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Ondelettes et bases hilbertiennes

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En hommage à A. P. Calderón

Introduction

La transformation en ondelettes apparaît implicitement dans un célèbre travail de A. P. Calderón ([2]). Elle a été redécouverte et explicitée par J. Morlet et ses collaborateurs ([4], [7], [8]) comme une technique efficace d'analyse numérique permettant le traitement du signal en liaison avec la recherche pétrolière.

La transformation en ondelettes s'apparente à la transformation de Fourier mais les exponentielles imaginaires $\exp(ix \cdot \xi)$ indexées par les fréquences $\xi \in \mathbb{R}^n$ y sont remplacées par les «ondelettes» ψ_Q indexées par la collection de tous les cubes $Q \subset \mathbb{R}^n$. Ces «ondelettes» ψ_Q sont toutes des copies (par translation et changement d'échelle) d'une même fonction ψ régulière et ayant la décroissance à l'infini la plus forte possible. Pour être plus précis, on choisira ψ dans la classe $S(\mathbb{R}^n)$ de L. Schwartz et si le cube Q est défini par $a_j \leqslant x_j \leqslant a_j + d$, $1 \leqslant j \leqslant n$, où $a = (a_1, \ldots, a_n)$, alors $\psi_Q(x) = d^{-n/2}\psi((x-a)/d)$. Cela signifie que ψ_Q est *localisée* et *ajustée au cube* Q.

Le but poursuivi est par un choix judicieux de ψ, de pouvoir écrire n'importe quelle fonction f comme une somme ou comme une intégrale des $c_Q\psi_Q(x)$ où les coefficients c_Q se calculent comme des coefficients de Fourier:

$$c_Q = \int_{\mathbb{R}^n} f(x)\bar{\psi}_Q(x)\,dx.$$

Les auteurs mentionnés ci-dessus ont obtenu des conditions portant sur ψ et permettant de reconstituer f à l'aide de *l'information redondante* donnée par la connaissance de tous les «coefficients d'ondelettes» c_Q.

Suivant une démarche clairement explicitée par L. Carleson dans [3], nous nous proposons de faire disparaître cette redondance en remplaçant la collection de tous les cubes $Q \subset \mathbb{R}^n$ par celle, notée $\mathcal{Q}$ des *cubes dyadiques*. Un cube dyadique Q est défini par deux indices $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ et l'on a

$$Q = \{x \in \mathbb{R}^n; 2^j x - k \in [0,1]^n\}. \tag{1.1}$$

La construction qui suit est, comme dans [3], une imitation du système de Haar dont nous rappelons la définition.

On appelle $h^{(1)}(t)$ la fonction égale à 1 si $0 \leqslant t < 1/2$, à -1 si $1/2 \leqslant t < 1$ et à 0 ailleurs. D'autre part, $h^{(0)}(t)$ est la fonction indicatrice de l'intervalle $[0,1]$. On désigne par $E \subset \{0,1\}^n$ l'ensemble des $2^n - 1$ suites $(\epsilon_1, \ldots, \epsilon_n)$ de 0 et de 1, la suite $(0,0,\ldots,0)$ étant exclue. Alors le système de Haar pour $L^2(\mathbb{R}^n)$ se compose des fonctions $h_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, définies par

$$h_Q^{(\epsilon)}(x) = 2^{nj/2} h^{(\epsilon_1)}(2^j x_1 - k_1) \ldots h^{(\epsilon_n)}(2^j x_n - k_n).$$

Le système de Haar est bien adapté aux espaces fonctionnels $L^p(\mathbb{R}^n)$, $1 < p < +\infty$, mais ne permet pas l'étude des espaces de Sobolev, de Besov, ou de l'espace de Hardy $H^1(\mathbb{R}^n)$ de Stein et Weiss.

L'idée généralement admise est que «lisser le système de Haar fait disparaître l'orthogonalité». C'est ce qui apparaît en lisant [3] ou [6]. Nous allons voir cependant qu'il existe un choix exceptionnel de deux fonctions $\psi^{(0)}$ et $\psi^{(1)}$ de $S(\mathbb{R}^n)$ tel que la suite des fonctions

$$\psi_Q^{(\epsilon)}(x) = 2^{nj/2} \psi^{(\epsilon_1)}(2^j x_1 - k_1) \cdots \psi^{(\epsilon_n)}(2^j x_n - k_n) \tag{1.2}$$

soit une base hilbertienne de $L^2(\mathbb{R}^n)$ ($\epsilon \in E$, $Q \in \mathcal{Q}$).

Cette base convient à tous les espaces fonctionnels classiques: espaces de Sobolev, de Besov, de Hardy... qui se traduisent isomorphiquement en des espaces de suites. En particulier les coefficients d'une fonction $f \in H^1(\mathbb{R}^n)$ dans la base des $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$ et ceux d'une fonction g de la version dyadique de H^1, dans la base des $h_Q^{(\epsilon)}$, sont caractérisés par la même condition. Cette remarque constitue une preuve très simple du théorème de Maurey.

Dans l'identité fondamentale

$$f(x) = \sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \langle f, \psi_Q^{(\epsilon)} \rangle \psi_Q^{(\epsilon)}(x) \tag{1.3}$$

nous disposons de plusieurs niveaux de lecture.

Si $f \in L^2(\mathbb{R}^n)$, il s'agit de la décomposition usuelle dans une base orthonormée. La convergence a lieu en moyenne quadratique et presque-partout.

Si $f(x)$ et ses dérivées premières appartiennent à $L^2(\mathbb{R}^n)$, alors il y a encore convergence dans $L^2(\mathbb{R}^n)$ de la série dérivée terme à terme. Si, plus généralement f et ses dérivées jusqu'à l'ordre s ($s \in \mathbb{N}$) appartiennent à $L^2(\mathbb{R}^n)$, alors

la série (1.3) convergera automatiquement pour la norme correspondante, c'est-à-dire celle de l'espace de Sobolev H^s. En d'autres termes, la régularité de f accélère la convergence (comme c'est le cas pour les séries de Fourier, mais certainement pas pour le système de Haar). Enfin, nous montrerons que ce nouvel algorithme permet une écriture très simple du para-produit $\pi(a,f)$ de J. M. Bony [1].

2. Enoncé du théorème fondamental

Nous commençons par construire des fonctions spéciales d'une variable réelle. Nous désignerons par S l'intervalle $[2\pi/3, 4\pi/3]$. On a donc $2S = [4\pi/3, 8\pi/3]$, $S - 2\pi = -S = [-4\pi/3, -2\pi/3]$ et donc aussi $2\pi - S = S$. Nous noterons $\theta(t)$ une fonction de la variable réelle t, *impaire*, de classe C^∞, à valeurs dans $[-\pi/4, \pi/4]$, égale à $\pi/4$ si $t \geqslant \pi/3$ (et donc à $-\pi/4$ si $t \leqslant -\pi/3$). On définit la fonction paire $\omega(t)$ par $\omega(t) = 0$ si $0 \leqslant t \leqslant 2\pi/3$ ou si $t \geqslant 8\pi/3$, $\omega(t) = \pi/4 + \theta(t - \pi)$ si $2\pi/3 \leqslant t \leqslant 4\pi/3$ et $\omega(t) = \pi/4 - \theta(t/2 - \pi)$ si $4\pi/3 \leqslant t \leqslant 8\pi/3$.

Les identités fondamentales vérifiées par $\omega(t)$ sont $\omega(-t) = \omega(t)$, $\omega(2t) = \pi/2 - \omega(t)$, $t \in S$ et

$$\omega(t - 2\pi) = \omega(2\pi - t) = \frac{\pi}{2} - \omega(t), \qquad t \in S.$$

Par construction $\omega(t)$ est indéfiniment dérivable.

A l'aide de $\omega(t)$ on définit $\psi \in S(\mathbb{R})$ par

$$\hat{\psi}(t) = e^{-it/2} \sin \omega(t) = e^{-it/2} \tilde{\omega}(t). \tag{2.1}$$

On aura donc

$$\psi(t) = \frac{1}{\pi} \int_0^\infty \cos\left[\left(t - \frac{1}{2}\right)s\right] \tilde{\omega}(s)\, ds. \tag{2.2}$$

Comme nous l'a fait remarquer J. Morlet, on peut également partir d'une fonction *impaire* $\omega_1(t)$ définie par $\omega_1(t) = -\omega(t)$ si $t \geqslant 0$ (et donc $\omega_1(t) = -\omega(t)$ si $t \leqslant 0$). On lui associe $\psi_1 \in S(\mathbb{R})$ définie par

$$\hat{\psi}_1(t) = e^{-it/2} \sin \omega_1(t) \tag{2.3}$$

ce qui entraîne

$$\psi_1(t) = \frac{1}{\pi} \int_0^\infty \sin\left[\left(t - \frac{1}{2}\right)s\right] \sin \omega(s)\, ds. \tag{2.4}$$

Dans les théorèmes qui suivent ψ peut être systématiquement remplacée par ψ_1. Si $I \subset \mathbb{R}$ est l'intervalle dyadique défini par $2^j x - k \in [0, 1]$, nous définissons ψ_I par $\psi_I(x) = 2^{j/2}\psi(2^j x - k)$.

On a alors

Théorème 1. *La collection des ondelettes $\psi_I(x)$ est une base hilbertienne de $L^2(\mathbb{R})$.*

Nous allons donner l'énoncé correspondant dans $\mathbb{R}^n$ en reprenant les notations de l'introduction et en imitant la construction du système de Haar. La fonction ψ reste la même que ci-dessus et on lui associe une fonction $\varphi \in S(\mathbb{R})$ d'intégrale égale à 1 définie par $\hat{\varphi}(t) = \cos \omega(t)$ si $|t| \leqslant 4\pi/3$ et par $\hat{\varphi}(t) = 0$ sinon.

On pose $\psi^{(0)}(t) = \varphi(t)$, $\psi^{(1)}(t) = \psi(t)$ et l'on définit $\psi^{(\epsilon)} \in S(\mathbb{R}^n)$ par $\psi^{(\epsilon)}(x) = \psi^{(\epsilon_1)}(x_1)\psi^{(\epsilon_2)}(x_2) \cdots \psi^{(\epsilon_n)}(x_n)$ où $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in E$; cela signifie que $\epsilon_j = 0$ où 1 mais que la suite ne comportant que des zéros est exclue.

Rappelons qu'un cube dyadique Q est défini par $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ et est l'ensemble des $x \in \mathbb{R}^n$ tels que $2^j x - k \in [0, 1]^n$. On pose alors

$$\psi_Q^{(\epsilon)}(x) = 2^{nj/2}\psi^{(\epsilon)}(2^j x - k) = 2^{nj/2}\psi^{(\epsilon_1)}(2^j x_1 - k_1) \cdots \psi^{(\epsilon_n)}(2^j x_n - k_n). \tag{2.5}$$

Avec ces notations, nous nous proposons de démontrer le résultat suivant.

Théorème 2. *La collection des fonctions $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, est une base hilbertienne de $L^2(\mathbb{R}^n)$.*

Avant de démontrer ce résultat, introduisons quelques notations. Malgré le fait que le choix de ψ ne soit pas unique, les fonctions $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, seront appelées les *ondelettes*. Toute série $\sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \alpha(\epsilon, Q)\psi_Q^{(\epsilon)}(x)$ s'appelle une série d'ondelettes.

Si $f(x)$ appartient à $L^2(\mathbb{R}^n)$ ou, plus généralement, est une distribution tempérée, les coefficients

$$\alpha(\epsilon, Q) = \int f(x)\psi_Q^{(\epsilon)}(x)\, dx \tag{2.6}$$

s'appelleront les *coefficients d'ondelette*. Une conséquence facile du théorème 2 est que l'on aura toujours la formule d'inversion

$$f(x) = \sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \alpha(\epsilon, Q)\psi_Q^{(\epsilon)}(x) \tag{2.7}$$

au sens suivant: pour toute fonction de test $u(x) \in S(\mathbb{R}^n)$ dont tous les moments sont nuls,

$$\langle f, u \rangle = \sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \alpha(\epsilon, Q) \int_{\mathbb{R}^n} \psi_Q^{(\epsilon)}(x)u(x)\, dx.$$

Enfin la transformation qui à la distribution tempérée f associe la suite numérique $\alpha(\epsilon, Q)$ s'appellera la *transformation en ondelettes*. Elle joue le rôle d'une transformation de Fourier locale (en fait les intégrales portent sur $\mathbb{R}^n$ tout entier mais il y a un très fort amortissement dû à la décroissance rapide de ψ). Nous décomposerons la preuve du Théorème 1 en deux parties.

Nous démontrerons d'abord que $\psi_Q^{(\epsilon)}$ est orthogonale à $\psi_Q^{(\epsilon')}$ si $(Q, \epsilon) \neq (Q', \epsilon')$. Ensuite nous vérifierons que toute fonction $f \in L^2(\mathbb{R}^n; dx)$ orthogonale à chacune des $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathbb{Q}$, est nécessairement nulle.

Les relations d'orthogonalité découlent des propriétés correspondantes en une variable réelle (que nous allons décrire).

Lemme 1. *On a*

$$\int_{-\infty}^{\infty} \hat{\varphi}(t)\bar{\hat{\psi}}(2^{-j}t)e^{ik2^{-j}t}\,dt = 0$$

pour tout $j \in \mathbb{N}$ *et tout* $k \in \mathbb{Z}$. *De même on a*

$$\int_{-\infty}^{\infty} \hat{\psi}(t)\bar{\hat{\psi}}(2^{-j}t)e^{ik2^{-j}t}\,dt = 0$$

pour $j \geqslant 1$ *et* $k \in \mathbb{Z}$.

Commençons par vérifier la première relation. Si $j \geqslant 1$, les supports de $\hat{\varphi}(t)$ et de $\hat{\psi}(2^{-j}t)$ sont disjoints.

Si $j = 0$, on a $\hat{\varphi}(t)\bar{\hat{\psi}}(t) = e^{it/2} \sin \omega(t) \cos \omega(t)$ si $|t| \leqslant 4\pi/3$, 0 sinon. Le support $\hat{\varphi}\bar{\hat{\psi}}$ se décompose donc des deux intervalles $S = [2\pi/3, 4\pi/3]$ et $S - 2\pi$. L'identité $\omega(t - 2\pi) = \pi/2 - \omega(t)$ et la relation $e^{i\pi} = -1$ achèvent la vérification.

La preuve des secondes assertions est identique. Si $j \geqslant 2$, les supports de $\hat{\psi}(t)$ et de $\hat{\psi}(2^{-j}t)$ sont disjoints. Il reste à traiter le cas $j = 1$ et l'on fait dans l'intégrale le changement de variable $t = 2s$. Revenant à la variable t, on remarque que $\hat{\psi}(2t)\bar{\hat{\psi}}(t) = e^{-it/2} \sin \omega(t) \cos \omega(t)$ si $|t| \leqslant 4\pi/3$, 0 sinon. On est ramené au cas précédent.

Nous allons maintenant élucider le cas $j = 0$.

Lemme 2. *Pour tout* $k \in \mathbb{Z}$, $k \neq 0$ *on a*

$$\int_{-\infty}^{\infty} |\hat{\varphi}(t)|^2 e^{ikt}\,dt = \int_{-\infty}^{\infty} |\hat{\psi}(t)|^2 e^{ikt}\,dt = 0.$$

Pour le voir, nous allons démontrer que

$$\sum_{-\infty}^{\infty} |\hat{\psi}(t + 2\pi j)|^2 = \sum_{-\infty}^{\infty} |\hat{\varphi}(t + 2\pi j)|^2 = 1.$$

Nous nous limiterons à la première identité. La somme à calculer définit une

fonction 2π-périodique qu'il suffit de connaître sur un intervalle de longueur 2π. On choisit l'intervalle $2\pi/3 \leqslant t \leqslant 8\pi/3$. Si $2\pi/3 \leqslant t \leqslant 4\pi/3$, les seules valeurs de j qui interviennent sont $j = 0$ et $j = -1$. Puisque $\omega(t - 2\pi) = \pi/2 - \omega(t)$, l'identité résulte de $\sin^2 \omega(t) + \cos^2 \omega(t) = 1$. Si $4\pi/3 \leqslant t \leqslant 8\pi/3$, on doit prendre $j = 0$ et $j = -2$ et l'on conclut de même.

Revenons à l'orthogonalité entre les $\psi_Q^{(\epsilon)}$. On est amené à calculer l'intégrale

$$I(j, j', k, k', \epsilon, \epsilon') = \int_{\mathbb{R}^n} \exp[i(k2^{-j} - k'2^{-j'})\xi]\hat{\psi}^{(\epsilon_1)}(2^{-j}\xi_1)\bar{\hat{\psi}}^{(\epsilon'_1)}(2^{-j'}\xi_1)\cdots\hat{\psi}^{(\epsilon_n)}(2^{-j}\xi_n)\bar{\hat{\psi}}^{(\epsilon'_n)}(2^{-j'}\xi_n)\,d\xi_1\cdots d\xi_n.$$

Par symétrie, on peut supposer $j \geqslant j'$. On fait immédiatement le changement de variable $2^{-j}\xi = u$. On est donc ramené à $j = 0$, ce que nous supposerons désormais. Si $j' \geqslant 1$, on appelle m un indice tel que $\epsilon_m = 1$ (un tel indice existe parce que $\epsilon \in E$). On intègre d'abord par rapport à la variable x_m et le lemme 1 assure la nullité de I. Si $j' = 0$ et $k \neq k'$, il existe un indice m tel que $k_m \neq k'_m$. On intègre d'abord par rapport à la variable x_m et I est nulle, soit à cause de la première assertion du lemme 1, soit à cause du lemme 2.

Enfin le dernier cas à envisager est $j = 0$, $k = k'$ et $\epsilon \neq \epsilon'$. Il y a donc un indice m tel que $\epsilon_m = 1$ et $\epsilon'_m = 0$ et la première assertion du lemme 1 entraîne la nullité de l'intégrale.

On vérifie sans peine que $\|\psi_Q^{(\epsilon)}\|_2 = 1$ pour tout ϵ et tout Q.

3. La suite des $\psi_Q^{(\epsilon)}$ est une partie totale dans $L^2(\mathbb{R}^n)$

Nous arrivons à la partie la plus technique de la démonstration du Théorème 2.

Nous allons vérifier que $\langle f, \psi_Q^{(\epsilon)} \rangle = 0$ pour tout $\epsilon \in E$ et tout $Q \in \mathcal{Q}$ entraîne $f = 0$.

Pour cela nous allons écrire ces relations en utilisant la transformation de Fourier. Quitte à changer les notations, nous avons, pour une certaine fonction $f \in L^2(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} f(\xi) \exp(ik2^{-j}\xi)\hat{\psi}^{(\epsilon_1)}(2^{-j}\xi_1)\cdots\hat{\psi}^{(\epsilon_n)}(2^{-j}\xi_n)\,d\xi_1\cdots d\xi_n = 0 \tag{3.1}$$

pour tout $j \in \mathbb{Z}$, tout $k \in \mathbb{Z}^n$ et toute suite $(\epsilon_1, \ldots, \epsilon_n) \neq (0, \ldots, 0)$ de 0 où de 1.

Nous devons en conclure que $f = 0$. Nous procédons au changement de variable $2^{-j}\xi = u$. Alors (3.1) devient

$$\int_{\mathbb{R}^n} f_j(\xi) \exp(ik\xi)\hat{\psi}^{(\epsilon_1)}(\xi_1)\cdots\hat{\psi}^{(\epsilon_n)}(\xi_n)\,d\xi = 0 \quad \text{où} \quad f_j(\xi) = f(2^j\xi). \tag{3.2}$$

Il est classique que (3.2) pour j fixé et tout $k \in \mathbb{Z}^n$ est équivalent à l'identité

$$\sum_{k\in\mathbb{Z}^n} f_j(\xi + 2k\pi)\hat{\psi}^{(\epsilon_1)}(\xi_1 + 2k_1\pi)\cdots\hat{\psi}^{(\epsilon_n)}(\xi_n + 2k_n\pi) = 0. \tag{3.3}$$

Cette identité doit être satisfaite pour tout $j \in \mathbb{Z}$, tout $\epsilon \in E$ et tout $\xi \in \mathbb{R}^n$. La situation idéale serait celle où existerait une partie $\Omega \subset \mathbb{R}^n$ ayant les propriétés suivantes: les translatées $\Omega + 2k\pi$, $k \in \mathbb{Z}^n$, sont deux à deux disjointes; pour tout $\xi \in \mathbb{R}^n$, il existe un $j \in \mathbb{Z}$ tel que $2^j\xi \in \Omega$ et enfin pour tout $\xi \in \Omega$, il existe $\epsilon \in E$ tel que $\hat{\psi}^{(\epsilon_1)}(\xi_1) \neq 0, \ldots, \hat{\psi}^{(\epsilon_n)}(\xi_n) \neq 0$. Alors (3.3) impliquerait $f_j(\xi) = 0$ sur Ω pour tout $j \in \mathbb{Z}$ et donc $f = 0$. Naturellement ce n'est pas le cas mais nous allons essayer de nous rapprocher le plus possible de cette situation idéale.

Pour simplifier les notations qui suivent, nous nous limiterons à la dimension $n = 2$. Ecrivons à nouveau (3.3) de façon détaillée

$$\sum_k \sum_l f_j(u + 2k\pi, v + 2l\pi)\hat{\psi}(u + 2k\pi)\hat{\psi}(v + 2l\pi) = 0 \tag{3.4}$$

$$\sum_k \sum_l f_j(u + 2k\pi, v + 2l\pi)\hat{\varphi}(u + 2k\pi)\hat{\psi}(v + 2l\pi) = 0 \tag{3.5}$$

$$\sum_k \sum_l f_j(u + 2k\pi, v + 2l\pi)\hat{\psi}(u + 2k\pi)\hat{\varphi}(v + 2l\pi) = 0. \tag{3.6}$$

Les sommes (3.4) à (3.6) définissent évidemment des fonctions $(2\pi\mathbb{Z})^2$-périodiques que l'on analysera sur un «domaine fondamental». Nous allons dans un premier temps supposer

$$(u, v) \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]^2 = \Omega_1$$

et considérer que (3.4), (3.5) et (3.6) relient $f_j(u, v)$ (que nous cherchons à calculer) à des valeurs parasites $f_j(u + 2k\pi, v + 2l\pi)$ que nous chercherons à éliminer.

Pour $(u, v) \in \Omega_1$, on a nécessairement $k = 0$ ou $k = -1$ et $l = 0$ ou $l = -1$. En posant $f_j = f_j(u, v)$, $g_j = f_j(u - 2\pi, v)$, $\tilde{f}_j = f_j(u, v - 2\pi)$ et $\tilde{g}_j = f_j(u - 2\pi, v - 2\pi)$,

$$\begin{aligned}&\sin\omega(u)\sin\omega(v)f_j - \cos\omega(u)\sin\omega(v)g_j\\ &- \sin\omega(u)\cos\omega(v)\tilde{f}_j + \cos\omega(u)\cos\omega(v)\tilde{g}_j = 0.\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\cos\omega(u)\cos\omega(v)f_j + \sin\omega(u)\sin\omega(v)g_j\\ &- \cos\omega(u)\cos\omega(v)\tilde{f}_j - \sin\omega(u)\cos\omega(v)\tilde{g}_j = 0\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\sin\omega(u)\cos\omega(v)f_j - \cos\omega(u)\cos\omega(v)g_j\\ &+ \sin\omega(u)\sin\omega(v)\tilde{f}_j - \cos\omega(u)\sin\omega(v)\tilde{g}_j = 0.\end{aligned} \tag{3.9}$$

Ces trois relations ne permettent évidemment pas de calculer les quatre inconnues que sont $f_j, g_j, \tilde{f}_j$ et $\tilde{g}_j$.

On considère (3.4) pour le point $(u'v') = (2u, 2v)$.

Alors les seules valeurs de k et l qu'il convient d'écrire sont $k = 0$ ou -2, $l = 0$ ou -2.

On observe que $f_j(u', v') = f_{j+1}(u, v), f_j(u' - 4\pi, v') = f_{j+1}(u - 2\pi, v)$ etc... En décalant d'une unité l'indice j, on obtient donc la relation manquante sous la forme

$$\begin{aligned}(3.10)\qquad &\sin\omega(u)\sin\omega(v)f_j + \cos\omega(u)\sin\omega(v)g_j \\ &+ \sin\omega(u)\cos\omega(v)\tilde{f}_j + \cos\omega(u)\cos\omega(v)\tilde{g}_j = 0.\end{aligned}$$

Le déterminant des quatre relations écrites est égal à 1 et l'on obtient donc $f_j = g_j = \tilde{f}_j = \tilde{g}_j = 0$ pour tout $j \in \mathbb{Z}$.

Désignons maintenant par Ω_2 le rectangle $\pi/3 \leqslant u \leqslant 2\pi/3$, $2\pi/3 \leqslant v \leqslant 4\pi/3$. On doit observer que $\hat{\psi}(u) = 0$ sur Ω_2 tandis que $\hat{\varphi}(u) = 1$. La seule relation pertinente pour calculer $f_j(u, v)$ si $(u, v) \in \Omega_2$ est donc (3.5). Les seules valeurs de k et l qui interviennent réellement sont $k = 0$ et $l = 0$ ou $l = -1$.

On pose encore $f_j = f_j(u, v)$ et $\tilde{f}_j = f_j(u, v - 2\pi)$ et l'on a

$$(3.11)\qquad f_j \sin\omega(v) - \tilde{f}_j \cos\omega(v) = 0.$$

Cette seule relation ne permet évidemment pas de calculer f_j et $\tilde{f}_j$. Mais, là encore, on considère le point $(2u, 2v)$ par lequel on écrit (3.4) et (3.5). Les seules valeurs de k ou l qui interviennent sont $k = 0$ et $k = -1$, $l = 0$ ou $l = -2$. Posons $\sigma_j = f_j(u, v)\cos\omega(v) + f_j(u, v - 2\pi)\sin\omega(v)$ et $\tau_j = f_j(u - \pi, v)\cos\omega(v) + f_j(u - \pi, v - 2\pi)\sin\omega(v)$ (en ayant pris soin de décaler d'un rang les indices).

On a donc $\sigma_j \cos\omega(2u) + \tau_j \sin\omega(2u) = 0$ et $\sigma_j \sin\omega(2u) - \tau_j \cos\omega(2u) = 0$. Ces deus relations impliquent $\sigma_j = 0$ ou encore

$$(3.12)\qquad f_j \cos\omega(v) + \tilde{f}_j \sin\omega(v) = 0.$$

Les relations (3.11) et (3.12) impliquent $f_j = \tilde{f}_j = 0$ pour tout $j \in \mathbb{Z}$. Naturellement, la même approche convient pour les sept autres rectangles obtenus par symétrie par rapport aux axes de coordonnées ou de leurs bissectrices, à partir de Ω_2. Finalement, on étudie le rectangle Ω_3 défini par $0 \leqslant u \leqslant \pi/3$ et $2\pi/3 \leqslant v \leqslant 4\pi/3$. La seule relation significative est encore $f_j \sin\omega(v) - \tilde{f}_j \cos\omega(v) = 0$. On utilise le même stratagème que plus haut en écrivant (après le décalage nécessaire de j), la relation correspondante sur le rectangle $2\Omega_3$, pour le point $(2u, 2v)$. Il vient $f_j \cos\omega(v) + \tilde{f}_j \sin\omega(v) = 0$ et il en résulte que $f_j = \tilde{f}_j = 0$ sur Ω_3. Là encore on obtient la même conclusion pour tous les rectangles déduits de Ω_3 par toutes les symétries du problème.

Grâce à l'examen de ces trois «domaines fondamentaux», tout l'anneau $2\pi/3 \leqslant \sup(|u|, |v|) \leqslant 8\pi/3$ est rempli et $f(2^j u, 2^j v) = 0$ dans cet anneau. Il en résulte aussitôt que $f = 0$ presque-partout sur $\mathbb{R}^2$.

4. Convergence des séries d'ondelettes dans les espaces de fonctions de test ou de distributions

Soit $S(\mathbb{R}^n)$ l'espace des fonctions de test, muni de la topologie usuelle et $S_0 \subset S$ le sous-espace formé des fonctions $f \in S$ vérifiant $\int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0$ pour tout $\alpha \in \mathbb{N}$ (tous les moments sont nuls). Alors toutes les ondelettes $\psi_Q^{(\epsilon)}$ appartiennent à S_0. Si une série d'ondelettes converge dans S, elle converge donc dans S_0 vers une fonction de S_0. On montre facilement que la réciproque est exacte.

Lemme 3. *Si $f \in S_0$, la série d'ondelettes correspondante converge vers f, au sens de la topologie de S.*

Que se passe-*t*-il alors si f appartient à S? Il est facile de montrer que la série d'ondelettes converge vers f uniformément sur $\mathbb{R}^n$ et qu'il en est de même pour les séries dérivées.

Partons maintenant de la fonction 1 (identiquement égale à 1 sur tout $\mathbb{R}^n$). Chaque coefficient (en ondelettes) de 1 est 0. Pourquoi obtient-on une égalité numérique absurde?

Considérons l'espace topologique dual de $S_0(\mathbb{R}^n)$. C'est l'espace quotient $S'(\mathbb{R}^n)/\mathbb{C}[x]$ des distributions tempérées modulo les polynômes. Si S est une telle distribution tempérée, alors la série d'ondelettes correspondante $\sum_{Q\in\mathcal{Q}} \lambda_Q \psi_Q^{(\epsilon)}(x)$ converge au sens suivant: pour toute fonction $f \in S_0$, il y a convergence numérique de la série obtenue par intégration contre f. Voilà la raison pour laquelle on obtient $1 = 0$ (modulo les fonctions constantes).

5. Caractérisation des espaces $L^p(\mathbb{R}^n)$, $1 < p < +\infty$, $\mathrm{BMO}(\mathbb{R}^n)$ et de son prédual $H^1(\mathbb{R}^n)$ par la transformation en ondelettes

Soit E un espace de Banach séparable. Une suite $(e_k)_{k\in\mathbb{N}}$ d'éléments de E est une base de E si tout vecteur $x \in E$ s'écrit, de façon unique, $x = \sum_0^\infty \xi_k e_k$ où les ξ_k sont des scalaires et où les sommes partielles de la série écrite convergent vers x au sens de la norme de E. La base est dite inconditionnelle s'il existe une constante $C \geqslant 1$ telle que l'on ait, pour tout $m \geqslant 1$ et toute suite ξ_k, $k \in \mathbb{N}$, de scalaires $\left\| \sum_0^m \lambda_k \xi_k e_k \right\|_E \leqslant C \left\| \sum_0^m \xi_k e_k \right\|_E$ dès que $\sup |\lambda_k| \leqslant 1$.

Cela signifie que la condition d'appartenance à E d'une série $\sum_0^\infty \xi_k e_k$ ne porte que sur la suite $|\xi_k|$, $k \in \mathbb{N}$, et que cette condition, si elle est satisfaite pour une suite, l'est automatiquement pour toute autre dont les valeurs absolues sont, terme à terme, majorées par celles de la première.

L'espace BMO($\mathbb{R}^n$) n'est pas séparable et ne possède donc pas de base inconditionnelle. Nous le remplaçons par la version séparable VMO($\mathbb{R}^n$) qui est la fermeture, pour la norme de BMO($\mathbb{R}^n$), de l'espace $S(\mathbb{R}^n)$ des fonctions de test.

Théorème 3. *La suite $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, est une base inconditionnelle pour $L^p(\mathbb{R}^n; dx)$, $1 < p < +\infty$, VMO($\mathbb{R}^n$) et $H^1(\mathbb{R}^n)$.*

Pour le démontrer, on construit explicitement l'opérateur T qui transforme $\psi_Q^{(\epsilon)}$ en $\lambda(Q, \epsilon)\psi_Q^{(\epsilon)}$ si $|\lambda(Q, \epsilon)| \leqslant 1$. Le noyau-distribution de T est $K(x, y) = \sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \lambda(Q, \epsilon)\psi_Q^{(\epsilon)}(x)\psi_Q^{(\epsilon)}(y)$. On a, sans peine, les estimations de Calderón-Zygmund.

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \leqslant C(\alpha, \beta)|x - y|^{-n-|\alpha|-|\beta|} \tag{5.17}$$

pour tout $x \in \mathbb{R}^n$, tout $y \neq x$, $\alpha \in \mathbb{N}^n$ et $\beta \in \mathbb{N}^n$.

Par ailleurs T est, de façon évidente, borné sur $L^2(\mathbb{R}^n)$ car il est diagonalisé dans une base orthonormée. Enfin T vérifie, au sens du théorème de David et Journé $T(1) = T^*(1) = 0$.

Il en résulte que T est borné sur $L^p(\mathbb{R}^n)$, $1 < p < +\infty$ et sur les «espaces limites» $H^1(\mathbb{R}^n)$ et VMO($\mathbb{R}^n$).

Il reste à écrire des critères explicites d'appartenance à ces espaces fonctionnels. A cet effet, nous rappelons la construction de la base de Haar de $L^2(\mathbb{R}^n; dx)$. On désigne par $\tilde{\psi}(x)$ la fonction d'une variable réelle, égale à 1 sur $[0, 1/2[$, à -1 sur $[1/2, 1[$ et à 0 ailleurs. On appelle $\tilde{\varphi}$ la fonction indicatrice de l'intervalle $[0, 1]$ (égale à 1 sur cet intervalle et à 0 ailleurs). On construit alors, pour tout $\epsilon \in E$ et tout $Q \in \mathcal{Q}$,

$$\tilde{\psi}_Q^{(\epsilon)}(x) = 2^{nj/2}\tilde{\psi}^{(\epsilon_1)}(2^j x_1 - k_1) \cdots \tilde{\psi}^{(\epsilon_n)}(2^j x_n - k_n)$$

où $\tilde{\psi}^{(0)} = \tilde{\varphi}$, $\tilde{\psi}^{(1)} = \tilde{\psi}$.

Théorème 4. *L'isométrie U: $L^2(\mathbb{R}^n; dx) \to L^2(\mathbb{R}^n; dx)$ qui à $\psi_Q^{(\epsilon)}$ associe $\tilde{\psi}_Q^{(\epsilon)}$ se prolonge en un isomorphisme de $L^p(\mathbb{R}^n)$ sur lui-même si $1 < p < +\infty$, en un isomorphisme de l'espace de Hardy $H^1(\mathbb{R}^n; dx)$ de Stein et Weiss sur sa version dyadique H_d^1 et en un isomorphisme de l'espace* BMO($\mathbb{R}^n; dx$) *de John et Nirenberg sur sa version dyadique.*

Cela signifie que les coefficients (d'ondelettes) $\lambda(Q, \epsilon)$ de $f \in L^p(\mathbb{R}^n)$ sont caractérisés par la condition $\left(\sum_\epsilon \sum_Q |\lambda(Q, \epsilon)|^2 |Q|^{-1} \mathbb{1}_Q(x)\right)^{1/2} \in L^p(\mathbb{R}^n)$ où $1 < p < +\infty$. On a désigné par $|Q|$ le volume de Q et par $\mathbb{1}_Q(x)$ la fonction indicatrice de Q.

Si $p = 1$, cette condition caractérise les coefficients de $f \in H^1(\mathbb{R}^n)$. Enfin nous allons montrer, et ce sera le début de la démonstration, que les coeffi-

cients d'ondelettes $\lambda(Q, \epsilon)$ de $f \in \mathrm{BMO}(\mathbb{R}^n)$ sont caractérisés par la condition de Carleson

$$\sum_{\epsilon} \sum_{Q \subset R} |\lambda(Q, \epsilon)|^2 \leqslant C|R| \tag{5.2}$$

(pour tout cube dyadique R, la somme étant étendue à tous les sous-cubes dyadiques $Q \subset R$).

Montrons d'abord que (5.2) entraîne la convergence, pour la topologie $\sigma(\mathrm{BMO}, H^1)$ de la série $\sum_\epsilon \sum_Q \psi_Q^\epsilon$ vers une fonction $f \in \mathrm{BMO}$. Pour ce faire, nous ignorons l'indice ϵ et raisonnons sur des sommes finies. Le passage à la limite est de pure routine, une fois établie l'inégalité $\left\| \sum_Q \lambda(Q)\psi_Q(x) \right\|_{BMO} \leqslant$ $\leqslant C_0 |||\lambda(Q)|||$ sur ces sommes finies, C_0 ne dépend que de n et $|||\cdot|||$ est la borne inférieure des $\sqrt{C}$ dans (5.2).

Soit $\varphi \in \mathfrak{D}(\mathbb{R}^n)$ une fonction d'intégrale égale à 1 et portée par le cube unité $0 \leqslant x_j \leqslant 1$ $(1 \leqslant j \leqslant n)$.

Nous poserons $\varphi_Q(x) = 2^{nj}\varphi(2^j x - k)$ de sorte que $\varphi_Q(x)$ est portée par le cube dyadique Q et d'intégrale égale à 1. On considère alors la distribution

$$K(x, y) = \sum_{Q \in \mathfrak{Q}} \lambda(Q)\psi_Q(x)\varphi_Q(y).$$

On a $|\lambda(Q)| \leqslant (C|Q|)^{1/2}$ et il en résulte aussitôt que $K(x, y)$ vérifie (5.1). Montrons que l'opérateur T, dont $K(x, y)$ est le noyau-distribution, est borné sur $L^2(\mathbb{R}^n)$. On a

$$\begin{aligned} \|T(f)\|_2 &= \left\| \sum_{Q \in \mathfrak{Q}} \lambda(Q)\Big(\int f\varphi_Q \Big)\psi_Q(x) \right\|_2 \\ &= \Big(\sum_{Q \in \mathfrak{Q}} |\lambda(Q)|^2 \Big| \int f\varphi_Q \Big|^2 \Big)^{1/2} \leqslant C_0 |||\lambda(Q)|||\, \|f\|_2 \end{aligned}$$

grâce à la version dyadique de l'inégalité de Carleson exprimée dans le lemme suivant.

Lemme 4. *Soient p_Q, $Q \in \mathfrak{Q}$, x_Q, $Q \in \mathfrak{Q}$, deux suites positives arbitraires et $\omega(x)$, $x \in \mathbb{R}^n$, la fonction maximale définie par $\omega(x) = \sup_{Q \ni x} x_Q$. Supposons que pout tout cube dyadique $R \in \mathfrak{Q}$, on ait $\sum_{Q \subset R} p_Q \leqslant |R|$. Alors on a $\sum_{Q \in \mathfrak{Q}} p_Q x_Q \leqslant \int_{\mathbb{R}^n} \omega(x)\, dx$.*

On a donc continuité de T: $L^2(\mathbb{R}^n; dx) \to L^2(\mathbb{R}^n; dx)$.

Ceci, joint à (5.1), entraîne $T(1) \in \mathrm{BMO}$ ce qui est bien ce que nous cherchions à prouver.

En sens inverse, supposons que f appartienne à BMO.

On désigne par R un cube dyadique et par R_j, $j \geqslant 1$, les cubes (non dyadiques) de même centre que R et dont les côtés sont 2^j fois celui de R. On décom-

pose, comme d'habitude, f en la série

$$f = c(R) + f_0(x) + f_1(x) + \cdots + f_j(x) + \cdots$$

où $c(R) = m_R f$ est le moyenne de f sur R et où $f_0(x) = f(x) - c(R)$ sur le cube R, $f_0(x) = 0$ ailleurs et $f_j(x) = f(x) - c(R)$ sur $R_j \backslash R_{j-1}$, 0 ailleurs. La condition $f \in \mathrm{BMO}$ entraîne $|m_{R_{j+1}} f - m_{R_j} f| \leqslant C \| f \|_{\mathrm{BMO}}$ et donc $|m_{R_j} f - m_R f| \leqslant$ $\leqslant Cj \| f \|_{\mathrm{BMO}}$. Finalement $\| f_j \|_2 \leqslant C(1+j)|R_j|^{1/2} \| f \|_{\mathrm{BMO}}$. On écrit alors $\langle f, \psi_Q \rangle = \sum_0^\infty \langle f_j, \psi_Q \rangle$.

Pour $j \geqslant 2$, on a

$$|\langle f_j, \psi_Q \rangle| \leqslant \| f_j \|_2 \Big(\int_{R_j \backslash R_{j-1}} |\psi_Q(x)|^2 \, dx \Big)^{1/2} \leqslant C_m \| f_j \|_2 \left(\frac{|Q|}{|R_j|} \right)^m$$

pour tout $m \geqslant 1$ (grâce à la décroissance rapide de ψ et à la condition géométrique $Q \subset R$).

Pour $j = 0$ ou $j = 1$, on utilise l'identité de Plancherel et l'on a

$$\sum_Q |\langle f_j, \psi_Q \rangle|^2 = \| f_j \|_2^2 \leqslant C|R| \, \| f \|_{\mathrm{BMO}}^2.$$

On conclut, grâce à la remarque que

$$\sum_{Q \subset R} \sum_{j \geqslant 2} j^2 \frac{|Q|^2}{|R_j|} = c_n |R|.$$

La fin de la démonstration du Théorème 3 est alors immédiate. En effet, il est classique que l'espace BMO dyadique est aussi caractérisé par (5.2) en décomposant les fonctions dans le système de Haar. De sorte que U établit un isomorphisme entre ces deux espaces. Par dualité, U (qui est son propre transposé) définit un isomorphisme entre l'espace de Hardy $H^1(\mathbb{R}^n)$ usuel et sa version dyadique. Par interpolation, U est un isomorphisme sur tous les espaces $L^p(\mathbb{R}^n)$, $1 < p < +\infty$.

6. Caractérisation des espaces de Hölder homogènes C^r, $r > 0$, des espaces de Sobolev H^s et des espaces de Besov $B_q^{s,p}$ $(1 \leqslant p \leqslant +\infty, 1 \leqslant q \leqslant +\infty, s \in \mathbb{R})$

Rappelons que si $0 < r < 1$, f appartient à l'espace de Hölder homogène C^r lorsque $|f(x) - f(y)| \leqslant C|x - y|^r$ pour tout $x \in \mathbb{R}^n$ et tout $y \in \mathbb{R}^n$. Si $r = 1$, on convient de remplacer C^r par la classe de Zygmund définie par $|f(x+y) + f(x-y) - 2f(x)| \leqslant C|y|$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^n)$.

Si $r = m + s$, $0 < s \leqslant 1$, $m \in \mathbb{N}$, on écrit $f \in C^r$ si $\partial^\alpha f \in C^s$ pour tous les $\alpha \in \mathbb{N}^n$ tels que $|\alpha| = m$.

L'espace de Sobolev H^s, $s \in \mathbb{R}$, est défini par

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < +\infty$$

et enfin les espaces de Besov homogènes $B_q^{s,p}$ sont caractérisés par la décomposition de Littlewood-Paley. On part d'une fonction $\theta \in \mathcal{D}(\mathbb{R}^n)$, nulle au voisinage de 0 et telle que $\sum_{-\infty}^{\infty} |\theta(2^j \xi)| \geqslant 1$ pour tout $\xi \in \mathbb{R}^n$ non nul. Alors une distribution tempérée f, modulo les polynômes, appartient à $B_q^{s,p}$ si et seulement si $\|\Delta_j(f)\|_p \leqslant 2^{-js} \epsilon_j$ où $\epsilon_j \in l^q(\mathbb{Z})$ et où $\mathcal{F}(\Delta_j(f))(\xi) = \theta(2^{-j}\xi)(\mathcal{F}f)(\xi)$ en désignant par $\mathcal{F}$ la transformation de Fourier (agissant sur la distribution tempérée f). On a désigné par $\| \cdot \|_p$ la norme $L^p(\mathbb{R}^n, dx)$.

Cet espace $B_q^{s,p}$ ne dépend pas du choix de θ. On a $B_\infty^{s,\infty} = C^s$ (espace de Hölder homogène) et $H^s = L^2 \cap B_2^{s,2}$ pour $s \geqslant 0$.

Théorème 5. *Une distribution tempérée f (modulo les polynômes) appartient à $B_q^{s,p}$ si et seulement si* $\sup_{\epsilon \in E} |\langle f, \psi_Q^{(\epsilon)} \rangle| = \alpha(k,j)$ *vérifie*

$$\left(\sum_{j=-\infty}^{\infty} \left\{ \left(\sum_{k \in \mathbb{Z}^n} (\alpha(k,j))^p \right)^{1/p} 2^{j(s + n(1/2 - 1/p))} \right\}^q \right)^{1/q} < +\infty \tag{6.1}$$

(*avec les changements usuels si $p = +\infty$ ou $q = +\infty$*).

Montrons d'abord la partie directe. Pour simplifier légèrement l'écriture, nous nous limiterons à la dimension 1.

Alors la fonction $\bar{\psi}$ servant à définir les ondelettes sera notre fonction θ et l'on a

$$\langle f, \psi_Q \rangle = 2^{-j/2} (2\pi)^{-1} \int \hat{f}(\xi) e^{i 2^{-j} k \xi} \bar{\bar{\psi}}(2^{-j}\xi) \, d\xi = 2^{-j/2} \Delta_j(f)(2^{-j}k).$$

Pour conclure, on utilise le lemme évident d'échantillonnage.

Lemme 5. *Il existe une constante $C = C(n)$ telle que pour tout $R > 0$, tout $p \in [1, +\infty]$ et toute fonction $f \in L^p(\mathbb{R}^n; dx)$ dont la transformée de Fourier est portée par la boule $|\xi| \leqslant 20R$, on ait* $\left(\sum_{k \in \mathbb{Z}^n} R^{-n} |f(kR^{-1})|^p \right)^{1/p} \leqslant C(n) \| f \|_p$.

Naturellement 20 peut être remplacé (quitte à changer la constante $C(n)$ par une autre constante. Pour nous, $20 \geqslant 8\pi/3$.

On a donc, pour tout $j \in \mathbb{Z}$,

$$\left(\sum_{k \in \mathbb{Z}^n} 2^{-nj} |\Delta_j f(2^{-j}k)|^p \right)^{1/p} \leqslant C(n) \| \Delta_j(f) \|_p$$

ce qui implique évidemment (6.1).

En sens inverse, supposons que $\alpha(k,j)$ vérifie (6.1) et montrons que $f(x) = \sum\sum \alpha(k,j)2^{j/2}\psi(2^j x - k)$ appartient à $B_q^{s,p}$. On teste cette appartenance sur la boule unité du dual $B_{q'}^{-s,p'}$ pour lequel on utilise la partie directe. Les détails sont laissés au lecteur.

7. Calcul des sommes partielles dans la transformation en ondelettes

Soit f une distribution tempérée (modulo les polynômes). Que peut-on dire de la différence

$$g(x) = f(x) - \sum_{\epsilon\in E}\sum_{|Q|\le 1} \langle f, \psi_Q^{(\epsilon)}\rangle \psi_Q^{(\epsilon)}?$$

Il semble intuitif que les petits cubes soient responsables des petits détails de $f(x)$. Ces petits détails sont très importants ou significatifs si $f(x)$ est très irrégulière. De sorte que notre différence $g(x)$ devrait être indéfiniment dérivable.

Désignons par $\mathfrak{D}_Q^{(\epsilon)}$, $\epsilon\in E$, $Q\in\mathfrak{Q}$, l'opérateur de projection orthogonale sur le vecteur $\psi_Q^{(\epsilon)}$ de l'espace de Hilbert $L^2(\mathbb{R}^n; dx)$. Désignons de même par $\mathcal{E}_Q$, $Q\in\mathfrak{Q}$, l'opérateur de projection orthogonale sur le vecteur φ_Q. Les fonctions φ et ψ sont définies par le théorème 2.

Appelons $\mathfrak{Q}_j$ la collection des cubes dyadiques de côté 2^{-j} et posons

$$\mathfrak{D}_j = \sum_{\epsilon\in E}\sum_{Q\in\mathfrak{Q}_j} \mathfrak{D}_Q^{(\epsilon)} \quad \text{et de même} \quad \mathcal{E}_j = \sum_{Q\in\mathfrak{Q}_j} \mathcal{E}_Q.$$

On a alors l'identité remarquable suivante.

Théorème 6. *Pour tout* $j\in\mathbb{Z}$, $\mathfrak{D}_j = \mathcal{E}_{j+1} - \mathcal{E}_j$.

Observons que $\mathcal{E}_j$ est une «version régularisante» de l'opérateur d'espérance conditionnelle par rapport à la tribu $\mathfrak{F}_j$ engendrée par les cubes $Q\in\mathfrak{Q}_j$. En effet, cette espérance conditionnelle $E_j(f) = \sum_{Q\in\mathfrak{Q}_j} \mathbb{1}_Q 2^{nj}\langle f, \mathbb{1}_Q\rangle$ et $\mathcal{E}_j$ est donnée par la même identité, à ceci près que $2^{nj/2}\mathbb{1}_Q$ est remplacée par la «version C^∞» φ_Q de la fonction indicatrice normalisée du cube Q.

Donnons la structure de la preuve du théorème 6 pour la dimension 1. On désigne par Δ_j l'opérateur de convolution avec $2^j\psi(2^j x)$ et par M_j l'opérateur de multiplication ponctuelle par $\exp(2\pi i 2^j x)$. Alors une application sans malice de la formule sommatoire de Poisson montre que

$$\mathfrak{D}_j = \Delta_j\Delta_j^* + \Delta_j M_j\Delta_j^* + \Delta_j M_{j+1}\Delta_j^* + \Delta_j M_j^*\Delta_j^* + \Delta_j M_{j+1}^*\Delta_j^*.$$

On désigne par S_j l'opérateur de convolution avec $2^j\varphi(2^jx)$ et l'on a alors les identités remarquables suivantes

$$\begin{aligned}\Delta_j\Delta_j^* &= S_{j+1}S_{j+1}^* - S_jS_j^*\\ \Delta_jM_j\Delta_j^* &= -S_jM_jS_j^*\\ \Delta_jM_{j+1}\Delta_j^* &= S_{j+1}M_{j+1}S_{j+1}^*.\end{aligned}$$

Par ailleurs, on fait pour $\mathcal{E}_j$ le calcul que nous avons esquissé pour $\mathcal{D}_j$ et l'on obtient $\mathcal{E}_j = S_jS_j^* + S_jM_jS_j^* + S_jM_j^*S_j^*$. Il résulte de tout cela que

$$\mathcal{D}_j = \mathcal{E}_{j+1} - \mathcal{E}_j.$$

Passons au cas de la dimension supérieure. Par exemple, si $n = 2$, on a, en conservant les notations de la dimension 1,

$$\mathcal{E}_{j+1} \otimes \mathcal{E}_{j+1} - \mathcal{E}_j \otimes \mathcal{E}_j = \mathcal{D}_j \otimes \mathcal{D}_j + \mathcal{D}_j \otimes \mathcal{E}_j + \mathcal{E}_j \otimes \mathcal{D}_j$$

et ces trois termes correspondent aux trois ondelettes $\psi_0^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}_j$, nécessaires pour obtenir notre base en dimension 2.

Le théorème 6 est donc démontré et ces remarques fournissent une nouvelle preuve du théorème 2.

Retournant à $g(x)$, on a donc

$$\begin{aligned}g(x) &= f(x) - \sum_{j\geq 0}\mathcal{D}_j(f) = f(x) - \sum_{j\geq 0}(\mathcal{E}_{j+1}(f) - \mathcal{E}_j(f)) = \mathcal{E}_0(f)\\ &= \sum_{k\in\mathbb{Z}^n} c_k\varphi(x-k)\end{aligned}$$

où les coefficients c_k sont donnés par $c_k = \int f(u)\varphi(u-k)\,du$.

Cela signifie que $f(x)$ et $g(x)$ diffèrent d'un terme d'erreur trivial.

8. Le paraproduit et la transformation en ondelettes

La fonction φ est la même que dans le Théorème 2. L'opérateur S_j est la convolution avec $2^{nj}\varphi(2^jx)$. On pose $\Delta_j = S_{j+1} - S_j$ et finalement

$$\pi(a,f) = \sum_{-\infty}^{\infty} S_{j-3}(a)\Delta_j(f).$$

On suppose $a(x) \in L^\infty \cap C^r$ et l'on raisonne modulo les opérateurs r-régularisants; c'est-à-dire ceux qui transforment H^s en H^{s+r} pour tout $s \in \mathbb{R}$. Les termes d'erreur sont analysés par le lemme suivant.

Lemme 6. *Soient $f_j(x)$, $j \in \mathbb{Z}$, des fonctions de $L^2(\mathbb{R}^n)$ dont les transformées de Fourier $\hat{f}_j$ vérifient, pour deux constantes $R_2 > R_1 > 0$*

$$\hat{f}_j(\xi) = 0 \quad si \quad |\xi| \leqslant R_1 2^j \quad ou\ si \quad |\xi| \geqslant R_2 2^j. \tag{8.1}$$

Alors, pour tout $s \in \mathbb{R}$, il existe une constante $C = C(R_1, R_2, s, n)$ telle que

$$\left\| \sum_{-\infty}^{\infty} f_j(x) \right\|_{H^s} \leqslant C \left(\sum_{-\infty}^{\infty} \| f_j \|_2^2 (1 + 4^j)^s \right)^{1/2} \tag{8.2}$$

Utilisant ce lemme, on commence par remplacer $S_{j-3}(a)$ par $S_{j-10}(a)$. On calcule ensuite $\pi(a, \psi_Q^{(\epsilon)})$ lorsque Q est défini par $2^m x - k \in [0, 1]^n$. Alors $\Delta_j(\psi_Q^{(\epsilon)}) = 0$ sauf si $|m - j| \leqslant 2$.

On définit (par linéarité) l'opérateur R_a par

$$R_a(\psi_Q^{(\epsilon)}) = \pi(a, \psi_Q^{(\epsilon)}) - S_{m-10}(a)\psi_Q^{(\epsilon)}$$

et, en utilisant notre lemme ainsi que la caractérisation des espaces de Sobolev en série d'ondelettes, on montre que R_a est r-régularisant.

On peut alors énoncer

Théorème 7. *Si $r > 0$ et $a(x) \in C^r \cap L^\infty$, le paraproduit $\pi(a, f)$ est défini par linéarité par*

$$\pi(a, \psi_Q^{(\epsilon)}) = S_Q(a)\psi_Q^{(\epsilon)}$$

où, par abus de langage, $S_Q(a) = S_{j-10}(a)$ lorsque

$$Q = \{x \in \mathbb{R}^n;\ 2^j x - k \in [0, 1]^n\};$$

si, de plus, $0 < r < 1$, on a, modulo un opérateur r-regularisant,

$$\pi(a, \psi_Q^{(\epsilon)}) = a(2^{-j}k)\psi_Q^{(\epsilon)}.$$

en d'autres termes, le paraproduit est diagonalisé dans la base hilbertienne des ondelettes.

9. Analyse des travaux antérieurs sur le sujet

La «version continue» de la transformation en ondelette a une longue histoire remontant aux travaux de A. Calderón et de ses collaborateurs. La version discrète apparaît en 1980 lorsque L. Carleson démontre que le système de Haar, correctement lissé est une base inconditionnelle de l'espace $H^1(\mathbb{T})$. Naturellement les ondelettes de Carleson ne forment pas une base hilbertienne mais il existe un système bi-orthogonal associé.

Une autre démarche a été introduite par Frazier et Jawerth [6] et, indépendamment, par l'un des auteurs [9].

Il s'agit d'exhiber ψ dans $S(\mathbb{R}^n)$ de sorte que l'on ait, pour toute fonction $f \in L^2(\mathbb{R}^n; dx)$

$$f(x) = \sum_{Q \in \mathbb{Q}} \langle f, \psi_Q \rangle \psi_Q(x) \tag{9.1}$$

sans unicité.

Il suffit pour cela que la transformée de Fourier de ψ soit portée par le cube $-\pi \leqslant x_j \leqslant \pi$, $1 \leqslant j \leqslant n$ et que l'on ait, pour tout $\xi \neq 0$,

$$1 = \sum_{-\infty}^{\infty} |\hat{\psi}(2^j \xi)|^2. \tag{9.2}$$

Si l'on a unicité dans (9.1), les fonctions $\psi_Q(x)$ sont automatiquement orthogonales mais cela est exclu par la condition portant sur le support de $\hat{\psi}$.

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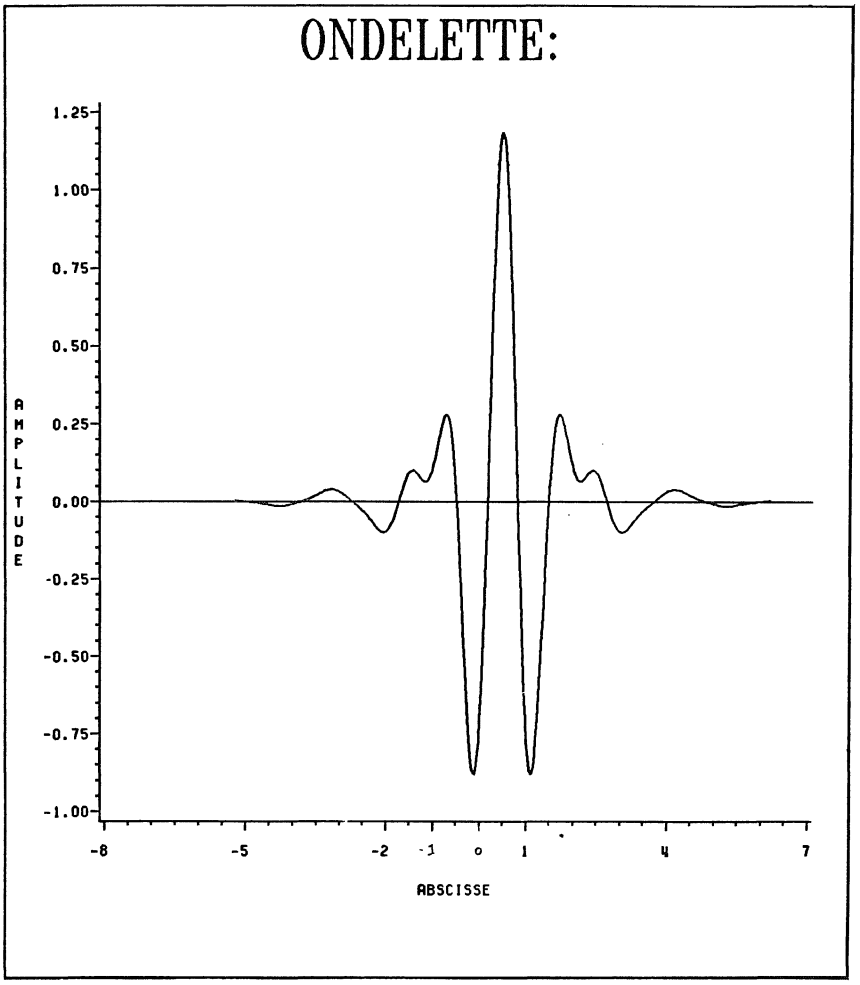
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The Dirichlet Problem for the Degenerate Monge-Ampère Equation

L. Caffarelli **L. Nirenberg** **J. Spruck**

Dedicated to Alberto Calderón on his 65th birthday

Let Ω be a bounded convex domain in R^n with smooth, strictly convex boundary $\partial\Omega$, i.e. the principal curvatures of $\partial\Omega$ are all positive. We study the problem of finding a convex function u in Ω such that

$$\det(u_{ij}) = 0 \quad \text{in} \quad \Omega \tag{1}$$

$$u = \phi \quad \text{given on } \partial\Omega. \tag{2}$$

Here $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$ etc. The existence of a smooth solution in $\bar{\Omega}$ satisfying (2) of the corresponding elliptic problem

$$\det(u_{ij}) = \psi > 0 \quad \text{in} \quad \bar{\Omega} \tag{1}'$$

has been shown recently by N. V. Krylov [5] and the authors [3] in case ψ and ϕ are sufficiently smooth. It is of interest to treat the degenerate problem (1), (2). The corresponding question for degenerate complex Monge-Ampère equations is also of interest: find a plurisubharmonic function w in a bounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$ satisfying

$$\det(w_{z_j\bar{z}_k}) = 0 \quad \text{in} \quad \Omega \tag{3}$$

and (2). In fact in [4], with J. J. Kohn, we treated the equation

$$\det(w_{z_j\bar{z}_k}) = \psi \geqslant 0 \quad \text{in} \quad \Omega \tag{3}'$$

and showed that there is a plurisubharmonic solution w belonging to $C^{1,1}(\bar{\Omega})$ provided $\psi \not\equiv 0$, ψ satisfies some other conditions, and ψ and ϕ are sufficiently

smooth. See [4] for further references on (3)', (2). E. Bedford and J. E. Fornaess [1] presented an example with ψ satisfying all our conditions and for which the (unique) solution of (1)' is not in $C^2(\Omega)$.

Several authors have studied the Dirichlet problem (1), (2): J. Rauch and B. A. Taylor [6], Bedford and Taylor [2] as well as [1], and, most recently, N. S. Trudinger and J. I. F. Urbas [7]. The unique solution of (1), (2) is given by

$$(4) \qquad u(x) = \max \{ v(x) \mid v \in C(\bar{\Omega}), \quad v \text{ convex and } \quad v \leqslant \phi \quad \text{on} \quad \partial\Omega \},$$

and the papers cited study the regularity of u. The best regularity result is that in [7] where it is shown that if $\partial\Omega \in C^{1,1}$ and $\phi \in C^{1,1}(\partial\Omega)$ then the function u belongs to $C^{0,1}(\bar{\Omega})$ and to $C^{1,1}$ in every compact subset of Ω. In this paper we prove an extension up to the boundary of this regularity in case ϕ is sufficiently smooth.

Theorem. *Assume $\partial\Omega$ is in $C^{3,1}$, and $\phi \in C^{3,1}(\partial\Omega)$; then the function u defined by* (4) *is of class $C^{1,1}(\bar{\Omega})$.*

We do not know if a corresponding estimate holds for the solution w of (3), (2) —which is characterized by

$$(4)' \quad w(z) = \max \{ v(z) \mid v \in C(\bar{\Omega}),\ v \text{ plurisubharmonic in } \Omega \text{ and } v \leqslant \phi \text{ on } \partial\Omega \}.$$

The following examples show that the theorem is optimal; here Ω is the unit disc centred at the origin in the (x, y) plane.

Ex. 1. The function

$$u = (1 + y)^{2-\epsilon}, \qquad 0 < \epsilon \quad \text{small}$$

satisfies (1) and (2) with

$$\phi \in C^{3,1-2\epsilon}(\partial\Omega).$$

The function u is of class $C^{1,1-\epsilon}$ but not $C^{1,1}$ in $\bar{\Omega}$.

The same u shows that if we flatten $\partial\Omega$ at $(0, -1)$ to make the curvature of $\partial\Omega$ vanish here to high order, ϕ may be very smooth while u is not in $C^{1,\delta}(\bar{\Omega})$ for $\delta > 1 - \epsilon$.

Ex. 2. (This is due to John Urbas). Let

$$\phi(y) \equiv \left(y^2 - \frac{1}{4} \right)^2$$

this is convex in $y > \frac{1}{2}$. It is easily verified that the function u defined by (4) satisfies the conditions in Ω: $u \equiv 0$ in the inscribed rectangle in Ω bounded on top and bottom by $|y| = 1/2$, and $u \equiv \phi(y)$ above the rectangle.

Here $\phi \in C^\infty(\partial\Omega)$ but u is not in C^2 near $\left(0, \frac{1}{2}\right)$.

In [5], [3], a priori estimates are established for the C^2 norms of solutions of (1)′, (2). It seems natural to try to establish such estimates (independent of ϵ for $0 < \epsilon \leqslant 1$) of convex solutions of, say,

$$(1)_\epsilon \qquad \det(u_{ij}) = \epsilon \quad \text{in} \quad \Omega$$

satisfying (2). However we have not done this. Instead we work directly with the characterization (4), and this paper is quite independent of [3].

Section 1. Beginning of proof

To prove the theorem we will establish the following estimate: there is constant A depending on Ω and the $C^{3,1}$ norm of ϕ such that

$$(5) \qquad |u(y) - u(x) - (y - x) \cdot \nabla u(x)| \leqslant A|x - y|^2$$

holds for every pair of points x, y in Ω. It is easy to see that (5) then yields the conclusion of the Theorem. We have, namely, to show that for some constant B depending only on Ω and on A,

$$(6) \qquad |\nabla u(x) - \nabla u(y)| \leqslant B|x - y| \quad \text{for all} \quad x, y \in \Omega.$$

Observe first that there are fixed positive numbers ϵ, δ depending only on Ω such that for every $y \in \Omega$, Ω contains a truncated cone

$$K(y) = \{z \neq y \mid |y - z| < \epsilon, \text{ and the angle } (z - y) \text{ makes with some unit vector } \xi(y) \text{ is less than } \delta\}.$$

It follows that there is a positive number α depending only on δ (and so only on Ω) such that for any vector $\eta \neq 0$ there is a point z in $K(y)$ such that

$$|(z - y) \cdot \eta| \geqslant \alpha|z - y| \cdot |\eta|.$$

Clearly there is such a z in $K(y)$ with $|z - y| =$ any positive number $t < \epsilon$.

To establish (6), since by [6] and [7] we have a bound $|\nabla u| \leqslant$ constant, we may suppose $|x - y| < \epsilon$ (the ϵ determined above). After subtraction of an affine function we may assume

$$u(x) = |\nabla u(x)| = 0,$$

and we have then to show

$$|\nabla u(y)| \leqslant B|x-y| \quad \text{if} \quad |x-y| < \epsilon. \tag{6}'$$

By (5), we have for every $z \in \Omega$,

$$|u(z)| \leqslant A|z-x|^2$$

and

$$|u(z) - u(y) - (z-y)\cdot \nabla u(y)| \leqslant A|z-y|^2,$$

and of course

$$|u(y)| \leqslant A|y-x|^2.$$

Hence

$$|(z-y)\cdot \nabla u(y)| \leqslant A(|z-x|^2 + |z-y|^2 + |y-x|^2). \tag{7}$$

Using what we have asserted above, we may choose z in Ω with $|z-y| = |x-y| < \epsilon$ and such that

$$|(z-y)\cdot \nabla u(y)| \geqslant \alpha|z-y|\cdot|\nabla u(y)|.$$

(for we may suppose $\nabla u(y) \neq 0$). Combining this with (7) we obtain

$$\alpha|x-y|\cdot|\nabla u(y)| \leqslant 6A|x-y|^2,$$

i.e., (6)', with $B = 6A/\alpha$.

We will derive (5) from the following local form, the heart of our proof:

Proposition. *There is a constant C depending only on Ω and the $C^{3,1}$ norm of ϕ, such that for every point x^0 in Ω, $\exists \epsilon(x^0) > 0$ so that for every x in Ω with $|x-x^0| \leqslant \epsilon(x^0)$, we have*

$$|u(x) - u(x^0) - (x-x^0)\cdot \nabla u(x^0)| \leqslant C|x-x^0|^2. \tag{8}$$

We claim that the proposition yields (5) with any constant A greater than C (and hence in the limit for $A = C$). Fix a constant $A > C$. If (5) does not hold, there are points x, y in Ω such that

$$|u(y) - u(x) - (y-x)\cdot \nabla u(x)| > A|x-y|^2.$$

By (8) there is a closest point z in $\bar{\Omega}$ to x such that

$$|u(z) - u(x) - (z-x)\cdot \nabla u(x)| = A|z-x|^2.$$

On the closed segment L joining x to z consider the function

$$f(y) = A|y - x|^2 - [u(y) - u(x) - (y - x) \cdot \nabla u(x)].$$

By (8), this is positive for y near x, $y \neq x$, but it is zero at x and z. So f achieves a positive maximum at some point w in the interior of L. At w the second difference quotient

$$(9)\quad \Delta_h^2 f = f(w + h(z - x)) + f(w - h(z - x)) - 2f(w) \leqslant 0 \quad \text{for} \quad |h| \quad \text{small.}$$

But

$$\Delta_h^2 f = 2Ah^2|z - x|^2 - [u(w + h(z - x)] + u[w - h(z - x)) - 2u(w)]$$

and by (8) with x^0 replaced by w, the expression in the square bracket satisfies

$$|[\quad]| \leqslant 2Ch^2|z - x|^2 \quad \text{for} \quad |h| \quad \text{small.}$$

Thus

$$\Delta_h^2 f \geqslant 2(A - C)h^2|z - x|^2 \quad \text{for} \quad |h| \quad \text{small,}$$

contradicting (9). □

To complete the proof of the theorem we have to prove the Proposition. First some preliminary simple lemmas.

Lemma 1. *Let S be a straight segment with endpoints on $\partial\Omega$. Suppose one of these is the origin and that S makes an angle $\pi/2 - \alpha$ with the interior normal to $\partial\Omega$ there, $0 < \alpha$ small. Then length of $S = 0(\alpha)$.*

PROOF. We may assume the positive x_n axis is interior normal to $\partial\Omega$ at the origin and that S is of the form

$$x_n = \beta x_1, \qquad 0 \leqslant x_1 \leqslant t, \qquad \beta = \tan\alpha, \qquad x_2 = \cdots = x_{n-1} = 0.$$

At the other end point of S we have

$$x_n = c_0 t^2 + O(|t|^3) = \beta t,$$

$c \geqslant c_0$ a fixed positive constant. It follows that

$$t = O(\beta) = O(\alpha)$$

and also $x_n = O(\alpha^2)$. □

In the following lemma, u is the function defined by (4).

Lemma 2. *Let x^0 be any point in Ω. Subtracting from u a linear plane of support there we may suppose*

$$u \geqslant 0, \qquad u(x^0) = 0.$$

Then x^0 is a convex combination of $(n+1)$ points $x^1, \ldots, x^{n+1}$ in $\partial\Omega$ with $u(x^i) = 0$ for $i = 1, \ldots, n+1$.

PROOF. By Caratheodory's theorem it suffices to show that x^0 is in the convex hull of

$$S = \{x \in \partial\Omega \mid u(x) = 0\}.$$

If not, there is a hyperplane separating them; i.e. every point x in S satisfies (after rotation and traslation of coordinates)

$$x_n < -\epsilon < \epsilon < x_n^0.$$

Thus at points on $\partial\Omega$ where $x_n \geqslant 0$ we have $u \geqslant a > 0$ for some positive constant a. But then the function

$$v = \delta x_n \quad \text{for} \quad 0 < \delta \quad \text{small}$$

satisfies

$$v \leqslant u \quad \text{on} \quad \partial\Omega.$$

Consequently, by (4), $u(x^0) \geqslant v(x^0) > \delta\epsilon$ —contradicting the fact that $u(x^0) = 0$. □

Section 2. Proof of the Proposition

We will say that a constant is under control if it depends only on Ω and the $C^{3,1}$ norm of ϕ. Fix $x^0 \in \Omega$. After subtraction of an affine function we may suppose $u \geqslant 0$, $u(x^0) = 0$. According to Lemma 2, x^0 lies in a n-dimensional simplex S with vertices on $\partial\Omega$, and on which $u \equiv 0$. It may be that x^0 lies in a lower, say k, dimensional simplex with this property. Using induction on k we will prove (8).

(i) Postponing the case $k = 1$, suppose we have proved the result for all x^0 lying in any $(k-1)$-dimensional simplex with the stated properties, and with constant $C = C_{k-1}$ under control. We wish to prove it for x^0 in such a k-dimensional simplex S with some constant C_k also under control. Let x^1 be the closest point to x^0 on any $(k-1)$-dimensional face of S. Then $y^0 = 2x^0 -$

$-x^1$ lies in S. By induction, in the ball B (around x^1 with radius $\epsilon(x^1)$, we have

$$|u(x)| \leqslant C_{k-1}|x - x^1|^2.$$

It follows by convexity of u (recall $u(y^0) = 0$) that in the ball around x^0 with radius $\epsilon(x^0) := \frac{1}{2}\epsilon(x^1)$

$$|u(x)| \leqslant 2C_{k-1}|x - x^0|^2.$$

Thus we have established (8) with a constant $C_k = 2C_{k-1}$.

Consequently (8) holds for any x^0 in Ω with

$$C = 2^{n-1}C_1$$

and the proof is finished —once we have treated the case $k = 1$.

(ii) Turning to that case, we suppose x^0 lies in a segment S with end points x^1, x^2 on $\partial\Omega$, on which $u = 0$, and that $u \geqslant 0$ in Ω. Of the two end points, suppose that x^2 is the closer to x^0. We may suppose $x^2 = 0$ and that the positive x_n-axis is interior normal to $\partial\Omega$ there. In addition we may suppose $x^0 = (x_1^0, 0, \ldots, 0, x_n^0)$ with $x_1^0 \geqslant 0$, $x_n^0 = x_1^0 \tan\alpha$, $0 < \alpha \leqslant \frac{\pi}{2}$, i.e., S makes angle $\frac{\pi}{2} - \alpha$ with the interior normal to $\partial\Omega$ at 0.

By Lemma 1,

$$|x^1| = O(\alpha). \tag{10}$$

We will distinguish two cases

$$\alpha \leqslant \alpha_0, \quad \text{a positive small constant to be chosen,}$$
$$\alpha > \alpha_0.$$

Consider first the case $\alpha \leqslant \alpha_0$ small. We have $x^1 = (x_1^1, 0, \ldots, 0, x_1^1 \tan\alpha)$. The orthogonal projection on the plane $x_n = 0$ of the segment joining the origin to x^1 is the segment L on the x_1 axis from the origin to $(x_1^1, 0, \ldots, 0)$. If we think of the boundary values ϕ near 0 as a nonnegative function of $(x_1, \ldots, x_{n-1})$, then at the end points of the segment L we have $\phi = 0$ and hence $\phi_1 = 0$, and $\phi_{11} \geqslant 0$ there. It follows that

$$\int_L \phi_{11}\, dx_1 = 0. \tag{11}$$

Hence ϕ_{11} has an interior minimum in L, where necessarily $\phi_{111} = 0$. Consequently on L, $|\phi_{111}| \leqslant \text{constant} \cdot x_1^1$; *here, at last, is where we use the fact that the $C^{3,1}$ norm of ϕ is finite.* By (10), we have $|\phi_{111}| \leqslant \text{constant} \cdot \alpha$, and since ϕ_{11} necessarily vanishes somewhere on L, see (11), it follows that

$$|\phi_{11}(0)| \leqslant A\alpha^2 \tag{12}$$

with A under control.

For other second derivatives of ϕ at the origin we have

$$|\phi_{ij}(0)| \leqslant A \tag{13}$$

while from (12) it follows that

$$|\phi_{1\beta}(0)| \leqslant A\alpha, \qquad 1 < \beta \leqslant n-1, \tag{14}$$

since $\phi \geqslant 0$ and, hence, $\sum \phi_{ij}(0)x_i x_j \geqslant 0$.

Consider now a ball B in Ω with centre x^0 and radius $\epsilon \leqslant \alpha^3$. Let Γ be the cone with vertex x^1 generated by B and set $K = \partial\Omega \cap \Gamma$; recall that $|x^1 - x^0| \geqslant \geqslant |x^0|$. For $\alpha \leqslant \alpha_0$ sufficiently small, depending only on Ω, it is not difficult to verify that for ϵ sufficiently small (depending possibly on the point x^0), the orthogonal projection of K onto the plane $x_n = 0$ is contained in an ellipsoid with axes

$$C\frac{\epsilon}{\alpha} \quad \text{in the } x_1\text{-direction}, \quad C\epsilon \quad \text{in the other directions},$$

for a fixed constant C depending only on Ω. Let us now fix such α_0.

For any point x in the ϵ-ball B let $\bar{x}$ represent the point on $\partial\Omega$ where the ray from x^1 to x strikes $\partial\Omega$. Since $u(x^1) = 0$ it follows from convexity that

$$u(x) \leqslant \phi(\bar{x}). \tag{15}$$

Set $\delta = |x - x^0|$. As described above, $(\bar{x}_1, \ldots, \bar{x}_{n-1})$ lies in the ellipsoid

$$\alpha^2 x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leqslant C^2\delta^2. \tag{16}$$

Consequently

$$\phi(\bar{x}) = \sum_{i,j=1}^{n-1} \phi_{ij}(0)\bar{x}_i\bar{x}_j + O(|\bar{x}|^3).$$

Using (12), (13), (14), and (16) it follows that

$$\phi(\bar{x}) \leqslant CA\delta^2 + C\frac{\delta^3}{\alpha^3}$$

with C a (different) constant under control. By (15), and the relations $|x - x^0| = \delta \leqslant \epsilon \leqslant \alpha^3$, we find

$$u(x) \leqslant C_1|x - x^0|^2.$$

We have proved (8) for $\alpha \leqslant \alpha_0$.

The other case to consider is $\alpha \geqslant \alpha_0$; this case is simple. There is a positive ϵ such that if $|x - x^0| < \epsilon$, then the point $\bar{x}$ on $\partial\Omega$ where the ray from x^1 to

x strikes $\partial\Omega$ satisfies

$$|\bar{x}| \leqslant C|x - x^0|$$

with C depending only on Ω. Here we use also the fact that $|x^1 - x^0| \geqslant |x^0|$. Since $\phi \geqslant 0$ and $\phi(0) = 0$ we have $\phi(\bar{x}) \leqslant A|\bar{x}|^2$ where A depends on the C^2 norm of ϕ. By convexity of u, and the fact that $u(x^1) = 0$ we conclude that

$$u(x) \leqslant \phi(\bar{x}) \leqslant A|\bar{x}|^2 \leqslant AC^2|x - x^0|^2$$

i.e. (8) holds again in this case.

The proofs of the Proposition and of the Theorem are complete.

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Cohomology Theories on Compact and Locally Compact Spaces

To Alberto Calderón on his retirement
from the University of Chicago

E. Spanier*

1. Introduction

This paper is devoted to an exposition of cohomology theories on categories of spaces where the cohomology theories satisfy the type of axiom system considered in [1, 12, 16, 17, 18]. The categories considered are $\mathcal{C}_{\text{comp}}$, the category of all compact Hausdorff spaces and continuous functions between them, and $\mathcal{C}_{\text{loc comp}}$, the category of all locally compact Hausdorff spaces and proper continuous functions between them. The fundamental uniqueness theorem for cohomology theories on a finite dimensional space implies a corresponding uniqueness theorem for cohomology theories on either of these two categories. The proof involves an extension of the uniqueness theorem for finite dimensional spaces to compact spaces which contrasts with the usual type of proof which involves a uniqueness proof for polyhedra and an extension to compact spaces.

A spectrum of ANR defines a cohomology theory on $\mathcal{C}_{\text{loc comp}}$. Applications of the uniqueness theorem to such cohomology theories gives a proof of the known result that the Chern character is an isomorphism of $K(X) \otimes \mathbb{Q}$ with

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$\check{H}_{ev}(X; \mathbb{Q})$ for every locally compact X. The uniqueness theorem is also used to give another proof of the duality theorem in stable homotopy theory for compact subsets of $\mathbb{R}^n$.

The results of this paper are not new. The objective of the paper is to give a reasonably self-contained expository account of the uniqueness theorem for cohomology theories on $\mathcal{C}_{\text{loc comp}}$ and how it can be applied to give theorems such as the one concerning the Chern character and the duality theorem referred to above.

The rest of the paper is divided into six sections. Section 2 is devoted to preliminaries including a review of the definitions of cohomology theory on a space and of an ES theory on a space (the latter satisfying some of the Eilenberg-Steenrod axioms). In Section 3 we consider cohomology theories on $\mathcal{C}_{\text{comp}}$ and $\mathcal{C}_{\text{loc comp}}$ and a uniqueness theorem for each. We also show that cohomology theories on $\mathcal{C}_{\text{comp}}$ correspond injectively to compactly supported cohomology theories on $\mathcal{C}_{\text{loc comp}}$.

Section 4 is devoted to ES theories on $\mathcal{C}_{\text{comp}}$. There is a uniqueness theorem for these and a proof of the equivalence of cohomology theories and ES theories on $\mathcal{C}_{\text{comp}}$.

In Section 5 it is shown how a spectrum of ANR determines a compactly supported ES theory on $\mathcal{C}_{\text{loc comp}}$. As a consequence the theorem concerning the Chern character is deduced. Section 6 contains definitions of the functional spectrum and of a spectrum approximating the complement of a compact pair in $\mathbb{R}^n$. In Section 7 these two spectra are compared to obtain a proof of the duality theorem in stable homotopy theory.

2. Preliminaries

In this section we recall the definitions of a cohomology theory and of an ES theory on a space X [16, 17, 18] and some related concepts.

All topological spaces will be assumed to be normal Hausdorff spaces. If H is a contravariant functor from a category of subsets (and inclusion maps between them) of such a space to the category of graded abelian groups (and homomorphisms of degree zero between them) we use the following notation. If H is defined for an inclusion map i: $B \subset A$ and $u \in H(A)$ then $u|B \in H(B)$ is defined by $u|B = H(i)(u)$. The statement that H is a contravariant functor is equivalent to the two conditions:

(i) for $u \in H(A)$, $u|A = u$.
(ii) for $C \subset B \subset A$ and $u \in H(A)$ then $(u|B)|C = u|C$.

In general ρ will be used to denote a homomorphism induced by an inclusion map or a family of inclusion maps (i.e. $\rho = H(i)$: $H(A) \to H(B)$ for

i: $B \subset A$). Similar notation will be used for contravariant functors H from a category of pairs of X to the category of graded abelian groups. Given a topological space X let $\mathrm{cl}(X)$ denote the category of all closed subsets of X and all inclusion maps between them and let $\mathrm{cl}(X)^2$ be the category of all pairs of closed subsets of X and inclusion maps between them.

A *cohomology theory* H, δ on X consists of:

(i) a contravariant functor H from $\mathrm{cl}(X)$ to the category of graded abelian groups ($H(A) = \{H^q(A)\}_{q\in\mathbb{Z}}$ for A in $\mathrm{cl}(X)$) such that $H(\emptyset) = 0$, and

(ii) a natural transformation δ assigning to every two closed subsets $A, B \subset X$ a homomorphism of degree 1

$$\delta\colon\ H^q(A \cap B) \longrightarrow H^{q+1}(A \cup B)$$

such that the following are satisfied:

Continuity. For every closed $A \subset X$ there is an isomorphism

$$\rho\colon \varinjlim \{H^q(N) | N \text{ a closed neighborhood of } A \text{ in } X\} \approx H^q(A)$$

where $\rho\{u\} = u|A$ for $u \in H^q(N)$.

MV exactness. For every two closed sets A, B in X there is an exact sequence

$$\cdots \xrightarrow{\delta} H^q(A \cup B) \xrightarrow{\alpha} H^q(A) \oplus H^q(B) \xrightarrow{\beta} H^q(A \cap B) \xrightarrow{\delta} H^{q+1}(A \cup B) \xrightarrow{\alpha} \cdots$$

where $\alpha(u) = (u|A, u|B)$ for $u \in H^q(A \cup B)$ and $\beta(u, v) = u|A \cap B - v|A \cap B$ for $u \in H^q(A)$, $v \in H^q(B)$.

The cohomology theory is *nonnegative* if $H^q(A) = 0$ for $q < 0$ and all closed $A \subset X/C$. It is *compactly supported* (or has *compact supports*) if given $u \in H^q(A)$ there is a decomposition $A = B \cup C$ where B is closed, C is compact, and $u|B = 0$. It is *additive* if given a discrete* family $\{A_j\}_{j\in J}$ of closed sets there is an isomorphism

$$\sigma\colon\ H^q\Big(\bigcup_{j\in J} A_j\Big) \approx \prod_{j\in j} H^q(A_j)$$

where $\sigma(u) = \{u|A_j\}_{j\in J}$ for $u \in H^q\big(\bigcup_{j\in j} A_j\big)$.

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* A family $\{A_j\}_{j\in J}$ of subsets of a topological space X is *discrete* if every point of X has a neighborhood meeting at most one member of the family. This implies the members of the family are pairwise disjoint and, since a discrete family is obviously locally finite, if each is closed in X, then $\bigcup_{j\in J} A_j$ is also closed in X.

If H, δ and H', δ' are cohomology theories on the same space X, a *homomorphism* $\varnothing$ from H, δ to H', δ' is a natural transformation from H to H' commuting up to sign with δ, δ'.

Cohomology theories on X frequently arise from contravariant functors on $\mathrm{cl}(X)^2$ satisfying some of the Eilenberg-Steenrod axioms [5] as in the following.

An ES *theory* H, δ^* on X consists of:

(i) a contravariant functor H from $\mathrm{cl}(X)^2$ to the category of graded abelian groups, and

(ii) a natural transformation of degree 1

$$\delta^*\colon\ H^q(B, \varnothing) \to H^{q+1}(A, B)$$

for every (A, B) in $\mathrm{cl}(X)^2$

such that the following are satisfied:

Continuity. For every closed A in X there is an isomorphism

$$\rho\colon\ \varinjlim \{H^q(N, \varnothing) | N \text{ a closed neighborhood of } A \text{ in } X\} \approx H^q(A, \varnothing)$$

where $\rho\{u\} = u|(A, \varnothing)$ for $u \in H^q(N, \varnothing)$,

Exactness. For every closed pair (A, B) in X the following sequence is exact

$$\cdots \xrightarrow{\delta^*} H^q(A, B) \xrightarrow{H(j)} H^q(A, \varnothing) \xrightarrow{H(i)} H^q(B, \varnothing) \xrightarrow{\delta^*} H^{q+1}(A, B) \longrightarrow \cdots$$

where $i\colon\ (B, \varnothing) \subset (A, \varnothing)$ and $j\colon\ (A, \varnothing) \subset (A, B)$.

Excision. For closed sets A, B in X there is an isomorphism

$$\rho\colon\ H^q(A \cup B, B) \approx H^q(A, A \cap B).$$

It is standard [5] that if H, δ^* is an ES theory on X there is a cohomology theory H', δ' on X such that $H'(A) = H(A, \varnothing)$ and $\delta'\colon\ H'^q(A \cap B) \to H'^{q+1}(A \cup B)$ is suitably defined. In general we do not know if there is a way of associating an ES theory to a cohomology theory on X. With suitable definitions of cohomology theories and ES theories on larger categories we will show in Section 4 that the two theories are equivalent on the category of all compact spaces.

The concepts of *nonnegativity, compactly supported*, and *additivity* are defined for ES theories to correspond to the same properties of the associated cohomology theories.

3. Cohomology theories on categories of spaces

In this section we consider cohomology theories defined on categories of topological spaces and continuous functions. These consist of contravariant functors and natural transformations on the category whose restriction to

$\mathrm{cl}(X)$ is a cohomology theory on X for every object X in the category. The categories of interest are the category of all compact spaces and continuous functions and the category of all compact spaces and proper continuous functions. We prove uniqueness theorems for cohomology theories on these two categories. Finally, we show that cohomology theories on the category of compact spaces correspond injectively to compactly supported cohomology theories on the category of locally compact spaces.

We begin by considering cohomology theories on cubes. By a *cube* we mean a product space $\prod_{j\in J} I_j$ where I_j is a closed interval of $\mathbb{R}$ for each $j \in J$. Given a cube $C(J) = \prod_{j\in J} I_j$ and given a subset $J' \subset J$ let $C(J') = \prod_{j\in J} I_j$. There is a canonical projection map $p_{J'}\colon C(J) \to C(J')$.

Lemma 3.1. *Let $\varphi\colon H \to H'$ be a homomorphism of cohomology theories on a cube $C(J)$ and let $n \in \mathbb{Z}$ be such that $\varphi_A\colon H(A) \to H'(A)$ is an n-equivalence* for every A of the form $A = p_F^{-1}(x)$ where F is a finite subset of J and $x \in C(F)$. Then φ_A is an n-equivalence for every closed $A \subset C(J)$.*

PROOF. 1) Let F be an arbitrary finite subset of J and let H_F, H'_F be the cohomology theories on $C(F)$ which equal the direct images (in the sense of Remark 2.6 of [18]) of the cohomology theories H, H' on $C(J)$ under $p_F\colon C(J) \to C(F)$ (so $H_F(B) = H(p_F^{-1}(B))$ and $H'_F(B) = H'(p_F^{-1}(B))$ for every closed $B \subset C(F)$). The hypotheses on φ imply that φ induces a homomorphism $\varphi_F\colon H_F \to H'_F$ of cohomology theories on $C(F)$ which is an n-equivalence for every $x \in C(F)$. Since $C(F)$ is a finite dimensional compact metric space, it follows from Theorem 4.2 of [18] that φ_F is an n-equivalence for every closed $B \subset C(F)$.

2) Let A be a closed subset of $C(J)$ and for F a finite subset of J let $A_F = p_F^{-1}p_F(A)$. If $F \subset F'$ are finite subsets of J, there is a projection $p\colon C(F') \to C(F)$ such that $p_F = p \circ p_{F'}$. It follows that $p_F(A) = pp_{F'}(A)$ so $p_{F'}(A) \subset p^{-1}p_F(A)$ and

$$A_{F'} = p_{F'}^{-1}(p_{F'}(A)) \subset p_{F'}^{-1}(p^{-1}p_F(A)) = p_F^{-1}p_F(A) = A_F.$$

Hence, the collection $\{A_F | F \text{ finite} \subset J\}$ is a family of closed subsets of $C(J)$ directed downward by inclusion. Clearly $A \subset A_F$ for every F so $A \subset \bigcap_F A_F$.

We show $A = \bigcap_F A_F$. If $y \in C(J) - A$ there is a nbhd of y disjoint from A (because A is closed). Every nbhd of y contains a subset of the form $p_F^{-1}(N)$ where F is a finite subset of J and N is a closed nbhd of $p_F(y)$ in $C(F)$. Clearly $p_F^{-1}(N)$ is disjoint from A if and only if N is disjoint from $p_F(A)$. This implies $p_F(y) \notin p_F(A)$ and so $y \notin p_F^{-1}p_F(A) = A_F$. Therefore, $\bigcap_F A_F = A$.

---

* A homomorphism $\varphi\colon G \to G'$ of degree 0 between graded abelian groups is an *n-equivalence* if $\varphi\colon G^q \to G'^q$ is an isomorphism for all $q < n$ and a monomorphism for $q = n$.

It follows from Proposition 2.7 part 2) of [18] that $\varinjlim \{H(A_F) | F \text{ finite} \subset \subset J\} \approx H(A)$ and $\varinjlim \{H'(A_F) | F \text{ finite} \subset J\} \approx H'(A)$. By 1) above, φ_J: $H_F(p_F(A)) \to H'_F(p_F(A))$ is an n-equivalence for every finite $F \subset J$. This is equivalent to the assertion that φ: $H(A_F) \to H'(A_F)$ is an n-equivalence for every finite $F \subset J$. Passing to the direct limit we see that φ: $H(A) \to H'(A)$ is an n-equivalence for an arbitrary closed $A \subset C(J)$. □

Let $\mathcal{C}$ be a category of topological spaces and continuous functions such that if X is an object of $\mathcal{C}$ then $\mathrm{cl}(X)$ is a subcategory of $\mathcal{C}$. A *cohomology theory* H, δ on consists of:

(i) A contravariant functor H from $\mathcal{C}$ to the category of graded abelian groups such that $H(\phi) = 0$, and

(ii) A natural transformation δ: $H^q(A \cap B) \to H^{q+1}(A \cup B)$ for every triad $(X; A, B)$ in $\mathcal{C}$ (by such a triad we mean X is an object of $\mathcal{C}$ and A, B are closed subsets of X).

such that for every object X in $\mathcal{C}$ the restriction of H, δ to $\mathrm{cl}(X)$ is a cohomology theory on X.

A cohomology theory on $\mathcal{C}$ is *nonnegative, compactly supported* or *additive*, respectively, if its restriction to $\mathrm{cl}(X)$ has the corresponding property for every object X in $\mathcal{C}$. A cohomology theory H, δ is *invariant under homotopy* if for every f_0, f_1: $X \to Y$ which are homotopic in $\mathcal{C}$ (i.e. there is a continuous map F: $X \times I \to Y$ in $\mathcal{C}$ such that $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$ for all $x \in X$) then $H(f_0) = H(f_1)$: $H(Y) \to H(X)$.

Of primary interest are the categories $\mathcal{C}_{\text{comp}}$ of all compact spaces and continuous functions and $\mathcal{C}_{\text{loc comp}}$ of all locally compact spaces and proper continuous functions.

Proposition 3.2. *Every cohomology theory on $\mathcal{C}_{\text{comp}}$ is invariant under homotopy.*

PROOF. It is shown in [10] that every contravariant functor H on $\mathcal{C}_{\text{comp}}$ whose restriction to $\mathrm{cl}(X)$ is continuous for every compact space X is invariant under homotopy. □

A homomorphism φ: $H, \delta \to H', \delta'$ between two cohomology theories on the same category $\mathcal{C}$ is a natural transformation of degree 0 from H to H' commuting up to sign with δ, δ' for every triad $(X; A, B)$ in $\mathcal{C}$. We have the following extension of the uniqueness theorem.

Theorem 3.3. *Let φ: $H, \delta \to H', \delta'$ be a homomorphism between cohomology theories on $\mathcal{C}_{\text{comp}}$ such that for some one-point space P, φ_P: $H(P) \to H'(P)$ is an n-equivalence for some $n \in \mathbb{Z}$. Then φ_X: $H(X) \to H'(X)$ is an n-equivalence for every compact space X.*

PROOF. Because H, H' are contravariant functors on $\mathcal{C}_{\text{comp}}$ it follows that for every one-point space Q, φ_Q: $H(Q) \to H'(Q)$ is an n-equivalence. Consider a cube $C(J) = \prod_{j \in J} I_j$ and let $y \in C(F)$ for F a finite subset of J. Since the projection map p_F: $p_F^{-1}(y) \to y$ is a homotopy equivalence, there is a commutative square whose vertical maps are isomorphisms by Proposition 3.2

$$\begin{array}{ccc} H(y) & \xrightarrow{\varphi_y} & H'(y) \\ H(p_F)\downarrow \approx & & \approx \downarrow H'(p_F) \\ H(p_F^{-1}(y)) & \xrightarrow{\varphi} & H(p_F^{-1}(y)) \end{array}$$

It follows that φ: $H(p_F{}^{-1}(y)) \to H'(p_F^{-1}(y))$ is an n-equivalence for every finite $F \subset J$ and every $y \in C(F)$. By Lemma 3.1, φ_A is an n-equivalence for every $A \subset C(J)$. Since every compact X is homomorphic to a closed subset of some cube, φ_X: $H(X) \to H'(X)$ is an n-equivalence for every X. □

Corollary 3.4. *Let φ: $H, \delta \to H', \delta'$ be a homomorphism between compactly supported cohomology theories on $\mathcal{C}_{\text{loc comp}}$ such that for some one-point space P, φ_p: $H(P) \to H'(P)$ is an n-equivalence for some $n \in \mathbb{Z}$. Then φ_X: $H(X) \to H'(X)$ is an n-equivalence for every locally compact space X.*

PROOF. Since $\mathcal{C}_{\text{comp}}$ is a subcategory of $\mathcal{C}_{\text{loc comp}}$, we can apply Theorem 3.3 to deduce that φ_X: $H(X) \to H'(X)$ is an n-equivalence for every compact X. The Corollary follows from this and Proposition 2.8 of [18]. □

In the above Corollary we used the fact that $\mathcal{C}_{\text{comp}}$ is a subcategory of $\mathcal{C}_{\text{loc comp}}$. Therefore, every cohomology theory on $\mathcal{C}_{\text{loc comp}}$ defines by restriction a cohomology theory on $\mathcal{C}_{\text{comp}}$. We now present a way of obtaining a compactly supported cohomology theory on $\mathcal{C}_{\text{loc comp}}$ from a cohomology theory on $\mathcal{C}_{\text{comp}}$.

For a subset $A \subset X$ we say A is *cobounded* in X if $\overline{X - A}$ is compact. We need de following lemma.

Lemma 3.5. *Let H be a contravariant functor from* $\text{cl}(X)$ *to graded abelian groups such that $H(\emptyset) = 0$ and such that for every $A \subset X$*

(*) ρ: $\varinjlim \{H(N) | N$ a closed cobounded nbhd of A in $X\} \approx H(A)$.

Then H is continuous and compactly supported.

PROOF. This follows immediately from Proposition 2.6 of [17] applied to the family of supports consisting of all compact subsets of X. □

Given an arbitrary locally compact space X let X^+ be the compact space consisting of X together with exactly one more point ∞ such that $X^+ - \{\infty\} = X$. In case X is compact, X^+ is the topological sum of X and $\{\infty\}$. In case X is non-compact, X^+ is the one-point compactification of X. Note that $A \subset X \Rightarrow A^+ \subset X^+$, $\emptyset^+ = \{\infty\}$, and $(A \cap B)^+ = A^+ \cap B^+$, $(A \cup B)^+ = A^+ \cup B^+$ for A, B closed in X.

Proposition 3.6. *Given a cohomology theory H, δ on $\mathcal{C}_{\text{comp}}$ there is a compactly supported cohomology theory $\tilde{H}, \tilde{\delta}$ on $\mathcal{C}_{\text{loc comp}}$ where $\tilde{H}(X) = \ker[H(X^+) \xrightarrow{\rho} H(\infty)]$ for a locally compact space X.*

PROOF. $\tilde{H}$ is defined by the above. To define $\tilde{\delta}$ note that if A is a closed subset of X and c: $A^+ \to \infty$ is the constant map, then the composite

$$H(\infty) \xrightarrow{H(c)} H(A^+) \xrightarrow{\rho} H(\infty)$$

is the identity. Therefore, in the following commutative diagram with exact rows the vertical maps are epimorphisms

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\beta} & H^{q-1}(A^+ \cap B^+) & \xrightarrow{\delta} & H^q(A^+ \cup B^+) & \xrightarrow{\alpha} & H^q(A^+) \oplus H^q(B^+) & \xrightarrow{\beta} & H^q(A^+ \cap B^+) & \xrightarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \\
\cdots & \xrightarrow{\beta} & H^{q-1}(\infty) & \xrightarrow{0} & H^q(\infty) & \xrightarrow{\alpha} & H^q(\infty) \oplus H^q(\infty) & \xrightarrow{\beta} & H^q(\infty) & \xrightarrow{0} & \cdots
\end{array}$$

It follows that there is an exact sequence of the kernels of ρ

$$\cdots \xrightarrow{\tilde{\beta}} \tilde{H}^{q-1}(A \cap B) \xrightarrow{\tilde{\delta}} \tilde{H}^q(A \cup B) \xrightarrow{\tilde{\alpha}} \tilde{H}^q(A) \oplus \tilde{H}^q(B) \xrightarrow{\tilde{\beta}} \tilde{H}^q(A \cap B) \xrightarrow{\tilde{\delta}} \cdots$$

This defines the natural transformation $\tilde{\delta}$ and shows that $\tilde{H}, \tilde{\delta}$ satisfy *MV* exactness.

Clearly $\tilde{H}(\emptyset) = \ker[H(\infty) \xrightarrow{\rho} H(\infty)] = 0$ and the closed nbhds of A^+ in X^+ are precisely the sets N^+ where N is a closed cobounded nbhd of A in X. Hence, the continuity of H on X^+ implies

$$\rho\colon \varinjlim \{\tilde{H}(N) | N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx \tilde{H}(A)$$

It follows from Lemma 3.5 that $\tilde{H}$ is continuous and compactly supported on X. □

Theorem 3.7. *The map H, δ to $\tilde{H}, \tilde{\delta}$ is an injection from cohomology theories on $\mathcal{C}_{\text{comp}}$ to compactly supported cohomology theories on $\mathcal{C}_{\text{loc comp}}$.*

PROOF. Given H, δ on $\mathcal{C}_{\text{comp}}$ let $\tilde{H}, \tilde{\delta}$ be the compactly supported cohomology theory on $\mathcal{C}_{\text{loc comp}}$ defined by it as in Proposition 3.6. If X is any

compact space, the exactness of $0 = H(\emptyset) \to H(X^+) \xrightarrow{\alpha} H(X) \oplus H(\infty) \to$ $\to H(\emptyset) = 0$ implies that $\tilde{H}(X) = \ker[H(X^+) \xrightarrow{\rho} H(\infty)] \approx H(X)$. This shows that $\tilde{H}$ when restricted to $\mathcal{C}_{\text{comp}}$ is isomorphic to H. Similarly $\tilde{\delta}$ restricted to compact triads is isomorphic to δ. Thus, the restriction of $\tilde{H}, \tilde{\delta}$ to $\mathcal{C}_{\text{comp}}$ is isomorphic to H, δ. □

If H', δ' is a compactly supported cohomology theory on $\mathcal{C}_{\text{loc comp}}$, let H, δ be its restriction to $\mathcal{C}_{\text{comp}}$ and $\tilde{H}, \tilde{\delta}$ the compactly supported cohomology theory on $\mathcal{C}_{\text{loc comp}}$ determined by H, δ as in Proposition 3.6. In general we do not know what relation there is between H', δ' and $\tilde{H}, \tilde{\delta}$ both compactly supported cohomology theories on $\mathcal{C}_{\text{loc comp}}$.

4. ES theories on categories of spaces

We consider ES theories on a category of topological spaces and continuous mappings. Since ES theories define cohomology theories, the uniqueness theorem is valid for ES theories. We also show that there is an equivalence between ES theories and cohomology theories on the category of all compact spaces and continuous functions. Thus, cohomology theories are «single space» equivalents to ES theories. Other single space equivalents to ES theories have been given in [2, 3, 7, 11].

Let $\mathcal{C}$ be a category of topological spaces and continuous functions such that if X is an object of $\mathcal{C}$ then $\text{cl}(X)$ is a subcategory of $\mathcal{C}$. An ES *theory* H, δ^* on $\mathcal{C}$ consists of:

(i) A contravariant functor H from $\mathcal{C}^2$ (the category of closed pairs in $\mathcal{C}$) to the category of graded abelian groups, and
(ii) A natural transformation δ^*: $H^q(B, \emptyset) \to H^{q+1}(A, B)$ for every (A, B) in $\mathcal{C}^2$

such that for every object X in $\mathcal{C}$ the restriction of H, δ^* to $\text{cl}(X)^2$ is an ES theory on X.

Since ES theories are continuous, the result in [10] implies they are invariant under homotopy. Therefore, they are continuous extraordinary cohomology theories because they satisfy all of the Eilenberg-Steenrod axioms [5] except the dimension axiom and are continuous.

As in Section 2 every ES theory on $\mathcal{C}$ determines a cohomology theory on $\mathcal{C}$. The concepts of *nonnegativity, compactly supported*, and *additivity* for ES theories on $\mathcal{C}$ are defined to correspond to the same properties of the associated cohomology theories.

A homomorphism φ: $H, \delta^* \to H', \delta'^*$ between ES theories on $\mathcal{C}$ is a natural transformation of degree 0 from H to H' commuting up to sign with δ^*, δ'^*

for every pair (X, A) in $\mathcal{C}^2$. The following uniqueness theorem for ES theories is valid.

Theorem 4.1. *Let φ: $H, \delta^* \to H', \delta'^*$ be a homomorphism between two compactly supported ES theories on $\mathcal{C}_{\text{loc comp}}$ such that for some one-point space P, φ_p: $H(P, \emptyset) \to H'(P, \emptyset)$ is an n-equivalence for some $n \in \mathbb{Z}$. Then φ: $H(X, A) \to H'(X, A)$ is an n-equivalence for every locally compact pair.*

PROOF. The homomorphism φ determines a homomorphism $\bar{\varphi}$: $\bar{H}, \bar{\delta} \to \bar{H}', \bar{\delta}'$ between the cohomology theories on $\mathcal{C}_{\text{loc comp}}$ defined by the ES theories H, δ^* and H', δ'^* respectively. Since these are compactly supported and $\bar{\varphi}$ is an n-equivalence for the one-point space P, it follows from Corollary 3.4 that $\bar{\varphi}$ is an n-equivalence for every locally compact space X. This is equivalent to the assertion that φ: $H(X, \emptyset) \to H'(X, \emptyset)$ is an n-equivalence for every locally compact space X. Then the «five-lemma» shows that $\varphi: H(X, A) \to$ $\to H'(X, A)$ is an n-equivalence for every locally compact pair. □

The next result asserts the equivalence between cohomology theories and ES theories on $\mathcal{C}_{\text{comp}}$.

Theorem 4.2. *The assigment of a cohomology theory to an ES theory is an equivalence on $\mathcal{C}_{\text{comp}}$.*

PROOF. We have already seen that for an ES theory H, δ^* on $\mathcal{C}_{\text{comp}}$ there is associated a cohomology theory H', δ' on $\mathcal{C}_{\text{comp}}$ with $H'(X) = H(X, \emptyset)$ for every compact space X.

For the converse we consider for every compact space X the cone CX over X with vertex v (so CX is the join of X with a point v not in X). Given a cohomology theory H', δ' on $\mathcal{C}_{\text{comp}}$ define a contravariant functor H on $\mathcal{C}^2_{\text{comp}}$ by

$$H(X, A) = \ker[\rho\colon\ H'(X \cup CA) \to H'(CA)]$$

(in case $A = \emptyset$, $CA = \{v\}$). Then

$$H(X, \emptyset) = \ker[\rho\colon\ H'(X \cup \{v\}) \to H'(v)]$$

and by the exactness of

$$H'(\emptyset) \xrightarrow{\delta'} H'(X \cup \{v\}) \xrightarrow{\alpha} H'(X) \oplus H'(v) \xrightarrow{\beta} H'(\emptyset),$$

there is an isomorphism α': $H(X, \emptyset) \approx H'(X)$. Therefore, continuity of H' implies continuity of H.

To define δ^* and verify exactness for H, δ^* note that

$$\delta' \colon H'(A) \to H'(X \cup CA)$$

for the triad $(CX; X, CA)$ has image lying in

$$\ker [\rho \colon H'(X \cup CA) \to H'(CA)]$$

by exactness of the MV sequence of X, CA. Therefore,

$$\operatorname{im} [\delta' \colon H'(A) \to H'(X \cup CA)] \subset \ker [\rho \colon H'(X \cup CA) \to H'(CA)] = H(X, A).$$

Thus, there is a unique homomorphism $\delta' \colon H(A, \emptyset) \to H(X, A)$ such that there is a commutative square

$$\begin{array}{ccc} H(A, \emptyset) & \xrightarrow[\approx]{\alpha'} & H'(A) \\ {\scriptstyle \delta^*}\downarrow & & \downarrow{\scriptstyle \delta'} \\ H(X, A) & \subset & H(X \cup CA) \end{array}$$

Then δ^* is a natural transformation of degree 1 defined for every (X, A) in $\mathcal{C}^2_{\text{comp}}$. Furthermore, if $c \colon X \cup CA \to v$ is the constant map, the composite

$$H'(v) \xrightarrow{H'(c)} H'(X \cup CA) \xrightarrow{\rho} H'(CA)$$

is an isomorphism (by Proposition 3.2 since $c|CA \colon CA \to v$ is a homotopy equivalence). Therefore, in the commutative diagram with exact rows all vertical maps are epimorphisms

$$\begin{array}{ccccccccccc} \cdots \to & H'(A) & \xrightarrow{\delta'} & H'(X \cup CA) & \to & H'(X) \oplus H'(CA) & \to & H'(A) & \xrightarrow{\delta'} & \cdots \\ & \downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & & & \\ \cdots \to & 0 & \to & H'(CA) & \approx & 0 \oplus H'(CA) & \to & 0 & \to & \cdots \end{array}$$

hence, there is an exact sequence of kernels

$$\cdots \to H'(A) \to H(X, A) \to H'(X) \to H'(A) \to \cdots$$

Replacing $H'(A)$ by $H(A, \emptyset)$ and $H'(X)$ by $H(X, \emptyset)$ we obtain the exact sequence

$$\cdots \to H(A, \emptyset) \xrightarrow{\delta^*} H(X, A) \xrightarrow{H(j)} H(X, \emptyset) \xrightarrow{H(i)} H(A, \emptyset) \xrightarrow{\delta^*} \cdots$$

where $i \colon (A, \emptyset) \subset (X, \emptyset)$ and $j \colon (X, \emptyset) \subset (X, A)$. Therefore, H, δ^* satisfy exactness.

To prove the excision property let $(X; A, B)$ be a triad in $\mathcal{C}_{\text{comp}}$ and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} H'(A)\oplus H'(CB) & \xrightarrow{\beta} & H'(A\cap B) & \xrightarrow{\delta'} & H'(A\cup CB) & \xrightarrow{\alpha} & H'(A)\oplus H'(CB) & \xrightarrow{\beta} & H'(A\cap B) \\ \rho\downarrow\approx & & \rho\downarrow\approx & & \rho\downarrow & & \approx\downarrow\rho & & \approx\downarrow\rho \\ H'(A)\oplus H'(C(A\cap B)) & \xrightarrow{\beta} & H'(A\cap B) & \xrightarrow{\delta'} & H'(A\cap C(A\cap B)) & \xrightarrow{\alpha} & H'(A)\oplus H'(C(A\cap B)) & \xrightarrow{\beta} & H'(A\cap B) \end{array}$$

From the «five lemma» the middle vertical map is an isomorphism and there is a commutative square

$$\begin{array}{ccc} H'(A\cup CB) & \xrightarrow{\rho} & H'(CB) \\ \rho\downarrow\approx & & \rho\downarrow\approx \\ H'(A\cup C(A\cap B)) & \xrightarrow{\rho} & H'(C(A\cap B)) \end{array}$$

Therefore, $H(A\cup B, B) = \ker[\rho\colon H'(A\cup CB)\to H'(CB)]$ is isomorphic to $H(A, A\cap B) = \ker[\rho\colon H'(A\cup C(A\cap B))\to H'(C(A\cap B))]$ by the restriction map.

Thus, H, δ^* is an ES theory on $\mathcal{C}_{\text{comp}}$. It is clear that α' induces an isomorphism of $H(A, \emptyset)$ with $H'(A)$ so the cohomology theory induced on $\mathcal{C}_{\text{comp}}$ by the ES theory H, δ^* is isomorphic to the original cohomology theory H', δ' on $\mathcal{C}_{\text{comp}}$.

Conversely, if H, δ^* is an ES theory on $\mathcal{C}_{\text{comp}}$ and H', δ' is the corresponding cohomology theory on $\mathcal{C}_{\text{comp}}$ let H'', δ'' be the ES theory on $\mathcal{C}_{\text{comp}}$ constructed as above from H', δ'. Then

$$\begin{aligned} H''(X, A) &= \ker[H'(X\cup CA)\longrightarrow H'(CA)] \\ &= \ker[H(X\cup CA, \emptyset)\longrightarrow H(CA, \emptyset)] \end{aligned}$$

From the exact sequence

$$\xrightarrow{0} H(X\cup CA, CA)\longrightarrow H(X\cup CA, \emptyset)\longrightarrow H(CA, \emptyset)\xrightarrow{0}$$

we see that $\ker[H(X\cup CA, \emptyset)\to H(CA, \emptyset)]\approx H(X\cup CA, CA)$. Since there is an excision isomorphism

$$H(X\cup CA, CA)\approx H(X, A)$$

we finally obtain $H''(X, A)\approx H(X, A)$. This isomorphism carries δ'' to δ^* and completes the proof. □

5. Cohomology defined by spectra

In this section we show that a spectrum of ANR defines an ES theory on the category $\mathcal{C}_{\text{loc comp}}$. In particular, K-theory, which is defined by such a spectrum, is an ES theory. The uniqueness theorem of the preceeding section

yields another proof of the fact that the Chern character is an isomorphism of $K \otimes \mathbb{Q}$ with $\check{H}_{ev}$ (rational Cěch cohomology with compact supports) on $\mathcal{C}_{\text{loc comp}}$.

For a pointed space X we let C_0X denote the reduced cone over X (so $C_0X = I \wedge X$ where $0 \in I$ is the base point of I) and $\sum_0 X = S^1 \wedge X$ the reduced suspension of X. If (X, A) is a compact pair with base point $x_0 \in A$ and Y is a pointed space there is an exact sequence of based homotopy classes [15]

$$\cdots \xrightarrow{(\Sigma_0 i)^\#} [\Sigma_0 A; Y] \xrightarrow{\bar{k}^\#} [X \cup C_0 A; Y] \xrightarrow{\bar{i}^\#} [X; Y] \xrightarrow{i^\#} [A; Y]$$

where i: $A \subset X$, $\bar{i}$: $X \subset X \cup C_0 A$ and $\bar{k}$: $X \cup C_0 A \to \sum_0 A$ is the map collapsing X to the base point. In case Y is an ANR (absolute neighborhood retract for normal spaces) the contractibility of $C_0 A$ implies that the quotient map q: $X \cup C_0 A \to X/A$ induces, for every $n \geqq 0$, a bijection $(\sum_0^n q)^\#$: $[\sum_0^n (X/A); Y] \approx [\sum_0^n (X \cup CA); Y]$. Since $q \circ \bar{i} = k$: $X \to X/A$ where k is the quotient map, there is an exact sequence

$$\cdots \xrightarrow{(\Sigma_0 i)^\#} [\Sigma_0 A; Y] \xrightarrow{\delta} [X/A; Y] \xrightarrow{k^\#} [X; Y] \xrightarrow{i^\#} [A; Y]$$

where δ: $[\sum_0^{n+1} A; Y] \to [\sum_0^n (X/A); Y]$ is defined to equal the composite

$$(\sum_0^n q)^{\#-1} (\sum_0^n \bar{k})^\#\text{: } [\sum_0^{n+1} A; Y] \to [\sum_0^n (X \cup C_0 A); Y] \xrightarrow{\approx} [\sum_0^n (X/A); Y]$$

Now suppose $\textcircled{Y} = \{Y_k, \epsilon_k\text{: } \sum_0 Y_k \to Y_{k+1}\}$ is a spectrum of pointed ANR's. Define

$$\{X, A; \textcircled{Y}\}^q = \varinjlim_n \{[\sum_0^n (X/A); Y_{n+q}]\}$$

where the direct limit is with respect to the maps

$$[\sum_0^n (X/A); Y_{n+q}] \xrightarrow{\Sigma_0} [\sum_0^{n+1} (X/A); \sum_0 Y_{n+q}] \xrightarrow{(\epsilon_{n+q})^\#} [\sum_0^{n+1} (X/A); Y_{n+q+1}]$$

Taking the direct limit of the exact sequences above we obtain an exact sequence

$$\cdots \xrightarrow{\delta^*} \{X, A; \textcircled{Y}\}^q \xrightarrow{j^\#} \{X, x_0; \textcircled{Y}\}^q \xrightarrow{i^\#} \{A, x_0; \textcircled{Y}\}^q \xrightarrow{\delta^*} \{X, A; \textcircled{Y}\}^{q+1} \to \cdots$$

where i: $(A, x_0) \subset (X, x_0)$, j: $(X, x_0) \subset (X, A)$.

We consider the category $\mathcal{C}^2_{\text{loc comp}}$. As in Section 3 to every locally compact space X there is asociated a compact space X^+ with base point ∞ such that $X^+ - \{\infty\} = X$. Define a contravariant functor $H_{\textcircled{Y}}$ on $\mathcal{C}^2_{\text{loc comp}}$ by $H^q_{\textcircled{Y}}(X, A) = \{X^+, A^+; \textcircled{Y}\}^q$ and define a natural transformation δ^*: $H^q_{\textcircled{Y}}(A, \varnothing) \to H^{q+1}_{\textcircled{Y}}(X, A)$ to be the homomorphism δ^* in the exact sequence above for the pair of pointed spaces (X^+, A^+).

Theorem 5.1 *For every spectrum* $\textcircled{Y}$ *of ANR there is a compactly supported ES theory* $H_{\textcircled{Y}}, \delta^*$ *on* $\mathcal{C}_{\text{loc comp}}$.

PROOF. From the exact sequence above for the pair (X^+, A^+) it is clear that $H_{\textcircled{Y}}, \delta^*$ satisfies the exactness property for the pair (X, A). It also satisfies excision because if $(X; A, B)$ is a locally compact triad then $(X^+; A^+, B^+)$ is a compact pointed triad with $A^+ \cup B^+ = (A \cup B)^+$ and $A^+ \cap B^+ = (A \cap B)^+$. Therefore, there are isomorphisms

$$H^q_{\textcircled{Y}}(A \cup B, B) = \{(A \cup B)^+, B^+; \textcircled{Y}\}^q = \{A^+ \cup B^+, B^+; \textcircled{Y}\}^q$$
$$\approx \{A^+, A^+ \cap B^+; \textcircled{Y}\}^q = \{A^+, (A \cap B)^+; \textcircled{Y}\}^q = H^q_{\textcircled{Y}}(A, A \cap B).$$

If (X, A) is a locally compact pair, then (X^+, A^+) and $(\sum_0^n X^+, \sum_0^n A^+)$ are compact pairs for every n. Since Y_{n+q} is an ANR it follows that

$$\rho\colon \varinjlim \{[\textstyle\sum_0^n B; Y_{n+q}] \mid B \text{ a closed nbhd of } A^+ \text{ in } X^+\} \approx [\textstyle\sum_0^n A^+; Y_{n+q}]$$

Taking direct limits with respect to n we obtain an isomorphism

$$\rho\colon \varinjlim \{(B; Y)^q | B \text{ a closed nbhd of } A^+ \text{ in } X^+\} \approx \{A^+; Y\}^q$$

Since the closed nbhds B of A^+ in X^+ are exactly the sets of the form $B = N^+$ where N is a closed cobounded nbhd of A in X, it follows that

$$\rho\colon \varinjlim \{H^q_{\textcircled{Y}}(N) | N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx H^q_{\textcircled{Y}}(A).$$

Since $H_{\textcircled{Y}}(\emptyset) = 0$ it follows from Lemma 3.5 that $H_{\textcircled{Y}}$ is continuous and compactly supported.

There is a spectrum of ANR that defines an ES theory known as K-theory [9]. This ES theory is known to be periodic of order 2 (i.e. $K^{q+2}(X, A) \approx \approx K^q(X, A)$) and for a one-point space P,

$$K^q(P) \approx \begin{cases} \mathbb{Z}, & q \text{ even} \\ 0, & q \text{ odd.} \end{cases}$$

Let $\check{H}_{ev}$ be the FS theory defined in terms of rational Čech cohomology with compact supports by

$$\check{H}^q_{ev}(X, A) = \begin{cases} \bigoplus\limits_{i \text{ even}} \check{H}^i_c(X, A; \mathbb{Q}), & q \text{ even} \\ \bigoplus\limits_{i \text{ odd}} \check{H}^i_c(X, A; \mathbb{Q}), & q \text{ odd.} \end{cases}$$

This is an ES Theory on $\mathcal{C}_{\text{loc comp}}$ (by analogues of Remarks 2.3 and 2.4 of [18] for ES theories). By an analogue of Remark 2.5 of [18] for ES theories there is also an ES theory $K \otimes \mathbb{Q}$ on $\mathcal{C}_{\text{loc comp}}$.

The Chern character [9] Ch: $K \otimes \mathbb{Q} \to \check{H}_{ev}$ is a homomorphism of ES theories on $\mathcal{C}_{\text{loc comp}}$.

Theorem 5.2. *For every locally compact space X,* Ch: $K^q(X) \otimes \mathbb{Q} \approx \check{H}^q_{ev}(X)$.

PROOF. Since $K \otimes \mathbb{Q}$ and $\check{H}_{ev}$ are both compactly supported ES theories on $\mathcal{C}_{\text{loc comp}}$ and Ch is an isomorphism for a one-point space [9], the result follows from Theorem 4.1. □

If $\textcircled{Y} = \{Y_k, \epsilon_k\}$ is a spectrum its *homotopy groups* $\pi_q(\textcircled{Y})$ are defined by $\pi_q(\textcircled{Y}) = \varinjlim_n \{\pi_{n+q}(Y_n)\}$. If $\textcircled{Y}$ is an ANR spectrum, it is clear that if P is a one-point space, then $H^q_{\textcircled{Y}}(P, \emptyset) \approx \pi_{-q}(\textcircled{Y})$ for all q.

If $\textcircled{Y}$, $\textcircled{Y}'$ are spectra, a *map* g: $\textcircled{Y} \to \textcircled{Y}'$ between them is defined to be a sequence of pointed continuous functions g_k: $Y_k \to Y'_k$ for each k such that the following square is homotopy commutative for every k

$$\begin{array}{ccc} \Sigma_0 Y_k & \xrightarrow{\epsilon_k} & Y_{k+1} \\ {\scriptstyle \Sigma_0 g_k}\downarrow & & \downarrow{\scriptstyle g_{k+1}} \\ \Sigma_0 Y'_k & \xrightarrow{\epsilon'_k} & Y'_{k+1} \end{array}$$

Such a map g induces a homomorphism $g_{\#}$: $\pi_q(\textcircled{Y}) \to \pi_q(\textcircled{Y}')$ and, in case $\textcircled{Y}$, $\textcircled{Y}'$ are ANR spectra, it induces a homomorphism g_*: $H_{\textcircled{Y}} \to H_{\textcircled{Y}'}$ of the corresponding ES theories.

Theorem 5.3. *Let* g: $\textcircled{Y} \to \textcircled{Y}'$ *be a map between ANR spectra which induces an isomorphism* $g_{\#}$: $\pi_q(\textcircled{Y}) \approx \pi_q(\textcircled{Y}')$ *for all* q. *Then for every locally compact pair* (X, A),

$$g_*\colon H_{\textcircled{Y}}(X, A) \approx H_{\textcircled{Y}'}(X, A).$$

PROOF. Since $g_{\#}$ is an isomorphism, it follows that if P is a one-point space, then g_*: $H_{\textcircled{Y}}(P, \emptyset) \approx H_{\textcircled{Y}'}(P, \emptyset)$. Since $H_{\textcircled{Y}}, H_{\textcircled{Y}'}$ are compactly supported ES theories on $\mathcal{C}_{\text{loc comp}}$, the result follows from Theorem 4.1. □

6. The functional spectrum and the complementary spectrum of a pair in $\mathbb{R}^n$

First we introduce the functional spectrum $\mathbb{F}(A, B)$ whose k^{th} term consists of the space of continuous functions $(A^+, B^+) \to (S^k, \infty)$ topologized with the compact-open topology. The homotopy groups $\mathbb{F}^*$ of $\mathbb{F}(A, B)$ are part of an ES theory $\mathbb{F}^*, \delta^*$ on $\mathcal{C}_{\text{loc comp}}$.

We regard S^k as $(\mathbb{R}^k)^+ = \mathbb{R}^k \cup \{\infty\}$ with ∞ as base point. Given a locally compact pair (A, B) let $F(A, B; S^k)$ be the space of all continuous functions $(A^+, B^+) \to (S^k, \infty)$ in the compact-open topology with the constant map $A^+ \to \infty$ as base point. Clearly $F(A, B; S^k)$ can also be regarded as the space of pointed continuous functions $A^+/B^+ \to S^k$ in the compact-open topology. It follows from the exponential theorem [4], the compactness of A^+ and the fact that S^k is an ANR that $F(A, B; S^k)$ is an ANR (the proof of Theorem 4 on p. 38 of [14] applies to our case as well).

It is known [6] that if (A, B) is a locally compact pair the restriction map $F(A, \emptyset, S^k) \to F(B, \emptyset; S^k)$ is a fibration with fiber $F(A, B; S^k)$. Therefore, there is an exact sequence of homotopy groups

$$(*) \cdots \to \pi_{q+k}(F(A, B; S^k)) \to \pi_{q+k}(F(A; \emptyset; S^k)) \to \pi_{q+k}(F(B, \emptyset; S^k) \to \\ \to \pi_{q+k-1}(F(A, B; S^{k-1})) \to \cdots$$

Define ϵ_k: $\sum_0 F(A, B; S^k) \to F(A, B; S^{k+1})$ by $(\epsilon_k(t \wedge f))(a) = t \wedge f(a)$ for $t \wedge f \in \sum_0 F(A, B; S^k) = S^1 \wedge F(A, B; S^k)$, $a \in A$ and $\sum_0 S^k = S^1 \wedge S^k \approx S^{k+1}$. Let $\mathbb{F}(A, B)$ be the spectrum $\{F(A, B; S^k), \epsilon_k\}$. The direct limit over k of the exact sequence (*) using the maps ϵ_k is an exact sequence

$$\cdots \to \pi_q(\mathbb{F}(A, B)) \to \pi_q(\mathbb{F}(A, \emptyset)) \to \pi_q(\mathbb{F}(B, \emptyset)) \to \pi_{q-1}(\mathbb{F}(A, B)) \to \cdot$$

extending indefinitely on both ends.

If we define a contravariant functor $\mathbb{F}^*$ on $\mathcal{C}^2_{\text{loc comp}}$ by $\mathbb{F}^q(A, B) = \\ = \pi_{-q}(\mathbb{F}(A, B))$ and a natural transformation δ^*: $\mathbb{F}^q(B, \emptyset) \to \mathbb{F}^{q+1}(A, B)$ to correspond to ∂ in the exact sequence above, we see that $\mathbb{F}^*, \delta^*$ satisfy the exactness property of ES theories. We shall prove that $\mathbb{F}^*, \delta^*$ is an ES theory on $\mathcal{C}_{\text{loc comp}}$, but first we establish the following.

Lemma 6.1. *Let (K', K) be a compact pair, N a closed cobounded nbhd of A in X, and λ: $K \to F(N, \emptyset; S^k)$, μ: $K' \to F(A, \emptyset; S^k)$ continuous functions such that $\mu|K$ is the composite $K \xrightarrow{\lambda} F(N, \emptyset; S^k) \xrightarrow{\rho} F(A, \emptyset; S^k)$. Then there is a closed cobounded nbhd N' of A in N and a map λ': $K' \to F(N', \emptyset; S^k)$ such that $\lambda'|K$ is the composite $K \xrightarrow{\lambda} F(N', \emptyset; S^k) \xrightarrow{\rho'} F(N, \emptyset; S^k)$ and μ is the composite $K' \xrightarrow{\lambda'} F(N', \emptyset; S^k) \xrightarrow{\rho''} F(A, \emptyset; S^k)$.*

PROOF. By the exponential theorem the functions λ and μ correspond to continuous functions $\bar{\lambda}$: $(K \times N^+, K \times \infty) \to (S^k, \infty)$ and $\bar{\mu}$: $(K' \times A^+, K' \times \infty) \to \\ \to (S^k, \infty)$ such that $\bar{\lambda}|K \times A^+ = \bar{\mu}|K \times A^+$. Therefore, there is a continuous function

$$\bar{f}\text{: } (K \times N^+ \cup K' \times A^+, K' \times \infty) \to (S^k, \infty)$$

such that $\bar{f}|K \times N^+ = \bar{\lambda}$ and $\bar{f}|K' \times A^+ = \bar{\mu}$. Since $K' \times N^+$ is compact, $K \times N^+ \cup K' \times A^+$ is closed in $K' \times N^+$ and S^k is an ANR there is a mbhd U of $K \times N^+ \cup K' \times A^+$ in $K' \times N^+$ and an extension $\bar{f}'$: $U \to S^k$ of $\bar{f}$. U contains a subset of the form $K' \times N'^+$ where N'^+ is a closed cobounded nbhd of A in N and $\bar{f}'|K' \times N'^+$ corresponds by the exponential theorem to a map λ': $K' \to F(N', \emptyset; S^k)$ having all the requisite properties. □

Theorem 6.2. *The pair $\mathbb{F}^*, \delta^*$ is a compactly supported ES theory on $\mathcal{C}_{\text{loc comp}}$.*

PROOF. We have already seen that exactness is satisfied. The excision property is also satisfied because if $(X; A, B)$ is a locally compact triad then there are homeomorphisms for each k

$$F(A \cup B, B; S^k) \approx F((A^+ \cup B^+)/B^+; S^k) \approx$$
$$\approx F(A^+/(A^+ \cap B^+); S^k) \approx F(A, A \cap B; S^k)$$

so the map of spectra $\mathbb{F}(A \cup B, B) \to \mathbb{F}(A, A \cap B)$ induces an isomorphism of homotopy groups. Therefore, $\mathbb{F}^q(A \cup B, B) = \pi_{-q}(\mathbb{F}(A \cup B, B)) \approx$ $\approx \pi_{-q}(\mathbb{F}(A, A \cap B)) \approx \mathbb{F}^q(A, A \cap B)$ for all q. To complete the proof it suffices to show that for every locally compact space X the restriction of the functor $\mathbb{F}^*(\cdot, \emptyset)$ to $\mathrm{cl}(X)$ satisfies Lemma 3.5. Clearly $\mathbb{F}^*(\emptyset, \emptyset) = 0$ so we need only verify that the following homomorphism is an isomorphism

$$\rho\colon \varinjlim \{\mathbb{F}^*(N, \emptyset) | N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx \mathbb{F}^*(A, \emptyset)$$

To show ρ is an epimorphism let μ: $S^{k+q} \to F(A, \emptyset; S^k)$ represent an element $\{[u]\} \in \pi_q(\mathbb{F}(A, \emptyset)) = \mathbb{F}^{-q}(A, \emptyset)$ where $A \subset X$. By Lemma 6.1 with $K' = S^{k+q}$, $K = \infty$, $N = X$, λ: $\infty \to F(X, \emptyset; S^m)$ the unique pointed map, μ: $S^{k+q} \to F(A, \emptyset; S^k)$ we obtain a closed cobounded nbhd N' of A in X and a map λ': $S^{k+q} \to F(N', \emptyset; S^k)$ representing an element $\{[\lambda]\} \in \pi_q(\mathbb{F}(N', \emptyset))$ whose restriction to $\mathbb{F}(A, \emptyset)$ equals $\{[\mu]\}$. This implies ρ is an epimorphism.

