, t]}^{p} .
$$

Definition A.3. A functional $X=\left(1, X^{(1)}, \ldots, X^{(n)}\right)$ defined on $T^{(n)}\left(\mathbb{R}^{d}\right)$ where $n=[p]$ is said to have finite $p$-variation if there is a control function $\omega$ such that

$$
\begin{equation*}
\left|X_{s t}^{(i)}\right| \leq \frac{\omega(s, t)^{i / p}}{\beta\left(\frac{i}{p}\right)!}, \quad \text { for all }(s, t) \in \Delta, i=1, \ldots, n \tag{31}
\end{equation*}
$$

for some sufficiently large $\beta$ and $x!\triangleq \Gamma(x+1)$.
Theorem A. 1 ([12, Theorem 2.2.1]). Let $X^{(n)}$ be a multiplicative functional of degree $n$ which has finite $p$-variation, with $n \triangleq[p]$ ( $[p]$ denotes the integer part of $p$ ). Then for $m>n$ there is a unique multiplicative extension $X^{(m)}$ in $T^{(m)}\left(\mathbb{R}^{d}\right)$ which has finite p-variation.

Remark. The above theorem shows that once a sufficient number of low order integrals associated to a path $X_{t}$ have been defined, then the remaining iterated integrals of $X_{t}$ are defined.

Definition A.4. We call a multiplicative functional $X: \Delta \longrightarrow T^{(n)}\left(\mathbb{R}^{d}\right)$ geometric if there is a control function $\omega$ such that for any positive $\varepsilon$ there exists a classical multiplicative functional $Y(\varepsilon)$ which approximates $X$ in the following way

$$
\left|\left(X_{s t}-Y_{s t}(\varepsilon)\right)^{(i)}\right| \leq \varepsilon \omega(s, t)^{i / p}, \quad i=1, \ldots, n=[p] .
$$

We denote the class of geometric multiplicative functionals with finite p-variation by $\Omega G\left(\mathbb{R}^{d}\right)^{p}$.

Example A.1. Let $W_{t}$ be an $\mathbb{R}^{d}$-valued Brownian motion. Then the following functional $W$ defined on $T^{(2)}\left(\mathbb{R}^{d}\right)$ belongs to $\Omega G\left(\mathbb{R}^{d}\right)^{p}$ for any $p>2$.

$$
\begin{equation*}
W_{s t} \triangleq\left(1, W_{t}-W_{s}, \iint_{s<u_{1}<u_{2}<t} \circ d W_{u_{1}} \circ d W_{u_{2}}\right) \tag{32}
\end{equation*}
$$

where od $W_{u}$ denotes the Stratonovich integral. It should be noted that if one replaced the Stratonovich differential in (33) by the Itô differential then one would not get an element of $\Omega G\left(\mathbb{R}^{d}\right)^{p}$. This is due to the quadratic variation term which occurs in the symmetric part of the area process

$$
W_{s t}^{(2)}=\iint_{s<u_{1}<u_{2}<t} d W_{u_{1}} d W_{u_{2}}
$$

It was shown in [19] that one had sufficient control of the above functional to generate path-wise solutions to stochastic differential equations driven by a Brownian motion. This control was derived from a moment condition in the same spirit as Kolmogorov's criterion for Hölder continuous paths. The moment condition was verified for the above area by the use of known stochastic integral results, though one could also derive it from a construction depending on the linearly interpolated Brownian motion.

There are two stages to defining the integral against a geometric multiplicative functional. The first gives a functional which is almost multiplicative (see [12] for definition). The second associates, uniquely, a multiplicative functional to the almost multiplicative functional.

Theorem A.2. There is a unique geometric multiplicative functional $Y$ which we call the integral of the 1 -form $\theta$ against the geometric multiplicative functional $X$. We denote this by

$$
Y_{s t} \triangleq \int_{s}^{t} \theta\left(X_{u}\right) \delta X
$$

Corollary A.1. One has the following control on the p-variation of $Y$

$$
\begin{equation*}
\left|\left(\int_{s}^{t} \theta\left(X_{u}\right) \delta X\right)^{(i)}\right| \leq \frac{(C \omega(s, t))^{i / p}}{\beta\left(\frac{i}{p}\right)!}, \quad i=1, \ldots,[p] \tag{33}
\end{equation*}
$$

where $C$ depends on $p,\|\theta\|_{\operatorname{Lip}(\gamma)}, \gamma, \lambda, \beta, L$ and $[p]$.
The estimate is derived from estimating both the almost multiplicative functional and the difference of it from the integral.

We now state two lemmas which help prove that the solutions of (1) are homeomorphic flows when the initial condition is varied.

Lemma A.1. Let $X$ be in $\Omega G\left(\mathbb{R}^{d}\right)^{p}$ controlled by a regular $\omega_{0}$. Let $f: \mathbb{R}^{n} \longrightarrow \operatorname{hom}\left(\mathbb{R}^{d}, \mathbb{R}^{n}\right)$ be a $\operatorname{Lip}(\gamma)$ map for some $\gamma>p$. Let $Y_{s t}^{(i)}$, $i=1,2$ denote the element in $\Omega G\left(\mathbb{R}^{n}\right)^{p}$ which solves the rough integral equation

$$
Y_{s t}^{(i)}=\int_{s}^{t} f\left(Y^{(i)}\right) \delta X
$$

with initial condition $Y_{0}^{(i)}=a_{i}, i=1,2$. Let $W_{\text {st }}$ be the multiplicative functional which records the difference in the multiplicative functionals $Y_{s t}^{(1)}$ and $Y_{s t}^{(2)}$. Then

$$
\begin{equation*}
\left|W_{s t}^{(i)}\right| \leq \theta^{i} \frac{\omega(s, t)^{(i / p)}}{\beta\left(\frac{i}{p}\right)!}, \quad \text { for all } i \geq 1 \tag{34}
\end{equation*}
$$

where $\theta=\left|a_{1}-a_{2}\right|, \omega \triangleq C \omega_{0}$, the constant $C$ depends on $p,\|f\|_{\operatorname{Lip}(\gamma)}$, $\beta, \gamma$. The bound holds for all times $s \leq t$ on the interval $J \triangleq\{u$ : $\omega(0, u) \leq 1\}$.

Lemma A.2. With the assumptions of Lemma A. 1 one can estimate the difference of the increments of $Y_{s t}^{(1)}$ and $Y_{s t}^{(2)}$ for any pair of times $0 \leq s<t$ which satisfy $\omega(s, t) \leq 1$ as follows

$$
\left|Y_{s t}^{(1)}-Y_{s t}^{(2)}\right| \leq \theta \exp \left(\frac{1}{\beta\left(\frac{1}{p}\right)!}\left(\omega(0, s)+\omega(0, s)^{(1 / p)}\right)\right) \frac{\omega(s, t)^{(1 / p)}}{\beta\left(\frac{1}{p}\right)!} .
$$

In particular for any $t>0$ one has

$$
\begin{equation*}
\left|Y_{t}^{(1)}-Y_{t}^{(2)}\right| \leq\left|a_{1}-a_{2}\right| C(t) . \tag{35}
\end{equation*}
$$

Now we can prove that the solutions form a flow of homeomorphisms as the initial condition is varied.

Proof of Theorem 1.3. The continuity of solutions follows from Lemma A.2. It remains to show that the inverse map exists and is continuous. This can be checked by repeating all the previous arguments using the reversed path $\left(X_{t-s}\right)_{0 \leq s \leq t}$ as the integrator.

The induction part of the proof of Lemma A. 2 will require the following lemma about rescaling:

Lemma A. 3 ([12]). Let $X$ be a multiplicative functional in $T^{([p])}\left(\mathbb{R}^{d}\right)$ which is of finite p-variation controlled by $\omega$. Let $(X, Y)$ be an extension of $X$ to $T^{([p])}\left(\mathbb{R}^{d} \oplus \mathbb{R}^{n}\right)$ of finite $p$-variation controlled by $K \omega$. Then $(X, \phi Y)$ is controlled by

$$
\max \left\{1, \phi^{k p / i} K: 1 \leq k \leq i \leq[p]\right\} \omega,
$$

where $\phi \in \mathbb{R}$. In particular, if $\phi \leq K^{-[p] / p} \leq 1$ then $(X, \phi Y)$ is controlled by $\omega$.

Proof of Lemma A.1. We set up an iteration scheme of multiplicative functionals which we will bound uniformly, by induction. A projection of the sequence proves that a Picard iteration scheme converges to the solutions of (1) starting from $a_{1}$ and $a_{2}$. Another projection shows that the difference of these solutions is bounded.

Let $\varepsilon>0$ and $\eta>1$. Let $V_{s t}^{(1)}$ be the geometric multiplicative functional given by

$$
\begin{aligned}
V_{s t}^{(1)} \triangleq & \left(Z_{s t}^{(1)(1)}, Y_{s t}^{(1)(1)}, Y_{s t}^{(1)(0)}, Z_{s t}^{(2)(1)}, Y_{s t}^{(2)(1)}, Y_{s t}^{(2)(0)}, W_{s t}^{(1)}, \varepsilon^{-1} X_{s t}\right) \\
= & \left(\int_{s}^{t} f\left(a_{1}\right) \delta X-a_{1}, \int_{s}^{t} f\left(a_{1}\right) \delta X, a_{1}, \int_{s}^{t} f\left(a_{2}\right) \delta X-a_{2},\right. \\
& \left.\int_{s}^{t} f\left(a_{2}\right) \delta X, a_{2}, \int_{s}^{t} f\left(a_{1}\right)-f\left(a_{2}\right) \delta X, \varepsilon^{-1} X_{s t}\right) .
\end{aligned}
$$

The iteration step is a two stage process. Given $V^{(m)}$ we set

$$
\tilde{V}^{(m+1)}=\int k_{\theta}^{m}\left(V^{(m)}\right) \delta V^{(m)},
$$

where $k_{\theta}^{m}$ is the 1 -form on $\left(\left(\mathbb{R}^{n}\right)^{\oplus 7} \oplus \mathbb{R}^{d}\right)$ given by

$$
\begin{aligned}
& k_{\theta}^{m}\left(a_{1}, \ldots, a_{8}\right)\left(d A_{1}, \ldots, d A_{8}\right) \\
& \quad=\left(a_{1} g\left(a_{2}, a_{3}\right) d A_{8}, d A_{3}+\eta^{-m} d A_{1}, d A_{2}, a_{4} g\left(a_{5}, a_{6}\right) d A_{8},\right. \\
& \left.\quad d A_{6}+\eta^{-m} d A_{4}, d A_{5}, \theta^{-1} g\left(a_{2}, a_{4}\right) d A_{8}, d A_{8}\right) .
\end{aligned}
$$

$g(x, y)$ is the 1 -form appearing in [11, Lemma 3.2] which satisfies the following relation with respect to $f$

$$
f^{i}(x)-f^{i}(y)=\sum_{j}(x-y)^{j} g^{i j}(x, y) .
$$

$\widetilde{V}^{(m+1)}$ is well defined because $g$ and $k_{\theta}^{m}$ are both $\operatorname{Lip}(\gamma)$ for some $\gamma>p-1$.

We define $V^{(m+1)}$ to be the geometric multiplicative functional obtained by rescaling the first and fourth components of $\widetilde{V}^{(m+1)}$ by $\varepsilon \eta$ and the seventh component by $\varepsilon$.

The uniform bound on the iterates $\left(V^{(m)}\right)_{m>1}$ will be obtained by induction. $X$ is controlled by a regular $\omega_{0}$ so there exists a constant $C$ such that $V^{(1)}$ is controlled by $\omega \triangleq C \omega_{0}$. Suppose that $V^{(k)}(k \leq m)$ are controlled by $\omega$. From (Corollary A.1) there is a constant $C_{1}$ such that $\widetilde{V}^{(m+1)}$ is controlled by $C_{1} \omega$. If we choose $\varepsilon>0, \eta>1$ such that $\varepsilon \leq C_{1}^{-[p] / p}$ and $\varepsilon \eta \leq C_{1}^{-[p] / p}$, then Lemma A. 3 implies that $V^{(m+1)}$ is controlled by $\omega$, completing the induction step.

The uniform control on the iterates $V^{(m)}$ ensures the convergence of $\left\{Y^{(i)(m)}\right\}_{m \geq 1}$ to the solutions of

$$
d Y_{t}^{(i)}=f\left(Y_{t}^{(i)}\right) d X_{t}, \quad Y_{0}^{(i)}=a_{i}, i=1,2
$$

Through the definition of $\left\{{ }^{\theta} W^{(m)}\right\}_{m \geq 1}$, the sequence at the level of the paths will converge to the scaled difference of the two solutions $\theta^{-1}\left(Y^{(2)}-Y^{(1)}\right)$. For $s, t$ in $J$ one has

$$
\left|{ }^{\theta} W_{s t}^{(i)}\right| \leq \frac{\omega(s, t)^{i / p}}{\beta\left(\frac{i}{p}\right)!}, \quad i=1, \ldots,[p]
$$

which implies that

$$
\left|W_{s t}^{(i)}\right| \leq \theta^{i} \frac{\omega(s, t)^{i / p}}{\beta\left(\frac{i}{p}\right)!}, \quad i=1, \ldots,[p]
$$

Proof of Lemma A.2. We define the following set of times

$$
\begin{equation*}
t_{0} \triangleq 0 \quad \text { and } \quad t_{j} \triangleq \inf \left\{u>t_{j-1}: \omega\left(t_{j-1}, u\right)=1\right\}, \tag{36}
\end{equation*}
$$

for all $j \in\{1, \ldots, n(s)\}$, where $n(s) \triangleq \max \left\{j: t_{j} \leq s\right\}$ and $t_{n(s)+1}=s$.
We solve the differential equation starting from $s$ and use (34) to show that

$$
\left|W_{s t}^{k}\right| \leq K(s)^{k} \frac{\omega(s, t)^{(k / p)}}{\beta\left(\frac{k}{p}\right)!}
$$

where $K(s)$ is an upper bound on the supremum over all the possible differences of the paths $\left|Y_{s}^{(1)}-Y_{s}^{(2)}\right|$, at time $s$. The bound $K(s)$ is derived recursively by considering the analogous upper bound for the difference of the solutions to the differential equation over the time interval $\left[t_{i-1}, t_{i}\right]$ given below

$$
\begin{aligned}
\left|Y_{t_{i}}^{(1)}-Y_{t_{i}}^{(2)}\right| & \leq\left|Y_{t_{i-1}}^{(1)}-Y_{t_{i-1}}^{(2)}\right|+\left|W_{t_{i-1} t_{i}}\right| \\
& \leq\left|Y_{t_{i-1}}^{(1)}-Y_{t_{i-1}}^{(2)}\right|\left(1+\frac{\omega\left(t_{i-1}, t_{i}\right)^{(1 / p)}}{\beta\left(\frac{1}{p}\right)!}\right)
\end{aligned}
$$

which implies that

$$
K\left(t_{j}\right) \leq K\left(t_{j-1}\right)\left(1+\frac{\omega\left(t_{j-1}, t_{j}\right)^{(1 / p)}}{\beta\left(\frac{1}{p}\right)!}\right), \quad j=1, \ldots, n(s)+1
$$

Therefore

$$
\begin{aligned}
\left|W_{s t}^{k}\right| \leq & K\left(t_{0}\right)^{k} \prod_{j=1}^{n(s)+1}\left(1+\frac{\omega\left(t_{j-1}, t_{j}\right)^{(1 / p)}}{\beta\left(\frac{1}{p}\right)!}\right)^{k} \frac{\omega(s, t)^{(k / p)}}{\beta\left(\frac{k}{p}\right)!} \\
\leq & \theta^{k} \exp \left(k\left(\sum_{j=1}^{n(s)} \frac{\omega\left(t_{j-1}, t_{j}\right)^{(1 / p)}}{\beta\left(\frac{1}{p}\right)!}+\frac{\omega\left(t_{n(s)}, s\right)^{(1 / p)}}{\beta\left(\frac{1}{p}\right)!}\right)\right) \\
& \cdot \frac{\omega(s, t)^{(k / p)}}{\beta\left(\frac{k}{p}\right)!},
\end{aligned}
$$

noting that $\omega\left(t_{j-1}, t_{j}\right)=1$ and using the sub-additivity of $\omega$ we obtain

$$
\leq \theta^{k} \exp \left(\frac{k}{\beta\left(\frac{1}{p}\right)!}\left(\omega(0, s)+\omega(0, s)^{(1 / p)}\right)\right) \frac{\omega(s, t)^{(k / p)}}{\beta\left(\frac{k}{p}\right)!}
$$

By considering the above bound at the level of the paths $(k=1)$ and repeatedly using the triangle inequality one deduces (35).

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[^3]
# Branching process associated with 2d-Navier Stokes equation 

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#### Abstract

Omega\) being a bounded open set in $\mathbb{R}^{2}$, with regular boundary, we associate with Navier-Stokes equation in $\Omega$ where the velocity is null on $\partial \Omega$, a non-linear branching process ( $Y_{t} ; t \geq 0$ ). More precisely: $E_{\omega_{0}}\left(\left\langle h, Y_{t}\right\rangle\right)=\langle\omega, h\rangle$, for any test function $h$, where $\omega=\operatorname{rot} u$, $u$ denotes the velocity solution of Navier-Stokes equation. The support of the random measure $Y_{t}$ increases or decreases of one unit when the underlying process hits $\partial \Omega$; this stochastic phenomenon corresponds to the creation-annihilation of vortex localized at the boundary of $\Omega$.


## 0. Introduction.

We consider here the 2d-Navier-Stokes (N.S.) equation in a bounded open set $\Omega$ of $\mathbb{R}^{2}$. The aim is not the study of existence or uniqueness for the solution, when the initial data or the boundary of $\Omega$ are smooth enough. We suppose that there exists a unique smooth solution $u$ of the Navier-Stokes equation, and we represent $u$ and the vorticity ( $\omega=\operatorname{rot} u$ ) through a stochastic model.

We firstly introduce two dual nonlinear differential systems. We prove (see sections 2 and 3) that the (N.S.) equation is equivalent to each of the former nonlinear reflected stochastic differential equations. The nonlinear feature appears in two places:

- inside $\Omega$, through the singular kernel of Biot and Savart, defining the mean velocity of the stochastic particle, when it moves in $\Omega$,
- on the boundary of $\Omega$, via the local time; this process governing the local behaviour of the particle when it reaches $\partial \Omega$.

Since $\Omega$ is a subset of $\mathbb{R}^{2}$, the vorticity $\omega$ is a scalar function. Roughly speaking $\omega(t, \cdot)$ is the "density" of one of the two previous diffusion processes taken a time $t$. (Corollary 3.4).

Secondly we define a branching process $Y$, having again a double non linearity. By definition, $Y_{t}$ is a linear and random combination of Dirac measures. $\omega$ is expressed through $Y$ as follows (cf. Theorem 4.5)

$$
E\left[\left\langle h, Y_{t}\right\rangle\right]=E\left[\int_{\bar{\Omega}} h(x) Y_{t}(d x)\right]=\int_{\Omega} h(x) \omega(t, x) d x
$$

$h$ being a test function.
$\omega(t, x) d x$ can be interpreted as the mean value at time $t$ of the number of particles associated with $Y$, lying in a infinitesimal box located at $x$, with area $d x$. We heuristically describe the dynamic of branching of $Y$ :

- a single particle moves in $\Omega$ as a diffusion process introduced in the first step, all the particles being alive are independent,
- no particle is created when all of them lie in $\Omega$.
- Sometime (i.e. randomly), when a particle hits $F \subset \partial \Omega$ (respectively $F^{\dagger} \subset \partial \Omega$ ) the particle dies and give rise to two new independent particles, (resp. the particle dies), where $F \cup F^{\dagger}=\partial \Omega$. The branching mechanism taking on the boundary gives a stochastic interpretation of the creation or disappearing of vorticity on $\partial \Omega$.

We conjecture that the nonlinear branching process can be approximated by a system of interacting particles. Our algorithm does not coincide with those intoduced by A. Chorin ([C.M]).

Let in briefly detail the organisation of the paper:

- In Section 1, we study the connections between the Navier-Stokes equation and the equation verified by the vorticity $\omega$.
- In sections 2 and 3 , we prove the equivalence between the two reflected stochastic differential equations and the (N.S.) equation. Moreover we check that these diffusion processes are in duality.
- We detail in Section 4, the construction of the branching process associated with the (N.S.) equation.
- We describe in Section 5, our open question concerning the simulation of the nonlinear branching process through a system of particles.


## 1. A first approach to the Navier Stokes equation.

1) In this paper, $\Omega$ will denote a simply connected, bounded open subset of $\mathbb{R}^{2}$. We assume that the boundary $\partial \Omega$ is smooth. The Navier Stokes system in $\Omega$, with velocity vanishing at $\partial \Omega$, is

$$
\left\{\begin{array}{l}
\text { i) } \frac{\partial u}{\partial t}+(u \cdot \nabla) u=\nu \Delta u-\nabla p  \tag{N.S.}\\
\text { ii) } \operatorname{div} u=0 \\
\text { iii) } u(0, \cdot)=u_{0}(\cdot), \\
\text { iv) } u(t, x)=0, \quad \text { for all } t \geq 0, \text { for all } x \in \partial \Omega
\end{array}\right.
$$

$u=\left(u_{1}, u_{2}\right)$ is the velocity ( $u$ is a two-dimensional valued vector field), $u_{0}$ is the initial velocity, $p$ denotes the pressure ( $p$ is a scalar function), $\nu$ is the viscosity of the fluid ( $\nu$ will be taken without loss of generality equal to $1 / 2$ in the sequel). As usual

$$
\begin{aligned}
\nabla p & =\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) \\
\operatorname{div} u & =\frac{\partial u_{1}}{\partial x}+\frac{\partial u_{2}}{\partial y} \\
\Delta u & =\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}
\end{aligned}
$$

and

$$
u \cdot \nabla=u_{1} \frac{\partial}{\partial x}+u_{2} \frac{\partial}{\partial y} .
$$

Note that the first equation i) in the (N.S.) system has to be understood as an equation in $\mathbb{R}^{2}$. Condition ii), i.e. $\operatorname{div} u=0$, means that the fluid is incompressible.

We know that, if $u_{0}$ with $\operatorname{div} u_{0}=0$ and $\partial \Omega$ are smooth, then (N.S.) has a unique smooth solution defined on $\mathbb{R}^{+} \times \bar{\Omega}([\mathrm{La}],[\mathrm{Li}],[\mathrm{C}-\mathrm{F}])$ if $u_{0}$ is $C^{\infty}(\bar{\Omega})$ and ([Ka], $\left.[\mathrm{Ko}]\right)$ if $u_{0}$ is analytic.
2) In a first step we weaken iv) and consider
(N.S.) $\quad\left\{\begin{array}{l}\text { i) } \frac{\partial u}{\partial t}+(u \cdot \nabla) u=\frac{1}{2} \Delta u-\nabla p, \\ \text { ii) } \operatorname{div} u=0, \\ \text { iii) } u(0, \cdot)=u_{0}, \\ \text { iv) } u \cdot n(t, x)=0, \quad \text { for all } t \geq 0, \text { for all } x \in \partial \Omega .\end{array}\right.$
where $n(x)$ denotes the normalized outer normal vector at $x \in \partial \Omega$. Since iv) in (N.S.) is stronger than iv) in (N.S.) the solutions of (N.S.) ${ }^{\prime}$ are not unique.

If $w: \mathbb{R}^{2} \longrightarrow \mathbb{R}$ we set

$$
\begin{equation*}
\nabla^{\perp} w=\left(-\frac{\partial w}{\partial y},+\frac{\partial w}{\partial x}\right) \tag{1.1}
\end{equation*}
$$

Let us introduce the following operator $K$ ( $K$ is the Biot and Savart kernel associated with $\Omega$ )

$$
\begin{align*}
K f(t, z) & =\nabla_{z}^{\perp} \int_{\Omega} G\left(z, z^{\prime}\right) f\left(t, z^{\prime}\right) d z^{\prime} \\
& =\left\{\begin{array}{l}
-\frac{\partial}{\partial y} \int_{\Omega} G\left((x, y), z^{\prime}\right) f\left(t, z^{\prime}\right) d z^{\prime} \\
+\frac{\partial}{\partial x} \int_{\Omega} G\left((x, y), z^{\prime}\right) f\left(t, z^{\prime}\right) d z^{\prime}
\end{array}\right. \tag{1.2}
\end{align*}
$$

where $z=(x, y)$ and $G$ is the Green function of $\Delta$ on $\Omega$, i.e.

$$
\Delta_{z^{\prime}} G\left(z, z^{\prime}\right)=\delta_{z}\left(\delta_{z} \text { being the Dirac measure at } z\right),
$$

$$
\begin{equation*}
G\left(z, z^{\prime}\right)=0, \quad \text { if } z^{\prime} \in \partial \Omega \tag{1.3}
\end{equation*}
$$

$$
\begin{equation*}
G\left(z, z^{\prime}\right)=G\left(z^{\prime}, z\right) \tag{1.4}
\end{equation*}
$$

It is classical to replace the two-dimensional equation i) in (N.S.) (or (N.S.)') by an equivalent equation where the unknown parameter will be a real function $\omega$. $\omega$ is called the vorticity associated with $u$ and is defined by

$$
\begin{equation*}
\omega=\operatorname{rot} u:=\frac{\partial u_{2}}{\partial x}-\frac{\partial u_{1}}{\partial y} . \tag{1.5}
\end{equation*}
$$

Recall that it is supposed that $\Omega$ is a bounded simply connected open set.

Lemma 1.1. 1) Suppose $u: \bar{\Omega} \longrightarrow \mathbb{R}^{2}$ being a smooth function i) and ii) below are equivalent
i) $\operatorname{div} u=0$ in $\bar{\Omega}, u \cdot n=0$ in $\partial \Omega$.
ii) There exists $\omega: \bar{\Omega} \longrightarrow \mathbb{R}$ such that $u=K \omega$.

In this case $\omega=\operatorname{rot} u$.
2) a) Assume $(u, p)$ is a solution of (N.S.)', then $\omega=\operatorname{rot} u$ solves the vorticity equation

$$
\left\{\begin{array}{l}
\text { i) } \left.\frac{\partial \omega}{\partial t}=\frac{1}{2} \Delta \omega-K \omega \cdot \nabla \omega, \quad \text { in }\right] 0,+\infty[\times \Omega  \tag{1.8}\\
\text { ii) } \omega(0, \cdot)=\operatorname{rot} u_{0}:=\omega_{0}
\end{array}\right.
$$

b) Suppose that $\omega$ is a solution of (1.8); then there exists $p$ such that ( $u, p$ ) solves (N.S.)' where $u=K \omega$.

Proof. 1) $\Omega$ being a simply connected open set, the condition div $u=0$ implies the existence of a function $\psi$ (the stream function) such that,

$$
\begin{equation*}
u=\nabla^{\perp} \psi \tag{1.9}
\end{equation*}
$$

Obviously $\psi$ is defined up to an additive constant. If we take a parametrization of $\partial \Omega$, we easily verify

$$
u \cdot n(t, x)=0, \quad \text { for all } \mathrm{t} \geq 0, \text { for all } x \in \partial \Omega
$$

$$
\begin{equation*}
\text { if and only if } \psi(t, x)=c, \quad \text { for all } t \geq 0, \text { for all } x \in \partial \Omega . \tag{1.10}
\end{equation*}
$$

$\psi$ is unique if we choose $c=0$.
We set $\omega=\operatorname{rot} u$. By a straightforward calculation we obtain

$$
\begin{equation*}
\omega=\Delta \psi . \tag{1.11}
\end{equation*}
$$

Since $\psi$ vanishes on $\partial \Omega$, it can be expressed through $\omega$, via the Green function

$$
\begin{equation*}
\psi(t, z)=\int_{\Omega} G\left(z, z^{\prime}\right) \omega\left(t, z^{\prime}\right) d z^{\prime} \tag{1.12}
\end{equation*}
$$

(1.9) implies that $u=K \omega$.

We now analyze the converse. Suppose that $u=K \omega$. This means that $u=\nabla^{\perp} \psi$ when $\psi$ is defined by (1.12). Hence, $\operatorname{div} u=0$ in $\left[0,+\infty\left[\times \Omega\right.\right.$ and $\psi(t, x)=0$ for any $(t, x) \in \mathbb{R}_{+} \times \partial \Omega$. Then the equivalence (1.10) implies that $u \cdot n=0$.
2) a) For any smooth functions, $w: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$, and $q: \mathbb{R}^{2} \longrightarrow \mathbb{R}$, we have,
(1.13) $\operatorname{rot}(\nabla q)=0, \quad \operatorname{rot}((w \cdot \nabla) w)=w \cdot \nabla(\operatorname{rot} w)+(\operatorname{div} w) \operatorname{rot} w$.

Let $(u, p)$ be a solution of (N.S.) $)^{\prime}$. We take the rotational operator on both sides of (N.S.) ${ }^{\prime}$ i), we easily obtain (1.8) i).
b) Let us suppose that $\omega$ solves (1.8). We set $u=K \omega$. Then $\operatorname{div} u=0$, and $u \cdot n(t, x)=0$ for any $t \geq 0$ and $x \in \partial \Omega$. The former calculation tells us that

$$
\operatorname{rot}\left(\frac{\partial u}{\partial t}-(u \cdot \nabla) u-\frac{1}{2} \Delta u\right)=0 .
$$

Hence there exists a function $p$ such that

$$
\frac{\partial u}{\partial t}-(u \cdot \nabla) u-\frac{1}{2} \Delta u=-\nabla p .
$$

We have proved that $u$ solves (N.S.) ${ }^{\prime}$.
We have now to characterize among the solutions $\omega$ of (1.8), the unique function such that $u=K \omega$ solves (N.S.), i.e.

$$
\begin{equation*}
K \omega \cdot \tau(x)=0, \quad \text { for all } t \geq 0, \text { for all } x \in \partial \Omega, \tag{1.14}
\end{equation*}
$$

where $\tau(x)$ denotes the tangent vector at $x \in \partial \Omega$.
Lemma 1.2. Let $\omega$ be a solution of (1.8) and $u=K \omega$. The four following assertions are equivalent:
a) $u(t, x)=0$, for all $t \geq 0$, for all $x \in \partial \Omega$.
b) $\frac{\partial}{\partial n} \int_{\Omega} G\left(z, z^{\prime}\right) \omega\left(t, z^{\prime}\right) d z^{\prime}=0$, for all $t \geq 0$, for all $z \in \partial \Omega$.
c) $\int_{\Omega} h(z) \omega(t, z) d z=0$, for any bounded and harmonic function $h$ defined on $\Omega$.
d) $\int_{\Omega} h(t, z) \omega(t, z) d z=0$, for any $h: \mathbb{R}_{+} \times \bar{\Omega} \longrightarrow \mathbb{R}$ verifying $\frac{1}{2} \Delta h \pm K \omega \cdot \nabla h=0$.

Proof. 1) Recall that $u=K \omega$ verifies (N.S.)'. Moreover

$$
u \cdot \tau=\nabla^{\perp} \psi \cdot \tau= \pm \nabla \psi \cdot n= \pm \frac{\partial}{\partial n} \psi
$$

Since $\psi$ is given by (1.12), a) if and only if b) follows immediately.
2) Let $h: \mathbb{R}_{+} \times \bar{\Omega} \longrightarrow \mathbb{R}$, such that $\Delta h=\alpha K \omega \cdot \nabla h$ in $] 0, \infty[\times \Omega$, where $\alpha$ is a constant.

The Stokes formula and (1.11) tell us

$$
\begin{aligned}
& \int_{\Omega} h(t, z) \omega(t, z) d z \\
&=\int_{\Omega} h(t, z) \Delta \psi(t, z) d z \\
&=\int_{\Omega} \Delta h(t, z) \psi(t, z) d z+\int_{\partial \Omega} h \frac{\partial \psi}{\partial n}-\int_{\partial \Omega} \psi \frac{\partial h}{\partial n} \\
&=\alpha \int_{\Omega} K \omega \cdot \nabla h(t, z) \psi(t, z) d z+\int_{\partial \Omega} h \frac{\partial \psi}{\partial n}-\int_{\partial \Omega} \psi \frac{\partial h}{\partial n} .
\end{aligned}
$$

We calculate the first integral by integrating by parts, we obtain,

$$
\begin{aligned}
\int_{\Omega}(K \omega \cdot & \nabla h)(t, z) \psi(t, z) d z \\
& =-\int_{\Omega}(\psi h \operatorname{div}(K \omega))(t, z) d z-\int_{\Omega}(h(K \omega \cdot \nabla \psi))(t, z) d z
\end{aligned}
$$

But $\psi=0$ on $\mathbb{R}_{+} \times \partial \Omega, \operatorname{div} K \omega=\operatorname{div} u=0$ and $K \omega \cdot \nabla \psi=\nabla^{\perp} \psi \cdot \nabla \psi=$ 0 , as a result

$$
\int_{\Omega} h(t, z) \omega(t, z) d z=\int_{\partial \Omega} h(t, \cdot) \frac{\partial \psi}{\partial n} .
$$

It is now clear that b) if and only if c) if and only if d).
We will say that $u(\operatorname{or} \omega)$ is a solution of (N.S.) if,

$$
\begin{equation*}
u=K \omega, \omega \text { solving the vorticity equation (1.8) } \tag{1.15}
\end{equation*}
$$

(1.16) and $u$ verifies one of the four conditions of Lemma 1.2.

## 2. The vorticity equation as a Kolmogorov equation.

Let $u$ be the solution of the Navier Stokes system (N.S.), $\omega$ denotes the vorticity, $\omega=\operatorname{rot} u$.

The operator

$$
\mathcal{L}(f)=\frac{1}{2} \Delta f-u \cdot \nabla f
$$

is the generator of a diffusion process $D$. If $\Omega=\mathbb{R}^{2}$, it is classical to represent $\omega$ through $\omega_{0}$ and $D$, the crucial fact being that $\omega\left(t-s, D_{s}\right)$ is a martingale.

Here $\Omega$ is an open, bounded, simply connected open set. We suppose moreover that the boundary is smooth. Let us introduce the following reflected stochastic differential equation driven by a twodimensional Brownian motion ( $B_{t}, t \geq 0$ ), started at 0 , and defined on a probability space $\left(\Omega_{0},\left(\mathcal{F}_{t}\right)_{t \geq 0}, P\right)$
$u$ being a smooth function and $\partial \Omega$ being of class at less $C^{2}$, there exists a unique strong solution ( $\left.\widetilde{X}_{s}^{t, x} ; 0 \leq s \leq t\right)$ of (2.1), taking its values in $\bar{\Omega}$, for any $t>0$ and $x \in \bar{\Omega}(c f .[\mathrm{P}]$ or [SV]). Recall that $n(x)$ is the normalized outer normal vector at $x \in \partial \Omega$.

The solution of $(2.1)$ is denoted $\tilde{X}$, because we will see in Section 3 that there exists a process $X$ such that $X$ and $\widetilde{X}$ are dual processes.

The drift coefficient $-u$ corresponds to the mean velocity of $\widetilde{X}$; if $\Omega=\mathbb{R}^{2}$, it is easy to check that

$$
\begin{equation*}
u\left(t-s, x_{0}\right)=-\lim _{h \rightarrow 0_{+}} E\left[\left.\frac{\widetilde{X}_{s+h}^{t, x}-\widetilde{X}_{s}^{t, x}}{h} \right\rvert\, \mathcal{F}_{s}\right], \quad 0 \leq s \leq t \tag{2.2}
\end{equation*}
$$

Recall that if $u$ is a solution of (N.S.)' (the weak form of (N.S.)), then $u \cdot n=0$ on $\mathbb{R}_{+} \times \partial \Omega$, and $u$ solves (N.S.) if $u \cdot \tau=0$ on $\mathbb{R}_{+} \times \partial \Omega$. If
$\Omega=\mathbb{R} \times] 0,+\infty\left[\left(\Omega\right.\right.$ is not bounded), then $\partial \Omega=\left\{\left(x_{1}, x_{2}\right): x_{2}=0\right\}$. We choose $\rho_{0}\left(x_{1}, x_{2}\right)=x_{1}$. The analog of (2.2) would be

$$
\begin{equation*}
(u \cdot \tau)\left(t-s, x_{0}\right)=\lim _{h \rightarrow 0} E\left[\left.\frac{\rho_{0}\left(\widetilde{X}_{s+h}^{t, x}\right)-\rho_{0}\left(\widetilde{X}_{s}^{t, x}\right)}{h} \right\rvert\, \mathcal{F}_{s}\right] \tag{2.3}
\end{equation*}
$$

where $0 \leq s \leq t, x \in \partial \Omega$. Our situation is more complicated. The function $\rho_{0}$ is replaced by the set $\mathcal{V}$ of velocity test functions. $\rho: \bar{\Omega} \longrightarrow$ $\mathbb{R}$ belongs to $\mathcal{V}$, if $\rho$ is of class $C^{\infty}$ and

$$
\begin{gather*}
\frac{\partial \rho}{\partial n}(z)=0, \quad \text { for all } z \in \partial \Omega  \tag{2.4}\\
\frac{\partial \rho}{\partial \tau} \neq 0 \text { almost sure on } \partial \Omega \tag{2.5}
\end{gather*}
$$

We note that it is not possible to choose $\rho$ such that $(\partial \rho / \partial \tau)(z) \neq 0$ for all $z \in \partial \Omega$. If $(\partial \rho / \partial \tau)$ never vanishes on $\partial \Omega, \partial \rho / \partial \tau$ being continuous we suppose without loss of generality that $\partial \rho / \partial \tau>0$. Let $\gamma:[0,1] \longrightarrow \partial \Omega$ be a parametrization of $\partial \Omega$. Since $t \longrightarrow \rho(\gamma(t))$ is increasing, then $\rho(\gamma(0))<\rho(\gamma(1))$, this generates a contradiction with $\gamma(0)=\gamma(1)$.

Two objects play a crucial role in our approach. The first one is

$$
\begin{equation*}
\varphi_{c}(s, x)=\frac{\frac{\partial \omega}{\partial n}}{\omega+c}(s, x), \quad s \geq 0, x \in \partial \Omega \tag{2.7}
\end{equation*}
$$

where $c$ is a positive constant such that $\omega+c>0$.
The second one is the stochastic process

$$
\begin{equation*}
\widetilde{Z}_{c}^{t, x}(s)=\left(\omega\left(t-s, \widetilde{X}_{s}^{t, x}\right)+c\right) \exp \left(\int_{0}^{s} \varphi_{c}\left(t-r, \widetilde{X}_{r}^{t, x}\right) d \widetilde{A}_{r}^{t, x}\right) \tag{2.8}
\end{equation*}
$$

where $0 \leq s \leq t$.
We are now ready to state the martingale property concerning $\omega$.
Proposition 2.1. Suppose $t>0, x \in \bar{\Omega}$ and $\omega$ is the vorticity solution of (1.8).

1) a) $\left(\widetilde{Z}_{0}^{t, x}(s \wedge \widetilde{\xi}) ; 0 \leq s \leq t\right)$ is a continuous local martingale where

$$
\begin{equation*}
\widetilde{\xi}=\inf \left\{s \leq t: \widetilde{X}_{s}^{t, x} \in \partial \Omega \text { and } \omega\left(t-s ; \widetilde{X}_{s}^{t, x}\right)=0\right\} \wedge t \tag{2.9}
\end{equation*}
$$

(We assume the convention $\inf \phi=+\infty$ ).
b) If $c$ is large enough (i.e. $\left.c>C_{t, \Omega}\right),\left(\widetilde{Z}_{c}^{t, x}(s) ; 0 \leq s \leq t\right)$ is a square integrable continuous martingale, $C_{t, \Omega}$ being defined as

$$
\begin{equation*}
C_{t, \Omega}=-\min \{\omega(s, x): 0 \leq s \leq t, x \in \Omega\} \tag{2.10}
\end{equation*}
$$

2) For any positive $t, x$ in $\Omega$ and velocity test function $\rho$,

$$
\begin{equation*}
\frac{1}{h} E\left[\rho\left(\widetilde{X}_{(\widetilde{T}+h) \wedge t}^{t, x}-\rho\left(\widetilde{X}_{\widetilde{T}}^{t, x}\right) \mid \mathcal{F}_{\widetilde{T}}\right] \xrightarrow[h \rightarrow 0]{\text { a.s. }} 0\right. \tag{2.11}
\end{equation*}
$$

with,

$$
\begin{equation*}
\widetilde{T}=\inf \left\{s \leq t: \tilde{X}_{s}^{t, x} \in \partial \Omega\right\} \wedge t \tag{2.12}
\end{equation*}
$$

Proof. 1) $t$ and $x$ being fixed, we denote for simplicity $\widetilde{X}=\widetilde{X}^{t, x}$ and $\widetilde{A}=\widetilde{A}^{t, x}$.

We apply the Itô formula

$$
\begin{aligned}
\omega\left(t-s, \widetilde{X}_{s}\right)= & \omega(t, x)+\int_{0}^{s} \nabla \omega\left(t-r, \widetilde{X}_{r}\right) d B_{r}-\int_{0}^{s} \frac{\partial \omega}{\partial n}\left(t-r, \widetilde{X}_{r}\right) d \widetilde{A}_{r} \\
& +\int_{0}^{s}\left(-\frac{\partial \omega}{\partial t}-u \cdot \nabla \omega+\frac{1}{2} \Delta \omega\right)\left(t-r, \widetilde{X}_{r}\right) d r
\end{aligned}
$$

where $0 \leq s \leq t . \omega$ solves (1.8), therefore,

$$
\begin{align*}
\omega\left(t-s, \widetilde{X}_{s}\right)= & \omega(t, x)+\int_{0}^{s} \nabla \omega\left(t-r, \widetilde{X}_{r}\right) d B_{r}  \tag{2.13}\\
& -\int_{0}^{s} \frac{\partial \omega}{\partial n}\left(t-r, \widetilde{X}_{r}\right) d \widetilde{A}_{r} .
\end{align*}
$$

Let us introduce, for all integer $n \geq 1$

$$
\begin{equation*}
\widetilde{\xi}_{n}=\inf \left\{s \leq t:\left|\omega\left(t-s, \widetilde{X}_{s}\right)\right| \leq \frac{1}{n} \text { and } \widetilde{X}_{s} \in \partial \Omega\right\} \wedge t \tag{2.14}
\end{equation*}
$$

$\left\{\widetilde{\xi}_{n}\right\}_{n \geq 1}$ is an increasing sequence of stopping times converging to $\widetilde{\xi}$. We set

$$
\begin{equation*}
\widetilde{M}_{c}(s)=\exp \left(\int_{0}^{s} \varphi_{c}\left(t-r, \widetilde{X}_{r}\right) d \widetilde{A}_{r}\right) . \tag{2.15}
\end{equation*}
$$

Using again the Itô formula we get,

$$
\begin{aligned}
& \omega\left(t-s \wedge \widetilde{\xi}_{n}, \widetilde{X}\left(s \wedge \widetilde{\xi}_{n}\right)\right) \widetilde{M}_{0}\left(s \wedge \widetilde{\xi}_{n}\right) \\
& =\omega(t, x)+\int_{0}^{s \wedge \widetilde{\xi}_{n}} \widetilde{M}_{0}(r) \nabla \omega\left(t-r, \widetilde{X}_{r}\right) d B_{r} \\
& \quad+\int_{0}^{s \wedge \widetilde{\xi}_{n}}\left(-\widetilde{M}_{0}(r) \frac{\partial \omega}{\partial n}\left(t-r, \widetilde{X}_{r}\right)+\widetilde{M}_{0}(r)\left(\omega \varphi_{0}\right)\left(t-r, \widetilde{X}_{r}\right)\right) d \widetilde{A}_{r}
\end{aligned}
$$

But $\partial \omega / \partial n-\omega \varphi_{0}=0$, then

$$
\begin{equation*}
\widetilde{Z}_{0}^{t, x}\left(s \wedge \widetilde{\xi}_{n}\right)=\omega(t, x)+\int_{0}^{s \wedge \widetilde{\xi}_{n}} \widetilde{M}_{0}(r) \nabla \omega\left(t-r, \widetilde{X}_{r}\right) d B_{r} \tag{2.16}
\end{equation*}
$$

Part 1) a) follows immediately.
We note that $(\omega+c)(s, y) \geq \alpha>0$ for any $(s, y) \in[0, t] \times \Omega, \omega+c$ solves (1.8) and

$$
\frac{\partial(\omega+c)}{\partial n}-(\omega+c) \varphi_{c}=0
$$

the former proof tells us that $\left(\widetilde{Z}_{c}^{t, x}(s) ; 0 \leq s \leq t\right)$ is a local martingale and

$$
\begin{equation*}
\widetilde{Z}_{c}^{t, x}(s)=\omega(t, x)+c+\int_{0}^{s} \widetilde{M}_{c}(r) \nabla \omega\left(t-r, \widetilde{X}_{r}\right) d B_{r} . \tag{2.17}
\end{equation*}
$$

We write

$$
\nabla \omega=(\omega+c) \frac{\nabla \omega}{\omega+c}
$$

The function $\nabla \omega /(\omega+c)$ being bounded on $[0, t] \times \Omega$, making use of localization we have

$$
E\left[\left(\widetilde{Z}_{c}^{t, x}(s)\right)^{2}\right] \leq(\omega(t, x)+c)^{2}+K \int_{0}^{s} E\left[\left(\widetilde{Z}_{c}^{t, x}(r)\right)^{2}\right] d r .
$$

Gronwall lemma implies $E\left[\left(\widetilde{Z}_{c}^{t, x}(s)\right)^{2}\right] \leq(\omega(t, x)+c)^{2} e^{K s}, 0 \leq s \leq t$. This shows 1) b).
2) Let $\rho$ be a function of class $C^{\infty}$ verifying (2.4)-(2.6) (i.e. $\rho$ is a velocity test function). We apply the Itô formula to $\rho$

$$
\begin{aligned}
\rho\left(\widetilde{X}_{s}^{t, x}\right)= & \rho(x)+\int_{0}^{s} \nabla \rho\left(\widetilde{X}_{r}^{t, x}\right) d B_{r} \\
& -\int_{0}^{s}\left(u\left(t-r, \widetilde{X}_{r}^{t, x}\right) \cdot \nabla \rho\left(\widetilde{X}_{r}^{t, x}\right)-\frac{1}{2} \Delta \rho\left(\widetilde{X}_{r}^{t, x}\right)\right) d r .
\end{aligned}
$$

Consequently (for simplicity we write $T$ for $\widetilde{T}$ ).

$$
\begin{aligned}
& \rho\left(\widetilde{X}_{(T+h) \wedge t}^{t, x}\right)-\rho\left(\widetilde{X}_{T}^{t, x}\right) \\
& =\int_{T}^{(T+h) \wedge t} \nabla \rho\left(\widetilde{X}_{r}^{t, x}\right) d B_{r} \\
& \quad-\int_{T}^{(T+h) \wedge t}\left(u\left(t-r, \widetilde{X}_{r}^{t, x}\right) \cdot \nabla \rho\left(\widetilde{X}_{r}^{t, x}\right)-\frac{1}{2} \Delta \rho\left(\widetilde{X}_{r}^{t, x}\right)\right) d r, \\
& \quad \begin{array}{r}
\frac{1}{h} E\left[\rho\left(\widetilde{X}_{(T+h) \wedge t}^{t, x}\right)-\rho\left(\widetilde{X}_{T}^{t, x}\right) \mid \mathcal{F}_{T}\right] \\
\\
\quad=-\frac{1}{h} E\left[\int _ { T } ^ { ( T + h ) \wedge t } \left(u\left(t-r, \widetilde{X}_{r}^{t, x}\right) \cdot \nabla \rho\left(\widetilde{X}_{r}^{t, x}\right)\right.\right. \\
\\
\left.\left.\quad-\frac{1}{2} \Delta \rho\left(\widetilde{X}_{r}^{t, x}\right) d r\right) \mid \mathcal{F}_{T}\right] .
\end{array}
\end{aligned}
$$

On $\{T=t\} \in \mathcal{F}_{T}$, the integral is equal to 0 , therefore the limit is 0 . Suppose $\{T<t\}$. Recall that $\Delta \rho=0$ in a neighbourhood of $\partial \Omega$, $u$ and $\nabla \rho$ are continuous functions, then the almost sure limit of the right-hand side of $(2.18)$ is $-u\left(t-T, \widetilde{X}_{T}^{t, x}\right) \cdot \nabla \rho\left(\widetilde{X}_{T}^{t, x}\right)$. But on $\{T<t\}$, $\widetilde{X}_{T}^{t, x} \in \partial \Omega$ then $u\left(t-T, \widetilde{X}_{T}^{t, x}\right)=0$.

Applying the stopping theorem we get:
Corollary 2.2. Recall that $\widetilde{\xi}_{n}$ is the stopping time defined by (2.14). Then

$$
\omega(t, x)=E\left[\omega\left(t-s \wedge \widetilde{\xi}_{n}, \widetilde{X}^{t, x}\left(s \wedge \widetilde{\xi}_{n}\right)\right)\right.
$$

$$
\begin{align*}
\cdot & \left.\exp \left(\int_{0}^{s \wedge \widetilde{\xi}_{n}} \varphi_{0}\left(t-r, \widetilde{X}_{r}^{t, x}\right) d \widetilde{A}_{r}^{t, x}\right)\right]  \tag{2.19}\\
\omega(t, x)+c=E & {\left[\left(\omega\left(t-s, \widetilde{X}_{s}^{t, x}\right)+c\right)\right.} \\
& \left.\cdot \exp \left(\int_{0}^{s} \varphi_{c}\left(t-r, \widetilde{X}_{r}^{t, x}\right) d \widetilde{A}_{r}^{t, x}\right)\right] \tag{2.20}
\end{align*}
$$

$c$ being larger than $C_{t, \Omega}$.

We would like to define a self-contained nonlinear stochastic system -we call it the Stochastic Navier Stokes system (S.N.S.)- equivalent to the Navier Stokes system. Proposition 2.1, tells us that $\widetilde{X}$ is a good candidate concerning the stochastic part. It remains to express the drift term $K \omega$ through the underlying process $\widetilde{X}$.

Let us detail the (S.N.S.) system and its three conditions (S.N.S.1), (S.N.S.2) and (S.N.S.3). The unknown parameters are $\omega$, $\left\{\left(\widetilde{X}_{s}^{t, x} ; 0 \leq\right.\right.$ $\left.s \leq t),\left(\widetilde{A}_{s}^{t, x} ; 0 \leq s \leq t\right), t \geq 0, x \in \bar{\Omega}\right\}$.
(S.N.S.1) For any positive $t$ and $x$ in $\bar{\Omega}$, consider the following reflected stochastic differential equation

$$
\begin{gather*}
\widetilde{X}_{s}^{t, x}=x+B_{s}-\int_{0}^{s} K \omega\left(t-r, \widetilde{X}_{r}^{t, x}\right) d r  \tag{2.21}\\
-\int_{0}^{s} n\left(\widetilde{X}_{r}^{t, x}\right) d \widetilde{A}_{r}^{t, x}, \quad 0 \leq s \leq t, \\
\int_{0}^{s} 1_{\left\{\widetilde{X}_{r}^{t, x} \in \partial \Omega\right\}} d \widetilde{A}_{r}^{t, x}=\widetilde{A}_{s}^{t, x}, \quad \text { for all } s \in[0, t] . \tag{2.22}
\end{gather*}
$$

$\left(\widetilde{A}_{s}^{t, x} ; 0 \leq s \leq t\right)$ is the local time of $\widetilde{X}^{t, x}$ on the boundary.
Recall that $K \omega$ is the function defined by (1.2).
(S.N.S.2) The process $\left(\omega\left(t-s \wedge \widetilde{T}, \widetilde{X}^{t, x}(s \wedge \widetilde{T})\right)\right)$ is a martingale, for any $t>0$ and $x \in \bar{\Omega}$, where $\widetilde{T}$ is the first hitting time of the boundary

$$
\widetilde{T}=\inf \left\{s \leq t: \widetilde{X}^{t, x}(s) \in \partial \Omega\right\}
$$

(S.N.S.3) For any $t>0$ and velocity test function $\rho$

$$
\frac{1}{h} E\left[\rho\left(\widetilde{X}_{\widetilde{T}+h}^{t, x}\right)-\rho\left(\widetilde{X}_{\widetilde{T}}^{t, x}\right) \mid \mathcal{F}_{\widetilde{T}}\right] \underset{h \rightarrow 0}{\text { a.s. }} 0,
$$

for any $x$ in $\Omega$.
We just now state the converse of Proposition 2.1.
Proposition 2.3. Suppose that (S.N.S.) has a unique solution

$$
\left(\omega,\left\{\widetilde{X}_{s}^{t, x}, \widetilde{A}_{s}^{t, x} ; 0 \leq s \leq t\right\}, t \geq 0, x \in \bar{\Omega}\right),
$$

where $\omega$ is smooth. Then $u=K \omega$ is a solution of (N.S.) equation (i.e. verifies (1.15) and (1.16)).

Proof. 1) $\omega$ is bounded on $[0, t] \times \bar{\Omega}$, therefore $\left(\omega\left(t-s \wedge \widetilde{T}, \widetilde{X}^{t, x}(s \wedge\right.\right.$ $\widetilde{T}) ; 0 \leq s \leq t)$ ) is a bounded martingale. By the stopping theorem,

$$
\begin{equation*}
\omega(t, x)=E\left[\omega\left(t-s \wedge \widetilde{T}, \widetilde{X}^{t, x}(s \wedge \widetilde{T})\right)\right], \quad 0 \leq s \leq t \tag{2.23}
\end{equation*}
$$

The infinitesimal generator of $\left(\left(t-s ; \widetilde{X}_{s}^{t, x}\right) ; 0 \leq s \leq t\right)$ is $\mathcal{L}$, with

$$
\mathcal{L} f(s, x)=\left(-\frac{\partial f}{\partial s}+\frac{1}{2} \Delta f-u \cdot \nabla f\right)(t-s, x) .
$$

(2.23) implies that $\omega$ is the solution to the Dirichlet problem in $[0, t] \times \bar{\Omega}$ associated with $\mathcal{L}$. Consequently $\omega$ solves the vorticity equation (1.8).
2) Let $\rho$ be a velocity test function (recall that $\rho$ is of class $C^{\infty}$ and solves (2.4)-(2.6)). We are allowed to use relation (2.18)

$$
\begin{aligned}
\frac{1}{h} E\left[\rho\left(\widetilde{X}_{\widetilde{T}+h}^{t, x}\right)\right. & \left.-\rho\left(\widetilde{X}_{\widetilde{T}}^{t, x}\right) \mid \mathcal{F}_{\widetilde{T}}\right] \\
& =-\frac{1}{h} E\left[\int_{\widetilde{T}}^{\widetilde{T}+h} K \omega\left(t-r, \widetilde{X}_{r}^{t, x}\right) \cdot \nabla \rho\left(\widetilde{X}_{r}^{t, x}\right) d r \mid \mathcal{F}_{\widetilde{T}}\right]
\end{aligned}
$$

since for $h$ small enough $\Delta \rho\left(\widetilde{X}_{r}^{t, x}\right)=0, \widetilde{T} \leq r \leq \widetilde{T}+h$.
We take the limit $h \rightarrow 0_{+}$, the functions $K \omega$ and $\nabla \rho$ being bounded, we have

$$
\begin{equation*}
K \omega\left(t-\widetilde{T}, \widetilde{X}_{\widetilde{T}}^{t, x}\right) \cdot \nabla \rho\left(\widetilde{X}_{\widetilde{T}}^{t, x}\right)=0, \quad \text { almost sure . } \tag{2.24}
\end{equation*}
$$

From part 1) of Lemma 1.1, we know that $(K \omega) \cdot n=0$ on $[0, t] \times \partial \Omega$. Consequently (2.24) is equivalent to

$$
\frac{\partial K \omega}{\partial n}\left(t-\widetilde{T}, \widetilde{X}_{\widetilde{T}}^{t, x}\right) \frac{\partial \rho}{\partial n}\left(\widetilde{X}_{\widetilde{T}}^{t, x}\right)=0, \quad \text { almost sure } .
$$

Since $x$ belongs to $\Omega$, conditionnally to $\{\widetilde{T}<t\}$, the distribution of $\left(\widetilde{T}, \widetilde{X}_{\widetilde{T}}^{\tau},{ }_{\widetilde{\alpha}}^{t, x}\right)$ is absolutely continuous with respect to $\left(\mathbf{1}_{[0, t]}(u) d u\right) \otimes \widetilde{\lambda}(d v)$, where $\widetilde{\lambda}$ denotes the Lebesgue measure on $\partial \Omega$. Assumption (2.5) implies

$$
\frac{\partial K \omega}{\partial n}(t-s, y)=0, \quad \text { for almost every }(s, y) \in[0, t] \times \partial \Omega
$$

$\partial K \omega / \partial n$ being a continuous function, the former condition is equivalent to $(\partial K \omega / \partial n)(s, y)=0$ for any $s \in[0, t]$ and $y \in \partial \Omega$. $u$ is a solution of the (N.S.) equation because $K \omega$ vanishes on $\mathbb{R}_{+} \times \partial \Omega$.

Remarks 2.4. 1) (S.N.S.2) reveals the nonlinearity of $\tilde{X}$ inside $\Omega$. Indeed, (2.21) shows that $\widetilde{X}$ depends on $\omega$; and the martingale property (S.N.S.2) involving $\omega$ depends on $\widetilde{X}$.
2) (S.N.S.3) implies that the tangential component $u \cdot \tau$ of $u=K \omega$ is equal to 0 on the boundary of $\Omega$.
3) A priori, 1) a) of Proposition 2.1 seems a stronger condition than (S.N.S.2). We claim that these two conditions are equivalent.

We remark that $\widetilde{Z}_{0}^{t, x}(t \wedge \widetilde{T})=\omega\left(t-s \wedge \widetilde{T}, \widetilde{X}_{s \wedge \widetilde{T}}^{t, x}\right)$, where $\widetilde{T}$ is the first hitting time of $\partial \Omega$, and $\widetilde{T} \leq \widetilde{\xi}$ and $\widetilde{\xi}$ is the stopping time defined by (2.9). Therefore if $\omega$ solves the vorticity equation (1.8), $\left(\widetilde{Z}_{0}^{t, x}(s \wedge \widetilde{\xi}) ; 0 \leq s \leq t\right)$ is a local martingale, then (S.N.S.2) holds.

Let us analyze the converse. Suppose that the (S.N.S.) system has a solution. Hence $\omega$ solves (1.8). Applying the Itô formula we have,

$$
\begin{aligned}
\omega\left(t-s, \widetilde{X}_{s}^{t, x}\right)= & \omega(t, x)+\int_{0}^{s} \nabla \omega\left(t-r, \widetilde{X}_{r}^{t, x}\right) d B_{r} \\
& -\int_{0}^{s} \frac{\partial \omega}{\partial n}\left(t-r, \widetilde{X}_{r}^{t, x}\right) d \widetilde{A}_{r}^{t, x}
\end{aligned}
$$

Recall that

$$
\varphi_{0}=\frac{1}{\omega} \frac{\partial \omega}{\partial n},
$$

using again the Itô formula we obtain,

$$
\begin{aligned}
& \widetilde{Z}_{0}^{t, x}(s)=\omega\left(t-s, \widetilde{X}_{s}^{t, s}\right) \exp \int_{0}^{s} \varphi_{0}\left(t-r, \widetilde{X}_{r}^{t, x}\right) d \widetilde{A}_{r}^{t, x} \\
&=\omega(t, x)+\int_{0}^{s} \nabla \omega\left(t-u, \widetilde{X}_{u}^{t, x}\right) \\
& \cdot \exp \left(\int_{0}^{u} \varphi_{0}\left(t-r, \widetilde{X}_{r}^{t, x}\right) d \widetilde{A}_{r}^{t, x}\right) d B_{u},
\end{aligned}
$$

$s$ belonging to the stochastic interval $[0, \widetilde{\xi}]$.
This proves that $\left(\widetilde{Z}_{0}^{t, x}(s \wedge \widetilde{\xi}) ; 0 \leq s \leq \widetilde{\xi}\right)$ is a local martingale (i.e. 1) a) of Proposition 2.1 holds).

We analyze the integrability of $\widetilde{A}^{t, x}$. This plays an important role in Lemma 4.2.

Proposition 2.5. For any $\theta>0, t>0$ and $x \in \bar{\Omega}$,

$$
\begin{equation*}
E\left[\exp \left(\theta \widetilde{A}_{t}^{t, x}\right)\right]<\infty \tag{2.25}
\end{equation*}
$$

Moreover, for any $k$,

$$
\lim _{u \rightarrow 0} \sup _{\substack{0 \leq t \leq k \\ x \in \bar{\Omega}}} E\left[\widetilde{A}_{u}^{t, x}\right]=0 .
$$

Remarks. 1) In dimension 1, for the reflected Brownian motion, recall that the local time at $0, L_{t}^{0}$, has exponential moments, since $L_{t}^{0} \stackrel{(\mathrm{~d})}{=}$ $\sqrt{t}|N|$, where $N$ is a centered, unit variance, Gaussian random variable.
2) A similar estimation can be found in [S.V.].

Proof of Proposition 2.5. 1) Let $\lambda>0$ be fixed.
We choose $\gamma: \bar{\Omega} \longrightarrow \mathbb{R}$, a function of class $C^{2}$ such that,

$$
\begin{align*}
& \gamma(x) \geq 1, \quad \text { for all } x \in \bar{\Omega}  \tag{2.27}\\
& \text { i) } \frac{\partial \gamma}{\partial n}(x)=2 \lambda,  \tag{2.28}\\
& \text { ii) } \gamma(x)=2, \text { for any } x \in \partial \Omega .
\end{align*}
$$

A straightforward calculation based on the Itô formula and (2.28) shows ${ }_{\sim}^{\text {that }}\left(\underset{\sim}{U}{\underset{\sim}{s}}_{s} ; 0 \leq s \leq t\right)$ is a bounded martingale, where $\widetilde{X}_{s}=\widetilde{X}_{s}^{t, x}$, $\widetilde{A}_{s}=\widetilde{A}_{s}^{t, x}$ and

$$
\begin{gather*}
U_{s}=\gamma\left(\widetilde{X}_{s}\right) \exp \left(\lambda \widetilde{A}_{s}\right)-\int_{0}^{s} H(r) \exp \left(\lambda \widetilde{A}_{r}\right) d r  \tag{2.29}\\
H(r)=\frac{1}{2} \Delta \gamma\left(\widetilde{X}_{r}\right)-u\left(t-r, \widetilde{X}_{r}\right) \nabla \gamma\left(\widetilde{X}_{r}\right)
\end{gather*}
$$

$\gamma$ being of class $C^{2}, H(r)$ is a bounded process, then there exists a positive constant $k$ such that $|H(r)| \leq k$ for any $r$ in $[0, t]$. (2.27) implies that

$$
H(r) \leq|H(r)| \leq k \gamma\left(\widetilde{X}_{r}\right), \quad \text { for all } r \in[0, t] .
$$

Let $\left\{T_{n}\right\}_{n \geq 1}$ be a increasing sequence of stopping times, converging to $t$ such that $\widetilde{A}_{t \wedge T_{n}} \leq n$.

We replace $s$ by $s \wedge T_{n}$ in (2.29) and we take the expectation, we easily obtain the following inequality

$$
\begin{aligned}
\gamma(x) & =E\left[U_{s \wedge T_{n}}\right] \\
& \geq E\left[\gamma\left(\widetilde{X}_{s \wedge T_{n}}\right) \exp \left(\lambda \widetilde{A}_{s \wedge T_{n}}\right)\right]-k E\left[\int_{0}^{s \wedge T n} \gamma\left(\widetilde{X}_{r}\right) \exp \left(\lambda \widetilde{A}_{r}\right) d r\right] .
\end{aligned}
$$

We set $\alpha_{n}(s)=E\left[\gamma\left(\widetilde{X}_{s \wedge T_{n}}\right) \exp \left(\lambda \widetilde{A}_{s \wedge T_{n}}\right)\right]$. The former inequality is equivalent to

$$
\alpha_{n}(s) \leq \gamma(x)+k \int_{0}^{s} \alpha_{n}(u) d u, \quad \text { for all } s \in[0, t] .
$$

Using the Gronwall lemma and (2.27) we conclude that

$$
E\left[\exp \lambda \widetilde{A}_{s \wedge T_{n}}\right] \leq \gamma(x) e^{k s}
$$

We take the limit $n$ going to infinity,

$$
E\left[\exp \lambda \widetilde{A}_{s}\right] \leq \gamma(x) e^{k s}
$$

2) As for (2.26), we take $\gamma_{0}: \bar{\Omega} \longrightarrow \mathbb{R}$ of class $C^{2}$, such that

$$
\frac{\partial \gamma_{0}}{\partial n}(x)=1, \quad \text { for all } x \in \partial \Omega
$$

We apply the Itô formula and we take the expectation

$$
\begin{gathered}
E\left[\widetilde{A}_{s}^{t, x}\right]=E\left[\gamma_{0}(x)-\gamma_{0}\left(\widetilde{X}_{s}^{t, x}\right)\right]+\int_{0}^{s} h(r) d r, \\
h(r)=E\left[\frac{1}{2} \Delta \gamma_{0}\left(\widetilde{X}_{r}^{t, x}\right)-u\left(t-r, \widetilde{X}_{r}^{t, x}\right) \nabla \gamma_{0}\left(\widetilde{X}_{r}^{t, x}\right)\right] .
\end{gathered}
$$

Since $\gamma_{0}$ is of class $C^{2}, h$ is bounded, moreover the continuity of $(t, x, s)$ $\longrightarrow \widetilde{X}_{s}^{t, x}$ implies that (2.26) is verified.

Before ending this section we prove some properties concerning the distribution of $\widetilde{X}_{s}^{t, x}$ (respectively $\int_{0}^{t} H\left(r, \widetilde{X}_{r}^{t}\right) d \widetilde{A}_{r}^{t}$ ) when $\widetilde{X}_{0}^{t}$ is uniformly distributed on $\bar{\Omega}$.

Notations. 1) $\lambda$ is the normalized Lebesgue measure on $\bar{\Omega}: \lambda$ is proportional to the Lebesgue measure on $\bar{\Omega}$ and $\lambda(\bar{\Omega})=1$.
2) Let $h: \bar{\Omega} \longrightarrow \mathbb{R}$ and $F: \mathcal{C}([0, t]) \longrightarrow \mathbb{R}$, we set

$$
\begin{align*}
& E_{h \cdot \lambda}\left[F\left(\widetilde{X}_{s}^{t} ; 0 \leq s \leq t\right)\right] \\
& \quad=\int_{\Omega} E\left[F\left(\widetilde{X}_{s}^{t, x} ; 0 \leq s \leq t\right)\right] h(x) \lambda(d x) \tag{2.30}
\end{align*}
$$

(2.30) is meaningfull if for instance $h$ and $F$ are positive.

Proposition 2.6. 1) Suppose $f: \bar{\Omega} \longrightarrow \mathbb{R}$ and $F:[0, t] \times \partial \Omega \longrightarrow \mathbb{R}$, $H:[0, t] \times \partial \Omega \longrightarrow \mathbb{R}$ are bounded Borel functions. Then, for any $s$ in $[0, t]$,

$$
\begin{align*}
E_{\lambda}\left[f\left(\widetilde{X}_{s}^{t}\right)\right] & =\int_{\Omega} f(x)(d x)  \tag{2.31}\\
E_{\lambda}\left[\int_{0}^{s} H\left(r, \widetilde{X}_{r}^{t}\right) d \widetilde{A}_{r}^{t}\right] & =\frac{1}{2} \int_{0}^{s}\left(\int_{\partial \Omega} H(r, x) d x\right) d r \tag{2.32}
\end{align*}
$$

Proof of Proposition 2.6.1) The first identity is classical. Let $\widetilde{L}$ be the infinitesimal generator of $\widetilde{X}$ and $L$

$$
L(f)=\frac{1}{2} \Delta f+u \cdot \nabla f .
$$

$L$ and $\widetilde{L}$ are symmetric with respect to the probability measure $\lambda$ (see Section 3 ), therefore $\lambda$ is the invariant probability measure of $\widetilde{X}$.
2) We analyze (2.32). Let $g:[0, t] \times \bar{\Omega} \longrightarrow \mathbb{R}$ be of class $C^{2}$. We apply the Itô formula and we take the expectation

$$
\begin{aligned}
& E_{\lambda}\left[g\left(s, \widetilde{X}_{s}^{t}\right)\right] \\
& =\int_{\Omega} g(0, x) d \lambda(x)-E_{\lambda}\left[\int_{0}^{s} \frac{\partial g}{\partial n}\left(r, \widetilde{X}_{r}^{t}\right) d \widetilde{A}_{r}^{t}\right] \\
& \quad+E_{\lambda}\left[\int_{0}^{s}\left(\frac{1}{2} \Delta g(r, \cdot)-u(t-r, \cdot) \nabla g(r, \cdot)+\frac{\partial g}{\partial t}(r, \cdot)\right)\left(\widetilde{X}_{r}^{t}\right) d r\right] .
\end{aligned}
$$

We use the former identity (2.31)

$$
\left.E_{\lambda}\left[\int_{0}^{s} \frac{\partial g}{\partial n}\left(r, \widetilde{X}_{r}^{t}\right) d \widetilde{A}_{r}^{t}\right)\right]=A_{1}+A_{2}
$$

where

$$
\begin{gathered}
A_{1}=\int_{\Omega} g(0, x) d \lambda(x)-\int_{\Omega} g(s, x) d \lambda(x)+\int_{0}^{s}\left(\int_{\Omega} \frac{\partial g}{\partial t}(r, x) \lambda(d x)\right) d r \\
A_{2}=\int_{0}^{s}\left(\int_{\Omega}\left(\frac{1}{2} \Delta g(r, x)-u(t-r, x) \nabla g(t, x)\right) \lambda(d x)\right) d r
\end{gathered}
$$

It is obvious that $A_{1}=0$.
We transform $A_{2}$ through Stokes formula, and an integration by parts,

$$
\begin{gathered}
\int_{\Omega} \Delta g(r, x) \lambda(d x)=\int_{\partial \Omega} \frac{\partial g}{\partial n}(r, x) d x, \\
\int_{\Omega} u(t-r, x) \nabla g(r, x) \lambda(d x)=-\int_{\Omega} g(r, x) \operatorname{div} u(t-r, x) \lambda(d x)=0 .
\end{gathered}
$$

(Recall that $\operatorname{div} u=0$ ). Therefore

$$
E_{\lambda}\left[\int_{0}^{s} \frac{\partial g}{\partial n}\left(r, \widetilde{X}_{r}^{t}\right) d \widetilde{A}_{r}^{t}\right]=\frac{1}{2} \int_{0}^{s}\left(\int_{\partial \Omega} \frac{\partial g}{\partial n}(r, x) d x\right) d r .
$$

For any $g$, of class $C^{2}$. (2.32) is a direct consequence of the monotone class theorem.

## 3. Fokker-Planck interpretation of the vorticity equation.

Let $\omega$ be the vorticity associated with $u, u$ being the velocity solving the Navier Stokes system. Recall that $\omega$ solves (1.8).

We know that if $D$ is a diffusion process with drift term $b$, and coefficient of diffusion identically equal to 1 , the density $\varphi$ of $D$ verifies the Fokker-Planck equation

$$
\begin{equation*}
\frac{\partial \varphi}{\partial t}=\frac{1}{2} \Delta \varphi-b \nabla \varphi-(\operatorname{div} b) \varphi, \quad t>0, x \in \Omega \tag{3.1}
\end{equation*}
$$

Since $\operatorname{div} u=0$, if we choose $u=b$, we note that the vorticity equation can be written as a Fokker-Planck equation.

If $\Omega$ is equal to the whole space $\mathbb{R}^{2}, \omega$ is the "density" of $D, D$ starting with initial distribution $\omega_{0}(x) d x$

$$
\begin{equation*}
\int_{\mathbb{R}^{2}} h(x) \omega(t, x) d x=\int_{\mathbb{R}^{2}} \omega_{0}(x) E\left[h\left(D_{t}^{x}\right)\right] d x \tag{3.2}
\end{equation*}
$$

where $D^{x}$ solves,

$$
\begin{equation*}
D_{t}^{x}=x+B_{t}+\int_{0}^{t} u\left(s, D_{s}^{x}\right) d s \tag{3.3}
\end{equation*}
$$

$B$ being a two-dimensional Brownian motion, $B_{0}=0$.
In our context we guess that the underlying process associated with $\omega$ (or $u$ ) has to stay in $\bar{\Omega}$. It is a well-known problem solved by adding a local time process in the right-hand side of (3.3).

More precisely let $\mathcal{X}=\left(\left(X_{s}^{t, x} ; s \geq 0\right), x \in \bar{\Omega}, t \geq 0\right)$ be the family of diffusions, with normal reflection

$$
\left\{\begin{array}{l}
X_{s}^{t, x}=x+B_{s}+\int_{0}^{s} u\left(t+r, X_{r}^{t, x}\right) d r-\int_{0}^{s} n\left(X_{r}^{t, x}\right) d A_{r}^{t, x}  \tag{3.4}\\
A_{s}^{t, x}=\int_{0}^{s} \mathbf{1}_{\left\{X_{r}^{t, x} \in \partial \Omega\right\}} d A_{r}^{t, x},
\end{array}\right.
$$

$n(y)$ is the normalized outer normal vector at $y \in \partial \Omega,\left(A_{s}^{t, x} ; s \geq 0\right)$ is the local time process corresponding to a normal reflection at the boundary. We know that $X_{s}^{t, x}$ belongs to $\bar{\Omega}$ for any $(s, t, x)$ in $\mathbb{R}_{+}^{2} \times \bar{\Omega}$.

The aim of this section is double. We prove that $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are in duality, $\widetilde{\mathcal{X}}=\left(\left(\widetilde{X}_{s}^{t, x} ; 0 \leq s \leq t\right) ;(t, x) \in \mathbb{R}_{+} \times \bar{\Omega}\right)$ being the family of stochastic processes introduced in Section 2, and especially in (2.1). We also come back to the interpretation of $\omega$ as a "density" function.

We keep the same notation we have introduced in (2.30)

$$
\begin{gather*}
E_{h \cdot \lambda}\left[F\left(\widetilde{X}_{s}^{t} ; 0 \leq s \leq t\right)\right] \\
=\int_{\Omega} E\left[F\left(\widetilde{X}_{s}^{t, x} ; 0 \leq s \leq t\right)\right] h(x) \lambda(d x),  \tag{3.5}\\
E_{h \cdot \lambda}\left[F\left(X_{s}^{t} ; s \geq 0\right)\right]=\int_{\Omega} E\left[F\left(X_{s}^{t, x} ; s \geq 0\right)\right] h(x) \lambda(d x), \tag{3.6}
\end{gather*}
$$

where $h: \bar{\Omega} \longrightarrow \mathbb{R}$ and $F: \mathcal{C}\left(\mathbb{R}_{+}\right) \longrightarrow \mathbb{R}$ are Borel functions. $\lambda$ denotes the normalized Lebesgue measure on $\Omega$.

We begin with the duality between $\mathcal{X}$ and $\widetilde{\mathcal{X}}$.
Proposition 3.1. i) Suppose $t>0$ and $F: \mathcal{C}([0, t]) \rightarrow \mathbb{R}$ be a Borel bounded function, then

$$
\begin{equation*}
E_{\lambda}\left[F\left(X_{s}^{0} ; 0 \leq s \leq t\right)\right]=E_{\lambda}\left[F\left(\widetilde{X}_{t-s}^{t} ; 0 \leq s \leq t\right)\right] \tag{3.7}
\end{equation*}
$$

ii) In particular if $0=s_{0}<s_{1}<\cdots<s_{n} \leq t$ and $f_{0}, \ldots, f_{n}: \bar{\Omega} \longrightarrow \mathbb{R}$ are bounded Borel functions we have

$$
\begin{align*}
E_{\lambda}\left[f_{0}\left(X_{s_{0}}^{0}\right) f_{1}\left(X_{s_{1}}^{0}\right)\right. & \left.\cdots f_{n}\left(X_{s_{n}}^{0}\right)\right] \\
& =E_{\lambda}\left[f_{0}\left(\widetilde{X}_{t-s_{0}}^{t}\right) f_{1}\left(\widetilde{X}_{t-s_{1}}^{t}\right) \cdots f_{n}\left(\widetilde{X}_{t-s_{n}}^{t}\right)\right] \tag{3.8}
\end{align*}
$$

Proof. The result is well-known if $X$ or $\widetilde{X}$ are homogeneous Markov processes. Here they are not, therefore we briefly indicate the main steps of the proof. The monotone class theorem implies that it is sufficient to verify (3.8). Using the Markov property and induction we can reduce to $n=1$. We set

$$
\begin{aligned}
& \delta=E_{\lambda}\left[f_{0}\left(X_{s_{0}}^{0}\right) f_{1}\left(X_{s_{1}}^{0}\right)\right], \\
& \widetilde{\delta}=E_{\lambda}\left[f_{0}\left(\widetilde{X}_{t-s_{0}}^{t}\right) f_{1}\left(\widetilde{X}_{t-s_{1}}^{t}\right)\right], \\
& 0=s_{0}<s_{1} \leq t .
\end{aligned}
$$

Let $\left(\wedge_{r, s}\right)_{0 \leq r<s}$ (respectively $\left.\left(\widetilde{\wedge}_{r, s}^{t}\right)_{0<r<s \leq t}\right)$ be the non homogeneous semigroup associated with $X$ (respectively $\widetilde{X}^{t}$ ).

We denote by $L_{r}$ and $\widetilde{L}_{r}^{t}$ the infinitesimal generators of $\wedge$ and $\widetilde{\wedge}$

$$
\begin{gathered}
L_{r} f(x)=\frac{1}{2} \Delta f(x)+u(r, x) \nabla f(x), \\
\widetilde{L}_{r}^{t} f(x)=\frac{1}{2} \Delta f(x)-u(t-r, x) \nabla f(x),
\end{gathered}
$$

for any $f$ of class $C^{2}$ in $\bar{\Omega}$, and verifying

$$
\frac{\partial f}{\partial n}=0, \quad \text { on } \partial \Omega
$$

1) We claim that

$$
\begin{equation*}
\left\langle g, L_{r} f\right\rangle_{\lambda}=\left\langle f, \widetilde{L}_{t-r}^{t} g\right\rangle_{\lambda}, \quad 0<r<t \tag{3.9}
\end{equation*}
$$

where $f$ and $g$ are of class $C^{2}$,

$$
\frac{\partial f}{\partial n}=\frac{\partial g}{\partial n}=0
$$

and

$$
\begin{equation*}
\langle f, g\rangle_{\lambda}=\int_{\Omega} f(x) g(x) \lambda(d x) . \tag{3.10}
\end{equation*}
$$

We integrate by part, making use of Stokes formula

$$
\begin{aligned}
\int_{\Omega} g(x) L_{r} f(x) d x= & \frac{1}{2}\left(\int_{\Omega} f(x) \Delta g(x) d x+\int_{\partial \Omega} g \frac{\partial f}{\partial n}-\int_{\partial \Omega} f \frac{\partial g}{\partial n}\right) \\
& -\int_{\Omega} g(x) f(x) \operatorname{div} u(r, x) d x \\
& -\int_{\Omega} f(x)(u(r, x) \nabla g(x)) d x+\int_{\partial \Omega} u g f .
\end{aligned}
$$

Since $\partial f / \partial n, \partial g / \partial n, u_{\mid \partial \Omega}$ and $\operatorname{div} u$ cancel, we obtain (3.9).
2) In this second step, we prove

$$
\begin{equation*}
\left\langle f, \tilde{\wedge}_{t-s, t-r}^{t} g\right\rangle_{\lambda}=\left\langle\wedge_{r, s} f, g\right\rangle_{\lambda}, \quad 0 \leq r \leq s \leq t \tag{3.11}
\end{equation*}
$$

for any $f$ and $g$ of class $C^{2}, \partial f / \partial n=\partial g / \partial n=0$.
We set

$$
\alpha(h)=\left\langle\wedge_{r, h} f, \widetilde{\wedge}_{t-s, t-h}^{t} g\right\rangle_{\lambda}, \quad h \in[r, s] .
$$

We take the derivative of $\alpha$

$$
\alpha^{\prime}(h)=\left\langle L_{h} \wedge_{r, h} f, \widetilde{\wedge}_{t-s, t-h}^{t} g\right\rangle_{\lambda}-\left\langle\wedge_{r, h} f, \widetilde{L}_{t-h} \widetilde{\wedge}_{t-s, t-h}^{t} g\right\rangle_{\lambda},
$$

because

$$
\frac{\partial}{\partial h} \wedge_{r, h} f=L_{h} \wedge_{r, h} f \quad \text { and } \quad \frac{\partial}{\partial h} \widetilde{\wedge}_{t-s, t-h}^{t}=-\widetilde{L}_{t-h} \widetilde{\wedge}_{t-s, t-h}^{t}
$$

(3.9) implies that $\alpha^{\prime}(h)=0$, for all $h \in[r, s]$. Hence $\alpha$ is constant

$$
\alpha(r)=\left\langle f, \widetilde{\wedge}_{t-s, t-r}^{t} g\right\rangle_{\lambda}=\alpha(s)=\left\langle\wedge_{r, s} f, g\right\rangle_{\lambda} .
$$

3) We come back to $\delta$ and $\widetilde{\delta}$. We have

$$
\begin{gathered}
\delta=\left\langle f_{0}, \wedge_{0, s_{1}} f_{1}\right\rangle_{\lambda}, \\
\widetilde{\delta}=E_{\lambda}\left[f_{1}\left(\widetilde{X}_{t-s_{1}}^{t}\right) \widetilde{\wedge}_{t-s_{1}, t}^{t} f_{0}\left(\widetilde{X}_{t-s_{1}}^{t}\right)\right]=\left\langle 1, \widetilde{\wedge}_{0, t-s_{1}}^{t}\left(f_{1}\left(\widetilde{\wedge}_{t-s_{1}, t}^{t} f_{0}\right)\right)\right\rangle_{\lambda}
\end{gathered}
$$

We apply twice (3.11),

$$
\widetilde{\delta}=\left\langle 1, f_{1} \widetilde{\wedge}_{t-s_{1}, t}^{t} f_{0}\right\rangle_{\lambda}=\left\langle f_{1}, \widetilde{\wedge}_{t-s_{1}, t}^{t} f_{0}\right\rangle_{\lambda}=\left\langle\wedge_{0, s_{1}} f_{1}, f_{0}\right\rangle_{\lambda}=\delta .
$$

Propositions 2.5 and 2.6 admit a dual version.
Proposition 3.2. 1) Let $f: \bar{\Omega} \longrightarrow \mathbb{R}$ and $H:[0, t] \times \partial \Omega \longrightarrow \mathbb{R}$ two Borel bounded functions then,

$$
\begin{gather*}
E_{\lambda}\left[f\left(X_{t}^{0}\right)\right]=\int_{\Omega} f(x) \lambda(d x),  \tag{3.12}\\
E_{\lambda}\left[\int_{0}^{t} H\left(r, X_{r}^{0}\right) d A_{r}^{0}\right]=\frac{1}{2} \int_{0}^{t}\left(\int_{\partial \Omega} H(r, x) d x\right) d r  \tag{3.13}\\
E_{\lambda}\left[\exp \left(\theta A_{t}^{0}\right)\right]<\infty, \quad \text { for any } \theta>0,  \tag{3.14}\\
\lim _{u \rightarrow 0_{+}} \sup _{\substack{x \in \bar{\Omega} \\
0 \leq t \leq k}} E\left[A_{u}^{t, x}\right]=0, \quad(c f .(2.26)) . \tag{3.15}
\end{gather*}
$$

We omit the proof of the Proposition 3.2.
Before stating the analog of Proposition 2.1, let us introduce $\left(\mathcal{H}_{u}^{t}\right)_{0 \leq u \leq t}$ the natural filtration generated by ( $X_{t-u} ; 0 \leq u \leq t$ )

$$
\begin{equation*}
\underline{\mathcal{H}}_{u}^{t}=\sigma\left(X_{v} ; t-u \leq v \leq t\right), \quad \mathcal{H}_{u}^{t}=\bigcap_{v<u} \underline{\mathcal{H}}_{v}^{t} \tag{3.16}
\end{equation*}
$$

Proposition 3.3. Suppose that $t>0$ and $\omega$ is a solution of (1.8).

1) a) For any $h: \bar{\Omega} \longrightarrow \mathbb{R}$ Borel bounded function,

$$
\begin{align*}
& \left(h\left(X_{t}^{0}\right) \omega\left(t-s \wedge \xi, X^{0}(t-s \wedge \xi)\right)\right. \\
& \left.\quad \cdot \exp \left(\int_{t-s \wedge \xi}^{t} \varphi_{0}\left(r, X_{r}^{0}\right) d A_{r}\right) ; 0 \leq s \leq t\right) \tag{3.17}
\end{align*}
$$

is a $\left(P_{\lambda}, \mathcal{H}^{t}\right)$ continuous local martingale, where $\varphi_{c}$ is defined by (2.7) and,
(3.18) $\xi=\inf \left\{s \leq t: \omega\left(t-s, X^{0}(t-s)\right)=0\right.$ and $\left.X^{0}(t-s) \in \partial \Omega\right\} \wedge t$.
b) If $c>C_{t, \Omega}\left(C_{t, \Omega}\right.$ is defined by (2.10)),

$$
\begin{align*}
& \left(h\left(X_{t}\right)\left(\omega\left(t-s, X^{0}(t-s)\right)+c\right)\right. \\
& \left.\quad \cdot \exp \left(\int_{t-s}^{t} \varphi_{c}\left(r, X_{r}^{0}\right) d A_{r}^{0}\right) ; 0 \leq s \leq t\right) \tag{3.19}
\end{align*}
$$

is a $\left(P_{\lambda}, \mathcal{H}^{t}\right)$ square integrable continuous martingale.
2) The tangential component of the velocity of $X^{0}$ vanishes on the boundary.

Let $\rho$ be a velocity test function, then

$$
\begin{equation*}
\frac{1}{h} E_{\lambda}\left[\rho\left(X_{T+h}^{0}\right)-\rho\left(X_{T}^{0}\right) \mid \mathcal{H}_{T}^{t}\right] \xrightarrow[h \rightarrow 0_{+}]{\text {a.s. }} 0, \tag{3.20}
\end{equation*}
$$

where $T$ is the $\mathcal{H}^{t}$-stopping time,

$$
\begin{equation*}
T=\inf \left\{s \leq t: X^{0}(t-s) \in \partial \Omega\right\} \wedge t \tag{3.21}
\end{equation*}
$$

Proof. For simplicity we write $X_{t}$ instead of $X_{t}^{0}$. We set, for any $c \geq 0$

$$
\begin{align*}
Z_{c}(s)= & h\left(X_{t}\right)\left(\omega\left(t-s, X_{t-s}\right)+c\right) \\
& \cdot \exp \left(\int_{t-s}^{t} \varphi_{c}\left(r, X_{r}\right) d A_{r}\right), \quad 0 \leq s \leq t,  \tag{3.22}\\
\xi_{n}= & \inf \left\{s \leq t: \omega(t-s) X(t-s) \leq \frac{1}{n}\right.  \tag{3.23}\\
& \text { and } X(t-s) \in \partial \Omega\} \wedge t, \quad n \in \mathbb{N} .
\end{align*}
$$

$\left\{\xi_{n}\right\}_{n \geq 1}$ is an increasing sequence of $\mathcal{H}^{t}$-stopping times converging, as $n$ goes to infinity, to $\xi$.

Let $0 \leq u_{0}<u_{1} \leq t, 0 \leq s_{0}<s_{1}<\cdots<s_{n} \leq u_{0}, \Gamma_{0}, \Gamma_{1}, \ldots, \Gamma_{n}$ Borel subsets of $\mathbb{R}$ and

$$
A=\left\{X_{t-s_{0}} \in \Gamma_{0}, X_{t-s_{1}} \in \Gamma_{1}, \ldots, X_{t-s_{n}} \in \Gamma_{n}\right\}
$$

Using the duality property (3.7) we obtain

$$
E_{\lambda}\left[Z_{c}\left(u_{1} \wedge \xi_{n}\right) \mathbf{1}_{A}\right]=E_{\lambda}\left[\widetilde{Z}_{c}^{t}\left(u_{1} \wedge \widetilde{\xi}_{n}\right) \mathbf{1}_{\widetilde{A}}\right]
$$

with $\widetilde{A}=\left\{\widetilde{X}_{s_{0}}^{t} \in \Gamma_{0}, \widetilde{X}_{s_{1}}^{t} \in \Gamma_{1}, \ldots, \widetilde{X}_{s_{n}}^{t} \in \Gamma_{n}\right\}$. Recall that $\widetilde{Z}_{c}^{t}$ and $\widetilde{\xi}_{n}$ were introduced in (2.8), respectively (2.14).

We apply 1) a) of Proposition 2.1

$$
E_{\lambda}\left[\widetilde{Z}_{c}^{t}\left(u_{1} \wedge \widetilde{\xi}_{n}\right) \mathbf{1}_{\widetilde{A}}\right]=E_{\lambda}\left[\widetilde{Z}_{c}^{t}\left(u_{0} \wedge \widetilde{\xi}_{n}\right) \mathbf{1}_{\widetilde{A}}\right] .
$$

The duality property implies (3.17).
A similar approach and (2.18) estimate prove (3.19). As for (3.20) we mimic the proof of (2.11).

Recall that $\left\{\xi_{n}\right\}_{n \geq 1}$ is the increasing sequence of stopping time, converging to $\xi$. Using the stopping theorem and (3.12) we obtain:

Corollary 3.4. For any bounded Borel function $h, t>0$ and $n \geq 1$, we have

$$
\begin{align*}
& \langle h, \omega(t, \cdot)\rangle_{\lambda}=\int_{\Omega} h(x) \omega(t, x) d \lambda(x) \\
& =E_{\lambda}\left[h\left(X_{t}^{0}\right) \omega\left(t-\xi_{n}, X^{0}\left(t-\xi_{n}\right)\right)\right.  \tag{3.24}\\
& \left.24) \cdot \exp \left(\int_{t-\xi_{n}}^{t} \varphi_{0}\left(t, X_{r}^{0}\right) d A_{r}^{0}\right)\right] \\
& \langle h,(\omega(t, \cdot)+c)\rangle_{\lambda}= \\
& 25)  \tag{3.25}\\
& \text { 25) } h(x)(\omega(t, x)+c) \lambda(d x) \\
& =
\end{aligned} \begin{aligned}
& E_{\lambda}\left[\omega_{0}\left(X_{0}^{0}\right) h\left(X_{t}^{0}\right) \exp \left(\int_{0}^{t} \varphi_{c}\left(r, X_{r}^{0}\right) d A_{r}^{0}\right)\right] .
\end{align*}
$$

Remark 3.5. 1) A priori we are not allowed to drop $\xi_{n}$ in (3.24) since we do not know if

$$
\begin{aligned}
& \left(Z_{0}(s) ; 0 \leq s \leq t\right) \\
= & \left(h\left(X_{t}^{0}\right) \omega\left(t-s, X^{0}(t-s)\right) \exp \left(\int_{t-s}^{t} \varphi_{0}\left(r, X_{r}^{0}\right) d A_{r}^{0}\right) ; 0 \leq s \leq t\right)
\end{aligned}
$$

is a $\left(P_{\lambda}, \mathcal{H}^{t}\right)$-martingale.
We note that if $\omega_{0}$ is analytic, $\omega$ is also an analytic function defined on $\mathbb{R}_{+} \times \bar{\Omega}([\mathrm{Ko}])$. Therefore $\left\{(t, x) \in \mathbb{R}_{+} \times \partial \Omega: \omega(t, x)=0\right\}$ is a finite union of $C^{\infty}$ curves. $X$ being a nice diffusion process, it does not visit
this set: for any $t>0$, almost surely, $\omega\left(t, X_{r}\right) \neq 0$ for all $r \in[0, t]$. Hence,

$$
\begin{equation*}
\int_{0}^{r}\left|\varphi_{0}\left(r, X_{r}\right)\right| d A_{r}<+\infty, \quad \text { almost sure } \tag{3.26}
\end{equation*}
$$

We conjecture that

$$
\begin{equation*}
\langle h, \omega(t, \cdot)\rangle_{\lambda}=E_{\lambda}\left[\omega_{0}\left(X_{0}^{0}\right) h\left(X_{t}^{0}\right) \exp \left(\int_{0}^{t} \varphi_{0}\left(r, X_{r}^{0}\right) d A_{r}\right)\right] . \tag{3.27}
\end{equation*}
$$

2) Following the convention (3.6) we rather write the former identity

$$
\begin{equation*}
\langle h, \omega(t, \cdot)\rangle_{\lambda}=E_{\omega_{0} \cdot \lambda}\left[h\left(X_{t}^{0}\right) \exp \left(\int_{0}^{t} \varphi_{0}\left(r, X_{r}^{0}\right) d A_{r}\right)\right] . \tag{3.28}
\end{equation*}
$$

Recall (see Lemma 1.1) that $u$ can be expressed through $\omega$ via an integral

$$
u(t, x)=K \omega(t, x)=\int_{\Omega} \nabla_{x}^{\perp} G(x, z) \omega(t, z) d z
$$

$G$ being the Green function of $\Delta$ on $\Omega$. Therefore we have the formal expression of $u$

$$
\begin{equation*}
u(t, x)=E_{\omega_{0} \cdot \lambda}\left[\nabla_{x}^{\perp} G\left(x, X_{t}\right) \exp \left(\int_{0}^{t} \varphi_{0}\left(r, X_{r}\right) d A_{r}\right)\right] \tag{3.29}
\end{equation*}
$$

We point out that the right-hand side of (3.29) is a double integral with respect the probability measure $\lambda \otimes P$. It seems difficult to check that this integral is convergent in some sense.

Let us define the stochastic differential system (S.N.S*.) based on $x$ :
(S.N.S*.1) Suppose $x \in \bar{\Omega}$. Let us consider the following reflected stochastic differential equation in $\bar{\Omega}$,

$$
\begin{align*}
X_{s}^{0, x}=x+B_{s}+\int_{0}^{s} K \omega\left(r, X_{r}^{0, x}\right) d r \\
-\int_{0}^{s} n\left(X_{r}^{0, x}\right) d A_{r}^{0, x}, \quad s \geq 0  \tag{3.30}\\
\int_{0}^{s} \mathbf{1}_{\left\{X_{r}^{0, x} \in \partial \Omega\right\}} d A_{r}^{0, x}=A_{s}^{0, x}, \quad s \geq 0 \tag{3.31}
\end{align*}
$$

$K \omega$ being defined by (1.2).
(S.N.S*.2) For any $h: \bar{\Omega} \longrightarrow \mathbb{R}$ Borel bounded function,
$\left(\left(h\left(X_{t}^{0}\right) \omega\left(t-s \wedge \xi, X^{0}(t-s \wedge \xi)\right) \exp \left(\int_{t-s \wedge \xi}^{t} \varphi_{0}\left(r, X_{r}^{0}\right) d A_{r}^{0}\right) ; 0 \leq s \leq t\right)\right.$
is a $\left(P_{\lambda}, \mathcal{H}^{t}\right)$ continuous local martingale, where $\varphi_{c}$ is defined by (2.7) and $\xi$ by (3.18).
(S.N.S*.3) Let $\rho$ be a velocity test function,

$$
\frac{1}{h} E_{\lambda}\left[\rho\left(X_{T+h}^{0}\right)-\rho\left(X_{T}^{0}\right) \mid \mathcal{H}_{T}^{t}\right] \xrightarrow[h \rightarrow 0_{+}]{\text {a.s. }} 0,
$$

where $T$ is defined by (3.21).
We are able to state a second stochastic system equivalent to the (N.S.) one.

Proposition 3.6. Suppose that (S.N.S*.) (= (S.N.S*.1) + (S.N.S*.2) + (S.N.S*.3)) has a solution ( $\omega,\left\{\left(X_{s}^{t, x} ; 0 \leq s \leq t\right),\left(A_{s}^{t, x} ; 0 \leq s \leq\right.\right.$ $t) ; t \geq 0, x \in \bar{\Omega}\} \omega$ being a smooth function; then $u=K \omega$ is a solution of the (N.S.) equation. Conversly if $u=K \omega$ solves the (N.S.) equation, then the (S.N.S*.) system has a unique solution.

Proof. It is a direct consequence of duality (Proposition 3.1) and Proposition 2.3.

## 4. Branching particle system associated with the NavierStokes equation.

1) Heuristically (see the Remark 3.5) $\omega(t, \cdot)$ can be interpreted as a density function

$$
\langle h, \omega(t, \cdot)\rangle_{\lambda}=E_{\omega_{0} \cdot \lambda}\left[h\left(X_{t}\right) \exp \left(\int_{0}^{t} \varphi_{0}\left(r, X_{r}\right) d A_{r}\right)\right] .
$$

If the sign of $\varphi_{0}$ is constant and negative, $\omega(t, \cdot)$ is truly the density of $X_{t}$, starting with initial "distribution" $\omega_{0} \cdot d \lambda$, and killed with the multiplicative functional

$$
\left(\exp \int_{0}^{t} \varphi_{0}\left(r, X_{r}\right) d A_{r} ; t \geq 0\right)
$$

Here $\varphi_{0}$ is not negative. To take into account the sign of $\varphi_{0}$, a branching particule system $Y$ is very adapted. We keep in mind that $\varphi_{0}<0$ (respectively $\varphi_{0}>0$ ) corresponds to disappearing (respectively creation) of mass.

More precisely we know that $Y$ takes its values in the set of finite linear combinations of Dirac measures

$$
\begin{equation*}
Y_{t}=\sum_{i=1}^{N_{t}} \alpha_{t}^{i} \delta_{Y_{t}^{i}} \tag{4.1}
\end{equation*}
$$

where $\alpha_{t}^{i}$ belongs to $\mathbb{N}, Y_{t}^{i}$ is an element of $\bar{\Omega}$.
Then if $h: \bar{\Omega} \longrightarrow \mathbb{R}$ is a bounded Borel function, we set

$$
\begin{equation*}
\left\langle h, Y_{t}\right\rangle=\int_{\bar{\Omega}} h(x) d Y_{t}(x)=\sum_{i=1}^{N_{t}} \alpha_{t}^{i} h\left(Y_{t}^{i}\right) . \tag{4.2}
\end{equation*}
$$

The aim of this section is the construction of a branching particle system $Y$ such that

$$
\begin{equation*}
\langle h, \omega(t, \cdot)\rangle_{\lambda}=E\left[\left\langle h, Y_{t}\right\rangle\right] . \tag{4.3}
\end{equation*}
$$

$\omega$ appears as the mean value of the density of particles $\left(Y_{t}^{i}\right)$ associated with $Y$.
2) We follow the introduction of branching particle system given by Dynkin [D] and we adapt directly the general definitions to our context. Such a system is based on three ingredients:
a) a Markov process $\left(\left(X_{s}^{x, t} ; s \geq 0\right), x \in \bar{\Omega}, t \geq 0\right)$ coming from (3.4),
b) a positive continuous additive functional $C$ of $X$,
c) an offspring distribution $p=\left(p_{n}(t, x) ; t \geq 0, x \in \bar{\Omega}\right)_{n \geq 1}$ on $\mathbb{N}$, indexed by $\mathbb{R}_{+} \times \Omega$ : for any $n, p_{n}$ is a non negative Borel function and

$$
\begin{equation*}
\sum_{n \geq 0} p_{n}(t, x)=1 \tag{4.4}
\end{equation*}
$$

We denote by $\alpha$ the generating function associated with $\left(p_{n}(t, x) ; t \geq\right.$ $0, x \in \bar{\Omega})$

$$
\begin{equation*}
\alpha(t, x, u)=\sum_{n \geq 0} p_{n}(t, x) u^{n}, \quad u \in[0,1] . \tag{4.5}
\end{equation*}
$$

It is supposed, that

$$
\begin{equation*}
\beta(t, x)=\sum_{n \geq 0} n p_{n}(t, x) \text { is bounded } \tag{4.6}
\end{equation*}
$$

The description of the branching particle system $Y$ with parameters $X$, $C$ and $p$ (we note for simplicity $Y=(X, C, p)$ ) is easy to understand. Suppose that the system starts with on particle located at $x \in \bar{\Omega}$. We choose $\xi_{1}$ an exponential random variable with parameter one, independent of $X$. The dynamic of the initial particle is given by $X$ up to the first branching time $U_{1}=\inf \left\{s \geq 0: C_{s}>\xi_{1}\right\}$. At time $U_{1}$, the particle dies and a random number $N_{U_{1}}$ of new particles spring from the ancestor particle, according to $p$. The conditional distribution of $N_{U_{1}}$ given the past up to time $U_{1}$, is $\left(p_{n}\left(U_{1}, X_{U_{1}}\right) ; n \geq 0\right)$. The $N_{U_{1}}$ particles move independently off each other, as $X$, up to a second branching stopping time. A new branching occurs, and so on.
3) Let $X, C$ and $p$ be the parameters of $Y$.

We denote by $\left(Y_{s}^{t, x} ; s \geq 0\right)$ the branching particle system starting with one particle at $x \in \bar{\Omega}$, with dynamic ( $X_{s}^{t, x} ; s \geq 0$ ) and offspring distribution $\left(p_{n}(t+s, x) ; s \geq 0, x \in \bar{\Omega}\right)_{n \geq 1}$ and

$$
\begin{equation*}
W(t, x ; s)=E\left[\exp \left(-\left\langle h, Y_{s}^{t, x}\right\rangle\right)\right] \tag{4.7}
\end{equation*}
$$

for any $(s, x) \in \mathbb{R}_{+} \times \bar{\Omega}, h: \bar{\Omega} \longrightarrow \mathbb{R}_{+}$Borel positive bounded function.
$W$ solves the basic equation (see [D, (1.5)])

$$
\begin{align*}
W(t, x ; s)= & E\left[\int_{0}^{s} \bar{\alpha}\left(t+r, X_{r}^{t, x}, W\left(u, X_{r}^{t, x} ; s-r\right)\right) d C_{r}\right]  \tag{4.8}\\
& +E\left[\exp -h\left(X_{s}^{t, x}\right)\right], \quad s \geq 0
\end{align*}
$$

where

$$
\begin{equation*}
\bar{\alpha}(t, x ; u)=\alpha(t, x ; u)-u=\sum_{n \geq 0} p_{n}(t, x) u^{n}-u \tag{4.9}
\end{equation*}
$$

where $t \geq 0, x \in \bar{\Omega}, u \in[0,1]$.
4) We are interested by

$$
\begin{equation*}
v(t, x ; s)=E\left[\left\langle h, Y_{s}^{t, x}\right\rangle\right], \quad s \geq 0 \tag{4.10}
\end{equation*}
$$

$h$ being a positive and bounded Borel function.
Obviously $\left\langle h, Y_{s}^{t, x}\right\rangle \geq 0$, however we do not know if this positive random variable has a finite expectation. We are interested by branching processes $Y$ such that,

$$
\begin{equation*}
\sup _{x, 0 \leq s \leq t} E\left[\left\langle h, Y_{s}^{t, x}\right\rangle\right]<+\infty \tag{4.11}
\end{equation*}
$$

for all $h$ Borel positive bounded functions. It is clear that previous assumption is equivalent to

$$
\begin{equation*}
\sup _{x, 0 \leq s \leq t} E\left[\left\langle 1, Y_{s}^{t, x}\right\rangle\right]<+\infty \tag{4.12}
\end{equation*}
$$

We remark that $\left\langle 1, Y_{s}^{t, x}\right\rangle$ is the number of particles still living at time $s$.

Proposition 4.1. Let $Y$ be a branching process with parameters ( $X, C, p$ ). We suppose that (4.11) holds. Then the function $v$ defined by (4.10) solves the "integral" equation

$$
\begin{align*}
v(t, x ; s)= & E\left[\int_{0}^{s}\left(\beta\left(t+r, X_{r}^{t, x}\right)-1\right) v\left(t, X_{r}^{t, x} ; s-r\right) d C_{r}\right]  \tag{4.13}\\
& +E\left[h\left(X_{s}^{t, x}\right)\right]
\end{align*}
$$

where $h$ is a positive and bounded Borel function and $\beta$ defined by

$$
\begin{equation*}
\beta(s, x)=\sum_{k \geq 0} k p_{k}(s, x)<+\infty, \tag{4.14}
\end{equation*}
$$

and verify

$$
\begin{equation*}
\sup _{x, s \leq t} \beta(s, x)<+\infty \tag{4.15}
\end{equation*}
$$

Proof of Proposition 4.1. Since $t>0$ and $x \in \bar{\Omega}$ are supposed to be fixed we write $X$ (respectively $Y, v$ ) instead of $X^{t, x}$ (respectively $\left.Y^{t, x}, v(t, x ; \cdot)\right)$. Let $a$ be a positive number, and $W$ the function defined by (4.7), where $h$ is replaced by $a h$

$$
W(s)=E\left[\exp -a\left\langle h, Y_{s}\right\rangle\right] .
$$

Applying the dominated convergence theorem and (4.11), we have,

$$
\left.\frac{\partial}{\partial a} W(s)\right|_{a=0}=-E\left[\left\langle h, Y_{s}\right\rangle\right] .
$$

On the other hand, (4.15) and (4.11) imply,

$$
\begin{gathered}
\sup _{x, u, r \leq s}\left|\frac{\partial \bar{\alpha}}{\partial u}(t+r, x, u)\right| \leq 1+\sup _{x, r \leq t+s} \beta(r, x)<+\infty \\
\sup _{x, r \leq s}\left|\frac{\partial W}{\partial a}(r)\right| \leq \sup _{x, r \leq s} E\left[\left\langle h, Y_{s}\right\rangle\right]<+\infty
\end{gathered}
$$

Since $W$ solves (4.8), we are allowed to take the partial derivative with respect to $a$, in (4.8). If we choose $a=0$, we obtain immediately (4.13).

Lemma 4.2. Let $(X, C, p)$ be the parameters of a branching process $Y$. We suppose,

$$
\begin{equation*}
E^{x, t}\left[\exp \left(\theta C_{s}\right)\right]<+\infty \tag{4.16}
\end{equation*}
$$

for any $x \in \bar{\Omega}, s, t \geq 0$, and $\theta>0$, and

$$
\begin{equation*}
p_{k}(s, x)=0, \quad \text { if } k \geq 3 \tag{4.17}
\end{equation*}
$$

Then (4.11) holds.
Proof of Lemma 4.2.1) Let $\tau$ be the right inverse of $C: \tau_{t}=\inf \{s>$ $\left.0, C_{s}>t\right\}$. By a changing of time,

$$
\begin{equation*}
\left\langle 1, Y_{\tau_{s}}^{t, x}\right\rangle=\left\langle 1, \bar{Y}_{s}^{t, x}\right\rangle, \quad s \geq 0 \tag{4.18}
\end{equation*}
$$

where $\bar{Y}$ is the branching process associated with ( $X_{\tau}, t, p$ ).
Assumption (4.17) tells us that

$$
\begin{equation*}
\left\langle 1, \bar{Y}_{s}^{t, x}\right\rangle \leq \rho_{s} \tag{4.19}
\end{equation*}
$$

Where $\left(\rho_{s} ; s \geq 0\right)$ is the Yule process (each particle lives an exponential time and then splits into two particles). It is well know ([AN, p. 109]) that $\rho_{s}$ is geometrically distributed with parameter $e^{b s}, b$ being a positive constant

$$
P\left(\rho_{s}=k+1\right)=e^{-b s}\left(1-e^{-b s}\right)^{k}, \quad k \geq 0
$$

Therefore

$$
\begin{equation*}
E\left(\rho_{s}^{2}\right) \leq 2 e^{2 b s} \tag{4.20}
\end{equation*}
$$

2) We set $N_{s}=\left\langle\mathbf{1}, Y_{\tau_{s}}^{t, x}\right\rangle$.

We have,

$$
E\left(N_{s}\right)=\sum_{n \geq 0} E\left[N_{s} 1_{\left\{\tau_{n} \leq s<\tau_{n+1}\right\}}\right] .
$$

Moreover,
$E\left(N_{s} \mathbf{1}_{\left\{\tau_{n} \leq s \leq \tau_{n+1}\right\}}\right] \leq E\left(N_{\tau_{n+1}} \mathbf{1}_{\left\{s \geq \tau_{n}\right\}}\right) \leq\left(E\left[\left(N_{\tau_{n+1}}\right)^{2}\right] P\left(s \geq \tau_{n}\right)\right)^{1 / 2}$.
On the one hand, (4.18), (4.19) and (4.20) imply

$$
E\left[\left(N_{\tau_{n+1}}\right)^{2}\right] \leq E\left[\rho_{n+1}^{2}\right] \leq 2 e^{2 b(n+1)} .
$$

On the other hand,

$$
P\left(s \geq \tau_{n}\right)=P\left(C_{s} \geq n\right)=P\left(e^{\theta C_{s}} \geq e^{\theta n}\right) \leq e^{-\theta n} E\left[e^{\theta C_{S}}\right] \leq e^{-\theta n} E\left(e^{\theta C t}\right)
$$

$\theta$ being a positive number.
Let us take $\theta=3 b$, making use of (4.16), we obtain

$$
P\left(s \geq \tau_{n}\right) \leq C e^{-3 b n} .
$$

As a result

$$
\sup _{x, s \leq t} E\left(N_{s}\right) \leq C^{\prime}\left(\sum_{n \geq 0} e^{-b(n / 2)}\right)<\infty,
$$

(4.12) (or (4.11)) follows immediately.