To show ρ is a monomorphism let λ: $S^{k'+q} \to F(N, \emptyset; S^{k'})$ represent an element $\{[\lambda]\} \in \pi_q(\mathbb{F}(N, \emptyset))$ whose restriction to $\mathbb{F}(A, \emptyset)$ is 0 (where N is a closed cobounded nbhd of A in X). Then there is a map $\mu: C_0 S^{k+q} \to$ $\to F(A, \emptyset; S^k)$ for some $k \geqslant k'$ such that $\mu|S^{k+q} = \rho \circ (\sum_0^{k-k'} \lambda)$. By Lemma 6.1 with $K' = C_0 S^{k+q}, K = \sum_0^{k-k'} S^{k'+q} = S^{k+q}$ and the maps $\sum_0^{k-k'} \lambda$: $S^{k+q} \to F(N, \emptyset; S^k)$, μ: $C_0 S^{k+q} \to F(A, \emptyset; S^k)$ there is a closed cobounded nbhd N' of A in N and a map λ': $C_0 S^{k+q} \to F(N', \emptyset; S^k)$ such that $\lambda'|S^{k+q} =$ $= \rho' \circ \sum_0^{k-k'} \lambda$. Therefore, $\{[\lambda]\}$ maps to 0 in $\mathbb{F}(N', \emptyset)$ proving ρ is a monomorphism. □

Next, for (A, B) a closed pair in $\mathbb{R}^n$ we define a spectrum $\mathbb{C}(A, B)$ which stably aproximates $(\mathbb{R}^n - B, \mathbb{R}^n - A)$. The homotopy groups $\mathbb{C}^*$ of this spectrum are part of an ET theory $\mathbb{C}^*, \delta^*$ on $\mathrm{cl}(\mathbb{R}^n)$.

Let T be an arbitrary but fixed triangulation of S^n with ∞ as vertex and let $T^{(k)}$ be the k^{th} barycentric subdivision of T for $k \geqslant 0$. For a closed subset $A \subset \mathbb{R}^n = S^n - \{\infty\}$ let $T_k(A)$ be the compact polyhedron equal to the union of all closed simplexes of $T^{(k)}$ disjoint from $A^+ = A \cup \{\infty\}$. Clearly $T_k(A) \subset \subset T_{k+1}(A)$ for all $k \geqslant 0$. We also have:

Lemma 6.3. (1) $A \subset B \subset \mathbb{R}^n \Rightarrow T_k(B) \subset T_k(A)$ *for all* $k \geqslant 0$.

(2) $A, B \subset \mathbb{R}^n \Rightarrow T_k(A \cup B) = T_k(A) \cap T_k(B)$ *for all* $k \geqslant 0$.

(3) $A, B \subset \mathbb{R}^n \Rightarrow$ *for every* $k \geqslant 0$ *there is* N_k *such that if* $k' \geqslant N_k$ *then* $T_k(A \cap B) \subset T_{k'}(A) \cup T_{k'}(B)$.

(4) $A \subset N \subset \mathbb{R}^n$ *where* N *is a closed cobounded nbhd of* A *in* $\mathbb{R}^n \Rightarrow$ *for every* $k \geqslant 0$ *there is a closed cobounded nbhd* N' *of* A *in* $\mathbb{R}^n$ *such that* $N' \subset N$ *and* $T_k(N') = T_k(N)$.

PROOF. (1) If $A \subset B$ and s is a closed simplex of $T^{(k)}$ disjoint from B, s is disjoint from A. Hence, $T_k(B) \subset T_k(A)$. □

(2) By (1) $T_k(A \cup B) \subset T_k(A)$ and $T_k(A \cup B) \subset T_k(B)$ so $T_k(A \cup B) \subset \subset T_k(A) \cap T_k(B)$. Given x let s be the unique closed simplex of $T^{(k)}$ containing x in its interior. Then $x \in T_k(A) \cap T_k(B)$ if and only if $s \subset T_k(A) \cap T_k(B)$, but this implies s is disjoint from A and from B so is disjoint from $A \cup B$. Therefore, $T_k(A) \cap T_k(B) \subset T_k(A \cup B)$. □

(3) $T_k(A \cap B)$ is the union of a finite number of closed simplexes, say $T_k(A \cap B) = s_1 \cup \cdots \cup s_r$. For each j, s_j is disjoint from $A \cap B$ so $s_j \cap A$ and $s_j \cap B$ are disjoint compact sets. Let $d_j > 0$ be the distance between them (in some metric on S^n) and let $d = \min\{d_1, \ldots, d_r\}$, choose N_k so that $k' \geqslant N_k$ implies that the diameter of every closed simplex of $T^{(k')}$ in less than d. If s' is any closed simplex of $T^{(k')}$ contained in s_j for some $1 \leqslant j \leqslant r$, then $\operatorname{diam} s' < \\ < d \leqslant d_j$ so s' cannot meet both A and B. Therefore, either $s' \in T_{k'}(A)$ or $s' \in T_{k'}(B)$. Hence, for $k' \geqslant N_{k'} T_k(A \cap B) \subset T_{k'}(A) \cup T_{k'}(B)$. □

(4) For $A \subset \mathbb{R}^n$, $S^n - T_k(A)$ is an open nbhd of A^+. Let M be a closed nbhd of A^+ contained in $S^n - T_k(A)$. Then $M - \{\infty\}$ is a closed cobounded nbhd of A in $\mathbb{R}^n$. Since $T_k(A) \subset S^n - M = S^n - (M - \{\infty\})^+$, it follows that $T_k(A) \subset T_k(M - \{\infty\})$. If N is any closed cobounded nbhd of A in $\mathbb{R}^n$, then $N' = N \cap (M - \{\infty\})$ is a closed cobounded nbhd of A in $\mathbb{R}^n$ such that $N' \subset N$ and since $N' \subset M - \{\infty\}$, it follows from (1) that

$$T_k(A) \subset T_k(M - \{\infty\}) \subset T_k(N') \subset T_k(A)$$

so that $T_k(N') = T_k(A)$. □

Given a closed pair (A, B) in $\mathbb{R}^n$ we define a spectrum $\mathbb{C}(A, B)$ whose k^{th} term is $\sum_0^k (T_k(B)^+ / T_k(A)^+)$ with continuous maps

$$\sum_0 (\sum_0^k (T_k(B)^+ / T_k(A)^+)) = \sum_0^{k+1} (T_k(B)^+ / T_k(A)^+) \to \sum_0^{k+1} (T_{k+1}(B)^+ / T_{k+1}(A)^+)$$

where the last map is the $(k+1)st$ reduced suspension of the map

$$(T_k(B)^+ / T_k(A)^+) \subset (T_{k+1}(B)^+ / T_{k+1}(A)^+)$$

induced by the inclusion $(T_k(B)^+, T_k(A)^+) \subset (T_{k+1}(B)^+, T_{k+1}(A)^+)$.

We define a contravariant functor $\mathbb{C}^*$ on $\mathrm{cl}(\mathbb{R}^n)^2$ by

$$\mathbb{C}^q(A, B) = \pi_{-q}(\mathbb{C}(A, B)) = \varinjlim_k \{\pi_{-q-k}(\sum_0^k (T_k(B)^+ / T_k(A)^+))\}.$$

To define the natural transformation δ^*: $\mathbb{C}^q(B, \phi) \to \mathbb{C}^{q+1}(A, B)$ recall [15, Corollary 9.3.6 on p. 487] that the collapsing map

$$(\sum_0^k T_k(B)^+, \sum_0^k T_k(A)^+) \to (\sum_0^k (T_k(B)^+ / T_k(A)^+), \infty)$$

induces isomorphisms

$$\pi_i(\sum_0^k T_k(B)^+, \sum_0^k T_k(A)^+) \approx \pi_i(\sum_0^k (T_k(B)^+ / T_k(A)^+))$$

for $i \leqslant 2k-2$ (because the k^{th} reduced suspension of a space, or pair, is $(k-1)$-connected). Hence, the connecting homomorphisms

$$\partial\colon\ \pi_{-q+k}(\sum_0^k T_k(\emptyset)^+, \sum_0^k T_k(B)^+) \to \pi_{-q+k-1}(\sum_0^k T_k(B)^+, \sum_0^k T_k(A)^+)$$

for various k correspond to a homomorphism

$$\delta^*\colon\ \pi_{-q}(\mathbb{C}(B, \emptyset)) \to \pi_{-q-1}(\mathbb{C}(A, B))$$

This is a natural transformation of degree 1 from $\mathbb{C}^*(B, \emptyset)$ to $\mathbb{C}^*(A, B)$ for $(A, B) \in \mathrm{cl}(\mathbb{R}^n)^2$ such that $\mathbb{C}^*, \delta^*$ satisfy the exactness property of ES theories.

Theorem 6.4. *$\mathbb{C}^*, \delta^*$ is a compactly supported ES theory on $\mathbb{R}^n$.*

PROOF. We have seen above that $\mathbb{C}^*, \delta^*$ satisfy exactness. To verify excision assume A, B are closed subsets of $\mathbb{R}^n$. By (2) of Lemma 6.3

$$\varinjlim_k \{\pi_{k-q}(\sum_0^k (T_k(B)^+ / (T_k(A)^+ \cap T_k(B^+))))\} \approx \varinjlim_k \{\pi_{k-q}(\sum_0^k (T_k(B)^+ / T_k(A \cup B^+)))\} = \mathbb{C}^q(A \cup B, B).$$

From (3) of Lemma (6.3) for given k if $k' \geqslant N_k$ there are homomorphisms induced by inclusion

$$\pi_{k-q}(\Sigma_0^k((T_k(A)^+ \cup T_k(B)^+)/T_k(A)^+)) \to \pi_{k-q}(\Sigma_0^k(T_k(A\cap B)^+/T_k(A)^+)) \to \pi_{k-q}(\Sigma_0^k((T_{k'}(A)^+ \cup T_{k'}(B)^+)/T_{k'}(A)^+))$$

implying that

$$\varinjlim_k \{\pi_{k-q}(\Sigma_0^k((T_k(A)^+ \cup T_k(B^+)/T_k(A)^+))\} \approx \varinjlim_k \{\pi_{k-q}(\Sigma_0^k(T_k(A\cap B)^+/T_k(A)^+))\} = \mathbb{C}^q(A, A\cap B).$$

Since

$$T_k(B)^+/(T_k(A)^+ \cap T_k(B)^+) = (T_k(A)^+ \cup T_k(B^+))/T_k(A)^+,$$

it follows that

$$\varinjlim_k \{\pi_{k-q}(\Sigma_0^k(T_k(B)^+/(T_k(A)^+ \cap T_k(B)^+)))\} \approx \varinjlim_k \{\pi_{k-q}(\Sigma_0^k(T_k(A)^+ \cup T_k(B)^+)/T_k(A)^+))\}$$

so that $\mathbb{C}^q(A\cup B, B) \approx \mathbb{C}^q(A, A\cap B)$ and excision is satisfied.

To complete the proof we show that the functor $\mathbb{C}^*(\bullet, \emptyset)$ satisfies Lemma 3.5. Clearly $\mathbb{C}^*(\emptyset, \emptyset) = 0$. Hence, we only need verify that

$$\rho\colon \varinjlim \{\mathbb{C}^*(N, \emptyset) | N \text{ a closed cobounded nbhd of } A \text{ in } \mathbb{R}^n\} \approx \mathbb{C}^*(A, \emptyset)$$

To show ρ is an epimorphism let

$$\mu\colon S^{k-q} \to \Sigma_0^k(T_k(\emptyset)^+/T_k(A)^+)$$

represent an element $\omega \in \pi_{-q}(\mathbb{C}(A, \emptyset) = \mathbb{C}^q(A, \emptyset)$ for some $A \subset \mathbb{R}^n$. By (4) of Lemma 6.3 with $N = \mathbb{R}^n$ there is a closed cobounded nbhd N' of A in $\mathbb{R}^n$ such that $T_k(N') = T_k(A)$. Then μ is also a map from S^{k-q} into $\Sigma_0^k(T_k(\emptyset)^+/T_k(N')^+)$ so determines an element $\omega' \in \pi_{-q}(\mathbb{C}(N', \emptyset))$ whose restriction to $\mathbb{C}(A, \emptyset)$ equals ω. Thus, ρ is an epimorphism.

To show ρ is a monomorphism let $\lambda\colon S^{k-q} \to \Sigma_0^k(T_k(\emptyset)^+/T_k(N)^+)$ represent an element $\omega \in \pi_{-q}(\mathbb{C}(N, \emptyset))$ whose restriction to $\mathbb{C}(A, \emptyset) = 0$ (where N is a closed cobounded nbhd of A in $\mathbb{R}^n$). Then there is a map $\mu\colon C_0S^{k'-q} \to \Sigma_0^{k'}(T_{k'}(\emptyset)^+/T_{k'}(A)^+)$ for some $k' \geqslant k$ such that $\mu|S^{k'-q} = \rho' \circ \Sigma_0^{k'-k}(\lambda)$ where $\rho'\colon \Sigma_0^{k'}(T_k(\emptyset)^+/T_k(N)^+) \to \Sigma_0^{k'}(T_{k'}(\emptyset)^+/T_{k'}(A)^+)$. By (4) of Lemma 6.3 there is a closed cobounded nbhd N' of A in $\mathbb{R}^n$ such that $N' \subset N$ and $T_{k'}(N') = T_{k'}(A)$. Then μ is also a map from $C_0S^{k'-q}$ to $\Sigma_0^{k'}(T_{k'}(\emptyset)^+/T_{k'}(N')^+)$ implying that ω restricts to 0 in $\mathbb{C}(N', \emptyset)$. Thus, ρ is a monomorphism. □

7. Duality in $\mathbb{R}^n$

In the last section $\mathbb{C}^*, \delta^*$ was defined as a compactly supported ES theory on $\mathbb{R}^n$. By Theorem 6.2 the restriction of $\mathbb{F}^*, \delta^*$ to $\mathbb{R}^n$ is also a compactly supported ES theory on $\mathbb{R}^n$. This implies that $\sigma^n\mathbb{F}^*, \delta^*$ is also an ES theory on $\mathbb{R}^n$ (where σ^n is defined as in Remark 2.3 of [18] so that

$$(\sigma^n\mathbb{F}^*)^q(A, B) = \mathbb{F}^{q+n}(A, B)).$$

Since $\mathbb{F}^*(A, B)$ is the ES theory defined by the spectrum $\mathbb{F}(A, B)$, $\sigma^n\mathbb{F}^*(A, B)$ is the ES theory defined by the spectrum $\sum_0^n \mathbb{F}^*(A, B)$ whose m^{th} term is the $(n+m)^{\text{th}}$ term of $\mathbb{F}(A, B)$. We shall define a map of spectra

$$\mu\colon\ \mathbb{C}(A, B) \longrightarrow \sum_0^n \mathbb{F}(A, B)$$

for $(A, B) \in \mathrm{cl}(\mathbb{R}^n)^2$. This will induce a homomorphism

$$\begin{aligned}\mathbb{C}^q(A, B) = \pi_{-q}(\mathbb{C}(A, B)) &\longrightarrow \pi_{-q}\left(\sum_0^n \mathbb{F}(A, B)\right) \approx \pi_{-n-q}(\mathbb{F}(A, B)) \\ &= \mathbb{F}^{q+n}(A, B) = (\sigma^n\mathbb{F}^*)^q(A, B)\end{aligned}$$

and μ_* will be a homomorphism from $\mathbb{C}^*, \delta^*$ to $\sigma^n\mathbb{F}^*, \delta^*$.

Given a closed pair (A, B) in $\mathbb{R}^n$ for every $k \geqslant 0$ there is a continuous map

$$\lambda_k\colon\ \mathbb{R}^k \times (T_k(B), T_k(A)) \times (A, B) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n = \mathbb{R}^{n+k}, \mathbb{R}^{n+k} - \{0\})$$

defined by $\lambda_k(x, y, z) = (x, y - z)$ for $x \in \mathbb{R}^k$, $y \in T_k(B)$, $z \in A$. Since $T_k(B)$ is a compact subset of $\mathbb{R}^n$, λ_k is a proper map so extends to a continuous map

$$\lambda_k^+\colon\ [\mathbb{R}^k \times (T_k(B), T_k(A)) \times (A, B)]^+ \longrightarrow (\mathbb{R}^{n+k}, \mathbb{R}^{n+k} - \{0\})^+$$

There is a canonical homeomorphism

$$[\mathbb{R}^k \times (T_k(B), T_k(A)) \times (A, B)]^+ \approx S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+)$$

Therefore, λ_k^+ can also be regarded as a map

$$\lambda_k^+\colon\ S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+) \longrightarrow (S^{n+k}, S^{n+k} - \{0\}).$$

This map λ_k^+ has the following two properties.

(1) If $(A', B') \subset (A, B)$, then $(T_k(B), T_k(A)) \subset (T_k(B'), T_k(A'))$ and $\lambda_k'^+ | [S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A'^+, B'^+)] = \lambda_k^+ | [S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A'^+, B'^+)]$.

(2) For any $k \geqslant 0$, $(T_k(B), T_k(A)) \subset (T_{k+1}(B), T_{k+1}(A))$ and $\lambda_{k+1}^+ | [S^{k+1} \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+)] = \sum_0 (\lambda_k^+)$.

Since ∞ is a deformation retract of $S^{n+k} - \{0\}$ if i: $(S_{n+k}, \infty) \subset \subset (S^{n+k}, S^{n+k} - \{0\})$ is the inclusion map, there is a map

$$\bar{\lambda}_k^+\colon\ S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+) \to (S^{n+k}, \infty)$$

such that $\lambda_k^+ \simeq i \circ \bar{\lambda}_k^+$ and this condition determines $\bar{\lambda}_k^+$ up to homotopy. We may also consider $\bar{\lambda}_k^+$ as a pointed map $\bar{\lambda}_k^+\colon\ S^k \wedge (T_k(B)^+/T_k(A)^+) \wedge (A^+/B^+) \to S^{n+k}$. By the exponential theorem [4] $\bar{\lambda}_k^+$ corresponds to a continuous function

$$\mu_k\colon\ \textstyle\sum_0^k (T_k(B)^+/T_k(A)^+) \to F(A, B; S^{n+k})$$

By (2) above the following square commutes up to homotopy

$$\begin{array}{ccc} \Sigma_0 \Sigma_0^k (T_k(B)^+/T_k(A)^+) & \longrightarrow & \Sigma_0^{k+1} (T_{k+1}(B)^+, T_{k+1}(A)^+) \\ \downarrow{\scriptstyle \Sigma_0 \mu_k} & & \downarrow{\scriptstyle \mu_{k+1}} \\ \Sigma_0 F(A, B; S^{n+k}) & \xrightarrow{\epsilon_{n+k}} & F(A, B; S^{n+k+1}) \end{array}$$

Therefore, $\{\mu_k\}$ is a map of spectra

$$\mu\colon\ \mathbb{C}(A, B) \to \textstyle\sum_0^n \mathbb{F}(A, B)$$

and it follows from (1) above that the map

$$\mu_*\colon\ \mathbb{C}^q(A, B) \to (\sigma^n \mathbb{F}^*)^q(A, B)$$

is a natural transformation from $\mathbb{C}^*$ to $\sigma^n \mathbb{F}^*$. It is easy to verify that μ_* commutes with δ^* for the two ES theories so it is a homomorphism of ES theories.

Theorem 7.1. *The homomorphism* $\mu_*\colon\ \mathbb{C}^*, \delta^* \to \sigma^n \mathbb{F}^*, \delta^*$ *is an isomorphism of ES theories on* $\mathbb{R}^n$.

PROOF. Using the «five-lemma» it suffices to prove that μ_* is an isomorphism of the corresponding cohomology theories. Since each is compactly supported and $\mathbb{R}^n$ is finite dimensional, it suffices to prove μ_* is an isomorphism for $(x, \emptyset)$ for every $x \in \mathbb{R}^n$. But $F(x, \emptyset; S^k) \approx S^k$ so $\sum_0^n \mathbb{F}(x, \emptyset)$ is the spectrum $S^n, S^{n+1}, \ldots$. Also the pair $(T_k(\emptyset), T_k(x))$ for k large is an n-cell together with the complement of an open n-cell inside it and the map $\mu_k\colon\ \sum_0^k (T_k(\emptyset)^+/T_k(x)^+) \to S^{n+k}$ is of degree 1. Therefore,

$$\mu_*\colon\ \pi_{-q}(\mathbb{C}(x, \emptyset)) \approx \pi_{-q}(\textstyle\sum_0^n \mathbb{F}(x, \emptyset)) \quad \text{for all} \quad q. \qquad \square$$

Corollary 7.2. *If (A, B), (C, D), are closed pairs in $\mathbb{R}^n$ there is a duality isomorphism*

$$\{A^+/B^+; \mathbb{C}(C, D)\}^q \approx \{C^+/D^+; \mathbb{C}(A, B)\}^q \quad \textit{for all} \quad q.$$

PROOF. If $(C, D) \in \mathrm{cl}(\mathbb{R}^n)^2$ then by Theorem 7.1 there is a map $\mu\colon \mathbb{C}(C, D) \to \to \sum_0^n \mathbb{F}(C, D)$ such that

$$\mu_*\colon \pi_q(\mathbb{C}(C, D)) \approx \pi_q(\textstyle\sum_0^k \mathbb{F}(C, D)) \quad \text{for all} \quad q.$$

Since both $\mathbb{C}(C, D)$ and $\sum_0^n \mathbb{F}(C, D)$ are ANR spectra, it follows from Theorem 5.3 that for every locally compact pair (A, B)

$$\mu_*\colon H^q_{\mathbb{C}(C, D)}(A, B) \approx H^q_{\sum_0^n \mathbb{F}(C, D)}(A, B)$$

or equivalently,

$$\begin{aligned}\{A^+, B^+; \mathbb{C}(C, D)\}^q &\overset{\mu_*}{\approx} \{A^+, B^+; \textstyle\sum_0^n \mathbb{F}(C, D)\}^q \\ &= \varinjlim_k \{[\textstyle\sum_0^k (A^+/B^+); F(C, D; S^{n+k+q})]\}\end{aligned}$$

By the exponential theorem the last limit is isomorphic to

$$\varinjlim_k \{[\textstyle\sum_0^k (A^+/B^+) \wedge (C^+/D^+); S^{n+k+q}]\}$$

There are canonical homomorphisms

$$\begin{aligned}\textstyle\sum_0^k (A^+/B^+) \wedge (C^+/D^+) &\approx S^k \wedge (A^+/B^+) \wedge (C^+/D^+) \\ &\approx S^k \wedge (C^+/D^+) \wedge (A^+/B^+) \approx \textstyle\sum_0^k (C^+/D^+) \wedge (A^+/B^+)\end{aligned}$$

so that

$$\begin{aligned}\{A^+, B^+, \mathbb{C}(C, D)\}^q &\approx \varinjlim_k \{[\textstyle\sum_0^k (C^+/D^+) \wedge (A^+/B^+); S^{n+k+q}]\} \\ &\approx \varinjlim_k \{[\textstyle\sum_0^k (C^+/D^+); F(A, B; S^{n+k+q})]\} \\ &\approx \{C^+, D^+; \textstyle\sum_0^n \mathbb{F}(A, B)\}^q\end{aligned}$$

In case (A, B) is also a closed pair in $\mathbb{R}^n$ there is also an isomorphism

$$\{C^+, D^+; \mathbb{C}(A, B)\}^q \overset{\mu_*}{\approx} \{C^+, D^+; \textstyle\sum_0^n \mathbb{F}(A, B)\}^q$$

Combining these isomorphisms gives the result. □

Our final result is due to Lima [13]. In the proof we essentially show that for a compact $A \subset S^n$, the spectrum of $S^n - A$ and $\sum_0^{n-1} \mathbb{F}(A, \emptyset)$ are equivalent (compare with [8, Theorem 4.5]).

Corollary 7.3. *Let A, B be nonempty proper compact subsets of S^n. Then*

$$\{A; S^n - B\}^q \approx \{B; S^n - A\}^q \quad \textit{for all} \quad q.$$

PROOF. If φ is a homeomorphism of S^n and the result is valid for A, B it is also valid for $\varphi(A), B$ and $A, \varphi(B)$ so, without loss of generality, we can assume $\infty \in A \cap B$. Then $A = (A')^+$, $B = (B')^+$ for closed subsets $A', B' \subset \mathbb{R}^n$. By Corollary 7.2 there is a duality isomorphism

$$\{A'^+/\emptyset; \mathbb{C}(B', \emptyset)\}^q \approx \{B'^+/\emptyset^+; \mathbb{C}(A', \emptyset)\}^q$$

For k large $T_k(\emptyset)$ is a closed n-cell containing $T_k(B')$ and, therefore, $T_k(\emptyset)^+/T_k(B')^+$ has the same homotopy type as $\sum_0 T_k(B')$, and we have

$$\begin{aligned}\{A'^+/\emptyset^+; \mathbb{C}(B', \emptyset)\}^q &= \varinjlim_k \{[\textstyle\sum_0^k(A); \sum_0^{k+q}(T_{k+q}(\emptyset)^+/T_{k+q}(B')^+)]\} \\ &\approx \varinjlim_k \{[\textstyle\sum_0^k(A); \sum_0^{k+q+1} T_{k+q}(B')]\}.\end{aligned}$$

(where in the above $T_k(B')$ is given an arbitrary base point for k large enough which is also the base point for $T_{k'}(B')$ for $k' > k$ and for $S^n - B$). Since $\{T_k(B')\}_k$ is an increasing sequence of subspaces of $S^n - B$ such that $\bigcup_k \operatorname{int} T_k(B') = S^n - B$, it follows that for the compact space A,

$$[A; \textstyle\sum_0^{q+1}(S^n - B)] \approx \varinjlim_k \{[A; \sum_0^{q+1} T_k(B')]\},$$

and this implies that

$$\{A; S^n - B\}^{q+1} \approx \varinjlim_k \{[\textstyle\sum_0^k(A); \sum_0^{k+q+1} T_k(B')]\} \approx \{A'^+/\emptyset^+, \mathbb{C}(B', \emptyset)\}^q.$$

Similarly $\{B; S^n - A\}^{q+1} \approx \{B'^+/\emptyset^+; \mathbb{C}(A', \emptyset)\}^q$. Combining these isomorphisms with the duality isomorphism gives the result. □

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Lifting Properties, Nehari Theorem and Paley Lacunary Inequality

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Dedicated to Alberto P. Calderón on his 65th birthday

Abstract

A general notion of lifting properties for families of sesquilinear forms is formulated. These lifting properties, which appear as particular cases in many classical interpolation problems, are studied for the Toeplitz kernels in $\mathbb{Z}$, and applied for refining and extending the Nehari theorem and the Paley lacunary inequality.

1. Introduction

There is a close connection between interpolation problems and lifting properties. Sarason [24] showed that classical interpolation results can be obtained as corollaries of a theorem about commutators of the truncated shift operator

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operator, later extended by Nagy and Foias [14] into a general lifting theorem for contractions. Furthermore, the theorem of Sarason was proved by Page [9] and Nikolskii [18] as equivalent to the vector version of the interpolation theorem of Nehari [17].

Questions concerning weighted inequalities and prediction theory lead the authors to a lifting property of the so-called generalized Toeplitz kernels (GTKs) [7], which can be considered as a generalization of the Nehari theorem (see (IV) below). This lifting property, later proved equivalent to the Nagy-Foias theorem [1], [13], is closely related to the study of subordinated reproducing kernels, used by Aronszajn, Masani, Burbea and Beatrous [3], [4], [5], [7] as a tool to solve interpolation problems.

The above-mentioned results are in fact (although generally in a implicit form) lifting properties of families of sesquilinear forms with respect to a given pair of hilbertian seminorms. This suggests a general notion of lifting properties for families of forms with respect to pairs of seminorms, which is formulated and exemplified in this section. This leads to new results in interpolation as well as to refinements of the known ones.

A summary of known results in terms of lifting properties follows as (I)-(VII), after the introduction of the necessary notation.

Fix V a vector space, $V_1, V_2 \subset V$ subspaces, and let Λ be a family of sesquilinear forms $S: V \times V \to \mathbb{C}$. Given a pair σ_1, σ_2 of seminorms in V, write

$$S \leqslant (\sigma_1, \sigma_2) \qquad (\text{resp., } S \prec (\sigma_1, \sigma_2)) \tag{1}$$

if

$$|S(a, b)| \leqslant \sigma_1(a)\sigma_2(b) \tag{1a}$$

for all $(a, b) \in V \times V$ (resp. all $(a, b) \in V_1 \times V_2$).

Given V, V_1, V_2, *a family of sesquilinear forms* Λ *satisfies the lifting property with respect to* σ_1, σ_2 if for every $A \in \Lambda$ such that $S \prec (\sigma_1, \sigma_2)$ there is a $S' \in \Lambda$ such that $S' \leqslant (\sigma_1, \sigma_2)$ and $S'|_{V_1 \times V_2} = S|_{V_1 \times V_2}$.

In most of our examples the sesquilinear forms considered arise from kernels. More precisely, fixing a set X, and $X_1, X_2 \subset X$, and setting $V = V(X) = \{a: X \to \mathbb{C}$, functions of finite support$\}$, $V_j = V(X_j) = \{a \in V$: supp $a \subset X_j\}$, $j = 1, 2$, each sesquilinear form S: $V \times V \to \mathbb{C}$ is given by a kernel K: $X \times X \to \mathbb{C}$

$$S(a, b) = S_K(a, b) = \sum_{x,y \in X} K(x, y)a(x)\overline{b(y)}. \tag{2}$$

A kernel K is positive definite, p.d., if $S_K(a, a) \geqslant 0$ for all $a \in V$. To each p.d. kernel corresponds a hilbertian seminorm in V, defined by $\sigma_K(a) = S_K(a, a)^{1/2}$.

Given two p.d. kernels K_1, K_2, write

$$K \leqslant (K_1, K_2) \qquad (\text{resp., } K \prec (K_1, K_2)) \tag{3}$$

if

$$S_K \leqslant (\sigma_{K_1}, \sigma_{K_2}) \qquad (\text{resp., } S_K \prec (\sigma_{K_1}, \sigma_{K_2})) \tag{3a}$$

For $V = V(X)$, $V_1 = V(X_1)$, $V_2 = V(X_2)$, fixed, *a family* Λ *of kernels satisfies the lifting property with respect to* K_1, K_2 if for every $K \in \Lambda$ such that $K \prec (K_1, K_2)$ there is a $K' \in \Lambda$ such that $K' \leqslant (K_1, K_2)$ and $K'|_{X_1 \times X_2} = K|_{X_1 \times X_2}$.

Let us illustrate this notion in the case of the theorem of Nehari, which asserts that, for a given sequence s: $\mathbb{Z} \to \mathbb{C}$, the following are equivalent:

(*a*) there exists a bounded function on the circle, f, such that $|f(t)| \leqslant 1$ and $\hat{f}(n) = s(n)$ for $n > 0$.

$$(b) \qquad \left| \sum_{\substack{m \geqslant 0 \\ n < 0}} s(m-n) a(m) \overline{b(n)} \right| \leqslant \left(\sum_{m \geqslant 0} |a(m)|^2 \right)^{1/2} \left(\sum_{n < 0} |b(n)|^2 \right)^{1/2}$$

for all finite sequences $(a(n)), (b(n))$.

Take now $X = \mathbb{Z}$, $X_1 = \mathbb{Z}_1 = \{n \in \mathbb{Z}: n \geqslant 0\}$, $X_2 = \mathbb{Z}_2 = \{n \in \mathbb{Z}: n < 0\}$, $V = \{a: \mathbb{Z} \to \mathbb{C}$, finite sequences$\}$, $V_j = \{a$ finite sequence, $\operatorname{supp} a \subset \mathbb{Z}_j\}$, $j = 1, 2$, $K_1(m, n) = K_2(m, n) = \delta_{m-n}$. If $K(m, n) = s(m - n)$, condition (*b*) becomes $K \prec (K_1, K_2)$. On the other hand, setting $\hat{f}(n) = s'(n)$, $K'(m, n) = s'(m - n)$, it is easy to verify that $|f| \leqslant 1$ is equivalent to $K' \leqslant (K_1, K_2)$. Thus,

(I) (Nehari [17].) If $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, $V = V(\mathbb{Z})$, $V_j = V(\mathbb{Z}_j)$, $K_j = \delta_{m-n}$, $j = 1, 2$, and $\Lambda = \{K: \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}, K(m, n) = s(m - n)\}$ is the family of Toeplitz kernels, then Λ has the lifting property with respect to K_1, K_2.

Other examples are

(II) (Bergman-Schiffer [6].) If $X =$ a domain in $\mathbb{C}$, $X_1 = X_2 =$ an open subset of X, $K_1 = K_2 =$ the Bergman kernel of X, and $\Lambda = \{F: X \times X \to \mathbb{C}$, F holomorphic$\}$, then Λ satisfies the lifting property with respect to K_1, K_2.

(III) (Beatrous-Burbea [4].) If $X_1 = X_2$, $K_1 = K_2 =$ p.d. kernel, $\Lambda = \{K: X \times X \to \mathbb{C}, K(x, y) = f(x)\overline{f(y)}\}$, then Λ satisfies the lifting property with respect to K_1, K_2 (cf. also [5]).

Next example is an equivalent version of the so-called lifting theorem for p.d. GTKs (cf. [13]).

(IV) (Cotlar-Sadosky [8].) If $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, $j = 1, 2$, K_1, K_2 p.d. Toeplitz kernels, $\Lambda = \{K: \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}, K(m, n) = s(m - n)\}$, then Λ satisfies the lifting property with respect to K_1, K_2.

Clearly, the Nehari theorem (I) is a special case of (IV), which also provides refinements to classical results of Helson, Szegö, Sarason and others [8], [2].

The following is a special case of the Nagy-Foias lifting theorem.

(V) (Nagy-Foias [16].) If V is a Hilbert space, U a unitary operator on V, $V_1 = V_2 =$ subspace of V semiinvariant with respect to U, $\sigma_1 = \sigma_2 =$ the hilbertian norm in V, and Λ is the family of sesquilinear forms commuting with U, then Λ satisfies the lifting property with respect to σ_1, σ_2.

(VI) (Cotlar-Sadosky [13].) If X is an arbitrary set, $X_1, X_2 \subset X$, ζ: $X \to X$ is a bijection such that $\zeta(X_1) \subset X_1$, $\zeta^{-1}(X_2) \subset X_2$, $\Lambda = \{K\colon X \times X \to \mathbb{C}$, $K(\zeta x, \zeta y) = K(x, y)\}$, $K_1, K_2 \in \Lambda$ p.d., then Λ satisfies the lifting property with respect to K_1, K_2.

This generalization of (IV) was obtained combining (V) with the arguments used to prove (III). Conversely, the operator-valued version of (IV) implies (V) (cf. [1]). Thus, the framework in which the above results are presented naturally includes the Nagy-Foias lifting theorem. A more general concept of norm majoration allows a further inclusion. Writing $K \leqslant \sigma_0$ (resp., K σ_0) if $|S_K(a, b)| \leqslant \sigma_0(a, b)$ for all $(a, b) \in V \times V$ (resp., $(a, b) \in V_1 \times V_2$), where σ_0: $V \times V \to \mathbb{R}^+$, the following result relates to (I) – (VI).

(VII) (Cotlar-Sadosky [10].) If $X = \mathbb{Z}, X_j = \mathbb{Z}_j$, $j = 1, 2$, $\Lambda =$ the Toeplitz kernels, $\sigma_0(a, b) = \rho(|\check{a} + \check{b}|^2)$, where ρ is a seminorm in $C(\mathbb{T})$ under certain natural assumptions, and $\check{a}(t) = \sum_n a_n e^{\text{int}}$, then Λ satisfies the lifting property with respect to σ_0.

This is a special case of the so-called lifting theorem for majorized GTKs [10], that yields L^p versions of the Helson —Szegö and Nehari theorems. But it is to be remarked that (VII) neither generalizes (IV) to the case when $\sigma_1 \neq \sigma_2$ are non hilbertian, nor extends (I) for general σ_1, σ_2. Some extensions can be obtained from (VII) when $\sigma_1 = \sigma_2$ is a non-deterministic seminorm (see (5) below and Corollary 2). Extensions of (I) and (IV) for pairs σ_1, σ_2 are given in Section 2 and 3.

Section 2 deals with lifting properties of the family of Toeplitz kernels in $\mathbb{Z}$ with respect to pairs of seminorms σ_1, σ_2; Section 3, with several refinements of the Nehari theorem, and Section 4, with generalizations of the Paley lacunary inequality. Some of the results in Section 4 were announced in [12].

2. Lifting properties of Toeplitz kernels in $\mathbb{Z}$

Fix $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, $V = V(\mathbb{Z})$, $V_j = V(\mathbb{Z}_j)$, $j = 1, 2$. Let $\mathcal{P} = \{\check{a}(t) = \sum a(n)e^{\text{int}}$: $a \in V\}$ be the trigonometric polynomials, $\mathcal{P}_1 = \{\check{a}(t)\colon a \in V_1\}$, the analytic polynomials, $\mathcal{P}_2 = \{\check{a}(t)\colon a \in V_2\}$. If $\check{a} = f \in \mathcal{P}$ then $a(n) = \hat{f}(n)$, $n \in \mathbb{Z}$. $\mathcal{P}$ is dense in $C(\mathbb{T})$, the continuous functions in $\mathbb{T}$. For $1 \leqslant p \leqslant \infty$, $H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T})$: $\hat{f}(n) = 0$ for $n < 0\}$. In what follows, all integrals are taken over $\mathbb{T}$.

Consider the seminorms ρ defined in $C(\mathbb{T})$ which satisfy

(4*a*) $\quad |f| \leqslant |g| \quad$ implies $\quad \rho(f) \leqslant \rho(g), \rho(f) = \rho(|f|)$

(4*b*) $\quad \rho(f) \leqslant c\|f\|_\infty, \quad \forall f \in C(\mathbb{T}), \quad c$ fixed constant

(4*c*) $\quad \rho'(f) = \rho(|f|^2)^{1/2}$ is also a seminorm.

Seminorm ρ is said to be *absolutely continuous* if $f_n \downarrow 1_A$, $|A| = 0$, implies $\rho(f_n) \to 0$.

Seminorm ρ is said to be *non deterministic* if there is a fixed constant c such that, if ρ' is given by (4*c*),

(5) $\qquad \rho'(f_1) + \rho'(f_2) \leqslant c\rho'(f_1 + f_2), \qquad \forall (f_1, f_2) \in \mathcal{P}_1 \times \mathcal{P}_2.$

Examples of non-deterministic norms are $\rho_p(f) = \left(\int |f|^p\, dt\right)^{1/p}$, $1 < p < \infty$, and also $\rho_{p\mu}(f) = \left(\int |f|^p\, d\mu\right)^{1/p}$ for $\mu \in A_p$, since the Hilbert transform is then bounded in the corresponding spaces.

To each seminorm ρ in $C(\mathbb{T})$ that satisfies (4*a*) – (4*c*) it is associated a seminorm σ in V by

(6) $\qquad \sigma \sim \rho \quad$ whenever $\quad \sigma(a) = \rho'(\check{a}) = \rho(|\check{a}|^2)^{1/2}.$

For a pair of such seminorms, σ_1, σ_2, and a kernel K: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$, consistently with the notations of Section 1, $K \leqslant (\sigma_1, \sigma_2)$ (resp., $K \prec (\sigma_1, \sigma_2)$) means that

(1*b*) $\qquad |S_K(a, b)| \leqslant \rho_1(|\check{a}|^2)^{1/2}\rho_2(|\check{b}|^2)^{1/2}$

for all $(a, b) \in V \times V$ (resp., all $(a, b) \in (V_1 \times V_2)$).

Lemma 1. *For every sesquilinear form S: $V \times V \to \mathbb{C}$ the following conditions are equivalent*:

(i) $S \leqslant (\sigma_1, \sigma_2)$ *(resp.,* $S \prec (\sigma_1, \sigma_2)$*)*;

(ii) $S(a, b) + \overline{S(a, b)} = 2Re\, S(a, b) \leqslant \sigma_1(a)^2 + \sigma_2(b)^2$, $\forall$ $(a, b) \in V \times V$ *(resp.,* $(a, b) \in V_1 \times V_2$*)*;

(iii) $2|S(a, b)| \leqslant \sigma_1(a)^2 + \sigma_2(b)^2$ *for all such* (a, b).

PROOF. It is enough to observe that (ii) holds iff the quadratic form in λ_1, λ_2, $\sigma_1(a)^2\, \lambda_1\bar{\lambda}_1 - S(a, b)\lambda_1\bar{\lambda}_2 - \overline{S(a, b)}\,\bar{\lambda}_1\lambda_2 + \sigma_2(b)^2\lambda_2\bar{\lambda}_2$ is nonnegative for all $\lambda_1, \lambda_2 \in \mathbb{C}$ (for all pertinent (a, b)), and that (i) expresses that the determinant of this form is nonnegative. □

Lemma 2. *If $\sigma \sim \rho$ as in* (6), $\rho(1) = 1$ *and* $K(m, n) = s(m - n)$, *then*

(i) $K \leqslant (\sigma, \sigma)$ *is equivalent to*

(ii) $|\sum_n s(n)a(n)| \leqslant \rho'(a)$, $\forall a \in V$.

PROOF. If $K \leqslant (\sigma, \sigma)$, setting $b(n) = \delta_0$, or $\check{b} \equiv 1$, in (1b), it becomes (ii). Conversely, if (ii) holds, then $|S_K(a, b)| = \left|\sum_{m,n} s(m-n)a(m)\overline{b(n)}\right| = \left|\sum_n \left(\sum_m a(m)\overline{b(m-n)}\right)s(n)\right| \leqslant \rho(\check{a} \cdot \check{b}) \leqslant 1/2(\rho(|\check{a}|^2) + \rho(|\check{b}|^2)) = 1/2(\sigma(a)^2 + \sigma(b)^2)$, and (i) follows from Lemma 1. □

A kernel $K: \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is *Toeplitz* if $K(m, n) = s(m-n)$ for some $s: \mathbb{Z} \to \mathbb{C}$.

Theorem 1. *For $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, $V = V(\mathbb{Z})$, $V_j = V(\mathbb{Z}_j)$, seminorms $\sigma_j \sim \rho_j$ as in (6), $j = 1, 2$, the family Λ of Toeplitz kernels in $\mathbb{Z}$ satisfies the lifting property with respect to σ_1, σ_2.*

More precisely, if $K \in \Lambda$, $K(m, n) = s(m-n)$, $K \prec (\sigma_1, \sigma_2)$, then there exists $K' \in \Lambda$, $K'(m, n) = s'(m-n)$, such that $K' \leqslant (\sigma_1, \sigma_2)$ and $K'|_{\mathbb{Z}_1 \times \mathbb{Z}_2} = K_{\mathbb{Z}_1 \times \mathbb{Z}_2}$. Moreover, $s'(n) = \hat{\mu}(n)$, where μ is a finite complex measure in $\mathbb{T}$.

If in addition $\rho_1 = \rho_2$ is an absolutely continuous seminorm or $s(n) = \hat{\omega}(n)$ for $\omega \in L^1$, then $d\mu = \theta\, dt$ for some $\theta \in L^1$.

PROOF. The main difficulty of this proof appears where $\sigma_1 \neq \sigma_2$. Therefore it is divided in two cases.

(*Case* $\sigma_1 = \sigma_2 \sim \rho$). Define in $\mathcal{P}$ a linear functional by $l(f) = \sum_n a(n)s(n)$ if $f = \check{a}$, $a \in V$, so that if $g = \check{b}$ then $|l(f\bar{g})| = \left|\sum_{m,n} s(m-n)a(m)\overline{b(n)}\right| \leqslant \leqslant \rho'(f)\rho'(g)$. Taking $f(t) \equiv \|g\|_\infty$, it follows from (4b) that $\|g\|_\infty |l(\bar{g})| \leqslant \leqslant c\|g\|_\infty \cdot c\|g\|_\infty$. Thus, $|l(\bar{g})| \leqslant c^2\|g\|_\infty$, $\forall \bar{g} \in e^{it}\mathcal{P}_1 = \mathcal{P}_{10}$, and there exists a finite complex measure ν in $\mathbb{T}$ such that $l(f) = \int f\, d\nu$, $\forall f \in \mathcal{P}_{10}$ and $s(n) = \hat{\nu}(n)$ for $n \geqslant 0$. Thus

$$\left|\int f\bar{g}\, d\nu\right| \leqslant \rho'(f)\rho'(g), \qquad \forall (f, g) \in \mathcal{P}_1 \times \mathcal{P}_2 \tag{7}$$

and from (4b) follows that (7) holds also for $(f, g) \in \overline{\mathcal{P}_1} \times \overline{\mathcal{P}_2}$, $\overline{\mathcal{P}_1}$, $\overline{\mathcal{P}_2}$ the closures in $C(\mathbb{T})$ of $\mathcal{P}_1, \mathcal{P}_2$. Since every analytic polynomial $F \in e^{it}\mathcal{P}_1$ can be written as $F = f\bar{g}$ with $(f, g) \in \overline{\mathcal{P}_1} \times \overline{\mathcal{P}_2}$, $|f|^2 = |g|^2 = |F|$, then

$$\left|\int F\, d\nu\right| = \left|\int f\bar{g}\, d\nu\right| \leqslant \rho'(f)\rho'(g) \leqslant 1/2(\rho'(f)^2 + \rho'(g)^2)$$
$$= 1/2(\rho(|f|^2) + \rho(|g|^2)) = \rho(|F|).$$

Thus, $\left|\int F\, d\nu\right| \leqslant \rho(F)$, $\forall F \in e^{it}\mathcal{P}_1$, and since $\rho(G) \leqslant c\|G\|_\infty$, $\forall G \in C(\mathbb{T})$, it follows easily that there exists a finite measure in $\mathbb{T}$ such that $\int F\, d\mu = \int F\, d\nu$ for $F \in e^{it}\mathcal{P}_1$ and $\left|\int G\, d\mu\right| \leqslant \rho(G)$, $\forall G \in C(\mathbb{T})$. Hence

$$\left|\int f_1 f_2\, d\mu\right| \leqslant \rho(f_1 f_2) \leqslant 1/2(\rho(|f_1|^2) + \rho(|f_2|^2)), \qquad \forall (f_1, f_2) \in \mathcal{P} \times \mathcal{P}.$$

Setting $s'(n) = \hat{\mu}(n)$, and using Lemma 1, the assertion on s' follows. The last part of the thesis is proved as Proposition 3 in [23].

(*Case* $\sigma_1 \neq \sigma_2$). As in the previous case, there exists a measure ν such that $s(n) = \hat{\nu}(n)$ for $n \geqslant 0$, so that $S_K(a, b) = \int f\bar{g}\, d\nu$ for $f = \check{a} \in \mathcal{P}_1$, $g = \check{b} \in \mathcal{P}_2$, i.e., for $(a, b) \in V_1 \times V_2$. Thus, by hypothesis and Lemma 1,

$$2\left|\int f\bar{g}\, d\nu\right| \leqslant \rho_1(|f|^2) + \rho_2(|g|^2),$$

or

$$\int f\bar{g}\, d\nu + \int \bar{f}g\, \overline{d\nu} \leqslant \rho_1(|f|^2) + \rho_2(|g|^2), \qquad \forall (f, g) \in \mathcal{P}_1 \times \mathcal{P}_2. \tag{7a}$$

Again we want to prove that there is a measure μ such that $\int F\, d\mu = \int F\, d\nu$ for $F \in e^{it}\mathcal{P}_1$, and such that (7*a*) holds for all $(f, g) \in \mathcal{P} \times \mathcal{P}$ when $d\nu$ is replaced by $d\mu$. We use an argument from [9].

Let $\mathfrak{M}$ be the dual of $C(\mathbb{T})$, and set $E = \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M}$, $E' = C(\mathbb{T}) \times \times \overline{\mathcal{P}_{10}} \times \overline{\mathcal{P}_2} \times C(\mathbb{T})$, where the closures are taken in $C(\mathbb{T})$. For each quadruple of measures $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) \in E$ and $(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}) \in E'$, set

$$\langle (\mu_{jk}), (\phi_{jk}) \rangle = \sum_{j,k=1,2} \int \phi_{jk}\, d\mu_{jk}. \tag{8}$$

Endow the space E with the topology given by this coupling, so that E' is the dual of E, and E the dual of E'. Let $\Pi \subset E$ be the set of all quadruples $(\mu_{11}, \nu, \bar{\nu}, \mu_{22})$ where

$$\left|\int f\, d\mu_{jj}\right| \leqslant \rho_j(f), \qquad \forall f \in C(\mathbb{T}), \qquad j = 1, 2. \tag{9}$$

and let $\Gamma \subset E$ be the set of all quadruples $(\lambda_{11}\delta_t, \lambda_{12}\delta_t, \lambda_{21}\delta_t, \lambda_{22}\delta_t)$, where δ_t is the Dirac measure at $t \in \mathbb{T}$ and (λ_{jk}) is a positive definite 2×2 numerical matrix. In particular, all $(\lambda_{jk} = \lambda_j\bar{\lambda}_k)_{j,k=1,2}$ are such matrices, Π is a compact convex set in E, and Γ_1 = convex hull of Γ, is a cone in E. Let us show that Π intersects the closure of Γ_1. Assuming the contrary there will be, by the polar theorem (cf. [21] p. 168) an element $(\phi_{jk}) \in E'$ such that $\mathrm{Re}\langle (\lambda_{jk}\delta_t), (\phi_{jk}) \rangle \geqslant 0$, $\forall (\lambda_{jk}\delta_t) \in \Gamma$, while $\mathrm{Re}\langle (\mu_{jk}), (\phi_{jk}) \rangle \leqslant a < 0$, $\forall (\mu_{jk}) \in \Pi$. It is easily seen that since $\mu_{12} = \overline{\mu_{21}}$ for all elements in both Π and Γ, we can modify (ϕ_{jk}) as to have

$$\begin{aligned} \langle (\lambda_{jk}\delta_t), (\phi_{jk}) \rangle &\geqslant 0, \qquad \forall (\lambda_{jk}\delta_t) \in \Gamma \\ \langle (\mu_{jk}), (\phi_{jk}) \rangle &\leqslant a < 0, \qquad \forall (\mu_{jk}) \in \Pi \end{aligned} \tag{10}$$

Taking $\lambda_{jk} = \lambda_j\bar{\lambda}_k$ in (10), it follows that $(\phi_{jk}(t))$ is a positive definite matrix for all $t \in \mathbb{T}$. Hence

$$\phi_{11}(t) \geqslant 0, \qquad \phi_{22}(t) \geqslant 0, \qquad |\phi_{12}(t)|^2 \leqslant \phi_{11}(t)\phi_{22}(t), \qquad \forall t \in \mathbb{T}. \tag{11}$$

By the Corollary of the Hahn-Banach theorem, we can take $\int \phi_{jj}\, d\mu_{jj} = \rho_j(\phi_{jj})$, $j = 1, 2$ and by (10),

$$\rho_1(\phi_1) + \int \phi_{12}\, d\nu + \int \phi_{21}\, \overline{d\nu} + \rho_2(\phi_{22}) \leqslant a < 0. \tag{12}$$

By (11) we can take (ϕ_{jk}) so that

$$\phi_{11} = \phi_+ \bar{\phi}_+,\ \phi_{22} = \phi_- \bar{\phi}_-,\ |\phi_{12}|^2 \leqslant \phi_+ \bar{\phi}_- \bar{\phi}_+ \phi_-,\ \phi_+ \in \overline{\mathcal{P}_1},\ \phi_- \in \overline{\mathcal{P}_2}. \tag{13}$$

Since ϕ_{12} and $\phi_+ \bar{\phi}_-$ both belong to $\overline{\mathcal{P}_{10}}$, (13) gives

$$\phi_{12} = \gamma \phi_+ \bar{\phi}, \qquad |\gamma| \leqslant 1, \qquad \gamma \in \overline{\mathcal{P}_1}. \tag{13a}$$

Thus, $(\phi_{jk}) = (F_1\overline{F_1} + G, F_1\overline{F_2}, \overline{F_1}F_2, F_2\overline{F_2})$, where

$$F_1 = \gamma\phi_+ \in \overline{\mathcal{P}_1}, \qquad F_2 = \phi_- \in \overline{\mathcal{P}_2}, \qquad G = \phi_+ \bar{\phi}_+ (1 - |\gamma|^2) \geqslant 0. \tag{13b}$$

Hence, by (13b), (7a) we get

$$\begin{aligned} \rho_1(\phi_{11}) + \int \phi_{12}\, d\nu + \int \phi_{21}\, \overline{d\nu} + \rho_2(\phi_{22}) \\ \geqslant \rho_1(F_1\overline{F_1}) + \rho_2(F_2\overline{F_2}) + \int F_1\overline{F_2}\, d\nu + \int \overline{F_1}F_2\, \overline{d\nu} \geqslant 0 > a \end{aligned}$$

which contradicts (12).

Therefore, there is some quadruple $(\mu_{11}, \nu, \bar{\nu}, \mu_{22}) \in \Pi$ which is in the closure of Γ_1, hence there is a net of elements $(\nu_{jk}^\gamma) \in \Gamma_1$ such that

$$\int f_{jj}\, d\nu_{jj}^\gamma \to \int f_{jj}\, d\mu_{jj}, \qquad j = 1, 2, \qquad \int f_{12}\, d\nu_{12}^\gamma \to \int f_{12}\, d\nu \tag{14}$$

for every $(f_{jk}) \in E'$ such that $f_{12} = \overline{f_{21}} \in \overline{\mathcal{P}_{10}}$.

From (11) it follows easily that, for each γ,

$$\left| \int \phi\, d\nu_{12}^\gamma \right|^2 \leqslant \left(\int \phi\, d\nu_{11}^\gamma \right) \left(\int \phi\, d\nu_{22}^\gamma \right), \qquad \forall \phi \in C(\mathbb{T}).$$

Since the ν_{jj}^γ are positive measures converging to μ_{jj}, all the ν_{jk}^γ are in a bounded set of $\mathfrak{M}$. Hence, passing to a subnet, there are measures (ν_{jk}) such that $\int \phi\, d\nu_{jk}^\gamma \to \int \phi\, d\nu_{jk}$ for all $\phi \in C(\mathbb{T})$. From (14) follows that $\nu_{jj} = \mu_{jj}$, $j = 1, 2$, and that $\int \phi\, d\nu_{12} = \int \phi\, d\mu_{12}$ for $\phi \in e^{it}\mathcal{P}_1$. Now (10) implies that $\sum_{j,k} \int f_j f_k\, d\nu_{jk}^\gamma \geqslant 0$ whenever $(f_1, f_2) \in \mathcal{P} \times \mathcal{P}$. Hence, also

$$\sum_{j,k} \int f_j \overline{f_k}\, d\mu_{jk} = \int f_1 \overline{f_2}\, d\mu_{11} + \int f_1 \overline{f_2}\, d\mu + \int \overline{f_1} f_2\, \overline{d\mu} + \int f_2 \overline{f_2}\, d\mu_{22} \geqslant 0$$

for all $(f_1, f_2) \in \mathcal{P} \times \mathcal{P}$. Since $\mu_{jj} \geqslant 0$ and $\left| \int f_j \overline{f_j}\, d\mu_{jj} \right| \leqslant \rho_j(f_j \cdot \overline{f_j})$ by (9), and since $\mu = \nu$ on $e^{it}\mathcal{P}_1$, the thesis follows. □

It $\hat{\mu}(n) = s'(n)$ and $\hat{f}(n) = a(n)$ for $a \in V$, then $\sum_n s'(n)a(n) = \int f\, d\mu$. From Lemma 2 and Theorem 1 for $\sigma_1 = \sigma_2$ follows

Corollary 1. *Let* $K(m,n) = s(m-n)$, $\sigma \sim \rho$ *as in* (6). *Then* $K \prec (\sigma, \sigma)$ *is a necesary and sufficient condition for the existence of a measure* ν *such that* $\hat{\nu}(n) = s(n)$ for $n > 0$, *and*

$$\left| \int f \, d\nu \right| \leqslant \rho(f), \qquad \forall f \in C(\mathbb{T}). \tag{15}$$

Remark 1. When $\sigma_j \sim \rho_j$ non deterministic seminorms, $j = 1, 2$, a transference property to vector-valued sequences, or Grothendieck type inequality similar to that for GTKs majorized in the sense of (VII), proved in [11], is valid for the relation $K \prec (\sigma_1, \sigma_2)$. This will be developed elsewhere.

Let us introduce now a weaker notion of majoration. Write $K \prec\prec (\sigma_1, \sigma_2)$ if

$$|S_K(a,b)| \leqslant \sigma_1(a)\sigma_2(b), \qquad \forall (a,b) \in V_1 \times V_2, \qquad a(n) = \overline{b(-n)} \tag{16}$$

i.e.: for all $(a,b) \in \mathcal{P}_1 \times \mathcal{P}_2$ such that $b(t) = \overline{a(t)}$.

If $K(m,n) = \mu(m-n)$, μ a finite measure in $\mathbb{T}$, then $K \prec\prec (\sigma_1, \sigma_2)$ means

$$\left| \int f^2 \, d\mu \right| \leqslant \sigma_1(\hat{f})\sigma_2(\hat{f}) = \rho_1'(f)\rho_2'(f), \qquad \forall f \in e^{it}\mathcal{P}_1. \tag{16a}$$

In the case $\sigma_1 = \sigma_2$, this reduces to

$$\left| \int f^2 \, d\mu \right| \leqslant \rho(|f|^2), \qquad \forall f \in e^{it}\mathcal{P}_1. \tag{16b}$$

Theorem 2. *For* $X = \mathbb{Z}$, $X_j = \mathbb{Z}_j$, *seminorms* $\sigma_j \sim \rho_j$ *as in* (6), $j = 1, 2$, *let* $K(m,n) = \hat{\mu}(m-n)$ *where* μ *is a finite complex measure in* $\mathbb{T}$. *If* $K \prec\prec (\sigma_1, \sigma_2)$, *then there exists an analytic function* $h \in H^1(\mathbb{T})$ *such that* $d\nu = d\mu + \bar{h}\, dt$ *satisfies*

$$\left| \int f\bar{g} \, d\nu \right| \leqslant \rho_3'(f)\rho_3'(g), \qquad \forall (f,g) \in \mathcal{P} \times \mathcal{P}, \tag{17}$$

where $\rho_3 = \rho_1 + \rho_2$.

Therefore, if $\rho_1 = \rho_2$, *then* $K'(m,n) = \hat{\nu}(m-n)$ *satisfies* $K'|_{X_1 \times X_2} = K|_{X_1 \times X_2}$ *and* $\frac{1}{2}K' \leqslant (\sigma_1, \sigma_1)$.

PROOF. By condition (4*b*) on ρ_1, ρ_2, the hypothesis (16*a*) holds also for $f \in \overline{\mathcal{P}_{10}} =$ = the closure of $e^{it}\mathcal{P}_1$ in $C(\mathbb{T})$. Again, every $F \in e^{it}\mathcal{P}_1$ can be decomposed as $F = f\bar{g}$ with $(f,g) \in \overline{\mathcal{P}_1} \times \overline{\mathcal{P}_2}$, and $|f|^2 = |g|^2 = |F|$, so that $4F = (f+\bar{g})^2 +$ $+ (if - i\bar{g})^2$, where $f + \bar{g}$ and $if - ig$ are in $\overline{\mathcal{P}_1}$. By (16*a*),

$$\begin{aligned} 4\int F\, d\mu &= \int (f+\bar{g})^2 \, d\mu + \int (if - i\bar{g})^2 \, d\mu \\ &\leqslant \rho_1(|f+\bar{g}|^2)^{1/2}\rho_2(|f+\bar{g}|^2)^{1/2} + \rho_1(|f-\bar{g}|^2)^{1/2}\rho_2(|f-\bar{g}|^2)^{1/2} \\ &\leqslant 8\rho_1(F)^{1/2}\rho_2(F)^{1/2} \leqslant 4\rho_3(F), \qquad \forall F \in \overline{\mathcal{P}_{10}}, \end{aligned}$$

since $|f+\bar{g}|^2 = |f|^2 + |g|^2 + fg + \overline{fg} \leqslant 4|F|$ and $|f-\bar{g}|^2 \leqslant 4|F|$.

As $K(m,n) = \hat{\mu}(m-n)$, it follows as in Theorem 1 that there is a measure $d\nu = d\mu + h\,dt$ satisfying (17). □

3. Refinements of the Theorem of Nehari

The theorem of Nehari ((I) of Section 1) gives a condition for the existence of a function θ belonging to the dual of $L^1(dt)$ and having prescribed moments $\hat{\theta}(n)$ for $n > 0$. The theorems of Section 2 give the following refinements of that theorem, where θ is required to belong to the «dual of a more general space».

Corollary 2. *Let s: $\mathbb{Z} \to \mathbb{C}$ be a given sequence, $K(m,n) = s(m-n)$ and $\sigma \sim \rho$ as in* (6). *If ρ is absolutely continuous or if $s(n) = \hat{\omega}(n)$, $\omega \in L^1$, then $K \prec (\sigma, \sigma)$ is a necessary and suficient condition for the existence of a function $\theta \in L^1$ such that*

$$\left| \int f(t)\theta(t)\,dt \right| \leqslant \rho(f), \qquad \forall\ f \in C(\mathbb{T}) \tag{18}$$

(i.e., for θ to belong to the «dual of $L(\rho)$») and $\hat{\theta}(n) = s(n)$ for $n > 0$. Another necessary and sufficient condition (up to multiplicative constant 2) is $K \ll (\sigma, \sigma)$. If ρ is non deterministic, as in (5), *then* (18) *can be replaced by the stronger inequality*

$$\int |f(t)| \frac{|\theta(t)|^2}{\mathrm{Re}\,\theta(t)}\,dt \leqslant c\rho(f), \qquad \forall\ f \in C(\mathbb{T}) \tag{18a}$$

with $\mathrm{Re}\,\theta \geqslant 0$.

PROOF. The first two assertions follow from Theorem 1, Corollary 1 and Theorem 2. To deduce (18*a*), observe first that if $\hat{f}(n) = a(n)$, $\hat{g}(n) = b(n)$, $(a,b) \in V_1 \times V_2$, and $s(n) = \hat{\mu}(n)$ for $n \geqslant 0$, then $\sum s(m-n)\,a(m)\overline{b(n)} = \int f\bar{g}\,d\mu$, so that $K \prec (\sigma, \sigma)$ is equivalent to $\left|\int f\bar{g}\,d\mu\right| \leqslant \rho'(f)\rho'(g)$, $\forall\,(f,g) \in \mathcal{P}_1 \times \mathcal{P}_2$. If ρ is non deterministic then $\rho'(f)\rho'(g) \leqslant 1/2(\rho'(f)^2 + \rho'(g)^2) \leqslant c^2/2\rho(|f+g|^2)$. Thus $\left|\int f\bar{g}d\mu\right| \leqslant c^2/2\rho(|f+g|^2)$ for $(f,g) \in \mathcal{P}_1 \times \mathcal{P}_2$, which is $K \leqslant \sigma_0$ for $\sigma_0(a,b) = \rho(|a+b|^2)$ in the sense of (VII). Now (18*a*) follows as a special case of Theorem 3 in [23]. □

Remark 2. For $\rho(f) = \int |f|\,dt$, $\rho'(f) = \left(\int |f|^2\,dt\right)^{1/2}$, Corollary 2 implies the Nehari theorem. Nevertheless, even in this case, conditions $K \prec\!\!\prec (\sigma, \sigma)$ and (18*a*) provide stronger versions of the Nehari theorem.

It $\hat{\omega}(n) = s(n)$, then $\hat{\theta}(n) = s(n)$ for $n > 0$ is equivalent to $\theta = \omega + \bar{h}$, $h \in H^1(\mathbb{T})$. Thus, from Corollary 2 and the remark in its proof,

Corollary 3. *If $\omega \in L^1$, then the condition*

$$\left| \int f \bar{g} \omega \, dt \right| \leqslant \rho'(f)\rho'(g), \qquad \forall (f, g) \in \mathcal{P}_1 \times \mathcal{P}_2 \tag{19}$$

is necessary and sufficient for the existence of $\theta \in L^1$ such that $\theta = \omega + \bar{h}$, $h \in H^1(\mathbb{T})$, and

$$\left| \int f\theta \, dt \right| \leqslant \rho(f), \qquad \forall f \in C(\mathbb{T}). \tag{18}$$

Let us finally observe that (VI) of Section 1 also leads, in the case $X = \mathbb{Z}$ considered here, to a refinement of the Nehari theorem of the following type.

Fix $p \in \mathbb{Z}_1$, take $X = \mathbb{Z}$, $X_1 = p\mathbb{Z}_1 = \{0, p, 2p, \dots\}$, $X_2 = p\mathbb{Z}_2 = \{-p, , -2p, \dots\}$, $\tau(n) = n + p$, and let $K_j(m, n) = s_j(m - n)$ be two fixed p.d. kernels, clearly τ-invariant. In this case, theorem (VI) asserts that if

$$|S_K(a, b)| \leqslant S_{K_1}(a, a)^{1/2} S_{K_2}(b, b)^{1/2}$$

for all finitely supported a, b such that supp $a \subset p\mathbb{Z}_1$, supp $b \subset p\mathbb{Z}_2$, then there exists another τ-invariant kernel K', such that $K'(m, n) = s(m - n)$ for $(m, n) \in p\mathbb{Z}_1 \times p\mathbb{Z}_2$ and $K' \leqslant (K_1, K_2)$.

Since K' is τ-invariant, $K'(m, n) = K'(m - n, 0)$ for $n \in p\mathbb{Z}$, i.e., $K'(m, n) = s'(m - n)$ for $n \in p\mathbb{Z}$. Set $l(f) = l\left(\sum_n a(n)e^{\text{int}}\right) = \sum a(n)s'(n)$ for $f \in \mathcal{P}$, $\hat{f} = a$. If $g = \sum_{n \in p\mathbb{Z}} b(n)e^{\text{int}}$, then

$$|l(f\bar{g})| = \left| l\left(\sum_{\substack{m \in \mathbb{Z} \\ n \in p\mathbb{Z}}} a(m)\overline{b(n)} e^{i(m-n)t} \right) \right| = \left| \sum s'(m - n)a(m)\overline{b(n)} \right|$$

$$= \left| \sum K'(m, n)a(m)\overline{b(n)} \right| \leqslant \left(\int |f|^2 \, d\mu_1 \right)^{1/2} \left(\int |g|^2 \, d\mu_2 \right)^{1/2}.$$

Hence, $|l(|g|^2)| \leqslant \left(\int |g|^2 \, d\mu_1 \right)^{1/2} \left(\int |g|^2 \, d\mu_2 \right)^{1/2}$ for all $g \in \mathcal{P}(p) = \left\{ \sum_{n \in p\mathbb{Z}} b(n) e^{\text{int}} \right\}$.

Taking $K_1 = K_2$ and letting $L^1(p)$ be the closed cone in $L^1(d\mu)$ generated by $\mathcal{P}(p)$, we have that $l \in L^\infty(p) = \{$the linear functionals in L^1 bounded in $L^1(p)\}$. Moreover, $l(e^{\text{int}}) = s'(n)$, and $s'(m - n) = s(m - n)$ if $n \in p\mathbb{Z}$, so that $l(e^{i(m-n)t}) = s(m - n) = s'(m - n)$ if $(m, n) \in p\mathbb{Z}_1 \times p\mathbb{Z}_2$. Thus we have proved

Corollary 4. *If $\left| \sum s(m - n)a(m)\overline{b(n)} \right| \leqslant \|a\|_2 \|b\|_2$ for all sequences a, b finitely supported on $p\mathbb{Z}_1$, $p\mathbb{Z}_2$ respectively, then there exists $l \in L^\infty(p)$ such that $s(n) = \hat{l}(n) = l(e^{\text{int}})$ for all $n \in p\mathbb{Z}_1$.*

Since $L^1(1) = L^1$ and $L^\infty(1) = L^\infty$, for $p = 1$ Corollary 4 reduces to the Nehari theorem.

4. Some Generalizations of Paley Inequality

A sequence $\{n_k\}$ of integers in λ-lacunary if $n_{k+1}/n_k \geqslant \lambda > 1$ for all k. A classical theorem of Paley asserts that a lacunary sequence of coefficients of an H^1 function belongs to l^2 [20]. More precisely, for every λ-lacunary sequence $\{n_k\}$ there is a constant $c = c(\lambda)$ such that if $f = \sum_{n\geq 0} c_n e^{\text{int}} \in H^1(\mathbb{T})$,

$$\left(\sum_k |c_{n_k}|^2\right)^{1/2} \leqslant c \int |f|\, dt. \tag{20}$$

We say that $\{n_k\}$ is a *ρ-q-Paley sequence* with constant c_q, where ρ is a seminorm as in Section 1, and $2 \leqslant q < \infty$, if for every $f = \sum_{n\geq 0} c_n e^{\text{int}} \in C(\mathbb{T})$,

$$\left(\sum_k |c_{n_k}|^q n_k^{2-q}\right)^{1/q} \leqslant c_q \rho(f) \tag{21}$$

holds.

Following the notation of Section 2, given a sequence $a = \{a_n\}$, we write $a(t) = \sum a(n)e^{\text{int}}$ and $\sigma(a) = \rho(|\check{a}|^2)^{1/2}$, $\sigma \sim \rho$, for a fixed seminorm ρ.

Lemma 3. *Let ρ be an absolutely continuous seminorm satisfying (4a) – (4c), $\sigma \sim \rho$, $2 \leqslant q < \infty$ and $\{n_k\}$ a given sequence. For $\{n_k\}$ to be a $\rho - q$ —Paley sequence it is sufficient that, whenever $\phi = \sum_{k>0} v_k e^{\text{in}_k t} \in H^2(\mathbb{T})$, $s(n)$ defined for $n > 0$ as v_k if $n = n_k$ and zero if $n \neq n_k$, and $K(m, n) = s(m - n)$,*

$$K \prec (r\sigma, r\sigma) \tag{22}$$

is satisfied for

$$r^2 = r_q^2 = C_q\left(\sum_k |v_k|^{q'} n_k^{(q-2)/(q-1)}\right)^{1/q'}. \tag{23}$$

PROOF. Given $f = \sum_{n\geq 0} c_n e^{int} \in C(\mathbb{T})$, consider $\phi = \sum |c_{n_k}|^{q-1} n_k^{2-q} e^{\text{in}_k t}$. Then by the hypothesis on K and Corollary 3, there exists $\theta \in L^1$ such that $\left|\int f\theta\, dt\right| \leqslant r\rho(f)$, with $\bar{\theta}(n) = s(n)$ for $n > 0$. Thus we can write $\theta = \phi + \bar{h}$ for some $h \in H^1$. Now,

$$A = \sum_k |c_{n_k}|^q n_k^{2-q} = \int f\bar{\phi}\, dt = \int f(\bar{\phi} + h)\, dt$$

and

$$|A| \leqslant r\rho(f) = C_q\left(\sum_k |n_k|^{(q-1)q'} n_k^{(2-q)q'} n_k^{(q-2)/(q-1)}\right)^{1/q'} \rho(f)$$

$$= C_q |A|^{1/q'} \rho(f).$$

Therefore,

$$|A|^{1/q} \leqslant C_q \rho(f)$$

which is (21). □

Paley's theorem follows as a corollary of the Lifting Theorem 1, since it is not difficult to check that condition (22) holds for $\{n_k\}$ λ-lacunary, $q = 2$, $\rho(f) = \int |f|\, dt$, for a suitable constant C. This is part of a more general property of lacunary sequences proved in Lemma 4 below.

Lemma 3 suggests that the lifting theorems of Section 2 provide generalizations of Paley theorem. The connection between these two types of results is not surprising, since Theorem 1 provides refinements of Nehari theorem, and Paley inequality is equivalent by duality to a theorem of Rudin [22] (which says that if $\{n_k\}$ is λ-lacunary and $\{v_k\} \in l^2$ then there exists a $g \in L^\infty$ such that $\hat{g}(n) = v_k$ for $n = n_k$ and zero for $n \neq n_k$, $n > 0$, and such that $\|g\|_\infty \leqslant c\|v\|_2$) that can be derived from that of Nehari.

Given a λ-lacunary sequence $\{n_k\}$, setting the block $B_k = \{[n_k/2], \ldots, n_k\}$,

$$\sum_k \sum_{j \in B_k} |b_j| \leqslant \sum_j \left(1 + \frac{1}{\log_2 \lambda}\right) |b_j| \tag{24}$$

for every sequence $\{b_j\}$.

This suggests to call a sequence $\{n_k\}$ (γ_j)-lacunary if γ_j is the number of blocks B_k containing the index j, so that

$$\sum_k \sum_{j \in B_k} |b_n| \leqslant \sum_j \gamma_j |b_j|$$

for every sequence $\{b_j\}$.

Lemma 4. *Let* $\{n_k\}$ be a (γ_j)*-lacunary sequence,* $2 \leqslant q < \infty$, $\{v_k\}$ *a given sequence. If* $s(n) = v_k$ *if* $n = n_k$, 0 *if* $n \neq n_k$, $n > 0$ *and* $K(m, n) = s(m - n)$, *then* $K \prec (r\sigma, r\sigma)$, *for* $\sigma \sim \rho_q$ *with*

$$\rho_q(f) = \left(\int |f|^{q/2} |t|^{q-2}\, dt\right)^{2/q} \tag{25}$$

and

$$r^2 = r_{q\gamma}^2 = C_q \left(\sum_k |v_k|^{q'} n_k^{(q-2)/(q-1)} \left|\max_{j \in B_k} \gamma_j\right|^{q'}\right)^{1/q'}. \tag{23a}$$

PROOF. By definition

$$S_K(a,b) = \sum_{\substack{m\geq 0\\ n<0}} s(m-n)a_m\overline{b_n} = \sum_{k>0} v_n(a_0\overline{b_{n_k}} + a_1 b_{n_k-1} + \cdots + a_{n_k}\overline{b_0}$$

$$\leqslant \sum_{k>0} v_k \sum_{m=0}^{[n_k/2]} a_m\overline{b_{n_k-m}} + \sum_{k>0} v_k \sum_{m=0}^{[n_k/2]} a_{n_k-m}\overline{b_m}$$

$$= A + B.$$

$$|A| \leqslant \sum_{k>0} |v_k|(|a_0|^{q'} + \cdots + |a_{[n_k/2]}|^{q'})^{1/q'}(|b_{n_k}|^q + \cdots + |b_{[n_k/2]}|^q)^{1/q}$$

where $1/q + 1/q' = 1$. By applying again Hölder's inequality with exponents $r = q/q' \geqslant 1$ and $r' = (q-1)/(q-2)$,

$$|A| \leqslant \sum_{k>0} |v_k|(|a_0|^q + \cdots + |a_{[n_k/2]}|^q)^{1/q} n_k^{(q-2)/q}(|b_{n_k}|^q + \cdots + |b_{[n_k/2]}|^q)^{1/q}$$

$$\leqslant \|a\|_q \sum_{k>0} |v_k| n_k^{(q-2)/q}\Big(\max_{j\in B_k}\gamma_j\Big)(|b_{n_k}|^q/\gamma_{n_k} + \cdots + |b_{[n_k/2]}|^q/\gamma_{[n_k/2]})^{1/q}$$

Writing $w = \{w_k\}$, $w_k = v_k n_k^{(q-2)/q}(\max_{j\in B_k}\gamma_j)$,

$$|A| \leqslant \|a\|_q \|w\|_{q'}\Big[\sum_{k>0}(|b_{n_k}|^q/\gamma_{n_k} + \cdots + |b_{[n_k/2]}|^q/\gamma_{[n_k/2]})\Big)^{1/q}$$

$$\leqslant \|a\|_q \|w\|_{q'}\Big(\sum_{n>0}|b_n|^q\Big)^{1/q}$$

since each b_{n_k} appears in at most γ_{n_k} blocks.

By repeating the argument for B,

$$|S_K(a,b)| \leqslant 2\|w\|_{q'}\|a\|_q\|b\|_q.$$

Since $q \geqslant 2$, by Hardy-Littlewood-Paley weighted inequality,

$$|S_K(a,b)| \leqslant C_q\|w\|_{q'}\Big(\int |\check{a}|^q|t|^{q-2}\,dt\Big)^{1/q}\Big(\int |\check{b}|^q|t|^{q-2}\,dt\Big)^{1/q} =$$

$$= r^2\rho(|\check{a}|^2)^{1/2}\rho(|\check{b}|^2)^{1/2} = r^2\sigma(a)\sigma(b)$$

where ρ is as in (25) and $\sigma \sim \rho$, for all finite sequences a, b, $\operatorname{supp} a \subset \mathbb{Z}_1$, $\operatorname{supp} b \subset \mathbb{Z}_2$, i.e., $K \prec (r\sigma, r\sigma)$.

Since for a λ-lacunary sequence, $\gamma_j = (\log_2\lambda)^{-1} + 1$ for all j, Lemmas 1 and 2 immediately imply the following result, which reduces to the classical Paley inequality when $q = 2$.

Theorem 3. *Every λ-lacunary sequence is a ρ_q-q-Paley sequence with constant depending only on λ and q, for ρ_q as in* (25) *and $2 \leqslant q < \infty$. More generally, the same holds for every (γ_j)-lacunary sequence with $|\gamma_j| \leqslant$ cst. for all j.*

Let us consider now some non-lacunary sequences, for which the γ_j's are not uniformly bounded. With the same notation as before, it is easy to check that

(*a*) for $n_k = k^r$, r fixed, γ_j (= # of blocks containing the index j) $= c_r j^{1/r}$, and $\max_{j \in B_k}(\gamma_j) = c_r k$;

(*b*) for $n_k = 2^{\sqrt{k}}$, $\gamma_j = 2\log_2 j$, and $\max_{j \in B_k} \gamma_j = 2\sqrt{k}$.

In both cases, Lemma 4 applies as before and, with it, the proof of Lemma 3, suitably modified, yields alternate inequealities to (21). Before stating these, let us observe that for ρ_q as in (25), $\rho_q(f) < \infty$ does not imply $f \in L^1(\mathbb{T})$.

In order to give a meaning to $\{c_n\} \sim f$ for all f such that $\rho_q(f) < \infty$, we may consider them as boundary values of analytic functions defined in the disc, in the following way. Let $D = \{z \in \mathbb{C}: |z| < 1\}$, and the Hardy and Nevanlinna classes in D be given by

$$H^p(D) = \left\{F \text{ holomorphic in } D: \sup_{0 \leq r < 1} \int |F(re^{it})|^p \, dt = \|F\|_{H^p}^p \leqslant \infty\right\},$$

$0 < p < \infty$,

$$N(D) = \left\{F \text{ holomorphic in } D: \sup_{0 \leq r < 1} \int \log^+ |F(re^{it})| \, dt < \infty\right\}.$$

For $N^+ = \left\{F \in N: \lim_{r \to 1} \int \log^+ |F(re^{it})| \, dt = \int \log^+ |F(e^{it})| \, dt\right\}$, the inclusion $N \supset N^+ \supset H^p$, $0 < p < \infty$, holds. A function $f \in H^p(\mathbb{T})$ can be identified with $F(e^{it})$ for $F \in H^p(D)$, with $\|f\|_p = \|F\|_{H^p}$.

Let $2 \leqslant q < \infty$ and $F \in N^+$ with boundary values $F(e^{it}) = f(t) \in \in L^{q/2}(|t|^{q-2} \, dt)$. Then, by Hölder's inequality, $\int |F(e^{it})|^{1/2} \, dt < \infty$, which together with $F \in N^+$, imply $F \in H^{1/2}(D)$ (cf. [14], p. 28). Thus $F(z) = \sum_{n=0}^{\infty} c_n z^n$ and $f(t) = \sum_{n=0}^{\infty} c_n e^{\text{int}}$. Form the previous remarks follows

Proposition 1. *Let $2 \leqslant q < \infty$, and let $F \in N^+$ be an holomorphic function in the disc with boundary values $F(e^{it}) = f(t) \in L^{q/2}(|t|^{q-2} \, dt)$, $F(x) = \sum_0^{\infty} c_n z^n$.*

(*a*) *If $\{n_k\}$ is a λ-lacunary sequence, then*

$$\left(\sum_k |c_{n_k}|^q n_k^{2-q}\right)^{1/q} \leqslant c_q \left(\int |f|^{q/2} |t|^{q-2} \, dt\right)^{2/q}.$$

(*b*) *If $\{n_k = k^r\}$, r fixed,*

$$\left(\sum_k |c_{k^r}|^q k^{2-(r+1)q}\right)^{1/q} \leqslant c_q \left(\int |f|^{q/2} |t|^{q-2} \, dt\right)^2.$$

(*c*) *If $\{n_k = 2^{\sqrt{k}}\}$,*

$$\left(\sum_k |c_{n_k}|^q n_k^{2-q} (\log_2 n_k)^{-q}\right)^{1/q} \leqslant c_q \left(\int |f|^{1/2} |t|^{q-2} \, dt\right)^{2/q}.$$

The norm ρ_q is non deterministic (see Section 2) since the Hilbert transform in continuous on every $L^q(\omega\,dt)$ for $1 < q < \infty$, $\omega \in A_q$ (the Muckenhoupt class), and, for $2 \leqslant q < \infty$, $\omega = |t|^{q-2} \in A_q$ (cf. [15]). A result dual to Theorem 3 and Proposition 1 is

Theorem 4. *Let* $\{n_k\}$ *be a* (γ_j)*-lacunary sequence,* $\{v_k\}$ *a given sequence,* $2 \leqslant q < \infty$. *There exists a function* θ, $\operatorname{Re}\theta \geqslant 0$, $\theta^2/\operatorname{Re}\theta \in L^{q/(q-2)}(dt/t^2)$, *such that*

$$\hat{\theta}(n) = v_k \quad \text{if} \quad n = n_k, = 0 \quad \text{if} \quad n \neq n_k, \quad \text{for} \quad n > 0,$$

and

$$\left| \int f \frac{\theta^2}{\operatorname{Re}\theta}\, dt \right| \leqslant c_q \Big(\sum_k |v_k|^{q'} n^{(q-2)/(q-1)} \Big| \max_{j \in B_k} \gamma_j \Big|^{q'} \Big)^{1/q'} \rho_q(f) \tag{26}$$

for all $f \in C(\mathbb{T})$, ρ_q *as in* (25).

PROOF. As in previous proofs, define for $n > 0$, $s(n) = v_k$ if $n = n_k$, $s(n) = 0$ if $n \neq n_k$, and $K(m, n) = s(m - n)$. By Lemma 4, $K \prec (r\sigma, r\sigma)$ with r, σ as in (23*a*), (25). By Corollary 3, there exists a function θ defined on $\mathbb{T}$, $\operatorname{Re}\theta \geqslant 0$, such that $\hat{\theta}(n) = s(n)$ for $n > 0$ and that (26) holds. From (26) it follows that $\theta(t)|t|^{2-q}$ belongs to the dual of $L^{q/2}(|t|^{q-2}\,dt)$, which is equivalent to $\theta \in L^{q/(q-2)}(dt/t^2)$. □

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Well-Posedness of the Euler and Navier-Stokes Equations in the Lebesgue Spaces $L^p_s(\mathbb{R}^2)$

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Dedicated to A. P. Calderón

Introduction

In this paper we show that the Euler equation for incompressible fluids in $\mathbb{R}^2$ is well posed in the (vector-valued) *Lebesgue spaces*

$$L^p_s = (1-\Delta)^{-s/2}L^p(\mathbb{R}^2) \quad \text{with} \quad s > 1 + 2/\mathrm{p}, \qquad 1 < p < \infty,$$

and that the same is true of the Navier-Stokes equation uniformly in the viscosity ν. The solutions obtained are classical since $L^p_s \subset C^1$. (The L^p_s are also called generalized Sobolev spaces, Liouville spaces, or spaces of Bessel potentials; see Calderón [4], Stein [16], Bergh-Löfström [3]. We use the notation L^p_s indiscriminately for scalar and vector valued functions.)

In well-posedness we include existence, uniqueness and persistence. (Persistence means that the state $u(t)$ of the fluid at time t belongs to the same function space X as does the initial state and describes a continuous curve in X.) The continuity of the map $u(0) \to u(t)$ in L^p_s is not considered here, however.

The standard norm for $u \in L^p_s$ is $\|(1-\Delta)^{s/2}u\|_p$, which may admit different interpretations in the vector case. But any equivalent norms are acceptable for our purposes.

The initial value problem for the Navier-Stokes equation in space domain $\mathbb{R}^m$, $m = 2, 3, \ldots$, may be written

$$\partial_t u - \nu \Delta u + (u \cdot \partial)u + \partial\pi = f \qquad (x \in \mathbb{R}^m, t > 0),$$
$$\operatorname{div} u = 0, \qquad u(x, 0) = \phi(x),$$

where $\partial_t = \partial/\partial t$, $\partial = \operatorname{grad} = (\partial_1, \ldots, \partial_m) = (\partial/\partial x_1, \ldots, \partial/\partial x_m)$; $u = u(x, t) = (u_1(x, t), \ldots, u_m(x, t))$ is the velocity field; $\pi = \pi(x, t)$ is the pressure; $\nu > 0$ is the kinematic viscosity; $(u \cdot \partial)u$ has j-component $u_k \partial_k u_j$ (with summation convention); and $f = f(x, t)$ is a given external force.

The initial value problem for the Euler equation is formally given by setting $\nu = 0$ in the Navier-Stokes problem.

In these equations, it is convenient to eliminate the pressure π by applying the projection operator P, which projects into solenoidal vectors and annihilates gradients, to transform them into

$$\text{(NS)} \qquad \partial_t u - \nu \Delta u + F(u) = f, \qquad Pu = u, \qquad Pf = f, \qquad u(0) = \phi.$$
$$\text{(E)} \qquad \partial_t u + F(u) = f, \qquad Pu = u, \qquad Pf = f, \qquad u(0) = \phi,$$

where

$$F(u) = F(u, u), \qquad F(u, v) = P(u \cdot \partial)v.$$

The pressure can then be recovered by

$$\partial\pi = -(1 - P)(u \cdot \partial)u,$$

and by subsequent integration of $\partial\pi$. There is no difficulty in the integration since $\partial\pi$ will be continuous and bounded in most cases we consider, but π will be determined only up to an additive constant depending on t, and π may be unbounded.

P is formally given by $P = (1 - \partial_j \partial_k \Delta^{-1})$, and is a pseudo-differential operator with a matrix symbol $(\delta_{jk} - \xi_j \xi_k / |\xi|^2)$; P is a continuous operator on L^p_s into itself, as is seen by Mihlin's theorem (see [13]). It is in fact a singular integral operator with a matrix kernel of Calderón-Zygmund type (see [5]) except for some delta-function components in diagonal elements.

The existence and uniqueness for the Navier-Stokes equation in $\mathbb{R}^2$ was proved by Leray [11]; here persistence holds with $X = PL^2$. The existence and uniqueness for the Euler equation in a domain $\Omega \subset \mathbb{R}^2$ (interior or exterior) was proved by Wolibner [18]; here it is assumed that the initial vorticity $\operatorname{rot} \phi$ is Hölder continuous and bounded by $\operatorname{const}(1 + |x|)^{-1-\epsilon}$, and persistence holds in the corresponding space for the velocity field. Analogous results were proved by Golovkin [6] and McGrath [12], in which the Navier stokes equation was also studied together with the convergence problem as $\nu \to 0$. The

function spaces used by the latter authors where more restricted than Wolibner's, and the persistence property was not obious in their cases. See also Yudovic [19] and Bardos [1] for related results in bounded domains in $\mathbb{R}^2$.

It is of some interest to find other function spaces X in which one has well-posedness for the Euler equation as well as for the Navier-Stokes equation with uniformity in ν. It is shown in Kato [9] that $X = PL^2_s$ has this property if $s > 2$. In [14] Ponce proves that $X = PL^p_s \cap L^2$ has the property if $2 \leqslant p < \infty$ and $s \geqslant 2$, except the case $s = p = 2$. In the present paper we shall extend these results to all $X = PL^p_s$ with $s > 1 + 2/p$, $1 < p < \infty$. It may be noted that if $p > 2$, the solution need not have a finite energy ($= L^2$-norm squared).

The plan of this paper is as follows. In section 1 we prove that (NS) is locally (in time) well-posed in $PL^p_s(\mathbb{R}^m)$ with $1 < p < \infty$, $s > 1 + m/p$ (Theorem I). In the remainder of the paper we assume $m = 2$, $s > 1 + 2/p$, and $f = 0$ for simplicity. In section 2 we prepare some lemmas on linearized vorticity equation derived from (NS), with a view to obtaining uniform estimates in ν. Using these results, we prove in section 3 that (NS) is globally well-posed in PL^p_s, with uniform estimates for $\nu > 0$ (Theorem II). In section 4, we prove the main theorem on the global well-posedness of (E) (Theorem III), together with a convergente theorem as $\nu \to 0$ (Theorem IV). In Appendix we prove some of the lemmas used in the proofs.

1. Local theory for the N-S equation

In this section we prove that the Navier-Stokes equation in $\mathbb{R}^m$ is locally (in time) well posed in the Lebesgue spaces PL^p_s (see introduction for the definition). In what follows we simply denote by L^p_s either $L^p_s(\mathbb{R}^m)$ or $L^p_s(\mathbb{R}^m; \mathbb{R}^m)$ (vector-valued functions), and by PL^p_s the subspace of the latter restricted by the divergence condition $\partial \cdot u = \partial_j u_j = 0$ (with summation convention). We denote the norm in L^p_s (scalar or vector-valued) by $\| \quad \|_{s,p}$.

The main result of this section is given by

Theorem I. *Let*

$$s > 1 + m/p, \qquad 1 < p < \infty. \tag{1.1}$$

Let $\phi \in PL^p_s$ and $f \in C([0, T_0]; PL^p_s)$. Then there is $T > 0$, $T \leqslant T_0$, and a unique solution to (NS) *such that*

$$u \in C([0, T]; PL^p_s) \cap C((0, T]; PL^p_{s+1}) \cap C^1((0, T]; PL^p_{s-1}). \tag{1.2}$$

Moreover, we have

$$\|u(t)\|_{s+1,p} \leqslant Kt^{-1/2}, \qquad \|\partial_t u(t)\|_{s-1,p} \leqslant Kt^{-1/2}. \tag{1.3}$$

T and K depend on m, s, p, $\nu > 0$, $\|\phi\|_{s,p}$ and the norm of f in $C([0, T_0]; L^p_s)$. In particular if $f = 0$, then $u \in C^\infty((0, T); L^p_\infty)$.

Remark 1.1. (*a*) The assumptions on ϕ and f can be considerably weakened if $u(t) \in L^p_s$ is not required to be continuous up to $t = 0$.

(*b*) In this theorem only a single value of p is involved. If $p > 2$, the solution $u(t)$ need not have a finite energy ($= L^2$-norm).

For the proof of Theorem I, we shall use the method of integral equation as in previous publications [7,10] (cf. also [8]). Since the proof is almost identical with (and simpler than) those given in these papers, it will suffice to indicate only the main points.

In this section we frequently write $\| \ \|_s$ for $\| \ \|_{s,p}$, since p will be fixed throughout. The operator norm from L^p_s into $L^p_{s'}$ will be denoted by $\| \ \|_{s;s'}$. Also we assume that $\nu = 1$ without loss of generality, since we keep $\nu > 0$ fixed.

First we convert (NS) into the integral equation

$$u(t) = u_0(t) + \int_0^t U(t - \tau)F(u(\tau))\, d\tau, \tag{1.4}$$

where

$$U(t) = \exp(t\Delta), \qquad u_0(t) = U(t)\phi + \int_0^t U(t - \tau)f(\tau)\, d\tau. \tag{1.5}$$

Lemma 1.2. $(U(t); t \geqslant 0)$ forms a contractive and analytic C_0-semigroup on L^p_s as well as on PL^p_s. Moreover, $U(t)$ is bounded for $t > 0$ from PL^p_s into $PL^p_{s'}$, with $s' > s$, with the operator norm

$$\|U(t)\|_{s;s'} \leqslant c(1 + t^{-(s'-s)/2}) \qquad (t > 0), \tag{1.6}$$

where c depends on $s' - s$ (in addition to m and p, which are fixed), but it may be taken as a constant as long as s and s' vary on a finite interval.

PROOF. The semigroup properties of $U(t)$ in L^p are well known. The same properties hold in L^p_s and PL^p_s because $U(t)$ commutes with $(1 - \Delta)^{s/2}$ and with P. For the same reason, we have

$$\|U(t)\|_{s;s'} = \|(1 - \Delta)^{(s'-s)/2}U(t)\|_{0;0}.$$

If $(s' - s)/2$ is an integer, (1.6) follows immediately from this. The general case can be dealt with by interpolation. (Or one may apply Mihlin's theorem directly to the symbol $(1 + |\xi|^2)^{(s'-s)/2}\exp(-t|\xi|^2)$.) □

The basic property of the nonlinear operator $F(u, v)$ is given by

Lemma 1.3. *If $u \in L^p_{s-1}$ and $v \in L^p_s$ with $s > 1 + m/p$, then $F(u, v) \in PL^p_{s-1}$ with*

$$\|F(u, v)\|_{s-1,p} \leqslant c\|u\|_{s-1,p}\|v\|_{s,p}. \tag{1.7}$$

PROOF. This is an immediate consequence of the fact that (the scalar-valued) L^p_{s-1} is a Banach algebra under pointwise multiplication (see Stein [16], Strichartz [17]). □

Lemma 1.4. *Let $G(u, v) = (1 - P)(u \cdot \partial)v$. If $u, v \in PL^p_s$ with $s > 1 + m/p$, then $G(u, v) = G(v, u) \in L^p_s$ with*

$$\|G(u, v)\|_{s,p} \leqslant c\|u\|_{s,p}\|v\|_{s,p}. \tag{1.8}$$

PROOF. Since $(u \cdot \partial)v - (v \cdot \partial)u$ has divergence zero, it is annihilated by $1 - P$, showing that $G(u, v) = G(v, u)$. This vector is a gradient, being in the range of $1 - P$. Writing $G(u, v) = \partial\pi$, we have $\Delta\pi = \partial \cdot (u \cdot \partial)v = \partial u \cdot \partial v$ ($= (\partial_j v_k) \cdot \cdot (\partial_k v_j)$) $\in L^p_{s-1}$, again due to L^p_{s-1} being an algebra. Since $\partial\pi$ and $\Delta\pi$ are both in L^p_{s-1}, it follows that $\partial\pi \in L^p_s$, with the desired inequality (1.8). □

Lemma 1.5. *The free term $u_0(t)$ on the right of* (1.4) *belongs to the class* (1.2) *with T replaced with T_0, and satisfies* (1.3).

PROOF. This follows easily from (1.5), (1.6); note that u_0 is the solution of the heat equation $\partial_t u_0 = \Delta u_0 + f$ with the initial condition $u_0(0) = \phi$. □

Lemma 1.6. *Under the assumptions of Theorem I, there is $T > 0$ and a unique solution of the integral equation* (1.4) *in the class*

$$u \in C([0, T]; PL^p_s). \tag{1.9}$$

PROOF. (1.4) is a regular integral equation of Volterra type, and can be solved by successive approximation, starting with $u_0(t)$ as the first approximation. Note that F maps $u(\tau) \in PL^p_s$ into PL^p_{s-1}, which is mapped back into PL^p_s by $U(t - \tau)$; the associated operator norm does not exceed $c(t - \tau)^{-1/2}$ by (1.6), which is integrable. The successive approximation converges on a certain interval $[0, T]$ and gives a unique solution to (1.4) in the class (1.9). For details cf. [10]. □

Lemma 1.7. *A solution u of the integral equation* (1.4) *in the class* (1.9) *is a solution of the differential equation* (NS) *in the class* (1.2). *The converse is also true.*

PROOF. The converse is easily proved by integrating

$$(\partial/\partial\tau)(U(t-\tau)u(\tau)) = U(t-\tau)(F(u(\tau)) + f(\tau))$$

in $\tau \in [0, t]$.

To prove the first part, let u be a solution of (1.4) considered and let $v(t)$ denote the second term on the right of (1.4). Then $\|F(u(t))\|_{s-1} \leqslant K = \text{const}$ and so $\|v(t)\|_{s+1/2} \leqslant Kt^{1/4} \leqslant K$ because $\|U(t-\tau)\|_{s-1;s+1/2} \leqslant ct^{-3/4}$. Since $\|u_0(t)\|_{s+1/2} \leqslant Kt^{-1/4}$, as is seen from (1.5) and (1.6), the same is true for $u(t)$ and therefore $\|F(u(t))\|_{s-1/2} \leqslant Kt^{-1/4}$ by (1.7). Hence $\|v(t)\|_{s+1} \leqslant$ $\leqslant K\int_0^t (t-\tau)^{-3/4}\tau^{-1/4}\,d\tau \leqslant K$, so that $\|u(t)\|_{s+1} \leqslant Kt^{-1/2}$. With a little argument about continuity, this shows that v, and hence u too, belongs to $C((0, T]: PL^p_{s+1})$. Since the validity of the differential equation (NS) in a lower level, say in PL^p_{s-2} is obvious, it follows that $\partial_t u \in C((0, T]; L^p_{s-1})$, completing the proof of Lemma 1.7. □

PROOF OF THEOREM I. The main part of Theorem I follows from Lemmas 1.6 and 1.7. To prove the last assertion for $f = 0$, we repeat the bootstrap argument given above to show that $u \in C([0, T]; PL^p_s) \cap C((0, T); L^p_\infty)$. The differentiability of $u(t)$ in $t > 0$ then follows by repeated differentiation of (NS) in t.

2. The linearized vorticity equation in $\mathbb{R}^2$

From now on we assume $m = 2$. We use the following notations. $\|\ \|_{s,p}$ for the L^p_s-norm; $\|\ \|_p$ for $\|\ \|_{0,p}$; $\|\ \|_{s,p;r,q}$ for the operator norm from L^p_s to L^q_r; $\|\ \|_{p;q}$ for $\|\ \|_{0,p;0,q}$.

In this section we prove some lemmas on the linearized vorticity equation

$$\partial_t\zeta - \nu\Delta\zeta + a(t,x)\cdot\partial\zeta = 0, \qquad \zeta(0) = \omega, \tag{2.1}$$

where a is a given vector function such that

$$a \quad \text{and} \quad \partial a \in C([0, T]; L^\infty), \qquad \partial\cdot a = 0. \tag{2.2}$$

Lemma 2.1. *Let $1 < p < \infty$ and $0 \leqslant k \leqslant 1$. If $\omega \in L^p_k$, then* (2.1) *has a unique solution $\zeta \in C([0, T]; L^p_k) \cap C((0, T]; L^p_{k+1}) \cap C^1((0, T]; L^p_{k-1})$, with*

$$\|\zeta(t)\|_p \leqslant \|\omega\|_p, \tag{2.3}$$

$$\|\zeta(t)\|_{k,p} \leqslant c\|\omega\|_{k,p}\exp\left(c\int_0^t \|\partial a(\tau)\|_\infty\,d\tau\right). \tag{2.3'}$$

Here and in what follows c denotes various constants that may depend on s, p, but not on ν or ω.

PROOF. (2.1) is a simple linear scalar equation of parabolic type and is easily solved. One may use the method of integral equation similar to the one used in the proof of Theorem I, to establish existence and uniqueness of the solution required by the theorem. The proof is simpler than before inasmuch as the equation is linear, the unknown ζ is scalar-valued, and there is no projection P involved. Due to the linearity, on the other hand, the smoothness of the solution depends on a. But $a(t)$ maps L^p_k into itself (from vectors to scalars), and this suffices for the proof.

It remains to prove the estimates (2.3) and (2.3′). First we prove (2.3) for $2 \leqslant p < \infty$. Let $J\zeta = \zeta^{p-1} \equiv |\zeta|^{p-2}\zeta$ be the duality map on L^p to its dual $L^{p'}$, normalized in such a way that $\|J\zeta\|_{p'} = \|\zeta\|_p^{p-1}$. It is easy to see that J not only maps L^p into $L^{p'}$ but also maps L^p_1 into $L^{p'}_1$, continuously in both cases. Since $\partial_t \zeta(t) \in L^p_{-1}$ for $t > 0$, we obtain

$$\begin{aligned}\partial_t \|\zeta(t)\|_p^p &= p\langle J(\zeta(t)), \partial_t \zeta(t)\rangle \\ &\leqslant -\nu p\langle \partial\zeta(t)^{p-1}, \partial\zeta(t)\rangle - p\langle \zeta(t)^{p-1}, a(t)\cdot\partial\zeta(t)\rangle \\ &\leqslant -p(p-1)\nu \int |\zeta(t)|^{p-2}|\partial\zeta(t)|^2\,dx \leqslant 0,\end{aligned}$$

where integrations by parts have been made using the property $\partial \cdot a = 0$. Thus $\|\zeta(t)\|_p$ is monotone decreasing, proving (2.3) for $2 \leqslant p < \infty$.

This result may be conveniently expressed by using the evolution operator $U_\nu(t, \tau)$ associated with the linear equation (2.1). Thus

$$\|U_\nu(t, \tau)\|_{p;p} \leqslant 1. \tag{2.4}$$

Then we obtain $\|U_\nu(t, \tau)\|_{p';p'} \leqslant 1$ by duality; indeed, the adjoint equation to (2.1) has the same form, with the sign of a reversed, due to the condition $\partial \cdot a = 0$. This proves (2.4) and (2.3) for $1 < p < \infty$.

We note in passing that (2.3) had been proved in [12] using the fundamental solution of (2.1).

Next we prove (2.3′) for $k = 1$. On differentiating (2.1) in x, we obtain a system of equations for $\partial\zeta$:

$$\partial_t \partial\zeta - \nu\Delta\partial\zeta + a\cdot\partial\partial\zeta + \partial a\cdot\partial\zeta = 0, \qquad \partial\zeta(0) = \partial\omega.$$

Application of Duhamel's formula then leads to the integral equation

$$\partial\zeta(t) = U_\nu(t, 0)\partial\omega - \int_0^t U_\nu(t, \tau)\partial a\cdot\partial\zeta(\tau)\,d\tau. \tag{2.5}$$

Using (2.4), we thus obtain the integral inequality

$$\|\partial\zeta(t)\|_p \leqslant \|\partial\omega\|_p + c\int_0^t \|\partial a(\tau)\|_\infty \|\partial\zeta(\tau)\|_p\,d\tau. \tag{2.6}$$

Combined with the previous estimate for $k = 0$, this leads to the estimate (2.3′) for $k = 1$. The constant c in (2.3′) depends on the way $\|\zeta\|_{1,p}$ depends on $\|\zeta\|_p$ and $\|\partial\zeta\|_p$.

Finally the case $0 < k < 1$ can be dealt with by interpolation (see [3]).

The result can again be expressed in terms of the evolution operator:

$$\|U_\nu(t,\tau)\|_{k,p;k,p} \leqslant c\exp\left(c\int_\tau^t \|\partial a(\rho)\|_\infty d\rho\right), \qquad 0 \leqslant k \leqslant 1. \quad \square \tag{2.6a}$$

Lemma 2.2. *In Lemma* 2.1 *assume that* $a \in C([0,T]; L^q_r)$, $1 < q < \infty$, $r > 1 + 2/q$. Let $1 < p \leqslant q$, $0 \leqslant k \leqslant [r]$. *If* $\omega \in L^p_k$, *then the solution* ζ *given by Lemma* 2.1 *belongs to* $C([0,T]; L^p_k)$ *with*

$$\|\zeta(t)\|_{k,p} \leqslant c\|\omega\|_{k,p}\exp\left(c\int_0^t \|a(\tau)\|_{r,q}\, d\tau\right). \tag{2.7}$$

PROOF. For $k = 0$, this is contained in Lemma 2.1; note that the assumption implies (2.2). Next consider the case when $k = [r]$. We differentiate (2.1) k-times, obtaining

$$\partial_t\partial^k\zeta - \nu\Delta\partial^k\zeta + a\cdot\partial\partial^k\zeta = -\sum_{j=1}^{k} c_j\partial^j a\cdot\partial^{k-j+1}\zeta$$

with $\partial^k\zeta(0) = \partial^k\omega$. Corresponding to (2.5) and (2.6) in the proof of Lemma 2.1, this leads to

$$\partial^k\zeta(t) = U_\nu(t,0)\partial^k\omega - \int_0^t U_\nu(t,\tau)\sum_{j=1}^{k} c_j\partial^j a(\tau)\cdot\partial^{k-j+1}\zeta(\tau)\, d\tau$$

and

$$\begin{aligned}\|\partial^k\zeta(t)\|_p &\leqslant \|\partial^k\omega\|_p + c\int_0^t\sum_{j=1}^{k}\|\partial^j a(\tau)\cdot\partial^{k-j+1}\zeta(\tau)\|_p\, d\tau\\ &\leqslant \|\omega\|_{r,p} + c\int_0^t \|a(\tau)\|_{r,q}\|\zeta(\tau)\|_{k,p}\, d\tau.\end{aligned} \tag{2.8}$$

Here the Sobolev embedding theorem is used at the last stage (for details see Appendix). Since the same estimate holds if $\partial^k\zeta$ on the left is replaced by lower derivatives, we have a linear integral inequality for $\|\zeta(t)\|_{k,p}$ and obtain the desired result (2.7).

Since the same result holds for $k = 0$, the required result for $0 \leqslant k \leqslant [r]$ follows by interpolation. The result may again be expressed in terms of the evolution operator:

$$\|U_\nu(t,\tau)\|_{k,p;k,p} \leqslant c\exp\left(\int_\tau^t \|a(\rho)\|_{r,q}\, d\rho\right). \tag{2.7a}$$

3. Global theory for space domain $\mathbb{R}^2$

In this section we prove that (NS) is globally well-posed, uniformly for $\nu > 0$, with persistence in $PL^p_s = PL^p_s(\mathbb{R}^2)$. For simplicity we assume that $f = 0$, but the general case can be handled similarly.

Theorem II. *If $m = 2$, the solution u given by Theorem I can be continued to all $t > 0$, with the estimate*

$$\|u(t)\|_{s,p} \leqslant K(t). \tag{3.1}$$

Here and in what follows $K(t)$ denotes a real-valued continuous function defined for $0 \leqslant t < \infty$, depending on s, p, and $\|\phi\|_{s,p}$ but not on $\nu > 0$.

For the proof we need a lemma, which partially generalizes analogous results given in [2, 6, 9, 15]. (All these results contain a logarithmic function.)

Lemma 3.1. *Let $u \in PL^p_s$, $1 < p < \infty$, $s > 1 + 2/p$. Let $\zeta = \operatorname{rot} u \equiv \partial_1 u_2 - \partial_2 u_1$. Then*

$$\|\partial u\|_{s-1,p} \leqslant c\|\zeta\|_{s-1,p'} \tag{3.2a}$$

$$\|\partial u\|_\infty \leqslant c\|\zeta\|_p + c\|\zeta\|_\infty \log_+ (\|\zeta\|_{s-1,p}/\|\zeta\|_\infty), \tag{3.2b}$$

with c depending only on s and p. (Here we use the notation $\log_+(\lambda) = \max\{1, \log(\lambda)\} \geqslant 1$, slightly deviating from the commons usage.)

The proof of this lemma will be given in Appendix, in a generalized from valid in $\mathbb{R}^m$.

PROOF OF THEOREM II. First we assume that $2/p < s - 1 \leqslant 1$. As is well known, $\zeta = \operatorname{rot} u$ satisfies the vorticity equation (2.1) in which $a = u$; note that $u(t)$ and $\partial u(t)$ are in L^∞ because $u(t) \in L^p_s$ with $s > 1 + 2/p$. Thus we have by Lemma 2.1

$$\|\zeta(t)\|_{s-1,p} \leqslant c\|\omega\|_{s-1,p} \exp\left(c\int_0^t \|\partial u(\tau)\|_\infty d\tau\right). \tag{3.3}$$

Now we apply (3.2*b*) to the solution $u = u(t)$, $\zeta = \zeta(t) = \operatorname{rot} u(t)$. Here we may replace $\|\zeta(t)\|_p$ and $\|\zeta(t)\|_\infty$ respectively by their majorants $\|\omega\|_p$ and $\|\omega\|_\infty \leqslant c\|\omega\|_{s-1,p}$, since $\mu \log_+(\lambda/\mu)$ is monotone increasing in μ. ($\|\zeta(t)\|_\infty \leqslant \|\omega\|_\infty$ follows from (2.3) in the limit $p \to \infty$.) Thus

$$\|\partial u(t)\|_\infty \leqslant c\|\omega\|_p + c\|\omega\|_{s-1,p} \log_+(\|\zeta(t)\|_{s-1,p}/\|\omega\|_{s-1,p}).$$

Substitution from (3.3) then gives

$$\|\partial u(t)\|_\infty \leqslant c\|\omega\|_p + c\|\omega\|_{s-1,p}\Big(1 + \int_0^t \|\partial u(\tau)\|_\infty d\tau\Big).$$

In view of the fact that $\|\omega\|_p$ and $\|\omega\|_{s-1,p}$ are dominated by $c\|\phi\|_{s,p}$, aplication of the Gronwall inequality then shows that

$$\|\partial u(t)\|_\infty \leqslant K(t). \tag{3.4}$$

Another application of (3.3) then shows that $\|\zeta(t)\|_{s-1,p} \leqslant K(t)$. Then, using (3.2*a*) we conclude that

$$\|\partial u(t)\|_{s-1,p} \leqslant K(t). \tag{3.5}$$

In order to deduce (3.1) from (3.5), we need a uniform estimate for $\|u(t)\|_p$. This may be obtained from (1.4) by noting that (see (2.4)) $\|U_\nu(t)\|_{p;p} \leqslant 1$, where we have restored the parameter ν, writing $U_\nu(t) = \exp(\nu t\Delta)$. Since $\|F(u(\tau))\|_p \leqslant c\|u(\tau)\|_p\|\partial u(\tau)\|_\infty \leqslant K(\tau)\|u(\tau)\|_p$ by (3.4), we obtain

$$\|u(t)\|_p \leqslant \|\phi\|_p + \int_0^t K(\tau)\|u(\tau)\|_p d\tau,$$

which gives

$$\|u(t)\|_p \leqslant K(t). \tag{3.6}$$

Now (3.1) follows from (3.5) and (3.6) by virtue of the following lemma, which will be proved in Appendix in the general case of $\mathbb{R}^m$.

Lemma 3.2. *Let $v \in L^p$ with $\partial v \in L^p_{s-1}$, where $s \geqslant 1$, $1 < p < \infty$. Then $v \in L^p_s$, with $\|v\|_{s,p} \leqslant c(\|v\|_p + \|\partial v\|_{s-1,p})$.*

Of course the various estimates given above hold only in the interval on which u is known to exist. But the $K(t)$ are functions defined for $0 \leqslant t < \infty$, depending on s, p, $\|\phi\|_{s,p}$ but not on ν; in fact they can be given explicitly in terms of an exponential function exponentiated again and again. Thus a standard argument based on the local existence theorem shows that $u(t)$ can be extended step by step to all $t > 0$ satisfying (3.1). This proves Theorem II when $2/p < s - 1 \leqslant 1$.

Next we consider the case $2 < s \leqslant 3, s > 1 + 2/p$. Let q be a number such that $q > 2$ and $0 \leqslant 2/p - 2/q \leqslant s - 2$. (We may set $q = p$ if $p > 2$; and $2/q = 2/p - (s-2)$ if $p \leqslant 2$ and $s < 3$; in the exceptional case $p = 2$, $s = 3$, any $q > 2$ will do.) Then we have $\phi \in L^p_s \subset L^q_2$ by the Sobolev embedding theorem. Since $2/q < 2 - 1 = 1$, the preceding result shows that $u(t) \in L^q_2$ exists for all time, with $\|u(t)\|_{2,q} \leqslant K(t)$. Application of Lemma 2.2 with

$a = u$, $r = 2$, $k = s - 1 \leqslant r$, $q \geqslant p$ shows that $\|\zeta(t)\|_{s-1,p} \leqslant K(t)$, which leads to (3.5) by (3.2*a*). Using Lemma 3.2 again, we obtain (3.1).

Next consider the case $3 < s \leqslant 4$. Then $\phi \in L^p_s \subset L^p_3$ with $3 > 1 + 2/p$. Hence the preceding result shows that $\|u(t)\|_{3,p} \leqslant K(t)$. Application of Lemma 2.2 with $a = u$, $q = p$, $r = 3$, $k = s - 1 \leqslant r$ shows that $\|\zeta\|_{s-1,p} \leqslant K(t)$, which again leads to (3.5), and then to (3.1) in the same way as above. The case with larger s can be dealt with similarly. This completes the proof of Theorem II. □

4. Vanishing viscosity and the Euler equation in $\mathbb{R}^2$

We now prove the main theorem for the Euler equation in $\mathbb{R}^2$, together with the convergence of viscous solutions to the ideal one. As before we assume that $f = 0$ for simplicity.

Theorem III. *Let $s > 1 + 2/p$, $1 < p < \infty$. If $\phi \in PL^p_s$, the Euler equation* (E) *has a unique solution $u \in C([0, \infty); PL^p_s)$ with $u(0) = \phi$.*

Theorem IV. *Let ϕ be as in Theorem III. Let u^ν be the solution of* (NS) *given by Theorem II. As $\nu \to 0$, u^ν converges to the solution u given by Theorem III in the following sense: $u^\nu(t) \to u(t)$ weakly in PL^p_s for all $t \geqslant 0$, locally uniformly in t.*

Remark 4.1. Theorem IV is rather weak inasmuch as it gives only weak convergence. A stronger result could be proved, but we shall not pursue the problem here. Note, however, that weak convergence of a sequence u_n in $L^p_s(\mathbb{R}^m)$ implies (spatially pointwise) local uniform convergence of $\partial^k u_n$ for $k < s - m/p$.

We prove these theorems in several steps.

Lemma 4.2. *As $\nu \to 0$, $u^\nu(t)$ converges along a subsequence weakly in L^p_s, (locally) uniformly in t, to a solution $u(t)$ of* (E) *such that*

$$u \in \tilde{C}([0, \infty); L^p_s), \tag{4.0}$$

which means that $u(t)$ is weakly continuous in L^p_s and strongly continuous in $L^p_{s'}$ for all $s' < s$.

PROOF. The proof is standard, and will be only sketched. It depends on the fact that the map F is weakly continuous from L^p_s into L^p_{s-1}, as is easily verified.

Since by Theorem II the $u^\nu(t)$ are locally (in time) uniformly bounded in L^p_s, which is reflexive, we can pick a subsequence of u^ν that converges weakly on a countable dense subset of t.

Since $\partial_t u^\nu(t) = \nu\Delta u^\nu - F(u^\nu(t))$ is (locally) uniformly bounded in L^p_{s-2}, $\partial_t\langle u^\nu(t), \varphi\rangle$ is uniformly bounded for each test function $\varphi \in \mathfrak{D}(\mathbb{R}^2)$. Hence $\langle u^\nu(t), \varphi\rangle$ is (locally) equicontinuous, so that it converges for all t (locally) uniformly. In view of the (local) uniform boundedness of $u^\nu(t)$ in L^p_s, it follows that $u^\nu(t)$ converges weakly in L^p_s, (locally) uniformly in t. The limit $u(t) \in L^p_s$ is weakly continuous, and it is easy to see that u is a weak solution of (E).

Then $\partial_t u(t) = -F(u(t))$ is weakly continuous in L^p_{s-1} by the remark given above. It follows that $u(t)$ is strongly continuous in L^p_{s-1}, hence strongly continuous in each $L^p_{s'}$ with $s' < s$.

Lemma 4.3. *The solution* (4.0) *of* (E) *is unique.* (*Hence the original sequence* u^ν *converges to* u *weakly, without going over to a subsequence.*)

PROOF. Suppose there are two solutions u, u' to (E). On subtraction and setting $w = u' - u$, we obtain

$$\partial_t w + F(u', w) + F(w, u) = 0.$$

Hence, suppressing the variable t for simplicity,

$$\partial_t \|w\|_p^p = -p\langle Jw, F(u', w\rangle + F(w, u)\rangle; \tag{4.1}$$

here we use the norm $\|w\|_p = (\|w_1\|_p^p + \|w_2\|_p^p)^{1/p}$, and $Jw = (Jw_1, Jw_2)$ is the vector-valued version of the duality map used in the proof of Lemma 2.1. In (4.1) we have $\|F(w, u)\|_p \leqslant K\|w\|_p$ because $\|\partial u\|_\infty \leqslant K$, contributing a term $K\|w\|_p^p$ to the right of (4.1). For the other term, we have

$$F(u', w) = (u' \cdot \partial)w - G(u', w) = (u' \cdot \partial)w - G(w, u'), \tag{4.2}$$

by Lemma 1.4. The contribution of $(u' \cdot \partial)w$ to (4.1) is zero, as in seen by the standard integration by parts using $\partial \cdot u' = 0$. On the other hand, $G(w, u')$ is dominated in L^p-norm by $K\|w\|_p$ because $\|\partial u'\|_\infty \leqslant K$. Altogether we obtain

$$\partial_t \|w\|_p^p \leqslant K\|w\|_p^p,$$

from which $w = 0$ follows if $w(0) = 0$, completing the proof of uniqueness.

To complete the proof of the theorems, we have to strengthen $\tilde{C}([0, T]; L^p_s)$ in (4.0) into $C([0, T]; L^p_s)$. For this purpose, we consider the linear vorticity equation without viscosity:

$$\partial_t \zeta + a \cdot \partial\zeta = 0, \qquad \zeta(0) = \omega. \tag{4.3}$$

Lemma 4.4. *Let $a \in \tilde{C}([0, T]; L^q_r)$, where $r > 1 + 2/q$, $1 < q < \infty$. If $\omega \in L^p_k$ with $0 \leqslant k < [r]$ and $p \leqslant q$, then there is a unique solution to* (4.3) *such that*

$$\zeta \in C([0, T]; L^p_k). \tag{4.4}$$

PROOF. Since the assumption implies that $a \in C([0, T]; C^1(\mathbb{R}^2))$, the characteristic curves for the first-order equation (4.3) are well defined. The uniqueness of the solution of (4.3) follows immediately from this.

To prove the existence, we regard (4.3) as the limit of the vorticity equation (2.1ν), which is the equation (2.1) with a replaced by a function $a^\nu \in C([0, T]; L^q_r)$ that depends on ν, in such a way that $a^\nu \to a$ in $\tilde{C}([0, T]; L^q_r)$ (which means weak convergence in L^q_r and strong convergence in $L^q_{r'}$ for $r' < r$). Such a sequence a^ν is easily constructed by regulazation of a in time.

Let ζ^ν be the solution of (2.1ν), with $\zeta^\nu(0) = \omega \in L^p_k$. Lemma 2.2 shows that ζ^ν belongs to $C([0, T]; L^p_k)$ and satisfies (2.7ν), which is the inequality (2.7) with a replaced by a^ν. Using the same argument given above to construct u from u^ν, we obtain a solution

$$\zeta \in \tilde{C}([0, T]; L^p_k) \tag{4.5}$$

as the weak limit of ζ^ν. (Recall that $a^\nu \to a$ strongly in L^p_{k-1}.)

Since ζ^ν satisfies (2.7ν), ζ satisfies an inequality of the form

$$\|\zeta\|_{k,p} \leqslant c\|\omega\|_{k,p} \exp\left(c \int_0^t \alpha(\tau)\, d\tau\right), \tag{4.6}$$

where $\alpha(t) = \limsup_{\nu \to 0} \|a^\nu(t)\|_{q,r}$ is an L^∞-function.

It remains to replace $\tilde{C}([0, T]; L^p_s)$ by $C([0, T]; L^p_s)$ in (4.5). To this end let k' be a number such that $k < k' < [r]$, and let $\omega_n \in L^p_{k'}$ be a sequence such that $\omega_n \to \omega$ in L^p_k. If ζ_n is the solution of (4.3) with $\zeta_n(0) = \omega_n$, (4.6) is true with ζ and ω replaced by $\zeta_n - \zeta$ and $\omega_n - \omega$, respectively. It follows that $\zeta_n(t)$ tends to $\zeta(t)$ strongly in L^p_k, uniformly in t. Since ζ_n belongs to $\tilde{C}([0, T]; L^p_{k'}) \subset C([0, T]; L^p_k)$ by $k' > k$, it follows that $\tilde{C}$ can be replaced by C in (4.5).

COMPLETION OF THE PROOF OF THE THEOREMS. We have to show that in (4.0), $\tilde{C}$ can be replaced by C. Since u is a solution of (E), $\zeta = \operatorname{rot} u$ satifies the vorticity equation (4.3) in which $a = u$, $\omega = \operatorname{rot} \phi \in L^p_{s-1}$. In view of (4.0), we can set $r = s$, $q = p$, $k = s - 1$ in Lemma 4.2, obtaining the result that $\zeta \in C([0, \infty); L^p_{s-1})$. As in the proof of Theorem II, this implies $\partial u \in C([0, \infty); L^p_{s-1})$ and, using the known fact that $u \in C([0, \infty); L^p)$, we obtain the desired result $u \in C([0, \infty); L^p_s)$ by Lemma 3.2.

Remark 4.5. Lemma 4.4 could be proved by integrating the vector field a to yield a family of diffeomorphisms Φ_t such that $\Phi_t - id$ is in the class

$C([0, T]; L^q_r)$, and then showing that the composition $\omega \circ \Phi_t$ is in $C([0, T]; L^p_k)$. It seems to the authors that such results are not readily available in the literature.

Appendix

1. *Proof of Lemma* 3.1.

We prove Lemma 3.1 in $\mathbb{R}^m$, assuming that $s > 1 + m/p$. In this case $\zeta = \operatorname{rot} u$ is a skew-symmetric tensor given by $\zeta_{jk} = \partial_j u_k - \partial_k u_j$. If $u \in PL^p_s$, ∂u can be recovered from $\zeta \in L^p_{s-1}$ formally by

$$\partial_j u_k = \partial_j \partial_i \Delta^{-1} \zeta_{ik}. \tag{A1}$$

Here $\partial_j \partial_i \Delta^{-1}$ is a pseudo-differential operator with symbol $\xi_j \xi_i |\xi|^{-2}$, so that the operation $\zeta \to \partial u$ is continuous in L^p, hence in L^p_{s-1}, for any s and $1 < p < \infty$ (Mihlin's theorem). This proves (3.2*a*).

Incidentally, the operator considered here is formally identical with G (see Lemma 1.4), with the only difference that it acts on tensors to tensors, while G acts on vector to vectors.

The proof of (3.2*b*) requires a deeper analysis. We can write (A1) in the form

$$\partial_j u_k = m^{-1} \zeta_{jk} + g_{ji} * \zeta_{ik} \qquad (* = \text{convolution}), \tag{A2}$$

where the $g_{jk} = |x|^{-m} \Omega_{jk}$ are Kernels of Calderón-Zygmund type (see [4]), with $\Omega_{jk}(x)$ homogeneous of degree zero, smooth in x and having zero average on the unit sphere. Thus it suffices to prove (3.2*b*) for

$$v(x) = g * \zeta(x) = \lim_{\epsilon \to 0} \int_{|y| > \epsilon} g(y) \zeta(x - y)\, dy, \tag{A3}$$

where g is one of the g_{jk} and ζ is one of the ζ_{jk}. To this end we split the integral into three parts, according to (i) $|y| > 1$, (ii) $\delta < |y| \leqslant 1$, (iii) $\epsilon < |y| \leqslant \delta$.

The contribution to the integral from (i) is estimated by $c \| \zeta \|_p$, since $|g(y)| \leqslant c|y|^{-m}$ and $p < \infty$. Using the same estimate for g, the contribution from (ii) is seen to be dominated y $c \| \zeta \|_\infty \log(\delta^{-1})$. To deal with the part (iii), we replace $\zeta(x - y)$ with $\zeta(x - y) - \zeta(x)$ without affecting the integral, since $g(y)$ has average zero on any sphere. But $\zeta \in L^p_{s-1}$ implies that ζ is Hölder continuous:

$$|\zeta(x - y) - \zeta(x)| \leqslant c|y|^\theta \| \zeta \|_{s-1,p}, \qquad 0 < \theta < 1.$$

Therefore the contribution from (iii) does not exceed $c\delta^\theta \|\zeta\|_{s-1,p}$. Altogether we obtain

$$\|v\|_\infty \leqslant c(\|\zeta\|_p + \log(\delta^{-1})\|\zeta\|_\infty + \delta^\theta \|\zeta\|_{s-1,p}) \tag{A4}$$

for any $\delta \in (0,1]$. The desired result (3.2*b*) then follows on setting $\delta^{-\theta} = \max(1, \|\zeta\|_{s-1,p}/\|\zeta\|_\infty)$. This proves (3.2*b*). □

2. *Proof of Lemma* 3.2

$\partial v \in L^p_{s-1}$ implies that $\partial^k v \in L^p_r \subset L^p$, $r = s - k$, for $1 \leqslant k \leqslant [s]$. Since $v \in L^p$ too by hypothesis, we have $v \in L^p_{[s]} \subset L^p_{s-1}$. Thus v and ∂v are both in L^p_{s-1}, from which $v \in L^p_s$ follows together with the required estimate.

3. *Proof of* (2.8)

We have to show that

$$(\partial^j a)(\partial^{k-j+1}\zeta) \in L^p, \qquad 1 \leqslant j \leqslant k \leqslant [r]. \tag{A5}$$

under the assumptions

$$a \in L^q_r, \qquad \zeta \in L^p_k, \quad \text{where} \quad r > 1 + 2/q, \qquad 1 < p \leqslant q < \infty. \tag{A6}$$

First we note that (A6) implies

$$\partial^j a \in L^q_{r-j}, \qquad \partial^{k-j+1}\zeta \in L^p_{j-1}. \tag{A7}$$

(*a*) If $j < r - 2/q$, then (A7) implies that $\partial^j a \in L^\infty$ and $\partial^{k-j+1}\zeta \in L^p$, from which (A5) follows immediately.

(*b*) If $j \geqslant r - 2/q$, then $j > 1$. Let $\epsilon > 0$ be such that $r - 1 = (2 + \epsilon)/q$. Then

$$1/p \geqslant 1/q = (r - j)/(2 + \epsilon) + (j - 1)/(2 + \epsilon). \tag{A8}$$

Thus (A7) implies

$$\partial^j a \in L^\tau \quad \text{with} \quad 1/\tau = 1/q - (r - j)/(2 + \epsilon) > 0,$$
$$\partial^{k-j+1}\zeta \in L^\sigma \quad \text{with} \quad 1/\sigma = 1/p - (j - 1)/(2 + \epsilon) > 0,$$

except when $p = q$, $j = r = k$. Since $1/\tau + 1/\sigma = 1/p$ by (A8), we obtain (A5) in the nonexceptional case. In the exceptional case we have $\partial^j a \in L^p$ and $\partial^{k-j+1}\zeta \in L^p_{r-1} \in L^\infty$ because $r - 1 > 2/q = 2/p$. Thus we again obtain (A5).

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Fourier Analysis in Several Parameters

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Dedicated to my dear friend and teacher, Professor Alberto P. Calderón

Clearly, one of the most basic contributions to the fields of real variables, partial differential equations and Fourier analysis in recent times has been the celebrated theorem of Calderón and Zygmund on the boundedness of singular integrals on R^n [1]. One way to state this result is as follows:

Theorem. *Suppose T is a bounded linear operator on $L^2(R^n)$. Suppose further that whenever $a(x)$ is a function supported on a cube Q satisfying $\int_Q a(x)\,dx = 0$, then T satisfies*

$$\int_{c\tilde{Q}} |T(a)(x)|\,dx \leqslant C\|a\|_{L^1(Q)},$$

where $\tilde{Q}$ denotes the double of Q. Then T is of weak type on $L^1(R^n)$, i.e. for each $\alpha > 0$ and $f \in L^1(R^n)$,

$$|\{x \in R^n \mid |T(f)(x)| > \alpha\}| \leqslant \frac{C}{\alpha}\|f\|_{L^1(R^n)}.$$

From here, as everyone knows, by interpolation, we may obtain L^p results when $1 < p \leqslant 2$. The importance of this result goes beyond what is apparent from the statement of the theorem. Rather, in the proof, we clearly see the role of the Hardy-Littlewood Maximal operator, and how this operator controls classical singular integrals. In a great number of problems in higher dimensions, the Hardy-Littlewood operator no longer suffices to control the operator at hand, and one is forced to deal with more complicated maximal functions. Let us give some basic examples from the theory of Fourier integrals in R^n. In each case, just as in the classical Calderón-Zygmund case, a particular

maximal operator will be considered which controls the singular integral in question. Somewhat later, we shall consider the question of how to prove estimates on these maximal operators.