We now investigate uniqueness in (4.13).
Lemma 4.3. Let $t>0$, and $h$ be a bounded and positive Borel function.
We suppose that (4.15) holds and

$$
\begin{equation*}
\lim _{s \rightarrow 0_{+}} \sup _{x \in \bar{\Omega}} E^{x, t}\left(C_{s}\right)=0 \tag{4.21}
\end{equation*}
$$

Then there exists at most one bounded function $v$ solving (4.13).

Proof of Lemma 4.3. Suppose that $v_{1}$ and $v_{2}$ are two bounded solutions of (4.13). We set $v=v_{1}-v_{2}$. Then

$$
\begin{equation*}
v(t, x ; s)=E\left[\int_{0}^{s}\left(\beta\left(t+r, X_{r}^{t, x}\right)-1\right) v\left(t, X_{r}^{t, r}, s-r\right) d C_{r}\right] \tag{4.22}
\end{equation*}
$$

(4.15) implies that,

$$
\lambda:=1+\sup _{x, s \leq t} \beta(s, x)<+\infty .
$$

Consequently, (4.21) implies that, there exists $0<t_{0} \leq t$ such that

$$
\sup _{x \in \bar{\Omega}} E^{x, t}\left(C_{s}\right) \leq \frac{1}{2 \lambda}, \quad \text { for any } s \leq t_{0}
$$

We come back to (4.22)

$$
\sup _{x, s \leq t_{o}}|v(t, x ; s)| \leq \lambda\left(\sup _{x, s \leq t_{0}}|v(t, x ; s)|\right) \frac{1}{2 \lambda} .
$$

Since $v$ is bounded, the former inequality says that $v(t, x ; \cdot)$ vanishes on $\left[0, t_{0}\right]$.

By the same way, $v(t, x ; \cdot)=0$ on $\left[t_{0}, 2 t_{0}\right] \cap[0, t]$.
This shows by induction that $v(x, t ; \cdot)=0$ on $[0, t]$.
5) Let $\left(\left(X_{s}^{t, x} ; s \geq 0\right), x \in \bar{\Omega}, t \geq 0\right)$ be a diffusion process, taking its values in $\bar{\Omega}$, and $C$ be a continuous, non-decreasing additive functional, vanishing at 0 verifying (4.16) and (4.21).

We introduce a new additive functional based on a Borel function $a: \mathbb{R}_{+} \times \bar{\Omega} \longrightarrow \mathbb{R}$

$$
\begin{equation*}
C^{(a)}(s)=\int_{0}^{s}\left|a\left(t+r, X_{r}^{t, x}\right)\right| d C_{r}, \quad s \geq 0 \tag{4.23}
\end{equation*}
$$

It is supposed

$$
\begin{equation*}
\sup _{x, s \leq t}|a(x, s)|<+\infty \tag{4.24}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
C^{(a)}(s) \leq \lambda_{t} C_{s}, \quad \text { for all } s \leq t \tag{4.25}
\end{equation*}
$$

where $\lambda_{t}$ is a positive constant, independent of $x$. We define the offspring distribution $p^{(a)}$ associated with $a$

$$
\begin{array}{ll}
p_{k}^{(a)}(t, x)=0, & \text { if } k \neq 0 \text { or } k \neq 2, \\
p_{0}^{(a)}(t, x)=1, & \text { if } a(t, x)<0,  \tag{4.26}\\
p_{2}^{(a)}(t, x)=1, & \text { if } a(t, x) \geq 0 .
\end{array}
$$

In other words,

$$
\begin{equation*}
p^{(a)}(t, x)=\mathbf{1}_{\{a(t, x)<0\}} \delta_{0}+\mathbf{1}_{\{a(t, x) \geq 0\}} \delta_{2} . \tag{4.27}
\end{equation*}
$$

Theorem 4.4. Let $Y$ be the branching process, $Y=\left(X, C^{(a)}, p^{(a)}\right)$. We suppose that (4.16) and (4.21) hold. Then for any $t \geq 0, x \in \bar{\Omega}, h$ Borel, positive, bounded function

$$
\begin{equation*}
E\left[h\left(X_{s}^{t, x}\right) \exp \left(\int_{0}^{s} a\left(t+r, X_{r}^{t, x}\right) d C_{r}\right)\right]=E\left[\left\langle h, Y_{s}^{t, x}\right\rangle\right] \tag{4.28}
\end{equation*}
$$

Proof of Theorem 4.4. 1) Let $t \geq 0, x \in \bar{\Omega}$ and $h \geq 0$ be fixed. We set

$$
\begin{equation*}
\widetilde{v}(t, x ; s)=E\left[h\left(X_{s}^{t, x}\right) \exp \left(\int_{0}^{s} a\left(t+r, X_{r}^{t, x}\right) d C_{r}\right)\right] \tag{4.29}
\end{equation*}
$$

and

$$
\begin{equation*}
v(t, x ; s)=E\left[\left\langle h, Y_{s}^{t, x}\right\rangle\right] . \tag{4.30}
\end{equation*}
$$

(4.16) and (4.25) imply that $C^{(a)}$ also verifie (4.16).

Obviously (4.17) is realized, therefore Lemma 4.2 tells us that $Y$ verifies (4.11). Applying Proposition 4.1, $v$ solves

$$
\begin{align*}
& v(t, x ; s)=E[ \int_{0}^{s}\left(\beta^{(a)}\left(t+r, X_{r}^{t, x}\right)-1\right) \\
&\left.\cdot v\left(t, X_{r}^{t, x} ; s-r\right) d C_{r}^{(a)}\right]+E\left[h\left(X_{s}^{t, x}\right)\right]  \tag{4.31}\\
& \beta^{(a)}(s, x)=\sum_{k \geq 0} k p_{k}^{(a)}(s, x)
\end{align*}
$$

but

$$
\begin{gathered}
\beta^{(a)}(s, x)-1=2 \cdot \mathbf{1}_{\{a(s, x) \geq 0\}}-1=\operatorname{sgn}(a(s, x)), \\
d C_{r}^{(a)}=\left|a\left(t+r, X_{r}^{t, x}\right)\right| d C_{r},
\end{gathered}
$$

(4.31) can be reduced as follows,

$$
\begin{align*}
v(t, x ; s)= & E\left[\int_{0}^{s} a\left(t+r, X_{r}^{t, x}\right) v\left(t, X_{r}^{t, x} ; s-r\right) d C_{r}\right]  \tag{4.32}\\
& +E\left[h\left(X_{s}^{t, x}\right)\right]
\end{align*}
$$

2) Suppose that $\widetilde{v}$ solves (4.32).

We have already remarked that $Y$ verifies (4.11), then $\widetilde{v}$ is bounded. $C^{(a)}$ has the property (4.21) (it is an easy consequence of (4.25) and (4.21)).

Applying Lemma 4.3, we can conclude that $v=\widetilde{v}$. This means that (4.28) is verified.
3) We have to prove that $\widetilde{v}$ solves (4.32).

We set

$$
\rho=E\left[\int_{o}^{s} a\left(t+r, X_{r}^{t, x}\right) \widetilde{v}\left(t, X_{r}^{t, x}, s-r\right) d C_{r}\right] .
$$

Using the definition of $\widetilde{v}$, and Markovian notations,

$$
\begin{aligned}
\rho=E\left[\int_{0}^{s}\right. & a\left(t+r, X_{r}^{t, x}\right) \\
& \left.\cdot E^{t, X_{r}^{t, x}}\left[h\left(X_{s-r}\right) \exp \left(\int_{0}^{s-r} a\left(t+u, X_{u}\right) d C_{u}\right)\right] d C_{r}\right] .
\end{aligned}
$$

A straightforward application of the Markov property gives

$$
\begin{aligned}
\rho=E[ & \int_{0}^{s} a\left(t+r, X_{r}^{t, x}\right) h\left(X_{s}^{t, x}\right) \\
& \left.\cdot\left(\exp \left(\int_{0}^{s-r} a\left(t+r+u, X_{u+r}^{t, x}\right) d C_{u+r}\right)\right) d C_{r}\right] .
\end{aligned}
$$

It is convenient to introduce the following multiplicative functional,

$$
M_{r}=\exp \left(\int_{0}^{r} a\left(t+u, X_{u}^{t, x}\right) d C_{u}\right)
$$

We have

$$
\begin{aligned}
M_{s}= & M_{r} \exp \left(\int_{r}^{s} a\left(t+u, X_{u}^{t, x}\right) d C_{u}\right) \\
= & M_{r} \exp \left(\int_{0}^{s-r} a\left(t+r+u, X_{u+r}^{t, x}\right) d C_{u+r}\right), \quad r \leq s, \\
& d M_{r}=a\left(t+r, X_{r}^{t, x}\right) M_{r} d C_{r}
\end{aligned}
$$

Hence

$$
\begin{gathered}
\rho=E\left[M_{s} h\left(X_{s}^{t, x}\right)\left(\int_{0}^{s} \frac{1}{\left(M_{r}\right)^{2}} d M_{r}\right)\right] \\
\rho=E\left[M_{s} h\left(X_{s}^{t, x}\right)\left(1-\frac{1}{M_{s}}\right)\right]=E\left[M_{s} h\left(X_{s}^{t, x}\right)\right]-E\left[h\left(X_{s}^{t, x}\right)\right] .
\end{gathered}
$$

But $\widetilde{v}(t, x ; s)=E\left[M_{s} h\left(X_{s}^{t, x}\right)\right]$, then $\widetilde{v}$ verifies (4.32).
We are now able to prove that $\omega+c$ can be interpreted as the "density" of a branching process $Y_{c}, \omega$ denoting the solution of the (N.S.) equation. We apply the Theorem 4.4. We have to define the underlying process and the functions $a, p$ and $C$.
6) Let $\left(\left(X_{s}^{t, x} ; s \geq 0\right) ; x \in \bar{\Omega}, t \geq 0\right)$ be the family of diffusions defined by (3.4).

- $c$ denotes a constant, $c>C_{t, \Omega}$. (Recall that $C_{t, \Omega}$ is defined by (2.10) and $\omega(s, x)+c>0$ for any $s \in[0, t]$ and $x \in \bar{\Omega})$.
- We define the function $a$ as follows

$$
\begin{equation*}
a(s, x)=\left(\frac{\frac{\partial \omega}{\partial n}}{\omega+c}\right)(s, x) \mathbf{1}_{\{x \in \partial \Omega\}} \tag{4.33}
\end{equation*}
$$

- $p$ is the offspring distribution based on $a(c f .(4.26))$,

$$
\begin{array}{ll}
p_{k}(s, x)=0, & \text { if } k \neq 0 \text { or } k \neq 2 \text { or } x \notin \partial \Omega,  \tag{4.34}\\
p_{0}(t, x)=1, & \text { if } \frac{\partial \omega}{\partial n}(t, x)<0 \text { and } x \in \partial \Omega, \\
p_{2}(t, x)=1, & \text { if } \frac{\partial \omega}{\partial n}(t, x) \geq 0 \text { and } x \in \partial \Omega .
\end{array}
$$

We will say that ( $s, x$ ) is a annihilation (respectively creation) point of the vortex if $x \in \partial \Omega$ and $(\partial \omega / \partial n)(s, x)<0$ (respectively $(\partial \omega / \partial n)$ $(s, x) \geq 0)$.

- $C^{(a)}$ coincides with $A^{(a)}$, where $A$ is the local time process associated with $X$ (see (3.4)), namely

$$
\begin{equation*}
C_{s}^{(a)}=\int_{0}^{s}\left|\frac{\partial \omega}{\partial n}\left(r, X_{r}^{t, x}\right)\right| \frac{1}{\omega\left(r, X_{r}^{t, x}\right)+c} d A_{r}^{t, x} \tag{4.37}
\end{equation*}
$$

Before stating the main result of this paper, we recall a notation (see for instance (2.30))

$$
\begin{equation*}
E_{h \cdot \lambda}\left[\left\langle f, Y_{s}\right\rangle\right]=\int_{\Omega} h(x) E\left[\left\langle f, Y_{s}^{x}\right\rangle\right] \lambda(d x) \tag{4.38}
\end{equation*}
$$

where $h$ and $f$ are two Borel and positive functions, $\lambda$ is the normalized Lebesgue measure on $\Omega,\left(Y_{s}^{x} ; s \geq 0\right)$ is the branching process starting at $\delta_{x}$ associated with $X^{0, x}, C$, and $p$ ).

Theorem 4.5. Let $t>0, c>C_{t, \Omega}, \omega$ be the vorticity solution of the (1.8) system. Then

$$
\begin{equation*}
E_{\left(\omega_{0}+c\right) \cdot \lambda}\left[\left\langle h, Y_{s}\right\rangle\right]=\int_{\Omega} h(x)(\omega(s, x)+c) \lambda(d x), \tag{4.39}
\end{equation*}
$$

for any $s \leq t$, and $h$ Borel and positive function.
Proof of Theorem 4.5. We apply Theorem 4.4

$$
E\left[h\left(X_{s}^{0, x}\right) M_{s}\right]=E\left[\left\langle h, Y_{s}^{0, x}\right\rangle\right], \quad s \leq t
$$

where $h \geq 0$ and

$$
M_{s}=\exp \left(\int_{0}^{s}\left(\frac{1}{\omega+c} \frac{\partial \omega}{\partial n}\right)\left(r, X_{r}^{0, x}\right) d A_{r}^{0, x}\right)
$$

We multiply the former equality by $\left(\omega_{0}(x)+c\right)$, we integrate with respect to $\lambda(d x), \omega_{0}+c$ being non-negative, we have

$$
E_{\left(\omega_{0}+c\right) \cdot \lambda}\left[\left\langle h, Y_{s}^{0}\right\rangle\right]=E_{\left(\omega_{0}+c\right) \cdot \lambda}\left[h\left(X_{s}^{0}\right) M_{s}\right] .
$$

Let $\gamma$ be equal to the right hand-side of the previous identity. Using duality ( $c f$. Proposition 3.1), we have
$\gamma=E_{\lambda}\left[\left(\omega_{0}+c\right)\left(X_{0}^{0}\right) h\left(X_{s}^{0}\right) M_{s}\right]=\int_{\Omega} h(x) E\left[\left(\omega_{0}+c\right)\left(\widetilde{X}_{s}^{0, x}\right) \widetilde{M}_{s}\right] \lambda(d x)$.
Obviously $\widetilde{M}_{s}=M_{s}, \widetilde{M}_{s}\left(\omega_{0}+c\right)\left(X_{s}^{x}\right)=\widetilde{Z}_{c}^{s, x}(s),\left(\widetilde{Z}_{c}^{s, x}(r), 0 \leq r \leq s\right)$ being the process defined by (2.8), then

$$
\gamma=\int_{\Omega} h(x) E\left[\widetilde{Z}_{c}^{s, x}(s)\right] \lambda(d x) .
$$

$\operatorname{But}\left(\widetilde{Z}_{c}^{s, x}(r) ; 0 \leq r \leq s\right)$ is a martingale (cf. 1) b)), Proposition 2.1), therefore,

$$
E\left[\widetilde{Z}_{c}^{s, x}(s)\right]=E\left(\widetilde{Z}_{c}^{s, x}(0)\right)=\omega(s, x) .
$$

This achieves the proof of (4.38).
Remark 4.6. We have proved,

$$
\begin{align*}
& E_{\left(\omega_{0}+c\right) \cdot \lambda}\left[\left\langle h, Y_{s}\right\rangle\right] \\
& \quad=E_{\left(\omega_{0}+c\right) \cdot \lambda}\left[h\left(X_{s}^{0}\right) \exp \left(\int_{0}^{s} \varphi_{c}\left(r, X_{r}^{0}\right) d A_{r}\right)\right], \tag{4.40}
\end{align*}
$$

where

$$
\varphi_{c}=\frac{1}{\omega+c} \frac{\partial \omega}{\partial n} .
$$

## 5. The particle algorithm associated with the branching process.

In sections 3 and 4, we suppose that the solution $u$ (or $\omega$ ) of the (N.S.) system is given, and then we defined two nonlinear stochastic processes $X$ and $\widetilde{X}$, and a branching process $Y$. We proved that $u$ and $\omega$ can be expressed through $X, \widetilde{X}$ and $Y$. In this nonlinear context, it is classical $[\mathrm{McK}]$ to introduce a particle algorithm having the propagation of chaos property. Our closed formulas allow us to guess the dynamic of the particle system associated with the (N.S.) equations. Unfortunately we are not able to check the convergence. We are convince that is it interesting to write it out, it will appear as a program.

1) Let $N \geq 1$ be a fixed integer ( $N$ will go to infinity later). Recall that $u_{0}$ is the initial data in (N.S.), and $\omega_{0}=\operatorname{rot} u_{0}$.
a) $X_{0}^{1}, \ldots, X_{0}^{N}$ denote $N$ independent and equidistributed random variables taking its values in $\bar{\Omega}$ with common density

$$
\frac{1}{\int_{\Omega}\left(\omega_{0}(x)+\gamma\right) d x}\left(\omega_{0}+\gamma\right),
$$

where $\gamma$ is a constant, supposed to be large enough. In particular $\omega_{0}(x)+\gamma>0$, for all $x \in \bar{\Omega}$.
b) We define the underlying system of particles, up to the first branching time in the McKean's sens.
$X=\left(X_{t}^{i, N} ; t \geq 0 ; 1 \leq i \leq N\right)$ is the $\bar{\Omega}^{N}$-values diffusion solving the (linear) reflected stochastic differential equation

$$
\begin{align*}
X_{t}^{i, N}= & X_{0}^{i}+B_{t}^{i}+\int_{0}^{t} u_{N}\left(s, X_{s}^{i, N}\right) d s \\
& -\int_{0}^{t} n\left(X_{s}^{N}\right) d A_{s}^{N}, \quad 0 \leq i \leq N, t \geq 0,  \tag{5.1}\\
A_{t}^{N}= & \int_{0}^{t} \mathbf{1}_{\left\{X_{s}^{N} \in \partial \Omega^{N}\right\}} d A_{s}^{N}, \quad t \geq 0 .
\end{align*}
$$

$n$ denotes the outer normal of $\partial \Omega^{N},\left(B^{i} ; 1 \leq i \leq N\right)$ are $N$ independent two dimensional Brownian motions and $B_{0}^{i}=0$, independant of $\left(X_{0}^{i} ; 1 \leq i \leq N\right)$.

The function $u_{N}$ will be defined in c). It corresponds to some approximation of $u$.
c) Let $\mu_{N}$ be the empirical measure,

$$
\begin{equation*}
\mu_{N}(t)=\frac{1}{N} \sum_{i=1}^{N} \delta_{X_{t}, N} \tag{5.2}
\end{equation*}
$$

Recall (Lemma 1.1) that

$$
\begin{equation*}
u(t, z)=\int_{\Omega} \nabla_{z}^{\perp} G\left(z, z^{\prime}\right) \omega\left(t, z^{\prime}\right) d z^{\prime} \tag{5.3}
\end{equation*}
$$

$G$ being the Green function of $\Omega$ (see (1.3)).
If $t$ is small (lower than the first branching time) the branching process $Y_{t}$ reduces to $\delta_{X t}$, therefore $\mu_{N}(t)$ is a good candidate to approximate $\omega(t, x) d x$.

We set

$$
\begin{align*}
\widetilde{u}_{N}(t, z) & =E\left[\int_{\Omega} \nabla_{z}^{\perp} G\left(z, z^{\prime}\right) \mu_{N}\left(t, d z^{\prime}\right)\right] \\
& =\frac{1}{N} \sum_{i=1}^{N} E\left[\nabla_{z}^{\perp} G\left(z, X_{t}^{i, N}\right)\right] \tag{5.4}
\end{align*}
$$

Unfortunately $z \longrightarrow \nabla_{z}^{\perp} G\left(z, z^{\prime}\right)$ has a singularity at $z=z^{\prime}$, therefore we regularize $\widetilde{u}_{N}$, by replacing $\widetilde{u}_{N}$ by $\widetilde{u}_{N} * V_{N}$, where $V_{N}(z) d z$ converges (in $\mathbb{R}^{2}$ ) to $\delta_{0}$ (choose for instance, $V_{N}(z)=N^{2} V(N z), V \geq 0$, $\int_{\mathbb{R}^{2}} V(x) d x=1, V$ of class $C^{\infty}$, with compact support)

$$
\begin{equation*}
u_{N}(t, z)=V_{N} * \widetilde{u}_{N}(t, \cdot)(z)=\frac{1}{N} \sum_{i=1}^{N} E\left[V_{N} * \nabla^{\perp} G\left(\cdot, X_{t}^{i, N}\right)(z)\right] \tag{5.5}
\end{equation*}
$$

Hence $u_{N}$ is $C^{\infty}$, and $x \rightarrow u_{N}(t, x)$ is of class $C^{1}$, therefore the $2 N$ dimensional stochastic differential equation (5.1) has a unique and strong solution. It is meaningful to set

$$
\begin{equation*}
\omega_{N}=\operatorname{rot} u_{N} . \tag{5.6}
\end{equation*}
$$

2) The first branching time.
a) Let $\xi_{1}, \xi_{2}, \ldots, \xi_{N}$ be $N$ independent, and exponential random variable (with unit parameter), independent of the previous system of particles. The first branching time $T$ is defined as follows

$$
\begin{gather*}
T=\inf \left\{T_{i}: 1 \leq i \leq N\right\}  \tag{5.7}\\
T_{i}=\inf \left\{t \geq 0: \int_{0}^{t}\left(\frac{1}{\omega_{N}+\gamma}\left|\frac{\partial \omega_{N}}{\partial n}\right|\right)\left(s, X_{s}^{i}\right) d A_{s}^{N} \geq \xi_{i}\right\} . \tag{5.8}
\end{gather*}
$$

b) Suppose that $T=T_{i_{0}}$. Then $X_{T}^{i_{0}, N} \in \partial \Omega$. We have the alternative

$$
(5.9)+
$$

$$
\begin{align*}
& a_{N}\left(T, X_{T}^{i_{0}, N}\right)<0,  \tag{5.9}\\
& a_{N}\left(T, X_{T}^{i_{0}, N}\right)>0,
\end{align*}
$$

where

$$
\begin{equation*}
a_{N}(s, x)=\left(\frac{1}{\omega_{N}+\gamma} \frac{\partial \omega_{N}}{\partial n}\right)(s, x) \tag{5.10}
\end{equation*}
$$

$a_{N}$ is some approximation of $a, a$ being defined by (4.33).
i) In the negative case (5.9) - according to (4.34)-(4.35), the particle $i_{0}$ is killed at time $T=T_{i_{0}}$. The $N-1$ remaining particles start afresh

$$
\left(X_{T}^{1, N}, \ldots, X_{T}^{i_{0}-1, N}, X_{T}^{i_{0}+1, N}, \ldots, X_{T}^{N, N}\right)
$$

and move as (5.1) with drift coefficient $u_{N}^{(1)}$,

$$
\begin{equation*}
u_{N}^{(1)}(t, z)=\frac{1}{N} \sum_{\substack{i=1 \\ i \neq i_{0}}}^{N} E\left[V_{N} * \nabla^{\perp} G\left(\cdot, X_{t}^{1, i, N}\right)(z)\right] . \tag{5.11}
\end{equation*}
$$

$u_{N}^{(1)}$ is associated with the empirical measure

$$
\mu_{N}^{(1)}(t, \cdot)=\frac{1}{N} \sum_{\substack{i=1 \\ i \neq i_{0}}}^{N} \delta_{X_{t}^{1, i, N}} .
$$

Note that the factor of normalization is $1 / N$ and not $1 /(N-1)$.
We define as in (5.2) the second branching time and the branching dynamic.
ii) If (5.9)+ holds, the particle $i_{0}$ dies and has two descendants. Then the $N+1$ processes move after $T$, as previously.

After having generated $N-1$ or $N+1$ particles, a second branching time is defined by the same way.
3) Conjectures. We claim that the offsprings of one particle (for example, the first one), $Y_{N}$ coming from the former procedure converges in law, as $N$ goes to infinity, to the branching process $Y$.

Another open question is : can we take $\gamma=0$ ? In this situation we introduce in the algorithm signed particles. Particle $i$ is said positive (respectively negative) if $\omega_{0}\left(X_{0}^{i, N}\right)>0$ (respectively $\omega_{0}\left(X_{0}^{i, N}>0\right)$, the density of $X_{0}^{i, N}$ being equal to

$$
\frac{\left|\omega_{0}(x)\right|}{\int_{\Omega}\left|\omega_{0}(x)\right| d x} .
$$

The sign $\xi_{0}^{i, N}$ of $X_{t}^{i, N}$ remains constant, for any time $t . \mu_{N}$ is replaced by the signed measure,

$$
\mu_{N}(t, \cdot)=\frac{1}{N} \sum_{i=1}^{N} \xi_{0}^{i, N} \delta_{X_{t}^{i, N}} .
$$

When $\Omega=\mathbb{R}^{2}$, Marchioro and Pulverenti [MP] has introduced signed particles in order to take into account the non-positivity of $\omega$.

If $\gamma=0$, is this algorithm converging?

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# An X-ray transform estimate in $\mathbb{R}^{n}$ 

## Izabella Łaba and Terence Tao

Abstract. We prove an x-ray estimate in general dimension which is a stronger version of Wolff's Kakeya estimate [12]. This generalizes the estimate in [13], which dealt with the $n=3$ case.

## 1. Introduction.

Let $n \geq 3$ be an integer. Let $B^{n-1}(0,1)$ be the unit ball in $\mathbb{R}^{n}$, and for all $x, v \in B^{n-1}(0,1)$ define the line segment $l(x, v) \in \mathbb{R}^{n}$ by

$$
l(x, v)=\{(x+v t, t): t \in[0,1]\},
$$

where we have parameterized $\mathbb{R}^{n}$ as $\mathbb{R}^{n-1} \times \mathbb{R}$ in the usual manner. Let $\mathcal{G}$ be the set of all such line segments; this space is thus identified with $B^{n-1}(0,1) \times B^{n-1}(0,1)$. If $l \in \mathcal{G}$, we write $x(l)$ and $v(l)$ for the values of $x$ and $v$ respectively such that $l=l(x, v)$.

For any function $f$ on $\mathbb{R}^{n}$, define the x-ray transform $X f$ on $\mathcal{G}$ by

$$
X f(l)=\int_{l} f .
$$

We consider the question of determining the exponents $1 \leq p, q, r \leq \infty$ and $\alpha \geq 0$ such that we have the bound

$$
\begin{equation*}
\|X f\|_{L_{v}^{q} L_{x}^{r}} \lesssim\|f\|_{L_{\alpha}^{p}}, \tag{1}
\end{equation*}
$$

where $L_{\alpha}^{p}$ is the Sobolev space $(1+\sqrt{-\Delta})^{-\alpha} L^{p}$.

From scaling considerations (or by letting $f$ be a bump function adapted to a small ball) we have the necessary condition

$$
\begin{equation*}
1+\frac{n-1}{r} \geq \frac{n}{p}-\alpha \tag{2}
\end{equation*}
$$

while if one lets $f$ be adapted to a tubular neighbourhood of a line segment $l \in \mathcal{G}$, we obtain the condition

$$
\begin{equation*}
\frac{n-1}{q}+\frac{n-1}{r} \geq \frac{n-1}{p}-\alpha . \tag{3}
\end{equation*}
$$

From the Besicovitch set construction we have

$$
\begin{equation*}
(r, \alpha) \neq(\infty, 0) . \tag{4}
\end{equation*}
$$

It was conjectured by Drury [7] and Christ [5] that these three necessary conditions are in fact sufficient. In [5] this conjecture was shown to be true when $p \leq(n+1) / 2$.

By Hölder, Sobolev, and interpolation with trivial estimates, the full conjecture is equivalent (modulo endpoints) to the Kakeya conjecture, which asserts that (1) holds for $q=n, r=\infty, p=n$, and $\alpha=\varepsilon$ for arbitrarily s mall $\varepsilon$.

Wolff [12] showed (1) was true when

$$
q=\frac{(n-1)(n+2)}{n}, \quad r=\infty, \quad p=\frac{n+2}{2}, \quad \alpha=\frac{n-2}{n+2}+\varepsilon,
$$

this can of course be interpolated with the results in [5] to yield further estimates. However, this is not the best one can do in the $p=(n+2) / 2$ case. From (2) and (3) one expects to have (1) for

$$
\begin{align*}
& q=\frac{(n-1)(n+2)}{n}, \quad r=\frac{(n-1)(n+2)}{n-2}, \\
& p=\frac{n+2}{2}, \quad \alpha=0, \tag{5}
\end{align*}
$$

this would imply the results of [12] by Sobolev embedding in the $v$ variable. Although we are not able to get that sharp result, we are able to obtain the following interpolant, which is our main result.

Theorem 1.1. For any $\varepsilon>0$, we have (1) for

$$
\begin{align*}
& q=\frac{(n-1)(n+2)}{n}, \quad r=2(n+2), \\
& p=\frac{n+2}{2}, \quad \alpha=\frac{n-3}{2(n+2)}+\varepsilon . \tag{6}
\end{align*}
$$

This result was obtained in the three dimensional case $n=3$ by Wolff [13], and the result is sharp up to endpoints for that value of $n$ and $p$. Our arguments shall be based on those in [13], with some mild simplifications based on the bilinear approach in [10].

Theorem 1.1 can be stated in a discretized adjoint form, which is more convenient for applications. Namely ${ }^{1}$ :

Theorem 1.2. Let $\varepsilon>0,0<\delta \ll 1$, and $1 \leq m \lesssim \delta^{1-n}$. Let $\mathcal{E}$, $\mathcal{E}^{\prime}$ be $\delta$-separated subsets of $B^{n-1}(0,1)$, and let $\mathcal{A} \subset \widetilde{\mathcal{E}} \times \mathcal{E}^{\prime} \subset \mathcal{G}$ be a collection of line segments such that

$$
\begin{equation*}
|\{l \in \mathcal{A}: v(l)=v\}| \leq m, \tag{7}
\end{equation*}
$$

for all $v \in \mathcal{E}$. Then we have

$$
\begin{equation*}
\left\|\sum_{l \in \mathcal{A}} \chi_{T_{l}}\right\|_{p^{\prime}} \lesssim \delta^{-n / p+1-\varepsilon} m^{1 / q-1 / r}\left(\delta^{n-1}|\mathcal{A}|\right)^{1 / q^{\prime}} \tag{8}
\end{equation*}
$$

where $p, q, r$ are as in (6).
As observed in [12], an x-ray estimate of this form reveals some information on Besicovitch sets in $\mathbb{R}^{n}$. Namely, such sets have Minkowski dimension at least $(n+2) / 2$, and if the dimension is exactly $(n+2) / 2$ then the line segments w hich comprise the set must be "sticky" in a certain sense. This observation was made rigorous in [8], where the results of [13] were applied (together with those of [3] and some additional arguments) in the three-dimensional case to improve slightly upon the Minkowski bound just stated. We will use Theorem 1.1 to achieve a similar result in higher dimensions [9]. Fortunately, one does not need a sharp value of $r$ in (6) to obtain this type of observation, as long as $r$ is finite of course.

[^4]To illustrate the connection between x-ray estimates and Besicovitch sets, we note the following simple application of Theorem 1.2:

Corollary 1.3. Let $0 \leq \alpha \leq n-1$, and $E$ be a bounded subset of $\mathbb{R}^{n}$ such that, for each direction $\omega \in S^{n-1}$, $E$ contains a family of unit line segments parallel to $\omega$, whose union has Minkowski dimension $\alpha+1$. Then the Minkowski dimension of $E$ is at least $(n+2) / 2+\alpha / 4$.

The proof follows standard discretization arguments (see e.g. [1], [2]) and will be omitted. A similar result holds when Minkowski dimension is replaced by Hausdorff. This corollary is stronger than the corresponding corollary of the Kakeya estimate in [12], which covers the $\alpha=0$ case. If one had an x-ray estimate for (5) then one would be able to improve the $\alpha / 4$ term to the optimal $\alpha(n-2) /(2 n-2)$.

## 2. Notation.

We use $0<\delta \ll 1$ and $0<\varepsilon \ll 1$ to denote certain small numbers, and $N \gg 1$ denotes a certain large integer. If $l$ is a line segment in $\mathcal{G}$, we use $T_{l}$ to denote the $\delta$-neighbourhood of $l$, which is thus a $\delta \times 1$ tube.

We write $A \lesssim B$ for $A \leq C B, A \ll B$ for $A \leq C^{-1} B$, and $A \lesssim B$ for $A \leq C(\log (1 / \delta))^{\nu} B$, and $C, \nu$ are quantities which vary from line to line and are allowed to depend on $\varepsilon$ and $N$ but not on $\delta$. We write $A \sim B$ for $A \lesssim B \lesssim B$ and $A \approx B$ for $A \lesssim B \lesssim B$.

Our argument will require the introduction of many quantities, which measure various angles or cardinalities in a collection of tubes. For purposes of visualizing the argument we recommend that one sets the values of these quantities as follows

$$
\begin{aligned}
|\mathcal{E}| \sim\left|\mathcal{E}_{i}\right| \sim \delta^{1-n}, & |\mathcal{A}| \sim \delta^{1-n} m, \\
\lambda & \sim \theta \sim \sigma \sim 1,
\end{aligned} \quad \rho \sim \mathbf{p}_{i} \sim w, ~ l
$$

for $i=1,2$. The treatment of this case can be done while avoiding the more technical tools in the argument such as the two-ends and bilinear reductions, and most of the uniformization theory, while still capturing the core ideas of the argument. To improve the value of $r$ in (6) one would probably start by considering this case.

## 3. Derivation of Theorem 1.1 from Theorem 1.2.

Assume that Theorem 1.2 holds. In this section we shall see how Theorem 1.1 follows. The argument is standard (cf. [1], [2], [12], [13], [10]).

By a Littlewood-Paley decomposition, and giving up an epsilon in the $\alpha$ index, one may assume that $f$ has Fourier support in an annulus $\left\{\xi:|\xi| \sim \delta^{-1}\right\}$. The case $\delta \gtrsim 1$ is easy to handle, so we assume henceforth that $0<\delta \ll 1$.

Fix $\delta$. It is then well known that (1) follows from the variant

$$
\left\|X_{\delta} f\right\|_{L_{v}^{q} L_{x}^{r}} \lesssim \delta^{-\alpha}\|f\|_{p}
$$

where

$$
X_{\delta} f(l)=\delta^{1-n} \int_{T_{l}} f
$$

By duality this is equivalent to

$$
\left\|X_{\delta}^{*} F\right\|_{p^{\prime}} \lesssim \delta^{-\alpha}\|F\|_{L_{v}^{q^{\prime}} L_{x}^{r^{\prime}}},
$$

for all $F$ on $\mathcal{G}$, where $X_{\delta}^{*}$ is the adjoint x-ray transform

$$
X_{\delta}^{*} F=\delta^{1-n} \int_{\mathcal{G}} F(l) \chi_{T_{l}} d x d v
$$

Let $\mathcal{E}, \mathcal{E}^{\prime}$ by any $\delta$-separated subsets of $B^{n-1}(0,1)$. By discretization it suffices to show that

$$
\begin{aligned}
\| \delta^{n-1} \sum_{v \in \mathcal{E}} \sum_{x \in \mathcal{E}^{\prime}} & F(l(x, v)) \chi_{T_{l(x, v)}} \|_{p^{\prime}} \\
& \lesssim \delta^{-\alpha}\left(\delta^{n-1} \sum_{v \in \mathcal{E}}\left(\delta^{n-1} \sum_{x \in \mathcal{E}^{\prime}}|F(l(x, v))|^{r^{\prime}}\right)^{q^{\prime} / r^{\prime}}\right)^{1 / q^{\prime}}
\end{aligned}
$$

uniformly in $\mathcal{E}, \mathcal{E}^{\prime}$.
Fix $\mathcal{E}, \mathcal{E}^{\prime}$. By pigeonholing and positivity it suffices to verify this when $F$ is a characteristic function $F=\chi_{\mathcal{A}}$ for some $A \subseteq \mathcal{E} \times \mathcal{E}^{\prime}$, so that we reduce to

$$
\begin{aligned}
& \left\|\sum_{l \in \mathcal{A}: v(l) \in \mathcal{E}} \chi_{T_{l}}\right\|_{p^{\prime}} \\
& \quad \lesssim \delta^{(n-1)(1-1 / r-1 / q)-\alpha}\left(\sum_{v \in \mathcal{E}}|\{l \in \mathcal{A}: v(l)=v\}|^{q^{\prime} / r^{\prime}}\right)^{1 / q^{\prime}} .
\end{aligned}
$$

By a further pigeonholing and refining of $\mathcal{E}$, we may assume that there exists $1 \leq m \lesssim \delta^{1-n}$ such that

$$
\begin{equation*}
\frac{m}{2} \leq|\{l \in \mathcal{A}: v(l)=v\}| \leq m \tag{9}
\end{equation*}
$$

for all $v \in \mathcal{E}$. Our task is then to show that

$$
\left\|\sum_{l \in \mathcal{A}: v(l) \in \mathcal{E}} \chi_{T_{l}}\right\|_{p^{\prime}} \lesssim \delta^{(n-1)(1-1 / r-1 / q)-\alpha} m^{1 / r^{\prime}}|\mathcal{E}|^{1 / q^{\prime}} .
$$

From (9) we then have $|\mathcal{A}| \sim m|\mathcal{E}|$. The claim then follows from Theorem 1.2 and the fact that (2) is almost satisfied with equality.

It thus remains to prove Theorem 1.2.

## 4. A three-dimensional estimate.

For any collection $\mathcal{A}$ of line segments, we follow Wolff [13] (see also [14]) and define the plate number $\mathbf{p}(\mathcal{A})$ by

$$
\begin{equation*}
\mathbf{p}(\mathcal{A})=\sup _{R} \frac{\left|\left\{l \in \mathcal{A}: T_{l} \subset R\right\}\right|}{\frac{w}{\delta}}, \tag{10}
\end{equation*}
$$

where $R$ ranges over all rectangles of dimension $C \times C w \times C \delta \times \cdots \times C \delta$. By considering the $w \sim \delta$ case we see that $\mathbf{p}(\mathcal{A}) \gtrsim 1$ for any non-empty $\mathcal{A}$.

The purpose of this section is to prove the following distributional estimate on a set $E$ assuming that the directions of $\mathcal{A}$ are effectively constrained to a two-dimensional slab, and the intersection of the tubes $T_{l}$ with $E$ satisfy a certain "two-ends" condition of the type used in [11], [12]. This lemma will be key in the main argument, and also employs several techniques, notably a hairbrush argument and a uniformization argument (both due to Wolff), which will re-appear in slightly different form in the sequel.

Lemma 4.1 ([13]). Let $N \gg 1$ be an integer, $\delta^{C} \lesssim \lambda \leq 1$, E be a subset of $\mathbb{R}^{n}$, and let $\mathcal{A} \subset \mathcal{E} \times \mathcal{E}^{\prime}$ be a collection of lines satisfying (7) which satisfy the uniform density estimate

$$
\begin{equation*}
\left|T_{l} \cap E\right| \approx \lambda \delta^{n-1} \tag{11}
\end{equation*}
$$

and the two-ends condition

$$
\begin{equation*}
\left|T_{l} \cap E \cap B\left(x, \delta^{1 / N}\right)\right| \lesssim \delta^{\varepsilon / 2 N} \lambda \delta^{n-1}, \tag{12}
\end{equation*}
$$

for all $l \in \mathcal{A}, x \in \mathbb{R}^{n}$. Suppose also that the set of directions $\{v(l): l \in$ $\mathcal{A}\}$ is contained in a $C \times C \rho \times C \delta \times \cdots \times C \delta$ box in $B^{n-1}(0,1)$ for some $\delta \lesssim \rho \lesssim 1$. Then, if $\delta$ is sufficiently small depending on $\varepsilon$ and $N$, we have

$$
\begin{equation*}
|E| \gtrsim \delta^{C / N} \lambda^{2}|\mathcal{A}| m^{-1 / 2} \rho^{-1 / 2} \mathbf{p}(\mathcal{A})^{-1 / 2} \delta^{n-1 / 2} . \tag{13}
\end{equation*}
$$

Proof. We repeat the argument in [13]. We may assume that $\mathcal{A}$ is non-empty, and that $E$ is contained in $\bigcup_{l \in \mathcal{A}} T_{l}$.

For every $l \in \mathcal{A}$ and dyadic $\delta \lesssim \sigma \lesssim 1,1 \leq \mu \lesssim \delta^{-C}$, we let $Y_{l, \mu, \sigma, \mathcal{A}} \subset T_{l} \cap E$ denote the set
$Y_{l, \mu, \sigma, \mathcal{A}}$
(14)

$$
=\left\{x \in T_{l} \cap E: \sum_{l^{\prime} \in \mathcal{A}: \delta+\left|v(l)-v\left(l^{\prime}\right)\right| \sim \sigma} \chi_{T_{l^{\prime}}}(x) \approx \sum_{l^{\prime} \in \mathcal{A}} \chi_{T_{l^{\prime}}}(x) \approx \mu\right\} .
$$

In other words, $Y_{l, \mu, \sigma, \mathcal{A}}$ consists of those points $x$ in $T_{l} \cap X$ which lies in about $\mu$ tubes from $\mathcal{A}$, most of which make an angle of about $\sigma$ with $T_{l}$. From the pigeonhole principle we see that

$$
\begin{equation*}
T_{l} \cap E=\bigcup_{\delta \lesssim \sigma \lesssim 1} \bigcup_{1 \leq \mu \lesssim \delta^{-C}} Y_{l, \mu, \sigma, \mathcal{A}} . \tag{15}
\end{equation*}
$$

We now prove a technical lemma which allows us to uniformize $\mu$ and $\sigma$. This type of argument will also be used in the sequel. (For a more general formulation of this type of argument, see [13]). A somewhat similar lemma appears in [4].

Lemma 4.2. Let the notation be as above. Then there exist quantities $\delta \lesssim \sigma \lesssim 1,1 \leq \mu \lesssim \delta^{-C}$ and sets

$$
\mathcal{A}^{(2)} \subseteq \mathcal{A}^{(1)} \subseteq \mathcal{A}^{(0)}=\mathcal{A}
$$

and for each $i=1,2, l \in \mathcal{A}^{(i)}$ there exists a set

$$
Y_{l}^{(i)} \subseteq T_{l} \cap E
$$

such that

$$
\begin{align*}
\left|\mathcal{A}^{(i)}\right| & \approx|\mathcal{A}|  \tag{16}\\
\left|Y_{l}^{(i)}\right| & \approx \lambda \delta^{n-1} \tag{17}
\end{align*}
$$

and

$$
\begin{equation*}
Y_{l}^{(i)} \subseteq Y_{l, \mu^{(i)}, \sigma^{(i)}, \widetilde{\mathcal{A}}^{(i)}}, \tag{18}
\end{equation*}
$$

for some set $\mathcal{A}^{(i)} \subseteq \widetilde{\mathcal{A}}^{(i)} \subseteq \mathcal{A}^{(i-1)}$ and $\mu^{(i)}, \sigma^{(i)}$ satisfying

$$
\begin{align*}
& \delta^{C / N} \mu \lesssim \mu^{(i)} \lesssim \delta^{-C / N} \mu  \tag{19}\\
& \delta^{C / N} \sigma \lesssim \sigma^{(i)} \lesssim \delta^{-C / N} \sigma .
\end{align*}
$$

The implicit constants may depend on $N$.
Proof. The first stage shall be to construct sequences

$$
\begin{gathered}
\mathcal{A}=\mathcal{A}_{0} \supseteq \mathcal{A}_{1} \supseteq \cdots \supseteq \mathcal{A}_{N^{2}}, \\
T_{l} \cap E=Y_{l, 0} \supset Y_{l, 1} \supseteq \cdots \supseteq Y_{l, N^{2}},
\end{gathered}
$$

and quantities $\mu_{k}, \sigma_{k}$ for all $1 \leq k \leq N^{2}$ and $l \in \mathcal{A}_{k}$, such that

$$
\begin{gather*}
\left|\mathcal{A}_{k}\right| \approx|\mathcal{A}|  \tag{21}\\
\left|Y_{l, k}\right| \approx \lambda \delta^{n-1} \tag{22}
\end{gather*}
$$

and

$$
\begin{equation*}
Y_{l, k} \subseteq Y_{l, \mu_{k}, \sigma_{k}, \mathcal{A}_{k-1}} \tag{23}
\end{equation*}
$$

for all $1 \leq k \leq N^{2}$.
To do this, suppose inductively that $0 \leq k<N^{2}$ is such that $\mathcal{A}_{k}$ and $Y_{l, k}$ have been constructed for all $l \in \mathcal{A}_{k}$. From (15) we have

$$
Y_{l, k} \subseteq \bigcup_{\delta \lesssim \sigma \lesssim 1} \bigcup_{1 \leq \mu \delta^{-C}} Y_{l, \mu, \sigma, \mathcal{A}_{k}} .
$$

By the pigeonhole principle, for every $l \in \mathcal{A}_{k}$ one can thus find $\mu_{k+1}(l)$, $\sigma_{k+1}(l)$ such that

$$
\left|Y_{l, k+1}\right| \approx\left|Y_{l, k}\right|,
$$

where

$$
Y_{l, k+1}=Y_{l, k} \cap Y_{l, \mu_{k+1}(l), \sigma_{k+1}(l), \mathcal{A}_{k}}
$$

By the pigeonhole principle again, there exists $\mu_{k+1}, \sigma_{k+1}$ independent of $l$ such that the set

$$
\mathcal{A}_{k+1}=\left\{l \in \mathcal{A}_{k}: \quad \mu_{k+1}(l)=\mu_{k+1}, \sigma_{k+1}(l)=\sigma_{k+1}\right\}
$$

satisfies (21). It is clear that this construction gives the desired properties.

By the pigeonhole principle, there must exist $1 \leq k_{1}<k_{2} \leq N^{2}$ and $\sigma, \mu$ such that

$$
\delta^{C / N} \mu \lesssim \mu_{k_{i}} \lesssim \delta^{-C / N} \mu
$$

and

$$
\delta^{C / N} \sigma \lesssim \sigma_{k_{i}} \lesssim \delta^{-C / N} \sigma
$$

for $i=1,2$. The claim then follows by setting $\mathcal{A}^{(i)}=\mathcal{A}_{k_{i}}$ and $Y_{l}^{(i)}=$ $Y_{l, k_{i}}$.

Let the notation be as in the above lemma. From (17) and (16) we have

$$
\sum_{l \in \mathcal{A}^{(2)}}\left|Y_{l}^{(2)}\right| \approx \lambda \delta^{n-1}|\mathcal{A}|
$$

which we rewrite as

$$
\int_{E} \sum_{l \in \mathcal{A}^{(2)}} \chi_{Y_{l}^{(2)}} \approx \lambda \delta^{n-1}|\mathcal{A}|
$$

From (18), the nesting $\mathcal{A}^{(2)} \subseteq \widetilde{\mathcal{A}}^{(1)}$, and (19), the integrand is bounded by $\delta^{-C / N} \mu$. We thus see that $\lambda$ and $\mu$ are naturally related by the estimate

$$
\begin{equation*}
|E| \mu \gtrsim \delta^{C / N} \lambda \delta^{n-1}|\mathcal{A}| . \tag{24}
\end{equation*}
$$

One can reverse the inequality in (24), but we shall not need to do so here.

From (21), $\mathcal{A}^{(2)}$ is non-empty. Let $l_{0}$ be an arbitrary element of $\mathcal{A}^{(2)}$. Consider the "hairbrush" $\mathcal{A}_{\text {brush }}^{l_{0}}$ defined by
$\mathcal{A}_{\text {brush }}^{l_{0}}=\left\{l \in \mathcal{A}^{(1)}: T_{l_{0}} \cap T_{l} \neq \varnothing, \delta^{C / N} \sigma \lesssim \delta+\left|v\left(l_{0}\right)-v(l)\right| \lesssim \delta^{-C / N} \sigma\right\}$.
From (18), (19), (20) we see that

$$
\sum_{l \in \mathcal{A}_{\mathrm{brush}}^{l_{0}}} \chi_{T_{l}}(x) \gtrsim \delta^{C / N} \mu
$$

for all $x \in Y_{l_{0}}^{(2)}$. Integrating this using (17), we obtain

$$
\sum_{l \in \mathcal{A}_{b r u s h}^{l_{0}}}\left|T_{l} \cap Y_{l_{0}}^{(2)}\right| \gtrsim \delta^{C / N} \mu \lambda \delta^{n-1}
$$

From elementary geometry we see that

$$
\left|T_{l} \cap Y_{l_{0}}^{(2)}\right| \leq\left|T_{l} \cap T_{l_{0}}\right| \lesssim \delta^{-C / N} \delta^{n} \sigma^{-1}
$$

so we conclude that

$$
\begin{equation*}
\left|\mathcal{A}_{\text {brush }}^{l_{0}}\right| \gtrsim \delta^{C / N} \mu \lambda \sigma \delta^{-1} . \tag{25}
\end{equation*}
$$

We will shortly combine (25) with (17) and (12) to prove the estimate

$$
\begin{equation*}
\left|\bigcup_{l \in \mathcal{A}_{\text {brush }}^{I_{0}}} Y_{l}^{(1)}\right| \gtrsim \delta^{C / N} \mu \lambda^{3} \sigma \mathbf{p}(\mathcal{A})^{-1} \delta^{n-2} \tag{26}
\end{equation*}
$$

Assuming this bound for the moment, let us complete the proof of (13). From (18) and (20) we have

$$
\sum_{l^{\prime} \in \mathcal{A}: \delta+\left|v(l)-v\left(l^{\prime}\right)\right| \AA^{-C / N} \sigma} \chi_{T_{l^{\prime}} \cap E}(x) \gtrsim \delta^{C / N} \mu,
$$

for all $l \in \mathcal{A}_{\text {brush }}^{l_{0}}$ and $x \in Y_{l}^{(1)}$. From the definition of $\mathcal{A}_{\text {brush }}^{l_{0}}$ and the triangle inequality we thus see that

$$
\sum_{l^{\prime} \in \mathcal{A}: \delta+\left|v\left(l^{\prime}\right)-v\left(l_{0}\right)\right| \lesssim \delta^{-C / N} \sigma} \chi_{T_{l^{\prime}} \cap E}(x) \gtrsim \delta^{C / N} \mu,
$$

for all $x$ in the set in (26). Integrating this and using (26), we thus obtain

$$
\sum_{l^{\prime} \in \mathcal{A}: \delta+\left|v\left(l^{\prime}\right)-v\left(l_{0}\right)\right| \lesssim \delta^{-C / N} \sigma}\left|T_{l^{\prime}} \cap E\right| \gtrsim \delta^{C / N} \mu^{2} \lambda^{3} \sigma \mathbf{p}(\mathcal{A})^{-1} \delta^{n-2}
$$

From (11) we thus have

$$
\begin{aligned}
&\left|\left\{l^{\prime} \in \mathcal{A}: \delta+\left|v\left(l^{\prime}\right)-v\left(l_{0}\right)\right| \lesssim \delta^{-C / N} \sigma\right\}\right| \lambda \delta^{n-1} \\
& \gtrsim \delta^{C / N} \mu^{2} \lambda^{3} \sigma \mathbf{p}(\mathcal{A})^{-1} \delta^{n-2} .
\end{aligned}
$$

However, from (7) and the fact that $v\left(l^{\prime}\right)$ is constrained to a $C \times C \rho \times$ $C \delta \times \cdots \times C \delta$ box, we see from elementary geometry that

$$
\left|\left\{l^{\prime} \in \mathcal{A}: \delta+\left|v\left(l^{\prime}\right)-v\left(l_{0}\right)\right| \lesssim \delta^{-C / N} \sigma\right\}\right| \lesssim \delta^{-C / N} \sigma \rho \delta^{-2} m .
$$

Combining these two estimates we obtain (after some algebra)

$$
\mu \lesssim \delta^{-C / N} \rho^{1 / 2} \delta^{-1 / 2} \lambda^{-1} \mathbf{p}(\mathcal{A})^{1 / 2} m^{1 / 2},
$$

and the claim (13) follows after some algebra from this and (24).
It remains to prove (26). We first deal with a trivial case when $\sigma \lesssim \delta^{-C / N} \delta$. In this case we simply use the bound

$$
\left|\bigcup_{l \in \mathcal{A}_{\mathrm{brush}}^{l_{0}}} Y_{l}^{(1)}\right| \geq Y_{l}^{(1)} \gtrsim \delta^{C / N} \lambda \delta^{n-1}
$$

from (17) and the fact from (25) that $\mathcal{A}_{\mathrm{brush}}^{l_{0}}$ is non-empty, and (26) follows since $\mathbf{p}(\mathcal{A}), \mu \gtrsim 1$ and $\lambda \lesssim 1$.

Now assume $\sigma \gg \delta^{-C / N} \delta$. To prove (26) we will in fact prove the stronger bound

$$
\begin{equation*}
\left|E^{\prime}\right| \gtrsim \delta^{C / N} \mu \lambda^{3} \sigma \delta^{n-2} \mathbf{p}(\mathcal{A})^{-1} \tag{27}
\end{equation*}
$$

where

$$
E^{\prime}=\bigcup_{l \in \mathcal{A}_{\text {brush }}^{l_{0}}} Y_{l}^{(1)} \cap \Omega
$$

and

$$
\Omega=\left\{x \in \mathbb{R}^{n}: \delta^{C / N} \sigma \lesssim \operatorname{dist}\left(x, l_{0}\right) \lesssim \delta^{-C / N} \sigma\right\}
$$

From (17), (12), and elementary geometry we have

$$
\left|T_{l} \cap E^{\prime}\right| \approx \lambda \delta^{n-1}
$$

for all $l \in \mathcal{A}_{\text {brush }}^{l_{0}}$. Summing this in $l$ we obtain

$$
\sum_{l \in \mathcal{A}_{\text {brush }}^{l_{0}}}\left|T_{l} \cap E^{\prime}\right| \approx\left|\mathcal{A}_{\text {brush }}^{l_{0}}\right| \lambda \delta^{n-1}
$$

which we rewrite as

$$
\int_{E^{\prime}} \sum_{l \in \mathcal{A}_{\mathrm{brush}}^{l_{0}}} \chi_{T_{l} \cap \Omega} \approx\left|\mathcal{A}_{\mathrm{brush}}^{l_{0}}\right| \lambda \delta^{n-1} .
$$

We now use Córdoba's argument (see e.g. [6]). From Cauchy-Schwarz and the above we have

$$
\left|E^{\prime}\right|^{1 / 2}\left\|\sum_{l \in \mathcal{\mathcal { A } _ { \mathrm { brush } } ^ { l _ { 0 } }}} \chi_{T_{l} \cap \Omega}\right\|_{2} \gtrsim\left|\mathcal{A}_{\mathrm{brush}}^{l_{0}}\right| \lambda \delta^{n-1} .
$$

From this and (25), suffices to show that

$$
\begin{equation*}
\left\|\sum_{l \in \mathcal{A}_{\text {brush }}^{l_{0}}} \chi_{T_{l} \cap \Omega}\right\|_{2}^{2} \lesssim \delta^{-C / N}\left|\mathcal{A}_{\mathrm{brush}}^{l_{0}}\right| \delta^{n-1} \mathbf{p}(\mathcal{A}), \tag{28}
\end{equation*}
$$

since (27) then follows from algebra.
To prove (28), we expand the left-hand side as

$$
\sum_{l \in \mathcal{A}_{\text {brush }}^{l_{0}}} \sum_{l^{\prime} \in \mathcal{A}_{\text {brush }}^{l_{0}}}\left|T_{l} \cap T_{l^{\prime}} \cap \Omega\right|,
$$

which we break up further as

$$
\sum_{\delta \lesssim \tau \lesssim 1} \sum_{l \in \mathcal{A}_{\text {brush }}^{l_{0}}} \sum_{l^{\prime} \in \mathcal{A}_{\text {brush }}^{l_{\mathrm{o}}}: T_{l} \cap T_{l^{\prime}}^{\prime} \cap \Omega \neq \varnothing, \delta+\left|v(l)-v\left(l^{\prime}\right)\right| \sim \tau}\left|T_{l} \cap T_{l^{\prime}} \cap \Omega\right| .
$$

From elementary geometry we have

$$
\left|T_{l} \cap T_{l^{\prime}}\right| \lesssim \delta^{n} \tau^{-1} .
$$

It thus suffices to show that
$\left|\left\{l^{\prime} \in \mathcal{A}_{\text {brush }}^{l_{0}}: T_{l} \cap T_{l^{\prime}} \cap \Omega \neq \varnothing, \delta+\left|v(l)-v\left(l^{\prime}\right)\right| \sim \tau\right\}\right| \lesssim \delta^{-C / N} \mathbf{p}(\mathcal{A}) \frac{\tau}{\delta}$,
for each $l, \tau$.
Fix $l, \tau$. The conditions $l^{\prime} \in \mathcal{A}_{\text {brush }}^{l_{0}}$ and $T_{l} \cap T_{l^{\prime}} \cap \Omega \neq \varnothing$ force $l^{\prime}$ to lie in a $\delta^{1-C / N}$-neighbourhood of the 2-plane spanned by $l_{0}$ and (a slight translate of) $l$. Together with the condition $\delta+\left|v(l)-v\left(l^{\prime}\right)\right| \sim \tau$, this constrains $T_{l^{\prime}}$ to live in one of $O\left(\delta^{-C / N}\right)$ boxes, each of dimension $C \times C \tau \times C \delta \times \cdots \times C \delta$. The claim then follows from (10).

## 5. The bilinear reduction.

We now begin the proof of Theorem 1.2.
Fix $0<\varepsilon \ll 1$. For each $0<\delta \ll 1$, let $A(\delta)=A_{\varepsilon}(\delta)$ denote the best constant such that

$$
\begin{equation*}
\left\|\sum_{l \in \mathcal{A}} \chi_{T_{l}}\right\|_{p^{\prime}} \leq A(\delta) \delta^{-n / p+1-\varepsilon} m^{1 / q-1 / r}\left(\delta^{n-1}|\mathcal{A}|\right)^{1 / q^{\prime}} \tag{29}
\end{equation*}
$$

for all choices of $m, \mathcal{E}, \mathcal{E}^{\prime}$ and $\mathcal{A}$ satisfying (7). Clearly $A(\delta)$ is finite for each $\delta$; to prove Theorem 1.2, we need to show

$$
\begin{equation*}
A(\delta) \lesssim 1 \tag{30}
\end{equation*}
$$

It will be convenient to denote the right-hand side of $(29)$ as $\mathbf{Q}(\delta, \mathcal{A})$, thus

$$
\begin{equation*}
\mathbf{Q}(\delta, \mathcal{A})=A(\delta) \delta^{-n / p+1-\varepsilon} m^{1 / q-1 / r}\left(\delta^{n-1}|\mathcal{A}|\right)^{1 / q^{\prime}} \tag{31}
\end{equation*}
$$

By an inductive argument it suffices to prove (30) assuming that

$$
\begin{equation*}
A(\delta) \sim \sup _{\delta \leq \delta^{\prime} \ll 1} A\left(\delta^{\prime}\right) \tag{32}
\end{equation*}
$$

Fix $\delta$ so that (32) holds. We may find $m, \mathcal{E}$, and $\mathcal{A}$ such that

$$
\begin{equation*}
\left\|\sum_{l \in \mathcal{A}: v(l) \in \mathcal{E}} \chi_{T_{l}}\right\|_{p^{\prime}} \sim \mathbf{Q}(\delta, \mathcal{A}) . \tag{33}
\end{equation*}
$$

The estimate (33) states that $\mathcal{A}$ is essentially an optimal configuration. This has several consequences, at least heuristically. Firstly, it implies
that the generic angle between two lines in $\mathcal{A}$ is $\sim 1$. Secondly, it implies a "two-ends" condition, which roughly asserts that the contribution of the generic tube $T_{l}$ to (33) is not concentrated on a short interval. We make these claims rigorous in the following sections, together with a technical uniformization reduction; these preliminaries will simplify the ensuing argument. We remark that one needs $\varepsilon>0$ in order to obtain these reductions.

We begin with the assertion that the generic angle between two lines is $\sim 1$. This is accomplished by

Proposition 5.1. There exist subsets $\mathcal{E}_{1}, \mathcal{E}_{2}$ of $\mathcal{E}$ such that

$$
\begin{equation*}
\operatorname{dist}\left(\mathcal{E}_{1}, \mathcal{E}_{2}\right) \sim 1 \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(\sum_{l \in \mathcal{A}: v(l) \in \mathcal{E}_{1}} \chi_{T_{l}}\right)\left(\sum_{l^{\prime} \in \mathcal{A}: v\left(l^{\prime}\right) \in \mathcal{E}_{2}} \chi_{T_{l^{\prime}}}\right)\right\|_{p^{\prime} / 2}^{1 / 2} \sim \mathbf{Q}(\delta, \mathcal{A}) . \tag{35}
\end{equation*}
$$

Without (34), one could simply take $\mathcal{E}_{1}=\mathcal{E}_{2}=\mathcal{E}$ in the above proposition. The point of this proposition is that it allows one to restrict one's attention to pairs of tubes which intersect at large angle. This bilinear reduction allows us to avoid many (but not all) of the difficulties involving small angle intersections, which we have already encountered when managing the $\sigma$ and $\tau$ parameters in the previous section.

Proof. By squaring (33) we have

$$
\begin{equation*}
\left\|\sum_{l, l^{\prime} \in \mathcal{A}} \chi_{T_{l}} \chi_{T_{l^{\prime}}}\right\|_{p^{\prime} / 2} \sim \mathbf{Q}(\delta, \mathcal{A})^{2} . \tag{36}
\end{equation*}
$$

Now let $0<c_{0}<1$ be a small number to be chosen later, and consider the quantity

$$
\begin{equation*}
\left\|_{l, l^{\prime} \in \mathcal{A}:\left|v(l)-v\left(l^{\prime}\right)\right|<c_{0}} \chi_{T_{l}} \chi_{T_{l^{\prime}}}\right\|_{p^{\prime} / 2} . \tag{37}
\end{equation*}
$$

Cover $\mathcal{E}$ by finitely overlapping sets $\mathcal{E}=\bigcup_{\alpha} \mathcal{E}_{\alpha}$ where each $\mathcal{E}_{\alpha}$ has diameter $O\left(c_{0}\right)$, and such that for every $v, v^{\prime} \in \mathcal{E}$ with $\left|v-v^{\prime}\right| \leq c_{0}$ there exists an $\alpha$ such that $v, v^{\prime} \in \mathcal{E}_{\alpha}$. We thus have

$$
\sum_{l, l^{\prime} \in \mathcal{A}:\left|v(l)-v\left(l^{\prime}\right)\right|<c_{0}} \chi_{T_{l}} \chi_{T_{l^{\prime}}} \leq \sum_{\alpha}\left(\sum_{l \in \mathcal{A}_{\alpha}} \chi_{T_{l}}\right)^{2},
$$

where $\mathcal{A}_{\alpha}=\left\{l \in \mathcal{A}: v(l) \in \mathcal{E}_{\alpha}\right\}$. Since $p^{\prime} / 2<1$, we have the quasitriangle inequality

$$
\begin{equation*}
\left\|\sum_{\alpha} f_{\alpha}^{2}\right\|_{p^{\prime} / 2} \leq\left(\sum_{\alpha}\left\|f_{\alpha}^{2}\right\|_{p^{\prime} / 2}^{p^{\prime} / 2}\right)^{2 / p^{\prime}}=\left(\sum_{\alpha}\left\|f_{\alpha}\right\|_{p^{\prime}}^{p^{\prime}}\right)^{2 / p^{\prime}}, \tag{38}
\end{equation*}
$$

(see e.g. [10]), and so we may estimate (37) by

$$
\begin{equation*}
\left(\sum_{\alpha}\left\|\sum_{l \in \mathcal{A}_{\alpha}} \chi_{T_{l}}\right\|_{p^{\prime}}^{p^{\prime}}\right)^{2 / p^{\prime}} . \tag{39}
\end{equation*}
$$

We now claim that

$$
\begin{equation*}
\left\|\sum_{l \in \mathcal{A}_{\alpha}} \chi_{T_{l}}\right\|_{p^{\prime}} \lesssim c_{0}^{-(n-1) / p^{\prime}} \mathbf{Q}\left(\frac{\delta}{c_{0}}, \mathcal{A}_{\alpha}\right) \tag{40}
\end{equation*}
$$

To see this, first apply a mild affine map to make $\mathcal{E}_{\alpha}$ centered at the origin, and apply the dilation $\left(\underline{x}, x_{n}\right) \rightarrow\left(\underline{x} / c_{0}, x_{n}\right)$, and then apply (29) to the result; cf. [10].

Since our choice of $p, q$ satisfy the scaling condition $q=(n-1) p^{\prime}$, we may simplify (40) using (32) and (31) to

$$
\left\|\sum_{l \in \mathcal{A}_{\alpha}} \chi_{T_{l}}\right\|_{p^{\prime}} \lesssim c_{0}^{\varepsilon}\left(\frac{\left|\mathcal{A}_{\alpha}\right|}{|\mathcal{A}|}\right)^{1 / q^{\prime}} \mathbf{Q}(\delta, \mathcal{A})
$$

Inserting this back into (37) and using the elementary inequality

$$
\sum_{\alpha}\left(\frac{\left|\mathcal{A}_{\alpha}\right|}{|\mathcal{A}|}\right)^{p^{\prime} / q^{\prime}} \leq\left(\sum_{\alpha} \frac{\left|\mathcal{A}_{\alpha}\right|}{|\mathcal{A}|}\right)^{p^{\prime} / q^{\prime}} \leq 1
$$

which follows since $p^{\prime}>q^{\prime}$, we obtain

$$
(37) \lesssim\left(c_{0}^{\varepsilon} \mathbf{Q}(\delta, \mathcal{A})\right)^{2}
$$

Comparing this with (36) we see that

$$
\left\|\sum_{l, l^{\prime} \in \mathcal{A}:\left|v(l)-v\left(l^{\prime}\right)\right| \geq c_{0}} \chi_{T_{l}} \chi_{T_{l^{\prime}}}\right\|_{p^{\prime} / 2} \sim \mathbf{Q}(\delta, \mathcal{A})^{2}
$$

if we choose $c_{0}$ to be a sufficiently small number depending only on $n$ and $\varepsilon$ (so $c_{0} \sim 1$ ).

Now cover $\mathcal{E}$ by $O\left(c_{0}^{1-n}\right)$ balls of diameter $c_{0} / 4$. By the pigeonhole principle and the above estimate we see that there must exist at least one pair $\mathcal{E}_{1}, \mathcal{E}_{2}$ of such balls with $\operatorname{dist}\left(\mathcal{E}_{1}, \mathcal{E}_{2}\right) \geq c_{0} / 2$ such that

$$
\left\|\sum_{l, l^{\prime} \in \mathcal{A}: v(l) \in \mathcal{E}_{1}, v\left(l^{\prime}\right) \in \mathcal{E}_{2}} \chi_{T_{l}} \chi_{T_{l^{\prime}}}\right\|_{p^{\prime} / 2} \gtrsim c_{0}^{C} \mathbf{Q}(\delta, \mathcal{A})^{2} .
$$

The claim follows.
Note that the above argument is not restricted to this particular choice of $p, q, r$. See [8], [10], [11] for variants of this argument. The arguments in the next three sections are similarly not restricted to the exponent choices in (6).

Henceforth $\mathcal{E}_{1}, \mathcal{E}_{2}$ will be fixed.

## 6. Uniformity of multiplicity and density.

Let $\mathcal{A}$ be a subset of $\mathcal{E} \times \mathcal{E}^{\prime}$ satisfying (7), and let $E$ be a subset of $\mathbb{R}^{n}$. It would be convenient if we could ensure some uniformity on the multiplicity function $\sum_{l \in A} \chi_{T_{l}}$ and the density function $\left|T_{l} \cap E\right|$, as in Lemma 4.2. This is achieved by

Lemma 6.1. Let $\mathcal{A}$ be a subset of $\mathcal{E} \times \mathcal{E}^{\prime}$ satisfying (7), and let $E$ be a subset of $\mathbb{R}^{n}$. Let $\lambda, \mu>0$ be quantities satisfying

$$
\begin{equation*}
\mu|E|=\lambda \delta^{n-1}|\mathcal{A}| . \tag{41}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu|E|^{1 / p^{\prime}} \gtrsim \mathbf{Q}(\delta, \mathcal{A}) \tag{42}
\end{equation*}
$$

Suppose $E^{\prime} \subset E, \mathcal{A}^{\prime} \subset \mathcal{A}$ are such that

$$
\begin{equation*}
\int_{E^{\prime}} \sum_{l \in \mathcal{A}^{\prime}} \chi_{T_{l}} \approx \mu|E| \tag{43}
\end{equation*}
$$

or equivalently that

$$
\begin{equation*}
\sum_{l \in \mathcal{A}^{\prime}}\left|T_{l} \cap E^{\prime}\right| \approx \lambda \delta^{n-1}|\mathcal{A}| \tag{44}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\int_{x \in E^{\prime}: \sum_{l \in \mathcal{A}^{\prime}} \chi_{T_{l}}(x) \approx \mu} \sum_{l \in \mathcal{A}^{\prime}} \chi_{T_{l}}(x) \approx \mu|E| \tag{45}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{l \in \mathcal{A}^{\prime}:\left|T_{l} \cap E^{\prime}\right| \approx \lambda \delta^{n-1}}\left|T_{l} \cap E^{\prime}\right| \approx \lambda \delta^{n-1}|\mathcal{A}| . \tag{46}
\end{equation*}
$$

Equivalently, we have

$$
\left|\left\{x \in E^{\prime}: \sum_{l \in \mathcal{A}^{\prime}} \chi_{T_{l}}(x) \approx \mu\right\}\right| \approx|E|
$$

and

$$
\left|\left\{l \in \mathcal{A}^{\prime}:\left|T_{l} \cap E^{\prime}\right| \approx \lambda \delta^{n-1}\right\}\right| \approx|\mathcal{A}|
$$

The condition (41) is quite natural; cf. (24). The condition (42) is a variant of (33), and states that $\mu|E|^{1 / p^{\prime}}$ is essentially as large as possible. Although this lemma is not phrased in a bilinear way, we will be able to combine it with the bilinear reduction (and the two-ends reduction in the next section) in Section 8.

Proof. We first prove (45). Let $B=(\log (1 / \delta))^{\nu}$, where $\nu$ is a large constant to be chosen later. We trivially have

$$
\int_{x \in E^{\prime}: \sum_{l \in \mathcal{A}^{\prime}} \chi_{T_{l}}(x) \lesssim B^{-1} \mu} \sum_{l \in \mathcal{A}^{\prime}} \chi_{T_{l}}(x) \lesssim B^{-1} \mu|E| .
$$

We now claim that

$$
\begin{equation*}
\int_{x \in E^{\prime}: \sum_{l \in \mathcal{A}^{\prime}} \chi_{T_{l}}(x) \gtrsim B \mu} \sum_{l \in \mathcal{A}^{\prime}} \chi_{T_{l}}(x) \lesssim B^{-\left(p^{\prime}-1\right)} \mu|E|, \tag{47}
\end{equation*}
$$

the claim then follows by subtracting these two estimates from (43) and choosing $\nu$ suitably.

To prove (47), we first observe that the left-hand side is bounded by

$$
\lesssim(B \mu)^{1-p^{\prime}} \int\left(\sum_{l \in \mathcal{A}} \chi_{T_{l}}\right)^{p^{\prime}}
$$

By (29) and (42), this is bounded by

$$
\lesssim(B \mu)^{1-p^{\prime}}\left(\mu|E|^{1 / p^{\prime}}\right)^{p^{\prime}}
$$

and (47) follows.
Now we prove (46), which is a dual of (45); the last two claims in the lemma then follow easily.

As before we have

$$
\sum_{l \in \mathcal{A}^{\prime}:\left|T_{l} \cap E^{\prime}\right| \lesssim B^{-1} \lambda \delta^{n-1}}\left|T_{l} \cap E^{\prime}\right| \lesssim B^{-1} \lambda \delta^{n-1}|\mathcal{A}|
$$

It suffices to show that

$$
\begin{equation*}
\sum_{l \in \mathcal{A}^{\prime \prime}}\left|T_{l} \cap E^{\prime}\right| \lesssim B^{\prime-(q-1)} \lambda \delta^{n-1}|\mathcal{A}| \tag{48}
\end{equation*}
$$

for all $B^{\prime} \geq B$, where

$$
A^{\prime \prime}=\left\{l \in \mathcal{A}^{\prime}:\left|T_{l} \cap E^{\prime}\right| \approx B^{\prime} \lambda \delta^{n-1}\right\},
$$

by summing this for all dyadic $B^{\prime} \geq B$ and using the exponential decay of the $B^{\prime-(q-1)}$ we can obtain the analogue of (47).

Fix $B^{\prime}$. By definition of $A^{\prime \prime}$ we have

$$
\int_{E^{\prime}} \sum_{l \in \mathcal{A}^{\prime \prime}} \chi_{T_{l}}=\sum_{l \in \mathcal{A}^{\prime \prime}}\left|T_{l} \cap E^{\prime}\right| \approx B^{\prime} \lambda \delta^{n-1}\left|\mathcal{A}^{\prime \prime}\right|
$$

From Hölder we thus have

$$
\begin{equation*}
|E|^{1 / p}\left\|\sum_{l \in \mathcal{A}^{\prime \prime}} \chi_{T_{l}}\right\|_{p^{\prime}} \gtrsim B^{\prime} \lambda \delta^{n-1}\left|\mathcal{A}^{\prime \prime}\right| \tag{49}
\end{equation*}
$$

From (29) we have

$$
\left\|\sum_{l \in \mathcal{A}^{\prime \prime}} \chi_{T_{l}}\right\|_{p^{\prime}} \leq \mathbf{Q}\left(\delta, \mathcal{A}^{\prime \prime}\right),
$$

from (31) and (42) we thus have

$$
\left\|\sum_{l \in \mathcal{A}^{\prime \prime}} \chi_{T_{l}}\right\|_{p^{\prime}} \lesssim \mu|E|^{1 / p^{\prime}}\left(\frac{\left|\mathcal{A}^{\prime \prime}\right|}{|\mathcal{A}|}\right)^{1 / q^{\prime}}
$$

Inserting this into (49) and using (41) we obtain

$$
\lambda \delta^{n-1}|\mathcal{A}|\left(\frac{\left|\mathcal{A}^{\prime \prime}\right|}{|\mathcal{A}|}\right)^{1 / q^{\prime}} \gtrsim B \lambda \delta^{n-1}\left|\mathcal{A}^{\prime \prime}\right|
$$

which simplifies to

$$
\left|\mathcal{A}^{\prime \prime}\right| \lesssim B^{\prime-q}|\mathcal{A}|,
$$

and (48) follows from the definition of $\mathcal{A}^{\prime \prime}$.

## 7. The two ends reduction.

In order to apply Lemma 4.1 we need (among other things) to obtain the conditions (11) and (12). The condition (11) can essentially be guaranteed by Lemma 6.1, but this lemma does not give us the twoends condition (12). To obtain this we shall use the following lemma.

Lemma 7.1. Let $N \gg 1, E$ be a subset of $\mathbb{R}^{n}$, and let $\mathcal{A}$ be a subset of $\mathcal{E} \times \mathcal{E}^{\prime}$ satisfying (7), and such that for every $l \in \mathcal{A}$ there exists an $x \in \mathbb{R}^{n}$ such that

$$
\left|T_{l} \cap E \cap B\left(x, \delta^{1 / N}\right)\right| \gtrsim \delta^{\varepsilon / 2 N}\left|T_{l} \cap E\right| .
$$

Then we have

$$
\begin{equation*}
\sum_{l \in \mathcal{A}}\left|T_{l} \cap E\right| \lesssim \delta^{\varepsilon / 2 N}|E|^{1 / p} \mathbf{Q}(\delta, \mathcal{A}) \tag{50}
\end{equation*}
$$

The factor of $\delta^{\varepsilon / N}$ in the above argument will allow us to conclude that for most tubes, the set $\left|T_{l} \cap E\right|$ is not concentrated in a short end of the tube. This type of "two-ends condition" first appears in [11], [12].

Proof. Cover [0, 1] by $\sim \delta^{-1 / N}$ finitely overlapping intervals $I_{\alpha}$ of width $\sim \delta^{1 / N}$, and let $S_{\alpha}$ denote the slab $\mathbb{R}^{n-1} \times I_{\alpha}$. For each $l \in \mathcal{A}$, we can then find an $\alpha=\alpha(l)$ such that

$$
\left|T_{l} \cap E \cap S_{\alpha}\right| \gtrsim \delta^{\varepsilon / 2 N}\left|T_{l} \cap E\right| .
$$

It thus suffices to show that

$$
\begin{equation*}
\sum_{\alpha} \sum_{l \in \mathcal{A}_{\alpha}}\left|T_{l} \cap S_{\alpha} \cap E\right| \lesssim \delta^{\varepsilon / N}|E|^{1 / p} \mathbf{Q}(\delta, \mathcal{A}), \tag{51}
\end{equation*}
$$

where

$$
\mathcal{A}_{\alpha}=\{l \in \mathcal{A}: \quad \alpha(l)=\alpha\} .
$$

Partition $\mathcal{E}$ into about $\delta^{(1-n) / N}$ refinements $\mathcal{E}_{\beta}$, each of which is $\delta^{1-1 / N_{-}}$ separated. We can split the left-hand side of (51) as

$$
\sum_{\alpha} \sum_{\beta} \int_{S_{\alpha} \cap E} \sum_{l \in \mathcal{A}_{\alpha, \beta}} \chi_{T_{l} \cap E}
$$

where

$$
\mathcal{A}_{\alpha, \beta}=\left\{l \in \mathcal{A}_{\alpha}: v(l) \in \mathcal{E}_{\beta}\right\} .
$$

By Hölder, we may estimate this by

$$
\begin{equation*}
\sum_{\alpha} \sum_{\beta}\left|S_{\alpha} \cap E\right|^{1 / p}\left\|\sum_{l \in \mathcal{A}_{\alpha, \beta}} \chi_{T_{l} \cap S_{\alpha} \cap E}\right\|_{p^{\prime}} \tag{52}
\end{equation*}
$$

The sets $T_{l} \cap S_{\alpha}$ in the innermost sum can be rescaled to form a collection of $\delta^{1-1 / N} \times 1$ tubes which continue to satisfy (7). Also, the set of directions $\mathcal{E}_{\beta}$ satisfies the correct separation condition for the scale $\delta^{1-1 / N}$. By a rescaled version of (29) and (31), we can therefore bound the norm in (52) by

$$
\lesssim \delta^{n / N p^{\prime}} \mathbf{Q}\left(\delta^{1-1 / N}, \mathcal{A}_{\alpha, \beta}\right),
$$

which can be estimated using (31), (32) and algebra by

$$
\lesssim \delta^{\varepsilon / N} \mathbf{Q}(\delta, \mathcal{A}) \delta^{(n-1) / q N}\left(\frac{\left|\mathcal{A}_{\alpha, \beta}\right|}{|\mathcal{A}|}\right)^{1 / q^{\prime}}
$$

Inserting this back into (52), we may estimate the left-hand side of (51) as

$$
\lesssim \delta^{\varepsilon / N} \mathbf{Q}(\delta, \mathcal{A}) \sum_{\alpha}\left|S_{\alpha} \cap E\right|^{1 / p} \delta^{(n-1) / q N} \sum_{\beta}\left(\frac{\left|\mathcal{A}_{\alpha, \beta}\right|}{|\mathcal{A}|}\right)^{1 / q^{\prime}}
$$

Since we have $O\left(\delta^{(1-n) / N}\right) \beta$ 's, we can use Hölder to obtain

$$
\delta^{(n-1) / q N} \sum_{\beta}\left(\frac{\left|\mathcal{A}_{\alpha, \beta}\right|}{|\mathcal{A}|}\right)^{1 / q^{\prime}} \lesssim\left(\frac{\left|\mathcal{A}_{\alpha}\right|}{|\mathcal{A}|}\right)^{1 / q^{\prime}} .
$$

We can thus bound the left-hand side of (51) as

$$
\lesssim \delta^{\varepsilon / N} \mathbf{Q}(\delta, \mathcal{A}) \sum_{\alpha}\left|S_{\alpha} \cap E\right|^{1 / p}\left(\frac{\left|\mathcal{A}_{\alpha}\right|}{|\mathcal{A}|}\right)^{1 / q^{\prime}}
$$

By Hölder again, we bound this by

$$
\lesssim \delta^{\varepsilon / N} \mathbf{Q}(\delta, \mathcal{A})\left(\sum_{\alpha}\left|S_{\alpha} \cap E\right|^{q / p}\right)^{1 / q}
$$

Since $q>p$, we can bound this by

$$
\lesssim \delta^{\varepsilon / N} \mathbf{Q}(\delta, \mathcal{A})\left(\sum_{\alpha}\left|S_{\alpha} \cap E\right|\right)^{1 / p},
$$

and (51) follows.

## 8. Plate number uniformization.

We now combine the tools developed in the previous three sections to obtain the following technical uniformization lemma, which is analogous to Lemma 4.2. We use $\mathcal{A}_{i, 0}$ for $i=1,2$ to denote the set

$$
\mathcal{A}_{i, 0}=\left\{l \in \mathcal{A}: v(l) \in \mathcal{E}_{i}\right\} .
$$

Lemma 8.1. Let the notation be as in the previous sections, and let $N \gg 1$ be a large number. Then, if $\delta$ is sufficiently small depending on $\varepsilon$ and $N$, there exist numbers $\mu, \lambda, \mathbf{p}_{1}, \mathbf{p}_{2}>0$ and sets

$$
\begin{equation*}
\mathcal{A}_{i}^{(3)} \subset \mathcal{A}_{i}^{(2)} \subset \mathcal{A}_{i}^{(1)} \subset \mathcal{A}_{i}^{(0)}=\mathcal{A}_{i, 0}, \quad \text { for } i=1,2, \tag{53}
\end{equation*}
$$

and

$$
\begin{equation*}
E^{(3)} \subset E^{(2)} \subset E^{(1)} \subset E^{(0)} \subset \mathbb{R}^{n} \tag{54}
\end{equation*}
$$

such that

$$
\begin{equation*}
\left|E^{(0)}\right| \mu=|\mathcal{A}| \lambda \delta^{n-1}, \tag{55}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu\left|E^{(0)}\right|^{1 / p^{\prime}} \approx \mathbf{Q}(\delta, \mathcal{A}) \tag{56}
\end{equation*}
$$

Furthermore, one has

$$
\begin{equation*}
\left|T_{l} \cap E^{(j-1)}\right| \approx \lambda\left|T_{l}\right|, \tag{57}
\end{equation*}
$$

$$
\begin{equation*}
\left|T_{l} \cap E^{(j-1)} \cap B\left(x, \delta^{1 / N}\right)\right| \lesssim \delta^{\varepsilon / 2 N} \lambda\left|T_{l}\right| \tag{58}
\end{equation*}
$$

for all $l \in \mathcal{A}_{i}^{(j)}, i=1,2, j=1,2,3, x \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\sum_{l \in \mathcal{A}_{i}^{(j)}} \chi_{T_{l}}(x) \approx \mu, \quad \text { for all } x \in E^{(j)}, i=1,2, j=0,1,2,3 \tag{59}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta^{C / N} \mathbf{p}_{i} \lesssim \mathbf{p}_{i}\left(\mathcal{A}_{i}^{(j)}\right) \lesssim \delta^{-C / N} \mathbf{p}_{i}, \quad \text { for } i=1,2, j=1,2,3 \tag{60}
\end{equation*}
$$

The implicit constants in these estimates may depend on $N$.
Proof. The first step is to find $\mu$ and $E^{(0)}$.
Let $\mu_{1}, \mu_{2}$ range over all dyadic integers from 1 to $\delta^{-C}$. Let $E^{(0)}\left(\mu_{1}, \mu_{2}\right)$ denote the set

$$
E^{(0)}\left(\mu_{1}, \mu_{2}\right)=\left\{x: \sum_{l \in \mathcal{A}_{i, 0}} \chi_{T_{l}}(x) \sim \mu_{i} \text { for } i=1,2\right\} .
$$

Clearly we have
(61) left hand side of (35) $\sim\left(\sum_{\mu_{1}} \sum_{\mu_{2}} \mu_{1}^{p^{\prime} / 2} \mu_{2}^{p^{\prime} / 2}\left|E^{(0)}\left(\mu_{1}, \mu_{2}\right)\right|\right)^{1 / p^{\prime}}$.

Since the number of $\mu_{1}$ and $\mu_{2}$ is $\approx 1$, we can use the pigeonhole principle and conclude that there exist $\mu_{1}, \mu_{2}$ for which (56) holds with $E^{(0)}=E^{(0)}\left(\mu_{1}, \mu_{2}\right)$ and $\mu=\left(\mu_{1} \mu_{2}\right)^{1 / 2}$.

Fix this choice of $\mu_{i}, \mu$ and $E^{(0)}$; this also fixes $\lambda$. By construction we have

$$
\left\|\sum_{l \in \mathcal{A}_{i, 0}} \chi_{T_{l}}\right\|_{p^{\prime}} \gtrsim \mu_{i}\left|E^{(0)}\right|^{1 / p^{\prime}} .
$$

Combining this with (29) we have

$$
\mu_{i}\left|E^{(0)}\right|^{1 / p^{\prime}} \lesssim \mathbf{Q}(\delta, \mathcal{A}),
$$

for $i=1,2$. Combining this with (29) we see that

$$
\mu_{i} \lesssim \mu
$$

From the definition of $\mu$ we thus have $\mu_{i} \approx \mu$. Since $\mu \lesssim \delta^{-C}$, we see from (56), (55) that $\left|E^{(0)}\right|, \lambda \gtrsim \delta^{C}$.

We now produce sets

$$
E^{(0)}=E_{0} \supset E_{1} \supset \cdots \supset E_{N^{2}}
$$

and

$$
\mathcal{A}_{i, 0} \supset \mathcal{A}_{i, 1} \supset \cdots \supset \mathcal{A}_{i, N^{2}}
$$

with the properties that

$$
\begin{equation*}
\left|E_{k}\right| \approx\left|E_{0}\right|, \quad \text { for all } 0 \leq k \leq N^{2}, \tag{62}
\end{equation*}
$$

$$
\begin{equation*}
\left|T_{l} \cap E_{k-1}\right| \approx \lambda\left|T_{l}\right| \tag{63}
\end{equation*}
$$

$$
\begin{equation*}
\left|T_{l} \cap E_{k-1} \cap B\left(x, \delta^{1 / N}\right)\right| \lesssim \delta^{\varepsilon / 2 N} \lambda\left|T_{l}\right|, \tag{64}
\end{equation*}
$$

for all $l \in \mathcal{A}_{i, k}, i=1,2,1 \leq k \leq N^{2}, x \in \mathbb{R}^{n}$, and

$$
\begin{equation*}
\sum_{l \in \mathcal{A}_{i, k}} \chi_{T_{l}}(x) \approx \mu, \quad \text { for all } x \in E_{k}, i=1,2,0 \leq k \leq N^{2} \tag{65}
\end{equation*}
$$

Clearly (62) and (65) hold for $k=0$. Now suppose inductively that $0 \leq k<N^{2}$ is such that $E_{k}, \mathcal{A}_{1, k}, \mathcal{A}_{2, k}$ have been constructed satisfying (65) and (62) for this value of $k$.

We perform a certain sequence of dance steps. From (62) and (65) we have

$$
\int_{E_{k}} \sum_{l \in \mathcal{A}_{1, k}} \chi_{T_{l}} \approx \mu\left|E_{0}\right|
$$

which by (55) implies

$$
\sum_{l \in \mathcal{A}_{1, k}}\left|T_{l} \cap E_{k}\right| \approx|\mathcal{A}| \lambda \delta^{n-1}
$$

By Lemma 6.1 (noting that $\mathbf{Q}\left(\delta, \mathcal{A}_{1, k}\right) \leq \mathbf{Q}(\delta, \mathcal{A})$; we shall need similar observations in the sequel), we thus have

$$
\begin{equation*}
\sum_{l \in \mathcal{A}_{1, k}^{\prime}}\left|T_{l} \cap E_{k}\right| \approx|\mathcal{A}| \lambda \delta^{n-1}, \tag{66}
\end{equation*}
$$

where $\mathcal{A}_{1, k}^{\prime} \subseteq \mathcal{A}_{1, k}$ is the set

$$
\mathcal{A}_{1, k}^{\prime}=\left\{l \in \mathcal{A}_{1, k}:\left|T_{l} \cap E_{k}\right| \approx \lambda \delta^{n-1}\right\}
$$

Now define the set $\mathcal{A}_{1, k+1} \subseteq \mathcal{A}_{1, k}^{\prime}$ by

$$
\left.\begin{array}{rl}
\mathcal{A}_{1, k+1}=\left\{l \in \mathcal{A}_{1, k}^{\prime}:\right. & \left|T_{l} \cap E_{k} \cap B\left(x, \delta^{1 / N}\right)\right| \\
& \leq
\end{array} \delta^{\varepsilon / 2 N}\left|T_{l} \cap E_{k}\right| \text { for all } x \in \mathbb{R}^{n}\right\} . ~ \$
$$

From Lemma 7.1 we have

$$
\sum_{l \in \mathcal{A}_{1, k}^{\prime} \backslash \mathcal{A}_{1, k+1}}\left|T_{l} \cap E_{k}\right| \lesssim \delta^{\varepsilon / 2 N}\left|E_{0}\right|^{1 / p} \mathbf{Q}(\delta, \mathcal{A})
$$

by (56) and (55) we thus have

$$
\sum_{l \in \mathcal{A}_{1, k}^{\prime} \backslash \mathcal{A}_{1, k+1}}\left|T_{l} \cap E_{k}\right| \lesssim \delta^{\varepsilon / 2 N}|\mathcal{A}| \lambda \delta^{n-1} .
$$

Combining this with (66) we obtain (if $\delta$ is sufficiently small)

$$
\sum_{l \in \mathcal{A}_{1, k+1}}\left|T_{l} \cap E_{k}\right| \approx|\mathcal{A}| \lambda \delta^{n-1}
$$

We may rewrite this using (55) as

$$
\int_{E_{k}} \sum_{l \in \mathcal{A}_{1, k+1}} \chi_{T_{l}}(x) \approx \mu\left|E_{0}\right| .
$$

By Lemma 6.1, we have

$$
\left|E_{k}^{\prime}\right| \approx\left|E_{0}\right|
$$

where $E_{k}^{\prime} \subseteq E_{k}$ is the set

$$
E_{k}^{\prime}=\left\{x \in E_{k}: \sum_{l \in \mathcal{\mathcal { A } _ { 1 , k + 1 }}} \chi_{T_{l}}(x) \approx \mu\right\} .
$$

In particular, from (65) with $i=2$, we have

$$
\int_{E_{k}^{\prime}} \sum_{l \in \mathcal{A}_{2, k}} \chi_{T_{l}}(x) \approx \mu\left|E_{0}\right|
$$

By (55), we may rewrite this as

$$
\sum_{l \in \mathcal{A}_{2, k}}\left|T_{l} \cap E_{k}^{\prime}\right| \approx|\mathcal{A}| \lambda \delta^{n-1}
$$

By Lemma 6.1 again, this implies

$$
\sum_{l \in \mathcal{A}_{2, k}^{\prime}}\left|T_{l} \cap E_{k}^{\prime}\right| \approx|\mathcal{A}| \lambda \delta^{n-1}
$$

where

$$
\mathcal{A}_{2, k}^{\prime}=\left\{l \in \mathcal{A}_{2, k}:\left|T_{l} \cap E_{k}^{\prime}\right| \approx \lambda \delta^{n-1}\right\}
$$

Defining

$$
\begin{aligned}
\mathcal{A}_{2, k+1}=\left\{l \in \mathcal{A}_{1, k}^{\prime}:\right. & \left|T_{l} \cap E_{k} \cap B\left(x, \delta^{1 / N}\right)\right| \\
& \left.\leq \delta^{\varepsilon / 2 N}\left|T_{l} \cap E_{k}\right| \text { for all } x \in \mathbb{R}^{n}\right\}
\end{aligned}
$$

we apply Lemma 7.1, (56), (55) and the preceding estimate as before to conclude

$$
\sum_{l \in \mathcal{A}_{2, k+1}}\left|T_{l} \cap E_{k}^{\prime}\right| \approx|\mathcal{A}| \lambda \delta^{n-1}
$$

By (55) again, we rewrite this as

$$
\int_{E_{k}^{\prime}} \sum_{l \in \mathcal{A}_{2, k+1}} \chi_{T_{l}}(x) \approx \mu\left|E_{0}\right|
$$

By Lemma 6.1 we have

$$
\left|E_{k+1}\right| \approx\left|E_{0}\right|
$$

where

$$
E_{k+1}=\left\{x \in E_{k}^{\prime}: \sum_{l \in \mathcal{\mathcal { A } _ { 2 , k + 1 }}} \chi_{T_{l}}(x) \approx \mu\right\}
$$

This completes the dance sequence. One can easily verify that (62), (64) and (65) are all satisfied for $k+1$ and $i=1,2$. One now replaces $k$
by $k+1$, and repeats the above dance. Of course, the implicit constants in the bounds will depend on $k$ and hence on $N$.

The quantities $\mathbf{p}_{i}\left(\mathcal{A}_{i, k}\right)$ are clearly monotone decreasing, and satisfy the trivial estimates $1 \lesssim \mathbf{p}_{i}\left(\mathcal{A}_{i, k}\right) \lesssim \delta^{-C}$. By the pigeonhole principle one can then find $1<k<N^{2}-1$ such that

$$
\mathbf{p}_{i}\left(\mathcal{A}_{i, k+2}\right) \geq \delta^{C / N} \mathbf{p}_{i}\left(\mathcal{A}_{i, k}\right), \quad \text { for } i=1,2 .
$$

The lemma then follows by setting $E^{(j)}=E_{k+j-1}, \mathcal{A}_{i}^{(j)}=\mathcal{A}_{i, k+j-1}$, and $\mathbf{p}_{i}=\mathbf{p}_{i}\left(\mathcal{A}_{i, k}\right)$ for $j=1,2,3$ and $i=1,2$.

This argument can be extended to create arbitrarily longer sequences than the ones in the above lemma, but we shall not need to do so here.

## 9. Estimates for a slab.

Let the notation be as in Lemma 8.1. Define a $\theta$-slab to be a $\theta / 2$-neighbourhood of a 2 -plane in $\mathbb{R}^{n}$.

In the sequel we shall prove two propositions.
Proposition 9.1. Let $\delta \lesssim \theta \lesssim 1$, and let $S$ be a $\theta$-slab. Then we have

$$
\begin{equation*}
\left|E^{(1)} \cap S\right| \lesssim \theta^{1 / 2} \lambda^{7 / 2-n}|\mathcal{A}|^{(n-2) /(n-1)} m^{1 /(n-1)} \delta^{n-2} \mu^{-1} . \tag{67}
\end{equation*}
$$

Proposition 9.2. There exists a $\delta \lesssim \theta \lesssim 1$ and a $\theta$-slab $S$ such that

$$
\begin{equation*}
\left|E^{(1)} \cap S\right| \gtrsim \delta^{C / N} \mu \lambda^{7 / 2} m^{-1 / 2} \theta^{1 / 2} \delta^{n-2} . \tag{68}
\end{equation*}
$$

Suppose for the moment that both propositions were true. Then we would have

$$
\delta^{C / N} \mu \lambda^{7 / 2} m^{-1 / 2} \delta^{n-2} \lesssim \lambda^{7 / 2-n}|\mathcal{A}|^{(n-2) /(n-1)} m^{1 /(n-1)} \delta^{n-2} \mu^{-1}
$$

If one uses (55) to eliminate $\lambda$, this becomes (using (31), (6) and a lot of algebra)

$$
\mu\left|E^{(0)}\right|^{1 / p^{\prime}} \lesssim \delta^{-C / N} \delta^{\varepsilon} \mathbf{Q}(\delta, \mathcal{A}) \mathcal{A}(\delta)^{-1}
$$

Comparing this with (56) one obtains (30) if $N$ is chosen sufficiently large depending on $\varepsilon$.

It remains to prove the Propositions.

## 10. Proof of Proposition 9.1.

We now prove Proposition 9.1. The estimate (67) is not best possible; it was chosen primarily so that it cancelled nicely against (68). Accordingly, our techniques shall be quite crude.

Fix $\theta$ and $S$. From (59) we have

$$
\left|E^{(1)} \cap S\right| \approx \mu^{-1} \int_{E^{(1)} \cap S} \sum_{l \in \mathcal{A}_{1}^{(0)}} \chi_{T_{l}}
$$

We can rewrite the right-hand side as

$$
\mu^{-1} \sum_{l \in \mathcal{A}_{1}^{(1)}}\left|E^{(1)} \cap S \cap T_{l}\right| \leq \mu^{-1} \sum_{l \in \mathcal{A}_{1}^{(1)}}\left|E^{(0)} \cap S \cap T_{l}\right|
$$

For each $l$, let $\alpha(l)$ denote the quantity

$$
\alpha(l)=\theta+\angle(l, S),
$$

where $\angle(l, S)$ is the angle between $l$ and the plane in the middle of $S$. From elementary geometry we have

$$
\left|S \cap T_{l}\right| \lesssim \delta^{n-1} \theta \alpha(l)^{-1}
$$

and so by (57) we have

$$
\left|E^{(0)} \cap S \cap T_{l}\right| \lesssim \delta^{n-1} \min \left\{\theta \alpha(l)^{-1}, \lambda\right\} \lesssim \delta^{n-1} \theta^{1 / 2} \alpha(l)^{-1 / 2} \lambda^{1 / 2} .
$$

Combining all these estimates we obtain

$$
\left|E^{(1)} \cap S\right| \lesssim \mu^{-1} \delta^{n-1} \sum_{l \in \mathcal{A}_{1}^{(0)}} \theta^{1 / 2} \alpha(l)^{-1 / 2} \lambda^{1 / 2}
$$

From (57) we have $\lambda \lesssim 1$, so that $\lambda^{1 / 2} \lesssim \lambda^{7 / 2-n}$. It thus suffices to show that

$$
\begin{equation*}
\sum_{l \in \mathcal{A}_{1}^{(0)}} \alpha(l)^{-1 / 2} \lesssim \delta^{-1}|\mathcal{A}|^{(n-2) /(n-1)} m^{1 /(n-1)} \tag{69}
\end{equation*}
$$

We can estimate the left-hand side of (69) by

$$
\begin{equation*}
\sum_{\delta \lesssim \alpha \lesssim 1} \sum_{l \in \mathcal{A}: \alpha(l) \sim \alpha} \alpha^{-1 / 2} \sim \sum_{\delta \lesssim \alpha \lesssim 1} \alpha^{-1 / 2}|\{l \in \mathcal{A}: \quad \alpha(l) \sim \alpha\}|, \tag{70}
\end{equation*}
$$

where $\alpha$ ranges over the dyadic numbers. From (7) and the $\delta$-separated nature of $\mathcal{E}$ we have

$$
|\{l \in \mathcal{A}: \alpha(l) \sim \alpha\}| \lesssim \alpha^{n-2} \delta^{1-n} m .
$$

Interpolating this with the trivial bound of $|\mathcal{A}|$ we obtain

$$
|\{l \in \mathcal{A}: \alpha(l) \sim \alpha\}| \lesssim \alpha^{(n-2) /(n-1)} \delta^{-1} m^{1 /(n-1)}|\mathcal{A}|^{(n-2) /(n-1)} .
$$

Inserting this back into (70) we obtain (69) since $(n-2) /(n-1) \geq 1 / 2$. This concludes the proof of Proposition 69.

It is clear that there is plenty of slack in the above estimate. Indeed, the only time when (67) is efficient is when $\lambda, \alpha, \theta \approx 1$, and when $|\mathcal{E}| \approx \delta^{1-n}$. These phenomena seems to be a typical consequence of the two ends and bilinear reductions respectively.

## 11. Proof of Proposition 9.2.

We now prove Proposition 9.2. This shall be a modified version of the hairbrush argument in [13].

By symmetry we may assume

$$
\begin{equation*}
\mathbf{p}_{1} \geq \mathbf{p}_{2} \tag{71}
\end{equation*}
$$

Since $\mathbf{p}\left(\mathcal{A}_{1}^{(3)}\right) \gtrsim \delta^{C / N} \mathbf{p}_{1}$ by (60), we see from (10) that one can find a $\delta \lesssim w \lesssim 1$ and a $C \times C w \times C \delta \times \cdots \times C_{\delta}$ rectangle $R$ such that

$$
\begin{equation*}
\frac{\left|\mathcal{A}_{R}\right|}{w / \delta} \gtrsim \delta^{C / N} \mathbf{p}_{1} \tag{72}
\end{equation*}
$$

where

$$
\mathcal{A}_{R}=\left\{l \in \mathcal{A}_{1}^{(3)}: T_{l} \subset R\right\} .
$$

This rectangle $R$ shall form the stem of a hairbrush in $S \cap E^{1}$. Let $l_{R}$ denote the line generated by the first direction of $R$, and $\pi_{R}$ be the

2-plane generated by the first two directions of $R$; thus $R$ lies in the $C \delta$ neighbourhood of $\pi_{R}$ and in the $C w$-neighbourhood of $l_{R}$.

By refining $\mathcal{A}_{R}$ slightly if necessary, we may assume that $w \ll \delta^{1 / N}$; this may worsen the power of $\delta^{1 / N}$ in (72), but is otherwise harmless. From (34) we thus have

$$
\begin{equation*}
\left|v\left(l_{R}\right)-v(l)\right| \sim 1, \quad \text { for all } l \in \mathcal{A}_{2}^{(2)} . \tag{73}
\end{equation*}
$$

Since $\mathcal{A}_{1}^{(3)} \subset \mathcal{E} \times \mathcal{E}^{\prime}$, we have from elementary geometry that

$$
\left|\mathcal{A}_{R}\right| \lesssim\left(\frac{w}{\delta}\right)^{2}
$$

Combining this with (72) we see that

$$
\begin{equation*}
w \gtrsim \delta^{C / N} \mathbf{p}_{1} \delta \tag{74}
\end{equation*}
$$

From (57) we see that

$$
\begin{equation*}
\left|T_{l} \cap E^{(2)}\right| \approx \lambda \delta^{n-1} \tag{75}
\end{equation*}
$$

for all $l \in \mathcal{A}_{R}$. From this we conclude the following:
Lemma 11.1. We have

$$
\begin{equation*}
\left|E^{(2)} \cap R\right| \gtrsim \delta^{C / N} \lambda^{3 / 2} w^{1 / 2} \mathbf{p}_{1}^{1 / 2} \delta^{n-3 / 2} . \tag{76}
\end{equation*}
$$

Proof. Firstly, from (72) and elementary geometry we see that $\mathcal{A}_{R}$ must contain at least $\delta^{C / N} \mathbf{p}_{1}$ parallel lines, which with (75) and (71) gives

$$
\left|E^{(2)} \cap R\right| \gtrsim \delta^{C / N} \lambda \mathbf{p}_{1} \delta^{n-1} .
$$

It thus suffices to show

$$
\left|E^{(2)} \cap R\right| \gtrsim \delta^{C / N} \lambda^{2}\left|\mathcal{A}_{R}\right| \mathbf{p}_{1}^{-1} \delta^{n-1}
$$

since (76) follows by taking the geometric mean of these estimates and then using (72).

To prove this estimate we invoke Córdoba's argument as in the proof of (27). Summing (75) over all $l \in \mathcal{A}_{R}$ we obtain

$$
\sum_{l \in \mathcal{A}_{R}}\left|T_{l} \cap E^{(2)}\right| \approx \lambda \delta^{n-1}\left|\mathcal{A}_{R}\right|
$$

which we rewrite as

$$
\int_{E^{(2)} \cap R} \sum_{l \in \mathcal{A}_{R}} \chi_{T_{l}} \approx \lambda \delta^{n-1}\left|\mathcal{A}_{R}\right|
$$

By the Cauchy-Schwarz inequality we thus have

$$
\left|E^{(2)} \cap R\right|^{1 / 2}\left\|\sum_{l \in \mathcal{A}_{R}} \chi_{T_{l}}\right\|_{2} \gtrsim \lambda \delta^{n-1}\left|\mathcal{A}_{R}\right| .
$$

It thus suffices to show that

$$
\begin{equation*}
\left\|\sum_{l \in \mathcal{A}_{R}} \chi_{T_{l}}\right\|_{2}^{2} \lesssim \delta^{-C / N}\left|\mathcal{A}_{R}\right| \mathbf{p}_{1} \delta^{n-1} \tag{77}
\end{equation*}
$$

Repeating the derivation of (28), we may estimate the left-hand side by

$$
\sum_{\delta \lesssim \tau \lesssim 1} \sum_{l \in \mathcal{A}_{R}} \sum_{l^{\prime} \in \mathcal{A}_{R}: T_{l} \cap T_{l^{\prime}} \neq \varnothing, \delta+\left|v(l)-v\left(l^{\prime}\right)\right| \sim \tau} \delta^{n} \tau^{-1}
$$

and the claim follows from the observation

$$
\left|\left\{l^{\prime} \in \mathcal{A}_{R}: T_{l} \cap T_{l^{\prime}} \neq \varnothing, \delta+\left|v(l)-v\left(l^{\prime}\right)\right| \sim \tau\right\}\right| \lesssim \delta^{-C / N} \delta^{-1} \tau \mathbf{p}_{1}
$$

which follows from (10) and elementary geometry.
Thus $E^{(2)}$ has a large intersection with $R$. We now wish to conclude that there are many tubes from $\mathcal{A}_{2}^{(2)}$ passing through $R$.

Combining (76) with (59) and (71) we have

$$
\int_{R} \sum_{l \in \mathcal{A}_{2}^{(2)}} \chi_{T_{l}}(x) \gtrsim \lambda^{3 / 2} \mu w^{1 / 2} \mathbf{p}_{2}^{1 / 2} \delta^{n-3 / 2}
$$

which we rewrite as

$$
\sum_{l \in \mathcal{A}_{2}^{(2)}}\left|T_{l} \cap R\right| \gtrsim \lambda^{3 / 2} \mu w^{1 / 2} \mathbf{p}_{2}^{1 / 2} \delta^{n-3 / 2}
$$

For each dyadic $\delta \lesssim \theta \lesssim 1$, let $\mathcal{A}_{\text {brush }}^{\theta}$ denote the set

$$
\mathcal{A}_{\mathrm{brush}}^{\theta}=\left\{l \in \mathcal{A}_{2}^{(2)}: T_{l} \cap R \neq \varnothing, \delta / w+\angle l, \pi_{R} \sim \theta\right\} .
$$

We thus have

$$
\sum_{\delta / w \lesssim \theta \lesssim 1} \sum_{l \in \mathcal{A}_{\text {brush }}^{\theta}}\left|T_{l} \cap R\right| \gtrsim \lambda^{3 / 2} \mu w^{1 / 2} \mathbf{p}_{2}^{1 / 2} \delta^{n-3 / 2} .
$$

By the pigeonhole principle, there must therefore exist a $\delta / w \lesssim \theta \lesssim 1$ such that

$$
\sum_{l \in \mathcal{\mathcal { A } _ { \mathrm { brush } } ^ { \theta }}}\left|T_{l} \cap R\right| \gtrsim \lambda^{3 / 2} \mu w^{1 / 2} \mathbf{p}_{2}^{1 / 2} \delta^{n-3 / 2} .
$$

Fix this $\theta$. From (73) and the definition of $\mathcal{A}_{\text {brush }}^{\theta}$, we see from elementary geometry that $\left|T_{l} \cap R\right| \lesssim \delta^{n} \theta^{-1}$. Combining this with the previous, we see that

$$
\begin{equation*}
\left|\mathcal{A}_{\mathrm{brush}}^{\theta}\right| \gtrsim \lambda^{3 / 2} \mu w^{1 / 2} \mathbf{p}_{2}^{1 / 2} \theta \delta^{-3 / 2} \tag{78}
\end{equation*}
$$

Thus to prove (68) it suffices to show that

$$
\left|E^{(1)} \cap S\right| \gtrsim \delta^{C / N} \lambda^{2}\left|\mathcal{A}_{\mathrm{brush}}^{\theta}\right| m^{-1 / 2} \theta^{-1 / 2} w^{-1 / 2} \mathbf{p}_{2}^{-1 / 2} \delta^{n-1 / 2} .
$$

We will in fact show the slightly stronger

$$
\begin{equation*}
\left|E^{(1)} \cap S \cap \Omega\right| \gtrsim \delta^{C / N} \lambda^{2}\left|\mathcal{A}_{\text {brush }}^{\theta}\right| m^{-1 / 2} \theta^{-1 / 2} w^{-1 / 2} \mathbf{p}_{2}^{-1 / 2} \delta^{n-1 / 2}, \tag{79}
\end{equation*}
$$

where $\Omega$ denotes the region $\Omega=\left\{x \in \mathbb{R}^{n}: \delta^{1 / N} \lesssim \operatorname{dist}\left(x, l_{R}\right) \lesssim 1\right\}$. We now foliate the hairbrush into three-dimensional regions in order to apply Lemma 4.1.

Let $S^{n-3}$ denote the portion of the unit sphere $S^{n-1}$ in $\mathbb{R}^{n}$ which is orthogonal to $\pi_{R}$, and let $\Gamma$ be a maximal $C^{-1} \delta$-separated subset of $S^{n-3}$. For each $\omega \in \Xi$, let $V_{\omega}$ denote the set

$$
V_{\omega}=\pi_{R}+\mathbb{R} \omega+B^{n}(0, C \delta),
$$

these sets are $C \delta$-neighbourhoods of 3 -spaces. From elementary geometry we may cover

$$
\mathcal{A}_{\text {brush }}^{\theta}=\bigcup_{\omega \in \Gamma} \mathcal{A}_{\text {brush }}^{\theta, \omega},
$$

where $\mathcal{A}_{\text {brush }}^{\theta, \omega}=\left\{l \in \mathcal{A}_{\text {brush }}^{\theta}: T_{l} \subset V_{\omega}\right\}$. The sets $V_{\omega} \cap \Omega$ have an overlap of at most $O\left(\delta^{-C / N}\right)$ as $\omega$ varies. Thus

$$
\left|E^{(1)} \cap S \cap \Omega\right| \gtrsim \delta^{C / N} \sum_{\omega \in \Gamma}\left|E^{(1)} \cap S \cap V_{\omega} \cap \Omega\right| .
$$

To show (79), it thus suffices to show that

$$
\begin{align*}
\mid E^{(1)} \cap S & \cap V_{\omega} \cap \Omega \mid \\
& \gtrsim \delta^{C / N} \lambda^{2}\left|\mathcal{A}_{\text {brush }}^{\theta, \omega}\right| m^{-1 / 2} \theta^{-1 / 2} w^{-1 / 2} \mathbf{p}_{2}^{-1 / 2} \delta^{n-1 / 2}, \tag{80}
\end{align*}
$$

for each $\omega \in \Gamma$.
Fix $\omega$. The region $S \cap V_{\omega} \cap \Omega$ is essentially a $C \times C \times C \theta \times C \delta \times$ $\cdots \times C \delta$ box. We cover this box by about $w^{-1}$ smaller boxes $B_{\alpha}$ of dimensions $C \times C \times C w \theta \times C \delta \times \cdots \times C \delta$ such that $l_{R}$ is contained in the plane generated by the first two directions of this box. Note that $w \theta \gtrsim \delta$ from the construction of $\theta$. From elementary geometry we see that for each $l \in \mathcal{A}_{\mathrm{brush}}^{\theta, \omega}$ there exists a box $B_{\alpha}$ such that $T_{l} \subset B_{\alpha}$. Also, the boxes $B_{\alpha}$ have an overlap of $O\left(\delta^{-C / N}\right)$. Thus, by the same argument as before, it suffices to show that

$$
\begin{equation*}
\left|E^{(1)} \cap B_{\alpha}\right| \gtrsim \delta^{C / N} \lambda^{2}\left|\mathcal{A}_{\text {brush }}^{\theta, \omega, \alpha}\right| m^{-1 / 2} \theta^{-1 / 2} w^{-1 / 2} \mathbf{p}_{2}^{-1 / 2} \delta^{n-1 / 2}, \tag{81}
\end{equation*}
$$

where $\mathcal{A}_{\text {brush }}^{\theta, \omega, \alpha}=\left\{l \in \mathcal{A}_{\text {brush }}^{\theta, \omega}: T_{l} \subset B_{\alpha}\right\}$. From (57), (58) and elementary geometry we note that

$$
\left|T_{l} \cap E^{(1)} \cap B_{\alpha}\right| \approx \lambda \delta^{n-1}, \quad \text { for all } l \in \mathcal{A}_{\text {brush }}^{\theta, \omega, \alpha} .
$$

Also, from elementary geometry we see that the set of directions $\{v(l)$ : $\left.l \in \mathcal{A}_{\text {brush }}^{\theta, \omega, \alpha}\right\}$ is contained in a $C \times C w \theta \times C \delta \times \cdots \times C \delta$ box in $B^{n-1}(0,1)$. The claim (81) now follows from Lemma 4.1, and we are done.

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# Dynamical instability of symmetric vortices 

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#### Abstract

Using the Maxwell-Higgs model, we prove that linearly unstable symmetric vortices in the Ginzburg-Landau theory are dynamically unstable in the $H^{1}$ norm (which is the natural norm for the problem).


In this work we study the dynamic instability of the radial solutions of the Ginzburg-Landau equations in $\mathbb{R}^{2}$,

$$
\left\{\begin{array}{l}
\operatorname{curl}^{2} A+\frac{i}{2}(\bar{\phi} D \phi-\phi \overline{D \phi})=0  \tag{1}\\
-D^{2} \phi+\frac{\lambda}{2}\left(|\phi|^{2}-1\right) \phi=0
\end{array}\right.
$$

where $\phi: \mathbb{R}^{2} \longrightarrow \mathbb{C}$ is the Higgs field, or condensed wave function ( $|\phi|^{2}$ is proportional to the local density of Cooper pairs), and $A$ is the gauge potential 1-form (it can also be seen as the vector potential of the magnetic field). The covariant derivative is $D \phi=\nabla \phi-i A \phi$, with $i=\sqrt{-1}$. The electric field is absent in the stationary model, and $H=\operatorname{curl} A$ is the magnetic field. The dimensionless coupling constant $\lambda$ is positive, $\lambda<1$ corresponding to superconductors of type I and $\lambda>1$ to those of type II.

Solutions of (1) are critical points of the Helmholtz free energy associated to the Ginzburg-Landau model, which we may write as

$$
\begin{equation*}
\mathcal{E}=\int_{\mathbb{R}^{2}}\left(\frac{1}{2}|\operatorname{curl} A|^{2}+\frac{1}{2}|D \phi|^{2}+\frac{\lambda}{8}\left(|\phi|^{2}-1\right)^{2}\right) . \tag{2}
\end{equation*}
$$

There is a vortex number (charge) associated with every finite energy solution of (1). It can be defined as

$$
n:=\frac{1}{2 \pi} \int_{\mathbb{R}^{2}} H=\frac{1}{2 \pi} \lim _{N \rightarrow \infty} \int_{|x|=N} A d x .
$$

This number, which is always an integer, has a topological meaning it is the winding number of the Higgs field $\phi$ (see, for instance, [7]).

In the early seventies, Nielson and Olesen ([8]) interpreted the finite energy solutions of (1) as string-like field configurations and, soon after, a family of topologically non-trivial solutions (one for every integer value of the topological degree $n$ and positive real value of the parameter $\lambda$ ) was constructed mathematically by Berger and Chen ([2]) and Plohr ([9]). In polar coordinates $(r, \theta) \in \mathbb{R}^{2}$ these radial solutions $(a, \eta)$ are of the form

$$
\begin{equation*}
a(r, \theta)=a_{1} d x_{1}+a_{2} d x_{2}=n S(r) d \theta, \quad \eta(r, \theta)=R(r) e^{i n \theta} \tag{3}
\end{equation*}
$$

where $n \in \mathbb{Z}$ and $\lambda>0$ are arbitrary. Here $r=\sqrt{\left(x^{1}\right)^{2}+\left(x^{2}\right)^{2}}$ and $\theta=\tan ^{-1}\left(x^{2} / x^{1}\right)$.

For studying the dynamics, one should consider the action on the Minkowski space-time $\mathbb{R}^{1+2}$ (we add the time coordinate, $t$, which will also be denoted by $x^{0}$ ), with a metric $g^{\mu \nu}, \mu, \nu=0,1,2$, with signature $(-,+,+)$. The action is then given by (see [7, 1.9.a) and b)])

$$
\mathcal{A}=\frac{1}{2} \int_{\mathbb{R}^{1+2}}\left(\left|F_{12}\right|^{2}-\left|D_{0} \phi\right|^{2}+\sum_{j=1}^{2}\left(\left|D_{j} \phi\right|^{2}-\left|F_{0 j}\right|^{2}\right)+\frac{\lambda}{4}\left(|\phi|^{2}-1\right)^{2}\right)
$$

$$
\begin{equation*}
=\frac{1}{2} \int_{\mathbb{R}^{1+2}}\left(g^{i j} g^{k l} F_{i k} F_{j l}+g^{i j} D_{i} \phi \overline{D_{j} \phi}+\frac{\lambda}{4}\left(|\phi|^{2}-1\right)^{2}\right), \tag{4}
\end{equation*}
$$

where, in the last expression we used the Einstein convention for summing over repeated indices (we will continue to do so below). Here, $A$ is now the electro-magnetic potential - it has also an electric potential component $A_{0}$. We denote partial derivatives by $\partial_{\mu}=\partial_{x^{\mu}}$, for $\mu=0,1,2$. Then, since $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$, for $0 \leq \mu, \nu \leq 2$, the electric field is given by $F_{0 j}, j=1,2$, and $-F_{12}$ is the magnetic field. We denoted covariant derivatives with respect to space or time variables by $D_{\mu}=\partial_{\mu}-i A_{\mu}$. As usual, we will raise and lower indices by using the metric $g$. For instance, $D^{i}:=g^{i j} D_{j}$, and thus $D^{0}=-D_{0}$, while $D^{1}=D_{1}$.

Dynamical stability is then investigated using the Maxwell-Higgs system, which can be written as

$$
\left\{\begin{array}{l}
\partial^{\mu} F_{\mu \nu}=-j_{\nu}  \tag{5}\\
D_{\mu} D^{\mu} \phi+\frac{\lambda}{2}\left(|\phi|^{2}-1\right) \phi=0
\end{array}\right.
$$

The charge and current densities are given by

$$
j_{\nu}=\operatorname{Im}\left(\phi \overline{D_{\nu} \phi}\right)=-\frac{i}{2}\left(\phi \overline{D_{\nu} \phi}-\bar{\phi} D_{\nu} \phi\right),
$$

for $\mu=0,1,2$, and the conserved energy is

$$
\frac{1}{2} \int_{\mathbb{R}^{2}}\left(\left|F_{\mu \nu}\right|^{2}+\left|D_{\mu} \phi\right|^{2}+\frac{\lambda}{4}\left(|\phi|^{2}-1\right)^{2}\right) d x^{1} d x^{2}
$$

The Maxwell-Higgs model is invariant under a gauge transformation

$$
\left\{\begin{array}{l}
A_{\mu} \longrightarrow A_{\mu}+\partial_{\mu} \chi \\
\phi \rightarrow e^{i \chi} \phi
\end{array}\right.
$$

We will work under the temporal gauge condition $A_{0} \equiv 0$, and thus we just need to consider the variations of the spatial components of $A$ (we will be back to working with a 2 -dimensional $A$, with real components $A_{1}$ and $A_{2}$, and a $\mathbb{C} \simeq \mathbb{R}^{2}$ valued Higgs field $\left.\phi\right)$. Let $v=(W, \psi)^{T} \in \mathbb{R}^{4}$ be a perturbation of the radially symmetric vortex $(a, \eta)$, where $W=$ $A-a$ and $\psi=\phi-\eta$. The full nonlinear Maxwell-Higgs system, in terms of $v$, can be written as

$$
\left\{\begin{array}{l}
\partial_{t}\left(\partial_{1} W_{1}+\partial_{2} W_{2}\right)-\frac{i}{2} \partial_{t}(\eta \bar{\psi}-\bar{\eta} \psi)=\frac{i}{2}\left(\psi \partial_{t} \bar{\psi}-\bar{\psi} \partial_{t} \psi\right),  \tag{6}\\
\frac{d^{2} v}{d t^{2}}+\mathcal{E}^{\prime \prime}{ }_{(a, \eta)} v=N(v) .
\end{array}\right.
$$

Here, $\mathcal{E}^{\prime \prime}{ }_{(a, \eta)}$ denotes the second order variation of (2) around the vortices $(a, \eta)$, and the nonlinear term $N(v)$ is equal to

$$
\left(\begin{array}{c}
\frac{i}{2}\left(\psi \partial_{k} \bar{\psi}-\bar{\psi} \partial_{k} \psi\right)-W_{k}|\psi|^{2}-W_{k}(\eta \bar{\psi}+\bar{\eta} \psi)-a_{k}|\psi|^{2} \\
-i W_{j} \partial_{j} \psi-i \partial_{j}\left(W_{j} \psi\right)-W_{j}^{2} \psi-2 a_{j} W_{j} \psi \\
-\eta W_{j}^{2}-\frac{\lambda}{2}\left(|\psi|^{2}(\psi+2 \eta)+\psi^{2} \bar{\eta}\right)
\end{array}\right)
$$

where $1 \leq k \leq 2$, and we have an implicit sum over $1 \leq j \leq 2$.
Ever since the construction of these stationary radially symmetric vortices, their stability against initial perturbations with the same charge has been an interesting problem for both mathematicians and physicists. In a classical paper of [3], the question of stability was addressed by numerical and formal analysis. The study of the linear operator $\mathcal{E}^{\prime \prime}{ }_{(a, \eta)}$ indicates that for $\lambda \leq 1$ and all charges $n$ the vortices are linearly stable. On the other hand, for $\lambda>1$ and $|n| \geq 2$, the vortices are linearly unstable. Unlike in the finite dimensional dynamical system, the passage form linear growing modes to a genuine nonlinear instability in an infinite dimensional partial differential equation is quite delicate. This is due to the possible presence of the continuous spectrum for the linearized operator and to severe high order perturbations arising from the nonlinearity.

In previous works ([4] and [1]), the dynamical instability of vortices with large coupling constant $\lambda$ was proven in the norm

$$
\|f\|_{X}=\|f\|_{H^{1}\left(\mathbb{R}^{2}\right)}+\|f\|_{\infty}
$$

The $\|\cdot\|_{\infty}$ was needed to control the $H^{2}$ growth estimate.
In this work, we improve the passage from linear instability to nonlinear dynamical instability in the more natural $H^{1}$ norm by a refined bootstrap argument. Let the initial perturbation be of the order $\delta$. Within a time-interval of the order of $|\ln \delta|$ we can estimate the $H^{2}$ norm of the perturbation only by its $H^{1}$ norm (without any extra assumptions on its $L^{\infty}$ norm). A similar argument has also been used in [5].

In fact, in Theorem 1, we show that if the linear operator $\mathcal{E}^{\prime \prime}{ }_{(a, \eta)}$ has a negative direction, then the vortex is dynamically unstable in $H^{1}$ norm.

For given positive constants $\Omega$ and $\varepsilon_{0}$, and for any small parameter $\delta>0$, we define the associated escape time $T^{\delta}$ by

$$
\begin{equation*}
\delta e^{\Omega T^{\delta}}=\varepsilon_{0} \tag{7}
\end{equation*}
$$

For a fixed appropriately chosen $\varepsilon_{0}$, as $\delta \longrightarrow 0$ the dynamical instability will occur within $0 \leq t \leq T^{\delta}$.

Lemma 1. Let $v(t)$ be a solution of the full Maxwell-Higgs system (6). Assume

$$
\begin{gather*}
\|v(0)\|_{H^{2}}+\left\|v_{t}(0)\right\|_{H^{1}} \leq C_{0} \delta  \tag{8}\\
\|v(t)\|_{H^{1}}+\left\|v_{t}(t)\right\|_{2} \leq C_{0} e^{\Omega t} \delta \tag{9}
\end{gather*}
$$

for $0 \leq t \leq T$, where $\Omega>0$ and $C_{0}$ is independent of $t$. Then, there exist $C_{1}, \varepsilon_{0}>0$ such that if $0 \leq t \leq \min \left\{T, T^{\delta}\right\}$ then

$$
\begin{equation*}
\|v(t)\|_{H^{2}}+\left\|v_{t}(t)\right\|_{H^{1}} \leq C_{1} \delta e^{\Omega t} \leq C_{1} \varepsilon_{0}, \tag{10}
\end{equation*}
$$

where $T^{\delta}$ is defined in (7).
Proof. We shall estimate $\|v\|_{H^{2}}$ in terms of $\|v\|_{H^{1}}$ by energy type estimates. Taking one spatial derivative $\partial_{l}=\partial_{x^{l}}$ through both equations in (6) we obtain

$$
\left\{\begin{align*}
\partial_{t}\left(\partial_{1} \partial_{l} W_{1}\right. & \left.+\partial_{2} \partial_{l} W_{2}\right)-\frac{i}{2} \partial_{t}\left(\eta \partial_{l} \bar{\psi}-\bar{\eta} \partial_{l} \psi\right)  \tag{11}\\
& =\frac{i}{2} \partial_{t}\left(\partial_{l} \eta \bar{\psi}-\partial_{l} \bar{\eta} \psi\right)+\frac{i}{2} \partial_{l}\left(\psi \partial_{t} \bar{\psi}-\bar{\psi} \partial_{t} \psi\right), \\
\frac{d^{2}}{d t^{2}}\left(\partial_{l} v\right) & -L\left(\partial_{l} v\right)=L_{1}(v)+\partial_{l} N(v) .
\end{align*}\right.
$$

Here $L_{1}(v)$ is

$$
\left(\begin{array}{c}
\frac{i}{2}\left(\partial_{l} \eta \partial_{k} \bar{\psi}+\psi \partial_{k l} \bar{\eta}-\partial_{l} \bar{\eta} \partial_{k} \psi-\bar{\psi} \partial_{k l} \eta\right) \\
-\partial_{l} a_{k}(\eta \bar{\psi}+\bar{\eta} \psi)-a_{k}\left(\partial_{l} \bar{\psi}+\partial_{l} \bar{\eta} \psi\right)-W_{k} \partial_{l}|\eta|^{2}, \\
-2 i \partial_{l} a_{j} \partial_{j} \psi-2 i W_{j} \partial_{j l} \mid \eta-i \partial_{j} W_{j} \partial_{l} \eta-\partial_{l}\left(|a|^{2}+\lambda|\eta|^{2}\right) \psi \\
-2 W_{j} \partial_{l}\left(a_{j} \eta\right)-\frac{\lambda}{2} \partial_{l}\left(\eta^{2}\right) \bar{\psi}
\end{array}\right) .
$$

Define $y(t):=\|v(t)\|_{H^{2}}^{2}+\left\|v_{t}(t)\right\|_{H^{1}}^{2}$. Using estimate (3.18) in [4] with sufficiently small $\varepsilon$, we have that

$$
\begin{align*}
y(t) \leq & \left(C\left(\|v\|_{X}+\|v\|_{X}^{2}\right)+\frac{3 \Omega}{16}\right) \int_{0}^{t} y(\tau) d \tau \\
& +C_{\Omega} \int_{0}^{t}\left(\|v\|_{H^{1}}^{2}+\left\|v_{t}\right\|_{2}^{2}\right) d \tau  \tag{12}\\
& +C\|v(t)\|_{H^{1}}^{2}+C\left(\|v(0)\|_{H^{2}}^{2}+\left\|v_{t}(0)\right\|_{H^{1}}^{2}\right) .
\end{align*}
$$

We notice that due to typographical errors, the square was omitted in (3.18) for both $\|v(t)\|_{H^{1}}$ and the initial data $\|v(0)\|_{H^{2}}+\left\|v_{t}(0)\right\|_{H^{1}}$.

Since $\|v\|_{X} \leq C\|v\|_{H^{2}}$,

$$
\begin{align*}
y(t) \leq & \left(\bar{C}\left(\|v\|_{H^{2}}+\|v\|_{H^{2}}^{2}\right)+\frac{3 \Omega}{16}\right) \int_{0}^{t} y(\tau) d \tau \\
& +C_{\Omega} \int_{0}^{t}\left(\|v\|_{H^{1}}^{2}+\left\|v_{t}\right\|_{2}^{2}\right) d \tau  \tag{13}\\
& +C\|v(t)\|_{H^{1}}^{2}+C\left(\|v(0)\|_{H^{2}}^{2}+\left\|v_{t}(0)\right\|_{H^{1}}^{2}\right) .
\end{align*}
$$

We now define

$$
\begin{align*}
& T^{*}=\sup \{t: \text { for all } s \in[0, t] \\
& \left.\qquad\|v(s)\|_{H^{2}}+\left\|v_{t}(s)\right\|_{H^{1}} \leq \min \left\{\frac{\Omega}{8 \bar{C}}, 1\right\}\right\} \tag{14}
\end{align*}
$$

For $0 \leq t \leq \min \left\{T, T^{*}\right\}$, since $\|v(t)\|_{H^{2}} \leq 1$, we have

$$
\bar{C}\left(\|v\|_{H^{2}}+\|v\|_{H^{2}}^{2}\right) \leq \frac{\Omega}{4} .
$$

Moreover, from (9),

$$
\begin{aligned}
& C_{\Omega} \int_{0}^{t}\left(\|v\|_{H^{1}}^{2}+\left\|v_{t}\right\|_{2}^{2}\right) d \tau+C\|v(t)\|_{H^{1}}^{2} \\
& \leq C_{\Omega} \int_{0}^{t}\left(C_{0} \delta e^{\Omega \tau}\right)^{2} d \tau+C \delta^{2} e^{2 \Omega t} \\
& \leq C \delta^{2} e^{2 \Omega t}
\end{aligned}
$$

Therefore, using (8), we obtain from (13)

$$
\begin{equation*}
y(t) \leq \frac{\Omega}{2} \int_{0}^{t} y(\tau) d \tau+C \delta^{2} e^{2 \Omega t} \tag{15}
\end{equation*}
$$

Now, proceeding as in the proof of the Gronwall inequality, we deduce that

$$
\left(e^{-(\Omega / 2) t} \int_{0}^{t} y(\tau) d \tau\right)^{\prime} \leq C \delta^{2} e^{2 \Omega t-(1 / 2) \Omega t}=C \delta^{2} e^{(3 / 2) \Omega t}
$$

Integrating over $t$, we obtain

$$
e^{-(\Omega / 2) t} \int_{0}^{t} y(\tau) d \tau \leq C \delta^{2} \int_{0}^{t} e^{(3 / 2) \Omega s} d s=C \delta^{2} e^{(3 / 2) \Omega t}
$$

Therefore,

$$
\int_{0}^{t} y(\tau) d \tau \leq C \delta^{2} e^{2 \Omega t}
$$

And plugging this into (15) yields

$$
\begin{equation*}
\|v(t)\|_{H^{2}}+\left\|v_{t}(t)\right\|_{H^{1}} \leq C_{1} \delta e^{\Omega t} \tag{16}
\end{equation*}
$$

for $0 \leq t \leq \min \left\{T, T^{*}\right\}$, where $C_{1}$ is some fixed constant which depends on $\Omega$ and $C_{0}$, but is independent of $\delta$.

We now define $T^{\delta}$ as in (7), choosing $\varepsilon_{0}$ such that

$$
\begin{equation*}
C_{1} \varepsilon_{0}<\min \left\{\frac{\Omega}{8 \bar{C}}, 1\right\} \tag{17}
\end{equation*}
$$

Then, if $T^{\delta} \leq \min \left\{T, T^{*}\right\}$, clearly the lemma follows. On the other hand, if $T^{\delta}>\min \left\{T, T^{*}\right\}$, we claim that $T \leq T^{*}$. It thus follows that $\min \left\{T, T^{*}\right\}=T$ and, once more, the lemma follows easily.

To prove $T \leq T^{*}$, we argue by contradiction. If not, we would have $T>T^{*}$ and therefore $\min \left\{T, T^{*}\right\}=T^{*}$. Letting $t=T^{*}$ in (16) would yield

$$
\left\|v\left(T^{*}\right)\right\|_{H^{2}}+\left\|v_{t}\left(T^{*}\right)\right\|_{H^{1}} \leq C_{1} \delta e^{\Omega T^{*}}<C_{1} \delta e^{\Omega T^{\delta}}=C_{1} \varepsilon_{0}
$$

by the definition of $T^{\delta}$. However, this is impossible by the choice (17) since it would contradict the definition of $T^{*}$ in (14).

Now, we may prove our main result.
Theorem 1. Let $(a, \eta)$ be a vortex such that

$$
\begin{equation*}
\left\langle\mathcal{E}^{\prime \prime}{ }_{(a, \eta)}\left(v_{1}\right), v_{1}\right\rangle<0, \tag{18}
\end{equation*}
$$

for some $v_{1} \in H^{1}\left(\mathbb{R}^{2}\right)$. Then, there exist constants $\varepsilon_{0}>0, C>0$, so that for any small $\delta>0$ there exists a family of solutions $v^{\delta}(t)$ of the Maxwell-Higgs system (6) such that the vortex number of $W^{\delta}(0)$ is zero, and

$$
\left\|v^{\delta}(0)\right\|_{H^{2}}+\left\|v_{t}^{\delta}(0)\right\|_{H^{1}} \leq C \delta
$$

but

$$
\sup _{\{0 \leq t \leq C|\ln \delta|\}}\left\|v^{\delta}(t)\right\|_{H^{1}}+\left\|v_{t}^{\delta}(t)\right\|_{L^{2}} \geq \frac{\varepsilon_{0}}{2} .
$$

Proof. By [4, Theorem 1.3], there exists a dominant growing mode $v_{0} e^{\omega t}$ of the linearized Maxwell-Higgs system with $\omega>0$ and $v_{0} \in$ $H^{2}\left(\mathbb{R}^{2}\right)$. We normalize $v_{0}$ such that

$$
\begin{equation*}
\left\|v_{0}\right\|_{H^{1}}+\left\|\omega v_{0}\right\|_{L^{2}}=1 . \tag{19}
\end{equation*}
$$

Moreover, we assume that

$$
\left\|v_{0}\right\|_{H^{2}}+\left\|\omega v_{0}\right\|_{H^{1}}=r<\infty .
$$

Now we solve the Maxwell-Higgs system with a family of initial data $\left.v\right|_{t=0}=\delta v_{0}$ and $\left.v_{t}\right|_{t=0}=\delta \omega v_{0}$. Notice that the vortex number (charge) of $a+W$ is the same as that of $a$. We denote the corresponding $H^{2}$ solutions by $v^{\delta}(t)$. They can be written as

$$
\begin{equation*}
v^{\delta}(t)=\delta e^{\omega t} v_{0}+\int_{0}^{t} \mathcal{L}(t-\tau) \mathcal{N}\left(v^{\delta}\right) d \tau \tag{20}
\end{equation*}
$$

where $\mathcal{L}$ is the solution operator for the linearized Maxwell-Higgs system, and

$$
\mathcal{N}\left(v^{\delta}\right)=\left(\frac{i}{2}\left(\psi^{\delta} \partial_{t} \bar{\psi}^{\delta}-\bar{\psi}^{\delta} \partial_{t} \psi^{\delta}\right), N\left(v^{\delta}\right)\right) .
$$

Let $\omega \leq \Omega<2 \omega$, and

$$
T=\sup \{s: \text { for all } t \in[0, s],
$$

$$
\begin{equation*}
\left.\left\|v^{\delta}(t)-\delta v_{0} e^{\omega t}\right\|_{H^{1}}+\left\|v_{t}^{\delta}(t)-\omega \delta v_{0} e^{\omega t}\right\|_{2} \leq \frac{1}{2} \delta e^{\omega t}\right\} . \tag{21}
\end{equation*}
$$

Using the triangle inequality, we see that for $0 \leq t \leq T$,

$$
\begin{aligned}
\left\|v^{\delta}(t)\right\|_{H^{1}}+\left\|v_{t}^{\delta}(t)\right\|_{2} \leq & \left\|\delta v_{0} e^{\omega t}\right\|_{H^{1}}+\left\|\omega \delta v_{0} e^{\omega t}\right\|_{2} \\
& +\left\|v^{\delta}(t)-\delta v_{0} e^{\omega t}\right\|_{H^{1}}+\left\|v_{t}^{\delta}(t)-\omega \delta v_{0} e^{\omega t}\right\|_{2} \\
\leq & \frac{3}{2} \delta e^{\omega t}\left(\left\|v_{0}\right\|_{H^{1}}+\left\|\omega v_{0}\right\|_{2}\right) \\
= & \frac{3}{2} e^{\omega t} \delta .
\end{aligned}
$$

For any $\varepsilon>0$, from the sharp linear estimate for $\mathcal{L}$ as in [4, Theorems 1.3 and 1.5], we know that the solutions of the linearized GinzburgLandau equation grow no faster than $e^{(\omega+\varepsilon) t}$. By Lemma 1 with $\Omega=\omega$,
$C_{0}=\max \{3 / 2, r\}$, there exist constants $C_{1}, \varepsilon_{0}>0$ such that for $0 \leq t \leq \min \left\{T, T^{\delta}\right\}$,

$$
\left\|v^{\delta}(t)\right\|_{H^{2}}+\left\|v_{t}^{\delta}(t)\right\|_{2} \leq C_{1} \delta e^{\omega t} \leq C_{1} \varepsilon_{0} .
$$

From the Sobolev embedding,

$$
\left\|v^{\delta}(t)\right\|_{\infty} \leq C\left\|v^{\delta}(t)\right\|_{H^{2}} \leq C \delta e^{\omega t} \leq C C_{1} \varepsilon_{0},
$$

for $0 \leq t \leq \min \left\{T, T^{\delta}\right\}$, where $\delta e^{\omega T^{\delta}}=\varepsilon_{0}$. From the linearized estimate (20) together with the linear estimate in [4, Theorem 1.5], for $0 \leq t \leq \min \left\{T, T^{\delta}\right\}$,

$$
\begin{align*}
& \left\|v^{\delta}(t)-\delta e^{\omega t} v_{0}\right\|_{H^{1}}+\left\|v_{t}^{\delta}(t)-\delta \omega e^{\omega t} v_{0}\right\|_{2} \\
& \leq C \int_{0}^{t} e^{(3 / 2) \omega(t-\tau)}\left(\left\|\frac{i}{2}\left(\psi^{\delta} \partial_{t} \bar{\psi}^{\delta}-\bar{\psi}^{\delta} \partial_{t} \psi^{\delta}\right)\right\|_{2}+\left\|N\left(v^{\delta}\right)\right\|_{2}\right) d \tau \\
& \leq C \int_{0}^{t} e^{(3 / 2) \omega(t-\tau)}\left(\left\|v^{\delta}(\tau)\right\|_{\infty}\left\|v^{\delta}(\tau)\right\|_{H^{1}}+\left\|v^{\delta}(\tau)\right\|_{\infty}^{2}\left\|v^{\delta}(\tau)\right\|_{2}\right) d \tau \\
& \leq C \int_{0}^{t} e^{(3 / 2) \omega(t-\tau)}\left\|v^{\delta}(\tau)\right\|_{\infty}\left\|v^{\delta}(\tau)\right\|_{H^{1}} d \tau \tag{23}
\end{align*}
$$

$$
\begin{aligned}
& \leq C \int_{0}^{t} e^{(3 / 2) \omega(t-\tau)}\left(\delta e^{\omega \tau}\right)\left(\delta e^{\omega \tau}\right) d \tau \\
& =C \delta^{2} e^{(3 / 2) \omega t} \int_{0}^{t} e^{(1 / 2) \omega \tau} d \tau \\
& \leq C_{2}\left(\delta e^{\omega t}\right)^{2},
\end{aligned}
$$

where $C_{2}$ is a constant.
Notice that both Lemma 1 and (23) remain valid with the same constants $C_{1}$ and $C_{2}$, respectively, for all smaller $\varepsilon_{0}$, as long as we take the corresponding $T^{\delta}$ in (7). In particular, if necessary we can fix $\varepsilon_{0}$ sufficiently small so that

$$
\begin{equation*}
C_{2} \varepsilon_{0} \leq \frac{1}{2} \tag{24}
\end{equation*}
$$

Now, clearly $T^{\delta} \leq T$. Otherwise, we would have $T<T^{\delta}$, and from (19), (23), (22) and (24) it would follow that

$$
\left\|v^{\delta}(T)\right\|_{H^{1}}+\left\|v_{t}^{\delta}(T)\right\|_{2} \leq \delta e^{\omega T}+\left(C_{2} \delta e^{\omega T^{\delta}}\right) \delta e^{\omega T}<\frac{3}{2} \delta e^{\omega T}
$$

which would contradict the definition of $T$ in (21).
Once we know $T^{\delta} \leq T$, using (7), (19) and (23) again, we see that

$$
\begin{aligned}
\left\|v^{\delta}\left(T^{\delta}\right)\right\|_{H^{1}}+\left\|v_{t}^{\delta}\left(T^{\delta}\right)\right\|_{2} \geq & \delta e^{\omega T^{\delta}}\left(\left\|v_{0}\right\|_{H^{1}}+\left\|\omega v_{0}\right\|_{2}\right) \\
& -\left\|v^{\delta}\left(T^{\delta}\right)-\delta v_{0} e^{\omega T^{\delta}}\right\|_{H^{1}} \\
& -\left\|v_{t}^{\delta}\left(T^{\delta}\right)-\omega \delta v_{0} e^{\omega T^{\delta}}\right\|_{2} \\
\geq & \delta e^{\omega T^{\delta}}-C_{2} \varepsilon_{0} \delta e^{\omega T^{\delta}} \\
\geq & \frac{\varepsilon_{0}}{2} \\
> & 0
\end{aligned}
$$

and the proof is complete.
Remark. In [4] and [1], we had already proved that when $|n|>1$, such negative directions exist for large $\lambda$. Very recently, Gustafson and Sigal ([6]) proved that for all $\lambda>1$ and $|n|>1$, there exists a $v_{1}$ such that $\left\langle\mathcal{E}^{\prime \prime}{ }_{(a, \eta)}\left(v_{1}\right), v_{1}\right\rangle<0$. Thus, we obtain as an easy corollary of Theorem 1, the dynamic instability of symmetric vortices when $|n|>1$ and $\lambda>1$.

Corollary 1. For all $\lambda>1$, radially symmetric solutions $(a, \eta)$ of Ginzburg-Landau equations (i.e. solutions of (1) of the form (3)) with $|n|>1$, are dynamically unstable in $H^{1}$ norm, in the sense of Theorem 1.

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# Convexity and uniqueness in a free boundary problem arising in combustion theory 

## Arshak Petrosyan

Abstract. We consider solutions to a free boundary problem for the heat equation, describing the propagation of flames. Suppose there is a bounded domain $\Omega \subset Q_{T}=\mathbb{R}^{n} \times(0, T)$ for some $T>0$ and a function $u>0$ in $\Omega$ such that

$$
\begin{aligned}
u_{t} & =\Delta u, \\
u=0 \text { and }|\nabla u| & =1, \\
u(\cdot, 0)=u_{0}, & \text { on } \Gamma:=\partial \Omega \cap Q_{T}, \\
& \text { on } \bar{\Omega}_{0},
\end{aligned}
$$

where $\Omega_{0}$ is a given domain in $\mathbb{R}^{n}$ and $u_{0}$ is a positive and continuous function in $\Omega_{0}$, vanishing on $\partial \Omega_{0}$. If $\Omega_{0}$ is convex and $u_{0}$ is concave in $\Omega_{0}$, then we show that $(u, \Omega)$ is unique and the time sections $\Omega_{t}$ are convex for every $t \in(0, T)$, provided the free boundary $\Gamma$ is locally the graph of a Lipschitz function and the fixed gradient condition is understood in the classical sense.

## 1. Introduction and main result.

In this paper we consider solutions to a free boundary problem for the heat equation. Suppose there is a domain $\Omega \subset Q_{T}:=\mathbb{R}^{n} \times(0, T)$
for some $T>0$ and a positive smooth function $u$ in $\Omega$ such that

$$
\begin{gather*}
u_{t}=\Delta u, \quad \text { in } \Omega,  \tag{1}\\
u=0 \text { and }|\nabla u|=1, \quad \text { on } \Gamma,  \tag{2}\\
u(\cdot, 0)=u_{0}, \quad \text { on } \bar{\Omega}_{0}, \tag{3}
\end{gather*}
$$

where $\Gamma:=\partial \Omega \cap Q_{T}$ is the (free) lateral boundary of $\Omega, \Omega_{0} \subset \mathbb{R}^{n}$ is the initial domain and $u_{0}$ is a prescribed positive continuous function in $\Omega_{0}$, that vanishes continuously on $\Gamma_{0}:=\partial \Omega_{0}$. Then we say the pair $(u, \Omega)$ or, when there is no ambiguity, $\Omega$ to be a solution to problem $(\mathbf{P})$. This problem, in mathematical framework, was introduced by L. A. Caffarelli and J. L. Vázquez [CV]. It describes propagation of socalled premixed equi-diffusional flames in the limit of high activation energy. In this problem the time sections

$$
\begin{equation*}
\Omega_{t}=\left\{x \in \mathbb{R}^{n}:(x, t) \in \Omega\right\} \tag{4}
\end{equation*}
$$

represent the unburnt (fresh) zone in time $t, \Gamma_{t}:=\partial \Omega_{t}$ corresponds to the flame front, and $u=c\left(T_{c}-T\right)$ is the normalized temperature. For further details in combustion theory we refer to paper [V] of J. L. Vázquez.

The existence of weak solutions to problem ( $\mathbf{P}$ ) as well as their regularity under suitable conditions on the data were established in [CV]. However, we should not expect any uniqueness result unless we impose some special geometrical restrictions. In this paper we study the case when the initial domain $\Omega_{0}$ is bounded and convex, and the initial function $u_{0}$ is concave. Throughout the paper we make the following assumptions concerning solutions $(u, \Omega)$ to problem (P). First, the boundary of $\Omega$ consists of three parts

$$
\begin{equation*}
\partial \Omega=\Omega_{0} \cup \bar{\Gamma} \cup \Omega_{T} \tag{5}
\end{equation*}
$$

where $\Omega_{T}$ is a nonvoid open set in the plane $t=T$. The presence of nonempty $\Omega_{T}$ excludes the extinction phenomenon in time $t \in[0, T]$. This assumption is rather of technical character, that can be avoided with the following simple procedure. Consider the extinction time

$$
\begin{equation*}
T_{\Omega}=\sup \left\{t: \Omega_{t} \neq \varnothing\right\} \tag{6}
\end{equation*}
$$

Then every domain $\Omega^{(\tau)}=\Omega \cap\{0<t<\tau\}, \tau \in\left(0, T_{\Omega}\right)$, has nonempty "upper bound" $\Omega_{\tau}$. Therefore we can consider first $\Omega^{(\tau)}$ instead of $\Omega$ and then let $\tau \longrightarrow T_{\Omega}$.

Next, we assume that for every $\left(x_{0}, t_{0}\right) \in \bar{\Gamma}$ there exists a neighborhood $V$ in $\mathbb{R}^{n} \times \mathbb{R}$ such that (after a suitable rotation of $x$-axes)

$$
\begin{equation*}
\Omega \cap V=\left\{(x, t)=\left(x^{\prime}, x_{n}, t\right): x_{n}>f\left(x^{\prime}, t\right)\right\} \cap V \cap Q_{T}, \tag{7}
\end{equation*}
$$

where $f$ is a Lipschitz function, defined in

$$
V^{\prime}=\left\{\left(x^{\prime}, t\right): \text { there exists } x_{n} \text { with }\left(x^{\prime}, x_{n}, t\right) \in V\right\} .
$$

Further, for $u$ we assume that it is continuous up to the boundary $\partial \Omega$ and can be extended smoothly through $\Omega_{T}$. The gradient condition in (2) is understood in the classical sense

$$
\begin{equation*}
\lim _{\Omega_{t} \ni y \rightarrow x}|\nabla u(y, t)|=1, \tag{8}
\end{equation*}
$$

for every $x \in \partial \Omega_{t}, 0<t \leq T$.
The main result of this paper is as follows.
Theorem 1. In problem ( $\mathbf{P}$ ) let $\Omega_{0}$ be a bounded convex domain and $u_{0}$ be a concave function in $\Omega_{0}$. Suppose that $(u, \Omega)$ is a solution to this problem in the sense described above. Then $(u, \Omega)$ is a unique solution. Moreover, the time sections $\Omega_{t}$ of $\Omega$ are convex for every $t \in(0, T)$.

The plan of the paper is as follows. In Section 2 we prove a theorem on the convexity of level sets of solutions to a related Dirichlet problem. In Section 3 we recall some properties of caloric functions in Lipschitz domains. And finally in Section 4 we prove Theorem 1.

## 2. Convexity of level sets.

In this section we establish some auxiliary results, which are, however, of independent interest.

Let $u_{0}$ and $\Omega_{0}$ be as in problem ( $\mathbf{P}$ ) and a domain $\Omega \subset Q_{T}$ meets conditions (5) and (7). Then by the Petrowski criterion $[\mathrm{P}] \Omega$ is a regular domain for the Dirichlet problem for the heat equation (in the Perron sense), and its parabolic boundary is given by

$$
\begin{equation*}
\partial_{p} \Omega=\bar{\Omega}_{0} \cup \Gamma . \tag{9}
\end{equation*}
$$

We fix one such domain $\Omega$ and denote by $u$ the solution to the Dirichlet problem

$$
\begin{cases}u_{t}=\Delta u, &  \tag{10}\\ u=u_{0}, & \text { on } \bar{\Omega}_{0}, \\ u=0, & \text { on } \Gamma .\end{cases}
$$

Theorem 2. Let the time sections $\Omega_{t}$ of the domain $\Omega$ be convex for $t \in[0, T]$. Let also $u_{0}$ be a concave function on $\bar{\Omega}_{0}$, positive in $\Omega_{0}$ and vanishing on $\partial \Omega_{0}$. Then the level sets

$$
\begin{equation*}
\mathcal{L}_{s}(u(\cdot, t))=\left\{x \in \Omega_{t}: u(x, t)>s\right\} \tag{11}
\end{equation*}
$$

are convex for every fixed $s>0$ and $t \in(0, T)$, where $u$ is the solution to the Dirichlet problem (10).

The proof is based on the Concavity maximum principle originally due to N . Korevaar [K1] and [K2]. For a function $v$ on $\Omega$ set

$$
\begin{equation*}
\mathcal{C}(x, y, t)=\frac{v(x, t)+v(y, t)}{2}-v\left(\frac{x+y}{2}, t\right) . \tag{12}
\end{equation*}
$$

The function $\mathcal{C}$ is defined on an open subset $D$ of the fiber product

$$
\widetilde{\Omega}=\{(x, y, t):(x, t),(y, t) \in \Omega\}
$$

Note that $D=\widetilde{\Omega}$ if the time sections of $\Omega$ are convex. Note also that if $v$ is extended to the "upper bound" $\Omega_{T}$ of $\Omega$, then $\mathcal{C}$ is extended to the "upper bound" $D_{T}$ of $D$. We denote $\partial_{p} D=\bar{D} \backslash\left(D \cup D_{T}\right)$.

Lemma (Concavity maximum principle). Let $v$ in $C_{x, t}^{2,1}(\Omega) \cap C\left(\Omega \cup \Omega_{T}\right)$ satisfy to a parabolic equation

$$
\begin{equation*}
v_{t}=a^{i j}(t, \nabla v) v_{i j}+b(t, x, v, \nabla v), \quad \text { in } \Omega \tag{13}
\end{equation*}
$$

with smooth coefficients and such that $b$ is nonincreasing in $v$ and jointly concave in $(x, v)$. Then either $\mathcal{C} \leq 0$ in $D \cup D_{T}$ or

$$
0<\sup _{(x, y, t) \in D \cup D_{T}} \mathcal{C}(x, y, t)=\limsup _{(x, y, t) \rightarrow \partial_{p} D} \mathcal{C}(x, y, t) .
$$

Proof. See [K2], the proof of Theorem 1.6. Though the result is proved there for cylindrical domains, the proof is valid also in our case.

Remark. There are several formulations of this principle in the elliptic case. The strongest version states that it is sufficient to require harmonic concavity of $b$ in $(x, v)$ instead of concavity; see B. Kawohl [Ka], and A. Greco and G. Porru [GP]. In the parabolic case, in order to use such an extension, it seems necessary to assume also the nonnegativeness of $v_{t}$; see A. Kennington [Ke].

Proof of Theorem 2. Assume first, that the functions $f$ in the local representations (7) of $\Omega$ are smooth in ( $x^{\prime}, t$ ) and strictly convex in $x^{\prime}$ and that $u_{0}$ is smooth. These assumptions imply the smoothness up to $\partial \Omega$ of the solution $u$ to (10). Also, the positivity of $u_{0}$ implies the positivity of $u$. Define now $v=\log (u)$. We claim then that $v(\cdot, t)$ are concave functions in $\Omega_{t}$ for every $t \in(0, T]$. Clearly, this will imply the statement of the theorem. For this purpose, we consider the concavity function $\mathcal{C}$, defined above, and show that $\mathcal{C} \leq 0$ on $D \cup D_{T}$. Suppose the contrary. Then take a maximizing sequence $\left(x_{k}, y_{k}, t_{k}\right) \in D \cup D_{T}$ such that

$$
\begin{equation*}
\lim \mathcal{C}\left(x_{k}, y_{k}, t_{k}\right)=\sup _{D \cup D_{T}} \mathcal{C}>0 \tag{14}
\end{equation*}
$$

Without loss of generality we may assume that there exists limit

$$
\left(x_{0}, y_{0}, t_{0}\right)=\lim \left(x_{k}, y_{k}, t_{k}\right) .
$$

Direct calculation shows, that $v$ satisfies

$$
\begin{equation*}
v_{t}=\Delta v+|\nabla v|^{2}, \tag{15}
\end{equation*}
$$

in $\Omega$ and hence the Concavity maximum principle is applicable. Hence we may assume $\left(x_{0}, y_{0}, t_{0}\right) \notin D \cup D_{T}$. We want to exclude also the other possibilities. First, the case $t_{0}=0$ and $x_{0}, y_{0} \in \Omega_{0}$ is impossible, since $v(\cdot, 0)=\log \left(u_{0}\right)$ is concave in $\Omega_{0}$. Next, $x_{0} \in \Gamma_{t_{0}}$ but $y_{0} \neq x_{0}$ is also excluded by the strict convexity of $\Omega_{t}$ 's, since then $\mathcal{C}\left(x_{k}, y_{k}, t_{k}\right) \longrightarrow$ $-\infty$. So, it remains to consider the last case $x_{0}=y_{0} \in \Gamma_{t_{0}}$. We observe now that by the boundary point lemma, the outward spatial normal derivatives $u_{\nu}<0$ on $\Gamma \cup \Gamma_{T}$. Besides, $u_{\nu}<0$ also on $\Gamma_{0}$ since $u_{0}$ is concave and positive in $\Omega_{0}$ and vanishes on $\Gamma_{0}$. By the smoothness assumption we have therefore $u_{\nu} \leq-\varepsilon_{0}<0$ on $\bar{\Gamma}$. Hence we can carry
out the same reasonings as in [CS, Proof of Lemma 3.1] (see also [GP, Lemma 3.2]) to obtain that $\lim \inf \mathcal{C}\left(x_{k}, y_{k}, t_{k}\right)<0$, which contradicts (14). Therefore $\mathcal{C} \leq 0$ in $D \cup D_{T}$ and $v(\cdot, t)$ is concave in $\Omega_{t}$ for every $t \in(0, T]$. This proves the theorem in the considering case.

To prove the theorem in the general case, we use approximation of $\Omega$ by domains with smooth lateral boundary and with strictly convex time sections, and relevant smooth concave approximations of $u_{0}$.

## 3. On caloric functions.

In this section we recall some properties of caloric functions in Lipschitz domains. They will be used in the next section, where we prove Theorem 1. The main reference here is the paper [ACS] by I. Athanasopoulos, L. Caffarelli and S. Salsa.

As in the previous section we consider a domain $\Omega$, satisfying conditions (5) and (7). Let also $u$ be the solution to (10). Consider a neighborhood $V$ of a point $\left(x_{0}, t_{0}\right) \in \Gamma$, where (7) holds. The function $u$ vanishes on $\Gamma \cap V$, is positive and satisfies the heat equation in $\Omega \cap V$. In other words, $u$ is caloric.

We start with the following lemma from [ACS], which states that a caloric function $u$ is "almost harmonic" in time sections near the lateral boundary $\Gamma$.

Lemma 4 ([ACS, Lemma 5]). There exist $\varepsilon>0$ and a neighborhood $Q$ of the point $\left(x_{0}, t_{0}\right) \in \Gamma$ such that the functions

$$
\begin{equation*}
w_{+}=u+u^{1+\varepsilon}, \quad w_{-}=u-u^{1+\varepsilon} \tag{16}
\end{equation*}
$$

are respectively sub- and superharmonic in $Q \cap \Omega \cap\left\{t=t_{0}\right\}$.
We will need also the following lemma on asymptotic development of $u$ near the boundary point $\left(x_{0}, t_{0}\right)$.

Lemma 5 [ACS, Lemma 6]. Suppose there exists an $n$-dimensional ball $B \subset \Omega^{c} \cap\left\{t=t_{0}\right\}$ such that $\bar{B} \cap \Gamma=\left\{\left(x_{0}, t_{0}\right)\right\}$. Then near $x_{0}$ in $\Omega_{t_{0}}$

$$
\begin{equation*}
u\left(x, t_{0}\right)=\alpha\left(x-x_{0}, \nu\right)^{+}+o\left(\left|x-x_{0}\right|\right), \tag{17}
\end{equation*}
$$

for some $\alpha \in[0, \infty)$ and where $\nu$ denotes the outward radial direction of $B$ at $\left(x_{0}, t_{0}\right)$.

In the next lemma we show that $\alpha$ in (17) is in fact the nontangential limit of $\left|\nabla u\left(y, t_{0}\right)\right|$ as $y \longrightarrow x_{0}$.

Lemma 6. Under the conditions of Lemma 5, let also $K \subset \Omega_{t_{0}}$ be an $n$-dimensional truncated cone with the vertex at $\left(x_{0}, t_{0}\right)$ such that $\left|x-x_{0}\right| \leq c_{1} \operatorname{dist}\left(x, \Gamma_{t_{0}}\right)$ for every $x \in K$ and some constant $c_{1}$. Then

$$
\begin{equation*}
\lim _{K \ni y \rightarrow x_{0}} \nabla u\left(y, t_{0}\right)=\alpha \nu, \tag{18}
\end{equation*}
$$

where $\alpha$ and $\nu$ are as in the asymptotic development (17).
Proof. By [ACS, Corollary 4], there exists a neighborhood $V$ of the point $\left(x_{0}, t_{0}\right)$ such that

$$
\begin{equation*}
\left|u_{t}(x, t)\right| \leq c_{2} \frac{u(x, t)}{d_{x, t}}, \quad d_{x, t}=\operatorname{dist}\left(x, \Gamma_{t}\right) \tag{19}
\end{equation*}
$$

for all $(x, t) \in V \cap \Omega$. Take an arbitrary sequence $y_{k} \longrightarrow x_{0}, y_{k} \in K$, and consider the functions

$$
\begin{equation*}
v_{k}(z)=\frac{u\left(y_{k}+r_{k} z, t_{0}\right)}{r_{k}}, \quad r_{k}=\left|y_{k}-x_{0}\right| \tag{20}
\end{equation*}
$$

defined on the ball $B=B(0, \rho), \rho=1 /\left(2 c_{1}\right)$. Using (17) and (19), we obtain that for large $k$

$$
\begin{equation*}
\left|v_{k}(z)\right|<(\alpha+1)(1+\rho) \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\Delta v_{k}(z)\right|=r_{k}\left|\Delta u\left(y_{k}+r_{k} z, t_{0}\right)\right|=r_{k}\left|u_{t}\left(y_{k}+r_{k} z, t_{0}\right)\right| \leq 2 c_{1} c_{2}, \tag{22}
\end{equation*}
$$

uniformly in $B$. Then $C^{1, \beta}$ norms of $v_{k}$ are locally uniformly bounded in $B$ for a $\beta \in(0,1)$; see $e . g$. [LU]. Therefore a subsequence of $v_{k}$ converges locally in $C^{1}$ norm to a function $v_{0}$ in $B$. We may also assume that over this subsequence there exists $e_{0}=\lim e_{k}$, where $e_{k}=\left(y_{k}-x_{0}\right) /\left|y_{k}-x_{0}\right|$. Then, using (17), we can compute that $v_{0}(z)=\alpha(z, \nu)+\alpha\left(e_{0}, \nu\right)$ in $B$, hence $\nabla v_{0}(0)=\alpha \nu$. Therefore, over a subsequence, $\lim \nabla u\left(y_{k}, t_{0}\right)=$ $\lim \nabla v_{k}(0)=\nabla v_{0}(0)=\alpha \nu$. Since the sequence $y_{k} \longrightarrow x_{0}, y_{k} \in K$ was arbitrary, this proves the lemma.

## 4. Proof of the main theorem.

In this section $\Omega$ will be a solution to problem $(\mathbf{P})$, under conditions of Theorem 1. Denote by $\Omega^{*}$ the spatial convex hull of $\Omega$, in the sense that the time sections $\Omega_{t}^{*}$ are the convex hulls of $\Omega_{t}$ for every $t \in(0, T)$. Since $\Omega$ is assumed to satisfy (5) and (7), $\Omega^{*}$ will also satisfy similar conditions. In particular, we may apply the results of two previous sections to $\Omega^{*}$. The lateral boundary of $\Omega^{*}$ will be denoted by $\Gamma^{*}$ and the solution to the Dirichlet problem, corresponding to (5), by $u^{*}$.

In the proof of Theorem 1 we use ideas of A. Henrot and H. Shahgholian [HS]. The key step is to prove the following lemma.

Lemma 7. For every $x_{0} \in \Gamma_{t_{0}}^{*}, 0<t_{0} \leq T$,

$$
\begin{equation*}
\liminf _{\Omega_{t_{0}}^{*} \ni y \rightarrow x_{0}}\left|\nabla u^{*}\left(y, t_{0}\right)\right| \geq 1 . \tag{23}
\end{equation*}
$$

Proof. From Lemma 4 it follows that there are $\varepsilon$ and $s_{0}$ such that the function $w_{+}(y)=u^{*}\left(y, t_{0}\right)+u^{*(1+\varepsilon)}\left(y, t_{0}\right)$ is subharmonic in the ringshaped domain $\left\{u^{*}\left(\cdot, t_{0}\right)<s_{0}\right\}$. Let now $y \in \Omega_{t_{0}}^{*}$ and $u^{*}\left(y, t_{0}\right)=$ $s<s_{0}$. Then $y \in \ell_{s}^{*}=\partial \mathcal{L}_{s}\left(u^{*}\left(\cdot, t_{0}\right)\right)$. By Theorem $2, \mathcal{L}_{s}^{*}=\mathcal{L}_{s}\left(u^{*}\left(\cdot, t_{0}\right)\right)$ is convex and therefore there exists a supporting plane in $\mathbb{R}^{n}$ to $\mathcal{L}_{s}^{*}$ at the point $y$. After a suitable translation and rotation in spatial variable we may assume that $y=0$, the supporting plane is $x_{1}=0$, and $\mathcal{L}_{s}^{*} \subset\left\{x_{1}<0\right\}$. Let $x^{*} \in \partial \Omega_{t_{0}}^{*}$ have the maximal positive $x_{1}$-coordinate. Since $\Omega_{t_{0}}^{*}$ is the convex hull of $\Omega_{t_{0}}$, there must be $x^{*} \in \partial \Omega_{t_{0}}^{*} \cap \partial \Omega_{t_{0}}$. Take now $\beta \in(0,1)$ and consider a function $v(x)=w_{+}(x)+\beta x_{1}$. Since $\Omega_{t_{0}}^{*} \cap\left\{x_{1}>0\right\} \subset\left\{u^{*}\left(\cdot, t_{0}\right)<s_{0}\right\}, v$ is subharmonic in $\Omega_{t_{0}}^{*} \cap\left\{x_{1}>0\right\}$ and therefore it must admit its maximum value on the boundary of this domain. Note that the maximum can be admitted either at $x^{*}$ or at $y=$ 0 . We show that the former case cannot occur. Indeed, the plane $x_{1}=$ $x_{1}^{*}$ is supporting to the convex set $\Omega_{t_{0}}^{*}$ and therefore there exists a ball $B \subset \Omega_{t_{0}}^{*} \subset \Omega_{t_{0}}^{c}$, "touching" both boundaries $\partial \Omega_{t_{0}}^{*}$ and $\partial \Omega_{t_{o}}$ at $x^{*}$ and with the outward radial direction $\nu=-e_{1}=(-1,0, \ldots, 0)$. Therefore from Lemma 5 we will have the following asymptotic developments for $u$ and $u^{*}$ near $x^{*}$ in $\Omega_{t_{0}}$ and $\Omega_{t_{0}}^{*}$ respectively

$$
\begin{align*}
u\left(x, t_{0}\right) & =\alpha\left(x_{1}^{*}-x_{1}\right)^{+}+o\left(\left|x-x^{*}\right|\right)  \tag{24}\\
u^{*}\left(x, t_{0}\right) & =\alpha^{*}\left(x_{1}^{*}-x_{1}\right)^{+}+o\left(\left|x-x^{*}\right|\right) . \tag{25}
\end{align*}
$$

Since (8) is satisfied at the point $x^{*}$, we conclude by Lemma 6 that $\alpha=1$. Next, $u^{*} \geq u$ in $\Omega$ and hence $\alpha^{*} \geq \alpha=1$. Observe now that $w_{+}$ admits the same representation as (25). Hence for the function $v(x)$ introduced above

$$
\begin{equation*}
v(x)=\left(\alpha^{*}-\beta\right)\left(x_{1}^{*}-x_{1}\right)+\beta x_{1}^{*}+o\left(\left|x-x^{*}\right|\right) . \tag{26}
\end{equation*}
$$

Let now $\nu^{\prime}$ be a spatial unit vector with $\left(\nu^{\prime}, e_{1}\right)<0$ such that $x^{*}+$ $h \nu^{\prime} \in \Omega_{t_{0}}$ for small $h>0$. The existence of such a $\nu^{\prime}$ follows from the local representation of $\partial \Omega_{t_{0}}$ as the graph of a Lipschitz function. Then $v\left(x^{*}+h \nu^{\prime}\right)>v\left(x^{*}\right)$ by (26) and consequently $v$ has no maximum at $x^{*}$. Therefore $v$ admits its maximum at the origin $y=0$. Hence

$$
\begin{equation*}
\left|\nabla w_{+}(0)\right|=\lim _{h \rightarrow 0+} \frac{w_{+}(0)-w_{+}\left(h e_{1}\right)}{h} \geq \lim _{h \rightarrow 0+} \frac{\beta h-0}{h}=\beta . \tag{27}
\end{equation*}
$$

Letting $\beta \longrightarrow 1$ we obtain that $\left|\nabla w_{+}(y)\right| \geq 1$, provided $u^{*}\left(y, t_{0}\right)<s_{0}$. Now observe that $\nabla w_{+}=\left(1+(1+\varepsilon) u^{* \varepsilon}\right) \nabla u^{*}$. This proves the lemma.

Proof of Theorem 1. Prove first that the domain $\Omega$ coincides with its spatial convex hull $\Omega^{*}$, studied above. For this purpose we apply the Lavrentiev principle. As a reference point we take $x_{\max } \in \Omega_{0}$, a maximum point for the initial function $u_{0}$. Without loss of generality we may assume that $x_{\max }=0$. Since $u_{0}$ is concave,

$$
\begin{equation*}
u_{0}(\lambda x) \leq u_{0}(x), \tag{28}
\end{equation*}
$$

for every $\lambda \geq 1$ and $x \in \Omega_{0}(\lambda)=\lambda^{-1} \Omega_{0}$. For $\lambda \geq 1$ define

$$
\begin{equation*}
u_{\lambda}^{*}(x, t)=u^{*}\left(\lambda x, \lambda^{2} t\right), \tag{29}
\end{equation*}
$$

in $\Omega^{*}(\lambda)=\left\{(x, t):\left(\lambda x, \lambda^{2} t\right) \in \Omega^{*}\right\}$. Suppose now that $\Omega^{*} \not \subset \Omega$. Then

$$
\begin{equation*}
\lambda_{0}=\inf \left\{\lambda: \Omega^{*}(\lambda) \subset \Omega\right\}>1, \tag{30}
\end{equation*}
$$

$\Omega^{*}\left(\lambda_{0}\right) \subset \Omega$, and there exists a common point $\left(x_{0}, t_{0}\right) \in \overline{\Gamma^{*}\left(\lambda_{0}\right)} \cap \Gamma$ with $t_{0} \in(0, T)$. Show that this leads to a contradiction. Indeed, by construction, $u_{\lambda_{0}}^{*}$ satisfies the heat equation in $\Omega^{*}\left(\lambda_{0}\right)$. Comparing the values of $u_{\lambda_{0}}^{*}$ and $u$ on the parabolic boundary $\partial_{p} \Omega^{*}\left(\lambda_{0}\right)$ (see (28)), we obtain that $u_{\lambda_{0}}^{*} \leq u$ in $\Omega^{*}\left(\lambda_{0}\right)$. Let now $\nu$ be the normal vector of a supporting plane in $\mathbb{R}^{n}$ to the convex domain $\Omega^{*}\left(\lambda_{0}\right)_{t_{0}}$ at the point $x_{0}$, pointing into $\Omega^{*}\left(\lambda_{0}\right)_{t_{0}}$. From lemmas 5, 6 and 7 and the definition
of $u_{\lambda}^{*}$ we conclude that $\nabla u_{\lambda_{0}}^{*}\left(x_{0}+h \nu, t_{0}\right) \longrightarrow \lambda_{0} \alpha^{*} \nu$ with $\alpha^{*} \geq 1$, as $h \longrightarrow 0+$. From elementary calculus there exists $\theta \in(0,1)$ such that

$$
\begin{equation*}
\frac{\frac{\partial}{\partial \nu} u\left(x_{0}+\theta h \nu, t_{0}\right)}{\frac{\partial}{\partial \nu} u_{\lambda_{0}}^{*}\left(x_{0}+\theta h \nu, t_{0}\right)}=\frac{u\left(x_{0}+h \nu, t_{0}\right)}{u_{\lambda_{0}}^{*}\left(x_{0}+h \nu, t_{0}\right)} \geq 1 \tag{31}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\limsup _{\Omega_{t_{0}} \ni y \rightarrow x_{0}} \frac{\partial}{\partial \nu} u\left(y, t_{0}\right) \geq \lim _{h \rightarrow 0+} \frac{\partial}{\partial \nu} u_{\lambda_{0}}^{*}\left(x_{0}+h \nu\right)=\lambda_{0} \alpha^{*}>1 \tag{32}
\end{equation*}
$$

which violates condition (8) at the point $\left(x_{0}, t_{0}\right)$. Therefore $\Omega^{*}=\Omega$, i.e. the time sections $\Omega_{t}$ are convex, for every $t \in(0, T)$.

It remains to prove the uniqueness of $\Omega$. For this we make the following observation. Let $\Omega^{\prime}$ be another solution. Then if everywhere in the proof of inclusion $\Omega^{*} \subset \Omega$ above we replace $\Omega^{*}$ by $\left(\Omega^{\prime}\right)^{*}$, but leave $\Omega$ unchanged, we will obtain that $\left(\Omega^{\prime}\right)^{*} \subset \Omega$. Since $\Omega$ and $\Omega^{\prime}$ are interchangeable, also we will have $\Omega^{*} \subset \Omega^{\prime}$. Therefore $\Omega^{\prime}=\Omega$ and the proof of Theorem 1 is completed.

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[^5]
# Parabolic equations involving $0^{\text {th }}$ and $1^{\text {st }}$ order terms with $L^{1}$ data 

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#### Abstract

This paper is devoted to general parabolic equations involving $0^{\text {th }}$ and $1^{\text {st }}$ order terms, in linear and nonlinear expressions, while the data only belong to $L^{1}$. Existence and entropic-uniqueness of solutions are proved.


## 1. Introduction.

In this paper, we are concerned with the following general parabolic equation

$$
\begin{cases}\partial_{t} u-\nabla \cdot(A(t, x) \nabla u) &  \tag{1.1}\\ +B(t, x, u, \nabla u)=f, & \text { in }(0, T) \times \Omega \\ u_{\mid t=0}=u_{0}, & \text { in } \Omega \\ u=0, & \text { on }(0, T) \times \partial \Omega\end{cases}
$$

where $\Omega$ is a regular open bounded set in $\mathbb{R}^{N}$ and $B$ involves the unknown $u$ and its first derivatives. Precisely, $B$ splits into terms which are linear with respect to $u$ and $\nabla u$ and a nonlinear term as follows

$$
\begin{equation*}
B(t, x, u, \nabla u)=b(t, x) \cdot \nabla u+d(t, x) u+g(t, x, u, \nabla u) . \tag{1.2}
\end{equation*}
$$

Here, $A, b$ and $d$ are given functions defined on $Q=(0, T) \times \Omega$ with values in $\mathbb{R}^{N} \times \mathbb{R}^{N}, \mathbb{R}^{N}$ and $\mathbb{R}$, respectively. Our basic requirement on $A, b, d$ is

$$
\begin{array}{lr}
A \in\left(L^{\infty}(Q)\right)^{N \times N}, \quad d \in L^{\infty}(Q), \\
b \in\left(L^{\infty}(Q)\right)^{N}, & \nabla \cdot b \in L^{\infty}(Q) . \tag{1.4}
\end{array}
$$

As usual, we also assume that there exists $\underline{a}>0$ such that the matrix $A$ satisfies

$$
\begin{equation*}
A(t, x) \xi \cdot \xi \geq \underline{a}|\xi|^{2} \tag{1.5}
\end{equation*}
$$

for almost every $(t, x) \in Q$ and for all $\xi \in \mathbb{R}^{N}$. The function $g$ : $Q \times \mathbb{R} \times \mathbb{R}^{N} \longrightarrow \mathbb{R}$ is measurable on $Q$ for all $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^{N}$, continuous with respect to $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^{N}$, almost everywhere in $Q$. Furthermore, $g$ is required to satisfy both a sign condition and a growth condition with respect to the gradient variable since we suppose that

$$
\begin{equation*}
\lambda g(t, x, \lambda, \xi) \geq 0 \tag{1.6}
\end{equation*}
$$

there exists $0 \leq \sigma<2$ such that

$$
\begin{equation*}
|g(t, x, \lambda, \xi)| \leq h(|\lambda|)\left(\gamma(t, x)+|\xi|^{\sigma}\right) \tag{1.7}
\end{equation*}
$$

holds for all $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^{N}$, and almost everywhere in $Q$, with $\gamma \in$ $L^{1}(Q) ; h$ being a non decreasing function on $\mathbb{R}^{+}$. Main difficulties in this work arise from the fact that we consider data which only belong to $L^{1}$, namely

$$
\begin{equation*}
u_{0} \in L^{1}(\Omega), \quad f \in L^{1}(Q) \tag{1.8}
\end{equation*}
$$

Many physicals models lead to elliptic and parabolic problems with $L^{1}$-data. For instance, in [10] the authors study the modelling of an electronical device. The derived elliptic system coupled the temperature (denoted $u$ ) and the electronical potential (denoted $\Phi$ ). The temperature equation is considered as an elliptic equation where the second member $f=|\nabla \Phi|^{2}$ belongs to $L^{1}(\Omega)$. In [11], a Fokker-Planck equation arising in populations dynamics is studied. The initial density of individuals, i.e. $u_{0}$, is considered to be positive and belongs to $L^{1}(\Omega)$.

Models of turbulent flows in oceanography and climatology also lead to such kind of problems (see [14] and the references therein).

Consider an incompressible flow described by a velocity field $u(t, x)=$ $\bar{u}+u^{\prime}$ where $\bar{u}$ is the mean field and $u^{\prime}$ is related to some fluctuations. Let $k=\left|\overline{u^{\prime}}\right|^{2}$. For small Reynolds number, the following academic model can be used as a simplification of more general $(k, \varepsilon)$ models

$$
\partial_{t} k+\bar{u} \cdot \nabla_{x} k-\operatorname{div}_{x}\left(\left(\nu+\nu_{t}\right) \nabla_{x} k\right)+k^{3 / 2}=\nu_{t}\left|\nabla_{x} \bar{u}+{ }^{t} \nabla_{x} \bar{u}\right|^{2},
$$

where $\nu_{t}$ can depend on $k$. It is quite natural to expect that the right hand side lies in $L^{1}(Q)$ and, for given $\nu, \nu_{t}$ and $\bar{u}$, the above equation can be considered as a simplified version of (1.1). In [14] more complicated and coupled models are dealt with.

In ([16, p. 110]), the author studies the Navier-Stokes equations completed by an equation for the temperature $(u=T)$. In this case, if we denote by $v$ the velocity of the fluid, then the temperature equation reduces to (1.1) with $b=v, d=\operatorname{div}(v)=0, g=0$ and $f=$ $\left(\partial_{i} v_{j}+\partial_{j} v_{i}\right)^{2} \in L^{1}(Q)$. Note that for compressible flows the divergence of the velocity does not vanish, and the temperature equation can be considered with linear terms having the form $b \cdot \nabla u+d u$. These linear terms introduce new difficulties in the sense that the compactness results developped in [3], [4], [16] do not apply directly to (1.1) which needs further technical investigations.

Assuming $B=0$, existence results for such parabolic problems with non regular data are established in [4] (see also [3], [10]) while uniqueness questions, in the sense of entropic or renormalized formulations, are considered in [17], [1]. Existence-uniqueness of renormalized solution for a linear parabolic equation involving a first order term with a free divergence coefficient is discussed in [16]. Taking into account the $g$ term, the corresponding elliptic problem, with an integrable source term, is treated in [9] when $\sigma<2$ and the critical case $\sigma=2$ is dealt with in [5]. In [6], the $g$ term appears in (1.1), still neglecting the linear terms involving $b$ and $d$, with the restriction that $g$ does not vanish for large value of $u$, which induces some regularizing effects in the equation. Note that in view of the quoted papers, our results extend to more general Leray-Lions operators ; however, to avoid technical complications and to emphasize the influence of the term $B$ we restrict our attention on a simple operator satisfying (1.2). Let us now introduce some definitions and give the statement of our main results.

For the sake of clarity, we dropped the dependence on $t, x$ of $A, b, d$ and $g$. When no confusion can arise, we will follow this convention in the sequel.

Definition 1. By weak solution of (1.1) we shall mean any function $u \in L^{q}\left(0, T ; W_{0}^{1, q}(\Omega)\right) \cap C^{0}\left(0, T ; L^{1}(\Omega)\right)$ such that $g(u, \nabla u)$ belongs to $L^{1}(Q)$ and satisfying

$$
\begin{align*}
& \int_{\Omega} u \phi(T, x) d x-\int_{\Omega} u_{0} \phi(0, x) d x-\int_{Q} u \partial_{t} \phi(t, x) d x d t \\
& +\int_{Q} A \nabla u \cdot \nabla \phi d x d t+\int_{Q}(g(u, \nabla u)+b \cdot \nabla u+d u) \phi d x d t  \tag{1.9}\\
& \quad=\int_{Q} f \phi d x d t
\end{align*}
$$

for all $T>0, \phi \in C^{0}\left(0, T ; W_{0}^{1, q^{\prime}}(\Omega)\right) \cap C^{1}\left(0, T ; L^{q^{\prime}}(\Omega)\right)$ and for all $q$ such that $1 \leq q<(N+2) /(N+1)$ and $1 / q+1 / q^{\prime}=1$.

All terms in (1.9) are clearly defined (by duality $L^{q}, L^{q^{\prime}}$ ), except those involving $g(u, \nabla u)$. However, since $1 \leq q<(N+2) /(N+1)$, we have $q^{\prime}=q /(q-1)>N$ and by Sobolev's embedding the test function $\phi$ actually lies in $L^{\infty}(Q)$ so that the integral of $g(u, \nabla u) \phi$ makes sense.

Theorem 1. Assume that (1.3)-(1.8) hold. Then, there exists a weak solution of (1.1), in the sense of Definition 1 .

Let us recall the definition of the truncated function $T_{k}$. Let $k \in$ $\mathbb{R}^{+}$. We set

$$
T_{k}(z)= \begin{cases}z, & \text { if }|z| \leq k  \tag{1.10}\\ k, & \text { if } z>k \\ -k, & \text { if } z<-k\end{cases}
$$

and we denote $S_{k}(z)=\int_{0}^{z} T_{k}(\tau) d \tau$.
Definition 2. Let $g=0$. We say that $u$ is a entropic solution of (1.1) if $u \in C^{0}\left(0, T ; L^{1}(\Omega)\right)$ satisfies $T_{k}(u) \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$ for all $k>0$, $\nabla u \in L^{1}(Q)$ and

$$
\begin{align*}
& \int_{\Omega} S_{k}(u-\psi)(T) d x-\int_{\Omega} S_{k}\left(u_{0}-\psi(0, \cdot)\right) d x \\
& +\int_{0}^{T}\left\langle\partial_{t} \psi, T_{k}(u-\psi)\right\rangle_{H^{-1}(\Omega), H_{0}^{1}(\Omega)} d t \\
& +\int_{Q} A \nabla u \cdot \nabla\left(T_{k}(u-\psi)\right) d x d t \tag{1.11}
\end{align*}
$$

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$$
\begin{gathered}
+\int_{Q}(d u+b \nabla u) T_{k}(u-\psi) d x d t \\
\quad \leq \int_{Q} f T_{k}(u-\psi) d x d t
\end{gathered}
$$

for all $k>0$ and $\psi \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right) \cap L^{\infty}(Q) \cap C^{0}\left(0, T ; L^{1}(\Omega)\right)$ with $\partial_{t} \psi \in L^{2}\left(0, T ; H^{-1}(\Omega)\right)$.

Obviously, $T_{k}(u-\psi)$ lies in $L^{\infty}(Q)$ and $S_{k}$ is $k$-Lipschitzian; hence with the requirements $\nabla u \in L^{1}(Q)$ and $T_{k}(u-\psi) \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$, both term in (1.11) clearly makes sense except the product $A \nabla u$. $\nabla\left(T_{k}(u-\psi)\right)$. Remark now that $\nabla\left(T_{k}(u-\psi)\right)=\chi_{|u-\psi| \leq k} \nabla(u-\psi)$ can be estimated by $\chi_{|u| \leq k+\|\psi\|_{L^{\infty}}}|\nabla u|+|\nabla \psi|=\left|\nabla T_{k+\|\psi\|_{L^{\infty}}}(u)\right|+|\nabla \psi|$ which belongs to $L^{2}$ since one chooses the test function $\psi$ in $L^{\infty}(Q)$. Therefore $A \nabla u \cdot \nabla\left(T_{k}(u-\psi)\right)$ is integrable.

Theorem 2. Let $g=0$. Assume that (1.3)-(1.8) hold. Then, there exists a unique entropic solution of (1.1).

The strategy we adopt is rather close to those introduced in [4]. However, new difficulties arise essentially related to the influence of the linear $0^{\text {th }}$ and $1^{\text {st }}$ order terms. Then, this paper is organized as follows. First, Section 2 is devoted to an independent preliminary result which will be used to derive a bound in $L^{q}$ on the gradient of the solutions, in despite of the perturbation induced by the additional terms of lower order. In Section 3, we deal with sequences $u_{\varepsilon}$ of approximate solutions. We establish some a priori estimates on these solutions and we translate the obtained bounds in terms of compactness properties. Then, we explain how we can pass to the limit as $\varepsilon \longrightarrow 0$ in the weak formulation satisfied by $u_{\varepsilon}$. In Section 4, we are concerned with the uniqueness of entropic solution. Finally, in Section 5, we slightly weaken the regularity assumption concerning the coefficient $b$.

## 2. A preliminary result.

The main idea in the proof of Theorem 1 consists in deriving a $L^{q}\left(0, T ; W_{0}^{1, q}(\Omega)\right)$ estimate on the solutions depending only on the $L^{1}$ norm of the data $f$ and $u_{0}$. Such an estimate will appear as a consequence of the following lemma.

Lemma 1. Let $u \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$ satisfy

$$
\begin{equation*}
\sup _{t \in(0, T)} \int_{\Omega}|u| d x \leq \beta, \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{B_{n}}|\nabla u|^{2} d x d t \leq C_{0}+C_{1} \int_{E_{n}}|\nabla u| d x d t, \quad \text { for all } n \in \mathbb{N} \tag{2.2}
\end{equation*}
$$

where

$$
B_{n}=\{(t, x) \in Q: n \leq|u(t, x)| \leq n+1\},
$$

and

$$
E_{n}=\{(t, x) \in Q:|u(t, x)|>n+1\} .
$$

Then, for all $1 \leq q<(N+2) /(N+1)$, there exists $C>0$, depending on $\beta, C_{0}, C_{1},|\Omega|, T$, and $q$ such that

$$
\begin{equation*}
\|u\|_{L^{q}\left(0, T ; W_{0}^{1, q}(\Omega)\right)} \leq C . \tag{2.3}
\end{equation*}
$$

Proof. In [4], [10] inequality (2.2) appears with $C_{1}=0$ and is used to derive (2.3). Here, the additional term is related to the influence of the first order term $b \cdot \nabla u$ in the equation as we shall see in next section (see Proposition 1). However, exploiting carefully the fact that the integral in the right hand side is only taken over the large values of the unknow, we can obtain (2.3) as a consequence of (2.2).

Let $1 \leq q<2$. From (2.2), we first notice that

$$
\begin{align*}
\int_{B_{n}}|\nabla u|^{2} d x d t & \leq C_{0}+C_{1}\left(\int_{E_{n}}|\nabla u|^{q} d x\right)^{1 / q}\left|E_{n}\right|^{(q-1) / q}  \tag{2.4}\\
& \leq C_{0}+C_{1}\|\nabla u\|_{L^{q(Q)}}\left|E_{n}\right|^{(q-1) / q}
\end{align*}
$$

holds by using Holder's inequality. Thus, applying again Holder's inequality, we obtain

$$
\begin{align*}
\int_{B_{n}}|\nabla u|^{q} d x d t & \leq\left|B_{n}\right|^{(2-q) / 2}\left(\int_{B_{n}}|\nabla u|^{2} d x d t\right)^{q / 2} \\
& \leq\left|B_{n}\right|^{(2-q) / 2}\left(C_{0}^{q / 2}+C_{1}^{q / 2}\|\nabla u\|_{L^{q}(Q)}^{q / 2}\left|E_{n}\right|^{(q-1) / 2}\right) \tag{2.5}
\end{align*}
$$

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by (2.4) and the elementary inequality $(a+b)^{q / 2} \leq a^{q / 2}+b^{q / 2}$. Let $r \geq 0$ to be chosen later. Clearly, one has

$$
\left\{\begin{array}{l}
\left|B_{n}\right| \leq \frac{1}{n^{r}} \int_{B_{n}}|u|^{r} d x d t  \tag{2.6}\\
\left|E_{n}\right| \leq \frac{1}{n^{r}} \int_{E_{n}}|u|^{r} d x d t \leq \frac{1}{n^{r}}\|u\|_{L^{r}(Q)}^{r}
\end{array}\right.
$$

Hence, (2.5) becomes

$$
\begin{align*}
& \int_{B_{n}}|\nabla u|^{q} d x d t \\
& \leq C_{0}^{q / 2}\left(\frac{1}{n}\right)^{r(2-q) / 2}\left(\int_{B_{n}}|u|^{r} d x d t\right)^{(2-q) / 2} \\
&+C_{1}^{q / 2}\|\nabla u\|_{L^{q}(Q)}^{q / 2}\|u\|_{L^{r}(Q)}^{r(q-1) / 2}\left(\frac{1}{n}\right)^{r / 2}  \tag{2.7}\\
& \cdot\left(\int_{B_{n}}|u|^{r} d x d t\right)^{(2-q) / 2}
\end{align*}
$$

Let $K \in \mathbb{N}$ to be determined. We split $\|\nabla u\|_{L^{q(Q)}}^{q}$ as follows

$$
\begin{equation*}
\int_{Q}|\nabla u|^{q} d x d t=\sum_{n=0}^{K} \int_{B_{n}}|\nabla u|^{q} d x d t+\sum_{n=K+1}^{\infty} \int_{B_{n}}|\nabla u|^{q} d x d t \tag{2.8}
\end{equation*}
$$

Since $\left|B_{n}\right| \leq T|\Omega|$ and $\left|E_{n}\right| \leq T|\Omega|$, we simply evaluate the first term in the right hand side of (2.8) as follows

$$
\begin{equation*}
\sum_{n=0}^{K} \int_{B_{n}}|\nabla u|^{q} d x d t \leq K C_{2}\left(1+\|\nabla u\|_{L^{q}(Q)}^{q / 2}\right) \tag{2.9}
\end{equation*}
$$

by $(2.5)$, where $C_{2}=\max \left\{C_{0}^{q / 2}(T|\Omega|)^{(2-q) / 2}, C_{1}^{q / 2}(T|\Omega|)^{1 / 2}\right\}$. Thus, by using Young's inequality in (2.8)-(2.9), we get

$$
\begin{equation*}
\|\nabla u\|_{L^{q}(Q)}^{q} \leq C(K)+\sum_{n=K+1}^{\infty} \int_{B_{n}}|\nabla u|^{q} d x d t \tag{2.10}
\end{equation*}
$$

where $C(K)$ tends to infinity as $K$ becomes large. It remains to proceed to the study of the series which appears in the right hand side.

Applying Holder's inequality on the series with exponents $2 /(2-q)$ and $2 / q$ and using (2.7), we have

$$
\begin{align*}
& \sum_{n=K+1}^{\infty} \int_{B_{n}}|\nabla u|^{q} d x d t \\
\leq & C_{0}^{q / 2}\left(\sum_{n=K+1}^{\infty} \frac{1}{n^{r(2-q) / q}}\right)^{q / 2}\left(\sum_{n=K+1}^{\infty} \int_{B_{n}}|u|^{r} d x d t\right)^{(2-q) / 2} \\
& +C_{1}^{q / 2}\|\nabla u\|_{L^{q}(Q)}^{q / 2}\|u\|_{L^{r}(Q)}^{r(q-1) / 2}\left(\sum_{n=K+1}^{\infty} \frac{1}{n^{r / q}}\right)^{q / 2}  \tag{2.11}\\
& \cdot\left(\sum_{n=K+1}^{\infty} \int_{B_{n}}|u|^{r} d x d t\right)^{(2-q) / 2} \\
\leq & C_{0}^{q / 2}\left(\sum_{n=K+1}^{\infty} \frac{1}{n^{r(2-q) / q}}\right)^{q / 2}\|u\|_{L^{r}(Q)}^{r(2-q) / 2} \\
& +C_{1}^{q / 2}\|\nabla u\|_{L^{q}(Q)}^{q / 2}\|u\|_{L^{r}(Q)}^{r / 2}\left(\sum_{n=K+1}^{\infty} \frac{1}{n^{r / q}}\right)^{q / 2} .
\end{align*}
$$

Note that the conditions

$$
\begin{equation*}
r \frac{2-q}{q}>1 \quad \text { and } \quad \frac{r}{q}>1 \tag{2.12}
\end{equation*}
$$

ensure the convergence of the series which appear in the right hand side of (2.11). Consequently, these terms become arbitrarily small when choosing $K$ large enough as soon as (2.12) is fulfilled.

With the convention that $\delta(K)$ denotes quantities which tend to 0 as $K$ goes to $\infty$, by combining (2.10) with (2.11), we get

$$
\begin{equation*}
\|\nabla u\|_{L^{q}(Q)}^{q} \leq C(K)+\delta(K)\left(\|u\|_{L^{r}(Q)}^{r(2-q) / 2}+\|\nabla u\|_{L^{q}(Q)}^{q / 2}\|u\|_{L^{r}(Q)}^{r / 2}\right) . \tag{2.13}
\end{equation*}
$$

Therefore, by using Young's inequality on the last term in the right side, it follows that

$$
\begin{equation*}
\|\nabla u\|_{L^{q}(Q)}^{q} \leq C(K)+\delta(K)\left(\|u\|_{L^{r}(Q)}^{r(2-q) / 2}+\|u\|_{L^{r}(Q)}^{r}\right), \tag{2.14}
\end{equation*}
$$

where we keep the notation $C(K), \delta(K)$ while the value of these terms may have changed, still with the meaning that $C(K) \longrightarrow \infty, \delta(K) \longrightarrow$ 0 when $K$ becomes large.

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We denote by $q_{\star}=N q /(N-q)$ the Sobolev conjugate of $q$. The Sobolev imbedding theorem implies that

$$
\begin{equation*}
\int_{0}^{T}\left(\int_{\Omega}|u|^{q_{\star}} d x\right)^{q / q_{\star}} d t \leq C \int_{Q}|\nabla u|^{q} d x d t \tag{2.15}
\end{equation*}
$$

Assume now $1<r<q_{\star}$ and set $1 / r=\theta+(1-\theta) / q^{\star}$ with $0<\theta<1$. For almost everywere $t \in(0, T)$, one has

$$
\begin{equation*}
\|u(t, \cdot)\|_{L^{r}(\Omega)}^{r} \leq\|u(t, \cdot)\|_{L^{1}(\Omega)}^{r \theta}\|u(t, \cdot)\|_{L^{q_{\star}(\Omega)}}^{r(1-\theta)} . \tag{2.16}
\end{equation*}
$$

Integrating (2.16) with respect to time and recalling the bound (2.1) in $L^{\infty}\left(0, T, L^{1}(\Omega)\right)$ yield

$$
\begin{equation*}
\|u\|_{L^{r}(Q)}^{r} \leq \beta^{r \theta} \int_{0}^{T}\left(\int_{\Omega}|u|^{q_{\star}} d x\right)^{r(1-\theta) / q_{\star}} d t . \tag{2.17}
\end{equation*}
$$

Choose now $r=q(N+1) / N$, noting that the convergence condition (2.12) is fulfilled as soon as $1 \leq q<(N+2) /(N+1)$. Combining (2.14)-(2.17) with Young's inequality (since $(2-q) / 2<1$ ) leads to

$$
\begin{align*}
& \int_{0}^{T}\left(\int_{\Omega}|u|^{q_{\star}} d x\right)^{q / q_{\star}} d t \\
& \leq C(K)+\delta(K)\left(\left(\int_{0}^{T}\left(\int_{\Omega}|u|^{q_{\star}} d x\right)^{q / q_{\star}} d t\right)^{(2-q) / 2}\right. \\
& \left.\quad+\int_{0}^{T}\left(\int_{\Omega}|u|^{q_{\star}} d x\right)^{q / q_{\star}} d t\right)  \tag{2.18}\\
& \leq C(K)+\delta(K) \int_{0}^{T}\left(\int_{\Omega}|u|^{q_{\star}} d x\right)^{q / q_{\star}} d t
\end{align*}
$$

We fix $K>0$ so that, for instance, $1-\delta(K)>1 / 2$. Hence, we deduce from (2.18) that

$$
\begin{equation*}
\|u\|_{L^{q}\left(0, T, L^{\left.q_{\star}(\Omega)\right)}\right.} \leq C \tag{2.19}
\end{equation*}
$$

holds, and the asserted estimate (2.3) follows easily from (2.17) and (2.14).

## 3. Proof of Theorem 1.

The proof falls naturally into several steps and we detail each of them separetely.

### 3.1. Approximate solutions.

We introduce the following smooth approximations of the data

$$
\begin{cases}u_{0, \varepsilon} \in C_{0}^{\infty}(\Omega), & f_{\varepsilon} \in C_{0}^{\infty}(\Omega)  \tag{3.1}\\ u_{0, \varepsilon} \longrightarrow u_{0} \text { in } L^{1}(\Omega), & f_{\varepsilon} \longrightarrow f \text { in } L^{1}(Q)\end{cases}
$$

with

$$
\begin{equation*}
\left\|u_{0, \varepsilon}\right\|_{L^{1}(\Omega)} \leq\left\|u_{0}\right\|_{L^{1}(\Omega)}, \quad\left\|f_{\varepsilon}\right\|_{L^{1}(\Omega)} \leq\|f\|_{L^{1}(\Omega)} \tag{3.2}
\end{equation*}
$$

Moreover, we regularize the function $g$ as follows

$$
\begin{equation*}
g_{\varepsilon}(u, \nabla u)=\frac{g(u, \nabla u)}{1+\varepsilon|g(u, \nabla u)|} . \tag{3.3}
\end{equation*}
$$

Note that $g_{\varepsilon}$ belongs in $L^{\infty}(Q)$ and satisfy the sign condition (1.6) and the growth condition in (1.7). Then, classical results, see e.g. [15], [12], [7], (or, in the linear case, use a Galerkin method), provide the existence of a sequence $u_{\varepsilon} \in C^{0}\left(0, T ; L^{2}(\Omega)\right) \cap L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$, with $\partial_{t} u_{\varepsilon} \in L^{2}\left(0, T ; H^{-1}(\Omega)\right)$, of solutions of (1.1) where $u_{0}, f$ and $g$ are replaced by $u_{0, \varepsilon}, f_{\varepsilon}$ and $g_{\varepsilon}$ respectively. We have

$$
\begin{align*}
& \left\langle\partial_{t} u_{\varepsilon}, \phi\right\rangle_{H^{-1}(\Omega), H_{0}^{1}(\Omega)}+\int_{\Omega} A \nabla u_{\varepsilon} \cdot \nabla \phi d x  \tag{3.4}\\
& +\int_{\Omega}\left(g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)+b \cdot \nabla u_{\varepsilon}+d u_{\varepsilon}\right) \phi d x=\int_{\Omega} f_{\varepsilon} \phi d x
\end{align*}
$$

for all $T>0$ and $\phi \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$.

### 3.2. A priori estimates.

In this section, we are concerned with a priori estimates satisfied by the sequence $u_{\varepsilon}$ of solutions of (3.4) which lead to compactness properties essential to the proof.

Proposition 1. Let $A, b, d, g$ satisfy (1.3)-(1.7). Then, there exist $\beta>0, C_{0}$ and $C_{1}$ depending only on $\left\|u_{0}\right\|_{L^{1}(\Omega)},\|f\|_{L^{1}(Q)},\|b\|_{L^{\infty}(\Omega)}$, $\|d\|_{L^{\infty}(Q)},|\Omega|$ and $T$ such that the sequence $u_{\varepsilon}$ of solutions of (3.4) satisfies

$$
\begin{equation*}
\sup _{\substack{\varepsilon>0 \\ t \in(0, T)}}\left\|u_{\varepsilon}(t)\right\|_{L^{1}(\Omega)} \leq \beta, \tag{3.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t \leq C_{0}+C_{1} \int_{E_{n}}\left|\nabla u_{\varepsilon}\right| d x d t \tag{3.6}
\end{equation*}
$$

In view of Lemma 1, we deduce immediately the following
Corollary 1. Let $A, b, d, g$ satisfy (1.3)-(1.7). Let $1 \leq q<(N+$ $2) /(N+1)$. Then, there exists $C>0$ depending only on the data, such that

$$
\begin{equation*}
\sup _{\varepsilon>0}\left\|u_{\varepsilon}\right\|_{L^{q\left(0, T ; W_{0}^{1, q}(\Omega)\right)}} \leq C . \tag{3.7}
\end{equation*}
$$

Proof of Proposition 1. Since $T_{k}$ is a Lipschitz function and $u_{\varepsilon} \in$ $L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$, one has $T_{k}\left(u_{\varepsilon}\right) \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$, see [19], [20], with, moreover,

$$
\nabla T_{k}\left(u_{\varepsilon}\right)=\chi_{\left|u_{\varepsilon}\right| \leq k} \nabla u_{\varepsilon},
$$

where $\chi_{\left|u_{\varepsilon}\right| \leq k}$ denotes the characteristic function of the set $\{(t, x) \in Q$ : $\left.\left|u_{\varepsilon}(t, x)\right| \leq k\right\}$. Thus, we choose $\phi=T_{k}\left(u_{\varepsilon}\right)$ as test function in (3.4). Writing $b \cdot \nabla u_{\varepsilon}=\nabla \cdot\left(b u_{\varepsilon}\right)-(\nabla \cdot b) u_{\varepsilon}$, one gets

$$
\begin{align*}
& \frac{d}{d t} \int_{\Omega} S_{k}\left(u_{\varepsilon}\right) d x+\int_{\Omega} \chi_{\left|u_{\varepsilon}\right| \leq k} A \nabla u_{\varepsilon} \cdot \nabla u_{\varepsilon} d x \\
& +\int_{\Omega} T_{k}\left(u_{\varepsilon}\right) g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right) d x \\
& =\int_{\Omega} f_{\varepsilon} T_{k}\left(u_{\varepsilon}\right) d x+\int_{\Omega} \chi_{\left|u_{\varepsilon}\right| \leq k} u_{\varepsilon} b \cdot \nabla u_{\varepsilon} d x  \tag{3.8}\\
& \quad+\int_{\Omega}((\nabla \cdot b)-d) u_{\varepsilon} T_{k}\left(u_{\varepsilon}\right) d x .
\end{align*}
$$

By using Holder's and Young's inequalities, one obtains

$$
\begin{align*}
& \left|\int_{\Omega} \chi_{\left|u_{\varepsilon}\right| \leq k} u_{\varepsilon} b \cdot \nabla u_{\varepsilon} d x\right| \\
& \leq \frac{a}{2} \int_{\Omega} \chi_{\left|u_{\varepsilon}\right| \leq k}\left|\nabla u_{\varepsilon}\right|^{2} d x+\frac{1}{2 \underline{a}}\|b\|_{L^{\infty}(Q)}^{2} \int_{\Omega} \chi_{\left|u_{\varepsilon}\right| \leq k}\left|u_{\varepsilon}\right|^{2} d x . \tag{3.9}
\end{align*}
$$

Moreover, $u_{\varepsilon} T_{k}\left(u_{\varepsilon}\right)$ is non negative and we assume that $d$ and $\nabla \cdot b$ belong to $L^{\infty}(Q)$. Hence, after integration of (3.8) with respect to $t$ and using (1.5), we are led to

$$
\begin{align*}
& \int_{\Omega} S_{k}\left(u_{\varepsilon}\right)(t) d x+\int_{0}^{t} \int_{\Omega} T_{k}\left(u_{\varepsilon}\right) g_{\varepsilon} d x d s \\
& +\frac{a}{2} \int_{0}^{t} \int_{\Omega} \chi_{\left|u_{\varepsilon}\right| \leq k}|\nabla u|^{2} d x d s \\
& \leq \int_{0}^{t} \int_{\Omega}\left|f_{\varepsilon} T_{k}\left(u_{\varepsilon}\right)\right| d x d s+\int_{\Omega} S_{k}\left(u_{0, \varepsilon}\right) d x  \tag{3.10}\\
& \quad+\frac{1}{2 \underline{a}}\|b\|_{L^{\infty}(Q)}^{2} \int_{0}^{t} \int_{\Omega} \chi_{\left|u_{\varepsilon}\right| \leq k}\left|u_{\varepsilon}\right|^{2} d x d s \\
& \quad+\left(\|d\|_{L^{\infty}(Q)}+\|\nabla \cdot b\|_{L^{\infty}(Q)}\right) \int_{0}^{t} \int_{\Omega} u_{\varepsilon} T_{k}\left(u_{\varepsilon}\right) d x d s
\end{align*}
$$

where, by the sign assumption (1.6) and the definition of $S_{k}$, all the terms in the left hand side of (3.10) are non negative. Next, we observe that

$$
\begin{align*}
0 & \leq z^{2} \chi_{|z| \leq k} \\
& \leq z T_{k}(z) \\
& =z^{2} \chi_{|z| \leq k}+k|z| \chi_{|z|>k}  \tag{3.11}\\
& \leq z^{2} \chi_{|z| \leq k}+\left(2 k|z|-k^{2}\right) \chi_{|z|>k} \\
& =2 S_{k}(z)
\end{align*}
$$

which yields

$$
\begin{align*}
\int_{\Omega} S_{k}\left(u_{\varepsilon}\right)(t) d x \leq & \int_{0}^{t} \int_{\Omega}\left|f_{\varepsilon} T_{k}\left(u_{\varepsilon}\right)\right| d x d s+\int_{\Omega} S_{k}\left(u_{0, \varepsilon}\right) d x \\
& +C(b, d) \int_{0}^{t} \int_{\Omega} S_{k}\left(u_{\varepsilon}\right) d x d s \tag{3.12}
\end{align*}
$$

Parabolic equations involving $0^{\text {th }}$ and $1^{\text {st }}$ order terms with $L^{1}$ data 445 where $C(b, d)$ stands for

$$
2\left(\|d\|_{L^{\infty}(Q)}+\|\nabla \cdot b\|_{L^{\infty}(Q)}\right)+\frac{1}{\underline{a}}\|b\|_{L^{\infty}(Q)}
$$

We set $z(t)=\int_{\Omega} S_{k}\left(u_{\varepsilon}\right)(t) d x$. Thus, dropping non negative terms, we have

$$
0 \leq z(t) \leq z(0)+\int_{0}^{t} \int_{\Omega}\left|f_{\varepsilon}\right|\left|T_{k}\left(u_{\varepsilon}\right)\right| d x d s+C(b, d) \int_{0}^{t} z(s) d s
$$

and we apply Gronwall's lemma to deduce that

$$
\begin{equation*}
z(t) \leq e^{C(b, d) T}\left(\int_{\Omega} S_{k}\left(u_{0, \varepsilon}\right) d x+\int_{Q}\left|f_{\varepsilon}\right|\left|T_{k}\left(u_{\varepsilon}\right)\right| d x d t\right) \tag{3.13}
\end{equation*}
$$

holds.
We set $k=1$ in (3.13). Remarking that $\left|T_{1}(z)\right| \leq 1$ and $0 \leq$ $S_{1}(z) \leq|z|$ leads to

$$
\int_{\Omega} S_{1}\left(u_{\varepsilon}\right)(t) d x \leq e^{C(b, d) T}\left(\left\|u_{0}\right\|_{L^{1}(\Omega)}+\|f\|_{L^{1}(\Omega)}\right)
$$

by (3.2). Therefore, we end the proof of (3.5) with the following observation

$$
\begin{aligned}
\int_{\Omega}\left|u_{\varepsilon}\right| d x & =\int_{\left|u_{\varepsilon}\right| \leq 1}\left|u_{\varepsilon}\right| d x+\int_{\left|u_{\varepsilon}\right|>1}\left|u_{\varepsilon}\right| d x \\
& \leq \int_{\left|u_{\varepsilon}\right| \leq 1} d x+\int_{\left|u_{\varepsilon}\right|>1}\left(S_{1}\left(u_{\varepsilon}\right)+\frac{1}{2}\right) d x \\
& \leq \frac{3}{2}|\Omega|+e^{C(b, d) T}\left(\left\|u_{0}\right\|_{L^{1}(\Omega)}+\|f\|_{L^{1}(Q)}\right) \\
& =\beta
\end{aligned}
$$

To achieve the proof of Proposition 1, we are left with the task of showing that (3.6) holds. According to [4], we introduce the function

$$
\phi_{n}(z)= \begin{cases}1, & \text { if } z \geq n+1  \tag{3.14}\\ z-k, & \text { if } n \leq z<n+1 \\ 0, & \text { if }-n<z<n \\ z+k, & \text { if }-n-1<z \leq-n \\ -1, & \text { if } z \leq-n-1\end{cases}
$$

and we set $\Psi_{n}(z)=\int_{0}^{z} \phi_{n}(\tau) d \tau$. We note that $\phi_{n}$ is a Lipschitz function. Thus, we have $\phi_{n}\left(u_{\varepsilon}\right) \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$, see [19], [20] with

$$
\nabla \phi_{n}\left(u_{\varepsilon}\right)=\chi_{B_{n}} \nabla u_{\varepsilon},
$$

$\chi_{B_{n}}$ denoting the characteristic function of the set $B_{n}=\{(t, x) \in Q$ : $\left.n \leq\left|u_{\varepsilon}(t, x)\right| \leq n+1\right\}$. Then, taking $\phi=\phi_{n}\left(u_{\varepsilon}\right) \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$ as test function in (3.4) gives

$$
\begin{gathered}
\frac{d}{d t} \int_{\Omega} \Psi_{n}\left(u_{\varepsilon}\right) d x+\int_{\Omega} \chi_{B_{n}} A \nabla u_{\varepsilon} \cdot \nabla u_{\varepsilon} d x+\int_{\Omega} g\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right) \phi_{n}\left(u_{\varepsilon}\right) d x \\
=\int_{\Omega} f_{\varepsilon} \phi_{n}\left(u_{\varepsilon}\right) d x-\int_{\Omega} d u_{\varepsilon} \phi_{n}\left(u_{\varepsilon}\right) d x-\int_{\Omega} b \cdot \nabla u_{\varepsilon} \phi_{n}\left(u_{\varepsilon}\right) d x
\end{gathered}
$$

Thus, integrating the above equation with respect to $t$, we have

$$
\begin{align*}
& \int_{\Omega} \Psi_{n}\left(u_{\varepsilon}\right)(t) d x+\int_{0}^{t} \int_{\Omega} \chi_{B_{n}} A \nabla u_{\varepsilon} \cdot \nabla u_{\varepsilon} d x d t \\
& +\int_{0}^{t} \int_{\Omega} g\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right) \phi_{n}\left(u_{\varepsilon}\right) d x d t \\
& =\int_{\Omega} \Psi_{n}\left(u_{0, \varepsilon}\right) d x+\int_{0}^{t} \int_{\Omega} f_{\varepsilon} \phi_{n}\left(u_{\varepsilon}\right) d x d t  \tag{3.15}\\
& \quad-\int_{0}^{t} \int_{\Omega} d u_{\varepsilon} \phi_{n}\left(u_{\varepsilon}\right) d x d t \\
& \quad-\int_{0}^{t} \int_{\Omega} b \cdot \nabla u_{\varepsilon} \phi_{n}\left(u_{\varepsilon}\right) d x d t
\end{align*}
$$

Since $\left|\phi_{n}(z)\right| \leq 1$, and taking into account the estimate (3.5) we have

$$
\begin{equation*}
\left|\int_{\Omega} d u_{\varepsilon} \phi_{n}\left(u_{\varepsilon}\right) d x\right| \leq \beta\|d\|_{L^{\infty}(Q)} . \tag{3.16}
\end{equation*}
$$

Furthermore, we remark that $u_{\varepsilon} \phi_{n}\left(u_{\varepsilon}\right) \geq 0$. Then, the third term in the left side is non negative. From the coercivity of $A$ (see (1.5)), the positivity of $\Psi_{n}(\cdot)$ and (3.15) we deduce that

$$
\begin{aligned}
\underline{a} \int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t \leq & \int_{Q}\left|\phi_{n}\left(u_{\varepsilon}\right) f_{\varepsilon}\right| d x d t+\int_{\Omega} \Psi_{n}\left(u_{0, \varepsilon}\right) d x \\
& +\beta\|d\|_{L^{\infty}(Q)}+\|b\|_{L^{\infty}(Q)} \int_{Q}\left|\nabla u_{\varepsilon}\right|\left|\phi_{n}\left(u_{\varepsilon}\right)\right| d x d t \\
\leq & \|f\|_{L^{1}(Q)}+\left\|u_{0}\right\|_{L^{1}(\Omega)}+\beta\|d\|_{L^{\infty}(Q)} \\
& +\|b\|_{L^{\infty}(Q)} \int_{Q}\left|\nabla u_{\varepsilon}\right|\left|\phi_{n}\left(u_{\varepsilon}\right)\right| d x d t .
\end{aligned}
$$

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Let us split the last integral in (3.17) as follows

$$
\begin{align*}
\int_{Q}\left|\nabla u_{\varepsilon}\right|\left|\phi_{n}\left(u_{\varepsilon}\right)\right| d x d t & =\int_{B_{n}}\left|\nabla u_{\varepsilon}\right|\left|\phi_{n}\left(u_{\varepsilon}\right)\right| d x d t+\int_{E_{n}}\left|\nabla u_{\varepsilon}\right| d x d t \\
& \leq \int_{B_{n}}\left|\nabla u_{\varepsilon}\right| d x d t+\int_{E_{n}}\left|\nabla u_{\varepsilon}\right| d x d t \tag{3.18}
\end{align*}
$$

since $\left|\phi_{n}\left(u_{\varepsilon}\right)\right|=1$ on $E_{n}=\{(t, x) \in Q:|u(t, x)|>n+1\}$ and $\phi_{n}\left(u_{\varepsilon}\right)=0$ if $\left|u_{\varepsilon}(t, x)\right|<n$.

Using the fact $0 \leq \Psi_{n}(z) \leq|z|$ and (3.18), we deduce from (3.17) that

$$
\begin{aligned}
\underline{a} \int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t \leq & \|f\|_{L^{1}(Q)}+\left\|u_{0}\right\|_{L^{1}(\Omega)}+\beta\|d\|_{L^{\infty}(Q)} \\
& +\|b\|_{L^{\infty}(Q)}\left(\int_{B_{n}}\left|\nabla u_{\varepsilon}\right| d x d t+\int_{E_{n}}\left|\nabla u_{\varepsilon}\right| d x d t\right)
\end{aligned}
$$

By using Holder's and Young's inequalities, we have
$\underline{a} \int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t$

$$
\leq C+\frac{a}{2} \int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t+\frac{1}{2 \underline{a}}\|b\|_{L^{\infty}(Q)}^{2} T|\Omega|+\int_{E_{n}}\left|\nabla u_{\varepsilon}\right| d x d t
$$

where $C=\|f\|_{L^{1}(Q)}+\left\|u_{0}\right\|_{L^{1}(\Omega)}+\beta\|d\|_{L^{\infty}(Q)}$. This finishes the proof of (3.6) with $C_{0}$ and $C_{1}$ depending on $\|f\|_{L^{1}(Q)},\left\|u_{0}\right\|_{L^{1}(Q)},\|b\|_{L^{\infty}(Q)}$, $\|d\|_{L^{\infty}(Q)}, \underline{a},|\Omega|, T$ and the bound $\beta$.

Now, we are interested in the nonlinear term $g_{\varepsilon}$. We have

Lemma 2. Suppose $A, b, d, g$ satisfy (1.3)-(1.7) and let $u_{\varepsilon}$ be a sequence of solutions of (3.4). Then, there exists $C>0$ depending only on the data such that the sequence $g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)$ satisfies

$$
\begin{gather*}
\sup _{\varepsilon>0}\left\|g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)\right\|_{L^{1}(Q)} \leq C  \tag{3.19}\\
\lim _{k \rightarrow \infty} \sup _{\varepsilon>0} \int_{\left|u_{\varepsilon}\right|>k}\left|g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)\right| d x d t=0 . \tag{3.20}
\end{gather*}
$$

Proof. It is clear that

$$
\begin{aligned}
\int_{\left|u_{\varepsilon}\right|>n+1}\left|g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)\right| d x d t & =\int_{\left|u_{\varepsilon}\right|>n+1} \phi_{n}\left(u_{\varepsilon}\right) g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right) d x d t \\
& \leq \int_{Q} \phi_{n}\left(u_{\varepsilon}\right) g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right) d x d t,
\end{aligned}
$$

since we recall that $\left|\phi_{n}(z)\right|=1$ when $|z|>n+1$ and $\phi_{n}\left(u_{\varepsilon}\right) g_{\varepsilon}$ is non negative by the sign condition (1.6). In the sequel, we will often write $g_{\varepsilon}=g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)$ when no confusion can arise. From the positivity of the first and the second terms in (3.15), we obtain

$$
\begin{align*}
0 \leq & \int_{Q} \phi_{n}\left(u_{\varepsilon}\right) g_{\varepsilon} d x d t \\
\leq & \left|\int_{Q} \phi_{n}\left(u_{\varepsilon}\right) f_{\varepsilon} d x d t\right|+\left|\int_{\Omega} \Psi_{n}\left(u_{0, \varepsilon}\right) d x\right| \\
& +\left|\int_{Q} d \phi_{n}\left(u_{\varepsilon}\right) u_{\varepsilon} d x d t\right|+\left|\int_{Q} \phi_{n}\left(u_{\varepsilon}\right) b \cdot \nabla u_{\varepsilon} d x d t\right|  \tag{3.22}\\
\leq & \left\|f_{\varepsilon}\right\|_{L^{1}(Q)}+\left\|u_{0, \varepsilon}\right\|_{L^{1}(\Omega)}+\|d\|_{L^{\infty}(Q)} \int_{Q}\left|u_{\varepsilon}\right| d x d t \\
& +\|b\|_{L^{\infty}(Q)} \int_{Q}\left|\nabla u_{\varepsilon}\right| d x d t
\end{align*}
$$

since $\left|\phi_{n}(z)\right| \leq 1$ and $0 \leq \Psi_{n}(z) \leq|z|$. By (3.2), (3.5) and the estimate (3.7) with $q=1$, we deduce

$$
\begin{equation*}
\int_{\left|u_{\varepsilon}\right|>n+1}\left|g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)\right| d x d t \leq C \tag{3.23}
\end{equation*}
$$

It remains to evaluate the integral over $\left\{\left|u_{\varepsilon}\right|<n+1\right\}$. Assumption (1.6) yields

$$
\begin{align*}
\int_{\left|u_{\varepsilon}\right|<n+1} & \left|g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)\right| d x d t \\
& \leq h(n+1) \int_{\left|u_{\varepsilon}\right|<n+1}\left(\left|\nabla u_{\varepsilon}\right|^{\sigma}+\gamma(t, x)\right) d x d t, \tag{3.24}
\end{align*}
$$

Parabolic equations involving $0^{\text {th }}$ and $1^{\text {st }}$ order terms with $L^{1}$ data 449 where we estimate as follows

$$
\begin{align*}
\int_{\left|u_{\varepsilon}\right|<n+1}\left|\nabla u_{\varepsilon}\right|^{\sigma} d x d t & =\sum_{j=0}^{n} \int_{B_{j}}\left|\nabla u_{\varepsilon}\right|^{\sigma} d x d t  \tag{3.25}\\
& \leq \sum_{j=0}^{n}\left|B_{j}\right|^{1-\sigma / 2}\left(\int_{B_{j}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t\right)^{\sigma / 2}
\end{align*}
$$

By using (3.6), (3.7) and Holder's inequality, we get

$$
\begin{align*}
& \int_{\left|u_{\varepsilon}\right|<n+1}\left|\nabla u_{\varepsilon}\right|^{\sigma} d x d t \\
& \quad \leq(T|\Omega|)^{1-\sigma / 2} \sum_{j=0}^{n}\left(C_{0}+C_{1} \int_{Q}\left|\nabla u_{\varepsilon}\right| d x d t\right)^{\sigma / 2} \leq C \tag{3.26}
\end{align*}
$$

Combining (3.26) with (3.24) and (3.23), we conclude that $g_{\varepsilon}$ is bounded in $L^{1}(Q)$ uniformly in $\varepsilon$.