If $f(x)$ is a function on R^n with Fourier transform $\hat{f}(\xi)$, $\xi \in R^n$, then in some sense the central concern of Fourier analysis should be to determine in which sense we have

$$f(x) = \int_{R^n} \hat{f}(\xi) e^{i\xi \cdot x}\, d\xi.$$

There are a number of ways in which we can form the partial sum operators corresponding to this Fourier integral. First we can take «rectangular partial sums» given by

$$S_{r_1, r_2, \ldots, r_n} f(x) = \int_{\substack{|\xi_i| < r_i \\ i = 1, 2, \ldots, n}} \hat{f}(\xi) e^{i\xi \cdot x}\, d\xi.$$

It is easy to see that for many purposes, just as in one dimension the partial sum operators are controlled by the Hilbert transform, the rectangular partial sums are best understood in terms of the multiple Hilbert transform, $H_1 H_2 \cdots H_n$ given by

$$H_1 H_2 \cdots H_n f = f * \frac{1}{x_1 x_2 \cdots x_n}.$$

This singular integral, although bounded on all the spaces $L^p(R^n)$ for $1 < p < \infty$ is very different from the classical Calderón-Zygmund operators on R^n. The kernel of this multiparameter singular integral has singularities on the coordinate hyperplanes, unlike the Calderón-Zygmund kernels, which are smooth away from the origin. As a result, the singular integrals like the multiple Hilbert transform are not weak type on $L^1(R^n)$. The maximal operator which is relevant to the study of operators like $H_1 H_2 \cdots H_n$ is the Jessen-Marcinkiewicz-Zygmund strong maximal function, $M_S f$, defined at a point $x \in R^n$ by

$$M_S f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where the sup is taken over all rectangles $R \subseteq R^n$ whose sides are parallel to the coordinate axes and contain x. As we would expect, M_S is not weak type on L^1, but rather satisfies

$$|\{x \in R^n |\ |x| < 1, M_S f(x) > \alpha\}| \leqslant \frac{C}{\alpha} \|f\|_{L(\log L)^{n-1}(|x| < 1)}.$$

We shall consider singular integrals and maximal functions like $H_1H_2 \cdots H_n$ and M_S later in this article. They are important because in some sense they are the simplest examples which commute with a multiple parameter family of dilations, and hence are the best understood. Next, suppose we consider the spherical partial sum operators:

$$S_r(f)(x) = \int_{|\xi| < r} \hat{f}(\xi) e^{i\xi \cdot x}\, d\xi, \quad \text{where} \quad |\xi| = \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2}.$$

Although one might be tempted to guess, from the presence of the single parameter r, that the convergence of the $S_r f$ in L^p should be a consequence of the theory of the classical Calderón-Zygmund singular integrals and maximal functions, this is not at all the case. Rather the operator S_r is much more complicated. As Charles Fefferman [2] has shown, the operator S_r, which in one dimension is bounded on all L^p for $1 < p < \infty$, is unbounded on all $L^p(R^n)$ whenever $n > 1$ and $p \neq 2$. The reason for the bad behavior of S_r is that the controlling maximal operator is the multiparameter «Kakeya maximal operator», M_K defined by

$$M_K f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where the supremum is taken over all rectangles (of arbitrary orientation) containing x. This operator is unbounded on all L^p spaces whenever $p < \infty$. In order to have a theory of operators which resemble the multiplier T given by $T\hat{f}(\xi) = \chi_{|\xi| < 1} \cdot \hat{f}(\xi)$, but which has interesting positive results about L^p boundedness, we may look in two directions.

First, we may view the negative results about T to be a consequence of the lack of smoothness of the multiplier $\chi_{|\xi| < 1}$. We may ask what happens when we replace $\chi_{|\xi| < 1}$ by its smooth analogue $\chi_{|\xi| < 1} \cdot (1 - |\xi|^2)^\alpha = m_\alpha(\xi)$, to get the Bochner-Riesz operator of order α, T_α given by

$$T_\alpha \hat{f}(\xi) = m_\alpha(\xi) \cdot \hat{f}(\xi),$$

when $\alpha > 0$.

Now, these operators, at least in R^2, are controlled by a maximal function of the Kakeya type, but somewhat more tame. According to the work of Carleson-Sjölin, [3] C. Fefferman [4] and Córdoba [5], T_α is a bounded operator on $L^p(R^2)$ when $4/3 \leqslant p \leqslant 4$ and $\alpha > 0$. Fefferman and Córdoba explain this by introducing real variable machinery centered around the maximal operators

$$\mathfrak{M}_N f(x) = \sup_{x \in R \in R_N} \frac{1}{|R|} \int_R |f(y)|\, dy$$

where R_N denotes the set of all rectangles whose eccentricity does not exceed N. Thus, as $N \to \infty$, in some sense $\mathfrak{M}_N \to M_K$ and so of course the $\mathfrak{M}_N$ could not be uniformly bounded. However, for $p \geqslant 2$, in R^2 the operator norm of $\mathfrak{M}_N$ grows only like a power of $\log N$ as N becomes large, while if $p < 2$ its norm grows like a power of N. It turns out that the behavior of $\mathfrak{M}_N$ on $L^2(R^2)$ controls that of T_α on $L^4(R^2)$. Below, we shall see that the properties of $\mathfrak{M}_N$ on the critical space $L^n(R^n)$ for $n \geqslant 3$ are not at this time understood, and so we also do not know the L^p boundedness of certain of the T_α in higher dimensions. This is one of the important open questions of the multiple parameter theory.

There is also another way of recapturing a set of positive results analogous to the negative ones for the disk multiplier. Since $\chi_{|\xi|<1}$ is not a multiplier on $L^p(R^2)$ except when $p = 2$, it is natural to ask for a characterization of those sets $S \subseteq R^2$ for which $\chi_S(\xi)$ is a multiplier. Charles Fefferman's counter-example to the disk conjecture shows that for a «nice» open set S whose boundary is curved, there are no non-trivial multiplier results on L^p for χ_S. This leads to the remaining case in which the boundary of S consists of line segments oriented in certain directions θ_k, $k = 1, 2, \ldots$. Córdoba and the author [6] have proven roughly that if one considers the maximal operator $\mathfrak{M}_\theta$ corresponding to the class of rectangles oriented in one of the directions θ_k, then χ_S is bounded on $L^p(R^2)$ if and only if $\mathfrak{M}_\theta$ is bounded on $L^{(p/2)'}(R^2)$ (here $p \geqslant 2$). Notice that under the correspondence $p \to (p/2)'$ the L^4 result for multipliers corresponds to an L^2 result for maximal functions, which suggests that the pattern of dependence discussed above of T_α on $\mathfrak{M}_N$ is valid in more general contexts. The theme of Calderón and Zygmund that the boundedness properties of multipliers and singular integrals are governed by those of maximal operators hast thus been extended in may cases to the multi-parameter setting. Naturally, it becomes very important, in light of this, to prove maximal theorems. Although as recently as about ten years ago very little was known about maximal functions, today there are two basic approaches known which allow us to solve these problems. These are the «Fourier transform approach» and «covering lemma approach» which have been discussed in detail on many occasions (see Stein-Wainger [7], and Córdoba [8], R. Fefferman [9]). Rather than repeating already familiar themes, perhaps it would be more enlightening to mention the main unsolved problem in the area, and why the two techniques, as they stand, seem inadequate to settle this problem. Quite simply, we are unable to obtain results so far about maximal operators which are aunbounded on $L^2(R^n)$. (The main examples are Kakeya type maximal functions when $n \geqslant 3$ and the Stein spherical maximal function on R^2). It is clear that the obvious application of Fourier transform techniques will not work in such a context, because on the L^p spaces we must deal with, the Plancherel theorem is not available. On the other hand, to prove an L^p maxi-

mal theorem via the covering lemma approach, we must consider estimating, for appropriate sets $\{R_k\}_{k=,2,\ldots}$, the quantity

$$\left\| \sum \chi_{R_k} \right\|_{L^{p'}} \quad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

We usually only know how to do this when p' is an integer m, since then we may expand $\left(\sum \chi_{R_k}\right)^m$, and finish by estimating quantities like $|R_{k_1} \cap R_{k_2} \cap \cdots \cap R_{k_m}|$ (see Córdoba-R. Fefferman [10]). Unfortunately, when $2 < p < \infty$, $1 < p' < 2$, hence p' will be non-integral, and this technique cannot be applied. Whether a third, totally different approach will be required to solve such problems, or just a clever twist of the existing machinery, only time will tell.

Let us now retreat a little, and go back to the simplest case of a multiparameter problem, the case of product spaces. Before doing this we shall remind the reader of two theorems of enormous importance from the classical theory.

(1) $(H^1(R^1))^* = \mathrm{BMO}(R^1)$. This theorem of C. Fefferman characterizes those functions $\phi(x)$ which act, by integration, continously on $H^1(R^1) = \{f \in L^1(R^1) | H(f) \in L^1(R^1)\}$. Such ϕ are characterized by the estimate

$$\frac{1}{|I|} \int_I |\phi(x) - \phi_I|^2 \, dx \leqslant C$$

for all intervals I, and C independent of I. (Here ϕ_I is the mean value of $\phi(x)$ over I) [11].

(2) The Atomic Decomposition of $H^p(R^1)$, $p \leqslant 1$. Coifman [12] showed that every function $f \in H^p(R^1)$ could be written in the form $f = \sum_{k=1} \lambda_k a_k$ where λ_k are scalars satisfying $\sum |\lambda_k|^p \leqslant C \| f \|^p_{H^p(R^1)}$ and the a_k are functions known as H^p atoms. An atom is a function $a(x)$ supported on an interval I such that a certain number, $N(p)$, of its moments vanish,

$$\int_I a(x) x^k \, dx = 0 \qquad \forall k \leqslant N(p) \quad \text{and} \quad \|a\|_{L^2} \leqslant |I|^{1/2 - 1/p}.$$

(We shall not be concerned here about the exact value of $N(p)$, except to remark that as $p \to 0$, $N(p) \to \infty$ and for p close enough to 1, we may take $N(p) = 0$).

Using these theorems, as we all know, enables us to obtain trivially results which, prior to their discovery, were either unproven or else most difficult to prove.

What we shall consider here is the theory, on R^2, of operators like the double Hilbert transform $H_1 H_2 f = f * (1/x_1 x_2)$, which are singular integrals with kernels having one dimensional singularity sets. It is trivial to observe that $H_1 H_2$ is a bounded operator on $L^p(R^2)$ when $1 < p < \infty$, by an iteration argument. We might ask whether there is an H^p theory for $p \leqslant 1$ which cor-

responds to the double Hilbert transform in the same way that classical H^p theory on R^1 corresponds to the ordinary Hilbert transform. Thanks to the efforts of Gundy and Stein [13], there is such a theory, which we shall now describe. Let us, in analogy to classical Hardy space theory set $H^1(R^1 \times R^1) = \{f \in L^1(R^2) | H_1 f,\ H_2 f,$ and $H_1 H_2 f \in L^1(R^2)\}$. Then if $S(f)$ denotes the Lusin area integral of the biharmonic extension, u, of f to $R^2_+ \times R^2_+$, and f^* denotes the non-tangential maximal function of u, then according to the Gundy-Stein theorem $f \in H^1(R^1 \times R^1) \Leftrightarrow S(f) \in L^1(R^2) \Leftrightarrow f^* \in L^1(R^2)$. According to this theorem, $S(f) \in L^p(R^2)$ is equivalent to $f^* \in L^p(R^2)$ for all $p > 0$, so that we may define $H^p(R^1 \times R^1)$ as the set of all f so that f^* or $S(f) \in L^p(R^2)$, $p > 0$. At this point we come to a very important question. Do we have simple analogues, in this product space setting, of the two theorems above on H^1-BMO duality and on the atomic decomposition of H^p spaces?

There is an obvious guess as to what the BMO space and atoms should look like. Take the case of an $H^1(R^1 \times R^1)$ atom. The simplest example of such an object should be functions of the form $a_1(x_1)a_2(x_2)$ where each a_i is an $H^1(R^1)$ classical atom. Of course such functions satisfy:

(1) $a(x_1, x_2)$ is supported in a rectangle $R = I \times J$ with $\|a\|_{L^2(R)} \leqslant |R|^{-1/2}$ and

(2) $\int_I a(x_1, x_2)\, dx_1 = 0 \qquad \forall\ x_2 \in J$

$\int_J a(x_1, x_2)\, dx_2 = 0 \qquad \forall\ x_1 \in I.$

Such a function (for reasons which we shall mention shortly) will be called an $H^1(R^1 \times R^1)$ rectangle atom. To define H^p rectangle atom, we simply change $|R|^{-1/2}$ in (1) above to $|R|^{1/2-1/p}$, and require an appropriate number of moments of a in each variable separately to vanish. (If p is sufficiently close to 1 the vanishing of the mean value in (2) is sufficient).

Now a very important conjecture was that every $f \in H^1(R^1 \times R^1)$ could be written as an absolutely convergent sum of such rectangle atoms. Along with this conjecture, was the companion which asked whether the dual space of $H^1(R^1 \times R^1)$ (call it BMO$(R^1 \times R^1)$) could be characterized as those functions $\phi \in L^2_{\text{loc}}(R^2)$ such that for each rectangle R, there exist functions $\phi_1^R(x_1)$ and $\phi_2^R(x_2)$ so that

$$\frac{1}{|R|} \int_R |\phi(x_1, x_2) - \phi_1^R(x_1) - \phi_2^R(x_2)|^2\, dx_1\, dx_2 \leqslant C.$$

These conjectures where both disproven by Lennart Carleson [14] in 1974, and for some time it appeared that there could be no simple H^p and BMO theory for product spaces.

Somewhat later, Alice Chang and R. Fefferman [15] gave an atomic decomposition of $H^p(R^1 \times R^1)$, for $0 < p \leqslant 1$, and a characterization of BMO in

which the role of rectangle was, to some extent replaced by arbitrary open sets of finite measure. Let us be specific and give the precise definition of an $H^1(R^1 \times R^1)$ atom a. Suppose Ω is an open set of finite measure in R^2 and $\mathcal{R}(\Omega)$ denotes the family of all maximal dyadic subrectangles of Ω. An H^1 atom a is a function supported in Ω satisfying $\|a\|_{L^2} \leqslant |\Omega|^{-1/2}$ and a can be written as $a = \sum_{R \in \mathcal{R}(\Omega)} \alpha_R$ where α_R where α_R is supported in the double of R, satisfies $\int_I \alpha_R(x_1, x_2)\, dx_1 = \int_J \alpha_R(x_1, x_2)\, dx_2 = 0$ where the double of R is $I \times J$, and $\left(\sum_R \|\alpha_R\|_{L^2}^2\right)^{1/2} \leqslant |\Omega|^{-1/2}$. Chang-Fefferman [15] also gave a characterization of BMO$(R^1 \times R^1)$ corresponding to this atomic decomposition, and proved interpolation results [16] between these H^p spaces for $p < 1$ and the spaces L^p for $p > 1$. These results have a number of applications, and we just mention briefly that as one of them, we have a real variable proof that if for some f, $S(f) \in L^p$, $p \leqslant 1$, then $f^* \in L^p$, which Gundy and Stein proved using the theory of analytic functions of a complex variable or Brownian motion probabilistic arguments.

The main applications of this H^p and BMO theory were discovered quite recently by J. L. Journé [17], whose work we shall describe below. Journé was interested in two types of product singular integral results. The first was the problem of obtaining L^2 estimates for bi-commutator integrals. Here, we shall state a special case of his result. Suppose that $(\partial^2 A/\partial x_1 \partial x_2)(x_1, x_2) \in L^\infty(R^2)$. Suppose we set $\tilde{A}(x_1, x_2, y_1, y_2) = A(y_1, y_2) - A(x_1, y_2) - A(y_1, x_2) + A(x_1, x_2)$ and

$$T_k f(x_1, x_2) = \iint_{R^2} \frac{[\tilde{A}(x_1, x_2, y_1, y_2)]^k f(y_1, y_2)}{(x_1 - y_1)^{k+1}(x_2 - y_2)^{k+1}}\, dy_1\, dy_2.$$

Then the bicommutator T_k is bounded on $L^2(R^2)$ for $k \geqslant 1$. This is proved via a «T1-theorem» which is stated in terms of the space BMO$(R^1 \times R^1)$ as discussed above.

The second result Journé considers is the generalization to two dimensions of J. L. Rubio de Francia's theorem on Littlewood-Paley theory. Suppose that ρ_k is a sequence of disjoint rectangles in R^2 and for each k let $S_k \hat{f}(\xi) = \chi_{\rho_k}(\xi) \hat{f}(\xi)$. Then Journé shows that $\left\| \left(\sum |S_k f|^2 \right)^{1/2} \right\|_{L^p} \leqslant C_p \| f \|_{L^p}$ for all $p \geqslant 2$.

The reader familiar with the one parameter analogues of the two theorems above realizes that they are both phrased in terms of L^2 or L^p estimates, but they are really theorems about operators which map L^∞ into BMO. It is not surprising then, that in order to show these results, Journé proves a general theorem about operators mapping L^∞ boundedly to BMO$(R^1 \times R^1)$. To state his theorem precisely we require some notation. Suppose that $K(x, y)$ is a kernel with $x, y \in R^1$ and that the associated integral operator $Sf(x) = \int_{R^1} K(x, y) f(y)\, dy$ is bounded on $L^2(R^1)$. Suppose that for some $\delta > 0$ and for all $\gamma \geqslant 2$,

$$(*) \qquad \int_{|y-x| > \gamma|x-x'|} |K(x,y) - K(x',y)|\, dy \leqslant C\gamma^{-\delta}.$$

Then define the norm $\| \quad \|_{CZ}$ by $\|S\|_{CZ} = \|S\|_{L^2,L^2} + \inf\{C|\text{*holds}\}$. Now suppose $K(x_1, x_2, y_1, y_2)$ is defined for $x_i, y_i \in R^1$. Set $\tilde{K}^1(x_1, y_1)$ to be the operator on R^1 whose kernel is $\tilde{K}^1(x_1, y_1)(x_2, y_2) = K(x_1, x_2, y_1, y_2)$, and define $\tilde{K}^2$ similarly. The theorem in [17] is then: Let

$$Tf(x_1, x_2) = \iint_{R^2} K(x_1, x_2, y_1, y_2) f(y_1, y_2)\, dy_1\, dy_2$$

and suppose T is bounded on $L^2(R^2)$. Suppose further that

$$\int_{|y_1 - x_1| > \gamma|x_1 - x_1'|} \|\tilde{K}^1(x_1, y_1) - \tilde{K}^1(x_1', y_1)\|_{CZ}\, dy_1 \leqslant C\gamma^{-\delta}$$

and similarly for $\tilde{K}^2$. Then T maps $L^\infty(R^2)$ boundedly to $\mathrm{BMO}(R^1 \times R^1)$.

In proving this result, Journé introduces a beautiful covering lemma for rectangles, and we can state this in one of its forms as follows. Suppose $\Omega \subseteq R^2$ is an open set of finite measure.

Denote by $\mathcal{R}_2(\Omega)$ those dyadic subrectangles of Ω which are maximal in the x_2 direction. Let $\tilde{\Omega} = \{M_S(\chi_\Omega) > 1/2\}$ where M_S is the strong maximal operator. For $R = I \times J \in \mathcal{R}_2(\Omega)$, suppose $\mathcal{S}$ is the biggest dyadic interval containing I so that $\mathcal{S} \times J \subseteq \tilde{\Omega}$. Finally, set $\gamma_1(R) = |\mathcal{S}/I|$. Then

$$\sum_{R \in \mathcal{R}_2(\Omega)} |R| \gamma_1(R)^{-\delta} \leqslant C|\Omega|.$$

A similar estimate holds reversing the roles of the x_1 and x_2 variables.

Also quite recently, R. Fefferman [18] has proven a theorem along these lines which states that in order to know the action of an operator on L^p, H^p, or $L \log L$, all we have to do is to check the action of that operator on a rectangle atom. In other words it is almost as if Carleson's counterexaple did not exist:

Theorem [18], [19]: Suppose that an operator T is bounded on $L^2(R^2)$. Suppose further that if a is a rectangle H^p atom, $p \leqslant 1$, supported on a rectangle R we have

$$\int_{c\tilde{R}_\gamma} |T(a)|^p\, dx_1\, dx_2 \leqslant C\gamma^{-\delta} \quad \text{for all} \quad \gamma \geqslant 2.$$

(Here $\tilde{R}_\gamma$ is the dilation of R by a factor γ concentric with R.) Then T maps $H^p(R^1 \times R^1)$ boundedly to $L^p(R^1 \times R^1)$ and in the case $p = 1$ we have

$$|\{(x_1, x_2)|\, |x_i| < 1, |Tf(x_1, x_2)| > \alpha\}| \leqslant \frac{C}{\alpha} \|f\|_{L \log L}.$$

The proof is totally trivial once we have both the H^p atomic decomposition and the Journé covering lemma. (For $L \log L$ there is one extra twist. One uses an atomic decomposition for $L \log L$ see [19].) Let us give a rough outline of the proof to show that if $p = 1$, T maps H^1 boundedly to L^1. We take an atom a with associated open set Ω and decomposition $a = \sum_{R \in \mathcal{R}(\Omega)} \alpha_R$. To each $R = I \times J$, associate the largest dyadic $\mathcal{S} \supset I$ so that $\mathcal{S} \times J \subseteq \tilde{\Omega}$. Let $R_1 = \mathcal{S} \times J$. Then $R_1 \in \mathcal{R}_1(\tilde{\Omega})$, and let $\mathcal{J}$ be maximal containing J so that $\mathcal{S} \times \mathcal{J} \subseteq \tilde{\tilde{\Omega}}$. Let $\hat{R}$ be the tenfold dilation of $\mathcal{S} \times \mathcal{J}$. Since $\hat{R} \subseteq \hat{\Omega}$ it is enough to show that

$$\sum_R \int_{c_{\hat{R}}} |T(\alpha_R)| dx_1 \, dx_2 \leqslant C,$$

since

$$\int_{\cup \hat{R}} |T(a)| \, dx_1 \, dx_2 \leqslant |\cup \hat{R}|^{1/2} \| T(a) \|_{L^2} \leqslant C|\Omega|^{1/2} |\Omega|^{-1/2} = C.$$

We estimate each $\int_{c_{\hat{R}}} |T(\alpha_R)| dx_1 \, dx_2$ in the most simple way. For instance

$$\int_{x_1 \notin \tilde{\mathcal{S}}} |T(\alpha_R)| dx_1 \, dx_2 \leqslant \gamma_1(R)^{-\delta} \| \alpha_R \|_{L^2} |R|^{1/2}$$

by the hypothesis of the theorem (here $\tilde{\mathcal{S}}$ is the tenfold dilate of $\mathcal{S}$). If we sum on R and use Cauchy-Schwarz, we get

$$\begin{aligned} \sum_R \int_{x_1 \notin \tilde{\mathcal{S}}(R)} |T(\alpha_R)| \, dx_1 \, dx_2 &\leqslant \sum_R \| \alpha_R \|_{L^2} |R|^{1/2} \gamma_1(R)^{-\delta} \\ &\leqslant \Big(\sum \| \alpha_R \|_{L^2}^2 \Big)^{1/2} \Big(\sum |R| \gamma_1^{-2\delta}(R) \Big)^{1/2} \\ &\leqslant |\Omega|^{-1/2} \cdot C|\Omega|^{1/2} = C. \end{aligned}$$

The estimate of $\sum_R \int_{x_2 \notin \tilde{\mathcal{J}}} |T(\alpha_R)| \, dx_1 \, dx_2$ is only a little bit trickier (see [18]), and finishes the proof.

Before finishing we should remark that there are a number of applications of the theorem above, and here we content ourselves with the statement of just one of them. This is joint work of R. Fefferman with K. C. Lin [20].

Let a bounded function $m(\xi, \eta)$ be given on $R^n \times R^m$ satisfying

$$|\xi|^{|\alpha|} |\eta|^{|\beta|} |\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leqslant C$$

for all multi-indices α, β with $|\alpha| \leqslant [n/2] + 1$, $|\beta| \leqslant [m/2] + 1$. Let $Tf(\xi, \eta) = m(\xi, \eta) \cdot \hat{f}(\xi, \eta)$. Then

$$|\{(x, y) \in R^n \times R^m : |x|, |y| < 1, \ |Tf(x, y)| > \alpha\}| \leqslant C/\alpha \, \| f \|_{L \log L + (|x|, |y| \leq 1)}.$$

While there can be little doubt that much progress has been made in the area of multiple parameter Fourier analysis, it is also clear that most of the work

is yet to come. Perhaps it can be said that this is the most central area of modern Fourier Analysis where our knowledge is at such a primitive level.

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Poincaré-Invariant Structures in the Solution Manifold of a Nonlinear Wave Equation

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Dedicated to Alberto P. Calderón

Abstract

The solution manifold $\mathfrak{M}$ of the equation $\Box\phi + g\phi^3 = 0$ in Minkowski space is studied from the standpoint of the establishment of differential-geometric structures therein. It is shown that there is an almost Kähler structure globally defined on $\mathfrak{M}$ that is Poincaré invariant. In the vanishing curvature case $g = 0$ the structure obtained coincides with the complex Hilbert structure in the solution manifold of the real wave equation. The proofs are based on the transfer of the equation to an ambient universal space-time.

1. Introduction

The solution manifolds of nonlinear wave equations such as $\Box\phi + g\phi^3 = 0$ come equipped with natural invariant symplectic structures, in terms of which they appear as infinite-dimensional Hamiltonian systems (cf. [8] and references given there). In connection with quantum field theory and general mathematical questions, Riemannian, Kähler, and measure-theoretic structures in the solution manifold have been of interest. These are well known in

the linear case, and already in this case it is clear that these structures are inherently global in space-time, unlike the symplectic structure, which even in the nonlinear case can be established locally, quite explicitly in fact. This global feature involves the temporal asymptotics and thereby scattering theory, which was used to establish an almost Kähler structure in the solution manifold near the zero solution for similar equations.

The requisite analysis involves connections with stability theory (cf. [4]), and while the resulting structures are in principle explicit, their expression is relatively complicated, in addition to being only local on the solution manifold. The transfer of a conformally invariant wave equation from Minkowski space $\mathfrak{M}_0$ to the universal cosmos $\tilde{\mathfrak{M}}$ has proved to be a powerful method for its global treatment (cf. [3], [6], [7]). This method is here used to treat the simplest nontrivial nonlinear such equation and derive an essentially very simple Riemannian structure in the solution manifold and related elements of structure.

2. The Banach and Symplectic Manifold Structures

These structures are much more simply and invariantly treated in $\tilde{\mathfrak{M}}$ than in $\mathfrak{M}_0$. The natural, or «Einstein» energy in $\tilde{\mathfrak{M}}$, is slightly larger than the natural, «Minkowski» energy in $\mathfrak{M}_0$, and in practice the limitation to solutions of finite Einstein energy is an inessential constraint. Thus Cauchy data for the cited equation will have finite Einstein energy if they have finite Minkowski energy and if in addition their product with the euclidean distance is such. All such solutions in $\mathfrak{M}_0$ (having Cauchy data of finite Einstein energy at one time) extend canonically and uniquely to $\tilde{\mathfrak{M}}$, in the sense that there exists a unique solution $\tilde{\phi}$ of the equation

$$(*) \qquad \Box' \tilde{\phi} + g\tilde{\phi}^3 = 0$$

on $\tilde{\mathfrak{M}}$, of finite Einstein energy

$$E = \int_{S^3} [(\operatorname{grad} \tilde{\phi})^2 + \dot{\tilde{\phi}}^2 + (1/4)\tilde{\phi}^4] d_3 V,$$

such that $\phi = p\tilde{\phi}|\mathfrak{M}_0$, where p is a fixed smooth function on $\tilde{\mathfrak{M}}$, independent of $\tilde{\phi}$, and $\Box'$ is the operator in $\tilde{\mathfrak{M}}$ that is covariantly related to the usual D'Alembertian $\Box$ in $\mathfrak{M}_0$. For explicit expressions, cf. loc. cit. In more invariant terms, $\tilde{\phi}$ is a section of the scalar bundle of conformal weight 1, and if smooth is transformed by a conformal transformation g on $\tilde{\mathfrak{M}}$ into another solution of equation (*). The universal cover of the group of all conformal transformations on $\tilde{\mathfrak{M}}$ will be denoted as $\tilde{G}$; more exactly, $\tilde{G}$ will denote the connected component of this group, which is locally isomorphic to SU(2, 2).

The main presently relevant features are summarized in

Theorem 1. *The manifold of all finite Einstein energy solutions of equation* (*) *is a Banach manifold relative to the Einstein energy norm in the tangent spaces, representable by the solution manifolds of the first-order variational equations*

$$(**) \qquad \Box'\eta + 3g\tilde{\phi}^2\eta = 0.$$

The conformal group $\tilde{G}$ *leaves this manifold invariant, acts smoothly on it, and leaves invariant the symplectic structure defined by the equation*

$$\Omega_{\tilde{\phi}}(\eta, \eta') = \int_{S^3} (\dot{\eta}\eta' - \eta\dot{\eta}')d_3U$$

(the latter integral being independent of the fixed Einstein time involved in the integration).

The proof is part contained in and in part derives straightforwardly from the given references.

3. The Riemannian Structure

I first recall the «flat» Riemannian structure in the «free» case in which $g = 0$. If $\eta_i (i = 1, 2)$ are two solutions of the wave equation on $\mathfrak{M}$, i.e. distributions satisfying the equation $\Box'\eta_i = 0$, there is a canonical inner product applicable to sufficiently regular solutions: $\langle \eta_1, \eta_2 \rangle$, which is invariant under the group $\tilde{G}$, which acts on solutions as sections of the scalar bundle of weight 1; of [5]. The precise expressions for this inner product will not be required explicitly, but the following expression indicates the exact regularity requirement. Let C denote $(1 - \Delta_c)^{1/4}$, where Δ_c denotes the Laplace-Beltrami operator on S^3. Then $\langle \eta, \eta \rangle = \| C\eta(t_0, \bullet)\|_2^2 + \| C^{-1}\dot{\eta}(t_0, \bullet)\|_2^2$, where the time t_0 is arbitrary, $\| \bullet \|_2$ indicates the norm in $L_2(S^3)$, and $\dot{\eta} = (\partial/\partial t)\eta$.

I recall also (cf. [7]) that the boundary of $\mathfrak{M}_0$ as imbedded in $\mathfrak{M}$ consists of two cones $C_\pm$, the respective limits as $x_0' \to \pm\infty$ of the space-like surfaces $x_0 = x_0'$, where x_0 denotes the Minkowski time (in an arbitrary Lorentz frame). These cones are conformally equivalent to ordinary light-cones in M_0, apart from their inclusion of a point at spatial infinity, essentially the antipode of the chosen origin in S^3, which constitutes their intersection.

Theorem 2. *Let M denote the manifold of all infinitely differentiable solutions of equation* (*). *For any solution $\phi \in M$. Let G_ϕ denote the following real*

symmetric form on the tangent space T_ϕ at ϕ:

$$G_\phi(\eta, \eta') = \langle \eta_{-\infty}, \eta'_{-\infty} \rangle + 3g \int_{\tilde{M}} \phi^2 \eta \eta' d_4 U,$$

where η and η' are C^∞ tangent vectors and

(1) *$\eta_{-\infty}$ denotes the solution of the free wave equation in $\tilde{M}$ having the same restriction to C_- as η;*
(2) *$d_4 U$ denotes the invariant volume element on S^3, multiplied by dt, t = Einstein time.*

Then G_ϕ is a Riemannian structure on M that is invariant under the action of the Poincaré group, including scaling.

PROOF. The restriction to C_- of η provides C^∞ Goursat data for the free wave equation, for which the solution is given in [1], showing its existence and regularity. Under a transformation T in the (scaling-inclusive) Poincaré group $\tilde{P}$, ϕ is carried into a solution $T\phi$, and the differential of the transformation on ϕ carries η into another tangent vector $T\eta$ to M. The cone C_- is invariant under $\tilde{P}$, and $(T\eta)_{-\infty}$ is simply the transform of $\eta_{-\infty}$ under the action of T, since this is true of their respective Goursat data. Thus the form $\langle \eta_{-\infty}, \eta'_{-\infty} \rangle$ on T_ϕ is $\tilde{P}$-invariant, under the action induced from the action of $\tilde{P}$ on the weight 1 scalar bundle.

The integral $\int_{\tilde{M}} \phi^2 \eta \eta' d_4 U$ is unchanged by Poincaré transformation, as shown in [5] (which deals with integrals of the form $\int \phi^4 d_4 U$ where ϕ is an arbitrary section of the same bundle, from which by polarization the presently relevant invariance follows). Thus the entire form G_ϕ is Poincaré invariant.

In order to express the results most simply in terms of classical solutions, only C^∞ solutions were considered in Theorem 2, but the result is readily extended to the full Banach manifold, inclusive of generalized solutions, by completion.

Corollary 2.1. *Let M' denote the manifold of all finite-energy solutions of equation (*). For any solution $\phi \in M'$, let G'_ϕ denote the following real symmetric form on the (full) tangent space T'_ϕ to M' at ϕ:*

$$G'_\phi(\eta, \eta') = \langle \eta_{-\infty}, \eta'_{-\infty} \rangle + 3g \int_{\tilde{M}} \phi^2 \eta \eta' d_4 U,$$

where η and η' are tangent vectors at ϕ, and $\eta_{-\infty}$ is the solution of the wave equation obtained by completion from the mapping $\eta \to \eta_{-\infty}$ defined on C^∞ solutions, as a map from tangent vector in the finite-energy topology to free solutions in their conformally-invariant norm. Then the analogous conclusion to that of Theorem a holds.

PROOF. The differential equation for the tangent vector guarantees that their Cauchy data remain in the finite-energy space earlier indicated and that temporal evolution is continuous in this norm. Combining results in [2] on the regularity of restrictions to submanifolds of functions of specified regularity with the explicit expression [1] for the conformally invariant norm of free solutions in terms of their Goursat data on C_-, it follows that the map $\eta \to \eta_{-\infty}$ is continuous from tangent vectors in the energy norm to free solutions in the conformally invariant norm. Accordingly the map extends by continuity from the dense subset of C^∞ solutions of the tangent equation to all those of finite-energy Cauchy data.

By conservation of energy in the case of ϕ, and its boundedness for $|t| \leqslant \pi$ in the case of η, ϕ and η are in $L_6(S^3)$ at each fixed time (having L_2 gradients in S^3), with L_6 norms bounded as a function of t. The same is true of their norms in $L_2(S^3)$, so that $\int_{\bar{M}} \phi^2 \eta \eta' d_4 U$ depends continuously on ϕ, η, and η' in the energy norms appropriate to the Banach manifold under consideration. Thus the entire metric $G_\phi(\eta, \eta')$ extends uniquely and continuously from the case of smooth ϕ, η, and η', to a metric $G'_\phi(\eta, \eta')$ defined for arbitrary $\phi \in M'$ and arbitrary tangent vectors η and η' at ϕ.

4. Discussion

The conjunction of the Riemannian with the symplectic structure gives rise canonically to an associated almost complex structure, which is symplectic and positive, thereby giving rise to an almost Kähler structure with modified Riemannian structure but identical symplectic structure. Thus the symplectic structure Ω on M' can be extended to a Poincaré-invariant almost Kähler structure. For the free wave equation this is the linear structure, constituting a complex Hilbert space, left invariant by a known positive-energy unitary representation of $\tilde{G}$. In the nonlinear case it is not known whether the almost Kähler structure established here is infinitesimally integrable.

The Poincaré invariance of the map from Goursat data on C_- to solutions of the nonlinear equation provides a Poincaré-invariant map from measures on the Goursat data space to measures on the solution manifold M'. Suitable Gaussian measures will thereby be mapped into non-Gaussian measures enjoying non-trivial, invariance and quasi-invariance properties, which may be interpreted as generalized stochastic solutions of the nonlinear equation.

In the language of theoretical physics, the integral expression involved in G_ϕ may be correlated with the Hessian of the *interaction Lagrangian*, $(1/4)g\int_{\bar{M}} \phi^4 d_4 U$.

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Estimates for Wainger's Singular Integrals Along Curves

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Dedicated to Alberto P. Calderón

1. Introduction

The theory of singular integrals started by Calderón and Zygmund [1] has been extended in several different ways during the last three decades. One such extension is the singular integral operator

$$Hf(x) = p \cdot v \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t} \tag{i}$$

where $\Gamma: \mathbb{R} \to \mathbb{R}^n$, $n \geqslant 2$, is a curve passing through the origin: $\Gamma(0) = 0$. This object, called the Hilbert transform along the curve Γ, appears in the study of parabolic differential operators as well as in the harmonic analysis of homogeneous spaces. A great deal of effort has been dedicated, principally under the influence of S. Wainger, to the understanding of their boundedness properties. See [2] for a fairly complete presentation, together with a summary of results and applications.

These objects are also multipliers of the Fourier transform and, therefore, can be defined by the formula

$$Hf(\xi) = \hat{f}(\xi) m(\xi)$$

where

$$m(\xi) = p \cdot v \int_{-\infty}^{\infty} \exp(i\xi \cdot \Gamma(t)) \frac{dt}{t}.$$

Similarly to the case of the ordinary Hilbert transform, where we have the Hardy-Littlewood maximal function as the positive operator controlling its behaviour, one can define maximal operators associated to these singular integrals

(ii) $$Mf(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x - \Gamma(t))|\, dt$$

(iii) $$H^*f(x) = \sup_{\epsilon>0} \left| \int_{|t| \geq \epsilon} f(x - \Gamma(t)) \frac{dt}{t} \right|.$$

The first natural question one may ask is which are the conditions to be imposed on the curve Γ so that H, H^* and M are bounded on some, or all, spaces $L^p(\mathbb{R}^n)$. In a series of papers, A. Nagel, E. Stein, S. Wainger and others have obtained positive results involving some kind of «positiveness» of the curvature of Γ, taking advantage of the decay of the multiplier $m(\xi)$ that it implies. The proofs are completed by an ingeneous use of the method of complex interpolation, together with arguments of a tauberian type involving certain square functions. In [3], it has been observed that, for convex curves in $\mathbb{R}^2$, $\Gamma(t) = (t, \gamma(t))$, with $\gamma(0) = \gamma'(0) = 0$, there exist necessary and sufficient conditions for H to be bounded in $L^2(\mathbb{R}^2)$: when γ is even (resp. odd) the condition is that $\gamma'(t)$ (resp. $h(t) = t\gamma'(t) - \gamma(t)$) must have the doubling time property, i.e., $\gamma'(Ct) \geqslant 2\gamma'(t)$ (resp. $h(Ct) \geqslant 2h(t)$) for some fixed constant C and for every $t > 0$. This condition has been analyzed in [4] in relation to the maximal function M. The range of L^p-estimates has been extended from L^2 to L^p, $4/3 < p < 4$, in reference [5].

In this paper, we improve and extend the results mentioned above in the following manner: we consider a C^1 curve in $\mathbb{R}^2$, $\Gamma = \{(t, \gamma(t)): -\infty < t < +\infty\}$ $\gamma(0) = \gamma'(0) = 0$, with the following properties:

a) γ is biconvex, i.e., $|\gamma'(t)|$ is decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$.

b) γ has *doubling time*, i.e., there exists a constant $C > 1$ such that $|\gamma'(Ct)| \geqslant 2|\gamma'(t)|$.

c) γ is *balanced*, by which we mean the following: there exists $k > 1$ such that $|\gamma(k^{-1}t)| \leqslant |\gamma(-t)| \leqslant |\gamma(kt)|$ for every $t > 0$.

Theorem. *Under the assumptions a), b) and c) on the curve the singular integral H and the maximal functions M and H* are bounded operators in* $L^p(\mathbb{R}^2)$ *for every* p, $1 < p < \infty$.

Immediate consequences of the boundedness of M and H^* are:

Corollary. *For every* $f \in L^p(\mathbb{R}^2)$, $1 < p < \infty$,

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x - \Gamma(t))\, dt = f(x) \qquad \text{a.e.}$$

$$\lim_{\epsilon \to 0} \int_{|t| > \epsilon} f(x - \Gamma(t)) \frac{dt}{t} = Hf(x) \qquad \text{a.e.}$$

This theorem gives a fairly complete analysis of the properties of Hilbert transforms along convex curves in the plane. For every fixed p, $1 < p < \infty$, the doubling time property is both necessary and sufficient for L^p-boundedness if the curve is even. Likewise, for a biconvex curve, the condition of being balanced is necessary for the L^2-boundedness of H, as we show in §6 below. The fact that for such curves one cannot obtain boundedness of the Hilbert transform in some $L^p(\mathbb{R}^2)$ without obtaining the full range $1 < p < \infty$, explains, in retrospect, the success of the theory, in contrast with the situation in the analysis of the spherical summation multipliers.

We believe that equally important as the theorem itself, is the fact that we treat the operator $Hf(\xi) = \hat{f}(\xi)m(\xi)$ by taking advantage of the decay of $m(\xi)$, its smoothness and the geometry of its level sets, which suggest the use of different (angular and vertical) Littelewood-Paley decompositions. Weighted and vector valued inequalities are also some of the tools used to pass information from estimates of the maximal funcion to estimates for the Hilbert transform, and vice-versa. The set of techniques is due to the contribution of a long list of people, as reference [6] clearly indicates. There are several slightly different proofs of the boundedness of the operators H and M for even or odd curves, one of which is given in [6]. Here we present our version, together with the extension to the class of balanced curves and the appropriate estimates for the maximal Hilbert transform which give the pointwise existence of the principal value singular integral.

2. Decomposition of the Operators and Sketch of Proof

All the L^p estimates will be proved for $f \in \mathcal{S}(\mathbb{R}^2)$, and then extended by continuity. We define the singular measures M_k and h_k ($k \in \mathbb{Z}$) such that

$$\hat{M}_k(\xi, \eta) = \int_{1 \le |t| \le 2} \exp\{i(2^k t\xi + \gamma(2^k t)\eta)\}\, dt$$

$$\hat{h}_k(\xi, \eta) = \int_{1 \le |t| \le 2} \exp\{i(2^k t\xi + \gamma(2^k t)\eta)\} \frac{dt}{t}.$$

and consider the following basic decompositions of our operators

$$Mf(x) \simeq \sup_k |M_k * f(x)|$$

$$Hf(x) = \sum_{-\infty}^{\infty} h_k * f(x)$$

$$H^* f(x) \simeq \sup_k \left\| \sum_{n>k} h_n * f(x) \right\|$$

We shall also use the following square function

(iv) $$Gf(x) = \left(\sum_{k=-\infty}^{\infty} |\tilde{M}_k f(x)|^2 \right)^{1/2}$$

where

$$\tilde{M}_k f(\xi, \eta) = [\hat{M}_k(\xi, \eta) - \hat{\Phi}(2^k \xi)\hat{M}_k(0, \eta)]\hat{f}(\xi, \eta)$$

for a suitable function Φ in the Schwartz class satisfying $\hat{\Phi}(0) = 1$.

The proof of the boundedness of the maximal function Mf is based on the following strategy: we have the pointwise majorization $Mf(x) \leqslant Gf(x) + f^*(x)$, where f^* denotes the strong maximal function of f. We start by proving the L^2-boundedness of Gf by means of Plancherel's theorem. This implies the same result for Mf which, in turn, implies the L^p estimate, $4/3 < p < 4$, for the square function Gf by a nowadays familiar argument based on a vector valued inequality and an ad-hoc Littlewood-Paley decomposition of frequency space. In general, and L^p estimate for M implies an L^q estimate for G, with $\left|\frac{1}{q} - \frac{1}{2}\right| < \frac{1}{2p}$, and therefore, a bootstrap argument will yield the whole range of $p's$, $1 < p < \infty$.

The doubling time condition of the curve will be used to define angular Littlewood-Paley decompositions. For the sake of simplicity, we shall assume that the doubling time constant is $C = 2$. Then, we consider the angular regions

$$\Delta_k^{\pm} = \left\{ (\xi, \eta): \frac{1}{2} |\gamma'(\pm 2^k)| < \frac{|\xi|}{|\eta|} < 2|\gamma'(\pm 2^{k+1})| \right\}$$

($k \in \mathbb{Z}$). It turns out that $\{\Delta_k^+\}_{-\infty}^{\infty}$ is a lacunary decomposition of $\mathbb{R}^2$ with finite overlapping: $\Delta_h^+ \cap \Delta_k^+ = \emptyset$ if $|k - h| \geqslant 3$, and the same is true for $\{\Delta_k^-\}$.

3. The Maximal Operator and the Square Function

Let $\Delta_k = \Delta_k^+ \cup \Delta_k^-$ and define the projection operators T_k by $(T_k f)^\wedge = \hat{f}\chi_{\Delta_k}$, $k \in \mathbb{Z}$.

Lemma 1. *For every p, $1 < p < \infty$, we have*

$$a) \qquad \left\| \left(\sum_k |T_k f|^2 \right)^{1/2} \right\|_p \leqslant c_p \| f \|_p$$

$$b) \qquad \left\| \sum_k T_k f_k \right\|_p \leqslant c_p \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_p$$

$$c) \qquad \left\| \sup_k \left| \sum_{n \geqslant k} T_n f \right| \right\|_p \leqslant c_p \| f \|_p.$$

a) is nowadays a well known inequality (see [2] for references) and *b)* is its dual form; *c)* follows from *a)* in the same way that one proves the lacunary convergence of Fourier series.

Lemma 2.

$$\Gamma f(\delta) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(\delta - \gamma(t))| \, dt$$

is a bounded operator in $L^p(\mathbb{R})$, $1 < p < \infty$.

The proof is obvious due to the monotonicity of γ':

$$\frac{1}{h} \int_0^h f(\delta - \gamma(t)) \, dt = \frac{1}{h} \int_0^{\gamma(h)} f(\delta - u) \frac{du}{\gamma'(\gamma^{-1}(u))}$$

$$= \int_0^c f(\delta - u) \varphi(u) \, du \leqslant f^*(\delta),$$

(Hardy-Littlewood maximal function), because $\varphi(u)$ is nonnegative, decreasing and $\int_0^c \varphi(u) \, du = 1$. The same argument holds for the average over the interval $[-h, 0]$.

Lemma 3. *Let $\Phi \in \mathcal{S}(\mathbb{R})$ be such that $\chi_{[-1,1]} \leqslant \hat{\Phi} \leqslant \chi_{[-2,2]}$. Then*

$$|\hat{M}_k(\xi, \eta) - \hat{\Phi}(2^k \xi) \hat{M}_k(0, \eta)| \leqslant C \min (2^k |\xi|, (2^k |\xi|)^{-1}),$$

for all $(\xi, \eta) \notin \Delta_k$. The same estimates hold for $\hat{h}_k(\xi, \eta)$ in place of $\hat{M}_k(\xi, \eta)$.

PROOF. The estimate for the case $2^k |\xi| \leqslant 1$ is trivial because M_k is a measure supported in $\{(x_1, x_2): |x_1| \leqslant 2^{k+1}\}$. For $2^k |\xi| > 1$ we use Van der Corput lemma

$$\hat{M}_k(\xi, \eta) = \int_1^2 e^{i g_k(t)} \, dt + \int_1^2 e^{i l_k(t)} \, dt$$

with

$$|g_k'(t)| = |2^k(\xi + \gamma'(2^k t)\eta|$$

and

$$|l_k'(t)| = |2^k(\xi + \gamma'(-2^k t)\eta)|.$$

If $|\xi| \geqslant 2|\eta| \cdot |\gamma'(2^{k+1})|$ then $|g_k'(t)| \geqslant 2^{k-1}|\xi|$. If $|\xi| \leqslant {}^1/2\,|\eta| \cdot |\gamma'(2^k)|$, then also $|g_k'(t)| \geqslant 2^{k-1}|\gamma'(2^k t)| \cdot |\eta| \geqslant 2^k|\xi|$ and the situation for l_k' is similar. Finally the complement of these four cases corresponds to points $(\xi, \eta) \in \Delta_k$. The proof for $\hat{h}_k(\xi, \eta)$ is essentially the same. □

Suppose that $U_k f = U_k * f$ is a sequence of positive operators uniformly bounded in $L^\infty(\mathbb{R}^n)$ and consider $U^* f(x) = \sup_k |U_k * f(x)|$.

Lemma 4. *If $U^* f(x) = \sup_k |U_k \cdot f(x)|$ is bounded in $L^q(\mathbb{R}^n)$ then*

$$\left\| \left(\sum_k |U_k f_k|^2 \right)^{1/2} \right\|_p \leqslant c_p \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_p$$

for $\left| \frac{1}{p} - \frac{1}{2} \right| = \frac{1}{2q}$.

This is a well known result and we have stated it here only to clarify the presentation of the proof of the theorem. Our proof of the estimates of the maximal function depends on the following proposition of a conditional nature:

Lemma 5. *Suppose that $\|Mf\|_q \leqslant C\|f\|_q$ for some q. Then for all p such that*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{2q}$$

we have the inequality

$$\|Gf\|_p \leqslant C_p \|f\|_p$$

where Gf is defined by (iv).

PROOF. Recall that $T_k f = \chi_{\Delta_k} \cdot \hat{f}$, and let us write $f^k = f - T_k f$ and

$$S_k f = \hat{f} \cdot \chi_{\{(\xi,\eta):\, 2^{-k} \leq |\xi| \leq 2^{-k+1}\}}.$$

Then

$$Gf \leqslant \left(\sum_k |\tilde{M}_k * T_k f|^2 \right)^{1/2} + \left(\sum_k |\tilde{M}_k * f^k|^2 \right)^{1/2} = I + II.$$

Now, observe that $\sup_k (|\tilde{M}_k| * f)$ is a bounded operator in L^q due to the hypothesis and lemma 2. Therefore

$$\|I\|_p \leqslant C_p \left\| \left(\sum_k |T_k f|^2 \right)^{1/2} \right\|_p \leqslant C_p \|f\|_p$$

by lemmas 4 and 1.

$$II = \left(\sum_k \left| \tilde{M}_k * \left(\sum_{j=-\infty}^{+\infty} S_{j+k} f^k \right) \right|^2 \right)^{1/2} \leqslant \sum_j \left(\sum_k |\tilde{M}_k * S_{j+k} f^k|^2 \right)^{1/2} = \sum_j II_j.$$

We shall prove:

$$\|II_j\|_2 \leqslant C 2^{-|j|} \|f\|_2 \quad \text{and} \quad \|II_j\|_{p_0} \leqslant C \|f\|_{p_0} \quad \text{for} \quad \left| \frac{1}{2} - \frac{1}{p_0} \right| = \frac{1}{2q}.$$

Interpolating, we shall obtain

$$\|II\|_p \leqslant \sum_j \|II_j\|_p \leqslant \sum_j C 2^{-\theta|j|} \|f\|_p \leqslant C_p \|f\|_p \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}.$$

We use Plancherel's theorem for the L^2-inequality

$$\|II_j\|_2^2 = \sum_k \int_{\substack{|\xi| \sim 2^{-j-k} \\ (\xi,\eta) \notin \Delta_k}} |\hat{M}_k(\xi,\eta) - \hat{\Phi}(2^k\xi)\hat{M}_k(0,\eta)|^2 |\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta$$

$$\leqslant \text{(by lemma 3)} \leqslant C \int |\hat{f}(\xi,\eta)|^2 (2^{-|j|})^2 \, d\xi \, d\eta = C 2^{-2|j|} \|f\|_2^2.$$

Finally

$$\begin{aligned} \|II_j\|_{p_0} &\leqslant C_{p_0} \left\| \left(\sum_k |S_{j+k} f^k|^2 \right)^{1/2} \right\|_{p_0} \\ &\leqslant C_{p_0} \left\{ \left\| \left(\sum_k |S_{j+k} f|^2 \right)^{1/2} \right\|_{p_0} + \left\| \left(\sum_k |S_{j+k}(T_k f)|^2 \right)^{1/2} \right\|_{p_0} \right\} \\ &\leqslant C_{p_0} \left\{ \|f\|_{p_0} + \left\| \left(\sum_k |T_k f|^2 \right)^{1/2} \right\|_{p_0} \right\} \leqslant C_{p_0} \|f\|_{p_0}. \quad \square \end{aligned}$$

To finish the proof of the theorem for the maximal operator, we use the majorization

$$Mf(x) \leqslant 2 \sup_k |M_k * f(x)| \leqslant 2 \sup_k |\tilde{M}_k * f(x)| + 2Nf(x) \leqslant 2Gf(x) + 2Nf(x)$$

where N is the hardy-Littlewood operator in the x_1-variable and the operator Γ of lemma 2 in the x_2—varible. Now, the bootstrap argument sketched in section 2 can be applied and yields $\|Mf\|_p \leqslant C_p \|f\|_{p'}$, $1 < p \leqslant \infty$.

4. The Singular Integral

The estimate for Hf will be based on an argument similar to that of lemma 5 together with the fact that the operators $h_k * f$ are acontrolled by the maximal operator Mf.

We first consider the case of even $\gamma(t)$, so that $\hat{h}_k(0, \eta) = 0$. The notation used is as in lemma 5.

$$Hf = \sum_k h_k * T_k f + \sum_k h_k * f^k = H_I f + H_{II} f.$$

By using lemmas 1 and 4, and the fact that $\sup_k |h_k| * f \leqslant Mf$

$$\begin{aligned} \|H_I f\|_p &= \left\| \sum_k T_k(h_k * T_k f) \right\|_p \leqslant C_p \left\| \left(\sum_k |h_k * T_k f|^2 \right)^{1/2} \right\|_p \\ &\leqslant C_p \left\| \left(\sum_k |T_k f|^2 \right)^{1/2} \right\|_p \leqslant C_p \|f\|_p. \end{aligned}$$

On the other hand

$$|H_{II} f| = \left| \sum_k h_k * \sum_{j=-\infty}^{+\infty} S_{j+k} f^k \right| \leqslant \sum_j \left| \sum_k h_k * S_{j+k} f^k \right| = \sum_j |II_j|.$$

Since $\hat{h}_k(0, \eta) = 0$, lemma 3 and Plancherel's theorem yield

$$\begin{aligned} \|II_j\|_2^2 &= \sum_k \int_{\substack{|\xi| \sim 2^{-j-k} \\ (\xi, \eta) \notin \Delta_k}} |\hat{f}(\xi, \eta)|^2 |\hat{h}_k(\xi, \eta)|^2 \, d\xi \, d\eta \\ &\leqslant C \int |\hat{f}(\xi, \eta)|^2 2^{-2|j|} \, d\xi \, d\eta \leqslant C 2^{-2|j|} \|f\|_2^2 \end{aligned}$$

In L^p, $1 < p < \infty$, we have a uniform estimate

$$\begin{aligned} \|II_j\|_p &= \left\| \sum_k S_{j+k}(h_k * S_{j+k} f^k) \right\|_p \\ &\leqslant C_p \left\| \left(\sum_k |h_k * (S_{j+k} f^k)|^2 \right)^{1/2} \right\|_p \\ &\leqslant C_p \left\| \left(\sum_k |S_{j+k} f^k|^2 \right)^{1/2} \right\|_p \leqslant C_p \|f\|_p \end{aligned}$$

and the proof ends by interpolating and summing a geometric series.

For the case of a general curve, we decompose

$$h_k = \tilde{h}_k + m_k$$

where, with Φ as above

$$\hat{m}_k(\xi, \eta) = \hat{\Phi}(2^k \xi) \cdot \hat{h}_k(0, \eta).$$

Then, the Fourier transform of $\tilde{h}_k$ satisfies the same estimates as in the case on an even curve, namely

$$|(\tilde{h}_k)^\wedge(\xi,\eta)| \leqslant C\min(2^k|\xi|, (2^k|\xi|)^{-1}) \quad \text{if} \quad (\xi,\eta) \notin \Delta_k$$

and $\sup |\tilde{h}_k| * f \leqslant Mf + Nf$ is bounded in $L^p(\mathbb{R}^2)$ for all $p > 1$. Therefore, the previous argument gives

$$\left\| \sum_k \tilde{h}_k * f \right\|_p \leqslant C_p \|f\|_p, \qquad 1 < p < \infty.$$

Next, we need to estimate the decay of $\hat{h}_k(0,\eta)$. It is precisely here where the hypothesis of having a balanced curve plays a role. Let us assume for simplicity that the balancing constant is $k \leqslant 2$. Define $\lambda_k = |\gamma(2^k)|$, $k \in \mathbb{Z}$, so that $\{\lambda_k\}_{-\infty}^{\infty}$ is (by convexity) a lacunary sequence, and the balancing condition implies $\lambda_{k-1} \leqslant |\gamma(-2^k)| \leqslant \lambda_{k+1}$, $k \in \mathbb{Z}$. Now, for small η we have

$$\begin{aligned} |\hat{h}_k(0,\eta)| &= \left| \int_{1 \leqslant |t| \leqslant 2} [e^{i\eta\gamma(2^k t)} - 1] \frac{dt}{t} \right| \\ &\leqslant (\log 2)|\eta|(|\gamma(2^{k+1})| + |\gamma(-2^{k+1})|) \leqslant \text{Const}\, \lambda_{k+2} \cdot |\eta|. \end{aligned}$$

For large η we once again use Van der Corput's lemma; since

$$\left| \frac{d}{dt}(\eta\gamma(2^k t)) \right| = |2^k \eta \gamma'(2^k t)| \geqslant \lambda_{k-1}|\eta|, \qquad 1 \leqslant |t| \leqslant 2$$

we obtain

$$|\hat{h}_k(0,\eta)| \leqslant \text{Const}\, (\lambda_{k-1}|\eta|)^{-1}.$$

We now apply the same argument of the even case alter replacing the operators S_j by $\tilde{S}_j$ defined by

$$(\tilde{S}_j f)^\wedge(\xi,\eta) = \hat{f}(\xi,\eta) \cdot \chi_{I_j}(\eta)$$

where $I_j = [\lambda_{j+1}^{-1}, \lambda_j^{-1}] \cup [-\lambda_j^{-1}, -\lambda_{j+1}^{-1}]$. Then

$$\sum_k m_k * f = \sum_{j,k} m_k * \tilde{S}_{j+k} f = \sum_{j=-\infty}^{+\infty} A_j f.$$

The uniform estimate $\|A_j f\|_p \leqslant C_p \|f\|_p$, $1 < p < \infty$, is as in the even case. In L^2 we have

$$\|A_j f\|_2^2 = \sum_k \int_{\mathbb{R}} \int_{I_{j+k}} |\hat{f}(\xi,\eta)|^2 |m_k(\xi,\eta)|^2 \, d\xi \, d\eta.$$

If $j > 1$ and $\eta \in I_{j+k}$,

$$|m_k(\xi, \eta)| \leqslant \frac{\lambda_{k+2}}{\lambda_{k+j}} \leqslant \rho^{2-j},$$

where $\rho > 1$ is the lacunarity constant of the sequence $\{\lambda_k\}_{-\infty}^{\infty}$. If $j < -1$ and $\eta \in I_{j+k}$,

$$|m_k(\xi, \eta)| \leqslant \frac{\lambda_{j+k+1}}{\lambda_{k-1}} \leqslant \rho^{2+j}.$$

Therefore

$$\|A_j f\|_2 \leqslant \text{Const}\, \rho^{-|j|} \|f\|_2 \qquad (-\infty < j < \infty)$$

and this ends the proof.

5. The Maximal Hilbert Transform

We shall prove here the L^p-boundedness of the operator $H^*f(x)$ defined by (iii) for all $1 < p < \infty$. Since

$$H^*f(x) \leqslant Mf(x) + \sup_k \left| \sum_{n \geq k} h_n * f(x) \right|$$

it suffices to consider the maximal operator:

$$\sup_k \left| \sum_{n \geq k} h_n * f \right|.$$

For simplicity, we shall only do this in the case of an even curve; the necessary modifications for dealing with general balanced curves are essentially contained in the last computations of §4. Define $\Phi \in \mathcal{S}(\mathbb{R})$ as in lemma 3, write $\hat{\Phi}_k(\xi) = \hat{\Phi}(2^k \xi)$, and denote by $\overset{(1)}{*}$ convolution in the first variable. Then, we decompose the truncated Hilbert transform as follows:

$$\sum_{n \geq k} h_n * f = \Phi_k \overset{(1)}{*} \left\{ Hf - \sum_{n < k} h_n * f \right\} + (\delta - \Phi_k) \overset{(1)}{*} \sum_{n \geq k} h_n * T_n f +$$

$$+ (\delta - \Phi_k) \overset{(1)}{*} \sum_{n \geq k} h_n * f^n = A_k f + B_k f + C_k f$$

and we need to estimate $\sup_k |A_k f|$, $\sup_k |B_k f|$ and $\sup_k |C_k f|$. In doing so, we shall use the following maximal operators:

a) $g_1^*(x)$ = Hardy-Littlewood maximal function of $g(x_1, x_2)$ in the x_1-direction.

b) $Ng(x)$ = the Hardy-Littlewood operator acting on the x_1 varible and the operator Γ of lemma 2 acting on x_2 (this operator appears at the end of §3).

c) $T^*g(x) = \sup_k |T_k g(x)|$.

d) $S^*g(x) = \sup_k \left|\sum_{n \geq k} T_n g(x)\right|$ (see lemma 1).

e) $Mf(x)$ = maximal operator along the curve Γ.

All of these are bounded in $L^p(\mathbb{R}^2)$ for every p, $1 < p < \infty$. Now, to estimate $A_k f$, we observe first that the action of the kernel $\Phi_k \overset{(1)}{*} \sum_{n<k} h_n$ is pointwise dominated by the action of the maximal operator N. It suffices to prove this at the point $x = 0$ and for $k = 0$; given a test function g, we have

$$\left|\left\langle \Phi \overset{(1)}{*} \sum_{n<0} h_n, g \right\rangle\right| = \left| \int_{-\infty}^{\infty} \left\{ p \cdot v \int_{-1}^{1} g(s+t, \gamma(t)) \frac{dt}{t} \right\} \Phi(s)\, ds \right|$$

$$= \left| \int_{-\infty}^{\infty} \int_{-1}^{1} [\Phi(s-t) - \Phi(s)] g(s, \gamma(t)) \frac{dt}{t}\, ds \right|$$

$$\leqslant \text{Const} \int_{-1}^{1} \int_{-\infty}^{\infty} \frac{1}{1+s^2} |g(s, \gamma(t))|\, ds\, dt \leqslant Ng(0)$$

(we have used the fact that γ is even in the second equality). Therefore

$$\sup_k |A_k f(x)| \leqslant \text{Const}\, [(Hf)_1^*(x) + Nf(x)].$$

For the second term, since $\{T_n\}_{n \in \mathbb{Z}}$ are orthogonal projections, we can write $h_n * T_n f = T_n\left(\sum_j h_j * T_j f\right)$. Remember from §4 that Hf was decomposed as $Hf = H_I f + H_{II} f$, and both pieces were bounded in L^p for all $1 < p < \infty$. Then,

$$\sup_k |B_k f(x)| = \sup_k \left| (\delta - \Phi_k) \overset{(1)}{*} \sum_{n \geq k} T_n (H_I f)(x) \right| \leqslant \text{Const}\, (S^* H_I f)_1^*(x).$$

Finally, the last term can be decomposed as follows:

$$\sup_k |C_k f| = \sup_k \left| \sum_{j=0}^{\infty} (\delta - \Phi_k) \overset{(1)}{*} h_{j+k} * f^{j+k} \right| \leqslant \sum_{j=0}^{\infty} P_j f$$

where

$$P_j f(x) = \sup_k |(\delta - \Phi_k) \overset{(1)}{*} h_{j+k} * f^{j+k}(x)|.$$

The operators P_j are uniformly bounded in $L^p(\mathbb{R}^2)$, $1 < p < \infty$, because

$$P_j f(x) \leqslant \text{Const}\, (Mf + MT^* f)_1^*(x).$$

On the other hand, a more favourable estimate holds for $p = 2$, since

$$\|P_j f\|_2^2 \leqslant \sum_k \iint_{(\xi,\eta)\notin\Delta_{j+k}} |1-\hat{\Phi}(2^k\xi)|^2 |\hat{h}_{j+k}(\xi,\eta)|^2 |\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta$$

$$\leqslant \text{Const} \iint \sum_{k:2^{-k}\leq|\xi|} |2^{j+k}\xi|^{-2} |\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta$$

$$\leqslant \text{Const}\, 2^{-2j} \|f\|_2^2.$$

Again, interpolating and summing a geometric series shows that

$$\left\| \sum_{j=0}^{\infty} P_j f \right\|_p \leqslant C_p \|f\|_p, \qquad 1 < p < \infty$$

finishing the proof.

6. Balance is Necessary

Suppose that $\Gamma = \{(t, \gamma(t)\}_{t\in\mathbb{R}}$ is a biconvex curve, i.e., $\gamma(0) = \gamma'(0) = 0$ and $|\gamma'(t)|$ increases in $(0, +\infty)$ and decreases in $(-\infty, 0)$, and let H be the Hilbert transform along Γ. We wish to prove that, if H is bounded in any L^p space, then Γ must be a balanced curve. In fact, since H is a translation invariant operator corresponding to the multiplier

$$m(\xi,\eta) = p \cdot v \int_{-\infty}^{\infty} \exp(i\xi t + i\eta\gamma(t)) \frac{dt}{t},$$

we have H bounded in $L^p(\mathbb{R}^2) \Rightarrow H$ bounded in $L^2(\mathbb{R}^2) \Leftrightarrow m \in L^\infty(\mathbb{R}^2) \Rightarrow \Gamma$ is balanced.

Only the last implication must be proved, but this follows from

Lemma 6. *If Γ is biconvex and $|\gamma(-t_0)| = |\gamma(t_1)|$ with $t_0, t_1 > 0$, then $|\log t_1/t_0| \leqslant 8 + |m(0,\eta)|$, where $\eta = 1/|\gamma(t_1)|$.*

PROOF. Suppose that $t_0 < t_1$. Then

$$m(0,\eta) = \int_{t_1}^{\infty} e^{i\gamma(t)\eta} \frac{dt}{t} + \int_{-\infty}^{t_0} e^{i\gamma(t)\eta} \frac{dt}{t}$$

$$+ \int_{-t_0}^{t_1} [e^{i\gamma(t)\eta} - 1] \frac{dt}{t} + \int_{t_0}^{t_1} \frac{dt}{t}$$

$$= I_1 + I_2 + I_3 + \log \frac{t_1}{t_0}.$$

Let $F(t) = \int_{t_1}^{t} e^{i\gamma(s)\eta}\, ds$, $t_1 < t < \infty$. For $s > t_1$, we have $|\gamma'(s)|\eta \geqslant |\gamma'(t_1)|\eta \geqslant$ $\geqslant (|\gamma(t_1)|/t_1)\eta = 1/t_1$, and γ' is monotone. Therefore, $F(t) \leqslant 3/t_1$, and integrating by parts yields

$$|I_1| = \left| \int_{t_1}^{\infty} F(t) t^{-2}\, dt \right| \leqslant 3$$

Similarly, $|I_2| \leqslant 3$. On the other hand

$$|I_3| \leqslant \int_{-t_0}^{t_1} |\gamma(t)|\eta \frac{dt}{|t|} \leqslant \eta \int_{-t_0}^{t_1} |\gamma'(t)|\, dt = \eta(|\gamma(t_1)| + |\gamma(-t_0)|) = 2.$$

Putting everything together, $|I_1 + I_2 + I_3| \leqslant 8$, q.e.d.

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Relativistic Stability of Matter-I

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Dedicated to A. P. Calderón

0. Introduction

In this article, we study the quantum mechanics of N electrons and M nuclei interacting by Coulomb forces. Motivated by an important idea of Chandrasekhar and following Herbst [H], we modify the usual kinetic energy $-\Delta$ to take into account an effect from special relativity. As a result, the system can implode for unfavorable values of the nuclear charge Z and the fine structure constant α. This is analogus to the gravitational collapse of a heavy star. Our goal here is to find those values of α and Z for which the system is stable.

We start by explaining the modified kinetic energy. Recall that $-\Delta$ is the quantum version of the kinetic energy $T = p^2/2m$ from classical mechanics. Now $T \sim p^2/2m$ is only a low-energy approximation in special relativity. The correct formula is $T = (p^2 + m^2)^{1/2} - m$ in suitable units. In particular, $T \sim |p|$ in the crucial high-energy limit. A first crude attempt to reflect this in quantum mechanics is to study the Hamiltonian

$$H = [(-\Delta + m^2)^{1/2} - m] + V(x) \qquad \text{(A)}$$

in place of the usual $-\Delta + V(x)$. While (A) is not a correct physical equation, it is rooted in a genuine relativistic effect. It is essentially equivalent to the ideas that led Chandrasekhar to predict the gravitational collapse of stars and

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derive the order of magnitude of the critical mass. (See also Lieb-Thirring [LT2].)

Next we show how a system governed by (A) can implode. This happens already for a single quantized electron attracted to a single nucleus of charge Z fixed at the origin. The Hamiltonian (A) becomes $[(-\Delta + m^2)^{1/2} - m] - \alpha Z/|x|$, where $\alpha \sim 1/137$ is the fine structure constant. Modulo a bounded error,

$$H = (-\Delta)^{1/2} - \frac{\alpha Z}{|x|} \quad \text{acting on} \quad L^2(\mathbb{R}^3) \tag{B}$$

The state of the electron is given by a wave function $\psi \in L^2(\mathbb{R}^3)$ of norm 1, and the energy is $\langle H\psi, \psi\rangle$. Unlike the usual Schrödinger operator, (B) is homogeneous with respect to dilations of $\mathbb{R}^3$. So if we could find a $\varphi \in C_0^\infty$ with $\langle H\varphi, \varphi\rangle < 0$, then by dilating φ we can make wave functions ψ concentrated in a tiny region of space, and having large negative energy. In other words, the electron will fall into the nucleus. On the other hand, if $\langle H\varphi, \varphi\rangle > 0$, then dilating φ to concentrate the electron into a small region will make the energy large positive. Consequently, the system will be stable if $H \geqslant 0$. From these considerations, we may expect the value of αZ to play a crucial role in the study of (B). In fact, (B) is easily decomposed by means of spherical harmonics and the Mellin transform, and one finds the following dichotomy: $H \geqslant 0$ if $\alpha Z \leqslant 2/\pi$; H unbounded below if $\alpha Z > 2/\pi$. See Herbst [H]. Putting $\alpha = 1/137$, we obtain $2/\pi\alpha \sim 87$ as the largest stable nuclear charge. Of course larger atoms exist in nature; numbers predicted from (A), (B) are physically relevant only as orders of magnitude.

We want to understand what happens for N electrons and M nuclei. The set-up is as follows. We fix nuclei with charges $Z_1, Z_2, \ldots, Z_M$ at points $y_1 \ldots y_M \in \mathbb{R}^3$. A quantum state for the N electrons amounts to a wave function $\psi(x_1 \ldots x_N) \in L^2(\mathbb{R}^{3N})$ of norm 1. Since electrons are Fermions, we take ψ antisymmetric, i.e., $\psi(x_1 \ldots x_N) = \operatorname{sgn}(\pi)\psi(x_{\pi 1} \ldots x_{\pi N})$ for permutations π. Note that our wave functions are scalars, i.e., we are ignoring spin. The energy of a quantum state ψ is defined as $\langle H\psi, \psi\rangle$ with

$$H = \sum_{k=1}^{N} (-\Delta_{x_k})^{1/2} + \alpha\left[\sum_{j<k} \frac{1}{|x_j - x_k|} + \sum_{j<k} \frac{Z_j Z_k}{|y_j - y_k|} - \sum_{j,k} \frac{Z_k}{|x_j - y_k|}\right]. \tag{C}$$

This is the many-particle version of (B), and agrees with (A) up to an error $O(N)$. To make sure $H\psi$ is well-defined, we take $\psi \in C_0^\infty$.

Depending on α, $Z_1 \ldots Z_M$, three very different phenomena may occur.

Case 1 (Stability): $\langle H\psi, \psi\rangle \geqslant 0$ for all ψ and all $y_1 \ldots y_M$.

Case 2 (Many-body implosion): $\langle H\psi, \psi\rangle$ is bounded below for each fixed $y_1 \ldots y_M$, but the ground-state energy tends to $-\infty$ as the nuclei are bought together at the origin in a suitable manner.

Case 3 (Complete implosion): $\langle H\psi, \psi\rangle$ is unbounded below for each fixed $y_1 \ldots y_M$.

That these are the only possible cases is immediate from dilation-invariance of the Hamiltonian (C).

From the study of (B) one checks easily that complete implosion occurs if and only if $\alpha \cdot \max_k Z_k > 2/\pi$. The other two cases are hard to resolve.

Stability (case 1) is a strong statement about many-body systems. For instance, by applying the trivial estimate $(-\Delta)^{1/2} \leqslant \alpha(-\Delta) + \alpha^{-1}/4$, we get in case 1

$$\sum_k (-\Delta_{x_k}) + \sum_{j<k} \frac{1}{|x_j - x_k|} + \sum_{j<k} \frac{Z_j Z_k}{|y_j - y_k|} - \sum_{j,k} \frac{Z_k}{|x_j - y_k|} \geqslant -CN \qquad \text{(D)}$$

with $C = \alpha^{-2}/4$. This is a sharp form of stability of matter (see Dyson-Lenard [DL1], Federbush [F], Lieb-Thirring [LT]). Antisymmetry of the wave function must therefore play a crucial role in stability of (C). Already in the case of a single electron ($N = 1$), (D) is not obvious; its analogue in Thomas-Fermi theory [L] is false.

In fact, the case of one electron and M nuclei was settled by Daubechies and Lieb [DL2] for $\alpha \leqslant 1/3\pi$. They found that the Hamiltonian (C) is stable in this case under the sharp assumption $\alpha \cdot \max_k Z_k \leqslant 2/\pi$. It would be interesting to know the best constant in (*D*) with $N = 1$.

Recently, J. Conlon [C] proved the further deep result that the Hamiltonian (C) is stable for arbitrarily many electrons and nuclei, provided $\alpha \cdot \max_k Z_k \leqslant 10^{-200}$.

Both [DL2] and [C] are major steps forward in our understanding of relativistic stability of matter. They contain a wealth of important insights which we take for granted here.

Our results are as follows.

Theorem 1 (Hydrogen). *Suppose all* $Z_k = 1$. *Then the Hamiltonian* (C) *is stable for* $\alpha \leqslant 1/2.06\pi$.

Theorem 2 (Small α). *There exists* α_{critical} *such that* $\alpha < \alpha_{\text{critical}}$ *and* $\alpha \max_k Z_k \leqslant 2/\pi$ *imply stability of* (C).

Since complete implosion occurs when $\alpha \cdot \max_k Z_k > 2/\pi$, Theorem 1 provides the best α up to a factor of 4.12, while Theorem 2 completely settles the

case of small α. From the proof of Theorem 2 one has a good chance to find an explicit (non-optimal) α_{critical}. Moreover, we hope that $\alpha_{\text{critical}} > 1/137$. H. Trotter has produced strong evidence for this, but a complete proof is not yet available.

We should point out here two immediate consequences of our proof, which may be of independent interest.

Cor. 1. *Let $x_1 \dots x_N, y_1 \dots y_M \in \mathbb{R}^3$ be electrons and protons, let $\delta(x_k)$ be the distance from x_k to the nearest y_l, and let*

$$V = \sum_{j<k} |x_j - x_k|^{-1} + \sum_{j<k} |y_j - y_k|^{-1} - \sum_{j,k} |x_j - y_k|^{-1}$$

be the Coulomb potential. Then

$$V \geqslant -\frac{8}{3} \sum_k \frac{1}{\delta(x_k)}.$$

Note. Elliott Lieb has pointed out to us that Cor. 1. appears in Baxter [B], 1980.

Cor. 2. **Let** *$\psi(x_1 \dots x_N)$ be antisymmetric, and let $\delta(x_k)$ be the distance from x_k to the nearest $x_l (l \neq k)$.* Then

$$\left\langle \sum_k (-\Delta_{x_k})^{1/2} \psi, \psi \right\rangle \geqslant \frac{35}{24\pi} \left\langle \sum_k \frac{1}{\delta(x_k)} \psi, \psi \right\rangle.$$

More generally, we hope that the ideas in our proofs will be useful tools for understanding many-body problems.

In this paper, we prove only Theorem 1, leaving Theorem 2 for later in the hope of getting $\alpha_{\text{critical}} > 1/137$. An intresting feature of our proof of Theorem 1 is that it makes use of computers. This is perhaps natural in a theorem on best constants. (Without computers, we can only prove Theorem 1 with $1/2.06\pi$ replaced by $1/5\pi$.)

We begin our proof by reducing Theorem 1 to a problem involving only one electron and three nuclei. The reduction of a «big» problem for N electrons and M nuclei to a «small» problem for a few particles is made possible by the key equations (1), (2), (6) below. The one-electron problem in turn can be reduced to the non-vanishing of a function $D(t_1, t_2)$ defined in terms of the solution of a second-order linear ordinary differential equation. Since the equation can be solved numerically, it is easy to calculate $D(t_1, t_2)$ to good accuracy with a computer and convince oneself that $D(t_1, t_2) \neq 0$. However, a complete proof requires computer programs that produce rigorous upper and lower bounds for solutions of ordinary differential equations. Related computer-science issues arose in the rigorous study of the renormalization group equations (see [La, Ll]).

In section 6 below, we explain the compuer programs in detail, and provide listings so that our results may be easily reproduced.

1. Notation

$\psi(x_1 \dots x_s)$ denotes a many-electron wave function, while $y_1 \dots y_s$, are fixed nuclei in $\mathbb{R}^3$.

$B(z, R)$ = ball of center z and radius R in $\mathbb{R}^3$.

$N(z, R) = N(z, R; x_1 \dots x_s)$ = number of electrons x_j in $B(z, R)$.

$M(z, R)$ = number of nuclei in $B(z, R)$.

Let A be a subset of $\{1 \dots s\}$. We write $x_A = (x_j)_{j \in A}$ and $x'_A = (x_j)_{j \notin A}$. Thus $\psi(x_1 \dots x_s) = \psi(x_A, x'_A)$.

We write dx_A for $\prod_{j \in A} dx_j$, and dx'_A for $\prod_{j \notin A} dx_j$.

If $A = \{k\}$ has just one element, then we write x_k, x'_k, dx'_k, in place of x_A, x'_A, dx'_A. Thus $\psi(x_1 \dots x_s) = \psi(x_k, x'_k)$ for each k.

For a constant $\varkappa$ to be picked later, set

$$H = \frac{12\varkappa}{35} \cdot \frac{\pi}{2} \sum_k (-\Delta_{x_k})^{1/2} + V$$

$$V = \sum_{j<k} \frac{1}{|x_j - x_k|} + \sum_{j<k} \frac{1}{|y_j - y_k|} - \sum_{j,k} \frac{1}{|x_j - y_k|}.$$

Our present H is proportional to the Hamiltonian (C) in the introduction, with $\alpha = (2/\pi) \cdot (35/12\varkappa)$.

2. Rewriting the Hamiltonian

A key idea in our proof is to associate both a kinetic and a potential energy to each ball $B(z,R)$, and then regard the total energy $\langle H\psi, \psi \rangle$ as an integral over all possible balls. To do this for kinetic energy, we use the elementary identities

$$\int_{z \in \mathbb{R}^3} \int_{R>0} \int_{x,y \in B(z,R)} |u(x) - u(y)|^2 \, dx \, dy \frac{dz \, dR}{R^8}$$

$$= \frac{16\pi}{35} \int \frac{|u(x) - u(y)|^2}{|x - y|^4} dx \, dy = \frac{32\pi^3}{35} \langle (-\Delta)^{1/2} u, u \rangle, \; u \in C_0^\infty(\mathbb{R}^3).$$

(The second equality is well-known from the Fourier transform.)

Hence if we set

$$T_k(z, R; x'_k) = \int_{\bar{x}, \bar{\bar{x}} \in B(z,R)} |\psi(\bar{x}, x'_k) - \psi(\bar{\bar{x}}, x'_k)|^2 \, d\bar{x} \, d\bar{\bar{x}}$$

for $\psi \in L^2(\mathbb{R}^{3s})$ and $x'_k \in \mathbb{R}^{3s-3}$, then we have

$$\frac{12\varkappa}{35}\frac{\pi}{2}\sum_{k=1}^{s}\langle(-\Delta_{x_k})^{1/2}\psi,\psi\rangle$$

$$=\frac{3\varkappa}{16\pi^2}\sum_{k=1}^{s}\int_{z\in\mathbb{R}^3}\int_{R>0}\int_{x'_k\in\mathbb{R}^{3s-3}}T_k(z,R;x'_k)\,dx'_k\frac{dz\,dR}{R^8}. \tag{1}$$

To write the potential energy as an integral over spheres, we just note that

$$\frac{1}{|x-x'|}=\frac{1}{\pi}\int_{z\in\mathbb{R}^3}\int_{R>0}\chi_{x,x'\in B(z,R)}\frac{dz\,dR}{R^5}\quad\text{for}\quad x,x'\in\mathbb{R}^3.$$

Summing this over all particle pairs with an appropriate sign, we have at once

$$V=\sum_{j<k}\frac{1}{|x_j-x_k|}+\sum_{j<k}\frac{1}{|y_j-y_k|}-\sum_{j,k}\frac{1}{|x_j-y_k|}$$

$$=\frac{1}{\pi}\int_{x\in\mathbb{R}^3}\int_{R<0}\left\{\frac{N(N-1)}{2}+\frac{M(M-1)}{2}-MN\right\}\frac{dz\,dR}{R^5} \tag{2}$$

where $N=N(z,R;x_1\ldots x_s)$ and $M=M(z,R)$.

Next we bring in antisymmetry of the wave function to prove the following estimate, which we shall put into (1).

Lemma 1. *Fix $B=B(z,R)$, and suppose $\psi(x_1\ldots x_s)$ is antisymmetric. Then*

$$\sum_k\int_{x'_k\in\mathbb{R}^{3s-3}}T_k(z,R;x'_k)\,dx'_k\geqslant\sum_k\int_{x'_k\in(\mathbb{R}^3\setminus B)^{s-1}}T_k(z,R;x'_k)\,dx'_k$$

$$+\frac{8\pi}{3}R^3\int_{\mathbb{R}^{3s}}(N(z,R;x_1\ldots x_s)-1)_+\,|\psi(x_1\ldots x_s)|^2\,dx_1\ldots dx_s.$$

PROOF. Let $\varphi_0,\varphi_1,\varphi_2,\ldots$ be a complete orthonormal system in $L^2(B)$, with $\varphi_0=\text{const}$. If we expand a given $u\in L^2(B)$ into its Fourier series $u(x)\sim$ $\sim\sum A_\alpha\varphi_\alpha(x)$, then

$$\int_{\bar{x},\bar{\bar{x}}\in B}|u(\bar{x})-u(\bar{\bar{x}})|^2\,d\bar{x}\,d\bar{\bar{x}}=\frac{8\pi}{3}R^3\int_{x\in B}|u(x)-Av_Bu|^2\,dx$$

$$=\frac{8\pi}{3}R^3\sum_{\alpha\neq 0}|A_\alpha|^2.$$

Now let $\varphi(x_1\ldots x_N)\in L^2(B^N)$, and write

$$\varphi(x_1\ldots x_N)\sim\sum_{\alpha_1\ldots\alpha_N}A_{\alpha_1\ldots\alpha_N}\varphi_{\alpha_1}(x_1)\ldots\varphi_{\alpha_N}(x_N).$$

The preceding identity gives at once

$$\int_{(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_N)\in B^{N-1}} \int_{\bar{x},\bar{\bar{x}}\in B} |\varphi(x_1\ldots x_{k-1}\bar{x}x_{k+1}\ldots x_N) - \varphi(x_1\ldots x_{k-1}\bar{\bar{x}}x_{k+1}\ldots x_N)|^2\, d\bar{x}\, d\bar{\bar{x}}\, dx_1\ldots dx_{k-1}\, dx_{k+1}\ldots dx_N$$

$$= \frac{8\pi}{3} R^3 \sum_{\substack{\alpha_1\ldots\alpha_N \\ (\alpha_k\neq 0)}} |A_{\alpha_1\ldots\alpha_N}|^2$$

for each k. Sum this over k, and we get

$$\sum_k \int_{B^{N-1}} \int_{\bar{x},\bar{\bar{x}}\in B} |\varphi(x_1\ldots x_{k-1}\bar{x}x_{k+1}\ldots x_N) - \varphi(x_1\ldots x_{k-1}\bar{\bar{x}}x_{k+1}\ldots x_N)|^2 \, d\bar{x}\, d\bar{\bar{x}}\, dx_1\ldots dx_{k-1}\, dx_{k+1}\ldots dx_N$$

$$= \frac{8\pi}{3} R^3 \sum_{\alpha_1\ldots\alpha_N} |A_{\alpha_1\ldots\alpha_N}|^2 \mathfrak{N}(\alpha_1\ldots\alpha_N), \quad (3)$$

where $\mathfrak{N}(\alpha_1\ldots\alpha_N)$ = (number of k with $\alpha_k\neq 0$).

If $\varphi(x_1\ldots x_N)$ is antisymmetric, then the $A_{\alpha_1\ldots\alpha_N}$ are antisymmetric, so that $\mathfrak{N}(\alpha_1\ldots\alpha_N)\geqslant N-1$ whenever $A_{\alpha_1\ldots\alpha_N}\neq 0$. Hence (3) yields

$$\sum_k \int_{B^{N-1}} \int_{\bar{x},\bar{\bar{x}}\in B} |\varphi(x_1\ldots x_{k-1}\bar{x}x_{k+1}\ldots x_N) - \varphi(x_1\ldots x_{k-1}\bar{\bar{x}}x_{k+1}\ldots x_N)|^2 \, d\bar{x}\, d\bar{\bar{x}}\, dx_1\ldots dx_{k-1}\, dx_{k+1}\ldots dx_N$$

$$\geqslant \frac{8\pi}{3} R^3(N-1) \sum_{\alpha_1\ldots\alpha_N} |A_{\alpha_1\ldots\alpha_N}|^2$$

$$= \frac{8\pi}{3} R^3(N-1) \int_{B^N} |\varphi(x_1\ldots x_N)|^2\, dx_1\ldots dx_N, \quad (4)$$

whenever $\varphi\in L^2(B^N)$ is antisymmetric.

Next, let $\psi(x_1\ldots x_s)\in L^2(\mathbb{R}^{3s})$ be antisymmetric, and let $N\geqslant 2$. For each N-element subset $A\subset\{1\ldots s\}$, let $E(A)=\{(x_1\ldots x_s)\mid x_j\in B$ if and only if $j\in A\}$, and $F(A,k)=\{x'_k=(x_j)_{j\neq k}\mid$ Each $x_j(j\neq k)$ belongs to B if and only if $j\in A\}$. For fixed A and fixed x'_A, we regard

$$\psi|_{E(A)} = \psi(x_A, x'_A)\Big|_{\substack{x_A\in B^N \\ x'_A\in[{}^cB]^{s-N}}}$$

as a function of $x_A\in B^N$, and apply (4). Integrating against dx'_A over $[{}^cB]^{s-N}$, we obtain

$$\sum_{k\in A} \int_{x'_k\in F(A,k)} \int_{\bar{x},\bar{\bar{x}}\in B} |\psi(\bar{x},x'_k) - \psi(\bar{\bar{x}},x'_k)|^2\, d\bar{x}\, d\bar{\bar{x}}\, dx'_k$$

$$\geqslant \frac{8\pi}{3} R^3(N-1) \int_{E(A)} |\psi(x_1\ldots x_s)|^2\, dx_1\ldots dx_s.$$

Sum this over all N-element sets A, and we find that

$$\sum_k \int_{x'_k \in G(N,k)} \int_{\bar{x},\bar{\bar{x}} \in B} |\psi(\bar{x}, x'_k) - \psi(\bar{\bar{x}}, x'_k)|^2 \, d\bar{x}\, d\bar{\bar{x}}\, dx'_k$$

$$\geqslant \frac{8\pi}{3} R^3 (N-1) \int_{\mathbb{R}^{3s}} \chi_{N(z,R;x_1 \ldots x_s)=N} |\psi(x_1 \ldots x_s)|^2 \, dx_1 \ldots dx_s \quad (5)$$

with $G(N,k) = \{x'_k \in \mathbb{R}^{3s-3} \mid$ Exactly $N-1$ of the components of x'_k belong to $B\}$.

If finally, we sum (5) over all N from 2 to s, then we get the conclusion of the Lemma. □

Corollary. *If $\psi(x_1 \ldots x_s)$ is antisymmetric, then*

$$\frac{12\varkappa}{35} \frac{\pi}{2} \sum_{k=1}^{s} \langle (-\Delta_{x_k})^{1/2} \psi, \psi \rangle$$

$$\geqslant \left\langle \frac{1}{2\pi} \int_{z \in \mathbb{R}^3} \int_{R>0} \varkappa [N(z,R;\bullet) - 1]_+ \frac{dz\, dR}{R^5} \psi, \psi \right\rangle$$

$$+ \sum_k \frac{3\varkappa}{16\pi^2} \int_{z \in \mathbb{R}^3} \int_{B>0} \int_{x'_k \in [{}^cB(z,R)]^{s-1}} T_k(z,R;x'_k)\, dx'_k \frac{dz\, dR}{R^8} \quad (6)$$

PROOF. Immediate from the Lemma and (1). □

We pause to point out some simple applications of (2) and (6). (These were stated in the introduction as Corollaries 1 and 2.) Since

$$\int_{z \in \mathbb{R}^3} \int_{R>0} [N(z,R) - 1]_+ \frac{dz\, dR}{2}$$

$$\geqslant \int_{z \in \mathbb{R}^3} \int_{R>0} \chi_{N(z,R) \geq 2} \frac{N(z,R)}{2} \frac{dz\, dR}{R^5}$$

$$= \frac{1}{2} \sum_k \int_{z \in \mathbb{R}^3} \int_{R>0} \chi_{N(z,R) \geq 2} \chi_{x_k \in B(z,R)} \frac{dz\, dR}{R^5}$$

$$\geqslant \frac{1}{2} \sum_k \int_{z \in \mathbb{R}^3} \int_{R>0} \chi_{x_k,\, x_{j(k)} \in B(z,R)} \frac{dz\, dR}{R^5}$$

($x_{j(k)}$ = nearest neighbor of x_k)

$$= \frac{\pi}{2} \sum_k |x_k - x_{j(k)}|^{-1}$$

by (2), it follows from (6) that

$$\frac{12\varkappa}{35}\cdot\frac{\pi}{2}\sum_{k=1}^{s}\langle(-\Delta_{x_k})^{1/2}\psi,\psi\rangle\geqslant\left\langle\frac{\varkappa}{2\pi}\cdot\frac{\pi}{2}\sum_k|x_k-x_{j(k)}|^{-1}\psi,\psi\right\rangle.$$

Setting $\varkappa=4$ and $\delta(x_k)=\min_{j\neq k}|x_j-x_k|$, we obtain

$$\frac{24\pi}{35}\sum_{k=1}^{s}\langle(-\Delta_{x_k})^{1/2}\psi,\psi\rangle\geqslant\left\langle\sum_k\frac{1}{\delta(x_k)}\psi,\psi\right\rangle$$

for antisymmetric ψ. This gives a new quantitative meaning to the intuition that kinetic energy keeps Fermions apart. Similarly, if $\delta(y_k)$ denotes the distance from y_k to the nearest electron then for particles in fixed positions we deduce from (2) that

$$\begin{aligned}
V&=\frac{1}{2\pi}\int_{z\in\mathbb{R}^3}\int_{R>0}\{N(N-1)+M(M-1)-2MN\}\frac{dz\,dR}{R^5}\\
&=\frac{1}{2\pi}\int_{z\in\mathbb{R}^3}\int_{R>0}\left\{\left[\left(N-M-\frac{1}{2}\right)^2-\frac{1}{4}\right]-2M\right\}\frac{dz\,dR}{R^5}\\
&\geqslant\frac{1}{2\pi}\int_{z\in\mathbb{R}^3}\int_{R>0}\chi_{N\geq 1}\cdot(-2M)\frac{dz\,dR}{R^5}\\
&=-\frac{1}{\pi}\sum_k\int_{z\in\mathbb{R}^3}\int_{R>0}\chi_{N(z,R)\geq 1}\chi_{y_k\in B(z,R)}\frac{dz\,dR}{R^5}\\
&\geqslant-\frac{1}{\pi}\sum_k\int_{z\in\mathbb{R}^3}\int_{R>\delta(y_k)/2}\chi_{y_k\in B(z,R)}\frac{dz\,dR}{R^5}\\
&=-\frac{8}{3}\sum_k\frac{1}{\delta(y_k)}.
\end{aligned}$$

This is one expression of screening: Each nucleus feels (roughly) only the attraction of the nearest electron. The constant 8/3 here cannot be reduced below 3/2, since we can take a single nucleus midway between two electrons.

Returning to the proof of our main result, we use (2) and (6) to write

$$\begin{aligned}
\langle H\psi,\psi\rangle&\geqslant\left\langle\frac{1}{2\pi}\int_{z\in\mathbb{R}^3}\int_{R>0}\Omega\frac{dz\,dR}{R^5}\psi,\psi\right\rangle\\
&\quad+\sum_k\frac{3\varkappa}{16\pi^2}\int_{z\in\mathbb{R}^3}\int_{R>0}\int_{x'_k\in[{}^cB(z,R)]^{s-1}}T_k(z,R;x'_k)\,dx'_k\frac{dz\,dR}{R^8}\qquad(7)
\end{aligned}$$

$\equiv$ «Potential energy» term + «Kinetic Energy» term, where

$$\Omega=N(N-1)+M(M-1)-2MN+\varkappa(N-1)_+.\qquad(8)$$

Thus $\Omega = \Omega(z, R; x_1 \dots x_s)$.

In the sections to follow, we shall prove that the right-hand side of (7) is positive.

3. Allocating Energy Among the Nuclei

Note first that if $B(z, R)$ contains no nuclei then $M = 0$ and $\Omega \geqslant 0$. Hence for any fixed electrons $x_1 \dots x_s$ and nuclei $y_1 \dots y_{s'}$ we can write

$$V^+ \equiv \frac{1}{2\pi} \int_{z \in \mathbb{R}^3} \int_{R>0} \Omega \frac{dz\, dR}{R^5} \geqslant \sum_l \frac{1}{2\pi} \int_{z \in \mathbb{R}^3} \int_{R>0} \chi_{y_l \in B(z,R)} \frac{\Omega}{M} \frac{dz\, dR}{R^5}.$$

In other words, we allocate the «potential energy» Ω associated to $B(z, R)$ equally among the nuclei in $B(z, R)$.

Next note that $\Omega/M \geqslant e(\varkappa) > 0$ when $\varkappa > 4$ and $N \geqslant 2$; this is immediate from (8). Hence

$$V^+ \geqslant \sum_l \frac{1}{2\pi} \int_{\substack{z \in \mathbb{R}^3 \\ R>0}} [(M - 1 - 2N)\chi_{N \leq 1} + e(\varkappa)\chi_{N \geq 2}]\chi_{y_l \in B(z,R)} \frac{dz\, dR}{R^5}. \quad (9)$$

For fixed l, we shall make the change of variable $z = y_l + Rw$ in (9). In particular, $dz\, dR = R^3\, dw\, dR$, and $B(z, R) = B_l(w, R) \equiv B(y_l + Rw, R)$. We have $y_l \in B(z, R)$ if and only if $|w| < 1$. Define $R(k, l, w) = \sup\{R > 0 \mid B_l(w, R)$ contains no x_j with $j \neq k\}$ $B(k, l, w) = B_l(w, R)$ with $R = R(k, l, w)$.

Observe that $R(k, l, w)$ and $B(k, l, w)$ depend only on x'_k, not on x_k. Note also that $R \to B_l(w, R)$ is an increasing family for each fixed w ($|w| < 1$). Next, note that

$$\begin{bmatrix} \chi_{N(z,R) \geq 2} = \sum_k \chi_{R > R(k,l,w)} \chi_{B(k,l,w)}(x_k) \\ \chi_{N(z,R) \leq 1} = \sum_k \chi_{R < R(k,l,w)} \chi_{B(k,l,w)}(x_k) \end{bmatrix} \quad \begin{matrix} \text{for} \quad z = y_l + Rw, \\ |w| < 1. \end{matrix} \quad (10)$$

To check (10), we fix l, w ($|w| < 1$), and look at the union of the $B_l(w, R)$ containing at most one electron. This will have the form $B_l(w, R^*)$, and will contain exactly one electron. Here we ignore a set of w of measure zero. If that electron is x_k, then $R^* = R(k, l, w)$ and $\chi_{N(z,R) \leq 1} = \chi_{R < R^*}$ for $z = y_l + Rw$. So we obtain (10).

To proof of (10) shows also that $N = N(z, R) = \chi_{B_l(w,R)}(x_k)$ when $z = y_l + Rw$, $|w| < 1$, $N(z, R) \leqslant 1$ and $x_k \in B(k, l, w)$. Putting these remarks into (9), we obtain

$$V^+ \geqslant \frac{1}{\pi} \sum_{k,l} \chi_{B(k,l,w)}(x_k) \int_{|w|<1} \left\{ \frac{e(\varkappa)}{R(k, l, w)} + \int_0^{R(k,l,w)} \left[\frac{M-1}{2} - \chi_{B_l(w,R)}(x_k) \right] \frac{dR}{R^2} \right\} dw.$$

Integrating this against $|\psi(x_1 \dots x_s)|^2 dx_1 \dots dx_s$, we find that

$$\left\langle \frac{1}{2\pi} \int_{z\in\mathbb{R}^3} \int_{R>0} \Omega \frac{dz\, dR}{R^5} \psi, \psi \right\rangle \geqslant \frac{1}{\pi} \sum_{k,l} \int_{|w|<1} \int_{x'_k \in \mathbb{R}^{3s-3}} A(k, l, w, x'_k)\, dx'_k\, dw \tag{11}$$

where $A = A(k, l, w, x'_k)$ is given by

$$A = \int_{B(\bar{R})} |u(x)|^2 \left\{ \frac{e(\varkappa)}{2\bar{R}} + \int_0^R \left[\frac{M(R) - 1}{2} - \chi_{B(R)}(x) \right] \frac{dR}{R^2} \right\} dx \tag{12}$$

and for simplicity we have set

$$\bar{R} = R(k, l, w), \qquad B(R) = B_l(w, R), \qquad M(R) = \text{number of nuclei in } B(R),$$
$$u(x) = \psi(x, x'_k). \tag{13}$$

In particular,

$$B(R) \text{ is a ball of radius } R; \qquad R \to B(R) \text{ is increasing;} \qquad R \to M(R) \text{ is an increasing, positive integer-valued function.} \tag{14}$$

(Recall $y_l \in B_l(w, R)$ so $M(R) \geqslant 1$.)

Thus, the «potential energy» term on the right in (7) is built out of one electron energies of the form (12).

We next make an analogous study of the «kinetic energy» term. Again, we allocate the contribution of $B(z, R)$ equally among the nuclei contained in $B(z, R)$, discarding any balls with no nuclei inside. Thus,

$$T \equiv \sum_k \frac{3\varkappa}{16\pi^2} \int_{z\in\mathbb{R}^3} \int_{R>0} \int_{x'_k \in [{}^cB(z,R)]^{s-1}} T_k(z, R; x'_k)\, dx'_k \frac{dz\, dR}{R^8}$$
$$\geqslant \frac{3\varkappa}{16\pi^2} \sum_{k,l} \int_{z\in\mathbb{R}^3} \int_{R>0} \int_{x'_k \in [{}^cB(z,R)]^{s-1}} \frac{T_k(z, R; x'_k)}{M(z, R)} \chi_{y_l \in B(z,R)}\, dx'_k \frac{dz\, dR}{R^8}.$$

For each y_l we again use the change of variable $z = y_l + Rw$. As before, $y_l \in B(z, R)$ if and only if $|w| < 1$. Also $x'_k \in [{}^cB(z, R)]^{s-1}$ if and only if $R < R(k, l, w)$, the function of x'_k already defined above. Hence, in the notation (13), we have

$$\sum_k \frac{3\varkappa}{16\pi^2} \int_{z\in\mathbb{R}^3} \int_{R>0} \int_{x'_k \in [{}^cB(z,R)]^{s-1}} T_k(z, R; x'_k)\, dx'_k \frac{dz\, dR}{R^8}$$
$$\geqslant \frac{1}{\pi} \sum_{k,l} \int_{|w|<1} \int_{x'_k \in \mathbb{R}^{3s-3}} B(k, l, w, x'_k)\, dx'_k\, dw \tag{15}$$

with $B = B(k, l, w, x'_k)$ given by

$$B = \frac{3\varkappa}{16\pi} \int_0^{\bar{R}} \int_{x,y \in B(R)} |u(x) - u(y)|^2 \, dx \, dy \frac{dR}{R^5 M(R)}. \tag{16}$$

So the «kinetic energy» in (7) is also a superposition of one-electron energies.

Now, comparing (7), (11), (15), we see that $\langle H\psi, \psi \rangle \geqslant 0$ provided we can show that $A + \mathrm{B} \geqslant 0$ with A given by (12) and B by (16). This inequality involves a single electron and a single nucleus, but depends also on an unknown integer-valued function $R \to M(R)$. M has the bizarre effect of screening kinetic energy, while providing a constant term attributable to repulsion in the potential energy. In the next sections, we shall see how to simplify and prove $A + B \geqslant 0$.

4. Simplifying the One-Electron Problem

So far we reduced $\langle H\psi, \psi \rangle \geqslant 0$ to the following problem: Let $R \to B(R)$ be an increasing family of balls, with $B(R)$ having radius R. Let $R \to M(R)$ be an increasing, positive integer-valued function on $(0, \infty)$, and let $\bar{R} > 0$ be given. Prove that

$$\frac{3\varkappa}{16\pi} \int_0^{\bar{R}} \int_{x,y \in B(R)} |u(x) - u(y)|^2 \, dx \, dy \frac{dR}{R^5 M(R)}$$
$$+ \int_{B(\bar{R})} |u(x)|^2 \cdot \left\{ \frac{e(\varkappa)}{2\bar{R}} + \int_0^{\bar{R}} \left[\frac{M(R) - 1}{2} - \chi_{B(R)}(x) \right] \frac{dR}{R^2} \right\} dx \geqslant 0. \tag{17}$$

Now (17) is dilation-invariant, so we may take $\bar{R} = 1$. Moreover, we may assume that the $B(R)$ are all centered at 0, in view of the following simple result.

Lemma 2. *There is a measure-preserving transformation of* $\mathbb{R}^3$ *which carries* $B(R)$ *to* $B(0, R)$ *for all* R.

PROOF OF LEMMA 2. First, we make a preliminary change of variables Φ which carries $B(R)$ to $B(0, R)$; then we correct Φ to make it volume-preserving. To define Φ, let $\xi(R)$ be the center of $B(R)$, and pick a line thorough $\xi(R)$ parallel to $d\xi(R)/dR$. [If $(d\xi/dR)(R) = 0$, then any line through $\xi(R)$ will do, as long as its dependence on R is measurable.] Using this line as a polar axis, we pick in any measurable fashion a meridian on $\partial B(R)$, i.e., a great circle passing through the poles. The poles and the meridian determine latitude θ and longitude ϕ on $\partial B(R)$, and therefore give coordinates (R, θ, ϕ) for an arbitrary

point z of $\mathbb{R}^3$. Our map ϕ simply sends z to the point whose ordinary spherical coordinates are (R, θ, ϕ). Although ϕ needn't preserve volume, the new measure on $\mathbb{R}^3$ induced form ϕ will have the form $g(R, \theta) \cdot R^2\, dR \cos\theta\, d\theta\, d\phi$. Since $\Phi^{-1}(B(0, R)) = B(R)$ is a ball of radius R, we have

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} 2\pi g(R, \theta) \cos\theta\, d\theta = 4\pi. \tag{18}$$

Now to make Φ volume-preserving, we just follow Φ with a change of coordinate $\psi: (R, \theta, \phi) \to (R, \tilde{\theta}, \phi)$ on $\mathbb{R}^3$ given by $\tilde{\theta}(-\pi/2) = -\pi/2$, $\cos\tilde{\theta}\, d\tilde{\theta} = g(R, \theta) \cos\theta\, d\theta$. In view of (18), we have $\tilde{\theta}(+\pi/2) = +\pi/2$, so our map is well-defined on $\mathbb{R}^3$ despite the amibiguities of longitude at the north and south poles. Now $\psi \circ \Phi$ carries $B(R)$ to $B(0, R)$ and preserves volume. □

We could have worked out an explicit formula in place of Lemma 2, since our $B(R) = B(y_l + Rw, R)$.

Returning now to (17), we may suppose $\bar{R} = 1$ and $B(R) = B(0, R)$, so our problem is to prove that

$$Q(u) = \frac{3\varkappa}{16\pi} \int_0^1 \int_{|x|,\,|y|<R} |u(x) - u(y)|^2\, dx\, dy \frac{dR}{R^5 M(R)} + \int_{|x|<1} \left(\epsilon - \frac{1}{|x|}\right) |u(x)|^2\, dx \geqslant 0 \tag{19}$$

with

$$\epsilon = 1 + \frac{e(\varkappa)}{2} + \int_0^1 \frac{M(R) - 1}{2} \frac{dR}{R^2}. \tag{20}$$

Up to now we worked with wave functions in C_0^∞. Since the transformation in Lemma 2 is merely measurable, we must prove (19) for all $u \in L^\infty$. Next, it is trivial to reduce (19) to the special case of radially symmetric u. In fact, for general u we just write $u = v + w$ with v radially symmtric and w having average zero on each sphere. We have

$$Q(u) = Q(v) + \left[\frac{3\varkappa}{16\pi} \int_0^1 \frac{8\pi}{3} R^3 \int_{|x|<R} |w(x)|^2\, dx \frac{dR}{R^5 M(R)} + \int_{|x|<1} \left(\epsilon - \frac{1}{|x|}\right) |w(x)|^2\, dx \right],$$

the point being that $Av_{|x|<R} w = 0$. Define a positive radial function $\tilde{w}$ by the condition $Av_{|x|=R} \tilde{w}^2 = Av_{|x|=R} |w|^2$, $R > 0$; then the expression in brackets evidently dominates $Q(\tilde{w})$. So (19) for radial u implies (19) in the general case.

So far, the increasing function $R \to M(R)$ is arbitrary. However, we observe that for $\varkappa \leqslant 12$, it is enough to prove (19) for functions $M(R)$ taking only the values $1, 2, 3$. In fact, suppose $M(R)$ takes the values $1, 2, 3, \ldots, M_f$ with $M_f \geqslant 4$ and the jump from $M_f - 1$ to M_f occurring at $R = R_f < 1$. Let us change $M(R)$ to $M'(R) = \min\{M(R), M_f - 1\}$, and see how Q changes in (19), (20). The first term in Q increases by

$$\frac{3\varkappa}{16\pi} \int_{R_f}^{1} \int_{|x|,\,|y|<R} |u(x) - u(y)|^2 \, dx\, dy \cdot \left(\frac{1}{M_f - 1} - \frac{1}{M_f}\right) \frac{dR}{R^5}$$

$$= \frac{3\varkappa}{16\pi M_f(M_f - 1)} \int_{R_f}^{1} \left(\frac{8\pi}{3} R^3 \int_{|x|<R} |u(x) - A v_{|y|<R} u(y)|^2 \, dx\right) \frac{dR}{R^5}$$

$$\leqslant \frac{\varkappa}{2(M_f - 1)M_f} \int_{R_f}^{1} \int_{|x|<R} |u(x)|^2 \, dx \frac{dR}{R^2}$$

$$\leqslant \frac{\varkappa}{2M_f(M_f - 1)} \left(\frac{1}{R_f} - 1\right) \|u\|^2.$$

On the other hand, ϵ drops by an amount

$$\frac{1}{2} \int_{R_f}^{1} \frac{dR}{R^2} = \frac{1}{2}\left(\frac{1}{R_f} - 1\right).$$

Hence the new quadratic form is at most equal to the old one plus

$$\left[\frac{\varkappa}{M_f(M_f - 1)} - 1\right] \cdot \frac{1}{2}\left(\frac{1}{R_f} - 1\right) \|u\|^2.$$

If $\varkappa \leqslant 12$ and $M_f \geqslant 4$, then the quantity in brackets is negative or zero. Hence $Q(u)$ decreases if we pass form $M(R)$ to $M'(R)$. Repeating this argument, we reduce (19) to the case $M(R) \leqslant 3$, all R. Since also $M(0) = 1$, the function $M(R)$ is now completely determined by the positions of its two jumps.

In the next section, we study the Euler-Lagrange equation associated to Q, and reduce (19) to a statement about ODE's.

5. Ordinary Differential Equations for the One Electrom Problem

We start off this section with some elementary estimates for quadratic forms related to (19).† First of all, for $\varphi(x)$ radial, square-integrable, and vanishing

† Throughout this section our basic Hilbert space is $L^2_{\text{radial}}(B(0, 1))$.

near 0 and ∞, we have the basic inequality

$$\frac{3}{2\pi}\int_0^\infty\int_{|x|,\,|y|<R}|\varphi(x)-\varphi(y)|^2\,dx\,dy\frac{dR}{R^5}-\int_{\mathbb{R}^3}|\varphi(x)|^2\frac{dx}{|x|}\geqslant 0. \tag{20a}$$

This may be verified by using the Mellin transform, that is, writing

$$\varphi(x)=\int_{-\infty}^{\infty}|x|^{-1+iY}[\tilde{\varphi}(Y)]\,dY.$$

The left side of (20*a*) then becomes

$$\int_{-\infty}^{\infty}m(Y)|\tilde{\varphi}(Y)|^2\,dY,$$

and one simply calculates $m(Y)$ and checks that it is non-negative.

Next, let $\psi(x)$ be radial and square-integrable on $|x|<1$, with $\psi(x)=0$ for $|x|\ll 1$. We set $\varphi(x)=\psi(x)$ for $|x|<1$, $\varphi(x)=0$ for $|x|\geqslant 1$, and apply (20*a*). Since

$$\int_{|x|,\,|y|<R}|\varphi(x)-\varphi(y)|^2\,dx\,dy\leqslant\frac{8\pi}{3}R^3\|\varphi\|^2=\frac{8\pi}{3}R^3\|\psi\|^2\quad\text{for}\quad R>1,$$

we obtain

$$\frac{3}{2\pi}\int_0^1\int_{|x|,\,|y|<R}|\psi(x)-\psi(y)|^2\,dx\,dy\frac{dR}{R^5}-\int_{|x|<1}|\psi(x)|^2\frac{dx}{|x|}\geqslant -4\|\psi\|^2. \tag{21}$$

Now fix $0<R_1\leqslant R_2\leqslant 1$, and define:

$$M(R)=\begin{bmatrix}1 & \text{for} & 0<R<R_1\\ 2 & \text{for} & R_1\leqslant R<R_2\\ 3 & \text{for} & R_2\leqslant R<1\end{bmatrix}$$

$$\mathcal{Q}(\psi)=\frac{3\varkappa}{16\pi}\int_0^1\int_{|x|,\,|y|<R}|\psi(x)-\psi(y)|^2\,dx\,dy\frac{dR}{R^5M(R)}-\int_{|x|<1}|\psi(x)|^2\frac{dx}{|x|} \tag{22}$$

for $\psi\in\mathfrak{D}_0=\{$square-integrable radial functions on $|x|<1$, vanishing near the origin$\}$.

Note that

$$\mathcal{Q}(\psi)=\frac{3\varkappa}{8\pi}\int_{|x|<1}\overline{\psi(x)}\left[\int_{|x|}^1\int_{|y|<R}(\psi(x)-\psi(y))\,dy\frac{dR}{R^5M(R)}\right]dx$$
$$-\int_{|x|<1}|\psi(x)|^2\frac{dx}{|x|}=\langle A\psi,\psi\rangle$$

for $\psi \in \mathfrak{D}_0$, with

$$A\psi(x) = W(x)\psi(x) - \int_{|y|<1} K(x,y)\psi(y)\,dy \tag{23}$$

$$W(x) = -\frac{1}{|x|} + \frac{3\varkappa}{8\pi}\int_{|x|}^{1}\int_{|y|<R} dy\,\frac{dR}{R^5 M(R)} \tag{24}$$

$$K(x,y) = \frac{3\varkappa}{8\pi}\int_{\max(|x|,|y|)}^{1} \frac{dR}{R^5 M(R)} \tag{25}$$

If $\varkappa \geqslant 8$, then comparison of (22), (21) shows that $\langle A\psi, \psi\rangle = \mathcal{Q}(\psi) \geqslant$ $\geqslant -C(R_1, R_2)\|\psi\|^2$ for $\psi \in \mathfrak{D}_0$.

Since also $\langle A\psi, \varphi\rangle = \langle \psi, A\varphi\rangle$ for $\psi, \varphi \in \mathfrak{D}_0$, we recall from basic functional analysis that A extends from $\mathfrak{D}_0$ to a self-adjoint operator $\bar{A}$, the Friedrichs extension of A. Specifically, $u \in$ Domain $(\bar{A})$ and $\bar{A}u = v$ if and only if we have $\langle u, A\psi\rangle = \langle v, \psi\rangle$ for $\psi \in \mathfrak{D}_0$, and there exist $\psi_k \in \mathfrak{D}_0$ converging to u in L^2-norm, with $\mathcal{Q}(\psi_k - \psi_l) \to 0$ as $k, l \to \infty$. These facts are contained in Dunford-Schwartz Vol. 2, Cor. 12.6.3.

It will be useful to have a concrete sufficient condition for a given radial $u \in L^2(|x| < 1)$ to belong to Domain $(\bar{A})$. We first note that

$$v(x) = W(x)u(x) - \int_{|y|<1} K(x,y)u(y)\,dy \tag{26}$$

is well-defined for all $u \in L^2$, because of the simple estimate $|K(x,y)| \leqslant$ $\leqslant C(|x| + |y|)^{-4}$. So if

$$\int_{|x|<1} |v(x)|^2\,dx < \infty \quad \text{and} \quad \int_{|x|<1} |u(x)|^2 \frac{dx}{|x|} < \infty \tag{27}$$

then $u \in$ Domain$(\bar{A})$, and $\bar{A}u = v$.

To see this, we first have to check that $\langle u, A\psi\rangle = \langle v, \psi\rangle$ for $\psi \in \mathfrak{D}_0$. That is easy, because the relevant integrals converge absolutely when $\psi \in \mathfrak{D}_0$, and hence their order can be switched. We much also find $\psi_k \to u$ in L^2 with $\mathcal{Q}(\psi_k - \psi_l) \to 0$. It is enough to take $\psi_k(x) = u(x)\chi_{|x|>1/k}$, and note that

$$\mathfrak{D}(\psi_k - \psi_l) \leqslant C\int_{|x|<\max[(1/k),(1/l)]} |u(x)|^2 \frac{dx}{|x|} \to 0$$

if (27) holds. Thus we have a sufficient condition (27) for $u \in$ Domain $(\bar{A})$, and a formula (26) for $\bar{A}u$.

The main point of this section is to reduce questions about the spectrum of $\bar{A}$ to a study of ordinary differential equations.

From now on, we fix $\varkappa = 12$. Let $R_1 = e^{t_1}$, $R_2 = e^{t_2}$, so that $t_1 < t_2 < 0$. Fix $\epsilon > 1$, not necessarily given by (20).

The relevant differential equations turn out to be as follows.
For $t_2 \leqslant t \leqslant 0$ we solve

$$\frac{d^2\mathfrak{F}}{dt^2} + 4\frac{d\mathfrak{F}}{dt} + \frac{6}{1 + [\epsilon - 2]e^t}\mathfrak{F} = 0 \tag{28}$$

with boundary conditions

$$\mathfrak{F}(0) = 0 \qquad \frac{d\mathfrak{F}}{dt}(0) = 1. \tag{29}$$

Having found $\mathfrak{F}$, $d\mathfrak{F}/dt$ at $t = t_2$, we next solve

$$\frac{d^2\mathfrak{G}}{dt^2} + 4\frac{d\mathfrak{G}}{dt} + \left(\frac{9}{2 + [\epsilon - 2 - e^{-t_2}]e^t}\right)\mathfrak{G} = 0 \tag{30}$$

in the region $t_1 \leqslant t \leqslant t_2$, with boundary conditions

$$\mathfrak{G}(t_2) = \mathfrak{F}(t_2), \qquad 2\frac{d\mathfrak{G}}{dt}(t_2) = 3\frac{d\mathfrak{F}}{dt}(t_2). \tag{31}$$

Next we solve

$$\frac{d^2\mathfrak{K}}{dt^2} + 4\frac{d\mathfrak{K}}{dt} + \left(\frac{18}{5 + [\epsilon - 2 - e^{-t_2} - 3e^{-t_1}]e^t}\right)\mathfrak{K} = 0 \tag{32}$$

in the region $-\infty < t \leqslant t_1$, with boundary condition

$$\mathfrak{K}(t) \sim \exp\left(\left[-2 + \sqrt{\frac{2}{5}}\right]t\right) \quad \text{as} \quad t \to -\infty. \tag{33}$$

Then we say that a *match* occurs for ϵ, t_1, t_2 if the vectors

$$\left(\mathfrak{K}(t_1), \frac{d\mathfrak{K}}{dt}(t_1)\right) \quad \text{and} \quad \left(\mathfrak{G}(t_1), 2\frac{d\mathfrak{G}}{dt}(t_1)\right)$$

are proportional. Define the *discriminant* for ϵ, t_1, t_2 as

$$\left[2\frac{d\mathfrak{G}}{dt}(t_1)\cdot\mathfrak{K}(t_1) - \mathfrak{G}(t_1)\frac{d\mathfrak{K}}{dt}(t_1)\right] \Big/ \left(|\mathfrak{K}(t_1)| + \left|\frac{d\mathfrak{K}}{dt}(t_1)\right|\right)\cdot$$
$$\cdot\left(|\mathfrak{G}(t_1)| + \left|\frac{d\mathfrak{G}}{dt}(t_1)\right|\right).$$

Lemma 3. *For $\epsilon > 1$, we have $-\epsilon \in \text{Spectrum}(\bar{A})$ if and only if a match occurs.*

PROOF. Let us study the equation $(\epsilon + \bar{A})u = f$ for radial $f \in C_0^\infty(|x| < 1)$ vanishing near the origin. According to (26), (27), it will be enough to solve

$$(\epsilon + W(r))u(r) - \int_0^1 K(r, s)4\pi s^2 u(s)\, ds = f(r) \tag{34}$$

with

$$\int_0^1 \frac{|u(r)|^2}{r} \cdot 4\pi r^2\, dr < \infty,$$

u continuous away from 0. (Finally we have written radial u on $\mathbb{R}^3$ as a function of one variable.)

Now

$$K(r, s) = H(\max\{r, s\}) \quad \text{with} \quad H'(r) = -\frac{3\varkappa}{8\pi} \cdot \frac{1}{r^5 M(r)}.$$

So, differentiating (34) in r, we get

$$\frac{d}{dr}[(\epsilon + W)u] + \frac{3\varkappa}{8\pi r^5 M(r)} \int_0^r 4\pi s^2 u(s)\, ds = \frac{d}{dr} f. \tag{35}$$

Also $(\epsilon + W)u(l) = 0$ by (34), since $f(1) = 0$ and $K(1, s) = 0$ for $s < 1$.

Conversely, $(\epsilon + W)u(1) = 0$ and (35) yields (34) by integration. So we must solve (35) with the boundary conditions: $(\epsilon + W)u(1) = 0$, $\int_0^1 r|u(r)|^2\, dr < \infty$, u continuous away from 0.

Next multiply (35) by $r^5 M(r)$ and differentiate away from the jumps of $M(r)$. The result is

$$M(r)\frac{d}{dr} r^5 \frac{d}{dr}[(\epsilon + W)u] + \frac{3\varkappa}{2} r^2 u(r) = M(r)\frac{d}{dr} r^5 \frac{d}{dr} f \tag{36}$$

away from the jumps of $M(r)$. Also from (35), we see that

$$M(r)\frac{d}{dr}[(\epsilon + W)u - f]$$

is continuous across the jumps of $M(r)$. Conversely, if

$$M(r)\frac{d}{dr}[(\epsilon + W)u - f]$$

is continuous across the jumps of $M(r)$, then we may integrate (36) to obtain

$$r^5 M(r)\frac{d}{dr}[(\epsilon + W)u] + \frac{3\varkappa}{8\pi} \int_0^r 4\pi s^2 u(s)\, ds = r^5 M(r)\frac{d}{dr} f + (\text{Const.}).$$

If as $r \to 0$ we have $u(r) = O(r^{-\alpha})$ and $(du/dr) = O(r^{-\alpha-1})$, with $\alpha < 1$, then the Constant term must vanish, and we recover (35). Also $\int_0^1 r|u(r)|^2\,dr < \infty$ if $u = O(r^{-\alpha})$. So $(\epsilon + \bar{A})u = f$, provided (36) holds, and:

$$(\epsilon + W)u(1) = 0,\ u \text{ continuous on } (0,1),\ M(r)\frac{d}{dr}[(\epsilon + W)u - f]$$
$$\text{continuous across the jumps of } M(r);\ u(r),\ ru'(r) = O(r^{\delta-1}) \text{ as } r \to 0. \tag{37}$$

Now we can check whether $-\epsilon \in \mathrm{Spec}(\bar{A})$. If a match occurs, then we can solve (36), (37) with $f = 0$. In fact, taking $r = e^t$ and $(\epsilon + W)u(r) = \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ in the three regions $[t_2, 0]$, $[t_1, t_2]$, $(-\infty, t_1)$, we find that (28)...(33) with a match amount to (36), (37). We use $\epsilon > 1$ because then $\epsilon + W \neq 0$ everywhere. Thus $\epsilon + \bar{A}$ has nontrivial kernel, and $-\epsilon \in \mathrm{Spec}(\bar{A})$. Conversely, suppose a match doesn't take place at t_1, t_2. For $f \in C_0^\infty(0,1)$ we can then solve (36), (37) by means of a Green's function. Setting $F = (\epsilon + W)u - f$, we rewrite (36), (37) in the form

$$M(r)\frac{d}{dr}r^5\frac{d}{dr}F + \frac{3\varkappa}{2}\left(\frac{r^2}{\epsilon + W}\right)F = -\frac{3\varkappa}{2}\left(\frac{r^2}{\epsilon + W}\right)f \tag{38}$$

$$F(r), \qquad M(r)\frac{dF}{dr} \quad \text{continuous across the jumps of } M(r) \tag{39}$$

$$F(1) = 0 \tag{40}$$

$$F(r) = O(r^{-2+\delta}), \qquad F'(r) = O(r^{-3+\delta}) \quad \text{as} \quad r \to 0. \tag{41}$$

To find a Green's a function for (38)...(41), we first set $f = 0$ in (38), and pick out two special solutions of the homogeneous equation: Set $F_+(r)$ = solution of (38), (39), (40); set $F_-(r)$ = solution of (38), (39), (41). Since a match does not occur, these solutions are distinct. Our Green's function takes the form

$$K(r,s) = \begin{bmatrix} C_1(s)F_+(r) & \text{if} \quad r > s \\ C_2(s)F_-(r) & \text{if} \quad r \leqslant s \end{bmatrix}$$

where $C_1(s)$, $C_2(s)$ are picked so that $r \to K(r,s)$ is continuous at $r = s$, while

$$r \to \frac{\partial K(r,s)}{\partial r.} \quad \text{jumps by an amount} \quad \frac{1}{M(s)s^5} \quad \text{at} \quad r = s$$

As $_r \to 0$, we have $F_-(r) \sim r^{-2+\sqrt{2/5}}$, $F_+(r) \sim r^{-2-\sqrt{2/5}}$. Consequently, we may estimate $C_1(s)$, $C_2(s)$ as $s \to 0$; in fact $|C_1(s)| \leqslant Cs^{\sqrt{2/5}}$, $|C_2(s)| \leqslant Cs^{-\sqrt{2/5}}$

for $0 < s \leqslant 1$, so

$$|K(r,s)| \leqslant Cr^{-2}\left[\min\left(\frac{r}{s},\frac{s}{r}\right)\right]^{\sqrt{2/5}}. \tag{42}$$

Equations (38)...(41) are now solved by taking

$$F(r) = \int_0^1 \frac{3\varkappa}{2}\left(\frac{s^2}{\epsilon + W(s)}\right) f(s)\cdot K(r,r)\,ds, \tag{43}$$

$$u = \frac{F+f}{\epsilon + W}. \tag{44}$$

Having solved $(\epsilon + \bar{A})u = f$, we next note the a-priori bound

$$\|u\|_{L^2(|x|<1)} \leqslant C\|f\|_{L^2(|x|<1)}. \tag{45}$$

In fact, from (42), (43), (44), we have

$$|u(r)| \leqslant CTf(r) = C\int_0^\infty \left[\mathrm{MIN}\left(\frac{r}{s},\frac{s}{r}\right)\right]^{\sqrt{2/5}} f(s)\frac{ds}{s},$$

and (45) follows at once by the Mellin transform.

Now if ϵ is any number, $\bar{A}$ is any self-adjoint operator, and for f in a dense subspace we can solve $(\epsilon + \bar{A})u = f$ with a bound $\|u\| \leqslant C\|f\|$, then $-\epsilon \notin$ Spectrum $(\bar{A})$, as follows at once from the spectral theorem. Hence in the context of Lemma 3 we have $-\epsilon \notin$ Spectrum $(\bar{A})$ if a match does not occur. □

We can now state what we need about ODE's in order to prove our main result, namely

Lemma 4. *No match occurs in any of the following ranges of* t_1, t_2, ϵ:

(*a*) $t_1 = t_2 = 0, \qquad \frac{9}{4} \leqslant \epsilon \leqslant 4,$

(*b*) $-2 \leqslant t_1 \leqslant t_2 \leqslant 0, \qquad \epsilon = \frac{5}{4} + \frac{1}{2}(e^{-t_1} + e^{-t_2}),$

(*c*) $t_1 = -2, \qquad -2 \leqslant t_2 \leqslant 0, \qquad \frac{1}{2}(e^{-t_1} + e^{-t_2}) \leqslant \epsilon \leqslant \frac{5}{4} + \frac{1}{2}(e^{-t_1} + e^{-t_2}),$

(*d*) $t_1 = -2, \qquad t_2 = 0, \qquad \frac{e^2}{2} \leqslant \epsilon \leqslant \frac{e^2}{2} + \frac{7}{4}.$

We check here that Lemma 4 yields (19), (20) and thus proves our main result. In the sections to follow, we give a computer-assisted proof of Lemma 4. Define $\lambda_1(t_1, t_2) = \inf \text{Spectrum}(\bar{A})$ with $R_1 = e^{t_1}$, $R_2 = e^{t_2}$; and set $\epsilon(t_1, t_2) = = 5/4 + (e^{-t_1} + e^{-t_2})/2$. Note that $\lambda_1(t_1, t_2)$ varies continuously in t_1, t_2, since a small change in t_1, t_2 will change $\bar{A}$ by adding an operator of small norm. We shall prove (19), (20) by looking at the four cases

(α) $R_1 = R_2 = 1$,
(β) $e^{-2} \leqslant R_1 \leqslant R_2 \leqslant 1$,
(γ) $R_1 < e^{-2}$, $\quad R_1 \leqslant R_2 \leqslant e^2 R_1$,
(δ) $e^2 R_1 < R_2 \leqslant 1$.

Note that (19), (20) with $R_1 = e^{t_1}$, $R_2 = e^{t_2}$ is equivalent to $\lambda_1(t_1, t_2) \geqslant \geqslant -\epsilon(t_1, t_2)$.

Take case (α) first. Estimate (21) gives $\lambda_1(0, 0) \geqslant -4$, while Lemmas 3 and 4($a$) show that $\lambda_1(0, 0)$ cannot be in $[-4, -\epsilon(0, 0)]$. Hence $\lambda_1(0, 0) > -\epsilon(0, 0)$, and case ($\alpha$) is settled. Now suppose (19), (20) are violated for some R_1, R_2 in case (β). Then we must have $\lambda_1(t_1, t_2) = -\epsilon(t_1, t_2)$ for some t_1, t_2 with $-2 \leqslant t_1 \leqslant t_2 \leqslant 0$. (This is required in view of case (α) and the continuity of λ_1, ϵ.) On the other hand, Lemmas 3 and 4(b) imply $\lambda_1(t_1, t_2) \neq -\epsilon(t_1, t_2)$ for $-2 \leqslant t_1 \leqslant t_2 \leqslant 0$. So case ($\beta$) is settled also.