We turn to the proof of (3.20). Obviously, one has

$$
\begin{equation*}
\int_{\left|u_{\varepsilon}\right|>k}\left|g_{\varepsilon}\right| d x d t \leq \frac{1}{k} \int_{Q} T_{k}\left(u_{\varepsilon}\right) g_{\varepsilon} d x d t \tag{3.27}
\end{equation*}
$$

Similarly, replacing $\phi_{n}\left(u_{\varepsilon}\right)$ by $T_{k}\left(u_{\varepsilon}\right)$ in (3.15) we obtain, similarly to (3.22), that

$$
\begin{align*}
0 \leq & \int_{Q} T_{k}\left(u_{\varepsilon}\right) g_{\varepsilon} d x d t \\
\leq & \int_{Q}\left|f_{\varepsilon} T_{k}\left(u_{\varepsilon}\right)\right| d x d t+\int_{\Omega} S_{k}\left(u_{0, \varepsilon}\right) d x  \tag{3.28}\\
& +\|d\|_{L^{\infty}(Q)} \int_{Q} T_{k}\left(u_{\varepsilon}\right) u_{\varepsilon} d x d t \\
& +\|b\|_{L^{\infty}(Q)} \int_{Q}\left|T_{k}\left(u_{\varepsilon}\right)\right|\left|\nabla u_{\varepsilon}\right| d x d t
\end{align*}
$$

holds. Let $M>0$. According to [16], we use the following trick

$$
\left\{\begin{array}{l}
0 \leq S_{k}(z) \leq M^{2}+k|z| \chi_{|z|>M}  \tag{3.29}\\
\left|T_{k}(z)\right| \leq M+k \chi_{|z|>M}
\end{array}\right.
$$

which gives

$$
\begin{aligned}
\int_{\left|u_{\varepsilon}\right|>k}\left|g_{\varepsilon}\right| d x d t \leq & \frac{M}{k}\left\|f_{\varepsilon}\right\|_{L^{1}(Q)}+\int_{\left|u_{\varepsilon}\right|>M}\left|f_{\varepsilon}\right| d x d t \\
& +\frac{M^{2}}{k}\left\|u_{0, \varepsilon}\right\|_{L^{1}(\Omega)}+\int_{\left|u_{0, \varepsilon}\right|>M}\left|u_{0, \varepsilon}\right| d x \\
& +\frac{M}{k}\|d\|_{L^{\infty}(Q)}\left\|u_{\varepsilon}\right\|_{L^{1}(Q)}+\int_{\left|u_{\varepsilon}\right|>M}\left|u_{\varepsilon}\right| d x d t \\
& +\frac{M}{k}\|b\|_{L^{\infty}(Q)}\left\|\nabla u_{\varepsilon}\right\|_{L^{1}(Q)}+\int_{\left|u_{\varepsilon}\right|>M}\left|\nabla u_{\varepsilon}\right| d x d t
\end{aligned}
$$

by (3.28) and (3.29). Since, on the one hand, $u_{\varepsilon}$ is bounded in $L^{q}(0, T$, $W_{0}^{1, q}(\Omega)$ ) for some $q>1$ and $f_{\varepsilon}, u_{0, \varepsilon}$ are convergent sequences in $L^{1}(Q)$, $L^{1}(\Omega)$ respectively, and, on the other hand,

$$
\sup _{\varepsilon>0} \text { meas }\left\{(t, x) \in Q:\left|u_{\varepsilon}(t, x)\right|>M\right\} \leq \frac{1}{M} \sup _{\varepsilon>0}\left\|u_{\varepsilon}\right\|_{L^{1}(Q)} \leq \frac{\beta T}{M}
$$

tends to 0 as $M$ goes to $\infty$, we can choose $M$ large enough so that the terms

$$
\begin{aligned}
& \sup _{\varepsilon>0} \int_{\left|u_{\varepsilon}\right|>M}\left|f_{\varepsilon}\right| d x d t \\
& \sup _{\varepsilon>0} \int_{\left|u_{0, \varepsilon}\right|>M}\left|u_{0, \varepsilon}\right| d x \\
& \sup _{\varepsilon>0} \int_{\left|u_{\varepsilon}\right|>M}\left|u_{\varepsilon}\right| d x d t \\
& \sup _{\varepsilon>0} \int_{\left|u_{\varepsilon}\right|>M}\left|\nabla u_{\varepsilon}\right| d x d t
\end{aligned}
$$

are arbitrarily smalls, which, achieves the proof of (3.20).
Let the assumptions of Proposition 1 be fulfilled. Then, $u_{\varepsilon}$ is bounded in $L^{q}\left(0, T ; W_{0}^{1, q}(\Omega)\right), g_{\varepsilon}$ is bounded in $L^{1}(Q)$ which imply, in view of the equation satisfied by $u_{\varepsilon}$ that $\partial_{t} u_{\varepsilon}$ is bounded in $L^{1}(0, T$; $\left.W^{-1, q}(\Omega)\right)+L^{1}(Q)$. Therefore, possibly at the cost of extracting sub-

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sequences, see e.g. [18], [20] we can assume that

$$
\begin{cases}u_{\varepsilon} \longrightarrow u, & \text { strongly in } L^{q}(Q)  \tag{3.31}\\ & \text { and almost everywhere in } Q \\ \left|u_{\varepsilon}(t, x)\right| \leq \Gamma(t, x), & \text { almost everywhere in } Q \\ & \text { with } \Gamma \in L^{q}(Q) \\ \nabla u_{\varepsilon} \rightharpoonup \nabla u, & \text { weakly in } L^{q}(Q)\end{cases}
$$

### 3.3. Convergence almost everywhere of the gradients.

The weak convergence of the gradients is clearly insufficient to pass to the limit when $\varepsilon \longrightarrow 0$ in nonlinear terms. Then, we claim

Lemma 3. Let the assumptions of Proposition 1 be fulfilled and let $u_{\varepsilon}$ satisfy (3.31). Then, the sequence $\left\{\nabla u_{\varepsilon}\right\}_{\varepsilon}$ converges to $\nabla u$ almost everywhere as $\varepsilon$ goes to zero.

Proof. It suffices to show that $\left\{\nabla u_{\varepsilon}\right\}_{\varepsilon}$ is a Cauchy sequence in measure, see [8], i.e. for all $\mu>0$

$$
\begin{equation*}
\text { meas }\left\{(t, x) \in Q:\left|\nabla u_{\varepsilon^{\prime}}-\nabla u_{\varepsilon}\right|>\mu\right\} \longrightarrow 0, \tag{3.32}
\end{equation*}
$$

as $\varepsilon^{\prime}, \varepsilon \longrightarrow 0$. Let us denote by $\mathcal{A}$ the subset of $Q$ involved in (3.32). Let $k>0$ and $\delta>0$. Following [17], we remark that

$$
\begin{equation*}
\mathcal{A} \subset \mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{A}_{3} \cup \mathcal{A}_{4}, \tag{3.33}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{A}_{1}=\{(t, x) \in Q: & \left.\left|\nabla u_{\varepsilon}\right| \geq k\right\}, \\
\mathcal{A}_{2}=\{(t, x) \in Q: & \left.\left|\nabla u_{\varepsilon^{\prime}}\right| \geq k\right\}, \\
\mathcal{A}_{3}=\{(t, x) \in Q: & \left.\left|u_{\varepsilon}-u_{\varepsilon^{\prime}}\right| \geq \delta\right\},  \tag{3.34}\\
\mathcal{A}_{4}=\{(t, x) \in Q: & \left|\nabla u_{\varepsilon}-\nabla u_{\varepsilon^{\prime}}\right| \geq \mu:\left|\nabla u_{\varepsilon}\right| \leq k, \\
& \left.\left|\nabla u_{\varepsilon^{\prime}}\right| \leq k,\left|u_{\varepsilon}-u_{\varepsilon^{\prime}}\right| \leq \delta\right\} .
\end{align*}
$$

By Corollary 1 and (3.31), we conclude easily for the three first sets. Indeed, one has

$$
\left|\mathcal{A}_{1}\right| \leq \frac{1}{k}\left\|\nabla u_{\varepsilon}\right\|_{L^{1}(Q)} \leq \frac{C}{k}
$$

and an analoguous estimate holds for $\mathcal{A}_{2}$. Hence, by choosing $k$ large enough, $\left|\mathcal{A}_{1}\right|+\left|\mathcal{A}_{2}\right|$ is arbitrarily small. Similarly, one gets

$$
\left|\mathcal{A}_{3}\right| \leq \frac{1}{\delta}\left\|u_{\varepsilon}-u_{\varepsilon^{\prime}}\right\|_{L^{1}(Q)}
$$

which, for $\delta>0$ fixed, tends to 0 when $\varepsilon, \varepsilon^{\prime} \longrightarrow 0$ since, by (3.31), $u_{\varepsilon}$ is a Cauchy sequence in $L^{1}(Q)$. Then, the proof is completed by choosing $\delta$ so that $\left|\mathcal{A}_{4}\right|$ is given arbitrarily small, uniformly with respect to $\varepsilon, \varepsilon^{\prime}$. To this end, we shall use the equations satisfied by $u_{\varepsilon}$ and $u_{\varepsilon^{\prime}}$. Indeed, we observe that

$$
\begin{align*}
\left|\mathcal{A}_{4}\right| & \leq \frac{1}{\mu^{2}} \int_{\mathcal{A}_{4}}\left|\nabla u_{\varepsilon}-\nabla u_{\varepsilon^{\prime}}\right|^{2} d x d t \\
& \leq \frac{1}{\mu^{2}} \int_{\left|u_{\varepsilon}-u_{\varepsilon^{\prime}}\right| \leq \delta}\left|\nabla u_{\varepsilon}-\nabla u_{\varepsilon^{\prime}}\right|^{2} d x d t  \tag{3.35}\\
& =\frac{1}{\mu^{2}} \int_{Q}\left|\nabla\left(T_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right)\right)\right|^{2} d x d t .
\end{align*}
$$

Substracting the relations obtained with $\phi=T_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right)$ as test funtion in equation (3.4) satisfied successively by $u_{\varepsilon}$ and $u_{\varepsilon^{\prime}}$ leads to

$$
\begin{align*}
& \frac{d}{d t} \int_{\Omega} S_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right) d x+\int_{\Omega} A\left(\nabla u_{\varepsilon}-\nabla u_{\varepsilon^{\prime}}\right) \nabla T_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right) d x \\
& =\int_{\Omega}\left(f_{\varepsilon}-f_{\varepsilon^{\prime}}\right) T_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right) d x  \tag{3.36}\\
& \quad-\int_{\Omega}\left(d\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right)-b \nabla\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right)\right) T_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right) d x \\
& \quad-\int_{\Omega}\left(g_{\varepsilon}-g_{\varepsilon^{\prime}}\right) T_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right) d x
\end{align*}
$$

Since $\left|T_{\delta}(z)\right| \leq \delta$ and $0 \leq S_{\delta}(z) \leq \delta|z|$, integrating (3.36) with respect to $t$ and using the coercivity of $A$ (see (1.5)) yield

$$
\begin{align*}
& \underline{a} \int_{Q}\left|\nabla\left(T_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right)\right)\right|^{2} d x d t \\
& \leq \delta\left(\left\|f_{\varepsilon}-f_{\varepsilon^{\prime}}\right\|_{L^{1}(Q)}+\left\|u_{0, \varepsilon}-u_{0, \varepsilon^{\prime}}\right\|_{L^{1}(\Omega)}\right.  \tag{3.37}\\
& \quad+\|d\|_{L^{\infty}(Q)}\left\|u_{\varepsilon}-u_{\varepsilon^{\prime}}\right\|_{L^{1}(Q)}+\|b\|_{L^{\infty}(Q)}\left\|\nabla\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right)\right\|_{L^{1}(Q)} \\
& \left.\quad+\left\|g_{\varepsilon}-g_{\varepsilon^{\prime}}\right\|_{L^{1}(Q)}\right)
\end{align*}
$$

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Therefore, by using (3.2) and the bounds (3.7), uniform in $\varepsilon$, on $\left\|u_{\varepsilon}\right\|_{L^{1}(Q)},\left\|\nabla u_{\varepsilon}\right\|_{L^{1}(Q)}$ and on $\left\|g_{\varepsilon}\right\|_{L^{1}(Q)}$, we deduce from (3.37) that

$$
\begin{align*}
\underline{a} \int_{Q}\left|\nabla\left(T_{\delta}\left(u_{\varepsilon}-u_{\varepsilon^{\prime}}\right)\right)\right|^{2} d x d t \leq & 2 \delta\left(\|f\|_{L^{1}(Q)}+\left\|u_{0}\right\|_{L^{1}(\Omega)}\right) \\
& +2 \delta C\left(1+\|b\|_{L^{\infty}(Q)}+\|d\|_{L^{\infty}(Q)}\right), \tag{3.38}
\end{align*}
$$

goes to zero as $\delta$ goes to zero, uniformly in $\varepsilon, \varepsilon^{\prime}$. This completes the proof of Lemma 3 .

Having disposed of the proof of Lemma 3, let us consider the behaviour of $g_{\varepsilon}$ as $\varepsilon$ goes to 0 , when it is assumed that $0 \leq \sigma<2$.

Corollary 2. Let the assumptions of Proposition 1 be fulfilled and let $u_{\varepsilon}$ satisfy (3.31). Then, (up to subsequences) the sequence $\left\{g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)\right\}_{\varepsilon}$ converges to $g(u, \nabla u)$ almost everywhere in $Q$ and strongly in $L^{1}(Q)$.

Proof. This result is similar to those obtained in [9] in the context of elliptic problems. For the sake of completeness, we sketch the proof. By combining Lemma 3 and (3.31), it is clear that

$$
g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right) \longrightarrow g(u, \nabla u)
$$

almost everywhere in $Q$ as $\varepsilon$ tends to 0 , since $g(t, x, \lambda, \xi)$ is a continuous function with respect to $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^{N}$. Thus, by classical results, see e.g. [8], the sequence $g_{\varepsilon}$ will be actually strongly convergent in $L^{1}(Q)$ if one shows that $g_{\varepsilon}$ lies in weakly compact set in $L^{1}(Q)$. This property follows from (3.20) since $0 \leq \sigma<2$. Indeed, let $\mathcal{A}$ be a measurable set in $Q$. We split

$$
\begin{equation*}
\int_{\mathcal{A}}\left|g_{\varepsilon}\right| d x d t=\int_{\mathcal{A} \cap\left\{\left|u_{\varepsilon}\right| \leq k\right\}}\left|g_{\varepsilon}\right| d x d t+\int_{\mathcal{A} \cap\left\{\left|u_{\varepsilon}\right|>k\right\}}\left|g_{\varepsilon}\right| d x d t \tag{3.39}
\end{equation*}
$$

where it is clear that

$$
\int_{\mathcal{A} \cap\left\{\left|u_{\varepsilon}\right|>k\right\}}\left|g_{\varepsilon}\right| d x d t \leq \int_{\left|u_{\varepsilon}\right|>k}\left|g_{\varepsilon}\right| d x d t
$$

tends to 0 , uniformly in $\varepsilon$ as $k \longrightarrow \infty$, by (3.20). Moreover, the growth
condition (1.6) and Holder's inequality yield

$$
\begin{align*}
& \int_{\mathcal{A} \cap\left\{\left|u_{\varepsilon}\right| \leq k\right\}}\left|g_{\varepsilon}\right| d x d t \\
& \leq h(k)\left(\int_{\mathcal{A} \cap\left\{\left|u_{\varepsilon}\right| \leq k\right\}}\left|\nabla u_{\varepsilon}\right|^{\sigma} d x d t+\int_{\mathcal{A} \cap\left\{\left|u_{\varepsilon}\right| \leq k\right\}} \gamma(t, x) d x d t\right) \tag{3.40}
\end{align*}
$$

$$
\begin{aligned}
& \leq h(k)\left(\int_{\left|u_{\varepsilon}\right| \leq k}\left|\nabla u_{\varepsilon}\right|^{2} d x d t\right)^{\sigma / 2}|\mathcal{A}|^{(2-\sigma) / 2}+h(k) \int_{\mathcal{A}} \gamma(t, x) d x d t \\
& \leq h(k) C_{k}|\mathcal{A}|^{(2-\sigma) / 2}+h(k) \int_{\mathcal{A}} \gamma(t, x) d x d t
\end{aligned}
$$

by using (3.6) and (3.7) as in (3.26). Since $\sigma<2$ and $\gamma \in L^{1}(Q)$, the right hand side of this last inequality goes to 0 as $|\mathcal{A}| \longrightarrow 0$. We conclude that

$$
\lim _{|\mathcal{A}| \rightarrow 0} \sup _{\varepsilon>0} \int_{\mathcal{A}}\left|g_{\varepsilon}\right| d x d t=0
$$

which completes the proof of Corollary 2.

### 3.4. Cauchy property in $C^{0}\left(0, T ; L^{1}(\Omega)\right)$ and passage to the limit.

We end our review of the properties of the sequence $u_{\varepsilon}$ with the following result.

Lemma 4. Let the assumptions of Proposition 1 be fulfilled. We assume that the sequence $\left\{u_{\varepsilon}\right\}_{\varepsilon}$ satisfies (3.31). Then, $\left\{u_{\varepsilon}\right\}_{\varepsilon}$ is a Cauchy sequence in $C^{0}\left(0, T ; L^{1}(\Omega)\right)$.

Proof. We set $w_{\varepsilon, \varepsilon^{\prime}}=u_{\varepsilon}-u_{\varepsilon^{\prime}}, F_{\varepsilon, \varepsilon^{\prime}}=f_{\varepsilon}-f_{\varepsilon^{\prime}}$ and $G_{\varepsilon, \varepsilon^{\prime}}=g_{\varepsilon}-g_{\varepsilon^{\prime}}$. We multiply the equations (3.4) satisfied respectively by $u_{\varepsilon}$ and $u_{\varepsilon^{\prime}}$ by $T_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right)$. Substracting the obtained relations yields

$$
\begin{align*}
& \frac{d}{d t} \int_{\Omega} S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x+\int_{\left|w_{\varepsilon, \varepsilon^{\prime}}\right| \leq 1} A \nabla w_{\varepsilon, \varepsilon^{\prime}} \cdot \nabla w_{\varepsilon, \varepsilon^{\prime}} d x \\
& =\int_{\Omega} F_{\varepsilon, \varepsilon^{\prime}} T_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x-\int_{\Omega} G_{\varepsilon, \varepsilon^{\prime}} T_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x  \tag{3.41}\\
& \quad-\int_{\Omega}\left(b \cdot \nabla w_{\varepsilon, \varepsilon^{\prime}}+d w_{\varepsilon, \varepsilon^{\prime}}\right) T_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x
\end{align*}
$$

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Since $0 \leq z T_{1}(z)=z^{2} \chi_{|z| \leq 1}+|z| \chi_{|z|>1} \leq z^{2} \chi_{|z| \leq 1}+(2|z|-1) \chi_{|z|>1}=$ $2 S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right)$, one gets

$$
\left|\int_{\Omega} d w_{\varepsilon, \varepsilon^{\prime}} T_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x\right| \leq 2\|d\|_{L^{\infty}(Q)} \int_{\Omega} S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x .
$$

Moreover, one has $\left|T_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right)\right| \leq 1$. Then, integrating (3.41) between 0 and $t$ and from the positivity of $A$, it follows

$$
\begin{aligned}
& \int_{\Omega} S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right)(t) d x \\
& \leq \int_{\Omega} S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}^{0}\right) d x+\int_{0}^{t} \int_{\Omega}\left|F_{\varepsilon, \varepsilon^{\prime}}\right| d x d s+\int_{0}^{t} \int_{\Omega}\left|G_{\varepsilon, \varepsilon^{\prime}}\right| d x d s
\end{aligned}
$$

$$
\begin{equation*}
+\|b\|_{L^{\infty}(Q)} \int_{0}^{t} \int_{\Omega}\left|\nabla w_{\varepsilon, \varepsilon^{\prime}}\right| d x d s+2\|d\|_{L^{\infty}(Q)} \int_{0}^{t} \int_{\Omega} S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x d s \tag{3.42}
\end{equation*}
$$

where $w_{\varepsilon, \varepsilon^{\prime}}^{0}=u_{0, \varepsilon}-u_{0, \varepsilon^{\prime}}$. Hence, Gronwall's lemma implies that

$$
\begin{equation*}
\int_{\Omega} S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x \leq a_{\varepsilon, \varepsilon^{\prime}} \tag{3.43}
\end{equation*}
$$

where $a_{\varepsilon, \varepsilon^{\prime}}$ stands for

$$
\begin{aligned}
& a_{\varepsilon, \varepsilon^{\prime}}=e^{C T}\left(\int_{\Omega} S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}^{0}\right) d x+\int_{Q}\left|F_{\varepsilon, \varepsilon^{\prime}}\right| d x d t\right. \\
&\left.+\int_{Q}\left|G_{\varepsilon, \varepsilon^{\prime}}\right| d x d t+\int_{Q}\left|\nabla w_{\varepsilon, \varepsilon^{\prime}}\right| d x d t\right) \\
& \leq e^{C T}\left(\left\|u_{0, \varepsilon}-u_{0, \varepsilon^{\prime}}\right\|_{L^{1}(\Omega)}+\left\|f_{\varepsilon}-f_{\varepsilon^{\prime}}\right\|_{L^{1}(Q)}\right. \\
&\left.+\left\|g_{\varepsilon}-g_{\varepsilon^{\prime}}\right\|_{L^{1}(Q)}+\left\|\nabla u_{\varepsilon}-\nabla u_{\varepsilon^{\prime}}\right\|_{L^{1}(Q)}\right)
\end{aligned}
$$

since $S_{1}(z) \leq|z|$. By (3.1), $u_{0, \varepsilon}$ and $f_{\varepsilon}$ are convergent sequences in $L^{1}(\Omega)$ and $L^{1}(Q)$, respectively and by Corollary $2, g_{\varepsilon}$ is a convergent sequence in $L^{1}(Q)$. Furthermore, by Corollary 1 and Lemma 3, $\nabla u_{\varepsilon}$ is both bounded in $L^{q}(Q)$ and almost everywhere in $Q$ convergent, which implies that $\nabla u_{\varepsilon}$ is actually strongly convergent in $L^{p}(Q)$ for $1 \leq p<q$, and in particular in $L^{1}(Q)$. Hence, it is clear that $a_{\varepsilon, \varepsilon^{\prime}}$ tends to 0 as
$\varepsilon, \varepsilon^{\prime} \longrightarrow 0$. Finally, by Holder's inequality, we have

$$
\begin{aligned}
& \int_{\Omega}\left|w_{\varepsilon, \varepsilon^{\prime}}\right| d x \\
& =\int_{\left|w_{\varepsilon, \varepsilon^{\prime}}\right| \leq 1}\left|w_{\varepsilon, \varepsilon^{\prime}}\right| d x+\int_{\left|w_{\varepsilon, \varepsilon^{\prime}}\right|>1}\left|w_{\varepsilon, \varepsilon^{\prime}}\right| d x \\
& \leq\left(\int_{\left|w_{\varepsilon, \varepsilon^{\prime}}\right| \leq 1}\left|w_{\varepsilon, \varepsilon^{\prime}}\right|^{2} d x\right)^{1 / 2}\left(\int_{\left|w_{\varepsilon, \varepsilon^{\prime}}\right| \leq 1} \mathbf{1} d x\right)^{1 / 2}+\int_{\left|w_{\varepsilon, \varepsilon^{\prime}}\right|>1}\left|w_{\varepsilon, \varepsilon^{\prime}}\right| d x \\
& \leq \sqrt{|\Omega|}\left(\int_{\left|w_{\varepsilon, \varepsilon^{\prime}}\right| \leq 1} 2 S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x\right)^{1 / 2}+\int_{\left|w_{\varepsilon, \varepsilon^{\prime}}\right|>1} 2 S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) d x
\end{aligned}
$$

since

$$
\frac{|z|}{2} \chi_{|z|>1} \leq\left(\frac{|z|}{2}+\frac{|z|-1}{2}\right) \chi_{|z|>1}=S_{1}\left(w_{\varepsilon, \varepsilon^{\prime}}\right) \chi_{|z|>1}
$$

and

$$
\frac{|z|^{2}}{2} \chi_{|z| \leq 1}=S_{1}(z) \chi_{|z| \leq 1}
$$

By (3.43), we deduce that

$$
\int_{\Omega}\left|u_{\varepsilon}-u_{\varepsilon^{\prime}}\right| d x=\int_{\Omega}\left|w_{\varepsilon, \varepsilon^{\prime}}\right| d x \leq \sqrt{2|\Omega|} \sqrt{a_{\varepsilon, \varepsilon^{\prime}}}+2 a_{\varepsilon, \varepsilon^{\prime}}
$$

tends to 0 as $\varepsilon, \varepsilon^{\prime} \longrightarrow 0$ which proves that $u_{\varepsilon}$ is a Cauchy sequence in $C^{0}\left(0, T ; L^{1}(Q)\right)$.

Finally, we achieve the proof of Theorem 1 by passing easily to the limit $\varepsilon \longrightarrow 0$ in the following weak formulation

$$
\begin{align*}
& \int_{\Omega} u_{\varepsilon} \phi(t) d x-\int_{\Omega} u_{0, \varepsilon} \phi(0, x) d x \\
& -\int_{0}^{t} \int_{\Omega} u_{\varepsilon} \partial_{t} \phi d x d t+\int_{0}^{t} \int_{\Omega} A \nabla u_{\varepsilon} \cdot \nabla \phi d x d t \\
& +\int_{0}^{t} \int_{\Omega}\left(b \cdot \nabla u_{\varepsilon}+d u_{\varepsilon}+g_{\varepsilon}\left(u_{\varepsilon}, \nabla u_{\varepsilon}\right)\right) \phi d x d t  \tag{3.44}\\
& =\int_{0}^{t} \int_{\Omega} f_{\varepsilon} \phi d x d t,
\end{align*}
$$

with $\phi \in C^{0}\left(0, T, W_{0}^{1, q^{\prime}}(\Omega)\right) \cap C^{1}\left(0, T, L^{q^{\prime}}(\Omega)\right)$, obtaining in this way that the limit $u$ is a solution of (1.1) in the sense of (1.9).

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Remark 1. We point out the fact that the assumption on the derivatives of the coefficient $b$ is useful uniquely to obtain the uniform bound (3.5) in $L^{\infty}\left(0, T ; L^{1}(\Omega)\right)$.

Remark 2. A similar existence result may be obtained if the strong convergences in (3.1) are replaced by weak $L^{1}$ convergences.

## 4. Entropic solutions: End of proof of Theorem 2.

In this Section, we assume $g=0$. First, we prove that, besides the weak "natural" formulation (1.9), the limit $u$ of the sequence of approximate solutions $u_{\varepsilon}$ also satisfies the entropic relation (1.11). Having disposed of the existence of such a solution, we show that $u$ is unique in the class of entropic solutions.

### 4.1. Existence of entropic solution.

Let us recall the convergence properties obtained in Section 3 on the sequence $u_{\varepsilon}$, after suitable extraction of subsequences. First, $u_{\varepsilon}$ converges to $u$ strongly in $L^{q}(Q)$, with $1 \leq q<(N+2) /(N+1)$, in $C^{0}\left(0, T ; L^{1}(\Omega)\right)$, almost everywhere in $Q$ and is dominated. Moreover, $\nabla u_{\varepsilon}$ is bounded in $L^{q}(Q)$ and converges almost everywhere in $Q$ to $\nabla u$; thus, the convergence actually holds strongly in $L^{p}(Q)$, for $1 \leq$ $p<q$ and in particular in $L^{1}(Q)$. We can also assume that $\nabla u_{\varepsilon}$ is dominated. Let $k>0$. Since $T_{k}$ is continuous and bounded by $k$, $T_{k}\left(u_{\varepsilon}\right)$ converges almost everywhere in $Q$ and, by Lebesgue's theorem, strongly in $L^{2}(Q)$ to $T_{k}(u)$. Furthermore, from (3.10) it is easy to see that $\nabla T_{k}\left(u_{\varepsilon}\right)$ is bounded in $L^{2}(Q)$ (uniformly in $\varepsilon$, the bound depending on $k$ ). Therefore, we may suppose that $\nabla T_{k}\left(u_{\varepsilon}\right) \rightharpoonup \nabla T_{k}(u)$ weakly in $L^{2}(Q)$.

Fix $k>0$ and let $\psi \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right) \cap L^{\infty}(Q)$ with $\partial_{t} \psi \in$ $L^{2}\left(0, T ; H^{-1}(\Omega)\right)$. We set $P=\|\psi\|_{L^{\infty}(Q)}$. It is clear that $\left|T_{k}\left(u_{\varepsilon}-\psi\right)\right| \leq$ $k$ and

$$
\begin{equation*}
\left|\nabla T_{k}\left(u_{\varepsilon}-\psi\right)\right|=\chi_{\left|u_{\varepsilon}-\psi\right| \leq k}\left|\nabla\left(u_{\varepsilon}-\psi\right)\right| \leq \chi_{\left|u_{\varepsilon}\right| \leq k+P}\left|\nabla u_{\varepsilon}\right|+|\nabla \psi|, \tag{4.1}
\end{equation*}
$$

which implies that $T_{k}\left(u_{\varepsilon}-\psi\right)$ belongs to (a bounded set in) $L^{2}(0, T$;
$\left.H_{0}^{1}(\Omega)\right)$. Then, plugging $\phi=T_{k}\left(u_{\varepsilon}-\psi\right)$ in (3.4) gives

$$
\begin{align*}
& \int_{\Omega} S_{k}\left(u_{\varepsilon}-\psi\right)(T) d x-\int_{\Omega} S_{k}\left(u_{0, \varepsilon}-\psi(0, \cdot)\right) d x \\
& +\int_{0}^{T}\left\langle\partial_{t} \psi, T_{k}\left(u_{\varepsilon}-\psi\right)\right\rangle d s \\
& +\int_{Q} A \nabla u_{\varepsilon} \nabla T_{k}\left(u_{\varepsilon}-\psi\right) d x d s  \tag{4.2}\\
& +\int_{Q}\left(b \cdot \nabla u_{\varepsilon}+d u_{\varepsilon}\right) T_{k}\left(u_{\varepsilon}-\psi\right) d x d s \\
& =\int_{Q} f_{\varepsilon} T_{k}\left(u_{\varepsilon}-\psi\right) d x d s
\end{align*}
$$

We shall study the behaviour of (4.2) when we let $\varepsilon$ go to 0 . Since $S_{k}$ is $k$-Lipschitz, one has

$$
\left|\int_{\Omega} S_{k}\left(u_{\varepsilon}-\psi\right)-S_{k}(u-\psi) d x\right| \leq k \int_{\Omega}\left|u_{\varepsilon}-u\right| d x
$$

for all $t \in[0, T]$ where the right hand side tends to 0 as $\varepsilon \longrightarrow 0$. Next, since we have assumed that $\partial_{t} \psi$ lies in $L^{2}\left(0, T ; H^{-1}(\Omega)\right)$, we have to prove that

$$
\begin{equation*}
T_{k}\left(u_{\varepsilon}-\psi\right) \rightharpoonup T_{k}(u-\psi), \quad \text { in } L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right) \tag{4.3}
\end{equation*}
$$

Obviously, this convergence holds in $L^{2}(Q)$ since $u_{\varepsilon}$ converges to $u$ almost everywhere in $Q$ and $T_{k}$ is continuous and bounded by $k$. Derivating $T_{k}\left(u_{\varepsilon}-\psi\right)$ leads to

$$
\begin{align*}
\nabla T_{k}\left(u_{\varepsilon}-\psi\right) & =\nabla T_{k}\left(T_{k+P}\left(u_{\varepsilon}\right)-\psi\right) \\
& =\chi_{\left|T_{k+P}\left(u_{\varepsilon}\right)-\psi\right| \leq k}\left(\nabla T_{k+P}\left(u_{\varepsilon}\right)-\nabla \psi\right) \tag{4.4}
\end{align*}
$$

where, by the above mentioned convergences, $\nabla T_{k+P}\left(u_{\varepsilon}\right)$ converges weakly in $L^{2}(Q)$ to $\nabla T_{k+P}(u)$ which proves (4.3). We also deal easily with the terms involving $b, d$ and $f_{\varepsilon}$ since it appears in these integrals a product of the sequence $T_{k}\left(u_{\varepsilon}-\psi\right)$ which converges almost everywhwere in $Q$ and is bounded in $L^{\infty}(Q)$ with a sequence which converges at least weakly in $L^{1}(Q)$. Finally, it remains to show that

$$
\begin{equation*}
\int_{Q} A \nabla u \cdot \nabla T_{k}(u-\psi) d x d s \leq \liminf _{\varepsilon \rightarrow 0} \int_{Q} A \nabla u_{\varepsilon} \cdot \nabla T_{k}\left(u_{\varepsilon}-\psi\right) d x d s \tag{4.5}
\end{equation*}
$$

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By using (4.4), we split the integral in the right hand side as follows

$$
\begin{aligned}
\int_{Q} A \nabla u_{\varepsilon} & \cdot \nabla T_{k}\left(u_{\varepsilon}-\psi\right) d x d s \\
= & \int_{Q} \chi_{\left|T_{k+P}\left(u_{\varepsilon}\right)-\psi\right| \leq k} A \nabla u_{\varepsilon} \cdot \nabla T_{k+P}\left(u_{\varepsilon}\right) d x d s \\
& -\int_{Q} \chi_{\left|T_{k+P}\left(u_{\varepsilon}\right)-\psi\right| \leq k} A \nabla u_{\varepsilon} \cdot \nabla \psi d x d s \\
= & A_{\varepsilon}-B_{\varepsilon},
\end{aligned}
$$

where, by the same argument as above, we have

$$
\lim _{\varepsilon \rightarrow 0} B_{\varepsilon}=\int_{Q} \chi_{\left|T_{k+P}(u)-\psi\right| \leq k} A \nabla u \cdot \nabla \psi d x d s
$$

Therefore (4.5) is a consequence of Fatou's lemma, applied by combining (4.3) with $T_{k+P}$ and the positiveness property (1.5). Finally, letting $\varepsilon \longrightarrow 0$ in (4.2), one gets (1.11).

### 4.2. Uniqueness.

Let $v$ be an entropic solution. To obtain the uniqueness, we will show that $v=u, u$ still being the solution obtained by approximation. To this end, it would be natural to choose $T_{h}\left(u_{\varepsilon}\right)$ as test function $\psi$ in (1.11). However, as pointed out in [17], $T_{h}$ is not regular enough which leads to difficulties in order to write the term involving the time derivative of the test function. Then, it is necessary to regularize the truncation. Let $\nu>0$. We introduce $T_{h}^{\nu} \in C^{2}(\mathbb{R}, \mathbb{R})$ satisfying

$$
\begin{cases}\left(T_{h}^{\nu}\right)^{\prime}(z)=0, & \text { if }|z| \geq h  \tag{4.6}\\ \left(T_{h}^{\nu}\right)^{\prime}(z)=1, & \text { if }|z| \leq h-\nu \\ 0 \leq\left(T_{h}^{\nu}\right)^{\prime}(z) \leq\left(T_{h}\right)^{\prime}(z) \leq 1 . & \end{cases}
$$

Note that $\left|T_{h}^{\nu}(z)\right| \leq\left|T_{h}(z)\right|$, and $\left(T_{h}^{\nu}\right) "(z)=0$ when $|z| \geq h$ or $|z| \leq$ $h-\nu$.

In the sequel, let us denote

$$
L(f, u)=f-b \cdot \nabla u-d u
$$

We take $\psi=T_{h}^{\nu}\left(u_{\varepsilon}\right)$ as test function in the entropic formulation (1.11) satisfied by $v$, we have

$$
\begin{align*}
& {\left[\int_{\Omega} S_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right) d x\right]_{0}^{t}} \\
& +\int_{0}^{t}\left\langle\partial_{t} u_{\varepsilon},\left(T_{h}^{\nu}\right)^{\prime}\left(u_{\varepsilon}\right) T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right)\right\rangle d s \\
& +\int_{0}^{t} \int_{\Omega} A \nabla v \nabla T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right) d x d s  \tag{4.7}\\
& \leq \int_{0}^{t} \int_{\Omega} L(f, v)\left(T_{h}^{\nu}\right)^{\prime}\left(u_{\varepsilon}\right) T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right) d x d s .
\end{align*}
$$

By using (3.4), we write the term involving the time derivative of the test function as follows

$$
\begin{equation*}
\int_{0}^{t}\left\langle\partial_{t} u_{\varepsilon}, \phi\right\rangle d s=\int_{0}^{t} \int_{\Omega}\left(L\left(f_{\varepsilon}, u_{\varepsilon}\right) \phi-A \nabla u_{\varepsilon} \nabla \phi\right) d x d s \tag{4.8}
\end{equation*}
$$

where $\phi=\left(T_{h}^{\nu}\right)^{\prime}\left(u_{\varepsilon}\right) T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right)$ and, consequently,

$$
\nabla \phi=\nabla u_{\varepsilon}\left(T_{h}^{\nu}\right)^{\prime \prime}\left(u_{\varepsilon}\right) T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right)+\left(T_{h}^{\nu}\right)^{\prime}\left(u_{\varepsilon}\right) \nabla\left(T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right)\right) .
$$

By (4.8), the entropic formulation (4.7) is equivalent to

$$
\begin{align*}
& {\left[\int_{\Omega} S_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right) d x\right]_{0}^{t}} \\
& +\int_{0}^{t} \int_{\Omega} A\left(\nabla v-\left(T_{h}^{\nu}\right)^{\prime}\left(u_{\varepsilon}\right) \nabla u_{\varepsilon}\right) \nabla T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right) d x d s  \tag{4.9}\\
& -\int_{0}^{t} \int_{\Omega} A \nabla u_{\varepsilon} \nabla u_{\varepsilon}\left(T_{h}^{\nu}\right)^{\prime \prime}\left(u_{\varepsilon}\right) T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right) d x d s \\
& \leq \int_{0}^{t} \int_{\Omega}\left(L(f, v)-L\left(f_{\varepsilon}, u_{\varepsilon}\right)\left(T_{h}^{\nu}\right)^{\prime}\left(u_{\varepsilon}\right)\right) T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right) d x d s
\end{align*}
$$

Now, according to [17], let successively $\nu \longrightarrow 0, \varepsilon \longrightarrow 0$ and $h \longrightarrow \infty$. Difficulties only arise from the third integal in the left hand, denoted by $I_{\nu}$ which involves the second derivative of $T_{h}^{\nu}$; the remaining integrals being treated by using the Lebesgue theorem. Indeed, it is clear that

$$
\begin{gathered}
\left|S_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right)\right| \leq k|v|+k h \\
\left|T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right)\right| \leq k \\
\left|\left(T_{h}^{\nu}\right)^{\prime}\left(u_{\varepsilon}\right) T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right)\right| \leq k,
\end{gathered}
$$

and

$$
\left|\nabla T_{k}\left(v-T_{h}^{\nu}\left(u_{\varepsilon}\right)\right)\right| \leq\left(\left|\nabla T_{k+h}(v)\right|+\left|\nabla T_{h}\left(u_{\varepsilon}\right)\right|\right)
$$

Next, we wish to obtain an estimate on $I_{\nu}$. Following [17] (see also [2]), we define another $C^{2}$ function $R_{h}^{\nu}$, satisfying for $z \geq 0:\left(R_{h}^{\nu}\right)^{\prime}(z)=$ $1-\left(T_{h}^{\nu}\right)^{\prime}(z), R_{h}^{\nu}(0)=0, R_{h}^{\nu}(-z)=R_{h}^{\nu}(z)$.

Take $\left(R_{h}^{\nu}\right)^{\prime}\left(u_{\varepsilon}\right)$ as test function in (3.4). By using the positivity of $R_{h}^{\nu}$ and the fact that $\left(R_{h}^{\nu}\right)^{\prime \prime}(z)=\left|\left(T_{h}^{\nu}\right)^{\prime \prime}(z)\right|$, we obtain according to [17] the following estimate

$$
\begin{align*}
\left|I_{\nu}\right| \leq & k \int_{0}^{t} \int_{\Omega}\left|L\left(f_{\varepsilon}, u_{\varepsilon}\right)\right| \chi_{\left|u_{\varepsilon}\right|>h-\nu} d x d s  \tag{4.10}\\
& +k \int_{\Omega}\left|u_{0, \varepsilon}\right| \chi_{\left|u_{0, \varepsilon}\right|>h-\nu} d x .
\end{align*}
$$

By Lebesgue's theorem, we can pass to the limit $\nu \longrightarrow 0$ in the right hand side of (4.10), obtaining without difficulties

$$
\limsup _{\nu \rightarrow 0}\left|I_{\nu}\right| \leq k \int_{0}^{t} \int_{\Omega}\left|L\left(f_{\varepsilon}, u_{\varepsilon}\right)\right| \chi_{\left|u_{\varepsilon}\right|>h} d x d s+k \int_{\Omega}\left|u_{0, \varepsilon}\right| \chi_{\left|u_{0, \varepsilon}\right|>h} d x .
$$

Collecting these results, we get from (4.9) the following estimate

$$
\begin{align*}
& {\left[\int_{\Omega} S_{k}\left(v-T_{h}\left(u_{\varepsilon}\right)\right) d x\right]_{0}^{t}} \\
& +\int_{0}^{t} \int_{\Omega} A \nabla\left(v-T_{h}\left(u_{\varepsilon}\right)\right) \nabla T_{k}\left(v-T_{h}\left(u_{\varepsilon}\right)\right) d x d s \\
& \leq \int_{0}^{t} \int_{\Omega}\left(L(f, v)-L\left(f_{\varepsilon}, u_{\varepsilon}\right)\left(T_{h}\right)^{\prime}\left(u_{\varepsilon}\right)\right) T_{k}\left(v-T_{h}\left(u_{\varepsilon}\right)\right) d x d s  \tag{4.11}\\
& \quad+k \int_{0}^{t} \int_{\Omega}\left|L\left(f_{\varepsilon}, u_{\varepsilon}\right)\right| \chi_{\left|u_{\varepsilon}\right|>h} d x d s+k \int_{\Omega}\left|u_{0, \varepsilon}\right| \chi_{\left|u_{0, \varepsilon}\right|>h} d x .
\end{align*}
$$

The assumptions on the sequence of data and the properties of $u_{\varepsilon}$ recalled above allow us to apply the Lebesgue theorem to pass to the limit as $\varepsilon \longrightarrow 0$ in the first term of the left hand side as well as in the right hand side. In addition, the coercivity of $A$ (see (1.5)) and the following almost everywhere convergence

$$
\begin{aligned}
\nabla T_{k}\left(v-T_{h}\left(u_{\varepsilon}\right)\right)=\chi_{\left|v-T_{h}\left(u_{\varepsilon}\right)\right| \leq k} & \left(\nabla v-\chi_{\left|u_{\varepsilon}\right| \leq h} \nabla u_{\varepsilon}\right) \\
& \longrightarrow \chi_{\left|v-T_{h}(u)\right| \leq k}\left(\nabla v-\chi_{|u| \leq h} \nabla u\right),
\end{aligned}
$$

permit us to apply Fatou's lemma on the second term in the left hand side.

It remains to deal with $h \longrightarrow \infty$ in the following relation

$$
\begin{align*}
& {\left[\int_{\Omega} S_{k}\left(v-T_{h}(u)\right) d x\right]_{0}^{t}} \\
& +\int_{0}^{t} \int_{\Omega} A \nabla T_{k}\left(v-T_{h}(u)\right) \nabla T_{k}\left(v-T_{h}(u)\right) d x d s  \tag{4.12}\\
& \leq \int_{0}^{t} \int_{\Omega}\left(L(f, v)-L(f, u)\left(T_{h}\right)^{\prime}(u)\right) T_{k}\left(v-T_{h}(u)\right) d x d s \\
& \quad+k \mathcal{O}(h)
\end{align*}
$$

where $\mathcal{O}(h)$ stands for

$$
\begin{aligned}
& \int_{0}^{t} \int_{\Omega}\left(|f|+\|b\|_{L^{\infty}(Q)}|\nabla u|+\|d\|_{L^{\infty}(Q)}|u|\right) \chi_{|u|>h} d x d s \\
&+\int_{\Omega}\left|u_{0}\right| \chi_{\left|u_{0}\right|>h} d x
\end{aligned}
$$

which goes to 0 as $h \longrightarrow \infty$ because $f, u, \nabla u$ belong to $L^{1}(Q)$.
We search for another expression of the integral in the right hand side of (4.12). We write, on the one hand,

$$
\begin{align*}
L(f, v)- & L(f, u)\left(T_{h}\right)^{\prime}(u) \\
& =f-b \cdot \nabla v-d v-(f-b \cdot \nabla u-d u)\left(T_{h}\right)^{\prime}(u)  \tag{4.13}\\
& =L(f, u)\left(1-\left(T_{h}\right)^{\prime}(u)\right)-b \cdot \nabla(v-u)-d(v-u)
\end{align*}
$$

and, on the other hand

$$
\begin{align*}
\int_{0}^{t} \int_{\Omega}(b \cdot \nabla(v-u)) & T_{k}\left(v-T_{h}(u)\right) d x d s \\
=-\int_{0}^{t} \int_{\Omega} & (v-u) b \cdot \nabla T_{k}\left(v-T_{h}(u)\right)  \tag{4.14}\\
& +(\nabla \cdot b)(v-u) T_{k}\left(v-T_{h}(u)\right) d x d s
\end{align*}
$$

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By (4.13)-(4.14), inequality (4.12) becomes

$$
\begin{align*}
& {\left[\int_{\Omega} S_{k}\left(v-T_{h}(u)\right) d x\right]_{0}^{t}} \\
& +\int_{0}^{t} \int_{\Omega} A \nabla T_{k}\left(v-T_{h}(u)\right) \nabla T_{k}\left(v-T_{h}(u)\right) d x d s \\
& \leq \int_{0}^{t} \int_{\Omega}\left|(v-u) b \cdot \nabla T_{k}\left(v-T_{h}(u)\right)\right| d x d s  \tag{4.15}\\
& \quad+\int_{0}^{t} \int_{\Omega}\left|(d-(\nabla \cdot b))(v-u) T_{k}\left(v-T_{h}(u)\right)\right| d x d s \\
& \quad+\int_{0}^{t} \int_{\Omega}\left|L(f, u)\left(1-\left(T_{h}\right)^{\prime}(u)\right) T_{k}\left(v-T_{h}(u)\right)\right| d x d s \\
& \quad+k \mathcal{O}(h) .
\end{align*}
$$

We remark that $1-\left(T_{h}\right)^{\prime}(u)$ tends to 0 as $h \longrightarrow \infty$, and by Lebegue's theorem the third term of the right hand side can be included in the general expression $k \mathcal{O}(h)$ which tends to 0 as $h \longrightarrow \infty$.

Proceeding as in Section 4 leads to

$$
\begin{aligned}
& \int_{\Omega} S_{k}\left(v-T_{h}(u)\right)(t) d x \\
& +\frac{a}{2} \int_{0}^{t} \int_{\Omega}\left|\nabla T_{k}\left(v-T_{h}(u)\right)\right|^{2} d x d s \\
& \leq \int_{\Omega} S_{k}\left(v-T_{h}(u)\right)(0) d x \\
& \quad+\frac{1}{a}\|b\|_{L^{\infty}(Q)} \int_{0}^{t} \int_{\Omega} \chi_{\left|v-T_{h}(u)\right|<k}|v-u|^{2} d x d s \\
& \quad+\left(\|d\|_{L^{\infty}(Q)}+\|\nabla \cdot b\|_{L^{\infty}(Q)}\right) \\
& \quad \cdot \int_{0}^{t} \int_{\Omega}\left|(v-u) T_{k}\left(v-T_{h}(u)\right)\right| d x d s \\
& \quad+k \mathcal{O}(h) .
\end{aligned}
$$

In classical way, by Lebesgue's theorem and Fatou's lemma, letting $h$ go to $\infty$, we are led to inequality (4.16) where $T_{h}(u)$ is replaced by $u$ and
the last term in the right hand side vanishes. By using $0 \leq z T_{k}(z) \leq$ $2 S_{k}(z), 0 \leq z^{2} \chi_{|z| \leq k} \leq 2 S_{k}(z)$, we deduce as in Section 2 that

$$
\begin{align*}
& \int_{\Omega} S_{k}(v-u)(t) d x+\frac{a}{2} \int_{0}^{t} \int_{\Omega}\left|\nabla T_{k}(v-u)\right|^{2} d x d s  \tag{4.17}\\
& \quad \leq \int_{\Omega} S_{k}(v-u)(0) d x+C(b, d) \int_{0}^{t} \int_{\Omega} S_{k}(v-u) d x d s
\end{align*}
$$

where

$$
C(b, d)=2\|\nabla b\|_{L^{\infty}(Q)}+2\|d\|_{L^{\infty}(Q)}+\frac{1}{\underline{a}}\|b\|_{L^{\infty}(Q)}
$$

Therefore, it suffices to apply Gronwall's lemma to deduce that

$$
\int_{\Omega} S_{k}(v-u)(t) d x=0
$$

since $v_{0}=u_{0}$, which gives $v=u$.

## 5. Lower regularity requirement on $b$.

Our aim in this section is to weaken the regularity requirement on $b$, replacing the $L^{\infty}(Q)$ condition by $b \in L^{s}(Q)$ for $s>q^{\prime}$; precisely one has

Theorem 3. Let $A, d, g$ satisfy (1.3)-(1.5) and let $b \in L^{s}(Q)$ with $s>$ $q^{\prime}=q /(q-1)($ recall that $1 \leq q<(N+2) /(N+1))$ and $\nabla \cdot b \in L^{\infty}(Q)$. Then, there exists a weak solution of (1.1) in the sense of Definition 1.

Proof. The outline of the proof is the same of Theorem 1. Consider the approximate solution of (3.4). In the first step, we show, according to (3.5), that

$$
\begin{equation*}
u_{\varepsilon} \text { is uniformly bounded in } L^{\infty}\left(0, T ; L^{1}(\Omega)\right) . \tag{5.1}
\end{equation*}
$$

Reproducing the proof of Proposition 1, we take $\phi=T_{k}\left(u_{\varepsilon}\right)$ as test function in (3.4), and we find (3.8). All terms are treated as above except those involving $u_{\varepsilon} b \cdot \nabla T_{k}\left(u_{\varepsilon}\right)$ which becomes

$$
\begin{aligned}
& \left|\int_{Q} u_{\varepsilon} b \cdot \nabla T_{k}\left(u_{\varepsilon}\right) d x d t\right| \\
& \quad \leq \frac{a}{2} \int_{Q}\left|\nabla T_{k}\left(u_{\varepsilon}\right)\right|^{2} d x d t+\frac{1}{2 \underline{a}} \int_{Q} \chi_{\left|u_{\varepsilon}\right| \leq k}\left|b u_{\varepsilon}\right|^{2} d x d t
\end{aligned}
$$

by using Holder's and Young's inequalities. Since $s>2$ the last integral is bounded by $(1 / 2 \underline{a}) k^{2}(T|\Omega|)^{s /(s-2)}\|b\|_{L^{s}(Q)}^{2}$. Then, from (3.12), we deduce that

$$
\int_{\Omega} S_{k}\left(u_{\varepsilon}\right)(t) d x \leq \alpha_{0}+\alpha_{1} \int_{0}^{t} \int_{\Omega} S_{k}\left(u_{\varepsilon}\right) d x d s
$$

holds where $\alpha_{0}$ depends on $\|f\|_{L^{1}(Q)},\left\|u_{0}\right\|_{L^{1}(\Omega)}$ and $\|b\|_{L^{s}(Q)}$ and $\alpha_{1}$ depends on $\|d\|_{L^{\infty}(Q)},\|\nabla b\|_{L^{\infty}(Q)}$. Gronwall's Lemma permits us to conclude as in Proposition 1 and leads to (5.1).

To establish an estimate on the solutions in $L^{q}\left(0, T, W_{0}^{1, q}(\Omega)\right)$, we follow step by step the proofs of estimate (3.6) and of Lemma 1 which need to be adapted. For that, take $\phi=\phi_{n}\left(u_{\varepsilon}\right)$ in (3.4). We deduce from (3.15)

$$
\begin{aligned}
\underline{a} \int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t \leq & \int_{Q}\left|\phi_{n}\left(u_{\varepsilon}\right) f_{\varepsilon}\right| d x d t+\int_{\Omega} \Psi_{n}\left(u_{0, \varepsilon}\right) d x \\
& +\beta\|d\|_{L^{\infty}(Q)}+\int_{E_{n}}\left|b \cdot \nabla u_{\varepsilon}\right|\left|\phi_{n}\left(u_{\varepsilon}\right)\right| d x d t \\
\leq & \|f\|_{L^{1}(Q)}+\left\|u_{0}\right\|_{L^{1}(\Omega)}+\beta\|d\|_{L^{\infty}(Q)} \\
& +\left(\int_{E_{n}}|b|^{q^{\prime}} d x d t\right)^{1 / q^{\prime}}\left(\int_{Q}\left|\nabla u_{\varepsilon}\right|^{q} d x d t\right)^{1 / q} .
\end{aligned}
$$

Since $s>q^{\prime}$, using Holder's inequality, with exponents $s / q^{\prime}$ and $s /(s-$ $\left.q^{\prime}\right)$, yields the following substitute to (3.6)

$$
\begin{equation*}
\int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t \leq C_{0}+C_{1}\left\|\nabla u_{\varepsilon}\right\|_{L^{q}(Q)}\left|E_{n}\right|^{\left(s-q^{\prime}\right) /\left(s q^{\prime}\right)} \tag{5.2}
\end{equation*}
$$

where $C_{0}$ stands for

$$
\frac{\frac{1}{a}}{\underline{a}}\left(\|f\|_{L^{1}(Q)}+\left\|u_{0}\right\|_{L^{1}(\Omega)}+\beta\|d\|_{L^{\infty}(Q)}\right)
$$

and $C_{1}=\|b\|_{L^{s}(Q)}$.
Recall that $q<2$. Therefore, Holder's inequality yields

$$
\begin{align*}
\int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{q} d x d t \leq & \left|B_{n}\right|^{(2-q) / 2}\left(\int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{2} d x d t\right)^{q / 2} \\
\leq & \left|B_{n}\right|^{(2-q) / 2}  \tag{5.3}\\
& \cdot\left(C_{0}^{q / 2}+C_{1}^{q / 2}\left\|\nabla u_{\varepsilon}\right\|_{L^{q}(Q)}^{q / 2}\left|E_{n}\right|^{\left(\left(s-q^{\prime}\right) /\left(s q^{\prime}\right)\right)(q / 2)}\right) .
\end{align*}
$$

Let $r$ and $K$ as in Lemma 1. By using (2.6), one gets

$$
\begin{aligned}
& \int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{q} d x d t \\
& \leq C_{0}^{q / 2} \frac{1}{n^{r(2-q) / 2}}\left(\int_{B_{n}}\left|u_{\varepsilon}\right|^{r} d x d t\right)^{(2-q) / 2} \\
&+C_{1}^{q / 2}\left\|\nabla u_{\varepsilon}\right\|_{L^{q}(Q)}^{q / 2}\left\|u_{\varepsilon}\right\|_{L^{r}(Q)}^{r\left(\left(s-q^{\prime}\right) /\left(s q^{\prime}\right)\right)(q / 2)} \\
& \quad \frac{1}{n^{r\left(\left(s-q^{\prime}\right) /\left(s q^{\prime}\right)\right)(q / 2)+r(2-q) / 2}}\left(\int_{B_{n}}\left|u_{\varepsilon}\right|^{r} d x d t\right)^{(2-q) / 2} .
\end{aligned}
$$

Repeated use of Holder's inequality, as in (2.11), implies

$$
\begin{aligned}
& \sum_{n=K+1}^{\infty} \int_{B_{n}}\left|\nabla u_{\varepsilon}\right|^{q} d x d t \\
& \leq C_{0}^{q / 2}\left(\sum_{n=K+1}^{\infty} \frac{1}{n^{r(2-q) / q}}\right)^{q / 2}\left\|u_{\varepsilon}\right\|_{L^{r}(q)}^{r(2-q) / 2} \\
&4)+C_{1}^{q / 2}\left\|\nabla u_{\varepsilon}\right\|_{L^{q}(Q)}^{q / 2}\left\|u_{\varepsilon}\right\|_{L^{r}((q)}^{\left.r\left(\left(s-q^{\prime}\right) /\left(s q^{\prime}\right)\right)(q / 2)+(2-q) / 2\right)} \\
& \cdot\left(\sum_{n=K+1}^{\infty} \frac{1}{n^{r\left(\left(\left(s-q^{\prime}\right) /\left(s q^{\prime}\right)\right)+(2-q) / q\right)}}\right)^{q / 2} .
\end{aligned}
$$

The conditions

$$
\begin{equation*}
r \frac{2-q}{q}>1 \quad \text { and } \quad r\left(\frac{s-q^{\prime}}{s q^{\prime}}+\frac{2-q}{q}\right)>1 \tag{5.5}
\end{equation*}
$$

ensure the convergence of the series. As in Section 2, we deduce from (2.8), that

$$
\begin{aligned}
& \left\|\nabla u_{\varepsilon}\right\|_{L^{q}(Q)}^{q} \\
& \leq \\
& C(K) \\
& \quad+\delta(K)\left(\left\|u_{\varepsilon}\right\|_{L^{r}(Q)}^{r(2-q) / 2}+\left\|\nabla u_{\varepsilon}\right\|_{L^{q}(Q)}^{q / 2}\left\|u_{\varepsilon}\right\|_{L^{r}(Q)}^{r\left(\left(\left(s-q^{\prime}\right) /\left(s q^{\prime}\right)\right)(q / 2)+(2-q) / 2\right)}\right)
\end{aligned}
$$

holds where $\delta(K)$ tends to zero as $K$ goes to infinity. Therefore, Young's inequality yields

$$
\begin{align*}
\left\|\nabla u_{\varepsilon}\right\|_{L^{q}(Q)}^{q} \leq & C(K) \\
& +\delta(K)\left(\left\|u_{\varepsilon}\right\|_{L^{r}(Q)}^{r(2-q) / 2}+\left\|u_{\varepsilon}\right\|_{L^{r}(Q)}^{r\left(\left(s-q^{\prime}\right) /\left(s q^{\prime}\right) q+2-q\right)}\right) . \tag{5.6}
\end{align*}
$$

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If we choose $r=q(N+1) / N$, estimate $(2.17)$ becomes

$$
\left\|u_{\varepsilon}\right\|_{L^{r}(Q)}^{r} \leq C\left\|u_{\varepsilon}\right\|_{L^{q}\left(0, T ; L^{q^{\star}}(\Omega)\right)}^{q} .
$$

Using Sobolev's theorem, as in Section 2, we derive the following estimate on $u_{\varepsilon}$ in $L^{q}\left(0, T ; L^{q^{\star}}(\Omega)\right)$

$$
\begin{aligned}
& \left\|u_{\varepsilon}\right\|_{L^{q}\left(0, T ; L^{q^{\star}}(\Omega)\right)}^{q} \\
& \quad \leq C(K)+\delta(K)\left(\left\|u_{\varepsilon}\right\|_{L^{q}\left(0, T ; L^{q^{\star}}(\Omega)\right)}^{q(2-q) / 2}+\left\|u_{\varepsilon}\right\|_{L^{q}\left(0, T ; L^{q^{\star}}(\Omega)\right)}^{q\left(\left(s-q^{\prime}\right) /\left(s q^{\prime}\right) q+2-q\right)}\right) .
\end{aligned}
$$

Since $(2-q) / 2<1$ and $q\left(s-q^{\prime}\right) /\left(s q^{\prime}\right)+2-q<1$, we can use again Young's inequality which, choosing $K$ large enough, leads to a bound on $u_{\varepsilon}$ in $L^{q}\left(0, T ; L^{q^{\star}}(\Omega)\right)$ and, thus, in $L^{q}\left(0, T ; W_{0}^{1, q}(\Omega)\right)$. Finally, let us verify the compatibility of conditions (5.5). For

$$
r=q \frac{N+1}{N},
$$

the first condition is equivalent to

$$
1 \leq q<\frac{N+2}{N+1}
$$

and the second condition means that

$$
s>\frac{(N+1) q^{\prime}}{q^{\prime}-1}
$$

which is clearly satisfied since it is yet required $s>q^{\prime}$.
Finally, one can easily verify that Lemma 2, Lemma 3, Corollary 2 and Lemma 4 are valid in the context of Theorem 3 and the proof follows.

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# Liouville type theorems for $\varphi$-subharmonic functions 

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#### Abstract

In this paper we presents some Liouville type theorems for solutions of differential inequalities involving the $\varphi$-Laplacian. Our results, in particular, improve and generalize known results for the Laplacian and the $p$-Laplacian, and are new even in these cases. PhragmenLindeloff type results, and a weak form of the Omori-Yau maximum principle are also discussed.


## 0. Introduction.

Let $(M,\langle\cdot, \cdot\rangle)$ be a smooth, connected, non-compact, complete Riemannian manifold of dimension $m$. We fix an origin $o$, and denote by $r(x)$ the distance function from $o$, and by $B_{t}=\{x \in M: r(x)<t\}$ and $\partial B_{t}=\{x \in M: r(x)=t\}$ the geodesic ball and sphere of radius $t>0$ centered at $o$.

To avoid inessential technical difficulties we will assume that $\partial B_{t}$ is a regular hypersurface. This is certainly the case if $o$ is a pole of $M$; in the general case one could overcome the problem using a Gaffney regularized distance instead of the Riemannian distance function $r(x)$.

We denote by vol $B_{t}$ and $\operatorname{vol} \partial B_{t}$ the Riemannian measure of $B_{t}$ and the induced measure of $\partial B_{t}$, respectively. Integrating in polar coordinates then gives

$$
\operatorname{vol} B_{t}=\int_{0}^{t} \operatorname{vol} \partial B_{s} d s
$$

In this paper, we will always denote with $\varphi$ a real valued function in $C^{1}((0,+\infty)) \cap C^{o}([0,+\infty))$ satisfying the following structural conditions
i) $\varphi(0)=0$,
ii) $\varphi(t)>0, \quad$ for all $t>0$,
iii) $\varphi(t) \leq A t^{\delta}, \quad$ for all $t \geq 0$,
for some positive constants $A$ and $\delta$.
We will focus our attention on the differential operator defined for $u \in C^{1}(M)$ by

$$
\begin{equation*}
\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \tag{0.2}
\end{equation*}
$$

and which could be referred to as the $\varphi$-Laplacian. Of course, if the the vector field in brackets is not $C^{1}$, then the divergence in ( 0.2 ) must be considered in distributional sense. Note that the vector field in consideration may fail to be $C^{1}$ at the points where $\nabla u=0$, even if $u$ is assumed to be $C^{2}$.

We also note that the $\varphi$-Laplacian arises naturally when writing the Euler-Lagrange equation associated to the energy functional

$$
\Lambda(u)=\int \Phi(|\nabla u|)
$$

where $\Phi(t)=\int_{0}^{t} \varphi(s) d s$.
As important natural examples we mention:

1) the Laplace-Beltrami operator, $\Delta u$, corresponding to $\varphi(t)=t$;
2) or, more generally, the $p$-Laplacian, $\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right), p>1$, corresponding to $\varphi(t)=t^{p-1}$;
3) the generalized mean curvature operator, $\operatorname{div}\left(\nabla u /\left(1+|\nabla u|^{2}\right)^{\alpha}\right)$, $\alpha>0$, corresponding to $\varphi(t)=t /\left(1+t^{2}\right)^{\alpha}$.

The general philosophy is to explore the mutual interactions between the behavior of solutions of differential equations/inequalities involving the $\varphi$-Laplacian, and geometric properties of the underlying manifold. As it will become clear in the sequel, many of the results we
present can be generalized to a slightly more general class of operators including, for instance, the $\mathcal{A}$-Laplace operators as defined in [HeKM]. For some related results in this setting we also refer to some recent work by I. Holopainen [Ho]. We have decided to concentrate on operators of the form given in (0.2) since all the main ideas appear already, and the techniques are more transparent in this setting. In any case, our results have interesting consequences in non-linear potential theory, and many of them appear to be new even for the Laplacian.

We introduce some notation. A function $u \in C^{1}(M)$ is said to be $\varphi$-subharmonic if

$$
\begin{equation*}
\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq 0, \quad \text { on } M \tag{0.3}
\end{equation*}
$$

Reversing the inequality, or replacing the inequality with an equality one obtains the definition of $\varphi$-super harmonic, and $\varphi$-harmonic function, respectively. Note that the notion of $\varphi$-(sub, super)harmonicity is unaffected by adding a constant to $u$. In accordance with what remarked after (0.2), if the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is not $C^{1}$, the inequality in (0.3) must be understood in weak sense. Explicitly, $u \in C^{1}$ is $\varphi$-subharmonic if

$$
\left.-\left.\int\langle\nabla \psi,| \nabla u\right|^{-1} \varphi(|\nabla u|) \nabla u\right\rangle \geq 0
$$

for all $0 \leq \psi \in C_{c}^{\infty}(M)$, or equivalently, for all nonnegative compactly supported Lipschitz functions on $M$.

One of the basic problems is to determine sufficient conditions so that ( 0.3 ) has only constant solutions. In the case of the LaplaceBeltrami operator, typically one considers the problem where $u$ belongs to two main function classes: $\left\{u \in C^{2}(M): \sup u<+\infty\right\}$ and $\left\{u \in C^{2}(M): u \geq 0\right\} \cap L^{q}(M), q>1$. The fact that the only solutions of (0.3) in these two cases are constant amounts to the parabolicity of the manifold $(M,\langle\cdot, \cdot\rangle)$, and to an $L^{q}$-type Liouville property, respectively. It is well known, see for instance the recent survey paper by A. Grigor'yan, [Gr1], that

$$
\begin{align*}
& \text { i) } \frac{r}{\operatorname{vol}\left(B_{r}\right)} \notin L^{1}(+\infty), \\
& \text { ii) } \frac{r}{\int_{B_{r}} u^{q}} \notin L^{1}(+\infty), \tag{0.4}
\end{align*}
$$

are sufficient to guarantee parabolicity or the validity of an $L^{q}$-type Liouville property, respectively. However, both conditions are far from being necessary, as shown by a counterexample due to R. Greene and quoted in [V3].

On the other hand, it may be shown that parabolicity is equivalent to

$$
\begin{equation*}
\frac{1}{\operatorname{vol}\left(\partial B_{r}\right)} \notin L^{1}(+\infty) \tag{0.5}
\end{equation*}
$$

if the manifold $(M,\langle\cdot, \cdot\rangle)$ is a model in the sense of Greene and Wu , [GW], but the equivalence fails in general (see for instance [Gr1, example 7.9, p. 40]).

In this connection, we note that (0.5) is always implied by (0.4) i), for instance. Further, it is easy to construct examples of manifolds of exponential volume growth where (0.5) holds, while (0.4) i) obviously does not. We shall therefore concentrate on conditions involving $\operatorname{vol} \partial B_{r}$, as in (0.5), rather than vol $B_{r}$ itself.

Following the classical terminology, we shall say that $(M,\langle\cdot, \cdot\rangle)$ is $\varphi$ parabolic if the only bounded above solutions of ( 0.3 ) are constant. As a consequence of the results presented in Section 1 below (see Theorem 1.5) we have:

Theorem A. Let $(M,\langle\cdot, \cdot\rangle)$ be a complete manifold, let $\varphi$ and $\delta$ be as in (0.1) and assume that

$$
\left(\operatorname{vol}\left(\partial B_{r}\right)^{1 / \delta}\right)^{-1} \notin L^{1}(+\infty) .
$$

Then $M$ is $\varphi$-parabolic.
Note that the same $\delta$ may correspond to different operators. For instance, $\delta=1$ may be associated both to the Laplacian and to the mean curvature operator $\operatorname{div}\left(\nabla u / \sqrt{1+|\nabla u|^{2}}\right)$. It follows that if (0.5) holds, then $M$ is parabolic both in the usual sense, and with respect to the mean curvature operator.

As for $L^{q}$-type Liouville Theorems we have:
Theorem B. Let $(M,\langle\cdot, \cdot\rangle)$ be a complete manifold, let $\varphi$ and $\delta$ be as in (0.1) and let $u$ be a $C^{1}$, non-negative $\varphi$-subharmonic function. If

$$
\left(\int_{\partial B_{r}} u^{q}\right)^{-1 / \delta} \notin L^{1}(+\infty),
$$

for some $q>\delta$, then $u$ is constant.
Theorem B generalizes two recent results for the Laplacian and the $\mathcal{A}$-Laplacian (see the above remark) due to K. T. Sturm, [St], and Holopainen, [Ho], respectively. In these papers, constancy of $u$ is established assuming that the following stronger condition holds

$$
\left(\frac{r}{\int_{B_{r}} u^{q}}\right)^{1 / \delta} \notin L^{1}(+\infty)
$$

More interestingly, our result extends to solutions of a large class of differential inequalities, see Theorem 2.2, and, in a different direction, Theorem 4.3. It should be pointed out that Sturm and Holopeinen have a version of Theorem B for nonnegative ( $p$-)superharmonic functions satisfying the above growth condition with $q<\delta$. In Proposition 2.3 we show that our techniques are flexible enough to recover their result.

Note that the case $\delta=q$ is quite special. Indeed, for a rather long time it was not known whether an $L^{1}$-Liouville property was true on an arbitrary Riemannian manifold, even in the case of the LaplaceBeltrami operator. At the beginning of the ' 80 's, reference was made to a preprint by L. O. Chung where the first example of a complete Riemannian manifold admitting a non-trivial integrable harmonic function $u$ was constructed. A further example was published by P. Li and R. Schoen, [LS] in 1984. However, the constancy of integrable harmonic functions can be obtained provided we impose some further condition, for instance an appropriate bound on the growth. This was first observed by N. S. Nadirashvili, [N]. The following result may be viewed as a generalization and an improvement of [ N , Theorem 2], even for the Laplace-Beltrami operator.

Theorem C. Let $(M,\langle\cdot, \cdot\rangle)$ be a complete manifold, and let $\varphi$ and $\delta$ be as in (0.1). Let $u$ be a $C^{1}$, non-negative $\varphi$-subharmonic function. If

$$
\begin{equation*}
\text { i) } \int_{\partial B_{r}} u^{\delta} \leq \frac{C}{r \log ^{b} r} \quad \text { and } \quad \text { ii) } u(x) \leq C \exp \left(r(x)^{1+1 / \delta}\right) \text {, } \tag{0.6}
\end{equation*}
$$

for some positive constants $b$ and $C$, and $r(x)$ sufficiently large, then $u$ is constant.

We point out that the assumptions do not force $u$ to belong to $L^{\delta}(M)$, and refer to Section 1 below for a more precise statement and a detailed discussion.

As mentioned above, many of our results extend and improve previous results valid for the Laplacian and the $p$-Laplacian. In many instances the latter have been obtained using capacity techniques. These techniques in general depend on the solvability of the Dirichlet problem (at least on annuli), and the fact that $p$-harmonic functions are minimizers of the appropriate energy integral. Underlying the method is the even more basic relationship between the energy density $\Phi(|\nabla u|)$ and the expression $\varphi(|\nabla u|)|\nabla u|$, which is crucial when applying the divergence theorem. In the case of the $p$-Laplacian the two expressions coincide and are equal to $|\nabla u|^{p}$. Since none of these facts holds in the general case of the $\varphi$-Laplacian, a capacity approach to $\varphi$-parabolicity appears to be infeasible, and alternative methods must be devised.

In the last section of the paper we show that, under suitable geometric assumptions, a weak version of the Omori-Yau maximum principle holds for the $\varphi$-Laplacian.

Theorem D. Assume that

$$
\begin{equation*}
\liminf _{r \rightarrow+\infty} \frac{\log \operatorname{vol} B_{r}}{r^{1+\delta}}<+\infty \tag{0.7}
\end{equation*}
$$

and let $u$ be a smooth function on $M$ with $u^{*}=\sup u<+\infty$. Suppose further that the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least $C^{1}$. Then there exists a sequence $\left\{x_{n}\right\} \subset M, n=1,2, \ldots$, such that

$$
\begin{cases}u\left(x_{n}\right) \longrightarrow u^{*}, & \text { as } n \longrightarrow+\infty,  \tag{0.8}\\ \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right)\left(x_{n}\right) \leq \frac{1}{n}\end{cases}
$$

Observe that the regularity condition in the statement is certainly satisfied in the case of the Laplacian, of the $p$-Laplacian ( $p \geq 2$ ), or the generalized mean curvature operator, once $u$ is assumed to be at least $C^{2}$.

We also note that in a recent paper, K. Takegoshi [T] asserted that, if (0.7) holds with $\delta=1$, then the Laplacian satisfies the full strength Omori-Yau maximum principle, i.e., there exists a sequence $\left\{x_{n}\right\}$ satisfying (0.8) and the additional condition $\left|\nabla u_{n}\right| \leq 1 / n$. However, there seems to be a gap in his proof, which the present authors have not been able to fill.

The paper is organized as follows: in Section 1 we prove Theorem A, Theorem C and some related results. Section 2 is devoted to a generalization of Theorem B. In Section 3 we presents some extensions of Theorem A. Examples show that the main results in each of these sections are fairly sharp. In Section 4 we state some Phragmen-Lindelöff type results, and make some further comments. In Section 5 we discuss the weak Omori-Yau maximum principle, and some related topics.

## 1. Proof of theorems A, C and related results.

We keep the notation of the Introduction; in particular the constants $A$ and $\delta$ refer to the structural conditions (0.1) satisfied by $\varphi$.

The following observation will be repeatedly used in the sequel. Assume that $\Omega$ is a bounded domain in $M$ with smooth boundary $\partial \Omega$, and outward unit normal $\nu$. Denote by $\rho(x)$ the distance function from $\partial \Omega$ (with the convention that $\rho(x)$ is $>0$ if $x \in \Omega$ and $<0$ if $x \notin \Omega$ ), so that $\rho$ is the radial coordinate for the Fermi coordinates relative to $\partial \Omega$. By Gauss Lemma, $|\nabla \rho|=1$ and, $\nabla \rho=-\nu$ on $\partial \Omega$. Finally, let $\Omega_{\varepsilon}=\{x \in \Omega: \rho(x)>\varepsilon\}$, and let $\psi_{\varepsilon}$ be the Lipschitz function defined by

$$
\psi_{\varepsilon}(x)= \begin{cases}1, & \text { if } x \in \Omega_{\varepsilon} \\ \frac{1}{\varepsilon} \rho(x), & \text { if } x \in \Omega \backslash \Omega_{\varepsilon} \\ 0, & \text { if } x \in \Omega^{c}\end{cases}
$$

Given a continuous vector field $Z$ defined on $\bar{\Omega}$, the following version of the divergence theorem holds

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0+}\left\langle\operatorname{div} Z, \psi_{\varepsilon}\right\rangle=\int_{\partial \Omega}\langle Z, \nu\rangle . \tag{1.1}
\end{equation*}
$$

Indeed, by definition of weak divergence, and by the co-area formula,

$$
\left\langle\operatorname{div} Z, \psi_{\varepsilon}\right\rangle=-\frac{1}{\varepsilon} \int_{\Omega \backslash \Omega_{\varepsilon}}\langle Z, \nabla \rho\rangle=\frac{1}{\varepsilon} \int_{0}^{\varepsilon} d t \int_{\partial \Omega_{t}}\langle Z, \nabla \rho\rangle,
$$

and (1.1) follows letting $\varepsilon \longrightarrow 0$. With slight abuse of notation, we will refer to (1.1) as the divergence theorem, and write, even in this case,

$$
\int_{\Omega} \operatorname{div} Z=\int_{\partial \Omega}\langle Z, \nu\rangle .
$$

Assume now that the differential inequality $\operatorname{div} Z \geq \lambda$ holds in weak sense for some real valued continuous function $\lambda$ defined on $\bar{\Omega}$. Substituting into (1.1) yields

$$
\int_{\Omega} \lambda=\lim _{\varepsilon \rightarrow 0} \int_{\Omega} \lambda \psi_{\varepsilon} \leq \lim _{\varepsilon \rightarrow 0}\left\langle\operatorname{div} Z, \psi_{\varepsilon}\right\rangle=\int_{\partial \Omega}\langle Z, \nu\rangle .
$$

This observation will allow us to deal with continuous vector fields satisfying weak differential inequalities as if we were working with smooth vector fields satisfying pointwise inequalities.

The next simple technical lemma, and its companion Lemma 2.1, are key ingredients in the proofs of our main results.

Lemma 1.1. Let $f \in C^{o}(\mathbb{R})$, and let $u$ be a non-constant $C^{1}$ solution of the differential inequality

$$
\begin{equation*}
\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq \varphi(|\nabla u|)|\nabla u| f(u) . \tag{1.2}
\end{equation*}
$$

Assume that there are functions $\alpha \in C^{1}(I)$ and $\beta \in C^{o}(I)$ defined on an interval $I \supset u(M)$ such that

$$
\begin{gather*}
\alpha(u) \geq 0  \tag{1.3}\\
\alpha^{\prime}(u)+f(u) \alpha(u) \geq \beta(u)>0, \tag{1.4}
\end{gather*}
$$

on $M$. Then there exist $R_{o}$ depending only on $u$ and a constant $C>0$ independent of $\alpha$ and $\beta$, such that, for every $r>R \geq R_{o}$,

$$
\begin{equation*}
\left(\int_{B_{R}} \beta(u) \varphi(|\nabla u|)|\nabla u|\right)^{-1} \geq C\left(\int_{R}^{r}\left(\int_{\partial B_{t}} \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{-1 / \delta}\right)^{\delta} . \tag{1.5}
\end{equation*}
$$

Proof. Let $Z$ be the continuous vector field defined by

$$
Z=\alpha(u)|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u .
$$

Observing that $\alpha(u)$ is $C^{1}$, we compute the distributional divergence of $Z$, and use our assumptions on $u, \alpha$, and $\beta$ to obtain

$$
\operatorname{div} Z \geq\left(\alpha(u) f(u)+\alpha^{\prime}(u)\right) \varphi(|\nabla u|)|\nabla u| \geq \beta(u) \varphi(|\nabla u|)|\nabla u| .
$$

Integrating over $B_{t}$ and applying the divergence theorem gives

$$
\begin{equation*}
\int_{\partial B_{t}}\langle Z, \nabla r\rangle \geq \int_{B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u| \tag{1.6}
\end{equation*}
$$

On the other hand, using Schwarz inequality, the assumed positivity of $\beta(u)$, Hölder inequality with conjugate exponents $1+\delta$ and $1+1 / \delta$, and the inequality $\varphi(t)^{1+1 / \delta} \leq A^{1 / \delta} \varphi(t) t$, we estimate

$$
\begin{align*}
\int_{\partial B_{t}}\langle Z, \nabla r\rangle \leq & \int_{\partial B_{t}}|Z| \\
= & \int_{\partial B_{t}} \alpha(u) \varphi(|\nabla u|) \\
\leq & A^{1 /(1+\delta)}\left(\int_{\partial B_{t}} \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{1 /(1+\delta)}  \tag{1.7}\\
& \cdot\left(\int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u|\right)^{1+1 / \delta}
\end{align*}
$$

Combining (1.6) and (1.7) yields

$$
\begin{align*}
\int_{B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u| \leq & A^{1 /(1+\delta)}\left(\int_{\partial B_{t}} \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{1 /(1+\delta)}  \tag{1.8}\\
& \cdot\left(\int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u|\right)^{1+1 / \delta}
\end{align*}
$$

Denoting by $H(t)$ the left hand side of (1.8), and noting that, by the co-area formula

$$
H^{\prime}(t)=\int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u|
$$

we may rewrite (1.8) in the form

$$
\begin{equation*}
H(t) \leq A^{1 /(1+\delta)}\left(\int_{\partial B_{t}} \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{1 /(1+\delta)}\left(H^{\prime}(t)\right)^{\delta /(1+\delta)} \tag{1.9}
\end{equation*}
$$

Further, since $u$ is non-constant, $\beta(u)>0$ and $\varphi(t)>0$ if $t>0$, we deduce that there exists $R_{o}$ such that $H(t)>0$ for every $t \geq R_{o}$. It follows that the right hand side of (1.9) is also strictly positive for $t \geq R_{o}$. Rearranging we finally obtain

$$
H(t)^{-1-1 / \delta} H^{\prime}(t) \geq A^{-1 / \delta}\left(\int_{\partial B_{t}} \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{-1 / \delta}, \quad t \geq R_{o}
$$

whence, integrating between $R$ and $r, R_{o} \leq R<r$, yields
$H(R)^{-1 / \delta} \geq H(R)^{-1 / \delta}-H(r)^{-1 / \delta} \geq \frac{1}{\delta A^{1 / \delta}} \int_{R}^{r}\left(\int_{\partial B_{t}} \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{-1 / \delta} d t$,
and (1.5) follows with $C=\delta^{-\delta} A^{-1}$.
Theorem 1.2. Let $u \in C^{1}(M)$ be a non-negative $\varphi$-subharmonic function. If there exists $b>0$ such that

$$
\begin{align*}
& \left(\int_{\partial B_{t}} u^{\delta}(1+\log (1+u))^{\delta}\left(1+\log ^{b+\delta}(1+\log (1+u))\right)\right)^{-1 / \delta} \\
& 0) \quad \notin L^{1}(+\infty), \tag{1.10}
\end{align*}
$$

then $u$ is constant.
Proof. We argue by contradiction, and assume that $u$ is not constant. For every integer $n \geq 1$, let $\alpha_{n}$ be the function defined for $t \geq 0$ by $\alpha_{n}(t)=\log ^{b}(1+\log (1+1 / n+t))$, and let

$$
\beta_{n}=\alpha_{n}^{\prime}(t)=\frac{b \log ^{b-1}(1+\log (1+1 / n+t))}{(1+\log (1+1 / n+t))(1+1 / n+t)}, \quad \text { for all } t \geq 0
$$

It is readily verified that $\alpha_{n}$ and $\beta_{n}$ satisfy the conditions in the statement of Lemma 1.1, with $f \equiv 0$, so that

$$
\begin{align*}
& \left(\int_{B_{R}} \beta_{n}(u) \varphi(|\nabla u|)|\nabla u|\right)^{-1}  \tag{1.11}\\
& \quad \geq C\left(\int_{R}^{r}\left(\int_{\partial B_{t}} \frac{\alpha_{n}(u)^{1+\delta}}{\beta_{n}(u)^{\delta}}\right)^{-1 / \delta}\right)^{\delta}
\end{align*}
$$

with $C=\delta^{-\delta} A^{-1}$ independent of $n$. It is also easy to verify that there exists a positive constant $\gamma$ which depends only on $\delta$ and $b$ such that

$$
(1+s)^{\delta} \log ^{b+\delta}(1+\log (1+s)) \leq \gamma s^{\delta}\left(1+\log ^{b+\delta}(1+\log (1+s))\right)
$$

for all $s \geq 0$, and therefore (using $s=1 / n+u$ )

$$
\begin{aligned}
\frac{\alpha_{n}^{1+\delta}(u)}{\beta_{n}^{\delta}(u)} \leq & \gamma\left(\frac{1}{n}+u\right)^{\delta}\left(1+\log \left(1+\frac{1}{n}+u\right)\right)^{\delta} \\
& \cdot\left(1+\log ^{b+\delta}\left(1+\log \left(1+\frac{1}{n}+u\right)\right)\right) b^{-\delta}
\end{aligned}
$$

on $M$. It follows that the expression in (1.11) is bounded below by a multiple of

$$
\begin{aligned}
\left(\int _ { R _ { o } } ^ { r } \left(\int_{\partial B_{t}}\right.\right. & \left(\frac{1}{n}+u\right)^{\delta}\left(1+\log \left(1+\frac{1}{n}+u\right)\right)^{\delta} \\
& \left.\left.\cdot\left(1+\log ^{b+\delta}\left(1+\log \left(1+\frac{1}{n}+u\right)\right)\right)\right)^{-1 / \delta}\right)^{\delta}
\end{aligned}
$$

We substitute into (1.11), let $n$ tend to infinity in the resulting inequality, and apply the monotone convergence, and dominated convergence theorems to conclude that

$$
\begin{aligned}
& \left(\int_{B_{R}} \frac{\varphi(|\nabla u|)|\nabla u|}{(1+u)(1+\log (1+u)) \log ^{1-b}(1+\log (1+u))}\right)^{-1} \\
& \geq C_{1}\left(\int _ { R } ^ { r } \left(\int_{\partial B_{t}} u^{\delta}(1+\log (1+u))^{\delta}\right.\right. \\
& \left.\left.\cdot\left(1+\log ^{b+\delta}(1+\log (1+u))\right)\right)^{-1 / \delta}\right)^{\delta}
\end{aligned}
$$

with $C_{1}=b^{1+\delta} /\left(A \gamma \delta^{\delta}\right)$. Letting $r$ tend to infinity, we contradict assumption (1.3).

Proof (of Theorem C). We use conditions (0.6) to deduce that

$$
\left(\int_{\partial B_{t}} u^{\delta}(1+\log (1+u))^{\delta}\left(1+\log ^{b+\delta}(1+\log (1+u))\right)\right)^{-1 / \delta} \geq \frac{C}{r \log r}
$$

for large enough $r$. Thus (1.10) holds, and the conclusion follows from Theorem 1.2.

Remark. We define, for $t \geq 0, L_{1}(t)=1+\log (1+t)$, and, for $k \geq 2$, $L_{k}(t)=1+\log L_{k-1}(t)$. It is a simple matter to verify that condition (1.10) in the statement of Theorem 1.2 can be replaced by the weaker

$$
\begin{equation*}
\left(\int_{\partial B_{t}} u^{\delta}\left(\prod_{k=1}^{n-1} L_{k}^{\delta}(t)\right)\left(1+\log ^{b+\delta} L_{n-1}(u)\right)\right)^{-1 / \delta} \notin L^{1}(+\infty) \tag{1.12}
\end{equation*}
$$

for some $n \geq 2$ and $b>0$.
It follows that Theorem C may be correspondingly improved. Indeed, denoting by $\ell_{k}$ the $k$ th-composition power of log, so that $\ell_{k}(t)=$
$\log \left(\ell_{k-1}(t)\right)$, for sufficiently large $t>0$, conditions (0.6) in the statement of Theorem C can be replaced by

$$
\begin{equation*}
\text { i) } \int_{\partial B_{r}} u^{\delta} \leq \frac{C}{r \ell_{n}^{b}(r)} \quad \text { and } \quad \text { ii) } u(x) \leq C \exp \left(r(x)^{1+1 / \delta}\right) \tag{1.13}
\end{equation*}
$$

for some integer $n \geq 1$, some positive constants $b$ and $C$, and sufficiently large $r$.

The following example shows that Theorem C is rather sharp. For the sake of simplicity, we restrict our considerations to the case of the $p$-Laplacian, $p>1$. This corresponds to the values $A=1$ and $\delta=p-1$ in (0.1).

Let $\sigma \in C^{\infty}([0,+\infty))$ be a positive function such that $\sigma(t)=t$ for $t \in[0,1]$, and define

$$
\langle\cdot, \cdot\rangle=d r^{2}+\sigma^{2}(r) d \theta^{2}
$$

where $(r, \theta)$ are the polar coordinates on $\mathbb{R}^{m} \backslash\{0\}=(0,+\infty) \times S^{m-1}$, and $d \theta^{2}$ denotes the standard metric on $S^{m-1}$. Clearly, $\langle\cdot, \cdot\rangle$ extends to a smooth complete metric on $\mathbb{R}^{m}$. Next, let $a \in C^{o}([0,+\infty))$ be a non-negative function such that, for $t \in[0,1]$

$$
a(t)= \begin{cases}1, & \text { if } 1<p<2 \\ t^{p-2}, & \text { if } p \geq 2\end{cases}
$$

We define the non-negative function

$$
\begin{equation*}
u(x)=\int_{0}^{r(x)} \sigma(t)^{-(m-1) /(p-1)}\left(\int_{0}^{t} a(s) \sigma(s)^{m-1} d s\right)^{1 /(p-1)} d t \tag{1.14}
\end{equation*}
$$

where $r(x)$ denotes the distance function from 0 . It is easily verified that $u$ is $C^{2}$, and satisfies

$$
\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)(x)=a(r(x)),
$$

on ( $\left.\mathbb{R}^{m},\langle\cdot, \cdot\rangle\right)$. Thus $u$ is not constant and $p$-subharmonic. Since $u$ is radial, for ease of notation we will write $u(r)$.

To construct the required example we fix $T_{o}>1$, and choose the functions $a(t)$ and $\sigma(t)$ so as to satisfy the further conditions

$$
\begin{equation*}
a(t)=0 \quad \text { and } \quad \sigma(t)=t^{-1 /(m-1)} \exp \left(-\frac{(p-1) t^{p /(p-1)}}{m-1}\right) \tag{1.15}
\end{equation*}
$$

on $\left[T_{o},+\infty\right)$. Inserting these in the definition of $u$, we deduce that there exist constants $C_{1}, C_{2}$ such that

$$
\begin{aligned}
u(r) & =C_{1}+C_{2} \int_{T_{o}}^{r} \sigma^{-(m-1) /(p-1)}(t) d t \\
& =C_{1}+C_{2} \int_{T_{o}}^{r} t^{1 /(p-1)} \exp \left(t^{p /(p-1)}\right) d t
\end{aligned}
$$

Thus there exist constants $C_{i}>0$ such that

$$
u(r) \leq C_{3} \exp \left(r^{p /(p-1)}\right),
$$

and

$$
\int_{\partial B_{r}} u^{p-1}=C_{4} \sigma^{m-1}(r) u^{p-1}(r) \sim \frac{C_{5}}{r}, \quad \text { as } r \longrightarrow+\infty
$$

showing that (0.6) ii) is satisfied, while (0.6) i) barely fails to hold.
On the other hand, let $\varepsilon>0$ and choose

$$
\sigma(t)=t^{-1 /(m-1)}(\log t)^{-\varepsilon(p-1) /(m-1)} \exp \left(-\frac{(p-1) t^{p /(p-1)}\left(\log ^{\varepsilon} t\right)}{m-1}\right),
$$

on $\left[T_{o},+\infty\right)$. Then

$$
u(r) \sim C_{6} \exp \left(r^{p /(p-1)} \log ^{\varepsilon} r\right)
$$

and

$$
\int_{\partial B_{r}} u^{p-1} \sim \frac{C_{7}}{r \log ^{\varepsilon(p-1)} r},
$$

as $r \longrightarrow+\infty$, so that, in this case, (0.6) i) holds, while (0.6) ii) does not.

We also observe that if $\varepsilon>1 /(p-1)$, then $u$ belongs to $L^{p-1}(M)$. In particular, in the case of the Laplacian, where $p=2$, this gives a further example, in the spirit of [LS] quoted in the Introduction, of an integrable non-negative subharmonic function. We note that in this case, the manifold ( $M,\langle\cdot, \cdot\rangle$ ) has finite volume.

We now show how to recover Theorem 2 of Nadirashvili, [ N ], from Theorem 1.2. For this, and for later comparison, we first state the following

Proposition 1.3. Let $(M,\langle\cdot, \cdot\rangle)$ be a complete Riemannian manifold, let $h \in C^{o}(M), h \geq 0$, and set

$$
v(t)=\int_{B_{t}} h
$$

so that

$$
v^{\prime}(t)=\int_{\partial B_{t}} h
$$

Fix $R>0$, and let $r>R$. Then for any $\delta>0$,

$$
\begin{equation*}
\int_{R}^{r}\left(\frac{t-R}{v(t)}\right)^{1 / \delta} d t \leq C \int_{R}^{r} \frac{d t}{v^{\prime}(t)^{1 / \delta}}, \tag{1.16}
\end{equation*}
$$

for some constant $C>0$ independent of $r$. In particular,

$$
\begin{equation*}
\left(\frac{t}{v(t)}\right)^{1 / \delta} \notin L^{1}(+\infty) \quad \text { implies } \quad \frac{1}{v^{\prime}(t)^{1 / \delta}} \notin L^{1}(+\infty) . \tag{1.17}
\end{equation*}
$$

We remark that the reverse implication in (1.17) does not hold in general. In some interesting cases, the two conditions can be equivalent. For instance, it was showed by Varopoulos, [V1], that if $(M,\langle\cdot, \cdot\rangle)$ is a regular cover of a compact manifold, then $r / \operatorname{vol} B_{r} \notin L^{1}(+\infty)$ is equivalent to $1 / \operatorname{vol} \partial B_{r} \notin L^{1}(+\infty)$. The same is true if we impose curvature conditions, for instance if the Ricci curvature is non-negative (see [V2]). For further results in this direction, see [LT].

Proof. Proposition 1.3 is well known. We provide an elementary proof for completeness and the convenience of the reader. Fix $\varepsilon>0$, and set

$$
v_{\varepsilon}(t)=\int_{B_{t}} h+\varepsilon,
$$

so that, by the co-area formula,

$$
v_{\varepsilon}^{\prime}(t)=\int_{\partial B_{t}} h+\varepsilon .
$$

Applying Hölder inequality with conjugate exponents $1+\delta$ and $1+1 / \delta$ yields

$$
\int_{R}^{r}\left(\frac{t-R}{v_{\varepsilon}(t)}\right)^{1 / \delta} d t
$$

$$
\begin{equation*}
\leq C\left(\int_{R}^{r}\left(\frac{t-R}{v_{\varepsilon}(t)}\right)^{1+1 / \delta} v_{\varepsilon}^{\prime}(t)\right)^{1 /(1+\delta)}\left(\int_{R}^{r} \frac{d t}{v_{\varepsilon}^{\prime}(t)^{1 / \delta}}\right)^{\delta /(1+\delta)} . \tag{1.18}
\end{equation*}
$$

Integrating by parts the first integral on the right hand side we get

$$
\begin{aligned}
\int_{R}^{r}\left(\frac{t-R}{v_{\varepsilon}(t)}\right)^{1+1 / \delta} v_{\varepsilon}^{\prime}(t) & =-\delta \frac{(r-R)^{1+1 / \delta}}{v_{\varepsilon}(r)^{1 / \delta}}+(1+\delta) \int_{R}^{r}\left(\frac{t-R}{v_{\varepsilon}(t)}\right)^{1+1 / \delta} \\
& \leq(1+\delta) \int_{R}^{r}\left(\frac{t-R}{v_{\varepsilon}(t)}\right)^{1 / \delta} d t
\end{aligned}
$$

whence, substituting into (1.18),

$$
\begin{equation*}
\int_{R}^{r}\left(\frac{t-R}{v_{\varepsilon}(t)}\right)^{1 / \delta} d t \leq(1+\delta)^{1 / \delta} \int_{R}^{r} \frac{d t}{v_{\varepsilon}^{\prime}(t)^{1 / \delta}} \tag{1.19}
\end{equation*}
$$

By dominated convergence, as $\varepsilon \longrightarrow 0, v_{\varepsilon}$ and $v_{\varepsilon}^{\prime}$ decrease to $v$ and $v^{\prime}$, respectively. Inequality (1.16) follows by applying the monotone convergence theorem to both sides of (1.19).

Since

$$
\left(\frac{t-R}{v(t)}\right)^{1 / \delta} \geq 2^{-1 / \delta}\left(\frac{t}{v(t)}\right)^{1 / \delta}, \quad \text { for } t \geq 2 R
$$

it is clear that (1.17) follows from (1.16).
Proposition 1.3 shows that condition (1.10) in Theorem 1.2 may be replaced by the stronger

$$
\left(\frac{r}{\int_{B_{t}} u^{\delta}(1+\log (1+u))^{\delta}\left(1+\log ^{b+\delta}(1+\log (1+u))\right)}\right)^{1 / \delta}
$$

$$
\begin{equation*}
\notin L^{1}(+\infty) \tag{1.20}
\end{equation*}
$$

for some $b>0$.
Assume now that $u$ is a non-negative $\varphi$-subharmonic function satisfying $u \in L^{\delta}(M)$ and $u(x) \leq C \exp \left(r(x)^{1+1 / \delta-\varepsilon}\right)$, for some $\varepsilon>0$ and $C>0$, as in [ N , Theorem 2]. It is easy to verify that the left hand side of (1.20) is bounded below by a multiple of $r^{-1+\varepsilon} \log ^{-1-b / \delta}$, which is not integrable at infinity. In light of what remarked above, Theorem 1.2 applies and $u$ is necessarily constant. This shows that Theorem 1.2 extends the work of Nadirashvili. The case where the assumption of non-negativity of $u$ is replaced by the condition that there
exists $x_{o} \in M$ such that $u\left(x_{o}\right)>0$, may be treated using similar techniques and will be taken up in Section 4 below (see Theorem 4.3 and the comment thereafter).

At this point, it also looks natural to consider the case of a nonnegative $\varphi$-subharmonic function $u \in C^{1}(M)$ satisfying $u \in L^{q}(M)$, with $0<q<\delta$. It turns out that to obtain constancy of $u$ we need to impose some additional conditions on $\varphi$ and on the geometry of $M$.

As far as $\varphi$ is concerned, one could consider two kinds of conditions, namely that there exists $B>0$ such that

$$
\begin{equation*}
B t^{\delta} \leq \varphi(t), \quad \text { on }[0,+\infty) \tag{1.21}
\end{equation*}
$$

or that there exist constants $c_{o}$ and $c_{1}$ such that

$$
\begin{equation*}
c_{o} \leq \frac{t \varphi^{\prime}(t)}{\varphi(t)} \leq c_{1} \tag{1.22}
\end{equation*}
$$

Note that both condition are satisfied in the case of the $p$-Laplacian, while neither of them holds for the mean curvature operator.

We briefly consider the case where (1.21) is satisfied, leaving the case where (1.22) holds to the interested reader, who may refer to [Lb] for the general theory of operators satisfying this kind of conditions.

We are going to be sketchy since the arguments are standard. The starting point is the following Caccioppoli type inequality. Arguing as in the proof of Lemma 1.1 with the vector field

$$
W=\psi^{1+\delta}(u+\varepsilon)^{q-\delta}|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u,
$$

one shows (no additional assumption on $\varphi$ is needed here) that if $\psi$ is a smooth, compactly supported function and $u$ is a $C^{1}$, non-negative $\varphi$-subharmonic function, then, for every $\widetilde{q}>\delta$,

$$
\begin{align*}
\int_{B_{r}} \psi^{1+\delta} u^{\widetilde{q}-\delta-1} & \varphi(|\nabla u|)^{1+1 / \delta} \\
& \leq \frac{A^{1+1 / \delta}(1+\delta)^{1+\delta}}{(\widetilde{q}-\delta)^{1+\delta}} \int_{B_{r}} u^{\widetilde{q}}|\nabla \psi|^{1+\delta} \tag{1.23}
\end{align*}
$$

Let $0<\rho<r$ and apply (1.23) when $\psi$ is a smooth cutoff function such that

$$
\begin{align*}
& \psi= \begin{cases}1, & \text { on } B_{\rho}, \\
0, & \text { on } B_{r} \backslash B_{\rho},\end{cases}  \tag{1.24}\\
& |\nabla \psi| \leq \frac{C_{o}}{r-\rho},
\end{align*}
$$

with $C_{o}$ independent of $r$ and $\rho$. Further, assume that (1.21) holds, and that the Sobolev inequality

$$
\begin{equation*}
\left(\int_{B_{r}(o)}|f|^{k(1+\delta)}\right)^{1 / k(1+\delta)} \leq S_{k, 1+\delta}(r)\left(\int_{B_{r}(o)}|\nabla f|^{1+\delta}\right)^{1 /(1+\delta)} \tag{1.25}
\end{equation*}
$$

is valid for some $k>1$, and every $r>0$ and $f \in C_{o}^{2}\left(B_{r}(o)\right)$. Then one deduces the fundamental inequality
$\left(\int_{B_{\rho}} u^{\widetilde{q} k}\right)^{1 / k} \leq C S_{k, 1+\delta}^{1+\delta}(r)\left(\left(\frac{A}{B}\right)^{1+1 / \delta}\left(\frac{\widetilde{q}}{\widetilde{q}-\delta}\right)^{\delta}+1\right)(r-\rho)^{-1-\delta} \int_{B_{r}} u^{\widetilde{q}}$,
which holds for every $0<\rho<r$ with a constant $C$ that depends only $\delta$ and on the constant $C_{o}$ in (1.24).

The Möser iteration procedure allows to deduce that for every $q>0$ there exists a constant $C$ which depends only on $\delta, k, q, A, B$ and $C_{o}$ such that, for every $0<\bar{R}<R$

$$
\begin{equation*}
\sup _{B_{\bar{R}^{(o)}}} u \leq C\left(S_{k, 1+\delta}(R)(R-\bar{R})^{-1}\right)^{k(1+\delta) /((k-1) q)}\left(\int_{B_{R}} u^{q}\right)^{1 / q} . \tag{1.26}
\end{equation*}
$$

Note now that if $M$ satisfies the doubling condition

$$
\begin{equation*}
\operatorname{vol}\left(B_{2 r}(o)\right) \leq C \operatorname{vol}\left(B_{r}(o)\right), \tag{1.27}
\end{equation*}
$$

for every $r>0$ and $o \in M$, and the (weak) Poincaré inequality

$$
\begin{align*}
& \int_{B_{r}(o)}\left|f-f_{B_{r}(o)}\right|  \tag{1.28}\\
& \quad \leq C r \operatorname{vol}\left(B_{2 r}(o)\right)^{1-1 /(1+\delta)}\left(\int_{B_{2 r}(o)}|\nabla f|^{1+\delta}\right)^{1 /(1+\delta)},
\end{align*}
$$

for each $r>0, o \in M$ and $f \in C^{\infty}(M)$, where $f_{B_{r}(o)}$ denotes the average of $f$ over $B_{r}(o)$, then, by [HK, Theorem 1], the Sobolev inequality (1.25) holds for some $k>1$ and for every $o \in M$ and $r>0$, with

$$
\begin{equation*}
S_{k, 1+\delta}(r) \leq C\left(\operatorname{vol}\left(B_{r}(o)\right)\right)^{-(k-1) /(k(1+\delta))}, \tag{1.29}
\end{equation*}
$$

and $C$ depending only on $\delta, k$ and the constants in the doubling condition and in the weak Poincaré inequality.

We remark that (1.27) implies that $(M,\langle\cdot, \cdot\rangle)$ has at most polynomial growth.

Setting $\bar{R}=R-1$, and inserting (1.29) into (1.26) yield the following

Theorem 1.4. Assume that $\varphi \in C^{1}((0,+\infty)) \cap C^{o}([0,+\infty))$ satisfies the structural conditions

$$
\varphi(0)=0, \quad \text { and } \quad B t^{\delta} \leq \varphi(t) \leq A t^{\delta}, \quad \text { for all } t>0
$$

for some $0<B \leq A$. Let $M,\langle\cdot, \cdot\rangle$ be a complete Riemannian manifold satisfying the doubling condition (1.27) and the weak Poincaré inequality (1.28). Let $u \in C^{1}(M)$, be a non-negative $\varphi$-subharmonic function on $M$. Then, either $u \equiv 0$ or, for every $q>0$,

$$
\liminf _{r \rightarrow+\infty} \frac{1}{\operatorname{vol}\left(B_{r}(o)\right)} \int_{B_{r}(o)} u^{q}>0 .
$$

We present now the following further application of Lemma 1.1, from which Theorem A follows immediately. Related, and somewhat stronger, results are presented in Section 3.

Theorem 1.5. Let $u \in C^{1}(M)$ be a solution of the differential inequality

$$
\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq \varphi(|\nabla u|)|\nabla u| f(u),
$$

where $f \in C^{o}(\mathbb{R})$ is such that

$$
\inf _{M} f(u)>-\sigma,
$$

for some $\sigma \in \mathbb{R}$. If

$$
\begin{equation*}
\left(\int_{\partial B_{t}} e^{\sigma u}\right)^{-1 / \delta} \notin L^{1}(+\infty) \tag{1.30}
\end{equation*}
$$

then $u$ is constant.
Proof. If $u$ were not constant, one could apply Lemma 1.1 with $\alpha(t)=$ $e^{\sigma t}$ and $\beta(t)=\mu e^{\sigma t}, \mu=\inf _{M} f(u)+\sigma$, and contradict assumption (1.30).

We end this section observing that the conclusion (1.5) of Lemma 1.1 holds if $M$ is a manifold with smooth boundary $\partial M$, with the only additional assumption that $\partial u / \partial \nu \leq 0$, where $\nu$ denotes the outward unit normal to $\partial M$. Correspondingly, one obtains a version of Theorem 1.5 for manifolds with boundary.

In analogy with the situation of the Laplacian, we may define a manifold with boundary $M$ to be $\varphi$-parabolic if the only $\varphi$-subharmonic functions on $M$ which are bounded above and satisfy $\partial u / \partial \nu \leq 0$ on $\partial M$ are the constants. Applying the version of Theorem 1.5 for manifolds with boundary, we then conclude that if $\operatorname{vol}\left(\partial B_{r}\right)^{-1} \notin L^{1}(+\infty)$, then $M$ is $\varphi$-parabolic.

## 2. Proof of Theorem B and related results.

The same reasoning used in the proof of Lemma 1.1 yields the following:

Lemma 2.1. Let $f \in C^{o}(M)$ let $u$ be a non-constant $C^{1}$ solution of the differential inequality

$$
\begin{equation*}
u \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq \varphi(|\nabla u|)|\nabla u| f(u) \tag{2.1}
\end{equation*}
$$

Assume that for some functions $\alpha \in C^{1}(I)$ and $\beta \in C^{o}(I)$ defined in an interval $I \supset u(M)$

$$
\begin{gather*}
\alpha(u) \geq 0  \tag{2.2}\\
u \alpha^{\prime}(u)+(1+f(u)) \alpha(u) \geq \beta(u)>0 \tag{2.3}
\end{gather*}
$$

on $M$. Then there exist $R_{o}$ which depends only on $u$, and a constant $C>0$ independent of $\alpha$ and $\beta$ such that, for $r>R \geq R_{o}$ we have

$$
\begin{align*}
& \left(\int_{B_{R}} \beta(u) \varphi(|\nabla u|)|\nabla u|\right)^{-1}  \tag{2.4}\\
& \quad \geq C\left(\int_{R}^{r}\left(\int_{\partial B_{t}} \frac{|u \alpha(u)|^{1+\delta}}{\beta(u)^{\delta}}\right)^{-1 / \delta}\right)^{\delta}
\end{align*}
$$

Remark. As in Lemma 1.1, if the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is not $C^{1}$ on $M$, the differential inequality (2.1) must be considered in the weak sense. Namely,

$$
\left.-\left.\int_{M}\langle | \nabla u\right|^{-1} \varphi(|\nabla u|) \nabla u, \nabla(u \psi)\right\rangle \geq \int_{M} \psi \varphi(|\nabla u|)|\nabla u| f(u),
$$

must hold for every non-negative, compactly supported Lipschitz continuous function $\psi$.