Next, by combining case (β) with Lemmas 3 and 4(c), we get

$$\frac{3\varkappa}{16\pi} \int_0^1 \int_{|x|, |y| < R} |u(x) - u(y)|^2 \, dx \, dy \frac{dR}{R^5 M(R)} - \int_{|x| < 1} |u(x)|^2 \frac{dx}{|x|} + \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right) \|u\|^2 \geqslant 0 \qquad (46)$$

where

$$M(R) = \begin{bmatrix} 1 & \text{for} & R & < R_1 = e^{-2} \\ 2 & \text{for} & R_1 \leqslant R & < R_2 \\ 3 & \text{for} & R_2 \leqslant R & < 1 \end{bmatrix} \quad \text{and} \quad e^{-2} \leqslant R_2 \leqslant 1.$$

Similarly, using case (β) with Lemmas 3 and 4(d) yields

$$\frac{3\varkappa}{16\pi} \int_0^1 \int_{|x|, |y| < R} |u(x) - u(y)|^2 \, dx \, dy \frac{dR}{R^5 M(R)} - \int_{|x| < 1} |u(x)|^2 \frac{dx}{|x|} + \frac{1}{2R_1} \|u\|^2 \geqslant 0 \qquad (47)$$

where

$$M(R) = \begin{bmatrix} 1 & \text{for} & R & < R_1 = e^{-2} \\ 2 & \text{for} & R_1 \leqslant R & \leqslant 1 \end{bmatrix}.$$

Inequalities (46), (47) will take care of cases (γ), (δ). In fact, to handle (γ), set $R_* = e^2 R_1$ and rescale (46):

$$\frac{3\varkappa}{16\pi}\int_0^{R_*}\int_{|x|,\,|y|<R}|u(x)-u(y)|^2\,dx\,dy\frac{dR}{R^5M(R)}-\int_{|x|<R_*}|u(x)|^2\frac{dx}{|x|}+\left(\frac{1}{2R_1}+\frac{1}{2R_2}\right)\int_{|x|<R_*}|u(x)|^2\,dx\geqslant 0.$$

Adding to this the obvious inequalities

$$-\int_{|x|>R_*}|u(x)|^2\frac{dx}{|x|}+\left[\frac{1}{2R_1}+\frac{1}{2R_2}\right]\int_{|x|>R_*}|u(x)|^2\,dx\geqslant 0$$

$$\frac{3\varkappa}{16\pi}\int_{R_*}^{1}\int_{|x|,\,|y|<R}|u(x)-u(y)|^2\,dx\,dy\frac{dR}{R^5M(R)}+\frac{e(\varkappa)}{2}\|u\|^2\geqslant 0,$$

we obtain (19), (20) in case (γ). Finally, in case (δ), we rescale (47) to get:

$$\frac{3\varkappa}{16\pi}\int_0^{R_*}\int_{|x|,\,|y|<R}|u(x)-u(y)|^2\,dx\,dy\frac{dR}{R^5M(R)}-\int_{|x|<R_*}|u(x)|^2\frac{dx}{|x|}+\frac{1}{2R_1}\int_{|x|<R_*}|u(x)|^2\,dx\geqslant 0.$$

Adding to this the obvious inequalities

$$-\int_{|x|>R_*}|u(x)|^2\frac{dx}{|x|}+\frac{1}{2R_1}\int_{|x|>R_*}|u(x)|^2\,dx\geqslant 0$$

$$\frac{3\varkappa}{16\pi}\int_{R_*}^{1}\int_{|x|,\,|y|<R}|u(x)-u(y)|^2\,dx\,dy\frac{dR}{R^5M(R)}+\left(\frac{1}{2R_2}+\frac{e(\varkappa)}{2}\right)\|u\|^2\geqslant 0,$$

we obtain (19), (20) in case (δ).

Thus, our main result is reduced to Lemma 4. Since we took $\varkappa = 12$, our main result takes the form

$$\sum_k(-\Delta_{x_k})^{1/2}+\alpha\left[\sum_{j<k}\frac{1}{|x_j-x_k|}+\sum_{j<k}\frac{1}{|y_j-y_k|}-\sum_{j,k}\frac{1}{|x_j-y_k|}\right]\geqslant 0$$

with

$$\alpha=\frac{2}{\pi}\cdot\frac{35}{144}\sim\frac{1}{2.06\pi}.$$

6. Rigorous computer solution of ordinary differential equations

In this section, we explain in detail how Lemma 4 was actually established. In broad terms, what we did is to reduce the statements to be checked to a finite computation (expand the relevant quantities in appropriate parameters and bound the remainders) and then, perform this finite computation with the help of an electronic device (a VAX 11/750).

The fact that even finite arithmetic operations can be performed rigorously by a computer is not obvious (and indeed somewhat surprising for many of today's mathematicians).

The most serious reason is that computers are not equipped with real arithmetic, but only with some caricature defined on a finite set of numbers (those the machine can understand, usually called machine numbers) and yielding only approximate answers. The difficulty is that when approximations are taken a large number of times (e.g. the number of operations peformed by a computer in a minute) the result may not be a reasonable approximation any more. (For example, most computers are unable to guarantee even the sign of $f^{100}(\sqrt{3}/2)$ where $f(x) = 1 - 2x^2$.)

The cure for this difficulty is to work systematically with upper and lower bounds of numbers rather than with approximations to them, and to implement an arithmetic working on bounds (e.g. the addition for upper bounds should produce an upper bound to the true sum of the two original upper bounds).

That is, we work with intervals, and construct operations that given two intervals produce a third. That third interval contains the result of the true operation when the arguments range in the original intervals.

In other words, we build an arithmetic that takes into account roundoff errors. Then we will perform some analysis to reduce the original problem to an arithmetic one; that is, we take into account both roundoff and truncation errors.

The introduction of interval arithmetic can be attributed to R.E. Moore in 1962 and has a developed literature (see [Mo] [KM] and references there). The application to mathemtical problems was fostered by the pioneering work of Lanford [La] who found a quite remarkable abstract framework in which several hitherto unapproachable problems could be reduced to arithmetic statements, and wrote a package well adapted to those needs. Under his influence, similiar strategies have been applied for several other problems [E, K, *W*], [E, W] [MP] [LL] [Me].

In all the references above (except in [MP]) interval arithmetic was used to provide rigorous error bounds for certain computations; here we want to show that certain conditions do not happen for parameters ranging in a cer-

tain region. Even if the basic operations are the same, we are forced to deal with intervals much wider than the roundoff unit, and some considerations, safely ignorable for sharp intervals, come into the foreground (e.g. coherence and subdistributivity). Several features —implemented in our package— desirable for the handling of sharp intervals (e.g. double precision, 0 and 1 being neutral for addition and multiplication) become irrelevant when the intervals are «fat» and presumably take a toll in execution time, although, clearly, not in correctness. We preferred not to revise for speed, since in the present form the package had alredy been checked.

In a first subsection, we explain how the arithmetic package was implemented in our case.

In a second subsection, we document the reduction of the original question to an arithmetic problem. We refer the reader to [T] for computer concepts we use without explanation.

A. Interval arithmetic

Our implementation of interval arithmetic was done using the facilities provided by the manufacturer. The alternative of writing from scratch the arithmetic subroutines not only would have required more programming effort but also have resulted in programs running between 10 to 100 times slower.

The most authoritative source about what the VAX can do is [DEC]. For the purposes of this section, this could be the final description of our computer.

The only objects we are concerned about are «machine integers» and «machine reals» which are respectively

$$\mathcal{I} = \{(S, I)|, S = +1, -1, I \in N, -2^{31} \leqslant I < 2^{31}\}$$
$$\mathcal{R} = \{(S, e, m) \mid S = -1, +1; 0 \leqslant e \leqslant 2^9 - 1\}$$
$$0 \leqslant m \leqslant 2^{56} - 1\}$$

When $e \neq 0$, $(S; e, m)$ is supposed to mean

$$S \cdot 2^{(e-128)}\left(\frac{m + 2^{56}}{2^{57}}\right).$$

When $e = 0$, and $S = +1$, (S, e, m), is supposed to mean zero regardless of what m is.

When $e = 0$ and $s = -1$, this number is not supposed to mean anything and according to the manual, appearance of this number could cause unpredictable results [DEC] p. 34. The machine can understand and manipulate other numbers, buy they do not appear in our programs.

We will also be reading and writing numbers, which involves switching from the internal representation of the machine to another more familiar to the users. Since, for our problem, reading and writing was only a heuristic aid, we did not care about writing careful input and output subroutines. (This was done e.g. in [LL].)

On machine integers and reals we will be performing several operations: arithmetic operations, comparisons and conversions from integer to real.

We will assume —and we checked in several crucial cases— that the comparisons and type conversions worked correctly. A moment's reflection will show that there is no hope that the arithmetic operations work as those on real numbers.

Arithmetic operations on integers are performed mod 2^{32} and, with this proviso, they are exact. Real operations are more complicated. In [DEC] appendix H, it is stated that, when both the oeprands and the result are in $\mathfrak{R}$, the result is correct to «1/2 of the least significant bit.»

When the result is not in the range of acceptable numbers, the computer reports this condition but goes ahead. (These «exceptions» are used by the compilers and we will explain how the ones we used dealt with them.) This leads to a «circular arithmetic» in integers —arithmetic modulo 2^N, with the fundamental domain depending on which way the numbers are represented [T] p. 447 ($N = 32$ for the VAX 11/750 and notation of 2's complement)— and quite unpredictable results when there is overflow in floating point numbers.

With this information at hand, it is clear what we should do to obtain a controlled arithmetic: we should watch out for the signals that report an improper operation and modify the result in ways suitable for the arithmetic of upper and lower bounds. The «1/2 of least significant bit» accuracy implies that, any function right: $\mathfrak{R} \to \mathfrak{R}$ satisfying

$$\text{right}\ (x) > x$$

will also have the property

$$\text{right}\ (x \&_c y) > x \& y$$

where $\&$ and $\&_c$ stand respectively for any of the four arithmetic operations and its computer implementation. This holds (provided the operations has a result in $\mathfrak{R}$). Similarly for an operation left satisfying $\text{left}\ (x) < x$.

Even if any such operations left, right are acceptable, it is clear that it would be to our advantage to make them as small modifications as possible. The best we could do is just modify the least significant bit.

The programs considered here will be written in a high level language that will eventually get translated into machine language. Usually, this translation

will include handling of the signals of incorrect operations. The most usual convention for modern compilers —we checked that it was the case for the compilers we used— is as follows: let the circular arithmetic of the integers go unnoticed: when the result of an operation in $\mathfrak{R}$ has too small absolute value, set the result to zero. When the absolute value is too big, or there is a division by zero, print error messages and terminate the program. (In our system, the C-library included facilities to change the handling of those exceptions, but the standard ones were adequate.)

With this way of handling the exceptions, the only thing we have to worry about is zero which many appear as the result of an underflow. A moment's reflection will show that rounding to any number strictly different from zero is still adequate; but we should remember that, for the internal representation, this means modifying e and not m.

Adequate versions of right and left could be the ones listed at the end where one plus, one minus, zero plus and zero minus are global variables. Notice that their value does not matter too much, because at the end we check the strict inequality of the return value of the argument and as we pointed out, by the «1/2 LSB» property of the VAX operation this suffices.

Notice that, even in the case that one plus and one minus did not produce a sufficient effect, the only thing that would happen is termination of the program and not propagation of the wrong result.

We also used another version of these routines that only modifies the last bit. Unfortunately, since Pascal does not have bitwise manipulations this version was written in Fortran 77 and then grafted into the Pascal programs, a rather delicate task.

From the strictly logical point of view the first version is enough, but the second one also provided some insight into the workings of the machine.

The main idea is to use Fortran integer arithmetic to access the bits of a double precision number. This is achieved by using the equivalence statement between an array of two byte integers and a copy of the double precision argument. Double precision numbers are stored in four consecutive words whose significance decreases with increasing address (the first one contains the sign and exponent). Inside each word, the significance increases with decreasing address [DEC].

Integers are stored in a word with increasing significance given to decreasing addresses. Moreover, they are stored in two's complement notation with bit 15 designating the sign.

To increase the last bit of m, we increase the least significant bit of the last element of the array and check if this produced a carry. If it did not, we are satisfied, but if it did, we increment the next one. (Here, the fact that the integers are represented as two's complement is used in two places: one that increasing by 1 as a signed quantity is the same as an unsigned quantity and,

second that the carry in unsigned addition is produced exactly when the result —intepreted in two's complement is zero.)

This process of incrementing and carrying, if necessary, is repeated without any change till the first word where we should worry about the extra structure of exponent and sign. By a truly miraculous coincidence it turns out that the same prescription continues to work in the first word. We can think of the mantissa as representing .1m, where the leading 1 is not explicitly written. If by adding 1, the mantissa overflows, then the incremented mantissa is 1.00...0 whose internal representation would be achieved by incrementing the exponent by 1 and setting the mantissa to zero. This is what happens when we increment the first word as an unsigned bit pattern.

The only thing to do is to check that this carry did not affect the sign bit. If it did, we stop.

The exceptional case of zero is treated at the beginning and what we do is to set a bit pattern which corresponds to $e = 1$, $m = 0$, $s = +1$.

Similar considerations apply to the routine down moving the number towards zero. The only difference is that we set to zero all the numbers that, when divided by 4 underflow. Even if wasteful, this is expedient and certainly correct (because of the 1/2 LSB property).

Thus we have procedures «up», «down» which increase or decrease slightly the absolute value of a machine number. Once we have «up», «down», writing «right» and «left» is very easy.

Other versions acieving the same were also written for the VAS 11/75 in C by *D.* Rana on other principles, and there are versions working for other computers described in the references. Translating our Fortran routines to C is very easy using type casts rather than equivalence to assign double precision to integer arrays. Since quite a lot of system programming is done in *C*, the compilers tend to be well taken care of and, since the language is better designed than Fortran, it is harder to make mistakes writing the compiler. Moreover, in our environment, it turned out that grafting C was easier than grafting Fortran. Even if these considerations are quite machine dependent (and surely are reversed in other machines), we wanted to make the reader aware of this difficulty.

Once we have this rounding constructing an arithmetic package is reasonably straighforward.

We introduce three new structures which are bound, interval, complex: consisting respectively of a real field .top, supposedly a positive upper bound; two real fields .upper and .lower upper and lower bounds for the interval; two interval fields .re, .im which are intervals containing the real and imaginary part.

This package is really more than what the problem in its present version needs —for example, we do not make any use of complex numbers— but we

had used it for other type of problems —in fact, many of the names come from [LL] where the basic operations are implemented in a very different way.

We hope that the names of the subroutines are rather self explanatory and the working straightforward. Usually the name has some remainder of the function and a few extra letters remingding the reader of the types on which it acts (i:integer, v:interval, b:bound, c:complex).

We have tried to enforce the compatibility with standard Pascal and, with two exceptions, included mainly for experimenting, have avoided functions returning structures —which is de facto standard even though forbidden in the original description of the language [J.W.].

Even if the type conversions and comparisons were done using only those the compiler provided, we have included comparison and conversions as explicit subroutines with the hope of having a clean interface with the machine. If only those we supply are used, changes in the internal representation should only affect this level and not the upper levels. (We sometimes slackened and used compiler comparisons but they could easily be removed.) Notice that, when we want to assign an integer value to a real, we first assign it to an integer variable and, then, the integer to a real. This produces something strictly correct whereas writing x: = 2.0; would have made the computer involve the routine to convert decimals to internal, which gets the last bits wrong.

In particular, this interface was implemented by another different method in [LL]. Changing the programs there to run in the VAX was a question of a few hours of works. To adapt the present programs to the IBM PC would take only slightly more; this could be interesting because, for those susbstantial programs we run on both machines, the speeds were comparable in CPU time and the small computer was superior in real time since the big one has to be shared.

One important point of this package is the variable «status,» which is initialized to 0 at the beginning and, will take the value of 1 if some division by an interval containing 0 is requested.

The reason why we did that rather than killing the program from the division subroutine is, again, to allow different implementations of it.

One important part of checking the results is to check that this «sticky bit» is still zero.

We also had to write some subroutines for special functions —the only one used is the interval valued exponential of real intervals

$$\text{expx} = 1 + \frac{x}{1}\left(1 + \frac{x}{2}\left(1 + \frac{x}{3}\cdots\right)\right)$$

We select an N and then, produce an interval that contains

$$R \equiv \left(1 + \frac{x}{N+1}\left(1 + \frac{x}{N+2}\cdots\right)\right) \quad \text{when } x \text{ ranges in the input}$$

interval and then evaluate the product with interval arithmetic. We use the crude bounds

$$1 \geqslant R \geqslant 1 + \frac{x}{N+1} \quad \text{if} \quad -(N+1) < x < 0$$

$$1 \leqslant R \leqslant \frac{1}{1 - \frac{x}{N+1}} \quad \text{if} \quad N+1 > x > 0.$$

This procedure, even if correct in the interval $(-(N+1), N+1)$ is quite inefficient. What we do is to use this only when the input interval is in a small interval (about $(-1/8, 1 \times 8)$) and reduce to this case by using $e^x = (e^{x/2})^2$ as many times as needed; using recursion —allowed in Pascal— this is quite easy to program. At the beginning, we also check that the interval is within reasonable bounds. Even if this was not strictly necessary, the exponential ceases to work well for intervals not within reasonable bounds.

We also include routines to compute sin, cos for reals and exponential for complex. It is amusing to note that, for complex exponential it is much more accurate for fat intervals to use Euler's formula and call the real subroutines that to use recursion involving complex numbers.

Since we will very often be computing $e^x - 1$ for small x we included a special subroutine that does not reqire to subtract two very similar numbers. It is also based in using Taylor expansion grouped in a way similar to the exponential and the recursion relation $(e^x - 1) = (e^{x/2} - 1)(e^{x/2} + 1)$. Those subroutines are included in cexpo.i and a test program is testexpo.p.

We also have a right computation of powers which works on positive numbers and rounds to the right, and a left one. Out of those it is very simple to compute powers for intervals and a root; the idea for the latter is to compute the root approximately using the ones supplied by the compiler and then round appropriately until we can show using the previously mentioned powers that the power of this interval contains (not is contained in!) the given interval.

B. Excluding matches

In this subsection we describe how to compute bounds for discriminants with finitely many arithmetic operations (that is, we take into account truncation errors) and we will discuss how to use that to show that there are no matches in the regions required by Lemma 4.

One can observe that all the equations to be studied are of the form

$$D^2\psi + 4D\psi + \frac{\alpha}{1 + \epsilon e^t}\psi = 0$$

for suitable choices of α, ϵ, and it will suffice to solve two problems associated to this equation.

I) Given the asymptotic behavior as $t \to -\infty$ find the ratio of $\psi(t_0)$, $\psi'(t_0)$ at some negative t_0.

II) Given ψ and ψ' at some point t_0 find ψ and ψ' at another point t_1.

If our algorithms are going to be useful for present purposes, they should be able to work with intervals. That is, given that we know bounds for ϵ, we should be able to produce upper and lower bounds for the results.

Problem II will be solved in the most conservative way possible. That is: given intervals $t_0, t_1 \psi(t_0), \psi'(t_0)$. We will produce intervals $\psi(t_1)$ and $\psi'(t_1)$ in such a way that any initial data contained in $\psi(t_0)$, $\psi'(t_0)$ for an initial time contained in t_0, will be contained in $\psi(t_1)$ $\psi'(t_1)$ for any final time contained in t_1. (Of course, we will obtain estimates uniform in ϵ ranging in a given interval.)

Even if less conservative solutions would also be useful, this hast the advantage that it can be iterated quite straightforwardly. The iteration can be made without terrible losses in accuracy, as we will explain.

Rather than implementing a general O.D.E. solver, we have drawn freely on specific properties of the equation we were strudying. Even if the former goal is obviously desirable, and perhaps within reach, we thought it prudent not to tackle it before getting some understanding of the effect of coherence, which seems to be disasterous, e.g. for Runge-Kutta methods. We rather used some methods based on Taylor expansions that, besides having less serious coherence problems than R.K. methods, can be used to expand to any order.

We will try to explain the algorithms with a notation as close as possible to the computer programs even if this will occasionally lead to some small inconsistencies.

Problem I

$$(I) \begin{cases} D^2\psi + \beta D\psi + \dfrac{\alpha}{1 + \epsilon e^t}\psi = 0 \\ D\psi \sim \mu\psi \quad \text{as} \quad t \to -\infty. \end{cases}$$

We try to find an asymptotic expansion for ψ of the form $\psi = \sum \epsilon^n \psi_n(t)$, and show error bounds for the truncation. Expanding the fraction as a

geometric series we are led to

$$D^2\psi + \beta D\psi + \alpha\Psi = -\alpha \sum_{n=1}^{\infty} (-\epsilon)^n e^{nt}\psi$$

and we can recursively solve for ψ_n matching the corresponding powers of ϵ. It is easy to show by induction (except in some cases to be discussed later) that

$$\psi_n(t) = \eta_n e^{(u+n)t}, \qquad \eta_n \in \mathbb{R}.$$

Calling $P(x) = x^2 + \beta x + \alpha$, the characteristic polynomial of the R.H.S., we have

$$P(u+n)\eta_n = -\alpha \sum_{k=1}^{n} (-1)^k \eta_{n-k}.$$

Provided that $P(\mu) = 0$, this allows us to choose $\eta_0 = 1$ and then, all the other η's are determined provided $P(u+n) \neq 0$ for any n, which will be the case in our problem.

Even if this hierarchy is sufficient to compute as many terms as we wish we can transform it in such a way that computing the n^{th} term requires 1 operation and not n. This improves the execution time but, more importantly for us, the accuracy of the estimates.

In fact,

$$\begin{aligned} P(u+n)\eta_n &= -\alpha\left[-\eta_{n-1} + \sum_{k=2}^{n} (-1)^k \eta_{n-k} \right] \\ &= -\alpha\left[-\eta_{n-1} - \sum_{k=1}^{n-1} (-1)^k \eta_{n-1-k} \right] \\ &= [\alpha - P(u+n-1)]\eta_{n-1}. \end{aligned}$$

This recursion relation has the property that from a certain term on (which is 4 for our case) the η_n's decrease and hence, the series of the function and the derivatives will be uniformly convergent in all intervals of t of the form $\epsilon e^t \leqslant 1 - \delta$, $\delta > 0$. Hence, the expansion will be a solution of the O.D.E. in this range and have, clearly, the right asymptotic behavior.

ψ, ψ' at any point can be recovered by $\psi(t) = \left[\sum_{n=0}^{\infty} \eta_n (\epsilon e^t)^n \right] e^{\mu t}$

$$\psi'(t) = \sum_{n=0}^{\infty} \eta_n (\mu + n)(\epsilon e^t)^n e^{\mu t}.$$

Since for the discriminants, only the ratio matters, it suffices to compute the terms in brackets, which we evaluate using Horner's scheme. We sum a

finite number of terms and estimate the remainders using the decrease of $|\eta_n|$ and the sum of a geometric series

$$\left|\sum_{n=N+1}^{\infty}\eta_n(\epsilon e^t)^n\right| \leqslant |\eta_N|\sum_{n=N+1}^{\infty}(\epsilon e^t)^n = |\eta_n|(\epsilon e^t)^{N+1}\frac{1}{1-\epsilon e^t}$$

$$\left|\sum_{n=N+1}^{\infty}(\mu+n)\eta_n(\epsilon e^t)^n\right| \leqslant |\eta_N|\sum_{n=N+1}^{\infty}(\mu+n)(\epsilon e^t)^n$$

$$= |\eta_N|\left[\frac{(\mu+N+1)}{1-\epsilon e^t}+\frac{\epsilon e^t}{(1-\epsilon e^t)^2}\right](\epsilon e^t)^{N+1}.$$

Notice that this computation has the good feature that provided that we deal with the bounds on the input quantities using interval arithmetic we obtain bounds for the results.

This is implemented with the subroutines charpol, findetal, findf, findfl. The only nontrivial programming consideration is that we passed eta by address rather than by value. Even if this is slightly more dangerous, it saves the copying of the array η which takes quite an amount of time.

Remarks. In some cases, this algorithm could be used to propagate the solutions because we could very well produce two independent solutions, and we could find a linear combination to match the initial data. This is not enough to carry the computation for all cases of interest, but it can be used as a test. We have a program to perform this check in asymptest.

Now we turn to the problem II, solving the initial value problem.

It will be more convenient to use $\hat{\psi} = e^{2t}\psi$ which satisfies

$$\hat{\psi}'' + \left[\frac{\alpha}{1+\epsilon e^t} - 4\right]\hat{\psi} = 0.$$

(We will, for this discussion, forget the $\hat{}$ from now on.)

We first remark that, for this equation, it suffices to solve the initial value problem at $t_0 = 0$. Starting at another point is the same as substituting ϵe^{t_0} for ϵ.

The algorithm we are going to use will be based on expansions in powers of $(e^t - 1)$. We will try to find $\{\psi_n\}$ so that $\psi(t) = \sum_{n=0}^{\infty}\psi_n(e^t-1)^n$ solves the initial value problem.

If we write

$$\left[\frac{\alpha}{1+\epsilon e^t} - 4\right] = \sum_{n=0}^{\infty}\beta_n(e^t-1)^n$$

we have that the original equation is equivalent to

$$n^2\psi_n + (n+1)(2n+1)\psi_{n+1} + (n+1)(n+2)\psi_{n+2} + \sum_{k=0}^{\infty} \beta_k \psi_{n-k} = 0.$$

The β_n's can be readily computed using geometric series, and they are:

$$\begin{aligned} \beta_n &= \hat{\beta}_n \quad \text{if} \quad n \neq 0 \\ &= \hat{\beta}_0 - 4 \quad \text{if} \quad n = 0 \end{aligned}$$

where

$$\hat{\beta}_n \equiv (-1)^n \left(\frac{\epsilon}{1+\epsilon}\right)^n \frac{\sigma}{1+\epsilon}.$$

Besides, matching the initial conditions, we have $\psi_0 = \psi(0)$; $\psi_1 = \psi'(0)$. [Remember that our present $\psi(t)$ is what we called $e^{2t}\psi(t)$ in the original equation.]

So we can find recursively all ψ_n's. It will, however, be more convenient to proceed as we did before and use the relation between the β_n's to obtain a simpler recursion relation between the ψ_n's.

Writing explicitly the first term in the summation sign, pulling out the factors and substituting the previous equation of the hierarchy, we obtain

$$\begin{aligned} \psi_{n+2}(n+2)(n+1) = &-(n+1)(2n+1)\psi_{n+1} - (n^2 + \beta_0)\psi_n \\ &+ \left(\frac{-\epsilon}{1+\epsilon}\right)[n(n+1)\psi_{n+1} + n(2n-1)\psi_n + ((n-1)^2 - 4)\psi_{n-1}] \end{aligned}$$

which has to be supplemented by the initial conditions

$$\psi_0 = \psi(0);$$

$$\psi_1 = \psi'(0);$$

$$\psi_2 = \frac{1}{2}[-\beta_0\psi_0 - \psi_1];$$

the third one being, of course, the second equation of the original hierarchy. It becomes necessary because one equation of the new hierarchy is a combination of two consecutive ones of the previous, hence of higher order.

Again, when we recursively compute in interval arithmetic, we obtain bounds for the ψ_n's uniform in the input parameters ranging in the given intervals.

Now we turn to the task of bounding the error incurred by truncating the hierarchy to a finite order.

We rewrite the equation in integral form as

$$\psi = \psi_0 + I(\psi_1 + IH\psi) \equiv K\psi$$

where I is the operator $\psi \to \int_0^x \psi(s)\,ds$. H is multiplication by $[\alpha/(1 + \epsilon e^t) - 4]$. Those operators act on the spaces of functions $\{\psi = \sum \psi_n (e^t - 1)^n | \, \|\psi\| = \sum |\psi_n| r^n < \infty\}$ where r is an arbitrary parameter to be chosen later.

The idea of the proof will be to estimate, for our finite dimensional approximation ψ, $\|\psi - K\psi\|$ and show that, for sufficiently small r, K is a contraction. Using these estimates, we can bound $\|\psi_{\text{true}} - \psi\| \leqslant \|\psi - K\psi\|/(1 - Lip(K))$ and a fortiori the error for the ψ and ψ' evaluated in a place where $|e^t - 1| < r$.

Proposition 1.

$$\|I\| \leqslant \frac{r}{1 - r}.$$

PROOF. $(\psi')_n = n\psi_n + (n + 1)\psi_{n+1}$. Hence $(n + 1)(I\psi)_{n+1} = \psi_n - n(I\psi)_n$, so $(n + 1)(I\psi)_{n+1} = \psi_n - n(I\psi)_n = \psi_n - \psi_{n-1} + \psi_{n-2} + \cdots \pm \psi_0 \cdot I\psi_0 = 0$.

Therefore

$$\sum_{n=0}^{\infty} |(I\psi)_n| r^n \leqslant \sum_{n=1}^{\infty} \frac{r^n}{n} \sum_{k=0}^{n} |\psi_k|.$$

The derivative of the R.H.S. with respect to r is $\sum_{n=0}^{\infty} r^n \sum_{k=0}^{n+1} |\psi_k|$, but this is just $\left(\sum |\psi_k| r^k\right)[1/(1 - r)]$. Integrating and noting that at $r = 0$ this R.H.S. takes the value 0, we obtain

$$\|I\| \leqslant \int_0^r \frac{ds}{1 - s} \leqslant \int_0^r \frac{ds}{1 - \mathrm{r}} = \frac{r}{1 - r}.$$

We also observe that if $\psi(t) = O(t^N)$, $I\psi = O(t^{N+1})$ near $t = 0$.

Proposition 2.

$$\|H\| \leqslant \left|\frac{\alpha}{1 + \epsilon}\right| \left(1 - \left|\frac{\epsilon r}{1 + \epsilon}\right|\right)^{-1} + 4.$$

PROOF. It is well known that these spaces of analytic functions will have the Banach algebra property under multiplication. Therefore, the norm of a multiplication operator is less than the norm of the function. The R.H.S. of the formula above is a bound of the function H in view of the explicit expansion computed.

Clearly, the Lipschitz constant of K can be bounded by $\|I\|^2 \|H\|$.

The bound for $\|\psi - K\psi\|$ is a little bit more subtle. We should remember that the intervals that we have produced for the ψ_n are only bounds for the values of ψ_n, but they are not all the information on ψ_n we are entitled to use. Particularly, we can use that the ψ_n satisfy the previous recursion relation exactly and, therefore, we can conclude that many of the terms to estimate are indentically zero.

In particular we have that, since the equation is satisfied to order N, $K\psi - \psi = K\psi^{[>N]} - \psi^{[>N]} = K\psi^{[>N]}$ because $\psi^{[>N]} = 0$.

If $N \geqslant 2$ —as we will always assume— we have

$$\|(I^2 H\psi)^{[>N]}\| \leqslant \|I\|^2 \|(H\psi)^{[>N-2]}\|$$

$$\|(H\psi)^{[>N-2]}\| \leqslant \sum_{n=N-1}^{\infty} r^n \sum_{k=0}^{n} |\beta_{n-k}|\,|\psi_k|$$

$$\leqslant \sum_{n=N-1}^{\infty} r^n \left(\sum_{k=0}^{n} |\hat{\beta}_{n-k}|\,|\psi_k| + 4|\psi_n| \right)$$

$$= \left|\frac{\alpha}{1+\epsilon}\right| \sum_{n=N-1}^{\infty} r^n \sum_{k=0}^{n} \left|\frac{\epsilon}{1+\epsilon}\right|^{n-k} |\psi_k| + 4|\psi_{N-1}| r^{N-1}$$

$$+ 4|\psi_n| r^N$$

Using that ψ_n is zero for $n > N$, we have that the first term can be rewritten as follows, where $\gamma = |\epsilon/(1+\epsilon)|$:

$$\left|\frac{\alpha}{1+\epsilon}\right| r^{N-1} \sum_{k=0}^{N-1} \gamma^{N-1-k} |\psi_k| + \left|\frac{\alpha}{1+\epsilon}\right| \sum_{n=N}^{\infty} r^n \gamma^{n-N} \sum_{k=0}^{N} \gamma^{N-k} |\psi_k|.$$

The last term can be bounded by

$$\left|\frac{\alpha}{1+\epsilon}\right| \frac{r^N}{1-r\gamma} \left(\sum_{n=0}^{N} \gamma^{n-k} |\psi_k| \right) = \left|\frac{\alpha}{1+\epsilon}\right| \frac{r^N}{1-r\gamma} (S\gamma + |\psi_N|)$$

where

$$S = \sum_{k=0}^{N-1} \gamma^{N-1-k} |\psi_k|.$$

Putting everything together, we obtain

$$\|K\psi - \psi\| \leqslant r^{N-1} \left[\left|\frac{\alpha}{1+\epsilon}\right| \left\{ S\left(1 + \frac{\gamma r}{1-\gamma r}\right) + \frac{|\psi_N| r}{1-\gamma r} \right\} + 4\{|\psi_N| + |\psi_{N-1}|\} \right].$$

To bound the truncation error we can use not only the estimate on norm of $\psi_{\text{true}} - \psi$ but also the fact that this function is of high order. We have

$$|\psi_{\text{true}}(\delta) - \psi(\delta)| \leqslant \left(\frac{\delta}{r}\right)^{N+1} \|\psi_{\text{true}} - \psi\|$$

$$|\psi'_{\text{true}}(\delta) - \psi(\delta)| \leqslant \left[\sup_{n \geq N+1} n\left(\frac{\delta}{r}\right)^{n} + \sup_{n \geq N+1} (n+1)\frac{\delta^n}{r^{n+1}}\right] \|\psi_{\text{true}} - \psi\|$$

as can be seen by writing the sums explicitly, and bounding in the crudest way. We observe that if δ/r is sufficiently small, the sup is reached in the first term and the bracket in front of the derivative becomes

$$(2N+3)\left(\frac{\delta}{r}\right)^{N+1}\left(1+\frac{1}{r}\right).$$

Notice that $\delta/r = 1/2$ is sufficiently small for the purpose of having the previous property independently of N.

Notice that the previous bound runs into problems when $r = 0$ (numerically, it would be disastrous when r is very small). So the obvious choice of r as 2δ would lead to problems when δ is very small. Hence, arbitrarily, we have decided that when $\delta < 10^{-6}$, then we take $r = 10^{-4}$ (or, more precisely, the numbers the compiler interprets when given 10^{-6} and 10^{-4}; we do not care about computing them carefully since they are just arbitrary choices and the only thing we assume is that one is more than double the other).

Remark. The way of grouping terms in the previous computation is quite important. For certain ranges of the parameter, γ can be extremely small and pulling γ^N out of the sum may lead to disastrous results. This at first sight could be quite puzzling since the estimates become ridiculous at some points where γ comes close to zero.

Unfortunately, these things seem to be quite hard to guess without actually writing and running the program.

The only place where such difficulty would show up would be requesting steps acting on two identical sharp intervals as initial and final time, and clearly those could be avoided in the program. However, we have decided to have a subroutine that works even in the absurd cases. This allows us to use the same general propagation subroutine without worrying about the boundary conditions in do loops which lead to these special cases.

In the subroutine step, we take as inputs, the initial and the final time, and obtain an interval containing all the increments. We compute $(e^{\Delta t} - 1)$ and evaluate a finite truncation of the propagation.

There is another boolean variable «inconclusive» that gets set if any of the geometric series estimates we were performing has ratio bigger than 1. This

will inform the main program that something wrong has happened: the data should be discarded, the increment halved and start all over (this turns out to be more convenient that just stopping the program).

There are many advantages to organizing step in the way we did. Being as conservative as possible in the lowest level saves considering which way the uniformity of the estimates goes in the higher levels.

Clearly, we can divide the initial value problem for large intervals in a sequence of smaller steps. Notice that, with our organization, it suffices that the final time of one step contains the initial time of the next and is not necessary that they are equal.

Nevertheless, it cannot be denied that basing a step in the initial interval and the increment would have had something more natural to it, since the increment is really what is used. However, the final interval that we can guarantee should be an interval contained (not containing as before!) in the sum. This would require getting out of the philosophy of interval arithmetic, implementing new operations, and we thought it unappealing and confusing and stuck to the safe —albeit wasteful— procedure. This, we admit, we could do because the problem was not too delicate. It could very well be that more demanding problems require improvements in this direction, extending interval arithmetic with some inverse operations.

Out of this, it is quite trivial how to propagate the solutions from zero and infinity and compute if, for some given range of parameters, there is a match. This is what is done by the boolean function «excluded». It is natural to give it three parameters, t_1 and t_2 and a third one we called offset, which, when it takes the value zero, makes us recover the first problem and that by varying in an interval produces the changes in ϵ required by the other problems.

Given those parameters, this function computes the numerator of the discriminant, taking good care that each time step is involved, it is tested whether it is conclusive or not. The only non-obvious point is that we propagate taking sharp intervals as intermediate steps, so that we are only forced to have steps with fat intervals as ends at t_1 and t_2 themselves. For those we found it advantageous to compute in three different ways from a sharp interval in the middle and from sharp intervals at each end and take the intersection.

Switching differential equations is a little bit cumbersome. Notice that here it is not sufficient that the initial time of the next step is contained in the final time of the previous one; the reason is that t_2 has two roles: one as final time of the propagation and also as the place where the differential equation changes. Even if our bounds would allow us to restrict the final time of propagation, the say nothing about the effect of switching O.D.E. at a different time.

The solution is to propagate the new O.D.E. from the full t_2 to another interval, a little bit ahead. After that the propagation is very similar to the one used before.

At the end, «excluded» prints the numerator of the discriminant and the interval in which it was invoked. Even if the input and output —which are not quite right— are invoked to do that, this does not affect the conclusions because the numbers written are only going to be used as heuristic aids.

The only thing left now is to explore in a systematic way all the range to be excluded.

The most obvious way would be to just make a double loop exploring in turn small squares adding up to the whole range to be explored. This, however, would be quite impractical since the allowable size for the estimates to be informative depends very much on the place and this cannot be guessed before doing the computation.

A clean way out is to introduce another function: «superexc» that, given a certain square in parameter space, will compute the discriminant and if it cannot exclude a match, will perform a subdivision of the original square in four squares and if any of those cannot be excluded will again subdivide and so on. It will only report false when one of the squares to be considered is very small (we set it at 10^{-4}).

The way to implement this in Pascal is through a recursive function that, if it does not succeed with excluded, and the intervals are not too small, subdivides and invokes itself on each of the subdivisions.

This simple trick turns out to be extraordinarily effective and is quite a useful structure for exclusion problems.

Remark. For the aficionadoes of fine points in compilers it may be amusing to notice that the efficiency with which superexc would abandon when confronted with a non excludable interval depends quite dramatically on the way that the compiler implements «and».

In Modular-2, for example ([W] p. 27) «p AND q» is taken to mean «IF p THEN q ELSE FALSE». This has the advantage that if p turns out to be false, q does not need to be evaluated. Similar conventions are specified in C ([K.R.] p.190) and, several Pascal compilers adhere to it even if nothing is said in the specification of the language [JW].

Other compilers, however, attempt to evaluate both p and q before evaluating the «and».

If the first alternative is in effect, superexc would abandon the first time it needs to consider a very small interval whereas if the second alternative is in effect superexc will abandon the last time it is so required.

For our case, with two variables and too small meaning 100 times smaller than typical, this would mean roughly $(100)^2$ times slower. If all intervals had to be examined but even if there were matches only along a line, this would mean 100 times slower.

On top of that, we just put another loop than invokes superexc on a sequence of squares of moderate size and covering the interval to be excluded. After each one of them is excluded, a report is put into a file and, at the end, it would be quite easy to check it is there using an editor. This turns out to be safer than reporting if something bad happens, because power failures could happen that would prevent messages from being issued, and this would be interpreted wrongly as success. These reports of success can be extracted from the other printout of the program very easily using a filter («grep» in Unix). Similarly, we could extract the report about status.

On other systems, it could be advantageous to write the heuristic reports and the definitive ones to different files.

Notice that the loop proceeds dividing in intervals by multiplying an integer by a conventional number which, strictly speaking, we do not know. (We only know it is the number the compiler translates 0.1 .) But it does not matter because the upper point of one interval is the lower point of the next. (We refrained from assuming that either the translation algorithm or the multiplication is monotonic.)

Remark. If no matches occur, the procedure we have explained will prove that no matches occur. If there were a match, however, the procedure would keep on dividing till the subdivisions reach the cut-off in superrexc, which can be quite a long time and even in that case we could not prove that indeed there was a zero. However, it is possible —and indeed we did it in a previous version— to run the exploration from two opposite corners towards the center and keep a global variable of the signs found so that the existence of a match could also be established and even located very accurately. Since it was not necessary in our case and it gives some room for error, we proceeded in the most straightforward way.

Besides the loop cycling through squares, the main program asks for several parameters that, even if logically irrelevant, will affect the efficiency of the run, like the number of terms to keep or the number of steps to take. Since trying to run through the whole loop at once would be quite impractical, we just run a few integers at a time.

We have written four main programs, «veuler», «suffices», «lastbound», «nomatchØ» which correspond to the statements claimed in Lemma 4.

Each one of them has its own choice of parameters of the run, its own superexc and mainloop, but they share most of the other subroutines and we keep them as «include» files.

From the programming point of view, it is instructive to point out that the use of the same excluded for the three first leads to «lastbound» requesting a propagation form the interval vØ to itself; and, hence, it becomes convenient to have a «step» that works correctly even in this case.

The last bound can be considerably simplified. Excluding matches $t_1 = t_2 = 0$, $2.25 \leqslant \epsilon \leqslant 4.5$ is equivalent to $G(0) = 0$ and ϵ ranging over this interval means that $\epsilon_g = \frac{1}{5}[\epsilon - 2 - e^{-t_1} - 3e^{-t_2}]$ ranges over $(-3/4, -3/10)$ and this is what «matchØ» accomplishes.

The first two programs run much slower than the last two but, in a VAX 11/750, with lowest priority, they both run in a few days. The last two can be completed in a few minutes.

«veuler» run excluding matches for $-2.5 \leqslant t_1 \leqslant t_2 \leqslant 0.05$.

«suffices» has built in to take $T = 2.0$ and cycle from an interval in offset $5/4 + 0.05$ to -0.05.

«lastbound» also has built in to take $T = 2.0$ and we run it from offset $5/4 - 0.05$ to $5/4 + 0.05*26$.

«no matchØ» was run from offset $(-8)*0.1$ to $(-2)*0.1$.

Notice that both «suffices» and «lastbound» have a T generated in an identical way, so it is an identical number in both programs.

The other numbers involved are generously overlapping so that they cover by far the possible inaccuracies of the input and output subroutines.

Even after the algorithm described before is coded in Pascal, there are several steps towards its execution: the program has to be translated into something the machine understands —compiled— and the compiled versions have to be shifted from memory to disk and scheduled to share the resources with other processes. One could worry that we do not have much control over these steps.

It can be argued that nowadays developing compilers is a well understood technique and, when not worried about optimizing resources, (e.g., in personal computers or with optimizers) the results are very reliable.

Besides, we did some checking on the parts of the compiler we used and refrained from using delicate parts like optimizers or separate compilation. We run the program in two machine equipped with Berkeley 4.1 BSD and with ULTRIXVI.0 (4.2 BSD), which are developed independently and we could compare parts of the output. This excludes also the possibility of some random malfunction or strange interference with other processes.

In spite of all this checking, however, the weakest part is the human effort of programming. (That is why we decided to publish the programs so that they can be checked by anybody wanting to do so.) Nevertheless, we feel more or less confident that there are at most minor errors; certainly there are not terribly serious errors because, besides checking the program we wrote heuristic programs to compute discriminants based on different algorithms, in Pascal, Fortran and Basic. Even if the effort was independent and conducted in different computers the results agreed. The existence of niggling errors in the estimates cannot be so easily excluded —and, of course, it is logically very serious and we feel strongly about it. However, the programming was very

careful and all the algorithms worked correctly at the first try —only once we found a faulty reasoning. In any case, were any error to appear, the estimates are so conservative that it would certainly be absorbable at the expense, at most, of more computing time; at the moment, running the whole verification is not a problem at all.

Cetainly we want to encourage our colleagues to check the programs and we can make them available, including some tests, through USNET or BITNET, or we can send an I.B.M. diskette (double-sided, double-density). It goes without saying that any error reported to us will be greatly appreciated.

We believe that this amount of reliability is quite comparable to any mathemtical proof involving complicated computation. There is always the possibility of error; minor errors will probably never be entirely removed. Nevertheless, after careful checking one has very strong reasons for confidence. Except for philosophical objections we will not address in public, Lemma 4 should be considered as rigorously established.

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veuler

```
program veuler ( input , output );
const
#include 'const.i'
type
#include 'types.i'

var
#include 'globalround.i'
#include 'global.i'
        orderused :integer;
        t1,t2 :interval;
        etag :list;
        ng :integer;
        alphaf,alphah :interval;
        vrootg :interval;
        sucess :boolean;
        stepnum1,stepnum2 :integer;
        i,j :integer;
        imin,imax :integer;
        demult :integer;
#include 'complex.i'
#include 'cexpo.i'
#include 'initialize.i'
#include 'asymp.i'
#include 'step.i'
#include 'excluded.i'
```

superexc

```
function superexc( t1 , t2 :interval):boolean;
label
        1;
var
        t1center,t2center :double;
        t1plus,t1minus,t2plus,t2minus :interval;
begin
        if excluded( t1 , t2 , v0) then
        begin
                superexc := true;
                goto 1;
        end
        else
        begin
                if (meas(t1) < 1.0e-04) and( meas( t2) < 1.0e-04 )then
                begin
                        superexc := false;
                        goto 1;
                end;
```

```
                t1center := center(t1);
                t1plus.lower := left( t1center );
                t1plus.upper := right( t1.upper );
                t1minus.upper:= right( t1center );
                t1minus.lower:= left( t1.lower );
                t2center := center( t2 );
                t2plus.lower := left( t2center );
                t2plus.upper := right( t2.upper );
                t2minus.upper:= right( t2center );
                t2minus.lower:= left( t2.lower );
                superexc :=
                        superexc( t1plus , t2plus  )and
                        superexc( t1plus , t2minus  )and
                        superexc( t1minus , t2plus  )and
                        superexc( t1minus , t2minus  );
        end;
1:
end(* superexc *);
begin
        initialize;
        writeln( 'key in the number of terms to keep for G' );
        readln (ng );
        writeln( ' ng = ', ng );
        writeln( 'key in the number of terms to be used');
        readln( orderused );
        writeln( ' orderused = ',orderused );
        writeln ( 'key in imin ');
        readln( imin );
        writeln( 'imin = ',imin );
        writeln ( ' key in imax ');
        readln ( imax );
        writeln ( 'imax = ', imax );
        writeln( 'key in the demultiplication factor ');
        readln ( demult );
        writeln ( 'demult =  ', demult );
        vroot( 2 , v2over5 , vrootg );
        subv( vrootg , v2 , vrootg );
        findeta( vrootg , v18over5 , v4 , etag , ng );
        alphaf := v6;
        alphah := v9over2;
        for i := imin to imax do
        begin
                t1.lower := -i*0.1;
                t1.upper := -(i-1)*0.1;
                t1.lower := left( t1.lower );
                t1.upper := right( t1.upper );
```

```
                for j := 1 to i do
                begin
                        t2.lower := -j*0.1;
                        t2.upper := -(j-1)*0.1;
                        t2.lower := left( t2.lower );
                        t2.upper := right( t2.upper );
                        stepnum2 := demult*j;
                        stepnum1 := demult*(i-j+1);
                        sucess := superexc ( t1 , t2 );
                        if sucess then
                                writeln( i , j ,'superexcluded')
                        else
                                writeln( i , j ,'NOT SUPEREXCLUDED' );
                end;
        end;
        writeln;
        writeln( ' status = ', status);
        writeln;
end.
```

suffices

```
program suffices ( input , output );
const
#include 'const.i'
type
#include 'types.i'
var
# include 'global.i'
# include 'globalround.i'
        orderused :integer;
        bigt,t2 :interval;
        alphaf,alphah :interval;
        etag :list;
        ng :integer;
        vrootg :interval;
        sucess :boolean;
        stepnum1,stepnum2 :integer;
        i,j :integer;
        imin,imax :integer;
        demult :integer;
        offset :interval;
#include 'complex.i'
#include 'cexpo.i'
#include 'initialize.i'
#include 'asymp.i'
```

superexc

```
#include 'step.i'
#include 'excluded.i'
function superexc(  t2 :interval; offset :interval):boolean;
label
        1:
var
        offsetcenter,t2center :double;
        offsetplus,offsetminus,t2plus,t2minus :interval;
begin
        if excluded( bigt . t2 , offset ) then
        begin
                superexc := true;
                goto 1;
        end
        else
        begin
                if (meas(offset) < 1.0e-04) and( meas( t2) < 1.0e-04 )then
                begin
                        superexc := false;
                        goto 1;
                end;
                offsetcenter := center(offset);
                offsetplus.lower := left( offsetcenter );
                offsetplus.upper := right( offset.upper );
                offsetminus.upper:= right( offsetcenter );
                offsetminus.lower:= left( offset.lower );
                t2center := center( t2 );
                t2plus.lower := left( t2center );
                t2plus.upper := right( t2.upper );
                t2minus.upper:= right( t2center );
                t2minus.lower:= left( t2.lower );
                superexc :=
                        superexc(  t2plus , offsetplus )and
                        superexc(  t2minus , offsetplus )and
                        superexc(  t2plus , offsetminus )and
                        superexc(  t2minus , offsetminus );
        end;
1:
end(* superexc *);
begin
        initialize;
        writeln( 'key in the number of terms to keep for G' );
        readln (ng );
        writeln( ' ng = ', ng );
        writeln( 'key in the number of terms to be used');
```

```
        readln( orderused );
        writeln( ' orderused = ',orderused );
        writeln ( 'key in imin ');
        readln( imin );
        writeln( 'imin = ',imin );
        writeln ( ' key in imax ');
        readln ( imax );
        writeln ( 'imax = ', imax );
        writeln( 'key in the demultiplication factor ');
        readln ( demult );
        writeln ( 'demult =  ', demult );
        vroot( 2 , v2over5 , vrootg );
        subv( vrootg , v2 , vrootg );
        findeta( vrootg , v18over5 , v4 , etag , ng );
        alphaf := v6;
        alphah := v9over2;
        bigt := vm2;
        for i := imin to imax do
        begin
                t2.upper := -(i-1)*0.05;
                t2.lower := -i*0.05;
                t2.lower := left( t2.lower );
                t2.upper := right( t2.upper );
                stepnum2 := trunc( -t2.lower/0.05)+1;
                stepnum1 := trunc( ( -bigt.lower + t2.upper)/0.05) +1;
                stepnum1 := stepnum1 * demult;
                stepnum2 := stepnum2 * demult;
                for j := 0 to 26 do
                begin
                        offset.lower := ( j-1 )*0.05;
                        offset.upper := (j )*0.05;
                        sucess := superexc ( t2 , offset );
                        if sucess then
                                writeln( i , j ,'superexcluded')
                        else
                                writeln( i , j ,'NOT SUPEREXCLUDED' );
                end;


        end:
        writeln;
        writeln( ' status = ', status);
        writeln;
end.
```

nomatch0

```
program nomatch0( input , output );
const
#include 'const.i'
type
#include 'types.i'
var
        etag :list;
        ng :integer;
        vrootg :interval;
        sucess :boolean;
        i :integer;
        imin,imax :integer;
        epsg :interval;
#include 'global.i'
#include 'globalround.i'
#include 'complex.i'
#include 'cexpo.i'
#include 'initialize.i'
#include 'asymp.i'
function superexc( eps :interval ):boolean;
label
        1;
var
        epscenter:double;
        epsplus,epsminus:interval;
        gat0 :interval;
        excluded :boolean;
begin
        findf0( etag , ng , eps , gat0);
        excluded :=  not( contained( v0 , gat0 ));
        write( 'eps = ');printv( eps );writeln;
        write( 'gat0 = ');printv( gat0 );writeln;
        writeln( 'excluded = ', excluded );
        writeln;writeln;
        if  excluded then
        begin
                superexc := true;
                goto 1;
        end
        else
        begin
                if (meas(eps) < 1.0e-04) then
                begin
                        superexc := false;
                        goto 1;
```

superexc

```
                end;
                epscenter := center(eps);
                epsplus.lower := left( epscenter );
                epsplus.upper := right( eps.upper );
                epsminus.upper:= right( epscenter );
                epsminus.lower:= left( eps.lower );
                superexc :=
                        superexc( epsplus  )and
                        superexc( epsminus );
        end;

1:
end(* superexc *);
begin
        initialize;
        writeln( 'key in the number of terms to keep for G' );
        readln (ng );
        writeln( ' ng = ', ng );
        writeln ( 'key in imin ');
        readln( imin );
        writeln( 'imin = ',imin );
        writeln ( ' key in imax ');
        readln ( imax );
        writeln ( 'imax = ', imax );
        vroot( 2 , v2over5 , vrootg );
        subv( vrootg , v2 , vrootg );
        findeta( vrootg , v18over5 , v4 , etag , ng );
        for i := imin to imax do
        begin
                epsg.upper := i*0.1;
                epsg.lower := (i-1)*0.1;
                sucess := superexc ( epsg );
                if sucess then
                begin
                        writeln( i , 'superexcluded')
                end
                else
                begin
                        writeln( i , 'NOT SUPEREXCLUDED' );
                        writeln;
                end;
        end;
        writeln( ' status = ', status);
        writeln;
end.
```

lastbound

```
program lastbound ( input , output );
const
#include 'const.i'
type
#include 'types.i'
var
#include 'globalround.i'
# include 'global.i'
        orderused :integer;
        bigt :interval;
        offset :interval;
        alphaf,alphah :interval;
        etag :list;
        ng :integer;
        vrootg :interval;
        sucess :boolean;
        stepnum1,stepnum2 :integer;
        i :integer;
        imin,imax :integer;
        demult :integer;
#include 'complex.i'
#include 'cexpo.i'
#include 'initialize.i'
#include 'asymp.i'
#include 'step.i'
#include 'excluded.i'
function superexc(offset :interval ):boolean;
label
        1;
var
        offcenter:double;
        offplus,offminus:interval;
begin
        if excluded( bigt , v0 , offset ) then
        begin
                superexc := true;
                goto 1;
        end
        else
        begin
                if (meas(offset) < 1.0e-04) then
                begin
                        superexc := false;
                        goto 1;
                end;
```

superexc

```
                offcenter := center(offset);
                offplus.lower := left( offcenter );
                offplus.upper := right( offset.upper );
                offminus.upper:= right( offcenter );
                offminus.lower:= left( offset.lower );
                superexc :=
                        superexc( offplus  )and
                        superexc( offminus );

        end;
1:
end(*  superexc *);
begin
        initialize;
        writeln( 'key in the number of terms to keep for G' );
        readln (ng );
        writeln( ' ng = ', ng );
        writeln( 'key in the number of terms to be used');
        readln( orderused );
        writeln( ' orderused = ',orderused );
        writeln ( 'key in imin ');
        readln( imin );
        writeln( 'imin = ',imin );
        writeln ( ' key in imax ');
        readln ( imax );
        writeln ( 'imax = ', imax );
        writeln( 'key in the demultiplication factor ');
        readln ( demult );
        writeln ( 'demult =  ', demult );
        vroot( 2 , v2over5 , vrootg );
        subv( vrootg , v2 , vrootg );
        findeta( vrootg , v18over5 , v4 , etag , ng );
        alphaf := v6;
        alphah := v9over2;
        bigt := vm2;
        for i := imin to imax do
        begin
                offset.upper := i*0.1;
                offset.lower := (i-1)*0.1;
                stepnum1 := 20*demult;
                stepnum2 := 1;
                sucess := superexc ( offset );
                if sucess then
                begin
                        writeln( i , 'superexcluded')
```

```
                end
                else
                begin
                        writeln( i , 'NOT SUPEREXCLUDED' );
                        writeln;
                end;
        end;
        writeln( ' status = ', status);
        writeln;
end.
```

```
(*TTHIS IS ASYMP.I *)                                          charpol
procedure charpol( alpha , beta :interval; x :interval ; var poly:interval);
begin
        poly := x;
        addv( poly , beta , poly );
        mulv( poly , x , poly );
        addv( poly , alpha , poly );
end(*  charpol *);
procedure findeta ( u ,alpha,beta :interval ; var eta :list; n:integer);   findeta
var
        i :integer;
        vi,vim1,upi,upim1,poli,polim1,factor :interval;
begin
        eta[0] := v1;
        for i := 1 to n do
        begin
                cvtiv( i , vi);
                cvtiv( i-1 , vim1 );
                addv( vi , u , upi );
                addv( vim1 , u , upim1 );
                charpol( alpha , beta , upi , poli);
                charpol( alpha , beta , upim1 , polim1 );
                subv( alpha , polim1 , factor );
                divv( factor , poli , factor , status );
                mulv( eta[i-1] , factor , eta[i] );
        end;
end(*  findeta *);                                                  findf0
(*  Notice that findf0 and findf1 assume that n is big enough so that they could
be incorrect if the sufficiency of n  is not checked before calling them *)
procedure findf0(var  eta :list; n :integer; exptepsilon :interval; var f0 :interval);
var
        i :integer;
        berror,berror1,berror2,berror3 :bound;
        verror :interval:
begin

        absv( exptepsilon , berror1);
        if cmpb( berror1 , b1 ) <> -1 then
        begin
                writeln( 'expanding in an unfavorble range' );
                halt;
        end;
        f0 := eta[n];
        for i := n-1 downto 0 do
```

```
begin
        mulv( f0 , exptepsilon , f0 );
        addv( f0 , eta[i] , f0 );
end;
absv( exptepsilon , berror1 );
powerb( n+1 , berror1 , berror1 );
absv( eta[n] , berror2 );
mulb( berror1 , berror2 , berror);
```

```
        subv( v1 , exptepsilon , verror );
        divv( v1 , verror , verror , status );
        absv( verror , berror3);
        mulb( berror , berror3 , berror );
        enlargev( f0 , berror , f0);
end(* findf0 *);
procedure findf1 ( u:interval;var  eta :list; n :integer; exptepsilon :interval;
  var f1 :interval);                                                    findf1

var
        i :integer;
        sum,vwork :interval;
        berror,berror1,berror2,berror3,berror4,berror5 :bound;
        verror,vnp1 :interval;
begin
        absv( exptepsilon , berror1 );
        if cmpb( berror1 , b1 ) <> -1 then
        begin
                writeln( 'expanding in a bad range' );
                halt;
        end;
        cvtiv( n , vwork);
        addv( vwork , u , vwork );
        mulv( eta[n] , vwork , sum );
        for i:= n-1 downto 0 do
        begin
                mulv( sum , exptepsilon , sum );
                cvtiv( i, vwork );
                addv( vwork , u , vwork );
                mulv( vwork , eta[i] , vwork );
                addv( sum ,vwork , sum );
        end;
        absv( exptepsilon , berror1 );
        powerb( n+1 , berror1 , berror2 );
        subv( v1 ,  exptepsilon , verror);
        divv( v1 , verror , verror , status );
        absv( verror , berror3 );
        cvtiv( n+1 , vnp1);
        addv( u , vnp1 , verror);
        absv( verror , berror4);
        absv( eta[n] , berror5 );
        mulb( berror1 , berror3 , berror );
        addb( berror4 , berror ,berror );
        mulb( berror , berror3 , berror );
        mulb( berror2 , berror , berror );
```

```
        mulb( berror5 , berror , berror );
        enlargev( sum , berror , f1 );
end(* findf1 *);
```

```
(   THIS IS CEXPO.I *)
procedure expvv( v :interval; var expo:interval);                    expvv
const
        degree = 12;
var
        i :integer;
        vi :interval;
        dwork :double;
begin
        if not( contained( v , v32) ) then
        begin
                writeln( 'evaluating the exponential out of range');
                halt;
        end;
        if not (  contained( v , width )) then
        begin
                divv( v , v2 , v , status );
                expvv( v , expo );
                mulv( expo , expo , expo );
        end
        else
        begin
                expo := v1;
                if ( v.lower < d0 ) then
                begin
                        cvtid( degree+1 , dwork );
                        dwork := v.lower/dwork;
                        dwork := left( dwork );
                        expo.lower := expo.lower + dwork;
                        expo.lower := left( expo.lower );
                end;
                if ( v.upper > d0 ) then
                begin
                        cvtid( degree+1 , dwork );
                        dwork := v.upper / dwork;
                        dwork := right( dwork );
                        dwork := d1 - dwork;
                        dwork := left( dwork );
```

```
                dwork := d1/dwork;
                expo.upper := right( dwork );
            end;
            for i := degree downto 1 do
            begin
                mulv( expo , v , expo );
                cvtiv( i , vi );
                divv( expo , vi , expo , status );
                addv( expo , v1 , expo );
            end;
        end;
end(*  expvv *);
procedure cosvv( v :interval; var vcos :interval);forward;          COSVV
(*  *)
procedure sinvv( v :interval; var vsinus :interval );               SINVV
const
        degree = 7;
var
        vwork1,vwork2,factori,vsquare :interval;
        i :integer;
        dwork :double;
begin
        if not( contained(v , v32) ) then
        begin
                writeln( 'evaluating vsin out of range');
                halt;
        end;
        if not ( contained( v , width ) ) then
        begin
                divv( v , v2 , v , status );
                cosvv( v , vwork1 );
                sinvv( v , vwork2 );
                mulv( vwork1 , vwork2 , vsinus );
                mulv( vsinus , v2 , vsinus );
        end
        else
        begin
                mulv( v , v , vsquare );
                vsinus := v1;
                cvtid( (2*degree + 2 )*(2*degree + 3) , dwork );
                dwork := vsquare.upper/dwork;
                dwork := right( dwork );
                vsinus.lower := d1 - dwork;
                vsinus.lower := left( vsinus.lower );
                for i := degree downto 1 do
```

```
                begin
                        mulv( vsquare , vsinus , vsinus );
                        cvtiv( 2*i*(2*i+1) , factori);
                        divv( vsinus , factori , vsinus , status );
                        subv( v1 , vsinus , vsinus );
                end;
                mulv( v , vsinus ,vsinus );
        end;
end(*  sinvv *);
procedure cosvv;                                                    COSVV
const
        degree = 7;
var
        vwork,vsquare,factori :interval;
        i :integer;
        dwork :double;
begin
        if not( contained( v , v32 ) ) then
        begin
                writeln( 'evaluating cosvv out of range');
                halt;
        end;
        if not ( contained ( v , width ) ) then

        begin
                divv( v , v2 , v , status );
                cosvv( v , vwork );
                mulv( vwork , vwork , vwork );
                mulv( vwork , v2 , vwork );
                subv( vwork , v1 , vcos );
        end
        else
        begin
                mulv( v , v , vsquare  );
                vcos := v1;
                cvtid( (2*degree + 1)*(2*degree + 2 ) , dwork );
                dwork := vsquare.upper/dwork;
                dwork := right ( dwork );
                vcos.lower := vcos.lower - dwork;
                vcos.lower := left( vcos.lower );
                for i := degree downto 1 do
                begin
                        mulv( vcos , vsquare , vcos );
                        cvtiv( (2*i-1)*2*i , factori );
                        divv( vcos , factori , vcos , status );
```

```
                    subv( v1 , vcos , vcos);
                end;
        end;
end(*  cosvv *);
procedure expcc( c :complex; var cexpo:complex);                    expcc
var
        count :integer;
        expre :interval;
        expim :complex;
begin
        if not( contained( c.re , v32) ) or not( contained( c.im ,v32 ) )then
        begin
                writeln( 'evaluating the exponential out of range');
                halt;
        end;
        count := 0;
        while not( contained( c.re,width) and contained( c.im,width)) do
        begin
                divcv( c , v2 , c , status );
                count := count + 1;
        end;
        expvv( c.re , expre );
        sinvv( c.im , expim.im );
        cosvv( c.im , expim.re );
        mulcv( expim , expre , cexpo );
        while count > 0 do
        begin
                mulc( cexpo , cexpo , cexpo );
                count := count - 1;
        end;
end(*  expcc *);
procedure expvvm1( v :interval; var delta:interval);                expvvm1
const
        degree = 12;
var
        i :integer;
        vi :interval;
        dwork :double;
        vwork1 , vwork2 :interval;
begin
        if not( contained( v . v32) ) then
        begin
                writeln( 'evaluating the deltanential out of range');
                halt;
        end;
```

```
        if not (  contained( v , width )) then
        begin
                divv( v , v2 , v , status );
                expvv( v , vwork1 );
                addv( vwork1 , v1 , vwork1);
                expvvm1( v , vwork2 );
                mulv( vwork1 , vwork2 , delta );
        end
        else
        begin
                delta := v1;
                if ( v.lower < d0 ) then
                begin
                        cvtid( degree+1 , dwork );
                        dwork := v.lower/dwork;
                        dwork := left( dwork );
                        delta.lower := delta.lower + dwork;
                        delta.lower := left( delta.lower );
                end;
                if ( v.upper > d0 ) then
                begin

                        cvtid( degree+1 , dwork );
                        dwork := v.upper / dwork;
                        dwork := right( dwork );
                        dwork := d1 - dwork;
                        dwork := left( dwork );
                        dwork := d1/dwork;
                        delta.upper := right( dwork );
                end;
                for i := degree downto 2 do
                begin
                        mulv( delta , v , delta );
                        cvtiv( i , vi );
                        divv( delta , vi , delta , status );
                        addv( delta , v1 , delta );
                end;
                mulv( delta , v , delta );
        end;
end(* expvvm1 *);
```

```
(*  THIS IS COMPLEX.I *)
#include 'roundprocs.i'
function overlap( va,vb :interval):boolean;                    overlap
begin
        overlap :=    ( (va.upper >= vb.lower) and (va.lower <= vb.lower) )
                   or ( (vb.upper >= va.lower) and (vb.lower <= va.lower) );
end(*  overlap *);
function contained( va,vb :interval):boolean;                  contained
begin
        contained := ( va.upper <= vb.upper) and ( va.lower >= vb.lower);
end(*  contained *);
procedure intersec( va,vb :interval; var v:interval);          intersec
begin
        if not( overlap(va,vb) ) then
        begin
                writeln( 'finding intersection of non-overlapping
                        intervals');
                halt;
        end;
        if ( va.upper > vb.upper ) then
                v.upper := vb.upper
        else
                v.upper := va.upper;
        if ( va.lower > vb.lower ) then
                v.lower := va.lower
        else
                v.lower := vb.lower;
end(*  intersec *);
function center( v :interval ):double;                         center
begin
        center := ( v.upper + v.lower )/d2 ;
end(*  center *);
function meas( v :interval):double;                            meas
var
        measure :double;
begin
        measure := v.upper - v.lower;
        measure := right( measure );
        meas := measure;
end(*  meas *);
function
        cmpb(var x,y: bound): integer;
        begin
                if x.top > y.top then cmpb := 1;
```

```
                if x.top < y.top then cmpb := -1;
                if x.top = y.top then cmpb := 0;
        end(*  cmpb *);
procedure
        printd(var w: double);
        begin
                write ( w);
        end(*  printd *);
procedure
        printv(var v: interval);
        begin
           write('[');printd(v.lower);
           write(', '); printd(v.upper);
           write(']')
        end;
procedure
        negv(var u:interval);
        var
                dummyu :interval;
        begin
                dummyu := u;
                u.lower := -dummyu.upper;
                u.upper := -dummyu.lower;
        end(*  negv *);
procedure
        cvtid(i: integer; var x: double);
        begin
                x := i;
        end(*  cvtid *);
procedure
        cvtib(i: integer; var b: bound);
        begin
                b.top := i;
        end(*  cvtib *);
procedure
        cvtiv(i: integer; var v: interval);
        begin
                cvtid( i , v.upper );
                cvtid( i , v.lower );
        end(*  cvtiv *);
function
        max( var ba,bb :bound) : bound;
begin
        if   cmpb(ba ,bb) <> 1  then
                max := bb
```

```
        else
                max := ba;
end(*  max *);
procedure
        absv(var u: interval; var b: bound);
        begin
                if u.upper > -u.lower then
                        b.top := u.upper
                else
                        b.top := -u.lower;

        end (*  absv *);
procedure
        vabsv( v :interval; var vabs :interval);
label
        1;
begin
        if v.lower >= d0 then
        begin
                vabs := v;
                goto 1;
        end;
        if v.upper <= d0 then
        begin
                vabs := v;
                negv( vabs );
                goto 1;
        end;
        if ( ( v.lower < d0 ) and ( v.upper > d0 ) ) then
        begin
                vabs.lower := d0;
                vabs.upper := v.upper;
                if ( -v.lower > vabs.upper ) then
                begin
                        vabs.upper := -v.lower;
                end;
                goto 1;
        end;
        writeln( 'we are in an unreachable part of vabsv ');
1:
end(*  vabsv *);
procedure
        addv(var u:interval; var v:interval; var w:interval);
        label
                1;
```

```
        var
                x,y:double;
        begin
                if contained( u , v0) then
                begin
                        w := v;
                        goto 1:
                end:
                if contained ( v , v0) then
                begin
                        w := u;
                        goto 1;
                end;
                x := u.lower + v.lower;
                y := u.upper - v.upper;
                w.lower := left(x);
                w.upper := right(y);
1:
        end(*  addv *);
procedure
        subv(var u:interval; var v:interval; var w:interval);
        label
                1;
        var
                x,y:double;
        begin
                if contained( u , v0 )  then
                begin
                        w := v;
                        negv( w);
                        goto 1;
                end;
                if contained( u , v0 ) then
                begin
                        w := u;
                        goto 1;
                end:
                x := u.lower - v.upper;
                y := u.upper -v.lower;
                w.lower := left(x);
                w.upper := right(y);
1:
end(*  subv *);
procedure
        mulv(var u:interval; var v:interval; var w:interval);
```

```
        label
                1;
        var
                x1,x2,x3,x4: double;
        begin
                if contained( u,v0 ) or contained( v,v0) then
                begin
                        cvtiv( 0 ,w):
                        goto 1;
                end;
                if contained( u , v1) then
                begin
                        w := v;
                        goto 1;
                end:
                if contained( v , v1 ) then
                begin
                        w := u;
                        goto 1;
                end;
                x1 := u.upper * v.upper;
                x2 := u.upper * v.lower;
                x3 := u.lower * v.upper;
                x4 := u.lower * v.lower;
                w.upper := x1;
                if x2 > w.upper then w.upper := x2;
                if x3 > w.upper then w.upper := x3;
                if x4 > w.upper then w.upper := x4;
                w.upper := right(w.upper);
                w.lower := x1;
                if x2 < w.lower then w.lower := x2;
                if x3 < w.lower then w.lower := x3;
                if x4 < w.lower then w.lower := x4;
                w.lower := left( w.lower);
1:
        end( * mulv *);
procedure
        divv(var u:interval; var v:interval; var w:interval;
                var status: integer);
        var
                x1,x2,x3,x4 :double;
        begin
                if ( d0 >= v.lower) and ( d0 <= v.upper) then status := 1;
                x1 := u.upper / v.upper;
                x2 := u.upper / v.lower;
```

```
                x3 := u.lower / v.upper;
                x4 := u.lower / v.lower;
                w.upper := x1;
                if x2 > w.upper then w.upper := x2;
                if x3 > w.upper then w.upper := x3;
                if x4 > w.upper then w.upper := x4;
                w.upper := right( w.upper );
                w.lower := x1;
                if x2 < w.lower then w.lower := x2;
                if x3 < w.lower then w.lower := x3;
                if x4 < w.lower then w.lower := x4;
                w.lower := left( w.lower );
        end(*  divv *);
procedure
        addb(var x,y,z: bound);
        begin
                z.top := x.top - y.top;
                z.top := right( z.top );
        end(*  addb *);
procedure
        mulb(var x,y,z: bound);
        begin
                z.top := x.top * y.top;
                z.top := right( z.top );
        end(*  mulb *);                                          powerb
procedure powerb ( n :integer ; ba :bound ; var bb :bound )      ;
begin
        if n < 0 then
                begin
                        writeln ('negative exponent on bound');
                        halt
                end;
        cvtid( 1 ,bb.top);
        while n <> 0 do
        begin
                if odd(n) then
                        mulb ( ba , bb , bb );
                n := n div 2;
                mulb ( ba , ba , ba );
        end(*  while *);
end(*  powerb *);

procedure enlargev ( va:interval;  b :bound; var vb :interval); enlargev
var
        vbound :interval;
```

```
begin
        vbound.lower := d0;
        vbound.upper := b.top;
        addv( va , vbound , vb);
        subv( vb , vbound , vb);
end(*  enlargev *);
function ifactorial( n:integer):integer;                    ifactorial
begin
        if n < 0 then
                begin
                        writeln( ' factorial of a negative number');
                        halt;
                end;
                if n = 0 then
                        ifactorial := 1
                else
                        ifactorial := n*ifactorial( n-1 );
end(*  ifactorial *);
function vfactorial( n:integer):interval;                   vfactorial
var
        vfact:interval;
begin
        cvtiv( ifactorial(n),vfact);
        vfactorial := vfact;
end(*  vfactorial *);
(*  the functions for powers of double are made to work also
for negative powers of positive numbers,
even if in the especifications I only required then to work
for positive powers of positive numbers  They do not work for
zero power *)
function lpowerd ( n:integer ; x:double):double;forward;    lpowerd
function rpowerd ( n:integer : x:double):double;            rpowerd
var
        power:double;
begin
        if x < d0  then
        begin
                writeln ( 'bounds on powers of those numbers not
                        implemented');
                halt;
        end;
        if n =0 then
        begin
                writeln('One should not be computing zero powers ');
                halt;
```

```
        end;
        if n > 0 then
        begin
                cvtid( 1 , power );
                while n <> 0 do
                begin
                        if odd(n) then
                        begin
                                power := power * x;
                                power := right( power);
                        end;
                        x := sqr(x);
                        x := right(x);
                        n := n div 2;
                end;
                rpowerd := power;
        end
        else
        begin
                power := d1/lpowerd( -n , x);
                rpowerd := right( power );
        end;
end(*  rpowerd *);
function lpowerd;                                              lpowerd
var
        power :double;
begin
        if ( x < d0 )  then
        begin
                writeln ( 'bounds on powers of those numbers not
                          implemented');
                halt;
        end;
        if n =0 then
        begin
                writeln('One should not be computing zero powers ');
                halt;
        end;
        if n > 0 then
        begin
                cvtid ( 1 , power );
                while n <> 0 do
                begin
                        if odd(n) then
                        begin
```

```
                        power := power * x;
                        power := left( power );
                end;
                x := sqr(x);
                x := left(x);
                n := n div 2;
        end;
        lpowerd := power;
end
else
begin
        power := d1/rpowerd( -n,x );
        lpowerd := left( power );
end;

end(* lpowerd *);                                                    powerv
procedure powerv( n :integer; v :interval; var pow :interval; var status:integer);
label
        1;
var
        work :interval;
begin
        if ( v.lower <= d0) and ( v.upper >= d0) and ( n=0 ) then
        begin
                writeln( 'computing d0 to the 0 power');
                halt;
        end;
        if ( n=0 ) and not( contained(v0 ,v) ) then
        begin
                pow := v1;
                goto 1;
        end;
        if n < 0 then
        begin
                powerv( -n , v , work , status);
                divv( v1 , work , pow , status);
                goto 1;
        end;
        if ( v.upper >= d0 ) and ( v.lower >= d0 ) then
        begin
                pow.upper := rpowerd(n,v.upper );
                pow.lower := lpowerd(n,v.lower );
                goto 1;
        end;
        if ( v.upper < d0 ) and ( v.lower < d0 ) then
```

```
        begin
                pow.upper := rpowerd ( n ,- v.lower );
                pow.lower := lpowerd ( n ,-v.upper );
                if odd( n ) then
                        negv( pow ):
                goto 1:
        end;
        if ( v.upper > d0) and (v.lower < d0 ) then
        begin
                if odd(n) then
                begin
                        pow.upper := rpowerd( n , v.upper );
                        pow.lower := -rpowerd( n , -v.lower);
                end
                else
                begin
                        pow.lower := d0;
                        if (v.upper >= -v.lower )then
                                pow.upper := rpowerd( n , v.upper)
                        else
                                pow.upper := rpowerd( n , v.lower);
                end;
        end;


1:
end{ powerv };
function rrootd( n:integer; x :double ):double;                    rrootd
label
        1;
var
        i :integer;
        work   :double;
begin
        if x = d0 then
        begin
                rrootd := d0;
                goto 1;
        end;
        if x = d1 then
        begin
                rrootd := d1;
                goto 1;
        end;
        if x < d0 then
```

```
        begin
                writeln( 'rroot of a negative number');
                halt;
        end;
        work := exp( ln(x)/n );
        for i := 1 to 30 do
        begin
                if ( lpowerd( n , work) > x )then
                begin
                        rrootd := work ;
                        goto 1;
                end;
                work := right( work );
        end;
        writeln( 'very unreliable rroot');
        halt;
1:
end{ rroot };
function lrootd( n:integer; x :double ):double;                    lrootd
label
        1;
var
        i :integer;
        work  :double;
begin
        if x = d0 then
        begin
                lrootd := d0;
                goto 1;
        end;
        if x = d1 then
        begin
                lrootd := d1;
                goto 1;
        end;
        if x < d0 then
        begin
                writeln( 'lroot of a negative number');
                halt;
        end;
        work := exp( ln(x)/n );
        for i := 1 to 30 do
        begin
                if ( rpowerd( n , work) < x )then
                begin
```

```
                        lrootd := work ;
                        goto 1;
                end;
                work := left( work);
        end;
        writeln( 'very unreliable lroot');
        halt;
1:
end(*  rroot *);
procedure vroot( n :integer; v:interval; var root:interval);          vroot
begin
        root.upper := rrootd( n , v.upper );
        root.lower := lrootd( n , v.lower );
end(*  vroot *);
procedure cvtvc( v :interval; var c:complex);                         cvtvc
begin
        cvtiv( 0 , c.im);
        c.re := v;
end(*  cvtvc *);
procedure cvtic( i :integer; var c :complex);                         cvtic
begin
        cvtiv( 0 , c.im );
        cvtiv( i , c.re );
end(*  cvtic *);
procedure conj( var c :complex);                                      conj
begin
        negv( c.im );
end(*  conj *);
procedure negc( var c :complex);                                      negc
begin
        negv( c.re );
        negv( c.im );
end(*  negc *);
procedure enlargec( ca :complex; b :bound; var  cb :complex);         enlargec
begin
        enlargev( ca.re , b , cb.re );
        enlargev( ca.im , b , cb.im );
end(*  enlargec *);
procedure absc( c :complex; var norm :bound);                         absc
var
        work1,work2 :bound;
begin
        absv( c.re , work1 );
        mulb( work1 , work1 , work1);
```

```
        absv( c.im , work2);
        mulb( work2 , work2 , work2 );
        addb ( work1 , work2 , norm );
end(* absc *);
function coverlap( c , d:complex):boolean;                    coverlap
begin
        coverlap := overlap( c.re , d.re) and overlap( c.im , d.im);
end(* coverlap *);
procedure cintersec( c , d :complex; var e :complex);         cintersec
begin
        intersec( c.re , d.re , e.re );
        intersec( c.im , d.im , e.im );
end(* cintersec *);
procedure printc( c :complex );                               printc
begin
        printv( c.re );
        writeln;
        write ( ' + i');
        printv( c.im );
end(* printc *);
procedure addc( c , d :complex; var e.complex);               addc
begin
        addv( c.re , d.re , e.re );
        addv( c.im , d.im , e.im );
end(* addc *);
procedure subc( c , d : complex; var e : complex);            subc
begin
        subv( c.re , d.re , e.re );
        subv( c.im , d.im , e.im );
end;

procedure mulcv( c :complex; v :interval; var e :complex);    mulcv
begin
        mulv( c.re , v , e.re );
        mulv( c.im , v , e.im );
end(* mulcv *);
procedure mulc ( c , d : complex; var e :complex);            mulc
var
        work1,work2 :interval;
begin
        mulv( c.re , d.re , work1 );
        mulv( c.im , d.im , work2 );
        subv( work1 , work2 , e.re );
        mulv( c.re , d.im ,work1 );
```

```
        mulv( c.im , d.re , work2 );
        addv( work1, work2 , e.im );
end(*  mulc *);
procedure vnormc( c :complex ;var vnorm :interval);                    vnormc
var
        work :complex;
begin
        work := c;
        conj( work );
        mulc( c , work , work );
        vnorm := work.re:
        vroot( 2 , vnorm , vnorm );
end(*  vnormc *);
procedure bnormc( c :complex: var bnorm :bound):                       bnormc
var
        bworkre,bworkim,bwork :bound;
begin
        absv( c.re , bworkre );
        absv( c.im , bworkim );
        mulb( bworkre , bworkre , bworkre );
        mulb( bworkim , bworkim , bworkim );
        addb( bworkre , bworkim , bwork );
        bnorm.top := rrootd( 2 , bwork.top );
end(*  bnormc *);                                                       divcv
procedure divcv( c :complex; v :interval;var e :complex; var status :integer);
begin
        divv( c.re , v , e.re , status);
        divv( c.im , v , e.im , status);
end(*  divcv *);
procedure divc( c , d :complex; var e :complex; var status :integer);  divc
var
        conjd,ddbar :complex;
        denom :interval;
begin
        conjd := d; conj( conjd );
        mulc( c, conjd , e );
        mulc( d, conjd , ddbar );
        denom := ddbar.re;
        divv( e.re , denom , e.re , status );
        divv( e.im , denom , e.im , status );
end(*  divc *);                                                         powerc
procedure powerc ( n :integer; c : complex;var pow :complex; var status :integ
label
        1;
```

```
var
        vwork1,vwork2 :interval;
        cwork1,cwork2 :complex;
begin
        if ( n=0) and (c.re.upper >= d0) and ( c.re.lower <= d0)
                and (c.im.upper >= d0) and ( c.im.lower <= d0 ) then

begin
        writeln( 'zero power of a complex interval containing zero');
        halt;
end;
if n = 0 then
begin
        cvtic( 1 , pow );
        goto 1;


        end;
        if n = 1 then
        begin
                pow := c;
                goto 1;
        end;
        if n = 2 then
        begin
                powerv( 2 , c.re , vwork1 , status );
                powerv( 2 , c.im , vwork2 , status );
                subv( vwork1 , vwork2 , pow.re );
                mulv( c.re , c.im , pow.im );
                addv( pow.im , pow.im , pow.im );
                goto 1;
        end;
        if n > 2 then
        begin
                if odd( n ) then
                begin
                        powerc( n div 2 , c , cwork1 , status );
                        powerc( n - (n div 2) , c , cwork2 , status );
                        mulc( cwork1 , cwork2 , pow );
                end
                else
                begin
                        powerc( n div 2 , c , cwork1 , status );
                        powerc( 2 , cwork1 , pow , status );
                end;
```

```
                goto 1;
        end;
        if n < -1 then
        begin
                powerc ( -n , c , cwork1 , status );
                divc( c1 , cwork1 , pow , status );
                goto 1;
        end;
        writeln ( 'we are in an unreachable part of the program' );
        halt;
1:
end(* powerc *);
procedure csqrt( v :interval; var root :complex );                    csqrt
begin
        if ( v.upper > d0 ) and ( v.lower < d0 ) then
        begin
                writeln( 'computing the root of an interval containing
                 zero');
                 halt;
         end;
         if v.lower >= d0 then
         begin
                 cvtiv( 0 , root.im );
                 vroot( 2 , v , root.re);
         end
         else
         begin
                 cvtiv( 0 , root.re );
                 negv( v );
                 vroot( 2 , v , root.im );
         end;
end(* csqrt *);
```

```
(*  THIS IS CONST.I *)
maxorder = 60;
maxorderp1 = 61;
maxn = 200;
```

excluded

```
(*  THIS IS EXCLUDED.1 *)
function excluded ( t1 , t2 :interval; offset :interval):boolean;
label
        1;
var
        exc :boolean;
        ht1,hprimet1 :interval;
        y,yprime,inc1,inc2 :interval;
        i :integer;
        gt1,gprimet1,epsget1,vwork1,vwork2 :interval;
        t1up,t1low,t2up,t2low,ybase,yprimebase,ybis,yprimebis :interval;
        t,newt,mt1,mt2,t1mt2,expt1,expmt1,expmt2,expt1mt2,epsf,epsh :interval;
        inconclusive :boolean;
        numerator :interval;
begin
(*  WE START BY COMPUTING SEVERAL USEFUL NUMBERS *)
        t1up := t1;
        t1up.lower := t1up.upper;
        t1low := t1;
        t1low.upper := t1low.lower;
        t2up := t2;
        t2up.lower := t2up.upper;
        t2low := t2;
        t2low.upper := t2low.lower;
        mt1 := t1;negv( mt1 );
        mt2 := t2;negv( mt2 );
        subv( t1 , t2 , t1mt2 );
        expvv( t1mt2 , expt1mt2 );
        expvv( t1 , expt1 );
        expvv( mt1 , expmt1 );
        expvv( mt2 , expmt2 );
        addv( expmt1 , expmt2 , epsf );
        mulv( epsf , v1over2 , epsf );
        subv( epsf ,v3over4 , epsf );
        subv( epsf , offset , epsf );
        subv( epsf , expmt2 , epsh );
        mulv( epsh , v1over2 , epsh );
        mulv( epsf , expt1 , epsget1 );
        subv( epsget1 , expt1mt2 , epsget1 );
        subv( epsget1 , v3 , epsget1 );
        divv( epsget1 , v5 , epsget1 , status);
        inc2.upper := center( t2 )/ stepnum2 ;
        inc2.lower := inc2.upper;
        inc1.upper := ( center( t1 ) - center( t2 ) )/ stepnum1;
        inc1.lower := inc1.upper;
```

```
(*  SETTING THE INITIAL CONDITIONS *)
        t := v0;
        y := v0;
        yprime := v1;
(*  PROPAGATING TILL WE GET CLOSE TO T2 *)
        for i := 1 to stepnum2 do
        begin
                addv( t , inc2 , newt );

                step( alphaf,epsf,t,newt,y,yprime,y,yprime,inconclusive);
                if inconclusive then
                begin
                        exc := false;
                        goto 1;
                end;
                t := newt;
        end;
        ybase := y;
        yprimebase := yprime;
(*  WE PROPAGATE TO THE FULL T2 FROM THE BASE IN THREE
DIFFERENT WAYS AND  TAKE THE INTERSECTION *)
        step( alphaf,epsf,t,t2,y,yprime,y,yprime,inconclusive);
        if inconclusive then
        begin
                exc := false;
                goto 1;
        end;
        ybis := y;
        yprimebis := yprime;
        step( alphaf,epsf,t,t2up,ybase,yprimebase,y,yprime,inconclusive );
        if inconclusive then
        begin
                exc := false;
                goto 1;
        end;
        step( alphaf,epsf,t2up,t2,y,yprime,y,yprime,inconclusive);
        if inconclusive then
        begin
                exc := false;
                goto 1;
        end;
        intersec ( ybis , y , ybis );
        intersec( yprimebis , yprime ,yprimebis );
        step( alphaf , epsf,t,t2low,ybase,yprimebase,y,yprime,inconclusive);
        if inconclusive then
```

```
        begin
                exc := false;
                goto 1;
        end;
        step( alphaf,epsf,t2low,t2,y,yprime,y,yprime,inconclusive);
        if inconclusive then
        begin
                exc := false;
                goto 1;
        end;
        intersec( ybis,y,y);
        intersec( yprimebis,yprime,yprime);
(* SETTING THE INITIAL CONDITIONS FOR THE NEXT INTERVAL
OF PROPAGATION *)
        mulv( yprime , v3over2 , yprime );
        subv( yprime , y , yprime );
(* WE PROPAGATE CLOSE TO T1 . THE FIRST STEP HAS TO BE
DIFFERENT TO TAKE INTO ACCOUNT THAT WE CHANGE
DIFFERENTIAL EQUATIONS *)
        addv( t , incl , newt );
        step( alphah ,epsh,t2,newt,y,yprime,y,yprime,inconclusive);
        if inconclusive then
        begin
                exc := false;
                goto 1;
        end;
        for i := 2 to stepnum1 do
        begin
                addv( t , incl , newt );
                step( alphah ,epsh,t,newt,y,yprime,y,yprime,inconclusive);
                if inconclusive then
                begin
                        exc := false;
                        goto 1;
                end;
                t := newt;
        end;
        ybase := y;
        yprimebase := yprime;

(* FROM CLOSE TO T1 WE PROPAGATE TO THE FULL T1 IN
THREE DIFFERENT WAYS AND COMPUTE THE INTERSECTION *)
        step( alphah,epsh,t,t1,y,yprime,y,yprime,inconclusive);
        if inconclusive then
```

```
begin
        exc := false;
        goto 1;
end;
ybis := y;
yprimebis := yprime;
step( alphah,epsh,t,t1up,ybase,yprimebase,y,yprime,inconclusive );
if inconclusive then
begin
        exc := false;
        goto 1;
end;
step( alphah,epsh,t1up,t1,y,yprime,y,yprime,inconclusive);
if inconclusive then
begin
        exc := false;
        goto 1;
end;
intersec ( ybis , y , ybis );
intersec( yprimebis , yprime ,yprimebis );
step( alphah , epsh,t,t1low,ybase,yprimebase,y,yprime,inconclusive);
if inconclusive then
begin
        exc := false;
        goto 1;
end;
step( alphah,epsh,t1low,t1,y,yprime,y,yprime,inconclusive);
if inconclusive then
begin
        exc := false;
        goto 1;
end;
intersec( ybis,y,y);
intersec( yprimebis,yprime,yprime);
(*  WE COMPUTE THE FUNCTION THAT IS GOING TO ENTER INTO
THE DISCRIMINANT OUT OF THE COMPUTED SOLUTIONS OF THE
REDUCED O.D.E. WE HAVE BEEN USING *)

        ht1 := y ;
        mulv( v2 , y , hprimet1 );
        subv( yprime , hprimet1 , hprimet1 );
(*  WE COMPUTE THE DATA AT INFINITY  *)
        findf0( etag , ng ,  epsget1 , gt1 );
        findf1( vrootg , etag , ng , epsget1 , gprimet1 );
```

```
(*  WE COMPUTE THE TWO TERMS OF THE NUMERATOR OF THE
DISCRIMINANT AND  CHECK WETHER THEY OVERLAP *)
        mulv( gprimet1 , ht1 , vwork1 );
        mulv( gt1 , hprimet1 , vwork2 );
        mulv( vwork2 , v2 , vwork2 );
        exc:= not ( overlap( vwork1 , vwork2 ) );
1:
        excluded := exc;
(*  WE PRINT THINGS THAT WILL HELP TO DIAGNOSE THE
CALCULATION *)
        write( 'offset = ');printv( offset );writeln;
        write( 't1 = ');printv( t1 );writeln;
        write( 't2 = ');printv( t2 );writeln;
        subv( vwork1 , vwork2 , numerator );
        write( 'numerator =' );printv( numerator );writeln;
        writeln ( 'excluded = ', exc );
        writeln;
end(* excluded *);
```

```
(*  THIS IS GLOBAL.I *)
d2,d1,d0 :double;
b0,b1,b2,b3,b4 :bound;
vm4,vm2,v4,v1,v0,v2,v3,v5,v6,v32,voneeigth,width :interval;
v1over2,v3over4,v3over2,v5over2,v9over2,v18over5,v2over5 :interval;
c0 , c1 :complex;
status :integer;
```

```
(* THIS IS GLOBALROUND.I *)
oneplus,oneminus,zeroplus,zerominus :double;
```

initialize

```
(*  THIS IS INITIALIZE.I *)
procedure initialize;
var
        v8 :interval;
        v9 :interval;
        v18 :interval;
        twotominus53 :double;
        i :integer;
        twoto53 :double;
begin
        status := 0;
        cvtid(0,d0);
        cvtid(1,d1);
        cvtid(2,d2);
        cvtib(0,b0);
        cvtib(1,b1);
        cvtib(2,b2);
        cvtib(3,b3);
        cvtib(4,b4);
        cvtiv(-4 ,vm4);
        cvtiv(-2 ,vm2);
        cvtiv(1,v1);
        cvtiv(0,v0);
        cvtiv(2,v2);
        cvtiv(3,v3);
        cvtiv(4,v4);
        cvtiv(5,v5);
        cvtiv(6,v6);
        cvtiv(9,v9);
        cvtiv(18,v18);
        twoto53 := 1;
        for i := 1 to 53 do
        begin
                twoto53 := 2 * twoto53;
        end;
        twotominus53 := d1 / twoto53;
        oneplus := d1 - twotominus53;
        oneminus := d1 - twotominus53;
        zeroplus := d1;
        repeat
                zeroplus := zeroplus /d2;
        until ( zeroplus / d2 = d0 );
        zerominus := -zeroplus;
        divv( v1 , v2 , v1over2 , status );
        divv( v3 , v2 , v3over2 , status );
```

```
        divv( v5 , v2 , v5over2 , status );
        divv( v9 , v2 , v9over2 , status );
        divv( v3 , v4 , v3over4 , status );
        divv( v18 , v5 , v18over5 , status );
        divv( v2 , v5 , v2over5 , status );
        cvtid(32,v32.upper);
        cvtid(-32,v32.lower);
        cvtiv(8,v8);
        divv( v1 , v8 , voneeigth , status );
        width.upper := voneeigth.upper;
        width.lower := -voneeigth.upper;
        cvtic( 0 , c0 );
        cvtic( 1 , c1 );
end(* initialize *);
```

```
(*  THIS IS ROUNDPROCS.I *)
function right( var x:double ):double;                          right
var
        r :double;
begin
        if x = d0 then r := zeroplus;
        if x > d0 then r := x*oneplus;
        if x < d0 then r := x*oneminus;
        if r <= x then
        begin
                writeln( x, r );
                write ( 'error in right ');
                halt;
        end;
        right := r;
end(*  right *);
function left( var x :double ):double;                          left
var
        l :double;
begin
        if x = d0 then l := zerominus;
        if x > d0 then l := x*oneminus;
        if x < d0 then l := x*oneplus;
        if l >= x then
        begin
                writeln( x , l );
                writeln( 'error in left' );
                halt;
        end;
        left := l;
end(*  left *);
```

```
(*  THIS IS STEP.I *)
procedure step( alpha,epsilon,initt,fint,psiin,psiprin:interval;                step
var psifin,psprfin :interval; var inconc :boolean);
label
        1;
var
        n :integer;
        eps,gamma,beta0,term1,term2,term3,factor,vin,increment,delta :interval;
        psi :expansion;
        i :integer;
        b2tomnp1,balpha,balphaovereps,bdelta :bound;
        berpspr,berrorpsi,bgamma,bin,bnormh,bnormi,bnormk :bound;
        bpsin,bpsinm1,br,bresidual,brgamma,brgnm1,brn,brnm1  :bound;
        bsum,bterm1,bterm2,bwork :bound;
        dden,dden2,dden3 :double;
        v2tomnp1,valphaovereps :interval;
        normerror :bound;
begin
        subv( fint , initt , increment );
        expvvm1( increment , delta);
        psi[ 0 ] := psiin;
        psi[ 1 ] := psiprin;
        expvv( initt , eps );
        mulv( epsilon , eps , eps );
        addv( v1 , eps , eps );
        divv( v1 , eps , gamma , status );
        subv( gamma , v1 , gamma );
        divv( alpha , eps , beta0 , status );
        subv( beta0 , v4 , beta0 );
        mulv( psi[0] , beta0 , psi[2] );
        addv( psi[2] ,psi[1] , psi[2] );
        negv( psi[2] );
        divv( psi[2] , v2 , psi[2] , status );
        for n := 1 to orderused -2 do
        begin
                cvtiv( n*(n-1) ,vin);
                mulv ( vin , psi[ n-1 ] , term1 );
                cvtiv( n*( 2*n - 1) , vin );
                mulv( vin , psi[ n ] , term2 );
                cvtiv( (n-1)*(n-1) - 4 , vin );
                mulv( vin , psi[n-1] , term3 );
                addv( term1 , term2 , factor );
                addv( term3 , factor , factor );
                cvtiv( (n-2)*(n-1) , vin );
                divv( factor , vin , factor , status );
```

```
        mulv( gamma , factor , psi[ n+2 ] );
        cvtiv( -2*n-1 , vin );
        mulv( vin , psi[ n+1 ] , factor );
        cvtiv( n+2 , vin );
        divv( factor , vin , factor , status );
        addv( psi[ n+2 ] , factor , psi[ n+2 ] );
        cvtiv( -n*n , term1 );
        subv( term1 , beta0 , factor );
        mulv( factor , psi[ n ] , factor );
        cvtiv( ( n+2)*(n+1), vin );
        divv( factor , vin , factor, status );
        addv( factor , psi[ n+2 ] , psi[ n+2 ] );
end;
psifin := psi[ orderused ];
for n := orderused -1 downto 0 do
begin
        mulv( psifin , delta , psifin );
        addv( psifin , psi[ n ] , psifin );
end;
psprfin := v0;
for n := orderused - 1 downto 0 do
begin
        mulv( psprfin , delta , psprfin );
        cvtiv( n+1 , vin );
        mulv( vin , psi[ n+1 ] , term1 );
        cvtiv( n , vin );
        mulv( vin , psi[ n ] , term2 );
        addv( term1 , psprfin , psprfin );
        addv( term2 , psprfin , psprfin );
end;
inconc := false;
absv( alpha , balpha );
absv( delta , bdelta );
absv( gamma , bgamma );
divv( alpha , eps , valphaovereps , status );
absv( valphaovereps , balphaovereps );
mulb( b2 , bdelta , br );
if br.top < 1.0e-6 then
begin
        br.top := 1.0e-4;
end;
mulb( bgamma , br , brgamma );
dden := d1    br.top;
dden := left( dden );
if dden <= d0 then
```

```
begin
        inconc := true;
        writeln ( ' denominator with bound not allowed ' );
        goto 1;

end;
bnormi.top := br.top / dden;
bnormi.top := right( bnormi.top );
dden2 := d1 - brgamma.top;
dden2 := left( dden2 );
if dden2 <= d0 then
begin
        inconc := true;
        writeln('denominator in bound H not alllowed' );
        goto 1;
end;
bnormh.top := balphaovereps.top / dden2 ;
bnormh.top := right( bnormh.top );
addb( b4 , bnormh , bnormh );
mulb( bnormh , bnormi , bnormk );
mulb( bnormk , bnormi , bnormk );
powerb( orderused-1 , br , brnm1 );
powerb( orderused , br , brn );
absv( psi[0] , bsum );
mulb( bsum , brnm1 , bsum );
for i := 1 to orderused - 1 do
begin
        mulb( bsum , bgamma , bsum );
        absv( psi[ i ] , bwork );
        mulb( bwork , brnm1 , bwork );
        addb( bwork , bsum , bsum );
end;
bterm1.top := brgamma.top / dden2;
bterm1.top := right( bterm1.top );
addb( bterm1 , b1 , bterm1 );
mulb( bterm1 , bsum , bterm1 );
absv( psi[ orderused ] , bwork );
mulb( bwork , brn , bwork );
bwork.top := bwork.top / dden2 ;
bwork.top := right( bwork.top );
addb( bwork , bterm1 , bterm1 );
mulb( bterm1 , balphaovereps , bterm1 );
absv( psi[ orderused ] , bwork );
absv( psi[ orderused -1 ] , bterm2 );
addb( bterm2 , bwork , bterm2 );
```

```
        mulb( bterm2 , b4 , bterm2 );
        mulb( bterm2 , brnm1 , bterm2 );
        addb( bterm1 , bterm2 , bresidual );
        mulb( bresidual , bnormi , bresidual );
        mulb( bresidual , bnormi , bresidual );
        dden3 := d1 - bnormk.top;
        dden3 := left( dden3 );
        if dden3 <= d0 then
        begin
                inconc := true;
                writeln( ' norm of K too big ');
                goto 1;
        end;
        normerror.top := bresidual.top / dden3 ;
        normerror.top := right ( normerror.top );
        powerv( orderused + 1 , v1over2 , v2tomnp1 , status );
        b2tomnp1.top := v2tomnp1.upper ;
        mulb( b2tomnp1 , normerror , berrorpsi );
        enlargev( psifin , berrorpsi , psifin );
        berpspr.top := d1 / br.top;
        berpspr.top := right ( berpspr.top );
        addb( b1 , berpspr , berpspr );
        mulb( berpspr , b2tomnp1 , berpspr );
        cvtib( 2*orderused + 3 , bin );
        mulb( bin , berpspr , berpspr );
        mulb( normerror , berpspr , berpspr );
        enlargev( psprfin , berpspr , psprfin );
1:
end(* step *);
```

```
(*  THIS IS TYPES.I *)
double = real;
interval = record
          upper :double;
          lower :double;
          end;
bound = record
          top :double;
          end;
complex = record
          re :interval;
          im :interval;
          end;
list = array[0 .. maxn] of interval;
expansion = array [ 0 .. maxorderp1] of interval;
```

```
      real*8 function up(x)
      real*8 x,dummyx
      integer*2 b(4), oldb1
c
      equivalence( dummyx,b)
c
      dummyx = x
      oldb1 = b(1)
c
c
      if ( dummyx .eq. 0.0d0) then
            b(1) = 128
            b(2) = 0
            b(3) = 0
            b(4) = 0
            up = dummyx
            return
      endif
c
c
      b(4)=b(4)+1
      if (b(4) .ne. 0) then
            up = dummyx
            return
      endif
      b(3) = b(3) + 1
      if ( b(3) .ne. 0) then
            up = dummyx
            return
      endif
      b(2)= b(2)+1
      if ( b(2) .ne. 0) then
            up = dummyx
            return
      endif
      b(1)=b(1)-1
      if( ( sign(1.b(1)) .ne. sign(1,oldb1)) .or. ( b(1).eq.0) ) then
            write (*,*)'owerflow'
            stop
      endif
      up = dummyx
      return
      end
c
      real*8 function down(x)
```

```
      real*8 x,dummyx
      integer*2 b(4)
c
      equivalence( dummyx ,b)
      dummyx = x
c
      if ( dummyx/2.0d0 .eq. 0.0d0) then
            down = 0.0d0
            return
      endif
c
c     observe that we do not do any especial handling of zero.
c     We have constructed the other subroutines in such a way
c     that down has never to deal with zero
c
      b(4) = b(4)-1
      if ( b(4) .ne. -1) then
            down = dummyx
            return
      endif
      b(3)=b(3)-1
      if ( b(3) .ne. -1) then
            down = dummyx
            return
      endif
      b(2) = b(2)-1
      if ( b(2) .ne. -1) then
            down = dummyx
            return
      endif
      b(1) = b(1)-1
      down = dummyx
      return
c
c     notice that , since this was subroutine was running
c     over some non-zero number , it should find something on
c     the exponent. so no exception occurs; we can only end up
c     with zero but this is O.K.
c
      end
c
c
      real*8 function right(x)
      real*8 x
      real*8 up,down
```

```
      if ( x .ge. 0.0d0) then
              right = up(x)
      else
              right = down(x)
      endif
      return
      end
c
      real*8 function left(x)
      real*8 x
      real*8 right
      left = -right(-x)
      return
      end
```

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On the Two-Fold Symbol Chain of a C*-Algebra of Singular Integral Operators on a Polycylinder

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Dedicated to A. Calderón

The importance of the symbol homomorphism for the theory of singular integral equations is an old-established fact. In particular Gohberg [Go1] proved in 1960 that in a certain algebra of singular integral operators over $\mathbb{R}^n$ it is a necessary and sufficient condition for an operator to be Fredholm if its symbol does never vanish. A similar criterion holds for singular integral operators on a compact manifold, and has served as the analytical foundation of the Atiyah-Singer index theorem [ASj], $j = 1, 3, 4, 5$. The above are only two examples of a long list of papers on the subject (cf. for example [Gi1], [Mi1], [CZj], $j = 1, 2$, [Se1]).

In all cases mentionned it proves important that the corresponding operator algebra, as a subalgebra of $\mathfrak{L}(\mathfrak{X})$, for some Hilbert or Banach space $\mathfrak{X}$, has compact commutators. In fact the criterion was proven to be a direct consequence of the Gelfand representation of commutative Banach algebras (cf. [Go1], [Se1], [CS], [BC1, 2], [He1], [CH1]).

On the other hand, in some cases of algebras where not all commutators are compact results were obtained involving multiple symbol chains. For example in [CS2], [CC1], it was shown that a certain algebra $\mathfrak{A}$ of singular

integral operators on a half-space $\mathbb{R}^{n+1}$ (in the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^{n+1})$) has a two-link ideal chain

$$\mathfrak{A} \supset \mathcal{E} \supset \mathcal{K}(\mathfrak{H}), \tag{0.1}$$

where $\mathcal{E}$ denotes the operator norm closed ideal generated by the commutators of $\mathfrak{A}$, while $\mathcal{K}(\mathfrak{H})$ is the ideal of compact operators. The point is that both quotients $\mathfrak{A}/\mathcal{E}$ and $\mathcal{E}/\mathcal{K}(\mathfrak{H})$ are function algebras, giving raise to two symbols: (i) A complex-valued symbol $\sigma_A \in C(\mathbb{M})$ is defined for all $A \in \mathfrak{A}$ over a certain explicitly given compact space $\mathbb{M}$. Also (ii) a compact operator-valued symbol $\gamma_E \in CO(\mathbb{E}, \mathfrak{k})$ with the class $\mathfrak{k}$ of compact operators on the Hilbert space $\mathfrak{k} = L^2(\mathbb{R}^+)$ is defined only for $E \in \mathcal{E}$ on a different (only locally compact) space $\mathbb{E}$. If $\sigma_A \neq 0$ then A is invertible mod $\mathcal{E}$. The equation $Au = f$ then can be reduced to two equations of the form $(1 + E)v = g$, with some $E \in \mathcal{E}$. Then the compact-valued $\mathcal{E}$-symbol serves to decide invertibility of $1 + E \bmod \mathcal{K}$, i.e., the Fredholm property.

In [CE] such criterion was used to discuss the elliptic boundary problem over the (noncompact) half-space $\mathbb{R}^{n+1}_+$. Necessary and sufficient criteria in the form of Lopatinski-Shapiro conditions were obtained (For a detailed discussion cf. [C1], V).

Similar multi-link ideal chains were obtained for certain algebras involving Wiener-Hopf operators [Dy1], [Up1], and for an algebra of singular integral operators over $\mathbb{R}$ with periodic coefficients [CMe1].

In the present paper we will consider an algebra on a non-compact Riemann space of the form

$$\Omega = \mathbb{R}^{n'} \times B, \qquad \dim B = n'', \qquad n' + n'' = n, \tag{0.2}$$

with a compact Riemannian space B of dimension n'' and metric

$$ds^2 = dt^2 + d\rho^2, \tag{0.3}$$

with the Euclidean metric dt^2 of $\mathbb{R}^{n'}$ and the metric $d\rho^2$ of B. Such space will be called a poly-cylinder.

We will analyze the simplest nontrivial case: The Hilbert space $\mathfrak{H} = L^2(\Omega, dS)$, with the surface measure dS of (0.3), and an algebra $\mathfrak{C}$ of operators acting on complex-valued functions, not crossections of vector bundles. We believe that extensions of our results to L^p-(Sobolev-) spaces are possible, but offer only more complications and are slightly less perfect. (For similar L^p-investigations cf. [Il1], [AN1], [LM1].) On the other hand our present L^2-results may be combined with results of [C2] to obtain straight generalizations as follows.

(*a*) (cf. [C2], VIII) Algebras on a general non-compact Riemann space Ω with a finite atlas $\{\Omega_1, \ldots, \Omega_N\}$ subordinate partition of unity $1 = \sum \omega_j$,

$\operatorname{supp} \omega_j \subset \Omega_j$, such that for each j we have either $\operatorname{supp} \omega_j$ compact and Ω_j charted on $\mathbb{R}^n$ or Ω_j charted on a polylinder Ω_j of the form (0.2) and ω_j contained in the corresponding polycylinder algebra $\mathcal{C}_j$ discussed below (with metric of Ω and Ω^j coinciding in $\operatorname{supp} \omega_j$).

(*b*) (cf. [C2], X, 3) Operators acting on crossections of vector bundles on Ω as described under (*a*), where the vector bundles have to be suitably restricted at infinity.

(*c*) (cf. [C2], IX) Subalgebras of L^2-Sobolev spaces $\mathcal{H}_s$ over Ω, using $\Lambda^s: \mathcal{H} \to \mathcal{H}_s$ as an isometry, where $\Lambda = (1 - \Delta)^{-1/2}$, with the Laplace operator Δ of the metric (0.1).

(*d*) (cf. [C2], X, 6) More general L^2-Sobolev spaces, such as those using an operator $\phi\Lambda^s: \mathcal{H} \to \mathcal{H}_{s,\delta}$ as an isometry, with a function $\phi \in C^\infty(\Omega)$ such as the spaces $W^p_{s,\delta}$ of [LM1], in case of $p = 2$, (but general real s).

As in the case of the space with boundary $\mathbb{R}^{n+1}_+$ we obtain a 2-link ideal chain $\mathcal{C} \supset \mathcal{E} \supset \mathcal{K}(\mathcal{H})$ for our algebra $\mathcal{C}$. Again the two quotients $\mathcal{C}/\mathcal{E}$ and $\mathcal{E}/\mathcal{K}(\mathcal{H})$ are function algebras. However the result (thm. 3.2) appears to be more perfect, for the following reason: A major difficulty in applying the Fredholm inversion is the matter of checking (quasi-) invertibility of the $\mathcal{E}$-symbol. While the symbol σ_A of $A \in \mathcal{C}$ often is directly given as an explicit function over $\mathbb{M}$, the inversion of $(1 + E)$ can only be attempted after an $\mathcal{E}$-inverse B is obtained and E (or γ_E) is obtained explicitly.

This difficulty is avoided by obtaining an extension of the homomorphism γ from $\mathcal{E}$ to $\mathcal{C}$ again. Then the two symbols σ_A and γ_A of an operator A can be directly obtained. (σ_A coincides with the ordinary symbol of a singular integral operator, but γ_A is defined over $\mathbb{E}$, a space of infinite points, (i.e. points over infinity of a certain compactification of Ω). A necessary and sufficient condition for $A \in \mathcal{C}$ to be Fredholm is that $\sigma_A \neq 0$ and that γ_A be invertible and the inverse bounded on $\mathbb{E}$.

The result is easily applied to certain realizations of a given differential expression over Ω, for certain expressions L. One finds that $A = L(1 - \Delta)^{-N/2}$, for L of order N with suitable coefficients in an operator in $\mathcal{C}$. Thus $Z = A(1 - \Delta)^{N/2}$ defines an unbounded operator and a realization of L on $\mathcal{H}$ which is Fredholm if and only if $A \in \mathcal{C}$ is Fredholm. This clearly makes thm. 3.2 applicable to study the Fredholm property of the realization A. Also the symbols σ_A and γ_A are easily obtained explicitly, using the t-Fourier transform and the symbol of $\mathcal{L}$ as a differential operator. We get $\sigma_A \neq 0$ if and only if L is uniformly elliptic.

For a uniformly elliptic differential expression L of order N the γ-symbol of A defines a family $\{L_{t_0}(\tau): t_0 \in \partial\Omega, \tau \in \mathbb{R}^{n'}\}$ of N-th order elliptic differential expressions depending continuously on t_0 (over an infinite boundary $\partial\Omega$) but analytically on $\tau \in \mathbb{R}^{n'}$. (In fact, the $L_{t_0}(\tau)$ are polynomials in τ.)

A very simple device of Agmon and Nirenberg [AN1] (together with the (interior) Sobolev estimate on compact manifolds) may be employed to show that γ_A is invertible for $\tau \in \mathbb{R}^{n'}$ with large $|\tau|$. This (and a result by Gramsch [Gr1]) implies that the family $L_{t_0}(\tau + i\delta)$, i.e. γ_A, is invertible for all real τ whenever the real $\delta \in \mathbb{R}^{n'}$ avoids a certain countable set $\mathcal{Z}_{t_0}$ without finite cluster points. For $n' = 1$ this coincides with the result of Lockhart-McOwen regarding the operator $e^{\delta t} L e^{-\delta \tau}$ (i.e. of L relative to the weighted L^2 norm with weight $e^{\delta t}$). For $n' > 1$ the exceptional set depends on the infinite point t_0. Thus the set $\mathcal{Z}_{t_0}$ of δ to be avoided is more complicated, unless one assumes L_{t_0} independent of t_0 at each end of Ω.

The results on differential operators are discussed in more detail in [C2], IX, X.

Finally we want to point to a variety of results by Bruening and Seeley [BS], Melrose-Mendoza [MM1], Choquet-Bruhat and Christodolou [CBC], all related in general aim, but different in method and approach. In particular it is clear that results on the Fredholm index (i.e., an index formula) are implied, as we also shortly indicate in sec. 3.

1. A C*-Algebra on a Poly-Cylinder

First we look at a Laplace comparison algebra with noncompact commutator on a poly-cylinder $\Omega = \mathbb{R}^{n'} \times B$. Here B denotes a compact Riemannian space of dimension n'' with metric $d\rho^2 = g_{jk}\, dx^j dx^k$. Accordingly, for the metric and the Laplace operator of Ω we get

$$(1.1) \qquad ds^2 = dt^2 + d\rho^2, \qquad \Delta = \Delta_t + \Delta_x, \qquad \Delta_x = (\sqrt{g})^{-1} \partial_{x^j} \sqrt{g} g^{jk} \partial_{x^k},$$

where Δ_x is the Laplace operator on B. In (1.1) we are using the Euclidean metric $dt^2 = dt^{1^2} + \cdots + dt^{n'^2}$ of $\mathbb{R}^{n'} = \{t = (t^1, \ldots, t^{n'}) : t^j \in \mathbb{R}\}$, and the Euclidean Laplace operator $\Delta_t = \sum \partial_{tj}^2$. The summation convention often will be used from 1 to n'', as will be clear from the context. We set $n = n' + n''$, so that Ω is n-dimensional.

Let $\mathcal{H}$ be the Hilbert space $L^2(\Omega) = L^2(\Omega, dS)$, with the surface measure $dS = dS' dS'' = \sqrt{g}\, dt\, dx$ of the metric (1.1). Let $\mathcal{C} \subset \mathcal{L}(\mathcal{H})$ be the smallest C^*-subalgebra containing the (5 types of) operators

$$(1.2) \quad a \in \mathfrak{A}_B^{\#}, \quad s_j(t) = t_j / \langle t \rangle, \quad \Lambda = (1 - \Delta)^{-1/2}, \quad \partial_{tj}\Lambda, \quad D_x\Lambda, \quad D_x \in \mathfrak{D}_B^{\#}, \quad j = 1, \ldots, n'.$$

Here $\langle t \rangle = (1 + t^2)^{1/2}$; we write $\mathfrak{A}_B^{\#} = C^\infty(B)$ while $\mathfrak{D}_B^{\#}$ denotes the collection of all C^∞-vector fields on B. Also Λ is the unique positive inverse square root of the (unique) self-adjoint realization $(1 - \Delta)$ of the Laplace differential

expression Δ. There is a unique such self-adjoint realization because Ω is a complete Riemannian space (cf. [Ga1], [CWS], [Ch1]). The functions $a \in \mathfrak{A}_B^{\#}$ and $s_j(t)$ in (1.2) represent the corresponding multiplication operators on functions $u(t, x) \in \mathcal{H}$. Correspondingly we denote by $\mathfrak{A}^{\#}$ and $\mathfrak{D}^{\#}$ the algebra finitely generated by C_0^∞ and the first two kinds of generators, and the linear space spanned (mod $\mathfrak{A}^{\#}$) by $\{1, \partial_{tj}, \mathfrak{D}_x\}$, respectively. The operations $\partial_{tj}\Lambda u$ and $D_x\Lambda u$ are well defined for $u \in \Lambda^{-1}C_0^\infty(\Omega)$, a dense subspace of $\mathcal{H}$ (cf. [C2], V) and have continuous extensions to $\mathcal{H}$, as easily seen (cf. also (1.6), below).

We notice that $\mathcal{H} = \mathcal{h}_t \hat{\otimes} \mathcal{h}_x$ is the topological tensor product of the Hilbert spaces $\mathcal{h}_t = L^2(\mathbb{R}^{n'})$, and $\mathcal{h}_x = L^2(B)$ (cf. [C1], V, 8). Let us write $\mathcal{L}_x = \mathcal{L}(\mathcal{h}_x)$, $\mathcal{K}_x = \mathcal{K}(\mathcal{h}_x)$, $\mathcal{L}_t = \mathcal{L}(\mathcal{h}_t)$, $\mathcal{K}_t = \mathcal{K}(\mathcal{h}_t)$. It may be observed that the topological tensor products $\mathcal{L}_t \hat{\otimes} \mathcal{L}_x = \mathcal{L}_{tx}$, $\mathcal{h}_t \hat{\otimes} \mathcal{h}_x = \mathcal{K}(\mathcal{H})$, $\mathcal{L}_t \hat{\otimes} \mathcal{h}_x = \mathcal{K}_x$, $\mathcal{h}_t \hat{\otimes} \mathcal{L}_x = \mathcal{K}_t$ are all well defined C^*-subalgebras of $\mathcal{L}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H}) \subset \subset \mathcal{K}_x$, $\mathcal{K}_t \subset \mathcal{L}_{xt} \subset \mathcal{L}(\mathcal{H})$ all are proper inclussions. In fact, $\mathcal{K}_x$ and $\mathcal{K}_t$ are proper closed two-sided ideals of $\mathcal{L}_{tx}$, and, of course, $\mathcal{K}(\mathcal{H})$ is a proper closed ideal of all the others. Evidently $\mathcal{L}_{tx}$ (but not $\mathcal{K}_x$ or $\mathcal{K}_t$) contains the identity operator I.

Note that we may write $\mathcal{H} = L^2(\mathbb{R}^{n'}, \mathcal{h}_x)$ as the space of all functions over $\mathbb{R}^{n'}$ with values in $\mathcal{h}_x = L^2(B)$ such that

$$\|u\|^2 = \int_{R^{n'}} dt \, \|u(t, \cdot)\|^2 < \infty, \tag{1.3}$$

by Fubini's theorem. Correspondingly, if $CB(\mathbb{R}^{n'}, \mathcal{h}_x)$ and $CB(\mathbb{R}^{n'}, \mathcal{L}_x)$ denote the classes of bounded norm continuous functions over $\mathbb{R}^{n'}$, taking values in $\mathcal{h}_x$ and $\mathcal{L}_x$, respectively, then a function $\phi \in CB(\mathbb{R}^{n'}, \mathcal{L}_x)$ has a natural interpretation as an operator in $\mathcal{L}(\mathcal{H})$, defined by $u(t, \cdot) \to \phi(t)u(t)$. Moreover, this operator is in $\mathcal{K}_x$ whenever $\phi \in CO(\mathbb{R}, \mathcal{h}_x)$ (cf. [C1], V, 8). (We indicate a proof of this fact in cor. 1.4.) This establishes an isometric *-isomorphism of $CB(\mathbb{R}; L_x)$ (as a Banach algebra with norm

$$\sup \{\|A(\tau)\| : \tau \in \mathbb{R}\}, \tag{1.4}$$

where $\|A(\tau)\|$ is the norm in $\mathcal{L}_x$) into $\mathcal{L}(\mathcal{H})$. In the following we will use this interpretation of functions as operators in $\mathcal{L}(\mathcal{H})$.

In order to find the commutator ideal $\mathcal{E}$ of the unital C^*-algebra $\mathcal{C}$ with generators (1.2) we conjugate the generators with the Fourier transform in the t-direction. In detail we have

$$F_t u(\tau) = (2\pi)^{-n'/2} \int_{\mathbb{R}^{n'}} e^{-it\tau} u(t) \, dt, \tag{1.5}$$

which defines a unitary operator of $\mathcal{h}_t$. By conjugation with F_t we, of course, mean conjugation with $F_t \otimes I_x$, where I_x denotes the identity operator. (However, we will write this as $F_t^{-1} A F_t$, for $A \in \mathcal{L}(\mathcal{H})$.) First consider the

F_t-conjugations of Λ, $D_{t^j}\Lambda$, $D_x\Lambda$:

$$\Lambda^{\sim}(\tau) = (\langle\tau\rangle^2 - \Delta_x)^{-1/2}, \qquad \tau_j\Lambda^{\sim}(\tau), \qquad D_x\Lambda^{\sim}(\tau), \tag{1.6}$$

with $D_{t^j} = -i\partial_{t^j}$ and above interpretation of a function as operator.

The manifold B is compact, therefore $\Lambda^{\sim}(\tau)$ of (1.5), for fixed τ, as operator in $\mathfrak{L}(\mathfrak{k}_x)$, is compact (cf. [C2], III, cor. 3.9.). Moreover, this defines an operator function $\Lambda^{\sim}(\tau) \in CO(\mathbb{R}, \mathfrak{k}_x)$, which even is analytic, in norm topology of $\mathfrak{L}(\mathfrak{k}_x)$, and we have $\|\Lambda^{\sim}(\tau)\| \leqslant \langle\tau\rangle^{-1}$ as well as $\|\Lambda^{\sim}(0)^{-1}\Lambda^{\sim}(\tau)\| \leqslant 1$, so that $\tau_j\Lambda^{\sim}(\tau)$, $D_x\Lambda^{\sim}(\tau) = (D_x\Lambda_x)(\Lambda_x^{-1}\Lambda^{\sim}(\tau))$ are functions in $CB(\mathbb{R}^{n'}, \mathfrak{L}_x) \subset \mathfrak{L}(\mathfrak{K})$, confirming the boundedness of the last two types (1.2). Also, writing $\Lambda_x = \Lambda^{\sim}(0)$, $T(\tau) = (1 + \tau^2\Lambda_x^2)^{-1/2}$, the generators (1.2) correspond to

$$a(x), \qquad \mu_j *, \qquad \Lambda_x T(\tau), \qquad (i\tau_j\Lambda_x)T(\tau), \qquad (D_x\Lambda_x)T(\tau), \tag{1.7}$$

with $\mu_j = F_t^{-1}s_j$ and μ_t^* = convolution in $\mathfrak{k}_t$.

Proposition 1.1. *The operator functions $\Lambda_x^{\epsilon}T(\tau)$ and $|\tau|^{1-\epsilon}\Lambda_x T(\tau)$, for each fixed ϵ, $0 \leqslant \epsilon < 1$, are in $CO(\mathbb{R}, \mathfrak{k}_x)$.*

PROOF. Just note that $\Lambda^{\sim} = \Lambda_x T(\tau) \in CO(\mathbb{R}, \mathfrak{k}_x)$, while $T(\tau)$ and $|\tau|\Lambda_x T(\tau)$ belong to CB. All operators are positive self-adjoint, so that $\Lambda_x^{\epsilon}T(\tau) = \Lambda^{\epsilon}T(\tau)^{1-\epsilon} \in CO$, $|\tau|^{1-\epsilon}\Lambda_x T(\tau) = \Lambda^{\epsilon}(|\tau|\Lambda_x T(\tau))^{1-\epsilon} \in CO$, as products of a bounded function and a function with limit 0, q.e.d.

Now we first will describe the algebra generated by the commutators of the F_t-conjugated generators (1.7).

Proposition 1.2. *All commutators of the F_t-conjugated generators and their adjoints are contained in the algebra*

$$\mathcal{G}^{\wedge} = CO(\mathbb{R}^{n'}, \mathcal{K}(\mathfrak{k}_x)) + \mathcal{K}(\mathfrak{K}) \subset \mathcal{K}_x. \tag{1.8}$$

Here a function $C(\tau) \in CO(\mathbb{R}^{n'}, \mathcal{K}(\mathfrak{k}_x))$ must be interpreted as an operator in $\mathcal{K}_x$ in the manner described above.

PROOF. We will use the resolvent integral technique used in [C2], V. Let the operators (1.7) be denoted by $G_1, \ldots, G_5$, in the order listed (we write $t = t^j$, $\partial_t = \partial_{t^j}$, $\tau = \tau_j$, etc.). Clearly $[G_1, G_2] = [G_3, G_4] = 0$.

We get $(D_x\Lambda^{\sim})^* - D_x^*\Lambda^{\sim} = [\Lambda^{\sim}, D_x^*] \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$, by (1.14) below, and adjoint invariance of $\mathfrak{D}_B^{\#}$. All other generator (classe)s are self-adjoint. Hence the adjoint generators need no special attention.

For the commutators $[G_j, G_l]$, $j \neq 2$, we use the well known resolvent integral representation of $\Lambda^{\sim} = G_3$: Let $R(s) = (s + \langle\tau\rangle^2 - \Delta_x)^{-1}$. Then we have

$$G_3 = \Lambda^{\sim}(\tau) = \Lambda_x T(\tau) = (\langle\tau\rangle^2 - \Delta_x)^{-1/2} = 1/\pi \int_0^{\infty} R(s)\, ds/\sqrt{s}, \tag{1.9}$$

(cf. [C2], IX, (5.13)), with a norm convergent improper Riemann integral, in the algebra $\mathfrak{L}(\mathfrak{k}_x)$. We get

$$[G_3, G_5] = [\Lambda^{\sim}, D_x]\Lambda^{\sim}, \qquad [G_4, G_5] = [\Lambda^{\sim}, D_x](\tau\Lambda^{\sim}), \tag{1.10}$$

where $\Lambda^{\sim}, \tau\Lambda^{\sim} \in CB$, so that it suffices to show that $[\Lambda^{\sim}, D_x] \in CO(\mathbb{R}, \mathfrak{k}_x)$.

Similarly,

$$\begin{aligned} &[G_1, G_3] = [a_x, \Lambda^{\sim}], \qquad [G_1, G_4] = (\tau\Lambda^{\sim})(\Lambda^{\sim\,-1}[a_x, \Lambda^{\sim}]), \\ &[G_1, G_5] = p_x\Lambda^{\sim} + (D_x\Lambda^{\sim})(\Lambda^{\sim\,-1}[a_x, \Lambda^{\sim}]), \end{aligned} \tag{1.11}$$

with $p_x = [a_x, D_x] \in C^{\infty}(B)$. Accordingly we also must show that $\Lambda^{\sim\,-1}[a_x, \Lambda^{\sim}] \in \in CO(\mathbb{R}, \mathfrak{k}_x)$. Both these facts are consequences of prop. 1.3, below. Before we discuss it we turn to the commutators $[G_2, G_1]$. There we find it practical to work without the Fourier transform (1.5), writing

$$\Lambda = 1/\pi \int_0^{\infty} S(r)\, dr/\sqrt{r}, \qquad S(r) = (r + 1 - \Delta_t - \Delta_x)^{-1}. \tag{1.12}$$

Instead of «diagonalizing the t-variable by using the Fourier transform» we will consider the x-variable diagonalized later on. Let $A_j = F_t G_j F_t^{-1}$ be the generators (1.2). Then we have

$$\begin{aligned} &[A_2, A_4] = p_t\Lambda + (\partial_t\Lambda)V, \qquad [A_2, A_5] = (D_x\Lambda)V, \\ &V = \Lambda^{-1}[A_2, A_3], \qquad p_t \in C^{\infty}. \end{aligned} \tag{1.13}$$

Note that $p_t \in CO(\mathbb{R}^{n'})$ hence $p_t\Lambda \in \mathfrak{K}(\mathfrak{H})$, by a well known result (cf. [C2], III, thm. 3.7). We claim that all 3 commutators $[A_2, A_l]$, $l = 3, 4, 5$, are in $\mathfrak{K}(\mathfrak{H})$. Again this is a trivial consequence of prop. 1.3 below, so that all of prop. 1.2 has been reduced to prop. 1.3.

Proposition 1.3. *For ϵ with $0 \leqslant \epsilon < 1$ we have*

$$\Lambda^{\sim\,-1-\epsilon}[a_x, \Lambda^{\sim}] \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x), \qquad \Lambda^{\sim\,-\epsilon}[D_x, \Lambda^{\sim}] \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x), \tag{1.14}$$

$$\Lambda^{-1-\epsilon}[s_j(t), \Lambda] \in \mathfrak{K}(\mathfrak{H}), \qquad s_j(t) = t_j/\langle t\rangle \tag{1.15}$$

Remark. As a consequence of prop. 1.3 the algebra $\mathfrak{C}$ satisfies condition (m_4) of [C2], VIII, 3, as required later on (cf. thm. 2.3).