Proof. The proof follows the lines of that of Lemma 1.1. Applying the divergence theorem to the continuous vector field

$$
Z=u \alpha(u)|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u,
$$

and using Hölder inequality we deduce that

$$
H(t) \leq A^{1 /(1+\delta)}\left(\int_{\partial B_{t}} \frac{|u \alpha(u)|^{1+\delta}}{\beta(u)^{\delta}}\right)^{1 /(1+\delta)}\left(H^{\prime}(t)\right)^{\delta /(1+\delta)},
$$

where, as in Section 1, we have set

$$
H(t)=\int_{B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u| .
$$

Since $u$ is not constant, there exists $R_{o}$ which depends only on $u$, such that $H(t)>0$ for $t \geq R_{o}$. Thus we also have

$$
\int_{B_{t}} \frac{|u \alpha(u)|^{1+\delta}}{\beta(u)^{\delta}}>0 \quad \text { and } \quad H^{\prime}(t)>0
$$

for $t \geq R_{o}$. Rearranging and integrating the resulting differential inequality yield the required conclusion.

Theorem 2.2. Let $f \in C^{o}(\mathbb{R})$, and let $u \in C^{1}(M)$ be a solution of the differential inequality (2.1) on $M$, with

$$
\begin{equation*}
\inf _{M} f(u)>-1 \tag{2.5}
\end{equation*}
$$

Let $q \in \mathbb{R}$ be such that

$$
\begin{equation*}
q>\delta-\inf _{M} f(u) \tag{2.6}
\end{equation*}
$$

If

$$
\begin{equation*}
\left(\int_{\partial B_{r}}|u|^{q}\right)^{-1 / \delta} \notin L^{1}(+\infty) \tag{2.7}
\end{equation*}
$$

then $u$ is constant. If $u>0$, the same conclusion holds without assuming (2.5).

Proof. Assume by contradiction that $u$ is not constant. For every integer $n \geq 1$, let $\alpha_{n}(t)=\left(t^{2}+1 / n\right)^{(q-\delta-1) / 2}$. Then

$$
\begin{aligned}
u \alpha_{n}^{\prime}(u) & +(1+f(u)) \alpha_{n}(u) \\
= & \left(u^{2}+\frac{1}{n}\right)^{(q-\delta-1) / 2-1}\left((q-\delta+f(u)) u^{2}+\frac{1}{n}(1+f(u))\right)
\end{aligned}
$$

and assumptions (2.5) and (2.6) imply that the second factor on the right hand side is bounded below by

$$
\left(q-\delta+\inf _{M} f(u)\right) u^{2}+\frac{1}{n}\left(1+\inf _{M} f(u)\right) \geq C\left(u^{2}+\frac{1}{n}\right)
$$

with $C=\min \left\{q-\delta-\inf _{M} f(u), 1+\inf _{M} f(u)\right\}$. We therefore conclude that

$$
u \alpha_{n}^{\prime}(u)+(1+f(u)) \alpha_{n}(u) \geq \beta_{n}(u)>0
$$

with

$$
\beta_{n}(t)=C\left(t^{2}+\frac{1}{n}\right)^{(q-\delta-1) / \delta}
$$

We apply Lemma 2.1 and deduce that there exist $R_{o}>0$ independent of $n$ and $C_{1}>0$ independent of $n$ and $r$ such that, for every $r>R_{o}$

$$
\begin{aligned}
&\left(\int_{B_{R_{o}}}\left(u^{2}+\frac{1}{n}\right)^{(q-\delta-1) / 2} \varphi(|\nabla u|)|\nabla u|\right)^{-1} \\
& \geq C_{1}\left(\int_{R_{o}}^{r}\left(\int_{\partial B_{t}}|u|^{1+\delta}\left(u^{2}+\frac{1}{n}\right)^{(q-\delta-1) / 2}\right)^{-1 / \delta}\right)^{\delta}
\end{aligned}
$$

Letting $n \longrightarrow+\infty$, and using the dominated and monotone convergence theorems we conclude that for every $r>R_{o}$

$$
\left(\int_{B_{R_{o}}}|u|^{(q-\delta-1) / 2} \varphi(|\nabla u|)|\nabla u|\right)^{-1} \geq C_{1}\left(\int_{R_{o}}^{r}\left(\int_{\partial B_{t}}|u|^{q}\right)^{-1 / \delta}\right)^{\delta}
$$

which contradicts (2.7).
If we assume that $u>0$, we can repeat the reasoning using $\alpha(t)=$ $t^{q-\delta-1}$ and $\beta(t)=C t^{q-\delta-1}$, with $C=q-\delta+\inf _{M} f(u)$.

Theorem B in the Introduction is an immediate consequence of Theorem 2.2. We also note that in the case of subharmonic and $p$-subharmonic functions, we can compare with L. Karp, [K1] and Holopainen, [Ho], respectively. Indeed, using Proposition 1.3, assumption (2.7) can be replaced by any of the following

$$
\begin{gather*}
\left(\frac{t}{\int_{B_{r}}|u|^{q}}\right)^{1 / \delta} \notin L^{1}(+\infty),  \tag{2.8}\\
\liminf _{r \rightarrow+\infty} \frac{1}{r^{1+\delta}} \int_{B_{r}}|u|^{q}<+\infty,  \tag{2.9}\\
\limsup _{r \rightarrow+\infty} \frac{1}{r^{1+\delta} F(r)^{\delta}} \int_{B_{r}}|u|^{q}<+\infty, \tag{2.10}
\end{gather*}
$$

where $F(t)$ is a positive function defined for sufficiently large values of $t$, and such that $1 /(t F(t))$ is not integrable at infinity, the remaining assumptions of Theorem 2.2 being unchanged. It is easily verified that both (2.9) and (2.10) imply (2.8).

Lemma 2.1 also allows to obtain the following Liouville type result for $p$-superharmonic function, which compares with Sturm, [St], in the case of the Laplacian, and with Holopainen, $[\mathrm{Ho}]$, in the case of the $\mathcal{A}$ Laplacian. This is also an instance of a situation where the differential inequality (2.1) arises naturally.

Proposition 2.3. Let $u \in C^{1}(M)$ be p-superharmonic and non-negative on M. If

$$
\begin{equation*}
\left(\int_{\partial B_{r}} u^{q}\right)^{-1 /(p-1)} \notin L^{1}(+\infty), \tag{2.11}
\end{equation*}
$$

for some $q \in \mathbb{R}, q<p-1$, then $u$ is constant.
Proof. For every integer $n \geq 1$, let $v_{n}=(u+1 / n)^{-1}$. Then $\nabla v_{n}=$ $-v_{n}^{2} \nabla u$ and
$\operatorname{div}\left(\left|\nabla v_{n}\right|^{p-2}\left|\nabla v_{n}\right|\right)=-v_{n}^{2(p-1)} \operatorname{div}\left(|\nabla u|^{p-2}|\nabla u|\right)+2(p-1) v_{n}^{-1}\left|\nabla v_{n}\right|^{p}$.

Since $u$ is $p$-superharmonic and $v_{n}>0$, it follows that

$$
v_{n} \operatorname{div}\left(\left|\nabla v_{n}\right|^{p-2} \nabla v_{n}\right) \geq 2(p-1)\left|\nabla v_{n}\right|^{p}
$$

showing that $v_{n}$ satisfies (2.1) with $\varphi(t)=t^{p-1}$, and $f(t) \equiv 2(p-1)$.
The proof now follows the lines of that of Theorem 2.2. If we set $\alpha(t)=t^{-p-q}$, we have

$$
t \alpha^{\prime}(t)+(1+f(t)) \alpha(t)=(p-1-q) t^{-p-q}, \quad \text { for all } t>0,
$$

so that (2.3) is verified with $\beta(t)=(p-1-q) t^{-p-q}$.
Assume by contradiction that $u$ is not constant. By Lemma 2.1 we conclude that there exist $C$ and $R_{o}>0$, such that

$$
\left(\int_{B_{R_{o}}} v_{n}^{-p-q}\left|\nabla v_{n}\right|^{p}\right)^{-1} \geq C\left(\int_{R_{o}}^{r}\left(\int_{\partial B_{t}} v_{n}^{-q}\right)^{-1 /(p-1)} d t\right)^{p-1},
$$

for every $r>R_{o}$. Note that both $C$ and $R_{o}$ are independent of $n$, as it can be easily verified from the proof of the Lemma. Indeed, $C$ depends only on the structural constants in (0.1), in the case at hand, $A=1$ and $\delta=p-1$, while $R_{o}$ is the infimum of the values $t$ such that the function

$$
H_{n}(t)=\int_{B_{t}} v_{n}^{-p-q}\left|\nabla v_{n}\right|^{p}=\int_{B_{t}}\left(u+\frac{1}{n}\right)^{p+q}|\nabla u|^{p}
$$

is positive. It is clear that the right hand side is bounded below by $H_{1}(t)$.

Rewriting the main inequality in terms of $u$, letting $n \longrightarrow+\infty$, and using the monotone and dominated convergence theorems we obtain

$$
\left(\int_{B_{R_{o}}} u^{p+q} \varphi(|\nabla u|)|\nabla u|^{p}\right)^{-1} \geq C\left(\int_{R}^{r}\left(\int_{\partial B_{t}} u^{q}\right)^{-1 /(p-1)} d t\right)^{p-1}
$$

Letting $r \longrightarrow+\infty$ we contradict (2.11).
The following easy consequence of Theorem 2.2 will be useful to show its sharpness.

Corollary 2.4. Assume that

$$
\begin{equation*}
\operatorname{vol} \partial B_{r} \leq C r^{\eta-1} \tag{2.12}
\end{equation*}
$$

for some $\eta \geq 0, C>0$, and sufficiently large $r$. Let $u \in C^{1}(M)$ be a non-negative $\varphi$-subharmonic function on $M$. If there exist $q>\delta$ and $a$ constant $C_{1}>0$ such that

$$
\begin{equation*}
u(x)^{q} \leq C_{1} r(x)^{\delta-\eta+1} \log ^{\delta}(r(x)) \tag{2.13}
\end{equation*}
$$

for $r(x)$ sufficiently large, then $u$ is constant.
Remark. Assumption (2.13) deserves some further comment. Indeed, if $\delta-\eta+1<0$, then $u$ tends to zero at infinity and the validity of the maximum principle would force $u$ to vanish identically, with no need for (2.12). Therefore, this begs the question: when does the $\varphi$-Laplacian, $\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right)$, satisfy a maximum, or at least a comparison principle? The following elementary result answers in the affirmative if $\varphi$ is non-decreasing (see also [PSZ]).

Proposition 2.5. Let $\varphi$ satisfy conditions (0.1) i) and ii), i.e., $\varphi(0)=$ 0 and $\varphi(t)>0$ if $t>0$, and assume moreover that $\varphi$ is non-decreasing on $[0,+\infty)$. Let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$, and let $u$ and $v \in C^{1}(\bar{\Omega})$ satisfy

$$
\begin{aligned}
\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) & \geq \operatorname{div}\left(|\nabla v|^{-1} \varphi(|\nabla v|) \nabla v\right), \quad \text { on } \Omega, \\
u & \leq v, \quad \text { on } \partial \Omega .
\end{aligned}
$$

Then $u \leq v$ on $\Omega$.
Proof. We choose $\alpha \in C^{1}(\mathbb{R})$ such that

$$
\text { i) } \alpha(t)=0 \text { on }(-\infty, 0], \quad \text { ii) } \alpha^{\prime}(t)>0 \text { on }(0,+\infty),
$$

and consider the vector field $W$ defined on $\bar{\Omega}$ by

$$
W=\alpha(u-v)\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u-|\nabla v|^{-1} \varphi(|\nabla v|) \nabla v\right) .
$$

A computation that uses the properties of $u, v$ and $\alpha$, shows that

$$
\operatorname{div} W \geq \alpha^{\prime}(u-v) h, \quad \text { on } \bar{\Omega},
$$

where

$$
\left.h(x)=\left.\langle | \nabla u\right|^{-1} \varphi(|\nabla u|) \nabla u-|\nabla v|^{-1} \varphi(|\nabla v|) \nabla v, \nabla u-\nabla v\right\rangle(x) .
$$

Applying the divergence theorem (see the observation at the beginning of Section 1) and noting that $u \leq v$ on $\partial \Omega$ implies $\alpha(u-v)=0$ there, we obtain

$$
\begin{equation*}
\int_{\Omega} \alpha^{\prime}(u-v) h \leq 0 . \tag{2.14}
\end{equation*}
$$

Observe now that a simple computation shows that $h(x)$ is equal to

$$
\begin{aligned}
& (\varphi(|\nabla u|)-\varphi(|\nabla v|))(|\nabla u|-|\nabla v|)(x) \\
& \quad+\left(|\nabla u|^{-1} \varphi(|\nabla u|)+|\nabla v|^{-1} \varphi(|\nabla v|)\right)(|\nabla u||\nabla v|-\langle\nabla u, \nabla v\rangle)(x) .
\end{aligned}
$$

Since $\varphi$ is non-decreasing, we deduce from Schwarz inequality that $h(x) \geq 0$ for every $x$, with equality if and only if $\nabla u(x)=\nabla v(x)$. Therefore, it follows from (2.14) that $\alpha^{\prime}(u-v) h$ vanishes identically on $\Omega$.

Next, we assume by contradiction that

$$
\mathcal{O}=\{x \in \Omega: u(x)>v(x)\} \neq \varnothing .
$$

Since $\alpha^{\prime}(u-v)>0$ on $\mathcal{O}$, we must have $\nabla u=\nabla v$ on $\mathcal{O}$, so that $u-v$ is constant on each connected component of $\mathcal{O}$. But $u \leq v$ on $\partial \mathcal{O}$ (indeed, $u(z)=v(z)$ if $z \in \partial \mathcal{O} \cap \Omega$ by definition of $\mathcal{O}$, while $u(z) \leq v(z)$ by assumption if $z \in \partial \mathcal{O} \cap \partial \Omega$ ) and therefore $u \leq v$ on $\mathcal{O}$, contradicting the definition of $\mathcal{O}$.

We explicitly observe that the structural condition (0.1) iii) was not used in Proposition 2.5. Since constants are $\varphi$-harmonic, the Proposition easily implies that if $\varphi$ is non-decreasing, then a $\varphi$-subharmonic function on $\Omega$ attains its maximum on $\partial \Omega$. In particular, a nonnegative, $\varphi$-subharmonic function on $M$ that vanishes at infinity is necessarily identically zero. Indeed, under the further assumption that $\liminf _{t \rightarrow 0+} t \varphi^{\prime}(t) / \varphi(t)>0$, a slight modification of the proof of [PW, Theorem 5, pp. 61-64] shows that the usual strong maximum principle holds, namely, $u$ cannot attain an interior maximum unless it is constant $([\mathrm{P}])$. For a version of the strong maximum principle valid under slightly different, and somewhat weaker, assumptions see also [PSZ, Theorem 1].

To show that Corollary 2.4 is sharp we proceed as in Section 1. We keep the notation used there, and consider the case of the $p$-Laplacian. Here $\varphi(t)=t^{p-1}, p>1$, is increasing, and therefore we only need to consider the case where assumption (2.13) holds with $p \geq \eta \geq 0$.

Given any $q>p-1$, choose $a(t)$ as in (1.15), and

$$
\sigma(t)=t^{\mu(p-1) /(m-1)}(\log t)^{\nu(p-1) /(m-1)},
$$

on $\left[T_{o},+\infty\right)$, with constants $\mu$ and $\nu$ to be specified later. Then

$$
\operatorname{vol} \partial B_{r}=C_{m} \sigma^{m-1}(r)=C_{m} r^{\mu(p-1)}(\log t)^{\nu(p-1)}
$$

for $r \geq T_{o}$. Proceeding as in Section 1 it is easy to verify that if $u$ is defined in (1.14) then

$$
\begin{aligned}
u(r) & =C_{1}+C_{2} \int_{T_{o}}^{r} t^{-\mu}(\log t)^{-\nu} d t \\
& \sim C \begin{cases}r^{1-\mu}(\log r)^{-\nu}, & \text { if } \mu<1, \\
(\log r)^{-\nu+1}, & \text { if } \mu=1, \nu<1, \\
\log (\log r), & \text { if } \mu=1=\nu,\end{cases}
\end{aligned}
$$

as $r \longrightarrow+\infty$.
Consider first the case $p>\eta$. Let $q(1-\mu)=p-\eta$ and $-\nu q=p-1$, i.e., $\mu=(q-p+\eta) / q<1$ and $\nu=-(p-1) / q$. Then the non-constant $p$-subharmonic function $u$ satisfies

$$
u(r)^{q} \sim C r^{p-\eta}(\log r)^{p-1}, \quad \text { as } r \longrightarrow+\infty
$$

and condition (2.13) is met. On the other hand,

$$
\operatorname{vol} \partial B_{r}=C r^{(p-1)(\eta-p+q) / q}(\log r)^{-(p-1)^{2} / q} .
$$

The exponent of $r$ on the right hand side is greater than $\eta-1$ for every $q>p-1$, and tends to $\eta-1$ as $q$ tends $p-1$, showing that (2.12) barely fails.

Turning things around, if we take $\mu=(\eta-1) /(p-1)<1$ and $\nu=0$, then (2.12) is satisfied, while

$$
u(r)^{q} \sim C r^{q(p-\eta) /(p-1)}, \quad \text { as } r \longrightarrow+\infty
$$

Again, the exponent of $r$ on the right hand side is greater than $p-\eta$ for every $q>p-1$, and tends to $p-\eta$ as $q$ tends to $p-1$, showing that the non-constant $p$-subharmonic function narrowly fails to satisfy (2.13).

The case $p=\eta$ is dealt with similarly. To show that if (2.12) fails, then there are non-constant $p$-subharmonic functions satisfying (2.13), it suffices to take $\mu=1$, and $\nu=(q-p+1) / q$. Then

$$
u(r)^{q} \sim C \log ^{p-1} r, \quad \text { as } r \longrightarrow+\infty
$$

while

$$
\operatorname{vol} \partial B_{r}=C_{m} r^{p-1}(\log r)^{(p-1)(q-p+1) / q}, \quad r \geq T_{o}
$$

so that (2.12) is off only by a logarithmic term. On the other hand, if we take $\mu=1$ and $\nu=0$, then (2.12) holds, while

$$
u(r)^{q} \sim C \log ^{q} r, \quad \text { as } r \longrightarrow+\infty,
$$

so that (2.13) is not satisfied for every $q>p-1$.

## 3. Further results.

Lemmas 1.1 and 2.1 give estimates from above for the quantity

$$
H(t)=\int_{B_{R}} \beta(u) \varphi(|\nabla u|)|\nabla u| .
$$

The next lemma provides an estimate from below. By combining the two estimates, we will obtain new results.

Lemma 3.1. Let $f \in C^{o}(\mathbb{R})$, and let $u \in C^{1}(M)$ be a solution of the differential inequality

$$
\begin{equation*}
u \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq \varphi(|\nabla u|)|\nabla u| f(u), \quad \text { on } M \tag{3.1}
\end{equation*}
$$

Assume that there exist functions $\rho \in C^{1}(I)$ and $\beta \in C^{o}(I)$ defined in an interval $I \supset u(M)$ such that

$$
\begin{gather*}
\beta(u)>0  \tag{3.2}\\
\rho(u) \geq 0  \tag{3.3}\\
\frac{|u \rho(u)|}{\beta(u)} \leq L<+\infty  \tag{3.4}\\
u \rho^{\prime}(u)+(1+f(u)) \rho(u)>0 \tag{3.5}
\end{gather*}
$$

on $M$. Then there exist $R_{o}>0$ and a constant $C>0$ such that, for every $r>R \geq R_{o}$,

$$
\begin{equation*}
\int_{B_{r} \backslash B_{R}} \beta(u) \varphi(|\nabla u|)|\nabla u| \geq C \int_{R}^{r}\left(\int_{\partial B_{t}} \beta(u)^{\delta}\right)^{-1 / \delta} d t . \tag{3.6}
\end{equation*}
$$

Proof. Note first of all that, using the structural condition $\varphi(t) \leq$ $A t^{\delta}$, we may estimate

$$
\int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u| \geq A^{-1 / \delta} \int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)^{1+1 / \delta} .
$$

Now, by Hölder inequality with conjugate exponents $1+\delta$ and $1+1 / \delta$, we have

$$
\begin{aligned}
& \int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|) \\
& \leq\left(\int_{\partial B_{t}} \beta(u)\right)^{1 /(1+\delta)}\left(\int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)^{1+1 / \delta}\right)^{\delta /(1+\delta)}
\end{aligned}
$$

whence, using $\beta(u)>0$, rearranging and substituting, we obtain

$$
\begin{align*}
& \int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u| \\
& \quad \geq A^{-1 / \delta}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta}\left(\int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)\right)^{1+1 / \delta} . \tag{3.7}
\end{align*}
$$

Next, we consider the continuous vector field $X$ defined by

$$
X=u \rho(u)|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u
$$

and set

$$
\gamma(t)=\int_{\partial B_{t}}\langle X, \nabla r\rangle,
$$

so that, by Schwarz inequality and assumptions (3.2) and (3.4) we get

$$
\begin{equation*}
\gamma(t) \leq \int_{\partial B_{t}}|u \rho(u)| \varphi(|\nabla u|) \leq L \int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|) . \tag{3.8}
\end{equation*}
$$

On the other hand, computing the divergence of $X$ and using the assumption $\rho(u) \geq 0$ we estimate

$$
\operatorname{div} X \geq\left(u \rho^{\prime}(u)+(1+f(u)) \rho(u)\right) \varphi(|\nabla u|)|\nabla u|
$$

so that, by the divergence theorem,

$$
\gamma(t)=\int_{B_{t}} \operatorname{div} X \geq \int_{B_{t}}\left(u \rho^{\prime}(u)+(1+f(u)) \rho(u)\right) \varphi(|\nabla u|)|\nabla u| .
$$

Since $u$ is not constant, and (3.5) holds, there exist $R_{o}$ and a constant $C_{o}>0$, both depending on $\rho$ and $f$ only through the quantity $u \rho^{\prime}(u)+$ $(1+f(u)) \rho(u)$, such that

$$
\gamma(t) \geq C_{o}, \quad \text { for all } t \geq R_{o} .
$$

Combining this with (3.8) and inserting into (3.7) yield

$$
\begin{aligned}
\int_{\partial B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u| & \geq A^{-1 / \delta}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta}\left(L^{-1} \gamma(t)\right)^{1+1 / \delta} \\
& \geq C\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta}
\end{aligned}
$$

with $C=A^{-1 / \delta}\left(C_{o} / L\right)^{1+1 / \delta}$. Integrating over $[R, r], R_{o} \leq R<r$, and using the co-area formula we obtain (3.6).

Remark. In some applications it is crucial to avoid the explicit dependence on $\beta$ and $\rho$ of the quantity $R_{o}$ and the constant $C$ in (3.6). It is clear from the above proof that this may be achieved if we assume that $L$ is independent of $\beta$ and $\rho$ and replace (3.5) with

$$
\begin{equation*}
u \rho^{\prime}(u)+(1+f(u)) \rho(u) \geq \varepsilon \tag{3.9}
\end{equation*}
$$

for some absolute constant $\varepsilon>0$.
Putting together the estimate from below just obtained with the estimate from above provided by Lemma 2.1 we obtain

Lemma 3.2. Let $f \in C^{o}(\mathbb{R})$, and let $u \in C^{1}(M)$ be a solution of the differential inequality (3.1). Assume that there exist functions $\beta \in$ $C^{o}(I)$ and $\alpha, \rho \in C^{1}(I)$ defined in an interval $I \supset u(M)$ such that

$$
\begin{gather*}
\beta(u)>0, \quad \alpha(u), \rho(u) \geq 0  \tag{3.10}\\
u \alpha^{\prime}(u)+(1+f(u)) \alpha(u) \geq \beta(u),  \tag{3.11}\\
u \rho^{\prime}(u)+(1+f(u)) \rho(u)>0,  \tag{3.12}\\
\frac{|u \rho(u)|}{\beta(u)} \leq L<+\infty \tag{3.13}
\end{gather*}
$$

on $M$. Then there exist $R_{o}>0$ and a constant $C>0$ such that, for every $r>R \geq R_{o}$,

$$
\begin{equation*}
\frac{1}{\sup _{B_{r}}\left|\frac{u \alpha(u)}{\beta(u)}\right|} \int_{R}^{r}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta} d t \leq C \tag{3.14}
\end{equation*}
$$

Remark. As it will become clear from the proof below, if we can guarantee that the constants appearing in the conclusion (3.6) of Lemma 3.1 do not depend explicitly on $\beta$ and $\rho$, then the quantity $R_{o}$ and the constants $C$ above do not depend explicitly on $\alpha, \beta$ and $\rho$. In particular, this is the case if we assume that $L$ is independent of $\beta$ and $\rho$ and replace (3.12) with (3.9). This will be used in Theorem 3.6 below.

Proof. The assumptions of Lemma 2.1 and Lemma 3.1 are satisfied, so there exist $R_{o}$ and constants $C_{1}, C_{2}>0$ such that for every $r>R \geq R_{o}$ (2.4) and (3.6) hold with constant $C_{1}$ and $C_{2}$, respectively.

Denote as above

$$
H(t)=\int_{B_{t}} \beta(u) \varphi(|\nabla u|)|\nabla u|
$$

and let $r>\bar{R} \geq R \geq R_{o}$. It follows from (2.4) that

$$
H(\bar{R})^{-1} \geq \frac{C_{1}}{\sup _{B_{r}}\left|\frac{u \alpha(u)}{\beta(u)}\right|^{1+\delta}}\left(\int_{\bar{R}}^{r}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta} d t\right)^{\delta}
$$

while (3.6) yields

$$
H(\bar{R})-H(R) \geq C_{2} \int_{R}^{\bar{R}}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta} d t
$$

Combining the two inequalities we deduce that

$$
\begin{aligned}
1 \geq & \frac{H(\bar{R})-H(R)}{H(\bar{R})} \\
\geq & C_{3}\left(\sup _{B_{r}}\left|\frac{u \alpha(u)}{\beta(u)}\right|\right)^{-(1+\delta)} \\
& \cdot\left(\int_{\bar{R}}^{r}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta} d t\right)^{\delta} \int_{R}^{\bar{R}}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta} d t
\end{aligned}
$$

for every $r>\bar{R} \geq R \geq R_{o}$, with $C_{3}=C_{1} C_{2}$. We claim that we can choose $\bar{R}$ in such a way that the product of the two integrals on the right hand side is equal to

$$
\frac{\delta^{\delta}}{(1+\delta)^{\delta}}\left(\int_{R}^{r}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta} d t\right)^{1+\delta} .
$$

Indeed, having set

$$
B=\int_{R}^{r}\left(\int_{\partial B_{t}} \beta(u)\right)^{-\delta}, \quad x=\int_{\bar{R}}^{r}\left(\int_{\partial B_{t}} \beta(u)\right)^{-\delta},
$$

the claim amounts to finding a solution $x_{o} \in(0, B)$ to the equation

$$
x^{\delta}(B-x)=\frac{\delta^{\delta}}{(1+\delta)^{\delta}} B^{1+\delta},
$$

and it is easily verified that the (unique) solution $x_{o}$ in $(0, B)$ to the given equation is

$$
x_{o}=\frac{\delta}{1+\delta} B
$$

We conclude that, for every $r>R \geq R_{o}$

$$
1 \geq C_{4}\left(\sup _{B_{r}}\left|\frac{u \alpha(u)}{\beta(u)}\right|\right)^{-(1+\delta)}\left(\int_{\bar{R}}^{r}\left(\int_{\partial B_{t}} \beta(u)\right)^{-1 / \delta} d t\right)^{1+\delta}
$$

with

$$
C_{4}=C_{3} \frac{\delta^{\delta}}{(1+\delta)^{1+\delta}}
$$

whence, rearranging, we obtain (3.14).
As a first consequence of Lemma 3.2 we have
Theorem 3.3. Let $u \in C^{1}(M)$ be a solution of the differential inequality

$$
\begin{equation*}
u \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq 0 \tag{3.15}
\end{equation*}
$$

on M. If

$$
\begin{equation*}
\liminf _{r \rightarrow+\infty} \frac{\sup _{B_{r}}|u|}{\int_{R}^{r}\left(\operatorname{vol} \partial B_{t}\right)^{-1 / \delta} d t}=0 \tag{3.16}
\end{equation*}
$$

for some $R>0$ sufficiently large, then $u$ is constant.
Proof. Assume by contradiction that $u$ is not constant. We apply Lemma 3.2 with $\alpha=\beta=1, \rho(t)=\left(1+t^{2}\right)^{-1 / 2}$ (and $f \equiv 0$ ) to conclude that there exist $R_{o}>0$ and a constant $C>0$ such that for every $r>R \geq R_{o}$,

$$
\frac{1}{\sup _{B_{r}}|u|} \int_{R}^{r}\left(\operatorname{vol} \partial B_{t}\right)^{-1 / \delta} d t \geq C .
$$

It is clear that this contradicts our assumption (3.16).
Remark. If $u$ is non-negative we can replace (3.16) with

$$
\begin{equation*}
\liminf _{r \rightarrow+\infty} \frac{\sup _{B_{r}} u}{\int_{R}^{r}\left(\operatorname{vol} \partial B_{t}\right)^{-1 / \delta} d t}=0 \tag{3.17}
\end{equation*}
$$

for some $R>0$ sufficiently large. Observe that, if $u \not \equiv 0$, then (3.17) implies that

$$
\begin{equation*}
\left(\operatorname{vol} \partial B_{t}\right)^{-1 / \delta} \notin L^{1}(+\infty) \tag{3.18}
\end{equation*}
$$

This in particular implies that a non-negative $\varphi$-subharmonic function $u$ satisfying (3.17) is necessarily constant. In this connection we remark that in [RSV, Theorem 3], it was shown, with a different proof, that the same conclusion holds without any sign condition on $u$ if (3.17) and (3.18) hold. This also follows from the results presented in Section 4 below. We note however that the proof in [RSV] does not seem to adapt to the case of solutions of the differential inequality (3.15), and therefore does not yield the further consequences of Lemma 3.2 presented below.

The following corollary is the companion of Corollary 2.4 and will be useful to show the sharpness of Theorem 3.3.

Corollary 3.4. Assume that

$$
\begin{equation*}
\operatorname{vol} \partial B_{r} \leq C r^{\eta-1} \tag{3.19}
\end{equation*}
$$

for some $\eta \geq 0, C>0$, and sufficiently large $r$. Let $u \in C^{1}(M)$ be a non-negative $\varphi$-subharmonic function on $M$. If

$$
\begin{gather*}
u(x)^{\delta}=o\left(r(x)^{\delta-\eta+1}\right), \quad \text { if } \delta>\eta-1,  \tag{3.20}\\
u(x)=o(\log r(x)), \quad \text { if } \delta=\eta-1, \tag{3.21}
\end{gather*}
$$

as $r(x) \longrightarrow+\infty$, then $u$ is constant.
As in the examples in sections 1 and 2 , we consider the case of the $p$-Laplacian, and keep the notation used there. In particular, $\delta=p-1$, $a(t)$ is defined in (1.15), and $u$ is the $p$-subharmonic function defined in (1.14). As in Section 2, it suffices to consider the case $p \geq \eta$. The construction done there provides examples of manifolds satisfying (3.19), and admitting non-constant $p$-subharmonic functions which barely fail to satisfy (3.20) or (3.21) respectively in the case $p>\eta$ and $p=\eta$.

On the other hand, if we choose

$$
\sigma(t)= \begin{cases}t^{(\eta-1) /(m-1)}(\log t)^{\nu(p-1) /(m-1)}, & \text { if } p>\eta, \\ t^{(\eta-1) /(m-1)}(\log \log t)^{\nu(\eta-1) /(m-1)}, & \text { if } p=\eta\end{cases}
$$

on $\left[T_{o},+\infty\right)$, for some $\nu>0$, then we have

$$
\operatorname{vol} \partial B_{r}=C_{m} \begin{cases}r^{\eta-1}(\log r)^{\nu(p-1)}, & \text { if } p>\eta \\ r^{\eta-1}(\log \log r)^{\nu(p-1)}, & \text { if } p=\eta\end{cases}
$$

and

$$
u(r) \sim C \begin{cases}r^{(p-\eta) /(p-1)}(\log r)^{-\nu}, & \text { if } p>\eta \\ \log r(\log \log r)^{-\nu}, & \text { if } p=\eta\end{cases}
$$

as $r \longrightarrow+\infty$. Thus $u$ satisfies condition (3.20) or (3.21), respectively, while (3.19) barely fails to hold.

Lemma 3.2 also yields the following
Theorem 3.5. Let $f \in C^{\circ}(\mathbb{R})$, and let $u \in C^{1}(M)$ be a solution of the differential inequality (3.1) on $M$ satisfying

$$
u>0, \quad \inf _{M} f(u)>-\gamma,
$$

for some $\gamma>0$. If

$$
\begin{equation*}
\liminf _{r \rightarrow+\infty} \frac{\left(\sup _{B_{r}} u\right)^{1+\gamma / \delta}}{\int_{R}^{r}\left(\operatorname{vol} \partial B_{t}\right)^{-1 / \delta} d t}=0 \tag{3.22}
\end{equation*}
$$

for some $R>0$ sufficiently large, then $u$ is constant.

Proof. Assuming by contradiction that $u$ is not constant, we set

$$
\mu=\gamma+\inf _{M} f(u)>0
$$

and let $\alpha, \beta, \rho$ be the functions defined on $(0,+\infty)$ by $\alpha(t)=t^{\gamma}$, $\beta(t)=\mu t^{\gamma}$ and $\rho(t)=t^{\gamma-1}$. It is easy to verify that the assumptions of Lemma 3.2 are satisfied, and we deduce that there exist $R_{o}>0$ and a constant $C>0$ such that for every $r>R \geq R_{o}$

$$
\frac{\mu^{1+\gamma / \delta}}{\sup _{B_{r}} u} \int_{R}^{r}\left(\int_{\partial B_{t}} u^{\gamma}\right)^{-1 / \delta} d t \leq C
$$

which contradicts (3.22).
When $u$ is not assumed to be positive, we have the following version of the above result.

Theorem 3.6. Let $f \in C^{o}(\mathbb{R})$, and let $u \in C^{1}(M)$ be a solution of the differential inequality (3.1) on $M$ satisfying

$$
\inf _{M} f(u)>-1
$$

If

$$
\begin{equation*}
\liminf _{r \rightarrow+\infty} \frac{\left(\sup _{B_{r}}|u|\right)^{1+1 / \delta}}{\int_{R}^{r}\left(\operatorname{vol} \partial B_{t}\right)^{-1 / \delta} d t}=0 \tag{3.23}
\end{equation*}
$$

for some $R>0$ sufficiently large, then $u$ is constant.
Proof. Again, we assume by contradiction that $u$ is not constant. Let $S=1+\inf _{M} f(u)>0$, and, for every integer $n \geq 1$, define $\alpha_{n}(t)=$ $\left(t^{2}+1 / n\right)^{1 / 2}, \beta_{n}=S \alpha_{n}$ and $\rho_{n}(t)=\rho(t) \equiv 1$. Then

$$
\begin{gathered}
u \alpha_{n}^{\prime}(u)+(1+f(u)) \alpha_{n}(u) \geq \beta_{n}(u) \\
u \rho^{\prime}(u)+(1+f(u)) \rho(u)=1+f(u) \geq S>0 .
\end{gathered}
$$

Moreover

$$
\frac{|u \rho(u)|}{\beta_{n}(u)}=\frac{|u|}{S\left(u^{2}+1 / n\right)^{1 / 2}} \leq \frac{1}{S}, \quad \text { on } M
$$

independently of $n$. By Lemma 3.2 and the remark thereafter, there exist $R_{o}>0$ and a constant $C>0$ independent of $n$ such that, for every $r>R \geq R_{o}$,

$$
\frac{1}{\sup _{B_{r}}|u|} \int_{R}^{r}\left(\int_{\partial B_{t}}\left(u^{2}+\frac{1}{n}\right)^{1 / 2}\right)^{-1 / \delta} d t \leq C .
$$

Letting $n \longrightarrow+\infty$, and using the monotone and dominated convergence theorems we deduce that, for every $r>R \geq R_{o}$,

$$
\frac{1}{\sup _{B_{r}}|u|} \int_{R}^{r}\left(\int_{\partial B_{t}}|u|\right)^{-1 / \delta} d t \leq C,
$$

and this contradicts assumption (3.23).

## 4. Phragmen-Lindelöff type results.

Lemma 4.1. Let $u \in C^{1}(\Omega) \cap C^{o}(\bar{\Omega})$ be a $\varphi$-subharmonic function on an unbounded domain $\Omega \subset M$, and assume that $u \leq \Gamma$ on $\partial \Omega$, for some $\Gamma \in \mathbb{R}$. Given $B>\Gamma$, define

$$
\Omega_{B}=\{x \in \Omega: u(x)>B\},
$$

and suppose that $\Omega_{B}$ is not empty with boundary $\partial \Omega_{B}$. Let $\alpha \in C^{1}$ and $\beta \in C^{o}$ be defined in $[B,+\infty)$ and such that $\alpha(u) \geq 0, \alpha^{\prime}(u) \geq \beta(u)>0$ on $\bar{\Omega}_{B}$. Let also $\lambda \in C^{1}(\mathbb{R})$ be such that $\lambda(t)=0$ for $t \leq B, \lambda(t)>0$ for $t>B$ and $\lambda^{\prime}(t) \geq 0$. Then there exist $R_{o}$ and a constant $C>0$ independent of $\alpha$ and $\beta$ and $\lambda$ such that, for every $r>R \geq R_{o}$

$$
\begin{aligned}
& \left(\int_{B_{R} \cap \Omega_{B}} \lambda(u) \beta(u) \varphi(|\nabla u|)|\nabla u|\right)^{-1} \\
& \quad \geq C\left(\int_{R}^{r}\left(\int_{\partial B_{t} \cap \Omega_{B}} \lambda(u) \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{-1 / \delta}\right)^{\delta} .
\end{aligned}
$$

Proof. Observe first of all that since $B>\Gamma$, then $\bar{\Omega}_{B} \subset \Omega$. Thus $u=$ $B$ on $\partial \Omega_{B}$, and it follows that $u$ cannot be constant on any component of $\Omega_{B}$. In particular $\nabla u$ does not vanish identically on $\Omega_{B}$.

The argument now follows the lines of the proof of Lemma 1.1. Let $\widetilde{Z}$ be the vector field on $\bar{\Omega}_{B}$ defined by

$$
\widetilde{Z}=\lambda(u) \alpha(u)|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u .
$$

Note that $\widetilde{Z}$ can be extended to a continuous vector field on $M$ by setting it equal to 0 on $\Omega_{B}^{c}$. Similarly, we can and will similarly extend to all of $M$ every product containing a factor $\lambda(u)$. Set also

$$
\widetilde{H}(t)=\int_{B_{t}} \lambda(u) \beta(u) \varphi(|\nabla u|)|\nabla u|,
$$

so that, by the co-area formula,

$$
\widetilde{H}^{\prime}(t)=\int_{\partial B_{t}} \lambda(u) \beta(u) \varphi(|\nabla u|)|\nabla u| .
$$

Since

$$
\operatorname{div} Z \geq \lambda(u) \beta(u) \varphi(|\nabla u|)|\nabla u|, \quad \text { on } M
$$

integrating over $B_{t}$, applying the divergence theorem, Hölder inequality with exponents $1+\delta$ and $1+1 / \delta$, and using the structural condition $\varphi(t)^{1 / \delta} \leq A^{1 / \delta} t$, we obtain

$$
\begin{equation*}
\widetilde{H}(t) \leq A^{1 /(1+\delta)}\left(\int_{\partial B_{t}} \lambda(u) \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{1 /(1+\delta)}\left(\widetilde{H}^{\prime}(t)\right)^{\delta /(1+\delta)} . \tag{4.1}
\end{equation*}
$$

Since $\nabla u$ is not identically zero on $\Omega_{B}$, and $\beta(u)$ and $\lambda(u)$ are there strictly positive, there exists $R_{o}>0$ (independent of $\alpha, \beta$ and $\lambda$ ) such that $\widetilde{H}(t)>0$ if $t \geq R_{o}$. It follows that the right hand side of (4.1) is also strictly positive for $t \geq R_{o}$. In particular, $\Omega_{B}$ is necessarily unbounded.

Rearranging and integrating between $R$ and $r, R_{o} \leq R<r$, we obtain

$$
\widetilde{H}(R)^{-1 / \delta} \geq \frac{1}{\delta A^{1 / \delta}} \int_{R}^{r}\left(\int_{\partial B_{t}} \lambda(u) \frac{\alpha(u)^{1+\delta}}{\beta(u)^{\delta}}\right)^{-1 / \delta} d t
$$

To conclude we only have to observe that, since $\lambda(u)=0$ off $\Omega_{B}$, the integrals over $B_{t}$ and $\partial B_{t}$ may be replaced with integrals over $B_{t} \cap \Omega_{B}$ and $\partial B_{t} \cap \Omega_{B}$, respectively.

We remark that if in the above proof $\Omega$ is assumed to be bounded, then for $t$ sufficiently large (4.1) leads to a contradiction. This in turn, forces $\Omega_{B}=\varnothing$, and we conclude that $u \leq \Gamma$ on $\Omega$. In other words, if $u$ is $C^{1}$ in a bounded domain $\Omega$, continuous up to the boundary, and
$\varphi$-subharmonic in $\Omega$, then $u$ attains its maximum on $\partial \Omega$. Of course, if $\varphi$ is non-decreasing, then this follows also from Proposition 2.5.

As a consequence of Lemma 4.1 we have the following PhragmenLindelöff type result:

Theorem 4.2. Let $\Omega$ be an unbounded domain in $M$ and let $u \in$ $C^{1}(\Omega) \cap C^{o}(\bar{\Omega})$, be a non-negative $\varphi$-subharmonic function on $\Omega$ such that $u \leq \Gamma$ on $\partial \Omega$. Assume that, for some $q>\delta$,

$$
\begin{equation*}
\left(\int_{\partial B_{r} \cap \Omega} u^{q}\right)^{-1 / \delta} \notin L^{1}(+\infty) . \tag{4.2}
\end{equation*}
$$

Then $u \leq \Gamma$ on $\Omega$.
Proof. Assume by contradiction that $\{x \in \Omega: u(x)>\Gamma\} \neq \varnothing$, and choose $B>\Gamma \geq 0$ sufficiently close to $\Gamma$ that $\Omega_{B}=\{x \in \Omega: u(x)>$ $B\} \neq \varnothing$. We apply Lemma 4.1 with the choices

$$
\alpha(t)=t^{q-\delta}, \quad \beta(t)=\alpha^{\prime}(t)=(q-\delta) t^{q-\delta-1}, \quad t \geq B,
$$

and $\lambda \in C^{1}(\mathbb{R})$ satisfying the conditions in the statement of the lemma, and $\sup _{\mathbb{R}} \lambda(t)=1$. It follows that there exist $R_{o}$ and $C>0$ such that, for every $r>R \geq R_{o}$,

$$
\left(\int_{B_{R} \cap \Omega_{B}} \lambda(u) u^{q-\delta-1} \varphi(|\nabla u|)|\nabla u|\right)^{-1} \geq C\left(\int_{R}^{r}\left(\int_{\partial B_{t} \cap \Omega} u^{q}\right)^{-1 / \delta}\right)^{\delta} .
$$

By virtue of (4.2), letting $r \longrightarrow+\infty$ this yields the required contradiction.

Lemma 4.1 also allows us to prove the following, slightly more general version of Theorem B.

Theorem 4.3. Let $u \in C^{1}(M)$ be a $\varphi$-subharmonic function on $M$. Assume that there exists $x_{o} \in M$ such that $u\left(x_{o}\right)>0$ and let $u_{+}(x)=$ $\max \{0, u(x)\}$. If there exists $q>\delta$ such that

$$
\left(\int_{\partial B_{r}} u_{+}^{q}\right)^{-1 / \delta} \notin L^{1}(+\infty),
$$

then $u$ is constant on $M$.

Proof. Arguing as above, assume by contradiction that $u$ is not constant, and let $B>0$ be sufficiently small that $\Omega_{B}=\{x \in M: u(x)>$ $B\}$ is a non-empty set with boundary $\partial \Omega_{B}$. Applying Lemma 4.1 with the same choice of $\alpha, \beta$ and $\lambda$ as in Theorem 4.2, we obtain

$$
\left(\int_{B_{R} \cap \Omega_{B}} \lambda(u) u^{q-\delta-1} \varphi(|\nabla u|)|\nabla u|\right)^{-1} \geq C\left(\int_{R}^{r}\left(\int_{\partial B_{t} \cap \Omega_{B}} u^{q}\right)^{-1 / \delta}\right)^{\delta},
$$

for every $r>R \geq R_{o}$. Since the surface integral on the right hand side is bounded above by $\int_{\partial B_{t}} u_{+}^{q}$, a contradiction is reached letting $r \longrightarrow+\infty$.

Similarly, applying Lemma 4.1 with $\alpha(t)=\log ^{b}(1+\log (1+t))$ and $\beta(t)=\alpha^{\prime}(t)$, and arguing as above, one proves a version of Theorem 1.2 valid for functions of arbitrary sign. Namely, if $u$ is a $\varphi$-subharmonic function such that $u\left(x_{o}\right)>0$ for some $x_{o} \in M$ and (1.10) holds with $u_{+}$instead of $u$ then $u$ is necessarily constant. This, in turn, yields a version of Theorem C valid for functions of arbitrary sign.

The next lemma is a version of Lemma 3.2 on a domain.
Lemma 4.4. Let $f \in C^{\circ}(\mathbb{R})$. Let $\Omega \subset M$ be an unbounded domain, and let $u \in C^{1}(\Omega) \cap C^{o}(\bar{\Omega})$, satisfy

$$
u \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq \varphi(|\nabla u|)|\nabla u| f(u), \quad \text { on } \Omega,
$$

and assume that $u \leq \Gamma$ on $\partial \Omega$. Given $B>\Gamma$, define $\Omega_{B}=\{x \in$ $\Omega: u(x)>B\}$, and suppose that $\Omega_{B}$ is not empty with boundary $\partial \Omega_{B}$. Assume that there exist functions $\beta \in C^{o}([B,+\infty))$ and $\alpha, \rho \in$ $C^{1}([B,+\infty))$ such that

$$
\begin{gathered}
\beta(u)>0, \quad \alpha(u), \rho(u) \geq 0 \\
u \alpha^{\prime}(u)+(1+f(u)) \alpha(u) \geq \beta(u), \\
u \rho^{\prime}(u)+(1+f(u)) \rho(u)>0, \\
\frac{|u \rho(u)|}{\beta(u)} \leq L<+\infty
\end{gathered}
$$

on $\Omega_{B}$. Finally, let $\lambda \in C^{1}(\mathbb{R})$ be such that $\lambda(t)=0$ for $t \leq B, \lambda(t)>0$ for $t>B$ and $\lambda^{\prime}(t) \geq 0$. Then there exist $R_{o}$ and a constant $C>0$
such that, for every $r>R \geq R_{o}$,

$$
\frac{1}{\sup _{B_{r} \cap \Omega_{B}} \frac{|u \alpha(u)|}{\beta(u)}} \int_{R}^{r}\left(\int_{\partial B_{t} \cap \Omega_{B}} \lambda(u) \beta(u)\right)^{-1 / \delta} d t \leq C .
$$

Remark. As in Lemma 3.2, if we assume that $L$ is independent of $\beta$ and $\rho$, and that $u \rho^{\prime}(u)+(1+f(u)) \rho(u) \geq \varepsilon>0$ on $\Omega_{B}$, then $R_{o}$ and $C$ do not depend explicitly on $\alpha, \beta, \lambda$ and $\rho$.

Proof. The proof is modeled after that of Lemma 3.2. Set

$$
\widehat{H}(t)=\int_{B_{R} \cap \Omega_{B}} \lambda(u) \beta(u) \varphi(|\nabla u|)|\nabla u| .
$$

Arguing as in Lemma 2.1, one shows that there exist $R_{1}$ and $C_{1}>0$ (independent of $\alpha, \beta$ and $\lambda$ ) such that, for every $r>\bar{R} \geq R_{1}$,

$$
\begin{aligned}
\widehat{H}(\bar{R})^{-1} & \geq C_{1}\left(\int_{\bar{R}}^{r}\left(\int_{\partial B_{t} \cap \Omega_{B}} \lambda(u) \frac{|u \alpha(u)|^{1+\delta}}{\beta(u)^{\delta}}\right)^{-1 / \delta}\right)^{\delta} \\
& \geq \frac{C_{1}}{\sup _{B_{r} \cap \Omega_{B}}\left(\frac{|u \alpha(u)|}{\beta(u)}\right)^{1+\delta}}\left(\int_{\bar{R}}^{r}\left(\int_{\partial B_{t} \cap \Omega_{B}} \lambda(u) \beta(u)\right)^{-1 / \delta}\right)^{\delta} .
\end{aligned}
$$

On the other hand, a minor modification of the proof of Lemma 3.1 shows that there exist $R_{o} \geq R_{1}$ and $C_{2}>0$ such that for every $\bar{R}>$ $R \geq R_{o}$,

$$
\widehat{H}(\bar{R})-\widehat{H}(R) \geq C_{2} \int_{R}^{\bar{R}}\left(\int_{\partial B_{t}} \lambda(u) \beta(u)\right)^{-1 / \delta} d t
$$

The required conclusion follows as in the final part of the proof of Lemma 3.2.

Theorem 4.5. Let $\Omega$ be an unbounded domain in $M$, and let $u \in$ $C^{1}(\Omega) \cap C^{o}(\bar{\Omega})$ be a $\varphi$-subharmonic function on $\Omega$ such that $u \leq \Gamma$ on $\partial \Omega$. Assume that

$$
\operatorname{vol}\left(\partial B_{r} \cap \Omega\right)^{-1 / \delta} \notin L^{1}(+\infty),
$$

and

$$
\begin{equation*}
\liminf _{r \rightarrow+\infty} \frac{\sup _{B_{r} \cap \Omega} u}{\int_{R}^{r} \operatorname{vol}\left(\partial B_{r} \cap \Omega\right)^{-1 / \delta}}=0 \tag{4.3}
\end{equation*}
$$

for some $R>0$ sufficiently large. Then $u \leq \Gamma$ on $\Omega$.

Proof. Note first that if $C$ is a constant and $v=u+C$, then $v$ is $\varphi$-subharmonic on $\Omega, v \leq \Gamma+C$ on $\partial \Omega$, and $v$ satisfies (4.3). Clearly, $u \leq \Gamma$ on $\Omega$ if and only if $v \leq \Gamma+C$ on $\Omega$. Without loss of generality, we can therefore assume that $\Gamma>0$.

Assume by contradiction that $\{x \in \Omega: u(x)>\Gamma\} \neq \varnothing$, and choose $B>\Gamma$ close enough to $\Gamma$ that $\Omega_{B}=\{x \in \Omega: u(x)>B\}$ is not empty. We apply Lemma 4.4 with the choices

$$
\alpha(t)=1, \quad \beta(t)=1, \quad \text { and } \quad \rho(t)=\left(1+t^{2}\right)^{-1 / 2}
$$

and with $\lambda$ satisfying the further condition $\sup _{\mathbb{R}} \lambda=1$. We conclude that there exist $R_{o}$ and $C>0$ such that, for $r>R \geq R_{o}$,

$$
\frac{1}{\sup _{B_{r} \cap \Omega_{B}} u} \int_{R}^{r} \operatorname{vol}\left(\partial B_{t} \cap \Omega_{B}\right)^{-1 / \delta} d t \leq C
$$

Since $\operatorname{vol}\left(\partial B_{t} \cap \Omega\right)^{-1 / \delta} \leq \operatorname{vol}\left(\partial B_{t} \cap \Omega_{B}\right)^{-1 / \delta}$ and $\sup _{B_{r} \cap \Omega} u=$ $\sup _{B_{r} \cap \Omega_{B}} u$, this clearly contradicts (4.3).

We observe that if $\varphi$ is non-decreasing, by Proposition 2.5, $\sup _{B_{r} \cap \Omega} u$ may be replaced by $\sup _{\partial B_{r} \cap \Omega} u$.

We conclude this section by showing how Theorem 4.5 allows us to recover the conclusion of [RSV, Theorem 3], quoted in Section 3.

Corollary 4.6. Assume that

$$
\operatorname{vol}\left(\partial B_{t}\right)^{-1 / \delta} \notin L^{1}(+\infty)
$$

and let $u \in C^{1}(M)$ be a $\varphi$-subharmonic function on $M$. If

$$
\liminf _{r \rightarrow+\infty} \frac{\sup _{B_{r}} u}{\int_{R}^{r} \operatorname{vol}\left(\partial B_{r}\right)^{-1 / \delta}}=0
$$

then $u$ is constant on $M$.
Proof. Assume that $u$ is not constant, and choose $\Gamma<\sup u$ in such a way that $\varnothing \neq \Omega=\{x: u(x)>\Gamma\}$ and $\partial \Omega$ is of class $C^{1}$. Since $\operatorname{vol}\left(\partial B_{t}\right)^{-1 / \delta} \leq \operatorname{vol}\left(\partial B_{t} \cap \Omega\right)^{-1 / \delta}$ and $\sup _{B_{r} \cap \Omega} u=\sup _{B_{r} \cap \Omega} u$, both the assumptions of Theorem 4.5 hold, and we conclude that $u \leq \Gamma$ on $\Omega$, contradicting the definition of $\Omega$.

## 5. A weak maximum principle.

We begin by proving a weak maximum principle asserting that, under suitable volume growth conditions, given a smooth function $u$ which is bounded above on $M$, the set where $u$ is close to its supremum and $\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right)$ is less that any given positive constant is nonempty. A special case of this result for the Laplacian was proved by Karp in [K2, Theorem 2.3]. His proof made use of the stochastic completeness of the underlying manifold. In our general setting, such an approach is clearly not feasible. The proof presented below is direct and based on elementary considerations. We recall that $A$ and $\delta$ are the constants that appear in the structural condition (0.1) iii). Throughout this section it will be assumed that the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least $C^{1}$. As mentioned in the Introduction, if $u$ is $C^{2}$, this is certainly the case for the Laplacian, or the $p$-Laplacian with $p \geq 2$ and for the (generalized) mean curvature operators.

Theorem 5.1. Let $u \in C^{2}(M)$ be such that $u^{*}=\sup _{M} u<+\infty$, and assume that the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least $C^{1}$. Given $\alpha<u^{*}$, let $\Omega_{\alpha}=\{x \in M: u(x)>\alpha\}$, and assume that

$$
\begin{equation*}
\liminf _{r \rightarrow+\infty} \frac{\log \operatorname{vol}\left(B_{r} \cap \Omega_{\alpha}\right)}{r^{1+\delta}}<+\infty \tag{5.1}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\inf _{\Omega_{\alpha}} \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \leq 0 \tag{5.2}
\end{equation*}
$$

Proof. According to (5.1), there exist $0<\gamma<+\infty$, a constant $C_{o}>0$, and a sequence $R_{k} \nearrow+\infty$ such that

$$
\begin{equation*}
\operatorname{vol}\left(B_{R_{k}} \cap \Omega_{\alpha}\right) \leq C_{o} e^{\gamma R_{k}^{1+\delta}} \tag{5.3}
\end{equation*}
$$

for every $k=1,2, \ldots$. Fix $\eta>0$ and define $v=u-u^{*}+\eta$, so that

$$
\Omega_{\alpha}=\left\{x \in M: v(x)>\alpha-u^{*}+\eta\right\} .
$$

We are going to prove that

$$
\begin{align*}
\inf _{\Omega_{\alpha}} \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) & =\inf _{\Omega_{\alpha}} \operatorname{div}\left(|\nabla v|^{-1} \varphi(|\nabla v|) \nabla v\right)  \tag{5.4}\\
& \leq A \gamma \delta^{\delta} 2^{1+\delta} \eta^{\delta},
\end{align*}
$$

whence the required conclusion follows letting $\eta \longrightarrow 0^{+}$.
To prove (5.4), we assume by contradiction that for some $\rho>0$
(5.5) $\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq A \gamma \delta^{\delta} 2^{1+\delta} \eta^{\delta}(1+\rho)=B, \quad$ on $\Omega_{\alpha}$.

We choose a $C^{1}$ function $\lambda$ such that $\lambda(t)=0$ if $t \leq \alpha-u^{*}+\eta, \lambda(t)>0$ if $t>\alpha-u^{*}+\eta, \lambda^{\prime} \geq 0$, and $\sup \lambda(t)=1$. Fix $\varepsilon>0$, and let $\sigma=\gamma / \eta$, $\zeta=2 \gamma+\varepsilon$ and $\theta=(\gamma / \gamma+\varepsilon)^{1 /(1+\delta)}$, so that $0<\theta<1$. Finally, choose a smooth cutoff function $h=h_{k}$ such that $h=1$ on $B_{\theta R_{k}}, h=0$ off $B_{R_{k}}$ and $|\nabla h| \leq C_{1} /\left((1-\theta) R_{k}\right)$, for some $C_{1}>0$ independent of $k$, and define the vector field

$$
W=h^{1+\delta} \lambda(v) e^{(\sigma v-\zeta) r^{1+\delta}}|\nabla v|^{-1} \varphi(|\nabla v|) \nabla v
$$

We compute the divergence of $W$, and use (5.5), $\lambda^{\prime} \geq 0$, Schwarz inequality, $|\nabla r|=1$, the inequality $\sigma v-\zeta \leq 0$, and the structural condition $|\nabla v| \geq A^{-1 / \delta} \varphi(|\nabla v|)^{1 / \delta}$, to obtain, after some computations,
$\operatorname{div} W$

$$
\begin{aligned}
\geq- & (1+\delta) h^{\delta} \lambda(v) e^{(\sigma v-\zeta) r^{1+\delta}} \varphi(|\nabla v|)|\nabla h|+h^{1+\delta} \lambda(v) e^{(\sigma v-\zeta) r^{1+\delta}} \\
& \cdot\left(B-(1+\delta) \zeta r^{\delta} \varphi(|\nabla v|)+\sigma A^{-1 / \delta} r^{1+\delta} \varphi(|\nabla v|)^{1+1 / \delta}\right) .
\end{aligned}
$$

We claim that, if $\varepsilon>0$ is small enough,

$$
B-(1+\delta) \zeta r^{\delta} \varphi(|\nabla v|)+\sigma A^{-1 / \delta} r^{1+\delta} \varphi(|\nabla v|)^{1+1 / \delta}
$$

$$
\begin{equation*}
\geq \Lambda r^{1+\delta} \varphi(|\nabla v|)^{1+1 / \delta} \tag{5.6}
\end{equation*}
$$

with $\Lambda=\Lambda(\varepsilon)>0$.
Postponing the proof of the claim, we insert (5.6) into the above inequality, integrate over $\Omega_{\alpha} \cap B_{R_{k}}$, and apply the divergence theorem.

Since every factor containing $\lambda(v)$ vanishes off $\Omega_{\alpha}$, while every product containing $h$ vanishes off $B_{R_{k}}$ (so that we may equivalently integrate over $B_{R_{k}}$ thus avoiding possible problems due to the non-smoothness of the boundary of $\Omega_{\alpha} \cap B_{R_{k}}$ ), we obtain

$$
\begin{aligned}
& \int_{\Omega_{\alpha} \cap B_{R_{k}}} h^{1+\delta} \lambda(v) e^{(\sigma v-\zeta) r^{1+\delta}} r^{1+\delta} \varphi(|\nabla v|)^{1+1 / \delta} \\
& \leq \frac{1+\delta}{\Lambda} \int_{\Omega_{\alpha} \cap B_{R_{k}}} h^{\delta} \lambda(v) e^{(\sigma v-\zeta) r^{1+\delta}} \varphi(|\nabla v|)|\nabla h| .
\end{aligned}
$$

Applying Hölder inequality with conjugate exponents $1+\delta$ and $1+1 / \delta$ to estimate the right hand side, rearranging and using the properties of the cutoff function $h$ and the inequality $\lambda(v) \leq 1$, we conclude that

$$
\begin{align*}
& \int_{\Omega_{\alpha} \cap B_{\theta R_{k}}} r^{1+\delta} e^{(\sigma v-\zeta) r^{1+\delta}} \lambda(v) \varphi(|\nabla v|)^{1+1 / \delta}  \tag{5.7}\\
& \leq \frac{C_{2}}{R_{k}^{(1+\delta)^{2}}} \int_{B_{R_{k} \backslash B_{\theta R_{k}}}} e^{(\sigma v-\zeta) r^{1+\delta}}
\end{align*}
$$

with $C_{2}=\left((1+\delta) C_{1} /\left(\Lambda(1-\theta) \theta^{\delta}\right)\right)^{1+\delta}$. Now, using the definitions of $\sigma, \gamma, \zeta$, and $\sup v=\eta$, we have $\sigma v-\zeta \leq-(\gamma+\varepsilon)$. Thus

$$
\begin{aligned}
\int_{B_{R_{k}} \backslash B_{\theta R_{k}}} e^{(\sigma v-\zeta) r^{1+\delta}} & \leq \int_{B_{R_{k}} \backslash B_{\theta R_{k}}} e^{-(\gamma+\varepsilon) r^{1+\delta}} \\
& \leq e^{-(\gamma+\varepsilon)\left(\theta R_{k}\right)^{1+\delta}} \operatorname{vol} B_{R_{k}}
\end{aligned}
$$

and it follows from the definition of $\theta$ and (5.3) that that the right hand side is bounded above by $C_{o}>0$. Inserting into (5.7), we deduce that

$$
\int_{\Omega_{\alpha} \cap B_{\theta R_{k}}} r^{1+\delta} e^{(\sigma v-\zeta) r^{1+\delta}} \lambda(v) \varphi(|\nabla v|)^{1+1 / \delta} \leq \frac{C_{3}}{R_{k}^{(1+\delta)^{2}}} .
$$

Letting $k \longrightarrow+\infty$, we conclude that the integrand vanishes identically in $\Omega_{\alpha}$. Since $\lambda(v)>0$ in $\Omega_{\alpha}$, and $\varphi(t)>0$ if $t>0$, this implies that $v$ is constant on every connected component of $\Omega_{\alpha}$, and this contradicts assumption (5.5).

To conclude it remains to prove (5.6). Setting $x^{-1}=r^{\delta} \varphi(|\nabla v|)$ this amounts to showing that

$$
\Lambda=\inf _{x>0}\left\{B x^{1+1 / \delta}-(1+\delta) \zeta x^{1 / \delta}+\sigma A^{-1 / \delta}\right\}>0 .
$$

It is easily verified that the function in braces attains its minimum on $(0,+\infty)$ at $x_{o}=\mu(B(1+\delta))^{-1}$ where it is equal to

$$
\frac{\sigma}{A^{1 / \delta}}-\frac{\delta \zeta^{1+1 / \delta}}{B^{1 / \delta}}
$$

Recalling the definitions of the quantities involved, an easy computation shows that the latter quantity is equal to

$$
\frac{\gamma}{A^{1 / d} \eta}\left(1-\frac{(1+\varepsilon / 2 \gamma)^{1+1 / \delta}}{(1+\rho)^{1 / \delta}}\right),
$$

which is strictly positive if $\varepsilon$ is small enough.
Theorem 5.1 immediately yields the following weak version of the Omori-Yau maximum principle for the $\varphi$-Laplacian.

Corollary 5.2. Assume that

$$
\begin{equation*}
\liminf _{r \rightarrow+\infty} \frac{\log \operatorname{vol} B_{r}}{r^{1+\delta}}<+\infty \tag{5.8}
\end{equation*}
$$

and let $u$ be a smooth function on $M$ with $u^{*}=\sup u<+\infty$, such that $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least $C^{1}$. For every $n$ the set

$$
Z_{n}=\left\{y \in M: u(y)>u^{*}-\frac{1}{n}, \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right)<\frac{1}{n}\right\} \neq \varnothing
$$

In particular, this yields Theorem D in the Introduction.
Proof. We define the sets $A_{n}=\left\{y: u(y)>u^{*}-1 / n\right\}$ and $B_{n}=\{y$ : $\left.\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right)(y)<1 / n\right\}$ so that $Z_{n}=A_{n} \cap B_{n}$.

Clearly, $A_{n} \neq \varnothing$, and, by Theorem 5.1 we also have $B_{n} \neq \varnothing$. We may therefore define $u_{n}^{*}=\sup _{B_{n}} u$, and the required conclusion follows from $u_{n}^{*}=u^{*}$.

Indeed, assume by contradiction that $u_{n}^{*}<u^{*}$, and let $\Omega_{u_{n}^{*}}=\{x$ : $\left.u(x)>u_{n}^{*}\right\} \neq \varnothing$. By definition, $\Omega_{u_{n}^{*}} \subset B_{n}^{c}$, and therefore

$$
\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq \frac{1}{n}, \quad \text { on } \Omega_{u_{n}^{*}},
$$

and this contradicts Theorem 5.1.

In a forthcoming paper we shall see how the validity of this weak form of the maximum principle for the Laplacian is tightly related to stochastic completeness (see [PRS]).

The next theorem provides a version of the weak maximum principle when the boundedness of the function is replaced by a suitable condition on the growth at infinity. We note that Theorem 5.1 corresponds to the limit case $b=1+\delta$. For the case of the Laplacian, our result generalizes [K2, Theorem 2.2] (where it is considered the case $b=1$ ).

Theorem 5.3. Assume that, for some $1 \leq b<1+\delta$,

$$
\liminf _{r \rightarrow+\infty} \frac{\log \operatorname{vol} B_{r}}{r^{b}}=\gamma_{o}<+\infty
$$

Let $u \in C^{2}(M)$ be such that the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least $C^{1}$, and assume that

$$
\begin{equation*}
\limsup _{r(x) \rightarrow+\infty} \frac{u(x)}{r(x)^{(1+\delta-b) / \delta}} \leq a_{o} \tag{5.9}
\end{equation*}
$$

for some $a_{o} \geq 0$. Then

$$
\begin{equation*}
\inf _{M} \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \leq \frac{A \gamma_{o}\left(a_{o} \delta\right)^{\delta}(2 b)^{1+\delta}}{(1+\delta)^{1+\delta}} . \tag{5.10}
\end{equation*}
$$

Proof. The proof is a modification of that of Theorem 5.1. Let $a>a_{o}$, and $\gamma>\gamma_{o}$. We are going to show that

$$
\begin{equation*}
\inf _{M} \operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \leq \frac{A \gamma(a \delta)^{\delta}(2 b)^{1+\delta}}{(1+\delta)^{1+\delta}}, \tag{5.11}
\end{equation*}
$$

whence the required conclusion follows letting $a \longrightarrow a_{o}$ and $\gamma \longrightarrow \gamma_{o}$.
By adding a suitable constant to $u$, it may be assumed that
i) $\frac{u(x)}{(1+r(x))^{(1+\delta-b) / \delta}}<a, \quad$ on $M$,
ii) there exists $x_{o} \in M$ such that $u\left(x_{o}\right)>0$.

Clearly it suffices to show that (5.11) holds when the infimum is taken over the set $\Omega=\{x: u(x)>0\}$ instead of $M$. To prove this, we assume by contradiction that for some $\rho>0$,

$$
\begin{equation*}
\operatorname{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq \frac{A \gamma(a \delta)^{\delta}(2 b)^{1+\delta}(1+\rho)}{(1+\delta)^{1+\delta}}=B>0 \tag{5.13}
\end{equation*}
$$

on $\Omega$. Let $\lambda \in C^{1}(\mathbb{R})$ such that $\lambda(t)=0$ if $t \leq 0, \lambda(t)>0$ if $t>0$, $\lambda^{\prime}(t) \geq 0$ and $\sup \lambda=1$. Let also $\varepsilon>0$, and define

$$
\sigma=\frac{\gamma}{a}, \quad \theta=\left(\frac{\gamma}{\gamma+\varepsilon}\right)^{1 / b}, \quad \text { and } \quad \zeta=2 \gamma+\varepsilon .
$$

By the volume growth assumption and the inequality $\gamma>\gamma_{o}$, there exist a sequence $R_{k} \nearrow+\infty$ and a constant $C_{o}>0$ such that

$$
\begin{equation*}
\operatorname{vol} B_{R_{k}} \leq C_{o} e^{\gamma R_{k}^{b}} \tag{5.14}
\end{equation*}
$$

For every $k$, we let $h=h_{k}$ be a smooth cutoff function such that $h=1$ on $B_{\theta R_{k}}, h=0$ off $B_{R_{k}}$, and $|\nabla h| \leq C_{1} /\left((1-\theta) R_{k}\right)$ with $C_{1}>0$ independent of $k$. Finally, let $W$ be the vector field defined by

$$
W=h^{1+\delta} \lambda(u) \Xi(u)|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u,
$$

where we have set, for notational convenience,

$$
\Xi(u)=\exp \left((1+r)^{b}\left(\frac{\sigma u}{(1+r)^{(1+\delta-b) / \delta}}-\zeta\right)\right) .
$$

A computation that uses (5.13), $\lambda^{\prime} \geq 0$, and Schwarz inequality, shows that

$$
\begin{aligned}
\operatorname{div} W \geq & h^{1+\delta} \lambda(u) \Xi(u) B-(1+\delta) h^{\delta} \lambda(u) \Xi(u) \varphi(|\nabla u|)|\nabla h| \\
& +\sigma(1+r)^{(b-1)(1+1 / \delta)} h^{1+\delta} \lambda(u) \Xi(u) \varphi(|\nabla u|)|\nabla u| \\
+ & h^{1+\delta} \lambda(u) \Xi(u)|\nabla u|^{-1} \varphi(|\nabla u|)\langle\nabla u, \nabla r\rangle \\
& \cdot(1+r)^{b-1}\left((b-1)\left(1+\frac{1}{\delta}\right) \frac{\sigma u}{(1+r)^{(1+\delta-b) / \delta}}-b \zeta\right) .
\end{aligned}
$$

Using (5.12) i), it is easily verified that the quantity in braces on the right hand side is negative on $\Omega$. It follows that the last term in the above inequality is bounded below by

$$
\begin{aligned}
h^{1+\delta} \lambda(u) \Xi(u) & \varphi(|\nabla u|)(1+r)^{b-1}(\cdots) \\
& \geq-h^{1+\delta} \lambda(u) \Xi(u) \varphi(|\nabla u|)(1+r)^{b-1} b \zeta, \quad \text { on } \Omega,
\end{aligned}
$$

where we have used the fact that $u \geq 0$ on $\Omega$. Substituting, and using the structural condition (0.1) iii), we obtain
$\operatorname{div} W$

$$
\begin{aligned}
& \geq-(1+\delta) h^{\delta} \lambda(u) \Xi(u) \varphi(|\nabla u|)|\nabla h| \\
&+h^{1+\delta} \lambda(u) \Xi(u)\left(B-b \zeta(1+r)^{b-1} \varphi(|\nabla u|)\right. \\
&\left.+\sigma A^{-1 / \delta} \varphi(|\nabla u|)^{(\delta+1) / \delta}(1+r)^{(b-1)(1+1 / \delta)}\right) .
\end{aligned}
$$

Setting $x^{-1}=(1+r)^{b-1} \varphi(|\nabla u|)$ and arguing exactly as in the proof of the claim in Theorem 5.1 one shows that for $\varepsilon$ sufficiently small the quantity in braces on the right hand side is bounded from below by

$$
\Lambda(1+r)^{(b-1)(1+1 / \delta)} \varphi(|\nabla u|)^{1+1 / \delta}
$$

At this point the proof proceeds as in Theorem 5.1. Integrating div $W$ over $\Omega \cap B_{R_{k}}$, applying the divergence theorem, Hölder inequality with exponents $1+\delta$ and $1+1 / \delta$, and using the properties of the cutoff function $h$ and $\sup \lambda=1$ we obtain

$$
\begin{align*}
& \int_{\Omega \cap B_{\theta R_{k}}} \lambda(u) \Xi(u)(1+r)^{(b-1)(1+1 / \delta)} \varphi(|\nabla u|)^{1+1 / \delta}  \tag{5.15}\\
& \leq \frac{C_{2}}{R_{k}^{b(1+\delta)}} \int_{B_{R_{k}} \backslash B_{\theta R_{k}}} \Xi(u),
\end{align*}
$$

for some constant $C_{2}>0$ independent of $k$.
It follows from (5.12) i), and from the definition of the quantities involved that

$$
\Xi(u) \leq \exp \left((1+r)^{b}(\sigma a-\zeta)\right)=\exp \left(-(\gamma+\varepsilon)(1+r)^{b}\right),
$$

so that, using the volume estimate (5.14), we deduce that the integral on the right hand side of (5.15) is bounded above by $C_{o}$ for every $k$. Thus

$$
\int_{\Omega \cap B_{\theta R_{k}}} \lambda(u) \Xi(u)(1+r)^{(b-1)(1+1 / \delta)} \varphi(|\nabla u|)^{1+1 / \delta} \leq \frac{C_{3}}{R_{k}^{b(1+\delta)}},
$$

with $C_{3}$ independent of $k$. Letting $k \longrightarrow+\infty$ we conclude that the integrand must be identically equal to zero on $\Omega$, and therefore that $u$
is constant on every connected component of $\Omega$. But this contradicts (5.13), as required to finish the proof.

We conclude by showing that Theorems 5.1 and 5.3 are sharp with respect to the volume growth conditions in their statement. We consider the $p$-Laplacian, and keep the notation introduced in Section 1. For $1 \leq b \leq p$, let $\sigma(t)$ satisfy

$$
\sigma(t)=\exp \left(t^{b} \log ^{\nu} t\right), \quad \text { for all } t \geq T_{o},
$$

for some $T_{o}>1$, and let $u$ be the $p$-subharmonic function defined in (1.14), with $a(t)=1$ for every $t$. Then $\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)=1$ on $M$, and there exist constants $C_{1}$ and $C_{2}$ such that, for $r>T_{o}$,

$$
u(r)=C_{1}+\int_{T_{o}}^{r} \sigma(t)^{-(m-1) /(p-1)}\left(C_{2}+\int_{T_{o}}^{t} \sigma(s)^{m-1} d s\right)^{1 /(p-1)} d t
$$

It is easy to verify that
$\int_{T_{o}}^{t} \sigma(s)^{m-1} d s \sim C t^{1-b}(\log t)^{-\nu} \exp \left((m-1) t^{b} \log ^{\nu} t\right), \quad$ as $t \longrightarrow+\infty$,
and therefore

$$
\frac{\log \operatorname{vol} B_{r}}{r^{b}} \sim C(\log r)^{\nu}, \quad \text { as } r \longrightarrow+\infty
$$

Furthermore,

$$
\begin{aligned}
& \sigma(t)^{-(m-1) /(p-1)}\left(C_{2}+\int_{T_{o}}^{t} \sigma(s)^{m-1} d s\right)^{1 /(p-1)} \\
& \sim \frac{C}{t^{(b-1) /(p-1)}(\log t)^{\nu /(p-1)}}
\end{aligned}
$$

To show that Theorem 5.1 is sharp, we choose $b=p$ and $\nu>p-1$. Then $u$ is bounded, and the conclusion of the theorem clearly does not hold. Since $u(r)$ is increasing, the set $\Omega_{\alpha}(\alpha<\sup u)$ is a ball, and $\log \operatorname{vol}\left(\Omega_{\alpha} \cap B_{r}\right) \sim \log \operatorname{vol} B_{r}$, showing that the volume growth condition in the statement of the theorem fails by a log factor.

On the other hand, if we take $1 \leq b<p$, and $\nu>0$ then, the volume growth condition (5.8) fails (again by a $\log$ term). In this case

$$
u(r) \sim C r^{1+1 /(p-1)-b /(p-1)}, \quad \text { as } r \longrightarrow+\infty
$$

so that assumption (5.9) is satisfied with $a_{o}=0$, while (5.10) clearly is not.

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# Endpoint multiplier theorems of Marcinkiewicz type 

Terence Tao and James Wright

Abstract. We establish sharp ( $H^{1}, L^{1, q}$ ) and $\operatorname{local}\left(L \log ^{r} L, L^{1, q}\right)$ mapping properties for rough one-dimensional multipliers. In particular, we show that the multipliers in the Marcinkiewicz multiplier theorem map $H^{1}$ to $L^{1, \infty}$ and $L \log ^{1 / 2} L$ to $L^{1, \infty}$, and that these estimates are sharp.

## 1. Introduction.

Let $m$ be a bounded function on $\mathbb{R}$, and let $T_{m}$ be the associated multiplier

$$
\widehat{T_{m} f}(\xi)=m(\xi) \widehat{f}(\xi) .
$$

There are many multiplier theorems which give conditions under which $T_{m}$ is an $L^{p}$ multiplier. We will be interested in the mapping behaviour of $T_{m}$ near $L^{1}$. Specifically, we address the following questions:

- For which $1 \leq q \leq \infty$ does $T_{m}$ map the Hardy space $H^{1}$ to the Lorentz space $L^{1, q}$ ?
- We say that $T_{m}$ locally maps the Orlicz space $L \log ^{r} L$ to $L^{1, q}$ if

$$
\left\|T_{m} f\right\|_{L^{1, q}(K)} \leq C_{K}\|f\|_{L \log ^{r} L(K)},
$$

for all compact sets $K$ and all functions $f$ on $K$. For which $r \geq 0$ and $1 \leq q \leq \infty$ does $T_{m}$ locally map $L \log ^{r} L$ to $L^{1, q}$ ?

Standard interpolation theory (see e.g. [1]) shows that if $T_{m}$ locally maps $L \log ^{r} L$ to $L^{1, q}$, then it locally maps $L \log ^{\widetilde{r}}$ to $L^{1, \widetilde{q}}$ whenever $\widetilde{q} \leq q$ and $\widetilde{r} \geq r+1 / \widetilde{q}-1 / q$. Also, extrapolation theory ([14], [13]) shows that $T_{m}$ maps $L \log ^{r} L$ to $L^{1}$ if and only if the $L^{p}$ operator norm of $T_{m}$ grows like $O\left((p-1)^{-r-1}\right)$ as $p \longrightarrow 1$.

Here and in the sequel, $\eta$ is an even bump function adapted to $\pm[1 / 2,4]$ which equals 1 on $\pm[3 / 4,3]$.

Definition 1.1. If $m$ is a symbol and $j$ is an integer, we define the $j^{\text {th }}$ frequency component $m_{j}$ of $m$ to be the function

$$
m_{j}(\xi)=\eta(\xi) m\left(2^{j} \xi\right)
$$

We say that $T_{m}$ is a Hörmander multiplier if the frequency components $m_{j}$ are in the Sobolev space $L_{1 / 2+}^{2}$ uniformly in $j$. These multipliers are Calderón-Zygmund operators and hence map $H^{1}$ to $L^{1}$ (and even to $H^{1}$ ), and $L^{1}$ to $L^{1, \infty}$; see e.g. [11]. By interpolation one then sees that $T_{m}$ locally maps $L \log ^{r} L$ to $L^{1, q}$ whenever $r \leq 1 / q$.

We now consider multipliers not covered by the Hörmander theory. We say that $T_{m}$ is a Marcinkiewicz multiplier if the frequency components $m_{j}$ have bounded variation uniformly in $j$. The Marcinkiewicz multiplier theorem (see e.g. [11]) shows that $T_{m}$ is bounded on $L^{p}$.

Our first result characterizes the endpoint behaviour of Marcinkiewicz multipliers:

Theorem 1.2. Marcinkiewicz multipliers map $H^{1}$ to $L^{1, \infty}$, and locally map $L \log ^{r} L$ to $L^{1, q}$ whenever $r \geq 1 / 2+1 / q$. Conversely, there exist Marcinkiewicz multipliers which do not map $H^{1}$ to $L^{1, q}$ for any $q<\infty$, and do not locally map $L \log ^{r} L$ to $L^{1, q}$ for any $r<1 / 2+1 / q$.

We can generalize the notion of a Marcinkiewicz multiplier as follows.

Definition 1.3 ([3]). Let $X$ denote the set of all functions of the form

$$
m=\sum_{I} c_{I} \chi_{I}
$$

where I ranges over a collection of disjoint intervals in $\pm[1 / 2,4]$, and the $c_{I}$ are square summable coefficients

$$
\begin{equation*}
\left(\sum_{I}\left|c_{I}\right|^{2}\right)^{1 / 2} \leq 1 \tag{1}
\end{equation*}
$$

Let $\bar{X}$ denote the Banach space generated by using the elements of $X$ as atoms; note that this space includes all functions of bounded variation on $\pm[1 / 2,4]$. We say that $T_{m}$ is a $R_{2}$ multiplier if the frequency components $m_{j}$ are in $\bar{X}$ uniformly in $j$.

This class is more general than the Marcinkiewicz and Hörmander classes. In [3] it was established that $R_{2}$ multipliers are bounded on all $L^{p}, 1<p<\infty$.

We can extend the positive results of Theorem 1.2 as follows.
Theorem 1.4. All the statements in Theorem 1.2 continue to hold for $R_{2}$ multipliers.

One can also show the $L^{p}$ norms of these multipliers grow like $\max \left\{p, p^{\prime}\right\}^{3 / 2}$ by converse extrapolation theorems (see [13]). This is sharp. Theorem 1.4 also has an easy corollary to multipliers of bounded $s$-variation as studied in [3]; we detail this in Section 8.

We now consider another multiplier class which is slightly smoother than the $R_{2}$ multiplier class.

Definition 1.5 ([9]). Let $X^{\prime}$ denote the set of all functions of the form

$$
m=\sum_{I} c_{I} \psi_{I}
$$

where $I, c_{I}$ are as in the definition of $X$, and the $\psi_{I}$ are $C^{1} 0$ bump functions adapted to $I$. Let $\overline{X^{\prime}}$ be the atomic Banach space generated by $X^{\prime}$. We say that $m$ is in $R_{1 / 2,2}^{2}$ if

$$
\begin{equation*}
\left\|\psi m\left(2^{-j}\right)\right\|_{\overline{X^{\prime}}} \lesssim 1 \tag{2}
\end{equation*}
$$

for all integers $j$, where $\psi$ is a bump function adapted to $\pm[1 / 2,4]$ which equals 1 on $\pm[1,2]$. We say that $T_{m}$ is a $R_{1 / 2,2}^{2}$ multiplier if the frequency components $m_{j}$ are in $\overline{X^{\prime}}$ uniformly in $j$.

This class was first studied in [9]; it contains the Hörmander class, is contained in the $R_{2}$ class, and is not comparable with the Marcinkiewicz class. In [9, Theorem 2.2] the $R_{1 / 2,2}^{2}$ multipliers were shown to map $H^{1}$ to $L^{1, \infty}$; we can improve this to

Theorem 1.6. $R_{1 / 2,2}^{2}$ multipliers map $H^{1}$ to $L^{1,2}$, and locally map $L \log ^{r} L$ to $L^{1, q}$ whenever $r \geq \max \{1 / 2,1 / q\}$. Conversely, there exist $R_{1 / 2,2}^{2}$ multipliers which do not map $H^{1}$ to $L^{1, q}$ for any $q<2$, and do not map $L \log ^{r} L$ to $L^{1, q}$ whenever $r<\max \{1 / 2,1 / q\}$.

The converse extrapolation theorem in [13] thus shows that these operators have an $L^{p}$ operator norm of $O\left(\max \left\{p, p^{\prime}\right\}\right)$, and this is sharp.

Thus, to summarize our main results, $R_{2}$ multipliers map both $H^{1}$ and $L \log L^{1 / 2}$ to $L^{1, \infty}$, while the smoother $R_{1 / 2,2}^{2}$ multipliers map both $H^{1}$ and $L \log L^{1 / 2}$ to $L^{1,2}$, with all exponents being best possible.

From the classical study [6] of the multipliers

$$
\begin{equation*}
m(\xi)=\frac{e^{i|\xi|^{\alpha}}}{\left(1+|\xi|^{2}\right)^{\beta / 2}} \tag{3}
\end{equation*}
$$

it is known that the condition (1) cannot be replaced with a weaker $l^{q}$ condition, $q>2$, if the intervals $I$ are the same size. However, even if the intervals are different sizes, one still cannot relax this condition, as the following result shows.

Definition 1.7 ([9]). For any $1 \leq q \leq \infty$, let $X_{q}^{\prime}$ be defined as in $X^{\prime}$ but with (1) replaced by

$$
\left(\sum_{k}\left(\sum_{I:|I| \sim 2^{k}} c_{I}^{2}\right)^{q / 2}\right)^{1 / q} \leq 1
$$

Let $\overline{X_{q}^{\prime}}$ be the atomic Banach space generated by $X_{q}^{\prime}$. We say that $T_{m}$ is a $R_{1 / 2, q}^{2}$ multiplier if the frequency components $m_{j}$ are in $\overline{X_{q}^{\prime}}$ uniformly in $j$.

Theorem 1.8. For any $q>2$, there exist $R_{1 / 2, q}^{2}$ multipliers which are unbounded on $L^{p}$ for $|1 / 2-1 / p|>1 / q$. In particular, there are no mapping properties near $L^{1}$.

One can obtain positive ( $L^{p}, L^{p}$ ) or ( $L^{p}, L^{p, 2}$ ) mapping results when $2<q \leq \infty$ for these operators by complex interpolation between Theorem 1.6 and trivial $L^{2}$ estimates ( $c f$. [4]), but we shall not do so here.

The space $H^{1}$ has of course appeared countless times in endpoint multiplier theory, but the appearance of the Orlicz space $L \log ^{1 / 2} L$
space is more unusual. This space first appeared in work of Zygmund [15], who showed the inequality

$$
\begin{equation*}
\left(\sum_{j=0}^{\infty}\left|\widehat{f}\left(2^{j}\right)\right|^{2}\right)^{1 / 2} \lesssim\|f\|_{L \log ^{1 / 2} L} \tag{4}
\end{equation*}
$$

for all $f$ on the unit circle $S^{1}$. This inequality can be viewed as a rudimentary prototype of the multiplier theorems described above (indeed, one can derive (3) from either of the above theorems by transplanting the results to the circle, and considering multipliers supported on the dyadic frequencies $\left.2^{j}\right)$. As we shall see in Section 4 , the space $L \log ^{1 / 2} L$ is in fact very similar to the Hardy space $H^{1}$ in that it has an associated square function which is integrable.

The space $L^{1,2}$ has appeared in recent work of Seeger and Tao [10] Very roughly speaking, just as the space $L^{1, \infty}$ is natural for maximal functions and $L^{1}$ is natural for sums, the space $L^{1,2}$ is natural for certain square functions. A concrete version of this principle appears in Lemma 7.1.

This paper is organized as follows. After some notational preliminaries we detail the negative results to the above Theorems in Section 3. In Section 4 and the Appendix we show how both $H^{1}$ and $L \log ^{1 / 2} L$ functions are associated with an integrable square function. In Sections $5,6,7$ we then show how control of this square function leads to $L^{1,2}$ and $L^{1, \infty}$ multiplier estimates. Finally, we discuss the $V_{q}$ class in Section 8.

## 2. Notation.

We use $C$ to denote various constants, and $A \lesssim B, A=O(B)$, or " $B$ majorizes $A$ " to denote the estimate $A \leq C B$. We use $A \sim B$ to denote the estimate $A \lesssim B \lesssim A$.

Here and in the sequel, $\Delta_{j}$ denotes the Littlewood-Paley multiplier with symbol $\eta\left(2^{-j}.\right)$, where $\eta$ is as in the introduction. For integers $j$, we use $\phi_{j}$ to denote the weight function

$$
\begin{equation*}
\phi_{j}(x)=2^{j}\left(1+2^{2 j}|x|^{2}\right)^{-3 / 4} . \tag{5}
\end{equation*}
$$

Similarly, for intervals $I$ we use $\phi_{I}$ to denote the weight

$$
\begin{equation*}
\phi_{I}(x)=|I|\left(1+|I|^{2}|x|^{2}\right)^{-3 / 4} . \tag{6}
\end{equation*}
$$

These weights are thus smooth and decay like $|x|^{-3 / 2}$ at infinity. Many quantities in our argument will be controlled using the $\phi_{j}, \phi_{I}$; the reason why the decay is so weak is because we are forced at one point to use the Haar wavelet system, which has very poor moment conditions. (The exact choice of $3 / 2$ has no significance, any exponent strictly between 1 and 2 would have sufficed).

## 3. Negative results.

In this section we detail the counter-examples which yield the negative results stated in the introduction. In all of these examples $N$ is a large integer which will eventually be sent to infinity, $\left\{e_{j}\right\}_{j \in \mathbb{Z}}$ is the standard basis of $l^{2}(\mathbb{Z})$, and $\psi$ is a non-negative even bump function supported on $\{|\xi| \ll 1\}$ which equals 1 at the origin and has a non-negative Fourier transform. Some of our counter-examples will be vector-valued, but one can obtain scalar-valued substitutes by replacing $e_{j}$ with randomized signs $\varepsilon_{j}= \pm 1$ and using the Lorentz-space version of Khinchin's inequality; we omit the details.
3.1. Marcinkiewicz multipliers and $R_{2}$ multipliers need not $\operatorname{map} H^{1}$ to $L^{1, q}$ for any $q<\infty$.

Consider the symbol

$$
\begin{equation*}
m_{0}(\xi)=\chi_{[1, \infty)}(\xi) \psi(\xi-1) \tag{7}
\end{equation*}
$$

The convolution kernel $\widehat{m_{0}}$ of this function is bounded for $|x| \lesssim 1$, and can be estimated via stationary phase as

$$
\begin{equation*}
\widehat{m_{0}}(x)=\frac{e^{2 \pi i x}}{x}+O\left(|x|^{-2}\right) \tag{8}
\end{equation*}
$$

for $|x| \gg 1$. If we then test this multiplier against a bump function $f$ with $\widehat{f}(0)=0$ and $\widehat{f}(1) \neq 0$, we see that $f$ is in $H^{1}$, but $\left|T_{m_{0}} f(x)\right| \sim 1 / x$ as $|x| \longrightarrow \infty$, so $T_{m_{0}} f$ is not in $L^{1, q}$ for any $q<\infty$.

### 3.2. Marcinkiewicz multipliers and $R_{2}$ multipliers need not locally map $L \log ^{r} L$ to $L^{1, q}$ for any $r<1 / 2+1 / q$.

Define the vector-valued multiplier

$$
m_{N}(\xi)=\sum_{j=0}^{N} e_{j} m_{0}\left(\frac{\xi}{2^{j}}\right)
$$

where $m_{0}$ is defined in (7); this multiplier satisfies the requirements of both Theorems.

By testing $T_{m_{N}}$ against a function $f$ whose Fourier transform is a bump function which equals 1 on $\left[-2^{N}, 2^{N}\right]$ and is adapted to a slight dilate of this interval, (so that $\|f\|_{L \log ^{r} L} \sim N^{1 / r}$ ) we see that we must have

$$
\left\|\widehat{m_{N}}\right\|_{L^{1, q}([0,1])} \lesssim N^{1 / r}
$$

in order for $T_{m_{N}}$ to locally ${ }^{1}$ map $L \log ^{r} L$ to $L^{1, q}$. However, by (8) we have

$$
\left|\widehat{m_{N}}(x)\right| \sim \frac{\log (1 /|x|)^{1 / 2}}{|x|}
$$

for $2^{N} \ll|x| \ll 1$, and the necessary condition $r<1 / 2+1 / q$ follows by a routine computation.
3.3. $R_{2}^{2,1 / 2}$ multipliers need not map $H^{1}$ to $L^{1, q}$ for any $q<2$.

We use the multiplier

$$
m_{N}^{\prime}(\xi)=N^{-1 / 2} \sum_{j=0}^{N} \psi\left(2^{j}(\xi-1)-1\right)
$$

This multiplier is in the class of Theorem 1.6. Now suppose for contradiction that $T_{m_{N}^{\prime}}$ mapped $H^{1}$ to $L^{1, q}$. Since $m_{N}^{\prime}$ is supported in a single dyadic scale, we may factor $T_{m_{N}^{\prime}}=T_{m_{N}^{\prime}} S_{0}$ where $S_{0}$ is a LittlewoodPaley projection to frequencies $|\xi| \sim 1$. From the Littlewood-Paley square-function characterization we see that $S_{0}$ maps $H^{1}$ to $L^{1}$, hence

[^6]$T_{m_{N}^{\prime}} \operatorname{maps} L^{1}$ to $L^{1, q}$. In particular, the kernel $\widehat{m_{N}^{\prime}}$ must be in $L^{1, q}$. However, a computation shows that
$$
\widehat{m_{N}^{\prime}}(x) \left\lvert\, \lesssim \frac{N^{-1 / 2}}{|x|}\right.
$$
for $1 \ll|x| \ll 2^{N}$, which contradicts the assumption that $q<2$.
3.4. $R_{1 / 2,2}^{2}$ multipliers need not locally $\operatorname{map} L \log ^{r} L$ to $L^{1, q}$ for any $r<1 / 2$.

We consider the vector-valued multiplier

$$
m_{N}^{\prime \prime}(\xi)=\sum_{j=0}^{N} e_{j} \psi\left(\xi-2^{j}\right)
$$

this is a multiplier in the class of Theorem 1.6. By repeating the argument with the $m_{N}$ multipliers, we must have

$$
\left\|\widehat{m_{N}^{\prime \prime}}\right\|_{L^{1, q}([0,1])} \lesssim N^{1 / r}
$$

However, a computation shows that

$$
\left|\widehat{m_{N}^{\prime \prime}}(x)\right| \sim \sqrt{N}
$$

for $|x| \ll 1$, and this contradicts the assumption $r<1 / 2$.
3.5. $R_{1 / 2,2}^{2}$ multipliers need not locally $\operatorname{map} L \log ^{r} L$ to $L^{1, q}$ for any $r<1 / q$.

We consider the Hilbert transform $H$, which of course is of the class in Theorem 1.6, and test it against the function $f=2^{N} \chi_{\left[0,2^{-N}\right]}$. Clearly $f$ has a $L \log ^{r} L$ norm of $N^{r}$ but the Hilbert transform of this function has a local $L^{1, q}$ norm of about $N^{1 / q}$, hence the claim.
3.6. $R_{1 / 2, q}^{2}$ multipliers need not be bounded on $L^{p}$ for $\mid 1 / 2$ $-1 / p \mid>1 / q$.

By duality it suffices to show unboundedness when $1 / p-1 / 2>1 / q$. We define the vector-valued multiplier

$$
m_{N}^{\prime \prime \prime}(\xi)=N^{-1 / q} \sum_{j=N / 100}^{N / 10} e_{j} \psi\left(2^{j}\left(\xi-\frac{j}{N}\right)\right)
$$

This multiplier is in the class of Theorem 1.8. We test this against the function

$$
f(x)=\sum_{|k|<2^{N}} \psi(x-N k) .
$$

We expand

$$
\begin{aligned}
& T_{m_{N}^{\prime \prime \prime}} f(x) \\
& =N^{1 / q} \sum_{j=N / 100}^{N / 10} e_{j} \sum_{|k|<2^{N}} \int \psi(x-y-N k) e^{2 \pi i j y / N} 2^{-j} \widehat{\psi}\left(2^{-j} y\right) d y
\end{aligned}
$$

Making the change of variables $y \longrightarrow y-N k$, this becomes

$$
N^{1 / q} \sum_{j=N / 100}^{N / 10} e_{j} \sum_{|k|<2^{N}} \int \psi(x-y) e^{2 \pi i j y / N} 2^{-j} \widehat{\psi}\left(2^{-j}(y+N k)\right) d y
$$

The function $e^{2 \pi i j y / N}$ has real part bounded away from zero, so

$$
\begin{aligned}
& \left|T_{m_{N}^{\prime \prime \prime}} f(x)\right| \\
\sim & N^{-1 / q}\left(\sum_{j=N / 100}^{N / 10}\left(\int \psi(x-y) 2^{-j} \sum_{|k| \leq K} \widehat{\psi}\left(2^{-j}(y+N k)\right) d y\right)^{2}\right)^{1 / 2} .
\end{aligned}
$$

If $|x| \ll 2^{N}$, then $|y| \ll 2^{N}$ and the inner sum is $\sim 2^{j} / N$ (note that $N 2^{N} \gg 2^{j} \gg N$ ). Thus we have

$$
\left|T_{m_{N}^{\prime \prime \prime}} f(x)\right| \sim N^{1 / q}\left(\sum_{j=N / 100}^{N / 10}\left(\int N^{-1} \psi(x-y) d y\right)^{2}\right)^{1 / 2} \sim N^{-1 / q-1 / 2}
$$

for $|x| \ll 2^{N}$. Thus

$$
\left\|T_{m_{N}^{\prime \prime}} f\right\|_{p} \gtrsim N^{-1 / q-1 / 2} 2^{N / p}
$$

On the other hand, an easy computation shows

$$
\|f\|_{p} \sim N^{-1 / p} 2^{N / p}
$$

which demonstrates unboundedness when $1 / p-1 / 2>1 / q$.

## 4. The spaces $H^{1}$ and $L \log ^{1 / 2} L$.

Our positive results involve the spaces $H^{1}$ and $L \log ^{1 / 2} L$. As is well known, $L \log ^{1 / 2} L$ functions are in general not in $H^{1}$ and thus do not have an integrable Littlewood-Paley square function. However, there is a substitute square function for these functions which are indeed integrable, which is why all our results for $H^{1}$ also extend to $L \log ^{1 / 2} L$. More precisely:

Proposition 4.1. Let $f$ be a function which is either in the unit ball $H^{1}(\mathbb{R})$, or in the unit ball of $L \log ^{1 / 2} L([-C, C])$ and with mean zero. Then there exists non-negative functions $F_{j}$ for each integer $j$ such that we have the pointwise estimate

$$
\begin{equation*}
\left|\Delta_{j} f(x)\right| \lesssim F_{j} * \phi_{j}(x) \tag{9}
\end{equation*}
$$

for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}$, and the square function estimate

$$
\begin{equation*}
\left\|\left(\sum_{j}\left|F_{j}\right|^{2}\right)^{1 / 2}\right\|_{1} \lesssim 1 \tag{10}
\end{equation*}
$$

This proposition is easy to prove when $f$ is in $H^{1}$. Indeed, one simply chooses $F_{j}=\left|\widetilde{\Delta}_{j} f\right|$, where $\widetilde{\Delta}_{j}$ is a slight enlargement of $\Delta_{j}$ such that $\Delta_{j}=\Delta_{j} \widetilde{\Delta}_{j}$. The claim (9) follows from pointwise control on the kernel of $\Delta_{j}$, while (10) follows from the square function characterization of $H^{1}$.

The corresponding claim for $L \log ^{1 / 2} L$ is much more delicate. We remark that this claim implies Zygmund's inequality (4). To see this, we first observe that we may assume $f$ satisfies the conditions of the above

Proposition, in which case $\widehat{f}\left(2^{j}\right)$ can be estimated by $\left\|\Delta_{j} f\right\|_{1} \lesssim\left\|F_{j}\right\|_{1}$. The claim then follows from (10) and the Minkowski inequality

$$
\left(\sum_{j}\left\|F_{j}\right\|_{1}^{2}\right)^{1 / 2} \leq\left\|\left(\sum_{j}\left|F_{j}\right|^{2}\right)^{1 / 2}\right\|_{1}
$$

The same argument shows that $L \log L^{1 / 2}$ cannot be replaced by any weaker Orlicz norm. However, the Proposition is substantially stronger than Zygmund's inequality.

As an example of the Proposition, let $f=2^{N} N^{-1 / 2} \psi_{N}$, where $N$ is a large integer and $\psi_{N}$ is a bump function of mean zero adapted to the interval $\left[-2^{-N}, 2^{-N}\right]$. This function is normalized in $L \log ^{1 / 2} L$ and has mean zero, but is not in $L^{1}$. Indeed, if one lets $F_{j}=\left|\widetilde{\Delta}_{j} f\right|$ as before, then for each $1 \ll j \ll N, F_{j}$ is comparable to $2^{j} N^{-1 / 2} \psi_{j}$ on the interval $\left[-2^{-j}, 2^{-j}\right]$, and is rapidly decreasing outside of this interval. From this we see that the left hand side of (10) is too large (about $N^{1 / 2}$ ). The problem here is that the functions $F_{j}$ have very different supports, and so their contributions to (10) add up in $l^{1}$ rather than $l^{2}$. To get around thi s we can redistribute the mass of the $F_{j}$, setting $F_{j}=2^{N} N^{-1 / 2} \chi_{\left[-2^{-N}, 2^{-N}\right]}$ for each $1 \ll j \ll N$; one verifies that (9) is still satisfied, and that (10) is now satisfied because the $F_{j}$ are summing in $l^{2}$ rather than $l^{1}$. (The frequencies $j \leq 1$ or $j \geq N$ can be handled by the original assignment $F_{j}=\left|\Delta_{j} f\right|$ without difficulty).

To handle the general case we shall follow a similar philosophy, namely that each $F_{j}$ shall be a redistribution of $\left|\Delta_{j} f\right|$, whose supports overlap so much that their contributions to (10) are summed in $l^{2}$ rather than $l^{1}$. To do this for general functions $f$ we will use a delicate recursive algorithm. In order to control the error terms in this algorithm we shall be forced to move to the dyadic (Haar wavelet) setting, and also to reduce $f$ to a characteristic function.

The argument is somewhat lengthy, and the methods used are not needed anywhere else in the paper. Because of this, we defer the argument to an Appendix, and proceed to the key estimate in the proofs of Theorems 1.4, 1.6 in the next section.

## 5. Positive results: the main estimate.

In this section we summarize the main estimate we will need to prove in order to achieve the positive results in theorems 1.4 and 1.6.
(The positive results in Theorem 1.2 follow immediately from those in Theorem 1.4).

By interpolation with the trivial $L^{2}$ boundedness results coming from Plancherel's theorem, it suffices to show that the operators in Theorem 1.4 map $H^{1}$ and $L \log ^{1 / 2} L$ to $L^{1, \infty}$, and the operators in Theorem $1.6 \mathrm{map} H^{1}$ and $L \log ^{1 / 2} L$ to $L^{1,2}$.

We will use two key results to obtain these boundedness properties. The first is the square function estimate obtained above in Proposition 4.1. The second is an endpoint multiplier result associated to an arbitrary collection of intervals, which we now state.

Proposition 5.1. Let $N \geq 1$ be an integer, and let $\{I\}$ be a collection of intervals in $\mathbb{R}$ which overlap at most $N$ times in the sense that

$$
\begin{equation*}
\left\|\sum_{I} \chi_{I}\right\|_{\infty} \leq N . \tag{11}
\end{equation*}
$$

For each $I$, we assign a function $f_{I}$, a non-negative function $F_{I}$, and a multiplier $T_{m_{I}}$ with the following properties.

- For each $I, m_{I}$ is supported on $I$, there exists a $\xi_{I} \in I$ such that the symbol $m_{I}\left(\cdot+\xi_{I}\right)$ is a standard symbol of order 0 in the sense of e.g. [11].
- For any $I \in \mathbf{I}$ and $x \in \mathbb{R}$ we have the pointwise estimate

$$
\begin{equation*}
\left|f_{I}(x)\right| \lesssim F_{I}(x) * \phi_{I}(x), \tag{12}
\end{equation*}
$$

where $\phi_{I}$ was defined in (6).
Then we have

$$
\begin{equation*}
\left\|\sum_{I} T_{m_{I}} f_{I}\right\|_{L^{1, \infty}} \lesssim N^{1 / 2}\left\|\left(\sum_{I}\left|F_{I}\right|^{2}\right)^{1 / 2}\right\|_{1} . \tag{13}
\end{equation*}
$$

If we strengthen the condition on $m_{I}$ and assume that the $m_{I}$ are actually bump functions adapted to $I$ uniformly in $I$, then we may strengthen (13) to

$$
\begin{equation*}
\left\|\sum_{I} T_{m_{I}} f_{I}\right\|_{L^{1,2}} \lesssim N^{1 / 2}\left\|\left(\sum_{I}\left|F_{I}\right|^{2}\right)^{1 / 2}\right\|_{1} . \tag{14}
\end{equation*}
$$

We will prove this proposition in sections 6, 7. For now, we see how this proposition and Proposition 4.1 imply the desired mapping properties on $R_{2}$ and $R_{1 / 2,2}^{2}$ multipliers.

Let us first make the preliminary reduction that to prove the $L \log ^{1 / 2} L$ local mapping properties on $T_{m}$ it suffices to prove global estimates on $T_{m} f$ assuming that $f$ is supported in $[0,1]$, is normalized in $L \log ^{1 / 2} L$, and has mean zero. The normalization to $[0,1]$ follows from dilation and translation invariance; the mean zero assumption comes by subtracting off a bump function and observing from the $L^{2}$ theory that $T_{m}$ applied to a bump function is locally in $L^{2}$, hence locally in $L^{1, \infty}$ and $L^{1,2}$.

Our task is now to show that any $f$ satisfying either of the conditions in Proposition 4.1, we have

$$
\begin{equation*}
\left\|T_{m} f\right\|_{L^{1, \infty}} \lesssim 1, \tag{15}
\end{equation*}
$$

for $R_{2}$ multipliers and

$$
\begin{equation*}
\left\|T_{m} f\right\|_{L^{1,2}} \lesssim 1 \tag{16}
\end{equation*}
$$

for $R_{1 / 2,2}^{2}$ multipliers.
Fix $f$, and let $F_{j}$ be as in Proposition 4.1. We first prove (15). We may assume without loss of generality that $m$ is supported in $\bigcup_{j \text { even }}\left[2^{j}, 2^{j+1}\right]$ (The case of odd $j$ is similar and is omitted). By a limiting argument we may assume that only finitely many of the frequency components $m_{j}$ are non-zero for even $j$. By a further limiting argument we may assume that each $m_{j}$ for even $j$ is a rational linear combination of elements in $X$, e.g. $m_{j}=\sum_{i=1}^{N_{j}} \alpha_{j, i} m_{j, i}$ where the $m_{j, i}$ are uniformly in $X$ and the $\alpha_{j, i}$ are non-negative rational numbers. By placing the rational $\alpha_{j, i}$ under a common denominator $N$, and repeating each $m_{j, i}$ with a multiplicity equal to $N \alpha_{j, i}$, we may thus write

$$
m=\frac{1}{N} \sum_{i=1}^{N} m^{(i)}
$$

where the frequency components $m_{j}^{(i)}$ are uniformly in $X$ for even $j$. In particular, this implies that

$$
m=\sum_{I} c_{I} \chi_{I}
$$

where each interval $I$ belongs to $\left[2^{j_{I}}, 2^{j_{I}+1}\right]$ for some even $j_{I}$, the intervals $I$ satisfy (11), and

$$
\begin{equation*}
\sum_{I: j_{I}=j} c_{I}^{2} \lesssim N^{-1} \tag{17}
\end{equation*}
$$

for each $j$. We may assume that $|I| \ll 2^{j_{I}}$ for all $I$. We split $\chi_{I}$ as

$$
\begin{equation*}
\chi_{I}(\xi)=\psi_{I} \psi_{I}^{l} H\left(\xi-\xi_{I}^{l}\right)+\psi_{I} \psi_{I}^{r} H\left(\xi_{I}^{r}-\xi\right), \tag{18}
\end{equation*}
$$

where $H=\chi_{(0, \infty)}$ is the Heaviside function, $\xi_{I}^{l}$ and $\xi_{I}^{r}$ are the left and right endpoints of $I$, and $\psi_{I}^{l}, \psi_{I}^{r}, \psi_{I}$ are bump functions adapted to $\left[\xi_{l}-|I|, \xi_{l}+|I|\right],\left[\xi_{r}-|I|, \xi_{r}+|I|\right]$, and $5 I$ respectively.

We thus need to prove

$$
\left\|\sum_{I} c_{I} T_{\psi_{I}} T_{\psi_{I}^{l} H\left(-\xi_{I}^{l}\right)} f\right\|_{L^{1, \infty}} \lesssim 1,
$$

together with the analogous estimate with the $l$ index replaced by $r$. We show the displayed estimate only, as the other estimate is proven similarly.

Write $m_{I}=\psi_{I}^{l} H\left(\cdot-\xi_{I}^{l}\right), \xi_{I}=\xi_{I}^{l}, f_{I}=c_{I} T_{\psi_{I}} f$, and $F_{I}=$ $\left|c_{I}\right| F_{j_{I}}$. The estimate (12) follows from eqreffj-support, the identity $T_{\psi_{I}}=T_{\psi_{I}} \Delta_{j_{I}}$ and kernel estimates on $T_{\psi_{I}}$. Applying (13) we thus see that

$$
\left\|\sum_{I} c_{I} T_{\psi_{I}} T_{\psi_{I}^{l} H\left(\cdot-\xi_{I}^{l}\right)} f\right\|_{L^{1, \infty}} \lesssim N^{1 / 2}\left\|\left(\sum_{I}\left|F_{I}\right|^{2}\right)^{1 / 2}\right\|_{1} .
$$

The claim then follows from the definition of $F_{I},(17)$, and (10). This proves (15).

The proof of (16) is similar, but with $\chi_{I}$ replaced by a bump function $\widetilde{\psi}_{I}$ adapted to $I$. The only change is that the splitting (18) is replaced by $\widetilde{\psi}_{I}=\psi_{I} \widetilde{\psi}_{I}$, where $\psi_{I}$ is a bump function adapted to $5 I$ which equals 1 on $I$, and that (14) is used instead of (13).