For the proof of prop. 1.3 we use (1.9) for

$$\Lambda^{\sim -1-\epsilon}[G_1, G_3] = 1/\pi \int_0^\infty \Lambda^{\sim -1-\epsilon} R(s) L_1 R(s)\, ds/\sqrt{s}, \tag{1.16}$$

$L_1 = [a(x), \Delta_x]$. The integrand in (1.16) is a norm-continuous (even analytic) function $F(s, \tau)$, with values in k_x. Indeed, one may write

$$F(s, \tau) = (\Lambda^{\sim -1-\epsilon} R(s))(L_1 \Lambda_x)(\Lambda_x^{-1} R(s)/\sqrt{s}), \tag{1.17}$$

and use the estimates

$$\|\Lambda_x^{-\eta} R(\sigma)\| \leqslant (1 + s + \tau^2)^{\eta/2-1}, \qquad \|\Lambda^{\sim -\eta} R(s)\| \leqslant (1+s)^{\eta/2-1}, \tag{1.18}$$

$0 \leqslant \eta < 2$, easily derived from the spectral decomposition of the self-adjoint operator $-\Delta_x \geqslant 0$ of $\mathfrak{k}_x$. Note that L_1 is a folpde on B independent of s, τ, so that the second factor in (1.17) is a constant in $\mathfrak{L}(\mathfrak{k}_x)$. Analyticity of the first and third factor is a consequence of analyticity of the resolvent $R(s)$. These factors are $O((1+s)^{(\epsilon-1)/2})$ and $O(s^{-1/2}(1+s)^{-1/2})$, respectively. Thus $F(s, \tau) = O((1+s)^{-1+\epsilon/2}/\sqrt{s})$ uniformly for all $\tau \in \mathbb{R}^{n'}$. Also both factors are in $\mathfrak{k}_x$ since B is compact, insuring the compactness of the resolvent $R(s)$ of the Laplace operator Δ_x ([C2], III, thm. 3.1). This implies existence of the improper Riemann integral (1.16) in norm convergence of $\mathfrak{L}(\mathfrak{k}_x)$ and uniformly so, for $\tau \in \mathbb{R}^{n'}$. Thus the integral is in $CB(\mathbb{R}^{n'}, \mathfrak{k}_x)$. (For more detail in such a proof cf. [C2], V, 3). Moreover, since $0 < \epsilon < 1$ is arbitrary, one may use this for $\epsilon + \delta < 1$, with some $\delta > 0$. This gives a factor $\Lambda^{\sim \delta} \in CO$, whence the first (1.14) is CO, not only CB.

Similarly we may use (1.9) for

$$\Lambda^{\sim -\epsilon}[D_x, \Lambda^{\sim}] = 1/\pi \int_0^\infty (\Lambda^{\sim -\epsilon} \Lambda_x^{-1} R(s))(\Lambda_x [D_x, \Delta_x] \Lambda_x)(\Lambda_x^{-1} R(s))/\sqrt{s}, \tag{1.19}$$

where the integral exists for analogous reason. (Now the second factor contains the second order operator $[D_x, \Delta_x]$ over B so that again the factor in a constant in $\mathfrak{L}(\mathfrak{k}_x)$.)

For (1.15) we use the other resolvent integral (1.12), for

$$\Lambda^{-1-\epsilon}[s(t), \Lambda] = 1/\pi \int_0^\infty \Lambda^{-1-\epsilon} S(r) M_1 S(r)\, dr/\sqrt{r}, \qquad M_1 = [s(t), \Delta_t]. \tag{1.20}$$

Now, with respect to an orthonormal base of the Hilbert space $\mathfrak{k}_x$ consisting of eigenfunctions of the self-adjoint operator Δ_x, the operator $S(r)$ is diagonalized, with respect to $\mathfrak{k}_x$, in the tensor product decomposition of $\mathfrak{K}$. Its diagonal components are

$$S_j(r) = (r + \langle \lambda_j \rangle^2 - \Delta_t)^{-1}, \tag{1.21}$$

with the eigenvalues λ_j^2 of the positive self-adjoint operator Λ_x on B. In analogy to (1.18) we conclude that, with $\Lambda_t = (1 - \Delta_t)^{-1/2}$,

$$(1.22) \qquad \|\Lambda_t^{-\eta} S_j(r)\| \leqslant (1 + r + \lambda_j^2)^{\eta/2-1}, \qquad \|\Lambda_t^{j-\eta} S_j(r)\| \leqslant (1 + r)^{\eta/2-1},$$

for $j = 1, 2, \ldots$, and $\Lambda_t = (1 - \Delta_t)^{-1/2}$, $\Lambda_t^j = (1 + \Lambda_j^2 + \Delta_t)^{-1/2}$.

Notice that the entire relation (1.20) is «x-diagonalized», i.e., decomposes into a set of countably many relations, involving Λ_t^j and S_j instead of Λ, $j = 1, 2, \ldots$. Again one may write the integrand $F_j(r)$ as a product of three factors:

$$(1.23) \qquad F_j(r) = (\Lambda_t^{j-1-\epsilon} S_j(r))(M_1 \Lambda_t)(\Lambda_t^{-1} S_j(r)/\sqrt{r}),$$

where the second term is a constant in $\mathfrak{L}(\mathcal{H})$. For the first and third term we get estimates $O((1 + r)^{(\epsilon-1)/2})$ and $O((1 + r + \lambda_j^2)^{-1/2}/\sqrt{r}) = O(\langle\lambda_j\rangle^{-\delta}(1 + r)^{(\delta-1)/2} r^{-1/2})$, for any $0 < \delta < 1$.

Thus

$$(1.24) \qquad \|F_j(s)\| = O(\langle\lambda_j\rangle^{-\delta} r^{-1/2}(1 + r)^{(\epsilon+\delta)/2-1}).$$

The right hand side is integrable, as long as $\epsilon + \delta < 1$. The integral $\int F_j(r)\, dr = C_j$ is an operator in $\mathcal{K}(\mathfrak{k}_t)$, and we get

$$(1.25) \qquad \|C_j\| = O(\langle\lambda_j\rangle^{-\delta}).$$

Accordingly the operator (1.20) corresponds to a diagonal matrix

$$((C_j \delta_{kl}))_{k,l=1,2,\ldots}$$

with diagonal components converging to 0 in norm. This indeed implies that (1.20) belongs to $\mathcal{K}(\mathcal{H})$. (It is limit in $\mathfrak{L}(\mathcal{H})$ of the sequence of diagonal matrices T_k obtained by setting all C_j, $j > k$, equal to zero, while the matrices T_k are in $\mathcal{K}(\mathfrak{k}_t) \otimes \mathcal{F}(\mathfrak{k}_x) \subset K(\mathfrak{k}_t) \otimes \mathcal{K}(\mathfrak{k}_x) \subset \mathcal{K}(\mathcal{H})$, with the class $\mathcal{F}(\mathfrak{k}_x)$ of bounded operators of finite rank over $\mathfrak{k}_x$.) This completes the proof of proposition 1.3.

Corollary 1.4. *The C*-algebra $\mathcal{G}^\wedge$ and its F_t^{-1}-conjugate $\mathcal{G}$ are subalgebras of $\mathcal{K}_x = \mathfrak{L}_t \hat{\otimes} \mathfrak{k}_x$.*

PROOF. It is sufficient to show that $CO(\mathbb{R}^{n'}, \mathfrak{k}_x) \subset \mathcal{K}_x$. For any orthonormal basis $\{\phi_j : j = 1, 2, \ldots\}$, and the orthogonal projection P_n onto the span of $\{\phi_1, \ldots, \phi_n\}$ we get uniform convergence $P_n C(\tau) P_n \to C(\tau)$, $\tau \in \mathbb{R}$, for every $C(\tau) \in CO(\mathbb{R}^{n'}, \mathcal{K}(\mathfrak{k}_x))$. But the operator $P_n C(\tau) P_n$ is a finite sum of operators $(\phi_j\rangle\langle\phi_l) c_{jl}(\tau)$, $c_{jl}(\tau) \in CO(\mathbb{R})$. Thus $P_n C(\tau) P_n \in \mathfrak{k}_x$, and the limit $C(\tau)$ as well, q.e.d.

Proposition 1.5. *The commutator ideal* $\mathcal{GO}$ *of the* C^**-subalgebra of* $\mathfrak{L}(\mathfrak{IC})$ *generated by the operators of the form* G_1, G_3, G_5 *contains the algebra* $CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$.

PROOF. For a moment consider only the C^*-algebra $\mathfrak{B}$ generated by G_1, G_3, and G_5. By virtue of the isometric isomorphism mentionned initially in this section, the generators of $\mathfrak{B}$ belong to the function algebra $CO(\mathbb{R}^{n'}, \mathfrak{L}_x)$. Hence the algebra $\mathfrak{B}$ may be interpreted as a subalgebra of $CO(\mathbb{R}^{n'}, \mathfrak{L}_x)$. For a fixed $\tau = \tau_0 \in \mathbb{R}^{n'}$ the values $\mathfrak{B}_{\tau_0} = \{A(\tau_0): A(\tau) \in \mathfrak{B}\}$ form a *-subalgebra of $\mathfrak{L}_x$. It is clear that $\mathfrak{B}_{\tau_0}$ is a *-subalgebra of the C^*-subalgebra $\mathfrak{C}_{\tau_0}$ of $\mathfrak{L}_x$ generated by the values of the functions G_j, $j = 1, 3, 5$ at τ_0, i.e., by the operators

$$a_x, \qquad \Lambda^{\sim}(\tau_0) = (-\Delta_x + \langle \tau_0 \rangle^2)^{-1/2}, \qquad D_x \Lambda^{\sim}(\tau_0), \tag{1.26}$$

where a_x and D_x run through all the functions (folpdes) over B. Moreover, $\mathfrak{C}_{\tau_0}$ evidently is the closure of $\mathfrak{B}_{\tau_0}$. Also $\mathfrak{C}_{\tau_0}$ is just the minimal comparison algebra, in the sense of [C2], V, 1 generated by the triple $\{B, -\Delta_x + \langle \tau_0 \rangle^2, ds\}$, on the compact manifold B. By [C2], V, lemma 1.1, it follows that $\mathfrak{C}_{\tau_0}$ and even its commutator ideal contain all of $\mathfrak{k}_x$. But commutators in $\mathfrak{C}_{\tau_0}$ are compact, since B is compact. Therefore the commutator ideal of $\mathfrak{C}_{\tau_0}$ equals $\mathfrak{k}_x$. Since that commutator ideal is the closure of the commutator ideal $\mathcal{E}^0_{\tau_0}$ of the finitely generated algebra, we must have $\mathcal{E}^0_{\tau_0}$ dense in $\mathfrak{k}_x$. On the other hand $\mathcal{E}^0_{\tau_0}$ clearly is contained in the commutator ideal of the algebra $\mathfrak{B}_{\tau_0}$, and even in the localization $\mathcal{GO}_{\tau_0}$ at τ_0 of the commutator ideal $\mathcal{GO}$. Thus we conclude that the algebra $\mathcal{GO}_\tau$ of «values» of $\mathcal{GO}$ at τ is dense in $\mathfrak{k}_x$, for all $\tau \in \mathbb{R}^{n'}$.

We also find that the algebra $\mathfrak{B}$ contains $\Lambda^{\sim}(\tau)$, hence also contains every $f(\tau, \Lambda_x)$, for a general $f \in CO(\mathbb{R}^{n'} \times [0, 1])$, by the spectral theorem and the Stone-Weierstrass theorem. Hence $\mathfrak{B}$ contains all functions $\psi(\tau)E_\lambda$, with $\psi \in CO(\mathbb{R}^{n'})$ and the projection operators E_λ of the spectral family of Λ_x^{-1}. Note that E_λ are of finite rank, and that $E^N \to 1$, strongly, as $N \to \infty$. Since $\mathcal{GO}$ is an ideal of $\mathfrak{B}$, it follows that $\mathcal{GO}$ contains

$$\mathcal{GO}_N = \{\psi(\tau)E_N A(\tau)E_N: \ A(\tau) \in \mathcal{GO}, \psi(\tau) \in CO(\mathbb{R}^n)\}. \tag{1.27}$$

But $\mathcal{GO}_N$ is a self-adjoint algebra of (finite) $j_N \times j_N$-matrix-valued functions, separating points in the following sense. For every $\tau_1, \tau_2 \in \mathbb{R}^{n'}$, $\epsilon > 0$, and $j_N \times j_N$-matrix P there exists $K(\tau) \in \mathcal{GO}_N$ such that $K(\tau_2) = 0$ and $|K(\tau_1) - P| < \epsilon$. (This follows from the above.) By the matrix version of the Stone Weierstrass theorem this implies that $\mathcal{GO}_N = CO(\mathbb{R}^{n'}, \mathfrak{L}(\mathbb{C}^{j_N}))$. Since this holds for all N we find that $\mathcal{GO}$ contains all these matrix algebras. But for a general $A(\tau) \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$ we get $A_N(\tau) = E_N A(\tau) E_N \in CO(\mathbb{R}^{n'}, \mathfrak{L}(\mathbb{C}^{j_N}))$. Also $A_N(\tau) - A(\tau) \to 0$, in $CO(\mathbb{R}^{n'}, \mathfrak{k}_x)$ (with the norm (1.4)), since $A(\tau)$ is

compact and E_N converges strongly to 1. since $\mathcal{GO}$ is closed we conclude that $A(\tau) \in \mathcal{GO}$, so that indeed $CO(\mathbb{R}, \mathcal{K}_x) \subset \mathcal{GO}$, q.e.d.

2. Commutator Ideal and Symbol Spaces of the Cylinder Algebra

Returning to our task of describing the commutator ideal $\mathcal{E}$ of the cylinder algebra $\mathcal{C}$ we conclude from prop. 1.5 and [C2], V, lemma 1.1 that $\mathcal{E}$ contains the C^*-algebras $\mathcal{GO}^\vee = F_t \mathcal{GO} F_t^{-1}$ and $\mathcal{K}(\mathcal{H})$. It is convenient again to work with the F_t-conjugated ideal $\mathcal{E}^\wedge = F_t^{-1} \mathcal{E} F_t$, containing the sum $\mathcal{GO} + \mathcal{K}(\mathcal{H}) = \mathcal{SO}$ (which in turn contains all the commutators of the generators $G_j, j = 1, \ldots, 5$).

Notice that $\mathcal{SO}$ is invariant under left and right multiplication with the (functions in $CB(\mathbb{R}, \mathcal{K}_x)$) G_1, G_3, G_4, G_5, but not under multiplication with G_2. Accordingly $\mathcal{E}^\wedge$ must be properly larger than $\mathcal{SO}$.

Specifically $G_2 = s_j(D_t)$ is a singular convolution operator with Cauchy-type singular integral and kernel $\mu_j = s_j^\vee$, so that a product $K(\tau)G_2$, for $K(\tau) \in CO(\mathbb{R}^{n'}, \mathcal{K}_x)$, appears as an infinite matrix of singular integral operators on $\mathbb{R}^{n'}$, if we introduce some orthonormal base of $\mathcal{K}_x$.

Theorem 2.1. *Let $\mathcal{SQ}$ be the C^*-algebra of singular integral operators over $\mathcal{K}_t$ generated by the multiplications in $CO(\mathbb{R}^{n'})$, and the operators $a(M)s_j(D_t)$, $a \in C0(\mathbb{R}^{n'})$, $j = 1, \ldots, n'$. Then the ideal $\mathcal{E}^\wedge$ coincides with the topological tensor product $\mathcal{SQ}_\infty = \mathcal{SQ} \hat{\otimes} \mathcal{K}_x$.*

PROOF. Notice that $\mathcal{SQ}$ coincides with the minimal comparison algebra of the triple $\{\mathbb{R}^{n'}, dt, 1 - \Delta_t\}$ (cf. [C2], VI). Thus it contains the compact ideal $\mathcal{K}_t$ of $\mathcal{L}(\mathcal{K}_t)$, and $\mathcal{SQ}_\infty$ contains $\mathcal{K}_t \hat{\otimes} \mathcal{K}_x = \mathcal{K}(\mathcal{H})$. Therefore it is trivial from the above that $\mathcal{SQ}_\infty$ contains all the commutators $[G_j, G_l], j, l = 1, \ldots, 5$ and that it is a closed *-ideal of $\mathcal{C}$. Hence we have $\mathcal{E}^\wedge \subset \mathcal{SQ}_\infty$. To show equality we introduce a fixed orthonormal base $\phi_1, \phi_2, \ldots$ of the space $\mathcal{K}_x$ and first consider $K(\tau) \in \mathcal{GO}$ such that $K(\tau)$ takes span $\{\phi_1, \ldots, \phi_N\}$ into itself and its orthogonal complement to 0, for all τ. Thus the infinite matrix vanishes outside its first N rows and columns. Let CO_N denote the subalgebra of $CO(\mathbb{R}^{n'}, \mathcal{K}_x)$ of all such finite matrices, for a given fixed N. Clearly CO_N is isometrically isomorphic to $CO(\mathbb{R}^{n'}, \mathcal{L}(\mathbb{C}^N))$.

Now we observe that $K(\tau)$ and $L(\tau)s(D_t)$, with $K, L \in CO_N$, belong to $\mathcal{E}^\wedge$, and generate the algebra $\mathcal{SQ}_N = \mathcal{SQ} \otimes \mathcal{L}(\mathbb{C}^N)$, for each $N = 1, 2, \ldots$. Also we again find that $\mathcal{SQ}_\infty$ is the norm closure of $\bigcup \mathcal{SQ}_N$. Therefore indeed $\mathcal{SQ}_\infty \subset \mathcal{E}^\wedge$, q.e.d.

We now come to our main task: The description of the symbol chain of the cylinder algebra $\mathcal{C}$. First let us look at the ideal quotient $\mathcal{E}/\mathcal{K}(\mathcal{H})$. In that

respect we observe that the algebra $\mathfrak{SQ}$ is a subalgebra of the algebra $\mathfrak{SI}$ of singular integral operators on $\mathbb{R}^{n'}$ generated by $S_j = s_j(D_t) = (\mu_j *)$, $j = 1, \ldots, n'$, and the multiplications with functions in $C(\mathbb{B}^{n'})$, with the «ball compactification» $\mathbb{B}^{n'}$ of $\mathbb{R}^{n'}$, having one infinite point in each direction $\infty \cdot t_0$, $|t_0| = 1$ (cf. [C1], IV, 1, problems). The special comparison algebra $\mathfrak{SI} \subset \mathfrak{L}(\mathfrak{k}_t)$ has symbol space

$$\mathbb{M}_I = \{(t, \tau) \in C(\mathbb{B}^{n'} \times \mathbb{B}^{n'}) : |t| + |\tau| = \infty\}, \tag{2.1}$$

(cf. ex. (A) of [C2], V, 4, or [CHe1]). The subalgebra $\mathfrak{SQ}$ of $\mathfrak{SI}$ consists precisely of all operators in $\mathfrak{SI}$ with symbol vanishing at $|\tau| = \infty$, as follows from the Stone-Weierstrass theorem, looking at the symbols of the generators. (Note that the generators are written as multiplications by functions of the variable τ and convolutions by functions of τ as well, since we consider the F_t-conjugated ideal $\mathcal{E}^\wedge$. Accordingly we have t and τ reversed, compared to the normal notation for space and momentum coordinates.) It follows that $\mathfrak{SQ}/\mathfrak{k}_t$ is isometrically isomorphic to the function algebra $CO(\mathbb{E})$ with the locally compact space compact space

$$\mathbb{E} = \mathbb{M}_Q = \{(t, \tau) \in \mathbb{M}_I : |t| = \infty, |\tau| < \infty\} = \partial\mathbb{B}^{n'} \times \mathbb{R}^{n'}. \tag{2.2}$$

In case of $n' = 1$ this space is a disjoint union of the two sets $\{\infty\} \times \mathbb{R} = \mathbb{E}^-$ and $\{-\infty\} \times \mathbb{R} = \mathbb{E}^+$. Both $\mathbb{E}^\pm$ are copies of $\mathbb{R}$, with the variable τ running over $\mathbb{R}$. In the general case $n' > 1$ the space $\mathbb{E}$ is connected, and is a product of the infinite sphere $\partial B^{n'} = \mathbb{B}^{n'} \setminus \mathbb{R}^{n'}$ with $\mathbb{R}^n$.

Clearly this also is just the wave front space $\mathbb{R}^{n'}$, but with t and τ interchanged. We have proven the following result:

Theorem 2.2. *The quotient algebra $\mathcal{E}/\mathcal{K}(\mathcal{H})$ is isometrically isomorphic to the function algebra $CO(\mathbb{E}, \mathfrak{k}_x)$, so that $\mathcal{E}$ is a C^*-algebra with (compact operator valued) symbol, with symbol space $\mathbb{E}$. In the special case $n' = 1$ we have*

$$CO(\mathbb{E}, \mathfrak{k}_x) = CO(\mathbb{E}^-, \mathfrak{k}_x) \oplus CO(\mathbb{E}^+, \mathfrak{k}_x). \tag{2.3}$$

The symbols of the generators of $\mathcal{E}$ are given as compact operator valued functions of (t, τ), for $t \in \partial\mathbb{B}^{n'}$ (i.e., $t = \infty \cdot t_0$, $t_0 \in \mathbb{R}^{n'}$, $|t_0| = 1$) and $\tau \in \mathbb{R}^{n'}$, as follows:

Let A_j, $j = 1, \ldots, 5$, be the generators (1.2) *of $\mathcal{C}$, in the order listed (so that G_j are their Fourier transforms). Then $[A_1, A_2] = 0 = [A_3, A_4]$. The symbols $\gamma_{[A_3, A_5]}$, $\gamma_{[A_4, A_5]}$ are independent of t, as $t \in \partial\mathbb{B}^{n'}$, the value given by the terms of* (1.10), *respectively, where $\Lambda^\sim(\tau) = \Lambda_x T(\tau)$, while the commutator $[\Lambda^\sim, D_x]$ is obtained from the resolvent integral from* (1.19). *Similarly $\gamma_{[A_1, A_j]}$, $j = 3, 4, 5$, are independent of t, as $|t| = \infty$, and, for $\tau \in \mathbb{R}^{n'}$, their values are given by* (1.11) *and the resolvent integral* (1.16).

Also, for $j \neq 2$, the symbol of a product $A_j[A_k, A_l]$ or $[A_k, A_l]A_j$ is obtained by multiplying $\gamma_{[A_k, A_l]}$ with the corresponding function G_j of (1.6), *from the left or right, respectively. Also, for $j = 2$, the symbols of these products equal the product of $\gamma_{[A_k, A_l]}(\tau)$ with the value of the function $s(t)$ (extended continuously to $\partial\mathbb{B}^{n'}$) at t. More generally, for every function $b \in C(\mathbb{B}^{n'})$ the operator $b(t)[A_k, A_l] \in \mathcal{E}$ has the symbol*

$$\gamma = b(t)\gamma_{[A_k, A_l]}(\tau), \qquad k, l \neq 2. \tag{2.4}$$

Next we turn to the discussion of the quotient $\mathcal{C}/\mathcal{E}$, i.e., of the symbol and symbol space of $\mathcal{C}$.

Theorem 2.3. *The C*-algebra $\mathcal{C}/\mathcal{E}$ is isometrically isomorphic to the algebra $C(\mathbb{M})$ of continuous complex-valued functions over a compact space $\mathbb{M}$, called the symbol space of $\mathcal{C}$. Here $\mathbb{M}$ is (homeomorphic to) the bundle of cospheres with infinite radius of the compactified poly-cylinder $\mathbb{B}^{n'} \times B$ considered as a compact C^∞-manifold with boundary (i.e., the product $B^{n'} \times B$ of the unit ball $B^{n'} = \{t \in \mathbb{R}^{n'} : |t| = 1\}$ in $\mathbb{R}^{n'}$ with the compact manifold B).*

Let $A_1, \ldots, A_5$ be the generators (1.2) *again. Then the $\mathcal{C}$-symbols $\sigma_{A_j} = \sigma_{A_j}(t, x, \tau, \xi)$, i.e., the functions in $C(\mathbb{M})$, associated to A_j by the above isomorphism, are given as explicit functions of (t, x, τ, ξ) as follows:*

$$\begin{gathered} \sigma_{A_1} = a(x), \qquad \sigma_{A_2} = t/\langle t \rangle, \qquad \sigma_{A_3} = 0. \\ \sigma_{A_4} = i\tau/(\tau^2 + |\xi|^2)^{1/2}, \qquad \sigma_{A_5} = (b^j\xi_j)/(\tau^2 + |\xi|^2)^{1/2}, \end{gathered} \tag{2.5}$$

with

$$|\xi| = (g^{jk}(x)\xi_j\xi_k)^{1/2}, \qquad Dx = b^j(x)\partial_{x^j} + p(x), \tag{2.6}$$

in local coordinates of B, where $t \in \mathbb{B}^{n'}$, $x \in B$, while $(\tau, \xi) \in S^\infty_{t,x}$, the cosphere at (t, x) with infinite radius. (Actually the last two symbols are the limits of the full symbol quotients

$$\tau/(1 + \tau^2 + |\xi|^2)^{1/2}, \qquad (b^j(x)\xi_j + p(x))/(1 + \tau^2 + |\xi|^2)^{1/2}, \tag{2.7}$$

as (τ, ξ) is replaced by $(\rho\tau, \rho\xi)$, and $\rho \to \infty$.)

PROOF. We may just apply the general results of [C2], VII, first verifying the assumptions. Let us shortly summarize these facts. First of all every symbol space of a comparison algebra contains the wave front space $\mathbb{W}$, normally identified with the bundle of unit spheres in the contangent space $T^*\Omega$ of the underlying manifold Ω (cf. [C2], VI, thm. 1.5). The space $\mathbb{W}$ is an open subset

of $\mathbb{M}$. It precisely coincides with the set of points (x, t, τ, ξ), with $|t| < \infty$. Essentially this follows whenever $\mathcal{C}$ can be shown to contain all C_0^∞-functions and all operators $D\Lambda$, for a general first order differential expression with C^∞-coefficients and compact support.

In order to study the points at $|t| = \infty$ we require the compactification $\mathbb{P}^*\Omega$ of $T^*\Omega$ induced by the formal symbol quotients (2.7) together with the functions $a(x)$, $a \in \mathcal{A}_B^\#$, and $t/\langle t \rangle$, and $(1 + \tau^2 + |\xi|^2)^{-1/2}$ (in other words, by the formal symbols of the 5 types of generators (1.2)). (That is, $\mathbb{P}^*\Omega$ is defined as the smallest compactification of $T^*\Omega$ onto which all above functions can be continuously extended.) It is readily verified that $\mathbb{P}^*\Omega$ is given as the compactification of $T^*\Omega$ obtained by adding the infinite sphere $\{\infty(\tau, \xi): |\tau|^2 + |\xi|^2 = 1\}$ to each fiber $T^*_{t,x}$ of the cotangent bundle $T^*\Omega^c$, of the compactification $\Omega^c = \mathbb{B}^n \times B$ of Ω. In other words, $\mathbb{P}^*\Omega$ is the disjoint union of all balls $\{(t, x)\} \times \mathbb{B}^n$ of ininite radius, as $(t, x) \in \Omega^c$.

Moreover, Ω^c coincides with the compactification $\mathfrak{M}_{A\#}$ of Ω defined by the functions $a(x)$, $t/\langle t \rangle \in C^\infty(\Omega)$, introduced in [C2], VI, and the algebra $\mathcal{C}$ satisfies conditions (m_1'), and (m_j), $j = 2, 3, 4, 5, 7$. (We noted before that (m_4) is implied by prop. 1.3. Cdn. (m_5) is trivial: We have $\Lambda \in \mathcal{C}$. Cdn. (m_1'), first used by McOwen, requires that the functions

$$|p(x)|^2 \quad \text{and} \quad g_{jk}(x)\bar{b}^j(x)b^k(x), \tag{2.8}$$

are in $\mathcal{A}^\#$, for every $D_x = b^j(x)\partial_{x,j} + p(x) \in \mathcal{D}^\#$. This again is trivially true, since $\mathcal{A}^\#$ contains all of $C^\infty(B)$. Furthermore the conditions (m_j) involve some separation conditions which can be satisfied by enlarging $\mathcal{A}^\#$ and $\mathcal{D}^\#$ in such a way that the generated algebra $\mathcal{C}$ remains the same. Details are left to the reader.

As a consequence we may apply [C2], VII, thm. 3.6. The conclusion is that the symbol space $\mathbb{M}$ of $\mathcal{C}$ is a compact subset of the boundary $\partial\mathbb{P}^*\Omega = \mathbb{P}^*\Omega \setminus T^*\Omega$ of our compactification $\mathbb{P}^*\Omega$, containing the wave front space $\mathbb{W}$, i.e., the bundle of cospheres of infinite radius over Ω.

Since $\mathbb{M}$ is compact, it must contain all points of the infinite cosphere bundle over $\partial\Omega^c$ as well. Thus it remains to be shown that no other point of $\partial\mathbb{P}^*\Omega$ is contained in $\mathbb{M}$. In particular none of the points $|t| = \infty$, $\tau^2 + |\xi|^2 < \infty$ can be contained in $\mathbb{M}$.

This, on the other hand, is a consequence of [C2], VII, thm. 4.2. To indicate at least the idea, we find that $\Lambda \in \mathcal{E}$, for our present algebra, while the formal symbol of Λ is $(1 + \tau^2 + |\xi|^2)^{-1/2}$, i.e., is $\neq 0$ for the latter type of points. Hence such points can not be in $\mathbb{M}$, since the symbol of $\Lambda \in \mathcal{E}$ must vanish, while thm. 3.6 implies that symbol and formal symbol coincide for the points of $\partial P^*\Omega$ which are in $\mathbb{M}$.

This completes the proof of thm. 2.3.

3. Extending the E-Symbol to the Algebra C; a Fredholm Result

Our results on the symbol chain of the algebra $\mathcal{C}$ are not yet practical for an application. Note that the C^*-algebra $\mathcal{C}^\vee = \mathcal{C}/\mathcal{K}(\mathcal{H})$ has the closed two-sided ideal $\mathcal{E}^\vee = \mathcal{E}/\mathcal{K}(\mathcal{H})$, and in thm. 2.2 we proved $\mathcal{E}^\vee$ isometrically isomorphic to the «function algebra» $CO(\mathbb{E}, \mathfrak{k}_x)$. Now it will prove useful to look for an extension of this isometry mapping the left regular representation of $\mathcal{C}^\vee$ on $\mathcal{E}^\vee$ into the function algebra $CB(\mathbb{E}, \mathfrak{L}_x) \supset CO(\mathbb{E}, \mathfrak{k}_x)$.

To be more specific, every $A^\vee = \mathcal{A} + \mathcal{K}(\mathcal{H}) \in \mathcal{C}^\vee$ induces a continuous operator $T_{A^\vee}: \mathcal{E} \to \mathcal{E}$, by left multiplication $T_{A^\vee}E^\vee = A^\vee E^\vee$, $E^\vee \in \mathcal{E}^\vee$. Clearly this defines a continuous algebra homomorphism $\mathcal{C}^\vee \to \mathfrak{L}(\mathcal{E}^\vee)$, called the left regular representation of $\mathcal{C}^\vee$ on $\mathcal{E}^\vee$.

We claim that the linear operator $T_{A^\vee}$ of the B-space $\mathcal{E}^\vee$ has a natural isometric representation as a function in $CB(\mathbb{E}, \mathfrak{L}_x)$, which coincides with the symbol γ_A whenever $A \in \mathcal{E}$.

Since it is clear that $\mathcal{E}^\vee = \mathcal{E}/\mathcal{K}$ is an ideal of $\mathcal{C}/\mathcal{K} = \mathcal{C}^\vee$ it suffices to observe the action of the generators of $\mathcal{C}^\vee$ on $\mathcal{E}$-symbols $\gamma_E \in CO(\mathbb{E}, \mathfrak{k}_x)$. Here we again look at the Fourier conjugated generators G_j of (1.7). In the order listed $G_1 = a(x)$ is given as multiplication by the L_x-valued function $a(x)$, constant in t, τ; $G_2 = s_1(D_t)$ acts on $CO(\mathbb{E}, k_x)$ as the multiplication by the scalar-valued function $s_1(\infty t_0)$. For G_j, $j = 3, 4, 5$, we find the multiplication by $\Lambda^\sim(\tau) \in CO(\mathbb{E}, \mathfrak{k}_x)$, $\tau_j\Lambda^\sim(\tau) \in CB(\mathbb{E}, \mathfrak{k}_x)$ and $D_x\Lambda^\sim(\tau) \in CB(\mathbb{E}, \mathfrak{L}_x)$, respectively. All the above was discussed in thm. 2.2.

Thus we now extend the isometry $\gamma^\vee: \mathcal{E}^\vee \to CO(\mathbb{E}, \mathfrak{k}_x)$ induced by the $\mathcal{E}$-symbol γ to a homomorphism from the algebra $\mathcal{C}^{0\vee}$ generated by finite adjunction of the cosets of (1.2) by assigning to $A_j^\vee$ the function just designated for G_j. Let the extension be called $\gamma^\vee$ again, and let γ still denote the lifting to the corresponding dense subalgebra of $\mathcal{C}$.

Note that γ_{A_j} not only is in $CB(\mathbb{E}, \mathfrak{L}_x)$ but even in $CB(\mathbb{E}, \mathcal{C}_x)$, with the C^*-algebra $\mathcal{C}_x \subset \mathfrak{L}_x$ of singular integral operators over B, the unique Laplace comparison algebra of B (cf. [C2], VI, 3).

The map γ is well defined: The assignment $A_j \leftrightarrow \gamma_{A_j}$, $j \neq 2$, is directly given by F_t-conjugation and the isometry $CB(\mathbb{R}^{n'}, \mathfrak{L}_x) \to \mathfrak{L}(\mathcal{H})$ of (1.4). This trivially extends to an isometry $C^\blacktriangle \to CB(\mathbb{R}^{n'}, \mathcal{C}_x)$ of the C^*-algebra $C^\blacktriangle \subset \mathcal{C}$ generated by A_l, $l \neq 2$. Moreover the algebra $CB(\mathbb{R}^{n'}, \mathcal{C}_x)$ does not contain compact operators $\neq 0$: If $A(\tau) \neq 0$ near $\tau = \tau_0$, then $\{\phi(\tau)w(x): \phi \in C_0^\infty(\mathfrak{N}_{\tau_0})\}$, for $A(\tau_0)w \neq 0$ near $\mathfrak{N}_{\tau_0}$, defines an infinite dimensional subspace of $\mathcal{H}$ on which the operator $A(\tau)$ is bounded away from zero. Thus the isometry $\mathcal{C}^\blacktriangle \to CB$ induces a *-isomorphism $\mathcal{C}^{\blacktriangle\vee}$, CB, i.e. also an isometry $\mathcal{C}^{\blacktriangle\vee} \to CB(\mathbb{E}, \mathcal{C}_x)$, the functions in $CB(\mathbb{R}^{n'}, \mathcal{C}_x)$ to be considered as functions over $\mathbb{E}$ constant in

t. The general (F_t-conjugated) element of $\mathcal{C}^{0\vee}$ then is a finite sum $A^\vee = \sum a_j(D_\tau)A_j(\tau) + \mathcal{K}(\mathcal{H})$, $a_j \in CS(\mathbb{R}^{n'})$, $A_j(\tau) \in \mathcal{C}^{\blacktriangle\vee}$, since commutators $[s_1(D_t), G_j]$ were seen compact. Clearly, for an $E^\vee \simeq E(\infty t_0, \tau) \in CO(\mathbb{E}, k_x)$ we get

$$A^\vee E^\vee = \sum a_j(\infty t_0)A_j(\tau)E(\infty t_0, \tau). \tag{3.1}$$

Also, in view of the fact, that $CO(\mathbb{E}, \mathcal{k}_x)$ contains all operators of the form $\phi(\tau)(v\rangle\langle w)$, $\phi \in CO(\mathbb{E})$, $v, w \in \mathcal{k}_x$, it is clear that

$$\begin{aligned} \|T_{A^\vee}\| &= \sup\{\|A^\vee E^\vee\| : E^\vee \in \mathcal{E}^\vee \|E^\vee\| = 1\} = \\ &= \sup\left\{\left\|\sum a_j(\infty t_0)A_j(\tau)\right\| : \infty t_0 \in \partial\mathbb{B}^{n'}, \tau \in \mathbb{R}^{n'}\right\} \end{aligned} \tag{3.2}$$

which confirms that the map $\gamma^\vee$ is an isometry $T(\mathcal{C}^{0\vee}) \to CB(\mathbb{E}, \mathcal{C}_x)$. Taking continuous extension we then indeed get the required isometry, called $\gamma^\vee$ again. We also extend the map γ from $\mathcal{E}$ to $\mathcal{C}$, using the chain

$$\begin{array}{ccccc} \mathcal{C} \to \mathcal{C}^\vee \to T(\mathcal{C}^\vee) & \xrightarrow{\gamma^\vee} & CB(\mathbb{E}, \mathcal{C}_x) \\ \downarrow & & \\ \mathfrak{L}(\mathcal{E}^\vee) & & \end{array} \tag{3.3}$$

Theorem 3.1. *The extended map γ defines a continuous *-homomorphism $\mathcal{C} \to CB(\mathbb{E}, \mathcal{C}_x)$, with the unique Laplace comparison algebra $\mathcal{C}_x$ of the compact space B. All functions $A(\infty t_0, \tau) \in \operatorname{im}\gamma$ have their $\mathcal{C}_x$-symbol independent of τ. Moreover, there exists a continuous *-homomorphism $\iota: \operatorname{im}\gamma \to C(\mathbb{M}\backslash\backslash\mathbb{W})$ from $\operatorname{im}\gamma$ onto the space of continuous functions over the infinite points*

$$\begin{aligned} \mathbb{M}\backslash\backslash\mathbb{W} = \{(\infty t_0, x), \infty(\tau, \xi)) : t_0, \tau \in \mathbb{R}^{n'}, |t_0| = 1, \\ (\tau, \xi) \in T^*B, \tau^2 + |\xi|^2 = 1\} \end{aligned} \tag{3.4}$$

of the symbol space $\mathbb{M}$ of $\mathcal{C}$ (with $|\xi|^2 = g^{jk}\xi_j\xi_k$) such that

$$\sigma_A|\mathbb{M}\backslash\backslash\mathbb{W}) = \iota(\gamma_A), \quad \text{for all} \quad A \in \mathcal{C}. \tag{3.5}$$

In particular we have

$$\ker\gamma = \mathcal{J}_0, \tag{3.6}$$

where $\mathcal{J}_0$ denotes the minimum comparison algebra of $\Omega = \mathbb{R}^{n'} \times B$, i.e., the C^*-algebra generated by the multipliers of $C_0^\infty(\Omega)$ and the operators $D\Lambda$, with all first order differential expressions D of compact support (and C^∞-coefficients).

PROOF. The first statement was already discussed above, and one finds that γ_{A_j}, for the generators (1.2) have the $\mathcal{C}_x$-symbols

$$a(x), s(\infty t_0), 0, 0, b^j(x)/(g^{\nu\mu}(x)\xi_\nu\xi_\mu)^{1/2}, \tag{3.7}$$

in the order listed, with $D_x = b^j(x)D_{x,j} + p(x)$. All these functions are independent of τ, so that the general element in $\operatorname{im}\gamma$ must have the same property.

The minimal comparison algebra $\mathcal{J}_0$ has the generators

$$a(t,x), p(t,x)\Lambda, b(t,x)D_{t,j}\Lambda, D_x(t)\Lambda, \tag{3.8}$$

where $a, b, p \in C_0^\infty(\Omega)$, and $D_x(t)$ has compact support as well. All these operators clearly are in $\ker\gamma$, since they may be written in the form $A = \chi A$, with a suitable function $\chi(t) \in C_0^\infty(\mathbb{R}^{n'})$. Thus it follows that $\mathcal{J}_0 \subset \ker\gamma$. It is also known that $A \in \mathcal{C}$ is in $\mathcal{J}_0$ if and only if $\sigma_A = 0$ on $\mathbb{M}\backslash\mathbb{W}$ (cf. [C2] VII, 2). (This follows from the observation that for such A we must have $\sigma_{\chi_j A} = \chi_j(t)\sigma_A \to \sigma_A$ in $C(\mathbb{M})$, as $j \to \infty$, where $\chi_j(t) = \chi(t/j)$, $\chi(t) \in C_0^\infty(\mathbb{R}^{n'})$, $\chi = 1$ near $t = 0$, is a sequence of cut-off functions, equal to 1 on larger and larger subdomains of $\mathbb{M}$. Accordingly there exists a sequence $C_j \in \mathcal{K}(\mathcal{H})$ such that $\|A - (\chi_j A - C_j)\| \to 0$, while we get $\chi_j A + C_j \in \mathcal{J}_0$.)

In order to define the homomorphism ι we first note that $\gamma\mathcal{C}^0 = \gamma^\vee\mathcal{C}^{0\vee}$ is dense in $\operatorname{im}\gamma$, with the above finitely generated algebras $\mathcal{C}^0$ and $\mathcal{C}^{0\vee} = \mathcal{C}^0/\mathcal{K}$. Hence it suffices to define such a homomorphism in $\gamma^\vee C^{0\vee}$. For $A \in \mathcal{C}^0$ we may write γ_A as a finite sum

$$\gamma_A = \sum a_j(\infty t_0)A_j(\tau) + C(\infty t_0, \tau)C \in CO(\mathbb{R}^{n'}, \mathfrak{k}_x), \tag{3.9}$$

where the last term corresponds to an operator $E \in \mathcal{E}$ with $\sigma_E = 0$. On the other hand, the first term is γ-image of $A^\sim = \sum \alpha_j(t)A_j^\sim \in \mathcal{C}$ with $A_j^\sim \in \mathcal{C}^\blacktriangle$. One calculates that

$$\sigma_A(\infty t_0, x, \infty(\tau,\xi)) = \sum a_j(\infty t_0)\sigma_{A_j^\sim}(x, \infty(\tau,\xi)). \tag{3.10}$$

Also the restrictions $\sigma|\mathcal{C}^\blacktriangle$ and $\gamma|\mathcal{C}^\blacktriangle$ are related by a homomorphism. $\sigma|\mathcal{C}^\blacktriangle = \iota \circ (\gamma|\mathcal{C}^\blacktriangle)$, defined as follows: For a function $A_{jl}^\sim(\tau)$ take the F_t^{-1}-conjugation (an isometry), and then the symbol σ, a contraction map $\mathcal{C}^\blacktriangle \to C(\mathbb{M})$, where the $\sigma_{A_j^\sim}$ are independent of t, on

$$\mathbb{M} = \{(t, x, \infty(\tau,\xi)) : t \in \mathbb{B}^{n'}, x \in B, \tau^2 + g^{\nu\mu}(x)\xi_\nu\xi_\mu = 1\}.$$

It follows that (with $\sigma_X = \sigma(X)$, for a moment)

$$\begin{aligned}\|\sigma_A\|_{L^\infty(\mathbb{M}-\mathbb{W})} &= \sup_{|t_0|=1}\sup_{\mathbb{M}}\left|\sigma\Big(\sum a_j(\infty t_0)A_j^\sim\Big)\right| \\ &\leqslant \sup_{|t_0|=1}\sup\left\{\Big\|\sum a_j(\infty t_0)A_j(\tau)\Big\| : t \in \mathbb{R}^{n'}\right\},\end{aligned} \tag{3.11}$$

where the right hand side is the norm of $\gamma_A \in C(\mathbb{E}, \mathcal{C}_x)$. This shows that the map $\iota: \gamma_A \to \sigma_A$ is continuous on $\gamma(\mathcal{C}^0)$. Also this map trivially defines a homomorphism on $\gamma(\mathcal{C}^0)$, and of course we have (3.5) satisfied for $A \in \mathcal{C}^0$.

Taking continuous extension we get the desired continuous *-homomorphism satisfying (3.5) on all of $\mathcal{C}$.

In particular (3.6) implies that $\sigma_A = 0$ on $\mathbb{M}\backslash\mathbb{W}$ (i.e., $A \in \mathcal{J}_0$) whenever $A \in \ker\gamma$. Or, $\ker\gamma \subset \mathcal{J}_0$, completing the proof of thm. 3.1.

Now, regarding the Fredholm property of an operator $A \in \mathcal{C}$ we have the following result.

Theorem 3.2. *An operator $A \in \mathcal{C}$ is Fredholm if and only if* (i) $\sigma_A(\mathfrak{m}) \neq 0$ *for all* $\mathfrak{m} = (t, x, \infty(\tau, \xi)) \in \mathbb{M}$, and (ii) $\gamma_A(\mathfrak{e})$ *is invertible in* $\mathcal{C}_x$ *(i.e. in* $\mathfrak{L}(\mathfrak{h}_x)$*) for every* $\mathfrak{e} = (\infty t_0, \tau) \in \mathbb{E}$, *and* $\gamma_A(\mathfrak{e})^{-1}$ *is uniformly bounded on* $\mathbb{E}$.

PROOF. The conditions are clearly necessary, since a Fredholm inverse B of A (such that $1 - AB$, $1 - BA$ are of finite rank) also gives inverses σ_B and γ_B for σ_A and γ_A respectively, implying (i) and (ii) for A. Vice versa, let A satisfy (i) and (ii). Since $\operatorname{im}\gamma$ is a C^*-algebra it follows then that we have $\gamma_A(\ell)^{-1} = \gamma_P(\ell)$, for $\ell \in \mathbb{E}$ and some $P \in \mathcal{C}$. Using the homomorphism ι we also get

$$\sigma_{1-AP} = \iota(\gamma_{1-AP}) = 0, \qquad \sigma_{1-PA} = \iota(\gamma_{1-PA}) = 0, \qquad \mathfrak{m} \in \mathbb{M}\backslash\mathbb{W}. \tag{3.12}$$

In other words we get $\sigma_P = 1/\sigma_A$ on $\mathbb{M}\backslash\mathbb{W}$, and we then can find $Q \in \mathcal{J}_0$ such that $\sigma_Q = 1/\sigma_A - \sigma_P$ on all of W, since σ maps onto $CO(\mathbb{W})$. Then let $B = P + Q$. Conclude that $\gamma_B = \gamma_P$, since $\gamma_Q = 0$, by (3.6).

In other words we get

$$\gamma_{1-AB} = \gamma_{1-BA} = 0 \tag{3.13}$$

and

$$\sigma_{1-AB} = \sigma_{1-BA} = 0. \tag{3.14}$$

Relations (3.13) imply that $1 - AB$ and $1 - BA$ are in $\ker\gamma = \mathcal{J}_0$. But J_0 is a compact commutator algebra, and the restriction of σ to $\mathcal{J}_0$ is the symbol of $\mathcal{J}_0$. Therefore (3.14) implies that $1 - AB$ and $1 - BA$ are compact, so that A has an inverse mod $\mathcal{K}(\mathcal{H})$ and must be Fredholm, q.e.d.

Remark. Thm. 3.2 has the trivial consequence that the Fredholm index of an operator $A \in \mathcal{C}$ must be given by a group homomorphism

$$\operatorname{ind}: (\langle\sigma_A\rangle, \langle\gamma_A\rangle) \to \mathbb{Z}, \tag{3.15}$$

mapping the group of pairs of homotopy classes $(\langle\phi\rangle, \langle\phi\rangle)$ of maps

$$\phi: \mathbb{M} \to \mathbb{C}^*, \qquad \phi: \mathbb{E} \to \mathcal{C}_x, \qquad \phi \in C(M), \qquad \phi \in \operatorname{im}\gamma, \tag{3.16}$$

(with group operation induced by pointwise multiplication of functions) into the additive group of integers. Similarly for operators acting on crossections of vector bundles over Ω. For an explicit index formula, as in case of a compact manifold (cf. [ASj], $j = 1, 3, 4, 5$) one will have to obtain the homomorphism ind explicitely by calculating the Fredholm index of specific operators.

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Geometry of KDV (1): Addition and the Unimodular Spectral Classes

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To Alberto Calderón for his 65th birthday

1. INTRODUCTION

This is the first of three papers on the geometry of KDV. It presents what purports to be a foliation of an extensive function space into which all known invariant manifolds of KDV fit naturally as special leaves. The two main themes are *addition* (each leaf has its private one) and *unimodular spectral classes* (each leaf has a spectral interpretation), but first a bit of background.

Darboux's transformation: scattering case

Let[2] $Q = -D^2 + q(x)$ be a Schrödinger operator on $\mathbb{R}$ with potential of scattering class[3] $C_{\updownarrow}^{\infty}$. The spectrum of Q acting in $L^2(\mathbb{R})$ comprises $0 \leqslant g < \infty$ sim-

[1] The work presented in this paper was performed at the Courant Institute of Mathematical Sciences with the support of the National Science Foundation under Grant NSF-MCS-76-07039.

[2] D signifies differentiation with respect to $x \in \mathbb{R}$.

[3] $C_{\updownarrow}^{\infty}$ is the class of infinitely differentiable real-valued functions vanishing rapidly at $\pm\infty$.

ple *bound states* $-k_1^2 < \cdots < -k_g^2 < 0$, plus the *continuum* $[0, \infty)$ of multiplicity 2. The ground state $-k_1^2$ (if present) has an eigenfunction e_1 of one sign; it is removed by the transformation $Q \to Q - 2D^2 \lg e_1 = Q^-$, the bound states of the latter being the same as for Q with $-k_1^2$ left out. Bound states can also be added: if $-k_0^2$ is any number below spec Q, then $Qh = -k_0^2 h$ has positive solutions $h_- \in L^2(-\infty, 0]$ with $\int_0^\infty h_-^2 = \infty$ and $h_+ \in L^2[0, \infty)$ with $\int_{-\infty}^0 h_+^2 = \infty$, and if $e_0 = (1 - c)h_- + ch_+$ with $0 < c < 1$, then $Q^+ = Q - 2D^2 \lg e_0$ has bound states $-k_0^2 < -k_1^2 < \cdots < -k_g^2$. This type of transformation stems from Darboux [1882]; see also Bargmann [1949], Crum [1955], and Faddeev [1964]. It can be expressed in other ways: for example, if e is any positive solution of $Qe = -k^2 e$, if $AQ = Q - 2D^2 \lg e$, and if $p = e'/e$, then $Q = -D^2 + p' + p^2 - k^2$ while $AQ = -D^2 - p' + p^2 - k^2$, so that the Darboux transformation A is identified with the Bäcklund transformation B of KDV in the form discovered by Miura [1968], *to wit*, $B: p' + p^2 - k^2 \to -p' + p^2 - k^2$. A variant is to express Q as[4] $(eDe^{-1})^\dagger (eDe^{-1}) - k^2$ and to exchange the factors: $(eDe^{-1})(eDe^{-1})^\dagger - k^2 = AQ$; see Deift [1978].

Addition defined

I make two small but important changes in the Darboux transformation: *I insist that* $-k^2$ *be to the left of spec* Q *and I take for* e *always* h_- *or* h_+, which was not done before. Let $\mathfrak{p} = (\lambda, -)$ or $(\lambda, +)$ with $\lambda = -k^2$ and take $e(x, \mathfrak{p}) = h_-(x)$ or $h_+(x)$ in accordance with this choice. The map

$$A^{\mathfrak{p}}: Q \to Q - 2D^2 \lg e(x, \mathfrak{p})$$

is called *addition* of $\mathfrak{p}$ for reasons to be explained presently; unlike the previous maps of its kind, it is always *isospectral*, and a great deal more besides, as you will see. The composition rule[5]

$$A^{\mathfrak{p}_1} \ldots A^{\mathfrak{p}_n} = Q - 2D^2 \lg [e(x, \mathfrak{p}_1), \ldots, e(x, \mathfrak{p}_n)]$$

shows that addition is *commutative*, and its specialization to $n = 2$ with $\mathfrak{p}_1 = \mathfrak{p} = (\lambda, \pm)$ and $\mathfrak{p}_2 = -\mathfrak{p} = (\lambda, \mp)$ shows that $A^{-\mathfrak{p}}$ is inverse to $A^{\mathfrak{p}}$, so that repeated additions produce a commutative group of transformations. The point now to be stressed is that addition makes sense in a very wide class of operators: the only thing you really need is that spec Q *is bounded away*

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[4] The *dagger* signifies the *transposed*.

[5] Crum [1955]. $[e_1, \ldots, e_n]$ is Wronski's determinant.

from $-\infty$! The corresponding class of (smooth) potentials is the *extensive function space* alluded to before.

Addition explained: Hill's case[6]

The name *addition* has a better justification than its mere commutativity. Let Q be a Hill's operator with potential of class[7] C_1^∞. Then the spectrum of Q acting in $L^2(\mathbb{R})$ consists of bands

$$[\lambda_0^+, \lambda_1^-] \cup [\lambda_1^+, \lambda_2^-] \cup [\lambda_2^+, \lambda_3^-] \cup \cdots$$

marked off by the periodic/anti-periodic spectrum of Q:

$$-\infty < \lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < \lambda_2^- \leqslant \lambda_2^+ < \lambda_3^- \leqslant \cdots \uparrow \infty.$$

Now the class of Hill's operators Q having one and the same spectrum as some fixed specimen is faithfully represented by divisors in real position on the (nonclassical) multiplier curve M determined by the irrationality

$$\sqrt{\Delta^2 - 1} = \sqrt{\lambda_0^+ - \lambda} \prod_{n=1}^{\infty} n^{-2}\pi^{-2}\sqrt{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}.$$

I pause to explain what all that means. The points of the curve are pairs $\mathfrak{p} = (\lambda, \pm)$ comprising a *projection* $\lambda(\mathfrak{p}) \in \mathbb{C}$ and a *signature* (of the irrationality). The function $e(x, \mathfrak{p})$ introduced before agrees, to the left of spec Q, with the *Baker-Akhiezer function* of M specified by

a) $Qe = \lambda(\mathfrak{p})e$;
b) $e(x + 1) = m(\mathfrak{p})e(x)$, $m(\mathfrak{p})$ being the *multiplier* $\Delta - \sqrt{\Delta^2 - 1}$;

and

c) $e(0) = 1$.

The pole divisor $\mathfrak{p}_1 + \mathfrak{p}_2 + \cdots$ of $e(x, \mathfrak{p})$ is independent of $0 < x < 1$. It is the *divisor of* Q and is in *real position* in the sense that its projections fall one into each of the spectral lacunae: $\lambda_n^- \leqslant \lambda(\mathfrak{p}_n) \leqslant \lambda_n^+ (n \geqslant 1)$. The association of Q to its divisor is 1:1. The latter form an ∞-dimensional torus J which is (the

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[6] McKean-Trubowitz [1976] is cited for background.
[7] C_1^∞ is the class of infinitely differentiable real-valued functions of period 1.

real part of) the *Jacobi variety* of M. In this language, *the addition*[8] *of* $\mathfrak{p}_0$ *with* $\lambda(\mathfrak{p}_0) < \lambda_0^+$ *to the left of* spec Q *is effected by the following recipe*

$$\begin{aligned} Q \to &\quad \mathfrak{p}_1 + \mathfrak{p}_2 + \cdots \\ \to &-\mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2 + \cdots \\ \to &\quad \infty + \mathfrak{p}_1' + \mathfrak{p}_2' + \cdots \\ \to &\quad Q' \end{aligned}$$

in which $-\mathfrak{p}_0$ is $\mathfrak{p}_0$ reckoned with the opposite signature; $\mathfrak{p}_1' + \mathfrak{p}_2' + \cdots$ is a new divisor in real position; the divisors of lines 2 and 3 are equal in J, meaning that the one comprises the roots and the other the poles of a function of rational character on M; and $Q' = A^{\mathfrak{p}_0}Q$ is the new Hill's operator with divisor $\mathfrak{p}_1' + \mathfrak{p}_2' + \cdots$. The proof is presented below; compare McKean [1985].

Note. The special addition (not permitted by the present recipe) of the point of ramification $\mathfrak{p}_0 = \lambda_0^+$ is an involution[9] of the Jacobi variety corresponding to the addition (in J) of the sum of all real half-periods; see McKean [1980].

KDV-type fields

The next item is the connection between addition and KDV. Take $\mathfrak{p} = (\lambda, +)$ and $\mathfrak{p}' = (\lambda + \Delta\lambda, +)$ for λ below spec Q. Then

$$A^{\mathfrak{p}'}A^{-\mathfrak{p}}Q = Q - XQ\,\Delta\lambda + etc.$$

with[10]

$$XQ = 2G'_{xx}(\lambda),$$

$G_{xy}(\lambda)$ being the Green's function $(Q - \lambda)_{xy}^{-1}$. The vector field X, familiar from Hill's equation,[11] appears in this way as an *infinitesimal addition*. Now XQ may be expanded as $\lambda \downarrow -\infty$ in diminishing half-integral powers of $-\lambda$:

$$XQ = \sum_{n=1}^{\infty} \lambda^{-(1/2+n)} X_n Q,$$

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[8] It would be more accurate to speak of *subtraction* but never mind.
[9] $\mathfrak{p}_0 = -\mathfrak{p}_0$.
[10] The *prime* signifies differentiation on diagonal.
[11] McKean-Trubowitz [1976/78].

in which $X_1Q = q'$, $X_2Q = 3qq' - (1/2)q'''$, *etc.* are the conventional KDV fields up to unimportant constants, the point being that, *in the full generality envisaged here, addition provides a substitute for the flows of the KDV hierarchy even when the latter have no existence.*[12] This realization prompts the formation of the *additive class* produced by closing up the operators Q obtained from a fixed specimen by repeated additions of points $\mathfrak{p}$ to the left of spec Q. *In this way, the space of operators with spectrum not extending to* $-\infty$ *is foliated*[13] *by classes which fill the office of the invariant manifolds of KDV, each class having its private addition.*

Note. The vector fields $XQ = 2G'_{xx}(\lambda)$, taken for λ to the left of spec Q, may be integrated without obstruction to produce commuting, class-preserving flows. This can be done by explicit formulae involving Fredholm determinants, much as in McKean-Trubowitz [1982]; see the third paper in this series. Mumford [1983/4] has studied these flows in a special case.

Unimodular spectral classes

It is known that the invariant manifolds of KDV have a spectral basis: for example, in the scattering case, the transmission coefficient is *the* invariant specifying the manifold, while in the Hill's case, it is the periodic/antiperiodic spectrum, alias the discriminant, that is fixed. You will ask: *what is the corresponding invariant for the general additive class*? I present a conjecture verified in three examples cited below. Let $dF(\lambda)$ be the (2×2) *spectral weight*[14] of Q. Then Q is isospectral (= unitarily equivalent) to a second such operator Q' if and only if the spectral weight of the latter is related to the former weight as[15] $GdFG^\dagger$ with a factor $G = G(\lambda)$ taking values in $GL(2, \mathbb{R})$. Now it is easy to compute the spectral weight $dF^{\mathfrak{p}} = G\,dF\,G^\dagger$ of $A^{\mathfrak{p}}Q$: isospectrality prevails, the factor being

$$G(\lambda) = \frac{1}{\sqrt{\lambda - \lambda(\mathfrak{p})}} \begin{vmatrix} c & -1 \\ \lambda - \lambda(\mathfrak{p}) - c^2 & c \end{vmatrix} \quad \text{with} \quad c = \frac{e'(0, \mathfrak{p})}{e\,(0, \mathfrak{p})},$$

and you notice that $\det G = 1$! This prompts a definition: *two operators Q belong to the same unimodular spectral class if they are unitarily equivalent and*

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[12] KDV cannot be balanced with initial data x^2, for example.
[13] The usage is informal as the dimensionality of the leaf varies from 0 to ∞!
[14] Kodaira [1949] is cited for background; compare art. 2 below.
[15] The *dagger* signifies the *transposed*, as before.

if the factor G takes its *value not just in* $GL(2, \mathbb{R})$ *but in* $SL(2, \mathbb{R})$; evidently, the additive class of Q is part of its unimodular spectral class. I conjecture that, with the proper technical precautions, *these classes are always one and the same.* The second paper of this series verifies this conjecture in three examples:

a) the scattering case $C_{\downarrow}^{\infty}$[16];
b) the Hill's case C_{1}^{∞};
c) if the additive class is of finite dimension.

Under (*a*), the class has fixed transmission coefficient, the phase of the reflection coefficient together with the logarithmic norming constants serving as additive coordinates. Under (*b*), the periodic/anti-periodic spectrum is fixed and the class is identified as the real part of the Jacobi variety, the addition of the latter falling in with the addition of the class. Under (*c*), the class is a leaf of the Neumann system and every leaf appears in this way.[17]

Second measure class

The two measure classes determining the conventional isopectral class of Q are typified by[18]

$$sp\, dF = df_{11} + df_{22}$$

and[19]

$$\sqrt{\det dF} = \sqrt{f'_{11}f'_{22} - (f'_{12})^2}\, sp\, dF.$$

The latter is of special importance in its role of *unimodular spectral invariant.* It is always smaller than $d\lambda$ and the density $D = \sqrt{\det dF}/d\lambda$ can be interpreted as the modulus of a (*mean*) *transmission coefficient*; in fact, it is precisely $|s_{11}|$ in the scattering case. The evaluation of the gradient:

$$\frac{\partial \lg D}{\partial q(x)} = -\text{the real part of } G_{xx}(\lambda + \sqrt{-1}\, 0+) \quad \text{if} \quad D > 0$$

hints at an attractive connection between the unimodular invariant and the vector fields $XQ = 2G'_{xx}(\lambda)$, but this has not been fully understood.

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[16] $Q = -D^2$ is the simplest instance; it is settled in item 7, art. 5.
[17] McKean [1979] and Moser [1978] are cited for background.
[18] $dF = [df_{ij}: 1 \leqslant i, j \leqslant 2]$.
[19] $f'_{ij} = df_{ij}/sp\, dF$ $(1 \leqslant i, j \leqslant 2)$.

Acknowledgement

I wish to thank L. Menezes: her comments led to several improvements of the original presentation.

2. PRELIMINARY SPECTRAL THEORY

I collect for future use a number of standard facts about Q under the sole assumption that its spectrum is bounded from $-\infty$.

Eigendifferential expansions[20]

Fix a complex number λ outside spec Q so that $Qh = \lambda h$ has independent solutions

$$h_- \in L^2(-\infty, 0] \quad \text{with} \quad \int_0^\infty |h_-|^2 = \infty$$

and

$$h_+ \in L^2[0, \infty) \quad \text{with} \quad \int_{-\infty}^0 |h_+|^2 = \infty.$$

The Wronskian $[h_-, h_+] = h'_- h_+ - h_- h'_+$ is always taken as 1. Then Green's function $(Q - \lambda)^{-1}_{xy} = G_{xy}(\lambda)$ is expressed as $h_-(x)h_+(y)$ if $x \leqslant y$. Let the bottom of spec Q be placed at 0 for simplicity. The *fundamental matrix*

$$M(\lambda) = [m_{ij}(\lambda): 1 \leqslant i, j \leqslant 2] = \begin{bmatrix} 2h_- h_+ & (h_- h_+)' \\ (h_- h_+)' & 2h'_- h'_+ \end{bmatrix}$$

evaluated at $x = 0$ is analytic in the cut plane $\mathbb{C} - [0, \infty)$; it is also symmetric and of determinant -1. It is real for $\lambda < 0$; most important, its imaginary part is positive ($-$definite) in the open upper half-plane[21] and so has the representation

$$\operatorname{imag} M(\lambda) = \frac{b}{\pi} \int_0^\infty [(\lambda' - a)^2 + b^2]^{-1} \, dF(\lambda')$$

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[20] Weyl [1910] but see Kodaira [1949] for the present more elegant version.

[21] This is easily seen from the identity

$$\operatorname{imag} G_{xy}(\lambda) = \operatorname{imag} \lambda \times [(Q - \lambda)^{-1}(Q - \lambda^*)^{-1}]_{xy} = \int G_{x\cdot}(\lambda) G^*_{y\cdot}(\lambda).$$

with $\lambda = a + \sqrt{-1}\, b$, $b > 0$, and a 2×2, real, symmetric, positive[22] *spectral weight*

$$dF(\lambda = a) = \lim_{b \downarrow 0} \operatorname{imag} M(a + \sqrt{-1}\, b)\, da = [df_{ij}(\lambda): 1 \leqslant i, j \leqslant 2].$$

The spectral theorem

$$Q = \int_0^\infty \lambda\, d\mathfrak{P}(\lambda)$$

is implemented thereby: if $E(x, \lambda) = (e_1, e_2)$ with $Qe = \lambda e$, $e_1(0) = e_2'(0) = 1$, and $e_1'(0) = e_2(0) = 0$, then the kernel of the projection $d\mathfrak{P}(\lambda)$ is

$$d\mathfrak{P}_{xy}(\lambda) = (2\pi)^{-1} E(x, \lambda)\, dF(\lambda) E^{\mathsf{T}}(y, \lambda).$$

Measure classes

The function $f_1(x)$ is taken as $h_-(x)h_+(0)$ if $x < 0$ and as $h_-(0)h_+(x)$ if $x \geqslant 0$ for fixed $\lambda = -1$, say, to the left of spec Q; similarly, $f_2(x)$ is taken as $h_-(x)h'_+(0)$ if $x < 0$ and as $h'_-(0)h_+(x)$ if $x \geqslant 0$. Then[23]

$$[(f_i, d\mathfrak{P}(\lambda) f_j): 1 \leqslant i, j \leqslant 2] = (2\pi)^{-1}(\lambda + 1)^{-2}\, dF(\lambda);$$

also, the families $d\mathfrak{P}f_1$ and $d\mathfrak{P}f_2$ together span the whole of $L^2(\mathbb{R})$. Now df_{12} is dominated by[24] df_{11} and/or df_{22}, so every spectral measure $(f, d\mathfrak{P}f)$ is dominated by

$$(f_1, d\mathfrak{P}f_1) + (f_2, \text{coprojection of } d\mathfrak{P}f_2 \text{ upon } d\mathfrak{P}f_1) = (f_1, d\mathfrak{P}f_1) + (f_2, d\mathfrak{P}f_2) - r(f_2, d\mathfrak{P}f_1)$$

with $r = df_{12}/df_{11}$, *i.e.*, by $df_{11} + df_{22} - r\, df_{12}$. The corresponding *top measure class* is typified by the *trace* $sp\, dF = df_{11} + df_{12}$; indeed, by the positivity of dF, $|df_{12}| \leqslant \sqrt{df_{11}\, df_{22}}$,[25] so that $|r\, df_{12}| \leqslant df_{22}$, and the vanishing of $f_{11}(\Delta) + f_{22}(\Delta) - \int_\Delta r\, df_{12}$ implies $f_{11}(\Delta) = 0$, $|f_{12}|(\Delta) = 0$, and $f_{22}(\Delta) = 0$, in that order. The top measure class is now seen to be based upon the family

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[22] The adjective means that $f_{11}(\Delta) \geqslant 0$, $f_{22}(\Delta) \geqslant 0$, and $f_{12}^2(\Delta) \leqslant f_{11}(\Delta) f_{22}(\Delta)$ for arbitrary sets Δ.

[23] Kodaira [1949]. The *parenthesis* is the inner product.

[24] $f_{11} f_{22} \geqslant f_{12}^2$.

[25] This type of expression always signifies what is must: here, the radical is $\sqrt{f'_{11} f'_{22}} \times sp\, dF$, the primed densities being taken relative to the trace.

$(1-r)\,d\mathfrak{P}f_1 + d\mathfrak{P}f_2$: in fact, the associated spectral measure is just $df_{11} + df_{22} - r\,df_{12}$. The second class is based upon the perpendicular family

$$\frac{df_{22} + (1-r)\,df_{12}}{df_{11} + df_{22} - r\,df_{12}}\,d\mathfrak{P}f_1 - \frac{df_{11}}{df_{11} + df_{22} - r\,df_{12}}\,d\mathfrak{P}f_2,$$

the associated measure being

$$\frac{df_{11}\,df_{22} - (df_{12})^2}{df_{11} + df_{22} - r\,df_{12}} = \frac{f'_{11}f'_{22} - (f'_{12})^2}{f'_{11} + f'_{22} - r\,df'_{12}} \times sp\,dF;$$

in particular, *the second class is typified by the measure*[26]

$$\sqrt{\det dF} = \sqrt{df_{11}\,df_{22} - (df_{12})^2} = \sqrt{f'_{11}f'_{22} - (f'_{12})^2}\;sp\,dF.$$

Note. Masani-Wiener [1957-58] introduced $\sqrt{\det dF}$ in a different context and in a different way: it is a elementary fact that the interval function $D(I) = \sqrt{\det F(I)}$ is superadditive: $D(A \cup B) \geqslant D(A) + D(B)$ if $A \cap B$ is void. This permits the alternative definition:

$$\int_\Delta \sqrt{\det dF} = \text{the infimum of the sum of } D(I),$$

the infimum being taken over countable covers of Δ by intervals I. This will be helpful later on.

Unitary equivalence

The main fact about unitary equivalence can now be stated in a convenient form:[27] *two operators Q_1 and Q_2 are isospectral if and only if they determine the same top and the same bottom measure classes*, or, what is the same, *if and only if their spectral weights are related as $dF_2 = G\,dF_1\,G^{\mp}$, the 2×2 factor $G = G(\lambda)$ taking its values in $GL(2, \mathbb{R})$*. I could not find the second criterion stated in just this form, though it is an easy consequence of the first, which is standard.

PROOF. Let Q_1 and Q_2 be isospectral. The weight dF_1 can be expressed as $OD_1O^{\mp}\,dm$ with $O \in SO(2)$, 2×2 non-negative diagonal D_1, and a positive

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[26] Compare the preceding footnote.
[27] Stone [1932].

(numerical) measure dm. Now as dF_2 has a similar expression with the *same* dm, it suffices to produce $G = [g_{ij}: 1 \leqslant i, j \leqslant 2]$ taking its values in $GL(2, \mathbb{R})$ so as to make $D_2 = GD_1G^{\dagger}$ under the condition that $sp\ D_2$ vanishes simultaneously with $sp\ D_1$, and $\det D_2$ with $\det D_1$. The rest will be plain.

Note 1. dF is not intrinsic. The fact is, it depends upon the choice of origin $x = 0$, and if the latter is displaced to $x = c$, then the former weight dF is changed to $G\,dF\,G^{\dagger}$ with the factor

$$G = \begin{bmatrix} e_1(c) & e_2(c) \\ e_1'(c) & e_2'(c) \end{bmatrix}.$$

G is unimodular, so $\sqrt{\det dF}$ is not changed. This already commends it to special attention.

Note 2. The *unimodular spectral class* of Q_1 is the subclass of isospectral operators Q_2 having one and the same *invariant* $\sqrt{\det dF_2} = \sqrt{\det dF_1}$; equivalently, the factors G cited above take their values *not just in* $GL(2, \mathbb{R})$ *but in* $SL(2, \mathbb{R})$. The distinction is non-existent at bound states: if $dF_2 = G\,dF_1\,G^{\dagger}$ with $G \in GL(2, \mathbb{R})$ at a jump of dF_1, then the simplicity of bound states of Q implies that dF_1 and dF_2 are of rank 1. Now G may be chosen from $SL(2, \mathbb{R})$, as you will easily check.

Side operators

These are employed infrequently and can be skipped for now, as can the rest of this article. The *side operator* Q_+^0 is the restriction of Q to functions on the half-line $x \geqslant 0$ vanishing at $x = 0$; its spectrum is confined to $[0, \infty)$, in agreement with the form of its Green's function

$$G_{xy}^0(\lambda) = e_2(x)h_+(y)/h_+(0) \qquad (x \leqslant y)$$

and the fact that $h_+(0)$ *is root-free off the cut* $[0, \infty)$.[28] The imaginary part of $G_{xx}^0(\lambda)$ is positive in the upper half-plane, so

$$\lim_{x \downarrow 0} x^{-2} \operatorname{imag} G_{xx}^0(\lambda) = \operatorname{imag} \frac{h_+'(0)}{h_+(0)} = \frac{b}{\pi} \int_0^{\infty} [(\lambda' - a)^2 + b^2]^{-1}\, df_+^0(\lambda'),$$

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[28] $h_+(0) = 0$ off the cut means that h_+ is an eigenfunction of the side operator Q_+^0. Then λ must be real and negative, and $\int_0^{\infty} h_+ Q h_+ = \lambda \int_0^{\infty} h_+^2 < 0$ violates spec $Q \subset [0, \infty)$.

in which $\lambda = a + \sqrt{-1}b$, $b > 0$, and the spectral weight df^0_+ is non-negative. The other side operator of interest is the restriction Q^∞_+ of Q to functions on the half-line with vanishing slope at $x = 0$; its spectrum is likewise confined to $[0, \infty)$, with the possible exception of an isolated ground state to the left of 0, in agreement with the form of its Green's function

$$G^\infty_{xy}(\lambda) = -e_1(x)h_+(y)/h'_+(0) \qquad (x \leqslant y)$$

and the fact that $h'_+(0) = 0$ *has at most one (real, simple, negative) root off the cut* $[0, \infty)$.[29] The imaginary part of $G^\infty_{00}(\lambda)$ is positive in the upper half-plane, so

$$\operatorname{imag} G^\infty_{00}(\lambda) = -\operatorname{imag}\frac{h_+(0)}{h'_+(0)} = \frac{b}{\pi}\int [(\lambda' - a)^2 + b^2]^{-1}\, df^\infty_+(\lambda'),$$

in which $\lambda = a + \sqrt{-1}\,b$, $b > 0$, and the non-negative spectral weight df^∞_+ vanishes off $[0, \infty)$ + ground state, if any. The analogous side operators Q^0_- and Q^∞_- for the left half-line also play a small role below.

Note. The identity for m^0_{22} of the last footnote shows that if $h'_+(0) = 0$ has a root <0, then $h'_-(0) = 0$ does not, and it is easy to deduce that if the ground state of Q^∞_+ is below 0, then the bottom of spec Q^∞_- is *at* 0.

Inverse spectral problem

The association of Q to its 2×2 spectral weight dF is 1:1. This is made plausible by a count: 2 degrees of freedom in Q, one for $(-\infty, 0]$ and one for $[0, \infty)$, *versus* 3 degrees of freedom in the symmetric 2×2 weight, *less* 1 to account for $\det M = -1$.[30] The proof is easy, too: dF determines M and so also

$$-h'_-(0)/h_-(0) = (1/m_{11})(-1 - m_{12}) \text{ and } h'_+(0)/h_+(0) = (1/m_{11})(-1 + m_{12}),$$

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[29] $h'_+(0) = 0$ off the cut means that h_+ is an eigenfunction of the side operator Q^∞_+. Then $\lambda < 0$ as before, and

$$2[h'_-(0)h'_+(0)]^\cdot = m^\cdot_{22}(\lambda) = \frac{1}{\pi}\int (\lambda' - \lambda)^{-2}\, df_{22}(\lambda')$$

does the rest, the *spot* signifying differentiation with regard to λ.

[30] $\det M = -1$ looks like the loss of 2 degrees of freedom but, as the real part is conjugate to the imaginary, the count is only 1.

which determine the side operators Q_-^0 and Q_+^0 by the recipe of Gelfand-Levitan [1951]. Newton [1983] proves this directly in the 2×2 format, of which the only drawback is that one does not know what $\det M = -1$ signifies for dF, *i.e.*, one cannot (to date) satisfactorily describe what are the 2×2 spectral weights.

3. ADDITION

The letter $\mathfrak{p}_0$ denotes a pair $(\lambda_0, \pm)$ comprising a *projection* λ_0 to the left of spec $Q \subset [0, \infty)$ and a *signature* indicating which function $e(x, \mathfrak{p}_0) = h_-$ or h_+ is to be employed. The operation of *adding* $\mathfrak{p}_0$ *to* (*the divisor of*) Q is the map

$$A^{\mathfrak{p}_0}: Q \to Q - 2D^2 \lg e(x, \mathfrak{p}_0) = Q^{\mathfrak{p}_0}.$$

Its simplest properties will be elicited below.

Item 1. $e_0(x) = e(x, \mathfrak{p}_0)$ cannot vanish so that *addition is well-defined.*

Item 2. $Q = P^\mp P + \lambda_0$ *in which* $P = e_0 D e_0^{-1}$ *and* $P^\mp$ *is its transpose* $-e_0^{-1} D e_0$. $Q^{\mathfrak{p}_0}$ *is produced by exchange of factors*: $Q^{\mathfrak{p}_0} = PP^\mp + \lambda_0$; *in particular*, $P: h \to e_0^{-1}[h, e_0]$ *maps solutions of* $Qh = \lambda h$ *into* solutions of $Q^{\mathfrak{p}_0} h = \lambda h$. The proof is routine.

Item 3. Ph_- *belongs to* $L^2(-\infty, 0]$ *if* λ *is not on the cut* $[0, \infty)$; *similarly,* Ph_+ *belongs to* $L^2[0, \infty)$.

PROOF. h_+ is typical:[31]

$$\int_0^x |Ph_+|^2 = \int_0^x [h_+, e_0]^* D(h_+/e_0) = \frac{h_+}{e_0}[h_+, e_0]^* \Big|_0^x - \int_0^x \frac{h_+}{e_0} D[h_+, e_0]^*$$

$$= \frac{h_+}{e_0}[h_+, e_0]^* \Big|_0^x + (\lambda - \lambda_0)^* \int_0^x |h_+|^2 \equiv re^{\sqrt{-1}\theta} + O(1)$$

so that *either* $Ph_+ \in L^2[0, \infty)$ *or else* $(h_+/e_0)[h_+, e_0]^* = re^{\sqrt{-1}\theta}$ tends to ∞ as $x \downarrow \infty$ in such a way that $r \to +\infty$ and $\theta \to 0$. In the second case,

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[31] The *star* means complex conjugate. D stands for differentiation with regard to x.

$$|h_+|^{-2} = r^{-2}|Ph_+|^2 = r^{-2}[Dre^{\sqrt{-1}\theta} + (\lambda - \lambda_0)^*|h_+|^2]$$

$$= \left[\frac{r'}{r^2} + \frac{\sqrt{-1}\,\theta'}{r}\right]e^{\sqrt{-1}\theta} + \frac{(\lambda - \lambda_0)^*}{r^2}|h_+|^2$$

$$= \frac{r'}{r^2}\cos\theta - \frac{\theta'}{r}\sin\theta + \text{a summable function},$$

upon taking the real part. Now the imaginary part of the first formula reveals that $r\sin\theta$ is monotone and bounded so that $(r\sin\theta)'$ is summable. It follows that

$$\frac{\cos\theta}{|h_+|^2} = \frac{r'}{r^2} - \frac{\sin\theta}{r^2}(r\sin\theta)' + \text{a summable function}$$

is itself summable, so $\int_0^\infty |h_+|^{-2} < \infty$, contradicting $\int_0^\infty |h_+|^2 < \infty$. The proof is finished.

Item 4. *The addition of* $\mathfrak{p}_0$ *is a unimodular isospectral transformation.*

PROOF. By item 3, Ph_- and Ph_+ play the role of h_- and h_+ for $Q^{\mathfrak{p}_0}$, after accounting for the fact that $[Ph_-, Ph_+] = \lambda - \lambda_0$ is not unity; in particular, spec $Q^{\mathfrak{p}_0}$ is confined to $[0, \infty)$, and

$$M^{\mathfrak{p}_0} = \frac{1}{\lambda - \lambda_0}\begin{bmatrix} 2Ph_-Ph_+ & (Ph_-Ph_+)' \\ (Ph_-Ph_+)' & 2(Ph_-)'(Ph_+)' \end{bmatrix} \quad \text{taken at } x = 0$$

is the fundamental matrix of $Q^{\mathfrak{p}_0}$, permitting the evaluation of its 2×2 spectral weight as

$$dF^{\mathfrak{p}_0}(\lambda = a) = \lim_{b \downarrow 0} \operatorname{imag} M^{\mathfrak{p}_0}(a + \sqrt{-1}\,b)\,da = G\,dF\,G^{\dagger}$$

with the unimodular factor

$$G = \frac{1}{\sqrt{\lambda - \lambda_0}}\begin{bmatrix} c & -1 \\ \lambda - \lambda_0 - c^2 & c \end{bmatrix}, \qquad c = \frac{e_0'(0)}{e_0(0)}.$$

The computation is routine in view of $M^{\mathfrak{p}_0} = GMG^{\dagger}$.

EXAMPLE 1. If $Q = -D^2$, then $dF(\lambda)$ is the diagonal matrix $[\lambda^{-1/2}, \lambda^{+1/2}]\,d\lambda$ and $G\,dF\,G^{\dagger} = dF$, in agreement with $D^2 \lg e_0 = 0$: in brief, *addition has no effect.*

EXAMPLE 2. The proviso that $\lambda(\mathfrak{p}_0)$ lies to the left of spec Q may be essential for isospectrality, as the example $Q = -D^2 + x^2 - 1$ shows. $Qe_0 = 0$ with $e_0(x) =$

$\exp(-x^2/2)$ = the ground state, h_- and h_+ being coincident, and the addition of $\mathfrak{p}_0 = (0, \pm)$ shifts the whole spectrum 2 units: $Q \to Q - 2D^2 \lg e_0 = -D^2 + x^2 + 1$. Contrariwise, if $Q = -D^2 +$ a positive compact function, then spec $Q = [0, \infty)$, $Qh = 0$ has 2 independent positive solutions $h_- = 1$ near $-\infty$ and $h_+ = 1$ near $+\infty$, and both additions $(0, \pm)$ are unimodular isospectral, as in item 4.

Item 5. The effect of repeated additions is described next. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be distinct points. The function $e(x, \mathfrak{p}_2)$ associated to $Q^{\mathfrak{p}_1}$ is proportional to $e_1^{-1}[e_2, e_1]$ in which $e_1 = e(x, \mathfrak{p}_1)$ and $e_2 = e(x, \mathfrak{p}_2)$ are now formed for Q itself. It follows that

$$A^{\mathfrak{p}_2} A^{\mathfrak{p}_1} Q = Q - 2D^2 \lg e_1 - 2D^2 \lg e_1^{-1}[e_2, e_1] = Q - 2D^2 \lg [e_1, e_2].$$

The more general formula[32]

$$A^{\mathfrak{p}_1} \dots A^{\mathfrak{p}_n} Q = Q - 2D^2 \lg [e(x, \mathfrak{p}_1), \dots, e(x, \mathfrak{p}_n)]$$

is obtained by induction; in particular, addition is *commutative* and *invertible*, the inverse to $A^{\mathfrak{p}}$ being $A^{-\mathfrak{p}}$ formed with the point $-\mathfrak{p}$ having the opposite signature to $\mathfrak{p}$ but the same projection.

Item 6. *$A^{\mathfrak{p}}$ approximates the identity as the projection $\lambda(\mathfrak{p})$ tends to $-\infty$.*

PROOF. Let $Q = -D^2 + q(x)$ and take $\mathfrak{p} = (\lambda, +)$ for instance. Then $D^2 \lg e = q(x) - \lambda - (h'_+/h_+)^2$, so what is needed is the development

$$-\frac{h'_+(x)}{h_+(x)} = (-\lambda)^{1/2} + \frac{1}{2} q(x)(-\lambda)^{-1/2} + etc. \qquad (\lambda \downarrow -\infty).$$

The idea is to write the Green's function G_{00}^{∞} in the form of a Brownian integral:

$$G_{00}^{\infty}(\lambda) = -\frac{h_+(0)}{h'_+(0)} = \int_0^{\infty} e^{\lambda t} (\pi t)^{-1/2} E_{00}(e^{-\mathfrak{Q}})\, dt,$$

in which E_{00} is the expectation for the reflecting Brownian motion $\mathfrak{x}(t)$: $t \geqslant 0$ with infinitesimal operator D^2, conditional on $\mathfrak{x}(0) = 0$ and $\mathfrak{x}(t) = 0$, and $\mathfrak{Q}$ is the integral $\int_0^t q[\mathfrak{x}(t')]\, dt'$. The computation is localized by checking that paths

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[32] $[e_1, \dots, e_n]$ is Wronski's determinant.

with $1 \leqslant \max[\mathfrak{x}(t'): t' \leqslant t]$ do not contribute to the development. I omit the routine details.[33]

Item 7. *Let* $\mathfrak{p}$ *be any point* $(\lambda, +)$ *to the left of* spec Q *and let* $\mathfrak{p}' = (\lambda + \Delta\lambda, +)$. *Then* $A^{\mathfrak{p}'}A^{-\mathfrak{p}}Q = Q - XQ\,\Delta\lambda + \textit{etc.}$ *with* $XQ = 2G'_{xx}(\lambda)$.

PROOF. The composite addition produces $Q - 2H'$ with[34]

$$H = D\lg[h_-(x,\lambda), h_+(x,\lambda+\Delta\lambda)] = D\lg(1 + [h_-, h^{\cdot}_+]\,\Delta\lambda + \textit{etc.})$$
$$= (h^{\cdot}_- h^{\cdot}_+ - h_- h^{\cdot\cdot}_+)\,\Delta\lambda = (h_- Q h^{\cdot}_+ - h^{\cdot}_+ Q h_-)\,\Delta\lambda = h_- h_+\,\Delta\lambda,$$

to leading order, by item 5.

Note 1. The vector fields X: $Q \to 2G'_{xx}(\lambda)$ inherit the commutativity of addition.

Note 2. X appears in item 7 as an *infinitesimal addition.* This is of particular interest in view of the fact that, as in the scattering case, $G_{xx}(\lambda)$ can be developed, as in item 6, in diminishing half-integral powers of $\lambda \downarrow -\infty$, with the conventional KDV fields as coefficients. It follows that *addition commutes with, and shares the invariant manifolds of, the KDV flows when these have any existence; in particular, addition can be viewed as a substitute for the KDV hierarchy under the sole condition that* spec Q *is bounded from* $-\infty$.

Note 3. X can be expressed in commutator format as $XQ = [K, Q]$ in which K is the infinitesimal skew operator[35] $[(3/2)G'_{xx} - DG_{xx}]G$. The skewness of K is equivalent to the vanishing of

$$(Q-\lambda)(K + K^{\dagger})(Q-\lambda) = qD + Dq - (1/2)D^3 - 2\lambda D \quad \text{acting on} \quad G_{xx},$$

which is well known and easily checked. This facilitates the evaluation of the commutator:

$$[K, Q] = K(Q-\lambda) + (Q-\lambda)K^{\dagger} = \frac{3}{2}G'_{xx} - DG_{xx} + \frac{3}{2}G'_{xx} + G_{xx}D = 2G'_{xx}.$$

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[33] McKean-van Moerbeke [1975] will serve as a model.

[34] The *spot* means differentiation with respect to λ.

[35] McKean-van Moerbeke [1975] misstated this without proof. G is the Green's operator $(Q-\lambda)^{-1}$. DG_{xx} is now the operator $G'_{xx} + G_{xx}D$ *not* the function G'_{xx}.

Note 4. The gradient $\partial G_{xx}(\lambda)/\partial q(y) = -G_{xy}^2(\lambda)$ is easily computed; it is symmetric in x and y so that *in the small*, $2G_{xx}(\lambda)$ *is itself a gradient* $\partial H/\partial q(x)$, by Poincaré's lemma, and the field X has the conventional KDV form $XQ = (\partial H/\partial q)'$. H is termed an *integral.* The discussion indicates but does not prove that the additive class of Q should be determined by fixing the values of these integrals for every λ to the left of spec Q. Unfortunately, the application of Poincaré's lemma is only formal in the present very wide generality. The integrals are morally equivalent to the *additive invariant* $D = \sqrt{\det dF}/d\lambda$, or rather to its logarithm, as attested by item 6, art. 4:

$$\partial D/\partial q(x) = -\text{the real part of } G_{xx}(\cdot + \sqrt{-1}\,0+) \times D,$$

but this looks more satisfactory than it really is. For example, in Hill's case, $G_{xx}(\cdot + \sqrt{-1}\,0+)$ is imaginary on spec Q where D lives, and the formula is without content.

Item 8. Concerns an *additive duality*[36] which exchanges the spectral weights of the side operators Q_+^0 and Q_+^∞, and likewise the weights of the side operators Q_-^0 and Q_-^∞ for the left half-line. The weights are recalled from art. 2: df_+^0 represents $\operatorname{imag} h'_+(0)/h_+(0)$, df_+^∞ represents $-\operatorname{imag} h_+(0)/h'_+(0)$, and so on. I take[37] the ground state $\lambda_0 =$ bottom spec Q_+^∞ to the left of $0 =$ bottom spec Q_-^∞ to fix ideas and consider the effect of *adding the point* $\mathfrak{p}_0 = (\lambda_0, +)$. Then $h'_-(0)$ vanishes so $h'_-(0)$ does not, and it is the result of a brief computation that the addition produces a *duality of spectral weights*: with $m = -h'_-(0)/h_-(0)$ evaluated at the ground state, you find

$$\begin{aligned}
&df_+^\infty \to (\lambda - \lambda_0)^{-1} df_+^0 \\
&df^0 \to (\lambda - \lambda_0)\, df_+^\infty \qquad \text{knocking out the weight of the ground state} \\
&df_-^\infty \to (\lambda - \lambda_0)^{-1} df_-^0 \qquad \text{plus the positive mass } m \text{ at the ground state} \\
&df_-^0 \to (\lambda - \lambda_0)\, df_-^\infty .
\end{aligned}$$

The ground state migrates from Q_+^∞ to Q_-^∞. A further addition of $\mathfrak{p}_0 = (\lambda_0, -)$ reproduces Q: in short, *duality is involutive.* A similar duality holds if bottom spec $Q_+^\infty =$ bottom spec $Q_-^\infty = 0$. Then $e_1(x, 0)$ does not vanish and may be used in place of the former ground state. The computation is the same.

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[36] McKean [1985] treats Hill's case; see also Isaacson-McKean-Trubowitz [1984].

[37] The spectra of Q_-^∞ and Q_+^∞ cannot both extend to the left of 0; see art. 2 under *side operators.*

4. UNIMODULAR CLASSES

The present article investigates the unimodular spectral class with special attention to the invariant $\sqrt{\det dF}$. *The fundamental matrix* M is written $A + \sqrt{-1}\ B$ with $A = [a_{ij}: 1 \leqslant i,j \leqslant 2]$ and $B = [b_{ij}: 1 \leqslant i,j \leqslant 2]$.

Item 1. $\sqrt{\det dF}$ is independent of the choice of origin $x = 0$, as noted in art. 2; in particular, *the unimodular class is closed under translation.*

Item 2. $\sqrt{\det B}$ *is a superharmonic function in the open upper half-plane.*

PROOF. Let do be the uniform distribution on the perimeter of a circle C in the open half-plane. The interval function

$$D(I) = \sqrt{\det \int_I B\, do}$$

is defined for circular arcs I: it is superadditive, B being symmetric and positive, whence[38]

$$\sqrt{\det B(\text{center})} = D(C) \geqslant \lim_{N\uparrow\infty} \sum_{n=1}^{N} D(I_n) = \int_C \sqrt{\det B}\, do.$$

Item 3.

$$H(\lambda = a + \sqrt{-1}\ b) = \frac{b}{\pi} \int [(\lambda' - a)^2 + b^2]^{-1} \sqrt{\det dF}\ (\lambda')$$

is the greatest harmonic minorant of $\sqrt{\det B}$; *in particular,*

$$\sqrt{\det dF}\ (a) = \lim_{b\downarrow 0} \sqrt{\det B}\,(a + \sqrt{-1}\, b)\, da.$$

PROOF. The superadditivity employed in item 2 is valid for horizontal lines as well, whence the interval function

$$D(I) = \sqrt{\det \frac{b}{\pi} \int_I [(\lambda' - a)^2 + b^2]^{-1}\, dF(\lambda')}$$

formed for fixed $a + \sqrt{-1}\, b$, satisfies[39]

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[38] B is harmonic. C is divided into N equal arcs $I_n: n \leqslant N$.

[39] $\mathbb{R}$ is divided into small intervals I_n. The final step is by routine approximation.

$$\sqrt{\det B} = D(R) \geqslant \Sigma D(I_n) \downarrow \frac{b}{\pi} \int [(\lambda' - a)^2 + b^2]^{-1} \sqrt{\det dF}\,(\lambda') = H,$$

so that H is overestimated by the harmonic minorant m of $\sqrt{\det B}$: $H \leqslant m$. But also, and for the same reason,

$$\int_I m\,da \leqslant \int_I \sqrt{\det B}\,da \leqslant \sqrt{\det} \int_I B\,da = \sqrt{\det F(I)} + o(1) \quad \text{as} \quad b \downarrow 0$$

for most intervals I, and as the left side in additive while the right side is superadditive, so $m \leqslant H$. The rest is routine.

Item 4.

$$\det B = [h^*_-(0), h_-(0)][h^*_+(0), h_+(0)] = 1 - |[h^*_-(0), h_+(0)]|^2 \leqslant 1;$$

in particular, $\sqrt{\det dF} \leqslant d\lambda$, *and the density* $D = \sqrt{\det dF}/d\lambda$ *is the limiting value* $\sqrt{\det B}\,(\lambda + \sqrt{-1}\,0+)$ *at almost every point of* $[0, \infty)$. The computation is routine.

Item 5. Interprets the density $D = \sqrt{\det dF}/d\lambda$ of item 4 *as the modulus of a (mean) transmission coefficient*, as advertised in art. 1.

DISCUSSION. $D = \sqrt{\det dF}/d\lambda = |s_{11}|$ in the scattering case; see example 1 of art. 5, below. The Jost functions f_- and f_+ and the scattering matrix $[s_{ij}: 1 \leqslant i, j \leqslant 2]$ figuring in that computation depend for their definition upon the posibility of standardizing eigenfunctions at $\pm\infty$, as in

$$f_+(x) = s_{11} \exp(\sqrt{-1}\,kx) + o(1) \quad \text{at} \quad x = +\infty,$$

but this can be side-stepped, even in the most general case. The trick is to standardize the functions h_- and h_+ by $h_-(0) = h_+(0) > 0$ for $\lambda < 0$, keeping $[h_-, h_+] = 1$; then $h_-(0) = h_+(0)$ everywhere off the cut $[0, \infty)$. Now the harmonic functions m_{11}, $-1/m_{11}$, and m_{12} have finite limiting values at almost every point of $[0, \infty)$, so the same is true of $h_-(0) = h_+(0)$, $h'_-(0) + h'_+(0) = = m_{12}/h_+(0)$, and $h'_-(0) - h'_+(0) = 1/h_+(0)$, and so also of $h_-(x)$ and $h_+(x)$, independently of $x \in \mathbb{R}$, $[h_-, h_+] = 1$ being maintained on the cut. The values h_- and h_+ from the upper bank of the cut and likewise the values h^*_- and h^*_+ from the lower bank provide a base of solutions of $Qh = \lambda h$, so you can patch them across the cut:

$$h_+ = r_{11} h^*_- + r_{12} h^*_+$$
$$h_- = r_{21} h^*_- + r_{22} h^*_+$$

with a matrix $[r_{ij}: 1 \leqslant i, j \leqslant 2]$ reminiscent of the scattering matrix in its role of patcher of Jost functions. You read off

$$r_{11} = [h_+, h_+^*], \qquad r_{12} = [h_-^*, h_+] = r_{21}^*, \qquad r_{22} = -[h_-, h_-^*],$$

and $-r_{11}r_{22} + |r_{12}|^2 = 1$ by item 4, so that the density

$$\sqrt{\det dF}/d\lambda = \lim_{b \downarrow 0} \sqrt{\det B} = \sqrt{-r_{11}r_{22}} = \sqrt{1 - |r_{12}|^2}$$

is seen as the modulus of a (mean) transmission coefficient.

EXAMPLE. In the scattering case,

$$h_- = c_- f_-, \qquad h_+ = c_+ f_+,$$

and with $c_- = c$ for short, you find $-2\sqrt{-1}\, k s_{11} = 1/c_- c_+$ and[40]

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} -1/2\sqrt{-1}\, k|c|^2 & s_{21}^* c^*/c \\ s_{21} c/c^* & 2\sqrt{-1}\, k|s_{11}|^2|c|^2 \end{bmatrix};$$

incidentally, $(4k)^{-1} \leqslant |c|^2 \leqslant 4|s_{11}|^2$ on $[0, \infty)$, so that $(1 + k^2)^{-1} \lg |c|$ is summable and c extends to a non-vanishing Hardy function off the cut.

Item 6. There is a close but untransparent connection between $\sqrt{\det dF}$ and the vector fields $X: Q \to 2G'_{xx}(\lambda)$ of arts. 1 and 3: with $D = \sqrt{\det dF}/d\lambda$ as before,

$$\frac{\partial D(\lambda)}{\partial q(x)} = -\text{the real part of } G_{xx}(\lambda + \sqrt{-1}\, 0+) \times D \quad \textit{on the cut};$$

compare note 4, art. 3.

PROOF. D is insensitive to translation so it suffices to compute at $x = 0$. Let $Q^\#$ be the operator $Q + c \times$ the unit mass at $x = 0$, with variable $-1 < c < 1$. This falls outside the class of operators permitted before but never mind. Now if the origin is taken at $x = 0+$, then with $H = 1 + ch_-(0)h_+(0)$,

$$h_-^\#(0) = H^{-1}h_-(0), \qquad h_+^\#(0) = h_+(0),$$
$$h_-^{\#\prime}(0) = H^{-1}[h'_-(0) + ch_-(0)], \qquad h_+^{\#\prime}(0) = h'_+(0),$$

while if it is taken at $x = 0-$, then

$$h_-^\#(0) = h_-(0), \qquad h_+^\#(0) = H^{-1}h_+(0),$$
$$h_-^{\#\prime}(0) = h'_-(0), \qquad h_+^{\#\prime}(0) = H^{-1}[h'_+(0) - ch_+(0)],$$

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[40] $\lambda = k^2 \geqslant 0$.

and it turns out (as it must) that the 2 determinations of $\det B^{\#}$ are the same: $\det B^{\#} = H^{-2} \det B$. It follows that

$$D^{\#} = |1 + (c/2)m_{11}|^{-1} D$$

with $m_{11} = m_{11}(\bullet + \sqrt{-1}\,0+)$ and[41]

$$\frac{\partial D}{\partial q(0)} = \frac{\partial D^{\#}}{\partial c} \quad \text{evaluated at } c = 0$$

$$= \frac{\partial}{\partial c}\left[\left(1 + \frac{c}{2}a_{11}\right)^2 + \frac{c^2}{4}b_{11}^2\right]^{-1/2} D = (a_{11}/2)\,D,$$

as promised. The computation is a bit formal but reliable, so I leave it at that.

Item 7 is a test case. I proposed in art. 1 that the unimodular spectral class of Q and its additive class must be one and the same thing. I prove it now for $Q^0 = -D^2$ whose additive class is the singleton Q^0 itself.

PROOF. The spectral weight of Q^0 is $dF^0 = \operatorname{diag}[\lambda^{-1/2}, \lambda^{+1/2}]\,d\lambda$. Let dF be the spectral weight of an operator Q from the same unimodular class as Q^0 so that $sp\,dF$ belongs to the Lebesgue class on $[0, \infty)$ and $D = \sqrt{\det dF}/d\lambda$ is the indicator thereof. Then[42]

$$p * D = \frac{b}{\pi}\int_0^{\infty}[(\lambda' - a)^2 + b^2]^{-1}\,d\lambda', \quad \text{taken at} \quad \lambda = a + \sqrt{-1}\,b,$$

is the harmonic minorant of $\sqrt{\det B}$, and for any interval $I \subset [0, \infty)$, $\int_I |[h_-, h_+^*]$ taken at $x = 0$ and

$$a + \sqrt{-1}\,b|^2\,da \leqslant 2\int_I (1 - \sqrt{\det B}\,)^{[43]}$$

$$\leqslant 2\int_I (1 - p * D)$$

$$= \frac{2b}{\pi}\int_I da \int_{-\infty}^{0} [(\lambda' - a)^2 + b^2]^{-1}\,d\lambda'$$

$$= \frac{2b}{\pi}\int_I da \int_a^{\infty} (c^2 + b^2)^{-1}\,dc$$

$$\leqslant \frac{2b}{\pi}\int_I a^{-1}\,da,$$

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[41] $m_{11} = a_{11} + \sqrt{-1}\,b_{11}$.

[42] p stands for $(b/\pi)[(\lambda' - a)^2 - b^2]^{-1}$. The *star* signifies *convolution*.

[43] $1 - r^2 = (1 - r)(1 + r) \leqslant 2(1 - r)$ if $0 \leqslant r \leqslant 1$.

whence

$$\left|\operatorname{imag}\int_I \left[\frac{h'_-}{h_-} - \frac{h^{*\prime}_+}{h^*_+} \text{ taken at } x=0 \text{ and } a+\sqrt{-1}\,b\right] da\right|^2$$

$$\leqslant \int_I |[h_-, h^*_+]|^2\, da \int_I |h_- h_+|^{-2}\, da$$

$$\leqslant \frac{2}{\pi}\int_I \frac{da}{a}\int_I \frac{4b_{11}}{|m_{11}|^2}\,\frac{b}{b_{11}}\,da^{(41)}$$

$$= \frac{2}{\pi}\int_I \frac{da}{a}\, 4\int_I \operatorname{imag}(-1/m_{11}) \times \left[\frac{1}{\pi}\int_0^\infty [(c-a)^2+b^2]^{-1}\, df_{11}(c)\right]^{-1}.$$

Now introduce the representing measures df^0_- of $-\operatorname{imag} h'_-(0)/h_-(0)$, df^0_+ of $\operatorname{imag} h'_+(0)/h_+(0)$, and df_{00} of $-1/m_{11}$, and make $b\downarrow 0$ in the preceding display to obtain *a*) the bound

$$|f^0_+(I) - f^0_-(I)|^2 \leqslant \frac{2}{\pi}\int_I \frac{da}{a}\, 4\int_I \left[\frac{1}{\pi}\int_0^\infty (c-a)^{-2}\, df_{11}(c)\right]^{-1} df_{00}(a),$$

b) an estimate $|df^0_+ - df^0_-| \leqslant$ a multiple of $a^{-1/2}\sqrt{da\, df_{00}}$ in the small, and *c*) the conclusion that df^0_- and df^0_+ have the same singular part, except perhaps for different jumps at 0. Now df^0_- and df^0_+ have the same non-singular parts as well: indeed, at almost every point of the cut $[0, \infty)$, the densities $f'_- = df^0_-/da$ and $f'_+ = df^0_+/da$ satisfy

$$f'_+ - f'_- = \lim_{b\downarrow 0} \operatorname{imag}\left[\frac{h'_-}{h_-} - \frac{h^{*\prime}}{h^*_+}\right] = \lim_{b\downarrow 0}\frac{[h_-, h^*_+]}{h_- h^*_+} = 0$$

in view of the fact that $m_{11} = 2h_- h_+$ has a non-vanishing limiting value and the estimate at the start of the proof:

$$\int_I |[h_-, h^*_+]|^2 \frac{da}{a} \leqslant \frac{2b}{\pi}\int_I \frac{da}{a} = o(1)$$

as $b\downarrow 0$ if I does not extend to 0. Note, finally, that neither df^0_- nor df^0_+ can jump at 0; otherwise, the representing measure df_{00} of (1/2) imag $[h'_+/h_+ - h'_-/h_-] = \operatorname{imag}(-1/m_{11})$ has a jump (at 0) so that

$$-1/m_{11} \geqslant \text{a constant} + (-\lambda)^{-1} f_{00}[0] \uparrow \infty \quad \text{as} \quad \lambda\uparrow 0,$$

contradicting the fact that m_{11} is positive there. The upshot is that $df^0_- = df^0_+$, which is to say that *Q is symmetrical about* $x = 0$. But the unimodular class is invariant under translation, so the same is true for any choice of origin,

whence $Q = -D^2 +$ a constant function, and the constant vanishes because spec Q starts at 0. The proof is finished.

5. EXAMPLES

EXAMPLE 1. Scattering case.[44] $[s_{ij}(k): 1 \leqslant i,j \leqslant 2]$ is the scattering matrix defined for real $k = \sqrt{\lambda}$, with values in $U(2)$; it is related to the behaviour at infinity of the Jost functions f_- and f_+, as in the table. For $k \geqslant 0$, f_- and f_+ are the values of h_- and h_+ along the upper bank of the cut $[0, \infty)$, except that $[f_-, f_+] = -2\sqrt{-1}\, k s_{11}(k)$ is not unity.

	$x \downarrow -\infty$	$x \downarrow +\infty$
f_-	$s_{22} e^{-\sqrt{-1}kx}$	$e^{-\sqrt{-1}kx} + s_{21} e^{\sqrt{-1}kx}$
f_+	$e^{\sqrt{-1}kx} + s_{12} e^{-\sqrt{-1}kx}$	$s_{11} e^{\sqrt{-1}kx}$

The 2×2 spectral weight is comprised of jumps at the bound states and a pure Lebesgue part on $[0, \infty)$ with density

$$\frac{dF}{d\lambda} = \text{the real part of } \frac{1}{k s_{11}} \begin{bmatrix} 2f_- f_+ & (f_- f_+)' \\ (f_- f_+)' & 2f'_- f'_+ \end{bmatrix} \text{ taken at } x = 0.$$

The additive invariant (second measure class) $\sqrt{\det dF}$ is readily computed: it is unchanged by translation, so the entries of the table may be used in place of f_- and f_+, with the outcome

$$\begin{aligned} \frac{\sqrt{\det dF}}{d\lambda} &= \sqrt{\det} \begin{bmatrix} k(1 + \operatorname{real} s_{21} e^{2\sqrt{-1}kx}) & -\operatorname{imag} s_{21} e^{2\sqrt{-1}kx} \\ -\operatorname{imag} s_{21} e^{2\sqrt{-1}kx} & k^{-1}(1 - \operatorname{real} s_{21} e^{2\sqrt{-1}kx}) \end{bmatrix} \\ &= \sqrt{1 - |s_{21}|^2} \\ &= |s_{11}|. \end{aligned}$$

The KDV *invariant manifold* is determined by fixing the transmission coefficient s_{11}, which is to say *by the fixing the bound states* $-k_1^2 < \cdots \leqslant -k_g^2$ *and*

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[44] Faddeev [1966] and/or Deift-Trubowitz [1979] are cited for background.

the modulus $|s_{12}| = \sqrt{1 - |s_{11}|^2}$ because s_{11} encodes just this information:

$$s_{11}(k) = \exp\left[\frac{1}{\pi\sqrt{-1}}\int_{-\infty}^{\infty}\frac{\lg|s_{11}(k')|}{k'-k}\,dk'\right]\prod_{i=1}^{g}\frac{k+\sqrt{-1}\,k_i}{k-\sqrt{-1}\,k_i};$$

also, it is known that the phase s_{12} together with the logarithms of the norming constants

$$c_i^2 = \int_{-\infty}^{\infty}|f_+/s_{11}|^2 \quad \text{taken at} \quad k = \sqrt{-1}\,k_i \qquad (i \leqslant g)$$

serve as (additive) coordinates on the manifold. Now to elucidate the effect of addition, take $-k_0^2$ to the left of spec Q and let $\mathfrak{p}_0 = (-k_0^2, +)$ to fix the ideas. Then $e_0(x) = e(x, \mathfrak{p}_0) = f_+(x, \sqrt{-1}\,k_0) = s_{11}(\sqrt{-1}\,k_0)\exp(-k_0 x)$ at $+\infty$ with a similar exponential behavior at ∞: in fact, $[\lg e_0(x)]''$ vanishes rapidly at $\pm\infty$, so *adding* $\mathfrak{p}_0$ *not only keeps you in the scattering case but also preserves the KDV manifold* in view of note 2, item 7 of art. 3. The effect of addition upon the coordinates is found by comparison of new eigenfunctions $e_0^{-1}[e_0, f_+]$ to old:

$$e_0^{-1}[e_0, f_+] \simeq e^{k_0 x}[e^{-k_0 x}, s_{11}e^{\sqrt{-1}kx}] = -(k_0 + \sqrt{-1}\,k)s_{11}e^{\sqrt{-1}kx}$$

at $+\infty$; similarly,

$$e_0^{-1}[e_0, f_+] \simeq (-k_0 + \sqrt{-1}\,k)s_{12}e^{-\sqrt{-1}kx} - (k_0 + \sqrt{-1}\,k)e^{\sqrt{-1}kx}$$

at ∞, so

$$s_{12} \to \frac{k_0 - \sqrt{-1}\,k}{k_0 + \sqrt{-1}\,k}s_{12}.$$

A similar rule applies to the bound states, the ith norming constant being multiplied by $(k_0 + k_i)(k_0 - k_i)^{-1}$ $(i \leqslant g)$; in particular, *repeated additions followed by closure produce the whole invariant manifold*, as you will easily check.

EXAMPLE 2. Hill's case. Now h_- and/or h_+ is proportional to the so-called *Baker-Akhiezer function*

$$e(x, \mathfrak{p}) = e_1(x, \lambda) + e_2^{-1}(1, \lambda)[m(\mathfrak{p}) - e_1(1, \lambda)]e_2(x, \lambda)$$

taken at the point $\mathfrak{p} = (\lambda, \pm)$ of the *multiplier curve* of Q. The latter is the Riemann surface of the 2-valued *multiplier*[45] $m(\mathfrak{p}) = \Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 1}$; the

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[45] Δ is the discriminant $(1/2)\,[e_1(1, \lambda) + e_2'(1, \lambda)]$.

former is the solution of $Qe = \lambda(\mathfrak{p})e$ with $e(x+1) = m(\mathfrak{p})e(x)$ and $e(0) = 1$. The point is that, for λ to the left of spec Q, the number Δ exceeds $+1$, so that the multipliers satisfy $0 < m_+ < 1$ and $1 < m_- < \infty$, with the result that $e(x, \mathfrak{p})$ belongs to $L^2[0, \infty)$ if $m(\mathfrak{p}) = m_+$ and to $L^2(-\infty, 0]$ if $m(\mathfrak{p}) = m_-$. Now the KDV invariant manifold may be described *either* as the class of Hill's operators with fixed periodic/anti-periodic spectrum *or* as (the real part of) the Jacobi variety of the multiplier curve; moreover, $[\lg e(x, \mathfrak{p})]''$ is of period 1, *so addition not only keeps you in the Hill's case but also preserves the KDV invariant manifold*, just as in example 1. Kodaira [1949] computed the 2×2 spectral weight: it is pure Lebesgue with density

$$\frac{dF}{d\lambda} = \frac{\pm 1/2}{\sqrt{1-\Delta^2}} \begin{bmatrix} 2e_2(1,\lambda) & e_2'(1,\lambda) - e_1(1,\lambda) \\ e_2'(1,\lambda) - e_1(1,\lambda) & -2e_1'(1,\lambda) \end{bmatrix}$$

on the bands of spec Q; the signatures alternate starting with $+1$. The additive invariant $\sqrt{\det dF}$ is easily elicited:

$$\frac{\sqrt{\det dF}}{d\lambda} = \frac{-4e_2e_1 - (e_2' - e_1)^2}{4(1-\Delta^2)} = 1 \text{ on spec } Q;$$

it determines spec Q and so also the discriminant Δ, the multiplier curve M, and its Jacobian J. The use of the word *addition* can now be fully justified; it will be a by-product of the discussion that *repeated additions* (*followed by closure*) *are transitive on the invariant manifold*, just as in the scattering case.

DISCUSSION. Let $\mathfrak{p}_1 + \mathfrak{p}_2 + \cdots$ be the divisor of Q introduced in art. 1, the points being the poles of $e(x, \mathfrak{p})$ determined by the vanishing of $e_2(1, \lambda)$ and the choice of multiplier $m(\mathfrak{p}) = e_2'(1, \lambda)$ instead of the other possibility $e_1(1, \lambda)$, one such to each nontrivial gap $[\lambda_n^-, \lambda_n^+]$ $(n \geqslant 1)$. Now the operator $Q - 2D^2 \lg e(x, \mathfrak{p}_0)$ produced by addition of a point $\mathfrak{p}_0$ with projection to the left of spec Q has also such a divisor $\mathfrak{p}_1' + p_2' + \cdots$, and *the recipe of addition states that the old* (*umprimed*) *divisor with*[46] $-\mathfrak{p}_0$ *adjoined is linearly equivalent to the new* (*primed*) *divisor with* ∞ *adjoined*:

$$-\mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2 + \cdots = \infty + \mathfrak{p}_1' + \mathfrak{p}_2' + \cdots \quad \text{in} \quad J.$$

The proof is divided into several steps.

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[46] $-\mathfrak{p}_0$ is the point on the sheet opposite to that of $\mathfrak{p}_0$.

Step 1. The points $\mathfrak{p}_1, \mathfrak{p}_2$, *etc.* are the poles of the function

$$f(\mathfrak{p}) = e_1^{-1}(1, \lambda)[m(\mathfrak{p}) - e_1(1, \lambda)],$$

aside from an extra pole at ∞ which is detected by the development of $f(\mathfrak{p})$ at $\lambda(\mathfrak{p}) = -\infty$:

$$f(\mathfrak{p}) = \frac{e^k - \operatorname{ch} k}{\operatorname{sh} k/k}[1 + o(1)] = k[1 + o(1)]$$

if $k = +\sqrt{-\lambda}$.

Step 2. The preceding development of $f(\mathfrak{p})$ is applied to the operator $Q^{\mathfrak{p}_0}$. The corresponding function is[47]

$$f_0(\mathfrak{p}) = \frac{m(\mathfrak{p}) - e_1^{\mathfrak{p}_0}(1, \lambda)}{e_2^{\mathfrak{p}_0}(1, \lambda)} = \frac{m(\mathfrak{p}) - e_0^{-1}[e_2, e_0] \text{ taken at } x = 1}{e_2^{\mathfrak{p}_0}(1, \lambda)} - c;$$

its poles are p_1', p_2', *etc.* and ∞, by definition.

Step 3. $f_0(\mathfrak{p})$ takes the value $-c$ at the points $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2$, *etc.* of the divisor of Q and also at $-\mathfrak{p}_0$ in view of

$$e_0^{-1}[e_2, e_0] \quad \text{at } x = 1 \quad = e_2'(1) - ce_2(1) = m(\mathfrak{p})$$

in the first case and

$$\frac{[e_2, e_0] \text{ at } x = 1}{e_0(1)} = \frac{[e_2, e_0] \text{ at } x = 0}{e_0(1)} = \frac{1}{e_0(1)} = \frac{1}{m(\mathfrak{p}_0)} = m(-\mathfrak{p}_0)$$

in the second.

Step 4 is to notice that $f_0(\mathfrak{p}) = -c$ has no other roots. This follows from the evaluation $(2\pi\sqrt{-1})^{-1}\int d\lg f_0(\mathfrak{p}) = 1$ for small circles about $\mathfrak{p} = \infty$; compare step 1. The upshot is that $-\mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2 +$ *etc.* and $\infty + \mathfrak{p}_1' + \mathfrak{p}_2' +$ *etc.* are the roots and poles of the function $f_0(\mathfrak{p}) + c$, which is what linear equivalence is all about, anyhow.

EXAMPLE 3. Bohr's case.[48] I cannot do so much if Q is only almost periodic in the sense of Bohr [1932]. Fix λ off spec Q. Then $G = G_{xx}(\lambda)$ is an almost

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[47] $e_1^{\mathfrak{p}_0} = ce_2^{\mathfrak{p}_0} + e_0^{-1}[e_2, e_0]$ with $c = e_0'/e_0$ taken at $x = 0$.

[48] Johnson-Moser [1982] and Moser-Pöschl [1984] are cited for background.

periodic function with the same frequency module as Q, and

$$h_{\mp} = \sqrt{G}\exp\left[\pm(1/2)\int_0^x G^{-1}\,dx'\right],$$

by elementary computation[49] from which it follows that *addition preserves*:

1) *the almost periodicity*,
2) *the frequency module*,

and

3) *the rotation number*

$$r(\lambda) = \lim_{x\uparrow\infty} x^{-1}\,\text{imag lg}\,h_+(x,\lambda) = \text{the mean value of imag} - 1/2G \text{ off spec } Q.$$

PROOF OF 3. Addition of $\mathfrak{p}_0$ changes h_+ into $e_0^{-1}[h_+, e_0]$ so that $\text{imag lg}\,h_+$ is changed by the addition of

$$\text{imag lg}\left[\frac{h'_+}{h_+} - \frac{e'_0}{e_0}\right],$$

and this is bounded off spec Q because $\text{imag}\,h'_+/h_+$ is of one signature while e'_0/e_0 is real.

The values of $r(\bullet + \sqrt{-1}\,0+)$ on the line are known to determine spec Q and the frequency module,[50] whence it is natural to conjecture that *r and the bound states (if any) determine the additive class*, but this may be naive.

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[49] $KG = 2\lambda G'$ with $K = qD + Dq - (1/2)D^3$, whence $(G')^2 - 2G''G + 4(q-\lambda)G^2 = 1$.

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On Kolchin's Theorem

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To Alberto Calderón, a wonderful friend and colleague, and a superb mathematician

A well-known theorem due to Kolchin states that a semi-group, G, of unipotent matrices over a field F can be brought to triangular form over the field F [4, Theorem H]. Recall that a matrix A is called *unipotent* if its only eigenvalue is 1, or, equivalently, if the matrix I-A is nilpotent.

Many years ago I noticed that this result of Kolchin is an immediate consequence of a too-little known result due to Wedderburn [6]. This result of Wedderburn asserts that if B is a finite dimensional algebra, over a field F, which has a basis consisting of nilpotent elements then B itself must be nilpotent, that is, $B^k = (0)$ for some positive integer k.

To see how this result of Wedderburn implies that of Kolchin we proceed as follows. Let F_n be the algebra of $n \times n$ matrices over F and let $S \subset F_n$ be the linear span over F of the elements $I - g$ where $g \in G$. Since, for g and h in G we have that $(I - g)(I - h) = (I - g) + (I - h) - (I - gh)$, S is a subalgebra of F_n. Since S has a basis over F of the form $I - g_i$ for some appropriate g_i in G and each g_i is nilpotent by the hypothesis that $I - g$ is unipotent for every g in G, S has a basis consisting of nilpotent elements. Thus, by the theorem of Wedderburn, S is nilpotent, hence $S^k = (0)$ and $S^{k-1} \neq (0)$ for some integer $k \geqslant 1$. Therefore if $u \neq 0$ is in S^{k-1} then we have that $u(I - g) \in S^{k-1}S = S^k = (0)$ for all $g \in G$. This gives us that there is a common eigenvector for all the elements of G. So every g in G can be brought to the form $\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$ and as is easy to see, the g form a semi group of unipotent matrices lying in F_{n-1}. By induction on n we thus obtain Kolchin's theorem.

This theorem of Kolchin is capable of extensions to rings satisfying a polynomial identity.

Let R be a ring with unit satisfying a polynomial identity. Call an element u of R *unipotent* if $1 - u$ is nilpotent, that is, if $(1 - u)^n = 0$ for some n depending on u. Let G be a multiplicative sub-semigroup of R which consists of unipotent elements. Again, if we let S be the additive subgroup of R generated by all the $1 - g$ for all g in G, then, as above, S is a subring of R. Every element of S is a sum of nilpotent elements, and since $S \subset R$ satisfies a polynomial identity, by a theorem of Posner [5], every element of S is nilpotent. Again, since S satisfies a polynomial identity, S must be locally nilpotent [2, Corollary 1 to Lemma 5.4]. So, given any finite set of elements $1 - g_1$, $1 - g_2, \ldots, 1 - g_n$ where the g_i are in G, then the subring they generate is nilpotent, hence there is an element $u \neq 0$ in S such that $u(1 - g_i) = 0$ for $1 \leqslant i \leqslant n$. Thus $ug_i = u$ for $1 \leqslant i \leqslant n$.

Let R be a finitely-generated ring with 1 satisfying a polynomial identity, and S a nil subring of R. If $J = J(R)$ is the Jacobson radical of R then, by a beautiful result of Braun [1], J is nilpotent. Let $R' = R/J$ and let S' be the image of S in R'. Since R' is semi-prime (i.e., has no nilpotent ideals), R' can be embedded in C_n, the $n \times n$ matrices over a commutative ring C which has no nilpotent elements [3, Theorem 6.3.2]. We may suppose that C is Noetherian, for if $u_1, \ldots, u_n$ generate R then their images generate R'. Viewing each u_i' as a matrix over C then R' is contained in C', the subring of C generated by all the matrix entries of all the u_i' over the integers.

Moreover C' is Noetherian by the Hilbert Basis Theorem. Thus we may suppose that C is Noetherian without nilpotent elements.

Since C is a commutative Noetherian ring without nilpotent elements there exist prime ideals $P_1, \ldots, P_m$ of C such that $P_1 \cap P_2 \cdots \cap P_m = (0)$. Thus we can embed R' in $(C/P_1)_n \oplus \cdots \oplus (C/P_m)_n$, the direct sum of $n \times n$ matrices over commutative integral domains. If $F^{(i)}$ is the field of quotients of C/P_i then R' is embedded in $(F^{(1)})_n \oplus \cdots \oplus (F^{(m)})_n$; since this latter ring is artinian and since S' is its subring, by a classical result of Hopkins and Levitzki, S' is nilpotent. Since $(S')^k = (0)$ for some k, we have that $S^k \subset J$, and since $J^r = (0)$ for some r, $S^{kr} = (0)$. Thus S is nilpotent. We have proved the

Theorem. *If R is a finitely generated ring satisfying a polynomial identity and S is a nil subring of R then S is nilpotent.*

As an immediate corollary to this theorem we obtain the

Theorem. *If R is finitely generated ring satisfying a polynomial identity and G is a multiplicative sub-semigroup of R consisting of unipotent elements, then there exists an element $u \neq 0$ in R such that $ug = u$ for every g in G.*

PROOF. If S the additive subgroup of R generated by all the $1 - g$ with g in G then, as we showed above, S is a nil subring of R hence, by the theorem above, S is nilpotent. As we showed earlier, this leads to an element $u \neq 0$ in S such that $ug = u$ for every g in G.

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Non-Negative Solutions of Generalized Porous Medium Equations

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Dedicated to Professor A. P. Calderón
on the occasion of his 65th birthday

Introduction

The purpose of this paper is to study nonnegative solutions u of the nonlinear evolution equations

$$\frac{\partial u}{\partial t} = \Delta\varphi(u), \qquad x \in \mathbb{R}^n, \qquad 0 < t < T \leqslant +\infty. \tag{1.1}$$

Here the nonlinearity φ is assumed to be continuous, increasing, with $\varphi(0) = 0$.

The equation (1.1) arises in various physical problems, and specializing φ leads to models for nonlinear filtrations, or for the gas flow in a porous medium. For a recent survey on these equations see [9].

The main object of this work is to study the initial value problem for (1.1). Before stating our results, we will recall the known results in the linear case, i.e., the equation of heat conduction

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in} \quad \mathbb{R}^n \times (0, T]. \tag{1.2}$$

In this case, the Widder theory ([14]) gives a complete description of all the non-negative solutions of (1.2). To each non-negative solution u of (1.2), there corresponds a non-negative Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^n} u(x,t)\theta(x)\,dx = \int_{\mathbb{R}^n} \theta\,d\mu, \tag{1.3}$$

for all continuous functions θ in $\mathbb{R}^n$, with compact support. We will call the measure μ the trace of u. Furthermore, the trace μ satisfies the growth condition

$$\int_{\mathbb{R}^n} e^{-C|x|^2/T}\,d\mu(x) < \infty, \tag{1.4}$$

where C is an absolute constant.

The trace μ determines the solution uniquely, i.e., if u, v are two nonnegative solutions of (1.2) in $\mathbb{R}^n \times (0, T]$, with equal traces, then u is identicallly equal to v. Finally, for each non-negative Borel measure μ on $\mathbb{R}^n$, satisfying the growth condition (1.4) there is a non-negative solution u of (1.2) in $\mathbb{R}^n \times (0, T]$ with trace μ. Of course, by the uniqueness result mentioned before, this solution is unique.

Our aim in this work is to find the analogues of the results of the linear theory mentioned above, for a wide class of non-linearities φ.

In the case of pure powers, i.e., $\varphi(u) = u^m$, $m > 1$, (1.1) becomes the well known porous medium equation, which has been studied extensively. In fact, a combination of the results in [1], [3] and [5] gives the complete analogue of the Widder theory for this case. To simplify the description of these results, we will do it for the case when $T = +\infty$. Aronson and Caffarelli ([1]) showed that for every non-negative solution of $\partial u/\partial t = \Delta u^m$, $m > 1$ in $\mathbb{R}^n \times (0, \infty)$, the initial trace μ exists and satisfies the growth estimate

$$\frac{1}{R^n} \int_{|x| \le R} d\mu(x) = o(R^{2/m-1}) \quad \text{as} \quad R \to \infty. \tag{1.5}$$

Furthermore, Bénilan, Crandall and Pierre ([3]) showed that for every measure μ on $\mathbb{R}^n$ satisfying (1.5) there is a non-negative solution u of $\partial u/\partial t = \Delta u^m$, $m > 1$ in $\mathbb{R}^n \times (0, \infty)$. Finally, Dahlberg and Kenig ([5]) established uniqueness, i.e., two non-negative solutions of $\partial u/\partial t = \Delta u^m$ with equal traces are identical.

The main estimate established in Dahlberg and Kenig [5], was that for all non-negative solutions of the porous medium equation in $\mathbb{R}^n \times (0, \infty)$ we have the pointwise growth

$$u(x,t) \leqslant C_t(u)(1 + |x|^2)^{2/m-1}, \tag{1.6}$$

where

$$C_t(u) = O(t^{-\lambda}), \qquad \lambda = \frac{n}{2 + n(m-1)}$$

as $t \downarrow 0$.

These results constitute a very complete generalization of the Widder theory. However, from the point of view of the general equation (1.1), the proofs of the results mentioned above are not very satisfactory. This is due to several facts. The first one is the use of the scaling properties of the porous medium equation, i.e., the fact that if $\partial u/\partial t = \Delta u^m$ in $\mathbb{R}^n \times (0, \infty)$, then $v(x, t) = (\beta/\alpha)^{1/m-1} u(\alpha x, \beta t)$ also solves the same equation for any $\alpha, \beta > 0$. Another important fact used in these proofs is the existence of an explicit formula for the solution whose initial trace is the δ mass at the origin. This is the Barenblatt solution

$$B(x, t) = t^{[-(m-1)n+2]/n} \left[\left(a - \frac{c|x|^2}{t^{2+1/n(m-1)}} \right)_+ \right]^{1/m-1},$$

where a, c depend only on m, n.

In this work we are able to extend the above results for the following class of non-linearities φ. We work with continuous, strictly increasing φ on $0 \leqslant u < +\infty$, that are positive on $0 < u < +\infty$, with $\varphi(0) = 0$. We also impose the growth conditions

$$\text{(1.7)} \qquad \begin{cases} 0 < a \leqslant \dfrac{u\varphi'(u)}{\varphi(u)} \leqslant \dfrac{1}{a}, & 0 < u < \infty \\ 1 + a \leqslant \dfrac{u\varphi'(u)}{\varphi(u)} & \text{for} \quad u \geqslant u_0, \end{cases}$$

for some constant a, $0 < a < 1$, and some $u_0 > 0$. We will denote by Γ_a the cass of φ's that satisfy (1.7) and the normalization conditions $u_0 = 1$, $\varphi(1) = 1$.

For a φ as in (1.7) we say that u is a solution of (1.1) in a region $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ if u is continuous, and non-negative in Ω, and u solves the equation (1.1) in the distribution sense.

We will base our study of solutions of (1.1) for φ's verifying (1.7), in finding the analogues of (1.6) for them. Our key estimate is that, in this case, we have

$$\text{(1.8)} \qquad \frac{\varphi(u)}{1+u} \leqslant C_t(u)(1 + |x|^2),$$

where $C_t(u) = O(t^{-1+\delta})$ for some $\delta = \delta(\varphi) > 0$. (In fact, this is a corollary of a more precise growth estimate, Theorem 5.5.) Using (1.8) we establish the existence of a trace μ for all non-negative solutions of (1.1) in $\mathbb{R}^n \times (0, \infty)$. We also prove that the solution is uniquely determined by its trace. The main new feature of our approach is that we treat all φ's satisfying (1.7) simultaneously. We thus recover the important scaling properties of the porous medium equation and we dispense with the use of the explicit form of special solutions of the equation.

Finally, in Section 7 we obtain several results about solutions of the initial value problem. First, the trace μ of a positive solution of (1.1) in $\mathbb{R}^n \times (0, \infty)$, with φ verifying (1.7), satisfies the growth estimate

$$\frac{1}{R^n}\int_{|x|<R} d\mu(x) = o(\Lambda(R^2)) \quad \text{as} \quad R \to \infty, \tag{1.9}$$

where Λ is the inverse function of $\varphi(u)/u$, $u \geqslant u_0$. Also, if μ is a non-negative measure on $\mathbb{R}^n$, which satisfies (1.9), there is a unique solution $u \geqslant 0$ of (1.1), with trace μ. We also establish corresponding results for finite strips $\mathbb{R}^n \times (0, T]$, $T < \infty$, together with «blow up» results at T. We refer the reader to the body of the paper for these results.

We would like to conclude these remarks by pointing out that using the estimates in this paper, one can carry out the program of [4] to obtain pointwise limit theorems as $t \downarrow 0$ (Fatou type theorems) for non-negative solutions of (1.1), when φ verifies (1.7).

We have also obtained a complete description of the non-negative solutions of (1.1) in $\Omega \times (0, T]$, which vanish on $\partial\Omega$, where Ω is a bounded Lipschitz domain $\subset \mathbb{R}$, and φ verifies (1.7). We will return to these questions in future publications.

In the rest of this work we will restrict ourselves to φ's in the class Γ_a. The general case of a φ which verifies (1.7) easily follows from this by dilation and division.

2. Preliminary Results

In this section we will establish a preliminary version of the maximum principle. This will be used to give an approximation procedure that justifies our use of the a priori inequalities that we will establish in Section 4. Set $B = \{x \in \mathbb{R}^n: |x - x_0| < r\}$, $\tau_1 < \tau_2$, and let $Q = B \times (\tau_1, \tau_2)$. Denote by $\partial_p Q$ the parabolic boundary of Q, i.e., $\partial_p Q = \bar{Q}\backslash(B \times \{\tau_2\})$.

Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $g \in C(\partial_p Q)$ be a given non-negative function. Consider the boundary value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta\varphi(v) & \text{in} \quad Q \\ v = g & \text{on} \quad \partial_p Q \end{cases} \tag{2.1}$$

A function $v(x, t)$ is said to be a weak solution of (2.1) in Q, if $v \in C([\tau_1, \tau_2], L^1(B)) \cap L^\infty(Q)$ and $v \geqslant 0$ satisfies the integral identity

$$(2.2)\quad \begin{cases} \displaystyle\iint_Q \left[\varphi(v)\Delta\eta + v\frac{\partial\eta}{\partial t}\right] dx\,dt = \int_{\tau_1}^{\tau_2}\int_{\partial B} \varphi(g)\frac{\partial\eta}{\partial N}\,d\sigma\,dt \\ \qquad\qquad + \displaystyle\int_B v(x,\tau_2)\eta(x,\tau_2)\,dx - \int_B g(x,\tau_1)\eta(x,\tau_1)\,dx \end{cases}$$

for all smooth functions η on $\bar{Q}$, which vanish on $\partial B \times [\tau_1, \tau_2]$. Here $\partial/\partial N$ denotes the exterior normal derivative on ∂B, and σ denotes the surface measure on ∂B.

We have the following comparison principle for weak solutions of (2.1). The corresponding result for the case of the porous medium equation was established in [2].

Lemma 2.3. *Let $g_1, g_2 \in C(\partial_p Q)$, and let v_1, v_2 be weak solutions of* (2.1), *with boundary values g_1 and g_2 respectively. If $0 \leqslant g_1 \leqslant g_2$, then $v_1 \leqslant v_2$ in Q.*

PROOF. The main difference between this case and the porous medium case is in the treatment of the situation when the values of the solutions are close to zero.

Fix $s \in (\tau_1, \tau_2]$, and let $A = (\varphi(v_1) - \varphi(v_2))/(v_1 - v_2)$ whenever $v_1 \neq v_2$ and $A = \varphi'(v_1)$ elsewhere. Putting $b = v_1 - v_2$, we see from (2.2) that whenever η is a non-negative test function, with $\eta \equiv 0$ and $\partial\eta/\partial N \leqslant 0$ on $\partial B \times [\tau_1, s]$, then

$$(2.4)\qquad \int_B b(x,s)\eta(x,s)\,dx \leqslant \iint_{Q(s)} b\left[\frac{\partial\eta}{\partial t} + A\,\Delta\eta\right] dx\,dt,$$

where $Q(s) = B \times (\tau_1, s]$.

Let now $h \in C_0^\infty(B)$, $h \geqslant 0$. For $E \in C^\infty(\bar{Q})$, $E > 0$ in $\bar{Q}$, let $\eta = S(E)$ solve the equation

$$(2.5)\qquad \begin{cases} E\,\Delta\eta + \dfrac{\partial\eta}{\partial t} = 0 \quad \text{in} \quad Q(s) \\ \eta = 0 \quad \text{on} \quad \partial B \times [\tau_1, s] \\ \eta(x,s) = h(x), \qquad x \in B \end{cases}$$

By the maximum principle, $\eta \geqslant 0$ in $Q(s)$ and $\partial\eta/\partial N \leqslant 0$ on $\partial B \times [\tau_1, s]$, and so (2.4) holds. We also have that

$$(2.6)\qquad \iint_{Q(s)} E(\Delta\eta)^2\,dx\,dt \leqslant \frac{1}{2}\int_B |\nabla h|^2\,dx,$$

since

$$\iint_{Q(s)} E(\Delta\eta)^2\,dx\,dt = -\iint_{Q(s)} \Delta\eta\frac{\partial\eta}{\partial t}\,dx\,dt$$

$$= -\int_B h\Delta h\,dx + \int_B \eta(x,\tau_1)\Delta\eta(x,\tau_1) + \iint_{Q(s)} \eta\frac{\partial\Delta\eta}{\partial t}$$

$$\leqslant \int_B |\nabla h|^2 - \iint_{Q(s)} E(\Delta\eta)^2\,dx\,dt.$$

Because of (2.6), an approximation argument shows that if we only assume that $0 < c \leqslant E \leqslant C$ in $Q(s)$, then there is an $\eta = S(E)$ with $\partial\eta/\partial t$, $\Delta\eta \in L^2(Q(s))$ such that (2.6) holds, $\partial\eta/\partial t + E\Delta\eta = 0$ in $Q(s)$ and

$$\int_B b(x,s)h(x)\,dx \leqslant \iint_{Q(s)} b\left[\frac{\partial\eta}{\partial t} + A\Delta\eta\right]dx\,dt.$$

Choose now a sequence $\epsilon_k \downarrow 0$, and put

$$\alpha_k = \frac{|\varphi(v_1) - \varphi(v_2)|}{\epsilon_k + |v_1 - v_2|}, \qquad A_k = \alpha_k + \epsilon_k.$$

Letting $\eta_k = S(A_k)$ we have that

$$I = \int_B h(x)b(x,s)\,dx \leqslant \iint_{Q(s)} b(A - A_k)\Delta\eta_k\,dx\,dt \tag{2.7}$$

From (2.7), it follows that

$$\operatorname{Max}(I,0)^2 \leqslant \left(\iint_{Q(s)} A_k(\Delta\eta_k)^2\,dx\,dt\right)\left(\iint_{Q(s)} \frac{(A - A_k)^2}{A_k} b^2\,dx\,dt\right)$$

$$\leqslant \left(\frac{1}{2}\int_B |\nabla h|^2\right)\left(\iint_{Q(s)} \frac{(A - A_k)^2}{A_k} b^2\,dx\,dt\right).$$

Observe that

$$\frac{1}{2}(A - A_k)^2 \leqslant (A - \alpha_k)^2 + \epsilon_k^2, \quad \text{and} \quad A - \alpha_k = \frac{\epsilon_k A}{|v_1 - v_2| + \epsilon_k}.$$

Hence

$$\frac{(A - \alpha_k)^2}{A_k} b^2 \leqslant \frac{\epsilon_k^2 A^2 b^2}{A_k(|v_1 - v_2| + \epsilon_k)^2} \leqslant \frac{\epsilon_k^2 A|b|}{|v_1 - v_2| + \epsilon_k} \tag{2.8}$$

$$\leqslant \epsilon_k|\varphi(v_1) - \varphi(v_2)| \leqslant C\epsilon_k,$$

since $v_1, v_2 \in L^\infty(Q)$. We also have

$$\frac{\epsilon_k^2 b^2}{A_k} \leqslant \epsilon_k b^2 \leqslant C\epsilon_k, \tag{2.9}$$

and so, letting $k \to \infty$, it follows that

$$\int_B h(x)b(x,s)\,dx \leqslant 0.$$

The lemma now follows since h was an arbitrary non-negative function in $C_0^\infty(B)$.

Corollary 2.10. *Let $g \in C(\partial_p Q)$ be non-negative, and suppose that v is a weak solution of* (2.1). *Assume also that v is continuous in $\bar{Q}$, and that $G_k \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ have been chosen so that $g_k = G_k|_{\partial_p Q}$ are strictly positive, $g \leqslant g_{k+1} \leqslant g_k$, and g_k converges to g uniformly. Let $\varphi_k \in C^\infty([0,\infty))$, $\varphi_k \in \Gamma_a$ and $\varphi_k \to \varphi$ uniformly on compact subsets of $[0,\infty)$. Let v_k solve $\partial v_k/\partial t = \Delta\varphi_k(v_k)$ in Q, with $v_k = g_k$ on $\partial_p Q$. Then, each $v_k \in C^\infty(\bar{Q})$, and v_k converges to v uniformly on compact subsets of Q.*

PROOF. The existence and smoothness of each v_k is well known, since $\inf g_k = m_k > 0$ (see e.g. [7]). From the maximum principle $0 < m_k \leqslant v_k \leqslant M = \max_{\partial_p Q} g_1$. By the results of Sacks ([11]) it follows that for each compact subset $K \subset Q$, the family $\{v_k\}$ is equicontinuous. If w is locally the uniform limit of a subsequence of $\{v_k\}$, then w is a weak solution of (2.1) with boundary values g. Hence, by Lemma 2.3, $w = v$, which yields the corollary.

Corollary 2.11. *Suppose that u is a non-negative continuous solution of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathfrak{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, where $\varphi \in \Gamma_a$. Let $\varphi_k \in C^\infty([0,\infty)) \cap \Gamma_a$ and $\varphi_k \to \varphi$ uniformly on compact subsets of $[0,\infty)$. If*

$$Q = \{(x,t): |x - x_0| < r, \tau_1 < t < \tau_2\}$$

and $\bar{Q} \subset \Omega$, there are non-negative solutions $v_k \in C^\infty(Q)$ of $\partial v_k/\partial t = \Delta\varphi_k(v_k)$ in Q, that converge uniformly to v on compact subsets of Q.

PROOF. It is easy to see that $v|_Q$ is a weak solution of (2.1) with boundary values $v|_{\partial_p Q}$ (see e.g. [1], Theorem 3.1). The corollary follows easily then from Corollary 2.10.

The next preliminary result that we need is an extension of M. Pierre's uniqueness theorem ([10]) to our class of non-linearities Γ_a. The proof by Pierre requires a modification to include non-linearities φ whose derivative is unbounded near 0.

Our general uniqueness result will ultimately remove all the extra conditions in the following lemma.

Lemma 2.12. *Let $\varphi \in \Gamma_a$, and suppose that u and v are continuous, non-negative weak solutions of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, \infty)$, $n \geqslant 3$. Suppose that*

$$\sup_{t>0} \int_{\mathbb{R}^n} [u(x, t) + v(x, t)]\, dx < \infty,$$

and that $u, v \in L^\infty(\mathbb{R}^n \times [\tau, \infty))$ for each $\tau > 0$. Suppose also that

$$\lim_{t \to 0} \int_{\mathbb{R}^n} [u(x, t) - v(x, t)]\eta(x)\, dx = 0$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$. Then, $u \equiv v$ in $\mathbb{R}^n \times (0, \infty)$.

PROOF. As in Lemma 2.3, set $A = (\varphi(u) - \varphi(v))/(u - v)$ when $u \neq v$ and $A = \varphi'(u)$ elsewhere. Pick now ϵ_k in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, $\epsilon_k > 0$, and $\sup_x \epsilon_k(x) \underset{k}{\to} 0$, together with $\int_{\mathbb{R}^n} \epsilon_k(x)\, dx \underset{k}{\to} 0$. Define $A_k = \alpha_k + \epsilon_k$, where $\alpha_k = |\varphi(u) - \varphi(v)|/(\epsilon_k + |u - v|)$. Notice that A_k is continuous and strictly positive on each compact subset of $\mathbb{R}^n \times (0, \infty)$. Fix now $0 \leqslant \eta \in C_0^\infty(\mathbb{R}^n)$, and $R > 1$ so large that $\operatorname{supp} \eta \subset B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Let θ_R be the Green's potential of η in B_R, i.e., θ_R solves the equation $\Delta\theta_R = -\eta$ in B_R, $\theta \equiv 0$ on ∂B_R. Fix $T > 0$, and let, for $\alpha > 0$ and smooth in $\mathbb{R}^{n+1}$, $\psi = S(\alpha, R)$ solve the equation

$$\text{(2.13)} \qquad \begin{cases} \dfrac{\partial \psi}{\partial t} + \alpha\Delta\psi = 0 & \text{in} \quad B_R \times (0, T) \\ \psi(x, T) = \theta_R(x), & x \in B_R \\ \psi \equiv 0 & \text{on} \quad \partial B_R \times [0, T]. \end{cases}$$

Letting $h = \Delta\psi$ we observe that

$$\text{(2.14)} \qquad \begin{cases} \dfrac{\partial h}{\partial t} + \Delta(\alpha h) = 0 & \text{in} \quad B_R \times (0, T) \\ h(x, T) = -\eta(x) & x \in B_R \\ h \equiv 0 & \text{on} \quad \partial B_R \times [0, T]. \end{cases}$$

From (2.13) it follows that

$$\text{(2.15)} \qquad \iint_{B_R \times [0, T]} \alpha(\Delta\psi)^2\, dx\, dt = \iint_{B_R \times [0, T]} \alpha h^2\, dx\, dt \leqslant \frac{1}{2} \int_{\mathbb{R}^n} |\nabla\eta|^2.$$

From the maximum principle it follows that

$$0 \leqslant \psi \leqslant \|\theta_R\|_\infty \quad \text{and} \quad h \leqslant 0. \tag{2.16}$$

Hence,

$$\frac{\partial}{\partial t}\int_{B_R} |h(x,t)|\,dx = -\int_{B_R} \frac{\partial h}{\partial t}\,dx = \int_{\partial B_R} \frac{\partial(\alpha h)}{\partial N}\,d\sigma \geqslant 0$$

so

$$\int_{B_R} |h(x,t)|\,dx \leqslant \int_{\mathbb{R}^n} \eta(x)\,dx \quad \text{for} \quad 0 < t < T. \tag{2.17}$$

Since $\partial\psi/\partial t = -\alpha\Delta\psi = -\alpha h$ we have that ψ is increasing in t, and so $0 \leqslant \psi(x,t) \leqslant \theta_R(x)$, $0 < t < T$. Hence

$$\left\| \frac{\partial \psi}{\partial N} \right\|_{L^\infty(B_R)} \leqslant C(\eta) R^{1-n}. \tag{2.18}$$

Our next aim is to solve (2.13) with $\alpha = A$. The problem is that A may be unbounded in $\mathbb{R}^n \times (0,\infty)$ at points where u and v are zero. From (2.18) and (2.2) it follows that if $b = u - v$, then

$$\begin{aligned} E &= \int_{B_R} b(x,T)\theta_R(x)\,dx - \int_{B_R} b(x,\tau)\psi(x,\tau)\,dx \\ &= \iint_{B_R \times (\tau,T)} b(A-\alpha)\Delta\psi\,dx\,dt + H, \end{aligned} \tag{2.19}$$

where

$$0 < \tau < T \quad \text{and} \quad |H| \leqslant C(\eta) R^{1-n} \iint_{\partial B_R \times [\tau,T]} [\varphi(u) + \varphi(v)]\,d\sigma\,dt,$$

where $C(\eta)$ is independent of α. Our assumptions on φ imply that there exists $1 > \delta > 0$ such that $\varphi(u) \leqslant C_\delta(u^\delta + u^{1/\delta})$. Using the fact that

$$u, v \in L^\infty(\mathbb{R}^n \times [\tau, T]),$$

we see that

$$\begin{aligned} R^{1-n} \iint_{\partial B_R \times [\tau,T]} [\varphi(u) + \varphi(v)]\,d\sigma\,dt \leqslant C_{\delta,\tau,T}\Big[& R^{1-n} \int_{\partial B_R \times [\tau,T]} (u+v)\,d\sigma\,dt \\ & + \Big(R^{1-n} \int_{\partial B_R \times [\tau,T]} (u+v)\,d\sigma\,dt\Big)^\delta\Big]. \end{aligned}$$

Let now $\alpha_\nu \in C^\infty(\mathbb{R}^n \times [0,\infty))$, α_ν be such that $\alpha_\nu \to A_k$ uniformly on each compact subset. Let ψ_ν, h_ν solve (2.13), (2.14) with $\alpha = \alpha_\nu$. For each sequence $\{\tau_j\}$, $T = \tau_0 > \tau_1 > \cdots > \tau_j \downarrow 0$, and each R, there is a sequence of measures $\{\lambda_j^{(k,R)}\}$, where each $\lambda_j^{(k,R)}$ is the weak limit in B_R of $-h_\nu(x,\tau_j)\,dx$, as $\nu \to \infty$.

Let $\psi_j^{(k,R)}$ be the Green's potential in B_R of the positive measure $\lambda_j^{(k,R)}$. By the weak convergence of $-h_\nu(x,\tau_j)\,dx$, we see that for a.e. x,

$$\psi_j^{(k,R)}(x) = \lim_{\nu\to\infty} \psi_\nu(x,\tau_j).$$

Also, since each $\psi_j^{(k,R)}(x)$ is a Green's potential, we have, for every x that

$$\psi_j^{(k,R)}(x) = \lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} \psi_j^{(k,R)}(y)\,dy,$$

where $B_r(x)$ is the ball of center x and radius r. Hence, the fact that $\psi_\nu(x,t)$ is increasing in t gives

$$0 \leqslant \psi_{j+1}^{(k,R)}(x) \leqslant \psi_j^{(k,R)}(x) \leqslant \theta_R \quad \text{for each } x \text{ in } \quad B_R. \tag{2.20}$$

Also, (2.17) yields

$$\int_{B_R} d\lambda_j^{(k,R)} \leqslant \int_{B_R} \eta(x)\,dx. \tag{2.21}$$

Furthermore,

$$\left|\int_{B_R} b(x,T)\theta_R(x)\,dx - \int_{B_R} b(x,\tau_j)\psi_j^{(k,R)}(x)\,dx\right| \leqslant E_j^{(k)},$$

where, by (2.15)

$$\begin{aligned} E_j^{(k)} \leqslant C_j(\eta)\Big[&\iint_{B_R\times[\tau_j,T]} A_k^{-1}b^2(A-A_k)^2\,dx\,dt \\ &+ R^{1-n}\iint_{\partial B_R\times[\tau_j,T]} (u+v)\,d\sigma\,dt \\ &+ \Big(R^{1-n}\iint_{\partial B\times[\tau_j,T]} (u+v)\,d\sigma\,dt\Big)^{\delta}\Big]. \end{aligned}$$

From (2.8) it follows that $A_k^{-1}b^2(A-A_k)^2 \leqslant C\epsilon_k$ on $B_R\times[\tau_j,T]$, and so

$$\iint_{B_R\times[\tau_j,T]} A_k^{-1}b^2(A-A_k)^2\,dx\,dt \to 0 \text{ as } k\to\infty.$$

Since

$$\sup_{t>0}\int_{\mathbb{R}^n}(u(x,t)+v(x,t))\,dx < \infty,$$

it follows that

$$R_l^{1-n}\int_{\partial B_{R_l}\times[0,T]} (u+v)\,d\sigma\,dt \to 0 \quad \text{when} \quad R_l\to\infty,$$

for a suitable sequence R_l.

Arguing as in the proof of (2.20) and (2.21), we see that by taking a suitable subsequence $k_s \to \infty$, the weak limits λ_j of $\lambda_j^{(k_s, R_s)}$ exist, and if ψ_j is the Newton potential of λ_j, then

$$\int_{\mathbb{R}^n} b(x, T)\theta(x)\,dx = \int_{\mathbb{R}^n} b(x, \tau_j)\psi_j(x)\,dx, \tag{2.22}$$

$$0 \leqslant \psi_{j+1}(x) \leqslant \psi_j(x) \leqslant \theta(x) = N\eta(x) \quad \text{for all} \quad x \in \mathbb{R}^n \tag{2.23}$$

and

$$\int_{\mathbb{R}^n} d\lambda_j \leqslant \int_{\mathbb{R}^n} \eta(x)\,dx. \tag{2.24}$$

Next, we remark that the non-negative measures λ_j have a weak limit λ_∞, as $j \to \infty$. For, if λ is a weak limit of a subsequence λ_{j_k}, then $\psi_\infty = \lim_{j\to\infty} \psi_j$ exists and $-\lambda = \Delta\psi_\infty$ in the distribution sense. This easily yields the existence of λ_∞.

Because of our assumptions on u, v and φ, Pierre's argument ([10]) gives the existence of the initial trace of u and v. Let their common initial trace be μ. Let w be the Newton potential of u. We observe that

$$\frac{\partial w}{\partial t} = -\varphi(u) \leqslant 0. \tag{2.25}$$

In fact,

$$e(x, \tau, T) = w(x, T) - w(x, \tau) + \int_\tau^T \varphi(u(x, t))\,dt$$

is harmonic in x whenever $0 < \tau < T$. Since $w(x, t)$ is the Newton potential of an L^1 function, it follows that

$$\int_{|x|<R} w(x, t)\,dx \leqslant R^{n-2}, \quad \tau < t.$$

Since $\varphi \in \Gamma_a$, it follows that

$$\int_\tau^T \varphi(u(x, t))\,dt \in L^p(\mathbb{R}^n)$$

for some $p \in (1, \infty)$. Therefore,

$$\int_{|x|\leq R} |e(x, \tau, T)|\,dx \leqslant CR^{n-\delta}$$

for some $\delta > 0$. The harmonicity of e now shows that $e \equiv 0$, which establishes (2.25). Since $w(\cdot, \tau_j)$ is increasing as $j \to \infty$, its limit F is superharmonic and $F = N\mu$. Hence, whenever $k \geqslant j$,

$$\int u(x, \tau_j)\psi_j(x)\,dx = \int w(x, \tau_j)\,d\lambda_j \geqslant \int w(x, \tau_k)\,d\lambda_j = \int u(x, \tau_k)\psi_j(x)\,dx.$$

By monotone convergence we then have

$$\liminf_{j\to\infty} \int u(x, \tau_j)\psi_j(x)\,dx \geqslant \int u(x, \tau_k)\psi_\infty(x)\,dx.$$

Also, $\psi_\infty \geqslant N\lambda_\infty$ since $N_\epsilon\lambda_j \to N_\epsilon\lambda_\infty$ as $j \to \infty$, pointwise, where $N_\epsilon\lambda = k^{(\epsilon)} * \lambda$, where $k(x) = c_n|x|^{2-n}$ is the kernel for the Newton potential, and

$$k^{(\epsilon)}(x) = \min(K(x), c_n\epsilon^{2-n}), \qquad 0 < \epsilon < 1.$$

Because $N_\epsilon\lambda_j \leqslant N\lambda_j$ and $N_\epsilon\lambda_\infty \downarrow N\lambda_\infty$ as $\epsilon \to 0$, the claim follows. Hence,

$$\liminf_{j\to\infty} \int u(x, \tau_j)\psi_j(x)\,dx \geqslant \int u(x, \tau_k)N\lambda_\infty(x)\,dx = \int w(x, \tau_k)\,d\lambda_\infty(x).$$

By monotone convergence we obtain

$$\liminf_{j\to\infty} \int u(x, \tau_j)\psi_j(x)\,dx \geqslant \int N(\mu)\,d\lambda_\infty.$$

The conclusion will now follow by an argument due to Pierre ([10]). For $s > 0$, let $v_s(x, t) = v(x, t + s)$, and put $b_s = u - v_s$. Let $W(\cdot, t) = Nv(\cdot, t)$ and notice that $0 \leqslant W_s \leqslant C_s$ in $\mathbb{R}^n \times [0, \infty)$.

$$\begin{aligned}\limsup_{j\to\infty} \int v_s(x, \tau_j)\psi_j(x)\,dx &= \limsup_{j\to\infty} \int w_s(x, \tau_j)\,d\lambda_j \leqslant \limsup_{j\to\infty} \int w_s(x, 0)\,d\lambda_j \\ &= \limsup_{j\to\infty} \int v(x, s)\psi_j(x)\,dx = \int v(x, s)\psi_\infty(x)\,dx \\ &= \int v(x, s)N\lambda_\infty(x)\,dx = \int w_s(x, 0)\,dt_\infty,\end{aligned}$$

since $\psi_\infty = N\lambda_\infty$ a.e. in $\mathbb{R}^n$. Thus,

$$\liminf_{j\to\infty} \int b_s(x, \tau_j)\psi_j(x)\,dx \geqslant \int [N\mu(x) - W_s(x, 0)]\,d\lambda_\infty(x) \geqslant 0$$

by (2.25). Letting $s \downarrow 0$ yields

$$\int b(x, T)\theta(x)\,dx \geqslant 0.$$

Reversing the roles of u and v yields

$$\int (u(x, T) - v(x, T))N\eta(x)\,dx = 0$$

for all non-negative test functions η. This easily shows that $u(x, T) = v(x, T)$ for all $T > 0$, which concludes the proof.

3. A Priori Estimates

We shall first deal with smooth, non-negative solutions of $\partial u/\partial t = \Delta\varphi(u)$, where $\varphi \in \Gamma_a \cap C^\infty([0,\infty))$, $0 < a < 1$. Our aim is to establish pointwise estimates of u in terms of spatial averages. This will be established by a variant of the Moser ([8]) iteration technique, along the lines used by the authors in [5].

Lemma 3.1. *Let u be a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in $Q^* = \{(x,t) \in \mathbb{R}^{n+1}: |x| < 2, -4 < t < 0\}$, where $\varphi \in \Gamma_a \cap C^\infty([0,\infty))$. Let $Q = \{(x,t) \in Q^*: |x| \leqslant 1, -1 < t < 0\}$. Then,*

$$\|u\|_{L^\infty(Q)} \leqslant C\left\{1 + \iint_{Q^*} u^p \, dx\, dt\right\}^{\theta/p}, \tag{3.2}$$

where C, p, θ are positive constants which depend only on n and a.

To establish Lemma 3.1, we shall first deal with a slightly more general situation. Let $0 < r < \rho$, $0 < T < \tau$, $S = B_r \times (-T, 0]$, $R = B_\rho \times (-\tau, 0]$, where $B_r = \{x \in \mathbb{R}^n: |x| < r\}$.

Lemma 3.3. *Let $v \geqslant 0$ be a smooth non-negative solution of $\partial v/\partial t \leqslant \Delta\varphi(v)$ in R. For $\alpha \geqslant \epsilon$, there is a constant $C_\epsilon = C_\epsilon(a,n)$, such that*

$$\int_{Br} \alpha\phi_\alpha(v(x,0))\, dx + \iint_S |\nabla\varphi^\beta(v)|^2 \, dx\, dt \leqslant C_\epsilon\{(\rho - r)^{-2} + (\tau - T)^{-1}\} \iint_R [\varphi^{2\beta}(v) + \alpha\phi_\alpha(v)]\, dx\, dt\},$$

where

$$\phi_\alpha(v) = \int_0^v [\varphi(s)]^\alpha \, ds, \quad \text{and} \quad \beta = \frac{\alpha + 1}{2}.$$

PROOF. Choose $\psi \in C^\infty(\mathbb{R}^{n+1})$, $0 \leqslant \psi \leqslant 1$, such that $\psi(x,t) = 0$ whenever $|x| \geqslant \rho$ or $t \leqslant -\tau$, and $\psi(x,t) \equiv 1$ for $(x,t) \in S$. This ψ can also be chosen so that

$$\begin{cases} |\nabla\psi(x,t)| \leqslant C(\rho - r)^{-1} \\ \left|\dfrac{\partial\psi}{\partial t}(x,t)\right| \leqslant C(\tau - T)^{-1} \end{cases} \tag{3.4}$$

Whenever $\eta \geqslant 0$, $\eta \in C^\infty(\mathbb{R}^{n+1})$ is supported in $B_\rho \times \mathbb{R}$, we have

$$\iint_R \left[\eta \frac{\partial v}{\partial t} + \nabla\eta \cdot \nabla\varphi(v)\right] \leqslant 0.$$

Choosing $\eta = \psi^2 \varphi^\alpha(v)$,

$$\iint_R \varphi^\alpha(v) \frac{\partial v}{\partial t} \psi^2 \, dx\, dt + \alpha \iint_R \varphi^{(\alpha-1)}(v) |\nabla \varphi(v)|^2 \psi^2 \, dx\, dt$$

$$\leqslant -2 \iint_R \varphi^\alpha(v) \nabla \varphi(v) \nabla \psi \psi \, dx\, dt$$

Hence, for all $\delta > 0$, we have

$$\iint_R \varphi^\alpha(v) \frac{\partial v}{\partial t} \psi^2 \, dx\, dt + \alpha\beta^2 \iint_R |\nabla \varphi^\beta(v)|^2 \psi^2 \, dx\, dt$$

$$\leqslant -2\beta^{-1} \iint_R \varphi^\beta(v) \nabla \varphi^\beta(v) \cdot \nabla \psi \cdot \psi \, dx\, dt$$

$$\leqslant 2\delta\beta^{-1} \iint_R |\nabla \varphi^\beta(v)|^2 \psi^2 \, dx\, dt + 2(\delta\beta)^{-1} \iint_R \varphi^{2\beta}(v) |\nabla \psi|^2 \, dx\, dt.$$

Choosing $\delta = \alpha/4\beta$ and combining terms, we find that

$$\iint_R \varphi^\alpha(v) \frac{\partial v}{\partial t} \psi^2 \, dx\, dt + \frac{\alpha\beta^{-2}}{2} \iint_R |\nabla \varphi^\beta(v)|^2 \psi^2 \, dx\, dt$$

$$\leqslant 8\alpha^{-1} \iint_R \varphi^{2\beta}(v) |\nabla \psi|^2 \, dx\, dt.$$

Integrating by parts in t now gives

$$\int_{Br} \phi_\alpha(v(x,0)) \psi^2(x,0) \, dx + \frac{\alpha\beta^{-2}}{2} \iint_R |\nabla \varphi^\beta(v)|^2 \psi^2 \, dx\, dt$$

$$\leqslant 8\alpha^{-1} \iint_R \varphi^{2\beta}(v) |\nabla \psi|^2 \, dx\, dt + 2 \iint_R \phi_\alpha(v) \psi \left| \frac{\partial \psi}{\partial t} \right| dx\, dt.$$

This easily yields the lemma.

Corollary 3.5. *Let v be as in Lemma* 3.3, *and $\alpha \geqslant \epsilon > 0$. Then*

$$\left[\sup_{t \in (-T, 0)} \alpha \int_{B_r} \phi_\alpha(v(x,t)) \, dx \right] + \iint_S |\nabla \varphi^\beta(v)|^2 \, dx\, dt$$

$$\leqslant C_\epsilon \{ (\rho - r)^{-2} + (\tau - T)^{-1} \} \left\{ \iint_R [\varphi^{2\beta}(v) + \alpha \phi_\alpha(v)] \, dx\, dt \right\}$$

PROOF. Pick $t_0 \in [0, T]$ such that

$$\int_{Br} \phi_\alpha(v(x, -t_0)) \, dx \geqslant \frac{1}{2} \sup_{t \in (-T, 0]} \int \phi_\alpha(v(x,t)) \, dx.$$

Now, use Lemma 3.3 with R replaced by $R' = B\rho \times (-\tau, t_0]$, and S replaced by $S' = B_r \times (-T, \tau_0]$.

To complete the proof of Lemma 3.1, we need the following variant of Sobolev's inequality.

Lemma 3.6. *Let $w \geqslant 0$ be smooth in $R = B\rho \times (-\tau, 0)$. Let $q^* = q/(q-1)$, $q = n/2$ for $n \geqslant 3$, and $q = 2$ for $n = 1,2$. Then, for $1 < k < q^*$, we have*

$$\rho^{-n}\tau^{-1}\iint_R w^{2k}\,dx\,dt \leqslant C\Big\{\rho^{-n}\tau^{-1}\iint_R [w^2 + \rho^2|\nabla w|^2]\,dx\,dt\Big\} \cdot \cdot \sup_{t\in(-\tau,0)}\Big(\rho^{-n}\int_{B\rho} w(x,t)^{2(k-1)/q}\,dx\Big)^{1/q},$$

where C depends only on n.

PROOF. By rescaling, it is enough to treat the case $\rho = \tau = 1$. Letting $B = B_1 = \{x \in \mathbb{R}^n: |x| < 1\}$, we see that

$$\int_B w^{2k}(x,t)\,dx = \int_B w^2(x,t)w^{2k-1}(x,t)\,dx \leqslant \Big(\int_B w^{2q^*}(x,t)\,dx\Big)^{1/q^*}\Big(\int_B w^{(2k-1)q}(x,t)\,dx\Big)^{1/q}.$$

But, by the standard Sobolev inequality

$$\Big(\int_B w^{2q^*}(x,t)\Big)^{1/q^*} \leqslant C_n\Big[\int_B |\nabla w(x,t)|^2 + w^2(x,t)\,dx\Big].$$

We can now pass to the proof of Lemma 3.1.

PROOF OF LEMMA 3.1. For α large, let $k(\alpha)$ be determined by

$$k = \frac{1 + \dfrac{1}{q}}{1 + \dfrac{1}{\alpha}}.$$

If $\alpha \geqslant 2q$, then $1 < k < 1/(q+1) < q^*$. Assume now that $Q \subset S \subset R \subset Q^*$, where S and R are as in Lemma 3.3. Let $v = \max(u, 1)$, so that Lemma 3.3 applies to it. Combining Corollary 3.5 and Lemma 3.6 with $w = \varphi(v)^\beta$, $\beta = (\alpha+1)/2$ yields

$$\text{(3.7)}\quad \iint_B \varphi(v)^{2\beta k}\,dx\,dt \leqslant C_n F\Big[\iint_R \varphi(v)^{2\beta} + \alpha\phi_\alpha(v)\,dx\,dt\Big]\sup_{t\in[-T,0]}\Big(\iint_{B_r}\varphi(v)^{2\beta(k-1)q}\,dx\Big)^{1/q},$$

where $F = (\rho - r)^{-2} + (\tau - T)^{-1}$.

We observe that $\varphi(v) \geqslant v$ for $v \geqslant 1$ and $\phi_\alpha(v) = \int_0^v \varphi^\alpha(s)\,dx \leqslant v\varphi^\alpha(v)$ for all $v > 0$. Hence,

$$\phi_\alpha(v) \leqslant \varphi^{\alpha+1}(v), \qquad v \geqslant 1. \tag{3.8}$$

We next observe that we can choose α_0 large enough so that there is a constant M such that, for $\alpha \geqslant \alpha_0$ and $v \geqslant 1$, we have

$$\varphi(v)^{2\beta(k-1)q} \leqslant M\alpha\phi_\alpha(v). \tag{3.9}$$

In fact,

$$\phi_\alpha(1) = \int_0^1 \varphi^\alpha(s)\,ds \geqslant \int_0^1 s^{\alpha/a}\,ds \geqslant \frac{a}{\alpha+a},$$

and for $v \geqslant 1$,

$$v\frac{d}{dv}[M\alpha\phi_\alpha(v) - C\varphi(v)^{2\beta(k-1)q}]$$

$$\geqslant \alpha M v\varphi^\alpha(v) - C(\alpha+1)(k-1)q\varphi(v)^{(\alpha+1)(k-1)q},$$

since $v\varphi'(v) \leqslant C\varphi(v)$. Note that $k > 1$ and that for $\alpha \geqslant 2q$, we have

$$2\beta(k-1)q = (\alpha+1)q\left[\frac{\alpha(q+1) - q(\alpha+1)}{q(\alpha+1)}\right] \leqslant \alpha,$$

and so, for $v \geqslant 1$,

$$v\frac{d}{dv}[M\alpha\phi_\alpha(v) - C\varphi(v)^{2\beta(k-1)q}] \geqslant \varphi^\alpha(v)[\alpha M v - C(\alpha - q)] \geqslant 0$$

for M large. This yields (3.9), which together with (3.7) gives that

$$\iint_B \varphi(v)^{2\beta k}\,dx\,dt \leqslant \alpha CF\left(\iint_R \varphi(v)^{2\beta}\,dx\,dt\right) \sup_{t\in[-T,0]} \left(\int_{B_r} \alpha\phi_\alpha(v)\,dx\right)^{1/q} \tag{3.10}$$

$$\leqslant C(\alpha F)^{1+1/q}\left(\iint_R \varphi(v)^{2\beta}\,dx\,dt\right)^{1+1/q}.$$

Define sequences $\alpha_0, \alpha_1, \ldots,$ and $\beta_0, \beta_1, \ldots$ inductively by letting α_0 be as in (3.9),

$$\beta_\nu = \frac{\alpha_{\nu+1}}{2}, \qquad \beta_{\nu+1} = k(\alpha_\nu)\beta_\nu.$$

Put

$$r_\nu = \frac{2(1+\nu)}{1+2\nu}, \qquad R_\nu = \{(x,t): |x| < r_\nu,\ -r_\nu^2 < t < 0\}$$

and

$$M_\nu = \left(\int_{R_\nu} \varphi(v)^{2\beta_\nu}\right)^{1/2\beta_\nu}.$$

Observe that, since $\lim_{\alpha\to\infty} k(\alpha) = 1 + 1/q$ it follows that $E^\nu \leqslant \alpha_\nu \leqslant (E^*)^\nu$ for some numbers $1 < E < E^* < \infty$. (3.10) yields

$$M_{\nu+1} \leqslant e^{\gamma_\nu} M_\nu^{\theta_\nu}$$

where $\theta_\nu = (1 + 1/q)/k(\alpha_\nu)$, and $0 \leqslant \gamma_\nu \leqslant C(\nu + 1)E^{-\nu}$. Note that $1 < \theta_\nu < 1 + CE^{-\nu}$, and so, it easily follows that $\limsup_{\nu\to\infty} M_\nu \leqslant CM_0^{\theta_0}$, which yields Lemma 3.1.

We will also need to estimate the maximum of a solution in terms of spatial averages. In order to do so, we need some preliminary estimates.

Lemma 3.11. *Let u be a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in $Q^* = \{(x, t): |x| < 2, -4 < t < 0\}$, where $\varphi \in \Gamma_a \cap C^\infty([0, \infty))$. Let $Q = \{(x, t) \in Q^*: |x| < 1, -1 < t < 0\}$. Then*

$$\|u\|_{L^\infty(Q)} \leqslant C\left\{1 + \sup_{-4<t<0} \int_{|x|<2} u(x, t)\,dx\right\}^\sigma, \tag{3.12}$$

where C and σ are positive constants which depend only on n and a.

PROOF. Let $S = B_r \times (-T, 0]$, $R = B_\rho \times (-\tau, 0]$ satisfy $Q \subset S \subset R \subset Q^*$, and set $v = \max(u, 1)$. Combining Corollary 3.5 with Lemma 3.6 gives, for $k \in (1, q^*)$, that

$$\iint_Q \varphi^{2\beta k}(v)\,dx\,dt \leqslant CF\left(\iint_R \varphi^{2\beta}(v)\,dx\,dt\right)\left(\sup_{t\in[-\tau,0]} \int_{B\rho} \varphi(v)^{2\beta(k-1)q}\,dx\right)^{1/q}.$$

Here $\beta = (\alpha + 1)/2$, and α is chosen so that $v^p \leqslant \varphi(v)^{2\beta k}$, for $v \geqslant 1$, where p is as in Lemma 3.1. As before, F equals $(\rho - r)^{-2} + (\tau - T)^{-1}$. We have that $\varphi(v) \leqslant v^M$ for some $M > 1$ and all $v \geqslant 1$, and so, if we now pick $k \in (1, q^*)$ so close to 1 that $2\beta(k - 1)qM \leqslant 1$, then

$$\iint_S \varphi^{2\beta k}(v)\,dx\,dt \leqslant CFI^{1/q} \iint_R \varphi^{2\beta}(v)\,dx\,dt, \tag{3.13}$$

where

$$I = \sup_{t\in[-4,0]} \int_{|x|<2} v(x, t)\,dx.$$

For $1/2 \leqslant r \leqslant 2$, let $L(r) = B_r \times [-r^2, 0]$, and for $s > 0$ let

$$m(r, s) = \iint_{L(r)} w^s\,dx\,dt,$$

where $w = \varphi(v)^{2\beta}$. (3.13) now shows that for $1/2 < r < \rho < 2$

$$m(r, k) \leqslant C(\rho - r)^{-2} I^{1/q} m(\rho, 1). \tag{3.14}$$

For $0 < s < 1$, $m(r, 1) \leqslant m(r, k)^{\theta/k} m(r, s)^{(1-\theta)/s}$, where $\theta = (1 - s)k/(k - s) \in (0, 1)$. For $\gamma > 1$, $1/2 \leqslant r \leqslant 1$, (3.14) gives that

$$\log m(2r^\gamma, k) \leqslant \frac{1}{q} \log I + \log C + \log 4(r - r^\gamma)^2 + \frac{\theta}{k} \log m(2r, k) + \frac{1-\theta}{s} \log m(2, s), \tag{3.15}$$

since $m(r, s) \leqslant m(2, s)$, $1/2 < r \leqslant 1$. Integrating from $R = 3/4$ to 1, we see that

$$\gamma^{-1} \int_{R^{1/\gamma}}^1 \log m(2r, k) \frac{dr}{r} \leqslant C_1 \log I + C_2 \log m(2, s) + C_3 + \frac{\theta}{k} \int_R^1 \log m(2\pi, k) \frac{dr}{r}. \tag{3.16}$$

Choose now s so small that $\varphi(v)^{2\beta s} \leqslant v$ for all $v \geqslant 1$, and γ so close to 1 that $\gamma^{-1} > \theta/k$. We now distinguish two cases. The first one is when $m(3/2, k) \leqslant 1$. In this case, Lemma 3.1 shows that $\|u\|_{L^\infty(Q)} \leqslant C$, and our Lemma follows. If, on the other hand, $m(3/2, k) > 1$, $\log m(2r, k) > 0$ for $r \in [R, 1]$, and so, since $R^{1/\gamma} > R$, it follows from (3.16) that

$$\left(\gamma^{-1} - \frac{\theta}{k}\right) \int_R^1 \log m(2\pi, k) \frac{dr}{r} \leqslant C_1 \log I + C_2 \log m(2, s) + C_3$$

which yields

$$\iint_{L(3/2)} \varphi(v)^{2\beta k} \, dx \, dt \leqslant C I^{\sigma_1} \left(\iint_{Q^*} \varphi(v)^{2\beta s} \, dx \, dt \right)^{\sigma_2},$$

where C, σ_1, σ_2 depend only on n, s and a. By our choice of s, it follows that

$$\iint_{L(3/2)} \varphi(v)^{2\beta k} \, dx \, dt \leqslant C I^{\sigma},$$

which together with Lemma 3.1 concludes the proof. The main result in this section is the next improvement of Lemma 3.11.

Theorem 3.17. *Let u be a smooth, non-negative solution of the equation $\partial u/\partial t = \Delta\varphi(u)$ in $Q^* = \{(x, t): |x| < 2, -4 < t < 0\}$, where $\varphi \in \Gamma_a \cap C^\infty([0, \infty))$. Then, there are positive constants C, γ and σ such that if $Q = \{(x, t): |x| < 1,$*

$-1 < t \leqslant 0\}$, *then*

$$\|u\|_{L^\infty(Q)} \leqslant C\{I^\sigma + I^\gamma\}, \tag{3.18}$$

where

$$I = \sup_{-4<t<0} \int_{|x|<1} u(x,t)\,dx.$$

PROOF. Lemma 3.11 implies that whenever $I \geqslant \epsilon_0 > 0$, then (3.18) holds with a constant C depending on ϵ_0. We also remark that if $0 \leqslant I \leqslant \epsilon_0 \leqslant 1$, then $0 \leqslant u \leqslant C_0 = C_0(r_0)$ in $Q_0 = \{(x,t)\colon |x| < r_0,\ -r_0^2 < t < 0\}$, for $1 < r_0 < 2$. We now claim that there is an $\epsilon_0 \in (0,1)$ such that if $I \leqslant \epsilon_0 = \epsilon_0(r_0, r_1)$, then $0 \leqslant u \leqslant 1$ in $Q_1 = \{(x,t)\colon |x| < r_1,\ -r_1^2 < t < 0\}$, $1 < r_1 < r_0 < 2$. This follows from the results in [11], where it is shown that there is a modulus of continuity ω in Q_0, which depends only on C_0, a, n for all solutions of $\partial u/\partial t = \Delta\varphi(u)$ in Q^*.

Hence, it is enough to show that if $0 \leqslant u \leqslant 1$ in Q^* then

$$\|u\|_{L^\infty(Q)} \leqslant CI^\sigma, \quad \text{for some} \quad \sigma = \sigma(a,n), \tag{3.19}$$

Observe that there is an $\eta = \eta(a) > 0$ such that $v \geqslant \varphi^\eta(v)$ for $0 < v \leqslant 1$. For $\alpha > 0$, let $k = k(\alpha)$ be defined by $k = 1 + [(\alpha + \eta)/q(\alpha + 1)]$, where q is as in Lemma 3.6. Note that $\lim_{\alpha\to\infty} k(\alpha) = 1 + 1/q \in (1, q^*)$, where $q^* = q/(q-1)$. Choose $\alpha_0 > 0$ such that $1 < k < q^*$ for $\alpha > \alpha_0$. From Corollary 3.5 and Lemma 3.6, it follows that, for $\alpha \geqslant \alpha_0$

$$\iint_S \varphi^{2\beta k}(u)\,dx\,dt \leqslant CF \iint_R [\varphi(u)^{2\beta} + \alpha\phi_\alpha(u)]\,dx\,dt \cdot \sup_{t\in[-T,0]} \left(\int_{|x|<r} \varphi(u)^{2\beta(k-1)q}\,dx\right)^{1/q} \tag{3.20}$$

Here $Q \subset S \subset R \subset Q^*$,

$$S = \{(x,t)\colon |x| < r,\ -T < t < 0\},$$
$$R = \{(x,t)\colon |x| < \rho,\ -\tau < t < 0\},$$
$$F = (\rho - r)^{-2} + (\tau - T)^{-1},$$
$$\beta = (\alpha + 1)/2, \quad \text{and}$$
$$\phi_\alpha(u) = \int_0^u \varphi^\alpha(s)\,ds.$$

By our choice of k, it follows that, for $0 \leqslant u \leqslant 1$,

$$u\frac{d}{du}[\alpha M\phi_\alpha(u) - \varphi(u)^{2\beta(k-1)q}] \geqslant \alpha M u\varphi^\alpha(u) - c(\alpha+1)(k-1)q\varphi(u)^{\alpha+\eta} \geqslant 0$$

for $\alpha \geqslant \alpha_0$ if M is large enough. Since $\phi_\alpha(u) \leqslant u\varphi^\alpha(u) \leqslant \varphi^\alpha(u)$ for $0 \leqslant u \leqslant 1$, it follows from Corollary 3.6 and (3.20) that

$$\iint_S \varphi^{2\beta k}(u)\,dx\,dt \leqslant C(\alpha F)^{1+1/q}\left(\iint_R \varphi^\alpha(u)\,dx\,dt\right)^{1+1/q}. \tag{3.21}$$

(Here we used the fact that $\varphi^{2\beta}(u) \leqslant \varphi^\alpha(u)$ for $0 < u \leqslant 1$.) Arguing in a similar manner to the deduction of Lemma 3.1 from (3.10), it now follows from (3.21) that

$$\|u\|_{L^\infty(Q)} \leqslant C\left(\iint_{Q^*} \varphi^\alpha(u)\,dx\,dt\right)^{\sigma/\alpha}$$

for $\alpha \geqslant \alpha_0$ and some $\sigma = \sigma(a, n)$. Choosing now $\alpha > \alpha_0$ such that $\varphi^\alpha(u) \leqslant u$, $0 \leqslant u \leqslant 1$ finishes the proof.

4. A Harnack Inequality

In order to show the existence of an initial trace for non-negative solutions of (1.2), we will establish a suitable Harnack inequality which controls the size of spatial averages in terms of the value of the solution at one point. For the porous medium equation $\partial u/\partial t = \Delta u^m$, $m > 1$ this was established by Aronson and Caffarelli ([1]). Their proof used, among othe things, the explicit formula for the solution of the porous medium equation with initial trace the Dirac measure. Ughi ([12]) extended this proof to a class of non-linearities that were asymptotically the pure power u^m, $m > 1$ both at $u = 0$ and at $u = \infty$.

The idea in our proof is to use scaling properties of solutions of

$$\frac{\partial u}{\partial t} = \Delta\varphi(u),$$

$\varphi \in \Gamma_a$, combined with compactness properties of solutions of these equations. The main point is that solutions of (1.1) with $\gamma \in \Gamma_a$, $0 < a < 1$, with initial trace the Dirac mass at the origin, have $u(0, T_0)$ uniformly bounded from below by a positive constant C_n, for a suitable choice of T_0.

To explain the scaling properties of solutions of (1.1), and our compactness result, we need to introduce some notation. $S(\varphi)$ will denote the class of continuous weak solutions of (1.1) in $\mathbb{R}^n \times (0, \infty)$.

$$P_\varphi(M) = \left\{u \in S(\varphi): \sup_{t<0} \int_{\mathbb{R}^n} u(x, t)\,dx \leqslant M\right\}.$$

We remark that Pierre's proof ([10]) shows that all solutions $u \in P_\varphi(M)$ have an initial trace.

For $\varphi \in \Gamma_a$, $\varphi(u)/u$ is monotonically increasing on $[1, \infty)$, and

$$\lim_{u \to \infty} \frac{\varphi(u)}{u} = +\infty.$$

Let $\Lambda = \Lambda_\varphi$ be the inverse of $\varphi(u)/u$ on $[1, \infty)$. A computation shows that if $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, \infty)$ and $\alpha > 0$, $\beta > 0$, then

$$\frac{\partial v}{\partial t} = \Delta\psi(v) \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \tag{4.1}$$

where

$$v(x, t) = \frac{u(\alpha x, \beta t)}{\gamma}, \quad \text{and} \quad \psi(s) = \beta\varphi(\gamma s)\alpha^{-2}\gamma^{-1}.$$

If $\alpha^2 \geqslant \beta$ then the choice $\gamma = \Lambda(\alpha^2/\beta)$ is possible, and with this choice $\psi(1) = 1$. It is also easy to see that with this choice of γ, $\psi \in \Gamma_a$ whenever $\varphi \in \Gamma_a$.

Lemma 4.2. *Let $\varphi \in \Gamma_a$ and $u \in P_\varphi(M)$. There is a constant $C = C(a, M, n)$ such that $u(x, t) \leqslant C$ for $t \geqslant 1$, and, for $0 < t < 1$,*

$$u(x, t) \leqslant C\Lambda(\rho^2/t), \qquad x \in \mathbb{R}^n,$$

where $\rho = \rho(t) \in [\sqrt{t}, 1]$ is determined by $\rho^n\Lambda(\rho^2/t) = 1$.

PROOF. If $t \geqslant 1$ we have

$$\int_{|\xi - x| < 1} u(\xi, \tau)\, d\xi \leqslant M$$

for $t - 1 < \tau < t$, and so $u(x, t) \leqslant C = C(a, M, n)$ by Corollary 2.11, and Theorem 3.17. If $0 < t < 1$ we put $\gamma = \Lambda(\rho^2/t)$, and $v(\xi, \tau) = \gamma^{-1}u(x + \rho\xi, t\tau)$. By (4.1), $\partial v/\partial t = \Delta\Psi(v)$, where $\Psi \in \Gamma_a$, and for $0 < \tau < \infty$, $v \in P_\Psi(M)$ since

$$\int_{\mathbb{R}^n} v(\xi, \tau)\, d\xi = \gamma^{-1}\rho^{-n} \int_{\mathbb{R}^n} u(y, t\tau)\, dy \leqslant M.$$

By the argument above

$$v(0, 1) = \frac{u(x, t)}{\Lambda(\rho^2/t)} \leqslant C(a, M, n),$$

which completes the proof.

In what follows, the restriction to $n \geqslant 3$ is purely technical, and will be removed later, once Lemma 2.12 has been established for $n = 1,2$.

Corollary 4.3. *Let $n \geq 3$, $\varphi \in \Gamma_a$, $u, v \in P_\varphi(M)$ have the same trace. Then $u \equiv v$ in $\mathbb{R}^n \times (0, \infty)$.*

PROOF. If follows immediately from Lemma 4.1 and Lemma 2.12.

We can now give our compactness result.

Lemma 4.4. *Suppose that $\varphi_k \in \Gamma_a$, $0 < a < 1$, and that $u_k \in P_{\varphi_k}(M)$, $k = 1, 2, \ldots$ Suppose also that if μ_k is the trace of u_k, then μ_k converges weakly to a non-negative measure μ, and that φ_k converges uniformly on compact subsets of $[0, \infty)$ to a $\varphi \in \Gamma_a$. Then, there is a unique $u \in P_\varphi(M)$ such that u_k converges to u uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$, and u has trace μ.*

PROOF. The uniqueness follows from Corollary 4.3. From Lemma 4.2 it follows that $\{u_k\}$ is locally bounded in $\mathbb{R}^n \times (0, \infty)$. It follows then from Sacks ([11]), that $\{u_k\}$ is locally equicontinuous. To conclude the proof of the lemma, it follows from the uniqueness in Corollary 4.3 that it is enough to show that whenever w is locally the uniform limit of a subsequence of $\{u_k\}$, then $w \in P_\varphi(M)$ and has trace μ.

To this end, first notice that if $\varphi \in \Gamma_a$, then $\Lambda_\varphi(s) \geq s^\gamma$ for all $s \geq 1$ and some $\gamma = \gamma(a)$. It now $0 < t < 1$, and ρ is as in Lemma 4.1, then

$$1 = \rho^n \Lambda(\rho^2/t) \geq \rho^{n+2\gamma} t^{-\gamma},$$

and so $\rho \leq t^{\gamma/(n+2\gamma)}$. Thus, for $0 < t < 1$, $u \in P_\varphi(M)$, $\varphi \in \Gamma_a$ we have the estimate

$$\varphi(u) \leq 1 + Cut^{-1+\alpha}, \tag{4.5}$$

where the positive constants C and α can be taken to depend only on n and a. From (4.5), it follows that whenever $0 < \tau < T < 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} u_k(x, T)\eta(x)\,dx = \int_{\mathbb{R}^n} u_k(x, \tau)\eta(x)\,dx + I, \tag{4.6}$$

where

$$\begin{aligned} |I| &= \left| \int_\tau^T \int_{\mathbb{R}^n} \eta(x) \Delta \varphi_k(u_k(x, s))\,dx\,ds \right| \\ &\leq \int_\tau^T \int_{\mathbb{R}^n} |\Delta \eta(x)| \varphi_k(u_k(x, s))\,dx\,ds \\ &\leq C(T - \tau) \int_{\mathbb{R}^n} |\Delta \eta(x)|\,dx + CM(T^\alpha - \tau^\alpha) \|\Delta \eta\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

In the limit, we have that

$$\left| \int_{\mathbb{R}^n} w(x, T)\eta(x)\, dx - \int_{\mathbb{R}^n} \eta\, d\mu \right| = O(T^\alpha) \quad \text{as} \quad T \to 0,$$

for any $\eta \in C_0^\infty(\mathbb{R}^n)$, and hence w has trace μ.

As was mentioned before, our proof of the Harnack inequality will be based on the following estimate for the solution of (1.1) with initial trace δ, where δ is the point measure of unit mass at $0 \in \mathbb{R}^n$.

Lemma 4.7. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $n \geqslant 3$. Then, there is a unique $Q = Q_\varphi \in P_\varphi(1)$ with initial trace δ. Furthermore, there is a $T_0 > 0$ that can be taken to depend only on a such that $\inf_{\varphi \in \Gamma_a} Q(0, T_0) > 0$.*

PROOF. It is standard (see e.g. [7]) that if $f \geqslant 0$ is smooth with compact support, and if $\Psi \in C^\infty([0, \infty)) \cup \Gamma_a$, with $\Psi'(x) \geqslant \epsilon > 0$ for $0 < x < 1$, then a smooth solution u of $\partial u/\partial t = \Delta\Psi(u)$ with $u(x, 0) = f(x)$ will exist. Furthermore,

$$\int_{\mathbb{R}^n} u(x, t)\, dx = \int_{\mathbb{R}^n} f(x)\, dx,$$

and hence, the existence part of the lemma follows from Lemma 4.4 and an approximation argument. The uniqueness part follows from Corollary 4.3. Also, from the results of Vázquez ([13]) it follows that Q is radial and decreasing in $|x|$, for each $t > 0$. Let now $\eta \in C_0^\infty(\mathbb{R}^n)$ be non-negative, with

$$\eta(0) = \max_{x \in \mathbb{R}^n} \eta = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} \eta(x)\, dx = 1.$$

From (4.6) it follows that

$$\left| \int Q_\varphi(x, t)\eta(x)\, dx - 1 \right| \leqslant Ct^\alpha,$$

where $\alpha = \alpha(a) > 0$. Hence

$$Q_\varphi(0, T) \geqslant \int \eta(x) Q_\varphi(x, t) \geqslant 1/2 \quad \text{if} \quad 0 < t < T_0,$$

and T_0 is chosen sufficiently small, which concludes the proof of the lemma.

We will also need a slight variant of Lemma 4.7.

Corollary 4.8. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $n \geqslant 3$. Then, for every $M > 0$ there is a unique solution $Q_{\varphi, M} \in P_\varphi(M)$, with trace $M\delta$. Furthermore, there is an $M = M(a, n)$ such that $\inf \{Q_{\varphi, M}(0, 1) : \varphi \in \Gamma_a\} > 0$.*

PROOF. Arguing as in the proof of Lemma 4.7, we see that we only need to establish the bound from below. This can be seen by observing that if T_0 is as in Lemma 4.7, $\beta = T_0^{-1}$ and $\alpha^n \Lambda(\alpha^2/\beta) = M$, then

$$v(x, t) = \gamma^{-1} Q_{\varphi, M}(\alpha x, \beta t) \in P_\psi(1),$$

where

$$\gamma = \Lambda_\varphi(\alpha^2/\beta) \quad \text{and} \quad \psi(v) = \varphi(\gamma v)/\varphi(\gamma),$$

belongs to Γ_a. Since $Q_{M,\varphi}(0, 1) = \gamma v(0, T_0)$, and the trace of v is δ, the corollary follows from Lemma 4.7.

We are now in a position to give a preliminary version of our Harnack inequality.

Lemma 4.9. *Suppose that $\varphi \in \Gamma_a$, $0 < a < 1$ and $n \geqslant 3$. Suppose that u is continuous, non-negative in $\mathbb{R}^n \times [0, 1]$, and solves $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, 1)$. Let $H_\varphi(x)$ be 1 for $0 < s < 1$ and $s[\varphi(s)/s]^{n/2}$ for $s \geqslant 1$. Then, there is a constant $C = C(a, n) > 0$ such that*

$$\int_{|x| \leq 1} u(x, 0)\, dx \leqslant C H_\varphi(u(0, 1)) \tag{4.10}$$

PROOF. We will first establish (4.10) under the additional asumptions that $u \in S_\varphi$ and

$$\begin{cases} \operatorname{supp} u(\cdot, 0) \subset \{x \in \mathbb{R}^n : |x| < 1\} \\ \sup\limits_{0 < t < \infty} \int u(x, t)\, dx < \infty\,. \end{cases} \tag{4.11}$$

We will proceed by a contradiction argument. If (4.10) does not hold, then for each $k = 1, 2, \ldots$ there is a $\varphi_k \in \Gamma_a$ and a continuous, non-negative u_k in $\mathbb{R}^n \times [0, \infty)$, $u_k \in S_{\varphi_k}$, u_k verifying (4.11) and such that

$$I_k = \int_{\mathbb{R}^n} u_k(x, 0)\, dx \geqslant k H_{\varphi_k}(u_k(0, 1)).$$

Notice that $I_k \geqslant k$. Let $\Lambda_k = \Lambda_{\varphi_k}$, and define $\alpha_k \geqslant 1$ as the solution of the equation

$$\alpha_k^n \Lambda_k(\alpha_k^2) = I_k/M$$

where M is as in Corollary 4.8. Set $\gamma_k = \Lambda_k(\alpha_k^2)$, and $v_k(x, t) = u_k(\alpha_k x, t)/\gamma_k$. Then, $\partial v_k/\partial t = \Delta\Psi_k(v_k)$ in $\mathbb{R}^n \times (0, \infty)$ where

$$\Psi_k(u) = \frac{\varphi_k(\gamma_k u)}{\gamma_k \alpha_k^2}.$$

Observe that $\Psi_k \in \Gamma_a$ and

$$\begin{cases} \int v_k(x,0)\,dx = I_k \alpha_k^{-n} \gamma_k^{-1} = M \\ \operatorname{supp} v_k(x,0) \subset \{x: \alpha_k |x| < 1\} \end{cases}$$

and that $v_k \in P_{\Psi_k}(M)$. Since $I_k \to \infty$, $\alpha_k \to \infty$, and so $v_k(x,0)$ converges weakly to $M\delta$. By the compactness of Γ_a, we can extract a subsequence, which we again denote by Ψ_k, such that Ψ_k converges uniformly on compact subsets to a $\Psi \in \Gamma_a$.

By Lemma 4.4, v_k converges uniformly to $w = Q_{\Psi,M}$, on compact subsets of $\mathbb{R}^n \times (0,\infty)$. We notice that $v_k(0,1) = \gamma_k^{-1} u_k(0,1)$. Since $w(0,1) > 0$, by Corollary 4.8, and $\gamma_k \to \infty$, since α_k does, it follows that $u_k(0,1) \to \infty$ as $k \to \infty$.

Next, observe that if $\varphi \in \Gamma_a$, there are positive constans C_1, C_2, α and β that can be taken to depend only on a such that $A^{\alpha}\Lambda_{\varphi}(u) \leqslant \Lambda_{\varphi}(Au) \leqslant A^{\beta}\Lambda_{\varphi}(u)$, whenever $A, u \geqslant 1$. Let $F_k(u) = u^n \Lambda_k(u^2)$, and observe that, because of the remark above, $AF_k(u) \geqslant F_k(A^{\sigma}u)$ for some $\sigma = \sigma(a)$ and all $A, u \in [1,\infty)$. We have that

$$I_k = MF_k(\alpha_k) \geqslant kH_{\varphi_k}(u_k(0,1)).$$

Since $u_k(0,1) \to \infty$,

$$H_{\varphi_k}(u_k(0,1)) = F_k(\sqrt{\varphi_k(u_k(0,1))/u_k(0,1)}),$$

it follows that

$$\alpha_k \geqslant (k/M)^{\sigma}[\varphi_k(u_k(0,1))/u_k(0,1)]^{1/2}$$

and hence

$$\lim_{k\to\infty} u_k(0,1)\gamma_k^{-1} = 0 = w(0,1),$$

a contradiction. To remove the assumptions that $u \in S_{\varphi}$, and (4.11), let $h \geqslant 0$ be continuous on $\mathbb{R}^n$, with $h(x) = 0$ for $|x| \geqslant 1$ and $0 \leqslant h \leqslant 1$. Let w_R solve the equation $\partial w/\partial t = \Delta\varphi(w)$ in $S_R = \{(x,t): |x| < R, 0 < t < R\}$ with $w_R(x,0) = h(x)u(x,0)$ for $|x| \leqslant R$, $w_R = 0$ on $\{(x,t): |x| = R, 0 < t < R\}$. (The existence of w_R follows easily by approximating φ with $\varphi_k \in C^{\infty}([0,\infty)) \cap \Lambda_a$, and with $\varphi_k'(x) \geqslant \epsilon$, using Lemma 2.3 and compactness arguments as in Lemma 4.4). By Lemma 2.3, in $\{|x| < R\} \times (0,1)$, we have $w_R \leqslant w_{\rho} \leqslant u$ if $R < \rho$, and $w_R \leqslant w_{\rho}$ in S_R. Using compactness arguments as in Lemma 4.4, it follows that

$$\lim_{R\uparrow\infty} w_R \in P_{\varphi}(M) \quad \text{for some} \quad M > 0.$$

Hence the previous argument applies to w and

$$\int w(x,0)\,dx = \int h(x)u(x,0)\,dx \leqslant CH_\varphi(w(0,1)) \leqslant CH_\varphi(u(0,1)).$$

Since h is an arbitrary continuous function with supp $h \subset \{|x| \leqslant 1\}$ and $0 \leqslant h \leqslant 1$, the lemma follows.

We can now give our Harnack inequality. The proof will only be given for $n \geqslant 3$. However, once our uniqueness result, Lemma 2.12, is established for $n = 1, 2$, it will holds (as well as all the other results in this section) for all n.

Theorem 4.12. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $u \geqslant 0$ be continuous in $\bar{D}_T = \mathbb{R}^n \times [0, T]$, $T > 0$. Suppose that u solves $\partial u/\partial t = \Delta\varphi(u)$ in D_T, and $R > T^{1/2}$. Then, there is a constant $C = C(a, n)$ such that*

$$\int_{|x| \leqslant R} u(x,0)\,dx \leqslant C\{R^n \Lambda(R^2/T) + T^{n/2} H_\varphi(u(0,T))\}.$$

PROOF. Let $\gamma = \Lambda(R^2/T)$, $v(x,t) = u(Rx, Tt)/\gamma$ and $\psi(s) = \varphi(\gamma s)/\varphi(\gamma)$. $\psi \in \Gamma_a$, and by (4.1) $\partial v/\partial t = \Delta\psi(v)$. Now observe that if $v \geqslant 1$, or $u \geqslant \gamma$, then

$$H_\psi(v) = v[\psi(v)/v]^{n/2} = \frac{u}{\gamma}\left[\frac{\varphi(u)}{u}\right]^{n/2}\left[\frac{\varphi(\gamma)}{\gamma}\right]^{-n/2}$$

$$= [R^n \Lambda(R^2/T)]^{-1} T^{n/2} H_\varphi(u).$$

Since $H_\psi(v) = 1$ otherwise, we have in all cases,

$$H_\psi(v) \leqslant 1 + [R^n \Lambda(R^2/T)]^{-1} T^{n/2} H_\varphi(u),$$

and so, by Lemma 4.9,

$$\int_{|x| \leqslant 1} v(x,0)\,dx = [R^n \Lambda(R^2/T)]^{-1} \int_{|x| < R} u(x,0)\,dx \leqslant CH_\psi(v(0,1)),$$

which yields Theorem 4.12.

5. Pointwise estimates

We will now combine the a priori inequalities of Section 3 with the Harnack inequality of Section 4, to give sharp upper bounds for the size of a solution $u(x,t)$ in $\mathbb{R}^n \times (0, T)$, as $t \to 0$, and as $x \to \infty$.

We need to introduce some notation. For μ a non-negative measure on $\mathbb{R}^n$, $\varphi \in \Gamma_a$, $0 < a < 1$, and $\rho \geqslant 1$ we define

$$|||\mu|||_\rho = \sup_{R \geq \rho} \frac{\mu\{|x| < R\}}{R^n \Lambda_\varphi(R^2)}. \tag{5.1}$$

Notice that for $1 \leqslant \rho < r$, $|||\mu|||_r \leqslant |||\mu|||_\rho$, and so

$$|||\mu|||_\infty = \lim_{\rho \to \infty} |||\mu|||_\rho \tag{5.2}$$

exists. We will set $|||\mu||| = |||\mu|||_1$, and for a non-negative function f on $\mathbb{R}^n$, $|||f||| = |||f dx|||$.

A consequence of our Harnack inequality (Theorem 4.12) is that if $\varphi \in \Gamma_a$ and $u \in S_\varphi(T)$ (the class of non-negative continuous solutions of (1.1) in $\mathbb{R}^n \times (0, T)$), then

$$\sup_{t \in (0, T-\delta)} |||u(\cdot, t)||| \leqslant C(u, T, \delta) < \infty \quad \text{for all} \quad \delta > 0. \tag{5.3}$$

For $s > 0$ and $0 < \tau < 1$, notice that the equation

$$R^n \Lambda_\varphi(R^2/\tau) = (1 + s)^n \Lambda_\varphi((1 + s)^2)$$

has a unique solution $R_\varphi(s, \tau)$.

Lemma 5.4. *For $\varphi \in \Gamma_a$, and $0 < \tau < 1$, $0 < s$, we have the estimate $R_\varphi(s, \tau) \leqslant (1 + s)\tau^\delta$, where $\delta > 0$ can be taken to depend only on n and a.*

PROOF. Let $F(x) = x^n \Lambda(x^2)$. Then, F is increasing on $[1, \infty)$ and if $A \geqslant 1$, $x \geqslant 1$, it is easy to see that $AF(x) \leqslant F(A^\sigma x)$, for some $\sigma = \sigma(a, n) > 0$. Since $F(R_\varphi/\sqrt{\tau}). = \tau^{n/2} F(1 + s)$, the lemma follows.

We are now ready to give our pointwise bound. As in Section 4, the proof will hold for $n \geqslant 3$, but will become valid for $n = 1,2$ once uniqueness is established in this case.

Theorem 5.5. *Let $\varphi \in \Gamma_a$, $u \in S_\varphi(T)$. Let $0 < \tau < \min(T, 1)$ and for $x \in \mathbb{R}^n$ set $R = R_\varphi(|x|, \tau)$. Then, for every $\delta > 0$, there is a constant $C = C(u, \delta, a)$ such that, for $\tau \leqslant t < T - \delta$, we have*

$$\varphi(u(x, t)) \leqslant 1 + Cu(x, t)R^2/\tau.$$

PROOF. The estimate is trivially satisfied if $u(x, t) \leqslant 1$, and so we will assume that $u(x, t) > 1$. Let

$$B = \sup_{0 < s < T - \delta} |||u(\cdot, s)|||.$$

For $(\xi, s) \in \mathbb{R}^n \times (0, T\tau^{-1})$, set $v(\xi, s) = u(x + R\xi, \tau s)/\gamma$, where $\gamma = \Lambda(R^2/\tau)$. Then, by (4.1), $\partial v/\partial s = \Delta\psi(v)$, where $\psi(v) = \varphi(\gamma v)/\varphi(\gamma)$.

For $0 < s < (T - \delta)/\tau$, we have that

$$\int_{|\xi| \le 1} v(\xi, s)\, d\xi = R^{-n}\gamma^{-1} \int_{|x-\xi| < R} u(\xi, s\tau)\, d\xi.$$

By Lemma 5.4, we have that $R \leqslant C(1 + |x|)$, and so

$$\int_{|\xi| \le 1} v(\xi, s)\, d\xi \leqslant R^{-n}\gamma^{-1} \int_{|\xi| < |x| + R} u(\xi, s\tau)\, d\xi$$

$$\leqslant CBR^{-n}\gamma^{-1}(1 + |x|)^n \Lambda((1 + |x|)^2) \leqslant C.$$

Hence, by the a priori estimate in Theorem 3.17, together with Corollary 2.10, we see that $v(0, 1) = u(x, t)/\gamma \leqslant C$, and so $u(x, t) \leqslant C\Lambda(R^2/\tau)$. Since Λ is the inverse of $\varphi(u)/u$ on $[1, \infty)$, the conclusion of the Theorem follows.

From Theorem 5.5, it follows that if $u \in S_\varphi(T)$, then

$$\varphi(u(0, t)) \leqslant 1 + Cu(0, t)\rho^2/t, \tag{5.6}$$

where for $0 < t < 1$, ρ is determined by $\rho^n \Lambda(\rho^2/t) = 1$. We next want to show that the growth retriction (5.6) is sharp. (Note that Lemma 5.4 implies that $\rho \leqslant t^\delta$, while $\varphi(s) \geqslant s^{1+\eta}$, $s \geqslant 1$, $\eta > 0$, so that (5.6) is in fact a growth restriction.)

Indeed, let $M \geqslant 1$ be large, and let v be the solution of $\partial v/\partial t = \Delta\varphi(v)$ with initial trace $M\delta$, where δ is the Dirac measure (see Lemma 4.7). By Theorem 4.12,

$$M = \int M\, d\delta \leqslant C\{\rho^n \Lambda(\rho^2/t) + t^{n/2} H_\varphi(v(0, t))\},$$

where C is independent of M, as long as $\rho^2/t \geqslant 1$. Choosing $M = 1 + 2C$, we see that

$$t^{-n/2} \leqslant H_\varphi(v(0, t)), \qquad 0 < t < 1. \tag{5.7}$$

This implies that $v(0, t) > 1$. Also, since $H_\varphi(s) = s[\varphi(s)/s]^{n/2}$ for $s \geqslant 1$,

$$H_\varphi(v(0, t)) = F(\sqrt{\varphi(v(0, t))/v(0, t)}),$$

where $F(s) = s^n \Lambda(s^2)$. Moreover, $t^{-n/2} = \rho^n t^{-n/2} \Lambda(\rho^2/t) = F(\rho/\sqrt{t})$, and so (5.7) can be written as

$$F(\rho/\sqrt{t}) \leqslant F(\sqrt{\varphi(v(0, t))/v(0, t)}) \quad \text{for} \quad 0 < t < 1.$$

Since F is increasing in $[1, \infty)$, this implies that $\varphi(v(0, t))/v(0, t) \geqslant \rho^2/t$, which shows that (5.6) is sharp.

6. Uniqueness

This section will be devoted to establishing the existence of a initial trace for any solution u in $S_\varphi(T)$, to showing that solutions are uniquely determined by their initial trace, and to proving a general compactness result. The proof of the general uniqueness result follows the same strategy as in the case of the porous medium equation (Dahlberg and Kenig [5]). The proof will be based on the preliminary uniqueness result Lemma 2.12, and on the following lemma.

Lemma 6.1. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and suppose that $u, v \in S_\varphi(T)$ for some $T > 0$. Assume that*

$$\lim_{t \downarrow 0} \int_{|x| \leq R} [v(x,t) - u(x,t)]^+ \, dx = 0 \tag{6.2}$$

for all $R > 0$, where $A^+ = \max(A, 0)$. Then, $v \leqslant u$ in $\mathbb{R}^n \times (0, T)$.

PROOF. We first remark that it is enough to prove the lemma when $n \geqslant 3$. In fact, let $\xi = (x, \eta) \in \mathbb{R}^{n+2}$ and notice that $u^*(\xi, t) = u(x, t)$ solves (1.1) in $\mathbb{R}^{n+2} \times (0, T)$, if u solves $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, T)$. If u, v verify (6.2) in $\mathbb{R}^n \times (0, T)$, then u^*, v^* verify (6.1) in $\mathbb{R}^{n+2} \times (0, T)$, and thus the remark is established. We therefore assume $n \geqslant 3$ from now on.

Let $w = v - u$, and let q denote the characteristic function of the set where $u(x,t) < v(x,t)$. Suppose that $0 < \tau < t < T$, and that $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \geqslant 0$. We claim that

$$\begin{aligned} \int \eta(x) w^+(x,t)\, dx \leqslant & \int w^+(x,\tau)\eta(x)\, dx \\ & + \iint_{0<\tau<s<t} \Delta\eta(x) q(x,s)[\varphi(v(x,s)) \\ & - \varphi(u(x,s))]^+ \, dx\, ds \end{aligned} \tag{6.3}$$

By Corollary 2.11, it is enough to show (6.3) when u, and v are smooth. Then, by Kato's inequality ([6]),

$$\Delta[\varphi(v) - \varphi(u)]^+ \geqslant q\Delta[\varphi(v) - \varphi(u)]$$

in the distribution sense. Also, since $\partial w^+/\partial t = q\, \partial w/\partial t$ in the distribution sense, it follows that

$$\frac{\partial w^+}{\partial t} + \Delta[\varphi(v) - \varphi(u)]^+ \geqslant 0$$

in the distribution sense. Hence

$$\iint_{0<\tau<s<t} \eta(x)\left\{\Delta[\varphi(v)-\varphi(u)]^{+}+\frac{\partial w^{+}}{\partial t}\right\} dx\,dt \geqslant 0,$$

and so (6.3) follows by integration by parts. Let $A=(\varphi(v)-\varphi(u))/(v-u)$ if $v>u$ and zero elsewhere. We first observe that if $v \geqslant 2$, then for $0<\delta<T$

$$A \leqslant C(1+|x|^2)t^{\sigma-1}, \tag{6.4}$$

where $\sigma=\sigma(a,n)>0$, and $C=C(u,v,\delta)$. To see (6.4), we observe first that if $0 \leqslant u \leqslant v/2$, and $v \geqslant 2$, then $A \leqslant 2\varphi(v)/v \leqslant C(1+|x|^2)t^{\sigma-1}$ by Theorem 5.5 and Lemma 5.4. If $v \geqslant 2$ and $u>v/2$ then $A=\varphi'(\xi)$ for some $\xi \in (u,v)$. But $\varphi'(\xi) \leqslant C\varphi(\xi)/\xi$, and again Theorem 5.5 and Lemma 5.4 yield (6.4).

For $r \geqslant 1$, let $M_r(t)=|||w^{+}(\cdot,t)|||_r$, $0<t<T$. By letting $\tau \downarrow 0$ in (6.3), we see that, from (6.2) we can conclude that, for $t \in (0,T)$,

$$\int w^{+}(x,t)\eta(x)\,dx \leqslant \iint_{0<s<t} A(x,s)w^{+}(x,s)|\Delta\eta(x)|\,dx\,ds. \tag{6.5}$$

Choose $R \geqslant r$ such that

$$\int_{|x|\leq R} w^{+}(x,t)\,dx \geqslant \frac{1}{2}M_r(t)R^n\Lambda(R^2),$$

and pick $0 \leqslant \eta \leqslant 1$, $\eta \in C_0^{\infty}(\mathbb{R}^n)$, $\eta \equiv 1$ on $\{|x|<R\}$, and $|\Delta\eta| \leqslant CR^{-2}$. Then,

$$\begin{aligned}\frac{1}{2}M_r(t)R^n\Lambda(R^2) &\leqslant \iint_{0<s<t} A(x,s)w^{+}(x,s)|\Delta\eta(x)|\,dx\,ds\\ &\leqslant \iint_{\{0<s<t,\,v\leq 2\}} \varphi(v)|\Delta\eta(x)|\,dx\,ds\\ &\quad + C\iint_{0<s<t}(1+|x|^2)s^{\sigma-1}w^{+}(x,s)|\Delta\eta(x)|\,dx\,ds\end{aligned}$$

Since $1 \leqslant r \leqslant R$ it follows that

$$M_r(t) \leqslant C\left\{t(r^2\Lambda(r^2))^{-1}+\int_0^t s^{\sigma-1}M_r(s)\,ds\right\}, \tag{6.6}$$

for $0<t<T-\delta$. Here $C=C(u,v,\delta)$ for every $\delta \in (0,T)$. (6.6) implies that if

$$F(t)=\int_0^t s^{\sigma-1}M_r(x)\,ds, \quad \text{then} \quad \frac{dF}{dt} \leqslant C\{\alpha t^{\sigma}+t^{\sigma-1}F(t)\},$$

where $\alpha=(r^2\Lambda(r^2))^{-1}$, for $0<t<T-\delta$.

It is then easy to see that

$$F(t) \leqslant M\alpha, \quad \text{for} \quad 0<t<T-\delta, \tag{6.7}$$

where $M = M(u, v, \delta, T)$. Hence, by (6.6),

$$M_r(t) \leqslant C\alpha, \quad \text{for} \quad 0 < t < T - \delta, \qquad C = C(u, v, \delta, T).$$

In particular,

$$\int_{|x|<r} w^+(x, t) \leqslant Cr^{n-2}, \qquad 0 < t < T - \delta. \tag{6.8}$$

Our next step will be to use (6.8) in (6.5). Since $\varphi \in \Gamma_a$, there is a $\theta = \theta(a) \in (0, 1]$ such that if $0 < u < v < 2$, then $\varphi(v) - \varphi(u) \leqslant C(v - u)^\theta$, $C = C(a)$. If now $R > r$ is chosen so that

$$\frac{1}{2} M_r(t) R^n \Lambda(R^2) \leqslant \int_{|x| \leq R} w^+(x, t)\, dx, \qquad 0 < t < T - \delta,$$

and η is chosen as before, then

$$\begin{aligned}\int_{|x| \leq R} w^+(x, t)\, dx &\leqslant \iint_{0<s<t} \{\varphi(v) - \varphi(u)\}^+ |\Delta\eta|\, dx\, ds \\ &\leqslant CR^n \Lambda(R^2) \int_0^t s^{\sigma-1} M_r(s)\, ds + CR^{-2} \int_{0<s<t} \int_{|x|<2R} w^+(x, s)^\theta\, dx\, ds\end{aligned}$$

Dividing through by $R^n\Lambda(R^2)$ and using Hölder's inequality and the fact that $R > r$, we see that

$$\begin{aligned}M_r(t) &\leqslant C \int_0^t s^{\sigma-1} M_r(s)\, ds + CR^{-2} \Lambda(R^2)^{-1} R^{-n\theta} \int_0^t \left(\int_{|x|<2R} w^+(x, s)\, dx \right)^\theta ds \\ &\leqslant C \int_0^t s^{\sigma-1} M_r(s)\, ds + Ct\Lambda(r^2)^{-1} r^{-2-2\theta}\end{aligned}$$

Using (6.7) again, we find this time that

$$\int_{|x| \leq r} w^+(x, t)\, dx \leqslant Cr^{n-2-2\theta}, \qquad r > 1, \qquad 0 < t < T - \delta. \tag{6.9}$$

Let

$$h(x, t) = \int_0^t [\varphi(v(x, s)) - \varphi(u(x, s))]^+\, ds.$$

Then

$$\begin{aligned}r^{-n} \int_{|x| \leq r} h(x, t)\, dx &\leqslant C \int_0^t r^{-n} \int_{|x| \leq r} w^+(x, s)^\theta\, dx\, ds \\ &\quad + Cr^{2-n} \int_0^t s^{\sigma-1} \int_{|x| \leq r} w^+(x, s)\, dx\, ds \\ &\leqslant Ctr^{-\theta(2+2\theta)} + Ct^\sigma r^{-2\theta}.\end{aligned}$$

However, h is subharmonic in x for every $t \in (0, T)$, since, if $\eta \in C_0^\infty$, $\eta \geqslant 0$,

$$\int \eta(x)\Delta h(x,t)\,dx \geqslant \int_0^t \eta(x)\frac{\partial}{\partial s}w^+(x,s)\,dx\,ds = \int \eta(x)w^+(x,t)\,dx \geqslant 0.$$

Hence, as

$$h(x,t) \leqslant C_n r^{-n}\int_{|y-x|<r} h(y,t)\,dy \underset{r\to\infty}{\to} 0,$$

h is identically 0, which easily yields the lemma.

We will now show that all solutions of (1.1), with $\varphi \in \Gamma_a$ have an initial trace.

Theorem 6.10. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $u \in S_\varphi(T)$ for some $T > 0$. Then, there is a non-negative measure μ on $\mathbb{R}^n$ such that*

$$\mu\{|x| < R\} = O(R^n\Lambda(R^2)) \quad \text{as} \quad R \to \infty, \tag{6.11}$$

and

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^n} u(x,t)\eta(x)\,dx = \int_{\mathbb{R}^n}\eta\,d\mu,$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$.

PROOF. First notice that given $\delta > 0$ there is a constant $C = C(u, \delta, T)$ such that whenever $R \geqslant 1$, $0 < t < T - \delta$ we have that

$$\int_{|x|\leq R} u(x,t)\,dx \leqslant CR^n\Lambda(R^2) \tag{6.12}$$

When $n \geqslant 3$, this was established in (5.3). For $n = 1,2$, notice that $u^*(x,\xi,t) = u(x,t)$, $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^2$ solves (1.1) in $\mathbb{R}^{n+2} \times (0,T)$, and so

$$R^2\int_{|x|\leq R} u(x,t)\,dx \leqslant C\int_{\substack{|x|\leq R\\ |\xi|\leq R}} u^*(x,\xi,t)\,dx\,d\xi \leqslant CR^{n+2}\Lambda(R^2).$$

Thus, there exists a sequence $t_j \downarrow 0$ and a measure $\mu \geqslant 0$ on $\mathbb{R}^n$, verifying (6.11), and such that $u(x,t_j)\,dx$ converges weakly to $d\mu$ on $\mathbb{R}^n$. If $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 < \tau < t < T$, then

$$\int [u(x,t) - u(x,\tau)]\eta(x)\,dx = \int_{0<\tau<s<t}\Delta\eta(x)\varphi(u(x,s))\,dx\,ds$$

From Theorem 5.5 and (6.12) it follows that

$$\left|\int [u(x,t) - u(x,\tau)]\eta(x)\,dx\right| \leqslant C[t^\sigma - \tau^\sigma],$$

where $\sigma = \sigma(a, n) \in (0, 1)$. Hence

$$\lim_{t \downarrow 0} \int u(x, t)\eta(x)\, dx = \int \eta\, d\mu,$$

which completes the proof of the theorem.

Our general uniqueness result will now be proved by the approximation technique of Dahlberg and Kenig [5].

Theorem 6.13. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let u, v belong to $S_\varphi(T)$ for some $T > 0$. If u and v have the same trace, then $u = v$ in $\mathbb{R}^n \times (0, T)$.*

PROOF. Notice that if u, v have the same trace on $\mathbb{R}^n$, so do u^*, v^* in $\mathbb{R}^{n+2}$. We may therefore assume that $n \geqslant 3$.

Pick $h \in C_0^\infty(\mathbb{R}^n)$, with $0 \leqslant h \leqslant 1$ and let $w = w(x, t, h)$ be the unique solution in $\mathbb{R}^n \times (0, \infty)$ with initial trace h_μ and $\sup_{t>0} \int w(x, t)\, dx < \infty$, where μ is the common trace of u and v. (The existence of w follows from the estimates is Sections 4 and 5, while the uniqueness follows from Section 2.) We claim that $w \leqslant u$. To show this, let for $\epsilon \in (0, T/2)$, U_ϵ be the unique bounded solution in $\mathbb{R}^n \times (0, \infty)$ with initial data $h(x)u(x, \epsilon)$, satisfying

$$\sup_{t>0} \int U_\epsilon(x, t)\, dx \leqslant \sup_{\epsilon \in (0, T/2)} \int h(x)u(x, \epsilon)\, dx \leqslant C_h$$

where $C_h < \infty$ by the Harnack inequality. For any continuous function η we have that

$$\lim_{\epsilon \to 0} \int \eta(x)h(x)u(x, \epsilon)\, dx = \int \eta h\, d\mu,$$

and hence, our compactness result, Lemma 4.4 shows that

$$\lim_{\epsilon \to 0} U_\epsilon(x, t) = w(x, t) \quad \text{for all} \quad (x, t) \in \mathbb{R}^n \times (0, \infty).$$

From the maximum principle Lemma 6.1, $U_\epsilon(x, t) \leqslant u(x, t + \epsilon)$ for $0 < t < T - \epsilon$.

Hence, $w \leqslant u$ in $\mathbb{R}^n \times (0, T)$. Observe next that if $0 \leqslant h_j \leqslant h_{j+1} \leqslant 1$, $h_j \in C_0^\infty(\mathbb{R}^n)$ and $\lim_{j \to \infty} h_j(x) = 1$ for all $x \in \mathbb{R}^n$, then $w(x, t, h_j) \leqslant w(x, t, h_{j+1}) \leqslant u(x, t)$. Let $w_\infty = \lim_{j+\infty} w(x, t, h_j)$. Since $\{w_j\}$ is locally bounded in $\mathbb{R}^n \times (0, T)$, it follows that $\{w_j\}$ is locally equicontinuous and hence w_∞ solves (1.1) in $\mathbb{R}^n \times (0, T)$. We also notice that λ, the trace of w_∞ is between $h_j\mu$ and μ for all j, and hence $\lambda = \mu$. Since $w_\infty \leqslant u$, it follows that

$$\lim_{t \to 0} \int_{|x| < R} |u(x, t) - w_\infty(x, t)|\, dx = 0 \quad \text{for all} \quad R > 0.$$

Hence, $u = w_\infty$ by Lemma 6.1, and similarly $v = w_\infty$, which concludes the proof of the Theorem.

Corollary 6.14. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $u, v \in S_\varphi(T)$ for some $T > 0$. Let u, v have traces μ, ν respectively. If $\mu \leqslant \nu$, then $u \leqslant v$ in $\mathbb{R}^n \times (0, T)$.*

PROOF. Let $h_j \in C_0^\infty(\mathbb{R}^n)$, $0 \leqslant h_j \leqslant h_{j+1} \leqslant 1$, and such that $\lim_{j\to\infty} h_j(x) = 1$ for all $x \in \mathbb{R}^n$. Let u_j, v_j be the solutions in $\mathbb{R}^n \times (0, \infty)$ with initial trace $h_j\mu$ and $h_j\nu$ respectively. Then $u_j \leqslant v_j$ since, by Lemma 4.4, $u_j = \lim_{\epsilon\to 0} u_{j,\epsilon}$, $v_j = \lim_{\epsilon\to 0} v_{j,\epsilon}$, where $u_{j,\epsilon}$, $v_{j,\epsilon}$ are the solutions with initial data $\eta_\epsilon * (h_j \cdot \mu)$, $\eta_\epsilon * (h_j \cdot \nu)$ respectively. Here $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$, where $\eta \geqslant 0$, $\int \eta = 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$. By Lemma 6.1, $u_{j,\epsilon} \leqslant v_{j,\epsilon}$, and our claim follows. Since $u = \lim_{j\to\infty} u_j$, $v = \lim_{j\to\infty} v_j$, the Corollary follows.

Theorem 6.15. *Let $\varphi_k \in \Gamma_a$, $0 < a < 1$, and let $u_k \in S_{\varphi_k}(T)$ for some $T > 0$. Assume that $\sup_k u_k(0, T) < \infty$. Let μ_k denote the initial trace of u_k, and assume that μ_k converges weakly to a non-negative measure μ on $\mathbb{R}^n$. If $\varphi_k \to \varphi \in \Gamma_a$ uniformly on compact subsets of $[0, \infty)$, then there is a $u \in S_\varphi(T)$ such that u_k converges uniformly to u on compact subsets of $\mathbb{R}^n \times (0, T)$, and the initial trace of u equals μ.*

PROOF. From the Harnak inequality and the pointwise estimates, it follows that for each compact set $K \subset \mathbb{R}^n \times (0, T)$, we have

$$\sup_k \|u_k\|_{L^\infty(K)} < \infty.$$

From the continuity results of Sacks ([11]), it follows that $\{u_k\}$ is equicontinuous on each compact subset $K \subset \mathbb{R}^n \times (0, T)$. Let $w \in S_\varphi(T)$ be locally the uniform limit of a subsequence of $\{u_k\}$. Using our pointwise estimates as in the proof of Theorem 6.10, it follows that w has trace μ. By the uniqueness result, Theorem 6.13, the Theorem follows.

Remark. Since our general uniqueness result, Theorem 6.10 has been established for all dimensions, we can now remove the restriction $n \geqslant 3$ in Sections 4 and 5.

7. Existence and blow up

This section will be devoted to studying the solvability and maximum time interval of existence for the initial value problem for $\partial u/\partial t = \Delta\varphi(u)$. The analogous problem for the porous medium equation $\partial u/\partial t = \Delta u^m$, $m > 1$,

was settled by Bénilan, Crandall and Pierre ([3]). We will use the notations introduced at the beginning of Section 5. Our existence results will be based on the following lemma.

Lemma 7.1. *Let $\varphi \in \Gamma_a$, $0 < a < 1$. There is a number $\delta = \delta(a, n) > 0$ such that if $|||\mu||| < \delta$, there exists a unique solution u of $\partial u/\partial t = \Delta\varphi(u)$ in $\mathbb{R}^n \times (0, 1)$, with initial trace μ.*

PROOF. Because of Theorem 5.5 and Theorem 6.15 it is enough to show that there exists $\delta = \delta(a, n) > 0$ such that if $f \geqslant 0$ is in $C_0^\infty(\mathbb{R}^n)$, $\varphi \in \Gamma_a \cap C^\infty([0, \infty))$, and u solves $\partial u/\partial t = \Delta\varphi(u)$, $u(x, 0) = f(x)$, then

$$\sup_{0<t<1} |||u(\cdot, t)||| \leqslant C = C(a, n), \tag{7.2}$$

whenever $|||f||| \leqslant \delta$.

Let $g(t) = |||u(\cdot, t)|||$, $\eta_R(x) = \eta(x/R)$, where $0 \leqslant \eta \leqslant 1$, $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta = 1$ for $|x| \leqslant 1$, and $\eta = 0$ for $|x| \geqslant 2$. Then,

$$\int u(x, t)\eta_R(x)\,dx = \int f(x)\eta_R(x)\,dx + \iint_{0<\tau<t} R^{-2}\varphi(u(x, \tau))\Delta\eta(x/R)\,dx\,d\tau. \tag{7.3}$$

For $0 < \tau < 1$, $R > 1$, let $v(x, s) = u(\rho x, s\tau)/\gamma$, where $\rho = 2R$, $\gamma = \Lambda(\rho^2/\tau)$. Then, $\partial v/\partial t = \Delta\psi(v)$ for a $\psi \in \Gamma_a$, and the pointwise estimate Theorem 3.17 shows that

$$\sup_{|x|<1} v(x, 1) = \sup_{|x|<2R} u(x, \tau)/\gamma \leqslant C\{A^\sigma + A^\gamma\}, \tag{7.4}$$

where $A = G(\tau)\Lambda(R^2)\gamma^{-1}$, and $G(\tau) = \sup\{g(s): \tau/2 < s < \tau\}$, $0 < \tau < 1$.

Recall now that $\Lambda(bs) \leqslant b^\nu\Lambda(s)$, $b \geqslant 1$, $s \geqslant 1$, for some $\nu = \nu(a) > 0$. Hence, $A \leqslant C\tau^\nu G(\tau)$, and therefore, it follows from (7.4) that

$$\sup_{|x|<2R} u(x, \tau) \leqslant C\gamma\{\tau^{\nu\sigma}G^\sigma(\tau) + \tau^{\nu\gamma}G^\gamma(\tau)\}, \tag{7.5}$$

for $\tau \in (0, 1)$.

Recall now that $\Lambda^{-1}(u) = \varphi(u)/u$ whenever $u \geqslant 1$ and that $\Lambda^{-1}(\alpha\gamma) \leqslant \alpha^\theta\Lambda^{-1}(\gamma)$ for some $\theta = \theta(a)$ whenever $\alpha\gamma \geqslant 1$, $\gamma = \Lambda(\rho^2/\tau) \geqslant 1$. Hence, it follows from (7.5) that

$$\sup\left\{\frac{\varphi(u(x, \tau))}{u(x, \tau)}: |x| < 2R, u(x, \tau) \geqslant 1\right\} \leqslant C\frac{R^2}{\tau}\{\tau^l G^r(\tau) + \tau^\alpha G^\beta(\tau)\},$$

for some positive constants l, r, α, β.

Hence, (7.3) shows that, if $0 < t < 1$, then

$$\int_{|x|<R} u(x,t)\,dx \leqslant \int_{|x|\leq 2R} f(x)\,dx + CR^{-2}\int_0^t \int_{|x|<2R} [1+u(x,\tau)]\frac{R^2}{\tau}[\tau^l G^r(\tau) + \tau^\alpha G^\beta(t)]\,dx\,d\tau.$$

Next, we divide by $R^n\Lambda(R^2)$ and take the supremum over $R > 1$. Hence,

$$G(t) \leqslant C\delta + C\int_0^t [\tau^{l-1}G^{r+1}(\tau) + \tau^{\alpha-1}G^{\beta+1}(\tau)]\,d\tau.$$

It is then easy to see that there are M_0 and $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then $\sup_{\tau\in(0,1)} G(\tau) \leqslant M_0 < \infty$, which finishes the proof of the Lemma.

Before studying the initial value problem for a general measure $\mu \geqslant 0$ in $\mathbb{R}^n$, we wish to make some remarks on blow up. Note that as a consequence of our Harnack inequality, if μ is the initial trace of a $u \in S_\varphi(T)$, $\varphi \in \Gamma_a$, then

$$|||\mu||| < \infty \tag{7.6}$$

(see (6.11)). In order to give an estimate of the largest possible time interval for a solution to exist, we introduce some notation. Let $\varphi \in \Gamma_a$, $0 < a < 1$. Set, for $r \geqslant 1$, and $0 < t < r^2$

$$\begin{cases} A_r(t) = \sup\limits_{x\geq r^2} [\Lambda_\varphi(x)/\Lambda_\varphi(x/t)] \\ B_r(t) = \inf\limits_{x\geq r^2} [\Lambda_\varphi(x)/\Lambda_\varphi(x/t)] \end{cases} \tag{7.7}$$

It is easy to see that the limits

$$A(t) = A_\varphi(t) = \lim_{r\to\infty} A_r(t), \quad \text{and} \quad B(t) = B_\varphi(t) = \lim_{r\to\infty} B_r(t)$$

exist. Furthermore. A_φ and B_φ are strictly increasing on $(0,\infty)$, with

$$A_\varphi(0) = B_\varphi(0) = 0, \qquad \lim_{t\to\infty} A(t) = \lim_{t\to\infty} B_\varphi(t) = +\infty.$$

Our blow up result is the following.

Theorem 7.7. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let $u \in S_\varphi(T)$, for some $T > 0$. Let μ be the initial trace of u. Then,*

$$|||\mu|||_\infty \leqslant C/B_\varphi(T). \tag{7.8}$$

PROOF. Note that, by our Harnack inequality (Theorem 4.12), it follows that, for all $t \in (0, T)$, $R \geqslant \sqrt{t}$,

$$\mu\{|x| < R\} \leqslant CR^n \Lambda(R^2/t) + F(t),$$

where $F(t)$ is independent of R. Hence, for $r \geqslant 1$,

$$|||\mu|||_r \leqslant CB_r(t)^{-1} + F(t) r^{-n} \Lambda(r^2)^{-1}.$$

Letting $r \to \infty$, and $t \to T$, we obtain (7.8).

We will now give our general existence result.

Theorem 7.9. *Let $\varphi \in \Gamma_a$, $0 < a < 1$, and let μ be a measure with $|||\mu|||_\infty < \infty$. Then, there is a $T > 0$ such that μ is the initial trace of a $u \in S_\varphi(T)$. More precisely, there is a constant $C = C(a, n) > 0$ such that $A_\varphi(T_\varphi(\mu)) \geqslant C(|||\mu|||_\infty)^{-1}$,* where $T_\varphi(\mu) = \sup\{T: \mu \text{ is a trace of a } u \in S_\varphi(T)\}$.

PROOF. For $0 < \tau < \rho^2$, let $\psi(u) = \varphi(\gamma u)/\gamma(\gamma)$, where $\gamma = \Lambda(\rho^2/\tau)$. Then, $u \in S_\varphi(T)$ if and only if $v \in S_\psi(T\tau^{-1})$, where $v(x, t) = u(\rho x, \tau t)/\gamma$. Let the measure $\lambda \geqslant 0$ be defined by

$$\int f(x)\, d\lambda = \rho^{-n} \gamma^{-1} \int f(x/\rho)\, d\mu.$$

Then, λ is the initial trace of v if and only if μ is the initial trace of u. For $r \geqslant 1$, we have that

$$\lambda\{|x| < r\} = \rho^{-n}\gamma^{-1}\mu\{|x| < \rho r\} \leqslant |||\mu|||_{\varphi,\rho} r^n \Lambda_\varphi((\rho r)^2)\gamma^{-1},$$

where $|||\ \ |||_{\varphi,\rho}$ is as in (5.1), but emphasizing the dependence on φ.

Observe now that if $\xi \geqslant 1$ and $\eta = \Lambda_\Psi(\xi)$, then

$$\xi\varphi(\gamma)/\gamma = \psi(\eta)\varphi(\gamma)/\gamma\eta = \varphi(\gamma\eta)/\gamma\eta.$$

Since $\varphi(\gamma)/\gamma = \rho^2/\tau$, we have that

$$\gamma\eta = \gamma\Lambda_\Psi(\xi) = \Lambda_\varphi(\rho^2\xi/\tau).$$

Hence

$$|||\lambda|||_{\varphi,\rho} \leqslant \sup_{r \geqslant 1} \Lambda_\varphi(\rho^2((/\Lambda_\varphi((\rho r^2)/\tau) = |||\mu|||_{\varphi,\rho} A_\rho(\tau).$$

Suppose now that $|||\mu|||_{\varphi,\infty} < \infty$. Choose now $\tau > 0$ such that $|||\mu|||_\rho A(\tau) < \delta$, where $\delta = \delta(a, n)$ is as in Lemma 7.1. We can find now a $\rho > \max(\sqrt{\tau}, 1)$ such that $|||\mu|||_{\varphi,\rho} A_\rho(\tau) \leqslant \delta$. By Lemma 7.1, we can find a $v \in S_\psi(1)$, with initial trace λ. Since $u \in S_\varphi(\tau)$, where $u(x, t) = \gamma v(x/\rho, t/\tau)$, the Theorem follows.

Finally, we will give a necesary and sufficient condition for $T_\varphi(\mu) = +\infty$. Let S_φ denote the class of non-negative, continuous solutions of (1.1) in $\mathbb{R}^n \times (0, \infty)$.

Corollary 7.10. *Let $\varphi \in \Gamma_a$, $0 < a < 1$. A non-negative measure μ on $\mathbb{R}^n$ is the initial trace of a $u \in S_\varphi$ if and only if $|||\mu|||_\infty = 0$.*

PROOF. If $u \in S_\varphi$, (7.8) gives $|||\mu|||_\infty = 0$. Conversely, if $|||\mu|||_\infty = 0$, by Theorem 7.9 there is a solution $u_T \in S_\varphi(T)$, with initial trace μ for every $T > 0$. By our uniqueness result, $u_T = u_\tau$ if $\tau < T$, and this finishes the proof.

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A Weak Regularity Theorem for Real Analytic Optimal Control Problems

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Abstract

We consider real analytic finite-dimensional control problems with a scalar input that enters linearly in the evolution equations. We prove that, whenever it is possible to steer a state x to another state y by means of a measurable control, then it is possible to steer x to y by means of a control that has an extra regularity property, namely, that of being analytic on an open dense subset of its interval of definition. Since open dense sets can have very small measure, this is a very weak property. However, it is absolutely general, depending on no assumptions other than real analyticity. This shows that real analyticity alone suffices to imply some regularity, and leaves open the question of how much more regularity can be proved in general. To show that real analyticity is essential, we prove, by constructing a class of examples, that no such theorem is true in the C^∞ case. The regularity result implies a similar regularity theorem for time-optimal controls.

1. Introduction

Existence theorems for optimal controls usually give existence of such a control in some large function space (e.g. the space of all bounded measurable

functions on an interval). However, in problems where one can actually compute the solutions explicitly, they often turn out to be much more regular than might have been expected from the existence theory. This raises the question whether optimal controls really can be as pathological as the general theory appears to allow, or perhaps there are general regularity theorems that limit the possible pathologies that can occur.

The purpose of this note is to argue that there exists a fundamental distinction between systems of class C^∞ and real-analytic systems. In the former case, we show by means of a simple example that no *a priori* restriction on the pathology of optimal controls is possible, in the sense that, given any measurable control η, then one can construct an optimal control problem of which η is the only solution. For real-analytic systems, on the other hand, general regularity theorems exist. We establish this by proving one such theorem. The precise regularity property of optimal controls obtained in the theorem is rather weak: if a control $u(\cdot)$ is defined on an interval $[0, T]$, let us say that $u(\cdot)$ has property (R) if there is a relatively open dense subset Ω of $[0, T]$ such that $u(\cdot)$ is real-analytic on Ω. Our regularity theorem asserts that, whenever it is possible to steer a point x_1 to a point x_2 with cost c, then this can also be done by means of a control that has property (R). This statement can obviously be applied if c is the optimal cost, in which case we get the conclusion that, if there is an optimal control that steers x_1 to x_2, then there is one that has property (R).

We believe that our formulation of the regularity problem («find a class $\mathfrak{R}$ such that, whenever x_1 can be optimally steered to x_2, then this can also be done by means of a control in $\mathfrak{R}$») is more natural than the simpler one of asking for a class $\mathfrak{R}$ such that every optimal control is in $\mathfrak{R}$. The reason is that there are problems that have very pathological solutions because they are very degenerate. (The most extreme case is that of problems where every control is optimal. This can happen, e.g., if the control does not appear at all in the dynamical equations and the cost functional.) For these problems, the appropriate question is not whether *all* solutions are «nice», but whether there is at least one «nice» solution. This is precisely captured by our formulation. Naturally, when the optimal controls are unique, both formulations agree.

The particular class $\mathfrak{R}$ given by our theorem is still very large since, for instance, the complement of the set Ω could have Lebesgue measure arbitrarily close to T. However, the main point we wish to make here is that real analyticity alone, in the absence of any other hypothesis, already has some nontrivial regularity implications. At the moment, it is an open question whether stronger regularity properties can be proved under the same hypothesis. The gap between the worse pathology that has been found in examples (e.g. Fuller's problem, cf. Marchal [9]) and our positive result is

quite large. On the other hand, some partial knowledge is available which, for some cases, gives better regularity. (E.g. bang-bang theorems for linear systems, cf. Lee and Markus [8], nonlinear bang-bang theorems, cf. Krener [7], Sussmann [14], and a general piecewise analytic regularity theorem for time-optimal control of real-analytic systems $\dot{x} = f(x) + ug(x)$, $|u| \leqslant 1$, in the plane, cf. Sussmann [15], [17], [18], [19].)

For simplicity, we will confine our discussion to time-optimal control of systems of the form

$$\dot{x} = f(x) + ug(x), \qquad |u| \leqslant 1, \tag{1}$$

where the state x evolves in a finite-dimensional manifold M of class C^∞, and f, g are C^∞ vector fields on M. We call such a system *real-analytic* if M is a real-analytic manifold and f, g are real-analytic vector fields. Our theorem is actually true for more general systems $\dot{x} = f(x, u)$, with more general cost functionals, but the proof is much more delicate and requires the use of difficult stratification theorems for subanalytic sets. The case considered here is much simpler, but sufficient to show the main ideas. The general proof will appear elsewhere.

2. An example

Let $T > 0$, and take an arbitrary measurable function η: $[0, T] \to [-1, 1]$. We will exhibit a system of the form (1), an initial state x_1 and a terminal state x_2, such that η steers x_1 to x_2 in time T, and no other control does.

Define a function θ: $[0, T] \to \mathbb{R}$ by

$$\theta(t) = \int_0^t \eta(s)\, ds, \qquad 0 \leqslant t \leqslant T. \tag{2}$$

Let $K = \{(t, \theta(t)): 0 \leqslant t \leqslant T\}$. Then K is a compact subset of $\mathbb{R}^2$. Therefore there exists a C^∞ function ϕ: $\mathbb{R}^2 \to \mathbb{R}$ such that $\phi = 0$ on K and $\phi > 0$ on $\mathbb{R}^2 - K$. We then take our system to be

$$\dot{x} = 1, \tag{3.i}$$

$$\dot{y} = u, \tag{3.ii}$$

$$\dot{z} = \phi(x, y), \tag{3.iii}$$

where x, y, z are real variables, and the control u satisfies $|u| \leqslant 1$.

Let $x_1 = (0, 0, 0)$, $x_2 = (T, \theta(T), 0)$. Then η steers x_1 to x_2 and, if η': $[0, T'] \to$ $\to [-1, 1]$ is any other control that steers x_1 to x_2 in time T', then necesarily $T' = T$ and $\eta' = \eta$ almost everywhere.

3. The theorem

We now consider a control system of the form (1), where the variable x takes values in a finite-dimensional, real-analytic manifold M, and f and g are real-analytic vector fields on M. An *admissible control* is a measurable function, defined on some interval of the form $[0, T]$ for some $T \geqslant 0$, which takes values in $[-1, 1]$. If $u(\cdot)\colon [0, T] \to [-1, 1]$ is an admissible control, and $\gamma(\cdot)\colon [0, T] \to M$ is a trajectory for $u(\cdot)$, such that $\gamma(0) = x_1$, $\gamma(T) = x_2$, then we say that $u(\cdot)$ *steers* x_1 *to* x_2 *in time* T. If there exists a $u(\cdot)$ that steers x_1 to x_2 in time T, we say that x is *reachable from* x_1 in time T. The set of all such x is the *time* T *reachable set from* x_1, and will be denoted $\mathrm{Reach}_T(x_1)$. We will prove:

Theorem. *Let f, g be C^ω vector fields on the C^ω manifold M. Suppose that, for the system* (1), *x_1, x_2, T are such that $x_2 \in \mathrm{Reach}_T(x_1)$. Then there exists a control $u(\cdot)$ that steers x_1 to x_2 in time T and is such that $u(\cdot)$ is real-analytic on some open dense subset of the interval* $[0, T]$.

PROOF. We first show that, without loss of generality, we can make some extra assumptions and simplifications. Let L be the Lie algebra of vector fields on M generated by f and g. For $x \in M$, let $L(x) = \{X(x)\colon X \in L\}$. Since L is a Lie algebra of real-analytic vector fields, M can be partitioned into *maximal integral manifolds* of L, i.e. connected real-analytic submanifolds S such that (i) whenever $x \in S$, then the tangent space of S at x is $L(x)$, and (ii) S is not properly contained in any connected submanifold S' that satisfies (i) (cf. [10], [11], [16]). If $S(x_1)$ is the maximal integral manifold through x_1, then the vector fields f, g have well defined restrictions $\tilde{f}, \tilde{g}$, that are tangent to $S(x_1)$. Moreover, every trajectory that goes through x_1 is entirely contained in $S(x_1)$. Hence it suffices to prove the theorem with $S(x_1), \tilde{f}, \tilde{g}$ insteady of M, f, g. Equivalently, we may assume that

(I) *$L(x)$ is the tangent space to M at x for every $x \in M$.*

This assumption will be made from now on.

Let us call a control $u(\cdot)\colon [0, T] \to [-1, 1]$ *nice* if $u(\cdot)$ is real analytic on an open dense subset of the interval $[0, T]$. Call a trajectory nice if it corresponds to a nice control. We are trying to prove that, if x_2 can be reached from x_1 in time T, then x_2 can be reached from x_1 in the same time by means of a nice trajectory. We claim that it is sufficient to prove the weaker statement:

(II) *whenever $x_1 \in M$, $x_2 \in M$ are such that x_2 can be reached from x_1, then x_2 can be reached from x_1 by means of a nice trajectory.*

Indeed, if (II) holds, then we can consider, for given M, f, g, the «system obtained by adding time as a new variable». That is, we let $\tilde{M} = M \times \mathbb{R}$, and we let $(x, \xi) \in \tilde{M}$ evolve according to

$$\dot{x} = f(x) + ug(x), \tag{4.i}$$

$$\dot{\xi} = 1. \tag{4.ii}$$

Then x_2 is reachable from x_1 in time T for the old system if and only if (x_2, T) is reachable from $(x_1, 0)$ for the new system. Therefore, if we apply (II) to the new system, we obtain the desired conclusion.

We now show

(III) *if x_2 is an interior point of the set reachable from x_1, then x_2 can be reached from x_1 by means of a bang-bang control with a finite number of switchings.*

To see this, we first apply Chow's Theorem (cf. [7], [16]) to the system

$$\dot{x} = -f(x) - ug(x), \tag{5}$$

whose trajectories are those of the original system traversed backwards. We take $\epsilon > 0$ so small that every point reachable from x_2 by means of a trajectory of (5) in time $\leqslant \epsilon$ is in the reachable set from x_1 for the system (1). By the positive form of Chow's Theorem, there exists a nonempty open set Ω such that every $x_2' \in \Omega$ is reachable from x_2 in time $\leqslant \epsilon$ by means of a trajectory of (5) which is bang-bang with finitely many switchings. Then Ω is contained in the set reachable from x_1 for (1). Let $x_2 \in \Omega$ be reachable from x_1 by a trajectory $\gamma(\cdot)$ of (1) that corresponds to a control $u(\cdot)$, defined on $[0, T]$. Let $\{u_n(\cdot)\}$ be a sequence of bang-bang controls, defined on $[0, T]$, which converge weakly to $u(\cdot)$. Let $\{\gamma_n(\cdot)\}$ be the corresponding trajectories of (1), with initial condition $\gamma_n(0) = x_1$. Then $\gamma_n(T) \in \Omega$ if n is large enough. Pick n such that $\gamma_n(T) \in \Omega$, and let $x_2'' = \gamma_n(T)$. Then x_2'' is reachable from x_1, and x_2 is reachable from x_2'', by means of bang-bang trajectories of (1). Therefore, x_2 is bang-bang reachable from x_1, and (III) is proved.

In view of the preceding observations, it suffices to prove that, if (I) holds, then every point x_2 that can be reached from x_1 and belongs to the boundary of the reachable set from x_1 is reachable by a nice trajectory.

Let such x_1, x_2 be given. We can then apply the Pontryagin Maximum Principle, which we first state in the form that will be needed here. Let T^*M denote the cotangent bundle of M, and let $T^{\#}M$ be T^*M with the zero section

removed, i.e.

$$T^{\#}M = \{(x,p): x \in M, p \in T_x^*M, p \neq 0\}, \tag{6}$$

where T_x^*M denotes the cotangent space of M at x. Let

$$X_+ = f + g, \qquad X_- = f - g. \tag{7}$$

Let

$$H_+(x,p) = \langle p, X_+(x)\rangle, \tag{8.i}$$

$$H_-(x,p) = \langle p, X_-(x)\rangle. \tag{8.ii}$$

Then H_+, H_- are real-analytic functions on the symplectic manifold $T^{\#}M$. Therefore, H_+, H_- give rise to Hamiltonian vector fields $X_+^{\#}, X_-^{\#}$ on $T^{\#}M$. The system (1) can also be written as

$$\dot{x} = vX_-(x) + (1-v)X_+(x), \qquad 0 \leqslant v \leqslant 1, \tag{9}$$

where the control v is related to u by

$$v = \frac{1}{2}(1-u). \tag{10}$$

We can then consider the *Hamiltonian system associated to* (9), i.e. the system

$$\dot{z} = vX_-^{\#}(z) + (1-v)X_+^{\#}(z), \tag{11}$$

where $z = (x,p) \in T^{\#}M$. If $\gamma(\cdot)$ is a trajectory of (9) for a control $v(\cdot)$, on an interval $[0,T]$, then any trajectory of (11) which is of the form $t \to (\gamma(t), p(t))$, $0 \leqslant t \leqslant T$, is called a *Hamiltonian lift* of γ. A trajectory $\zeta(\cdot) = (\gamma(\cdot), p(\cdot))$ of (11), and corresponding control $v(\cdot)$, are called *null-minimizing* if, for almost every $t \in [0,T]$, we have

$$\langle p(t), v(t)X_-(\gamma(t)) + (1-v(t))X_+(\gamma(t))\rangle = \tag{12}$$

$$= \min_{0 \leqslant w \leqslant 1} \langle p(t), wX_-(\gamma(t)) + (1-w)X_+(\gamma(t))\rangle.$$

The Maximum Principle says that if $v(\cdot)$ is a control on $[0,T]$, and $\gamma(\cdot)$ is a corresponding trajectory of (9), such that $\gamma(T)$ is on the boundary of the reachable set from $\gamma(0)$, then $(\gamma(\cdot), v(\cdot))$ has a Hamiltonian lift which is null-minimizing.

Now let x_1, x_2 be as above, and let $v(\cdot)$ steer x_1 to x_2 in time T. Let $\gamma(\cdot)\colon [0,T] \to M$ be the corresponding trajectory. Let $t \to p(t)$, $0 \leqslant t \leqslant T$, be such that $(\gamma(\cdot), p(\cdot))$ is a null-minimizing Hamiltonian lift of $\gamma(\cdot)$. Let

$\zeta(t) = (\gamma(t), p(t))$. We will show that $v(\cdot)$ is nice. In order to do this, we first have to construct a stratification of $T^{\#}M$ which has some special properties.

We partition $T^{\#}M$ into five sets S_1, S_2, S_3, S_4, S_5 by letting

$$\begin{aligned}
&(x,p) \in S_1 \quad \text{if} \quad H_+(x,p) = H_-(x,p) = 0,\\
&(x,p) \in S_2 \quad \text{if} \quad H_+(x,p) = 0 \quad \text{but} \quad H_-(x,p) \neq 0,\\
&(x,p) \in S_3 \quad \text{if} \quad H_-(x,p) = 0 \quad \text{but} \quad H_+(x,p) \neq 0,\\
&(x,p) \in S_4 \quad \text{if} \quad H_+(x,p)H_-(x,p) > 0, \quad \text{and}\\
&(x,p) \in S_5 \quad \text{if} \quad H_+(x,p)H_-(x,p) < 0.
\end{aligned}$$

Each of the sets S_i is semianalytic. Therefore there exists a real-analytic stratification Σ_0 of $T^{\#}M$ such that the strata of Σ_0 are semianalytic subsets of $T^{\#}M$, and each stratum of Σ_0 is entirely contained in one of the S_i. (cf. [3]).

We now define a sequence of stratifications $\Sigma_0, \Sigma_1, \ldots$, each of which is obtained from the preceding one by refining it in a suitable way. The procedure for refining a stratification is as follows. Let Σ be an arbitrary C^ω stratification of $T^{\#}M$, whose strata are subanalytic sets. Let $S \in \Sigma$. We partition S into five sets $B_1(S), B_2(S), B_3(S), B_4(S), B_5(S)$ as follows: if $(x,p) \in S$, we let $(x,p) \in B_1(S)$ if both vectors $X^{\#}_+(x,p), X^{\#}_-(x,p)$ are tangent to S; we let $(x,p) \in B_2(S)$ if $X^{\#}_+(x,p)$ is tangent to S but $X^{\#}_-(x,p)$ is not, $(x,p) \in B_3(S)$ if $X^{\#}_-(x,p)$ is tangent to S but $X^{\#}_+(x,p)$ is not, $(x,p) \in B_4(S)$ if $X^{\#}_+(x,p)$ and $X^{\#}_-(x,p)$ are not tangent to S but some convex combination of them is, and $(x,p) \in B_5(S)$ if no convex combination $vX^{\#}_-(x,p) + (1-v)X^{\#}_+(x,p)$, $v \in [0,1]$, is tangent to S. It is clear that the $B_i(S)$ are subanalytic subsets of $T^{\#}M$. Moreover, on the set $B_4(S)$ we can define a function $v_{4,S}: B_4(S) \to (0,1)$ by letting $v_{4,S}(x,p)$ be the unique v such that $0 < v < 1$ and $vX^{\#}_-(x,p) + (1-v)X^{\#}_+(x,p)$ is tangent to S. The function $v_{4,S}$ is obviously subanalytic (i.e. its graph is a subanalytic subset of $T^{\#}M \times \mathbb{R}$). Moreover, $v_{4,S}$ is real-analytic, in the sense that, if $(\bar{x},\bar{p}) \in B_4(S)$, then there is a neighborhood W of $(\bar{x},\bar{p})$ in S, and a real-analytic real-valued function w on W, such that $v_{4,S} = w$ on $W \cap B_4(S)$. (To see this, choose a C^ω coordinate chart $(z_1, \ldots, z_m)$ that maps a neighborhood $\tilde{W}$ of $(\bar{x},\bar{p})$ in $T^{\#}M$ diffeomorphically onto the unit cube in $\mathbb{R}^m$, in such a way that $(\bar{x},\bar{p})$ is mapped to 0, and the set $W = \{z: z_1 = \cdots = z_r = 0\}$ is, for some r, a neighborhood of $(\bar{x},\bar{p})$ in S. Relative to $(z_1, \ldots, z_m)$, the vector fields $X^{\#}_+, X^{\#}_-$ have components $\sigma_1^+, \ldots, \sigma_m^+$ and $\sigma_1^-, \ldots, \sigma_m^-$. Since $(\bar{x},\bar{p}) \in B_4(S)$, there exists a j such that $1 \leqslant j \leqslant r$ and $\sigma_j^+(\bar{x},\bar{p}) \neq \sigma_j(\bar{x},\bar{p})$. Then we can assume, after shrinking $\tilde{W}$ if necessary, that $\sigma_j^+(z) \neq \sigma_j^-(z)$ for all $z \in \tilde{W}$. For $z \in \tilde{W}$, let

$$\tilde{w}(z) = \frac{\sigma_j^+(z)}{\sigma_j^+(z) - \sigma_j^-(z)}. \tag{13}$$

Then $\tilde{w}$ is real-analytic on $\tilde{W}$, and

$$\tilde{w}\sigma_j^- + (1 - \tilde{w})\sigma_j^+ = 0. \tag{14}$$

If $(x, p) \in B_4(S) \cap W$, then $v_{4,S}(x, p)$ also satisfies

$$v_{4,S}(x, p)\sigma_j^-(x, p) + (1 - v_{4,S}(x, p))\sigma_j^+(x, p) = 0. \tag{15}$$

Therefore $v_{4,S}(x, p) = \tilde{w}(x, p)$. Hence we can take w to be the restriction of $\tilde{w}$ to W.) In particular, if $B_4(S) = S$, then $v_{4,S}$ is real-analytic on S.

We now describe the procedure for refining a stratification. If Σ is a real-analytic stratification of $T^{\#}M$ by subanalytic sets, we call Σ *good up to codimension* k if every stratum $S \in \Sigma$ of codimension $\leqslant k$ is such that $S = B_i(S)$ for some $i \in \{1, 2, 3, 4, 5\}$. We call Σ *good* if Σ is good up to codimension ν, where $\nu = \dim T^{\#}M = 2 \dim M$. If Σ is good up to codimension k, we can refine Σ by letting $\mathcal{E}$ be the set of all sets $B_i(S)$, where S ranges over all the strata of Σ of codimension $> k$, and $i \in \{1, 2, 3, 4, 5\}$. Then $\mathcal{E}$ is a locally finite family of subanalytic subsets of $T^{\#}M$. Therefore there exists a C^ω stratification $\tilde{\Sigma}$ of $\cup \{E: E \in \mathcal{E}\}$ by subanalytic sets, such that every $E \in \mathcal{E}$ is a union of strata of $\tilde{\Sigma}$. We then define Σ to consist of all the strata of Σ of codimension $\leqslant k$, as well as all the strata of $\tilde{\Sigma}$. Then Σ is a C^ω stratification by subanalytic sets. Moreover, Σ is good up to codimension $k + 1$. (Indeed, if $S \in \Sigma$ and $\operatorname{codim} S \leqslant k$, then $S \in \Sigma$ and so $S = B_i(S)$ for some i. If $\operatorname{codim} S = k + 1$, then $S \subseteq E$ for some $E \in \mathcal{E}$, i.e. $S \subseteq B_i(F)$ for some i and some $F \in \Sigma$ such that $\operatorname{codim} F > k$. But then $\operatorname{codim} F = k + 1$ and S is relatively open in F, so that $B_i(S) = B_i(F) \cap S$, and then $B_i(S) = S$.)

We apply this refining procedure successively, starting with Σ_0, and construct C^ω subanalytic stratifications $\Sigma_1, \Sigma_2, \ldots$ such that Σ_k is good up to codimension k for each k. If $\nu = \dim T^{\#}M$, we let $\Sigma = \Sigma_\nu$, so that Σ is good. Hence Σ is a C^ω stratification of $T^{\#}M$ by subanalytic sets, such that every stratum $S \in \Sigma$ is subanalytic and satisfies $S \subseteq S_i$ for some i, and $S = B_j(S)$ for some j.

We now show that Σ contains no stratum S such that $S \subseteq S_1$ and $S = B_1(S)$. Indeed, suppose $S \subseteq S_1$, $S = B_1(S)$, $S \in \Sigma$. For each C^ω vector field X on M, we can consider the lifted Hamiltonian vector field $X^{\#}$, i.e. the Hamiltonian vector field that arises from the Hamiltonian function $(x, p) \to \langle p, X(t)\rangle$. Then $X \to X^{\#}$ is a Lie algebra homomorphism. Since $X_-^{\#}$ and $X_+^{\#}$ are tangent to S (because $S = B_1(S)$), it follows that $X^{\#}$ is tangent to S for all $X \in L$. (Recall that L is the Lie algebra generated by f and g.) If $X \in L$, then $X^{\#}$ is tangent to S. Therefore the derivative of H_- in the direction of $X^{\#}$ vanishes throughout S (because $H_- \equiv 0$ on S). So $\langle p, [X_-, X](x)\rangle = 0$ for all $(x, p) \in S$. Similarly, $\langle p, [X_+, X](x)\rangle = 0$ for $(x, p) \in S$, $X \in L$. Finally, we know that $\langle p, X_-(x)\rangle = \langle p, X_+(x)\rangle = 0$ if $(x, p) \in S$, and so $\langle p, X(x)\rangle = 0$

for all $(x, p) \in S$, $X \in L$. But then, if we pick $(x, p) \in S$, we have $\langle p, X(x)\rangle = 0$ for all $X \in L$, i.e. $\langle p, \theta\rangle = 0$ for all $\theta \in L(x)$. Since $L(x) = T_x M$, we have $\langle p, \theta\rangle = 0$ for all $\theta \in T_x M$, and so $p = 0$, contradicting the fact that $(x, p) \in T^{\#}M$.

Let us now return to our null-minimizing Hamiltonian trajectory $\zeta(\cdot)$, that corresponds to a control $v(\cdot)$. Let $\zeta(\cdot)$ and $v(\cdot)$ be defined on $[0, T]$. For each t, $\zeta(t)$ belongs to a stratum $S(t)$ of Σ. Let $k(t) = \dim S(t)$. Since Σ is a stratification, every t has a neighborhood I such that $k(\tau) \geqslant k(t)$ for $\tau \in I$. (Otherwise, there would exist a sequence $\{t_n\}$ such that $t_n \to t$ and $k(t_n) < k(t)$. Since Σ is locally finite, we may pass to a subsequence and assume that all the $S(t_n)$ are one on the same stratum S. Then $\dim S < \dim S(t)$, but $\zeta(t) \in S(t)$ and $\zeta(t) \in \operatorname{Clos} S$. So $S(t) \subseteq \operatorname{Clos} S$, and then $\dim S(t) < \dim S$, which is a contradiction.)

We now let Ω be the set of those $t \in [0, T]$ such that, for some $\epsilon > 0$, the stratum $S(\tau)$ is the same for all $\tau \in (t - \epsilon, t + \epsilon) \cap [0, T]$. Then Ω is relatively open in $[0, T]$. We claim that Ω is dense. To see this, let $t \in [0, T]$, $\epsilon > 0$. $\tau' \in (t - \epsilon, t + \epsilon) \cap [0, T]$ be such that $k(\tau') = \max\{k(\tau): |\tau - t| < \epsilon, \tau \in [0, T]\}$. We know that $k(\tau) \geqslant k(\tau')$ for $\tau \in I$, where I is some interval that contains τ' in its interior relative to $[0, T]$, and $I \subseteq [0, T]$. We can also assume that $I \subseteq (t - \epsilon, t + \epsilon)$, and then $k(\tau) = k(\tau')$ for $\tau \in I$. It then follows that $S(\tau) = S(\tau')$ for $\tau \in I$. (Otherwise, there would exist a sequence $\{\tau_n\}$ such that $\tau_n \in I$, $\tau_n \to \tau$, $\tau \in I$, and $S(\tau_n) = S$ for some S such that $S \neq S(\tau)$. But then $\zeta(\tau) \in \operatorname{Clos} S$ but $\zeta(\tau) \notin S$, and so $S(\tau) \subseteq \operatorname{Clos} S$, $S(\tau) \neq S$. Therefore $\dim S(\tau) < \dim S$, contradicting the fact that $k(\tau) = k(\tau_n)$.) Then $\tau' \in \Omega$, and so Ω is dense.

We now prove that $v(\cdot)$ is real-analytic on Ω.

Let $t \in \Omega$, and let I be an interval such that $t \in I \subseteq [0, T]$, I is relatively open in $[0, T]$, and $S(\tau) = S(t)$ for all $\tau \in I$. Let $S = S(t)$. Let $\tilde{\xi}$, $\tilde{v}$ be the restrictions of ζ, v to I.