It remains only to prove (13) and (14). This shall be done in the next two sections.

## 6. Proof of (13).

Fix $I, N, f_{I}, F_{I}, m_{I}$; we may assume by limiting arguments that the collection of $I$ is finite. From (12) we can find bounded functions $a_{I}$ for each $I \in \mathbf{I}$ such that

$$
f_{I}=a_{I}\left(F_{I} * \phi_{I}\right)
$$

Our task is then to show that

$$
\left|\left\{\left|\sum_{I} T_{m_{I}}\left(a_{I}\left(F_{I} * \phi_{I}\right)\right)\right| \gtrsim \alpha\right\}\right| \lesssim \alpha^{-1} N^{1 / 2}\|F\|_{1},
$$

where $F$ denotes the vector $F=\left(F_{I}\right)_{I \in\{I\}}$.
We now perform a standard vector-valued Calderón-Zygmund decomposition on $F$ at height $N^{-1 / 2} \alpha$ as

$$
F=g+\sum_{J} b_{J},
$$

where $g=\left(g_{I}\right)_{I \in \mathbf{I}}$ satisfies the $L^{2}$ estimate

$$
\begin{equation*}
\|g\|_{2}^{2} \lesssim N^{-1 / 2} \alpha\|F\|_{1} \tag{19}
\end{equation*}
$$

while the bad functions $b_{J}$ are supported on $J$, satisfy the moment condition $\int_{J} b_{J}=0$, and the $L^{1}$ estimate

$$
\left\|b_{J}\right\|_{1} \lesssim N^{-1 / 2} \alpha|J| .
$$

Finally, the intervals $J$ satisfy

$$
\sum_{J}|J| \lesssim \alpha^{-1} N^{1 / 2}\|F\|_{1}
$$

Consider the contribution of the good function $g$. By Chebyshev, it suffices to prove the $L^{2}$ estimate

$$
\begin{equation*}
\left\|\sum_{I} T_{m_{I}}\left(a_{I}\left(g_{I} * \phi_{I}\right)\right)\right\|_{2}^{2} \lesssim \alpha N^{1 / 2}\left\|\left(\sum_{I}\left|F_{I}\right|^{2}\right)^{1 / 2}\right\|_{1} . \tag{20}
\end{equation*}
$$

From Plancherel, the overlap condition on the $I$, and Cauchy-Schwarz, we have the basic inequality

$$
\begin{equation*}
\left\|\sum_{I} T_{m_{I}} h_{I}\right\|_{2}^{2} \leq N \sum_{I}\left\|T_{m_{I}} h_{I}\right\|_{2}^{2}, \tag{21}
\end{equation*}
$$

for any $h_{I}$. We may thus estimate the left-hand side of (20) by

$$
\begin{aligned}
N \sum_{I}\left\|T_{m_{I}}\left(a_{I}\left(g_{I} * \phi_{I}\right)\right)\right\|_{2}^{2} & \lesssim N \sum_{I}\left\|a_{I}\left(g_{I} * \phi_{I}\right)\right\|_{2}^{2} \\
& \lesssim N \sum_{I}\left\|g_{I} * \phi_{I}\right\|_{2}^{2} \\
& \lesssim N \sum_{I}\left\|g_{I}\right\|_{2}^{2} \\
& \lesssim N N^{-1 / 2} \alpha\left\|\left(\sum_{I}\left|F_{I}\right|^{2}\right)^{1 / 2}\right\|_{1}
\end{aligned}
$$

as desired.
It remains to deal with the bad functions $b_{J}$. It suffices to show that

$$
\left|\left\{\left|\sum_{I} \sum_{J} T_{m_{I}}\left(a_{I}\left(b_{J, I} * \phi_{I}\right)\right)\right| \gtrsim \alpha\right\}\right| \lesssim \sum_{J}|J| .
$$

From uncertainty principle heuristics we expect the contribution of the case $|I||J| \leq 1$ to be easy. Indeed, this case can be treated almost exactly like the good function $g$. As before, it suffices to show the $L^{2}$ estimate

$$
\left\|\sum_{I, J:|I||J| \leq 1} T_{m_{I}}\left(a_{I}\left(b_{J, I} * \phi_{I}\right)\right)\right\|_{2}^{2} \lesssim \alpha^{2} \sum_{J}|J| .
$$

By repeating the previous calculation, the left-hand side is majorized by

$$
N \sum_{I}\left\|\sum_{J:|I||J| \leq 1} b_{J, I} * \phi_{I}\right\|_{2}^{2} .
$$

From the triangle inequality, it thus suffices to show that

$$
\sum_{I}\left\|\sum_{J:|I||J|=2^{-m}} b_{J, I} * \phi_{I}\right\|_{2}^{2} \leq 2^{-2 m} N^{-1} \alpha^{2} \sum_{J}|J|,
$$

for all $m \geq 0$. This in turn follows if we can show

$$
\begin{equation*}
\sum_{I:|I|=2^{-m-j}}\left\|\sum_{J:|J|=2^{j}} b_{J, I} * \phi_{I}\right\|_{2}^{2} \leq 2^{-2 m} N^{-1} \alpha^{2} \sum_{J:|J|=2^{j}}|J|, \tag{22}
\end{equation*}
$$

for all $m \geq 0$ and $j \in \mathbb{Z}$.
Fix $m, j$, and observe from (5) that $\phi_{I}=\phi_{-m-j}$. By moving the $I$ summation inside the norm, we can estimate the left-hand side of (22) by

$$
\left\|\sum_{J:|J|=2^{j}} b_{J} * \phi_{-m-j}\right\|_{2}^{2},
$$

where $*$ is now a vector-valued convolution. From the normalization and moment condition on $b_{J}$ we have

$$
b_{J} * \phi_{-m-j} \lesssim N^{-1 / 2} \alpha \chi_{J} * \phi_{-m-j}
$$

Inserting this into the previous, the claim then follows from Young's inequality and the $L^{1}$ normalization of the $\phi_{-m-j}$.

It remains to treat the case $|I||J|>1$. We split

$$
b_{J, I} * \phi_{I}=\chi_{2 J}\left(b_{J, I} * \phi_{I}\right)+\left(1-\chi_{2 J}\right)\left(b_{J, I} * \phi_{I}\right) .
$$

The contribution of the latter terms can be dealt with in a manner similar to that of the $|I||J| \leq 1$ case. As before, it suffices to show the $L^{2}$ estimate

$$
\left\|\sum_{I, J:|I||J|>1} T_{m_{I}}\left(a_{I}\left(1-\chi_{2 J}\right)\left(b_{J, I} * \phi_{I}\right)\right)\right\|_{2}^{2} \lesssim \alpha^{2} \sum_{J}|J| .
$$

As before, the left-hand side is majorized by

$$
\begin{equation*}
N \sum_{I}\left\|\sum_{J:|I||J|>1}\left(1-\chi_{2 J}\right)\left(b_{J, I} * \phi_{I}\right)\right\|_{2}^{2} . \tag{23}
\end{equation*}
$$

A computation shows the pointwise estimate

$$
\left|\left(1-\chi_{2 J}\right)\left(b_{J, I} * \phi_{I}\right)\right| \lesssim\left\|b_{J, I}\right\|_{1}|J|^{-1}\left(M \chi_{J}\right)^{3 / 2}
$$

(In fact there is an additional decay if $|I||J|$ is large, but we shall not exploit this). Inserting this estimate into (23) and moving the $I$ summation back inside, we can majorize (23) by

$$
N\left\|\left(\left.\left.\sum_{I}\left|\sum_{J}\left\|b_{J, I}\right\|_{1}\right| J\right|^{-1}\left(M \chi_{J}\right)^{3 / 2}\right|^{2}\right)^{1 / 2}\right\|_{2}^{2}
$$

Using the triangle inequality for $l^{2}$ we may move the $I$ square-summation inside the $J$ summation. If one then applies Minkowski's inequality

$$
\begin{equation*}
\left(\sum_{I}\left\|b_{J, I}\right\|_{1}^{2}\right)^{1 / 2} \leq\left\|b_{J}\right\|_{1} \lesssim N^{-1 / 2} \alpha|J| \tag{24}
\end{equation*}
$$

we can thus majorize (23) by

$$
\alpha^{2}\left\|\sum_{J}\left(M \chi_{J}\right)^{3 / 2}\right\|_{2}^{2}
$$

The claim then follows from the Fefferman-Stein vector-valued maximal inequality [4].

It remains to show that

$$
\begin{equation*}
\left|\left\{\left|\sum_{I, J:|I||J|>1} T_{m_{I}} B_{J, I}\right| \gtrsim \alpha\right\}\right| \lesssim \sum_{J}|J|, \tag{25}
\end{equation*}
$$

where

$$
B_{J, I}=a_{I} \chi_{2 J}\left(b_{J, I} * \phi_{I}\right)
$$

For future reference we note from (24) that the $B_{J, I}$ are supported on $2 J$ and satisfy

$$
\begin{equation*}
\sum_{I}\left\|B_{J, I}\right\|_{1}^{2} \lesssim N^{-1} \alpha^{2}|J|^{2} \tag{26}
\end{equation*}
$$

for all $J$.
For each $I, J$ in (25), let $P_{J, I}$ be a multiplier whose symbol is a bump function which equals 1 on the interval $\left[\xi_{I}-|J|^{-1}, \xi_{I}+|J|^{-1}\right]$, and is adapted to a dilate of this interval. We split

$$
T_{I}=T_{I} P_{J, I}+Q_{J, I},
$$

where $Q_{J, I}=T_{I}\left(1-P_{J, I}\right)$. The point is that even though the kernel of $T_{I}$ decays very slowly, the operators $P_{J, I}$ and $Q_{J, I}$ have kernels which are essentially supported on an interval of width $|J|$.

We first consider the contribution of the $T_{I} P_{J, I}$. It suffices as before to prove an $L^{2}$ estimate

$$
\begin{equation*}
\left\|\sum_{I, J:|I||J|>1} T_{m_{I}} P_{J, I} B_{J, I}\right\|_{2}^{2} \lesssim \alpha^{2} \sum_{J}|J| . \tag{27}
\end{equation*}
$$

By (21) again, the left-hand side of (27) is majorized by

$$
N \sum_{I}\left\|\sum_{J:|I||J|>1} P_{J, I} B_{J, I}\right\|_{2}^{2} .
$$

From kernel estimates on $P_{I, J}$ we have the pointwise estimates

$$
\left|P_{J, I} B_{J, I}\right| \lesssim\left\|B_{J, I}\right\|_{1}|J|^{-1}\left(M \chi_{J}\right)^{3 / 2} .
$$

The contribution of the $T_{I} P_{J, I}$ is thus acceptable by repeating the arguments used to treat (23), and using (26) instead of (24).

It remains to consider the contribution of the $Q_{J, I}$. For this final contribution we will not use $L^{2}$ estimates, but the more standard $L^{1}$ estimates outside an exceptional set

$$
\left\|\sum_{I, J:|I||J|>1} Q_{J, I} B_{J, I}\right\|_{L^{1}\left(\left(\cup_{J} C J\right)^{c}\right)} \lesssim \alpha \sum_{J}|J| .
$$

By the triangle inequality it suffices to prove this for each $J$ separately

$$
\left\|\sum_{I:|I||J|>1} Q_{J, I} B_{J, I}\right\|_{L^{1}\left((C J)^{c}\right)} \lesssim \alpha|J| .
$$

By translation and scale invariance we may set $J=[0,1]$. Let $\varphi$ denote a bump function which equals 1 on $[-1,1]$ and is adapted to $[-2,2]$. Let $r_{I}$ denote the symbol

$$
r_{I}=q_{J, I}-q_{J, I} * \varphi,
$$

where $q_{J, I}$ is the symbol of $Q_{J, I}$. Observe that $Q_{J, I} B_{J, I}=T_{r_{I}} B_{J, I}$ outside of $C J$. Thus it suffices to show that

$$
\left\|\sum_{I:|I|>1} T_{r_{I}} B_{J, I}\right\|_{L^{1}\left((C J)^{c}\right)} \lesssim \alpha
$$

By Hölder's inequality it suffices to show the global weighted $L^{2}$ estimate

$$
\left\|x \sum_{I:|I|>1} T_{r_{I}} B_{J, I}(x)\right\|_{2} \lesssim \alpha
$$

By Plancherel, this becomes

$$
\left\|\sum_{I:|I|>1}\left(r_{I} \widehat{B_{J, I}}\right)^{\prime}\right\|_{2} \lesssim \alpha
$$

where the prime denotes differentiation.
The function $\widehat{B_{J, I}}$ is very smooth, in fact it satisfies the estimates

$$
\left\|\widehat{B_{J, I}}\right\|_{C^{1}} \lesssim\left\|B_{J, I}\right\|_{1}
$$

for all $I$. A computation using the construction of $Q_{J, I}$ and $r_{I}$ shows that the symbol $r_{I}$ satisfies the estimates

$$
\left|r_{I}(\xi)\right|,\left|r_{I}^{\prime}(\xi)\right| \lesssim\left(1+\left|\xi-\xi_{I}\right|\right)^{-10}
$$

Combining these two estimates we see the pointwise estimate

$$
\left|\left(r_{I} \widehat{B_{J, I}}\right)^{\prime}\right| \lesssim\left\|B_{J, I}\right\|_{1}\left(M \chi_{\left[\xi_{I}-1, \xi_{I}+1\right]}\right)^{2}
$$

From the Fefferman-Stein vector-valued maximal inequality [4] it thus suffices to show that

$$
\left\|\sum_{I:|I|>1}\right\| B_{J, I}\left\|_{1} \chi_{\left[\xi_{I}-1, \xi_{I}+1\right]}\right\|_{2} \lesssim \alpha
$$

However from (11) and the hypothesis $|I|>1$ we see that the characteristic functions $\chi_{\left[\xi_{I}-1, \xi_{I}+1\right]}$ overlap at most $O(N)$ times at any given point. The claim then follows from Cauchy-Schwarz and (26). This completes the proof of (13).

We remark that the one can modify this argument so that one does not need the full power of Proposition 4.1 in the $L \log ^{1 / 2} L$ case, using a rescaled version of Zygmund's estimate (4) (for arbitrary lacunary frequencies, not just the powers of 2 ) as a substitute; we omit the details. On the other hand, the ( $L \log ^{1 / 2} L, L^{1,2}$ ) result in Proposition 1.6 seems to require the full strength of Proposition 4.1.

## 7. Proof of (14).

We now prove (14). As before we fix $I, N, m_{I}, f_{I}, F_{I}$, and assume that the collection of $I$ is finite. We may also assume that the functions $F_{I}$ are smooth.

To prove (14) it suffices to prove the stronger estimate

$$
\begin{equation*}
\left\|\sum_{I} T_{m_{I}} f_{I}\right\|_{L^{1,2}} \lesssim N^{1 / 2}\left\|\left(\sum_{I}\left|F_{I} * \phi_{I}\right|^{2}\right)^{1 / 2}\right\|_{L^{1,2}} . \tag{28}
\end{equation*}
$$

This is because of the following lemma, which illustrates the natural role of the Lorentz space $L^{1,2}$.

Lemma 7.1. Let $I$ be an arbitrary collection of intervals, and $F_{I}$ an arbitrary collection of non-negative functions. Then

$$
\left\|\left(\sum_{I}\left|F_{I} * \phi_{I}\right|^{2}\right)^{1 / 2}\right\|_{L^{1,2}} \lesssim\left\|\left(\sum_{I}\left|F_{I}\right|^{2}\right)^{1 / 2}\right\|_{1}
$$

Proof. The desired estimate is the $p=2$ case of the more general estimate

$$
\left\|\left(\sum_{I}\left|F_{I} * \phi_{I}\right|^{p}\right)^{1 / p}\right\|_{L^{1, p}} \lesssim\left\|\left(\sum_{I}\left|F_{I}\right|^{p}\right)^{1 / p}\right\|_{1} .
$$

This estimate is trivial for $p=1$ by Young's inequality and the integrability of the $\phi_{I}$. For $p=\infty$ the claim follows from the Hardy-Littlewood maximal inequality and the pointwise estimates

$$
\left|F_{I} * \phi_{I}(x)\right| \lesssim M F_{I}(x) \lesssim M\left(\sup _{I} F_{I}\right)(x) .
$$

The complex interpolation theorem of Sagher [7] for Lorentz spaces then allows one to obtain the $p=2$ estimate. Alternatively, one can interpolate manually by writing $F_{I}=|F| a_{I}$, where $|F|=\left(\sum_{I}\left|F_{I}\right|^{2}\right)^{1 / 2}$, and exploiting the Cauchy-Schwarz inequality

$$
\left|F_{I} * \phi_{I}(x)\right|^{2} \leq\left(\left(F a_{j}^{2}\right) * \phi_{j}(x)\right)\left(|F| * \phi_{j}(x)\right) \lesssim|F| a_{I}^{2} * \phi_{I}(x) M|F|(x)
$$

and the Hölder inequality for Lorentz spaces [6]

$$
\left\|(f g)^{1 / 2}\right\|_{L^{1,2}} \lesssim\|f\|_{1}^{1 / 2}\|g\|_{L^{1, \infty}}^{1 / 2} .
$$

We omit the details.

It remains to prove (28). Let $G$ denote the square function

$$
G=\left(\sum_{I}\left|F_{I} * \phi_{I}\right|^{2}\right)^{1 / 2}
$$

Note that $G$ is continuous from our a priori assumptions. It would be nice if the distributional estimate

$$
\left|\left\{\left|\sum_{I} T_{m_{I}} f_{I}\right| \sim 2^{j}\right\}\right| \lesssim\left|\left\{G \sim N^{-1 / 2} 2^{j}\right\}\right|
$$

held for all $j$, as this easily implies (28). While this is not quite true, we are able to prove the substitute

$$
\begin{equation*}
\left|\left\{\left|\sum_{I} T_{m_{I}} f_{I}\right| \gtrsim 2^{j}\right\}\right| \lesssim 2^{-2 j} N\left\|\min \left\{G, N^{-1 / 2} 2^{j}\right\}\right\|_{2}^{2} \tag{29}
\end{equation*}
$$

for all $j$. Indeed, if (29) held, then we have

$$
\begin{aligned}
2^{j} \mid\left\{\left|\sum_{I} T_{m_{I}} f_{I}\right|\right. & \left.\sim 2^{j}\right\} \mid \\
& \lesssim N^{1 / 2} \sum_{s} 2^{-|s|} N^{-1 / 2} 2^{j+s}\left|\left\{G \sim N^{-1 / 2} 2^{j+s}\right\}\right|
\end{aligned}
$$

the claim then follows by square-summing this in $j$, using the estimate

$$
\|F\|_{L^{1,2}} \sim\left(\sum_{j}\left(2^{j}\left|\left\{F \sim 2^{j}\right\}\right|\right)^{2}\right)^{1 / 2}
$$

and using Young's inequality.
It remains to prove (29). Fix $j$, and consider the set $\Omega=\{G>$ $\left.N^{-1 / 2} 2^{j}\right\}$. Since $G$ is continuous, $\Omega$ is an open set, and we may decompose it into intervals $\Omega=\bigcup_{J} J$ such that $G(x)=N^{-1 / 2} 2^{j}$ on the endpoints of $J$. Note that

$$
\begin{equation*}
\sum_{J}|J|=|\Omega| \leq 2^{-2 j} N\left\|\min \left\{G, N^{-1 / 2} 2^{j}\right\}\right\|_{2} \tag{30}
\end{equation*}
$$

We can therefore split

$$
\begin{align*}
\sum_{I} T_{m_{I}} f_{I}= & \sum_{I} T_{m_{I}}\left(f_{I} \chi_{\Omega^{c}}\right)+\sum_{I, J:|I||J| \leq 1} T_{m_{I}}\left(f_{I} \chi_{J}\right)  \tag{31}\\
& +\sum_{I, J:|I||J|>1} T_{m_{I}}\left(f_{I} \chi_{J}\right)
\end{align*}
$$

To treat the contribution of the first term in (31) we use $L^{2}$ estimates. By Chebyshev it suffices to show that

$$
\left\|\sum_{I} T_{m_{I}}\left(f_{I} \chi_{\Omega^{c}}\right)\right\|_{2}^{2} \lesssim N\left\|\min \left\{G, N^{-1 / 2} 2^{j}\right\}\right\|_{2}^{2} .
$$

However, by (21) the left-hand side is majorized by

$$
\begin{aligned}
N \sum_{I}\left\|f_{I} \chi_{\Omega^{c}}\right\|_{2}^{2} & =N\left\|\left(\sum_{I}\left|f_{I}\right|^{2}\right)^{1 / 2} \chi_{\Omega^{c}}\right\|_{2}^{2} \\
& \lesssim N\left\|\left(\sum_{I}\left|F_{I} * \phi_{I}\right|^{2}\right)^{1 / 2} \chi_{\Omega^{c}}\right\|_{2}^{2} \\
& \leq N\left\|\min \left\{G, N^{1 / 2} 2^{j}\right\}\right\|_{2}^{2},
\end{aligned}
$$

as desired.
To treat the second term in (31) we also use $L^{2}$ estimates. As before, it suffices to show

$$
\begin{equation*}
\left\|\sum_{I} T_{m_{I}}\left(\sum_{J:|I||J| \leq 1} f_{I} \chi_{J}\right)\right\|_{2}^{2} \lesssim N\left\|\min \left\{G, N^{-1 / 2} 2^{j}\right\}\right\|_{2}^{2} . \tag{32}
\end{equation*}
$$

Using (21) as before, we can majorize the left-hand side of (32) by

$$
N \sum_{I}\left\|\sum_{J:|I||J| \leq 1}\left(F_{I} * \phi_{I}\right) \chi_{J}\right\|_{2}^{2} .
$$

Since the $J$ are all disjoint, we may re-arrange this as

$$
N \sum_{J} \sum_{I:|I||J| \leq 1}\left\|F_{I} * \phi_{I}\right\|_{L^{2}(J)}^{2}
$$

For each $J$ let $x_{J}^{r}$ be the right endpoint of $J$, so that $G\left(x_{J}^{r}\right) \leq N^{-1 / 2} 2^{j}$. Now we exploit the assumption $|I||J| \leq 1$ to observe that

$$
\left|F_{I} * \phi_{I}(x)\right| \lesssim\left|F_{I} * \phi_{I}\left(x_{J}^{r}\right)\right|,
$$

for all $x \in J$. Applying this to the previous, we can thus majorize (32) by

$$
N \sum_{J}|J| \sum_{I}\left|F_{I} * \phi_{I}\left(x_{J}^{r}\right)\right|^{2}=N \sum_{J}|J| G\left(x_{J}^{r}\right)^{2} \leq 2^{2 j} \sum_{J}|J| .
$$

The claim then follows from (30).
It remains to treat the third term in (31). By Chebyshev and (30) it suffices to prove an $L^{1}$ estimate outside the exceptional set $\bigcup_{J} C J$

$$
\left\|\sum_{I, J:|I||J|>1} T_{m_{I}}\left(f_{I} \chi_{J}\right)\right\|_{L^{1}\left(\left(\cup_{J} C J\right)^{c}\right)} \lesssim 2^{j} \sum_{J}|J| .
$$

By the triangle inequality it suffices to prove this for each $J$ separately

$$
\left\|\sum_{I:|I||J|>1} T_{m_{I}}\left(f_{I} \chi_{J}\right)\right\|_{L^{1}\left(C J^{c}\right)} \lesssim 2^{j}|J| .
$$

We now adapt the arguments in the previous section. By dilation and translation invariance we may set $J=[0,1]$. Define $\varphi$ as before, and let $r_{I}$ be the multipliers

$$
r_{I}=m_{I}-m_{I} * \varphi .
$$

Then we have $T_{m_{I}}\left(f_{I} \chi_{J}\right)=T_{r_{I}}\left(f_{I} \chi_{J}\right)$ on $(C J)^{c}$, and it suffices to show that

$$
\left\|\sum_{I:|I|>1} T_{r_{I}}\left(f_{I} \chi_{J}\right)\right\|_{L^{1}\left(C J^{c}\right)} \lesssim 2^{j}
$$

By Hölder as before, it suffices to show the global weighted $L^{2}$ estimate

$$
\left\|x \sum_{I:|I|>1} T_{r_{I}}\left(f_{I} \chi_{J}\right)(x)\right\|_{2} \lesssim 2^{j} .
$$

By Plancherel, this becomes

$$
\begin{equation*}
\left\|\sum_{I:|I|>1}\left(r_{I} \widehat{f_{I} \chi_{J}}\right)^{\prime}\right\|_{2} \lesssim 2^{j} . \tag{33}
\end{equation*}
$$

The multipliers $r_{I}$ can be estimated as

$$
\left|r_{I}(\xi)\right|,\left|r_{I}^{\prime}(\xi)\right| \lesssim|I|^{10}\left(M \chi_{\left[\xi_{I}-1, \xi_{I}+1\right]}\right)^{10}
$$

The functions $\widehat{f_{I} \chi_{J}}$ can similarly be estimated as

$$
\left\|\widehat{f_{I} \chi_{J}}\right\|_{C^{1}} \lesssim\left\|f_{I} \chi_{J}\right\|_{1} \lesssim\left\|F_{I} * \phi_{I}\right\|_{L^{1}([0,1])}
$$

From the positivity of $F_{I}$ we have

$$
F_{I} * \phi_{I}(x) \lesssim|I|^{-10} F_{I} * \phi_{I}(0)
$$

and so we thus have

$$
\left\|\widehat{f_{I} \chi_{J}}\right\|_{C^{1}} \lesssim|I|^{-10}\left(F_{I} * \phi_{I}\right)(0)
$$

We can thus majorize the left-hand side of (33) by

$$
\left\|\sum_{I:|I|>1}\left(F_{I} * \phi_{I}\right)(0)\left(M \chi_{\left[\xi_{I}-1, \xi_{I}+1\right]}\right)^{10}\right\|_{2}
$$

By the Fefferman-Stein vector-valued maximal inequality [4], (11), and Cauchy-Schwarz as in the previous section, this is majorized by

$$
N^{1 / 2}\left(\sum_{I}\left(F_{I} * \phi_{I}\right)(0)^{2}\right)^{1 / 2}=N^{1 / 2} G(0)=2^{j}
$$

as desired. This completes the proof of (29) and hence (14).

## 8. Remarks on multipliers of bounded $s$-variation.

Let $1 \leq s<\infty$. For any function $f$ supported on an interval $[a, b]$, we define the $s$-variation of $f$ to be the supremum of the quantity

$$
\left(\sum_{i=0}^{N}\left|f\left(a_{i+1}\right)-f\left(a_{i}\right)\right|^{s}\right)^{1 / s}
$$

where $a=a_{0}<a_{1}<\cdots<a_{N}=b$ ranges over all partitions of $[a, b]$ of arbitrary length. We say that a multiplier $T_{m}$ is a $V_{s}$ multiplier if the frequency component $m_{j}$ have bounded $s$-variation uniformly in $j$.

Clearly the Marcinkiewicz class is the same as the $V_{1}$ class, but for $s>1$ the $V_{s}$ class contains multipliers not covered by the Marcinkiewicz multiplier theorem.

In [2] it was shown that the $V_{s}$ class was contained in the $R_{2}$ class for $s<2$. In particular, they showed that $V_{s}$ multipliers were bounded on $L^{p}$ for $1<p<\infty$ and $s<2$. From Theorem 1.2 and Theorem 1.4 we have the sharp endpoint version of this result when $s<2$

Corollary 8.1. Let $1 \leq s<2$. Then the statements of Theorem 1.2 (both positive and negative) continue to hold when the Marcinkiewicz class is replaced by the $V_{s}$ class.

Now consider the case $s>2$. By complex interpolation it was shown in [2] (see also earlier work in [5]) that $V_{s}$ multipliers were bounded in $L^{p}$ when

$$
\left|\frac{1}{2}-\frac{1}{p}\right|<\frac{1}{s}
$$

From the study [5] of the multipliers (3) it is known that this restriction on $p$ is sharp up to endpoints. However, the endpoint problem remains unresolved. The most interesting case is when $s=2$. From the counterexamples in Section 3 we see that negative results in Theorem 1.2 hold for $V_{2}$ multipliers, and so one may conjecture that these multipliers also map both $H^{1}$ and $L \log ^{1 / 2} L$ locally to $L^{1, \infty}$. If this were true, th en for $s>2$ the $V_{s}$ multiplier class would map $L^{p}$ to $L^{p, p^{\prime}}$ when $1 / p=1 / s+1 / 2$ by complex interpolation ( $c f$. [3]). However, we have been unable to prove these estimates using the techniques in this paper. A natural model case would be when the frequency components $m_{j}$ not only have bounded 2-variation, but have the stronger property of Hölder continuity of order $1 / 2$ uniformly in $j$. (In [2] it was shown that a general function of bounded 2 -variation can be transformed into a Hölder continuous function of order $1 / 2$ by a change of variables).

In [2] $V_{2}$ multipliers were shown to be bounded on $L^{p}$ for all $1<$ $p<\infty$. By going through their argument carefully one can show that the $L^{p}$ operator norm grows like $O\left(1 /(p-1)^{C}\right)$ for some constant $C$ as $p \longrightarrow 0$, so by extrapolation they map $L \log ^{C} L$ to $L^{1}$ locally for some sufficiently large $C$. However these results are far from best possible.

## 9. Appendix: proof of Proposition 4.1.

We now prove Proposition 4.1 when $f$ is in $L \log ^{1 / 2} L([-C, C])$ and has mean zero.

It will be convenient to move to the dyadic setting ${ }^{2}$ as we will need to perform a delicate induction shortly. Accordingly, we introduce the Haar wavelet system

$$
\psi_{I}=|I|^{-1 / 2}\left(\chi_{I_{l}}-\chi_{I_{r}}\right)
$$

defined for all dyadic intervals $I$ in $[0,1]$, where $I_{l}, I_{r}$ are the left and right halves of $I$ respectively.

The dyadic analogue of Proposition 4.1 is
Proposition 9.1. Let $f$ be a function on $[0,1]$ such that

$$
\int|f| \log ^{1 / 2}(2+|f|) \lesssim 1
$$

Then for each integer $j \geq 0$ we may find a non-negative function $f_{j}$ supported on $[0,1]$ such that

$$
\begin{equation*}
\left|\left\langle f, \psi_{I}\right\rangle\right| \leq|I|^{-1 / 2} \int_{I} f_{j} \tag{34}
\end{equation*}
$$

for all $j \geq 0$ and dyadic intervals $I \subset[0,1]$ of length $2^{-j}$, and that

$$
\begin{equation*}
\left\|\left(\sum_{j \geq 0}\left|f_{j}\right|^{2}\right)^{1 / 2}\right\|_{1} \lesssim 1 \tag{35}
\end{equation*}
$$

We now show that Proposition 9.1 implies Proposition 4.1. The idea is to use an averaging over translations to smooth out the dyadic singularities of the Haar wavelet system.

Let $f$ be as in Proposition 4.1; we may assume that $f$ is supported on the interval $[1 / 3,2 / 3]$. For negative $j$, we define $F_{j}=\left|\widetilde{\Delta}_{j} f\right|$ as in the $H^{1}$ theory, so that (9) holds as before. From the mean zero condition of $f$ we see that $\left\|F_{j}\right\|_{1} \lesssim 2^{j}$, so the contribution of these $j$ to (10) is acceptable.

[^7]For all $-1 / 3 \leq \theta \leq 1 / 3$, let $f^{\theta}$ denote the translated function $f^{\theta}(x)=f(x-\theta)$. These functions all satisfy the requirements of Proposition 9.1, with the associated functions $f_{j}^{\theta}$. We now define $F_{j}$ for $j \geq 0$ by

$$
F_{j}(x)=\sum_{k \geq 0} 2^{-|j-k| / 2} \int_{-1 / 3}^{1 / 3} f_{k}^{\theta}(x+\theta) d \theta
$$

We now verify (9). Fix $x \in[0,1]$ and $j \geq 0$. We say that a number $-1 / 3 \leq \theta \leq 1 / 3$ is normal with respect to $x$ and $j$ if

$$
\operatorname{dist}\left(x+\theta, 2^{-k} \mathbb{Z}\right) \geq \frac{1}{100} 2^{-|j-k| / 10} 2^{-k}
$$

for all integers $0 \leq k \leq j$.
Let $\Theta_{x, j}$ denote the set of all normal $\theta$; it is easy to see that $\left|\Theta_{x, j}\right| \sim 1$. Let $\theta$ be any element of $\Theta_{x, j}$. We compute

$$
\begin{aligned}
\left|\Delta_{j} f(x)\right| & =\left|\Delta_{j} f^{\theta}(x+\theta)\right| \\
& =\left|\sum_{I}\left\langle f^{\theta}, \psi_{I}\right\rangle \Delta_{j} \psi_{I}(x+\theta)\right| \\
& \leq \sum_{k} \sum_{I:|I|=2^{-k}}\left(\int_{I} f_{k}^{\theta}\right)|I|^{-1 / 2}\left|\Delta_{j} \psi_{I}(x+\theta)\right| .
\end{aligned}
$$

If $k \geq j$, then a computation shows that

$$
\begin{aligned}
|I|^{-1 / 2}\left|\Delta_{j} \psi_{I}(x+\theta)\right| & \lesssim 2^{2 j-k}\left(1+2^{k} \operatorname{dist}(x+\theta, I)\right)^{-100} \\
& \lesssim 2^{-|k-j| / 2} 2^{j}\left(1+2^{j} \operatorname{dist}(x+\theta, I)\right)^{-3 / 2}
\end{aligned}
$$

and thus that

$$
\sum_{I:|I|=2^{-k}}\left(\int_{I} f_{j}^{\theta}\right)|I|^{-1 / 2}\left|\Delta_{j} \psi_{I}(x+\theta)\right| \lesssim 2^{-|k-j| / 2} f_{k}^{\theta} * \phi_{j} .
$$

Now suppose that $k<j$. A computation using the normality of $\theta$ shows that

$$
|I|^{-1 / 2}\left|\Delta_{j} \psi_{I}(x+\theta)\right| \lesssim 2^{-100|k-j|} 2^{j}\left(1+2^{j} \operatorname{dist}(x+\theta, I)\right)^{-100}
$$

and hence that

$$
\sum_{I:|I|=2^{-k}}\left(\int_{I} f_{j}^{\theta}\right)|I|^{-1 / 2}\left|\Delta_{j} \psi_{I}(x+\theta)\right| \lesssim 2^{-|k-j| / 2} f_{k}^{\theta} * \phi_{j}
$$

Combining these estimates and then averaging over $\Theta_{x, j}$ we obtain (9) as desired.

Now we show (10) for the non-negative $j$. From Young's inequality and Minkowski's inequality we see the pointwise estimate

$$
\begin{aligned}
\left(\sum_{j}\left|F_{j}(x)\right|^{2}\right)^{1 / 2} & \lesssim\left(\sum_{k}\left|\int_{-1 / 3}^{1 / 3} f_{k}^{\theta}(x+\theta)^{2} d \theta\right|^{2}\right)^{1 / 2} \\
& \leq \int_{-1 / 3}^{1 / 3}\left(\sum_{k} f_{k}^{\theta}(x+\theta)^{2}\right)^{1 / 2} d \theta
\end{aligned}
$$

The claim then follows from Fubini's theorem and (35).
It remains to prove Proposition 9.1. To do this, we first reduce to the case when $f$ is a characteristic function. More precisely, we shall show

Proposition 9.2. Let $N \geq 0$ be an integer, $I_{0}$ be a dyadic interval, and let $\mathbf{I}_{0}$ be the collection of all dyadic intervals in $I_{0}$ of side-length at least $2^{-N}\left|I_{0}\right|$. Let $E$ be the union of some intervals in $\mathbf{I}$. Then for each dyadic interval $I \subseteq I_{0}$ of length at least $2^{-N}\left|I_{0}\right|$, we may find a non-negative function $f_{I}$ supported on $I$ such that

$$
\begin{equation*}
\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right| \leq|I|^{-1 / 2}\left\|f_{I}\right\|_{1}, \tag{36}
\end{equation*}
$$

for all such $I$, and that ${ }^{3}$

$$
\begin{equation*}
\left\|\left(\sum_{I \in \mathbf{I}_{0}}\left|f_{I}\right|^{2}\right)^{1 / 2}\right\|_{1} \leq A|E| \log \left(2+\frac{\left|I_{0}\right|}{|E|}\right)^{1 / 2} \tag{37}
\end{equation*}
$$

for some absolute constant $A$.
Indeed, by setting $I_{0}=[0,1]$ and $N \longrightarrow \infty$, we see that Proposition 9.2 immediately implies Proposition 9.1 for the $L \log ^{1 / 2} L$-normalized functions $|E|^{-1} \log (1 /|E|)^{-1 / 2} \chi_{E}$ for any set $E$ with measure $0<|E| \ll$ 1. A general $L \log ^{1 / 2} L$ function can be written as a convex linear combination of such functions (see e.g. [12]), so the general case of Proposition 9.1 obtains (observing that the $L^{1}\left(l^{2}\right)$ space appearing in (35) is a Banach space).

[^8]It remains to prove Proposition 9.2. This shall be done by induction on $N$. Clearly the claim is true for $N=0$ simply by setting $f_{I_{0}}=\chi_{E}$. We warn the reader in advance that the inductive nature of the argument will require some delicate estimates in which one cannot afford to lose constant factors in the main terms.

Now fix $N>0, m>0, I_{0}, E$, and suppose the claim holds for all smaller values of $N$. We may rescale $I_{0}$ to be the unit interval $[0,1]$.

Let $0<\varepsilon \ll 1$ be a small absolute constant to be chosen later. We first prove the claim in the easy case $|E| \geq \varepsilon$. In this case we set

$$
f_{I}=|I|^{-1 / 2}\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right| \chi_{I} .
$$

The estimate (36) is trivial. To verify (37), we use Hölder's inequality and the orthonormal nature of the Haar basis

$$
\begin{aligned}
\left\|\left(\sum_{I \in \mathbf{I}_{0}}\left|f_{I}\right|^{2}\right)^{1 / 2}\right\|_{1} & \leq\left\|\left(\sum_{I \in \mathbf{I}_{0}}\left|f_{I}\right|^{2}\right)^{1 / 2}\right\|_{2} \\
& =\left(\sum_{I \in \mathbf{I}_{0}}\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} \\
& \leq\left\|\chi_{E}\right\|_{2} \\
& \lesssim|E| \log \left(2+\frac{1}{|E|}\right)^{1 / 2}
\end{aligned}
$$

as desired (if $A$ is sufficiently large depending on $\varepsilon$.)
Now suppose $|E|<\varepsilon$. Let $\mathbf{I}$ denote the set of all intervals $I \in \mathbf{I}_{0}$ such that

$$
\begin{equation*}
\varepsilon|E||I| \leq|E \cap I| \geq 2|E||I| \tag{38}
\end{equation*}
$$

holds, where $0<\varepsilon \ll 1$ is an absolute constant to be chosen later. Let $\mathbf{J}$ denote the set of all intervals not in $\mathbf{I}$ which are maximal with respect to set inclusion. From our assumptions on $E$ we see that $\mathbf{J}$ is a partition of $[0,1]$ into disjoint intervals, and each interval $J \in \mathbf{J}$ satisfies

$$
2^{-N}<|J|<1
$$

Let $J$ be any element of $\mathbf{J}$. From the induction hypothesis we can associate a function $f_{I}$ to each $I \in \mathbf{I}_{0}, I \subseteq J$ such that

$$
\left\langle\chi_{E}, \psi_{I}\right\rangle=\left\langle\chi_{E \cap J}, \psi_{I}\right\rangle \leq|I|^{-1 / 2} \int_{I} f_{I},
$$

for all such $I$, and

$$
\begin{equation*}
\left\|F_{J}\right\|_{1} \leq A|E \cap J| \log \left(2+\frac{|J|}{|E \cap J|}\right)^{1 / 2} \tag{39}
\end{equation*}
$$

where we have written $F_{J}$ for the function

$$
F_{J}=\left(\sum_{I \in \mathbf{I}_{0}: I \subseteq J}\left|f_{I}\right|^{2}\right)^{1 / 2}
$$

We have now defined the $f_{I}$ for all intervals contained in one of the intervals $J \in \mathbf{J}$. It remains to assign functions $f_{I}$ to the intervals $I$ in I.

Let $\mathbf{I}^{*}$ denote those intervals $I$ in $\mathbf{I}$ such that $|E \cap I|>0$. We will set $f_{I}=0$ for all $I \in \mathbf{I} \backslash \mathbf{I}^{*}$; note that (36) holds vacuously for these $I$. For $I \in \mathbf{I}^{*}$, we define $f_{I}$ by the formula

$$
f_{I}=|I|^{1 / 2}\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right| \sum_{J \in \mathbf{J}: J \subset I} \frac{|E \cap J|}{|E \cap I|} \frac{F_{J}}{\left\|F_{J}\right\|_{1}}
$$

Since $I$ is the union of the intervals $J \in \mathbf{J}$ contained inside it, we see that

$$
\left\|f_{I}\right\|_{1}=|I|^{1 / 2}\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right| \sum_{J \in \mathbf{J}: J \subset I} \frac{|E \cap J|}{|E \cap I|}=|I|^{1 / 2}\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right|,
$$

so that (36) holds for these $I$.
We now verify (37). For any $J \in \mathbf{J}$ and $x \in J$, we have

$$
\begin{aligned}
\sum_{I \in \mathbf{I}_{0}}\left|f_{I}(x)\right|^{2} & =\left(\sum_{I \in \mathbf{I}_{0}: I \subseteq J}\left|f_{I}(x)\right|^{2}+\sum_{I \in \mathbf{I}^{*}: I \supset J}\left|f_{I}(x)\right|^{2}\right)^{1 / 2} \\
& =F_{J}(x)^{2}+\sum_{I \in \mathbf{I}^{*}: I \supset J}|I|\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right|^{2} \frac{|E \cap J|^{2}}{|E \cap I|^{2}} \frac{F_{J}^{2}(x)}{\left\|F_{J}\right\|_{1}^{2}} \\
& =\frac{F_{J}(x)^{2}}{\left\|F_{J}\right\|_{1}^{2}}\left(\left\|F_{J}\right\|_{1}^{2}+\sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|^{2}}{|E \cap I|^{2}}\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right|^{2}\right) .
\end{aligned}
$$

Taking the square root of this and integrating, we obtain

$$
\begin{align*}
& \left\|\left(\sum_{I \in \mathbf{I}_{0}}\left|f_{I}\right|^{2}\right)^{1 / 2}\right\|_{1} \\
& \quad=\sum_{J \in \mathbf{J}}\left(\left\|F_{J}\right\|_{1}^{2}+\sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|^{2}}{|E \cap I|^{2}}\left|\left\langle\chi_{E}, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} . \tag{40}
\end{align*}
$$

Now define the function

$$
g=\sum_{J \in \mathbf{J}}|E \cap J| \frac{\chi_{J}}{|J|} .
$$

For all $I \in \mathbf{I}^{*}$ we see that $\psi_{I}$ is constant on intervals in $\mathbf{J}$, and hence that $\left\langle g, \psi_{I}\right\rangle=\left\langle\chi_{E}, \psi_{I}\right\rangle$. Thus

$$
\begin{equation*}
(40)=\sum_{J \in \mathbf{J}}\left(\left\|F_{J}\right\|_{1}^{2}+\sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|^{2}}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} . \tag{41}
\end{equation*}
$$

For future reference we observe from the construction of $\mathbf{J}$ and $g$ that $\|g\|_{1}=|E|$ and $\|g\|_{\infty} \leq 4|E|$, hence

$$
\begin{equation*}
\sum_{I \in \mathbf{I}^{*}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2} \leq\|g\|_{2}^{2} \leq\|g\|_{1}\|g\|_{\infty} \lesssim|E|^{2} . \tag{42}
\end{equation*}
$$

To estimate (41), we define

$$
\begin{gathered}
\mathbf{J}_{1}=\{J \in \mathbf{J}: 2|E||J| \leq|E \cap J| \leq 4|E||J|\}, \\
\mathbf{J}_{2}=\left\{J \in \mathbf{J}:|E|^{10}|J| \leq|E \cap J| \leq \varepsilon|E||J|\right\}, \\
\mathbf{J}_{3}=\left\{J \in \mathbf{J}:|E \cap J|<|J||E|^{10}\right\},
\end{gathered}
$$

note from (38) and the construction of $\mathbf{J}$ that $\mathbf{J}=\mathbf{J}_{1} \cup \mathbf{J}_{2} \cup \mathbf{J}_{3}$. Thus (40) is the sum of

$$
\begin{equation*}
\sum_{J \in \mathbf{J}_{1} \cup \mathbf{J}_{2}}\left(\left\|F_{J}\right\|_{1}^{2}+\sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|^{2}}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} \tag{43}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{J \in \mathbf{J}_{3}}\left(\left\|F_{J}\right\|_{1}^{2}+\sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|^{2}}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} \tag{44}
\end{equation*}
$$

We first consider (44), the contribution of the very sparsely occupied intervals. In this case we use crude estimates. From the estimate ( $a^{2}+$ $b)^{1 / 2} \leq a+b^{1 / 2}$ we have

$$
(44) \leq \sum_{J \in \mathbf{J}_{3}}\left\|F_{J}\right\|_{1}+\sum_{J \in \mathbf{J}_{3}}\left(\sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|^{2}}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} .
$$

To estimate the first term, we observe from (39) that

$$
\left\|F_{J}\right\|_{1} \lesssim A|E|^{10}|J| \log \left(\frac{1}{|E|}\right)^{1 / 2}
$$

and so

$$
\sum_{J \in \mathbf{J}_{3}}\left\|F_{J}\right\|_{1} \lesssim A|E|^{10} \log \left(\frac{1}{|E|}\right)^{1 / 2} \lesssim A|E|^{9}
$$

since we of course have

$$
\begin{equation*}
\sum_{J \in \mathbf{J}_{3}}|J| \leq 1 \tag{45}
\end{equation*}
$$

To estimate the second term, we use Cauchy-Schwarz and (45), to obtain

$$
(44) \leq C A|E|^{9}+\left(\sum_{J \in \mathbf{J}_{3}}|J|^{-1} \sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|^{2}}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} .
$$

Using the estimate $|J|^{-1}|E \cap J| \leq|E|^{10}$, and then interchanging summations, we obtain

$$
(44) \leq C A|E|^{9}+\left(\sum_{I \in \mathbf{I}^{*}} \sum_{J \in \mathbf{J}: J \subset I}|E|^{10}|I| \frac{|E \cap J|}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} .
$$

Performing the $J$ summation, this becomes

$$
(44) \leq C A|E|^{9}+|E|^{5}\left(\sum_{I \in \mathbf{I}^{*}} \frac{|I|}{|E \cap I|}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} .
$$

Applying (38) and then (42) we thus obtain

$$
\begin{equation*}
(44) \leq C A|E|^{9}+|E|^{5}\left(|E|^{-1}|E|^{2}\right)^{1 / 2} \lesssim A|E|^{2} . \tag{46}
\end{equation*}
$$

Now we turn to the more interesting term (43). From (39) we have

$$
\begin{aligned}
(43) \leq & \sum_{J \in \mathbf{J}_{1} \cup \mathbf{J}_{2}}( \\
& \left(A|E \cap J| \log \left(2+\frac{|J|}{|E \cap J|}\right)^{1 / 2}\right)^{2} \\
& \left.+\sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|^{2}}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2}\right)^{1 / 2} .
\end{aligned}
$$

Using the inequality

$$
\sqrt{a^{2}+b} \leq \sqrt{a^{2}+b+\frac{b^{2}}{4 a^{2}}}=a+\frac{b}{2 a},
$$

for $a, b>0$, we thus have

$$
(43) \leq(47)+(48),
$$

where (47) and (48) are given by

$$
\begin{equation*}
\sum_{J \in \mathbf{J}_{1} \cup \mathbf{J}_{2}} A|E \cap J| \log \left(2+\frac{|J|}{|E \cap J|}\right)^{1 / 2} \tag{47}
\end{equation*}
$$

and

$$
\begin{align*}
\sum_{J \in \mathbf{J}_{1} \cup \mathbf{J}_{2}} & \frac{1}{2}  \tag{48}\\
& \cdot \sum_{I \in \mathbf{I}^{*}: I \supset J}|I| \frac{|E \cap J|}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2} .
\end{align*}
$$

Let us first estimate the error term (48). Since $J \in \mathbf{J}_{1} \cup \mathbf{J}_{2}$, we see that

$$
\log \left(2+\frac{|J|}{|E \cap J|}\right)^{1 / 2} \sim \log \left(\frac{1}{|E|}\right)^{1 / 2}
$$

Applying this, re-arranging the summation, and simplifying, we obtain

$$
(48) \lesssim \log \left(\frac{1}{|E|}\right)^{-1 / 2} \sum_{I \in \mathbf{I}^{*}} \sum_{J \in \mathbf{J}: J \subset I}|I| \frac{|E \cap J|}{|E \cap I|^{2}}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2} .
$$

Performing the $J$ summation, we obtain

$$
(48) \lesssim \log \left(\frac{1}{|E|}\right)^{-1 / 2} \sum_{I \in \mathbf{I}^{*}} \frac{|I|}{|E \cap I|}\left|\left\langle g, \psi_{I}\right\rangle\right|^{2} .
$$

From (38) and (42) we thus have

$$
\begin{equation*}
(48) \lesssim|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2} . \tag{49}
\end{equation*}
$$

It remains to treat (47), which is the main term. We split this as $(47)=(50)-(51)+(52)$, where $(50),(51),(52)$ are given by

$$
\begin{equation*}
\sum_{J \in \mathbf{J}_{1} \cup \mathbf{J}_{2}} A|E \cap J| \log \left(2+\frac{1}{|E|}\right)^{1 / 2} \tag{50}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{J \in \mathbf{J}_{1}} A|E \cap J|\left(\log \left(2+\frac{1}{|E|}\right)^{1 / 2}-\log \left(2+\frac{|J|}{|E \cap J|}\right)^{1 / 2}\right) \tag{51}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{J \in \mathbf{J}_{2}} A|E \cap J|\left(\log \left(2+\frac{|J|}{|E \cap J|}\right)^{1 / 2}-\log \left(2+\frac{1}{|E|}\right)^{1 / 2}\right) \tag{52}
\end{equation*}
$$

Note that (50), (51), (52) are all non-negative. We can estimate (50) by

$$
(50) \leq A|E| \log \left(2+\frac{1}{|E|}\right)^{1 / 2}
$$

which is exactly the quantity needed for the induction hypothesis. Collecting all the terms and using (46), (49) we see that we have to show that

$$
\begin{equation*}
(51) \geq(52)+C A|E|^{2}+C|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2} \tag{53}
\end{equation*}
$$

We thus seek good lower bounds on (51) and good upper bounds on (52).

We first deal with (51). We may write this as

$$
(51)=A \sum_{J \in \mathbf{J}_{1}}|E \cap J| \frac{\log (2+1 /|E|)-\log (2+|J| /|E \cap J|)}{\left(\log (2+1 /|E|)^{1 / 2}+\log (2+|J| /|E \cap J|)^{1 / 2}\right.}
$$

Both terms in the denominator are comparable to $\log (1 /|E|)^{1 / 2}$, while the numerator is bounded from below by

$$
\log \left(2+\frac{1}{|E|}\right)-\log \left(2+\frac{1}{2|E|}\right) \sim 1 .
$$

Thus we have

$$
(51) \sim A \log \left(\frac{1}{|E|}\right)^{1 / 2} \sum_{J \in \mathbf{J}_{1}}|E \cap J| .
$$

To obtain lower bounds for this, we observe that

$$
\sum_{J \in \mathbf{J}_{1}}|E \cap J|=|E|-\sum_{J \in \mathbf{J}_{2} \cup \mathbf{J}_{3}}|E \cap J|
$$

and

$$
\sum_{J \in \mathbf{J}_{2} \cup \mathbf{J}_{3}}|E \cap J| \leq \sum_{J \in \mathbf{J}} \varepsilon|E||J|=\varepsilon|E| .
$$

Thus

$$
(51) \gtrsim A|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2} .
$$

Now we attend to (52). As before, we may write

$$
(52)=A \sum_{J \in \mathbf{J}_{1}}|E \cap J| \frac{\log (2+|J| /|E \cap J|)-\log (2+1 /|E|)}{\left(\log (2+1 /|E|)^{1 / 2}+\log (2+|J| /|E \cap J|)^{1 / 2}\right.}
$$

Again, the denominator is comparable to $\log (1 /|E|)^{1 / 2}$, while the numerator is comparable to $\log (|E||J| /|E \cap J|)$. Thus

$$
(52) \lesssim A \log \left(\frac{1}{|E|}\right)^{-1 / 2} \sum_{J \in \mathbf{J}:|E \cap J| \leq \varepsilon|E||J|}|E \cap J| \log \left(\frac{|E||J|}{|E \cap J|}\right) .
$$

We estimate this dyadically as

$$
\begin{aligned}
(52) \lesssim & A \log \left(\frac{1}{|E|}\right)^{-1 / 2} \\
& \cdot \sum_{k: 2^{-k} \lesssim \varepsilon} \sum_{J \in \mathbf{J}:|E \cap J| \sim 2^{-k}|E||J|}|E \cap J| \log \left(\frac{|E||J|}{|E \cap J|}\right) \\
& \lesssim A \log \left(\frac{1}{|E|}\right)^{-1 / 2} \sum_{k: 2^{-k} \lesssim \varepsilon} \sum_{J \in \mathbf{J}} 2^{-k}|E||J| k \\
& \lesssim A|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2} \sum_{k: 2^{-k} \lesssim \varepsilon} 2^{-k} k \\
& \lesssim A|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2} \sum_{k: 2^{-k} \lesssim \varepsilon} 2^{-k / 2} \\
\lesssim & A \varepsilon^{1 / 2}|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2} .
\end{aligned}
$$

Thus (53) resolves to

$$
\begin{aligned}
& C^{-1} A|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2} \\
& \quad \geq C A \varepsilon^{1 / 2}|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2}+C A|E|^{2}+C|E| \log \left(\frac{1}{|E|}\right)^{-1 / 2}
\end{aligned}
$$

and this is achieved if $\varepsilon$ is chosen sufficiently small (recall that $|E| \leq \varepsilon$ ), and then $A$ is chosen sufficiently large depending on $\varepsilon$.

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# Morera type problems in Clifford analysis 

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#### Abstract

The Pompeiu and the Morera problems have been studied in many contexts and generality. For example in different spaces, with different groups, locally, without an invariant measure, etc. The variations obtained exhibit the fascination of these problems.

In this paper we present a new aspect: we study the case in which the functions have values over a Clifford Algebra. We show that in this context it is completely natural to consider the Morera problem and its variations. Specifically, we show the equivalence between the Morera problem in Clifford analysis and Pompeiu problem for surfaces in $\mathbb{R}^{n}$. We also show an invariance theorem. The non-commutativity of the Clifford algebras brings in some peculiarities.

Our main result is a theorem showing that the vanishing of the first moments of a Clifford valued function implies Clifford analyticity. The proof depends on results which show that a particular matrix system of convolution equations admits spectral synthesis.


## 0. Introduction.

The framework provided by Clifford Algebras has proven to be very useful to generalize many aspects of one variable complex analysis to $\mathbb{R}^{n}$. The subject has come to be known as Clifford Analysis. Unexpected links to classical harmonic analysis, several complex variables and representation theory have been discovered. Many books on the subject have recently appeared [11], [15], [16], [23], [24] and it has grown
to be an important area of research.
It is therefore completely natural to ask which aspects of the Morera problem in the complex plane are valid in this context. Let us point out that the non-commutativity of the Clifford Algebras brings many peculiarities to Clifford Analysis. In particular many familiar properties are not valid in this context. Nevertheless we will show a positive result for the Clifford Morera problem.

The plan of the paper is as follows. In the first section, we give a short survey on the Pompeiu problem and on the Morera problem. We include the results and examples that we will use later on. We also comment a little about the methods involved to prove this results.

In the second section, we set up the framework of Clifford analysis. We reproduce the most fundamental results for the Clifford holomorphic functions or regular functions. This includes the corresponding versions of the Stokes formula, the Cauchy representation formula and the Morera theorem. The Vahlen-Ahlfors representation of Moebius transformations in $\mathbb{R}^{n}$ is also presented.

After these two preliminary sections we start our study properly. In the third section we present first the equivalence of the Morera problem and the Pompeiu problem for surfaces in $\mathbb{R}^{n}$. Although this is an easy fact to prove it has many consequences. We discuss these consequences in a sequence of corollaries. Then we show a non-invariant version of the Morera problem.

Section Four, our main contribution, deals with the statement and proof of a First Moments Theorem. Roughly speaking, this correspond to proving that a matrix system of convolution equations admits spectral synthesis. It turns out that the determinant minors of this matrix satisfy the Hörmander condition and the theorem follows. We note that in most Euclidean cases of the Pompeiu problem a reduction to the fundamental theorem of mean periodic function is made. This is not the case here.

Finally, in the last section, we discuss some problems for future research. The advantage of being able to carry specific calculations was important to prove the moments result but for generic surfaces we do not know how to proceed. The easy proof for one complex variable proof cannot be adapted to this context.

## 1. Preliminaries about the Pompeiu and Morera problems.

### 1.1. Notation.

As usual, let $\mathcal{E}\left(\mathbb{R}^{n}\right)$ denote the space of all infinitely differentiable functions on $\mathbb{R}^{n}$ with the topology of uniform convergence of all derivatives on compact subsets of $\mathbb{R}^{n}$. Let $\mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$ be its dual space of distributions with compact support.

Also let $\mathcal{C}\left(\mathbb{R}^{n}\right)$ denote the space of all continuous functions on $\mathbb{R}^{n}$ with the usual topology of uniform convergence on compact sets. We will denote the Fourier transform of a function or a distribution $f$ by $\widehat{f}$ or by $\mathcal{F}(f)$.

Let us also recall that the algebra $\widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$ can be characterized as the space of all holomorphic functions $F: \mathbb{C}^{n} \longrightarrow \mathbb{C}$ satisfying the Paley-Wiener estimates: for some constants $C, A, N$ greater than zero and all $z$ in $\mathbb{C}^{n}, z=\operatorname{Re} z+i \operatorname{Im} z$

$$
|F(z)| \leq C(1+\|z\|)^{N} e^{A|\operatorname{Im} z|}
$$

### 1.2. The Pompeiu problem.

A general version of the Pompeiu problem can be formulate as follows [10]: Let $X$ be a locally compact space, $\mu$ a non-negative Radon measure on $X,\left\{C_{i}\right\}_{i=1}^{N}$ a finite family of compact subsets of $X$, and $G$ a topological group acting on $X$ and keeping $\mu$ invariant. The Pompeiu map

$$
P: C(X) \longrightarrow(C(G))^{N}
$$

is defined by

$$
\left(P_{i} f\right)(g):=\int_{g C_{i}} f d \mu
$$

where $P_{i}$ is the ith component of $P$ and we denote by $g x$ the action of the element $g \in G$ on the point $x \in X$.

We say that the family $\left\{C_{i}\right\}$ has the Pompeiu property if $P$ is injective. The Pompeiu problem consists of deciding as explicitly as possible whether the family has the Pompeiu property. For a historical introduction to these problems as well as their ramifications, generalizations, progress and a complete bibliography we refer to [31], [30], [5],
[10], [28], [12]. In [10] a general method is explained and some theorems are proved for symmetric spaces of real rank 1 .

When $G$ is a separable unimodular Lie group, the Pompeiu map may be interpreted as a system of convolution equations on $\mathcal{E}^{\prime}(G)$, the space of distributions of compact support on $G$. Further reduction is made rewriting the problem as a problem of spectral analysis. We illustrate this line of reasoning in the case when $X=\mathbb{R}^{n}, G=M(n)$, and $\mu=d x$, where $M(n)$ is the group of orientation preserving rigid motions, that is, the group generated by all translations and by all rotations in $S O(n)$, and $d x$ is Lebesgue measure.

A translation invariant subspace $\mathcal{M}$ of $\widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$ is said to admit spectral analysis if $\mathcal{M}$ contains an exponential. If the exponential polynomials belonging to $\mathcal{M}$ are dense in $\mathcal{M}$ we say that $\mathcal{M}$ admits spectral synthesis.

To decide whether the map $P$ is injective one can assume by a standard approximation argument that $f$ is a smooth function. Now for smooth $f$, we rewrite the conditions $P f=0$

$$
\int_{g C_{i}} f d x=0, \quad g \in M(n), \quad i=0, \ldots N
$$

where $g(x)=\sigma x+y$ with $\sigma \in S O(n)$ and $y \in \mathbb{R}^{n}$, as the (infinite) system of convolution equations in $\mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$

$$
\check{\chi}_{\sigma C_{i}} * f=0, \quad \sigma \in S O(n), \quad i=1, \ldots, N
$$

where $\chi_{C}$ denotes the characteristic function on the set $C$ and $\check{h}(x)=$ $h(-x)$.

Consider the convolution ideal $\mathbb{I}$ in $\mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$ generated by the $\check{\chi}_{\sigma C_{i}}$. If $\mathbb{I}$ is dense in $\mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$, then for any solution $f \in \mathcal{E}\left(\mathbb{R}^{n}\right)$ of the system and a generic element in $\mathbb{I}, \sum g_{\alpha} * \check{\chi}_{\sigma C_{i}}$, we have

$$
\left(\sum g_{\alpha} * \check{\chi}_{\sigma C_{i}}\right) * f=\sum g_{\alpha} *\left(\check{\chi}_{\sigma C_{i}} * f\right)=0
$$

thus by the density

$$
f=\delta * f=0
$$

A necessary condition for $\mathbb{I}$ to be dense is that the Fourier transforms $\widehat{\chi}_{\sigma C_{i}}$ have no common zeroes. Moreover if $x_{0}$ is the common zero, then $f(x)=e^{i x \cdot x_{0}}$ is a non-zero solution of the system since

$$
\check{\chi}_{\sigma C_{i}} * f=f \cdot \widehat{\chi}_{C_{i}}\left(x_{0}\right)=0 .
$$

In the real case ( $n=1$ ) the condition is also sufficient. This result is a consequence of the Schwartz spectral synthesis theorem. Unfortunately the theorem is not true in $\mathbb{R}^{n}, n>1,[17]$. Nevertheless, under certain symmetric conditions for the sets $C_{i}$, if their Fourier transforms $\widehat{\chi}_{C_{i}}$ have no common zeroes, a reduction to the Schwartz theorem can be made.

In the case of a single set $C$, the above discussion can be carried further, [12], to prove that $C$ has the Pompeiu property if and only if $\widehat{\chi}_{C_{i}}$ does not vanish identically on any of the analytic varieties

$$
C_{\alpha}=\left\{z \in \mathbb{C}^{n}: z_{1}^{2}+z_{2}^{2}+\cdots+z_{n}^{2}=\alpha\right\}, \quad \alpha \neq 0
$$

Note that no ball has the Pompeiu property [28]. We now state some of the known results [28], [30].

Theorem 1.1 (Two balls Theorem). Let $B_{i}$ denote the closed ball of radius $r_{i}$. Then $\left\{B_{1}, B_{2}\right\}$ has the Pompeiu property with respect to Lebesgue measure if and only if $r_{1} / r_{2} \notin Z_{n}=\{\xi / \mu: \xi, \mu$ non zero roots of the Bessel equation $\left.J_{n / 2}(z)=0\right\}$.

Theorem 1.2 (Two spheres Theorem). Let $S_{i}$ denote a sphere of radius $r_{i}$. Then $\left\{S_{1}, S_{2}\right\}$ has the Pompeiu property with respect to surface measure if and only if $r_{1} / r_{2} \notin Z_{n-2}$.

In the case when $X$ is a irreducible symmetric space of rank 1 , there are analogues to the two balls and two spheres theorems above [10].

In the case we discuss below, a link to overdetermined problems is given in [27]. It has proven to be very important. When $C=\bar{\Omega}$, for $\Omega$ a bounded open set in $R^{n}$, if $C^{c}$ is connected, then the failure of the Pompeiu property for $C$ is equivalent to the existence of an eigenvalue for a overdetermined Neumann boundary value problem. Namely,

Theorem 1.3. Let $C=\bar{\Omega}$, where $\Omega$ is a bounded open set, $C^{c}$ is connected and $\partial C$ is (at least) Lipschitz. Then $C$ fails to have the Pompeiu property if and only if there is an eigenvalue $\alpha$ and a function $u$ on $\Omega$ satisfying the overdetermined Neumann problem

Theorem 1.4. Let $\Omega$ be as above. If $\partial \Omega$ is Lipschitz but not real analytic everywhere then $\Omega$ has the Pompeiu property.

### 1.3. The Morera problem.

There is already a discussion on Morera type theorems [6], but new results and different aspects keep appearing. We will mention only the results that we will try to generalize.

Let $\Gamma$ be a Jordan curve in $\mathbb{C}$. We say that $\Gamma$ has the Morera property if each continuous complex valued function $f$ on $\mathbb{C}$ which satisfies

$$
\int_{\sigma(\Gamma)} f(z) d z=0
$$

for every rigid motion $\sigma$ of $\mathbb{C}$ is entire.
A similar definition holds for a family of Jordan curves $\left\{\Gamma_{i}\right\}$. The Morera problem is to decide as explicitly as possible whether the family has this property. We can also consider the hyperbolic case in which the function is defined only in the unit disk and the group is the Moebius group.

The Morera and Pompeiu problems are equivalent in the following situation [28], [12].

Theorem 1.5. Suppose that $\left\{\Gamma_{i}\right\}$ is a family of Jordan curves and $\Omega_{i}=\overline{\operatorname{int}\left(\Gamma_{i}\right)}$ is a family of Jordan domains. Then the family $\left\{\Omega_{i}\right\}$ has the Pompeiu property if and only if the family $\left\{\Gamma_{i}\right\}$ has the Morera property.

This theorem follows from the following version of the Green formula

$$
\frac{d}{d \bar{z}} \chi_{\Omega}=\chi_{\partial \Omega}
$$

taken in the distributional sense. Because of this equivalence and Theorem 1.4, many classes of curves satisfy the Morera property.

As the example of the circle shows, one single curve is not in general enough to solve the Morera problem. The following Theorem, [1], solves the problem of giving necessary and sufficient conditions for a single curve to determine holomorphycity.

Theorem 1.6. (Moments Theorem). Let $f \in C(\mathbb{C})$, and let $\Gamma$ be a piecewise smooth Jordan curve. Then $f$ is entire if and only if

$$
\int_{\sigma(\Gamma)} z^{k} f(z) d z=0, \quad k=0,1, \ldots,
$$

for every rigid transformation $\sigma$ of $\mathbb{C}$.

Remark 1.7. 1. This result at first sight seem obvious, since for every $\sigma$ the vanishing of the moment implies that the function can be extended holomorphically inside the region bounded by $\sigma(\Gamma)$. But we do not know that these extensions agree on overlaps.
2. The proof follows from an averaging argument and the argument principle.
3. A similar result is true in the unit disk $\mathbb{D}$.
4. The proof of 3 follows from the maximality of invariant algebras of functions in $\mathbb{D}$ under Moebius transformations, [1].
5. Actually, it is enough to request that the moments do not grow too fast [29].

In the case of a circle only 2 moments are required [29].
Theorem 1.8 (Two Moments Theorem). Let $f \in C(\mathbb{C})$ and let $r>0$, $n>1$ be fixed. Suppose that

$$
\int_{\partial B(z, r)} f(\zeta) d \zeta=\int_{\partial B(z, r)}(z-\zeta)^{n} f(\zeta) d \zeta=0
$$

for all $z \in \mathbb{C}$. Then $f$ is an entire function.

Remark 1.9. This result follows from rewriting the hypothesis as two convolution equations and appealing to the Schwartz spectral synthesis Theorem.

The last result is true if we consider functions defined in the unit disk but it is interesting that the following variation of the Morera Problems gives different results. Suppose $f \in C(\mathbb{D})$ satisfies

$$
\int_{\Gamma} f(\sigma(z)) d z=0
$$

for all Moebius transformation $\sigma$ in $\mathbb{D}$. Is it true that $f$ is holomorphic in $\mathbb{D}$ ?

Observe that the measure $d z_{\left.\right|_{\Gamma}}$ is not invariant under the action of the Moebius group $\mathcal{M} \simeq S U(1,1)$ also note that now we are moving the values of the function. The following Theorems [4] give the answer to this problem in the circular case and in the general case.

Theorem 1.10 (Circular Morera Theorem). Let $r>0$ and let $f \in$ $C(\mathbb{D})$ satisfy

$$
\int_{\partial B(c, r)} f(\sigma(z)) d z=0
$$

for every Moebius transformation $\sigma$ in $\mathbb{D}$.
a) If $c \neq 0$ then $f$ is holomorphic on $\mathbb{D}$.
b) If $c=0$ then $f$ is not necessarily holomorphic on $\mathbb{D}$ (There are counterexamples).

Theorem 1.11. Let $\Omega \subset \mathbb{D}$ be a Jordan domain of class $C^{2, \varepsilon}$ for some $\varepsilon>0$ and suppose that the Jordan curve $\Gamma=\partial \Omega$ is not real analytic. Assume $f \in C(\mathbb{D})$ satisfies

$$
\int_{\Gamma} f(\sigma(z)) d z=0
$$

for every $\sigma \in \mathcal{M}$. Then $f$ is holomorphic on $\mathbb{D}$.

## 2. Rudiments of Clifford analysis.

### 2.1. Basic results.

The goal of this section is to present the basic definitions in Clifford Algebras and the basic concepts and results in Clifford Analysis as we will need them later on. For a complete development of the subject we refer to the books [11], [15], [24], [16].

We consider the real $2^{n}$ dimensional Clifford algebra $\mathbb{A}_{n}$ generated out of $\mathbb{R}^{n}$ as follows: let $e_{1}, \ldots, e_{n}$ be an orthonormal basis for $\mathbb{R}^{n}$. Then $\mathbb{A}_{n}$ is defined by the anti-commutation relationship

$$
e_{i} e_{j}+e_{j} e_{i}=-2 \delta_{i j}
$$

where $\delta_{i j}$ is the Kronecker delta function. Consequently, the algebra $\mathbb{A}_{n}$ has as basis elements

$$
1, e_{1}, \ldots, e_{n}, \ldots, e_{j_{1}} \ldots e_{j_{r}}, \ldots, e_{1} \ldots e_{n}
$$

where $j_{1}<\cdots<j_{r}$ and $1 \leq r \leq n$. Hence for an element $a \in \mathbb{A}_{n}$ we write

$$
a=\sum_{\alpha} a_{\alpha} e_{\alpha},
$$

where $a_{\alpha} \in \mathbb{R}$ and where we identify $e_{\alpha}$ with $e_{j_{1}}, \ldots, e_{j_{r}}$ for $\alpha=$ $\left\{j_{1}, \ldots, j_{r}\right\}$ and $e_{\varnothing}$ with 1.

Note that if $x \in \mathbb{R}^{n}$ we have that $x^{2}=-\|x\|^{2}$. It follows that every non zero $x \in \mathbb{R}^{n}$ is invertible with inverse $x^{-1}=-x /\|x\|^{2}$. Observe that $\mathbb{A}_{1}=\mathbb{C}$, and $\mathbb{A}_{2}=\mathbb{H}$, the quaternionic division algebra. For $n \geq 3, \mathbb{A}_{n}$ is no longer a division algebra.

We will use the following two involutions. First the anti-automorphism defined by

$$
\sim: \mathbb{A}_{n} \longrightarrow \mathbb{A}_{n}: e_{j_{1}} \cdots e_{j_{r}} \longrightarrow e_{j_{r}} \cdots e_{j_{1}}
$$

For an element $a \in \mathbb{A}_{n}$, we write $\tilde{a}$ instead of $\sim(a)$. Second the antiautomorphism defined by

$$
-: \mathbb{A}_{n} \longrightarrow \mathbb{A}_{n}: e_{j_{1}} \cdots e_{j_{r}} \longrightarrow(-1)^{r} e_{j_{r}} \cdots e_{j_{1}} .
$$

Again we write $\bar{a}$ for -(a). This anti-automorphism is a generalization of complex conjugation.

The Clifford algebra $\mathbb{A}_{n}$ becomes a Hilbert space and a Banach Algebra when the inner product on $\mathbb{A}_{n}$ is defined by putting for any $a, b \in \mathbb{A}_{n}$,

$$
\langle a, b\rangle=\sum_{\alpha} a_{\alpha} b_{\alpha} .
$$

Note that for $x, y$ vectors (i.e. $x, y \in \mathbb{R} \oplus \mathbb{R}^{n}$ ), we have $\langle x, y\rangle=$ $(x \bar{y}+y \bar{x}) / 2$. In particular, $\|x\|^{2}=x \bar{x}$ and $\|x y\|=\|x\|\|y\|$, but for general $a, b \in \mathbb{A}_{n},\|a\|^{2} \neq a \bar{a}$ and $\|a b\| \neq\|a\|\|b\|$.

We will consider the space $\mathcal{E}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$ of smooth $\mathbb{A}_{n}$ valued functions, which is an $\mathbb{A}_{n}$ module under pointwise multiplication. The topology we will consider in $\mathcal{E}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$ is the one of uniform convergence of all derivatives over compact subsets. Similar considerations are made for the space of continuous $\mathbb{A}_{n}$ valued functions $\mathcal{C}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$.

Two basic definitions are
i) The Dirac operator is the differential operator

$$
D=\sum_{i=1}^{n} e_{i} \frac{\partial}{\partial x_{i}} .
$$

ii) Let $f, g \in \mathcal{C}^{1}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$ be differentiable functions. Then $f$ is called left regular if

$$
D f=\sum_{i=1}^{n} e_{i} \frac{\partial f}{\partial x_{i}}=\sum_{i=1}^{n} \sum_{\alpha} e_{i} e_{\alpha} \frac{\partial f_{\alpha}}{\partial x_{i}}=0,
$$

and $g$ is called right regular if

$$
g D=\sum_{i=1}^{n} \frac{\partial g}{\partial x_{i}} e_{i}=\sum_{i=1}^{n} \sum_{\alpha} e_{\alpha} e_{i} \frac{\partial g_{\alpha}}{\partial x_{i}}=0
$$

In the literature left regular, left monogenic or left Clifford holomorphic are used indistinctly. Note that since $\tilde{D f}=-\tilde{f} D$, a function $f$ is left regular if and only if $\tilde{f}$ is right regular. Also note that if $f(x)$ is a left regular function then so is $f(x) a$ for any $a \in \mathbb{A}_{n}$ but not in general for $a f(x)$.

An important property is that $D^{2}=-\triangle$, the Laplacian over $\mathbb{R}^{n}$, hence, each component of a left or right regular function is harmonic. The function

$$
G(x)=\frac{1}{\omega_{n}} \frac{-x}{\|x\|^{n}}=\frac{1}{\omega_{n}} \frac{x^{-1}}{\|x\|^{n+2}}
$$

where $\omega_{n}$ is the surface area of the unit sphere in $\mathbb{R}^{n}$ is left and right regular. This function $G(x)$ plays the role of the Cauchy kernel.

The Green Formula can be formulated in the framework of Clifford algebra valued functions as follows [11], [15].

Theorem 2.1. Let $f$ and $g$ be Clifford algebra valued functions defined in a domain $U \subset \mathbb{R}^{n}$ and let $M$ be a bounded domain in $U$ with Lipschitz boundary. Then

$$
\int_{\partial M} g(x) n(x) f(x) d S(x)=\int_{M}((g D)(x) f(x)+g(x)(D f)(x)) d v(x) .
$$

Note that, here and in the following theorems, $d S$ is the canonical surface measure, $n(x)$ stands for the outward unit normal to $\partial M$ regarded as a Clifford algebra-valued function, $d v$ is the volumen element, and the integrands are interpreted in the sense of Clifford algebra multiplication.

The Borel-Pompeiu formula for Clifford valued functions is the following.

Theorem 2.2. Let $M$ be a bounded domain with Lipschitz boundary. Then for $f \in C^{1}\left(U, \mathbb{A}_{n}\right)$ and $x \in M$,

$$
f(x)=\int_{\partial M} G(y-x) n(y) f(y) d S(y)-\int_{M} G(y-x) D f(y) d v(y) .
$$

The Cauchy integral formula is given by the following theorem.
Theorem 2.3. Let $M$ be a bounded domain in $U$ with Lipschitz boundary. If $f$ is a left regular function on $U$, then for each $x$ in $M$,

$$
f(x)=\int_{\partial M} G(y-x) n(y) f(y) d S(y) .
$$

We also have the Morera theorem.
Theorem 2.4. If $f$ is a Clifford algebra valued continuous function on the domain $U$ such that

$$
\int_{\partial M} n(y) f(y) d S(y)=0
$$

for every bounded domain $M$ in $U$ with Lipschitz boundary, then $f$ is left regular.

Of course there are similar versions of this theorems for right regular functions. Taylor series where the polynomial are regular functions are also possible [11]. In this paper, we will use only the polynomials

$$
P_{i}(x)=x_{i} e_{1}+x_{1} e_{i}, \quad i=2, \ldots, n,
$$

which are a basis for both the right (left) module of homogeneous left (right) regular polynomials of degree 1.

### 2.2. Vahlen matrices.

We now introduce the Vahlen matrices. The collection of all products of non-zero vectors in $\mathbb{R}^{n}$ form a group $\mathbb{A}_{n}^{*}$ lying in $\mathbb{A}_{n}$. Let $\mathbb{V}(n)$ be the set of $2 \times 2$ matrices $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ such that
i) $a, b, c, d \in \mathbb{A}_{n}^{*}$.
ii) $a \tilde{c}, c \tilde{d}, d \tilde{b}$ and $d \tilde{a} \in \mathbb{R}^{n}$.
iii) $a \tilde{d}-b \tilde{c}= \pm 1$.

A matrix $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \mathbb{V}(n)$ is called a Vahlen matrix. The usefulness of this concept is given by the following theorem [2].

Theorem 2.5. Let $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \mathbb{V}(n)$. Then the function $\phi(x)=(a x+$ b) $(c x+d)^{-1}$ defines a Moebius transformation over $\mathbb{R}^{n} \cup\{\infty\}$. Moreover this representation gives a surjective group homomorphism from $\mathbb{V}(n)$ with matrix multiplication to the orientation preserving Moebius group over $\mathbb{R}^{n} \cup\{\infty\}$ with kernel $\pm \mathbb{I}$.

A computation shows that the Jacobian of $\phi(x)=(a x+b)(c x+$ $d)^{-1}$ is given by

$$
\operatorname{Jac}(\phi(x))=\frac{1}{\|c x+d\|^{2 n}}
$$

The following theorem can be seen as a change of variable for Clifford valued functions under Moebius transformations [22].

Theorem 2.6. Suppose that $y=\phi(x)=(a x+b)(c x+d)^{-1}$ is a Moebius transformation and $f$ and $g$ are Clifford valued functions. If $S$ is a closed, bounded and oriented surface then

$$
\begin{aligned}
\int_{S} g(y) n(y) & f(y) d S(y) \\
& =\int_{\phi^{-1}(S)} g(\phi(x)) \widetilde{J(\phi, x)} n(x) J(\phi, x) f(\phi(x)) d S(x),
\end{aligned}
$$

where

$$
J(\phi, x)=\frac{\widetilde{c x+d}}{\|c x+d\|^{n}} .
$$

The factor $J(\phi, x)$ is called the covariance of $\phi(x)$.
The Dirac operator and the covariance are intertwined as follows (see for example [22]).

Theorem 2.7. Let $f$ be a Clifford valued function and $\phi(x)=(a x+$ b) $(c x+d)^{-1}$ a Moebius transformation. Then

$$
D J(\phi, x) f(\phi(x))=J_{-1}(\phi, x) D f(\phi(x)),
$$

where

$$
J_{-1}(\phi, x)=\frac{\widetilde{c x+d}}{\|c x+d\|^{n+2}} .
$$

As the composition or product of regular functions is not regular, the following theorem provides a kind of substitute [22].

Theorem 2.8. Let $y=\phi(x)$ be a Moebius transformation and $f(y)$ a Clifford valued function. Then $f(y)$ is left regular if and only if $J(\phi, x) f(\phi(x))$ is left regular.

Finally note that

$$
\tilde{J_{-1}}(\phi, x) J(\phi, x)=\operatorname{Jac}(\phi(x))
$$

This end our summary on the basic facts in Clifford Analysis. We are now ready to start our study properly.

## 3. First results.

### 3.1. Equivalence of Morera and Pompeiu.

In this section we give the results which are easy to prove and similar to the complex case.

By a Jordan surface $S$ we be will mean a Lipschitz embedding of the $(n-1)$-sphere in $\mathbb{R}^{n}$ (i.e. $S$ is homeomorphic to the ( $n-1$ )-sphere by a Lipschitz function). Let $M=\overline{\operatorname{int} S}$. We say that a Jordan surface $S$ in $\mathbb{R}^{n}$ (or a collection of them $\left\{S_{j}\right\}$ ), has the Morera property if any $f \in \mathcal{C}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$ satisfying

$$
\begin{equation*}
\int_{\sigma S} n(x) f(x) d S(x)=0 \tag{1}
\end{equation*}
$$

for every rigid motion $\sigma \in M(n)$ is left regular. Note that here as in the rest of the section, the integrals and product are considered in the Clifford analysis setting.

The Morera problem consist of deciding as explicitly as possible whether a surface (or a family of them) has the Morera property.

More generally we can state the Morera problem on a different space or with a different group or with a more general surface. For example we can take the space as the unit ball in $\mathbb{R}^{n}$ and the group as the group of Moebius transformation of the ball.

Remark 3.1. The Morera problem is stated for the case in which the function is continuous but it is equivalent to the case in which the function is smooth. This follows from a standard smoothing argument. We reproduce it in here for the sake of completeness.

Suppose that $f \in \mathcal{E}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$, satisfying (1) implies that $f$ is left regular. Let $g \in \mathcal{C}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$ satisfies (1). Let $\phi$ be a (real value) approximate identify of compact support. Then $g * \phi \in \mathcal{E}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$ and satisfies (1). Therefore $g * \phi$ is left regular. Now since

$$
g * \phi_{\varepsilon_{n}} \longrightarrow g
$$

uniformly on compact sets as $\varepsilon_{n} \longrightarrow 0$, we conclude that $g$ is left regular. Therefore, we will assume from now on that the function is smooth.

Let $\left\{S_{j}\right\}$ be a collection of Jordan surfaces and let $M_{j}=\overline{\operatorname{int} S_{j}}$. As in the complex case we have:

Theorem 3.2. $\left\{S_{j}\right\}$ has the Morera property in $\mathbb{A}_{n}$ if and only if $\left\{M_{j}\right\}$ has the Pompeiu property in $\mathbb{R}^{n}$.

Proof. Let $g \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}\right)$. Then there is a Clifford valued function $f$ such that $f$ solves the Dirac equation $D f=g$ ([11, Theorem 19.2]). Then by the Green formula (Section 2, Theorem 2.1), for every rigid motion $\sigma \in M(n)$, we have

$$
\int_{\sigma(M)} g(x) d v(x)=\int_{\operatorname{int}(\sigma(M))} n(x) f(x) d S(x)=\int_{\sigma(S)} n(x) f(x) d S(x) .
$$

Hence, if $S$ satisfy 3.1 then $M$ has the Pompeiu property.
Conversely if $f \in \mathcal{E}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$, then by Stokes Theorem and the Pompeiu property for $M$ we have that $D f \equiv 0$, so $f$ is left regular and $S$ has the Morera property.

This equivalence has several consequences. Using the results of Section 2 we get at once the following corollaries.

## Corollary 3.3.

1. No sphere has the Morera property.
2. Two spheres have the Morera property if and only if their radii $r_{1}, r_{2}$ satisfy the condition in Theorem 1.1 of Section 1, namely $r_{1} / r_{2} \notin$ $Z_{n}=\left\{\xi / \mu: \xi, \mu\right.$ non zero roots of the Bessel equation $\left.J_{n / 2}(z)=0\right\}$.
3. We have a condition for the Morera property in terms of the Fourier transform of the characteristic function of $M$.

Note the difference with the two spheres Theorem of Section 1.
Among the concrete examples for which the Morera property holds are ellipsoids, tori, and some surfaces of revolution, [13].

Corollary 3.4. If the Jordan surface $S$ is Lipschitz but not real analytic everywhere then $S$ has the Morera property.

It follows that polygonal surfaces have the Morera property, e.g. $n$-cubes, polyhedra, etc.

Another corollary to the equivalence of Morera and Pompeiu problem is the study of the local situation. This is what can we say if the function is defined only on a domain $D \subset \mathbb{R}^{n}$ and the vanishing of the integrals is required only when $\sigma S \subset D$. It turns out that the local Pompeiu problem is a harder question [7]. As before we get the following corollary.

Corollary 3.5. Let $r_{1}, r_{2}>0$ be such $r_{1} / r_{2} \notin Z_{n}$, and let $R>r_{1}+r_{2}$. If $f \in \mathcal{C}\left(B(R, 0), \mathbb{A}_{n}\right)$ satisfies

$$
\int_{\partial B\left(y, r_{i}\right)} n(x) f(x) d S(x)=0, \quad i=1,2,
$$

for all $y \in \mathbb{R}^{n}$ such that $\partial B\left(y, r_{i}\right) \subset B(R, 0)$, then $f$ is left regular. Moreover the condition is sharp.

### 3.2. Non-invariant measures.

We now study a non-invariant measure variant of the Morera problem. Using the result of section 2 , we can state the problem as follows. Let $\phi \in \mathcal{M}$, where $\mathcal{M}$ is the group of Mobius transformations of the unit ball $\mathbb{B}$ in $R^{n}$. We know that $\phi(x)=(a x+b)(c x+d)^{-1}$ with $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \mathbb{V}(n)$ a Vahlen matrix. If $f$ is a regular function defined in $\mathbb{B}$, then $J(\phi, x) f(\phi(x))$ is also left regular in $\mathbb{B}$. Therefore by the Cauchy Theorem,

$$
\int_{S} J(\tilde{\phi}, x) n(x) J(\phi, x) f(\phi(x)) d S(x)=0
$$

for every surface $S$ in $\mathbb{B}$. The problem is to determine whether for a fixed surface $S$ and a continuous function $f$ the above condition implies that $f$ is left regular.

The next proposition shows that the above problem could be reduced to the Pompeiu Problem for the unit ball and the Moebius group.

Proposition 3.6. Let $f$ be a continuous Clifford valued function defined in the unit ball $\mathbb{B}$ in $R^{n}$ and let $S$ be a Jordan surface in $\mathbb{B}$. If

$$
\int_{S} J(\tilde{\phi}, x) n(x) J(\phi, x) f(\phi(x)) d S(x)=0
$$

for every $\phi \in \mathcal{M}$, where $\mathcal{M}$ is the group of Mobius transformations of the ball, then $f$ is left regular if and only if $M=\overline{\operatorname{int} S}$ has the Pompeiu property with respect to $\mathcal{M}$.

Proof. By the Clifford algebra version of Stokes Theorem we have that

$$
\begin{aligned}
\int_{S} J(\tilde{\phi}, x) n(x) J(\phi, x) f(\phi(x) & d S(x) \\
& =\int_{M} J(\tilde{\phi}, x) D(J(\phi, x) f(\phi(x))) d v
\end{aligned}
$$

Now using Theorem 2.2 of Section 2 we get that the integral is equal to

$$
\int_{M} J(\tilde{\phi}, x) J_{-1}(\phi, x) D f(\phi(x)) d v
$$

By using that $\tilde{J}_{-1}(\phi, x) J(\phi, x)=\operatorname{Jac}(\phi(x))$ we get that the last integral is equal to

$$
\int_{M} \operatorname{Jac}(\phi(x) D f(\phi(x)) d v
$$

Now by a change of variable this integral is equal to

$$
\int_{\phi^{-1}(M)} D f(y) d v(y) .
$$

Using that $d \mu=d v /\left(1-\|y\|^{2}\right)^{2}$ is the hyperbolic measure for the ball the last integral is equal to

$$
\int_{\phi^{-1}(M)} D f(y)\left(1-\|y\|^{2}\right)^{2} d \mu(y)
$$

Hence that we get the Pompeiu problem for the function $D f(y)(1-$ $\|y\|^{2}$ ) in the ball $\mathbb{B}$ with the group $\mathcal{M}$. The conclusion follows.

## 4. The moment condition for Clifford valued functions.

### 4.1. Introduction.

In this section we show that a sphere has the Morera property if we add the first Clifford moments. Namely we show that a continuous function in $\mathbb{R}^{n}$ with values in the Clifford Algebra $\mathbb{A}_{n}$, whose first moments over all spheres of fixed radius $r$ vanish is a regular function. We prove this result by first reducing the problem to a overdetermined matrix system of convolution equations in $\mathbb{R}^{n}$. Then we need to see than spectral synthesis holds for this kind of system.

It turns out that the determinants of the maximal minors of this matrix of convolution operators satisfy the Hörmander conditions and thus spectral synthesis holds.

### 4.2. Statement of the problem.

We saw in Section 3 than no sphere has the Morera property. The natural question is to look for the extra conditions needed. In the spirit of the Two Moment theorem of Section 1 we found the following Theorem.

Theorem 4.1. Let $f: \mathbb{R}^{n} \longrightarrow \mathbb{A}_{n}$ be a continuous function with Clifford values. Let $r>0$ be fixed. If for each $x \in \mathbb{R}^{n}$,

$$
\int_{\partial B(x, r)} n(y) f(y) d S(y)=0
$$

and

$$
\int_{\partial B(x, r)} P_{i}(y-x) n(y) f(y) d S(y)=0,
$$

for

$$
P_{i}(x)=e_{1} x_{i}+e_{i} x_{1}, \quad i=2, \ldots, n,
$$

then $f$ is a left regular function.
Of course the integrals are understood in the sense of Section 2.
Proof. As before we can assume that $f$ is smooth. Applying the Clifford version of the Green formula (2.1 in Section 2) and using that $P_{i}$ is right regular, we get that

$$
\int_{B(x, r)} D f(y) d V(y)=0
$$

and

$$
\int_{B(x, r)} P_{i}(y-x) D f(y) d V(y)=0
$$

for each $x$ in $\mathbb{R}^{n}$. Let $g=D f$. Then the above conditions can be rewritten as

$$
\chi_{r} * \check{g}=0
$$

and

$$
P_{i} \chi_{r} * \check{g}=0, \quad i=2, \ldots, n,
$$

where $\chi_{r}$ denotes the characteristic function on the ball of radius $r$ and the (Clifford) convolutions are understood in the natural way.

We have a system of convolution equations for Clifford valued functions. We want to show that $g=0$ is the only solution to the system. In order to do that we first need to have a short discussion about general systems of convolution equations and present some properties of Bessel functions. We will do that in the next two sections, and then come back to the system.

### 4.3. Spectral synthesis for modules.

Given an $r$-tuple of functions $F_{1}, \ldots, F_{r} \in \widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$, the Hörmander condition, [18], gives a necessary and sufficient condition to guarantee
that the $r$-tuple generate this algebra, i.e., that there exist $G_{1}, \ldots, G_{r} \in$ $\widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$ such that $\sum G_{i} F_{i}=1$. Namely, there must exist $\varepsilon, L, B>0$, such that all $z \in \mathbb{C}^{n}$,

$$
\begin{equation*}
\left|F_{1}(z)\right|+\cdots+\left|F_{r}(z)\right| \geq \varepsilon \frac{e^{-B|\operatorname{Im} z|}}{(1+\|z\|)^{L}} . \tag{2}
\end{equation*}
$$

Given a matrix system of convolution equations

$$
\left\{\begin{array}{c}
\mu_{11} * \check{f}_{1}+\mu_{12} * \check{f}_{2}+\cdots+\mu_{1 N} * \check{f}_{N}=0 \\
\mu_{21} * \check{f}_{1}+\mu_{22} * \check{f}_{2}+\cdots+\mu_{2 N} * \check{f}_{N}=0 \\
\vdots \ddots \\
\mu_{m 1} * \check{f}_{1}+\mu_{m 2} * \check{f}_{2}+\cdots+\mu_{m N} * \check{f}_{N}=0
\end{array}\right.
$$

where $\mu_{j, i} \in \mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$ and $f_{i} \in \mathcal{E}\left(\mathbb{R}^{n}\right)$, for $i=1, \ldots, N$ and $j=1, \ldots, m$. Let $T=\left[\mu_{j, i}\right]$ be the $m \times N$ matrix of convolution operators and $f$ the vector with components $f_{i}$. We represent the above system as $T f=0$.

The representation of solutions of convolution equations is a very deep, big and delicate subject as the survey [9] shows. Here we just need a condition which guarantees that the only solution to the matrix system is $f_{i}=0$. Under technical conditions, the solutions of convolution equations have an integral Fourier representation. For us the following particular case will be sufficient.

Suppose than we can solve the equation

$$
R T=\delta \mathbb{I},
$$

where $R$ and $T$ are respectively $N \times m$ and $m \times N$ matrices with coefficients in $\mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$, i.e. $R$ is a left inverse of $T$. Then clearly in this case, the only solution to $T f=0$ is $f$ identically zero.

The above equation becomes, via Fourier transform in each entry of the matrices in the Bezout equation,

$$
M F=\mathbb{I}_{n},
$$

where $M$ and $F$ are the matrices with coefficient in $\widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$.
The existence of a solution to the Bezout equation is given by the following theorem from [19] (cf. [8]).

Theorem 4.2. Let $F$ be a $m \times N$ matrix with coefficients in the ring $\widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$. If the $N \times N$ minors of $F$ generate $\widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$, then there exists a solution $M$ of the Bezout equation $M F=\mathbb{I}_{N}$.

### 4.4. Some lemmas about Bessel functions.

Here we collect some properties of the Bessel functions and its zeros that will be used further on. Our references are [26], [14]. We assume $v>1$.

For the Bessel function $J_{v}(z)$ of real order $v$, we consider its normalized function $j_{v}(z)=J_{v}(z) / z^{v}$. Note that $j_{v}(z)$ is an entire even function and that $z=0$ is not a zero of $j_{v}(z)$. For $z \in \mathbb{C}^{n}$ we write $z^{2}=z_{1}^{2}+z_{2}^{2}+\cdots+z_{n}^{2}$.

Lemma 4.3. Let $Q(x)$ be a homogeneous, harmonic polynomial of degree $k$ in $\mathbb{R}^{n}$. Then the complexified Fourier transform of $Q \chi_{r}$ is given by

$$
\left.\mathcal{F}\left(Q \chi_{r}\right)(z)=\kappa r^{2 \mu} Q(z) j_{n / 2+k}\left(r \sqrt{z^{2}}\right)\right),
$$

where $\kappa$ is a constant depending only on $k$ and $n$; and $\mu=n / 2+k$.
Proof. The proof follows from [25, Theorem 3.10, p. 158] and a simple computation.

Remark 4.4.

1. The Macmahon's asymptotic development of the positive zeros $\alpha_{k, v}$ of $J_{v}(z):$

$$
0<\alpha_{1, v}<\alpha_{2, v}<\cdots,
$$

is given by

$$
\alpha_{k, v}=(2 k+1) \frac{\pi}{2}+(2 v+1) \frac{\pi}{4}+O\left(\frac{1}{k}\right) .
$$

2. The positive zeros of $J_{v}(z)$ are interlaced with those of $J_{v+1}(z)$

$$
0<\alpha_{1, v}<\alpha_{1, v+1}<\alpha_{2, v}<\alpha_{2, v+1}<\cdots,
$$

The next lemma will estimate the growth of $j_{v}(z)$ away from its zeroes $V_{v}$. Let $d(z, V)=\min \{1, \operatorname{dist}(z, V)\}$.

Lemma 4.5. Let $\varepsilon>0$ be given. If $d\left(z, V_{v}\right)>\varepsilon$ and $|z|$ is big enough, then

$$
\left|j_{v}(z)\right| \geq \frac{e^{|\operatorname{Im} z|}}{8 \pi e \sqrt{2 \pi}|z|^{v+3 / 2}}
$$

Proof. We use the following asymptotic development of the Bessel function $J_{v}(z)$ (see [14]),

$$
\left|J_{v}(z)-\sqrt{\frac{2}{\pi z}} \cos \left(z-\frac{\pi}{4}(2 v+1)\right)\right| \leq \frac{3 e}{8} \sqrt{\frac{\pi}{2}}\left(4 v^{2}-1\right) \frac{e^{|\operatorname{Im} z|}}{|z|^{3 / 2}}
$$

which is valid when $|z| \geq(\pi / 8)\left(4 v^{2}-1\right)$. On the other hand, the cosine satisfies the Lojasiewicz inequality

$$
|\cos z| \geq \frac{1}{\pi e} d(z, V) e^{|\operatorname{Im} z|},
$$

where $V=\{(2 l+1) \pi / 2: l \in \mathbb{Z}\}$.
It follows that if $d\left(z, V_{v}\right)>\varepsilon$, then

$$
\left|\sqrt{\frac{2}{\pi z}} \cos \left(z-\frac{\pi}{4}(2 v+1)\right)\right| \geq \sqrt{\frac{2}{\pi}} \frac{1}{\pi e} \frac{\varepsilon e^{|\operatorname{Im} z|}}{|z|^{1 / 2}} .
$$

After subtracting the bounds above and taking $|z|$ big enough, we get the desired conclusion.

### 4.5. Proof of the Moments Theorem.

Let us recall that we want to solve the system

$$
\chi_{r} * \check{g}=0
$$

and

$$
P_{i} \chi_{r} * \check{g}=0, \quad i=2, \ldots, n,
$$

where the $P_{i}$ are the regular polynomials and $g$ is a Clifford valued function.

In order to do that first we consider $\mathbb{A}_{n}$ as the matrix subalgebra of $M\left(2^{n} \times 2^{n}, \mathbb{R}\right)$. In this way we will see the system of Clifford valued convolution equations as an overdetermined matrix system of convolution equations.

First we view $\mathbb{A}_{n}$ as a matrix subalgebra of $M\left(2^{n} \times 2^{n}, \mathbb{R}\right)$ as follows [20], [15]. Consider the matrices $e_{j}:=E_{j}^{n}, j=1, \ldots, n$, where for each $1 \leq k \leq n,\left\{E_{j}^{k}\right\}_{j=1}^{k}$ are inductively defined by

$$
E_{1}^{1}:=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right),
$$

and in general for $1 \leq k \leq n-1$, and $1 \leq j \leq k$

$$
E_{j}^{k+1}:=\left(\begin{array}{cc}
E_{j}^{k} & 0 \\
0 & -E_{j}^{k}
\end{array}\right)
$$

and

$$
E_{k+1}^{k+1}:=\left(\begin{array}{cc}
0 & -\mathbb{I}_{2^{k}} \\
\mathbb{I}_{2^{k}} & 0
\end{array}\right)
$$

Then it is easy to check that the generator relations hold. Thus $\mathbb{A}_{n}$ is isomorphic to the subalgebra of $M\left(2^{n} \times 2^{n}, \mathbb{R}\right)$ consisting of all matrices generated by the $E_{j}^{n}$.

It is important to note that under this representation, the Clifford conjugation corresponds to the transposition of matrices. In particular, if $a \in \mathbb{A}_{n}$ is such that $a \bar{a} \in \mathbb{R}$ (for example for vectors), then the determinant of the corresponding matrix A is given by

$$
\operatorname{Det}(A)=(a \bar{a})^{2^{n-1}}
$$

It follows using the representation in $M\left(2^{n} \times 2^{n}, \mathbb{R}\right)$ that the system of Clifford valued convolution equations is equivalent to a matrix system of convolution equations in $\mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$. Indeed, let $T$ be the ( $n 2^{n}$ ) $\times 2^{n}$ matrix of convolution operators whose blocks $T_{i}$ are the matrices corresponding to the distributions $\chi_{r}(i=1)$ and $P_{i} \chi_{r}$ for $i=2, \ldots, n$. Let $G$ be the matrix corresponding to $g$. Thus we can write the system as

$$
T G=0,
$$

where $T \in M\left(\left(n 2^{n}\right) \times 2^{n}, \mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)\right)$ and $G \in M\left(2^{n} \times 2^{n}, \mathcal{E}\left(\mathbb{R}^{n}\right)\right)$.
Let $F$ be the $\left(n 2^{n}\right) \times 2^{n}$ matrix obtained from $T$ via Fourier Transform in each entry. We will show that the minors of $F$ generate $\widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$.

Note that the blocks $F_{i}$ correspond to the Fourier transform of the matrix representation of $P_{i} \chi_{r}$. Then from the form of $P_{i}$, Lemma 4.3 and the note above about determinants, we get that

$$
\operatorname{Det}\left(F_{i}\right)=\left(\lambda\left(z_{1}^{2}+z_{i}^{2}\right) j_{n / 2+1}\left(r \sqrt{z^{2}}\right)\right)^{2^{n-1}},
$$

for $i=2, \ldots, n$, where $\lambda$ is a constant depending only on $r$ and $n$. Similarly for the distribution $\chi_{r}$ the determinant of the matrix $F_{1}$ is given by

$$
\operatorname{Det}\left(F_{1}\right)=\left(\lambda j_{n / 2}\left(r \sqrt{z^{2}}\right)\right)^{2^{n-1}}
$$

for a constant $\lambda$ as above.
So far we have obtained the determinant of $n$ minors of $T$, we will need only one more. Note that taking a minor of $T$ is equivalent to taking a linear combination of the $P_{i} \chi_{r}$. In other words, since the $P_{i}$ are a basis of the left regular homogeneous polynomial of degree 1 , any left regular homogeneous polynomials of degree 1 can be obtained as a minor of $T$. Hence, we can repeat the argument used for the $P_{i} \chi_{r}$ said for $q \chi_{r}$ with $q=e_{2} x_{3}+e_{3} x_{2}$. We then get that the determinant of this minor $F_{n+1}$ is given by

$$
\operatorname{Det}\left(F_{n+1}\right)=\left(\lambda\left(z_{2}^{2}+z_{3}^{2}\right) j_{n / 2+1}\left(r \sqrt{z^{2}}\right)\right)^{2^{n-1}}
$$

We will drop the exponent $2^{n-1}$ from these functions as they are not relevant.

It is clear that the functions $f_{i}:=\operatorname{Det}\left(F_{i}\right)$ for $i=1, \ldots, n+1$, have no common zeros because the two Bessel functions which appear have no common, zeros and the polynomials have no common zeros. Moreover, we claim that the set $\left\{f_{i}\right\}$ generate $\widehat{\mathcal{E}}^{\prime}\left(\mathbb{R}^{n}\right)$.

Since the zeroes of $j_{n / 2}$ and $j_{n / 2+1}$ interlace, and they are separated from each other by a fixed number (see Remark 4.4), we can find an estimate as in Lemma 4.5 that works for the sum of the two functions. Thus for all $w \in \mathbb{C}$,

$$
\left|j_{n / 2}(r w)\right|+\left|j_{n / 2+1}(r w)\right| \geq \frac{\kappa e^{-|\operatorname{Im} w|}}{(1+|w|)^{n+5 / 2}}
$$

where $\kappa$ is a positive constant. Now note that for $z \in \mathbb{C}^{n}$,

$$
\left|\operatorname{Im} \sqrt{z^{2}}\right| \leq|\operatorname{Im} z|
$$

It follows that for all $z \in \mathbb{C}^{n}$,

$$
\left|j_{n / 2}\left(r \sqrt{z^{2}}\right)\right|+\left|j_{n / 2+1}\left(r \sqrt{z^{2}}\right)\right| \geq \frac{\kappa e^{-|\operatorname{Im} z|}}{(1+\|z\|)^{n+5 / 2}}
$$

Now for a set of polynomials, the Hörmander condition (4.3) is equivalent to that the polynomials have no common zeros. In that case, we
can take $B=0$. It then follows from this and the above inequality that the set of functions $\left\{f_{i}\right\}$ satisfies the Hörmander condition.

Applying Theorem 4.2, the proof is completed.
Remark 4.6. We need all the first moments in the theorem. Indeed if we have less of the $P_{i} \chi_{r}$, the respective convolution system will have a non-zero solution.

Remark 4.7. It follows from the proof of the theorem that for the moments of order greater than one, what we need is that the corresponding minors have no common zeros. This will follow from dimensionality.

## 5. Conclusions.

There are many directions for which the type of problems we have considered could be investigated. This includes the study in other spaces, other operators of Dirac type or more concrete surfaces.

As we showed in Section Four some of the results in the plane generalize to the Clifford analysis setting but the proofs are more involved than the ones for the case of the plane. Hence some difficulties are expected for the other variations. Of course, it would not be possible to recover all the results in the plane in part because there is no Riemann Mapping Theorem when $n>2$. For instance, for the case of higher order moments we can only offer the following remarks.

Using the Premelj formulas [21] and the Taylor series expansion for regular functions (see [11]), it is easy to show that if $S$ is a Jordan surface and $f$ is a continuous function defined on $S$ with Clifford values then $f$ can be extended to a left regular function inside $S$ if and only if

$$
\int_{S} V_{l_{1}, \ldots, l_{k}}(x) n(x) f(x) d S(x)=0
$$

for every $k$ and for every homogeneous regular polynomial $V_{l_{1}, \ldots, l_{k}}(x)$ of degree $k$. This means that a function could be extended to be left regular inside a surface if and only if all its Clifford moments vanish. Using this we can formulate the general version of the moments problem as follows.

Let $S$ be a Jordan surface and let $f \in \mathcal{C}\left(\mathbb{R}^{n}, \mathbb{A}_{n}\right)$. Suppose that for every $\sigma \in M(n), f$ can be extended to be left regular inside $\sigma S$. Does
it follow that $f$ is left regular? that is if

$$
\int_{\sigma S} V_{l_{1}, \ldots, l_{k}}(x) n(x) f(x) d S(x)=0
$$

for every $k$, and for every homogeneous regular polynomial $V_{l_{1}, \ldots, l_{k}}(x)$ of degree $k$, and for every $\sigma \in M(n)$ is then $f$ left regular? As we mention in Section One, the proof for the complex case relies on the argument principle. But in Clifford analysis there is no argument principle.

It is shown in [3] that we do not need vanishing of moments but only that they do not grow too fast. Whether or not this is true in the situation of Section Four is another interesting problem.

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# Non-symmetric hitting distributions on the hyperbolic half-plane and subordinated perpetuities 

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#### Abstract

We study the law of functionals whose prototype is $\int_{0}^{+\infty} e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}$, where $B^{(\nu)}, W^{(\mu)}$ are independent Brownian motions with drift. These functionals appear naturally in risk theory as well as in the study of invariant diffusions on the hyperbolic half-plane. Emphasis is put on the fact that the results are obtained in two independent, very different fashions (invariant diffusions on the hyperbolic half-plane and Bessel processes).


## 1. Introduction.

Let $W_{t}, B_{t}$ be two independent one-dimensional Brownian motions, and set

$$
W_{t}^{(\mu)}=W_{t}-\mu t, \quad B_{t}^{(\nu)}=B_{t}-\nu t,
$$

where $\nu>0$ and $\mu \in \mathbb{R}$. In this paper we prove some results concerning the distribution of the random variable.

$$
\begin{equation*}
\int_{0}^{+\infty} e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)} \tag{1.1}
\end{equation*}
$$

First we prove that it has a density given by

$$
\begin{equation*}
f(x)=c_{\mu, \nu} \frac{e^{-2 \mu \arctan x}}{\left(1+x^{2}\right)^{\nu+1 / 2}} \tag{1.2}
\end{equation*}
$$

which belongs to the type IV family of Pearson distributions. The functional (1.1) has been much studied because it appears in risk theory. The density (1.2) was derived in [P, Example 3.1], with a proof for $\nu>1$ only; easy derivations for $\nu>0$ in the particular case $\mu=0$ can be found in [BCF, Remark 4.1] (if, in addition, $\nu$ is a half integer see also [AG, p. 32]). Interestingly, random variables as in (1.1) also appear in connection with invariant diffusions on the hyperbolic halfplane $\Pi=\{z \in \mathbb{C}: \operatorname{Im} z>0\}$.