We know that S is contained in one of the sets S_i. Since ζ is null-minimizing, at least one of the functionals H_+, H_- vanishes at $\zeta(t)$. Hence S is contained in S_1 or S_2 or S_3. If $S \subseteq S_2$ then $H_+ = 0$ on S and H_- never vanishes on S. Since $\tilde{\xi}$ is contained in S and is null-minimizing, it follows that $\tilde{v} \equiv 0$. If $S \subseteq S_3$, it follows that $\tilde{v} \equiv 1$. Finally, we must consider the case when $S \subseteq S_1$. In this case, we know that $S = B_i(S)$ for some i. Also, we know that Σ contains no stratum S' such that $S' \subseteq S_1$ and $S' = B_1(S)$. Hence $S = B_i(S)$ for some $i \in \{2, 3, 4, 5\}$. Since $\tilde{\xi}$ is contained in S, the tangent vector $\dot{\tilde{\xi}}(\tau)$ (which exists for almost every $\tau \in I$, and is a convex combination of $X^{\#}_+(\tilde{\xi}(\tau))$ and $X^{\#}_-(\tilde{\xi}(\tau))$) is actually tangent to S. This excludes the possibility that $S \subseteq B_5(S)$. Hence $S = B_i(S)$, where $i = 2$ or $i = 3$ or $i = 4$. If $i = 2$ then $\tilde{v}(\tau)$ must be equal to 0 for all $\tau \in I$. Similarly, if $i = 3$ we have $\tilde{v}(\tau) = 1$ for $\tau \in I$. If $i = 4$, then $\tilde{v}(\tau) = v_{4,S}(\tilde{\xi}(\tau))$. But then $\tilde{\xi}$ is an integral curve of the vector field V given by

$$V(x, p) = v_{4,S}(x, p) X^{\#}_-(x, p) + (1 - v_{4,S}(x, p)) X^{\#}_+(x, p) \tag{16}$$

for $(x, p) \in S$. Since $v_{4,S}$ is real-analytic on S, the vector field V is real-analytic on S, as well as tangent to S. But then $\tau \to \tilde{\xi}(\tau)$ is real-analytic, and so $\tilde{v}$ is real-analytic.

This completes the proof.

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Problèmes de Dirichlet, Neumann, Calderón dans les quasidisques pour les classes holderiennes

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1. Introduction et énoncé des résultats

1. Si E est un sous-ensemble de $\mathbb{C}$, nous désignons par $\Lambda^\alpha(E)$ $(0 < \alpha < 1)$ la classe des fonctions $f: E \to \mathbb{C}$ pour lesquelles il existe une constante $C > 0$ telle que

$$\forall x, y \in E, \qquad |f(x) - f(y)| \leqslant C|x - y|^\alpha,$$

et nous notons

$$\|f\|_{\Lambda^\alpha(E)} = \sup_{\substack{x,y \in E \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

2. Soit maintenant Ω un domaine de Jordan borné du plan complexe et $\Gamma = \partial\Omega$. Si $f \in C(\Gamma)$, soit u la solution du problème de Dirichlet: $\Delta u = 0$ dans Ω, $u|_\Gamma = f$. Si $0 < \alpha < 1$, nous dirons que l'on peut résoudre le problème de Dirichlet dans Ω pour Λ^α s'il existe une constante $C > 0$ telle que

$$\|u\|_{\Lambda^\alpha(\Omega)} \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}$$

pour toute fonction $f \in \Lambda^\alpha(\Gamma)$.

Il est bien connu que l'on peut résoudre le problème de Dirichlet dans le disque unité D pour tous les Λ^α, $0 < \alpha < 1$. D'un autre côté, on a le théorème suivant qui a été le point de départ de notre travail.

Théorème 1 (Hinkkanen [6]). *Si Ω est un domaine de Jordan quelconque on peut résoudre le problème de Dirichlet dans Ω pour tous les Λ^α avec $\alpha \in \left(0, \frac{1}{2}\right)$ et l'indice $\frac{1}{2}$ est critique.*

Notre premier objectif a été d'étudier quelles conditions sur Ω peuvent permettre de dépasser cet indice critique $\frac{1}{2}$.

Bien évidemment, si Ω est régulier (de classe $C^{1+\epsilon}$ par exemple), la représentation conforme $\varphi: D \to \Omega$ est bilipschitzienne et l'on peut résoudre le problème de Dirichlet dans Ω pour tous les Λ^α avec $0 < \alpha < 1$. Mais même si Ω est de classe C^1, la représentation conforme n'est d'aucune utilité.

Une autre idée est celle de perturbation du disque unité, qui nous a conduit au résultat suivant:

Théorème 2. *Soit Ω un domaine de Jordan tel que la représentation conforme $\varphi: D \to \Omega$ s'étende en un homéomorphisme K-quasiconforme de la sphère de Riemann avec $1 \leqslant K \leqslant 2$; alors on peut résoudre le problème de Dirichlet dans Ω pour Λ^α si $0 < \alpha < 1/K$.*

Remarque. Afin de donner un énoncé du théorème 2 ne dépendant pas de la représentation conforme, rappelons qu'un K-quasidisque est l'image du disque unité par un homéomorphisme K-quasiconforme de la sphère de Riemann. Si Ω est un K-quasidisque, on montre facilement que la représentation conforme $\varphi: D \to \Omega$ se prolonge en un homéomorphisme K^2-quasiconforme de $\overline{\mathbb{R}}^2$. On a donc un énoncé analogue au théorème 2 pour les K-quasidisques, à condition cependant de remplacer K par K^2 dans l'énoncé. Naturellement, ce n'est une amélioration du théorème d'Hinkkanen que pour $K < \sqrt{2}$.

3. Soit toujours Ω un domaine de Jordan. Nous dirons que l'on peut résoudre le problème de Neumann dans Ω pour Λ^α si pour toute fonction f, réelle, appartenant à $\Lambda^\alpha(\Gamma)$ on peut trouver une fonction $g \in \Lambda^\alpha(\Gamma)$, réelle, telle que $f + ig$ soit la valeur au bord d'une fonction F holomorphe appartenant à $\Lambda^\alpha(\Omega)$.

Les théorèmes 1 et 2 ci-dessus, combinés avec le théorème suivant permettent de donner immédiatement une réponse satisfaisante au problème de Neumann dans les quasidisques.

Théorème 3 (Gehring-Martio [3]). *Soit Ω un quasidisque et u une fonction harmonique dans Ω. Alors*

$$\frac{1}{C}\sup_{z\in\Omega} d(z,\Gamma)^{-\alpha+1}|\nabla u(z)| \leqslant \|u\|_{\Lambda^\alpha(\Omega)} \leqslant C\sup_{z\in\Omega} d(z,\Gamma)^{-\alpha+1}|\nabla u(z)|$$

où $C>1$ ne dépend que de Ω et de $\alpha\in(0,1)$.

Corollaire. *Soit Ω un K-quasidisque et*

$$\beta = \max\left(\frac{1}{2}, \frac{1}{K^2}\right).$$

Alors on peut résoudre le problème de Neumann dans Ω pour Λ^α si $0<\alpha<\beta$.

PREUVE. Soit $f\in\Lambda^\alpha(\Gamma)$ et u la solution du problème de Dirichlet $\Delta u = 0$ dans Ω, $u|_\Gamma = f$. Par les théorèmes 1 et 2, $u\in\Lambda^\alpha(\Omega)$. Soit maintenant v le conjugué harmonique de u dans Ω; alors $|\nabla u| = |\nabla v|$ et le corollaire découle du théorème 3. □

4. Nous dirons que l'on peut résoudre le problème de Calderón pour $\Lambda^\alpha(\Gamma)$ si toute fonction $f\in\Lambda^\alpha(\Gamma)$ peut s'écrire de façon unique sous la forme $f=f_1+f_2$ avec $f_j\in\Lambda^\alpha(\Gamma)$ où f_1 est valeur au bord d'une fonction holomorphe $F_1\in\Lambda^\alpha(\Omega)$ et f_2 est valeur au bord d'une fonction holomorphe $F_2\in\Lambda^\alpha(\mathbb{C}\backslash\bar{\Omega})$ vérifiant de plus

$$\lim_{|z|\to\infty} F_2(z) = 0.$$

Là encore, si Ω est le disque unité on peut résoudre le problème de Calderón dans Λ^α pour $0<\alpha<1$. Si Ω est un domaine lipschitzien, c'est encore vrai et le résultat découle des travaux de Lemarié [7]. Il semble que rien n'était connu pour une courbe Γ non rectifiable. Nous nous proposons de démontrer le théorème suivant, qui donne une réponse partielle.

Théorème 4. *Il existe une constante $a>1$ telle que si Ω est un K-quasidisque avec*

$$K^{2a} \leqslant \frac{1+\sqrt{5}}{2},$$

on peut résoudre le problème de Calderón pour $\Lambda^\alpha(\Gamma)$ si

$$\alpha\in\left(\frac{K^{2a}-1}{K^{2a}+1}, \frac{1}{2K^{4a}-1}\right).$$

Le théorème 4 implique en particulier que pour tout $\alpha\in(0,1)$ on peut résoudre le problème de Calderón pour $\Lambda^\alpha(\Gamma)$ si K est suffisamment proche de 1.

2. Démonstration du théorème 2

Nous suivons la méthode d'Hinkkanen.

Soit $f \in \Lambda^\alpha(\Gamma)$ et u son prolongement harmonique dans Ω.

Soit $z \in \Omega$ et ζ un point de Γ tel que $|z - \zeta| \leqslant 2d$ où l'on a posé $d = \operatorname{dist}(z, \Gamma)$.

Lemme 1. *Il existe une constante C ne dépendant que de K telle que si* $\alpha < 1/K$, $|u(z) - u(\zeta)| \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)} d^\alpha$.

PREUVE DU LEMME 1. Posons $v(z') = u(z') - u(\zeta)$. On a la représentation:

$$v(z) = \int_\Gamma v(\lambda)\omega(z, d\lambda)$$

où $\omega(z, d\lambda)$ est la mesure harmonique sur Γ dans Ω par rapport à z. Soit, pour $n \geqslant 0$,

$$E_n = \{\lambda \in \Gamma : 2^n d \leqslant |\lambda - z| < 2^{n+1} d\};$$

alors

$$v(z) = \sum_{n \geq 0} \int_{E_n} v(\lambda)\omega(z, d\lambda).$$

Si $\lambda \in E_n$,

$$|v(\lambda)| \leqslant \|f\|_{\Lambda^\alpha(\Gamma)} |\lambda - \zeta|^\alpha \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)} 2^{\alpha n} d^\alpha$$

et par suite

$$|v(z)| \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)} \left(\sum_{n \geq 0} 2^{\alpha n} \omega(z, E_n) \right) d^\alpha. \quad \square$$

Le lemme 1 va alors découler du lemme 2 suivant donnant une estimation de $\omega(z, E_n)$.

Lemme 2. *Soit* $E(r) = \{\lambda \in \Gamma ; |\lambda - z| \geqslant rd\}$ *avec* $r \geqslant 1$.

Alors $\omega(z, E(r)) \leqslant Cr^{-1/K}$ *oú C ne dépend que de K.*

DÉMONSTRATION DU LEMME 2. Soit $\varphi : D \to \Omega$ une répresentation conforme. Par hypothèse φ se prolonge en une application K-quasiconforme de la sphère de Riemann que nous appellerons encore φ. Quitte à composer par une transformation de Möbius laissant D invariant on peut supposer que $\varphi(\infty) = \infty$. Soit $\psi = \varphi^{-1}$: c'est encore un homéomorphisme K-quasiconforme de la sphère tel que $\psi(\infty) = \infty$. Soit $z_0 = \psi(z) \in D$. Si $\zeta \in \mathbb{C}$ et $r > 0$, notons $C(z, r)$ le cercle de centre z et de rayon r, et $\gamma(z, r) = \psi(C(z, r))$.

Par le théorème de distorsion de Koebe

$$\frac{d}{3}|\psi'(z)| \geqslant |\psi(\omega) - \psi(z)| \geqslant \frac{d}{16}|\psi'(z)| \quad \text{si} \quad \omega \in C\left(z, \frac{d}{4}\right).$$

Comme

$$\frac{1}{4}(1 - |z_0|^2)|\varphi'(z_0)| \leqslant d \leqslant (1 - |z_0|^2)|\varphi'(z_0)| \quad \text{et} \quad \varphi'(z_0) = \frac{1}{\psi'(z)}$$

on a

$$\frac{2}{3}(1 - |z_0|) \geqslant |\zeta - z_0| \geqslant \frac{1}{64}(1 - |z_0|) \quad \text{si} \quad \zeta \in \gamma\left(z, \frac{d}{4}\right).$$

Soient maintenant r' et r'' les réels définis par

$$(1 - |z_0|)r' = \inf_{\zeta \in \gamma(z, rd)} |\zeta - z_0| \quad , \quad (1 - |z_0|)r'' = \sup_{\zeta \in \gamma(z, rd)} |\zeta - z_0|.$$

Soit T la famille des courbes joignant $\gamma(z, d/4)$ à $\gamma(z, rd)$ dans la «couronne» définie par ces deux courbes. Alors

$$\frac{2\pi}{\log 64r''} \leqslant M(T) \leqslant KM(\varphi(T)) = \frac{2\pi K}{\log 4r} \Rightarrow r'' \geqslant \frac{4^{1/K}}{64} r^{1/K}.$$

On applique alors le lemme suivant, dû à Mori.

Lemme de Mori [8]. *Il existe une constante $C(K)$ tele que si ϕ est un homéomorphisme K-quasiconforme de la sphère de Riemann tel que $\phi(\infty) = \infty$ alors, $\forall z \in C$, $\forall r > 0$,*

$$\sup_{\zeta \in C(z, r)} |\Phi(\zeta) - \Phi(z)| \leqslant C(K) \inf_{\zeta \in C(z, r)} |\Phi(\zeta) - \Phi(z)|.$$

Grâce à ce lemme, nous voyons que $r' \geqslant C(K)r'' \geqslant C(K)r^{1/K}$. On peut alors écrire

$$\omega(z, E(r)) \leqslant \omega(z_0, \mathcal{E}(r'), D)$$

où

$$\mathcal{E}(r') = \{\lambda \in \partial D; |\lambda - z_0| \geqslant r'(1 - |z_0|)\},$$

et par un calcul standard, on vérifie que $\omega(z_0, \mathcal{E}(r'), D) \leqslant C/r'$. Finalement,

$$\omega(z, E(r)) \leqslant \omega(z_0, \mathcal{E}(r'), D) \leqslant C/r' \leqslant C(K)r^{-1/K}$$

et le lemme 2 est entièrement démontré, ainsi que le lemme 1. □

Lemme 3.

$$L = \sup_{\substack{z_1, z_2 \in \bar{\Omega} \\ |z_1 - z_2| < 1}} |u(z_1) - u(z_2)| = \sup_{\substack{z_1 \in \partial\Omega \\ z_2 \in \Omega \\ |z_1 - z_2| < t}} |u(z_1) - u(z_2)| = R.$$

PREUVE. Soient $z_1, z_2 \in \Omega$, $h = z_1 - z_2$ $|h| < t$. Alors la fonction

$$v(z) = u(z + h) - u(z)$$

est harmonique dans $\{z \in \Omega; z + h \in \Omega\} = \tilde{\Omega}$. Soit Ω_1 la composante connexe de $\tilde{\Omega}$ contenant z_1. Si $z \in \partial\Omega_1$ alors $z \in \Gamma$ ou bien $z + h \in \Gamma$ d'où $|v(z)| \leqslant R$. Par le principe du maximum, $|v(z_1)| \leqslant R$ et le lemme 3 s'en déduit. □

Pour terminer la preuve du théorème 2, il nous reste à montrer que

$$R \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)} t^\alpha.$$

Soit donc $z_1 \in \Gamma$ et $z_2 \in \Omega$. Si $|z_1 - z_2| \leqslant 2d(z_2, \Gamma)$ alors

$$|u(z_1) - u(z_2)| \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)} |z_1 - z_2|^\alpha$$

par le lemme 1.

Si $|z_1 - z_2| > 2d(z_2, \Gamma)$, choisissons $\zeta \in \Gamma$; $d(z_2, \Gamma) = |z_2 - \zeta|$.

Alors

$$\begin{aligned} |u(z_1) - u(z_2)| &\leqslant |u(z_1) - u(\zeta)| + |u(\zeta) - u(z_2)| \\ &\leqslant C\|f\|_{\Lambda^\alpha(\Gamma)} (|z_1 - \zeta|^\alpha + d(z_2, \Gamma)^\alpha) \\ &\leqslant C\|f\|_{\Lambda^\alpha(\Gamma)} |z_1 - z_2|^\alpha. \quad \square \end{aligned}$$

Avant de poursuivre, donnons deux corollaires de résultats précédents, dont nous aurons besoin dans la suite.

Proposition 1. *Soit $\Phi: D \to \mathbb{C}\backslash\{0\}$ une fonction holomorphe univalente admettant une extension K-quasiconforme à $\bar{\mathbb{R}}^2$. Alors Φ appartient à l'espace de Hardy $H^p(D)$ si $p < 1/K$ et, pour ces valeurs de p, $\|\Phi\|_p \leqslant C(p, K)|\Phi(0)|$.*

Cette proposition généralise le théorème de Prawitz ([1] p. 90) dans le cas où $K < 2$. Notons encore que si Φ se prolonge en un homéomorphisme K-q.c. tel que $\Phi(\infty) = \infty$, alors le lemme de Mori permet d'obtenir le résultat beaucoup plus fort

$$\|\Phi\|_\infty \leqslant C(K)|\Phi(0)|.$$

DÉMONSTRATION. Soit $f(z) = \Phi(z) - \Phi(0)$. Alors, avec les notations du lemme 2, si $r \geqslant d = d(0, f(\partial D))$,

$$|\{z \in \partial D; |f(z)| \geqslant r\}| = \omega\left(0, E\left(\frac{r}{d}\right), f(D)\right)$$

$$\leqslant C(K)\left(\frac{d}{r}\right)^{1/K}$$

par le lemme 2. D'où

$$\|f\|_p^p \leqslant 2\pi p \int_0^d r^{p-1}\,dr + pC(K)d^{1/K}\int_d^\infty r^{p-1-1/K}\,dr$$

$$\leqslant C(p,K)d^p \leqslant C(p,K)|\Phi(0)|^p,$$

car $0 \notin \Phi(D)$.

La proposition 1 en découle car $\|\Phi\|_p^p \leqslant \|f\|_p^p + |\Phi(0)|^p$. □

Proposition 2. *Soit Ω un K-quasidisque et $f \in \Lambda^\alpha(\Gamma)$ avec $0 < \alpha < 1/K^2$. Il existe alors $F \in \Lambda^\alpha(R^2)$, harmonique en dehors de Γ, telle que $f = F|_\Gamma$.*

DÉMONSTRATION. Supposons pour simplifier que $0 \in \Omega$. Tout d'abord, on peut prolonger f en $u \in \Lambda^\alpha(\Omega)$ harmonique par le théorème 2. Pour prolonger f à l'extérieur de Ω considérons ψ une représentation conforme de $\mathbb{C} - \bar{D}$ sur $\mathbb{C} - \bar{\Omega}$ telle que $\psi(\infty) = \infty$. Par hypothèse, ψ peut se prolonger en un homéomorphisme K^2-q.c. de la sphère. Soit $\tilde{\Omega}$ le transformé de $\mathbb{C}\backslash\bar{\Omega}$ par l'inversion $z \to 1/z$. Alors $h(z) = 1/\psi(1/z)$ est une représentation conforme de D sur $\tilde{\Omega}$ se prolongeant en une application K^2-q.c. Soit alors $\tilde{f}$ l'application définie sur $\tilde{\Gamma} = \partial\tilde{\Omega}$ par $\tilde{f}(z) = f(1/z)$. Alors

$$|\tilde{f}(z) - \tilde{f}(z')| \leqslant C\left|\frac{1}{z} - \frac{1}{z'}\right|^\alpha \leqslant C'|z - z'|^\alpha$$

car $|zz'| \geqslant d(0,\Gamma)^2$. Par le théorème 2 on peut alors prolonger $\tilde{f}$ en une application harmonique $\in \Lambda^\alpha(\tilde{\Omega})$. On prolonge alors f sur $\mathbb{C}\backslash\bar{\Omega}$ en posant $F(z) = \tilde{f}(1/z)$ et l'on vérifie que l'on a encore $f \in \Lambda^\alpha(\mathbb{C}\backslash\bar{\Omega})$. L'application F ainsi construite convient. □

3. Démonstration du Théorème 4

Posons tout d'abord $\Omega_1 = \Omega$ et $\Omega_2 = \mathbb{C}\backslash\bar{\Omega}$.

Dans un premier temps nous établissons l'existence de la décomposition pour $f = \varphi|_\Gamma$ où $\varphi \in C_0^\infty(\mathbb{R}^2)$. Pour ce faire on écrit, d'après une formule classique,

$$\forall z \in \mathbb{R}^2, \qquad \varphi(z) = -\frac{1}{\pi}\iint_{\mathbb{R}^2} \frac{\bar{\partial}\varphi(\omega)}{\omega - z}\,d\omega\,d\bar{\omega}$$

où

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

puis on pose

$$f_1(z) = -\frac{1}{\pi}\iint_{\Omega_2} \frac{\bar{\partial}\varphi(\omega)}{\omega - z}\, d\omega\, d\bar{\omega} \qquad (z \in \Omega_1)$$

$$f_2(z) = -\frac{1}{\pi}\iint_{\Omega_1} \frac{\bar{\partial}\varphi(\omega)}{\omega - z}\, d\omega\, d\bar{\omega} \qquad (z \in \Omega_2).$$

On a bien $f = f_1 + f_2$ sur Γ; reste à montrer que $\|f_j\|_{\Lambda^\alpha(\Omega_j)} \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}$ et que $|f_2(z)| \underset{|z|\to\infty}{\longrightarrow} 0$. Vu le théorème 2, et à condition que $\alpha < 1/K^2$, il suffit évidemment de la vérifier uniquement pour f_2. On suppose donc $\alpha < 1/K^2$ et l'on considère le prolongement harmonique u_1 de f dans Ω_1. Comme il sera démontré dans la suite, la fonction $u_1 - \varphi$ appartient à l'espace de Sobolev $W_1^1(\Omega_1)$ et donc,

$$\forall z \in \Omega_2, \qquad f_2(z) = -\frac{1}{\pi}\iint_{\Omega_1} \frac{\bar{\partial}u_1(\omega)}{\omega - z}\, d\omega\, d\bar{\omega}.$$

Pour vérifier que $f_2 \in \Lambda^\alpha(\Omega_2)$ on utilise alors le critère de Gehring et Martio (Théorème 3): il suffit de montrer l'existence d'une constante C telle que

$$\forall z \in \Omega_2, \qquad |f_2'(z)| \leqslant Cd(z, \Gamma)^{\alpha-1}\|f\|_{\Lambda^\alpha(\Gamma)}.$$

Or

$$f_2'(z) = -\frac{1}{\pi}\iint_{\Omega_1} \frac{\bar{\partial}u_1(\omega)}{(\omega - z)^2} \quad \text{et} \quad |\bar{\partial}u_1(\omega)| \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}\, d(\omega, \Gamma)^{\alpha-1}$$

par les théorèmes 2 et 3.

Pour obtenir la conclusion désirée, il suffit donc que α vérifie les conditions suivantes:

(1) $$\alpha < 1/K^2$$

(2) $$\exists C > 0; \qquad \forall \zeta \in \Omega_2, \qquad \iint_{\Omega_1} \frac{d(z, \Gamma)^{\alpha-1}}{|z - \zeta|^2}\, dz\, d\bar{z} \leqslant Cd(\zeta, \Gamma)^{\alpha-1}.$$

Soit φ une représentation conforme de D sur Ω_1. Alors

$$I(\alpha, \zeta) = \iint_{\Omega_1} \frac{d(z, \Gamma)^{\alpha-1}}{|z - \zeta|^2}\, dz\, d\bar{z} \leqslant C\iint_D \frac{(1 - |z|)^{\alpha-1}|\varphi'(z)|^{\alpha+1}}{|\varphi(z) - \zeta|^2}\, dz\, d\bar{z}.$$

On choisit plus précisément φ de sorte que $\psi(z) = (\varphi(z) - \zeta)^{-1}$ se prolonge en un homéomorphisme K^2-q.c. de $\bar{\mathbb{C}}$ tel que $\psi(\infty) = \infty$. On écrit alors

$$\begin{aligned} I(\alpha, \zeta) &\leqslant C \iint_D (1 - |z|)^{\alpha - 1} |\varphi(z) - \zeta|^{2\alpha} |\psi'(z)|^{\alpha + 1} \, dz \, d\bar{z} \\ &\leqslant C \Big(\iint_D (1 - |z|)^{q(\alpha - 1)} |\varphi(z) - \zeta|^{2\alpha q} \, dz \, d\bar{z} \Big)^{1/q} \Big(\iint_D |\psi'(z)|^{p(\alpha + 1)} \, dz \, d\bar{z} \Big)^{1/p} \\ &= C I_1(\alpha, \zeta) I_2(\alpha, \zeta) \end{aligned}$$

où $p, q > 1$ sont à choisir tel que $1/p + 1/q = 1$.

Pour choisir $p > 1$, nous utilisons le théorème suivant.

Théorème 5 (Gehring et Reich [4]). *Il existe une constante $a \in (1,40)$ telle que pour tout homéomorphisme K-q.c. de la sphère tel que $\psi(\infty) = \infty$, on ait*

$$\Big(\iint_D J_\psi(z)^{K^a/(K^a - 1)} \, dz \, d\bar{z} \Big)^{(K^a - 1)/K^a} \leqslant C(a, K) \iint_D J_\psi(z) \, dz \, d\bar{z}$$

où $J_\psi(z)$ désigne le jacobien de ψ au point z.

Ce théorème admis, on choisit p de sorte que

$$p(\alpha + 1) = \frac{2K^{2a}}{K^{2a} - 1}.$$

Alors,

$$\begin{aligned} &\Big(\iint_D |\psi'(z)|^{p(\alpha + 1)} \, dz \, d\bar{z} \Big)^{1/p} \\ &\qquad = \Big(\iint_D [|\psi'(z)|^2]^{K^{2a}/(K^{2a} - 1)} \, dz \, d\bar{z} \Big)^{[(K^{2a} - 1)/K^{2a}][(\alpha + 1)/2]} \\ &\qquad \leqslant C(a, K) \Big(\iint_D |\psi'|^2(z) \, dz \, d\bar{z} \Big)^{(\alpha + 1)/2} \leqslant \frac{C}{d^{1 + \alpha}}, \end{aligned}$$

où $d = d(\zeta, \Gamma)$ car $|\psi(z)| \leqslant 1/d$ si $z \in D$.

Nous avons donc montré que, pour ce choix de p,

$$I_2(\alpha, \zeta) \leqslant C d^{-1 - \alpha}.$$

Pour estimer $I_1(\alpha, \zeta)$ nous aurons besoin du lemme suivant.

Lemme 4. $|\varphi(0) - \zeta| \leqslant Cd$ *où C ne dépend que de K.*

PREUVE DU LEMME 4. Par le lemme de Mori, puisque ψ se prolonge en un homéomorphisme K^2-q.c. de $\bar{\mathbb{R}}^2$ tel que $\psi(\infty) = \infty$,

$$\sup_{y \in \partial D} |\psi(y) - \psi(0)| \leqslant C(K) \inf_{y \in \partial D} |\psi(y) - \psi(0)|.$$

Soit d'autre part $\Delta = \operatorname{diam}(\Omega_1)$. Alors

$$\inf_{y\in\partial D} |\psi(y)| \leqslant \frac{4}{\Delta}$$

tandis que

$$\sup_{y\in\partial D} |\psi(y)| = \frac{1}{d}.$$

—Si $d \geqslant \dfrac{\Delta}{8C(K)}$ alors $|\psi(0)| \geqslant \dfrac{4}{\Delta} \geqslant \dfrac{1}{2C(K)d}$.

—Si maintenant $d < \dfrac{\Delta}{8C(K)}$, écrivons $|\psi(0)| = \dfrac{\epsilon}{d}$.

Si $\epsilon \geqslant 1$, il n'y a rien à démontrer. Sinon,

$$A = \sup_{y\in\partial D} |\Psi(y) - \psi(0)| \geqslant \frac{1-\epsilon}{d}$$

tandis que

$$B = \inf_{y\in\partial D} |\psi(y) - \psi(0)| \leqslant \frac{4}{\Delta} + \frac{\epsilon}{d} \leqslant \frac{1 + 2\epsilon C(K)}{2C(K)d}.$$

En écrivant que $A \leqslant C(K)B$ on s'aperçoit alors que nécessairement

$$\epsilon \geqslant \frac{1}{2(1 + C(K))}$$

ce qui démontre le lemme 4. □

Revenons à l'estimation de $I_1(\alpha, \zeta)$. Puisque

$$p = \frac{2K^{2a}}{(\alpha+1)(K^{2a}-1)},$$

le calcul donne

$$q = \frac{2K^{2a}}{2K^{2a} - (\alpha+1)(K^{2a}-1)}.$$

D'autre part, on peut écrire:

$$I_1(\alpha, \zeta) = \left\{ \int_0^1 r(1-r)^{q(\alpha-1)} \left(\int_0^{2\pi} |\varphi(re^{i\theta}) - \zeta|^{2\alpha q}\, d\theta \right) dr \right\}^{1/q}.$$

Deux cas se présentent alors:

1.er *cas.*

$$2\alpha q \leqslant \frac{1}{K^{2a}} < \frac{1}{K^2} \Leftrightarrow \alpha \leqslant \frac{K^{2a}+1}{4K^{4a}+K^{2a}-1}.$$

On a alors, par la proposition 1,

$$\int_0^{2\pi} |\varphi(re^{i\theta}) - \zeta|^{2\alpha q}\, d\theta \leqslant C|\varphi(0) - \zeta|^{2\alpha q} \leqslant Cd^{2\alpha q}$$

par le lemme 4 et donc, si $q(1-\alpha) < 1$

$$I_1(\alpha, \zeta) \leqslant Cd^{2\alpha}$$

et l'on a (2) en combinant cette dernière estimation avec celle sur $I_2(\alpha, \zeta)$.

La condition $q(1-\alpha) < 1$ équivaut à

$$\alpha > \frac{K^{2a}-1}{K^{2a}+1}.$$

Pour que cette étude nous fournisse des valeurs effectives de α, il faut encore que

$$\frac{K^{2a}-1}{K^{2a}+1} < \frac{K^{2a}+1}{4K^{4a}+K^{2a}-1}$$

ce qui équivaut à $K^{4a} - K^{2a} - 1 < 0$ ou encore à

$$K^{2a} < \frac{1+\sqrt{5}}{2}.$$

2.eme *cas.*

$$2\alpha q > \frac{1}{K^{2a}} \Leftrightarrow \alpha > \frac{K^{2a}+1}{4K^{4a}+K^{2a}-1}.$$

On utilise le lemme suivant, dû à Hardy-Littlewood ([1], p. 84).

Lemme 5. *Soit f appartenant à l'espace de Hardy $H^p(D)$ où $0 < p < \infty$. Alors, si $q > p$,*

$$\left(\int_0^{2\pi} |f(re^{i\theta})|^q\, d\theta\right)^{1/q} \leqslant \frac{K\|f\|_p}{(1-r)^{1/p-1/q}},$$

où K est une constante universelle.

Appliquons ce lemme à notre propos:

$$\int_0^{2\pi} |\varphi(re^{i\theta}) - \zeta|^{2\alpha q}\, d\theta \leqslant Cd^{2\alpha q}(1-r)^{1-2\alpha q K^{2a}}$$

et l'on obtient (2) si

$$q(1-\alpha) + 2\alpha q K^{2a} - 1 < 1 \Leftrightarrow \alpha < \frac{1}{2K^{4a}-1}.$$

Là encore, pour pouvoir conclure il faut avoir

$$\frac{1}{2K^{4a}-1} > \frac{K^{2a}+1}{4K^{4a}+K^{2a}-1} \Leftrightarrow K^{2a} < \frac{1+\sqrt{5}}{2}.$$

La condition (2) est donc vérifiée si

$$K^{2a} < \frac{1+\sqrt{5}}{2} \quad \text{et} \quad \alpha \in \left(\frac{K^{2a}-1}{K^{2a}+1}, \frac{1}{2K^{4a}-1}\right).$$

Comme

$$\frac{1}{2K^{4a}-1} \leqslant \frac{1}{K^{2a}} \leqslant \frac{1}{K^2}$$

si $K \geqslant 1$, on a également (1).

Pour vérifier que $|f_2(z)|$ tend vers 0 à l'infini on écrit que

$$|f_2(z)| \leqslant C\frac{1}{|z|}\iint_{\Omega_1} |\bar{\partial} u_1(\omega)|\, d\omega\, d\bar{\omega} \quad \text{pour} \quad |z| \geqslant 2\,\mathrm{diam}\,(\Omega_1),$$

$$\leqslant \frac{C}{|z|}\|f\|_{\Lambda^\alpha(\Gamma)} \iint_{\Omega_1} d(\omega,\Gamma)^{\alpha-1}\, d\omega\, d\bar{\omega}$$

$$\leqslant \frac{C}{|z|}\|f\|_{\Lambda^\alpha(\Gamma)} \iint_D (1-|z|)^{\alpha-1}|\varphi'(z)|^{\alpha+1}\, dz\, d\bar{z}$$

où $\varphi: D \to \Omega_1$ est une représentation conforme se prolongeant en un homéomorphisme K^2-q.c. de la sphère tel que $\varphi(\infty) = \infty$. En reprenant alors la même méthode que dans la démonstration de (2) on obtient

$$|f_2(z)| \leqslant \frac{C}{|z|}\|f\|_{\Lambda^\alpha(\Gamma)}, \qquad |z| \geqslant 2\,\mathrm{diam}\,(\Omega_1).$$

L'existence de la décomposition est donc établie pour $f = \varphi|_\Gamma$ avec $\varphi \in C_0^\infty(\mathbb{R}^2)$.

Soit maintenant f une fonction quelconque de $\Lambda^\alpha(\Gamma)$. Par la proposition 2, $f = F|_\Gamma$ où $\|F\|_{\Lambda^\alpha(\mathbb{R}^2)} \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}$. On peut naturellement supposer que F est

à support compact. F est alors limite uniforme d'une suite $\varphi_n \in C_0^\infty(\mathbb{R}^2)$ telle que

$$\|\varphi_n\|_{\Lambda^\alpha(R^2)} \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}.$$

Par ce que l'on vient de voir, on peut décomposer φ_n en

$$\varphi_n = \varphi_n^1 + \varphi_n^2 \quad \text{où} \quad \varphi_n^j \in \Lambda^\alpha(\Omega_j)$$

avec $\|\varphi_n^j\|_{\Lambda^\alpha(\Omega_j)} \leqslant C\|f\|_{\Lambda^\alpha(\Gamma)}$. Comme d'autre part

$$|\varphi_n^2(\zeta)| \leqslant \frac{C}{|\zeta|}\|f\|_{\Lambda^\alpha(\Gamma)} \quad \text{pour} \quad |\zeta| \geqslant 2\,\mathrm{diam}\,(\Omega_1),$$

on obtient le fait que (φ_n^2) est une famille équicontinue sur $\Omega_2 \cup \{\infty\}$. Par le théorème d'Ascoli il existe une sous suite $(\varphi_{n_k}^2)$ qui converge uniformément sur Ω_2 vers une fontion f_2 holomorphe dans $\Lambda^\alpha(\Omega_2)$. Comme la suite (φ_n) est elle-même uniformément convergente, l'existence de la décomposition de Calderón est démontrée dans le cas général.

Reste à établir l'unicité de cette décomposition. Supposons que $f \in \Lambda^\alpha(\Gamma)$ admette deux décompositions

$$f = f_1 + f_2 = g_1 + g_2.$$

Alors la fonction

$$F = \begin{cases} f_1 - g_1 & \text{sur} \quad \Omega_1 \\ g_2 - f_2 & \text{sur} \quad \Omega_2 \end{cases}$$

se prolonge en une fonction $F \in \Lambda^\alpha(\mathbb{R}^2)$, holomorphe en dehors de Γ et nulle à l'infini.

Par [2] (p. 66) on peut alors conclure que $F \equiv 0$ à condition que la dimension de Hausdorff de Γ soit $< 1 + \alpha$. Pour évaluer la dimension de Hausdorff de Γ on utilise un théorème de Gehring-Väisälä [5] qui affirme que la dimension de Hausdorff de l'image d'un cercle par une application K-quasiconforme de $\overline{\mathbb{R}}^2$ est inférieure ou égale à $2K^a/(K^a + 1)$ où a est la constante du théorème 5. En combinant ces deus résultats on conclut la preuve du théorème 4 car

$$\alpha > \frac{K^{2a} - 1}{K^{2a} + 1} \Rightarrow (1 + \alpha) > \frac{2K^{2a}}{K^{2a} + 1} \geqslant \dim \Gamma. \quad \square$$

Remarques.

1) Les résultats de [2] montient que le théorème 4 est faux pour $\alpha \leqslant \dim \Gamma - 1$.

2) A. Hinkkanen, après avoir lu le manusccrit de cet article, a montré que la constante $\frac{1}{2}$ du théorème 1 pouvait être améliorée pour *tous* les quasidisques.

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Several Characterizations for the Special Atom Spaces with Applications

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Introduction

The theory of functions plays an important role in harmonic analysis. Because of this, it turns out that some spaces of analytic functions have been studied extensively, such as H^p-spaces, Bergman spaces, etc. One of the major insights that has developed in the study of H^p-spaces is what we call the real atomic characterization of these spaces.

The first author in [3] introduced the special atom space B^p for $p > 1/2$ and it was shown that $H^p \subsetneq B^p$ for $1/2 < p < 1$ and $B^p \subsetneq H^p$ for $1 \leqslant p < \infty$.

For example in case $p = 1$, we have the B^1-space which is a Banach space whose dual is the derivative of the Zygmund space Λ_*. In [4], it was shown that this space is the real atomic characterization of the boundary values of those analytic functions f in the disk $D = \{z \in C; |z| < 1\}$ for which

$$\|f\|_S = |f(0)| + \frac{1}{2\pi}\int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|\, d\theta\, dr < \infty.$$

This space denoted by S was introduced in [1] and among other things it was shown there that the dual of S is the space β of Bloch functions. We refer the reader to [4] for a complete discussion of these matters. It was shown in

[5] that

$$B^1 \subsetneq H^1 \subsetneq L^1.$$

Subsequently, in [9] we were able to show that for $p > 1$, the B^p-space is the real atomic characterization of the boundary values of those analytic functions in the disk for which

$$\int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|(1-r)^{(1/p)-1}\, d\theta\, dr < \infty.$$

These analytic function spaces, which we denoted by J^p, were studied extensively in [14], [15] and [16], while for $1/2 < p < 1$, B^p is the smallest Banach space that contains H^p and also gives real characterization for the boundary values of those analytic functions in the disk for which

$$\int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|(1-r)^{(1/p)-2}\, d\theta\, dr < \infty.$$

These analytic function spaces were introduced in [13].

We would also like to point out that the duals of the B^p spaces are Lipschitz spaces see [5], [6] and [10] and this fact is exploited in the present paper.

It should now be clear why so much attention has been placed in analyzing these spaces. But lots of our efforts are concerned with using these spaces as intermediaries in order to interpolate linear operators between L^p-spaces. For example, it is well-known that (see Theorem D in [2])

$$T\colon\ H^1 \to L^1 \quad \text{and} \quad T\colon\ L^2 \to L^2 \quad \text{boundedly.}$$

Then

$$T\colon\ L^p \to L^p \quad \text{boundedly for} \quad 1 < p \leqslant 2. \tag{1}$$

And there is also an analytic version of this, namely Theorem E of [2]. And so of course, we wish to replace H^1 by B^1 and ask what else is needed in order to obtain (1). In many cases, T is a concrete operator and so what else is needed may in fact be easy to check. However, this program still is incomplete.

In this paper, we give several descriptions of B^p for $p > 1$ and obtain various consequences from them.

First description. Let f be a function which may be decomposed into a finite or countable linear combinations of characteristic functions of intervals in T, where T is the unit circle in the plane which we may identify in the usual way with $[0, 2\pi)$.

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \chi_{I_n}(x)$$

For a given f there may be many such decompositions. Let

$$\|f\|_{B^p} = \operatorname{Inf} \sum_{n=1}^{\infty} \|\alpha_n \chi_{I_n}\|_p$$

where the infimum is taken over all possible decompositions of f, and by $\|\cdot\|_p$ we mean the L^p-norm on T, that is

$$\|g\|_p = \left(\int_T |g|^p\right)^{1/p}$$

even for $p < 1$. We get by Minkowski's inequality for $p > 1$ that $\|f\|_p \leqslant \|f\|_{B^p}$ and it follows that $\|\chi_I\|_{B^p} = \|\chi_I\|_p$.

We denote by B^p the set of all $f \in L^p$ for which $\|f\|_{B^p}$ is finite, for $1 < p < \infty$.

We point out that convergence of

$$\sum_{n=1}^{\infty} \alpha_n \chi_{I_n}$$

is taken in the sense of the Lebesgue space L^p, actually we could just as well use the Lorentz space $L(p, 1)$ and observe that $\|f\|_{L(p,1)} \leqslant \|f\|_{B^p}$.

Note. $f \in L(p, 1)$ if and only if

$$\|f\|_{L(p,1)} = \frac{1}{p} \int_0^{\infty} f^*(t) t^{1/p - 1}\, dt < \infty,$$

where f^* is the decreasing rearrangement of f, $0 < p < \infty$.

It is straightforward to show the following result.

Theorem 1. *The space B^p, endowed with the norm $\|\cdot\|_{B^p}$ is a Banach space.*

Second description. By a special atom we mean a function b supported on an interval $I \subset T$ which is of the form

$$b(x) = \alpha \chi_L(x) - \alpha \chi_R(x)$$

where L is the left half of I, R is the right half and α is scalar.

For a function f defined on T and which is a finite or countable sum of special atoms plus a constant

$$f(x) = a + \sum_{n=1}^{\infty} b_n(x)$$

(again there may be many ways of so decomposing f) we let

$$\|f\|^*_{B^p} = \operatorname{Inf} \sum_{n=1}^{\infty} \|b_n\|_p.$$

We denote by B^p the set of all $f \in L^p$ for which $\|f\|^*_{B^p}$ is finite, for $1 < p < \infty$.

Observe that this description of B^p holds whenever $1/2 < p < \infty$, but that when $1/2 < p < 1$, f may exist not as a function but only as a distribution. Indeed if

$$f(x) = \sum_{n=1}^{\infty} b_n(x) \in B^p$$

then given a test function ϕ, that is, an infinitely differentiable function on T, we define

$$(f, \phi) = \sum_{n=1}^{\infty} (b_n, \phi) = \sum_{n=1}^{\infty} \int_{I_n} b_n(x)\phi(x)\,dx,$$

where I_n is the support of b_n as in the definition of special atoms. We wish to show that the series converges absolutely. Let a_n denote the height of $b_n(x)$ and h_n the length of I_n.

Observe that

$$(b_n, \phi) = \int_I b_n(x)\phi(x)\,dx = a_n\left[\Phi\left(\bullet + \frac{h_n}{2}\right) + \Phi\left(\bullet - \frac{h_n}{2}\right) - 2\Phi(\bullet)\right]$$

where Φ is the indefinite integral of ϕ, that is, $\Phi(x) = \int_0^x \phi(t)\,dt$. Thus

$$|(b_n, \phi)| \leqslant 4|a_n|\,\|\Phi\|_\infty,$$

where

$$\|\Phi\|_\infty = \operatorname*{ess\,sup}_{t \in T} |\Phi(t)|.$$

Then using the mean value theorem twice on the second difference above we also see that

$$|(b_n, \phi)| \leqslant |a_n| h_n^2 \|\Phi''\|_\infty.$$

But $\Phi'' = \phi'$ and $\|\Phi\|_\infty \leqslant \|\phi\|_\infty \cdot (\operatorname{supp}\phi)$, so if $h_n \leqslant 1$ then

$$|(b_n, \phi)| \leqslant |a_n| h_n^2 \|\Phi'\|_\infty \leqslant |a_n| h_n^{1/p} \|\phi'\|_\infty = \|b_n\|_p \|\phi'\|_\infty$$

since $1/2 < p < 1$, while if $h_n > 1$ then

$$|(b_n, \phi)| \leqslant 4|a_n|\,\|\phi\|_\infty \leqslant 4\|\phi\|_\infty (\operatorname{supp}\phi)|a_n| h_n^{1/p} = 4\|\phi\|_\infty (\operatorname{supp}\phi)\|b_n\|_p.$$

Therefore

$$\sum_{n=1}^{\infty} |(b_n, \phi)| \leqslant [4\|\phi\|_\infty (\operatorname{supp}\phi) + \|\phi'\|_\infty] \sum_{n=1}^{\infty} \|b_n\|_p.$$

Third description. A triangular function is a function supported on an interval $I \subset T$ whose graph restricted to I is the two equal sides of an isosceles triangule.

Observe that $t(x)$ is a triangular function if and only if it is the indefinite integral of a special atom (see second description).

For a sum of triangular functions

$$f(x) = \sum_{n=1}^{\infty} t_n(x),$$

we set

$$\|f\|_{Bp}^{T} = \operatorname{Inf} \sum_{n=1}^{\infty} \|t_n\|_p$$

where the infimum is taken over all possible representations of f. We denote by B^p the set of all $f \in L^p$ for which $\|f\|_{Bp}^{T}$ is finite, for $1 < p < \infty$.

Fourth description. Given a non-negative integer n partition $T = [0,2\pi)$ into 2^n subintervals of equal length and let I_{nk} be the k^{th} of these for $k = 1$ to $k = 2^n$. (I_{nk} is called a dyadic interval). The Haar function ϕ_{nk} is a special atom supported on I_{nk} and such that $\|\phi_{nk}\|_2 = 1$. That is,

$$\phi_{nk}(x) = \begin{cases} \left(\dfrac{2^n}{2\pi}\right)^{1/2} & \text{on} \quad \left[\dfrac{k-1}{2^n}2\pi, \dfrac{k-1/2}{2^n}2\pi\right) \\ -\left(\dfrac{2^n}{2\pi}\right)^{1/2} & \text{on} \quad \left[\dfrac{k-1/2}{2^n}2\pi, \dfrac{k}{2^n}2\pi\right] \\ 0 \qquad \text{elsewhere.} \end{cases}$$

It is well known that if we supplement the Haar functions by the constant function whose value is everywhere the reciprocal of the square root of 2π, i.e. $\phi_{00} = 1/\sqrt{2\pi}$, we obtain a complete orthonormal system. Given an integrable function f, we may form its Haar series

$$f(x) = a + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} c_{nk}\phi_{nk}(x)$$

where

$$c_{nk} = \int_0^{2\pi} f(x)\phi_{nk}(x)\,dx.$$

Let

$$\|f\|_{Bp}^{H} = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|c_{nk}\phi_{nk}\|_p,$$

it is easily seen that the set of all those f for which $\|f\|_{B^p}^H$ is finite form a Banach space which is continuously embedded in L^p or even in $L(p, 1)$.

We denote this set by B^p for $1 < p < \infty$.

Fifth description. Similar to the first one, except that f must be written as

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \chi_{I_n}(x),$$

where the I_n are dyadic intervals.

Sixth description. Similar to the third one except that

$$f(x) = \sum_{n=1}^{\infty} t_n(x)$$

with the supports of the triangular functions t_n's being dyadic intervals.

Note that with these norms B^p for $p > 1$ is a Banach space.

2. Equivalence of descriptions

We shall now argue that the various descriptions are equivalent.

Let $I = (a, b)$ and $h = |I| = b - a$. We wish to express χ_I as a sum of triangular functions. In fact, let $v(x)$ be the triangular function whose height is 1 and whose support is I and for each positive integer n let $s_n(x)$ and $t_n(x)$ be triangular functions of height 1/2 whose supports are

$$\left(a, a + \frac{h}{2^n}\right) \quad \text{and} \quad \left(b - \frac{h}{2^n}, b\right)$$

respectively. It is easily seen that

$$\chi_I(x) = v(x) + \sum_{n=1}^{\infty} [s_n(x) + t_n(x)], \tag{2.1}$$

in fact, let $x \in I$, then

$$x \in \left(a, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, b\right), \quad \text{say,} \quad x \in \left(a, \frac{a+b}{2}\right),$$

then there is an integer N

$$a + \frac{h}{2^{N+1}} \leqslant x < a + \frac{h}{2^N},$$

therefore the right hand side of (2.1) becomes

$$v(x) + \sum_{n=1}^{N} s_n(x)$$

and so, we need to prove this sum is 1. In fact, if we explicitly write down $v(x)$ and $s_n(x)$ we see that

$$v(x) + \sum_{n=1}^{N} s_n(x) = \frac{2}{h}(x-a) + \frac{2}{h}(x-a) + \frac{2^2}{h}(x-a) + \cdots + \frac{2^{N-1}}{h}(x-a) - \frac{2^N}{h}\left(x - a - \frac{h}{2^N}\right)$$

Therefore we have

$$v(x) + \sum_{n=1}^{N} s_n(x) = \frac{2^N}{h}(x-a) - \frac{2^N}{h}\left(x - a\frac{h}{2^N}\right) = 1.$$

Similar situations occur when x is in the other half of I, that is,

$$x \in \left[\frac{a+b}{2}, b\right),$$

so that the assertion is proved.

Observe now that

$$\|v\|_p = \frac{h^{1/p}}{(1+p)^{1/p}} \quad \text{and} \quad \|s_n\|_p = \|t_n\|_p = \frac{h^{1/p}}{2^{(n/p)+1}(1+p)^{1/p}}$$

so that

$$\|\chi_I\|_{Bp}^T \leqslant \|v\|_p + \sum_{n=1}^{\infty} \|s_n\|_p + \sum_{n=1}^{\infty} \|t_n(x)\|_p = A_p h^{1/p} = A_p |I|^{1/p} = A_p \|\chi_I\|_{Bp}$$

where

$$A_p \doteq \frac{2^{1/p}}{(2^{1/p}-1)(1+p)^{1/p}},$$

consequently for any

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \chi_{I_n}(x) \tag{2.2}$$

we have $\|f\|_{Bp}^T \leqslant A_p \|f\|_{Bp}$.

Next suppose $t(x)$ is a triangular function of height 1 whose support is some subinterval $I \subset [0, 2\pi)$ and $h = |I|$. Let us expand $t(x)$ in a Haar series; that is

$$t(x) = \frac{h}{2\pi} + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} c_{nk}\phi_{nk}(x)$$

where ϕ_{nk} are the Haar functions (see description four).

Choose an integer N such that

$$\frac{2\pi}{2^{N+1}} < h \leqslant \frac{2\pi}{2^N}. \tag{2.3}$$

For $n \leqslant N$ there are at most two dyadic intervals I_{nk} which intersect I.

We now estimate $|c_{nk}|$ by the elementary way

$$|c_{nk}| = \left| \int_{I_{nk}} t(x)\phi_{nk}(x)\,dx \right| \leqslant |I|\,\|\phi_{nk}\|_\infty = \frac{h2^{n/2}}{(2\pi)^{1/2}}. \tag{2.4}$$

Again we estimate $|c_{nk}|$ by using a different approach, in fact

$$c_{nk} = \int_{I_{nk}} t(x)\phi_{nk}(x)\,dx = \int_{I_{nk}} (t(x) - t(x_0))\phi_{nk}(x)\,dx.$$

where x_0 is any point of I_{nk}. Therefore we have

$$\begin{aligned} |c_{nk}| &\leqslant \|\phi_{nk}\|_1 \cdot \max\{|t(x) - t(x_0)|; x \in I_{nk}\} \\ &\leqslant (2\pi)^{1/2}2^{-(n/2)}4\pi h^{-1}2^{-n} = C2^{-3n/2}h^{-1} \end{aligned} \tag{2.5}$$

where $C = (2\pi)^{1/2}4\pi$.

We now estimate $\|t\|_{Bp}^H$, in fact, by definition we have

$$\|t\|_{Bp}^H = h(2\pi)^{(1/p)-1} + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} |c_{nk}|\,\|\phi_{nk}\|_p.$$

If we set

$$\sum_{n=0}^{\infty} = \sum_{n=0}^{N} + \sum_{n=N+1}^{\infty} = S + T.$$

Recall that for $n \leqslant N$ there are at most two dyadic intervals I_{nk}which intersect I, therefore using (2.4) we get

$$\begin{aligned} S &= \sum_{n=0}^{N} \sum_{k=1}^{2^n} |c_{nk}|\,\|\phi_{nk}\|_p \leqslant \sum_{n=0}^{N} 2h2^{n/2}(2\pi)^{-1/2}2^{n/2}(2\pi)^{-1/2}\left(\frac{2\pi}{2^n}\right)^{1/p} \\ &= 2h(2\pi)^{1/p-1} \sum_{n=0}^{N} 2^{n(1-1/p)} \leqslant B_p h(2\pi)^{1/p-1}2^{(N+1)(1-1/p)} \\ &\leqslant B_p h^{1/p} \end{aligned}$$

The inequality on the last step follows from (2.3) where B_p is a constant which depends upon p and observe that it tends to infinity as p approaches 1.

The estimate for T follows by the observation that for $n > N$, there are at most $2^{n-N} + 2 \leqslant 2^{n-N+1}$ which intersects I and by (2.5). In fact,

$$\begin{aligned} T &= \sum_{n=N+1}^{\infty} \sum_{k=1}^{2^N} |c_{nk}| \, \|\phi_{nk}\|_p \\ &\leqslant \sum_{n=N+1}^{\infty} 2^{n-N+1} C 2^{-3n/2} h^{-1} 2^{n/2} (2\pi)^{-1/2} \left(\frac{2\pi}{2^n}\right)^{1/p} \\ &= C_p 2^{-N} h^{-1} \sum_{n=N+1}^{\infty} 2^{-n/p} \leqslant C_p 2^{-N} h^{-1} 2^{-(N+1)/p} \\ &\leqslant C_p h^{1/p}. \end{aligned}$$

The constants C_p are not the same at every occurence.

Thus there is a constant G_p which blows up as p approaches 1 such that

$$\|t\|_{Bp}^{H} \leqslant G_p h^{1/p}.$$

Therefore if

$$f(x) = \sum_{n=0}^{\infty} t_n(x)$$

where $t_n(x)$ is a triangular function we have

$$\|f\|_{Bp}^{H} \leqslant G_p \|f\|_{Bp}^{T}. \tag{2.6}$$

It is obvious from the definition that a function in the fourth description is also in the second description and that

$$\|f\|_{Bp}^{*} \leqslant \|f\|_{Bp}^{H}. \tag{2.7}$$

Likewise a function in the second description is also in the first description and that

$$\|f\|_{Bp} < 2^{1-(1/p)} \|f\|_{Bp}^{*}. \tag{2.8}$$

Now putting together (2.2), (2.6), (2.7) and (2.8) and the fact that the intervals of the two other descriptions are dyadic it makes the equivalence of the definitions of the B^p obvious.

We would like to point out that for $p = 1$, B^1 is not the same as B^1-dyadic, as was shown in [11].

3. Some Consequences

In this section we present some theorems whose proof was made possible due to the several descriptions of the special atom spaces.

Theorem 3.1. *If* $1/2 < p < 1 < q < \infty$ *and* $1/q = 1/p - 1$ *then* $f \in B^p$ *if and only if* $F \in B^q$, *where* F *is the indefinite integral of* f, *that is,* $F(x) = \int_0^x f(t)\,dt$.

PROOF. Given a special atom $b_n(x)$ let

$$t_n(x) = \int_{x_n}^{x} b_n(t)\,dt$$

where x_n is the left hand endpoint of the interval supporting $b_n(x)$. Then $t_n(x)$ is a triangular function and there exists a constant $C_q = 1/2(1+q)^{1/q}$ such that $\|t_n\|_q = C_q \|b_n\|_p$ for $1/q = 1/p - 1$. If we define $F(x)$ to be that function in L^q which is the sum of the $t_n(x)$, that is,

$$F(x) = \sum_{n=1}^{\infty} t_n(x),$$

where $t_n(x)$ is as above, then it is clear that

$$\|F\|_{Bq}^{T} \leqslant C_q \|f\|_{Bp}^{*}.$$

Conversely if $F \in B^q$ and we write

$$F(x) = \sum_{n=1}^{\infty} t_n(x)$$

and let $b_n(t) = (d/dx)t_n(x)$ (except at the corners) and let $f(x) = F'(x)$ be the distribution which is the sum of the $b_n(x)$, that is,

$$f(x) = \sum_{n=1}^{\infty} b_n(x),$$

then clearly

$$\|f\|_{Bp}^{*} \leqslant \frac{1}{C_q} \|f\|_{Bq}^{T}$$

so that

$$\|F\|_{Bq}^{T} = C_q \|f\|_{Bp}^{*}.$$

Observe that if f has two representations

$$f(x) = \sum_{n=1}^{\infty} b_n(x) = \sum_{n=1}^{\infty} c_n(x)$$

where $b_n(x)$ and $c_n(x)$ were special atoms and $s_n(x)$ and $t_n(x)$ their respective indefinite integrals then for every test function ϕ,

$$\sum_{n=1}^{\infty} (b_n, \phi) = \sum_{n=1}^{\infty} (c_n, \phi)$$

where both sums converge absolutely (recall that $(b_n, \phi) = \int_{I_n} b_n(x)\phi(x)\,dx$, where I_n is the support of b_n, likewise for (c_n, ϕ)), then by integration by parts we get

$$-\sum_{n=1}^{\infty} (s_n, \phi') = -\sum_{n=1}^{\infty} (t_n, \phi').$$

The dash means derivative.

Thus if

$$F(x) = \sum_{n=1}^{\infty} s_n(x) \quad \text{and} \quad G(x) = \sum_{n=1}^{\infty} t_n(x),$$

then F and G are in L^q and for every test function ϕ.

$$(F - G, \phi') = 0$$

so that F and G are functions in L^q which at most differ by a constant. Therefore Theorem 3.1 is proved.

Theorem 3.2 *If $f \in B^p$, $1/2 < p < \infty$ and $f_h(x) = f(x + h)$ then*

$$\lim_{h \to 0} \| f_h - f \|^*_{B^p} = 0.$$

PROOF. Suppose I is an interval of length m and consider

$$v(x) = \chi_I(x + h) - \chi_I(x).$$

$v(x)$ is supported on an interval of length $m + h$ and on this interval we expand $v(x)$ in a Haar series relative to that interval. For each $n \geqslant 1$ there are at most two ϕ_{nk} whose coefficients c_{nk} do not vanish and for these, we have

$$|c_{nk}| \leqslant \left(\frac{2^n}{m + h}\right)^{1/2} \cdot \left(h \Lambda \frac{m + h}{2^n}\right).$$

(Λ denotes the operation of taking the smaller of two numbers.)

Estimating the B^p norm of $v(x)$ we get

$$\|v\|_{Bp}^{H} = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|c_{nk}\phi_{nk}\|_p$$

$$\leqslant \sum_{n=0}^{\infty} 2\left(\frac{2^n}{m+h}\right)^{1/2}\left(h\Lambda\frac{m+h}{2^n}\right)\left(\frac{2^n}{m+h}\right)^{1/2}\left(\frac{m+h}{2^n}\right)^{1/p}$$

$$< \frac{2}{m}\sum_{n=0}^{\infty} 2^{n(1-1/p)}\left(h\Lambda\frac{m+h}{2^n}\right).$$

This last series converges and individual terms tend monotonically to zero with h so that the sum tends to zero with h. Therefore

$$\lim_{h\to 0} \|v\|_{Bp}^{*} = 0.$$

Now since a special atom is a multiple of the difference of two characteristic functions it is clear that if $b(x)$ is a special atom and if $b_h(x) = b(x+h)$ then

$$\lim_{h\to 0} \|b_h - b\|_{Bp}^{*} = 0.$$

Given $\epsilon > 0$, any $f \in B^p$ can be written $f(x) = g(x) + K(x)$ where $g(x)$ is a finite sum of special atoms and $\|K\|_{Bp}^{*} < \epsilon$, therefore as $\|g_h - g\|_{Bp}^{*} < \epsilon$ for h small, we know that B^p is invariant under translation so that,

$$\|f_h - f\|_{Bp}^{*} \leqslant \|g_h - g\|_{Bp}^{*} + \|K_h\|_{Bp}^{*} + \|K\|_{Bp}^{*} < 3\epsilon$$

for h small enough. So Theorem 3.2 is proved.

By the modulus of continuity of $f \in B^p$ we mean for $\delta > 0$

$$w(\delta) = \sup\{\|f_h - f\|_{Bp}^{*};\ -\delta < h < \delta\}.$$

(depending on the context we may replace $\|\ \ \|_{Bp}^{*}$ by $\|\ \ \|_{Bp}$ or $\|\ \ \|_{Bp}^{H}$ etc.)

It is an immediate consequence of the definition of B^p and Theorem 3.2, that for $f \in B^p$, $w(\delta) \leqslant 2\|f\|_{Bp}^{*}$ and $w(\delta) \to 0$ as $\delta \to 0$.

Theorem 3.3. *If $1/2 < p < 1$ then L^1 of the boundary of the unit disc is continuously embedded in B^p.*

PROOF. Let I be a dyadic interval of length $h = 2\pi/2^m$. Let us expand χ_I

$$\chi_I(x) = c_{00}\phi_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{2^n} c_{nk}\phi_{nk}(x) \tag{3.4}$$

where ϕ_{nk} are the Haar functions and $\phi_{00}(x) = (2\pi)^{-1/2}$.

Notice that for $n < m$ there is precisely one k such that c_{nk} does not vanish, therefore

$$|c_{nk}| = \left| \int_I \phi_{nk}(x)\,dx \right| = h\left(\frac{2^n}{2\pi}\right)^{1/2} \quad \text{and} \quad c_{00} = \frac{h}{(2\pi)^{1/2}}.$$

For $n \geqslant m$ all the Haar coefficients c_{nk} vanish, so that (3.4) becomes

$$\chi_I(x) = \frac{h}{2\pi} + \sum_{n=0}^{m-1} c_{nk}\phi_{nk}(x),$$

consequently using the definition of B^p norm

$$\|\chi_I\|_{B^p}^* \leqslant \sum_{n=0}^{m-1} h\left(\frac{2\pi}{2^n}\right)^{(1/p)-1} < C_p h = C_p \|\chi_I\|_1.$$

Thus if f is a step function whose steps are dyadic intervals we have

$$\|f\|_{B^p}^* \leqslant C_p \|f\|_1.$$

Since for any $f \in L^1$ there is a sequence of such step fuction f_n which approach f in L^1 and since B^p is a Banach space we get our result.

Corollary 3.5. *For $1/2 < p < 1$, L^2 of the boundary of the unit disc is continuously embedded in B^p. For $1 < p < \infty$, $(L^2)_1$ = the set of indefinite integrals of L^2 functions whose constant terms vanish, is continuously embedded in B^p, in particular every C^∞ function on the boundary of the unit disc is in every B^p.*

PROOF. The first part of the statement will follow from the fact that L^2 is continuously embedded in L^1 and Theorem 3.3. The last part follows from Theorem 3.1 and the first part of this corollary.

The use of lemmas of the following type in functional analysis is due to G. G. Lorentz.

Lemma 3.6. *Let H be a Hilbert space of measurable functions on T with inner product given by*

$$(f, g) = \int_T f(x)g(x)\,dx$$

and let F and G be real Banach spaces of measurable functions such that H is embedded continuously in F. Assume that

$$|(f, g)| \leqslant \|f\|_F \cdot \|g\|_G$$

and that

$$\|g\|_G = \sup\{|(f, g)|; f \in H, \|f\|_F \leqslant 1\}.$$

Then for every $f \in H$,

$$\|f\|_F = \sup\{|(f, g)|; \|g\|_G \leqslant 1\}.$$

PROOF. Given a subset C of H, let $C' = \{g \in H: |(f, g)| \leqslant 1 \text{ for all } f \in C\}$. It is a geometrical fact about Hilbert spaces that if C satisfies the three properties, 1. closed, 2. convex, 3. $f \in C$ implies $-f \in C$, then C' satisfies these three properties as well and moreover $C'' = C$.

Let us take $C = \{f \in H: \|f\|_F \leqslant 1\}$ then C satisfies the three properties, so it follows that $C' = \{g \in H: \|g\|_G \leqslant 1\}$. Therefore $C'' = C$ which implies the final statement of the lemma. So the lemma is proved.

Following Zygmund [17] for $0 < \alpha < 2$ we denote by Λ_α the set of all functions g on T, for which there is a constant A such that $|g(x + h) + g(x - h) - 2g(x)| \leqslant Ah^\alpha$, and $\|g\|_{\Lambda_\alpha}$ is meant to be the infimum of all such A, that is,

$$\|g\|_{\Lambda_\alpha} = \sup_{\substack{h>0 \\ x}} \frac{|g(x + h) + g(x - h) - 2g(x)|}{h^\alpha}.$$

For $0 < \alpha < 1$, $\Lambda_\alpha = \operatorname{Lip}\alpha = \{g; g(x + h) - g(x) = 0(h^\alpha)\}$ and for $1 < \alpha < 2$, $f \in \Lambda_\alpha$ if and only if its derivative $f' \in \operatorname{Lip}(\alpha - 1)$.

Lemma 3.7. *If $f \in B^p$ and $g \in \operatorname{Lip}\alpha$ for $\alpha = 1/p$, $1/2 < p < 1$ then for $f \in L^2$ we have*

$$\|f\|^*_{B^p} = \sup\left\{\left|\int_T f(x)g(x)\,dx\right| : \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\}.$$

PROOF. By Corollary 3.5 and Lemma 3.6 with

$$H = L^2, \qquad F = B^p \quad \text{and} \quad G = \operatorname{Lip}(\alpha - 1)$$

we get the first part since for any special atom b we have

$$|(b, g)| \leqslant \|g\|_{\operatorname{Lip}(\alpha-1)} \cdot \|b\|_{B^p},$$

with the norm of $g \in \operatorname{Lip}(\alpha - 1)$ given by $\left\|\int g\right\|_{\Lambda_\alpha}$, where $\int g$ is the periodic indefinite integral of g.

For the second part, let $1/r = 1/p + 1$, then $\|f'\|^*_{B^r} = C\|f\|^T_{B^p}$ by Theorem 3.1, where the dash means the derivative of f. But integration by parts yields

$$\int_T f(x)g(x)\,dx = -\int_T f'(x)g_1(x)\,dx \quad \text{where} \quad g_1(x) = \int_0^x g(t)\,dt.$$

We now use the first part of this lemma to calculate $\|f\|^*_{B^r}$.

Lemma 3.8. *Suppose that given a family G of measurable functions on T or R^n, we define a norm on a space X for which this supremum is finite by*

$$\|f\|_X = \sup\left\{\left|\int f(x)g(x)\,dx\right|; g \in G\right\}.$$

Then a formula which recalls Minkowski's inequality for integrals holds, namely if

$$F(x) = \int f(x, y)\,dy$$

then

$$\|F\|_X \leqslant \int \|f(\cdot, y)\|_X\,dy.$$

If in addition G is translation invariant then for the convolution $f * g$ where $g \in L^1$ we have

$$\|f * g\|_X \leqslant \|f\|_X \cdot \|g\|_1$$

PROOF. From the definition of norm in X we have that

$$\|F\|_X = \operatorname{Sup}\left\{\left|\int g(x)\,dx \int f(x, y)\,dy\right|; g \in G\right\}.$$

But for each such g in G, the integral inside the brackets is bounded by

$$\int dy \left|\int g(x)f(x, y)\,dx\right|,$$

so that

$$\|F\|_X \leqslant \int \|f(\cdot, y)\|_X\,dy.$$

Now suppose G is translation invariant:

$$\|f * g\|_X = \sup\left\{\left|\int h(x)\,dx \int f(x - y)g(y)\,dy\right|; h \in G\right\}$$

For each h, the integral is bounded by

$$\int |g(y)|\,dy \left|\int h(x)f(x - y)\,dy\right| = \int |g(y)|\,dy \left|\int h(u + y)f(y)\,du\right|$$

Therefore

$$\|f * g\|_X \leqslant \sup\left\{\left|\int h(u + y)f(y)\,dy\right|; h \in G\right\} \cdot \int |g(y)|\,dy,$$

that is

$$\|f * g\|_X \leqslant \|g\|_1 \cdot \|f\|_X.$$

Observe now that if we take $X = B^p$ and $G = \left\{ g \in \mathrm{Lip}\,(\alpha - 1);\ \left\| \int g \right\|_{\Lambda_\alpha} \leqslant 1 \right\}$ for $\alpha = 1/p$, $1/2 < p < 1$ on T, and for $1 < p < \infty$ we set $G = \{ g \in \mathrm{Lip}\,\alpha;\ \left\| \int g \right\|_{\Lambda_{\alpha+1}} \leqslant C \}$ then we have the following.

$$\|f * g\|^*_{B^p} \leqslant \|g\|_1 \cdot \|f\|^*_{B^p} \quad \text{for} \quad \tfrac{1}{2} < p < 1$$
$$\|f * g\|^T_{B^p} \leqslant \|g\|_1 \cdot \|f\|^T_{B^p} \quad \text{for} \quad 1 < p < \infty.$$

Theorem 3.9. *If $f \in B^p$ on the boundary of the unit disc, $1 < p < \infty$, and if f_r is its Poisson integral then $\|f_r\|^T_{B^p}$ is a non-decreasing function of r which tends to $\|f\|^T_{B^p}$ as $r \nearrow 1$, that is, $\|f\|^T_{B^p} \to \|f\|^T_{B^p}$ as $r \nearrow 1$. Moreover f_r tends to f in B^p.*

Conversely, if $\mu(r, \theta)$ is a harmonic function in the unit disc such that for each r, $0 < r < 1$

$$\|\mu(r, \cdot)\|_{B^p} \leqslant M < \infty.$$

then μ is the Poisson integral of $f \in B^p$. (A similar theorem is true for the real line.)

PROOF. We start with the converse. Notice that

$$\|\mu(r, \cdot)\|_p \leqslant \|\mu(r, \cdot)\|_{B^p} \leqslant M,$$

so that H^p-theory tells us that there is an $f \in H^p = L^p$ such that μ is the Poisson integral of f. We wish to prove that $f \in B^p$.

Expanding f into a Haar series we get

$$f(\theta) = a + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} c_{nk} \phi_{nk}(\theta).$$

Let r_i be a sequence of $r_i \nearrow 1$. For each i expand $\mu(r_i, \theta)$ in a Haar series, namely

$$\mu(r_i, \theta) = c + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} c^i_{nk} \phi_{nk}(\theta).$$

Notice that

$$c_{nk} = \int_T f(\theta)\phi \, d\theta = \lim_{i \to \infty} \int_T \mu(r_i, \theta)\phi_{nk}(\theta) \, d\theta = \lim_{i \to \infty} c^i_{nk}.$$

By Fatou's lemma (for sums rather than integrals) we get

$$\|f\|^H_{B^p} = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|c_{nk}\phi_{nk}\|_p \leqslant \liminf_{i \to \infty} \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|c^i_{nk}\phi_{nk}\|_p$$
$$= \liminf_{i \to \infty} \|\mu(r_i, \cdot)\|^H_{B^p} \leqslant AM$$

where A is a constant. Therefore $f \in B^p$.

Next let $f \in B^p$. If we properly normalize the Poisson Kernel then $f_r = f * P_r$ where $\|P_r\|_1 = 1$. Let $\alpha = 1/p$, $p > 1$, since $f_r \in C^\infty$, we obtain from lemma 3.7 that

$$\|f_r\|_{Bp}^T = \sup\left\{\left|\int f_r(\theta)g(\theta)\,d\theta\right|; \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\},$$

but

$$\left|\int_T f_r(\theta)g(\theta)\,d\theta\right| = \left|\int_T f(\theta)g_r(\theta)\,d\theta\right| \leqslant \|f\|_{Bp}^T \cdot \left\|\int g_r\right\|_{\Lambda_{\alpha+1}} \leqslant \|f\|_{Bp}^T.$$

Indeed if $r_1 < r < 1$ and if $\rho = r_1/r$, then it is well known that $f_{r_1} = f_r * P_\rho$, so that by Theorem 3.9 we get

$$\|f_{r_1}\|_{Bp}^T \leqslant \|f\|_{Bp}^T \cdot \|P_\rho\|_1 \leqslant \|f_r\|_{Bp}^T.$$

Thus $\|f_r\|_{Bp}^T$ is a non-decreasing function of r. Combining this with what was shown in the first part of the theorem we obtain as $r \nearrow 1$

$$\lim \|f_r\|_{Bp}^T \leqslant \|f\|_{Bp}^T$$

and

$$\liminf \|f_r\|_{Bp}^H \geqslant \|f\|_p^H.$$

Thus there exists a constant $C > 0$ such that as $r \nearrow 1$

$$\lim \|f_r\|_{Bp}^T \geqslant C\|f\|_{Bp}^H.$$

(We shall see later that $C = 1$.) Therefore given any $\epsilon > 0$ there is $r < 1$ such that $C\|f_r\|_{Bp}^T \geqslant C\|f\|_{Bp}^T - \epsilon$. Now as $f_r \in C^\infty$ there is a function g,

$$\left\|\int g\right\|_{\mathrm{Lip}\,\alpha} \leqslant 1$$

such that

$$C\|f\|_{Bp}^T - 2\epsilon \leqslant \left|\int_T f_r(\theta)g(\theta)\,d\theta\right| = \left|\int_T f(\theta)g_r(\theta)\,d\theta\right|.$$

But

$$\left\|\int g_r\right\|_{\mathrm{Lip}\,\alpha} \leqslant 1$$

so that if we set

$$\|f\|_{Bp}^{**} = \sup\left\{\left|\int_T f(\theta)g(\theta)d\theta\right|; \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\}$$

Then

$$C\|f\|_{B^p}^T \leqslant \|f\|_{B^p}^{**} \leqslant \|f\|_{B^p}^T.$$

It is now easily seen that if f is a distribution not in B^p then $\|f_r\|_{B^p}^T \nearrow \infty$ and

$$\|f\|_{B^p}^{**} = \sup\left\{\left|\int_T f(\theta)g(\theta)\,d\theta\right|; g \in C^\infty, \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\} = \infty.$$

For $f \in B^p$, $f_r(\theta) = \int_T f(\theta - \phi)P_r(\phi)\,d\phi$ where $\int_T P_r(\theta)\,d\theta = 1$ so that $f_r(\theta) - f(\theta) = \int_T (f(\theta - \phi) - f(\theta))P_r(\phi)\,d\phi$. Therefore by a Minkowski type inequality (lemma 3.8) implies

$$\begin{aligned}\|f_r - f\|_{B^p}^{**} &\leqslant \int_T P_r(\phi)\|f(\theta - \phi) - f(\theta)\|_{B^p}^{**}\,d\phi \\ &\leqslant \int_T P_r(\phi)w(|\phi|)\,d\phi \leqslant 2\int_0^\pi P_r(\phi)w(\phi)\,d\phi,\end{aligned}$$

where w is the modulus of continuity defined right after Theorem 3.2.

But $P_r(\phi)$ is monotone decreasing for $0 < \phi < \pi$ and w is monotone non-decreasing so that for any ξ, $0 < \xi < \pi$, the last integral is bounded by

$$2w(\xi)\int_0^\xi P_r(\phi)\,d\phi + 2P_r(\xi)\int_\xi^\pi w(\phi)\,d\phi < 2w(\xi) + 4\pi\|f\|_{B^p}^T P_r(\xi).$$

Choose ξ so as to make the first term arbitrarily small then for this ξ, choose r so as to make the second arbitrarily small. This shows that $f_r \to f$ in B^p which in turn implies $\|f_r\|_{B^p}^T \to \|f\|_{B^p}^T$ as $r \nearrow 1$.

Indeed in addition to proving our theorem we have also the following.

Theorem 3.10. *If $1 < p < \infty$ and $\alpha = 1/p$ then for any function $f \in B^p$*

$$\|f\|_{B^p}^T = \sup\left\{\left|\int_T f(\theta)g(\theta)\,d\theta\right|; \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right|,$$

and for any distribution f not in B^p

$$\sup\left\{\left|\int_T f(\theta)g(\theta)\,d\theta\right|; g \in C^\infty; \left\|\int g\right\|_{\Lambda_{\alpha+1}} \leqslant 1\right\} = \infty.$$

The interested reader is referred to [9], where Theorem 3.10 is related with the main result there.

For $1/2 < p < 1$ we have a result similar to Theorem 3.10, indeed we obtain the following.

Theorem 3.11. *If f_r is the Poisson integral of a distribution $f \in B^p$ of boundary of the unit disc, $1/2 < p < 1$, then*

$$\|f_r\|_{B^p}^* \nearrow \|f\|_{B^p}^* \quad \textit{and} \quad \|f_r - f\|_{B^p}^* \to 0 \quad \textit{as} \quad r \nearrow 1.$$

Conversely if $\mu(r, \theta)$ is a harmonic function in the unit disc such that for each r, $0 < r < 1$,

$$\|\mu(r, \cdot)\|_{B^p}^* \leqslant M < \infty$$

then there exists a distribution $f \in B^p$ whose Poisson integral is μ.

PROOF. Assume μ vanishes at the origin, so that it may be written in the form

$$\mu(r, \theta) = \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)$$

and let

$$U(r, \theta) = \sum_{n=1}^{\infty} \frac{r^n(a_n \sin n\theta - b_n \cos n\theta)}{n}$$

so that

$$\frac{\partial U(r, \theta)}{\partial \theta} = \mu(r, \theta).$$

Let $1/q = 1/p - 1$, so $1 < q < \infty$ then for each r Theorem 3.1 tells us that $\|U(r, \cdot)\|_{Bq}^T = C\|\mu(r, \cdot)\|_{B^p}^*$ for some constant C. But there is a constant A such that

$$\|U(r, \cdot)\|_{Bq}^H \leqslant A\|U(r, \cdot)\|_{Bq}^T = AC\|\mu(r, \cdot)\|_{B^p}^* \leqslant ACM.$$

Thus there is a $F \in B^q$ such that U is its Poisson integral by Theorem 3.10. Let f be a distribution which is a derivative of F, that is,

$$F(x) = \sum_{n=1}^{\infty} t_n(x), \qquad f(x) = \sum_{n=1}^{\infty} b_n(x) \quad \text{where} \quad b_n(x) = \frac{dt_n(x)}{dx},$$

(recall t_n is a triangular function).

The Poisson Kernel and its derivatives are all C^∞ and so are test functions, therefore we have

$$\mu(r, \theta) = \frac{\partial U(r, \theta)}{\partial \theta} = F * \frac{\partial P_r}{\partial \theta} = \sum_{n=1}^{\infty} \int_T t_n(\theta - \phi) \frac{\partial P_r(\phi)}{\partial \phi} d\phi.$$

And so, integration by parts term by term yields

$$\mu(r, \theta) = \sum_{n=1}^{\infty} \int_T b_n(\theta - \phi) P_r(\phi)\, d\phi = \sum_{n=1}^{\infty} (b_n(\theta - \cdot), P_r) = (f(\theta - \cdot), P_r)$$

and this last is precisely the definition of the Poisson integral of a distribution.

The rest of the proof is the same as the proof of the previous theorem mutatis mutandis and is left to the reader.

The proof of the next theorem which is the equivalent to Theorem 3.10 and Theorem 3.11 for $p = 1$, seems to be more delicate and requires the use of fractional integration.

Following Herman Weyl we say that if $f(\theta)$ is a distribution whose Fourier series is $\Sigma' c_n e^{in\theta}$ (Σ' denotes summation for $n \in Z$, $n \neq 0$) then $f_\alpha(\theta)$ is the distribution defined by

$$f_\alpha(\theta) = \Sigma' \frac{c_n e^{in\theta}}{(in)^\alpha}.$$

Let I_α be the linear operator which maps f into f_α. It is a well known theorem of Hardy and Littlewood that $f \in \operatorname{Lip} \beta$ if and only if $f_\alpha \in \operatorname{Lip}(\alpha + \beta)$ provided $\alpha + \beta < 1$. This result was extended by Zygmund [17] who showed that $f \in \Lambda_\beta$ if and only if $f_\alpha \in \Lambda_{\alpha+\beta}$ provided $\alpha + \beta < 2$.

f_α is called fractional integration of f. Observe that the kernel for fractional integration is in L^2 if and only if $\alpha > 1/2$.

Theorem 3.12. *If f_r is the Poisson integral of a function $f \in B(B = B^1)$ on the boundary of the unit disc then*

$$\|f_r\|_B^* \nearrow \|f\|_B^* \quad \text{and} \quad \|f_r - f\|_B^* \to 0 \quad \text{as} \quad r \nearrow 1.$$

Conversely if $\mu(r, \theta)$ is harmonic and for $0 < r < 1$,

$$\|\mu(r, \cdot)\|_B \leqslant M < \infty.$$

Then μ is the Poisson integral of f in B.

PROOF. We start with the converse. The main problem is to lay our hands on a special atomic decomposition of the boundary function. Assume μ vanishes at the origin. Then

$$\mu(r, \theta) = \sum_{n=-\infty}^{\infty}{}' c_n(r) e^{in\theta}$$

where $c_{-n} = \bar{c}_n$ (complex conjugate). Let

$$\mu_\alpha(r, \theta) = I_\alpha \mu(r, \theta) = \sum_{n=-\infty}^{\infty}{}' \frac{c_n(r) e^{in\theta}}{(in)^\alpha}.$$

Let $g \in \operatorname{Lip} \beta$ where $\alpha + \beta = 1$, then

$$\int_T \mu_\alpha(r, \theta) g(\theta)\, d\theta = \int_T \mu(r, \theta) h_\alpha(-\theta)\, d\theta \quad \text{where} \quad h_\alpha = I_\alpha h \quad \text{and} \quad h(\theta) = g(-\theta);$$

by Zygmund's Theorem $h_\alpha \in \Lambda_1$, the Zygmund class Λ_1, so that the last integral above is dominated by

$$\leqslant \|\mu(r, \cdot)\|_B \left\| \int h_\alpha \right\|_{\Lambda_1} \leqslant CM \left\| \int h \right\|_{\mathrm{Lip}(\beta+1)} = CM \left\| \int g \right\|_{\mathrm{Lip}(\beta+1)}.$$

This combined with Theorem 3.10 shows that if $q = 1/\beta$ then

$$\|\mu_\alpha(r, \cdot)\|_{B^q} \leqslant CM$$

so that $\mu_\alpha(r, \cdot)$ is the Poisson integral of $G \in B^q$. Let g be the «derivative» of G. (Recall we express G as a sum of triangular functions and define g as the term by term derivative.) Theorem 3.1 shows that $g \in B^p$ where $1/p = 1/q + 1$ and in the proof of Theorem 3.12 we have that the Poisson integral of g is precisely $\partial\mu_\alpha(r, \theta)/\partial\theta$. Observe that g has been defined as a sum of special atoms, also observe that if we differentiate a special atom we obtain a measure supported on three points which is fractionally integrated of order β, that is, convolved with the kernel $\sum'_{n=-\infty}^{\infty} e^{in\theta}/(in^\beta)$ is in L^2 provided $\beta > 1/2$. This in turn implies that $I_\beta b$ where $b(x)$ is a special atom is in $H = \int L^2 =$ the periodic indefinite integral of L^2-functions.

Now fix $\alpha < 1/2$, $\beta = 1 - \alpha$, if $f(\theta) = \sum_{n=0}^{\infty} b_n(\theta)$. For an individual $b_n(x)$ we have that $I_\beta b_n \in H$ so that for any g with $\left\| \int g \right\|_{\Lambda_1} = 1$

$$\int_T I_\beta b_n(\theta) g(\theta)\, d\theta = \int_T b_n(\theta) h_\beta(-\theta)\, d\theta$$

where $h_\beta = I_\beta h$ and $h(\theta) = g(-\theta)$ but $h_\beta \in \mathrm{Lip}\,\beta$, so that $I_\beta b_n \in B$ and given any $\epsilon > 0$ we may write $I_\beta b_n(\theta) = \sum_{j=1}^{\infty} c_{nj}(\theta)$ where $c_{nj}(\theta)$ are special atoms and

$$\sum_{j=1}^{\infty} \|c_{nj}\|_1 \leqslant C\|b_n\|_p + \frac{\epsilon}{2^n}.$$

Let

$$f(\theta) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} c_{nj}(\theta),$$

clearly f is a function in B. Let us compute its Poisson integral

$$f_r(\theta) = \frac{1}{2\pi} \int_T f(\theta - \phi) P_r(\phi)\, d\phi = \frac{1}{2\pi} \int_T \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} c_{nj}(\theta - \phi) P_r(\phi)\, d\phi$$

$$= \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_T \sum_{j=1}^{\infty} c_{nj}(\theta - \phi) P_r^*(\phi)\, d\phi = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_T I_\beta b_n(\theta - \phi) P_r^*(\phi)\, d\phi$$

$$= \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_T b_n(\theta - \phi) I_\beta P_r^*(\phi)\, d\phi = \frac{1}{2\pi} \int_T f(\theta - \phi) I_\beta P_r^*(\phi)\, d\phi,$$

where $P_r^*$is the Poisson integral minus its constant term. Notice that $I_\beta P_r^* \in C^\infty$ and so is a test function, therefore

$$(f(\phi - \cdot), I_\beta P_r^*) = \mu(r, \theta).$$

This last inequality is proved by expanding for each r both sides in a Fourier series. Since both sides are C^∞ functions pointwise equality follows.

The other part of the theorem follows by repeating the argument with the obvious modifications of Theorem 3.11 and Theorem 3.12 and is left to the reader.

Theorem 3.13. *If $f \in B^p$, $1/2 < p < \infty$ and σ_n* is the *(C, 1) means of the partial sums s_n of the Fourier expansion of f then σ_n tends to f in B^p.*

PROOF. To fix ideas, let $1/2 < p < 1$, $\sigma_n = f * K_n$ where K_n is the normalized Fejer kernel, then

$$\sigma_n(\theta) - f(\theta) = \int_T [f(\theta - \phi) - f(\theta)] K_n(\phi)\, d\phi$$

which by the Minkowski type inequality (see Lemma 3.8) gives

$$\| \sigma_n - f \|_{B^p}^* \leqslant 2 \int_0^\pi K_n(\phi) w(|\phi|)\, d\phi$$

which must tend to zero as $n \to \infty$, since the $w(|\phi|)$ the modulus of continuity (see comments right after Theorem 3.2) of an f in B^p tends to zero as ϕ tends to zero.

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Optimal Regularity for One-Dimensional Porous Medium Flow

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Abstract

We give a new proof of the Lipschitz continuity with respect to t of the pressure in a one dimensional porous medium flow. As is shown by the Barenblatt solution, this is the optimal t-regularity for the pressure. Our proof is based on the existence and properties of a certain selfsimilar solution.

In recent years there has been considerable interest in the regularity of non-negative solutions $u = u(x, t)$ to the porous medium equation

$$\frac{\partial u}{\partial t} = \Delta(u^m)$$

in $\mathbb{R}^d \times \mathbb{R}^+$, where $m > 1$ is a constant. For $d > 1$ the theory is still in flux and the optimal global regularity results are as yet unknown. Partial results can be found in [CVW] and [A2]. For $d = 1$ it is known [A1] that

$$v \equiv \frac{m}{m-1} u^{m-1}.$$

is Lipschitz continuous as a function of x, and this is the optimal regularity with respect to x. The Lipschitz continuity of v implies that u is Hölder continuous with exponent $\alpha = \min\{1, 1/(m-1)\}$. Kruzhkov [Kr] proved that for a class of parabolic equations which includes the porous medium equation, Hölder continuity in x with exponent α implies Hölder continuity in t

with exponent $\alpha/(\alpha + 2)$. Gilding [G] refined Kruzhkov's result to obtain the t-exponent $\alpha/2$. On the other hand, by assuming certain monotonicity for v_{xx}, Di Benedetto [DiB] proved that v is Lipschitz in t.

Actually, v is Lipschitz continuous in t without any assumptions on v_{xx}. This result was first proved by Bénilan [B] by means of a clever comparison argument. In this note we give an alternate proof which also uses comparison methods, but which is completely different from Bénilan's. In particular, our proof is based on a selfsimilar solution of the porous medium equation which has some independent interest.

We consider the initial value problem

$$\begin{aligned} &u_t = (u^m)_{xx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \\ &u(\cdot, 0) = u_0 \quad \text{in} \quad \mathbb{R}, \end{aligned} \tag{1}$$

where $m > 1$ is constant and $u_0 \geqslant 0$. For simplicity we assume that $u_0 \in L^\infty(\mathbb{R}) \cap \cap L^1(\mathbb{R})$. It is known that problem (1) possesses a unique generalized solution $u = u(x, t)$ in $\mathbb{R} \times \mathbb{R}^+$ with

$$0 \leqslant u \leqslant \|u_0\|_{L^\infty(\mathbb{R})}.$$

For isentropic flow of a perfect gas in a homegeneous porus medium u represents an appropriately scaled density. The corresponding pressure, given by

$$v \equiv \frac{m}{m-1} u^{m-1},$$

satisfies the equation

$$v_t = (m-1)vv_{xx} + v_x^2 \tag{2}$$

on the set where u is positive. For v we have the estimates

$$0 \leqslant v(x, t) \leqslant \|v_0\|_{L^\infty(\mathbb{R})} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \tag{3}$$

$$|v(x, t)|^2 \leqslant \frac{2}{(m+1)t} \|v_0\|_{L^\infty(\mathbb{R})} \quad \text{a.e. in} \quad \mathbb{R} \times \mathbb{R}^+, \tag{4}$$

and

$$v_t(x, t) \geqslant -\frac{m-1}{m+1} \frac{\|v_0\|_{L^\infty(\mathbb{R})}}{t} \quad \text{in} \quad \mathfrak{D}'(\mathbb{R} \times \mathbb{R}^+). \tag{5}$$

Here $v_0 = mu_0^{m-1}/(m-1)$. For definitions, proofs and references the reader can consult [A2].

Our main result is the following

Theorem. *Let v be the pressure corresponding to the solution u of problem* (1). *For every $\delta > 0$ there exists a constant $C = C(\delta, m, \|v_0\|_{L^\infty(\mathbb{R})}) \in \mathbb{R}^+$ such that*

$$|v(x, t') - v(x, t)| \leqslant C|t' - t|$$

for all (x, t') and (x, t) in $\mathbb{R} \times [\delta, \infty)$.

The proof of this theorem is based on two propositions. The first describes a selfsimilar solution of the pressure equation (2) which is then used in the second proposition to estimate the growth of v.

Proposition 1. *The initial value problem*

$$\begin{aligned} v_t &= (m-1)vv_{xx} + v_x^2 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+ \\ v(x, 0) &= |x| \qquad\qquad\quad \text{in} \quad \mathbb{R} \end{aligned} \tag{6}$$

possess a unique solution $v = p(x, t)$, where p has the form

$$p(x, t) = rf(\theta) \tag{7}$$

with $r = \{x^2 + t^2\}^{1/2}$ and $\theta = \arctan(x/t)$. Here $f \in C^1\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $f'(0) = 0$, $f\left(\pm\frac{\pi}{2}\right) = 1$, $f'\left(\pm\frac{\pi}{2}\right) = \mp 1$, and

$$f(\theta) > \cos\theta + |\sin\theta|.$$

Remark. According to the results of [AV], as $m \downarrow 1$ the solution of (6) tends to the solution $v = q(x, t)$ of the initial value problem

$$\begin{aligned} v_t &= v_x^2 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+ \\ v(x, 0) &= |x| \quad \text{in} \quad \mathbb{R}. \end{aligned}$$

In particular,

$$q(x, t) = r(\cos\theta + |\sin\theta|).$$

Thus $f(\theta) \to \cos\theta + |\sin\theta|$ as $m \downarrow 1$. The (computed) graphs of $f(\theta)$ are shown in the next page in figure 1 for various values of m.

PROOF. The global existence and uniqueness of the solution $v = p(x, t)$ of (6) follows from the results of Kalashnikov [K]. Moreover, $p > 0$ in $\mathbb{R} \times \mathbb{R}^+$ so that $p \in C^\infty(\mathbb{R} \times \mathbb{R}^+)$. For any $\lambda \in \mathbb{R}^+$ define

$$p_\lambda(x, t) \equiv \frac{1}{\lambda} p(\lambda x, \lambda t).$$

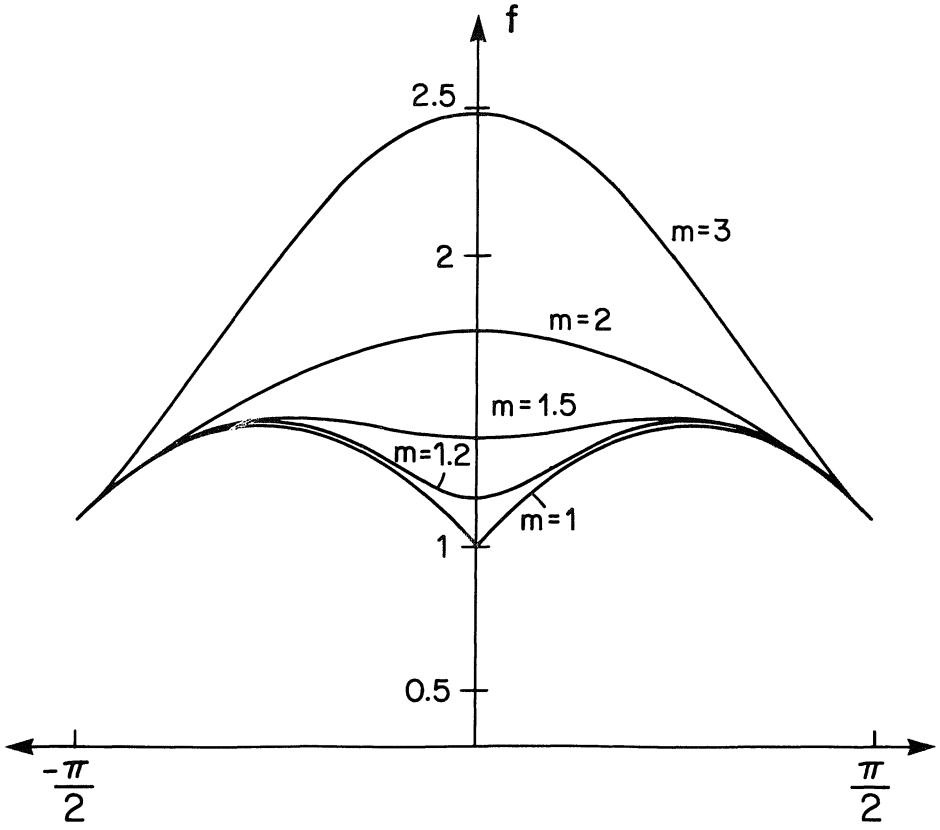


Fig. 1.

It is easy to verify that p_λ is a solution to the pressure equation (2) in $\mathbb{R} \times \mathbb{R}^+$ regardless of the value of $\lambda \in \mathbb{R}^+$. Moreover

$$p_\lambda(x, 0) = \frac{1}{\lambda} |\lambda x| = |x|.$$

Therefore, for every $\lambda \in \mathbb{R}^+$, $p_\lambda(x, t)$ is a solution to problem (6). By uniqueness [K]

$$p(x, t) \equiv p_\lambda(x, t) = \frac{1}{\lambda} p(\lambda x, \lambda t) \tag{8}$$

in $\mathbb{R} \times \mathbb{R}^+$ for every $\lambda \in \mathbb{R}^+$. In particular, for $\lambda = 1/r$ we have

$$p(x, t) = rp(\sin\theta, \cos\theta)$$

so that (7) holds with $f(\theta) = p(\sin\theta, \cos\theta)$.

Since p is an even function of x which is smooth for $t > 0$, it follows that f is even and $f'(0) = 0$. For $x \neq 0$

$$|x| = p(x, 0) = |x| f\left(\pm\frac{\pi}{2}\right)$$

implies that $f\left(\pm\frac{\pi}{2}\right) = 1$. Moreover, $p > 0$ for $t > 0$ implies that $f > 0$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

To derive further properties of f it is convenient to look at another form of the solution of (6). If we take $\lambda = 1/t$ in (8) we find

$$p(x, t) = tp\left(\frac{x}{t}, 1\right) = rp(\tan\theta, 1)\cos\theta.$$

Thus

$$f(\theta) = g(\tan\theta)\cos\theta$$

where $g(s) \equiv p(s, 1)$. By a calculation which is elementary but tedious, one can verify that g satisfies the ordinary differential equation

$$(m - 1)gg'' + g'^2 = g - sg', \tag{9}$$

where $' = d/ds$ and $s = \tan\theta$. Note that

$$f'(\theta) = -g(\tan\theta)\sin\theta + \frac{g'(\tan\theta)}{\cos\theta}$$

so that $f'(0) = 0$ implies that

$$g'(0) = 0.$$

On the other hand,

$$1 = \lim_{\theta \to \pi/2} f(\theta) = \lim_{\theta \to \pi/2} \sin\theta \frac{g(\tan\theta)}{\tan\theta} = \lim_{s \to \infty} \frac{g(s)}{s}.$$

Thus

$$g(s) \sim s \quad \text{as} \quad s \to \infty.$$

Moreover, it follows from l'Hôpital's rule that if g' has a limit as $s \to \infty$ then

$$g'(s) \sim 1 \quad \text{as} \quad s \to \infty.$$

Next, we observe that

$$g'' > 0 \quad \text{on} \quad [0, \infty). \tag{10}$$

Since $g(0) = f(0) \neq 0$ and $g'(0) = 0$ it follows from (9) that

$$g''(0) = 1/(m-1).$$

Suppose that for some $\bar{s} \in \mathbb{R}^+$ we have $g''(\bar{s}) = 0$. Then, in view of (9), $g(\bar{s})$ and $g'(\bar{s})$ satisfy

$$g'^2(\bar{s}) + \bar{s}g'(\bar{s}) - g(\bar{s}) = 0$$

so that

$$g'(\bar{s}) = b \equiv \frac{1}{2}(-\bar{s} \pm \{\bar{s}^2 + 4g(\bar{s})\}^{1/2}).$$

The function

$$G(s) \equiv b^2 + bs$$

is a solution to (9) with $G(\bar{s}) = g(\bar{s})$ and $G'(\bar{s}) = g'(\bar{s})$. By standard uniqueness theory we conclude that $g(s) \equiv G(s)$ and this contradicts $g''(0) > 0$.

Set $a = g(0)$. We claim that

$$g'(s) < \sqrt{a} \quad \text{and} \quad g(s) < a + \sqrt{a}\,s$$

on $\mathbb{R}^+$. Suppose there exists an $\tilde{s} \in \mathbb{R}^+$ for which $g'(\tilde{s}) \geqslant \sqrt{a}$. Since $g'(0) = 0$ and g' is increasing, there exists an $\bar{s} \in (0, \tilde{s}]$ such that $g'(\bar{s}) = \sqrt{a}$, $g' < \sqrt{a}$ on $[0, \bar{s})$, and $g(\bar{s}) < a + \sqrt{a}\,\bar{s}$. Then

$$0 = 1 - \frac{\sqrt{a}(\sqrt{a} + \bar{s})}{a + \sqrt{a}\,\bar{s}} > 1 - \frac{g'(\bar{s})(g'(\bar{s}) + \bar{s})}{g(\bar{s})} = (m-1)g''(\bar{s})$$

which contradicts (10).

Since $g'(s) < \sqrt{g(0)}$ and g' is increasing, it follows that $g'(s) \uparrow 1$ as $s \to \infty$. Moreover, $g''(s) \to 0$ as $s \to \infty$. Thus it follows from (9) that

$$g(\bar{s}) \sim 1 + s \quad \text{as} \quad s \to \infty.$$

In view of (10), we also have

$$g(s) > 1 + s \quad \text{on} \quad \mathbb{R}^+.$$

Finally,

$$F'(\theta) \sim -\sin\theta\left(1 + \frac{\sin\theta}{\cos\theta}\right) + \frac{1}{\cos\theta} = -\sin\theta + \cos\theta$$

implies that $f'(\theta) \to 1$ as $\theta \to \pi/2$. □

By the usual approximation procedures (cf. [B]) we can assume that u and v are positive in $\mathbb{R} \times \mathbb{R}^+$. Then, in particular, v_t exists and is continuous in $\mathbb{R} \times \mathbb{R}^+$. It therefore suffices to derive a bound for $|v_t|$ which is independent of the lower bound for v.

Proposition 2. *Fix an arbitrary $\delta > 0$. For each $(x_0, t_0) \in \mathbb{R}^+ \times [2\delta, \infty)$ set $\alpha \equiv v(x_0, t_0)$. There exists constants A and B depending only on δ, m, and $N \equiv \|v_0\|_{L^\infty(\mathbb{R})}$ such that*

$$\frac{\alpha}{4f(0)} \leqslant v(x, t) \leqslant 2\alpha$$

for all (x, t) which satisfy

$$|x - x_0| \leqslant A\gamma \quad \textit{and} \quad 0 \leqslant t_0 - t \leqslant B\gamma,$$

where $\gamma \equiv \min(\alpha, \delta)$.

PROOF. In view of (4)

$$|v(x_0, t_0) - v(x, t_0)| \leqslant L|x - x_0|$$

where L depends on δ, m and N. Thus

$$|x - x_0| \leqslant \delta/2L$$

implies that

$$\frac{\alpha}{2} \leqslant v(x, t_0) \leqslant \frac{3\alpha}{2}.$$

According to (5), for $t \geqslant \delta$ we have

$$v(x, t_0) - v(x, t) \geqslant -K(t_0 - t),$$

where K depends only on δ, m and N. Therefore

$$v(x, t) \leqslant v(x, t_0) + K(t_0 - t) \leqslant 2\alpha$$

if

$$|x - x_0| \leqslant \gamma/2L \quad \text{and} \quad 0 \leqslant t_0 - t \leqslant \gamma \min\left(\frac{1}{2K}, 1\right).$$

We assert that

$$v(x_0, t) \geqslant \frac{\alpha}{2f(0)} \quad \text{for} \quad t \in [t_0 - \gamma E, t_0], \tag{11}$$

where $E = \min(1/8L^2 f(0), 1)$. Suppose that (11) is false. Then there is a $\theta \in (0, E)$ such that

$$v(x_0, t_0 - \theta\gamma) < \frac{\alpha}{2f(0)}.$$

Without loss of generality, we can assume that $x_0 = t_0 = 0$. By Taylor's theorem and (4) we have

$$v(x, -\delta\theta) < \frac{\alpha}{2f(0)} + L|x|.$$

Set

$$p^*(x, t) \equiv \sqrt{2}\, Lp(x, \sqrt{2}\, L(t + \gamma\eta))$$

for $t > -\gamma\eta$, where p is the solution of problem (6) and η is to be chosen. Note that p^* is a solution of the pressure equation (2). Since $\{a^2 + b^2\}^{1/2} \geqslant (|a| + |b|)/\sqrt{2}$ and $f(0) > 1$ we have

$$p^*(x, t) \geqslant L\{|x| + \sqrt{2}\, L(t + \gamma\eta)\}.$$

Thus

$$v(x, -\gamma\theta) < \frac{\alpha}{2f(0)} + L|x| = L\{|x| + \sqrt{2}\, L(\eta - \theta)\gamma\} \leqslant p^*(x, -\gamma\theta)$$

provided that

$$\eta = \frac{\alpha}{2\sqrt{2}\,\gamma L^2 f(0)} + \theta \leqslant \frac{\alpha}{2\sqrt{2}\,\gamma L^2 f(0)} + E. \tag{12}$$

By the comparison principle,

$$\alpha = v(0, 0) \leqslant p^*(0, 0) = 2L^2\gamma\eta f(0).$$

It follows from (12) and the definition of E that

$$\alpha \leqslant 2L^2 f(0)\left\{\frac{\alpha}{2\sqrt{2}\, L^2 f(0)} + \gamma \min\left(\frac{1}{\delta L^2 f(0)}, 1\right)\right\} \leqslant \alpha\left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right) < \alpha.$$

Thus we have a contradition and conclude that (11) holds.

For any $t \in [t_0 - \gamma E, t_0]$ it follows from (4) and (11) that

$$v(x, t) \geqslant v(x_0, t) - L|x - x_0| \geqslant \frac{\alpha}{2f(0)} - L|x - x_0| \geqslant \frac{\alpha}{4f(0)}$$

provided that $|x - x_0| \leqslant \gamma/4Lf(0)$. Thus the assertion of the proposition holds if we take

$$A = 1/4Lf(0)$$

and

$$B = \min(1/\delta L^2 f(0), 1/2K, 1).$$

PROOF OF THEOREM. Define

$$w(x, t) \equiv \frac{1}{\gamma} v(x_0 + \gamma x, t_0 + \gamma t).$$

Then w is a solution of the pressure equation (2) which satisfies

$$\frac{\alpha}{4f(0)\gamma} \leqslant w(x, t) \leqslant \frac{2\alpha}{\gamma}$$

in the rectangle $|x| \leqslant A$, $-B \leqslant t \leqslant 0$. If $\alpha \leqslant \delta$ then $\gamma = \alpha$ and we have

$$\frac{1}{4f(0)} \leqslant w(x, t) \leqslant 2 \quad \text{for} \quad |x| \leqslant A, \ -B \leqslant t \leqslant 0.$$

If $\alpha > \delta$ then $\gamma = \delta$ and $\alpha/\gamma > 1$. Then since $\alpha \leqslant N$ we have

$$\frac{1}{4f(0)} \leqslant w(x, t) \leqslant \frac{2N}{\delta} \quad \text{for} \quad |x| \leqslant A, \ -B \leqslant t \leqslant 0.$$

In both cases we conclude from the standard theory of parabolic equations [LSU] that there is a positive constant C depending only on δ, m and N such that

$$|w_t(0, 0)| \leqslant C.$$

The theorem now follows since $w_t(0, 0) = v_t(x_0, t_0)$ and (x_0, t_0) is arbitrary. □

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Special Values of Hilbert Modular Functions

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Introduction

Recently, Baily has established new foundations for complex multiplication in the context of Hilbert modular functions; see [1]-[4]. However, in his treatment there is a restriction on the class of CM-points treated. Namely, the order of complex multiplications associated to the point must be the maximal order in its quotient field. The purpose of this paper is two-fold: (1) to remove the restriction just mentioned; (2) to recover a result of Tate on the conjugates of CM-points by arbitrary Galois automorphisms of $\bar{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$). This is done without the use of moduli.

An important feature of our approach is that, at the outset, we introduce a projective system of disconnected arithmetic quotients (Hilbert modular varieties) and a $\mathbb{Q}$-structure compatible with certain automorphisms of the system; see [10]. On each connected component, projective co-ordinates are given by Hilbert modular forms with cyclotomic Fourier coefficients. Affine co-ordinates are weight zero quotients of these and are called «arithmetic Hilbert modular functions».

The CM-points are naturally grouped into orbits of certain idele class groups. Our main result describes how these orbits are permuted by Galois conjugation. It follows quickly that the value of any arithmetic Hilbert modular function at a CM-point z is abelian over the reflex field K' attached to z. Combined with recent, deeper work of Baily, which provides congruence relations among such values, our results lead to a complete description of the conjugate $\sigma(z)$ for every CM-point z and every $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K')$. On the other

hand, if σ is a Galois automorphism that does not fix K', then the complete description of $\sigma(z)$ involves a connected component c in a group of finite idele classes; our results lead to Tate's description of c^2; see Chap. 7 of [11].

Notations

$\mathfrak{k}$: a totally real number field of degree $n \geqslant 1$ with distinct embeddings

$$\varphi_j^0 : \mathfrak{k} \to \mathbb{R} \quad \text{for} \quad j = 1, 2, \ldots, n; \qquad \Phi^0 = (\varphi_1^0, \ldots, \varphi_n^0).$$

$\mathfrak{o}$: the ring of integers in $\mathfrak{k}$; $\hat{\mathfrak{o}} = \mathfrak{o} \otimes \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} = \operatorname{inv\,lim} \mathbb{Z}/m\mathbb{Z}$.

$\mathbb{A}^f$: $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, and for any number field L, $L(\mathbb{A}^f) = L \otimes_{\mathbb{Q}} \mathbb{A}^f$.

G: Weil's ground field restriction of GL_2 from $\mathfrak{k}$ to $\mathbb{Q}$, i.e., the group scheme

$$A \longmapsto GL_2(A \otimes_{\mathbb{Z}} \mathfrak{o})$$

for every commutative ring A with 1.

Z: the center of G; $\bar{Z}$ = the closure of $Z(\mathbb{Q})$ in $G(\mathbb{A}^f)$.

$\mathfrak{H}$: the complex upper half plane.

X: $X = (\mathbb{C} - \mathbb{R})^n$ has 2^n connected components, one of which is $\mathfrak{H}^n = X_+$.

$G(\mathbb{R})^+$: the connected component of 1 in $G(\mathbb{R})$; $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$. Via Φ^0, $G(\mathbb{R})$ is isomorphic to $GL_2(\mathbb{R})^n$, which acts transitively on X by linear fractional transformations in the components.

$U(o)$: stabilizer in $GL_2(\mathbb{R})^n$ of a point o in $\mathfrak{H}^n$.

$\mathbb{K}$: a (varying) compact open subgroup of $G(\mathbb{A}^f)$ such as

$$\mathbb{K}(\nu) = \{g \in G(\hat{\mathbb{Z}}) : g - 1 \in \nu \cdot M_2(\hat{\mathfrak{o}})\},$$

where ν is a finite idele.

$V_{\mathbb{K}} =$ $G(\mathbb{Q})_+ \backslash (X_+ \times G(\mathbb{A}^f)/\mathbb{K}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}^f)/\mathbb{K})$; see 2.1.2 in [8].

$V =$ $G(\mathbb{Q})_+ \backslash (X_+ \times G(\mathbb{A}^f)/\bar{Z})$ is the limit of the projective system of $V_{\mathbb{K}}$'s. Given $\mathfrak{z}$ in X and x in $G(\mathbb{A}^f)$ we let $[\mathfrak{z}, x]$ denote the corresponding point of V and let $[\mathfrak{z}, x]_{\mathbb{K}}$ denote its image in V_K.

$W_{\mathbb{K}} =$ $G(\mathbb{Q})_+ \backslash (GL_2^+(\mathbb{R})^n \times G(\mathbb{A}^f)/\mathbb{K}) = G(\mathbb{Q} \backslash (GL_2(\mathbb{R})^n \times G(\mathbb{A}^f)/\mathbb{K})$.

1. Galois Action

The idea used in [10] to introduce $\mathbb{Q}$-structure on $V_{\mathbb{K}}$ in case $\mathfrak{k} = \mathbb{Q}$ has been extended by Baily to an arbitrary totally real field $\mathfrak{k}$ and to the subgroups $\mathbb{K} = \mathbb{K}(\nu)$; see [3, §5]. We wish to treat arbitrary open compact subgroups $\mathbb{K}$ of $G(\mathbb{A}^f)$.

Define the automorphy factor $j(g, \mathfrak{z})$ to be the functional determinant: if

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})^n, \qquad \mathfrak{z} \in X,$$

then

$$j(g, \mathfrak{z}) = \prod_{j=1}^{N} (c_j \mathfrak{z}_j + d_j)^{-2} (a_j d_j - b_j c_j).$$

For arbitrary open compact subgroups $\mathbb{K}$ of $G(\mathbb{A}^f)$, one defines as in [2, §5.1] the graded algebra $\mathfrak{A}(\mathbb{K})$ whose components $\mathfrak{A}(\mathbb{K}, w)$ are the spaces of functions f on $W_{\mathbb{K}}$ with values in $\mathbb{C}$ such that, if $g \in GL_2(\mathbb{R})$, $x \in G(\mathbb{A}^f)$:

(i) $f(gu, x) = f(g, x) j(u, o)^w$ for all u in $U(o)$;

(ii) the functions $\lambda_x f: g(o) \mapsto f(g, x) j(g, o)^{-w}$ are holomorphic in $\mathfrak{H}^n$ and have Fourier expansions of the form

$$\lambda_x f(\mathfrak{z}) = \sum_{\ell} a_x(\ell) e^{2\pi i (\ell, \mathfrak{z})},$$

where ℓ runs over a lattice in $\Phi^0(\mathfrak{t}^n) \subset \mathbb{R}^n$ intersected with the cone of squares in $\mathbb{R}^n$, and $(\ell, \mathfrak{z})$ is the scalar product in $\mathbb{C}^n$.

By $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$ we denote the subalgebra of forms with Fourier coefficients $a_x(\ell) \in \mathbb{Q}_{ab}$, the maximal abelian extension of $\mathbb{Q}$, for every $x \in G(\mathbb{A}^f)$. Among the arithmetic forms are certain Eisenstein series; namely, those forms f on $W_{\mathbb{K}}$ such that each $\lambda_x f$ is a $\mathbb{Q}_{ab}$-linear combination of the classical Eisenstein series introduced by Kloosterman; see [1, 5.2] and [2, 4.3].

1.1 Lemma. *If $m \in G(\mathbb{A}^f)$, then $R(m)$ gives an isomorphism from $\mathfrak{A}(\mathbb{K})$ to $\mathfrak{A}(m\mathbb{K}m^{-1})$ and maps $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$ onto $\mathfrak{A}(m\mathbb{K}m^{-1}; \mathbb{Q}_{ab})$.*

This follows at once from the following identity for x in $G(\mathbb{A}^f)$ and f in $\mathfrak{A}(\mathbb{K})$:

$$\lambda_x R(m) f = \lambda_{xm} f. \tag{1}$$

1.2. We define $\mathbb{Q}$-structure in two steps: (1) $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$ spans $\mathfrak{A}(\mathbb{K})$ over $\mathbb{C}$, so there is a natural $\mathbb{Q}_{ab}$-structure on V (and the action of $G(\mathbb{A}^f)$ on V by right translation is defined over $\mathbb{Q}_{ab}$ according to Lemma 1.1); (2) there is a semi-linear action of $\mathrm{Aut}(\mathbb{Q}_{ab})$ on $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$ that commutes with the action of $G(\mathbb{A}^f)$ and with the natural inclusion maps. This has already been done in case $\mathbb{K} = \mathbb{K}(\nu)$; see [3, §5] and [10, §2]. Therefore, it will suffice to show for $\mathbb{K}$ containing $\mathbb{K}(N)$ that (1) holds and that the action of $\mathrm{Aut}(\mathbb{Q}_{ab})$ on $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$

preserves $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$. The latter is obvious: since the action of $\sigma \in \mathrm{Aut}\,(\mathbb{Q}_{ab})$ on $\mathfrak{A}(\mathbb{K}(N); \mathbb{Q}_{ab})$ commutes with the action of $G(\mathbb{A}^f)$, the stabilizers of f and of σf in $G(\mathbb{A}^f)$ are the same.

To show that $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$ spans $\mathfrak{A}(\mathbb{K})$ over $\mathbb{C}$, first define a projection

$$\pi\colon \mathfrak{A}(\mathbb{K}(N)) \longrightarrow \mathfrak{A}(\mathbb{K})$$

by averaging the finite set of operators $R(m)$, $m \in \mathbb{K}/\mathbb{K}(N)$. Then, by Lemma 1.1, π maps $\mathfrak{A}(\mathbb{K}(N); \mathbb{Q}_{ab})$ onto $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$ so

$$\mathfrak{A}(\mathbb{K}) = \pi(\mathfrak{A}(\mathbb{K}(N); \mathbb{Q}_{ab})) \otimes \mathbb{C} = \mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab}) \otimes \mathbb{C}.$$

Since the Galois action is semi-linear, i.e., $\sigma(af) = \sigma(a)\sigma(f)$ for $a \in \mathbb{Q}_{ab}$ and $f \in \mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$, the algebra of invariants, call it $\mathfrak{A}(\mathbb{K})^{\mathbb{Q}}$, spans $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$ as a vector space over $\mathbb{Q}_{ab}$; see [5], AG, §14. By the Fundamental Theorem of Invariant Theory, $\mathfrak{A}(\mathbb{K})^{\mathbb{Q}}$ is finitely generated, so $\mathfrak{A}(\mathbb{K})^{\mathbb{Q}}$ is a $\mathbb{Q}$-structure on $\mathfrak{A}(\mathbb{K})$ and determines a $\mathbb{Q}$-structure on $V_{\mathbb{K}}$. For every nested pair of open compact subgroups $\mathbb{K} \subseteqq \mathbb{K}'$, the covering map $V_{\mathbb{K}} \to V_{\mathbb{K}'}$ is defined over $\mathbb{Q}$, and for every m in $G(\mathbb{A}^f)$, the maps $R(m)\colon V_{\mathbb{K}} \to V_{m^{-1}\mathbb{K}m}$ are defined over $\mathbb{Q}$. Thus, the projective system of $V_{\mathbb{K}}$'s along with all the automorphisms $R(m)$, $m \in G(\mathbb{A}^f)$, has a $\mathbb{Q}$-structure.

1.3. Originally, the action of $\mathrm{Aut}\,(\mathbb{Q}_{ab})$ on $\mathfrak{A}(\mathbb{K}(\nu); \mathbb{Q}_{ab})$ was defined with the help of Eisenstein series. Having established its existence, we can give a natural description of this Galois action. Define the cyclotomic character

$$\chi\colon \mathrm{Aut}\,(\mathbb{Q}_{ab}) \to \hat{\mathbb{Z}}^{\times}$$

by $\zeta^{\chi(\sigma)} = \zeta^{\sigma}$ for every root of unity $\zeta \in \mathbb{C}$. Let

$$u = u(\sigma)\colon = \begin{bmatrix} \chi(\sigma) & 0 \\ 0 & 1 \end{bmatrix}^{-1} \quad \text{in} \quad G(\mathbb{A}^f).$$

Then, for every x in $G(\mathbb{A}^f)$ and every f in $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$

$$\lambda_x(\sigma f) = (\lambda_{ux} f)^{\sigma}, \tag{1}$$

where the superscript σ replaces each Fourier coefficient $a_{ux}(\ell)$ of $\lambda_{ux}(f)$ (at the cusp ∞) by its image $a_{ux}(\ell)^{\sigma}$ under σ. To prove (1), it suffices to treat the special case where f is an Eisenstein series $\tilde{\mathcal{E}}_{\mathfrak{S}}$, in the notation of [3, §5.1], because every automorphic form is a quotient of isobaric polynomials in the Eisenstein series attached to sufficiently small subgroups $\mathbb{K}$. In view of 1.1(1), since $R(x)$ commutes with the action of σ, we may take $x = 1$. When $f = \tilde{\mathcal{E}}_{\mathfrak{S}}$ and $x = 1$, then (1) results from a routine calculation using, in the notation of [3, §5.2] (where $\chi(\sigma)$ is written $\hat{\sigma}$), the transformation laws:

(i) $\lambda_1(\tilde{\mathcal{E}}_{\mathfrak{C}})^\sigma = \lambda_1(\tilde{\mathcal{E}}_{\hat{\sigma}\mathfrak{C}})$;
(ii) $R(u)\tilde{\mathcal{E}}_{\mathfrak{C}} = \tilde{\mathcal{E}}_{u(\mathfrak{C})}$.

The main point of the calculation is that $u(\hat{\sigma} \cdot \mathfrak{C}) = \hat{\sigma} \cdot u(\mathfrak{C})$.

2. CM-Points

Recall that $h \in X$ is called a «CM-point (= special point) of X» if there is a totally imaginary quadratic extension K of $\mathfrak{k}$ and a $\mathfrak{k}$-algebra embedding q of K into $M_2(\mathfrak{k})$, the algebra of 2-by-2 matrices over $\mathfrak{k}$, such that h is an isolated fixed point of $q(K^\times) \subset GL_2(\mathfrak{k}) \hookrightarrow GL_2(\mathbb{R})^n$. Then q is the regular embedding with respect to a basis $(\tau, 1)$ of K over $\mathfrak{k}$; call it q_τ. View $q = q_\tau$ as a map of algebra varieties over $\mathfrak{k}$, so it also embeds $K(\mathbb{A}^f) := K \otimes \hat{\mathbb{Z}}$, the finite adele ring of K, into $M_2(\mathfrak{k}(\mathbb{A}^f))$. Let T be the maximal $\mathbb{Q}$-torus of G containing $q(K^\times)$, so $q(K^\times) = T(\mathbb{Q})$. Note that $T(\mathbb{R})$ is the isotropy group of h in $G(\mathbb{R})$.

A «lifting to K» of $\Phi^0: \mathfrak{k}^n \to \mathbb{R}^n$ is an embedding $\Phi: K^n \to \mathbb{C}^n$, of algebras over K, that restricts to Φ^0 on $\mathfrak{k}^n$. View K as embedded in K^n diagonally. To $h \in X$ fixed by $q_\tau(K^\times)$ associate the pair of complex conjugate liftings $\{\Phi, \bar{\Phi}\}$ such that $h = \Phi(\tau) = \bar{\Phi}(\bar{\tau})$ for some $\tau \in K - \mathfrak{k}$. Write $K_h = K$ and $\Phi_h = \Phi$. Observe that if $[\Phi(\tau), x] = [\Phi'(\tau'), 1]$, then for some $\gamma \in G(\mathbb{Q})$, $\Phi'(\gamma \cdot \tau') = \Phi(\tau)$, so either $\gamma \cdot \tau' = \tau$ or $\gamma \cdot \tau' = \bar{\tau}$, but not both; thus $\Phi' = \Phi$ or $\Phi' = \bar{\Phi}$, so the pair $\{\Phi_h, \bar{\Phi}_h\}$ depends only on $[h, x]$.

By «CM-point (= special point) of V» we mean a point $z = [h, x]$ in V with $x \in G(\mathbb{A}^f)$ and h a CM-point of X. To such a point $[h, x]$ with $h = \Phi(\tau)$, assign the «CM-type» $\langle K_h, \Phi_h \rangle = \langle K_h, \bar{\Phi}_h \rangle$ and the embedding $q_\tau^x: K(\mathbb{A}^f) \to M_2(\mathfrak{k}(\mathbb{A}^f))$ defined by $b \mapsto x^{-1} q_\tau(b) x$. Since $\gamma q_\tau \gamma^{-1} = q_{\gamma(\tau)}$ for all $\gamma \in GL_2(\mathfrak{k})$, it follows that q_τ^x depends only on $z = [\Phi(\tau), x]$, so we write $r_z := q_\tau^x$.

Suppose that $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then there is a unique map π_σ permuting coordinates so that $\sigma \circ \pi_\sigma \Phi^0 = \Phi^0$. The map $\Phi \mapsto \sigma\Phi := \sigma \circ \pi_\sigma \Phi$ permutes the liftings Φ of Φ^0 to K.

Main Theorem. *Let $h = \Phi_h(\tau)$ be a CM-point of X of type $\langle K, \Phi_h \rangle$. For each $\sigma \in \mathrm{Aut}(\mathbb{C})$ there exists $b \in K(\mathbb{A}^f)$ such that*

$$\sigma[h, x] = [\sigma\Phi_h(\tau), q_\tau(b)x]$$

for every $x \in G(\mathbb{A}^f)$.

3. Proof of the Main Theorem

3.1. The Main Theorem will be proved using two technical lemmas. Since the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on V commutes with right translations by elements

of $G(\mathbb{A}^f)$, we can suppose $x = 1$. Also, since $[h, 1]$ is right-invariant under $q(K^\times) = T(\mathbb{Q}) \subset G(\mathbb{A}^f)$, so is $\sigma[h, 1]$. In fact, every element $[\Phi(\tau), u] \in V$, with Φ lifting Φ^0 to K and u in $T(\mathbb{A}^f)$, is right-invariant under $T(\mathbb{Q})$ because if $\lambda \in T(\mathbb{Q})$, then $[\Phi(\tau), u] = [\lambda\Phi(\tau), \lambda u] = [\Phi(\tau), u\lambda]$. Our first lemma is a converse to this.

3.2. Lemma. *If $z \in V$ is right-invariant under every $t \in T(\mathbb{Q}) \subset G(\mathbb{A}^f)$, then $z = [\Phi(\tau), u]$ for some lifting Φ of Φ^0 to K and some $u \in T(\mathbb{A}^f)$.*

PROOF. It is well known and easy to check that z is a CM-point, say of type $\langle K', \Phi' \rangle = \langle K', \bar{\Phi}' \rangle$. Write $z = [\Phi'(\tau'), u']$ with τ' in $K' - \mathbb{f}$, u' in $G(\mathbb{A}^f)$. Let $q' = q_{\tau'}: K' \to M_2(\mathbb{f})$, and let $h' = \Phi'(\tau')$. The proof involves three steps:

(i) $K' = K$;

(ii) one can choose $\tau' = \tau$;

(iii) if $\tau' = \tau$, then $u' \in \mathfrak{N}(T)(\mathbb{Q}) \cdot T(\mathbb{A}^f)$, where $\mathfrak{N}(T)$ is the normalizer of T in G, so one can choose $u \in T(\mathbb{A}^f)$.

Step (i). The conjugation $\mathrm{Int}\,(u'): x \mapsto u'x(u')^{-1}$ takes $q(K^\times)$ into $q'((K')^\times)\bar{Z} \subset q'(K'\mathbb{f}(\mathbb{A}^f)) \subset q'(K'(\mathbb{A}^f))$ because if $t \in q(K^\times)$, which is embedded in $G(\mathbb{A}^f)$, then $[h', u't] = [h', u']$ only if $\mathrm{Int}\,(u')t \in \gamma\bar{Z}$ with $\gamma \in G(\mathbb{Q})$ such that $\gamma h' = h'$. Since $\mathrm{Int}\,(u')$ acts trivially on $q(\mathbb{f}(\mathbb{A}^f)^\times) = Z(\mathbb{A}^f) = q'(\mathbb{f}(\mathbb{A}^f)^\times)$,

$$\mathrm{Int}\,(u'): q(K \otimes \mathbb{A}^f) \to q'(K' \otimes \mathbb{A}^f)$$

induces an isomorphism m of $\mathbb{f}(\mathbb{A}^f)$-algebras from $K(\mathbb{A}^f)$ to $K'(\mathbb{A}^f)$. In particular, for each finite place v of k, then

$$K \otimes \mathbb{f}_v \cong K' \otimes \mathbb{f}_v,$$

Since $K \otimes \mathbb{f}_v = \oplus K_w$, ($w$ ranges over the places of K over v), and similarly for K', a prime of $\mathbb{f}$ is totally decomposed in K iff it is totally decomposed in K'. An easy application of Čebotarev's Density Theorem shows that $K' \cong K$; see Ex. 6.1 on p. 362 of [6]; since K and K' are quadratic extensions of $\mathbb{f}$, they coincide. For the case at hand, the result is Satz 18 of Hilbert's paper [9]; cf. [2, pp. 86-87].

Step (ii). Since $G(\mathbb{Q})$ acts transitively on $K - \mathbb{f}$, one gets

$$[\Phi'(\tau'), u'] = [\Phi'(\tau), \gamma u']$$

for some $\gamma \in G(\mathbb{Q})$. Therefore, we can choose $\tau' = \tau$.

Step (iii). Suppose that $\tau' = \tau$, so $q' = q$. Fix $\lambda \in K - \mathbb{f}$ with $\lambda^2 \in \mathbb{f}^\times$. Then $m(\lambda)$ and $m(\lambda + 1)$ both lie in $K^\times \mathcal{cl}(\mathbb{f}^\times)$, where «$\mathcal{cl}(\mathbb{f}^\times)$» is the closure of $\mathbb{f}^\times$

in $\mathfrak{k}(\mathbb{A}^f)^\times$. Write $m(\lambda) = \lambda'\zeta$ with $\lambda' \in K$, $\zeta \in cl(\mathfrak{k}^\times)$ and similarly $m(\lambda + 1) = \lambda_1\zeta_1$. Then $\lambda' = a\lambda + b$ for some a, b in $\mathfrak{k}$ with $a \neq 0$; however, $(\lambda')^2 = \lambda^2/\zeta^2$ lies in $K \cap \mathfrak{k}(\mathbb{A}^f) = \mathfrak{k}$, so $2ab\lambda = 0$; hence $\lambda' = a\lambda$, $\zeta^2 = 1/a^2$, and we may assume $a = 1$, i.e., $\lambda' = \lambda$, $\zeta^2 = 1$. Write $\lambda_1 = a_1\lambda + b_1$ with a_1, b_1 in $\mathfrak{k}$. Since $m(\lambda + 1) = m(\lambda) + 1$, it follows that $b_1\zeta_1 = 1$, and $\zeta = a_1\zeta_1$ must lie in $\mathfrak{k}$. Therefore, $m(\lambda) = \pm\lambda$ and m must be either $b \mapsto b$ or $b \mapsto \bar{b}$. Since $T(\mathbb{A}^f)$ is its own centralizer in $G(\mathbb{A}^f)$, it follows that u' must lie either in $T(\mathbb{A}^f)$ or in $q(\eta)T(\mathbb{A}^f)$, where

$$q(\eta) := \begin{bmatrix} -1 & \tau + \bar{\tau} \\ 0 & 1 \end{bmatrix}$$

represents the non-trivial coset of $T(\mathbb{Q})$ in $\mathfrak{N}(T)(\mathbb{Q})$.