On $\Pi$ consider the diffusion process associated to the infinitesimal generator

$$
L=\frac{y^{2}}{2} \Delta-\mu y \frac{\partial}{\partial x}-\left(\nu-\frac{1}{2}\right) y \frac{\partial}{\partial y},
$$

where the real coefficients $\mu$ and $\nu-1 / 2$ measure the horizontal, respectively vertical component of the drift (positive for leftward and downward drift, negative for rightward and upward drift). The differential operator $L$ is invariant under the orientation-preserving isometries of $\Pi$ that fix the point at infinity $\infty$, that is, under the real affine transformations $z \longmapsto a z+b$ with $a>0$ and $b \in \mathbb{R}$. The diffusion process associated to $L$ corresponds to the stochastic differential equation

$$
\left\{\begin{array}{l}
d X_{t}=Y_{t} d W_{t}-\mu Y_{t} d t  \tag{1.3}\\
d Y_{t}=Y_{t} d B_{t}-\left(\nu-\frac{1}{2}\right) Y_{t} d t
\end{array}\right.
$$

where, as before, $W_{t}, B_{t}$ are independent one-dimensional Brownian motions. The solution of (1.3) with starting point $i y=(0, y)$ is

$$
\left\{\begin{array}{l}
Y_{t}=y e^{B_{t}^{(\nu)}}  \tag{1.4}\\
X_{t}=\int_{0}^{t} y e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}
\end{array}\right.
$$

Consider the hitting distribution of the diffusion associated to $L$ and starting at $x+i y$ on any horizontal line $H_{a}=\{\operatorname{Im} z=a\}$ with $0 \leq a<$ $y$. For $a=0$ the line $H_{a}$ is the boundary portion $\partial \Pi \backslash\{\infty\}$ (in this case the expression "hitting distribution" is a slight abuse of terminology),
while for $a>0$ it is a horocycle through $\infty$. Thus the law of the random variable (1.1) is the hitting distribution of the diffusion associated to $L$ in $H_{0}$ and starting at $i$. If $a>0$ the hitting distribution is given by the law of the random variable

$$
\int_{0}^{\tau_{a}} y e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}
$$

where $\tau_{a}=\inf \left\{t \geq 0: Y_{t}=a\right\}$ is the hitting time on $H_{a}$.
In this paper we prove (1.2) and compute the characteristic function of $f$ in two different fashions.

One is based on a computation of the Poisson kernel of the infinitesimal generator associated to the process. Exploiting the invariance, this kernel can be written in terms of a single function of one real variable that satisfies a second-order linear ordinary differential equation and is determined explicitly. Conjugating by the inverse Fourier transform another second-order linear ordinary differential equation is obtained whose solution is a confluent hypergeometric function and the characteristic function of the hitting distribution. This is done in Section 2.

The second method uses probabilistic techniques (mostly classical properties of Bessel processes) and is the object of Section 4. It is based on the representation formulae (1.3), (1.4), and uncovers interesting relations between Brownian exponential functionals and previous work of Ph. Biane, J. Pitman, and the fourth-named author on Bessel processes (see [PY1], [PY2], and the references therein).

In Section 3 we discuss an alternate derivation of the ordinary differential equation satisfied by the characteristic function, by means of the Feynman-Kac formula.

In Section 5 we prove that, as the parameters $\mu, \nu$ as well as the coordinates of the starting point of the process take their admissible values (namely $\mu \in \mathbb{R}, \nu>0$ and $\operatorname{Im} z>0$ ), the corresponding hitting distributions belong to the domain of attraction (extended domain of attraction for $\nu=1$ ) of nearly all stable laws with exponent $\alpha=\min \{2,2 \nu\}$, for $0<\alpha \leq 2$.

Finally, Section 6 is devoted to the study of the hitting distribution on $H_{a}$ for $y>a>0$. Using the invariance properties of the diffusion process and the strong Markov property it is possible to derive an expression for the characteristic function of this distribution, and to prove that it still belongs to the domain of attraction of a stable law with exponent $\alpha=2 \nu$. However, in this case we are not able to give an explicit expression for the density.

## 2. The hitting distribution on $H_{0}$ and its characteristic function.

We perform computations both on the hyperbolic half-plane $\Pi$ and the Poincaré disk $D$. They are isomorphic via the Cayley map $z=$ $i(1-w) /(1+w)$ (where $z \in \Pi$ and $w \in D)$, which corresponds to $\xi=\tan (\phi / 2)$ on the boundaries, with $\xi \in \mathbb{R} \cup\{\infty\}=\partial \Pi$ and $\left\{e^{i \phi}:\right.$ $-\pi<\phi \leq \pi\}=\partial D$.

The density $P(\xi, z)$ at $\xi$ of the hitting distribution on $\mathbb{R}$ of the process associated to the operator $L$ and starting at $z \in \Pi$ is called the Poisson kernel of $L$ in the domain $\Pi$, and satisfies the following conditions:

1) $L_{x, y} P(\xi, x+i y) \equiv 0$ for all $\xi \in \mathbb{R}$;
2) $P(\xi, z)>0$ for all $\xi \in \mathbb{R}$ and $z \in \Pi$;
3) $\int_{\mathbb{R}} P(\xi, z) d \xi=1$ for all $z \in \Pi$;
4) $\lim _{y \rightarrow 0^{+}} P(\xi, x+i y)=0$ if $\xi \neq x$ and $\xi, x \in \mathbb{R}$.

Since $L$ is invariant under the maps $z \longmapsto a z+b$, then so is the measure $P(\xi, z) d \xi$ on $\mathbb{R}$ for the diagonal action of the same maps, that is, $P(\xi, z)=a P(a \xi+b, a z+b)$. Setting $f(x)=P(x, i)$, we therefore have

$$
\begin{equation*}
P(\xi, x+i y)=\frac{1}{y} P\left(\frac{\xi-x}{y}, i\right)=\frac{1}{y} f\left(\frac{\xi-x}{y}\right) . \tag{2.1}
\end{equation*}
$$

In other words, the hitting distribution with arbitrary starting point is obtained, by a simple rescaling, from the one starting at $i$.

The differential operator on $D$ corresponding to $L$ is invariant under the maps $w \longmapsto((1+a+i b) w+(1-a+i b)) /((1-a-i b) w+$ $(1+a-i b))$. Its Poisson kernel $Q$ satisfies

$$
\begin{aligned}
Q(\phi, w)= & \frac{a\left(1+\tan ^{2} \frac{\phi}{2}\right)}{1+\left(a \tan \frac{\phi}{2}+b\right)^{2}} \\
& \cdot Q\left(2 \arctan \left(a \tan \frac{\phi}{2}+b\right), \frac{(1+a+i b) w+(1-a+i b)}{(1-a-i b) w+(1+a-i b)}\right),
\end{aligned}
$$

so that, if $g(\phi)=Q(\phi, 0)$, then

$$
Q(\phi, w)=\frac{1-|w|^{2}}{\left|e^{i \phi}-w\right|^{2}} g\left(2 \arctan \frac{|1+w|^{2} \tan \frac{\phi}{2}-2 \operatorname{Im} w}{1-|w|^{2}}\right)
$$

Furthermore $f(x)=2 g(2 \arctan x) /\left(1+x^{2}\right)$.
Condition 1) can be translated for $f$ using (2.1), then setting $\xi=0$ (since $L$ is autonomous in $x$ this may also be done beforehand) and $y=$ 1. The result is the second-order linear ordinary differential equation $M f=0$, with

$$
\begin{align*}
M f(x) & =\frac{1+x^{2}}{2} f^{\prime \prime}(x)+\left(\mu+\left(\nu+\frac{3}{2}\right) x\right) f^{\prime}(x)+\left(\nu+\frac{1}{2}\right) f(x)  \tag{2.2}\\
& =\frac{d}{d x}\left(\frac{d}{d x}\left(\frac{1+x^{2}}{2} f(x)\right)+\left(\mu+\left(\nu-\frac{1}{2}\right) x\right) f(x)\right)
\end{align*}
$$

proportional to

$$
\frac{d}{d \phi}\left(\left(g^{\prime}(\phi)+\left(\mu+\left(\nu-\frac{1}{2}\right) \tan \frac{\phi}{2}\right) g(\phi)\right) \cos ^{2} \frac{\phi}{2}\right)
$$

if $\phi=2 \arctan x$. The first-order linear equation obtained by equating the expression in square brackets to a constant multiple of $\cos ^{-2}(\phi / 2)$ is solved by
$g(\phi)=\left(\frac{c}{2}+k \int_{0}^{\phi} e^{\mu \phi} \cos ^{-2 \nu-1} \frac{\phi}{2} d \phi\right) e^{-\mu \phi} \cos ^{2 \nu-1} \frac{\phi}{2}, \quad$ with $c, k \in \mathbb{R}$.
Since $P$, whence $f, Q, g$, must be positive by condition 2 ) and since for $\phi \in(-\pi, \pi)$ the above integral takes arbitrarily large values of either sign because $\nu>0$, then $k=0$ and $f$ is given by (1.2).

Since $\int_{-\pi}^{\pi} g=1$ as a consequence of condition 3), then by [GR, 3.892 .2 and 8.384.1] and the basic properties of the Euler Gamma function we have

$$
\begin{align*}
c & =c_{\mu, \nu} \\
& =2\left(\int_{-\pi}^{\pi} e^{-\mu \phi} \cos ^{2 \nu-1} \frac{\phi}{2} d \phi\right)^{-1}  \tag{2.3}\\
& =\frac{2^{2 \nu-1}\left|\Gamma\left(\frac{1}{2}+\nu-i \mu\right)\right|^{2}}{\pi \Gamma(2 \nu)}
\end{align*}
$$

In particular, by [GR, 8.332.2-3]

$$
c_{\mu, 1 / 2}=\frac{\mu}{\sinh \mu \pi}, \quad c_{\mu, 1}=\frac{\frac{1}{2}+2 \mu^{2}}{\cosh \mu \pi}
$$

and, more generally,
$c_{\mu, \nu}= \begin{cases}\frac{2^{2 \nu-1}}{(2 \nu-1)!\mu \sinh \mu \pi} \prod_{j=0}^{\nu-1 / 2}\left(j^{2}+\mu^{2}\right), & \text { if } \nu=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \\ \frac{2^{2 \nu-1}}{(2 \nu-1)!\cosh \mu \pi} \prod_{j=0}^{\nu-1}\left(\left(j+\frac{1}{2}\right)^{2}+\mu^{2}\right), & \text { if } \nu=1,2,3, \ldots,\end{cases}$
as can also be checked by elementary means from the integral expression of $c_{\mu, \nu}$. On the other hand, for $\mu=0$ from [GR, 8.335.1] we have

$$
c_{0, \nu}=\frac{\Gamma\left(\frac{1}{2}+\nu\right)}{\sqrt{\pi} \Gamma(\nu)} .
$$

We now compute the characteristic function of the hitting distribution. Again by invariance, the expression for an arbitrary starting point $x+$ $i y \in \Pi$ can be derived from the special case of starting point $i$. Indeed, if $u=\mathcal{F}^{-1} f$ is the inverse Fourier transform of $f$, then by (2.1) the required characteristic function is $\mathcal{F}^{-1} P(\cdot, x+i y)=e^{i \lambda x} u(\lambda y)$.

We have $u(-\lambda)=\overline{u(\lambda)}$ because $f$ is real-valued, and $u(0)=1$ by condition 3). Moreover, for $k=0,1,2$ the function $x^{k} f^{(k)}(x)$ is integrable, whence $\lambda^{k} u^{(k)}(\lambda)$ (exists and) is continuous, and vanishes at infinity; in particular, $u$ is continuous on $\mathbb{R}$ and twice continuously differentiable outside 0 , and vanishes at infinity. Thus $u$ is in the kernel of the operator $N=\mathcal{F}^{-1} M \mathcal{F}$, given by

$$
\begin{equation*}
N u(\lambda)=\frac{\lambda^{2}}{2} u^{\prime \prime}(\lambda)-\left(\nu-\frac{1}{2}\right) \lambda u^{\prime}(\lambda)-\left(\frac{\lambda^{2}}{2}+i \mu \lambda\right) u(\lambda) . \tag{2.4}
\end{equation*}
$$

With the change of variables $v(w)=e^{w / 2} u(w / 2)$ the equation $N u=0$ becomes

$$
\begin{align*}
& w v^{\prime \prime}(w)+(b-w) v^{\prime}(w)-a v(w)=0, \\
& \text { where }\left\{\begin{array}{l}
a=\frac{1}{2}-\nu+i \mu, \\
b=1-2 \nu
\end{array}\right. \tag{2.5}
\end{align*}
$$

This is a confluent hypergeometric equation in one of its standard forms [EMOT, Chapter VI], [T], and its solutions are called confluent hypergeometric functions. One solution for $w>0$ is the Tricomi $\Psi$-function, defined as in [EMOT, 6.11.(13)] by
$\Psi(a, b ; w)=\frac{2^{1-b} \Gamma(1-a) e^{w / 2}}{\pi} \int_{0}^{\pi / 2} \cos \left(\frac{w}{2} \tan \theta+(2 a-b) \theta\right) \cos ^{-b} \theta d \theta$.

For this formula to hold it is required that $\operatorname{Re} b<1$ and $a$ is not a positive integer, both of which hold for $a, b$ given in (2.5). Since $\Psi(a, b ; w)$ has a finite non-zero limit for $w \rightarrow 0^{+}$and since

$$
\lim _{w \rightarrow+\infty} e^{-w / 2} \Psi(a, b ; w)=0
$$

[EMOT, 6.13.(1)], then the solution of $N u=0$ we are looking for is, for $\lambda>0$, a multiple of $e^{-\lambda} \Psi(a, b ; 2 \lambda)$. After some obvious manipulations (2.3) gives

$$
\frac{1}{2 c_{\mu, \nu}}=\int_{0}^{\pi / 2} \cos (2 i \mu \theta) \cos ^{2 \nu-1} \theta d \theta
$$

so that

$$
\begin{aligned}
\Psi\left(\frac{1}{2}-\nu+i \mu,\right. & -2 \nu ; 0) \\
& =\frac{2^{2 \nu} \Gamma\left(\frac{1}{2}+\nu-i \mu\right)}{\pi} \int_{0}^{\pi / 2} \cos (2 i \mu \theta) \cos ^{2 \nu-1} \theta d \theta \\
& =\frac{1}{2 c_{\mu, \nu} \pi} 2^{2 \nu} \Gamma\left(\frac{1}{2}+\nu-i \mu\right) \\
& =\frac{\Gamma(2 \nu)}{\Gamma\left(\frac{1}{2}+\nu+i \mu\right)}
\end{aligned}
$$

We summarize the results of this section:
Proposition 2.1. For every $\nu>0$ and $\mu \in \mathbb{R}$, as $t \longrightarrow+\infty$ the distribution of $X_{t}$ with starting point $i$ converges to the probability defined by the density (1.2). Its characteristic function is

$$
\begin{equation*}
\frac{\Gamma\left(\frac{1}{2}+\nu+i \mu\right)}{\Gamma(2 \nu)} e^{-\lambda} \Psi\left(\frac{1}{2}-\nu+i \mu, 1-2 \nu ; 2 \lambda\right) \tag{2.6}
\end{equation*}
$$

where $\Psi$ is the Tricomi $\Psi$-function.

## 3. Use of the Feynman-Kac formula.

We now prove in a different way that the characteristic function $u$ of $X_{\infty}$ with starting point $\left(X_{0}, Y_{0}\right)=(0,1)$, i.e., thanks to (1.1), the function

$$
u(\lambda)=\mathbf{E}_{0,1}\left[e^{i \lambda X_{\infty}}\right]=\mathbf{E}\left[\exp \left(i \lambda \int_{0}^{\infty} e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}\right)\right]
$$

is in the kernel of the ordinary differential operator (2.4). Henceforth we denote by $P_{x, y}$ the law of the diffusion with starting point $\left(X_{0}, Y_{0}\right)=(x, y)$ and by $\mathbf{E}_{x, y}$ the corresponding expectation. Since the $Y$-component is independent from $W$, then

$$
\begin{aligned}
u(\lambda) & =\mathbf{E}_{0,1}\left[\exp \left(i \lambda \int_{0}^{\infty} Y_{s} d W_{s}-i \mu \lambda \int_{0}^{\infty} Y_{s} d s\right)\right] \\
& =\mathbf{E}_{0,1}\left[\exp \left(-\frac{\lambda^{2}}{2} \int_{0}^{\infty} Y_{s}^{2} d s-i \mu \lambda \int_{0}^{\infty} Y_{s} d s\right)\right] \\
& =\mathbf{E}\left[\exp \int_{0}^{\infty}-\left(\frac{\left(\lambda e^{B_{s}^{(\nu)}}\right)^{2}}{2}+i \mu \lambda e^{B_{s}^{(\nu)}}\right) d s\right] \\
& =\mathbf{E}_{0, \lambda}\left[\exp \int_{0}^{\infty} G\left(Y_{s}\right) d s\right]
\end{aligned}
$$

where $G(y)=-y^{2} / 2-i \mu y$.
Proposition 3.1. Let $N$ be given by (2.4), and let $u(\lambda)$ be a solution for $\lambda>0$ of $N u=0$ such that

$$
\begin{aligned}
& \lim _{\lambda \rightarrow 0^{+}} u(\lambda)=1 \\
& \lim _{\lambda \rightarrow \infty} u(\lambda)=0
\end{aligned}
$$

Extend $u$ to the negative half-line by setting $u(\lambda)=\overline{u(-\lambda)}$ for $\lambda<0$. Then $u(\lambda)=\mathbf{E}_{0,1}\left[e^{i \lambda X_{\infty}}\right]$.
(Unlike in the previous section, we require $\lim _{\lambda \rightarrow \infty} \lambda^{k} u(\lambda)=0$ only for $k=0$.)

Proof. For $0<a<1$ let

$$
\sigma_{a}=\inf \left\{t \geq 0: Y_{t}=a \text { or } Y_{t}=\frac{1}{a}\right\}
$$

be the exit time of $Y_{t}$ from the interval $(a, 1 / a)$. Then the Feynman-Kac formula gives

$$
u(\lambda)=\mathbf{E}_{0, \lambda}\left[u\left(Y_{\sigma_{a}}\right) \exp \int_{0}^{\sigma_{a}} G\left(Y_{s}\right) d s\right]
$$

As $a \longrightarrow 0$ we have $\sigma_{a} \longrightarrow+\infty$ and $u\left(Y_{\sigma_{a}}\right) \longrightarrow u(0)=1$ almost surely.

## 4. A probabilistic computation of the hitting distribution.

We shall now compute again the law of $\int_{0}^{\infty} e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}$, as a consequence of the following three simple observations.

1) For a fixed real number $x$ (the starting point), consider the two processes

$$
\begin{aligned}
& X_{t}^{(\mu, \nu)}=e^{B_{t}^{(\nu)}} x+\int_{0}^{t} e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)} \\
& \widetilde{X}_{t}^{(\mu, \nu)}=e^{B_{t}^{(\nu)}}\left(x+\int_{0}^{t} e^{-B_{s}^{(\nu)}} d W_{s}^{(\mu)}\right) .
\end{aligned}
$$

Then $X_{t}^{(\mu, \nu)}, \widetilde{X}_{t}^{(\mu, \nu)}$ have the same law for every fixed $t$ (although the two processes do not have the same law). More generally this holds whenever $B, W$ are independent Lévy processes [CPY, Lemma 2.3].
2) The process $\left(\widetilde{X}_{t}^{(\mu, \nu)}, t \geq 0\right)$ is a diffusion process with generator

$$
M^{*}=\frac{1+x^{2}}{2} \frac{d^{2}}{d x^{2}}-\left(\mu+\left(\nu-\frac{1}{2}\right) x\right) \frac{d}{d x}
$$

the adjoint of the operator $M$ given in (2.2).
3) The distribution at time $t$ of this diffusion process converges to the invariant distribution, whose density $f(x)$ is given in (1.2).

Proof of 2). By Itô's formula

$$
\begin{aligned}
d \widetilde{X}_{t}^{(\mu, \nu)} & =\widetilde{X}_{t}^{(\mu, \nu)}\left(d B_{t}^{(\nu)}+\frac{d t}{2}\right)+e^{B_{t}^{(\nu)}} e^{-B_{t}^{(\nu)}} d W_{t}^{(\mu)} \\
& =\widetilde{X}_{t}^{(\mu, \nu)} d B_{t}+d W_{t}-\left(\mu+\left(\nu-\frac{1}{2}\right) \widetilde{X}_{t}^{(\mu, \nu)}\right) d t
\end{aligned}
$$

from which one derives easily that $\widetilde{X}^{(\mu, \nu)}$ is a diffusion process with a generator as stated.

Proof of 3). From 1), $\widetilde{X}_{t}^{(\mu, \nu)}$ converges in law as $t \longrightarrow+\infty$, since $\widetilde{X}_{t}^{(\mu, \nu)} \stackrel{\text { law }}{\sim} X_{t}^{(\mu, \nu)}$, and $X_{t}^{(\mu, \nu)} \longrightarrow X_{\infty}^{(\mu, \nu)}$. It is easy to see that the limit distribution is invariant, that is, it is annihilated by $M$, whence it necessarily coincides with $f$.

As remarked in Section 3, the hitting distribution on $H_{0}$ under $P_{0, y}$ is the law of the random variable $\int_{0}^{\infty} y e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}$. If we set

$$
A_{\infty}^{(\nu)}=\int_{0}^{\infty} e^{2 B_{s}^{(\nu)}} d s, \quad A_{\infty}^{(\nu, 1)}=\int_{0}^{\infty} e^{B_{s}^{(\nu)}} d s
$$

then $X_{\infty}$ can be written in the form of a subordinated perpetuity as

$$
X_{\infty}=\gamma_{A_{\infty}^{(\nu)}}-\mu A_{\infty}^{(\nu, 1)},
$$

where $\gamma$ is a Brownian motion independent of $B, W$. It is thus clear that the law of $X_{\infty}$ is the same as that of

$$
Z \sqrt{A_{\infty}^{(\nu)}}-\mu A_{\infty}^{(\nu, 1)}
$$

where $Z$ is an $N(0,1)$ random variable, independent of $B, W$. If $h$ is any bounded Borel function on $\mathbb{R}$, then

$$
\begin{aligned}
\mathbf{E}\left[h\left(X_{\infty}\right)\right] & =\mathbf{E}\left[h\left(Z \sqrt{A_{\infty}^{(\nu)}}-\mu A_{\infty}^{(\nu, 1)}\right)\right] \\
& =\int_{\mathbb{R}} \frac{e^{-z^{2} / 2}}{\sqrt{2 \pi}} \mathbf{E}\left[h\left(z \sqrt{A_{\infty}^{(\nu)}}-\mu A_{\infty}^{(\nu, 1)}\right)\right] d z \\
& =\mathbf{E}\left[\int_{\mathbb{R}} \frac{h(x)}{\sqrt{2 \pi A_{\infty}^{(\nu)}}} e^{-\left(x+\mu A_{\infty}^{(\nu, 1)}\right)^{2} /\left(2 A_{\infty}^{(\nu)}\right)} d x\right] .
\end{aligned}
$$

Since $h$ is arbitrary, this yields a representation formula for the density $f$ of $X_{\infty}$ :

Theorem 4.1. We have

$$
f(x)=\mathbf{E}\left[\frac{1}{\sqrt{2 \pi A_{\infty}^{(\nu)}}} e^{-\left(x+\mu A_{\infty}^{(\nu, 1)}\right)^{2} /\left(2 A_{\infty}^{(\nu)}\right)}\right]
$$

Next comes a representation formula of the density $f$ in terms of Bessel processes. The main tool is Lamperti's representation formula for the geometric Brownian motion [RY, Exercise 11.1.28], which states that

$$
\begin{equation*}
e^{B_{s}^{(\nu)}}=R_{A_{s}^{(\nu)}}^{(-\nu)} \tag{4.1}
\end{equation*}
$$

where $R^{(-\nu)}$ is a Bessel process with index $-\nu$. Taking $s \longrightarrow+\infty$ in this relation, since the left-hand side tends to 0 one has $R_{A_{\infty}^{(\nu)}}^{(-\nu)}=0$, so that $A_{\infty}^{(\nu)}$ coincides with the first passage time $T_{0}\left(R^{(-\nu)}\right)$ of $R^{(-\nu)}$ by 0 . Moreover one can write

$$
A_{\infty}^{(\nu, 1)}=\int_{0}^{\infty} e^{B_{s}^{(\nu)}} d s=\int_{0}^{\infty} \frac{d A_{s}^{(\nu)}}{e^{B_{s}^{(\nu)}}}=\int_{0}^{\infty} \frac{d A_{s}^{(\nu)}}{R_{A_{s}^{(\nu)}}^{(-\nu)}}=\int_{0}^{T_{0}\left(R^{(-\nu)}\right)} \frac{d u}{R_{u}^{(-\nu)}}
$$

This can be summarized by stating that

$$
\begin{equation*}
\left(A_{\infty}^{(\nu)}, A_{\infty}^{(\nu, 1)}\right) \stackrel{\text { law }}{\simeq}\left(T_{0}\left(R^{(-\nu)}\right), \int_{0}^{T_{0}\left(R^{(-\nu)}\right)} \frac{d u}{R_{u}^{(-\nu)}}\right) . \tag{4.2}
\end{equation*}
$$

Theorem 4.2. Denote by $P^{(\nu)}$ the law of the transient Bessel process $\left(R_{u}^{(\nu)}, u \geq 0\right)$ with dimension $d=2(\nu+1)$ starting at 0 , and define

$$
H_{s}=\int_{0}^{s} \frac{d u}{R_{u}^{(\nu)}} .
$$

Then

$$
f(x)=\sqrt{\frac{2}{\pi}} \mathbf{E}^{(\nu)}\left[\frac{\nu}{R_{1}^{(\nu)}} e^{-\left(x R_{1}^{(\nu)}+\mu H_{1}\right)^{2} / 2}\right] .
$$

In the particular case $\nu=1 / 2$ we have that $\left(R_{u}^{(\nu)}, u \geq 0\right)$ is the 3 -dimensional Bessel process starting at 0 .

Proof. Recall D. Williams' time reversal result, which states that

$$
\left(R_{T_{0}\left(R^{(-\nu)}\right)-u}^{(-\nu)}, u \leq T_{0}\left(R^{(-\nu)}\right)\right) \stackrel{\text { law }}{\simeq}\left(R_{u}^{(\nu)}, u \leq L_{1}\left(R^{(\nu)}\right)\right),
$$

where the right-hand side denotes the Bessel process with index $\nu$ starting from 0 , and $L_{1}$ its last passage time by 1 . Thus (4.2) may be written

$$
\left(A_{\infty}^{(\nu)}, A_{\infty}^{(\nu, 1)}\right) \stackrel{\text { law }}{\sim}\left(L_{1}\left(R^{(\nu)}\right), \int_{0}^{L_{1}\left(R^{(\nu)}\right)} \frac{d u}{R_{u}^{(\nu)}}\right)
$$

It is now sufficient to use a result of absolute continuity between the laws of

$$
\left(\frac{R_{u L_{1}}^{(\nu)}}{\sqrt{L_{1}}}, u \leq 1\right)
$$

and of $\left(R_{u}^{(\nu)}, u \leq 1\right)$, a transient Bessel process starting at 0 [BLY, Théorème 3], [Y, sections 2 and 4].

Techniques based on Bessel processes give also an alternative proof of expression (2.6) for the characteristic function of the hitting distribution.

We assume $y=1$. Lamperti's representation formula (4.1) implies also that $A_{\tau_{x}}^{(\nu)}=T_{x}\left(R^{(-\nu)}\right)$, where we denote by $\tau_{x}$ the hitting time in $x$ of $Y_{t}=e^{B_{t}^{(\nu)}}$ and $T_{x}\left(R^{(-\nu)}\right)$ the hitting time in $x$ of the Bessel process $R^{(-\nu)}$. The same arguments leading to (4.2) give

$$
\left(A_{\tau_{x}}^{(\nu)}, A_{\tau_{x}}^{(\nu, 1)}\right) \stackrel{\text { law }}{\sim}\left(T_{x}\left(R^{(-\nu)}\right), \int_{0}^{T_{x}\left(R^{(-\nu)}\right)} \frac{d u}{R_{u}^{(-\nu)}}\right) .
$$

Thus for $\theta \in \mathbb{R}$, using [PY1, Proposition 12.2, p. 363] (see [PY2] for more information) and the expression of the density of the law of a Bessel process with index $-\nu$ with respect to the law of a Bessel process with index 0 (see, e.g., [RY, Exercise 11.1.18]), we have

$$
\begin{aligned}
\mathbf{E}\left[e^{i \theta X_{\tau_{x}}}\right] & =\mathbf{E}\left[\exp \left(i \theta \int_{0}^{\tau_{x}} e^{B_{s}^{(\nu)}} d \gamma_{s}-i \theta \mu \int_{0}^{\tau_{x}} e^{B_{s}^{(\nu)}} d s\right)\right] \\
& =\mathbf{E}\left[\exp \left(-\frac{\theta^{2}}{2} \int_{0}^{\tau_{x}} e^{2 B_{s}^{(\nu)}} d s-i \theta \mu \int_{0}^{\tau_{x}} e^{B_{s}^{(\nu)}} d s\right)\right] \\
& =\mathbf{E}_{1}^{(-\nu)}\left[\exp \left(-\frac{\theta^{2}}{2} T_{x}\left(R^{(-\nu)}\right)-i \theta \mu \int_{0}^{T_{x}\left(R^{(-\nu)}\right)} \frac{d u}{R_{u}^{(-\nu)}}\right)\right]
\end{aligned}
$$

$$
=x^{-\nu+1 / 2} \frac{W_{-i \mu, \nu}(2 \theta)}{W_{-i \mu, \nu}(2 \theta x)}
$$

where $W$., . denotes the Whittaker functions. This gives the characteristic function of the hitting distribution on $H_{x}$. Recalling the relation between the Whittaker and Tricomi $\Psi$-functions [T, 2.4.(5)]

$$
W_{k, \nu}(z)=z^{\nu+1 / 2} e^{-z / 2} \Psi\left(\frac{1}{2}+\nu-k, 1+2 \mu ; z\right)
$$

and a functional property of the $\Psi$ function [ $\mathrm{T}, 2.3 .(9)$ ]

$$
\Psi(a-c+1,2-c ; z)=z^{c-1} \Psi(a, c ; z),
$$

we get

$$
\begin{aligned}
W_{-i \mu, \nu}(2 \theta) & =(2 \theta)^{\nu+1 / 2} e^{-\theta} \Psi\left(\frac{1}{2}+\nu+i \mu, 1+2 \nu ; 2 \theta\right) \\
& =(2 \theta)^{-\nu+1 / 2} e^{-\theta} \Psi\left(\frac{1}{2}-\nu+i \mu, 1-2 \nu ; 2 \theta\right),
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathbf{E}\left[e^{i \theta X_{\tau_{x}}}\right] & =x^{-\nu+1 / 2} \frac{(2 \theta)^{-\nu+1 / 2} e^{-\theta} \Psi\left(\frac{1}{2}-\nu+i \mu, 1-2 \nu ; 2 \theta\right)}{(2 \theta x)^{-\nu+1 / 2} e^{-\theta x} \Psi\left(\frac{1}{2}-\nu+i \mu, 1-2 \nu ; 2 \theta x\right)} \\
& =\frac{e^{-\theta} \Psi\left(\frac{1}{2}-\nu+i \mu, 1-2 \nu ; 2 \theta\right)}{e^{-\theta x} \Psi\left(\frac{1}{2}-\nu+i \mu, 1-2 \nu ; 2 \theta x\right)}
\end{aligned}
$$

This gives the characteristic function of the hitting distribution on the horocycle $H_{x}$. Taking $x \longrightarrow 0^{+}$one gets easily

$$
\mathbf{E}\left[e^{i \theta X_{\infty}}\right]=\frac{e^{-\theta} \Psi\left(\frac{1}{2}-\nu+i \mu, 1-2 \nu ; 2 \theta\right)}{\Psi\left(\frac{1}{2}-\nu+i \mu, 1-2 \nu ; 0\right)}
$$

which is consistent with Proposition 2.1.

## 5. Hitting distributions and stable laws.

The definitions and the theorem below are taken from $[\mathrm{H}$, sections 5.18 and 5.25].

Definition 5.1. A probability distribution is stable if and only if its characteristic function $\phi$ is of the following form $S(z, c, \alpha, \gamma)$

$$
\phi(t)=\left\{\begin{array}{l}
\exp \left(i z t+c|t|^{\alpha}\left(1+i \gamma \operatorname{sgn}(t) \tan \frac{\alpha \pi}{2}\right)\right) \\
\text { if } 0<\alpha \leq 2 \text { and } \alpha \neq 1, \\
\exp \left(i z t+c|t|^{\alpha}\left(1+i \gamma \operatorname{sgn}(t) \frac{2}{\pi} \log |t|\right)\right), \\
\text { if } \alpha=1,
\end{array}\right.
$$

where $c>0,1 \leq \gamma \leq 1$, and $z \in \mathbb{R}$.
Definition 5.2. A probability law $m_{0}$ is said to belong to the domain of attraction of a stable law $m$ if there exist two sequences of real numbers $\left\{a_{n}\right\}_{n},\left\{b_{n}\right\}_{n}$ such that

$$
\frac{X_{1}+\cdots+X_{n}-b_{n}}{a_{n}} \xrightarrow{\text { law }} m, \quad \text { as } n \longrightarrow \infty
$$

where $\left\{X_{n}\right\}_{n}$ is a sequence of independent and identically distributed random variables with common law equal to $m_{0}$.

Define

$$
C(\alpha)= \begin{cases}-\Gamma(-\alpha) \cos \frac{\alpha \pi}{2}, & \text { if } 0<\alpha<2 \text { and } \alpha \neq 1, \\ \frac{\pi}{2}, & \text { if } \alpha=1 .\end{cases}
$$

Note that $C(\alpha)>0$ whenever $0<\alpha<2$.

Theorem 5.3. Let $\left\{X_{n}\right\}_{n}$ be a sequence of independent and identically distributed random variables and assume that

$$
\lim _{x \rightarrow \infty} x^{\alpha} P\left(X_{1}>x\right)=a, \quad \lim _{x \rightarrow \infty} x^{\alpha} P\left(X_{1}<-x\right)=b,
$$

where $0<\alpha<2$, and $a, b \geq 0$ with $a+b>0$. Set

$$
\begin{gathered}
m_{n}=\left\{\begin{array}{ll}
0, & \text { if } 0<\alpha<1, \\
n \mathbf{E}\left[\sin \left(\frac{1}{n} X_{1}\right)\right], & \text { if } \alpha=1, \\
\mathbf{E}\left[X_{1}\right], & \text { if } 1<\alpha<2, \\
c=\alpha(a+b) C(\alpha), \\
\quad \gamma=\frac{b-a}{b+a} .
\end{array} .\right.
\end{gathered}
$$

Then

$$
\frac{X_{1}+\cdots+X_{n}-n m_{n}}{n^{1 / \alpha}} \xrightarrow{\text { law }} m, \quad \text { as } n \longrightarrow \infty
$$

where $m$ is $S(0, c, \alpha, \gamma)$.
We have determined in the previous sections that the density $f$ of the random variable $X_{\infty}$ is given by (1.2) if the starting point is $i=(0,1)$. If the starting point is $(0, y)$, then the hitting distribution is the same as the law of $y Z$ where $Z$ is distributed according to $f$. We now check that such a distribution belongs to the domain of attraction of a stable law, of which we determine the parameters.

Assume first that $0<\nu<1$. We have

$$
\lim _{x \rightarrow \infty} x^{2 \nu} P(Z>x)=\frac{c_{\mu, \nu}}{2 \nu} e^{-\mu \pi}, \quad \lim _{x \rightarrow \infty} x^{2 \nu} P(Z<-x)=\frac{c_{\mu, \nu}}{2 \nu} e^{\mu \pi}
$$

so that, if the starting point is $i y=(0, y)$, then

$$
\begin{aligned}
& a=\lim _{x \rightarrow \infty} x^{2 \nu} P_{0, y}\left(X_{\infty}>x\right)=\frac{c_{\mu, \nu} y^{2 \nu}}{2 \nu} e^{-\mu \pi} \\
& b=\lim _{x \rightarrow \infty} x^{2 \nu} P_{0, y}\left(X_{\infty}<-x\right)=\frac{c_{\mu, \nu} y^{2 \nu}}{2 \nu} e^{\mu \pi}
\end{aligned}
$$

Thus the assumption of Theorem 5.3 is satisfied and this density belongs to the domain of attraction of the stable law $S(0, c, 2 \nu, \gamma)$.

Let us investigate the possible values of the parameters $c, \gamma$. Clearly

$$
c=2 c_{\mu, \nu} y^{2 \nu} C(2 \nu) \cosh \mu \pi, \quad \gamma=\tanh (-\mu \pi) .
$$

Thus the parameter $\gamma$ can take all the values in the range $(-1,1)$, that is, all possible values except the extremal ones $\pm 1$. Finally, by tuning the value of $y$, one can make $c$ take any positive value.

It is clear that if $\nu>1$ then the hitting distribution, having a finite second order moment, belongs to the domain of attraction of a Gaussian law. If $\nu=1$ then a finite second moment does not exist, but it is known [GK, Theorem 35.1] that a probability law $\mu$ belongs to the extended domain of attraction of a Gaussian distribution if and only if

$$
\lim _{x \rightarrow+\infty} \frac{x^{2}\left(1-\int_{-x}^{x} \mu(d y)\right)}{\int_{-x}^{x} y^{2} \mu(d y)}=0 .
$$

It is immediate to check that the above condition is satisfied for the density (1.2) with $\nu=1$. This means that there exist two sequences $\left\{a_{n}\right\}_{n},\left\{b_{n}\right\}_{n}$ of real numbers, with $a_{n}>0$, such that if $\left\{X_{n}\right\}_{n}$ is a sequence of independent, identically distributed random variables with density (1.2) for $\nu=1$, then

$$
\frac{X_{1}+\cdots+X_{n}}{a_{n}}-b_{n} \xrightarrow{\text { law }} N(0,1)
$$

(although $a_{n}$ is not necessarily equal to $n^{1 / \alpha}$ ).

## 6. The case $a>0$.

Recall that we denote by $\tau_{a}$ the first hitting time of the diffusion associated to $L$ on the horocycle $H_{a}$, with $a>0$. We now show how the characteristic function of the hitting distribution on $H_{a}$ can be derived from that of $X_{\infty}$. This will allow us to prove that the hitting distribution on $H_{a}$ is still in the domain of attraction of a stable law with exponent $2 \nu$, but we are not able to give its density. Denote by $K$ the Fourier transform of the hitting distribution on $H_{0}$ with starting point $i$, that is, with the notation of Section 5, the distribution of $X_{\infty}$ under $P_{0,1}$. Then the characteristic function with starting point $i y$ is $t \longrightarrow K(y t)$. By conditioning with respect to the $\sigma$-algebra $\mathcal{F}_{\tau_{a}}$ and using the strong Markov property, for $a<y$ one has

$$
\begin{aligned}
K(y t) & =\mathbf{E}\left[\exp \left(i t \int_{0}^{+\infty} y e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}\right)\right] \\
& =\mathbf{E}_{0, y}\left[\exp \left(i t \int_{0}^{+\infty} Y_{s} d W_{s}^{(\mu)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& =\mathbf{E}_{0, y}\left[\exp \left(i t \int_{0}^{\tau_{a}} Y_{s} d W_{s}^{(\mu)}\right)\right] \mathbf{E}_{0, a}\left[\exp \left(i t \int_{0}^{+\infty} Y_{s} d W_{s}^{(\mu)}\right)\right] \\
& =\mathbf{E}\left[\exp \left(i t \int_{0}^{\tau_{a}} y e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}\right)\right] \mathbf{E}\left[\exp \left(i t \int_{0}^{+\infty} a e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}\right)\right] \\
& =\mathbf{E}\left[\exp \left(i t \int_{0}^{\tau_{a}} y e^{B_{s}^{(\nu)}} d W_{s}^{(\mu)}\right)\right] K(a t) .
\end{aligned}
$$

Thus, if we denote by $K_{a, y}$ the characteristic function of the hitting distribution on $H_{a}$, starting at $i y$, then

$$
K_{a, y}(t)=\frac{K(y t)}{K(a t)} .
$$

We already know that there exist sequences $\left\{a_{n}\right\}_{n},\left\{b_{n}\right\}_{n}$ of real numbers, with $a_{n}>0$, such that

$$
e^{-i b_{n} t} K\left(\frac{t}{a_{n}}\right)^{n} \underset{n \rightarrow \infty}{\longrightarrow} \phi(t),
$$

where $\phi$ is the characteristic function of a stable law, as described at the beginning of Section 5 . Thus we have

$$
e^{-i(y-a) b_{n} t} K_{a, y}\left(\frac{t}{a_{n}}\right)^{n}=\frac{e^{-i b_{n} y t} K\left(\frac{y t}{a_{n}}\right)^{n}}{e^{-i b_{n} a t} K\left(\frac{a t}{a_{n}}\right)^{n}} \longrightarrow \frac{\phi(y t)}{\phi(a t)} .
$$

It is easy to check now that, if $\phi$ is the characteristic function of a stable law $S(z, c, \alpha, \gamma)$, then $t \longrightarrow \phi(y t) / \phi(a t)$ is the characteristic function of a stable law

$$
\begin{cases}S\left(z(y-a), c\left(y^{\alpha}-a^{\alpha}\right), \alpha, \gamma\right), & \text { if } \alpha \neq 1 \\ S(z(y-a)+c \gamma(y \log |y|-a \log |a|), c(y-a), 1, \gamma), & \text { if } \alpha=1\end{cases}
$$

Thus the law of $X_{\tau_{a}}$ is still in the domain of attraction of a stable law with exponent $\alpha=2 \nu$. More precisely, if $\nu<1$, if $\left\{X_{n}\right\}_{n}$ is a sequence of independent, identically distributed random variables with the same law as $X_{\tau_{a}}$, and if $m_{n}$ is defined as in Theorem 5.3, then

$$
\frac{X_{1}+\cdots+X_{n}-n m_{n}}{n^{1 / \alpha}} \xrightarrow{\text { law }} m, \quad \text { as } n \longrightarrow \infty
$$

where $m$ is a stable law $S(0, c, \alpha, \gamma)$ with

$$
\begin{gathered}
\alpha=\min \{2,2 \nu\}, \\
c=2 c_{\mu, \nu}\left(y^{2 \nu}-a^{2 \nu}\right) C(2 \nu) \cosh \mu \pi, \\
\gamma=\tanh (-\mu \pi) .
\end{gathered}
$$

We omit the, otherwise obvious, statement for $\nu=1$.

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# Weak slice conditions, product domains, and quasiconformal mappings 

Stephen M. Buckley and Alexander Stanoyevitch

Abstract. We investigate geometric conditions related to Hölder imbeddings, and show, among other things, that the only bounded Euclidean domains of the form $U \times V$ that are quasiconformally equivalent to inner uniform domains are inner uniform domains.

## 0. Introduction.

Two Euclidean domains are $K$-quasiconformally equivalent if there is a $K$-quasiconformal mapping from one onto the other. Determining what domains are quasiconformally equivalent to a ball or other nice Euclidean domain is an important and open problem when $n \geq 3$. Some partial results are known, notably those of Gehring and Väisälä [GV], [V4]; see also [R]. In [V4], Väisälä classifies cylinders in $\mathbb{R}^{3}$ that are quasiconformally equivalent to a ball.

Inner uniform domains, as defined by Väisälä [V5], satisfy a uniformity condition with respect to the inner Euclidean metric. These domains form a class intermediate between uniform and John domains and, in particular, they include all Lipschitz domains; see Section 1 for definitions. We prove the following theorem which indicates that this class is well suited to the study of quasiconformal equivalence.

Theorem 0.1. Suppose that $\Omega=U \times V \subset \mathbb{R}^{n} \times \mathbb{R}^{m}$ is a bounded domain, $n, m \in \mathbb{N}$. The following are equivalent:
i) $\Omega$ is quasiconformally equivalent to an inner uniform domain.
ii) $\Omega$ is an inner uniform domain.
iii) Both $U$ and $V$ are inner uniform domains.

In particular, since balls are inner uniform, a bounded product domain $\Omega=U \times V$ must be inner uniform if it is quasiconformally equivalent to a ball (this criterion alone, however, is not sufficient as we explain in Remark 4.11).

The following two theorems show that among product domains, inner uniformity is closely connected with the concept of broadness, as introduced by Väisälä [V4]; the inner 0-wSlice ${ }^{+}$condition, defined in Section 2, is a technical assumption satisfied in particular by inner uniform domains and their quasiconformal images.

Theorem 0.2. If $\Omega=U \times V \subset \mathbb{R}^{n} \times \mathbb{R}^{m}, n, m \in \mathbb{N}$, is a bounded inner $0-w$ Slice ${ }^{+}$domain, then $\Omega$ is broad if and only if it in iner uniform.

Theorem 0.3. Suppose that $\Omega=U \times V \subset \mathbb{R}^{n} \times \mathbb{R}^{m}$ is bounded, and quasiconformally equivalent to a broad inner $0-w S l i c e^{+}$domain $G$. Then $\Omega$ is inner uniform.

Obviously, one can remove every instance of the word "inner" from the above theorems if $\Omega$ is assumed to be quasiconvex (i.e., the Euclidean and inner Euclidean metrics are comparable). However it is easy to construct non-quasiconvex counterexamples to the non-inner versions of these theorems. In the case of Theorem 0.2, though, the counterexamples are for one implication only since inner uniform domains are always broad [BHK, Example 6.5(b)].

The rest of the paper is organized as follows. After some preliminaries, we introduce the slice conditions in Section 2. In Section 3, we show that a large class of domains satisfy the various weak slice conditions. In Section 4, we classify bounded product domains satisfying weak slice conditions and prove the above theorems. We examine some further results in Section 5 and, finally, we discuss some open problems in Section 6.

## 1. Preliminaries.

### 1.1. Notation.

We adopt two common conventions. First, we drop parameters if we do not wish to specify their values; for instance, we define $C$ uniform domains, but often talk about uniform domains. Second, we write $C=C(x, y, \ldots)$ to mean that a constant $C$ depends only on the parameters $x, y, \ldots$.

If $S \subset \mathbb{R}^{n}$ is measurable, then $|S|$ is the Lebesgue measure of $S$, and $u_{S}$ is the average value of a function $u$ on $S$. We write $A \lesssim B$ if $A \leq C B$ for some constant $C$ dependent only on allowed parameters; we write $A \approx B$ if $A \lesssim B \lesssim A$. We write $A \wedge B$ and $A \vee B$ for the minimum and maximum, respectively, of the quantities $A$ and $B$. Unless otherwise stated, $\Omega$ and $G$ are proper subdomains of $\mathbb{R}^{n}$.

Let $x, y \in U \subsetneq \mathbb{R}^{n}$. We denote by $\delta_{U}(x)$ the distance from $x$ to $\partial U$, and by $\Gamma_{U}(x, y)$ the class of rectifiable paths $\lambda:[0, t] \longrightarrow U$ for which $\lambda(0)=x, \lambda(t)=y$. If $\alpha \in \mathbb{R}, \gamma$ is a rectifiable path in $U$, and $d s$ is arclength measure, we define

$$
\begin{aligned}
\operatorname{len}_{\alpha, U}(\gamma) & =\int_{\gamma} \delta_{U}^{\alpha-1}(z) d s(z) \\
d_{\alpha, U}(x, y) & =\inf _{\gamma \in \Gamma_{U}(x, y)} \operatorname{len}_{\alpha, U}(\gamma)
\end{aligned}
$$

Of course, $d_{\alpha, U}(x, y)=\infty$ if $x, y$ lie in different path components of $U$. We are mainly interested in $d_{\alpha, U}$ when $\alpha \in[0,1]$ and $U$ is a domain; $d_{\alpha, U}$ is then a metric. Note that $d_{\alpha, U^{-}}$-geodesics may fail to exist if $\alpha>0$ [BS, Proposition 1.2], but they do exist when $U$ is a domain and $\alpha=0$ [GO].

We write len in place of $\operatorname{len}_{1, U}$, the Euclidean length of a path. Note that $\operatorname{len}_{0, U}$ and $d_{0, U}$ are the well-known quasihyperbolic length and distance, and $d_{1, U}$ is the inner Euclidean metric. For brevity, we abuse notation by writing, for instance, $\operatorname{len}_{\alpha, U}(\gamma \cap S)$ for the $d_{\alpha, U}$-length of those parts of a path $\gamma$ lying in a subset $S$ of $U$. We write $[x, y]$ for the line segment joining a pair of points in $\mathbb{R}^{n}$, and $[x \rightarrow y]$ for the path parametrized by arclength that goes from $x$ to $y$ along $[x, y]$.

Given $x \in U, E, F \subset U$, and a metric $\rho$ on $U$, we write $d_{\rho}(E, F)$ for the $\rho$-distance between $E$ and $F, \operatorname{dia}_{\rho}(E)$ for the $\rho$-diameter of $E$, and $B_{\rho}(x, r)=\left\{y \in U: d_{\rho}(x, y)<r\right\}$. If $\rho=d_{1, U}$, we instead
write $d_{U}(E, F)$, $\operatorname{dia}_{U}(E)$, and $B_{U}(x, r)$ for these concepts, while if $\rho$ is the Euclidean metric (and so $U=\mathbb{R}^{n}$ ), we write $d(E, F)$, $\operatorname{dia}(E)$, and $B(x, r)$. We write $d_{U}=d_{1, U}$; in particular, $d_{\mathbb{R}^{n}}$ is the Euclidean metric. Note that distance to the boundary of $U$ is the same with respect to $d_{\mathbb{R}^{n}}$ and $d_{U}$, and that $B_{U}(x, r)=B(x, r)$ if $r \leq \delta_{U}(x)$. We define the inradius of $U, r(U)=\sup _{x \in U} \delta_{U}(x)$.

### 1.2. Uniform domains and mean cigar domains.

Let $C \geq 1$ and let $d$ be the Euclidean metric. A domain $G$ is a $C$-uniform domain if for every $x, y \in G$, there is a $C$-uniform path, i.e., a path $\gamma \in \Gamma_{G}(x, y)$ of length $l$ and parametrized by arclength for which $l \leq C d(x, y)$, and $t \wedge(l-t) \leq C \delta_{G}(\gamma(t))$. An inner $C$-uniform domain is defined similarly except that $d=d_{G}$. Uniform domains include all bounded Lipschitz domains, as well as some domains with fractal boundary, such as the interior of a von Koch snowflake. All uniform domains are inner uniform, and a slit disk is a standard example of an inner uniform domain that is not uniform. For more on inner uniform domains, see [V5].

Suppose that $0 \leq \alpha \leq 1 \leq C$ and let $d: G \times G \longrightarrow[0, \infty)$. We say that $G$ is an $(\alpha, C ; d)$-mCigar domain if for every pair $x, y \in G$, there is a $(\alpha, C ; d)$-mCigar path, i.e., a path $\gamma \in \Gamma_{G}(x, y)$ such that

$$
\begin{gathered}
\operatorname{len}_{\alpha, G}(\gamma) \leq C d(x, y)^{\alpha}, \quad 0<\alpha \leq 1 \\
\operatorname{len}_{0, G}(\gamma) \leq C \log \left(\frac{1+d(x, y)}{\delta_{G}(x) \wedge \delta_{G}(y)}\right), \quad \alpha=0
\end{gathered}
$$

In particular, if $d$ is the Euclidean metric, we simply say that $G$ is an $(\alpha, C)$-mCigar domain, while if $d=d_{G}$, we say that $G$ is an inner ( $\alpha, C$ )-mCigar domain. $\alpha$-mCigar conditions for $0<\alpha<1$ imply the existence of a path $\lambda$ that satisfies a type of cigar condition on average; see [BK2, Lemma 2.2] and Lemma 4.6 below. In practice we shall not use this terminology for $\alpha=1$ : we prefer to use the more common term $C$-quasiconvex domain rather than ( $1, C$ )-mCigar domain.

Uniform domains are $\alpha$-mCigar domains for all $\alpha$. Gehring and Osgood [GO] showed that the classes of 0-mCigar domains and uniform domains coincide, and Väisälä [V4; 2.33] showed that the classes of inner $0-\mathrm{mCigar}$ and inner uniform domains coincide. The class of (inner) $\alpha^{\prime}$-mCigar domains includes the class of (inner) $\alpha$-mCigar domains if
and only if $\alpha \leq \alpha^{\prime}$. The Euclidean version is dealt with in [L] and [BK2]; inclusion follows similarly in the inner case and the counterexamples in [L] also handle the inner version. Thus mCigar domains include domains with rough (even fractal) boundary. Note that the class of inner uniform and inner mCigar domains contain their Euclidean analogues (strictly, since a planar slit disk is in all of the inner classes but none of the Euclidean classes).

We refer the reader to [BK2], [GM], and [L] for more information about $\alpha$-mCigar domains; these domains are called "weak cigar domains" in [BK2] and "Lip ${ }_{\alpha}$ extension domains" in [GM] and [L] when $\alpha>0$. The last name derives from the fact that for $\alpha>0, G$ is $\alpha$-mCigar if and only if all functions defined on $G$ which are locally Lipschitz of order $\alpha$ are globally Lipschitz of order $\alpha$; see [GM].

## 2. Slice domains.

The conditions defined in Section 1 rather strongly restrict the geometry. For instance, among planar domains, inner uniform domains cannot have external cusps, while uniform and mCigar domains can have neither internal nor external cusps. By contrast, the slice conditions that we define in this section are all quite weak, at least in two dimensions: they are satisfied by any domain quasiconformally equivalent to a uniform domain and hence by all simply-connected planar domains.

We first discuss weak slice conditions, as first defined in [BS]. The adjective "weak" refers to the fact that for all $\alpha$, an $\alpha$-wSlice condition is implied by the analogous "strong" slice condition which we define later; see [BS, Lemma 2.8].

Suppose $0 \leq \alpha<1 \leq C$ and let $d$ be a metric on $G$ satisfying $d_{\mathbb{R}^{n}} \leq d \leq d_{G}$. Then $G$ is an $(\alpha, C ; d)$-wSlice domain if every pair $x, y \in G$ satisfies the following $(\alpha, C ; d)$-wSlice condition: there exist a path $\gamma \in \Gamma_{G}(x, y)$, pairwise disjoint open subsets $\left\{S_{i}\right\}_{i=1}^{m}$ of $G, m \geq 0$, and numbers $d_{i} \in\left[\operatorname{dia}_{d}\left(S_{i}\right), \infty\right)$ such that for all $1 \leq i \leq m$

$$
\begin{align*}
& \operatorname{len}\left(\lambda \cap S_{i}\right) \geq \frac{d_{i}}{C}, \quad \text { for all } \lambda \in \Gamma_{G}(x, y)  \tag{WS-1}\\
& \operatorname{len}_{\alpha, G}(\gamma) \leq C\left(\delta_{G}^{\alpha}(x)+\delta_{G}^{\alpha}(y)+\sum_{i=1}^{m} d_{i}^{\alpha}\right) \tag{WS-2}
\end{align*}
$$

$$
\begin{equation*}
\left(B\left(x, \frac{\delta_{G}(x)}{C}\right) \cup B\left(y, \frac{\delta_{G}(y)}{C}\right)\right) \cap S_{i}=\varnothing . \tag{WS-3}
\end{equation*}
$$

If $d$ is the Euclidean metric, we say that $G$ is an $(\alpha, C)$-wSlice domain, while if $d=d_{G}$, we say that $G$ is an inner $(\alpha, C)$-wSlice domain; these are the two metrics that mainly interest us. Note that the metric $d$ enters the definition only in limiting the size of the numbers $\left\{d_{i}\right\}$, and that for $\alpha=0$, (WS-2) simply says that $\operatorname{len}_{0, G}(\gamma) \leq C(2+m)$.

Roughly speaking, a wSlice condition for a pair of points $x, y$ limits the amount of floating boundary and slab-shaped regions in the domain that lie "between" $x$ and $y$; by a "slab-shaped" region, we mean a piece of the domain which is much larger in two coordinate directions than a third such as $(0,1) \times(0,1) \times(0, \varepsilon)$ for some small $\varepsilon>0$. The "tolerance level" of an $\alpha$-wSlice domain for floating boundary components and slab-shaped regions is lower for smaller $\alpha$. In particular it follows from Theorem 4.1 that the product of an externally cusped domain and an interval is never an $\alpha$-wSlice domain for any $\alpha \in[0,1)$. The reader should feel more comfortable with the geometry of this condition after working through the examples in Section 6, and reading the statements of results in Section 4.

As discussed in [BS, 2.1], we can essentially take $d_{i}=\operatorname{dia}_{d}\left(S_{i}\right)$ in the definition, but allowing inequality is sometimes convenient. A significant difference between the $\alpha=0$ and $\alpha>0$ cases is that, whereas (WS-3) is an essential part of the definition for $\alpha=0$ (lest every domain be a $(0 ; d)$-wSlice domain), it can be dropped when $\alpha>0$ (as shown in Theorem 5.1). Modulo a change in the value of $C$ by a factor at most 4 , it is shown in [BS, 2.1] that we may add the following condition to the definition of an $(\alpha, C ; d)$-wSlice condition for $x, y$ (and all $1 \leq i \leq m)$

$$
\begin{equation*}
\operatorname{len}_{\alpha, G}\left(\gamma \cap S_{i}\right) \leq C d_{i}^{\alpha} \tag{WS-4}
\end{equation*}
$$

Given a path $\lambda$ intersecting a slice $S_{i}$, let $\lambda^{i}$ denote the component of $\lambda \cap S_{i}$ with maximal $d$-diameter. We define an $(\alpha, C ; d)$-wSlice ${ }^{+}$ domain to be an ( $\alpha, C ; d$ )-wSlice domain in which the slice data satisfy the following extra pair of conditions for all $1 \leq i \leq m$

$$
\begin{equation*}
\operatorname{dia}_{d}\left(\lambda^{i}\right) \geq \frac{d_{i}}{C}, \quad \text { for all } \lambda \in \Gamma_{G}(x, y) \tag{+}
\end{equation*}
$$

$$
\begin{equation*}
\text { exists } z_{i} \in S_{i}: \quad B_{i} \equiv B\left(z_{i}, \frac{d_{i}}{C}\right) \subset S_{i} \tag{WS-5}
\end{equation*}
$$

$(\alpha, C)-w$ Slice $^{+}$and inner $(\alpha, C)-w$ Slice ${ }^{+}$domains are defined in the obvious way. (For (WS-5), $d_{i}$ comes from the inner metric but the ball
is Euclidean.) We need the extra conditions (WS-1 ${ }^{+}$) and (WS-5) for some of our proofs but intuitively one should think of wSlice ${ }^{+}$domains as being very similar to wslice domains. In fact, we believe it likely that the classes of $(\alpha ; d)$-wSlice and $(\alpha ; d)$-wSlice ${ }^{+}$domains coincide; see the discussion before Open Problem A in Section 6.

We recall [BS, Lemma 2.3].
Lemma 2.1. If the data $\gamma,\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$ satisfy (WS-1) and (WS-3) for the pair $x, y \in G$, and $d_{i}>0$, then $\operatorname{dia}\left(S_{i}\right) \geq 2 \delta_{G}(z) /(C+1)$, for all $z \in S_{i}$ and $1 \leq i \leq m$. Furthermore, if $d_{i} \geq \operatorname{dia}\left(S_{i}\right)$ and $|x-y| \geq\left(\delta_{G}(x)+\delta_{G}(y)\right) / 2$, then there exists a constant $C^{\prime}=C^{\prime}(C, \alpha)$ such that

$$
\begin{equation*}
\delta_{G}^{\alpha}(x)+\delta_{G}^{\alpha}(y)+\sum_{k=1}^{m} d_{i}^{\alpha} \leq C^{\prime} \operatorname{len}_{\alpha, G}(\lambda), \quad \lambda \in \Gamma_{G}(x, y) . \tag{2.2}
\end{equation*}
$$

We next define "strong" slice conditions. Suppose $C \geq 1$ and let $d$ be a metric on $G$ satisfying $d_{\mathbb{R}^{n}} \leq d \leq d_{G}$. Then $G$ is a $(C ; d)$ Slice domain if every pair $x, y \in G$ satisfies the following $(C ; d)$-Slice condition: there exist a path $\gamma \in \Gamma_{G}(x, y)$ and pairwise disjoint open subsets $\left\{S_{i}\right\}_{i=0}^{j}$ of $G$, with $d_{i} \equiv \operatorname{dia}_{d}\left(S_{i}\right)<\infty$, such that:
i) $x \in S_{0}, y \in S_{j}$, and $x$ and $y$ are in different components of $G \backslash \overline{S_{i}}$, for all $0<i<j$.
ii) $\operatorname{len}\left(\lambda \cap S_{i}\right) \geq d_{i} / C$, for all $0<i<j$ and $\lambda \in \Gamma_{G}(x, y)$.
iii) For all $t \in[0,1]$, we have $B\left(\gamma(t), C^{-1} \delta_{G}(\gamma(t))\right) \subset \bigcup_{i=0}^{j} \overline{S_{i}}$. Also, for all $0 \leq i \leq j$, there exists $x_{i} \in \gamma_{i}$, such that $x_{0}=x, x_{j}=y$, and $B\left(x_{i}, C^{-1} \delta_{G}\left(x_{i}\right)\right) \subset S_{i}$.
iv) For all $0 \leq i \leq j$ and $z \in \gamma_{i} \equiv \gamma([0,1]) \cap S_{i}$, we have $d_{i} \leq$ $C \delta_{G}(z)$.

If $d$ is the Euclidean metric, we say that $G$ is a $C$-Slice domain, while if $d=d_{G}$, we say that $G$ is an inner $C$-Slice domain. The (Euclidean) Slice condition was defined in [BK2, Definition 3.1] (for a fixed $y$ but uniformly in $x$ ).

The $d$-Slice condition for a pair of points implies an $(\alpha ; d)$-wSlice condition ${ }^{1}$ for the same pair of points, quantitatively; see [BS, Lemma 2.8]. However, if $\alpha>0$, then there are $\alpha$-wSlice domains which are

[^10]not Slice domains; see [BS, Proposition 4.5]. The $d$-Slice condition is quite similar to the $(0, d)$-wSlice but even less tolerant of "slab-shaped" regions, as discussed after Open Problem C in Section 6. Although a 0 -wSlice condition does not necessarily quantitatively imply an Slice condition, we suspect that the classes of Slice and 0 -wSlice domains coincide.

## 3. Inner uniform and inner slice domains.

In this section, we show that inner uniform domains and their quasiconformal images satisfy certain inner slice conditions.

Theorem 3.1. Let $\alpha \in[0,1)$ and let $f$ be a $K$-quasiconformal mapping from an inner $C$-uniform domain $G \subset \mathbb{R}^{n}$ onto $\Omega$. Then $\Omega$ is an inner $C^{\prime}$-Slice domain and an inner $\left(\alpha, C^{\prime}\right)$-wSlice ${ }^{+}$domain for some $C^{\prime}=C^{\prime}(C, n, K, \alpha)$.

Suppose that $E, F$ are disjoint subsets of a domain $G \subset \mathbb{R}^{n}$. The conformal modulus, $\bmod (E, F ; G)$, of the pair $E, F$ relative to $G$ is defined to be the infimum of $\int_{G} \rho^{n}$, as $\rho: G \longrightarrow[0, \infty]$ ranges over the class of Borel functions for which every line integral over a path $\gamma:[0,1] \longrightarrow G$ joining $E$ and $F$ is at least 1 . We refer the reader to [V2] for the fundamentals of conformal modulus and quasiconformal mappings.

We say that a domain $G \subset \mathbb{R}^{n}$ is $\phi$-broad ${ }^{2}$ if

$$
\phi(t) \equiv \inf \left\{\bmod (E, F ; G): \Delta_{G}(E, F) \leq t\right\}>0, \quad t>0,
$$

where $E, F$ designate non-degenerate disjoint continua in $G$ and

$$
\Delta_{G}(E, F) \equiv \frac{d_{G}(E, F)}{\operatorname{dia}_{G}(E) \wedge \operatorname{dia}_{G}(F)}
$$

denotes the relative inner distance between $E$ and $F$.
Before proving Theorem 3.1, we need some lemmas. The first is a special case of results of Bonk, Heinonen and Koskela (see Example $6.5(\mathrm{~b})$ in $[\mathrm{BHK}]$ ); in the terminology of that paper, $G$ is broad if it is Loewner with respect to $d_{G}$.

[^11]Lemma 3.2. An inner $C$-uniform domain $G \subset \mathbb{R}^{n}$ is $\phi$-broad, with $\phi$ dependent only on $C$ and $n$.

Lemma 3.3. Suppose $G \subset \mathbb{R}^{n}$ is a domain and $E, F \subset G$ are disjoint compact subsets in $G$ and $\Delta_{G}(E, F) \geq 2$. Then there exists a constant $C=C(n)$ such that

$$
\bmod (E, F ; G) \leq C\left(\log \Delta_{G}(E, F)\right)^{-n+1}
$$

Proof. Without loss of generality, we assume that $\operatorname{dia}_{G}(E) \leq \operatorname{dia}_{G}(F)$. Let us fix a point $x \in E$ and write $r=\operatorname{dia}_{G}(E), R=d_{G}(E, F)$, so that $\Delta_{G}(E, F)=R / r \geq 2$. Let $N=\left\lfloor\log _{2} R / r\right\rfloor$, let $A_{i}=B_{G}\left(x, 2^{i} r\right) \backslash$ $B_{G}\left(x, 2^{i-1} r\right)$ for each $1 \leq i \leq n$ and define the function $\rho: G \longrightarrow[0, \infty)$ by the equation

$$
\rho(x)= \begin{cases}\frac{1}{2^{i-1} N r}, & x \in A_{i}, 1 \leq i \leq N \\ 0, & \text { otherwise }\end{cases}
$$

Clearly $\rho$ is an allowable modulus test function and, since $\left|A_{i}\right|$ is dominated by the measure of a Euclidean ball of radius $2^{i} r$, it follows that $\bmod (E, F ; G) \lesssim N^{-n+1}$. The lemma now follows readily.

Our next lemma implies that an inner $\alpha$-mCigar domain is an inner $\alpha$-wSlice ${ }^{+}$domain and, if $\alpha=0$, it is also an inner Slice domain.

Lemma 3.4. Suppose that $0 \leq \alpha<1$ and that $G \subsetneq \mathbb{R}^{n}$. If there is an inner $\left(\alpha, C_{1}\right)$-mCigar path for the points $x, y \in G$, then the pair $x, y$ satisfies an inner $\left(\alpha, C_{2}\right)$-wSlice ${ }^{+}$condition for some $C_{2}=C_{2}\left(C_{1}, \alpha, n\right)$. If $\alpha=0, x, y$ also satisfies an inner $C_{3}$-Slice condition for some $C_{3}=$ $C_{3}\left(C_{1}, n\right)$.

Proof. Without loss of generality, $\delta_{G}(y) \leq \delta_{G}(x)$. We write $B_{w}=$ $B\left(w, \delta_{G}(w) / 2\right)$ for $w \in G$. Suppose that $z \in B_{x}$. If $\alpha>0$, then

$$
\begin{aligned}
d_{\alpha, G}(x, z) & \leq \operatorname{len}_{\alpha, G}([x \rightarrow z]) \\
& <\int_{\delta_{G}(x) / 2}^{\delta_{G}(x)} t^{\alpha-1} d t \\
& =\left(\frac{\delta_{G}(x)}{2}\right)^{\alpha}\left(\frac{2^{\alpha}-1}{\alpha}\right) \\
& <\delta_{G}^{\alpha}(x) .
\end{aligned}
$$

By a separate calculation, we see that $d_{\alpha, G}(x, z)<\delta_{G}^{\alpha}(x)$ even for $\alpha=0$. Thus if $B_{x}$ and $B_{y}$ overlap, then $d_{\alpha, G}(x, y)<\delta_{G}^{\alpha}(x)+\delta_{G}^{\alpha}(y)$, so $x, y$ satisfy an inner $(\alpha, 1)$-wSlice ${ }^{+}$condition with zero slices. We may therefore suppose that $B_{x}$ and $B_{y}$ are disjoint and so $d_{G}(x, y) \geq$ $\left(\delta_{G}(x)+\delta_{G}(y)\right) / 2$.

Define annuli $S_{i} \equiv B_{G}\left(y, 2^{i-2} \delta_{G}(y)\right) \backslash \overline{B_{G}\left(y, 2^{i-3} \delta_{G}(y)\right)}$ for every $i \in \mathbb{N}$. Let $m \geq 2$ be the smallest integer for which $S_{i+1}$ intersects $B\left(x, \delta_{G}(x) / 2\right)$, and let $d_{i}=2^{i-1} \delta_{G}(y)$. Consider the slice data $\gamma$, $\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$, where $\gamma$ is any inner $\left(\alpha, C_{1}\right)$-mCigar path for $x, y$. First (WS-3) is automatically true, and (WS-1+) is true because the $d_{G^{-}}$ diameter of each annulus is comparable to its thickness.

Suppose $\alpha>0$. Since $\gamma$ is an inner ( $\alpha, C_{1}$ )-mCigar path, we have

$$
\operatorname{len}_{\alpha, G}(\gamma) \leq C_{1} d_{G}(x, y)^{\alpha}<C_{1}\left(d_{m}+\delta_{G}(x)\right)^{\alpha},
$$

which implies (WS-2). If instead $\alpha=0$, note that

$$
m \geq \log _{2}\left(\frac{8 d_{G}(x, y)}{\delta_{G}(y)}\right)
$$

Since $d_{G}(x, y)>\delta_{G}(y) / 2$, the inner 0-mCigar property of $\gamma$ then implies (WS-2).

We have now proved that all conditions other than (WS-5) hold with some preliminary constant value $C=C_{4}$. To prove (WS-5) we shall discard some of the slices, leaving enough of them that (WS-2) remains true with $C=2 C_{4}$. For $1 \leq i \leq m$, let $f_{i}: G \longrightarrow \mathbb{R}$ be defined by $f_{i}(z)=d_{G}(y, z) / 2^{i-3} \delta_{G}(y)$. Thus $S_{i}=f_{i}^{-1}((1,2))$, and we also define the thinner annuli $S_{i}^{\prime}=f_{i}^{-1}([4 / 3,5 / 3]) \subset S_{i}$ and their "inner and outer boundaries", $I_{i}=f_{i}^{-1}(4 / 3), O_{i}=f_{i}^{-1}(5 / 3)$. Since $I_{i}$ and $O_{i}$ are separated by an inner Euclidean distance $d_{i} / 12$, and $\gamma$ must pass from one to the other on its way through $S_{i}^{\prime}$, we see that

$$
\begin{equation*}
\operatorname{len}_{\alpha, G}\left(\gamma \cap S_{i}^{\prime}\right) \geq M_{i}^{\alpha-1} \frac{d_{i}}{12} \tag{3.5}
\end{equation*}
$$

where $M_{i}$ is the maximum value of $\delta_{G}$ on $S_{i}^{\prime}$. Let $z=z_{i} \in S_{i}^{\prime}$ be any point for which $\delta_{G}(z)=M_{i}$; this will be the point $z_{i}$ in (WS-5) for appropriate $i$.

We partition the set of integers $i \in[1, m]$ into two sets: the set of good indices $\mathcal{G}$ for which $d_{i} / M_{i} \leq K$, and the set of bad indices $\mathcal{B}$ for which $d_{i} / M_{i}>K$, where the cut-off value $K$ equals $\left(2^{\alpha} \cdot 24 C_{1}\right)^{1 /(1-\alpha)}$.

Since (WS-5) readily follows for any value of $i$ for which $d_{i} / M_{i} \lesssim 1$, it suffices to find a value $K$, dependent on allowable parameters, such that (WS-2) remains true with $C=2 C_{4}$ if we sum up only over good indices on the right-hand side.

Consider first the case $\alpha>0$. We may as well assume that $\delta_{G}^{\alpha}(x) \leq$ $\sum_{i=1}^{m} d_{i}^{\alpha}$ since otherwise $x, y$ satisfy an inner ( $\alpha, 2 C_{4}$ )-wSlice ${ }^{+}$condition (with $m=0$ ). By simple geometry, we see that $d_{G}(x, y) \leq d_{m}+$ $\delta_{G}(x) / 2 \leq 2 d_{m}$, and so

$$
\begin{aligned}
2^{\alpha} \cdot 24 C_{1} \sum_{i \in \mathcal{B}} d_{i}^{\alpha} & =K^{1-\alpha} \sum_{i \in \mathcal{B}} d_{i}^{\alpha} \\
& <\sum_{i \in \mathcal{B}} M_{i}^{\alpha-1} d_{i} \\
& \leq 12 \operatorname{len}_{\alpha, G}(\gamma) \\
& \leq 12 C_{1} d_{G}(x, y)^{\alpha} \\
& \leq 12 C_{1}\left(2 \sum_{i=1}^{m} d_{i}\right)^{\alpha} \\
& \leq 2^{\alpha} \cdot 12 C_{1} \sum_{i=1}^{m} d_{i}^{\alpha},
\end{aligned}
$$

where the second inequality follows from (3.5), and the third from the $\alpha$-mCigar condition. It follows that $\sum_{i \in \mathcal{B}} d_{i}^{\alpha} \leq \sum_{i \in \mathcal{G}} d_{i}^{\alpha}$, and so (WS2) holds $C=2 C_{4}$ for the set of good indices $\mathcal{G}$ alone.

As for the case $\alpha=0$, we have $d_{G}(x, y) \leq 2 d_{m} \leq 2^{m} \delta_{G}(y)$, and so since $m \geq 2$,

$$
\begin{aligned}
\left(12 C_{1}\right)^{-1} \sum_{i=1}^{m} M_{i}^{-1} d_{i} & \leq C_{1}^{-1} \operatorname{len}_{0, G}(\gamma) \\
& \leq \log \left(1+\frac{d_{G}(x, y)}{\delta_{G}(y)}\right) \\
& \leq \log \left(1+2^{m}\right) \\
& \leq m .
\end{aligned}
$$

It follows that (WS-2) holds with $C=2 C_{4}$ for the set of good indices alone.

We omit the proof of the last statement of the lemma, as it merely involves making straightforward adjustments to the proof for the Euclidean case, which is [BK2, Lemma 3.3(a)].

Theorem 3.6. Suppose $f$ is a $K$-quasiconformal mapping from a $\phi$ broad inner $(0, C)$-wSlice ${ }^{+}$(or inner $C$-Slice $)$ domain $G \subset \mathbb{R}^{n}$ onto $\Omega$. Then $\Omega$ is an inner $\left(0, C^{\prime}\right)$-wSlice ${ }^{+}$domain (or inner $C^{\prime}$-Slice, respectively) domain for some $C^{\prime}=C^{\prime}(C, \phi, n, K)$.

If $f, G$, and $\Omega$ are as in Theorem 3.1, then Lemma 3.2 tells us that $G$ is broad and the $\alpha=0$ case of Lemma 3.4 tells us that $\Omega$ is an inner 0 wSlice ${ }^{+}$domain. Thus Theorem 3.1 follows from Theorem 3.6, at least when $\alpha=0$. The $\alpha>0$ case requires little extra effort. First, according to [BS, Lemma 2.8], an inner Slice domain is an inner $\alpha$-wSlice domain, quantitatively, for all $\alpha \in[0,1)$, so the Slice part already implies most of the $\alpha$-wSlice ${ }^{+}$part of Theorem 3.1. It remains to verify (WS-5) and (WS-1+). The former immediately follows from the Slice condition, while the latter is implicit in the proof of the Slice part of Theorem 3.6.

Recall that 0 -wSlice ${ }^{+}$domains are 0 -wSlice domains that satisfy two extra conditions, (WS-1 ${ }^{+}$) and (WS-5). (WS-1 ${ }^{+}$) will play an important role in the proof of Theorem 3.6 but, by contrast, the proof would work as well if (WS-5) were not part of the definition of an inner 0 -wSlice ${ }^{+}$domain; it will, however, play an important role in Section 4 when proving the theorems stated in the introduction.

In proving Theorem 3.6, we will make use of a few basic properties of quasiconformal mappings which we describe here. Suppose that $f$ is a $K$-quasiconformal mapping from $G$ onto $\Omega$, where $G, \Omega$ are domains in $\mathbb{R}^{n}$. Then $f^{-1}$ is $K^{\prime}$-quasiconformal, where $K^{\prime}=K^{\prime}(K, n)$. If $B=B(x, r) \subset G$ with $r=C \delta(B, \partial G)$ for some $1>C>0$, then for any $y \in f B$, we have $c^{\prime} \delta_{\Omega}(y) \leq \operatorname{dia}_{\Omega} f B \leq C^{\prime} \delta_{\Omega}(y)$ and $B\left(f(x), c^{\prime} \delta_{\Omega}(f(x))\right) \subset f B$, where $c^{\prime}$ and $C^{\prime}$ depend only on $C, K, n$; furthermore we can choose $c^{\prime}, C^{\prime}$ tending to 0 as $C \longrightarrow 0$. Briefly, quasiconformal mappings send Whitney balls to Whitney type objects. $K$ quasiconformal mappings quasipreserve conformal modulus (i.e., they preserve it up to a multiplicative constant dependent on $K$ and $n$ ) and they also quasipreserve large quasihyperbolic distance, in the sense that $1+d_{0, G}(x, y)$ and $1+d_{0, \Omega}(f(x), f(y))$ are comparable. For details of these and other properties of quasiconformal mappings, we refer the reader to Theorem 18.1 and other parts of [V2], [V3, 2.4], and [GO, Theorem 3].

Proof of Theorem 3.6. Given $x^{\prime}, y^{\prime} \in \Omega$, let $\gamma,\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$ be $(0, C)$ wSlice ${ }^{+}$data for the pair $x, y$, where $x \equiv f^{-1}\left(x^{\prime}\right)$ and $y \equiv f^{-1}\left(x^{\prime}\right)$. Since we are working with an $\alpha$-wSlice ${ }^{+}$condition with $\alpha=0$, we
may a fortiori take $d_{i}=\operatorname{dia}_{G}\left(S_{i}\right)$. Here and throughout the proof, our notation for objects associated with $G$ and corresponding objects associated with $\Omega$ differs only by the use of superscript primes in the latter case.

Multiplying the size of $C$ by 4 if necessary, we may also assume that (WS-4) holds. If $m=0$, then $x^{\prime}, y^{\prime}$ satisfy a 0 -wSlice ${ }^{+}$condition with $m^{\prime}=0$ (since $f$ quasipreserves large quasihyperbolic distance). We may therefore assume that $m>0$. Let $\gamma_{i}=\gamma([0,1]) \cap S_{i}$, let $\gamma^{i}$ be a component of $\gamma_{i}$ of inner diameter at least $d_{i} / C$, as guaranteed by (WS-1 ${ }^{+}$), and let $\delta_{i}=\delta_{G}\left(z^{i}\right)$ for some fixed but arbitrary point $z^{i} \in \gamma^{i}$. By elementary estimation, we see that the quasihyperbolic length of any component $K$ of $\gamma_{i}$ must be at least $\log \left(\delta_{G}\left(z^{\prime}\right) / \delta_{G}\left(z^{\prime \prime}\right)\right)$ for any pair of points $z^{\prime}, z^{\prime \prime} \in K$. Thus (WS-4) implies that $\delta_{G}(z) \approx \delta_{i}, z \in \gamma^{i}$. By (WS-1 ${ }^{+}$) and (WS-4), it follows that $d_{i} / \delta_{i} \lesssim \operatorname{len}_{0, G}\left(\gamma^{i}\right) \leq C$, while the first statement in Lemma 2.1 says that $d_{i} / \delta_{i} \gtrsim 1$. Consequently, $\delta_{G}(z) \approx d_{i}, z \in \gamma^{i}$.

Fix $x_{i} \in \gamma^{i}$ and let $x_{i}^{\prime}=f\left(x_{i}\right)$, for each $1 \leq i \leq m$. For a constant $C_{0}^{\prime}>3$ to be chosen later, let $B_{i}^{\prime}=B_{\Omega}\left(x_{i}^{\prime}, C_{0}^{\prime} \delta_{\Omega}\left(x_{i}^{\prime}\right)\right)$ and $S_{i}^{\prime}=f\left(S_{i}\right) \cap B_{i}^{\prime}$. Writing $m^{\prime}=m, d_{i}^{\prime}=\operatorname{dia}_{\Omega}\left(S_{i}^{\prime}\right)$, and choosing $\gamma^{\prime}$ to be a quasihyperbolic geodesic in $\Omega$, we claim that $\gamma^{\prime},\left\{S_{i}^{\prime}, d_{i}^{\prime}\right\}_{i=1}^{m^{\prime}}$ are $\left(0, C^{\prime}\right)$-wSlice ${ }^{+}$data for $x^{\prime}, y^{\prime}$, as long as $C^{\prime}>C_{0}^{\prime}$ are both suitably large.

Since $f$ maps Whitney balls to Whitney type objects, the slice data for $x^{\prime}, y^{\prime}$ inherit the conditions (WS-3) and (WS-5) from $G$ (in general not with the same constant, of course). Since $f$ quasipreserves large quasihyperbolic distance, the slice data for $x^{\prime}, y^{\prime}$ inherit condition (WS-2) from $G$. It remains to prove (WS-1+).

We claim that $x^{\prime}$ and $y^{\prime}$ lie in separate components of $\Omega \backslash S_{i}^{\prime}$, provided that $C_{0}^{\prime}$ is large enough. Suppose that they lie in the same component, and so there exists a path $\lambda^{\prime} \in \Gamma_{\Omega}\left(x^{\prime}, y^{\prime}\right)$ which does not intersect $S_{i}^{\prime}$. Let $\lambda=f^{-1} \circ \lambda^{\prime}$, let $\lambda^{i}$ be as in (WS-1 ${ }^{+}$), and define $F \equiv \overline{\lambda^{i}}, F^{\prime}=f F, E=\overline{B\left(x_{i}, c \delta_{G}\left(x_{i}\right)\right)}$, and $E^{\prime}=f E$, where $c=$ $c(K, n)$ is the largest constant in $(0,1 / 2]$ for which $E^{\prime} \subset \overline{B\left(x_{i}^{\prime}, \delta_{\Omega}\left(x_{i}^{\prime}\right) / 2\right)}$. Then $\operatorname{dia}_{G}(F) \approx d_{i}, d_{G}(E, F) \lesssim d_{i}$, and by the quasiconformality of $f, \operatorname{dia}_{G}(E) \approx d_{i}$. Thus $\Delta_{G}(E, F) \lesssim 1$, and so $\bmod (E, F ; G) \geq \varepsilon=$ $\varepsilon(\phi, C, n, K)>0$.

Now $d_{\Omega}\left(E^{\prime}, F^{\prime}\right) \geq\left(C_{0}^{\prime}-1 / 2\right) \delta_{\Omega}\left(x_{i}^{\prime}\right)$ and $\operatorname{dia}_{\Omega}\left(E^{\prime}\right) \leq \delta_{\Omega}\left(x_{i}^{\prime}\right)$, and so $\Delta_{\Omega}\left(E^{\prime}, F^{\prime}\right) \geq C_{0}^{\prime}-1 / 2$. Thus by Lemma 3.3 and the quasiconformality of $f$,

$$
\bmod (E, F ; G) \approx \bmod \left(E^{\prime}, F^{\prime} ; \Omega\right) \lesssim\left(\log \left(C_{0}^{\prime}-1 / 2\right)\right)^{-n+1}
$$

Since $\bmod (E, F ; G) \geq \varepsilon$, we get an upper bound for $C_{0}^{\prime}$ in terms of $\phi$, $C, n$, and $K$; we may assume that this upper bound is at least 3 . For any $C_{0}^{\prime}$ larger than this bound, the claim follows.

We fix $C_{0}^{\prime}$ to be a little more than twice as large as this bound, so that $f\left(S_{i}\right) \cap(1 / 2) B_{i}^{\prime}$ separates $x^{\prime}$ from $y^{\prime}$. Let $\lambda^{\prime} \in \Gamma_{\Omega}\left(x^{\prime}, y^{\prime}\right), \lambda \equiv f^{-1} \circ$ $\lambda^{\prime}$, let $\lambda^{i}$ be as in (WS-1 ${ }^{+}$), and define $F=\overline{\lambda^{i}}, F^{\prime}=f F$. We wish to show that $\operatorname{dia}_{\Omega}\left(F^{\prime}\right) \gtrsim d_{i}^{\prime}$. We may assume that $F^{\prime} \subset S_{i}^{\prime}$ since otherwise $F^{\prime}$ contains points in both $\Omega \backslash B_{i}^{\prime}$ and $(1 / 2) B_{i}^{\prime}$, and so $\operatorname{dia}_{\Omega}\left(F^{\prime}\right) \geq$ $C_{0}^{\prime} \delta_{\Omega}\left(x_{i}^{\prime}\right) / 2 \geq d_{i}^{\prime} / 4$.

Now for each $z \in \gamma^{i}$, we have $d_{i} \approx \delta_{G}(z) \lesssim \operatorname{dia}_{G}\left(\gamma^{i}\right)$, so we can choose a connected compact subset $E_{0}$ of $\gamma^{i}$ for which

$$
\delta_{G}(z) \lesssim \operatorname{dia}_{G}\left(E_{0}\right) \leq \frac{\delta_{G}(z)}{2},
$$

for all $z \in E_{0}$. Letting $E_{0}^{\prime}=f E_{0}$, it follows that $d_{i}^{\prime} \approx \delta_{\Omega}\left(z^{\prime}\right) \approx$ $\operatorname{dia}_{\Omega}\left(E_{0}^{\prime}\right)$ for each $z^{\prime} \in E_{0}^{\prime}$. We choose continua $E_{1}^{\prime}, E_{2}^{\prime} \subset E_{0}^{\prime}$ such that $\operatorname{dia}_{\Omega}\left(E_{1}^{\prime}\right), \operatorname{dia}_{\Omega}\left(E_{2}^{\prime}\right) \geq \operatorname{dia}_{\Omega}\left(E_{0}^{\prime}\right) / 4$ and $d_{\Omega}\left(E_{1}^{\prime}, E_{2}^{\prime}\right) \geq \operatorname{dia}_{\Omega}\left(E_{0}^{\prime}\right) / 4$. If $d_{\Omega}\left(F^{\prime}, E_{j}^{\prime}\right) \leq \operatorname{dia}_{\Omega}\left(E_{0}^{\prime}\right) / 10$ for $j=1,2$, then $\operatorname{dia}_{\Omega}\left(F^{\prime}\right) \gtrsim \operatorname{dia}_{\Omega}\left(E_{0}^{\prime}\right) \approx d_{i}^{\prime}$ as required. Suppose therefore that $d_{\Omega}\left(F^{\prime}, E_{j}^{\prime}\right)>\operatorname{dia}\left(E_{0}^{\prime}\right) / 10$ for some $j \in\{1,2\}$. We write $E^{\prime}=E_{j}^{\prime}, E=f^{-1} E^{\prime}$. Note that $\operatorname{dia}_{G}(F) \approx d_{i}$, $d_{G}(E, F) \lesssim d_{i}$, and by quasiconformality of $f, \operatorname{dia}_{G}(E) \approx d_{i}$. Thus by Lemma 3.2 we obtain

$$
\bmod \left(E^{\prime}, F^{\prime} ; \Omega\right) \approx \bmod (E, F ; G) \gtrsim 1 .
$$

But $\operatorname{dia}_{\Omega}\left(E^{\prime}\right) \approx d_{i}^{\prime}, d_{\Omega}\left(E^{\prime}, F^{\prime}\right) \gtrsim d_{i}^{\prime}$, and so by Lemma 3.3,

$$
\operatorname{dia}_{\Omega}\left(F^{\prime}\right) \gtrsim d_{i}
$$

The proof for the Slice version is similar, so we omit it.

## 4. Product domains.

One of the main lessons of this section is that (inner) slice conditions are rather restrictive when imposed upon product domains. This stands in contrast to Section 3, where we saw that the various slice conditions are very weak, at least in the plane. We note that simplyconnected planar counterexamples are easily constructed to each of the product domain results in this section if we remove the product domain hypothesis.

Our main theorem in this section is as follows.
Theorem 4.1. Suppose that $0 \leq \alpha<1$ and that $\Omega=U \times V \subset \mathbb{R}^{n} \times \mathbb{R}^{N}$ is a bounded domain, $n, N \in \mathbb{N}$. The following are equivalent:
i) $\Omega$ is an inner $\left(\alpha, C_{1}\right)$-wSlice ${ }^{+}$domain.
ii) Both $U$ and $V$ are inner ( $\alpha, C_{2}$ )-mCigar domains.
iii) $\Omega$ is an inner $\left(\alpha, C_{3}\right)$-mCigar domain.

The constants $C_{i}$ depend only on each other and on $\alpha$, $\operatorname{dia}_{\Omega}(\Omega) / r(\Omega), n$, and $N$.

A result of Lappalainen [L, 6.7] says that, for every $0<\alpha<\beta<1$, there exists a planar domain $D^{*}$ which is an (inner) $\beta$-mCigar domain but not an (inner) $\alpha$-mCigar domain; $D^{*}$ happens to be bounded, quasiconvex, and simply-connected. Lappalainen's result extends to the case $0=\alpha<\beta<1$ since a 0 -mCigar domain is a uniform domain and so any $\beta$-mCigar domain which is not a $(\beta / 2)$-mCigar domain is certain not a 0 -mCigar domain. Taking $U=D^{*}$ and letting $V$ be the unit ball in $\mathbb{R}^{n-2}$, we thus get the following corollary of Theorem 4.1.

Corollary 4.2. For any $0 \leq \alpha<\beta<1$ and $3 \leq n \in \mathbb{N}$, there exists an (inner) $\beta$-wSlice ${ }^{+}$domain $\Omega \subsetneq \mathbb{R}^{n}$ which is not an (inner) $\alpha$-wSlice ${ }^{+}$ domain but is homeomorphic to a ball.

Note that $\alpha$-wSlice domains may be inner unbounded even if they are bounded (e.g., many simply-connected planar domains with a spiralling cusp). If however $\Omega$ is assumed to be inner bounded in Theorem 4.1, then the reader can verify from the proof that the inner $\alpha$-wSlice ${ }^{+}$condition in this theorem can be weakened to an inner $\alpha$ wSlice condition. Lappalainen's examples are certainly inner bounded, so the same examples show that for any $0 \leq \alpha<\beta<1$ and $3 \leq n \in \mathbb{N}$, there exists an (inner) $\beta$-wSlice domain $\Omega \subset \mathbb{R}^{n}$ which is not an (inner) $\alpha$-wSlice domain but is homeomorphic to a ball.

To prove Theorem 4.1, we shall need some lemmas.
Lemma 4.3. Let $\Omega$ be an inner ( $\alpha, C$ )-mCigar domain, $0 \leq \alpha<1$. For every $x, y \in \Omega$, there exists an inner $(\alpha, C)$-mCigar path $\gamma$ such that all initial and final segments of $\gamma$ are inner ( $\alpha, 2 C$ )-mCigar paths (for the segment endpoints).

Proof. Fixing $x, y \in \Omega$, we may assume that $|x-y| \geq \delta_{\Omega}(x) \vee$ $\delta_{\Omega}(y)$, since otherwise $[x \rightarrow y]$ has minimal $d_{\alpha, \Omega}$-length among all paths connecting $x$ and $y$, and so all segments of this line segment are ( $\alpha, C$ )mCigar paths. Let $B_{x} \equiv B\left(x, \delta_{\Omega}(x) / 2\right)$ and $B_{y} \equiv B\left(y, \delta_{\Omega}(y) / 2\right)$.

By symmetry, it suffices to prove the result only for initial segments. Consider first the case $\alpha>0$. Let $\varepsilon=\varepsilon_{x} \wedge \varepsilon_{y}$, where $\varepsilon_{z}=$ $\left((3 / 2)^{\alpha}-1\right) \delta_{\Omega}^{\alpha}(z) / 2 \alpha$ for $z \in\{x, y\}$. The desired path $\gamma$ will be an inner $(\alpha, C)$-mCigar path for $x, y$ with some extra properties. First, we assume that $\operatorname{len}_{\alpha, \Omega}(\gamma)<d_{\alpha, \Omega}(x, y)+\varepsilon$. Since the $d_{\alpha, \Omega}$-mimimal length paths from $x$ to any $x_{1} \in \partial B_{x}$, and from $y$ to any $y_{1} \in \partial B_{y}$, are line segments, we may also assume that the only subarc of $\gamma$ lying in either $B_{x}$ or $B_{y}$ is a single line segment. Finally by reparametrization, we may assume that $\left.\gamma\right|_{[0,1 / 4]}$ and $\left.\gamma\right|_{[3 / 4,1]}$ are the line segments in question, from $x$ to $x^{\prime} \in \partial B_{x}$ and from $y^{\prime} \in \partial B_{y}$ to $y$, respectively, and that both of these line segments are traversed by $\gamma$ at a constant Euclidean speed.

By direct calculation, it is easy to check that $\left.\gamma\right|_{[0, t]}$ is an inner $\left(\alpha, f_{\alpha}(t)\right)$-mCigar path for $t \leq 1 / 4$, with $f_{\alpha}(t)=\left(1-(1-2 t)^{\alpha}\right) / \alpha(2 t)^{\alpha}$; this largest constant is attained by picking $x^{\prime}$ so that $\delta_{\Omega}\left(x^{\prime}\right)=\delta_{\Omega}(x) / 2$. Since $f_{\alpha}$ is increasing on $[0,1 / 4]$, we have $f_{\alpha}(t) \leq f_{\alpha}(1 / 4)=\left(2^{\alpha}-1\right) / \alpha$, $t \in[0,1 / 4]$. By calculus, we see that $f_{\alpha}(1 / 4)<f_{1}(1 / 4)=1, \alpha \in(0,1)$. Thus these initial segments are (inner) ( $\alpha, 1$ )-mCigar paths.

To go from $x$ to $\gamma(t), t>1 / 4$, one must first exit $B_{x}$, and so $d_{\alpha, \Omega}(x, \gamma(t)) \geq \min _{u \in \partial B_{x}} d_{\alpha, \Omega}(x, u) \geq 2 \varepsilon_{x}$. Suppose for the purposes of contradiction that $\left.\gamma\right|_{[0, t]}$ is not an inner $(\alpha, 2 C)$-mCigar path for the pair $x, \gamma(t)$. The $d_{\alpha, \Omega}$-length of an inner $(\alpha, C)$-mCigar path for $x, \gamma(t)$ is less than half that of $\left.\gamma\right|_{[0, t]}$, and so shorter by an amount in excess of $\varepsilon_{x}$. Thus splicing the (reparametrized) shorter path into $\gamma$ in place of $\gamma \mid[0, t]$, we get a new path, contradicting the near-minimal $d_{\alpha, \Omega}$-length of $\gamma$.

Taking $\varepsilon=\log \sqrt{3 / 2}$, the proof when $\alpha=0$ is similar, so we omit it. Alternatively, it follows from the fact that quasihyperbolic geodesics in an inner $\left(0, C_{1}\right)$-mCigar domain are inner $C_{2}$-uniform paths for some $C_{2}=C_{2}\left(C_{1}\right)$; see [V4, 2.29].

Lemma 4.4. If $0 \leq \alpha<1$ and $\Omega=U \times V \subset \mathbb{R}^{n} \times \mathbb{R}^{N}$ is a bounded inner $(\alpha, C)$-wSlice ${ }^{+}$domain, then $\Omega$ is also inner bounded, and $\operatorname{dia}_{\Omega}(\Omega) \leq$ $C^{\prime} \operatorname{dia}(\Omega)$, where $C^{\prime}=C^{\prime}(\alpha, C, \operatorname{dia}(\Omega) / r(\Omega), n+N)$.

Proof. Without loss of generality, we may assume that $C \geq 4$ and that $\operatorname{dia}(\Omega)=1$ (the latter because of the scale invariance of the hypotheses
and conclusion). By symmetry it suffices to prove that $\operatorname{dia}_{U}(U) \lesssim 1$. We choose $v_{0} \in V$ such that $\delta_{0} \equiv \delta_{V}\left(v_{0}\right)=r(V)$. Note that $r(\Omega) \leq$ $\delta_{0} \leq 1 / 2$ and that $d_{U}\left(u_{1}, u_{2}\right)=d_{\Omega}\left(\left(u_{1}, v_{0}\right),\left(u_{2}, v_{0}\right)\right)$.

Suppose that there exist points $u, w \in U$ such that $d_{U}(u, v)>1$. Writing $x=\left(u, v_{0}\right), y=\left(w, v_{0}\right)$, we assume that inner $(\alpha, C)$-wSlice ${ }^{+}$ data for $x, y$ are $\gamma,\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$, with the indexing chosen so that $\left\{d_{i}\right\}_{i=1}^{m}$ is non-decreasing. It is also convenient to define $d_{0}=0$ and $d_{m+1}=\infty$. Let $m_{0} \in[0, m]$ be the unique integer for which $d_{m_{0}}<\delta_{0} / 2 \leq d_{m_{0}+1}$.

Using only (WS-1), we claim that $d_{1} \geq 2\left(\delta_{\Omega}(x) \wedge \delta_{\Omega}(y)\right) / C$, and that there exist constants $C_{1}, t>0$, dependent only on $C$, such that $d_{i} \geq C_{1} 2^{(i-j) t} d_{j}$ whenever $j<i \leq m_{0}$. We first construct two paths $\lambda^{+}$and $\lambda^{-}$from $x$ to $y$, each consisting of three segments. The first segment of $\lambda^{+}$is $\left[x \rightarrow x^{\prime}\right]$, where $x^{\prime}=\left(u, v_{0}+v^{\prime}\right) \in \partial B\left(x, \delta_{0} / 2\right)$. The second segment, from $x^{\prime}$ to $y^{\prime}=\left(w, v_{0}+v^{\prime}\right)$, has constant $V$-component, and the final segment is (a reparametrization of) $\left[y^{\prime} \rightarrow y\right]$. The path $\lambda^{-}$is defined in a similar fashion except that we replace $v^{\prime}$ by $-v^{\prime}$ throughout.

Let $i \leq m_{0}$. By (WS-1), both $\lambda^{+}$and $\lambda^{-}$intersect $S_{i}$ on a set of length at least $d_{i} / C$; we denote the sets of intersection by $S_{i}^{+}$and $S_{i}^{-}$, and write $S_{i}^{\prime}=S_{i}^{+} \cup S_{i}^{-}$. Since $d_{i}<\delta_{0} / 2<1 / 4$, it follows that $S_{i}^{\prime}$ (in fact, all of $\left.S_{i}\right)$ is contained in either $B\left(x, \delta_{0} / 2\right)$ or $B\left(y, \delta_{0} / 2\right)$. The argument is the same in both cases, so we assume that $S_{i}^{\prime} \subset B\left(x, \delta_{0} / 2\right)$. Since $S_{i}^{+}$and $S_{i}^{-}$lie outside $B\left(x, \delta_{\Omega}(x) / C\right)$, and on opposite sides of ( $u, v_{0}$ ), we have $d_{i} \geq 2 \delta_{\Omega}(x) / C$, giving the first half of our claim. For the same reason, we actually have $S_{i}^{\prime} \subset B\left(x, d_{i}\right)$. In particular, if $d_{i} \in(a / 2, a]$ for some positive number $a \leq \delta_{0} / 2$, then $S_{i}$ intersects both $\lambda^{+}$and $\lambda^{-}$on sets of length at least $a / 2 C$ lying in $B(x, a) \cup B(y, a)$. Slices are disjoint, and the total intersection of either $\lambda^{+}$or $\lambda^{-}$with $B(x, a) \cup B(y, a)$ has length $2 a$, so there can be at most $4 C$ such slices $S_{i}$. The second half of our claim now follows.

Suppose that $\alpha>0$. To prove inner boundedness of $U$, we find a bound for len $(\gamma)$. Since $\operatorname{dia}(\Omega)=1$, we have len $(\gamma) \leq \operatorname{len}_{\alpha, \Omega}(\gamma)$. Thus it suffices to bound $\sum_{i=1}^{m} d_{i}^{\alpha}$. The geometric growth of $\left\{d_{i}\right\}_{i=1}^{m_{0}}$ ensures that

$$
\sum_{i=1}^{m_{0}} d_{i}^{\alpha} \lesssim d_{m_{0}}^{\alpha} \leq\left(\frac{\delta_{0}}{2}\right)^{\alpha}
$$

By (WS-5), we have

$$
\left(\frac{d_{i}}{C}\right)^{n+N} \leq\left|S_{i}\right| \leq|\Omega| \leq 1,
$$

and so $d_{i} \leq C$ for all $i$. If $i \geq m_{0}$ then $d_{i} \geq \delta_{0} / 2$ and so $\left|S_{i}\right| \geq$ $\left(\delta_{0} / 2\right)^{n+N}$. Since the slices are disjoint, we deduce that $m-m_{0} \leq$ $\left(2 / \delta_{0}\right)^{n+N}$. Thus

$$
\sum_{i=m_{0}+1}^{m} d_{i}^{\alpha} \leq\left(m-m_{0}\right) d_{m}^{\alpha} \leq 2^{n+N} \delta_{0}^{-n-N} C^{\alpha} .
$$

It follows that $\sum_{i=1}^{m} d_{i}^{\alpha} \lesssim 1$, as desired.
Suppose instead that $\alpha=0$. It is not hard to show that

$$
\begin{equation*}
d_{0, \Omega}(x, y) \gtrsim \log \left(1+\frac{d_{\Omega}(x, y)}{\delta_{\Omega}(x) \wedge \delta_{\Omega}(y)}\right) \tag{4.5}
\end{equation*}
$$

For the Euclidean version of this inequality, see [GP, Lemma 2.1], whose proof also handles this inner version; see also [V4, 2.5]. As in the case $\alpha>0$, we have $m-m_{0} \leq 2^{n+N} / \delta_{0}^{n+N}$. The size and growth properties of $\left\{d_{i}\right\}_{i=1}^{m_{0}}$ obtained above imply that $m_{0} \lesssim 1+\log \left(1 /\left(\delta_{\Omega}(x) \wedge \delta_{\Omega}(y)\right)\right)$. Thus
$d_{0, \Omega}(x, y) \lesssim 2+m \lesssim 1+\log \left(\frac{1}{\delta_{\Omega}(x) \wedge \delta_{\Omega}(y)}\right) \lesssim \log \left(1+\frac{1}{\delta_{\Omega}(x) \wedge \delta_{\Omega}(y)}\right)$.
Comparing this last inequality with (4.5), we deduce that $d_{\Omega}(x, y) \lesssim 1$.
The Euclidean version of the next lemma is part of the $\beta=\alpha$ case of [BK2, Lemma 2.2] ${ }^{3}$. We omit a proof, as it is entirely analogous to the Euclidean case.

Lemma 4.6. Let $0<\alpha<1$ and let $\gamma:[0, l] \longrightarrow \Omega$ be an inner ( $\alpha, C$ )-mCigar path, parametrized by arclength, for the points $x, y$ in a domain $\Omega \subsetneq \mathbb{R}^{n}$. Let us denote by $r:[0, l] \longrightarrow(0, \infty)$ the nondecreasing rearrangement of $t \longmapsto \delta_{\Omega}(\gamma(t))$. Then there exists exists a constant $C_{0}=C_{0}(C, \alpha)$ such that $\operatorname{len}(\gamma) \leq C_{0} d_{\Omega}(x, y)$ and $r(t) \geq$ $C_{0}^{-1}\left(t d_{\Omega}(x, y)^{-\alpha}\right)^{1 /(1-\alpha)}$. In particular, $r(c l) \geq C_{0}^{-1} c^{1 /(1-\alpha)} d_{\Omega}(x, y)$ for all $c>0$.

Proof of Theorem 4.1. i) implies ii). Assuming that $\Omega$ is an inner $\left(\alpha, C_{1}\right)$-wSlice ${ }^{+}$domain, it suffices by symmetry to prove that

[^12]$U$ is an inner $\alpha$-mCigar domain. Fix a point $v_{0} \in V$ such that $\delta_{0} \equiv$ $\delta_{V}\left(v_{0}\right)=r(V)$. Let $\gamma,\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$ be inner $\left(\alpha, C_{1}\right)$-wSlice ${ }^{+}$data for a pair of points $x=\left(u, v_{0}\right), y=\left(w, v_{0}\right) \in \Omega$. We may assume that $d_{\Omega}(x, y)>\left(\delta_{\Omega}(x) \vee \delta_{\Omega}(y)\right) / 2$, since otherwise a line segment satisfies an $\alpha$-mCigar condition. We index the slices so that $\left\{d_{i}\right\}_{i=1}^{m}$ is nondecreasing; it is also convenient to define $d_{0}=0$ and $d_{m+1}=\infty$. Let $m_{0} \in[0, m]$ be the unique integer for which $d_{m_{0}}<s \leq d_{m_{0}+1}$, where $s=\left(\delta_{0} \wedge d_{\Omega}(x, y)\right) / 2$.

As in Lemma 4.4, we see that $d_{1} \geq 2\left(\delta_{\Omega}(x) \wedge \delta_{\Omega}(y)\right) / C_{1}$ and that there exist constants $C^{\prime}, t>0$, dependent only on $C$, such that $d_{i} \geq$ $C^{\prime} 2^{(i-j) t} d_{j}$ whenever $j<i \leq m_{0}$. Fixing a path $\lambda \in \Gamma_{\Omega}(x, y)$ such that $\operatorname{len}(\lambda) \leq 2 d_{\Omega}(x, y)$, (WS-1) implies that

$$
\begin{equation*}
\sum_{i=1}^{m} d_{i} \leq 2 C d_{\Omega}(x, y) \tag{4.7}
\end{equation*}
$$

Thus $m-m_{0} \leq 2 C d_{\Omega}(x, y) / s \leq 2 C \operatorname{dia}_{U}(U) / \delta_{0}$. Thus by Lemma 4.4, $m-m_{0} \lesssim 1$.

Consider the case $\alpha=0$. By the size and growth properties of $\left\{d_{i}\right\}_{i=1}^{m_{0}}$, we see that

$$
m_{0} \lesssim 1+\log \left(\frac{d_{\Omega}(x, y)}{\delta_{\Omega}(y) \wedge \delta_{\Omega}(x)}\right) \lesssim \log \left(1+\frac{d_{\Omega}(x, y)}{\delta_{\Omega}(y) \wedge \delta_{\Omega}(x)}\right) .
$$

Since also $m-m_{0} \lesssim 1$, (WS-2) now implies an inner 0 -mCigar condition for $x, y$. When we project from $\Omega$ to $U$, Euclidean length cannot increase, and distance to the boundary cannot decrease. Therefore we deduce an inner ( $0, C_{2}$ )-mCigar condition for $u$, $w$, with $C_{2}=C_{2}(T)$, where $T$ denotes the data $\left(\alpha, C_{1}, \operatorname{dia}_{\Omega}(\Omega) / r(\Omega), n+N\right)$.

Consider next the case $\alpha>0$. Suppose, for the purposes of contradiction that $U$ is not an inner $\alpha$-mCigar domain. For each $k \in \mathbb{N}$, there exist points $u_{k}$ and $w_{k}$ for which $d_{\alpha, U}\left(u_{k}, w_{k}\right) \geq k d_{U}\left(u_{k}, w_{k}\right)^{\alpha}$; also let $x_{k}=\left(u_{k}, v_{0}\right)$ and $y=\left(w_{k}, v_{0}\right)$. Regardless of the values of $k, \alpha$, we must have $2 d_{\Omega}\left(x_{k}, y_{k}\right) \geq \delta_{\Omega}\left(x_{k}\right) \vee \delta_{\Omega}\left(y_{k}\right)$ since otherwise by consideration of the segment $\left[x_{k} \rightarrow y_{k}\right]$, the points $u_{k}$ and $w_{k}$ would violate the previous inequality. But

$$
\begin{equation*}
d_{\alpha, \Omega}\left(x_{k}, y_{k}\right) \geq d_{\alpha, U}\left(u_{k}, w_{k}\right) \geq k d_{U}\left(u_{k}, w_{k}\right)^{\alpha}=k d_{\Omega}\left(x_{k}, y_{k}\right)^{\alpha} \tag{4.8}
\end{equation*}
$$

and so $d_{\alpha, \Omega}\left(x_{k}, y_{k}\right) \geq k 2^{-\alpha}\left(\delta_{\Omega}\left(x_{k}\right) \vee \delta_{\Omega}\left(y_{k}\right)\right)$. Let $\gamma,\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$, be inner $\left(\alpha, C_{1}\right)$-wSlice ${ }^{+}$data for the pair $x_{k}, y_{k}$, with $\left\{d_{i}\right\}_{i=1}^{m}$ non-decreasing; for ease of notation, the dependence on $k$ is implicit. Taking
$k>3 \cdot 2^{\alpha} C_{1}$, it follows from (WS-2) that $\operatorname{len}_{\alpha, \Omega}(\gamma) \leq 3 C_{1} \sum_{i=1}^{m} d_{i}^{\alpha}$. Combining this inequality with (4.7) and (4.8), we get

$$
\begin{align*}
\frac{\sum_{i=1}^{m} d_{i}^{\alpha}}{\left(\sum_{i=1}^{m} d_{i}\right)^{\alpha}} & \geq \frac{\operatorname{len}_{\alpha, \Omega}(\gamma)}{3 C_{1}\left(\sum_{i=1}^{m} d_{i}\right)^{\alpha}}  \tag{4.9}\\
& \geq \frac{\operatorname{len}_{\alpha, \Omega}(\gamma)}{3 C_{1}^{1+\alpha} \cdot 2^{\alpha} \cdot d_{\Omega}\left(x_{k}, y_{k}\right)^{\alpha}} \\
& \geq \frac{k}{3 C_{1}^{1+\alpha} \cdot 2^{\alpha}}
\end{align*}
$$

But the growth rate of the $\left\{d_{i}\right\}_{i=1}^{m_{0}}$ and the bound on $m-m_{0}$ imply that both $\sum_{i=1}^{m_{0}} d_{i}^{\alpha}$ and $\sum_{i=m_{0}+1}^{m} d_{i}^{\alpha}$ are no more than a constant multiple of $\left(\sum_{i=1}^{m} d_{i}\right)^{\alpha}$. Taking $k$ to be larger than some constant $C_{2}=C_{2}(T)$, we get the desired contradiction to (4.9).
ii) Implies iii). Assume first that $\alpha>0$. By the triangle inequality, it suffices to verify the inner $\alpha$-mCigar condition for pairs of points $x, y \in \Omega$ with one common coordinate; by symmetry, we may assume that $x=(u, v), y=(w, v)$. Let us fix a point $v_{0} \in V$ such that $\delta_{0} \equiv \delta_{V}\left(v_{0}\right)=r(V)$. Let $\mu:[0,1] \longrightarrow V$ and $\gamma:[0,1] \longrightarrow U$ be inner $\left(\alpha, C_{2}\right)$-mCigar paths from $v$ to $v_{0}$ and from $u$ to $w$ respectively, where $\mu$ has the additional properties guaranteed by Lemma 4.3 (applied to $V)$. Letting $L=\operatorname{len}(\gamma), \gamma_{1}=\left.\gamma\right|_{[0,1 / 2]}, \gamma_{2}=\left.\gamma\right|_{[1 / 2,1]}, z=\gamma(1 / 2)$, we may assume that $\gamma$ is parametrized so that $\operatorname{len}\left(\gamma_{1}\right)=\operatorname{len}\left(\gamma_{2}\right)=L / 2$.