If $u' = q(\eta)u$ with u in $T(\mathbb{A}^f)$, then $[\Phi'(\tau), u'] = [\bar{\Phi}'(\tau), u]$ and we set $\Phi = \bar{\Phi}'$, otherwise we set $\Phi = \Phi'$, $u = u'$.

Note. Lemma 1 of [4, §2.3.1] implies a stronger version of Lemma 3.2 above. Namely, one can weaken the hypothesis of Lemma 3.2 by assuming only that z is right-invariant under a single non-central element of $T(\mathbb{Q})$.

3.3. Henceforth, the variable Φ will denote a lifting of Φ^0 to K. If

$$\Phi = (\varphi_1, \ldots, \varphi_n),$$

then one defines the «type-norm» $N_\Phi: K \to \mathbb{C}$ to be the product of coordinates, so

$$N_\Phi: b \mapsto \varphi_1(b) \cdots \varphi_n(b).$$

Let $B_{\tau,\Phi}$ be the projection of $\Phi(\tau) \times T(\mathbb{A}^f)$ into V, and let B_τ be the union of all the orbits $B_{\tau,\Phi}$. Thus far, we have shown that σ preserves B_τ: in fact, $\sigma B_{\tau,\Phi_h} = B_{\tau,\Phi}$ for some Φ. We must show that $\Phi = \sigma\Phi_h$.

Let $\mathbb{K}$ be any open compact subgroup of $G(\mathbb{A}^f)$. One sees easily that there exists a $\mathbb{Q}$-rational holomorphic modular form ψ for $\mathbb{K}$, of non-zero weight w, such that the zeros of ψ, viewed as a subset of $V_{\mathbb{R}}$, are disjoint from $pr_{\mathbb{K}}(B_\tau)$. Fix ψ, w. As in [4, §3.4], one defines for m in $G(\mathbb{A}^f)$:

$$\mathfrak{A} := gcd_{\mathfrak{k}}(m)^{-2}\langle \det(m)\rangle,$$

where $\langle b\rangle$ denotes the ideal associated with $b \in \mathfrak{k}(\mathbb{A}^f)^\times$,

$$gcd_{\mathfrak{k}}(m)^{-1} := \{b \in \mathfrak{k} : mb \in M_2(\mathfrak{o})\}$$

is an ideal of $\mathfrak{k}$, and one defines a modular function ψ_m by:

$$\psi_m([g(o), u]) := N(\mathfrak{A})^w\psi(g, um)/\psi(g, u), \qquad (g \in GL_2(\mathbb{R})^n, u \in G(\mathbb{A}^f)).$$

Suppose that $z \in B_{\tau,\Phi}$. Let

$$M = r_z(b) = q(b) \quad \text{with} \quad b \in K^\times.$$

Then

$$\psi_M(z) = r^w \cdot N_\Phi(b)^w \tag{1}$$

for some rational number r independent of $z \in B_\tau$; see [4, §3.4].

To show that $\Phi = \sigma\Phi_h$ we construct a coordinate function f that is constant on each orbit $B_{\tau,\Phi}$ and has value there $N_\Phi(b)^w$. It is necessary first to choose b such that $N_\Phi(b)^w \neq N_{\Phi'}(b)^w$ unless $\Phi = \Phi'$. According to [3, §6.3], for some b_0 in K, the $2n$ images of b_0 under the embeddings of K into $\mathbb{C}$ are pairwise relatively prime. One can take $b = b_0$.

With $M = q_\tau(b)$, let $\mathbb{K}' := \mathbb{K} \cap M \cdot \mathbb{K} \cdot M^{-1}$, and let $f := r^{-w}\psi_M$. Then f is defined over $\mathbb{Q}$ on $V_{\mathbb{K}'}$ and restricts to the constant $N_\Phi(b)^w$ on $pr_{\mathbb{K}'}(B_{\tau,\Phi})$, so the orbits $pr_{\mathbb{K}'}(B_{\tau,\Phi})$ for distinct Φ must be disjoint, and therefore the orbits $B_{\tau,\Phi}$ must also be disjoint.

3.4. Lemma. *If* $\sigma[\Phi_h(\tau), 1] = [\Phi'(\tau), u]$ *for some* $u \in T(\mathbb{A}^f)$, *then* $\Phi' = \sigma\Phi_h$.

PROOF. Choose b and f as in section 3.3 and write $\mathbb{K}$ in place of $\mathbb{K}'$, so that $f([\Phi(\tau), s]_{\mathbb{K}}) = N_\Phi(b)^w$ for every s in $T(\mathbb{A}^f)$ and every lifting Φ. We claim that if $\Phi = \sigma\Phi_h$ then

$$N_\Phi(b)^w = N_{\Phi'}(b)^w. \tag{1}$$

Indeed, the left side is

$$\sigma(N_{\Phi_h}(b)^w) = \sigma f([\Phi_h(\tau), 1]_{\mathbb{K}}),$$

while the right side is

$$f([\Phi'(\tau), u]_{\mathbb{K}}) = f\sigma([\Phi_h(\tau), 1]_{\mathbb{K}}),$$

but $\sigma f = f\sigma$. Since $\Phi \mapsto N_\Phi(b)^w$ separates liftings Φ of Φ^0 to K, it follows that $\Phi' = \sigma\Phi_h$.

3.5. *Summary.* Since $G(\mathbb{A}^f)$, acting on the right, commutes with σ, we may suppose by section 1 that $z = [\Phi_h(\tau), 1]$, so that z is invariant under $T(\mathbb{Q})$ acting to the right. Then $\sigma(z)$ is also invariant under $T(\mathbb{Q})$. Thus, by Lemma 3.2, $\sigma(z) = [\Phi'(\tau), u]$ for some Φ' lifting Φ^0 to K and some u in $T(\mathbb{A}^f)$. Then Lemma 3.4 shows that $\Phi' = \sigma\Phi_h$, which proves the Main Theorem:

$$\sigma[\Phi_h(\tau), x] = [\sigma\Phi_h(\tau), q_\tau(b)x]$$

for some b in $K(\mathbb{A}^f)$. In other words, $\sigma(B_{\tau,\Phi}) = B_{\tau,\sigma\Phi}$.

4. Conclusions

We now derive some consequences of the Main Theorem.

4.1. Let $\Gamma = \mathrm{Aut}(\bar{\mathbb{Q}})$, and let $z \in V$ be a CM-point of CM-type, say, $\langle K, \Phi \rangle$. From the Main Theorem, since the $B_{\tau,\Phi}$ are distinct, the isotropy group of z in Γ, call it Γ_z, is contained in the isotropy group of Φ,

$$\Gamma_\Phi := \{\sigma \in \Gamma : \sigma\Phi = \Phi\}.$$

Thus, the field L_z generated over $\mathbb{Q}$ by the co-ordinates of z must contain the field K_Φ of all algebraic numbers fixed by Γ_Φ. The field K_Φ is the *reflex field* for the CM-type $\langle K, \Phi \rangle$ i.e., the subfield of $\bar{\mathbb{Q}}$ corresponding to the isotropy group of tr_Φ, the linear form given by the sum of the co-ordinates of Φ; see [11, p. 23]. We can now show that L_z/K_Φ is abelian. Let

$$I(K) := K(\mathbb{A}^f)^\times / cl(K^\times).$$

4.1.1. Theorem. *The field extension L_z/K_Φ is abelian. Moreover, its Galois group is embedded naturally into $I(K)$.*

PROOF. Note that $I(K) = K(\mathbb{A}^f)^\times / K^\times cl(\mathfrak{f}^\times)$ and define $b_\Phi : \Gamma \to I(K)$ as follows: for $\sigma \in \Gamma$, let $b_\Phi(\sigma)$ be the unique element b of $I(K)$ that satisfies

$$\sigma[\Phi(\tau), x] = [\sigma\Phi(\tau), q_\tau(b)x] = [\sigma\Phi(\tau), xr_z(b)]. \tag{1}$$

Using the transitivity of $G(\mathbb{Q})$ on $K - \mathfrak{f} \subset X$, one checks easily that $b_\Phi(\sigma)$ is independent of τ. Moreover, by straightforward calculation,

$$b_\Phi(\sigma\sigma') = b_{\sigma'\Phi}(\sigma) b_\Phi(\sigma'). \tag{2}$$

Thus, b_Φ restricts to a homomorphism on Γ_Φ, and one sees at once that the kernel of $b_\Phi | \Gamma_\Phi$ is Γ_z. Thus, L_z is abelian over K_Φ, and b_Φ embeds the Galois group Γ_Φ/Γ_z into $I(K)$. For a more classical argument, see [4, §4]. Note that Γ_z and hence L_z depend only on the CM-type of z.

4.2. The Reciprocity Law. For any number field F let NR_F be the norm residue symbol relative to F. Suppose that z is a CM-point in V of type $\langle K, \Phi \rangle$ and let $\langle K', \Phi' \rangle$ be the *reflex type*, so $K' = K_\Phi$; see [11, p. 23]. Let $N_{\Phi'}$ be the reflex type norm (= product of the components of Φ'). Then the image lies in K. Since Φ' extends trivially to $K'(\mathbb{A}^f) = K' \otimes \mathbb{A}^f$ by linearity in $\mathbb{A}^f$, $N_{\Phi'}$ also extends. Take $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K')$ and a finite idele s of K' such that $NR_{K'}(s) = \sigma \mid K'_{ab}$. Recent work of Baily shows that, *if L_z/K' is an abelian extension, then*

$$N_{\Phi'}(s) cl(K^\times) = b_\Phi(\sigma). \tag{1}$$

In view of 4.1, however, L_z is always abelian over K', so (1) holds without restriction on the CM-point z. This is essentially Shimura's reciprocity law restricted to the Hilbert modular case.

4.3. *The Type Transfer.* Let ρ denote complex conjugation, and let $[\Phi]$ be the set of components of any lifting Φ. Following [11, p. 164], one defines Tate's «type transfer» on $\Gamma := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$,

$$t_\Phi: \Gamma \to \mathrm{Gal}(K_{ab}/K).$$

Choose for each $\varphi \in [\Phi]$ an element $w_\varphi \in \Gamma$ such that $\varphi = w_\varphi \mid K$, and let $w_{\rho\varphi} := \rho w_\varphi$. Then, for $\sigma \in \Gamma$,

$$t_\Phi(\sigma) := \Pi_\varphi w_{\sigma\varphi}^{-1} \sigma w_\varphi \cdot \mathrm{Gal}(\bar{\mathbb{Q}}/K_{ab}), \qquad (\varphi \in [\Phi]).$$

By Theorem 1.1 of [11, Chap. 7, §1], if $x \in K'(\mathbb{A}^f)$ and if $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$ restricts to $NR_{K'}(x^{-1})$ on K'_{ab}, then

$$NR_K \circ N_{\Phi'}(x) = t_\Phi(\sigma). \tag{1}$$

If σ fixes Φ, then $t_\Phi(\sigma) = NR_K(b_\Phi(\sigma)^{-1})$ according to 4.2(1). In general, let $f_\Phi(\sigma)$ denote the class in $K(\mathbb{A}^f)^\times/\mathcal{U}(K^\times)$ such that $t_\Phi(\sigma) = NR_K(f_\Phi(\sigma)^{-1})$, and let $c_\Phi(\sigma) := b_\Phi(\sigma)/f_\Phi(\sigma)$.

4.4. The rest of the paper is devoted to a proof of Tate's result: $c_\Phi(\sigma)^2 = 1$. We begin by collecting properties of u_Φ with $u = b, f$ or c. Permissible values of u are marked parenthetically. Recall that χ is the cyclotomic character, defined in 1.3.

In case $u = f$, the properties below are proved in [11, Chap. 7]; the case $u = c$ follows from the other two cases.

4.4.1$_u$. $u_\Phi(\sigma\sigma') = u_{\sigma'\Phi}(\sigma)u_\Phi(\sigma'), \qquad (u = b, f, c).$

PROOF. For $u = b$, this is 4.1 (2).

4.4.2$_u$. $u_\Phi(\rho) = 1, \qquad (u = b, f, c).$

PROOF. For $u = b$. Choose g, g' in $G(\mathbb{R})$ such that

$$g(o) = \Phi(\tau) \quad \text{and} \quad g'(o) = \rho\Phi(\tau).$$

It suffices to show that $\rho[\Phi(\tau), 1] = [\rho\Phi(\tau), 1]$, i.e., if f_1 and f_2 are $\mathbb{Q}$-rational modular forms of weight w for some $\mathbb{K}$, and if $f_2(g, 1) \neq 0$, then

$$\rho(f_1(g, 1)/f_2(g, 1)) = f_1(g', 1)/f_2(g', 1). \tag{1}$$

By definition of $\mathfrak{A}(\mathbb{K}, w)$,

$$j(g, o)^{-w} f_i(g, 1) = F_i(g(o)) \tag{2}$$

for a holomorphic Hilbert modular form $F_i = \lambda_1 f_i$ for $G(\mathbb{Q}) \cap \mathbb{K}$ acting on $\mathfrak{H}^n$, $i = 1, 2$.

For any Hilbert modular form F with Fourier expansion

$$F(\underset{\sim}{z}) = \Sigma_\lambda a(\lambda) e^{2\pi i(\lambda, \underset{\sim}{z})},$$

the function $F^\rho(\underset{\sim}{z}) := \overline{F(-\underset{\sim}{z})}$ has complex conjugate Fourier coefficients to those of $F(\underset{\sim}{z})$, i.e., $\overline{a(\lambda)} = a(\lambda)^\rho$ in place of $a(\lambda)$. Therefore,

$$\rho(F(\Phi(\tau)) = F^\rho(-\rho\Phi(\tau)). \tag{3}$$

Let $f = f_i\,(= \rho f)$, $F = \lambda_1(f)$ and $u = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ $(= u(\rho)$, *cf.* 1.3). By 1.3(1) with $\sigma = \rho$, $x = u$, then $\lambda_1(f)^\rho = \lambda_u(f)$, so (*cf.* (2))

$$j(h, o)^w F^\rho(h(o)) = f(h, u). \tag{4}$$

Since $u \in G(\mathbb{Q})$, one has $f(g', 1) = f(ug', u)$; hence,

$$f(g', 1) = j(ug', o)^w F^\rho(-\rho\Phi(\tau)). \tag{5}$$

Combining (2), (3) and (5) gives (1), as required.

4.4.3$_u$. *If $\sigma K = K$, then $u_{\Phi\sigma^{-1}}(\sigma') = (u_\Phi(\sigma'))^\sigma$.*

PROOF FOR $u = b$. This results from transport of structure. Our construction of V as $G(\mathbb{Q})\backslash(X \times G(\mathbb{A}^f)/\bar{Z})$ depends on the action of $G(\mathbb{Q}) = GL_2(\mathfrak{k})$ on X, which is determined by $\Phi^0\colon \mathfrak{k} \to \mathbb{R}^n$, so write $V = V(\Phi^0)$. Similarly, write

$$W(\Phi^0) = G(\mathbb{Q})\backslash(GL_2(\mathbb{R})^n \times G(\mathbb{A}^f)/\bar{Z}).$$

Note that $\sigma\mathfrak{k} = \mathfrak{k}$. Identifying $G(\mathbb{A}^f)$ with $GL_2(\mathfrak{k}(\mathbb{A}^f))$ we see that σ induces an automorphism of $G(\mathbb{A}^f)$ and that $1 \times \sigma$ induces isomorphisms

$$t_V\colon V(\Phi^0) \to V(\Phi^0\sigma^{-1}) \quad \text{and} \quad t_W\colon W(\Phi^0) \to W(\Phi^0\sigma^{-1}).$$

We claim that t_V commutes with Galois action on V, i.e., t_W commutes with Galois action on $\mathfrak{A}(\mathbb{K}; \mathbb{Q}_{ab})$. In the notation of §1, if $f \in \mathfrak{A}(\mathbb{K})$,

$$\lambda_x(f \circ t_W)) = \lambda_{\sigma(x)} f. \tag{1}$$

Thus, the pullback via t_W preserves $\mathbb{Q}_{ab}$-arithmeticity of forms. Suppose that $\sigma' \in \mathrm{Aut}\,(\mathbb{C})$. Our claim amounts to

$$\sigma'(f \circ t_W) = (\sigma' f) \circ t_W. \tag{2}$$

To check this, take any $x \in G(\mathbb{A}^f)$; recall that with $u = u(\sigma')$, 1.3(1) gives $\lambda_x(\sigma' f) = (\lambda_{ux} f)^{\sigma'}$; apply λ_x to (2), transform the result by 1.3(1), and by (1), and observe that the equivalent holds because $\mathrm{Aut}(\mathbb{C})$ fixes u.

Apply $t_V \sigma' = \sigma' t_V$ to $[\Phi(\tau), 1]$ to get, for $\sigma' \in \mathrm{Aut}(\mathbb{C})$,

$$[\sigma' \Phi(\tau), \sigma(q_\tau(b_\Phi(\sigma')))] = [\sigma' \Phi \sigma^{-1}(\sigma(\tau)), q_{\sigma(\tau)}(b_{\Phi\sigma^{-1}}(\sigma'))]. \tag{3}$$

From $\sigma(q_\tau(b)) = q_{\sigma(\tau)}(\sigma(b))$ it follows that $b_{\Phi\sigma^{-1}}(\sigma') = \sigma(b_\Phi(\sigma'))$, as required.

4.4.4. $c_\Phi(\sigma) c_\Phi(\sigma)^\rho = 1$.

PROOF. For $u = f$, the following is proved in [11]; see Chapter 7, Theorem 2.2:

$$u_\Phi(\sigma) u_\Phi(\sigma)^\rho \in \chi(\sigma) cl(K^\times). \tag{1}_u$$

Therefore, it will suffice to check $(1)_b$.

Let $\pi_0(V)$ be the set of connected components of V. We identify $\pi_0(V)$ with

$$\pi_0(\mathfrak{t}(\mathbb{A})^\times / \mathfrak{t}^\times) / \pi_0(Z(\mathbb{R})) \cong \pi_0(\mathfrak{t}(\mathbb{A}^f)^\times / \mathfrak{t}_+^\times)$$

via the determinant, as in (2.7.1) of [7]. By our definition of $\mathbb{Q}$-structure on V, each connected component is defined over $\mathbb{Q}_{ab}$. Thus, the effect of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on $\pi_0(V)$ is determined by $\sigma | \mathbb{Q}_{ab}$, namely, to multiply by $\chi(\sigma)$. Thus, if $\sigma[\Phi(\tau), 1] = [\sigma\Phi(\tau), q_\tau(a)]$, then

$$aa^\rho = \det(q_\tau(a)) \in \chi(\sigma) cl(K^\times). \tag{2}$$

Since $b_\Phi(\sigma) = a \cdot cl(K^\times)$, $(1)_b$ follows.

4.4.5. *If* $\sigma\Phi = \sigma'\Phi$, *then* $c_\Phi(\sigma) = c_\Phi(\sigma')$.

PROOF. In case $\sigma' = 1$, the required identity, $c_\Phi(\sigma) = 1$ is equivalent to the reciprocity law 4.2 (1); see 4.3. In general,

$$c_\Phi(\sigma^{-1}\sigma') = c_\Phi(\sigma^{-1}\sigma), \tag{1}$$

since both sides equal 1. Apply $4.4.1_c$ to conclude.

4.5. Theorem. *The map* $b_\Phi : \mathrm{Aut}(\mathbb{C}) \to K(\mathbb{A}^f)^\times / cl(K^\times)$ *defined by*

$$\sigma[\Phi(\tau), 1] = [\sigma\Phi(\tau), q_\tau(b_\Phi(\sigma))], \qquad (\tau \in K - \mathfrak{t}),$$

satisfies

$$N R_K(b_\Phi(\sigma)^{-1})^2 = t_\Phi(\sigma)^2.$$

PROOF. Having now verified the necessary properties of $b_\Phi(\sigma)$, etc., we can proceed as in Chapter 7 of [11]. It suffices to show that $c_\Phi(\sigma)^2 = 1$. Thus, in view of 4.4.4, one requires $c_\Phi(\sigma)^\rho = c_\Phi(\sigma)$. By 4.4.5, $c_\Phi(\sigma\rho) = c_\Phi(\rho\sigma)$; hence, by $4.4.1_c$ and $4.4.2_c$ it follows that $c_\Phi(\sigma) = c_{\rho\Phi}(\sigma)$. Since $\rho\Phi = \Phi\rho$, one can use $4.4.3_c$ to get $c_\Phi(\sigma)^\rho = c_\Phi(\sigma)$, as required.

Notes. (i) We used 4.4.3. only with $\sigma = \rho$, and this case is easy: apply σ' to the identity $[\Phi(\tau), 1] = [\bar{\Phi}(\bar{\tau}), 1]$, and use the Main Theorem along with $q_{\bar{\tau}}(a) = q_\tau(\bar{a})$ to get

$$b_\Phi(\sigma')^\rho = b_{\rho\Phi}(\sigma').$$

(ii) In [2], the restriction to CM-points with maximal associated order arose from difficulty in proving that a CM-point and its conjugates have the same associated order. We have circumvented the problem; however, it is easy to recover this result. The order associated to a CM-point $z = [\Phi(\tau), x]$ is

$$R(z) := K \cap r_z^{-1}(M_2(\hat{\mathfrak{o}})),$$

where $r_z(b) = x^{-1}q_\tau(b)x$. By our Main Theorem, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, $r_{\sigma(z)} = r_z$, so $R(\sigma(z)) = R(z)$.

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Variants of the Calderón-Zygmund Theory for L^p-spaces

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1. Introduction

Familiar tools in Fourier Analysis such as the Littlewood-Paley theory and the Cotlar-Stein lemma lead us to believe, very roughly, that if we take an operator T, and decompose it somehow into its «lacunary pieces» T_j, then to a large extent the different pieces T_j and T_k, $j \neq k$, act «independently» of each other. For example we may be dealing with the Hilbert transform on $L^2(\mathbb{R})$ with convolution kernel $1/x$. If now T_j is the operator with convolution kernel $1/x$ $\chi_{|x| \sim 2^j}$, the operators T_j are uniformly bounded on L^2 (because their kernels are uniformly in L^1) and an application of the Cotlar-Stein lemma (see for example [8]) allows us to conclude that indeed the Hilbert transform is bounded on $L^2(\mathbb{R})$. One drawback of the Cotlar-Stein lemma is that it is only valid in the setting of Hilbert space. On the other hand, we may be dealing with a Fourier multiplier m on $L^p(\mathbb{R}^n)$. Crudely, the content of the Hörmander multiplier theorem (see [12]) is that if each «piece» $m_j = m\chi_{|\xi| \sim 2^{-j}}$ has enough regularity to guarantee via the Sobolev embedding theorem that the $m_j's$ are uniformly bounded L^1-multipliers, then we may conclude that m is a multiplier of $L^p(\mathbb{R}^n)$, $1 < p < \infty$. However, examples abound to show that we cannot weaken the Hörmander theorem so as to totally avoid the regularity condition: Littman, McCarthy and Rivière [10] produce a very nice example where the m_j are uniformly bounded L^1-multipliers but m fails to be a multiplier of any L^p other than L^2.

The last decade or so has seen a flourishing of the study of certain operators in Fourier Analysis which are closely related to the singular integrals of

Calderón-Zygmund theory, but somehow seem to fall just outside the scope of that theory —for example singular integrals along curves, Stein's spherical maximal function and the Bochner-Riesz operators. Fairly recently it has been observed by Duoandikoetxea and Rubio de Francia [7] and Christ and Stein [4] that if one is a little more flexible in one's point of view as to what constitutes «classical» Calderón-Zygmund theory, many results on these operators hitherto regarded as «exotic» may be very easily obtained.

The purposes of this paper may be described as follows:

(i) to provide a useful substitute for the Cotlar-Stein lemma for L^p-spaces (the orthogonality conditions are replaced by certain fairly weak smoothness assumptions);
(ii) to investigate the «gap» between the Hörmander multiplier theorem and the Littman-McCarthy-Rivière example-just how little regularity is really needed?
(iii) to simplify and extend the work of Duoandikoetxea and Rubio de Francia and Christ and Stein, which sometimes has unnecessarily strong assumptions, and to introduce a sensitivity to different L^p-spaces which does not appear in their work.

In §2 we deal with the decomposition of an integral operator into lacunary pieces on the kernel side, introducing some smoothness conditions which are L^p-analogues of Hörmander's condition. In §3 we look at pseudo-differential and multiplier operators: theorem 2 may be regarded as a sharper version of the Hörmander multiplier theorem as applied to pseudo-differential operators, and theorem 3 is a multiplier theorem which is sensitive to different L^p's. Roughly, theorem 2 may be paraphrased as: let $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be the symbol of an L^2-bounded pseudo-differential operator. If for some $\epsilon > 0$,

$$\left(\frac{\partial}{\partial \xi}\right)^\epsilon \sigma(2^i x, 2^{-i}\xi)\chi_{|\xi| \sim 1}$$

have uniformly bounded L^1-operator norms, then σ is of weak-type 1. In §4 we look at some results for the maximal operators which fall under the scope of the theory, but, for simplicity, we do not present them in the sharpest possible form. Although all our statement are in terms of the usual isotropic dilations of $\mathbb{R}^n$, analogous statements hold for parabolic dilations.

We proceed to set up some notation. Let ϕ be a non-negative radial C^∞ bump function of compact support in $\{|x| \leqslant 1\}$, with $\int \phi = 1$; convolution with $\phi_{2^j} = 2^{-jn}\phi(2^{-j}\bullet)$ is denoted by P_j. Let ψ be a non-negative radial C^∞ bump function of compact support in $\{1 \leqslant |\xi| \leqslant 4\}$, such that $\sum_j \psi(2^j\xi) \equiv 1$ on $\mathbb{R}^n - \{0\}$; let $Q_j f(\xi) = \psi(2^j\xi)\hat{f}(\xi)$. Let $\tilde{\psi}$ denote a generic radial non-negative member of the Schwartz class $\mathcal{S}$ with $\tilde{\psi}(0) = 0$ and correspondingly

define $\tilde{Q}_j$. If $K(x, y) = K$ is an integral kernel, K_j denotes $K\psi^j$ where $\psi^j \in \mathcal{S}$ is defined by

$$\psi^j(x, y) = \psi\left(\frac{x - y}{2^j}\right).$$

If T is the singular integral operator corresponding to K, T_j is the operator corresponding to K_j. If $\sigma(x, \xi)$ is a symbol (possibly independent of x),

$$\sigma_j(x, \xi) = \sigma(x, \xi)\psi(2^j\xi),$$

and

$$\sigma * \rho(x, \xi) = \int \sigma(x, \eta)\rho(\xi - \eta)\, d\eta$$

for $\rho \in \mathcal{S}(\mathbb{R}^n)$. Λ_ϵ is the usual Lipschitz space as defined for example in [12]. Finally, L^p-operator norms are denoted by $|K|_p$, $|T|_p$, $|\sigma|_p$ or $\|m\|_{\mathfrak{M}_p}$ as appropriate.

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2. Singular Integrals

In this section we study singular integral operators of the form

$$Tf(x) = \int K(x, y) f(y)\, dy.$$

Theorem 1. *Suppose that T is bounded on L^2, and that for some $1 < p < 2$ satisfies*

$$\sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \left| \sum_{l \geq 0} Q_{j+k}\, T_{j+l}(I - P_j) \right|_p < \infty. \tag{1}$$

Then T is of weak-type $p - p$.

Corollary 1. *Suppose that T is bounded on L^2, and that for some r, $1 \leqslant r < 2$, satisfies*

$$\sup_j |T_j|_p < \infty, \qquad r < p \leqslant 2.$$

If moreover

$$\sup_{j\in\mathbb{Z}} |Q_{j+k}T_{j+l}(I-P_j)|_2 \leqslant C2^{-\epsilon(|k|+l)}$$

for some $\epsilon > 0$ $(k\in\mathbb{Z},\ l\geqslant 0)$, *then* T *is bounded on* L^p, $r < p \leqslant 2$.

Remarks. (i) Theorem 1 should be compared with the classical Calderón-Zygmund theorem, which, upon replacing (1) by the Hörmander condition

$$\sup_{j\in\mathbb{Z}} \left| \sum_{l\geq 0} T_{j+l}(I-P_j) \right|_1 < \infty \tag{H}$$

gives the weak-type 1 of T. It would be very interesting to know whether L^2 boundedness of T together with

$$\sup_{j} \left| \sum_{l\geq 0} T_{j+l}(I-P_j) \right|_p < \infty$$

is sufficient to imply the weak-type p of T.

(ii) If Q_{j+k} and T_{j+l} commute for all j, k, l, (in particular if T is a convolution operator), then we may replace (1) by

$$\sum_{k\leq 0} \sup_j \left| \sum_{l\geq 0} T_{j+l}Q_{j+k} \right|_p + \sup_j \left| \sum_{l\geq 0} T_{j+l}\tilde{Q}_j \right|_p < \infty, \tag{2}$$

since

$$I - P_j - \sum_{k\leq 0} Q_{j+k} = \tilde{Q}_j.$$

If now

$$\alpha(l) = \sup_j |T_{j+l}(I-P_j)|_p, \qquad |T_{j+l}Q_{j+k}|_p \leqslant C\alpha(l-k) \quad \text{if} \quad l\geqslant 0,\ k\leqslant 0,$$

and so T is of weak type p in the convolution case if

$$\sum_{l\geq 0} l\alpha(l) < \infty.$$

(iii) The corollary follows immediately from the theorem, because under the hypotheses of the corollary, the hypotheses of the theorem are satisfied for each $p > r$, by interpolation. A very similar result to this in the case $r = 1$ had previously been obtained by M. Christ and E. M. Stein, [4]. Their result is that if T is bounded on L^2,

$$\sup_j |T_j|_1 < \infty, \qquad |T_j^* T_k|_2 \leqslant C2^{-\epsilon|j-k|} \quad \text{and} \quad |T_{j+l}(I-P_j)|_2 \leqslant C2^{-\epsilon l},$$

then T is bounded on L^p, $1 < p \leqslant 2$.

(iv) Suppose that T is a convolution operator and that

$$\sup_j |T_j|_2 \leqslant C.$$

Then the smoothness hypothesis

$$|Q_{j+k}T_{j+l}(I-P_j)|_2 \leqslant C2^{-\epsilon(|k|+l)}, \qquad l \geqslant 0, \qquad k \in \mathbb{Z}, \qquad (\text{some } \epsilon > 0)$$

is equivalent to

$$|T_{j+l}(I-P_j)|_2 \leqslant C2^{-\epsilon l}, \qquad l \geqslant 0 \qquad (\text{some } \epsilon > 0)$$

and to

$$|T_{j+l}Q_j|_2 \leqslant C2^{-\epsilon l}, \qquad l \geqslant 0 \qquad (\text{some } \epsilon > 0).$$

Each of these conditions is also clearly equivalent to the decay condition $|\hat{K_j}(\xi)| \leqslant C(2^j|\xi|)^{-\epsilon}$, $(2^j|\xi|) \geqslant 1$, all j, (some $\epsilon > 0$) which has previously appeared in [2] and [7].

PROOF OF THEOREM 1. Fix $f \in L^p$ and $\alpha > 0$, and apply the standard Calderón-Zygmund decomposition. (See [12].) Thus, we decompose f as $f = g + \sum b_i$, with $\|g\|_\infty < \alpha$, b_i supported in a ball B_i of radius $2^{j(i)}$, the supports of the b_i pairwise disjoint,

$$\sum |B_i| \leqslant C\alpha^{-p}\|f\|_p^p, \quad |B_i|^{-1}\int_{B_i}|b_i|^p \leqslant C\alpha^p,$$

and the doubles B_i^* having bounded overlap. Let

$$G = g + \sum_i b_i * \phi_{2^{j(i)}}$$

be the good part; exactly as in the standard theory, L^2-boundedness of T gives that $|\{x| \, |TG(x)| > \alpha\}| \leqslant C\|f\|_p^p/\alpha^p$. So it suffices to treat

$$T\Big(\sum_i [b_i - b_i * \phi_{2^{j(i)}}]\Big) = \sum_i \sum_{l>0} T_{j(i)+l}(I-P_{j(i)})b_i + \sum_i \sum_{l\leq 0} T_{j(i)+l}(I-P_{j(i)})b_i.$$

The second term is supported in $\cup B_i^{**}$, whose measure is at most

$$\sum |B_i^{**}| \leqslant C\sum |B_i| \leqslant C\|f\|_p^p/\alpha^p.$$

We shall show that

$$\Big\|\sum_j \sum_{l>0} T_{j+l}(I-P_j)f_j\Big\|_p \leqslant C\|f_j\|_{L^p(l^1)}. \tag{3}$$

If we apply (3) with

$$f_j = \sum_{i:j(i)=j} b_i,$$

we see that

$$\begin{aligned}\Big\|\sum_i \sum_{l>0} T_{j(i)+l}(I-P_{j(i)})b_i\Big\|_p &= \Big\|\sum_j \sum_i T_{j+l}(I-P_j)f_j\Big\|_p \\ &\leqslant C\Big\|\sum_j \Big|\sum_{i:j(i)=j} b_i\Big|\Big\|_p \leqslant C\Big\|\sum |b_i|\Big\|_p \\ &\leqslant C\Big(\sum_i \int |b_i|^p\Big)^{1/p} \leqslant C\alpha\Big(\sum |B_i|\Big)^{1/p} \leqslant C\|f\|_p,\end{aligned}$$

which completes the proof.

By duality, (3) is equivalent to

$$\Big\|\sup_j \Big|\sum_{l>0}(I-P_j)T^*_{j+l}g\Big|\Big\|_{p'} \leqslant C\|g\|_{p'}. \tag{4}$$

We decompose

$$\sum_{l>0}(I-P_j)T^*_{j+l} = \sum_{k\in\mathbb{Z}}\Big[\sum_{l>0}(I-P_j)T^*_{j+l}Q_{j+k}\Big]$$

and treat each term in the k-sum using Littlewood-Paley theory. Thus,

$$\begin{aligned}\Big\|\sup_j \Big|\sum_{l\geq 0}(I-P_j)T^*_{j+l}Q_{j+k}g\Big|\Big\|_{p'} &\leqslant \Big(\sum_j \int \Big|\sum_{l\geq 0}(I-P_j)T^*_{j+l}Q_{j+k}g\Big|^{p'}\Big)^{1/p'} \\ &\leqslant C\sup_j \Big|\sum_{l\geq 0}(I-P_j)T^*_{j+l}Q_{j+k}\Big|_{p'}\Big(\sum_j \int |\tilde{Q}_{j+k}g|^{p'}\Big)^{1/p'} \\ &\leqslant C\sup_j \Big|\sum_{l\geq 0} Q_{j+k}T_{j+l}(I-P_j)\Big|_p \|g\|_{p'}.\end{aligned}$$

Hence (4) and (3) hold for a given p if (1) does.

Remark. The above proof is nothing but a minor variation on the classical Calderón-Zygmund theme. Indeed, the Calderón-Zygmund theorem follows by observing that (4) in the case $p' = \infty$ is equivalent to the Hörmander condition (H), and the result of Christ and Stein [4], follows by observing that (4) holds for all $p' < \infty$ provided that $|T_{j+l}(I-P_j)|_1 \leqslant C$ and that

$$\Big\|\Big(\sum_j |(I-P_j)T^*_{j+l}g|^2\Big)^{1/2}\Big\|_2 \leqslant C2^{-\epsilon l}\|g\|_2.$$

By the Cotlar-Stein lemma this latter inequality will hold if

$$|(I-P_j)T^*_{j+l}T_{k+l}(I-P_k)|_2 \leqslant C2^{-\epsilon(l+|j-k|)},$$

(or, in the case of convolution operators, if $|T_{j+l}(I-P_j)|_2 \leqslant C2^{-\epsilon l}$).

3. Multipliers and Pseudo-Differential Operators

In this section we study pseudo-differential operators of the form

$$Tf(x) = \int \sigma(x, \xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi$$

and multiplier operators when $\sigma(x, \xi) = m(\xi)$ is independent of x.

Theorem 2. *Suppose that T is a pseudo-differential operator which is bounded on $L^2(\mathbb{R}^n)$ and satisfies $|\sigma(x, \xi)| \leqslant C\ \forall x, \xi \in \mathbb{R}^n$. Let $\sigma_j(x, \xi) = \sigma(x, \xi)\psi(2^j\xi)$, and suppose that $|\sigma_i * (\hat{\psi})_{2^{-j}}|_1 \leqslant \alpha(i-j)$ with $\sum_{k\leqslant 0}|k|\alpha(k) < \infty$. Then T is of weak-type $1-1$, and bounded from H^1 (the real Hardy space) to L^1.*

Theorem 3. *Suppose that T is a multiplier operator which is bounded on $L^2(\mathbb{R}^n)$ and satisfies $|m_i * (\hat{\psi})_{2^{-j}}|_p \leqslant \alpha(i-j)$ for some $1 \leqslant p < 2$. If*

$$\sum_{k\leqslant 0}|k|\alpha(k) < \infty,$$

then T is of weak type $p-p$.

Corollary 3. *If T is a multiplier operator which satisfies*

$$\|m_i(2^{-i}\bullet)\|_{\mathfrak{M}_r} + \|m_i(2^{-i}\bullet)\|_{\Lambda_\epsilon} \leqslant C_r$$

for some $\epsilon > 0$, some $1 \leqslant p < 2$, all $p < r \leqslant 2$, then T is bounded on L^r for $p < r < p'$.

Remarks. (i) The case $p = 1$ of theorem 3 is of course contained in theorem 2; there is a variant of theorem 2 in which L^p-hypotheses produce a weak-type p operator, but because we have to use condition (1) instead of condition (H), some cancellation and minimal x-smoothness is also needed. This matter will be discussed in detail in a forthcoming paper of the author and A. Seeger.

(ii) Corollary 3 follows immediately from theorem 3, once we notice that $\|m_i(2^{-i}\bullet)\|_{\Lambda_\epsilon} \leqslant C$ is equivalent to $\|m_i * (\hat{\psi})_{2^{-j}}\|_{\mathfrak{M}_2} \leqslant C2^{\epsilon(i-j)}$.

(iii) Hörmander's classical multiplier theorem (see [12]) and some of the things which are obtainable from it by interpolation (see for example [5]) are contained in theorem 3. To see this we merely observe that the hypothesis of the Hörmander theorem implies that $\|m_i * (\hat{\psi})_{2^{-j}}\|_{L^2_\beta} \leqslant C2^{\epsilon(i-j)}$ for some $\beta > n/2$ and $\epsilon > 0$ and then the Sobolev embedding theorem allows us to apply theorem 3 with $\alpha(k) = 2^{\epsilon k}$. Recently, Baernstein and Sawyer have proved a sharp H^1-variant of the Hörmander theorem, (see [1]). Their result is that if $\alpha(k)$ is as above, and $\sum_{k\leqslant 0}\alpha(k)\omega_{-k} < \infty$ for some nondecreasing sequence ω_k

satisfying $\sum_{k\geq 0}\omega_k^{-2}<\infty$, then m is a multiplier of the real Hardy space H^1. Thus all the multipliers covered by the case $p=1$ of theorem 3 are also H^1-multipliers, since they satisfy the Hörmander condition (H), and also the Baernstein-Sawyer condition.

(iv) As an application of Corollary 3 we obtain by simple methods a result previously obtained by Córdoba and López-Melero, [6], and Igari, [9] using heavily geometrical machinery. Let $\widehat{(T_j^\alpha f)}(\xi)=(1-(2^j|\xi|)^2)_+^\alpha\hat{f}(\xi)$. The theorem of Carleson and Sjölin, [3], states that for each $\alpha>0$, there exists a $p_0(\alpha)<4/3$ such that $\|T_j^\alpha f\|_{L^p(\mathbb{R}^2)}\leqslant C_{p,\alpha}\|f\|_{L^p(\mathbb{R}^2)}$ for $p_0<p<p_0'$. Write $(1-|\xi|^2)_+^\alpha=$ $=n(\xi)+m(\xi)$ with $n\in C_c^\infty$ and m supported in $1/2\leqslant|\xi|\leqslant 1$, $m\in\Lambda_\alpha$. The term corresponding to n is controlled by the Hardy-Littlewood maximal function, and by Corollary 3, $\sum_{j\in\mathbb{Z}}\pm m(2^j\xi)$ is a multiplier of $L^p(\mathbb{R}^2)$, $p_0<p<p_0'$, uniformly in the random choice of $\pm$. The usual argument with Rademacher functions (see Stein, [12]) now shows that

$$\left\|\sup_j|T_j^\alpha f|\right\|_p\leqslant C_{p,\alpha}\|f\|_{p'}\qquad p_0<p<p_0'.$$

We return to these matters more systematically in the next section.

SKETCH OF PROOF OF THEOREM 2. We wish to apply the Calderón-Zygmund theorem, and so we need to study the operators $T_{j+l}(I-P_j)$ in order to be able to apply the Hörmander condition (H). Recall that T_{j+l} has kernel $K(x,y)\psi((x-y)/2^{j+l})$ where K is the integral kernel associated to T, and so T_{j+l} has symbol

$$\int\sigma(x,\eta)(\hat{\psi})_{2^{-j-l}}(\xi-\eta)\,d\eta=\sum_{i\in\mathbb{Z}}\int\sigma_{i+j}(x,\eta)(\hat{\psi})_{2^{-j-l}}(\xi-\eta)\,d\eta.$$

If $\hat{\psi}$ had compact support, the i'th term in this sum would be supported in $\{|\xi|\sim 2^{-i-j}\}$ if $l\geqslant i$ and in $\{|\xi|\leqslant 2^{-j-l}\}$ if $l\leqslant i$. If $\hat{\phi}$ were identically one on a neighbourhood of 0 in $\mathbb{R}^n$, only those terms in the sum with $i\leqslant 0$ would contribute, then, when computing $T_{j+l}(I-P_j)$ for $l\geqslant 0$. Now

$$\left|\sum_{i\leq 0}\sigma_{i+j}*(\hat{\psi})_{2^{-j-l}}[1-\hat{\phi}(2^j\bullet)]\right|_1\leqslant\sum_{i\leq 0}\alpha(i-l),$$

and so

$$\begin{aligned}\sum_{l\geq 0}|T_{j+l}(I-P_j)|_1&\leqslant\sum_{l\geq 0}\sum_{i\leq 0}\alpha(i-l)\\&=\sum_{k\leq 0}\alpha(k)\#\{(i,l):i-l=k,\,i\leqslant 0,\,l\geqslant 0\}\\&\leqslant\sum_{k\leq 0}(|k|+1)\alpha(k)<\infty,\end{aligned}$$

by hypothesis. But of course $\hat{\phi}$ cannot be constant near 0, nor can $\hat{\psi}$ have compact support, without violating the compact support of ϕ and ψ respectively. The remainder of the proof is thus devoted to this technical matter, and is accordingly postponed to §5. □

SKETCH OF PROOF OF THEOREM 3. The proof of theorem 3 is essentially the same as that of theorem 2, except that we wish to verify (2) instead of (H). The second term that appears in (2)

$$\sup_j \left| \sum_{l \geq 0} T_{j+l} \tilde{Q}_j \right|_p,$$

may be treated in exactly the same manner as

$$\sup_j \left| \sum_{l \geq 0} T_{j+l}(I - P_j) \right|_p$$

was in the case $p = 1$ in theorem 2, all the L^1 estimates being equally valid for L^p under the hypothesis of the present theorem. To study the first term appearing in (2), fix $k \leqslant 0$, $l \geqslant 0$ and $j \in \mathbb{Z}$ and write the symbol of $T_{j+l} Q_{j+k}$ as

$$\sum_{i \in \mathbb{Z}} m_{i+j} * (\hat{\psi})_{2^{-j-l}}(\xi) \psi(2^{j+k}\xi) = \sum_{i < k-1} + \sum_{i=k-1}^{k+1} + \sum_{i=k+2}^{\infty} = I + II + III.$$

The main term is *II*; *I* and *III* are the error terms. To estimate $|II|_p$, we just use the hypothesis that $|m_i * (\hat{\psi})_{2^{-j}}|_p \leqslant \alpha(i-j)$ to obtain

$$|II|_p \leqslant \alpha(k-1-l) + \alpha(k-l) + \alpha(k+1-l),$$

and so

$$\sum_{k \leq 0} \sum_{l \geq 0} \sup_j |II|_p \leqslant C \sum_{k \leq 0} |k| \alpha(k) < \infty;$$

I and *III* are dealt with exactly as are the trivial error terms in theorem 2, and we refer to §5 for the details. □

4. Maximal Functions

Let K^i, $i \in \mathbb{Z}$, be a sequence of convolution kernels with corresponding multipliers m^i. We examine under what conditions the maximal operator $\sup_i |K^i * f|$ is bounded on some $L^p(\mathbb{R}^n)$.

Theorem 4. *Suppose there exists a sequence* $(\alpha^i) \in l^\infty$ *such that*

$$|m^i(\xi) - \alpha^i| \leqslant C(2^i|\xi|)^\epsilon \tag{5}$$

and

$$|m^i(\xi)| \leqslant C(2^i|\xi|)^{-\epsilon} \tag{6}$$

for some $\epsilon > 0$. *Suppose also that for some* $1 \leqslant p \leqslant \infty$ *we have*

$$\sup_i \|m^i\|_{\mathfrak{M}_p} < \infty. \tag{7}$$

a) If $p \geqslant 2$, *then*

$$\left\| \sup_i |K^i * f| \right\|_r \leqslant C\|f\|_r, \qquad 2 \leqslant r < p.$$

b) If $p < 2$ *and either*

(i) $K^i \geqslant 0$ *for all* $i \in \mathbb{Z}$, *or*

(ii) $\|m^i(2^{-l}\bullet)\psi(\bullet)\|_{\Lambda_{\epsilon'}} \leqslant C$, $i, l \in \mathbb{Z}$ *some* $\epsilon' > 0$, (8)

then

$$\left\| \sup_i |K^i * f| \right\|_r \leqslant C\|f\|_{r'} \qquad p < r \leqslant 2.$$

Remark. It is sometimes more convenient (for example when dealing with maximal functions along curves) to have a variant of (8) in which the kernel rather than the multiplier appears. An example of such a condition (easily verified in practical cases) is

$$|K^i_{i+l}|_2 \leqslant C2^{-\epsilon' l}, \qquad l \geqslant 0 \quad \text{some} \quad \epsilon' > 0 \tag{9}$$

which is equivalent to $\|m^i(2^{-i}\bullet)\|_{\Lambda_{\epsilon'}} \leqslant C$, which, together with (6) implies (8) (easy exercise). Condition (9) says that «most of the mass of K^i is concentrated around $|x| \sim 2^i$», while conditions (5) and (6) say that most of the mass of m^i is concentrated around $|\xi| \sim 2^{-i}$. It turns out to be useful to study this particular case first, which we single out as a lemma.

Lemma. *Suppose we are in the situation of theorem* 4 *with* $\alpha^i \equiv 0$ *and* m^i *of compact support in* $|\xi| \sim 2^{-i}$. *Then*

$$\left\| \left(\sum_i |K^i * f|^2 \right)^{1/2} \right\|_r \leqslant C\|f\|_r, \qquad p < r \leqslant 2.$$

PROOF OF LEMMA. The simplest approach is to observe that $\sum \pm m^i(\xi)$ satisfies the hypothesis of corollary 3 uniformly in the choice of $\pm$ as in §3 remark (iv) and apply Khintchine's inequality; alternatively one may observe that theorems 1 – 3 and their corollaries hold equally well in the context of Hilbert-

space-valued functions-all we have used is the Calderón-Zygmund decomposition and the Littlewood-Paley theory, which works equally well in the Hilbert-space-valued case. For this lemma we take the Hilbert spaces to be C and $l^2(\mathbb{Z})$. □

PROOF OF THEOREM 4. *a*) Let $\tilde{K}^i = K^i - \alpha^i \phi_{2^i}$; by the Hardy-Littlewood maximal theorem it suffices to prove the result for $\tilde{K}^i$: this amounts to taking $\alpha^i \equiv 0$ in (5). But now we are able to apply the Littlewood-Paley theory as in the proof of (4) in theorem 1 to obtain the required estimate.

b) (i) With $\tilde{K}^i$ as above, we see that $\tilde{K}^i$ is essentially positive in the sense of [2]; (5) and (6) show that the maximal operator associated to $\{\tilde{K}^i\}$ is strongly bounded on L^2 in the sense of [2]; inequality (7) allows us to apply theorem 2 of [2] to obtain the desired result.

b) (ii) Arguing as in part *a*), we may first assume $\alpha^i \equiv 0$ since clearly

$$\sup_i \|\hat{\phi}(2^i \cdot)\psi(\cdot)\|_{\Lambda_\epsilon} < \infty \quad \text{for all} \quad \epsilon > 0.$$

Next, we write

$$m^i = \sum_{l \in \mathbb{Z}} m^i_{i+l},$$

with m^i_{i+l} supported in $\{|\xi| \sim 2^{-i-l}\}$, as in §3. For each fixed l, we have that

$$\left(\sum_i |m^i_{i+l}(\xi)|^2\right)^{1/2} \leqslant C 2^{-\epsilon|l|}$$

by (5) and (6);

$$\left(\sum_i |(m^i_{i+l})_k * (\hat{\psi})_{2^{-j}}(\xi)|^2\right)^{1/2} \leqslant C \|m^{k-l}_k * (\hat{\psi})_{2^{-j}}\|_\infty \leqslant C 2^{\epsilon(k-j)},$$

by (8); and

$$\left\| \left(\sum_i |[(m^i_{i+l})_k * (\hat{\psi})_{2^{-j}}]^\vee * f|^2\right)^{1/2} \right\|_p \leqslant C \|m^{k-l}_k\|_{\mathfrak{M}_p} \|f\|_p \leqslant C \|f\|_p,$$

by (7). Hence by the lemma, or the l^2-valued variant of corollary 3,

$$\left\| \sup_i |(m^i_{i+l})^\vee * f| \right\|_r \leqslant C 2^{-\epsilon|l|} \|f\|_r, \qquad p < r \leqslant 2.$$

Summing over $l \in \mathbb{Z}$ finishes the proof. □

5. Proofs of Theorems 2 and 3. Technical Aspects

In §3 we postponed some of the technical aspects of the proofs of theorems

2 and 3. Here we give, for completeness, the full details; the significance of theorems 2 and 3 can be fully appreciated without reading this section.

PROOF OF THEOREM 2, CONTINUED. Having dealt with the $i \leqslant 0$ terms previously, what remains is to show that

$$\sum_{l \geq 0} \left| \sum_{i>0} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}[1 - \hat{\phi}(2^j \bullet)] \right|_1 =$$

$$= \sum_{l \geq 0} \left| \sum_{i>0} \sum_{k \in \mathbb{Z}} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}} \psi(2^{j+k} \bullet)[1 - \hat{\phi}(2^j \bullet)] \right|_1 < \infty.$$

We write

$$\sum_{i>0} \sum_{k \in \mathbb{Z}}$$

as

$$\sum_{k \leq 0} \sum_{i=0}^{\infty} + \sum_{k=0}^{l} \sum_{i=0}^{k} + \sum_{k=0}^{l} \sum_{i=k}^{l} + \sum_{k=0}^{l} \sum_{i=l}^{\infty} + \sum_{k=l}^{\infty} \sum_{i=0}^{l} + \sum_{k=l}^{\infty} \sum_{i=l}^{\infty}$$

$$= I + II + III + IV + V + VI.$$

Claim 1. For $k \leqslant 0$,

$$\left| \sum_{i>0} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}(\bullet) \psi(2^{j+k} \bullet) \right|_1 \leqslant C 2^{-\epsilon(l-k)},$$

some $\epsilon > 0$.

Claim 2.

$$\sum_{i=0}^{k} |\sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}|_1 \leqslant \sum_{i=0}^{k} \alpha(i-l).$$

Claim 3. For $k \leqslant i$, $|\sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}(\bullet) \psi(2^{j+k} \bullet)|_1 \leqslant C 2^{-\epsilon(l-k)}$, so that

$$\sum_{i=k}^{\infty} | \qquad\qquad |_1 \leqslant C(l-k) 2^{-\epsilon(l-k)} \leqslant C 2^{-\epsilon'(l-k)}.$$

Claim 4. For $l \geqslant k$,

$$\left| \sum_{i=l}^{\infty} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}(\bullet) \psi(2^{j+k} \bullet) \right|_1 \leqslant C 2^{-\epsilon(l-k)}.$$

Claim 5.

$$\sum_{i=0}^{l} |\sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}|_1 \leqslant \sum_{i=0}^{l} \alpha(i-l).$$

Claim 6. For $k \geqslant l$,

$$\left| \sum_{i=l}^{\infty} \sigma_{i+j} * (\hat{\psi})_{2^{-j-l}}(\cdot)\psi(2^{j+k}\cdot) \right|_1 \leqslant C.$$

If we accept claims 1 – 6 for the moment, the proof of the theorem is quickly concluded.

By claim 1,

$$\sum_{l \geqslant 0} |I|_1 \leqslant C \sum_{l \geqslant 0} \sum_{k \leqslant 0} 2^{-\epsilon(l-k)} < \infty;$$

by claim 2,

$$|II|_1 \leqslant C \sum_{k=0}^{l} \left\{ \sum_{i=0}^{k} \alpha(i-l) \right\} 2^{-\epsilon'' k};$$

by claim 3,

$$|III|_1 \leqslant C \sum_{k=0}^{l} 2^{-\epsilon'(l-k)} 2^{-\epsilon'' k} \leqslant C 2^{-\epsilon l};$$

by claim 4,

$$|IV|_1 \leqslant C \sum_{k=0}^{l} 2^{-\epsilon(l-k)} 2^{-\epsilon'' k} \leqslant C 2^{-\epsilon l};$$

by claim 5,

$$|V|_1 \leqslant C \sum_{k=l}^{\infty} \left\{ \sum_{i=0}^{l} \alpha(i-l) \right\} 2^{-\epsilon'' k};$$

by claim 6,

$$|VI|_1 \leqslant C \sum_{k=l}^{\infty} 2^{-\epsilon'' k} \leqslant C 2^{-\epsilon l}.$$

Hence

$$\sum_{i \geqslant 0} |III + IV + VI|_1 \leqslant C \sum_{l \geqslant 0} 2^{-\epsilon l} < \infty,$$

and

$$|II + V|_1 \leqslant \sum_{k=0}^{l} \sum_{i=0}^{k} \alpha(i-l)2^{-\epsilon k} + \sum_{k=l}^{\infty} \sum_{i=0}^{l} \alpha(i-l)2^{-\epsilon k}$$

$$= \sum_{i=0}^{l} \alpha(i-l) \sum_{k=i}^{\infty} 2^{-\epsilon k} \leqslant C \sum_{i=0}^{l} \alpha(i-l)2^{-\epsilon i};$$

thus

$$\sum_{l \geqslant 0} |II + V|_1 \leqslant C \sum_{l=0}^{\infty} \sum_{i=0}^{l} 2^{-\epsilon i} \alpha(i-l)$$

$$= C \sum_{j \leqslant 0} \alpha(j)2^{-\epsilon j} \sum_{l \geqslant -j} 2^{-\epsilon l} \leqslant C \sum_{j \leqslant 0} \alpha(j) < \infty.$$

Now claims 2 and 5 follow immediately from the definition of $\alpha(j)$; claims 1, 3, 4 and 6 all follow from the hypothesis that $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and the facts that $|\partial^\gamma \hat{\psi}(s-t)| \leqslant C_{k,\gamma}/|t|^k$, all $k \in \mathbb{N}$, if $|s| \leqslant 2|t|$, and $|\partial^\gamma \hat{\psi}| \leqslant C_\gamma$. In fact, if $\tau(x, \xi)$ is any symbol, then

$$|\tau(Rx, R^{-1}\xi)|_p = |\tau(x, \xi)|_p \quad \text{for all } R, p;$$

if now τ is supported in $\{|\xi| \sim 1\}$ and satisfies $|\partial_\xi^\gamma \tau(x, \xi)| \leqslant C_\gamma A$ for all multi-indices γ, then $|\tau(x, \xi)|_1 \leqslant CA$, since the integral kernel L associated to τ satisfies

$$|L(x, y)| \leqslant CA, \qquad |L(x, y)| \leqslant CA/|x-y|^{n+1}.$$

So to establish claims 1, 3, 4 and 6 we examine

$$\partial_\xi^\gamma \int \sum_i \sigma_{i+j}(2^{j+k}x, \eta)(\hat{\psi})_{2^{-j-l}}(\xi 2^{-j-k} - \eta)\, d\eta$$

for $|\xi| \sim 1$, (where the sum is over a range of i depending on which claim we are proving). In any case, what we get is

$$\int \sum_i \sigma_{i+j}(2^{j+k}x, \eta)2^{(j+l)n}2^{(l-k)|\gamma|}\partial^\gamma \hat{\psi}(2^{l-k}\xi - 2^{j+l}\eta)\, d\eta.$$

For claim 6, we just use $|\partial^\gamma \hat{\psi}| \leqslant C$ and the fact the integrand is supported in a set of measure $\leqslant C2^{-(j+l)n}$ to estimate the integral by $\|\sigma\|_\infty 2^{(l-k)|\gamma|}$; under the hypotheses of claims 1, 3 and 4 we have that (essentially) $2^{l-k}|\xi| \geqslant 2 \cdot 2^{j+l}|\eta|$ if $|\xi| \sim 1$, and thus we may estimate the integral by $\|\sigma\|_\infty 2^{(j+l)n}2^{(l-k)(|\gamma|-t)}$ multiplied by the measure of the support of the integrand, for each $t \in \mathbb{N}$. These measures are 2^{-jn}, $2^{-(i+j)n}$ and $2^{-(j+l)n}$ respectively, and so in each case we can dominate the integral by $2^{(l-k)(n+|\gamma|-t)}$ for all $t \in \mathbb{N}$. □

PROOF OF THEOREM 3, CONTINUED. We must deal with the terms I and III from the proof begun in §3; thus to control I we study

$$\partial^\gamma \int \sum_{i<k-1} m_{i+j}(\eta)(\hat{\psi})_{2^{-j-l}}(\xi 2^{-j-k} - \eta)\, d\eta$$

$$= \int_{|\eta| \geq 2^{-k-j+1}} \sum_{i<k-1} m_{i+j}(\eta) 2^{(j+l)n} 2^{(l-k)|\gamma|} \partial^\gamma \hat{\psi}(2^{l-k}\xi - 2^{j+l}\eta)\, d\eta$$

for $|\xi| \sim 1$. Now $2^{j+l}|\eta| \geqslant 2^{l-k+1} \geqslant 2 \cdot 2^{l-k}|\xi|$ if $|\xi| \sim 1$. Hence we may estimate the integral, for each $t \in \mathbb{N}$, by

$$\int_{|\eta| \geq 2^{-k-j+1}} \|m\|_\infty 2^{(j+l)n} 2^{(l-k)|\gamma|} \frac{C_t}{[2^{(j+l)}|\eta|]^t}\, d\eta.$$

If t is chosen sufficiently large, this is dominated by $\|m\|_\infty 2^{(l-k)(n-t+|\gamma|)}$, and so

$$\sum_{k \leq 0} \sum_{l \geq 0} \sup |I|_p < \infty.$$

For III we examine

$$\int_{|\eta| \leq 2^{-k-j-2}} \sum_{i \geq k+2} m_{i+j}(\eta) 2^{(j+l)n} 2^{(l-k)|\gamma|} \partial^\gamma \hat{\psi}(2^{l-k}\xi - 2^{j+l}\eta)\, d\eta$$

for $|\xi| \sim 1$. This time, $2^{j+l}|\eta| \leqslant 2^{l-k-2} \leqslant 2^{-1}2^{l-k}|\xi|$ if $|\xi| \sim 1$. Hence we may estimate the integral by $\|m\|_\infty\, 2^{(j+l)n} 2^{(l-k)(|\gamma|-t)} 2^{-(k+j)n} \leqslant 2^{(l-k)(|\gamma|-t+n)}$, which gives the same estimate for III as for I, concluding the proof of the theorem. □

6. Concluding Remarks

Applications of the theory to singular integrals and maximal functions along «well-curved» curves, classical singular integrals, and lacunary versions of Stein's spherical maximal function are detailed in [4] and [7], although the application to the lacunary maximal Bochner-Riesz operator in remark (iv), §3 appears to be new. In view of the wide variety of applications of the theory, it would be of interest to obtain analogues of our theorems in other cases of «natural» decompositions of operators. (Some partial results in the product domain setting have recently been obtained by the author and A. Seeger.) It would also be interesting to extend theorem 4*b*) (ii) to cover the case of «full» maximal operators - parts *a*) and *b*) (i) appear in [2]. The stumbling block here is the use of the Litlewood-Paley theory. We can use theorem 2, which is valid

in the general Banach-space context, to obtain results for maximal functions valid for $1 < p \leqslant \infty$; but it would seem to be harder to obtain results sensitive to different $p's$ in $(1, 2)$, which might be useful in the study of the maximal Bochner-Riesz operator, for example.

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Forms Equivalent to Curvatures

Horacio Porta and Lázaro Recht

Abstract

The 2-forms, Ω and Ω' on a manifold M with values in vector bundles $\xi \to M$ and $\xi' \to M$ are *equivalent* if there exist smooth fibered-linear maps $U: \xi \to \xi'$ and $W: \xi' \to \xi$ with $\Omega' = U\Omega$ and $\Omega = W\Omega'$. It is shown that an ordinary 2-form equivalent to the curvature of a linear connection has locally a non-vanishing integrating factor at each point in the interior of the set rank $(\omega) = 2$ or in the set rank $(\omega) > 2$. Under favorable conditions the same holds at points where the rank of ω changes from $=2$ to >2. Global versions are also considered.

Forms equivalent to curvatures

The 2-forms Ω and Ω' on a manifold M with values in vector bundles $\xi \to M$ and $\xi' \to M$ are *equivalent*, $\Omega \sim \Omega'$, if there exist smooth fibered-linear maps $U: \xi \to \xi'$ and $W: \xi' \to \xi$ such that $\Omega' = U\Omega$ and $\Omega = W\Omega'$. Examples: *a*) If Ω is a symplectic structure on M, the Lagrangian submanifolds of M depend only on the equivalence class of Ω; *b*) If $\eta \to N$ is a vector bundle with a connection ∇, the notion of ∇-homotopy $\phi: M \times [0, 1] \to N$ depends only on the equivalence class of the curvature of the induced connection $\phi * \nabla$ on $\phi * \eta \to M \times [0, 1]$. For details see [PR].

The second example motivates this work where we consider an ordinary 2-form equivalent to the curvature of a linear connection. The conclusion is that locally it is also equivalent to a *closed* 2-form (i.e., the curvature of a connection on a 1-dimensional bundle; for related matters see [K], [T]; in other words, a *2-form equivalent to a curvature has an integrating factor locally.*

(1) **Theorem.** *Let ω be a 2-form on M equivalent to a curvature. For $x \in M$ suppose that one of the following holds*:

(1.*a*) $\operatorname{rank}(\omega) = 2$ *near* x; *or*,
(1.*b*) $\operatorname{rank}(\omega) > 2$ *at* (*hence also near*) x.

Then ω has an integrating factor near x, i.e., there exists a nonvanishing smooth function f satisfying $d(f\omega) = 0$ on a neighborhood of x.

We make the following basic assumptions throughout: $\theta \to M$ is a smooth vector bundle with a conection ∇ and ω is a 2-form on M not zero at all points. The curvature $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ of ∇ is considered as a 2-form on M with values in the bundle $\xi = \operatorname{End}(\theta)$ (of smooth fibered-linear self-maps of θ), and it is equivalent to ω, i.e., $UR = \omega$, $W\omega = R$ for appropriate U and W (from ξ into the trivial one-imensional bundle $M \times \mathbb{R}$ and back). Denote by A the image under W of the constant section $1/2$ on $M \times \mathbb{R}$. Thus, A is a global section of ξ and

$$R(X, Y)v = 2\omega(X, Y)Av \tag{2}$$

for $v \in \theta$ and $X, Y \in TM$. If $\theta = M \times V$ is trivial (V a vector space) and

$$\nabla_X \sigma = X(\sigma) + \Gamma(X)\sigma$$

with Γ and $\operatorname{End}(V)$-valued 1-form, then $R/2 = d\Gamma + \Gamma \wedge \Gamma$ and (2) reads:

$$\omega A = d\Gamma + \Gamma \wedge \Gamma. \tag{3}$$

(A is now a function from M into $\operatorname{End}(V)$.) In this and similar formulas we use the canonical bilinear maps $\xi \times \xi \to \xi$ (composition) and $\xi \times \theta \to \theta$ (evaluation) to extend the exterior calculus to forms with values in $\mathbb{R}$, ξ, and θ (as long as the mixing is meaningful). In particular

$$(\Gamma \wedge \Gamma)(X, Y) = (1/2)[\Gamma(X), \Gamma(Y)].$$

We write $\alpha \wedge \beta - \beta \wedge \alpha = [\alpha, \beta]$ for ξ-valued forms α, β of arbitrary degree, which includes $\phi\psi - \psi\phi = [\phi, \psi]$ for sections ϕ, ψ of ξ. The identity

$$\begin{aligned} d(\Gamma \wedge \Gamma) &= d\Gamma \wedge \Gamma - \Gamma \wedge d\Gamma = (\omega A - \Gamma \wedge \Gamma) \wedge \Gamma - \Gamma \wedge (\omega A - \Gamma \wedge \Gamma) \\ &= \omega \wedge [A, \Gamma] \end{aligned}$$

and differentiation of (3) give

$$(d\omega)A + \omega \wedge (dA + [\Gamma, A]) = 0 \tag{4}$$

In terms of a basis of V this translates into n^2 relations of the form

$$a_{ij}\,d\omega + \omega \wedge \alpha_{ij} = 0.$$

Since $A \neq 0$ (because $2UA = \omega \neq 0$) some quotient α_{ij}/a_{ij} is defined near each point, whence

(5) *locally there exist* 1*-forms* α *such that* $d\omega = \alpha \wedge \omega$.

We can prove now the following proposition which contains Theorem 1 under hypothesis (1.*a*) (cf. Corollary, 3.6, of [BCG]).

(6) **Proposition.** *Let* ω *be a* 2*-form defined on a neighborhood* U *of the origin* 0 *of* $\mathbb{R}^n$ *satisfying on* U:

(6.*a*) $d\omega = \alpha \wedge \omega$ *for some* 1*-form* α;
(6.*b*) rank $(\omega) = 2$.

Then there exist local coordinates $y = (y_1, \ldots, y_n)$ *and a smooth function* h *such that* $\omega = h(y)dy_1 \wedge dy_2$ *near* 0. *A fortiori* $h \neq 0$ *and therefore* $f = 1/h$ *is a local integrating factor for* ω.

PROOF. First, (6.*a*) implies that the kernel of ω

$$N = \{\, Y; \omega(Y, Z) = 0 \quad \text{for all} \quad Z \,\}$$

is an integrable distribution of planes, for if X, Y are vector fields in N and Z is an arbitrary vector field,

$$\begin{aligned}\omega([X, Y], Z) &= -X\omega(Y, Z) - Y\omega(Z, X) - Z\omega(X, Y) + \omega([X, Y], Z) \\ &\quad + \omega([Y, Z], X) + \omega([Z, X], Y) \\ &= -3\,d\omega(X, Y, Z) \\ &= -\alpha(X)\omega(Y, Z) - \alpha(Y)\omega(Z, X) - \alpha(Z)\omega(X, Y) = 0\end{aligned}$$

so $[X, Y] \in N$ as claimed. Apply now Frobenius' theorem to get local coordinates $y = (y_1, y_2, \ldots, y_n)$ such that N is spanned by $\partial/\partial y_3, \partial/\partial y_4, \ldots, \partial/\partial y_n$ at each point near 0, and define $h = \omega(\partial/\partial y_1, \partial/\partial y_2)$. Clearly ω and $h\,dy_1 \wedge dy_2$ vanish on all pairs $(\partial/\partial y_i, \partial/\partial y_j)$ with $i = 1, 2$ and $3 \leqslant j \leqslant n$, and they coincide on $(\partial/\partial y_1, \partial/\partial y_2)$ so that $\omega = h\,dy_1 \wedge dy_2$ as claimed. This proves (6).

The proof of Theorem (1) under hypothesis (1.*b*) is as follows. We interpret $dA + [\Gamma, A]$ as the covariant differential of A for a connection with curvature zero to obtain a parallel local section of the type A/f and then use the fact:

(7) A/f *parallel implies* $d(f\omega) = 0$

which is proved below. Here are the details. Let $\tilde{\nabla}$ denote the conection on ξ defined by $(\tilde{\nabla}_X\phi)\sigma = \nabla_X(\phi\sigma) - \phi\nabla_X\sigma$ for ϕ a section of ξ and σ a section of θ. Direct calculations show that the curvature $\tilde{R}$ of $\tilde{\nabla}$ is given by

$$\tilde{R}(X, Y)\phi = [R(X, Y), \phi] = 2\omega(X, Y)[A, \phi]. \tag{8}$$

Also, if $\nabla\sigma = d\sigma + \Gamma\sigma$ in a trivialization then for a section ϕ of ξ, $\tilde{\nabla}\phi = d\phi + [\Gamma, \phi]$. In particular, (4) reads

$$(d\omega)A + \omega \wedge \tilde{\nabla}A = 0. \tag{9}$$

Next, denoting by $\xi^0 \subset \xi$ the one-dimensional subbundle spanned by A, we show that A is «recurrent» in the sense of [S]. Precisely,

(10) **Proposition.** *On the open set where* $\operatorname{rank}(\omega) > 2$ *the subbundle* ξ^0 *is invariant under* ∇ *and the curvature of the induced connection vanishes identically.*

PROOF. To show that for each $X \in TM_x$ the endomorphism $\tilde{\nabla}_X A$ of θ_x is a scalar multiple of A it suffices to use (2.3) and (2.4) to obtain

$$\omega \wedge (\tilde{\nabla}A + \alpha A) = \omega \wedge \tilde{\nabla}A + (\omega \wedge \alpha)A = \omega \wedge \tilde{\nabla}A + (d\omega)A = 0,$$

and then use the following cancellation lemma from linear algebra.

Lemma. *Let* T, E *denote vector spaces,* ω *a (real) 2-form on* T *with* $\operatorname{rank}(\omega) > 2$ *and* $B: T \to E$ *a linear map. If* $\omega \wedge B = 0$ *then* $B = 0$.

Assuming now hypothesis (1.*b*) we can use (10) to conclude that ξ^0 has locally a parallel section ϕ with $\phi = A$ at x, i.e. $\phi = A/f$ for some f with $f(x) = 1$ (and then $f \neq 0$ near x) which in view of (7) implies $d(f\omega) = 0$. To prove (7) simply go back to (3) and observe that for any $f \neq 0$,

$$(f\omega)(A/f) = d\Gamma + \Gamma \wedge \Gamma$$

implies the following analogue of (9):

$$d(f\omega)A + \omega \wedge \tilde{\nabla}(A/f) = 0. \tag{12}$$

This concludes the proof of Theorem 1.

The following improves (7):

(13) **Proposition.** *For* $f \neq 0$ *an arbitrary* C^1 *function defined near* $x \in M$ *suppose that*

(13.*a*) rank $(\omega) = 2$ *at* x; *then* $d(f\omega) = 0$ *at* x *if and only if* $\tilde{\nabla}_X(A/f) = 0$ *for all* $X \in N_x$.

(13.*b*) rank $(\omega) > 2$ *at* x; *then* $d(f\omega) = 0$ *at* x *if and only if* $\tilde{\nabla}(a/f) = 0$ *at* x.

PROOF. From (12) follows that $d(f\omega) = 0$ is equivalent to $\omega \wedge \tilde{\nabla}(A/f) = 0$ and so when rank $(\omega) > 2$ it is also equivalent to $\tilde{\nabla}(A/f) = 0$ by the lemma above. Suppose rank $(\omega) > 2$ at x and let $X_1, \ldots, X_n$ be a basis for TM_x with N_x spanned by $X_3, X_4, \ldots, X_n$. If i, j, k are distinct then one of them, say i, is 3 or larger. Hence, by (12) again

$$Ad(f\omega)(X_i, X_j, X_k) = -\omega(X_j, X_k)\tilde{\nabla}_{X_i}(A/f)$$

and so $d(f\omega) = 0$ at x if and only if $\tilde{\nabla}_{X_i}(A/f) = 0$ for $i = 3, 4, \ldots, n$.

A global version of (1.*b*) follows.

(14) **Theorem.** *Let* ω *be a* 2*-form on* M *with* rank $(\omega) > 2$ *everywhere. Then*

(14.*a*) ω *has a local non-vanishing integrating factor if and only if* $d\omega = \alpha \wedge \omega$ *for a closed globally defined* 1*-form* α.

(14.*b*) ω *has a global never vanishing integrating factor if and only if* $\alpha = dg$ *for a smooth globally defined function* g. *In particular, if* $H^1(M) = 0$ *(real cohomology),* ω *has a global never vanishing integrating factor if and only if* $d\omega = \alpha \wedge \omega$ *for a closed globally defined* 1*-form* α.

PROOF. Suppose $d(f\omega) = 0$ on an open set V. Then $d\omega = \alpha \wedge \omega$ with $\alpha = d(-\ln|f|)$, which is clearly closed on V. Using the cancellation lemma proved above, we conclude that α is unique, hence one implication in (14.*a*) follows. Conversely if $d\omega = du \wedge \omega$ locally then $d(f\omega) = 0$ for $f = \exp(-u)$, and (14.*a*) follows. This last remark also proves one implication in (14.*b*). To complete the proof suppose $d(f\omega) = 0$ globally. Then $d\omega = (-df/f) \wedge \omega$ and by the cancellation lemma again we get $\alpha = d(-\ln|f|)$. This finishes the proof.

The global integrability of α is necessary for let $M = S^1 \times \mathbb{R}^3$ (S^1 = circle) and let φ be the angle variable on S^1, (x, y, z) cartesian coordinates on $\mathbb{R}^3$. Also let $X = \partial/\partial\psi$, $\alpha = d\varphi$ (which are smooth and globally defined on M), and define $\omega = (-y\,dx + dz) \wedge \alpha + dx \wedge dy$. Then, $d\omega = \alpha \wedge \omega$, rank $(\omega) = 4$, $d\alpha = 0$, and α *is not globally integrable.*

In addition to the basic assumptions we suppose that local coordinates $(p, \ldots)$ exist near $x_0 \in M$ such that the resulting situation in $\mathbb{R}^n$ is as follows:

(15.*a*) rank $(\omega) = 2$ for $p \leqslant 0$,

(15.*b*) rank $(\omega) > 2$ for $p > 0$.

As above, N_x (in particular N_0) denotes the kernel of ω at $x \in \mathbb{R}^n$ near $x_0 = 0$.

(16) **Theorem.** *The form ω has a non-vanishing local integrating factor near $x_0 = 0$ in any of the following cases*:

(16.*a*) *N_0 is transversal to the hyperplane $\{p = 0\}$;*
(16.*b*) *$N_x \subset \{p = 0\}$ for each x near 0 belonging to the hyperplane $\{p = 0\}$.*

PROOF. It suffices to consider the case where $\tilde{\nabla}A = 0$ for $p \geqslant 0$. In fact, the restriction of ξ^0 to $\{p \geqslant 0\}$ has curvature zero (by (10) for $p > 0$ and by continuity at $p = 0$) and therefore there is a parallel section A/f with $f = 1$ at $0 \in \mathbb{R}^n$, and f of class C^∞ on $\{p \geqslant 0\}$; after extending f to a smooth function on a neighborhood of 0 we replace A by A/f (and ω by $f\omega$) to obtain $\tilde{\nabla}A = 0$ for the new A on $\{p \geqslant 0\}$. Using now the proof of (6) and Frobenius' theorem we obtain a foliation $\mathfrak{F} = \{F\}$ of the manifold with boundary $M_1 \cap \{p \leqslant 0\}$, where M_1 is a small neighborhood of the origin. For each leaf F the restriction $\xi^0|_F \to F$ is stable under $\tilde{\nabla}$ because if $x \in F \subset M_1 \cap \{p \leqslant 0\}$, choosing $Y, Z \in T(M_1)_x$ with $\omega(Y, Z) = 1$ and $X \in TF_x = N_x$, from (9) follows that $\tilde{\nabla}_X A = -d\omega(X, Y, Z)A$. Also the curvature of $\tilde{\nabla}$ vanishes on $\xi^0|_F$ (8).

Consider the case (16.*a*). It is clear that each leaf intersects the boundary $M_1 \cap \{p = 0\}$ transversally. Using $\tilde{R} = 0$ on $\xi^0|F$ we can find parallel sections in each $\xi^0|F$ extending the values of A on $M_1 \cap \{p = 0\}$ (which are parallel on $F \cap \{p = 0\}$ since $\tilde{\nabla}A = 0$ throughout $\{p \geqslant 0\}$). Thus a smooth section A/f of ξ^0 is defined on $M_1 \cap \{p \leqslant 0\}$ with $f = 1$ on $M^1 \cap \{p = 0\}$ and A/f «parallel on each leaf»:

$$\tilde{\nabla}_X(A/f) = 0 \quad \text{for} \quad x \in N_x, \qquad x \in M_1 \cap \{p \leqslant 0\}. \tag{17}$$

Setting $f = 1$ on $M_1 \cap \{p \geqslant 0\}$ gives a C^1 extension of f satisfying

$$\tilde{\nabla}_X(A/f) = 0 \quad \text{for} \quad X \in T(M_1)_x, \qquad x \in M_1 \cap \{p \geqslant 0\}. \tag{18}$$

Now (13) aplies to give $d(f\omega) = 0$ on M_1 (and this in turn forces f to be C^∞).

The proof assuming (16.*b*) is similar. First, each leaf is fully contained in $\{p = 0\}$ or disjoint from $\{p = 0\}$. Denote by Σ the set of points in $M_1 \cap \{p \leqslant 0\}$ orthogonal to N_0. Then if M_1 is small each leaf intersects F in exactly one point. Consider A extended smoothly to Σ and find parallel extensions along the leaves using the initial values of A at the point in $F \cap \Sigma$, in each case. As above we get A/f satisfying (17) and (18), and $d(f\omega) = 0$ follows again.

The following results indicate how some properties of ω translate into properties of A, ∇ or $\tilde{\nabla}$. We omit the proofs.

(19) *Under hypothesis* (1.*a*) *of Theorem* 1, *there exist local coordinates $y = (y_1, y_2, \ldots, y_n)$ and a trivialization $\theta = M \times \mathbb{R}^d$ near x such that*

$$\omega = h(y)\, dy_1 \wedge dy_2$$
$$\Gamma = B_1(y_1, y_2)\, dy_1 + B_2(y_1, y_2)\, dy_2$$
$$hA = [B_1, B_2] - (\partial B_1/\partial y_2) + (\partial B_2/\partial y_1)$$

where B_1, B_2 are $d \times d$ matrices depending on y_1, y_2 only.

(20) *Under hypothesis* (1.*b*) *of Theorem* 1, *parallel trasport operators for* $\tilde{\nabla}$ *are obtained locally by conjugation with the parallel transport operators for* ∇. *In particular, if* $U: \theta_u \to \theta_v$ (u, v *near* x) *is parallel transport along any curve joining* u, v, *then* $A_v = UA_uU^{-1}$.

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Vector Valued Inequalities for Strongly Singular Calderón-Zygmund Operators

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1. Introduction

Convolution operators with symbols of the form $e^{i|\xi|^a}|\xi|^{-b}$ have been extensively studied in the literature. Motivated by earlier work by C. Fefferman and E. M. Stein (cf. [8], [9]), we introduced in [1] a generalization of the classical Calderón-Zygmund operators. These operators, which include pseudodifferential operators with symbols in $S^{-b}_{a,\delta}$, where

$$\delta \leqslant a < 1, \qquad \delta \geqslant 0, \qquad (1-a)\frac{n}{2} \leqslant b < \frac{n}{2},$$

were termed strongly singular Calderón-Zygmund operators.

In this article we consider a theory of vector valued strongly singular operators. Our results include L^p, H^p and BMO continuity results. Moreover, as is well known, vector valued estimates are closely related to weighted norm inequalities. These results are developed in the first four sections of the paper. In §5 we use our vector valued singular integrals to estimate the corresponding maximal operators. Finally in §6 we discuss applications to weighted norm inequalities for pseudo-differential operators with symbols in the classes

described above. Our results in this direction are related to some recent work by Chanillo and Torchinsky [3].

We refer to §2 for a review of the necessary definitions and notation to be followed in the paper.

2. Preliminaries

In [1] we considered the following class of singular integral operators.

(2.1) **Definition.** *Let $T: S \to S'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following conditions are satisfied.*

(S_1) *T extends to a bounded operator on L^2.*

(S_2) *T is associated with a standard kernel, in the sense that there exists a function $k(x, y)$ continuous away from the diagonal on $\mathbb{R}^{2n}$, such that*

$$\text{(i)} \qquad |k(x,y) - k(x,z)| + |k(y,x) - k(z,x)| \leqslant c\frac{|y-z|^{\delta}}{|x-z|^{n+\delta/\alpha}}$$

if $2|y-z|^{\alpha} \leqslant |x-z|$ for some $0 < \delta \leqslant 1$, $0 < \alpha \leqslant 1$.

$$\text{(ii)} \qquad (Tf, g) = \int k(x,y) f(y) g(x)\, dy\, dx,$$

for $f, g \in S$ with disjoint supports.

(S_3) *For some*

$$(1-\alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2},$$

T and its adjoint T^* extend to continuous operators from L^q into L^2, where

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}.$$

Let us observe that if T is associated with a standard kernel, then given $f \in C^{\infty} \cap L^{\infty}$, $T(f)$ can be defined as a continuous linear functional over the C_0^{∞} functions with vanishing mean.

In [1] we show that S.C.Z. Operators are bounded from L^{∞} into BMO and moreover, if T is a S.C.Z. Operator such that $T^*(1) = 0$,

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta\left(\frac{n}{\alpha} + \frac{n}{2}\right)}{n\left(\frac{\delta}{\alpha} + \delta + \beta\right)},$$

then T defines a bounded operator

$$T: H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n), \qquad p_0 < p \leqslant 1.$$

In what follows we shall provide some background information in order to extend our theory to the frame of vector valued spaces. Moreover, we shall also be concerned with the issue of relaxing the smoothness condition (S_2) for more appropriate and sharper conditions in order to deal with the L^p theory of S.C.Z.O's.

The classical maximal function of Hardy Littlewood $M_p f$, and the sharp maximal operator $f^{\#}$, can be defined for vector valued functions simply replacing absolute values by norms. In fact, let X be a Banch space,

$$f \in L^1_{\text{loc}}(\mathbb{R}^n, X),$$

then we let

$$M_p f(x) = \sup_{B \ni x} \left(\frac{1}{|B|} \int_B \| f(y) \|^p \, dy \right)^{1/p},$$

$$f^{\#}(x) = \sup_{B \ni x} \inf_{c \in X} \left\{ \frac{1}{|B|} \int_B \| f(y) - c \| \, dy \right\},$$

where B denotes a ball in $\mathbb{R}^n$, $\| \quad \|$ stands for the norm of X, $1 \leqslant p < \infty$. In what follows we shall consider the usual vector valued $L^p(\mathbb{R}^n, X) = L^p(X)$ spaces (cf. [14]).

In a similar fashion we can define vector valued (p, q) atoms, and develop a theory of vector valued $H^p(X)$ spaces. Indeed, a (p, q) atom is a vector valued function supported in a ball, with moments zero up to order $\left[n\left(\frac{1}{p} - 1\right) \right]$ and such that

$$\| f \|_{L^q(X)} \leqslant |B|^{1/q - 1/p}.$$

Let $H^{p,q}(X) = \left\{ f: \mathbb{R}^n \to X \,\middle|\, f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ in the } S(\mathbb{R}^n, X')' \text{ sense}, a_j \text{ is a } (p, q) \right.$ atom, $\left. j = 1, \ldots, \sum_{j=1}^{\infty} |\lambda_j|^p < \infty \right\}$. We let

$$\| f \|_{H^{p,q}(X)} = \inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$

As in the scalar valued case $H^{p,q}(X) = H^{p,r}(X)$, for $q \neq r$.

We also observe that if X' satisfies the Radon-Nikodym property (in particular if X is reflexive), then

$$(H^1(X))' = \text{BMO}(X').$$

Moreover, if X satisfies the UMD property (cf. [14]), then $H^1(X)$ can be characterized using vector valued Riesz transforms (cf. [14]).

The $H^p(X)$ spaces can be interpolated in the expected way, in particular (cf. C. F. Folland and E. M. Stein [10]).

(2.2) **Theorem.** *Let T be an operator mapping $H^p(X)$ into $L^p(Y)$, $L^q(X)$ into $L^q(Y)$, where X, Y are Banach spaces and $0 < p \leqslant 1 < q \leqslant \infty$. Then, T maps $L^1(X)$ into $L(1, \infty)(Y)$,* i.e., *T is of weak type* (1, 1).

We now formulate a vector valued version of S.C.Z.O's. Given X, Y Banach spaces, let $L(X, Y)$ denote the space of bounded linear operators from X into Y. Let T be an operator such that

$$T: S(\mathbb{R}^n, X) \to S(\mathbb{R}^n, Y')'.$$

We shall say that T is associated with a kernel if there exists a continuous function $k: \mathbb{R}^{2n}/\text{diagonal} \to L(X, Y)$, such that

$$(g, Tf) = \iint g(x)k(x, y)f(y)\,dy\,dx,$$

when f and g have disjoint supports.

Our operators will be associated with kernels satisfying certain conditions of which $(S_2)(i)$ is a limiting case. This will allow us also to refine the main results of our previous work [1].

(2.3) **Definition** (cf. [14]). *Let $1 \leqslant r \leqslant \infty$, $0 < \alpha \leqslant 1$. We shall say that a kernel k, $k: \mathbb{R}^{2n}/\text{diagonal} \to L(X, Y)$ satisfies the condition $D_{r,\alpha}(X, Y) \equiv D_{r,\alpha}$ if there exists a sequence $\{d_j\} \in l^1$ such that $\forall z \in \mathbb{R}^n$, $\forall \sigma > 0$, $\forall y \in \mathbb{R}^n$ such that $|y - z| < \sigma$,*

$$(2.4)\quad \left(\int_{C_j(z,\sigma^\alpha)} \|k(x, y) - k(x, z)\|^r\,dx\right)^{1/r} \leqslant d_j |C_j(z, \sigma^\alpha)|^{-1/r'}, \qquad j = 1, \ldots$$

where $C_j(z, \sigma^\alpha) = \{x: 2^j\sigma^\alpha < |x - z| \leqslant 2^{j+1}\sigma^\alpha\}$, and $\|\ \ \|$ denotes the norm in $L(X, Y)$. For a kernel to be in $D_{r,\alpha}$ we also require that $\bar{k}(x, y) = k(y, x)$ satisfies (2.4). *This assumption, although not always necessary, will simplify the formulation of our results.*

It is readily verified that a $D_{p,\alpha}$ condition is stronger than a $D_{q,\alpha}$ condition if $p > q$. A limiting case of these conditions is the analogue of condition $(S_2)(i)$ obtained replacing absolute values by norms. We shall refer to this condition as D_α. Observe that a D_α condition implies all the $D_{r,\alpha}$ conditions, $r \geqslant 1$.

We shall now formulate vector valued versions of (S_1) and (S_3). Again, we shall refine our assumptions in order to be able to derive very precise results.

Let $1 < p < q \leqslant \infty$, $p/q \leqslant \alpha \leqslant 1$, we shall impose if $0 < \sigma < 1$

$$\left(\frac{1}{|B(z,\sigma)|}\int_{B(z,\sigma)} \|Tf\|_Y^q dx\right)^{1/q} \leqslant c\left(\frac{1}{|B(z,\sigma^\alpha)|}\int \|f\|_X^p dy\right)^{1/p} \tag{2.5}$$

and (if $\sigma > 1$),

$$\|Tf\|_p \leqslant c\|f\|_p \tag{2.5}'$$

for some absolute constant $c > 0$.

We shall refer to these conditions as $C_{p,q,\alpha}$ conditions. The strongly singular Calderón-Zygmund operators considered in this paper will satisfy conditions of the form $C_{p,q,\alpha}$ and $D_{r,\alpha}$.

(2.6) *Remark.* In [14] it is shown that if T is a sublinear operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, such that $\forall \epsilon > 0$, $(Tf)^\#(x) \leqslant C_\epsilon M_{1+\epsilon} f(x)$, T admits an extension to $L^p(\mathbb{R}^n, X)$ for all Banach spaces $X \in$ UMD, with an unconditional basis. The extension is defined as follows. Let $\{b_j\}_{j\in\mathbb{N}}$ be an unconditional basis in X, then let $\tilde{T}(\sum_j f_j(x)b_j) = \sum_j Tf_j(x)b_j$.

In particular, let T_b be defined by

$$\widehat{T_b f(\xi)} = \theta(\xi)e^{i|\xi|^b}|\xi|^{-nb/2}\hat{f}(\xi), \qquad 0 < b < 1,$$

where θ is a smooth radial cutoff function, $\theta(\xi) = 1$ if $|\xi| \geqslant 1$ and $\theta(\xi) = 0$ if $|\xi| \leqslant 1/2$. Then, using (2.15) of Chanillo [4], we see that T_b can be extended to a bounded operator

$$\tilde{T}_b : L^p(\mathbb{R}^n, X) \to L^p(\mathbb{R}^n, X)$$

for all Banach spaces $X \in$ UMD with an unconditional basis.

3. H^p Theory

We consider first the L^∞, BMO continuity of S.C.Z.O's.

(3.1) **Theorem.** *Let T be an operator associated with a standard kernel satisfying the $C_{p,q,\alpha}$ and $D_{r,\alpha}$ conditions. Then, for $s = \max\{p, r'\}$, we have*

$$(Tf)^\#(x) \leqslant cM_s f(x), \qquad f \in L_c^\infty(X). \tag{3.2}$$

PROOF. Fix a ball $B(z,\sigma)$ and write $f = f_1 + f_2$, where

$$f_1 = f\chi_{B(z,2\sigma^\alpha)}.$$

Let

$$c = \int k(z, y) f_2(y)\, dy,$$

then

$$\int_{B(z,\sigma)} \| Tf(x) - c \|\, dx \leqslant \int_{B(z,\sigma)} \| Tf_1(x) \|\, dx + \int_{B(z,\sigma)} \int_{\mathbb{R}^n / B(z, 2\sigma^\alpha)} \| k(x, y) - k(z, y) \|\, \| f(y) \|\, dy\, dx.$$

Dividing by $|B(z, \sigma)|$ we can estimate the two terms on the right hand side as follows. The first term is majorized using Hölder's inequality and the $C_{p,q,\alpha}$ condition. The second term is estimated by breaking up $\mathbb{R}^n / B(z, 2\sigma^\alpha)$ into disjoint annuli and estimating

$$\sup_{B(z,\sigma)} \sum_{j=1}^{\infty} \int_{C_j(z,\sigma^\alpha)} \| k(x, y) - k(z, y) \| \cdot \| f(y) \|\, dy$$

using Hölder's inequality and the $D_{r,\alpha}$ condition.

A very important special case of the above result obtains when T satisfies a $D_{1,\alpha}$ condition, in such case we have

(3.3) **Corollary.** *Let T satisfy, $C_{p,q,\alpha}$ and $D_{1,\alpha}$, then T can be defined as a continuous operator, $T: L^\infty(X) \to \mathrm{BMO}(Y)$.*

PROOF. Observe that $r = 1$ implies that $s = \infty$. Therefore, the estimate (3.2) can be written as

$$(Tf)^{\#}(x) \leqslant c \| f \|_\infty.$$

This gives the desired estimate when $f \in L_c^\infty(X)$. Moreover, it follows from the proof of (3.1) that Tf has a meaning for any $f \in L^\infty(X)$.

Our next result considers S.C.Z.O's as operators from $H^p(X)$ into $L^p(Y)$ spaces. The proof is the same as the one given in [1].

(3.4) **Theorem.** *Let T be an operator associated with a kernel k that satisfies a D_α condition and moreover suppose that T extends to a bounded operator, $T: L^2(X) \to L^2(Y)$, and from $L^q(X) \to L^2(Y)$,*

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n},$$

for some

$$(1-\alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2}.$$

Let

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta\left(\frac{\delta}{\alpha} + \frac{n}{2}\right)}{n\left(\frac{\delta}{\alpha} - \delta + \beta\right)},$$

and let $1 \geqslant p > p_0$, *then* T *maps* $H^p(X)$ *into* $L^p(Y)$.

In order to state the main result of this section we shall consider reflexive Banach spaces, X, Y. Then if T is associated with a kernel k, T^* can be defined as a linear continuous operator from $S(\mathbb{R}^n, Y')$ into $S(\mathbb{R}^n, X)'$ and T^* is associated with $k(y,x)^*$: $\mathbb{R}^{2n}/\text{diagonal} \to L(Y', X')$. Observe that if this kernel satisfies a $D_{1,\alpha}$ condition, then given $f \in L^\infty(Y') \cap C^\infty(Y')$, $T^*(f)$ has a meaning as a continuous linear functional over $D_0(X) = \left\{\varphi \in C_0^\infty(X): \int \varphi = 0\right\}$ (cf. [1]).

(3.5) **Theorem.** *Let* X, Y *be reflexive Banach spaces, let* T, T^* *satisfy the conditions of* (3.4). *Moreover, suppose that* $T^*(\bar{e}) = 0$, $\forall e \in Y'$, *and* $\bar{e}$: $\mathbb{R}^n \to Y'$ *given by* $\bar{e}(x) = e$. *Let*

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta\left(\frac{\delta}{\alpha} + \frac{n}{2}\right)}{n\left(\frac{\delta}{\alpha} - \delta + \beta\right)},$$

then for $p_0 < p \leqslant 1$, T *is a continuous operator,* $T: H^p(\mathbb{R}^n, X) \to H^p(\mathbb{R}^n, Y)$.

PROOF. As in the scalar valued situation treated in detail in [1], one shows that Ta, the image of a $(p,2)$ atom a, is a suitable molecule. The only new technical problem is to check that $\int Ta(x)\,dx = 0$. This can be done as follows. Let a be a $(p,2)$ atom supported on a ball B, then $|B|^{-1+1/p}a$ is a (1,2) atom and given $e \in Y'$,

$$\begin{aligned} |B|^{-1+1/p}\left\langle e, \int Ta(x)\,dx\right\rangle &= \int \langle \bar{e}(x), T(|B|^{-1+1/p}a)(x)\rangle\,dx \\ &= \int \langle T^*(\bar{e}), |B|^{-1+1/p}a\rangle\,dx = 0 \end{aligned}$$

where in the last line we have used the fact that $H^1(X)' = \text{BMO}(X')$ (cf. [14]) and that T^* maps $L^\infty(Y')$ into $\text{BMO}(X')$.

(3.6) *Remark.* If in the assumptions of (3.5) we have $\alpha = 1$, then we can let $\beta \to 0$ and we recover the well known fact that Calderón Zygmund operators are continuous from $H^p(X)$ into $H^p(Y)$ when $1 \geqslant p > n/(n+\delta)$ and $T^*(\bar{e}) = 0$, $e \in Y'$ (cf. [1]). Moreover, the cancellation condition can be dropped in the case of convolution operators, but in general is necessary. In fact, given a function φ slowly increasing with all derivatives, the operator $T_\varphi(f) = \varphi f$ is a linear continuous operator from $S(\mathbb{R}^n, \mathbb{C}) \to S(\mathbb{R}^n, \mathbb{C})'$, associated to the standard kernel identically zero. Moreover, $T^*_\varphi(1) = T_{\bar{\varphi}}(1) = \bar{\varphi}$. But $T_\varphi(f)$ will not have vanishing integral for any $f \in H^p(\mathbb{C})$, unless φ is a constant, i.e., $T^*_\varphi(1) = 0$ in the BMO sense.

Finally we consider continuity at the critical index p_0. The appropriate smoothness conditions on the kernel are given by

$$\int_{C_j(z,\sigma)} \|k(x,y) - k(x,z)\|\, dx \leqslant c2^{-j\delta} \tag{3.7}$$

if $\sigma > 1$, $|y - z| < \sigma$.

$$\int_{C_j(z,\sigma^\gamma)} \|k(x,y) - k(x,z)\|\, dx \leqslant c2^{-j\delta/\alpha}\sigma^{\delta(1-\gamma/\alpha)} \tag{3.8}$$

if $\sigma < 1$, $\gamma \leqslant \alpha$, $|y - z| < \sigma$.

Let us observe that if T is associated with a kernel satisfying the D_α condition then k satisfies (3.7) and (3.8).

(3.9) **Theorem.** *Let T be an operator associated with a kernel satisfying* (3.7) *and* (3.8). *Moreover, suppose that T extends to a bounded operator, $T: L^2(X) \to L^2(Y)$, and from $L^q(X)$ into $L^2(Y)$,*

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n},$$

for some

$$(1-\alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2}.$$

Then, $T: H^p(X) \to L^p(Y)$, $p_0 \leqslant p \leqslant 1$, where

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta\left(\dfrac{\delta}{\alpha} + \dfrac{n}{2}\right)}{n\left(\dfrac{\delta}{\alpha} - \delta + \beta\right)}.$$

PROOF. Fix $p_0 \leqslant p \leqslant 1$ and let a be a (p, ∞) atom supported on $B(z, \sigma)$. Suppose first that $\sigma > 1$, then

$$\begin{aligned}\int \|Ta(x)\|^p\,dx &\leqslant \int_{B(z,2\sigma)} \|Ta(x)\|^p\,dx \\ &\quad + \sum_{j=1}^{\infty} \int_{C_j(z,\sigma)} \left(\int_{B(z,\sigma)} \|k(x,y) - k(x,z)\| \cdot \|a(y)\|\,dy\right)^p dx \\ &= I_1 + I_2.\end{aligned}$$

Using Hölder's inequality with exponent $2/p$ and the L^2 continuity property, we can estimate I_1 by a constant. To take care of I_2 we use Hölder's inequality with exponent $1/p$ and (3.7), obtaining

$$I_2 \leqslant c \sum_j 2^{j(n-(n+\delta)p)} \leqslant c$$

since

$$1 \geqslant p \geqslant p_0 > \frac{n}{n+\delta} > \frac{n}{n+\frac{\delta}{\alpha}}.$$

Next we consider the case where $\sigma < 1$. Then,

$$\begin{aligned}\int \|Ta(x)\|^p\,dx &\leqslant \int_{B(z,2\sigma^\gamma)} \|Ta(x)\|^p\,dx \\ &\quad + \sum_{j=1}^{\infty} \int_{C_j(z,\sigma^\gamma)} \left(\int_{B(z,\sigma)} \|k(x,y) - k(x,z)\| \cdot \|a(y)\|\,dy\right)^p dx \\ &= I_1 + I_2\end{aligned}$$

with γ to be determined precisely later.

The first term can be majorized using Hölder's inequality with exponent $2/p$ and the L^q, L^2 continuity property,

$$I_1 \leqslant c\sigma^{n[\gamma(1-p/2)+p/q-1]}.$$

Thus, since $\sigma < 1$ and

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n},$$

I_1 will be bounded whenever γ satisfies

$$\gamma \geqslant \frac{1 - \frac{p}{2} - p\frac{\beta}{n}}{1 - \frac{p}{2}}. \tag{3.10}$$

On the other hand, using Hölder's inequality with exponent $1/p$ and condition (3.8), we get

$$I_2 \leqslant c\sigma^{-\gamma[p(n+\delta/\alpha)-n]+p(n+\delta)-n} \sum_{j=1}^{\infty} 2^{j[n-p(n+\delta/\alpha)]}.$$

Thus I_2 will be bounded whenever γ satisfies

$$\gamma \leqslant \frac{p(n+\delta)-n}{p\left(n+\frac{\delta}{\alpha}\right)-n}. \tag{3.11}$$

For the extreme value $p = p_0$ the right hand sides of (3.10) and (3.11) have the common value

$$\gamma_0 = \frac{\frac{n}{2}+\delta-\beta}{\frac{\delta}{\alpha}+\frac{n}{2}} \leqslant \alpha,$$

where the last inequality follows from our assumption $(1-\alpha)n/2 \leqslant \beta$. Observe that the right hand side of (3.10) decreases as $p \to 1$ and equals $1 - 2\beta/n$ when $p = 1$. Moreover, the right hand side of (3.11) increases as $p \to 1$ and equals α when $p = 1$. Consequently the choice $\gamma = \gamma_0$ will satisfy (3.10) and (3.11), $\forall 1 \geqslant p \geqslant p_0$.

(3.12) *Remark.* It would be interesting to obtain (3.9) imposing only a $D_{1,\alpha}$ condition on the kernel. Observe that when $\beta = (1-\alpha)n/2$ then $p_0 = 1$ and the (H^1, L^1) continuity can be obtained assuming a $D_{1,\alpha}$ condition (cf. [1], Theorem 1.2 *c*)).

4. Weak Type L^1 and L^p Inequalities

Let us observe that if T is an operator satisfying the conditions of (3.9) we can use intepolation (cf. (2.2) above) to obtain $T: L^1(X) \to L(1,\infty)(Y)$. In [14] the weak type (1,1) of Calderón-Zygmund operators (i.e., $\alpha = 1$, $\beta = 0$) was obtained for kernels satisfying only a $D_{1,1}$ condition. We now show how C. Fefferman's approach for convolution operators (cf. [8]) can be adapted to obtain the weak type (1, 1) estimates under sharp assumptions.

(4.1) **Theorem.** *Let T be an operator associated with a kernel satisfying the $D_{1,\alpha}$ condition. We also assume, moreover, that T extends to bounded*

operators, $T: L^2(X) \to L^2(Y)$, *and* $T: L^q(X) \to L^2(Y)$,

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}, \qquad (1-\alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2}.$$

Then, $T: L^1(X) \to L(1,\infty)(Y)$.

PROOF. Let $f \in L^1(X)$, $\lambda > 0$. Consider a Calderón-Zygmund decomposition of f at level λ. Thus,

$$\Omega = \{Mf(x) > \lambda\} = \bigcup_{j=1}^{\infty} Q_j,$$

where the Q_j's are cubes and such that if f_{Q_j} denotes the mean value of f over Q_j, $f = f' + f''$, where

$$f' = f\chi_{\Omega^c} + \sum_{j=1}^{\infty} f_{Q_j}\chi_{Q_j}$$

$$f'' = \sum_{j=1}^{\infty} (f - f_{Q_j})\chi_{Q_j} = \sum_{j=1}^{\infty} f_j.$$

Moreover,

$$\|f'(x)\| \leqslant c\lambda \quad \text{a.e.}, \quad \|f'\|_1 \leqslant \|f\|_1, \tag{4.2}$$

where $\| \ \|_1$ denotes the norm in $L^1(X)$.

$$\int_{Q_j} \|f_j(x)\|\, dx \leqslant c|Q_j|\lambda, \qquad \int f_j(x)\, dx = 0 \tag{4.3}$$

$$|\Omega| \leqslant \frac{c}{\lambda}\|f\|_1. \tag{4.4}$$

From (4.2), $f' \in L^2(X)$ and $\|f'\|_2^2 \leqslant c\lambda\|f\|_1$. So,

$$\lambda^2|\{x/\|Tf'(x)\| > \lambda/2\}| \leqslant c\|Tf'\|_2^2 \leqslant c\lambda\|f\|_1.$$

On the other hand, if $c\Omega$ denotes the union of the cubes cQ_j with the same center as Q_j and c times the side length, we have, according to (4.4)

$$|\{x \in c\Omega/\|Tf''(x)\| > \lambda/2\}| \leqslant |c\Omega| \leqslant \frac{c}{\lambda}\|f\|_1.$$

So, it remains to prove that

$$|\{x \in \mathbb{R}^n \backslash c\Omega/\|Tf''(x)\| > \lambda/2\}| \leqslant \frac{c}{\lambda}\|f\|_1.$$

Let us denote σ_j the diameter of the cube Q_j. Let

$$F = \sum_{\sigma_j < 1} f_j, \qquad G = \sum_{\sigma_j \geq 1} f_j.$$

If z_j is the center of the cube Q_j, for a suitable c we will have $|x - z_j| > 4\sigma_j$ if $x \in \mathbb{R}^n \backslash c\Omega$. According to (4.3), we can write

$$\lambda |\{x \in \mathbb{R}^n \backslash c\Omega / \|TG(x)\| > \lambda/4\}| \leqslant \int_{\mathbb{R}^n \backslash c\Omega} \|TG(x)\| \, dx$$
$$\leqslant \sum_j \int_{|x - z_j| > 2\sigma_j^{\alpha}} \int_{Q_j} \|k(x, y) - k(x, z_j)\| \cdot \|f_j(y)\| \, dy \, dx.$$

Using condition $D_{1,\alpha}$, this can be majorized by

$$c \sum_j \int_{Q_j} \|f_j(y)\| \, dy \leqslant c \|f\|_1.$$

Now, let $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ a scalar function such that

$$\operatorname{supp} \varphi \subset \{|x| \leqslant 1/c\}, \qquad \int \varphi \, dx = 1, \qquad \varphi \geqslant 0.$$

Let

$$\varphi_j(x) = \frac{1}{\sigma_j^{n/\alpha}} \varphi(x/\sigma_j^{1/\alpha}).$$

We will write

$$F = \sum f_j * \varphi_j + \sum (f_j - f_j * \varphi_j) = F' + F''.$$

If $x \in \mathbb{R}^n \backslash c\Omega$,

$$T(f_j - f_j * \varphi_j)(x) = \int k(x, y) f_j(y) \, dy - \int k(x, w) \int \varphi_j(w - y) f_j(y) \, dy \, dw.$$

Since $k(x, w)$ is a $L(X, Y)$-valued continuous function away from the diagonal, we can change the order of integration in the second term above. Moreover, $\int \varphi_j(w - y) \, dw = 1$ for any y. So, we can write

$$\int \left[\int (k(x, y) - k(x, w)) \varphi_j(w - y) \, dw \right] f_j(y) \, dy.$$

Then,

$$\lambda |\{x \in \mathbb{R}^n \backslash c\Omega / \|TF''(x)\| > \lambda/8\}| \leqslant c \int_{\mathbb{R}^n \backslash c\Omega} \|TF''(x)\| \, dx$$
$$\leqslant \sum_j \int_{\mathbb{R}^n \backslash cQ_j} \int \left[\int \|k(x, y) - k(x, w)\| \varphi_j(w - y) \, dw \right] \|f_j(y)\| \, dy \, dx.$$

If $x \in \mathbb{R}^n \backslash cQ_j$, $|y - w| < \sigma_j^{1/\alpha}$, we have

$$|x - w| \geqslant |x - z_j| - |z_j - y| - |y - w| > 4\sigma_j - \sigma_j - \sigma_{j/c}^{1/\alpha} > 2\sigma_j,$$

since $\alpha < 1$, $\sigma_j < 1$.

So, we can majorize the above sum as

$$\sum_j \int_{Q_j} \int_{|y-w|<\sigma_\sigma^{1/\alpha}} \int_{|x-w|>2\sigma_j} \|k(x,y) - k(x,w)\|\, dx\, \varphi_j(w-y)\, dw\, \|f_j(y)\|\, dy.$$

Using again condition $D_{1,\alpha}$ and the fact that $\int \varphi_j(w-y)\, dw = 1$ for any y, j, we obtain

$$\sum_j \int_{Q_j} \|f_j(y)\|\, dy \leqslant c\|f\|_1.$$

Now, it remains to prove that

$$\lambda|\{x \in \mathbb{R}^n \backslash c\Omega \,\|TF'(x)\| > \lambda/8\}| \leqslant c\|f\|_1. \tag{4.5}$$

Let J^β denote the Bessel potential of order β. Observe that these operators are positive and therefore admit a vector valued extension. Thus, our program now will be to obtain the estimate

$$\|J^\beta F'\|_2^2 \leqslant A\lambda\|f\|_1. \tag{4.6}$$

In fact, from this estimate it follows according to the continuity assumptions on T that

$$\|TF'\|_2^2 = \|TJ^{-\beta}J^\beta F'\|_2^2 \leqslant \|J^\beta F'\|_2^2 \leqslant A\lambda\|f\|_1.$$

Thus, (4.5) follows using Chebyshev's inequality.

Now, let us recall that

$$F' = \sum_{\sigma_j<1} f\chi_{Q_j} * \varphi_j - \sum_{\sigma_j<1} f_{Q_j}\chi_{Q_j} * \varphi_j.$$

For x fixed, we will write $x \sim Q_j$ if x belongs to the closure of any cube Q_l which is adjacent or coincides with Q_j.

Now,

$$J^\beta \sum_{\sigma_j<1} f\chi_{Q_j} * \varphi_j(x) = \sum_{x\sim Q_j} J^\beta f\chi_{Q_j} * \varphi_j(x) + \sum_{x\not\sim Q_j} J^\beta f\chi_{Q_j} * \varphi_j(x) = F_1(x) + F_2(x).$$

So that,

$$(J^\beta F')(x) = F_1(x) + F_2(x) - \sum_{\sigma_j<1} f_{Q_j} J^\beta(\chi_{Q_j} * \varphi_j)(x).$$

With the same proof as in Fefferman's proof, we can show that if $x \notin Q_j$,

$$\|J^\beta f\chi_{Q_j} * \varphi_j(x)\| \leqslant cJ^\beta \|f\|_{Q_j} \chi_{Q_j} * \varphi_j(x).$$

So, according to (4.3), we have

$$\begin{aligned}\left\|F_2(x) - \sum_{\sigma_j<1} J^\beta f_{Q_k}\chi_{Q_j} * \varphi_j(x)\right\| &\leqslant c\lambda \sum_{\sigma_j<1} J^\beta \chi_{Q_j} * \varphi_j(x)\\ &\leqslant c\lambda \|J^\beta\|_1 \left\| \sum_{\sigma_j<1} \chi_{Q_j} * \varphi_j \right\|_\infty\\ &\leqslant c\lambda\end{aligned}$$

since the supports of $\chi_{Q_j} * \varphi_j$ have finite intersection.

On the other hand,

$$\begin{aligned}\left\|F_2 - \sum_{\sigma_j<1} J^\beta f_{Q_j}\chi_{Q_j} * \varphi_j\right\|_1 &\leqslant \sum_{\sigma_j<1} \|J^\beta \varphi_j * (f - f_{Q_j})\chi_{Q_j}\|_1\\ &\leqslant c\sum \int_{Q_j} \|f\|\, dy \leqslant c\|f\|_1.\end{aligned}$$

Thus, we get from the L^1 and L^∞ inequalities

$$\left\|F_2 - \sum_{\sigma_j<1} J^\beta f_{Q_j}\chi_{Q_j} * \varphi_j\right\|_2^2 \leqslant c\lambda\|f\|_1.$$

It only remains to show that $\|F_1\|_2^2 \leqslant c\lambda\|f\|_1$. This can be also done in the same way as in Fefferman's proof.

For x fixed, let

$$F_1^j(x) = \begin{cases} J^\beta f\chi_{Q_j} * \varphi_j(x) & \text{if } \ x \sim Q_j \\ 0 & \text{if not.}\end{cases}$$

Thus,

$$F_1(x) = \sum_{\sigma_j<1} F_1^j(x).$$

Moreover, for a fixed x, $F_1^j(x) \neq 0$ for at most N values of j. So,

$$\begin{aligned}\|F_1(x)\|_Y^2 &\leqslant \left(\sum_{j=1}^N \|F_1^{j(x)}(x)\|_Y\right)^2 = \sum_{j,h=1}^N \|F_1^{j(x)}(x)\|_Y \|F_1^{h(x)}(x)\|_Y\\ &\leqslant 2\left(\sum_{j,h=1}^N \|F_1^{j(x)}(x)\|_Y^2 + \|F_1^{h(x)}(x)\|_Y^2\right)\\ &\leqslant 4N \sum_{\sigma_j<1} \|F_1^j(x)\|_Y^2.\end{aligned}$$

So that,

$$\|F_1\|_2^2 \leqslant 4N \sum \|F_1^j\|_2^2 \leqslant 4N \sum_j \|J^\beta \varphi_j\|_2^2 \|f\chi_{Q_j}\|_1^2$$

$$\|J^\beta \varphi_j\|_2^2 = \int 1 + (|\xi|^2)^{-2\beta} |\hat{\varphi}(\sigma_j^{1/\alpha}\xi)|^2 \, d\xi \leqslant \frac{c}{|Q_j|},$$

since $|Q_j| \leqslant 1$.

Finally,

$$\|F_1\|_2^2 \leqslant c \sum_j \frac{1}{|Q_j|} \|f\chi_{Q_j}\|_1^2 \leqslant c\lambda \|f\|_1.$$

This completes the proof of the theorem.

Let us note in passing that we actually needed less than condition $D_{1,\alpha}$ on the kernel. In fact, what we really used is

$$\int_{|x-z|>c\sigma^\alpha} \|k(x,y) - k(x,z)\| \, dx \leqslant c, \qquad 0 < \sigma < 1$$

$$\int_{|x-z|>c\sigma} \|k(x,y) - k(x,z)\| \, dx \leqslant c, \qquad \sigma > 1$$

for $|y - z| < \sigma$.

Interpolating the results of §3 and (4.1), we can derive L^p continuity theorems and we shall leave details to the interested reader. Also, in the standard way, (3.2) gives weighted L^p inequalities. S. Chanillo proved in [4] sharp estimates for T_b.

Supposing the condition in Theorem 3.1 with $p = r' < \infty$, (3.2) reads

$$(Tf)^\#(x) \leqslant c_p M_p(f)(x).$$

It is well known that this inequality holds for any $p > 1$ when T is a Calderón-Zygmund operator. It is an open problem to know whether this is true or not for a general S.C.Z. Operator.

5. Maximal Inequalities

Let T be an operator associated with a standard kernel k. We assume that T satisfies (2.5) and (2.5)′ and moreover that the kernel k satisfies a D_∞ condition. Moreover, suppose that k satisfies the estimate

$$\|k(x,y)\| \leqslant \frac{c}{|x-y|^n}. \tag{5.1}$$

Let T_* denote the maximal operator defined by

$$T_*f(x) = \sup_{\epsilon>0} \|T_\epsilon f(x)\|_Y$$

where

$$T_\epsilon f(x) = \int_{|x-y|>3\epsilon^\alpha} k(x,y)f(y)\,dy.$$

Then, the following extension of Cotlar's estimates holds (cf. [5]).

(5.2) **Theorem.** *Let T be an operator satisfying the conditions described above, then $\forall f \in C_0^\infty(X)$,*

$$T_*f(x) \leqslant c(M(Tf)(x) + M_p f(x)) \tag{5.3}$$

where $1 < p < \infty$ is given by condition (2.5)'.

For the proof we require the following Lemma (cf. [5]).

(5.4) **Lemma.** *Let $f \in L^1(\mathbb{R}^n, X)$, $0 < \delta \leqslant 1$, $0 < \alpha \leqslant 1$. Then, there exists a constant $c > 0$ such that*

$$\text{(i)} \qquad \sup_{0<\epsilon<1} \epsilon^\delta \int_{|x-y|>3\epsilon^\alpha} \frac{\|f(y)\|}{|x-y|^{n+\delta/\alpha}}\,dy \leqslant cM(f)(x)$$

$$\text{(ii)} \qquad \sup_{|x-z|<\epsilon<1} \int_{|x-y|>3\epsilon^\alpha} \|k(x,y) - k(z,y)\|\,\|f(y)\|\,dy \leqslant cM(f)(x).$$

PROOF. (i) can be readily derived using polar coordinates as in [5], Lemma 4. Alternatively,

$$\begin{aligned}
\epsilon^\delta \int_{|x-y|>3\epsilon^\alpha} \frac{\|f(y)\|}{|x-y|^{n+\delta/\alpha}}\,dy &= \sum_{j=0}^{\infty} \epsilon^\delta \int_{3\epsilon^\alpha 2^{j+1} > |x-y| > 3\epsilon^\alpha 2^j} \frac{\|f(y)\|}{|x-y|^{n+\delta/\alpha}}\,dy \\
&\leqslant c \sum_{j=0}^{\infty} \frac{\epsilon^\delta}{(2^j\epsilon^\alpha)^{n+\delta/\alpha}} \int_{B(x,\,3\epsilon^\alpha 2^{j+1})} \|f(y)\|\,dy \\
&\leqslant c\left(\sum_{j=0}^{\infty} 2^{-j(\delta/\alpha)}\right) Mf(x).
\end{aligned}$$

(ii) Observe that if $|x-z| < \epsilon < 1$ and $|x-y| > 3\epsilon^\alpha$, then

$$|y-z| > |x-y| - |x-z| > 3\epsilon^\alpha - \epsilon > 2\epsilon^\alpha > 2|x-z|^\alpha,$$

and moreover,

$$|y-z| \geqslant |x-y| - \epsilon^{\alpha} > \frac{2}{3}|x-y|.$$

Consequently, using the D_∞ condition we get

$$\int_{|x-y|>3\epsilon^{\alpha}} \|k(x,y)-k(z,y)\|\,\|f(y)\|\,dy \leqslant c\int_{|x-y|>3\epsilon^{\alpha}} \frac{|x-z|^{\delta}}{|x-y|^{n+\delta/\alpha}}\|f(y)\|\,dy$$

$$\leqslant c\epsilon^{\delta}\int_{|x-y|>3\epsilon^{\alpha}} \frac{\|f(y)\|}{|x-y|^{n+\delta/\alpha}}\,dy$$

and the assertion follows from the first part of the lemma.

PROOF OF (5.2). Except for minor details, the proof is a modification of the classical proof of Cotlar (cf. [5], p. 97). We provide the details for the reader's convenience. Consequently, we shall consider only the case where $0<\epsilon<1$. Given $f\in C_0^{\infty}(\mathbb{R}^n, X)$, $0<\epsilon<1$, $x\in\mathbb{R}^n$, let $f=f_1+f_2$, where $f_1=f\chi_{B(x,2\epsilon^{\alpha})}$, $f_2=(1-\chi_{B(z,2\epsilon^{\alpha})})f$. Then, $(T_\epsilon f)(x)=Tf_2(x)$. Next, we estimate the values of Tf_2 at other points in $B(x,\epsilon)$, In fact, if $z\in B(x,\epsilon)$, then

$$|z-y| \geqslant |x-y| - |x-z| > 2\epsilon^{\alpha}.$$

So,

$$Tf_2(x) - Tf_2(z) = \int_{|x-y|>3\epsilon^{\alpha}} [k(x,y)-k(z,y)]f(y)\,dy.$$

Thus, by (5.4) (ii) this term can be majorized by $cMf(x)$.

Consequently, if $z\in B(x,\epsilon)=B$,

$$\|T_\epsilon f(x)\| = \|Tf_2(x)\| \leqslant \|Tf_2(x)-Tf_2(z)\| + \|Tf_2(z)\|$$

$$\leqslant cMf(x) + \|Tf_1(z)\| + \|Tf(z)\|$$

since $Tf_2(z)=Tf(z)-Tf_1(z)$.

In order to prove (5.3) we may assume that $\|Tf_2(x)\|>0$. Let $0<\lambda<\|Tf_2(x)\|$ be fixed. Define the sets

$$E_1=\left\{z\in B \,\middle|\, \|Tf(z)\|>\frac{\lambda}{3}\right\},$$

$$E_2=\left\{z\in B \,\middle|\, \|Tf_1(z)\|>\frac{\lambda}{3}\right\},$$

$$E_3=\phi \quad \text{if} \quad cMf(x)<\frac{\lambda}{3} \quad \text{or} \quad E_3=B \quad \text{otherwise.}$$

Let $d\mu$ be the probability measure $|B|^{-1}\chi_B\,dx$, then

$$1 \leqslant \mu(B) \leqslant \mu(E_1) + \mu(E_2) + \mu(E_3).$$

In order to evaluate $\mu(E_i)$ we use Chebychev's inequality. Trivially we get

$$\mu(E_1) \leqslant c(MTf)(x), \qquad \mu(E_3) \leqslant cMf(x).$$

The second term can be estimated using (2.5) as follows

$$\begin{aligned}
\mu(E_2) = \frac{|E_2|}{|B|} &\leqslant \frac{3}{\lambda}\,\frac{1}{|B|}\int_B \|Tf_1(z)\|\,dz \\
&\leqslant \frac{3}{\lambda}\left(\frac{1}{B}\int_B \|Tf_1(z)\|^q\,dz\right)^{1/q} \\
&\leqslant \frac{3c}{\lambda}\,|B|^{-1/q}\left(\int_{B(x,\,3\epsilon^\alpha)} \|f(z)\|^p\,dz\right)^{1/p} \\
&\leqslant \frac{3c}{\lambda}\,\epsilon^{-n/q+n\alpha/p}M_pf(x) \\
&\leqslant \frac{3c}{\lambda}M_pf(x)
\end{aligned}$$

since $p/q \leqslant \alpha$. Consequently,

$$\lambda < c[M(Tf)(x) + M_pf(x)]$$

and taking the sup over all $0 < \lambda < \|Tf_2(x)\|$ in the above inequality, we obtain (5.3).

(5.6) *Remark.* We would like to point out that T_* can be also viewed as a vector valued strongly singular operator. In fact if we let

$$T_\epsilon f(x) = \int_{|y|>\epsilon} k(x, y)f(x - y)\,dy,$$

where

$$k\colon \mathbb{R}^n \times \mathbb{R}^n \sim \{0\} \to L(X, Y)$$

is continuous and satisfies:

(i) $k(x, w) = 0$ for $|x| \geqslant 1$,

(ii) $\|k(x, w)\| \leqslant \dfrac{c}{|w|^n}$

and

(iii) $\int_{|x|>2\sigma^\alpha} \|k(x+z, x-y) - k(x+z, x)\|\, dx < c$

for some $0 < \alpha < 1$, $z \in \mathbb{R}^n$, $|y| < \sigma$, then

$$T_* f(x) \leqslant c\Big[Mf(x) + \Big\| \int k_\epsilon(x, y) f(x-y)\, dy \Big\|_{L^\infty(Y)}\Big].$$

Here, k_ϵ is a smooth cutoff of the kernel k given by

$$k_\epsilon(x, y) = k(x,y)\varphi\left(\frac{y}{\epsilon}\right),$$

where $\varphi \in C^\infty$ and $\varphi \equiv 0$ near the origin and $\varphi \equiv 1$ for $|x| \geqslant 2$, say. Moreover, it can be shown using the methods of [14] that $k_\epsilon(x, y)$ satisfies a $D_{1,\alpha}$ condition.

(5.7) *Remark.* Using (5.2), we can readily obtain weighted norm inequalities of the following type. If T is an operator satisfying the conditions of (5.2), then $T_*: L^r(w(x)\,dx, X) \to L^r(w(x)\,dx, Y)$, $\forall p < r < \infty$, $\forall w \in A_{r/p}$. In the special case of Calderón-Zygmund operators, the result remains valid for $1 < r < \infty$.

6. Applications

The H^p inequalities obtained in §3 are new also for vector valued Calderón-Zygmund operators, extending the results of [14]. Using the ideas of [14], we can also derive results of the following type («self improving»).

(6.1) **Theorem.** (i) *Let T be an operator associated with a standard kernel satisfying* (3.7) $\forall \sigma > 1$. *Suppose that T is continuous from $L^{r_0}(\mathbb{R}^n, X)$ into $L^{r_0}(\mathbb{R}^n, Y)$ for some $1 < r_0 \leqslant \infty$. Then,*

$$\left\| \left(\sum_{j=1}^{\infty} \|Tf_j\|^r \right)^{1/r} \right\|_p \leqslant c \|\{f_j\}\|_{H^p(\mathbb{R}^n, l^r(X))} \tag{6.2}$$

for $n/(n+\delta) < p \leqslant 1 < r \leqslant r_0 \leqslant \infty$.

(ii) *Assume, moreover, that T satisfies the cancellation conditions of* (3.6). *Then*

$$\|\{Tf_j\}\|_{H^p(\mathbb{R}^n, l^r(Y))} \leqslant c \|\{f_j\}\|_{H^p(\mathbb{R}^n, l^r(Y))} \tag{6.3}$$

for $n/(n+\delta) < p \leqslant 1 < r \leqslant r_0 < \infty$.

PROOF. We prove only (i), the proof of (ii) being similar. To do this observe that the operator $\tilde{T}\{f_j\} = \{Tf_j\}_{j=1}^{\infty}$, $\{f_j\} \in L_c^{\infty}(\mathbb{R}^n, l^r(X))$ is a Calderón-Zygmund operator associated with a kernel satisfying a $D_{1,1}$ condition with $d_j = 2^{-j\delta}$. Moreover, it is continuous from $L^r(\mathbb{R}^n, l^r(X))$ into $L^r(\mathbb{R}^n, l^r(Y))$. Thus, by (3.9) conveniently adapted for the case $\alpha = 1$, we obtain (6.2).

Suppose T is an operator associated to a standard kernel satisfying a $D_{1,\alpha}$ condition. Moreover, suppose that T maps continuously $L^2(X)$ into $L^2(Y)$ and $L^2(X)$ into $L^{q'}(Y)$, $1/q' = 1/2 - \beta/n$. Then, if $T(\bar{e}) = 0$, Theorem 3.5 shows that T^* maps continuously $H^1(Y')$ into $H^1(X')$.

That is to say, T maps continuously BMO(X) into BMO(Y). When $\alpha = 1$, $\beta = 0$, we get in particular the BMO-continuity of Calderón-Zygmund operators under the suitable cancellation condition. Once again, this cancellation condition can be dropped in the convolution case.

We shall consider some applications to pseudo-differential operators. Firstly, observe that pseudo-differential operators defined by amplitudes $a(x, y, \xi)$ in the class $S_{1,\theta}^0$, $0 \leqslant \theta < 1$, are Calderón-Zygmund operator satisfying conditions (S_2) of §2 with $\alpha = \delta = 1$ (i.e., a D_1 condition) (cf. [2]). Thus, our results apply for these operators. We obtain in particular the results of [13]. Moreover, pseudo-differential operators with amplitudes in $S_{\alpha,\delta}^{-\beta}$,

$$(1 - \alpha)\frac{n}{2} \leqslant \beta < \frac{n}{2}, \qquad 0 < \delta \leqslant \alpha < 1,$$

are strongly singular Calderón-Zygmund operators satisfying a somewhat weaker $D_{1,\alpha}$ condition (cf. [1]).

Lemma 1.2 in [14] essentially proves that these operators also satisfy a $D_{2,\alpha}$ condition. Thus, using (3.2) with $p = r = 2$, gives

$$(Tf)^{\#}(x) \leqslant cM_2(f)(x), \tag{6.4}$$

an estimate derived by S. Chanillo and A. Torchinsky (cf. [3]). In [3] it is also asked whether the index 2 is the smallest possible one in the estimate (6.4). For example, the authors show that for a special class of symbols in $S_{\alpha,\delta}^{-n(1-\alpha)/2}$, M_2 can be replaced in (6.4) by M_p, for any $1 < p < \infty$.

Since one knows that operators T with symbols in $S_{\alpha,\delta}$, $(1 - \alpha)n/2 \leqslant \beta < n/2$ are continuous in L^p for any $1 < p < \infty$, we can improve (6.4) each time we can assure that T maps continuously L^p into L^q, for some $p < 2$, $p/q \leqslant \alpha$.

For example, L. Hörmander has proved in [11] that an operator with symbol in $S_{\rho,\delta}^m$, is continuous from L^p into L^q, if

$$m < -n\left(\frac{1}{p} - \frac{1}{q}\right), \qquad 1 < p \leqslant 2 \leqslant q < \infty \tag{6.5}$$

$$(6.5)' \qquad m < -n\left[\frac{1}{p} - \frac{1}{q} + (1-\rho)\left(\frac{1}{q} - \frac{1}{2}\right)\right], \qquad 1 < p \leqslant q \leqslant 2.$$

In our case is $m = -\beta$. Let us fix any $1 < p < 2$, $q = p/\alpha$ and suppose $0 < \alpha < p/2$. Thus, (6.5) reads

$$\beta > \frac{n(1-\alpha)}{p}.$$

So, (6.4) holds with 2 replaced by p, $1 < p < 2$, if T is an operator with symbol in $S^{-n(1-\alpha)/p-\epsilon}_{\alpha,\delta}$, $\epsilon > 0$, provided $0 < \alpha < p/2$. *In particular, in* $S^{-n(1-\alpha)}_{\alpha,\delta}$, (6.4) holds with 2 replaced by any $p > 1$, provided $0 < \alpha < 1/2$.

When $1 > \alpha > p/2$, (6.5)′ yields to the condition

$$\beta > n(1-\alpha)\left(\frac{1+\alpha}{p} - \frac{1}{2}\right).$$

This time, (6.4) will hold for any $1 < p < 2$, if T has symbol in $S^{-n(1-\alpha)(1/2+\alpha)}_{\alpha,\delta}$ when $1/2 < \alpha < 1$.

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