Suppose also that $L \leq 2 \operatorname{len}(\mu)$. We wish to define an inner $\alpha$ mCigar path $\Lambda \in \Gamma_{\Omega}(x, y)$. We choose $\Lambda(t)=(\gamma(t), \lambda(t))$, where $\lambda$ is a path in $V$ which starts and finishes at $v$ but, in between times, moves along $\mu$ and back. More precisely, for $0 \leq t \leq 1 / 2, \lambda$ coincides with a reparametrized initial segment of $\mu$, with the parametrization chosen so that $\operatorname{len}\left(\left.\gamma\right|_{[0, t]}\right)=\operatorname{len}\left(\left.\lambda\right|_{[0, t]}\right)$. For $1 / 2 \leq t \leq 1, \lambda$ traces its way back along the curve of $\mu$ in such a way that $\operatorname{len}\left(\left.\gamma\right|_{[t, 1]}\right)=\operatorname{len}\left(\left.\lambda\right|_{[t, 1]}\right)$.

Since $\delta_{\Omega}((a, b))=\delta_{U}(a) \wedge \delta_{V}(b), a \in U, b \in V$, we obtain

$$
\begin{aligned}
\operatorname{len}_{\alpha, \Omega}(\Lambda) & <\int_{\Lambda} \delta_{U}(\gamma(t))^{\alpha-1} d s(t)+\int_{\Lambda} \delta_{V}(\lambda(t))^{\alpha-1} d s(t) \\
& =\sqrt{2}\left(\operatorname{len}_{\alpha, U}(\gamma)+\operatorname{len}_{\alpha, V}(\lambda)\right)
\end{aligned}
$$

Now $\operatorname{len}_{\alpha, U}(\gamma) \leq C_{2} d_{U}(u, w)^{\alpha}=C_{2} d_{\Omega}(x, y)^{\alpha}$. By Lemma 4.6 we may assume that $L \lesssim d_{U}(u, w)=d_{\Omega}(x, y)$, and so by Lemma 4.3 applied to the segments $\gamma_{1}$ and $\gamma_{2}$,

$$
\frac{\operatorname{len}_{\alpha, V}(\lambda)}{4 C_{2}} \leq d_{V}\left(v, \lambda\left(\frac{1}{2}\right)\right)^{\alpha} \leq\left(\frac{L}{2}\right)^{\alpha} \lesssim d_{\Omega}(x, y)^{\alpha}
$$

The inner $\alpha$-mCigar condition for $\Lambda$ now follows.
The construction for $L>2$ len $(\mu)$ is similar: $\lambda(t)$ moves along $\mu([0,1])$ from $v$ at the same speed as before, except now it reaches $v_{0}$ at some $t=t_{0}<1 / 2$. Similarly, there is some number $t_{1}>1 / 2$ such that $\operatorname{len}\left(\left[t_{1}, 1\right]\right)=\operatorname{len}(\mu)$. The path $\lambda$ is now continued so that $\lambda(t)=v_{0}$ for that $t_{0} \leq t \leq t_{1}$, and finally for $t_{1} \leq t \leq 1, \lambda(t)$ moves back along $\mu$ to $v$ at the same speed as before. The estimates are the same as before except for

$$
\int_{\left.\Lambda\right|_{\left[t_{0}, t_{1}\right]}} \delta_{V}(\lambda(t))^{\alpha-1} d s(t) \leq L \delta_{V}\left(v_{0}\right)^{\alpha-1} \lesssim L^{\alpha} .
$$

We must still consider the $\alpha=0$ case. Since the 0 -mCigar condition is quantitatively equivalent to uniformity [V4, 2.33], it suffices to verify that if $U$ and $V$ are uniform, then $\Omega$ is uniform. Let $v_{0} \in V$ be as in the $\alpha>0$ case, but now we seek to find a uniform path between a pair of points $\left(u_{1}, v_{1}\right)$ and $\left(u_{2}, v_{2}\right)$. Let $\gamma:\left[0, l_{1}\right] \longrightarrow U$ and $\mu:\left[0, l_{2}\right] \longrightarrow V$ be uniform paths parametrized by arclength for the pairs of points $u_{1}, u_{2}$ and $v_{1}, v_{2}$ in their respective domains; without loss of generality $l_{1} \geq l_{2}$. Let $\nu:\left[0, l_{3}\right] \longrightarrow V$ be a uniform path in $V$, parametrized by arclength, for the pair $\mu\left(l_{2} / 2\right), v_{0}$. We now define a new path $\lambda:\left[0, l_{1}\right] \longrightarrow V$ linking $v_{1}$ and $v_{2}$. If $l_{1} \leq l_{2}+2 l_{3}$, then

$$
\lambda(t)= \begin{cases}\mu(t), & 0 \leq t \leq \frac{l_{2}}{2} \\ \nu\left(t-\frac{l_{2}}{2}\right), & \frac{l_{2}}{2} \leq t \leq \frac{l_{1}}{2} \\ \nu\left(l_{1}-\frac{l_{2}}{2}-t\right), & \frac{l_{1}}{2} \leq t \leq l_{1}-\frac{l_{2}}{2} \\ \mu\left(t-l_{1}+l_{2}\right), & l_{1}-\frac{l_{2}}{2} \leq t \leq l_{1}\end{cases}
$$

while if $l_{1} \geq l_{2}+2 l_{3}$, then the definition is similar except that $\lambda$ "rests" at $v_{0}$ for an interval of length $l_{1}-l_{2}-2 l_{3}$ before turning back. We leave
it to the reader to verify that the path $\Lambda=(\gamma, \lambda)$ is a uniform path in $\Omega$ for the pair ( $u_{1}, v_{1}$ ) and ( $u_{2}, v_{2}$ ), with quantitative dependence only on allowed parameters, namely $\operatorname{dia}_{\Omega}(\Omega) / r(\Omega), n, N$, and the uniformity constants for $U, V$.
iii) IMPLIES i). This follows from Lemma 3.4.

Proof of Theorems 0.1, 0.2 , and 0.3 . We first prove Theorem 0.1. Trivially ii) implies i). Since an inner 0-mCigar domain is just an inner uniform domain, the equivalence of ii) and iii) follows from Theorem 4.1. If $\Omega$ is $K$-quasiconformally equivalent to an inner $C$-uniform domain then Theorem 3.1 ensures that it is an inner $\left(0, C_{1}\right)$-wSlice ${ }^{+}$domain, with $C_{1}=C_{1}(C, n+N, K)$, and so Theorem 4.1 tells us that i) implies ii).

Theorem 0.3 follows similarly by combining Theorem 3.6 and Theorem 4.1. As for Theorem 0.2 , one direction is given by Lemma 3.2, while the other follows from Theorem 0.3 with $G=\Omega$.

Remark 4.10. The implication i) implies ii) of Theorem 0.1 also follows from recent work of Bonk, Heinonen, and Koskela; see [BHK, Remark 7.34]. Their methods (based around Gromov hyperbolicity) are however quite different and do not apply to the $\alpha>0$ case of Theorem 4.1.

Remark 4.11. Theorem 0.1 does not tell us what product domains are quasiconformally equivalent to a ball. In fact, Väisälä [V4] showed that if $G$ is a simply-connected proper subdomain of the plane, then $G \times \mathbb{R}$ is quasiconformally equivalent to a ball if and only if there is a BLD (bounded length distortion) mapping from $G$ to a disk or a halfplane. It is not hard to modify his proof to show that for a bounded domain $G, G \times(0,1)$ is quasiconformally equivalent to a ball if and only if there is a BLD (bounded length distortion) mapping from $G$ to a disk. It follows that there are inner uniform domains of product type that are not quasiconformally equivalent to a ball. For instance, the planar domain $U$ bounded by a von Koch snowflake is a uniform domain but, because its boundary is not locally rectifiable, no such BLD mapping can exist and consequently $\Omega=U \times \mathbb{R}$ is uniform but not quasiconformally equivalent to a ball.

Remark 4.12. The 0 -wSlice ${ }^{+}$hypothesis cannot be removed from

Theorems 0.2 and 0.3 . For example, let $B^{k}$ denote the unit ball in $\mathbb{R}^{k}$, let $n>1$, and consider the product domain $\Omega=B \backslash N$, where $B=B^{n} \times B^{m}$ and $N=A \times B^{m}, A=\bigcup_{j=1}^{\infty} A_{j}$, and $A_{j}$ consists of $(j!)^{n-1}$ points on the sphere $S_{j}=\left\{|z|=1-2^{-j}\right\} \subset \mathbb{R}^{n}$, spaced so that the distance from any $x \in S_{j}$ to $A_{j}$ is at most $C / j$ !, for some $C=C(n)$. Clearly $B$ is broad, and we claim that $\Omega$ is also broad. To see this, let $E, F$ be non-degenerate disjoint continua in $\Omega$. Since $B \backslash \Omega$ has Hausdorff dimension at most $m<n+m-1$, it is an null set for extremal distance $[\mathrm{V} 1]$ and so $\bmod (E, F ; B)=\bmod (E, F, \Omega)$. The restriction of $d_{B}$ to $\Omega \times \Omega$ coincides with $d_{\Omega}$, and so $\Delta_{B}(E, F)=$ $\Delta_{\Omega}(E, F)$. The claim now follows readily. However $B^{n} \backslash A$, and hence $\Omega$, is not inner uniform since a path from the origin to a point $x$ of norm nearly 1 must pass through very narrow bottlenecks as it approaches $x$.

## 5. Further results.

We first use some of the ideas developed in the last section to prove, as promised in Section 2, that (WS-3) can be removed from the definition of (inner) $\alpha$-wSlice conditions when $\alpha>0$ without changing the class of domains; we shall need this result in the final section.

Theorem 5.1. Suppose that $0<\alpha<1, x, y \in G \subset \mathbb{R}^{n}$, and that $\gamma \in$ $\Gamma_{G}(x, y),\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$ satisfy (WS-1) and (WS-2), with $d_{i} \geq \operatorname{dia}_{d}\left(S_{i}\right)$ for some metric satisfying $d_{\mathbb{R}^{n}} \leq d \leq d_{G}$. Then $x, y$ satisfy an $\left(\alpha, C^{\prime} ; d\right)$ $w$ Slice condition for some $C^{\prime}=C^{\prime}(C, \alpha)$, with slice data $\gamma^{\prime},\left\{T_{i}, e_{i}\right\}_{i=1}^{M}$ satisfying

$$
C^{\prime}\left(\delta_{G}^{\alpha}(x)+\delta_{G}^{\alpha}(y)+\sum_{i=1}^{M} e_{i}^{\alpha}\right) \geq C\left(\delta_{G}^{\alpha}(x)+\delta_{G}^{\alpha}(y)+\sum_{i=1}^{m} d_{i}^{\alpha}\right) .
$$

Proof. Without loss of generality $\delta_{G}(x) \geq \delta_{G}(y)$. We may assume that $|x-y|>\delta_{G}(x)$, since otherwise the conclusion is true with $M=0$ and $\gamma^{\prime}=[x \rightarrow y]$. Writing $B_{z}=B\left(z, \delta_{G}(z) / 16 C\right)$ and $\widetilde{B}_{z}=B\left(z, \delta_{G}(z) / 2\right)$, for $z \in\{x, y\}$, we note that $\widetilde{B}_{x}$ and $\widetilde{B}_{y}$ are disjoint. The first step is to define new slices $S_{i}^{\prime}=S_{i} \backslash \overline{B_{x} \cup B_{y}}$, and leave the numbers $d_{i}$ unchanged. Certainly, these new slices satisfy (WS-3), but (WS-1) may now fail. We discard any slice $S_{i}^{\prime}$ for which (WS-1) still fails even after we replace $C$ by $2 C$. Renumbering the remaining pairs $\left(S_{i}^{\prime}, d_{i}\right)$, we get new slice data $\gamma^{\prime} \equiv \gamma,\left\{T_{i}, e_{i}\right\}_{i=1}^{M}$.

By construction, the new data satisfy (WS-3), and (WS-1) with constant $2 C$. It remains to prove (WS-2) (with $C$ replaced by some $C^{\prime}$ ). If $S_{i}^{\prime}$ is a discarded slice then there must exist some path $\lambda \in \Gamma_{G}(x, y)$ whose intersection with $S_{i}^{\prime}$ has length less than $d_{i} / 2 C$. Now (WS-1) for $S_{i}$ tells us that $\operatorname{len}\left(\lambda \cap S_{i} \cap \overline{\left.B_{x} \cup B_{y}\right)}>d_{i} / 2 C\right.$. If we alter $\lambda$ so that for $z=x, y$ the only segment of $\gamma$ lying in $\overline{B_{z}}$ is a single line segment (of length $\delta_{G}(z) / 16 C$ ), but otherwise leave $\lambda$ unchanged, this inequality must remain true. Thus $\left(\delta_{G}(x)+\delta_{G}(y)\right) / 16 C \geq d_{i} / 2 C$, and so $d_{i} \leq \delta_{G}(x) / 4$. If $S_{i} \cap \overline{B_{x}}$ is non-empty, then $S_{i}$ must lie fully in $\widetilde{B}_{x}$. On the other hand, if $S_{i} \cap \overline{B_{x}}$ is empty, then $S_{i} \cap \overline{B_{y}}$ is non-empty and $d_{i} \lesseqgtr \delta_{G}(y) / 4$. In either case, we deduce that $S_{i}$ lies fully in either $\widetilde{B}_{x}$ or $\widehat{B}_{y}$.

Let us enumerate the discarded slices and the corresponding numbers as $\left\{S_{j_{i}}^{\prime}, d_{i}^{\prime}\right\}_{i=1}^{k}$, with $d_{i}^{\prime} \equiv d_{j_{i}}$. We choose the enumeration so that $\left\{d_{i}^{\prime}\right\}_{i=1}^{k}$ is non-decreasing. As in the proof of Lemma 4.4, we obtain the growth estimate

$$
d_{i}^{\prime} \geq C_{1} 2^{(i-j) t} d_{j}^{\prime}, \quad \text { for } 1 \leq j<i \leq k
$$

In fact to get this estimate, the two paths used should be as follows. The first one, $\lambda^{+}$, starts off as any line segment of length $\delta_{\Omega}(x) / 2$ emanating from $x$, and ends as any line segment of length $\delta_{\Omega}(y) / 2$ ending at $y$, the middle part of the path being any path joining the outer endpoints of these two segments in $\Omega$ which stays outside $\widetilde{B}_{x} \cup \widetilde{B}_{y}$. The second path $\lambda^{-}$has the same construction except that the initial and final line segments are in directions opposite to those of the $\lambda^{+}$.

The growth estimate and (WS-1) now give

$$
\sum_{i=1}^{k}\left(d_{i}^{\prime}\right)^{\alpha} \lesssim\left(d_{k}^{\prime}\right)^{\alpha} \sum_{j=1}^{\infty} 2^{-t j} \lesssim\left(d_{k}^{\prime}\right)^{\alpha} \lesssim \delta_{\Omega}^{\alpha}(x)+\delta_{\Omega}^{\alpha}(y)
$$

Thus

$$
\delta_{G}^{\alpha}(x)+\delta_{G}^{\alpha}(y)+\sum_{i=1}^{M} e_{i}^{\alpha} \gtrsim \delta_{G}^{\alpha}(x)+\delta_{G}^{\alpha}(y)+\sum_{i=1}^{m} d_{i}^{\alpha}
$$

and so if we replace $C$ by an appropriate $C^{\prime}$, then the remaining slices satisfy all three conditions (WS-1), (WS-2), and (WS-3).

Next we wish to state a John-Separation version of Theorem 4.1, but let us begin with two definitions that we need.

Let us fix a constant $C \geq 1$ and a point $x_{0}$ in the domain $G$. A $C$-John path for $x$ (with respect to $x_{0}$ ) is a path $\gamma \in \Gamma_{G}\left(x, x_{0}\right), \gamma$ : $[0, l] \rightarrow G$, which is parametrized by arclength such that $\delta(\gamma(t)) \geq t / C$ for all $t \in[0, l]$. We say that $G$ is a $C$-John domain (with respect to $x_{0}$ ) if there exists a $C$-John path (with respect to $x_{0}$ ) for all $x \in G$.

Let $C, x_{0}$ be as above and let $B_{z}=B\left(z, C \delta_{G}(z)\right), z \in G$. As defined in [BK1], a $C$-Separation path for $x$ (with respect to $x_{0}$ ) is a path $\gamma:[0,1] \longrightarrow G, \gamma \in \Gamma_{G}\left(x, x_{0}\right)$, such that for each $t \in[0,1]$, any path from a point in $\gamma([0, t]) \backslash B_{\gamma(t)}$ to $x_{0}$ must intersect $\partial B_{\gamma(t)}$. We say that $G$ is a $C$-Separation domain (with respect to $x_{0}$ ) if there exists a $C$-Separation path (with respect to $x_{0}$ ) for all $x \in G$. A $C$-John domain is a $C$-Separation domain (since $\gamma([0, t]) \backslash B_{\gamma(t)}$ is empty) but there are many more Separation domains, including all quasiconformal images of uniform domains [BK1].

Theorem 5.2. Suppose that $\Omega=U \times V \subset \mathbb{R}^{n} \times \mathbb{R}^{N}$ is a bounded domain, $x_{0}=\left(u_{0}, v_{0}\right) \in \Omega$, and $n, N \in \mathbb{N}$. The following are equivalent:
i) $\Omega$ is a $C_{1}$-Separation domain with respect to $x_{0}$.
ii) Both $U$ and $V$ are $C_{2}$-John domains with respect to $u_{0}$ and $v_{0}$ respectively.
iii) $\Omega$ is a $C_{3}$-John domain with respect to $x_{0}$.

The constants $C_{i}$ depend only on each other and on $n, N$, and $\operatorname{dia}_{\Omega}(\Omega) / d_{\Omega}\left(x_{0}\right)$.

Proof. We omit the easy verifications of the implications ii) implies iii) implies i). Supposing that $\Omega$ satisfies i), we shall prove ii). We may assume that $C_{1}>2$ and, by symmetry, it suffices to show that $U$ is a John domain with respect to $u_{0}$. We claim that the first coordinate projection $\gamma_{1}$ of any $C_{1}$-separation path $\gamma$ for the point $x=\left(u, v_{0}\right)$ must be a $C_{2}$-John path for $u$, with $C_{2}=C_{2}\left(C_{1}, \operatorname{dia}(\Omega) / \delta_{\Omega}\left(x_{0}\right)\right)$. To see this, we write $r(t)=C_{1} \delta_{\Omega}(\gamma(t)), B_{t}=B(\gamma(t), r(t))$. If $\gamma([0, t]) \subset \overline{B_{t}}$, then $\gamma_{1}$ satisfies the $2 C_{1}$-John condition for $x$ at $\gamma_{1}(t)$, so we shall assume that $\gamma([0, t]) \not \subset \overline{B_{t}}$. We may also assume that $r(t)<\delta_{V}\left(v_{0}\right) / 6$ since otherwise the claim follows with $C_{2}=6 C_{1} \operatorname{dia}(V) / \delta_{V}\left(x_{0}\right) \leq 6 C_{1} \operatorname{dia}(\Omega) / \delta_{\Omega}\left(x_{0}\right)$.

If $\left|\gamma(t)-x_{0}\right| \leq \delta_{\Omega}\left(x_{0}\right) / 2$, then $r(t) \geq C_{1} \delta_{\Omega}\left(x_{0}\right) / 2$. On the other hand, if $\left|\gamma(t)-x_{0}\right|>\delta_{\Omega}\left(x_{0}\right) / 2$ and $x_{0} \in \overline{B_{t}}$, then $r(t) \geq \delta_{\Omega}\left(x_{0}\right) / 2$. Both of these contradict the bound on $r(t)$, so we conclude that $x_{0} \notin \overline{B_{t}}$.

Suppose that $x \notin \overline{B_{t}}$. We construct two paths $\lambda^{+}, \lambda^{-} \in \Gamma_{\Omega}\left(x, x_{0}\right)$ by first moving in a straight line from $\left(u, v_{0}\right)$ to points $x^{+}=\left(u, v_{0}+w\right)$,
$x^{-}=\left(u, v_{0}-w\right)$, respectively, where $w \in \mathbb{R}^{N}$ is chosen so that $2 r(t)<$ $|w|<\delta_{V}\left(v_{0}\right)$. The second segment of each paths has constant second coordinate and rectifiable first coordinate finishing at a point with first coordinate $u_{0}$, and the last segment of each is a straight line segment back to $\left(u_{0}, v_{0}\right)$.

Now $\partial B_{t}$ must intersect both $\lambda^{+}$and $\lambda^{-}$. But $\partial B_{t}$ cannot intersect the middle segment of either path since its distance from the other path exceeds $2 r(t)$. Neither can it intersect the first segments, or the last segments, of both paths, since it would then follow that either $x \in B_{t}$ or $x_{0} \in B_{t}$. Finally suppose that $\partial B_{t}$ intersects the first segment of one path at $\left(u, v_{1}\right)$, say, and the last segment of the other at $\left(u_{0}, v_{2}\right)$, say. Thus

$$
2 r(t)>\left|v_{1}-v_{2}\right|>2\left(\left|v_{1}-v_{0}\right| \wedge\left|v_{2}-v_{0}\right|\right),
$$

and so $2 B_{t}$ contains either $x$ or $x_{0}$. The claim follows as before.
We are left to consider the case where $x \in \overline{B_{t}}$. By assumption, there is a point $\widehat{x} \in \gamma([0, t]) \backslash \overline{B_{t}}$, which by continuity we may assume to lie in the annulus $2 B_{t} \backslash \overline{B_{t}}$. We now define a pair of paths $\lambda^{+}, \lambda^{-} \in$ $\Gamma_{\Omega}\left(\widehat{x}, x_{0}\right)$ by first moving in a straight line from $\widehat{x}$ to points $x^{+}=$ $\widehat{x}+(0, w), x^{-}=\widehat{x}+(0,-w)$, respectively, where $w \in \mathbb{R}^{N}$ is chosen so that $4 r(t)<|w|<\delta_{V}\left(v_{0}\right)-2 r(t)$. As before, the second segment of each path has constant second coordinate and rectifiable first coordinate finishing at a point with first coordinate $u_{0}$, and the last segment of both paths is a straight line segment back to $x_{0}$. This claim now follows as in the previous case.

We now discuss the case of unbounded domains $\Omega \subsetneq \mathbb{R}^{n}$. As we shall see below, most of the implications in Theorem 4.1 fail if we simply drop the boundedness assumption, but we do have the following theorem.

Theorem 5.3. Suppose that $0 \leq \alpha<1$ and that $\Omega=U \times V$ where $U \subsetneq \mathbb{R}^{n}, V \subsetneq \mathbb{R}^{N}, r(U)=r(V)=\infty$, and $n, N \in \mathbb{N}$. The following are equivalent:
i) $\Omega$ is is an inner $\left(\alpha, C_{1}\right)$-wSlice ${ }^{+}$domain.
ii) Both $U$ and $V$ are inner ( $\alpha, C_{2}$ )-mCigar domains.
iii) $\Omega$ is an inner $\left(\alpha, C_{3}\right)$-mCigar domain.

The constants $C_{i}$ depend only on each other and on $\alpha, n$, and $N$.

Sketch of proof. Let $\Omega$ satisfy the hypotheses and that it is an inner $(\alpha, C)$-wSlice ${ }^{+}$domain. By symmetry, it suffices to prove an inner $\alpha$-mCigar condition for $U$. We fix points $u, w \in U$, and choose a path $\gamma \in \Gamma_{U}(u, w)$ such that $L_{\alpha} \equiv \operatorname{len}_{\alpha, U}(\gamma) \leq 2 d_{\alpha, U}(u, w)$. Let $L_{1}=\operatorname{len}(\gamma)$ and let $M$ denote the largest value of $\delta_{U}(z)$ on the image of $\gamma$. Now choose $v_{0}$ so that $\delta_{0} \equiv \delta_{V}\left(v_{0}\right)>M \vee 2 C L_{1}$. Let $x=\left(u, v_{0}\right)$, $y=\left(w, v_{0}\right)$, and define the path $\Lambda$ by $\Lambda(t)=\left(\gamma(t), v_{0}\right)$. It follows that $\operatorname{len}_{\alpha, \Omega}(\Lambda)=L_{\alpha}$ and that $d_{\alpha, \Omega}(x, y)=d_{\alpha, U}(u, w)$. We deduce that the pair $x, y$ possesses $(\alpha, 2 C)$-wSlice ${ }^{+}$data of the form $\Lambda,\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$, with the indexing chosen so that $\left\{d_{i}\right\}_{i=1}^{m}$ is non-decreasing. By (WS1 ), we see that $d_{m} \leq \delta_{0} / 2$. Arguing as in Lemma 4.4, it then follows that the numbers $d_{i}$ satisfy a geometric growth condition and the inner $\alpha$-mCigar condition for $U$ now follows as before.

As for the implication ii) implies iii), assume $u, v, w$ are as in the corresponding part of the proof of Theorem 4.1, and let $\gamma$ be an $\left(\alpha, C_{2}\right)$ mCigar path from $u$ to $w$ of length $L$, say. Choosing $v_{0}$ so that $\left|v-v_{0}\right|$ exceeds $L / 2$, the proof then follows as before. The implication iii) implies i) follows from Lemma 3.4.

The assumption $r(U)=r(V)=\infty$ can be weakened in the above theorem, although it cannot be dropped since we shall give counterexamples in the case where only one domain is unbounded (it might suffice for both domains to be unbounded but we cannot prove this). The assumption $r(U)=r(V)=\infty$ can be dropped altogether from the implication iii) implies i), and for ii) implies iii) above, it suffices that $U$ and $V$ are both unbounded (but this is hardly more general, since it is easy to see that an inner $\alpha$-mCigar domain must have infinite inradius if it has infinite inner diameter). Finally for i) implies ii), the following substitute assumption suffices (we leave to the reader the straightforward task of adapting the proof).

The following condition is satisfied by both $W=U$ and $W=V$ for some constant $c \in(0,1)$ : for every $A>0$, there exists a point $w_{0} \in W$ and paths $\lambda^{+}, \lambda^{-}$parametrized by arclength and of total length $A$, such that $\lambda^{+}(0)=\lambda^{-}(0)=w_{0}$, and for every $t \in(0, A]$, the distances from $\lambda^{+}(t)$ to the image of $\lambda^{-}$, and from $\lambda^{-}(t)$ to the image of $\lambda^{+}$are both at least $c t$.

Of course any domain $W$ satisfying such a condition but having finite inradius is certainly not an inner $\alpha$-mCigar domain. For a typical example of such a domain, we first let $\mu^{+}, \mu^{-}$be the Archimidean spirals given in polar coordinates by $\mu^{+}(\theta)=(\theta, \theta), \mu^{-}(\theta)=(\theta, \theta+\pi)$,
both for all $t \geq 0$, and let $W$ be the planar domain consisting of all points in the unit disk together with all points within a distance $1 / 10$ of the union of the images of $\mu^{+}$and $\mu^{-}$. Then for each $A>0$, we can take $w_{0}=0$ and $\lambda^{+}, \lambda^{-}$to be suitably reparametrized initial segments of $\mu^{+}, \mu^{-}$.

For an arbitrary pair of domains $U \subsetneq \mathbb{R}^{n}, V \subsetneq \mathbb{R}^{N}$, the implications iii) implies i) and iii) implies ii) hold (the former because of Lemma 3.4, while the latter is easy), but we now give three counterexamples which show that the other four possible implications fail. In all examples, $\Omega \equiv U \times V$, and $U$ is the open interval $(0,1)$, which is of course an inner $\alpha$-mCigar domain for every $\alpha \in[0,1)$.

First, we see that $V=(0, \infty)$ is uniform, and so an inner $\alpha$-mCigar domain for every $\alpha \in[0,1)$. Moreover $\Omega$ is simply connected, and so an inner $\alpha$-wSlice ${ }^{+}$domain by Theorem 3.1 and the Riemann mapping theorem. However $\Omega$ is not an inner $\alpha$-mCigar domain for any such $\alpha$. This neither i) nor ii) imply iii).

Next taking $V=(0,1) \times(0, \infty)$, we see that $V$ is not an inner $\alpha$-mCigar domain. However $\Omega$ is an inner $\alpha$-wSlice ${ }^{+}$domain for every $\alpha$. In fact if $x, y \in \Omega$ and $|x-y|<6$, then zero slices suffice. Suppose instead that $|x-y| \geq 6$, with $x_{3}<y_{3}$, where $x_{3}, y_{3}$ are the third coordinates of $x, y$ respectively. Then $y_{3} \geq x_{3}+4$ and we take as slices all cylinders $(0,1) \times(0,1) \times(i, i+1), i \in \mathbb{N}$, for which $x_{3}+1 \leq i \leq y_{3}-2$; we leave the verifications to the reader. Thus i) does not imply ii).

Finally, $V=(0, \infty) \times(0, \infty)$ is uniform and so an inner $\alpha$-mCigar domain for every $\alpha$. However $\Omega$ is not an inner $\alpha$-wSlice domain. In fact for any constant $C$, the points $\left(u, v_{1}\right)$ and $\left(u, v_{2}\right)$ fail to satisfy an $(\alpha, C)$-wSlice condition if $u=1 / 2, v_{1}=(t, t), v_{2}=(t, 2 t)$, and $t \geq t_{0}$ for some sufficiently large number $t_{0}=t_{0}(C)$. We leave the verification of this to the reader, with the hint that the techniques of Lemma 4.4 can again be adapted to this purpose. Thus ii) does not imply i).

## 6. Open problems.

In this final section, we discuss the basic relationships between the various slice ${ }^{4}$ conditions. We use the term zero-point implications for implications between slice conditions for a fixed pair of points. Note

[^13]that all slice conditions hold for a fixed pair of points if we choose a sufficiently large constant, so zero-point implications are only of interest if we insist that the implied slice constant depends quantitatively only on the assumed slice constant and other reasonable parameters such as the dimension. We also discuss one-point implications involving one-point slice conditions, where the slice condition is assumed to be true uniformly for one fixed point $x=x_{0}$ and all $y$ in the domain; we call the classes of domains satisfying such conditions one-sided slice domains. Finally, we discuss two-point implications involving two-point slice conditions, where the slice condition is assumed to be true uniformly for all pairs $x, y$ in the domain; as in previous sections, we use the term slice domains to refer to the associated domains.

We shall first note some quantitative zero-point implications; these immediately imply the corresponding one- and two-point implications. Most other quantitative zero-point implications will be seen to be false and the corresponding one-point implications are also false. Actually, these facts are essentially equivalent since a counterexample to a onepoint implication immediately gives a counterexample to a quantitative zero-point implication, while the opposite direction involves the usual trick of gluing successively worse appendages either to each other or to a central subdomain. By contrast, we have few answers as to whether or not the corresponding two-point implications are true.

As a convenient reference, we include the following diagram of some of the basic quantitative zero-point implications among the various slice conditions that have been used in this paper.


As mentioned at the end of Section 2, the two left-to-right implications were established in [BS]. The remaining implications are immediate consequences of the definitions. The authors conjecture that the second and third columns of this diagram coincide; see Open Problem A below and the accompanying discussion. Eliminating the third column, the counterexamples in this section together with those in [BS, Section 5] show that the four remaining one-point implications cannot be reversed (and so the zero-point implications cannot be reversed with quantitative dependence). In fact, we even have a one-sided Slice domain which is
not a one-sided inner $\alpha$-wSlice (after Open Problem D) and a one-sided inner $\alpha$-wSlice domain that is not a one-sided Slice domain ([BS] for $\alpha>0$; the example after Open Problem B for $\alpha=0$ ). We also know that the one-sided $\alpha$-wSlice $(+)$ conditions are incomparable for different values of the slice parameter $\alpha$ (see the example after Open Problem B for one direction and Corollary 4.2 for the other).

At this point, the authors know much less in terms of being able to reverse the two-point versions of the four implications discussed above. We do know that the two left to right implications cannot be reversed when $\alpha>0$ (a counterexample appears in [BS]) but we conjecture that these arrows can be reversed in case $\alpha=0$. The same examples in [BS] prove the diagonal non-implications

$$
\begin{aligned}
\alpha \text {-wSlice } & \not \not ㇒ \text { Inner Slice }, & & \alpha>0, \\
\text { Inner } \alpha \text {-wSlice } & \nRightarrow \text { Slice }, & & \alpha>0 .
\end{aligned}
$$

But when $\alpha=0$, we again conjecture that these implications are valid. Below we give some more details on these open questions and related examples.

For any constants $C, C^{\prime}$, it is not hard to concoct a set of slices for a pair of points $x, y$ that satisfies the $(\alpha, C)$-wSlice condition, but not the $\left(\alpha, C^{\prime}\right)$-wSlice ${ }^{+}$condition. For instance, let us begin with annular slices $\left\{S_{i}\right\}_{i=1}^{m}$, as given by Lemma 3.4, for a pair $x, y$ in a ball. Cut each annulus $S_{i}$ into $2 N$ equally thin subannuli for some $N \in \mathbb{N}$, and redistribute each of these subannuli in alternating order into two new slices $S_{i}^{\prime}$ and $S_{i}^{\prime \prime}$. The set of new slices $\left\{S_{i}^{\prime}, S_{i}^{\prime \prime}\right\}_{i=1}^{m}$ still satisfy the $\alpha$ wSlice condition (although we must double the size of $C$ to ensure (WS1)) but no longer satisfy the extra wSlice ${ }^{+}$conditions with any given constant if $N$ is very large. However in this and all other examples we have constructed, there always exists a "better" set of slices which demonstates that the pair $x, y$ satisfies an $\alpha$-wSlice ${ }^{+}$condition. We suspect that in fact that the logically weaker $\alpha$-wSlice condition implies the $\alpha$-wSlice ${ }^{+}$condition quantitatively, but this seems hard to prove.

Open Problem A. If the pair $x, y \in G \subset \mathbb{R}^{n}$ satisfies an $(\alpha, C)$-wSlice condition, show that it also satisfies an $\left(\alpha, C^{\prime}\right)$-wSlice ${ }^{+}$condition for some $C^{\prime}=C^{\prime}(C, \alpha, n)$.

Let $0 \leq \alpha, \beta<1$. For the class of $\alpha$-mCigar domains to contain the class of $\beta$-mCigar domains it is necessary and sufficient that $\alpha \geq \beta$; see [L] and [BK2]. One might suspect that an analogous result might
be true for wSlice ${ }^{+}$(or wSlice) domains. Indeed, Corollary 4.2 gives us necessity. We suspect that sufficiency is also valid, but proving this appears to be difficult.

Open Problem B. Suppose $0 \leq \beta<\alpha<1$. Show that a $(\beta, C)$ wSlice $^{+}$domain is an $\left(\alpha, C^{\prime}\right)$-wSlice ${ }^{+}$domain for some $C^{\prime}=$ $C^{\prime}(C, \alpha, \beta, n)$.

The analogous result for mCigar domains is rather easy. In fact, an $\beta$-mCigar condition for a fixed pair $u, v \in G$ implies an $\left(\alpha, C^{\prime}\right)$-mCigar condition for $u, v$, with $C^{\prime}=C^{\prime}(C, \alpha, \beta, n)$, as can be seen from the proof of [BK2, Proposition 2.4]. By contrast, the wSlice ${ }^{+}$variant for a fixed pair of points cannot be true. Indeed, given $0<\beta<\alpha<1$, we now describe a bounded domain $G \subset \mathbb{R}^{3}$ which is a one-sided $\beta$-wSlice ${ }^{+}$ domain (with respect to $x_{0} \in G$ ), but it is not a one-sided $\alpha$-wSlice ${ }^{+}$ domain (with respect to $x_{0}$ ). It is not hard to modify this example to handle also the case $\beta=0$.

Our counterexample $G \subset \mathbb{R}^{3}$ is got by gluing together a sequence of open rectangular boxes $F_{n}, L_{n}(n \geq 0)$ of dimensions $R_{n} \times R_{n} \times r_{n}$ and $S_{n} \times s_{n} \times s_{n}$, respectively, where $R_{n}=2^{-n}, r_{n}=2^{-n(1-\alpha)^{-1}}$, $S_{n}=2^{-2 n}, s_{n}=2^{-n\left((1-\alpha)^{-1}+(1-\beta)^{-1}\right)}$; note that for large $n, r_{n}$ is much smaller than $R_{n}$ and $s_{n}$ is much smaller than $S_{n}$ so that $F_{n}$ is a flat box and $L_{n}$ is a long box. For each $n$, we choose a line segment of length $R_{n}$ (and $S_{n}$ ) linking the centers of opposite faces of $F_{n}$ (and $L_{n}$, respectively) and call this the main axis of this box. $G$ is then defined by gluing these boxes together according to the order $F_{0}, L_{0}, F_{1}, L_{1}, \ldots, F_{n}, L_{n}, \ldots$, so that all the main axes line up to form a single main axis (of symmetry) for $G$. Let $f_{k, t}$ and $l_{k, t}$ denote the $d_{t, G}$-length of the main axis of $F_{k}$ and $L_{k}$ respectively, for $k \geq 0$.

We claim that $l_{k, \beta} \approx f_{k, \beta} \approx 2^{k(\alpha-\beta) /(1-\alpha)}$, whereas $f_{k, \alpha} \approx 1$ and $l_{k, \alpha} \approx 2^{-k(\alpha-\beta) /(1-\beta)}$. Let us first consider $l_{n, \beta}$, for large $n$. It is easy to see that if we define a truncated box by chopping off a cube (of sidelength $s_{n}$ ) from both ends of $L_{n}$, then the $d_{\beta, G}$-length of the part of the main axis lying in the truncated box is comparable to $s_{n}^{\beta-1} S_{n} \approx l_{n, \beta}$. The length of the parts of the axis that were chopped off is at most comparable to

$$
\int_{0}^{r_{n}} t^{\beta-1} d t \approx r_{n}^{\beta}
$$

which is much smaller. The estimate for $l_{n, \alpha}$ is similar. For $f_{n, \alpha}$ and $f_{n, \beta}$, the estimates are derived in a similar fashion once we chop off a box of size $r_{n} \times R_{n} \times r_{n}$ from both ends of $F_{n}$ in such a way that these
little boxes cover the ends of the main axis of $F_{n}$. This establishes our claim.

The choice of $x_{0}$ is not important; we may as well take it to be the center of $F_{0}$. There is no difficulty in choosing slices for $x_{0}, x$ when $x \in F_{n} \cup L_{n}$ for some small $n$, since $d_{t, G}\left(x, x_{0}\right)$ is bounded in such cases ( $t=\alpha$ or $t=\beta$ ), so zero slices will suffice. Let us look at the case where $x=x_{n}$ is the center of $F_{n}$ for large $n$; it is easy to adjust the arguments to handle other points. Notice that the $d_{\beta, G}$ geodesic $\gamma_{n}$ from $x_{0}$ to $x_{n}$ is simply $\left[x_{0} \rightarrow x_{n}\right]$. For the $\beta$-wSlice ${ }^{+}$condition, we slice up the boxes $L_{k}, 0 \leq k<n$, perpendicular to their main axes into cubes of sidelength $s_{n}$, discarding any remnant at one end of $L_{i}$ which is too small to make another cube. Gathering together all these slices, it is easy to see that (WS-1+), (WS-3), and (WS-5) hold. Almost all the $d_{\beta, G^{-}}$length of $\gamma_{n} \cap L_{k}, 0 \leq k<n$, lies in some slice. Since also $l_{k, \beta} \approx f_{k, \beta}$, (WS-2) follows easily.

Suppose for the purposes of contradiction that an $\alpha$-wSlice ${ }^{+}$condition also holds for the pair $x_{0}, x_{n}$, uniformly in $n$. We show that this is untenable for large $n$. This is rather tricky but the idea is simple: flat boxes, unlike long boxes, cannot be "nicely sliced", which causes a problem since most of the $d_{\alpha, G}$-distance between $x_{0}$ and $x_{n}$ consists of flat boxes.

We denote by $F_{k}^{\prime}$ and $L_{k}^{\prime}$ the parts of a box $F_{k}$ or $L_{k}$, respectively, that lie within a distance $s_{k} / 2$ of a face of that box that is glued to another box, and by $T_{k}^{-}$and $T_{k}^{+}$the transitional part of $\overline{L_{k} \cup F_{k}} \cap G$ or $\overline{L_{k} \cup F_{k+1}} \cap G$ that lies within a distance $s_{k}$ of a glued face of one of its component boxes. We first modify the slices so that there only two types of slices: nice slices which are contained in a single $F_{k}^{\prime}$ or $L_{k}^{\prime}$, and transitional slices that are contained in either $T_{k}^{-}$or $T_{k}^{+}$for some $k$. This can be done (with a controlled change in the slice constant) by replacing each original slice $S$ with $S \cap F_{k}, S \cap L_{k}, S \cap T_{k}^{-}$, or $S \cap T_{k}^{+}$, for some $k$; we leave the details to the reader.

Let us fix a box $B$ from among the boxes intersecting $\left[x_{0}, x_{n}\right]$. Take $x$ to be the point in the box to the immediate left of $B$ which lies on [ $x_{0}, x_{n}$ ] and whose Euclidean distance from $B$ equals $r(B)$ (note that $r(B)$ is $r_{k}$ or $s_{k}$ for some $0 \leq k \leq n$, depending on whether $B$ is a flat or a long box), and take $y$ to be the corresponding point in the box to the immediate right of $B$. There are two endpoint cases where these definitions do not make sense: if $B=F_{0}$, instead let $x=x_{0}$ and if $B=F_{n}$, instead let $y=x_{n}$. The $d_{\alpha, G}$-length of the line segment joining $x$ and $y$ is easily seen to be comparable to the $d_{\alpha, G}$-length of
the main axis of $B$, which we call $l_{\alpha}(B)$ for short. By construction, the nice slices in $B$ also satisfy (WS-1) and (WS-3) for the pair $x, y$, so Lemma 2.1 implies that the contribution to the sum in (WS-2) of the numbers $d_{i}$ that correspond to these slices is at most some constant multiple of $l_{\alpha}(B)$. Similarly, for large $n$, the transitional slices between two adjacent boxes $B_{1}, B_{2}$ cannot contribute more than a small multiple of $l_{\alpha}\left(B_{1}\right)+l_{\alpha}(B+2)$.

Since $f_{k, \alpha} \approx 1$ is much larger than $l_{k, \alpha}$ for large $k$, the last estimates imply that the contributions of the nice slices contained in $F_{k}$ must be bounded below, at least for some fixed fraction of the numbers $0 \leq k \leq$ $n$. However (WS-1) implies that nice slices in $F_{k}$ must have diameter comparable with $R_{k}$. It follows that their number is bounded and that their total contribution can be at most comparable with $R_{k}^{\alpha}$. Since $R_{k}^{\alpha}$ is much smaller than 1 , we get a contradiction.

Note that above we have only used the wSlice conditions, not (WS$1^{+}$) or (WS-5), so as to emphasise that the peculiarity of this example is not because of the latter extra conditions. The proof that an $\alpha$-wSlice ${ }^{+}$ condition does not uniformly hold for pairs $x_{0}, x_{n}$ is a little easier if we use (WS-5). Also note that $G$ is not a $\beta$-wSlice ${ }^{+}$domain, as can be shown by considering the $\beta$-wSlice ${ }^{+}$condition for points near either end of $F_{n}$ for large $n$.

Open Problem C. Show that a $(0, C)$-wSlice domain is a $C^{\prime}$-Slice domain for some $C^{\prime}=C^{\prime}(C, n)$.

According to Corollary 4.2, the classes of $\alpha$-wSlice ${ }^{+}$domains are distinct for all $\alpha>0$, and according to [BS, Proposition 4.5] there are domains that are $\alpha$-wSlice ${ }^{+}$domains for all $\alpha>0$, but not Slice domains. However, even if the first two open problems can be made into theorems, Open Problem C remains unresolved. Furthermore, taking $0=\beta<\alpha=1 / 2$, the counterexample $G$ to the one-point variant of Open Problem B is also a counterexample to the one-point variant of this problem since, as mentioned in Section 2, any Slice condition implies an $\alpha$-wSlice condition quantitatively.

Open Problem D. Suppose $0 \leq \alpha<1$. Show that an $(\alpha, C)$-wSlice domain (or $C$-Slice domain) is an inner ( $\alpha, C^{\prime}$ )-wSlice domain (or inner $C^{\prime}$-Slice domain, respectively) for some $C^{\prime}=C^{\prime}(C, n)$.

Note that if this can be shown then the class of $(\alpha ; d)$-wSlice domains is the same for every metric $d$ lying between the Euclidean and inner Euclidean metrics.

Yet again, there are counterexamples for the one-point variant of this problem. Consider for example the planar domain $G=(0,1)^{2} \cup$ $\left(\bigcup_{k=1}^{\infty} R_{k}\right)$, where
$R_{k}=\left(\left(2^{-k}-2^{-s k}, 2^{-k}+2^{-s k}\right) \times\left[1,1+2^{-k}\right)\right) \backslash\left(\left\{2^{-k}\right\} \times\left[1,1+2^{-k-1}\right]\right)$
for some $s>2$; note that $G$ consists of the unit square with disjoint narrow slitted rectangles attached. Taking $u_{0}=(1 / 2,1 / 2)$ and $v$ to be arbitrary, we claim that the pair $u_{0}, v$ satisfies any of the Euclidean slice conditions with a constant independent of $v=\left(v_{1}, v_{2}\right)$, but that it does not uniformly satisfy any inner Slice condition, nor any inner $\alpha$-wSlice condition if $s>1 /(1-\alpha)$.

In the positive direction, we sketch only the $\alpha$-wSlice ${ }^{+}$condition for $\alpha>0$; the case $\alpha=0$ and the Slice condition are left as exercises. The cases where $v \in(0,1)^{2}$, or $v \in R_{k}$ with $v_{2}-1 \lesssim 2^{-s k}$, are easily handled since $d_{\alpha, G}\left(u_{0}, v\right)$ is then bounded so $u_{0}, v$ satisfy an $\alpha$-wSlice ${ }^{+}$condition with zero slices. Suppose instead that $v \in R_{k}$ and $v_{2}>1+4 \cdot 2^{-s k}$. For each $i \in \mathbb{N}$ define

$$
S_{i} \equiv R_{k} \cap\left(\mathbb{R} \times\left(1+2^{-s k+1}(i-1), 1+2^{-s k+1} i\right)\right), \quad i \in \mathbb{N} .
$$

Letting $\gamma \in \Gamma_{G}\left(u_{0}, v\right)$ be such that $\operatorname{len}_{\alpha, G}(\gamma)<2 d_{\alpha, G}\left(u_{0}, v\right)$, and letting $m$ be the integer such that $v \in S_{m+2}$, it is straightforward to verify a uniform $\alpha$-wSlice ${ }^{+}$condition for $u_{0}, v$ with slice data $\gamma$, $\left\{S_{i}, \operatorname{dia}\left(S_{i}\right)\right\}_{i=1}^{m}$.

For the negative results, it suffices to show that for every $\alpha \in[0,1)$, $s>1 /(1-\alpha)$, and $C>1$, there always exists $v \in G$ such that the pair $u_{0}=(1 / 2,1 / 2), v$ fails to satisfy the inner $(\alpha, C)$-wSlice condition. We consider only the case $\alpha>0$; the case $\alpha=0$ is left as an exercise. We write $v_{k}=\left(2^{-k}+2^{-s k-1}, 1+2^{-k-1}\right), k \in \mathbb{N}$. We claim that if the data $\gamma,\left\{S_{i}, d_{i}\right\}_{i=1}^{m}$ satisfies (WS-1) and (WS-3) for the pair $u_{0}, v_{k} \in G$, and $d_{i} \geq d_{G}\left(S_{i}\right)$, then $\Sigma \equiv \sum_{i=1}^{m} d_{i}^{\alpha} \lesssim 1$. Since $d_{\alpha, G}\left(u_{0}, v_{k}\right) \approx 2^{k(s(1-\alpha)-1)}$ grows arbitrarily large as $k \rightarrow \infty$, it follows from this claim that pairs $u_{0}, v_{k}$ cannot uniformly satisfy any inner $\alpha$-wSlice condition.

We may as well assume that the slices $S_{i}$ are contained in $(0,1)^{2} \cup$ $R_{k}$, since if we remove those parts of $S_{i}$ lying in $R_{j}, j \neq k$, it follows that (WS-1) must still be true with the same constant $C$. Let $\Sigma_{1}$ be the subsum of $\Sigma$ corresponding to those slices contained entirely in

$$
A_{k} \equiv G \cap\left(\mathbb{R} \times\left(0,1+2^{-s k+1}\right)\right)
$$

The subset of slices contained in $A_{k}$, together with the corresponding numbers $d_{i}$, forms a set of data satisfying (WS-1) and (WS-3) for the
pair of points $u_{0}, v_{k}^{\prime}$, with $v_{k}^{\prime}=\left(2^{-k}+2^{-s k-1}, 1+2^{-s k+2}\right)$. Since $d_{\alpha, G}\left(u_{0}, v_{k}^{\prime}\right) \lesssim 1$, it follows from Lemma 2.1 that $\Sigma_{1} \lesssim 1$.

Next let $\Sigma_{2}$ be the subsum corresponding to those slices contained entirely in $R_{k}$. Since we can move from $u_{0}$ to $v_{k}$ by going up either side of the slit in $R_{k}$, an argument similar to that used in the proof of Theorem 5.1 shows that the numbers $d_{i}$ satisfy a geometric growth condition. It readily follows that $\Sigma_{2} \lesssim 1$.

Finally, we consider slices $S_{i}$ that intersect both $(0,1)^{2}$ and $R_{k} \backslash A_{k}$. If by replacing $S_{i}$ by $S_{i} \cap A_{k}$ (but leaving $d_{i}$ unchanged) we get a wouldbe slice that satisfies (WS-1) with $C$ replaced by $2 C$, then we can include the term $d_{i}^{\alpha}$ in $\Sigma_{1}$. Assume instead that $S_{i} \cap A_{k}$ fails to satisfy (WS-1) even with $C$ replaced by $2 C$. Let $w_{0}=\left(x_{0}, 1\right)$ be the point of first entry into $R_{k}$ of a path $\lambda_{0}$ for which this version of (WS-1) fails. Since $\operatorname{len}\left(\lambda_{0} \cap S_{i}\right) \geq d_{i} / C$ and $\operatorname{len}\left(\lambda_{0} \cap S_{i} \cap A_{k}\right)<d_{i} / 2 C$, it follows that

$$
\operatorname{len}\left(\lambda_{0} \cap S_{i} \cap(0,1)^{2}\right)<\frac{d_{i}}{2 C}-2^{-s k+1}
$$

Suppose that there exists $\lambda_{1} \in \Gamma_{G}\left(u_{0}, v_{k}\right)$ such that $\operatorname{len}\left(\lambda_{1} \cap S_{i} \cap R_{k}\right)<$ $d_{i} / 2 C$. We define a path $\lambda \in \Gamma_{G}\left(u_{0}, v_{k}\right)$ as follows: $\lambda$ coincides with $\lambda_{0}$ as far as the point $w_{0}$, then it traverses a path in $G \cap[(0,1) \times(0,1]]$ of length at most $2^{-s k+1}$ from $w_{0}$ to the last point of entry of $\lambda$ into $R_{k}$, and finally it traverses the final segment of $\lambda_{1}$. Such a path $\lambda$ would satisfy len $\left(\lambda \cap S_{i}\right)<d_{i} / C$ in contradiction to (WS-1). Thus if we replace $S_{i}$ by $S_{i} \cap R_{k}$, and $C$ by $2 C$, then (WS-1) is still satisfied, and so we may include the term $d_{i}^{\alpha}$ in $\Sigma_{2}$. There are no remaining terms, so our claim is proved and we are done.

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[^14]
# Quasicircles modulo bilipschitz maps 

Steffen Rohde


#### Abstract

We give an explicit construction of all quasicircles, modulo bilipschitz maps. More precisely, we construct a class $\mathcal{S}$ of planar Jordan curves, using a process similar to the construction of the van Koch snowflake curve. These snowflake-like curves are easily seen to be quasicircles. We prove that for every quasicircle $\Gamma$ there is a bilipschitz homeomorphism $f$ of the plane and a snowflake-like curve $S \in \mathcal{S}$ with $\Gamma=f(S)$. In the same fashion we obtain a construction of all bilipschitzhomogeneous Jordan curves, modulo bilipschitz maps, and determine all dimension functions occuring for such curves. As a tool, we construct a variant of the Konyagin-Volberg uniformly doubling measure on $\Gamma$.


## 1. Introduction.

Quasicircles are images of circles under quasiconformal maps of the plane, see Section 2 for definitions and basic properties. They appear in many different settings in analysis, for instance as Julia sets of some rational maps, as limit sets of some Kleinian groups, or as boundaries of those domains for which every BMO-function extends. There are a large number of characterizations of quasicircles, see [G]. In this paper we present a simple construction of Jordan curves that yields all quasicircles, up to applying a bilipschitz map of the plane.

To give a rough description of our snowflake-like curves $S$, proceed as in the inductive construction of the standard van-Koch snowflake,
with the main difference that there are two replacement options instead of just one: Each of the $4^{n}$ line segments of the $n$-th generation can be replaced by a rescaled and rotated copy of one of the two polygonal arcs of Figure 1.1. The sidelength $p$ of the first alternative is a parameter that is fixed throughout the construction of each individual $S$. See Section 3 for a more precise description. Denote $\mathcal{S}$ the collection of all curves $S$ obtained in this way.


Figure 1.1. The two polygonal arcs allowed in forming a snowflake-like curve.

Theorem 1.1. A Jordan curve $\Gamma \subset \mathbb{R}^{2}$ is a quasicircle if and only if there are $S \in \mathcal{S}$ and a bilipschitz map $f$ of $\mathbb{R}^{2}$ such that

$$
\Gamma=f(S)
$$

If $\Gamma$ is a $K$-quasicircle, then there is $p=p(K)$ and a bilipschitz $f$ with $\Gamma=f(S)$. If in addition diam $\Gamma=1$, then the bilipschitz norm of $f$ depends on $K$ only.

As a possible application, consider a domain property that is invariant under bilipschitz maps. To decide if such a property holds for all quasidiscs (domains bounded by quasicircles), it is sufficient to test all snowflake-like curves. To illustrate what we have in mind, notice that the domains bounded by our snowflake-like curves are easily seen to be John domains (every point $x$ in the boundary can be joined to an interior point $x_{0}$ by a curve $\gamma$ such that for every point $y \in \gamma$, the distance of $y$ to the boundary is comparable to the diameter of the arc of $\gamma$ between $x$ and $y$ ). Since this John property is obviously preserved under bilipschitz maps, we conclude from Theorem 1.1 the (well-known) fact that quasidiscs are John-domains.

The proof of Theorem 1.1 is based on the construction of a uniformly doubling measure $\mu$ on $\Gamma$ which, in a scaling invariant way, is bounded above resp. below by 1 -dimensional respectively $\alpha$-dimensional Hausdorff content, where $\alpha<2$. More specifically, we prove

Theorem 1.2. Let $\Gamma$ be a $K$-quasicircle. Then there are a probability measure $\mu$ on $\Gamma$ and constants $C>0, \alpha<2$ depending only on $K$ such that

$$
C^{-1} \frac{r}{s} \leq \frac{\mu(B(x, r))}{\mu(B(x, s))} \leq C\left(\frac{r}{s}\right)^{\alpha},
$$

for all $s<r \leq \operatorname{diam} \Gamma=1$ and all $x \in \Gamma$.

Measures satisfying the upper bound have been constructed in arbitrary metric spaces by Konyagin and Volberg [KV], with any exponent larger than the Assouad dimension of the space. A simpler construction for arbitrary compact sets in $\mathbb{R}^{n}$ was given by $\mathrm{Wu}[\mathrm{W}]$. It is clear that measures having the lower bound do not exist in such generality, a minimal (though not sufficient) requirement being that the Hausdorff dimension of $\Gamma$ is 1 .

A Problem. Our construction of the measure of Theorem 1.2 is not canonical. Natural measures such as harmonic measure or Hausdorff measures don't work in general. Is there a natural (for instance Möbius invariant) construction?

The idea of the proof of Theorem 1.1 is as follows: Given $\Gamma$ and $\mu$ as above, we obtain a quasisymmetric homeomorphism $f: \mathbb{T} \longrightarrow$ $\Gamma$ such that $|I| \asymp \mu(f(I))$ for all arcs $I \subset \mathbb{T}$, where $\mathbb{T}$ is the unit circle and $|I|$ denotes normalized length. Here and in what follows we write $a \asymp b$ if the ratio $a / b$ is bounded above and below away from zero. We construct a snowflake-like curve $S$ together with a natural parametrization $g: \mathbb{T} \longrightarrow S$ satisfying $|I| \asymp \mu(g(I))$. Then we use the trivial but useful observation that quasiconformal maps are determined by their Jacobian determinant, up to composition by bilipschitz maps, Lemma 2.1 below. Applied to extensions of $f$ and $g$ this shows that $f \circ g^{-1}$ is a bilipschitz homeomorphism mapping $S$ to $\Gamma$.

The same idea can be applied to bilipschitz-homogeneous curves. A Jordan curve $\Gamma$ is called bilipschitz-homogeneous if there is a constant $L$ such that for every pair of points $a, b \in \Gamma$ there is a L-bilipschitz homeomorphism $f: \Gamma \longrightarrow \Gamma$ satisfying $f(a)=b$. These curves have been extensively studied by Mayer [M], Ghamsari and Herron [GH], [HM]. Recently Bishop [B] succeeded in proving that they are always quasicircles. Now consider the class $\mathcal{H S}$ of homogeneous snowflake-like curves $S$ defined by requiring that during the construction of $S$ all of the $4^{n}$ line segments of the $n$-th generation are replaced by the same
(rescaled and rotated) polygonal arc of Figure 1.1. Our next theorem says that these curves are precisely the bilipschitz-homogeneous curves, modulo bilipschitz maps.

Theorem 1.3. Let $\Gamma \subset \mathbb{R}^{2}$ be a Jordan curve. Then the following statements are equivalent:
i) $\Gamma$ is bilipschitz-homogeneous.
ii) There is $S \in \mathcal{H S}$ and a bilipschitz map $f$ of $\mathbb{R}^{2}$ such that $\Gamma=$ $f(S)$.
iii) There is a quasiconformal map $F$ of $\mathbb{R}^{2}$ with $\Gamma=F(\mathbb{T})$ such that the Jacobian determinant JF satisfies

$$
C^{-1} \leq \frac{J F(w)}{J F(z)} \leq C\left(\frac{1-|z|}{1-|w|}\right)^{\alpha}
$$

for some constants $C>0,0 \leq \alpha<1$ and all $z, w \in \mathbb{D}$ with $|z| \leq|w|$.
iv) There is a quasiconformal map $F$ of $\mathbb{R}^{2}$ with $\Gamma=F(\mathbb{T})$ such that $J F$ is almost radial (i.e. $J F(x) \asymp J F(|x|)$ for all $\left.x \in \mathbb{R}^{2}\right)$.

It is an open problem to characterize Jacobian determinants of quasiconformal maps (up to a bounded factor, say). David and Semmes conjectured that a weight $\omega: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{+}$is comparable to a Jacobian determinant if and only if $\omega$ is a strong $A^{\infty}$-weight. In this context, part iv) of Theorem 1.3 can be viewed as a characterization of sufficiently regular almost radial Jacobian determinants of quasiconformal maps:

Corollary 1.4. Let $\omega:[0,1) \longrightarrow \mathbb{R}^{+}$be non-decreasing. There is a quasiconformal map $F$ of $\mathbb{R}^{2}$ with $J F(z) \asymp \omega(|z|)$ in $\mathbb{D}$ if and only if

$$
C^{-1} \leq \frac{\omega(s)}{\omega(r)} \leq C\left(\frac{1-r}{1-s}\right)^{\alpha}
$$

for some $C>0, \alpha<1$ and all $0 \leq r \leq s<1$.
For a compact set $A \subset \mathbb{R}^{2}$, denote $N_{A}(r)$ the minimal number of discs of radius $r$ needed to cover $A$. Then $\delta(r)=N_{A}(r)^{-1}$ is a canonical choice of a dimension function in order to obtain a Hausdorff measure supported on $A$. Part iii) of Theorem 1.3 solves the problem posed in $[\mathrm{HM}]$ about characterizing the dimension functions $\delta:[0,1] \longrightarrow[0,1]$ that can occur for bilipschitz homogeneous curves:

Corollary 1.5. Let $\delta:[0,1] \longrightarrow[0,1]$ be non-decreasing. Then $\delta$ is comparable to $N_{\Gamma}(r)^{-1}$ for a bilipschitz homogeneous curve $\Gamma$ if and only if

$$
\frac{\delta(s)}{\delta(r)} \leq C\left(\frac{s}{r}\right)^{\beta}
$$

for some $C>0, \beta<2$ and all $0<r \leq s \leq 1$.
Organization of the paper. Section 2 provides the (well-known) background concerning quasiconformal maps. The snowflake-like curves and their parametrizations are described in Section 3. Section 4 contains the construction of the doubling measure and is independent from the rest of the paper. Theorem 1.1 is proved in Section 5. Section 6 is devoted to bilipschitz homogeneous curves. There we prove Theorem 1.4 and the corollaries.

## 2. Quasiconformal maps and their Jacobians.

In this section we collect the facts about quasiconformal maps needed throughout the rest of the paper. The expert may safely skip it. Let $K \geq 1$ and consider an orientation preserving homeomorphism $f: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$. Then $f$ is K-quasiconformal if $f \in W_{\text {loc }}^{1,2}$ (first order distributional derivatives being locally square-integrable) and if the inequality $|D f(x)|^{2} \leq K J f(x)$ between the operator-norm of the derivative $D f$ and the Jacobian determinant $J f$ holds almost everywhere. We have $K>1$ unless $f$ is conformal. The standard references to the basic theory are [A] and [LV].

Recall that homeomorphisms $f$ of $\mathbb{R}^{2}$ are called L-bilipschitz if

$$
\frac{1}{L}|x-y| \leq|f(x)-f(y)| \leq L|x-y|
$$

for all $x, y \in \mathbb{R}^{2}$. The smallest such $L$ is refered to as the bilipschitz norm of $f$. It is clear that bilipschitz maps are quasiconformal, whereas the converse is false in general.

Quasiconformal maps are quasisymmetric (if $|x-y|=|x-z|$, then $|f(x)-f(y)| \leq C|f(x)-f(z)|)$ and vice versa. If $f: \mathbb{T} \longrightarrow f(\mathbb{T})$ is quasisymmetric, then there is a quasiconformal extension $f: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ such that

$$
\begin{equation*}
\operatorname{diam} f(I) \asymp|I||D f(x)|, \tag{2.1}
\end{equation*}
$$

for every $\operatorname{arc} I \subset \mathbb{T}$ and every point $x \in \mathbb{R}^{2}$ for which $\operatorname{dist}(x, \mathbb{T}) \asymp$ dist $(x, I) \asymp|I|$. We may further assume that $|D f(x)| \asymp 1$ for $|x|>2$.

Lemma 2.1. If $f, g$ are quasiconformal homeomorphisms of $\mathbb{R}^{2}$ and if

$$
J f(x) \asymp J g(x), \quad \text { almost everywhere },
$$

then

$$
F=f \circ g^{-1}
$$

is bilipschitz.

Proof. By the chainrule $J F(x) \asymp 1$ almost everywhere. Since $F$ is quasiconformal, we obtain $|D F| \asymp 1$ almost everywhere. The lemma follows from $F \in W_{\text {loc }}^{1,2}$.

The images of circles under quasiconformal maps of the plane are called quasicircles. A simple closed curve (Jordan curve) $\Gamma$ is a quasicircle if and only if

$$
\begin{equation*}
\sup _{x, y \in \Gamma} \frac{\operatorname{diam} \Gamma(x, y)}{|x-y|}<\infty, \tag{2.2}
\end{equation*}
$$

where $\Gamma(x, y)$ denotes the subarc between $x$ and $y$ of smaller diameter. This is the Ahlfors three-point condition.

## 3. Snowflake-like curves.

To describe the construction, fix a parameter $1 / 4 \leq p<1 / 2$ defining the first arc $\gamma$ of Figure 1.1. Denote by $\gamma^{\prime}$ the second arc (the line segment) of Figure 1.1. Inductively define polygons $S_{n}$ consisting of $4^{n}$ line segments as follows: Denote the unit square by $S_{1}$. To pass from $S_{n}$ to $S_{n+1}$, for each of the $4^{n}$ edges $[x, y]$ of $S_{n}$ replace $[x, y]$ by a scaled copy of $\gamma$ or $\gamma^{\prime}$. Here we assume that $x$ follows $y$ in the positive orientation of $S_{n}$, that the scaling map is orientation preserving, and that it maps the left endpoint of $\gamma$ respectively $\gamma^{\prime}$ onto $x$. See Figure 3.1 for a possible $S_{3}$.


Figure 3.1. A possible $S_{3}$.
For a given $S_{n}$ there are $2^{4^{n}}$ possibilities for choosing $S_{n+1}$. It is clear that each sequence $S_{n}$ thus obtained converges (geometrically) to a closed limit curve $S$. Below we will show that these limit curves are quasicircles, in particular they are Jordan curves. Denote $\mathcal{S}(p)$ the collection of all limit curves $S$, and set

$$
\mathcal{S}=\bigcup_{1 / 4 \leq p<1 / 2} \mathcal{S}(p) .
$$

Next, consider the class

$$
\mathcal{H S}=\bigcup_{1 / 4 \leq p<1 / 2} \mathcal{H S}(p)
$$

of homogeneous snowflake-like curves defined as follows: A curve $S \in$ $\mathcal{S}(p)$ belongs to $\mathcal{H S}(p)$ if and only if the approximating curve $S_{n+1}$ is formed from $S_{n}$ by replacing all edges of $S_{n}$ by a scaled copy of the same arc $\gamma$ or $\gamma^{\prime}$. Hence there are only two choices of $S_{n+1}$ for a given $S_{n}$.

Lemma 3.1. Every curve $S \in \mathcal{S}(p)$ is a $K$-quasicircle, with $K$ depending on $p$ only.

Proof. For an edge $I$ of some $S_{n}$, denote $T(I)$ the isosceles triangle with base $I$ and height $\sqrt{p-1 / 4}|I|$. So $T(I)$ is the convex hull of the rescaled arc $\gamma$. Then the (smaller) arc $S(I)$ of $S$ with the same endpoints
as $I$ is contained in $T(I)$. If $J$ is another edge (of some $S_{m}$ ), one easily proves by induction that either $I \cap J \neq \varnothing$ or

$$
\operatorname{dist}(T(I), T(J)) \geq c_{p} \min \{\operatorname{diam} T(I), \operatorname{diam} T(J)\}
$$

Using the Ahlfors three-point condition (2.2), the lemma easily follows.
Next we will describe a one to one correspondence between $\mathcal{S}(p)$ and certain labelled graphs. Let $G=(V, E)$ be the infinite planar graph depicted in Figure 3.2. It is obtained from a rooted homogeneous tree of degree 7 by cyclically joining the $4^{n}$ vertices $v \in V$ of graph-distance $d(v)=d\left(v, v_{0}\right)=n$ from the root $v_{0}$.


Figure 3.2. The graph $G$.
The correspondence between a vertex $v$ and an $\operatorname{arc} S(v)$ of $S$ is characterized by the following four properties:
i) $S\left(v_{0}\right)=S$.
ii) If $d(v)=n$, then $S(v)$ is an arc obtained from an edge of $S_{n}$.
iii) If $d(v)=d\left(v^{\prime}\right)=n$ and if $v, v^{\prime}$ are adjacent in $G$, then $S(v)$ and $S\left(v^{\prime}\right)$ have a common endpoint.
iv) If $v^{\prime}$ is a descendent of $v$ (i.e. $\left.d\left(v, v^{\prime}\right)=d\left(v^{\prime}\right)-d(v)\right)$ then $S\left(v^{\prime}\right) \subset S(v)$.

Define the labelling $\ell_{S}: V \longrightarrow \mathbb{R}_{+}$by

$$
\begin{equation*}
\ell_{S}(v)=\operatorname{diam} S\left(v^{\prime}\right), \tag{3.1}
\end{equation*}
$$

where $v^{\prime}$ is any child of $v$ (that is $d\left(v, v^{\prime}\right)=d\left(v^{\prime}\right)-d(v)=1$ ).
This process of passing from $S$ to $\ell$ clearly is reversible: If $\ell: V \longrightarrow$ $\mathbb{R}_{+}$is given and has the property that

$$
\begin{equation*}
\frac{\ell\left(v^{\prime}\right)}{\ell(v)} \in\left\{p, \frac{1}{4}\right\}, \tag{3.2}
\end{equation*}
$$

whenever $v^{\prime}$ is a child of $v$, then there is a curve $S=S_{\ell} \in \mathcal{S}(p)$, unique up to rotation, such that $\ell=\ell_{S}$.

Given $S \in \mathcal{S}(p)$, there is a canonical homeomorphism $\phi_{S}: S_{1} \longrightarrow$ $S$, where $S_{1}$ is the unit square. It is the map that sends a four-adic interval $S_{1}(v)$ on $S_{1}$ onto the corresponding arc $S(v)$. More formally, the labelling $\ell_{1}(v) \equiv 4^{-d(v)}$ satisfies the above assumption (3.2) and obviously yields $S_{1}=S_{\ell_{1}}$. With this interpretation, $\phi_{S}$ is given by

$$
\begin{equation*}
\phi_{S}\left(S_{1}(v)\right)=S(v), \tag{3.3}
\end{equation*}
$$

for every $v \in V$.
The next lemma can be proved in the same way as Lemma 3.1.
Lemma 3.2. Given $S \in \mathcal{S}(p)$, the homeomorphism

$$
\phi_{S}: S_{1} \longrightarrow S
$$

is quasisymmetric if and only if there is $C$ such that

$$
C^{-1} \leq \frac{\ell\left(v^{\prime}\right)}{\ell(v)} \leq C
$$

for all adjacent vertices $v, v^{\prime} \in V$.
Notice that for every $S \in \mathcal{S}(p)$ there exists a quasisymmetric parametrization $\phi: S_{1} \longrightarrow S$. But the natural parametrization described above need not be quasisymmetric.

## 4. The doubling measure.

This section is devoted to the proof of Theorem 1.2. Throughout this section $\Gamma$ is a K- quasicircle. We are first going to show that the uniform metric dimension (Assouad dimension) of $\Gamma$ is bounded away from 2, depending only on K. More precisely, we have

Lemma 4.1. There are constants $C(K)>0$ and $a(K)<2$ such that, for every $q>0$, every arc $I \subset \Gamma$ contains at most

$$
n \leq \frac{C}{q^{a}}
$$

disjoint subarcs $I_{1}, \ldots, I_{n}$ of diameters $d_{m} \geq q \operatorname{diam} I$.
Proof. It is well-known (and easily follows from quasisymmetry) that quasicircles are porous: There is a constant $c(K)$ such that for every disc $D(x, r)$ there is a disc $D(y, c r) \subset D(x, r) \backslash \Gamma$. Let $S$ be a square of sidelength $l$, subdivided into $k^{2}$ subsquares $S_{j}$ with sidelength $l / k$. Then porosity and induction shows that $\Gamma$ meets at most $C k^{a}$ of the $S_{j}$, where $a<2$ depends only on $c$. Setting $l=\operatorname{diam} I$ and $k=[1 / q]$, the lemma follows from the fact that only a bounded number of the $\operatorname{arcs} I_{m}$ can meet a fixed $S_{j}$, by the three-point property.

Proof of Theorem 1.2. We may assume $\operatorname{diam} \Gamma=1$. Let $a$ be the constant from Lemma 4.1, pick any $a<b<2$ and choose a sufficiently small number $q<1$, specified during the course of the proof.

First choose a sequence $\mathcal{I}_{n}=\left\{I_{n, j}\right\}$ of subdivisions of $\Gamma$ into disjoint half-open arcs $I_{n, j}$ with the following two properties:
a) $q^{n} \leq \operatorname{diam} I_{n, j}<2 q^{n}$ for all $n, j$.
b) For $I \in \mathcal{I}_{n}$ and $J \in \mathcal{I}_{n+1}$, either $J \subset I$ (in this case we write $J<I$ ), or $J \cap I=\varnothing$.

Such a sequence is easy to find by successive "bisection" of arcs. Next, define a sequence $\mu_{n}$ of probability measures on $\Gamma$ by specifying $\mu_{n}\left(I_{n, j}\right)$ for each $n, j$. Our measure $\mu$ will be the weak limit of $\mu_{n}$. The $\mu_{n}$ will have the following properties:

1) For all $n$ and all pairs of adjacent $\operatorname{arcs} I, I^{\prime} \in \mathcal{I}_{n}$,

$$
\frac{1}{10} \leq \frac{\mu_{n}(I)}{\mu_{n}\left(I^{\prime}\right)} \frac{\operatorname{diam} I^{\prime}}{\operatorname{diam} I} \leq 10 .
$$

2) For all $n$ and all $\operatorname{arcs} I \in \mathcal{I}_{n}$, the mass $\mu_{n}(I)$ is distributed over its "children" $J<I$, i.e. no mass from $I$ is transported away from $I$

$$
\sum_{J<I} \mu_{n+1}(J)=\mu_{n}(I) .
$$

3) For all $n$, all $\operatorname{arcs} I \in \mathcal{I}_{n}$ and all $\operatorname{arcs} J<I$ we have

$$
\frac{\operatorname{diam} I}{\operatorname{diam} J} \leq \frac{\mu_{n}(I)}{\mu_{n+1}(J)} \leq q^{-b} .
$$

It is immediate from 2) and a) above that $\mu_{n}$ weakly converges to a measure $\mu$ on $\Gamma$. Before we proceed with the construction of $\mu_{n}$, let us show that $\mu$ will have the required properties. To this end, consider $x \in \Gamma$ and $0<r<R \leq 1$. By a) and b) there are $\operatorname{arcs} I \in \mathcal{I}_{n}$ and $J \in \mathcal{I}_{m}$ with $x \in J \subset I$ and $\operatorname{diam} I \asymp R \asymp q^{n}, \operatorname{diam} J \asymp r \asymp q^{m}$. It easily follows from the three-point property, together with 1) and (2), that $\mu(B(x, R)) \asymp \mu(I)$ and $\mu(B(x, r)) \asymp \mu(J)$. Now 3) implies

$$
\frac{\operatorname{diam} I}{\operatorname{diam} J} \leq \frac{\mu(I)}{\mu(J)} \leq q^{-b(m-n)} \asymp\left(\frac{R}{r}\right)^{b}
$$

proving the theorem.
Now we describe the inductive construction of $\mu_{n}$. Set

$$
\mu_{1}\left(I_{1, j}\right)=\frac{1}{\# \mathcal{I}_{1}},
$$

for all $j$, where \# denotes cardinality. Then 1 ) is clear from a), and 2), 3) are void.

To obtain $\mu_{n+1}$ from $\mu_{n}$, let $I \in \mathcal{I}_{n}$ and let $J_{1}, \ldots, J_{r}$ denote the children of $I$ (i.e. $J_{l}<I$ ), where $r=r(I)$ is the number of children. We assume the labeling is such that $J_{l}$ and $J_{l+1}$ are adjacent for all $l$. A first attempt is to set

$$
m_{l}=m\left(J_{l}\right)=\mu_{n}(I) \frac{\operatorname{diam} J_{l}}{\sum_{k=1}^{r} \operatorname{diam} J_{k}}
$$

and to $\operatorname{try} \mu_{n+1}\left(J_{l}\right)=m_{l}$. Notice that $m_{l} \asymp \mu_{n}(I) / r$ so that we would roughly equidistribute the mass of $I$ over its children. But then there is no reason for the ratio in 1) to remain bounded after some generations. To fix this, we proceed similarly to [W] and define

$$
\mu_{n+1}\left(J_{l}\right)=w_{l} m_{l}
$$

with weights $w_{l}=w\left(J_{l}\right)$ described below.

Let us denote $I^{-}$and $I^{+}$the two arcs of $\mathcal{I}_{n}$ adjacent to $I$, and by $J_{0}<I^{-}, J_{r+1}<I^{+}$the arcs of $\mathcal{I}_{n+1}$ adjacent to $J_{1}$ respectively $J_{r}$. Set $m_{0}=m\left(J_{0}\right)$ and $m_{r+1}=m\left(J_{r+1}\right)$. Notice that $m_{0} \asymp \mu_{n}\left(I^{-}\right) / r\left(I^{-}\right) \asymp$ $\mu_{n}(I) / r\left(I^{-}\right)$.

We first define $w_{1}$ and $w_{r}$ : Set

$$
Q\left(J, J^{\prime}\right)=\frac{m(J)}{m\left(J^{\prime}\right)} \frac{\operatorname{diam} J^{\prime}}{\operatorname{diam} J}
$$

and let $w_{1}=1$ if $Q\left(J_{1}, J_{0}\right) \geq 1 / 10$, and $w_{1}=1 /\left(10 Q\left(J_{1}, J_{0}\right)\right)$ if $Q\left(J_{1}, J_{0}\right)<1 / 10$. In the same way define $w_{r}=1$ if $Q\left(J_{r}, J_{r+1}\right) \geq 1 / 10$, else $w_{r}=1 /\left(10 Q\left(J_{r}, J_{r+1}\right)\right)$. This definition applies to all those $J \in$ $\mathcal{I}_{n+1}$ that have an endpoint in common with their parent $J<I \in \mathcal{I}_{n}$. In particular we have defined $w_{0}$ and $w_{r+1}$.

Notice that $w_{1} \geq 1$, and that $w_{0}=1$ if $w_{1}>1$ since $Q\left(J_{0}, J_{1}\right)=$ $Q\left(J_{1}, J_{0}\right)^{-1}$. Next, set $w_{2}=\cdots=w_{r-1}=1$ if $w_{1}=w_{r}=1$. Otherwise we may assume $w_{1} \geq w_{r}$ and choose a sequence $w_{2}, \ldots, w_{r-1}$ in such a way that

$$
\begin{gather*}
\sum_{j=1}^{r} w_{j} m_{j}=\mu_{n}(I),  \tag{4.1}\\
\frac{1}{2} \leq \frac{w_{j}}{w_{j+1}} \leq 2 \tag{4.2}
\end{gather*}
$$

and that

$$
\begin{equation*}
\varepsilon \leq w_{j} \leq w_{1} \tag{4.3}
\end{equation*}
$$

for $j=1, \ldots, r-1$ and some universal constant $\varepsilon$. The existence of such a sequence is easy to establish if $q$ is sufficiently small: Indeed, from Lemma 4.1 we have $w_{1} \asymp r(I) / r\left(I^{-}\right) \leq C q^{1-a} \leq C^{\prime} r(I)^{a-1}$. Hence $w_{1} m_{1} \asymp w_{1} r(I)^{-1} \mu_{n}(I)=o\left(\mu_{n}(I)\right)$ as $q \longrightarrow 0$. Now define $w_{j}=2^{-j+1} w_{1}$ for $j=1,2, \ldots, j_{0}$, let the $w_{j}$ have a constant value $w$ for $j_{0}+1 \leq j \leq j_{1}$ and finally set $w_{j}=2^{j-r(I)} w_{r}$ for $j_{1}+1 \leq j \leq r(I)$. It is clear that $j_{0}, j_{1}$ and $w$ can be chosen so that (4.1) and (4.2) are fulfilled. Since the contribution to $\sum_{j=1}^{r} w_{j} m_{j}$ from $1 \leq j \leq j_{0}$ and from $j_{1} \leq j \leq r$ is $o\left(\mu_{n}(I)\right)$ as $q$ decreases, $w$ is bounded away from 0 and we have (4.3).

It remains to verify that $\mu_{n+1}\left(J_{l}\right)=w_{l} m_{l}$ satisfies 1)-3) above. To see 1) for the pair ( $J_{0}, J_{1}$ ), just notice that

$$
\frac{\mu_{n+1}\left(J_{1}\right)}{\mu_{n+1}\left(J_{0}\right)} \frac{\operatorname{diam} J_{0}}{\operatorname{diam} J_{1}}=\frac{w_{1}}{w_{0}} Q\left(J_{1}, J_{0}\right)=\frac{1}{10}, \quad Q\left(J_{1}, J_{0}\right) \text { or } 10,
$$

if $Q\left(J_{1}, J_{0}\right)<1 / 10, \in[1 / 10,10]$ or greater to 10 respectively. Similarly, 1) holds for ( $J_{r}, J_{r+1}$ ). For the pairs $\left(J_{k}, J_{k+1}\right)$ with $1 \leq k \leq r-1$, the ratio in 1) equals $w_{k} / w_{k+1}$ which is bounded above and below by $1 / 2$ and 2.

Property 2) is immediate from (4.1).
To check the lower bound of 3 ), let us begin with $J=J_{1}$ : If $w_{1}=1$ (the case $\left.Q\left(J_{1}, J_{0}\right) \geq 1 / 10\right)$ this follows at once from the triangle inequality $\operatorname{diam} I \leq \sum_{1}^{r} \operatorname{diam} J_{l}$. Otherwise we have $w_{1}=$ $1 /\left(10 Q\left(J_{1}, J_{0}\right)\right)>1$ and $w_{0}=1$ Hence we have the lower estimate of 3) for $J_{0}$,

$$
\frac{\operatorname{diam} I^{-}}{\operatorname{diam} J_{0}} \frac{\mu_{n+1}\left(J_{0}\right)}{\mu_{n}\left(I^{-}\right)} \leq 1
$$

Using property 1) for $\mathcal{I}_{n}$, we obtain

$$
\begin{aligned}
\frac{\mu_{n}(I)}{\mu_{n+1}\left(J_{1}\right)} & =\frac{1}{\operatorname{diam} J_{1}} \frac{10 \operatorname{diam} J_{0}}{m_{0}} \mu_{n}(I) \\
& \geq \frac{1}{\operatorname{diam} J_{1}} \frac{10 \operatorname{diam} J_{0}}{\mu_{n+1}\left(J_{0}\right)} \frac{\mu_{n}\left(I^{-}\right) \operatorname{diam} I}{10 \operatorname{diam} I^{-}} \\
& \geq \frac{\operatorname{diam} I}{\operatorname{diam} J_{1}}
\end{aligned}
$$

To prove the lower bound of 3 ) for $J_{l}$ with $2 \leq l \leq r$, use $w_{l} \leq w_{1}$ to obtain

$$
\frac{\mu_{n}(I)}{\mu_{n+1}\left(J_{l}\right)} \geq \frac{\mu_{n}(I)}{w_{1} m_{l}}=\frac{\mu_{n}(I)}{\mu_{n+1}\left(J_{1}\right)} \frac{\operatorname{diam} J_{1}}{\operatorname{diam} J_{l}} \geq \frac{\operatorname{diam} I}{\operatorname{diam} J_{l}} .
$$

The upper bound of 3) easily follows from Lemma 4.1 if $q$ is small enough, since the $w_{j}$ are bounded below (independently of $q$ ) by (4.3).

## 5. The proof of Theorem 1.1.

Proof of Theorem 1.1. Given a quasicircle $\Gamma$, apply Theorem 1.2 to obtain the probability measure $\mu$ on $\Gamma$. Use $\mu$ to define a homeomorphism

$$
\phi: S_{1} \longrightarrow \Gamma
$$

between the unit square $S_{1}$ and $\Gamma$ in such a way that the push-forward under $\phi$ of length on $S_{1}$ is $\mu$ : Fix points $a \in S_{1}$ and $b \in \Gamma$, and for
$x \in S_{1}$ define $\phi(x)$ to be the unique point on $\Gamma$ such that the (oriented) $\operatorname{arc} \Gamma(\phi(x))$ from $b$ to $\phi(x)$ has

$$
\mu(\Gamma(\phi(x)))=\frac{\operatorname{length} S_{1}(x)}{\operatorname{length} S_{1}}
$$

where $S_{1}(x)$ is the arc from $a$ to $x$.
From $\phi$ we obtain a function (labelling) $\ell: V \longrightarrow \mathbb{R}_{+}$in the canonical way, compare (3.1) and (3.3): For vertices $v \in V$ set

$$
\ell(v)=\operatorname{diam} \phi\left(S_{1}(v)\right) .
$$

We first observe that

$$
\begin{equation*}
\ell(v) \asymp \ell\left(v^{\prime}\right), \tag{5.1}
\end{equation*}
$$

if $v$ and $v^{\prime}$ are adjacent. To see this, just notice that the $\operatorname{arcs} \Gamma(v)=$ $\phi\left(S_{1}(v)\right)$ and $\Gamma\left(v^{\prime}\right)$ have measure $\asymp 4^{-d(v)}$, that $\Gamma$ is a quasicircle, and use the doubling property of $\mu$ (no uniformity is needed yet).

Next, let $\alpha<2$ be the exponent from Theorem 1.2, set

$$
A=4^{1 / \alpha}>2
$$

and observe that for all $v \in V$ and all descendents $v^{\prime}$ of $v$ we have

$$
\begin{equation*}
C^{-1} 4^{-d\left(v, v^{\prime}\right)} \leq \frac{\ell\left(v^{\prime}\right)}{\ell(v)} \leq C A^{-d\left(v, v^{\prime}\right)} . \tag{5.2}
\end{equation*}
$$

To see this, observe that the four-adic interval $S_{1}\left(v^{\prime}\right)$ is contained in $S_{1}(v)$ and has length $S_{1}\left(v^{\prime}\right)=4^{-d}$ length $S_{1}(v)$, where $d=d\left(v, v^{\prime}\right)$. Then (5.2) is obtained from Theorem 1.2, applied to any $x \in \phi\left(S\left(v^{\prime}\right)\right)$, by choosing $r, s$ comparable to the diameters of $\phi(S(v))$ and $\phi\left(S\left(v^{\prime}\right)\right)$.

For every labelling $\ell$ satisfying (5.1) and (5.2) there is a labelling

$$
\begin{equation*}
\ell^{\prime} \asymp \ell \tag{5.3}
\end{equation*}
$$

(that is $\ell^{\prime}(v) \asymp \ell(v)$ for all $v \in V$ ) satisfying (3.2) with $p=A^{-1}$ : Just set $\ell^{\prime}\left(v_{0}\right)=1$ and inductively define

$$
\ell^{\prime}\left(v^{\prime}\right)= \begin{cases}\frac{1}{4} \ell^{\prime}(v), & \text { if } \ell^{\prime}(v) \geq \ell(v)  \tag{5.4}\\ \frac{1}{A} \ell^{\prime}(v), & \text { if } \ell^{\prime}(v)<\ell(v)\end{cases}
$$

if $v^{\prime}$ is a child of $v$. Then (3.2) is obvious and $\ell^{\prime} \asymp \ell$ is easy.
From (3.2) we obtain a snowflake-like curve $S$ with $\ell_{S}=\ell^{\prime}$. Now $\ell_{S} \asymp \ell$ together with (5.1) and Lemma 3.2 imply that both $\phi: S_{1} \longrightarrow \Gamma$ and $\phi_{S}: S_{1} \longrightarrow S$ are quasisymmetric. Let $\Phi$, respectively $\Phi_{S}$ be quasiconformal extensions to the plane satisfying (2.1) with $\mathbb{T}$ replaced by $S_{1}$ (the disc in (2.1) can be replaced by any chord-arc domain, as can be seen by applying a bilipschitz homeomorphism of the plane). Then (2.1) together with (5.3) implies $|D \Phi(x)| \asymp\left|D \Phi_{S}(x)\right|$ in $\mathbb{R}^{2}$, and the theorem follows from Lemma 2.1.

## 6. Bilipschitz homogeneous curves.

Proof of Theorem 1.3. By [HM] and [B], our definition of bilip-schitz-homogeneity coincides with the one used in [M] (existence of a bilipschitz group acting transitively on $\Gamma$.) To prove i) implies ii) we use [M, Theorem 1.1]. Hence there is a parametrization $h: \mathbb{T} \longrightarrow \Gamma$ satisfying

$$
|h(x)-h(y)| \leq C|h(u)-h(v)|
$$

whenever $|x-y| \leq|u-v|$. Set

$$
a_{n}=\min \left\{|h(x)-h(y)|:|x-y|=4^{-n}\right\}
$$

and consider the labelled graph $(G, V)$ of Figure 3.2 with $l(v)=a_{n}$ if $d(v)=n$. We proceed as in the proof of Theorem 1.1. First we claim that there is a labelling $\ell^{\prime} \asymp \ell$ satisfying (3.2). Now $|h(x)-h(y)| \asymp$ $N(r)^{-1}$, where $N(r)$ is the minimal number of discs of radius $r$ needed to cover $\Gamma$. So (5.2) follows from Lemma 4.1, and $\ell^{\prime}$ can be constructed by (5.4) as in the proof of Theorem 1.1. Since $\ell^{\prime}(v)=\ell^{\prime}\left(v^{\prime}\right)$ whenever $d(v)=d\left(v^{\prime}\right)$, the curve $S \in \mathcal{S}$ with $\ell_{S}=\ell^{\prime}$ belongs to $\mathcal{H S}$. As in the proof of Theorem 1.1 we observe that the Jacobian determinant of the extension of $h$ is comparable to $J \Phi_{S}$ and we obtain ii).

Now we show ii) implies iii). Let $S \in \mathcal{H S}(p)$. By Lemma 3.2, the canonical homeomorphism $\Phi_{S}: S_{1} \longrightarrow S$ constructed in Section 3 is quasisymmetric. Denote its quasiconformal extension satisfying (2.1) for $x \in \mathbb{D}$ by $\Phi_{S}$, too. Given $z, w \in \mathbb{D}$ with $|z| \leq|w|$, consider the four-adic intervals $I, J$ with $1-|z| \asymp \operatorname{dist}(z, I) \asymp|I| \asymp 4^{-n}$ and $1-|w| \asymp \operatorname{dist}(w, J) \asymp|J| \asymp 4^{-m}$. It follows from (2.1) that

$$
\begin{aligned}
C^{-1}\left(\frac{1}{4}\right)^{m-n}(1-|z|)\left|D \Phi_{S}(z)\right| & \leq(1-|w|)\left|D \Phi_{S}(w)\right| \\
& \leq C p^{m-n}(1-|z|)\left|D \Phi_{S}(z)\right|
\end{aligned}
$$

We obtain iii) with $\alpha=\log (4 p) / \log 2<1$.
Since iii) trivially implies iv), it remains to show iv) implies i). Pick two points $x, y$ on $\Gamma$ and let $R$ denote the rotation

$$
R(z)=\frac{F^{-1}(x)}{F^{-1}(y)} z .
$$

By assumption we have $J F \asymp J(F \circ R)$, and by Lemma 2.1 $F \circ R^{-1} \circ F^{-1}$ is bilipschitz, fixing $\Gamma$ and sending $x$ to $y$.

Proof of Corollary 1.4. First, let $\omega$ as in the Corollary be given. Define $\eta(s)=s \omega(1-s)^{1 / 2}$ for $0 \leq s<1$. Similar to the proof of i) implies ii) above, set $a_{n}=\eta\left(4^{-n}\right)$ and consider the labelled graph $(G, V)$ of Fig. 3.2 with $l(v)=a_{n}$ if $d(v)=n$. Then

$$
\frac{\ell\left(v^{\prime}\right)}{\ell(v)} \leq C 4^{-(1-\alpha / 2) d\left(v, v^{\prime}\right)}
$$

Proceeding as above ( $c f$. (5.2)), we obtain a quasiconformal map $\Phi_{S}$ onto a bilipschitz-homogeneous curve $S \in \mathcal{H S}(p)$ with $p=4^{\alpha / 2-1}$ such that $J \Phi_{S} \asymp \omega$ in $\mathbb{D}$.

The converse easily follows from Lemma 4.1.
Proof of Corollary 1.5. Given a bilipschitz-homogeneous $\Gamma$, let $F$ be the quasiconformal parametrization from Theorem 1.3 (iii) and set $\omega(s)=J F(s)$ for $0<s<1$. From $(1-s) \omega(s)^{1 / 2} \asymp \operatorname{dist}(F(s), \Gamma)$ (quasisymmetry and (2.1)) we conclude

$$
\delta(\rho) \asymp \frac{1}{1-s} \quad \text { if and only if } \quad(1-s) \omega(s)^{1 / 2} \asymp \rho .
$$

Thus the Corollary follows from Corollary 1.4.

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# Moduli of certain Fano 4-folds 

Walter L. Baily, Jr.

Dedicated to the Memory of Professor Kunihiko Kodaira


#### Abstract

In this brief note we give a proof that a certain family of Fano 4 -folds, described below, is complex (locally) complete and effectively parametrized in the sense of Kodaira-Spencer [Ko-Sp].


In this note we consider a specific family of Fano 4 -folds which is analogous in some ways to the family of nodal Enriques surfaces, considered in my article which appeared in the March, 2000, issue of the Asian Journal of Mathematics, dedicated to the late Professor Kunihiko Kodaira. This is all part of a program intended to investigate certain moduli problems of a very special nature related to trying to find some family of algebraic varieties whose moduli would be related to the exceptional 27-dimensional domain associated to the real form $E_{7(-25)}$ of the exceptional Lie group $E_{7}$. In this note we settle a conjecture left open in my Asian Journal article. Numerous authors have considered Fano 3-folds and their classification, notably Iskovskikh, Kollar, Mori, and others. Little has been known about Fano 4 -folds, and the current article is intended to break some new ground here. Actually I strongly suspect that the methods used here could be used to investigate a much wider family of Fano varieties, and suggesting such possibilities is one of its aims. The immediate background of the current note is to extend the results of the Asian Journal article and also of my article [AA] in the volume "Algebra and Analysis", Eds. Arslanov, Parshin, and Shafarevich, dedicated to N. G. Chebotarev.

At this point we shall state the main result of this note. In it we prove that a certain family of Fano 4 -folds, to be described below, is everywhere locally complex-analytically complete and effectively
parametrized. At this point we can say little about whether these Fano 4 -folds have a global complex moduli space. This still seems to be a difficult problem. But we believe that the main result here is new and rather interesting. In achieving it we have had a lot of help from Dr. Brendan Hassett, now at the Chinese University of Hong Kong, Professor Yujiro Kawamata of Tokyo University, and Professor Thomas Peternell of the University of Bayreuth. We acknowledge this help gratefully.

Notation. Let $Y=G(2,6)=\mathbf{G}(1,5)$, the variety of planes in the 6-dimensional linear space or the variety of projective lines $\mathbf{P}^{1}$ in $\mathbf{P}^{5}$.

We embed $Y$ into $\mathbf{P}^{14}$ by Plücker coordinates. Let $Q$ be, at first, a generic quadric hypersurface in $\mathbf{P}^{14}$ and $\mathbf{P}^{11}$ be a generic linear subspace of codimension 3 in $\mathbf{P}^{14}$. Thus, $\mathcal{Q}=\mathbf{P}^{11} \cap Q$ is a generic quadric of codimension 4 in $\mathbf{P}^{11}$, while $\operatorname{dim} Y=8$; therefore $X=Y \cap \mathcal{Q}$ is a smooth variety of dimension 4, and the canonical class $K_{X}$ of $X$ is $\{-L\}$, where $L$ is a hyperplane section of $X$; thus, the anti-canonical class of $X$ is very ample. Moreover, $X$ is easily seen to be simplyconnected. Therefore $X$ is a Fano 4 -fold and is clearly of index 1 , and $\operatorname{Pic}(X)=\mathbf{Z} \cdot L$. Let $T_{X}$ be the sheaf of germs of holomorphic crosssections of the holomorphic tangent bundle on $X$.

We shall prove that the family of all $X=Q \cap \mathbf{P}^{11} \cap Y$, assuming the intersection to be proper, is complete and effectively parametrized. To achieve this, it is sufficient by Kodaira and Spencer's criterion [Ko$\mathrm{Sp}],[\mathrm{Ko}]$ to show that

$$
H^{2}\left(X, T_{X}\right)=\{0\}
$$

and

$$
H^{0}\left(X, T_{X}\right)=\{0\} .
$$

Thanks to a communication from Prof. T. Peternell, the vanishing of $H^{2}\left(X, T_{X}\right)$ can be seen as follows: From Serre duality one has $(n=4)$

$$
H^{2}\left(X, T_{X}\right) \cong H^{n-2}\left(X, \Omega_{X}^{1} \otimes K_{X}\right)
$$

Now apply the vanishing criterion of Kodaira-Akizuki-Nakano for $L=$ $-K_{X}$ : If $L$ is ample on $X=X^{(n)}$, then

$$
H^{(p, q)}\left(X, L^{*}\right)=\{0\},
$$

for $p+q<n$. (Here $n=4$ and $L^{*}$ is the sheaf dual to $L$, namely $L^{*}=\Omega_{X}^{1} \otimes(-L)$.)

Thus it remains to prove that $H^{0}\left(X, T_{X}\right)=\{0\}$. Henceforth, if $A$ is a coherent analytic sheaf on $X$, we shall write simply $H^{k}(A)=$ $H^{k}(X, A)$.

First we make a calculation of the number of independent complex parameters needed to describe $X$ as a subvariety of $\mathbf{P}^{14}$. We can easily see that the space of codimension 3 subspaces $\mathbf{P}^{11}$ has dimension equal to

$$
\operatorname{dim} G(11,14)=(11+1)(14-11)=36
$$

The dimension of the space of homogeneous quadrics in $\mathbf{P}^{14}$ is equal to the dimension of the space of $15 \times 15$ symmetric matrices $=120$; this is the linear homogeneous dimension. The ideal of quadric relations among the Plücker coordinates, i.e., the ideal of quadric relations vanishing on $\mathbf{G}(1,5)$, has dimension 15 . The dimension of the space of quadrics belonging to the ideal generated by 3 linear forms, i.e., the dimension of the space of quadrics vanishing on $\mathbf{P}^{11}$, is 42 . Moreover, $\operatorname{dim}(\operatorname{Aut}(G(2,6)))=\operatorname{dim} P G L(6)=35$. Then the number of effective parameters to describe $X$ as a subvariety of $\mathbf{P}^{14}$ is $36+120-15-42-1-35=63$, where we have subtracted 1 to account for the change from linear to projective coordinates in calculating the number of quadrics in $\mathbf{P}^{14}$.

It has been shown by O. Küchle [Ku2], and by Borcea [Bor2], that every sufficiently small deformation of $X$ is described by the same type of equations as a subvariety of $\mathbf{G}(1,5)$. In fact, $X$ can be described as the projection on $Y=G(2,6)$ of the set of zeros of a cross-section of a rank 4 vector bundle $E$ over $Y$. Specifically, according to [Bor2], let $H$ be a smooth irreducible divisor on $Y$ which generates Pic $(Y)$, let $d_{1}=d_{2}=d_{3}=1$ and $d_{4}=2$, let $E$ be the rank 4 vector bundle on $Y$ given by

$$
E=[H] \oplus[H] \oplus[H] \oplus[2 H],
$$

where [D] is the complex line bundle over $Y$ associated to the divisor $D$. Then $X$ belongs to the family $\mathcal{F}$ of smooth, global complete intersections in $Y$, parametrized by the open set of $H^{0}(Y, E)$ consisting of sections of $E$ transversal to the zero-section. According to [Bor2, Theorem], $\mathcal{F}$ is complex-analytically complete, and every small deformation of $X$ is contained in $Y$ and is obtained by a small change in the crosssection defining $X$; i.e., by small changes in $\mathbf{P}^{11}$ and in $\mathcal{Q}$, modulo the relations described above. Since $H^{2}\left(T_{X}\right)=\{0\}$, it is well known that the versal deformation space $V_{X}$ of $X$ is smooth, has dimension equal to $\operatorname{dim} H^{1}\left(T_{X}\right)$, and by the above calculations, this dimension is 63 .

Now the calculation of $h^{0}\left(T_{X}\right)$ is based on the three articles by Küchle, [ $\mathrm{Ku} 1,2,3$ ], numerous communications and conversations with Brendan Hassett, email correspondence with Prof. T. Peternell, and conversations with Y. Kawamata.

Brendan Hassett has noted the following facts. In order to compute

$$
H^{i}\left(\operatorname{Gr}(2,6), I_{X} \otimes T_{\operatorname{Gr}(2,6)}\right)
$$

we take the Koszul resolution of $I_{X}$, the homogeneous ideal of $X$ in $Y=\operatorname{Gr}(2,6)$, which is

$$
\begin{aligned}
0 & \longrightarrow \mathcal{O}_{\mathrm{Gr}}(-5) \longrightarrow \mathcal{O}_{\mathrm{Gr}}(-4)^{\oplus 3} \oplus \mathcal{O}_{\mathrm{Gr}}(-3) \\
& \longrightarrow \mathcal{O}_{\mathrm{Gr}}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathrm{Gr}}(-3)^{\oplus 3} \longrightarrow \mathcal{O}_{\mathrm{Gr}}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathrm{Gr}}(-2) \\
& \longrightarrow I_{X} \longrightarrow 0,
\end{aligned}
$$

$\mathcal{O}_{\text {Gr }}$ being the structural sheaf on Gr. By examining the roots and simple roots of $S L_{6}$ and applying Bott's Theorem IV (see [Bot]), we obtain

$$
H^{i}\left(\mathrm{Gr}, T_{\mathrm{Gr}}(-1)\right)=0
$$

for all $i$ and

$$
H^{i}\left(\mathrm{Gr}, T_{\mathrm{Gr}}(-n)\right)=0
$$

for $i<8$ and $n>1$. Since $T_{\mathrm{Gr}}$, the holomorphic tangent bundle to Gr , is a locally free sheaf, we can tensor the above resolution by it to obtain another exact sequence which implies, finally, that

$$
\begin{equation*}
H^{i}\left(\operatorname{Gr}(2,6), I_{X} \otimes T_{Y}\right)=0, \quad i=0,1 \tag{a}
\end{equation*}
$$

Now one has the exact sequence

$$
\begin{equation*}
0 \longrightarrow T_{X} \longrightarrow T_{Y} \mid X \longrightarrow N_{X} \longrightarrow 0, \tag{b}
\end{equation*}
$$

where $N_{X}$ is the normal bundle to $X$ in $Y$, and

$$
\begin{equation*}
N_{X} \cong \mathcal{O}_{X}(+1)^{\oplus 3} \oplus \mathcal{O}_{X}(+2), \tag{c}
\end{equation*}
$$

so that

$$
\begin{equation*}
\operatorname{dim} H^{0}\left(N_{X}\right)=3 h^{0}\left(\mathcal{O}_{X}(+1)+h^{0}\left(\mathcal{O}_{X}(+2)\right)\right. \tag{d}
\end{equation*}
$$

Now by $[\mathrm{Ku} 3,1.3]$ one has by Riemann-Roch, for the Fano 4 -fold $X$ of index 1 ,
$h^{0}\left(-m K_{X}\right)=1+\frac{m(m+1)}{24}\left(-K_{X}\right)^{2} c_{2}(X)+\frac{m^{2}(m+1)^{2}}{24}\left(-K_{X}\right)^{4}$.
Since $X$ is a smooth 4 -fold of degree 28 in $\mathbf{P}^{11}$, where the class of its hyperplane section is $\left(-K_{X}\right)$, one has

$$
h^{0}\left(-K_{X}\right)=12
$$

and so

$$
\left(-K_{X}\right)^{4}=28
$$

leading to

$$
12=1+\frac{1}{12}\left(-K_{X}\right)^{2} c_{2}(X)+\frac{28}{6},
$$

or $\left(-K_{X}\right)^{2} c_{2}(X)=76$, hence $h^{0}\left(-2 K_{X}\right)=62$. Finally,

$$
h^{0}\left(N_{X}\right)=3.12+62=98 .
$$

Now Prof. Peternell has called attention to the exact sequence

$$
0 \longrightarrow I_{X} \otimes T_{Y} \longrightarrow T_{Y} \longrightarrow T_{Y} \mid X \longrightarrow 0,
$$

and this leads to a long exact cohomology sequence of which a part is

$$
\begin{aligned}
0 & \longrightarrow H^{0}\left(Y, T_{Y} \otimes I_{X}\right) \longrightarrow H^{0}\left(T_{Y}\right) \\
& \longrightarrow H^{0}\left(T_{Y} \mid X\right) \longrightarrow H^{1}\left(Y, T_{Y} \otimes I_{X}\right) \longrightarrow
\end{aligned}
$$

of which we now know that the second and last terms are $\{0\}$; namely,

$$
H^{i}\left(Y, T_{Y} \otimes I_{X}\right)=\{0\}, \quad i=0,1
$$

and hence

$$
h^{0}\left(T_{Y} \mid X\right)=h^{0}\left(T_{Y}\right)=\operatorname{dim} \operatorname{Aut}(Y)=35 .
$$

Now by [Ku2, 1.3b] we have

$$
H^{1}\left(X, T_{Y} \mid X\right)=\{0\}
$$

so from the exact sequence

$$
\begin{aligned}
0 & \longrightarrow H^{0}\left(T_{X}\right) \longrightarrow H^{0}\left(T_{Y} \mid X\right) \longrightarrow H^{0}\left(N_{X}\right) \\
& \longrightarrow H^{1}\left(T_{X}\right) \longrightarrow H^{1}\left(X, T_{Y} \mid X\right)=\{0\}
\end{aligned}
$$

and from $h^{1}\left(T_{X}\right)=63$ we obtain

$$
63=h^{0}\left(N_{X}\right)-h^{0}\left(T_{Y} \mid X\right)+h^{0}\left(T_{X}\right)=98-35+h^{0}\left(T_{X}\right) .
$$

Therefore, $h^{0}\left(T_{X}\right)=\operatorname{dim} H^{0}\left(T_{X}\right)=0$ as claimed earlier. Thus, by Theorem 6.4 on page 306 of Kodaira's Springer text [Ko], the family of $X$ is complete and effectively parametrized at all its points and the dimension of the base $\Delta$ of any complete, effectively parametrized complex analytic fiber system $\left(M_{\Delta}, \Delta, \varpi\right)$ with $X$ as one of its fibers, $X=\varpi^{-1}(\delta), \delta \in \Delta$, is $m(X)=63$. What remains open is whether this family has a global moduli space.

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[^4]:    ${ }^{1}$ The notation in the theorem will be explained shortly.

[^5]:    * The author was supported by the Swedish Institute.

[^6]:    1 Strictly speaking, $f$ is not quite compactly supported, but the error incurred because of this is extremely rapidly decreasing in $N$ and can be easily dealt with.

[^7]:    2 We remark that Zygmund's original proof of (4) also proceeded via a dyadic model.

[^8]:    ${ }^{3}$ If $|E|=0$, we adopt the convention that $|E| \log \left(2+\left|I_{0}\right| /|E|\right)^{1 / 2}=0$.

[^9]:    Partially supported by research funds of the Italian MURST, "Processi stocastici 1999".

[^10]:    1 We suspect but cannot prove that a $d$-Slice condition implies an $(\alpha, d)$-wSlice ${ }^{+}$ condition; it certainly implies (WS-5) because of iii) and the fact that the slices are left unchanged in the proof that a Slice condition implies an $\alpha$-wSlice condition.

[^11]:    2 This concept was introduced by Väisälä [V4]. Our definition looks a little different but is equivalent to Väisälä's in the Euclidean setting according to [HK, Theorem 3.6].

[^12]:    3 The version there is stated for $\alpha$-mCigar domains $\Omega$ but the proof merely uses the fact that that there exists an $\alpha$-mCigar path for a particular pair of points. Also $\Omega$ is assumed to be bounded, but this is not used.

[^13]:    4 Below, the term slice is used to refer generically to Slice, $\alpha$-wSlice, $\alpha$-wSlice ${ }^{+}$, and all other slice conditions.

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